Complete Blow-up for a semi-linear parabolic problem with a localized nonlinear term

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Contents

Résumé de la thèse

Ce travail est consacré á l'étude de problèmes de 'blow-up' pour une équation de type de la chaleur avec un terme source uniforme fonction de la température instantanée en un point x_0 du domaine spatial Ω. Pour simplifier, on suppose que la (variation de) température u satisfait des conditions aux limites de Dirichlet homogènes.

Rappelons la définition d'un 'blow-up' en temps fini. Soit

$$T_{\max} = \sup\{\, T > 0 \text{ tel que } u(x,t) \text{ est borné dans } \Omega \times (0,T)\,\}$$

Si $T_{\max} = \infty$, il n'y a pas de 'blow-up' en temps fini la solution u le l'équation parabolique semi linéaire est dite globale. Si $T_{\max} < \infty$, alors

$$\lim_{t \to T_{\max}} \|u(\cdot,t)\|_{L^\infty(\Omega)} \to \infty$$

et on dit qu' il y a 'blow-up' en un temps fini $T_{\max}$. L'ensemble de 'blow-up' est alors :

$$\begin{aligned} B \;\; = \;\; & \{x \in \Omega \text{ by tel que } \exists\,\{x_n,t_n\} \subset \Omega \times (0,T_{\max});\;\; \{x_n,t_n\} \to \{x,T_{\max}\} \\ & \text{et } u(x_n,t_n) \to \infty \text{ quand } n \to \infty.\} \end{aligned}$$

Tout point x de B est appelé point de 'blow-up'. Si $B = \overline{\Omega}$, on dit que le 'blow-up' est total, si B se réduit à un singleton on parle de 'blow-up' en un seul point.

In 2000, C. Y. Chan and J. Yang [1] ont considéré l'équation semi-linéaire parabolique

$$\begin{gathered} x^q u_t - u_{xx} = f(u(x_0,t)), \quad (x,t) \in (0,1) \times (0,T) \\ u(0,t) = u(1,t) = 0, \quad t \in (0,T) \\ u(x,0) = u_0(x), \quad x \in (0,1) \\ x_0 \in (0,1) \end{gathered}$$

Sous certaines conditions sur u_0 et f, ils ont montré qu'il y avait 'blow-up' en temps fini et que le 'blow-up' était total.

Le propos de cette these est de généraliser l'étude de C.Y. Chan et J. Yang à :

$$\begin{gathered} k(x)u_t - (\operatorname{div}(p(x)\nabla u) = k(x)f(u(x_0,t)), \quad (x,t) \in \Omega \times (0,T), \\ u(x,t) = 0, \text{ sur } \partial\Omega\, \forall t \in (0,T), \\ u(x,0) = u_0(x), x \in \Omega, \end{gathered} \tag{0.0.1}$$

où Ω est un domaine de $\mathbb{R}^N$, x_0 est un point donné de Ω, k, μ, f et u_0 sont des fonctions données.

Cette thèse est divisée en six chapitres. Un rappel historique des problèmes de 'blow-up' constitue le chaptre 1.

Les chapitres 2 et 3 traitent le problème monodimensionnel sous la condition de stricte positivité de k et p sur tout $\overline{\Omega}$. La différence entre le chapitre 2 et le chapitre 3 est qu'au chapitre 2 l'existence d'une solution avec 'blow-up' est établie par une méthode d'analyse fonctionnelle, i.e la méthode des semi-groupes d'opérateurs linéaires dans un espace de Hilbert, alors qu'au chapitre 3 cela est prouvé par une méthode d'analyse plus classique : la méthode des fonctions de Green. Le chapitre 4 concerne une extension des résultats obtenus aux cas de dimensions $N \leq 3$ en utilisant la méthode des semi-groupes. Avant d'examiner, en dimension 1, le cas où $k(0) = p(0) = 0$ avec $k(x)$, $p(x) > 0$ sur $(0, 1]$, on se fait la main au chapitre 5 avec $k(x) = x^\alpha$, $p(x) = x^\beta$, α, $\beta > 0$. Le chapitre suivant traite le cas général pour k et p avec, comme au chapitre 5, une méthode de fonctions de Green.

Pour plus de détails, au chapitre 2, en vue d'appliquer la théorie des semi-groupes on transforme le problème en une équation d'évolution du type :

$$u_t(t) - Au(t) = F(u(t)) \text{ pour } t > 0 \text{ et } u(0) = u_0, \tag{0.0.2}$$

où A est l'opérateur linéaire non borné de $D(A)$, le domaine de A, vers $L^2(I)$ défini par :

$$\begin{aligned} D(A) \quad = \quad & \Big\{ u \in H_0^1(I) \text{ tel que } \exists ! w \in L^2(I) \text{ et} \\ & \left. \int_I k(x)w(x)\varphi(x)dx = -\int_I p(x)D_x u(x)D_x\varphi(x)dx, \ \forall \varphi \in H_0^1(I) \right\}, \end{aligned}$$

$Au = w$ pour tout $u \in D(A)$ où D_x est la dérivée au sens des distributions sur I. L'opérateur (non linéaire) F appliquaant $D(A)$ dans $L^2(I)$ est défini par

$$F(u) = f(u(x_0, t)).$$

Rappelons que $L^2(I) = \{\, v$ est une fonction mesurable telle que $\int_I k(x) |v(x)|^2 dx < \infty\}$ est un espace de Hilbert équippé du produit scalaire et de la norme :

$$\langle u, v\rangle_{L^2(I)} = \int_I k(x)u(x)v(x)dx, \ \text{et} \ |u|_{L^2(I)} = \left(\int_I k(x) |u(x)|^2 dx\right)^{1/2},$$

respectivement et $H^1(I) = \left\{v \in L^2(I) \text{ tel que } D_x v \in L^2(I)\right\}$ est un espace de Hilbert de carré de norme

$$|v|^2_{H^1(I)} = |v|^2_{L^2(I)} + \int_I p(x) |D_x v(x)|^2 dx$$

tandis que le sous-espace fermé $H_0^1(I) = \left\{v \in H^1(I) \text{ tel que } v(0) = 0 = v(1)\right\}$ est équippé de la norme équivalente de carré :

$$|v|^2_{H_0^1(I)} = \int_I p(x) |D_x v(x)|^2 dx.$$

Notre principal résultat consiste en les 4 théorèmes :

Théorème 2.2.1 Il existe un nombre positif T tel que le problème d'évolution (0.0.2) ait une unique solution classique (au sens de la théorie des semi-groupes, i.e $u \in C([0, T], D(A)) \cap C^1([0, T], L^2(I))$) définie par :

$$u(t) = S(t)u_0 + \int_0^t S(t-\tau)F(u(\tau))d\tau$$

où $S(t)$ est le semi-groupe analytique engendré par A.

Théorème 2.2.2 Si $[0, T_{\max})$ est l'intervalle fini maximal pour lequel la solution u de (0.0.2) est bornée, alors $|u(x_0, t)|$ tend vers l'infini quand t tend vers $T_{\max}$.

Théorème 2.2.3 L'ensemble de 'blow-up' est $\overline{I}$.

Théorème 2.2.4 Si,

1 u_0 atteint son maximum en x_0,

2 $f(\xi) \geq b\xi^p$ avec $b > 0$ et $p > 1$,

3 $H(0) > \left(\frac{\lambda_1}{b}\right)^{\frac{1}{p-1}}$ avec $H(t) = \int_I k(x)\phi_1(x)u(x,t)dx$ où λ_1 est la première valeur propre et ϕ_1 la fonction propre associée de

$$\frac{d}{dx}\left(p(x)\frac{d}{dx}\phi(x)\right) = \lambda k(x)\phi(x) \text{ pour } x \in I \text{ et } \phi(0) = 0 = \phi(1),$$

alors il y a 'blow-up' en temps fini pour (0.0.1).

On a utilisé le fait que A est m-dissipatif autoadjoint et que F est Hölder-continue d'exposant $\alpha \in (0, 1)$.

Au chapitre 3, où on étudie le même problème, on prouve le 'blow-up' par une méthode de fonctions de Green.

Pour construire la fonction de Green, on considère le problème (régulier) de valeurs propres associé :

$$\frac{d}{dx}\left(p(x)\frac{d}{dx}\phi(x)\right) = \lambda k(x)\phi(x) \text{ pour } x \in I \text{ et } \phi(0) = 0 = \phi(1) \tag{0.0.3}$$

La propriété de complétion des fonctions propres ϕ_n de (0.0.3), implique que la fonction de Green est définie par :

$$G(x, t, \xi, \tau) = \sum_{n=1}^{\infty} \phi_n(x)\phi_n(\xi)e^{-\lambda_n(t-\tau)} \text{ pour } x, \xi \in I \text{ et } t > \tau.$$

En utilisant le theéorème de Green, l'équation intégrale correspondant au problème (0.0.1) est alors :

$$u(x,t) = \int_0^t \int_0^1 k(\xi)G(x,t,\xi,\tau)f(u(x_0,\tau))d\xi d\tau + \int_0^1 k(\xi)G(x,t,\xi,0)u_0(\xi)d\xi. \tag{0.0.4}$$

Pour prouver qu'il existe un réel positif T tel que (0.0.4) ait une solution u continue sur $[0, T]$ pour tout $x \in \overline{I}$, nous construisons une suite $\{w_n\}$ avec $w_0(x,t) = u_0(x)$ par :

$$\left.\begin{array}{c} k(t)(w_n)_t - (p(x)(w_n)_x)_x = k(x)f(w_{n-1}(x_0,t)),\ (x,t) \in I \times (0,\infty), \\ w_n(0,t) = 0 = w_n(1,t),\ t > 0, \\ w_n(x,0) = u_0(x),\ x \in \overline{I}. \end{array}\right\} \tag{0.0.5}$$

Ensuite, on montre que

1 la suite $\{w_n\}$ est une fonction non décroissante de t,

2 il existe $T > 0$ tel que $\{w_n\}$ converge ponctuellement vers u sur $[0, T]$ pour tout $x \in \overline{I}$.

Aussi, par le théorème de Dini, on peut conclure que $\{w_n\}$ converge uniformément vers u sur $[0,T]$ pour tout $x \in \bar{I}$. Ainsi, (0.0.4) a une unique solution u continue sur $[0,T]$ pour tout $x \in \bar{I}$. Avec cette idée, on peut prouver le résultat suivant :

Théorème 3.3.3 Il existe $T > 0$ tel que (0.0.4) a une solution unique non négative continue sur $[0,T]$ pour tout $x \in \bar{I}$, $u(x,t) \geq u_0(x)$ pour tout $(x,t) \in \bar{I} \times [0,T]$ et u est une fonction non décroissante de t.

Soit $T_{\max}$ le supremum des T tels que (0.0.4) ait une solution non négative.

Théorème 3.3.4 Si $T_{\max}$ est fini, alors $u(x_0,t)$ est non borné quand $t \to T_{\max}$.

Comme au chapitre 2, nous donnons la condition suffisante pour garantir l'existence d'un 'blow-up' eu temps fini.

Théorème 3.4.1 Si

$$\int_{H(0)}^{\infty} \frac{ds}{H(s) - \lambda_1 s} < \infty$$

où $H(s) = \int_0^1 k(x)u(x,s)\phi_1(x)dx$, λ_1 la première valeur propre de (0.0.3) et ϕ_1 la fonction propre associée, alors la solution de (0.0.2) présente un 'blow-up' en temps fini.

Théorème 3.5 Si une solution de (0.0.2) présente un 'blow-up' en temps fini, alors l'ensemble des points de 'blow-up' est $\bar{I}$.

Au chapitre 4 nous étendons les résultats au cas N-dimensionnel ($N \leq 3$) avec des conditions de stricte positivité pour k et p en tenant compte, grâce à la théorie de l'interpolation et les injections de Sobolev, de

$$D(A) \hookrightarrow D((-A^s)) \hookrightarrow H^{2s}(\Omega) \hookrightarrow C(\overline{\Omega})$$

avec $s \in (s_N, 1)$ et $s_N = N/4$ pour tout $N \leq 3$.

Au chapitre 5 est examiné un cas dégénéré pour le problème (0.0.1) où $k(x) = x^\alpha$, $p(x) = x^\beta$ $\alpha, \beta > 0$. Le problème aux valeurs propres associé

$$\frac{d}{dx}(x^\alpha \frac{d\phi}{dx}) = \lambda x^\beta \phi(x) \text{ pour } x \in I, \phi(0) = \phi(1) = 0 \tag{0.0.6}$$

est singulier, mais on dispose de propriétés suffisantes sur les valeurs propres et les fonctions propres pour obtenir à propos du 'blow-up' des résultats dans l'esprit du chapitre 3.

Le dernier chapitre est consacré au cas dégénéré plus général $k(0) = p(0)$, k et $p > 0$ sur $(0,1]$. Diverses conditions sont introduites en vue de l'existence, la complétion et la bornitude des fonctions propres. Comme au chapitre 3, le problème (0.0.1) est transformé en une équation intégrale, mais l'existence d'un 'blow-up' va être obtenue ici en utilisant le théorème de point fixe de Banach dans un ensemble

$$E(M,T) = \{\, u \in C(\bar{I} \times [0,T] \text{ tel que } \max_{(x,t)\in \bar{I}\times[0,T]} |u(x,t)| \leq M \,\}$$

Pour conclure, on peut dire que chacune des 2 méthodes : semi-groupes et fonction de Green présente des avantages ou des inconvénients quant à leur mise en oeuvre. L'intérêt de la méthode semi-groupes réside dans sa généralité et dans la panoplie de théorèmes concernant les équations d'évolution semi-linéaires. Il s'agit essentiellement de trouver le bon cadre fonctionnel et les bonnes propriétés sur f générant de bonnes propriétés sur F pour conclure quant à l'existence de 'blow-up' et la nature de l'ensemble des points de 'blow-up'. L'avantage de la méthode des fonctions de Green est que cet outil fondamental en théorie des équations aux dérivées partielles est enseigné dans tous les cours classiques et donc a été facile à comprendre et à utiliser pour le problème considéré dans cette thèse.

Une limitation de la méthode des fonctions de Green est dans la construction même de celles-ci, ce qui peut arriver Lorsqu'on manque d'informations sur les valeurs propres ou fonctions propres. Ce fut le cas au chapitre 6 où nous n'avons pas trouvé dans la litérature des résultats concernant des cas dégénérés pour k et p très généraux. En réalité, cette limitation de connaissance de résultats spectraux va se traduire dans la méthode des semi-groupes par la difficulté à choisir un bon cadre fonctionnel assurant de bonnes propriétés pour F. On est amené à travailler dans un espace L^2 avec le poids k et à remplacer H_0^1 par un espace V complété pour l'intégrale du carré de la dérivée avec le poids p, nous ne savons pas dans les cas généraux quand V est compact dans L_k^2.

Références

[1]. Chan C.Y., Yang J. Complete blow-up for degenerate semilinear parabolic equations. J. Comp. and Appl. 2000;113:353–364.

Chapter 1

Introduction

There has been a tremendous amount of recent activity due to the subjects of solutions to partial differential equations blowing up in finite time. The mathematical theory for this is extensive and reviews may be found in Levine (1990) and Samarskii et al. (1994). Finite time blow-up occurs in situations in mechanics and other areas of applied mathematics, and studies of these phenomena have very recently been gaining momentum.

The simplest form of spontaneous singularities in nonlinear problems appears when the variable or variables tend to infinity when time approaches a certain finite limit $T > 0$. This is what we call a blow -up phenomenon. Blow-up occurs in an elementary form in the theory of ordinary differential equations (ODEs), and the simplest example is the following initial-value problem:

$$u_t = u^2, \ t > 0,$$
$$u(0) = a,$$

with $u = u(t)$ and $a > 0$. It then is immediate that a unique solution u exists in the time interval $0 \leq t < T = 1/a$. Since the solution u is given by the formula $u(t) = 1/(T - t)$, one see that u is a smooth function for $0 \leq t < T$ and also that $u(t) \to \infty$ as $t \to T^-$. We can say that the solution u of this problem blows up in finite time at $t = T$ and also that u has blow-up at that time. Blow-up is referred to in Latin languages as explosion. Starting from this example, the concept of blow-up can be widely generalized as the phenomenon whereby solutions cease to exist globally in time. Thus, a first step is given by ODE's of the form $u_t = u^p$, with $p > 1$ and, more generally,

$$u_t = f(u),$$

where f is positive and continuous under the condition

$$\int_1^\infty \frac{ds}{f(s)} < \infty.$$

This Osgood's condition in the ODE theory established in 1898 [22] is necessary and sufficient for the occurrence of blow-up in finite time for any solution with positive initial data. More generally, we can think of systems $u_t = f(t, u)$ for a vector variable $u \in \mathbb{R}^n$. In this case we may have blow-up due to the same mechanism if f is super-linear with respect to u for $|u|$ large, and also blow-up due to the singular character of f with respect to t at certain given times.

The subject of blow-up was posed in the 1940's and 50's in the context of Semenov's chain reaction theory, adiabatic explosion and combustion theory, see [16] and [26]. A strong influence was also due to blow-up singularities in gas dynamics, the intense explosion (focusing) problem with second kind self-similar solutions considered by Bechert, Guderley and Sedov in the 1940's [4], p. 127. An essential increase of attention to blow-up research in gas dynamics, laser fusion and combustion in the 70's was initiated by the numerical results [21] on the possibility of the laser blow-up like compression of deuterium-tritium (DT) drop to super-high densities without shock waves. The problem of localization of blow-up solutions in reaction-diffusion equations was first proposed by Kurdyumov [19] in 1974.

The mathematical theory has been investigated by researches in the 60's mainly after approaches to blow-up by Kaplan [18], Fujita [14], [15], Friedman [13] and some others. There are two classical scalar models. One of them is the exponential reaction model

$$u_t = \Delta u + \lambda e^u,\ \lambda > 0,$$

which is important in combustion theory [26] under the name of solid-fuel model (Frank-Kamenetsky equation) and also in other areas. The occurrence and type of blow-up depends on the parameter $\lambda > 0$, the initial data and the domain. The other classical blow-up equation is

$$u_t = \Delta u + u^p,\ p > 1.$$

Both semilinear equations were studied in the pioneering works by Fujita.

To define the phenomenon of blow-up in finite time, let u be a solution to a first order in time partial differential equation, say

$$u_t = Lu, \tag{1.0.1}$$

for some partial derivative operator L which involves spatial derivatives. Suppose this equations is defined on a domain $\Omega \subset \mathbb{R}^N$, for some positive range of times $t > 0$. The solution to (1.0.1) will also be required to satisfy suitable boundary and initial data. The definition of blow-up in finite time is given if we define a number T^* by

$$T^* = \sup\left\{T > 0 : u(x,t) \text{ is bounded in } \Omega \times (0,T),\ \text{where } u \text{ satisfies (1.0.1)}\right\}.$$

If $T^* = +\infty$,then blow-up in finite time does not occur and solutions are said to be global. If $T^* < \infty$, then

$$\limsup_{t \to T^*} \|u(t)\|_\infty = \infty$$

and one says the solution blows up at time T^*. Furthermore, we can define the blow-up set, denoted by B, by

$$B = \left\{x \in \Omega : \exists\{x_n, t_n\} \subset \Omega \times (0,T),\ t_n \to T^-,\ x_n \to x \text{ and } u(x_n, t_n) \to \infty\right\}.$$

Its points in B are the blow-up points.

My inspiration comes from studying following papers. J. M. Chadam, A. Peirce and H. M. Yin [5] in 1992 studied the blow-up property of solutions to the problem

$$\begin{gathered} u_t - \triangle u = f(u(x_0,t)),\ (x,t) \in \Omega \times (0,T) \\ u(x,t) = 0,\ (x,t) \in \partial\Omega \times (0,T) \\ u(x,0) = u_0(x),\ x \in \overline{\Omega}, \end{gathered}$$

where T is a positive number, Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$ while x_0 is a fixed interior point of Ω. They showed that under some conditions the solution blows up in finite time and the blow-up set is the whole region. In 2000, C. Y. Chan and J. Yang [9] studied the same question for the degenerate semilinear parabolic initial-boundary value problem

$$\begin{gathered} x^q u_t - u_{xx} = f(u(x_0,t)),\ (x,t) \in I \times (0,T), \\ u(0,t) = 0 = u(1,t),\ t \in (0,T), \\ u(x,0) = u_0(x),\ x \in \overline{I}, \end{gathered}$$

where q is any nonnegative real number, f and u_0 are given functions. By using Green function method, they proved that with suitable conditions, u blows up in finite time, and the blow-up set is the entire interval $\overline{I}$.

In this work, we study a semilinear parabolic problem with a localized nonlinear term, $u_t - \frac{1}{k(x)}(p(x)u_x)_x = f(u(x_0,t))$ with k and $p > 0$ and x_0 be in the domain of x, which satisfies the Dirichlet boundary conditions and nonhomogeneous initial condition.We show the existence and uniqueness of a blow-up solution by semigroup theory and the blow-up set of such a solution. Furthermore, we give the sufficient condition to blow-up in finite time. We also consider a degenerate semilinear parabolic problem, $x^\alpha u_t - (x^\beta u_x)_x = f(u(x_0,t))$, which satisfies the Dirichlet boundary conditions and nonhomogeneous initial condition. The existence and uniqueness of a blow-up solution is established by Green's function method. Moreover, the sufficient condition for occurrence of blow-up in finite time is shown. We finally extend our degenerate semilinear parabolic problem in the form, $u_t - \frac{1}{k(x)}(p(x)u_x)_x = f(u(x_0,t))$ with $k(0) = 0 = p(0)$ and $k, p > 0$. We still obtain the same results as previous problems.

Chapter 2

Complete blow-up for a semilinear parabolic problem with a localized nonlinear term via functional method

2.1 Introduction

Let x_0 be a fixed point in $I = (0,1)$ and denote its closure by $\overline{I}$. We study the semilinear parabolic initial-boundary value problem with a localized nonlinear term

$$\left.\begin{array}{c} u_t(x,t) - \frac{1}{k(x)}(p(x)u_x(x,t))_x = f(u(x_0,t)),\ (x,t) \in I \times (0,\infty), \\ u(0,t) = 0 = u(1,t),\ t > 0, \\ u(x,0) = u_0(x),\ x \in \overline{I}, \end{array}\right\} \tag{2.1.1}$$

where $k \in L^{\infty}(I)$, $p \in W^{1,\infty}(I)$, $u_0 \in H^2(I) \cap H_0^1(I)$ and $f \in C^2([0,\infty))$. Our study is exclusively concerned with the question of existence and uniqueness of the blow-up solution of problem (2.1.1) and the blow-up point of such solution.

Our objective of this chapter is to show existence, uniqueness and blow-up for a classical solution of problem (2.1.1) by using semigroup theory. Throughout this chapter, we assume the following:

(H1) $k \in L^{\infty}(I)$ and $\exists\ k_m, k_M \in (0, +\infty)$ such that $k_m < k(x) < k_M$ a.e. $x \in I$,

(H2) $p \in W^{1,\infty}(I)$ and $\exists\ p_m, p_M, \beta_1, \beta_2 \in (0, +\infty)$ such that $p_m \leq p(x) \leq p_M$ and $\beta_1 \leq p'(x) \leq \beta_2$ a.e. $x \in I$,

(H3) $f \in C^2([0,\infty))$ is convex with $f(0) = 0$ and $f(s) > 0$ for $s > 0$.

(H4) $u_0 \in H^2(I) \cap H_0^1(I)$ are nontrivial and nonnegative on I and satisfies

$$\frac{d}{dx}\left(p(x)\frac{du_0(x)}{dx}\right) + f(u_0(x_0)) \geq \zeta_1 u_0(x) \text{ in } I$$

for some positive constant ζ_1,

In order to obtain existence and uniqueness of a solution of problem (2.1.1), we will consider its formally equivalent formulation in terms of a nonlinear evolution equation in the Hilbert space $L^2(I)$:

$$\left.\begin{aligned} \frac{du(t)}{dt} - Au(t) = F(u) \quad &\text{for } t > 0, \\ u(0) = u_0, \quad & \end{aligned}\right\} \tag{2.1.2}$$

where A is the linear unbounded operator from $D(A)$, the domain of A, to $L^2(I)$ with

$$\begin{aligned} D(A) = \Big\{ & v \in H_0^1(I) \mid \exists!\ w \in L^2(I) \text{ s.t.} \\ & \int_I k(x)w(x)\varphi(x)\,dx = -\int_I p(x)D_x v(x) D_x\varphi(x)\,dx,\ \forall \varphi \in H_0^1(I) \Big\}, \end{aligned}$$

and $Av(x) = w(x)$ for all $v \in D(A)$ and where F is defined by

$$u \in D(A) \longmapsto F(u) = f(u(x_0, t)) \in L^2(I).$$

It will be shown before showing proposition 2.3.1.6 that the definition of F is meaningful.

2.2 Main results

Our results comprise the following four theorems. The first one involves existence and uniqueness of a solution u of problem (2.1.2) (in the sense of semigroup theory) whereas the last three theorems deal with the blow-up time of u, blow-up set and sufficient condition to blow-up in finite time, respectively.

Theorem 2.2.1 *There exists a finite positive constant T such that the evolution problem (2.1.2) has a unique solution $u \in C([0,T], D(A)) \cap C^1([0,T], L^2(I))$ defined by*

$$u(t) = S(t)u_0 + \int_0^t S(t-\tau)F(u(\tau))d\tau$$

where $S(t)$ is an analytic semigroup generated by A.

Theorem 2.2.2 *If $[0, T_{\max})$ is the finite maximal time interval in which a continuous solution u of problem (2.1.2) exists, then $|u(x_0,t)|$ is unbounded as t tends to $T_{\max}$.*

Theorem 2.2.3 *The blow-up set of a solution u of problem (2.1.1) is $\overline{I}$.*

Theorem 2.2.4 *Assume that*

1 u_0 attains its maximum at the point x_0,

2 $f(\xi) \geq b\xi^p$ with $b > 0$ and $p > 1$ and

3 $H(0) > \left(\frac{\lambda_1}{b}\right)^{\frac{1}{p-1}}$ where the operator H defined by (2.3.4).

Then the solution u of problem (2.1.1) blows up in finite time.

2.3 The proof of main results

Hereafter we use an inner product and a norm, equivalent to the usual one, on $L^2(I)$ by

$$\langle v, w\rangle = \int_I k(x)v(x)w(x)\,dx, \text{ and } |v|_{L^2(I)} = \left(\int_I k(x)\,|v(x)|^2\,dx\right)^{1/2}.$$

If $D_x v$ denotes the distributional derivative with respect to x of the distribution $v \in \mathcal{D}'(I)$, we recall that

$$H^1(I) = \left\{ v \in L^2(I) \middle| D_x v \in L^2(I) \right\}.$$

The Hilbert space $H^1(I)$ here is equipped with the norm (equivalent to the usual one):

$$|v|^2_{H^1(I)} = |v|^2_{L^2(I)} + \int_I p(x)\,|D_x v(x)|^2\,dx$$

whereas its closed subspace $H^1_0(I) = \left\{ v \in H^1(I) \middle| v(0) = 0 = v(1) \right\}$ is equipped with

$$|v|^2_{H^1_0(I)} = \int_I p(x)\,|D_x v(x)|^2\,dx;$$

the norm induced by $|\cdot|_{H^1(I)}$.

2.3.1 The proof of Theorem 2.2.1

To get existence and uniqueness of a solution of problem (2.1.2), we need the following propositions referred to [17].

Proposition 2.3.1.1 *If A is self-adjoint and generates a C_0 uniformly bounded semigroup $S(t)$ and g is Hölder continuous of exponent $\alpha \in (0, 1]$. Then the evolution equation:*

$$\frac{du(t)}{dt} = Au(t) + g(t) \text{ with } u(0) = u_0 \in D(A)$$

has a unique solution u such that

$$u \in C^1([0, \infty), L^2(I)) \cap C([0, \infty), D(A))$$

which can be expressed as

$$u(t) = S(t)u_0 + \int_0^t S(t - \tau)g(\tau)d\tau.$$

Observe that the operator A of problem (2.1.2) is given by

$$Av(x) = \frac{1}{k(x)} D_x(p(x)D_x v(x)).$$

To apply proposition 2.3.1.1 to such an operator, we show first that

Proposition 2.3.1.2 *The operator A of problem (2.1.2) is m-dissipative and self-adjoint in $L^2(I)$.*

Proof. An m-dissipative property of A in $L^2(I)$ is an immediate consequence of these two conditions:

1 $\langle Av, v\rangle \leq 0$ for all $v \in D(A)$, and

2 for any $\lambda > 0$, $R(I - \lambda A) = L^2(I)$, where $R(I - \lambda A)$ and I denote the range of $I - \lambda A$ and the identity operator on $L^2(I)$ respectively.

Condition 1. follows directly from definition of A. To obtain condition 2., letting $g \in L^2(I)$ and $\lambda > 0$, we need to give an existence of $v \in H_0^1(I)$ with the property:

$$\frac{1}{\lambda}\int_I k(x)v(x)\varphi(x)\,dx + \int_I p(x)D_x v(x)D_x\varphi(x)\,dx = \frac{1}{\lambda}\int_I k(x)g(x)\varphi(x)\,dx$$

for each $\varphi \in H_0^1(I)$. Such an existence is guaranteed by Lax-Milgram theorem [3], and thus, A is m-dissipative.

In order to prove that A is a self-adjoint operator in $L^2(I)$, since A is m-dissipative in $L^2(I)$, it suffices to prove that A is symmetric, that is, $\langle Av, \varphi\rangle = \langle v, A\varphi\rangle$ for all v and φ in $D(A)$. Indeed, definitions of $D(A)$, Av and $A\varphi$ yield

$$\langle Av, \varphi\rangle = -\int_I p(x)D_x v(x)D_x\varphi(x)dx = \langle v, A\varphi\rangle\,.$$

We note that by an m-dissipative property of A, the operator A generates a C_0 semigroup $S(t)$. To solve problem (2.1.2), it is convenient to introduce the square root of $-A$, $(-A)^{\frac{1}{2}}$. An elementary way to define $(-A)^{\frac{1}{2}}$ is by considering the eigenvalues and eigenfunctions of $-A$. The operator $(\lambda I - A)^{-1}$ is a bounded well-defined operator on $L^2(I)$ with values in $H_0^1(I)$ so that Rellich theorem (the embedding of $H_0^1(I)$ into $L^2(I)$ is compact) implies that $(\lambda I - A)^{-1}$ is a compact operator on $L^2(I)$.

The following proposition is referred from [11].

Proposition 2.3.1.3 (The spectral theory of self-adjoint compact operator) *There exists a sequence $(\lambda_n, \phi_n) \subset (0, +\infty) \times H_0^1(I)$ such that*

1 $A\phi_n = -\lambda_n\phi_n$.

2 $\int_I k(x)\phi_n(x)\phi_m(x)dx = \delta_{nm}$.

3 $\int_I p(x)D_x\phi_n(x)D_x\phi_m(x)dx = \lambda_n\delta_{nm}$.

4 $v(x) = \sum_{n\in\mathbb{N}} \langle v, \phi_n\rangle\, \phi_n(x)$ *for all* $v \in L^2(I)$.

5 $|v|^2_{L^2(I)} = \sum_{n\in\mathbb{N}} \langle v, \phi_n\rangle^2$.

6 $D(A) = \left\{ v \in L^2(I) \;\middle|\; \sum_{n\in\mathbb{N}} \lambda_n^2 \langle v, \phi_n\rangle^2 < +\infty \right\}$ *and* $Av = -\sum_{n\in\mathbb{N}} \lambda_n \langle v, \phi_n\rangle\, \phi_n$ *for each* $v \in D(A)$.

7 $S(t)v = \sum_{n\in\mathbb{N}} e^{-\lambda_n t} \langle v, \phi_n\rangle\, \phi_n$ *for all* $(v, t) \in L^2(I) \times [0, \infty)$.

Now, we can define domain of $(-A)^{\frac{1}{2}}$ by

$$D((-A)^{\frac{1}{2}}) = \left\{ v \in L^2(I) \;\middle|\; \sum_{n\in\mathbb{N}} \lambda_n \langle v, \phi_n\rangle^2 < +\infty \right\}$$

and the unbounded self-adjoint operator $(-A)^{\frac{1}{2}}$ in $L^2(I)$ by

$$(-A)^{\frac{1}{2}}v = \sum_{n\in\mathbb{N}} \lambda_n^{\frac{1}{2}} \langle v, \phi_n\rangle\, \phi_n$$

for all $v \in D((-A)^{\frac{1}{2}})$. Moreover, we obtain the following propositions.

Proposition 2.3.1.4

1 $D((-A)^{\frac{1}{2}}) = H_0^1(I)$ *and* $\left|(-A)^{\frac{1}{2}} v\right|_{L^2(I)} = |v|_{H_0^1(I)}$ *for any* $v \in D((-A)^{\frac{1}{2}})$.

2 If $v \in D((-A)^{\frac{1}{2}})$, *then* $S(t)v \in D((-A)^{\frac{1}{2}})$ *and*

$$\left|(-A)^{\frac{1}{2}} S(t)v\right|_{L^2(I)} = \left|S(t)(-A)^{\frac{1}{2}} v\right|_{L^2(I)} \leq \left|(-A)^{\frac{1}{2}} v\right|_{L^2(I)}.$$

Proof. Let us prove result 1. first.

If $v = \sum_{n\in\mathbb{N}} \langle v, \phi_n\rangle \phi_n$ for $\phi_n \in H_0^1(I)$, we have in the distributional sense:

$$D_x v = \sum_{n\in\mathbb{N}} \langle v, \phi_n\rangle D_x \phi_n$$

so that $\sum_{n\in\mathbb{N}} \lambda_n \langle v, \phi_n\rangle^2 = \int_I p(x) |D_x v(x)|^2 dx = |v|^2_{H_0^1(I)} < +\infty$. Conversely, if $v \in D((-A)^{\frac{1}{2}})$, the sequence (V_N), where

$$V_N = \sum_{n=1}^{N} \langle v, \phi_n\rangle \phi_n,$$

is Cauchy in $H_0^1(I)$ because if $N < M$, then

$$\begin{aligned} |V_N - V_M|^2_{H_0^1(I)} &= \int_I p(x) \left| \sum_{n=N+1}^{M} \langle v, \phi_n\rangle D_x\phi_n(x) \right|^2 dx \\ &= \sum_{n=N+1}^{M} \langle v, \phi_n\rangle^2 \int_I p(x) |D_x\phi_n(x)|^2 dx \\ &= \sum_{n=N+1}^{M} \lambda_n \langle v, \phi_n\rangle^2. \end{aligned}$$

Hence it converges to some V in $H_0^1(I)$ ($H_0^1(I)$ is a Hilbert space) and to v in $L^2(I)$ so that $v = V \in H_0^1(I)$. The remaining equality has already been proven.

For result 2., because $\sum_{n\in\mathbb{N}} \lambda_n e^{-2\lambda_n t} \langle v, \phi_n\rangle^2 \leq \sum_{n\in\mathbb{N}} \lambda_n \langle v, \phi_n\rangle^2$ for all $t \geq 0$, proposition 2.3.1.3 yields: if $v \in D((-A)^{\frac{1}{2}})$, then $S(t)v \in D((-A)^{\frac{1}{2}})$ and $(-A)^{\frac{1}{2}} S(t)v = S(t)(-A)^{\frac{1}{2}} v$ for $t \geq 0$.

Proposition 2.3.1.5 *There exists a* $C_0 > 0$ *such that*

$$\left|(-A)^{\frac{1}{2}} S(t)v\right|_{L^2(I)} = |S(t)v|_{H_0^1(I)} \leq \frac{C_0}{t^{1/2}} |v|_{L^2(I)}$$

for all $(v, t) \in L^2(I) \times (0, +\infty)$.

Proof. It is not difficult to see that $\left|(-A)^{\frac{1}{2}} S(t)v\right|_{L^2(I)} = |S(t)v|_{H_0^1(I)}$ for any $v \in L^2(I)$. Let $v \in L^2(I)$. Since the function $s \in \mathbb{R}^+ \longmapsto se^{-2s} \in \mathbb{R}^+$ is bounded, we have that there is a $C_0 > 0$ such that

$$t \sum_{n\in\mathbb{N}} \lambda_n e^{-2\lambda_n t} \langle v, \phi_n\rangle^2 \leq C_0 \sum_{n\in\mathbb{N}} \langle v, \phi_n\rangle^2 = C_0 |v|^2_{L^2(I)}.$$

Therefore, the definition of $(-A)^{\frac{1}{2}}$ yields that $S(t)v \in D((-A)^{\frac{1}{2}})$ and that the estimate involved in proposition 2.3.1.5 is true.

Note that the previous result implies that $S(t)v \in D((-A)^{\frac{1}{2}})$ for all $t > 0$ and all $v \in L^2(I)$, which, a priori, is not obvious for a standard semigroup $T(t)$ on $L^2(I)$: usually $T(t)v$ belongs to $L^2(I)$ only but due to the self-adjointness of A, the semigroup $S(t)$ is analytic (holomorphic) and consequently $S(t)v \in D(A)$ for all $t > 0$ and all $v \in L^2(I)$.

Presently, we are in a position to solve the evolution problem (2.1.2). Firstly, we define a mapping F by:

$$v \in H_0^1(I) \longmapsto F(v) = f(v(x_0)) \in L^2(I). \tag{2.3.1}$$

Note that this definition is meaningful because $v \in H_0^1(I)$ implies that v is continuous on $\overline{I}$ so that $v(x_0)$ has a meaning and $F(v)$ is a constant on I and therefore belongs to $L^2(I)$.

Proposition 2.3.1.6 *The mapping F defined by (4.3.2) is locally Lipschitz from $D((-A)^{\frac{1}{2}})$ $(= H_0^1(I))$ to $L^2(I)$.*

Proof. Let $v, w \in H_0^1(I)$ $(\hookrightarrow C(\overline{I}))$ such that $|v|_{C(\overline{I})}, |w|_{C(\overline{I})} \leq M$ with M being a positive constant. Then (H3) implies:

$$\begin{aligned} |F(v) - F(w)|^2_{L^2(I)} &\leq k_M |f(v(x_0)) - f(w(x_0))|^2 \\ &\leq k_M L_M^2 |v(x_0) - w(x_0)|^2 \\ &\leq k_M L_M^2 |v - w|^2_{C(\overline{I})} \\ &\leq k_M L_M^2 C_s^2 |v - w|^2_{H_0^1(I)}, \end{aligned}$$

where C_s is the constant involved in the Sobolev embedding $H_0^1(I) \hookrightarrow C(\overline{I})$.

Next, due to proposition 2.3.1.4, we introduce a concept of mild solution for the evolution problem (2.1.2).

Definition *A function u is said to be a mild solution of problem (2.1.2) if there exists $u \in C([0,\infty), H_0^1(I))$ such that*

$$u(t) = S(t)u_0 + \int_0^t S(t-\tau)F(u(\tau))d\tau, \ \forall t \in [0,\infty),$$

u_0 being assumed to belong to $H_0^1(I)$.

We modify the proof of Theorem 2.5.1 of [27] to obtain the following result.

Proposition 2.3.1.7 *There exists a $T > 0$ such that problem (2.1.2) has a unique mild solution. Moreover, let $u(t)$, $\widetilde{u}(t)$ be mild solutions corresponding to u_0 and $\widetilde{u}_0$, respectively. Then, for all $t \in [0,T]$, the following estimate holds:*

$$|u(t) - \widetilde{u}(t)|_{H_0^1(I)} \leq |u_0 - \widetilde{u}_0|_{H_0^1(I)} e^{2C_0 C_s k_M^{\frac{1}{2}} L_M T^{\frac{1}{2}}}.$$

Proof. Let $M = |u_0|_{H_0^1(I)} + 1$ and L_M be the Lipschitz constant of f. Let T be a positive constant such that $T < \frac{1}{4k_M C_0^2 C_s^2 L_M^2}$. We define a mapping Φ by:

$$v \in E \mapsto \Phi(v) = S(t)u_0 + \int_0^t S(t-\tau)F(v(\tau))d\tau$$

where

$$E = \left\{ v \in C([0,T], H_0^1(I)) \text{ such that } |v(t)|_{H_0^1(I)} \leq M \text{ for all } t \in [0,T] \right\},$$

equipped with the norm:

$$|v|_E = \sup_{t\in[0,T]} |v(t)|_{H_0^1(I)}\,.$$

We note that E is a closed convex subset of a Banach space $C([0,T],H_0^1(I))$. We would like to prove that $\Phi(v)\in E$ for any $v\in E$ and Φ is a contraction in E. Propositions 2.3.1.4, 2.3.1.5 and 2.3.1.6 imply:

$$\begin{aligned}
|\Phi(v)|_E &= \sup_{t\in[0,T]} \left| S(t)u_0 + \int_0^t S(t-\tau)F(v(\tau))d\tau \right|_{H_0^1(I)} \\
&\le |u_0|_{H_0^1(I)} + \sup_{t\in[0,T]} \int_0^t |S(t-\tau)F(v(\tau))|_{H_0^1(I)}\, d\tau \\
&\le |u_0|_{H_0^1(I)} + \sup_{t\in[0,T]} \int_0^t \frac{C_0}{(t-\tau)^{\frac12}} \left(|f(0)|_{L^2(I)} + k_M^{\frac12} L_M C_s |v|_{H_0^1(I)} \right) d\tau \\
&\le |u_0|_{H_0^1(I)} + \left(C_0 |f(0)|_{L^2(I)} + C_0 k_M^{\frac12} L_M C_s M \right) \sup_{t\in[0,T]} \int_0^t \frac{d\tau}{(t-\tau)^{\frac12}} \\
&\le |u_0|_{H_0^1(I)} + 2C_0 \left(|f(0)|_{L^2(I)} + k_M^{\frac12} L_M C_s M \right) T^{\frac12}.
\end{aligned}$$

If T is chosen in such a way that

$$T < \min\left\{ \frac{1}{4k_M C_0^2 C_s^2 L_M^2}, \frac{1}{4C_0^2 \left(|f(0)|_{L^2(I)} + k_M^{\frac12} L_M C_s M \right)^2} \right\},$$

then $\Phi(v)$ is in E for any $v\in E$. Moreover, for any $v_1, v_2 \in E$

$$\begin{aligned}
|\Phi(v_1)-\Phi(v_2)|_E &= \sup_{t\in[0,T]} \left| \int_0^t S(t-\tau)\left(F(v_1(\tau)) - F(v_2(\tau))\right) d\tau \right|_{H_0^1(I)} \\
&\le C_0 \sup_{t\in[0,T]} \int_0^t \frac{1}{(t-\tau)^{\frac12}} \left|\left(F(v_1(\tau)) - F(v_2(\tau))\right)\right|_{L^2(I)} d\tau \\
&\le C_0 k_M^{\frac12} L_M C_s \sup_{t\in[0,T]} \int_0^t \frac{1}{(t-\tau)^{\frac12}} d\tau\, |v_1 - v_2|_E \\
&\le 2C_0 k_M^{\frac12} L_M C_s T^{\frac12} |v_1-v_2|_E\,.
\end{aligned}$$

That is, Φ is a contraction in E. Thus, Φ has a fixed point that is the mild solution to problem (2.1.2) in E. To show that the uniqueness also holds in $C([0,T],H_0^1(I))$, let $u_1, u_2 \in C([0,T],H_0^1(I))$ be two solutions of problem (2.1.2) and let $u = u_1 - u_2$. Then

$$u(t) = \int_0^t S(t-\tau)\left(F(u_1(\tau)) - F(u_2(\tau))\right) d\tau.$$

Propositions 2.3.1.4, 2.3.1.5 and 2.3.1.6 imply:

$$\begin{aligned}
|u(t)|_{H_0^1(I)} &= \left| \int_0^t S(t-\tau)\left(F(u_1(\tau)) - F(u_2(\tau))\right) d\tau \right|_{H_0^1(I)} \\
&\le C_0 C_s k_M^{\frac12} L_M \int_0^t \frac{1}{(t-\tau)^{\frac12}} |u_1(\tau) - u_2(\tau)|_{H_0^1(I)}\, d\tau.
\end{aligned}$$

By Gronwall inequality, we immediately conclude that $|u(t)|_{H_0^1(I)} = 0$, that is, the uniqueness in $C([0,T],H_0^1(I))$ is proven. As before, we have

$$u(t) - \widetilde{u}(t) = S(t)(u_0 - \widetilde{u}_0) + \int_0^t S(t-\tau)\left(F(u(\tau)) - F(\widetilde{u}(\tau))\right) d\tau.$$

Therefore,

$$\begin{aligned}
&\left|u(t)-\widetilde{u}(t)\right|_{H_0^1(I)} \\
\leq\ & \left|u_0-\widetilde{u}_0\right|_{H_0^1(I)}+C_0C_sk_M^{\frac{1}{2}}L_M\int_0^t\frac{1}{(t-\tau)^{\frac{1}{2}}}\left|u(\tau)-\widetilde{u}(\tau)\right|_{H_0^1(I)}d\tau.
\end{aligned}$$

Gronwall inequality implies:

$$\begin{aligned}
\left|u(t)-\widetilde{u}(t)\right|_{H_0^1(I)} &\leq \left|u_0-\widetilde{u}_0\right|_{H_0^1(I)}e^{C_0C_sk_M^{\frac{1}{2}}L_M\int_0^t\frac{1}{(t-\tau)^{\frac{1}{2}}}d\tau} \\
&\leq \left|u_0-\widetilde{u}_0\right|_{H_0^1(I)}e^{2C_0C_sk_M^{\frac{1}{2}}L_MT^{\frac{1}{2}}}.
\end{aligned}$$

Hence, this proposition is proven.

By modifying the proof of Corollary 2.5.1 of [27] we establish the following result.

Proposition 2.3.1.8 *The mild solution u is Hölder continuous of exponent α $(=\frac{1}{2})$ in t from $[0,T]$ toward $H_0^1(I)$ for any $u_0\in D(A)(=H^2(I)\cap H_0^1(I))$.*

Proof. Let $u_0\in D(A)$. For any $h>0$. Let $\widetilde{u}(t)=u(t+h)$. Then, we see that $\widetilde{u}$ is a mild solution of problem (2.1.2) with initial data $u_0=u(h)$. Then,

$$\begin{aligned}
\left|u(t+h)-u(t)\right|_{H_0^1(I)} &= \left|\widetilde{u}(t)-u(t)\right|_{H_0^1(I)} \\
&\leq \left|u(h)-u_0\right|_{H_0^1(I)}e^{2C_0C_sk_M^{\frac{1}{2}}L_Mt^{\frac{1}{2}}}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\left|u(h)-u_0\right|_{H_0^1(I)} \\
\leq\ & \left|S(h)u_0-u_0\right|_{H_0^1(I)}+\int_0^h\left|S(h-\tau)F(u(\tau))\right|_{H_0^1(I)}d\tau \\
\leq\ & \left|\int_0^hS(\tau)Au_0d\tau\right|_{H_0^1(I)}+\int_0^h\frac{C_0}{(h-\tau)^{\frac{1}{2}}}\left|F(u(\tau))\right|_{H_0^1(I)}d\tau \\
\leq\ & \int_0^h\left|S(\tau)Au_0\right|_{H_0^1(I)}d\tau \\
&+\int_0^h\frac{C_0}{(h-\tau)^{\frac{1}{2}}}\left(\left|F(u_0)\right|_{H_0^1(I)}+k_M^{\frac{1}{2}}L_MC_s\left|u(\tau)-u_0\right|_{H_0^1(I)}\right)d\tau \\
\leq\ & 2C_0\left(\left|Au_0\right|_{L^2(I)}+\left|F(u_0)\right|_{H_0^1(I)}\right)h^{\frac{1}{2}} \\
&+C_0k_M^{\frac{1}{2}}L_MC_s\int_0^h\frac{\left|u(\tau)-u_0\right|_{H_0^1(I)}}{(h-\tau)^{\frac{1}{2}}}d\tau.
\end{aligned}$$

By Gronwall inequality, we have

$$\left|u(h)-u_0\right|_{H_0^1(I)}\leq 2C_0\left(\left|Au_0\right|_{L^2(I)}+\left|F(u_0)\right|_{H_0^1(I)}\right)h^{\frac{1}{2}}e^{2C_0k_M^{\frac{1}{2}}L_MC_sh^{\frac{1}{2}}}.$$

Thus, for any $t_1,t_2\in[0,T]$ such that $t_1+h=t_2$

$$\begin{aligned}
&\left|u(t_1)-u(t_2)\right|_{H_0^1(I)} \\
\leq\ & 2C_0\left(\left|Au_0\right|_{L^2(I)}+\left|F(u_0)\right|_{H_0^1(I)}\right)e^{4C_0C_sk_M^{\frac{1}{2}}L_MT^{\frac{1}{2}}}\left|t_1-t_2\right|^{\frac{1}{2}}.
\end{aligned}$$

Hence u is Holder continuous of exponent $\alpha=\frac{1}{2}$ in t.

Now we are in a position to prove theorem 2.2.1.

Proof of Theorem 2.2.1. Since F is locally Lipschitz and u is Hölder continuous of exponent $\alpha = \frac{1}{2}$ in t, F is also Hölder continuous of exponent $\alpha = \frac{1}{2}$ in t. Hence, the result is a consequence of proposition 2.3.1.1.

2.3.2 The proof of Theorem 2.2.2

Let us modify the proof of theorem 2.5.5 of [27] to obtain the following result.

Proposition 2.3.2.1 *Let $[0, T_{\max})$ be the maximal time interval in which the mild solution u of the evolution problem (2.1.2) exists. If $T_{\max}$ is finite, then the solution u of problem (2.1.2) blows up in finite time, that is,*

$$\lim_{t \to T_{\max}} |u(t)|_{H_0^1(I)} = +\infty.$$

Proof. We will use the contraction argument to prove proposition 2.3.2.1. Suppose that there is a finite positive constant M and a sequence (t_n) such that

$$|u(t_n)|_{H_0^1(I)} \leq M \text{ as } t_n \to T_{\max}.$$

Consider the following problem:

$$\frac{dv(t)}{dt} = Av(t) + F(v) \text{ and } v(0) = u(t_n).$$

By proposition 2.3.1.7, the above problem has a unique local mild solution in $[0, \delta]$ with δ depending on M. We choose n large enough so that $t_n + \delta > T_{\max}$. Let

$$\widetilde{u}(t) = \begin{cases} u(t), & \text{for } 0 \leq t \leq t_n, \\ v(t - t_n), & \text{for } t_n \leq t \leq t_n + \delta. \end{cases}$$

We next would like to show that $\widetilde{u}(t)$ is a mild solution of problem (2.1.2) in $[0, t_n + \delta]$, i.e., $\widetilde{u}(t)$ satisfies the integral equation

$$\widetilde{u}(t) = S(t)u_0 + \int_0^t S(t - \tau)F(\widetilde{u}(\tau))d\tau \text{ for } 0 \leq t \leq t_n + \delta. \tag{2.3.2}$$

From

$$u(t) = S(t)u_0 + \int_0^t S(t - \tau)F(u(\tau))d\tau \text{ for } 0 \leq t \leq t_n,$$

and

$$v(t) = S(t)u(t_n) + \int_0^t S(t - \tau)F(v(\tau))d\tau \text{ for } 0 \leq t \leq \delta,$$

it is clear that for $t \in [0, t_n]$, $\widetilde{u}(t)$ satisfies (2.3.2). For $t \in [0, \delta]$,

$$\begin{aligned}
\widetilde{u}(t+t_n) &= v(t) \\
&= S(t+t_n)u_0 + \int_0^{t_n} S(t+t_n-\tau)F(u(\tau))d\tau + \int_0^t S(t-\tau)F(v(\tau))d\tau \\
&= S(t+t_n)u_0 + \int_0^{t_n} S(t+t_n-\tau)F(u(\tau))d\tau \\
&\quad + \int_{t_n}^{t+t_n} S(t+t_n-\tau)F(v(\tau-t_n))d\tau \\
&= S(t+t_n)u_0 + \int_0^{t_n} S(t+t_n-\tau)F(\widetilde{u}(\tau))d\tau \\
&\quad + \int_{t_n}^{t+t_n} S(t+t_n-\tau)F(\widetilde{u}(\tau))d\tau \\
&= S(t+t_n)u_0 + \int_0^{t+t_n} S(t+t_n-\tau)F(\widetilde{u}(\tau))d\tau.
\end{aligned}$$

Hence, $\widetilde{u}$ is a mild solution of problem (2.1.2) in $[0, t_n+\delta]$ with $t_n + \delta > T_{\max}$. This contradicts the definition of $T_{\max}$. Therefore, the proof of proposition 2.3.2.1 is complete.

We next prove Theorem 2.2.2

Proof of Theorem 2.2.2. Suppose that there is a positive constant M such that $|u(x_0,t)| \le M$ as $t \to T_{\max}$. Since

$$\begin{aligned}
u(t) &= S(t)u_0 + \int_0^t S(t-\tau)F(u(\tau))d\tau \\
&= S(t)u_0 + \int_0^t f(u(x_0,\tau))S(t-\tau)\mathbf{1}d\tau
\end{aligned}$$

where $\mathbf{1}$ is a function in $L^2(I)$ such that $\mathbf{1}(x) = 1\ \forall x \in I$. Then, from proposition 3.1.4, we have

$$\begin{aligned}
|u(t)|_{H_0^1(I)} &\le |u_0|_{H_0^1(I)} + (|f(0)| + ML_M) \int_0^t \frac{1}{(t-\tau)^{\frac{1}{2}}} |\mathbf{1}|_{L^2(I)}\, d\tau \\
&= |u_0|_{H_0^1(I)} + 2\,(|f(0)| + ML_M)\, |\mathbf{1}|_{L^2(I)}\, t^{\frac{1}{2}}.
\end{aligned}$$

Thus, as $t \to T_{\max}$, $|u(t)|_{H_0^1(I)}$ is bounded. This contradicts proposition 2.3.2.1. Hence, theorem 2.2.2 is proven.

2.3.3 The proof of Theorem 2.2.3

Before showing the blow-up set of a solution u of problem (2.1.1), we will give a following lemma.

Lemma 2.3.3.1 *For all $x \in I$, there exists $c(x) > 0$ such that $(S(t)\mathbf{1})(x) > c(x)$ for any $t \in [0, T_{\max})$.*

Proof. The proof of this lemma results from lemma 3.3.3.1 in chapter 3 because this lemma is the particular case of lemma 3.3.3.1.

We next prove theorem 2.2.3.

Proof of Theorem 2.2.3. Let M be a fixed positive constant with $M > \max_{x \in \overline{I}} u_0(x)$. Since $\lim_{t \to T_{\max}} |u(x_0,t)| \to$

$+\infty$, there is a positive t_M such that $|u(x_0,t)| > M$ for all $t \geq t_M$. Let us consider:

$$\begin{aligned} |u(x_0,t)| &= |u(t)(x_0)| \\ &\leq |(S(t)u_0)(x_0)| + \int_0^{t_M} |(S(t-\tau)\mathbf{1})(x_0)|\, |f(u(x_0,\tau))|\, d\tau \\ &\quad + \int_{t_M}^{t} |(S(t-\tau)\mathbf{1})(x_0)|\, |f(u(x_0,\tau))|\, d\tau. \end{aligned}$$

By locally Lipschitz continuity of f, we have

$$\begin{aligned} |u(x_0,t)| &\leq |(S(t)u_0)(x_0)| + \int_0^{t_M} |(S(t-\tau)\mathbf{1})(x_0)|\, (f(0) + L_M\, |u(x_0,\tau)|)\, d\tau \\ &\quad + \int_{t_M}^{t} |(S(t-\tau)\mathbf{1})(x_0)|\, |f(u(x_0,\tau))|\, d\tau, \end{aligned}$$

where L_M is a positive constant depending on M. Thus, there are three positive constants c_1, c_2 and c_3 such that

$$\begin{aligned} |u(x_0,t)| &\leq c_1 + c_2 + c_3 \int_{t_M}^{t} |f(u(x_0,\tau))|\, d\tau \\ &= c_1 + c_2 + c_3 \int_{t_M}^{t} f(u(x_0,\tau))d\tau. \end{aligned}$$

Taking $t \to T_{\max}$, we obtain that

$$\lim_{t \to T_{\max}} \int_{t_M}^{t} f(u(x_0,\tau))d\tau \to +\infty. \tag{2.3.3}$$

For any $x \in I$, we have

$$u(x,t) = (S(t)u_0)(x) + \int_0^t (S(t-\tau)\mathbf{1})(x) f(u(x_0,\tau))d\tau.$$

Then, by lemma 2.3.3.1, there are two constant $\widetilde{c}_1(x)$ and $\widetilde{c}_2(x)(>0)$ such that

$$u(x,t) \geq \widetilde{c}_1(x) + \widetilde{c}_2(x) \int_0^t f(u(x_0,\tau))d\tau.$$

Hence, as $t \to T_{\max}$, $u(x,t) \to +\infty$ for any $x \in I$. For $x \in \{0,1\}$, we can find a sequence $\{(x_n,t_n)\}$ such that $(x_n,t_n) \to (x, T_{max})$ and $\lim_{n\to\infty} u(x_n,t_n) \to \infty$. Therefore, the set of blow-up points of a solution u of problem (2.1.1) is $\overline{I}$.

2.3.4 The proof of Theorem 2.2.4

Here, we will give the sufficient condition for occurrence of blow-up in finite time. In order to obtain our result, we need the following lemmas.

Proposition 2.3.4.1 *Let v be a classical solution of the problem: for any $T \in (0, \infty)$,*

$$\begin{gathered} v_t - \tfrac{1}{k(x)}(p(x)v_x)_x \geq B(x,t)v(x_0,t), \ (x,t) \in I \times (0,T), \\ v(0,t) \geq 0 \text{ and } v(1,t) \geq 0, \\ v(x,0) \geq 0, \ x \in \overline{I}, \end{gathered}$$

where $B(x,t)$ is nonnegative and bounded on $\overline{I} \times [0,T]$. Then $v(x,t) \geq 0$ on $\overline{I} \times [0,T]$.

Proof. If $B(x,t) = 0$ on $\overline{I} \times [0,T]$, then it follows from Maximum principle that $v(x,t) \geq 0$ on $\overline{I} \times [0,T]$. We assume that $B(x,t)$ is positive on $\overline{I} \times [0,T]$. Let η be any positive real number and

$$V(x,t) = v(x,t) + \eta(1+x^2)e^{ct}$$

where c is a positive constant with

$$c = \frac{2}{k_m}(p_M + p'_M) + (1+x_0^2) \max_{\overline{I} \times [0,T]} B(x,t).$$

We then consider that for any $(x,t) \in I \times (0,T]$,

$$\begin{aligned} & V_t(x,t) - \frac{1}{k(x)}(p(x)V_x)_x - B(x,t)V(x_0,t) \\ = \ & u_t - \frac{1}{k(x)}(p(x)u_x)_x + c\eta(1+x^2)e^{ct} - \frac{2\eta e^{ct}}{k(x)}(xp(x))_x - B(x,t)V(x_0,t) \\ \geq \ & c\eta(1+x^2)e^{ct} - \frac{2\eta e^{ct}}{k(x)}[xp'(x) + p(x)] - \eta B(x,t)(1+x_0^2)e^{ct} \\ = \ & \eta e^{ct}\left(c(1+x^2) - \frac{2}{k(x)}[xp'(x)+p(x)] - B(x,t)(1+x_0^2)\right) \\ \geq \ & \eta e^{ct}\left(c - \frac{2}{k_m}(p_M + p'_M) - (1+x_0^2)\max_{\overline{I} \times [0,T]} B(x,t)\right) \\ > \ & 0. \end{aligned}$$

We see that $V(x,t) \geq 0$ on $\{0,1\} \times (0,T] \cup \overline{I} \times \{0\}$. We next would like to show that $V(x,t) > 0$ for any $(x,t) \in \overline{I} \times [0,T]$. Suppose that there exists a point $(x_1,t_1) \in I \times (0,T)$ such that $V(x_1,t_1) \leq 0$. We define the set A by

$$A = \{t : V(x,t) \leq 0 \text{ for some } x \in I\}.$$

It's clear that set A is nonempty. Let $t^* = \inf A$. Since $V(x,0) = u_0(x) + \eta(1+x^2) > 0$ for $x \in I$, we obtain that $t^* > 0$ and, additionally, by the definition of V, $V(x_0,t^*) \geq 0$. Indeed, if $V(x_0,t^*) < 0$, then, by continuity of V, there exists a $t_2 (< t^*)$ such that $V(x,t_2) \leq 0$ for some $x \in I$ which contradicts definition of t^*. Since A is closed, by definition of t^*, there exists a point $x_2 \in I$ such that

$$V(x_2,t^*) = 0, \ V_t(x_2,t^*) \leq 0 \text{ and } V_x(x_2,t^*) = 0.$$

Furthermore, since V attains its local minimum at the point x_2, we have that $V_{xx}(x_2,t^*) \geq 0$. Thus, we have that

$$0 \geq V_t(x_2,t^*) \geq V_t(x_2,t^*) - \frac{1}{k(x_2)}(p(x_2)V_x(x_2,t^*))_x - B(x_2,t^*)V(x_0,t^*) > 0.$$

Therefore, we get a contradiction. This shows that $V(x,t) > 0$ for any $(x,t) \in \overline{I} \times [0,T]$. Since η is arbitrary, we let $\eta \to 0^+$ and then we obtain the desired result.

Proposition 2.3.4.2 *For any $(x,t) \in \overline{I} \times [0,T_{\max})$, $u(x,t) \geq u_0(x)$ and $u_t(x,t) \geq 0$.*

Proof. Let $z(x,t) = u(x,t) - u_0(x)$ for any $(x,t) \in \overline{I} \times [0,T_{\max})$. Let us consider that

$$z_t - \frac{1}{k(x)}(p(x)z_x)_x = f(u(x_0,t)) + \frac{1}{k(x)}\frac{d}{dx}\left(p(x)\frac{du_0(x)}{dx}\right).$$

Condition (A3) implies

$$\frac{d}{dx}\left(p(x)\frac{du_0(x)}{dx}\right) \geq -f(u_0(x_0)).$$

Thus, we obtain that

$$z_t - \frac{1}{k(x)}(p(x)z_x)_x \geq f(u(x_0,t)) - f(u_0(x_0)) = f'(\xi_1)z(x_0,t),$$

where ξ_1 is between $u(x_0,t)$ and $u_0(x_0)$. Additionally, $z(x,t) = 0$ for any $(x,t) \in \{0,1\} \times (0,T) \cup \overline{I} \times \{0\}$. Lemma 2.3.4.1 yields that $z \geq 0$ or $u(x,t) \geq u_0(x)$ for any $(x,t) \in \overline{I} \times [0,T_{\max})$.

Let h be any positive constant less than $T_{\max}$ and

$$w(x,t) = u(x,t+h) - u(x,t) \text{ for any } (x,t) \in \overline{I} \times [0,T_{\max}).$$

We then have that for any $(x,t) \in I \times (0,T_{\max})$,

$$w_t - \frac{1}{k(x)}(p(x)w_x)_x = f(u(x_0,t+h)) - f(u(x_0,t)) = f'(\xi_2)w(x_0,t)$$

for ξ_2 between $u(x,t+h)$ and $u(x,t)$. For any $(x,t) \in \{0,1\} \times (0,T) \cap \overline{I} \times \{0\}$, $w \geq 0$. Then, lemma 2.3.4.1 implies that $w \geq 0$ or $u(x,t+h) \geq u(x,t)$ for any $(x,t) \in \overline{I} \times [0,T_{\max})$. Hence, this shows that $u_t \geq 0$ on $\overline{I} \times [0,T_{\max})$.

Before blow-up occurs, there exists a $M(>0)$ such that $|u(x,t)| \leq M$ for any $(x,t) \in \overline{I} \times [0,T_{\max})$.Locally Lipschitz continuous property of f implies that there is a positive constant $K(M)$ depending on M such that for any u and v with $|u| \leq M$ and $|v| \leq M$,

$$|f(u) - f(v)| \leq K(M)\,|u - v|\,.$$

We note that before blow-up occurs, $|u(x_0,t)| \leq M$ and then we obtain that there exists a $K(M)$ such that $f(u(x_0,t) \leq K(M)u(x_0,t)$ for $t \in [0,T_{\max})$.

Lemma 2.3.4.3 *Before blow-up occurs, $u_t(x,t) \geq K(M)u(x,t)$ for any $\overline{I} \times [0,T_{\max})$ if $f'(u_0(x_0)) \geq K(M)$.*

Proof. Let $z(x,t) = u_t(x,t) - K(M)u(x,t)$ for any $\overline{I} \times [0,T_{\max})$. We then consider that for any $I \times (0,T_{\max})$,

$$z_t - \frac{1}{k(x)}(p(x)z_x)_x = f'(u(x_0,t))u_t - K(M)f(u(x_0,t)).$$

It follows from locally Lipschitz continuous property of f that $f(u(x_0,t)) \leq K(M)u(x_0,t)$. We then have that, by lemma 2.3.4.2,

$$\begin{aligned} z_t - \frac{1}{k(x)}(p(x)z_x)_x &\geq f'(u(x_0,t))u_t(x_0,t) - K^2(M)u(x_0,t) \\ &\geq f'(u_0(x_0)u_t(x_0,t) - K^2(M)u(x_0,t) \\ &\geq K(M)u_t(x_0,t) - K^2(M)u(x_0,t) \\ &= K(M)z(x_0,t) \end{aligned}$$

for any $I \times (0, T_{\max})$. By lemma 2.3.4.2, we have that $z(0,t) = u_t(0,t) - K(M)u(0,t) = u_t(0,t) = 0$ and $z(1,t) = u_t(1,t) - K(M)u(1,t) = u_t(1,t) = 0$ for $t \in (0, T_{\max})$. As t tends to 0, condition (H4) implies that $z(x,0) = \lim_{t\to 0} u_t(x,t) - K(M)u(x,0) = \frac{1}{k(x)}\frac{d}{dx}\left(p(x)\frac{du_0(x)}{dx}\right) + f(u_0(x_0)) - K(M)u_0(x) \geq 0$ for $x \in I$. Then, lemma 2.3.4.1 implies that $u_t(x,t) \geq K(M)u(x,t)$ for any $\overline{I} \times [0, T_{\max})$.

Lemma 2.3.4.4 *If $u_0(x_0) \geq u_0(x)$ for all $x \in \overline{I}$, then $u(x_0,t) \geq u(x,t)$ for any $(x,t) \in \overline{I} \times [0, T_{\max})$.*

Proof. Let $z(x,t) = u(x_0,t) - u(x,t)$ for any $(x,t) \in I \times (0, T_{\max})$. Then, for any $(x,t) \in I \times (0, T_{\max})$, lemma 2.3.4.3 implies that

$$\begin{aligned} z_t - \frac{1}{k(x)}(p(x)z_x)_x &= u_t(x_0,t) - f(u(x_0,t)) \\ &\geq u_t(x_0,t) - K(M)u(x_0,t) \\ &\geq 0. \end{aligned}$$

On boundary, for any $(x,t) \in \{0,1\} \times (0,T) \cap \overline{I} \times \{0\}$, $z(0,t) = u(x_0,t) - u(0,t) = u(x_0,t) \geq u_0(x_0) \geq 0$ and $z(1,t) = u(x_0,t) \geq u_0(x_0) \geq 0$ for any $t \in (0, T_{\max})$ and $z(x,0) = u(x_0,0) - u(x,0) = u_0(x_0) - u(x_0) \geq 0$ because u_0 attains its maximum at point x_0. Therefore, these lemma is proved.

Let ϕ_1 be the first eigenfunction of the eigenvalue problem:

$$A\phi = \frac{1}{k(x)}\frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right) = -\lambda\phi \text{ for } x \in I = (0,1) \text{ and } \phi(0) = 0 = \phi(1),$$

λ_1 its corresponding eigenvalue and

$$\int_I k(x)\phi_1(x)dx = 1.$$

We construct the function H as

$$H(t) = \langle u(t), \phi_1 \rangle = \int_I k(x)u(x,t)\phi_1(x)dx. \tag{2.3.4}$$

Proof of Theorem 2.2.4. Let us consider: from the self-adjointness of A,

$$\begin{aligned} H'(t) &= \left\langle \frac{d}{dt}u(t), \phi_1 \right\rangle \\ &= \langle Au(t) + Fu(t), \phi_1 \rangle \\ &= \langle u(t), A\phi_1 \rangle + \langle Fu(t), \phi_1 \rangle \\ &= -\lambda_1 H(t) + \int_I k(x)f(u(x_0,t))\phi_1(x)dx. \end{aligned}$$

It follows from lemma 2.3.4.4 that

$$\begin{aligned} H'(t) &\geq -\lambda_1 H(t) + \int_I k(x)f(u(x_0,t))\phi_1(x)dx \\ &\geq -\lambda_1 H(t) + \int_I k(x)f(u(x,t))\phi_1(x)dx. \end{aligned}$$

Condition 2 of theorem 2.2.4 yields that

$$H'(t) \geq -\lambda_1 H(t) + b\int_I k(x)u^p(x,t)\phi_1(x)dx. \tag{2.3.5}$$

Hölder inequality implies that

$$\int_I k(x)u(x,t)\phi_1(x)dx \leq \left(\int_I k\phi_1(x)dx\right)^{\frac{p-1}{p}} \left(\int_I k\phi_1(x)u^p(x,t)dx\right)^{\frac{1}{p}}$$

or

$$\int_I k\phi_1(x)u^p(x,t)dx \geq \left(\int_I k(x)u(x,t)\phi_1(x)dx\right)^p.$$

From (2.3.5), we have that

$$H'(t) \geq -\lambda_1 H(t) + bH^p(t). \tag{2.3.6}$$

Dividing both sides by H^p, we have

$$H^{-p}(t)F'(t) + \lambda_1 H^{1-p}(t) \geq b.$$

Multiplying both sides by $1-p$, we obtain

$$\begin{aligned} (1-p)H^{-p}(t)H'(t) + \lambda_1(1-p)H^{1-p}(t) &\leq (1-p)b \\ \frac{dH^{1-p}(t)}{dt} + \lambda_1(1-p)H^{1-p}(t) &\leq (1-p)b. \end{aligned}$$

Multiplying both sides by $e^{\lambda_1(1-p)t}$, we get

$$\begin{aligned} e^{\lambda_1(1-p)t}\frac{dH^{1-p}(t)}{dt} + \lambda_1(1-p)e^{\lambda_1(1-p)t}H^{1-p}(t) &\leq (1-p)be^{\lambda_1(1-p)t} \\ \frac{d}{dt}\left(e^{\lambda_1(1-p)t}H^{1-p}(t)\right) &\leq (1-p)be^{\lambda_1(1-p)t}. \end{aligned}$$

Integrating both sides, we have

$$\begin{aligned} \int_0^t \frac{d}{ds}\left(e^{\lambda_1(1-p)s}H^{1-p}(s)\right) &\leq (1-p)b\int_0^t e^{\lambda_1(1-p)s}ds \\ e^{\lambda_1(1-p)t}H^{1-p}(t) - H^{1-p}(0) &\leq \frac{b}{\lambda_1}e^{\lambda_1(1-p)t} - \frac{b}{\lambda_1} \\ H^{1-p}(t) &\leq \frac{b}{\lambda_1} + \left[H^{1-p}(0) - \frac{b}{\lambda_1}\right]e^{-\lambda_1(1-p)t}. \end{aligned} \tag{2.3.7}$$

From (2.3.7), we see that

$$H^{p-1}(t) \geq \frac{1}{\frac{b}{\lambda_1} + \left[H^{1-p}(0) - \frac{b}{\lambda_1}\right]e^{-\lambda_1(1-p)t}}.$$

By condition 3 of theorem 2.2.4, we have that $H^{1-p}(0) < \frac{b}{\lambda_1}$. Thus, there exists some positive t_1 such that H tends to infinity. By the definition of H, we see that

$$H(t) = \int_I k(x)u(x,t)\phi_1(x)dx \leq \left(\int_I k(x)\phi_1(x)dx\right)u(x_0,t) = u(x_0,t).$$

Hence, there exists a finite time $t_1 > 0$ such that the solution u blows up in finite time.

Note that this chapter was the object of the article :

P. Sawangtong. C. Licht, B. Novaprateep and S. Orankitjaroen. Existence and uniqueness of a blow-up solution for a parabolic problem with a localized nonlinear term via semigroup theory, East-West Journal of Mathematics, to appear.

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Chapter 3

Complete blow-up for a semilinear parabolic problem with a localized nonlinear term via classical method

3.1 Introduction

In this chapter we still consider the same semilinear parabolic problem with a localized nonlinear term as the problem in previous chapter. But we use the classical method, Green's function method, investigate a blow-up solution of such a problem. Before starting our process, we recall that we are studying the semilinear parabolic problem with a localized nonlinear term in the form. Let T be any positive real number, $D = (0, 1)$ and $\Omega_T = (0, 1) \times (0, T)$ Let $\overline{D}$ and $\overline{\Omega}_T$ be the closure of D and Ω_T, respectively.

$$\left.\begin{array}{c} k(x)u_t - (p(x)u_x)_x = k(x)f(u(x_0, t)) \text{ for } (x, t) \in \ \Omega_T, \\ u(x, 0) = \psi(x) \text{ for } x \in \overline{D}, \\ u(0, t) = 0 = u(1, t) \text{ for } t \in (0, T), \end{array}\right\} \tag{3.1.1}$$

where x_0 is a fixed point in D and k, p, f and ψ are given functions. In order to obtain a blow-up solution of problem (3.1.1), we need the following assumptions.

(A1) $p(x)$, $k(x)$, $p'(x)$ and $k'(x)$ are real-valued and continuous for $x \in \overline{D}$, and $p(x)$ and $k(x)$ are positive for $x \in \overline{D}$, i.e.,

$$\begin{aligned} 0 < k_{\min} = \min_{x\in\overline{D}} k(x) \le k(x) \le \max_{x\in\overline{D}} k(x) = k_{\max}, \\ 0 < p_{\min} = \min_{x\in\overline{D}} p(x) \le p(x) \le \max_{x\in\overline{D}} p(x) = p_{\max}, \\ k'_{\min} = \min_{x\in\overline{D}} k'(x) \le k'(x) \le \max_{x\in\overline{D}} k'(x) = k'_{\max}, \\ p'_{\min} = \min_{x\in\overline{D}} p'(x) \le p'(x) \le \max_{x\in\overline{D}} p'(x) = p'_{\max}. \end{aligned}$$

(A2) $\psi(x) \in C^2(\overline{D})$, ϕ is nontrivial and nonnegative on D, $\psi(0) = 0 = \psi(1)$, and $\psi(x_0) \ge \psi(x)$ for any $x \in \overline{D}$ and the function ψ satisfies

$$\frac{d}{dx}\left(p(x)\frac{d\psi}{dx}\right) + f(\psi(x_0)) \ge \zeta_1\psi(x) \text{ on } D$$

for some positive constant ζ_1,

(A3) $f(s) \in C^2([0,\infty))$, f is positive, increasing and convex on $[0,\infty)$.

3.2 Green's function

We will begin this section with finding Green's function corresponding to semilinear parabolic problem (3.1.1). Let us construct the corresponding Green's function $G(x,t,\xi,\tau)$ and denote that

$$L \equiv k(x)\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\left[p(x)\frac{\partial}{\partial x}\right].$$

The corresponding Green's function is determined by the following system: for each $(x,t) \in \Omega_T$,

$$\begin{aligned} LG(x,t,\xi,\tau) &= \delta(x-\xi)\delta(t-\tau), \\ G(x,t,\xi,\tau) &= 0, \quad \text{for } t < \tau, \\ G(0,t,\xi,\tau) &= 0 = G(1,t,\xi,\tau), \end{aligned} \tag{3.2.1}$$

where $\delta(x)$ is the Dirac delta function. From [6], we will use the eigenfunction expansion method to construct the Green's function, so we let

$$G(x,t,\xi,\tau) = \sum_{n=1}^{\infty} a_n(t) g_n(x), \tag{3.2.2}$$

where g_n is the eigenfunction of the corresponding regular eigenvalue problem to semilinear parabolic problem (3.1.1)

$$\frac{d}{dx}\left[p(x)\frac{d}{dx}g_n(x)\right] + \lambda_n k(x) g_n(x) = 0, \tag{3.2.3}$$

and the boundary conditions

$$g_n(0) = 0 = g_n(1),$$

where λ_n is the eigenvalue associating to g_n and has a property

$$0 < \lambda_1 < \lambda_2 < \lambda_3 < ... < \lambda_n < \lambda_{n+1} < ...$$

and $\lambda_n = O(n^2)$ for sufficiently large n. Moreover the set $\{g_n(x)\}$ is a maximal (that is, complete) orthonormal set with the weight function $k(x)$, that is,

$$\int_0^1 k(x) g_n^2(x) dx = 1 \text{ for } x \in D$$

and g_n is bounded for any $x \in \overline{D}$. By substituting (3.2.2) into (3.2.1), we obtain that

$$\begin{aligned} \sum_{n=1}^{\infty} k(x) a_n'(t) g_n(x) - \sum_{n=1}^{\infty} a_n(t) \frac{\partial}{\partial x}\left[p(x)\frac{\partial}{\partial x}g_n(x)\right] &= \delta(x-\xi)\delta(t-\tau), \\ \sum_{n=1}^{\infty} k(x) a_n'(t) g_n(x) + \lambda_n \sum_{n=1}^{\infty} k(x) a_n(t) g_n(x) &= \delta(x-\xi)\delta(t-\tau), \\ \sum_{n=1}^{\infty} k(x) g_n(x)\left[a_n'(t) + \lambda_n a_n(t)\right] &= \delta(x-\xi)\delta(t-\tau). \end{aligned}$$

Multiplying both sides by $g_n(x)$ and integrating from 0 to 1 with respect to x, we have

$$\int_0^1 g_n(x)\sum_{n=1}^{\infty} k(x)g_n(x)\left[a_n'(t)+\lambda_n a_n(t)\right]dx = \int_0^1 g_n(x)\delta(x-\xi)\delta(t-\tau)dx.$$

Using the orthogonal property of the eigenfunction g_n, we have

$$\left[a_n'(t)+\lambda_n a_n(t)\right]\int_0^1 k(x)g_n^2(x)dx = g_n(\xi)\delta(t-\tau).$$

Multiplying both sides by $\exp(\lambda_n t)$ and using the property of g_n, we have

$$\begin{aligned}
\left[a_n'(t)+\lambda_n a_n(t)\right]\exp(\lambda_n t) &= g_n(\xi)\delta(t-\tau)\exp(\lambda_n t),\\
\frac{d}{dt}\left[a_n(t)\exp(\lambda_n t)\right] &= g_n(\xi)\delta(t-\tau)\exp(\lambda_n t).
\end{aligned}$$

Integrating from τ^- to t and applying the property of the green's function, $G(x,t,\xi,\tau)=0$ for $t<\tau$, we have

$$\begin{aligned}
\int_{\tau^-}^{t}\frac{d}{ds}\left[a_n(s)\exp(\lambda_n s)\right] &= \int_{\tau^-}^{t} g_n(\xi)\delta(s-\tau)\exp(\lambda_n s)ds,\\
a_n(t)\exp(\lambda_n t)-a_n(\tau^-)\exp(\lambda_n\tau^-) &= g_n(\xi)\int_{\tau^-}^{t}\delta(s-\tau)\exp(\lambda_n s)ds,\\
a_n(t)\exp(\lambda_n t) &= g_n(\xi)\exp(\lambda_n\tau),\\
a_n(t) &= g_n(\xi)\exp\left[-\lambda_n(t-\tau)\right].
\end{aligned}$$

Therefore, we obtain that the Green's function is in the form, for $t>\tau$

$$G(x,t,\xi,\tau)=\sum_{n=1}^{\infty} g_n(x)g_n(\xi)\exp\left[-\lambda_n(t-\tau)\right],$$

or

$$G(x,t,\xi,\tau)=\sum_{n=1}^{\infty} g_n(x)g_n(\xi)\exp\left[-\lambda_n(t-\tau)\right]H(t-\tau),$$

where the function H is the Heaviside unit-step function. We would like to show that the Green's function exists, i.e. the infinite series representing the Green's function converges. Then we consider that for $t>\tau$

$$\begin{aligned}
G(x,t,\xi,\tau) &= \sum_{n=1}^{\infty} g_n(x)g_n(\xi)\exp\left[-\lambda_n(t-\tau)\right]\\
&\le \left[\max_{x\in\overline{D}} g_n(x)\right]^2\sum_{n=1}^{\infty}\exp\left[-\lambda_n(t-\tau)\right].
\end{aligned}$$

It ia easy to show that this series $\sum_{n=1}^{\infty}\exp\left[-\lambda_n(t-\tau)\right]$ converges, and then we obtain that the infinite series $\sum_{n=1}^{\infty} g_n(x)g_n(\xi)\exp\left[-\lambda_n(t-\tau)\right]$ converges uniformly. Therefore, the Green's function exists.

Lemma 3.2.1 $\sum_{n=1}^{\infty} k(x)g_n(x)g_n(\xi)=\delta(x-\xi)$.

Proof. Using the completeness property of eigenfunction g_n, we can write the function $\delta(x-\xi)$ in the terms of the function g_n, that is

$$\delta(x-\xi) = \sum_{n=1}^{\infty} k(x) c_n g_n(x).$$

Multiplying both sides by $g_n(x)$, we have

$$g_n(x)\delta(x-\xi) = g_n(x) \sum_{n=1}^{\infty} k(x) c_n g_n(x).$$

Integrating both sides from 0 to 1 and using the orthogonal property of the eigenfunction g_n, we have

$$\begin{aligned}
\int_0^1 g_n(x)\delta(x-\xi)dx &= \int_0^1 g_n(x) \sum_{n=1}^{\infty} k(x) c_n g_n(x) dx \\
g_n(\xi) &= c_n \int_0^1 k(x) g_n(x) g_n(x) dx \\
c_n &= g_n(\xi).
\end{aligned}$$

Therefore, we get the result.

We next check that the Green's function we just construct satisfies the problem (3.2.1). We begin by computing

$$\begin{aligned}
\frac{\partial G}{\partial t} &= -\left[\sum_{n=1}^{\infty} \lambda_n g_n(x) g_n(\xi) \exp\left[-\lambda_n(t-\tau)\right]\right] H(t-\tau) \\
&\quad + \left[\sum_{n=1}^{\infty} g_n(x) g_n(\xi) \exp\left[-\lambda_n(t-\tau)\right]\right] \delta(t-\tau).
\end{aligned}$$

Using the property of the Dirac delta function, $f(t)\delta(t-\tau) = f(\tau)\delta(t-\tau)$, we have

$$\begin{aligned}
k(x)\frac{\partial G}{\partial t} &= -k(x)\left[\sum_{n=1}^{\infty} \lambda_n g_n(x) g_n(\xi) \exp\left[-\lambda_n(t-\tau)\right]\right] H(t-\tau) \\
&\quad + k(x)\left[\sum_{n=1}^{\infty} g_n(x) g_n(\xi)\right] \delta(t-\tau), \\
&= -k(x)\left[\sum_{n=1}^{\infty} \lambda_n g_n(x) g_n(\xi) \exp\left[-\lambda_n(t-\tau)\right]\right] H(t-\tau) \\
&\quad + \delta(x-\xi)\delta(t-\tau).
\end{aligned}$$

So,

$$\begin{aligned}
& k(x)\frac{\partial G}{\partial t} - \frac{\partial}{\partial x}\left(p(x)\frac{\partial G}{\partial x}\right) \\
= & -k(x)\left[\sum_{n=1}^{\infty} \lambda_n g_n(x) g_n(\xi) \exp\left[-\lambda_n(t-\tau)\right]\right] H(t-\tau) \\
& -\sum_{n=1}^{\infty}\left[\frac{\partial}{\partial x}\left(\left[p(x)\frac{\partial}{\partial x} g_n(x)\right]\right) g_n(\xi) \exp\left[-\lambda_n(t-\tau)\right]\right] H(t-\tau) \\
& + \delta(x-\xi)\delta(t-\tau).
\end{aligned}$$

We can rewrite in the form,

$$\begin{aligned} & k(x)\frac{\partial G}{\partial t} - \frac{\partial}{\partial x}\left(p(x)\frac{\partial G}{\partial x}\right) \\ = & -\sum_{n=1}^{\infty}\left(g_n(\xi)\left[\frac{\partial}{\partial x}\left(p(x)\frac{\partial}{\partial x}g_n(x)\right) + \lambda_n k(x)g_n(x)\right]\exp\left[-\lambda_n(t-\tau)\right]\right)H(t-\tau) \\ & +\delta(x-\xi)\delta(t-\tau). \end{aligned}$$

Using the equation $\frac{d}{dx}\left[p(x)\frac{d}{dx}g_n(x)\right] + \lambda_n k(x)g_n(x) = 0$, we finally obtain

$$LG = \delta(x-\xi)\delta(t-\tau),$$

and then by direct computation and definition of the Heaviside unit-step function, we can get the conditions $G(x,t,\xi,\tau) = 0$, for $t < \tau$ and $G(0,t,\xi,\tau) = 0 = G(1,t,\xi,\tau)$.

To derive the integral equation from the problem (3.1.1), let us consider the adjoint operator L^*, which is given by $L^* = -k(x)\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\left(p(x)\frac{\partial}{\partial x}\right)$. Using the Green's theorem, we obtain the integral equation

$$u(x,t) = \int_0^t\int_0^1 k(\xi)G(x,t,\xi,\tau)f(u(x_0,\tau))d\xi d\tau + \int_0^1 k(\xi)G(x,t,\xi,0)\psi(\xi)d\xi. \tag{3.2.4}$$

We state the properties of the Green's function corresponding to the problem (3.1.1) in below lemma.

Lemma 3.2.2 *In the set* $\{(x,t,\xi,\tau) : x \text{ and } \xi \text{ are in } D, 0 \le \tau < t < T\}$, $G(x,t,\xi,\tau) > 0$.

Proof. Let $A = \{(x,t,\xi,\tau) : x$ and ξ are in $D, 0 \le \tau < t < T\}$. Suppose that there exists a point (x_0,t_0,ξ_0,τ_0) in A such that $G\ (x_0,t_0,\xi_0,\tau_0) < 0$. Since the function G is continuous, there exists a positive constant ε such that $G(x,t,\xi,\tau) < 0$ in the set

$$W_0 = (x_0-\varepsilon, x_0+\varepsilon)\times(t_0-\varepsilon,t_0+\varepsilon)\times(\xi_0-\varepsilon,\xi_0+\varepsilon)\times(\tau_0-\varepsilon,\tau_0+\varepsilon)$$

in A. Let $W_1 = (\xi_0-\varepsilon,\xi_0+\varepsilon)\times(\tau_0-\varepsilon,\tau_0+\varepsilon)$ and $W_2 = (\xi_0-\varepsilon/2,\xi_0+\varepsilon/2)\times(\tau_0-\varepsilon/2,\tau_0+\varepsilon/2)$. We would like to show that there exists a function $h \in C^2$ such that $h \equiv 1$ on $\overline{W_2}$, $h \equiv 0$ outside W_1, and $0 \le h \le 1$ in $W_1\backslash W_2$. We will construct the function h in a sequence of steps.

First step: we define the function f_1 by

$$f_1(s) = \begin{cases} 0 & , s \le 0, \\ exp(-s^{-2}) & , s > 0, \end{cases}$$

which belongs to $C^2(R)$, vanishes for $s \le 0$, is positive for $s > 0$, and is monotone increasing.

Second step: we define the function f_2 by

$$f_2(s) = f_1(s)f_1(1-s),$$

which belongs to $C^2(R)$, vanishes for $s \le 0$ and $s \ge 1$, and is positive for $0 < s < 1$.

Third step: we define the function f_3 by

$$f_3(s) = \frac{\int_0^s f_2(t)dt}{\int_0^1 f_2(t)dt},$$

which belongs to $C^\infty(R)$, vanishes for $s \le 0$, is monotone increasing, equals to 1 for $s \ge 1$, and satisfies $0 < f_3(s) < 1$ for all $s \in D$.

Last step: we define the function $h(x,t)$ by

$$h(x,t) = f_3\left(\frac{\varepsilon - |x - x_0|}{\varepsilon/2}\right) f_3\left(\frac{\varepsilon - |t - t_0|}{\varepsilon/2}\right),$$

which is in $C^2(R^2)$, $h \equiv 1$ on $\overline{W_2}$, $h \equiv 0$ outside W_1, and $0 \le h \le 1$ in $W_1 \backslash W_2$. Hence the solution of the problem $Lu(x,t) = h(x,t)$ in Ω_α, where $t_0 < \alpha$, with u satisfying zero initial and boundary conditions is given by

$$u(x,t) = \int_{\tau_0-\varepsilon}^{\tau_0+\varepsilon} \int_{\xi_0-\varepsilon}^{\xi_0+\varepsilon} G(x,t,\xi,\tau) h(\xi,\tau) d\xi d\tau.$$

Since we have that $G(x,t,\xi,\tau) < 0$ in W_0, $0 \le h(\xi,\tau) \le 1$ in W_1, and $h \equiv 1$ on $\overline{W_2}$, it follows that $u(x,t) < 0$ for (x,t) in $(x_0 - \varepsilon, x_0 + \varepsilon) \times (t_0 - \varepsilon, t_0 + \varepsilon)$. On the other hand , $h(x,t) \ge 0$ in Ω_α implies that $u(x,t) \ge 0$ by the weak maximum principle [12]. Therefore we get a contradiction and hence $G(x,t,\xi,\tau) \ge 0$ in A. Next we will show that $G(x,t,\xi,\tau) \ne 0$ in A. Suppose that there exists a point (x_1,t_1,ξ_1,τ_1) in A such that $G(x_1,t_1,\xi_1,\tau_1) = 0$. Using the property of Green′s function, $G(x,t,\xi,\tau) < 0$ for $t < \tau$, we have that $G(x,t,\xi_1,\tau_1) = 0$ in $D \cap \{(x,t,\xi_1,\tau_1) : x \in D,\ t \le t_1\}$. On the other hand, $G(\xi_1,t_1,\xi_1,\tau_1) = \sum\limits_{n=1}^{\infty} g_n^2(\xi_1) \exp\left[-\lambda_n(t_1 - \tau_1)\right]$, which is positive. We again have a contradiction. This shows that $G(x,t,\xi,\tau)$ is positive in A.

Lemma 3.2.3 *If $r \in C([0,T])$, then $\int\limits_0^t \int\limits_0^1 G(x,t,\xi,\tau) r(\tau) d\xi d\tau$ is continuous for $x \in \overline{D}$ and $t \in [0,T]$.*

Proof. Let ε be any positive number such that $t - \varepsilon > 0$. For $x \in \overline{D}$ and $\tau \in [0, t-\varepsilon]$, we multiply the equation

$$G(x,t,\xi,\tau) = \sum_{n=1}^{\infty} g_n(x) g_n(\xi) \exp\left[-\lambda_n(t-\tau)\right]$$

by the function $r(\tau)$, and then we obtain

$$\begin{aligned} G(x,t,\xi,\tau) r(\tau) &= \sum_{n=1}^{\infty} g_n(x) g_n(\xi) \exp\left[-\lambda_n(t-\tau)\right] r(\tau) \\ &\le \max_{0 \le \tau \le T} r(\tau) \sum_{n=1}^{\infty} g_n(x) g_n(\xi) \exp\left[-\lambda_n(t-\tau)\right] \\ &\le \max_{0 \le \tau \le T} r(\tau) \left[\max_{x \in \overline{D}} g_n(x)\right]^2 \sum_{n=1}^{\infty} \exp\left[-\lambda_n(t-\tau)\right]. \end{aligned}$$

It is easy to show that the series $\sum\limits_{n=1}^{\infty} \exp\left[-\lambda_n(t-\tau)\right]$ converges and then we have that $\sum\limits_{n=1}^{\infty} g_n(x) g_n(\xi) \exp\left[-\lambda_n(t-\tau)\right] r(\tau)$ converges uniformly. Therefore we have

$$\int_0^{t-\varepsilon} \int_0^1 G(x,t,\xi,\tau) r(\tau) d\xi d\tau = \sum_{n=1}^{\infty} \int_0^{t-\varepsilon} \int_0^1 g_n(x) g_n(\xi) \exp\left[-\lambda_n(t-\tau)\right] r(\tau) d\xi d\tau.$$

Let us consider that

$$\begin{aligned}
&\sum_{n=1}^{\infty}\int_0^{t-\varepsilon}\int_0^1 g_n(x)g_n(\xi)\exp\left[-\lambda_n(t-\tau)\right]r(\tau)d\xi d\tau\\
\leq\ & \max_{0\leq\tau\leq T} r(\tau)\left[\max_{x\in\overline{D}} g_n(x)\right]^2\sum_{n=1}^{\infty}\int_0^{t-\varepsilon}\int_0^1 \exp\left[-\lambda_n(t-\tau)\right]d\xi d\tau\\
=\ & \max_{0\leq\tau\leq T} r(\tau)\left[\max_{x\in\overline{D}} g_n(x)\right]^2\sum_{n=1}^{\infty}\int_0^{t-\varepsilon} \exp\left[-\lambda_n(t-\tau)\right]d\tau\\
=\ & \max_{0\leq\tau\leq T} r(\tau)\left[\max_{x\in\overline{D}} g_n(x)\right]^2\sum_{n=1}^{\infty}\lambda_n^{-1}\left[\exp(-\lambda_n\varepsilon)-\exp(-\lambda_n t)\right]\\
\leq\ & \max_{0\leq\tau\leq T} r(\tau)\left[\max_{x\in\overline{D}} g_n(x)\right]^2\sum_{n=1}^{\infty}\lambda_n^{-1}.
\end{aligned}$$

Since the series $\sum_{n=1}^{\infty}\lambda_n^{-1}$ converges, we have

$$\sum_{n=1}^{\infty}\int_0^{t-\varepsilon}\int_0^1 g_n(x)g_n(\xi)\exp\left[-\lambda_n(t-\tau)\right]r(\tau)d\xi d\tau$$

converges uniformly with respect to x, t, and ε. Since the uniform convergence also holds for $\varepsilon = 0$, it follows that

$$\sum_{n=1}^{\infty}\int_0^{t-\varepsilon}\int_0^1 g_n(x)g_n(\xi)\exp\left[-\lambda_n(t-\tau)\right]r(\tau)d\xi d\tau$$

is a continuous function of x, t, and $\varepsilon \geq 0$. Therefore

$$\int_0^t\int_0^1 G(x,t,\xi,\tau)r(\tau)d\xi d\tau = \lim_{\varepsilon\to 0}\sum_{n=1}^{\infty}\int_0^{t-\varepsilon}\int_0^1 g_n(x)g_n(\xi)\exp\left[-\lambda_n(t-\tau)\right]r(\tau)d\xi d\tau$$

is a continuous function of x and t.

Lemma 3.2.3 *Given any $x \in D$ and any finite time T, there exist two positive numbers C_1 (depending on x and T) and C_2 (depending on T) such that for $0 \leq t \leq T$,*

$$C_1 < \int_0^1 G(x,t,\xi,0)d\xi \ \textit{and}\ \int_0^1 G(x_0,t,\xi,0)d\xi < C_2.$$

Proof. Let us consider the following auxiliary problem

$$\begin{aligned}
k(x)\frac{\partial v}{\partial t}-\frac{\partial}{\partial x}\left[p(x)\frac{\partial v}{\partial x}\right] &= 1 \text{ in } \Omega_T,\\
v(x,0) &= 0 \text{ on } x\in\overline{D},\\
v(0,t) &= 0 = v(1,t) \text{ for } 0<t<T. \qquad (3.2.5)
\end{aligned}$$

The problem (3.2.5) has a solution v given by

$$
\begin{aligned}
v(x,t) &= \int_0^t \int_0^1 G(x,t,\xi,\tau)d\xi d\tau \\
&= \int_0^t \int_0^1 G(x,t-\tau,\xi,0)d\xi d\tau \\
&= \int_0^t \int_0^1 G(x,\tau,\xi,0)d\xi d\tau,
\end{aligned}
$$

and then we differentiate with respect to t,

$$
v_t(x,t) = \int_0^1 G(x,t,\xi,0)d\xi > 0, \text{ i.e. } v \text{ is increasing function.}
$$

We consider at the time $t=0$, for any $x \in D$

$$
\begin{aligned}
v_t(x,0) &= \int_0^1 G(x,0,\xi,0)d\xi \\
&= \int_0^1 \sum_{n=0}^{\infty} g_n(x)g_n(\xi)d\xi \\
&= \int_0^1 \sum_{n=0}^{\infty} \frac{k(x)}{k(x)} g_n(x)g_n(\xi)d\xi \\
&\geq \frac{1}{\max\limits_{x\in\overline{D}} k(x)} \int_0^a \sum_{n=0}^{\infty} k(x)g_n(x)g_n(\xi)d\xi \\
&= \frac{1}{\max\limits_{x\in\overline{D}} k(x)} \int_0^a \delta(x-\xi)d\xi \\
&= \frac{1}{\max\limits_{x\in\overline{D}} k(x)}.
\end{aligned}
$$

Thus for any finite time T, there exists a positive constant C_1(depending on x and T) such that

$$
C_1 < \int_0^1 G(x,t,\xi,0)d\xi, \text{ for } 0 \leq t \leq T.
$$

Since we know that $\int_0^1 G(x,t,\xi,0)d\xi < \infty$, there exists a positive constant C_2(depending on T) such that

$$
\int_0^1 G(x_0,t,\xi,0)d\xi < C_2, \text{ for } 0 \leq t \leq T.
$$

3.3 Existence of a blow-up solution

After we know important properties of the corresponding Green'function to semilinear parabolic problem (3.1.1), we will show that that there exists a positive real number t_1 such that the integral equation (5.3.2) has a unique continuous solution u on $[0, t_1]$ for any $x \in \overline{D}$ before a blow-up occurs. To do so, let us construct the sequence $\{u_n\}$ with $u_0(x,t) = \psi(x)$ for $n = 0, 1, 2, ...$, and consider the following problem

$$\begin{aligned} Lu_{n+1}(x,t) &= k(x)f(u_n(x_0,t)), \text{ in } (x,t) \in \Omega_T, \\ u_{n+1}(x,0) &= \psi(x), \text{ for } x \in \overline{D}, \\ u_{n+1}(0,t) &= 0 = u_{n+1}(1,t), \text{ for } 0 < t < T. \end{aligned} \tag{3.3.1}$$

Lemma 3.3.1 *The sequence $u_n \geq \psi$ for $n = 0, 1, 2, ...$*

Proof. We will show by using the principle of mathematical induction. By using a property of ψ, $\frac{d}{dx}[p(x)\frac{d}{dx}\psi(x)] + k(x)f(\psi(x_0)) \geq 0$ for $x \in \overline{D}$, we have

$$\begin{aligned} L(u_1 - u_0)(x,t) &= k(x)f(u(x_0,t)) + \frac{d}{dx}[p(x)\frac{d}{dx}\psi(x)] \\ &\geq k(x)f(u(x_0,t)) - k(x)f(\psi(x_0)) \\ &= k(x)f(\psi(x_0)) - k(x)f(\psi(x_0)) \\ &= 0, \end{aligned}$$

and the initial and boundary conditions

$$\begin{aligned} (u_1 - u_0)(x,0) &= 0, \text{ for } x \in \overline{D}, \\ (u_1 - u_0)(0,t) &= 0 = (u_1 - u_0)(1,t), \text{ for } 0 < t < T. \end{aligned}$$

Applying the maximum principle for parabolic type [12], we obtain that $u_1 - u_0 \geq \min_{x \in \overline{\Omega}_T} (u_1 - u_0) = 0$, $u_1 \geq u_0$ in $\overline{\Omega}_T$. Next, we assume that for any positive j

$$\psi \leq u_1 \leq u_2 \leq ... \leq u_{n-1} \leq u_n \text{ in } \overline{\Omega}_T.$$

By using that f is an increasing function and $u_{n-1} \leq u_n$, we have

$$L(u_{n+1} - u_n) = k(x)f(u_n(x_0,t)) - k(x)f(u_{n-1}(x_0,t)) \geq 0, \text{ in } \Omega_T,$$

and the initial and boundary conditions

$$\begin{aligned} (u_{n+1} - u_n)(x,0) &= 0, \text{ for } x \in \overline{D}, \\ (u_{n+1} - u_n)(0,t) &= 0 = (u_{n+1} - u_n)(1,t), \text{ for } 0 < t < T. \end{aligned}$$

Applying the maximum principle for parabolic type [12], we obtain that $u_{n+1} \geq u_n$ for all n. Therefore, we can conclude that, by the principle of mathematical induction, $u_n \geq \psi$ in $\overline{\Omega}_T$ for each positive n.

Lemma 3.3.2 *The sequence $\{u_n\}$ is a non-decreasing function of t.*

Proof. Let us define the sequence $\{w_n\}$ by for $n = 0, 1, 2, ...$

$$w_n(x,t) = u_n(x, t+h) - u_n(x,t)$$

where h is any positive number such that $0 < t + h < T$. Thus we also have

$$w_0(x,t) = u_0(x,t+h) - u_0(x,t) = 0.$$

Let us consider the equation

$$Lw_1(x,t) = 0, \text{ in } \Omega_{T-h},$$

and the initial and boundary conditions

$$\begin{aligned} w_1(x,0) &= u_1(x,h) - u_1(x,0) = u_1(x,h) - \psi(x) \geq 0, \text{ for } x \in \overline{D} \\ w_1(0,t) &= 0 = w_1(1,t), \text{ for } 0 < t < T-h. \end{aligned}$$

Applying the maximum principle for parabolic type [12], we obtain that $w_1 \geq 0$ for $\overline{\Omega}_{T-h}$. Let us assume that for each n, $w_n \geq 0$ for $\overline{\Omega}_{T-h}$. By using the Mean Value Theorem, we obtain

$$\begin{aligned} Lw_{n+1}(x,t) &= L(u_{n+1}(x,t+h) - u_{n+1}(x,t)) \\ &= k(x)f(u_n(x_0,t+h)) - k(x)f(u_n(x_0,t)) \\ &= k(x)f'(u_n(x_0,t_1)(u_n(x_0,t+h) - u_n(x,t)) \\ &= k(x)f'(u_n(x_0,t_1)w_n(x_0+h,t) \\ &\geq 0, \end{aligned}$$

for some $t_1 \in (t, t+h)$, and the initial and boundary conditions

$$\begin{aligned} w_{n+1}(x,0) &= u_{n+1}(x,h) - u_{n+1}(x,0) = u_{n+1}(x,h) - \psi(x) \geq 0, \text{ for } x \in \overline{D}, \\ w_{n+1}(0,t) &= 0 = w_{n+1}(1,t), \text{ for } 0 < t < T-h. \end{aligned}$$

Applying the maximum principle for parabolic type [12], we obtain that $w_{n+1} \geq 0$ on $\overline{\Omega}_{T-h}$. Therefore, we can conclude that, by the principle of mathematical induction, $w_n \geq 0$ on $\overline{\Omega}_{T-h}$ for each positive n, i.e. u_n is a non-decreasing function of t.

Theorem 3.3.3 *There exists some $\widetilde{T}$ such that the integral equation (5.3.2) has a unique non-negative continuous solution $u \geq \psi(x)$ for $0 \leq t \leq \widetilde{T}$, and u is a non-decreasing function of t.*

Proof. Let us consider the following auxiliary problem

$$\begin{aligned} Lv(x,t) &= 0, \text{ in } \Omega_T, \\ v(x,0) &= \psi(x), \text{ for } x \in \overline{D}, \\ v(0,t) &= 0 = v(1,t), \text{ for } 0 < t < T. \end{aligned}$$

Then the solution of the above problem is

$$v(x,t) = \int_0^1 k(x)G(x,t,\xi,0)\psi(\xi)d\xi.$$

Since the functions k, G, and ϕ are non-negative, we have that $v \geq 0$ in $\overline{\Omega}_T$. By the maximum principle for parabolic type [12], we know that v attains its maximum $K = \max_{x \in \overline{D}} \psi(x)$ in $\overline{D} \times \{0\}$. We claim that for any given positive constant $M \geq K$, there exists some positive constant t_1 such that $u_n \leq M$ for $0 < t < t_1$. Let us consider the sequence $\{u_n\}$ which is constructed from problem (3.3.1). By (3.2.4), we obtain

$$u_n(x,t) = \int_0^t \int_0^1 k(\xi)G(x,t,\xi,\tau)f(u_{n-1}(x_0,\tau))d\xi d\tau + \int_0^1 k(\xi)G(x,t,\xi,0)\psi(\xi)d\xi. \quad (3.3.2)$$

As $t \to 0$, we can see that

$$\begin{aligned}
\lim_{t\to 0} u_n &= \lim_{t\to 0} \int_0^1 k(\xi)G(x,t,\xi,0)\phi(\xi)d\xi \\
&= \int_0^1 \lim_{t\to 0} k(\xi)G(x,t,\xi,0)\psi(\xi)d\xi \\
&= \int_0^1 \left(\sum_{i=1}^{\infty} k(\xi)g_i(x)g_i(\xi)\right)\psi(\xi)d\xi \\
&= \int_0^1 \psi(\xi)\delta(x-\xi)d\xi \\
&= \psi(x).
\end{aligned}$$

By lemma 3.3.2, there exists t_1 such that $u_n \leq M$ for $0 \leq t \leq t_1$. Let u denote $\lim_{n\to\infty} u_n$, and then we show that the sequence $\{u_n\}$ converges uniformly to u for $0 \leq t \leq t_1$. We consider

$$u_{n+1}(x,t) - u_n(x,t) = \int_0^t \int_0^1 k(\xi)G(x,t,\xi,\tau)\left(f(u_i(x_0,\tau)) - f(u_{i-1}(x_0,\tau))\right)d\xi d\tau. \tag{3.3.3}$$

Let $S_n = \max_{\overline{\Omega}_{t_1}} |u_n(x,t) - u_{n-1}(x,t)|$. Using the Mean Value Theorem, we have

$$f(u_n(x_0,\tau)) - f(u_{n-1}(x_0,\tau)) = f'(\mu)(u_n(x_0,\tau) - u_{n-1}(x_0,\tau)),$$

where μ is between $u_n(x_0,\tau)$ and $u_{n-1}(x_0,\tau)$. Since we know that $u_n \leq M$ for all n and $f''(s) > 0$ for $s > 0$, we have

$$\begin{aligned}
f(u_n(x_0,\tau)) - f(u_{n-1}(x_0,\tau)) &\leq f'(M)(u_n(x_0,\tau) - u_{n-1}(x_0,\tau)) \\
&\leq f'(M)S_n
\end{aligned}$$

From (5.3.5), we obtain

$$
\begin{aligned}
|u_{n+1}-u_n| &\leq k_{max}f'(M)S_n \int_0^t \int_0^1 G(x,t,\xi,\tau)d\xi d\tau \\
&= k_{max}f'(M)S_n \int_0^t \int_0^1 \sum_{i=1}^{\infty} g_i(x)g_i(\xi)\exp\left[-\lambda_i(t-\tau)\right] d\xi d\tau \\
&\leq k_{max}f'(M)\left[\max_{x\in\overline{D}} g_i(x)\right]^2 S_n \int_0^t \int_0^1 \sum_{i=1}^{\infty} \exp\left[-\lambda_i(t-\tau)\right] d\xi d\tau \\
&= k_{max}f'(M)\left[\max_{x\in\overline{D}} g_i(x)\right]^2 S_n \sum_{i=1}^{\infty}\int_0^t \int_0^1 \exp\left[-\lambda_i(t-\tau)\right] d\xi d\tau \\
&= k_{max}f'(M)\left[\max_{x\in\overline{D}} g_i(x)\right]^2 S_n \sum_{i=1}^{\infty}\int_0^t \exp\left[-\lambda_i(t-\tau)\right] d\tau \\
&= k_{max}f'(M)\left[\max_{x\in\overline{D}} g_i(x)\right]^2 S_n \sum_{i=1}^{\infty}\lambda_i^{-1}(1-\exp(-\lambda_i t)).
\end{aligned}
$$

We can see that $\sum_{i=1}^{\infty}\lambda_i^{-1}(1-\exp(-\lambda_i t)) \leq \sum_{i=1}^{\infty}\lambda_i^{-1}$, a convergent series. Therefore the series $\sum_{i=1}^{\infty}\lambda_i^{-1}(1-\exp(-\lambda_i t))$ converges uniformly. Claim that there exists a positive $\sigma_1(>0)$ such that

$$
k_{max}f'(M)\left[\max_{x\in\overline{D}} g_i(x)\right]^2 \left[\sum_{n=1}^{\infty}\lambda_n^{-1}(1-\exp(-\lambda_n t))\right] < 1 \text{ for } t\in[0,\sigma_1]. \tag{3.3.4}
$$

Let us consider that

$$
\lim_{t\to 0}\sum_{i=1}^{\infty}\lambda_i^{-1}(1-\exp(-\lambda_i t)) = \sum_{i=1}^{\infty}\lim_{t\to 0}\lambda_i^{-1}(1-\exp(-\lambda_i t)) = 0.
$$

By the definition of limit, there exists a positive $\sigma_1 > 0$ such that

$$
\left|\sum_{i=1}^{\infty}\lambda_i^{-1}(1-\exp(-\lambda_i t))\right| < \frac{1}{k_{max}f'(M)\left[\max_{x\in\overline{D}} g_i(x)\right]^2}, \text{ for } t\in[0,\sigma_1].
$$

Therefore we get the result and we also have that the sequence $\{u_n\}$ converges uniformly to u for $0\leq t\leq\sigma_1$. Similarly for $\sigma_1\leq t\leq t_1$, we substitute $u(\xi,\sigma_1)$ to the place of $\psi(\xi)$ in the integral equation (3.3.2), so we have

$$
u_n(x,t) = \int_{\sigma_1}^t \int_0^1 k(\xi)G(x,t,\xi,\tau)f(u_{i-1}(x_0,\tau))d\xi d\tau + \int_0^1 k(\xi)G(x,t,\xi,0)u(\xi,\sigma_1)d\xi.
$$

Moreover, we also have

$$
u_{n+1}(x,t)-u_n(x,t) = \int_{\sigma_1}^t \int_0^1 k(\xi)G(x,t,\xi,\tau)\left(f(u_n(x_0,\tau))-f(u_{n-1}(x_0,\tau))\right)d\xi d\tau.
$$

In the same way, since $S_n = \max\limits_{\overline{D}\times[\sigma_1,t_1]} |u_n - u_{n-1}|$, it follows from the Mean Value Theorem

$$f(u_n(x_0,\tau)) - f(u_{n-1}(x_0,\tau)) \leq f'(M)S_n.$$

From (5.3.5), we obtain

$$\begin{aligned}
|u_{n+1} - u_n| &\leq k_{max} f'(M) S_n \int\limits_{\sigma_1}^{t} \int\limits_{0}^{1} G(x,t,\xi,\tau) d\xi d\tau \\
&\leq k_{max} f'(M) \left[\max_{x\in\overline{D}} g_i(x)\right]^2 S_n \int\limits_{\sigma_1}^{t} \int\limits_{0}^{1} \sum_{i=1}^{\infty} \exp\left[-\lambda_i(t-\tau)\right] d\xi d\tau \\
&= k_{max} f'(M) \left[\max_{x\in\overline{D}} g_i(x)\right]^2 \left[\sum_{i=1}^{\infty} \lambda_i^{-1}\left[1-\exp(-\lambda_i(t-\sigma_1))\right]\right] S_n.
\end{aligned}$$

Thus there exists $\sigma_2 = \min\{\sigma_1, t_1 - \sigma_1\} > 0$ such that

$$k_{max} f'(M) \left[\max_{x\in\overline{D}} g_i(x)\right]^2 \left[\sum_{i=1}^{\infty} \lambda_i^{-1}\left[1-\exp(-\lambda_i(t-\sigma_1))\right]\right] < 1, \qquad (3.3.5)$$

for $t \in [\sigma_1, \min\{2\sigma_1, t_1\}]$. Hence the sequence $\{u_n\}$ converges uniformly to u for $t \in [\sigma_1, \min\{2\sigma_1, t_1\}]$. By proceeding in this way the sequence $\{u_i\}$ converges uniformly to u for $0 \leq t \leq t_1$. Therefore we can conclude that the integral equation (3.2.4) has a continuous solution u for $0 \leq t \leq t_1$. We would like to show that the solution u is unique for $t \in [0, t_1]$. Suppose that the integral equation (3.2.4) has two distinct the solution u and $\widetilde{u}$ for $t \in [0, t_1]$, and let $\Phi = \max\limits_{(x,t)\in\overline{\Omega}_{t_1}} |u - \widetilde{u}| > 0$. Since u and $\widetilde{u}$ are the solution of the problem, we have

$$u(x,t) - \widetilde{u}(x,t) = \int\limits_{0}^{t} \int\limits_{0}^{1} k(\xi) G(x,t,\xi,\tau) \left[f(u(x_0,\tau)) - f(\widetilde{u}(x_0,\tau))\right] d\xi d\tau.$$

Using the same idea, we obtain that

$$\Phi \leq k_{max} f'(M) \left[\max_{x\in\overline{D}} g_n(x)\right]^2 \left[\sum_{n=1}^{\infty} \lambda_n^{-1}(1-\exp(-\lambda_n t))\right] \Phi \text{ for } t \in [0,\sigma_1].$$

This implies that

$$k_{max} f'(M) \left[\max_{x\in\overline{D}} g_n(x)\right]^2 \left[\sum_{n=1}^{\infty} \lambda_n^{-1}(1-\exp(-\lambda_n t))\right] \geq 1, \text{ for } t \in [0,\sigma_1].$$

which contradicts to the equation (3.3.4). Hence the solution u is unique for $0 \leq t \leq \sigma_1$. Using the same idea, we can show that for $t \in [\sigma_1, \min\{2\sigma_1, t_1\}]$,

$$\Phi \leq k_{max} f'(M) \left[\max_{x\in\overline{D}} g_n(x)\right]^2 \left[\sum_{n=1}^{\infty} \lambda_n^{-1}\left[1-\exp(-\lambda_n(t-\sigma_1))\right]\right] \Phi.$$

This implies that for $t \in [\sigma_1, \min\{2\sigma_1, t_1\}]$,

$$k_{max} f'(M) \left[\max_{x\in\overline{D}} g_n(x)\right]^2 \left[\sum_{n=1}^{\infty} \lambda_n^{-1}\left[1-\exp(-\lambda_n(t-\sigma_1))\right]\right] \geq 1,$$

which contradicts to the equation (3.3.5). Hence the solution u is unique for $\sigma_1 \leq t \leq \min\{2\sigma_1, t_1\}$. By proceeding in this way, the integral equation u is unique continuous for $0 \leq t \leq t_1$. Since u_n is a non-decreasing function of t, u is a non-decreasing function of t.

Let T_{max} be the supremum of all t_1 that the integral equation (3.2.4) has a unique non-negative continuous solution u.

Theorem 3.3.4 *If T_{max} is finite, then $u(x_0, t)$ is unbounded as $t \to T_{\max}$.*

Proof. By lemma 2.3.4.4, we have that $u(x_0, t) \geq u(x, t)$ for all $(x, t) \in \overline{\Omega}_{T_{max}}$. Suppose that $u(x_0, t)$ is bounded on $[0, T_{max}]$ We consider the integral equation of the solution u for $[T_{max}, T]$ with the initial condition $u(x, 0)$ replaced by $u(x, T_{max})$,

$$u(x_0, t) = \int_{\widetilde{t}}^{t} \int_0^1 k(\xi)G(x_0, t, \xi, \tau)f(u(x_0, \tau))d\xi d\tau + \int_0^1 k(\xi)G(x_0, t, \xi, \widetilde{t})u(\xi, T_{max})d\xi.$$

For any positive constant $N > u(x_0, T_{max})$, an argument as before shows that there exists some positive t_2 such that the integral equation of the solution u is unique and continuous on the interval $[T_{max}, t_2]$. This contradicts to the definition of T_{max}. Hence this theorem is proven

Theorem 3.3.5 *The semilinear parabolic problem (3.1.1) has a unique solution u.*

Proof. Since $\int_0^t \int_0^1 k(\xi)G(x, t, \xi, \tau)f(u(x_0, \tau))d\xi d\tau < \infty$ for $x \in D$ and t in any compact subset of $[0, T_{max})$, we have that for any $x \in D$ and $t_2 \in (0, t)$,

$$\begin{aligned}
&\int_0^t \int_0^1 k(\xi)G(x, t, \xi, \tau)f(u(x_0, \tau))d\xi d\tau \\
= &\lim_{n\to\infty} \int_0^{t-1/n} \int_0^1 k(\xi)G(x, t, \xi, \tau)f(u(x_0, \tau))d\xi d\tau \\
= &\lim_{n\to\infty} \int_{t_2}^{t} \frac{\partial}{\partial \zeta} \int_0^{\zeta-1/n} \int_0^1 k(\xi)G(x, \zeta, \xi, \tau)f(u(x_0, \tau))d\xi d\tau d\zeta \\
&+ \lim_{n\to\infty} \int_0^{t_2-1/n} \int_0^a G(x, t_2, \xi, \tau)f(u(x_0, \tau))d\xi d\tau.
\end{aligned}$$

Let us consider the following problem

$$\begin{aligned}
Lw &= 0, \text{ for } (x, t) \in \Omega_{T_{max}}, \\
w(0, t, \xi, \tau) &= 0 = w(1, t, \xi, \tau), \text{ for } 0 \leq \tau < t < T_{max}, \\
\lim_{t\to\tau^+} k(x)w(x, t, \xi, \tau) &= \delta(x - \xi).
\end{aligned}$$

Then we obtain that

$$w(x, t, \xi, \tau) = \int_0^1 k(\alpha)G(x, t, \alpha, \tau)\frac{\delta(\alpha - \xi)}{k(\alpha)}d\alpha = G(x, t, \xi, \tau) \text{ for } t > \tau.$$

It means that $\lim_{t \to \tau^+} k(x)G(x,t,\xi,\tau) = \delta(x-\xi)$. Therefore we have

$$\begin{aligned}
&\int_0^t \int_0^1 k(\xi)G(x,t,\xi,\tau)f(u(x_0,\tau))d\xi d\tau \\
= &\int_0^{t_2} \int_0^1 k(\xi)G(x,t_2,\xi,\tau)f(u(x_0,\tau))d\xi d\tau \\
&+ \lim_{n\to\infty} \int_{t_2}^t \frac{\partial}{\partial \zeta} \int_0^{\zeta-1/n} \int_0^1 k(\xi)G(x,\zeta,\xi,\tau)f(u(x_0,\tau))d\xi d\tau d\zeta
\end{aligned}$$

$$\begin{aligned}
= &\int_0^{t_2} \int_0^1 k(\xi)G(x,t_2,\xi,\tau)f(u(x_0,\tau))d\xi d\tau \\
&+ \lim_{n\to\infty} \int_{t_2}^t \int_0^{\zeta-1/n} \int_0^1 k(\xi)G_\zeta(x,\zeta,\xi,\tau)f(u(x_0,\tau))d\xi d\tau d\zeta \\
&+ \lim_{n\to\infty} \int_{t_2}^t \int_0^1 k(\xi)G(x,\zeta,\xi,\zeta-1/n)f(u(x_0,\tau))d\xi d\zeta \\
= &\int_0^{t_2} \int_0^1 k(\xi)G(x,t_2,\xi,\tau)f(u(x_0,\tau))d\xi d\tau \\
&+ \lim_{n\to\infty} \int_{t_2}^t \int_0^{\zeta-1/n} \int_0^1 k(\xi)G_\zeta(x,\zeta,\xi,\tau)f(u(x_0,\tau))d\xi d\tau d\zeta \\
&+ \int_{t_2}^t \int_0^1 \delta(x-\xi)f(u(x_0,\zeta))d\xi d\zeta \\
= &\int_0^{t_2} \int_0^1 k(\xi)G(x,t_2,\xi,\tau)f(u(x_0,\tau))d\xi d\tau \\
&+ \lim_{n\to\infty} \int_{t_2}^t \int_0^{\zeta-1/n} \int_0^1 k(\xi)G_\zeta(x,\zeta,\xi,\tau)f(u(x_0,\tau))d\xi d\tau d\zeta + \int_{t_2}^t f(u(x_0,\zeta))d\zeta.
\end{aligned}$$

We let that

$$g_n(x,\zeta) = \int_0^{\zeta-1/n} \int_0^1 k(\xi)G_\zeta(x,\zeta,\xi,\tau)f(u(x_0,\tau))d\xi d\tau.$$

Without loss of generality, let $n > m$, so we have

$$g_n(x,\zeta) - g_m(x,\zeta) = \int_{\zeta-1/m}^{\zeta-1/n} \int_0^1 k(\xi)G_\zeta(x,\zeta,\xi,\tau)f(u(x_0,\tau))d\xi d\tau.$$

Since $k(x)G_t(x,t,\xi,\tau) \in C(\overline{D} \times (\tau, T))$ and $f(u(x_0,t))$ is a monotone function of t, we use the Second Mean Value Theorem for Integral and then we have that for any $x \neq \xi$ and any ζ in any compact subset of $(0, T_{max})$, there exists some real number υ such that $\zeta - \nu \in (\zeta - 1/m, \zeta - 1/n)$ and

$$\begin{aligned} g_n(x,\zeta) - g_m(x,\zeta) &= f(u(x_0, \zeta - 1/m)) \int_{\zeta-1/m}^{\zeta-\nu} \int_0^1 k(\xi) G_\zeta(x,\zeta,\xi,\tau) d\xi d\tau \\ &\quad + f(u(x_0, \zeta - 1/n)) \int_{\zeta-\nu}^{\zeta-1/n} \int_0^1 k(\xi) G_\zeta(x,\zeta,\xi,\tau) d\xi d\tau. \end{aligned}$$

It is easy to show that $G_\zeta(x,\zeta,\xi,\tau) = -G_\tau(x,\zeta,\xi,\tau)$ and then we have

$$\begin{aligned} & g_n(x,\zeta) - g_m(x,\zeta) \\ = & -f(u(x_0, \zeta - 1/m)) \int_{\zeta-1/m}^{\zeta-\nu} \int_0^1 k(\xi) G_\tau(x,\zeta,\xi,\tau) d\xi d\tau \\ & -f(u(x_0, \zeta - 1/n)) \int_{\zeta-\nu}^{\zeta-1/n} \int_0^1 k(\xi) G_\tau(x,\zeta,\xi,\tau) d\xi d\tau \\ = & -f(u(x_0, \zeta - 1/m)) \left[\int_0^1 k(\xi) G(x,\zeta,\xi,\zeta-\nu) d\xi - \int_0^1 k(\xi) G(x,\zeta,\xi,\zeta - 1/m) d\xi \right] \\ & -f(u(x_0, \zeta - 1/n)) \left[\int_0^1 k(\xi) G(x,\zeta,\xi,\zeta-1/n) d\xi - \int_0^1 k(\xi) G(x,\zeta,\xi,\zeta - \nu) d\xi \right] \\ = & [f(u(x_0, \zeta - 1/n)) - f(u(x_0, \zeta - 1/m))] \int_0^1 k(\xi) G(x,\zeta,\xi,\zeta-\nu) d \\ & +f(u(x_0, \zeta - 1/m)) \int_0^1 k(\xi) G(x,\zeta,\xi,\zeta-1/m) d\xi \\ & -f(u(x_0, \zeta - 1/n)) \int_0^1 k(\xi) G(x,\zeta,\xi,\zeta-1/n) d\xi \end{aligned}$$

Since, for $x \neq \xi$, $k(x)G(x,\zeta,\xi,\zeta-\varepsilon) = k(x)G(x,\varepsilon,\xi,0)$ converges uniformly to 0 with respect to ζ as $\varepsilon \to 0$, it follow that, for $x \neq \xi$,the sequence $\{g_n\}$ is a Cauchy sequence, and hence the sequence $\{g_n\}$ converges uniformly with respect to ζ in any compact subset of $(0, T_{max})$. Therefore for $x \neq \xi$

$$\begin{aligned} & \lim_{n\to\infty} \int_{t_2}^{t} \int_0^{\zeta-1/n} \int_0^1 k(\xi) G_\zeta(x,\zeta,\xi,\tau) f(u(x_0,\tau)) d\xi d\tau d\zeta \\ = & \int_{t_2}^{t} \lim_{n\to\infty} \int_0^{\zeta-1/n} \int_0^1 k(\xi) G_\zeta(x,\zeta,\xi,\tau) f(u(x_0,\tau)) d\xi d\tau d\zeta \\ = & \int_{t_2}^{t} \int_0^{\zeta} \int_0^1 k(\xi) G_\zeta(x,\zeta,\xi,\tau) f(u(x_0,\tau)) d\xi d\tau d\zeta. \end{aligned}$$

In the case $x = \xi$, we have

$$-k(x)G_\zeta(x,\zeta,\xi,\tau)f(u(x_0,\tau)) = \sum_{i=1}^{\infty} k(\xi)g_i^2(\xi)\lambda_i \exp[-\lambda_i(\zeta-\tau)]f(u(x_0,\tau)),$$

which is positive. Therefore the sequence $\{-g_n\}$ is a non-decreasing sequence of non-negative function with respect to ζ. By the Monotone Convergence Theorem,

$$\begin{aligned}
&\lim_{n\to\infty}\int_{t_2}^{t}\int_0^{\zeta-1/n}\int_0^1 k(\xi)G_\zeta(x,\zeta,\xi,\tau)f(u(x_0,\tau))d\xi d\tau d\zeta \\
=& \int_{t_2}^{t}\lim_{n\to\infty}\int_0^{\zeta-1/n}\int_0^1 k(\xi)G_\zeta(x,\zeta,\xi,\tau)f(u(x_0,\tau))d\xi d\tau d\zeta \\
=& \int_{t_2}^{t}\int_0^{\zeta}\int_0^1 k(\xi)G_\zeta(x,\zeta,\xi,\tau)f(u(x_0,\tau))d\xi d\tau d\zeta.
\end{aligned}$$

Thus, we obtain that

$$\begin{aligned}
&\frac{\partial}{\partial t}\int_0^t\int_0^1 k(\xi)G(x,t,\xi,\tau)f(u(x_0,\tau))d\xi d\tau \\
=& \int_0^1 k(\xi)G(x,t,\xi,t)f(u(x_0,t))d\xi + \int_0^t\int_0^1 k(\xi)G_t(x,t,\xi,\tau)f(u(x_0,\tau))d\xi d\tau \\
=& f(u(x_0,t))\int_0^1 \delta(x-\xi)d\xi + \int_0^t\int_0^1 k(\xi)G_t(x,t,\xi,\tau)f(u(x_0,\tau))d\xi d\tau \\
=& f(u(x_0,t)) + \int_0^t\int_0^1 k(\xi)G_t(x,t,\xi,\tau)f(u(x_0,\tau))d\xi d\tau.
\end{aligned}$$

We would like to show that by using the Leibnitz rule, we have for any $x \in D$ and t in any compact subset of $(0, T_{max})$,

$$\begin{aligned}
\frac{\partial}{\partial x}\int_0^{t-\varepsilon}\int_0^1 G(x,t,\xi,\tau)f(u(x_0,\tau))d\xi d\tau &= \int_0^{t-\varepsilon}\int_0^1 G_x(x,t,\xi,\tau)f(u(x_0,\tau))d\xi d\tau, \\
\frac{\partial}{\partial x}\int_0^{t-\varepsilon}\int_0^1 p(x)G_x(x,t,\xi,\tau)f(u(x_0,\tau))d\xi d\tau &= \int_0^{t-\varepsilon}\int_0^1 \frac{\partial}{\partial x}\left[p(x)G_x(x,t,\xi,\tau)\right]f(u(x_0,\tau))d\xi d\tau.
\end{aligned}$$

Let us consider that for any $x_1 \in D$,

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_0^{t-\varepsilon}\int_0^1 G(x,t,\xi,\tau)f(u(x_0,\tau))d\xi d\tau \\
= &\lim_{\varepsilon\to 0}\int_{x_1}^{x}\left[\frac{\partial}{\partial\eta}\int_0^{t-\varepsilon}\int_0^1 G(\eta,t,\xi,\tau)f(u(x_0,\tau))d\xi d\tau\right]d\eta \\
&+\lim_{\varepsilon\to 0}\int_0^{t-\varepsilon}\int_0^1 G(x_1,t,\xi,\tau)f(u(x_0,\tau))d\xi d\tau \\
= &\lim_{\varepsilon\to 0}\int_{x_1}^{x}\int_0^{t-\varepsilon}\int_0^1 G_\eta(\eta,t,\xi,\tau)f(u(x_0,\tau))d\xi d\tau d\eta \\
&+\int_0^{t}\int_0^1 G(x_1,t,\xi,\tau)f(u(x_0,\tau))d\xi d\tau.
\end{aligned}$$

Claim that

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_{x_1}^{x}\int_0^{t-\varepsilon}\int_0^1 G_\eta(\eta,t,\xi,\tau)f(u(x_0,\tau))d\xi d\tau d\eta \\
= &\int_{x_1}^{x}\lim_{\varepsilon\to 0}\int_0^{t-\varepsilon}\int_0^1 G_\eta(\eta,t,\xi,\tau)f(u(x_0,\tau))d\xi d\tau d\eta. \qquad (3.3.6)
\end{aligned}$$

By using the Fubini Theorem

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_{x_1}^{x}\int_0^{t-\varepsilon}\int_0^1 G_\eta(\eta,t,\xi,\tau)f(u(x_0,\tau))d\xi d\tau d\eta \\
= &\lim_{\varepsilon\to 0}\int_0^{t-\varepsilon} f(u(x_0,\tau))\int_{x_1}^{x}\int_0^1 G_\eta(\eta,t,\xi,\tau)d\xi d\eta d\tau \\
= &\lim_{\varepsilon\to 0}\int_0^{t-\varepsilon} f(u(x_0,\tau))\left[\int_0^1 G(x,t,\xi,\tau)d\xi - \int_0^1 G(x_1,t,\xi,\tau)d\xi\right]d\tau \\
= &\int_0^{t} f(u(x_0,\tau))\left[\int_0^1 G(x,t,\xi,\tau)d\xi - \int_0^1 G(x_1,t,\xi,\tau)d\xi\right]d\tau,
\end{aligned}$$

which exists because $\int\limits_0^t \int\limits_0^1 G(x,t,\xi,\tau) f(u(x_0,\tau)) d\xi d\tau$ is continuous. Therefore we have

$$\begin{aligned}
&\int\limits_0^t f(u(x_0,\tau)) \left[\int\limits_0^1 G(x,t,\xi,\tau) d\xi - \int\limits_0^1 G(x_1,t,\xi,\tau) d\xi \right] d\tau \\
=& \int\limits_0^t \int\limits_0^1 G(x,t,\xi,\tau) f(u(x_0,\tau)) d\xi d\tau - \int\limits_0^t \int\limits_0^1 G(x_1,t,\xi,\tau) f(u(x_0,\tau)) d\xi d\tau \\
=& \int\limits_{x_1}^x \frac{\partial}{\partial \eta} \left[\int\limits_0^t \int\limits_0^1 G(\eta,t,\xi,\tau) f(u(x_0,\tau)) d\xi d\tau \right] d\eta \\
=& \int\limits_{x_1}^x \int\limits_0^t \int\limits_0^1 G_\eta(\eta,t,\xi,\tau) f(u(x_0,\tau)) d\xi d\tau d\eta \\
=& \int\limits_{x_1}^x \lim_{\varepsilon \to 0} \int\limits_0^{t-\varepsilon} \int\limits_0^1 G_\eta(\eta,t,\xi,\tau) f(u(x_0,\tau)) d\xi d\tau d\eta.
\end{aligned}$$

thus, we have (3.3.6). Therefore we also have

$$\begin{aligned}
&\frac{\partial}{\partial x} \int\limits_0^t \int\limits_0^1 G(x,t,\xi,\tau) f(u(x_0,\tau)) d\xi d\tau \\
=& \frac{\partial}{\partial x} \left[\int\limits_{x_1}^x \int\limits_0^t \int\limits_0^1 G_\eta(\eta,t,\xi,\tau) f(u(x_0,\tau)) d\xi d\tau d\eta \right. \\
& \left. + \int\limits_0^t \int\limits_0^1 G(x_1,t,\xi,\tau) f(u(x_0,\tau)) d\xi d\tau \right] \\
=& \int\limits_0^t \int\limits_0^1 G_x(x,t,\xi,\tau) f(u(x_0,\tau)) d\xi d\tau.
\end{aligned}$$

Let us consider that for any $x_2 \in D$,

$$\begin{aligned}
& \lim_{\varepsilon\to 0}\int_0^{t-\varepsilon}\int_0^1 p(x)G_x(x,t,\xi,\tau)f(u(x_0,\tau))d\xi d\tau \\
= & \lim_{\varepsilon\to 0}\int_{x_2}^{x}\frac{\partial}{\partial\eta}\left[\int_0^{t-\varepsilon}\int_0^1 p(\eta)G_\eta(\eta,t,\xi,\tau)f(u(x_0,\tau))d\xi d\tau\right]d\eta \\
& +\lim_{\varepsilon\to 0}\int_0^{t-\varepsilon}\int_0^1 p(x_2)G_\eta(x_2,t,\xi,\tau)f(u(x_0,\tau))d\xi d\tau \\
= & \lim_{\varepsilon\to 0}\int_{x_2}^{x}\int_0^{t-\varepsilon}\int_0^1\frac{\partial}{\partial\eta}\left[p(\eta)G_\eta(\eta,t,\xi,\tau)\right]f(u(x_0,\tau))d\xi d\tau d\eta \\
& +\int_0^{t}\int_0^1 p(x_2)G_\eta(x_2,t,\xi,\tau)f(u(x_0,\tau))d\xi d\tau. \qquad (3.3.7)
\end{aligned}$$

Claim that

$$\begin{aligned}
& \lim_{\varepsilon\to 0}\int_{x_2}^{x}\int_0^{t-\varepsilon}\int_0^1\frac{\partial}{\partial\eta}\left[p(\eta)G_\eta(\eta,t,\xi,\tau)\right]f(u(x_0,\tau))d\xi d\tau d\eta \\
= & \int_{x_2}^{x}\lim_{\varepsilon\to 0}\int_0^{t-\varepsilon}\int_0^1\frac{\partial}{\partial\eta}\left[p(\eta)G_\eta(\eta,t,\xi,\tau)\right]f(u(x_0,\tau))d\xi d\tau d\eta. \qquad (3.3.8)
\end{aligned}$$

By using the Fubini Theorem

$$\begin{aligned}
& \lim_{\varepsilon\to 0}\int_{x_2}^{x}\int_0^{t-\varepsilon}\int_0^1\frac{\partial}{\partial\eta}\left[p(\eta)G_\eta(\eta,t,\xi,\tau)\right]f(u(x_0,\tau))d\xi d\tau d\eta \\
= & \lim_{\varepsilon\to 0}\int_0^{t-\varepsilon} f(u(x_0,\tau))\int_{x_2}^{x}\int_0^1\frac{\partial}{\partial\eta}\left[p(\eta)G_\eta(\eta,t,\xi,\tau)\right]d\xi d\eta d\tau \\
= & \lim_{\varepsilon\to 0}\int_0^{t-\varepsilon} f(u(x_0,\tau))\left[p(x)\int_0^1 G_x(x,t,\xi,\tau)d\xi - p(x_2)\int_0^1 G_x(x_2,t,\xi,\tau)d\xi\right]d\tau \\
= & \int_0^{t} f(u(x_0,\tau))\left[p(x)\int_0^1 G_x(x,t,\xi,\tau)d\xi - p(x_2)\int_0^1 G_x(x_2,t,\xi,\tau)d\xi\right]d\tau,
\end{aligned}$$

which exists $G_x(x,t,\xi,\tau)$ because is continuous. Therefore we have

$$\begin{aligned}
&\int_0^t f(u(x_0,\tau))\left[p(x)\int_0^1 G_x(x,t,\xi,\tau)d\xi - p(x_2)\int_0^1 G(x_2,t,\xi,\tau)d\xi\right]d\tau \\
=&\int_0^t\int_0^1 p(x)G_x(x,t,\xi,\tau)f(u(x_0,\tau))d\xi d\tau - \int_0^t\int_0^1 p(x_2)G_x(x_2,t,\xi,\tau)f(u(x_0,\tau))d\xi d\tau \\
=&\int_{x2}^x \frac{\partial}{\partial\eta}\left[\int_0^t\int_0^1 p(\eta)G_\eta(\eta,t,\xi,\tau)f(u(x_0,\tau))d\xi d\tau\right]d\eta \\
=&\int_{x_2}^x\int_0^t\int_0^1 \frac{\partial}{\partial\eta}\left[p(\eta)G_\eta(\eta,t,\xi,\tau)\right]f(u(x_0,\tau))d\xi d\tau d\eta \\
=&\int_{x_2}^x \lim_{\varepsilon\to 0}\int_0^{t-\varepsilon}\int_0^1 \frac{\partial}{\partial\eta}\left[p(\eta)G_\eta(\eta,t,\xi,\tau)\right]f(u(x_0,\tau))d\xi d\tau d\eta.
\end{aligned}$$

thus, we have (3.3.8). Therefore we also have

$$\begin{aligned}
&\frac{\partial}{\partial x}\left[\int_0^t\int_0^1 p(x)G_x(x,t,\xi,\tau)f(u(x_0,\tau))d\xi d\tau\right] \\
=&\frac{\partial}{\partial x}\left[\int_{x_2}^x\int_0^t\int_0^1 \frac{\partial}{\partial\eta}\left[p(\eta)G_\eta(\eta,t,\xi,\tau)\right]f(u(x_0,\tau))d\xi d\tau d\eta\right. \\
&\left.+\int_0^t\int_0^1 p(x_2)G_\eta(x_2,t,\xi,\tau)f(u(x_0,\tau))d\xi d\tau\right] \\
=&\int_0^t\int_0^1 \frac{\partial}{\partial x}\left[p(x)\frac{\partial}{\partial x}G(x,t,\xi,\tau)\right]f(u(x_0,\tau))d\xi d\tau,
\end{aligned}$$

for any x in D and t in any compact subset of $(0,T_{max})$. By using the Leibnitz rule, we have that for any x in D and t in any compact subset of $(0,T_{max})$,

$$\begin{aligned}
\frac{\partial}{\partial t}\int_0^1 G(x,t,\xi,0)\psi(\xi)d\xi &= \int_0^1 G_t(x,t,\xi,0)\psi(\xi)d\xi, \\
\frac{\partial}{\partial x}\int_0^1 G(x,t,\xi,0)\psi(\xi)d\xi &= \int_0^1 G_x(x,t,\xi,0)\psi(\xi)d\xi, \\
\frac{\partial}{\partial x}\int_0^1 p(x)G_x(x,t,\xi,0)\psi(\xi)d\xi &= \int_0^1 \frac{\partial}{\partial x}\left[p(x)G_x(x,t,\xi,0)\right]\psi(\xi)d\xi.
\end{aligned}$$

From the integral equation (3.2.4) we have for $x \in D$ and $0 < t < T_{max}$

$$\begin{aligned}
Lu(x,t) &= \left[k(x)\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\left(p(x)\frac{\partial}{\partial x}\right)\right] u(x,t) \\
&= k(x)\frac{\partial}{\partial t}\left[\int_0^t \int_0^1 k(\xi)G(x,t,\xi,\tau)f(u(x_0,\tau))d\xi d\tau\right] \\
&\quad + k(x)\frac{\partial}{\partial t}\left[\int_0^1 k(\xi)G(x,t,\xi,0)\psi(\xi)d\xi\right] \\
&\quad - \frac{\partial}{\partial x}\left(p(x)\frac{\partial}{\partial x}\right)\left[\int_0^t \int_0^1 k(\xi)G(x,t,\xi,\tau)f(u(x_0,\tau))d\xi d\tau\right] \\
&\quad - \frac{\partial}{\partial x}\left(p(x)\frac{\partial}{\partial x}\right)\left[\int_0^1 k(\xi)G(x,t,\xi,0)\psi(\xi)d\xi\right] \\
&= k(x)f(u(x_0,t)) + k(x)\int_0^t \int_0^1 k(\xi)G_t(x,t,\xi,\tau)f(u(x_0,\tau))d\xi d\tau \\
&\quad + k(x)\int_0^1 k(\xi)G_t(x,t,\xi,0)\psi(\xi)d\xi \\
&\quad - \int_0^t \int_0^1 k(\xi)\frac{\partial}{\partial x}\left[p(x)\frac{\partial}{\partial x}G(x,t,\xi,\tau)\right] f(u(x_0,\tau))d\xi d\tau \\
&\quad - \int_0^1 k(\xi)\frac{\partial}{\partial x}\left[p(x)\frac{\partial}{\partial x}G(x,t,\xi,0)\right]\psi(\xi)d\xi
\end{aligned}$$

$$\begin{aligned}
&= k(x)f(u(x_0,t)) + \int_0^1 k(\xi)\left(k(x)G_t(x,t,\xi,0) - \frac{\partial}{\partial x}\left[p(x)\frac{\partial}{\partial x}G(x,t,\xi,0)\right]\right)\psi(\xi)d\xi \\
&\quad + \int_0^t \int_0^1 k(\xi)\left(k(x)G_t(x,t,\xi,\tau) - \frac{\partial}{\partial x}\left[p(x)\frac{\partial}{\partial x}G(x,t,\xi,\tau)\right]\right) f(u(x_0,\tau))d\xi d\tau \\
&= k(x)f(u(x_0,t)) + \delta(t)\int_0^1 k(\xi)\delta(x-\xi)\psi(\xi)d\xi \\
&\quad + \int_0^t \int_0^1 k(\xi)\delta(x-\xi)\delta(t-\tau)f(u(x_0,\tau))d\xi d\tau \\
&= k(x)f(u(x_0,t)),
\end{aligned}$$

and the initial condition of the solution u can compute from for and $x \in \overline{D}$

$$\begin{aligned}
\lim_{t\to 0} u(x,t) &= \lim_{t\to 0} \int_0^1 k(\xi)G(x,t,\xi,0)\psi(\xi)d\xi \\
&= \int_0^1 \lim_{t\to 0} k(\xi)G(x,t,\xi,0)\psi(\xi)d\xi \\
&= \int_0^1 \delta(x-\xi)\phi(\xi)d\xi \\
&= \psi(x),
\end{aligned}$$

since we know that $G(0,t,\xi,\tau) = 0 = G(1,t,\xi,\tau)$, we can compute directly and obtain the boundary condition $u(0,t) = 0 = u(1,t)$. Therefore u defined by (3.2.4) is a solution of the problem.

3.4 A sufficient condition to blow-up in finite time

In this section, we give a sufficient condition to guarantee occurrence of blow-up in finite time for the solution u of semilinear parabolic problem (3.1.1). Let λ_1 be the first eigenvalue of a singular eigenvalue problem (3.2.3) and let $g_1 > 0$ be its corresponding eigenfunction. Without loss of generality, we assume that

$$\int_0^1 k(x)g_1(x)dx = 1. \tag{3.4.1}$$

We then define a function Q by

$$Q(t) = \int_0^1 k(x)g_1(x)u(x,t)dx. \tag{3.4.2}$$

Theorem 3.4.1 *If*

$$\int_{Q_0}^{\infty} \frac{ds}{Q(s) - \lambda_1 s} < \infty$$

with $Q_0 = Q(0) = \int_0^1 k(x)\psi_1(x)g_1(x)dx$*, then a solution u of semilinear parabolic problem (3.1.1) blows up in finite time.*

Proof. Suppose that a solution u of semilinear parabolic problem (3.1.1) exists for all $t > 0$. Multiplying both sides of problem (3.1.1) by g_1 and then integrating both sides of problem (3.1.1) with respect to x

over its domain, we obtain

$$\begin{aligned}\frac{dQ}{dt} &= \int_0^1 k(x)u_t(x,t)g_1(x)dx \\ &= \int_0^1 \left[\frac{\partial}{\partial x}\left(p(x)\frac{\partial u}{\partial x}\right) + k(x)f(u(x_0,t))\right] g_1(x)dx \\ &= \int_0^1 \left[\frac{\partial}{\partial x}\left(p(x)\frac{\partial u}{\partial x}\right)\right] g_1(x)dx + \int_0^1 k(x)f(u(x_0,t))g_1(x)dx.\end{aligned}$$

Using the integration by part for the first term in right hand side and then using the boundary condition of regular eigenvalue problem (3.2.3), we have

$$\begin{aligned}\frac{dQ}{dt} &= \left[p(x)\frac{\partial u}{\partial x}g_1(x)\right]_0^1 - \int_0^1 p(x)\frac{\partial u}{\partial x}\frac{\partial g_1}{\partial x}dx + \int_0^1 k(x)f(u(x_0,t))g_1(x)dx \\ &= -\int_0^1 p(x)\frac{\partial u}{\partial x}\frac{\partial g_1}{\partial x}dx + \int_0^1 k(x)f(u(x_0,t))g_1(x)dx.\end{aligned}$$

Using the integrate by part again for the first term in right hand side and then using the boundary condition of semilinear parabolic problem (3.1.1), we get

$$\begin{aligned}\frac{dQ}{dt} &= -\left[p(x)u(x,t)\frac{\partial g_1}{\partial x}\right]_0^1 + \int_0^1 u(x,t)\frac{\partial}{\partial x}\left[p(x)\frac{\partial \psi_1}{\partial x}\right]dx + \int_0^1 k(x)f(u(x_0,t))g_1(x)dx \\ &= \int_0^1 u(x,t)\frac{\partial}{\partial x}\left[p(x)\frac{\partial g_1}{\partial x}\right]dx + \int_0^1 k(x)f(u(x_0,t))g_1(x)dx \\ &= -\lambda_1\int_0^1 k(x)u(x,t)g_1(x)dx + \int_0^1 k(x)f(u(x_0,t))g_1(x)dx \\ &= -\lambda_1 Q(t) + \int_0^1 k(x)f(u(x_0,t))g_1(x)dx\end{aligned}$$

It follows from lemma 2.3.4.4 that

$$\frac{dQ}{dt} \geq -\lambda_1 Q(t) + \int_0^1 k(x)f(u(x,t))g_1(x)dx \tag{3.4.3}$$

Furthermore, from Jensen's inequality for convex functions, we apply to the second term in the right hand side 0f (3.4.3) and then we obtain

$$\begin{aligned}\int_0^1 k(x)f(u(x,t))g_1(x)dx &\geq f\left(\int_0^1 k(x)u(x,t)g_1(x)dx\right) \\ &= f\left(Q(t)\right).\end{aligned} \tag{3.4.4}$$

From (3.4.3) and (3.4.4), we have the inequality

$$\frac{dQ(t)}{dt} \geq -\lambda_1 Q(t) + f\left(Q(t)\right). \tag{3.4.5}$$

It follows from

$$Q'(t) = \int_0^1 k(x)u_t(x,t)g_1(x)dx \geq 0$$

that $Q(t) \geq Q_0$ for all $t > 0$ and since $\lim_{s\to\infty} \frac{f(s)}{s}$ tends to infinity, there exists a positive constant N with $N \geq Q_0$ such that

$$f(s) - \lambda_1 s \geq 0 \text{ for any } s \geq N \geq Q_0.$$

by above sentence we can rewrite the inequality (3.4.5) in the form

$$\int_{Q_0}^{Q(t)} \frac{ds}{Q(s) - \lambda_1 s} \geq t,$$

or

$$t \leq \int_{Q_0}^{Q(t)} \frac{ds}{Q(s) - \lambda_1 s} < \int_{Q_0}^{\infty} \frac{ds}{Q(s) - \lambda_1 s}.$$

By assumption of theorem, t is finite. We thus get a contradiction. Hence the solution u of semilinear parabolic problem (3.1.1) blow-up in finite time.

3.5 The blow-up set

In the last section of this chapter, we investigate the blow-up of solution u of semilinear parabolic problem (3.1.1).

Theorem 3.5.1 *If a solution u of semilinear parabolic problem (3.1.1) blows up in a finite time, then the blow-up set of such a u is $\overline{D}$.*

Proof. Assume that u blows up in a finite time T_{max}. A solution u of semilinear parabolic problem (3.1.1) is given by

$$u(x,t) = \int_0^t \int_0^1 k(\xi)G(x,t,\xi,\tau)f(u(x_0,\tau))d\xi d\tau + \int_0^1 k(\xi)G(x,t,\xi,0)\psi(\xi)d\xi.$$

By lemma 3.2.3 we obtain

$$\begin{aligned} u(x,t) &= \int_0^t \int_0^1 k(\xi)G(x,t,\xi,\tau)f(u(x_0,\tau))d\xi d\tau + \int_0^1 k(\xi)G(x,t,\xi,0)\psi(\xi)d\xi \\ &= \int_0^t \int_0^1 k(\xi)G(x,t,\xi,0)f(u(x_0,t-\tau))d\xi d\tau + \int_0^1 k(\xi)G(x,t,\xi,0)\psi(\xi)d\xi \\ &\leq C_2 k_{max} \int_0^t f(u(x_0,t-\tau))d\tau + C_2 k_{max} \max_{x\in\overline{D}} \psi(x). \end{aligned}$$

Since u blows up in finite time, we have that as t converges to T_{max},

$$\int_0^{T_{max}} f(u(x_0, t-\tau))d\tau = \infty.$$

On the other hand, let us consider that for any $(x,t) \in \Omega_{T_{max}}$

$$\begin{aligned} u(x,t) &= \int_0^t \int_0^1 k(\xi)G(x,t,\xi,\tau)f(u(x_0,\tau))d\xi d\tau + \int_0^1 k(\xi)G(x,t,\xi,0)\phi(\xi)d\xi \\ &\geq C_1 k_{min} \int_0^t f(u(x_0,t-\tau))d\tau + \int_0^1 k(\xi)G(x,t,\xi,0)\phi(\xi)d\xi \\ &\geq C_1 k_{min} \int_0^t f(u(x_0,t-\tau))d\tau, \end{aligned}$$

which tends to infinity for any $x \in D$ as t approaches T_{max}. For $x \in \{0,1\}$, we can find a sequence $\{(x_n, t_n)\}$ such that $(x_n, t_n) \to (x, T_{max})$ and $\lim_{n\to\infty} u(x_n, t_n) \to \infty$. Thus, the blow-up set of a solution u of semilinear parabolic problem (3.1.1) is $\overline{D}$.

Note that this chapter was the object of the communication :

P. Sawangtong and B. Novaprateep Complete blow-up for a semilinear parabolic problem with a localized nonlinear term, Thai national conference on Mathematics, Bangkok, Thailand, May 2008.

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Chapter 4

Complete blow-up for a semilinear parabolic problem with a localized nonlinear term in several dimensions

4.1 Introduction

Let Ω be an open bounded subset of $\mathbb{R}^N$ with a smooth boundary $\partial\Omega$ and $\overline{\Omega}$ be its closure. In this chapter, we consider the following semilinear parabolic problem

$$\left.\begin{array}{c} u_t - \frac{1}{k(x)}\operatorname{div}\left(p(x)\nabla u(x,t)\right) = f(u(b,t)) \text{ for } (x,t) \in \Omega \times (0,\infty), \\ u(x,t) = 0 \text{ for } (x,t) \in \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x) \text{ for } x \in \overline{\Omega}, \end{array}\right\} \tag{4.1.1}$$

where k, p, f, and u_0 are given functions, b is a fixed point in Ω.

Through this chapter, we assume that

(A1) $k \in L^\infty(\Omega)$ and $0 < k_0 \leq k(x) \leq k_1$ a.e. $x \in \Omega$ for some constants k_0 and k_1,

(A2) $p \in L^\infty(\Omega)$, $0 < p_0 \leq p(x) \leq p_1$ a.e. $x \in \Omega$ for some constants p_0 and p_1 and p satisfies the following condition: there exist positive constants c_0 and c_1 such that, for any real vector ξ,

$$c_0 \left|\xi\right|^2 \leq p(x) \sum_{i,j=1}^{N} \xi_i \xi_j \leq c_1 \left|\xi\right|^2$$

for all $(x,t) \in \Omega \times (0,\infty)$,

(A3) f is locally Lipschitz continuous, $f(0) = 0$ and $f(s) > 0$ for $s > 0$ and

(A4) $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$, u_0 is nontrivial and nonnegative on Ω.

4.2 Main results

Our main results in this chapter are as follows.

Theorem 4.2.1 *Before blow-up occurs, there exists a positive constant T such that the semilinear parabolic problem (4.1.1) has a unique continuous solution u on $\overline{\Omega} \times [0, T]$.*

Theorem 4.2.2 *Let $T_{\max}$ be the supremum of all T such that the semilinear parabolic problem (4.1.1) has a unique continuous solution u on $\overline{\Omega} \times [0, T]$. If $T_{\max}$ is finite, then $|u(b, t)|$ is unbounded as t tends to $T_{\max}$.*

Theorem 4.2.3 *If $T_{\max}$ is finite, then the blow-up set of a solution u of problem (4.1.1) is Ω.*

4.3 The proof of main results

Let us define the space $L^2(\Omega)$ by

$$L^2(\Omega) = \left\{ u : \Omega \to \mathbb{R} \text{ Lebesgue measurable such that } \int_\Omega k(x)u^2(x)dx < \infty \right\}.$$

The space $L^2(\Omega)$ equipped with the inner product

$$\langle u, v \rangle_{L^2(\Omega)} = \int_\Omega k(x)u(x)v(x)dx$$

is a Hilbert space and its corresponding norm is given by

$$\|u\|_{L^2(\Omega)} = \left(\int_\Omega k(x)u^2(x)dx \right)^{\frac{1}{2}}.$$

The space $H^1(\Omega)$ defined by

$$H^1(\Omega) = \left\{ u \in L^2(\Omega) : D_{x_i}u \in L^2(\Omega) \text{ for } i = 1, 2, \ldots, N \right\},$$

where $D_{x_i}u$ is partial differentiation of u with respect to x_i in the distributional sense, is a Hilbert space equipped with the inner product

$$\langle u, v \rangle_{H^1(\Omega)} = \int_\Omega \left(k(x)u(x)v(x) + p(x) \sum_{i=1}^{N} D_{x_i}uD_{x_i}v \right) dx$$

and the corresponding norm

$$\|u\|_{H^1(\Omega)} = \left[\int_\Omega \left(k(x)u^2(x) + p(x) \sum_{i=1}^{N} (D_{x_i}u)^2 \right) dx \right]^{\frac{1}{2}}.$$

Finally, we define a Hilbert space $H_0^1(\Omega)$ by

$$H_0^1(\Omega) = \left\{ u \in H^1(\Omega) : u(x) = 0 \text{ for } x \in \partial\Omega \right\}$$

where its inner product and norm are given by

$$\langle u, v \rangle_{H_0^1(\Omega)} = \int_\Omega p(x) \sum_{i=1}^{N} D_{x_i}uD_{x_i}vdx$$

and

$$\|u\|_{H^1(\Omega)} = \left(\int\limits_{\Omega} p(x) \sum_{i=1}^{N} (D_{x_i} u)^2 \, dx \right)^{\frac{1}{2}},$$

respectively. In order to obtain our main results, we will transform the semilinear parabolic problem to the following equivalent semilinear evolution problem

$$\frac{du(t)}{dt} - Au(t) = F(u) \text{ for } t > 0 \text{ and } u(0) = u_0, \tag{4.3.1}$$

where A is an operator mapping from $D(A)$, domain of A, to $L^2(\Omega)$ with

$$\begin{aligned} D(A) \quad = \quad & \{ u \in H_0^1(\Omega) : \text{ there exists an unique element } w \in L^2(\Omega) \text{ such that} \\ & \int\limits_{\Omega} k(x)w(x)\varphi(x)dx = -\int\limits_{\Omega} p(x) \sum_{i=1}^{N} D_{x_i} u D_{x_i} \varphi dx \text{ for any } \varphi \in H_0^1(\Omega) \}, \end{aligned}$$

and $Au = \frac{1}{k(x)}(p(x)u_x)_x = w$ for all $u \in D(A)$ and where the operator $F : D(A) \to L^2(\Omega)$ is defined by

$$F(u) = f(u(b)) \text{ for any } u \in D(A). \tag{4.3.2}$$

4.3.1 The proof of theorem 4.2.1

Existence and uniqueness of a solution u of the equivalent semilinear evolution problem (4.3.1) result from the next proposition referred to [17].

Proposition 4.3.1.1 *If $B : D(B) \to L^2(\Omega)$ is m-dissipative and self-adjoint and G is Hölder continuous of exponent $\alpha \in (0,1)$, then an semilinear evolution problem,*

$$\frac{dv(t)}{dt} - Bv(t) = G(t) \text{ for } t > 0 \text{ and } v(0) = v_0 \in D(B),$$

has an unique solution $v \in C([0,\infty), D(B)) \cap C^1([0,\infty), L^2(\Omega))$ which can be expressed as

$$v(t) = H(t)v_0 + \int\limits_0^t H(t-\tau)G(\tau)d\tau$$

where $H(t)$ is an analytic semigroup generated by B.

By modifying the proof of proposition 2.3.1.1 in chapter 2, we get the following lemma.

Lemma 4.3.1.2 *The operator A defined by (4.3.1) is m-dissipative and self-adjoint.*

Since an operator $(\lambda I - A)^{-1}$ is bounded well-defined operator on $L^2(\Omega)$ with its values in $H_0^1(\Omega)$, Rellich theorem yeilds that $(\lambda I - A)^{-1}$ ia a compact on $L^2(\Omega)$. Next proposition gives well-known results of self-adjoint compact operators, the spectral theory of self-adjoint compact operators referred from [11].

Proposition 4.3.1.3 *For any $n \in \mathbb{N}$, there exists a sequence $(\lambda_n, \phi_n) \subset (0,\infty) \times H_0^1(\Omega)$ such that*

1 $A\phi_n = -\lambda_n \phi_n$.

2 $\int\limits_{\Omega} k(x)\phi_n(x)\phi_m(x)dx = \delta_{nm}$ *with* $\delta_{nm} = \begin{cases} 1 & \text{if } n = m, \\ 0 & \text{if } n \neq m. \end{cases}$

3 $\int\limits_{\Omega} p(x) \sum\limits_{i=1}^{N} D_{x_i}\phi_n(x) D_{x_i}\phi_m(x)dx = \lambda_n \delta_{nm}$

4 For any $u \in L^2(\Omega)$, $u = \sum_{n=1}^{\infty} \langle u, \phi_n \rangle \phi_n$.

5 For any $u \in L^2(\Omega)$, $\|u\|^2_{L^2(\Omega)} = \sum_{n=1}^{\infty} \langle v, \phi_n \rangle^2$.

6 $Av = -\sum_{n=1}^{\infty} \lambda_n \langle v, \phi_n \rangle \phi_n$ *for any* $v \in D(A)$ *with*

$$D(A) = \left\{ v \in L^2(\Omega) : \sum_{n=1}^{\infty} \lambda_n^2 \langle v, \phi_n \rangle^2 < \infty \right\}$$

7 $S(t)v = \sum_{n=1}^{\infty} e^{-\lambda_n t} \langle v, \phi_n \rangle \phi_n$ *for all* $(v,t) \in L^2(\Omega) \times [0, +\infty)$.

Let $s \in \mathbb{R}$ with $0 < s < 1$. By using eigenfunctions and eigenvalues of $-A$, we define an operator $(-A)^s : D((-A)^s) \to L^2(\Omega)$ with

$$D((-A)^s) = \left\{ u \in H_0^1(\Omega) : \sum_{n=1}^{\infty} \lambda_n^{2s} \langle v, \phi_n \rangle^2 < \infty \right\}$$

by

$$(-A)^s u = \sum_{n=1}^{\infty} \lambda_n^s \langle v, \phi_n \rangle \phi_n.$$

We note that $D((-A)^s)$ is a Banach space equipped with a square of norm

$$\|u\|^2_{D((-A)^s)} = \sum_{n=1}^{\infty} \lambda_n^{2s} \langle u, \phi_n \rangle^2 = \|(-A)^s u\|^2_{L^2(\Omega)}$$

and $D(A) \hookrightarrow D((-A)^s)$. Let $N \in \mathbb{N}$ with $N \leq 3$, we set

$$s_N = \frac{N}{4}.$$

Hereafter, let $s \in (s_N, 1)$. We then note that by interpolation theory and Sobolev embedding referred from [24],

$$D(A) \hookrightarrow D((-A)^s) \hookrightarrow H^{2s}(\Omega) \hookrightarrow C(\overline{\Omega}), \tag{4.3.3}$$

and that the definition of F given by (4.3.2) is meaningful. Moreover, by equation (4.3.3), there exist positive constant c_0, c_1 and c_2 such that for any $u \in D(A)$,

$$\begin{aligned} \|u\|_{C(\overline{\Omega})} &\leq c_0 \|u\|_{H^{2s}(\Omega)}, \\ \|u\|_{H^{2s}(\Omega)} &\leq c_1 \|u\|_{D((-A)^s)} \end{aligned}$$

and

$$\|u\|_{D((-A)^s)} \leq c_2 \|u\|_{D(A)}.$$

From (4.3.3), we obtain the following lemma.

Proposition 4.3.1.4 *The operator F defined by (4.3.2) is locally Lipshitz continuous from $D(A)$ to $L^2(\Omega)$.*

Proof. Let $u, v \in D(A)$. From (4.3.3), there exists a $M > 0$ such that $|u| \leq M$ and $|v| \leq M$. By locally Lipschitz continuous property of f, there exists a positive constant L depending on M with

$$\begin{aligned}
\|F(u) - F(v)\|^2_{L^2(\Omega)} &= \int_\Omega k(x) \left|F(u(x)) - F(v(x))\right|^2 dx \\
&= \int_\Omega k(x) \left|f(u(b)) - f(v(b))\right|^2 dx \\
&\leq L \int_\Omega k(x) \left|u(b) - v(b)\right|^2 dx \\
&\leq k_1 L \left|\Omega\right| \|u - v\|^2_{C(\overline{\Omega})} \\
&\leq c_0 k_1 L \left|\Omega\right| \|u - v\|^2_{H^{2s}(\Omega)} \\
&\leq c_0 c_1 k_1 L \left|\Omega\right| \|u - v\|^2_{D((-A)^s)} \\
&\leq c_0 c_1 c_2 k_1 L \left|\Omega\right| \|u - v\|^2_{D(A)} .
\end{aligned}$$

Therefore, the proof is complete.

Moreover, we obtain the following results by modifying proofs of proposition 2.3.1.4.

Lemma 4.3.1.5 *Let $v \in D((-A)^s)$ and $t > 0$.*

1 $S(t)v \in D((-A)^s)$

2 $\|(-A)^s S(t)v\|_{L^2(\Omega)} = \|S(t)(-A)^s v\|_{L^2(\Omega)} \leq \|(-A)^s v\|_{L^2(\Omega)}$.

Proof. Let $v \in D((-A)^s)$. Proposition 3.3.1.4 gives:

$$\begin{aligned}
\sum_{n=1}^{\infty} \lambda_n^{2s} \left\langle S(t)v, \phi_n \right\rangle^2 &= \sum_{n=1}^{\infty} \lambda_n^{2s} e^{-2\lambda_n t} \left\langle v, \phi_n \right\rangle^2 \\
&\leq \sum_{n=1}^{\infty} \lambda_n^{2s} \left\langle v, \phi_n \right\rangle^2 \\
&= \|v\|^2_{D((-A)^s)}
\end{aligned}$$

which shows that $S(t)v \in D((-A)^s)$ and $\|S(t)v\|_{D((-A)^s)} \leq \|v\|_{D((-A)^s)}$. Moreover,

$$S(t)(-A)^s v = \sum_{n=1}^{\infty} \lambda_n^s \left\langle v, \phi_n \right\rangle e^{-2\lambda_n t} = (-A)^s S(t)v.$$

Since

$$\sum_{n=1}^{\infty} \lambda_n^{2s} \left\langle v, \phi_n \right\rangle^2 e^{-2\lambda_n t} \leq \sum_{n=1}^{\infty} \lambda_n^{2s} \left\langle v, \phi_n \right\rangle^2 \text{ for } t \in (0, T],$$

we have

$$\|(-A)^s S(t)v\|_{L^2(\Omega)} = \|S(t)(-A)^s v\|_{L^2(\Omega)} \leq \|v\|_{D((-A)^s)} \text{ for } t \in (0, T].$$

The proof then is complete.

Lemma 4.3.1.6 *There is a positive constant c_3 such that*

$$\|(-A)^s S(t)v\|_{L^2(\Omega)} = \|S(t)v\|_{D((-A)^s)} \leq \frac{c_3}{t^s} \|v\|_{L^2(\Omega)}$$

for any $(v,t) \in L^2(\Omega) \times (0,\infty)$.

Proof. Let $v \in L^2(\Omega)$. It is not difficult to see that $\|(-A)^s S(t)v\|_{L^2(\Omega)} = \|S(t)v\|_{D((-A)^s)}$. Since the function

$$y \in \mathbb{R}^+ \to y^{2s} e^{-2y} \in \mathbb{R}^+$$

is bounded, there exists a $c > 0$ such that

$$t^{2s} \sum_{n=1}^{\infty} \lambda_n^{2s} \langle v, \phi_n \rangle^2 e^{-2\lambda_n t} \leq c \sum_{n=1}^{\infty} \lambda_n^{2s} \langle v, \phi_n \rangle^2 = c \|v\|_{L^2(\Omega)}^2.$$

Hence, we get this lemma.

We next give the definition of a mild solution of a semilinear parabolic problem (4.3.1).

Definition *u is a mild solution of the semilinear parabolic problem (4.3.1) if there exists $u \in C([0,\infty), D((-A)^s))$ such that*

$$u(t) = S(t)u_0 + \int_0^t S(t-\tau)F(u(\tau))d\tau \text{ for all } t \in [0,T]$$

where u_0 is assumed to belong to $D((-A)^s)$.

Local existence of a mild solution u of a semilinear evolution problem (4.3.1) is shown in the next lemma which base on the proof of proposition 2.3.1.7.

Lemma 4.3.1.7 *Let $u_0 \in D((-A)^s)$. There is a positive constant T such that the equivalent semilinear evolution problem (4.3.1) has a unique mild solution on $[0,T]$. Moreover, let $u(t)$ and $\widetilde{u}(t)$ be the mild solutions corresponding to u_0 and $\widetilde{u}_0$. Then for all $t \in [0,T]$,*

$$\|u(t) - \widetilde{u}(t)\|_{D((-A)^s)} \leq \|u_0 - \widetilde{u}_0\|_{D((-A)^s)} e^{\frac{c_3(c_0c_1k_1L|\Omega|)^{1/2}T^{1-s}}{1-s}}.$$

Proof. Let

$$M = \|u_0\|_{D((-A)^s)} + 1$$

and

$$T < \min\left\{ \left(\frac{1-s}{c_3 M (c_0c_1k_1L|\Omega|)^{1/2}}\right)^{\frac{1}{1-s}}, \left(\frac{1-s}{c_3(c_0c_1k_1L|\Omega|)^{1/2}}\right)^{\frac{1}{1-s}} \right\} \tag{4.3.4}$$

We then define the set E by

$$E = \left\{ u \in C([0,T], D((-A)^s)) \text{ such that } \|u(t)\|_{D((-A)^s)} \leq M \text{ for all } t \in [0,T] \right\}$$

equipped with the norm

$$\|u\|_E = \sup_{t\in[0,T]} \|u(t)\|_{D((-A)^s)}.$$

Clearly, E is a Banach space. Let

$$\Phi(u) = S(t)u_0 + \int_0^t S(t-\tau)F(u(\tau))d\tau. \tag{4.3.5}$$

We now show that the operator Φ defined by (4.3.5) maps E into itself and the mapping is a contraction. For any $u \in E$, by proposition 4.3.1.4 and lemma 4.3.1.5 and 4.3.1.6, we have

$$\begin{aligned}
\|\Phi(u)\|_E &= \sup_{t\in[0,T]} \left\| S(t)u_0 + \int_0^t S(t-\tau)F(u(\tau))d\tau \right\|_{D((-A)^s)} \\
&\leq \sup_{t\in[0,T]} \|S(t)u_0\|_{D((-A)^s)} + \sup_{t\in[0,T]} \left\| \int_0^t S(t-\tau)F(u(\tau))d\tau \right\|_{D((-A)^s)} \\
&\leq \|u_0\|_{D((-A)^s)} + \sup_{t\in[0,T]} \int_0^t \|S(t-\tau)F(u(\tau))\|_{D((-A)^s)}\, d\tau \\
&\leq \|u_0\|_{D((-A)^s)} + c_3 \sup_{t\in[0,T]} \int_0^t \frac{\|Fu(\tau)\|_{L^2(\Omega)}}{(t-\tau)^s} d\tau \\
&\leq \|u_0\|_{D((-A)^s)} + c_3(c_0c_1k_1L\,|\Omega|)^{1/2} \sup_{t\in[0,T]} \int_0^t \frac{\|u(\tau)\|_{D((-A)^s)}}{(t-\tau)^s} d\tau \\
&\leq \|u_0\|_{D((-A)^s)} + c_3M(c_0c_1k_1L\,|\Omega|)^{1/2} \sup_{t\in[0,T]} \int_0^t \frac{1}{(t-\tau)^s} d\tau \\
&\leq \|u_0\|_{D((-A)^s)} + \frac{c_3M(c_0c_1k_1L\,|\Omega|)^{1/2}}{1-s} T^{1-s}.
\end{aligned}$$

Thus, by (4.3.4), $\Phi(u) \in E$. For any $u_1, u_2 \in E$, we have

$$\begin{aligned}
\|\Phi(u_1) - \Phi(u_2)\|_E &= \sup_{t\in[0,T]} \left\| \int_0^t S(t-\tau)\left[F(u_1(\tau)) - F(u_2(\tau))\right] d\tau \right\|_{D((-A)^s)} \\
&\leq \sup_{t\in[0,T]} \int_0^t \|S(t-\tau)\left[F(u_1(\tau)) - F(u_2(\tau))\right]\|_{D((-A)^s)}\, d\tau \\
&\leq c_3 \sup_{t\in[0,T]} \int_0^t \frac{\|F(u_1(\tau)) - F(u_2(\tau))\|_{L^2(\Omega)}}{(t-\tau)^s} d\tau \\
&\leq c_3(c_0c_1k_1L\,|\Omega|)^{1/2} \sup_{t\in[0,T]} \int_0^t \frac{\|u_1(\tau) - u_2(\tau)\|_{D((-A)^s)}}{(t-\tau)^s} d\tau \\
&\leq c_3(c_0c_1k_1L\,|\Omega|)^{1/2} \left(\sup_{t\in[0,T]} \int_0^t \frac{d\tau}{(t-\tau)^s} \right) \|v_1 - v_2\|_E \\
&\leq \frac{c_3(c_0c_1k_1L\,|\Omega|)^{1/2}T^{1-s}}{1-s} \|v_1 - v_2\|_E\,.
\end{aligned}$$

By (4.3.4), Φ is strict contraction on E. Therefore, by the contraction mapping theorem, Φ has a fixed point in E, that is, there exists a unique $u \in E$ such that

$$u(t) = S(t)u_0 + \int_0^t S(t-\tau)F(u(\tau))d\tau$$

which is a mild solution of equivalent semilinear evolution problem (4.3.1). To show that uniqueness also holds in $C([0,T], D((-A)^s))$, let $u_1, u_2 \in C([0,T], D((-A)^s))$ be two solutions of (4.3.1) and let $u = u_1 - u_2$. Then

$$u(t) = \int_0^t S(t-\tau)\,[F(u_1(\tau)) - F(u_2(\tau))]\,d\tau.$$

It follows from proposition 4.3.1.4 and lemma 4.3.1.6 that

$$\begin{aligned}
\|u(t)\|_{D((-A)^s)} &= \left\| \int_0^t S(t-\tau)\,[F(u_1(\tau)) - F(u_2(\tau))]\,d\tau \right\|_{D((-A)^s)} \\
&\leq \int_0^t \|S(t-\tau)\,[F(u_1(\tau)) - F(u_2(\tau))]\|_{D((-A)^s)}\,d\tau \\
&\leq c_3 \int_0^t \frac{\|F(u_1(\tau)) - F(u_2(\tau))\|_{L^2(\Omega)}}{(t-\tau)^s}\,d\tau \\
&\leq c_3 (c_0 c_1 k_1 L\,|\Omega|)^{1/2} \int_0^t \frac{\|u_1(\tau) - u_2(\tau)\|_{D((-A)^s)}}{(t-\tau)^s}\,d\tau \\
&= c_3 (c_0 c_1 k_1 L\,|\Omega|)^{1/2} \int_0^t \frac{\|u(\tau)\|_{D((-A)^s)}}{(t-\tau)^s}\,d\tau,
\end{aligned}$$

By the Gronwall inequality, $\|u(t)\|_{D((-A)^s)} = 0$ for all $t \in [0,T]$, i.e., the uniqueness in $C([0,T], D((-A)^s))$. Moreover, we have

$$u(t) - \widetilde{u}(t) = S(t)(u_0 - \widetilde{u}_0) + \int_0^t S(t-\tau)\,[F(u(\tau)) - F(\widetilde{u}(\tau))]\,d\tau.$$

Then

$$\begin{aligned}
&\|u(t) - \widetilde{u}(t)\|_{D((-A)^s)} \\
= &\left\| S(t)(u_0 - \widetilde{u}_0) + \int_0^t S(t-\tau)\,[F(u(\tau)) - F(\widetilde{u}(\tau))]\,d\tau \right\|_{D((-A)^s)} \\
\leq &\|S(t)(u_0 - \widetilde{u}_0)\|_{D((-A)^s)} + \left\| \int_0^t S(t-\tau)\,[F(u(\tau)) - F(\widetilde{u}(\tau))]\,d\tau \right\|_{D((-A)^s)} \\
\leq &\|u_0 - \widetilde{u}_0\|_{D((-A)^s)} + c_3 \int_0^t \frac{\|F(u(\tau)) - F(\widetilde{u}(\tau))\|_{L^2(\Omega)}}{(t-\tau)^s}\,d\tau \\
\leq &\|u_0 - \widetilde{u}_0\|_{D((-A)^s)} + c_3 (c_0 c_1 k_1 L\,|\Omega|)^{1/2} \int_0^t \frac{\|u(\tau) - \widetilde{u}(\tau)\|_{D((-A)^s)}}{(t-\tau)^s}\,d\tau.
\end{aligned}$$

The Gronwall inequality implies:

$$\begin{aligned}
\|u(t)-\widetilde{u}(t)\|_{D((-A)^s)} &\leq \|u_0-\widetilde{u}_0\|_{D((-A)^s)}\, e^{c_3(c_0c_1k_1L|\Omega|)^{1/2}\int\limits_0^t \frac{1}{(t-\tau)^s}d\tau} \\
&\leq \|u_0-\widetilde{u}_0\|_{D((-A)^s)}\, e^{\frac{c_3(c_0c_1k_1L|\Omega|)^{1/2}T^{1-s}}{1-s}}.
\end{aligned}$$

Therefore, this theorem is proven.

Proposition 4.3.1.8 *Let $u_0 \in D((-A)^s)$. The mild solution u of semilinear evolution problem (4.3.1) is Hölder continuous of exponent $1-s$ in t from $[0,T]$ to $D((-A)^s)$.*

Proof. Let $u_0 \in D((-A)^s)$. Let $\widetilde{u}(t) = u(t+h)$ for any $h > 0$ and $0 \leq t \leq T-h$. Then, $\widetilde{u}$ is a mild solution of problem (4.3.1) with the initial data $\widetilde{u}(0) = u(h)$. Thus

$$\begin{aligned}
\|u(t+h)-u(t)\|_{D((-A)^s)} &= \|\widetilde{u}(t)-u(t)\|_{D((-A)^s)} \\
&\leq \|u(h)-u_0\|_{D((-A)^s)}\, e^{\frac{c_3(c_0c_1k_1L|\Omega|)^{1/2}T^{1-s}}{1-s}}.
\end{aligned}$$

On the other hand, we have that

$$\begin{aligned}
&\|u(h)-u_0\|_{D((-A)^s)} \\
=\ & \left\| S(h)u_0 - u_0 + \int\limits_0^h S(h-\tau)F(u(\tau))d\tau \right\|_{D((-A)^s)} \\
\leq\ & \|S(h)u_0-u_0\|_{D((-A)^s)} + \left\| \int\limits_0^h S(h-\tau)F(u(\tau))d\tau \right\|_{D((-A)^s)} \\
\leq\ & \left\| \int\limits_0^h S(\tau)Au_0 d\tau \right\|_{D((-A)^s)} + c_3 \int\limits_0^h \frac{\|F(u(\tau))\|_{L^2(\Omega)}}{(h-\tau)^s} d\tau \\
\leq\ & \int\limits_0^h \|S(\tau)Au_0\|_{D((-A)^s)}\, d\tau \\
& + c_3 \int\limits_0^h \frac{\|F(u_0)\|_{L^2(\Omega)} + (c_0c_1k_1L\,|\Omega|)^{1/2}\, \|u(\tau)-u_0\|_{D((-A)^s)}}{(h-\tau)^s} d\tau \\
\leq\ & c_3 \int\limits_0^h \frac{\|Au_0\|_{L^2(\Omega)}}{(h-\tau)^s} d\tau + c_3 \int\limits_0^h \frac{\|F(u_0)\|_{L^2(\Omega)}}{(h-\tau)^s} d\tau \\
& + c_3(c_0c_1k_1L\,|\Omega|)^{1/2} \int\limits_0^h \frac{\|u(\tau)-u_0\|_{D((-A)^s)}}{(h-\tau)^s} d\tau \\
\leq\ & \frac{c_3\left(\|Au_0\|_{L^2(\Omega)} + \|F(u_0)\|_{L^2(\Omega)}\right)h^{1-s}}{1-s} \\
& + c_3(c_0c_1k_1L\,|\Omega|)^{1/2} \int\limits_0^h \frac{\|u(\tau)-u_0\|_{D((-A)^s)}}{(h-\tau)^s} d\tau.
\end{aligned}$$

Gronwall inequality implies:

$$
\begin{aligned}
&\|u(h) - u_0\|_{D((-A)^s)} \\
\leq\ & \frac{c_3\left(\|Au_0\|_{L^2(\Omega)} + \|F(u_0)\|_{L^2(\Omega)}\right)}{1-s} e^{c_3(c_0c_1k_1L|\Omega|)^{1/2}\int_0^h \frac{1}{(h-\tau)^s}d\tau} h^{1-s} \\
\leq\ & \frac{c_3\left(\|Au_0\|_{L^2(\Omega)} + \|F(u_0)\|_{L^2(\Omega)}\right)}{1-s} e^{\frac{c_3(c_0c_1k_1L|\Omega|)^{1/2}T^{1-s}}{1-s}} h^{1-s}.
\end{aligned}
$$

Then, for any $t_1, t_2 \in [0,T]$ such that $t_1 + h = t_2$,

$$
\begin{aligned}
&\|u(t_2) - u(t_1)\|_{D((-A)^s)} \\
\leq\ & \frac{c_3\left(\|Au_0\|_{L^2(\Omega)} + \|F(u_0)\|_{L^2(\Omega)}\right)}{1-s} e^{\frac{c_3(c_0c_1k_1L|\Omega|)^{1/2}T^{1-s}}{1-s}} |t_2 - t_1|^{1-s}.
\end{aligned}
$$

Hence, u is Hölder continuous of exponent $1-s$ in t.

It follows from $D(A) \hookrightarrow D((-A)^s)$ and lemma 4.3.1.7 that we obtain the local existence of a classical solution of the semilinear evolution problem (4.3.1).

Lemma 4.3.1.9 *There exists a positive constant T such that the equivalent semilinear evolution problem (4.3.1) has a unique classical solution $u(t) \in C([0,T], D((-A)^s)) \cap C^1([0,T], L^2(\Omega))$ given by*

$$u(t) = S(t)u_0 + \int_0^t S(t-\tau)F(u(\tau))d\tau$$

where $S(t)$ is an analytic semigroup generated by A and $u_0 \in D(A)$.

Proof of theorem 4.2.1 The proof of theorem 4.2.1 then follows directly from proposition 4.3.1.1.

4.3.2 The proof of theorem 4.2.2

Let $T_{\max}$ be the supremum of all T such that equivalent semilinear evolution problem (4.3.1) has an unique mild u on $[0,T]$. By modifying the proof of proposition 2.3.2.1, we have the following results.

Proposition 4.3.2.1 *Let $u_0 \in D(A)$. If $T_{\max}$ is finite, then $\|u(t)\|_{D((-A)^s)}$ is unbounded as t tends to $T_{\max}$.*

Proof. Suppose that there exists a positive constant M and a sequence $\{t_n\}$ such that

$$\|u(t_n)\|_{D((-A)^s)} \leq M \text{ as } t_n \to T_{\max}.$$

Let us consider the following semilinear evolution problem

$$\frac{dv}{dt} - Av(t) = F(v(t)) \text{ for } t > 0 \text{ and } v(0) = u(t_n). \tag{4.3.6}$$

We then have that by lemma 4.3.1.9, there exists a positive constant γ such that problem (4.3.6) has a unique mild solution v on the interval $[0,\gamma]$. We choose n large enough with $t_n + \gamma > T_{\max}$. We then define the function $\widetilde{u}$ by

$$\widetilde{u}(t) = \begin{cases} u(t) & \text{for } 0 \leq t \leq t_n, \\ v(t-t_n) & \text{for } t_n \leq t \leq t_n + \gamma. \end{cases}$$

We would like to show that $\widetilde{u}$ is a mild solution of the equivalent semilinear evolution problem (4.3.1) on $[0, t_n + \gamma]$, that is, $\widetilde{u}$ satisfies that

$$\widetilde{u}(t) = S(t)u_0 + \int_0^t S(t-\tau)F(\widetilde{u}(\tau))d\tau \text{ for } t \in [0, t_n + \gamma].$$

Clearly, $\widetilde{u}$ is a mild solution of the equivalent semilinear evolution problem (4.3.1) on $[0, t_n]$. We thus consider that for $t \in [0, \gamma]$,

$$\begin{aligned}
& \widetilde{u}(t + t_n) \\
= \ & v(t) \\
= \ & S(t+t_n)u_0 + \int_0^{t_n} S(t+t_n-\tau)F(u(\tau))d\tau + \int_0^t S(t-\tau)F(v(\tau))d\tau \\
= \ & S(t+t_n)u_0 + \int_0^{t_n} S(t+t_n-\tau)F(\widetilde{u}(\tau))d\tau + \int_{t_n}^{t+t_n} S(t+t_n-\tau)F(v(\tau-t_n))d\tau \\
= \ & S(t+t_n)u_0 + \int_0^{t_n} S(t+t_n-\tau)F(\widetilde{u}(\tau))d\tau + \int_{t_n}^{t+t_n} S(t+t_n-\tau)F(\widetilde{u}(\tau))d\tau \\
= \ & S(t+t_n)u_0 + \int_0^{t+t_n} S(t+t_n-\tau)F(\widetilde{u}(\tau))d\tau.
\end{aligned}$$

Therefore, $\widetilde{u}$ is a mild solution of the equivalent semilinear evolution problem (4.3.1) on $[t_n, t_n+\gamma]$. Hence, the proof is complete.

Proof of Theorem 4.2.2 Suppose that there is a positive constant M such that $|u(b,t)| \leq M$ for $t \in [0, T_{\max})$. By lemma 4.3.1.9, we have that

$$u(t) = S(t)u_0 + \int_0^t S(t-\tau)F(u(\tau))d\tau \text{ for any } t \in [0, T_{\max}).$$

Then, for any $t \in [0, T_{\max}]$,

$$\begin{aligned}
\|u(t)\|_{D((-A)^s)} &= \left\| S(t)u_0 + \int_0^t S(t-\tau)F(u(\tau))d\tau \right\|_{D((-A)^s)} \\
&\leq \|u_0\|_{D((-A)^s)} + \left\| \int_0^t S(t-\tau)f(u(b,\tau))d\tau \right\|_{D((-A)^s)} \\
&\leq \|u_0\|_{D((-A)^s)} + f(M) \left\| \int_0^t S(t-\tau)\cdot 1 d\tau \right\|_{D((-A)^s)} \\
&\leq \|u_0\|_{D((-A)^s)} + c_3 f(M) \int_0^t \frac{\|1\|_{L^2(\Omega)}}{(t-\tau)^s} d\tau \\
&= \|u_0\|_{D((-A)^s)} + c_3 f(M) \frac{t^{1-s}}{1-s}.
\end{aligned}$$

Then, we have that $\|u(t)\|_{D((-A)^s)}$ is bounded as $t \to T_{\max}$ which contradict to proposition 4.3.2.1.

4.3.3 The proof of theorem 4.2.3

Before proving theorem 3.2.3, we need the following lemma.

Lemma 4.3.3.1 *For any $x \in I$, there exists a positive real number $c(x)$ depending on x such that*

$$(S(t)\mathbf{1})(x) \geq c(x) \text{ for any } t \in [0, T_{\max}).$$

Proof. Let x_0 be arbitrary in I and φ a $C^\infty(\mathbb{R}^N)$ function with support valued in the ball $B(0,1)$ and such that

$$\int_{\mathbb{R}^N} \varphi(x)dx = 1.$$

Let φ_ε defined by

$$\varphi_\varepsilon(x) = \varepsilon^{-N}\varphi\left(\frac{x - x_0}{\varepsilon}\right).$$

The maximum principle yields that for any $(x,t) \in \Omega \times [0,\infty)$,

$$(S(t)\mathbf{1})(x) \geq \frac{\varepsilon^N}{|\varphi|_\infty}(S(t)\varphi_\varepsilon)(x).$$

Moreover, for any $(x,t) \in \Omega \times [0,\infty)$,

$$(S(t)\varphi_\varepsilon)(x) = \sum_{n=1}^{\infty} e^{-\lambda_n t}\langle\varphi_\varepsilon, \phi_n\rangle\phi_n(x).$$

But

$$\begin{aligned}\left|\int_\Omega \varphi_\varepsilon\phi_n dx - \phi_n(x_0)\right| &\leq \sup\{|\phi_n(x) - \phi_n(x_0)|,\ x \in B(0,1)\} \\ &\leq \varepsilon \sup\{\nabla\phi_n(x),\ x \in B(0,1)\}.\end{aligned}$$

Using the spectral theory and may be some additional properties of regularity for p and k we have a suitable power of λ_n say $(\lambda_n)^{r_N}$ (r_N because it depends on N but not on n). Hence,

$$\frac{\varepsilon^N}{|\varphi|_\infty}(S(t)\varphi_\varepsilon)(x_0) \geq \frac{\varepsilon^N}{|\varphi|_\infty}\left(\sum_{n=1}^{\infty} e^{-\lambda_n t}\phi_n^2(x_0) - \varepsilon\sum_{n=1}^{\infty} e^{-\lambda_n t}\lambda_n^{r_N+s_N}\right)$$

s_N being such that $|\phi_n| \leq c\lambda_n^{s_N}$ (such relationship exists [see spectral theory] and suitable smooth assumptions on Ω and p,k). The series $\sum_{n=1}^{\infty} e^{-\lambda_n t}\lambda_n^{r_N+s_N}$ converges because $\lambda_n \sim n^{\frac{2}{n}}$, thus the results is proven by choosing ε small enough.

Proof of Theorem 4.2.3 From theorem 3.2.2, that is, $|u(b,t)|$ is unbounded as t tends to $T_{\max}$, it means that there exists a t^* with $0 < t^* < T_{\max}$ such that $|u(b,t)| \geq M$ where M is a fixed positive constant for any $t > t^*$. Let us consider that

$$\begin{aligned}
|u(b,t)| &= |u(t)(b)| \\
&\leq |(S(t)u_0)(b)| + \int_0^t |(S(t-\tau)\mathbf{1})(b)|\,|f(u(b,\tau))|\,d\tau \\
&= |(S(t)u_0)(b)| + \int_0^{t^*} |(S(t-\tau)\mathbf{1})(b)|\,|f(u(b,\tau))|\,d\tau \\
&\quad + \int_{t^*}^t |(S(t-\tau)\mathbf{1})(b)|\,|f(u(b,\tau))|\,d\tau.
\end{aligned}$$

Locally Lipschitz continuity of f implies that there exists a positive constant L depending on M such that

$$|f(u(b,t))| \leq L\,|u(b,t)| \text{ for any } t \leq t^*$$

and then we have that

$$\begin{aligned}
|u(b,t)| &\leq |(S(t)u_0)(b)| + \int_0^{t^*} |(S(t-\tau)\mathbf{1})(b)|\,(L\,|u(b,t)|)d\tau \\
&\quad + \int_{t^*}^t |(S(t-\tau)\mathbf{1})(b)|\,|f(u(b,\tau))|\,d\tau \\
&\leq c_4 + c_5 \int_{t^*}^t |f(u(b,\tau))|\,d\tau,
\end{aligned}$$

where c_4 and c_5 are some positive constants. Then, it follows from theorem 4.2.2, we obtain that $\int_{t^*}^t |f(u(b,\tau))|\,d\tau$ is unbounded as t tends to $T_{\max}$. On the other hand, we consider that

$$\begin{aligned}
u(x,t) &= u(t)(x) \\
&= (S(t)u_0)(x) + \int_0^t (S(t-\tau)\mathbf{1})(x) f(u(b,\tau))d\tau
\end{aligned}$$

for any $(x,t) \in \Omega \times (0, T_{\max})$. From lemma 4.3.3.1, there exist two positive constant c_6 and c_7 such that

$$u(x,t) \geq c_6 + c_7 \int_0^t f(u(b,\tau))d\tau.$$

Since non-negativity of u_0 and positivity of f imply that u is nonnegative, we obtain that

$$\int_{t^*}^t f(u(b,\tau))d\tau \text{ is unbounded as } t \text{ converges to } t^*.$$

Hence, as t approaches to $T_{\max}$, the solution u of semilinear parabolic problem (4.1.1) will be blow-up at every point x in Ω.

References

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Chapter 5

Complete blow-up for a degenerate semilinear parabolic problem with a localized nonlinear term

5.1 Introduction

Let α and β be constants with $\alpha \geq 0$, $0 \leq \beta < 1$ and $\alpha+\beta \neq 0$ and let $D = (0,1)$, $\Omega_T = D \times (0,T)$ and $\overline{D}$, $\overline{\Omega}_T$ be the closure of D and Ω_T, respectively. Let us consider the following degenerate parabolic first initial-boundary value problem,

$$\left.\begin{array}{c} Lu(x,t) = f(u(x_0,t)), \text{ for } (x,t) \in \Omega_T, \\ u(x,0) = \phi(x), \text{ for } x \in \overline{D}, \\ u(0,t) = 0 = u(1,t), \text{ for } t \in (0,T], \end{array}\right\} \tag{5.1.1}$$

where $x_0 \in D$, $Lu = x^\alpha u_t - (x^\beta u_x)_x$ and u_t denotes the derivative of u with respect to t. We assume throughout this chapter that

(A) $f \in C^2([0,\infty))$ is convex with $f(0) = 0$ and $f(s) > 0$ for $s > 0$.

(B) $\phi \in C^2(\overline{D})$, ϕ is nontrivial and nonnegative, $\phi(0) = 0 = \phi(1)$, and

$$(x^\beta \phi'(x))' + f(\phi(x_0)) \geq 0 \text{ for } x \in D. \tag{5.1.2}$$

Since the coefficients of u_t, u_x, and u_{xx} may tend to 0 as x tends to 0, we can regard the equation as degenerate.

This chapter is organized as follows: in section 4.2, we show properties of eigenvalues and their corresponding eigenfunctions of (5.1.1); in section 4.3, we also give properties of the corresponding Green's function of (5.1.1) and show the existence and uniqueness of the solution of (5.1.1); in section 4.4, we give a criteria for the solution of (5.1.1) to blow up in a finite time; in the last section, we prove that the set of blow-up points is the whole interval $\overline{D}$.

5.2 Eigenvalues and eigenfunctions

Using separation of variables on the homogeneous problem corresponding to (5.1.1), we obtain the following singular eigenvalue problem:

$$(x^{\beta}g'(x))' + \lambda x^{\alpha}g(x) = 0, \text{ for } x \in D \text{ and } g(0) = 0 = g(1), \tag{5.2.1}$$

We set $g(x) = x^{\frac{1-\beta}{2}}y(x)$. Then, we obtain that

$$\begin{aligned}\frac{dg}{dx} &= \frac{d}{dx}\left(x^{\frac{1-\beta}{2}}y(x)\right)\\ &= \frac{1-\beta}{2}x^{\frac{-1-\beta}{2}}y(x) + x^{\frac{1-\beta}{2}}y'(x)\end{aligned}$$

and

$$\begin{aligned}\frac{d}{dx}\left(x^{\beta}\frac{dg}{dx}\right) &= \frac{d}{dx}\left(\frac{1-\beta}{2}x^{\frac{\beta-1}{2}}y(x) + x^{\frac{\beta+1}{2}}y'(x)\right)\\ &= -\left(\frac{1-\beta}{2}\right)^2 x^{\frac{\beta-3}{2}}y(x) + x^{\frac{\beta-1}{2}}y'(x) + x^{\frac{\beta+1}{2}}y''(x).\end{aligned}$$

Thus, equation (5.2.1) becomes

$$\left.\begin{array}{c} x^2y''(x) + xy'(x) + \left[\lambda x^{\alpha-\beta+2} - \frac{(1-\beta)^2}{4}\right]y(x) = 0 \text{ for } x \in D,\\ y(0) \text{ is finite and } y(1) = 0.\end{array}\right\}$$

Let $x = z^{\frac{2}{\alpha-\beta+2}}$. We then have

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{\alpha-\beta+2}{2}z^{\frac{\alpha-\beta}{\alpha-\beta+2}}\frac{dy}{dz}$$

and

$$\frac{d^2y}{dx^2} = \left(\frac{\alpha-\beta+2}{2}\right)^2 z^{\frac{2\alpha-2\beta}{\alpha-\beta+2}}\frac{d^2y}{dz^2} + \left(\frac{\alpha-\beta+2}{2}\right)^2\left(\frac{\alpha-\beta}{\alpha-\beta+2}\right)z^{\frac{\alpha-\beta-2}{\alpha-\beta+2}}\frac{dy}{dz}.$$

Thus, we get the following problem:

$$\left.\begin{array}{c} z^2\frac{d^2y}{dz^2} + z\frac{dy}{dz} + \left[\frac{4\lambda z^2}{(\alpha-\beta+2)^2} - \frac{(1-\beta)^2}{(\alpha-\beta+2)^2}\right]y(z) = 0 \text{ for } z \in I,\\ z(0) \text{ is finite and } z(1) = 0.\end{array}\right\} \tag{5.2.2}$$

Equation (5.2.2) is a Bessel equation. Its general solution is given by

$$y(z) = AJ_{\mu}(\omega z) + BJ_{-\mu}(\omega z),$$

or

$$g(x) = x^{(1-\beta)/2}\left\{AJ_{\mu}(\omega x^{(\alpha-\beta+2)/2}) + BJ_{-\mu}(\omega x^{(\alpha-\beta+2)/2})\right\},$$

where $\mu = \frac{1-\beta}{\alpha-\beta+2}$, $\omega = \frac{2\lambda^{1/2}}{\alpha-\beta+2}$, A and B are arbitrary constants, and J_{μ} denote the Bessel function of the first kind of order μ (> 0). Turning to the boundary conditions, at $z = 0$ leads to $B = 0$. The boundary condition at $z = 1$ gives the following equation

$$J_{\mu}(\omega) = 0. \tag{5.2.3}$$

Consequently, the appropriate eigenfunctions of (5.2.1) are

$$g_n(x) = Ax^{(1-\beta)/2}J_{\mu}(\omega_n x^{(\alpha-\beta+2)/2}),$$

where ω_n is the nth root of (5.2.3). We next use orthogonality of Bessel functions, that is,

$$\int_0^b xJ_p(k_n x)J_p(k_m x)dx = \begin{cases} \frac{1}{2}b^2 J_{p+1}^2(k_n b) & \text{if } m = n, \\ 0 & \text{if } m \neq n, \end{cases}$$

to determine value of A and to obtain the orthonormal property of g_n with the weight function x^α,

$$\int_0^1 x^\alpha g_n(x)g_m(x)dx = \begin{cases} 1 & \text{if } n = m, \\ 0 & \text{if } n \neq m. \end{cases}$$

To do so, let us consider the following:

$$\int_0^1 x^\alpha g_n^2(x)dx = A^2 \int_0^1 x^{\alpha-\beta+1} J_\mu^2(\omega_n x^{\frac{\alpha-\beta+2}{2}})dx. \tag{5.2.4}$$

Let $y = x^{\frac{\alpha-\beta+2}{2}}$. Then,

$$dy = \frac{\alpha-\beta+2}{2} x^{\frac{\alpha-\beta}{2}} dx.$$

Thus, we have that

$$\begin{aligned} \int_0^1 x^{\alpha-\beta+1} J_\mu^2(\omega_n x^{\frac{\alpha-\beta+2}{2}})dx &= \frac{2}{\alpha-\beta+2} \int_0^1 yJ_\mu^2(\omega_n y)dy \\ &= \frac{1}{\alpha-\beta+2} J_{\mu+1}^2(\omega_n). \end{aligned}$$

From (5.2.4), we obtain that

$$\int_0^1 x^\alpha g_n^2(x)dx = \frac{A^2}{\alpha-\beta+2} J_{\mu+1}^2(\omega_n). \tag{5.2.5}$$

Since the right-hand side of (5.2.5) must equal to 1, the value of A is determined by

$$A = \frac{(\alpha-\beta+2)^{1/2}}{|J_{\mu+1}(\omega_n)|}.$$

We then obtain

$$g_n(x) = \frac{(\alpha-\beta+2)^{1/2} x^{(1-\beta)/2} J_\mu\left(\omega_n x^{(\alpha-\beta+2)/2}\right)}{|J_{\mu+1}(\omega_n)|}.$$

We note that by [1], $\lambda_n = O(n^2)$ as $n \to \infty$. For convenience, we state the following properties of eigenfunctions.

Lemma 5.2.1 *For some positive constant k_0, $|g_n(x)| \leq k_0 x^{-(\alpha+\beta)/4}$ for $x \in (0,1]$.*

Proof. The asymptotic formula of $J_\mu(z)$ [2] is

$$J_\mu(z) \sim \left(\frac{2}{\pi z}\right)^{1/2} \cos\left(z - \frac{\mu\pi}{2} - \frac{\pi}{4}\right)$$

for z (> 0) sufficiently large. Thus for sufficiently large $\frac{2\lambda_n^{1/2}}{\alpha-\beta+2} x^{(\alpha-\beta+2)/2}$, we have

$$J_\mu\left(\omega_n x^{(\alpha-\beta+2)/2}\right) \leq \left(\frac{\alpha-\beta+2}{\pi\lambda_n^{1/2} x^{(\alpha-\beta+2)/2}}\right)^{1/2}.$$

It follows from [8] that we get

$$\frac{1}{|J_{\mu+1}(\omega_n)|} \leq \left(\frac{\pi\lambda_n^{1/2}}{\alpha-\beta+2}\right)^{1/2} k_0. \tag{5.2.6}$$

where k_0 is some positive constant. Then we get the result.

Lemma 5.2.2 *For some positive constant k_1, $|g_n(x)| \leq k_1 x^{(1-\beta)/2}\lambda_n^{1/4}$ for $x \in \overline{D}$.*

Proof. By the upper bounds [2], i.e.,

$$\left|J_\mu\left(\omega_n x^{(\alpha-\beta+2)/2}\right)\right| \leq 1, \text{ for any } \mu > 0.$$

and (5.2.6), we get that $|g_n(x)| \leq k_1 x^{(1-\beta)/2}\lambda_n^{1/4}$ for some constant k_1.

Lemma 5.2.3 *For any $x_1 > 0$ and for all $x \in [x_1, 1]$, there exists k_2 depending on x_1 such that $|g_n'(x)| \leq k_2\lambda_n^{1/4}$.*

Proof. Since $z = x^{(\alpha-\beta+2)/2}$, we obtain

$$g_n(z) = (\alpha-\beta+2)^{1/2} z^\mu J_\mu(\omega_n z) \Big/ |J_{\mu+1}(\omega_n)|.$$

By the property of Bessel functions

$$\frac{d}{dy}(y^\nu J_\nu(y)) = y^\nu J_{\nu-1}(y),$$

we have

$$\begin{aligned} g_n'(z) &= \frac{(\alpha-\beta+2)^{1/2}}{|J_{\mu+1}(\omega_n)|}\frac{d}{dz}(z^\mu J_\mu(\omega_n z)) \\ &= \frac{(\alpha-\beta+2)^{1/2}}{|J_{\mu+1}(\omega_n)|}\omega_n z^\mu J_{\mu-1}(\omega_n z). \end{aligned}$$

Since $z = x^{(\alpha-\beta+2)/2}$, $\omega_n = 2\lambda_n^{1/2}/(\alpha-\beta+2)$, and $\mu = (1-\beta)/(\alpha-\beta+2)$, we have

$$g_n'(x) = [(\alpha-\beta+2)\lambda_n]^{1/2} x^{(\alpha-2\beta+1)/2} J_{\mu-1}\left(\omega_n x^{(\alpha-\beta+2)/2}\right) \Big/ |J_{\mu+1}(\omega_n)|. \tag{5.2.7}$$

Since

$$\left|J_{\mu-1}\left(\omega_n x^{(\alpha-\beta+2)/2}\right)\right| \leq \left(\frac{\alpha-\beta+2}{\pi\lambda_n^{1/2}}\right)^{1/2} x^{-(\alpha-\beta+2)/4}.$$

From (5.2.6) and (5.2.7), for any $x \in [x_1, 1]$ with $x_1 > 0$

$$|g_n'(x)| \leq k_2\lambda_n^{1/4},$$

where k_2 is some positive constant. Hence we get the result.

5.3 Existence and uniqueness

Green's function $G(x, t, \xi, \tau)$ corresponding to (5.1.1) is determined by the following system for each x and ξ in D, and t and τ in $(0, T]$,

$$\left.\begin{aligned} LG(x,t,\xi,\tau) &= \delta(x-\xi)\delta(t-\tau), \\ G(x,t,\xi,\tau) &= 0, \quad \text{for } t < \tau, \\ G(0,t,\xi,\tau) &= 0 = G(1,t,\xi,\tau), \end{aligned}\right\} \tag{5.3.1}$$

where $\delta(x)$ is the Dirac delta function. By the method of eigenfunction expansion,

$$G(x,t,\xi,\tau) = \sum_{n=1}^{\infty} g_n(x) g_n(\xi) \exp\left[-\lambda_n (t-\tau)\right], \text{ for } t > \tau.$$

where λ_n and $g_n(x)$ are the eigenvalues and their corresponding eigenfunctions to (5.1.1).

We will give the following properties of $G(x,t,\xi,\tau)$.

Lemma 5.3.1 *For any $t > \tau$, $G(x,t,\xi,\tau)$ is continuous for $(x,t,\xi) \in \overline{D} \times (\tau, T] \times (0,1]$.*

Proof. By lemma 5.2.2,

$$\begin{aligned} \left| \sum_{n=1}^{\infty} g_n(x) g_n(\xi) \exp\left[-\lambda_n (t-\tau)\right] \right| &\le k_1^2 x^{(1-\beta)/2} \xi^{(1-\beta)/2} \sum_{n=1}^{\infty} \lambda_n^{1/2} \exp\left[-\lambda_n (t-\tau)\right] \\ &\le k_1^2 \sum_{n=1}^{\infty} \lambda_n^{1/2} \exp\left[-\lambda_n (t-\tau)\right], \end{aligned}$$

which converges uniformly, $G(x,t,\xi,\tau)$ is continuous for $(x,t,\xi,\tau) \in \left(\overline{D} \times (0,T]\right) \times ((0,1] \times [0,T))$.

Note that from lemma 5.3.1, the Green's function exists.

Lemma 5.3.2 *For each fixed $(\xi,\tau) \in \overline{D} \times [0,T)$, $G_t(x,t,\xi,\tau) \in C(\overline{D} \times (\tau,T])$.*

Proof. By lemma 5.2.2,

$$\begin{aligned} & \left| \sum_{n=1}^{\infty} \frac{\partial}{\partial t} g_n(x) g_n(\xi) \exp\left[-\lambda_n (t-\tau)\right] \right| \\ \le & \sum_{n=1}^{\infty} \left| g_n(x) \right| \left| g_n(\xi) \right| \lambda_n \exp\left[-\lambda_n (t-\tau)\right] \\ \le & \; k_1^2 \sum_{n=1}^{\infty} \lambda_n^{3/2} \exp\left[-\lambda_n (t-\tau)\right], \end{aligned}$$

which converges uniformly with respect to $x \in \overline{D}$ and t in any compact subset of $(\tau, T]$. This proves lemma.

Lemma 5.3.3 *For each fixed $(\xi,\tau) \in \overline{D} \times [0,T)$, $G_x(x,t,\xi,\tau)$ and $G_{xx}(x,t,\xi,\tau)$ are in $C((0,1] \times (\tau,T])$.*

Proof. By lemma 5.2.2 and 5.2.3,

$$\begin{aligned} & \left| \sum_{n=1}^{\infty} g_n'(x) g_n(\xi) \exp\left[-\lambda_n (t-\tau)\right] \right| \\ \le & \sum_{n=1}^{\infty} \left| g_n'(x) \right| \left| g_n(\xi) \right| \exp\left[-\lambda_n (t-\tau)\right] \\ \le & \; k_1 k_2 \sum_{n=1}^{\infty} \lambda_n^{1/2} \exp\left[-\lambda_n (t-\tau)\right], \end{aligned}$$

which converges uniformly with respect to x in any compact subset of $(0,1]$ and t in any compact subset of $(\tau, T]$. Thus $G_x(x,t,\xi,\tau)$ is continuous.

By equation (5.2.1) and lemma 5.2.3, we then have that for some positive constants k_3 and k_4,

$$\begin{aligned} & \left| \sum_{n=1}^{\infty} g_n''(x) g_n(\xi) \exp[-\lambda_n(t-\tau)] \right| \\ \leq & \sum_{n=1}^{\infty} \frac{\beta}{x} |g_n'(x)| \, |g_n(\xi)| \exp[-\lambda_n(t-\tau)] \\ & + \sum_{n=1}^{\infty} x^{\alpha-\beta} \lambda_n \, |g_n(x)| \, |g_n(\xi)| \exp[-\lambda_n(t-\tau)] \\ \leq & \; k_3 \sum_{n=1}^{\infty} \lambda_n^{1/2} \exp[-\lambda_n(t-\tau)] + k_4 \sum_{n=1}^{\infty} \lambda_n^{3/2} \exp[-\lambda_n(t-\tau)], \end{aligned}$$

which converges uniformly with respect to x in any compact subset of $(0, 1]$ and t in any compact subset of $(\tau, T]$. This lemma then is proved.

Lemma 5.3.4 *If $r(t)$ is a nonnegative, bounded, and continuous function on $[0, T]$, then $\int_0^t \int_0^1 G(x, t, \xi, \tau) r(\tau) d\xi d\tau$ is continuous for x in any compact subset of $(0, 1]$ and $t \in [0, T]$.*

Proof. Let ε be any positive number such that $t - \varepsilon > 0$. For x in any compact subset of $(0, 1]$, i.e., for any $x \in [x_2, 1]$ with $x_2 > 0$, and for $\tau \in [0, t-\varepsilon]$, by using lemma 5.2.1, 5.2.2 and letting $r_\infty = \max_{0 \leq \tau \leq T} r(\tau)$, we then obtain for $t > \tau$,

$$\begin{aligned} & \left| \sum_{n=1}^{\infty} g_n(x) g_n(\xi) \exp\left[-\lambda_n(t-\tau)\right] r(\tau) \right| \\ \leq & \; k_0 k_1 x_2^{-(\alpha+\beta)/4} \xi^{(1-\beta)/2} r_\infty \sum_{n=1}^{\infty} \lambda_n^{1/4} \exp\left[-\lambda_n(t-\tau)\right] \end{aligned}$$

which converges uniformly. Therefore we have

$$\int_0^{t-\varepsilon} \int_0^1 G(x, t, \xi, \tau) r(\tau) d\xi d\tau = \sum_{n=1}^{\infty} \int_0^{t-\varepsilon} \int_0^1 g_n(x) g_n(\xi) \exp\left[-\lambda_n(t-\tau)\right] r(\tau) d\xi d\tau.$$

Let us consider that

$$\begin{aligned} & \sum_{n=1}^{\infty} \int_0^{t-\varepsilon} \int_0^1 g_n(x) g_n(\xi) \exp\left[-\lambda_n(t-\tau)\right] r(\tau) d\xi d\tau \\ \leq & \; k_0 k_1 x_2^{-(\alpha+\beta)/4} r_\infty \sum_{n=1}^{\infty} \int_0^{t-\varepsilon} \int_0^1 \lambda_n^{1/4} \exp\left[-\lambda_n(t-\tau)\right] d\xi d\tau \\ \leq & \; k_0 k_1 x_2^{-(\alpha+\beta)/4} r_\infty \sum_{n=1}^{\infty} \lambda_n^{-3/4}, \end{aligned}$$

which converges (uniformly with respect to x, t, and ε) since $\lambda_n = O(n^2)$ as $n \to \infty$. Then

$$\sum_{n=1}^{\infty} \int_0^{t-\varepsilon} \int_0^1 g_n(x) g_n(\xi) \exp\left[-\lambda_n(t-\tau)\right] r(\tau) d\xi d\tau$$

converges uniformly with respect to x, t, and ε. Since the uniform convergence also holds for $\varepsilon \to 0$, it follows that

$$\sum_{n=1}^{\infty} \int_0^{t-\varepsilon} \int_0^1 g_n(x) g_n(\xi) \exp\left[-\lambda_n (t-\tau)\right] r(\tau) d\xi d\tau$$

is a continuous function of x, t, and $\varepsilon \geq 0$. Therefore

$$\int_0^t \int_0^1 G(x,t,\xi,\tau) r(\tau) d\xi d\tau = \lim_{\varepsilon \to 0} \sum_{n=1}^{\infty} \int_0^{t-\varepsilon} \int_0^1 g_n(x) g_n(\xi) \exp\left[-\lambda_n (t-\tau)\right] r(\tau) d\xi d\tau$$

is a continuous function of x in any compact subset of $(0,1]$and $t \in [0,T]$.

A proof similar to that of lemma 4.c of [7] gives the following additional property of the Green's function in the following lemma.

Lemma 5.3.5 *In the set $\{(x,t,\xi,\tau) : x$ and ξ are in $D, 0 \leq \tau < t \leq T\}$, $G(x,t,\xi,\tau) > 0$.*

To derive the integral equation of (5.1.1), let us consider the adjoint operator L^*, which is given by

$$L^* = -x^\alpha \frac{\partial}{\partial t} - \frac{\partial}{\partial x}\left(x^\beta \frac{\partial}{\partial x}\right).$$

Applying Green's second formula, we finally obtain the representation formular of (5.1.1)

$$u(x,t) = \int_0^t \int_0^1 G(x,t,\xi,\tau) f(u(x_0,\tau)) d\xi d\tau + \int_0^1 \xi^\alpha G(x,t,\xi,0)\phi(\xi) d\xi. \tag{5.3.2}$$

We state an additional property of the Green's function in the next lemma.

Lemma 5.3.6 *For each fixed $(\xi,\tau) \in \overline{D} \times [0,T)$, $\lim_{t \to \tau^+} x^\alpha G(x,t,\xi,\tau) = \delta(x-\xi)$.*

Proof. Let us consider the problem,

$$\begin{gathered} Lw(x,t,\xi,\tau) = 0 \text{ for } x, \xi \in D,\ 0 < \tau < t, \\ w(0,t,\xi,\tau) = 0 \ = w(1,t,\xi,\tau) \text{ for } 0 < \tau < t, \\ \lim_{t \to \tau^+} x^\alpha w(x,t,\xi,\tau) = \delta(x-\xi). \end{gathered}$$

From the representation formula (5.3.2),

$$\begin{aligned} w(x,t,\xi,\tau) &= \int_0^1 \zeta^\alpha G(x,t,\zeta,\tau) \zeta^{-\alpha} \delta(\zeta - \xi) d\zeta \\ &= G(x,t,\xi,\tau) \text{ for } t > \tau. \end{aligned}$$

It follows that $\lim_{t \to \tau^+} x^\alpha G(x,t,\xi,\tau) = \delta(x-\xi)$.

Next, we will give the blow-up results of the solution of (5.1.1)

Theorem 5.3.7 *There exists some $t_1 > 0$ such that the integral equation (5.3.2) has a unique nonnegative continuous solution $u \geq \phi(x)$ for x in any compact subset of $(0,1]$ and $0 \leq t \leq t_1$, and u is a nondecreasing function of t. Let t_b be the supremum of such t_1 that the integral equation (5.3.2) has a unique nonnegative continuous solution u. If t_b is finite, then $u(x_0,t)$ is unbounded as $t \to t_b$.*

Proof. Construct a sequence $\{u_n\}$ in Ω_T by $u_0(x,t) = \phi(x)$ for $n = 0, 1, 2, ...$, and consider the equation

$$Lu_{n+1}(x,t) = f(u_n(x_0,t)), \text{ for } (x,t) \in \Omega_T,$$
$$u_{n+1}(x,0) = \phi(x), \text{ for } x \in \overline{D},$$
$$u_{n+1}(0,t) = u_{n+1}(1,t) = 0, \text{ for } 0 < t \le T.$$

Claim 1. $u_n \ge u_0$ for each positive n.

We will show by using the principle of mathematical induction. By (5.1.2), we have

$$L(u_1 - u_0)(x,t) = f(\phi(x_0)) + (x^\beta \phi'(x))' \ge 0, \text{ for } (x,t) \in \Omega_T,$$
$$(u_1 - u_0)(x,0) = 0, \text{ for } x \in \overline{D},$$
$$(u_1 - u_0)(0,t) = 0 = (u_1 - u_0)(1,t), \text{ for } 0 < t \le T.$$

Maximum principle in [12] implies that $u_1 \ge u_0$ in Ω_T.

Next, we assume that for any positive n

$$\phi \le u_1 \le u_2 \le ... \le u_{n-1} \le u_n \text{ in } \Omega_T.$$

Since f is increasing and $u_{n-1} \le u_n$, we have

$$L(u_{n+1} - u_n) = f(u_n(x_0,t)) - f(u_{n-1}(x_0,t)) \ge 0, \text{ for } (x,t) \in \Omega_T,$$
$$(u_{n+1} - u_n)(x,0) = 0, \text{ for } x \in \overline{D},$$
$$(u_{n+1} - u_n)(0,t) = 0 = (u_{n+1} - u_n)(1,t), \text{ for } 0 < t < T.$$

It follows from Maximum principle that $u_{n+1} \ge u_n$ for all n. Therefore, we can conclude that, by the principle of mathematical induction, $u_n \ge \phi$ in $\overline{\Omega}_T$ for each positive n.

Claim 2. The sequence $\{u_n\}$ is a nondecreasing function of t.

Let us define the sequence $\{w_n\}$ for $n = 0, 1, 2, ...$ by

$$w_n(x,t) = u_n(x, t+h) - u_n(x,t)$$

where h is any positive number such that $0 < t + h < T$. Thus, we also have

$$w_0(x,t) = u_0(x,t+h) - u_0(x,t) = 0.$$

Let us consider the equation

$$Lw_1(x,t) = 0, \text{ for } (x,t) \in \Omega_{T-h},$$
$$w_1(x,0) \ge 0, \text{ for } x \in \overline{D}$$
$$w_1(0,t) = 0 = w_1(1,t), \text{ for } 0 < t < T - h.$$

Maximum principle yields that $w_1 \ge 0$ for Ω_{T-h}.

Let us assume that for each positive number n, $w_n \ge 0$ for Ω_T. By using the Mean Value Theorem, we obtain

$$Lw_{n+1}(x,t) = f'(u_n(x_0,t_1))w_n(x_0,t) \ge 0, \text{ in } \Omega_{T-h},$$
$$w_{n+1}(x,0) = u_{n+1}(x,h) - \phi(x) \ge 0, \text{ for } x \in \overline{D},$$
$$w_{n+1}(0,t) = w_{n+1}(1,t) = 0, \text{ for } 0 < t \le T - h.$$

for some $t_1 \in (t, t+h)$. By Maximum principle, we obtain that $w_{n+1} \ge 0$ for $\overline{\Omega}_{T-h}$. Therefore, we can conclude that, by the principle of mathematical induction, $w_n \ge 0$ in $\overline{\Omega}_T$ for each positive n, i.e. u_n is a nondecreasing function of t.

Claim 3. Before a blow-up occurs, the integral equation (5.3.2) has a unique continuous solution u.

Let us consider the following problem,

$$\left.\begin{aligned} Lv(x,t) = 0,\ \text{for } (x,t) \in \Omega_T, \\ v(x,0) = \phi(x),\ \text{for } x \in \overline{D}, \\ v(0,t) = 0 = v(1,t),\ \text{for } 0 < t \leq T. \end{aligned}\right\} \tag{5.3.3}$$

Then the solution of (5.3.3) is

$$v(x,t) = \int_0^1 x^\alpha G(x,t,\xi,0)\phi(\xi)d\xi.$$

Since the functions G and ϕ are nonnegative, we have that $v \geq 0$ in Ω_T. By the maximum principle, we know that v attains its maximum $k = \max_{x\in\overline{D}} \phi(x)$ in $\overline{D} \times \{0\}$.

For a given positive constant $M > k$, let us consider

$$u_n(x,t) = \int_0^t \int_0^1 G(x,t,\xi,\tau) f(u_{n-1}(x_0,\tau))d\xi d\tau + \int_0^1 \xi^\alpha G(x,t,\xi,0)\phi(\xi)d\xi, \tag{5.3.4}$$

as $t \to 0$, we see that

$$\lim_{t\to 0} u_n(x,t) = \int_0^1 \lim_{t\to 0} \xi^\alpha G(x,t,\xi,0)\phi(\xi)d\xi = \phi(x) < M.$$

This shows that there exists t_1 such that $u_n(x,t) \leq M$ for $0 \leq t \leq t_1$ and $n = 1,2,...$ In fact, t_1 satisfies

$$f(M) \int_0^{t_1} \int_0^1 G(x,t_1,\xi,\tau)d\xi d\tau + \int_0^1 \xi^\alpha G(x,t_1,\xi,0)\phi(\xi)d\xi \leq M.$$

Next, we denote $\lim_{n\to\infty} u_n$ by u.

Subclaim 3.1 The sequence $\{u_n\}$ converges uniformly to u for x in any compact subset of $(0,1]$ and $0 \leq t \leq t_1$.

Let us consider that for each x in any compact subset of $(0,1]$, i.e., for each $x \in [x_2, 1]$ with $x_2 > 0$ and from (5.3.4),

$$u_{n+1}(x,t) - u_n(x,t) = \int_0^t \int_0^1 G(x,t,\xi,\tau)\left(f(u_n(x_0,\tau)) - f(u_{n-1}(x_0,\tau))\right)d\xi d\tau. \tag{5.3.5}$$

Let $S_n = \max_{(x,t)\in[x_2,1]\times[0,t_1]} |u_n(x,t) - u_{n-1}(x,t)|$. Using the Mean Value Theorem, we have

$$f(u_n(x_0,\tau)) - f(u_{n-1}(x_0,\tau)) = f'(\mu)(u_n(x_0,\tau) - u_{n-1}(x_0,\tau)),$$

where μ is between $u_n(x_0,\tau)$ and $u_{n-1}(x_0,\tau)$. Since $u_n \leq M$ for all n and $f''(s) > 0$ for $s > 0$, we have

$$f(u_n(x_0,\tau)) - f(u_{n-1}(x_0,\tau)) \leq f'(M)S_n.$$

From (5.3.5), we obtain

$$
\begin{aligned}
S_{n+1} &\leq f'(M) S_n \int_0^t \int_0^1 \sum_{i=1}^{\infty} g_i(x) g_i(\xi) \exp\left[-\lambda_i (t-\tau)\right] d\xi d\tau \\
&\leq k_0 k_1 x_2^{-(\alpha+\beta)/4} f'(M) S_n \int_0^t \int_0^1 \sum_{i=1}^{\infty} \lambda_i^{1/4} \exp\left[-\lambda_i (t-\tau)\right] d\xi d\tau \\
&\leq k_0 k_1 x_2^{-(\alpha+\beta)/4} f'(M) S_n \sum_{i=1}^{\infty} \lambda_i^{-3/4} (1 - \exp(-\lambda_i t)).
\end{aligned}
$$

Since $\sum_{i=1}^{\infty} \lambda_i^{-3/4}(1-\exp(-\lambda_i t))$ converges uniformly because of $\lambda_i = O(i^2)$ as $i \to \infty$, we have $\lim_{t\to 0} \sum_{i=1}^{\infty} \lambda_i^{-3/4}(1-\exp(-\lambda_i t)) = 0$. Hence, there exists some positive $\sigma_1 > 0$ such that

$$
k_0 k_1 x_2^{-(\alpha+\beta)/4} f'(M) S_n \sum_{i=1}^{\infty} \lambda_i^{-3/4} (1 - \exp(-\lambda_i t)) < 1 \text{ for } t \in [0, \sigma_1]. \tag{5.3.6}
$$

Thus, $S_{n+1} < S_n$ and the sequence $\{u_i\}$ converges uniformly to u for x in any compact subset of $(0,1]$ and $0 \leq t \leq \sigma_1$.

Similarly for $\sigma_1 \leq t \leq t_1$, we replace $\phi(\xi)$ in the integral equation (5.3.4) by $u(\xi, \sigma_1)$ to obtain

$$
u_n(x,t) = \int_{\sigma_1}^t \int_0^1 G(x,t,\xi,\tau) f(u_{n-1}(x_0,\tau)) d\xi d\tau + \int_0^1 \xi^{\alpha} G(x,t,\xi,0) u(\xi,\sigma_1) d\xi.
$$

Moreover, we also have

$$
u_{n+1}(x,t) - u_n(x,t) = \int_{\sigma_1}^t \int_0^1 G(x,t,\xi,\tau) \left(f(u_n(x_0,\tau)) - f(u_{n-1}(x_0,\tau)) \right) d\xi d\tau.
$$

and

$$
\begin{aligned}
S_{n+1} &\leq f'(M) S_n \int_{\sigma_1}^t \int_0^1 G(x,t,\xi,\tau) d\xi d\tau \\
&\leq k_0 k_1 x_2^{-(\alpha+\beta)/4} f'(M) S_n \left[\sum_{i=1}^{\infty} \lambda_i^{-3/4} \left[1 - \exp(-\lambda_i (t - \sigma_1))\right] \right].
\end{aligned}
$$

Thus there exists $\sigma_2 = \min\{\sigma_1, t_1 - \sigma_1\} > 0$ such that for $t \in [\sigma_1, \min\{2\sigma_1, t_1\}]$.

$$
k_0 k_1 x_2^{-(\alpha+\beta)/4} f'(M) \left[\sum_{i=1}^{\infty} \lambda_i^{-3/4} \left[1 - \exp(-\lambda_i (t - \sigma_1))\right] \right] < 1, \tag{5.3.7}
$$

Hence the sequence $\{u_n\}$ converges uniformly to u for x in any compact subset of $(0,1]$ and $t \in [\sigma_1, \min\{2\sigma_1, t_1\}]$. By proceeding in this way the sequence $\{u_n\}$ converges uniformly to u for x in any compact subset of $(0,1]$ and $0 \leq t \leq t_1$. Therefore we can conclude that the integral equation (5.3.2) has a continuous solution u for x in any compact subset of $(0,1]$ and $0 \leq t \leq t_1$.

To prove claim 3, we suppose that the integral equation (5.3.2) has two distinct solution u and $\widetilde{u}$ for x in any compact subset of $(0,1]$ and $t \in [0,t_1]$. Let $\Phi = \max_{(x,t)\in[x_2,1]\times[0,t_1]} |u-\widetilde{u}| > 0$. Since u and $\widetilde{u}$ are the solution of (5.3.2),

$$u(x,t)-\widetilde{u}(x,t) = \int_0^t\int_0^1 G(x,t,\xi,\tau)\left[f(u(x_0,\tau))-f(\widetilde{u}(x_0,\tau))\right]d\xi d\tau.$$

Then

$$\Phi \leq k_0k_1x_2^{-(\alpha+\beta)/4}f'(M)\left[\sum_{i=1}^{\infty}\lambda_i^{-3/4}(1-\exp(-\lambda_i t))\right]\Phi, \text{ for } t\in[0,\sigma_1],$$

which implies that

$$k_0k_1x_2^{-(\alpha+\beta)/4}f'(M)\left[\sum_{i=1}^{\infty}\lambda_i^{-3/4}(1-\exp(-\lambda_i t))\right] \geq 1, \text{ for } t\in[0,\sigma_1].$$

We have a contradiction to (5.3.6). Hence, the solution u is unique for x in any compact subset of $(0,1]$ and $0 \leq t \leq \sigma_1$.

We can show in a similar fashion that solution u is unique for x in any compact subset of $(0,1]$ and $\sigma_1 \leq t \leq \min\{2\sigma_1, t_1\}$. By proceeding in this way, the integral equation u is unique continuous for x in any compact subset of $(0,1]$ and $0 \leq t \leq t_1$. Therefore we conclude that since u_n is a nondecreasing function of t, u is a nondecreasing function of t.

Let t_b be the supremum of such t_1 that the integral equation (5.3.2) has a unique continuous solution u. We would like to show that if t_b is finite, then $u(x_0,t)$ is unbounded as $t \to t_b$.

Suppose that $u(x_0,t)$ is bounded in $[0,t_b)$.We consider the integral equation of the solution u for $[t_b,T)$ with the initial condition $u(x,0)$ replaced by $u(x,t_b)$,

$$u(x_0,t) = \int_{t_b}^t\int_0^1 G(x_0,t,\xi,\tau)f(u(x_0,\tau))d\xi d\tau + \int_0^1 \xi^\alpha G(x_0,t,\xi,t_b)u(\xi,t_b)d\xi.$$

For any positive constant $N > u(x_0,t_b)$, an argument as before shows that there exists some positive t_2 such that the integral equation (5.3.2) has the unique continuous solution u on $[t_b,t_2]$. This contradicts to the definition of t_b. Therefore If t_b is finite, then $u(x_0,t)$ is unbounded in $[0,t_b)$.

The following theorem show that u is the solution of (5.1.1).

Theorem 5.3.8 *Before blow-up occurs, the problem (5.1.1) has a unique solution u.*

Proof. By lemma 5.3.4, we have that for any $x \in D$ and any $t_2 \in (0, t)$

$$
\begin{aligned}
& \int_0^t \int_0^1 x^\alpha G(x,t,\xi,\tau) f(u(x_0,\tau)) d\xi d\tau \\
= & \int_0^{t_2} \int_0^1 x^\alpha G(x,t_2,\xi,\tau) f(u(x_0,\tau)) d\xi d\tau \\
& + \lim_{n\to\infty} \int_{t_2}^t \frac{\partial}{\partial \zeta} \int_0^{\zeta-1/n} \int_0^1 x^\alpha G(x,\zeta,\xi,\tau) f(u(x_0,\tau)) d\xi d\tau d\zeta \\
= & \int_0^{t_2} \int_0^1 x^\alpha G(x,t_2,\xi,\tau) f(u(x_0,\tau)) d\xi d\tau \\
& + \lim_{n\to\infty} \int_{t_2}^t \int_0^{\zeta-1/n} \int_0^1 x^\alpha G_\zeta(x,\zeta,\xi,\tau) f(u(x_0,\tau)) d\xi d\tau d\zeta + \int_{t_2}^t f(u(x_0,\zeta)) d\zeta.
\end{aligned}
$$

Let

$$
g_n(x,\zeta) = \int_0^{\zeta-1/n} \int_0^1 x^\alpha G_\zeta(x,\zeta,\xi,\tau) f(u(x_0,\tau)) d\xi d\tau.
$$

Without loss of generality, let $n > m$, thus we have

$$
g_n(x,\zeta) - g_m(x,\zeta) = \int_{\zeta-1/m}^{\zeta-1/n} \int_0^1 x^\alpha G_\zeta(x,\zeta,\xi,\tau) f(u(x_0,\tau)) d\xi d\tau.
$$

Since $x^\alpha G_t(x,t,\xi,\tau) \in C(\overline{D} \times (\tau, T])$ and $f(u(x_0,\tau))$ is a monotone function of τ, it follows from Second Mean Value Theorem for Integration and then we have that for any x and $\xi \in D$ and any τ in any compact subset $[t_3, t_4]$ of $(0, t_b)$, there exists some real number υ such that $\zeta - \nu \in (\zeta - 1/m, \zeta - 1/n)$ and

$$
\begin{aligned}
g_n(x,\zeta) - g_m(x,\zeta) &= f(u(x_0,\zeta-1/m)) \int_{\zeta-1/m}^{\zeta-\nu} \int_0^1 x^\alpha G_\zeta(x,\zeta,\xi,\tau) d\xi d\tau \\
& \quad + f(u(x_0,\zeta-1/n)) \int_{\zeta-\nu}^{\zeta-1/n} \int_0^1 x^\alpha G_\zeta(x,\zeta,\xi,\tau) d\xi d\tau.
\end{aligned}
$$

Since $G_\zeta(x,\zeta,\xi,\tau) = -G_\tau(x,\zeta,\xi,\tau)$, we have

$$\begin{aligned}
& g_n(x,\zeta) - g_m(x,\zeta) \\
= \; & [f(u(x_0,\zeta-1/n)) - f(u(x_0,\zeta-1/m))] \int_0^1 x^\alpha G(x,\zeta,\xi,\zeta-\nu)d\xi \\
& + f(u(x_0,\zeta-1/m)) \int_0^1 x^\alpha G(x,\zeta,\xi,\zeta-1/m)d\xi \\
& - f(u(x_0,\zeta-1/n)) \int_0^1 x^\alpha G(x,\zeta,\xi,\zeta-1/n)d\xi.
\end{aligned}$$

Since $\int_0^1 x^\alpha G(x,\zeta,\xi,\zeta-\varepsilon)d\xi = 1$ as $\varepsilon \to 0$, it follow that, the sequence $\{g_n\}$ is a Cauchy sequence, and hence the sequence $\{g_n\}$ converges uniformly with respect to ζ in any compact subset $[t_3,t_4]$ of $(0,t_b)$. Then we obtain that

$$\begin{aligned}
& \lim_{n\to\infty} \int_{t_2}^{t} \int_0^{\zeta-1/n} \int_0^1 x^\alpha G_\zeta(x,\zeta,\xi,\tau) f(u(x_0,\tau))d\xi d\tau d\zeta \\
= \; & \int_{t_2}^{t} \lim_{n\to\infty} \int_0^{\zeta-1/n} \int_0^1 x^\alpha G_\zeta(x,\zeta,\xi,\tau) f(u(x_0,\tau))d\xi d\tau d\zeta \\
= \; & \int_{t_2}^{t} \int_0^{\zeta} \int_0^1 x^\alpha G_\zeta(x,\zeta,\xi,\tau) f(u(x_0,\tau))d\xi d\tau d\zeta.
\end{aligned}$$

Thus, we obtain that

$$\begin{aligned}
& \frac{\partial}{\partial t} \int_0^t \int_0^1 x^\alpha G(x,t,\xi,\tau) f(u(x_0,\tau))d\xi d\tau \\
= \; & \frac{\partial}{\partial t} \left(\int_0^{t_2} \int_0^1 x^\alpha G(x,t_2,\xi,\tau) f(u(x_0,\tau))d\xi d\tau + \int_{t_2}^{t} f(u(x_0,\zeta))d\zeta \right. \\
& \left. + \int_{t_2}^{t} \int_0^{\zeta} \int_0^1 x^\alpha G_\zeta(x,\zeta,\xi,\tau) f(u(x_0,\tau))d\xi d\tau d\zeta \right) \\
= \; & f(u(x_0,t)) + \int_0^t \int_0^1 x^\alpha G_t(x,t,\xi,\tau) f(u(x_0,\tau))d\xi d\tau.
\end{aligned}$$

We would like to show that by using the Leibnitz rule, we have for any $x \in D$ and t in any compact subset $[t_3,t_4]$ of $(0,t_b)$,

$$\frac{\partial}{\partial x}\int_0^{t-\varepsilon}\int_0^1 G(x,t,\xi,\tau)f(u(x_0,\tau))d\xi d\tau = \int_0^{t-\varepsilon}\int_0^1 G_x(x,t,\xi,\tau)f(u(x_0,\tau))d\xi d\tau,$$
$$\frac{\partial}{\partial x}\int_0^{t-\varepsilon}\int_0^1 x^\beta G_x(x,t,\xi,\tau)f(u(x_0,\tau))d\xi d\tau = \int_0^{t-\varepsilon}\int_0^1 (x^\beta G_x(x,t,\xi,\tau))_x f(u(x_0,\tau))d\xi d\tau.$$

Let us consider that for any $x_1 \in D$,

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_0^{t-\varepsilon}\int_0^1 G(x,t,\xi,\tau)f(u(x_0,\tau))d\xi d\tau \\
&= \lim_{\varepsilon\to 0}\int_{x_1}^{x}\left[\frac{\partial}{\partial\eta}\int_0^{t-\varepsilon}\int_0^1 G(\eta,t,\xi,\tau)f(u(x_0,\tau))d\xi d\tau\right]d\eta \\
&\quad +\lim_{\varepsilon\to 0}\int_0^{t-\varepsilon}\int_0^1 G(x_1,t,\xi,\tau)f(u(x_0,\tau))d\xi d\tau \\
&= \lim_{\varepsilon\to 0}\int_{x_1}^{x}\int_0^{t-\varepsilon}\int_0^1 G_\eta(\eta,t,\xi,\tau)f(u(x_0,\tau))d\xi d\tau d\eta \\
&\quad +\int_0^{t}\int_0^1 G(x_1,t,\xi,\tau)f(u(x_0,\tau))d\xi d\tau.
\end{aligned}$$

Claim that

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_{x_1}^{x}\int_0^{t-\varepsilon}\int_0^1 G_\eta(\eta,t,\xi,\tau)f(u(x_0,\tau))d\xi d\tau d\eta \\
&= \int_{x_1}^{x}\int_0^{t}\int_0^1 G_\eta(\eta,t,\xi,\tau)f(u(x_0,\tau))d\xi d\tau d\eta. \qquad (5.3.8)
\end{aligned}$$

By using the Fubini Theorem

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_{x_1}^{x}\int_0^{t-\varepsilon}\int_0^1 G_\eta(\eta,t,\xi,\tau)f(u(x_0,\tau))d\xi d\tau d\eta \\
&= \lim_{\varepsilon\to 0}\int_0^{t-\varepsilon}\left(\int_{x_1}^{x}\int_0^1 G_\eta(\eta,t,\xi,\tau)d\xi d\eta\right)f(u(x_0,\tau))d\tau \\
&= \lim_{\varepsilon\to 0}\int_0^{t-\varepsilon}\left[\int_0^1 G(x,t,\xi,\tau)d\xi - \int_0^1 G(x_1,t,\xi,\tau)d\xi\right]f(u(x_0,\tau))d\tau \\
&= \int_0^{t}\int_0^1 G(x,t,\xi,\tau)f(u(x_0,\tau))d\xi d\tau - \int_0^{t}\int_0^1 G(x_1,t,\xi,\tau)f(u(x_0,\tau))d\xi d\tau,
\end{aligned}$$

which exists because of lemma 5.3.4. Therefore we have

$$
\begin{aligned}
&\int_0^t \int_0^1 G(x,t,\xi,\tau) f(u(x_0,\tau)) d\xi d\tau - \int_0^t \int_0^1 G(x_1,t,\xi,\tau) f(u(x_0,\tau)) d\xi d\tau \\
= &\int_{x_1}^x \frac{\partial}{\partial \eta} \left[\int_0^t \int_0^1 G(\eta,t,\xi,\tau) f(u(x_0,\tau)) d\xi d\tau \right] d\eta \\
= &\int_{x_1}^x \int_0^t \int_0^1 G_\eta(\eta,t,\xi,\tau) f(u(x_0,\tau)) d\xi d\tau d\eta.
\end{aligned}
$$

Thus, we have (5.3.8). Therefore we also have

$$
\begin{aligned}
&\frac{\partial}{\partial x} \int_0^t \int_0^1 G(x,t,\xi,\tau) f(u(x_0,\tau)) d\xi d\tau \\
= &\frac{\partial}{\partial x} \left[\int_{x_1}^x \int_0^t \int_0^1 G_\eta(\eta,t,\xi,\tau) f(u(x_0,\tau)) d\xi d\tau d\eta \right. \\
&\left. + \int_0^t \int_0^1 G(x_1,t,\xi,\tau) f(u(x_0,\tau)) d\xi d\tau \right] \\
= &\int_0^t \int_0^1 G_x(x,t,\xi,\tau) f(u(x_0,\tau)) d\xi d\tau.
\end{aligned}
$$

Let us consider that for any $x_2 \in D$,

$$
\begin{aligned}
&\lim_{\varepsilon \to 0} \int_0^{t-\varepsilon} \int_0^1 x^\beta G_x(x,t,\xi,\tau) f(u(x_0,\tau)) d\xi d\tau \\
= &\lim_{\varepsilon \to 0} \int_{x_2}^x \frac{\partial}{\partial \eta} \left[\int_0^{t-\varepsilon} \int_0^1 \eta^\beta G_\eta(\eta,t,\xi,\tau) f(u(x_0,\tau)) d\xi d\tau \right] d\eta \\
&+ \lim_{\varepsilon \to 0} \int_0^{t-\varepsilon} \int_0^1 x_2^\beta G_\eta(x_2,t,\xi,\tau) f(u(x_0,\tau)) d\xi d\tau \\
= &\lim_{\varepsilon \to 0} \int_{x_2}^x \int_0^{t-\varepsilon} \int_0^1 (\eta^\beta G_\eta(\eta,t,\xi,\tau))_\eta f(u(x_0,\tau)) d\xi d\tau d\eta \\
&+ \int_0^t \int_0^1 x_2^\beta G_\eta(x_2,t,\xi,\tau) f(u(x_0,\tau)) d\xi d\tau. \qquad (5.3.9)
\end{aligned}
$$

Claim that

$$\lim_{\varepsilon\to 0}\int_{x_2}^{x}\int_{0}^{t-\varepsilon}\int_{0}^{1}(\eta^\beta G_\eta(\eta,t,\xi,\tau))_\eta f(u(x_0,\tau))d\xi d\tau d\eta$$

$$= \int_{x_2}^{x}\int_{0}^{t}\int_{0}^{1}(\eta^\beta G_\eta(\eta,t,\xi,\tau))_\eta f(u(x_0,\tau))d\xi d\tau d\eta. \quad (5.3.10)$$

By using the Fubini Theorem

$$\lim_{\varepsilon\to 0}\int_{x_2}^{x}\int_{0}^{t-\varepsilon}\int_{0}^{1}(\eta^\beta G_\eta(\eta,t,\xi,\tau))_\eta f(u(x_0,\tau))d\xi d\tau d\eta$$

$$= \lim_{\varepsilon\to 0}\int_{0}^{t-\varepsilon}\left(\int_{x_2}^{x}\int_{0}^{1}(\eta^\beta G_\eta(\eta,t,\xi,\tau))_\eta d\xi d\eta\right)f(u(x_0,\tau))d\tau$$

$$= \int_{0}^{t}\int_{0}^{1}x^\beta G_x(x,t,\xi,\tau)f(u(x_0,\tau))d\xi d\tau - \int_{0}^{1}x_2^\beta G_x(x_2,t,\xi,\tau)f(u(x_0,\tau))d\xi d\tau,$$

which exists because of lemma 5.3.3. Therefore we have

$$\int_{0}^{t}\int_{0}^{1}x^\beta G_x(x,t,\xi,\tau)f(u(x_0,\tau))d\xi d\tau - \int_{0}^{1}x_2^\beta G_x(x_2,t,\xi,\tau)f(u(x_0,\tau))d\xi d\tau$$

$$= \int_{x2}^{x}\frac{\partial}{\partial\eta}\left[\int_{0}^{t}\int_{0}^{1}\eta^\beta G_\eta(\eta,t,\xi,\tau)f(u(x_0,\tau))d\xi d\tau\right]d\eta$$

$$= \int_{x_2}^{x}\int_{0}^{t}\int_{0}^{1}(\eta^\beta G_\eta(\eta,t,\xi,\tau))_\eta f(u(x_0,\tau))d\xi d\tau d\eta.$$

Thus, we have (5.3.10). Therefore we also have

$$\frac{\partial}{\partial x}\left[\int_{0}^{t}\int_{0}^{1}x^\beta G_x(x,t,\xi,\tau)f(u(x_0,\tau))d\xi d\tau\right]$$

$$= \frac{\partial}{\partial x}\left[\int_{x_2}^{x}\int_{0}^{t}\int_{0}^{1}(\eta^\beta G_\eta(\eta,t,\xi,\tau))_\eta f(u(x_0,\tau))d\xi d\tau d\eta\right.$$

$$\left.+\int_{0}^{t}\int_{0}^{1}x_2^\beta G_x(x_2,t,\xi,\tau)f(u(x_0,\tau))d\xi d\tau\right]$$

$$= \int_{0}^{t}\int_{0}^{1}(x^\beta G_x(x,t,\xi,\tau))_x f(u(x_0,\tau))d\xi d\tau,$$

for any x in D and t in any compact subset $[t_3,t_4]$ of $(0,t_b)$. By using the Leibnitz rule, we have that for any x in D and t in any compact subset $[t_3, t_4]$ of $(0, t_b)$,

$$\begin{aligned} x^\alpha \frac{\partial}{\partial t}\int_0^1 \xi^\alpha G(x,t,\xi,0)\phi(\xi)d\xi &= x^\alpha \int_0^1 \xi^\alpha G_t(x,t,\xi,0)\phi(\xi)d\xi, \\ \frac{\partial}{\partial x}\int_0^1 \xi^\alpha G(x,t,\xi,0)\phi(\xi)d\xi &= \int_0^1 \xi^\alpha G_x(x,t,\xi,0)\phi(\xi)d\xi, \\ \frac{\partial}{\partial x}\int_0^1 \xi^\alpha \left(x^\beta G_x(x,t,\xi,0)\right)\phi(\xi)d\xi &= \int_0^1 \xi^\alpha (x^\beta G_x(x,t,\xi,0))_x \phi(\xi)d\xi. \end{aligned}$$

From the integral equation (5.3.2), for $x \in D$ and $0 < t < T$

$$\begin{aligned} & Lu(x,t) \\ = \; & f(u(x_0,t)) + \int_0^1 \xi^\alpha \left[x^\alpha G_t(x,t,\xi,0) - (x^\beta G_x(x,t,\xi,0))_x\right]\phi(\xi)d\xi \\ & + \int_0^t \int_0^1 \left[x^\alpha G_t(x,t,\xi,\tau) - (x^\beta G_x(x,t,\xi,\tau))_x\right] f(u(x_0,\tau))d\xi d\tau \\ = \; & f(u(x_0,t)) + \delta(t)\int_0^1 \xi^\alpha \delta(x-\xi)\phi(\xi)d\xi \\ & + \lim_{\varepsilon\to 0}\int_0^{t-\varepsilon}\int_0^1 \delta(x-\xi)\delta(t-\tau)f(u(x_0,\tau))d\xi d\tau \\ = \; & f(u(x_0,t)), \end{aligned}$$

and the initial condition of u can compute from for $x \in \overline{D}$

$$\lim_{t\to 0} u(x,t) = \int_0^1 \lim_{t\to 0} \xi^\alpha G(x,t,\xi,0)\phi(\xi)d\xi = \phi(x),$$

since $G(0,t,\xi,\tau) = 0 = G(1,t,\xi,\tau)$, we can compute directly and obtain the boundary condition $u(0,t) = 0 = u(1,t)$. Therefore u is a solution of the problem (5.1.1).

5.4 A sufficient condition to blow-up in finite time

In this section, we will give a blow-up criterion for the solution u to blow-up in a finite time. Let us denoted by $\lambda_1 > 0$ the principal (smallest) eigenvalue of the problem

$$(x^\beta \psi_1'(x))' = -\lambda_1 x^\alpha \psi_1(x) \text{ for } x \in D, \text{ and } \psi_1(0) = 0 = \psi_1(1) \tag{5.4.1}$$

and by $\psi_1(x)$ the corresponding (first) eigenfunction. Let $\psi_1(x) > 0$ be normalized such that

$$\|\psi_1\| = \int_0^1 x^\alpha \psi_1(x)dx = 1.$$

Since $\lim\limits_{s\to\infty} \frac{f(s)}{s} \to \infty$, there is a $z_0 > 0$ such that $f(s) - \lambda_1 s > 0$ for each $s > z_0$.

Proposition 5.4.1 *Suppose that* $\int\limits_{z_0}^{\infty} \frac{ds}{f(s)-\lambda_1 s}$ *is finite. Let us consider the following initial-boundary value problem,*

$$\left.\begin{array}{c} Lu(x,t) = f(u),\ \textit{for}\ (x,t) \in \Omega_T, \\ u(x,0) = u_0(x) \geq 0,\ \textit{for}\ x \in \overline{D}, \\ u(0,t) = 0 = u(1,t),\ \textit{for}\ 0 < t \leq T. \end{array}\right\} \tag{5.4.2}$$

If $\int\limits_0^1 x^\alpha u_0(x)\psi_1(x)dx \geq z_0$*, then the solution* u *of (5.4.2) blows up in finite time.*

Proof. Assume that the solution u of problem (5.4.2) exists for all t. Construct the function $E(t)$ by

$$E(t) = \int\limits_0^1 x^\alpha u(x,t)\psi_1(x)dx.$$

Then $E(0) = \int\limits_0^1 x^\alpha u_0(x)\psi_1(x)dx \geq z_0$ which means that the initial condition $u_0(x)$ have to be sufficiently large. Multiplying $Lu(x,t) = f(u)$ by ψ_1 and integrating from 0 to 1, we have

$$\frac{dE}{dt} = -\lambda_1 E(t) + \int\limits_0^1 f(u)\psi_1(x)dx \tag{5.4.3}$$

Applying Jensen's inequality to the second term in the right hand side, we then obtain

$$\int\limits_0^1 f(u)\psi_1(x)dx \geq \int\limits_0^1 x^\alpha f(u)\psi_1(x)dx \geq f\left(E(t)\right). \tag{5.4.4}$$

From (5.4.3) and (5.4.4), we obtain that

$$\frac{dE}{dt} \geq -\lambda_1 E(t) + f\left(E(t)\right) > 0.$$

Thus we obtain that

$$t \leq \int\limits_{E(0)}^{E(t)} \frac{ds}{f(s) - \lambda_1 s} \leq \int\limits_{z_0}^{E(t)} \frac{ds}{f(s) - \lambda_1 s} < \infty \tag{5.4.5}$$

which contradicts to assumption that the solution u of problem (5.4.2) exists for all t. Therefore, the solution $u(x,t)$ of (5.4.2) blows up in a finite time.

In order to obtain the sufficient condition to blow-up in finite time of (5.1.1), we need the following lemma.

Lemma 5.4.2 *Let* $u(x,t)$ *be a classical solution of the following problem*

$$\begin{array}{c} Lu(x,t) \geq b(x,t)u(x_0,t),\ \textit{for}\ (x,t) \in \Omega_T, \\ u(x,0) \geq 0,\ \textit{for}\ x \in \overline{D}, \\ u(0,t) \geq 0\ \textit{and}\ u(1,t) \geq 0,\ \textit{for}\ 0 < t \leq T, \end{array}$$

where $b(x,t)$ *is nonnegative and bounded on* $\overline{\Omega}_T$*, then* $u(x,t) \geq 0$ *in* $\overline{\Omega}_T$.

Proof. If $b(x,t) \equiv 0$, then by the weak maximum principle u attains its minimum on the parabolic boundary, i.e., $u(x,t) \geq 0$ in $\overline{\Omega}_T$.

For the case $b(x,t)$ being nonnegative and nontrivial, let $\beta' \in (\beta, 1)$ be a positive constant and

$$v(x,t) = u(x,t) + \eta(1 + x^{\beta'-\beta})e^{ct},$$

where $\eta > 0$ is sufficiently small and c is a positive constant to be determined. Then $v(x,t) > 0$ on the parabolic boundary $\partial\Omega_T$, and

$$\begin{aligned} & Lv(x,t) - b(x,t)v(x_0,t) \\ \geq\ & x^\alpha c\eta(1 + x^{\beta'-\beta})e^{ct} + \frac{\eta(\beta'-\beta)(1-\beta')e^{ct}}{x^{2-\beta'}} - \eta(1 + x_0^{\beta'-\beta})e^{ct}b(x,t) \\ \geq\ & \eta e^{ct}\left[cx^\alpha + \frac{(\beta'-\beta)(1-\beta')}{x^{2-\beta'}} - (1 + x_0^{\beta'-\beta}) \max_{(x,t)\in\overline{\Omega}_T} b(x,t)\right]. \end{aligned} \tag{5.4.6}$$

If $\max_{(x,t)\in\overline{\Omega}_T} b(x,t) \leq (\beta'-\beta)(1-\beta')/(1 + x_0^{\beta'-\beta})$, then from (5.4.6)

$$\begin{aligned} & Lv(x,t) - b(x,t)v(x_0,t) \\ >\ & \eta e^{ct}\left[\frac{(\beta'-\beta)(1-\beta')}{x^{2-\beta'}} - (1 + x_0^{\beta'-\beta}) \max_{(x,t)\in\overline{\Omega}_T} b(x,t)\right] \\ \geq\ & 0. \end{aligned}$$

On the other hand, assume that $\max_{(x,t)\in\overline{\Omega}_T} b(x,t) > (\beta'-\beta)(1-\beta')/(1 + x_0^{\beta'-\beta})$. Let s be the positive root of the algebraic equation

$$(1 + x_0^{\beta'-\beta}) \max_{(x,t)\in\overline{\Omega}_T} b(x,t) = (\beta'-\beta)(1-\beta')/x^{2-\beta'},$$

and let $c > 0$ be sufficiently large such that

$$c > (1 + x_0^{\beta'-\beta}) \max_{(x,t)\in\overline{\Omega}_T} b(x,t)/s^\alpha.$$

Then if $x \leq s$, then from (5.4.6)

$$\begin{aligned} & Lv(x,t) - b(x,t)v(x_0,t) \\ >\ & \eta e^{ct}\left[\frac{(\beta'-\beta)(1-\beta')}{x^{2-\beta'}} - (1 + x_0^{\beta'-\beta}) \max_{(x,t)\in\overline{\Omega}_T} b(x,t)\right] \\ \geq\ & \eta e^{ct}\left[\frac{(\beta'-\beta)(1-\beta')}{s^{2-\beta'}} - (1 + x_0^{\beta'-\beta}) \max_{(x,t)\in\overline{\Omega}_T} b(x,t)\right] \\ =\ & 0. \end{aligned}$$

On the other hand, if $x > s$, then from (5.4.6)

$$\begin{aligned} & Lv(x,t) - b(x,t)v(x_0,t) \\ >\ & \eta e^{ct}\left[cx^\alpha - (1 + x_0^{\beta'-\beta}) \max_{(x,t)\in\overline{\Omega}_T} b(x,t)\right] \\ >\ & \eta e^{ct}\left[(1 + x_0^{\beta'-\beta}) \max_{(x,t)\in\overline{\Omega}_T} b(x,t)((x/s)^\alpha - 1)\right] \\ \geq\ & 0. \end{aligned}$$

Therefore we have

$$Lv(x,t) - b(x,t)v(x_0,t) > 0 \text{ for } (x,t) \in \Omega_T. \tag{5.4.7}$$

We would like to show that $v(x,t) > 0$ in Ω_T, Suppose not , i.e., $v(x,t) \leq 0$ in Ω_T. We define the set

$$A = \{t : v(x,t) \leq 0 \text{ for some } x \in D\},$$

is non-empty. Let $\bar{t}$ denote its infimum. Then there exists some $x_1 \in D$ such that $v(x_1,\bar{t}) = 0$, $v_t(x_1,\bar{t}) \leq 0$, and $v_x(x_1,\bar{t}) = 0$. Since $\bar{t}$ is the infimum of the set A, we have that $v(x,t) > 0$ for $t < \bar{t}$ and by using the continuity of the function v we also have that $v(x,\bar{t}) \geq 0$ for all x. Since we have that $v(x_1,\bar{t}) = 0$, we obtain that $v(x_1,\bar{t})$ is local minimum. This means that $v_{xx}(x_1,\bar{t}) \geq 0$. Therefore we have

$$0 \geq x_1^\alpha v_t(x_1,\bar{t}) \geq Lv(x_1,\bar{t}) - b(x_1,\bar{t})v(x_0,\bar{t}) > 0,$$

which contradicts to (5.4.7). As $\eta \to 0^+$, $u(x,t) \geq 0$ in Ω_T.

The following theorem gives a sufficient condition for the solution u to blow-up in a finite time.

Theorem 5.4.3 *If $\phi(x)$ is sufficiently large in a neighborhood of x_0, then the solution of (5.1.1) blows up in a finite time.*

Proof. Let us consider the following problem,

$$\left.\begin{array}{c} Lv(x,t) = f(v), \text{ for } (x,t) \in (x_0-\delta, x_0+\delta) \times (0,T], \\ v(x,0) = v_0(x) \geq 0, \text{ for } x \in [x_0-\delta, x_0+\delta], \\ v(x_0-\delta,t) = v(x_0+\delta,t) = 0, \text{ for } 0 < t \leq T, \end{array}\right\} \tag{5.4.8}$$

where $v_0(x) > 0$ on $(x_0-\delta, x_0+\delta)$, $v_0(x_0-\delta) = 0 = v_0(x_0+\delta)$ and $v_0(x)$ is symmetric and attains its maximum at the point $x = x_0$. Since $\lim\limits_{s\to\infty} f(s)/s = \infty$, there exists a positive constant $k_4 > z_0$ such that

$$\frac{f(s)}{s} \geq \frac{2}{\delta^2}\left((x_0+\delta)^\beta + \frac{\delta\beta}{(x_0-\delta)^{1-\beta}}\right), \text{ for } s > k_4. \tag{5.4.9}$$

By proposition 5.4.1, the solution v of (5.4.8) blows up at the point $x = x_0$ in a finite time, provided that $v_0(x)$ is large enough. Since $v_0(x)$ is symmetric at the point $x = x_0$, the solution $v(x,t)$ have its maximum at the point x_0 and then we have

$$Lv(x,t) = f(v(x,t)) \leq f(v(x_0,t)) \text{ for } (x,t) \in (x_0-\delta, x_0+\delta) \times (0,T].$$

Next, we can choose a positive constant $k_5 \geq k_4\delta^{-2}$ big enough such that

$$w_0(x) = k_5\,[x - (x_0-\delta)]\,[(x_0+\delta) - x] \geq v_0(x), \quad \text{for } x \in [x_0-\delta, x_0+\delta].$$

Consider for each $x \in (x_0-\delta, x_0+\delta)$

$$\begin{aligned} & (x^\beta w_0'(x))' + f(w_0(x_0)) \\ = \;& -2k_5\left[x^\beta + \frac{\beta}{x^{1-\beta}}(x-x_0)\right] + f(k_5\delta^2) \\ \geq \;& -2k_5\left[(x_0+\delta)^\beta + \frac{\delta\beta}{(x_0-\delta)^{1-\beta}}\right] + f(k_5\delta^2) \\ \geq \;& 0. \end{aligned}$$

Then

$$(x^\beta w_0'(x))' + f(w_0(x_0)) \geq 0, \text{ for } x \in (x_0-\delta, x_0+\delta).$$

We consider the following problem

$$\left.\begin{array}{c} Lw(x,t) = f(w(x_0,t)),\ \text{for } (x,t) \in (x_0-\delta, x_0+\delta) \times (0,T], \\ w(x,0) = w_0(x) \geq 0,\ \text{for } x \in [x_0-\delta, x_0+\delta], \\ w(x_0-\delta,t) = 0 = w(x_0+\delta,t),\ \text{for } 0 < t \leq T. \end{array}\right\} \tag{5.4.10}$$

Therefore, we have for $(x,t) \in (x_0-\delta, x_0+\delta) \times (0,T]$,

$$\begin{aligned} L(w-v) &= f(w(x_0,t)) - f(v(x,t)) \\ &\geq f(w(x_0,t)) - f(v(x_0,t)) \\ &= f'(\eta)\left[w(x_0,t) - v(x_0,t)\right], \end{aligned}$$

where η lies between $w(x_0,t)$ and $v(x_0,t)$, and the initial and boundary conditions

$$\begin{aligned} w(x,0) - v(x,0) &= w_0(x) - v_0(x) \geq 0,\ \text{for } x \in [x_0-\delta, x_0+\delta], \\ w(x,t) - v(x,t) &= 0,\ \text{for } (x,t) \in \{x_0-\delta, x_0+\delta\} \times (0,T]. \end{aligned}$$

Therefore, it follows from lemma 5.4.2 that $w(x,t) \geq v(x,t)$ in $(x,t) \in [x_0-\delta, x_0+\delta] \times [0,T]$. Therefore the solution w of (5.4.10) blows up in a finite time. By the same way, we can choose that the function $\phi(x)$ is sufficiently large such that $\phi(x) \geq w_0(x)$, for $x \in [x_0-\delta, x_0+\delta]$, so we conclude that the solution $u(x,t)$ blows up in a finite time.

5.5 The blow-up set

The next lemma give an additional property of Green's function.

Lemma 5.5.1 *Given any $x \in D$ and any finite time T, there exist two positive numbers k_6 and k_7 such that*

$$k_6 < \int_0^1 G(x,t,\xi,0)d\xi < k_7 \text{ for } 0 \leq t \leq T.$$

Proof. Let us consider the following auxiliary problem

$$\left.\begin{array}{c} Lv(x,t) = 1,\ \text{for } (x,t) \in \Omega_T, \\ v(x,0) = 0,\ \text{for } x \in \overline{D}, \\ v(0,t) = 0 = v(1,t),\ \text{for } 0 < t \leq T. \end{array}\right\} \tag{5.5.1}$$

The solution of (5.5.1) is given by

$$v(x,t) = \int_0^t \int_0^1 G(x,t,\xi,\tau)d\xi d\tau = \int_0^t \int_0^1 G(x,\tau,\xi,0)d\xi d\tau.$$

It follows that

$$v_t(x,t) = \int_0^1 G(x,t,\xi,0)d\xi > 0.$$

Since for any $x \in D$

$$v_t(x,0) = \int_0^1 G(x,0,\xi,0)d\xi = \frac{1}{x^\alpha}.$$

there exists a positive k_6 such that

$$k_6 < \int_0^1 G(x,t,\xi,0)d\xi, \text{ for } 0 \le t \le T.$$

Furthermore, since $v_t(x,t)$ is continuous in $D \times [0,T]$, there exists a positive k_7 such that

$$\int_0^1 G(x,t,\xi,0)d\xi < k_7 \text{ for } 0 \le t \le T.$$

We finally show that the set of blow-up points of (5.1.1) is the whole interval $[0,1]$ in the following theorem.

Theorem 5.5.2 *If the solution u of (5.1.1) blows up in a finite time, then the set of blow-up points of (5.1.1) is $\overline{D}$.*

Proof. The solution u of (5.1.1) is given by

$$u(x,t) = \int_0^t \int_0^1 G(x,t,\xi,\tau) f(u(x_0,\tau)) d\xi d\tau + \int_0^1 \xi^\alpha G(x,t,\xi,0)\phi(\xi)d\xi. \tag{5.5.2}$$

It follows from theorem 5.3.7 that u blows up at least at the point $x = x_0$ as $t \to t_b$. From (5.5.2) and lemma 5.5.1,

$$\begin{aligned} u(x_0,t) &= \int_0^t \int_0^1 G(x_0,t,\xi,0) f(u(x_0,t-\tau)) d\xi d\tau + \int_0^1 \xi^\alpha G(x_0,t,\xi,0)\phi(\xi)d\xi \\ &\le k_7 \int_0^t f(u(x_0,t-\tau))d\tau + k_7 \left(\max_{x \in \overline{D}} \phi(x) \right) \end{aligned}$$

Since $u(x_0,t) \to \infty$ as $t \to t_b$, we have

$$\int_0^{t_b} f(u(x_0,t_b-\tau))d\tau = \infty.$$

On the other hand, let us consider that for any $(x,t) \in \Omega_T$,

$$\begin{aligned} u(x,t) &\ge k_6 \int_0^t f(u(x_0,t-\tau))d\tau + \int_0^1 \xi^\alpha G(x,t,\xi,0)\phi(\xi)d\xi \\ &\ge k_6 \int_0^t f(u(x_0,t-\tau))d\tau. \end{aligned}$$

As t approaches t_b^-, it follows from $\int_0^{t_b} f(u(x_0,t_b-\tau))d\tau \to \infty$ that $u(x,t)$ tends to infinity. Thus, the set of blow-up points is D. For $\widetilde{x} \in \{0,1\}$, we can find a sequence $\{(x_n,t_n)\}$ such that $(x_n,t_n) \to (\widetilde{x},t_b)$ and $\lim_{n\to\infty} u(x_n,t_n) \to \infty$. Therefore, the set of blow-up points of (5.1.1) is $\overline{D}$.

Note that this chapter was the object of the article :

P. Sawangtong. B. Novaprateep and W. Jumpen. Blow-up solutions for a Degenerate Parabolic Problems with a Localized Nonlinear Term, WSEAS Transactions on Heat and Mass Transfer, issue 3, vol. 5, 2010, p. 178-189.

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Chapter 6

Complete blow-up for a generalized degenerate semilinear parabolic problem with a localized nonlinear term

6.1 Introduction

Without loss of generality and for simplicity, we take the interval of x to be $[0,1]$. Let $I = (0,1)$, $Q_T = I \times (0,T)$, $\overline{I}$ and $\overline{Q}_T$ be the closure of I and Q_T, respectively. We here study the following degenerate semilinear parabolic problem with a localized nonlinear term:

$$\left.\begin{array}{c} Lu(x,t) = f(u(x_0,t)) \text{ for } (x,t) \in Q_T, \\ u(0,t) = 0 = u(1,t) \text{ for } t \in (0,T), \\ u(x,0) = u_0(x) \text{ for } x \in \overline{I}, \end{array}\right\} \tag{6.1.1}$$

where $x_0 \in I$, $Lu(x,t) = u_t - \frac{1}{k(x)}(p(x)u_x)_x$, and k, p, f and u_0 are given functions.

The purpose of this chapter is to prove that before blow-up occurs, there exists a T_1 such that problem (6.1.1) has a unique nonnegative continuous solution u on the interval $[0, T_1]$ for any $x \in \overline{I}$. Moreover, a sufficient condition to blow-up in finite time and the blow-up set of such a solution u of problem (6.1.1) are shown.

In order to obtain our results, we need the following assumptions.

(A) $p \in C^1(\overline{I})$, $p(0) = 0$, p is positive on $(0,1]$.

(B) $k \in C(\overline{I})$, $k(0) = 0$, k is positive on $(0,1]$.

(C) $f \in C^2([0,\infty))$ is convex with $f(0) = 0$ and $f(s) > 0$ for $s > 0$.

(D) $u_0 \in C^2(\overline{I})$, $u_0(0) = 0 = u_0(1)$, u_0 is nonnegative on I, $u_0(x_0) > 0$ and u_0 satisfies

$$\frac{1}{k(x)}\frac{d}{dx}\left(p(x)\frac{du_0(x)}{dx}\right) + f(u_0(x_0)) \geq \zeta u_0(x) \text{ on } I$$

for some positive constant ς.

By separation of variables, we get the corresponding singular eigenvalue problem to (6.1.1) defined by

$$\left.\begin{array}{c}\frac{d}{dx}\left(p(x)\frac{d\phi(x)}{dx}\right)+\lambda k(x)\phi(x)=0 \text{ on } I,\\ \phi(0)=0=\phi(1).\end{array}\right\} \tag{6.1.2}$$

We note that conditions (A) and (B) yield that the point $x = 0$ is a singular point of a singular eigenvalue problem (6.1.2) and, by proposition 2.1 of [20], condition (C) implies that f is increasing and locally Lipschitz on $[0, \infty)$. We rewrite a singular eigenvalue problem (6.1.2) in an equivalent form:

$$\left.\begin{array}{c}x^2\phi''(x)+x\left[x\frac{p'(x)}{p(x)}\right]\phi'(x)+\lambda\left[x^2\frac{k(x)}{p(x)}\right]\phi(x)=0 \text{ on } I,\\ \phi(0)=0=\phi(1).\end{array}\right\} \tag{6.1.3}$$

We have to add some conditions on functions p and k to make the point $x = 0$ a regular singular point, that is,

(E) The limit of $\frac{xp'(x)}{p(x)}$ and $\frac{x^2k(x)}{p(x)}$ are finite as $x \to 0$ and $\frac{xp'(x)}{p(x)}$ and $\frac{x^2k(x)}{p(x)}$ are analytic at $x = 0$.

We note that theorem 5.7.1 of [25] yields existence of eigenfunctions ϕ_n and their corresponding eigenvalues λ_n of problem (6.1.3). By [25], completeness of eigenfunctions ϕ_n of problem (6.1.3) results from next assumption.

(F) $\int_0^1\int_0^1 H(x,\xi)^2k(x)k(\xi)d\xi dx$ is finite where H is the corresponding Green's function to problem (6.1.3).

In order to obtain the existence of the corresponding Green's function defined by (6.2.2) to problem (6.1.1), we have to assume additional conditions on eigenvalues λ_n and their associating eigenfunctions ϕ_n.

(G) $\lambda_n = O(n^s)$ for some $s > 1$ as $n \to \infty$ and there exists some positive constant K such that $|\phi_n(x)| \le K\lambda_n^d$ for $d > 0$ and for any $x \in \overline{I}$.

6.2 Local existence and uniqueness

This section deals with the local existence and uniqueness of the nonnegative continuous solution u of problem (6.1.1). Next lemma states the well-known properties of eigenfunctions ϕ_n and eigenvalues λ_n of problem (6.1.2) referred to [25].

Lemma 6.2.1

1 $\int_0^1 k(x)\phi_n(x)\phi_m(x)dx = \begin{cases} 1 & \textit{for } n = m, \\ 0 & \textit{for } n \neq m. \end{cases}$

2 All eigenvalues are real and positive.

3 Eigenfunctions are complete with the weight function k.

4 $\lambda_1 < \lambda_2 < \ldots$ *and* $\lim_{n\to\infty} \lambda_n = \infty$.

5 $\int_0^1 p(x)\phi_n'(x)\phi_m'(x)dx = \begin{cases} \lambda_n & \textit{for } n = m, \\ 0 & \textit{for } n \neq m. \end{cases}$

6 For any $n \in \mathbb{N}$, $\phi_n \in C^\infty((0,1])$.

Let us construct Green's function $G(x,t,\xi,\tau)$ corresponding to problem (6.1.1). It is determined by the following system: for $x,\xi \in I$ and $t,\tau \in (0,T)$,

$$\left.\begin{array}{c} LG(x,t,\xi,\tau) = \delta(x-\xi)\delta(t-\tau), \\ G(0,t,\xi,\tau) = 0 = G(1,t,\xi,\tau), \\ G(x,t,\xi,\tau) = 0 \text{ for } t < \tau, \end{array}\right\} \tag{6.2.1}$$

where δ is the Dirac delta function. By eigenfunction expansion, the corresponding Green' s function G to problem (6.1.1) is defined by

$$G(x,t,\xi,\tau) = \sum_{n=1}^{\infty} \phi_n(x)\phi_n(\xi)e^{-\lambda_n(t-\tau)} \text{ for } x,\xi \in \overline{I} \text{ and } 0 \le \tau < t \le T. \tag{6.2.2}$$

To derive the equivalent integral equation of problem (6.1.1), let us consider the adjoint operator L^*, which is given by $L^*u = -u_t - \frac{1}{k(x)}(p(x)u_x)_x$. Using Green's second identity, we obtain

$$u(x,t) = \int_0^1 k(\xi)G(x,t,\xi,0)u_0(\xi)d\xi + \int_0^t \int_0^1 k(\xi)G(x,t,\xi,\tau)f(u(x_0,\tau))d\xi d\tau. \tag{6.2.3}$$

The following lemma is due to properties of Green's function G.

Lemma 6.2.2

1 G is continuous for $x,\xi \in \overline{I}$ and $0 \le \tau < t \le T$.

2 G is positive for $x,\xi \in \overline{I}$ and $0 \le \tau < t \le T$.

3 $\lim_{t \to \tau^+} k(x)G(x,t,\xi,\tau) = \delta(x-\xi)$.

4 For any fixed $\tau \in (0,T)$, there exist a positive constant c_0 such that

$$\int_0^1 k(x)G(x,t,\xi,\tau)d\xi \le c_0 \textit{ for any } (x,t) \in I \times (\tau,T).$$

Proof. By modifying proof of lemma 4.a and 4.c [7], we obtain the proof of 1 and 2, respectively. For proof of 3, let us consider the following problem:

$$\begin{array}{c} Lw(x,t,\xi,\tau) = 0 \text{ for } x,\xi \in I \text{ and } 0 < \tau < t < T, \\ w(0,t,\xi,\tau) = 0 = w(1,t,\xi,\tau) \text{ for } 0 < \tau < t < T, \\ \lim_{t \to \tau^+} k(x)w(x,t,\xi,\tau) = \delta(x-\xi). \end{array}$$

By equation (6.2.3), we have that for any $t > \tau$,

$$w(x,t,\xi,\tau) = \int_0^1 k(\eta)G(x,t,\eta,\tau)\frac{1}{k(\eta)}\delta(\eta-\xi)d\eta = G(x,t,\xi,\tau).$$

Hence, we get the proof of 3. We next prove 4. For any $t > \tau$. Let us consider the series

$$\sum_{n=1}^{\infty} \int_0^1 k(\xi)\phi_n(x)\phi_n(\xi)e^{-\lambda_n(t-\tau)}d\xi.$$

Since $\left|\int\limits_0^1 k(\xi)\phi_n(x)\phi_n(\xi)e^{-\lambda_n(t-\tau)}d\xi\right| \le \left(\max\limits_{x\in\overline{I}} k(x)\right) K^2\lambda_n^{2d}e^{-\lambda_n(t-\tau)}$ and, by assumption, the series $\sum\limits_{n=1}^{\infty}\lambda_n^{2d}e^{-\lambda_n(t-\tau)}$ converges, $\sum\limits_{n=1}^{\infty}\int\limits_0^1 k(\xi)\phi_n(x)\phi_n(\xi)e^{-\lambda_n(t-\tau)}d\xi$ converges uniformly for any $(x,t,\tau) \in I\times(\tau,T)\times(0,T)$. Hence, we obtain the proof of 4. Therefore, the proof of this lemma is complete.

Next theorem says to local existence of a solution u of the equivalent integral equation (6.2.3).

Theorem 6.2.3 *There exists a T_1 with $0 < T_1 < T$ such that the equivalent integral equation (6.2.3) has a unique continuous solution u on $\overline{Q}_{T_1}$.*

Proof. We will use the fixed point theorem to prove existence of a continuous solution u of the equivalent integral equation (6.2.3). Since $\int\limits_0^1 k(\xi)G(x,t,\xi,0)d\xi \le K_1$ for some positive K_1, we let $M = (K_1+1)\max\limits_{x\in\overline{I}}|u_0(x)|+1$. Locally Lipschitz property of f implies that there exists a positive constant L_M depending on M such that for any $x, y \in (0,\infty)$ with $|x| \le M$ and $|y| \le M$,

$$|f(x)-f(y)| \le L_M\,|x-y|\,.$$

We then choose

$$T_1 < \min\left\{\frac{1}{c_0 f(M)}, \frac{1}{c_0 L_M}\right\}. \tag{6.2.4}$$

Define a set E by

$$E = \left\{u \in C(\overline{Q}_{T_1}) \text{ such that } \max_{(x,t)\in\overline{Q}_{T_1}} |u(x,t)| \le M.\right\} \tag{6.2.5}$$

Then, E is a Banach space equipped with the norm $|u|_E = \max\limits_{(x,t)\in\overline{Q}_{T_1}} |u(x,t)|$. Let

$$\begin{aligned}\Lambda u(x,t) &= \int_0^1 k(\xi)G(x,t,\xi,0)u_0(\xi)d\xi \\ &\quad + \int_0^t\int_0^1 k(\xi)G(x,t,\xi,\tau)f(u(x_0,\tau))d\xi d\tau.\end{aligned} \tag{6.2.6}$$

for any $u \in E$. We next show that the operator Λ defined by (6.2.6) maps E into itself and that Λ is contractive. Let $u, v \in E$. We then have that

$$\begin{aligned}|\Lambda u(x,t)| &\le \left|\int_0^1 k(\xi)G(x,t,\xi,0)u_0(\xi)d\xi\right| \\ &\quad + \left|\int_0^t\int_0^1 k(\xi)G(x,t,\xi,\tau)f(u(x_0,\tau))d\xi d\tau\right|.\end{aligned} \tag{6.2.7}$$

From (6.2.7) and lemma 6.2.2.4,

$$\begin{aligned}|\Lambda u(x,t)| &\le K_1\max_{x\in\overline{I}}|u_0(x)| + f(M)\int_0^t\int_0^1 k(\xi)G(x,t,\xi,\tau)d\xi d\tau \\ &\le K_1\max_{x\in\overline{I}}|u_0(x)| + f(M)c_0T_1.\end{aligned}$$

By definition of T_1, $\Lambda u \in E$ for any $u \in E$. Since

$$\begin{aligned}
|\Lambda u(x,t) - \Lambda v(x,t)| &\leq \left| \int_0^t \int_0^1 k(\xi)G(x,t,\xi,\tau)\left[f(u(x_0,\tau)) - f(v(x_0,\tau))\right] d\xi d\tau \right| \\
&\leq L_M \int_0^t \int_0^1 k(\xi)G(x,t,\xi,\tau)\left|u(x_0,\tau) - v(x_0,t)\right| d\xi d\tau \\
&\leq L_M \int_0^t \int_0^1 k(\xi)G(x,t,\xi,\tau) d\xi d\tau \left|u - v\right|_E \\
&\leq c_0 L_M T_1 \left|u - v\right|_E , \qquad (6.2.8)
\end{aligned}$$

definition of T_1 and (6.2.8) yield that Λ is contractive. The fixed point then implies that there exists a unique $u \in E$ satisfying equation (6.2.3). Therefore, the proof of this theorem is complete.

Lemma 6.2.4 *Let v be a classical solution of the following problem:*

$$\left. \begin{array}{c} Lv(x,t) \geq B(x,t)v(x_0,t) \text{ for } (x,t) \in Q_T, \\ v(0,t) \geq 0 \text{ and } v(1,t) \geq 0 \text{ for } t \in (0,T), \\ v(x,0) = u_0(x) \text{ for } x \in \overline{I}, \end{array} \right\} \qquad (6.2.9)$$

where B is a nonnegative and bounded function on $\overline{Q}_T$. Then $v(x,t) \geq 0$ for any $(x,t) \in \overline{Q}_T$.

Proof. By modifying the proof of proposition 2.3.4.1, we obtain the proof of this lemma.

Next lemma gives additional properties of a solution u of problem (6.1.1).

Lemma 6.2.5 *Let u be a continuous solution of problem (6.1.1). Then $u(x,t) \geq u_0(x)$ and $u_t(x,t) \geq 0$ for any $(x,t) \in \overline{Q}_T$.*

Proof. Let $z(x,t) = u(x,t) - u_0(x)$ on $\overline{Q}_T$. Let us consider that for any $(x,t) \in Q_T$,

$$Lz(x,t) = f(u(x_0,t)) + \frac{1}{k(x)}\frac{d}{dx}\left(p(x)\frac{du_0(x)}{dx}\right).$$

Condition (D) implies that

$$\frac{1}{k(x)}\frac{d}{dx}\left(p(x)\frac{du_0(x)}{dx}\right) \geq -f(u_0(x_0)) \text{ on } I$$

and then we obtain that, by the second mean value theorem, for any $(x,t) \in Q_T$,

$$Lz(x,t) \geq f(u(x_0,t)) - f(u_0(x_0)) = f'(\eta_1)z(x_0,t)$$

where η_1 is between $u(x_0,t)$ and $u_0(x_0)$. Moreover, for any $(x,t) \in \{0,1\} \times (0,T) \cup \overline{I} \times \{0\}$, $z(x,t) = 0$. Lemma 5.2.4 implies that $z \geq 0$ on $\overline{Q}_T$ or $u \geq u_0$ on $\overline{Q}_T$. Let h be any positive constant less than T and $w(x,t) = u(x,t+h) - u(x,t)$ on $\overline{Q}_{T-h}$. We then have that, by the second mean value theorem, for any $(x,t) \in Q_{T-h}$,

$$Lw(x,t) = f(u(x_0,t+h)) - f(u(x_0,t)) = f'(\eta_2)w(x_0,t)$$

where η_2 is between $u(x_0,t+h)$ and $u(x_0,t)$. Furthermore, $w = 0$ on $\{0,1\} \times (0,T-h)$ and $w \geq 0$ on $\overline{I} \times \{0\}$. It then follows from lemma 5.2.4 that $w \geq 0$ on $\overline{Q}_{T-h}$. This shows that $u_t \geq 0$ on $\overline{Q}_T$.

We note that before blow-up occurs, there exists a positive constant M such that $|u(x,t)| \leq M$ for all $(x,t) \in \overline{Q}_{T_1}$. Locally Lipschitz continuity of f implies that there exists a positive constant L_M depending on M such that

$$|f(u(x_0,t))| \leq L_M\,|u(x_0,t)| \text{ for any } t \in [0,T_1].$$

Lemma 6.2.6 *If $f'(u_0(x_0)) \geq L_M$, then $u_t(x,t) \geq L_M u(x,t)$ on $\overline{Q}_{T_1}$.*

Proof. Let $z(x,t) = u_t(x,t) - L_M u(x,t)$ on $\overline{Q}_{T_1}$. We then have that for $(x,t) \in Q_{T_1}$,

$$Lz(x,t) = f'(u(x_0,t))u_t(x_0,t) - L_M f(u(x_0,t)).$$

Locally Lipschitz continuity of f implies that for $(x,t) \in Q_{T_1}$,

$$\begin{aligned} Lz(x,t) &\geq f'(u(x_0,t))u_t(x_0,t) - L_M^2 u(x_0,t) \\ &\geq f'(u_0(x_0))u_t(x_0,t) - L_M^2 u(x_0,t) \\ &\geq L_M u_t(x_0,t) - L_M^2 u(x_0,t) \\ &= L_M z(x_0,t). \end{aligned}$$

From lemma 6.2.5, $z(0,t) = u_t(0,t) = 0$ and $z(1,t) = u_t(1,t) = 0$ for $t \in (0,T_1)$. If, in condition (D), we set $\zeta = L_M$, then condition (D) implies that for any $x \in I$,

$$\begin{aligned} z(x,0) &= \lim_{t\to 0} u_t(x,t) - L_M u_0(x) \\ &= \frac{1}{k(x)}\frac{d}{dx}\left(p(x)\frac{du_0(x)}{dx}\right) + f(u_0(x_0)) - L_M u_0(x) \\ &\geq 0. \end{aligned}$$

Therefore, by lemma 6.2.4, the proof of this lemma is complete.

Lemma 6.2.7 *If $u_0(x_0) \geq u_0(x)$ for all $x \in \overline{I}$, then $u(x_0,t) \geq u(x,t)$ for any $(x,t) \in \overline{Q}_{T_1}$.*

Proof. Let $z(x,t) = u(x_0,t) - u(x,t)$ on $\overline{Q}_{T_1}$. We have that, on $\overline{Q}_{T_1}$, lemma xx and locally Lipschitz property of f yield that

$$\begin{aligned} Lz(x,t) &= u_t(x_0,t) - f(u(x_0,t)) \\ &\geq u_t(x_0,t) - L_M u(x_0,t) \\ &\geq 0. \end{aligned}$$

Since $z(0,t) = u(x_0,t) \geq u_0(x_0) \geq 0$, $z(1,t) = u(x_0,t) \geq u_0(x_0) \geq 0$ for $t \in (0,T_1)$ and $z(x,0) = u_0(x_0) - u_0(x) \geq 0$ for $x \in \overline{I}$, by lemma 6.2.4, we get the proof of this lemma.

Theorem 6.2.8 *Let $T_{\max}$ be the supremum of all T_1 such that a continuous solution u of an equivalent integral equation (6.2.3) exists. If $T_{\max}$ is finite, then $u(x_0,t)$ is unbounded as t tends to $T_{\max}$.*

Proof. Suppose that $u(x_0,T_{\max})$ is finite. Let $N = u(x_0,T_{\max}) + 1$. By theorem xx and a fact that u is nondecreasing in t, there exists a finite time $\widetilde{T}(> T_{\max})$ depending on N such that the equivalent integral equation (6.2.3) has a unique continuous solution u on the time interval $[0,\widetilde{T}]$ for any $x \in \overline{I}$. By the definition of $T_{\max}$, we get a contradiction.

6.3 A sufficient condition to blow-up in finite time

Let λ_1 be the first eigenvalue of a singular eigenvalue problem (6.1.2) and let ϕ_1 be its corresponding eigenfunction. Without loss of generality, we assume that

$$\int_0^1 k(x)\phi_1(x)dx = 1. \tag{6.3.1}$$

We then define a function H by

$$H(t) = \int_0^1 k(x)\phi_1(x)u(x,t)dx. \tag{6.3.2}$$

Theorem 6.3.1 *Assume that*

1 u_0 *attains its maximum at point* x_0*.*

2 $f(s) \geq bs^p$ *with* $b > 0$ *and* $p > 1$*.*

3 $H(0) > \left(\frac{\lambda_1}{b}\right)^{\frac{1}{p-1}}$.

Then a solution u *of problem (6.1.1) blows up in finite time.*

Proof. Multiplying equation (6.1.1) by $k(x)\phi_1(x)$ and integrating equation (6.1.1) from 0 to 1 with respect to x yield

$$\frac{dH(t)}{dt} = -\lambda_1 H(t) + \int_0^1 k(x)f(u(x_0,t))\phi_1(x)dx.$$

By lemma 6.2.7 and assumption 2, we have

$$\begin{aligned}\frac{dH(t)}{dt} &\geq -\lambda_1 H(t) + \int_0^1 k(x)f(u(x,t))\phi_1(x)dx \\ &\geq -\lambda_1 H(t) + b\int_0^1 k(x)u^p(x,t)\phi_1(x)dx. \end{aligned} \tag{6.3.3}$$

Hölder inequality implies that

$$\int_0^1 k(x)\phi_1(x)u(x,t)dx \leq \left(\int_0^1 k(x)\phi_1(x)dx\right)^{\frac{p-1}{p}} \left(\int_0^1 k(x)\phi_1(x)u^p(x,t)dx\right)^{\frac{1}{p}}.$$

From (6.3.1), we get

$$\int_0^1 k(x)\phi_1(x)u^p(x,t)dx \geq \left(\int_0^1 k(x)\phi_1(x)u(x,t)dx\right)^p = H^p(t). \tag{6.3.4}$$

From equation (6.3.3) and (6.3.4), we obtain

$$H'(t) \geq -\lambda_1 H(t) + bH^p(t)$$

or

$$H^{p-1}(t) \geq \frac{1}{\frac{b}{\lambda_1} + \left[H^{1-p}(0) - \frac{b}{\lambda_1}\right] e^{-\lambda_1(1-p)t}}.$$

It then follows from assumption 3 that there exists a $\widehat{T}(>0)$ such that H tends to infinity as t converges to $\widehat{T}$. By the definition of H (6.3.2), we find that

$$H(t) \leq \left(\int_0^1 k(x)\phi_1(x)dx \right) u(x_0,t) = u(x_0,t).$$

Therefore, a solution u of problem (6.1.1) blows up at point x_0 as t tends to $\widehat{T}$.

6.4 The blow-up set

Theorem 6.4.1 *The blow-up set of a solution u of problem (6.1.1) is I.*

Proof. From (6.2.3), we have that for $t \in (0, T_{\max})$,

$$\begin{aligned} u(x_0,t) &= \int_0^1 k(\xi)G(x_0,t,\xi,0)u_0(\xi)d\xi \\ &\quad + \int_0^t \int_0^1 k(\xi)G(x_0,t,\xi,\tau)f(u(x_0,\tau))d\xi d\tau \\ &\leq \max_{x \in \overline{I}} u_0(x) + c_0 \int_0^t f(u(x_0,\tau))d\tau. \end{aligned} \tag{6.4.1}$$

By theorem 6.2.8, we obtain that as t tends to $T_{\max}$,

$$\int_0^{T_{\max}} f(u(x_0,\tau))d\tau = \infty. \tag{6.4.2}$$

On the other hand, by positivity of k, G and u_0, we get that for any $(x,t) \in Q_{T_{\max}}$,

$$u(x,t) \geq \int_0^t \int_0^1 k(\xi)G(x,t,\xi,\tau)f(u(x_0,\tau))d\xi d\tau.$$

Since there exists a positive constant c_1 such that $\int_0^1 k(\xi)G(x,t,\xi,\tau)d\xi \geq c_1$, we obtain that

$$u(x,t) \geq c_1 \int_0^t f(u(x_0,\tau))d\tau \text{ for all } (x,t) \in Q_{T_{\max}}.$$

Hence, the solution u tends to infinity for all $x \in I$ as t approaches to $T_{\max}$. Therefore the proof of this theorem is complete.

Note that this chapter was the object of the article :

P. Sawangtong. B. Novaprateep and W. Jumpen. Complete blow-up for a degenerate semilinear parabolic problem with a localized nonlinear term, Proceeding of International Conference on Fluid Mechanics and Heat and Mass Transfer, Corfu Island, Greece, 22-24 July 2010, p. 95-99.

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Chapter 7

Conclusions

The motivation of this thesis is that the author, major advisor and co-advisors have studied the paper by Chan and Yang [9]. In [9] authors considered the semilinear parabolic problem with a localized nonlinear term: let q be any positive real number: $x^q u_t - u_{xx} = f(u(x_0, t))$ with nonhomogeneous initial and Dirichlet boundary conditions and x_0 is a fixed point in the domain of x. They proved that their problem has a blow-up solution and the blow-up of such a blow-up solution is the whole domain of x. They furthermore gave a condition to guarantee occurrence for blow-up in finite time of their problem. Applications of this kind of problems are mentioned in the first chapter.

The purpose of this thesis is to generalize the results of Chan and Yang [9] to a more general form and obtain the same results as in [9], that is, existence and uniqueness of a blow-up solution, blow-up set of such a blow-up solution and the sufficient condition to blow-up in finite time. Our generalized problem is $k(x)u_t - (p(x)u_x)_x = k(x)f(u(x_0, t))$ with nonhomogeneous initial and Dirichlet boundary conditions and x_0 is a fixed point in the domain of x.

This thesis is divided into six chapters as follows. The history of the study of blow-up problems is given in the first chapter. Chapter 2 and 3 deal with our problem in 1-dimension of variable x in the case that k and p are positive functions on the whole domain of x. The difference of chapters 2 and 3 is that in chapter 2 existence of a blow-up solution is shown by the functional method, i.e., the semigroup method, but in chapter 3 it is proven by a classical method, i.e., the Green's function method. Chapter 4 is concerned with the extended problem of the previous problem to N dimensions in the variable x by the using semigroup method. Before studying the case that $k(0) = 0 = p(0)$ and k and p are positive on the whole domain of x except for the point $x = 0$, we study the particular problem which results from replacing function k with x^α and function p with x^β in chapter 5. In the last chapter, we investigate a blow-up solution of our problem in such a case by using the Green's function method.

The advantage of the semigroup method is that before applying the semigroup method to our problem, we have to transform our problem into the equivalent evolution problem and then since, in the semigroup theory, there are many theorems on the existence of solutions of evolution problems, it is convenient to use a suitable theorem in the semigroup theory to show existence of solutions to our problem. On the other hand, the difficulty is that, in using such a suitable theorem, we have to make conditions in our equivalent evolution problem satisfy assumptions of such a suitable theorem.

The advantage of the Green's function method is that the Green's function method is a fundamental method in the topic of partial differential equations to find solutions of P.D.E. problems and furthermore it is included in elementary courses at the undergraduate level. This is why the Green's function method is easy to understand and apply to our problem.

On the other hand, there are limitations to the application of semigroup method and Green's function method to our problem. Firstly, we mention limitations in using the Green's function method. Before applying the Green's function method to our problem, we have to find eigenvalues and eigenfunctions of the corresponding eigenvalue problem in order to construct the Green's function associated with our problem. Different assumptions of k and p in chapters 3 and 6 allow us to obtain the regular eigenvalue problem in chapter 3 and the singular eigenvalue problem in chapter 6. For chapter 3, since there are many text books on Partial Differential Equations (P.D.E.) concerning the general regular eigenvalue problem, we have the asymptotic property of eigenvalues, $\lambda_n = O(n^2)$ as $n \to \infty$, and boundedness of eigenfunctions. With these facts, we can obtain desired results in chapter 3. But chapter 6 deals with the general singular eigenvalue problem. However, there are no books written on eigenvalues and eigenfunctions of the general singular eigenvalue problem. This is the reason why we must construct condition (G) in chapter 6.

As previously discussed, if we want to apply the semigroup method to our problem, we have to transform our problem into the equivalent evolution problem. In order to show that the operator F is Hölder continuous in chapter 2, we need proposition 2.3.1.3. Proposition 2.3.1.3 follows from the Rellich theorem. The embedding of H_0^1 into L^2 is compact. In the case that $k(0) = 0 = p(0)$ and k and p are positive on the whole domain of x except for the point $x = 0$. We can define the spaces H_0^1 and L^2 similar to those in section 2.3 of chapter 2. The limitation of applying semigroup method to our problem is that we don't know whether the embedding of H_0^1 into L^2 is compact in such a case.

In order to achieve our objective to apply the semigroup method to our problem in the case that $k(0) = 0 = p(0)$ and k and p are positive on the whole domain of x except for the point $x = 0$, we may complete this problem completely in the future.

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Résumé de la thèse : On étudie l'existence de 'blow-up' et l'ensemble des points de 'blow-up' pour une équation de type chaleur dégénérée ou non avec un terme source uniforme fonction non linéaire de la température instantanée en un point fixé du domaine. L'étude est conduite par les méthodes d'analyse classique (fonction de Green, développements en fonctions propres, principe du maximum) ou fonctionnelle (semi-groupes d'opérateurs linéaires).

TITRE en anglais : COMPLETE BLOW-UP FOR A SEMI-LINEAR PARABOLIC PROBLEM WITH A LOCALIZED NONLINEAR TERM.

ŔESUME en anglais : We study existence of blow-up and blow-up sets for a (degenerate or not) Heat-like equation with a uniform source term nonlinear function of the instantaneous temperature at a given point of the domain. The techniques are relevant from either classical analysis (Green function, eigenfunction expansion, maximum principle) or function analysis (semi-group of linear operators).

DISCIPLINE : MECANIQUE.

MOTS-CLES : 'Blow-up', Problèmes paraboliques semi-linéaires, Semi-groupes, Fonctions de Green.

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