Construction and multifractal analysis of random functions and their graphs

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N° d'ordre: 9745

THÈSE

Présentée pour obtenir

LE GRADE DE DOCTEUR EN SCIENCES
DE L'UNIVERSITÉ PARIS-SUD XI

Spécialité: Mathématiques

par

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Construction et analyse multifractale de fonctions aléatoires et de leurs graphes

Soutenue le 14 Janvier 2010 devant la Commission d'examen:

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Thèse préparée au
Projet SISYPHE
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Résumé

Cette thèse est consacrée à la construction et l'analyse multifractale de fonctions aléatoires et de leurs graphes.

La construction de ces objets se fait dans le cadre de la théorie des **T**-martingales de Kahane, et plus spécifiquement des $[0,1]$-martingales. Cette théorie est fréquemment utilisée pour construire des martingales à valeurs dans les mesures de Borel positives dont la limite soit presque sûrement singulière par rapport à la mesure de Lebesgue. Ceci se fait en perturbant cette dernière à l'aide d'une suite de densités aléatoires qui sont des martingales positives d'espérance 1. Ici, nous autorisons ces martingales à prendre des valeurs complexes, et plutôt que des martingales à valeurs dans les mesures, nous considérons des martingales à valeurs dans les fonctions continues à valeurs complexes, puis la question de leur convergence uniforme presque sûre. Nous obtenons une condition suffisante de convergence pour les éléments d'une large classe de $[0,1]$-martingales complexes. Les limites non dégénérées sont toutes candidates à être des fonctions multifractales. L'étude de leur nature multifractale révèle de nouvelles diffiultés. Nous la menons de façon complète dans le cas des "cascades b-adiques indépendantes" complexes. Ceci conduit à de nouveaux phénomènes. En particulier, nous construisons des fonctions continues statistiquement autosimilaires dont le spectre de singularité est croissant et entièrement supporté par l'intervalle $[0,\infty]$.

Nous considérons également de nouveaux spectres de singularité associés au graphe, à l'image, ainsi qu'aux ensembles de niveau d'une fonction multifractale f donnée. Ces spectres s'obtiennent de la façon suivante. Soit E_h l'ensemble iso-Hölder de f associé à l'exposant h. Soit Γ_h le sous-ensemble du graphe de f obtenu en y relevant E_h. Pour tout h, on cherche la dimension de Hausdorff de Γ_h, celle de $f(E_h)$, et celle des ensembles du type $\Gamma_h \cap L^y$, où L^y est l'ensemble de niveau y de f. Pour les cascades b-adiques indépendantes non conservatives à valeurs réelles, nous obtenons presque sûrement les spectres associés au graphe et à l'image, et pour les spectres associés aux ensembles de niveau, nous obtenons un résultat en regardant des lignes de niveau dans "Lebesgue presque toute direction".

Enfin, nous considérons les mêmes questions que précédemment pour une autre classe de foncions aléatoires multifractales obtenues comme séries d'ondelettes pondérées par des mesures de Gibbs. Nous obtenons presque sûrement les spectres associés au graphe et à l'image.

Mots-clefs : Fonctions aléatoires, chaos multiplicatif, dimension de Hausdorff, analyse multifractale, graphe, image, ensembles de niveaux.

CONSTRUCTION AND MULTIFRACTAL ANALYSIS OF RANDOM FUNCTIONS AND THEIR GRAPHS

Abstract

This thesis deals with the construction and multifractal analysis of random functions and their graphs.

At first, we contribute to Kahane's **T**-martingale theory by considering complex $[0,1]$-martingales. While until now this is done with positive $[0,1]$-martingales, in particular in order to build singular measures with respect to the Lebesgue, we construct complex continuous function-valued martingales and consider the question of their almost sure uniform convergence. We get a general sufficient condition for such a convergence to hold for the elements of a large subclass of $[0,1]$-martingales. All the non-degenerate limit functions are candidates to be multifractal. Their multifractal analysis reveals new difficulties. We conduct this multifractal analysis for complex "b-adic independent cascade functions". This study leads to new interesting phenomena. In particular, we build statistically self-similar continuous functions whose singularity spectrum is left-sided and supported by the whole interval $[0,\infty]$.

Further, we consider new singularity spectra associated with the graph, range and level sets of multifractal functions. These spectra consist in calculating the Hausdorff dimension of the iso-Hölder sets put on the graph, range and level sets. For real-valued b-adic independent cascade functions, with probability 1, we obtain the graph and range singularity spectra, as well as the level set singularity spectrum in "Lebesgue almost every directions", for the so-called non-conservative b-adic independent cascade functions.

Finally, we consider the same question for another class of random multifractal functions, namely random wavelet series built from Gibbs measures. Under suitable assumptions, we obtain the graph and range singularity spectra almost surely.

Keywords : Random functions, multiplicative chaos, Hausdorff dimension, multifractal analysis, graph, range, level sets.

Remerciements

First of all I would like to express my deepest gratitude to my supervisor, Professor Julien Barral, for his supervision, advice, and guidance. I gratefully thank him for his continuous encouragement and support during the last three years. I would also like to thank him for providing me with many interesting problems, ideas and inspirations, as well as his great efforts in achieving this thesis. I learnt so much from him, and his enthusiasm in mathematics greatly inspires me.

I would like to thank gratefully my another supervisor, Professor Zhiying Wen, for his supervision, advice, and guidance. I would also like to thank him for bringing me into the fascinating "fractal world" and keeping guiding me along my life's way.

I gratefully thank Professor Jacques Peyrière for his great efforts to help me doing my thesis in France. Without him, I would not have had the opportunity to learn excellent mathematics in such a great country.

Many thanks also go to INRIA and the members of Project SISYPHE, for providing me with a very good environment to do mathematical research and write my thesis. I would like to thank especially the head of Project SISYPHE, Michel Sorine, for his kind support and help during my stay at INRIA.

I would like to thank Kenneth Falconer and Davar Khoshnevisan for having accepted to be the referees of my thesis. I gratefully thank them for their careful reading of the manuscript and their many useful comments. I would also like to thank Kenneth Falconer, Aihua Fan, Grégory Miermont, Jacques Peyrière and Stéphane Seuret for accepting to be members of the jury.

During my thesis I have visited Professor Dejun Feng at The Chinese University of Hong Kong. I gratefully thank him for his invitation, hospitality and advice. I would also like to thank Doctor Yanhui Qu for valuable discussions and great helps during his visit at INRIA.

Finally I would like to bring my warmest thanks to my friends and my family, their unconditional support is priceless.

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Chapitre 1

Introduction

Cette thèse est consacrée à la construction et l'analyse multifractale de fonctions aléatoires et de leurs graphes.

La construction de ces objets se fait dans le cadre de la théorie des **T**-martingales de Kahane, et plus spécifiquement des $[0,1]$-martingales. Cette théorie est fréquemment utilisée pour construire des martingales à valeurs dans les mesures de Borel positives dont la limite soit presque sûrement singulière par rapport à la mesure de Lebesgue. Ceci se fait en perturbant cette dernière à l'aide d'une suite de densités aléatoires qui sont des martingales positives d'espérance 1. Ici, nous autorisons ces martingales à prendre des valeurs complexes, et plutôt que des martingales à valeurs dans les mesures, nous considérons des martingales à valeurs dans les fonctions continues à valeurs complexes, puis la question de leur convergence uniforme presque sûre. Nous obtenons une condition suffisante de convergence pour les éléments d'une large classe de $[0,1]$-martingales complexes. Les limites non dégénérées sont toutes candidates à être des fonctions multifractales. L'étude de leur nature multifractale révèle de nouvelles diffiultés. Nous la menons de façon complète dans le cas des "cascades b-adiques indépendantes" complexes. Ceci conduit à de nouveaux phénomènes. En particulier, nous construisons des fonctions continues statistiquement autosimilaires dont le spectre de singularité est croissant et entièrement supporté par l'intervalle $[0,\infty]$.

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L'introduction qui suit est organisée de telle sorte que le lecteur accède rapidement aux résultats, une place plus importante étant laissée aux commentaires dans les sections dédiées aux démonstrations.

1.1 Avant propos sur l'analyse multifractale

L'analyse multifractale a été introduite par les physiciens dans les années 80 pour l'étude de la turbulence développée (voir [37] pour un état de l'art récent). On peut la résumer ainsi : calculer la dimension de Hausdorff de l'ensemble des points où l'on observe une singularité höldérienne, ou exposant de hölder donné. De façon plus précise, soit (X, d) un espace métrique et $M : 2^X \mapsto \mathbb{R}_+$ une fonction positive et croissante définie sur l'ensemble des parties de X. Soit $\mathrm{Supp}(M) = \{x \in X : \forall\, r > 0, M(B(x,r)) > 0\}$, le support de M, où $B(x,r) = \{y \in X : d(x,y) \leq r\}$ est la boule fermée de centre x et de rayon r. L'exposant ponctuel de M en x peut être défini comme

$$h_M(x) = \liminf_{r \to 0^+} \frac{\log M(B(x,r))}{\log(r)}, \quad x \in \mathrm{Supp}(M).$$

Pour $h \geq 0$, l'ensemble de niveau h de h_M est défini par

$$E_M(h) = h_M^{-1}(\{h\}) = \{x \in X : h_M(x) = h\}.$$

L'analyse multifractale de M consiste à calculer le spectre des singularités

$$d_M : h \in \mathbb{R}_+ \mapsto \dim_H E_M(h),$$

où $\dim_H$ désigne la dimension de Hausdorff. Nous dirons que M est multifractale s'il existe au moins deux valeurs de $h \geq 0$ telles que $E_M(h)$ soit non vide ; sinon, M sera dite monofractale.

Le L^q spectre de M est défini par :

$$\tau_M(q) = \liminf_{r \to 0} \frac{\log \sup \left\{\sum_i M(B_i)^q\right\}}{\log(r)}, \quad q \in \mathbb{R},$$

la borne supérieure étant prise sur toutes les familles de boules fermées disjointes de rayon r centrées sur $\mathrm{Supp}(M)$. La transformée de Legendre de τ_M at $h \geq 0$ est donnée par

$$\tau_M^*(h) = \inf_{q \in \mathbb{R}} qh - \tau_M(q).$$

Si X possède la propriété de recouvrement de Besicovich, un argument de grandes déviations montre que pour tout $h \geq 0$,

$$d_M(h) \leq \tau_M^*(h),$$

une dimension négative signifiant que l'ensemble $E_M(h)$est vide. On dit que le formalisme multifractal est valide pour M en h si $d_M(h) = \tau_M^*(h)$.

On rencontre fréquemment les deux exemples suivants de fonctions M en analyse multifractale :

(i) $M = \mu$ est une mesure de Borel positive et finie sur X, et l'exposant de hölder ponctuel est

$$h_\mu(x) = \liminf_{r\to 0^+} \frac{\log \mu(B(x,r))}{\log(r)},$$

c'est à dire la dimension inférieure locale de μ en x. L'analyse multifractale de μ consiste donc dans le calcul de $d_\mu : h \geq 0 \mapsto \dim_H E_\mu(h)$, et on dit que le formalisme multifractal est valide pour μ en $h \geq 0$ si $d_\mu(h) = \tau_\mu^*(h)$. Bien que ce formalisme soit mis en défaut en général, il est valide pour de nombreuses classes naturelles de mesures telles que les mesure de Gibbs sur les espaces symboliques ou les répulseurs conformes, ainsi que pour les mesures auto-similaires sur les attracteurs de systèmes de fonctions itérés satisfaisant certaines hypothèses de séparation, et les contreparties aléatoires de ces mesures [134, 40, 43, 49, 69, 124, 137, 2, 117, 122, 52, 127, 128, 68, 54, 11, 19, 28, 66, 154, 15, 57, 22, 24, 56, 58].

(ii) $M = \mathrm{Osc}_f$ est l'oscillation d'une fonction $f : X \mapsto \mathbb{R}^n$, i.e., $\mathrm{Osc}_f(B) = \sup_{x,y\in B} |f(x) - f(y)|$ pour tout ensemble $B \subset X$, et l'exposant ponctuel est ici

$$h_f(x) = \liminf_{r\to 0^+} \frac{\log \mathrm{Osc}_f(B(x,r))}{\log(r)}.$$

Une définition alternative pour l'exposant ponctuel de Hölder de $f : \mathbb{R}^m \mapsto \mathbb{C}$ est

$$\bar{h}_f(x_0) = \sup\{h \geq 0 : \exists P_{x_0} \in \mathbb{C}[x], |f(x) - P_{x_0}(x)| = O(|x - x_0|^h), x \to x_0\}.$$

Quand f est uniformément höldérienne, étant donnée une base d'ondelettes dont la mère a suffisamment de moments nuls, on peut définir

$$M(B(x,r)) = L_{[-\log_2 r]}(x),\ x \in \mathbb{R}^m,\ r \in (0,1),$$

où $L_{[-\log_2 r]}(x)$ est le "wavelet leader" de f de generation $[-\log_2 r]$ en x (voir Section 5.1.1 pour les définitions). Alors, d'après le corollaire 1 de [81], on a $h_M = \bar{h}_f$. Pour for $h \geq 0$, l'ensemble de niveau $E_f(h) = h_f^{-1}(\{h\})$ ou $E_f(h) = \bar{h}_f^{-1}(\{h\})$ s'appelle ensemble iso-Hölder de f, et la recherche du spectre de singularité $d_f : h \geq 0 \mapsto \dim_H E_F(h)$ s'appelle analyse multifractale de f. Ce spectre de singularité a été déterminé pour certaines classes de fonctions incluant la fonction nulle part dérivable de Riemann, les processus de Lévy, les fonctions auto-similaires, certaines séries d'ondelettes, et les fonctions génériques de certains espaces de Besov et de Sobolev [74, 75, 76, 77, 78, 41, 80, 5, 81, 82, 23, 60, 25, 143]. Notons que lorsque $X = [0,1]$, l'analyse multifractale d'une mesure μ est essentiellement celle de la fonction $f(x) = \mu([0,x])$, $x \in [0,1]$.

Si τ_M est dérivable en $q \in \mathbb{R}$, alors $\tau_M^*(\tau_M'(q)) = q\tau_M'(q) - \tau_M(q)$. Si $\tau_M^*(\tau_M'(q)) > 0$, pour que le formalisme multifractale soit valide pour M en $\tau_M'(q)$, il suffit d'exhiber une mesure de Borel positive et finie μ_q sur X telle que pour μ_q-presque tout $x \in X$, pour r assez petit,

$$\mu_q(B(x,r)) \leq M(B(x,r))^q \cdot r^{-\tau_M(q)+o(1)}.$$

Pour trouver une telle mesure, il est nécessaire de connaître davantage de propriétés de X et M. Par exemple, si X est l'espace symbolique $\{0,1\}^{\mathbb{N}_+}$ muni de la métrique standard et $M = \mu$ est une mesure de probabilité sur X, l'existence d'une telle mesure

μ_q pour tout $q \in \mathbb{R}$ et la dérivabilité de τ_μ sont assurées si μ est quasi-Bernoulli : il existe $C > 0$ tel que

$$C^{-1}\mu([u])\mu([v]) \leq \mu([uv]) \leq C\mu([u])\mu([v]) \text{ pour tous mots } u, v,$$

où $[u]$, $[v]$ et $[uv]$ sont les cylindres enracinés en u, v, et uv. Pour en savoir plus sur le formalisme multifractal, voir [40, 124, 28].

1.2 Processus multifractals

En premier lieu nous voulons construire de nouveaux processus stochastiques multifractals. La motivation première provient du besoin de modéliser les signaux empiriques pour lesquelles l'estimation de quantités telles que le L^q-spectre révèle des propriétés d'invariance d'échelle frappantes. Ces signaux émanent de phénomènes physiques ou sociaux intermittents, comme la dissipation d'énergie en turbulence [107, 108, 109, 61], la répartition des précipitations (en météorologie) [65], la variabilité du rythme cardiaque [145, 120], le trafic internet [136, 63] et la variabilité des cours de la bourses [111, 7]. L'une des approches principales pour modéliser ces signaux consiste à construire des mesures statistiquement auto-similaires μ sur $[0, 1]$ (ou $\mathbb{R}_+$). Elles sont obtenues comme limites faibles presque sûres de martingales à valeurs dans les mesures absolument continues par rapport à la mesure de Lebesgue, $(\mu_n)_{n\geq 1}$, dont les densités $(Q_n)_{n\geq 1}$ sont des martingales engendrées par des processus multiplicatifs (voir [108, 109, 19, 115, 6, 20]). Ces objets sont ensuite utilisés pour construire des processus stochastiques multifractals non monotone comme suit : (a) on considère un mouvement brownien fractionnaire ou un processus de Lévy stable X, et indépendamment une mesure μ, puis on fait le changement de temps multifractal $X(\mu([0, t]))$ [111, 6, 135, 44, 25] ; (b) on intègre Q_n par rapport au mouvement brownien et on fait tendre n vers ∞ [6, 44] ; (c) on utilise μ pour spécifier la covariance d'un processus gaussien [106] ; (d) on construit une série d'ondelettes aléatoire dont les coefficients sont bâtis à partir de μ [3, 23].

Rappelons la construction des mesures statistiquement auto-similaires standardes que sont les cascades canoniques b-adiques de Mandelbrot sur l'intervalle $[0, 1]$. Soit $b \geq 2$ un entier et l'alphabet $\mathscr{A} = \{0, \cdots, b-1\}$. Les sous-intervalles b-adiques de $[0, 1]$ sont naturellement codés par les noeuds de l'arbre $\mathscr{A}^* = \bigcup_{n\geq 0} \mathscr{A}^n$, avec la convention que $\mathscr{A}^0$ contient la racine de $\mathscr{A}^*$ notée $\varnothing$. A chaque élément w de $\mathscr{A}^*$, on associe une variable aléatoire positive $W(w)$, ces poids étant i.i.d. avec une variable aléatoire W telle que $\mathbb{E}(W) = 1$. On obtient alors une suite de densités $(Q_n)_{n\geq 1}$ ainsi : soit I_w l'intervalle b-adique semi-ouvert à droite codé par $w = w_1 w_2 \cdots w_n$, i.e. $I_w = [\sum_{k=1}^n w_k b^{-k}, b^{-n} + \sum_{k=1}^n w_k b^{-k})$. On pose

$$Q_n(t) = W(w_1)W(w_1w_2)\cdots W(w_1w_2\cdots w_n) \text{ pour } t \in I_w.$$

Soit $(\mu_n)_{n\geq 1}$ la suite de mesures dont les densités par rapport à la mesure de Lebesgue sont données par $(Q_n)_{n\geq 1}$. Cette suite est la martingale à valeur mesure introduite dans [108, 109]. L'étude de ces martingales et leurs limites a conduit à de très nombreux développements en théorie des probabilités et en théorie géométrique de la mesure [92, 48, 88, 90, 64, 91, 45, 69, 50, 122, 2, 10, 11, 104, 126, 53, 103, 105, 24]. Ces objets, et les autres mesures statistiquement auto-similaires mentionnées plus haut sont des cas particuliers du modèle général que sont les "T-martingales" introduites dans [88, 90], afin de rendre rigoureuses la construction et les résultats présentés dans l'article fondateur [107] sur le chaos multiplicatif log-normal.

1.2.1 Construction de nouveaux processus multifractals

Rappelons ce que sont les $[0,1]$-martingales. Soit $(\Omega, \mathcal{A}, \mathbb{P})$ un espace probabilisé. L'intervalle $[0,1]$ est muni de la tribu borélienne $\mathcal{B}$ et le produit $[0,1] \times \Omega$ de la tribu produit $\mathcal{B} \otimes \mathcal{A}$. Soit $(\mathcal{A}_n)_{n\geq 1}$ une suite croissante de sous-tribus de $\mathcal{A}$. Soit $(Q_n)_{n\geq 1}$ une suite de fonctions positives mesurables définies sur $[0,1] \times \Omega$, telles que pour tout $t \in [0,1]$, $\{Q_n(t,\cdot), \mathcal{A}_n\}_{n\geq 1}$ soit une martingale d'espérance 1. Une telle suite s'appelle une $[0,1]$-martingale. Etant donnée une mesure de Radon λ sur $[0,1]$, pour tout $n \geq 1$ nous pouvons définir la mesure aléatoire μ_n dont la densité par rapport à λ est donnée par Q_n. La convergence faible presque sûre de la suite $(\mu_n)_{n\geq 1}$ est alors une conséquence directe du théorème de convergence des martingales positives et du théorème de représentation de Riesz ([88, 90]).

Il est naturel d'étendre la construction précédente en autorisant la $[0,1]$-martingale $(Q_n)_{n\geq 1}$ à prendre des valeurs négatives ou complexes, et en considérant la martingale à valeurs dans les fonctions continues

$$\Big(F_n : t \in [0,1] \mapsto \int_0^t Q_n(s)\,\mathrm{d}\lambda(s)\Big)_{n\geq 1}.$$

Quand les Q_n sont positifs, on retrouve le cas des mesures en considérant F_n'. Quand les Q_n ne sont plus positives, les martingales $(Q_n(t))_{n\geq 1}$ ne sont plus nécessairement bornées dans L^1, et la variation totale de F_n' peut diverger et donc la sequence $(F_n')_{n\geq 1}$ en faire de même dans le dual de $\mathcal{C}([0,1])$, l'espace des fonctions continues sur $[0,1]$ et à valeurs complexes.

La première question qui se pose dans le cas où $(Q_n)_{n\geq 1}$ prend des valeurs non positives est : en quel sens la suite $(F_n)_{n\geq 1}$ converge-t-elle presque sûrement, et ce vers une limite non triviale, i.e. non nulle avec probabilité strictement positive ? Nous considérons la convergence uniforme de $(F_n)_{n\geq 1}$ pour une sous-classe de $[0,1]$-martingales complexe, notée $\mathcal{M}$ (voir Section 2.1.1 pour sa définition), telle que pour $(Q_n)_{n\geq 1} \in \mathcal{M}$, nous obtenons une condition suffisante générale pour la convergence presque sûre uniforme de $(F_n)_{n\geq 1}$ vers une limite non triviale avec probabilité strictement positive, ainsi qu'un résultat de régularité höldérienne globale. Précisément, pour $p \in \mathbb{R}_+$ soit

$$\varphi(p) = \liminf_{n\to\infty} -\frac{1}{n}\log_b\left(\sum_{w\in\mathscr{A}^n} \lambda(I_w)^{p-1}\int_{I_w} \mathbb{E}(|Q_n(s)|^p)\,\mathrm{d}\lambda(s)\right)$$

Théorème 1.1 (Barral, Jin, Mandelbrot) *Supposons que* $(Q_n)_{n\geq 1} \in \mathcal{M}$.

1. *Supposons que* $\varphi(p) > 0$ *pour un nombre* $p \in (0,1)$, *et qu'il existe une fonction* $\psi : \mathbb{N}_+ \to \mathbb{R}_+$ *telle que* $\psi(n) = o(n)$ *et* $\mathbb{E}(\sup_{t\in I_w} |Q_n(t)|^p) \leq e^{\psi(n)}\mathbb{E}(|Q_n(t)|^p)$ *pour tous* $n \geq 1$, $w \in \mathscr{A}^n$ *et* $t \in I_w$. *Alors, presque sûrement,* F_n *converge uniformément vers 0 quand* $n \to \infty$.

2. *Soit* $p \in (1,2]$. *Supposons que* $\varphi(p) > 0$. *Les fonctions* F_n *convergent uniformément, presque sûrement et en norme* L^1, *vers une limite* F, *quand* $n \to \infty$. *De plus,* F *est* γ*-höldérienne pour tout* $\gamma \in (0, \max_{q\in(1,p]} \varphi(q)/q)$. *Enfin,* $\mathbb{E}(\|F\|_\infty^p) < \infty$.

Le Théorème 1.1 permet de construire les extensions aux fonctions complexes des exemples fondamentaux de mesures statistiquement auto-similaires telles que les cascades canoniques b-adiques [108, 109], les cascades de Poisson composées [19, 13], et les cascades infiniment divisibles [6, 44]. Nous pensons qu'en général les fonctions F obtenues dans le Théorème 1.1.2. sont multifractales. Cependant, l'étude de leur nature multifractale révèle de nouvelles difficultés que nous n'avons pu lever que pour une classe spéciale, qui contient en fait plus que des $[0,1]$-martingales. Elle correspond à l'extension aux fonctions complexes des cascades multiplicatives introduites comme modèle de turbulence dans [108, 109]. Pour être précis, fixons un vecteur aléatoire complexe $W = (W_0, \cdots, W_{b-1})$ tel que $\mathbb{E}(\sum_{i=0}^{b-1} W_i) = 1$, et considérons une suite $\{W(w)\}_{w\in\mathscr{A}^*}$ de copies de $W(\varnothing) = W$ indépendantes. Pour tout $w = w_1 \cdots w_n \in \mathscr{A}^n$ posons

$$Q_n(t) = b^n \cdot W_{w_1}(\varnothing) W_{w_2}(w_1) \cdots W_{w_n}(w_1 \cdots w_{n-1}),\ t \in I_w.$$

Prenons λ égale à la mesure de Lebesgue sur $[0,1]$ et considérons la suite de fonctions

$$\Big(F_n : t \in [0,1] \mapsto \int_0^t Q_n(s)\,\mathrm{d}s\Big)_{n\geq 1}.$$

On voit aisément que lorsque $\mathbb{E}(W_i) = 1/b$ pour tout i, $(Q_n)_{n\geq 1}$ est une $[0,1]$-martingale. Pour $p \in \mathbb{R}_+$ soit

$$\varphi_W(p) = -\log_b \mathbb{E}(\sum_{i=0}^{b-1} |W_i|^p).$$

Nous avons l'extension suivante du Théorème 1.1.

Théorème 1.2 (Barral, Jin, Mandelbrot)

(i) **(Cas Non-conservatif)** *Supposons que* $\mathbb{P}(\sum_{i=0}^{b-1} W_i \neq 1) > 0$, *et qu'il existe* $p > 1$ *tel que* $\varphi_W(p) > 0$. *Supposons de plus que* $p \in (1,2]$, *ou* $\varphi_W(2) > 0$. *Alors,*

1. $(F_n)_{n\geq 1}$ *converge uniformément, presque sûrement et en norme* L^p, *quand* n *tends to* ∞, *vers une function* $F = F_W$, *croissante si* $W \geq 0$. *De plus, la fonction* F *est* γ*-höldérienne pour tout* $\gamma \in (0, \max_{q\in(1,p]} \varphi_W(q)/q)$.

2. F *satisfait la propriété d'invariance d'échelle en loi suivante :*

$$F = \sum_{i=0}^{b-1} \mathbf{1}_{[i/b,(i+1)/b]} \Big(F(i/b) + W_i\, F_i \circ S_i^{-1}\Big),$$

où $S_i(t) = (t+i)/b$, *les objets aléatoires* W, $F_0, \ldots, F_{b-1}$ *sont indépendants, les* F_i *ont la même loi que* F, *et l'égalité a lieu presque sûrement.*

(ii) **(Cas conservatif)** *Supposons que* $\mathbb{P}(\sum_{i=0}^{b-1} W_i \neq 1) = 0$.

1. *S'il existe* $p > 1$ *tel que* $\varphi_W(p) > 0$, *les mêmes conclusions qu'en* (i) *sont vraies.*

2. **(Cas critique)** *Supposons que* $\lim_{p\to\infty} \varphi_W(p) = 0$ *(en particulier* φ_W *est croissante et* $\varphi_W(p) < 0$ *pour tout* $p > 1$*). Cela équivaut à dire que* $\mathbb{P}(\forall\ 0 \leq i \leq b-1,\ |W_i| \leq 1) = 1$ *et* $\sum_{i=0}^{b-1} \mathbb{P}(|W_i| = 1) = 1$.

   Supposons également que $\mathbb{P}(\#\{i : |W_i| = 1\} = 1) < 1$*, et qu'il existe* $\gamma \in (0,1)$ *tel que presque sûrement, l'une des deux propriétés suivantes soit satisfaite pour chaque* $0 \leq i \leq b-1$ *:*

$$\begin{cases} \textit{ou bien } |W_i| \leq \gamma, \\ \textit{ou bien } |W_i| = 1 \textit{ et } \left(\sum_{k=0}^{i-1} W_i, \sum_{k=0}^{i} W_i\right) \in \{(0,1),(1,0)\} \end{cases}.$$

   Alors, $(F_n)_{n\geq 1}$ *converge presque sûrement uniformément vers une limite* $F = F_W$ *qui n'est pas uniformément Hölder, et satisfait la partie 2. de (i).*

Remarque 1.1 *Si* $p_i = \mathbb{E}(W_i) > 0$ *pour chaque* i*, soit* ν *la mesure multinomiale engendrée par* $(p_0, \ldots, p_{b-1})$*. Alors* $\widetilde{Q}_n(t) = Q_n(t)/(p_{w_1} \cdots p_{w_n})$ *est une* $[0,1]$*-martingale et sous les hypothèses du théorème précédent, avec probabilité 1,* $\widetilde{F}_n(t) = \int_0^t \widetilde{Q}_n(s)\,\mathrm{d}\nu(s)$ *converge uniformément vers la même limite que* F_n.

On trouvera d'autres résultats sur le comportement asymptotique des cascades b-adiques dans le Chapitre 2.

1.2.2 Analyse multifractale des cascades b-adiques complexes

Discutons à présent de l'analyse multifractale des limites non dégénérées de cascades b-adiques complexes. Cela nécessite d'introduire quelques notions supplémentaires d'analyse multifractale des fonctions.

Etant donnée $f : I \mapsto \mathbb{C}$, où I est un intervalle non triviale, rappelons la définition de l'exposant de Hölder ponctuel $\bar{h}_f$ donnée dans la Section 1.1 :

$$\bar{h}_f(x_0) = \sup\{h \geq 0 : \exists P_{x_0} \in \mathbb{C}[x], |F(x) - P_{x_0}(x)| = O(|x - x_0|^h), x \to x_0\},$$

qui est l'exposant communément utilisé en analyse multifractale des fonctions. Lorsque f est globalement höldérienne, on montre dans [73, 81] que l'exposant $\bar{h}_f(x_0)$ peut se déduire du comportement asymptotique des coefficients d'ondelettes de f localisés dans un voisinage de x_0, si l'ondelette est suffisamment régulière. Le développement en série d'ondelette a ainsi été utilisé avec succès pour caractériser les ensembles iso-Hölder $E_f(h) = \bar{h}_f^{-1}(\{h\})$ $(h \geq 0)$, de larges classes de fonctions [74, 76, 29, 79, 80, 5, 23, 47], parfois directement construites commes séries d'ondelettes (le développement dans la base de Schauder est également exploité dans [83]). Pour la plupart de ces fonctions, le spectre des singularités s'obtient comme la transformée de Legendre d'une fonction d'énergie libre reflétant le comportement asymptotique global de la distribution des coefficients d'ondelettes.

En pratique, il se peut qu'il soit difficile d'extraire une bonne caractérisation des ensembles $E_f(h)$ à partir d'une décomposition en ondelette. Cela conduit à chercher d'autres méthodes d'estimation de $\bar{h}_f(t)$ ou d'exposants proches de $\bar{h}_f(t)$ et éventuellement plus simples à estimer. L'alternative la plus naturelle est l'exposant d'oscillation d'ordre 1 de f définie par

$$h_f^{(1)}(x) = h_f(x) = \liminf_{r \to 0^+} \frac{\log \mathrm{Osc}_f(B(x,r))}{\log(r)}.$$

Si cet exposant n'est pas entier, alors il est égal à $\bar{h}_f(x)$. Quand la fonction f peut se décomposer en une fonction monofractale B en temps multifractal $\widetilde{f}$, le spectre des singularités de f associé à l'exposant $h_f^{(1)}$ peut souvent se déduire aisément de celui de $\widetilde{f}$.

Il s'avère que pour les limites de cascades b-adiques, en général la décomposition en ondelettes n'est pas suffisamment tractable pour donner des informations précises sur les ensembles iso-Hölder. De plus, tandis que certaines de ces fonctions se décomposent naturellement en une fonction monofractale et un changement de temps multifractal, cette décomposition n'est pas possible dans certains cas, comme le cas conservatif critique.

Nous avons adopté une approche inspirée des idées développées dans [77, 81], et calculé le spectre des singularités en utilisant l'analyse multifractale des exposants d'oscillations de tous ordres. A notre connaisance, cette approche n'avait pas encore été utilisée pour traiter un exemple non trivial.

Si J est un sous-intervalle non trivial de I, pour $m \geq 1$, soit

$$\mathrm{Osc}_f^{(m)}(J) = \sup_{[x,x+\delta] \subset J} |\Delta_\delta^m f(t)|,$$

où $\Delta_\delta^1 f(t) = f(x+\delta) - f(x)$ et pour $m \geq 2$, $\Delta_\delta^m f(x) = \Delta_\delta^{m-1} f(x+\delta) - \Delta_\delta^{m-1} f(x)$ (notons que $\mathrm{Osc}_f^{(1)}(J) = \mathrm{Osc}_f(J)$). Alors, l'exposant ponctuel d'oscillation d'ordre $m \geq 1$ de f en $x \in \mathrm{Supp}(\mathrm{Osc}_f^{(m)})$ est défini par

$$h_f^{(m)}(x) = \liminf_{r \to 0^+} \frac{\log \mathrm{Osc}_f^{(m)}(B(x,r))}{\log r}.$$

Nous ne considérons que les points de $\mathrm{Supp}(\mathrm{Osc}_f^{(m)})$, car du point de vue de la régularité ponctuelle, $\mathrm{Supp}(\mathrm{Osc}_f^{(m)})$ est l'ensemble sur lequel on peut voir des informations non triviale grâce à $h_f^{(m)}$. En effet, en dehors de cet ensemble fermé, la fonction f est localement égale à un polynôme de degré au plus $m-1$, donc f est C^∞.

L'exposant ponctuel $\bar{h}_f$ porte une information non-triviale en tout point où f n'est pas localement égale à un polynome, i.e., en tout point de l'ensemble $\bigcap_{m \geq 1} \mathrm{Supp}(\mathrm{Osc}_f^{(m)})$. Si $x \in \bigcap_{m \geq 1} \mathrm{Supp}(\mathrm{Osc}_f^{(m)})$, il est clair que la suite $(h_f^{(m)}(x))_{m \geq 1}$ est croissante. En fait, $\sup_{m \geq 1} h_f^{(m)}(x) = \bar{h}_f(x)$. C'est une conséquence du théorème de Whitney sur l'approximation locale des fonctions bornées par des polynômes [150, 141] : pour tout $m \geq 1$, il existe $C_m > 0$ (indépendant de f) tel que pour tout sous-intervalle J de I, il existe une fonction polynomiale P de degré au plus $m-1$ telle que

$$|f(x) - P(x)| \leq C_m \mathrm{Osc}_f^{(m)}(J).$$

Ceci, combiné à la définition de $\bar{h}_f$ donne l'énoncé suivant, également prouvé dans [81] en utilisant une décomposition en ondelettes quand f est uniformément höldérienne :

Proposition 1.1 *Si $f : I \to \mathbb{C}$ est continue, alors pour $t \in \bigcap_{m\geq 1} \mathrm{Supp}(f^{(m)})$, $h_f^{(m)}(t)$ converge vers $\bar{h}_f(t)$ quand m tend vers l'infini. De plus, si $\bar{h}_f(t) < \infty$, alors $h_f^{(m)}(t) = \bar{h}_f(t)$ pour tout $m > \bar{h}_f(t)$.*

Maintenant, l'analyse multifractale de f consiste aussi à calculer les spectres de singularités de la forme

$$h \geq 0 \mapsto \dim_H E_f^{(m)}(h),$$

où pour $h \geq 0$ et $m \geq 1$,

$$E_f^{(m)}(h) = \left\{x \in \mathrm{Supp}(\mathrm{Osc}_f^{(m)}) : h_f^{(m)}(x) = h\right\},$$

et pour $h \geq 0$,

$$E_f^{(\infty)}(h) = \left\{x \in \bigcap_{m\geq 1} \mathrm{Supp}(\mathrm{Osc}_f^{(m)}) : h_f^{(\infty)}(x) = h\right\}, \quad \big(\text{où } h_f^{(\infty)}(x) = \bar{h}_f(x)\big).$$

La Proposition 1.1 donne

$$E_f^{(\infty)}(h) = E_f^{(m)}(h) \quad (\forall h \geq 0,\ \forall\ m > h).$$

Comme dans la Section 1.1, pour tout $m \geq 1$ nous pouvons considérer le L^q-spectre de $\mathrm{Osc}_f^{(m)}$:

$$\tau_f^{(m)}(q) = \liminf_{r\to 0} \frac{\log \sup \left\{\sum_i \mathrm{Osc}_f^{(m)}(B_i)^q\right\}}{\log(r)},$$

où la borne supérieure est prise sur toutes les familles de boules fermées de rayon r centrées sur $\mathrm{Supp}(\mathrm{Osc}_f^{(m)})$. Pour tous $h \geq 0$ et $m \geq 1$, nous avons (Proposition 3.2)

$$\dim_H E_f^{(m)}(h) \leq (\tau_f^{(m)})^*(h) = \inf_{q\in\mathbb{R}} hq - \tau_f^{(m)}(q),$$

et grâce à la Proposition 1.1,

$$\dim_H E_f^{(\infty)}(h) \leq (\tau_f^{(\infty)})^*(h) := \inf_{m>h} (\tau_f^{(m)})^*(h),$$

une dimension négative sigifiant que $E_f^{(m)}(h)$ est vide. Nous dirons que le formalisme multifractal est valide pour f et $m \in \mathbb{N}_+ \cup \{\infty\}$ en $h \geq 0$ si $E_f^{(m)}(h)$ est non vide et $\dim_H E_f^{(m)}(h) = (\tau_f^{(m)})^*(h)$.

Quand $m = \infty$, l'exposant $h_f^{(m)}$ est par construction stable par addition d'une fonction C^∞, et il en est de même de la validité du formalisme multifractal associé à cet exposant. Ce n'est pas le cas pour $m < \infty$ (voir le Corollaire 3.1 pour une illustration).

Comme nous l'avons dit, notre approche du formalisme multifractal s'inspire de l'"oscillation method" introduite dans [77, 81] pour les fonctions uniformément höldériennes. Dans ces travaux, des quandités du type $\tau_f^{(m)}$ sont définies en utilisant des boules centrées sur des grilles régulières de plus en plus fines, pour $q \geq 0$. Ainsi, notre choix pour $\tau_f^{(m)}$ est plus intrinsèque, bien qu'équivalent. Le choix de ne considérer que le cas $q \geq 0$ dans [81] est lié à l'introduction d'espaces fonctionnels qui via les fonctions $\tau_f^{(m)}$ permettent de lier les approches par ondelettes et par oscillations dans le cadre du formalisme multifractal quand $q \geq 0$. Il est intéressant de noter que grâce à ce lien, pour tout $q \geq 0$, si nous définissons n_q comme le plus petit entier n tel que $nq - 1 \geq \tau_f^{(n)}(q)$, alors pour tout $n \geq n_q$, la fonction $\tau_f^{(n)}$ coïncide sur $[q, \infty]$ avec la *fonction d'échelle* τ_f^W associée aux *wavelet leaders* dans [77, 81]. Ainsi, pour $h \geq 0$ tel que le formalisme multifractal est valide en h pour $m = \infty$, même si $E_f^{(n)}(h) = E_f^{(\infty)}(h)$ pour tout $n \geq [h]+1$, $\dim_H E_f^{(\infty)}(h)$ peut n'être égal à $(\tau_f^{(n)})^*(h)$ que pour $n \gg [h]+1$ quand h tend vers $(\tau_f^W)'(0^+)$.

Commençons à présent l'analyse multifractale des limites de cascades b-adiques. Nous traitons d'emblée un modèle plus général en considérant un couple (W, L) de vecteurs aléatoires à valeurs dans $\mathbb{C}^b \times (\mathbb{R}_+^*)^b$. Nous supposons que W and L satisfont les mêmes propriétés que W dans la section précédente : $\mathbb{E}(\sum_{i=0}^{b-1} W_i) = 1 = \mathbb{E}(\sum_{i=0}^{b-1} L_i)$. Nous considérons une suite $\{(W, L)(w)\}_{w \in \mathscr{A}^*}$ de copie de (W, L) indépedantes, et supposons aussi que W et L satisfont the les hypothèses du Theorem 1.2. Cela donne presque sûrement deux fonctions continues F_W et F_L, la seconde étant strictement croissante. Nous étudierons sur $[0, F_L(1)]$ la fonction

$$F = F_W \circ F_L^{-1}.$$

Si les composantes de W et L sont des nombres réels déterministes et $\sum_{i=0}^{b-1} W_i = 1 = \sum_{i=0}^{b-1} L_i$, nous retrouvons les fonctions auto-affines construites dans [26]. L'analyse multifractale de ces fonctions est faite dans [76] en utilisant leur décomposition en ondelettes (cependant, les points extrêmaux du spectres ne sont pas étudiés). Il est aussi possible d'utiliser l'alternative consistant à montrer que F_W peut-être représentée comme fonction monofractale en temps multifractal [112, 143], puis à considérer l'exposant $h_F^{(1)}$ plutôt que $\overline{h}_F$. Il s'avère qu'une telle décomposition existe aussi dans le cas aléatoire sous des hypothèses fortes sur W, qui incluent le cas déterministe (Théorème 2.4). C'est utile puisque, comme nous l'avons dit, nos calculs montrent qu'en général il semble difficile de s'en tirer grâce aux ondelettes. Cependant, cette approche ne pourrait pas couvrir tous les cas puisqu'il n'y a pas de telle représentation naturelle dans le cas conservatif critique. De plus, dans ce dernier cas, les fonctions obtenues sont nulle part localement höldériennes et n'appartiennent à aucun espcace de Besov critique (leur spectre de singularité a une pente infinie en 0), donc il y a peu d'espoir de finir par réussir à utiliser une approche par ondelettes.

L'emploi des exposants ponctuels d'oscillation s'avère ici très efficace. Nous obtenons les résultats suivants. Par soucis de simplicité, nous reportons à la section 3.2.7 la discussion d'extensions de ces résultat sous des hypothèses plus faibles. Nous excluons le cas trivial où $W = L$, pour lequel $F = \mathrm{Id}_{[0, F_L(1)]}$. Nous supposons aussi que

$$\varphi_L > -\infty \text{ sur } \mathbb{R} \text{ et } 0 < L_i < 1 \text{ presque sûrement.}$$

Le premier résultat concerne les fonctions F à spectre en forme de cloche. Dans le cas conservatif critique, le plus petit exposant est nul et la pente du spectre en 0 est infinie. C'est un nouveau comportement dans l'analyse multifractale des fonctions continues statistiquement auto-similaires.

Théorème 1.3 (Barral, Jin) (Spectres en cloche)

Supposons que $\mathbb{P}(\sum_{i=0}^{b-1}\mathbf{1}_{\{W_i\neq 0\}}\geq 2)=1$ *et* $\varphi_W>-\infty$ *sur* $\mathbb{R}$. *Pour* $q\in\mathbb{R}$, *soit* $\tau(q)$ *l'unique solution de* $\mathbb{E}(\sum_{i=0}^{b-1}\mathbf{1}_{\{W_i\neq 0\}}|W_i|^q L_i^{-t})=1$. *La fonction* τ *est concave et analytique. Avec probabilité 1,*

1. $\mathrm{Supp}(\mathrm{Osc}_F^{(m)})=\mathrm{Supp}(\mathrm{Osc}_F)$ *pour tout* $m\geq 1$ *et* $\dim_H \mathrm{Supp}(\mathrm{Osc}_F)=-\tau(0)$.
2. *Pour tous* $h\geq 0$ *et* $m\in\mathbb{N}_+\cup\{\infty\}$, $\dim_H E_F^{(m)}(h)=(\tau_F^{(m)})^*(h)=(\tau_F^{(1)})^*(h)$, *une dimension négative signifiant que* $E_F^{(m)}(h)$ *est vide. De plus,* $E_F^{(m)}(h)\neq\emptyset$ *si* $(\tau_F^{(1)})^*(h)=0$. *En d'autres termes, pour tout* $m\in\mathbb{N}_+\cup\{\infty\}$, F *satisfait le formalisme multifractal en tout* $h\geq 0$ *tel que* $(\tau_F^{(m)})^*(h)\geq 0$. *De surcroît, si* F_W *est construite comme dans le Théoorème 1.2 (ii) 2. (cas critique), le plus petit exposant de F est l'exposant 0, et l'ensemble de niveau correspondant est dense, de dimension de Hausdorff nulle.*
3. *Pour tout* $m\geq 1$, $\tau_F^{(m)}=\tau$ *sur l'intervalle* $J=\{q\in\mathbb{R}:\tau'(q)q-\tau(q)\geq 0\}$, *et si* $\overline{q}=\sup(J)<\infty$ *(resp.* $\underline{q}:=\inf(J)>-\infty$*) alors* $\tau_F^{(m)}(q)=\tau'(\overline{q})q$ *(resp.* $\tau'(\underline{q})q$*) sur* $[\overline{q},\infty)$ *(resp.* $(-\infty,\underline{q}]$*). De plus, s'il n'existe pas* $H\in(0,1)$ *tel que pour tout* $0\leq i\leq b-1$ *on ait* $|W_i|\in\{0,L_i^H\}$, *alors* τ *est strictement concave sur* J *; sinon,* $\tau(q)=qH+\tau(0)$ *et* F *est monofractale, d'exposant de Hölder égal à* H.

Notons que $-\tau(0)<1$ si et seulement si au moins l'une des composantes de W s'annule avec probabilité strictement positive, et dans ce cas le support de Osc_F est un ensemble de Cantor.

Dans le résultat suivant, nous obtenons des fonctions F satisfaisant le formalisme multifractal pour lesquels le spectre des singularités est strictement croissant et supporté par l'intervalle $[0,\infty]$ tout entier. C'est encore un nouveau comportement dans l'analyse multifractale des fonctions continues statistiquement auto-similaires. .

Théorème 1.4 (Barral, Jin) (Spectres croissants)

Supposons que $\mathbb{P}(\sum_{i=0}^{b-1}\mathbf{1}_{\{W_i\neq 0\}}\geq 2)=1$ *et* $\varphi_W(q)>-\infty$ *sur* $\mathbb{R}_+$. *Pour* $q\in\mathbb{R}_+$, *soit* $\tau(q)$ *comme dans le Théorème 1.3. La fonction* τ *est concave et analytique sur* $(0,\infty)$. *Supposons aussi que* $\mathbb{E}(\sum_{i=0}^{b-1}\mathbf{1}_{\{W_i\neq 0\}}L_i\log(|W_i|))=-\infty$, *i.e.* $\tau'(0)=\infty$. *Enfin, supposons qu'il existe* $\varepsilon>0$ *tel que* $\mathbb{E}\big((\max_{0\leq i\leq b-1}|W_i|)^{-\varepsilon}\big)<\infty$ $\varepsilon>0$.

Alors, les mêmes conclusions que dans le Théorème 1.3 sont vraies. De plus, le spectre des singularités est croissant, et $h_F^{(m)}=\infty$ *pour tout* $m\in\mathbb{N}_+\cup\{\infty\}$ *sur une partie de dimension pleine de* $\mathrm{Supp}(\mathrm{Osc}_F)$. *De surcroît, si* F_W *est construite comme dans le Théorème 1.2 (ii) 2. (cas critique case), le support du spectre est* $[0,\infty]$.

Des exemples de fonctions continues croissantes possédant des propriétés d'auto-similarité et un spectre croissant existent [110, 137, 113], mais leur régularité höldérienne minimale est strictement positive.

Il est intéressant de mentionner ici que dans certains espaces de Besov de fonctions continues, le spectre de singularités générique est croissant, supporté par un intervalle compact, et linéaire ; de plus, l'exposant minimal est nul pour les espaces de Besov critiques [80, 85].

Dans le cas conservatif critique, la pente du spectre de singularité est infinie en 0 du fait de la dualité entre $h = \tau'(q)$ et $q = (\tau^*)'(h)$, et $h \to 0$ correspond à $q \to \infty$.

Dans le cas où F est croissante (les composantes de W sont positives), des résultats pour l'analyse multifractale de la mesure $\mu = F'$ ont été obtenus dans plusieurs articles (qui traitent aussi des mesures sur $\mathbb{R}^d$). Pour le cas de la dimension 1, les énoncés précédents sont des améliorations substantielles de ces résultats pour les raisons suivantes :

(1) En premier lieu, tous ces travaux ne traitent que de l'exposant d'oscillation d'ordre 1, qui n'est parfois calculé que sur la grille distordue naturellement associée à F_L [69, 122, 11], et pas de la façon plus intrinsèque consistant à prendre des intervalles centrés. De plus, dans les articles qui considèrent le point de vue plus intrinsèque, les hypothèses sur W et L sont très fortes : leurs composantes doivent être comprises entre deux constantes a et b de $(0, 1)$ presque sûrement [2, 50] ; aussi les résultats sont vrais pour tout $h \geq 0$ tel que $\tau^*(h) > 0$ presque sûrement, et non presque sûrement pour tout $h \geq 0$ tel que $\tau^*(h) > 0$. Enfin, le cas des spectres croissants n'est pas traité.

(2) Un autre point important concerne le calcul du spectre aux bornes de son support, qui est une question délicate. En effet, il s'avère déjà non trivial de montrer que les ensembles iso-Hölder correspondants sont non vides. Notre résultat donne la description du spectre au bord de son support ${\tau_F^*}^{-1}(\mathbb{R}_+)$, sans restriction sur le comportement de τ. C'est une amélioration des résultats de [11] où le cas $\overline{q} = \infty$ (resp. $\underline{q} = -\infty$ et $\lim_{q\to\infty}$ (resp. $\lim_{q\to-\infty}$)$\tau'(q)q - \tau(q) = 0$ n'est pas traité (ce résultat est particulièrement important dans la cas conservatif critique), et les exposants de Hölder sont calculés à partir de l'oscillation d'odre 1 sur les intervalles de la grille associée à F_L. De plus, la méthode introduite ici peut être utilisée pour étudier les bords des spectres de singularité des éléments de la classe plus générale de mesures aléatoires considérée dans [20].

Dans les résultats précédents, tous les formalismes multifractals fournissent la même information. En particulier, notre discussion sur le lien entre les approches par oscillations et par ondelettes développées dans [81] montre que lorsque F est uniformément höldérienne, le formalisme multifractal utilisant les ondelettes est également valide dans la partie croissante du spectre, sans qu'il soit nécessaire de calculer de transformée en ondelette pour F.

Le résultat suivant illustre l'instabilité des exposants et des spectres associés aux oscillations d'ordre m par addition d'une fonction C^∞.

Corollaire 1.1 *Soit $f : \mathbb{R}_+ \to \mathbb{C}$ une fonction C^∞ telle que pour tout $m \in \mathbb{N}_+$ la fonction $f^{(m)}$ ne s'annule pas. Soit F comme dans le Théorème 1.4 et $G = F + f$. Les fonctions F et G ont le même comportement multifractal du point de vue de l'exposant ponctuel de Hölder $\overline{h}$..*

Pour $m \in \mathbb{N}_+$, soit q_m l'unique nombre réel tel que $\tau(q_m) = q_m m - 1$.

Presque sûrement, pour tout $m \in \mathbb{N}_+$, on a $\tau_G^{(m)} = \tau_F^{(m)} = \tau$ sur $[q_m, \infty)$, et $\tau_G^{(m)}(q) = qm - 1$ pour $0 \leq q < q_m$. De plus, pour tout $m \in \mathbb{N}_+$, le formalisme multifractal est valide en tout $h \in [0, \tau'(q_m)]$ tel que ${\tau_G^{(m)}}^*(h) \geq 0$ ainsi qu'en $h = m$, et pour tout $h \in (\tau'(q_m), m)$ on a $\dim_H E_G^{(m)}(h) = \tau^*(h) < (\tau_G^{(m)})^*(h)$.

1.3 Spectres de singularité associés au graphe, à l'image et aux ensembles de niveau des processus multifractals

Nous nous sommes aussi intéressés à de nouveaux spectres de singularité associés aux processus multifractals.

Soit f une fonction définie sur un intervalle I et à valeurs dans $\mathbb{R}$. Pour tout $E \subset I$ soit

$$G_f(E) = \{(x, f(x)) : x \in E\}, \ R_f(E) = \{f(x) : x \in E\}$$

les graphe et image de f au dessus de E ; pour tout $y \in R_f(E)$ soit

$$L_f^y(E) = \{(x, f(x)) : x \in E, f(x) = y\}$$

l'ensemble de niveau y de f au dessus de E. Quand $(f(t))_{t\in I}$ est un preocessus stochastique, le calcul des dimensions de Hausdorff de ces ensembles, $\dim_H S_f(E)$ pour $S \in \{G, R, L^y\}$, est un problème classique et important en théorie des probabilités et théorie géométrique de la mesure. Les premiers travaux sur ces questions remontent à 1953 et sont dûs à Lévy et Taylor [99, 146]. Ils concernent la dimension et la mesure de Hausdorff de l'image d'un mouvement brownien. Depuis, de nombreux progrès ont été faits, en particulier pour le mouvement brownien fractionnaire et les processus de Lévy stables, mais aussi pour bien d'autres processus et fonctions [35, 36, 147, 70, 131, 121, 125, 30, 67, 31, 89, 118, 132, 149, 27, 32, 97, 71, 72, 139, 55, 93, 46] (voir aussi l'article de survol [152] et ses références). Prenons l'exemple typique du mouvement brownien fractionnaire. Soit $\beta \in (0, 1)$ son indice. Dans son célèbre livre [89] Kahane étudie ce processus $(X(t))_{t\in\mathbb{R}_+}$, caractérisé par le fait qu'il s'agit de l'unique processus gaussien centré à trajectoires continues tel que $X(0) = 0$ et $\mathbb{E}(|X(s) - X(t)|^2) = |s - t|^{2\beta}$ pour tous $s, t \in \mathbb{R}_+$. Il montre que pour tout compact $E \subset \mathbb{R}$, presque sûrement

$$\dim_H G_X(E) = \frac{\dim_H E}{\beta} \wedge \big(\dim_H E + 1 - \beta\big), \ \dim_H R_X(E) = \frac{\dim_H E}{\beta} \wedge 1$$

et que si $\dim_H E - \beta > 0$, alors il existe un ensemble ouvert aléatoire $G \subset R_X(E)$ tel que $\mathbb{P}(G \neq \emptyset) > 0$ et

$$y \in G \Rightarrow \dim_H L_X^y(E) = \dim_H E - \beta.$$

Notons que β est l'(unique) exposant de Hölder de X et qu'il est donc uniforme sur E (X est monofractal). Quand la fonction f est multifractale, la discussion précédente conduit naturellement à considérer $E = E_f(h)$ et à chercher à savoir si

$$\dim_H G_f(E_f(h)) = \frac{\dim_H E_f(h)}{h} \wedge \big(\dim_H E_f(h) + 1 - h\big) \text{ et}$$

$$\dim_H R_f(E_f(h)) = \frac{\dim_H E_f(h)}{h} \wedge 1.$$

Cela conduit à définir les spectres de singularités suivants, associés respectivement au graphe, à l'image et aux ensembles de niveau de f :

$$d_f^S : h \geq 0 \mapsto \dim_H S_f(E_f(h)), \quad S \in \{G, R, L^y\}.$$

A notre connaissance, ces spectres n'ont pas été considérés auparavant.

1.3.1 Une majoration générale

En premier lieu, il est naturel de chercher des bornes supérieures générales pour ces spectres. Elles sont fournies par le résultat suivant, qui généralise le Lemme 8.2.1 de [1], le Théorème 6 du Chapitre 10 de [89], et le Lemme 2.2 de [151] :

Théorème 1.5 (Jin) *Soit E une partie de I. Supposons que* $\inf_{x\in E} h_f(x) = h > 0$.

1. *Notons* $\dim_P$ *la dimension de packing. Pour* $D \in \{H, P\}$ *on a*

$$\begin{aligned}\dim_D G_f(E) &\le \left(\frac{\dim_D E}{h} \wedge \big(\dim_D E + 1 - h\big)\right) \vee \dim_D E,\\ \dim_D R_f(E) &\le \frac{\dim_D E}{h} \wedge 1.\end{aligned}$$

2. *Soit μ une mesure de Borel positive sur $\mathbb{R}$. Pour tout $\gamma > 0$ soit $R_f^{\mu,\gamma}(E) := \{y \in R_f(E) : h_\mu(y) \ge \gamma\}$. Si $\mu(R_f^{\mu,\gamma}(E)) > 0$, alors pour μ-presque tout $y \in R_f^{\mu,\gamma}(E)$,*

$$\dim_H L_f^y(E) \le \big(\dim_H E - h\cdot\gamma\big) \vee 0.$$

Si nous remplaçons E par $E_f(h)$ pour $h > 0$, le Théorème 1.5 donne une borne supérieure générale pour $\dim_H S_f(h)$ for $S \in \{G, R, L^y\}$: pour tout $h > 0$, on a

$$\begin{aligned}d_f^G(h) &\le \left(\frac{d_f(h)}{h} \wedge \big(d_f(h) + 1 - h\big)\right) \vee d_f(h)\\ &\le \left(\frac{\tau_f^*(h)}{h} \wedge \big(\tau_f^*(h) + 1 - h\big)\right) \vee \tau_f^*(h),\\ d_f^R(h) &\le \frac{d_f(h)}{h} \wedge 1 \le \frac{\tau_f^*(h)}{h} \wedge 1\end{aligned}$$

et avec les notations du Théorème 1.5.2, pour μ-presque tout $y \in R_f^{\mu,\gamma}(h)$,

$$d_f^{L^y}(h) \le \big(d_f(h) - h\cdot\gamma\big) \vee 0 \le \big(\tau_f^*(h) - h\cdot\gamma\big) \vee 0.$$

Ces bornes supérieures appellent la question suivante : donnent-elles les valeurs exactes des dimensions, en particulier lorsque f satisfait le formalisme multifractal en h ?

En général la réponse est négative. Il suffit pour le voir de considérer une fonction monofractale f d'exposant de Hölder β dont le graphe $G_f = G_f(\beta)$ est irrégulier au sens où ses dimensions de Hausdorff et dimension inférieure de boîte diffèrent. Alors on a

$$1 - \beta + \tau_f^*(\beta) = 1 - \beta + d_f(\beta) = 2 - \beta \ge \underline{\dim}_B G_f(\beta) > \dim_H G_f(\beta).$$

On trouve de tels exemples dans [132, 149, 55]. Si au contraire le graphe de f est régulier (comme pour le mouvement brownien fractionnaire), on a $\dim_H G_f(\beta) = 1 - \beta + \tau_f^*(\beta)$.

Cela dit, la monofractalité concerne une classe très restreinte de fonctions. Comme fonction multifractale, considérons l'exemple suivant : Soit $f(x) = \mu([0, x])$ for $x \in [0, 1]$, où μ est une mesure de probabilité supportée par $[0,1]$, et supposons que f satisfait pleinement le formalisme multifractal associé à l'exposant $\tilde{h}_f(x)$défini par

$$\tilde{h}_f(x) = \lim_{r\to 0^+} \frac{\log \mathrm{Osc}_f(B(x,r))}{\log r},$$

quand cette limite existe. On peut prendre par exemple pour μ une mesure de Gibbs ou une la limite d'une cascade de Mandelbrot. Les résultats de [114] sur l'analyse multifractale de la mesure $\mu^* = \mu \circ f^{-1}$ portée par l'image de f (qui est toujours $[0,1]$ ici), permettent de montrer facilement que les bornes supérieures pour les spectres associés au graphe et à l'image donnent les dimensions exactes. Mais ici, le graphe de f est assez trivial.

Il est intéressant de trouver des exemples de processus stochastiques et de fonctions pour lesquels la détermination des nouveaux spectres que nous avons introduits est non triviale.

Nous considérerons deux classes de fonctions aléatoires multifractales : D'une part les limites de cascades b-adiques étudiées dans la Section 1.2.2, d'autre part des séries d'ondelettes aléatoires dont les coefficients sont construits à partir d'une mesure de Gibbs comme dans [23].

1.3.2 Spectres de singularités pour le graphe, l'image et les ensembles de niveau des limites de cascades b-adiques

Nous considérons la fonction $F = F_W \circ F_L^{-1}$construite à partir d'un couple de vecteurs aléatoires (W,L) dans la Section 1.2.2, et rappelons que pour $U \in \{W,L\}$, nous avons défini $\varphi_U(p) = -\log_b \mathbb{E}(\sum_{i=0}^{b-1}|U_i|^p)$. Nous supposons que les composantes de W sont réelles, et donc F est à valeurs réelles.

Pour $\theta \in [-\pi/2, \pi/2)$, notons l_θ la droite de $\mathbb{R}^2$ passant par l'origine et faisant un angle θ avec l'axe des y. Pour tout $y \in l_\theta$, soit $l^\perp_{y,\theta}$ la droite perpendiculaire à l_θ passant par y. Pour $h \geq 0$, soit $R_{F,\theta}(h)$ la projection orthogonale de $G_F(h)$ sur l_θ, et pour tout $y \in R_{F,\theta}(h)$ soit $L^y_{F,\theta}(h) = G_F(h) \cap l^\perp_{y,\theta}$.

Nous ferons les hypothèses suivantes :
(A1) $\mathbb{P}(\sum_{j=0}^{b-1} W_j = 1) < 1$ et il existe $p \in (1,2]$ tel que $\varphi_W(p) > 0$;
(A2) $\mathbb{P}(\forall\, j, L_j \in (0,1)) = 1$, $\mathbb{E}(\sum_{j=0}^{b-1} L_j \log L_j) < 0$ et φ_L est fini sur $\mathbb{R}$;
(A3) $\mathbb{P}(\forall\, j, |W_j| > 0) = 1$ et φ_W est fini sur $\mathbb{R}$.

Théorème 1.6 (Jin)

1. *Presque sûrement, pour tout* $h \in \{h > 0 : \tau_F^*(h) > 0\}$,

$$\begin{aligned}\dim_H G_F(h) &= \left(\frac{\tau_F^*(h)}{h} \wedge \big(\tau_F^*(h) + 1 - h\big)\right) \vee \tau_F^*(h),\\ \dim_H R_F(h) &= \frac{\tau_F^*(h)}{h} \wedge 1.\end{aligned}$$

De plus, si G_F désigne le graphe de F,

$$\dim_H G_F = \dim_P G_F = \dim_B G_F = 1 - \tau_F(1).$$

2. *Presque sûrement, pour Lebesgue presque tout $\theta \in [-\pi/2, \pi/2)$, pour tout $h \in \{h > 0 : \tau_F^*(h) > h\}$, pour $\mu_{h,\theta}^R$ presque tout $y \in R_{F,\theta}(h)$,*

$$\dim_H L_{F,\theta}^y(h) = \tau_F^*(h) - h,$$

où $\mu_{h,\theta}^R$ est une mesure de Borel positive portée par $R_{F,\theta}(h)$ et elle est absoluement continue par rapport à la mesure de Lebesgue sur la droite l_θ.

1.3.3 Spectres de singularités pour le graphe et l'image de séries d'ondelettes aléatoires

Soit ψ une ondelette mère r_0-régulière sur $\mathbb{R}$, $r_0 \geq 1$, telle que les fonctions $\{\psi_{j,k} = \psi(2^j \cdot - k)\}_{(j,k)\in\mathbb{Z}^2}$ forment une base orthogonale de $L^2(\mathbb{R})$ (voir [119] pour la définitions et la construction). Soit μ une measure de Gibbs sur $[0,1]$ (invariante par la transformation $T(x) = 2x \mod 1$) associée à un certain potentiel höldérien.

Soit $\{\pi_{j,k}\}_{j\geq 0, k=0,\cdots,2^j-1}$ une suite de variables aléatoires indépendantes. Définissons sur $[0,1]$

$$F_\mu^{\mathrm{pert}}(x) = \sum_{j\geq 0} \sum_{k=0}^{2^j-1} \pi_{j,k} \cdot (\pm 2^{-j(s_0-1/p_0)} \mu([k2^{-j}, (k+1)2^{-j}))^{1/p_0}) \cdot \psi_{j,k}(x),$$

où $s_0, p_0 > 0$ et $s_0 - 1/p_0 > 0$.

Nous assumerons les hypothèses :

(A1) Pour tout $q \in \mathbb{R}$, $\sup_{j\geq 0} \sup_{k=0,1,\cdots,2^j-1} \mathbb{E}(|\pi_{j,k}|^q) < \infty$.

(A2) L'ondelette ψ a un nombre fini de zéros dans $[0,1]$.

(A3) Chaque variable $\pi_{j,k}$ a une densité $f_{j,k}$ par rapport à la mesure de Lebesgue, et pour tout $\epsilon > 0$ on a $\sum_{j\geq 0} (\sup_{k=0,\cdots,2^j-1} \|f_{j,k}\|_\infty) \cdot 2^{-j\epsilon} < \infty$.

Théorème 1.7 (Jin) *Presque sûrement, pour tout $h > 0$ tel que $d_{F_\mu^{pert}}(h) \geq 0$,*

$$d_{F_\mu^{pert}}^G(h) = \left(\frac{d_{F_\mu^{pert}}(h)}{h} \wedge \left(d_{F_\mu^{pert}}(h) + 1 - h \right) \right) \vee d_{F_\mu^{pert}}(h),$$

et pour tout $h \in (0,1)$ tel que $d_{F_\mu^{pert}}(h) \geq 0$,

$$d_{F_\mu^{pert}}^R(h) = \frac{d_{F_\mu^{pert}}(h)}{h} \wedge 1.$$

1.3.4 Remarques sur le spectre de singularité associé au graphe

(1) Si nous considérons l'exposant

$$\tilde{h}_f(x) = \lim_{r\to 0} \frac{\log \mathrm{Osc}_f(B(x,r))}{\log(r)}$$

lorsque la limite existe et considérons les plus petits ensembles iso-Hölder

$$\widetilde{E}_f(h) := \{x \in I : \tilde{h}_f(x) = h\},$$

les résultats des Théorèmes 1.6.1 et 1.7 sont valides avec la dimension de packing si nous remplaçons $E_f(h)$ par $\widetilde{E}_f(h)$.

(2) Dans le cas conservatif, en particulier le cas déterministe, la calcul du spectre de singularité associé au graphe des limites de cascades b-adiques est un problème plus difficile que dans le cas non conservatif. Même pour le calcul de la dimension du graphe, il n'y a à notre connaissance de résultat général que pour la dimension de boîte [26] et la dimension de Hausdorff est connue dans des cas spéciaux [27, 59, 149, 9]. Ce problème est fortement lié au calcul explicite de la dimension des mesures auto-similaires avec overlaps, qui est toujours un problème ouvert. Dans certains cas particuliers comme la fonction auto-affine de Urbanski en temps multifractal (intégrale d'un produit de Bernoulli), nous avons quelque espoir d'obtenir le spectre de singularité associé au graphe, et ce spectre pourrait être donné par une formule différente de celle obtenue dans le cas aléatoire non conservatif. Cela nous conduit au point (3).

(3) Quand le graphe de la fonction est irrégulier, la borne supérieure générale que nous avons donnée pour le spectre de singularité associé au graphe peut être strictement supérieure à la valeur du spectre. Quelle peut bien être alors la bonne formule pour la dimension dans un cas pareil ? Remarquons que la formule obtenue dans les Théorèmes 1.6.1 et 1.7 peut être réécrite :

$$d_f^G(h) = (1-h)\cdot d_f^R(h) + d_f(h),\ h \leq 1.$$

En combinant ceci à la formule obtenue pour la dimension du graphe des fonctions auto-affines irrégulières dans [149, 55], nous devinons la formule variationnelle suivante :

$$d_f^G(h) = \sup\Big\{(1-h)\cdot \dim_H(f_*\mu) + \dim_H(\mu) : \mu \in \mathcal{M}(E_f(h))\Big\},\ h \leq 1,$$

où $\mathcal{M}(E_f(h))$ est l'ensemble des mesures de Borel positives et finies portées par $E_f(h)$.

Chapitre 2

Construction of new multifractal stochastic processes

This chapter contains two joint works [17, 18] with Julien Barral and Benoit Mandelbrot which study the convergence of complex [0,1]-martingales and complex b-adic independent cascades.

Preliminaries

Given an integer $b \geq 2$, we denote by $\mathscr{A}$ the alphabet $\{0, \ldots, b-1\}$ and define $\mathscr{A}^* = \bigcup_{n \geq 0} \mathscr{A}^n$ (by convention $\mathscr{A}^0$ is the set reduced to the empty word denoted $\varnothing$).

For every $n \geq 0$, the length of an element of $\mathscr{A}^n$ is by definition equal to n and we denote it by $|w|$.

Let $n \geq 1$ and $w = w_1 \cdots w_n \in \mathscr{A}^n$. Then for every $1 \leq k \leq n$, the word $w_1 \ldots w_k$ is denoted $w|_k$; if $k = 0$, then $w|_0$ stands for $\varnothing$.

For $w \in \mathscr{A}^*$, we define $t_w = \sum_{i=1}^{|w|} w_i b^{-i}$ and $I_w = [t_w, t_w + b^{-|w|})$. For $n \geq 1$ we define $T_n = \{t_w : w \in \mathscr{A}^n\} \cup \{1\}$ and then $T_* = \cup_{n \geq 1} T_n$.

For any $t \in [0, 1)$ and $n \geq 1$, we denote by $t|_n$ the unique word in $\mathscr{A}^n$ such that $t \in I_{t|_n}$, we also denote by $t|_0$ the empty word.

If $f \in \mathcal{C}([0, 1])$, the space of continuous complex-valued functions over $[0, 1]$, and I is a subinterval of $[0, 1]$, $\Delta f(I)$ denotes by the increment of f over I. Also, $\|f\|_\infty$ denotes the norm $\sup_{t \in [0,1]} |f(t)|$.

We denote by $(\Omega, \mathcal{A}, \mathbb{P})$ the probability space on which the random variables are defined. Write $U \equiv V$ to express that the random variable U and V have the same probability distribution. The probability distribution of a random variable V is denoted by $\mathcal{L}(V)$.

2.1 Convergence of complex $[0,1]$-martingales

Positive $\mathbf{T}$-martingales were developed as a general framework that extends the positive measure-valued martingales and are meant to model intermittent turbulence. We extend their scope by allowing the martingale to take complex values. We focus on martingales constructed on the interval $\mathbf{T} = [0,1]$, and replace random measures by random functions. We specify a large class of such martingales, one that contains the complex extension of b-adic canonical cascades, compound Poisson cascades, and more generally infinitely divisible cascades. For this class, we provide a general sufficient condition for almost sure uniform convergence to a non-trivial limit. Such limit yields new examples of naturally generated multifractal processes that may be of use in multifractal signals modeling.

2.1.1 A class of complex $[0,1]$-martingales

Consider a sequence of measurable complex functions

$$P_n : ([0,1]\times\Omega, \mathcal{B}([0,1])\otimes\mathcal{A}) \mapsto (\mathbb{C}, \mathcal{B}(\mathbb{C})),\ n\geq 1.$$

For $n\geq 1$ and I, a subinterval of $[0,1]$, let $\mathcal{A}_n^I$ be the σ-field generated in $\mathcal{A}$ by the family of random variables $\{P_m(t,\cdot)\}_{t\in I, 1\leq m\leq n}$. Also let $\overline{\mathcal{A}}_n^I$ be the σ-field generated in $\mathcal{B}$ by the family of random variables $\{P_m(t,\cdot)\}_{t\in I, m>n}$. The σ-fields $\mathcal{A}_n^{[0,1]}$ and $\overline{\mathcal{A}}_n^{[0,1]}$ are simply denoted by $\mathcal{A}_n$ and $\overline{\mathcal{A}}_n$.

(P1) For all $t\in[0,1]$, $P_n(t,\cdot)$ is integrable and $\mathbb{E}(P_n(t,\cdot)) = 1$.

(P2) For every $n\geq 1$, $\mathcal{A}_n$ and $\overline{\mathcal{A}}_n$ are independent.

(P3) There exist two integers $b\geq 2$ and $N\geq 1$ such that for every $n\geq 1$ and every family $\mathcal{G}$ of b-adic subintervals of $[0,1]$ of generation n such that $d(I,J)\geq Nb^{-n}$ for every $I\neq J\in\mathcal{G}$, the σ-algebra's $\overline{\mathcal{A}}_n^I$, $I\in\mathcal{G}$, are mutually independent, where $d(I,J) = \inf\{|t-s| : s\in I, t\in J\}$.

Under the properties **(P1)** and **(P2)**, for each $t\in(0,1)$ the sequence

$$Q_n(t,\cdot) = \prod_{k=1}^{n} P_k(t,\cdot)$$

is a martingale of expectation 1 with respect to the filtration $\{\mathcal{A}_n\}_{n\geq 1}$.

We denote by $\mathcal{M}$ the class of martingales $(Q_n)_{n\geq 1}$ obtained as above and which satisfy properties **(P1)**–**(P3)**.

We denote by $\mathcal{M}'$ the subclass of $\mathcal{M}$ of those $(Q_n)_{n\geq 1}$ which, in addition to **(P1)**–**(P3)**, satisfy the self-similarity property :

(P4) Let b be as in **(P3)**. For every closed b-adic subinterval I of $[0,1]$, let $n(I)$ and S_I respectively stand for the generation of I and the canonical affine map from $[0,1]$ onto I. The processes $\left(P_{n(I)+n}\circ S_I\right)_{n\geq 1}$ and $(P_n)_{n\geq 1}$ have the same probability distributions.

This property is referred to as statistical self-similarity.

Let λ be a Radon measure on $[0,1]$. If $(Q_n)_{n\geq 1}\in\mathcal{M}$, for $n\geq 1$, we define

$$F_n(t)=\int_0^t Q_n(u)\,\mathrm{d}\lambda(u). \tag{2.1}$$

2.1.2 Convergence theorem for $(F_n)_{n\geq 1}$

For $p\in\mathbb{R}_+$ we define

$$S(n,p)=\sum_{w\in\mathscr{A}^n}\lambda(I_w)^{p-1}\int_{I_w}\mathbb{E}(|Q_n(t)|^p)\,\mathrm{d}\lambda(t) \tag{2.2}$$

and then the concave function

$$\varphi(p)=\liminf_{n\to\infty}-\frac{1}{n}\log_b S(n,p). \tag{2.3}$$

We notice that $\varphi(0)\leq 0$ by construction, and that due to our assumption that $\mathbb{E}(Q_n(t))=1$, we also have $\varphi(1)\leq 0$.

Theorem 2.1

1. *Suppose that $\varphi(p)>0$ for some $p\in(0,1)$. Suppose, moreover, that there exists a function $\psi:\mathbb{N}_+\to\mathbb{R}_+$ such that $\psi(n)=o(n)$ and $\mathbb{E}(\sup_{t\in I_w}|Q_n(t)|^p)\leq e^{\psi(n)}\mathbb{E}(|Q_n(t)|^p)$ for all $n\geq 1$, $w\in\mathscr{A}^n$ and $t\in I_w$. Then, with probability 1, F_n converge uniformly to 0 as $n\to\infty$.*
2. *Let $p\in(1,2]$. Suppose that $\varphi(p)>0$. The functions F_n converge uniformly, almost surely and in L^1 norm, to a limit F, as $n\to\infty$. The function F is γ-Hölder continuous for all $\gamma\in(0,\max_{q\in(1,p]}\varphi(q)/q)$. Moreover, $\mathbb{E}(\|F\|_\infty^p)<\infty$.*

Theorem 2.1 provides a sufficient condition for the almost sure uniform convergence of F_n, as n tends to ∞, to a limit F such that $\mathbb{P}(F\neq 0)>0$. It invokes the growth rate of a kind of free energy function, and it is the extension of the condition introduced in Part II. of [20] to show that when $(Q_n)_{n\geq 1}$ is non-negative, the sequence of measures F_n' converge almost surely weakly to a random measure μ such that $\mathbb{P}(\mu\neq 0)>0$. In the non-negative case, the fact that the measures F_n' converges almost surely weakly [90] is a quite simple consequence of positive martingale convergence theorem and Riesz representation theorem. Moreover, this fact holds for any $[0,1]$-martingale. What is shown in [20] is that under our sufficient condition, when $(Q_n)_{n\geq 1}$ is non-negative and belongs to $\mathcal{M}$, the martingale $\|F_n'\|$ converges, almost surely and in L^p norm for some $p\geq 1$, hence $\mathbb{E}(\|\mu\|)=1$ and $\mathbb{P}(\mu\neq 0)>0$. When $(Q_n)_{n\geq 1}$ is not non-negative, the uniform convergence of F_n is a far more delicate issue.

2.1.3 Examples

Homogeneous b-adic independent cascades

Let b be an integer ≥ 2 and for every $k \geq 0$ let $W^{(k)} = (W_0^{(k)}, \dots, W_{b-1}^{(k)})$ be a vector such that each of its components is complex, integrable, and has an expectation equal to 1. Then, consider $\{W^{(|w|)}(w)\}_{w\in\mathscr{A}^*}$, a family of independent vectors such that for each $k \geq 0$ and $w \in \Sigma_k$ the vector $W^{(k)}(w)$ is a copy of $W^{(k)}$. A $[0,1]$-martingale is obtained as follows. For $t \in [0,1)$ and $n \geq 1$ we define

$$P_n(t) = W_{t_n}^{(n-1)}(t|_n - 1) \text{ and then } Q_n(t) = \prod_{k=1}^{n} P_k(t) = \prod_{k=1}^{n} W_{t_k}^{(k-1)}(t|_{k-1}).$$

By construction, $(Q_n)_{n\geq 1} \in \mathcal{M}$. If we suppose that λ is the generalized Bernoulli measure associated with a sequence of probability vectors $(\lambda^{(k)} = \lambda_0^{(k)}, \dots, \lambda_{b-1}^{(k)})_{k\geq 0}$, i.e., $\lambda(I_{w_1\cdots w_n}) = \prod_{k=1}^n \lambda_{w_k}^{(k-1)}$ for all $w_1 \cdots w_n \in \mathscr{A}^*$, then

$$\varphi(p) = \liminf_{n\to\infty} -\frac{1}{n} \sum_{k=0}^{n-1} \log_b \mathbb{E}\Big(\sum_{i=0}^{b-1} \big(\lambda_i^{(k)} \, |W_i^{(k)}|\big)^p \Big).$$

If we suppose that all the vectors $W^{(k)}$ have the same distribution as a vector W, and that all the vectors $\lambda^{(k)}$ are equal to a vector $(\lambda_0, \dots, \lambda_{b-1})$ (λ is a Bernoulli measure), then $(Q_n)_{n\geq 1}$ belongs to $\mathcal{M}'$ and

$$\varphi(p) = -\log_b \mathbb{E}\Big(\sum_{i=0}^{b-1} \big(\lambda_i \, |W_i|\big)^p \Big).$$

Canonical cascades correspond to the case where the components of W are i.i.d. and λ is the Lebesgue measure, see Figure 2.1 for illustration.

Remark 2.1 *If the components of W are non negative, we can fully recover the model for turbulence considered in [108] as limit of a $[0,1]$-martingale. There, a vector $W = (W_0, \dots, W_{b-1})$ with non-negative components and such that $\mathbb{E}(\sum_{i=0}^{b-1} W_i) = b$ is considered without the assumption that the components have the same expectation. Then, a sequence of independent copies $\{W(w)\}_{w\in\mathscr{A}^*}$ is fixed, and though in general for a given $t \in [0,1]$ the sequence $Q_n(t) = \prod_{k=1}^n W_{t_k}(t|_{k-1})$ is not a martingale, the sequence of measures μ_n whose density with respect to the Lebesgue measure is Q_n converges almost surely weakly to a measure μ. We can recover this model as a martingale by defining λ as the Bernoulli measure associated with the probability vector*

$$(\lambda_0 = \mathbb{E}(W_0)/b, \dots, \lambda_{b-1} = \mathbb{E}(W_{b-1})/b)$$

and replacing W by $\widetilde{W} = (W_0/b\lambda_0, \dots, W_{b-1}/b\lambda_{b-1})$ (with the convention $0 : 0 = 0$). Then, the weak limit of $(F_n')_{n\geq 1}$ is almost surely equal to μ. It seems that this fact had not been noticed before.

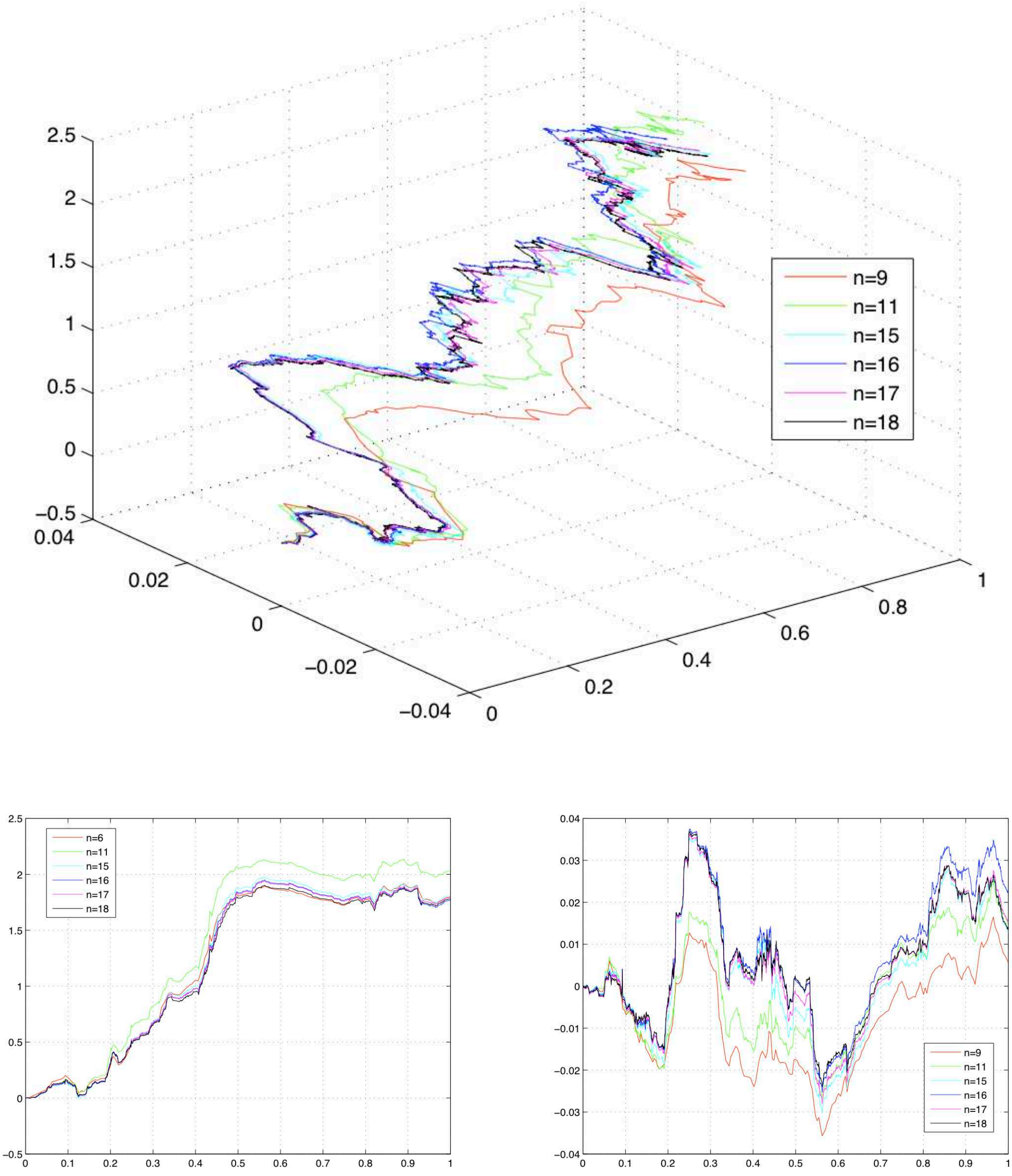

Figure 2.1 – A complex valued canonical cascade F_n (Top), as well as its real (left-bottom) and imaginary (right-bottom) parts for $n = 9,\ 11,\ 15,\ 16,\ 17,\ 18$.

Compound Poisson cascades

Let ν be a positive Radon measure over $(0,1]$ and denote by Λ the measure $\mathrm{Leb}\otimes\nu$, where Leb stands for the Lebesgue measure over $\mathbb{R}$. We consider a Poisson point process S of intensity Λ. To each point M of S, we associate a random variable W_M picked in a collection of random variables that are independent, independent of S, and are identically distributed with an integrable complex random variable W.

We fix $\beta > 0$, and for $\varepsilon \in (0,1]$ and $t \in [0,1]$ we define the truncated cone

$$\mathcal{C}_\varepsilon(t) = \big\{(t', r) : \varepsilon < r \leq 1,\ t - \beta r/2 \leq t' < t + \beta r/2\big\}.$$

Then, we obtain an element of $\mathcal{M}$ as follows. For $t \in [0,1)$ and $n \geq 1$ we define

$$P_n(t) = \exp\big(-\Lambda\big(\Delta\mathcal{C}_n(t)\big)(\mathbb{E}(W)-1)\big) \prod_{M\in S\cap\Delta\mathcal{C}_n(t)} W_M$$

and then $Q_n(t) = \displaystyle\prod_{k=1}^{n} P_k(t)$, where $\Delta\mathcal{C}_n(t) = \mathcal{C}_{b^{-n}}(t) \setminus \mathcal{C}_{b^{1-n}}(t)$. If λ is taken equal to the Lebesgue measure, we get

$$\varphi(p) = p - 1 + \widetilde{\beta}\big(p(\mathbb{E}(\Re W) - 1) - (\mathbb{E}(|W|^p) - 1)\big),\ \text{with } \widetilde{\beta} = \limsup_{n\to\infty} \frac{1}{n}\log_b \Lambda(\mathcal{C}_{b^{-n}}).$$

If, moreover, there exists $\delta > 0$ such that $\nu(\mathrm{d}r) = \delta\mathrm{d}r/r^2$, i.e. if Λ possesses scaling invariance properties, we have $\widetilde{\beta} = \beta\delta$ and $(Q_n)_{n\geq 1}$ belongs to $\mathcal{M}'$.

Representation as continuous time $[0,1]$-martingales

It is natural to consider the left-continuous martingale

$$\widetilde{Q}_\varepsilon(t) = \exp\big(-\Lambda(\mathcal{C}_\varepsilon(t))(\mathbb{E}(W)-1)\big) \prod_{M\in S\cap\mathcal{C}_\varepsilon(t)} W_M$$

and the functions $\widetilde{F}_\varepsilon(t) = \int_0^t \widetilde{Q}_\varepsilon(t)\,\mathrm{d}t$ rather than Q_n and $F_n = \widetilde{F}_{b^{-n}}$. Under the assumptions of Theorem 2.1, the fact that $\|F\|_\infty$ is integrable, as well as the left continuity of $\widetilde{F}_\varepsilon$ in ε and its martingale property ensure that $\widetilde{F}_\varepsilon$ converges uniformly to F, almost surely and in L^1 norm, as $\varepsilon \to 0$.

Remark 2.2 *Complex-valued compound cascades are the extension considered in [13] of the non-negative products of cylindrical pulses introduced in [19]. However, in [13] the uniform convergence of $\widetilde{F}_\varepsilon$ is not studied; only the martingale $\widetilde{F}_\varepsilon(1)$ is considered.*

Log-infinitely divisible cascades

This example is an extension of compound Poisson cascades when the weights W_M take the form $\exp(L_M)$, in particular the W_M do not vanish. We use the notations of the previous section and take $\beta = \delta = 1$. Let ψ be a characteristic Lévy exponent ψ defined on $\mathbb{R}^2$, i.e.

$$\psi : \xi \in \mathbb{R}^2 \mapsto i\langle \xi | a\rangle - Q(\xi)/2 + \int_{\mathbb{R}^2} \big(1 - e^{i\langle \xi | x\rangle} + i\langle \xi | x\rangle \mathbf{1}_{|x|\le 1}\big)\, \pi(\mathrm{d}x), \tag{2.4}$$

where $a \in \mathbb{R}^2$, Q is a non-negative quadratic form and π is a Radon measure on $\mathbb{R}^2 \setminus \{0\}$ such that $\int (1 \wedge |x|^2)\pi(\mathrm{d}x) < \infty$.

Then let $\rho = (\rho_1, \rho_2)$ be an independently scattered infinitely divisible random $\mathbb{R}^2$-valued measure on $\mathbb{R} \times \mathbb{R}_+^*$ with Λ as control measure and ψ as Lévy exponent (see [133] for the definition). In particular, for every Borel set $B \in \mathbb{R} \times \mathbb{R}_+^*$ and $\xi \in \mathbb{R}^2$ we have,

$$\mathbb{E}(e^{i\langle \xi | \rho(B)\rangle}) = \exp\big(\psi(\xi)\Lambda(B)\big),$$

and for every finite family $\{B_i\}$ of pairwise disjoint Borel subsets of $\mathbb{R} \times \mathbb{R}_+^*$ such that $\Lambda(B_i) < \infty$, the random variables $\rho(B_i)$ are independent.

Let I_1 be the interval of those $\xi_1 \in \mathbb{R}$ such that $\int_{|x|\ge 1} e^{\xi_1 x_1}\pi(\,\mathrm{d}x) < \infty$. The function ψ has a natural extension $\widetilde{\psi}$ to $\mathcal{D} = \mathbb{R}^2 \cup (-iI_1 \times \mathbb{R})$ given by the same expression as in (2.4) if we extend Q to an Hermitian form on $\mathbb{C}^2$. Then for every $\xi \in \mathcal{D}$ and every Borel subset of $\mathbb{R} \times \mathbb{R}_+^*$ we have $\mathbb{E}(e^{i\langle \xi | \rho(B)\rangle}) = \exp\big(\widetilde{\psi}(\xi)\Lambda(B)\big)$.

Now, we assume that $\xi_0 = (-i, 1) \in \mathcal{D}$, and without loss of generality we set

$$\widetilde{\psi} := \widetilde{\psi} - \widetilde{\psi}(\xi_0).$$

Then, with the same definition of cones as in the previous section, if $n \ge 1$ and $t \in [0,1]$ we define

$$P_n(t) = \exp\big[\langle \xi_0 | \rho\big(\Delta\mathcal{C}_n(t)\big)\rangle\big] = \exp\big[\rho_1\big(\Delta\mathcal{C}_n(t)\big) + i\rho_2\big(\Delta\mathcal{C}_n(t)\big)\big]$$

and $Q_n(t) = \prod_{k=1}^{n} P_n(t)$. If we take λ equal to the Lebesgue measure, and if $p \in \mathbb{R}$ is such that $(-ip, 0) \in \mathcal{D}$, then

$$\varphi(p) = p - 1 - \widetilde{\beta}\widetilde{\psi}(-ip, 0). \tag{2.5}$$

In the positive case, this construction has been proposed has an extension of compound Poisson cascades in [6] for the statistically self-similar case ($\nu(\mathrm{d}r) = \mathrm{d}r/r^2$) and in [44] for the general case.

If $\nu(\mathrm{d}r) = \mathrm{d}r/r^2$, then $(Q_n)_{n\ge 1}$ belongs to $\mathcal{M}'$. In [6], a modification of the cones $\mathcal{C}_\varepsilon$ is introduced, which yields a nice exact statistical scaling invariance property for the increments of the limit measure. This property, which is different from the statistical self-similarity imposed by **(P4)**, also holds for the complex extension.

Representation as continuous time $[0,1]$-martingales, and exact scaling property

Here, we use the different cones introduced in [6], namely

$$\widetilde{\mathcal{C}}_\varepsilon(t) = \left\{(t', r) : r \ge \varepsilon,\ \begin{cases} t - r/2 \le t' < t + r/2 & \text{if } r \le 1 \\ t - 1/2 \le t' < t + 1/2 & \text{otherwise} \end{cases}\right\}.$$

Then we define the left-continuous $\mathcal{C}([0,1])$-valued martingale

$$\widetilde{F}_\varepsilon(t) = \int_0^t \widetilde{Q}_\varepsilon(t)\,\mathrm{d}\lambda(t), \text{ with } \widetilde{Q}_\varepsilon(t) = \exp\big[\rho_1\big(\widetilde{\mathcal{C}}_\varepsilon(t)\big) + i\rho_2\big(\widetilde{\mathcal{C}}_\varepsilon(t)\big)\big].$$

If we set $Q_n(t) = \widetilde{Q}_{b^{-n}}(t)$, the function φ is still given by (2.5). Under the assumptions of Theorem 2.1, as $\varepsilon \to 0$, $\widetilde{F}_\varepsilon$ converges uniformly, almost surely and in L^1 norm, to the limit F of $F_n(\cdot) = \int_0^\cdot Q_n(u)\,\mathrm{d}u$.

Then, if we take $\nu(\mathrm{d}r) = \mathrm{d}r/r^2$, a direct verification shows that the arguments developed in [6] to obtain an exact statistical scaling invariance property in the positive case can be extended to the complex case. As a consequence, for every $\varepsilon \in (0,1]$ and every subinterval J of $[0,1]$, there exists a copy W_J of $\widetilde{Q}_{|J|}(0)$, as well as a copie F^J of F, independent of W_J, such that

$$F_{|_J} \equiv |J|\, W_J\, F^J \circ f_J^{-1},$$

where $F_{|_J}$ is the restriction of F to J, f_J is the unique increasing affine map from $[0,1]$ onto J, and $\equiv$ means equality in distribution. In particular, if $\varphi(p) > 0$ for some $p \in (1,2]$, then, for all $q \in [0,p]$ and $t \in [0,1]$,

$$\mathbb{E}(|F(t)|^q) = t^{1+\varphi(q)}\mathbb{E}(|F(1)|^q).$$

More elaborate constructions

An important property of $\mathcal{M}$ is that if $(Q_n)_{n\geq 1}$ and $(\widetilde{Q}_n)_{n\geq 1}$ belong to $\mathcal{M}$ and are independent, then their product still belongs to $\mathcal{M}$. This makes it possible to combine the previous constructions with some other proposed in [20].

2.2 Convergence of complex b-adic independent cascades

The familiar cascade measures are sequences of random positive measures obtained on $[0,1]$ via b-adic independent cascades. To generalize them, we allows the random weights invoked in the cascades to take negative or complex values. This yields sequences of random functions whose possible strong or weak limits are natural candidates for modeling multifractal phenomena. Their asymptotic behavior is investigated, yielding a sufficient condition for almost sure uniform convergence to non-trivial statistically self-similar limits.

2.2.1 Construction

Let $W = (W_0, \dots, W_{b-1})$ be a complex vector whose components are integrable and satisfy $\mathbb{E}(\sum_{i=0}^{b-1} W_i) = 1$. Then let $(W(w))_{w\in\mathscr{A}^*}$ be a sequence of independent copies of W and consider the sequence of random functions

$$F_n(t) = F_{W,n}(t) = \int_0^t b^n \prod_{k=1}^n W_{u_k}(u|_{k-1})\,\mathrm{d}u \tag{2.6}$$

(where each non b-adic point u is identified with the word $u_1 \cdots u_n \cdots$ defined by the b-adic expansion $u = \sum_{k\geq 1} u_k b^{-k}$, and we recall that $u|_{k-1} = u_1 \cdots u_{k-1}$). A special case playing an important role in the sequel is the conservative one, i.e. when $\sum_{i=0}^{b-1} W_i = 1$ almost surely. Remarkable functions obtained as limit of such deterministic sequences F_n are the self-affine functions considered for instance in [96, 26, 148] (these functions are called self-affine because their graphs are self-affine sets).

For $p \in \mathbb{R}_+$ let

$$\varphi_W(p) = -\log_b \mathbb{E}\Big(\sum_{i=0}^{b-1} |W_i|^p\Big). \tag{2.7}$$

Our assumptions imply $-1 \leq \varphi_W(0) \wedge \varphi_W(1) \leq \varphi_W(0) \vee \varphi_W(1) \leq 0$.

2.2.2 Strong uniform convergence

We give sufficient conditions for the almost sure uniform convergence of $(F_n)_{n\geq 1}$. We distinguish the conservative and non-conservative cases. Our results are illustrated in Figures 2.2 to 2.4.

Theorem 2.2 (Non-conservative case) *Suppose that* $\mathbb{P}\Big(\sum_{i=0}^{b-1} W_i \neq 1\Big) > 0$ *and there exists* $p > 1$ *such that* $\varphi_W(p) > 0$. *Suppose, moreover, that either* $p \in (1, 2]$ *or* $\varphi_W(2) > 0$.

1. *$(F_n)_{n\geq 1}$ converges uniformly, almost surely and in L^p norm, as n tends to ∞, to a function $F = F_W$, which is non decreasing if $W \geq 0$. Moreover, the function F is γ-Hölder continuous for all γ belonging to $(0, \max_{q\in(1,p]} \varphi_W(q)/q)$.*
2. *F satisfies the statistical scaling invariance property :*

$$F = \sum_{i=0}^{b-1} \mathbf{1}_{[i/b,(i+1)/b]}\Big(F(i/b) + W_i\, F_i \circ S_i^{-1}\Big), \tag{2.8}$$

where $S_i(t) = (t + i)/b$, the random objects W, $F_0, \ldots, F_{b-1}$ are independent, and the F_i are distributed like F and the equality holds almost surely.

Theorem 2.3 (Conservative case) *Suppose that* $\mathbb{P}\Big(\sum_{i=0}^{b-1} W_i \neq 1\Big) = 0$.

1. *If there exists $p > 1$ such that $\varphi_W(p) > 0$, then the same conclusions as in Theorem 2.2 hold.*
2. **(Critical case)** *Suppose that $\lim_{p\to\infty} \varphi_W(p) = 0$ (in particular φ_W is increasing and $\varphi_W(p) < 0$ for all $p > 1$). This is equivalent to the fact that $\mathbb{P}(\forall\ 0 \leq i \leq b-1,\ |W_i| \leq 1) = 1$ and $\sum_{i=0}^{b-1} \mathbb{P}(|W_i| = 1) = 1$.*

   Suppose also that $\mathbb{P}(\#\{i : |W_i| = 1\} = 1) < 1$, and there exists $\gamma \in (0, 1)$ such that, with probability 1, one of the two following properties holds for each $0 \leq i \leq b-1$

$$\begin{cases} \text{either } |W_i| \leq \gamma, \\ \text{or } |W_i| = 1 \text{ and } \Big(\sum_{k=0}^{i-1} W_i, \sum_{k=0}^{i} W_i\Big) \in \{(0,1),(1,0)\} \end{cases}. \tag{2.9}$$

   Then, with probability 1, $(F_n)_{n\geq 1}$ converges almost surely uniformly to a limit $F = F_W$ which is not uniformly Hölder and satisfies part 2. of Theorem 2.2.

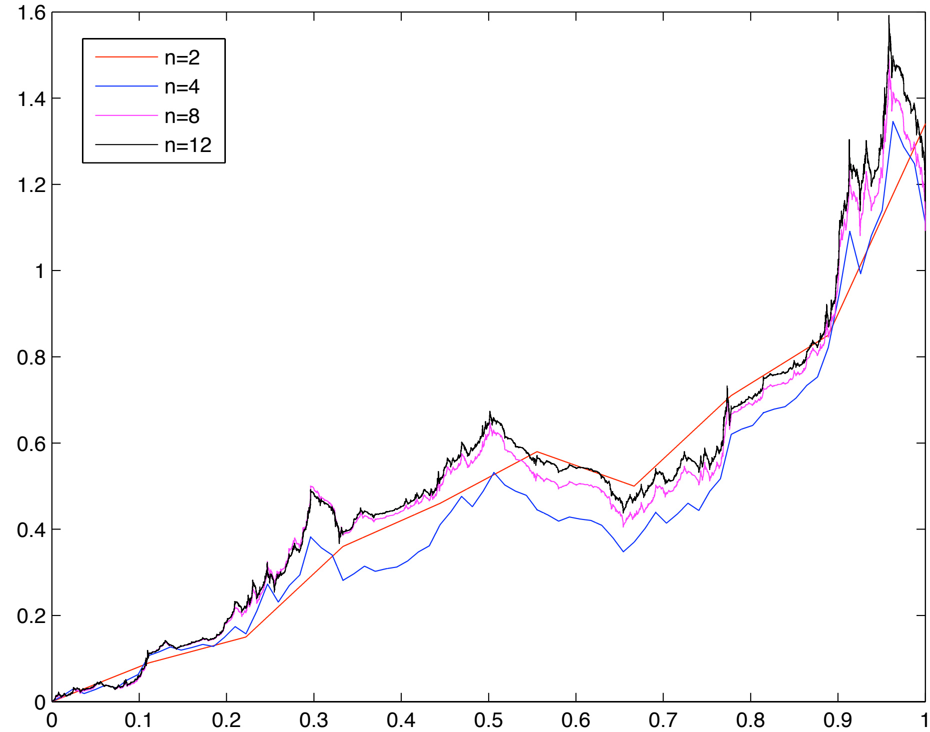

FIGURE 2.2 – Uniform convergence in the non-conservative case : $F_{W,n}$ for $n = 2,\ 4,\ 8,\ 12$ in the case $b = 3$ and $\varphi_W(\beta) = 0$ for $\beta \approx 1.395$.

FIGURE 2.3 – Uniform convergence in the non-critical conservative case : $F_{W,n}$ for $n = 2,\ 4,\ 8,\ 12$ in the case $b = 3$ and $\varphi_W(\beta) = 0$ for $\beta \approx 2.172$.

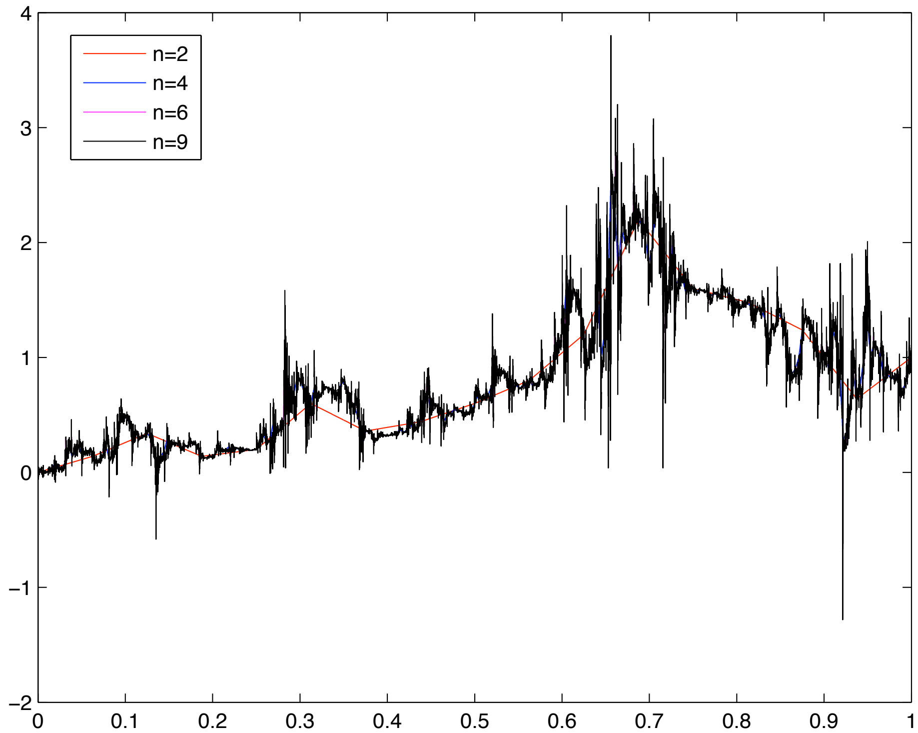

FIGURE 2.4 – Uniform convergence in the critical conservative case : $F_{W,n}$ for $n = 2,\ 4,\ 6,\ 9$ in the case $b = 4$ and $\varphi_W(p) < 0$ on $\mathbb{R}_+$, $\varphi_W(p) \to 0$ when $p \to \infty$. The limit is not uniformly Hölder.

Remark 2.3

(1) The sufficient condition for the convergence in L^p of complex-valued martingales like $(F_n(1))_{n\geq 1}$ is known in the context of martingales in the branching random walk ([33, 12]) ; however, the sequence of functions $(F_n)_{n\geq 1}$ is not considered in these papers. When W has non-negative components, it follows from [92] and [48] that this condition is necessary.

(2) Theorem 2.3.2 goes far beyond the construction of deterministic self-affine functions ([26, 148]) which all fall in Theorem 2.3.1.

(3) The following discussion will be useful for the statement and proof of Theorem 2.6. It is easily seen that F_n vanishes ($F_n = 0$) if and only if $\prod_{k=1}^n W_{w_k}(w|_{k-1}) = 0$ for all $w \in \mathscr{A}^n$, and in this case, $F_k = 0$ for all $k > n$. Thus, if we denote by $\mathscr{V}$ the event $\{\exists\, n \geq 1 :\ \ F_n = 0\}$, we have $\mathscr{V} = \liminf_{n\to\infty}\{F_n = 0\}$. Notice that $\mathbb{P}(\mathscr{V}) = 0$ in the conservative case.

By construction, there are b independent copies $(F_{i,n})_{n\geq 1}$ of $(F_n)_{n\geq 1}$, independent of W, and converging respectively to F_i almost surely, such that for $n \geq 1$ we can write

$$F_n = \sum_{i=0}^{b-1} \mathbf{1}_{[i/b,(i+1)/b]}\Big(F_n(i/b) + W_i\, F_{i,n-1} \circ S_i^{-1}\Big).$$

Thus, $\{F_n = 0\} = \bigcap_i \{W_i F_{i,n-1} = 0\}$. Similarly, $\{F = 0\} = \bigcap_i \{W_i F_i = 0\}$. It is then not difficult to see that under the assumptions of Theorem 2.2, $\mathbb{P}(\mathscr{V})$ and $\mathbb{P}(F = 0)$ are *equal to the unique fixed point distinct from 1 of the convex polynomial*

$$P(x) = \sum_{k=0}^{b} \mathbb{P}\big(\#\{0 \le i \le b-1 : |W_i| > 0\} = k\big)x^k.$$

Consequently, since $\mathscr{V} \subset \{F = 0\}$*, these events differ from a set of null probability.*

We can also interpret the set of words $w \in \mathscr{A}^n$ *such that* $\prod_{k=1}^n W_{w_k}(w|_{k-1}) = 0$ *as the nodes of generation* n *of a Galton-Watson tree whose offspring distribution is given by that of the integer* $N = \#\{0 \le i \le b-1 : |W_i| > 0\}$*. Then,* $P(x) = \mathbb{E}(x^N)$*.*

We end this section by introducing auxiliary non-negative b-adic independent cascades, which will play an important role in the rest of this chapter.

Definition 2.1 *If* $\beta > 0$ *and* $\varphi_W(\beta) > -\infty$*, then for* $w \in \mathscr{A}^*$ *let*

$$W^{(\beta)}(w) = b^{\varphi_W(\beta)}(|W_0(w)|^\beta, \dots, |W_{b-1}(w)|^\beta),$$

and simply denote $W^{(\beta)}(\emptyset)$ *by* $W^{(\beta)}$*. We have* $\varphi_{W^{(\beta)}}(p) = \varphi_W(\beta p) - p\varphi_W(\beta)$ *for all* $p > 0$*. In particular,* $\varphi_{W^{(\beta)}}(1) = 0$*. If* $\varphi_{W^{(\beta)}}(p) > 0$ *for some* $p \in (1, 2)$*, we denote by* $F_{W^{(\beta)}}$ *the non-decreasing function obtained in Theorem 2.2 as the almost sure uniform limit of* $F_{W^{(\beta)},n} : t \in [0,1] \mapsto \int_0^t b^n \prod_{k=1}^n W^{(\beta)}_{u_k}(u|_{k-1})\, du$*.*

2.2.3 Representation as a monofractal function in multifractal time

As explained in the introduction, in order to qualitatively compare the strong limit F_W obtained in Theorem 2.2 with other models of multifractal processes, it is important to study the possibility to decompose it as a monofractal function in multifractal time. Under the assumptions of Theorem 2.2 and Theorem 2.3.1 , if we denote by β the smallest solution of $\varphi_W(p) = 0$, the only natural choice at our disposal as time change is the function $F_{W^{(\beta)}}$ introduced in Definition 2.1. In the deterministic case, it is elementary to check that $B_{1/\beta} = F_W \circ F^{-1}_{W^{(\beta)}}$ is monofractal (Section 4.7 in [111]) in the strong sense, where by strong sense we mean the limit inferior of the pointwise Hölder exponet is a limit everywhere and the value is independent of the point. In the random case, this also true under strong assumptions on the moments of W as shows Theorem 2.4, which is illustrated in Figures 2.5 and 2.6. We do not know whether or not weaker assumptions on W lead to situations in which $B_{1/\beta}$ is not monofractal. However, the functions constructed in Theorem 2.3.2 provide simple examples of statistically self-similar continuous functions for which it seems to be impossible to find a natural decomposition as monofractal functions in multifractal time ; at least such a time change cannot be obtained as limit of a positive b-adic independent cascade.

Theorem 2.4 *Suppose that* $\mathbb{P}(W \in \mathbb{C}^b \setminus \mathbb{R}_+^b) > 0$ *and* $\sum_{i=0}^{b-1} \mathbb{E}(|W_i|^p) < \infty$ *for all* $p \in \mathbb{R}$*.*

Suppose also that the assumptions of Theorem 2.2 or Theorem 2.3.1 hold. Let β *be the smallest solution of* $\varphi_W(p) = 0$ *and suppose that* $\varphi_W(p) > 0$ *for all* $p > \beta$*. We have* $\beta > 1$*,* $\varphi_{W^{(\beta)}}(1) = 0$ *and* $\varphi_{W^{(\beta)}}(p) > 0$ *for all* $p > 1$*. Let* $B_{1/\beta} = F_W \circ F^{-1}_{W^{(\beta)}}$*.*

With probability 1, the function $B_{1/\beta}$ is a monofractal function in the strong sense, with constant pointwise Hölder exponent $1/\beta$.

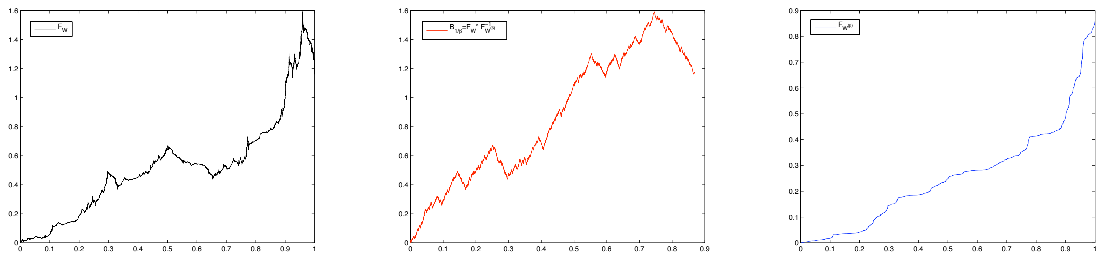

FIGURE 2.5 – The limit function F_W in Figure 2.2 (Left) can be written as the monofractal function $B_{1/\beta} = F_W \circ F^{-1}_{W^{(\beta)}}$ of exponent $H = 1/\beta \approx 0.7168$ (Middle) in multifractal time $F_{W^{(\beta)}}$ (Right).

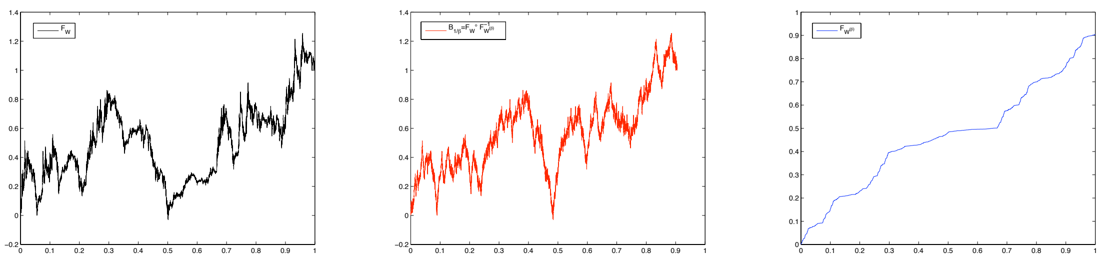

FIGURE 2.6 – The limit function F_W in Figure 2.3 (Left) can be written as the monofractal function $B_{1/\beta} = F_W \circ F^{-1}_{W^{(\beta)}}$ of exponent $H = 1/\beta \approx 0.4604$ (Middle) in multifractal time $F_{W^{(\beta)}}$ (Right).

Theorem 2.5 *Suppose that the assumptions of Theorem 2.3.2 hold and the components of W do not vanish. With probability 1, if the function F_W can be decomposed as a monofractal function B in a multifractal time G, then the Hölder exponent of B is 0, and G must be such that* $\dim\big([0,1]\setminus \overline{E}_G(\infty)\big) = 0$, *where*

$$\overline{E}_G(\infty) = \Big\{t \in [0,1] : \limsup_{r\to 0^+} \frac{\log(\mathrm{Osc}_G([t-r,t+r]))}{\log(r)} = \infty\Big\}.$$

Remark 2.4

(1) Notice that due to Theorems 2.2, $1/\beta$ *must belong to* $(1/2,1)$ *when* $\mathbb{P}(\sum_{k=0}^{b-1} W_k = 1) < 1$.

(2) When W is deterministic, we are necessarily in the conservative case $\sum_{i=0}^{b-1} W_i = 1$, *and Theorem 2.4.1 is well known (see for instance Section 4.7 in [111]). In this case, it is also the simplest illustration of the general result obtained in [143] regarding the representation of multifractal functions as monofractal functions (in the strong sense) in multifractal time (see also [84] for another illustration of this concept).*

(3) Under the assumptions of Theorem 2.4.1, it seems possible to obtain the result by using the general approach developed in [143]. However, this necessitates to use the extension to the present context of some sophisticated estimates developed for positive cascades in [24]. Thus, we will give a short and self-contained proof.

(4) The moments of $\dfrac{Osc_{F_W}([0,1])}{F_{W^{(\beta)}}(1)^{1/\beta}}$ *are all finite under the strong assumptions of Theorem 2.4. Suppose that we have found sufficient conditions under which there exists* $q \neq 0$ *such that* $\mathbb{E}\Big[\Big(\dfrac{Osc_{F_W}([0,1])}{F_{W^{(\beta)}}(1)^{1/\beta}}\Big)^q\Big] = \infty$*. Then, we know how to prove that the function* $B_{1/\beta}$ *is not monofractal if* $q > 0$*, and not strongly monofractal if* $q < 0$*. Thus, to finding information on the* $\dfrac{Osc_{F_W}([0,1])}{F_{W^{(\beta)}}(1)^{1/\beta}}$ *moments behavior under weak assumptions on* W *remains an important open question to complete the description of* $B_{1/\beta}$*.*

(5) As a consequence of Theorem 2.5, we get that the time change G *cannot be equivalent to the limit of a positive b-adic independent cascade obtained as in Theorem 2.2 or 2.3, in the sense that their derivative in the distribution sense are equivalent positive measures. Indeed, in this case the analysis of such a measure achieved in [92] or [10], implies that there exists* $D > 0$ *such that* $\lim_{n\to 0^+} \frac{\log_b(\mathrm{Osc}_G(I_n(t)))}{-n} = D$ *at each point* t *of a set* E *of Hausdorff dimension* D*.*

Also, we can notice that if a time change G *as described in Theorem 2.5 does exist, from the multifractal analysis point of view, the decomposition would not simplify the study of* F_W*, since it would necessitate to have a very fine description of the pointwise divergence of* $\frac{\log(\mathrm{Osc}_G([t-r,t+r]))}{\log(r)}$ *inside the set* $\overline{E}_G(\infty)$*.*

An example of increasing function G *such that* $\dim\big([0,1]\setminus \overline{E}_G(\infty)\big) = 0$ *is obtained as follows :*

$$G(t) = \int_0^t \sum_{n\geq 1}\sum_{0\leq k<b^n} b^{-n^2}\Big(b^{n^3}\mathbf{1}_{[kb^{-n},kb^{-n}+b^{-n^3}]}(u)\Big)\,du.$$

We leave the reader check that t *is not a b-adic number and if*

$$\delta_t = \limsup_{n\to\infty}\sup_{0\leq k<b^n}\frac{\log(|t-kb^{-n}|)}{-n} < \infty$$

then $\lim_{r\to 0^+}\frac{\log(\mathrm{Osc}_G([t-r,t+r]))}{\log(r)} = \infty$*. Moreover, it is clear that* $\dim\{t : \delta_t = \infty\} = 0$*.*

2.2.4 Degeneracy, divergence and weak uniform convergence

The following results are proven by Barral and Mandelbrot, here we only present the statements and the proofs can be found in [18].

Recall that φ_W is concave, $\varphi_W(0) < 0$ and $\varphi_W(1) \leq 0$. Let us define

$$p_0 = \sup\{p : \varphi_W'(p) \text{ exists and } \varphi_W'(p)p - \varphi_W(p) > 0\}.$$

In order to simplify the next discussion, we assume that $\varphi_W(p) > -\infty$ for all $p \geq 0$.

Since $\varphi_W(0) < 0$, we have $p_0 > 0$. Also, if $p_0 < \infty$ then $\varphi_W(p_0) = 0$ if and only if $\varphi_W(p_0) = \varphi_W'(p_0) = 0$. If $p_0 = \infty$, we define $\varphi_W(p_0) = \lim_{p\to\infty}\varphi_W(p)$.

In the previous section, we have dealt with the convergence of F_n in the following cases that we gather in the condition **(C)** :

(C) : One of the following three cases arise.

(1) There exists $p \in (1,2]$ such that $\varphi_W(p) > 0$.

(2) $\mathbb{P}(\sum_{k=0}^{b-1} W_k = 1) = 1$ and there exists $p > 1$ such that $\varphi_W(p) > 0$;

(3) $\mathbb{P}(\sum_{k=0}^{b-1} W_k = 1) = 1$, $\mathbb{P}(\forall\ 0 \le i \le b-1,\ |W_i| \le 1) = 1$, $\sum_{i=0}^{b-1} \mathbb{P}(|W_i| = 1) = 1$ and $\mathbb{P}(\#\{i : |W_i| = 1\} = 1) < 1$. Equivalently, $\lim_{p\to\infty} \varphi_W(p) = 0$ and $\mathbb{P}(\#\{i : |W_i| = 1\} = 1) < 1$.

Suppose that **(C)** does not hold. We cannot have simultaneously $p_0 \in (1,2]$ and $\varphi_W(p_0) > 0$. Also, $\varphi_W(2) \le 0$. Moreover, if $\mathbb{P}(\sum_{k=0}^{b-1} W_k = 1) = 1$, then $\varphi_W(p_0) \le 0$, and if $p_0 = \infty$ then $\sum_{i=0}^{b-1} \mathbb{P}(|W_i| = 1) > 1$ or $\mathbb{P}(\#\{i : |W_i| = 1\} = 1) = 1$ (see also Remark 2.5).

The following results concern the asymptotic behavior of F_n when **(C)** does not hold. Before stating it, we recall the discussion of Remark 2.3 (3).

Theorem 2.6 (Barral,Mandelbrot) (Degeneracy and divergence)

Suppose that **(C)** *does not hold and $\varphi_W(p) > -\infty$ for all $p \ge 0$.*

1. *Suppose that $p_0 \in (0,1]$. Then, for all $\alpha \le \varphi_W(p_0)/p_0$, $b^{n\alpha}F_n$ converges almost surely uniformly to 0, and for all $\alpha > \varphi_W(p_0)/p_0$, $b^{n\alpha}F_n$ is unbounded almost surely, conditionally on $\mathscr{V}^c$.*
2. *Suppose that $p_0 \in (1,2]$. We have $\varphi_W(p_0) \le 0$. Then, for all $\alpha > \varphi_W(p_0)/p_0$, $b^{n\alpha}F_n$ is unbounded almost surely, conditionally on $\mathscr{V}^c$. In particular, if we have $\varphi_W(p_0) < 0$, then F_n is unbounded almost surely, conditionally on $\mathscr{V}^c$.*
3. *Suppose that $p_0 > 2$ and $\mathbb{P}(\sum_{k=0}^{b-1} W_k = 1) = 1$. We have $\varphi_W(p_0) \le 0$. If $p_0 < \infty$, the same conclusions as in 2. hold. If $p_0 = \infty$, then $(F_n)_{n\ge1}$ diverges in $\mathcal{C}([0,1])$ almost surely.*

   Moreover, in both cases there is no sequence $(r_n)_{n\ge1}$ tending to 0 or ∞, as $n \to \infty$, such that r_nF_n converges in law to a non-trivial limit in $\mathcal{C}([0,1])$.
4. *Suppose that $p_0 > 2$ and $\mathbb{P}(\sum_{k=0}^{b-1} W_k \ne 1) > 0$. For $n \ge 1$, let $r_n = b^{n\varphi_W(2)/2}$ if $\varphi_W(2) < 0$ and $r_n = n^{-1/2}$ if $\varphi_W(2) = 0$. The probability distributions of the random functions r_nF_n form a tight sequence, and for all $\alpha > \varphi_W(2)/2$, $b^{n\alpha}F_n$ is unbounded almost surely, conditionally on $\mathscr{V}^c$.*

Remark 2.5

(1) Suppose that **(C)** *does not hold and $p_0 < \infty$. Thus $p_0 \in (1,2]$ and $\varphi_W(p_0) = 0$, or when $\mathbb{P}(\sum_{k=0}^{b-1} W_k = 1) = 1$, $p_0 \in (1,\infty)$ and $\varphi_W(p_0) = 0$. What we can only prove is that $\lim_{n\to\infty} \sup_{w\in\mathscr{A}^n} |\Delta F_n(I_w)| = 0$ and $\liminf_{n\to\infty} \dfrac{\log_b \sup_{w\in\mathscr{A}^n} |\Delta F_n(I_w)| = 0}{-n}$. This is not enough to decide whether or not F_n is convergent. It is mainly for the same reason*

that we cannot deal with the case $\alpha = \varphi_W(p_0)/p_0$ in Theorems 2.6.2 and 2.6.3 (when $p_0 < \infty$).

(2) When $p_0 = \infty$, Theorem 2.6 tells nothing about the case where the assumptions of Theorem 2.2.2 hold except that there is no $\gamma \in (0,1)$ such that (2.9) holds.

(3) The results obtained in [48, 64] when $W \geq 0$ show that in this case, when F_n converges almost surely uniformly to 0, there does not exist a sequence $(a_n)_{n\geq 1}$ such that F_n/a_n converge in law to a non-trivial process as n tend to ∞ (see the discussion in section VIII of [64]).

We now cease to assume that $\varphi_W(p) > -\infty$ for all $p \geq 0$, but assume that $\varphi_W(2) > -\infty$. Define

$$\sigma = \begin{cases} \sqrt{\dfrac{\mathbb{E}\left(\left|\sum_{i=0}^{b-1} W_i\right|^2\right) - 1}{\mathbb{E}\left(\sum_{i=0}^{b-1} |W_i|^2\right) - 1}} & \text{if } \varphi_W(2) < 0 \\ \sqrt{\sum_{i\neq j} \mathbb{E}(W_i \overline{W_j})} & \text{if } \varphi_W(2) = 0 \end{cases}.$$

Theorem 2.7 (Barral,Mandelbrot) (Tightness of $(\mathcal{L}(F_n/\sqrt{\mathbb{E}(F_n(1)^2)})_{n\geq 1}))$

Suppose that $\mathbb{P}(\sum_{k=0}^{b-1} W_k \neq 1) > 0$ and **(C)** *does not hold. In particular, $\varphi_W(2) \leq 0$.*

1. *The sequence $(F_n(1))_{n\geq 1}$ is unbounded in L^2 norm. Specifically, we have $\mathbb{E}(|F_n(1)|^2) \sim \sigma^2 b^{-n\varphi_W(2)}$ if $\varphi_W(2) < 0$ and $\mathbb{E}(|F_n(1)|^2) \sim \sigma^2 n$ if $\varphi_W(2) = 0$.*

2. *Suppose that $p_0 > 2$. Equivalently, $\varphi_W(p)/p > \varphi_W(2)/2$ near 2^+. For $n \geq 1$ let $Z_n = F_n/\sqrt{\mathbb{E}(F_n(1)^2)}$.*
   The sequence $(Z_n(1))_{n\geq 1}$ is bounded in L^p norm for all p such that $\varphi_W(p)/p > \varphi_W(2)/2$.
   Moreover, the probability distributions of the random continuous functions $Z_n = F_n/\sqrt{\mathbb{E}(F_n(1)^2)}$ form a tight sequence.
3. *Suppose that $p_0 > 2$. We have $\varphi_{W^{(2)}}(p) > 0$ near 1^+ (remember Definition 2.1). Suppose, moreover, that W is $\mathbb{R}^b$-valued and $(Z_n)_{n\geq 1}$ converges in law, as n tends to ∞. Then, the weak limit of Z_n is the Brownian motion in multifractal time $Z = B \circ F_{W^{(2)}}$, where B is a standard Brownian motion independent of $F_{W^{(2)}}$. Moreover, Z satisfies the statistical scaling invariance property :*

$$Z \equiv \sum_{i=0}^{b-1} \mathbf{1}_{[i/b,(i+1)/b]} \Big(Z(i/b) + b^{\varphi_W(2)/2} W_i \, Z_i \circ S_i^{-1} \Big), \tag{2.10}$$

   where $S_i(t) = (t+i)/b$, the random objects $W, Z_0, \ldots, Z_{b-1}$ are independent, and the Z_i are distributed like Z.

Theorem 2.8 (Barral,Mandelbrot) (Functional central limit theorem when F_n is unbounded)

Suppose that $\mathbb{P}(\sum_{k=0}^{b-1} W_k \neq 1) > 0$ and **(C)** *does not hold. Suppose, moreover, that W is $\mathbb{R}^b$-valued and $\varphi_W(p)/p > \varphi_W(2)/2$ over $(2,\infty)$ (equivalently φ_W is increasing, or $W_k \leq 1$ almost surely for all $0 \leq k \leq b-1$).*

Then, $(Z_n)_{n\geq 1}$ converges in law, as n tends to ∞, to the Brownian motion in multifractal time Z described in Theorem 2.7.3. Also, the probability distribution of $Z(1)$ is determined by its moments.

Theorem 2.9 (Barral,Mandelbrot) (Functional central limit theorem when F_n converges)

Suppose that $\mathbb{P}(\sum_{k=0}^{b-1} W_k \neq 1) > 0$ and **(C)** *holds. Suppose, moreover, that W is $\mathbb{R}^b$-valued and $\varphi_W(p)/p > \varphi_W(2)/2$ near 2^+.*

Then, $(F_n - F)/\sqrt{\mathbb{E}((F_n-F)(1)^2)}$ converges in law, as n tends to ∞, to the Brownian motion in multifractal time described in Theorem 2.7.3.

Moreover, $\sqrt{\mathbb{E}((F_n-F)(1)^2)} = b^{-n\varphi_W(2)/2}\sqrt{\mathbb{E}\big((1-F_W(1))^2\big)}$.

2.3 Proof of Theorem 2.1

2.3.1 Proof of Theorem 2.1.1

For any $w \in \mathscr{A}^*$ and $n \geq 1$, define

$$\Delta F_n(I_w) = F_n(t_w + b^{-n}) - F_n(t_w) = \int_{I_w} Q_n(t)\,\mathrm{d}\lambda(t). \tag{2.11}$$

We have $\mathbb{E}(\|F_n\|_\infty^p) \leq \mathbb{E}\Big(\big(\sum_{w\in\mathscr{A}^n} |\Delta F_n(I_w)|\big)^p\Big) \leq \mathbb{E}\Big(\sum_{w\in\mathscr{A}^n} |\Delta F_n(I_w)|^p\Big)$, where we have used the subadditivity of $x \geq 0 \mapsto x^p$ ($p \in (0,1]$). Thus

$$\begin{aligned}\mathbb{E}(\|F_n\|_\infty^p) &\leq \sum_{w\in\mathscr{A}^n} \mathbb{E}\left(\left|\int_{I_w} Q_n(t)\,\mathrm{d}\lambda(t)\right|^p\right) \leq \sum_{w\in\mathscr{A}^n} \lambda(I_w)^p \mathbb{E}(\sup_{t\in I_w} |Q_n(t)|^p) \\ &\leq \sum_{w\in\mathscr{A}^n} \exp(\psi(n))\lambda(I_w)^{p-1}\int_{I_w} \mathbb{E}(|Q_n(t)|^p)\,\mathrm{d}\lambda(t) = \exp(\psi(n))S(n,p).\end{aligned}$$

Due to the property of $\psi(n)$, we have $\limsup_{n\to\infty} \log_b\big(\mathbb{E}(\|F_n\|_\infty^p)\big)/n \leq -\varphi(p) < 0$. This implies the result.

2.3.2 Proof of Theorem 2.1.2

The two following crucial statements, which take natural and classical forms, will be proved at the end of the section.

Recall that for $n \geq 1$ we define $T_n = \{t_w : w \in \mathscr{A}^n\} \cup \{1\}$ and $T_* = \cup_{n\geq 1} T_n$.

Proposition 2.1 *There exists a constant $C_p > 0$ such that*

$$(\forall\ n \geq 2) \quad \mathbb{E}(\max_{t\in T_n} |F_n(t) - F_{n-1}(t)|^p) \leq C_p S(n,p). \tag{2.12}$$

Consequently, for every b-adic number $t \in T_*$, $F_n(t)$ converges almost surely and in L^p norm as $n \to \infty$.

Proposition 2.2 *Let $\gamma \in (0, \max_{q\in(1,p]} \varphi(q)/q)$. With probability 1, there exists $\eta_\gamma > 0$ such that for any $t, s \in T_*$ such that $|t-s| < \eta_\gamma$ we have*

$$\sup_{n\geq 1} |F_n(t) - F_n(s)| \leq C_\gamma |t-s|^\gamma. \tag{2.13}$$

where C_γ is a constant depending on γ only.

Since $F_n(0) = 0$ almost surely for all $n \geq 1$, it follows from Propositions 2.2 and Ascoli-Arzela's theorem that, with probability 1, the sequence of continuous functions $(F_n)_{n\geq 1}$ is relatively compact, and all the limit of subsequences of F_n are γ-Hölder continuous for all $0 < \gamma < \max_{q\in(1,p]} \varphi(q)/q$. Moreover, Proposition 2.1 learns us that, with probability 1, F_n is convergent over the dense countable subset T_* of $[0,1]$. This yields the uniform convergence of F_n and the Hölder regularity of the limit F.

Denote $\|\cdot\|_p = \mathbb{E}(|\cdot|^p)^{1/p}$. We then prove that $\|\|F(t)\|_\infty\|_p < \infty$.

For $n \geq 1$, let $M_n = \max_{t\in T_n} |F_n(t)|$. We have

$$M_{n+1} \leq M_n + \max_{t\in T^n} |F_{n+1}(t) - F_n(t)| + b \cdot \max_{w\in\mathscr{A}^{n+1}} |Y_{n+1}(w)|. \tag{2.14}$$

Then Minkowski's inequality yields

$$\|M_{n+1}\|_p \leq \|M_n\|_p + \|\max_{t\in T_n} |F_{n+1}(t) - F_n(t)|\|_p + b \cdot \|\max_{w\in\mathscr{A}^{n+1}} |Y_{n+1}(w)|\|_p.$$

Also, due to Proposition 2.1 we have $\sum_{n\geq 1} \|\max_{t\in T_n} |F_{n+1}(t) - F_n(t)|\|_p < \infty$. Moreover,

$$\|\max_{w\in\mathscr{A}^{n+1}} |Y_{n+1}(w)|\|_p \leq \Big(\sum_{w\in\mathscr{A}^{n+1}} \mathbb{E}(|Y_{n+1}(w)|^p)\Big)^{1/p} \leq S(n+1,p)^{1/p},$$

so $\sum_{n\geq 1} \|\max_{w\in\mathscr{A}^{n+1}} |Y_{n+1}(w)|\|_p < \infty$. This implies $\sup_{n\geq 1} \|M_n\|_p < \infty$, and since, with probability 1, F_n converges uniformly to F_∞ and T_* is dense in $[0,1]$, we get $\|\sup_{t\in[0,1]} |F(t)|\|_p \leq \liminf_{n\to\infty} \|M_n\|_p < \infty$. In particular, F belongs to L^1 and for every $n \geq 1$, the conditional expectation of F with respect to $\mathcal{A}_n$ is well defined and it converges almost surely and in L^1 norm to F (see Proposition V-2-6 in [123]). It remains to prove that $F_n = \mathbb{E}(F|\mathcal{A}_n)$ almost surely. For every $t \in T_*$, we have shown that the martingale $(F_n(t), \mathcal{A}_n)_{n\geq 1}$ is uniformly integrable, so $F_n(t) = \mathbb{E}(F(t)|\mathcal{A}_n)$ almost surely. Consequently, since T_* is countable, with probability 1, the restriction of $\mathbb{E}(F|\mathcal{A}_n)$ coincides with the function F_n over T_*. Moreover, these two random functions are continuous and T_* is dense in $[0,1]$ so, with probability 1, they are equal.

2.3.3 Proof of Proposition 2.1

Fix $n \geq 2$ and denote the elements of T_n by t_j , $0 \leq j \leq b^n$, where $0 = t_0 < t_1 < \cdots < t_{b^n} = 1$. Also define $J_j = [t_j, t_{j+1}]$ for $0 \leq j < b^n$. We can write

$$F_n(t_j) - F_{n-1}(t_j) = \sum_{k=0}^{j-1} \int_{J_k} U(t)V(t)\mathrm{d}\lambda(t)$$

with $U(t)=Q_{n-1}(t)$ and $V(t)=P_n(t)-1$. Then we divide the family $\{J_j\}_{0\le j<b^n}$ into bN sub-families, namely the $\{J_{bNk+i}\}_{k\ge 0, 0\le bNk+i<b^n}$, for $0\le i\le bN-1$. Also we define $M_n=\max_{0\le j\le b^n}|F_n(t_j)-F_{n-1}(t_j)|$ and remark that

$$M_n\le bN\max_{\substack{0\le j<b^n\\0\le i\le bN-1}}\Big|\sum_{\substack{k\ge 0\\0\le bNk+i\le j}}\int_{J_{bNk+i}}U(t)V(t)\mathrm{d}\lambda(t)\Big|.$$

By raising both sides of the previous inequality to the power p we can get

$$\begin{aligned}M_n^p &\le (bN)^p\max_{\substack{0\le j<b^n\\0\le i\le bN-1}}\Big|\int_{J_{bNk+i}}U(t)V(t)\mathrm{d}\lambda(t)\Big|^p\\ &\le (bN)^p\sum_{i=0}^{bN-1}\max_{0\le j\le b^n}\Big|\sum_{\substack{k\ge 0\\0\le bNk+i\le j}}\int_{J_{bNk+i}}U(t)V(t)\mathrm{d}\lambda(t)\Big|^p.\end{aligned}\tag{2.15}$$

We are going to use the following lemma. It is proved for real valued random variables in [144], and its extension to the complex case is immediate.

Lemma 2.1 *Let $p\in(1,2]$. There exists a constant $C_p>0$ such that for every $n\ge 1$ and every sequence $\{V_j\}_{1\le j\le n}$ of independent and centered complex random variables we have*

$$\mathbb{E}\Big(\max_{1\le k\le n}\Big|\sum_{j=1}^{k}V_j\Big|^p\Big)\le C_p\sum_{j=1}^{n}\mathbb{E}(|V_j|^p).$$

Due to **(P3)**, for each $0\le i\le bN-1$, the restrictions of the function $V(t)$ to the intervals J_{bNk+i} , $0\le bNk+i<b^n$, are centered and independent. Also, due to **(P2)**, the functions $U(t)$ and $V(t)$ are independent. Consequently, by taking the conditional expectation with respect to $\mathcal{A}_{n-1}$ in (2.15) and using Lemma 2.1 we get for each $0\le i\le bN-1$

$$\begin{aligned}&\mathbb{E}\left(\max_{0\le j\le b^n}\Big|\sum_{\substack{k\ge 0\\0\le bNk+i\le j}}\int_{J_{bNk+i}}U(t)V(t)\mathrm{d}\lambda(t)\Big|^p\Big|\mathcal{A}_{n-1}\right)\\ &\qquad\qquad\le C_p\sum_{\substack{k\ge 0\\0\le bNk+i<b^n}}\mathbb{E}\Big(\Big|\int_{J_{bNk+i}}U(t)V(t)\mathrm{d}\lambda(t)\Big|^p\Big|\mathcal{A}_{n-1}\Big).\end{aligned}$$

This implies

$$\mathbb{E}\Big(M_n^p\Big|\mathcal{A}_{n-1}\Big)\le\widetilde{C}_p\sum_{0\le j\le b^n}\mathbb{E}\Big(\Big|\int_{J_j}U(t)V(t)\mathrm{d}\lambda(t)\Big|^p\Big|\mathcal{A}_{n-1}\Big),\tag{2.16}$$

with $\widetilde{C}_p=C_p(bN)^{p+1}$. Now, since $p>1$, the Jensen inequality yields

$$\Big|\int_{I_j}U(t)V(t)\mathrm{d}\lambda(t)\Big|^p\le\lambda(I_j)^{p-1}\int_{I_j}|U(t)V(t)|^p\mathrm{d}\lambda(t).$$

Moreover, since $\mathbb{E}(|P_n(t)|)\ge 1$ and $p\ge 1$, we have

$$\mathbb{E}(|V(t)|^p)\le 2^{p-1}(1+\mathbb{E}(|P_n(t)|^p))\le 2^p\mathbb{E}(|P_n(t)|^p).\tag{2.17}$$

Thus, taking the expectation in (2.16) yields

$$\mathbb{E}(\max_{t\in T_n}|F_n(t)-F_{n-1}(t)|^p)\le 2^p\widetilde{C}_pS(n,p),$$

i.e. (2.12). If $\varphi(p)>0$, by definition of φ, $S(n,p)$ converges exponentially fast to 0, hence the series $\sum_{n\ge1}S(n,p)^{1/p}$ converge and, due to (2.12) and the fact that $T_*=\bigcup_{n\ge0}T_n$, $F_n(t)$ converges almost surely and in L^p norm as $n\to\infty$ for all $t\in T_*$.

2.3.4 Proof of Proposition 2.2

Recall (2.11). Let $q\in(1,p]$ such that $\varphi(q)>0$. It follows from **(P1)** that $(\Delta F_n(I_w))_{n\ge1}$ is a martingale, so Doob's and then Jensen's inequalities yield a constant C_q such that for $n\ge1$

$$\mathbb{E}(\max_{1\le k\le n}|\Delta F_k(I_w)|^q)\le C_q\mathbb{E}(|\Delta F_n(I_w)|^q)\le C_q\lambda(I_w)^{q-1}\int_{I_w}\mathbb{E}(|Q_n(t)|^q)\mathrm{d}\lambda(t).$$

Consequently

$$\sum_{w\in A^n}\mathbb{E}(\max_{1\le k\le n}|\Delta F_k(I_w)|^q)\le C_qS(n,q). \tag{2.18}$$

By using Markov's inequality as well as (2.18) and Proposition 2.1, we get

$$\begin{aligned}
&\mathbb{P}(\max_{w\in A^n}\max_{0\le k\le n}|\Delta F_k(I_w)|>b^{-n\gamma}\text{ or }\max_{t\in T_n}|F_n(t)-F_{n-1}(t)|>b^{-n\gamma})\\
\le\;&\sum_{w\in A^n}\mathbb{P}(\max_{0\le k\le n}|\Delta F_k(I_w)|>b^{-n\gamma})+\mathbb{P}(\max_{t\in T_n}|F_n(t)-F_{n-1}(t)|>b^{-n\gamma})\\
\le\;&\sum_{w\in A^n}b^{n\gamma q}\cdot\mathbb{E}(\max_{0\le k\le n}|\Delta F_k(I_w)|^q)+b^{n\gamma q}\cdot\mathbb{E}(\max_{t\in T_n}|F_n(t)-F_{n-1}(t)|^q)\\
\le\;&C_qb^{n\gamma q}S(n,q),
\end{aligned}$$

where C_q is another constant depending only on q. Since $\gamma\in(0,\varphi(q)/q)$, by definition of $\varphi(q)$ the series $\sum_{n\ge1}b^{n\gamma q}S(n,q)$ converges, and by the Borel-Cantelli lemma, with probability 1, there exists n_1 such that for all $n\ge n_1$,

$$\max_{w\in A^n}\max_{0\le k\le n}|\Delta F_k(I_w)|\le b^{-n\gamma}\text{ and }\max_{t\in T_n}|F_n(t)-F_{n-1}(t)|\le b^{-n\gamma}. \tag{2.19}$$

Now fix $n\ge n_1$. We are going to prove by induction that for all $M\ge n+1$ and $t,s\in T_M$ such that $0<t-s<b^{-n}$ we have

$$\Delta_M(t,s)\le 2b\sum_{m=n+1}^{M}b^{-m\gamma},\text{ where }\Delta_M(t,s)=\max_{0\le k\le M}|F_k(t)-F_k(s)|. \tag{2.20}$$

If $M=n+1$, then there exist i and i' with $0<i-i'<2b$ such that $t=ib^{-(n+1)}$ and $s=i'b^{-(n+1)}$, so due to (2.19) applied at generation $n+1$,

$$\Delta_{n+1}(t,s)\le(i-i')b^{-(n+1)\gamma}\le 2b\cdot b^{-(n+1)\gamma}.$$

Now let $M\ge n+1$ and suppose that (2.20) holds for all $n+1\le m\le M$. Let $t,s\in T_{M+1}$ such that $0<t-s<b^{-n}$. If there is no element of T_M between s and t, then (2.19) yields

$\Delta_{M+1}(t,s) \le (b-1)b^{-(M+1)\gamma}$. Otherwise, consider $\bar{t} = \max\{u \in T_M : u \le t\}$ and $\bar{s} = \min\{u \in T_M : u \ge s\}$. We have

$$s \le \bar{s} \le \bar{t} \le t\ ,\ t - \bar{t} \le (b-1)b^{-(M+1)}\ ,\ \bar{s} - s \le (b-1)b^{-(M+1)}\ ,\ \bar{t} - \bar{s} < b^{-n}.$$

Since $\bar{s}$ and $\bar{t}$ belong to $T_M \subset T_{M+1}$, we deduce from (2.19) that

$$\begin{cases} \max\left\{\Delta_{M+1}(t,\bar{t}), \Delta_{M+1}(\bar{s},s)\right\} \le (b-1)b^{-(M+1)\gamma} \\ \max\left\{|F_{M+1}(\bar{s}) - F_M(\bar{s})|, |F_{M+1}(\bar{t}) - F_M(\bar{t})|\right\} \le b^{-(M+1)\gamma} \end{cases}.$$

Also, due to (2.20) we have $\Delta_M(\bar{t},\bar{s}) \le 2b \sum_{m=n+1}^{M} b^{-m\gamma}$. Consequently,

$$\begin{aligned} & \Delta_{M+1}(t,s) \\ \le\ & \Delta_{M+1}(t,\bar{t}) + \Delta_{M+1}(\bar{s},s) + \Delta_M(\bar{t},\bar{s}) + |F_{M+1}(\bar{s}) - F_M(\bar{s})| + |F_{M+1}(\bar{t}) - F_M(\bar{t})| \\ \le\ & 2(b-1)b^{-(M+1)\gamma} + 2b \sum_{m=n+1}^{M} b^{-j\gamma} + 2b^{-(M+1)\gamma}, \end{aligned}$$

so (2.20) holds for $m = M+1$. Let $C_\gamma = 2b/(1-b^{-\gamma})$. Letting M tend to infinity in (2.20) yields that $\max_{k\ge1} |F_k(t) - F_k(s)| \le C_\gamma b^{-(n+1)\gamma}$ for all $n \ge n_1$ and $t,s \in T_*$ such that $|t-s| \le b^{-n}$. Now, for $t,s \in T_*$ with $|t-s| \le b^{-n_1}$, there is a unique $n \ge n_1$ such that $b^{-(n+1)} \le |t-s| < b^{-n}$ and $\max_{k\ge1} |F_k(t) - F_k(s)| \le C_\gamma b^{-(n+1)\gamma} \le C_\gamma |t-s|^\gamma$. The conclusion comes from the density of T_* in $[0,1]$ and the continuity of the F_k.

2.4 Proofs of Theorems 2.2, 2.3 and 2.4

We start with a remark. Under the assumptions of Theorems 2.2 and 2.3.1, we have $-1 \le \varphi_W(0) < \varphi_W(1) \le 0 < \varphi_W(p)$. Since φ_W is concave this implies that $\varphi_W(q) < q-1$ for all $q \in (1,p]$ except if $\varphi_W(q) = q-1$ for all q. This can happen only if the components of W are positive and equal to $1/b$ almost surely. In this case $F_n(t) = t$ for all $n \ge 1$ and $t \in [0,1]$ and the result obviously holds. We exclude this case in the rest of this section.

For $w \in \mathscr{A}^*$, we denote by $(F_n^{[w]})_{n\ge1}$ the copy of $(F_n)_{n\ge1}$ constructed with the random vectors $(W(w\cdot u))_{u\in\mathscr{A}^*}$:

$$F_n^{[w]}(t) = \int_0^t b^n \prod_{k=1}^{n} W_{u_k}(w \cdot u|_{|w|+k-1})\,\mathrm{d}u.$$

For $n > |w|$, the increment $\Delta F_n(I_w)$ of F_n over I_w takes the form

$$\Delta F_n(I_w) = Q(w) F^{[w]}_{n-|w|}(1), \tag{2.21}$$

where $Q(w) = \prod_{k=0}^{|w|-1} W_{w_{k+1}}(w|_k)$. This implies in particular that for every $n \ge 1$ we have

$$F_n = \sum_{i=0}^{b-1} \mathbf{1}_{[i/b,(i+1)/b]} \Big(F_n(i/b) + W_i F_{n-1}^{[i]} \circ S_i^{-1}\Big). \tag{2.22}$$

Moreover, $Q(w)$ and $F^{[w]}_{n-|w|}(1)$ are independent, and for each $p \ge 1$, the families $\{F_n^{[w]}\}_{n\ge1}$, $w \in \mathscr{A}^p$, are independent.

2.4.1 Proof of Theorems 2.2 and Theorems 2.3.1

If $\mathbb{P}(\sum_{i=0}^{b-1} W_i = 1) = 1$ then $F_k(1) = 1$ almost surely. If $p \in (1,2]$, the fact that the martingale $(F_k(1))_{k\geq 1}$ converges almost surely and in L^p norm is a consequence of Theorem 1 in [33], and the case $p > 2$ is a consequence of Theorem 1 in [12] (the positive case is treated in [92] and [48]). Then, equation (2.21) implies the almost sure convergence of the b-adic increments of F_n.

Now we establish the almost sure uniform convergence of F_n. When F_n can be interpreted as a $[0,1]$-martingale, that is when the components of W have positive expectations (see [20]), the proof provides a simpler alternative to the general proof given in [20].

Let $q \in (1,p]$ such that $\varphi_W(q) > 0$ and define $M_q = \mathbb{E}(\sup_{k\geq 1} |F_k(1)|^q)$. By using (2.21) as well as the martingale property of $(F_k(1))_{k\geq 1}$ and Doob's inequality we get

$$\begin{aligned}
\mathbb{E}(\sup_{n\geq 1} |\Delta F_n(I_w)|^q) &= \sum_{n=1}^{|w|} \mathbb{E}(|\Delta F_n(I_w)|^q) + \mathbb{E}(|Q(w)|^q)\mathbb{E}(\sup_{n>|w|} |F^{[w]}_{n-|w|}(1)|^q) \\
&\leq \sum_{n=1}^{|w|} b^{-(|w|-n)q}\mathbb{E}(|Q(w|_n)|^q) + C_q M_q \mathbb{E}(|Q(w)|^q)
\end{aligned}$$

for some constant C_q. Consequently, for $\gamma > 0$ and $N \geq 1$ we have

$$\begin{aligned}
&\mathbb{P}(\max_{w\in\mathscr{A}^N} \sup_{n\geq 1} |\Delta F_n(I_w)| > b^{-\gamma N}) \\
\leq\ & b^{\gamma N q} \sum_{w\in\mathscr{A}^N} \sum_{n=1}^{N} b^{-(N-n)q}\mathbb{E}(|Q(w|_n)|^q) + C_q M_q \mathbb{E}(|Q(w)|^q) \\
=\ & b^{\gamma N q} \left[\sum_{n=1}^{N} b^{-(N-n)(q-1)} b^{-n\varphi_W(q)} + C_q M_q b^{-N\varphi_W(q)} \right] \\
\leq\ & \left[\frac{b^{q-1-\varphi_W(q)}}{b^{q-1-\varphi_W(q)} - 1} + C_q M_q \right] b^{\gamma N q} b^{-N\varphi(q)},
\end{aligned}$$

where we used the fact that $\varphi_W(q) < q-1$. It follows that if $\gamma < \varphi_W(q)/q$, we have

$$\sum_{N\geq 1} \mathbb{P}(\max_{w\in\mathscr{A}^n} \sup_{n\geq 1} |\Delta F_n(I_w)| > b^{-\gamma N}) < \infty.$$

Due to the Borel-Cantelli lemma, we conclude that, with probability 1,

$$\text{for } N \text{ large enough, } \max_{w\in\mathscr{A}^n} \sup_{n\geq 1} |\Delta F_n(I_w)| \leq b^{-\gamma N}. \tag{2.23}$$

Next, we use the following classical property : for any continuous complex function f on $[0,1]$, one has

$$\omega(f,\delta) \leq 2b \sum_{n\geq -\frac{\log\delta}{\log b}} \sup_{w\in\mathscr{A}^n} \Delta f(I_w), \tag{2.24}$$

where, $\omega(f,\delta)$ stands for the modulus of continuity of f :

$$\omega(f,\delta) = \sup_{\substack{t,s\in[0,1] \\ |t-s|\leq\delta}} |f(t) - f(s)|.$$

Since $F_n(0) = 0$ almost surely for all $n \geq 1$, it follows from (2.23), (2.24) and Ascoli-Arzela's theorem that, with probability 1, the sequence of continuous functions $(F_n)_{n\geq 1}$ is relatively compact, and all the limit of subsequences of F_n are γ-Hölder continuous for all $\gamma < \max_{q\in(1,p]} \varphi(q)/q$. Moreover, by the self-similarity of the construction (2.21) we know that F_n converges almost surely on set of b-adic points. This yields the uniform convergence of F_n and the Hölder regularity property of the limit F.

To see that $(F_n)_{n\geq 1}$ converges in L^p norm, it is enough to prove that the sequence $\big(\mathbb{E}(\sup_{1\leq k\leq n} \|F_k\|_\infty^p)\big)_{n\geq 1}$ is bounded.

For $n \geq 1$ and $0 \leq i \leq b-1$ define

$$S_n = \sup_{1\leq k\leq n} \|F_k\|_\infty,\ S_n(i) = \sup_{1\leq k\leq n} \|F_k^{[i]}\|_\infty,\ \text{and}\ \widetilde{S}_n(i) = \sup_{1\leq k\leq n} |F_k(ib^{-1})|.$$

Due to (2.22) we have $S_n \leq \max_{0\leq i\leq b-1} \big[\widetilde{S}_n(i) + |W_i| S_{n-1}(i)\big]$, so

$$\mathbb{E}(S_n^p) \leq \sum_{i=0}^{b-1} \mathbb{E}\big([|W_i|S_{n-1}(i) + \widetilde{S}_n(i)]^p\big).$$

Denote by $\bar{p} \geq 2$ the unique integer such that $\bar{p} - 1 < p \leq \bar{p}$. By using the sub-additivity of the mapping $x \geq 0 \mapsto x^{p/\bar{p}}$ we get

$$\begin{aligned}
&\mathbb{E}((|W_i|S_{n-1}(i) + \widetilde{S}_n(i))^p) \leq \mathbb{E}\big((|W_i|S_{n-1}(i))^{p/\bar{p}} + \widetilde{S}_n(i)^{p/\bar{p}})^{\bar{p}}\big)\\
&\leq \mathbb{E}(|W_i|^p)\cdot\mathbb{E}(S_{n-1}(i)^p) + \mathbb{E}(\widetilde{S}_n(i)^p) + \sum_{m=1}^{\bar{p}-1}\binom{\bar{p}}{m}\mathbb{E}\big([|W_i|S_{n-1}(i)]^{mp/\bar{p}}[\widetilde{S}_{n-1}(i)]^{(\bar{p}-m)p/\bar{p}}\big).
\end{aligned}$$

Now let us make some remarks :

- The Hölder inequality yields for any pair of non-negative random variables (U, V) and $m \in [1, \bar{p}-1]$

$$\mathbb{E}(U^{mp/\bar{p}}V^{(\bar{p}-m)p/\bar{p}}) \leq \mathbb{E}(U^p)^{m/\bar{p}}\mathbb{E}(V^p)^{(\bar{p}-m)/\bar{p}}.$$

- The convergence in L^p norm of $F_n(1)$ implies that $(\widetilde{S}_n(i))_{n\geq 1}$ is bounded in L^p.

- The random variables $|W_i|$ and $S_{n-1}(i)$ are independent and $|W_i|$ is in L^p.

- Since the expectation of $F_k(1)$ is equal to 1 for all $k \geq 1$, we have $1 \leq \mathbb{E}(S_{n-1}^p)^{m/\bar{p}} \leq \mathbb{E}(S_{n-1}^p)^{(\bar{p}-1)/\bar{p}}$ for all $m \in [1, \bar{p}-1]$.

The previous remarks imply the existence of two constants A and B independent of i such that

$$\begin{aligned}
&\mathbb{E}((W_iS_{n-1}(i) + \widetilde{S}_n(i))^p)\\
\leq\ & \mathbb{E}(|W_i|^p)\mathbb{E}(S_{n-1}^p) + B + \sum_{m=1}^{\bar{p}-1}\binom{\bar{p}}{m} A^{m/\bar{p}}B^{(\bar{p}-m)/\bar{p}}\mathbb{E}(S_{n-1}^p)^{(\bar{p}-1)/\bar{p}}\\
=\ & \mathbb{E}(|W_i|^p)\mathbb{E}(S_{n-1}^p) + B + (A+B)^{\bar{p}}\mathbb{E}(S_{n-1}^p)^{(\bar{p}-1)/\bar{p}}.
\end{aligned}$$

Summing this inequality over i we find

$$\mathbb{E}(S_n^p) \leq b^{-\varphi_W(p)}\mathbb{E}(S_{n-1}^p) + b(A+B)^{\bar{p}}\mathbb{E}(S_{n-1}^p)^{(\bar{p}-1)/\bar{p}} + b\cdot B.$$

For $x \geq 0$ let $f(x) = b^{-\varphi_W(p)} \cdot x + b(A+B)^{\bar{p}} x^{(\bar{p}-1)/\bar{p}} + b \cdot B$. Since $b^{-\varphi_W(p)} < 1$ and $(\bar{p}-1)/\bar{p} < 1$, there exists x_0 such that $f(x) < x$ for any $x > x_0$, which implies $\mathbb{E}(S_n^p) \leq \max\{x_0, \mathbb{E}(S_{n-1}^p)\}$. This yields the conclusion.

Property (2.8) is a consequence of (2.22).

2.4.2 Proof of Theorem 2.3.2

By construction, if there exists $0 \leq i \leq b-1$ such that $\mathbb{P}(|W_i| > 1) > 0$, then $\lim_{p\to\infty} \varphi_W(p) = -\infty$. Otherwise, we have $\lim_{p\to\infty} \varphi_W(p) = -\log_b \sum_{i=0}^{b-1} \mathbb{P}(|W_i| = 1)$, and by concavity of φ_W we have $\varphi_W(p) < 0$ for all $p > 0$ and $\lim_{p\to\infty} \varphi_W(p) = 0$ if and only if $\sum_{i=0}^{b-1} \mathbb{P}(|W_i| = 1) = 1$.

For $n \geq 1$ let us define $m_n = \max_{w\in\mathscr{A}^n} |Q(w)|$. Then, due to (2.21), for $p \geq 1$, we have

$$\|F_{n+p} - F_n\|_\infty \leq \sup_{w\in\mathscr{A}^n} \sup_{t\in I_w} |Q(w)||F_p^{[w]}(t) - 1| \leq m_n \sup_{w\in\mathscr{A}^n} (1 + \|F_p^{[w]}\|_\infty).$$

We are going to prove that $\lim_{n\to\infty} m_n = 0$, and there exists $C > 0$ such that $\|F_p\|_\infty \leq C$ almost surely for all $p \geq 1$. Thus, $(F_n)_{n\geq 1}$ is almost surely a Cauchy sequence.

We start with the proof of $\lim_{n\to\infty} m_n = 0$. Due to the fact that the components of W are either equal to 1 or less than or equal to γ, the sequence $(m_n)_{n\geq 1}$ is non-increasing and $m_{n+1} = m_n$ or $m_{n+1} \leq \gamma m_n$. Thus if $\lim_{n\to\infty} m_n > 0$, we can find an infinite word $w_1 \cdots w_n \cdots$ in $\mathscr{A}^{\mathbb{N}_+}$ and $n_0 \geq 1$ such that or all $n \geq n_0$, $|W_{w_{n+1}}(w_1 \cdots w_n)| = 1$. By construction, such an infinite word must belong to the boundary of a Galton-Watson tree rooted at $w_1 \cdots w_{n_0}$, and whose offspring distribution generating function is given by $x \mapsto \sum_{k=0}^{b} \mathbb{P}(\#\{i : |W_i| = 1\} = k) x^k$. This tree is subcritical, since we assumed that $\sum_{k=1}^{b} k\mathbb{P}(\#\{i : |W_i| = 1\} = k) = \sum_{i=0}^{b-1} \mathbb{P}(|W_i| = 1) = 1$ and $\mathbb{P}(\#\{i : |W_i| = 1\} = 1) < 1$. Consequently its boundary is empty almost surely, and m_n tends to 0 as $n \to \infty$.

Now we prove by induction that, with probability 1, for all $p \geq 1$ and $w \in \mathscr{A}^p$, we have

$$\max\big(|F_p(t_w)|, |F_p(t_w + b^{-p})|\big) \leq 1 + \frac{b+1}{2} \sum_{l=1}^{p-k_w-1} \gamma^l + \frac{b-1}{2} \gamma^{p-k_w}, \tag{2.25}$$

where

$$k_w = \#\big\{1 \leq j \leq p : |W_{w_j}(w|j-1)| = 1\big\}.$$

This will imply that $\|F_p\|_\infty \leq 1 + b/(1-\gamma)$ almost surely.

For the case $p = 1$, that $|\sum_{i=0}^{j} W_i| \leq 1 + (b-1)\gamma/2$ for all $0 \leq j \leq b-1$ is a direct consequence of our assumptions $\mathbb{P}(\sum_{i=0}^{b-1} W_i = 1) = 1$, and almost surely either $|W_i| = 1$ and $(\sum_{k=0}^{i-1} W_i, \sum_{k=0}^{i} W_i) \in \{(0,1),(1,0)\}$, or $|W_i| \leq \gamma$. Thus $\|F_1\|_\infty \leq 1 + (b-1)\gamma/2$.

Now suppose that $p \geq 1$ and (2.25) holds. Let $w \in \mathscr{A}^p$. For every $1 \leq j \leq b-1$, let $s_j(w) = \sum_{i=0}^{j-1} W_i(w)$. By construction, we have

$$F_{p+1}(t_{wj}) \quad = \quad F_p(t_w) + \big(F_p(t_w + b^{-p}) - F_p(t_w)\big) s_j(w).$$

If $|W_j(w)| = 1$ then by our assumption we have $k_{wj} = k_w + 1$ and $s_j(w) \in \{0,1\}$, so $F_{p+1}(t_{wj}) \in \{F_p(t_w), F_p(t_w + b^{-p})\}$ and (2.25) holds. Otherwise, $k_{wj} = k_w$, and since $|Q(w)| \leq \gamma^{p-k_w}$ we get

$$\begin{aligned} |F_{p+1}(t_{wj})| &\leq |F_p(t_w)| + \gamma^{p-k_w}|s_j(w)| \\ &\leq 1 + \frac{b+1}{2}\sum_{l=1}^{p-k_w-1}\gamma^l + \frac{b-1}{2}\gamma^{p-k_w} + \gamma^{p-k_w} + \frac{b-1}{2}\gamma^{p+1-k_w} \\ &= 1 + \frac{b+1}{2}\sum_{l=1}^{p-k_{wj}}\gamma^l + \frac{b-1}{2}\gamma^{p+1-k_{wj}}. \end{aligned}$$

Now we prove that the limit F of $(F_n)_{n\geq 1}$ is not uniformly Hölder continuous. To see this, we can consider of any $q > 1$ the positive measure μ_q on $[0,1]$ obtained almost surely as the derivative in the distribution sense of $F_{W^{(q)}}$ (see Definition 2.1). It follows from the equality $Q(w) = \Delta F(I_w)$ and Theorem IVi) of [10] that $\lim_{n\to\infty} \log(|\Delta F(I_n(t))|)/-n\log(b) = \varphi'_W(q)$ for μ_q-almost every t, where $I_n(t)$ is the semi-open to the right b-adic interval of generation n containing t. Moreover, by construction we have $\lim_{q\to\infty}\varphi'_W(q) = 0$. This implies that, with probability 1, there exist a sequence $(t_k)_{k\geq 1}$ of points in $[0,1]$ such that $\lim_{k\to\infty} h_F(t_k) = 0$, hence F is not uniformly Hölder.

2.4.3 Proof of Theorem 2.4

The facts that $\beta \in (1,2)$ when $\mathbb{P}(\sum_{i=0}^{b-1} W_i = 1) < 1$, as well as the propeties of $W^{(\beta)}$ are immediate.

We will prove in the end of this section the following properties, which hold under the assumptions of Theorem 2.4.

Lemma 2.2 *Let X and X_β stand respectively for the oscillations of F_W and $F_{W^{(\beta)}}$ over $[0,1]$. For all $q \in \mathbb{R}$ we have $\mathbb{E}(X^q) < \infty$ and $\mathbb{E}(X_\beta^q) < \infty$.*

For every $w \in \mathscr{A}^*$, let $X(w)$ and $X_\beta(w)$ stand respectively for the oscillations of $F_W^{[w]}$ and $F_{W^{(\beta)}}^{[w]}$ over $[0,1]$. We deduce from the Lemma 2.2 that, with probability 1, for every $\varepsilon > 0$, there exists n_ε such that

$$\forall\, n \geq n_\varepsilon,\ \forall\, w \in \mathscr{A}^n,\ \forall\, Y \in \{X, X_\beta\},\ b^{-n\varepsilon} \leq Y(w) \leq b^{n\varepsilon}. \tag{2.26}$$

This implies that

$$\begin{aligned} b^{-n\varepsilon}|Q(w)| &\leq \mathrm{Osc}_{F_W}(I_w) = |Q(w)|X(w) \leq b^{n\varepsilon}|Q(w)| \\ b^{-n\varepsilon}|Q(w)|^\beta &\leq \mathrm{Osc}_{F_{W^{(\beta)}}}(I_w) = |Q(w)|^\beta X_\beta(w) \leq b^{n\varepsilon}|Q(w)|^\beta. \end{aligned}$$

Consequently,

$$b^{-n(1+1/\beta)\varepsilon} \leq \frac{\mathrm{Osc}_{F_W}(I_w)}{\mathrm{Osc}_{F_{W^{(\beta)}}}(I_w)^{1/\beta}} = \frac{X(w)}{X_\beta(w)^{1/\beta}} \leq b^{n(1+1/\beta)\varepsilon}.$$

Let $B = F_W \circ F_{W^{(\beta)}}^{-1}$. Let $J_w = F_{W^{(\beta)}}(I_w)$. We have $|J_w| = \mathrm{Osc}_{F_{W^{(\beta)}}}(I_w)$, and $\mathrm{Osc}_B(J_w) = \mathrm{Osc}_{F_W}(I_w)$, so the previous inequality is equivalent to

$$b^{-n(1+1/\beta)\varepsilon} \le \frac{\mathrm{Osc}_B(J_w)}{|J_w|^{1/\beta}} = \frac{X(w)}{X_\beta(w)^{1/\beta}} \le b^{n(1+1/\beta)\varepsilon}.$$

Under our assumptions, it is also true that (see Theorem 2.2.1)

$$\lim_{n\to\infty} \inf_{w\in\mathscr{A}^n} \frac{\log_b |J_w|}{-n} \ge \alpha_0$$

where $\alpha_0 = \sup_{p>0} \varphi_{W^{(\beta)}}(p)/p > 0$ (in fact the equality holds). Also, we have the following property (we postpone its proof to after that of Lemma 2.2).

Lemma 2.3 *With probability 1, for every $\varepsilon > 0$, there exists n_ε such that*

$$\forall\, n \ge n_\varepsilon,\ b^{-n\varepsilon} \le \inf_{w\in\mathscr{A}^n} \inf_{0\le i\le b-1} \frac{|J_{wi}|}{|J_w|} \le 1, \tag{2.27}$$

We can also choose the random integer n_ε so that for all $n \ge n_\varepsilon$ (2.27) holds as well as the property : $|J_w| \le b^{-n\alpha_0/2}$ for all $w \in \mathscr{A}^n$.

Let $t \in (0,1)$ and $0 < r < \min_{w\in\mathscr{A}_{n_\varepsilon+1}} |J_w|$. Let $w_2 \in \mathscr{A}^*$ such that $|w_2| > n_\varepsilon$, $t \in J_{w_2} \subset [t-r, t+r]$ and $|J_{w_2}|$ is maximal. Then let $(w_1, w_3) \in \mathscr{A}^* \times \mathscr{A}^*$ such that $\min(|w_1|, |w_3|) > n_\varepsilon$, $\min(|J_{w_1}|, |J_{w_3}|) \ge r$, the intervals J_{w_i}, $i \in \{1,2,3\}$ are adjacent, $[t-r, t+r] \subset J_{w_1} \cup J_{w_2} \cup J_{w_3}$, and $|J_{w_1}| + |J_{w_2}| + |J_{w_3}|$ is minimal. This constraint imposes that $|J_w| \le r b^{\varepsilon|w|}$ for $w \in \{w_1, w_3\}$. Otherwise, due to (2.27) we can replace J_w by one of its sons in the covering of $[t-r, t+r]$. Also, we have $2r \ge |J_{w_2}| \ge r b^{-\varepsilon|w_2|}$. Otherwise, since $t \in J_{w_2}$, due to (2.27) we can replace J_{w_2} by its father, hence $|J_{w_2}|$ is not maximal.

Since

$$\mathrm{Osc}_B(J_{w_2}) \le \mathrm{Osc}_B([t-r, t+r]) \le 3 \max_i \mathrm{Osc}_B(J_{w_i}),$$

we have

$$|J_{w_2}|^{1/\beta} b^{-|w_2|(1+1/\beta)\varepsilon} \le \mathrm{Osc}_B([t-r, t+r]) \le 3 \max_i |J_{w_i}|^{1/\beta} b^{|w_i|(1+1/\beta)\varepsilon}.$$

Now, we specify $\varepsilon < \alpha_0/4$. We can deduce from the constraints on the length of the intevals J_{w_i} that $b^{\varepsilon|w_i|} \le r^{-4\varepsilon/\alpha_0}$. Consequently, there exists a constant C depending on W only such that for r small enough,

$$r^{1/\beta + C\varepsilon} \le \mathrm{Osc}_B([t-r, t+r]) \le r^{1/\beta - C\varepsilon}.$$

Since this holds for all $0 < \varepsilon < \alpha_0/4$, almost surely for all $t \in (0,1)$, we have in fact that, with probability 1, for all $t \in (0,1)$, $\lim_{r\to 0^+} \frac{\log(\mathrm{Osc}_B([t-r,t+r]))}{\log(r)} = 1/\beta$, hence $h_B(t) = 1/\beta$.

Proof of Lemma 2.2

The part concerning the moments of positive orders is a consequence of Theorem 2.2.1 and the inequality $\mathrm{Osc}_f([0,1]) \leq 2\|f\|_\infty$, which holds for any continuous function f on $[0,1]$.

For the moments of negative orders, the case of X_β, which is the increment between 0 and 1 of the increasing function $F_{W^{(\beta)}}$, is treated for instance in any of [122, 10, 102]. For X, we just remark that we have

$$X \geq b^{-1} \sum_{k=0}^{b-1} |W_k| X(k). \tag{2.28}$$

Moreover, the event $\{X = 0\}$ is measurable with respect to $\bigcap_{n\geq 1} \sigma(W(w) : |w| \geq n)$ because the components of W do not vanish. Thus, this event have probability 0 or 1. Since the function F_W is not almost surely equal to 0, X is positive with probability 1. We can then use the inequality (2.28) in the same way as in [69, 122, 10] when W is positive to prove that all the moments of negative order of X are finite as soon as the same property holds for the random variables $|W_i|$.

Proof of Lemma 2.3

Let $(\varepsilon_k)_{k\geq 1}$ be a positive sequence converging to 0 at ∞. Since $|J_w| = |Q(w)|^\beta X_\beta(w)$, due to (2.26), for any $k \geq 1$, with probability 1, for n large enough we have $b^{-n\varepsilon_k} \leq \inf_{w\in\mathscr{A}^n} \inf_{0\leq i\leq b-1} |W_i(w)|$. This is an immediate consequence of the fact that all the moments of negative order of the random variables $|W_i|$ are finite. Since the set $\{\varepsilon_k : k \geq 1\}$ is countable, we have the conclusion.

2.4.4 Proof of Theorem 2.5

Suppose that there exist a continuous and increasing function G defined on $[0,1]$ with $G(0) = 0$, as well as a monofractal continuous function B defined on $[G(0), G(1)]$ such that $F = B \circ G$. We denote by H the Hölder exponent of B (notice that it may be random).

At first, suppose that $H \in (0,1]$. For every $\alpha \in (0, H)$ the function B is uniformly α-Hölder, so there exists $C > 0$ such that $\mathrm{Osc}_B(I) \leq C|I|^\alpha$. Consequently, for every $n \in \mathbb{N}_+$ and $q \geq 0$, we have

$$\sum_{w\in\mathcal{A}^n} \mathrm{Osc}_F(I_w)^q \leq C^q \sum_{w\in\mathcal{A}^n} |G(I_w)|^{\alpha q}.$$

Since G is increasing, taking $q = 1/\alpha$ yields $\sum_{w\in\mathcal{A}^n} \mathrm{Osc}_F(I_w)^{1/\alpha} \leq C^{1/\alpha} G(1)$, hence $\tau_F(1/\alpha) \geq 0$. This is in contradiction with the fact (established in Chapter 3) that $\tau_F = \varphi_W$ over $\mathbb{R}_+$ almost surely.

Now we suppose that $H = 0$. If $\dim\big([0,1]\setminus \overline{E}_G(\infty)\big) > 0$, then we have $h_{F_W}(t) = 0 = h_B(G(t))$ at each $t \in [0,1]\setminus \overline{E}_G(\infty)$. But since $\tau_F \geq \varphi_W$ over $\mathbb{R}_+$, we have $\dim E_F(0) \leq \tau_F^*(0) \leq \inf_{q\geq 0} -\varphi_W(q) = 0$. This yields a contradiction.

Chapitre 3

Multifractal Analysis of complex b-adic independent Cascade functions

This chapter contains a joint work [16] with Julien Barral, which studies the multifractal analysis of complex b-adic independent cascade functions constructed in Section 2.2.

Preliminaries

The coding space

We fix an integer $b \geq 2$. For every $n \geq 0$ we define $\mathscr{A}^n = \{0, \dots, b-1\}^n$ (by convention $\mathscr{A}^0$ contains the empty word denoted $\varnothing$), $\mathscr{A}^* = \bigcup_{n \geq 0} \mathscr{A}^n$, and $\mathscr{A}^{\mathbb{N}_+} = \{0, \dots, b-1\}^{\mathbb{N}_+}$.

If $n \geq 1$, and $w = w_1 \cdots w_n \in \mathscr{A}^n$ then for every $1 \leq k \leq n$, the word $w_1 \dots w_k$ is denoted $w|_k$, and if $k = 0$ then $w|_0$ stands for $\varnothing$. Also, if $t \in \mathscr{A}^{\mathbb{N}_+}$ and $n \geq 1$, $t|_n$ denotes the word $t_1 \cdots t_n$ and $t|_0$ the empty word.

The word obtained by concatenation of $u \in \mathscr{A}^*$ and $v \in \mathscr{A}^* \cup \mathscr{A}^{\mathbb{N}_+}$ is denoted $u \cdot v$ and sometimes uv. For every $w \in \mathscr{A}^*$, the cylinder with root w, i.e. $\{w \cdot t : t \in \mathscr{A}^{\mathbb{N}_+}\}$ is denoted $[w]$. The σ-algebra generated in $\mathscr{A}^{\mathbb{N}_+}$ by the cylinders, namely $\sigma([w] : w \in \mathscr{A}^*)$ is denoted $\mathcal{S}$. The set $\mathscr{A}^{\mathbb{N}_+}$ is endowed with the standard metric distance $d(t, s) = \inf\{b^{-n} : n \geq 0,\ \exists\, w \in \mathscr{A}^n,\ t, s \in [w]\}$. Then the Borel σ-algebra is equal to $\mathcal{S}$.

We denote by π the natural projection of $\mathscr{A}^{\mathbb{N}_+}$ onto $[0,1]$: If $t \in \mathscr{A}^{\mathbb{N}_+}$, $\pi(t) = \sum_{k=1}^{\infty} t_k b^{-k}$.

When $t \in [0,1]$ is not a b-adic point, we identify it with the element of $\mathscr{A}^{\mathbb{N}_+}$ which represent its b-adic expansion, namely the element of $\pi^{-1}(\{t\})$.

For every $n \geq 0$, the length of an element of $\mathscr{A}^n$ is by definition equal to n and we denote it $|w|$. For $w \in \mathscr{A}^*$, we define $I_w = [t_w, t_w + b^{-|w|})$ and $I_w^L = F_L(I_w)$. We denote by w^- or w^{-1} (resp. w^+ or w^{+1}) the unique element of $\mathscr{A}^{|w|}$ such that $t_w - t_{w^-} = b^{-|w|}$ (resp. $t_{w^+} - t_w = b^{-|w|}$) whenever $t_w \neq 0$ (resp. $t_w \neq 1 - b^{-|w|}$) . We also denote w by w^0.

Multifractal analysis of functions

Denote by $(f^{(m)})_{m\geq 1}$ the sequence of f derivatives in the distribution sense. If J is a non trivial compact subinterval of I, for $m \geq 1$, let

$$\mathrm{Osc}_f^{(m)}(J) = \sup_{[t,t+m\delta]\subset J} |\Delta_\delta^m f(t)|,$$

where $\Delta_\delta^1 f(t) = f(t+\delta) - f(t)$ and for $m \geq 2$, $\Delta_\delta^m f(t) = \Delta_\delta^{m-1} f(t+\delta) - \Delta_\delta^{m-1} f(t)$ (notice that $\mathrm{Osc}_f^{(1)}(J) = \sup_{s,t\in J} |f(s) - f(t)|$). Then, the pointwise oscillation exponent of order $m \geq 1$ of f at $t \in \mathrm{Supp}(f^{(m)})$ is defined as

$$h_f^{(m)}(t) = \liminf_{r\to 0^+} \frac{\log \mathrm{Osc}_f^{(m)}(B(t,r))}{\log r}.$$

If $t \in \bigcap_{m\geq 1} \mathrm{Supp}(f^{(m)})$, then the sequence $(h_f^{(m)}(t))_{m\geq 1}$ is non decreasing. In fact, $\sup_{m\geq 1} h_f^{(m)}(t) = h_f(t)$. Recall in Section 1.2.2 we have the following proposition :

Proposition 3.1 *If $f : I \to \mathbb{C}$ is continuous, then for $t \in \bigcap_{m\geq 1} \mathrm{Supp}(f^{(m)})$, $h_f^{(m)}(t)$ converges to $h_f(t)$. Moreover, if $h_f(t) < \infty$, then $h_f^{(m)}(t) = h_f(t)$ for all $m > h_f(t)$.*

The multifractal analysis of f consists in computing singularity spectra like

$$h \geq 0 \mapsto \dim_H E_f^{(m)}(h), \tag{3.1}$$

where for $h \geq 0$ and $m \in \mathbb{N}_+$,

$$E_f^{(m)}(h) = \left\{t \in \mathrm{Supp}(f^{(m)}) : h_f^{(m)}(t) = h\right\},$$

and for $h \geq 0$,

$$E_f^{(\infty)}(h) = \left\{t \in \bigcap_{n\geq 1} \mathrm{Supp}(f^{(n)}) : h_f^{(\infty)}(t) = h\right\}, \quad \big(\text{where } h_f^{(\infty)}(t) = h_f(t)\big).$$

Proposition 3.1 yields

$$E_f^{(\infty)}(h) = E_f^{(m)}(h) \quad (\forall h \geq 0,\ \forall\ m > h).$$

For each $m \geq 1$ the L^q-spectrum of f associated with the oscillations of order m is defined as

$$\tau_f^{(m)}(q) = \liminf_{r\to 0} \frac{\log \sup \left\{\sum_i \mathrm{Osc}_f^{(m)}(B_i)^q\right\}}{\log(r)},$$

where the supremum is taken over all the families of disjoint closed intervals B_i of radius r with centers in $\mathrm{Supp}(f^{(m)})$.

For all $h \geq 0$ and $m \geq 1$, we have (Proposition 3.2)

$$\dim_H E_f^{(m)}(h) \leq (\tau_f^{(m)})^*(h) = \inf_{q\in\mathbb{R}} hq - \tau_f^{(m)}(q),$$

and due to Proposition 3.1,

$$\dim_H E_f^{(\infty)}(h) \le (\tau_f^{(\infty)})^*(h) := \inf_{m>h} (\tau_f^{(m)})^*(h), \tag{3.2}$$

a negative dimension meaning that $E_f^{(m)}(h)$ is empty. We will say that the multifractal formalism holds for f and $m \in \mathbb{N}_+ \cup \{\infty\}$ at $h \ge 0$ if $E_f^{(m)}(h)$ is not empty and $\dim_H E_f^{(m)}(h) = (\tau_f^{(m)})^*(h)$.

Independent copies of F_W and F_L, and associated quantities

Let (W, L) be a couple of random vectors taking values in $\mathbb{C}^b \times {\mathbb{R}_+^*}^b$. Assume that W and L satisfy the assumptions in Theorem 1.2. Moreover, we assume that

$$\varphi_L > -\infty \text{ over } \mathbb{R} \text{ and } 0 < L_i < 1 \text{ almost surely },$$

where for $U \in \{W, L\}$ and $p \in \mathbb{R}$, $\varphi_U(p) = -\log_b \mathbb{E}(\sum_{i=0}^{b-1} \mathbf{1}_{\{U_i \neq 0\}} |U_i|^p)$.

If $w \in \mathscr{A}^*$, $n \ge 1$ and $U \in \{W, L\}$, we denote by $F_{U,n}^{[w]}$ the function constructed as $F_{U,n}$ in (2.6), but with the weights $(U(w \cdot v))_{v \in \mathscr{A}^*}$. By construction, $F_{U,n}^{[\varnothing]} = F_{U,n}$, and

$$F_{U,n}^{[w]}(t) = \int_0^t b^n \prod_{k=1}^n U_{u_k}(w \cdot u|_{|w|+k-1}) \, \mathrm{d}u.$$

We denote by $F_U^{[w]}$ the almost sure uniform limit of $(F_{U,n}^{[w]})_{n \ge 1}$ (ensured by Theorem 1.2). We also define

$$Q_U(w) = \prod_{k=1}^n U_{w_k}(w|_{k-1}).$$

For $m \ge 1$ we denote $\mathrm{Osc}_{F_U}^{(m)}([0,1])$ by $Z_U^{(m)}$ and more generally $\mathrm{Osc}_{F_U^{[w]}}^{(m)}([0,1])$ by $Z_U^{(m)}(w)$. Also, we denote $\mathrm{Osc}_{F_U}^{(m)}(I_w)$ by $\mathrm{O}_U^{(m)}(w)$. By construction, we have

$$\begin{aligned} \mathrm{Osc}_F^{(m)}(I_w^L) &= \mathrm{Osc}_{F_W}^{(m)}(I_w) = \mathrm{O}_W^{(m)}(w) = |Q_W(w)| Z_W^{(m)}(w), & (3.3)\\ |I_w^L| &= \mathrm{Osc}_{F_L}^{(1)}(I_w) = \mathrm{O}_L^{(1)}(w) = Q_L(w) Z_L^{(1)}(w). & (3.4) \end{aligned}$$

For $(q,t) \in \mathbb{R}^2$ let

$$\Phi(q,t) = \mathbb{E}\Big(\sum_{i=0}^{b-1} \mathbf{1}_{\{W_i \neq 0\}} |W_i|^q L_i^{-t}\Big) \text{ and } \Psi(q,t) = \mathbb{E}\big(\mathrm{Osc}_{F_w}([0,1])^q F_L(1)^{-t}\big). \tag{3.5}$$

Hausdorff dimension

If (X, d) is a locally compact metric space, for $D \in \mathbb{R}$, $\delta > 0$, and $E \subset X$, let

$$\mathcal{H}_\delta^D(E) = \inf\{\sum_{i \in I} |U_i|^D\},$$

where the infimum is taken over the set of all the at most countable coverings $\bigcup_{i\in I} U_i$ of E such that $0 \le |U_i| \le \delta$, where $|U_i|$ stands for the diameter of U_i and by convention $0^D = 0$. Then define

$$\mathcal{H}^D(E) = \lim_{\delta \searrow 0} \mathcal{H}^D_\delta(E)$$

($\mathcal{H}^D_\delta(E)$ is by construction a non-increasing function of δ). If $D \ge 0$, $\mathcal{H}^D(E)$ is called the D-dimensional Hausdorff measure of E. The Hausdorff dimension of E is the number

$$\dim_H E = \inf\{D : \mathcal{H}^D(E) < \infty\}.$$

It is clear that we have $\dim_H E < 0$ if and only if $\dim_H E = -\infty$ and E is the emptyset (see [51, 116] for more details).

If μ is a Borel measure on X, we define $\dim_H \mu$ as $\inf\{\dim_H E : \mu(E) > 0\}$.

We denote by $(\Omega, \mathcal{A}, \mathbb{P})$ the probability space on which the random variables are defined.

Finally, if f is a bounded $\mathbb{C}$-valued function over an interval I, then $\|f\|_\infty$ stands for $\sup_{t\in I} |f(t)|$.

3.1 Statements of results

Theorem 3.1 (Bell shaped spectra)

Suppose that $\mathbb{P}(\sum_{i=0}^{b-1} \mathbf{1}_{\{W_i \neq 0\}} \ge 2) = 1$ and $\varphi_W > -\infty$ over $\mathbb{R}$. For $q \in \mathbb{R}$, let $\tau(q)$ be the unique solution of $\mathbb{E}(\sum_{i=0}^{b-1} \mathbf{1}_{\{W_i\neq 0\}} |W_i|^q L_i^{-t}) = 1$. The function τ is concave and analytic. With probability 1,

1. $\mathrm{Supp}(F^{(m)}) = \mathrm{Supp}(F')$ *for all* $m \in \mathbb{N}_+$ *and* $\dim_H \mathrm{Supp}(F') = -\tau(0)$.
2. *For all $h \ge 0$ and $m \in \mathbb{N}_+ \cup \{\infty\}$, $\dim_H E_F^{(m)}(h) = (\tau_F^{(m)})^*(h) = (\tau_F^{(1)})^*(h)$, a negative dimension meaning that $E_F^{(m)}(h)$ is empty. Moreover, $E_F^{(m)}(h) \neq \emptyset$ if $(\tau_F^{(1)})^*(h) = 0$. In other words, for all $m \in \mathbb{N}_+ \cup \{\infty\}$, F obeys the multifractal formalism at every $h \ge 0$ such that $(\tau_F^{(m)})^*(h) \ge 0$. In addition, if F_W is built as in Theorem 2.3.2 (critical case), the left endpoint of these singularity spectra is the exponent* 0, *and the corresponding level set is dense, with Hausdorff dimension 0.*
3. *For all $m \in \mathbb{N}_+$, $\tau_F^{(m)} = \tau$ on the interval $J = \{q \in \mathbb{R} : \tau'(q)q - \tau(q) \ge 0\}$, and if $\overline{q} = \sup(J) < \infty$ (resp. $\underline{q} := \inf(J) > -\infty$) then $\tau_F^{(m)}(q) = \tau'(\overline{q})q$ (resp. $\tau'(\underline{q})q$) over $[\overline{q}, \infty)$ (resp. $(-\infty, \underline{q}]$).*
   Moreover, if there does not exist $H \in (0,1)$ such that for all $0 \le i \le b-1$ we have $|W_i| \in \{0, L_i^H\}$ then τ is strictly concave over J ; otherwise, $\tau(q) = qH + \tau(0)$ and F is monofractal with a Hölder exponent equal to H.

Theorem 3.2 (Left-sided spectra)

Suppose that $\mathbb{P}(\sum_{i=0}^{b-1} \mathbf{1}_{\{W_i\neq 0\}} \ge 2) = 1$ and $\varphi_W(q) > -\infty$ over $\mathbb{R}_+$. For $q \in \mathbb{R}_+$, let $\tau(q)$ be defined as in Theorem 3.1 . The function τ is concave, and analytic over $(0, \infty)$.

Suppose also that $\mathbb{E}(\sum_{i=0}^{b-1} \mathbf{1}_{\{W_i \neq 0\}} L_i \log(|W_i|)) = -\infty$, *i.e.* $\tau'(0) = \infty$. *Finally, suppose that* $\mathbb{E}\big((\max_{0\le i\le b-1}|W_i|)^{-\varepsilon}\big) < \infty$ *for some* $\varepsilon > 0$.

Then, the same conclusions as in Theorem 3.1 hold. Moreover, the singularity spectra are left-sided, and $h_F^{(m)} = \infty$ *for all* $m \in \mathbb{N}_+ \cup \{\infty\}$ *on a set of full dimension in* $\mathrm{Supp}(F')$. *In addition, if* F_W *is built as in Theorem 2.3.2 (critical case), the support of the spectra is* $[0, \infty]$.

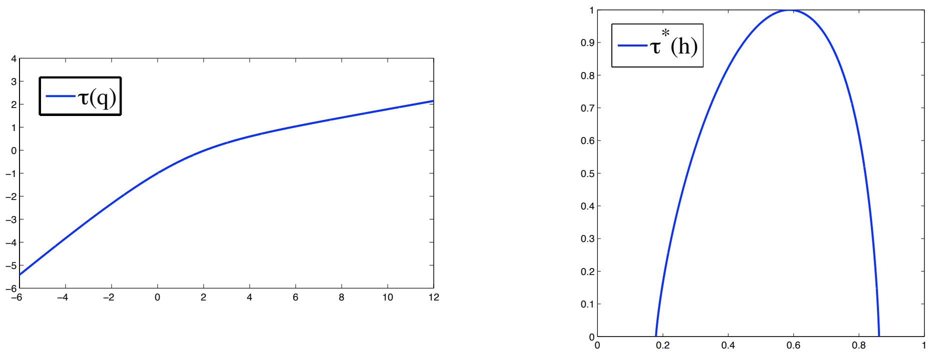

FIGURE 3.1 – Bell shaped spectrum in the case where the left endpoint is not 0.

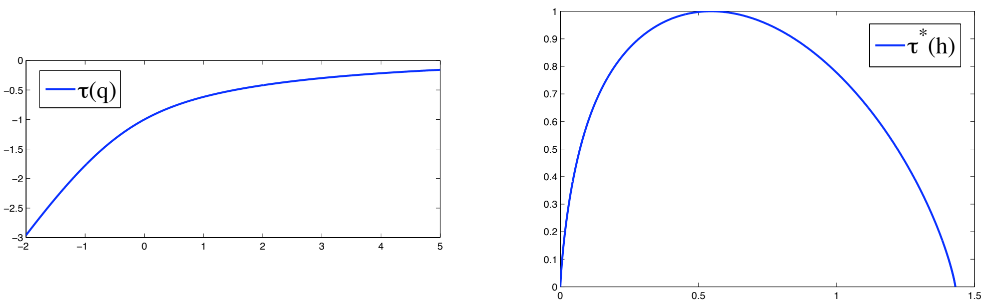

FIGURE 3.2 – Bell shaped spectrum in the critical case where the left endpoint is 0.

Corollary 3.1 *Let* f *be a complex valued* C^∞ *function over* $\mathbb{R}_+$ *such that for all* $m \in \mathbb{N}_+$ *the function* $f^{(m)}$ *does not vanish. Let* F *be as in Theorem 3.2 whose spectrum is supported on* $[0, \infty]$ *and let* $G = F + f$. *The functions* F *and* G *have the same multifractal behavior from the pointwise Hölder exponent point of view.*

For $m \in \mathbb{N}_+$, *let* q_m *be the unique real number such that* $\tau(q_m) = q_m m - 1$.

With probability 1, for all $m \in \mathbb{N}_+$, *we have* $\tau_G^{(m)} = \tau_F^{(m)} = \tau$ *over* $[q_m, \infty)$, *and* $\tau_G^{(m)}(q) = qm - 1$ *for* $0 \le q < q_m$. *Moreover, for all* $m \in \mathbb{N}_+$, *the multifractal formalism holds at every* $h \in [0, \tau'(q_m)]$ *such that* ${\tau_G^{(m)}}^*(h) \ge 0$ *as well as at* $h = m$, *and for all* $h \in (\tau'(q_m), m)$ *we have* $\dim_H E_G^{(m)}(h) = \tau^*(h) < (\tau_G^{(m)})^*(h)$.

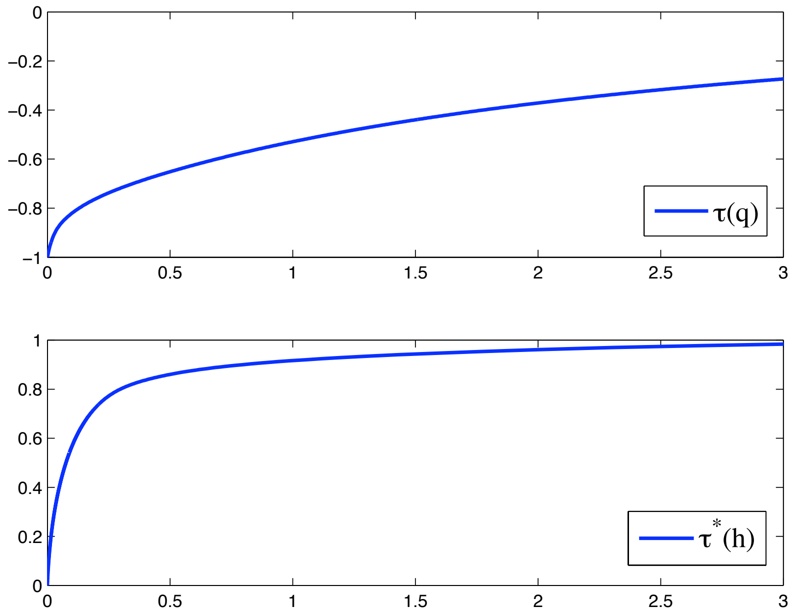

FIGURE 3.3 – Concave left-sided spectra with support $[0, \infty]$ in the critical case.

3.2 Proofs of Theorem 3.1, Theorem 3.2 and Corollary 3.1

The next three sections provide intermediate results yielding Theorem 3.1. Detailed proofs of these results are given in Section 3.3. The proof of Theorem 3.2 is almost the same as that of Theorem 3.1 and we outline it in Section 3.2.5. Corollary 3.1 is given in Section 3.2.6, and Section 3.2.7 provides weaker assumptions under which these result still hold, or partially hold.

In the next three sections we work under the assumptions of Theorem 3.1.

3.2.1 Upper bound for the singularity spectra

Let f be a measurable bounded function from $[0, 1]$ to $\mathbb{R}$.

Proposition 3.2 *Let $m \geq 1$. If $\mathrm{Supp}(f^{(m)}) \neq \emptyset$ then for every $h \geq 0$ we have*

$$\dim_H E_f^{(m)}(h) \leq (\tau_f^{(m)})^*(h),$$

a negative dimension meaning that $E_f(h)$ is empty. Also,

$$\dim_H \mathrm{Supp}(f^{(m)}) \leq \overline{\dim}_B \mathrm{Supp}(f^{(m)}) = -\tau_f^{(m)}(0),$$

where $\overline{\dim}_B$ stands for the upper box dimension (see [51] for the definition).

Remark 3.1 *When f is non-decreasing and $m = 1$, the L^q-spectrum $\tau_f^{(1)}$ is nothing but the L^q-spectrum of the measure f', and the inequality provided by Proposition 3.2 is*

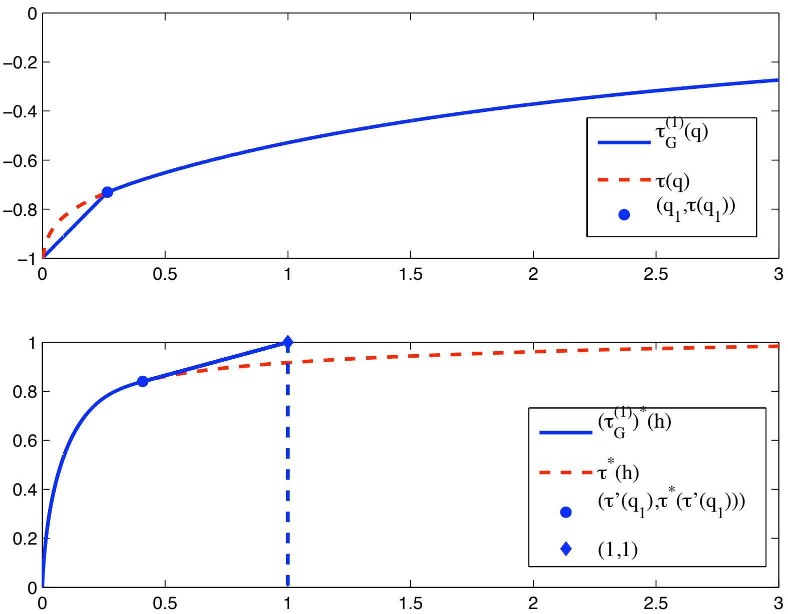

FIGURE 3.4 – Top : $\tau_G^{(1)}(q) = \min\{q-1, \tau(q)\}$ for $q \geq 0$. Bottom : $(\tau_G^{(1)})^*(h) = \tau^*(h)$ for $h \in [0, \tau'(q_1)]$, $(\tau_G^{(1)})^*(h) = \tau^*(\tau'(q_1)) + q_1(h - \tau'(q_1))$ for $h \in (\tau'(q_1), 1]$ and $(\tau_G^{(1)})^*(h) = 1$ elsewhere.

familiar from the multifractal formalism for measures. Though the proof of the inequality is similar for $m \geq 2$, for the reader's convenience we will give a proof of Proposition 3.2 in Section 3.4 (see also [81] for similar bounds).

We first need the following propositions.

Proposition 3.3 *With probability 1,* $\mathrm{Supp}(F') \neq \emptyset$*, and the function F is nowhere locally equal to a polynomial over the support of F'. Consequently,* $\mathrm{Supp}(F^{(m)}) = \mathrm{Supp}(F')$ *for all $m \geq 1$.*

Now for $n \geq 1$, and $(q,t) \in \mathbb{R}^2$ define

$$\theta_{F,n}^{(m)}(q,t) = \sum_{w \in \mathscr{A}^n} \mathrm{Osc}_{F_W}^{(m)}(I_w)^q |I_w^L|^{-t} \text{ and } \widetilde{\theta}_{F,n}^{(m)}(q,t) = \mathbb{E}\big(\theta_{F,n}^{(m)}(q,t)\big),$$

with the convention $0^q = 0$. Then define

$$\theta_F^{(m)}(q,t) = \limsup_{n\to\infty} \theta_{F,n}^{(m)}(q,t) \text{ and } \widetilde{\theta}_F^{(m)}(q,t) = \limsup_{n\to\infty} \widetilde{\theta}_{F,n}^{(m)}(q,t),$$

as well as

$$\tau_{F,b}^{(m)}(q) = \sup\{t \in \mathbb{R} : \theta_F^{(m)}(q,t) = 0\} \text{ and } \widetilde{\tau}_{F,b}^{(m)}(q) = \sup\{t \in \mathbb{R} : \widetilde{\theta}_{F,b}^{(m)}(q,t) = 0\}.$$

Proposition 3.4 *Let $m \geq 1$. With probability 1, for all $q \in \mathbb{R}_+$ we have $\tau_F^{(m)}(q) \geq \tau_{F,b}^{(1)}(q) \geq \widetilde{\tau}_{F,b}^{(1)}(q)$, and for all $q \leq \mathbb{R}_-^*$ we have $\tau_F^{(m)}(q) \geq \tau_{F,b}^{(m)}(q) \geq \widetilde{\tau}_{F,b}^{(m)}(q)$.*

Moreover, $\widetilde{\tau}_{F,b}^{(m)}(q) = \tau(q)$ for all $q < \widetilde{q}$, where $\widetilde{q} = \max\{p : \tau(p) = 0\}$ (by convention $\max(\emptyset) = \infty$*).*

Proof of the upper bound for the singularity spectra

Let $m \geq 1$. Recall that $J = \{q \in \mathbb{R} : \tau'(q)q - \tau(q) \geq 0\}$. Since τ is concave, we have $J \subset (-\infty, \widetilde{q}]$. Consequently, since $(\tau_F^{(m)})^*$ is concave, due to Proposition 3.4, with probability 1, for all $h \geq 0$ we may have $(\tau_F^{(m)})^*(h) \geq 0$ only if $\tau^*(h) \geq 0$. In this case, we have $\dim_H E_F^{(m)}(h) \leq (\tau_F^{(m)})^*(h) \leq \tau^*(h)$ by Proposition 3.2. Also, since 0 belongs to J, we have $\dim_H \operatorname{Supp}(F') \leq -\tau(0)$.

Lower bound for the singularity spectra

Let $I = \overline{\{\tau'(q) : q \in J\}}$. We are going to distinguish the case $h \in \operatorname{Int}(I)$ and the case $h \in \partial I$.

For $q \in J$ and $w \in \mathscr{A}^*$ we set

$$W_q(w) = \left(\mathbf{1}_{\{W_i(w)\neq 0\}}|W_i(w)|^q(w)L_i(w)^{-\tau(q)}\right)_{0\leq i\leq b-1}. \tag{3.6}$$

3.2.2 Lower bound in the case $h \in \operatorname{Int}(I)$

At first we introduce some auxiliary measures. If $q \in \operatorname{Int}(J)$, $w \in \mathscr{A}^*$, $n \geq 1$ and $v \in \mathscr{A}^n$ let

$$Q_q^{[w]}(v) = \prod_{k=1}^{n} W_{q,v_k}(w \cdot v|_{|w|+k-1}), \tag{3.7}$$

and simply denote $Q_q^{[\varnothing]}(v)$ by $Q_q(v) = \mathbf{1}_{\{Q_W(v)\neq 0\}}|Q_W(v)|^q Q_L(v)^{-\tau(q)}$. Let

$$Y_{q,n}(w) = \sum_{v\in\mathscr{A}^n} Q_q^{[w]}(v).$$

Proposition 3.5

1. *With probability 1, for all $q \in \operatorname{Int}(J)$ and $w \in \mathscr{A}^*$, the sequence $Y_{q,n}(w)$ converges to a positive limit $Y_q(w)$. Moreover, for every $n \geq 1$, $\sigma(\{Q_U(w) : w \in \mathscr{A}^{n-1}, U \in \{W, L\}\})$ and $\sigma(\{Y_q(w) : w \in \mathscr{A}^n\})$ are independent, and the random variables $Y_q(w)$, $w \in \mathscr{A}^n$, are independent copies of $Y_q(\emptyset)$, that we denote by Y_q.*
2. *For every compact subinterval K of $\operatorname{Int}(J)$, there exists $p_K > 1$ such that*

$$\mathbb{E}(\sup_{q\in K} Y_q^{p_K}) < \infty.$$

3. *With probability 1, for all $q \in \operatorname{Int}(J)$, the function*

$$\mu_q([w]) = Q_q(w)Y_q(w),\ w \in \mathscr{A}^* \tag{3.8}$$

defines a Borel measure on $\mathscr{A}^{\mathbb{N}_+}$.

Recall the definitions in the preliminaries.

For $m \geq 1$, $t \in \mathscr{A}^{\mathbb{N}_+}$, $U \in \{W, L\}$ and $\gamma \in \{-1, 0, +1\}$ let

$$\underline{\alpha}_U^{(m),\gamma}(t) \text{ (resp. } \overline{\alpha}_U^{(m),\gamma}(t)) = \liminf_{n\to\infty} \text{ (resp. } \limsup_{n\to\infty}) \ -\frac{\log_b \mathrm{Osc}_{F_U}^{(m)}((t|_n)^\gamma)}{n}.$$

The next proposition follows directly from the definition of the m^{th} oscillation.

Proposition 3.6 *Let* $t \in \mathscr{A}^{\mathbb{N}_+}$ *and* $\widetilde{t} = F_L(\pi(t))$.

1. *Let* $r \in (0,1)$ *and suppose that*

$$\exists\ n_r,\ n_r' \in \mathbb{N},\ I^L_{t|_{n_r}} \subset B(\widetilde{t}, r) \subset I^L_{t|^-_{n_r'}} \cup I^L_{t|_{n_r'}} \cup I^L_{t|^+_{n_r'}}. \tag{3.9}$$

Then

$$O_W^{(m)}(t|_{n_r}) \leq O_F^{(m)}(B(\widetilde{t}, r)) \leq 2^{m-1} O_F^{(1)}(B(\widetilde{t}, r)) \leq 2^{m-1} \sum_{w \in \{t|^-_{n_r'}, t|_{n_r'}, t|^+_{n_r'}\}} O_W^{(1)}(w). \tag{3.10}$$

2. *Suppose that* (3.9) *holds for all* $r > 0$ *small enough and* $\lim_{r\to 0^+} n_r/n_r' = 1$. *Then,*

$$\frac{\min\{\underline{\alpha}_W^{(1),\gamma}(t) : \gamma = -1, 0, +1\}}{\overline{\alpha}_L^{(1),0}(t)} \leq h_F^{(m)}(\widetilde{t}) \leq \frac{\overline{\alpha}_W^{(m),0}(t)}{\min\{\underline{\alpha}_L^{(1),\gamma}(t) : \gamma = -1, 0, +1\}}. \tag{3.11}$$

Recall that for $(q, t) \in \mathbb{R}^2$ we have defined

$$\Phi(q, t) = \mathbb{E}\Big(\sum_{i=0}^{b-1} \mathbf{1}_{\{W_i \neq 0\}} |W_i|^q L_i^{-t}\Big)$$

and $\tau(q)$ is the unique solution of $\Phi(q, \tau(q)) = 1$. By construction, we have

$$\tau'(q) = -\frac{(\partial\Phi/\partial q)(q, \tau(q))}{(\partial\Phi/\partial t)(q, \tau(q))} = \frac{\mathbb{E}\Big(\sum_{i=0}^{b-1} \mathbf{1}_{\{W_i \neq 0\}} |W_i|^q L_i^{-\tau(q)} \log(|W_i|)\Big)}{\mathbb{E}\Big(\sum_{i=0}^{b-1} \mathbf{1}_{\{W_i \neq 0\}} |W_i|^q L_i^{-\tau(q)} \log(L_i)\Big)}. \tag{3.12}$$

Proposition 3.7 *With probability 1, for all* $q \in \mathrm{Int}(J)$, *for* μ_q*-almost every* $t \in \mathrm{Supp}(\mu_q)$,

1. $\displaystyle\lim_{n\to\infty} \frac{\log |Q_W(t|_n)|}{-n} = -\frac{\partial\Phi}{\partial q}(q, \tau(q))$;
   $\displaystyle\lim_{n\to\infty} \frac{\log |Q_W((t|_n)^\gamma)|}{-n} \in \{-\frac{\partial\Phi}{\partial q}(q, \tau(q)), +\infty\}$, *for* $\gamma \in \{-1, 1\}$ *;*
2. $\displaystyle\lim_{n\to\infty} \frac{\log Q_L(t|_n)}{-n} = \lim_{n\to\infty} \frac{\log Q_L((t|_n)^\gamma)}{-n} = \frac{\partial\Phi}{\partial t}(q, \tau(q))$, *for* $\gamma \in \{-1, 1\}$ *;*
3. $\displaystyle\lim_{n\to\infty} \frac{\log_b Z_U^{(m)}(t|_n)}{n} = \lim_{n\to\infty} \frac{\log Z_U^{(m)}((t|_n)^\gamma)}{n} = 0$, *for all* $m \geq 1$, $U \in \{W, L\}$ *and* $\gamma \in \{-1, 1\}$.
4. $\displaystyle\liminf_{n\to\infty} \frac{\log Y_q(t|_n)}{-n} \geq 0$.

Proof of the lower bound

Due to (3.8) and Proposition 3.7 (1), (2) and (4), with probability 1, for all $q \in \mathrm{Int}(J)$, we have

$$\begin{aligned}\liminf_{n\to\infty} \frac{\log(\mu_q([t|_n]))}{-n} &\geq -q\frac{\partial \Phi}{\partial q}(q,\tau(q)) - \tau(q)\frac{\partial \Phi}{\partial t}(q,\tau(q)) \\ &= \big(q\tau'(q) - \tau(q)\big)\cdot \frac{\partial \Phi}{\partial t}(q,\tau(q)) > 0,\ \mu_q\text{-}a.e.\end{aligned}$$

($\frac{\partial \Phi}{\partial t}(q,\tau(q)) > 0$ due to our choice $L_i \in (0,1)$). Consequently, μ_q is atomless, and defining $\nu_q = \mu_q \circ \pi^{-1} \circ F_L^{-1}$, we have $\nu_q(I_w^L) = \mu_q([w])$ for all $w \in \mathscr{A}^*$. Thus,

$$\liminf_{n\to\infty} \frac{\log \nu_q(I_n^L(t))}{-n} \geq q\frac{\partial \Phi}{\partial q}(q,\tau(q)) + \tau(q)\frac{\partial \Phi}{\partial t}(q,\tau(q)),\ \nu_q\text{-almost everywhere,}$$

where $I_n^L(t)$ is the unique interval I_w^L of generation n containing t.

Now, Proposition 3.7 (2) and (3) as well as (3.4) also yield

$$\lim_{n\to\infty} \frac{\log|I_n^L(t)|}{-n} = \frac{\partial \Phi}{\partial t}(q,\tau(q)) > 0,\ \nu_q\text{-almost everywhere,}$$

hence

$$\liminf_{n\to\infty} \frac{\log \nu_q(I_n^L(t))}{\log|I_n^L(t)|} \geq q\tau'(q) - \tau(q),\ \nu_q\text{-almost everywhere.}$$

Consequently, we can apply the mass distribution principle ([130], Lemma 4.3.2) and we obtain $\dim_H(\nu_q) \geq q\tau'(q) - \tau'(q) = \tau^*(\tau'(q))$.

We can also deduce from Proposition 3.7 that for μ_q-almost every t, for all $m \geq 1$,

$$\begin{cases}\min\{\underline{\alpha}_W^{(1),\gamma}(t) : \gamma = -1,0,+1\} = \overline{\alpha}_W^{(m)}(t) = -\frac{\partial \Phi}{\partial q}(q,\tau(q))/\log(b),\\ \min\{\underline{\alpha}_L^{(1),\gamma}(t) : \gamma = -1,0,+1\} = \overline{\alpha}_L^{(1)}(t) = \frac{\partial \Phi}{\partial t}(q,\tau(q))/\log(b).\end{cases}$$

These properties imply that at ν_q-amost every $\widetilde{t}$, for $r \in (0,1)$ small enough, we can find integers n_r and n_r' such that (3.10) holds with $\lim_{r\to 0^+} n_r/n_r' = 1$, and we have for all $m \geq 1$ $h_F^{(m)}(t) = \tau'(q)$. Due to Proposition 3.1, we also have $h_F^{(\infty)}(t) = \tau'(q)$. Since $\dim_H(\nu_q) \geq \tau^*(\tau'(q))$ we have the desired lower bound for the dimensions of the sets $E_F^{(m)}(\tau'(q))$, $m \in \mathbb{N}\cup\{\infty\}$. The case $q = 0$ yields $\dim_H \mathrm{Supp}(F') \geq \dim_H E_F^{(1)}(\tau'(0)) \geq -\tau(0)$.

Combining this with Proposition 3.4 we obtain that, with probability 1, for all $m \in \mathbb{N}_+$, we have $(\tau^{(m)})_F^* = \tau^*$ over $\mathrm{Int}(I)$. Since we also have $\tau_F^{(m)} \geq \tau_{F,b}^{(m)} \geq \tau$ over J, this yields $\tau_F^{(m)} = \tau_{F,b}^{(m)} = \tau$ over J.

3.2.3 Lower bound in the case $h \in \partial I$

Recall that $\underline{q} = \inf J$ and $\overline{q} = \sup J$. Let

$$\underline{h} = \lim_{q\to\overline{q}} \tau'(q), \bar{h} = \lim_{q\to\underline{q}} \tau'(q), \bar{d} = \lim_{q\to\overline{q}} \tau'(q)q - \tau(q), \underline{d} = \lim_{q\to\underline{q}} \tau'(q)q - \tau(q).$$

Then $\partial I = \{\underline{h}, \bar{h}\}$, $\bar{d} = \tau^*(\underline{h})$ and $\underline{d} = \tau^*(\bar{h})$. Moreover, with probability 1, $\bar{d} = \tau^*(\underline{h}) = (\tau_F^{(m)})^*(\underline{h})$ and $\underline{d} = \tau^*(\bar{h}) = (\tau_F^{(m)})^*(\bar{h})$ for any $m \geq 1$.

The difficulty in the study of $E_F^{(m)}(h)$ when $h \in \{\underline{h}, \bar{h}\}$ comes from the fact that there is no simple choice of a measure carried by $E_F^{(m)}(h)$ and whose Hausdorff dimension is larger than or equal to $(\tau_F^{(m)})^*(h)$. Even, it is not obvious to construct a point belonging to $E_F^{(m)}(h)$. Nevertheless such a measure can be constructed.

A measure μ_q partly carried by $E_F^{(m)}(h)$, where $(q,h) \in \{(\underline{q}, \underline{h}), (\bar{q}, \bar{h})\}$

1. The case $q \notin \{-\infty, \infty\}$.

Recall (3.6). We have

$$\tau^*(\tau'(q)) = \varphi'_{W_q}(1) = -\mathbb{E}(\sum_{i=0}^{b-1} W_{q,i} \log_b W_{q,i}) = 0.$$

Moreover, $\varphi_{W_q}(p) > -\infty$ in a neighborhood of 1^+. Consequently, it follows from Theorem 2.5 of [101] that, with probability 1, for all $w \in \mathscr{A}^*$, the martingale

$$Y_{q,n}(w) = -\sum_{v \in \mathscr{A}^n} Q_q^{[w]}(v) \log Q_q^{[w]}(v)$$

converges to a limit $Y_q(w)$ ($Y_q(\emptyset) = Y_q$) as $n \to \infty$. Moreover, by construction, the branching property $Y_q(w) = \sum_{i=0}^{b-1} W_{q,i}(w) Y_q(wi)$ holds, the random variables $Y_q(w)$, $w \in \mathscr{A}^*$, are identically distributed, and for $\gamma > 0$ we have $\mathbb{E}(Y_q^\gamma) < \infty$ if and only if $\gamma < 1$.

We deduce from the branching property and our assumption on the probability that the components of W vanish that the event $\{Y_q = 0\}$ is measurable with respect to the tail σ-algebra $\bigcap_{N \geq 1} \sigma(W(w) : w \in \bigcup_{n \geq N} \mathscr{A}^n)$. Consequently, $\mathbb{P}(Y_q > 0) = 1$ since $\mathbb{E}(Y_q) > 0$, and with probability 1, the branching property makes it possible to define on $\mathscr{A}^{\mathbb{N}_+}$ a measure μ_q by the formula

$$\mu_q([w]) = Q_q(w) Y_q(w). \tag{3.13}$$

Proposition 3.8 *Let $h \in \{\bar{h}, \underline{h}\}$ and $q \notin \{-\infty, \infty\}$ such that $h = \tau'(q)$. With probability 1, there exists a Borel set $E_h \subset \mathscr{A}^{\mathbb{N}_+}$ of positive μ_q-measure such that for all $t \in E_h$ the same conclusions as in Proposition 3.7.1, 3.7.2 and 3.7.3 hold.*

Then, the same arguments as in Section 3.2.2 yield $\nu_q(E_F^{(m)}(h)) > 0$, hence $E_F^{(m)}(h)$ is not empty and we get the desired lower bound $\dim_H E_F^{(m)}(h) \geq 0$ since $\tau^*(h) = 0$.

2. The case $q \in \{-\infty, \infty\}$.

Let $(q_k)_{k \geq 0}$ be an increasing (resp. decreasing) sequence converging to q if $q = \infty$ (resp. $q = -\infty$). For every $k \geq 0$ and $w \in \mathscr{A}^k$, recall that by (3.6)

$$W_{q_k}(w) = \big(\mathbf{1}_{\{W_i(w) \neq 0\}} \cdot |W_i(w)|^{q_k} L_i(w)^{-\tau(q_k)}\big)_{0 \leq i \leq b-1}.$$

Then, for $w \in \mathscr{A}^*$, $n \geq 1$ and $v \in \mathscr{A}^n$, instead of (3.7) we define

$$Q_q^{[w]}(v) = \prod_{k=1}^{n} W_{q_{|w|+k-1}, v_k}(w \cdot v|_{|w|+k-1}),$$

and simply denote $Q_q^{[\varnothing]}(v)$ by $Q_q(v)$. Then let

$$Y_{q,n}(w) = \sum_{v \in \mathscr{A}^n} Q_q^{[w]}(v).$$

and simply denote $Y_{q,n}(\varnothing)$ by $Y_{q,n}$. The sequence $(Y_{q,n}(w))_{n\geq 1}$ is a non-negative martingale of expectation 1 which converges almost surely to a limit that we denote by $Y_q(w)$ (Y_q if $w = \emptyset$). Since the set $\mathscr{A}^*$ is countable, all these random variable are defined simultaneously. Moreover, the branching property $Y_q(w) = \sum_{i=0}^{b-1} Q_{q,i}(w) Y_q(wi)$ also holds. Notice that by construction, given $k \geq 1$, the random variables $Y_q(w)$, $w \in \mathscr{A}^k$, are independent and identically distributed.

Proposition 3.9 *The sequence $(q_k)_{k\geq 0}$ can be chosen so that there exists $a > 0$ such that for all $w \in \mathscr{A}^*$ the sequence $(Y_{q,n}(w))_{n\geq 1}$ converges in L^2 norm to a limit Y_q and $\|Y_q(w)\|_2 = O\big(b^{a|w|/\log(|w|)}\big)$.*

Fix a sequence $(q_k)_{k\geq 0}$ as in the previous proposition. For the same reason as in the case $q \notin \{-\infty, \infty\}$, we have $\mathbb{P}(Y_q > 0) = 1$ and with probability 1, the branching property makes it possible to define on $\mathscr{A}^{\mathbb{N}_+}$ a measure μ_q by the formula (3.13).

Proposition 3.10 *Let $h \in \{\bar{h}, \underline{h}\}$ and $q \in \{-\infty, \infty\}$ such that $h = \lim_{J \ni q' \to q} \tau'(q)$. Let $\nu_q = \mu_q \circ \pi^{-1} \circ F_L^{-1}$. With probability 1, for every $m \geq 1$, we have $h_F^{(m)}(t) = h$ ν_q-almost everywhere and $\dim_H(\nu_q) \geq \tau^*(h)$.*

Remark 3.2 *In the case $q \notin \{-\infty, \infty\}$, it is possible to construct μ_q as in the case $q \in \{-\infty, \infty\}$ by using a sequence $(J \ni q_k)_{k\geq 0}$ converging to q. This avoids to require to Theorem 2.5 of [101] which is a strong result. Nevertheless, we are able to use this alternative only if $\varphi_W(q) > -\infty$ for some $q < -1$. This is the case under the assumptions of Theorem 3.1, but this does not always hold under the weaker assumptions provided by Section 3.2.7.*

3.2.4 The L^q spectra of F

We have seen at the end of Section 3.2.2 that, with probability 1, for all $m \in \mathbb{N}$, $\tau_F^{(m)}(q) = \tau_{F,b}^{(m)}(q) = \tau(q)$ over $J = [\underline{q}, \overline{q}]$. It remains to show that $\tau_F^{(m)}$ is differentiable at $\overline{q}$ (resp. $\underline{q}$) and linear over $[\overline{q}, \infty)$ (resp. $(-\infty, \underline{q}]$) if $\overline{q} < \infty$ (resp. $\underline{q} > -\infty$). We treat the case $\overline{q} < \infty$ and leave the case $\underline{q} > -\infty$ to the reader.

At first we notice that the equality $\tau_F^{(m)} = \tau_{F,b}^{(m)} = \tau$ over J implies that $(\tau_F^{(m)})'(\overline{q}^-) = \tau_{F,b}^{(m)}(\overline{q})/\overline{q} = \tau(\overline{q})/\overline{q} = \underline{h}$. Also, by concavity of $\tau_F^{(m)}$, we have $\tau_F^{(m)}(q) \leq \tau_F^{(m)}(\overline{q}) + (\tau_F^{(m)})'(\overline{q}^-)(q - \overline{q}) = \tau(\overline{q}) + \tau'(\overline{q})(q - \overline{q}) = \underline{h}q$. To get the other inequality, and so the differentiability of $\tau_F^{(m)}$ at $\overline{q}$, we use a simple idea inspired by the work achieved in [122] which focuses on $\tau_{F,b}^{(1)}$ in the case when the components of W are non-negative and $L = (1/b, \dots, 1/b)$. If $q \geq \overline{q}$ and $t \in \mathbb{R}$, we have

$$\sum_{w\in\mathscr{A}^n} \mathrm{Osc}_{F_W}^{(m)}(I_w)^q |I_w^L|^{-t} \leq \Big[\sum_{w\in\mathscr{A}^n} \mathrm{Osc}_{F_W}^{(m)}(I_w)^{\overline{q}} |I_w^L|^{-\overline{q}t/q}\Big]^{q/\overline{q}},$$

because $q/\overline{q} \geq 1$. Consequently, by definition we have $\tau_{F,b}^{(m)}(q) \geq (q/\overline{q}) \cdot \tau_{F,b}^{(m)}(\overline{q}) = q\underline{h}$. This, together with Proposition 3.4, yields $\tau_F^{(m)}(q) \geq \tau_{F,b}^{(m)}(q) \geq \underline{h}q$ for $q \geq \overline{q}$.

It remains to discuss the strict concavity of τ over J. Suppose τ is affine over a non trivial sub-interval J' of J. The analyticity of τ implies that it is affine over J (in fact over $\mathbb{R}$ under our assumptions), which is equivalent to saying that for all $q, q' \in J$ and $\lambda \in [0,1]$ we have

$$\Phi\big(\lambda q + (1-\lambda)q', \lambda\tau(q) + (1-\lambda)\tau(q')\big) = 0, \tag{3.14}$$

where Φ is defined in (3.5). Let $\lambda \in (0,1)$ and $q \neq q' \in J$. Applying the Hölder inequality to $\sum_{i=0}^{b-1} \mathbf{1}_{\{W_i\neq 0\}} |W_i|^{\lambda q} L_i^{-\lambda\tau(q)} |W_i|^{(1-\lambda)q'} L_i^{-(1-\lambda)\tau(q')}$ shows that, in order to have (3.14) it is necessary and sufficient that there exists C such that

$$\mathbf{1}_{\{W_i\neq 0\}} |W_i|^q L_i^{-\tau(q)} = C\mathbf{1}_{\{W_i\neq 0\}} |W_i|^{q'} L_i^{-\tau(q')}$$

almost surely. Thus, there exists $H > 0$, the slope of τ, such that $|W_i| = L_i^H$ for all i, conditionally on $W_i \neq 0$. If the components of W are non-negative almost surely, by construction this implies $\mathbb{E}(\sum_{i=0}^{b-1} L_i^H) = 1$, hence $H = 1$ and $W = L$, the situation we have discarded. Otherwise, we have $\mathbb{E}(\sum_{i=0}^{b-1} L_i^H) > 1$ hence $H \in (0,1)$.

3.2.5 Proof of Theorem 3.2

We only have to deal with the exponent $h = \tau'(0) = \infty$. The rest of the study is similar to that achieved in the previous sections.

For $w \in \mathscr{A}^*$ let $\widetilde{W}(w) = \big(\mathbf{1}_{\{W_i(w)\neq 0\}} L_i(w)^{-\tau(0)}\big)_{0\leq i\leq b-1}$. By construction the components of $\widetilde{W}$ are non negative, we have $\varphi_{\widetilde{W}}(1) = 0$, and $\varphi'_{\widetilde{W}}(1) > 0$. Consequently, the Mandelbrot measure on $\mathscr{A}^{\mathbb{N}_+}$ defined as $\mu_0 = F'_{\widetilde{W}}$ (with the notations of Theorem 1.2) is positive with probability 1. Moreover, it follows from the study achieved in Chapter 2 that $\dim_H \nu_0 = -\tau(0)$, where $\nu_0 = \mu_0 \circ \pi^{-1} \circ F_L^{-1}$.

Now, for $a \in (0,1)$, we define $W^{(a)} = (|W_i| \wedge a)_{0\leq i\leq b-1}$. We have $h = \lim_{a\to 0} h(a)$, where $h(a) = -\mathbb{E}\big(\sum_{i=0}^{b-1} \mathbf{1}_{\{W_i\neq 0\}} L_i \log(W_i^{(a)})\big)$. By using the same techniques as in Section 3.3 we can prove that, with probability 1, for μ_0-almost every t, we have

$$\lim_{n\to\infty} \frac{\log |Q_{W^{(a)}}((t|_n)^\gamma)|}{-n} = h(a),\ \ \gamma \in \{-1, 0, 1\}.$$

Also, due to our assumptions and Proposition 3.14, we have $\lim_{n\to\infty} \dfrac{\log Z_W^{(1)}((t|_n)^\gamma)}{n} = 0$, for all $\gamma \in \{-1,0,1\}$. Consequently, for μ_0-almost every t, $\min(\underline{\alpha}_W^{(1),\gamma}(t) : \gamma \in \{-1,0,1\}) \geq$

$h(a)$. Since this holds for every $a \in (0,1)$, letting a tend to 0 yields $\min(\underline{\alpha}_W^{(1),\gamma}(t) : \gamma \in \{-1,0,1\}) = \infty$ for μ_0-almost every t. Since there exists $\overline{a} > 0$ such that $\overline{\alpha}_L^{(1),0}(t) \le \overline{a}$ for all t (see Lemma 3.1) we conclude thanks to Proposition 3.6 that for ν_0-almost every t we have $h_F^{(1)}(t) = \infty$.

3.2.6 Proof of Corollary 3.1

Fix $1 \le m \in \mathbb{N}$. Recall that q_m is the unique real number such that $\tau(q_m) = q_m m - 1$.

Let $C > 0$ such that $\mathrm{Osc}_f^{(m)}(B) \le C|B|^m$ for all subintervals B of $[0, F_L(1)]$.

For $r > 0$ let $\mathcal{B}_r$ be a family of disjoint closed intervals B of $[0, F_L(1)]$ of radius r with centers in $\mathrm{Supp}(F^{(m)})$. For any $q \in \mathbb{R}_+$ we have

$$\begin{aligned}\sum_{B\in\mathcal{B}_r} \mathrm{Osc}_{F+f}^{(m)}(B)^q \cdot r^{-t} &\le 2^q \sum_{B\in\mathcal{B}_r} \big(\mathrm{Osc}_F^{(m)}(B)^q + \mathrm{Osc}_f^{(m)}(B)^q\big) r^{-t} \\ &\le (2C)^q \cdot \Big(\sum_{B\in\mathcal{B}_r} \mathrm{Osc}_F^{(m)}(B)^q r^{-t} + \sum_{B\in\mathcal{B}_r} r^{qm-t}\Big).\end{aligned}$$

By the definition of $\tau_G^{(m)}(q)$ this yields $\tau_G^{(m)}(q) \ge \min(\tau_F^{(m)}(q), qm-1)$ so $(\tau_G^{(m)})^*(h) \le \tau^*(h)$ for $h \in [0, \tau'(q_m)]$ (we have used the equality $\tau_F^{(m)} = \tau$) and $(\tau_G^{(m)})^*(h) = 1$ for $h > m$.

On the other hand, since we assumed that $f^{(m)}$ does not vanish, we deduce from Theorem 3.2 that for any $t \in [0, F_L(1)]$ we have $h_G^{(m)}(t) = h_F^{(m)}(t)$ if $h_F^{(m)}(t) < m$ and $h_G^{(m)}(t) = m$ if $h_F^{(m)}(t) > m$. Thus

$$\big(\tau_G^{(m)}\big)^*(h) \ge \dim_H E_G^{(m)}(h) = \begin{cases} \tau^*(h), & \text{if } h \in [0,m); \\ 1, & \text{if } h = m. \end{cases}$$

This implies that $(\tau_G^{(m)})^*$ is equal to τ^* over $[0, \tau'(q_m)]$ and equal to $h \mapsto \tau^*(\tau'(q_m)) + q_m(h - \tau'(q_m))$ over $[\tau'(q_m), m]$. Taking the inverse Legendre transform implies that $\tau_G^{(m)}(q) = \min(\tau_F^{(m)}(q), qm-1)$ for all $q \ge 0$.

3.2.7 Weaker assumptions

Theorem 3.1

If we only assume that $\varphi_W > -\infty$ in a neighborhood $\widetilde{J}$ of $[0,1]$, then the multifractal formalisms holds for F at each $h = \tau'(q)$ for all $q \in \widetilde{J} \cap J$. Also, the functions $\tau_F^{(m)}$ and τ coincide over $\widetilde{J} \cap J$. If, moreover, there exists $q_0 \in \widetilde{J}$ such that $\tau^*(\tau'(q_0)) = 0$ then either $q_0 > 0$ and $\tau_F^{(m)}(q) = \tau'(q_0)q/q_0$ over $[q_0, \infty)$ or $q_0 < 0$ and $\tau_F^{(m)}(q) = \tau'(q_0)q/q_0$ over $(-\infty, q_0]$.

Theorem 3.2

The same discussion as for Theorem 3.1 holds, except that $\widetilde{J}$ is a neighbor of $[0,1]$ in $\mathbb{R}_+$.

3.3 Proofs of the intermediate results of Section 3.2

3.3.1 Proofs of the results of Section 3.2.1

Proof of Proposition 3.2

This is a consequence of Proposition 3.12.

Proof of Proposition 3.3

The result could be obtained after achieving the multifractal analysis using the first order oscillation exponent. Nevertheless we find valuable to have a proof only based on the the functional equation satisfied by the process F.

We assumed that $\mathbb{P}(\sum_{i=0}^{b-1}\mathbf{1}_{\{W_i\neq 0\}}\geq 2)=1$. Consequently, it follows from the definition of F_W that the event $\{Z_W^{(1)}=0\}$ is measurable with respect to the tail σ-algebra $\bigcap_{n\geq 0}\sigma(\{W(w):w\in\bigcup_{p\geq n}\mathscr{A}^p\})$ which contains only sets of probability 0 or 1. Since $\mathbb{E}(F_W(1))=1$, we have $Z_W^{(1)}>0$ with positive probability, hence almost surely. So $\mathrm{Supp}(F')\neq\emptyset$ almost surely.

Now we prove that F is nowhere locally equal to a polynomial function over the support $\mathrm{Supp}(F')$.

At first, suppose that there exists $0\leq i\leq b-1$ such that $\mathbb{P}(W_i=0)>0$. Then, with probability 1, the interior of $\mathrm{Supp}(F')$ is empty, since for every $w\in\mathscr{A}^*$ the probability that there exists $v\in\mathscr{A}^*$ such that $W_i(w\cdot v)=0$ is equal to 1. Thus F is nowhere locally equal to a polynomial function over $\mathrm{Supp}(F')$.

Now suppose that the components of W do not vanish and that there is a positive probability that there exists an interval I_w^L over which F is equal to a polynomial. Equivalently, $F^{[w]}=F_W^{[w]}\circ(F_L^{[w]})^{-1}$ is a polynomial function. Due to the statistical self-similarity of the construction, the probability that F be itself a polynomial function is positive. Moreover, F is almost surely the uniform limit of the sequence $(F_n=F_{W,n}\circ F_{L,n}^{-1})_{n\geq 1}$. The functions F_n are piecewise linear, and because we assumed $W\neq L$ and the vectors $(W(w),L(w))$, $w\in\mathscr{A}^*$, are independent, with probability 1, for every $w\in\mathscr{A}^*$, there are infinitely many n such that the restriction of F_n to I_w^L is not linear, thus non differentiable. Consequently, the event $\{F \text{ is a polynomial}\}$ is measurable with respect to the tail σ-algebra $\bigcap_{n\geq 0}\sigma(\{W(w),L(w):w\in\cup_{p\geq n}\mathscr{A}^p\})$, so it has a probability equal to 1. For $0\leq i\leq b-2$, let $x_i=F_L(i/b)$. By construction, we have $(W_i/L_i)(F^{[i]})'\big(F_L^{[i]}(1)\big)=F'(x_{i+1}^-)=F'(x_{i+1}^+)=(W_{i+1}/L_{i+1})(F^{[i+1]})'(0)$. Due to the independence between (W,L), $F^{[i]}$ and $F^{[i+1]}$, we see that all the terms in the previous equality must be deterministic, except if $F^{[i]\prime}\big(F_L^{[i]}(1)\big)=(F^{[i+1]})'(0)=0$ almost surely. In this later case, by statistical self-similarity we also have $F'\big(F_L(1)\big)\equiv F'(0)\equiv 0$, and by induction over $n\geq 0$ we see that F' vanishes at all the endpoints of the intervals I_w^L, $w\in\mathscr{A}^n$. Thus $F'\equiv 0$ and F is constant. This is in contradiction with $F(0)=0$ and $\mathbb{E}\big(F(F_L(1))\big)=\mathbb{E}\big(F_W(1)\big)=1$. Consequently, (W,L) must be deterministic. Since we supposed that $W\neq L$, the assumption $\sum_{i=0}^{b-1}W_i=1=\sum_{i=0}^{b-1}L_i$ implies that $|W_i|>L_i$ for some $0\leq i\leq b-1$. Let us write $|W_i|=L_i^H$ with $H<1$ (recall that $L_i<1$). Then, de-noting by $i^{\cdot n}$ the word consisting in n letters i, we have $\mathrm{Osc}^{(1)}(F, I^L_{i^{\cdot n}}) \geq |W_i|^n = |I^L_{i^{\cdot n}}|^H$ so F is not C^1. This is a new contradiction, hence F is nowhere locally equal to a polynomial function.

Proof of Proposition 3.4

We first establish the inequalities $\tau_F^{(m)} \geq \tau_{F,b}^{(1)}$ over $\mathbb{R}_+$ and $\tau_F^{(m)} \geq \tau_{F,b}^{(m)}$ over $\mathbb{R}_-^*$. By applying Theorem 2.4 in Chapter 2 to L we immediately have the following lemma :

Lemma 3.1 *There exist $\underline{a}, \overline{a} > 0$ such that, with probability 1, there exists $n_0 \in \mathbb{N}$ such that for $n \geq n_0$, $b^{-n\overline{a}} \leq \inf_{w\in\mathscr{A}^n} |I^L_w| \leq \sup_{w\in\mathscr{A}^n} |I^L_w| \leq b^{-n\underline{a}}$. Moreover, with probability 1, for every $\varepsilon > 0$, there exists n_ε such that*

$$\forall\, n \geq n_\varepsilon,\ b^{-n\varepsilon} \leq \inf_{w\in\mathscr{A}^n} \inf_{0\leq i\leq b-1} \frac{|I^L_{wi}|}{|I^L_w|} \leq 1. \tag{3.15}$$

For $\mathbb{P}$-almost every $\omega \in \Omega$, we fix $\varepsilon > 0$, n_0 and n_ε as in Lemma 3.1.

Let $n'_\varepsilon = \max(n_0, n_\varepsilon)$. Fix $0 < r \leq \min_{w\in\mathscr{A}^{\mathbb{N}_+}_{n'_\varepsilon+1}} |I^L_w|$.

Let $\mathcal{B}_r$ be a family of disjoint closed intervals B of radius r with centers in $\mathrm{Supp}(F')$. If $B \in \mathcal{B}_r$, by construction we can find three disjoint intervals $I^L_{w_k}$, $k = 1, 2, 3$, with $|w_k| \geq n'_\varepsilon + 1$ such that $B \subset I^L_{w_1} \cup I^L_{w_2} \cup I^L_{w_3}$ and $r \leq |I^L_{w_k}| \leq rb^{|w_k|\varepsilon}$. Also, $|I^L_{w_k}| \leq b^{-|w_k|\underline{a}}$ so $b^{|w_k|\varepsilon} \leq r^{-\varepsilon/\underline{a}}$. Thus $r \leq |I^L_{w_k}| \leq r^{1-\varepsilon/\underline{a}}$.

We have $O_F^{(m)}(B) \leq 2^{m-1} O_F^{(1)}(B) \leq 2^{m-1} \sum_{k=1}^3 O_F^{(1)}(I^L_{w_k})$, so for $q \geq 0$ and $t \in \mathbb{R}$ we have

$$O_F^{(m)}(B)^q r^{-t} f(t, r) \leq 2^{(m-1)q} 3^q \cdot \sum_{k=1}^{3} O_F^{(1)}(I^L_{w_k})^q |I^L_{w_k}|^{-t},$$

with $f(t, r) = 1$ if $t < 0$ and $f(t, r) = r^{t\varepsilon/\underline{a}}$ otherwise. Moreover, each so selected interval $I^L_{w_k}$ meets at most $1 + r^{-\varepsilon/\underline{a}}$ elements of $\mathcal{B}_r$. Consequently,

$$\sum_{B\in\mathcal{B}_r} O_F^{(m)}(B)^q r^{-t} f(t, r) \leq 2^{(m-1)q} 3^q (1 + r^{-\varepsilon/\underline{a}}) \sum_{n\geq n'_\varepsilon+1} \theta^{(1)}_{F,b,n}(q, t). \tag{3.16}$$

Suppose that $\tau^{(1)}_{F,b}(q) > -\infty$; otherwise there is nothing to prove. Due to the existence of $\underline{a}$, by definition of $\tau_F^{(1)}(q)$, if $t < \tau^{(1)}_{F,b}(q)$ then we have $\sum_{n\geq n'_\varepsilon+1} \theta^{(1)}_{F,n}(q, t) < \infty$. Then, it follows from (3.16) and the definition of $\tau_F^{(m)}(q)$ that $\tau_F^{(m)}(q) \geq t - (1 + |t|)\varepsilon/\underline{a}$. Since ε is arbitrary, we get $\tau_F^{(m)}(q) \geq \tau^{(1)}_{F,b}(q)$.

On the other hand, for each $B \in \mathcal{B}_r$ there exists I^L_w of maximal length included in B. We have $2rb^{-|w|\varepsilon} \leq |I^L_w| \leq 2r$. This yields $2r \leq b^{|w|(\varepsilon-\underline{a})}$ so $(2r)^{\varepsilon/(\underline{a}-\varepsilon)} \leq b^{-|w|\varepsilon}$ whenever $\underline{a} > \varepsilon$. consequently, for ε small enough, we have $(2r)^{1+\varepsilon/(\underline{a}-\varepsilon)} \leq |I^L_w| \leq 2r$. Thus, if $q < 0$ we have

$$O_F^{(m)}(B)^q (2r)^{-t} f(t, r) \leq O_F^{(m)}(I^L_w)^q |I^L_w|^{-t},$$

where $f(t,r)=1$ if $t\geq 0$ and $f(t,r)=(2r)^{-t\varepsilon/(\underline{a}-\varepsilon)}$ otherwise. Since the elements of $\mathcal{B}_r$ are pairwise disjoint, this implies

$$\sum_{B\in\mathcal{B}_r} O_F^{(m)}(B)^q(2r)^{-t}f(t,r)\leq \sum_{n\geq n'_\varepsilon+1}\theta_{F,n}^{(m)}(q,t) \tag{3.17}$$

and the same arguments as when $q\geq 0$ yield $\tau_F^{(m)}(q)\geq \tau_{F,b}^{(m)}(q)$.

To see that, with probability 1, $\tau_{F,b}^{(m)}\geq \widetilde{\tau}_{F,b}^{(m)}$, due to the concavity of $\tau_{F,b}^{(m)}$ and $\tau_{F,b}^{(m)}$, it is enough to show that given $q\in\mathbb{R}$, we have $\tau_{F,b}^{(m)}(q)\geq\widetilde{\tau}_{F,b}^{(m)}(q)$.

Let $(q,t)\in\mathbb{R}^2$, and suppose that $q<\max\{p:\varphi_W(p)=0\}$. Due to Proposition 3.14 we have $\psi(q,t)<\infty$. By using (3.3) we get $\widetilde{\theta}_{F,n}^{(m)}(q,t)=\Phi(q,t)^n\Psi(q,t)$ for al $n\geq 1$. This yields $\widetilde{\tau}_{F,b}^{(m)}(q)=\tau(q)$. Also, if $t<\tau(q)$ then $\Phi(q,t)<1$ and $\sum_{n\geq 1}\widetilde{\theta}_{F,n}^{(m)}(q,t)<\infty$ so $\sum_{n\geq 1}\theta_{F,n}^{(m)}(q,t)<\infty$ almost surely. This yields $t<\tau_{F,b}^{(m)}(q)$. Since t is arbitrary we get $\tau_{F,b}^{(m)}(q)\geq\widetilde{\tau}_{F,b}^{(m)}(q)$.

To finish the proof, we notice that by construction, we have $\tau(p)=0$ if and only if $\varphi_W(p)=0$.

3.3.2 Proofs of the results of Section 3.2.2

Proof of Proposition 3.5

This proof could be deduced from those of Lemma 4 and Corollary 5 of [11]. For reader's convenience, we provide it.

- *Proof of 1 and 2* : Recall (3.6) that for $q\in\mathrm{Int}(J)$ and $w\in\mathscr{A}^*$,

$$W_q(w)=\big(\mathbf{1}_{\{W_i(w)\neq 0\}}|W_i(w)|^qL_i(w)^{-\tau(q)}\big)_{0\leq i\leq b-1}.$$

The function Φ can be extended to an analytic function in a complex neighborhood of $J\times\mathbb{C}$ by

$$\Phi(z,t)=\mathbb{E}\Big(\sum_{i=0}^{b-1}\mathbf{1}_{\{W_i\neq 0\}}|W_i|^zL_i^{-t}\Big).$$

For each $q\in\mathrm{Int}(J)$ we have $\frac{\partial\Phi(q,\tau(q))}{\partial t}=-\mathbb{E}\Big(\sum_{i=0}^{b-1}W_{q,i}\log(L_i)\Big)>0$ and $\Phi(q,\tau(q))=1$, so there exists a neighborhood V_q of q in $\mathbb{C}$ such that for each $z\in V_q$ there exists a unique $\tau(z)$ such that $\Phi(z,\tau(z))=1$. Moreover, the mapping $z\mapsto\tau(z)$ is analytic. We define

$$W_z(w)=\big(\mathbf{1}_{\{W_i(w)\neq 0\}}|W_i(w)|^zL_i(w)^{-\tau(z)}\big)_{0\leq i\leq b-1}$$

as well as the mapping

$$(z,p)\in V_q\times[1,\infty)\mapsto M(z,p)=\sum_{i=0}^{b-1}\mathbb{E}(|W_{z,i}|^p).$$

The property $\tau^*(\tau'(q))>0$ is equivalent to $\frac{\partial M}{\partial p}(q,1^+)<0$, so there exists $p_q>1$ and a open neighborhood V'_q of q in J such that $\sup_{q'\in V'_q}M(q',p)<1$ for all $p\in(1,p_q]$

(because $p \mapsto M(q',p)$ is convex and $M(q',1)=1$). Now, we fix K a non-trivial compact subinterval of $\mathrm{Int}(J)$. It is covered by a finite number of such V'_{q_i} so that if $V'_K = \bigcup_i V'_{q_i}$ we have $\sup_{q\in V'_K} M(q,p_K) < 1$, where $p_K = \inf_i p_{q_i}$. By a comparable procedure we can now find a complex neighborhood V_K of V_K such $\sup_{z\in V_K} M(z,p_K) < 1$.

To prove the almost sure simultaneous convergence of the martingales $(Y_{q,n}(w))_{n\geq 1}$, $q\in K$, we are going to use the argument developed to get Theorem 2 in [33].

For $z\in V_K$ and $w\in \mathscr{A}^*$ let

$$Y_{z,n}(w) = \sum_{v\in\mathscr{A}^n} \prod_{k=1}^{n} W_{z,v_k}(w\cdot v|_{|w|+k-1})$$

and denote $Y_{z,n}(\emptyset)$ by $Y_{z,n}$. Applying Proposition 3.13 to $\{V(w)=W_z(w)\}_{w\in\mathscr{A}^*}$ yields for $n\geq 1$

$$\mathbb{E}(|Y_{z,n}-Y_{z,n-1}|^{p_K}) \leq C_{p_K} M(z,p_K)^n \leq C_{p_K}(\sup_{z\in V_K} M(z,p_K))^n,$$

where $Y_{z.0}=1$. Since, with probability 1, the functions $z\in V \mapsto Y_{z,n}$, $n\geq 0$, are analytic, if we fix a closed disc $D(z_0,2\rho)$ included in V, the Cauchy formula yields $\sup_{z\in D(z_0,\rho)} |Y_{z,n}-Y_{z,n-1}| \leq \rho^{-1}\int_{\partial D(z_0,2\rho)} |Y_{u,n}-Y_{u,n-1}|\,|\mathrm{d}u|/2\pi$, so by using Jensen's inequality an then Fubini's Theorem we get

$$\begin{aligned}\mathbb{E}\big(\sup_{z\in D(z_0,\rho)} |Y_{z,n}-Y_{z,n-1}|^{p_K}\big) &\leq 2^{p_K}\int_0^{2\pi} \mathbb{E}(|Y_{z_0+2\rho e^{it},n}-Y_{z_0+2\rho e^{it},n-1}|^{p_K})\,\frac{\mathrm{d}t}{2\pi}\\ &\leq 2^{p_K} C_{p_K}(\sup_{z\in V}\Phi(z,p_K))^n.\end{aligned}$$

This implies that, with probability 1, $z\mapsto Y_{z,n}$ converges uniformly over the compact $D(z_0,\rho)$ to a limit Y_z. This also implies that $\|\sup_{z\in D(z_0,\rho)} Y_z\|_{p_K} < \infty$. Since K can be covered by finitely many such discs, we get both the simultaneous convergence of $(Y_{q,n})_{n\geq 1}$ to Y_q for all $q\in K$ and 2.. Moreover, since $\mathrm{Int}(J)$ can be covered by a countable increasing union of compact subintervals, we get the simultaneous convergence for all $q\in \mathrm{Int}(J)$. The same holds simultaneously for all the functions $q\in\mathrm{Int}(J)\mapsto Y_{q,n}(w)$, $w\in\mathscr{A}^*$, because $\mathscr{A}^*$ is countable.

To finish the proof of (1) we need to establish that, with probability 1, $q\in K\mapsto Y_q$ does not vanish. Up to an affine transform, we can suppose that $K=[0,1]$. If I is a closed dyadic subinterval of $[0,1]$, we denote by E_I the event $\{\exists\, q\in I:\ Y_q=0\}$, and by I_0 and I_1 its two sons. At first, we note that since for each fixed $q\in K$ a component of W_q vanishes if and only the same component of W vanishes too, each E_I is a tail event. Consequently, if I is a closed dyadic subinterval of $[0,1]$ and $\mathbb{P}(E_I)=1$, then $\mathbb{P}(E_{I_j})=1$ for some $j\in\{0,1\}$. Suppose that $\mathbb{P}(E_{[0,1]})=1$. The previous remark yields a decreasing sequence $(I(n))_{n\geq 0}$ of nested closed dyadic intervals such that $\mathbb{P}(E_{I(n)})=1$. Let q_0 be the unique element of $\bigcap I(n)$. Since $q\mapsto Y_q$ is continuous, we have $\mathbb{P}(Y_{q_0}=0)=1$. This contradicts the fact that the martingale $(Y_{q_0,n})_{n\geq 1}$ converges to Y_{q_0} in L^{p_K} norm.

• *Proof of 3.* : This is a simple consequence of the fact that by construction we have for all $n\geq 1$ and $w\in\mathscr{A}^*$ the branching property

$$Y_{q,n+1}(w) = \sum_{i=0}^{b-1} W_{q,i}(w)\, Y_{q,n}(w\cdot i).$$

Proof of Proposition 3.7

1. We simply denote Q_W by Q and we define

$$\xi(q) = -\frac{\partial \Phi}{\partial q}(q,\tau(q)) = -\mathbb{E}\Big(\sum_{i=0}^{b-1} \mathbf{1}_{\{W_i\neq 0\}}|W_i|^q L_i^{-\tau(q)} \log(|W_i|)\Big).$$

If $\varepsilon > 0$, $n \geq 1$ and $\gamma \in \{-1,0,1\}$ we define

$$E^1_{q,n,\varepsilon}(\gamma) = \{t \in \mathscr{A}^{\mathbb{N}_+} : Q((t|_n)^\gamma) \neq 0 \text{ and } e^{n(\xi(q)-\varepsilon)}|Q((t|_n)^\gamma)| \geq 1\},$$

$$E^{-1}_{q,n,\varepsilon}(\gamma) = \{t \in \mathscr{A}^{\mathbb{N}_+} : Q((t|_n)^\gamma) \neq 0 \text{ and } e^{n(\xi(q)+\varepsilon)}|Q((t|_n)^\gamma)| \leq 1\}.$$

Our goal is to prove that for any compact subinterval K of $\mathrm{Int}(J)$ and $\varepsilon > 0$,

$$\mathbb{E}\Big(\sup_{q\in K}\sum_{n\geq 1}\mu_q(E^\lambda_{q,n,\varepsilon}(\gamma))\Big) < \infty \tag{3.18}$$

for all $\lambda \in \{-1,1\}$ and $\gamma \in \{-1,0,1\}$. Then, with probability 1, for all $q \in K$, $\lambda \in \{-1,1\}$ and $\gamma \in \{-1,0,1\}$, the series $\sum_{n\geq 1}\mu_q(E^\lambda_{q,n,\varepsilon}(\gamma))$ is finite. Since $\mathrm{Int}(J)$ can be written as a countable union of compact subintervals, this holds in fact for all $q \in \mathrm{Int}(J)$. Consequently, from the Borel-Cantelli lemma applied to $\mu_q/\|\mu_q\|$ we deduce that, with probability 1, for all $q \in \mathrm{Int}(J)$, for μ_q-almost every $t \in \mathscr{A}^{\mathbb{N}_+}$, there exists $N \geq 1$ such that for all $n \geq N$ and $\gamma \in \{-1,0,1\}$

$$\text{either } Q((t|_n)^\gamma) = 0 \text{ or } |Q((t|_n)^\gamma)| \in [e^{-n(\xi(q)+\varepsilon)}, e^{-n(\xi(q)-\varepsilon)}].$$

Notice that when $t \in \mathrm{Supp}(\mu_q)$, we have $Q((t|_n)^0) = Q(t|_n) \neq 0$. Consequently, with probability 1, for all $q \in K$, for μ_q-almost every t,

$$\lim_{n\to\infty}\frac{\log|Q(t|_n)|}{-n} \in [\xi(q)-\varepsilon, \xi(q)+\varepsilon],$$
$$\lim_{n\to\infty}\frac{\log|Q((t|_n)^\gamma)|}{-n} \in \{+\infty\}\cup[\xi(q)-\varepsilon, \xi(q)+\varepsilon] \text{ for } \gamma \in \{-1,1\}.$$

Since this holds for a sequence of positive ε tending to 0, we have the desired result.

Now we prove (3.18). Fix K, a non-trivial compact subinterval of $\mathrm{Int}(J)$. For $\eta \geq 0$, $q \in K$ and $\gamma \in \{-1,0,1\}$, by using a Markov inequality we get

$$\mu_q(E^1_{q,n,\varepsilon}(\gamma)) \leq \sum_{w\in\mathscr{A}^n}\mu_q([w])\mathbf{1}_{\{Q(w^\gamma)\neq 0\}}\Big(e^{n(\xi(q)-\varepsilon)}\cdot|Q(w^\gamma)|\Big)^\eta,$$
$$\mu_q(E^{-1}_{q,n,\varepsilon}(\gamma)) \leq \sum_{w\in\mathscr{A}^n}\mu_q([w])\mathbf{1}_{\{Q(w^\gamma)\neq 0\}}\Big(e^{n(\xi(q)+\varepsilon)}\cdot|Q(w^\gamma)|\Big)^{-\eta}.$$

Since $\mu_q([w]) = \mathbf{1}_{\{Q(w)\neq 0\}}|Q_q(w)|Y_q(w)$, for $\lambda \in \{-1,1\}$ and $\gamma \in \{-1,0,1\}$ we get

$$\mu_q(E^\lambda_{q,n,\varepsilon}) \leq \sum_{w\in\mathscr{A}^n} e^{n(\lambda\eta\xi(q)-\varepsilon\eta)}\mathbf{1}_{\{Q(w),Q(w^\gamma)\neq 0\}}Q_q(w)|Q(w^\gamma)|^{\lambda\eta}Y_q(w).$$

Now define

$$H_n^{\eta,\lambda}(q,\gamma) = \sum_{w\in\mathscr{A}^n} e^{n(\lambda\eta\xi(q)-\varepsilon\eta)}\mathbf{1}_{\{Q(w),Q(w^\gamma)\neq 0\}}Q_q(w)|Q(w^\gamma)|^{\lambda\eta}. \tag{3.19}$$

Write $K=[q_0,q_1]$. It follows from the independence between $\{H_n^{\eta,\lambda}(q,\gamma)\}_{q\in K}$ and $\{Y_q(w)\}_{w\in\mathscr{A}^n,q\in K}$ that for $n\geq 1$

$$\begin{aligned}\mathbb{E}(\sup_{q\in K}\mu_q(E^\lambda_{q,n,\varepsilon}(\gamma)) &\leq \mathbb{E}(\sup_{q\in K} Y_q)\mathbb{E}\big(\sup_{q\in K} H_n^{\eta,\lambda}(q,\gamma)\big)\\ &\leq \mathbb{E}(\sup_{q\in K} Y_q)\Big(\mathbb{E}\big(H_n^{\eta,\lambda}(q_0,\gamma)\big)+\int_{q_0}^{q_1}\mathbb{E}\big(\big|\frac{\mathrm{d}}{\mathrm{d}q}H_n^{\eta,\lambda}(q,\gamma)\big|\big)\mathrm{d}q\Big).\end{aligned}$$

Lemma 3.2 *Let $\lambda\in\{-1,1\}$ and $\gamma\in\{-1,0,1\}$. There exist constants C, $\delta>0$ and $\eta_*>0$ such that for any $q\in K$, $\eta\in(0,\eta_*)$, $\lambda\in\{-1,1\}$ and $n\geq 1$,*

$$\max\Big\{\mathbb{E}\big(H_n^{\eta,\lambda}(q,\gamma)\big),\mathbb{E}\big(\big|\frac{d}{dq}H_n^{\eta,\lambda}(q,\gamma)\big|\big)\Big\}\leq Cne^{-n\delta}.$$

Then (3.18) comes from the fact that $\mathbb{E}(\sup_{q\in K}Y_q)<\infty$ (see Proposition 3.5 2.).

Proof of Lemma 3.2

Recall that $\xi(q)=-\frac{\partial\Phi}{\partial q}(q,-\tau(q))$. Since Φ is twice continuously differentiable, we can chose $\eta_0>0$ such that for $\eta\in(0,\eta_0)$,

$$\delta_\eta=\inf_{q\in K}\varepsilon\eta-\lambda\eta\xi(q)-\log\big(\Phi(q+\lambda\eta,-\tau(q))\big)>0. \tag{3.20}$$

We now distinguish the cases $\gamma=0$ and $\gamma\in\{-1,1\}$.

• *The case* $\gamma=0$. Straighforward computations using the definition of $H_n^{\eta,\lambda}(q,0)$ and taking into account the independence in the b-adic cascade construction yield a constant C_K such that for all $q\in K$ and $n\geq 1$

$$\begin{aligned}\mathbb{E}\big(H_n^{\eta,\lambda}(q,0)\big) &= \Phi(q+\lambda\eta,-\tau(q))^n e^{n(\lambda\eta\xi(q)-\varepsilon\eta)}\leq e^{-n\delta_\eta}\\ \mathbb{E}\big(\big|\frac{\mathrm{d}}{\mathrm{d}q}H_n^{\eta,\lambda}(q,0)\big|\big) &\leq C_K n\Phi(q+\lambda\eta,-\tau(q))^n e^{n(\lambda\eta\xi(q)-\varepsilon\eta)}\leq C_K ne^{-n\delta_\eta}.\end{aligned}$$

• *The case* $\gamma=-1$. For $n\geq 1$ we have

$$\bigcup_{w\in\mathscr{A}^n}(w^-,w)=\bigcup_{m=0}^{n-1}\bigcup_{u\in A^m}\bigcup_{i=0}^{b-2}(u\cdot i\cdot g_{n-1-m},u\cdot(i+1)\cdot d_{n-1-m}) \tag{3.21}$$

where g_n (resp. d_n) is the word consisting of n times the letter $b-1$ (resp. 0). If $w=u\cdot(i+1)\cdot d_{n-1-m}$ and $w^-=u\cdot i\cdot g_{n-1-m}$ with $u\in\mathscr{A}_m$ and $Q(w)Q(w^-)\neq 0$ then

$$Q_q(w)|Q(w^-)|^{\lambda\eta}=Q_q(u)Q(u)^{\lambda\eta}W_{q,i}(u)|W_{i+1}(u)|^{\lambda\eta}\prod_{k=m+1}^{n-1}W_{q,0}(w|_k)|W_{b-1}(w^-|_k)|^{\lambda\eta}.$$

Again, simple computations yield $C_K > 0$ such that for all $q \in K$, $n \geq 1$ and $\eta \in (0, \eta_0)$ we have

$$\max\big(\mathbb{E}(H_n^{\eta,\lambda}(q,\gamma), \mathbb{E}\big(\big|\frac{\mathrm{d}}{\mathrm{d}q} H_n^{\eta,\lambda}(q,\gamma)\big|\big)\big) \leq C_K n\big(\Phi(q+\lambda\eta, \tau(q))e^{\lambda\eta\xi(q)-\varepsilon\eta}\big)^n S_n(q,\eta),$$

where

$$S_n(q,\eta) = \sum_{m=0}^{n-1} \Big[\frac{\mathbb{E}(W_{q,0})\mathbb{E}(|W_{b-1}|^{\lambda\eta})}{\Phi(q+\lambda\eta, \tau(q))}\Big]^m.$$

Due to (3.20), it is now enough to show that $S_n(q,\eta)$ is uniformly bounded with respect to n, $q \in K$ and η if η_0 is small enough. This is due to the fact that the mapping $(q,r) \mapsto \mathbb{E}(W_{q,0})\mathbb{E}(|W_{b-1}|^r)/\Phi(q+r,\tau(q))$ is continuous in a neighborhood of $J \times \{0\}$ and by definition of W_q and Φ it takes values less than 1 at points of the form $(q,0)$.

- *The case* $\gamma = 1$. It uses the same ideas as the case $\gamma = -1$.

2. The proof is similar to the proof of (1). The only difference is that the components of L are positive so the limit of $\frac{\log Q_L((t|_n)^\gamma)}{-n}$ cannot be infinite.

3. We denote $Z_U^{(m)}$ by Z. Fix K a non-trivial compact subinterval of $\mathrm{Int}(J)$, $\lambda \in \{1,-1\}$ and $\gamma \in \{-1,0,1\}$. For $a > 1$ and $n \geq 0$ let

$$E_{n,a}^\lambda(\gamma) = \{t \in \mathscr{A}^{\mathbb{N}_+} : (Z(t|_n)^\gamma))^\lambda > a^n\}.$$

It is enough that we show that

$$\mathbb{E}(\sup_{q\in K} \sum_{n\geq 0} \mu_q(E_{n,a}^\lambda)(\gamma)) < \infty. \tag{3.22}$$

Indeed, this implies that, with probability 1, for all $q \in K$, for μ_q-almost every t, if n is large enough then

$$-\log a \leq \liminf_{n\to\infty} \frac{\log Z((t|_n)^\gamma)}{n} \leq \limsup_{n\to\infty} \frac{\log Z((t|_n)^\gamma)}{n} \leq \log a.$$

Since this holds for a sequence of numbers a tending to 1, we have the conclusion.

We have

$$\begin{aligned}\sup_{q\in K} \mu_q(E_{n,a}^\lambda(\gamma)) &= \sup_{q\in K} \sum_{w\in\mathscr{A}^n} \mathbf{1}_{\{Z(w^\gamma)^\lambda > a^n\}} \cdot \mu_q([w]) \\ &= \sup_{q\in K} \sum_{w\in\mathscr{A}^n} Q_{q,n}(w) \cdot \mathbf{1}_{\{Z(w^\gamma)^\lambda > a^n\}} \cdot Y_q(w).\end{aligned}$$

By using the independence between $\sigma(Q(w) : w \in \mathscr{A}^n)$ and $\sigma(Z(w^\gamma), Y_q(w) : w \in \mathscr{A}^n, q \in K)$, as well as the equidistribution of the random variables $\mathbf{1}_{\{Z(w^\gamma)^\lambda > a^n\}} \cdot Y_q(w)$, we get

$$\mathbb{E}\big(\sup_{q\in K} \mu_q(E_{n,a}^\lambda(\gamma))\big) \leq \mathbb{E}\big(\mathbf{1}_{\{Z(w_0^\gamma)^\lambda > a^n\}} \cdot \sup_{q\in K} Y_q(w_0)\big)\mathbb{E}\big(\sup_{q\in K} H_n^{0,0}(q,\gamma)\big),$$

where $H_n^{0,0}(q,\gamma)$ is defined as in (3.19) and w_0 is any element of $\mathscr{A}^n$ such that w_0^γ is defined. We learn from our computations in proving (1) that there exists a positive number C_K such that $\mathbb{E}\big(\sup_{q\in K} H_n^{0,0}(q,\gamma)\big) \le C(1+|K|)n$. Moreover, the Hölder inequality yields

$$\begin{aligned}\mathbb{E}\big(\mathbf{1}_{\{Z(w_0^\gamma)^\lambda > a^n\}}\cdot \sup_{q\in K} Y_q(w_0)\big) &\le \|\sup_{q\in K} Y_q\|_{p_K}\mathbb{P}(Z^{\lambda\varepsilon} > a^{\varepsilon n})^{1-1/p_K}\\ &\le \|\sup_{q\in K} Y_q\|_{p_K}[\mathbb{E}(Z^{\lambda\varepsilon})]^{1-1/p_K} a^{-n\varepsilon(1-1/p_K)},\end{aligned}$$

where $\varepsilon > 0$ is chosen such that $\mathbb{E}(Z^{\lambda\varepsilon}) < \infty$ (this is possible thanks to Proposition 3.14). Finally, $\mathbb{E}\big(\sup_{q\in K}\mu_q(E_{n,a}^\lambda(\gamma))\big) = O(na^{-n\varepsilon(1-1/p_K)})$ (with $a > 1$), hence (3.22) holds.

4. Fix K a non-trivial compact subinterval of $\mathrm{Int}(J)$. For $a > 1$, $n \ge 0$ and $q \in K$ let

$$F_{n,a}^\lambda(q) = \{t \in \mathscr{A}^{\mathbb{N}_+} : Y_q(t|_n) > a^n\}.$$

For $\eta > 0$, we have

$$\begin{aligned}\sup_{q\in K}\mu_q(F_{n,a}(q)) &= \sup_{q\in K}\sum_{w\in\mathscr{A}^n}\mathbf{1}_{\{Y_q(w)>a^n\}}\cdot\mu_q([w])\\ &\le \sup_{q\in K} Q_{q,n}(w)|^q\cdot a^{-n\eta}Y_q(w)^{1+\eta}.\end{aligned}$$

Consequently, taking $\eta = p_K - 1$ and using the same kind of estimations as in the proof of (3) we obtain

$$\mathbb{E}\big(\sup_{q\in K}\mu_q(F_{n,a}(q))\big) \le a^{-n(p_K-1)}\mathbb{E}\big(\sup_{q\in K} H_n^{0,0}(q,\gamma)\big)\mathbb{E}(\sup_{q\in K} Y_q^{p_K}) = O(na^{-n(p_K-1)}),$$

hence the result.

3.3.3 Proofs of the results of Section 3.2.3

We only deal with the case $h = \underline{h}$, the case $h = \bar{h}$ being similar. Then $q = \overline{q} > 0$.

The case $q < \infty$: Proof of Proposition 3.8

At first, we specify a subset E_h of $E_F(h)$ of positive μ_q-measure. For $N \ge 1$, let $E_h(N) = \{t \in \mathscr{A}^{\mathbb{N}_+} : \forall n \ge N, \mu_q([t|_n]) \le 1\}$. With probability 1, there exists $N \ge 1$ such that $\mu_q(E_h(N)) > 0$, otherwise, μ_q is concentrated on a finite number of singletons with a positive probability, which is impossible since by construction $\mathrm{Supp}(\mu_q) = \mathrm{Supp}(F')$ and $\dim_H \mathrm{Supp}(F') = -\varphi(0) > 0$ almost surely. Thus, on a measurable set of probability 1, we can define the measurable function $N(\omega) = \inf\{N : \mu_q(E_h(N)) > 0\}$. Then we set $E_h(\omega) = E_h(N(\omega))$.

1. We denote Q_W by Q and $-\partial\Phi/\partial q(q,\tau(q))$ by $\xi(q)$. For $\varepsilon > 0$, $\gamma \in \{-1,0,1\}$ and $n \ge 1$ we define

$$E_{n,\varepsilon}^1(\gamma) = \{t \in E_h : Q(t|_n) \ne 0 \text{ and } e^{n(\xi(q)-\varepsilon)}|Q((t|_n)^\gamma)| \ge 1\},$$

$$E_{n,\varepsilon}^{-1}(\gamma) \quad = \quad \{t \in E_h : Q(t|_n) \neq 0 \text{ and } e^{n(\xi(q)+\varepsilon)}|Q((t|_n)^\gamma)|) \leq 1\}\}.$$

The result will follow if we show that for any $\varepsilon > 0$ and $\lambda \in \{-1, 0, 1\}$, with probability 1,

$$\sum_{n\geq 1} \mu_q(E_{n,\varepsilon}^\lambda(\gamma)) < \infty. \tag{3.23}$$

We deal with the case $\gamma = 0$. Let θ and η be two numbers in $(0, 1]$, that will be specified later. By using a Markov inequality and the definition of μ_h we can get

$$\begin{aligned}
\mu_q(E_{n,\varepsilon}^\lambda(0)) &\leq \sum_{\substack{w\in\mathscr{A}^n \\ \mu_q([w])\leq 1}} \mu_q([w])\mathbf{1}_{\{Q(w)\neq 0\}}\big(e^{n(\xi(q)-\lambda\varepsilon)} \cdot Q(w)\big)^{\lambda\eta} \\
&\leq \sum_{\substack{w\in\mathscr{A}^n \\ \mu_q([w])\leq 1}} \mu_q([w])^\theta\mathbf{1}_{\{Q(w)\neq 0\}}\big(e^{n(\xi(q)-\lambda\varepsilon)} \cdot Q(w)\big)^{\lambda\eta} = S_{n,\varepsilon}^\lambda(\theta, \eta),
\end{aligned}$$

where

$$S_{n,\varepsilon}^\lambda(\theta, \eta) = \sum_{w\in\mathscr{A}^n} e^{n(\lambda\eta\xi(q)-\varepsilon\eta)}\mathbf{1}_{\{Q(w)\neq 0\}}Q_q(w)^\theta Q(w)^{\lambda\eta}Y_q^\theta.$$

Consequently, (3.23) will follow if we show that

$$\sum_{n\geq 1} \mathbb{E}(S_{n,\varepsilon}^\lambda(\theta, \eta)) < \infty. \tag{3.24}$$

We have

$$\mathbb{E}(S_{n,\varepsilon}^\lambda(\theta, \eta)) = \mathbb{E}(Y_q^\theta)e^{n(\lambda\eta\xi(q)-\varepsilon\eta)}\Phi(\theta q + \lambda\eta, \theta\tau(q))^n.$$

Let $\widetilde{\xi}(q) = \frac{\partial\Phi}{\partial t}(q, \tau(q))$. By definition of $\xi(q)$, $\widetilde{\xi}(q)$ and $-\tau(q)$ we have

$$\begin{aligned}
&\lambda\eta\xi(q) + \log\Phi(\theta q + \lambda\eta, \theta\tau(q)) \\
= \;& -\xi(q)q(\theta - 1) + \widetilde{\xi}(q)(\theta - 1)\tau(q) + O([q(\theta - 1) + \lambda\eta]^2)
\end{aligned}$$

as $(\theta, \eta) \to (1^-, 0)$. Moreover, we have

$$-\xi(q)q + \widetilde{\xi}(q)\tau(q) = -\widetilde{\xi}(q)(\tau'(q)q - \tau(q)) = 0.$$

It follows that if we fix η small enough and θ close enough to 1^- we have

$$e^{n(\lambda\eta\xi(q)-\varepsilon\eta)}\Phi(\theta q + \lambda\eta, \theta\tau(q))^n \leq e^{-n\varepsilon\eta/2}.$$

Since $\mathbb{E}(Y_q^\theta) < \infty$, we get (3.24).

In the case $\gamma = -1$, we leave the reader check that like in the proof of Proposition 3.7.1 we can find a constant $C > 0$ such that for $\theta, \eta \in (0, 1]$ we have

$$\begin{aligned}
&\mu_q(E_{n,\varepsilon}^\lambda(-1)) \\
\leq\;& C \cdot \mathbb{E}(Y_q^\theta)e^{n(\lambda\eta\xi(q)-\varepsilon\eta)}\Phi(\theta q + \lambda\eta, \theta\tau(q))^n \sum_{m=0}^{n-1} \Big[\frac{\mathbb{E}(W_{q,0}^\theta)\mathbb{E}(|W_{b-1}|^{\lambda\eta})}{\Phi(\theta q + \lambda\eta, \theta\tau(q))}\Big]^m.
\end{aligned}$$

2. The proof is similar to that of 1.

3. We denote $Z_U^{(m)}$ by Z. Let $\theta, \eta \in (0,1]$. For $n \geq 1$, $\gamma \in \{-1,0,1\}$, $\lambda \in \{-1,1\}$ and $\varepsilon > 0$ let $E_{n,\varepsilon}^{\lambda}(\gamma) = \{t \in E_h : Z((t|_n)^{\gamma})^{\lambda} > e^{n\varepsilon}\}$. We have

$$\mu_q\big(E_{n,\varepsilon}^{\lambda}(\gamma))\}\big) \leq \sum_{\substack{w \in \mathscr{A}^n \\ \mu_q([w]) \leq 1}} \mu_q([w])^{\theta} e^{-n\varepsilon\eta} Z((t|_n)^{\gamma})^{\lambda\eta}.$$

Thus, $\mathbb{E}\Big(\mu_q\big(E_{n,\varepsilon}^{\lambda}(\gamma)\big)\Big) \leq e^{-n\varepsilon\eta}\Phi(\theta q, \theta\tau(q))^n \mathbb{E}(Y_q(w)^{\theta} Z(w^{\gamma})^{\lambda\eta})$. If we choose $\theta = 1-\eta^{2/3}$, then by using Hölder's inequality we get,

$$\mathbb{E}(Y_q^{\theta} Z(w)^{\lambda\eta}) \leq \|Y_q^{\theta}\|_{1+\eta^{3/4}} \big(\mathbb{E}(Z^{\lambda(1+\eta^{3/4})\eta^{1/4}})\big)^{\eta^{3/4}/(1+\eta^{3/4})} < \infty$$

since $\mathbb{E}(Y_q^{\theta}) < \infty$ for $\theta \in (0,1)$ and $\mathbb{E}(Z^{\lambda\beta}) < \infty$ if $|\beta|$ is small enough (see Proposition 3.14 2.). Moreover, by definition of $\tau(q)$, since the L_i are smaller than 1, we have $\Phi(\theta q, \theta\tau(q)) < 1$. Consequently, $\sum_{n\geq 1} \mathbb{E}\big(\mu(E_{n,\varepsilon}^{\lambda}(\gamma))\big) < \infty$. We conclude as in the proof of Proposition 3.7 3..

The case $q = \infty$: Proof of Proposition 3.9

First we have the following lemma.

Lemma 3.3 *If* $\displaystyle\sum_{n\geq 1} \prod_{k=0}^{n-1} \Phi\big(2q_k, 2\tau(q_k)\big)^{1/2} < \infty$ *then for every* $w \in \mathscr{A}^*$, $Y_{q,n}(w)$ *converges to* $Y_q(w)$ *in* L^1 *norm as* $n \to \infty$ *; in particular* $\mathbb{E}(Y_q(w)) = 1$.

Proof An application of Proposition 3.13 to $Y_{q,n} - Y_{q,n-1}$ and $p = 2$ yields

$$\sum_{n\geq 1} \|Y_{q,n} - Y_{q,n-1}\|_2 \leq C \sum_{n\geq 1} \prod_{k=0}^{n-1} \Phi\big(2q_k, 2\tau(q_k)\big)^{1/2}. \tag{3.25}$$

where we set $Y_{q,0} = 1$ and C is the supremum of the constants C_p invoked in Proposition 3.13. Then, since $\|Y_{q.n} - Y_{q,n-1}\|_1 \leq \|Y_{q,n} - Y_{q,n-1}\|_2$ we have the conclusion for $w = \varnothing$. Now, if $m \geq 1$ we have

$$Y_q = \sum_{w \in \mathscr{A}^m} Q_q(w)\, Y_q(w), \tag{3.26}$$

where the random variables $Y_q(w)$, $w \in \mathscr{A}^m$, are identically distributed, as well as the discrete processes $(Y_{q,n}(w))_{n\geq 1}$ converging to them. Consequently, if $Y_q(w)$ is not the limit of $(Y_{q,n}(w))_{n\geq 1}$ in L^1 for some $w \in \mathscr{A}^m$, then $\mathbb{E}(Y_q(w)) < 1$ and the same holds for all $w \in \mathscr{A}^m$. In particular, (3.26) yields $\mathbb{E}(Y_q) < 1$, which is in contradiction with the convergence in L^1 norm of $(Y_{q,n})_{n\geq 1}$.

Now we specify the sequence $(q_k)_{k\geq 0}$. We discard the obvious case where τ is affine and assume that τ is strictly concave.

The graph of the function τ has the asymptote line $l(q) = hq - \tau^*(h)$ with $\tau^*(h) \in [0,1)$. For $\delta \geq 0$, let $l_\delta(q) = l(q) - \delta$. We deduce from the strict concavity of τ that for any $\delta \in (0, -\tau(0) - \tau^*(h)]$ there is a unique $q(\delta) > 0$ such that $l_\delta(q(\delta)) = \tau(q(\delta))$. Moreover, $\delta \mapsto q(\delta)$ is continuous and strictly decreasing, and $q(\delta) \to \infty$ as $\delta \to 0$. Fix k_0 such that $\frac{1}{\log k_0} \in (0, -\tau(0) - \tau^*(h))$, and for $k \geq 0$ let $\delta_k = 1/\log(k_0 + k)$. Then choose $q_k = q(\delta_k)$ for $k \geq 0$. By using the definition of l_δ and $\delta(\cdot)$ as well as the concavity of τ, we obtain for all $k \geq 0$

$$\varepsilon_k = \tau(2q_k) - 2\tau(q_k) = \tau(2q_k) - 2l_{\delta_k}(q_k) \geq l_{\delta_k}(2q_k) - 2l_{\delta_k}(q_k) = \tau^*(h) + \delta_k. \tag{3.27}$$

We also have for any conjugate pair (α, α') such that $1/\alpha + 1/\alpha' = 1$

$$\begin{aligned} \Phi\big(2q_k, 2\tau(q_k)\big) &= \mathbb{E}\Big(\sum_{i=0}^{b-1} W_{2q_k,i} L_i^{\varepsilon_k}\Big) \\ &\leq \Big(\mathbb{E}\Big(\sum_{i=0}^{b-1} W_{2q_k,i}^{\alpha'}\Big)\Big)^{1/\alpha'} \Big(\mathbb{E}\Big(\sum_{i=0}^{b-1} L_i^{\varepsilon_k \alpha}\Big)\Big)^{1/\alpha}. \end{aligned}$$

Our assumption that τ as an asymptote at ∞ implies that $\tau(q)/q$ is increasing, so $\mathbb{E}\Big(\sum_{i=0}^{b-1} W_{q',i}^{\alpha'}\Big) = \Phi(\alpha' q', \alpha'\tau(q')) \leq \Phi(\alpha' q', \tau(\alpha' q')) = 1$ for all $q' > 0$ and $\alpha' > 1$. Also, the fact that the L_i belong to $(0,1)$ implies that $\varphi_L(q') \sim \bar{a}q' + c$ at ∞ with $\bar{a} > 0$, so by choosing α large enough we ensure $\mathbb{E}\Big(\sum_{i=0}^{b-1} L_i^{\varepsilon_k\alpha}\Big) = b^{-\varphi_L(\varepsilon_k \alpha)} \leq b^{-\bar{a}\varepsilon_k\alpha/2}$. Thus $\Phi\big(2q_k, 2\tau(q_k)\big) \leq b^{-\bar{a}\varepsilon_k/2}$. Consequently, for all $n \geq 1$, we have

$$\prod_{k=0}^{n-1} \Phi\big(2q_k, 2\tau(q_k)\big)^{1/2} \leq b^{-\bar{a}S_n/4},$$

where $S_n = \displaystyle\sum_{k=0}^{n-1} \varepsilon_k$. We have either $\tau^*(h) > 0$ and $S_n \geq n\tau^*(h)/2$ or $\tau^*(h) = 0$ and there exists $n_0 \geq 1$ such that $S_n \geq n/(4\log(n))$ for $n \geq n_0$. In both cases the conclusion of Lemma 3.3 holds.

The same arguments as those used in the proof of Lemma 3.3 show that for $w \in \mathscr{A}^*$ and $n \geq 1$ we have (with $Y_0(w) = 1$)

$$\|Y_{q,n}(w) - Y_{q,n-1}(w)\|_2 \leq C\, b^{-n\bar{a}(S_{|w|+n} - S_{|w|})/4}. \tag{3.28}$$

If $\tau^*(h) > 0$, by using (3.27) we get $\|Y_{q,n}(w) - Y_{q,n-1}(w)\|_2 \leq Cb^{-n\bar{a}\tau^*(h)/4}$, an upper bound which does not depend on w, and finally $\sup_{w \in \mathscr{A}^*} \|Y_q(w)\|_2 < \infty$.

If $\tau^*(h) = 0$, we have $\|Y_{q,n}(w) - Y_{q,n-1}(w)\|_2 \leq C\, b^{\bar{a}S_{|w|}/4} \cdot b^{-\bar{a}S_{|w|+n}/4}$. Thus,

$$\begin{aligned} \|Y_q(w)\|_2 &\leq 1 + \sum_{n\geq 1} \|Y_{q,n}(w) - Y_{q,n-1}(w)\|_2 \\ &\leq 1 + C \cdot b^{\bar{a}S_{|w|}/4} \sum_{n \geq 1} b^{-\bar{a}S_{|w|+n}/4} \leq 1 + C \cdot b^{\bar{a}S_{|w|}/4} L, \end{aligned}$$

where $L = \sum_{n\geq 1} b^{-\bar{a}S_n/4}$ is finite because $S_n \geq n/4\log(n)$ for n large enough.

Now, we notice that we also have by concavity of τ

$$\tau(2q_k) - 2\tau(q_k) = \tau(2q_k) - 2l_{\delta_k}(q_k) \leq l(2q_k) - 2l_{\delta_k}(q_k) = 2\delta_k. \tag{3.29}$$

Due to our choice for δ_k, this implies that for $|w|$ large enough we have $S_{|w|} \leq \frac{2|w|}{\log|w|}$. Finally, there exists $C' > 0$ such that for all $w \in \mathscr{A}^*$ we have $\|Y_q(w)\|_2 \leq C' b^{\frac{\bar{a}|w|}{2\log|w|}}$.

Proof of Proposition 3.10

We first prove the following proposition

Proposition 3.11 *Let $h \in \{\bar{h}, \underline{h}\}$ and $q \in \{-\infty, \infty\}$ such that $h = \lim_{J \ni q' \to q} \tau'(q)$.*

1. *Let $m \geq 1$, $\gamma_1, \gamma_2 \in \{-1, 0, 1\}$ and $\delta_1, \delta_2 \in \{0, 1\}$. With probability 1, for μ_q-almost every $t \in \mathrm{Supp}(\mu_q)$, $\lim_{n\to\infty} \dfrac{\log |Q_W(t|_n)| \cdot Z_W^{(m)}(t|_n^{\gamma_1})^{\delta_1}}{\log Q_L(t|_n) \cdot Z_L^{(1)}(t|_n^{\gamma_2})^{\delta_2}} = h$.*
2. *For $t \in \mathscr{A}^{\mathbb{N}_+}$, $i \in \{0, b-1\}$ and $n \geq 1$ we define $N_n(t) = \max\{0 \leq j \leq n : \forall\, 0 \leq k \leq j,\ t_{n-k} = i\}$. With probabilility 1, for μ_q-almost every t we have $N_n(t) = o(n)$.*
3. *Let $\varepsilon \in (0,1)$, $i \in \{0, b-1\}$ and $r \in \{0, ..., b-1\}$. There exists $\alpha(\varepsilon) \in (0, \varepsilon)$ such that, with probability 1, for μ_q-almost every t, for n large enough and $n(1-\alpha(\varepsilon)) \leq p \leq n-1$ such that $W_{n,p}(t|_n) := W_r(t|_{p-1}) \prod_{k=p+1}^n W_i(t|_{k-1}) \neq 0$, we have*

$$\log |W_{n,p}(t|_n)| / \log O_L(t|_n) \geq -\varepsilon.$$

Remark 3.3 *Notice that in such a case we have $h < \infty$ (but could be equal to 0), the case when $h = \infty$ is already discussed in Theorem 3.2.*

Proof

1. For $\gamma_1, \gamma_2 \in \{-1, 0, 1\}$, $\delta_1, \delta_2 \in \{0, 1\}$, $m \geq 1$ and $w \in \mathscr{A}^*$ we simply denote

$$\begin{cases} O_W(w) = Q_W(w) \cdot Z_W^{(m)}(w^{\gamma_1})^{\delta_1},\ O_L(w) = Q_L(w) \cdot Z_L^{(1)}(w^{\gamma_2})^{\delta_2}, \\ Z(w) = Z_L^{(1)}(w^{\gamma_2})^{\delta_2} / Z_W^{(m)}(w^{\gamma_1})^{\delta_1}. \end{cases}$$

For $\varepsilon > 0$, $n \geq 1$, and $\lambda \in \{-1, 1\}$ we define

$$E_{n,\varepsilon}^\lambda = \{t \in \mathscr{A}^{\mathbb{N}_+} : O_W(t|_n) \neq 0 \text{ and } O_W(t|_n)^\lambda < O_L(t|_n)^{\lambda h + \varepsilon}\}.$$

For any $\eta_n > 0$ we have

$$\begin{aligned} \mu_q(E_{n,\varepsilon}^\lambda) &\leq \sum_{w \in \mathscr{A}^n} \mu_q(w) |Q_W(w)|^{-\lambda\eta_n} Q_L(w)^{\lambda\eta_n h + \varepsilon\eta_n} Z(w)^{\lambda\eta_n} \\ &= \sum_{w \in \mathscr{A}^n} Y_q(w) Z(w)^{\lambda\eta_n} \prod_{k=1}^n W_{w_k}(w|_{k-1})^{q_{k-1} - \lambda\eta_n} L_{w_k}(w|_{k-1})^{\lambda\eta_n h + \varepsilon\eta_n - \tau(q_{k-1})}. \end{aligned}$$

This yields

$$\mathbb{E}(\mu_q(E_{n,\varepsilon}^\lambda)) \leq \mathbb{E}(Y_q \cdot Z^{\lambda\eta_n}) \prod_{k=1}^n \Phi(q_{k-1} - \lambda\eta_n, \tau(q_{k-1}) - \lambda h\eta_n - \varepsilon\eta_n). \tag{3.30}$$

Let us make the following observation. For any $q_k, \eta_n > 0$ we can write

$$\log \Phi(q_k - \lambda\eta_n, \tau(q_k) - \lambda h\eta_n - \varepsilon\eta_n)$$

$$\begin{aligned} &= \lambda\eta_n\xi(q_k) - (\lambda h\eta_n + \varepsilon\eta_n)\widetilde{\xi}(q_k) + \frac{1}{2}\Delta(\zeta_k)\eta_n^2 \\ &= -\varepsilon\eta_n\widetilde{\xi}(q_k) + \lambda\eta_n\widetilde{\xi}(q_k)(\tau'(q_k) - h) + \frac{1}{2}\Delta(\zeta_k)\eta_n^2, \end{aligned} \tag{3.31}$$

where $\zeta_k = (\zeta_1, \zeta_2) = s(q_k, \tau(q_k)) + (1-s)(q_k - \lambda\eta_n, \tau(q_k) - \lambda h\eta_n - \varepsilon\eta_n)$ for some $s \in [0, 1]$, and

$$\Delta(\zeta_k) = \lambda^2 \frac{\partial^2}{\partial q^2}\Phi(\zeta_k) + (\lambda h + \varepsilon)^2 \frac{\partial^2}{\partial t^2}\Phi(\zeta_k) + 2\lambda(\lambda h + \varepsilon)\frac{\partial^2}{\partial q \partial t}\Phi(\zeta_k).$$

Also, we have

$$\begin{aligned} & \log\Phi(q_k - \lambda\eta_n, \tau(q_k) - \lambda h\eta_n - \varepsilon\eta_n) \\ = \; & \log\Phi(q_k - \lambda\eta_n, \tau(q_k - \lambda\eta_n) + \lambda(\tau'(q_k) - h)\eta_n + \tau''(\widetilde{q}_k)\eta_n^2/2 - \varepsilon\eta_n), \end{aligned}$$

where $\widetilde{q}_k \in [q_k - \lambda\eta_n, q_k]$. Since τ is concave and $\tau'(q)$ tends to h at ∞, if η_n is small enough, then for k large enough we have

$$\lambda(\tau'(q_k) - h)\eta_n + \tau''(\widetilde{q}_k)\eta_n^2/2 - \varepsilon\eta_n \le -\varepsilon\eta_n/2 < 0,$$

hence $\log\Phi(q_k - \lambda\eta_n, \tau(q_k) - \lambda h\eta_n - \varepsilon\eta_n) < \log\Phi(q_k - \lambda\eta_n, \tau(q_k - \lambda\eta_n)) = 0$. Hence, due to (3.31)

$$0 \ge -\varepsilon\eta_n\widetilde{\xi}(q_k) + \lambda\eta_n\widetilde{\xi}(q_k)(\tau'(q_k) - h) + \frac{1}{2}\Delta(\zeta_k)\cdot\eta_n^2. \tag{3.32}$$

Moreover, under our assumptions, the multifractal analysis of the Mandelbrot measure $\mu_L = F_L'$ achieved in [11]) implies that for any random probability vector $\widetilde{W}$ with $\mathbb{E}(\sum_{i=0}^{b-1}\widetilde{W}_i) = 1$ and $\mathbb{E}(\sum_{i=0}^{b-1}\widetilde{W}_i\log\widetilde{W}_i) < 0$, $-\mathbb{E}(\sum_{i=0}^{b-1}\widetilde{W}_i\log_b L_i)$ is a Hölder exponent for μ_L, so it must belong to $[\underline{a}, \overline{a}]$, where

$$0 < \underline{a} = \lim_{q\to+\infty}\tau_{F_L}(q)/q \le \lim_{q\to-\infty}\tau_{F_L}(q)/q = \overline{a} < \infty.$$

Applying this with $\widetilde{W} = W_{q_k}$ yields $\widetilde{\xi}(q_k)/\log(b) \in [\underline{a}, \overline{a}]$ for all $k \ge 0$. Also, since $q_k \nearrow \bar{q} = +\infty$ we have $\tau'(q_k) = \xi(q_k)/\widetilde{\xi}(q_k) \searrow h < \infty$, so $\sup_{k\ge0}\xi(q_k) < \infty$. These properties together with (3.32) yield $c = \sup_k\Delta(\zeta_k) < \infty$.

For simplicity we define $\underline{a} := \log(b)\,\underline{a}$ and $\overline{a} := \log(b)\,\overline{a}$.

By using again the fact that $\tau'(q_k) - h \ge 0$ as well as (3.30) and (3.31) we get

$$\mathbb{E}(\mu_q(E_{n,\varepsilon}^\lambda)) \le e^{-\varepsilon\underline{a}n\eta_n + \overline{a}(\sum_{k=1}^n\tau'(q_k) - h)\eta_n + c\eta_n^2/2}\cdot\mathbb{E}(Y_q(w)\cdot Z^{\lambda\eta_n}(w)).$$

We take $\eta_n = 1/\sqrt{\log(n_m + n)}$ for all $n \ge 1$, where n_m is an integer large enough so that for any $\lambda \in \{-1, 1\}$ we have $\mathbb{E}((Z_W^{(m)})^{\frac{4\lambda}{\sqrt{\log(n_m+1)}}}) < \infty$ and $\mathbb{E}((Z_L^{(1)})^{\frac{4\lambda}{\sqrt{\log(n_m+1)}}}) < \infty$ (the existence of n_m comes from Propositioin 3.14). Then due to Proposition 3.9, for $n \ge 1$ and $w \in \mathscr{A}^n$ we have

$$\mathbb{E}(Y_q(w)\cdot Z^{\lambda\eta_n}) \le \|Y_q(w)\|_2\cdot\|(Z_L^{(1)})^{\lambda\delta_2\eta_n}\|_4\cdot\|(Z_W^{(m)})^{-\lambda\delta_1\eta_n}\|_4 = O(b^{2n/\log(n)}). \tag{3.33}$$

Notice that $\sum_{k=1}^n\tau'(q_k) - h = o(n)$, since $\tau'(q_k) - h \searrow 0$ when $k \to \infty$. Thus, due to our choice for η_n, for n large enough we have

$$-\varepsilon\underline{a}n\eta_n + \overline{a}(\sum_{k=1}^n\tau'(q_k) - h)\eta_n + c\eta_n^2 = -\varepsilon\underline{a}n\eta_n + o(n\eta_n). \tag{3.34}$$

Then (3.33) and (3.34) together yield $\mathbb{E}(\mu_q(E^\lambda_{n,\varepsilon})) = O(b^{-\varepsilon\underline{a}n/2\sqrt{\log(n)}})$.

2. Let us recall that $\mathbb{E}(\|\mu_q\|) = \mathbb{E}(Y_q) = 1$ and introduce on $\Omega \times \mathscr{A}^{\mathbb{N}_+}$ the "Peyrière probability measure" $\mathcal{Q}_q$ defined by

$$\mathcal{Q}_q(A) = \mathbb{E}\Big(\int \mathbf{1}_A(\omega,t)\mu_q^\omega(\mathrm{d}t)\,\mathbb{P}(\mathrm{d}(\omega))\Big),\ A \in \mathcal{A}\otimes\mathcal{S}.$$

Notice that "$\mathcal{Q}_q$-almost surely" means "with probability 1, μ_q-almost everywhere".

Without loss of generality, we can assume that for $i \in \{0, b-1\}$ the sequence

$$\big(\mathbb{E}(\mathbf{1}_{\{W_i\neq 0\}}|W_i|^{q_k}L_i^{-\tau(q_k)})\big)_{k\geq 0}$$

has a limit f_i as $k \to \infty$, since this sequence takes values in the bounded interval $[0,1]$.

Now for $n \geq 1$, $i \in \{0, b-1\}$ and $(\omega,t) \in \Omega \times \mathscr{A}^{\mathbb{N}_+}$ set $f_{i,n}(\omega,t) = \mathbf{1}_{\{i\}}(t_n)$. It is not difficult to show that the random variables $f_{i,n}$, $n \geq 1$ are $\mathcal{Q}_q$ independent. Moreover, we have

$$\widetilde{f}_{i,n} := \mathbb{E}_{\mathcal{Q}_q}(f_{i,n}) = \mathbb{E}_{\mathcal{Q}_q}(f_{i,n}^2) = \mathbb{E}(W_{q_{n-1},i}) = \mathbb{E}(|W_i|^{q_{n-1}}L_i^{-\tau(q_{n-1})}).$$

Indeed,

$$\begin{aligned}&\mathbb{E}_{\mathcal{Q}_q}(f_{i,n})\\ =\ &\sum_{w\in\mathscr{A}^n,\ w_n=i}\mathbb{E}(\mu_q([w])) = \sum_{w\in\mathscr{A}^n,\ w_n=i}\mathbb{E}(Q_q(w|_{n-1}))\mathbb{E}(W_{q_{n-1},i}(w|_{n-1}))\mathbb{E}(Y_q(w)).\end{aligned}$$

Consequently, $\widetilde{f}_{i,n}$ converges to f_i as $n \to \infty$, and on $(\Omega\times\mathscr{A}^{\mathbb{N}_+}, \mathcal{A}\otimes\mathcal{S}, \mathcal{Q}_q)$, the martingale $\sum_{k=1}^n (f_{i,k} - \mathbb{E}_{\mathcal{Q}_q}(f_{i,k}))/k$ is bounded in L^2 norm by $\sum_{k\geq 1}\widetilde{f}_{i,k}(1-f_{i,k})/k^2$. It follows that the series $\sum_{k\geq 1}(f_{i,k} - \mathbb{E}_{\mathcal{Q}_h}(f_{i,k}))/k$ converges $\mathcal{Q}_q$-almost surely, and the Kronecker lemma implies that $\sum_{k=1}^n f_{i,n}/n$ converges to f_i $\mathcal{Q}_q$-almost surely. This implies 2..

3. Let $\alpha \in (0,\varepsilon)$. Fix $\eta \in (0,1)$ and for n large enough let $p \in [(1-\alpha)n, n-1]$ be an integer. For $(\omega,t) \in \Omega\times\mathscr{A}^{\mathbb{N}_+}$, let $X_{n,p}(\omega,t) = \log|W_{n,p}(t|_n)|/\log O_L(t|_n)$. We have

$$\begin{aligned}\mathcal{Q}_q(X_{n,p} < -\varepsilon) &= \sum_{w\in\mathscr{A}^n}\mathbb{E}(\mathbf{1}_{\{\log|W_{n,p}(t|_n)|/\log O_L(t|_n)<-\varepsilon\}}\mu_q([w]))\\ &\leq \sum_{w\in\mathscr{A}^n}\mathbb{E}(Y_q(w)Z_L(w)^{\eta\varepsilon})\mathbb{E}(Q_q(w|_{p-1})Q_L(w|_{p-1})^{\eta\varepsilon})\\ &\qquad\cdot\mathbb{E}(W_{q_{p-1},w_p}(w|_{p-1})L_{w_p}(w|_{p-1})^{\eta\varepsilon}|W_r(w|_{p-1})|^{\eta})\\ &\qquad\cdot\prod_{k=p+1}^n\mathbb{E}(W_{q_{k-1},w_k}(w|_{k-1})L_{w_k}(w|_{k-1})^{\eta\varepsilon}|W_i(w|_{k-1})|^{\eta})\end{aligned}$$

for any $\eta > 0$. Applying the Cauchy-Schwarz inequality in the right hand side of the above inequality yields

$$\mathcal{Q}_q(X_{n,p} < -\varepsilon) \leq \prod_{k=1}^{p-1}\Phi(q_{k-1},\tau(q_{k-1})-\eta\varepsilon)\prod_{k=p}^{n}\Phi(2q_{k-1},2\tau(q_{k-1})-2\eta\varepsilon)$$

$$\cdot \|Y_q(w)\|_2 \cdot \|Z_L^{\eta\varepsilon}\|_2 \cdot \||W_r|^\eta\|_2 \cdot (\||W_i|^\eta\|_2)^{n-p}.$$

Also, by using the same arguments as in the proof of (1) we can get

$$\begin{cases} \log \Phi(q_{k-1}, \tau(q_{k-1}) - \eta\varepsilon) = -\widetilde{\xi}(q_{k-1})\eta\varepsilon + O(\eta\varepsilon^2) \\ \log \Phi(2q_{k-1}, 2\tau(q_{k-1}) - 2\eta\varepsilon) \leq \log \Phi(2q_{k-1}, \tau(2q_{k-1}) - 2\eta\varepsilon) = -\widetilde{\xi}(2q_{k-1})\eta\varepsilon + O(\eta\varepsilon^2). \end{cases}$$

It follows that

$$\mathcal{Q}_q(X_{n,p} < -\varepsilon) \leq C \cdot e^{n[-\underline{a}\eta\varepsilon + O(\eta^2)]} \cdot (\||W_i|^\eta\|_2)^{\alpha n}.$$

Since $\||W_i|^\eta\|_2 \to 1$ when $\eta \to 0$, then we can find η small enough and α small enough such that for n large enough :

$$\mathcal{Q}_q(X_{n,p} < -\varepsilon) \leq e^{-\underline{a}\eta\varepsilon n/2}, \quad \forall\ (1-\alpha)n \leq p \leq n.$$

Consequently, $\sum_{n\geq 1} \mathcal{Q}_q(\exists\ (1-\alpha)n \leq m \leq n : X_{n,p} < -\varepsilon) < \infty$, and the conclusion follows from the Borel-Cantelli lemma.

Proof of Proposition 3.10

• $h_F^{(m)}(t) = h$ *for* ν_q*-almost every* t. Due to Proposition 3.11.1, with probability 1, for μ_q almost every $t \in \mathscr{A}^{\mathbb{N}_+}$, (notice that $1/h$ can be infinite since h can be equal to 0),

$$\lim_{n\to\infty} \frac{\log |Q_W(t|_n)|}{\log Q_L(t|_n)} = \lim_{n\to\infty} \frac{\log |Q_W(t|_n)|}{\log \mathrm{Osc}^{(1)}_{F_L}(t|_n)} = h,\ \lim_{n\to\infty} \frac{\log Q_L(t|_n)}{\log \mathrm{Osc}^{(m)}_{F_W}(t|_n)} = \frac{1}{h},$$

and for $\gamma \in \{-1, 1\}$

$$\lim_{n\to\infty} \frac{\log Z_W^{(1)}((t|_n)^\gamma)}{\log \mathrm{Osc}^{(1)}_{F_L}(t|_n)} = \lim_{n\to\infty} \frac{\log Z_L((t|_n)^\gamma)}{\log \mathrm{Osc}^{(m)}_{F_W}(t|_n)} = 0.$$

Also, due to the Lemma 3.1 and the fact that all the moments of Z_L are finite, there exist $\varepsilon > 0$ such that for μ_q-almost every $t \in \mathscr{A}^{\mathbb{N}_+}$, there exists $n_{t,\varepsilon}$ such that for all $n \geq n_{t,\varepsilon}$ we have $Q_L(t|_n) \in [b^{-n(\bar{a}+\varepsilon)}, b^{-n(\underline{a}-\varepsilon)}]$. In particular, for n large enough we have

$$\frac{\log(Q_L(t|_n)/Q_L(t|_{n-N_n(t)}))}{\log Q_L(t|_n)} \in \Big[\frac{\underline{a}-\varepsilon}{\bar{a}+\varepsilon}\frac{N_n(t)}{n}, \frac{\bar{a}+\varepsilon}{\underline{a}-\varepsilon}\frac{N_n(t)}{n}\Big].$$

Consequently, since $N_n(t) = o(n)$ for μ_q-almost every $t \in \mathscr{A}^{\mathbb{N}_+}$ (Proposition 3.11.2), we have

$$\lim_{n\to\infty} \frac{\log Q_L(t|_{n-N_n(t)})}{\log Q_L(t|_n)} = 1,$$

and if $h \neq 0$, we have

$$1 = h \cdot 1 \cdot \frac{1}{h} = \lim_{n\to\infty} \frac{\log Q_W(t|_{n-N_n(t)})}{\log Q_L(t|_{n-N_n(t)})} \cdot \lim_{n\to\infty} \frac{\log Q_L(t|_{n-N_n(t)})}{\log Q_L(t|_n)} \cdot \lim_{n\to\infty} \frac{\log Q_L(t|_n)}{\log Q_W(t|_n)}$$

$$= \lim_{n\to\infty} \frac{\log Q_W(t|_{n-N_n(t)})}{\log Q_W(t|_n)}.$$

Moreover, let $i \in \{0, b-1\}$ and $r \in \{0, ..., b-1\}$, since $L_i \leq 1$, for any $p \leq n-1$ we have

$$\liminf_{n\to\infty} \frac{\log(L_r(t|_{p-1}) \prod_{k=p+1}^{n-1} L_i(t|_k))}{\log \mathrm{Osc}_{F_W}^{(m)}(t|_n)} \geq 0.$$

Then, due to Proposition 3.11.1 and .3, for μ_q-almost every $t \in \mathscr{A}^{\mathbb{N}_+}$, for $\gamma \in \{-1, 1\}$,

$$\text{either } \liminf_{n\to\infty} \frac{\log \mathrm{Osc}_{F_W}^{(1)}((t|_n)^\gamma)}{\log \mathrm{Osc}_{F_L}^{(1)}(t|_n)} = \infty \text{ or } \liminf_{n\to\infty} \frac{\log \mathrm{Osc}_{F_W}^{(1)}((t|_n)^\gamma)}{\log \mathrm{Osc}_{F_L}^{(1)}(t|_n)} \geq h;$$

where the inequality for the case $h = 0$ is automatically true, and

$$\liminf_{n\to\infty} \frac{\log \mathrm{Osc}_{F_L}^{(1)}((t|_n)^\gamma)}{\log \mathrm{Osc}_{F_W}^{(m)}(t|_n)} \geq \frac{1}{h}.$$

We conclude from the fact that due to (3.10), for $\widetilde{t} = F_L(\pi(t))$ we have

$$\begin{aligned}
\liminf_{r\to 0} \frac{\log \mathrm{Osc}_F^{(m)}(B(\widetilde{t}, r))}{\log r} &\geq \min_{\gamma=-1,0,1} \liminf_{n\to\infty} \frac{\log \mathrm{Osc}_{F_W}^{(1)}((t|_n)^\gamma)}{\log \mathrm{Osc}_{F_L}^{(1)}(t|_n)}; \\
\liminf_{r\to 0} \frac{\log r}{\log \mathrm{Osc}_F^{(m)}(B(\widetilde{t}, r))} &\geq \min_{\gamma=-1,0,1} \liminf_{n\to\infty} \frac{\log \mathrm{Osc}_{F_L}^{(1)}((t|_n)^\gamma)}{\log \mathrm{Osc}_{F_W}^{(m)}(t|_n)},
\end{aligned}$$

where in the last inequality we have used Lemma 3.1. Consequently,

$$\lim_{r\to 0} \frac{\log \mathrm{Osc}_F^{(m)}(B(\widetilde{t}, r))}{\log r} = h, \text{ for } \mu_q \circ \pi^{-1} \circ F_L^{-1}\text{-almost every } \widetilde{t}.$$

• *Almost surely* $\dim_H(\nu_q) \geq \tau^*(h)$. We only need to deal with the case where $\tau^*(h) > 0$. We are going to prove that, with probability 1, for μ_q-almost every $t \in \mathscr{A}^{\mathbb{N}_+}$,

$$\liminf_{n\to\infty} \frac{\log \mu_q([t|_n])}{\log O_L(t|_n)} \geq \tau^*(h).$$

Then, due to the last claim of Lemma 3.1, the mass distribution principle (see [130], Lemma 4.3.2 or Section 4.1 in [51]) yields the conclusion.

We set $d = \tau^*(h)$. Fix $\varepsilon > 0$, and for $n \geq 1$ define

$$E_{n,\varepsilon} = \{t \in \mathrm{Supp}(\mu_q) : O_L(t|_n)^{-d+\varepsilon} \cdot \mu_q([t|_n]) \geq 1\}.$$

For $n \geq 1$ let $\eta_n > 0$ and set $S_{n,\varepsilon} = \sum_{w\in\mathscr{A}^n} \mu_q([w])\Big(O_L(t|_n)^{-d+\varepsilon} \cdot \mu_q([w])\Big)^{\eta_n}$. For $n \geq 1$ we have $\mu_q(E_{n,\varepsilon}) \leq S_{n,\varepsilon} = \sum_{w\in\mathscr{A}^n} Q_L(w)^{-(d-\varepsilon)\eta_n} Q_q(w)^{1+\eta_n} Y_q(w)^{1+\eta_n} Z_L(w)^{\eta_n}$ and $\mathbb{E}(S_{n,\varepsilon}) = \prod_{k=0}^{n-1} \Phi(q_k + q_k\eta_n, \tau(q_k) + (\tau(q_k) + d - \varepsilon)\eta_n) \cdot \mathbb{E}(Y_q(w)^{1+\eta_n} Z_L(w)^{\eta_n})$. If we show that $\sum_{n\geq 1} \mathbb{E}(S_{n,\varepsilon}) < \infty$, then the series $\sum_{n\geq 1} \mu_q(E_{n,\varepsilon})$ converges almost surely and the conclusion follows from the Borel Cantelli lemma.

By an argument similar to those used in the proof of Proposition 3.8, we have

$$\begin{aligned} & \log \Phi(q_k + q_k\eta_n, \tau(q_k) + (\tau(q_k) + d - \varepsilon)\eta_n) \\ = \; & -\xi(q_k)q_k\eta_n + \widetilde{\xi}(q_k)(\tau(q_k) + d - \varepsilon)\eta_n + O(\eta_n^2) \\ = \; & -\widetilde{\xi}(q_k)(\tau^*(\tau'(q_k)) - d + \varepsilon)\eta_n + O(\eta_n^2) \le -\underline{a}\varepsilon\eta_n + O(\eta_n^2) \end{aligned}$$

Thus

$$\prod_{k=0}^{n-1} \Phi(q_k + q_k\eta_n, \tau(q_k) + (\tau(q_k) + d - \varepsilon)\eta_n) \le e^{-\underline{a}\varepsilon\eta_n n + O(n\eta_n^2)}.$$

Now since

$$\mathbb{E}(Y_q(w)^{1+\eta_n} Z_L(w)^{\eta_n}) \le \|Y_q(w)\|_2^{1+\eta_n} \cdot \|Z_L(w)^{\frac{\eta_n}{1-\eta_n}}\|_2^{1-\eta_n},$$

by taking $\eta_n = \frac{1}{\sqrt{n_0+n}}$ for n_0 large enough we will get $\mathbb{E}(S_{n,\varepsilon}) = O(b^{-\frac{\underline{a}\varepsilon n}{2\sqrt{\log(n)}}})$.

3.4 Appendix

Proposition 3.12 *Let M be a non-negative bounded and non-decreasing function defined over the subsets of $\mathbb{R}^d$. Let* $\mathrm{Supp}(M) = \{t : \forall r > 0, \ M(B(t,r)) > 0\}$ *be the closed support of M. Suppose that* $\mathrm{Supp}(M)$ *is a non-empty compact set and define the L^q-spectrum associated with M as the mapping namely*

$$\tau_M(q) = \liminf_{r\to 0} \frac{\log \sup \{\sum_i M(B_i)^q\}}{\log(r)},$$

where the supremum is taken over all the families of disjoint closed balls B_i of radius r with centers in $\mathrm{Supp}(M)$*. We have* $\overline{\dim}_B(\mathrm{Supp}(M)) = -\tau_M(0)$*, and for all $h \ge 0$,*

$$\dim_H E_M(h) := \left\{t \in \mathrm{Supp}(M) : \liminf_{r\to 0^+} \frac{\log(M(B(t,r))}{\log(r)} = h\right\} \le \tau_M^*(h),$$

a negative dimension meaning that $E_M(h)$ is empty.

Proof The equality $\overline{\dim}_B(\mathrm{Supp}(M)) = -\tau_M(0)$ is just the definition of the upper box dimension.

Let $h \ge 0$. Fix $\varepsilon > 0$. For every $t \in E_M(h)$, let $(r_{t,k})_{k\ge 0}$ be a decreasing sequence tending to 0 such that $r_{t,k}^{h+\varepsilon} \le M(B(t, r_{t,k})) \le r_{t,k}^{h-\varepsilon}$.

Fix $\delta > 0$, and for each $t \in E_M(h)$ let k_t be such that $r_{t,k_t} \le \delta$. Now, for every $n \ge 0$, let $A_n = \{t \in E_M(h) : 2^{-(n+1)} < r_{t,k_t} \le 2^{-n}\}$. By the Besicovich covering theorem (see Theorem 2.7 in [116]) there exists an integer N such that for every $n \ge 0$ we can find N disjoint subsets $A_{n,1}, \dots A_{n,N}$ of A_n such that each set $A_{n,j}$ is at most countable, the balls of the form $B(t, r_{t,k_t})$, $t \in A_{n,j}$, are pairwise disjoint, and $\bigcup_{n\ge 0} \bigcup_{j=1}^N \bigcup_{t\in A_{n,j}} B(t, r_{t,k_t})$ is a δ-covering of $E_M(h)$.

Suppose that $h \in [0, \tau_f'(0^+)]$. We have $\tau_M^*(h) = \inf_{q\in\mathbb{R}_+} hq - \tau_M(q)$. Fix $q \ge 0$ such that $\tau_M(q) > -\infty$ and then define $D_\varepsilon = (h+\varepsilon)q - \tau_M(q) + \varepsilon$. We have

$$\begin{aligned}\mathcal{H}_\delta^{D_\varepsilon}(E_M(h)) &\le \sum_{n\ge0}\sum_{j=1}^N\sum_{t\in A_{n,j}}(2r_{t,k_t})^{D_\varepsilon} \le 2^{D_\varepsilon}\sum_{n\ge0}\sum_{j=1}^N\sum_{t\in A_{n,j}} r_{t,k_t}^{(h+\varepsilon)q-\tau_M(q)+\varepsilon}\\ &\le 2^{D_\varepsilon}\sum_{n\ge0}\sum_{j=1}^N\sum_{t\in A_{n,j}} M(B(t,r_{t,k}))^q r_{t,k_t}^{-\tau_M(q)+\varepsilon}\\ &\le 2^{D_\varepsilon}2^{|\tau_M(q)|}\sum_{n\ge0}\sum_{j=1}^N\sum_{t\in A_{n,j}} M(B(t,2^{-n}))^q 2^{n(\tau_M(q)-\varepsilon)}.\end{aligned}$$

For each $1\le j\le N$, the family $\{B(t,2^{-n})\}_{t\in A_{n,j}}$ can be divided into two disjoint 2^{-n}-packing of $\mathrm{Supp}(M)$. Consequently, by definition of $\tau_M(q)$, for n large enough,

$$\sum_{t\in A_{n,j}} M(B(t,2^{-n}))^q \le 2\cdot 2^{-n(\tau_M(q)-\varepsilon/2)}$$

and $\mathcal{H}_\delta^{D_\varepsilon}(E_M(h)) = O\big(\sum_{n\ge0}2^{-n\varepsilon/2}\big)<\infty$. This yields $\dim_H E_M(h)\le D_\varepsilon$ for all $\varepsilon>0$, hence $\dim_H E_M(h)\le hq-\tau_M(q)$.

Now suppose that $h>\tau_f'(0^+)$. We have $\tau_M^*(h)=\inf_{q\in\mathbb{R}_-} hq-\tau_M(q)$. Fix $q\le 0$ such that $\tau_M(q)>-\infty$ and then $D_\varepsilon=(h-\varepsilon)q-\tau_M(q)+\varepsilon$. This time we have

$$\mathcal{H}_\delta^{D_\varepsilon}(E_M(h)) \le 2^{D_\varepsilon}2^{|\tau_M(q)|}\sum_{n\ge0}\sum_{j=1}^N\sum_{t\in A_{n,j}} M(B(t,2^{-(n+1)}))^q 2^{n(\tau_M(q)-\varepsilon)},$$

and for each $1\le j\le N$, the family $\{B(t,2^{-(n+1)})\}_{t\in A_{n,j}}$ is a $2^{-(n+1)}$-packing of $\mathrm{Supp}(M)$. We conclude as in the previous case.

Proposition 3.13 *Let $\big(V^{(n)}=(V_0^{(n)},\dots,V_{b-1}^{(n)})\big)_{n\ge1}$, be a sequence of random vectors taking values in $\mathbb{C}^b$, and such that $\mathbb{E}\big(\sum_{i=0}^{b-1}V_i^{(n)}\big)=1$. Let $\{V(w)\}_{w\in\mathscr{A}^*}$ be a sequence of independent vectors such that $V(w)$ is distributed as $V^{(|w|)}$ for each $w\in\mathscr{A}^*$. Define $Z_0=1$ and for $n\ge1$*

$$Z_n=\sum_{w\in\mathscr{A}^n}\prod_{k=1}^n V_{w_k}(w|_{k-1}).$$

Let $p\in(1,2]$. There exists a constant $C_p\le 2^p$ depending on p only such that for all $n\ge1$

$$\mathbb{E}(|Z_n-Z_{n-1}|^p)\le C_p\prod_{k=1}^n\mathbb{E}\Big(\sum_{i=0}^{b-1}|V_i^{(k)}|^p\Big).$$

See the proof of Theorem 1 in [12].

Proposition 3.14 *We work under the assumptions of Theorem 1.2. Let $m\ge1$ and $U\in\{W,L\}$.*

1. *If $q>1$ and $\varphi_U(q)>0$ then $\mathbb{E}((Z_U^{(m)})^q)<\infty$. Moreover, if W satisfies the assumptions of Theorem B(2) then $\operatorname{ess\,sup}\mathrm{Osc}_{F_W}^{(m)}([0,1])<\infty$.*
2. *Define $\psi_U^{(m)}(t)=\mathbb{E}(e^{-tZ_U^{(m)}})$ for $t\ge0$. Let $A_U=\max_{0\le i\le b-1}|U_i|$.*
   If $q>0$ and $\mathbb{E}(A_U^{-q})<\infty$ then $\psi_U^{(m)}(t)=O(t^{-p})$ for all $p\in(0,q)$. Consequently, $\mathbb{E}((Z_U^{(m)})^{-p})<\infty$ for all $p\in(0,q)$.

Proof of Proposition 3.14

1. Since $\mathrm{Osc}^{(m)}_{F_U}([0,1]) \leq 2^{m-1}\mathrm{Osc}^{(1)}(F_U,[0,1]) \leq 2^m\|F_U\|_\infty$, this is a direct consequence of Theorem 1.2 (that $\operatorname{ess\,sup}\mathrm{Osc}^{(m)}_{F_W}([0,1]) < \infty$ when W satisfies the assumptions of Theorem 2.3.2 is not stated in Chapter 2 but established in the proof of this theorem).

2. Since $\mathrm{Osc}^{(m)}_{F_U}([0,1]) \geq \mathrm{Osc}^{(m)}_{F_U}(I_i)$ for all $0 \leq i \leq b-1$, by using (3.3) we get

$$Z_U^{(m)} \geq b^{-1}\sum_{i=0}^{b-1}|U(i)|\cdot Z_U^{(m)}(i), \tag{3.35}$$

where the $Z_U^{(m)}(i)$ are independent copies of Z and they are independent of W.

Moreover, thanks to Proposition 3.3 applied to F_U, we know that $Z_U^{(m)} > 0$ almost surely for all $m \in \mathbb{N}_+$. Also, with probability 1, we can define $i_0 = \max\{0 \leq i \leq b-1 : |U_i| = \max_{0\leq k\leq b-1}|U_k|$ and $i_1 = \inf\{0 \leq i \leq b-1 : i \neq i_0,\ U_i \neq 0\}$, $A_0 = |U_{i_0}|$ and $A_1 = |U_{i_1}|$.

Suppose that $\mathbb{E}(A_0^{-q}) < \infty$. This clearly holds if $\varphi_U(-q) > -\infty$ or if there exists $a > 0$ such that $\max_{0\leq k\leq b-1}|U_k| \geq a$ almost surely (for instance $a = 1/b$ is convenient when U is conservative).

Set $\psi = \psi_U$. By definition of ψ, we deduce from (3.35) and the fact that $Z_U^{(m)}$ is almost surely positive that $\psi(t) \leq \mathbb{E}\big(\psi(A_0t)\psi(A_1t)\big)$ and $\lim_{t\to\infty}\psi(t) = 0$. Suppose that we have shown that $\psi(t) = O(t^{-p})$ at $+\infty$, for all $p \in (0,q)$. Then, for $x > 0$ we have $\mathbb{P}(Z^{(m)} \leq x) \leq e^{tx}\psi(t)$ and choosing $t = p/x$ yields $\mathbb{P}(Z \leq x) = O(x^p)$ at 0^+. Hence $\mathbb{E}(Z^{-p}) < \infty$ if $h \in (0,q)$.

Now we essentially use the elegant approach of [102] for the finiteness of the moments of negative orders of $F_U(1)$, when the components of W are non-negative (see also the references in [102] for this question). Let $r > 1$ and $\phi = \psi^r$. Due to the bounded convergence theorem we have $\lim_{t\to\infty}\mathbb{E}(\psi(A_1t)^{r/(r-1)}) = 0$, so the Hölder inequality yields $\phi(t) = o(\mathbb{E}(\phi(tA_0))$ at ∞. Let $\gamma \in (0,1)$ small enough to have $\gamma\mathbb{E}(A_0^{-p}) < 1$, and let $t_0 > 0$ such that

$$\phi(t) \leq \gamma\mathbb{E}(\phi(tA_0)),\ t \geq t_0. \tag{3.36}$$

Let $(\widetilde{A}_i)_{i\geq 1}$ be a sequence of independent copies of A_0 Since $\phi \leq 1$, for $t \geq t_0$ we can prove by induction using (3.36) the following inequalities valid for all $n \geq 2$:

$$\begin{aligned}
\phi(t) &\leq \gamma\mathbb{P}(A_0t < t_0) + \gamma\mathbb{E}\big(\mathbf{1}_{\{A_0t\geq t_0\}}\phi(A_0t)\big)\\
&\leq \gamma\mathbb{E}(A_0^{-p})(t_0/t)^p + \gamma^2\mathbb{E}\big(\mathbf{1}_{\{A_0t\geq t_0\}}\phi(A_0\widetilde{A}_1t)\big)\\
&\leq \gamma\mathbb{E}(A_0^{-p})(t_0/t)^p + \gamma^2\mathbb{E}\big(\phi(A_0\widetilde{A}_1t)\big)\\
&\leq \gamma\mathbb{E}(A_0^{-p})(t_0/t)^p + \gamma^2(\mathbb{E}(A_0^{-p}))^2(t_0/t)^p + \gamma^2\mathbb{E}\big(\mathbf{1}_{\{A_0\widetilde{A}_1t\geq t_0\}}\phi(A_0\widetilde{A}_1t)\big)\\
&\leq (t_0/t)^p\sum_{k=1}^{n}(\gamma\mathbb{E}(A_0^{-p}))^k + \gamma^n\mathbb{E}\big(\mathbf{1}_{\{A_0\widetilde{A}_1\cdots\widetilde{A}_{n-1}t\geq t_0\}}\phi(A_0\widetilde{A}_1\cdots\widetilde{A}_{n-1}t)\big).
\end{aligned}$$

Since $\psi \leq 1$, and both γ and $\gamma\mathbb{E}(A_0^{-p})$ belong to $(0,1)$, letting n tend to ∞ yields $\phi(t) = \psi(t)^r = O(t^{-p})$. Since r and p are arbitrary respectively in $(1,\infty)$ and $(0,q)$, we have the desired result.

Chapitre 4

Graph, range and level set singularity spectra of b-adic independent cascade functions

This chapter contains the work in [86] on the study of graph, range and level set singularity spectra of the b-adic independent cascade functions constructed in Section 2.2.

Preliminaries

The coding space

Given an integer $b \geq 2$. Let $b \geq 2$ be an integer and $\mathscr{A} = \{0, \cdots, b-1\}$. Then let $\mathscr{A}^* = \bigcup_{n\geq 0} \mathscr{A}^n$ (by convention $\mathscr{A}^0$ contains the empty word denoted $\varnothing$) the set of finite words over $\mathscr{A}$ and $\mathscr{A}^{\mathbb{N}_+} = \{0, \ldots, b-1\}^{\mathbb{N}_+}$ the set of infinite words.

If $u \in \mathscr{A}^*$ we denote by $|u|$ its length.

The word obtained by concatenation of $u \in \mathscr{A}^*$ and $v \in \mathscr{A}^* \cup \mathscr{A}^{\mathbb{N}_+}$ is denoted by $u \cdot v$ and sometimes uv. For every $u \in \mathscr{A}^*$, the cylinder with root u, i.e. $\{u \cdot v : v \in \mathscr{A}^{\mathbb{N}_+}\}$ is denoted by $[u]$.

The set $\mathscr{A}^{\mathbb{N}_+}$ is endowed with the standard metric distance $d(s,t) = \inf\{b^{-n} : n \geq 0,\ \exists\ u \in \mathscr{A}^n,\ s, t \in [u]\}$.

If $n \geq 1$ and $u = u_1 \cdots u_n \in \mathscr{A}^n$ then for every $1 \leq k \leq n$, the word $u_1 \ldots u_k$ is denoted by $u|_k$, and if $k = 0$ then $u|_0$ stands for $\varnothing$. Also, for any infinite word $t = t_1 t_2 \cdots \in \mathscr{A}^{\mathbb{N}_+}$ and $n \geq 1$, $t|_n$ denotes the word $t_1 \cdots t_n$ and $\mathbf{t}|_0$ the empty word.

We denote by $\pi(\cdot)$ the natural projection from $\mathscr{A}^* \cup \mathscr{A}^{\mathbb{N}_+}$ onto $[0,1]$: for $u \in \mathscr{A}^*$ we have $\pi(u) = \sum_{k=1}^{|u|} u_k b^{-k}$ and for $t = t_1 t_2 \cdots \in \mathscr{A}^{\mathbb{N}_+}$ we have $\pi(t) = \sum_{k=1}^{\infty} t_k b^{-k}$.

For any $t \in [0,1]$ and $n \geq 1$, we define $t|_n = t_1 \cdots t_n$ the element of $\mathscr{A}^n$ such that $\pi(t|_n) \leq t < \pi(t|_n) + b^{-n}$ if $t < 1$, as well as $1|_n = b-1 \cdots b-1$.

b-adic independent cascade function and multifractal analysis

Let $(W, L) : \Omega \mapsto \mathbb{R}^b \times \mathbb{R}_+^b$ be a couple of random vector satisfying the following three assumptions :

(A1) $\mathbb{P}(\sum_{j=0}^{b-1} W_j = 1) < 1$ and there exists $p \in (1, 2]$ such that $\varphi_W(p) > 0$;

(A2) $\mathbb{P}(\forall\ j, L_j \in (0, 1)) = 1$, $\mathbb{E}(\sum_{j=0}^{b-1} L_j \log L_j) < 0$ and φ_L is finite on $\mathbb{R}$;

(A3) $\mathbb{P}(\forall\ j, |W_j| > 0) = 1$ and φ_W is finite on $\mathbb{R}$.

where for $U \in \{W, L\}$ and $q \in \mathbb{R}$, $\varphi_U(q) = -\log_b \mathbb{E}(\sum_{j=0}^{b-1} \mathbf{1}_{\{U_j \neq 0\}} |U_j|^q)$.

Remark 4.1 *The condition (A1), (A2) ensures the existence of F_W and F_L. The extra condition $\mathbb{P}(\sum_{j=0}^{b-1} W_j = 1) = 1$ is for avoiding the non-conservative case, since in the proof we need the randomness of $F_W(1)$. The condition $\mathbb{P}(\forall\ j, |W_j| > 0) = 1$ implies that almost surely F is nowhere locally constant. The condition that φ_W is finite ensures that the probability distribution of $F_W(1)$ has a bounded density, in fact this condition can be weaken to the existence of a real number $q < -1$ such that $\varphi_W(q) > -\infty$. The existence of the bounded density of $F_W(1)$ is a key property in the proof.*

Let $\{(W, L)(w) = (W(w), L(w))\}_{w \in \mathscr{A}^*}$ be a family of independent copies of (W, L) encoded by the finite words. For $w, u \in \mathscr{A}^*$ define the products

$$W_u(w) = W_{u_1}(w) \cdot W_{u_2}(w \cdot u_1) \cdots W_{u_n}(w \cdot u_1 \cdots u_{n-1}); \tag{4.1}$$

$$L_u(w) = L_{u_1}(w) \cdot L_{u_2}(w \cdot u_1) \cdots L_{u_n}(w \cdot u_1 \cdots u_{n-1}). \tag{4.2}$$

For $w \in \mathscr{A}^*$, $n \geq 1$ and $t \in [0, 1]$ define

$$F_{W,n}^{[w]}(t) = \int_0^t b^n \cdot W_{u|_n}(w)\ \mathrm{d}u \quad \text{and} \quad F_{L,n}^{[w]}(t) = \int_0^t b^n \cdot L_{u|_n}(w)\ \mathrm{d}u.$$

Under assumptions (A1-3), due to Theorem 1.2, $(F_{W,n})_{n\geq 1}$ and $(F_{L,n})_{n\geq 1}$ converge almost surely and uniformly to two limit functions $F_W^{[w]}$ and $F_L^{[w]}$. Then the real-valued b-adic independent cascade function considered in this chapter is $F : F_W \circ F_L^{-1} : F_L([0, 1]) \mapsto \mathbb{R}$. See Figure 4.1 for illustrations.

The multifractal analysis of F is based on the construction of an uncountable family of statistically self-similar measures μ_q defined on the coding space $\mathscr{A}^{\mathbb{N}_+}$ with desired Hausdorff dimension. More precisely, for $(q, t) \in \mathbb{R}^2$ we define

$$\Phi(q, t) = \mathbb{E}(\sum_{j=0}^{b-1} \mathbf{1}_{\{W_j \neq 0\}} |W_j|^q \cdot L_j^{-t}). \tag{4.3}$$

Clearly $\Phi(q, t)$ is analytic on the rectangle $\{(q, t) \in \mathbb{R}^2 : |\varphi(q, t)| < \infty\}$. Since $L_j \in (0, 1)$ for $0 \leq j \leq b - 1$, for each $q \in Q := \{q \in \mathbb{R} : (q, t) \in \mathbb{R}^2, |\varphi(q, t)| < \infty \text{ for some } t\}$ there is a unique $\tau(q)$ such that $\Phi(q, \tau(q)) = 1$, and the function τ is easily seen to be concave and analytic over Q.

Define the interval $J = \{q \in Q : q\tau'(q) - \tau(q) > 0\}$. For $q \in J$, let

$$\xi(q) = -\frac{\partial}{\partial q}\Phi(q, \tau(q)), \text{ and } \widetilde{\xi}(q) = \frac{\partial}{\partial t}\Phi(q, \tau(q)). \tag{4.4}$$

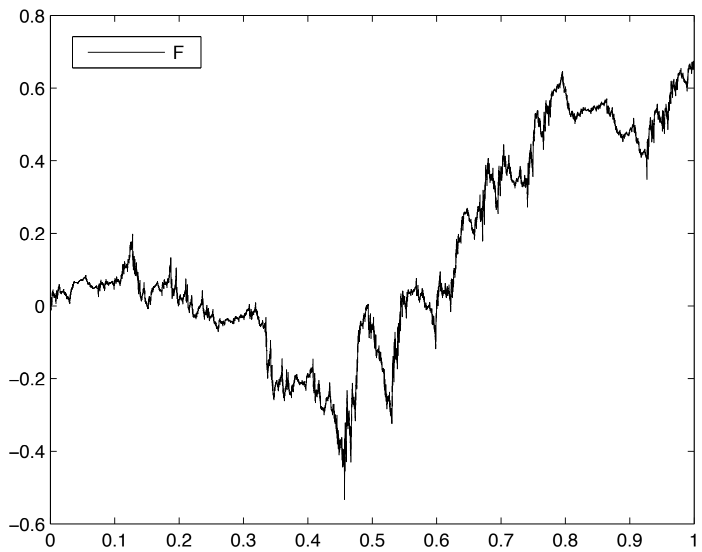

FIGURE 4.1 – F in the non-conservative case.

By construction $\tau'(q) = \xi(q)/\widetilde{\xi}(q)$.

For any $q \in J$ and $w \in \mathscr{A}^*$ we define the random vector

$$W_q(w) = \big(W_{q,j}(w) = \mathbf{1}_{\{W_j(w)\neq 0\}}|W_j(w)|^q \cdot L_j(w)^{-\tau(q)}\big)_{0\le j\le b-1}.$$

Due to (4.1) and (4.2), for $w, u \in \mathscr{A}^*$ we define the product :

$$W_{q,u}(w) = \mathbf{1}_{\{W_u(w)\neq 0\}}|W_u(w)|^q \cdot L_u(w)^{-\tau(q)}. \tag{4.5}$$

For $q \in J$, $w \in \mathscr{A}^*$ and $n \geq 1$ we define

$$Y_{q,n}(w) = \sum_{u\in\mathscr{A}^n} W_{q,u}(w).$$

Recall that if μ is a positive Borel measure on a compact metric space, its lower Hausdorff dimensions is defined as $\dim_H(\mu) = \inf\{\dim_H E : \mu(E) > 0\}$.

Due to the results in Chapter 3, we have the following proposition

Proposition 4.1

1. *With probability 1, for all $q \in J$ and $u \in \mathscr{A}^*$, the sequence $Y_{q,n}(u)$ converges to a positive limit $Y_q(u)$. For any $u \in \mathscr{A}^*$, the function $J \ni q \mapsto Y_q(u)$ is an independent copy of $J \ni q \mapsto Y_q(\varnothing) = Y_q$.*
2. *For every compact subinterval K of J and $u \in \mathscr{A}^*$ define*

$$Y_K(u) = \sup_{q\in K} Y_q(u). \tag{4.6}$$

Then there exists $p_K > 1$ such that $\mathbb{E}(Y_K(u)^{p_K}) < \infty$.

3. *With probability 1, for all $q \in J$, the function*

$$\mu_q([w]) = W_{q,w}(\varnothing) \cdot Y_q(w),\ \ w \in \mathscr{A}^* \tag{4.7}$$

defines a Borel measure on $\mathscr{A}^{\mathbb{N}_+}$ with

$$\dim_H(\mu_q) = \frac{\gamma(q)}{\log(b)},\ \ \textit{where } \gamma(q) = q\xi(q) - \tau(q)\widetilde{\xi}(q). \tag{4.8}$$

These statistically self-similar measures are the effective tools to study the multifractal behavior of $F = F_W \circ F_L^{-1}$. Specifically, with each μ_q we can induce a measure μ_q^D on the domain $D_F = [0, F_L(1)]$: for any Borel set $B \subset \mathbb{R}$,

$$\mu_q^D(B) = \mu_q \circ \pi_L^{-1}(B) = \mu_q\Big(\{t \in \mathscr{A}^{\mathbb{N}_+} : F_L(\bar{t}) \in B\}\Big),$$

where $\pi_L : \mathscr{A}^{\mathbb{N}_+} \ni t \mapsto F_L \circ \pi(t) \in D_F$ and $\pi : \mathscr{A}^{\mathbb{N}_+} \ni t \mapsto \sum_{i=1}^{\infty} t_i b^{-i}$ is the natural mapping from $\mathscr{A}^{\mathbb{N}_+}$ to $[0,1]$. Due to Chapter 3, with probability 1, for all $q \in J$, the measure μ_q^D is carried by the set $E_F(\tau'(q))$ and for μ_q^D-almost every $x \in E_F(\tau'(q))$,

$$\underline{\dim}_{\mathrm{loc}}\mu_q^D(x) = \liminf_{r\to 0^+} \frac{\log \mu_q^D(B(x,r))}{\log r} = q\tau'(q) - \tau(q) = \tau^*(\tau'(q)).$$

Consequently, $\dim_H E_F(\tau'(q)) \geq \dim_H(\mu_q^D) = \tau^*(\tau'(q))$. Then,

Proposition 4.2

1. *With probability* 1, $\tau_F = \tau$ *on the interval* J. *Moreover, if* φ_W *is finite on* $\mathbb{R}$, *then if* $\overline{q} = \sup(J) < \infty$ *(resp.* $\underline{q} := \inf(J) > -\infty$*) we have* $\tau_F(q) = \tau'(\overline{q})q$ *(resp.* $\tau'(\underline{q})q$*) over* $[\overline{q}, \infty)$ *(resp.* $(-\infty, \underline{q}]$*).*
2. *With probability* 1, $d_F = \tau^*$ *on the interval* $\tau'(J)$.

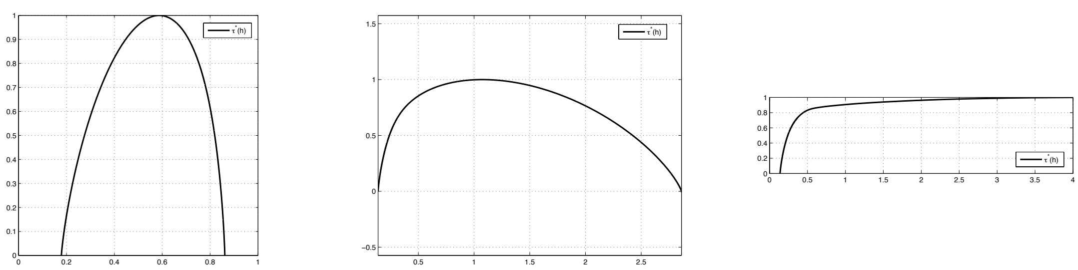

FIGURE 4.2 – τ^* in the case $\sup \tau'(J) < 1$ (Left), $\sup \tau'(J) \geq 1$ (Middle), and $\sup \tau'(J) = \infty$ (Right).

4.1 Statements of results

Let f be a real-valued function defined on an interval I. Recall that for any subset E of I, the graph and the range of f over the set E is given by

$$G_f(E) = \{(x, f(x)) : x \in E\} \text{ and } R_f(E) = \{f(x) : x \in E\}.$$

We should introduce here the level set of f over set E in θ-direction (since we can only show the level set singularity spectrum in "Lebesgue almost every direction") : For $\theta \in (-\pi/2, \pi/2)$ denote by l_θ the line in $\mathbb{R}^2$ passing through the origin and making an angle θ with the y-axis (clockwise). For any $y \in l_\theta$, denote by $l^{\perp}_{y,\theta}$ the line perpendicular to l_θ, passing through y. Denote by Proj_θ the orthogonal projection from $\mathbb{R}^2$ onto l_θ. Define $R_{f,\theta}(E) = \mathrm{Proj}_\theta(G_f(E))$. Then for each $y \in R_{f,\theta}(E)$, the level set of f over the set E in θ-direction is defined by $L^y_{f,\theta}(E) = G_f(E) \cap l^{\perp}_{y,\theta}$. Notice that the typical level set $L^y_f(E)$ is just the level set $L^y_{f,\theta}(E)$ when $\theta = 0$.

Thanks to the following generalization of Lemma 8.2.1 in [1], Theorem 6 of Chapter 10 in [89] and Lemma 2.2 in [151], we have

Theorem 4.1 *Let E be any subset of I. Suppose that* $\inf_{x\in E} h_f(x) = h > 0$.

1. *Let* $\dim_P$ *stand for the packing dimension (see [51]). For* $D \in \{H, P\}$ *we have*

$$\begin{aligned} \dim_D G_f(E) &\leq \dim_D E \vee \left((\dim_D E + 1 - h) \wedge \frac{1}{h} \dim_D E \right), \\ \dim_D R_f(E) &\leq 1 \wedge \frac{1}{h} \dim_D E. \end{aligned}$$

2. *Suppose* $h \leq 1$. *Fix* $\theta \in (-\pi/2, \pi/2)$. *Let* μ *be any positive Borel measure defined on* l_θ. *For any* $\gamma > 0$ *define the set* $R^{\mu,\gamma}_{f,\theta}(E) := \{y \in R_{f,\theta}(E) : h_\mu(y) \geq \gamma\}$. *If* $\mu(R^{\mu,\gamma}_{f,\theta}(E)) > 0$ *and* $\dim_H E - h \cdot \gamma > 0$, *then for* μ*-almost every* $y \in R^{\mu,\gamma}_{f,\theta}(E)$,

$$\dim_H L^y_{f,\theta}(E) \leq \dim_H E - h \cdot \gamma.$$

For each $h \geq 0$, the associated iso-Hölder set is defined by

$$E_f(h) = \{x \in I : h_f(x) = h\},$$

where

$$h_f(x) := \liminf_{r \to 0^+} \frac{1}{\log r} \log \mathrm{Osc}_f(B(x, r)) \tag{4.9}$$

is the pointwise oscillation exponent of f at x. The singularity spectrum of f is given by

$$d_f : h \geq 0 \mapsto \dim_H E_f(h).$$

From the multifractal analysis of functions we know that this spectrum has a general upper bound given by the Legendre transform of the L^q-spectrum of f, defined as

$$\tau_f(q) = \liminf_{r \to 0^+} \frac{1}{\log r} \log \sup \sum_i O_f(B_i)^q, \ q \in \mathbb{R}, \tag{4.10}$$

where the supremum is taken over all the families of disjoint closed intervals B_i of radius r with centers in the support of f' (in the sense of distributions). Thus, we have (see Proposition 3.12)

$$d_f(h) \leq \tau_f^*(h) = \inf_{q\in\mathbb{R}} hq - \tau_f(q) \quad (\forall h \geq 0),$$

a negative dimension meaning that $E_f(h)$ is empty. Define the graph, range and level set singularity spectra

$$d_f^S : h \geq 0 \mapsto \dim_H S_f(E_f(h)), \quad S \in \{G, R, L^y\}.$$

Then by replacing E to $E_f(h)$, we have the following corollary of Theorem 4.1 :

Corollary 4.1 *For any $h > 0$ we have*

$$\begin{aligned} d_f^G(h) &\leq d_f(h) \vee \left((d_f(h) + 1 - h) \wedge \frac{1}{h} d_f(h) \right) \\ &\leq \tau_f^*(h) \vee \left(\left(\tau_f^*(h) + 1 - h\right) \wedge \frac{1}{h}\tau_f^*(h) \right), \\ d_f^R(h) &\leq 1 \wedge \frac{1}{h} d_f(h) \leq 1 \wedge \frac{1}{h}\tau_f^*(h) \end{aligned}$$

and with the same notations as in Theorem 4.1.2, for μ-almost every $y \in R_{f,\theta}^{\mu,\gamma}(h)$,

$$\dim_H L_{f,\theta}^y(E_f(h)) \leq d_f(h) - h \cdot \gamma \leq \tau_f^*(h) - h \cdot \gamma.$$

Now we state our main result. For $h > 0$ denote $S_F(h) = S_F(E_F(h))$ for $S \in \{G, R\}$. For $\theta \in -\pi/2, \pi/2$ and $h > 0$ define $R_{F,\theta}(h) = \text{Proj}_\theta(G_F(h))$, and for each $y \in R_{F,\theta}(h)$ define $L_{F,\theta}^y(h) = G_F(h) \cap l_{y,\theta}^\perp$.

We have the following theorem :

Theorem 4.2 *Suppose that assumptions (A1)-(A3) hold.*

(a) Almost surely for all $h \in J_F = \{h > 0 : \tau_F^*(h) > 0\}$,

$$\begin{aligned} \dim_H G_F(h) &= \left(\frac{\tau_F^*(h)}{h} \wedge \left(\tau_F^*(h) + 1 - h\right) \right) \vee \tau_F^*(h), \\ \dim_H R_F(h) &= \frac{\tau_F^*(h)}{h} \wedge 1. \end{aligned}$$

Moreover, denote G_F the whole graph, then almost surely

$$\dim_H G_F = \dim_P G_F = \dim_B G_F = 1 - \tau_F(1).$$

(b) Almost surely for Lebesgue almost every $\theta \in (-\pi/2, \pi/2)$, for all $h \in (0, 1)$ such that $\tau_F^*(h) - h > 0$, for $\mu_{h,\theta}^R$ almost every $y \in R_{F,\theta}(h)$,

$$\dim_H L_{F,\theta}^y(h) = \tau_F^*(h) - h,$$

where $\mu_{h,\theta}^R$ is a positive Borel measure carried by $R_{F,\theta}(h)$ and it is absolutely continuous with respect to the one-dimensional Lebesgue measure on l_θ.

The rest of this chapter is organized as follows : in Section 4.2 we prove Theorem 4.2 with two intermediate results : Theorem 4.3 and Theorem 4.4, whose proofs are postponed to Section 4.4 ; in Section 4.3 we prove Theorem 4.1 and finally in Section 4.5 we prove Proposition 4.4, which is our essential tool for proving Theorem 4.3 and Theorem 4.4.

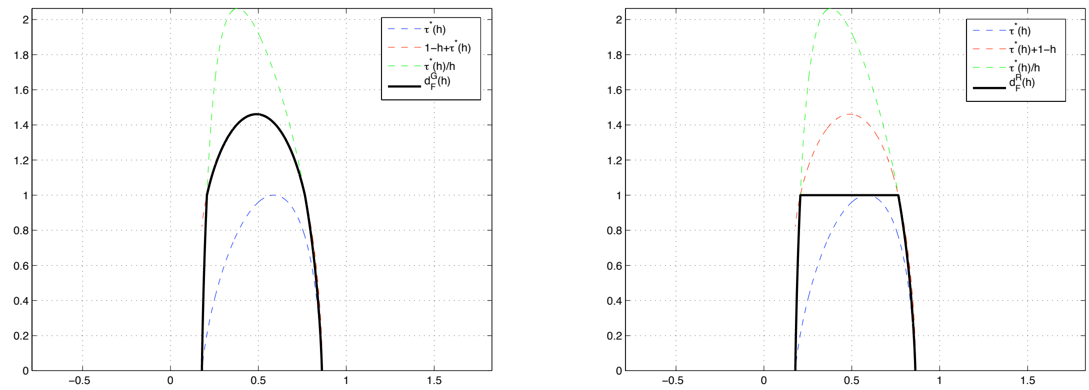

FIGURE 4.3 – d_F^G (Left) and d_F^R (Right) in case of $\sup J_F < 1$.

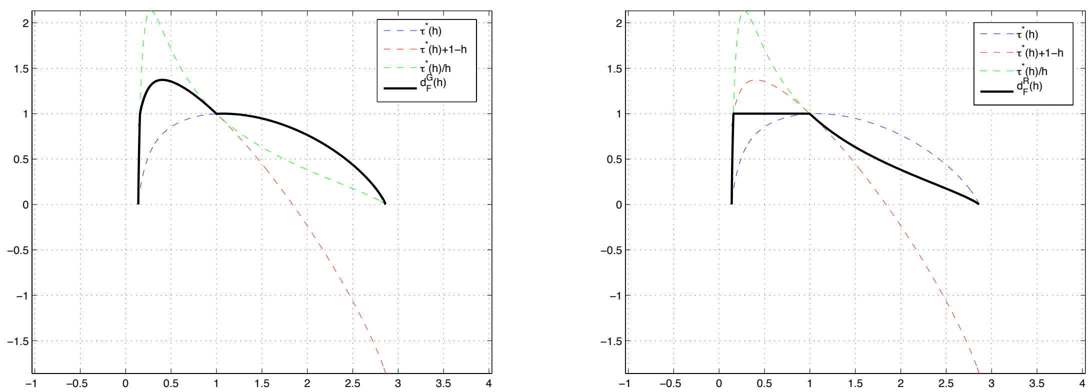

FIGURE 4.4 – d_F^G (Left) and d_F^R (Right) in case of $\sup J_F > 1$.

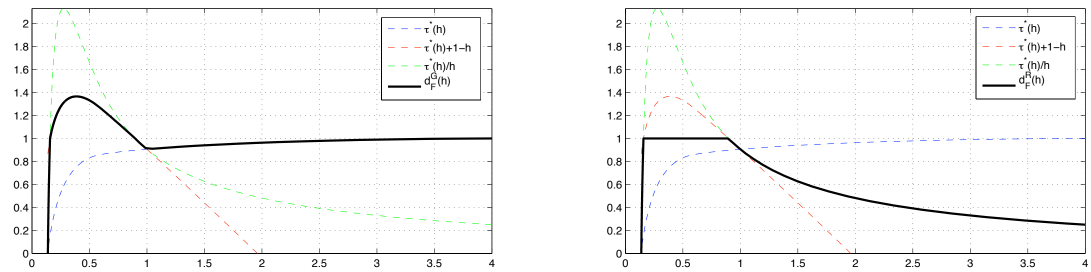

FIGURE 4.5 – d_F^G (Left) and d_F^R (Right) in case of $\sup J_F = \infty$.

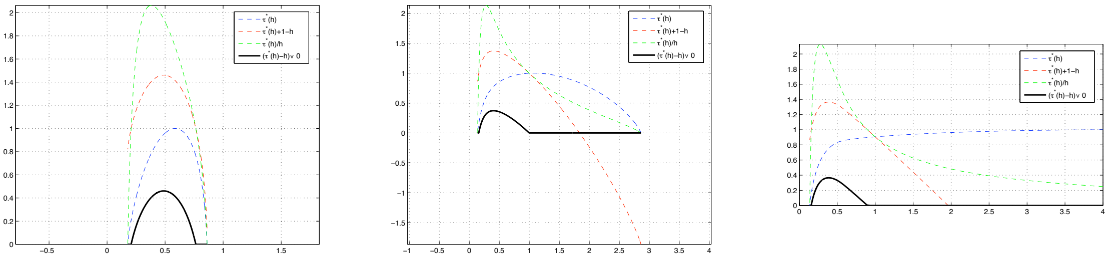

FIGURE 4.6 – Function $(\tau^*(h) - h) \vee 0$ in case of $\sup J_F < 1$ (Left), $\sup J_F > 1$ (Middle) and $\sup J_F = \infty$ (Right).

4.2 Proof of Theorem 4.2.

4.2.1 Proof of Theorem 4.2(a).

Since using μ_q is successful to describe the singularity spectrum of F, it is also worth trying to use these measures to study the graph and range singularity spectra of F.

For each $q \in J$, like μ_q^D, associated with F_L and F_W we can induce :
– a measure μ_q^G carried by the graph : for any Borel set $A \subset \mathbb{R}^2$,

$$\mu_q^G(A) = \mu_q\Big(\big\{t \in \mathscr{A}^{\mathbb{N}_+} : \big(F_L \circ \lambda(t), F_W \circ \lambda(t)\big) \in A\big\}\Big);$$

– a measure μ_q^R carried by the range : for any Borel set $A \subset \mathbb{R}$,

$$\mu_q^R(A) = \mu_q\Big(\big\{t \in \mathscr{A}^{\mathbb{N}_+} : F_W \circ (t) \in A\big\}\Big).$$

We focus on the lower Hausdorff dimension of the measures μ_q^G and μ_q^R. We show that these dimensions provide the graph and range singularity spectra. Our approach is based on the estimation of the energy of these two measures restricted on suitable random sets (see Remark 4.3). As an essential intermediate result, we have the following theorem :

Theorem 4.3 *Suppose that (A1)-(A3) hold. With probability* 1*, for all* $q \in J$ *we have* $\dim_H(\mu_q^G) = \gamma^G(q)$ *and* $\dim_H(\mu_q^R) = \gamma^R(q)$*, where*

$$\gamma^G(q) = \left(\frac{\tau^*(\tau'(q))}{\tau'(q)} \wedge \big(\tau^*(\tau'(q)) + 1 - \tau'(q)\big)\right) \vee \tau^*(\tau'(q)), \quad (4.11)$$

$$\gamma^R(q) = \frac{\tau^*(\tau'(q))}{\tau'(q)} \wedge 1. \quad (4.12)$$

Remark 4.2 *Notice that for* $q \in J$ *we have* $\dim_H(\mu_q^D) = \tau^*(\tau'(q))$*, and we can write*

$$\gamma^G(q) = \tau^*(\tau'(q)) + \gamma^R(q) \cdot (1 - \tau'(q)) \vee 0.$$

So Theorem 4.3 actually provides us with a Ledrappier-Young like formula [98] for the uncountable family of statistically self-similar measure $(\mu_q)_{q\in J}$ uniformly : with probability 1, for all $q \in J$,

$$\dim_H(\mu_q^G) = \dim_H(\mu_q^D) + \dim_H(\mu_q^R) \cdot (1-\tau'(q)) \vee 0.$$

Similar formula also appear in Theorem 12 in [27], Theorem 3 in [59] and Corollary 5.2 in [8] in the study of the Hausdorff dimension of self-affine measures and sets.

Remark 4.3 *It is worth noting that to prove Theorem 4.3 we are forced to calculate the energy of μ_q restricted to suitable Cantor-like random sets (see Section 4.4.1). If we did not use this restriction, for example for the measure μ_q^G in the case where its dimension is greater than 1, we would have to estimate the expectation of*

$$\iint_{s,t\in\mathscr{A}^{\mathbb{N}_+}} \frac{\mathrm{d}\mu_q(s)\mathrm{d}\mu_q(t)}{(|F_L\circ\lambda(s)-F_L\circ\lambda(t)|^2+|F_W\circ\lambda(s)-F_W\circ\lambda(t)|^2)^{\gamma/2}},\ \gamma>1,$$

which turns out to be finite only if $\Phi(2q-1,\gamma-1+2\tau(q))<1$, which is equivalent to saying that $\gamma < 1+\tau(2q-1)-2\tau(q)$. So the best lower bound we would get is :

$$\dim_H(\mu_q^G) \geq 1+\tau(2q-1)-2\tau(q). \tag{4.13}$$

Comparing this value with the exact dimension $1+(q-1)\tau'(q)-\tau(q)$, we find that (4.13) always provides a strick lower bound unless $q=1$. Thus, such an approach only provides the Hausdorff dimension of the whole graph.

Since μ_q^D is carried by the set $E_F(\tau'(q))$, by definition the measure μ_q^S is carried by the set $S_F(E_F(\tau'(q)))$ for $S\in\{G,R\}$. Then combining the results in Proposition 4.1, Proposition 4.2 and Theorem 4.3, we prove the results on the singularity spectra part of Theorem 4.2(a).

For the result on the dimension of the whole graph, let $I=[0,F_L(1)]$. For each $n\geq 1$ we divide I into b^n semi-open to the right intervals of the same length denoted by $I_{n,k}$, for $k=1,\cdots,b^n$.

Recall that $\mathrm{Osc}_F(I_{n,k})=\sup_{x,y\in I_{n,k}}|F(x)-F(y)|$, so for each interval $I_{n,k}$ we will need at most $[\frac{\mathrm{Osc}_F(I_{n,k})}{|I_{n,k}|}]+1$ many squares whose side length is $|I_{n,k}|$ to cover $G_F(I_{n,k})$. Then, by definition of the upper box-counting dimension and the definition of τ_F in (4.10) we get

$$\overline{\dim}_B G_F \leq \limsup_{n\to\infty} \frac{\log\sum_{j=1}^{b^n}([\frac{\mathrm{Osc}_F(I_{n,k})}{|I_{n,k}|}]+1)}{-\log(b^{-n}\cdot F_L(1))} \leq 1+(-\tau_F(1))\vee 0.$$

From Proposition 4.2 we know that almost surely $\tau_F(1)=\tau(1)\leq 0$ and applying Theorem 4.3 to $q=1$ we get with probability 1,

$$\dim_H G_F \geq \dim_H G_F(\tau_F'(1)) = 1-\tau_F(1).$$

Consequently, with probability 1,

$$\dim_H G_F = \dim_P G_F = \dim_B G_F = 1-\tau_F(1).$$

Remark 4.4 *If we consider the exponent*

$$\tilde{h}_F(x) = \lim_{r\to 0^+} \frac{1}{\log r} \log \mathrm{Osc}_f(B(x,r))$$

whenever the limit exisits, and consider the following smaller iso-Hölder sets

$$\widetilde{E}_f(h) := \{x \in I : \tilde{h}_f(x) = h\},$$

then we claim that the results in Theorem 4.2(a) also works for the packing dimension if we replace the mono-Hölder set $E_f(h)$ *by the set* $\widetilde{E}_f(h)$*, since in this case we have* $\dim_P \widetilde{E}_f(h) \leq \tau_f^*(h)$ *for* $h > 0$ *and, moreover, under the assumption* $\mathbb{P}(\forall j, |W_j| > 0) = 1$ *in (A3), the measure* μ_q^D *is actually carried by the set* $\widetilde{E}_F(\tau'(q))$ *for* $q \in J$ *(see [16]).*

4.2.2 Proof of Theorem 4.2(b).

To get measures on the level sets, in the same spirit as when one constructs the local times of certain stochastic processes, we could disintegrate the measures μ_q^G with respect to μ_q^R in order to obtain Radon measures μ_q^y carried by L_F^y for μ_q^R-almost every y, but such a disintegration turns out to be difficult to study. The reason is that the energy method we use does not provide the exact gauge function needed to describe the density of the measure μ_q^R with respect to Lebesgue. It only yields the lower Hausdorff dimension of μ_q^R. However, inspired by what is done in [125] to calculate the Hausdorff dimension of the level sets of Gaussian process by using classical Martrand theorem, and in [116] to deal with the the Hausdorff dimension of slices of sets, it is possible to solve this problem for Lebesgue almost every direction.

For $h > 0$ and $\theta \in (-\pi/2, \pi/2)$, recall that $R_{F,\theta}(h) = \mathrm{Proj}_\theta(G_F(h))$, and for each $y \in R_{F,\theta}(h)$ recall that $L_{F,\theta}^y(h) = G_F(h) \cap l_{y,\theta}^\perp$.

For $q \in J$, let $\mu_{q,\theta}^R$ be the orthogonal projection of the measure μ_q^G onto l_θ : $\mu_{q,\theta}^R(A) = \mu_q^G \circ \mathrm{Proj}_\theta^{-1}(A)$ for any Borel set $A \subset l_\theta$. Since μ_q^G is carried by $G_F(\tau'(q))$, so $\mu_{q,\theta}^R$ is carried by $R_{F,\theta}(\tau'(q))$.

We have the following theorem :

Theorem 4.4 *Suppose that (A1)-(A3) hold. With probability 1, for Lebesgue almost every* $\theta \in (-\pi/2, \pi/2)$*, for all* $q \in J$ *such that* $\dim_H(\mu_q^G) = \gamma^G(q) > 1$ *:*

(a) The projected measure $\mu_{q,\theta}^R$ *is absolutely continuous with respect to the one-dimensional Lebesgue measure on the* l_θ*.*

(b) For $\mu_{q,\theta}^R$*-almost every* $y \in l_\theta$*, the following limit :*

$$\lim_{r\to 0^+} \frac{1}{r} \int_{x\in\mathbb{R}^2, |x - l_{y,\theta}^\perp| \leq r} \psi(x)\ \mathrm{d}\mu_q^G(x)$$

exists for any continuous function $\psi : \mathbb{R}^2 \mapsto \mathbb{R}_+$*, so it defines a measure* $\mu_{q,\theta}^y$ *carried by* $L_{F,\theta}^y(\tau'(q))$*.*

(c) There exists a random set $R_{q,\theta} \subset R_{F,\theta}(\tau'(q))$ of full $\mu^R_{q,\theta}$-measure such that for any $y \in R_{q,\theta}$, the measure $\mu^y_{q,\theta}$ has lower Hausdorff dimension

$$\dim_H(\mu^y_{q,\theta}) = \dim_H(\mu^G_q) - 1 = \tau^*(\tau'(q)) - \tau'(q).$$

Theorem 4.2(b) is almost a direct consequence of Theorem 4.1(b) and Theorem 4.4, we only remark that $\gamma^G(q) > 1$ if and only if $\tau'(q) < 1$ and $\tau^*(\tau'(q)) - \tau'(q) > 0$.

Remark 4.5 *As mentioned in Remark 4.1, the condition "φ_W is finite on $\mathbb{R}$" can be weakened to "there exists $q < -1$ such that $\varphi_W(q)$ is finite". Under this weaker assumption, the results in Theorem 4.2 will still hold, but only for $h \in \{h > 0 : \tau^*(h) > 0\}$. The reason why we cannot conclude for $\{h > 0 : \tau^*_F(h) > 0\}$ is that under this weaker assumption we do not know the value of τ_F outside the interval $J = \{q : q\tau'(q) - \tau(q) > 0\}$. But if we assume "there exist $\underline{q} < \overline{q} \in J$ such that $\underline{q}\tau'(\underline{q}) - \tau(\underline{q}) = \overline{q}\tau'(\overline{q}) - \tau(\overline{q}) = 0$", then we will also obtain the same results as in Theorem 5.1.*

4.3 Proof of Theorem 4.1.

4.3.1 Results on the packing dimension.

Proof Without loss of generality we suppose $I = [0, 1]$. For any $x \in [0, 1]$ and $n \geq 1$, we denote by $I_{x,n}$ the unique interval $[k \cdot 2^{-n}, (k+1) \cdot 2^{-n})$, $k \in \mathbb{Z}$ that contains x.

Let $\{E_i \subset E, i \geq 1\}$ be any countable covering of E. The families $\{R_i = R_f(E_i) \subset R_f(E), i \geq 1\}$ and $\{G_i = G_f(E_i) \subset G_f(E, i \geq 1\}$ are countable coverings of $R_f(E)$ and $G_f(E)$ respectively, so

$$\dim_P R_f(E) = \sup_i \dim_P R_i, \quad \dim_P G_f(E) = \sup_i \dim_P G_i.$$

For any $i \geq 1$, $k \geq 0$ and $\epsilon > 0$ with $h - \epsilon > 0$ we define

$$E_{i,k} = \{x \in E_i : \forall\, n \geq k, O_f(I_{x,n}) \leq |I_{x,n}|^{h-\epsilon}\},$$

as well as $R_{i,k} = R_f(E_{i,k})$ and $G_{i,k} = G_f(E_{i,k})$. By the fact that for any $x \in E_i \subset E$,

$$\liminf_{n\to\infty} \frac{\log O_f(I_{x,n})}{\log |I_{x,n}|} \geq h_f(x) \geq h,$$

we have $\bigcup_{k\geq 0} E_{i,k} = E_i$, thus $\bigcup_{k\geq 0} R_{i,k} = R_i$ and $\bigcup_{k\geq 0} G_{i,k} = G_i$.

For $n \geq 1$ and $A \subset \mathbb{R}^2$, let $N_n(A)$ be the minimal number of dyadic squares of the form $[k \cdot 2^{-n}, (k+1) \cdot 2^n) \times [l \cdot 2^{-n}, (l+1) \cdot 2^{-n})$, $0 \leq k < 2^n - 1$, $l \in \mathbb{Z}$ necessary to cover A. By the construction of $E_{i,k}$, for any $n \geq k$ we have

$$N_n(R_{i,k}) \leq 2 \cdot N_{[\frac{n}{h-\epsilon}]+1}(E_{i,k}),\ N_n(G_{i,k}) \leq \begin{cases} 2^{n\cdot((1-h+\epsilon)\vee 0)} \cdot N_n(E_{i,k}); \\ 2 \cdot N_{[\frac{n}{(h-\epsilon)\wedge 1}]+1}(E_{i,k}). \end{cases}$$

This gives us

$$\overline{\dim}_B R_{i,k} \leq \frac{1}{h-\epsilon} \overline{\dim}_B E_{i,k},\ \overline{\dim}_B G_{i,k} \leq \begin{cases} (1-h+\epsilon) \vee 0 + \overline{\dim}_B E_{i,k}; \\ \overline{\dim}_B E_{i,k} / ((h-\epsilon) \wedge 1). \end{cases}$$

We know that the packing dimension is equal to the upper modified box-counting dimension (see Chapter 3, [51]), that is, for any set $A \subset \mathbb{R}^2$,

$$\dim_P A = \overline{\dim}_{MB} A := \inf \left\{ \sup_i \overline{\dim}_B A_i : A \subset \bigcup_i A_i \right\},$$

where the infimum is taken over all possible countable coverings of A. Since $\{R_{i,k}, k \geq 0\}$ and $\{G_{i,k}, k \geq 0\}$ are countable coverings of R_i and G_i, we have

$$\begin{aligned} \dim_P R_i &= \overline{\dim}_{MB} R_i \leq \sup_k \overline{\dim}_B R_{i,k} \leq \frac{1}{h-\epsilon} \sup_k \overline{\dim}_B E_{i,k}; \\ \dim_P G_i &= \overline{\dim}_{MB} G_i \leq \sup_k \overline{\dim}_B G_{i,k} \leq \begin{cases} (1-h+\epsilon) \vee 0 + \sup_k \overline{\dim}_{\mathbf{B}} E_{i,k}; \\ \sup_k \overline{\dim}_B E_{i,k}/((h-\epsilon) \wedge 1). \end{cases} \end{aligned}$$

Since for any $k \geq 0$ we have $E_{i,k} \subset E_i$ so that $\sup_k \overline{\dim}_B E_{i,k} \leq \overline{\dim}_B E_i$. Then we have shown that for any countable covering $\{E_i, i \geq 1\}$,

$$\begin{aligned} \dim_P R_f(E) &= \sup_i \dim_P R_i \leq \frac{1}{h-\epsilon} \sup_i \overline{\dim}_B E_i; \\ \dim_P G_f(E) &= \sup_i \dim_P G_i \leq \begin{cases} (1-h+\epsilon) \vee 0 + \sup_i \overline{\dim}_B E_i; \\ \sup_i \overline{\dim}_B E_i/((h-\epsilon) \wedge 1). \end{cases} \end{aligned}$$

By taking the infimum over all the possible $\{E_i, i \geq 1\}$ we get

$$\dim_P R_f(E) \leq \frac{1}{h-\epsilon} \dim_P E, \ \dim_P G_f(E) \leq \begin{cases} (1-h+\epsilon) \vee 0 + \dim_P E; \\ \dim_P E/((h-\epsilon) \wedge 1). \end{cases}$$

Letting ϵ tend to 0 yields the conclusion.

4.3.2 Results on the Hausdorff dimension.

Proof We fix $\epsilon \in (0, h)$ and $\theta \in (-\pi/2, \pi/2)$. Recall that Proj_θ is the orthogonal projection of $\mathbb{R}^2$ onto l_θ and $R_{f,\theta}(E) = \text{Proj}_\theta(G_f(E))$.

For $x \in \mathbb{R}$, $r > 0$ and $k \in \mathbb{Z}$ we define the interval $I_k(x,r) = [x+(2k-1)r, x+(2k+1)r]$ and the square $Q_k(x,r) = [x-r, x+r] \times I_k(f(x), r)$.

For any $r > 0$ let $n(r) = [r^{h-\epsilon-1}] + 1$ and $\mathcal{N}(r) = \{k \in \mathbb{Z} : 2|k| \leq [r^{h-\epsilon-1}]\}$. We have $\max\{r, r^{h-\epsilon}\} \leq n(r) \cdot r$ and $\#\mathcal{N}(r) \leq n(r)$.

For any $x \in E$ and $r > 0$ define the family $\mathcal{Q}(x,r) = \Big\{ Q_k(x,r) : k \in \mathcal{N}(r) \Big\}$ and for any $y \in R_{f,\theta}(E)$ define the family

$$\mathcal{Q}_\theta^y(x,r) = \Big\{ Q_k(x,r) : k \in \mathcal{N}(r), \ Q_k(x,r) \cap l_{y,\theta}^\perp \neq \emptyset \Big\}.$$

By simple calculation we know $\#\mathcal{Q}_\theta^y(x,r) \leq |\tan\theta| + 1$.

Let d stand for $\dim_H E$. By definition of the Hausdorff dimension we can find a decreasing sequence $(\delta_i)_{i \geq 1}$ tending to 0 and for each $i \geq 1$,

$$\mathscr{B}_i := \left\{ B_j^{(i)} = B\left(x_j^{(i)}, r_j^{(i)}\right) \right\}_{j \in \mathcal{J}_i},$$

a countable δ_i-covering of E such that $x_j^{(i)} \in E$ for $j \in \mathcal{J}_i$, and $\sum_{j\in\mathcal{J}_i}(r_j^{(i)})^{d+\epsilon} \leq 2^{-i}$.

For $x \in E$ and $r > 0$ denote the rectangle :

$$R(x,r) = B(x,r) \times \big[f(x) - n(r)\cdot r, f(x) + n(r)\cdot r\big],$$

and for any $B_j^{(i)} = B\left(x_j^{(i)}, r_j^{(i)}\right) \in \mathscr{B}_i$, we use the convention $R_j^{(i)} = R\left(x_j^{(i)}, r_j^{(i)}\right)$.

Let μ be a positive Borel measure defined on l_θ. For $\gamma > 0$ recall that

$$R_{f,\theta}^{\mu,\gamma}(E) = \{y \in R_{f,\theta}(E) : h_\mu(y) \geq \gamma\}.$$

Suppose that $\mu(R_{f,\theta}^{\mu,\gamma}(E)) > 0$. Then define a subset of $\mathcal{J}_i$:

$$\mathcal{J}_{i,\theta}^{\mu,\gamma} = \left\{j \in \mathcal{J}_i : \mu\Big(\mathrm{Proj}_\theta\big(R_j^{(i)}\big)\Big) \leq \Big(\big(2n(r_j^{(i)})\cdot\cos\theta + |\sin\theta|\big)\cdot r_j^{(i)}\Big)^{\gamma-\epsilon}\right\}.$$

We have the following lemma :

Lemma 4.1 *For any $N \geq 1$, let $\mathscr{C}_N^R = \bigcup_{i\geq N}\bigcup_{j\in\mathcal{J}_i}\left\{I_0(f(x_j^{(i)}), (r_j^{(i)})^{h-\epsilon})\right\}$,*

$$\mathscr{C}_N^G = \bigcup_{i\geq N}\bigcup_{j\in\mathcal{J}_i}\mathcal{Q}(x_j^{(i)}, r_j^{(i)}),\ \ \widetilde{\mathscr{C}_N^G} = \bigcup_{i\geq N}\bigcup_{j\in\mathcal{J}_i}\left\{Q_0(x_j^{(i)}, n(r_j^{(i)})\cdot r_j^{(i)})\right\},$$

and for any $y \in R_{f,\theta}^{\mu,\gamma}(E)$ let $\mathscr{C}_{\theta,N}^y = \bigcup_{i\geq N}\bigcup_{j\in\mathcal{J}_{i,\theta}^{\mu,\gamma}}\mathcal{Q}_\theta^y(x_j^{(i)}, r_j^{(i)})$.

Then $\mathscr{C}_N^R$, $\mathscr{C}_N^G$, $\widetilde{\mathscr{C}_N^G}$ and $\mathscr{C}_{\theta,N}^y$ form respectively a $(\delta_N)^{h-\epsilon}$-covering of $R_f(E)$, a δ_N-covering of $G_f(E)$, an $n(\delta_N)\cdot\delta_N$-covering of $G_f(E)$, and a δ_N-covering of $L_{f,\theta}^y(E)$.

Proof Fix $N \geq 1$. For any $x \in E$, since for any $i \geq N$ there are balls in $\mathscr{B}_i$ covering x and $\delta_i \searrow 0$, we can find a sequence of balls $\{B_l = B(x_l, r_l)\}_{l\geq 1} \subset \bigcup_{i\geq N}\mathscr{B}_i$ such that $x \in B_l$ for all $l \geq 1$ and $r_l \searrow 0$ as $l \to \infty$. For each $l \geq 1$, let $\bar{r}_l = |x - x_l| \leq r_l$ and $\bar{r}_l' = (2n(r_l)\cos\theta + |\sin\theta|)\cdot r_l$. Since

$$\liminf_{l\to\infty}\frac{\log O_f(B(x,\bar{r}_l))}{\log\bar{r}_l} \geq h_f(x) \geq h \ \text{ and}$$

$$\liminf_{l\to\infty}\frac{\log\mu\Big(l_\theta \cap B\big(\mathrm{Proj}_\theta(f(x)), \bar{r}_l'\big)\Big)}{\log\bar{r}_l'} \geq h_\mu\big(\mathrm{Proj}_\theta(f(x))\big) \geq \gamma,$$

if $\mathrm{Proj}_\theta(f(x)) \in R_{f,\theta}^{\mu,\gamma}(E)$, so we can find l_* (depending on x) such that for all $l \geq l_*$,

$$O_f(B(x,\bar{r}_l)) \leq (\bar{r}_l)^{h-\epsilon} \ \text{ and } \ \mu\Big(l_\theta \cap B\big(\mathrm{Proj}_\theta(f(x)), \bar{r}_l'\big)\Big) \leq (\bar{r}_l')^{\gamma-\epsilon}.$$

Now for any $l \geq l_*$ we have

$$|f(x_l) - f(x)| \leq O_f(B(x,\bar{x}_l)) \leq (\bar{r}_l)^{h-\epsilon} \leq (r_l)^{h-\epsilon} \leq n(r_l)\cdot r_l.$$

This implies that :

- $x \in B(x_l, r_l)$;
- $f(x) \in I_0(f(x_l), (r_l)^{h-\epsilon}) \subset I_0(f(x_l), n(r_l)r_l)$;

– $(x, f(x)) \in Q_0(x_l, n(r_l) \cdot r_l)$;
– $l_\theta \cap B\big(\mathrm{Proj}_\theta(f(x)), \bar{r}'_l\big) \supset \mathrm{Proj}_\theta(R(x_l, r_l))$,
which gives us the conclusion.

Now we are going to show that the coverings constructed in Lemma 4.1 lead to the expected upper bounds. In order to simplify the proof, we use the convention $|Q| = \frac{1}{2}\sup_{x,y\in Q}|x-y|$, the half-diameter of the set Q.

(i) Since we took $\epsilon \in (0, h)$ we have

$$\sum_{Q\in\mathscr{C}_N^R} |Q|^{\frac{d+\epsilon}{h-\epsilon}} = \sum_{i\geq N}\sum_{j\in\mathcal{J}_i}((r_j^{(i)})^{h-\epsilon})^{\frac{d+\epsilon}{h-\epsilon}} = \sum_{i\geq N}\sum_{j\in\mathcal{J}_i}(r_j^{(i)})^{d+\epsilon} \leq 2^{-N+1}.$$

(ii) If $h > 1$, if we take ϵ small enough so that $h - \epsilon > 1$, then $n(r) = 1$ for all $r < 1$, and for N large enough so that $\delta_N < 1$,

$$\sum_{Q\in\widetilde{\mathscr{C}_N^G}} |Q|^{d+\epsilon} = \sum_{i\geq N}\sum_{j\in\mathcal{J}_i}(n(r_j^{(i)})\cdot r_j^{(i)})^{d+\epsilon} = \sum_{i\geq N}\sum_{j\in\mathcal{J}_i}(r_j^{(i)})^{d+\epsilon} \leq 2^{-N+1}.$$

(iii) If $h \leq 1$, then $h - \epsilon - 1 \leq -\epsilon < 0$, thus for $r < 1$ we have $r^{h-\epsilon-1} > 1$, and this implies that $n(r) \leq 2\cdot r^{h-\epsilon-1}$, hence for N large enough so that $\delta_N < 1$,

$$\sum_{Q\in\widetilde{\mathscr{C}_N^G}} |Q|^{\frac{d+\epsilon}{h-\epsilon}} = \sum_{i\geq N}\sum_{j\in\mathcal{J}_i}(n(r_j^{(i)})\cdot r_j^{(i)})^{\frac{d+\epsilon}{h-\epsilon}} \leq 2^{\frac{d+\epsilon}{h-\epsilon}}\cdot\sum_{i\geq N}\sum_{j\in\mathcal{J}_i}(r_j^{(i)})^{d+\epsilon} \leq 2^{\frac{d+\epsilon}{h-\epsilon}}\cdot 2^{-N+1}.$$

(iv) If $h \leq 1$, for the same reason as (c), for N large enough such that $\delta_N < 1$,

$$\begin{aligned}\sum_{Q\in\mathscr{C}_N^G} |Q|^{d+1-h+2\epsilon} &= \sum_{i\geq N}\sum_{j\in\mathcal{J}_i}\sum_{k\in\mathcal{N}(r_j^{(i)})}|Q_k(x_j^{(i)}, r_j^{(i)})|^{d+1-h+2\epsilon}\\ &\leq \sum_{i\geq N}\sum_{j\in\mathcal{J}_i} n(r_j^i)\cdot(r_j^{(i)})^{d+1-h+2\epsilon} \leq 2\cdot\sum_{i\geq N}\sum_{j\in\mathcal{J}_i}(r_j^{(i)})^{d+\epsilon} \leq 2^{-N+2}.\end{aligned}$$

Now by letting N tend to infinity and then ϵ to 0 we obtain the desired upper bound for $\dim_H R_f(E)$ and $\dim_H G_f(E)$.

Next we prove the upper bound for the Hausdorff dimension of the level sets.

If $s = d + \epsilon - (\gamma - \epsilon)(h - \epsilon) > 0$ and $h \leq 1$, then for any N large enough we have

$$\begin{aligned}&\int_{y\in R_{f,\theta}^{\mu,\gamma}(E)}\sum_{Q\in\mathscr{C}_{\theta,N}^y}|Q|^s \mathrm{d}\mu(y)\\ \leq\ &\sum_{i\geq N}\sum_{j\in\mathcal{J}_{i,\theta}^{\mu,\gamma}}\sum_{Q\in\mathcal{Q}_\theta^y(x_j^{(i)}, r_j^{(i)})}|Q|^s\cdot\mu\Big(\mathrm{Proj}_\theta(Q)\Big)\\ \leq\ &\sum_{i\geq N}\sum_{j\in\mathcal{J}_{i,\theta}^{\mu,\gamma}}(r_j^{(i)})^s\cdot(|\tan\theta|+1)\cdot\mu\Big(\mathrm{Proj}_\theta\big(R(x_j^{(i)}, r_j^{(i)})\big)\Big)\\ &\qquad\Big(\text{since } |Q| = r_j^{(i)} \text{ and } \#\mathcal{Q}_\theta^y(x_j^{(i)}, r_j^{(i)}) \leq |\tan\theta| + 1\Big)\end{aligned}$$

$$\begin{aligned}
&\leq (|\tan\theta|+1)\sum_{i\geq N}\sum_{j\in\mathcal{J}_{i,\theta}^{\mu,\gamma}}(r_j^{(i)})^s\cdot\Big(\big(2n(r_j^{(i)})\cdot\cos\theta+|\sin\theta|\big)\cdot r_j^{(i)}\Big)^{\gamma-\epsilon}\\
&\leq C\sum_{i\geq N}\sum_{j\in\mathcal{J}_{i,\theta}^{\mu,\gamma}}(r_j^{(i)})^s\cdot\big((r_j^i)^{h-\epsilon-1}\cdot r_j^{(i)}\big)^{\gamma-\epsilon}\ \Big(\text{we used } h\leq 1 \text{ and } N \text{ large enough}\Big)\\
&= C\sum_{i\geq N}\sum_{j\in\mathcal{J}_{i,\theta}^{\mu,\gamma}}(r_j^{(i)})^{d+\epsilon}\leq C\sum_{i\geq N}\sum_{j\in\mathcal{J}_i}(r_j^{(i)})^{d+\epsilon}\leq 2^{-N+1+2\gamma},
\end{aligned}$$

where $C=2^{2(\gamma-\epsilon)}(|\tan\theta|+1)$. Due to Borel-Cantelli lemma we get for μ-almost every $y\in R_{f,\theta}^{\mu,\gamma}(E)$,

$$\dim_H L_{f,\theta}^y(E)\leq s=d+\epsilon-(\gamma-\epsilon)(h-\epsilon).$$

Applying this with a sequence $(\epsilon_n)_{n\geq 1}\searrow 0$ we get the conclusion.

4.4 Proof of Theorem 4.3 and Theorem 4.4.

From now on we assume that (A1)-(A3) hold.

4.4.1 Cantor-like subsets of $\mathscr{A}^{\mathbb{N}_+}$ carrying μ_q

For any $u\in\mathscr{A}^*$, define the b-adic interval $I_u=\{\pi(t):t\in[u]\}$. By construction we know that the limit functions F_W and F_L satisfy the following functional equation : For any $u\in\mathscr{A}^*$, $s,t\in I_u$ and $U\in\{W,L\}$,

$$F_U(s)-F_U(t)=U_u\cdot\Big(F_U^{[u]}(b^{|u|}\cdot(s-\pi(u)))-F_U^{[u]}(b^{|u|}\cdot(t-\pi(u)))\Big).\qquad(4.14)$$

For $u\in\mathscr{A}^*$ and $U\in\{W,L\}$ we define the oscillations $O_U(u)=\mathrm{Osc}_{F_U^{[u]}}([0,1])$. Then from (4.14) we deduce that for any $u\in\mathscr{A}^*$ and $U\in\{W,L\}$,

$$\mathrm{Osc}_{F_U}(I_u)=U_u\cdot O_U(u).\qquad(4.15)$$

Denote by u^- (resp. u^+) the unique element of $\mathscr{A}^{|u|}$ such that $\pi(u^-)=\pi(u)-b^{-|u|}$ (resp. $\pi(u^+)=\pi(u)+b^{-|u|}$) whenever $\pi(u)\neq 0$ (resp. $\pi(u)\neq 1-b^{-|u|}$).

Recall (4.4), the definition of $\xi(q)$ and $\widetilde{\xi}(q)$. For any $q\in J$, $\epsilon>0$, $u,v\in\mathscr{A}^*$ we define the following subsets of Ω :

$$\left\{\begin{array}{l}
\mathscr{W}_v^{[u]}(q,\epsilon)=\Big\{\omega\in\Omega\ :\ \Big\{W_v(u),\ W_{v^-}(u),\ W_{v^+}(u)\Big\}\subset\big[e^{-|v|(\xi(q)+\epsilon)},\ e^{-|v|(\xi(q)-\epsilon)}\big]\Big\};\\
\mathscr{L}_v^{[u]}(q,\epsilon)=\Big\{\omega\in\Omega\ :\ \Big\{L_v(u),\ L_{v^-}(u),\ L_{v^+}(u)\Big\}\subset\big[e^{-|v|(\widetilde{\xi}(q)+\epsilon)},\ e^{-|v|(\widetilde{\xi}(q)-\epsilon)}\big]\Big\};\\
\mathscr{O}_v^{[u]}(\epsilon)=\Big\{\omega\in\Omega\ :\ \Big\{O_U(uv),\ O_U(uv^-),\ O_U(uv^+),U\in\{L,W\}\Big\}\subset\big[e^{-|v|\epsilon},\ e^{|v|\epsilon}\big]\Big\}.
\end{array}\right.\qquad(4.16)$$

For $q\in J$, $\epsilon>0$, $u,v\in\mathscr{A}^*$ we define the indicator function :

$$\mathbf{1}_v^{[u]}(q,\epsilon)=\mathbf{1}_{\mathscr{L}_v^{[u]}(q,\epsilon)\cap\mathscr{W}_v^{[u]}(q,\epsilon)\cap\mathscr{O}_v^{[u]}(\epsilon)}.\qquad(4.17)$$

and for $q \in J$ and $\epsilon > 0$ we define the random subset of $\mathscr{A}^{\mathbb{N}_+}$:

$$\mathscr{A}_n^{\mathbb{N}_+}(q,\epsilon) = \{u \in \mathscr{A}^{\mathbb{N}_+} : \mathbf{1}_{u|_n}^{[\varnothing]}(q,\epsilon) = 1\},\ \mathscr{A}_n^{\mathbb{N}_+}(q,\epsilon)^c = \mathscr{A}^{\mathbb{N}_+} \setminus \mathscr{A}_n^{\mathbb{N}_+}(q,\epsilon).$$

From Chapter 3 we can deduce the following proposition :

Proposition 4.3 *Let K be a compact subinterval of J. Then for any $\epsilon > 0$ there exist constants $C = C(K) > 0$ and $\delta = \delta(\epsilon, K) > 0$ such that for any $n \geq 1$,*

$$\mathbb{E}\Big(\sup_{q\in K} \sum_{u\in\mathscr{A}^n} \mu_q\big([u] \cap \mathscr{A}_n^{\mathbb{N}_+}(q,\epsilon)^c\big)\Big) \leq C \cdot n \cdot b^{-n\delta}.$$

Now for $n \geq 1$ and $\epsilon > 0$ we define the random Cantor-like sets in $\mathscr{A}^{\mathbb{N}_+}$

$$\mathcal{C}_n(q,\epsilon) = \bigcap_{p\geq n} \mathscr{A}_p^{\mathbb{N}_+}(q,\epsilon),\ \text{and } \mathcal{C}(q) = \lim_{\epsilon\to 0}\lim_{n\to\infty} \mathcal{C}_n(q,\epsilon). \tag{4.18}$$

Then we can deduce from Proposition 4.3 that, with probability 1, for all $q \in K$, μ_q is supported by $\mathcal{C}(q)$, that is, $\mu_q(\mathcal{C}(q)) = \|\mu_q\| = Y_q > 0$.

It worth noting that by construction, for any $t \in \mathcal{C}(q)$, we have

$$\lim_{n\to\infty} \frac{\log W_{t|_n}(\varnothing)}{-n} = \lim_{n\to\infty} \frac{\log W_{t|_n^-}(\varnothing)}{-n} = \lim_{n\to\infty} \frac{\log W_{t|_n^+}(\varnothing)}{-n} = \xi(q),$$

$$\lim_{n\to\infty} \frac{\log L_{t|_n}(\varnothing)}{-n} = \lim_{n\to\infty} \frac{\log L_{t|_n^-}(\varnothing)}{-n} = \lim_{n\to\infty} \frac{\log L_{t|_n^+}(\varnothing)}{-n} = \widetilde{\xi}(q),$$

$$\lim_{n\to\infty} \frac{\log O_U(t|_n)}{-n} = \lim_{n\to\infty} \frac{\log O_U(t|_n^-)}{-n} = \lim_{n\to\infty} \frac{\log O_U(t|_n^+)}{-n} = 0,\ U \in \{W, L\}.$$

Moreover, due to (4.15), the above equalities imply that

$$\lim_{r\to\infty} \frac{\log \mathrm{Osc}_F(B(\lambda(t), r))}{\log r} = \xi(q)/\widetilde{\xi}(q) = \tau'(q).$$

4.4.2 Proof of Theorem 4.3

Proof Recall that for $t \in \mathscr{A}^* \cup \mathscr{A}^{\mathbb{N}_+}$ and $U \in \{W, L\}$ we use the convention $F_U(t) = F \circ \pi(t)$.

For any $s, t \in \mathscr{A}^* \cup \mathscr{A}^{\mathbb{N}_+}$ and $\gamma > 0$ we define the Riesz-like kernels :

$$\mathcal{K}_\gamma(s,t) = \begin{cases} (|F_L(s) - F_L(t)|^2 + |F_W(s) - F_W(t)|^2)^{-\gamma/2} \vee 1, & \text{if } \gamma \geq 1; \\ |F_W(s) - F_W(t)|^{-\gamma} \vee 1, & \text{if } \gamma < 1. \end{cases} \tag{4.19}$$

Recall that the definitions of $\gamma^G(q)$ and $\gamma^R(q)$ are given in (4.11) and (4.12) respectively (see also Remark 4.6). For $q \in J$, $S \in \{G, R\}$ and $\delta > 0$ we will use the notation

$$\mathcal{K}_{q,\delta}^S(s,t) = \mathcal{K}_{\gamma^S(q)-\delta}(s,t).$$

Recall (4.18), the definition of $\mathcal{C}_n(q,\epsilon)$. For $q \in J$, $\epsilon > 0$ and $\delta > 0$ we define the n-th energy for $n \geq 1$ and $S \in \{G, R\}$ by

$$I^S_{n,\delta}(q,\epsilon) = \iint_{s,t\in\mathcal{C}_n(q,\epsilon), s\neq t} \mathcal{K}^S_{q,\delta}(s,t)\ \mathrm{d}\mu_q(s)\mathrm{d}\mu_q(t).$$

Let K be any compact subinterval of J. We assume for a while that we have proved that there exists $\delta_K > 0$ such that for any $\delta \in (0, \delta_K)$, there exists $\epsilon_\delta > 0$ such that for any $n \geq 1$, $\epsilon \in (0, \epsilon_\delta)$ and $S \in \{G, R\}$,

$$\mathbb{E}\left(\sup_{q\in K} I^S_{n,\delta}(q,\epsilon)\right) < \infty. \tag{4.20}$$

The following lemma is a slight modification of Theorem 4.13 in [51] regarding the Hausdorff dimension estimate through the potential theoretic method.

Lemma 4.2 *Let μ be a Borel measure on $\mathbb{R}^n$ and let $E \subset \mathbb{R}^n$ be a Borel set such that $\mu(E) > 0$. For any $\gamma > 0$, if*

$$\iint_{x,y\in E, x\neq y} |x-y|^{-\gamma} \vee 1\ \mathrm{d}\mu(x)\mathrm{d}\mu(y) < \infty,$$

then

$$\mu\left(\left\{x \in E : \underline{\dim}_{\mathrm{loc}}\mu(x) = \liminf_{r\to 0^+} \frac{\log \mu(B(x,r))}{\log r} < \gamma\right\}\right) = 0.$$

Then, it easily follows from Proposition 4.3, (4.20) and Lemma 4.2 that, with probability 1, for all $q \in K$:

– for μ^G_q-almost every $x \in G_q := \{(F_L(t), F_W(t)) : t \in \mathcal{C}(q)\} \subset G_F(\tau'(q))$,

$$h_{\mu^G_q}(x) = \liminf_{r\to 0^+} \frac{1}{\log r} \log \mu^G_q(B(x,r)) \geq \gamma^G(q) - \delta;$$

– for μ^R_q-almost every $y \in R_q := \{F_W(t) : t \in \mathcal{C}(q)\} \subset R_F(\tau'(q))$,

$$h_{\mu^R_q}(x) = \liminf_{r\to 0^+} \frac{1}{\log r} \log \mu^R_q(B(y,r)) \geq \gamma^R(q) - \delta.$$

We can consider a countable sequence of compact subintervals $K_n \subset J$ such that $\bigcup K_n = J$ and a corresponding sequence $\delta_n \in (0, \delta_{K_n})$. Then the above facts imply that with probability 1, for any $q \in J$ and $S \in \{G, R\}$, for μ^S_q-almost every $x \in S_q$, $h_{\mu^S_q}(x) \geq \gamma^S(q)$, hence $\dim_H(\mu^S_q) \geq \gamma^S(q)$ (we use the mass distribution principle, see [51]).

To complete the proof, we use the fact that, with probability 1, for all $q \in J$, μ^D_q is carried by the iso-Hölder set $E_F(\tau'(q))$. Then, applying Theorem 4.1 to any set $E \subset \mathrm{Supp}(\mu^D_q) \cap E_F(\tau'(q))$ yields

$$\dim_H(\mu^G_q) \ \leq\ \dim_H(\mu^D_q) \vee \left((\dim_H(\mu^D_q) + 1 - \tau'(q)) \wedge \frac{\dim_H(\mu^D_q)}{\tau'(q)}\right),$$

$$\dim_H(\mu_q^R) \;\; \leq \;\; 1 \wedge \frac{\dim_H(\mu_q^D)}{\tau'(q)},$$

and the conclusion comes from the fact that $\dim_H(\mu_q^D) = \tau^*(\tau'(q))$ for all $q \in J$.

Now we prove (4.20).

For any $\bar{q} \in K$ and $\epsilon > 0$ we define the neighborhood of $\bar{q}$ in K :

$$U_\epsilon(\bar{q}) = \left\{ q \in K : \max_{\alpha \in \{\xi, \widetilde{\xi}, \tau, \tau', \gamma, \gamma^G, \gamma^R\}} |\alpha(q) - \alpha(\bar{q})| < \epsilon \right\}. \tag{4.21}$$

By continuity of these functions, the set $U_\epsilon(\bar{q})$ is open in K.

For any $u, v \in \mathscr{A}^* \cup \mathscr{A}^{\mathbb{N}_+}$ and $p \geq 2$, we define the indicator function

$$\mathbf{1}_p(u,v) = \mathbf{1}_{\{b^{-p+1} \leq |\pi(u) - \pi(v)| < b^{-p+2}\}}.$$

For any $u \in \mathscr{A}^*$ we use the notation

$$[u]_{q,\epsilon}^n = [u] \cap \mathcal{C}_n(q,\epsilon).$$

Notice that for $q \in K$, $\delta > 0$ and $S \in \{G, R\}$ the Riez-like kernels $\mathcal{K}_{q,\delta}^S$ are all positive functions and, moreover, by the continuity of F_W and F_L we have for any $s, t \in \mathscr{A}^{\mathbb{N}_+}$, $\lim_{m\to\infty} \mathcal{K}_{q,\delta}^S(s|_m, t|_m) = \mathcal{K}_{q,\delta}^S(s,t)$. Then by applying Fatou's lemma we get

$$\begin{aligned}
I_{n,\delta}^S(q,\epsilon) &= \iint_{s,t \in \mathcal{C}_n(q,\epsilon), s \neq t} \lim_{m\to\infty} \mathcal{K}_{q,\delta}^S(s|_m, t|_m) \; \mathrm{d}\mu_q(s)\mathrm{d}\mu_q(t) \\
&= \sum_{p\geq 2} \iint_{s,t \in \mathcal{C}_n(q,\epsilon); \mathbf{1}_p(s,t)=1} \lim_{m\to\infty} \mathcal{K}_{q,\delta}^S(s|_m, t|_m) \; \mathrm{d}\mu_q(s)\mathrm{d}\mu_q(t) \\
&\leq \sum_{p\geq 2} \liminf_{m\to\infty} \iint_{s,t \in \mathcal{C}_n(q,\epsilon); \mathbf{1}_p(s,t)=1} \mathcal{K}_{q,\delta}^S(s|_m, t|_m) \; \mathrm{d}\mu_q(s)\mathrm{d}\mu_q(t) \\
&= \sum_{p\geq 2} \liminf_{m\to\infty} \sum_{u,v \in \mathscr{A}^m; \mathbf{1}_p(u,v)=1} \mathcal{K}_{q,\delta}^S(u,v) \cdot \mu_q([u]_{q,\epsilon}^n)\mu_q([v]_{q,\epsilon}^n) \\
&\leq \sum_{p\geq 2} \liminf_{m\to\infty} \sum_{u,v \in \mathscr{A}^m; \mathbf{1}_p(u,v)=1} \mathcal{K}_{\bar{q},\delta+\epsilon}^S(u,v) \cdot \mu_q([u]_{q,\epsilon}^n)\mu_q([v]_{q,\epsilon}^n),
\end{aligned}$$

where the last inequality comes from the fact that due to (4.19) and (4.21), for any $\bar{q} \in K$, $\epsilon > 0$ and $u, v \in \mathscr{A}^* \cup \mathscr{A}^{\mathbb{N}_+}$, we have $\sup_{q \in U_\epsilon(\bar{q})} \mathcal{K}_{q,\delta}^S(u,v) \leq \mathcal{K}_{\bar{q},\delta+\epsilon}^S(u,v)$. Let

$$A_{p,m} = \sum_{u,v \in \mathscr{A}^m; \mathbf{1}_p(u,v)=1} \mathcal{K}_{\bar{q},\delta+\epsilon}^S(u,v) \cdot \mu_q([u]_{q,\epsilon}^n)\mu_q([v]_{q,\epsilon}^n).$$

Then,

$$\begin{aligned}
\sup_{q \in U_\epsilon(\bar{q})} I_{n,\delta}^S(q,\epsilon) &\leq \sup_{q \in U_\epsilon(\bar{q})} \sum_{p\geq 2} \liminf_{m\to\infty} \; A_{p,m} \\
&\leq \sup_{q \in U_\epsilon(\bar{q})} \sum_{p\geq 2} \Big(A_{p,m_p} + \sum_{m \geq m_p} |A_{p,m+1} - A_{p,m}| \Big)
\end{aligned}$$

$$\leq \sum_{p\geq 2}\Big(\sup_{q\in U_\epsilon(\bar q)} A_{p,m_p} + \sum_{m\geq m_p}\sup_{q\in U_\epsilon(\bar q)} |A_{p,m+1}-A_{p,m}|\Big), \quad (4.22)$$

where for $p\geq 2$, we can choose $m_p\geq 3$ to be any integer. We have

$$\sup_{q\in U_\epsilon(\bar q)} A_{p,m} \leq (I^{S,\delta}_{n,\bar q,\epsilon})_{p,m} \quad \text{and} \quad \sup_{q\in U_\epsilon(\bar q)} |A_{p,m+1}-A_{p,m}| \leq (\Delta I^{S,\delta}_{n,\bar q,\epsilon})_{p,m}, \quad (4.23)$$

where

$$(I^{S,\delta}_{n,\bar q,\epsilon})_{p,m} = \sum_{u,v\in\mathscr{A}^m;\mathbf{1}_p(u,v)=1} \mathcal{K}^S_{\bar q,\delta+\epsilon}(u,v)\sup_{q\in U_\epsilon(\bar q)}\mu_q([u]^n_{q,\epsilon})\mu_q([v]^n_{q,\epsilon}),$$

$$(\Delta I^{S,\delta}_{n,\bar q,\epsilon})_{p,m} = \sum_{\substack{u,v\in\mathscr{A}^m,u',v'\in\mathscr{A};\\ \mathbf{1}_p(u,v)=1}} \left|\mathcal{K}^S_{\bar q,\delta+\epsilon}(uu',vv') - \mathcal{K}^S_{\bar q,\delta+\epsilon}(u,v)\right| \sup_{q\in U_\epsilon(\bar q)}\mu_q([uu]^n_{q,\epsilon})\mu_q([vv]^n_{q,\epsilon}),$$

and we have used the equality $\mu_q([u]^n_{q,\epsilon}) = \sum_{u\in\mathscr{A}}\mu_q([uu]^n_{q,\epsilon})$ to get the second inequality.

Remark 4.6

(a) For technical reasons, we need to divide J into three parts, in which K will be chosen :

$$\begin{cases} J_1 = \{q\in J : \gamma^G(q) > 1\}, \\ J_2 = \{q\in J : \gamma^G(q)\leq 1,\ \tau'(q) < 1\}, \\ J_3 = \{q\in J : \gamma^G(q)\leq 1,\ \tau'(q)\geq 1\}. \end{cases}$$

Then, due to (4.11) *and* (4.12)*, with $h=\tau'(q)$ we have*

$$\gamma^G(q) = \begin{cases} \tau^*(h)+1-h & \text{if } q\in J_1,\\ \tau^*(h)/h & \text{if } q\in J_2,\\ \tau^*(h) & \text{if } q\in J_3 \end{cases} \quad \text{and } \gamma^R(q) = \begin{cases} 1 & \text{if } q\in J_1,\\ \tau^*(h)/h & \text{if } q\in J_2\cup J_3. \end{cases}$$

(b) Here we briefly explain why $\dim_H(\mu^G_q) = \gamma^G(q) = \tau^*(\tau'(q))$ when $q\in J_3$. Thus we will not consider this case in the rest of the proof.
From Chapter 3 we have $\dim_H(\mu^D_q) = \tau^*(\tau'(q))$ and μ^D_q is the orthogonal projection of μ^G_q onto the x-axis, so we automatically have $\dim_H(\mu^G_q)\geq \dim_H(\mu^D_q) = \tau^*(\tau'(q))$. Then to prove $\dim_H(\mu^G_q)=\gamma^G(q)$ we only need an upper bound estimate, but this estimate is actually a direct consequence of Theorem 4.1.

For any compact subinterval K of J there exists $c_K\in(0,1)$ such that for any $c<c_K$, $\gamma^G(q)-c>1$ if $K\subset J_1$ and $\gamma^R(q)-c>0$ if $K\subset J_2\cup J_3$. Let $\delta_K=\epsilon_K=c_K/2$. An essential tool in this chapter is the following proposition, whose proof is given in Section 4.5.

Proposition 4.4 *Let $S\in\{G,R\}$. Suppose that K is a subinterval of J_1 or J_2 if $S=G$, or a subinterval of J_1 or $J_2\cup J_3$ if $S=R$. Then there exists $\epsilon_*\in(0,\epsilon_K)$ such that for any $0<\delta<\delta_K$, we can find constants $\kappa_1,\kappa_2,\eta_1,\eta_2>0$ and $C>0$ such that for any $\bar q\in K$, $0<\epsilon\leq\epsilon_*$, $n\geq 1$, $p\geq 2$, and $m\geq 3\cdot(n\vee p)$*

$$\mathbb{E}\left((I^{S,\delta}_{n,\bar q,\epsilon})_{p,m}\right) \leq C\cdot b^{(n\vee p)-p+1}\cdot e^{\mathbf{1}_{\{p<n\}}\kappa_1\cdot n}\cdot e^{-\eta_1\delta\cdot(n\vee p)+\kappa_1\epsilon\cdot m};$$

$$\mathbb{E}\left((\Delta I^{S,\delta}_{n,\bar q,\epsilon})_{p,m}\right) \leq C\cdot b^{(n\vee p)-p+1}\cdot e^{\kappa_2\cdot(n\vee p)-\eta_2\cdot m}.$$

Now we may choose $m_p = \frac{\kappa_2+\frac{1}{2}\delta\eta_1}{\eta_2}\cdot(n\vee p)$ (by modifying a little η_2 we can always assume that $\frac{\kappa_2+\frac{1}{2}\delta\eta_1}{\eta_2} > 3$) and $\epsilon_\delta = \epsilon_* \wedge \frac{\frac{1}{2}\delta\eta_1\eta_2}{\kappa_1(\kappa_2+\frac{1}{2}\delta\eta_1)}$. Then by using Proposition 4.4 and (4.22), (4.23), for any $\delta < \delta_K$, $\bar{q}\in K$, $\epsilon < \epsilon_\delta$ and $S\in\{G,R\}$ we have

$$
\begin{aligned}
&\mathbb{E}\left(\sup_{q\in U_\epsilon(\bar{q})} I^S_{n,\delta}(q,\epsilon)\right)\\
\leq\ & \sum_{p\geq 2}\left(\mathbb{E}\left((I^{S,\delta}_{n,\bar{q},\epsilon})_{p,m_p}\right) + \sum_{m\geq m_p}\mathbb{E}\left((\Delta I^{S,\delta}_{n,\bar{q},\epsilon})_{p,m}\right)\right)\\
\leq\ & \sum_{p\geq 2} C\cdot b^{(n\vee p)-p+1}\cdot\left(e^{\mathbf{1}_{\{p<n\}}\kappa_1\cdot n}\cdot e^{-\eta_1\delta\cdot(n\vee p)+\kappa_1\epsilon\cdot m_p} + \sum_{m\geq m_p} e^{\kappa_2\cdot(n\vee p)-\eta_2\cdot m}\right)\\
\leq\ & C\cdot\sum_{p\geq 2} b^{(n\vee p)-p+1}\cdot\left(e^{\mathbf{1}_{\{p<n\}}\kappa_1\cdot n}\cdot e^{-\eta_1\delta\cdot(n\vee p)+\kappa_1\cdot\epsilon_\delta\cdot m_p} + e^{\kappa_2\cdot(n\vee p)-\eta_2\cdot m_p}\cdot\frac{1}{1-e^{-\eta_2}}\right)\\
\leq\ & \frac{C}{1-e^{-\eta_2}}\cdot\sum_{p\geq 2} b^{(n\vee p)-p+1}\cdot e^{\mathbf{1}_{\{p<n\}}\kappa_1\cdot n}\cdot\\
& \qquad\left(e^{-\eta_1\delta\cdot(n\vee p)+\kappa_1\cdot\frac{\frac{1}{2}\delta\eta_1\eta_2}{\kappa_1(\kappa_2+\frac{1}{2}\delta\eta_1)}\cdot\frac{\kappa_2+\frac{1}{2}\delta\eta_1}{\eta_2}\cdot(n\vee p)} + e^{\kappa_2\cdot(n\vee p)-\eta_2\cdot\frac{\kappa_2+\frac{1}{2}\delta\eta_1}{\eta_2}\cdot(n\vee p)}\right)\\
=\ & \frac{2C}{1-e^{-\eta_2}}\cdot\sum_{p\geq 2} b^{(n\vee p)-p+1}\cdot e^{\mathbf{1}_{\{p<n\}}\kappa_1\cdot n}\cdot e^{-\frac{1}{2}\delta\eta_1(n\vee p)}\\
=\ & \frac{2C}{1-e^{-\eta_2}}\cdot\left(\sum_{p=1}^{n-1} b^{n-p+1}\cdot e^{\kappa_1\cdot n}\cdot e^{-\frac{1}{2}\delta\eta_1 n} + \sum_{p\geq n} b\cdot e^{-\frac{1}{2}\delta\eta_1 p}\right)\\
=\ & \frac{2C}{1-e^{-\eta_2}}\cdot\left(b^{n+1}\cdot e^{\kappa_1\cdot n}\cdot e^{-\frac{1}{2}\delta\eta_1 n}\cdot\frac{1-b^{-n}}{1-b^{-1}} + b\cdot e^{-\frac{1}{2}\delta\eta_1 n}\cdot\frac{1}{1-e^{-\frac{1}{2}\delta\eta_1}}\right) < \infty.
\end{aligned}
$$

Since for any $0<\epsilon<\epsilon_\delta$, the family $\{U_\epsilon(\bar{q})\}_{\bar{q}\in K}$ forms an open covering of K, there exist $\bar{q}_1,\cdots,\bar{q}_N$ such that $\{U_\epsilon(\bar{q}_i)\}_{1\leq i\leq N}$ also covers K. This gives us the conclusion.

4.4.3 Proof of Theorem 4.4.

Proof The proof of Theorem 4.4 exploits the main idea developed in [116] to study the dimension of projections and sections of sets. Some complications come from the fact that we want results holding for uncountably many sets and measures simultaneously.

Through the proof we use the same notation as in Section 4.4.2. Moreover, for $n\geq 1$, $q\in J$ and $\epsilon>0$ we define

$$G_n(q,\epsilon) = \big\{(F_L(s),F_W(s)) : s\in\mathcal{C}_n(q,\epsilon)\big\}$$

and for $\theta\in[0,\pi)$ we define $R_{n,\theta}(q,\epsilon) = \mathrm{Proj}_\theta(G_n(q,\epsilon)) \subset l^\perp_{0,\theta}$.

For any $y\in l_\theta$ we define the lower derivative of the measure $\mu^R_{q,\theta}\big|_{R_{n,\theta}(q,\epsilon)}$ with respect to the one-dimensional Lebesgue measure on l_θ at y :

$$\underline{D}(\mu^R_{q,\theta}\big|_{R_{n,\theta}(q,\epsilon)}, y) = \liminf_{r\to 0^+}\frac{1}{r}\cdot\mu^R_{q,\theta}\Big(B(y,r)\cap R_{n,\theta}(q,\epsilon)\Big).$$

We fix a compact subset $K \subset J_1$ (recall Remark 4.6). For any $\bar{q} \in K$, we can choose $\delta \in (0, \delta_K)$ and ϵ_* such that the conclusions of Proposition 4.4 hold. Notice that for such δ and $\epsilon \in (0, \epsilon_*)$ we always have $\gamma^G(\bar{q}) - \delta - \epsilon > 1$.

For $s, t \in \mathscr{A}^* \cup \mathscr{A}^{\mathbb{N}_+}$ recall

$$\mathcal{K}_1(s,t)^{-1} = \left(|F_L(s) - F_L(t)|^2 + |F_W(s) - F_W(t)|^2\right)^{\frac{1}{2}} \wedge 1,$$

and for $\theta \in (-\pi/2, \pi/2)$ and $\gamma > 0$ we define

$$d^G_{\theta,\gamma}(s,t) = \mathcal{K}_\gamma(s,t)^{-1} \cdot |\sin(\theta + \theta_{s,t})|,$$

where $\theta_{s,t}$ stands for the angle between $(F_L(t), F_W(t)) - (F_L(s), F_W(s))$ and x-axis (clockwise). Notice that for any $r > 0$ and $\gamma > 1$ we always have

$$\mathbf{1}_{\{d^G_{\theta,1}(s,t)<r\}} \le \liminf_{m\to\infty} \mathbf{1}_{\{d^G_{\theta,1}(s|_m,\, t|_m)<2r\}} \le \liminf_{m\to\infty} \mathbf{1}_{\{d^G_{\theta,\gamma}(s|_m,\, t|_m)<2r\}}.$$

Now, recall (4.21). For simplicity we will also use the notation

$$M^n_{\bar{q},\epsilon}(u,v) = \sup_{q\in U_\epsilon(\bar{q})} \mu_q([u]^n_{q,\epsilon})\mu_q([v]^n_{q,\epsilon}). \tag{4.24}$$

Then an integration similar to that used in the proof of Theorem 9.7 of [116], as well as arguments similar to that used in Section 4.4.2 yield for any $\gamma > 1$ (by using Fatou's lemma repetitively)

$$\begin{aligned}
&\int_0^\pi \sup_{q\in U_\epsilon(\bar{q})} \int_{y\in R_{n,\theta}(q,\epsilon)} \underline{D}(\mu^R_{q,\theta}|_{R_{n,\theta}(q,\epsilon)}, y)\mathrm{d}\mu^R_{q,\theta}(y)\mathrm{d}\theta \\
= &\int_0^\pi \sup_{q\in U_\epsilon(\bar{q})} \int_{y\in R_{n,\theta}(q,\epsilon)} \liminf_{r\to 0^+} \frac{1}{r}\mu^R_{q,\theta}\Big(B(y,r)\cap R_{n,\theta}(q,\epsilon)\Big)\mathrm{d}\mu^R_{q,\theta}(y)\mathrm{d}\theta \\
\le &\int_0^\pi \sup_{q\in U_\epsilon(\bar{q})} \liminf_{r\to 0^+} \frac{1}{r} \int_{y\in R_{n,\theta}(q,\epsilon)} \int_{x\in G_n(q,\epsilon)} \mathbf{1}_{\{|x-l^\perp_{y,\theta}|\le r\}}\mathrm{d}\mu^G_q(x)\mathrm{d}\mu^R_{q,\theta}(y)\mathrm{d}\theta \\
= &\int_0^\pi \sup_{q\in U_\epsilon(\bar{q})} \liminf_{r\to 0^+} \frac{1}{r} \int_{s\in \mathcal{C}_n(q,\epsilon)} \int_{t\in \mathcal{C}_n(q,\epsilon)} \mathbf{1}_{\{d^G_{\theta,1}(s,t)\le r\}}\mathrm{d}\mu_q(t)\mathrm{d}\mu_q(s)\mathrm{d}\theta \\
\le &\int_0^\pi \sup_{q\in U_\epsilon(\bar{q})} \liminf_{r\to 0^+} \frac{1}{r} \sum_{p\ge 2} \iint_{\substack{s,t\in\mathcal{C}_n(q,\epsilon),\\ \mathbf{1}_p(s,t)=1}} \liminf_{m\to\infty} \mathbf{1}_{\{d^G_{\theta,1}(s|_m,t|_m)\le 2r\}}\mathrm{d}\mu_q(t)\mathrm{d}\mu_q(s)\mathrm{d}\theta \\
\le &\int_0^\pi \sup_{q\in U_\epsilon(\bar{q})} \liminf_{r\to 0^+} \frac{1}{r} \sum_{p\ge 2} \iint_{\substack{s,t\in\mathcal{C}_n(q,\epsilon),\\ \mathbf{1}_p(s,t)=1}} \liminf_{m\to\infty} \mathbf{1}_{\{d^G_{\theta,\gamma}(s|_m,t|_m)\le 2r\}}\mathrm{d}\mu_q(t)\mathrm{d}\mu_q(s)\mathrm{d}\theta \\
\le &\int_0^\pi \sup_{q\in U_\epsilon(\bar{q})} \liminf_{r\to 0^+} \frac{1}{r} \sum_{p\ge 2} \liminf_{m\to\infty} \iint_{\substack{s,t\in\mathcal{C}_n(q,\epsilon),\\ \mathbf{1}_p(s,t)=1}} \mathbf{1}_{\{d^G_{\theta,\gamma}(s|_m,t|_m)\le 2r\}}\mathrm{d}\mu_q(t)\mathrm{d}\mu_q(s)\mathrm{d}\theta \\
\le &\liminf_{r\to 0^+} \frac{1}{r} \sum_{p\ge 2} \int_0^\pi \sup_{q\in U_\epsilon(\bar{q})} \liminf_{m\to\infty} \sum_{\substack{u,v\in\mathscr{A}^m;\\ \mathbf{1}_p(u,v)=1}} \mathbf{1}_{\{d^G_{\theta,\gamma}(u,v)\le 2r\}}\mu_q([u]^n_{q,\epsilon})\mu_q([v]^n_{q,\epsilon})\mathrm{d}\theta \\
\le &\liminf_{r\to 0^+} \frac{1}{r} \sum_{p\ge 2} \Bigg(\sum_{\substack{u,v\in\mathscr{A}^{m_p};\\ \mathbf{1}_p(u,v)=1}} \int_0^\pi \mathbf{1}_{\{d^G_{\theta,\gamma}(u,v)\le 2r\}}\mathrm{d}\theta \cdot M^n_{\bar{q},\epsilon}(u,v) + \\
&\sum_{m\ge m_p} \sum_{\substack{u,v\in\mathscr{A}^m,u',v'\in\mathscr{A};\\ \mathbf{1}_p(u,v)=1}} \int_0^\pi \left|\mathbf{1}_{\{d^G_{\theta,\gamma}(uu',vv')\le 2r\}} - \mathbf{1}_{\{d^G_{\theta,\gamma}(u,v)\le 2r\}}\right|\mathrm{d}\theta \cdot M^n_{\bar{q},\epsilon}(uu',vv')\Bigg)
\end{aligned}$$

$$\leq \sum_{p\geq 2}\Big(\sum_{\substack{u,v\in\mathscr{A}^{m_p};\\ \mathbf{1}_p(u,v)=1}} \limsup_{r\to 0^+}\frac{1}{r}\int_0^\pi \mathbf{1}_{\{d^G_{\theta,\gamma}(u,v)\leq 2r\}}\mathrm{d}\theta\cdot M^n_{\bar{q},\epsilon}(u,v) + \sum_{m\geq m_p}$$
$$\sum_{\substack{u,v\in\mathscr{A}^{m};u',v'\in\mathscr{A};\\ \mathbf{1}_p(u,v)=1}} \limsup_{r\to 0^+}\frac{1}{r}\int_0^\pi \big|\mathbf{1}_{\{d^G_{\theta,\gamma}(uu',vv')\leq 2r\}} - \mathbf{1}_{\{d^G_{\theta,\gamma}(u,v)<2r\}}\big|\mathrm{d}\theta\cdot M^n_{\bar{q},\epsilon}(uu',vv')\Big),$$

where m_p is taken as in the proof of Theorem 4.3.

Notice that there exists a universal constant $C'>0$ such that for all $r>0$,

$$\int_0^\pi \mathbf{1}_{\{d^G_{\theta,\gamma}(w,u)\leq 2r\}}\mathrm{d}\theta = \int_0^\pi \mathbf{1}_{\{|\sin(\theta+\theta_{u,v})|\leq 2r\cdot\mathcal{K}_\gamma(u,v)\}}\mathrm{d}\theta \leq C'\cdot r\cdot \mathcal{K}_\gamma(u,v)$$

and

$$\int_0^\pi \big|\mathbf{1}_{\{d^G_{\theta,\gamma}(uu',vv')\leq 2r\}} - \mathbf{1}_{\{d^G_{\theta,\gamma}(u,v)\leq 2r\}}\big|\mathrm{d}\theta \leq C'\cdot r\cdot |\mathcal{K}_\gamma(uu',vv')-\mathcal{K}_\gamma(u,v)|.$$

Thus by taking $\gamma=\gamma^G(\bar{q})-\delta-\epsilon>1$, we deduce from Proposition 4.4 that

$$\mathbb{E}\left(\int_0^\pi \sup_{q\in U_\epsilon(\bar{q})}\int_{y\in R_{n,\theta}(q,\epsilon)} \underline{D}(\mu^R_{q,\theta}|_{R_{n,\theta}(q,\epsilon)},y)\mathrm{d}\mu^R_{q,\theta}(y)\mathrm{d}\theta\right)$$
$$\leq\; C'\cdot\mathbb{E}\left(\sum_{p\geq 2}(\mathcal{I}^{S,\delta}_{n,\bar{q},\epsilon})_{p,m_p} + \sum_{m\geq m_p}(\Delta\mathcal{I}^{S,\delta}_{n,\bar{q},\epsilon})_{p,m}\right)<\infty.$$

Then by using the same argument as in Section 4.4.2 we can conclude that with probability 1, for Lebesgue almost every $\theta\in(-\pi/2,\pi/2)$, for all $q\in J_1$, for $\mu^R_{q,\theta}$-almost every $y\in l_\theta$, the lower derivative $\underline{D}(\mu^R_{q,\theta},y)$ is finite, which is equivalent to saying that $\mu^R_{q,\theta}$ is absolutely continuous with respect to the one-dimensional Lebesgue measure on l_θ. This ensures that for $\mu^R_{q,\theta}$-almost every $y\in l_\theta$, the following limit :

$$\lim_{r\to 0^+}\frac{1}{r}\int_{|x-l^\perp_{y,\theta}|\leq r}\psi(x)\;\mathrm{d}\mu^G_q(x)$$

exists for any continuous function $\psi:\mathbb{R}^2\mapsto\mathbb{R}_+$ and thus defines a measure $\mu^y_{q,\theta}$ carried by $l^\perp_{y,\theta}\cap\{(F_L(t),F_W(t)):t\in\mathcal{C}(q)\}$.

Now for the lower bound of the Hausdorff dimension of $\mu^{y,\theta}_q$, we notice that like in Theorem 10.7 of [116], we have the following equality for the $\gamma^G(q)-\delta-1$ energy

$$\begin{aligned}&\mathcal{I}_{q,\theta}\\ :=\;& \int_{y\in R_{n,\theta}(q,\epsilon)}\iint_{x_1,x_2\in l^\perp_{y,\theta}\cap G_n(q,\epsilon)} 1\vee|x_1-x_2|^{-\gamma^G(q)-\delta-1}\mathrm{d}\mu^y_{q,\theta}(x_1)\mathrm{d}\mu^y_{q,\theta}(x_2)\mathrm{dLeb}_\theta(y)\\ =\;& \liminf_{r\to 0^+}\frac{1}{r}\iint_{s,t\in\mathcal{C}_n(q,\epsilon)}\mathbf{1}_{\{d^G_{\theta,1}(s,t)\leq r\}}d^G(s,t)^{-\gamma^G(q)-\delta-1}\mathrm{d}\mu_q(t)\mathrm{d}\mu_q(s),\end{aligned}$$

where Leb_θ stands for the one-dimensional Lebesgue measure on l_θ. By using the same method as above to establish the finiteness of $\underline{D}(\mu^R_{q,\theta},y)$, we can show that

$$\begin{aligned}&\int_0^\pi \sup_{q\in U_\epsilon(\bar q)} \mathcal{I}_{q,\theta}\ \mathrm{d}\theta\\ &\le \sum_{p\ge 2}\Big(\sum_{\substack{u,v\in\mathscr{A}^{m_p};\\ \mathbf{1}_p(u,v)=1}} \limsup_{r\to 0^+}\frac{1}{r}\int_0^\pi \mathbf{1}_{\{d^G_{\theta,1}(u,v)\le 2r\}}\mathcal{K}_{\widetilde\gamma}(u,v)\mathrm{d}\theta\cdot M^n_{\bar q,\epsilon}(u,v)+\sum_{m\ge m_p}\sum_{\substack{u,v\in\mathscr{A}^m;u',v'\in\mathscr{A};\\ \mathbf{1}_p(u,v)=1}}\\ &\quad \limsup_{r\to 0^+}\frac{1}{r}\int_0^\pi \Big|\mathbf{1}_{\{d^G_{\theta,1}(uu',vv')\le 2r\}}\mathcal{K}_{\widetilde\gamma}(uu',vv')-\mathbf{1}_{\{d^G_{\theta,1}(w,u)\le 2r\}}\mathcal{K}_{\widetilde\gamma}(u,v)\Big|\mathrm{d}\theta\cdot M^n_{\bar q,\epsilon}(u,v)\Big),\end{aligned}$$

where $\widetilde\gamma=\gamma-1=\gamma^G(\bar q)-\delta-\epsilon-1>0$.

Notice that for the same universal constant C', we have

$$\int_0^\pi \mathbf{1}_{\{d^G_{\theta,1}(u,v)\le 2r\}}\mathcal{K}_{\widetilde\gamma}(u,v)\mathrm{d}\theta\le C'\cdot r\cdot \mathcal{K}_{\widetilde\gamma+1}(u,v)=C'\cdot r\cdot\mathcal{K}_\gamma(u,v)$$

and

$$\begin{aligned}&\int_0^\pi \Big|\mathbf{1}_{\{d^G_{\theta,1}(uu',vv')\le 2r\}}\mathcal{K}_{\widetilde\gamma}(uu',vv')-\mathbf{1}_{\{d^G_{\theta,1}(u,v)\le 2r\}}\mathcal{K}_{\widetilde\gamma}(u,v)\Big|\mathrm{d}\theta\\ \le\ &\Big(\mathcal{K}_{\widetilde\gamma}(uu',vv')\vee\mathcal{K}_{\widetilde\gamma}(u,v)\Big)\cdot\int_0^\pi\Big|\mathbf{1}_{\{d^G_{\theta,1}(uu',vv')\le 2r\}}-\mathbf{1}_{\{d^G_{\theta,1}(u,v)\le 2r\}}\Big|\mathrm{d}\theta\\ &\qquad+|\mathcal{K}_{\widetilde\gamma}(uu',vv')-\mathcal{K}_{\widetilde\gamma}(u,v)|\int_0^\pi \mathbf{1}_{\{d^G_{\theta,1}(uu',vv')\le 2r\}}\vee\mathbf{1}_{\{d^G_{\theta,1}(u,v)\le 2r\}}\mathrm{d}\theta\\ \le\ & C'\cdot r\cdot\Delta\mathcal{K}_{\widetilde\gamma}(uu',vv'),\end{aligned}$$

where

$$\begin{aligned}\Delta\mathcal{K}_{\widetilde\gamma}(uu',vv')=&\Big(\mathcal{K}_{\widetilde\gamma}(uu',vv')\vee\mathcal{K}_{\widetilde\gamma}(u,v)\Big)\cdot|\mathcal{K}_1(uu',vv')-\mathcal{K}_1(u,v)|\\ &+\Big(\mathcal{K}_1(uu',vv')\vee\mathcal{K}_1(u,v)\Big)\cdot|\mathcal{K}_{\widetilde\gamma}(uu',vv')-\mathcal{K}_{\widetilde\gamma}(u,v)|.\end{aligned}$$

Then

$$\mathbb{E}\left(\int_0^\pi\sup_{q\in U_\epsilon(\bar q)}\mathcal{I}_{q,\theta}\ \mathrm{d}\theta\right)\le\mathbb{E}\left(\sum_{p\ge 2}(\mathcal{I}^{S,\delta}_{n,\bar q,\epsilon})_{p,m_p}+\sum_{m\ge m_p}(\widetilde{\Delta\mathcal{I}^{S,\delta}_{n,\bar q,\epsilon}})_{p,m}\right)<\infty, \tag{4.25}$$

where

$$(\widetilde{\Delta\mathcal{I}^{S,\delta}_{n,\bar q,\epsilon}})_{p,m}=\sum_{\substack{u,v\in\mathscr{A}^m;u',v'\in\mathscr{A};\\ \mathbf{1}_p(u,v)=1}}\Delta\mathcal{K}_{\widetilde\gamma}(uu',vv')\sup_{q\in U_\epsilon(\bar q)}\mu_q([uu']^n_{q,\epsilon})\mu_q([vv']^n_{q,\epsilon}).$$

The justification of the finiteness of the right hand side of (4.25) is postponed to Remark 4.7.

Now, by using the same arguments as in Section 4.4.2 again, we deduce that, with probability 1, for Lebesgue-almost every $\theta\in(-\pi/2,\pi/2)$, for all $q\in J_1$, for $\mu^R_{q,\theta}$-almost every $y\in l_\theta$, we have

$$\dim_H(\mu^y_{q,\theta})\ge\dim_H(\mu^G_q)-1=\tau^*(\tau'(q))-\tau'(q).$$

Then we get the conclusion by applying Theorem 4.1(b) to the measure $\mu^R_{q,\theta}$ since we know that the for $\mu^R_{q,\theta}$-almost every $y\in l_\theta$, the lower local dimension $h_{\mu^R_{q,\theta}}(y)$ is equal to 1.

4.5 Proof of Proposition 4.4

4.5.1 Main Proof.

Proof Due to (4.21) we always have

$$\bigcup_{q\in U_\epsilon(\bar q)} \mathcal{C}_n(q,\epsilon)\subset \mathcal{C}_n(\bar q,2\epsilon).$$

Let $\rho_K = 1\vee \sup_{q\in K}\{|q|+\xi(q)+\widetilde{\xi}(q)+|\tau(q)|+\gamma(q)\}<\infty$.

Due to (4.24), (4.7), (4.17), (4.21) and (4.6) we have

$$\begin{aligned}
M^n_{\bar q,\epsilon}(u,v) &= \sup_{q\in U_\epsilon(\bar q)} \mu_q([u]^n_{q,\epsilon})\mu_q([v]^n_{q,\epsilon})\\
&= \sup_{q\in U_\epsilon(\bar q)} \mathbf{1}_{\{[u]^n_{q,\epsilon}\neq\emptyset\}}\mathbf{1}_{\{[v]^n_{q,\epsilon}\neq\emptyset\}}|W_{q,u}(\varnothing)||W_{q,v}(\varnothing)|Y_q(u)Y_q(v)\\
&\le \sup_{q\in U_\epsilon(\bar q)} \mathbf{1}_{\{[u]^n_{q,\epsilon},[v]^n_{q,\epsilon}\neq\emptyset\}}e^{-(|u|+|v|)(\gamma(q)-2\rho_K\epsilon)}Y_q(u)Y_q(v)\\
&\le \mathbf{1}_{\{[u]^n_{\bar q,2\epsilon},[v]^n_{\bar q,2\epsilon}\neq\emptyset\}}e^{-(|u|+|v|)(\gamma(\bar q)-4\rho_K\epsilon)}Y_K(u)Y_K(v).
\end{aligned}$$

This gives us

$$(\mathcal{I}^{S,\delta}_{n,\bar q,\epsilon})_{p,m}\le e^{-2m(\gamma(\bar q)-4\rho_K\epsilon)}\cdot \sum_{u,v\in\mathscr{A}^m,\mathbf{1}_p(u,v)=1} \mathcal{K}^S_{\bar q,\delta+\epsilon}(u,v)\cdot \mathbf{1}_{\{[u]^n_{\bar q,2\epsilon},[v]^n_{\bar q,2\epsilon}\neq\emptyset\}}Y_K(u)Y_K(v), \quad (4.26)$$

$$(\Delta I^{S,\delta}_{n,\bar q,\epsilon})_{p,m}\le e^{-2(m+1)(\gamma(\bar q)-4\rho_K\epsilon)}\cdot \sum_{u,v\in\mathscr{A}^m;u',v'\in\mathscr{A};\mathbf{1}_p(u,v)=1} \left|\mathcal{K}^S_{\bar q,\delta+\epsilon}(uu',vv')-\mathcal{K}^S_{\bar q,\delta+\epsilon}(u,v)\right|\cdot \mathbf{1}_{\{[uu']^n_{\bar q,2\epsilon},[vv']^n_{\bar q,2\epsilon}\neq\emptyset\}}Y_K(uu')Y_K(vv'). \quad (4.27)$$

Now we deal with the individual terms of the above sums.

Fix p and n in $\mathbb{N}_+$, let $r=p\vee n$, and fix $m\ge 3r$.

Fix a pair $u,v\in\mathscr{A}^m$ with $\mathbf{1}_p(u,v)=1$, so $|\lambda(u)-\lambda(v)|\in[b^{-p+1},b^{-p+2})$.

Without loss of generality we suppose that $\lambda(u)<\lambda(v)$.

Since $\mathbf{1}_p(u,v)=1$, we have $\lambda(u)<\lambda(v|_p^-)\le\lambda(v|_r^-)$.

Let

$$\begin{cases} V := \mathcal{K}^S_{\bar q,\delta+\epsilon}(u,v)\mathbf{1}_{\{[u]^n_{\bar q,2\epsilon},[v]^n_{\bar q,2\epsilon}\neq\emptyset\}}Y_K(u)Y_K(v);\\[2ex] \Delta V := \left|\mathcal{K}^S_{\bar q,\delta+\epsilon}(uu',vv')-\mathcal{K}^S_{\bar q,\delta+\epsilon}(u,v)\right|\mathbf{1}_{\{[uu']^n_{\bar q,2\epsilon},[vv']^n_{\bar q,2\epsilon}\neq\emptyset\}}Y_K(uu')Y_K(vv'). \end{cases} \quad (4.28)$$

Let us state two elementary claims.

Claim 1. Recall that $[u]^n_{\bar{q},2\epsilon} = [u] \cap \mathcal{C}_n(\bar{q}, 2\epsilon)$. Due to (4.17) and (4.18), if $[u]^n_{\bar{q},2\epsilon} \neq \emptyset$, then for $l = r, \cdots, m$ we have $\mathbf{1}_{\mathscr{W}_{u|_l}(\bar{q},2\epsilon)} \cdot \mathbf{1}_{\mathscr{L}_{u|_l}(\bar{q},2\epsilon)} \cdot \mathbf{1}_{\mathscr{O}_{u|_l}(\epsilon)} = 1$. Define

$$\mathbf{1}^{(1)}_{u,v}(\bar{q},\epsilon) = \mathbf{1}_{\left\{O_{F_W}(I_{u|_r}) \vee O_{F_W}(I_{v|_r}) \leq e^{-r(\xi(\bar{q})-6\epsilon)},\ O_{F_L}(I_{v|_r^-}) \geq e^{-r(\tilde{\xi}(\bar{q})+6\epsilon)}\right\}}, \tag{4.29}$$

$$\mathbf{1}^{(2)}_{u}(\bar{q},\epsilon) = \mathbf{1}_{\mathscr{W}_{u|_r}(\bar{q},2\epsilon)} \mathbf{1}_{\mathscr{L}_{u|_r}(\bar{q},2\epsilon)} \cdot \mathbf{1}_{\mathscr{W}_{u}(\bar{q},2\epsilon)} \mathbf{1}_{\mathscr{L}_{u}(\bar{q},2\epsilon)} \mathbf{1}_{\mathscr{O}_{u}(\epsilon)}, \tag{4.30}$$

and $\mathbf{1}^{(2)}_{u,v}(\bar{q},\epsilon) = \mathbf{1}^{(2)}_{u}(\bar{q},\epsilon) \cdot \mathbf{1}^{(2)}_{v}(\bar{q},\epsilon)$. Also, $[uu']^n_{\bar{q},2\epsilon} \neq \emptyset$ implies $[u]^n_{\bar{q},2\epsilon} \neq \emptyset$. Then, due to (4.15) and (4.16), we have

$$\mathbf{1}_{\{[uu']^n_{\bar{q},2\epsilon}, [vv']^n_{\bar{q},2\epsilon} \neq \emptyset\}} \leq \mathbf{1}_{\{[u]^n_{\bar{q},2\epsilon}, [v]^n_{\bar{q},2\epsilon} \neq \emptyset\}} \leq \mathbf{1}^{(1)}_{u,v}(\bar{q},\epsilon) \cdot \mathbf{1}^{(2)}_{u,v}(\bar{q},\epsilon).$$

Claim 2. For $w \in \mathscr{A}^*$ we define $Z_W(w) = F_W^{[w]}(1)$. Then from (4.14) we have

$$F_W(\lambda(w) + b^{-|w|}) - F_W(\lambda(w)) = W_w(\varnothing) \cdot Z_W(w), \quad \forall\ w \in \mathscr{A}^*. \tag{4.31}$$

Due to (4.31) we have

$$\begin{aligned} & F_W(v) - F_W(u) \\ = \ & F_W(v) - F_W(v|_r) + F_W(v|_r) - F_W(v|_r^-) + F_W(v|_r^-) - F_W(u) \\ = \ & W_{v|_r^-}(\varnothing) \cdot Z_W(v|_r^-) + F_W(v|_r) - F_W(v|_r^-) + F_W(v|_r^-) - F_W(u). \end{aligned}$$

By construction we have $Z_W(v|_r^-)$ is measurable with respect to

$$\mathcal{A}(v|_r^-) := \sigma\big((W,L)(v|_r^- \cdot w) : w \in \mathscr{A}^*\big)$$

and is independent of

$$\mathcal{A}^c(v|_r^-) := \sigma\big((W,L)(w) : w \in \mathscr{A}^*, |w| < r \text{ or } w|_r \neq w|_r^-\big).$$

Also due to the statistical self-similarity (4.14) we have

$$\sigma\left(\left\{ \begin{array}{l} W_{v|_r^-}, F_W(v|_r) - F_W(v|_r^-) + F_W(v|_r^-) - F_W(u), \\ \mathbf{1}^{(2)}_{u,v}(\bar{q},\epsilon), Y_K(u), Y_K(uu'), Y_K(v), Y_K(vv') \end{array} \right.\right) \subset \mathcal{A}^c(u|_r^-).$$

Now, due to **Claim 1** and (4.28) we have

$$\begin{aligned} V &\leq \overline{\mathcal{K}^S_{\bar{q},\delta+\epsilon}}(u,v) \cdot \mathbf{1}^{(2)}_{u,v}(\bar{q},\epsilon) \cdot Y_K(u) Y_K(v), \text{ and} \\ \Delta V &\leq \overline{\Delta\mathcal{K}^S_{\bar{q},\delta+\epsilon}}(uu',vv') \cdot \mathbf{1}^{(2)}_{u,v}(\bar{q},\epsilon) \cdot Y_K(uu') Y_K(vv'), \end{aligned}$$

where

$$\begin{cases} \overline{\mathcal{K}^S_{\bar{q},\delta+\epsilon}}(u,v) = \mathcal{K}^S_{\bar{q},\delta+\epsilon}(u,v) \cdot \mathbf{1}^{(1)}_{u,v}(\bar{q},\epsilon); \\ \overline{\Delta\mathcal{K}^S_{\bar{q},\delta+\epsilon}}(uu',vv') = \left|\mathcal{K}^S_{\bar{q},\delta+\epsilon}(uu',vv') - \mathcal{K}^S_{\bar{q},\delta+\epsilon}(u,v)\right| \cdot \mathbf{1}^{(1)}_{u,v}(\bar{q},\epsilon). \end{cases} \tag{4.32}$$

Then due to **Claim 2** we have

$$\mathbb{E}\Big(V\Big|\mathcal{A}^c(v|_r^-)\Big) \leq \mathbb{E}\Big(\overline{\mathcal{K}^S_{\bar{q},\delta+\epsilon}}(u,v)\Big|\mathcal{A}^c(v|_r^-)\Big) \cdot \mathbf{1}^{(2)}_{u,v}(\bar{q},\epsilon) \cdot Y_K(u) Y_K(v), \tag{4.33}$$

$$\mathbb{E}\Big(\Delta V\Big|\mathcal{A}^c(v|_r^-)\Big) \le \mathbb{E}\Big(\overline{\Delta\mathcal{K}^S_{\bar q,\delta+\epsilon}}(uu',vv')\Big|\mathcal{A}^c(v|_r^-)\Big)\cdot \mathbf{1}^{(2)}_{u,v}(\bar q,\epsilon)\cdot Y_K(uu')Y_K(vv'). \quad (4.34)$$

Recall in Remark 4.6 we distinguished the cases $K\subset J_i$, $i=1,2,3$ according to whether or not the corresponding power on the kernel is greater than 1. Then, due to (4.19), once we have taken $\delta<\delta_K$ and $\epsilon<\epsilon_K$, only two situations are left :

$$\mathcal{K}^S_{\bar q,\delta+\epsilon}(u,v) = \begin{cases} (|F_L(u)-F_L(v)|^2+|F_W(u)-F_W(v)|^2)^{-\gamma/2}\vee 1, & \text{if } \gamma>1; \\ |F_W(u)-F_W(v)|^{-\gamma}\vee 1, & \text{if } \gamma<1, \end{cases}$$

where $\gamma=\gamma^S(\bar q)-\delta-\epsilon$.

We have the following lemma, whose proof is given in Section 4.5.2.

Lemma 4.3 *There exists a constant C_γ such that*

$$\mathbf{1}^{(2)}_{w,u}(\bar q,\epsilon)\cdot\mathbb{E}\Big(\overline{\mathcal{K}^S_{\bar q,\delta+\epsilon}}(u,v)\Big|\mathcal{A}^c(v|_r^-)\Big) \le C_\gamma\cdot\mathbf{1}^{(2)}_{w,u}(\bar q,\epsilon)\cdot\begin{cases} e^{r(\xi(\bar q)+6\epsilon-(\widetilde\xi(\bar q)+6\epsilon)(1-\gamma))}, & \text{if } \gamma>1\\ e^{r(\xi(\bar q)+6\epsilon-\mathbf{1}_{\{p\ge n\}}\cdot(\xi(\bar q)-6\epsilon)(1-\gamma))}, & \text{if } \gamma<1\end{cases}$$

$$\mathbf{1}^{(2)}_{w,u}(\bar q,\epsilon)\cdot\mathbb{E}\Big(\overline{\Delta\mathcal{K}^S_{\bar q,\delta+\epsilon}}(uu',vv')\Big|\mathcal{A}^c(v|_r^-)\Big) \le C_\gamma\cdot\mathbf{1}^{(2)}_{w,u}(\bar q,\epsilon)\cdot\begin{cases} e^{r(\widetilde\xi(\bar q)+6\epsilon)(1+\gamma)-m(\xi(\bar q)-6\epsilon)}, & \text{if } \gamma>1\\ e^{r(\xi(\bar q)+6\epsilon)-m(\xi(\bar q)-6\epsilon)(1-\gamma)}, & \text{if } \gamma<1\end{cases}$$

To complete the proof, it remains to count the average number of pairs (u,v) in $(\mathscr{A}^m)^2$ such that $\mathbf{1}_p(u,v)=1$ and $\mathbf{1}^{(2)}_{u,v}(\bar q,\epsilon)=1$. This is done in the next lemma, whose proof is given in Section 4.5.3.

Lemma 4.4 *For $m\ge 3r$, we have*

$$\mathbb{E}\Big(\sum_{u,v\in\mathscr{A}^m}\mathbf{1}_p(u,v)\cdot\mathbf{1}^{(2)}_{u,v}(\bar q,\epsilon)\Big)\le 2b^{r-p+1}\cdot e^{(2m-r)(\gamma(\bar q)+8\rho_K\epsilon)}.$$

Now, by using Remark 4.6 and the definition of γ, i.e. $\gamma=\gamma^S(\bar q)-\delta-\epsilon$ for $S=G,R$, we have to deal with the following three cases (i), (ii), (iii) :

$$1-\gamma=\begin{cases}\big(\gamma(\bar q)-\xi(\bar q)\big)/\widetilde\xi(\bar q)+\delta+\epsilon, & \text{if } \gamma>1, & \text{case (i)}\\ \delta+\epsilon, & \text{if } \gamma<1 \text{ and } K\subset J_1, & \text{case (ii)}\\ \big(\xi(\bar q)-\gamma(\bar q)\big)/\xi(\bar q)+\delta+\epsilon, & \text{if } \gamma<1 \text{ and } K\subset J_2 \text{ or } J_3, & \text{case (iii)}\end{cases}$$

Then, due to (4.26), (4.27), (4.33) and (4.34), since $\mathbf{1}^{(2)}_{u,v}(\bar q,\epsilon)$, $Y_K(u)$ and $Y_K(v)$ (resp. $Y_K(uu')$ and $Y_K(vv')$) are independent, taking the expectation of $Y_K(u)$ and $Y_K(v)$ (resp. $Y_K(uu')$ and $Y_K(vv')$), and using Lemmas 4.3 and 4.4, for cases (i), (ii), (iii) we have (C_K stands for $\mathbb{E}(Y_K)$, which is finite by Proposition 4.1(b)) :

$$\mathbb{E}\left((\mathcal{I}^{S,\delta}_{n,\bar q,\epsilon})_{p,m}\right) \le 2C_\gamma C_K^2\cdot e^{-2m(\gamma(\bar q)-4\rho_K\epsilon)}\cdot b^{r-p+1}\cdot e^{(2m-r)(\gamma(\bar q)+8\rho_K\epsilon)}$$

$$
\begin{aligned}
& \cdot\begin{cases} e^{r(\xi(\bar q)+6\epsilon-(\widetilde{\xi}(\bar q)+6\epsilon)(\frac{\gamma(\bar q)-\xi(\bar q)}{\widetilde{\xi}(\bar q)}+\delta+\epsilon))}, & \text{(i)}\\ e^{r\mathbf{1}_{\{p<n\}}\cdot(\xi(\bar q)-6\epsilon)(1-\gamma)}e^{r(\xi(\bar q)+6\epsilon-(\xi(\bar q)-6\epsilon)(\delta+\epsilon))}, & \text{(ii)}\\ e^{r\mathbf{1}_{\{p<n\}}\cdot(\xi(\bar q)-6\epsilon)(1-\gamma)}e^{r(\xi(\bar q)+6\epsilon-(\xi(\bar q)-6\epsilon)(\frac{\xi(\bar q)-\gamma(\bar q)}{\xi(\bar q)}+\delta+\epsilon))}, & \text{(iii)}\end{cases}\\
= \;& 2C_\gamma C_K^2\cdot b^{r-p+1}\cdot e^{\mathbf{1}_{\{p<n\}}[(\xi(\bar q)-6\epsilon)(1-\gamma)\vee 0]\cdot r}\\
& \cdot\begin{cases} e^{-(\widetilde{\xi}(\bar q)\delta-[6-8\rho_K-\widetilde{\xi}(\bar q)-6(\frac{\gamma(\bar q)-\xi(\bar q)}{\widetilde{\xi}(\bar q)}+\delta+\epsilon)]\epsilon)\cdot r+24\rho_K\epsilon\cdot m}, & \text{(i)}\\ e^{-(\gamma(\bar q)-\xi(\bar q)+\xi(\bar q)\delta-[6-8\rho_K-\xi(\bar q)+6(\delta+\epsilon)]\epsilon)\cdot r+24\rho_K\epsilon\cdot m}, & \text{(ii)}\\ e^{-(\xi(\bar q)\delta-[6-8\rho_K-\xi(\bar q)+6(\frac{\xi(\bar q)-\gamma(\bar q)}{\xi(\bar q)}+\delta+\epsilon)]\epsilon)\cdot r+24\rho_K\epsilon\cdot m}, & \text{(iii)}\end{cases}
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}\left((\Delta\mathcal{I}^{S,\delta}_{n,\bar q,\epsilon})_{p,m}\right) \;\le\;& 2C_\gamma C_K^2\cdot e^{-2(m+1)(\gamma(\bar q)-4\rho_K\epsilon)}\cdot b^2\cdot b^{r-p+1}\cdot e^{(2m-r)(\gamma(\bar q)+8\rho_K\epsilon)}\\
& \cdot\begin{cases} e^{r(\widetilde{\xi}(\bar q)+6\epsilon)(1+\gamma)-m(\xi(\bar q)-6\epsilon)}, & \text{(i)}\\ e^{r(\xi(\bar q)+6\epsilon)-m(\xi(\bar q)-6\epsilon)(1-\gamma)}, & \text{(ii) and (iii)}\end{cases}\\
=\;& 2C_\gamma C_K^2 e^{-2(\gamma(\bar q)-4\rho_K\epsilon)}b^2\cdot b^{r-p+1}\\
& \cdot\begin{cases} e^{[(\widetilde{\xi}(\bar q)+6\epsilon)(1+\gamma)-\gamma(\bar q)-8\rho_K\epsilon]\cdot r-(\xi(\bar q)+18\epsilon)\cdot m}, & \text{(i)}\\ e^{[\xi(\bar q)+6\epsilon-\gamma(\bar q)-8\rho_K\epsilon]\cdot r-((\xi(\bar q)-6\epsilon)(1-\gamma)+24\rho_K\epsilon)\cdot m}. & \text{(ii) and (iii)}\end{cases}
\end{aligned}
$$

Let $\eta_K = \inf_{q\in K}\xi(q)\wedge\widetilde{\xi}(q)$. Under our assumptions we have $\eta_K > 0$. Let

$$
\begin{cases}
\kappa_1 = \sup_{q\in K}\max\{6+8\rho_K+\widetilde{\xi}(q)+6\frac{|\gamma(q)-\xi(q)|}{\widetilde{\xi}(q)}+12, 6+8\rho_K+\xi(q)+6\frac{|\xi(q)-\gamma(q)|}{\xi(q)}+12, \xi(q)+1, 24\rho_K\},\\
\kappa_2 = \sup_{q\in K}\max\{3(\widetilde{\xi}(q)+6)+\gamma(q)+8\rho_K, \xi(q)+\gamma(q)+6+8\rho_K\},\\
\epsilon_* = \frac{\eta_K}{2\kappa_1+24(\rho_K\vee 1)}\wedge\epsilon_K,\ \eta_1 = \frac{\eta_K}{2},\ \eta_2 = \frac{\eta_K\delta}{2},\\
C = 2C_\gamma C_K^2 b^2.
\end{cases}
$$

Clearly those parameters are all positive and finite. Notice that

$$
\begin{cases}
1-\gamma\ge\delta, & \text{in case (ii) or (iii) ;}\\
\tau^*(\tau'(\bar q))/\tau'(\bar q)\ge 1 \text{ thus } \gamma(\bar q)-\xi(\bar q)\ge 0, & \text{in the case (ii) ;}\\
1+\gamma\le 3, \delta_K\vee\epsilon_K < 1 & \text{in all cases.}
\end{cases}
$$

Then, by construction, we get for any $\delta < \delta_K$ and $\epsilon < \epsilon_*$,

$$
\begin{aligned}
\mathbb{E}\left((\mathcal{I}^{S,\delta}_{n,\bar q,\epsilon})_{p,m}\right) &\le C\cdot b^{r-p+1}\cdot e^{\mathbf{1}_{\{p<n\}}\kappa_1\cdot r}\cdot e^{-\eta_1\delta\cdot r+\kappa_1\epsilon\cdot m}\\
\text{and}\quad \mathbb{E}\left((\Delta\mathcal{I}^{S,\delta}_{n,\bar q,\epsilon})_{p,m}\right) &\le C\cdot b^{r-p+1}\cdot e^{\kappa_2\cdot r-\eta_2\cdot m},
\end{aligned}
$$

which gives the conclusion.

4.5.2 Proof of Lemma 4.3.

Step 1. At first, we prove that the probability distribution of $Z_W = F_W(1)$ has a bounded density function f_W, with $\|f_W\|_\infty = C_W < \infty$.

Let $\phi(t) = \mathbb{E}(e^{itZ_W})$ be the characteristic function of Z_W. Since we have $Z_W = \sum_{j=0}^{b-1} W_j\cdot Z_W(j)$, where $\{W_j\}_j$ and $\{Z_W(j)\}_j$ are independent, and the $Z_W(j)$ are independent copies of Z_W, we have

$$
\phi(t) = \mathbb{E}\left(\mathbb{E}\left(e^{it\cdot\sum_{j=0}^{b-1}W_j\cdot Z_W(j)}\;\middle|\;\sigma(Z_W(j), 0\le j\le b-1)\right)\right) = \mathbb{E}\left(\prod_{j=0}^{b-1}\phi(W_j t)\right).
$$

Since $\mathbb{E}(|Z_W|) < \infty$, simultaneously we also get

$$\phi'(t) = \mathbb{E}(iZ_W \cdot e^{itZ_W}) = \mathbb{E}\left(\sum_{k=0}^{b-1} iW_k\phi'(W_k t)\prod_{j\neq k}\phi(W_j t)\right).$$

Notice that $|\phi(t)| = |\phi(-t)|$ and $|\phi'(t)| = |\phi'(-t)|$, so we have $|\phi(|t|)| = |\phi(t)| = |\phi(-t)|$ and $|\phi'(|t|)| = |\phi'(t)| = |\phi'(-t)|$, then

$$|\phi(t)| \leq \mathbb{E}\left(\prod_{j=0}^{b-1}|\phi(W_j t)|\right) = \mathbb{E}\left(\prod_{j=0}^{b-1}|\phi(|W_j|t)|\right); \tag{4.35}$$

$$|\phi'(t)| \leq \mathbb{E}\left(\sum_{k=0}^{b-1}|W_k||\phi'(|W_k|t)|\prod_{j\neq k}|\phi(|W_j|t)|\right); \tag{4.36}$$

Define $l = \limsup_{t\to\infty}|\phi(t)|$. Since $|\phi(t)| \leq 1$, we have $l \leq 1$. From Fatou's lemma and the fact that $\mathbb{P}(\forall\, j, W_j \neq 0) = 1$, we have

$$l \leq \limsup_{t\to\infty}\mathbb{E}\left(\prod_{j=0}^{b-1}|\phi(|W_j|t)|\right) \leq \mathbb{E}\left(\limsup_{t\to\infty}\prod_{j=0}^{b-1}|\phi(|W_j|t)|\right) = l^b.$$

This implies that $l = 0$ or 1. Since we are in the non-conservative case, Z_W is not almost surely a constant. Consequently, we can use the same approach as in the proof of Lemma 3.1 in [102] (which deals with the case $W \geq 0$), and using the fact that $\mathbb{E}(\max_{0\leq j\leq b-1}|W_j|^p) \leq \mathbb{E}(\sum_{i=0}^{b-1}|W_i|^p) < 1$ for some $p > 1$, we obtain $l = 0$.

Define $N = \min_{0\leq i\leq b-1, W_i\neq 0}|W_i|$. Due to assumption (A3) there exists a $q > 1$ such that $\mathbb{E}(N^{-q}) \leq \mathbb{E}(\sum_{i=0}^{b-1}\mathbf{1}_{\{W_i\neq 0\}}|W_i|^{-q}) < \infty$. Then by using (4.35), (4.36), the same arguments as in the proofs of Theorem 2.1 and 2.2 in [102] we can get $|\phi(t)| = O(t^{-q})$ and $|\phi'(t)| = O(t^{-(q+1)})$ when $t \to \infty$. Now as a consequence (Lemma 3 in [4]) we have that Z_W has a density function, which is bounded by $\int_{\mathbb{R}}|\phi(t)|\mathrm{d}t < \infty$.

Step 2. Recall that $\gamma = \gamma^S(\bar{q}) - \delta - \epsilon$ and $\mathcal{K}^S_{\bar{q},\delta+\epsilon}(u,v) = \mathcal{K}_\gamma(u,v)$, as well as

$$\begin{cases} \overline{\mathcal{K}^S_{\bar{q},\delta+\epsilon}}(u,v) = \mathcal{K}_\gamma(u,v)\cdot\mathbf{1}^{(1)}_{u,v}(\bar{q},\epsilon) := \overline{\mathcal{K}_\gamma}(u,v); \\ \overline{\Delta\mathcal{K}^S_{\bar{q},\delta+\epsilon}}(uu',vv') = \left|\mathcal{K}_\gamma(uu',vv') - \mathcal{K}_\gamma(u,v)\right|\cdot\mathbf{1}^{(1)}_{u,v}(\bar{q},\epsilon) := \overline{\Delta\mathcal{K}_\gamma}(uu',vv'). \end{cases}$$

Let us prove the desired estimates, i.e., there exists a constant $C_\gamma > 0$ such that

$$\mathbf{1}^{(2)}_{w,u}(\bar{q},\epsilon)\cdot\mathbb{E}\Big(\overline{\mathcal{K}_\gamma}(u,v)\Big|\mathcal{A}^c(v|_r^-)\Big) \leq C_\gamma\cdot\mathbf{1}^{(2)}_{w,u}(\bar{q},\epsilon)\cdot\begin{cases} e^{r(\xi(\bar{q})+6\epsilon-(\widetilde{\xi}(\bar{q})+6\epsilon)(1-\gamma))}, & \text{if } \gamma > 1; \\ e^{r(\xi(\bar{q})+6\epsilon-\mathbf{1}_{\{p\geq n\}}\cdot(\xi(\bar{q})-6\epsilon)(1-\gamma))}, & \text{if } \gamma < 1; \end{cases}$$

$$\mathbf{1}^{(2)}_{w,u}(\bar{q},\epsilon)\cdot\mathbb{E}\Big(\overline{\Delta\mathcal{K}_\gamma}(uu',vv')\Big|\mathcal{A}^c(v|_r^-)\Big) \leq C_\gamma\cdot\mathbf{1}^{(2)}_{w,u}(\bar{q},\epsilon)\cdot\begin{cases} e^{r(\widetilde{\xi}(\bar{q})+6\epsilon)(1+\gamma)-m(\xi(\bar{q})-6\epsilon)}, & \text{if } \gamma > 1; \\ e^{r(\xi(\bar{q})+6\epsilon)-m(\xi(\bar{q})-6\epsilon)(1-\gamma)}, & \text{if } \gamma < 1. \end{cases}$$

The σ-algebra $\mathcal{A}^c(v|_r^-)$ being defined as in **Claim 2**, we simplify the notations of the following quantities, which are measurable with respect to $\mathcal{A}^c(v|_r^-)$, hence constant given $\mathcal{A}^c(v|_r^-)$:

$$\begin{cases} A = W_{v|_r^-}; \\ B = F_W(v) - F_W(v|_r) + F_W(v|_r^-) - F_W(u); \\ C_1 = \begin{cases} 1, & \text{if } p < n; \\ 2e^{-p(\xi(\bar{q})-6\epsilon)}, & \text{if } p \geq n, \end{cases}; \\ C_2 = e^{-(n\vee p)(\widetilde{\xi}(\bar{q})+6\epsilon)}; \\ D_1 = F_W(vv') - F_W(uu') - (F_W(v) - F_W(u)); \\ D_2 = F_L(vv') - F_L(uu') - (F_L(v) - F_L(u)). \end{cases}$$

Let f_W stand for the bounded density of $Z_W(v_r^-)$ obtained in **Step 1**. Let

$$g_\gamma(s,t) = \Big(|F_W(v) - F_W(u) + s|^2 + |F_L(v) - F_L(u) + t|^2\Big)^{-\gamma/2}, \ \ s,t \in \mathbb{R}.$$

Define

$$\begin{aligned} \zeta_1(\gamma) &= \int_{\mathbb{R}} \frac{f_W(x)}{(|Ax+B|^2 + C_2^2)^{\gamma/2}} \mathrm{d}x; \\ \zeta_2(\gamma) &= \int_{|t|\leq|D_2|} \Big|\frac{\partial}{\partial t} g_\gamma(0,t)\Big| \mathrm{d}t + \int_{|s|\leq|D_1|} \sup_{|t|\leq|D_2|} \Big|\frac{\partial}{\partial s} g_\gamma(s,t)\Big| \mathrm{d}s; \\ \zeta_3(\gamma) &= \int_{|Ax+B|\leq C_1} \frac{f_W(x)}{|Ax+B|^\gamma} \mathrm{d}x; \\ \zeta_4(\gamma) &= \int_{\mathbb{R}} \Big|\frac{1}{|Ax+B+D_1|^\gamma} - \frac{1}{|Ax+B|^\gamma}\Big| f_W(x) \mathrm{d}x. \end{aligned}$$

From (4.29) and **Claim 2**, we have

$$\begin{cases} \mathbf{1}_{u,v}^{(1)}(\bar{q},\epsilon) \cdot \Big|F_W(v) - F_W(u)\Big| \wedge 1 \leq C_1; \\ \mathbf{1}_{u,v}^{(1)}(\bar{q},\epsilon) \cdot \Big(F_L(v) - F_L(u)\Big) \geq \mathbf{1}_{u,v}^{(1)}(\bar{q},\epsilon) \cdot C_2. \end{cases}$$

This implies

$$\mathbf{1}_{u,v}^{(2)}(\bar{q},\epsilon) \cdot \mathbb{E}\Big(\overline{\mathcal{K}_\gamma}(u,v)\Big|\mathcal{A}^c(v|_r^-)\Big) \leq \mathbf{1}_{u,v}^{(2)}(\bar{q},\epsilon) \cdot \begin{cases} 1+\zeta_1(\gamma), & \text{if } \gamma > 1; \\ 1+\zeta_3(\gamma), & \text{if } \gamma < 1; \end{cases} \tag{4.37}$$

$$\mathbf{1}_{u,v}^{(2)}(\bar{q},\epsilon) \cdot \overline{\Delta\mathcal{K}_\gamma}(u,v) \leq \mathbf{1}_{u,v}^{(2)}(\bar{q},\epsilon) \cdot \zeta_2(\gamma), \text{ if } \gamma > 1; \tag{4.38}$$

$$\mathbf{1}_{u,v}^{(2)}(\bar{q},\epsilon) \cdot \mathbb{E}\Big(\overline{\Delta\mathcal{K}_\gamma}(u,v)\Big|\mathcal{A}^c(v|_r^-)\Big) \leq \mathbf{1}_{u,v}^{(2)}(\bar{q},\epsilon) \cdot \zeta_4(\gamma), \text{ if } \gamma < 1, \tag{4.39}$$

where we have used the inequality $|x\vee 1 - y\vee 1| \leq |x-y|$ holds for any $x,y \geq 0$.

Now, we have the following inequalities :

(I) By using the change of variable $y = \frac{Ax+B}{C_2}$ we get

$$\mathbf{1}_{w,u}^{(2)}(\bar{q},\epsilon) \cdot \zeta_1(\gamma) \leq C_W |A|^{-1} C_2^{1-\gamma} \cdot \int_{\mathbb{R}} \frac{\mathrm{d}y}{(y^2+1)^{\frac{\gamma}{2}}};$$

(II) It is not difficult to check that when $F_L(v) - F_L(u) \geq C_2$ and $|t| \leq |D_2|$ we always have

$$\Big|\frac{\partial}{\partial t}g_\gamma(0,t)\Big| \vee \Big|\frac{\partial}{\partial s}g_\gamma(s,t)\Big| \leq \gamma \cdot |F_L(v) - F_L(u) + t|^{-\gamma-1} \leq \gamma \cdot ((C_2 - |D_2|) \vee 0)^{-\gamma-1}.$$

In fact, we have

$$\begin{aligned}
\Big|\frac{\partial}{\partial t}g_\gamma(0,t)\Big| &\leq \frac{\gamma \cdot |F_L(v) - F_L(u) + t|}{\Big(|F_W(v) - F_W(u)|^2 + |F_L(v) - F_L(u) + t|^2\Big)^{\frac{\gamma}{2}+1}} \\
&\leq \frac{\gamma \cdot |F_L(v) - F_L(u) + t|}{|F_L(v) - F_L(u) + t|^{\gamma+2}} = \gamma \cdot |F_L(v) - F_L(u) + t|^{-\gamma-1} \\
\Big|\frac{\partial}{\partial s}g_\gamma(s,t)\Big| &\leq \frac{\gamma \cdot |F_W(v) - F_W(u) + s|}{\Big(|F_W(v) - F_W(u) + s|^2 + |F_L(v) - F_L(u) + t|^2\Big)^{1+\frac{\gamma}{2}}} \\
&\leq \frac{\gamma \cdot |F_W(v) - F_W(u) + s| \cdot |F_L(v) - F_L(u) + t|^{-\gamma}}{|F_W(v) - F_W(u) + s|^2 + |F_L(v) - F_L(u) + t|^2} \\
&\leq \frac{\gamma \cdot |F_L(v) - F_L(u) + t|^{-\gamma}}{2|F_L(v) - F_L(u) + t|} \leq \gamma \cdot |F_L(v) - F_L(u) + t|^{-\gamma-1}
\end{aligned}$$

(where we have used that for any $a, b > 0$, $\frac{a}{a^2+b^2} \leq \frac{1}{2b}$). This together with the definition of $\zeta_2(\gamma)$ yields

$$\mathbf{1}^{(2)}_{u,v}(\bar{q},\epsilon) \cdot \zeta_2(\gamma) \leq 2((C_2 - |D_2|) \vee 0)^{-\gamma-1}(|D_1| + |D_2|); \tag{4.40}$$

(III) By using the change of variable $y = Ax + B$ when $\gamma < 1$ we get

$$\mathbf{1}^{(2)}_{u,v}(\bar{q},\epsilon) \cdot \zeta_3(\gamma) \leq C_W |A|^{-1} \int_{|u| \leq C_1} \frac{\mathrm{d}y}{|y|^\gamma} = 2C_W |A|^{-1} C_1^{1-\gamma};$$

(IV) By using the change of variable $y = \frac{Ax+B}{D_1}$ we get

$$\mathbf{1}^{(2)}_{u,v}(\bar{q},\epsilon) \cdot \zeta_4(\gamma) \leq C_W \frac{|D_1|^{1-\gamma}}{|A|} \int_{\mathbb{R}} \Big|\frac{1}{|y+1|^\gamma} - \frac{1}{|y|^\gamma}\Big| \mathrm{d}y.$$

Now we notice that

$$\int_{\mathbb{R}} (y^2+1)^{-\gamma/2} \mathrm{d}y \ (\gamma > 1) \text{ and } \int_{\mathbb{R}} \Big|\frac{1}{|y+1|^\gamma} - \frac{1}{|y|^\gamma}\Big| \mathrm{d}y \ (\gamma < 1)$$

are both finite and when $\mathbf{1}^{(2)}_{u,v}(\bar{q},\epsilon) = 1$, we have

$$\begin{cases} |A|^{-1} = |W_{v|_r^-}|^{-1} \leq b^{r(\xi(\bar{q})+2\epsilon)}, \\ D_2 \leq \mathrm{Osc}_{F_L}(I_v) + \mathrm{Osc}_{F_L}(I_u) \leq 2b^{-m(\widetilde{\xi}(\bar{q})-6\epsilon)}, \\ D_1 \leq \mathrm{Osc}_{F_W}(I_v) + \mathrm{Osc}_{F_W}(I_u) \leq 2b^{-m(\xi(\bar{q})-6\epsilon)}. \end{cases}$$

Moreover, when $\gamma > 1$ we have $\xi(\bar{q}) < \widetilde{\xi}(\bar{q})$, so if $m \geq 3r$ and $\widetilde{\xi}(\bar{q}) - 12\epsilon > 0$ then

$$\begin{aligned}
\zeta_2(\gamma) &\leq [b^{-r(\widetilde{\xi}(\bar{q})+6\epsilon)}(1 - b^{-(m-r)(\widetilde{\xi}(\bar{q})-6\frac{m+r}{m-r}\epsilon)})]^{-\gamma-1} 2(b^{-m(\xi(\bar{q})-6\epsilon)} + b^{-m(\widetilde{\xi}(\bar{q})-6\epsilon)}) \\
&\leq 4 \cdot b^{r(\widetilde{\xi}(\bar{q})+6\epsilon)(1+\gamma)} \cdot b^{-m(\xi(\bar{q})-6\epsilon)}.
\end{aligned}$$

Then by applying these inequalities to (4.37), (4.38), (4.39) we get the conclusion.

Remark 4.7 *At the end of the proof of Theorem 4.4, we define two quantities* $\Delta\mathcal{K}_{\widetilde{\gamma}}(uu',vv')$ *and* $(\widetilde{\Delta\mathcal{I}^{S,\delta}_{n,\bar{q},\epsilon}})_{p,m}$*, and we claim* (4.25)*. We justify this claim here. In fact,* $(\Delta\mathcal{I}^{S,\delta}_{n,\bar{q},\epsilon})_{p,m}$ *and* $(\widetilde{\Delta\mathcal{I}^{S,\delta}_{n,\bar{q},\epsilon}})_{p,m}$ *can be estimated from above similarly : we have (the first inequality is similar to* (4.38)*)*

$$\begin{aligned}\mathbf{1}^{(1)}_{u,v}(\bar{q},\epsilon)\cdot\mathbf{1}^{(2)}_{u,v}(\bar{q},\epsilon)\cdot\Delta\mathcal{K}_{\widetilde{\gamma}}(uu',vv') &\leq C_2^{-\widetilde{\gamma}}\cdot\zeta_2(1)+C_2^{-1}\zeta_2(\widetilde{\gamma})\\ &\leq 2(C_2-|D_2|)^{-2-\widetilde{\gamma}}(|D_1|+|D_2|),\end{aligned}$$

which is exactly the same bound as in (II) for $\mathbf{1}^{(2)}_{w,u}(\bar{q},\epsilon)\cdot\zeta_2(\gamma)$ *(we have used* (4.40) *with* $\gamma=1$ *and* $\gamma=\widetilde{\gamma}$*). Then, the upper bound estimation of* $(\widetilde{\Delta\mathcal{I}^{S,\delta}_{n,\bar{q},\epsilon}})_{p,m}$ *can be treated like that of* $(\Delta\mathcal{I}^{S,\delta}_{n,\bar{q},\epsilon})_{p,m}$*, and one obtains the same estimate as for* $\mathbb{E}\left((\Delta\mathcal{I}^{S,\delta}_{n,\bar{q},\epsilon})_{p,m}\right)$ *:*

$$\mathbb{E}\left((\widetilde{\Delta\mathcal{I}^{S,\delta}_{n,\bar{q},\epsilon}})_{p,m}\right)\leq C\cdot b^{r-p+1}\cdot e^{\kappa_2\cdot r-\eta_2\cdot m},$$

which is enough to get (4.25)*.*

4.5.3 Proof of Lemma 4.4.

Proof Let

$$S_{p,m}=\sum_{u,v\in\mathscr{A}^m}\mathbf{1}_p(u,v)\cdot\mathbf{1}_{\mathscr{W}_{u|_r}(\bar{q},2\epsilon)}\mathbf{1}_{\mathscr{W}_u(\bar{q},2\epsilon)}\cdot\mathbf{1}_{\mathscr{W}_{v|_r}(\bar{q},2\epsilon)}\mathbf{1}_{\mathscr{W}_v(\bar{q},2\epsilon)}\cdot$$
$$\mathbf{1}_{\mathscr{L}_{u|_r}(\bar{q},2\epsilon)}\mathbf{1}_{\mathscr{L}_u(\bar{q},2\epsilon)}\cdot\mathbf{1}_{\mathscr{L}_{v|_r}(\bar{q},2\epsilon)}\mathbf{1}_{\mathscr{L}_v(\bar{q},2\epsilon)}$$

Then by (4.30) we have $\sum_{u,v\in\mathscr{A}^m}\mathbf{1}_p(u,v)\cdot\mathbf{1}^{(2)}_{u,v}(\bar{q},\epsilon)\leq S_{p,m}$.

Recall that $r=p\vee n$ and $m\geq 3r$. For any $u\in\mathscr{A}^m$ we write $u=u|_r\cdot u'$ with $u'\in\mathscr{A}^{m-r}$. From (4.1) we have $W_u=W_{u|_r}\cdot W_{u'}(u|_r)$, so $\mathbf{1}_{\mathscr{W}_{u|_r}(\bar{q},2\epsilon)}\cdot\mathbf{1}_{\mathscr{W}_u(\bar{q},2\epsilon)}=1$ implies that

$$|W_{u'}(u|_r)|\in[e^{-(m-r)(\xi(\bar{q})+2\frac{m+r}{m-r}\epsilon)},e^{-(m-r)(\xi(\bar{q})-2\frac{m+r}{m-r}\epsilon)}].$$

Thus, when $m\geq 3r$, we have

$$\mathbf{1}_{\mathscr{W}_{u|_r}(\bar{q},2\epsilon)}\cdot\mathbf{1}_{\mathscr{W}_u(\bar{q},2\epsilon)}\leq\mathbf{1}_{\mathscr{W}_{u|_r}(\bar{q},2\epsilon)}\cdot\mathbf{1}_{\mathscr{W}^{[u|r]}_{u'}(\bar{q},4\epsilon)},$$

and, moreover, $\mathbf{1}_{\mathscr{W}_{u|_r}(\bar{q},2\epsilon)}$ and $\mathbf{1}_{\mathscr{W}^{[u|r]}_{u'}(\bar{q},4\epsilon)}$ are independent. Simultaneously we also have

$$\mathbf{1}_{\mathscr{W}_{v|r}(\bar{q},2\epsilon)}\cdot\mathbf{1}_{\mathscr{W}_v(\bar{q},2\epsilon)}\leq\mathbf{1}_{\mathscr{W}_{v|_r}(\bar{q},2\epsilon)}\cdot\mathbf{1}_{\mathscr{W}^{[v|r]}_{v'}(\bar{q},4\epsilon)},$$

$$\mathbf{1}_{\mathscr{L}_{u|r}(\bar{q},2\epsilon)}\cdot\mathbf{1}_{\mathscr{L}_u(\bar{q},2\epsilon)}\leq\mathbf{1}_{\mathscr{L}_{u|_r}(\bar{q},2\epsilon)}\cdot\mathbf{1}_{\mathscr{L}^{[u|r]}_{u'}(\bar{q},4\epsilon)},$$

$$\mathbf{1}_{\mathscr{L}_{v|r}(\bar{q},2\epsilon)}\cdot\mathbf{1}_{\mathscr{L}_v(\bar{q},2\epsilon)}\leq\mathbf{1}_{\mathscr{L}_{v|_r}(\bar{q},2\epsilon)}\cdot\mathbf{1}_{\mathscr{L}^{[v|r]}_{v'}(\bar{q},4\epsilon)}.$$

We can drop the terms $\mathbf{1}_{\mathscr{W}_{v|r}(\bar{q},2\epsilon)}$ and $\mathbf{1}_{\mathscr{L}_{v|r}(\bar{q},2\epsilon)}$ so that the remaining indicator functions on the right hand side of the above inequalities are independent. Since for each $u\in\mathscr{A}^m$, there are at most $2b^{r-p+1}$ many $v|_r$ such that $\mathbf{1}_p(u,v)=1$, we get

$$\mathbb{E}(N_{p,m})\ \leq\ 2b^{r-p+1}\mathbb{E}\Big(\sum_{u\in\mathscr{A}^r}\mathbf{1}_{\mathscr{W}_u(\bar{q},2\epsilon)}\mathbf{1}_{\mathscr{L}_u(\bar{q},2\epsilon)}\Big)\cdot\mathbb{E}\Big(\sum_{u\in\mathscr{A}^{m-r}}\mathbf{1}_{\mathscr{W}_u(\bar{q},4\epsilon)}\mathbf{1}_{\mathscr{L}_u(\bar{q},4\epsilon)}\Big)^2.$$

For $\bar{q}$ with $|\bar{q}| + \xi(\bar{q}) + \widetilde{\xi}(\bar{q}) + |\tau(\bar{q})| \leq \rho_K$ we always have

$$\mathbf{1}_{\mathscr{W}_u(\bar{q},2\epsilon)} \leq |W_u|^{\bar{q}} \cdot e^{|u|(\bar{q}\xi(\bar{q})+2\rho_K\epsilon)} \text{ and } \mathbf{1}_{\mathscr{L}_u(\bar{q},2\epsilon)} \leq L_u^{\tau(\bar{q})} \cdot e^{|u|(-\tau(\bar{q})\widetilde{\xi}(\bar{q})+2\rho_K\epsilon)}.$$

For $k \in \{r, m-r\}$ this yields

$$\mathbb{E}\left(\sum_{u\in\mathscr{A}^k} \mathbf{1}_{\mathscr{W}_u(\bar{q},4\epsilon)}\mathbf{1}_{\mathscr{L}_u(\bar{q},4\epsilon)}\right) \leq e^{k(\gamma(\bar{q})+8\rho_K\epsilon)},$$

where recall that $\gamma(\bar{q}) = \bar{q}\xi(\bar{q}) - \tau(\bar{q})\widetilde{\xi}(\bar{q})$. This gives us the conclusion.

Chapitre 5

Graph and range singularity spectra of random wavelet series built from Gibbs measures

This chapter deals with the graph and range singularity spectra of random wavelet series. We apply the method used in Chapter 4 and achieve the graph and range singularity spectra for random wavelet series built from Gibbs measures.

5.1 Introduction

5.1.1 Orthogonal wavelet basis and multifractal analysis

Let ψ be an r_0-smooth mother wavelet on $\mathbb{R}$, with $r_0 \in \mathbb{N}^*$, so that the functions $\{\psi_{j,k} = \psi(2^j \cdot -k)\}_{(j,k)\in\mathbb{Z}^2}$ form an orthogonal wavelet basis of $L^2(\mathbb{R})$ (see [119] for instance for the definition and construction). Each function $f \in L^2(\mathbb{R})$ can be written as

$$f(x) = \sum_{(j,k)\in\mathbb{Z}^2} d_{j,k} \cdot \psi_{j,k}(x), \text{ where}$$

where the wavelet coefficient $d_{j,k}$ is given by

$$d_{j,k} = 2^j \int_{\mathbb{R}} f(t) \cdot \psi_{j,k}(t)\, \mathrm{d}t.$$

It is known that the asymptotic behavior of the wavelet coefficients provides fine informations on the Hölder regularity of the function. For example, due to Proposition 4 in [81], if there exist constants $C_0 > 0$, $\epsilon_0 \in (0,1)$ such that $|d_{j,k}| \leq C_0 2^{-\epsilon_0 j}$ for each $j \geq 0$ and $k \in \mathbb{Z}$, then f is ϵ_0-Hölder continuous, that is there exist $C > 0$, $\delta > 0$ such that for any $x, y \in \mathbb{R}$ with $|x-y| \leq \delta$, we have $|f(x) - f(y)| \leq C|x-y|^{\epsilon_0}$. Moreover, once f is ϵ_0-Hölder continuous, one can also obtain the pointwise regularity of f from its wavelet coefficients : for each $(j,k) \in \mathbb{Z}^2$ define the wavelet leader

$$L_{j,k} = \sup\left\{|d_{j',k'}| : (j',k') \in \mathbb{Z}^2,\ [k'2^{-j'}, (k'+1)2^{-j'}) \subset [k2^{-j}, (k+1)2^{-j})\right\}$$

and for $x_0 \in \mathbb{R}$ and $j \geq 0$ define the coefficient

$$L_j(x_0) = \sup\left\{L_{j,k} : k \in \mathbb{Z},\ x_0 \in [(k-1)2^{-j}, (k+2)2^{-j})\right\}$$

and the exponent

$$\bar{h}_f(x_0) = \liminf_{j\to+\infty} -j^{-1}\log_2 L_j(x_0). \tag{5.1}$$

Then, due to Corollary 1 in [81], for any $x_0 \in \mathbb{R}$, if $[\bar{h}_f(x_0)] \leq r_0$ we have

$$\bar{h}_f(x_0) = \sup\left\{h > 0 : \exists P \in \mathbb{C}[x], |f(x) - P(x - x_0)| = O(|x - x_0|^h), x \to x_0\right\},$$

that is $\bar{h}_f(x_0)$ provides another very natural pointwise exponent for f at x_0, whose connection with $h_f(x_0)$ is explained in the following remark.

Remark 5.1 *By definition we have $h_f(x_0) \leq \bar{h}_f(x_0)$, and $h_f(x_0) = \bar{h}_f(x_0)$ if neither of them is an integer. The difference between these two exponents is that $\bar{h}_f(x_0)$ is not influenced by addition of a polynomial function, whereas $h_f(x_0)$ describes directly the oscillation of function f around x_0, and his sensible to the addition of a polynomial function.*

Wavelet expansion is thus an effective tool to study the local regularity of a function. It is also connected to the Hausdorff spectrum as follows. Define the scaling function of f as

$$\xi_f : q \in \mathbb{R} \mapsto \xi_f(q) = \liminf_{j\to+\infty} -j^{-1}\log_2 \sum_{k\in\mathbb{Z}:[k2^{-j},(k+1)2^{-j}]\subset[0,1],L_{j,k}\neq 0} |L_{j,k}|^q. \tag{5.2}$$

Then, if r_0 is large enough so that $\bar{h}_f(x) \leq r_0$ for all $x \in [0,1]$,we have

$$\dim_H\{x \in (0,1) : \bar{h}_f(x) = h\} \leq \xi_f^*(h) := \inf_{q\in\mathbb{R}} q \cdot h - \xi_f(q),\ h \geq 0, \tag{5.3}$$

where a negative dimension means that the set is empty [81, 82]. One says that the restriction of f to $[0,1]$ fulfills the multifractal formalism at $h \geq 0$ if the above inequality is an equality.

5.1.2 Random wavelet series built from multifractal measure

In [23], Barral and Seuret construct a class of wavelet series by directly taking the wavelet coefficients built from some well-known multifractal measures, in such a way that the Hausdorff spectrum of the wavelet series can be directly deduced from that of the measure.

Specifically, let μ be a positive Borel measure on $\mathbb{R}$ supported by the interval $[0,1]$, and define

$$F_\mu(x) = \sum_{j=0}^{\infty}\sum_{k=0}^{2^j-1} d_{j,k}\cdot\psi_{j,k}(x),\ \text{where } d_{j,k} = \pm 2^{-j(s_0-1/p_0)}\mu([k2^{-j},(k+1)2^{-j}))^{1/p_0},$$

with $s_0, p_0 > 0$ and $s_0 - 1/p_0 > 0$.

Denote by $I_{j,k} = [k2^{-j}, (k+1)2^{-j})$. Notice that in this setting, the wavelet leader $L_{j,k}$ is nothing but $|d_{j,k}| = 2^{-j(s_0-1/p_0)}\mu(I_{j,k})^{1/p_0}$, thus due to (5.2), we have

$$\xi_{F_\mu}(q) = q(s_0 - 1/p_0) + \tau_\mu(q/p_0),\ q \in \mathbb{R}, \tag{5.4}$$

where the so called Rényi entropy or L^q spectrum of μ is

$$\tau_\mu(q) = \liminf_{n\to\infty} -j^{-1}\log_2 \sum_{k=0,\cdots,2^j-1} \mathbf{1}_{\{\mu(I_{j,k})\neq 0\}} \cdot \mu(I_{j,k})^q,\ q \in \mathbb{R}. \tag{5.5}$$

By construction, F_μ is $(s_0 - 1/p_0)$-Hölder continuous and belongs to the Besov space $B^{s_0,\infty}_{p_0}(\mathbb{R})$ if $s_0 < r_0$. Moreover, if μ fulfills the multifractal formalism for measures at $\alpha \geq 0$ (in the sense of [40]), then the restriction of F_μ to $[0,1]$ fulfills the multifractal formalism described above at $h = s_0 - 1/p_0 + \alpha/p_0$ when $[h] \leq r_0$ (see [23]).

From now on, F_μ stands for the restriction of F_μ to $[0,1]$.

In [23], Barral and Seuret also considered some random multiplicative perturbation of F_μ. It consists in considering a sequence of independent random variables $\{\pi_{j,k}\}_{j\geq 0,\ k\in\{0,1,\cdots,2^j-1\}}$ and then the wavelet series F_μ^{pert} on $[0,1]$ whose coefficients are given by $d_{j,k}^{\text{pert}} = \pi_{j,k}\cdot d_{j,k}$:

$$F_\mu^{\text{pert}}(x) = \sum_{j=0}^{\infty}\sum_{k=0}^{2^j-1} \pi_{j,k}\cdot(\pm 2^{-j(s_0-1/p_0)}\mu(I_{j,k})^{1/p_0})\cdot\psi_{j,k}(x).$$

Under certain conditions on the moments of $\pi_{j,k}$, for example,

(A1) For any $q \in \mathbb{R}$ we have $\sup_{j\geq 0}\sup_{k=0,1,\cdots,2^j-1}\mathbb{E}(|\pi_{j,k}|^q) < \infty$.

they show that, with probability 1, F_μ^{pert} fulfills the multifractal formalism at h whenever F_μ does.

5.1.3 Main result

This chapter studies the graph and range singularity spectra for random wavelet series F_μ^{pert}, where μ is the canonical image on $[0,1]$ of a Gibbs measure μ_φ associated with a Hölder potential φ on a symbolic space Σ (see Section 5.2 and 5.3.1 for precise definitions).

The random perturbation of F_μ is essential to our approach based on the potential theoretic method for the estimation of Hausdorff dimensions (see Chapter 4 in [51]). The efficiency of the combination between randomness and potential theoretic method has been used to compute the Hausdorff dimension of the whole graph of classical processes [51], random Weierstrass function [72] and random wavelet series [139] (see also [77, 138] for questions related to the dimensions of the whole graph of wavelet series).

In addition to (A1), we assume :
(A2) The mother wavelet ψ has only finite many zeros on $[0,1]$.
(A3) Each random variable $\pi_{j,k}$ has a bounded density function $f_{j,k}$ and for any $\epsilon > 0$ we have $\sum_{j\geq 0}(\sup_{k=0,\cdots,2^j-1}\|f_{j,k}\|_\infty)\cdot 2^{-j\epsilon} < \infty$.

Under these assumptions we prove the following result :

Theorem 5.1 *With probability 1 for all $h \in (0,1)$ such that $\xi^*_{F_\mu}(h) > 0$,*

$$
\begin{aligned}
d^G_{F^{pert}_\mu}(h) &= \frac{d_{F^{pert}_\mu}(h)}{h} \wedge \Big(d_{F^{pert}_\mu}(h) + 1 - h\Big) = \frac{\xi^*_{F_\mu}(h)}{h} \wedge \Big(\xi^*_{F_\mu}(h) + 1 - h\Big),\\
d^R_{F^{pert}_\mu}(h) &= \frac{d_{F^{pert}_\mu}(h)}{h} \wedge 1 = \frac{\xi^*_{F_\mu}(h)}{h} \wedge 1.
\end{aligned}
$$

Remark 5.2

(1) Since we are dealing with sets on the graph and range, it is more convenient to use the oscillating exponent $h_{F^{\text{pert}}_\mu}$. But while transferring the local dimension of Gibbs measure μ to the Hölder exponent of wavelet series F^{pert}_μ, we have to use $\bar{h}_{F^{\text{pert}}_\mu}$. So to avoid complications, we only consider the iso-Hölder set $E_{F^{\text{pert}}_\mu}(h)$ for $h \in (0,1)$, since in this case $h_{F^{\text{pert}}_\mu}$ and $\bar{h}_{F^{\text{pert}}_\mu}$ are equal everywhere on the set $E_{F^{\text{pert}}_\mu}(h)$. For $h \geq 1$, it is clear that $\dim_H G_{F^{\text{pert}}_\mu}(h) = \dim_H E_{F^{\text{pert}}_\mu}(h)$, which provides us with the whole graph spectrum. But we have no result for the range singularity spectrum for $h \geq 1$.

(2) Notice that our result is uniform. It is valid almost surely for all $h \in (0,1)$ with $\xi^*_{F_\mu}(h) > 0$, and not just for each $h \in (0,1)$ almost surely.

Let us roughly explain our strategy to prove Theorem 5.1. We apply the potential theoretic method to families of images of Gibbs measures on the graph and range of F^{pert}_μ. We must consider the restrictions of the potentials $(q\varphi)_{q\in\mathbb{R}}$ on a sequence $\{X_k\}_{k\geq 1}$ of subshifts of finite type of Σ whose canonical projection $\widetilde{X}_k$ in $[0,1]$ has a positive Hausdorff distance to the set of zeros of ψ, which tends to 0 as k tends to ∞. We also need to consider the canonical projections on $[0,1]$ of the equilibrium states of these restricted potentials, that we denote by $\{(\mu^{(k)}_q)_{q\in\mathbb{R}}\}_{k\geq 1}$. Then for each $k \geq 1$, there exists an interval J_k such that for each $q \in J_k$, there exists an exponent $h^{(k)}_q \in (0,1)$ as well as $E^{(k)}_q \subset E_{F^{\text{pert}}_\mu}(h^{(k)}_q) \cap \widetilde{X}_k$ such that $\mu^{(k)}_q(E^{(k)}_q) > 0$, and two numbers $\gamma^{(k)}_{q,G}, \gamma^{(k)}_{q,R} > 0$ such that for any $\delta > 0$ small enough, almost surely, for all $q \in J_k$

$$
\iint_{s,t\in E^{(k)}_q} (|F^{\text{pert}}_\mu(s) - F^{\text{pert}}_\mu(t)|^2 + |s-t|^2)^{-(\gamma^{(k)}_{q,G}-\delta)/2} \mathrm{d}\mu^{(k)}_q(s)\mathrm{d}\mu^{(k)}_q(t) < \infty
$$

and

$$
\iint_{s,t\in E^{(k)}_q} |F^{\text{pert}}_\mu(s) - F^{\text{pert}}_\mu(t)|^{-\gamma^{(k)}_{q,R}+\delta} \mathrm{d}\mu^{(k)}_q(s)\mathrm{d}\mu^{(k)}_q(t) < \infty.
$$

This yields the almost sharp lower bounds

$$
d^G_{F^{\text{pert}}_\mu}(h^{(k)}_q) \geq \gamma^{(k)}_{q,G} - \delta \text{ and } d^R_{F^{\text{pert}}_\mu}(h^{(k)}_q) \geq \gamma^{(k)}_{q,R} - \delta,
$$

and by letting k tend to ∞ we can get the sharp lower bound in Theorem 5.1 (See Section 5.3.3 for details).

The reason why we must consider subshift of finite type that avoids zeros of ψ is that for proving the finiteness of the above integrals, we have to control from below the increment $|\psi(s)-\psi(t)|$ when $s \in E_q^{(k)}$ and t is far away from s. This is possible only if s is never too close to the zeros of ψ.

Here we must mention the results of Roueff in [139] which deals with the Hausdorff dimension of whole graph of random wavelet series. Briefly speaking, let $\{c_{j,k}\}_{j\geq 0, k=0,\cdots,2^j-1}$ be a sequence of real valued random variables whose laws are absolutely continuous with respect to Lebesgue measure. Let $\mathcal{T}(c_{j,k})$ stand for the L^∞ norm of the density of $c_{j,k}$. Roueff proves that (Theorem 1 in [139]) if ψ has finitely many zeros on $[0,1]$, then the Hausdorff dimension of the graph of the random wavelet series

$$F(x)=\sum_{j\geq 0}\sum_{k=0}^{2^j-1} c_{j,k}\cdot\psi_{j,k}(x)$$

is almost surely larger than or equal to

$$\limsup_{J\to\infty}\liminf_{j\to\infty}\frac{\log\min_{i=j}^{j+J}\left\{\sum_{k=0}^{2^i-1}\min\{1,\mathcal{T}(c_{i,k})\cdot 2^{-i}\}\cdot\nu(I_{i,k})^2\right\}}{-j\log 2},$$

where ν can be chosen as any probability measure on $[0,1]$ such that there exists a constant C and $s>0$ such that for any Borel sets $A\subset[0,1]$ and $B\subset A$ such that $\nu(A)>0$ we have $\nu(B)/\nu(A)\leq C(|B|/|A|)^s$, where $|B|$, $|A|$ stand for the diameters of A and B. Due to the scaling properties of the equilibrium state $\mu_{q\varphi}$ of each potential $q\varphi$, $q\in\mathbb{R}$, it is natural to try using Roueff's approach to our problem : for $j\geq 0$ and $k=0,\cdots,2^j-1$ we take

$$c_{j,k}=\pi_{j,k}\cdot d_{j,k},\ \text{where } d_{j,k}=\pm 2^{-j(s_0-1/p_0)}\mu(I_{j,k})^{1/p_0},$$

this, together with (A3) and the definition of $\mathcal{T}(c_{j,k})$, gives us

$$\mathcal{T}(c_{j,k})=\|f_{j,k}\|_\infty\cdot|d_{j,k}|^{-1}$$

Then, for $q\in\mathbb{R}$ and $\epsilon>0$ define

$$E_n(q,\epsilon)=\left\{x\in[0,1]:\forall\ j\geq n,\ \left\{\begin{array}{l}|d_{j,k_{j,x}}|\in[2^{-j(h_q+\epsilon)},2^{-j(h_q-\epsilon)}],\\ \mu_q(I_{j,k_{j,x}})\in[2^{-j(\xi^*_{F_\mu}(h_q)+\epsilon)},2^{-j(\xi^*_{F_\mu}(h_q)-\epsilon)}]\end{array}\right.\right\},$$

where μ_q is the canonical projection of $\mu_{q\varphi}$ on $[0,1]$, $k_{j,x}$ is the unique integer k such that $x\in I_{j,k}=[k2^{-j},(k+1)2^{-j})$ and $h_q=s_0-1/p_0+\tau'_\mu(q)/p_0$. Due to Section 5.2.3 and 5.3.1 we have for any $\epsilon>0$, $\mu_{q\varphi}(\lim_{n\to\infty}E_n(q,\epsilon))=1$. By continuity we can find an integer $N_{q,\epsilon}$ such that $\mu_{q\varphi}(E_{N_{q,\epsilon}}(q,\epsilon))>0$. Now we define ν by $\nu(B)=\mu_{q\varphi}(B\cap E_{N_{q,\epsilon}})$ for each Borel subset of $[0,1]$ (here $\nu(B)/\nu(A)\leq C(|B|/|A|)^s$ holds with some constant C, $s=\xi^*_{F_\mu}(h_q)-\epsilon$, and A and B dyadic intervals). Then for $j>N_{q,\epsilon}$, we have

$$\begin{aligned}
&\sum_{k=0}^{2^j-1}\min\{1,\mathcal{T}(c_{j,k})\cdot 2^{-j}\}\cdot\nu(I_{j,k})^2\\
\leq\ &\sum_{k=0}^{2^j-1}\min\{1,\|f_{j,k}\|_\infty\cdot|d_{j,k}|^{-1}\cdot 2^{-j}\}\cdot\mu_{q\varphi}(I_{j,k}\cap E_{N_{q,\epsilon}}(q,\epsilon))^2\\
\leq\ &\sum_{k=0}^{2^j-1}\min\{1,\|f_{j,k}\|_\infty\cdot 2^{-j(1-h_q-\epsilon)}\}\cdot 2^{-2j(\xi^*_{F_\mu}(h_q)-\epsilon)}\cdot\mathbf{1}_{\left\{\mu_{q\varphi}(I_{j,k})\geq 2^{-j(\xi^*_{F_\mu}(h_q)+\epsilon)}\right\}}
\end{aligned}$$

$$\leq \sum_{k=0}^{2^j-1} \min\{1, \|f_{j,k}\|_\infty \cdot 2^{-j(1-h_q-\epsilon)}\} \cdot 2^{-2j(\xi^*_{F_\mu}(h_q)-\epsilon)} \cdot 2^{j(\xi^*_{F_\mu}(h_q)+\epsilon)} \cdot \mu_{q\varphi}(I_{j,k})$$
$$\leq \min\{2^{j(1-h_q-\epsilon)}, \sup_{k=0,\cdots,2^j-1} \|f_{j,k}\|_\infty\} \cdot 2^{-j(\xi^*_{F_\mu}(h_q)+1-h_q-4\epsilon)} \sum_{k=0}^{2^j-1} \mu_{q\varphi}(I_{j,k})$$
$$\leq \big(\sup_{k=0,\cdots,2^j-1} \|f_{j,k}\|_\infty\big) \cdot 2^{-j(\xi^*_{F_\mu}(h_q)+1-h_q-4\epsilon)}$$

Then under assumptions (A1-3), due to the fact that μ_q is carried by the set $E_{F_\mu^{\text{pert}}}(h_q)$ almost surely, Roueff's result implies that, if $\xi^*_{F_\mu}(h_q) + 1 - h_q - 4\epsilon > 1$ (this is essential in his proof), then almost surely

$$d^G_{F_\mu^{\text{pert}}}(h_q) \geq \xi^*_{F_\mu}(h_q) + 1 - h_q - 4\epsilon.$$

By taking a sequence of ϵ tending to 0, we get the sharp lower bound for $d^G_{F_\mu^{\text{pert}}}(h_q)$ given by Theorem 5.1 when $d^G_{F_\mu^{\text{pert}}}(h_q) > 1$. But this result holds only "for each $q \in \mathbb{R}$ almost surely", so is not uniform like Theorem 5.1, and it seems that Roueff's method cannot yield such a result, nor the value of $d^G_{F_\mu^{\text{pert}}}(h_q)$ when $d^G_{F_\mu^{\text{pert}}}(h_q) \leq 1$ and $h_q \leq 1$.

The rest of the chapter is organized as follows : Section 5.2 gives some definitions and notations about subshift of finite type, Gibbs measure and its multifractal analysis. Sections 5.3 and 5.4 provide the proof of Theorem 5.1.

5.2 Subshift of finite types, Gibbs measures and multifractal analysis

5.2.1 Subshift of finite type

Let $\Sigma = \{0,1\}^{\mathbb{N}_+}$ and $\Sigma_* = \bigcup_{n\geq 0} \Sigma_n$, where $\Sigma_0 = \{\varnothing\}$ and $\Sigma_n = \{0,1\}^n$ for $n \geq 1$.

Denote the length of w by $|w| = n$ if $w \in \Sigma_n$, $n \geq 0$ and $|w| = \infty$ if $w \in \Sigma$.

For $w \in \Sigma_*$ and $t \in \Sigma_* \bigcup \Sigma$, the concatenation of w and t is denoted by $w \cdot t$ or wt.

For $w \in \Sigma_*$, the cylinder with root w, i.e. $\{w \cdot u : u \in \Sigma\}$ is denoted by $[w]$.

The set Σ is endowed with the standard metric distance

$$\rho(s,t) = \inf\{2^{-n} : n \geq 0,\ \exists\ w \in \Sigma_n \text{ such that } s,t \in [w]\}.$$

Then (Σ, ρ) is a compact metric space. Denote by $\mathcal{B}$ the Borel σ-algebra with respect to ρ. Clearly $\mathcal{B}$ can be generated by the cylinders $[w]$, $w \in \Sigma_*$.

If $n \geq 1$ and $w = w_1 \cdots w_n \in \Sigma_n$ then for every $0 \leq i \leq n$, we write $w|_i = w_1 \ldots w_i$, with the convention $w|_0 = \varnothing$. Also, for any infinite word $t = t_1 t_2 \cdots \in \Sigma$ and $i \geq 0$, we write $t|_i = t_1 \ldots t_i$, with the convention $t|_0 = \varnothing$.

For $t \in \Sigma$ define the left side shift $\sigma : \Sigma \mapsto \Sigma$ by

$$\sigma(t_1 t_2 \cdots) = t_2 t_3 \cdots .$$

A subshift is a σ-invariant compact set $X \subset \Sigma$, that is $\sigma(X) \subset X$.

A subshift X is said to be of finite type if there is an admissible set $A \subset \Sigma_n$ for some $n \geq 2$ such that

$$X = \{t \in \Sigma : \sigma^m(t)|_n \in A,\ \forall\, m \geq 0\}.$$

The admissible set A induces a transition matrix $B : \Sigma_{n-1} \times \Sigma_{n-1} \mapsto \{0, 1\}$ with $B(a_1 \cdots a_{n-1}, a_2 \cdots a_n) = 1$ if $a_1 \cdots a_n \in A$, and $B(i, j) = 0$ otherwise. Then X can be redefined as

$$X = \{t \in \Sigma : B(\sigma^m(t)|_{n-1}, \sigma^{m+1}(t)|_{n-1}) = 1,\ \forall\, m \geq 0\}.$$

The dynamical system (X, σ) is called topologically transitive (resp. mixing) if B is irreducible, that is for any $i, j \in \Sigma_{n-1}$ there is a $k \geq 1$ such that $B^k(i, j) > 0$ (resp. if B is primitive, that is there is a $k \geq 1$ such that $B^k(i, j) > 0$ for all $i, j \in \Sigma_{n-1}$).

5.2.2 Gibbs measure on topologically transitive subshift of finite type

Let φ be a Hölder continuous function defined on Σ, which will be mentioned as a Hölder potential in the following.

Let (X, σ) be a topologically transitive subshift of finite type of the full shift (Σ, σ).

For $n \geq 1$ the n^{th}-order Birkhoff sum of φ over σ is the function

$$S_n\varphi(t) = \sum_{i=0}^{n-1} \varphi \circ \sigma^i(t),\ t \in \Sigma.$$

The topological pressure of φ on X is defined by

$$P_X(\varphi) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{w \in \Sigma_n : [w] \cap X \neq \emptyset} \exp\left(\max_{t \in [w]} S_n\varphi(t)\right) \tag{5.6}$$

(the existence of the limit is ensured by sub-additivity properties of the logarithm on the right hand side).

It follows from the thermodynamic formalism developed by Sinai, Ruelle, Bowen and Walters [38, 140] that there exists a constant $C(\varphi)$ (independent of X), as well as a unique ergodic measure μ_φ on (X, σ), namely the equilibrium state or Gibbs measure of φ restricted to X, such that for any $t \in X$, $n \geq 0$ and $t' \in [t|_n]$,

$$C(\varphi)^{-1} \leq \frac{\mu_\varphi([t|_n])}{\exp(S_n\varphi(t') - nP_X(\varphi))} \leq C(\varphi), \tag{5.7}$$

and μ_φ possesses the quasi-Bernoulli property,

$$C(\varphi)^{-1}\mu_\varphi([w])\mu_\varphi([u]) \leq \mu_\varphi([wu]) \leq C(\varphi)\mu_\varphi([w])\mu_\varphi([u]), \quad \forall w, u \in \Sigma_*,\ [wu] \cap X \neq \emptyset. \tag{5.8}$$

5.2.3 Multifractal analysis of Gibbs measure

Here we follow [134, 14]. Consider a topologically transitive subshift X of finite type and a Hölder potential φ on X. Denote by μ_φ the equilibrium state on (X,σ) with potential φ.

Define the Rényi entropy or L^q spectrum of μ_φ as

$$\tau_{\mu_\varphi}(q) = \liminf_{n\to\infty} -\frac{1}{n}\log_2 \sum_{w\in\Sigma_n,\mu_\varphi([w])\neq 0} \mu_\varphi([w])^q,\ q\in\mathbb{R}. \tag{5.9}$$

It is easy to deduce from (5.6) and (5.7) that the above limit inferior is in fact a limit, and it is equal to

$$\tau_{\mu_\varphi}(q) = \frac{1}{\log 2}(qP_X(\varphi) - P_X(q\varphi)). \tag{5.10}$$

Due to Corollary 5.27 in [140], if (X,σ) is topologically transitive, then $q\mapsto P_X(q\varphi)$ is a convex analytic function on $\mathbb{R}$, thus τ_{μ_φ} is a concave analytic function on $\mathbb{R}$ and

$$\tau'_{\mu_\varphi}(q) = \frac{1}{\log 2}(P_X(\varphi) - \frac{\mathrm{d}}{\mathrm{d}q}P_X(q\varphi)). \tag{5.11}$$

Denote by $\tau^*_{\mu_\varphi} : \alpha\in\mathbb{R}\mapsto \inf_{q\in\mathbb{R}} q\alpha - \tau_{\mu_\varphi}(q)$ the Legendre transform of τ_{μ_φ}. Since τ_{μ_φ} is concave and analytic over $\mathbb{R}$, we have for any $q\in\mathbb{R}$,

$$\tau^*_{\mu_\varphi}(\tau'_{\mu_\varphi}(q)) = q\tau'_{\mu_\varphi}(q) - \tau_{\mu_\varphi}(q) = -q\frac{\mathrm{d}}{\mathrm{d}q}P_X(q\varphi) + P_X(q\varphi). \tag{5.12}$$

For $\alpha\in\{\tau'_{\mu_\varphi}(q) : q\in\mathbb{R}\}$ define the set

$$E_{\mu_\varphi}(\alpha) = \Big\{t\in X : \lim_{n\to\infty}\frac{\log\mu_\varphi([t|_n])}{\log 2^{-n}} = \alpha\Big\}$$

and

$$\widetilde{E}_{\mu_\varphi}(\alpha) = \Big\{t\in X : \lim_{n\to\infty}\frac{\log\mu_\varphi([t|_n])}{\log 2^{-n}} = \lim_{n\to\infty}\frac{\log\max_{w\in\mathcal{N}(t|_n)}\mu_\varphi([w])}{\log 2^{-n}} = \alpha\Big\},$$

where for any $w\in\Sigma_*$ define the set of neighbor words of w by

$$\mathcal{N}(w) = \Big\{u\in\Sigma_{|w|} : \Big|\sum_{i=1}^{|w|}(u_i - w_i)\cdot 2^{-i}\Big| \le 2^{-|w|}\Big\}. \tag{5.13}$$

By using (5.9), it is standard to check that

$$\dim_H \widetilde{E}_{\mu_\varphi}(\alpha) \le \dim_H E_{\mu_\varphi}(\alpha) \le \tau^*_{\mu_\varphi}(\alpha).$$

Moreover, one can prove that this is actually an equality : For $q\in\mathbb{R}$ denote by $\mu_{q\varphi}$ the equilibrium state of $q\varphi$ restricted to X. Then applying (5.7) to μ_φ and $\mu_{q\varphi}$, together with (5.10) we can easily get for any $t\in X$ and $n\ge 1$,

$$(C(\varphi)^{|q|}C(q\varphi))^{-1}\mu_\varphi([t|_n])^q e^{-n\tau_\varphi(q)} \le \mu_{q\varphi}([t|_n]) \le C(\varphi)^{|q|}C(q\varphi)\mu_\varphi([t|_n])^q e^{-n\tau_\varphi(q)}.$$

Then due to [14] and the fact that μ_φ is quasi-Bernoulli, we get for $\mu_{q\varphi}$-almost every $t \in X$,

$$\lim_{n\to\infty} \frac{\log \mu_\varphi([t|_n])}{\log 2^{-n}} = \lim_{n\to\infty} \frac{\log \max_{w\in\mathcal{N}(t|_n)} \mu_\varphi([w])}{\log 2^{-n}} = \tau'_{\mu_\varphi}(q)$$

and

$$\lim_{n\to\infty} \frac{\log \mu_{q\varphi}([t|_n])}{\log 2^{-n}} = \tau^*_{\mu_\varphi}(\tau'_{\mu_\varphi}(q)).$$

Due to the mass distribution principle, this implies that for any $q \in \mathbb{R}$,

$$\dim_H \widetilde{E}_{\mu_\varphi}(\tau'_{\mu_\varphi}(q)) \geq \dim_H(\mu_{q\varphi}) \geq \tau^*_{\mu_\varphi}(\tau'_{\mu_\varphi}(q)), \tag{5.14}$$

where for any positive Borel measure μ defined on a compact metric space, the lower Hausdorff dimension of μ is given by $\dim_H(\mu) = \inf\{\dim_H E : \mu(E) > 0\}$.

5.3 Proof of Theorem 5.1

From now on we fix a Hölder potential φ on the Σ and denote by μ the Gibbs measure on (Σ, σ) with potential φ. We avoid the trivial case that φ is a constant function.

5.3.1 The multifractal nature of F_μ and $F_\mu^{\mathbf{pert}}$.

Denote the canonical mapping

$$\lambda : w \in \Sigma_* \bigcup \Sigma \mapsto \lambda(w) = \sum_{i=1}^{|w|} w_i \cdot 2^{-i} \in [0, 1].$$

For $w \in \Sigma_*$ let

$$T_w : x \in \mathbb{R} \mapsto 2^{-|w|} \cdot x + \lambda(w) \text{ and } \psi_w = \psi \circ T_w^{-1}.$$

Consider the wavelet series

$$F_\mu(x) = \sum_{w\in\Sigma_*} d_w \cdot \psi_w(x), \text{ with } |d_w| = 2^{-|w|(s_0-1/p_0)} \mu([w])^{1/p_0}. \tag{5.15}$$

Up to the formal replacement of dyadic intervals by the cylinders of Σ, this is the wavelet series built from the image of μ by λ in Section 5.1.2. Recall (see (5.4)) that

$$\xi_{F_\mu}(q) = q(s_0 - 1/p_0) + \tau_\mu(q/p_0), \ q \in \mathbb{R}.$$

So for $\alpha \geq 0$ and $h = s_0 - 1/p_0 + \alpha/p_0$ such that $h \leq r_0$, we have

$$\dim_H\{x \in (0, 1) : \bar{h}_{F_\mu}(x) = h\} \leq \xi^*_{F_\mu}(h) = \tau^*_\mu(\alpha). \tag{5.16}$$

For $q \in \mathbb{R}$ denote by μ_q the equilibrium state of the potential $q\varphi$ on (Σ, σ).. Applying the results in Section 5.2.3 we have for any $q \in \mathbb{R}$, for μ_q-almost every $t \in \Sigma$,

$$\lim_{n\to\infty} \frac{\log \mu([t|_n])}{\log 2^{-n}} = \lim_{n\to\infty} \frac{\log \max_{w\in\mathcal{N}(t|_n)} \mu([w])}{\log 2^{-n}} = \tau'_\mu(q).$$

Together with Theorem 1 of [23], this implies that μ_q is carried by $\{x \in (0,1) : \bar{h}_{F_\mu}(x) = s_0 - 1/p_0 + \tau'_\mu(q)/p_0\}$. Then due to (5.16) and (5.14), we get that F_μ obeys the multifractal formalism at each $h = s_0 - 1/p_0 + \alpha/p_0$ such that $h \le r_0$ and $\alpha \in \{\tau'_\mu(q) : q \in \mathbb{R}\}$:

$$\dim_H\{x \in [0,1] : \bar{h}_{F_\mu}(x) = h\} = \xi^*_{F_\mu}(h) = \tau^*_\mu(\alpha).$$

The random perturbation F_μ^{pert} is obtained from F_μ and a sequence of independent random variables $\{\pi_w\}_{w\in\Sigma_*}$ as

$$F_\mu^{\text{pert}}(x) = \sum_{w\in\Sigma_*} \pi_w \cdot d_w \cdot \psi_w(x),$$

and our assumption (A1) is : For any $q \in \mathbb{R}$ we have $\sup_{w\in\Sigma_*} \mathbb{E}(|\pi_w|^q) < \infty$. We have seen in Section 5.1.2 that this implies that

$$\xi_{F_\mu^{\text{pert}}} = \xi_{F_\mu} \text{ and } \bar{h}_{F_\mu^{\text{pert}}} = \bar{h}_{F_\mu} \text{ over } (0,1) \text{ almost surely.} \tag{5.17}$$

Thus, F_μ^{pert} fulfills the multifractal formalism at h, whenever F_μ does.

5.3.2 Topologically transitive subshifts of finite type avoiding the set of zeros of ψ

For $k \ge 0$ and $x \in [0,1)$, let $x|_k$ be the unique word $w \in \Sigma_k$ such that

$$\lambda(w) \le x < \lambda(w) + 2^{-k},$$

as well as $1|_k = 1\cdots 1$ for $k \ge 1$.

Let $\mathcal{Z} = \psi^{-1}(\{0\}) \cap [0,1]$. We have assumed that $\mathcal{Z}$ is finite ((A2)).

For $k \ge 2$ define the set of forbidden words by

$$\mathcal{A}_k = \bigcup_{x\in\mathcal{Z}} \mathcal{A}_k(x),$$

where

$$\mathcal{A}_k(x) = \begin{cases} \{x|_k\}, & \text{if } x \notin \lambda(\Sigma_*), \\ \{w \in \Sigma_k : 0 \le \lambda(x|_k) - \lambda(w) \le 2^{-k}\}, & \text{otherwise.} \end{cases}$$

Define the subshift of finite type with respect to $\mathcal{A}_k$ by

$$X_k = \{t \in \Sigma : \sigma^m(t)|_k \notin \mathcal{A}_k,\ \forall\, m \ge 0\}.$$

Clearly for small k, the subshift X_k might be a empty set. But, since $\mathcal{Z}$ is a finite set, it is easy to see that X_k is not empty for all k large enough. In fact, denote by $\delta = \min\{|x-y| : x, y \in \mathcal{Z},\ x \ne y\} > 0$ and $k_0 = [-\log_2 \delta] + 3$. Then for any $x, y \in \mathcal{Z}$ with $x < y$, there exists at least one word $w \in \Sigma_{k_0-1}$ such that $x < \lambda(w) < y$ thus $\lambda(x|_{k_0-1}) < \lambda(w) < \lambda(y|_{k_0-1})$, since $y - x \ge \delta \ge 2^{-(k_0-2)}$. This ensures that for $k \ge k_0$, for all $w \in \mathcal{A}_k$, his brother w' (the unique $w' \in \Sigma_k$ such that $w'|_{k-1} = w|_{k-1}$, $w' \ne w$) is an admissible word. Thus for any $u \in \Sigma_{k-1}$, at least one of $u0, u1$ is allowed in X_k, which also implies that for each $u \in \Sigma_{k-1}$, there exists an infinite word $t \in X_k$ such that $t|_k = u0$ or $t|_k = u1$. So for $k \geq k_0$, the Hausdorff distance between X_k and Σ is not greater than 2^{-k}, that is

$$\mathrm{dist}_H(X_k, \Sigma) := \max\{\sup_{s\in X_k} \inf_{t\in\Sigma} \rho(s,t), \sup_{s\in\Sigma} \inf_{t\in X_k} \rho(s,t)\} \leq 2^{-k}, \tag{5.18}$$

thus it converges to 0 when $k \to \infty$.

Since X_k is a increasing sequence (it is easy to see that $\Sigma \setminus X_k \supset \Sigma \setminus X_{k+1}$), $\overline{\dim}_B X_k$ increases and converges to 1 as $k \to \infty$. Otherwise $\overline{\dim}_B \bigcup_k X_k < 1$, thus $\bigcup_k X_k$ is not dense in Σ, which is in contradiction with (5.18). Here $\overline{\dim}_B$ is the upper box-counting dimension (see [51] for the definition and properties).

It is known that any subshift of finite type can be decomposed into several disjoint closed sets $X_{k,1}, \cdots, X_{k,m}$, $m \geq 1$, and each of them is a topologically transitive subshift of finite type. This can be deduced from the non-negative matrix analysis that one can always decomposes reducible matrix into several irreducible pieces.

The finite stability of $\overline{\dim}_B$ (see Section 3.2 in [51]) implies

$$\overline{\dim}_B X_k = \max_{i=1,\cdots,m} \overline{\dim}_B X_{k,i},$$

so we can choose one of the $X_{k,i}$ such that $\overline{\dim}_B X_{k,i} = \overline{\dim}_B X_k$ and also denote it as X_k. Then we obtain a sequence of topologically transitive subshift of finite type $(X_k)_{k\geq 1}$ such that the upper box-counting dimension $\overline{\dim}_B X_k$ converges to 1. We prove that this sequence converge to Σ in the Hausdorff distance :

Suppose that it is not the case, then there exist an $\epsilon > 0$ and a subsequence $(X_{k_j})_{j\geq 1}$ such that $\mathrm{dist}_H(X_{k_j}, \Sigma) \geq \epsilon$ for $j \geq 1$. Fix an integer $N > -\log_2 \epsilon + 1$. Then $\mathrm{dist}_H(X_{k_j}, \Sigma) \geq \epsilon$ implies that there exist a $w_j \in \Sigma_N$ such that $X_{k_j} \cap [w_j] = \emptyset$. Since $\#\Sigma_N = 2^N$ is finite, then there exist $w_* \in \Sigma_N$ and a subsequence $(X_{k'_j})_{j\geq 1}$ of $(X_{k_j})_{j\geq 1}$ such that $X_{k'_j} \cap [w_*] = \emptyset$ for $j \geq 1$.

Since $X_{k'_j}$ is a subshift of finite type, $X_{k'_j} \cap [w_*] = \emptyset$ implies that

$$X_{k'_j} \subset X_* := \{t \in \Sigma : \sigma^m(t)|_N \neq w_*, \ \forall\, m \geq 0\}.$$

Denote by B_* the transition matrix of X_* and λ_* the maximal eigenvalue of B_*. Due to the standard Perron-Frobenius theory ([142], Thm 1.1), λ_* is strictly less than the maximal eigenvalue of the transition matrix of the full shift, which is equal to 2. This yields that $\overline{\dim}_B X_* = \log \lambda_* / \log 2 < 1$, which is in contradiction with the fact that $\overline{\dim}_B X_* \geq \lim_{j\to\infty} \overline{\dim}_B X_{k'_j} = 1$.

To end this section, since ψ is r_0-smooth, for each $k \geq k_0$ we can easily find a constant $c_{\psi,k} > 0$ such that for each $t \in X_k$,

$$|\psi(\lambda(\sigma^m(t)|_n))| \geq c_{\psi,k}, \ \forall\, m \geq 0, \ n \geq k. \tag{5.19}$$

This is the main property required in our proof, which clearly would not hold if we considered any $t \in \Sigma$.

5.3.3 Lower bound estimation

For $k \geq k_0$ and $q \in \mathbb{R}$, denote by $\mu_q^{(k)}$ the Gibbs measure on (X_k, σ) with potential $q\varphi$.

Apply (5.7) both to $\mu_q^{(k)}$ and μ, together with (5.10) we have for any $t \in X_k$ and $n \geq 1$,

$$(C(\varphi)^{|q|}C(q\varphi))^{-1}\mu([t|_n])^q e^{-n\tau_\varphi(q)} e^{-n\frac{P_\Sigma(q\varphi)-P_{X_k}(q\varphi)}{\log 2}} \leq \mu_q^{(k)}([t|_n])$$
$$\leq C(\varphi)^{|q|}C(q\varphi)\mu([t|_n])^q e^{-n\tau_\varphi(q)} e^{-n\frac{P_\Sigma(q\varphi)-P_{X_k}(q\varphi)}{\log 2}}.$$

By using large deviation method as in [14], it is standard to prove that for $\mu_q^{(k)}$-almost every $t \in X_k$,

$$\begin{aligned}\lim_{n\to\infty}\frac{\log\mu([t|_n])}{\log 2^{-n}} &= \lim_{n\to\infty}\frac{\log\max_{w\in\mathcal{N}(t|_n)}\mu([w])}{\log 2^{-n}}\\ &= \tau_\mu'(q) + \frac{\mathrm{d}}{\mathrm{d}q}\frac{P_\Sigma(q\varphi)-P_{X_k}(q\varphi)}{\log 2} := \alpha_q^{(k)},\end{aligned}\tag{5.20}$$

and

$$\begin{aligned}&\lim_{n\to\infty}\frac{\log\mu_q^{(k)}([t|_n])}{\log 2^{-n}}\\ =\ &\tau_\mu^*(\tau_\mu'(q)) - \frac{-q\frac{\mathrm{d}}{\mathrm{d}q}(P_\Sigma(q\varphi)-P_{X_k}(q\varphi)) + (P_\Sigma(q\varphi)-P_{X_k}(q\varphi))}{\log 2} := D_q^{(k)}.\end{aligned}\tag{5.21}$$

Let $h_q^{(k)} = s_0 - 1/p_0 + \alpha_q^{(k)}/p_0$. Then the above two equations together with Theorem 1 of [23] and (5.17) imply that

$$\mu_q^{(k)} \text{ is carried by } \{x\in[0,1] : \bar{h}_{F_\mu^{\text{pert}}}(x) = h_q^{(k)}\} \text{ and } \dim_H(\mu_q^{(k)}) \geq D_q^{(k)}. \tag{5.22}$$

We deduce from $\mu_q^{(k)}$ two Borel measures $\mu_{q,G}^{(k)}$, $\mu_{q,R}^{(k)}$ carried by the graph and range of F_μ^{pert} respectively in the following way :

– For any Borel set $A \subset G_{F_\mu^{\text{pert}}}([0,1])$, let

$$\mu_{q,G}^{(k)}(A) = \mu_q^{(k)}\left(t\in X_k : (\lambda(t), F_\mu^{\text{pert}}(\lambda(t))) \in A\right);$$

– For any Borel set $B \subset R_{F_\mu^{\text{pert}}}([0,1])$, we have

$$\mu_{q,R}^{(k)}(B) = \mu_q^{(k)}\left(t\in X_k : F_\mu^{\text{pert}}(\lambda(t)) \in B\right).$$

As the essential intermediate result of this chapter, we have the following theorem.

Theorem 5.2 *With probability* 1 *for all* $q\in\mathbb{R}$ *with* $0 < h_q^{(k)} < 1$, *we have*

$$\begin{aligned}\dim_H(\mu_{q,G}^{(k)}) &\geq \gamma_{q,G}^{(k)} := \frac{D_q^{(k)}}{h_q^{(k)}} \wedge \left(1 - h_q^{(k)} + D_q^{(k)}\right),\\ \dim_H(\mu_{q,R}^{(k)}) &\geq \gamma_{q,R}^{(k)} := \frac{D_q^{(k)}}{h_q^{(k)}} \wedge 1.\end{aligned}$$

Let us show how it makes it possible to conclude.

For $k \geq k_0$ let $I^{(k)} = \{h_q^{(k)} : q \in \mathbb{R}\} \cap (0,1)$ and $J^{(k)} = \bigcap_{p \geq k} I^{(p)}$. Also for $S \in \{G, R\}$ define the function

$$f_S^{(k)} : h_q^{(k)} \in I^{(k)} \mapsto \gamma_{q,S}^{(k)}.$$

Due to (5.22), $\mu_{q,S}^{(k)}$ is carried by $S_{F_\mu^{\text{pert}}}(h_q^{(k)})$. Thus Theorem 5.2 implies that, for each $k \geq k_0$, with probability 1 for all $h \in J^{(k)}$,

$$d^S_{F_\mu^{\text{pert}}}(h) \geq f_S^{(p)}(h), \ \forall\, p \geq k.$$

Then to end the proof of Theorem 5.1, it only remains to show that for each $q \in \mathbb{R}$,

$$h_q^{(k)} \to h_q = s_0 - 1/p_0 + \tau'_\mu(q)/p_0 \text{ and } D_q^{(k)} \to D_q = \tau^*_\mu(\tau'_\mu(q)) \text{ as } k \to \infty, \tag{5.23}$$

which implies that $\bigcup_{k \geq k_0} J^{(k)} = \{s_0 - 1/p_0 + \tau'_\mu(q)/p_0 : q \in \mathbb{R}\} \cap (0,1)$ and for any compact subset $I \subset \bigcup_{k \geq k_0} J^{(k)}$, the functions $f_S^{(k)}$, $S \in \{G, R\}$ restricted to I converge uniformly to

$$f_S : h_q \in I \mapsto \gamma_{q,S},$$

where

$$\gamma_{q,G} := \frac{D_q}{h_q} \wedge \Big(1 - h_q + D_q\Big) \ \text{ and } \ \gamma_{q,R} := \frac{D_q}{h_q} \wedge 1.$$

This implies that with probability 1 for all $h \in I$ and $\alpha = hp_0 + 1 - s_0 p_0$,

$$d^G_{F_\mu^{\text{pert}}}(h) \geq \frac{\tau^*_\mu(\alpha)}{h} \wedge \Big(1 - h + \tau^*_\mu(\alpha)\Big) \ \text{ and } \ d^R_{F_\mu^{\text{pert}}}(h) \geq \frac{\tau^*_\mu(\alpha)}{h} \wedge 1.$$

Together with Theorem A and Section 5.3.1, we get the conclusion by taking a sequence of I converging to $\{s_0 - 1/p_0 + \tau'_\mu(q)/p_0 : q \in \mathbb{R}\} \cap (0,1)$.

Now we prove (5.23). This can be done due to (5.20), (5.21) and the following lemma :

Lemma 5.1 *Given $q \in \mathbb{R}$, we have $\lim_{k \to \infty} P_{X_k}(q\varphi) = P_\Sigma(q\varphi)$. Consequently, since these functions are convex and analytic, $P_{X_k}(q\phi)$ and $\frac{\mathrm{d}}{\mathrm{d}q} P_{X_k}(q\phi)$ converge uniformly on compact intervals to $P_\Sigma(q\phi)$ and $\frac{\mathrm{d}}{\mathrm{d}q} P_\Sigma(q\phi)$ respectively.*

Proof The idea is borrowed from the proof of Proposition 2 in [62].

Assume that this is not the case for some $q \in \mathbb{R}$. Since $P_{X_k}(q\varphi) \leq P_\Sigma(q\varphi)$, let $P_*(q\varphi) = \liminf_{k \to \infty} P_{X_k}(q\varphi)$ and let $\delta = P_\Sigma(q\varphi) - P_*(q\varphi) > 0$.

Take a subsequence $(\mu_q^{(k_j)})_{j \geq 1}$ converging to some probability measure μ_q^* in the weak* topology. Due to Theorem B, for any $t \in \Sigma$ and $n \geq 1$,

$$(C(\varphi)^{|q|} C(q\varphi))^{-1} \leq \frac{\mu_q([t|_n])}{\exp(S_n q\varphi(t) - nP_\Sigma(q\varphi))} \leq (C(\varphi)^{|q|} C(q\varphi)).$$

Since X_k converges to Σ in sense of Hausdorff distance, then for all k large enough, we have $X_k \cap [t|_n] \neq \emptyset$, thus

$$(C(\varphi)^{|q|} C(q\varphi))^{-1} \leq \frac{\mu_q^{(k)}([t|_n])}{\exp(S_n q\varphi(t) - nP_{X_k}(q\varphi))} \leq (C(\varphi)^{|q|} C(q\varphi)).$$

This implies

$$\mu_q([t|_n]) \leq \mu_q^*([t|_n]) \cdot (C(\varphi)^{|q|} C(q\varphi))^2 \cdot \exp(-n\delta/2).$$

Taking n large enough sothat $(C(\varphi)^{|q|} C(q\varphi))^2 \cdot \exp(-n\delta/2) < 1$, this is in contradiction with the fact that both μ_q and μ_q^* are probability measures.

5.4 Proof of Theorem 5.2

From now on we fix a $k \geq k_0$ such that $X_k \neq \emptyset$.

5.4.1 Main proof

Proof Recall (5.13) that the set of neighbor words of $w \in \Sigma_*$ is

$$\mathcal{N}(w) = \Big\{ u \in \Sigma_{|w|} : \Big| \sum_{i=1}^{|w|} (u_i - w_i) \cdot 2^{-i} \Big| \leq 2^{-|w|} \Big\}.$$

For $p \geq 1$ let $\mathcal{P}_p$ be the subset of pairs of elements of Σ_{p+1} defined as

$$\mathcal{P}_p = \{(u, v) \in \Sigma_{p+1} \times \Sigma_{p+1} : v|_p \in \mathcal{N}(u|_p),\ v \notin \mathcal{N}(u)\}. \tag{5.24}$$

Then for any $s, t \in \Sigma$ with $|s - t| > 0$, there exists a unique $p \geq 1$ such that $\mathbf{1}_p(s, t) = 1$, where the indicator function is defined by

$$\mathbf{1}_p(s,t) = \begin{cases} 1, & \text{if } (s|_{p+1}, t|_{p+1}) \in \mathcal{P}_p; \\ 0, & \text{otherwise.} \end{cases}$$

By construction we know that if $\mathbf{1}_p(s, t) = 1$, then

$$\inf_{s' \in [s|_{p+1}], t' \in [t|_{p+1}]} |s' - t'| \geq 2^{-p-1} \text{ and } \sup_{s' \in [s|_{p+1}], t' \in [t|_{p+1}]} |s' - t'| \leq 2^{-p+1}.$$

Recall that for $w \in \Sigma_*$,

$$T_w : x \in \mathbb{R} \mapsto 2^{-|w|} \cdot x + \lambda(w).$$

Then for any $s, t \in \Sigma$ with $\mathbf{1}_p(s, t) = 1$, for any $m \geq 1$ we have

$$T^{-1}_{s|_{p+1+m}}(\lambda(s)) \in [0, 1] \text{ and } \big| T^{-1}_{s|_{p+1+m}}(\lambda(s)) - T^{-1}_{s|_{p+1+m}}(\lambda(t)) \big| \geq 2^m.$$

Since ψ decays at infinity, due to (5.19), there exists a large enough $N_{\psi,k} \geq 1$ such that for any $s, t \in X_k$ with $\mathbf{1}_p(s, t) = 1$, for any $m \geq 1$ and $n \geq k$

$$\big| \psi_{s|_{p+1+N_{\psi,k}+m}}(\lambda(s)|_{p+1+N_{\psi,k}+m+n}) - \psi_{s|_{p+1+N_{\psi,k}+m}}(\lambda(t)|_{p+1+N_{\psi,k}+m+n}) \big| \geq \frac{C_{\psi,k}}{2}. \tag{5.25}$$

Let $J_k = \{q \in \mathbb{R} : 0 < h_q^{(k)} < 1\}$.

For any $q \in J_k$, $\epsilon > 0$ and $w \in \Sigma_*$, define

$$\mathbf{1}_w^{(a)}(q,\epsilon) = \mathbf{1}_{\left\{|d_w| \in \left[2^{-|w|(h_q^{(k)}+\epsilon)}, 2^{-|w|(h_q^{(k)}-\epsilon)}\right]\right\}} \tag{5.26}$$

$$\mathbf{1}_w^{(b)}(q,\epsilon) = \mathbf{1}_{\left\{\mu_q^{(k)}([w]) \in \left[2^{-|w|(D_q^{(k)}+\epsilon)}, 2^{-|w|(D_q^{(k)}-\epsilon)}\right]\right\}} \tag{5.27}$$

$$\mathbf{1}_w^{(c)}(q,\epsilon) = \mathbf{1}_{\left\{\sup_{s,t \in \bigcup_{u \in \mathcal{N}(w)}[u]} \left|F_\mu^{\text{pert}}(s) - F_\mu^{\text{pert}}(t)\right| \le 2^{-|w|(h_q^{(k)}-\epsilon)}\right\}}. \tag{5.28}$$

For $n \geq 1$ define

$$\Sigma_n^{(k)}(q,\epsilon) = \left\{ w \in \Sigma_n : [w] \cap X_k \neq \emptyset \text{ and } \mathbf{1}_w^{(a)}(q,\epsilon) \cdot \mathbf{1}_w^{(b)}(q,\epsilon) \cdot \mathbf{1}_w^{(c)}(q,\epsilon) = 1 \right\}.$$

(In fact, $\mathbf{1}_w^{(b)}(q,\epsilon) = 1$ implies $[w] \cap X_k \neq \emptyset$). Then let

$$E_n^{(k)}(q,\epsilon) = \bigcap_{p \geq n} \bigcup_{w \in \Sigma_n^{(k)}(q,\epsilon)} [w] \quad \text{and} \quad E^{(k)}(q) = \lim_{\epsilon \to 0^+} \lim_{n \to \infty} E_n^{(k)}(q,\epsilon).$$

Due to (5.17), (5.20) and (5.21), with probability 1, for all $q \in J_k$,

$$E^{(k)}(q) \subset X_k \cap E_{F_\mu^{\text{pert}}}(h_q^{(k)}) \quad \text{and} \quad \mu_q^{(k)}(E^{(k)}(q)) = 1. \tag{5.29}$$

For $\gamma > 0$ define the Riesz-like kernel : for $s, t \in \Sigma_* \bigcup \Sigma$,

$$\mathcal{K}_\gamma(s,t) = \begin{cases} (|F_\mu^{\text{pert}}(\lambda(s)) - F_\mu^{\text{pert}}(\lambda(t))|^2 + |\lambda(s) - \lambda(t)|^2)^{-\frac{\gamma}{2}} \vee 1, & \text{if } \gamma \geq 1, \\ |F_\mu^{\text{pert}}(\lambda(s)) - F_\mu^{\text{pert}}(\lambda(t))|^{-\gamma} \vee 1, & \text{if } \gamma < 1. \end{cases} \tag{5.30}$$

For $q \in J_k$ recall that

$$\gamma_{q,G}^{(k)} = \frac{D_q^{(k)}}{h_q^{(k)}} \wedge \left(D_q^{(k)} + 1 - h_q^{(k)}\right) \quad \text{and} \quad \gamma_{q,R}^{(k)} = \frac{D_q^{(k)}}{h_q^{(k)}} \wedge 1. \tag{5.31}$$

For $q \in J_k$, $\delta > 0$ and $\epsilon > 0$ we define the n^{th}-energy for $n \geq 1$ and $S \in \{G, R\}$:

$$\mathcal{I}_{n,\delta}^S(q,\epsilon) = \iint_{s,t \in E_n^{(k)}(q,\epsilon), s \neq t} \mathcal{K}_{\gamma_{q,S}^{(k)} - \delta}(s,t) \, \mathrm{d}\mu_q^{(k)}(s) \mathrm{d}\mu_q^{(k)}(t).$$

Let K be any compact subinterval of J_k. We assume for a while that we have proved that for any δ small enough, there exists $\epsilon_\delta > 0$ such that for any $n \geq 1$, $\epsilon \in (0, \epsilon_\delta)$ and $S \in \{G, R\}$,

$$\mathbb{E}\left(\sup_{q \in K} \mathcal{I}_{n,\delta}^S(q,\epsilon)\right) < \infty. \tag{5.32}$$

The following lemma is a slight modification of Theorem 4.13 in [51] regarding the Hausdorff dimension estimate through the potential theoretic method.

Lemma 5.2 *Let ν be a Borel measure on $\mathbb{R}^d$ and let $E \subset \mathbb{R}^d$ be a Borel set such that $\nu(E) > 0$. For any $\gamma > 0$, if*

$$\iint_{x,y\in E, x\neq y} |x-y|^{-\gamma} \vee 1 \; \mathrm{d}\nu(x)\mathrm{d}\nu(y) < \infty,$$

then

$$\nu\left(\left\{x \in E : \underline{\dim}_{\mathrm{loc}}\nu(x) = \liminf_{r\to 0^+} \frac{\log \nu(B(x,r))}{\log r} < \gamma\right\}\right) = 0.$$

Then, it easily follows from (5.32) and Lemma 5.2 that, with probability 1, for all $q \in K$:

– For $\mu_{q,G}^{(k)}$-almost every $x \in \{(\lambda(t), F_\mu^{\mathrm{pert}}(\lambda(t))) : t \in E^{(k)}(q)\}$,

$$\underline{\dim}_{\mathrm{loc}}\mu_{q,G}^{(k)}(x) \geq \gamma_{q,G}^{(k)} - \delta;$$

– For $\mu_{q,R}^{(k)}$-almost every $y \in \{F_\mu^{\mathrm{pert}}(\lambda(t)) : t \in E^{(k)}(q)\}$,

$$\underline{\dim}_{\mathrm{loc}}\mu_{q,R}^{(k)}(y) \geq \gamma_{q,R}^{(k)} - \delta.$$

Since δ can be taken arbitrarily small, we get the conclusion by taking a countable sequence of compact subintervals $K_j \subset J_k$ such that $\bigcup K_j = J_k$.

Now we prove (5.32).

For any $\bar{q} \in K$ and $\epsilon > 0$ we define the neighborhood of $\bar{q}$ in K :

$$U_\epsilon(\bar{q}) = \left\{ q \in K : \max\left\{ \begin{array}{l} |q-\bar{q}|, |\alpha_q^{(k)} - \alpha_{\bar{q}}^{(k)}|, |h_q^{(k)} - h_{\bar{q}}^{(k)}|, \\ |D_q^{(k)} - D_{\bar{q}}^{(k)}|, |\gamma_{q,G}^{(k)} - \gamma_{\bar{q},G}^{(k)}|, |\gamma_{q,R}^{(k)} - \gamma_{\bar{q},R}^{(k)}| \end{array} \right\} < \epsilon \right\}. \quad (5.33)$$

By continuity of these functions, the set $U_\epsilon(\bar{q})$ is open in K.

Notice that for $q \in K$, $\delta > 0$ and $S \in \{G, R\}$ the Riesz-like kernels $\mathcal{K}_{\gamma_{q,S}^{(k)}-\delta}$ are positive functions and, moreover, by the continuity of F_μ^{pert} we have for any $s,t \in \Sigma$,

$$\lim_{m\to\infty} \mathcal{K}_{\gamma_{q,S}^{(k)}-\delta}(s|_m, t|_m) = \mathcal{K}_{\gamma_{q,S}^{(k)}-\delta}(s,t).$$

Then by applying Fatou's lemma we get for any $q \in U_\epsilon(\bar{q})$,

$$\begin{aligned}
&\mathcal{I}_{n,\delta}^S(q,\epsilon) \\
= \; &\iint_{s,t\in E_n^{(k)}(q,\epsilon), s\neq t} \lim_{m\to\infty} \mathcal{K}_{\gamma_{q,S}^{(k)}-\delta}(s|_m, t|_m) \; \mathrm{d}\mu_q^{(k)}(s)\mathrm{d}\mu_q^{(k)}(t) \\
= \; &\sum_{p\geq 1} \iint_{\substack{s,t\in E_n^{(k)}(q,\epsilon);\\ \mathbf{1}_p(s,t)=1}} \lim_{m\to\infty} \mathcal{K}_{\gamma_{q,S}^{(k)}-\delta}(s|_m, t|_m) \; \mathrm{d}\mu_q^{(k)}(s)\mathrm{d}\mu_q^{(k)}(t) \\
\leq \; &\sum_{p\geq 1} \liminf_{m\to\infty} \iint_{\substack{s,t\in E_n^{(k)}(q,\epsilon);\\ \mathbf{1}_p(s,t)=1}} \mathcal{K}_{\gamma_{q,S}^{(k)}-\delta}(s|_m, t|_m) \; \mathrm{d}\mu_q^{(k)}(s)\mathrm{d}\mu_q^{(k)}(t) \\
= \; &\sum_{p\geq 1} \liminf_{m\to\infty} \sum_{\substack{u,v\in\Sigma_m;\\ \mathbf{1}_p(u,v)=1}} \mathcal{K}_{\gamma_{q,S}^{(k)}-\delta}(u,v) \cdot \mu_q^{(k)}([u]\cap E_n^{(k)}(q,\epsilon))\mu_q^{(k)}([v]\cap E_n^{(k)}(q,\epsilon))
\end{aligned}$$

$$\leq \sum_{p\geq 1} \liminf_{m\to\infty} \sum_{\substack{u,v\in\Sigma_m;\\ \mathbf{1}_p(u,v)=1}} \mathcal{K}_{\gamma^{(k)}_{\bar{q},S}-\delta-\epsilon}(u,v)\cdot \mu_q^{(k)}([u]\cap E_n^{(k)}(q,\epsilon))\mu_q^{(k)}([v]\cap E_n^{(k)}(q,\epsilon)),$$

where the last inequality comes from the fact that due to (5.30) and (5.33), for any $\bar{q}\in K$, $\epsilon>0$ and $u,v\in\Sigma_*$, we have $\sup_{q\in U_\epsilon(\bar{q})}\mathcal{K}_{\gamma^{(k)}_{q,S}-\delta}(u,v)\leq \mathcal{K}_{\gamma^{(k)}_{\bar{q},S}-\delta-\epsilon}(u,v)$. Let

$$A_{p,m} = \sum_{\substack{u,v\in\Sigma_m;\\ \mathbf{1}_p(u,v)=1}} \mathcal{K}_{\gamma^{(k)}_{\bar{q},S}-\delta-\epsilon}(u,v)\cdot \mu_q^{(k)}([u]\cap E_n^{(k)}(q,\epsilon))\mu_q^{(k)}([v]\cap E_n^{(k)}(q,\epsilon)).$$

Then,

$$\begin{aligned}\sup_{q\in U_\epsilon(\bar{q})} \mathcal{I}^S_{n,\delta}(q,\epsilon) &\leq \sup_{q\in U_\epsilon(\bar{q})}\sum_{p\geq 1}\liminf_{m\to\infty} A_{p,m}\\ &\leq \sup_{q\in U_\epsilon(\bar{q})}\sum_{p\geq 1}\Big(A_{p,m_p}+\sum_{m\geq m_p}|A_{p,m+1}-A_{p,m}|\Big)\\ &\leq \sum_{p\geq 1}\Big(\sup_{q\in U_\epsilon(\bar{q})}A_{p,m_p}+\sum_{m\geq m_p}\sup_{q\in U_\epsilon(\bar{q})}|A_{p,m+1}-A_{p,m}|\Big), \qquad (5.34)\end{aligned}$$

where for $p\geq 1$, we can choose $m_p\geq 2$ to be any integer. We have

$$\sup_{q\in U_\epsilon(\bar{q})} A_{p,m}\leq B_{p,m} \quad \text{and} \quad \sup_{q\in U_\epsilon(\bar{q})}|A_{p,m+1}-A_{p,m}|\leq \Delta B_{p,m}, \qquad (5.35)$$

where

$$\begin{aligned} B_{p,m} &= \sum_{\substack{u,v\in\Sigma_m;\\ \mathbf{1}_p(u,v)=1}} \mathcal{K}_{\gamma^{(k)}_{\bar{q},S}-\delta-\epsilon}(u,v)\sup_{q\in U_\epsilon(\bar{q})}\mu_q^{(k)}([u]\cap E_n^{(k)}(q,\epsilon))\mu_q^{(k)}([v]\cap E_n^{(k)}(q,\epsilon)),\\ \Delta B_{p,m} &= \sum_{\substack{u,v\in\Sigma_m,u',v'\in\{0,1\};\\ \mathbf{1}_p(u,v)=1}} \big|\mathcal{K}_{\gamma^{(k)}_{\bar{q},S}-\delta-\epsilon}(uu',vv')-\mathcal{K}_{\gamma^{(k)}_{\bar{q},S}-\delta-\epsilon}(u,v)\big|\cdot\\ &\qquad \sup_{q\in U_\epsilon(\bar{q})}\mu_q^{(k)}([uu']\cap E_n^{(k)}(q,\epsilon))\mu_q^{(k)}([vv']\cap E_n^{(k)}(q,\epsilon)),\end{aligned}$$

and we have used the equality $\mu_q^{(k)}([u]\cap E_n^{(k)}(q,\epsilon))=\sum_{u'\in\{0,1\}}\mu_q^{(k)}([uu']\cap E_n^{(k)}(q,\epsilon))$ to get the second inequality.

Remark 5.3 For technical reasons, we need to divide J_k into two parts, in which K will be chosen :

$$J'_k=\{q\in J_k:\gamma^{(k)}_{q,G}>1\} \text{ and } J''_k=\{q\in J_k:\gamma^{(k)}_{q,G}\leq 1\}.$$

Then, due to (5.31), we have

$$\gamma^{(k)}_{q,G}=\begin{cases} D_q^{(k)}+1-h_q^{(k)} & \text{if } q\in J'_k,\\ D_q^{(k)}/h_q^{(k)} & \text{if } q\in J''_k\end{cases} \quad\text{and}\quad \gamma^{(k)}_{q,R}=\begin{cases}1 & \text{if } q\in J'_k,\\ D_q^{(k)}/h_q^{(k)} & \text{if } q\in J''_k.\end{cases}$$

For any compact subinterval K of J_k there exists $c_K>0$ such that for any $\epsilon<c_K$ and $q\in K$, $\gamma^{(k)}_{q,G}-\epsilon>1$ if $K\subset J'_k$ and $\gamma^{(k)}_{q,R}-\epsilon>0$ if $K\subset J''_k$.

Let $\delta_K=\epsilon_K=c_K/2$. We have the following key proposition :

Proposition 5.1 *Let $S \in \{G, R\}$. Suppose that K is a compact subinterval of J'_k or J''_k. For any $0 < \delta < \delta_K$ we can find constants $c_1, c_2 > 0$, $\kappa_1, \kappa_2, \eta_1, \eta_2 > 0$ and $\epsilon_* > 0$ such that for any $\bar{q} \in K$, $0 < \epsilon \le \epsilon_*$, $n \ge 1$, $p \ge 1$, and $m \ge (p \vee n) + 1 + N_{\psi,k} + k$,*

$$\begin{aligned}\mathbb{E}(B_{p,m}) &\le c_1 \cdot C_{p\vee n} \cdot 2^n \cdot 2^{c_2\cdot((p\vee n)-p)} \cdot 2^{-\eta_1\delta\cdot p+\kappa_1\epsilon\cdot m};\\ \mathbb{E}(\Delta B_{p,m}) &\le c_1 \cdot C_{p\vee n} \cdot 2^{c_2\cdot((p\vee n)-p)} \cdot 2^{\kappa_2\cdot p-\eta_2\cdot m},\end{aligned}$$

where $C_{p\vee n} = \sup_{w\in\Sigma_{p\vee n+4+N_{\psi,k}}} \|f_w\|_\infty$, here f_w is just the formal replacement of the bounded density function $f_{j,k}$ given in (A3).

Now fix any $n \ge 1$, and choose $m_p = \frac{\kappa_2+\frac{1}{2}\delta\eta_1}{\eta_2} \cdot (p \vee n)$ (by modifying a little η_2 we can always assume that $\frac{\kappa_2+\frac{1}{2}\delta\eta_1}{\eta_2}(p\vee n) > (p\vee n)+1+N_{\psi,k}+k$) and $\epsilon_\delta = \epsilon_* \wedge \frac{\frac{1}{2}\delta\eta_1\eta_2}{\kappa_1(\kappa_2+\frac{1}{2}\delta\eta_1)}$. Then by using Proposition 5.1 and (5.34), (5.35), for any $\delta < \delta_K$, $\bar{q} \in K$, $\epsilon < \epsilon_\delta$ and $S \in \{G, R\}$ we have

$$\begin{aligned}
&\mathbb{E}\left(\sup_{q\in U_\epsilon(\bar{q})} \mathcal{I}^S_{n,\delta}(q,\epsilon)\right)\\
\le\ & \sum_{p\ge1}\left(\mathbb{E}\left(B_{p,m_p}\right) + \sum_{m\ge m_p}\mathbb{E}\left(\Delta B_{p,m}\right)\right)\\
\le\ & \sum_{p\ge1} c_1 \cdot C_{p\vee n}\cdot 2^{c_2\cdot((p\vee n)-p)}\cdot\left(2^n\cdot 2^{-\eta_1\delta\cdot p+\kappa_1\epsilon\cdot m_p} + \sum_{m\ge m_p} 2^{\kappa_2\cdot p-\eta_2\cdot m}\right)\\
\le\ & c_1\cdot\sum_{p\ge1} C_{p\vee n}\cdot 2^{c_2\cdot((p\vee n)-p)}\cdot\left(2^n\cdot 2^{-\eta_1\delta\cdot p+\kappa_1\cdot\epsilon_\delta\cdot m_p} + 2^{\kappa_2\cdot p-\eta_2\cdot m_p}\cdot\frac{1}{1-2^{-\eta_2}}\right)\\
\le\ & \frac{c_1}{1-2^{-\eta_2}}\cdot\sum_{p\ge1} C_{p\vee n}\cdot 2^{c_2\cdot((p\vee n)-p)}\cdot 2^n\cdot\\
&\qquad\left(2^{-\eta_1\delta\cdot p+\kappa_1\cdot\frac{\frac{1}{2}\delta\eta_1\eta_2}{\kappa_1(\kappa_2+\frac{1}{2}\delta\eta_1)}\cdot\frac{\kappa_2+\frac{1}{2}\delta\eta_1}{\eta_2}\cdot(n\vee p)} + 2^{\kappa_2\cdot p-\eta_2\cdot\frac{\kappa_2+\frac{1}{2}\delta\eta_1}{\eta_2}\cdot p}\right)\\
=\ & \frac{2c_1}{1-2^{-\eta_2}}\cdot\sum_{p\ge1} C_{p\vee n}\cdot 2^{(c_2+\frac{1}{2}\eta_1\delta)\cdot((p\vee n)-p)}\cdot 2^n\cdot 2^{-\frac{1}{2}\delta\eta_1 p}\\
=\ & \frac{2^{n+1}c_1}{1-2^{-\eta_2}}\cdot\left(\left(\sup_{w\in\Sigma_{n+4+N_{\psi,k}}}\|f_w\|_\infty\right)\cdot\sum_{p=1}^{n} 2^{(c_2+\frac{1}{2}\eta_1\delta)\cdot(n-p)}\cdot 2^{-\frac{1}{2}\delta\eta_1 p} +\right.\\
&\qquad\left. 2^{\frac{1}{2}\delta\eta_1(4+N_{\psi,k})}\cdot\sum_{p=n+1}^{\infty}\left(\sup_{w\in\Sigma_{p+4+N_{\psi,k}}}\|f_w\|_\infty\right)\cdot 2^{-\frac{1}{2}\delta\eta_1(p+4+N_{\psi,k})}\right) < \infty,
\end{aligned}$$

where the finiteness is ensured by assumption (A3). Since for any $0 < \epsilon < \epsilon_\delta$, the family $\{U_\epsilon(\bar{q})\}_{\bar{q}\in K}$ forms an open covering of K, there exist $\bar{q}_1, \cdots, \bar{q}_N$ such that $\{U_\epsilon(\bar{q}_i)\}_{1\le i\le N}$ also covers K. This gives us the conclusion.

5.4.2 Proof of Proposition 5.1

Proof Due to (5.33) we always have

$$\bigcup_{q\in U_\epsilon(\bar q)} E_n^{(k)}(q,\epsilon)\subset E_n^{(k)}(\bar q,2\epsilon).$$

Then due to (5.27) we have

$$\begin{aligned}
&\sup_{q\in U_\epsilon(\bar q)} \mu_q^{(k)}([u]\cap E_n^{(k)}(q,\epsilon))\mu_q^{(k)}([v]\cap E_n^{(k)}(q,\epsilon))\\
\le\ &\sup_{q\in U_\epsilon(\bar q)} \mathbf{1}_{\left\{[u]\cap E_n^{(k)}(q,\epsilon)\neq\emptyset\right\}}2^{-|u|(D_q^{(k)}-\epsilon)}\cdot \mathbf{1}_{\left\{[v]\cap E_n^{(k)}(q,\epsilon)\neq\emptyset\right\}}2^{-|v|(D_q^{(k)}-\epsilon)}\\
\le\ &\mathbf{1}_{\left\{[u]\cap E_n^{(k)}(\bar q,2\epsilon)\neq\emptyset\right\}}\cdot \mathbf{1}_{\left\{[v]\cap E_n^{(k)}(\bar q,2\epsilon)\neq\emptyset\right\}}\cdot 2^{-(|u|+|v|)(D_{\bar q}^{(k)}-2\epsilon)}.
\end{aligned}$$

This gives us

$$B_{p,m}\le 2^{-2m(D_{\bar q}^{(k)}-2\epsilon)}\sum_{\substack{u,v\in\Sigma_m;\\ \mathbf{1}_p(u,v)=1}}\mathcal{K}_{\gamma_{\bar q,S}^{(k)}-\delta-\epsilon}(u,v)\cdot \mathbf{1}_{\left\{[u]\cap E_n^{(k)}(\bar q,2\epsilon)\neq\emptyset\right\}}\mathbf{1}_{\left\{[v]\cap E_n^{(k)}(\bar q,2\epsilon)\neq\emptyset\right\}},\tag{5.36}$$

$$\begin{aligned}
\Delta B_{p,m}\le 2^{-2(m+1)(D_{\bar q}^{(k)}-2\epsilon)}\cdot &\sum_{u,v\in\Sigma_m,u',v'\in\{0,1\};\mathbf{1}_p(u,v)=1}\\
&\left|\mathcal{K}_{\gamma_{\bar q,S}^{(k)}-\delta-\epsilon}(uu',vv')-\mathcal{K}_{\gamma_{\bar q,S}^{(k)}-\delta-\epsilon}(u,v)\right|\cdot \mathbf{1}_{\left\{[uu']\cap E_n^{(k)}(\bar q,2\epsilon)\neq\emptyset\right\}}\mathbf{1}_{\left\{[vv']\cap E_n^{(k)}(\bar q,2\epsilon)\neq\emptyset\right\}}.
\end{aligned}\tag{5.37}$$

Now we deal with each term of the above sums individually.

Fix p and n in $\mathbb{N}^*$, let $r=p\vee n$, and fix $m\ge r+1+N_{\psi,k}+k$.

Fix a pair $u,v\in\Sigma_m$ with $\mathbf{1}_p(u,v)=1$, so $(u|_{p+1},v|_{p+1})\in\mathcal{P}_p$.

Let

$$\begin{cases} V:=\mathcal{K}_{\gamma_{\bar q,S}^{(k)}-\delta-\epsilon}(u,v)\cdot \mathbf{1}_{\left\{[u]\cap E_n^{(k)}(\bar q,2\epsilon)\neq\emptyset\right\}}\mathbf{1}_{\left\{[v]\cap E_n^{(k)}(\bar q,2\epsilon)\neq\emptyset\right\}};\\ \Delta V:=\left|\mathcal{K}_{\gamma_{\bar q,S}^{(k)}-\delta-\epsilon}(uu',vv')-\mathcal{K}_{\gamma_{\bar q,S}^{(k)}-\delta-\epsilon}(u,v)\right|\cdot \mathbf{1}_{\left\{[uu']\cap E_n^{(k)}(\bar q,2\epsilon)\neq\emptyset\right\}}\mathbf{1}_{\left\{[vv']\cap E_n^{(k)}(\bar q,2\epsilon)\neq\emptyset\right\}}.\end{cases}$$

Due to (5.26), (5.27) and (5.28), if $[u]\cap E_n^{(k)}(\bar q,2\epsilon)\neq\emptyset$, then for $l=r,\cdots,m$ we have

$$\mathbf{1}_{u|_l}^{(a)}(\bar q,2\epsilon)\cdot\mathbf{1}_{u|_l}^{(b)}(\bar q,2\epsilon)\cdot\mathbf{1}_{u|_l}^{(c)}(\bar q,2\epsilon)=1.$$

Define

$$\mathbf{1}_{u,v}^{\mathrm{ran}}(\bar q,\epsilon)=\mathbf{1}_{u|_l}^{(a)}(\bar q,2\epsilon)\cdot\mathbf{1}_{u|_r}^{(c)}(\bar q,2\epsilon)\cdot\mathbf{1}_{u}^{(c)}(\bar q,2\epsilon)\cdot\mathbf{1}_{v}^{(c)}(\bar q,2\epsilon);\tag{5.38}$$

$$\mathbf{1}_{u,v}^{\mathrm{det}}(\bar q,\epsilon)=\mathbf{1}_{u|_r}^{(b)}(\bar q,2\epsilon)\cdot\mathbf{1}_{v|_r}^{(b)}(\bar q,2\epsilon)\cdot\mathbf{1}_{u}^{(b)}(\bar q,2\epsilon)\cdot\mathbf{1}_{v}^{(b)}(\bar q,2\epsilon),\tag{5.39}$$

where "ran" stands for random and "det" stands for deterministic.

Since $[uu']\cap E_n^{(k)}(\bar q,2\epsilon)\neq\emptyset$ implies $[u]\cap E_n^{(k)}(\bar q,2\epsilon)\neq\emptyset$, we have

$$\mathbf{1}_{\left\{[uu']\cap E_n^{(k)}(\bar q,2\epsilon)\neq\emptyset\right\}}\cdot \mathbf{1}_{\left\{[vv']\cap E_n^{(k)}(\bar q,2\epsilon)\neq\emptyset\right\}}\ \le\ \mathbf{1}_{\left\{[u]\cap E_n^{(k)}(\bar q,2\epsilon)\neq\emptyset\right\}}\cdot \mathbf{1}_{\left\{[v]\cap E_n^{(k)}(\bar q,2\epsilon)\neq\emptyset\right\}}$$

$$\leq \quad \mathbf{1}^{\mathrm{ran}}_{u,v}(\bar{q},\epsilon)\cdot \mathbf{1}^{\mathrm{det}}_{u,v}(\bar{q},\epsilon).$$

This implies

$$V \leq \bar{\mathcal{K}}_{\gamma^{(k)}_{\bar{q},S}-\delta-\epsilon}(u,v)\cdot \mathbf{1}^{\mathrm{det}}_{u,v}(\bar{q},\epsilon) \quad \text{and} \quad \Delta V \leq \Delta\bar{\mathcal{K}}_{\gamma^{(k)}_{\bar{q},S}-\delta-\epsilon}(u,v)\cdot \mathbf{1}^{\mathrm{det}}_{u,v}(\bar{q},\epsilon),$$

where

$$\begin{cases} \bar{\mathcal{K}}_{\gamma^{(k)}_{\bar{q},S}-\delta-\epsilon}(u,v) = \mathcal{K}_{\gamma^{(k)}_{\bar{q},S}-\delta-\epsilon}(u,v)\cdot \mathbf{1}^{\mathrm{ran}}_{u,v}(\bar{q},\epsilon); \\ \\ \Delta\bar{\mathcal{K}}_{\gamma^{(k)}_{\bar{q},S}-\delta-\epsilon}(u,v) = \left|\mathcal{K}_{\gamma^{(k)}_{\bar{q},S}-\delta-\epsilon}(uu',vv') - \mathcal{K}_{\gamma^{(k)}_{\bar{q},S}-\delta-\epsilon}(u,v)\right|\cdot \mathbf{1}^{\mathrm{ran}}_{u,v}(\bar{q},\epsilon). \end{cases} \tag{5.40}$$

Since $\mathbf{1}^{\mathrm{det}}_{u,v}(\bar{q},\epsilon)$ is deterministic, we have

$$\begin{aligned} \mathbb{E}(V) &\leq \mathbb{E}\Big(\bar{\mathcal{K}}_{\gamma^{(k)}_{\bar{q},S}-\delta-\epsilon}(u,v)\Big)\cdot \mathbf{1}^{\mathrm{det}}_{u,v}(\bar{q},\epsilon), \\ \mathbb{E}(\Delta V) &\leq \mathbb{E}\Big(\Delta\bar{\mathcal{K}}_{\gamma^{(k)}_{\bar{q},S}-\delta-\epsilon}(u,v)\Big)\cdot \mathbf{1}^{\mathrm{det}}_{u,v}(\bar{q},\epsilon). \end{aligned}$$

Recall that in Remark 5.3 we distinguished the cases $K \subset J'_k$ and $K \subset J''_k$ according to whether or not the corresponding power on the kernel is greater than 1. Then, due to (5.30), once we have taken $\delta < \delta_K$ and $\epsilon < \epsilon_K$, only two situations are left :

$$\mathcal{K}_\gamma(u,v) = \begin{cases} \left(\left|F^{\mathrm{pert}}_\mu(\lambda(u)) - F^{\mathrm{pert}}_\mu(\lambda(v))\right|^2 + |\lambda(u)-\lambda(v)|^2\right)^{\frac{\gamma}{2}} \vee 1, & \text{if } \gamma > 1, \\ \\ \left|F^{\mathrm{pert}}_\mu(\lambda(u)) - F^{\mathrm{pert}}_\mu(\lambda(v))\right|^\gamma \vee 1, & \text{if } \gamma < 1, \end{cases}$$

where $\gamma = \gamma^{(k)}_{\bar{q},S} - \delta - \epsilon$. *Notice that when we take K a compact subinterval of J'_k or J''_k, γ could never be equal to* 1.

Recall that $C_r = \sup_{w\in\Sigma_{r+4+N_{\psi,k}}} \|f_w\|_\infty$, where f_w is bounded density function of π_w given in (A3). We have the following two lemmas :

Lemma 5.3 *There exists a constant $c_\gamma > 0$ such that*

$$\begin{aligned} \mathbb{E}\Big(\bar{\mathcal{K}}_\gamma(u,v)\Big) &\leq c_\gamma \cdot C_r \cdot \begin{cases} 2^{r\cdot(h^{(k)}_{\bar{q}}+2\epsilon)-p(1-\gamma)}, & \text{if } \gamma > 1, \\ 2^n \cdot 2^{r\cdot(h^{(k)}_{\bar{q}}\gamma+4\epsilon)}, & \text{if } \gamma < 1, \end{cases} \\ \mathbb{E}\Big(\Delta\bar{\mathcal{K}}_\gamma(u,v)\Big) &\leq c_\gamma \cdot C_r \cdot \begin{cases} 2^{p\cdot 3-m\cdot(h^{(k)}_{\bar{q}}-2\epsilon)}, & \text{if } \gamma > 1, \\ 2^{r\cdot 3-m\cdot(h^{(k)}_{\bar{q}}-2\epsilon)}, & \text{if } \gamma < 1. \end{cases} \end{aligned}$$

Lemma 5.4

$$\sum_{u,v\in\Sigma_m} \mathbf{1}_p(u,v)\cdot \mathbf{1}^{\mathrm{det}}_{u,v}(\bar{q},\epsilon) \leq 3^{r-p+1}\cdot 2^{2m\cdot(D^{(k)}_{\bar{q}}+2\epsilon)-r\cdot(D^{(k)}_{\bar{q}}-2\epsilon)}.$$

Now, due to Remark 5.3, we have the following three expression of γ :

$$\begin{cases} \gamma = D_{\bar{q}}^{(k)} + 1 - h_{\bar{q}}^{(k)} - \delta - \epsilon > 1, & h_{\bar{q}}^{(k)} < D_{\bar{q}}^{(k)}, \quad \text{case (i)}, \\ \gamma = D_{\bar{q}}^{(k)}/h_{\bar{q}}^{(k)} - \delta - \epsilon < 1, & h_{\bar{q}}^{(k)} > D_{\bar{q}}^{(k)}, \quad \text{case (ii)}, \\ \gamma = 1 - \delta - \epsilon < 1, & h_{\bar{q}}^{(k)} < D_{\bar{q}}^{(k)}, \quad \text{case (iii)}. \end{cases}$$

Then, due to (5.36), (5.37), Lemma 5.3 and Lemma 5.4, we have

$$\begin{aligned} \mathbb{E}\left(B_{p,m}\right) &\le c_\gamma \cdot C_r \cdot 3^{r-p+1} \cdot 2^{-2m(D_{\bar{q}}^{(k)}-2\epsilon)} \cdot 2^{2m(D_{\bar{q}}^{(k)}+2\epsilon)-r(D_{\bar{q}}^{(k)}-2\epsilon)} \\ &\quad \cdot \begin{cases} 2^{r(h_{\bar{q}}^{(k)}+2\epsilon)-p(h_{\bar{q}}^{(k)}-D_{\bar{q}}^{(k)}+\delta+\epsilon)}, & \text{(i)} \\ 2^n \cdot 2^{r(D_{\bar{q}}^{(k)}-h_{\bar{q}}^{(k)}(\delta+\epsilon)+4\epsilon)}, & \text{(ii)} \\ 2^n \cdot 2^{r(h_{\bar{q}}^{(k)}-h_{\bar{q}}^{(k)}(\delta+\epsilon)+4\epsilon)}, & \text{(iii)} \end{cases} \\ &= c_\gamma \cdot C_r \cdot 3^{r-p+1} \cdot 2^{8m\epsilon} \cdot 2^{-(r-p)(D_{\bar{q}}^{(k)}-2\epsilon)} \cdot 2^{-p(D_{\bar{q}}^{(k)}-2\epsilon)} \\ &\quad \cdot \begin{cases} 2^{(r-p)(h_{\bar{q}}^{(k)}+2\epsilon)} \cdot 2^{p(D_{\bar{q}}^{(k)}-\delta+\epsilon)}, & \text{(i)} \\ 2^n \cdot 2^{(r-p)(D_{\bar{q}}^{(k)}-h_{\bar{q}}^{(k)}(\delta+\epsilon)+4\epsilon)} \cdot 2^{p(D_{\bar{q}}^{(k)}-h_{\bar{q}}^{(k)}(\delta+\epsilon)+4\epsilon)}, & \text{(ii)} \\ 2^n \cdot 2^{(r-p)(h_{\bar{q}}^{(k)}-h_{\bar{q}}^{(k)}(\delta+\epsilon)+4\epsilon)} \cdot 2^{p(h_{\bar{q}}^{(k)}-h_{\bar{q}}^{(k)}(\delta+\epsilon)+4\epsilon)}, & \text{(iii)} \end{cases} \\ &\le c_\gamma \cdot C_r \cdot 3 \cdot 2^{(r-p)(\log_2 3+2)} \cdot 2^n \cdot \begin{cases} 2^{-(\delta-3\epsilon)\cdot p+8\epsilon\cdot m}, & \text{(i)} \\ 2^{-(h_{\bar{q}}^{(k)}(\delta+\epsilon)-6\epsilon)\cdot p+8\epsilon\cdot m}, & \text{(ii)} \\ 2^{-(h_{\bar{q}}^{(k)}(\delta+\epsilon)-6\epsilon)\cdot p+8\epsilon\cdot m}, & \text{(iii)} \end{cases} \end{aligned}$$

The upper bound of $\mathbb{E}\left(\Delta B_{p,m}\right)$ is simpler, in all cases we have

$$\begin{aligned} &\mathbb{E}\left(\Delta B_{p,m}\right) \\ \le\ & c_\gamma \cdot C_r \cdot 3^{r-p+1} \cdot 4 \cdot 2^{-2(m+1)(D_{\bar{q}}^{(k)}-2\epsilon)} \cdot 2^{2m(D_{\bar{q}}^{(k)}+2\epsilon)-r(D_{\bar{q}}^{(k)}-2\epsilon)} \cdot 2^{(r-p)\cdot 3} \cdot 2^{p\cdot 3-m\cdot(h_{\bar{q}}^{(k)}-2\epsilon)} \\ \le\ & c_\gamma \cdot C_r \cdot 3 \cdot 2^{(r-p)(\log_2 3+3)} \cdot 2^{p\cdot 3-m\cdot(h_{\bar{q}}^{(k)}-10\epsilon)}. \end{aligned}$$

Notice that by construction we always have $h_{\bar{q}}^{(k)} \ge s_0 - 1/p_0 > 0$, then the existences of the parameters $c_1, c_2 > 0$, $\kappa_1, \kappa_2, \eta_1, \eta_2 > 0$ and $\epsilon_* > 0$ are direct consequences of what we have obtained.

5.4.3 Proof of Lemma 5.3.

Proof Let $l = r + 1 + N_{\psi,k}$. Due to (5.15), we have

$$\begin{aligned} F_\mu^{\text{pert}}(\lambda(u)) - F_\mu^{\text{pert}}(\lambda(v)) &= \sum_{w\in\Sigma_*} \pi_w \cdot d_w \cdot (\psi_w(\lambda(u)) - \psi_w(\lambda(v))) \\ &= \pi_{u|_l} \cdot A + B, \end{aligned} \tag{5.41}$$

where

$$\begin{aligned} A &= d_{u|_l} \cdot (\psi_{u|_l}(\lambda(u)) - \psi_{u|_l}(\lambda(v))); \qquad (5.42) \\ B &= \sum_{w\in\Sigma_*\setminus\{u|_l\}} \pi_w \cdot d_w \cdot (\psi_w(\lambda(u)) - \psi_w(\lambda(v))). \end{aligned}$$

By construction A is deterministic, and $\pi_{u|_l}$ and B are independent.

Since when $\mathbf{1}_u^{(b)}(\bar{q}, 2\epsilon) = 1$ we have $\mu_{\bar{q}}^{(k)}([u]) \neq 0$, then (5.25) and the fact that $|u| = m \geq l + k$ yield

$$\left|\psi_{u|_l}(\lambda(u)) - \psi_{u|_l}(\lambda(v))\right| \geq \frac{c_{\psi,k}}{2}. \tag{5.43}$$

For $u', v' \in \{0, 1\}$ we can write

$$\begin{aligned} & F_\mu^{\text{pert}}(\lambda(uu')) - F_\mu^{\text{pert}}(\lambda(vv')) \\ = \ & \eta \cdot (F_\mu^{\text{pert}}(\lambda(u)) - F_\mu^{\text{pert}}(\lambda(v))) + D = \eta \cdot (\pi_{u|_l} \cdot A + B) + D \end{aligned} \tag{5.44}$$

where

$$\begin{aligned} \eta &= \frac{\psi_{u|_l}(\lambda(uu')) - \psi_{u|_l}(\lambda(vv'))}{\psi_{u|_l}(\lambda(u)) - \psi_{u|_l}(\lambda(v))}; \\ D &= \sum_{w \in \Sigma_* \setminus \{u|_l\}} \pi_w \cdot d_w \cdot \big(\psi_w(\lambda(uu')) - \psi_w(\lambda(vv')) - \eta \cdot (\psi_w(\lambda(u)) - \psi_w(\lambda(v)))\big). \end{aligned}$$

We have that η is deterministic, and $\pi_{u|_l}$ and D are independent. Moreover, since ψ is r_0-smooth, there exists a constant C_ψ such that for any $x, y \in \mathbb{R}$ we have $|\psi(x) - \psi(y)| \leq C_\psi |s - t|$. Due to (5.43), this implies

$$\begin{aligned} |\eta - 1| &= \left|\frac{\psi_{u|_l}(\lambda(uu')) - \psi_{u|_l}(\lambda(u)) + \psi_{u|_l}(\lambda(v)) - \psi_{u|_l}(\lambda(vv'))}{\psi_{u|_l}(\lambda(u)) - \psi_{u|_l}(\lambda(v))}\right| \\ &\leq \frac{2C_\psi}{c_{\psi,k}} \left(\left|T_{u|_l}^{-1}(\lambda(uu')) - T_{u|_l}^{-1}(\lambda(u))\right| + \left|T_{u|_l}^{-1}(\lambda(v)) - T_{u|_l}^{-1}(\lambda(vv'))\right|\right) \\ &\leq \frac{2C_\psi}{c_{\psi,k}} \cdot 2^l \cdot 2^{-m}, \end{aligned} \tag{5.45}$$

where we have used $|\lambda(u) - \lambda(uu')| \vee |\lambda(v) - \lambda(vv')| \leq 2^{-m}$.

For $w \in \Sigma_*$ define the σ-algebra $\mathcal{A}_w = \sigma(\pi_u : u \in \Sigma_* \setminus \{w\})$.

By construction, B and D are $\mathcal{A}_{u|_l}$-measurable, thus are constant given $\mathcal{A}_{u|_l}$.

From assumption (A3) we know $\pi_{u|_l}$ has a bounded density function $f_{u|_l}$.

From (5.26), (5.42) and (5.43), we have

$$\mathbf{1}_{u|_l}^{\text{ran}}(\bar{q}, 2\epsilon) \cdot |A|^{-1} \leq \frac{2}{c_{\psi,k}} \cdot 2^{l(h_{\bar{q}}^{(k)} + 2\epsilon)}. \tag{5.46}$$

When $u, v \in \Sigma_m$ and $\mathbf{1}_p(u, v) = 1$, we have

$$\begin{cases} |\lambda(u) - \lambda(v) - (\lambda(uu') - \lambda(vv'))| \leq 2 \cdot 2^{-m} \\ |\lambda(u) - \lambda(v)| \wedge |\lambda(uu') - \lambda(vv')| \geq 2^{-p-1}. \end{cases} \tag{5.47}$$

Since $\mathbf{1}_p(u, v) = 1$ implies $v|_p \in \mathcal{N}(u|_p)$, by (5.28) we have

$$\mathbf{1}_{u|_p}^{(c)}(\bar{q}, 2\epsilon) \cdot \sup_{s,t \in \bigcup_{w \in \mathcal{N}(u|_p)} [w]} |F_\mu^{\text{pert}}(\lambda(s)) - F_\mu^{\text{pert}}(\lambda(t))| \leq 2^{-p(h_{\bar{q}}^{(k)} - 2\epsilon)}.$$

This implies, when $\mathbf{1}^{(c)}_{u|_r}(\bar{q},2\epsilon)=1$,

$$\Big(|F^{\mathrm{pert}}_\mu(\lambda(u))-F^{\mathrm{pert}}_\mu(\lambda(v))|\vee|F^{\mathrm{pert}}_\mu(\lambda(uu'))-F^{\mathrm{pert}}_\mu(\lambda(vv'))\big|\Big)\wedge 1 \le 2^{-\mathbf{1}_{\{p\ge n\}}\cdot r(h^{(k)}_{\bar{q}}-2\epsilon)}:=\alpha. \quad (5.48)$$

Also, for the same reason, when $\mathbf{1}^{(c)}_u(\bar{q},2\epsilon)\cdot\mathbf{1}^{(c)}_v(\bar{q},2\epsilon)=1$,

$$\Big(|F^{\mathrm{pert}}_\mu(\lambda(u))-F^{\mathrm{pert}}_\mu(\lambda(uu'))\big|\vee\big|F^{\mathrm{pert}}_\mu(t_v)-F^{\mathrm{pert}}_\mu(\lambda(vv'))\big|\Big) \le 2^{-m(h^{(k)}_{\bar{q}}-2\epsilon)}:=\beta. \quad (5.49)$$

These two inequalities with (5.39), (5.44) and (5.45) imply that when $\mathbf{1}^{\mathrm{ran}}_{u,v}(\bar{q},\epsilon)=1$,

$$\begin{aligned}
|D| &\le \Big|F^{\mathrm{pert}}_\mu(\lambda(uu'))-F^{\mathrm{pert}}_\mu(t_u)+F^{\mathrm{pert}}_\mu(t_v)-F^{\mathrm{pert}}_\mu(\lambda(vv'))\Big|+\Big|\eta-1\Big|\cdot\Big|F^{\mathrm{pert}}_\mu(t_u)-F^{\mathrm{pert}}_\mu(t_v)\Big| \\
&\le 2\beta+\frac{2C_\psi}{c_{\psi,k}}\cdot 2^l\cdot 2^{-m}\cdot\alpha \\
&= 2\cdot 2^{-m(h^{(k)}_{\bar{q}}-2\epsilon)}+\frac{2C_\psi}{c_{\psi,k}}\cdot 2^l\cdot 2^{-m}\cdot 2^{-\mathbf{1}_{\{p\ge n\}}\cdot r(h^{(k)}_{\bar{q}}-2\epsilon)} \\
&\le 2\left(\frac{2C_\psi}{c_{\psi,k}}\cdot 2^{1+N_{\psi,k}}\right)\cdot 2^{-m(h^{(k)}_{\bar{q}}-2\epsilon)+r(1-\mathbf{1}_{\{p\ge n\}}\cdot(h^{(k)}_{\bar{q}}-2\epsilon))} \\
&\le C_D\cdot 2^{-m(h^{(k)}_{\bar{q}}-2\epsilon)+r}, \qquad (5.50)
\end{aligned}$$

where $C_D=2\left(\frac{2C_\psi}{c_{\psi,k}}\cdot 2^{1+N_{\psi,k}}\right)$ and we have used $h^{(k)}_{\bar{q}}-2\epsilon\in(0,1)$.

Recall that $l=r+1+N_{\psi,k}$. Now we have

(I) When $\gamma>1$, (also $\gamma\le 2$), due to (5.40), (5.30), (5.46) and (5.47),

$$\begin{aligned}
&\mathbb{E}\Big(\bar{\mathcal{K}}_\gamma(u,v)\Big|\mathcal{A}_{u|_l}\Big) \\
&\le \int_{\mathbb{R}}\mathbf{1}_{\{\bar{\mathcal{K}}_\gamma(u,v)=1\}}\cdot f_{u|_l}(x)\,\mathrm{d}x+\int_{\mathbb{R}}\frac{\mathbf{1}^{\mathrm{ran}}_{u,v}(\bar{q},\epsilon)\cdot f_{u|_l}(x)}{(|A\cdot x+B|^2+|\lambda(u)-\lambda(v)|^2)^{\gamma/2}}\,\mathrm{d}x \\
&\le 1+\int_{\mathbb{R}}\mathbf{1}^{\mathrm{ran}}_{u,v}(\bar{q},\epsilon)\cdot|A|^{-1}|\lambda(u)-\lambda(v)|^{1-\gamma}\cdot\frac{f_{u|_l}(\frac{|\lambda(u)-\lambda(v)|z-B}{A})}{(|z|^2+1)^{\gamma/2}}\,\mathrm{d}z \\
&\le 1+\frac{2}{c_{\psi,k}}\cdot 2^{l(h^{(k)}_{\bar{q}}+2\epsilon)}\cdot 2^{-(p+1)(1-\gamma)}\cdot\|f_{u|_l}\|_\infty\cdot\int_{\mathbb{R}}\frac{1}{(|z|^2+1)^{\gamma/2}}\,\mathrm{d}z \\
&= 1+\left(\int_{\mathbb{R}}\frac{\mathrm{d}z}{(|z|^2+1)^{\gamma/2}}\cdot\frac{2^{(1+N_{\psi,k})(h^{(k)}_{\bar{q}}+2\epsilon)+\gamma}}{c_{\psi,k}}\right)\cdot\|f_{u|_l}\|_\infty\cdot 2^{r(h^{(k)}_{\bar{q}}+2\epsilon)-p(1-\gamma)} \\
&\le 2\left(\int_{\mathbb{R}}\frac{\mathrm{d}z}{(|z|^2+1)^{\gamma/2}}\cdot\frac{2^{(1+N_{\psi,k})(h^{(k)}_{\bar{q}}+2\epsilon)+\gamma}}{c_{\psi,k}}\right)\cdot C_r\cdot 2^{r(h^{(k)}_{\bar{q}}+2\epsilon)-p(1-\gamma)},
\end{aligned}$$

where we recall that $C_r=\sup_{w\in\Sigma_l}\|f_w\|_\infty$;

(II) When $\gamma>1$, let

$$\phi_\gamma(x,y)=(\big|F^{\mathrm{pert}}_\mu(\lambda(u))-F^{\mathrm{pert}}_\mu(\lambda(v))+x\big|^2+\big||\lambda(u)-\lambda(v)|+y\big|^2)^{-\gamma/2}.$$

Then due to (5.40), (5.30), (5.49) and (5.47), we have

$$\Delta\bar{\mathcal{K}}_\gamma(u,v) \le \int_{|y|\le 2\cdot 2^{-m}} \left|\frac{\partial}{\partial y}\phi_\gamma(0,y)\right| \,\mathrm{d}y + \int_{|x|\le\beta} \sup_{|y|\le 2\cdot 2^{-m}} \left|\frac{\partial}{\partial x}\phi_\gamma(x,y)\right| \,\mathrm{d}x, \quad (5.51)$$

where we have used that $|a\vee 1 - b\vee 1| \le |a-b|$ for any $a,b\ge 0$. It is not difficult to check that

$$\left|\frac{\partial}{\partial y}\phi_\gamma(0,y)\right| \vee \left|\frac{\partial}{\partial x}\phi_\gamma(x,y)\right| \le \gamma\cdot \big||\lambda(u)-\lambda(v)|+y\big|^{-\gamma-1}.$$

In fact, we have

$$\begin{aligned}\left|\frac{\partial}{\partial y}\phi_\gamma(0,y)\right| &\le \frac{\gamma\cdot\big||\lambda(u)-\lambda(v)|+y\big|}{\left(\big|F_\mu^{\mathrm{pert}}(\lambda(u))-F_\mu^{\mathrm{pert}}(\lambda(v))\big|^2+\big||\lambda(u)-\lambda(v)|+y\big|^2\right)^{1+\frac{\gamma}{2}}}\\ &\le \frac{\gamma\cdot\big||\lambda(u)-\lambda(v)|+y\big|}{\big||\lambda(u)-\lambda(v)|+y\big|^{2+\gamma}} = \gamma\cdot\big||\lambda(u)-\lambda(v)|+y\big|^{-1-\gamma};\end{aligned}$$

and

$$\begin{aligned}&\left|\frac{\partial}{\partial x}\phi_\gamma(x,y)\right|\\ \le\ & \frac{\gamma\cdot\big|F_\mu^{\mathrm{pert}}(\lambda(u))-F_\mu^{\mathrm{pert}}(\lambda(v))+x\big|}{\left(\big|F_\mu^{\mathrm{pert}}(\lambda(u))-F_\mu^{\mathrm{pert}}(\lambda(v))+x\big|^2+\big||\lambda(u)-\lambda(v)|+y\big|^2\right)^{1+\frac{\gamma}{2}}}\\ \le\ & \frac{\gamma\cdot\big|F_\mu^{\mathrm{pert}}(\lambda(u))-F_\mu^{\mathrm{pert}}(\lambda(v))+x\big|\cdot\big||\lambda(u)-\lambda(v)|+y\big|^{-\gamma}}{\big|F_\mu^{\mathrm{pert}}(\lambda(u))-F_\mu^{\mathrm{pert}}(\lambda(v))+x\big|^2+\big||\lambda(u)-\lambda(v)|+y\big|^2}\\ \le\ & \frac{\gamma\cdot\big||\lambda(u)-\lambda(v)|+y\big|^{-\gamma}}{2\big||\lambda(u)-\lambda(v)|+y\big|} \le \gamma\cdot\big||\lambda(u)-\lambda(v)|+y\big|^{-1-\gamma},\end{aligned}$$

where we have used that $\frac{a}{a^2+b^2}\le\frac{1}{2b}$ for any $a,b>0$. This together with $|\lambda(u)-\lambda(v)|\ge 2^{-p-1}$, $m>p+1$ and $\gamma\le 2$ yields

$$\begin{aligned}\Delta\bar{\mathcal{K}}_\gamma(u,v) &\le \gamma\cdot(2^{-p-1}-2\cdot 2^{-m})^{-\gamma-1}\cdot(2\cdot 2^{-m}+2\beta)\\ &\le \gamma\cdot 2^{(p+2)(\gamma+1)}\cdot(2\cdot 2^{-m}+2\cdot 2^{-m(h_{\bar{q}}^{(k)}-2\epsilon)})\\ &\le \gamma\cdot 2^{(p+2)(\gamma+1)}\cdot(4\cdot 2^{-m(h_{\bar{q}}^{(k)}-2\epsilon)}) \qquad (\text{since } h_{\bar{q}}^{(k)}-2\epsilon<1)\\ &\le \left(4\gamma 2^{2(\gamma+1)}\right)\cdot C_r\cdot 2^{p\cdot 3-m\cdot(h_{\bar{q}}^{(k)}-2\epsilon)}.\end{aligned}$$

(III) When $\gamma<1$, due to (5.40), (5.30), (5.46) and (5.48),

$$\begin{aligned}&\mathbb{E}\Big(\bar{\mathcal{K}}_\gamma(u,v)\Big|\mathcal{A}_{u|_l}\Big)\\ \le\ & \int_{|A\cdot x+B|\le\alpha}\mathbf{1}_{\{\bar{\mathcal{K}}_\gamma(u,v)=1\}}\cdot f_{u|_l}(x)\,\mathrm{d}x + \int_{|A\cdot x+B|\le\alpha}\mathbf{1}_{u,v}^{\mathrm{ran}}(\bar{q},\epsilon)\cdot\frac{f_{u|_l}(x)}{|A\cdot x+B|^\gamma}\,\mathrm{d}x\\ \le\ & 1+\int_{|z|\le\alpha}\mathbf{1}_{u,v}^{\mathrm{ran}}(\bar{q},\epsilon)\cdot|A|^{-1}\cdot\frac{f_{u|_l}(\frac{z-B}{A})}{|z|^\gamma}\,\mathrm{d}z\end{aligned}$$

$$
\begin{aligned}
&\leq\ 1+\frac{2}{c_{\psi,k}}\cdot 2^{l(h_{\bar q}^{(k)}+2\epsilon)}\cdot\|f_{u|_l}\|_\infty\cdot\int_{|z|\leq\alpha}\frac{1}{|z|^\gamma}\ \mathrm{d}z\\
&=\ 1+\frac{2}{c_{\psi,k}}\cdot 2^{l(h_{\bar q}^{(k)}+2\epsilon)}\cdot\|f_{u|_l}\|_\infty\cdot 2\alpha^{1-\gamma}\\
&\leq\ 2\left(\frac{2^{(1+N_{\psi,k})(h_{\bar q}^{(k)}+2\epsilon)}}{c_{\psi,k}}\right)\cdot\|f_{u|_l}\|_\infty\cdot 2^{r(h_{\bar q}^{(k)}+2\epsilon)-\mathbf{1}_{\{p\geq n\}}\cdot r(h_{\bar q}^{(k)}-2\epsilon)(1-\gamma)}\\
&=\ 2\left(\frac{2^{(1+N_{\psi,k})(h_{\bar q}^{(k)}+2\epsilon)}}{c_{\psi,k}}\right)\cdot\|f_{u|_l}\|_\infty\cdot 2^{\mathbf{1}_{\{p<n\}}r(h_{\bar q}^{(k)}-2\epsilon)(1-\gamma)}\cdot 2^{r(h_{\bar q}^{(k)}\gamma+(4-2\gamma)\epsilon)}\\
&\leq\ 2\left(\frac{2^{(1+N_{\psi,k})(h_{\bar q}^{(k)}+2\epsilon)}}{c_{\psi,k}}\right)\cdot\|f_{u|_l}\|_\infty\cdot 2^{\mathbf{1}_{\{p<n\}}r}\cdot 2^{r(h_{\bar q}^{(k)}\gamma+4\epsilon)}\\
&\leq\ 2\left(\frac{2^{(1+N_{\psi,k})(h_{\bar q}^{(k)}+2\epsilon)}}{c_{\psi,k}}\right)\cdot C_r\cdot 2^n\cdot 2^{r(h_{\bar q}^{(k)}\gamma+4\epsilon)};
\end{aligned}
$$

(IV) When $\gamma<1$, due to (5.40), (5.30), (5.46) and (5.50), by using again $|a\vee 1-b\vee 1|\leq|a-b|$ for any $a,b\geq 0$, we have

$$
\begin{aligned}
&\mathbb{E}\Big(\Delta\bar{\mathcal{K}}_\gamma(u,v)\Big|\mathcal{A}_{u|_l}\Big)\\
\leq\ &\int_{\mathbb{R}}\mathbf{1}_{u,v}^{\mathrm{ran}}(\bar q,\epsilon)\cdot\left|\frac{1}{|\eta(A\cdot x+B)+D|^\gamma}-\frac{1}{|A\cdot x+B|^\gamma}\right|f_{u|_l}(x)\ \mathrm{d}x\\
=\ &\int_{\mathbb{R}}\mathbf{1}_{u,v}^{\mathrm{ran}}(\bar q,\epsilon)\cdot|A|^{-1}\cdot|D|^{1-\gamma}\cdot\left|\frac{1}{|\eta\cdot z+1|^\gamma}-\frac{1}{|z|^\gamma}\right|f_{u|_l}(\frac{D\cdot z-B}{A})\ \mathrm{d}z\\
\leq\ &\frac{2}{c_{\psi,k}}2^{l(h_{\bar q}^{(k)}+2\epsilon)}\cdot(C_D\cdot 2^{-m(h_{\bar q}^{(k)}-2\epsilon)+r})^{1-\gamma}\cdot\|f_{u|_l}\|_\infty\cdot\int_{\mathbb{R}}\left|\frac{1}{|\eta\cdot z+1|^\gamma}-\frac{1}{|z|^\gamma}\right|\ \mathrm{d}z\\
=\ &\left(\frac{2^{1+(1+N_{\psi,k})(h_{\bar q}^{(k)}+2\epsilon)}C_D^{1-\gamma}}{c_{\psi,k}}\cdot\int_{\mathbb{R}}\left|\frac{1}{|\eta\cdot z+1|^\gamma}-\frac{1}{|z|^\gamma}\right|\ \mathrm{d}z\right)\\
&\qquad\cdot\|f_{u|_l}\|_\infty\cdot 2^{r(h_{\bar q}^{(k)}+2\epsilon+1-\gamma)-m(h_{\bar q}^{(k)}-2\epsilon)(1-\gamma)}\\
\leq\ &\left(\frac{2^{1+(1+N_{\psi,k})(h_{\bar q}^{(k)}+2\epsilon)}C_D^{1-\gamma}}{c_{\psi,k}}\cdot\int_{\mathbb{R}}\left|\frac{1}{|\eta\cdot z+1|^\gamma}-\frac{1}{|z|^\gamma}\right|\ \mathrm{d}z\right)\cdot C_r\cdot 2^{r\cdot 3-m(h_{\bar q}^{(k)}-2\epsilon)}.
\end{aligned}
$$

Now, since

$$
\int_{\mathbb{R}}\frac{\mathrm{d}z}{(|z|^2+1)^{\gamma/2}}\ (\gamma>1)\ \text{and}\ \int_{\mathbb{R}}\left|\frac{1}{|\eta\cdot z+1|^\gamma}-\frac{1}{|z|^\gamma}\right|\mathrm{d}z\ (\gamma<1)
$$

are both finite (notice that η is bounded away from 0 and infinity uniformly), and $h_{\bar q}^{(k)}$ are chosen between s_0-1/p_0 and 1, we can easily find a constant c_γ such that

$$
\max\left(\left\{\begin{array}{l}2\left(\int_{\mathbb{R}}\frac{\mathrm{d}z}{(|z|^2+1)^{\gamma/2}}\cdot\frac{2^{(1+N_{\psi,k})(h_{\bar q}^{(k)}+2\epsilon)+\gamma}}{c_{\psi,k}}\right),2\left(\frac{2^{(1+N_{\psi,k})(h_{\bar q}^{(k)}+2\epsilon)}}{c_{\psi,k}}\right),\\ \left(4\gamma 2^{2(\gamma+1)}\right),\left(\frac{2^{1+(1+N_{\psi,k})(h_{\bar q}^{(k)}+2\epsilon)}C_D^{1-\gamma}}{c_{\psi,k}}\cdot\int_{\mathbb{R}}\left|\frac{1}{|\eta\cdot z+1|^\gamma}-\frac{1}{|z|^\gamma}\right|\ \mathrm{d}z\right)\end{array}\right.\right)\leq c_\gamma.
$$

This gives us the conclusion.

5.4.4 Proof of Lemma 5.4.

Proof Recall (5.39) that

$$\mathbf{1}^{\det}_{u,v}(\bar{q},\epsilon) = \mathbf{1}^{(b)}_{u|_r}(\bar{q},2\epsilon)\cdot \mathbf{1}^{(b)}_{v|_r}(\bar{q},2\epsilon)\cdot \mathbf{1}^{(b)}_{u}(\bar{q},2\epsilon)\cdot \mathbf{1}^{(b)}_{v}(\bar{q},2\epsilon).$$

Let

$$S_{p,m} = \sum_{u,v\in\Sigma_m} \mathbf{1}_p(u,v)\cdot \mathbf{1}^{(b)}_{u|_r}(\bar{q},2\epsilon)\cdot \mathbf{1}^{(b)}_{v|_r}(\bar{q},2\epsilon)\cdot \mathbf{1}^{(b)}_{u}(\bar{q},2\epsilon)\cdot \mathbf{1}^{(b)}_{v}(\bar{q},2\epsilon).$$

Recall that $r = p\vee n$. For any $u\in\Sigma_m$ we write $u = u|_r\cdot u'$ with $u'\in\Sigma_{m-r}$. Since $\mathbf{1}_p(u,v)$ only depends on $u|_r$, $v|_r$, we can write

$$S_{p,m} = \sum_{u_r,v_r\in\Sigma_r} \mathbf{1}_p(u_r,v_r)\cdot \mathbf{1}^{(b)}_{u|_r}(\bar{q},2\epsilon)\cdot \mathbf{1}^{(b)}_{v|_r}(\bar{q},2\epsilon)\cdot \sum_{u',v'\in\Sigma_{m-r}} \mathbf{1}^{(b)}_{u|_r\cdot u'}(\bar{q},2\epsilon)\cdot \mathbf{1}^{(b)}_{v|_r\cdot v'}(\bar{q},2\epsilon).$$

Recall (see (5.27)) that

$$\mathbf{1}^{(b)}_{u|_r\cdot u'}(\bar{q},2\epsilon) = \mathbf{1}_{\left\{\mu^{(k)}_{\bar{q}}([u|_r\cdot u'])\in[2^{-m(D^{(k)}_{\bar{q}}+2\epsilon)},2^{-m(D^{(k)}_{\bar{q}}-2\epsilon)}]\right\}}.$$

Thus

$$\mathbf{1}^{(b)}_{u|_r\cdot u'}(\bar{q},2\epsilon) \le 2^{m(D^{(k)}_{\bar{q}}+2\epsilon)}\cdot \mu^{(k)}_{\bar{q}}([u|_r\cdot u']).$$

This implies that

$$\begin{aligned} &\sum_{u',v'\in\Sigma_{m-r}} \mathbf{1}^{(b)}_{u|_r\cdot u'}(\bar{q},2\epsilon)\cdot \mathbf{1}^{(b)}_{v|_r\cdot v'}(\bar{q},2\epsilon) \\ \le\;& 2^{2m(D^{(k)}_{\bar{q}}+2\epsilon)}\cdot \sum_{u',v'\in\Sigma_{m-r}} \mu^{(k)}_{\bar{q}}([u|_r\cdot u'])\cdot \mu^{(k)}_{\bar{q}}([v|_r\cdot v']) \\ \le\;& 2^{2m(D^{(k)}_{\bar{q}}+2\epsilon)}\cdot \mu^{(k)}_{\bar{q}}([u|_r])\cdot \mu^{(k)}_{\bar{q}}([v|_r]). \end{aligned}$$

Thus by the fact that given $u|_r$ in Σ_r, there are at most 3^{r-p+1} many $v|_r$ in Σ_r such that $\mathbf{1}_p(u|_r,v|_r) = 1$, we have

$$\begin{aligned} & S_{p,m} \\ \le\;& 2^{2m(D^{(k)}_{\bar{q}}+2\epsilon)}\cdot \sum_{u|_r,v|_r\in\Sigma_r} \mathbf{1}_p(u|_r,v|_r)\cdot \mathbf{1}^{(b)}_{u|_r}(\bar{q},2\epsilon)\cdot \mathbf{1}^{(b)}_{v|_r}(\bar{q},2\epsilon)\cdot \mu^{(k)}_{\bar{q}}([u|_r])\cdot \mu^{(k)}_{\bar{q}}([v|_r]) \\ \le\;& 2^{2m(D^{(k)}_{\bar{q}}+2\epsilon)-r(D^{(k)}_{\bar{q}}-2\epsilon)}\cdot \sum_{u|_r,v|_r\in\Sigma_r} \mathbf{1}_p(u|_r,v|_r)\cdot \mu^{(k)}_{\bar{q}}([u|_r]) \\ \le\;& 2^{2m(D^{(k)}_{\bar{q}}+2\epsilon)-r(D^{(k)}_{\bar{q}}-2\epsilon)}\cdot 3^{r-p+1}\cdot \sum_{u|_r\in\Sigma_r} \mu^{(k)}_{\bar{q}}([u|_r]) \\ =\;& 2^{2m(D^{(k)}_{\bar{q}}+2\epsilon)-r(D^{(k)}_{\bar{q}}-2\epsilon)}\cdot 3^{r-p+1} \end{aligned}$$

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