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Pongpol JUNTHAREE

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QUELQUES MODÈLES MATHÉMATIQUES DE JONCTIONS

JURY :

M.	Alain	LEGER	Laboratoire de Mécanique et d'Acoustique(LMA)	Rapporteur
M.	Mircea	SOFONEA	LAMPS, Université de Perpignan	Rapporteur
M.	Hedy	ATTOUCH	I3M, Université de Montpellier II	Examinateur
M.	Frédéric	LEBON	Laboratoire de Mécanique et d'Acoustique(LMA)	Examinateur
M.	Christian	LICHT	LMGC, Université de Montpellier II	Co-directeur
M.	Gérard	MICHAILLE	I3M, Université de Montpellier II	Co-directeur

Contents

List of Figures

Résumé de la thése en français

La plupart des structures du Génie Civil consistent en l'assemblage de corps déformables, ainsi il est important de disposer de modèles efficaces de jonctions entre des solides déformables. Les schémas classiques de la Mécanique des Milieux Continus conduisent à des problèmes aux limites mettant en jeu plusieurs parametres, un est essentiel : la faible épaisseur de la couche contenant l'adhésif. Pour les comportements usuels des adhérents et de l'adhésif, il est facile de prouver l'existence de solutions, mais l'obtention d'approximations numériques peut s'avérer délicate à cause de la faible épaisseur de l'adhésif impliquant alors un maillage par trop fin. En outre, les propriétes mécaniques très différentes des adhérents et de l'adhésif peuvent conduire à des systèmes très mal conditionnés. Donc, il est capital de proposer des modèles plus simples mais suffisamment précis. Une manière classique de les obtenir est de considérer les données géométriques et mécaniques réelles (épaisseur, rigidité,...) comme des paramètres et d'étudier le comportement asymptotique des problèmes aux limites paramétrisés quand ces paramètres tendent vers des limites naturelles (0 si la quantité est petite, $+\infty$ si elle est grande). Cela peut être fait de diverses manières : développements asymptotiques formels, perturbations singulières,... Ici, on choisit le point de vue rigoureux de l'analyse variationnelle en étudiant le comportement asymptotique des minimiseurs de la fonctionnelle énergie totale. On montre qu'ils convergent (pour une topologie induite par l'énergie mécanique) vers les solutions d'un problème de minimisation, qui, justement, sera notre proposition de modèle simplifié.

Deux principaux cas de jonctions élastiques ont été traités de cette façon:

i) les jonctions souples, où la rigidité de la jonction est bien plus petite que celles des adhérents (ce qui correspond è des joints collés souples), cf [1] et les références citées,

ii) les jonctions raides, où la rigidité de la jonction est bien plus grande que celles des adhérents (ce qui peut se produire pour certaines soudures), cf [21], [26] et les réferénces citées.

Rappelons la très grande différence quant à la nature des modèles asymptotiques ainsi obtenus. La jonction souple est remplacée par une liaison mécanique entre les adhérents dont la densité d'énergie surfacique est une fonction du déplacement relatif des adhérents le long de l'interface en laquelle se réduit la jonction. Au contraire, la jonction raide est remplacée par une surface matérielle parfaitement collée aux deux adhérents dont la densité d'énergie de déformation est une fonction du gradient surfacique du déplacement (ici, il n'y a pas de saut de déplacement a la traversée de l'interface).

De toutes façons, ces modèles sont plus simples que les originaux parce que une fonctionnelle intégrale de surface remplace une fonctionnelle intégrale définie sur la fine bande adhésive. Ils sont suffisamment précis a cause de résultats rigoureux de convergence : plus les paramètres sont proches de leurs limites naturelles, plus le modèle est précis.

Le but de cette thèse est de généraliser ces deux résultats fondamentaux dans plusieurs directions. La première partie de cette thèse est consacrée aux jonctions souples et deux extensions sont envisagées. On propose d'abord un modèle asymptotique pour un joint adhésif souple et imparfaitement collé. Pour simplifier, on n'a consideré que le cas d'un seul adhérent occupant un domaine Ω, inclus dans $\{x_3 > 0\}$ et de frontière Lipschitzienne dont l'intersection S avec $\{x_3 = 0\}$ est un domaine de $\mathbb{R}^2$. Il est lié à un support rigide $\{x_3 < \varepsilon\}$ par un adhésif occupant la bande $B_\varepsilon := S \times (-\varepsilon,\ 0)$. La densité d'énergie de déformation de l'adhérent est une fonction strictement convexe du tenseur de déformation linéarisé $e(u) - u$ est le deplacement-à croissance quadratique. L'adhérent est soumis à des densités de forces volumiques f et surfaciques φ et est fixé le long de $\Gamma_0 \subset \partial\Omega$. La densité d'énergie de déformation de l'adhésif est une fonction de la déformation du type :

$$\begin{aligned} W_{\mu_S,\mu_D}(e) := &\quad \mu_S W_1(tr(e)) + \mu_D W_2(dev(e)).\\ tr(e) := &\quad e_{11} + e_{22} + e_{33},\ dev(e) := e - \frac{1}{3} tr(e)\ I. \end{aligned}$$

qui, généralise sans difficultés mathématiques propres, la densité associée à un matériau isotrope linéairement élastique, $W_1,\ W_2$ étant strictement convexes et à croissance quadratique (mais, comme W, non nécessairement quadratiques). On demande aussi l'existence de fonctions de récession d'oder 2, $W_i^{\infty,2}$ suffisamment regulières. L'adhésif est libre de chargement, fixé sur le support rigide et lié mécaniquement à l'adhérent le long de S par une liaison dérivant d'une densité d'énergie surfacique h non négative, convexe, semi-continue dans $\mathbb{R}^3$ s'annulant en 0. Ainsi, peuvent être prises en considération des densités réalistes aussi bien régulières comme $\frac{1}{p}\,|\ .\ |^p$ qu'irrégulières comme des indicatrices de convexes fermés de $\mathbb{R}^3$. En supposant les densités de forces de classe L^2, il est clair qu'une configuration d'équilibre est donnée par l'unique solution $\overline{u}_s$ du problème suivant mettant en jeu le triplet $s := (\varepsilon,\ \mu_S, \mu_D)$:

$$\begin{aligned} V_s &= \left\{ v \in L^2(\Omega_\varepsilon;\mathbb{R}^3)\ ;\ v^+ := v_{\lfloor\Omega} \in H^1_{\Gamma_0}(\Omega;\mathbb{R}^3),\ v^- := v_{\lfloor B_\varepsilon} \in H^1_{S_{-\varepsilon}}(B_\varepsilon;\mathbb{R}^3) \right\}.\\ H^1_{S_{-\varepsilon}}(B_\varepsilon;\mathbb{R}^3) &= \left\{ v \in H^1(B_\varepsilon;\mathbb{R}^3)\ :\ v = 0 \text{ on } S_{-\varepsilon} \right\}.\\ L(v) &= \int_\Omega f(x)\ v(x)\ dx + \int_{\Gamma_1} \varphi(x)\ v(x)\ ds, \quad \text{le travail des forces extérieures.}\\ F_s(v) &= \int_\Omega W(e(v^+))\ dx + \int_{B_\varepsilon} W_{\mu_S,\mu_D}(e(v^-))\ dx + \int_S h([v](\widehat{x}))\ ds.\\ [v] &:= \gamma_0(v^+) - \gamma_0(v^-), \quad \text{le saut de déplacement.} \end{aligned}$$

(ou le déplacement relatif le long de S) où le même symbole $\gamma_0(w)$ désigne la trace de n'importe quel élément w d'à la fois $H^1(B_\varepsilon;\mathbb{R}^3)$ et $H^1(\Omega;\mathbb{R}^3)$.

Pour obtenir un modèle simplifié (propice aux calculs numériques), on étudie le comportement limite de $\overline{u}_s$ sous les conditions : il exist $\overline{s} \in \{0\} \times [0,\infty]^2$, $(\overline{\mu}_S,\ \overline{\mu}_D) \in [0,\infty]^2$ et un réal positif ε_0. telque

$$\overline{s} = \lim\ s,\ (\overline{\mu}_S,\ \overline{\mu}_D) = \lim\ (\frac{\mu_S}{\varepsilon},\ \frac{\mu_D}{\varepsilon}),\ 0 = \lim\ (\varepsilon\mu_S,\ \varepsilon\mu_D),\ 0 < \varepsilon < \varepsilon_0.$$

Dans [1], avait été mis en évidence la densité d'énergie surfacique

$$\overline{W}_{\bar{\mu}_S,\bar{\mu}_D}(v) = W^{\infty,2}_{\bar{\mu}_S,\bar{\mu}_D}(v \otimes_S e_3) := \bar{\mu}_S W_1^{\infty,2}(tr(v \otimes_S e_3)) + \bar{\mu}_D W_2^{\infty,2}(dev(v \otimes_S e_3)),\ \forall v \in \mathbb{R}^3.$$

où

$$a \otimes_S b = \frac{1}{2}\left(a \otimes b + b \otimes a\right),\ \forall a, b \in \mathbb{R}^3$$

qui définissait la densité d'énergie associée a la liaison mécanique le long de S qui remplacait le joint souple parfaitement collé aux adhérents. Dans le cas présent, on a montré que le joint imparfaitement collé va être remplacé par une liaison dont la densité d'énergie associée est l'inf-convolution g de h avec $\overline{W}_{\bar{\mu}_S,\bar{\mu}_D}$:

$$g(t) := h \,\#\, \overline{W}_{\bar{\mu}_S,\bar{\mu}_D}(t) := \inf\left\{h(t') + \overline{W}_{\bar{\mu}_S,\bar{\mu}_D}(t'')\ ;\ t = t' + t'',\ t',\ t'' \in \mathbb{R}^3\right\}$$

Ce qui correspond à la mise en série de la liaison mécanique initiale le long de S de densité h avec la liaison limite de densité $\overline{W}_{\bar{\mu}_S,\bar{\mu}_D}$. Précisément, on établit :

Quand s tend vers $\overline{s}$, alors $\overline{u}_{s\lfloor\Omega}$ converge fortement dans $H^1_{\Gamma_0}(\Omega;\mathbb{R}^3)$ vers l'unique solution $\overline{u}$ de

$$(\overline{\mathcal{P}}) : \qquad Min\left\{F(v) - L(v)\ ;\ v \in H^1_{\Gamma_0}(\Omega;\mathbb{R}^3)\right\}\ \text{ et } F(\bar{u}) = \lim_{s\to\bar{s}} F_s(\bar{u}_s),$$

où

$$F(v) := \begin{cases} \displaystyle\int_\Omega W(e(v))dx + \int_S g(\gamma_0(v))d\widehat{x}, & \text{quand } g(\gamma_0(v)) \in L^1(S), \\ +\infty & \text{sinon.} \end{cases}$$

Ce résultat est établi par les stratégies habituelles de la convergente variationnelle :

i) propriétes de compacité des suites d'énergie bornées,

ii) borné supérieure pour $F_s(u_s)$,

iii) borné inférieure pour $F_s(u_s)$.

Le point i) s'obtient par des estimations en function de ε des constantes intervenant dans les inégalites de Poincaré et de Korn et de continuité de l'opérateur trace sur S dans $H^1_{\Gamma_0}(B_\varepsilon;\mathbb{R}^3)$. Le point ii) s'obtient par un relèvement (similaire à celui de [1]) a B_ε du champ défini sur S et réalisant le minimum dans la définition de $g(\gamma_0(u))$ grâce à une propriété essentielle de Lipschitz-continuité de g, conséquence des propriétés de $W_i^{\infty,2}$. Enfin, l'inégalite sous-différentielle et une intégration par parties donnent le dernier point.

Ainsi, notre proposition de modèle est plus simple que le modèle original : la fonctionnelle intégrale définie sur le domaine tridimensionnel mince B_ε est remplacée par une fonctionnelle intégrale définie sur la surface S vers laquelle la bande adhésive se réduit. Et le résultat précédent de convergence montre que plus s est proche de $\overline{s}$, plus le modèle est précis. En pratique, il conviendrait de remplacer $\lim(\mu_D/\varepsilon)$, $\lim(\mu_S/\varepsilon)$ par les vraies données physiques μ_D/ε, μ_S/ε. On peut encore affiner le modèle par un résultat de type correcteurs en étudiant le comportement asymptotique du déplacement optimal dans l'adhésif. On a montré qu'il est énergétiquement équivalentè un champ fonction affine de x_3 dont la trace sur S est fournie par le minimiseur invoqué dans la définition de $g(\gamma_0(\overline{u}))$.

On donne ensuite divers exemples de densités réalistes h contenant celles traitées par [3] au moyen d'une zoom de la troisième coordonnée dans le joint et d'une formulation indirecte à deux champs (déplacements et contraintes).

Enfin, motivé par le concept tribologique de troisième corps, a été considéeré une variante ou la couche mince contient une couche beaucoup plus mince et plus souple au voisinage des adhérents.

La seconde extension concernant les junctions souples traite le cas où le joint est parfaitement collé aux deux adhérents mais est soumis à un chargement. Pour être réaliste, on considère un problème scalaire, l'inconnue étant, par exemple, la déflexion d'une membrane faite de trois parties et soumise à un chargement même dans l'étroite partie intérieure. Si $\Omega := \Sigma \times (-r, r)$, $r > 0$, où Σ est un domaine borné de $\mathbb{R}^2$, l'adhésif occupe $B_\varepsilon := \Sigma \times (-\varepsilon/2,\ \varepsilon/2)$, et les adhérents $\Omega_\varepsilon := \Omega \backslash \overline{B}_\varepsilon$. Les densités d'énergie de déformations sont respectivement εg et f, g et f sont strictement convexes et à croissances quadratique et on suppose l'existence d'une fonction de récession d'ordre 2 pour g. Les adhérents sont fixées sur $\Gamma_0 \subset \partial\Omega$ et soumis à des forces dont le travail L est une forme linéaire continue sur $H^1_{\Gamma_0}(\Omega_{\varepsilon_0})$. Au contraire, le travail du chargement appliqué à B_ε est défini à partir d'une forme linéaire $\mathcal{S}_\varepsilon$ continue sur $V(B) := \left\{u \in L^2(\Omega)\ ;\ \frac{\partial u}{\partial x_3} \in L^2(\Omega)\right\}$, $B := B_1$. Si τ_ε est l'opérateur de mise à l'échelle, continu de $V(B_\varepsilon)$ sur $V(B)$, défini par $\tau_\varepsilon(u)(\widehat{x}, x_N) := u(\widehat{x}, x_N/\varepsilon)\ \forall x = (\widehat{x}, x_N) \in B_\varepsilon$, alors le travail est la forme linéaire $u \mapsto \langle \mathcal{S}_\varepsilon,\ \tau_\varepsilon u\rangle$.

Ainsi la détermination des configurations d'équilibre conduit au problème

$$(\mathcal{P})_\varepsilon : \qquad Min\left\{F_\varepsilon(u)\ :\ u \in L^2(\Omega)\right\},$$

où $F_\varepsilon : L^2(\Omega) \longrightarrow \mathbb{R} \cup \{+\infty\}$ définie par,

$$F_\varepsilon(u)\ :\ = \begin{cases} \displaystyle\int_\Omega f(\nabla u)dx + \varepsilon \int_{B_\varepsilon} g(\nabla u))dx - \langle \mathcal{S}_\varepsilon,\ \tau_\varepsilon u\rangle - L(u) & \text{quand } u \in W^{1,2}_{\Gamma_0}(\Omega); \\ +\infty & \text{sinon.} \end{cases}$$

Clairement, le problème a une solution unique $\overline{u}_\varepsilon$ dont il s'agit d'analyser le comportement asymptotique lorsque ε tend vers zéro et en supposant que $\mathcal{S}_\varepsilon$ converge fortement vers $\mathcal{S}$ dans le dual de $V(B)$. Il est facile (en procédant comme dans [1]) d'établir que les suites d'énergies bornées sont relativement compactes dans $L^2(\Omega)$ et dans $H^1_{\Gamma_0}(\Omega_\eta)$ faible pour tout η positif. Le calcul de la $L^2(\Omega)-\Gamma$ *limite* des deux premiers termes de F_ε a été effectué en [1], mais comme $u \mapsto \langle \mathcal{S}_\varepsilon,\ \tau_\varepsilon u\rangle$ n'est pas une perturbation continue sur $L^2(\Omega)$, on peut s'attendre à ce que le problème limite mette en jeu un mixage des comportements limites de l'énergie de déformation de la bande et du travail du chargement au quel elle est soumise. Soit

$$\begin{aligned} V_0(B) &= \left\{ V(B)\ :\ u = 0 \text{ on } \Sigma \times \{\pm\frac{1}{2}\} \right\}. \\ G(u) &= \underset{\theta \in V_0(B)}{Min} \left\{ \int_B g^{\infty,2}(\widehat{0}, \frac{\partial \theta_{[u]}}{\partial x_N}(x) + [u](\widehat{x}))\, dx - \left\langle \mathcal{S}, \theta_{[u]} \right\rangle \right\} - \langle \mathcal{S}, \widetilde{u}\rangle \end{aligned}$$

où $\widetilde{u}(x) = x_3[u](\widehat{x}) + \dfrac{u^+(\widehat{x}) + u^-(\widehat{x})}{2}$.

On montre que : *Quand ε tend vers zéro et $\mathcal{S}_\varepsilon$ vers $\mathcal{S}$, alors $\overline{u}_\varepsilon$ converge fortement dans $L^2(\Omega)$ vers l'unique solution de*

$$(\mathcal{P})\ : \qquad Min \left\{ \int_\Omega f(\nabla u)\, dx + G(u) - L(u)\ :\ u \in H^1_{\Gamma_0}(\Omega \backslash \Sigma) \right\}.$$

Ainsi quand la source limite $\mathcal{S}$ est nulle, $G(u)$ se réduit à

$$G(u) = \int_\Sigma g^{\infty,2}(\widehat{0}, [u](\widehat{x}))\, d\widehat{x}$$

qui n'est autre que l'énergie de surface obtenue en [1]. Au contraire si $\mathcal{S}$ n'est pas nulle, G est de la forme

$$G(u) = \int_B g^{\infty,2}(\widehat{0}, \frac{\partial \theta_{[u]}}{\partial x_N}(x) + [u](\widehat{x}))\, dx - \left\langle \mathcal{S}, \theta_{[u]} \right\rangle - \langle \mathcal{S}, \widetilde{u}\rangle\,.$$

où $\theta_{[u]}$ est le minimizeur lié à $G(u)$, G est donc en général une fonctionnelle non locale non seulement du champ de sauts $[u]$ mais aussi des traces sur Σ des restrictions à $\Omega^\pm$ de u. L'intervention d'une variable interne θ vient du fait que $\tau_\varepsilon u_\varepsilon$ converge faiblement vers θ dans $V(B)$ et que, par conséquent une borne inférieure de $\varepsilon \int_{B_\varepsilon} g(\nabla u_\varepsilon)\, dx + \langle \mathcal{S}_\varepsilon, \tau_\varepsilon u_\varepsilon\rangle$ est $G(u)$. Pour obtenir la borné supérieure, on construit u_ε dans B_ε à partir de l'optimum $\theta_{[u]}$ mis en jeu par $G(u)$. Divers exemples de sources $\mathcal{S}_\varepsilon$ sont données du type

$$\mathcal{S}^\varepsilon_{\widehat{x}} = a_\varepsilon(\widehat{x}, .)\, dt + \sum_{n=-\infty}^{+\infty} b_{\varepsilon,n}(\widehat{x}) \delta_{t^\varepsilon_n(\widehat{x})}$$

correspondant à des sources distribuées ou concentrées.

Enfin, on examine la génération par $\nabla \overline{u}_\varepsilon$ d'une "gradient Young mesure de concentration" $\overline{\mu}$ que nous analysons dans l'esprit de [16]. En outre, on exprime le terme non local $G(u)$ en fontion de cette mesure et nous obtenons des bornes sur la mesure de probabilite $\overline{\mu}_{\widehat{x}}$ provenant de la désintégration de $\overline{\mu}$.

La seconde partie de cette thèse concerne la modèlisation de certaines soudures. Pour cela, nous avons reconsidéréles études [21], [26] consacrées aux jonctions raides. Dans [26], l'adhésif et les adhérents ont été considérés comme hyperélastiques avec des densités d'énergie de déformation d'un même ordre de croissance $p \in (1, \infty)$, l'ordre de grandeur de la rigidité de l'adhésif étant celui de l'inverse de son épaisseur. Les cordons de soudure envisagés présentant un caractère fragile, on a, dans un premier temps, considéré l'adhésif comme pseudo-plastique c'est à dire avec une densité d'énergie de déformation à croissance linéaire. Ainsi, du point de vue mathématique, deux difficultés apparaissent : la différence des ordres de croissance des densités d'énergie de déformation de l'adhésif et des adhérents et la croissance linéaire dans l'adhésif qui va impliquer d'utiliser des espaces de champs de déplacements à discontinuités libres. On utilise la même géométrie que précédemment, à savoir $\Omega := S \times (-r, r)$, $r > 0$. où S est un domaine borné de $\mathbb{R}^2$, l'adhésif occupe $B_\varepsilon := S \times (-\varepsilon/2, \varepsilon/2)$ et les adhérents $\Omega_\varepsilon := \Omega \backslash \overline{B}_\varepsilon$. La rigidité du matériau occupant la mince couche étant supposée d'ordre $1/\varepsilon$, on utilisera le cadre des petites perturbations pour modéliser l'adhésif. Sa densité d'énergie de déformation s'exprime comme $1/\varepsilon g(e(u))$, g étant une fonction convexe à croissance linéaire. Quant aux adhérents, il n'y a pas de difficultés mathématiques à supposer leur densité d'énergie de déformation plus généralement comme une fonction (du gradient de déplacement ∇u) f quasi-convexe à croissance d'ordre $p \in (1, \infty)$. La structure faite de l'adhésif parfaitement collé aux deux adhérents est fixée sur une part Γ_0 de $\partial\Omega$ et est soumise à des forces volumiques et surfaciques à support inclus dans $\overline{\Omega}_{\varepsilon_0}$ dont on note $L(.)$ le travail. Ainsi, la détermination des positions d'équilibre conduit au problème

$$(\mathcal{P}_\varepsilon) \; : \qquad \inf \left\{ F_\varepsilon(u) - L(u) : u \in L^1(\Omega, \mathbb{R}^3) \right\}.$$

où $F_\varepsilon : L^1(\Omega,\ \mathbb{R}^3) \longrightarrow \mathbb{R} \cup \{+\infty\}$ définie par,

$$F_\varepsilon(u) \ : \ = \begin{cases} \displaystyle\int_{\Omega_\varepsilon} f(\nabla u)\ dx + \frac{1}{\varepsilon}\int_{B_\varepsilon} g(e(u)\ dx & \text{quand} \quad u \in A_\varepsilon; \\ +\infty & \text{sinon.} \end{cases}$$

et

$$\begin{aligned} A_\varepsilon(\Omega) \ &: \ = \left\{ \ u \in LD(\Omega, \mathbb{R}^3) \ : \ u_{\lfloor \Omega_\varepsilon} \in \ W^{1,p}_{\Gamma_0}(\Omega_\varepsilon, \mathbb{R}^3) \ \right\}. \\ W^{1,p}_{\Gamma_0}(\Omega_\varepsilon, \mathbb{R}^3) \ &: \ = \left\{ \ u \in W^{1,p}(\Omega_\varepsilon, \mathbb{R}^3) \ : \ u = 0 \text{ on } \Gamma_0 \ \right\}. \\ LD(\Omega, \mathbb{R}^3) \ &: \ = \left\{ \ u \in L^1(\Omega, \mathbb{R}^3) \ : \ e(u) \in L^1(\Omega, \mathbb{M}^{3\times 3}_s) \ \right\}. \end{aligned}$$

A cause de la croissance linéaire de g, le problème peut ne pas avoir de solutions mais au moins des ε-minimiseurs. Il a au moins une solution dans

$$\overline{A}_\varepsilon(\Omega) := \left\{ \ u \in BD(\Omega) \ : \ u_{\lfloor \Omega_\varepsilon} \in \ W^{1,p}_{\Gamma_0}(\Omega_\varepsilon, \mathbb{R}^3) \ \right\}.$$

avec

$$BD(\Omega, \mathbb{R}^3) := \left\{ \ u \in L^1(\Omega, \mathbb{R}^3) \ : \ e(u) \in \mathcal{M}(\Omega, \mathbf{M}^{3\times 3}_s) \ \right\}.$$

qu'on notera aussi $\overline{u}_\varepsilon$.

Comme dans le cas [26] où la densité d'énergie de déformation de l'adhésif est à croissance super linéaire, lorsque ε tend vers zéro, l'adhésif va être remplacé par une surface matérielle dont la densité (surfacique) d'énergie de déformation est fonction de la déformation surfacique notée $e(\gamma_S(\widehat{u}))$, où $\gamma_S(\widehat{u})$ désigne la trace sur S des deux premières composantes de tout élément u de $W^{1,p}_{\Gamma_0}(\Omega,\mathbb{R}^3)$ et e est le gradient (au sens des distributions) symétrisé. Cette densité g_0 se déduit de g par :

$$g_0(\zeta) := \min\left\{ g(\xi) : \xi \in \mathbf{M}^{3\times 3}_s, \hat{\xi} = \zeta \right\}, \xi \in \mathbf{M}^{3\times 3}_s \mapsto \hat{\xi} \in \mathbf{M}^{2\times 2}_s;\ \hat{\xi}_{\alpha\beta} = \xi_{\alpha\beta}.$$

Mais, alors que dans le cas superlinéaire, les traces des limites des champs d'énergies bornées ont des tenseurs de déformations surfaciques dans $L^p(\Omega,\mathbf{M}^{2\times 2}_s)$, la croissance linéaire va impliquer que les tenseurs de déformations sont non pas dans $L^1(\Omega,\mathbf{M}^{2\times 2}_s)$ mais dans $\mathcal{M}(\Omega,\mathbf{M}^{2\times 2}_s)$. Plus précisément, si

$$\begin{aligned} A_0(\Omega) \ &: \ = \left\{ u \in W^{1,p}_{\Gamma_0}(\Omega_\varepsilon,\mathbb{R}^3) \ : \ \gamma_S(\widehat{u}) \in BD(S,\mathbb{R}^2) \right\}. \\ BD(\Omega,\mathbb{R}^2) \ &: \ = \left\{ u \in L^1(S,\mathbb{R}^2) \ : \ e(u) \in \mathcal{M}(S,\mathbf{M}^{2\times 2}_s) \right\}. \end{aligned}$$

la fonctionnelle énergie de déformation totale du modèle asymptotique va être :

$$\begin{aligned} F_0 \ &: \ L^1(\Omega,\mathbb{R}^3) \to \mathbb{R}\cup\{+\infty\} \text{ définie par,} \\ F_0(u) \ &: \ = \begin{cases} \displaystyle\int_\Omega f(\nabla u)\, dx + \int_S g_0(e(\gamma_s(\hat{u})) & \text{quand} \quad u \in A_0; \\ +\infty & \text{sinon.} \end{cases} \end{aligned}$$

le dernier terme étant compris au sens d'intégrale de fonction convexe de mesure, au vu du résultat de convergence obtenu : *lorsque ε tend vers zéro, il existe une sous-suite non renumérotée et $\overline{u}$ dans $W^{1,p}_{\Gamma_0}(\Omega,\mathbb{R}^3)$ tel que*

$$\begin{aligned} &\bar{u}_\varepsilon \rightharpoonup \bar{u} \ in\ BD(\Omega,\mathbb{R}^3); \\ &\bar{u}_\varepsilon \rightharpoonup \bar{u} \ in\ W^{1,p}_{\Gamma_0}(\Omega_\eta,\mathbb{R}^3)\ for\ every\ \eta > 0; \\ &\gamma_S(\hat{\bar{u}}) \in BD(S,\mathbb{R}^2). \end{aligned}$$

En outre, $\overline{u}$ est solution du problème

$$(\mathcal{P}) \ : \qquad \mathbf{M}in\Big\{F_0(u) - L(u) : u \in L^1(\Omega,\mathbb{R}^3)\Big\}$$

et

$$F_\varepsilon(\overline{u}_\varepsilon) - L(\overline{u}_\varepsilon) \to F_0(u) - L(u).$$

Dans ce modèle, les traces sur S des champs de déplacement solution peuvent présenter des discontinuités qu'on peut interpréter en termes de macrofissures et en défauts diffus ou fractures fractales. En fait, du aux immersions de Sobolev, les traces $\gamma_S(u)$ étant continues si $p > 3$, $\gamma_S(\widehat{u})$ en tant qu' élément de $BD(S,\mathbb{R}^2)$ ne présente pas de sauts mais seulement des singularités diffuses ou fractales. Il est à noter qu'alors que le modèle de départ peut présenter des fractures dans B_ε, le modèle limite (pour $p > 3$) ne présente que des défauts diffus ou des fractures fractales dans la surface matérielle qui remplace l'adhésif

Prenant en compte la géométrie de la bande, on montre aisément que pour toute suite d'énergie bornée u_ε il existe u dans $W^{1,p}_{\Gamma_0}(\Omega, \mathbb{R}^3)$ et une sous-suite non renumérotée tels que u_ε converge faiblement dans $BD(\Omega, \mathbb{R}^3)$ et $W^{1,p}_{\Gamma_0}(\Omega_\eta, \mathbb{R}^3)$ pour tout η positif vers u et que $\gamma_S(\widehat{u})$ appartient à $BD(S, \mathbb{R}^2)$ et est la limite faible dans $BD(S, \mathbb{R}^2)$ quotienté par l'ensemble des déplacements rigides de $\mathbb{R}^2$ de la moyenne selon x_3 de $\widehat{u}_\varepsilon$ dans B_ε. De cela et de la définition même de g_0, on déduit que F_0 est une borne inférieure possible quant à la Γ-convergence de F_ε vers F_0 pour la topologie forte de $L^1(\Omega, \mathbb{R}^3)$. Pour vérifier la borne supérieure, on montre d'abord que F_0 est la régularisée semi-continue inférieurement pour $L^1(\Omega, \mathbb{R}^3)$ d'une fonctionnelle $\widetilde{F}_0$ de même expression que F_0 mais vivant sur des champs réguliers ($\gamma_S(u) \in C^1(\overline{S}, \mathbb{R}^3)$). Ensuite est établi que $\widetilde{F}_0 \geq \Gamma - \limsup F_\varepsilon$ par le processus de relèvement à B_ε habituel et on conclut en prenant l'enveloppe s.c.i. des deux membres.

Ce résultat est ensuite étendu à une situation plus réaliste de soudage où le domaine occupé par la structure globale (adhérents plus adhésif) varie avec ε au moyen de translations dans la direction de x_3.

Il s'est agi ensuite d'affaiblir les hypothèses de quasiconvexité et de convexité pour f et g respectivement de manière à prendre en considération des matériaux susceptibles de subir des transformations de phase solide/solide réversibles. Un candidat raisonnable de fonctionnelle limite est $F_0 : L^1(\Omega, \mathbb{R}^3) \to \mathbb{R} \cup \{+\infty\}$ définie par,

$$F_0(u) \;:=\; \begin{cases} \displaystyle\int_\Omega Qf(\nabla u)\, dx + \int_S SQg_0(e(\gamma_s(\hat{u}))) & \text{quand } u \in A_0; \\ +\infty & \text{sinon.} \end{cases}$$

où Qf est l'enveloppe quasiconvexe de f et $SQg_0 : \mathbf{M}^{2\times 2}_s \to \mathbb{R}$ est l'enveloppe quasiconvexe symétrique définie par

$$SQg_0(\zeta) := \inf \Big\{ \frac{1}{|\hat{D}|} \int_{\hat{D}} g_0(\zeta + e(\varphi))\, d\hat{x} : \varphi \in \mathcal{C}^\infty_0(\hat{D}, \mathbb{R}^2) \Big\}.$$

Nous n'avons pu établir la borne inférieure que sur le sous ensemble $\widetilde{A}_0$ de A_0 défini par :

$$\tilde{A}_0 := \big\{ u \in W^{1,p}_{\Gamma_0}(\Omega, \mathbb{R}^3) : \gamma_S(\hat{u}) \in SBD(S, \mathbb{R}^2) \big\}$$

grâce à un argument supplémentaire pris dans [27]. Pour la borne supérieure, comme précédemment, on exhibe une fonctionnelle $\widetilde{F}_0$ dont F_0 est la régularisée s.c.i. La difficulté posée par la différence de croissance de f et g est surmontée par l'introduction d'une perturbation $\eta|.|^p$ de g_0.

Le dernier point, examine, à des fins numériques, les possibilités d'une régularisation à la Norton-Hoff de la fonctionnelle F_0 mise en jeu par le modèle limite. Si l'on renomme g_0 en h supposée positivement homogène de degré 1 et telle que $\exists \alpha, \beta > 0;\ \alpha|\xi| \leq h(\xi) \leq \beta|\xi|,\ \forall \xi \in \mathbf{M}^{2\times 2}_s$, on considère une suite $(h_q)_{q \in (1,p)}$ vérifiant.

i) $h_q : \mathbf{M}_s^{2\times 2} \to \mathbb{R}^+$ est convexe et positivement homogène de degré q;

ii) $h_q \to h$ simplement dans $\mathbf{M}_s^{2\times 2}$;

iii) $\exists \alpha > 0;\ \forall q > 1$, assez proche de 1, $h_q(\xi) \geq h(\xi)\ \ \forall \xi \in \mathbf{M}_s^{2\times 2},\ |\xi| \geq a$.

On montre alors que lorsque $q \to 1$ la fonctionnelle $\mathcal{F}_q : W^{1,p}_{\Gamma_0}(\Omega, \mathbb{R}^3) \longrightarrow \mathbb{R}^+ \cup \{+\infty\}$ définie par :

$$\mathcal{F}_q(u) = \begin{cases} \displaystyle\int_\Omega f(\nabla u)\ dx + \int_S h_q(\ e(\gamma_S(\hat{u}))\ )\ d\hat{x} & \text{quand } u \in \mathcal{B}_q, \\ +\infty & \text{sinon.} \end{cases}$$

où

$$\mathcal{B}_q := \left\{ u \in W^{1,p}_{\Gamma_0}(\Omega, \mathbb{R}^3) : h_q \circ (e(\gamma_S(\hat{u}))) \in L^1(\Omega) \right\}.$$

Γ-converge pour la topologie faible de $W^{1,p}_{\Gamma_0}(\Omega, \mathbb{R}^3)$ vers $\mathcal{F}_0 : W^{1,p}_{\Gamma_0}(\Omega, \mathbb{R}^3) \to \mathbb{R}^+ \cup \{+\infty\}$ définie par,

$$\mathcal{F}_0(u) = \begin{cases} \displaystyle\int_\Omega f(\nabla u)\ dx + \int_S h(e(\gamma_S(\hat{u}))) & \text{quand } u \in \mathcal{B}, \\ +\infty & \text{sinon.} \end{cases}$$

où

$$\mathcal{B} := \left\{ u \in W^{1,p}_{\Gamma_0}(\Omega, \mathbb{R}^3) : \gamma_S(\hat{u}) \in BD(S, \mathbb{R}^2) \right\}.$$

Ainsi, si la suite $(h_q)_{q\in(1,p)}$ satisfait en outre la condition de coercivité $\exists \alpha_q > 0;\ \alpha_q|\xi|^q \leq h_q(\xi),\ \forall \xi \in \mathbf{M}_s^{2\times 2}$, alors le problème $Min\,\{\mathcal{F}_q(u) - L(u) : u \in \mathcal{B}_q\}$ possède au moins une solution $\overline{u}_q$ et il existe une sous-suite extraite non renumérotée telle que $\overline{u}_q$ converge faiblement dans $W^{1,p}_{\Gamma_0}(\Omega, \mathbb{R}^3)$ vers $\overline{u}$, solution de $Min\,\{\mathcal{F}(u) - L(u) : u \in \mathcal{B}\}$. Comme les fonctions h_q sont convexes et positivement homogènes de degré q et peuvent être choisies différentiables, les méthodes numériques de l'optimisation convexe peuvent facilement fournir des approximations de $\overline{u}_q$ donc de $\overline{u}$.

Abstract in English

Most of the structures in Civil Engineering consists in assemblies of deformable bodies, thus it is of interest to dispose of efficient models of junctions between deformable solids. The classical schemes of Continuum Mechanics lead to boundary value problems involving several parameters, one being essential: the (low) thickness of the layer filled by the adhesive. For the usual behaviors of the adherents and the adhesive, it is not difficult to prove existence of solutions, but their numerical approximations may be difficult due to the rather low thickness of the adhesive implying a too fine mesh. Moreover, the mechanical properties of adherents and adhesive being very different, the involved systems may be very ill-conditioned. Hence, it is capital to propose simpler but accurate enough models. A classical way is to consider the real geometrical and mechanical data, like thickness, stiffness, etc..., as parameters and to study the asymptotic behavior of the parametrized boundary value problems when these parameters go to a natural limit (0 if the quantity is small, $+\infty$ if it is large). This may be done by various methods: formal asymptotic expansions, singular perturbations. Here, we chose the rigorous point of view of variational analysis by studying the asymptotic behavior of the minimizers of the total mechanical energy functional. We show that they converge (with respect to a topology induced by the mechanical energy) toward the solutions of a minimization problem which will be our proposal of simplified model.

Two main cases of elastic junctions have been treated in this way:

i) the soft junctions, where the stiffness of the junction is far lower than the ones of the adherents (it corresponds to soft adhesive bonded joints), see for instance [1] and the references therein,

ii) the hard junctions, where the stiffness of the junction is far larger than the ones of the adherents (which may occur in some situations of welding), see for instance [21], [26] and the references therein.

Let us recall that a big difference occurs in the nature of the asymptotic models. The soft adhesive junction is replaced by a mechanical constraint between the adherents whose surface energy is a function of the relative displacement of the adherents along the interface the junction shrinks toward. On the contrary, the hard junction is replaced by a material surface perfectly stuck to the adherents, with a surface strain energy density function of the surface gradient of the displacement (here, there is no jump of displacement across

the interface).

Anyway, these models are simpler than the genuine one because surface integral functionals are involved in place of integral functionals on a thin layer. They may be accurate enough due to the rigorous convergence results: the closer the parameters to their natural limits, the sharper the models.

The aim of this work is to generalize these two basic results in various directions. The first part of this thesis is devoted to soft junctions and two extensions are presented. First, we consider the case when the soft adhesive bonded joint is not perfectly stuck to the adherents. It will be shown that the asymptotic model involves the "inf-convolution" of the classical limit density supplied by the joint like in [1] with the densities corresponding to the constraints between the joint and the adherents. A second extension deals with the case when the joint, perfectly bonded to the adherents, is subjected to a loading. It will be shown that the asymptotic constraint incorporates the loading and is described through a non local functional acting on the fields of displacements of each adherent along the limit interface. This point is completed by a study of the gradient concentration phenomenon through a suitable tool of fields of measures. The second part of this thesis concerns hard junctions and is a first attempt to model some fracture phenomenon in soldered joint. So, the classical previous studies [21], [26] are revisited in assuming the joints to be pseudo-plastic, that is to say its bulk energy density has a linear growth. Hence, we have to tackle two new difficulties: the growths of the bulk energy densities in the adherents and in the adhesive are different and the linear growth in the adherent will imply to work in spaces of displacement fields with free discontinuities. First, we will consider the case when the bulk energies of the adherents are quasi-convex while the one of the junction is convex and next we will drop these conditions. Finally, for numerical purpose, we consider a variational regularization (in the Norton-Hoff spirit) of the limit functional involved in our model.

For the sake of simplicity, all the junctions considered here occupy layers of constant thickness and, like the adherents, are assumed to be elastic. Hence the starting equilibrium problems may be formulated in terms of minimization problems in some suitable function spaces and we systematically derive our asymptotic models through variational convergence methods.

Part I

Two mathematical models of bonded adhesive joints

Introduction

This part is devoted to the modeling of soft junctions: mainly, the classical modelling [1] of soft adhesive bonded joints is improved in two ways.

First, an asymptotic model for a thin, soft and imperfectly bonded elastic joint is proposed. Here, we consider that the joint and the adherents are not perfectly stuck together so that the reversible mechanical constraint between them is not pure adhesion as in [1], but is given by some smooth or non smooth convex surface energy densities h^1, h^2. We will show that the asymptotic model (obtained when thickness and stiffness of the joint go to zero) consists in replacing the joint by a mechanical constraint whose surface energy density is the "inf-convolution" of the densities h^1, h^2 and the limit energy $\overline{W}$ obtained in [1]. This corresponds to the connection in series of the classical limit constraint induced by the joint and the constraints between the joint and the adhesives. Hence clearly, $\overline{W}$ is intrinsically associated with the thin and soft elastic joint. The limit model is obtained by studying the variational convergence of the total energy functional of the structure made of the adhesive and the two adherents. For the sake of simplicity, the study is done in the framework of small perturbations (i.e., bulk energy densities as convex functions of the linearized strain tensor).

The second extension considers the case when the joint, perfectly stuck to the adherents, is subjected to body forces. Actually, to be realistic, we consider a scalar problem, the unknown being for instance the out-plane displacement of a membrane made of three parts and submitted to an external loading even in the narrow inner part. The main result (still deduced by variational convergence) is that the asymptotic model still consists in replacing the junction by a constraint, but the total surface energy functional associated with this constraint does incorporate the loading and, generally, is a non local functional not only of the relative displacement of the adherents along the interface the joint shrinks to, but also of the displacements of each adherent on the interface. This study is completed by an analysis of the gradient concentration phenomenon through a suitable tool of "gradient Young-concentration" measure as in [16].

Chapter 1

An asymptotic model for a thin, soft and imperfectly bonded elastic joint

1.1 Introduction

Many studies have been devoted to the asymptotic modeling of soft thin joints by considering the stiffness and the thickness of the joints as small parameters. In the static case, the mathematically rigorous variational approach consists in determining the asymptotic behavior of the minimizers of the total mechanical energy of the structure made of the adherents and the elastic joint when thickness and stiffness go to zero. This can be done in the classical framework of displacements fields [1] or in an extended one of fields of measures [2] which may supply additional pieces of information at lower scale. Anyway, the starting point is to assume that the joint and the adherents are perfectly stuck together so that the asymptotic model consists in replacing the physical joint by an abstract mechanical constraint between the two adherents. This constraint keeps the memory of the joint which disappears at the limit: its surface energy density $\overline{W}$ depends strongly on the relative behavior of the parameters but can be written in an unified way with a mathematical structure similar to the one of the bulk energy density of the genuine joint.

Here, under the assumption of small strains, we assume that the joint and the adherents are not perfectly stuck together so that the reversible mechanical constraint between them is not pure adhesion but is given by some smooth or non smooth convex surface energy densities h^1, h^2. We will show that the asymptotic model consists in replacing the joint by a constraint which, now, is the inf-convolution (or epigraphical sum) of the densities h^i and the limit density $\overline{W}$. This corresponds to the connection in series of the limit constraint induced by the joint and the constraints between the joint and the adherents.

For the sake of clarity, in Section 1.2, we introduce a simplified situation where the joint connects an elastic body to a rigid support: the joint is clamped on the rigid body while there is a mechanical constraint between it and the elastic body. This situation has been considered in [3] in the particular case of linearly elastic adhesive and adherent and of bilateral contact with Tresca's like sliding between them. Their asymptotic analysis uses a rescaling of the coordinates through the joint and a mixed formulation(with two fields: the displacement and the stress) in terms of variational inequality, so that the various limit mechanical constraints are explicited in terms of graphs only(relationships between the stress vector and the displacement). On the contrary, because it is obvious from the mechanical point of view to guess the structure of the energy density of the limit constraint(inf-convolution is the mathematical translation of connecting in series), we prefer to deal directly with the total energy functional and study its variational convergence; this is easily done by adapting the arguments of [1] to this framework, indeed simpler because convex. Then, in Section 1.3, we propose our limit model which takes into account the asymptotic behavior of the displacement inside the adherent and inside the adhesive. The proofs of our statements are given in Section 1.4. Section 1.5 is devoted to a variant important in Tribology(the concept of "third body"), where the thin adhesive layer contains a far thinner and softer layer in the viscinity of the elastic adherent.

Eventually, let us point out that a partial analysis using our method and assorted with numerical experiments may be found in [4] and that our analysis is also valuable when the stiffness of the adhesive is not necessarily low but not too large (say of order strictly lesser than the inverse of the thickness).

1.2 Introducing an elementary situation

As usual, we make no differences between $\mathbb{R}^3$ and the physical euclidean space whose an orthonormal basis is denoted $\{e_1, e_2, e_3\}$ and, for all $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$, $\widehat{\xi}$ stands for (ξ_1, ξ_2).

If Ω is a domain of $\mathbb{R}^3$ included in $\{x_3 > 0\}$ with a Lipschitz boundary $\partial\Omega$ whose intersection with $\{x_3 = 0\}$ is a domain S of $\mathbb{R}^2$, let $B_\varepsilon := S \times (-\varepsilon, 0)$ and $\Omega_\varepsilon := \Omega \cup S \cup B_\varepsilon$, where ε is a small positive number, the boundary of Ω_ε being assumed Lipschitz continuous.

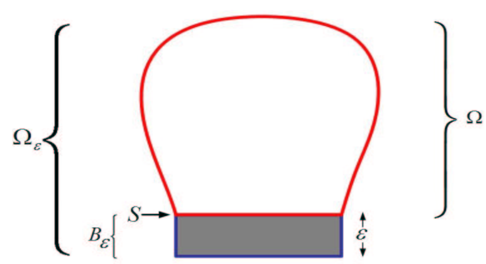

Figure 1.1: The physical domain

Actually, Ω is the reference configuration of an elastic body(the adherent) whose bulk energy density denoted W is a strictly convex function of the linearized strain tensor e which satisfies:

$$\exists \alpha > 0,\ \beta > 0,\ \text{s. t.}\ \ \alpha\,|\xi|^2 \leq W(\xi) \leq \beta\left(1 + |\xi|^2\right),\ \forall \xi \in S^3 \tag{1.2.1}$$

S^3 being the space of the 3×3 symmetric matrices. The body is clamped on $\Gamma_0 \subset \partial\Omega \backslash \bar{S}$ and subjected to body forces f and surface forces φ on $\Gamma_1 := \partial\Omega \backslash (\bar{S} \cup \Gamma_0)$, Γ_0 and Γ_1 being assumed of positive two dimensional Hausdorff measures. The set B_ε is the reference configuration of a thin adhesive layer made of an elastic material whose bulk energy density reads as:

$$\left.\begin{array}{l} W_{\mu_S,\mu_D}(e) := \mu_S\ W_1(tr(e)) + \mu_D\ W_2(dev(e)) \\ tr(e) := e_{11} + e_{22} + e_{33}, \quad dev(e) := e - \dfrac{1}{3}\ tr(e)\ I \end{array}\right\} \tag{1.2.2}$$

I is the identity matrix, μ_S, μ_D are positive numbers, W_1 and W_2 are strictly convex and satisfy:

$$\forall i \in \{1,2\}, \exists \alpha_i > 0,\ \exists \beta_i > 0,\ \text{s. t.}\ \ \alpha_i \left|\xi^i\right|^2 \leq W_i(\xi^i) \leq \beta_i \left(1 + \left|\xi^i\right|^2\right),\ \forall (\xi^1, \xi^2) \in \mathbb{R} \times S^3 \tag{1.2.3}$$

Moreover, we assume that there exists $W_i^{\infty,2}$, $i = 1,2$, strictly convex positively homogeneous of degree 2 on $\mathbb{R}$ and S^3 respectively such that: for each $i = 1,2$ there exist positive real numbers c_i, α_i^∞, β_i^∞ and $r_i \in [1,2)$ such that

$$\left.\begin{array}{ll} \left|W_i(\xi_i) - W_i^{\infty,2}(\xi_i)\right| \leq c_i\,(1 + |\xi_i|^{r_i}), & \forall (\xi_1,\xi_2) \in \mathbb{R} \times S^3 \\ (\xi_i^1 - \xi_i^2)\cdot(\zeta_i^1 - \zeta_i^2) \geq \alpha_i^\infty\ \ |\xi_i^1 - \xi_i^2|^2, & \forall \zeta_i^l \in \partial W_i^{\infty,2}(\xi_i^l),\ \forall (\xi_1^l, \xi_2^l) \in \mathbb{R} \times S^3,\ l = 1,2 \\ |\zeta_i^1 - \zeta_i^2| \leq \beta_i^\infty\ \ |\xi_i^1 - \xi_i^2|, & \forall \zeta_i^l \in \partial W_i^{\infty,2}(\xi_i^l),\ \forall (\xi_1^l, \xi_2^l) \in \mathbb{R} \times S^3,\ l = 1,2 \end{array}\right\} \tag{1.2.4}$$

Note that considering W_{μ_S,μ_D} in place of the classical $\dfrac{\lambda}{2}\,(tr(e))^2 + \mu\,|e|^2$ is a slight generalization of the isotropic linearly elastic case. The mathematical treatment being similar, this generalization seems to us useful because it may concern materials with different behaviors in stretching and squeezing. Actually, there should be minor changes by replacing the growth condition of order 2 by a growth condition of order p in $[1, \infty)$.

The adhesive is clamped along $S_{-\varepsilon} := -\varepsilon e_3 + S$ and not subjected to forces. Here we will consider the case when the mechanical constraint between the adhesive and the adherent is not necessarily pure adhesion but is described by a surface energy density satisfying:

(H) *h is a nonnegative convex lower semi-continuous function in $\mathbb{R}^3$ such that $h(0) = 0$.*

Note that this assumption is satisfied by either a continuous function like $\dfrac{1}{p}\ \ |\cdot|^p$, $1 \leq p < \infty$, or a non-smooth function like the indicator function I_C of a closed convex subset C of $\mathbb{R}^3$ containing 0.

In the following, for all domain G, $H^1_\gamma(G;\mathbb{R}^3)$ will denote the subspace of the Sobolev space $H^1(G;\mathbb{R}^3)$ whose elements vanish on a smooth enough part γ of the boundary ∂G of G.

Hence, if we assume that $(f,\varphi) \in L^2(\Omega;\mathbb{R}^3) \times L^2(\Gamma_1;\mathbb{R}^3)$, it is well-known that the equilibrium configuration is given by the unique solution $\bar{u}_s$ of the following problem involving the triplet $s := (\varepsilon, \mu_S, \mu_D)$ [1]:

$$(\mathcal{P}_s) \ : \quad Min\,\{F_s(v) - L(v)\ ;\ v \in V_s\}$$

with

$$V_s = \left\{ v \in L^2(\Omega_\varepsilon;\mathbb{R}^3)\ ;\ v^+ := v_{\lfloor\Omega} \in H^1_{\Gamma_0}(\Omega;\mathbb{R}^3),\ v^- := v_{\lfloor B_\varepsilon} \in H^1_{S_{-\varepsilon}}(B_\varepsilon;\mathbb{R}^3) \right\}; \tag{1.2.5}$$

$$L(v) = \int_\Omega f(x)\,v(x)\,dx + \int_{\Gamma_1} \varphi(x)\,v(x)\,ds, \text{ the work of the exterior loading;} \tag{1.2.6}$$

$$F_s(v) = \int_\Omega W(e(v^+))\,dx + \int_{B_\varepsilon} W_{\mu_S,\mu_D}(e(v^-))\,dx + \int_S h([v](\widehat{x}))\,d\widehat{x}. \tag{1.2.7}$$

$$[v] := \gamma_0(v^+) - \gamma_0(v^-), \text{ the jump of displacement across S}$$

(or the relative displacement along S) where the same symbol $\gamma_0(w)$ denotes the trace on S of any element w of both $H^1(B_\varepsilon;\mathbb{R}^3)$ and $H^1(\Omega;\mathbb{R}^3)$. To shorten notations, in the sequel we will drop the superscript $+$ and $-$ for any element v of V_s when no confusion is possible.

Actually, determining numerical approximations of $\bar{u}_s$ may be tricky because of the large number of degrees of freedom implied by the meshing of the very thin adhesive layer and the ill-conditioned system due to the low stiffness of the glue.

An important practical example of such constraints is when $h(v) = \gamma\Phi_p(\widehat{v}) + I_E(v_3)$, γ is a non negative number, E is a closed interval of $\mathbb{R}$ containing 0, and $\Phi_p = a_p\ |\cdot|^p$, $a_p > 0$ if $p \in [1,\infty)$, $\Phi_\infty = I_\mathcal{C}$, $\mathcal{C}$ a closed convex subset of $\mathbb{R}^2$ containing 0. The constraint implies bilateral contact when $E = \{0\}$, play(without penetration) when $E = [0,\ \delta]$, $\delta > 0$, and unilateral contact(without penetration) when $E = [0,\ \infty)$. Moreover, the case $\gamma = 0$ says that the tangential component of the stress vector always vanishes. When $\gamma \neq 0$, the sole realistic situation is $E = \{0\}$, then the constraint corresponds to bilateral contact with confined sliding when $p = \infty$ or sliding with resistance of Tresca type when $p = 1$ or Norton-Hoff type when $1 < p < \infty$. The particular case $p = 1$, $E = \{0\}$ has been treated in [3].

Thus, it is of interest to propose a simpler but accurate enough model for the structure. For that purpose, we will consider s as a triplet of small parameters and derive our modeling through a rigorous mathematical study of the asymptotic behavior of $\bar{u}_s$ when s goes to zero by due account of the low values of the thickness and the stiffness of the adhesive. Actually, the following analysis works also when $\varepsilon\mu_S$, $\varepsilon\mu_D$ goes to zero, that is

[1] Actually ε, μ_S and μ_D denote three sequences $(\varepsilon_n)_{n\in\mathbb{N}}$, $(\mu_{S,n})_{n\in\mathbb{N}}$ and $(\mu_{D,n})_{n\in\mathbb{N}}$.

to say also when the stiffness is not too high. Henceforth, we will make the assumption: there exists $\bar{s} \in \{0\} \times [0, \infty]^2$, $(\bar{\mu}_S, \bar{\mu}_D) \in [0, \infty]^2$ and a positive real number ε_0 such that

$$\bar{s} = \lim s, \quad (\bar{\mu}_S, \bar{\mu}_D) = \lim \left(\frac{\mu_S}{\varepsilon}, \ \frac{\mu_D}{\varepsilon} \right), \quad 0 = \lim (\varepsilon \ \mu_S \ , \varepsilon \ \mu_D), \quad 0 < \varepsilon < \varepsilon_0, \tag{1.2.8}$$

As usual, c or C will denote constants independent of s which may vary from line to line.

1.3 An asymptotic model

The theory initiated in [5] and completed in [1] says that the thin layer is, when s tends to zero, asymptotically equivalent to a mechanical constraint along S whose surface energy density reads as:

$$\overline{W}_{\bar{\mu}_S, \bar{\mu}_D}(v) = W^{\infty,2}_{\bar{\mu}_S, \bar{\mu}_D}(v \otimes_S e_3) := \bar{\mu}_S W^{\infty,2}_1(tr \, (v \otimes_S e_3)) + \bar{\mu}_D W^{\infty,2}_2(dev(v \otimes_S e_3)) \quad \text{for all } v \in \mathbb{R}^3 \tag{1.3.1}$$

where

$$a \otimes_S b = \frac{1}{2} (a \otimes b + b \otimes a) \quad \text{ for all } a, b \in \mathbb{R}^3$$

Of course when $\bar{\mu}_S, \bar{\mu}_D$ are not finite, $\bar{\mu}_S W^{\infty,2}_1$, $\bar{\mu}_D W^{\infty,2}_2$ is replaced by $I_{\{0\}}$. The convention $\infty \times 0 = 0$ makes then possible the unified writing (1.3.1). Note that

$$\begin{cases} tr(v \otimes_S e_3) = v_3 \\ dev(v \otimes_S e_3)_{11} = dev(v \otimes_S e_3)_{22} = -\dfrac{1}{2} dev(v \otimes_S e_3)_{33} = -\dfrac{v_3}{3}, \\ dev(v \otimes_S e_3)_{12} = 0, \ \ dev(v \otimes_S e_3)_{\alpha 3} = \dfrac{v_\alpha}{2}, \quad \alpha = 1, \, 2. \end{cases}$$

so that

$$\overline{W}_{\bar{\mu}_S, \bar{\mu}_D}(v) \geq \frac{\alpha_2}{2} \ \bar{\mu}_D \ |v|^2 \quad \text{ for all } v \in \mathbb{R}^3. \tag{1.3.2}$$

Thus we expect to be in face with the connecting in series of the initial mechanical constraint with surface energy density h and the limit one described by $\overline{W}_{\bar{\mu}_S, \bar{\mu}_D}$. Because the graph of the connecting in series of two constraints is obtained by taking the inverse of the addition of the inverse of the graph of each constraint, our asymptotic model should involve the inf-convolution or epigraphical sum $g : h \# \overline{W}_{\bar{\mu}_S, \bar{\mu}_D}$ of h and $\overline{W}_{\bar{\mu}_S, \bar{\mu}_D}$ in [6] defined by:

$$g(t) := h \# \overline{W}_{\bar{\mu}_S, \bar{\mu}_D}(t) := \inf \left\{ h(t') + \overline{W}_{\bar{\mu}_S, \bar{\mu}_D}(t'') \ ; \ t = t' + t'', \, t', \ t'' \in \mathbb{R}^3 \right\} \tag{1.3.3}$$

To deal easily with some singular cases, we introduce the additional assumption on h:

(H') *When $(\bar{\mu}_S, \bar{\mu}_D) \in \{\infty\} \times (0, \infty)$ or $(\bar{\mu}_S, \bar{\mu}_D) \in (0, \infty) \times \{0\}$, then $h(t) = \hat{h}(\hat{t}) + h_3(t_3)$ for all t in $\mathbb{R}^3$ with $\hat{h}$, h_3 two non negative convex lower semicontinuous functions in $\mathbb{R}^2$, $\mathbb{R}$ respectively and such that $\hat{h}(0) = h_3(0) = 0$,* which enables us to establish the following properties of g:

Proposition 1.3.1. *The inf-convolution $g = h \;\#\; \overline{W}_{\bar\mu_S,\bar\mu_D}$ is a non negative convex function defined in $\mathbb{R}^3$ such that $g(0) = 0$ and*

i) *when $\bar\mu_D = \infty$: $g = h$,*

ii) *when $\bar\mu_D \in (0,\ \infty)$ and*

a) *$\bar\mu_S \in [0,\ \infty)$: g is continuous on $\mathbb{R}^3$, $g \le Min\left(h,\ \beta_2\ |\cdot|^2\right)$; moreover for all t in $\mathbb{R}^3$ there exists a unique $z(t)$ in $\mathbb{R}^3$ such that $g(t) = h(\ t - z(t)\ ) + \overline{W}_{\bar\mu_S,\bar\mu_D}(z(t))$, z being a Lipschitz continuous function vanishing at 0,*

b) *$\bar\mu_S = \infty$: $g(t)$ is finite if and only if $h_3(t_3)$ is finite and then g satisfies $g(t) \le h_3(t_3) + C\ \left|\hat t\right|^2$; moreover for all $\hat t$ in $\mathbb{R}^2$, there exists a unique $\widehat z(\hat t)$ in $\mathbb{R}^2$ such that $g(t) = h(t_3) + \hat h(\ \hat t - \widehat z(\hat t)\ ) + \bar\mu_D W_2^{\infty,2}\left(\ dev\left(\ (\ \widehat z(\hat t), 0) \otimes_S e_3\ \right)\ \right)$, $\widehat z$ being a Lipschitz continuous function vanishing at 0, the previous constant c and the Lipschitz continuity constant of $\widehat z$ being independent of t_3.*

iii) *when $\bar\mu_D = 0$, and*

a) *$\bar\mu_S = 0$: $\ g = 0$*

b) *$\bar\mu_S \in (0,\ \infty)$: $\ g(t) = h_3 \;\#\; \bar\mu_S W_1^{\infty,2}(t_3)$ for all $t \in \mathbb{R}^3$, so that g is continuous on $\mathbb{R}^3$, $g(t) \le Min\left(h_3(t_3),\ C\ |t_3|^2\right)$ and there exists a unique $z_3(t_3)$ such that $g(t) = h_3(t_3 - z_3(t_3)) + \bar\mu_S W_1^{\infty,2}(z_3(t_3))$, z_3 being a Lipschitz function vanishing at 0,*

c) *$\bar\mu_S = \infty$: $\ g(t) = h_3(t_3)$ for all $t \in \mathbb{R}^3$.*

The proof of this capital proposition is given in Section 1.4. Hence, we are in a position to define a functional in $H^1_{\Gamma_0}(\Omega;\mathbb{R}^3)$ by:

$$F(v) := \begin{cases} \displaystyle\int_\Omega W(e(v))dx + \int_S g(\gamma_0(v))d\widehat{x}, & \text{when } g(\gamma^0(v)) \in L^1(S), \\ +\infty & \text{otherwise.} \end{cases}$$

Indeed to deal with the singular case $\bar\mu_S = \bar\mu_D = 0$, we, from now on, make the additional assumption:

$$(H'') \qquad \limsup_{s\to\bar s} \frac{\varepsilon^3}{\mu_S} = \limsup_{s\to\bar s} \frac{\varepsilon^3}{\mu_D} = 0;$$
when $\bar\mu_S = \bar\mu_D = 0$, $\quad h$ is coercive on $\mathbb{R}^3$,

which is essential, in this singular case, to establish following Proposition 1.3.3 that describes the asymptotic behavior of v_s^- for a sequence such that $F_s(v_s) < C$; so that our asymptotic model will be supplied by the convergence result:

Theorem 1.3.2. *When s goes to $\bar{s}$, $\bar{u}_s^+$ strongly converges in $H^1_{\Gamma_0}(\Omega;\mathbb{R}^3)$ towards the unique solution $\bar{u}$ of*

$$(\bar{\mathcal{P}}) \ : \qquad Min\{F(v) - L(v) \ ; \ v \in H^1_{\Gamma_0}(\Omega;\mathbb{R}^3)\}$$

and $F(\bar{u}) = \lim_{s\to\bar{s}} F_s(\bar{u}_s)$.

Actually, this result of variational convergence is a classical consequence of the following three propositions whose proofs can also be found in Section 1.4 :

Proposition 1.3.3. *For all sequences (v_s) in V_s such that $F_s(v_s) \leq C$, there exists a subsequence not relabelled such that:*

i) v_s^+ *weakly converges in* $H^1_{\Gamma_0}(\Omega;\mathbb{R}^3)$ *toward some* v*;*

ii) a) $\displaystyle\int_{B_\varepsilon} |v_s^-|^2\, dx \leq C\varepsilon^2 \left(\frac{1}{\mu_S} + \frac{1}{\mu_D}\right)$;

b) $\gamma_0(v_s^-)$ *weakly converges in* $L^2(S;\mathbb{R}^3)$ *towards some* ζ*; moreover* $\zeta_3 = 0$ *if* $\bar{\mu}_S = \infty$, $\zeta = 0$ *if* $\bar{\mu}_D = \infty$, *and when* $\bar{\mu}_S = \bar{\mu}_D = \infty$, $\gamma_0(v_s^-)$ *converges strongly in* $L^2(S;\mathbb{R}^3)$ *toward* 0.

Proposition 1.3.4. *For all u in $H^1_{\Gamma_0}(\Omega;\mathbb{R}^3)$, there exists a sequence (u_s) in V_s such that u_s^+ strongly converges in $H^1_{\Gamma_0}(\Omega;\mathbb{R}^3)$ toward u and $F(u) \geq \limsup_{s\to\bar{s}} F_s(u_s)$.*

Proposition 1.3.5. *For all u in $H^1_{\Gamma_0}(\Omega;\mathbb{R}^3)$ and all sequence (v_s) in V_s such that v_s^+ weakly converges in $H^1_{\Gamma_0}(\Omega;\mathbb{R}^3)$ toward u, we have:*

i) $\displaystyle J(u) := \int_\Omega W(e(u))\, dx \leq \liminf_{s\to\bar{s}} \int_\Omega W(e(v_s))\, dx$;

ii) $\displaystyle\int_S g(\gamma_0(u))\, d\widehat{x} \leq \liminf_{s\to\bar{s}} \left(\int_S h([v_s])\, d\widehat{x} + \int_{B_\varepsilon} W_{\mu_S,\mu_D}(e(v_s))\, dx\right)$;

iii) $F(u) \leq \liminf_{s\to\bar{s}} F_s(v_s)$.

Thus Theorem 1.3.2 describes the asymptotic behavior of the displacement field inside the adherent. Our limit model, problem$(\bar{\mathcal{P}})$, concerns the equilibrium of the elastic adherent subjected to body forces f and surface forces φ on Γ_1, clamped along Γ_0 and subjected to a mechanical constraint along S of energy density g which is the inf-convolution of the genuine energy h with the limit surface energy $\overline{W}_{\bar{\mu}_S,\bar{\mu}_D}$ stemming from the bulk energy of the thin adhesive layer.

When the adhesive is isotropic and linearly elastic i.e: $W_1 = \dfrac{1}{2}|\cdot|^2$, $W_2 = |\cdot|^2$, $\mu_S = \dfrac{3\lambda + 2\mu}{3}$, $\mu_D = \mu$, λ and μ being the classical Lame$'$ coefficients and $h_3 = I_{\{t_3=0\}}$, $\hat{h} = \dfrac{a_p}{p}|\cdot|^p$, $p \in [1,\infty)$, $\hat{h} = I_{\{|\hat{t}|\leq 1\}}$ if $p = \infty$, it is easy to establish that

i) $\lim \dfrac{\mu}{\varepsilon} = \infty$: $\quad g(t) = \dfrac{a_p}{p} \left|\widehat{t}\right|^p + I_{\{t_3=0\}}(t_3);$

ii) $\lim \dfrac{\mu}{\varepsilon} = \bar{\mu} \in (0, \infty);$

a) $\lim \dfrac{\lambda}{\varepsilon} = \bar{\lambda} \in (0, \infty)$:

$$\begin{array}{ll} p = 1, & g(t) = Min\left(\mu \ \left|\widehat{t}\right|^2, \, a_1 \left|\widehat{t}\right|\right) + \dfrac{\bar{\lambda}}{2}\, t_3^2; \\ 1 < p < \infty, & g(t) = h \,\#\, \overline{W}_{\bar{\mu}_S, \bar{\mu}_D}(t); \\ p = \infty, & g(t) = \mu \ \left(Min\left(\left|\widehat{t}\right| - 1, \, 0\right)\right)^2 + \dfrac{\bar{\lambda}}{2}\, t_3^2 \end{array}$$

b) $\lim \dfrac{\lambda}{\varepsilon} = \infty$:

$$\begin{array}{ll} p = 1, & g(t) = Min\left(\mu \ \left|\widehat{t}\right|^2, \, a_1 \left|\widehat{t}\right|\right) + I_{\{t_3=0\}}(t_3); \\ 1 < p < \infty, & g(t) = h \,\#\, \overline{W}_{\bar{\mu}_S, \bar{\mu}_D}(t); \\ p = \infty, & g(t) = \mu \ \left(Min\left(\left|\widehat{t}\right| - 1, \, 0\right)\right)^2 + I_{\{t_3=0\}}(t_3) \end{array}$$

iii) $\lim \dfrac{\mu}{\varepsilon} = 0$:

a) $\lim \dfrac{\lambda}{\varepsilon} = \bar{\lambda} \in (0, \infty),\ g(t) = \dfrac{\bar{\lambda}}{2}\, t_3^2;$

b) $\lim \dfrac{\lambda}{\varepsilon} = \infty,\ g(t) = I_{\{t_3=0\}}(t_3).$

From the mechanical point of view, it is interesting to also determine the graph of the previous constraints that is to say the relationship between the stress vector σ_n and the displacement u. We denote the stress and the unit normal outward Ω along S by σ and n, of course $n = -e_3$, so that the displacement and stress vector have tangential and normal component given by:

$$\sigma_T = \sigma_n - (\sigma_N)\ n,\ \sigma_N = \sigma_n \cdot n\ ;\ u_T = u - u_N\, n,\ u_N = u \cdot n.$$

Hence, the various constraints may also read as:

i) $\lim \dfrac{\mu}{\varepsilon} = \infty$: $\ u_N = 0$ and

- $p = 1,\ |\sigma_T| \leq a_1,\ |\sigma_T| < a_1 \Longrightarrow \sigma_T = 0,\ |\sigma_T| = a_1 \Longrightarrow \exists \lambda \geq 0\ ;\ u_T = -\lambda\, \sigma_T;$
- $1 < p < \infty,\ \sigma_T = -a_p\ |u_T|^{p-2}\ u_T;$
- $p = \infty,\ |u_T| \leq 1.$

  the constraint corresponds to a bilateral contact with sliding through Norton-Hoff like resistance($1 < p < \infty$) or Tresca like ($p = 1$) resistance or a confined sliding ($p = \infty$); the joint has no effect.

ii) $\lim \dfrac{\mu}{\varepsilon} = \bar{\mu} \in (0, \infty)$:

- $p = 1$, $|\sigma_T| \leq a_1$, $|\sigma_T| < a_1 \Longrightarrow \sigma_T = -2\mu\ u_T$, $|\sigma_T| = a_1 \Longrightarrow \exists \lambda > 0$; $u_T = -\lambda\ \sigma_T$;
- $1 < p < \infty$, $u_T = \left(\dfrac{1}{2\bar{\mu}} + a_p^{1-p'}\ |\sigma_T|^{p'-2} \right) \sigma_T$, $\frac{1}{p} + \frac{1}{p'} = 1$;
- $p = \infty$, $\sigma_T = \mu\ Min\,(|u_T| - 1,\ 0)\, u_T$;

and

a) $\lim \dfrac{\lambda}{\varepsilon} = \bar{\lambda} \in (0, +\infty)$:
$\sigma_N = -\left(\bar{\lambda} + 2\bar{\mu}\right) u_N$, the constraint corresponds to an elastic pull-back, non linear for the tangential component and linear for the normal one;

b) $\lim \dfrac{\lambda}{\varepsilon} = \infty$:
$u_N = 0$, the constraint corresponds to a bilateral contact with sliding through non linear resistance.

iii) $\lim \dfrac{\mu}{\varepsilon} = 0$:

a) $\lim \dfrac{\lambda}{\varepsilon} = \bar{\lambda} \in [0, +\infty)$:
$\sigma_T = 0$, $\sigma_N = \bar{\lambda}\ u_N$, the body is tangentially free and subjected to a normal linear elastic pull-back;

b) $\lim \dfrac{\lambda}{\varepsilon} = \infty$:
$\sigma_T = 0$, $u_N = 0$ that is to say free bilateral contact occurs.

It is worthwhile to recall that the laws of case ii) are used for numerical or theoretical regularization goals [7], [8].

This results were obtained in [3] for $p = 1$, but the structure of the density of the limit constraint as an epigraphical sum was not observed; this last point seems to us important for the mechanical interpretation of the mathematical analysis and its conclusions.

An other realistic example is when $h = 0$, $h_3 = I_{[0,T]}$, T very large which will supply a so-called normal compliance law which permits a slight penetration in the half-space $\{x_3 \leq 0\}$ but with a stiff normal pull-back; it corresponds to the penalization methods in numerical Contact Mechanics [8].

Remark 1.3.6. : *The coercivity assumption (H'') on h when $\bar{\mu}_S = \bar{\mu}_D = 0$ was made to get boundedness in $L^2(S, \mathbb{R}^3)$ of $\gamma_0(v_s^-)$ for any sequence such that $F_s(v_s) \leq c$. But this condition prohibits the so-frequent condition of unilateral contact where $h_3 = I_{\{t_3 \geq 0\}}$. It has been many times observed that the framework of small strains is not suitable to model soft adhesive joints (especially here where $\bar{\mu}_S = \bar{\mu}_D = 0$ suggests very large strains) because it does not take into account interpenetration condition. A first classical remedy* *is to remain in the frame work of small strains but to include in the formulation of* $(\mathcal{P}_s)$ *a global interpenetration condition like*

$$(\gamma_0(v^+))_3(\widehat{x}) \geq -\varepsilon \quad \text{for a.e. } \widehat{x} \in S. \tag{1.3.4}$$

With this additional assumption the case $h(t) = \hat{h}(\widehat{t}) + I_{\{t_3 \geq 0\}}(t_3)$ *may be treated because a sequence with bounded* $F_s(v_s)$ *will satisfy*

$$\gamma_0(v_s^+)_3 \geq \gamma_0(v_s^-)_3 \geq -\varepsilon.$$

so that the boundedness of $\int_\Omega |e(v_s^+)|^2\, dx$ *implies the boundedness of* $\gamma_0(v_s^+)_3$ *in* $L^2(S)$ *and consequently the one of* $\gamma_0(v_s^-)_3$. *Then the limit constraint with energy density* g *will be the inf-convolution of* h *and* $\overline{W}_{\bar{\mu}_S,\bar{\mu}_D}$ *augmented by* $I_{\{t_3 \geq 0\}}$.

It is interesting to improve the modeling by studying the asymptotic behavior of the adhesive layer, which was suggested in [3]. First, we have:

Proposition 1.3.7. *When* s *goes to* $\bar{s}$, $\gamma_0((\bar{u}_s)^-)$ *weakly converges in* $L^2(S;\mathbb{R}^3)$ *towards* $\bar{\zeta}$ *such that*

$$g(\gamma_0(\bar{u})(\widehat{x})) = h\left(\gamma_0(\bar{u})(\widehat{x}) - \bar{\zeta}(\widehat{x})\right) + \overline{W}_{\bar{\mu}_S,\bar{\mu}_D}(\bar{\zeta}(\widehat{x})) \quad \text{for a.e. } \widehat{x} \in S. \tag{1.3.5}$$

Moreover if h *is strictly convex and coercive, then* $\gamma_0((\bar{u}_s)^-)$ *strongly converges in* $L^2(S;\mathbb{R}^3)$.

Furthermore, it is shown in the proof of Proposition 1.3.4, that there exists $\overline{Z}$ in $H^1(B_{\varepsilon_0};\mathbb{R}^3)$ such that $\bar{\zeta} = \gamma_0(\overline{Z})$ and let $R_\varepsilon\bar{\zeta} \in H^1_{S_{-\varepsilon}}(B_\varepsilon;\mathbb{R}^3)$ defined by

$$R_\varepsilon\bar{\zeta}(x) = \overline{Z}(x)\left(1 + \frac{x_3}{\varepsilon}\right) \quad \text{for all } x \in B_\varepsilon. \tag{1.3.6}$$

Theorem 1.3.8. *With above definition*

$$\lim_{s\to\bar{s}} \int_{B_\varepsilon} W_{\mu_S,\mu_D}(e((\bar{u}_s)^-) - R_\varepsilon\bar{\zeta}))\, dx = \lim_{s\to\bar{s}} \varepsilon^{-2} \int_{B_\varepsilon} \left|(\bar{u}_s)^- - R_\varepsilon\bar{\zeta}\right|^2\, dx = 0 \tag{1.3.7}$$

Proposition 1.3.7 identifies the weak(strong) limit in $L^2(S;\mathbb{R}^3)$ of $\gamma_0((\bar{u}_s)^-)$ as the field which achieves the minimum defining the epigraphical sum g while Theorem 1.3.8 tells us that $(\bar{u}_s)^-$ is asymptotically equivalent to a field affine in x_3 with a profile on S given precisely by this limit $\bar{\zeta}$.

Hence, from a practical point of view to get easily a good approximation of $\bar{u}_s$, we suggest first to solve($\mathcal{P}$)(which is standard from a numerical point of view) where $\bar{\mu}_S, \bar{\mu}_D$ are replaced by the true real values $\dfrac{\mu_S}{\varepsilon}$, $\dfrac{\mu_D}{\varepsilon}$ and to replace $\bar{u}_s^+$ by the solution $\bar{\bar{u}}_s$. Next, $(\bar{u}_s)^-$ may be replaced by $R_\varepsilon\bar{\bar{\zeta}}_s$, $\bar{\bar{\zeta}}_s$ achieving the minimum in the definition of $\left(h \# \bar{W}_{\frac{\mu_S}{\varepsilon},\frac{\mu_D}{\varepsilon}}\right)(\gamma_0(\bar{\bar{u}}_s))$.

1.4 Proofs of the various results

1.4.1 Proof of Proposition 1.3.1:

The first claim and i) are obvious(see [6]). The growth condition in ii) a) is obtained by choosing $(t', t'') = (0, t)$, $(t', t'') = (t, 0)$ in the definition (1.3.3) of g and due account of (1.2.3). The existence and uniqueness of $z(t)$ for all t in $\mathbb{R}^3$ stems from the fact that $h(t - \cdot) + \overline{W}_{\mu_S,\mu_D}$ is a strictly convex coercive function on $\mathbb{R}^3$. Moreover

$$\begin{aligned} h\left(t_j - z(t_j)\right) &\geq h\left(t_i - z(t_i)\right) + \xi\left(z(t_i)\right).\left(t_j - z(t_j) - \left(t_i - z(t_i)\right)\right), \\ \xi\left(z(t_i)\right) &\in \partial \overline{W}_{\bar{\mu}_S,\bar{\mu}_D}\left(z(t_i)\right), i \neq j \in \{1,2\}, \end{aligned}$$

so that (1.2.4) yields that there exists $\bar{\alpha}$, $\bar{\beta}$ such that

$$\bar{\alpha}\ \left|z(t_i) - z(t_j)\right|^2 \leq \bar{c}\ \left|z(t_i) - z(t_j)\right|\ \left|t_i - t_j\right|.$$

The point ii) b) is obtained through the same reasoning but with due account of $\bar{\mu}_s\ W_1^{\infty,2}(v_3) = I_{\{0\}}(v_3)$. Eventually, iii-a) is obvious because $\overline{W}_{0,0} = 0$, and the splitting assumption $h(t) = \widehat{h}(\widehat{t}) + h_3(t_3)$ is just made in case b) to easily use the previous reasoning and in case c) to note that $g(t) = \inf\left\{\widehat{h}(\widehat{t} - \widehat{v}) + h_3(t_3);\ \widehat{v} \in \mathbb{R}^2\right\}$.

1.4.2 Proof of Proposition 1.3.3:

Point i) is an obvious consequence of the coercivity of W and of the Korn inequality. Moreover, if M_3 denotes the set of all 3×3 matrices, a Korn inequality like

$$\left|\nabla v\right|_{L^2(B_\varepsilon;M_3)} \leq C_K\ \left|e(v)\right|_{L^2(B_\varepsilon)} \quad \text{for all } v \in H^1_{S_{-\varepsilon}}(B_\varepsilon;\mathbb{R}^3) \tag{1.4.1}$$

with C_K independent of ε holds for all $\varepsilon < \varepsilon_0$, because it suffices to use the Korn inequality in $H^1_{S_{-\varepsilon_0}}(B_{\varepsilon_0})$ for the extension of v by 0 into $B_{\varepsilon_0} \backslash B_\varepsilon$. Hence, the standard inequality

$$\int_{B_\varepsilon} \left|v_s^-\right|^2\ dx \leq \varepsilon^2\ \int_{B_\varepsilon} \left|\partial_3 v_s^-\right|^2\ dx \tag{1.4.2}$$

implies

$$\int_{B_\varepsilon} \left|v_s^-\right|^2\ dx \leq C_K\ \varepsilon \left(\frac{\varepsilon}{3\mu_S}\frac{1}{\alpha_1}\int_{B_\varepsilon} \mu_S\ W_1(tr(e(v_s^-)))\ dx + \frac{\varepsilon}{\mu_D}\frac{1}{\alpha_2}\int_{B_\varepsilon} \mu_D\ W_2(dev(e(v_s^-)))\ dx\right) \tag{1.4.3}$$

that is ii-a); while the other standard one

$$\int_S \left|\gamma_0(v_s^-)\right|^2\ d\widehat{x} \leq \varepsilon\ \int_{B_\varepsilon} \left|\partial_3 v_s^-\right|^2\ dx \tag{1.4.4}$$

gives

$$\int_S \left|\gamma_0(v_s^-)\right|^2\ d\widehat{x} \leq C_K\ \left(\frac{\varepsilon}{3\mu_S}\frac{1}{\alpha_1}\int_{B_\varepsilon} \mu_S\ W_1(tr(e(v_s^-)))\ dx + \frac{\varepsilon}{\mu_D}\frac{1}{\alpha_2}\int_{B_\varepsilon} \mu_D\ W_2(dev(e(v_s^-)))\ dx\right) \tag{1.4.5}$$

which implies that $\gamma_0(v_s^-)$ is bounded in $L^2(S;\mathbb{R}^3)$ when $\bar{\mu}_S,\ \bar{\mu}_D \in (0,\infty]$. The same is true when $\bar{\mu}_S = \bar{\mu}_D = 0$ by due account of the additional coercivity property(H') of h and previous established point i). Clearly, (1.4.5) implies that $\gamma_0(v_s^-)$ strongly converges in $L^2(S;\mathbb{R}^3)$ to 0 when $\bar{\mu}_S = \bar{\mu}_D = \infty$, the remaining part is obtained by going to the limit in the identities:

$$\int_S p(\widehat{x})\ \Im^i e_3 \cdot \gamma_0(v_s^-)(\widehat{x})\ d\widehat{x} = \int_{B_\varepsilon} p(\widehat{x})\ \Im^i \cdot e(v_s^-)(x)\ dx + \int_{B_\varepsilon} div(p\ \Im^i)\cdot(v_s)(x)\ d\widehat{x},\ i = 1,4,$$

with

$$\Im^1 = I, \Im^2 = e_1 \otimes_S e_3, \Im^3 = e_2 \otimes_S e_3, \Im^4 = e_1 \otimes e_1 + e_2 \otimes e_2 - 2e_3 \otimes e_3.$$

p infinitely differentiable with compact support in S and the inequalities:

$$\left(\int_{B_\varepsilon} p\ \Im^1 \cdot e(v_s)\ dx\right)^2 \leq C(p)\frac{\varepsilon}{\mu_S}\frac{1}{\alpha_1}\int_{B_\varepsilon} \mu_S\ W_1(tr(e(v_s)))\ dx$$

$$\left(\int_{B_\varepsilon} p\ \Im^j \cdot e(v_s)\ dx\right)^2 \leq C(p)\frac{\varepsilon}{\mu_D}\frac{1}{\alpha_2}\int_{B_\varepsilon} \mu_D\ W_2(dev(e(v_s)))\ dx, \quad j = 2,4.$$

$$\begin{aligned}\left(\int_{B_\varepsilon} div(p\ \Im^j)\cdot v_s\ dx\right)^2 \ &\leq\ C(p)\ C_K^2\ \varepsilon^2 \left(\frac{\varepsilon}{\mu_S}\frac{1}{\alpha_1}\int_{B_\varepsilon} \mu_S\ W_1(tr(e(v_s)))\ dx\right.\\ &\qquad \left. + \frac{\varepsilon}{\mu_D}\frac{1}{\alpha_2}\int_{B_\varepsilon} \mu_D\ W_2(dev(e(v_s)))\ dx\right).\end{aligned}$$

1.4.3 Proof of Proposition 1.3.4:

It suffices to consider the case when $F(u) < \infty$. First, when $\bar{\mu}_D = \infty$ we may choose $u_s^+ = u$, $u_s^- = 0$. Next, when $(\bar{\mu}_D, \bar{\mu}_S) \in (0,\infty) \times [0,\infty)$ we proceed as follows. Any u in $H^1_{\Gamma_0}(\Omega;\mathbb{R}^3)$ can be continuously extended in $\mathbb{R}^3_+ := \{x \in \mathbb{R}^3\ ;\ x_3 > 0\}$ by a field $\widetilde{u}$ belonging to $H^1(\mathbb{R}^3_+;\mathbb{R}^3)$. From Proposition 1.3.1 ii-a) and a result of nonlinear interpolation [9][p. 137], $\zeta := Z(\widetilde{u})$ is an element of $H^{1/2}(S;\mathbb{R}^3)$ so that there exists a continuous lifting Z of ζ into $H^1_{S-\varepsilon_0}(B_{\varepsilon_0};\mathbb{R}^3)$. Let $R_\varepsilon\ \zeta$ defined by:

$$R_\varepsilon\ \zeta(x) = Z(x)\left(1 + \frac{x_3}{\varepsilon}\right) \quad \text{for all } x \in B_\varepsilon \tag{1.4.6}$$

clearly, $R_\varepsilon\ \zeta \in H^1_{S-\varepsilon}(B_\varepsilon)$ and

$$e\left(R_\varepsilon\ \zeta\right)(x) \ =\ \left(1 + \frac{x_3}{\varepsilon}\right) e(Z)(x) + Z(x) \otimes_S \frac{e_3}{\varepsilon} \quad \text{for all } x \in B_\varepsilon, \tag{1.4.7}$$

$$\int_{B_\varepsilon} |R_\varepsilon\ \zeta(x)|^2\, dx \ \leq\ C\left(\varepsilon\int_S |\zeta(\widehat{x})|^2\, d\widehat{x} + \varepsilon^2\int_{B_\varepsilon} |\nabla Z(x)|^2\, dx\right) \tag{1.4.8}$$

because of

$$\int_{B_\varepsilon} |Z(x)-\zeta(\widehat{x})|^2 \ dx \le \varepsilon^2 \int_{B_\varepsilon} |\nabla Z(x)|^2 \ dx \tag{1.4.9}$$

Let $u_s \in V_s$ such that $u_s^+ = u$, $u_s^- = R_\varepsilon \, \zeta$. The convexity and the growth conditions (1.2.3) satisfies by W_1, W_2 implies (see [10]) that there exists γ_1, γ_2 such that

$$\left|W_i(\xi_i^1)-W_i(\xi_i^2)\right| \le \gamma_i \ \left|\xi_i^1-\xi_i^2\right| \ \left(1+\left|\xi_i^1\right|+\left|\xi_i^2\right|\right) \quad \text{for all } (\xi_1^l,\xi_2^l)\in\mathbb{R}\times S^3,\ l=1,2 \tag{1.4.10}$$

and consequently

$$\left|W_i^{\infty,2}(\xi_i^1)-W_i^{\infty,2}(\xi_i^2)\right| \le \gamma_i \ \left|\xi_1^i-\xi_2^i\right| \ \left(\left|\xi_1^i\right|+\left|\xi_2^i\right|\right) \quad \text{for all } (\xi_1^l,\xi_2^l)\in\mathbb{R}\times S^3,\ l=1,2 \tag{1.4.11}$$

Hence, the Cauchy Schwarz inequality gives:

$$\begin{aligned}
&\left|\int_{B_\varepsilon}\left(W_{\mu_S,\mu_D}(e(u_s^-)) - W_{\mu_S,\mu_D}(\zeta(\widehat{x})\otimes_S \frac{e_3}{\varepsilon})\right)dx\right| \\
\le\ & C\max(\mu_S,\mu_D)\left\{\int_{B_\varepsilon}\left(\ \left|1+\frac{x_3}{\varepsilon}\right| e(Z)+(Z(x)-\zeta(\widehat{x}))\otimes_S \frac{e_3}{\varepsilon}\ \right)^2 \ dx\right\}^{\frac12} \\
& \times\left\{\varepsilon+\int_{B_\varepsilon}\left(1+\frac{x_3}{\varepsilon}\right)^2 \ |e(Z)|^2\,dx+\int_{B_\varepsilon}\frac{Z^2(x)}{\varepsilon^2}dx+\int_S\frac{|\zeta(\widehat{x})|^2}{\varepsilon}d\widehat{x}\right\}^{\frac12} \\
\le\ & C\max(\mu_S,\mu_D)\left(\int_{B_\varepsilon}|\nabla Z|^2\,dx\right)^{\frac12}\left(\varepsilon+\frac{\int_S|\zeta(\widehat{x})|^2\,d\widehat{x}}{\varepsilon}+\int_{B_\varepsilon}|\nabla Z(x)|^2\,dx\right)^{\frac12}
\end{aligned}$$

so that

$$\begin{aligned}
\lim_{s\to\bar{s}}\int_{B_\varepsilon} W_{\mu_S,\mu_D}(e(u_s^-))dx &= \lim_{s\to\bar{s}}\int_S W_{\mu_S,\mu_D}(\zeta(\widehat{x})\otimes_S\frac{e_3}{\varepsilon})dx \\
&= \int_S \overline{W}_{\bar{\mu}_S,\bar{\mu}_D}(\zeta(\widehat{x}))d\widehat{x}
\end{aligned}$$

Thus

$$\begin{aligned}
\int_S g\left(\gamma_0(u)\right)\ d\widehat{x} &= \int_S\left(\ h\left(\gamma_0(u)-\zeta\right)+\overline{W}_{\bar{\mu}_S,\bar{\mu}_D}(\zeta)\ \right)\ d\widehat{x} \\
&= \lim_{s\to\bar{s}}\left(\int_S h[u_s])d\widehat{x}+\int_{B_\varepsilon}W_{\mu_S,\mu_D}(e(u_s))dx\right) \\
\int_\Omega W(e(u))dx &= \lim_{s\to\bar{s}}\int_\Omega W(e(u_s^+))dx
\end{aligned}$$

When $(\bar{\mu}_S,\bar{\mu}_D)\in(0,\infty)\times\{\infty\}$, the proof is similar but, now, due to Proposition 1.3.1 ii) b) $u_s^-(x)=\left(\widehat{Z}(\widehat{x}),0\right)\left(1+\dfrac{x_3}{\varepsilon}\right)$ for all $x\in B_\varepsilon$ where $\widehat{Z}$ is a suitable lifting of $\widehat{Z}(\widehat{\gamma_0(u)})$.

Eventually, when $\bar{\mu}_D=0$ we proceed as previously but, according to Proposition 1.3.1 iii) with $\widehat{\zeta}=\widehat{\gamma_0(\widetilde{u})}$ and $\zeta_3=\gamma_0(\widetilde{u})_3$ when $\bar{\mu}_S=0$, $\zeta_3=Z_3\left(\gamma_0(\widetilde{u})_3\right)$ when $\bar{\mu}_S\in(0,\infty)$, $\zeta_3=0$ when $\bar{\mu}_S=\infty$.

1.4.4 Proof of Proposition 1.3.5:

The first assertion is a classical consequence of the convexity of W. Because iii) is an obvious consequence of i) and ii), it remains to prove ii). Furthermore we may assume that the lim inf is finite, so that, from Proposition 1.3.3, there exists $\zeta \in L^2(S,\mathbb{R}^3)$ such that $[v_s]$ weakly converges in $L^2(S,\mathbb{R}^3)$ toward $\gamma_0(u)-\zeta$ and due to the convexity of h

$$\int_S h\left(\gamma_0(u)-\zeta\right)d\widehat{x} \leq \liminf_{s\to\bar{s}} \int_S h\left([v_s]\right)d\widehat{x}.$$

Thus, ii) is true when $\bar{\mu}_D=\infty$ because Proposition 1.3.3 implies $\zeta=0$ and Proposition 1.3.1 $g=h$.

Let us now consider the case $(\bar{\mu}_D,\bar{\mu}_S)\in[0,\infty)^2$, the Hölder inequality and assumption (1.2.4) imply

$$\liminf_{s\to\bar{s}} \int_{B_\varepsilon} W_{\mu_S,\mu_D}(e(v_s))dx = \liminf_{s\to\bar{s}} \int_{B_\varepsilon} W^{\infty,2}_{\bar{\mu}_S,\bar{\mu}_{DS}}\left(e(v_s)\right)\ dx. \tag{1.4.12}$$

But, for all $\eta>0$ there exists ζ_η infinitely differentiable with compact support in S such that $\lim_{\eta\to 0}\ |\zeta_\eta-\zeta|_{L^2(S;\mathbb{R}^3)}=0$; let $u_{s,\eta}=R_\varepsilon\ \zeta_\eta$ (R_ε is defined in the proof of Proposition 1.3.4, then the subdifferential inequality yields:

$$\int_{B_\varepsilon} W^{\infty,2}_{\bar{\mu}_S,\bar{\mu}_D}\left(e(v_s)\right)dx \geq \int_{B_\varepsilon} W^{\infty,2}_{\bar{\mu}_S,\bar{\mu}_D}\left(e(R_\varepsilon\ \zeta_\eta)\right)\ dx + \int_{B_\varepsilon} \xi_{\varepsilon,\eta}\cdot e(v_s-R_\varepsilon\ \zeta_\eta)\ dx \tag{1.4.13}$$

with $\xi_{\varepsilon,\eta}\in\bar{\mu}_S\partial W^{\infty,2}_1\left(tr\ e(R_\varepsilon\ \zeta_\eta)\right)\ I_d+\bar{\mu}_D\partial W^{\infty,2}_2\left(dev\ e(R_\varepsilon\ \zeta_\eta)\right)$.

The previous proof of Proposition 1.3.4 implies

$$\lim_{s\to\bar{s}} \int_{B_\varepsilon} W^{\infty,2}_{\bar{\mu}_S,\bar{\mu}_D}\left(e(R_\varepsilon\ \zeta_\eta)\right)dx = \int_S \overline{W}_{\bar{\mu}_S,\bar{\mu}_D}(\zeta_\eta)\ d\widehat{x}$$

while (1.2.4) and (1.4.7)-(1.4.9) give:

$$\liminf_{s\to\bar{s}} \int_{B_\varepsilon} \xi_{\varepsilon\eta}\cdot e(v_s-R_\varepsilon\ \zeta_\eta)dx = \liminf_{s\to\bar{s}} \int_{B_\varepsilon} \bar{\xi}_n\cdot e(v_s-R_\varepsilon\ \zeta_\eta)dx$$

where $\bar{\xi}_n\in\bar{\mu}_S\partial W^{\infty,2}_1\left(\zeta_{\eta 3}\right)\ I_d+\bar{\mu}_D\partial W^{\infty,2}_2\left(dev\ (\zeta_\eta\otimes_S e_3)\right)$.

Therefore, (1.4.12), (1.4.13), Proposition 1.3.3 ii-a) and (1.4.7)-(1.4.9) yield:

$$\begin{aligned}\liminf_{s\to\bar{s}} \int_{B_\varepsilon} W_{\mu_S,\mu_D}\left(e(v_s)\right)\ dx \ &\geq\ \int_S \overline{W}_{\bar{\mu}_S,\bar{\mu}_D}\zeta_\eta)\ d\widehat{x} \\ &\quad + \liminf_{s\to\bar{s}}\left(-\int_{B_\varepsilon} div\bar{\zeta}_n\cdot(v_s-R_\varepsilon\ \zeta_\eta)dx+\int_S \bar{\zeta}_n e_3\cdot(v_s-\zeta_\eta)\ d\widehat{x}\right) \\ &=\ \int_S \overline{W}_{\bar{\mu}_S,\bar{\mu}_D}(\zeta_\eta)\ d\widehat{x}+\int_S \bar{\zeta}_n e_3\cdot(\zeta-\zeta_\eta)\ d\widehat{x}.\end{aligned}$$

Hence, according to (1.2.4), letting η tend to 0 gives:

$$\liminf_{s\to\bar{s}}\left(\int_S h\left([v_s]\right)\ d\widehat{x}+\int_{B_\varepsilon} W_{\mu_S,\mu_D}\left(e(v_s)\right)\ dx\right)\geq\int_S\big(\ h\left(\ \gamma_0(u)-\zeta\ \right)+\overline{W}_{\bar{\mu}_S,\bar{\mu}_D}(\zeta)\ \big)\ d\widehat{x}$$

which establishes $g\left(\gamma_0(u)\right)\in L^1(S)$ and ii).

Lastly, when $(\bar{\mu}_D,\bar{\mu}_S)\in[0,\infty)\times\{\infty\}$, Proposition 1.3.3 says that $\zeta_3=0$ so that the previous reasoning also establishes ii) by disregarding the terms involving μ_S.

1.4.5 Proof of Theorem 1.3.2:

Classically [6], gathering the previous propositions implies all the claims of Theorem 1.3.2 except the strong convergence in $H^1_{\Gamma_0}(\Omega,\mathbb{R}^3)$ which (see for instance [11] where such a more or less well-known argument is used) stems from the Korn inequality and the additional fact $J(\bar{u})=\lim_{s\to\bar{s}}J(u_s)$, $q\longmapsto\int_\Omega W(q)dx$ being strictly convex and coercive on $L^2(\Omega,S^3)$. In fact, Proposition 1.3.5 gives:

$$\begin{aligned}\lim_{s\to\bar{s}}J(\bar{u}_s) &= \limsup_{s\to\bar{s}}\left(F_s(\bar{u}_s)-\left\{\int_{B_\varepsilon}W_{\mu_S,\mu_D}(e(\bar{u}_s))dx+\int_S h([\bar{u}_s])d\widehat{x}\right\}\right)\\ &\leq \lim_{s\to\bar{s}}F_s(\bar{u}_s)-\liminf_{s\to\bar{s}}\left\{\int_{B_\varepsilon}W_{\mu_S,\mu_D}(e(\bar{u}_s))dx+\int_S h([\bar{u}_s])d\widehat{x}\right\}\\ &\leq F(\bar{u})-\int_S g(\gamma_0(u))d\widehat{x}\\ &= J(\bar{u})\leq\liminf_{s\to\bar{s}}J(\bar{u}_s).\end{aligned}$$

1.4.6 Proof of Proposition 1.3.7:

It has been established in the proof Proposition 1.3.5 that

$$\int_S\big(\ h(\ \gamma_0\left(\bar{u}\right)-\bar{\zeta}\ )+\overline{W}_{\bar{\mu}_S,\bar{\mu}_D}(\bar{\zeta})\ \big)\ d\widehat{x}\leq\liminf_{s\to\bar{s}}\left(\int_S h([\bar{u}_s])d\widehat{x}+\int_{B_\varepsilon}W_{\mu_S,\mu_D}(e(\bar{u}_s))\ dx\right),$$

but we also have:

$$\begin{aligned}\lim_{s\to\bar{s}}\left(\int_S h([\bar{u}_s])\ d\widehat{x}+\int_{B_\varepsilon}W_{\mu_S,\mu_D}(e(\bar{u}_s))\ dx\right) &= \lim_{s\to\bar{s}}\big(\ F_s(\bar{u}_s)-J(\bar{u}_s)\ \big)\\ &= F(u)-J(u)\\ &= \int_S g(\gamma_0(u))d\widehat{x},\end{aligned}$$

and the first result stems from the very definition of g.

Finally, in the proof of Proposition 1.3.5, it has also been shown that

$$\int_S \overline{W}_{\bar{\mu}_S,\bar{\mu}_D}(\bar{\xi})d\widehat{x} \leq \liminf_{s\to\bar{s}} \int_{B_\varepsilon} W_{\mu_S,\mu_D}(e(\bar{u}_s))dx,$$

hence

$$\begin{aligned}\limsup_{s\to\bar{s}} \int_S h([u_s])d\widehat{x} &= \lim_{s\to\bar{s}} \left(\int_S h([u_s])d\widehat{x} + \int_{B_\varepsilon} W_{\mu_S,\mu_D}(e(\bar{u}_s))dx \right) - \liminf_{s\to\bar{s}} \int_{B_\varepsilon} W_{\mu_S,\mu_D}(e(\bar{u}_s))dx \\ &\leq \int_S h(\gamma_0(\bar{u}) - \bar{\zeta})d\widehat{x} \\ &\leq \liminf_{s\to\bar{s}} \int_S h([u_s])d\widehat{x},\end{aligned}$$

which establishes the last claim since $\gamma_0(\bar{u}_s)$ converges strongly in $L^2(S,\mathbb{R}^3)$ toward $\gamma_0(\bar{u})$.

1.4.7 Proof of Theorem 1.3.8:

We confine to case $\bar{\mu}_S, \bar{\mu}_D \in (0,\infty)$, the other cases being obvious or directly deduced from this one. Let us consider the scaling $S_\varepsilon v$ of any field v, defined in B_ε, such that:

$$(S_\varepsilon v)_\alpha(\widehat{y}, y_3) = \frac{1}{\varepsilon}\, v_\alpha(\widehat{y}, \varepsilon y_3), \quad (S_\varepsilon v)_3(\widehat{y}, y_3) = v_3(\widehat{y}, \varepsilon y_3) \quad \text{for all } y \in B_1$$

and let $e(\varepsilon, W)$ defined by: for any $\alpha, \beta \in \{1,2\}$,

$$e(\varepsilon, W)_{\alpha\beta} = \varepsilon^2\, e_{\alpha\beta}(W), \;\; e(\varepsilon, W)_{\alpha 3} = \varepsilon\, e_{\alpha 3}(W) \;\text{ and }\; e(\varepsilon, W)_{33} = e_{33}(W).$$

Arguing as in the second part of the proof of Proposition 1.3.7 yields

$$\liminf_{s\to\bar{s}} \int_{B_\varepsilon} W_{\mu_S,\mu_D}(e(\bar{u}_s))dx = \int_S \overline{W}_{\bar{\mu}_S,\bar{\mu}_D}(\bar{\xi})d\widehat{x}$$

so that (1.2.4) implies

$$\int_{B_1} W^{\infty,2}_{\bar{\mu}_S,\bar{\mu}_D}(\bar{\xi} \otimes_S e_3)dy = \lim_{s\to\bar{s}} \int_{B_1} W^{\infty,2}_{\bar{\mu}_S,\bar{\mu}_D}(e(\varepsilon, S_\varepsilon \bar{u}_s))dy. \tag{1.4.14}$$

Using Proposition 1.3.3 and 1.3.5 with $v_s := \theta\,(\bar{u}_s)^- + (1-\theta)\; R_\varepsilon\,\bar{\zeta}, \quad 0 < \theta < 1,$ and the previous scaling give:

$$\begin{aligned}\int_{B_1} W^{\infty,2}_{\bar{\mu}_S,\bar{\mu}_D}(\bar{\zeta} \otimes_S e_3)dy &\leq \liminf_{s\to\bar{s}} \int_{B_1} W^{\infty,2}_{\bar{\mu}_S,\bar{\mu}_D}\big(\, e\varepsilon\,,\; S_\varepsilon \left(\, \theta\bar{u}_s + (1-\theta)\; R_\varepsilon\,\bar{\zeta}\,\right)\,\big)dy \\ &= \liminf_{s\to\bar{s}} \int_{B_1} W^{\infty,2}_{\bar{\mu}_S,\bar{\mu}_D}\big(\,\theta e \left(\, \varepsilon\,,\; S_\varepsilon\bar{u}_s + (1-\theta)\bar{\zeta} \otimes_S e_3 \,\right)\,\big)dy\end{aligned} \tag{1.4.15}$$

because of (1.4.11) and (1.4.7)-(1.4.9). Hence, (1.4.14), (1.4.15) and the strict convexity of $W^{\infty,2}_{\bar{\mu}_S,\bar{\mu}_D}$ yield that $e(\varepsilon, S_\varepsilon\bar{u}_s)$ converges almost every where in B_1 toward $\bar{\zeta} \otimes_S e_3$ (see

Lemma 4.7 and the proof of Theorem 4.9, page 272-273 in [11] for the details). Now, using the coercivity of $W_1^{\infty,2}$ and $W_2^{\infty,2}$(deduced from (1.2.3), (1.2.4)), Lemma 4.8 page 273 and again the proof of Theorem 4.9 of [11], we have

$$\lim_{s\to\bar{s}} \int_{B_1} \left| e\left(\varepsilon, S_\varepsilon(\bar{u}_s)\right) - \bar{\zeta} \otimes_S e_3 \right|^2 dx = 0.$$

A change of scale and (1.4.7)-(1.4.9) give:

$$\lim_{s\to\bar{s}} \int_{B_\varepsilon} W_{\mu_S,\mu_D}(e(\bar{u}_s) - R_\varepsilon\, \bar{\zeta})dx = 0,$$

while the last assertion stems from (1.4.3).

1.5 A variant

A trend in Tribology is to consider that a very thin "third body" is involved during the contact of two deformable bodies(see [12]). This third body is made from very small parts of matter pulled out from the bodies in the course of the first contacts that will occur. Of course, such a third body has mechanical strength far lesser than those of the genuine ones, so that the modeling of [1] involving a very soft layer may be also suitable to describe this situation and complements have been given in [13] where dissipative behaviors have been treated.

Actually, this trend is improved by considering that in the third body there are small layers near the viscinity of the bodies in contact with properties even more downgraded. To account for this point of view, we consider the following simplified situation where we assume that a third body lays between an elastic body and a rigid support.

We shall use the notations and assumptions introduced in the previous sections except if explicitly said. Now, B_ε is the reference configuration of the third body perfectly bounded along $S_{-\varepsilon}$ on a rigid support and perfectly stuck to the elastic body occupying Ω as reference configuration. We assume that the third body is made of two elastic parts perfectly bonded. The reference configuration of the previously mentioned very small layer inside the third body is B_δ, $0 < \delta << \varepsilon$, while its bulk energy density is $\delta^{p-1}k$ where k is a strictly convex function satisfying

$$\exists p > 1,\ \exists \alpha_3,\ \beta_3 > 0\ ;\ \alpha_3 \left|\xi\right|^p \le k(\xi) \le \beta_3 \left(1 + \left|\xi\right|^p\right) \quad \text{for all } \xi \in S^3 \tag{1.5.1}$$

$\exists k^{\infty,p}$ strictly convex and $p-$positively homogeneous such that

$$\exists r \in [1,p);\ \left|k(\xi) - k^{\infty,p}(\xi)\right| \le c\left(1 + \left|\xi\right|^r\right) \quad \text{for all } \xi \in S^3,$$

according to [1] the factor δ^{p-1} is chosen in order that a limit surface energy $k^{\infty,p}(\cdot \otimes_S e_3)$ be supplied.

The reference configuration of the remaining part of the third body is $B_\varepsilon \backslash \overline{B}_\delta$ and its bulk energy is the density W_{μ_S,μ_D} introduced Section 1.2. Hence, finding the equilibrium position of the structure made of the elastic adherent and the third body, submitted to the forces$(f,\ \varphi)$ and clamped along $\Gamma \cup S_{-\varepsilon}$ involves a new parameter δ. Let $\sigma = (s,\ \delta) = (\varepsilon,\ \mu_S, \mu_D,\ \delta)$. We now, assume:

$\exists \bar{\sigma} \in \{0\} \times [0,\infty]^2 \times \{0\},\ (\bar{\mu}_S, \bar{\mu}_D) \in [0,\infty]^2$, $\exists \varepsilon_0 > 0$ such that

$$\bar{\sigma} = \lim \sigma,\ (\bar{\mu}_S, \bar{\mu}_D) = \lim \left(\frac{\mu_S}{\varepsilon}, \frac{\mu_D}{\varepsilon}\right), 0 = \lim \left(\varepsilon \mu_S, \varepsilon \mu_D\right), 0 = \lim \frac{\delta}{\varepsilon},\ 0 < \varepsilon < \varepsilon_0. \tag{1.5.2}$$

Let the reflexive Banach space $W_{\sigma,p}$ defined by:

$$\left.\begin{array}{ll} p \geq 2, & W_{\sigma,p} = \{\ u \in H^1(\Omega_\varepsilon; \mathbb{R}^3)\ ;\ e(u) \in L^p(B_\delta; S^3)\} \\ p < 2, & W_{\sigma,p} = \{\ u \in W^{1,p}(\Omega_\varepsilon; \mathbb{R}^3)\ ;\ e(u) \in L^2(\Omega_\varepsilon \backslash B_\delta; S^3)\} \\ & |u|_{\sigma,p} = |e(u)|_{L^2(\Omega_\varepsilon \backslash B_\delta; S^3)} + |e(u)|_{L^p(B_\delta; S^3)} \end{array}\right\} \tag{1.5.3}$$

and the strictly convex, continuous and coercive function F'_σ on $W_{\sigma,p}$ such that:

$$F'_\sigma(v) = \int_\Omega W(e(v))\, dx + \delta^{p-1} \int_{B_\delta} k(e(v))\, dx + \int_{B_\varepsilon \backslash B_\delta} W_{\mu_S,\mu_D}(e(v)) dx \quad \text{for all } v \in W_{\sigma,p}.$$

Hence, the problem of finding an equilibrium position

$$(\mathcal{P}'_\sigma)\ :\quad Min\,\{F'_\sigma(v) - L(v)\ ;\ v \in W_{\sigma,p}\}$$

has a unique solution $\bar{u}'_\sigma$. We aim to study the asymptotic behavior of $\bar{u}'_\sigma$ when σ tends to $\bar{\sigma}$ to provide an asymptotic model simpler than $(\mathcal{P}'_\sigma)$ where, clearly, numerical difficulties may occur due to a kind of two-scales meshing.

Because, asymptotically, the energy inside the layer B_δ is equivalent to a surface energy $\int_S k^{\infty,p}([v] \otimes_S e_3) d\widehat{x}$ when δ goes to zero and δ is far lesser than ε, we guess that we are in a situation close to the one studied in Section 1.3 and 1.4, where h will be replaced by h'

$$h'(t) := k^{\infty,p}(t \otimes_S e_3) \quad \text{for all } t \in \mathbb{R}^3. \tag{1.5.4}$$

That is why, in order to simplify the mathematical analysis, we have considered here that there exists only one very thin layer inside the third body. A more realistic situation should involve a second layer occupy $B_\varepsilon \backslash B_{\varepsilon - \delta'}$, $0 < \delta' << \varepsilon$, and the analysis similar to the following but a little more technical on h' may be done. Henceforth, we use the previous assumptions $H' - H''$ made on h so that $g' := \overline{W}_{\bar{\mu}_S,\bar{\mu}_D} \#\ h'$ will have the same properties (see Proposition 1.3.1) as g and the asymptotic model is provided by

Theorem 1.5.1. *When σ goes to $\bar{\sigma}$, $\bar{u}'_\sigma$ strongly converges in $H^1_{\Gamma_0}(\Omega; \mathbb{R}^3)$ toward the unique solution $\bar{u}'$ of*

$$(\mathcal{P}') : \qquad Min\left\{\ F'(v) - L(v)\ ;\ v \in H^1_{\Gamma_0}(\Omega, \mathbb{R}^3)\ \right\}$$

where

$$F'(v) = \begin{cases} \displaystyle\int_\Omega W(e(v))\,dx + \int_S g'(\gamma_0(v))\,d\widehat{x} & \text{when} \quad g'(\gamma_0(v)) \in L^1(S); \\ +\infty & \text{otherwise.} \end{cases}$$

As in Section 1.2, this result of variational convergence is consequence of the following three propositions:

Proposition 1.5.2. *For all sequences (v_σ) in $W_{\sigma,p}$ such that $F'_\sigma(v_\sigma) \le C$, there exists a subsequence not relabelled such that:*

i) v_σ^+ *weakly converges in $H^1_{\Gamma_0}(\Omega;\mathbb{R}^3)$ toward some v,*

ii) a) $\displaystyle\int_{B_\varepsilon \setminus B_\delta} \left|v_\sigma^-\right|^2 \,dx \le C\varepsilon^2 \left(\frac{1}{\mu_S} + \frac{1}{\mu_D}\right)$

b) $\gamma_0^\delta(v_\sigma^-)$, *the trace of v_σ^- on $S_{-\delta}$ considered as an element of $L^2(S;\mathbb{R}^3)$ weakly converges in $L^2(S;\mathbb{R}^3)$ toward some ζ' moreover $\zeta'_3 = 0$ if $\bar\mu_S = \infty$, $\zeta' = 0$ if $\bar\mu_D = \infty$ and when $\bar\mu_S = \bar\mu_D = \infty$ the convergence occurs in $L^2(S;\mathbb{R}^3)$ strong.*

c) $\displaystyle\int_{B_\delta} |v_s|^{p_m} \,dx \le C\,\delta^{\min(1,\,p_m-1)}$ *where $p_m = \min(p,2)$.*

Proposition 1.5.3. *For all u in $H^1_{\Gamma_0}(\Omega;\mathbb{R}^3)$, there exists a sequences (u_σ) in $W_{\sigma,p}$ such that u_σ^+ strongly converges in $H^1_{\Gamma_0}(\Omega;\mathbb{R}^3)$ toward u and $F'(u) \ge \limsup_{\sigma\to\bar\sigma} F'_\sigma(u_\sigma)$.*

Proposition 1.5.4. *For all u in $H^1_{\Gamma_0}(\Omega;\mathbb{R}^3)$ and all sequences (v_σ) in $W_{\sigma,p}$ such that v_σ^+ weakly converges in $H^1_{\Gamma_0}(\Omega;\mathbb{R}^3)$ toward u, we have*

i) $\displaystyle J(u) := \int_\Omega w(e(u))\,dx \le \liminf_{\sigma\to\bar\sigma} J(v_\sigma^+),$

ii) $\displaystyle\int_S g'(\gamma_0(u))\,d\widehat{x} \le \liminf_{\sigma\to\bar\sigma} \left(\int_{B_\varepsilon} \delta^{p-1}\, k(e(v_\sigma))\,dx + \int_{B_\varepsilon\setminus B_\delta} W_{\mu_S,\mu_D}(e(v_\sigma))dx\right),$

iii) $F'(u) \le \liminf_{\sigma\to\bar\sigma} F'_\sigma(v_\sigma)$.

1.6 Proofs of the variant results

First, the proof of points i) and ii) a-b) of Proposition 1.5.2 follows the lines of the one of Proposition 1.3.3 with B_ε and S replaced by $B_\varepsilon\setminus B_\delta$ and $S_{-\delta}$ and due account of $0 = \displaystyle\lim_{\sigma\to\bar\sigma}\left(\frac{\delta}{\varepsilon}\right)$. Let $p_m = \min(p,2)$, we have

$$\int_{B_\delta} |v_s|^{p_m}\,dx \le 2^{p_m-1}\left(|S|\;\delta^{p_m-1}\int_S |\gamma_0(v_s)|^{p_m}\,d\widehat{x} + \delta^{p_m}\int_{B_\delta}|\nabla v_s|^{p_m}\,dx\right) \tag{1.6.1}$$

Introducing a smooth cut-off function such that

$\eta(x_3)=1$ if $x_3\le h$, $0\le\eta(x_3)\le 1$ if $0<h<x_3<2h$, $\eta(x_3)=0$ if $x_3\ge 2h$ with h small enough,

and reasoning on $\eta\ v_\sigma$ and $(1-\eta)\ v_\sigma$ it is easy to show that there exists C independent of ε and δ such that

$$\int_{B_\delta}|\nabla v_\sigma|^{p_m}\,dx\le C\int_{B_\delta\cup\Omega}|e(v_\sigma)|^{p_m}\,dx$$

which, with (1.6.1), proves point ii) c).

Next, to prove Proposition 1.5.3, it suffices, with respect to the proof of Proposition 1.3.4, to introduce liftings $\sqcup$ and Z' of $\gamma_0(u)$ and ζ' into $H^1(B_{\varepsilon_0};\mathbb{R}^3)$, ζ' achieving the minimum in the definition of $g'(\gamma_0(u))$ and to define $R'_{\varepsilon,\delta}(u,\zeta')$ by:

$$R'_{\varepsilon,\delta}(u,\zeta'):=\begin{cases}\left(1+\dfrac{x_3}{\delta}\right)\sqcup(x)-\dfrac{x_3}{\delta}Z'(x), & \text{if}\quad 0>x_3>-\delta\\[2ex] \left(1+\dfrac{x_3+\delta}{\varepsilon-\delta}\right)Z'(x), & \text{if}\quad -\delta>x_3>-\varepsilon\end{cases}$$

Then, if $u_\sigma^-=R'_{\varepsilon,\delta}(u,\zeta')$ the arguments used in the proof of Proposition 1.3.3 work to get

$$\lim_{\sigma\to\bar\sigma}\int_{B_\varepsilon\setminus B_\delta}W_{\mu_S,\mu_D}(e(u_s^-))\,dx=\int_S\overline{W}_{\bar\mu_S,\bar\mu_D}(\zeta')\,d\widehat{x}.$$

Moreover, because $e(u_\sigma^-)=(\sqcup-Z')\otimes_S\dfrac{e_3}{\delta}+\left(1+\dfrac{x_3}{\delta}\right)e\,(\sqcup)-\dfrac{x_3}{\delta}e\,(Z')$ the same arguments, but using now the convexity and the growth condition of order p for k (see for instance[1]) give

$$\lim_{\sigma\to\bar\sigma}\left(\int_{B_\varepsilon\setminus B_\delta}\delta^{p-1}\,k(e(u_\sigma^-))\,dx\right)=\int_S k^{\infty,p}\,\left((\gamma_0(u)-\zeta')\otimes_S e_3\right)\,d\widehat{x}=\int_S h'\left(\gamma_0(u)-\zeta'\right)\,d\widehat{x}$$

The proof is completed by taking $u_\sigma^+=u$.

Finally, concerning Proposition 1.5.4, as in the proof of Proposition 1.3.5, but considering $S_{-\delta}$ in place of S, the use of the subdifferential inequality, an integration by part and the estimate of $\int_{B_\varepsilon\setminus B_\delta}|v_\sigma|^2\ dx$ (see Proposition 1.5.2) yield:

$$\liminf_{\sigma\to\bar\sigma}\int_{B_\varepsilon\setminus B_\delta}W_{\mu_S,\mu_D}(\,e(v_\sigma)\,)\,dx\ge\int_S\overline{W}_{\bar\mu_S,\bar\mu_D}(\zeta')\,d\widehat{x}$$

ζ' being the weak limit in $L^2(S;\mathbb{R}^3)$ of $\gamma_0^\delta(v_\varepsilon)$(see Proposition 1.5.2). The remaining part

$$\liminf_{\sigma\to\bar\sigma}\int_{B_\delta}\delta^{p-1}\,k\,(\,e(v_\sigma)\,)\,\,dx\ge\int_S h'\,(\,\gamma_0(u)-\zeta'\,)\,\,d\widehat{x}$$

is also obtained by using the subdifferential inequality, an integration by part and the estimate of $\int_{B_\delta}|v_\sigma|^{p_m}\ dx$ obtained in Proposition 1.5.2.

As in Section 1.3, it is possible to determine the asymptotic behavior of $\bar u'_\sigma$ in the third body:

Theorem 1.6.1. *When σ goes to $\bar{\sigma}$*

i) $\gamma_0^\delta(\bar{u}'_\sigma)$ *weakly converges in* $L^2(S;\mathbb{R}^3)$ *towards* $\bar{\zeta}'$ *such that*

$$g'(\gamma_0(u))(\widehat{x}) = h'(\gamma_0(\bar{u}'))(\widehat{x}) - \bar{\zeta}'(\widehat{x}) + \bar{W}_{\bar{\mu}_S,\bar{\mu}_D}(\bar{\zeta}'(\widehat{x})) \textit{ for a.e. } \widehat{x} \in S.$$

ii)

$$\begin{aligned}
&\lim_{\sigma\to\bar{\sigma}} \left(\int_{B_\delta} \delta^{p-1}\, k(e(\bar{u}'_\sigma - R'_{\varepsilon,\delta}(\bar{u}',\bar{\zeta}'))) \, dx + \int_{B_\varepsilon \setminus B_\delta} W_{\mu_S,\mu_D}(e(\bar{u}'_\sigma - R'_{\varepsilon,\delta}(\bar{u}',\bar{\zeta}'))) \, dx \right) \\
= \; &\lim_{\sigma\to\bar{\sigma}} \left(\delta^{-p} \int_{B_\delta} \left|\bar{u}'_\sigma - R'_{\varepsilon,\delta}(\bar{u}',\bar{\zeta}')\right|^p \, dx + \varepsilon \int_{B_\varepsilon \setminus B_\delta} \left|\bar{u}'_\sigma - R'_{\varepsilon,\delta}(\bar{u}',\bar{\zeta}')\right|^2 \, dx \right)
\end{aligned}$$

whose proof is similar to the one of Proposition 1.3.7 and Theorem 1.3.8. Thus, to get easily a good approximation of $\bar{u}_\sigma$, we suggest first to solve $(\mathcal{P}')$ where $\bar{\mu}_S, \bar{\mu}_D$ are replaced by the true real values $\dfrac{\mu_S}{\varepsilon}$, $\dfrac{\mu_D}{\varepsilon}$ and to replace $\bar{u}^+_\sigma$ by the solution $\bar{\bar{u}}_\sigma$. Next, $\bar{u}^-_\sigma$ may be replaced by $R'_{\varepsilon,\delta}(\bar{u}_\sigma, \bar{\zeta}_\sigma)$, $\bar{\zeta}_\sigma$ achieving the minimum in the definition of $\left(h' \,\#\, \overline{W}_{\frac{\mu_S}{\varepsilon},\frac{\mu_D}{\varepsilon}}\right)(\gamma_0(\bar{\bar{u}}_\sigma))$.

Bibliography

[1]. C. Licht, G. Michaille. *A modelling of elastic adhesive bonded joints.* Advances in Mathematical Sciences and Applications, **7** (1997), no 2, 711-740.

[2]. C. Licht, G. Michaille, S. Pagano. *A model of elastic adhesive bonded joints through oscillation-concentration measures.* J. Math. Pures Appl. (9) 87 (2007), no. 4, 343–365.

[3]. G. Bayada, M. Chambat, K. Lhalouani. *Asymtotic analysis of a thin layer device with Tresca's contact law in elasticity.* Math. Meth. Appl. Sci, **22**(1999), 811-836.

[4]. Zaittouni F. *Ph.D. thesis University Montpellier 2, 2000.*

[5]. Aitmoussa A. *Modélisation et étude des singularités de contraintes d'un joint collé très mince.* Ph.D. Thesis, University Montpellier 2, France, 1989.

[6]. H. Attouch, G. Buttazzo, G. Michaille. *Variational analysis in Sobolev and BV space: application to PDEs and Optimization.* MPS-SIAM Book Series on Optimization, 2005.

[7]. P. Rabier, J. A. Martins, J. T. Oden, L. Campos. *Existence and local uniqueness of solutions to contact problems in elasticity with non linear friction laws.* Int. J. Engng. Sci. **24**(1986), 1755-1768.

[8]. M. Raous. *Friction and Instabilities. (Lectures from the Advanced Summer School on Friction and Instabilities held in Udine, July 3-7, 2000)/ed.* Martins J. A.C., Raous M., Springer-Verlag, 2002.

[9]. Tartar L. *An introduction to Sobolev spaces and Interpolation spaces.* Lecture Notes of the Unione of Mathematica Italiana, vol**3**., Springer, 2007.

[10]. Dacorogna B. *Direct methods in the Calculus of Variations.* Applied Mathematical Sciences, no.**78**., Springer-Verlag, Berlin, 1989.

[11]. Ball J., Massden J.E. *Quasi convexity at the boundary, Positivity of the Second Variation and Elastic Stability.* Arch. Rational Mech. Anal. 86, 1984, page 251-277.

[12]. Godet M. *Third bodies in Tribology.* Wear, vol.136, no.**1**(1990), 29-45.

[13]. C. Licht *Comportement asymtotique d'une bande dissipative mince de faille rigidité.* C.R. Acad. Sci. Paris, t. 317, Série I, 1993, page 429-433.

Chapter 2

The effect on the internal energy of concentrated sources in a junction and the gradient concentration phenomenon

2.1 Introduction

This work is concerned with a soft thin junction problem whose internal energy functional is perturbed by a source $\mathcal{S}_\varepsilon$ concentrated in the layer $B_\varepsilon := \Sigma \times (-\frac{\varepsilon}{2}, \frac{\varepsilon}{2})$, $\Sigma \subset \mathbb{R}^{N-1}$, i.e., the total energy of the physical system is of the form

$$F_\varepsilon(u) = \int_{\Omega\setminus B_\varepsilon} f(\nabla u)\ dx + \varepsilon \int_{B_\varepsilon} g(\nabla u)\ dx - \langle \mathcal{S}_\varepsilon, u\rangle$$

where $\Omega \subset \mathbb{R}^N$, $u : \Omega \longrightarrow \mathbb{R}$ runs through the Sobolev space $W^{1,2}_{\Gamma_0}(\Omega)$ of Sobolev functions satisfying $u = 0$ on a part Γ_0 of the boundary of Ω. The source $\mathcal{S}_\varepsilon$ rescaled on the rescaled layer $B := \Sigma \times (-\frac{1}{2}, \frac{1}{2})$, is assumed to strongly converge to some $\mathcal{S}$ in the dual of the space $V(B)$ made up of rescaled functions living on B_ε whose N^{th} distributional derivative belongs to $L^2(B_\varepsilon)$. In Section 2.4, we give a general example of such sources which are measures in B_ε. Obviously sources of the form $c\frac{1}{L(\varepsilon)}\mathbb{1}_{B_\varepsilon}$ where c is any constant and $L(\varepsilon) \sim \varepsilon$ is a trivial example of measure which satisfies this condition with $\mathcal{S} = \mathbb{1}_B$.

The Euler-Lagrange equation associated with the minimization problem

$$(\mathcal{P}_\varepsilon)\ :\qquad \min_{u\in W^{1,2}_{\Gamma_0}} F_\varepsilon(u)$$

is given by the Dirichlet problem

$$\begin{cases} -\mathrm{div}\nabla_\xi \sigma_\varepsilon(x, \nabla u) = \mathcal{S}_\varepsilon & \text{on } \Omega \\ u = 0 & \text{on } \Gamma_0, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \setminus \Gamma_0 \end{cases}$$

where $\sigma_\varepsilon(x,\xi) := \mathbb{1}_{\Omega\setminus B_\varepsilon} f(\xi) + \varepsilon \mathbb{1}_{B_\varepsilon} g(\xi)$. Among the physical motivations of $(\mathcal{P}_\varepsilon)$ one may mention various applications to heat conduction or electrostatic problems subjected to concentrated sources on a layer B_ε and whose conductivity in the layer is of order the small size of B_ε. One may also think about membrane problems with a exterior loading concentrated on a layer B_ε and whose stiffness in the layer is of order the small size of B_ε.

From the mathematical point of view, contrary to the various studies devoted to the asymptotic modelings of junction problems ([2, 8, 6, 9] and references therein), the source (or the loading) $\mathcal{S}_\varepsilon$ is a non L^2-continuous perturbation of the energy functional $\int_{\Omega\setminus B_\varepsilon} f(\nabla u)\, dx + \varepsilon \int_{B_\varepsilon} g(\nabla u)\, dx$. When the size ε of the layer goes to zero, we show that fields u_ε of finite energy develop a discontinuity through Σ and the variational limit problem in general gives rise to a non local effect. More precisely, at the variational limit, the internal energy functional $\varepsilon \int_{B_\varepsilon} g(\nabla u)\, dx$ of the junction and the external energy of the source $\langle \mathcal{S}_\varepsilon, u\rangle$ are combined into a functional of the type $\inf_{\theta\in X(u)} H(\theta)$ where $H(\theta) := \int_B g^{\infty,2}(\hat{0}, \frac{\partial\theta}{\partial x_N})\, dx - \langle \mathcal{S}, \theta\rangle$, $g^{\infty,2}$ is the 2-recession function of g and θ runs through a suitable subspace of $V(B)$ depending on the traces $u^\pm$ on Σ.

Such a junction problem with a source concentrated in the junction was considered in [3] in a one dimensional case in order to highlight and illustrate a gradient concentration phenomenon, but we were not able to express the variational limit problem. This chapter illustrates the same phenomenon with a complete description of the N-dimensional limit problem in the sense of the Γ-convergence (Theorem 2.3.3). We show that the sequence of gradient minimizers of $(\mathcal{P}_\varepsilon)$ converges to a minimizer $\bar{u}$ of the limit problem and generates a gradient Young-concentration measure $\bar{\mu}$ that we analyse in the spirit of [3]. Moreover we express the non local part $\inf_{\theta\in X(\bar{u})} H(\theta)$ of the total energy in terms of this measure (Theorem 2.5.3) and we obtain some bounds on the probability measure $\bar{\mu}_{\hat{x}}$ stemming from the disintegration of $\bar{\mu}$ (Corollary 2.5.4).

This work is organized as follows: in Section 2.2 we fix notations and describe in detail the problem $(\mathcal{P}_\varepsilon)$. Section 2.3 is devoted to the asymptotic analysis of $(\mathcal{P}_\varepsilon)$ in the sense of the Γ-convergence of the functional F_ε extended to $L^2(\Omega)$ equipped with its strong topology. In Section 2.4 we describe a large class of sources $\mathcal{S}_\varepsilon$ satisfying our suitable convergence condition. Finally Section 2.5 is concerned with the analysis of the gradient concentration phenomenon generated by the sequence of minimizers of $(\mathcal{P}_\varepsilon)$.

2.2 Description of the minimization problem

The reference configuration is a cylinder $\Omega := \Sigma \times (-r, r)$, $r > 0$ where Σ is a bounded domain in $\mathbb{R}^2$ with Lipschitz boundary and we define the following sets:

. $B_\varepsilon := \Sigma \times (-\frac{\varepsilon}{2}, \frac{\varepsilon}{2})$;

. $B := \Sigma \times (-1/2, 1/2)$;

. $\Omega_\varepsilon = \Omega \setminus \bar{B}_\varepsilon$;

. Γ_0 is a subset of the boundary $\partial\Omega$ of Ω such that $\text{dist}(\overline{\Gamma}_0, \overline{\partial B_\varepsilon \cap \partial\Omega}) > 0$;

. we write Ω_ε^-, Ω_ε^+, Ω^-, Ω^+, B_ε^+ and B_ε^- for the sets $\Omega_\varepsilon \cap [x_N < 0]$ and $\Omega_\varepsilon \cap [x_N > 0]$, $\Omega \cap [x_N < 0]$, $\Omega \cap [x_N > 0]$ and $B_\varepsilon \cap [x_N > 0]$, $B_\varepsilon \cap [x_N < 0]$ respectively;

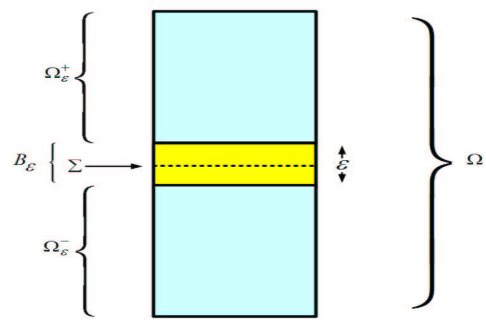

Figure 2.1: The physical domain

Writing,

. $W^{1,2}_{\Gamma_0}(\Omega_\varepsilon) := \{u \in W^{1,2}(\Omega_\varepsilon) : u = 0 \text{ on } \Gamma_0\}$.

. $W^{1,2}_{\Gamma_0}(\Omega) := \{u \in W^{1,2}(\Omega) : u = 0 \text{ on } \Gamma_0\}$.

. $W^{1,2}_{\Gamma_0}(\Omega \setminus \Sigma) := \{u \in W^{1,2}(\Omega \setminus \Sigma) : u = 0 \text{ on } \Gamma_0\}$.

We say that a function $h : \mathbb{R}^N \longrightarrow \mathbb{R} \cup \{+\infty\}$ satisfies a growth condition of order 2 if there exists α and β in $\mathbb{R}^+$ such that

$$\alpha\,|\xi|^2 \le h(\xi) \le \beta(1 + |\xi|^2) \text{ for all } \xi \in \mathbb{R}^N.$$

We consider two convex functions $f, g : \mathbb{R}^N \longrightarrow \mathbb{R}$ satisfying a growth condition of order 2, and we assume that there exists a positively 2-homogeneous function $g^{\infty,2}$ satisfying

$$\left|g(\xi) - g^{\infty,2}(\xi)\right| \le \beta(1 + |\xi|^{2-\delta}) \text{ for all } \xi \in \mathbb{R}^N \tag{2.2.1}$$

for some δ, $0 < \delta < 2$. Note that $g^{\infty,2}$ is the positively 2-homogeneous recession function of g, i.e.,

$$g^{\infty,2}(\xi) = \lim_{t \to +\infty} \frac{g(t\xi)}{t^2},$$

is convex and satisfies the same growth condition of order 2. We define the following space

$$V(B_\varepsilon) := \left\{ u \in L^2(B_\varepsilon) : \frac{\partial u}{\partial x_N} \in L^2(B_\varepsilon) \right\}$$

equipped with the norm

$$\|u\|_{V(B_\varepsilon)} := \Big(\int_{B_\varepsilon} |u|^2 \, dx + \int_{B_\varepsilon} |\frac{\partial u}{\partial x_N}|^2 \, dx \Big)^{\frac{1}{2}}$$

and we denote the duality bracket between the topological dual space $V'(B_\varepsilon)$ and $V(B_\varepsilon)$ by $\langle\ ,\ \rangle$. The total energy functional $F_\varepsilon : L^2(\Omega) \longrightarrow \mathbb{R}\cup\{+\infty\}$ considered in this chapter is defined by

$$F_\varepsilon(u) = \begin{cases} \displaystyle\int_{\Omega_\varepsilon} f(\nabla u)\ dx + \varepsilon \int_{B_\varepsilon} g(\nabla u)\ dx - \langle \mathcal{S}_\varepsilon, u\rangle \text{ if } u \in W^{1,2}_{\Gamma_0}(\Omega) \\ +\infty \text{ otherwise} \end{cases}$$

where $\mathcal{S}_\varepsilon$ is given in $V'(B_\varepsilon)$. Our aim is to describe the asymptotic behavior of the minimization problem

$$(\mathcal{P})_\varepsilon \ : \qquad \min\left\{F_\varepsilon(u) : u \in L^2(\Omega)\right\},$$

i.e., the limit of $\min\{F_\varepsilon(u) : u \in L^2(\Omega)\}$ together with the limit of the minimizer $\bar{u}_\varepsilon$, and to identify the limit problem. Therefore we are lead to compute the Γ-limit of the sequence of functionals $(F_\varepsilon)_{\varepsilon>0}$.

Let us consider the space $V(B) := \left\{u \in L^2(B) : \frac{\partial u}{\partial x_N} \in L^2(B)\right\}$ equipped with the norm

$$\|u\|_{V(B)} := \Big(\int_B |u|^2\ dx + \int_B \big|\frac{\partial u}{\partial x_N}\big|^2\ dx\Big)^{\frac{1}{2}},$$

the linear continuous operator

$$\tau_\varepsilon : V(B) \longrightarrow V(B_\varepsilon)$$

defined for every $x = (\hat{x}, x_N) \in B_\varepsilon$ by $\tau_\varepsilon(u)(\hat{x}, x_N) := u(\hat{x}, \frac{x_N}{\varepsilon})$ and the transposed operator

$${}^T\tau_\varepsilon : V'(B_\varepsilon) \longrightarrow V'(B)$$

defined for every $u \in V(B)$ by $\langle \mathcal{S}_\varepsilon, \tau_\varepsilon(u)\rangle = \langle {}^T\tau_\varepsilon \mathcal{S}_\varepsilon, u\rangle$ (for shorten notation, $\langle , \rangle$ denotes as well the duality bracket between $V'(B_\varepsilon)$ and $V(B_\varepsilon)$ as the duality bracket between $V'(B)$ and $V(B)$). Then we also can write the functional F_ε as follows:

$$F_\varepsilon(u) = \begin{cases} \displaystyle\int_{\Omega_\varepsilon} f(\nabla u)\ dx + \varepsilon^2 \int_B g(\hat{\nabla}\tau_\varepsilon^{-1} u, \frac{1}{\varepsilon}\frac{\partial \tau_\varepsilon^{-1} u}{\partial x_N})\ dx - \langle {}^T\tau_\varepsilon \mathcal{S}_\varepsilon, \tau_\varepsilon^{-1} u\rangle \text{ if } u \in W^{1,2}_{\Gamma_0}(\Omega) \\ +\infty \text{ otherwise.} \end{cases}$$

which is a good writing for computing the Γ-limit provided that we make the following hypothesis on the source $\mathcal{S}_\varepsilon$: there exists $\mathcal{S}$ in $V'(B)$ such that

$${}^T\tau_\varepsilon \mathcal{S}_\varepsilon \text{ strongly converges to } \mathcal{S} \text{ in } V'(B).$$

2.3 The variational asymptotic model

In all this chapter, C denotes a non negative constant which does not depend on ε and may vary from line to line. Moreover, we do not relabel the various considered subsequences and the symbols $\to$ and $\rightharpoonup$ denote various strong convergences and weak convergences respectively. Before addressing the variational convergence process, we begin by establishing some compactness properties of sequences with bounded energy. For this we will use the ε-translate operator T_ε from $W^{1,2}(\Omega)$ into $W^{1,2}(\Omega\setminus\Sigma)$ as follows: set $\widetilde{w}$ the extension by reflexion on $\Sigma\times(-2r,-r)\cup(r,2r)$ of any function $w\in W^{1,2}(\Omega)$, we define the ε-translate $T_\varepsilon w$ of w by

$$T_\varepsilon w(\hat{x},x_N)=\begin{cases}\widetilde{w}(\hat{x},x_N+\frac{\varepsilon}{2}) & \text{if } x\in\Omega^+;\\ \widetilde{w}(\hat{x},x_N-\frac{\varepsilon}{2}) & \text{if } x\in\Omega^-.\end{cases}$$

Lemma 2.3.1 (compactness). *Let $(u_\varepsilon)_{\varepsilon>0}$ be a sequence in $L^2(\Omega)$ such that $\sup_{\varepsilon>0}F_\varepsilon(u_\varepsilon)<+\infty$. Then*

(i)

$$\int_{B_\varepsilon}|u_\varepsilon|^2\,dx\le C\varepsilon\Big(\int_{\Omega_\varepsilon}|\nabla u_\varepsilon|^2\,dx+\varepsilon\int_{B_\varepsilon}|\frac{\partial u_\varepsilon}{\partial x_N}|^2\,dx\Big)\tag{2.3.1}$$

(ii)

$$\sup_{\varepsilon>0}\Big(\int_{\Omega_\varepsilon}|\nabla u_\varepsilon|^2\,dx+\varepsilon\int_{B_\varepsilon}\left|\frac{\partial u_\varepsilon}{\partial x_N}\right|^2dx\Big)<+\infty;\tag{2.3.2}$$

(iii) there exist $u\in W^{1,2}_{\Gamma_0}(\Omega\setminus\Sigma)$ and a subsequence of $(u_\varepsilon)_{\varepsilon>0}$ such that $u_\varepsilon\to u$ in $L^2(\Omega)$ and $u_\varepsilon\rightharpoonup u$ in $W^{1,2}_{\Gamma_0}(\Omega_\eta)$ for all $\eta>0$;

(iv) there exists $\theta\in V(B)$ and a subsequence such that $\tau_\varepsilon^{-1}u_\varepsilon\rightharpoonup\theta$ in $V(B)$, i.e.

$$\tau_\varepsilon^{-1}u_\varepsilon\rightharpoonup\theta \text{ in } L^2(B),\qquad \frac{\partial\tau_\varepsilon^{-1}u_\varepsilon}{\partial x_N}\rightharpoonup\frac{\partial\theta}{\partial x_N}\text{ in } L^2(B);$$

moreover $\varepsilon\hat{\nabla}\tau_\varepsilon^{-1}u_\varepsilon\rightharpoonup 0$ *in* $L^2(B,\mathbb{R}^2)$;

(v) $\theta(.,\pm\frac{1}{2})=u^\pm$.

Proof. Proof of (i). From

$$u_\varepsilon(\hat{x},x_N)=T_\varepsilon u_\varepsilon(\hat{x},0)+\int_{\frac{\varepsilon}{2}}^{x_N}\frac{\partial}{\partial x_N}u_\varepsilon(\hat{x},t)\,dt,$$

by using Cauchy-Schwarz inequality an easy calculation gives.

$$\begin{aligned}\int_{B_\varepsilon^+}|u_\varepsilon|^2\,dx &\le C\ \Big(\int_{\Omega^+}|\nabla T_\varepsilon u_\varepsilon|^2+\varepsilon\int_{B_\varepsilon^+}\left|\frac{\partial u_\varepsilon}{\partial x_N}\right|^2\,dx\Big)\\ &\le\ C\Big(\int_{\Omega_\varepsilon^+}|\nabla u_\varepsilon|^2+\varepsilon\int_{B_\varepsilon^+}\left|\frac{\partial u_\varepsilon}{\partial x_N}\right|^2\,dx\Big)\end{aligned}$$

The same calculation holds in B_ε^- and (2.3.1) follows.

Proof of (ii). From the coercivity conditions satisfied by f and g, from (2.3.1), and since ${}^T\tau_\varepsilon \mathcal{S}_\varepsilon$ strongly converges in $V(B)$, one has

$$\begin{aligned}
\alpha\Big(\int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2\,dx + \varepsilon\int_{B_\varepsilon}\left|\frac{\partial u_\varepsilon}{\partial x_N}\right|^2 dx\Big) &\le C + |\langle \tau_\varepsilon \mathcal{S}_\varepsilon, u_\varepsilon\rangle| \\
&= C + |\langle {}^T\tau_\varepsilon \mathcal{S}_\varepsilon, \tau_\varepsilon^{-1} u_\varepsilon\rangle| \\
&\le C + \left\|{}^T\tau_\varepsilon \mathcal{S}_\varepsilon\right\|_{V'(B)} \left\|\tau_\varepsilon^{-1} u_\varepsilon\right\|_{V(B)} \\
&= C + \left\|{}^T\tau_\varepsilon \mathcal{S}_\varepsilon\right\|_{V'(B)} \Big(\frac{1}{\varepsilon}\int_{B_\varepsilon} |u_\varepsilon|^2\,dx + \varepsilon\int_{B_\varepsilon}\left|\frac{\partial u_\varepsilon}{\partial x_N}\right|^2 dx\Big)^{\frac{1}{2}} \\
&\le C + C\Big(\int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2\,dx + \varepsilon\int_{B_\varepsilon}\left|\frac{\partial u_\varepsilon}{\partial x_N}\right|^2 dx\Big)^{1/2}.
\end{aligned}$$

Then, setting $X_\varepsilon := \Big(\int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2\,dx + \varepsilon\int_{B_\varepsilon}\left|\frac{\partial u_\varepsilon}{\partial x_N}\right|^2 dx\Big)^{1/2}$, estimates (2.3.2) follows from $\alpha X_\varepsilon^2 \le C + CX_\varepsilon$.

Proof of (iii).

Step 1. We claim that there exists $z \in W^{1,2}(\Omega\backslash\Sigma)$ and a subsequence of $(u_\varepsilon)_{\varepsilon>0}$ such that $T_\varepsilon u_\varepsilon \rightharpoonup z$ in $W^{1,2}(\Omega\backslash\Sigma)$ and strongly in $L^2(\Omega\backslash\Sigma)$. Clearly,

$$T_\varepsilon u_\varepsilon \text{ are in } W^{1,2}(\Omega\backslash\Sigma) \text{ and } \nabla T_\varepsilon u_\varepsilon = T_\varepsilon \nabla u_\varepsilon \ \text{ for all } \varepsilon > 0. \tag{2.3.3}$$

Indeed, by the Poincaré inequality, (2.3.2) and (2.3.3), we deduce

$$\sup_{\varepsilon>0} \|T_\varepsilon u_\varepsilon\|^2_{W^{1,2}(\Omega\backslash\Sigma,\mathbb{R}^N)} \le C \sup_{\varepsilon>0}\left(\int_{\Omega_\varepsilon} |\nabla u_\varepsilon(x)|^2\,dx + \varepsilon\int_{B_\varepsilon}\left|\frac{\partial u_\varepsilon}{\partial x_N}(x)\right|^2 dx\right) < +\infty.$$

Therefore, $(T_\varepsilon u_\varepsilon)_{\varepsilon>0}$ is bounded in $W^{1,2}(\Omega\backslash\Sigma)$, so that the claim follows immediately. We denote by z^+ and z^- the traces of z considered as a Sobolev function on Ω^+ and Ω^- respectively.

Step 2. We establish that there exists u in $L^2(\Omega)$ such that we can extract of the previous subsequence $(u_\varepsilon)_{\varepsilon>0}$ a subsequence which strongly converges to u in $L^2(\Omega)$. Precisely, we will prove that $\lim\limits_{\varepsilon\to 0}\int_\Omega |u_\varepsilon(x)|^2\,dx = \int_\Omega |z(x)|^2\,dx$. By applying

$$\int_{\Omega_\varepsilon} |u_\varepsilon(x)|^2\,dx = \int_{\Omega^+\cup\Omega^-} |T_\varepsilon u_\varepsilon(x)|^2\,dx - \int_{\Sigma\times\left(\left(r-\frac{\varepsilon}{2},r\right)\cup\left(-r,-r+\frac{\varepsilon}{2}\right)\right)} |u_\varepsilon(x)|^2\,dx,$$

we deduce

$$\|u_\varepsilon\|^2_{L^2(\Omega)} = \int_{\Omega^+\cup\Omega^-} |T_\varepsilon u_\varepsilon(x)|^2\,dx + \int_{B_\varepsilon} |u_\varepsilon(x)|^2\,dx - \int_{\Sigma\times\left(\left(r-\frac{\varepsilon}{2},r\right)\cup\left(-r,-r+\frac{\varepsilon}{2}\right)\right)} |u_\varepsilon(x)|^2\,dx. \tag{2.3.4}$$

From (2.3.1) $\lim\limits_{\varepsilon\to 0}\int_{B_\varepsilon}|u_\varepsilon(x)|^2\,dx = 0$. On the other hand from the fact that $T_\varepsilon u_\varepsilon \to z$ in $L^2(\Omega)$, we infer

$$\lim_{\varepsilon\to 0}\|u_\varepsilon\|^2_{L^2(\Omega)} = \int_\Omega |z(x)|^2\,dx.$$

It remains to establish

$$\lim_{\varepsilon\to 0}\int_{\Sigma\times\left(\left(r-\frac{\varepsilon}{2},r\right)\cup\left(-r,-r+\frac{\varepsilon}{2}\right)\right)}|u_\varepsilon(x)|^2\,dx = 0$$

which is an easy consequence of the strong convergence of $T_\varepsilon u_\varepsilon$ to z in $L^2(\Omega)$.

Step 3. We show that $u = z$. Since we have shown already that $u_\varepsilon \rightharpoonup u$ in $L^2(\Omega)$ and $T_\varepsilon u_\varepsilon \rightharpoonup z$ in $W^{1,2}(\Omega\backslash\Sigma)$, we have for any $\varphi \in C_c^\infty(\Omega)$,

$$\begin{aligned}\int_\Omega u(x)\varphi(x)dx &= \lim_{\varepsilon\to 0}\int_\Omega u_\varepsilon(x)\varphi(\widehat{x}, x_N - \frac{\varepsilon}{2})dx\\ &= \lim_{\varepsilon\to 0}\int_\Omega T_\varepsilon u_\varepsilon(x)\varphi(x)dx\\ &= \int_\Omega z(x)\varphi(x)dx.\end{aligned}$$

Then $u = z$, hence $u_\varepsilon \to u$ in $L^2(\Omega)$ and $u \in W^{1,2}(\Omega\backslash\Sigma)$.

Finally, we establish that for any $\eta > 0$, there exists a subsequence of $(u_\varepsilon)_{\varepsilon>0}$ with $u_{\varepsilon\lfloor\Omega_\eta} \rightharpoonup u_{\lfloor\Omega_\eta}$ in $W^{1,2}_{\Gamma_0}(\Omega_\eta)$. The fact that $u \in W^{1,2}_{\Gamma_0}(\Omega\backslash\Sigma)$ will follow immediately. Let $\eta > 0$, clearly, there exists $\varepsilon_0 > 0$ with $\varepsilon_0 < \eta$ and $\Omega_\eta \subseteq \Omega_\varepsilon$ for all $\varepsilon \le \varepsilon_0$. By the Poincaré inequality and $\sup\limits_{\varepsilon>0}\left(\int_{\Omega_\varepsilon}|\nabla u_\varepsilon(x)|^2\,dx + \varepsilon\int_{B_\varepsilon}\left|\frac{\partial u_\varepsilon}{\partial x_N}(x)\right|^2 dx\right) < +\infty$, we have

$$\sup_{\varepsilon>0}\|u_\varepsilon\|^2_{W^{1,2}(\Omega_\eta,\mathbb{R}^3)} \le C\sup_{\varepsilon>0}\left(\int_{\Omega_\varepsilon}|\nabla u_\varepsilon(x)|^2\,dx + \varepsilon\int_{B_\varepsilon}\left|\frac{\partial u_\varepsilon}{\partial x_N}(x)\right|^2 dx\right) < +\infty.$$

Thus, $(u_\varepsilon)_{\varepsilon>0}$ is bounded in $W^{1,2}_{\Gamma_0}(\Omega_\eta)$, and consequently, there exists $w \in W^{1,2}_{\Gamma_0}(\Omega_\eta)$ and a subsequence not relabelled of $(u_\varepsilon)_{\varepsilon>0}$ with $u_\varepsilon \to w$ in $L^2(\Omega_\eta)$ and $u_\varepsilon \rightharpoonup w$ in $W^{1,2}_{\Gamma_0}(\Omega_\eta)$. It is easily seen that $w = u_{\lfloor\Omega_\eta}$.

Proof of (iv). The weak convergence of $\tau_\varepsilon^{-1}u_\varepsilon$ to some θ in $V(B)$ follows from

$$\left\|\tau_\varepsilon^{-1}u_\varepsilon\right\|_{V(B)} = \left(\frac{1}{\varepsilon}\int_{B_\varepsilon}|u_\varepsilon|^2\,dx + \varepsilon\int_{B_\varepsilon}\left|\frac{\partial u_\varepsilon}{\partial x_N}\right|^2 dx\right)^{\frac{1}{2}}$$

which yields from (2.3.1)

$$\left\|\tau_\varepsilon^{-1}u_\varepsilon\right\|_{V(B)} \le CX_\varepsilon$$

and from (2.3.2)

$$\sup_{\varepsilon<0}\left\|\tau_\varepsilon^{-1}u_\varepsilon\right\|_{V(B)} < +\infty.$$

Now we deduce $\hat{\nabla}\tau^{-1}u_\varepsilon \rightharpoonup \hat{\nabla}\theta$ in the distributional sense so that $\varepsilon\hat{\nabla}\tau^{-1}u_\varepsilon \rightharpoonup 0$ in the distributional sense. But from coercivity $\varepsilon\hat{\nabla}\tau^{-1}u_\varepsilon$ weakly converges to some τ in $L^2(B,\mathbb{R}^2)$. Then $\varepsilon\hat{\nabla}\tau^{-1}u_\varepsilon \rightharpoonup 0$ in $L^2(B,\mathbb{R}^2)$.

Proof of (v). Note that $\theta(.,\pm\frac{1}{2})$ is well defined. Indeed, one has

$$V(B) \subset W^{1,2}\big((-\frac{1}{2},\frac{1}{2}),L^2(\Sigma)\big) \subset \mathcal{C}\big([-\frac{1}{2},\frac{1}{2}],L^2(\Sigma)\big).$$

With notation of the proof of (iii), clearly $\tau_\varepsilon^{-1}u_\varepsilon(\hat{x},\pm\frac{1}{2}) = (T_\varepsilon u_\varepsilon)^\pm(\hat{x})$ in the sense of traces on Σ of $W^{1,2}_{\Gamma_0}(\Omega\setminus\Sigma)$-functions so that $\tau_\varepsilon^{-1}u_\varepsilon(\hat{x},\pm\frac{1}{2}) \to u^\pm$ in $L^2(\Sigma)$. On the other hand, from

$$\tau_\varepsilon^{-1}u_\varepsilon(\hat{x},x_N) = \tau_\varepsilon^{-1}u_\varepsilon(\hat{x},\pm\frac{1}{2}) + \int_{\pm\frac{1}{2}}^{x_N}\frac{\partial\tau_\varepsilon^{-1}u_\varepsilon}{\partial x_N}(\hat{x},s)\,ds$$

for a.e. x in B, we infer that for all $\varphi\in\mathcal{C}_c(\Sigma)$,

$$\begin{aligned}\int_{-\frac{1}{2}}^{\frac{1}{2}}\int_\Sigma \tau_\varepsilon^{-1}u_\varepsilon(\hat{x},x_N)\varphi(\hat{x})\,dx \;=\; & \int_\Sigma (T_\varepsilon u_\varepsilon)^\pm(\hat{x})\varphi(\hat{x})d\hat{x} \\ & + \int_{-\frac{1}{2}}^{\frac{1}{2}}\int_\Sigma\int_{-\frac{1}{2}}^{x_N}\frac{\partial\tau_\varepsilon^{-1}u_\varepsilon}{\partial x_N}(\hat{x},s)\varphi(\hat{x})dsdx. \quad (2.3.5)\end{aligned}$$

Going to the limit in (2.3.5), we obtain

$$\int_{-\frac{1}{2}}^{\frac{1}{2}}\int_\Sigma \theta\varphi(\hat{x})\,dx = \int_\Sigma u^\pm(\hat{x})\varphi(\hat{x})d\hat{x} + \int_{-\frac{1}{2}}^{\frac{1}{2}}\int_\Sigma\int_{-\frac{1}{2}}^{x_N}\frac{\partial\theta}{\partial x_N}(\hat{x},s)\varphi(\hat{x})dsdx.$$

from which we deduce

$$\int_\Sigma u^\pm(\hat{x})\varphi(\hat{x})d\hat{x} = \int_\Sigma \theta(\hat{x},\pm\frac{1}{2})\varphi(\hat{x})d\hat{x},$$

thus $\theta(.,\pm\frac{1}{2}) = u^\pm$ a.e. in Σ. □

Let us consider the functional $H : V(B) \longrightarrow \mathbb{R}$ defined by

$$H(\theta) := \int_B g^{\infty,2}(\hat{0},\frac{\partial\theta}{\partial x_N})\,dx - \langle\mathcal{S},\theta\rangle.$$

We are going to show that F_ε Γ-converges to the functional $F_0 : L^2(\Omega) \longrightarrow \mathbb{R}\cup\{+\infty\}$ given by

$$F_0(u) = \begin{cases}\displaystyle\int_\Omega f(\nabla u)\,dx + \inf_{\theta\in X(u)} H(\theta) \text{ if } u\in W^{1,2}_{\Gamma_0}(\Omega\setminus\Sigma),\\ +\infty \text{ otherwise}\end{cases}$$

where $X(u) := \big\{\theta\in V(B) : \theta(.,\pm\frac{1}{2}) = u^\pm\big\}$. The following lemma used for proving the upper bound in the Γ-convergence of the sequence $(F_\varepsilon)_{\varepsilon>0}$ to the functional F_0 is a straightforward consequence of the direct method of the calculus of variations.

Lemma 2.3.2. *For any $u \in W^{1,2}_{\Gamma_0}(\Omega\backslash\Sigma)$, let $X(u)$ and $H(\theta)$ as in the previous setting. Then*

(i) for all θ in $X(u)$,

$$\|\theta\|^2_{V(B)} \leq C\left(1+\int_B\left|\frac{\partial\theta}{\partial x_N}(x)\right|^2 dx\right),$$

for some constant $C>0$, not depending on θ;

(ii) for all θ in $X(u)$,

$$H(\theta) \geq \alpha X_\theta^2 - C\,\|\theta\|_{V(B)} \quad \text{for some constant } C>0, \text{ not depending on } \theta$$

moreover,

$$H(\theta) \geq \frac{\alpha}{2}X_\theta^2 - C \quad \text{for some constant } C>0, \text{ not depending on } \theta$$

$$\text{where } X_\theta = \left(\int_B\left|\frac{\partial\theta}{\partial x_N}(x)\right|^2 dx\right)^{\frac{1}{2}};$$

(iii) $\inf\limits_{\theta\in X(u)} H(\theta) > -\infty$ and there exists $\theta(u)$ in $X(u)$ such that $\inf\limits_{\theta\in X(u)} H(\theta) = H(\theta(u))$.

Proof. Proof of (i). For any $\theta \in X(u)$, we have

$$\begin{aligned}\theta(\widehat{x},x_N) &= \theta(\widehat{x},-\frac{1}{2}) + \int_{-\frac{1}{2}}^{x_N}\frac{\partial\theta}{\partial x_N}(\widehat{x},t)dt \\ &= u^-(\widehat{x}) + \int_{-\frac{1}{2}}^{x_N}\frac{\partial\theta}{\partial x_N}(\widehat{x},t)dt. \end{aligned} \tag{2.3.6}$$

Actually, by integrating (2.3.6) with respect to x on B and by a straightforward calculation, we obtain

$$\|\theta\|^2_{V(B)} \leq C\left(1+\int_B\left|\frac{\partial\theta}{\partial x_N}\right|^2 dx\right).$$

Proof of (ii). Since $\mathcal{S} \in V'(B)$ and by the assumption on g, we infer that

$$\begin{aligned} H(\theta) &= \int_B g^{\infty,2}(\widehat{0},\frac{\partial\theta}{\partial x_N}(x))d\widehat{x} - \langle\mathcal{S},\theta\rangle \\ &\geq \alpha\int_B\left|\frac{\partial\theta}{\partial x_N}(x)\right|^2 dx - \|\mathcal{S}\|_{V'(B)}\|\theta\|_{V(B)} \\ &\geq \alpha\int_B\left|\frac{\partial\theta}{\partial x_N}(x)\right|^2 dx - C\ \|\theta\|_{V(B)}. \end{aligned} \tag{2.3.7}$$

By the following basic inequality

$$\|\theta\|_{V(B)} \leq \frac{1}{2}(\frac{(1)^2}{\delta} + \delta \|\theta\|^2_{V(B)}), \text{ valid for any } \delta > 0,$$

and choosing the suitable $\delta = \dfrac{\alpha}{2C}$, from (2.3.7) and assertion (i), we obtain

$$H(\theta) \geq \frac{\alpha}{2} \int_B \left| \frac{\partial \theta}{\partial x_N}(x) \right|^2 dx + C.$$

Proof of (iii). Obviously, $\inf_{\theta \in X(u)} H(\theta) > -\infty$ which follows from assertion (i). We purpose to establish existence of $\bar{\theta}$ in $X(u)$ such that $\inf_{\theta \in X(u)} H(\theta) = H(\bar{\theta})$ by using the so-called *direct method in the calculus of variations.* Setting $I(u) = \inf_{\theta \in X(u)} H(\theta)$.

Step 1. We will construct a sequence $(\theta_n)_{n\in\mathbb{N}}$ in $X(u)$ such that $\lim_{n\to\infty} H(\theta_n) = I(u)$. For each $n \in \mathbb{N}$, there exists θ_n in $L^2(B)$ such that

$$I(u) \leq H(\theta_n) \leq I(u) + \frac{1}{n} \tag{2.3.8}$$

Thus, clearly

$$\lim_{n\to\infty} H(\theta_n) = I(u) \tag{2.3.9}$$

Step 2. We establish that there exists a subsequence not relabelled of $(\theta_n)_{n\in\mathbb{N}}$ and function $\bar{\theta}$ in $L^2(B)$ with $\theta_n \rightharpoonup \bar{\theta}$ in $L^2(B)$ and $\dfrac{\partial \theta_n}{\partial x_N} \rightharpoonup \dfrac{\partial \bar{\theta}}{\partial x_N}$ in $L^2(B)$. According to assertion (ii), one has

$$1 + I(u) + C \geq \frac{\alpha}{2} \left\| \frac{\partial \theta_n}{\partial x_N} \right\|^2_{L^2(B)} \quad \text{for all } n \in \mathbb{N}, \tag{2.3.10}$$

then

$$\sup_{n\in\mathbb{N}} \left\| \frac{\partial \theta_n}{\partial x_N} \right\|^2_{L^2(B)} < +\infty \ . \tag{2.3.11}$$

By assertion (i), one has

$$\|\theta_n\|^2_{L^2(B)} \leq \|\theta_n\|^2_{V(B)} \leq C \left(1 + \left\| \frac{\partial \theta_n}{\partial x_N} \right\|^2_{L^2(B)} \right) \leq C \left(1 + \sup_{n\in\mathbb{N}} \left\| \frac{\partial \theta_n}{\partial x_N} \right\|^2_{L^2(B)} \right),$$

so that from (2.3.11), $(\theta_n)_{n\in\mathbb{N}}$ is bounded in $L^2(B)$. Then there exists $\bar{\theta} \in L^2(B)$ and a subsequence of $(\theta_n)_{n\in\mathbb{N}}$ such that $\theta_n \rightharpoonup \bar{\theta}$ in $L^2(B)$ and clearly $\dfrac{\partial \theta_n}{\partial x_N} \rightharpoonup \dfrac{\partial \bar{\theta}}{\partial x_N}$ in $L^2(B)$ and $\bar{\theta} \in V(B)$. It remain to show that

$$\inf_{\theta \in X(u)} H(\theta) = H(\bar{\theta}).$$

Obviously,

$$\inf_{\theta\in X(u)} H(\theta) \le H(\overline{\theta}). \tag{2.3.12}$$

Since $g^{\infty,2}$ is convex and $\theta_n \rightharpoonup \overline{\theta}$, $\dfrac{\partial\theta_n}{\partial x_N} \rightharpoonup \dfrac{\partial\overline{\theta}}{\partial x_N}$ weakly in $L^2(B)$, one has

$$\begin{aligned}
H(\overline{\theta}) &\le \liminf_{n\to\infty} \int_B g^{\infty,2}(\widehat{0}, \frac{\partial\theta_n}{\partial x_N}(x))dx + \lim_{n\to\infty}(-\langle \mathcal{S}, \theta_n\rangle) \\
&= \liminf_{n\to\infty} \int_B g^{\infty,2}(\widehat{0}, \frac{\partial\theta_n}{\partial x_N}(x))dx + \liminf_{n\to\infty}(-\langle \mathcal{S}, \theta_n\rangle) \\
&\le \liminf_{n\to\infty} \left(\int_B g^{\infty,2}(\widehat{0}, \frac{\partial\theta_n}{\partial x_N}(x))dx - \langle \mathcal{S}, \theta_n\rangle\right) \\
&\le \liminf_{n\to\infty} \left(I(u) + \frac{1}{n}\right) = I(u) = \inf_{\theta\in X(u)} H(\theta)
\end{aligned} \tag{2.3.13}$$

Thus, from (2.3.12) and (2.3.13), we obtain

$$\inf_{\theta\in X(u)} H(\theta) = \int_B g^{\infty,2}(\widehat{0}, \frac{\partial\overline{\theta}}{\partial x_N})(x)dx - \left\langle \mathcal{S}, \overline{\theta}\right\rangle = H(\overline{\theta}).$$

□

Then, in its domain $W^{1,2}_{\Gamma_0}(\Omega\setminus\Sigma)$, the functional F_0 may be written

$$F_0(u) = \int_\Omega f(\nabla u)\ dx + H(\theta(u)).$$

The main result of this section is

Theorem 2.3.3. *The sequence $(F_\varepsilon)_{\varepsilon>0}$ Γ-converges to the functional F_0 when $L^2(\Omega)$ is equipped with its strong topology.*

The proof is obtained by the two following propositions.

Proposition 2.3.4. *For every $u \in L^2(\Omega)$ and all $(u_\varepsilon)_{\varepsilon>0}$ strongly converging to u in $L^2(\Omega)$ one has*

$$F_0(u) \le \liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon).$$

Proposition 2.3.5. *For every $u \in L^2(\Omega)$ there exists $(v_\varepsilon)_{\varepsilon>0}$ strongly converging to u in $L^2(\Omega)$ satisfying*

$$F_0(u) \ge \limsup_{\varepsilon\to 0} F_\varepsilon(v_\varepsilon).$$

Proof of Proposition 2.3.4. Assuming $\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) < +\infty$, Lemma 2.3.1 yields $u \in W^{1,2}_{\Gamma_0}(\Omega\setminus\Sigma)$ and existence of $\theta \in X(u)$ satisfying $\tau_\varepsilon^{-1}u_\varepsilon \rightharpoonup \theta$ in $V(B)$. Since ${}^T\tau_\varepsilon\mathcal{S}_\varepsilon \to \mathcal{S}$ in $V'(B)$, one has

$$\lim_{\varepsilon\to 0}\langle {}^T\tau_\varepsilon\mathcal{S}_\varepsilon, \tau_\varepsilon^{-1}u_\varepsilon\rangle = \langle \mathcal{S}, \theta\rangle. \tag{2.3.14}$$

On the other hand, since from Lemma 2.3.1, $u_\varepsilon \rightharpoonup u$ in $W^{1,2}_{\Gamma_0}(\Omega_\eta)$ for all $\eta > 0$, one has

$$\liminf_{\varepsilon\to 0} \int_{\Omega_\varepsilon} f(\nabla u_\varepsilon)\ dx \geq \int_\Omega f(\nabla u)\ dx. \tag{2.3.15}$$

Finally from (iv) of Lemma 2.3.1 and a standard lower semicontinuity argument

$$\begin{aligned}
& \liminf_{\varepsilon\to 0} \varepsilon^2 \int_B g(\hat{\nabla}\tau_\varepsilon^{-1}u_\varepsilon, \frac{1}{\varepsilon}\frac{\partial(\tau_\varepsilon^{-1}u_\varepsilon)}{\partial x_N})\ dx \\
\geq\ & \liminf_{\varepsilon\to 0} \left(\varepsilon^2 \int_B g(\hat{\nabla}\tau_\varepsilon^{-1}u_\varepsilon, \frac{1}{\varepsilon}\frac{\partial(\tau_\varepsilon^{-1}u_\varepsilon)}{\partial x_N})\ dx - \int_B g^{\infty,2}(\varepsilon\hat{\nabla}\tau_\varepsilon^{-1}u_\varepsilon, \frac{\partial(\tau_\varepsilon^{-1}u_\varepsilon)}{\partial x_N})\ dx\right) \\
& + \liminf_{\varepsilon\to 0} \int_B g^{\infty,2}(\varepsilon\hat{\nabla}\tau_\varepsilon^{-1}u_\varepsilon, \frac{\partial(\tau_\varepsilon^{-1}u_\varepsilon)}{\partial x_N})\ dx \\
\geq\ & \liminf_{\varepsilon\to 0} \left(\varepsilon^2 \int_B g(\hat{\nabla}\tau_\varepsilon^{-1}u_\varepsilon, \frac{1}{\varepsilon}\frac{\partial(\tau_\varepsilon^{-1}u_\varepsilon)}{\partial x_N})\ dx - \int_B g^{\infty,2}(\varepsilon\hat{\nabla}\tau_\varepsilon^{-1}u_\varepsilon, \frac{\partial(\tau_\varepsilon^{-1}u_\varepsilon)}{\partial x_N})\ dx\right) \\
& + \int_B g^{\infty,2}(0,0,\frac{\partial\theta}{\partial x_N})\ dx \\
=\ & \int_B g^{\infty,2}(0,0,\frac{\partial\theta}{\partial x_N})\ dx
\end{aligned} \tag{2.3.16}$$

provided that we establish

$$\lim_{\varepsilon\to 0} \left(\varepsilon^2 \int_B g(\hat{\nabla}\tau_\varepsilon^{-1}u_\varepsilon, \frac{1}{\varepsilon}\frac{\partial(\tau_\varepsilon^{-1}u_\varepsilon)}{\partial x_N})\ dx - \int_B g^{\infty,2}(\varepsilon\hat{\nabla}\tau_\varepsilon^{-1}u_\varepsilon, \frac{\partial\theta}{\partial x_N})\ dx\right) = 0. \tag{2.3.17}$$

Since $g^{\infty,2}$ is positively homogeneous of degree 2, and from (2.2.1), we have

$$\begin{aligned}
& \int_B \left|\varepsilon^2 g(\widehat{\nabla}\tau_\varepsilon^{-1}u_\varepsilon, \frac{1}{\varepsilon}\frac{\partial\tau_\varepsilon^{-1}u_\varepsilon}{\partial x_N}) - g^{\infty,2}(\varepsilon\widehat{\nabla}\tau_\varepsilon^{-1}u_\varepsilon, \frac{\partial(\tau_\varepsilon^{-1}u_\varepsilon)}{\partial x_N})\right| dx \\
=\ & \varepsilon^2 \int_B \left|g(\widehat{\nabla}\tau_\varepsilon^{-1}u_\varepsilon, \frac{1}{\varepsilon}\frac{\partial\tau_\varepsilon^{-1}u_\varepsilon}{\partial x_N}) - g^{\infty,2}(\widehat{\nabla}\tau_\varepsilon^{-1}u_\varepsilon, \frac{1}{\varepsilon}\frac{\partial(\tau_\varepsilon^{-1}u_\varepsilon)}{\partial x_N})\right| dx \\
\leq\ & C\varepsilon^2 \int_B \left[1 + \left|\widehat{\nabla}\tau_\varepsilon^{-1}u_\varepsilon\right|^{2-\delta} + \left|\frac{1}{\varepsilon}\frac{\partial(\tau_\varepsilon^{-1}u_\varepsilon)}{\partial x_N}\right|^{2-\delta}\right]\ dx.
\end{aligned}$$

Thus, by using Hölder's inequality (take $p = \frac{2}{2-\delta},\ q = \frac{2}{\delta}$) we deduce

$$\int_B \left|\varepsilon^2 g(\widehat{\nabla}\tau_\varepsilon^{-1}u_\varepsilon, \frac{1}{\varepsilon}\frac{\partial\tau_\varepsilon^{-1}u_\varepsilon}{\partial x_N}) - g^{\infty,2}(\varepsilon\widehat{\nabla}\tau_\varepsilon^{-1}u_\varepsilon, \frac{\partial(\tau_\varepsilon^{-1}u_\varepsilon)}{\partial x_N})\right| dx \leq C\varepsilon^\delta$$

which proves (2.3.17). The conclusion of Proposition 2.3.4 follows by collecting (2.3.14), (2.3.15) and (2.3.16). □

Proof of Proposition 2.3.5. Let $u \in L^2(\Omega)$. We have to construct a sequence $(v_\varepsilon)_{\varepsilon>0}$ strongly converges to u in $L^2(\Omega)$ such that $\limsup_{\varepsilon\to 0} F_\varepsilon(v_\varepsilon) \leq F_0(u)$. If $F_0(u) = +\infty$, then

$u \in L^2(\Omega)\backslash W^{1,2}_{\Gamma_0}(\Omega\backslash\Sigma)$, and clearly, for any sequence $(v_\varepsilon)_{\varepsilon>0}$ converging to u, $\limsup_{\varepsilon\to 0} F_\varepsilon(v_\varepsilon) \leq F_0(u)$ is true. Now, for the harder part, we assume $F_0(u) < +\infty$, then $u \in W^{1,2}_{\Gamma_0}(\Omega\backslash\Sigma)$ and

$$F_0(u) = \int_\Omega f(\nabla u(x))dx + \inf_{\theta \in X(u)} H(\theta).$$

To complete the proof, we construct such a sequence $(v_\varepsilon)_{\varepsilon>0}$ obtained from $\overline{\theta}$ in Lemma 2.3.2 i.e., $H(\overline{\theta}) = \inf_{\theta\in X(u)} H(\theta)$. We divide the proof into four steps:

Step 1. Let us extend u and $\overline{\theta}$ by 0 in $(\mathbb{R}^{N-1} \setminus \Sigma) \times (-r, r)$ and write $\widetilde{u}$ and $\widetilde{\overline{\theta}}$ the extended functions respectively. For each $\delta > 0$, set

$$u_\delta = \rho_\delta * \widetilde{u} \text{ defined by } \rho_\delta * \widetilde{u}(\widehat{x}, x_N) = \int_{\mathbb{R}^{N-1}} \rho_\delta(\widehat{x} - \widehat{y}) * \widetilde{u}(\widehat{y}, x_N) d\widehat{y} \quad \text{for all } (\widehat{x}, x_N) \in \Omega;$$
$$\theta_\delta = \rho_\delta * \widetilde{\overline{\theta}} \text{ defined by } \rho_\delta * \widetilde{\overline{\theta}}(\widehat{x}, x_N) = \int_{\mathbb{R}^{N-1}} \rho_\delta(\widehat{x} - \widehat{y}) * \widetilde{\overline{\theta}}(\widehat{y}, x_N) d\widehat{y} \quad \text{for all } (\widehat{x}, x_N) \in \Omega.$$

where ρ_δ is a mollification function in $\mathcal{C}_c^\infty(\mathbb{R}^N)$.

Clearly,

- $\theta_\delta(\widehat{x}, \pm\frac{1}{2}) = u_\delta(\widehat{x}, 0)$ for all $\widehat{x} \in \Sigma$
- $u_\delta \to u$ in $L^2(\Omega)$ and $\theta_\delta \to \overline{\theta}$ in $L^2(B)$
- $u_\delta \in W^{1,2}(\Omega)$ and $\theta_\delta \in W^{1,2}(B)$

Next, for each $\delta > 0$, we defined the sequence $(v_{\delta,\varepsilon})_{\varepsilon>0}$ as follow:

$$v_{\delta,\varepsilon}(\widehat{x}, x_N) = \begin{cases} u_\delta(\widehat{x}, x_N \pm \dfrac{\varepsilon}{2}) & \text{on} \quad \Omega_\varepsilon^\mp \\ \\ \theta_\delta(\widehat{x}, \dfrac{x_N}{\varepsilon}) & \text{on} \quad B_\varepsilon \end{cases} \tag{2.3.18}$$

and obviously, $v_{\delta,\varepsilon}(\widehat{x}, x_N)$ belongs to $W^{1,2}(\Omega)$ and strongly converges to u_δ in $L^2(\Omega)$.

Step 2. We show that $\lim_{\varepsilon\to 0} F_\varepsilon(v_{\delta,\varepsilon}) = F_0(u_\delta) = \int_\Omega f(\nabla u_\delta)dx + H(\theta_\delta)$. In fact, we claim that

$$\lim_{\varepsilon\to 0} \int_{\Omega_\varepsilon} f(\nabla v_{\delta,\varepsilon})(x)dx = \int_\Omega f(\nabla u_\delta)(x)dx \tag{2.3.19}$$
$$\lim_{\varepsilon\to 0} \left(\varepsilon^2 \int_B g(\widehat{\nabla}\tau_\varepsilon^{-1} v_{\delta,\varepsilon}, \frac{1}{\varepsilon}\frac{\partial \tau_\varepsilon^{-1} v_{\delta,\varepsilon}}{\partial x_N})(x)dx - \left\langle T_{\tau_\varepsilon}\mathcal{S}_\varepsilon, \tau_\varepsilon^{-1} v_{\delta,\varepsilon}\right\rangle \right) = H(\theta_\delta). \tag{2.3.20}$$

Proof of (2.3.19): one has

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} f(\nabla v_{\delta,\varepsilon})(x)dx &= \lim_{\varepsilon\to 0}\left(\int_{\Omega_\varepsilon^+} f(\nabla u_\delta)(\widehat{x}, x_N-\frac{\varepsilon}{2})dx+\int_{\Omega_\varepsilon^-} f(\nabla u_\delta(\widehat{x}, x_N+\frac{\varepsilon}{2}))dx\right)\\ &= \int_{\Omega^+} f(\nabla u_\delta)(x)dx+\int_{\Omega^-} f(\nabla u_\delta)(x)dx\\ &= \int_{\Omega} f(\nabla u_\delta)(x)dx\end{aligned}$$

Proof of (2.3.20): Since $g^{\infty,2}$ is positively homogeneous of degree 2 and ${}^T\tau_\varepsilon\mathcal{S}_\varepsilon$ strongly converges to $\mathcal{S}$ in $V'(B)$, one has

$$\lim_{\varepsilon\to 0}\left(\varepsilon^2\int_B g(\widehat{\nabla}\theta_\delta, \frac{1}{\varepsilon}\frac{\partial\theta_\delta}{\partial x_N})(x)dx-\langle {}^T\tau_\varepsilon\mathcal{S}_\varepsilon, \theta_\delta\rangle\right)=\int_B g^{\infty,2}(\widehat{0}, \frac{\partial\theta_\delta}{\partial x_N})dx-\langle\mathcal{S}, \theta_\delta\rangle$$

Step 3. We establish that $\lim_{\delta\to 0} F_0(u_\delta)=F_0(u)$. Since

$$\begin{aligned}F_0(u_\delta) &= \int_\Omega f(\nabla u_\delta)dx+H(\theta_\delta)\\ &= \int_\Omega f(\nabla u_\delta)dx+\int_B g^{\infty,2}(\widehat{0}, \frac{\partial\theta_\delta}{\partial x_N})dx-\langle\mathcal{S}, \theta_\delta\rangle\end{aligned}$$

the result is a straightforward consequence of $u_\delta\to u$ in $L^2(\Omega)$ and $\theta_\delta\to\bar{\theta}$ in $L^2(B)$.

Step 4. By using a standard diagonalization argument, from step 2 and step 3, there exists a mapping $\varepsilon\mapsto\delta(\varepsilon)$ such that $v_{\delta(\varepsilon)}\longrightarrow u$ in $L^2(\Omega)$ and $\lim_{\varepsilon\to 0} F_\varepsilon(v_{\delta(\varepsilon)})=F_0(u)$. The sequence $(v_\varepsilon)_{\varepsilon>0}$ where $v_\varepsilon := v_{\delta(\varepsilon)}$ fullfils all the conditions except the boundary condition on Γ_0. By using De Giorgi's slicing method in a neighborhood of Γ_0, one can modify v_ε in Ω_ε into a function $\tilde{v}_\varepsilon$ satisfying the boundary condition, and from assumption $\mathrm{dist}(\overline{\Gamma}_0, \overline{\partial B_\varepsilon\cap\partial\Omega})>0$, which is equal to v_ε in B_ε, and satisfies $\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} f(\nabla v_\varepsilon)\ dx = \lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} f(\nabla\tilde{v}_\varepsilon)\ dx$. Still denoting by v_ε this new function, we have $\lim_{\varepsilon\to 0} F_\varepsilon(v_\varepsilon)=F_0(u)$. Hence the proof of this proposition is complete. □

Remark 2.3.6. In order to give an interpretation of the limit energy functional it is worthwhile to write

$$\inf_{\theta\in X(u)} H(\theta)=\inf_{\theta\in V_0(B)}\left\{\int_B g^{\infty,2}(\hat{0}, \frac{\partial\theta}{\partial x_N}(x)+[u](\hat{x}))\ dx-\langle\mathcal{S}, \theta\rangle\right\}-\langle\mathcal{S}, \tilde{u}\rangle \tag{2.3.21}$$

where $[u]=u^+-u^-$, $V_0(B)=\left\{u\in V(B) : u=0 \text{ on } \Sigma\times\{\pm\frac{1}{2}\}\right\}$ and $\tilde{u}(x)=x_N[u](\hat{x})+\frac{u^+(\hat{x})+u^-(\hat{x})}{2}$. Therefore when the limit source $\mathcal{S}$ vanishes, by using Jensen's inequality, $\inf_{\theta\in X(u)} H(\theta)$ reduces to

$$\int_\Sigma g^{\infty,2}(\hat{0}, [u](\hat{x}))\ d\hat{x}$$

which is nothing but the surface energy of the model obtained in [8]. In this case $G : u\mapsto \inf_{\theta\in X(u)} H(\theta)$ is a local functional with density h defined by $h(\hat{x})=g^{\infty,2}(\hat{0}, [u](\hat{x}))$. By contrast when the limit source is not trivial, the functional G is non local in general and of the form

$$G(u) = \int_B g^{\infty,2}(\hat{0}, \frac{\partial \theta_{[u]}}{\partial x_N}(x) + [u](\hat{x}))\ dx - \langle \mathcal{S}, \theta_{[u]} \rangle - \langle \mathcal{S}, \tilde{u} \rangle$$

where $\theta_{[u]}$ is the minimizer of (2.3.21). In this general case, the functional G is a non local functional, not only of the jump field $[u]$, but also of the trace fields u^+ and u^-.

2.4 Some Examples of measure sources $\mathcal{S}_\varepsilon$ concentrated in B_ε

The general form of elements of $V'(B)$ is given for every θ in $V(B)$ by $\langle \mathcal{S}, \theta \rangle = \int_B s_0 \theta\ dx + \int_B s_1 \frac{\partial \theta}{\partial x_N}\ dx$ where $(s_0, s_1) \in L^2(B) \times L^2(B)$. The limit sources $\mathcal{S}$ considered in this section are generated by measures $\mathcal{S}_\varepsilon$ in $\mathcal{M}(B_\varepsilon)$ whose slicing structure $\mathcal{H}^{N-1} \lfloor \Sigma \otimes \mathcal{S}^\varepsilon_{\hat{x}}$ is such that their slicing components $\mathcal{S}^\varepsilon_{\hat{x}}$ do not present a diffuse singular part in their Lebesgue-Nikodym decomposition in $\mathcal{M}(-\frac{\varepsilon}{2}, \frac{\varepsilon}{2})$, i.e., are of the general form

$$\mathcal{S}^\varepsilon_{\hat{x}} = a_\varepsilon(\hat{x}, .)\ dt + \sum_{n=-\infty}^{+\infty} b_{\varepsilon,n}(\hat{x}) \delta_{t^\varepsilon_n(\hat{x})}$$

where

$$\begin{cases} a_\varepsilon \in L^2(B_\varepsilon),\ b_{\varepsilon,n} \in L^2(\Sigma), \\ t^\varepsilon_n : \Sigma \longrightarrow (-\dfrac{\varepsilon}{2}, \dfrac{\varepsilon}{2}) \text{ is a Borel measurable map.} \end{cases}$$

We make the following additional assumptions:

(H1) there exists $a \in L^2(B)$ such that $\varepsilon \tau_\varepsilon^{-1} a_\varepsilon \to a$ in $L^2(B)$;

(H2) there exists $b_n \in L^2(\Sigma)$ such that $b_{\varepsilon,n} \to b_n$ in $L^2(\Sigma)$ when $\varepsilon \to 0$;

(H3) there exists $c_n \in \mathbb{R}^+$ such that $\|b_{\varepsilon,n}\|_{L^2(\Sigma)} \leq c_n$ and $\sum_{n=-\infty}^{+\infty} c_n < +\infty$;

(H4) there exists $(t_n)_{n \in \mathbf{Z}}$ in $(-1/2, 1/2)$ such that $t^\varepsilon_n = \varepsilon t_n$ for all $n \in \mathbf{Z}$.

It is easy to check that the measure ${}^T\tau_\varepsilon \mathcal{S}_\varepsilon$ of $\mathcal{M}(B)$ is given by: ${}^T\tau_\varepsilon \mathcal{S}_\varepsilon = \mathcal{H}^{N-1} \lfloor \Sigma \otimes ({}^T\tau_\varepsilon \mathcal{S}^\varepsilon)_{\hat{x}}$ where

$$({}^T\tau_\varepsilon \mathcal{S}^\varepsilon)_{\hat{x}} = \varepsilon \tau_\varepsilon^{-1} a_\varepsilon\ dt + \sum_{n=-\infty}^{+\infty} b_{\varepsilon,n}(\hat{x}) \delta_{t_n(\hat{x})}.$$

Proposition 2.4.1. *The measure ${}^T\tau_\varepsilon \mathcal{S}_\varepsilon$ strongly converges in $V'(B)$ to the measure $\mathcal{S}$ defined for every $\theta \in V(B)$ by*

$$\langle \mathcal{S}, \theta \rangle = \int_B a\theta\ dx + \sum_{n=-\infty}^{+\infty} \int_\Sigma b_n(\hat{x}) \theta(\hat{x}, t_n(\hat{x}))\ d\hat{x}.$$

Therefore, the functional F_ε Γ-converges to the functional $F_0 : L^2(\Omega) \longrightarrow \mathbb{R}\cup\{+\infty\}$ given by

$$F_0(u) = \begin{cases} \displaystyle\int_\Omega f(\nabla u)\ dx + \inf_{\theta \in X(u)} \left\{ \int_B g^{\infty,2}(\hat{0}, \frac{\partial \theta}{\partial x_N})\ dx - \int_B a\theta\ dx - \sum_{n=-\infty}^{+\infty} \int_\Sigma b_n(\hat{x})\theta(\hat{x}, t_n(\hat{x}))\ d\hat{x} \right\} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } u \in W^{1,2}_{\Gamma_0}(\Omega \setminus \Sigma) \\ +\infty \ \text{otherwise.} \end{cases}$$

Proof. The second assertion is a straightforward consequence of Theorem 2.3.3 provided that we establish the strong convergence of ${}^T\tau_\varepsilon \mathcal{S}_\varepsilon$ to $\mathcal{S}$ in $V'(B)$. For every $\theta \in V(B)$ we have

$$\langle {}^T\tau_\varepsilon \mathcal{S}_\varepsilon - \mathcal{S}, \theta \rangle = \int_B (\varepsilon \tau_\varepsilon^{-1} a_\varepsilon - a)\theta\ dx + \int_\Sigma \sum_{n=-\infty}^{+\infty} (b_{\varepsilon,n} - b_n)\theta(\hat{x}, t_n(\hat{x}))\ d\hat{x},$$

thus

$$\left|\langle {}^T\tau_\varepsilon \mathcal{S}_\varepsilon - \mathcal{S}, \theta \rangle\right| \leq \|\theta\|_{L^2(B)} \left\|\varepsilon \tau_\varepsilon^{-1} a_\varepsilon - a\right\|_{L^2(B)} + \sum_{n=-\infty}^{+\infty} \left[\|(b_{\varepsilon,n} - b_n\|_{L^2(\Sigma)} \left(\int_\Sigma |\theta(\hat{x}, t_n(\hat{x})|^2\ d\hat{x} \right)^{\frac{1}{2}} \right]. \tag{2.4.1}$$

But it is easy to establish that there exists a non negative constant C such that

$$\left(\int_\Sigma |\theta(\hat{x}, t_n(\hat{x})|^2\ d\hat{x} \right)^{\frac{1}{2}} \leq C \|\theta\|_{V(B)}$$

so that (2.4.1) yields

$$\left\|{}^T\tau_\varepsilon \mathcal{S}_\varepsilon - \mathcal{S}\right\|_{V'(B)} \leq \left\|\varepsilon \tau_\varepsilon^{-1} a_\varepsilon - a\right\|_{L^2(B)} + C \sum_{n=-\infty}^{+\infty} \|(b_{\varepsilon,n} - b_n\|_{L^2(\Sigma)}.$$

The conclusion follows from assumptions $(H1)$, $(H2)$ and $(H3)$. □

2.5 The gradient concentration phenomenon

We first recall the notion of gradient Young-concentration measure introduced in [3]. Let us denote the unit sphere $\{-1, 1\}$ of $\mathbb{R}$ by $\mathbb{S}^0$, and consider $S \subset\subset \Sigma$, $B'_\varepsilon := S \times (-\frac{\varepsilon}{2}, \frac{\varepsilon}{2})$. Then we define

Definition 2.5.1. *A pair $(v, \mu) \in L^2(\Omega) \times \mathcal{M}^+(\bar{\Omega} \times \mathbb{S}^0)$ is gradient Young-concentration measure (localized on S) iff there exists a sequence $(v_\varepsilon)_{\varepsilon>0}$ in $W^{1,2}_{\Gamma_0}(\Omega)$ satisfying $\sup_{\varepsilon>0} \int_{\Omega \setminus \mathbf{B}_\varepsilon} |\nabla v_\varepsilon|^2\ dx < +\infty$ such that*

$$\begin{cases} v_\varepsilon \to v \ \text{in } L^2(\Omega), \\ \mu_\varepsilon := \delta_{\frac{\partial v_\varepsilon}{\partial x_N} / \left|\frac{\partial v_\varepsilon}{\partial x_N}\right|\ (x)} \otimes \varepsilon \mathbb{1}_{B'_\varepsilon} \left|\frac{\partial v_\varepsilon}{\partial x_N}\right|^2\ dx \overset{*}{\rightharpoonup} \mu. \end{cases}$$

We say that the sequence $(v_\varepsilon)_{\varepsilon>0}$ generates the gradient Young-concentration measure (v, μ). We denote the set of gradient Young concentration measures by $\mathcal{YC}$.

Recall that the weak convergence $\overset{*}{\rightharpoonup}$ above is defined by

$$\int_{B'_\varepsilon} \varepsilon\theta(x)\tilde{\varphi}(\frac{\partial v_\varepsilon}{\partial x_N})\ dx \to \int_{\bar{\Omega}}\int_{\mathbb{S}^0} \theta(x)\varphi(\zeta)\ d\mu$$

for all $\theta \in \mathcal{C}(\bar{\Omega})$ and all $\varphi \in \mathcal{C}(\mathbb{S}^0)$, where the 2-homogeneous extension $\tilde{\varphi} : \mathbb{R}^m \to \mathbb{R}$ of $\varphi \in \mathcal{C}(\mathbb{S}^0)$ is defined for all $\zeta \in \mathbb{R}^m$ by

$$\tilde{\varphi}(\zeta) = \begin{cases} |\zeta|^2\varphi(\frac{\zeta}{|\zeta|}), & \text{if } \zeta \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Let denote by $\mu_x \otimes \pi$ the slicing decomposition of the limit measure μ. Then, whereas the measure π provides the intensity of the concentration on Σ generated by $\frac{\partial v_\varepsilon}{\partial x_N}$, the probability measure μ_x captures the change of sign of $\frac{\partial v_\varepsilon}{\partial x_N}$ through Σ.

Let us recall that the the gradient Young-concentration measures are characterized as follows (see [3] Theorem 3.1)

Theorem 2.5.2 (Characterization). *A pair $(v, \mu = \mu_x \otimes \pi)$ belongs to $\mathcal{YC}$ if and only if $v \in W^{1,2}_{\Gamma_0}(\Omega)$, π is concentrated on $\bar{S}$ and, for every $\varphi \in \mathcal{C}(\mathbb{S}^0)$ such that $\varphi^{**} > -\infty$,*

$$\begin{aligned} &\frac{d\pi}{d\mathcal{H}^{N-1}\lfloor S}(x)\int_{\mathbb{S}^0}\varphi(\zeta)\ d\mu_x \geq \varphi^{**}([v](x)) \quad \textit{for } \mathcal{H}^{N-1} \textit{ a. e. } x \in S \\ &\int_{\mathbb{S}^0}\varphi(\zeta)\ d\mu_x \geq 0 \quad \textit{for } \pi_s \textit{ a. e. } x \in \bar{S} \end{aligned} \tag{2.5.1}$$

where $\pi = \frac{d\pi}{d\mathcal{H}^{N-1}\lfloor S}\mathcal{H}^{N-1}\lfloor S + \pi_s$ is the Radon-Nikodym decomposition of π with respect to the measure $\mathcal{H}^{N-1}\lfloor S$.

We know (see Remark 2.5 in [3]) that every sequence $(v_\varepsilon)_{\varepsilon>0}$ satisfying (2.3.2) generates a gradient Young-concentration measure. In this section, under the condition $g^{\infty,2}(\hat{\xi}, \xi_3) \geq g^{\infty,2}(\hat{0}, \xi_3)$, we are going to characterize the gradient Young-concentration measures generated by $\bar{u}_\varepsilon \in \operatorname{argmin} F_\varepsilon$. From the next Theorem, combined with Theorem 2.5.2, we will deduce some bounds on these measures.

Theorem 2.5.3. *Let $\bar{u}_\varepsilon$ be a minimizer of $\min\{F_\varepsilon(v) : v \in L^2(\Omega)\}$, $(\bar{u}, \bar{\mu})$ a gradient Young-concentration measure localized on $S \subset\subset \Sigma$ generated by the sequence $(\bar{u}_\varepsilon)_{\varepsilon>0}$, and $\bar{\theta}$ satisfying $H(\bar{\theta}) = \inf_{\theta \in X(\bar{u})} H(\theta)$. Assume that $g^{\infty,2}$ satisfies the condition*

$$\forall \xi \in \mathbb{R}^3,\ g^{\infty,2}(\hat{\xi}, \xi_3) \geq g^{\infty,2}(\hat{0}, \xi_3), \tag{2.5.2}$$

then

$$\begin{aligned} &\bar{u}_\varepsilon \to \bar{u} \textit{ in } L^2(\Omega); \\ &F_\varepsilon(\bar{u}_\varepsilon) \to F_0(\bar{u}) = \min\left\{F_0(u) : u \in L^2(\Omega)\right\}; \\ &\int_{-\frac{1}{2}}^{\frac{1}{2}} g^{\infty,2}(\hat{0}, \frac{\partial\bar{\theta}}{\partial x_N})(\hat{x}, s)\ ds = \frac{d\bar{\pi}}{d\hat{x}}(\hat{x})\int_{\mathbb{S}^0} g^{\infty,2}(\hat{0}, \xi)\ d\bar{\mu}_{\hat{x}} \textit{ for a.e. } \hat{x} \textit{ in } S. \end{aligned} \tag{2.5.3}$$

Proof. According to the variational nature of the Γ-convergence, for a subsequence one has

$$\begin{aligned}
&\bar u_\varepsilon \to \bar u \text{ in } L^2(\Omega)\\
&\lim_{\varepsilon\to 0} F_\varepsilon(\bar u_\varepsilon) = F_0(\bar u) = \min\left\{F_0(v) : v \in L^2(\Omega)\right\}\\
&\qquad\qquad\qquad = \int_\Omega f(\nabla \bar u)\, dx + \inf_{\theta\in X(\bar u)} H(\theta).
\end{aligned} \tag{2.5.4}$$

On the other hand, for the subsequence associated with the gradient Young-concentration measure $(\bar u, \bar\mu)$, there exist a subsequence and a measure $\mu = \mu_{\hat x}\otimes \pi$ in $\mathcal{M}(\bar\Omega\times \mathbb{S}^0)$ with π concentrated in $\bar\Sigma$ such that

$$\begin{aligned}
\lim_{\varepsilon\to 0} \varepsilon^2 \int_B g\big(\widehat{\nabla}\tau_\varepsilon^{-1}\bar u_\varepsilon, \frac{1}{\varepsilon}\frac{\partial \tau_\varepsilon^{-1}\bar u_\varepsilon}{\partial x_N}\big)\, dx &= \lim_{\varepsilon\to 0}\int_B g^{\infty,2}\big(\varepsilon\widehat{\nabla}\tau_\varepsilon^{-1}\bar u_\varepsilon, \frac{\partial \tau_\varepsilon^{-1}\bar u_\varepsilon}{\partial x_N}\big)\, dx\\
&\geq \lim_{\varepsilon\to 0} \varepsilon\int_{B_\varepsilon} g^{\infty,2}\big(\hat 0, \frac{\partial \bar u_\varepsilon}{\partial x_N}\big)\, dx\\
&= \int_{\bar\Sigma}\Big(\int_{\mathbb{S}^0} g^{\infty,2}(\hat 0,\xi)\, d\mu_{\hat x}\Big)\, d\pi
\end{aligned} \tag{2.5.5}$$

where we have used (2.3.17) in the first equality and (2.5.2) in the inequality. Let $\bar\theta$ be the weak limit of $\tau_\varepsilon^{-1}\bar u_\varepsilon$ in $V(B)$ for the subsequence considered, then from (2.5.5), and since

$$\liminf_{\varepsilon\to 0}\int_{\Omega_\varepsilon} f(\nabla \bar u_\varepsilon)\, dx \geq \int_\Omega f(\nabla\bar u)\, dx \text{ and } \lim_{\varepsilon\to 0}\langle {}^T\tau_\varepsilon \mathcal{S}_\varepsilon, \tau_\varepsilon^{-1}\bar u_\varepsilon\rangle = \langle \mathcal{S}, \bar\theta\rangle,$$

we infer

$$\lim_{\varepsilon\to 0} F_\varepsilon(\bar u_\varepsilon) \geq \int_\Omega f(\nabla \bar u)\, dx + \int_{\bar\Sigma}\Big(\int_{\mathbb{S}^0} g^{\infty,2}(\hat 0,\xi)\, d\mu_{\hat x}\Big)\, d\pi - \langle \mathcal{S},\bar\theta\rangle. \tag{2.5.6}$$

Collecting (2.5.4) and (2.5.6) we obtain

$$\int_\Omega f(\nabla\bar u)\, dx + \inf_{\theta\in X(\bar u)} H(\theta) \geq \int_\Omega f(\nabla \bar u)\, dx + \int_{\bar\Sigma}\Big(\int_{\mathbb{S}^0} g^{\infty,2}(\hat 0,\xi)\, d\mu_{\hat x}\Big)\, d\pi - \langle \mathcal{S},\bar\theta\rangle,$$

in particular

$$\int_\Omega f(\nabla\bar u)\, dx + H(\bar\theta) \geq \int_\Omega f(\nabla \bar u)\, dx + \int_{\bar\Sigma}\Big(\int_{\mathbb{S}^0} g^{\infty,2}(\hat 0,\xi)\, d\mu_{\hat x}\Big)\, d\pi - \langle \mathcal{S},\bar\theta\rangle,$$

thus

$$\begin{aligned}
\int_B g^{\infty,2}\big(\hat 0, \frac{\partial\bar\theta}{\partial x_N}\big)\, dx &\geq \int_{\bar\Sigma}\Big(\int_{\mathbb{S}^0} g^{\infty,2}(\hat 0,\xi)\, d\mu_{\hat x}\Big)\, d\pi\\
&\geq \int_{\bar\Sigma} \frac{d\pi}{d\hat x}(\hat x)\Big(\int_{\mathbb{S}^0} g^{\infty,2}(\hat 0,\xi)\, d\mu_{\hat x}\Big)\, d\hat x.
\end{aligned} \tag{2.5.7}$$

But by a standard lower semicontinuity argument, for every $\varphi \in \mathcal{C}_c(\Sigma)$, $\varphi \geq 0$,

$$\begin{aligned}\liminf_{\varepsilon\to 0}\int_B \varphi(\hat{x}) g^{\infty,2}\big(\hat{0}, \frac{\partial \tau_\varepsilon^{-1}\bar{u}_\varepsilon}{\partial x_N}\big)\ dx &= \int_\Sigma \varphi(\hat{x})\Big(\int_{\mathbb{S}^0} g^{\infty,2}(\hat{0},\xi)\ d\mu_{\hat{x}}\Big)\ d\pi \\ &\geq \int_B \varphi(\hat{x}) g^{\infty,2}\big(\hat{0}, \frac{\partial \bar{\theta}}{\partial x_N}\big)\ dx\end{aligned}$$

so that

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} g^{\infty,2}\big(\hat{0}, \frac{\partial\bar{\theta}}{\partial x_N}\big)(\hat{x},s)\ ds \leq \frac{d\pi}{d\hat{x}}(\hat{x})\int_{\mathbb{S}^0} g^{\infty,2}(\hat{0},\xi)\ d\mu_{\hat{x}} \tag{2.5.8}$$

for a.e. $\hat{x}$ in Σ. Combining (2.5.7) and (2.5.8) we deduce

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} g^{\infty,2}\big(\hat{0}, \frac{\partial\bar{\theta}}{\partial x_N}\big)(\hat{x},s)\ ds = \frac{d\pi}{d\hat{x}}(\hat{x})\int_{\mathbb{S}^0} g^{\infty,2}(\hat{0},\xi)\ d\mu_{\hat{x}}$$

for a.e. $\hat{x}$ in Σ. Noticing that $\mu\lfloor \bar{S}\times\mathbb{S}^0 = \bar{\mu}$, we obtain

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} g^{\infty,2}\big(\hat{0}, \frac{\partial\bar{\theta}}{\partial x_N}\big)(\hat{x},s)\ ds = \frac{d\bar{\pi}}{d\hat{x}}(\hat{x})\int_{\mathbb{S}^0} g^{\infty,2}(\hat{0},\xi)\ d\bar{\mu}_{\hat{x}}$$

for a.e. $\hat{x}$ in S. It remains to show that $H(\bar{\theta}) = \inf\limits_{\theta\in X(\bar{u})} H(\theta)$. It suffices to notice that

$$\begin{aligned}\lim_{\varepsilon\to 0} F_\varepsilon(\bar{u}_\varepsilon) &= \int_\Omega f(\nabla\bar{u})\ dx + \inf_{\theta\in X(\bar{u})} H(\theta) \\ &\geq \liminf_{\varepsilon\to 0}\int_{\Omega_\varepsilon} f(\nabla\bar{u}_\varepsilon)\ dx + \liminf_{\varepsilon\to 0}\Big(\int_B g\big((\widehat{\nabla}\tau_\varepsilon^{-1}\bar{u}_\varepsilon, \frac{\partial\tau_\varepsilon^{-1}\bar{u}_\varepsilon}{\partial x_N}\big)\ dx - \langle {}^T\tau_\varepsilon\mathcal{S}_\varepsilon, \tau_\varepsilon^{-1}\bar{u}_\varepsilon\rangle\Big) \\ &\geq \int_\Omega f(\nabla\bar{u})\ dx + \int_B g^{\infty,2}\big(\hat{0}, \frac{\partial\bar{\theta}}{\partial x_N}\big)\ dx - \langle\mathcal{S},\bar{\theta}\rangle \\ &= \int_\Omega f(\nabla\bar{u})\ dx + H(\bar{\theta})\end{aligned}$$

which completes the proof. □

We define the following two constants associated with the function g:

$$c(g) := \min\Big(\frac{g^{\infty,2}(\widehat{0},-1)}{g^{\infty,2}(\widehat{0},1)}, \frac{g^{\infty,2}(\widehat{0},1)}{g^{\infty,2}(\widehat{0},-1)}\Big),\ C(g) = \frac{1}{c(g)} = \max\Big(\frac{g^{\infty,2}(\widehat{0},-1)}{g^{\infty,2}(\widehat{0},1)}, \frac{g^{\infty,2}(\widehat{0},1)}{g^{\infty,2}(\widehat{0},-1)}\Big)$$

Recall that

$$g^{\infty,2}(\widehat{0},\xi) = \begin{cases} g^{\infty,2}(\widehat{0},-1)\,|\xi|^2 & \text{if} \quad \xi\leq 0 \\ g^{\infty,2}(\widehat{0},1)\,|\xi|^2 & \text{if} \quad \xi > 0\end{cases}$$

and moreover, by the assumption on the function g, clearly

$$g^{\infty,2}(\widehat{0},1) > 0 \text{ and } g^{\infty,2}(\widehat{0},-1) > 0.$$

We make precise the probability measure $\bar{\mu}_{\hat{x}}$ localized on $S \subset\subset \Sigma$ as follows:

$$\bar{\mu}_{\hat{x}} := p(\hat{x})\delta_1 + q(\hat{x})\delta_{-1} \quad \text{with} \quad p(\hat{x}) + q(\hat{x}) = 1 \text{ a.e. } \hat{x} \in S.$$

Corollary 2.5.4. *With notation and assumption of Theorem 2.5.3, one has*

(i) for a.e. $\hat{x}$ in S

$$c(g) \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \frac{\partial \overline{\theta}}{\partial x_N}(\hat{x}, s) \right|^2 \, ds \le \frac{d\overline{\pi}}{d\hat{x}}(\hat{x}) \le C(g) \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \frac{\partial \overline{\theta}}{\partial x_N}(\hat{x}, s) \right|^2 \, ds, \tag{2.5.9}$$

and $\displaystyle \frac{d\overline{\pi}}{d\hat{x}}(\hat{x}) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \frac{\partial \overline{\theta}}{\partial x_N}(\hat{x}, s) \right|^2 \, ds$ *when* $g^{\infty,2}(\widehat{0}, -1) = g^{\infty,2}(\widehat{0}, 1)$*;*

(ii) $\displaystyle \frac{c(g)\ |[u](\hat{x})|^2}{\int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \frac{\partial \overline{\theta}}{\partial x_N}(\hat{x}, s) \right|^2 \, ds} \le p(\hat{x}) \le 1$ *for a.e. $\hat{x}$ such that $[u](\hat{x}) > 0$;*

(iii) $\displaystyle \frac{c(g)\ |[u](\hat{x})|^2}{\int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \frac{\partial \overline{\theta}}{\partial x_N}(\hat{x}, s) \right|^2 \, ds} \le q(\hat{x}) \le 1$ *for a.e. $\hat{x}$ such that $[u](\hat{x}) < 0$.*

Proof. Since $\bar{\mu}_{\hat{x}} = p(\widehat{x})\delta_1 + q(\widehat{x})\delta_{-1}$, we have $\int_{\mathbb{S}^0} g^{\infty,2}(\widehat{0}, \xi) d\overline{\mu}_x = p(\widehat{x}) g^{\infty,2}(\widehat{0}, 1) + q(\widehat{x}) g^{\infty,2}(\widehat{0}, -1)$ with $p(\widehat{x}) + q(\widehat{x}) = 1$ a.e. $\widehat{x}$ in S so that from (2.5.3), for a.e. in S, one has:

$$\begin{aligned} \int_{-\frac{1}{2}}^{\frac{1}{2}} g^{\infty,2}(\widehat{0}, \frac{\partial \overline{\theta}}{\partial x_N}(\widehat{x}, s)) ds &= \left(\int_{\mathbb{S}^0} g^{\infty,2}(\widehat{0}, \xi) d\overline{\mu}_x \right) \frac{d\overline{\pi}}{d\widehat{x}}(\widehat{x}) \\ &= \frac{d\overline{\pi}}{d\widehat{x}}(\widehat{x}) \left\{ p(\widehat{x}) g^{\infty,2}(\widehat{0}, 1) + q(\widehat{x}) g^{\infty,2}(\widehat{0}, -1) \right\}. \end{aligned} \tag{2.5.10}$$

We are going to establish

$$c(g) \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \frac{\partial \overline{\theta}}{\partial x_N}(\widehat{x}, s) \right|^2 ds \le \frac{d\overline{\pi}}{d\widehat{x}}(\widehat{x}) \le C(g) \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \frac{\partial \overline{\theta}}{\partial x_N}(\widehat{x}, s) \right|^2 ds.$$

From (2.5.10) we deduce

$$\begin{aligned} \min\left\{ g^{\infty,2}(\widehat{0}, -1),\ g^{\infty,2}(\widehat{0}, 1) \right\} \int_{-\frac{1}{\varepsilon}}^{\frac{1}{\varepsilon}} \left| \frac{\partial \overline{\theta}}{\partial x_N}(\widehat{x}, s) \right|^2 ds &\le \int_{-\frac{1}{\varepsilon}}^{\frac{1}{\varepsilon}} g^{\infty,2}(\widehat{0}, \frac{\partial \overline{\theta}}{\partial x_N}(\widehat{x}, s)) ds \\ &= \left(\int_{\mathbb{S}^0} g^{\infty,2}(\widehat{0}, \xi) d\overline{\mu}_x \right) \frac{d\overline{\pi}}{d\widehat{x}}(\widehat{x}) \\ &= \left\{ p(\widehat{x}) g^{\infty,2}(\widehat{0}, 1) + q(\widehat{x}) g^{\infty,2}(\widehat{0}, -1) \right\} \frac{d\overline{\pi}}{d\widehat{x}}(\widehat{x}) \\ &\le \max\left\{ g^{\infty,2}(\widehat{0}, -1),\ g^{\infty,2}(\widehat{0}, 1) \right\} \frac{d\overline{\pi}}{d\widehat{x}}(\widehat{x}) \end{aligned} \tag{2.5.11}$$

and

$$\begin{aligned}
\min\left\{g^{\infty,2}(\widehat{0},-1),\ g^{\infty,2}(\widehat{0},1)\right\}\frac{d\overline{\pi}}{d\widehat{x}}(\widehat{x}) &= \min\left\{g^{\infty,2}(\widehat{0},-1),\ g^{\infty,2}(\widehat{0},1)\right\}\{p(\widehat{x})+q(\widehat{x})\}\frac{d\overline{\pi}}{d\widehat{x}}(\widehat{x})\\
&\leq \left\{p(\widehat{x})g^{\infty,2}(\widehat{0},1)+q(\widehat{x})g^{\infty,2}(\widehat{0},-1)\right\}\frac{d\overline{\pi}}{d\widehat{x}}(\widehat{x})\\
&= \left(\int_{\mathbb{S}^0} g^{\infty,2}(\widehat{0},\xi)d\overline{\mu}_x\right)\frac{d\overline{\pi}}{d\widehat{x}}(\widehat{x})\\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} g^{\infty,2}(\widehat{0},\frac{\partial\overline{\theta}}{\partial x_N}(\widehat{x},s))ds\\
&\leq \max\left\{g^{\infty,2}(\widehat{0},-1),\ g^{\infty,2}(\widehat{0},1)\right\}\int_{-\frac{1}{2}}^{\frac{1}{2}}\left|\frac{\partial\overline{\theta}}{\partial x_N}(\widehat{x},s)\right|^2 ds \quad (2.5.12)
\end{aligned}$$

Then, from (2.5.11) and (2.5.12) we have

$$c(g)\left|\frac{\partial\overline{\theta}}{\partial x_N}(\widehat{x},s)\right|^2 ds = \frac{\min\left\{g^{\infty,2}(\widehat{0},-1),\ g^{\infty,2}(\widehat{0},1)\right\}}{\max\left\{g^{\infty,2}(\widehat{0},-1),\ g^{\infty,2}(\widehat{0},1)\right\}}\int_{-\frac{1}{2}}^{\frac{1}{2}}\left|\frac{\partial\overline{\theta}}{\partial x_N}(\widehat{x},s)\right|^2 ds \leq \frac{d\overline{\pi}}{d\widehat{x}}(\widehat{x})$$

and

$$\frac{d\overline{\pi}}{d\widehat{x}}(\widehat{x}) \leq \frac{\max\left\{g^{\infty,2}(\widehat{0},-1),\ g^{\infty,2}(\widehat{0},1)\right\}}{\min\left\{g^{\infty,2}(\widehat{0},-1),\ g^{\infty,2}(\widehat{0},1)\right\}}\int_{-\frac{1}{2}}^{\frac{1}{2}}\left|\frac{\partial\overline{\theta}}{\partial x_N}(\widehat{x},s)\right|^2 ds = C(g)\left|\frac{\partial\overline{\theta}}{\partial x_N}(\widehat{x},s)\right|^2 ds$$

from which we deduce

$$c(g)\int_{-\frac{1}{2}}^{\frac{1}{2}}\left|\frac{\partial\overline{\theta}}{\partial x_N}(\widehat{x},s)\right|^2 ds \leq \frac{d\overline{\pi}}{d\widehat{x}}(\widehat{x}) \leq C(g)\int_{-\frac{1}{2}}^{\frac{1}{2}}\left|\frac{\partial\overline{\theta}}{\partial x_N}(\widehat{x},s)\right|^2 ds$$

We are going to prove (ii) and (iii). According to Theorem 2.5.2, for every $\varphi \in \mathcal{C}(\mathbb{S}^0)$ such that $\varphi^{**} > -\infty$,

$$\frac{d\pi}{d\mathcal{H}^{N-1}_{\lfloor S}}(x)\int_{\mathbb{S}^0}\varphi(\zeta)\ d\mu_x \geq \varphi^{**}([v](x)) \quad \text{for } \mathcal{H}^{N-1}\text{a.e. } x\in S \tag{2.5.13}$$

where $\pi = \frac{d\pi}{d\mathcal{H}^{N-1}_{\lfloor S}}\mathcal{H}^{N-1}_{\lfloor S} + \pi_s$ is the Radon-Nikodym decomposition of π with respect to the measure $\mathcal{H}^{N-1}_{\lfloor S}$. We assume $[u](\hat{x}) > 0$ and show that

$$\frac{c(g)\ |[u](\hat{x})|^2}{\displaystyle\int_{-\frac{1}{2}}^{\frac{1}{2}}\left|\frac{\partial\overline{\theta}}{\partial x_N}(\hat{x},s)\right|^2\ ds} \leq p(\hat{x}) \leq 1.$$

By letting

$$\varphi(\xi) = \begin{cases} \varphi(1)\,|\xi|^2 & \text{if} \quad \xi > 0\\ 0 & \text{if} \quad \xi \leq 0\end{cases} \tag{2.5.14}$$

Clearly, $\varphi^{**}([u](\widehat{x})) = \varphi([u](\widehat{x})) = \varphi(1)\ \left|[u](\widehat{x})\right|^2$. From (2.5.13), it follows that

$$\begin{aligned}
\varphi(1)\ \left|[u](\widehat{x})\right|^2 &= \varphi^{**}([u](\widehat{x})) \\
&\leq \frac{d\overline{\pi}}{d\widehat{x}}(\widehat{x}) \left(\int_{\mathbb{S}^0} \varphi(\xi) d\overline{\mu}_x \right) \\
&\leq C(g) \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \frac{\partial \overline{\theta}}{\partial x_N}(\widehat{x}, s) \right|^2 ds. \int_{\mathbb{S}^0} \varphi(\xi) d\overline{\mu}_x \\
&= C(g) p(\widehat{x}) \varphi(1) \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \frac{\partial \overline{\theta}}{\partial x_N}(\widehat{x}, s) \right|^2 ds.
\end{aligned}$$

Then, we obtain

$$\frac{\left|[u](\hat{x})\right|^2}{C(g)\ \displaystyle\int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \frac{\partial \overline{\theta}}{\partial x_N}(\hat{x}, s) \right|^2\ ds} = \frac{c(g)\ \left|[u](\hat{x})\right|^2}{\displaystyle\int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \frac{\partial \overline{\theta}}{\partial x_N}(\hat{x}, s) \right|^2\ ds} \leq p(\hat{x}) \leq 1.$$

The proof of (iii) is similar. □

Bibliography

[1]. H. Attouch. *Variational Convergence for Functions and Operators.* Applicable Mathematics Series. Pitman Advanced Publishing Program, 1985.

[2]. D. Caillerie. *The Effect of a Thin Inclusion of High Rigidity in an Elastic Body.* Math. Meth. in the Appl. Sci, **2** (1980), 251-270.

[3]. G. Croce, C. Lacour, G. Michaille. *A characterization of gradient Young-concentration measures generated by solutions of Dirichlet-type problems with large sources* ESAIM: COCV **15** (2009), 818-838.

[4]. H. Federer. *Geometric Measure Theory.* Classic in Mathematics, Springer-Verlag 1969.

[5]. I. Fonseca, S. Müller, P. Pedregal. *Analysis of concentration and oscillation effects generated by gradients.* Siam J. Math. Anal. **29** (1998), no 3, 736-756.

[6]. G. Geymonat, F. Krasucki, S. Lenci. *Mathematical analysis of a bounded joint with a soft thin adhesive.* Mathematics and Mechanics of Solids, **4** (1999), 201-225.

[7]. D. Kinderlehrer, P. Pedregal. *Characterization of Young measures generated by gradients.* Arch. Rational Mech. Anal. **119** (1991), 329-365.

[8]. C. Licht, G. Michaille. *A modelling of elastic adhesive bonded joints.* Advances in Mathematical Sciences and Applications, **7** (1997), no 2, 711-740.

[9]. C. Licht, G. Michaille, S. Pagano. *A model of elastic adhesive bonded joints through oscillation-concentration measures.* J. Math. Pures Appl. (9) 87 (2007), no. 4, 343–365.

Part II

A mathematical model for a pseudo-plastic welding joint

Introduction

Motivated by the mathematical modeling of a problem of welding, we revisit previous studies ([1], [8]) devoted to the asymptotic behavior of a structure made of two adherents connected by a thin and strong adhesive layer. In [8], the adherents and the adhesive were modeled as hyperelastic by bulk energy densities with the same growth exponent p laying in $(1, +\infty)$, the stiffness of the adhesive being of the order of the inverse of its thickness. Here, our first attempt to account for some fracture phenomena in soldered joint is to model the adhesive as pseudo-plastic, that is to say, its behavior is described by a bulk energy density with linear growth. Hence, from the mathematical point of view, two difficulties appear: the growths of the the bulk energy in the adhesive and the adherents are different and the linear growth in the adhesive will imply to work in spaces of displacement fields with free discontinuities.

Chapter 3 is organized as follows. In Section 3.1, we describe a model problem with a simplified geometry directly connected to the study [8] where we assume that the bulk energy density of the adherents is quasiconvex and that of the adhesive is convex. In Section 3.2, a variational convergence result, when the thickness of the adhesive layer goes to zero, justifies our proposal of simplified but accurate enough model. The adhesive layer is replaced by a material pseudo-plastic surface. The case when f is not quasiconvex and g is not convex is considered in Chapter 4. In Section 3.3 of Chapter 3, we use the previous results to model a more realistic situation of welding. Eventually, in the spirit of ([7], [17]), we consider a variational regularization of the limit functional involved by our model in Chapter 5.

Chapter 3

First model(quasiconvexity and convexity of the density functions)

3.1 Description of the model

We make no difference between $\mathbb{R}^3$ and the three dimensional euclidean physical space whose orthogonal basis is denoted by (e_1, e_2, e_3), Greek coordinate indexes will run in $\{1, 2\}$ and Latin ones in $\{1, 2, 3\}$. For all $\zeta = (\zeta_1, \zeta_2, \zeta_3)$ of $\mathbb{R}^3$, $\hat{\zeta}$ stands for (ζ_1, ζ_2). Let S be a domain of $\mathbb{R}^2$ with a Lipschitz-continuous boundary ∂S and r a positive number. The cylindrical domain $\Omega := S \times (-r, r)$ is the reference configuration of a structure composed of two adherents and an adhesive which, respectively occupies $\Omega_\varepsilon^\pm := \{x \in \Omega : \pm x_3 > \frac{\varepsilon}{2}\}$ and $B_\varepsilon := \{x \in \Omega : |x_3| < \frac{\varepsilon}{2}\}$. We set $\Omega_\varepsilon := \Omega_\varepsilon^+ \cup \Omega_\varepsilon^-$. The structure is clamped on a part Γ_0 of the boundary Γ of Ω with a positive $\mathcal{H}^2$-measure and we assume that there exists $\varepsilon_0 > 0$ such that $dist(\bar{\Gamma}_0, \bar{B}_{\varepsilon_0}) > 0$. The structure is subjected to body forces of density Φ and to surface forces of density φ on the complementary part Γ_φ of Γ_0. We assume that the supports of φ and Φ lay outside of $\bar{B}_{\varepsilon_0}$. Obviously, one can consider other types of boundary conditions (e.g. a combination of some components of the stress vector and of the displacement). At last, adhesive and adherents are assumed to be perfectly stuck together. In Section 3.3 we will consider a more realistic structure (see figure 3.1 for the two geometrical structures).

The stiffness of the material occupying the thin layer B_ε is assumed to be of order $\dfrac{1}{\varepsilon}$ so that the strain in B_ε is expected to be small and we will use the framework of small perturbations to model the behavior of the adhesive. To account for possible fracture phenomena inside B_ε, we consider the adhesive as pseudo-plastic. Hence the behavior of the adhesive is described by a bulk energy density like $\dfrac{1}{\varepsilon}\, g(e(u))$ where g is a convex function with linear growth of the linearized strain $e(u)$ i.e., the symmetric part $(\nabla u)_s$ of the gradient displacement ∇u. By contrast, the deformations in the adherents may be large and they are modeled as hyperelastic with a continuous quasiconvex bulk energy density f, which is a function of the gradient displacement. More precisely, we assume that there exists $p > 1$, and two positive constants α, β such that

$$\begin{aligned}
&f,\ g : \mathbf{M}^{3\times 3} \to \mathbb{R};\\
&\alpha|\xi|^p \le f(\xi) \le \beta(1+|\xi|^p) \quad \text{for all } \xi \in \mathbf{M}^{3\times 3}; \qquad (3.1.1)\\
&\alpha|\xi| \le g(\xi) \le \beta(1+|\xi|) \qquad \text{for all } \xi \in \mathbf{M}^{3\times 3}_s. \qquad (3.1.2)
\end{aligned}$$

Here and in the sequel $\mathbf{M}^{n\times n}$ and $\mathbf{M}^{n\times n}_s$ stand for the set of $n\times n$ matrices and $n\times n$ symmetric matrices with real entries, respectively. It is well known that f and g satisfy the following locally Lipschitz conditions: there exists a positive constant L such that

$$|f(\xi)-f(\xi')| \le L|\xi-\xi'|\left(1+|\xi|^{p-1}+|\xi'|^{p-1}\right) \quad \text{for all } \xi,\ \xi' \in \mathbf{M}^{3\times 3}; \quad (3.1.3)$$

$$|g(\xi)-g(\xi')| \le L|\xi-\xi'| \quad \text{for all } \xi,\ \xi' \in \mathbf{M}^{3\times 3}_s. \qquad (3.1.4)$$

Thus, if $F_\varepsilon(u) := \int_{\Omega_\varepsilon} f(\nabla u)\, dx + \frac{1}{\varepsilon}\int_{B_\varepsilon} g(e(u)\, dx$ and $L(u) := \int_\Omega \Phi.u\, dx + \int_{\Gamma_\varphi} \varphi.u\, d\mathcal{H}^2$, respectively denote the total stored energy and the work of the external loading, determining the equilibrium configurations leads to the problem

$$\inf\left\{\int_{\Omega_\varepsilon} f(\nabla u)\, dx + \frac{1}{\varepsilon}\int_{B_\varepsilon} g(e(u))\, dx - L(u) : u \in A_\varepsilon\right\}$$

with

$$\begin{aligned}
A_\varepsilon &:= \left\{u \in LD(\Omega,\mathbb{R}^3) : u_{\lfloor\Omega_\varepsilon} \in W^{1,p}_{\Gamma_0}(\Omega_\varepsilon,\mathbb{R}^3)\right\};\\
W^{1,p}_{\Gamma_0}(\Omega_\varepsilon,\mathbb{R}^3) &:= \left\{u \in W^{1,p}(\Omega_\varepsilon,\mathbb{R}^3) : u = 0 \ \text{ on } \Gamma_0\right\};\\
LD(\Omega,\mathbb{R}^3) &:= \left\{u \in L^1(\Omega,\mathbb{R}^3) : e(u) \in L^1(\Omega,\mathbf{M}^{3\times 3}_s)\right\}.
\end{aligned}$$

We aim to propose a simplified but accurate model where qualitative and quantitative analysis are able to be done in an easier way than with the starting problem. For this, we will consider ε as a parameter going to zero and determine the asymptotic behavior of (approximate) solutions of the previous minimization problem by identifying the Γ-limit of F_ε extended into the fixed space $L^1(\Omega,\mathbb{R}^3)$. More precisely, we still denote by F_ε its extension outside A_ε given by: $F_\varepsilon : L^1(\Omega,\mathbb{R}^3) \to \mathbb{R}\cup\{+\infty\}$,

$$F_\varepsilon(u) := \begin{cases} \displaystyle\int_{\Omega_\varepsilon} f(\nabla u)\, dx + \frac{1}{\varepsilon}\int_{B_\varepsilon} g(e(u))\, dx & \text{if } u \in A_\varepsilon \\ +\infty & \text{otherwise} \end{cases}$$

Clearly, the previous minimization problem is equivalent to the following

$$(\mathcal{P}_\varepsilon)\ : \qquad \inf\left\{F_\varepsilon(u) - L(u) : u \in L^1(\Omega,\mathbb{R}^3)\right\}.$$

We will use the classical spaces

$$W^{1,p}_{\Gamma_0}(\Omega,\mathbb{R}^3) := \{\ u \in W^{1,p}(\Omega,\mathbb{R}^3) : u = 0 \quad \text{on } \Gamma_0\ \}\,;$$

$$BD(\Omega,\mathbb{R}^3) := \{\ v \in L^1(\Omega,\mathbb{R}^3) : e(v) \in \mathcal{M}(\Omega,\mathbf{M}^{3\times 3}_s)\ \}\,;$$

$$BD(S,\mathbb{R}^2) := \{\ v \in L^1(S,\mathbb{R}^2) : e(v) \in \mathcal{M}(S,\mathbf{M}^{2\times 2}_s)\ \}$$

and the set of "horizontal rigid motions" on S, i.e.

$$\begin{aligned} \widehat{\mathcal{R}_H} &:= \{\ v \in BD(S,\mathbb{R}^2) : e_{\alpha\beta}(v) = 0\ \} \\ &= \{\ v : v(x) = (a_1 - bx_2, a_2 + bx_1),\ (a_1,a_2) \in \mathbb{R}^2,\ b \in \mathbb{R}\ \}\,. \end{aligned}$$

For reasons clarified in Lemma 3.1.2 below, we define the limit admissible set A_0 by

$$A_0 := \left\{\ u \in W^{1,p}_{\Gamma_0}(\Omega,\mathbb{R}^3) : \gamma_S(\hat{u}) \in BD(S,\mathbb{R}^2)\ \right\}$$

and its subspace A_0^1 made of smooth elements

$$A_0^1 := \left\{ u \in\ A_0 : \gamma_S(u) \in \mathcal{C}^1(\bar{S},\mathbb{R}^3) \right\}.$$

For simplicity of notation, γ_S will denote indifferently the trace operator from $W^{1,p}(\Omega,\mathbb{R}^i)$ into $W^{1-\frac{1}{p},p}(S,\mathbb{R}^i)$ for $i \in \{1,2,3\}$.

For all $\xi \in \mathbf{M}^{3\times 3}_s$, define $\hat{\xi} \in \mathbf{M}^{2\times 2}_s$ by $(\hat{\xi})_{\alpha\beta} = \xi_{\alpha\beta}$ and consider the function $g_0 : \mathbf{M}^{2\times 2}_s \to \mathbb{R}$ defined by

$$g_0(\zeta) := \min\left\{\ g(\xi) : \xi \in \mathbf{M}^{3\times 3}_s,\, \hat{\xi} = \zeta\ \right\}.$$

It is easily seen that g_0 is a convex function on $\mathbf{M}^{2\times 2}_s$ and it will be sometimes convenient to express g_0 as stated in the next lemma:

Lemma 3.1.1. *For all* $3 \times 2-$*matrix* ξ, $\inf\limits_{\lambda\in\mathbb{R}^3} g((\xi|\lambda)_s) = g_0((\xi_{\alpha\beta})_s)$.

Proof. The conclusion is a straightforward consequence of the calculation

$$\begin{aligned} \min_{\lambda\in\mathbb{R}^3} g((\xi|\lambda)_s) &= \min_{\lambda\in\mathbb{R}^3} g\left(\begin{pmatrix} \xi_{11} & \frac{\xi_{12}+\xi_{21}}{2} & \frac{\lambda_1+\xi_{31}}{2} \\ \frac{\xi_{12}+\xi_{21}}{2} & \xi_{22} & \frac{\lambda_2+\xi_{32}}{2} \\ \frac{\lambda_1+\xi_{31}}{2} & \frac{\lambda_2+\xi_{32}}{2} & \lambda_3 \end{pmatrix}\right) \\ &= \min_{\lambda\in\mathbb{R}^3} g\left(\begin{pmatrix} \xi_{11} & \frac{\xi_{12}+\xi_{21}}{2} & \lambda_1 \\ \frac{\xi_{12}+\xi_{21}}{2} & \xi_{22} & \lambda_2 \\ \lambda_1 & \lambda_2 & \lambda_3 \end{pmatrix}\right) \end{aligned}$$

and the definition of g_0. □

In Section 3.2, we establish the Γ-convergence of the sequence $(F_\varepsilon)_{\varepsilon>0}$ to the functional $F_0 : L^1(\Omega,\mathbb{R}^3) \to \mathbb{R}\cup\{\ +\infty\ \}$ defined by:

$$F_0(u) := \begin{cases} \displaystyle\int_\Omega f(\nabla u)\ dx + \int_S g_0(e(\gamma_S(\hat{u})) & \text{if } u \in A_0 \\ +\infty \quad \text{otherwise} \end{cases}$$

when $L^1(\Omega, \mathbb{R}^3)$ is equipped with its strong topology. For the definition of the scalar measure $g_0(e(v))$, $v \in BD(S, \mathbb{R}^2)$, we refer the reader to ([12], [17], [6]). We recall that the integral over S of the measure $g_0(e(v))$ is given by

$$\int_S g_0(e(v)) = \int_S g_0(e_a(v))\, d\hat{x} + \int_S g_0^\infty\big(\frac{e_s(v)}{|e_s(v)|}\big)|e_s(v)|,$$

where $e(v) = e_a(v)\, d\hat{x} + e_s(v)$ is the Lebesgue decomposition of $e(v)$, $|e_s(v)|$ denotes the total variation of the singular measure $e_s(v)$, $\dfrac{e_s(v)}{|e_s(v)|}$ its Radon-Nikodym derivative, and $\xi \mapsto g_0^\infty(\xi) := \lim_{t\to+\infty} g_0(t\xi)/t$ is the recession function of g_0. In [3], Ambrosio, Coscia and Dal Maso proved that the singular measure $e_s(v)$ has the following structure: there exists a rectifiable set $S_v \subset S$ with normal ν_v and traces $v^\pm$ on both sides of S_v such that

$$e_s(v) = \frac{1}{2}\Big((v^+ - v^-) \otimes \nu_v + \nu_v \otimes (v^+ - v^-)\Big)\mathcal{H}^1 \lfloor S_v + Cv,$$

with Cv singular with respect to the Lebesgue measure and vanishing on Borel sets of σ-finite $\mathcal{H}^1$-measure. We will also denote by $[v] \otimes \nu_v$ the symmetrical tensor product $\frac{1}{2}\left((v^+ - v^-) \otimes \nu_v + \nu_v \otimes (v^+ - v^-)\right)$. The set S_v will represent the macroscopic cracks whereas the support of C_v deals with the diffuse defects or fractal cracks in S towards the layer shrinks.

We start by establishing a compactness result which justifies the introduction of the limit set A_0 of admissible functions. As usual, the arrows $\to$ and $\rightharpoonup$ will denote strong and weak convergences respectively.

Lemma 3.1.2 (Compactness lemma). *Let $(u_\varepsilon)_{\varepsilon>0}$ be a sequence in $L^1(\Omega, \mathbb{R}^3)$ such that $\sup_{\varepsilon>0} F_\varepsilon(u_\varepsilon) < +\infty$. Then, there exists $u \in L^1(\Omega, \mathbb{R}^3)$ and a subsequence not relabelled such that*

i) $u_\varepsilon \rightharpoonup u$ in $BD(\Omega, \mathbb{R}^3)$ and $u \in W^{1,p}_{\Gamma_0}(\Omega, \mathbb{R}^3)$;

ii) $u_\varepsilon \rightharpoonup u$ in $W^{1,p}_{\Gamma_0}(\Omega_\eta, \mathbb{R}^3)$ for every $\eta > 0$;

iii) $\gamma_S(\hat{u}) \in BD(S, \mathbb{R}^2)$;

iv) $\exists r_\varepsilon \in \widehat{\mathcal{R}_H}$ such that $\dfrac{1}{\varepsilon}\displaystyle\int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} \hat{u}_\varepsilon\, dx_3 + r_\varepsilon \rightharpoonup \gamma_S(\hat{u})$ in $BD(S, \mathbb{R}^2)$.

Proof. From now on, we do not relabel the various subsequences obtained in the proof, and C will denote a positive constant which may vary from line to line. We divide the proof into two steps.

Step 1. We establish (i) and (ii). According to the coerciveness condition (3.1.1) and because $u_\varepsilon = 0$ on Γ_0, $(u_\varepsilon)_{\varepsilon>0}$ is clearly bounded in $LD(\Omega, \mathbb{R}^3)$ so that there exist $u \in BD(\Omega, \mathbb{R}^3)$ and a subsequence satisfying $u_\varepsilon \rightharpoonup u$ in $BD(\Omega, \mathbb{R}^3)$ and $u_\varepsilon \to u$ in

$L^1(\Omega,\mathbb{R}^3)$. The weak convergence of u_ε to u in $W^{1,p}_{\Gamma_0}(\Omega_\eta,\mathbb{R}^3)$ for every $\eta>0$ is obvious. We are going to prove that $u\in W^{1,p}_{\Gamma_0}(\Omega,\mathbb{R}^3)$.

We extend every function $w\in W^{1,p}_{\Gamma_0}(\Omega,\mathbb{R}^3)$ by reflection on $S\times(\pm r,\pm 2r\mp\dfrac{\varepsilon}{2})$ so that the extended function, denoted by $\tilde{w}$, belongs to $W^{1,p}(S\times(\pm\dfrac{\varepsilon}{2},\pm 2r\mp\dfrac{\varepsilon}{2}),\mathbb{R}^3))$ and satisfies

$$\int_{S\times(\pm\frac{\varepsilon}{2},\,\pm 2r\mp\frac{\varepsilon}{2})}|\nabla\tilde{w}|^p\;dx\leq 2\int_{\Omega_\varepsilon^\pm}|\nabla w|^p\;dx.$$

Set $\Omega^\pm:=\Omega\cap[\pm x_3>0]$ and, for every function $w\in W^{1,p}_{\Gamma_0}(\Omega_\varepsilon,\mathbb{R}^3)$, define its ε-translate $T_\varepsilon w$ in $W^{1,p}(\Omega\setminus S,\mathbb{R}^3)$ by

$$T_\varepsilon w(\hat{x},x_3)=\begin{cases}\tilde{w}(\hat{x},x_3+\dfrac{\varepsilon}{2}), & \text{if } x\in\Omega^+\\[2ex] \tilde{w}(\hat{x},x_3-\dfrac{\varepsilon}{2}), & \text{if } x\in\Omega^-.\end{cases}$$

Because

$$\begin{aligned}\sup_{\varepsilon>0}\int_{\Omega\setminus S}|\nabla T_\varepsilon u_\varepsilon|^p\;dx &\leq \sup_{\varepsilon>0}\int_{S\times(\frac{\varepsilon}{2},2r-\frac{\varepsilon}{2})\cup S\times(-2r+\frac{\varepsilon}{2},-\frac{\varepsilon}{2})}|\nabla\tilde{u}_\varepsilon|^p\;dx\\ &\leq 2\sup_{\varepsilon>0}\int_{\Omega_\varepsilon}|\nabla u_\varepsilon|^p\;dx<+\infty,\end{aligned}$$

Poincaré's inequality implies that there exist $z\in W^{1,p}(\Omega\setminus S,\mathbb{R}^3)$ and a subsequence of $(u_\varepsilon)_{\varepsilon>0}$ such that $T_\varepsilon u_\varepsilon\rightharpoonup z$ in $W^{1,p}(\Omega\setminus S,\mathbb{R}^3)$ and $u_\varepsilon\to z$ in $L^p(\Omega,\mathbb{R}^3)$. Actually $u=z$ (so that $u\in W^{1,p}(\Omega\setminus S,\mathbb{R}^3)$) since for all $\psi\in\mathcal{D}(\Omega\setminus S,\mathbb{R}^3)$,

$$\begin{aligned}\int_{\Omega\setminus S}u.\psi\;dx &= \lim_{\varepsilon\to 0}\int_{\Omega\setminus S}u_\varepsilon.\psi(\hat{x},x_3-\frac{\varepsilon}{2})\;dx\\ &= \lim_{\varepsilon\to 0}\int_{\Omega\setminus S}T_\varepsilon u_\varepsilon.\psi\;dx\\ &= \int_{\Omega\setminus S}z.\psi\;dx.\end{aligned}$$

For all $w\in W^{1,p}(\Omega\setminus S,\mathbb{R}^3))$, we will denote the traces on S of w considered as a function of $W^{1,p}(\Omega^\pm,\mathbb{R}^3)$ by $w^\pm$ and its jump across S by $[w]:=w^+-w^-$. Take $\theta\in\mathcal{C}^\infty_c(S)$, Green's formula yields

$$2\int_{B_\varepsilon}e_{\alpha 3}(u_\varepsilon)\;\theta\;dx=\int_S\theta[T_\varepsilon u^\varepsilon_\alpha]\;d\hat{x}+I_\varepsilon \tag{3.1.5}$$

where

$$I_\varepsilon:=-\int_{B_\varepsilon}\frac{\partial u_3^\varepsilon}{\partial x_\alpha}\theta\;dx.$$

The left-hand side term of (3.1.5) tends to 0 by coercivity condition (3.1.2), and from

$$\begin{aligned} u_3^\varepsilon(\hat{x}, \pm|x_3|) &= u_3^\varepsilon(\hat{x}, \pm\frac{\varepsilon}{2}) + \int_{\pm\frac{\varepsilon}{2}}^{\pm|x_3|} \frac{\partial u_3^\varepsilon}{\partial x_3}(\hat{x}, t)\ dt \\ &= (T_\varepsilon u_3^\varepsilon)^\pm(\hat{x}) + \int_{\pm\frac{\varepsilon}{2}}^{\pm|x_3|} \frac{\partial u_3^\varepsilon}{\partial x_3}(\hat{x}, t)\ dt, \end{aligned}$$

we deduce

$$\begin{aligned} \int_{B_\varepsilon} |u_3^\varepsilon(\hat{x}, x_3)|\ dx &\leq \varepsilon \int_S \big(|(T_\varepsilon u_3^\varepsilon)^+(\hat{x})| + |(T_\varepsilon u_3^\varepsilon)^-(\hat{x})|\big)\ d\hat{x} + \varepsilon \int_{B_\varepsilon} |\frac{\partial u_3^\varepsilon}{\partial x_3}|\ dx \\ &\leq C\varepsilon(1+\varepsilon) \end{aligned} \tag{3.1.6}$$

so that

$$|I_\varepsilon| = \left| \int_{B_\varepsilon} u_3^\varepsilon \frac{\partial \theta}{\partial x_\alpha}\ dx \right| \leq C\varepsilon(1+\varepsilon),$$

and the right-hand side term of (3.1.5) tends to $\int_S \theta[u_\alpha]\ d\hat{x}$. Going to the limit on ε in (3.1.5) yields $[u_\alpha] = 0$ a.e. on S. Similarly, by letting $\varepsilon \to 0$ in

$$\int_{B_\varepsilon} \frac{\partial u_3^\varepsilon}{\partial x_3} \theta\ dx = \int_S \theta[T_\varepsilon u_3^\varepsilon]\ d\hat{x},$$

and by using coercivity condition (3.1.2), we obtain $[u_3] = 0$ a.e. on S.

Step 2. We establish (iii) and (iv). Coercivity condition (3.1.2) implies

$$\sup_{\varepsilon>0} \frac{1}{\varepsilon} \int_{B_e} |\widehat{e(u_\varepsilon)}| dx < +\infty$$

so that there exists a subsequence of $v_\varepsilon := \frac{1}{\varepsilon}\int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} \hat{u}_\varepsilon\ dx_3$, $r_\varepsilon \in \widehat{\mathcal{R}_H}$ and $v \in BD(S, \mathbb{R}^2)$ such that $v_\varepsilon + r_\varepsilon \rightharpoonup v$ in $BD(S, \mathbb{R}^2)$. It remains to prove that $v = \gamma_S(\hat{u})$. For arbitrary $\theta \in \mathcal{C}_c^\infty(S)$ and $x_3 \in (-\frac{\varepsilon}{2}, \frac{\varepsilon}{2})$, we have

$$u_\alpha^\varepsilon(\hat{x}, \pm|x_3|)\theta(\hat{x}) = (T_\varepsilon u_\alpha^\varepsilon)^\pm(\hat{x})\ \theta(\hat{x}) + \int_{\pm\frac{\varepsilon}{2}}^{\pm|x_3|} \frac{\partial u_\alpha^\varepsilon}{\partial x_3}(\hat{x}, t)\ \theta(\hat{x})\ dt$$

so that

$$\frac{1}{\varepsilon}\int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} \int_S u_\alpha^\varepsilon(\hat{x}, x_3)\ \theta(\hat{x})\ dx = \frac{1}{2}\int_S (T_\varepsilon u_\alpha^\varepsilon)^+(\hat{x}) + (T_\varepsilon u_\alpha^\varepsilon)^-(\hat{x})\ \theta(\hat{x})\ d\hat{x} + J_\varepsilon \tag{3.1.7}$$

where

$$J_\varepsilon := \frac{1}{\varepsilon}\int_{-\frac{\varepsilon}{2}}^{0} \int_S \int_{-\frac{\varepsilon}{2}}^{-|x_3|} \frac{\partial u_\alpha^\varepsilon}{\partial x_3}(\hat{x}, t)\ \theta(\hat{x})\ dt\ dx + \frac{1}{\varepsilon}\int_0^{\frac{\varepsilon}{2}} \int_S \int_{\frac{\varepsilon}{2}}^{|x_3|} \frac{\partial u_\alpha^\varepsilon}{\partial x_3}(\hat{x}, t)\ \theta(\hat{x})\ dt\ dx.$$

We claim that $\lim_{\varepsilon\to 0} J_\varepsilon = 0$. Indeed

$$\begin{aligned}\int_S\int_{\frac{\varepsilon}{2}}^{x_3} \frac{\partial u_\alpha^\varepsilon}{\partial x_3}\,\theta(\hat{x})\; dtd\hat{x} &= \int_S\int_{\frac{\varepsilon}{2}}^{x_3} \big(2e_{3\alpha}(u_\varepsilon) - \frac{\partial u_3^\varepsilon}{\partial x_\alpha}\big)\;\theta\; dtd\hat{x}\\ &= \int_S\int_{\frac{\varepsilon}{2}}^{x_3} \big(2e_{3\alpha}(u_\varepsilon)\;\theta + u_3^\varepsilon\,\frac{\partial \theta}{\partial x_\alpha}\big)\; dtd\hat{x}.\end{aligned}$$

Thus

$$|J_\varepsilon| \leq C\int_{B_\varepsilon} \big(|e_{3\alpha}(u_\varepsilon)| + |u_3^\varepsilon)|\big)dx,$$

and the claim follows from (3.1.6) and coercivity condition (3.1.2). Letting $\varepsilon \to 0$ in (3.1.7), we obtain

$$\int_S v_\alpha\;\theta\; d\hat{x} = \int_S \gamma_S(u_\alpha)\;\theta\; d\hat{x}.$$

Thus $v_\alpha = \gamma_S(u_\alpha)$ a.e. in S since θ is arbitrary.

□

3.2 The main convergence results

The main result of this section is the following theorem.

Theorem 3.2.1. *The sequence $(F_\varepsilon)_{\varepsilon>0}$ Γ-converges to the functional F_0.*

The proof consists in establishing Proposition 3.2.2 and Proposition 3.2.4 below, corresponding to the lower bound and the upper bound in the definition of the Γ_{τ_s}-convergence.

3.2.1 The lower bound for the $\Gamma_{\tau_{L^1}}-$convergence of $(F_\varepsilon)_{\varepsilon>0}$

Our purpose in this section is to prove the lower bound for $\Gamma_{\tau_{L^1}} -$ *convergence* of $(F_\varepsilon)_{\varepsilon>0}$, converging to F_0. That is for any sequence $(u_\varepsilon)_{\varepsilon>0}$ converging to u in $L^1(\Omega, R^3)$,it must be imply $F_0(u) \leq \liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon)$. Thus, the Proposition 3.2.2 is the goal in this subsection.

Proposition 3.2.2 (Lower bound). *For all u and all sequence $(u_\varepsilon)_{\varepsilon>0}$ in $L^1(\Omega, \mathbb{R}^3)$ such that $u_\varepsilon \xrightarrow{\tau_s} u$, the following inequality holds:*

$$F_0(u) \leq \liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon). \tag{3.2.1}$$

Proof. Clearly, we may assume that $\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) < +\infty$ so that from Lemma 3.1.2, u belongs to A_0, and $e\Big(\frac{1}{\varepsilon}\int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} \hat{u}_\varepsilon\; dx_3\Big) = \frac{1}{\varepsilon}\int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} \widehat{e(u_\varepsilon)}\; dx_3 \rightharpoonup e(\gamma_S(\hat{u}))$ in $\mathcal{M}(S, \mathbf{M}_s^{2\times 2})$. Thus from Jensen's inequality

$$\begin{aligned}\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) &= \liminf_{\varepsilon\to 0}\Big(\int_{\Omega_\varepsilon} f(\nabla u_\varepsilon)dx + \frac{1}{\varepsilon}\int_{B_\varepsilon} g(e(u_\varepsilon))dx\Big)\\ &\geq \liminf_{\varepsilon\to 0}\int_{\Omega_\varepsilon} f(\nabla u_\varepsilon)dx + \liminf_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{B_\varepsilon} g(e(u_\varepsilon))dx\\ &\geq \liminf_{\varepsilon\to 0}\int_{\Omega_\varepsilon} f(\nabla u_\varepsilon)dx + \liminf_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{B_\varepsilon} g_0(\widehat{e(u_\varepsilon)})dx\\ &\geq \int_\Omega f(\nabla u)dx + \int_S g_0(e(\gamma_S(\hat{u}))).\end{aligned}$$

In the last inequality, we used the weak lower semicontinuity of the second integral functional with respect to the weak convergence of measures. For the convergence of the first integral, we proceeded as follows: take $\eta > \varepsilon$, write $\int_{\Omega_\varepsilon} f(\nabla u_\varepsilon)dx \geq \int_{\Omega_\eta} f(\nabla u_\varepsilon)dx$, and apply the lower semicontinuity of the integral functional $u \mapsto \int_{\Omega_\eta} f(\nabla u)dx$ for the weak convergence in $W^{1,p}(\Omega_\eta, \mathbb{R}^3)$). Then let $\eta \to 0$.

□

3.2.2 The upper bound for the $\Gamma_{\tau_{L^1}}$−convergence of $(F_\varepsilon)_{\varepsilon>0}$

For proving the upper bound, we need to establish the following relaxation result.

Lemma 3.2.3 (Relaxation). *The functional F_0 is the l.s.c. regularization for the strong topology of the functional defined on $L^1(\Omega, \mathbb{R}^3)$ by*

$$\widetilde{F}_0(u) := \begin{cases} \displaystyle\int_\Omega f(\nabla u)\ dx + \int_S g_0(e(\gamma_S(\hat{u})))\ d\hat{x} & \text{if } u \in A_0^1,\\ +\infty & \text{otherwise.}\end{cases}$$

Proof. *Step 1.* By using standard lower semicontinuous arguments, it is easily seen that for every sequence $(u_n)_{n\in\mathbb{N}}$ strongly converging to u in $L^1(\Omega, \mathbb{R}^3)$, we have

$$F_0(u) \leq \liminf_{n\to+\infty} \widetilde{F}_0(u_n).$$

Step 2. We assume $u \in A_0$ and we construct a sequence $(u_\delta)_{\delta>0}$ in A_0^1 such that

$$\begin{cases} u_\delta \to u \text{ strongly in } W^{1,p}(\Omega, \mathbb{R}^3);\\ \gamma_S(\hat{u}_\delta) = v_\delta \to v := \gamma_S(\hat{u}) \text{ strongly in } L^1(S, \mathbb{R}^2);\\ \displaystyle\lim_{\delta\to 0}\int_\Omega f(\nabla u_\delta)\ dx = \int_\Omega f(\nabla u)\ dx;\\ \displaystyle\lim_{\delta\to 0}\int_S g_0(e(v_\delta))\ d\hat{x} = \int_S g_0(e(v)).\end{cases}$$

Consider the open cylinder $\tilde{\Omega} := \tilde{S} \times (-r, r)$ containing Ω, where $\tilde{S}$ is an open set of $\mathbb{R}^2$ strictly containing S and extend u into a function $\tilde{u}$ in $W^{1,p}(\tilde{\Omega}, \mathbb{R}^3)$. We also extend v

into a function $\tilde{v}$ in $BV(\tilde{S}, \mathbb{R}^3)$. More precisely, let P_3 denote the extension operators

$$P_3 : W^{1,p}(\Omega, \mathbb{R}^3) \to W^{1,p}(\tilde{\Omega}, \mathbb{R}^3),$$

and $\gamma_{\tilde{S}}$ the trace operator associated with the Sobolev space $W^{1,p}(\tilde{\Omega}, \mathbb{R}^3)$. We define $\tilde{v}$ in $BV(\tilde{S}, \mathbb{R}^3)$ by $\tilde{v} := \gamma_{\tilde{S}} \circ P_3(\hat{u})$, i.e., $\tilde{v} = \gamma_{\tilde{S}}(\tilde{\hat{u}})$.

For all $\delta > 0$, consider the open ball $B_{\eta(\delta)}(0)$ of $\mathbb{R}^2$ centered at 0 with a radius $\eta(\delta) > 0$ small enough so that $S + B_{\eta(\delta)}(0) \subset \tilde{S}$. Let ϕ in $\mathcal{C}_c^\infty(\tilde{S})$ with support $\mathrm{spt}(\phi)$, $\phi = 1$ on a neighborhood $S + B_{\eta(\delta)}(0)$ included in $\tilde{S}$ and $\rho_{\eta(\delta)}$ a standard mollifier with support $\bar{B}_{\eta(\delta)}(0)$. Set $v_\delta := \rho_{\eta(\delta)} * (\phi\tilde{v})$, $u_\delta := \rho_{\eta(\delta)} * (\phi\tilde{u})$ and choose $\eta(\delta)$ small enough so that:

$$\|v_\delta - v\|_{L^1(S,\mathbb{R}^2)} < \delta; \tag{3.2.2}$$

$$\Big| \int_{\mathbb{R}^2} g_0(\rho_{\eta(\delta)} * (\phi e(\tilde{v}))) - \int_{\mathbb{R}^2} g_0(\phi e(\tilde{v})) \Big| < \delta; \tag{3.2.3}$$

$$\|u_\delta - u\|_{W^{1,p}(\Omega,\mathbb{R}^3)} < \delta. \tag{3.2.4}$$

Estimates (3.2.2) and (3.2.4) are standard (note that the mollification of $\tilde{u}$ takes place only on the $\hat{x}$ argument). Estimate (3.2.3) is a straightforward consequence of the narrow convergence of the measure $g_0(\rho_{\eta(\delta)} * (\phi e(\tilde{v})))$ to the measure $g_0(\phi e(\tilde{v}))$ in $\mathbf{M}^+(\mathbb{R}^2)$ (see for instance Lemma 5.2 and Remark 5.1 in [17], or, if g_0 is positively homogeneous of degree 1, use Reshetnyak's continuity theorem, Theorem 2.39 in [4]).

Clearly $v_\delta \in \mathcal{C}^\infty(\bar{S}, \mathbb{R}^2)$ and, from (3.2.2), $v_\delta \to v$ in $L^1(S, \mathbb{R}^2)$. Moreover, from (3.2.3), and noticing that $\phi = 1$ on $S + B_{\eta(\delta)}(0)$,

$$\left| \int_S g_0(e(v_\delta)) - \int_S g_0(e(v)) \right| \le \delta + g_0(\ \rho_{\eta(\delta)} * (\phi e(\tilde{v})\ )\ (\ spt(\phi) \setminus S\ ) + g_0(\ \phi e(\tilde{v})\ )\ (\ spt(\phi) \setminus S\ ),$$

so that letting $\delta \to 0$ and letting $\mathrm{spt}(\phi) \setminus S \to \partial S$

$$\lim_{\delta \to 0} \int_S g_0(e(v_\delta)) = \int_S g_0(e(v)).$$

From (3.2.4), $u_\delta \to u$ in $W^{1,p}(\Omega, \mathbb{R}^3)$ and, from (3.1.3),

$$\lim_{\delta \to 0} \int_\Omega f(\nabla u_\delta)\ dx = \int_\Omega f(\nabla u)\ dx.$$

According to the definition of v_δ, u_δ, and from the fact that $\tilde{v} = \gamma_{\tilde{S}}(\tilde{\hat{u}})$, we have $\gamma_S(\hat{u}_\delta) = v_\delta$.

The sequence $(u_\delta)_{\delta > 0}$ fulfills all the conditions of the set A_0^1 except the boundary condition. By using De Giorgi's slicing method in a neighborhood of Γ_0 (see for instance Theorem 11.2.1 in [6]), we can modify u_δ into a new function $\tilde{u}_\delta \in A_0^1$ which has the same trace as its weak limit u on $\partial\Omega$, and satisfies

$$\limsup_{\delta \to 0} \int_\Omega f(\nabla \tilde{u}_\delta)\ dx \le \int_\Omega f(\nabla u)\ dx.$$

Thus, finally

$$\lim_{\delta\to 0}\int_\Omega f(\nabla \tilde u_\delta)\ dx=\int_\Omega f(\nabla u)\ dx.$$

Note that u_δ is not affected on a neighborhood of S by this modification because $\mathrm{dist}(\overline{\Gamma}_0,\overline{\partial B_\varepsilon\cap\Gamma})>0$. Thus $\tilde u_\delta$ that we denote now by u_δ is the expected sequence. □

Proposition 3.2.4 (Upper bound). *The following inequality holds in* $L^1(\Omega,\mathbb{R}^3)$

$$(\Gamma-\limsup F_\varepsilon)\le F_0. \tag{3.2.5}$$

Proof. Step 1. We establish $\Gamma-\limsup F_\varepsilon\le\tilde F_0$.

Take $u\in A_0^1$. In what follows we set $v:=\gamma_S(\hat u)$. For every fixed ξ in $\mathcal{D}(S,\mathbb{R}^3)$, consider the function v_ε in $W^{1,p}(B,\mathbb{R}^3)$, $B=S\times(-\frac12,\frac12)$ defined by

$$\begin{cases} v_\varepsilon^\alpha(\hat x,x_3) &:= & v^\alpha(\hat x)+\varepsilon x_3\xi^\alpha(\hat x)\\ v_\varepsilon^3(\hat x,x_3) &:= & \varepsilon u^3(\hat x,0)+\varepsilon^2x_3\xi^3(\hat x)\end{cases}$$

and set

$$u_\varepsilon(\hat x,x_3):=\begin{cases} u(\hat x,x_3-\frac{\varepsilon}{2})+\frac{\varepsilon}{2}\xi(\hat x) & \text{in}\quad \Omega_\varepsilon^+\\ v_\varepsilon(\hat x,\frac{x_3}{\varepsilon}) & \text{in}\quad B_\varepsilon\\ u(\hat x,x_3+\frac{\varepsilon}{2})-\frac{\varepsilon}{2}\xi(\hat x) & \text{in}\quad \Omega_\varepsilon^-.\end{cases}$$

Clearly, $u_\varepsilon\in A_\varepsilon$ except the boundary condition and $u_\varepsilon\to u$ in $L^1(\Omega,\mathbb{R}^3)$. An easy calculation and the local Lipschitz conditions (3.1.3), (3.1.4), yield

$$\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} f(\nabla u_\varepsilon)\ dx=\int_\Omega f(\nabla u)\ dx$$

and

$$\begin{aligned}\lim_{\varepsilon\to 0}\frac1\varepsilon\int_{B_\varepsilon} g(e(u_\varepsilon))\ dx &= \lim_{\varepsilon\to 0}\int_B g\big((\hat\nabla v_\varepsilon|\frac1\varepsilon\frac{\partial v_\varepsilon}{\partial x_3})_s\big)\ dx\\ &= \int_S g\big((\hat\nabla v|\xi)_s\big)\ d\hat x,\end{aligned}$$

so that

$$\lim_{\varepsilon\to 0}F_\varepsilon(u_\varepsilon)=\int_\Omega f(\nabla u)\ dx+\int_S g\big((\hat\nabla v|\xi)_s\big)\ d\hat x. \tag{3.2.6}$$

According to a well known interchange result between infimum and integral (see [2]), we have

$$\int_S g_0(e(v))\ d\hat x=\inf_{\xi\in\mathcal{D}(S,\mathbb{R}^3)}\int_S g\big((\hat\nabla v|\xi)_s\big)\ d\hat x. \tag{3.2.7}$$

By taking the infimum over all $\xi\in\mathcal{D}(S,\mathbb{R}^3)$ in (3.2.6) and by using Lemma 3.1.1 we deduce

$$\inf\Big\{\limsup_{\varepsilon\to 0}F_\varepsilon(u_\varepsilon):u_\varepsilon\to u\text{ in }L^1(\Omega,\mathbb{R}^3)\Big\}\le\tilde F_0(u),$$

i.e. $\Gamma - \limsup F_\varepsilon \leq \tilde{F}_0$.

Step 2. Taking the lower semicontinuous envelope of each two functionals for the strong topology of $L^1(\Omega, \mathbb{R}^3)$, the conclusion then follows from the lower semicontinuity of $\Gamma - \limsup F_\varepsilon$ and from Lemma 3.2.3. □

According to Lemma 3.1.2 and to variational properties of the Γ-convergence, we obtain :

Corollary 3.2.5. *Let $\bar{u}_\varepsilon$ be a ε-solution of $(\mathcal{P}_\varepsilon)$. Then there exist a subsequence of $(\bar{u}_\varepsilon)_{\varepsilon>0}$ and $\bar{u}$ in $W^{1,p}_{\Gamma_0}(\Omega, \mathbb{R}^3)$ such that*

$$\begin{aligned}
&\bar{u}_\varepsilon \rightharpoonup \bar{u} \text{ in } BD(\Omega, \mathbb{R}^3);\\
&\bar{u}_\varepsilon \rightharpoonup \bar{u} \text{ in } W^{1,p}_{\Gamma_0}(\Omega_\eta, \mathbb{R}^3) \text{ for every } \eta > 0;\\
&\gamma_S(\hat{\bar{u}}) \in BD(S, \mathbb{R}^2).
\end{aligned}$$

Moreover, $\bar{u}$ is solution of the minimization problem

$$(\mathcal{P}) \qquad \min\left\{F_0(u) - L(u) : u \in L^1(\Omega, \mathbb{R}^3)\right\}$$

and

$$F_\varepsilon(\overline{u}_\varepsilon) - L(\overline{u}_\varepsilon) \to \min\left\{F_0(u) - L(u) : u \in L^1(\Omega, \mathbb{R}^3)\right\}.$$

Thus, in this simplified case (see Section 3.3 for a realistic geometry), our proposal of model is given by the limit problem $(\mathcal{P})$ which describes the equilibrium of a structure made of two adherents perfectly stuck to a material surface. The reference configuration of the adherents are $\Omega^\pm := \Omega \cap [\pm x_3 > 0]$ while that the material surface is S. The adherents are hyperelastic with bulk energy density f and the material surface is pseudo-plastic with surface density g_0. Due to the linear growth of g_0, the displacement field solution of $(\mathcal{P})$ may present discontinuities in S which may be interpreted in terms of cracks. It is worthwhile to note that this situation with strong adhesive layer is completely different from the one considered in [15] with a soft adhesive: in the asymptotic model, the soft adhesive layer is replaced by a mechanical constraint between the adherents, whereas the strong adhesive layer is replaced by a material surface perfectly stuck to adherents. Another strategy proposed in [7] leads to a similar model.

3.3 A modeling of a welding assembly

An elementary situation in welding can be described as follows. Let Σ^+, Σ^- and S three domains of $\mathbb{R}^2$ with Lipschitz-continuous boundaries such that $S = \Sigma^+ \cap \Sigma^-$. Let r and ε two positive numbers such that $\varepsilon << r$ and $\Omega^\pm_\varepsilon := \Sigma^\pm \times (\pm\frac{\varepsilon}{2}, \pm r)$, $\Omega_\varepsilon = \Omega^+_\varepsilon \cup \Omega^-_\varepsilon$, $B_\varepsilon = S \times (-\frac{\varepsilon}{2}, \frac{\varepsilon}{2})$, and $S^\pm_\varepsilon = S^\pm \pm \frac{\varepsilon}{2}\, e_3$. Then $\mathcal{O}_\varepsilon := \Omega_\varepsilon \cup S^+_\varepsilon \cup S^-_\varepsilon \cup B_\varepsilon$ is the reference configuration of a structure made of two adherents and an adhesive (the soldered joint) which respectively occupies $\Omega_\varepsilon^\pm$ and B_ε (see Figure 3.1). The structure is clamped on a part Γ_0 of the boundary Γ of Ω with a positive $\mathcal{H}^2$-measure and we assume that there exists $\varepsilon_0 > 0$ such that $\mathrm{dist}(\bar{\Gamma}_0, \bar{B}_{\varepsilon_0}) > 0$. The structure is subjected to body forces of density Φ and to surface forces of density φ on the complementary part Γ_φ of Γ_0. We assume that the supports of φ and Φ lie outside of $\bar{B}_{\varepsilon_0}$. Obviously one can consider other type of boundary conditions (e.g. a combination of some components of the stress vector and of the displacement). At last, adhesive and adherents are assumed to be perfectly stuck together along $S^\pm$.

The adherents and the adhesive are modeled as in Section 3.1 so that determining the equilibrium configuration leads to the problem

$$(\mathcal{P}_\varepsilon) \qquad \inf\big\{F_\varepsilon(u) - L(u) : u \in A_\varepsilon\big\},$$

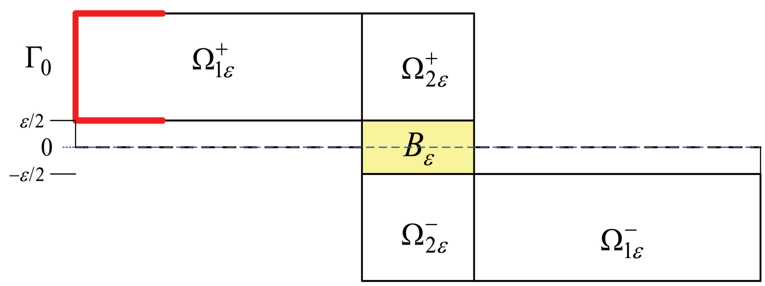

Figure 3.1: The reference configuration $\mathcal{O}_\varepsilon$. The reference configuration involved in Section 3.1 is $\Omega_{2\varepsilon}^+ \cup \Omega_{2\varepsilon}^- \cup B_\varepsilon \cup S_\varepsilon^+ \cup S_\varepsilon^-$.

where F_ε and L have the same expression as in Section 3.1 but with the new definitions of Ω_ε and B_ε, whereas A_ε now reads as:

$$A_\varepsilon := \big\{u \in LD(\mathcal{O}_\varepsilon, \mathbb{R}^3) : u_{\lfloor \Omega_\varepsilon} \in W^{1,p}_{\Gamma_0}(\Omega_\varepsilon, \mathbb{R}^3)\big\}.$$

Again, to propose a simplified but accurate model we consider ε as a parameter and study the asymptotic behavior, when ε goes to zero, of (approximate) solution of $(\mathcal{P}_\varepsilon)$. The essential difference from the model problem of Section 3.1 is that here the structure occupies a domain $\mathcal{O}_\varepsilon$ which *varies with* ε, which from the mathematical point of view is only of technical nature by simply modifying the kinds of convergences. That is why we have preferred to consider the model problem in total detail and to confine the stating of the sole results about this realistic problem of welding.

Let $\Omega := \Sigma^+ \times (0,r) \cup S \cup \Sigma^- \times (-r, 0)$ the set to which $\mathcal{O}_\varepsilon$ "converges". Let $I_\varepsilon := F_\varepsilon - L$ and $I_0 := F_0 - L$ with $F_0 : L^1(\Omega, \mathbb{R}^3) \to \mathbb{R} \cup \{+\infty\}$,

$$F_0(u) := \begin{cases} \displaystyle\int_\Omega f(\nabla u)\, dx + \int_S g_0(e(\gamma_s(\hat{u}))) & \text{if } u \in A_0 \\ +\infty & \text{otherwise,} \end{cases}$$

where we keep the same definition for γ_S and A_0 but with the new definition of Ω. Doing the same for the definition of the operator T_ε, our asymptotic model is supplied by:

Theorem 3.3.1. *Let $\bar{u}_\varepsilon$ be an ε-minimizer of $(\mathcal{P}_\varepsilon)$. Then there exist a subsequence of $(\bar{u}_\varepsilon)_{\varepsilon>0}$ and $\bar{u}$ in $W^{1,p}_{\Gamma_0}(\Omega,\mathbb{R}^3)$ such that*

$$\begin{array}{l} T_\varepsilon \bar{u}_\varepsilon \rightharpoonup \bar{u} \text{ weakly in } W^{1,p}(\Omega \setminus S,\mathbb{R}^3); \\ \bar{u}_{\varepsilon\lfloor S\times(-r,r)} \rightharpoonup \bar{u} \text{ weakly in } BD(S\times(-r,r),\mathbb{R}^3); \\ \gamma_S(\hat{\bar{u}}) \in BD(S,\mathbb{R}^2). \end{array}$$

Moreover, $\bar{u}$ is a solution of the minimization problem

$$(\mathcal{P}) \ : \qquad \min\left\{F_0(u) - L(u) : u \in L^1(\Omega,\mathbb{R}^3)\right\}$$

and

$$\inf\left\{F_\varepsilon(u) - L(u) : u \in L^1(\Omega,\mathbb{R}^3)\right\} \to \min\left\{F_0(u) - L(u) : u \in L^1(\Omega,\mathbb{R}^3)\right\}.$$

Sketch of the proof. The proof follows the line of the proof of Section 3.1 by considering simultaneously $T_\varepsilon \bar{u}_\varepsilon$ and its restriction to the fixed domain $S\times(-r,r)$. □

For the mechanical interpretation see the end of Section 3.2.

Chapter 4

Second model(with possibly potential wells)

4.1 Introduction

In this section, we drop the quasiconvex and convex assumptions on the density functions f and g, respectively. This is the case when the materials undergo reversible solid/solid phase transformations, for which the density functions present a multi-well structure (for f in the large deformation setting see [10], for g in the setting of small perturbations see [13]). However we assume that f and g satisfy the locally Lipschitz conditions (3.1.3), (3.1.4) and that g is positively 1-homogeneous.

4.2 The main convergence results

In this more general situation, we would like to show that the limit energy functional is given by $F_0 : L^1(\Omega, \mathbb{R}^3) \to \mathbb{R} \cup \{+\infty\}$,

$$F_0(u) := \begin{cases} \displaystyle\int_\Omega Qf(\nabla u)\ dx + \int_S SQg_0(e(\gamma_s(\hat{u}))) & \text{if } u \in A_0, \\ +\infty & \text{otherwise} \end{cases}$$

where Qf is the quasiconvex envelope of f and $SQg_0 : \mathbf{M}_s^{2\times 2} \to \mathbb{R}$ is the symmetric quasiconvex envelope of g_0 defined by

$$SQg_0(\zeta) := \inf\Big\{\frac{1}{|\hat{D}|}\int_{\hat{D}} g_0(\zeta + e(\varphi))\ d\hat{x} : \varphi \in \mathcal{C}_0^\infty(\hat{D}, \mathbb{R}^2)\Big\}.$$

Let denote the operator $\zeta \mapsto \zeta_s$ from $\mathbf{M}^{2\times 2}$ into $\mathbf{M}_s^{2\times 2}$ by $\mathcal{S}_2$. Thus, for every $\zeta \in \mathbf{M}_s^{2\times 2}$, $SQg_0(\zeta) = Q(g_0 \circ \mathcal{S}_2)(\zeta)$ where $Q(g_0 \circ \mathcal{S}_2)$ is the quasiconvex envelope of $g_0 \circ \mathcal{S}_2$. Note that the right-hand side term does not depend on the choice of the cube $\hat{D}$ of $\mathbb{R}^2$ and that SQg_0 is 1-homogeneous. With the notation of Section 3.1, the integral over S of the measure $SQg_0(e(v))$ is given by

$$\int_S SQg_0(e(v)) = \int_S SQg_0(e_a(v))\ d\hat{x} + \int_S SQg_0(\frac{e_s(v)}{|e_s(v)|})|e_s(v)|.$$

As in Section 3.1 of Chapter 3, we prove Proposition 4.2.4 and Proposition 4.2.7 below, corresponding to the lower and the upper bound in the definition of the Γ-convergence. Unfortunately, we establish the lower bound when u belongs to the subset $\tilde{A}_0$ of A_0 defined by

$$\tilde{A}_0 := \{u \in W^{1,p}_{\Gamma_0}(\Omega, \mathbb{R}^3) : \gamma_S(\hat{u}) \in SBD(S, \mathbb{R}^2)\}$$

where $SBD(S, \mathbb{R}^2)$ denotes the set of the elements u of $BD(S, \mathbb{R}^2)$ whose the Cantor part of the strain tensor $e(u)$ is zero. Then, the main result of this section is

Theorem 4.2.1. *The restriction of the Γ_{τ_s}-limit of F_ε to the set $\tilde{A}_0$ is given by*

$$F_0(u) = \int_\Omega Qf(\nabla u)\ dx + \int_S SQg_0(e_a(\gamma_S(\hat{u})))\ d\hat{x} + \int_S SQg_0([\gamma_s(\hat{u})] \otimes_s \nu_{\gamma_S(\hat{u})})d\mathcal{H}^1.$$

Remark 4.2.2. If we assume that an approximate minimizer of $(\mathcal{P}_\varepsilon)$ strongly converges to some $\bar{u}$ in $L^1(\Omega, \mathbb{R}^3)$ whose distributional gradient has no Cantor part, according to the variational nature of the Γ-convergence, we deduce that $\bar{u}$ is a solution of the limit problem

$$(\mathcal{P}) \ : \qquad \min\{F_0(u) - L(u) : u \in L^1(\Omega, \mathbb{R}^3)\}$$

and

$$\inf\{F_\varepsilon(u) - L(u) : u \in L^1(\Omega, \mathbb{R}^3)\} \to \min\{F_0(u) - L(u) : u \in \tilde{A}_0\}.$$

Therefore, under the assumption that some approximate minimizer is regular in the sense above, problem $(\mathcal{P})$ is a good model in the sense of Section 3.2, where the density functions are now Qf and SQg_0.

Remark 4.2.3. In the case when the deformations in the adhesive may be large, they are modeled as hyperelastic together with the deformations in the adherents. In this particular case, we obtain a complete description of the Γ-limit F_0 in the set

$$\tilde{A}'_0 := \{u \in W^{1,p}_{\Gamma_0}(\Omega, \mathbb{R}^3) : \gamma_S(u) \in BV(S, \mathbb{R}^3)\}.$$

More precisely F_0 is defined in $\tilde{A}'_0$ by

$$F_0(u) = \int_\Omega Qf(\nabla u)\ dx \ + \ \int_S Qg_0(\nabla(\gamma_S(u)))\ d\hat{x} + \int_S (Qg_0)^\infty(\frac{D_s\gamma_S(u)}{|D_s\gamma_S(u)|})|D_s\gamma_S(u)|d\mathcal{H}^1,$$

where $Du = \nabla u\ d\hat{x} + D_s u$ is the Lebesgue decomposition of the distributional derivative Du, $g_0(\zeta) = \min\{g(\xi) : \xi \in \mathbf{M}^{3\times3}, \hat{\xi} = \zeta\}$ for every $\zeta \in \mathbf{M}^{2\times2}_s$, and $\zeta \mapsto (Qg_0)^\infty(\zeta) := \lim_{t\to+\infty} Qg_0(t\zeta)/t$ is the recession function of Qg_0. The proof uses the relaxation theorem, Theorem 11.3.1 in [6], instead of Proposition 4.2.5 below and follows point by point the claims of Propositions 4.2.4, 4.2.7 below.

4.2.1 The lower bound for the $\Gamma_{\tau_{L^1}}$−convergence of $(F_\varepsilon)_{\varepsilon>0}$

Proposition 4.2.4 (Lower bound). *For all u in $\tilde{A}_0$ and all sequence $(u_\varepsilon)_{\varepsilon>0}$ in $L^1(\Omega,\mathbb{R}^3)$ such that $u_\varepsilon \xrightarrow{\tau_s} u$, the following inequality holds:*

$$F_0(u) \le \liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon). \tag{4.2.1}$$

Proof. We have

$$\begin{aligned}\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) &= \liminf_{\varepsilon\to 0}\Big(\int_{\Omega_\varepsilon} f(\nabla u_\varepsilon)dx + \frac{1}{\varepsilon}\int_{B_\varepsilon} g(e(u_\varepsilon))dx\Big)\\ &\ge \liminf_{\varepsilon\to 0}\int_{\Omega_\varepsilon} f(\nabla u_\varepsilon)dx + \liminf_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{B_\varepsilon} g(e(u_\varepsilon))dx\\ &\ge \liminf_{\varepsilon\to 0}\int_{\Omega_\varepsilon} f(\nabla u_\varepsilon)dx + \liminf_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{B_\varepsilon} g_0(\widehat{e(u_\varepsilon)})dx. \end{aligned}\tag{4.2.2}$$

For a.e. x in B, set $\hat{v}_\varepsilon(x) := \hat{u}_\varepsilon(\hat{x},\varepsilon x_3)$ and $v^3_\varepsilon(x) := \varepsilon u^3_\varepsilon(\hat{x},\varepsilon x_3)$. Set $B := S\times(-\frac{1}{2},\frac{1}{2})$. We have

$$\liminf_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{B_\varepsilon} g_0(\widehat{e(u_\varepsilon)})dx \ge \liminf_{\varepsilon\to 0}\int_B SQg_0((\widehat{e(v_\varepsilon)})dx. \tag{4.2.3}$$

Consider the function $h:\mathbf{M}^{3\times 3}_s\to\mathbb{R}$ defined for every $\xi\in\mathbf{M}^{3\times 3}_s$ by $h(\xi) := SQg_0(\hat{\xi})$. It is easily seen that h is symmetric quasiconvex, i.e. satisfies the inequality:

$$h(\xi) \le \frac{1}{|D|}\int_D h(\xi + e(\varphi))\ dx.$$

for every $\varphi\in\mathcal{C}^\infty_c(D,\mathbb{R}^3)$ where D is any cube of $\mathbb{R}^3$ (see [9] for the definition). Moreover, (4.2.3) can be written

$$\liminf_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{B_\varepsilon} g_0(\widehat{e(u_\varepsilon)})dx \ge \liminf_{\varepsilon\to 0}\int_B h(e(v_\varepsilon))dx. \tag{4.2.4}$$

Let us denote by $\mathrm{BD}(B,\mathbb{R}^3)$ the space of bounded deformation on B and by $\mathcal{R}_H$ the set of rigid motions on B. According to coercivity condition (3.1.2) we have

$$\sup_{\varepsilon>0}\int_B \Big|\begin{pmatrix} e_{\alpha\beta}(\hat{v}_\varepsilon) & \frac{1}{\varepsilon}e_{\alpha 3}(v_\varepsilon)\\ \frac{1}{\varepsilon}e_{3\alpha}(v_\varepsilon) & \frac{1}{\varepsilon^2}\frac{\partial v^\varepsilon_3}{\partial x_3}\end{pmatrix}\Big|\ dx \le \frac{1}{\alpha}\sup_{\varepsilon>0}\frac{1}{\varepsilon}\int_{B_\varepsilon} g(e(u_\varepsilon))dx < +\infty. \tag{4.2.5}$$

Thus, by using the arguments of the proof of Lemma 3.1.2, one can easily establish the existence of $v\in\mathrm{BD}(B,\mathbb{R}^3)$ and $r_\varepsilon\in\mathcal{R}_H$ such that $v_\varepsilon + r_\varepsilon \rightharpoonup v$ weakly in $BD(B,\mathbb{R}^3)$, $\hat{v} = \gamma_S(\hat{u})$, $v_3 = 0$ and $\frac{\partial v}{\partial x_3} = 0$. Combining (4.2.2), (4.2.4), a classical results in relaxation theory (Theorem 11.2.1 in [6]) and a relaxation result in $BD(S,\mathbb{R}^2)$ (see [9]), we infer

$$\begin{aligned}\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) \ge &\int_\Omega Qf(\nabla u)\ dx + \int_S SQg_0(e_a(\gamma_S(\hat{u})))\ d\hat{x}\\ &+ \int_S SQg_0([\gamma_s(\hat{u})]\otimes_s \nu_{\gamma_S(u)}(\hat{x}))d\mathcal{H}^1(\hat{x})\end{aligned}$$

which ends the proof. □

4.2.2 The upper bound for the $\Gamma_{\tau_{L^1}}$−convergence of $(F_\varepsilon)_{\varepsilon>0}$

For proving the upper bound, we need to establish the following relaxation result.

Proposition 4.2.5 (Relaxation). *The functional F_0 is the l.s.c. regularization for strong topology of the functional defined on $L^1(\Omega,\mathbb{R}^3)$ by*

$$\tilde{F}_0(u) := \begin{cases} \displaystyle\int_\Omega f(\nabla u)\,dx + \int_S g_0(e(\gamma_S(\hat{u})))\,d\hat{x} & \text{if } u \in A_0^1, \\ +\infty & \text{otherwise.} \end{cases}$$

In the proof of Proposition 4.2.5 we will use the following lemma.

Lemma 4.2.6. *Let $\eta > 0$. Then for every $\zeta \in \mathbf{M}_s^{2\times 2}$,*

$$\lim_{\eta\to 0} SQ(\,\eta\,|\,.\,|^p + g_0\,)(\zeta) = SQg_0(\zeta).$$

Proof. From the integral representations

$$SQ(\,\eta\,|\,.\,|^p + g_0\,)(\zeta) = \inf_{\phi\in C_c^\infty(D,\mathbb{R}^2)} \frac{1}{|D|}\int_D (\,\eta\,|\,.\,|^p + g_0\,)(\zeta + e(\phi))\,d\hat{x},$$

and

$$SQg_0(\zeta) = \inf_{\phi\in C_c^\infty(D,\mathbb{R}^2)} \frac{1}{|D|}\int_D g_0(\zeta + e(\phi))\,d\hat{x},$$

where D is an arbitrary cube of $\mathbb{R}^2$, we can write

$$\begin{aligned} & \lim_{\eta\to 0}\Big(\inf_{\phi\in C_c^\infty(D,\mathbb{R}^2)} \frac{1}{|D|}\int_D (\,\eta\,|\,.\,|^p + g_0\,)(\zeta + e(\phi))\,d\hat{x}\Big) \\ = \; & \inf_{\eta>0}\Big(\inf_{\phi\in C_c^\infty(D,\mathbb{R}^2)} \frac{1}{|D|}\int_D (\,\eta\,|\,.\,|^p + g_0\,)(\zeta + e(\phi))\,d\hat{x}\Big) \\ = \; & \inf_{\phi\in C_c^\infty(D,\mathbb{R}^2)}\Big(\inf_{\eta>0} \frac{1}{|D|}\int_D (\,\eta\,|\,.\,|^p + g_0\,)(\zeta + e(\phi))\,d\hat{x}\Big) \\ = \; & \inf_{\phi\in C_c^\infty(D,\mathbb{R}^2)}\Big(\lim_{\eta\to 0} \frac{1}{|D|}\int_D (\,\eta|\,.\,|^p + g_0\,)(\zeta + e(\phi))\,d\hat{x}\Big). \end{aligned}$$

We complete the proof by using the dominated convergence theorem. □

Proof of Proposition 4.2.5. Denote by $\overline{\widetilde{F}}_0$ the l.s.c. regularization of $\widetilde{F}_0$. By using standard l.s.c. results on integral functionals defined on Sobolev or BV-spaces, we can easily prove $F_0 \le \overline{\widetilde{F}}_0$ (see for instance [6], Chapter 11). We only establish the converse inequality $\overline{\widetilde{F}}_0 \le F_0$. Its proof is not easy because of the condition $u \in A_0$ and the fact that f and g do not fulfill the same growth conditions.

Let $u \in L^1(\Omega, \mathbb{R}^3)$ such that $F_0(u) < +\infty$. Then $u \in A_0$. We have to exhibit u_n in A_0^1 strongly converging to u in $L^1(\Omega, \mathbb{R}^3)$ such that $\lim_{n\to+\infty} \tilde{F}_0(u_n) = F_0(u)$. We proceed into two steps.

Step 1. We prove the thesis when $u \in A_0^1$. For shorten notation we write $v := \gamma_S(\hat{u})$.

Let $\eta > 0$ approach 0 and denote the constant involved in Korn's inequality by K:

$$\int_S |\nabla w|^p \, d\hat{x} \leq K\Big(\int_S |e(w)|^p + |w|^p\Big) \, d\hat{x}$$

for all functions w in $W^{1,p}(S, \mathbb{R}^2)$. From relaxation theory in Sobolev spaces, there exists a sequence of smooth functions $(u_n, v_n)_{n\in\mathbb{N}}$ in $W^{1,p}_{\Gamma_0}(\Omega, \mathbb{R}^3) \times W^{1,p}(S, \mathbb{R}^2)$ such that (see for instance [6], Theorem 11.2.1)

$$\left.\begin{array}{l} u_n \rightharpoonup u \text{ in } W^{1,p}_{\Gamma_0}(\Omega, \mathbb{R}^3),\ (|\nabla u_n|^p)_{n\in\mathbb{N}} \text{ uniformly integrable;} \\ v_n \rightharpoonup v \text{ in } W^{1,p}(S, \mathbb{R}^3); \\ \displaystyle\lim_{n\to\infty} \int_\Omega f(\nabla u_n) \, dx = \int_\Omega Qf(\nabla u) \, dx; \\ \displaystyle\lim_{n\to\infty} \int_S (\ 3K\beta\eta \mid . \mid^p + g_0\ )(e(v_n)) d\hat{x} = \int_S SQ(\ 3K\beta\eta \mid . \mid^p + g_0\ )(e(v)) \, d\hat{x}. \end{array}\right\} \tag{4.2.6}$$

The additional condition that $(|\nabla u_n|^p)_{n\in\mathbb{N}}$ may be assumed to be uniformly integrable comes from the following consideration. Consider the sequence $(\tilde{u}_n)_{n\in\mathbb{N}}$ whose gradients generate the same Young measure μ and such that $(|\nabla \tilde{u}_n|^p)_{n\in\mathbb{N}}$ is uniformly integrable (Lemma 11.4.1 in [6]). By using lower semicontinuity and continuity properties of Young measures (Proposition 4.3.3 and Theorem 4.3.3 in [6]), and standard lower semicontinuity results in Sobolev spaces, we have

$$\begin{aligned} \int_\Omega Qf(\nabla u) \, dx &= \lim_{n\to+\infty} \int_\Omega f(\nabla u_n) \, dx \\ &\geq \int_{\Omega\times\mathbf{M}^{3\times3}} f(\lambda) \, d\mu \\ &= \lim_{n\to+\infty} \int_\Omega f(\nabla \tilde{u}_n) \, dx \\ &\geq \int_\Omega Qf(\nabla u) \, dx, \end{aligned}$$

so that

$$\lim_{n\to+\infty} \int_\Omega f(\nabla \tilde{u}_n) \, dx = \lim_{n\to+\infty} \int_\Omega f(\nabla u_n) \, dx = \int_\Omega Qf(\nabla u) \, dx$$

which proves the thesis. In what follows, we still denote by $(u_n)_{n\in\mathbb{N}}$ the sequence $(\tilde{u}_n)_{n\in\mathbb{N}}$.

We start by modifying the function u_n near S so that $\gamma_S(u^n_\alpha) = v^n_\alpha$. Set $\Sigma_\eta := S \times (-\eta, \eta)$, $\Sigma_{2\eta} := S \times (-2\eta, 2\eta)$, consider a cut-off function φ_η in $\mathcal{C}^1(\mathbb{R})$ satisfying

$$\varphi_\eta = 1 \text{ on } \Omega \setminus \Sigma_{2\eta},\ \varphi_\eta = 0 \text{ on } \Sigma_\eta,\ 0 \leq \varphi_\eta \leq 1,\ |\frac{d\varphi_\eta}{dx_3}| \leq \frac{1}{\eta},$$

and define the function $u_{n,\eta}$ by

$$\begin{array}{l} u_\alpha^{n,\eta} := \varphi_\eta(u_\alpha^n - v_\alpha^n) + v_\alpha^n; \\ u_3^{n,\eta} := u_3^n. \end{array}$$

Clearly, $u_{n,\eta} \in W^{1,p}_{\Gamma_0}(\Omega, \mathbb{R}^3)$ and $\gamma_S(u_\alpha^{n,\eta}) = v_\alpha^n$. From the growth condition satisfied by f, we have

$$\begin{aligned} \int_\Omega f(\nabla u_{n,\eta})\, dx &= \int_{\Sigma_\eta} f(\nabla u_{n,\eta})\, dx + \int_{\Sigma_{2\eta}\setminus\Sigma_\eta} f(\nabla u_{n,\eta})\, dx + \int_{\Omega\setminus\Sigma_{2\eta}} f(\nabla u_n)\, dx \\ &\leq \beta \int_{\Sigma_\eta} (1 + |\nabla v_n|^p)\, dx + \beta \int_{\Sigma_{2\eta}\setminus\Sigma_\eta} (1 + |\nabla v_n|^p)\, dx \\ &\quad + C\Big[\int_{\Sigma_{2\eta}} |\nabla u_n|^p\, dx + \frac{1}{\eta^p}\int_{\Sigma_{2\eta}} |\hat{u}_n - v_n|^p\, dx\Big] + \int_\Omega f(\nabla u_n) dx \\ &= 3\beta\eta \int_S |\nabla v_n|^p\, d\hat{x} + C\Big[\eta + \int_{\Sigma_{2\eta}} |\nabla u_n|^p\, dx \\ &\quad + \frac{1}{\eta^p}\int_{\Sigma_{2\eta}} |\hat{u}_n - v_n|^p\, dx\Big] + \int_\Omega f(\nabla u_n) dx. \end{aligned}$$

Thus, according to Korn's inequality (note that $\sup_{n\in\mathbb{N}} \int_S |v_n|^p\, d\hat{x} < +\infty$),

$$\begin{aligned} \int_\Omega f(\nabla u_{n,\eta})\, dx \;\leq\; & 3\beta K\eta \int_S |(e(v_n)|^p\, d\hat{x} \;+\; C\Big[\eta \;+\; \int_{\Sigma_{2\eta}} |\nabla u_n|^p\, dx \\ & + \frac{1}{\eta^p}\int_{\Sigma_{2\eta}} |\hat{u}_n - v_n|^p\, dx\Big] \;+\; \int_\Omega f(\nabla u_n)\, dx. \end{aligned}$$

which yields

$$\begin{aligned} \int_\Omega f(\nabla u_{n,\eta)})\, dx + \int_S g_0(\, e(v_n)\,)\, d\hat{x} \;\leq\; & \int_\Omega f(\nabla u_n)\, dx + \int_S (\, 3\beta K\eta\, |.|^p + g_0\,)(e(v_n))\, d\hat{x} \\ & + C\Big[\eta + \int_{\Sigma_{2\eta}} |\nabla u_n|^p\, dx + \frac{1}{\eta^p}\int_{\Sigma_{2\eta}} |\hat{u}_n - v_n|^p\, dx\Big] \end{aligned}$$

and, from (4.2.6),

$$\begin{aligned} \limsup_{n\to+\infty} \widetilde{F}_0(u_{n,\eta}) \;\leq\; & \int_\Omega Qf(\nabla u) dx + \int_S SQ(\, 3\beta K\eta\, |.|^p + g_0\,)(e(v))\, d\hat{x} \\ & + C\Big[\,\eta + \sup_{n\in\mathbb{N}} \int_{\Sigma_{2\eta}} |\nabla u_n|^p\, dx + \frac{1}{\eta^p}\int_{\Sigma_{2\eta}} |\hat{u} - v|^p\, dx\,\Big]. \end{aligned}$$

But since $\gamma_S(\hat{u}) = v$, clearly one has

$$\int_{\Sigma_{2\eta}} |\hat{u} - v|^p dx \leq \eta^p \int_{\Sigma_{2\eta}} |\frac{\partial u}{\partial x_3}|^p dx$$

so that

$$\begin{aligned}\limsup_{n\to+\infty}\widetilde{F}_0(u_{n,\eta}) &\leq \int_\Omega Qf(\nabla u)dx+\int_S SQ(3\beta K\eta|\ |^p+g_0)(e(v))\ d\hat{x}\\ &\quad +C\Big[\eta+\sup_{n\in\mathbb{N}}\int_{\Sigma_{2\eta}}|\nabla u_n|^p\ dx+\int_{\Sigma_{2\eta}}|\frac{\partial u}{\partial x_3}|^p dx\Big].\end{aligned}$$

By letting $\eta\to 0$, from the uniform integrability of $(|\nabla u_n|^p)_{n\in\mathbb{N}}$, Lemma 4.2.6 and Lebesgue's dominated convergence theorem, we obtain

$$\lim_{\eta\to 0}\limsup_{n\to+\infty}\widetilde{F}_0(u_{n,\eta})\leq\int_\Omega Qf(\nabla u)dx+\int_S SQg_0(e(v))\ d\hat{x}.$$

By using a standard diagonalization argument, there exists a map $n\mapsto\eta(n)$ such that, setting $\tilde{u}_n:=u_{n,\eta(n)}$,

$$\limsup_{n\to+\infty}\widetilde{F}_0(\tilde{u}_n)\leq F_0(u).$$

It is easily seen that $\tilde{u}_n\to u$ in $L^1(\Omega,\mathbb{R}^3)$. Since classically $\liminf_{n\to+\infty}\widetilde{F}_0(\tilde{u}_n)\geq F_0(u)$, the proof of Step 1 is complete.

Step 2. We end the proof as Step 2 of the proof of Lemma 3.2.3. We only have to substitute

$$\left|\int_{\mathbb{R}^2} SQg_0(\rho_{\eta(\delta)}*(\phi e(\tilde{v})))\ d\hat{x}-\int_{\mathbb{R}^2} SQg_0(\phi e(v))\right|<\delta \tag{4.2.7}$$

for (3.2.3). Estimate (4.2.7) is a straightforward consequence of the weak convergence of the measure $\rho_{\eta(\delta)}*(\phi e(v))$ to the measure $\phi e(v)$ in $\mathcal{M}(\mathbb{R}^2)$ together with $\lim_{n\to+\infty}\int_{\mathbb{R}^2}|\ \rho_{\eta(\delta)}*(\phi e(v))\ |=\int_{\mathbb{R}^2}|\ \phi e(v)\ |$, and Reshetnyak's continuity theorem (see Theorem 2.39 in [4]). □

Then proceeding exactly like in the proof of Proposition 3.2.4 and using Lemma 4.2.5, we have

Proposition 4.2.7 (Upper bound). *The following inequality holds in* $L^1(\Omega,\mathbb{R}^3)$

$$(\Gamma-\limsup F_\varepsilon)\leq F_0. \tag{4.2.8}$$

Chapter 5

Regularization of the first model

5.1 Introduction

For the numerical solving of the optimization problem

$$\inf\left\{F_0(u) - L(u)\right\}$$

obtained in Section 3.2 of Chapter 3, we approximate, in a variational way, the functional of measure $u \mapsto \int_S g_0(e(\gamma_S(\hat{u})))$ by a suitable functional defined in the Sobolev space $W^{1,q}(S, \mathbb{R}^2)$, where q is close to 1 in the spirit of Norton-Hoff regularisation (cf [17]). The mathematical technics used here is an adaptation from those of [7]. In order to simplify the proofs, we assume that S is a finite union of cubes in $\mathbb{R}^2$.

We denote the limit density g_0 by h which we assume to be positively 1-homogeneous and fulfilling the growth conditions $\alpha|\xi| \leq h(\xi) \leq \beta|\xi|$. We consider a sequence $(h_q)_{q\in(1,p)}$ satisfying the following three conditions:

(i_q) $h_q : \mathbf{M}_s^{2\times 2} \to \mathbb{R}^+$ is convex and positively homogeneous of degree q;

(ii_q) $h_q \to h$ pointwise in $\mathbf{M}_s^{2\times 2}$;

(iii_q) there exists $a > 0$ such that for all $q > 1$ close enough to 1,

$$h_q(\xi) \geq h(\xi) \text{ for all } \xi \in \mathbf{M}_s^{2\times 2},\ |\xi| \geq a. \tag{5.1.1}$$

For instance, when $h = |\ .\ |$, $h_q = |\ .\ |^q$ satisfies these conditions with $a = 1$. Another natural example consists in choosing $h_q := h^q$. Condition (iii_q) is then satisfied by taking $a = \dfrac{1}{\alpha}$. In these two examples, h_q satisfies uniform growth conditions with respect to q (for the second example, $\dfrac{\alpha}{2}|\xi|^q \leq h_q(\xi) \leq 2\beta|\xi|^q$ for q closed to 1). Note that, according to $(iii)_q$, h_q fulfills the equi-coerciveness condition:

$$h_q(\xi) \geq \alpha|\xi|\ \forall \xi,\ |\xi| \geq a. \tag{5.1.2}$$

In what follows, the function h_q is not assumed to satisfy a uniform upper growth condition.

We consider the functional $\mathcal{F}_q : W^{1,p}_{\Gamma_0}(\Omega, \mathbb{R}^3) \to \mathbb{R}^+ \cup \{+\infty\}$ defined by:

$$\mathcal{F}_q(u) = \begin{cases} \displaystyle\int_\Omega f(\nabla u)\ dx + \int_S h_q(\ e(\gamma_S(\hat{u}))\ )\ d\hat{x} & \text{if } u \in \mathcal{B}_q, \\ \\ +\infty & \text{otherwise,} \end{cases}$$

where

$$\mathcal{B}_q := \big\{ u \in W^{1,p}_{\Gamma_0}(\Omega, \mathbb{R}^3) : h_q \circ (e(\gamma_S(\hat{u}))) \in L^1(\Omega) \big\}.$$

We are going to establish the Γ-convergence of $\mathcal{F}_q$ when $q \to 1$, when the space $W^{1,p}_{\Gamma_0}(\Omega, \mathbb{R}^3)$ is equipped with its weak topology. The expected limit is the functional $\mathcal{F}_0 : W^{1,p}_{\Gamma_0}(\Omega, \mathbb{R}^3) \to \mathbb{R}^+ \cup \{+\infty\}$ defined in the previous section, more precisely its restriction to $W^{1,p}_{\Gamma_0}(\Omega, \mathbb{R}^3)$ defined by

$$\mathcal{F}_0(u) = \begin{cases} \displaystyle\int_\Omega f(\nabla u)\ dx + \int_S h(e(\gamma_S(\hat{u}))) & \text{if } u \in \mathcal{B}, \\ \\ +\infty & \text{otherwise,} \end{cases}$$

where

$$\mathcal{B} := \big\{ u \in W^{1,p}_{\Gamma_0}(\Omega, \mathbb{R}^3) : \gamma_S(\hat{u}) \in BD(S, \mathbb{R}^2) \big\}.$$

5.2 The main convergence results

Lemma 5.2.1 (Compactness lemma). *Consider a sequence $(u_q)_{q\in(1,p)}$ in $W^{1,p}_{\Gamma_0}(\Omega, \mathbb{R}^3)$ such that $\sup_{q\in(1,p)} \mathcal{F}_q(u_q) < +\infty$. Then there exist a subsequence of $(u_q)_{q\in(1,p)}$ and $u \in \mathcal{B}$ such that $(u_q, \gamma_S(\hat{u}_q)) \rightharpoonup (u, \gamma_S(\hat{u}))$ weakly in $W^{1,p}_{\Gamma_0}(\Omega, \mathbb{R}^3) \times BD(S, \mathbb{R}^2)$ when q goes to 1.*

Proof. Since $\sup_{q\in(1,p)} \mathcal{F}_q(u_q) < +\infty$, one has $u_q \in \mathcal{B}_q$ and

$$\mathcal{F}_q(u_q) = \int_\Omega f(\nabla u_q))\ dx + \int_S h_q(e(v_q))\ d\hat{x}, \quad v_q := \gamma_S(\hat{u}_q).$$

Thus, from (5.1.2)

$$\begin{aligned} \sup_{q\in(1,p)} \int_S |e(v_q)|\ d\hat{x} &\leq \sup_{q\in(1,p)} \Big(a|S| + \int_{[|e(v_q)|\geq a]} |e(v_q)|\ \ d\hat{x} \Big) \\ &\leq a|S| + \frac{1}{\alpha} \sup_{q\in(1,p)} \int_S h_q(e(v_q))\ d\hat{x} < +\infty. \end{aligned} \tag{5.2.1}$$

On the other hand from the coercivity condition fulfilled by f, there exists a subsequence and $u \in W^{1,p}_{\Gamma_0}(\Omega, \mathbb{R}^3)$ such that $u_q \rightharpoonup u$ in $W^{1,p}_{\Gamma_0}(\Omega, \mathbb{R}^3)$. According to the continuity of the trace operator γ_S, we deduce that $v_q \to \gamma_S(\hat{u})$ in $L^p(S, \mathbb{R}^3)$, thus strongly in $L^1(S,\mathbb{R}^2)$ which, combined with (5.2.1), yields $v_q \rightharpoonup v = \gamma_S(\hat{u})$ in $BD(S,\mathbb{R}^2)$ and $u \in \mathcal{B}$. $\square$

5.2.1 The lower bound for the $\Gamma-$convergence of $\mathcal{F}_q$

Proposition 5.2.2 (Lower bound). *For every sequence $(u_q)_{q\in]1,p]}$ converging to u in $W^{1,p}_{\Gamma_0}(\Omega,\mathbb{R}^3)$, we have*

$$\mathcal{F}_0(u) \le \liminf_{q\to 1} \mathcal{F}_q(u_q).$$

Lemma 5.2.3. *Let $V_q \in L^q(S, \mathbf{M}_s^{2\times 2})$ and assume that the measure $\mu_q = V_q\ d\hat{x}$ weakly converges to μ in $\mathcal{M}(S,\mathbf{M}_s^{2\times 2})$ and that $|\mu_q| = |V_q|\ d\hat{x}$ weakly converges to ν in $\mathcal{M}^+(S)$. Then for all open subset ω of S such that $\nu(\partial\omega) = 0$, we have*

$$\liminf_{q\to 1} \int_\omega h_q(V_q)\ d\hat{x} \ge h\Big(\int_\omega d\mu\Big).$$

Proof. From Jensen's inequality and since h_q is positively q-homogeneous,

$$\int_\omega h_q(V_q)\ d\hat{x} \ge |\omega| h_q\Big(\frac{1}{|\omega|}\int_\omega V_q\ d\hat{x}\Big) = |\omega|^{1-q} h_q\Big(\int_\omega V_q\ d\hat{x}\Big).$$

It remains to establish

$$\liminf_{q\to 1} h_q\Big(\int_\omega V_q\ d\hat{x}\Big) \ge h\Big(\int_\omega d\mu\Big). \tag{5.2.2}$$

One may assume $\big|\int_\omega d\mu\big| > 0$, otherwise $\int_\omega d\mu = 0$ and (5.2.2) is trivially satisfied. Since $\nu(\partial\omega) = 0$, we have $\lim_{q\to 1}\int_\omega V_q\ d\hat{x} = \int_\omega d\mu$ (cf Corollary 4.2.1 in [6]). Then for $0 < \sigma < \big|\int_\omega d\mu\big|$, there exists q_0, $1 < q_0 \le p$ such that, for all q, $1 < q \le q_0$, $|\int_\omega V_q\ d\hat{x}| \ge \sigma$. We have

$$h_q\Big(\int_\omega V_q\ d\hat{x}\Big) = \big(\frac{\sigma}{a}\big)^q h_q(\Big(\frac{a}{\sigma}\int_\omega V_q\ d\hat{x}\Big)$$

with $\left|\dfrac{a}{\sigma}\displaystyle\int_\omega V_q\ d\hat{x}\right| \ge a$, so that from assumption (5.1.1),

$$\begin{aligned} h_q\Big(\int_\omega V_q\ d\hat{x}\Big) &\ge \big(\frac{\sigma}{a}\big)^q h(\Big(\frac{a}{\sigma}\int_\omega V_q\ d\hat{x}\Big) \\ &= \big(\frac{\sigma}{a}\big)^{q-1} h(\Big(\int_\omega V_q\ d\hat{x}\Big). \end{aligned}$$

Letting $q \to 1$, (5.2.2) follows from the lower semicontinuity of h and the fact that $\lim_{q\to 1}\displaystyle\int_\omega V_q\ d\hat{x} = \int_\omega d\mu$. $\square$

Proof of Proposition 5.2.2. We may assume $\mathcal{F}_q(u_q) < +\infty$ so that, from Lemma 5.2.1, one has $u \in \mathcal{B}$. We write v_q for $\gamma_S(\hat{u}_q)$ and v for $\gamma_S(\hat{u})$. According to a standard lower semicontinuity result in Sobolev spaces we have

$$\liminf_{q\to 1}\mathcal{F}_q(u_q) \ge \int_\Omega f(\nabla u)dx + \liminf_{q\to 1}\int_S h_q(e(v_q))\ d\hat{x}$$

and it remains to establish

$$\liminf_{q\to 1} \int_S h_q(e(v_q))\ d\hat{x} \geq \int_S h(e(v)). \tag{5.2.3}$$

For $\delta > 0$ and approaching 0, consider a standard mollifier ρ_δ and θ_δ in $\mathcal{C}_c^\infty(S)$ satisfying $0 \leq \theta_\delta \leq 1$, $\theta_\delta \to 1$ a.e. in S. For a subsequence not relabelled on q, clearly $\rho_\delta * \theta_\delta |e(v_q)|\ d\hat{x}$ weakly converges to some measure ν_δ in $\mathcal{M}^+(S)$. Moreover $\rho_\delta * \theta_\delta e(v_q)\ d\hat{x}$ weakly converges to the measure $\rho_\delta * \theta_\delta e(v)$ in $\mathcal{M}(S, \mathbf{M}_s^{2\times 2})$. For $\eta > 0$, consider a finite family $(\omega_i)_{i\in I_\eta}$ of pairwise disjoint open subsets of S, $|\omega_i| < \eta$, such that $|S \setminus \cup_{i\in I_\eta} \omega_i| = 0$ and a family $(\tilde{\omega}_\eta)_{i\in I_\eta}$ satisfying

$$\begin{aligned} &\Big| S \setminus \bigcup_{i\in I_\eta} \tilde{\omega}_i \Big| \leq \eta; \\ &\tilde{\omega}_i \subset \omega_i; \\ &\nu_\delta(\partial \tilde{\omega}_i) = 0. \end{aligned}$$

Such a family exists (use Lemma 4.2.1 of [6]). Note that this family depends on δ. By using the q-homogeneity of h_q, Jensen's inequality and standard convex duality principle, we have

$$\begin{aligned} \int_S h_q(e(v_q))\ d\hat{x} &\geq \int_S \theta_\delta^q h_q(e(v_q))\ d\hat{x} \\ &= \int_S h_q(\theta_\delta\ e(v_q))\ d\hat{x} \\ &\geq \int_S h_q(\rho_\delta * (\theta_\delta e(v_q))\ d\hat{x} \\ &\geq \sum_{i\in I_\eta} \int_{\tilde{\omega}_i} h_q(\rho_\delta * (\theta_\delta e(v_q)))\ d\hat{x}. \end{aligned}$$

Thus, from Lemma 5.2.3, 1-homogeneity and Lipschitz property of h,

$$\begin{aligned} \liminf_{q\to 1} \int_S h_q(e(v_q))\ d\hat{x} &\geq \liminf_{q\to 1} \sum_{i\in I_\eta} \int_{\tilde{\omega}_i} h_q(\rho_\delta * (\theta_\delta e(v_q)))\ d\hat{x} \\ &\geq \sum_{i\in I_\eta} h\Big(\int_{\tilde{\omega}_i} \rho_\delta * (\theta_\delta e(v)) \Big) \\ &\geq \sum_{i\in I_\eta} |\omega_i| h\Big(\frac{1}{|\omega_i|} \int_{\omega_i} \rho_\delta * (\theta_\delta e(v)) \Big) \\ &\qquad - L|\rho_\delta * (\theta_\delta e(v))|\ (S \setminus \bigcup_{i\in I_\eta} \tilde{\omega}_i). \end{aligned}$$

The first term of the second member is a Riemann sum. Since, moreover, $\rho_\delta * \theta_\delta e(v)$ is a smooth function, by letting $\eta \to 0$, we obtain

$$\liminf_{q\to 1} \int_S h_q(e(v_q))\ d\hat{x} \geq \int_S h(\rho_\delta * \theta_\delta e(v)).$$

Noticing that $\rho_\delta * \theta_\delta e(v) \rightharpoonup e(v)$ weakly in $\mathcal{M}(S, \mathbf{M}_s^{2\times 2})$, estimate (5.2.3) is obtained by letting $\delta \to 0$. □

5.2.2 The upper bound for the $\Gamma-$convergence of $\mathcal{F}_q$

Proposition 5.2.4 (Upper bound). *For every $u \in W^{1,p}_{\Gamma_0}(\Omega, \mathbb{R}^3)$, there exists u_q weakly converging to u in $W^{1,p}_{\Gamma_0}(\Omega, \mathbb{R}^3)$ such that*

$$\limsup_{q\to 1} \mathcal{F}_q(u_q) \le \mathcal{F}_0(u),$$

or, equivalently, $\Gamma - \limsup \mathcal{F}_q \le \mathcal{F}_0$.

Proof. Consider the following subset $\mathcal{B}^1$ of $\mathcal{B}$:

$$\mathcal{B}^1 := \left\{ u \in W^{1,p}_{\Gamma_0}(\Omega, \mathbb{R}^3) : \gamma_S(u) \in \mathcal{C}^1(\bar{S}, \mathbb{R}^3) \right\}$$

and the functional $\tilde{\mathcal{F}}_0 : W^{1,p}_{\Gamma_0}(\Omega, \mathbb{R}^3) \to \mathbb{R}^+ \cup \{+\infty\}$ defined by

$$\tilde{\mathcal{F}}_0(u) = \begin{cases} \displaystyle\int_\Omega f(\nabla u)\ dx + \int_S h(e(\gamma_S(u)))\ d\hat{x} & \text{if } u \in \mathcal{B}^1, \\ +\infty & \text{otherwise.} \end{cases}$$

Take $u \in \mathcal{B}^1$ and set $v := \gamma_s(u)$. Let $(v_n)_{n\in\mathbb{N}^*}$ be a sequence of continuous piecewise affine functions satisfying $\|v_n - v\|_{W^{1,1}(S,\mathbb{R}^2)} \le 1/n$, and consider a sequence $(u_n)_{n\in\mathbb{N}}$ weakly converging to u in $W^{1,p}_{\Gamma_0}(\Omega, \mathbb{R}^3)$ satisfying

$$\begin{aligned} &\lim_{n\to+\infty} \int_\Omega f(\nabla u_n)\ dx = \int_\Omega f(\nabla u)\ dx; \\ &\gamma_S(\hat{u}_n) = v_n. \end{aligned}$$

Such a sequence exists from Step 1 of the proof of Proposition 4.2.5 in Chapter 4 and u_n belongs to $\mathcal{B}_q$. Writing $e(v_n) = \sum_{i\in I_n} a_{i,n} \mathbb{1}_{S_{i,n}}$ where $(S_{i,n})_{i\in I_n}$ is a finite partition of S, and $a_{i,n} \in \mathbf{M}_s^{2\times 2}$, the following estimate holds:

$$\begin{aligned} \lim_{q\to 1} \Big(\int_\Omega f(\nabla u_n)\ dx + \int_S h_q(e(v_n))\ d\hat{x} \Big) &= \int_\Omega f(\nabla u_n)\ dx + \lim_{q\to 1} \sum_{i\in I_n} h_q(a_{i,n})\ |S_{i,n}| \\ &= \int_\Omega f(\nabla u_n)\ dx + \sum_{i\in I_n} h(a_{i,n})\ |S_{i,n}| \\ &= \int_\Omega f(\nabla u_n)\ dx + \int_S h(e(v_n))\ d\hat{x}. \quad (5.2.4) \end{aligned}$$

Letting $n \to +\infty$, (5.2.4) yields

$$\lim_{n\to+\infty} \lim_{q\to 1} \Big(\int_\Omega f(\nabla u_n)\ dx + \int_S h_q(e(v_n))\ d\hat{x} \Big) = \int_\Omega f(\nabla u)\ dx + \int_S h(e(v))\ d\hat{x}.$$

Then, by using a standard diagonalization argument, there exists a map $q \mapsto n(q)$ such that

$$\lim_{q \to 1} \mathcal{F}_q(u_{n(q)}) = \tilde{\mathcal{F}}_0(u);$$
$$u_{n(q)} \rightharpoonup u,$$

which implies

$$\inf \Big\{ \limsup_{q \to 1} \mathcal{F}_q(u_q) : u_q \rightharpoonup u \text{ in } W^{1,p}_{\Gamma_0}(\Omega, \mathbb{R}^3) \Big\} \leq \tilde{\mathcal{F}}_0(u)$$

for all $u \in W^{1,p}_{\Gamma_0}(\Omega, \mathbb{R}^3)$. Thus $\Gamma - \limsup \mathcal{F}_q \leq \tilde{\mathcal{F}}_0$. The conclusion of Proposition 5.2.4 follows by taking the lower semi-continuous envelope of each two functionals for the weak topology of $W^{1,p}_{\Gamma_0}(\Omega, \mathbb{R}^3)$ and by using Lemma 3.2.3. □

Corollary 5.2.5. *Assume that h_q satisfies the additional coerciveness condition: there exists $\alpha_q > 0$ such that $\alpha_q |\xi|^q \leq h_q(\xi)$ for all $\xi \in \mathbf{M}^{2\times 2}_s$. Then*

- *(i) The problem* $\min \big\{ \mathcal{F}_q(u) - L(u) \big\}$ *possesses at least a solution $\bar{u}_q$;*
- *(ii) There exists a subsequence of $(\bar{u}_q)_{q \in (1,p)}$ and $\bar{u}$ solution of the problem* $\min \big\{ \mathcal{F}_0(u) - L(u) \big\}$ *such that $\bar{u}_q \rightharpoonup \bar{u}$ in $W^{1,p}_{\Gamma_0}(\Omega, \mathbb{R}^3)$.*

Proof. The first assertion is obtained by using the direct method in the calculus of variations. The second assertion is a straightforward consequence of variational properties of the Γ-convergence. □

Bibliography

[1]. E. Acerbi and G. Buttazzo and D. Percivale. *Thin inclusions in linear elasticity: A variational approach.* J. Reine Angew. Math. **386** (1988), 99-115.

[2]. O. Anza Hafsa, J. P. Mandallena. *Interchange of infimum and integral.* Calc. Var. **18** (2003), 433-449.

[3]. L. Ambrosio, A. Coscia, G. Dal Maso. *Fine Properties of Functions in BD.* Arch. Rational Mech. Anal. **139** (1997), 201-238.

[4]. L. Ambrosio, N. Fusco, D. Pallara. *Functions of Bounded Variation and Free Discontinuity Problems.* Oxford Mathematical Monographs, Oxford University Press, New York, 2000.

[5]. H. Attouch. *Variational Convergence for Functions and Operators.* Applicable Mathematics Series. Pitman Advanced Publishing Program, 1985.

[6]. H. Attouch, G. Buttazzo, G. Michaille. *Variational Analysis in Sobolev and BV Space: Application to PDEs and Optimization.* MPS-SIAM Book Series on Optimization, 2005.

[7]. A. Laure Bessoud, F. Krasucki, G. Michaille. *Variational convergence of energy functionals for elastic materials with* ε *-thin strong inclusions growing as* $p(\varepsilon)$, $p(\varepsilon) \to 1^+$. In progress.

[8]. A. Laure Bessoud, F. Krasucki, G. Michaille. *Multi-material with strong interface: Variational modelings.* Asymptot. Anal.**61**(1) (2009), 1-19.

[9]. A. C. Barroso, I. Fonseca, R. Toader. *A Relaxation Theorem in the Space of Functions of Bounded Deformation.* Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), **29**(2000), 19-49.

[10]. M. Chipot, D. Kinderlehrer. *Equilibrium Configurations of Crystals.* Arch. Rat. Mech. Anal. **103** (1988), 237-277.

[11]. G. Dal Maso. *An Introduction to* Γ*-Convergence.* Birkäuser, Boston, 1993.

[12]. F. Demengel, R. Temam. *Convex functions of measures and applications.* Indiana Univ. Math. J. **33** (1984), 673-709.

[13]. R.V. Kohn. *The relaxation of a double-well energy.* Contin. Mech. Thermodyn. **3** (1991), 193-236.

[14]. H. Le Dret, A. Raoult. *The nonlinear membrane model as variational limit in nonlinear three-dimensional elasticity.* J. Math. Pures Appl., (9) **74** (1995), no 6, 549-578.

[15]. C. Licht, G. Michaille. *A modelling of elastic adhesive bonding joints.* Adv. Math. Sci. Appl., **7**, no. 2 (1997), 711-740.

[16]. C. Licht, G. Michaille. *A nonlocal energy functional in pseudo-plasticity.* Asymptot. Anal. **45** (2005), 313-339.

[17]. R.Temam. *Problèmes Mathématiques en Plasticité.* Gauthier-Villars, Paris 1983.

Bibliography

[1]. C. Licht, G. Michaille. *A modelling of elastic adhesive bonded joints.* Advances in Mathematical Sciences and Applications, **7** (1997), no 2, 711-740.

[2]. C. Licht, G. Michaille, S. Pagano. *A model of elastic adhesive bonded joints through oscillation-concentration measures.* J. Math. Pures Appl. (9) 87 (2007), no. 4, 343–365.

[3]. G. Bayada, M. Chambat, K. Lhalouani. *Asymtotic analysis of a thin layer device with Tresca's contact law in elasticity.* Math. Meth. Appl. Sci, **22**(1999), 811-836.

[4]. Zaittouni F. *Ph.D. thesis University Montpellier 2, 2000.*

[5]. Aitmoussa A. *Modélisation et étude des singularités de contraintes d'un joint collé très mince.* Ph.D. Thesis, University Montpellier 2, France, 1989.

[6]. H. Attouch, G. Buttazzo, G. Michaille. *Variational analysis in Sobolev and BV space: application to PDEs and Optimization.* MPS-SIAM Book Series on Optimization, 2005.

[7]. P. Rabier, J. A. Martins, J. T. Oden, L. Campos. *Existence and local uniqueness of solutions to contact problems in elasticity with non linear friction laws.* Int. J. Engng. Sci. **24**(1986), 1755-1768.

[8]. M. Raous. *Friction and Instabilities. (Lectures from the Advanced Summer School on Friction and Instabilities held in Udine, July 3-7, 2000)/ed.* Martins J. A.C., Raous M., Springer-Verlag, 2002.

[9]. Tartar L. *An introduction to Sobolev spaces and Interpolation spaces.* Lecture Notes of the Unione of Mathematica Italiana, vol**3**., Springer, 2007.

[10]. Dacorogna B. *Direct methods in the Calculus of Variations.* Applied Mathematical Sciences, no.**78**., Springer-Verlag, Berlin, 1989.

[11]. Ball J., Massden J.E. *Quasi convexity at the boundary, Positivity of the Second Variation and Elastic Stability.* Arch. Rational Mech. Anal. 86, 1984, page 251-277.

[12]. Godet M. *Third bodies in Tribology.* Wear, vol.136, no.**1**(1990), 29-45.

[13]. C. Licht *Comportement asymtotique d'une bande dissipative mince de faille rigidité.* C.R. Acad. Sci. Paris, t. 317, Série I, 1993, page 429-433.

[14]. H. Attouch. *Variational Convergence for Functions and Operators.* Applicable Mathematics Series. Pitman Advanced Publishing Program, 1985.

[15]. D. Caillerie. *The Effect of a Thin Inclusion of High Rigidity in an Elastic Body.* Math. Meth. in the Appl. Sci, **2** (1980), 251-270.

[16]. G. Croce, C. Lacour, G. Michaille. *A characterization of gradient Young-concentration measures generated by solutions of Dirichlet-type problems with large sources* ESAIM: COCV **15** (2009), 818-838.

[17]. H. Federer. *Geometric Measure Theory.* Classic in Mathematics, Springer-Verlag 1969.

[18]. I. Fonseca, S. Müller, P. Pedregal. *Analysis of concentration and oscillation effects generated by gradients.* Siam J. Math. Anal. **29** (1998), no 3, 736-756.

[19]. G. Geymonat, F. Krasucki, S. Lenci. *Mathematical analysis of a bounded joint with a soft thin adhesive.* Mathematics and Mechanics of Solids, **4** (1999), 201-225.

[20]. D. Kinderlehrer, P. Pedregal. *Characterization of Young measures generated by gradients.* Arch. Rational Mech. Anal. **119** (1991), 329-365.

[21]. E. Acerbi and G. Buttazzo and D. Percivale. *Thin inclusions in linear elasticity: A variational approach.* J. Reine Angew. Math. **386** (1988), 99-115.

[22]. O. Anza Hafsa, J. P. Mandallena. *Interchange of infimum and integral.* Calc. Var. **18** (2003), 433-449.

[23]. L. Ambrosio, A. Coscia, G. Dal Maso. *Fine Properties of Functions in BD.* Arch. Rational Mech. Anal. **139** (1997), 201-238.

[24]. L. Ambrosio, N. Fusco, D. Pallara. *Functions of Bounded Variation and Free Discontinuity Problems.* Oxford Mathematical Monographs, Oxford University Press, New York, 2000.

[25]. A. Laure Bessoud, F. Krasucki, G. Michaille. *Variational convergence of energy functionals for elastic materials with ε -thin strong inclusions growing as $p(\varepsilon)$, $p(\varepsilon) \to 1^+$.* In progress.

[26]. A. Laure Bessoud, F. Krasucki, G. Michaille. *Multi-material with strong interface: Variational modelings.* Asymptot. Anal.**61**(1) (2009), 1-19.

[27]. A. C. Barroso, I. Fonseca, R. Toader. *A Relaxation Theorem in the Space of Functions of Bounded Deformation.* Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), **29**(2000), 19-49.

[28]. M. Chipot, D. Kinderlehrer. *Equilibrium Configurations of Crystals.* Arch. Rat. Mech. Anal. **103** (1988), 237-277.

[29]. G. Dal Maso. *An Introduction to Γ-Convergence.* Birkäuser, Boston, 1993.

[30]. F. Demengel, R. Temam. *Convex functions of measures and applications.* Indiana Univ. Math. J. **33** (1984), 673-709.

[31]. R.V. Kohn. *The relaxation of a double-well energy.* Contin. Mech. Thermodyn. **3** (1991), 193-236.

[32]. H. Le Dret, A. Raoult. *The nonlinear membrane model as variational limit in nonlinear three-dimensional elasticity.* J. Math. Pures Appl., (9) **74** (1995), no 6, 549-578.

[33]. C. Licht, G. Michaille. *A nonlocal energy functional in pseudo-plasticity.* Asymptot. Anal. **45** (2005), 313-339.

[34]. R.Temam. *Problèmes Mathématiques en Plasticité.* Gauthier-Villars, Paris 1983.

Résumé : Cette thèse de doctorat est consacrée à l'étude mathématique de deux types de problèmes de jonction. Le premier modèle est obtenu comme limite variationnelle d'un problème d'élasticité avec jonction de faible épaisseur, l'adhésif occupant la jonction a une rigidité de l'ordre de l'épaisseur. La densité élastique W de l'adhésif est complétée par une densité d'énergie surfacique convexe h non nécessairement régulière. Cette densité traduit une contrainte mécanique entre adhérents et l'adhésif imparfaitement collés. On montre que le modèle limite consiste à remplacer la jonction par une contrainte qui est l'inf-convolution de h et de la densité surfacique limite de W. Dans un cadre scalaire on effectue l'analyse des concentrations de gradient à l'interface au moyen d'outils récents issus de la théorie de la mesure. Dans le second modèle, la rigidité de l'adhésif est de l'ordre inverse de l'épaisseur de le jonction, la densité élastique du matériaux adhérent a une croissance super-linéaire alors que celle de l'adhésif croit linéairement. Suivant la stratégie utilisée pour le premier problème on propose un modèle simplifié mais fiable comme limite variationnelle lorsque l'épaisseur tend vers zéro. A la limite la fine couche intermédiaire est remplacée par une interface pseudo-plastique prédisant la formation de fissures.

Titre : Quelques modèles mathématiques de jonctions.

Mots-clés : Gamma convergence, analyse asymptotique, jonctions.

Abstract : This thesis is devoted to the mathematical study of two kinds of junction problems. The first model is obtained as the variational limit of a junction problem in the scope of elasticity where the adhesive occupying the junction possesses a stiffness of the order of the thickness. One adds a smooth or non smooth convex surface energy density h to the elastic density W of the adhesive. This additional density conveys the fact that the joint and the adherents are not perfectly stuck together. We show that the asymptotic model consists in replacing the joint by a constraint which now is the inf-convolution of h and the limit density of W. In the scalar case we analyse the gradient concentration phenomenon at the interface by means of recent tools stemming from measure theory. In a second model, the stiffness of the adhesive is of the order of the inverse of the thickness of the junction. The bulk energy density of the adherents grows superlinearly while that of the adhesive grows linearly. Following the strategy used in the first problem, we propose a simplified but accurate model by studying the asymptotic behavior when the thickness goes to zero through a variational convergence method. At the limit the intermediate layer is replaced by a pseudo-plastic interface which allows cracks to appear.

Title : Some mathematical models of junctions.

Key words : Gamma convergence, asymptotic analysis, junctions.

Discipline : Mathématiques et modèlisation.

Laboratory : I3M UMR-CNRS 5149 and LMGC UMR-CNRS 5508, Université Montpellier 2.