On the Cremona group and it's subgroups

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Abstract

Ce travail peut être-divisé en trois partie:

1. Théorie des groupes. Il s'agit ici d'une étude de la structure du groupe T de Thompson. On explique la notion de la mutation linéaire par morceaux et on obtient la nouvelle présentation de ce groupe en termes des génerateurs et relations.

2. Géometrie birationnelle. On étudie en détail le groupe de Cremona qui est un groupe des automorphismes birationnels du plan projectif. En particulier on s'interesse à son sous-groupe $Symp$ des elements qui préserve le crochet de Poisson dit logarithmique, aussi bien qu'à un sous-groupe H engendré par $SL(2, \mathbf{Z})$ et par les mutations. On construit des limites projectives des surfaces sur lesquelles ces groupes agissent régulièrement, et on en déduit les répresentations linéaires de ces groupes dans les limites inductives des groupes de Picard des surfaces.

3. Algèbre homologique. A partir d'une variété algébrique on construit une catégorie triangulée qui ne dépend que de sa classe birationnelle. En utilisant la technique de quotient de dg-catégories, on calcule explicitement cette catégorie pour les surfaces rationnelles. Comme consequence on obtient l'action du groupe de Cremona sur une algébre non-commutative par les automorphismes extérieures.

On donne les applications de ces résultats aux formules des mutations des variables non-commutatives.

Abstract

The thesis consists of three parts:

1. Group theory. Here we emphasize the role of the mutation, which is some element of the Thompson group T. In particular using mutations we get a new presentation of this group in terms of generators and relations.

2. Birational geometry. We study the action of the Cremona group and some of it's subgroups on the projective plane. In particular we are interested in the subgroup $Symp$ of the Cremona group, that preserve the logarithmic Poisson bracket, and in it's subgroup H generated by cluster mutations and by $SL(2, \mathbf{Z})$. We construct a projective system of surfaces, on which this groups act by regular automorphisms, and then we deduce a linear presentation of H in the inductive limit of Picard groups of rational surfaces.

3. Homological algebra. For an algebraic variety we construct a triangulated category which depends only on it's birational class. Using the techniques of the quotients of dg-categories we compute such a triangulated category for a rational surface. As a consequence we obtain an action of the Cremona group on the non-commutative ring by outer automorphisms.

We give applications of this results to the formulas of non-commutative cluster mutations.

Résumé détaillé.

Le groupe de Cremona Cr est le groupe des automorphismes du corps commutatif $K = \mathbf{C}(x, y)$ de deux variables indépendants. C'est aussi un groupe des automorphismes birationnels de $\mathbf{CP}^2$, parce K est le corps des fonctions rationnelles sur $\mathbf{CP}^2$.

On applique à l'étude de ce groupe trois approches différentes:

1. Théorie des groupes, qui décrit les structures du groupe de Cremona et de ses sous-groupes.

2. Géometrie algébrique, où l'on se concentre sur l'action de Cr sur $\mathbf{CP}^2$ par les automorphismes birationnels.

3. Algébrique, où l'on étend l'action de Cr à un algébre non-commutative.

La motivation pour ce travail vient de la quantification par déformation ([12],[13],[16]), des prédictions conjecturales de la symétrie miroir ([14],[11],[10],[15]), et de la théorie des algèbres amassées ([23],[25],[24],[19],[20]).

La mutation basique des algèbres amassées est un automorphisme du corps K donné par une formule suivante:

$$P : (x, y) \to (y, \frac{1 + y}{x})$$

Cet automorphisme préserve une structure de Poisson logarithmique donnée par le crochet $\{x, y\} = xy$. Pour cette raison, on s'intéresse au groupe $Symp \subset Cr$ des automorphismes de K qui préserve ce crochet. On étudie aussi son sous-groupe $H \subset Symp$, engendré par un groupe $SL(2, \mathbf{Z})$ et par P.

Le groupe $Symp$ admet un morphisme surjectif dans un groupe T de Thompson. Dans la section (2.1) on donne les définitions équivalentes du groupe T et on dérive sa présentation en termes de générateurs et de relations. On insiste ici sur le rôle des mutations et sur le fait, qu'en prenant les mutations comme les générateurs, la présentation du groupe T est particulièrement simple. Concrètement dans le théorème(3.3.2) on prouve le résultat suivant:

Théorème 0.0.1. *Le groupe T de Thompson est engendré par $SL(2, \mathbf{Z})$ et*

par une mutation μ avec les relations suivantes:

$$T = <SL(2, \mathbf{Z}), \mu|(I^2\mu)^2 = U^{-1}, (I^{-1}\mu)^5 = (I\mu)^7 = 1>$$

Ici U, I sont les matrices comme suit: $I = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $U = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

Le rôle d'une mutation est plus explicite ici que dans l'article([21]) où elle est appellé 'half-transvection'.

La partie géométrique de ce travail concerne la géométrie des automorphismes du plan projectif qui sont les éléments des groupes $Symp$ et H. Dans la lemme(3.2.5) on identifie les points de $\mathbf{CP}^2$ qu'il faut éclater pour que ces automorphismes agissent de manière régulière.

En particulier, on obtient des systèmes projectifs des surfaces X_{Symp} et X_H sur lesquelles $Symp$ et H agissent régulièrement. Alors les limites inductives et les limites projectives des groupes de Picard de ces surfaces sont les groupes abéliens bien définies, et donc on obtient les représentations linéaires de $Symp$ et de H dans ces groupes.

On peut définir une mutation μ comme un automorphisme d'un groupe abélien libre $W = \oplus_{\alpha \in \mathbf{Z}^2 \backslash (0,0)} \mathbf{Z} p_\alpha$ par les formules suivantes :

$$p_{(x,y)} \mapsto p_{(x,y)} \text{ si } x > 0$$

$$p_{(x,y)} \mapsto p_{(x,y-x)} - x p_{(-1,0)} \text{ si } x < 0$$

$$p_{(0,y)} \mapsto p_{(0,y-1)} - p_{(0,-1)}$$

Si V_1 est un noyau d'une application linéaire $W \to \mathbf{Z}^2$ donnée par $p_\alpha \mapsto \alpha$, alors dans le théorème(3.3.1) on prouve:

Théorème 0.0.2. *V_1 est isomorphe à un sous-groupe de $Pic(X_H)$, tel que l'action induite de $I^*P^* \in H$ sur V_1 coïncide avec celle de la mutation μ.*

La partie algébrique de ce travail concerne le groupe de Cremona Cr tout entier. La structure de ce groupe en termes de générateurs et relations était décrite par Iskovskih dans l'article([9]). Il est aussi intéressant de comprendre la structure de ses sous-groupes qui préservent les structures des Poisson différents([5]). On s'attend à ce que si le groupe G préserve le crochet de Poisson, alors le groupe $\widetilde{G}$ des symétries d'une surface quantifiée est une extension de G par les conjugaisons. De ce fait, on prouve un résultat plus fort(4.3.1):

Théorème 0.0.3. *Il existe un groupe Cr^{nc} qui agit sur l'algèbre non-commutative A par les automorphismes. Une application de l'algèbre $A \to K$ induit une surjection des groupes $Cr^{nc} \to Cr$, et le noyau de cette surjection est composé des conjugaisons par les éléments de A.*

En particulier, le groupe Cr s'injecte dans le groupe des automorphismes extérieurs de A. Le corps des fonctions de la surface quantifiée peut être vu comme un quotient de A par un certain idéal. Cette approche d'un problème de la quantification explicite des surfaces rationnelles est un peu différente de celle abordée dans l'article([18]).

L'algèbre A est une algèbre non-commutative associative qui se construit, par une localisation consécutive, à partir d'une algèbre libre à deux générateurs. Sur chaque pas, on localise dans l'ensemble des éléments qui n'appartiennent pas à l'idéal engendré par les commutateurs. On donne deux preuves du fait que le groupe Cr agit sur A par les automorphismes extérieurs.

La première preuve est une construction directe. En fait, on définit explicitement les analogues non-commutatifs des générateurs de groupe de Cremona et ensuite, on vérifie que dans le cas non-commutatif les relations entre les générateurs ont lieu aux automorphismes intérieurs près.

La deuxième preuve passe par les quotients de dg-catégories comme les articles([8],[4],[1]) le développent. A une variété algébrique lisse X on associe une catégorie triangulée $\widetilde{C}(X)$, qui est un invariant birationnel de X. Plus précisément, si X est Y sont birationnellement isomorphes, alors $\widetilde{C}(X)$ et $\widetilde{C}(Y)$ sont équivalents comme les catégories triangulées. Prenons la catégorie dérivée bornée $D^b(X)$ d'une catégorie des faisceaux cohérents sur X([3],[2]). On considère la sous-catégorie pleine $\widetilde{D}(X) \subset D^b(X)$ qui est constituée par les objets orthogonaux à gauche à un faisceau de structure O_X. La catégorie $\widetilde{C}(X)$ est le quotient de $\widetilde{D}(X)$ par la sous-catégorie pleine des complexes des faisceaux qui ont la codimension de support au moins égale à 1.

Ensuite, on étudie en détail la structure d'une catégorie $\widetilde{C}(\mathbf{CP^2})$ et on prouve qu'elle est engendrée par un objet P qui est une image d'un faisceau $O(1)$. On vérifie que cet objet est préservé par les équivalences, et donc en conséquence le groupe de Cremona agit par les automorphismes extérieures sur un anneau d'endomorphismes de cet objet. On calcule que $RHom^0_{\widetilde{C}(\mathbf{CP^2})}(P,P) = A$ et on vérifie que l'action de Cr sur A ainsi obtenue coïncide avec celle donnée par les formules explicites(4.3.1).

Pour finir, on va donner des applications des résultats précédents aux formules des mutations des variables non-commutatives. Pour un entier positif

k on définit une mutation comme un automorphisme d'une algèbre A, donnée par la formule suivante:

$$\mu_k : (x, y) \mapsto (x^{-1}yx, x^{-1}(1 + y^k))$$

Dans la théorème(5.3.1) on montre que:

Théorème 0.0.4. *Le phénomène de Laurent non-commutatif a lieu pour $k = 2$.*

Autrement dit, $\mu_2^n(x)$ et $\mu_2^n(y)$ appartiennent à l'anneau

$$L = \mathbf{C} < x, x^{-1}, y, y^{-1} > \subset A$$

On l'appelle l'anneau des polynômes de Laurent non-commutatifs. Pour prouver ce résultat on étudie un autre automorphisme de A:

$$T : (X, Y) \mapsto (Y, (Y + Y^{-1})X^{-1}Y)$$

Soit M_n est une variété des pairs de matrices (X, Y) de la taille n. L'application T définit une système dynamique sur M_n. Grâce aux théorèmes(5.2.1,5.2.2) on prouve le résultat suivant:

Théorème 0.0.5. *Le système (M_n, T) est birationnellement equivalent à une système (S_n, P) avec les propriétés suivants:*
1. Il existe une projection $\pi : S_n \to B_n$.
2. B_n est une variété de la dimension $2n^2 - n$.
3. La fibre generique de π est un tore algébrique de dimension n.
4. P preserves les fibres de π.
5. L'action de P sur la fibre est donnée par une multiplication par un certain élément de ce tore.

Une autre application est une construction d'un morphisme des groupes $H^{nc} \to H$, tel que H^{nc} agit sur A en préservant $q = x^{-1}y^{-1}xy$. Comme conséquence du théorème(4.3.1) on prouve le lemme suivant(5.1.1):

Lemma 0.0.1. *Le noyau d'un morphisme $H^{nc} \to H$ est composé des conjugaisons par les éléments de $\mathbf{C}_1(q) \subset \mathbf{C}(q)$. En particulier, H^{nc} est une extension de H par un groupe abélien.*

Ici $\mathbf{C}_1(q) \subset \mathbf{C}(q)$ est un sous-anneau d'une corps des fonctions rationnelles en une variable q, qui sont bien définies en $q = 1$.
On vérifie aussi dans le lemme(5.1.2) que les relations qu'on connait dans H sont aussi vraies dans H^{nc} à conjugaison par q près.

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Chapter 1

Introduction

The Cremona group Cr is the group of automorphisms of the field $K = \mathbf{C}(x, y)$ of two independent commuting variables. Alternatively it is the group of birational automorphisms of $\mathbf{CP}^2$ or any other rational surface, because K is it's field of rational functions.

We study this group and it's subgroups from three different perspectives:

1. Group-theoretic, which deals with the structure of the Cremona group and it's various subgroups.

2. Algebraic geometry, where we concentrate on the action of Cr and it's subgroups by birational automorphisms of $\mathbf{CP}^2$.

3. Algebraic, where we extend an action of the Cr on $\mathbf{C}(x, y)$ to the action on the non-commutative algebra A.

The motivation for this questions is coming from the deformation quantization, as stated in ([12],[13],[16]), from the conjectural picture of the mirror symmetry ([14],[11],[10],[15]), and from cluster algebras([23],[25],[24],[19],[20]). The most simple mutation of a cluster algebra is the following automorphism of K:

$$P : (x, y) \rightarrow (y, \frac{1+y}{x})$$

It preserves the logarithmic Poisson structure given by the bracket $\{x, y\} = xy$. This motivates our interest in studying the group $Symp \subset Cr$ of automorphisms of K preserving this bracket. As well we are interested in it's subgroup $H \subset Symp$ which is generated by $SL(2, \mathbf{Z})$ and P.

The group $Symp$ admits a group morphism to a Thompson group T. In the section (2.1) we give equivalent definitions of this group and we derive it's alternative presentation in terms of generators and relations. We emphasize that our generators are the elements of $SL(2, \mathbf{Z})$ and mutations. So in terms of mutations, the presentation of the Thompson group T looks particularly simple. Concretely in the theorem (3.3.2) we prove the following result:

Theorem 1.0.1. *The Thompson group T has the following presentation in generators and relations:*

$$T = < SL(2, \mathbf{Z}), \mu | (I^2 \mu)^2 = U^{-1}, (I^{-1}\mu)^5 = (I\mu)^7 = 1 >$$

Here U, I are the following matrices: $I = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $U = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

The role of the mutation μ becomes more explicit than in ([21]), where the mutation is called half-transvection.

In the geometric part of this thesis we examine the geometry of automorphisms in $Symp$ and in H and specify in (3.2.5) the loci of points where we

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have to blow-up a surface $\mathbf{CP}^2$, so that the automorphism becomes regular. In particular, we get projective systems of surfaces X_{Symp} and X_H on which $Symp$ and H act regularly. It is not objects of the category of schemes, but still it makes sense to speak about their Picard groups, which are well-defined abelian groups. For a projective system of surfaces X_{Symp} we may consider inductive or projective limit of Picard groups of the surfaces in the system. Then $Symp$ will act by automorphisms on the both limits.

In particular we define a mutation μ on the free abelian group $W = \oplus_{\alpha \in \mathbf{Z}^2 \setminus (0,0)} \mathbf{Z} p_\alpha$ by the following formulas:

$$p_{(x,y)} \mapsto p_{(x,y)} \text{ if } x > 0$$

$$p_{(x,y)} \mapsto p_{(x,y-x)} - x p_{(-1,0)} \text{ if } x < 0$$

$$p_{(0,y)} \mapsto p_{(0,y-1)} - p_{(0,-1)}$$

Let V_1 be a kernel of the map $W \to \mathbf{Z}^2$ given by $p_\alpha \mapsto \alpha$, then in the theorem(3.3.1) we prove:

Theorem 1.0.2. *V_1 is isomorphic to a subgroup of $Pic(X_H)$, such that the induced action of $I^* P^* \in H$ on V_1 coincides with the restriction of the action of μ.*

This linear representation is very similar to the representation of the mapping class group of some cluster algebra, for which all the seeds are the same and the cluster variables of a seed are parameterized by $\mathbf{Z}^2 \setminus (0,0)$([23]).

The algebraic part of the thesis deals with the whole Cremona group Cr. The structure of the Cremona group Cr was described by Iskovskih in ([9]) in terms of generators and relations. It is also interesting to understand the presentation of it's subgroups, that preserve various Poisson brackets([5]). The expectation is that if the Poisson bracket have a group of symmetries G, than the corresponding quantized surface would have a group of symmetries $\widetilde{G}$, which is an extension of G by conjugation. Actually we prove a stronger theorem(4.3.1):

Theorem 1.0.3. *There exists a group Cr^{nc} which acts by automorphisms of a non-commutative algebra A. A map $A \to K$ induces a surjection $Cr^{nc} \to Cr$, and a kernel of this surjection consists of inner conjugations of A.*

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In particular Cr embeds in the group of outer automorphisms of A and the field of rational functions on the quantized surfaces may be seen as quotients of the algebra A by some ideals. This approach to the explicit quantization of rational surfaces is a bit different from the one considered in ([18]).

A non-commutative associative algebra A is constructed by consecutive localization of a free non-commutative algebra of two generators at the set of elements outside a commutator ideal. We give two proofs that the Cremona group acts on this non-commutative algebra by outer automorphisms.

First proof of this statement is a direct computation: we provide non-commutative analogs of generators of Cremona group and verify that the relations known in commutative setting also hold in a non-commutative up to inner automorphisms.

The second proof uses the quotients of dg-categories as developed in([8],[4],[1]). Namely for a smooth proper algebraic variety X we construct a triangulated category $\widetilde{C}(X)$, which is a birational invariant of X. By this we mean that if X and Y are birationally equivalent, then $\widetilde{C}(X)$ and $\widetilde{C}(Y)$ are equivalent as triangulated categories. The category $\widetilde{C}(X)$ is a quotient of the bounded derived category of coherent sheaves([3],[2]) that are left orthogonal to the structure sheaf O_X by the full subcategory of complexes of sheaves with the support of codimension at least 1.

Then we study the structure of $\widetilde{C}(\mathbf{CP^2})$ and prove that it is generated by one object P which is the image of $O(1)$. We also check that this object is preserved under equivalences, so the Cremona group will act by outer automorphisms on the ring of endomorphisms of this object. Then we compute that $RHom^0_{\widetilde{C}(\mathbf{CP^2})}(P, P) = A$, so we get an action of the Cremona group on the same non-commutative algebra as previous. Then we verify that two actions coincide.

At the end we give some applications of the previous results to non-commutative cluster mutations. Namely for a positive integer k we define a mutation to be the following automorphism of A:

$$\mu_k : (x, y) \mapsto (x^{-1}yx, x^{-1}(1 + y^k))$$

In the theorem(5.3.1) we prove:

Theorem 1.0.4. *The non-commutative Laurent phenomenon holds for $k = 2$.*

By this we mean that $\mu_2^n(x)$ and $\mu_2^n(y)$ belong to the sub-ring $L \subset A$ of elements that involve the division only of x and y. To prove this result study

the automorphism of A given by the formula:

$$T : (X, Y) \mapsto (Y, (Y + Y^{-1})X^{-1}Y)$$

It happens that if we consider the space of pairs of $n \times n$ matrices (X, Y), then this map T defines an integrable dynamical system. In the theorems(5.2.1,5.2.2) we prove that this dynamical system is birationally equivalent to the dynamical system (S_n, P) and it's properties are described in the following theorem:

Theorem 1.0.5. *There is a projection map $\pi : S_n \to B_n$, where B_n is a variety of dimension $2n^2 - n$ and the generic fiber of π is a n-dimensional algebraic torus. The map P preserves fibers of π and induces an action on the fiber which is a multiplication by some element of this torus.*

We construct a group morphism $H^{nc} \to H$, where H^{nc} acts on A preserving $q = x^{-1}y^{-1}xy$. As the consequence of the theorem(4.3.1) we prove the following lemma(5.1.1):

Lemma 1.0.2. *The kernel of the group morphism $H^{nc} \to H$ consists of conjugations by the elements of $\mathbf{C}_1(q)$. In particular H^{nc} is an extension of H by an abelian group.*

Here $\mathbf{C}_1(q) \subset \mathbf{C}(q)$ is a sub-ring of rational functions on q, which are well defined at $q = 1$.
It is curious to verify that the known relations in H hold in H^{nc} up to conjugation by q. We do this in lemma(5.1.2).

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Chapter 2

Thompson group T

2.1 Definitions of the Thompson group T

Here we gather some information about group T. The classical reference is [6].

This group may be interpreted in different ways, as:
- piecewise linear automorphisms of $\mathbf{Z}^2$
- piecewise linear dyadic homeomorphisms of circle
- pairs of binary trees with cyclic bijection between their leaves
- pairs of regular trivalent trees with fixed oriented edge and cyclic bijection between their leaves
- piecewise projective automorphisms of circle

Here we give five definitions of T, sketching the way to pass from one definition to another.

Let us say that the set $\{s_1, \ldots, s_n\} \subset S$ is cyclically ordered if an hour hand attached to $(0,0)$ will meet them in this order: $s_1, s_2, s_3, \ldots, s_n, s_1, s_2 \ldots$. They give a decomposition of $\mathbf{Z}^2$ into cones $\mathbf{Z}_{\geq 0} s_i + \mathbf{Z}_{\geq 0} s_{i+1}$. We require that $s_{i+1} \wedge s_i = 1$ and consider the bijections of S preserving cyclic order f, such that $a s_i + b s_{i+1}$ is mapped to $a f(s_i) + b f(s_{i+1})$, for a, b - non-negative. We call such bijection a piecewise linear automorphism of $\mathbf{Z}^2$.

Definition 1. The Thompson group T is a group of piecewise linear automorphisms of $\mathbf{Z}^2$.

Dyadic number is a number of the form $\frac{p}{2^q}$, $p, q \in \mathbf{Z}$. Circle is an interval $[0,1]$ with $0 = 1$. Dyadic automorphism of circle would be a piecewise-linear automorphism, that sends dyadic numbers to dyadic and whose derivatives on the intervals of linearity are 2^k, where k - integer numbers.

Definition 2. The Thompson group T is a group of dyadic automorphisms of a circle.

Actually if we put $S^1 \ni 0 = 1 \mapsto (1,0) \in S$, $\frac{1}{2} \mapsto (0,1)$, $\frac{3}{4} \mapsto (-1,-1)$, then we can extend this into bijection between dyadic points and S preserving cyclic ordering. Then it is possible to check that the automorphisms of the set of dyadic points, coming from dyadic automorphisms, and automorphisms of S, coming from PL automorphisms of $\mathbf{Z}^2$, coincide.

Binary tree has a root, and every vertex has either no or two descendants

- left and right. Vertices with no descendants are called leaves, they are naturally cyclically ordered. Such a tree gives a decomposition of the interval: we attach $[0,1]$ to the root. If vertex has an interval $[p,q]$ attached, we associate $[p,\frac{p+q}{2}]$ to it's left descendant and $[\frac{p+q}{2},q]$ to it's right descendant. Clearly enough all such intervals are of length $\frac{1}{2^k}$ and intervals attached to leaves give a decomposition of $[0,1]$.

Definition 3. The Thompson group T has elements represented by equivalence classes of pairs of binary trees (R,U) with the equal number of leaves, and a bijection between the leaves of R and the leaves of U, preserving the cyclic order. Attaching simultaneously two descendants to the leaf on R and to the corresponding by the bijection leaf of U gives an equivalent pair of trees, so it would give the same element of T. A composition of elements (R,S) and (M,N) is the following: find a binary tree Z, which contains all the vertices of S and M, then find representatives of the equivalence classes of (R,S) and (M,N), which look like (X,Z) and (Z,Y), by adding descendants to appropriate leaves, completing S and M to Z. The composition of (X,Z) and (Z,Y) is (X,Y), with the bijection on leaves induced.
If we attach intervals to a tree (R,U) as explained before, then mapping intervals of leaves of R to intervals of corresponding leaves of U gives a dyadic automorphism of a circle.

Looking at the binary tree as at the graph, let us remove a root and it's two adjacent edges and join root's left and right descendant by an oriented edge going from left to right. All other edges stay unoriented.
Let Υ be a set of connected planar trees with vertices of valence 3 or 1 and one oriented edge (vertices of valence one are called leaves and they are cyclically ordered). Consider the set Θ of pairs (A,B), $A,B \in \Upsilon$ with a bijective identification of leaves, preserving the cyclic order. Put an equivalence relation on Θ generated by the following: if we add two leaves to the leaf of A and two leaves to the corresponding leaf of B, the new pair of trees with naturally induced bijection of leaves is equivalent to (A,B).

Definition 4. The set of equivalence classes of Θ under this equivalence relation and composition given on representatives $(A,B) \cdot (B,C) \mapsto (A,C)$ has a well-defined structure of group, and is called the Thompson group T.

Given an element $A \in \Upsilon$, we will attach to it a union of triangles in the unit disc $D = \{|z| \leq 1\}$. So to every vertex of A we attach a triangle, and for any two vertices joined by an edge the corresponding triangles have

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a common edge. The edges of all such triangles will be geodesic and the vertices will be rational in the upper half-plane model of a disc.

To the vertex from which the oriented edge starts we associate the triangle with vertices $-i, i, -1$, to the vertex where the oriented edge goes we associate the triangle with vertices $-i, 1, i$. Next, suppose that we associate the triangle abc with a vertex, and we want to construct a triangle adjacent to the edge bc. We just need to choose a unique hyperbolic automorphism of the disc, that sends $-1, -i, i$ to a, b, c in this order, then c, b and the image of 1 will give the vertices of the new triangle. Triangles corresponding to leaves will have internal edges - those that correspond to the only edge of the leaf and by which this triangle is attached to previously constructed ones. Internal edges form a polygon, inscribed in the disc, so if we remove the polygon, the rest is the union of half discs, each arc having a point. So a pair of trees, corresponding to an element of Θ, gives a pair of sequences of half discs, thus a piecewise projective automorphism of a circle.

Definition 5. Thompson group T is a group of piecewise projective automorphisms of a circle.

Although we consider just finite binary trees, infinite full binary tree will correspond to the Farey trianguation of a disc.

2.2 Alternative description of the Thompson group T

Here we prove the following theorem:

Theorem 2.2.1. *The Thompson group T has the following presentation in terms of generators and relations:*

$$T = < L, C | I = LCL, C^3 = I^4 = L^5 = (CIL)^7 = 1, I^2 C = CI^2 >$$

Let us recall the standard description of the group T. It is given by generators: A, B, C
notations: $R = A^{-1}CB$, $X_2 = A^{-1}BA$, $P = A^{-1}RB$
and relations

$$R = B^{-1}C \tag{2.2.1}$$

$$RX_2 = BP \tag{2.2.2}$$

$$CA = R^2 \tag{2.2.3}$$

$$C^3 = 1 \tag{2.2.4}$$

$$[BA^{-1}, X_2] = 1 \qquad\qquad (2.2.5)$$

$$[BA^{-1}, A^{-1}X_2A] = 1 \qquad\qquad (2.2.6)$$

As a consequence we have the following relations:
$R^4 = 1$, $P^5 = 1$, $(P^2 X_2^{-1})^3 = 1$, $(PX_2)^4 = 1$, $(X_2^2 P^{-2})^4 = 1$.

As an intermediate step, we would find a presentation of T in terms of R and C.

Lemma 2.2.1. *The Thompson group T has the following presentation in terms of generators and relations:*

$$T = <R, C | \alpha = CRC, \beta = R^2 C^{-1} R^2,$$

$$R^4 = C^3 = (RC)^5 = [\alpha, R^2 \alpha R^2] = [\alpha, \beta^{-1} \alpha \beta] = 1 >$$

Remark The relation with the standard presentation is established by the group isomorphism $A \mapsto C^{-1}R^2$, $B \mapsto CR^{-1}$, $C \mapsto C$.

Proof. Consider isomorphic groups $G_1 = <A, B, C | R = B^{-1}C, CA = R^2 >$ and $G_2 = <R', C'>$. Isomorphism ϕ between them is given by $A \mapsto C'^{-1}R'^2$, $B \mapsto C'R'^{-1}$, $C \mapsto C'$. The inverse of it is clearly $C' \mapsto C$, $R' \mapsto R$. So ϕ send an element $R^{-1}A^{-1}CB$ to $R'^{-1}(R'^{-2}C')C'(C'R'^{-1}) = R'^{-3}C'^3R'^{-1}$, so the normal subgroup of G_1 generated by $<C^3, R^{-1}A^{-1}CB>$ via an isomorphism maps to a normal subgroup $<C'^3, R'^{-4}>$ of G_2. Let us take the quotient by this normal subgroups and introduce the groups $H_1 = <A, B, C | R = B^{-1}C, CA = R^2, C^3 = R^{-1}A^{-1}CB = 1 >$ and $H_2 = <R', C' | C'^3 = R'^4 = 1 >$ and let ϕ denote an induced isomorphism between them.
Now in the group H_1 we compute

$$X_2 = A^{-1}BA = R^{-2}CCR^{-1}C^{-1}R^2$$

$$P = A^{-1}RB = R^{-2}CRCR^{-1}$$

$$RX_2(BP)^{-1} = RR^{-2}C^2R^{-1}C^{-1}R^2(RC^{-1}R^{-1}C^{-1}R^2)RC^{-1} =$$

$$= (R^{-1}C^{-1})^5 = (CR)^{-5}$$

In the last equality we used the relation $C^3 = R^4 = 1$. So ϕ maps an element $RX_2(BP)^{-1} \in H_1$ to $(C'R')^{-5} \in H_2$. So the group generated by A, B, C with relations $(1) - (4)$ is isomorphic to the group $<R', C' | C'^3 = R'^4 = (C'R')^5 = 1 >$.

Now let us express the relations $(5), (6)$ in terms of R, C, for that we introduce $\alpha = BA^{-1} = CR^{-1}R^{-2}C = CRC$ and $\beta = R^2C^{-1}R^2$ in H_1.

Again

$$X_2 = A^{-1}BA = R^2(CRC)^{-1}R^2 = (R^2\alpha R^2)^{-1}$$

$$A^{-1}X_2A = R^2CR^2\alpha^{-1}R^2C^{-1}R^2 = \beta^{-1}\alpha\beta$$

So ϕ induces an isomorphism between the following subgroups of H_1 and H_2:

$$\phi :< RX_2(BP)^{-1}, [BA^{-1}, X_2], [BA^{-1}, A^{-1}X_2A] >\rightarrow$$

$$\rightarrow< (C'R')^5, [\alpha', R'^2\alpha'R'^2], [\alpha', (\beta'^{-1}\alpha'\beta')] >$$

Here α', β' are the images of α, β. Omitting the dashes and replacing $(CR)^5$ by it's conjugate $(RC)^5$ we conclude the proof of the lemma. $\qquad\square$

Introduce now the element $L = APA^{-1} \in T$.

Theorem 2.2.2. *The Thompson group T has the following presentation in terms of generators and relations:*

$$T =< L, C | I = LCL, C^3 = I^4 = L^5 = (CIL)^7 = 1, [I^2, C] = 1 >$$

Remark The relation with the standard presentation is given by the group isomorphism $A \mapsto C^{-1}L^{-2}C^{-1}L^{-2}C^{-1}$, $B \mapsto C^2L^2$, $C \mapsto C$. And the inverse isomorphism is given by $L \mapsto (CB)^{-2}$, $C \mapsto C$.

Proof. Again as in previous considerations let us start from isomorphic groups $G_1 =< R', C' | C'^3 = (R'C')^5 = 1 >$ and $G_2 =< L, C | C^3 = L^5 = 1 >$. The isomorphism ψ is given by $R' \mapsto L^{-2}C^{-1} = (CL^2)^{-1}$, $C' \mapsto C$ and the inverse is given by $L \mapsto (R'C')^2, C \mapsto C'$.

Make a notation $I = LCL$ in G_2. Isomorphism sends an element $R'^4 \in G_1$ to $(CL^2)^{-4}$, which is conjugate to the inverse of $L(CL^2)^4L^{-1} = (LCL)^4 = I^4$. In particular ψ induces an isomorphism between the factor group $H_1 = G_1/ < R'^4 >=< R', C' | C'^3 = R'^4 = (R'C')^5 = 1 >$ and the factor group $H_2 =< L, C | I = LCL, C^3 = L^5 = I^4 = 1 >$, which we will again denote by ψ.

In the following we use the conventions $[X, Y] = X^{-1}Y^{-1}XY$, $X^Y = Y^{-1}XY$. Also remark that from the definition of $I = LCL$ follows $LCI = ICL$. Remind that we introduced $\alpha' = C'R'C'$, $\beta' = R'^2C'^{-1}R'^2$ in H_1.

Let us now calculate the image of $[\alpha', R'^2\alpha'R'^2]$ in H_2 by the isomorphism ψ:

$$\psi(R'^2\alpha'R'^2) = L^{-2}C^{-1}L^{-2}C^{-1}CL^{-2}L^{-2}C^{-1}L^{-2}C^{-1} =$$

$$= L^{-2}C^{-1}L^{-1}C^{-1}L^{-2}C^{-1} = L^{-1}I^{-1}C^{-1}L(CL^{-2})^{-1}$$

$$\psi([\alpha', R'^2\alpha'R'^2]) = [CL^{-2}, L^{-1}I^{-1}C^{-1}L(CL^{-2})^{-1}] =$$

$$= [L^{-1}I^{-1}C^{-1}L, (CL^{-2})^{-1}] = [I^{-1}C^{-1}, (IL)^{-1}]^L =$$

$$= (CIILI^{-1}C^{-1}L^{-1}I^{-1})^L = (CI^2L(LCI)^{-1}I^{-1})^L =$$

$$= (CI^2L(ICL)^{-1}I^{-1})^L = [C^{-1}, I^{-2}]^L$$

so the normal subgroup of H_1 generated by $< [\alpha', R'^2\alpha'R'^2] >$ maps to the normal subgroup of H_2 generated by $< [C, I^2] >$.

Let us deal now with the second commutator, so let us calculate an element $\psi([\alpha', \beta'^{-1}\alpha'\beta']) \in H_2$:

$$\psi(R'^2) = \psi(R'^{-2}) = CL^2CL^2$$

$$\psi(\beta') = \psi(R'^2C'^{-1}R'^2) = CL^2CL^{-1}CL^2$$

$$\psi(\beta'^{-1}\alpha'\beta') = L^{-2}C^{-1}LC^{-1}L^{-2}C^{-1}CL^{-2}CL^2CL^{-1}CL^2 =$$

$$= L^{-1}(LCL)^{-1}L^2C^{-1}LCL^2CLL^{-3}LCLL = L^{-1}I^{-1}L^2C^{-1}I^2L^2IL =$$

$$= (L^2C^{-1}I^2L^2)^{IL}$$

Let us denote $V = C^{-1}I^2 = I^2C^{-1} \in H_2$. Notice that $\psi(\alpha')^{(IL)^{-1}} = ILCL^{-2}L^{-1}I^{-1} = I^2LI^{-1} = I^2C^{-1}L^{-1} = VL^{-1}$. So we can calculate in H_2:

$$\psi([\beta'^{-1}\alpha'\beta', \alpha'])^{(IL)^{-1}} =$$

$$= [\psi(\beta'^{-1}\alpha'\beta')^{(IL)^{-1}}, \psi(\alpha')^{(IL)^{-1}}] = [L^2C^{-1}I^2L^2, I^2C^{-1}L^{-1}] =$$

$$= L^{-2}V^{-1}L^{-2}LV^{-1}L^2VL^2VL^{-1}$$

Notice that $(CIL)^2 = CI(LCI)L = CI(ICL)L = VL^2$, so commutating now by L^{-2} we get

$$\psi([\beta'^{-1}\alpha'\beta', \alpha'])^{(IL)^{-1}L^{-2}} = V^{-1}L^{-1}V^{-1}L^2VL^2VL^2 = V^{-1}L^{-1}V^{-2}(VL^2)^3 =$$

$$CI^2(CL)^{-1}(CIL)^6 = (CIL)^7$$

It means that the normal subgroup of H_1 generated by $< [\beta'^{-1}\alpha'\beta', \alpha'] >$ maps via an isomorphism to the normal subgroup of H_2 generated by $< (CIL)^7 >$.

So the proof of the theorem is accomplished. $\square$

Chapter 3

Symplectomorphisms

3.1 Introduction

We study the structure of the group $Symp$ of birational automorphisms of $\mathbf{CP}^2$ preserving the meromorphic 2-form $\omega = \frac{dx \wedge dy}{xy}$. This group arises in several contexts, the most interesting for us being that of cluster mutations and deformation quantization.

The most simple cluster mutation for two variables is given by $P : (x, y) \mapsto (y, \frac{1+y}{x})$ and it preserves the 2-form ω, so P is the element of $Symp$. Also $Symp$ contains $SL(2, \mathbf{Z})$ with a multiplicative action on x, y (3.2.1). Motivated by cluster mutations, we study also a subgroup H of $Symp$, generated by P and $SL(2, \mathbf{Z})$.

First we examine the geometry of automorphisms in $Symp$ and in H and specify in (3.2.5) the loci of points where we have to blow-up a surface $\mathbf{CP}^2$, so that the automorphism becomes regular.

In particular, we get projective systems of surfaces X_{Symp} and X_H and on which $Symp$ and H act regularly. It is not objects of the category of schemes, but still it makes sense to speak about their Picard groups, which are well-defined abelian groups. For a projective system of surfaces X_{Symp} we may consider inductive or projective limit of Picard groups of the surfaces in the system. Then $Symp$ will act by automorphisms on the both limits. The main result of this paper is (3.3.3), where we write down explicitly the linear presentation of the group H in the inductive limit of picard groups. This linear representation is very similar to the representation of the mapping class group of some cluster algebra, for which all the seeds are the same and the cluster variables of a seed are parameterized by $\mathbf{Z}^2$.

The structure of H is closely related to the structure of the Thompson group T. We collected in the previous chapter equivalent definitions of T and gave it's particularly simple presentation in terms of generators and relations. This presentation is adapted to the cluster mutations language.

Actually there is a group morphism $Symp \to T$.

The chapter is organized in the following way: first we construct a morphism from the group $Symp$ of symplectic automorphisms of $\mathbf{CP}^2$ to the Thompson group T, then we gather some information about the later, next we find geometrically presentation of H, which is some natural subgroup of $Symp$.

3.1.1 Notations

For further use let us gather here some typical notations:

X,Y,Z homogeneous coordinates on $\mathbf{CP}^2$, we will mainly use affine coordinates $x = \frac{X}{Z}$, $y = \frac{Y}{Z}$.

$Symp$ - group of birational automorphisms of $\mathbf{CP}^2$ preserving the 2-form $\omega = \frac{dx \wedge dy}{xy}$, note that $\omega = d(\log x) \wedge d(\log y)$.

$P \in Symp$ is defined as $(x, y) \mapsto (y, \frac{1+y}{x})$.

For a form F on $\mathbf{CP}^2$ by (F) we mean it's zero locus.

T will denote the Thompson group T. It is the group of piecewise linear automorphisms of $\mathbf{Z}^2$, for precise definition and it's equivalent constructions see the previous chapter.

$\Gamma = SL(2, \mathbf{Z})$.

H -subgroup of $Symp$ generated by Γ and P.

S is the subset of $\mathbf{Z}^2$ consisting of vectors with co-prime coordinates, or in other words $S = \mathbf{Z}^2 \backslash \bigcup_{n \geq 2} n\mathbf{Z}^2$, the elements in this set we call *primitive vectors*.

$u \wedge v$ is an anti-symmetric bilinear product on $\mathbf{Z}^2$, normalized in a way that $(1, 0) \wedge (0, 1) = 1$. It also gives an orientation.

For a surface X we denote $Pic(X)$ the Picard group of X which is a quotient of the group of divisors by the linear equivalence.

μ is a piecewise linear transformation of $\mathbf{Z}^2$, defined as $(x, y) \mapsto (x - min(0, y), y)$.

We use standard notation for matrices $C = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$, $I = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $U = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

3.2 Birational automorphisms

3.2.1 Examples of symplectomorphisms

Let us define an action of P, $SL(2, \mathbf{Z})$ and $\lambda \in \mathbf{C}^*$ on $\mathbf{CP}^2$ by birational automorphisms in the following way:

$$P : (x, y) \mapsto (y, \frac{1+y}{x})$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : (x, y) \mapsto (x^a y^b, x^c y^d)$$

$$\lambda : (x, y) \mapsto (\lambda x, y)$$

All these automorphisms preserve the meromorphic 2-form $\omega = \frac{dx \wedge dy}{xy}$, which is easy to see if we write this 2-form as $\omega = d(\log x) \wedge d(\log y)$. By definition $Symp$ is a group of birational automorphisms of $\mathbf{CP}^2$ that preserve ω, so we have

$$P, SL(2, \mathbf{Z}), \lambda \in Symp$$

Also it is easy to see that any transformation of the sort $(x, y) \mapsto (Q(y)x, y)$ where Q is any rational function, may be obtained as a combination of these more primitive ones. This motivates the expectation that $Symp$ is generated by $P, SL(2, \mathbf{Z}), \mathbf{C}^*$.

To justify this particular form of P we want to observe that $P^5 = 1$.
Let us now introduce H a subgroup of $Symp$ generated $P, SL(2, \mathbf{Z})$. Although we don't introduce a topology on $Symp$, we may think of H as being a lattice.
We use the following notations for matrices in $SL(2, \mathbf{Z})$:

$$C = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$$

$$I = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Then it is easy to verify the following relations between the elements of H:

$$C^3 = I^4 = [C, I^2] = 1$$

$$PCP = I$$

$$P^5 = 1$$

One recognizes in the first row the defining relations of $\Gamma = SL(2, \mathbf{Z})$. We believe that this list of relations for H is complete, i.e. all the other relations are consequences of these, but we can't prove it so far.

3.2.2   Tropicalization morphism.

Here we construct a surjective group morphism $Symp \to T$, where T is a Thompson group T. One way to do it is to look at the assymptotic behavior of the automorphisms near the point $(x, y) = (0, 0)$ and we will explain it

below. Another equivalent way to construct this group morphism proceeds through the resolution of birational automorphism, so that it becomes regular, and then the way it acts on some special set of divisors defines an action on the $\mathbf{Z}^2$ in a piecewise linear way. It would be explained in lemma(3.2.5). Take an element $\gamma \in Symp$, it defines an automorphism

$$\gamma : x, y \mapsto \frac{P(x,y)}{Q(x,y)}, \frac{R(x,y)}{Q(x,y)}$$

P, Q, R - are polynomials and we have:

$$P(x,y) = \sum_{(m,n)\in P_0} p_{mn} x^m y^n$$

$$Q(x,y) = \sum_{(m,n)\in Q_0} q_{mn} x^m y^n$$

$$R(x,y) = \sum_{(m,n)\in R_0} r_{mn} x^m y^n$$

P_0 is a set of pairs of integers (m,n) for which the coefficient p_{mn} is non-zero. The notations for Q, R are similar. Now put $x = t^a$, $y = t^b$, for $t \in (0, +\infty)$, $a, b \in \mathbf{R}$. We are interested in the behavior of this automorphism when t tends to 0. The assymptotic of P, Q, R for small t is governed by the smallest powers of t. So for fixed (a,b) the minimal powers are

$$p = min_{(m,n)\in P_0}(am + bn)$$

$$q = min_{(m,n)\in Q_0}(am + bn)$$

$$r = min_{(m,n)\in R_0}(am + bn)$$

For those $(a,b) \in \mathbf{R}^2$ for which the former minimums are attained just by one element we have $\frac{P(t^a,t^b)}{t^p} = O(1), \frac{Q(t^a,t^b)}{t^q} = O(1), \frac{R(t^a,t^b)}{t^r} = O(1)$. So the birational automorphism γ gives a map $T_\gamma : \mathbf{R}^2 \to \mathbf{R}^2$ defined as $(a,b) \mapsto (p - q, r - q)$.

Lemma 3.2.1. T_γ *is a piecewise linear automorphism of* $\mathbf{Z}^2$ *and the map* $\gamma \mapsto T_\gamma$ *defines a group morphism* $Symp \to T$.

Proof. From the formula for T_γ it is clear that it is a continuous piecewise linear map and it maps $\mathbf{Z}^2$ to $\mathbf{Z}^2$. It may be non-linear on the rays for which we have two monomials $x^k y^l$ and $x^m y^n$ with non-zero coefficients in one of

the polynomials P, Q, R, such that $ak + bl = am + bn$. Clearly this equation defines an integer line in $\mathbf{R}^2$ and there is just finite number of such lines. Outside the set of such lines $lim_{t \to 0} \frac{P(t^a, t^b)}{Q(t^a, t^b)} / \frac{t^p}{t^q}$ is a non-zero constant, so in particular $(p - q)$ doesn't depend on the choice of P, Q.

The pullback of the form w by the map $f : (a, b) \mapsto (t^a, t^b)$ looks like $f^* \omega = (log(t))^2 da \wedge db$. Outside the finite number of rays as before we have for some non-zero locally constant c, d and small t:

$$\gamma \circ f : (a, b) \mapsto (t^e(c + o(1)), t^f(d + o(1)))$$

In this case we would say that the asymptotic of γ is given by $T_\gamma(a, b) = (e, f)$. So we also have:

$$f \circ T_\gamma : (a, b) \mapsto (t^e, t^f)$$

Comparing this two formulas we conclude that

$$lim_{t \to 0}(\gamma \circ f)^* \omega / (log(t))^2 = T_\gamma^*(de \wedge df)$$

So far our conclusions apply to any map γ. Now we use the fact that $\gamma^* \omega = \omega$, then the last formula implies that $T_\gamma^*(de \wedge df) = lim_{t \to 0} f^* \omega / (log(t))^2 = da \wedge db$, so T_γ has a non-degenerate Jacobian. Therefore it is locally invertible.

Suppose now we have two elements $\gamma_1, \gamma_2 \in Symp$. Let C_1, C_2 be the union of rays where γ_1, γ_2 are possibly non-linear, outside of this rays the automorphisms have the asymptotic, given by $T_{\gamma_1}, T_{\gamma_2}$. Because γ_1 is locally invertible outside of C_1, it follows that $\gamma^{-1}(C_2)$ is also a finite number of rays, so the asymptotic of $\gamma_2 \circ \gamma_1$ outside of $C_1 \bigcup \gamma_1^{-1}(C_2)$ as t tends to 0 is given by $T_{\gamma_2} \circ T_{\gamma_1}$. There is a unique way to extend this composition to the whole $\mathbf{R}^2$ by continuity, so

$$T_{\gamma_2 \circ \gamma_1} = T_{\gamma_2} \circ T_{\gamma_1}$$

In particular $T_\gamma \circ T_{\gamma^{-1}} = Id$, so T_γ is a bijection on $\mathbf{Z}^2$ and it defines a piecewise linear automorphism of $\mathbf{Z}^2$. This concludes the proof of the lemma.
$\square$

Example. Take $P : (x, y) \mapsto (y, \frac{1+y}{x})$. It sends (t^a, t^b) to $(t^b, \frac{1+t^b}{t^a})$ and the corresponding piecewise linear map is

$$(a, b) \mapsto (b, min(0, b) - a)$$

3.2.3 Geometry of the symplectic automorphism

Given a birational automorphism f the fact that it preserves the 2-form ω has some geometric consequences. For example the locus of indeterminacy of f is supported on the divisor $(XYZ = 0)$. Then we use standard algebraic geometry techniques to understand the geometry of a symplectic automorphism. Identifying the curves on surface with it's strict transform after blow-up, we may speak about all curves where the form ω has a pole. All these curves may be parameterized by the set S of primitive vectors of $\mathbf{Z}^2$, so the birational symplectomorphism just permutes these curves, and so defines a piecewise linear automorphism of $\mathbf{Z}^2$.

A blow-up of a smooth surface Y at the point p is a regular morphism $\pi : X \to Y$, such that $\pi^{-1}(p) = E$ is a smooth rational curve, i.e. isomorphic to $\mathbf{CP}^1$, X is smooth and π induces an isomorphism between $X \setminus E$ and $Y \setminus p$. E is called an exceptional curve. For any curve $C \subset Y$ we have $\pi^*(C) = \widetilde{C} \bigcup kE$ where k is a multiplicity of C at the point p and $\widetilde{C}$ is a strict transform of C.

We would use the following standard facts of algebraic geometry: if $f : X \to Y$ is regular morphism of smooth surfaces invertible outside a finite set of points, the f is a sequence of blow-ups. Another standard fact is that for any birational isomorphism $f : X \to Y$ of surfaces there exist sequences of blow-ups π_1, π_2 and a biregular isomorphism $\phi : X_1 \to X_2$ such that there is a commutative diagram:

$$
\begin{array}{ccc}
X_1 & \xrightarrow{\phi} & X_2 \\
\pi_1 \downarrow & & \downarrow \pi_2 \\
X & \xrightarrow{f} & Y
\end{array}
\qquad (3.2.1)
$$

We can blow-up X_1 at some point p and X_2 at the point $\phi(p)$, then we would obtain new surfaces Y_1, Y_2 and ϕ will induce regular isomorphism ϕ' between them, and the triple (Y_1, Y_2, ϕ') will represent the same birational automorphism f. It is also known that there is a unique minimal triple (X_1, X_2, ϕ) such that all the others are obtained as blow-ups of this one. We call it minimal resolution of f.

Let us now introduce the notion of the projective system of surfaces over $\mathbf{CP}^2$. It is the set $\mathcal{M}$ of pairs (X, π), where $\pi : X \to \mathbf{CP}^2$ is a sequence of blow-ups. We also ask that all the intermediate sequences of blow-ups be also included in $\mathcal{M}$, so if π factors through a non-trivial blow-up $\pi' : Y \to \mathbf{CP}^2$,

then $(Y, \pi') \in \mathcal{M}$. For a given pair $m = (Y, \pi')$ we denote $\mathcal{M}(m)$ a subset of $\mathcal{M}$ consisting of pairs (X, π) for which π factors through π', so that X is a sequence of blow-ups of Y.

If G is some group of birational automorphisms of $\mathbf{CP}^2$ then we say that a projective system of blow-ups $\mathcal{M}_G$ is <u>compatible</u> with G if the following two conditions hold:

1. For any $g \in G$ there exist elements $(X_1, \pi_1), (X_2, \pi_2) \in \mathcal{M}_G$ and regular isomorphism $\phi_g : X_1 \to X_2$, such that $g \circ \pi_1 = \pi_2 \circ \phi_g$.

2. ϕ_g induces regular isomorphisms between elements of the subsets $\mathcal{M}_G(X_1, \pi_1)$ and $\mathcal{M}_G(X_2, \pi_2)$.

Example. Consider a projective system of surfaces $\mathcal{X}_{Cr}$ of all possible sequences of blow-ups (X, π) of $\mathbf{CP}^2$. This projective system is compatible with the whole group Cr of birational automorphisms of $\mathbf{CP}^2$.

Here we will give a general construction of a projective system of surfaces V_G compatible with G for any group of birational automorphisms $G \subset Cr$. Let S_G be a set of pairs (X_1, π_1) which provide a minimal resolution(cf 3.2.3) of some element $g \in G$. Clearly if (X_1, X_2, ϕ) is a minimal resolution of g, then (X_2, X_1, ϕ^{-1}) is a minimal resolution of g^{-1}, so (X_2, π_2) also belongs to S_G. Given $(Y_1, \pi'_1), (Y_2, \pi'_2) \in X_{Cr}$, by (3.2.1) there exists a unique up to isomorphism minimal element (U, π_0), such that π_0 factors through π'_1 and π'_2. This implies that for any finite set $(Y_1, \pi'_1), .., (Y_n, \pi'_n) \in X_{Cr}$ there exist unique up to isomorphism (U, π), such that π factors through π'_i, and all the other (U', π_0) with this property factor through π. We call such (U, π) a fibered product of $(Y_1, \pi'_1), .., (Y_n, \pi'_n)$. Remind that U is smooth.

Take V_G to be the set of all fibered products of elements from S_G.

Lemma 3.2.2. *V_G is compatible with G.*

Proof. For any $g \in G$ we have (X_1, X_2, ϕ) a minimal resolution of g. (X_1, π_1) and (X_2, π_2) belong to S_G and thus to V_G, so the property (1) of compatibility with G is verified.

Take $(U, \pi) \in V_G$, so that π factors through π_1, then (U, π) is the fibered product of $(U_1, \pi'_1), .., (U_n, \pi'_n), (X_1, \pi_1)$. Let $(V, \psi) \in V_G$ be a fibered product of $(U_1, \pi'_1), .., (U_n, \pi'_n), (X_2, \pi_2)$. Then ϕ induces a regular isomorphism ϕ_0 between U and V, so the property (2) is verified. $\square$

From the construction it is clear that if K is a subgroup of G then V_K is a subsystem of V_G. We have natural inclusions of groups $SL(2, \mathbf{Z}) \subset H \subset Symp$. Now we would construct projective systems of surfaces compatible

with this groups
$$X_{SL(2,\mathbf{Z})} \subset X_H \subset X_{Symp}$$
We will give explicit constructions of this systems, and one may prove that they coincide with systems $V_{SL(2,\mathbf{Z})} \subset V_H \subset V_{Symp}$, but we will not need it further.

3.2.4 Projective system of toric surfaces

Here we would construct a projective system of surfaces $X_{SL(2,\mathbf{Z})}$ compatible with $SL(2,\mathbf{Z})$.

Actually $X_{SL(2,\mathbf{Z})}$ would consist of toric surfaces and the blow-ups would be taken at the fixed points of the toric action. We consider only smooth toric surfaces with a regular morphism to $\mathbf{CP}^2$ equivariant with respect to a torus action.

Let S be a set of primitive vectors in $\mathbf{Z^2}$, i.e. vectors $(m,n) \in \mathbf{Z}^2$ with m, n co-prime. A smooth proper toric surface is defined by it's fan, which is equivalent to a set of primitive vectors $u_1, .., u_n \in S$ up to a cyclic permutation. We ask that they are ordered in a counter-clockwise direction and such that $u_i \wedge u_{i+1} = 1$, and $u_n \wedge u_1 = 1$. Then the cone decomposition of a fan is given by simplicial cones $\mathbf{Z}_{\geq 0} u_i + \mathbf{Z}_{\geq 0} u_{i+1}$. Vectors u_i correspond to fixed divisors D_i. D_i intersects only with D_{i-1} and D_{i+1} with multiplicity 1. The blow-up at the point of intersection of D_i and D_{i+1} gives a new toric surface, which corresponds to a new set of primitive vectors $u_1, u_2, .., u_i, u_i + u_{i+1}, u_{i+1}, .., u_n$.

Let Λ be a set of such collections of vectors $u_1, .., u_n$ that contain vectors $(1,0), (0,1), (-1,-1)$. Then to any element $\lambda \in \Lambda$ corresponds a smooth proper toric surface X_λ together with a regular morphism $\pi_\lambda : X_\lambda \to \mathbf{CP}^2$. π_λ is a sequence of blow-ups at the fixed points of the torus action. We take $X_{SL(2,\mathbf{Z})}$ to be a projective system of surfaces (X_λ, π_λ) for all $\lambda \in \Lambda$.

Lemma 3.2.3. $X_{SL(2,\mathbf{Z})}$ *is compatible with* $SL(2,\mathbf{Z})$.

Proof. $SL(2,\mathbf{Z})$ is a group of symmetries of the action of $(\mathbf{C}^*)^2$ on itself. $\square$

3.2.5 Projective system of surfaces for Symp.

Here we will construct a projective system of surfaces compatible with $Symp$. It is convenient for us to introduce a notion <u>chain of rational curves</u>, which would be a set of smooth rational curves $D_1, D_2 \ldots, D_n$ on a surface, such that D_i intersects only with D_{i+1} and D_{i-1} transversally with multiplicity

1. D_n intersects with D_{n-1} and D_1. For example the set of all fixed divisors on a smooth proper toric surface is a chain of rational curves.

For a surface with a chain of rational curves (X, D) we define a <u>blow-up at a chain</u> as $(\widetilde{X}, \widetilde{D})$, were $\widetilde{X}$ is a blow-up of X at some point $p \in D$ and p is not a point of intersection of different components of the chain D. $\widetilde{D}$ is a strict transform of D, so it is again a chain.

Remind that as explained in (3.2.4) for any $\lambda \in \Lambda$ we have a (X_λ, π_λ) toric surface which is a sequence of blow-ups of $\mathbf{CP}^2$. Let D_λ be a chain of fixed divisors on X_λ. Consider a projective system of surfaces (Y, π), which are sequences of blow-ups at a chain(i.e. we blow-up only the points of a strict transform of toric divisors) of some (X_λ, π_λ) and denote this projective system by X_{Symp}.

Lemma 3.2.4. X_{Symp} *is compatible with Symp.*

Proof. As stated in the diagram (3.2.1) for any element $f \in Symp$ there exists a commutative diagram:

$$
\begin{array}{ccc}
X_1 & \xrightarrow{\phi} & X_2 \\
\pi_1 \downarrow & & \downarrow \pi_2 \\
\mathbf{CP}^2 & \xrightarrow{f} & \mathbf{CP}^2
\end{array}
$$

π_1, π_2 are sequences of blow-ups, ϕ is a regular isomorphism. Denote by E_1, E_2 exceptional set of π_1 and π_2 respectively, so that E_i is a union of $\mathbf{CP}^1$'s, $\pi_i(E_i)$ is a set of points and π_i induces bijection between $X_i \setminus E_i$ and $\mathbf{CP}^2 \setminus \pi_i(E_i)$. Remind that f is a well defined morphism except at a finite set of points(precisely at $\pi_1(E_1)$), so we may pull-back the 2-form w by f everywhere except this finite set. So the condition $f^*\omega = \omega$ is equivalent to the equality

$$
\pi_1^*\omega = \phi^*\pi_2^*\omega
$$

on $X_1 \setminus E_1$. There is a unique way to extend a meromorphic 2-form on E_1, so this equality holds on the whole X_1.

The 2-form $\omega = \frac{dx \wedge dy}{xy}$ doesn't have zero's on $\mathbf{CP}^2$ and it has pole of order 1 on the divisor $(XYZ = 0)$. First let us make local computation to see the behavior of ω under the blow-up. Fix u,v local coordinates and suppose that locally the form looks like $u^k v^l du \wedge dv$. After blow up this form will become $u^{k+l+1}(\frac{v}{u})^l du \wedge d(\frac{v}{u})$ which shows that the exceptional divisor will have order $k + l + 1$.

It follows that the pullback of w has order -1 at fixed divisors of toric surfaces, because they are obtained as blow-ups at points where $k = l = -1$. Divisors where the pull-back of w has order -1 then make up a chain of rational curves.

If we blow-up at a chain, the exceptional curve would have order 0, because in some coordinates we have $k = -1, l = 0$.

If we blow-up at a point, which doesn't lye on a chain, then the order of the pull-back of w would be positive on an exceptional curve. In particular let E_i^+ be a subset of E_i, where $\pi_i^* \omega$ has positive order. Then equality $\pi_1^* \omega = \phi^* \pi_2^* \omega$ implies that $\phi(E_1^+) = E_2^+$. So we may contract simultaneously X_i at E_i^+, so that ϕ is a lift of a regular isomorphism.

So we conclude, that the minimal resolution of an element $f \in Symp$ is obtained as a sequence of blow-ups of toric surfaces X_{λ_1}, X_{λ_2} at the chains of fixed divisors, and the resulting surfaces belong to X_{Symp}.

From the other side if X_1, X_2 give a minimal resolution of $f \in Symp$, then X_i carry a chain of rational curves D_i, where $\pi_i^* \omega$ has order -1. So regular isomorphism $\phi : X_1 \to X_2$ maps D_1 to D_2 and so elements of X_{Symp} that factor through π_1 and π_2 are identified, which proves that X_{Symp} is compatible with $Symp$. $\qquad\square$

Corollary. π_i involves blow-ups at two types of points:
1. At the points of intersection of curves of the chain, thus we enlarge chains;
2. At the interior points of curves of the chain.

Remind that S is the set of primitive vectors in $\mathbf{Z}^2$. As follows from lemma(3.2.4), the minimal resolution of $f \in Symp$ is given by $\phi : X_1 \to X_2$, where X_i is a sequence of blow-ups at chain of a toric surface X_{λ_i} and $\lambda_i \in \Lambda$. So λ_i is a cyclically ordered sequence of vectors $u_1, .., u_n \in S$, and λ_2 is a sequence $v_1, .., v_n$, such that consecutive pairs u_i, u_{i+1} and also v_i, v_{i+1} span simplicial cones. Vectors u_i and v_i correspond to fixed divisors on $X_{\lambda_1}, X_{\lambda_2}$ and their strict transforms on X_1, X_2 are exactly the divisors where $\pi^* w$ has order -1. So there is a cyclic order preserving bijection $\sigma(f)$ between $(u_1, u_2, .., u_n)$ and $(v_1, v_2, .., v_n)$, and by Definition 1 of the previous chapter $\sigma(f)$ defines an element of the Thompson group T.

Lemma 3.2.5. $\sigma : Symp \to T$ *is a well-defined morphism of groups.*

Proof. First let us prove that $\sigma(f)$ doesn't depend on the choice of resolution. By definition, it depends on the toric surfaces $X_{\lambda_1}, X_{\lambda_2}$, that underly

the surfaces X_1, X_2 in the minimal resolution of f. The toric divisors in X_{λ_1} correspond to vectors $u_1, .., u_n \in S$, and the toric divisors on X_{λ_2} correspond to $v_1, .., v_n \in S$. When we make blow-up of X_1 at the intersection point of divisors D_i, D_{i+1} that correspond to u_i, u_{i+1}, we obtain a new toric surface and exceptional divisor would correspond to a vector $u_i + u_{i+1}$. Similarly, when we blow-up X_2 at the intersection point of divisors E_i, E_{i+1} that correspond to v_i, v_{i+1}, we obtain a new toric surface and exceptional divisor would correspond to a vector $v_i + v_{i+1}$. So the vector $u_i + u_{i+1}$ would go through $\sigma(f)$ to $v_i + v_{i+1}$, which is coherent with the condition that $\sigma(f)$ is a linear function on the interior of the cone spanned by u_i, u_{i+1}.

The elements of $Symp$ act by permutation on the set of toric divisors, which in their turn correspond to the set S of primitive vectors. So a priori σ defines a group morphism from $Symp$ to the group of permutations of S. But as we've just seen the image of σ actually lies in T, which is a subgroup of piece-wise linear permutations. So we have a well defined group morphism $\sigma : Symp \to T$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\Box$

To be more explicit, the divisors $(Y = 0)$, $(X = 0)$, $Z = 0$ on $\mathbf{CP}^2$ correspond to vectors $(1, 0)$, $(0, 1)$, $(-1, -1)$ respectively.

In the Theorem(2.2.1) there is a presentation of T in terms of images of C and P. We can check, that $C, I \in SL(2, \mathbf{Z})$ go to corresponding elements in T.

Element $P : (x, y) \mapsto (y, \frac{1+y}{x})$ maps to L.

3.2.6 Projective system of surfaces for H

Given a smooth rational curve C, we call a regular map $f : C \to \mathbf{CP}^1$ a coordinate, if it's an isomorphism. Notice that any toric surface which is a sequence of blow-ups of $\mathbf{CP}^2$ has canonical coordinates on each irreducible divisor of a chain. First we take X/Y, Y/Z, Z/X to be coordinates on divisors $Z = 0$, $X = 0$, $Y = 0$ on $\mathbf{CP}^2$. Suppose we have two rational curves C, D on a smooth surface that intersect transversally at one point p and suppose also that they are equipped with coordinates a, b, such that at the point of intersection they take values: $a = 0, b = \infty$. We can choose local coordinates A, B at this point p, such that $A \mid_C = a$, $A \mid_D = 0$, $B \mid_C = 0$, $B \mid_D = 1/b$. Then we take A/B as canonical coordinate on the exceptional divisor E. If we made another choice of local coordinates A', B' with this properties, then functions $A - A'$ and $B - B'$ would be zero on the curves C and D, it means that $A' = A + ABf$, $B' = B + ABg$ and f, g are some

regular functions around p. The exceptional curve E is covered by two charts with local coordinates $A/B, B$ and $B/A, A$ and in both charts we have an equality of functions $A'/B' \mid_E = A/B \mid_E$. Moreover the value of this function at the intersection point of E with the strict transform of C is ∞, and at the intersection point of E with the strict transform of D is 0. So the canonical coordinate is well defined on any curve of the chain of rational curves on the blow-ups of $\mathbf{CP}^2$.

Let us understand the minimal resolution of P. Remind that we have the following formulas for P and it's inverse $L = P^{-1}$:

$$P : (x, y) \mapsto (y, \frac{1 + y}{x})$$

$$L : (x, y) \mapsto (\frac{1 + x}{y}, x)$$

In homogeneous coordinates X, Y, Z, such that $x = Z^{-1}X, y = Z^{-1}Y$ we have:

$$P : (X, Y, Z) \mapsto (YX, (Z + Y)Z, XZ)$$

$$L : (X, Y, Z) \mapsto ((Z + X)Z, XY, YZ)$$

P is undefined at points $(1, 0, 0), (0, 1, 0), (0, 1, -1)$ and L is undefined at points $(1, 0, 0), (0, 1, 0), (1, 0, -1)$ Let $A : (X, Y, Z) \mapsto (X, Y - Z, Z)$, $B : (X, Y, Z) \mapsto (Y, Z, X + Z)$ be linear automorphisms of $\mathbf{CP}^2$. Then $B \circ P \circ A : (X, Y, Z) \mapsto (YZ, XZ, XY)$ is a Cremona transformation. It's resolution is obtained by making a blow-up at three points and then contracting three curves, connecting this points.

Let $\pi_1 : X_1 \to \mathbf{CP}^2$ be a blow-up at the points $(1, 0, 0), (0, 1, 0), (0, 1, -1)$ and let E_1, E_2, E_3 be corresponding exceptional curves. We denote by D_X, D_Y, D_Z the strict transforms of the curves $(X = 0), (Y = 0), (Z = 0)$. So on the surface X_1 we have a chain of rational curves D_X, D_Y, E_1, D_Z, E_2. The rational curve E_3 intersects this chain transversally at one point on D_X.

Similarly we have $\pi_2 : X_2 \to \mathbf{CP}^2$ a blow-up at the points $(1, 0, 0), (0, 1, 0), (1, 0, -1)$ with exceptional curves E_1', E_2', E_3', and the strict transforms of $(X = 0), (Y = 0), (Z = 0)$ are denoted by D_X', D_Y', D_Z'. We have $D_X', D_Y', E_1', D_Z', E_2'$ a chain of rational curves on X_2.

There exists a regular isomorphism $\phi : X_1 \to X_2$ such that $\pi_2 \circ \phi = P \circ \pi_1$. In particular it maps the chain curves D_X, D_Y, E_1, D_Z, E_2 to the chain curves $E_2', D_X', D_Y', E_1', D_Z'$ in this order.

Lemma 3.2.6. *H preserves canonical coordinates on the exceptional divisors coming from chains.*

Proof. It is sufficient to prove this statement for P, C the generators of H. The canonical coordinates on the chains of surfaces from $X_{SL(2,\mathbf{Z})}$ may be written as $x^a y^b$ in affine coordinates, and $SL(2, \mathbf{Z})$ acts transitively and freely on the set of all chain curves preserving the canonical coordinates.

We should just check that P preserves them. Let us take surfaces X_1 and X_2 from the minimal resolution of P. They carry chains of rational curves D_X, D_Y, E_1, D_Z, E_2 and $E_2', D_X', D_Y', E_1', D_Z'$. The canonical coordinates on this curves are $y, 1/x, 1/y, x/y, x$ and $x, y, 1/x, 1/y, x/y$ respectively. Let us now look at the pullback of this canonical coordinates from X_2 to X_1:

$$P^* : (x, y, 1/x, 1/y, x/y) \mapsto (y, \frac{1+y}{x}, 1/y, \frac{x}{1+y}, \frac{xy}{1+y})$$

We claim that the obtained five functions define the same canonical coordinates on the rational curves as $y, 1/x, 1/y, x/y, x$.

The coordinates at the intersection point of D_Y and E_1 are $(u, v) = (1/x, y)$ and locally the curve D_Y is defined as $v = 0$, so $P^*(y) - 1/x = \frac{1+y}{x} - \frac{1}{x} = \frac{y}{x} = uv$ and it follows that $P^*(y) = 1/x$ on D_Y. Similarly at the intersection point of D_Z and E_2 the coordinates are $(u, v) = (x/y, 1/x)$. We have on D_Z that $\frac{1}{y} = uv = 0$ ant it implies that on D_Z:

$$P^*(1/y) - x/y = \frac{x}{1+y} - \frac{x}{y} = \frac{x}{y}(\frac{1}{1+\frac{1}{y}} - 1) = 0$$

The coordinates at the intersection point of E_2 and D_X are $(u, v) = (x, 1/y)$ and on E_2 we also have $\frac{1}{y} = uv = 0$, which implies that on E_2:

$$P^*(x/y) - x = x(\frac{1}{1+\frac{1}{y}} - 1) = 0$$

So we have verified that P preserves the canonical coordinates and so we conclude the proof of the lemma. $\square$

The corollary is that every chain curve in X_{Symp} has a canonical coordinate on it. So we may consider the set of points on chain curves Ω with coordinate -1. The automorphism $P : (x, y) \mapsto (y, \frac{1+y}{x})$ involved the blowups at the points in this set. So we may consider a projective system X_H of surfaces, obtained by sequences of blow-ups of surfaces in $X_{SL(2,\mathbf{Z})}$ at the points in the set Ω. In particular if we blow-up at a point $p \in \Omega$ which lies

on a chain curve D, then the next blow-up may be done at the intersection point of D with the exceptional divisor. If $s \in S$ corresponds to the divisor D, we denote by δ_s the exceptional divisor, by δ_s^2 the exceptional divisor of the blow-up at the intersection point of δ_s and D, and generally δ_s^k is the exceptional divisor of the intersection of δ_s^{k-1} and D. From what is said, it follows that X_H is compatible with H.

Let us now understand the action of P on the Picard group of it's minimal resolution. We remind that X_1 is a blow-up of $\mathbf{CP}^2$ at the points $(1,0,0), (0,1,0), (0,1,-1)$ and X_2 is a blow-up of $\mathbf{CP}^2$ at the points $(1,0,0), (0,1,0), (1,0,-1)$. Let X_P be a blow-up of $\mathbf{CP}^2$ at points $(1,0,0), (0,1,0), (0,1,-1), (1,0,-1)$ with exceptional curves D_1, D_2, C_1, C_2 , then clearly X_P dominates X_1 and X_2 in a natural way and P lifts to a regular automorphism $\Psi : X_P =\to X_P$. On the surface X_P we have a chain of rational curves D_X, D_Y, D_1, D_Z, D_2, moreover every curve in the chain have self-intersection -1. Their intersection matrix is:

$$\begin{pmatrix} -1 & 1 & 0 & 0 & 1 \\ 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 \\ 1 & 0 & 0 & 1 & -1 \end{pmatrix}$$

The rank of this intersection matrix is 5 and so the chain curves generate $Pic(X_P)$, which is also of rank 5. Then clearly Ψ^* permutes them in the following order:

$$\Psi^* : D_X \to D_Y \to D_1 \to D_Z \to D_2 \to D_X$$

Using the linear equivalence of divisors on the projective plane $(X = 0) = (Y = 0) = (Z = 0)$, we obtain an equality in the $Pic(X_P)$:

$$D_X + E_2 + C_1 = D_Y + E_1 + C_2 = D_Z + E_1 + E_2$$

In particular we can express C_1 and C_2:

$$C_1 = D_Z + E_1 - D_X$$

$$C_2 = D_Z + E_2 - D_Y$$

So we have $\Psi^*(C_1) = C_2, \Psi^*(C_2) = D_X + E_2 - E_1$.

3.3 Picard groups of a projective system of surfaces

Given a projective system of surfaces X_G compatible with a group G, we can construct inductive and projective limit of Picard groups of X_G. They both will be linear presentations of G.

Actually if $f : A \to B$ is the blow-up of a smooth surface B at a point then we have the pullback morphism $f^* : Pic(B) \to Pic(A)$ and if (E) is the class of the exceptional divisor we have $Pic(A) = f^* Pic(B) \oplus \mathbf{Z}(E)$ and we have also the projection morphism $f_* : Pic(A) \to Pic(B)$ which is just a forgetting of the exceptional set. Let us denote Pic an inductive limit over all elements $(X, \pi) \in X_G$ with connecting maps f^*:

$$Pic(X_G) = \varinjlim Pic(X)$$

If Bl_G is the set of all exceptional divisors, that appear in X_G, then as an abelian group

$$Pic(X_G) = Pic(\mathbf{CP}^2) \oplus \bigoplus_{E \in Bl_G} \mathbf{Z}$$

And denote $\widehat{Pic}$ a projective limit over all elements $(X, \pi) \in X_G$ with connecting maps f_*:

$$\widehat{Pic}(X_G) = \varprojlim Pic(X)$$

Then respectively we have an equality of abelian groups

$$Pic(X_G) = Pic(\mathbf{CP}^2) \oplus \prod_{E \in Bl_G} \mathbf{Z}$$

So from explicit formulas we conclude, that the inductive limit embeds in a projective limit, i.e. we have:

$$Pic \subset \widehat{Pic}$$

The intersection pairing on surfaces induce a perfect pairing between Pic and $\widehat{Pic}$, namely we have bilinear non-degenerate pairing:

$$(\bullet, \bullet) : Pic \times \widehat{Pic} \to \mathbf{Z}$$

It is well defined, because if $f : A \to B$ is a blow-up of surfaces and $a \in Pic(A)$, $b \in Pic(b)$, then we have an equality of the intersection pairings on A and on B:

$$(a, f^* b) = (f_* a, b)$$

There is a cone $Ample \subset Pic$ of ample divisors, and $Eff \subset \widehat{Pic}$ cone of effective divisors, which are dual to each other with respect to intersection pairing.

Because we described explicitly $X_{SL(2,\mathbf{Z})}, X_{Symp}, X_H$, we can easily compute it's inductive and projective Picard groups.
First let us deal with the Picard groups for $X_{SL(2,\mathbf{Z})}$. Recall that S is the set of points in $\mathbf{Z}^2$ with co-prime coordinates. It is a cyclically ordered set. Actually the curves in the chains correspond to elements in S. Let $Fun(S)$ be the space of all functions from S to $\mathbf{Z}$. A function $f : S \to \mathbf{Z}$ is said to be piece-wise linear if there is a finite cyclically ordered subset $v_1, .., v_n \in S$, such that for any $u \in S$ between v_i, v_{i+1} we have:

$$f(u) = af(v_i) + bf(v_{i+1})$$

and $a, b \in \mathbf{Q}$ are defined in a way that $av_i + bv_{i+1} = u$.
Let $PL(S) \subset Fun(S)$ be the space of piecewise linear function from S to $\mathbf{Z}$. We denote Lin the 2-dimensional space of linear functions.

Lemma 3.3.1. *We have the following isomorphisms of abelian groups:*

$$pl : Pic(X_{\mathbf{SL(2,Z)}}) \to PL/Lin$$

$$fun : \widehat{Pic}(X_{SL(2,\mathbf{Z})}) \to Fun/Lin$$

Proof. For a toric surface $(X, \pi) \in X_{SL(2,\mathbf{Z})}$, which is a blow-up of $\mathbf{CP}^2$, let $\lambda = \{s_1, .., s_n\}$ be a set parameterizing fixed divisors as defined in section(3.2.1). In particular $s_i \in S$ corresponds to a toric divisor D_i.
Any divisor D on a toric surface is lineally equivalent to a sum of fixed divisors, so $Pic(X)$ is generated by D_i. For a sum of fixed irreducible divisors $D = \sum n_i D_i$ on X we construct a function $f_D : \lambda \to \mathbf{Z}$:

$$f_D(s_i) := n_i$$

If $\pi : Y \to X$ is a blow-up at the point of intersection of D_i and D_{i+1}, then the exceptional divisor E corresponds to the vector $s_i + s_{i+1}$, and we have $\pi^* D_i = D_i + E$, $\pi^* D_{i+1} = D_{i+1} + E$ in the group of divizors $Div(Y)$. So $f_{\pi^* D}(s_i + s_{i+1}) = n_i + n_{i+1}$.
The principal divisors of the form $D = \sum n_i D_i$ are given by some function, which doesn't have zeros and poles on the open torus, i.e. the function is of the form $x^a y^b$, so the principal divisors are the pull-backs of principal

divisors from $\mathbf{CP}^2$.

Notice that in $Div(\mathbf{CP}^2)$ the subgroup of principal divisors is generated by $(x) = (X = 0) - (Z = 0)$ and $(y) = (Y = 0) - (Z = 0)$. In particular we can compute:

$$f_{(x)} : ((1,0),(0,1),(-1,-1)) = (1,0,-1)$$

$$f_{(y)} : ((1,0),(0,1),(-1,-1)) = (0,1,-1)$$

So principal divisors on X correspond exactly to linear functions on λ.

$Pic(X)$ is a quotient of the group of divisors by principal divisors, so we deduce that f defines isomorphism between $Pic(X)$ and $Fun(\lambda)/Lin(\lambda)$ - the quotient of all integer functions on λ by linear functions. Taking the inductive and projective limit over all (X, π) in $X_{SL(2,\mathbf{Z})}$ we conclude the lemma. $\qquad\qquad\square$

The action of $SL(2, \mathbf{Z})$ on $Pic(X_{SL(2,\mathbf{Z})})$ and on $\widehat{Pic}(X_{SL(2,\mathbf{Z})})$ is induced from the natural action on PL and Fun, space of linear functions and Lin is preserved.

The cone of effective divisors $Eff \subset \widehat{Pic}(X_{\mathbf{SL(2,Z)}})$ is generated by positive functions on S. The dual to it is the cone of ample divisors $Amp \subset Pic(X_{SL(2,\mathbf{Z})})$ which consists of convex functions. Both cones are preserved under the action of the group $SL(2, \mathbf{Z})$.

Let F be a piecewise linear function from S to $\mathbf{Z}$. For $a \in S$ we will introduce an index $d(F, a)$ which will measure how much F is far from being linear. So in the neighborhood of a choose u, v such that $u \wedge a = a \wedge v = 1$(so a is between u and v). We ask that F is not linear at most at a on this interval, which can be realized by adding a to u and v sufficiently many times. Then obviously $u + v = ka$ for $k \in \mathbf{Z}$. The <u>index</u> of F at a is the value:

$$d(F, a) := F(u) + F(v) - kF(a)$$

Easy to check that it does not depend on the choice of u, v, and it equals to 0 at all points where F is linear. d is linear in the first argument.

Let us describe a pairing between $Pic(X_{SL(2,\mathbf{Z})})$ and $\widehat{Pic}(X_{SL(2,\mathbf{Z})})$. Suppose F is a piecewise linear and G is any function on S, so F and G represent some elements from $Pic(X_{SL(2,\mathbf{Z})})$ and $\widehat{Pic}(X_{SL(2,\mathbf{Z})})$ respectively. The pairing is defined as follows:

$$< F.G >= \sum_{\alpha \in S} d(F, \alpha)G(\alpha)$$

3.3.1 Presentation of H

The group H acts linearly on $Pic(X_H)$ and on $\widehat{Pic}(X_H)$ and preserves the pairing between them. Actually $Pic(X_H)$ embeds in $\widehat{Pic}(X_H)$, so actually H acts by orthogonal transformations on $Pic(X_H)$. We would compute this action explicitly and find a vector subspace $V \subset Pic(X_H)$ preserved by this action.

Let $\oplus_{s \in S, k \geq 0} \mathbf{Z}\delta_s^k$ be a free abelian group generated by δ_s^k and $\prod_{s \in S, k > 0} \mathbf{Z}\delta_s^k$ be a free product over indices $s \in S$ and $k \in \mathbf{Z}_{>0}$. δ_s^k corresponds to an exceptional divisor which appears after a k'th blow-up of a point with coordinate -1 on a chain curve which corresponds to $s \in S$.
Then it follows from the structure of the projective system X_H as described in (3.2.6) that

$$Pic(X_H) = Pic(X_{SL(2,\mathbf{Z})}) \oplus \sum_{s \in S, k > 0} \mathbf{Z}\delta_s^k$$

$$\widehat{Pic}(X_H) = \widehat{Pic}(X_{SL(2,\mathbf{Z})}) \oplus \prod_{s \in S, k > 0} \mathbf{Z}\delta_s^k$$

The pairing between $Pic(X_H)$ and $\widehat{Pic}(X_H)$ is extended from that on $X_{SL(2,\mathbf{Z})}$ by saying that δ_s^k are orthogonal to each other and to $Pic(X_{SL(2,\mathbf{Z})})$ and $\delta_s^k.\delta_s^k = -1$.

Let us consider a subgroup $V' \subset \widehat{Pic}(X_{SL(2,\mathbf{Z})}) \subset \widehat{Pic}(X_H)$ generated by functions $F : S \to \mathbf{Z}$ which take value 1 at some point and 0 at all the others. So V' consists of functions with finite support. Geometrically they will correspond to some curve of the chain. As the group H preserves curves of the chain, the subspace of associated functions V' will also be preserved by H. So H will preserve the subgroup $V \subset Pic(X_H)$ dual to it. Let us describe explicitly V.
For any piecewise linear function $F \in PL(S)$ we associate an element

$$F' := F - \sum_i d(F, s_i)\delta_{s_i}^1 \in Pic(X_H)$$

Lemma 3.3.2. *The subgroup $V \subset Pic(X_H)$ dual to the subgroup $U \subset \widehat{Pic}(X_H)$ of chain curves is generated by F' and $\delta_s^k - \delta_s^{k+1}$ for all $F \in PL(S)$, $s \in S$, $k \in \mathbf{Z}_{>0}$. V is invariant under the action of H.*

Proof. Let $(X, \pi) \in X_H$ be a surface such that each chain curve D_i was blown-up exactly once at the point with canonical coordinate -1. Then we would have the following divisors $D_1, .., D_n, \delta^1_{s_1}, .., \delta^1_{s_n} \in Pic(X)$ and the rank of $Pic(X)$ is $2n - 2$. Here D_i are chain curves on X and $\delta^1_{s_i}$ are exceptional curves of blow-ups. Let $U(X) \subset Pic(X)$ be a subgroup generated by $D_1, .., D_n$. Because we made blow-ups at chain curves, it is not difficult to show that the rank of $U(X)$ is n. Remind that $f_i = D_i + \delta^1_{s_i}$ is a pullback of a divisor from a toric surface, so it is given by a function which we denote by the same letter: $f_i(s_i) = 1$ and $f_i(s_j) = 0$ for $j \neq i$. Using the formula for the pairing of piece-wise linear functions (3.3) we get $(f_i, f_i) = -k_i$, $(f_i, f_{i+1}) = 1$, where k_i is such that $k_i s_i = s_{i+1} + s_{i-1}$. Also we bear in mind that f_i is orthogonal to $\delta^1_{s_j}$ for all j. It follows that the divisor

$$f'_i = f_i + k_i \delta^1_{s_i} - \delta^1_{s_{i+1}} - \delta^1_{s_{i-1}} \in Pic(X)$$

is orthogonal to $D_j = f_j - \delta^1_{s_j}$ for all j. Moreover we notice that f_i's generate the subgroup of toric divisors on the toric surface underlying X, so the space $V(X)$ spanned by f'_i is of rank $n - 2$ and it is orthogonal to $U(X)$.
If we further blow-up a point with canonical coordinate -1, we increase the rank of Picard group by 1, but we then add an element of the sort $\delta^k_s - \delta^{k+1}_s$ to the group V, because it is orthogonal to D_i, so the co-rank of $V(X)$ always equals to n, which proves that the inductive limit V of $V(X)$ is exactly the orthogonal to chain curves and we finished the proof of the lemma. $\qquad \square$

Now group H generated by P and C is acting by orthogonal transformations on $Pic(X_H)$. We denote $T_P : S \to S$ the tropical action of P, i.e. it is an automorphism defined by the formula:

$$T_P : (a, b) \mapsto \begin{cases} (-b, a) & \text{if } a \geq 0 \\ (-b + a, a) & \text{if } a < 0. \end{cases}$$

Actually P^* permutes the chain curves, that are parameterized by vectors in S, so T_P is exactly the induced action of P^* on S.

Denote by A the piecewise linear function $A(x, y) = max(0, -y)$. Actually it is not linear only at $(1, 0)$ and $(-1, 0)$.

Lemma 3.3.3. *P^* acts as follows on $Pic(X_H) = PL(S)/Lin \oplus \bigoplus_{s,k} \delta^k_s$:*
$\delta^k_s \mapsto \delta^k_{(T_P(s))}$, *for $k \geq 1$, $s \neq (0, \pm 1)$*
$\delta^k_{(0,-1)} \mapsto \delta^{k+1}_{(1,0)}$

$\delta_{(0,1)}^k \mapsto \delta_{(-1,0)}^{k-1}$ *for* $k \geq 2$

$\delta_{(0,1)}^1 \mapsto -\delta_{(1,0)}^1 + A$

For $F \in PL(S)$ *such that* $F(0,1) = F(0,-1) = 0$ *we have* $F \mapsto F \circ T_P^{-1}$

$A \mapsto A \circ T_P^{-1} - \delta_{(1,0)}^1$

Proof. Remind that P lifts to a regular automorphism of X_P constructed in the section (3.3.3). We also described the action of P^* on $Pic(X_P)$. So we can use the decomposition

$$Pic(X_H) = Pic(X_P) \bigoplus \oplus_{blow-ups} \mathbf{Z}e_{exc}$$

The latter sum is taken over the blow-ups of X_P at the appropriate points. So the classes of exceptional curves of this blow-ups just get permuted, depending on where go the corresponding chain curves, which proves the first three statements of the lemma.

The chain curves D_X, D_Y, E_1, D_Z, E_2 on the surface X_P are parameterized by the points of the following set:

$$\lambda = ((1,0), (0,1), (-1,0), (-1,-1), (0,-1))$$

Respectively functions which are 1 at one point and 0 at all the others, defines an element of $Pic(X_P) \bigcap PL(S)/Lin$ which correspond to the divisors

$$D_X + \delta_{(1,0)}^1, D_Y + \delta_{(0,1)}^1, E_1, D_Z, E_2$$

In previous notations $\delta_{(1,0)}^1 = C_1$, $\delta_{(0,1)}^1 = C_2$. In particular we see that

$$P^*(\delta_{(0,1)}^1) = \Psi^*(C_2) = D_X + E_2 - E_1 = \phi - \delta_{(1,0)}^1$$

Here ϕ is a function that takes values in $(1, 0, -1, 0, 1)$. After adding to ϕ a linear function $(-1, 0, 1, 1, 0)$ it becomes A, which proves the fourth statement of the lemma.

Remind that

$$\Psi^*(D_X + \delta_{(1,0)}^1) = D_Y + \delta_{(0,1)}^1$$

$$\Psi^*(E_1) = D_Z$$

$$\Psi^*(D_Z) = E_2$$

If f is the function corresponding to any of this divisors then we have $\Psi^*(f) = f \circ T_P^{-1}$. Moreover such f together with functions corresponding to the new chain curves satisfy the property that $f(0,1) = f(0,-1) = 0$, so the fifth statement of the lemma is proved.

The function $A \in PL(S)$ corresponds to the divisor $D_Z + E_2$, so using the fact that the function $A \circ T_P^{-1}$ is given by $(1, 0, 0, 0, 1)$ we obtain:

$$\Psi^*(A) = \Psi^*(D_Z + E_2) = D_X + E_2 = A \circ T_P^{-1} - \delta_{(1,0)}^1$$

This concludes the lemma. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

Any piece-wise linear function on S up to linear functions is equal to the sum of A and the function that vanishes at $(0, \pm 1)$, so we described the action of P on the whole group $PL(S)/Lin$, but for convenience here is the general formula for $F \in PL(S)$:

$$\Psi^* : F \mapsto F \circ T_P^{-1} - F((0, -1))\delta_{(1,0)}^1 + F((0, 1))(-\delta_{(1,0)}^1 + A)$$

So $V \subset Pic(X_H)$ is invariant under group H, which may be also checked directly. The action of H looks quite simple in the appropriate basis. Let us introduce the following abelian groups:

$$B = \bigoplus_{\alpha \in S} b_\alpha$$

$$E = \bigoplus_{k=1, \alpha \in S}^{\infty} e_\alpha^k$$

There is a group morphism $\Upsilon : B \to \mathbf{Z}^2$ defined on the generators $\Upsilon(b_{(m,n)}) = (m, n)$, so we denote $B_0 = ker(\Upsilon)$, so that there is an exact sequence:

$$0 \to B_0 \to B \to \mathbf{Z}^2 \to 0$$

Let us now construct a group morphism $\Theta : PL(S) \to B$:

$$\Theta : F \to \sum_s d(F, s) b_s$$

$d(F, s)$ is the index of non-linearity of F at s.

Lemma 3.3.4. Θ *induces an isomorphism between $PL(S)/Lin$ and B_0.*

Proof. Lin is exactly the kernel of Θ because the index of non-linearity vanishes on linear functions. From the other side $PL(S)$ is generated by the functions f which are defined on vectors $u, u + v, v$ as $(0, 1, 0)$ for all pairs u, v such that $u \wedge v = 1$. But then $\Theta(f) = b_u + b_v - b_{u+v}$ generate B_0. $\quad\square$

We are now ready to construct a morphism

$$\Gamma : V \to B_0 \oplus E$$

Remind that the subgroup $V \subset Pic(X_P)$ is generated by the elements F' and $\delta_\alpha^k - \delta_\alpha^{k+1}$. So we set

$$\Gamma(\delta_\alpha^k - \delta_\alpha^{k+1}) = e_\alpha^k \in E$$

$$\Gamma(F') = \Theta(F) \in B_0$$

Then lemma implies that Γ is an isomorphism between V and $B_0 \oplus E$. The action of the group $SL(2, \mathbf{Z})$ on V induces an action on B_0 and E which is just a natural action on the indices of e and b.

For an element $v \in S$ let us define μ_v a <u>mutation of $\mathbf{Z}^2$ at v</u> by the following formula:

$$\mu_v : w \mapsto \begin{cases} w + (v \wedge w)v & \text{if } v \wedge w \geq 0 \\ w & \text{if } v \wedge w < 0. \end{cases}$$

Now let us define a mutation μ_v of $B \oplus E$ by the following formulas:

$$b_v \mapsto -b_{-v}$$

$$b_{-v} \mapsto e_{-v} + b_{-v}$$

$$b_w \mapsto b_{\mu_v(w)} \text{ if } w \wedge v > 0$$

$$b_w \mapsto b_{\mu_v(w)} + (v \wedge w)b_{-v} \text{ if } w \wedge v < 0$$

$$e_v \mapsto b_v + b_{-v}$$

$$e_v^k \mapsto e_v^{k-1}$$

$$e_{-v}^k \mapsto e_{-v}^{k+1}$$

$$e_w \mapsto e_{\mu_v(w)}$$

Lemma 3.3.5. *The action of I^*P^* coincides with the restriction of the action $\mu_{(0,1)}$ on $B_0 \oplus E$.*

Proof. We should just express the action of P as described in lemma (3.3.3) in the basis b_v, e_v. Notice that piecewise-linear action of I^*P^* on $\mathbf{Z}^2$ is given by $(x, y) \mapsto (x, y)$ for $x > 0$, and $(x, y) \mapsto (x, y - x)$ for $x < 0$, which is exactly the action of $\mu_{(0,1)}$.

So in particular we have

$$I^*P^* : e_w^k = \delta_w^k - \delta_w^{k+1} \mapsto e_{\mu_{(0,1)}(w)}^k$$

And also $e_{(0,1)}^k \mapsto e_{(0,1)}^{k-1}$ for $k > 1$, and $e_{(0,-1)}^k \mapsto e_{(0,-1)}^{k-1}$. This proves the last three statements of the lemma.

$$I^*P^*(e_{(0,1)}^1) = I^*P^*(\delta_{(0,1)}^1 - \delta_{(0,1)}^2) = (A \circ I - \delta_{(0,-1)}^1) - \delta_{(0,1)}^1 = (A \circ I)' = b_{(0,1)} + b_{(0,-1)}$$

From this calculation we conclude that $I^*P^*(e_{(0,1)}^1) = \mu_{(0,1)}(e_{(0,1)}^1)$.

Let us remind that

$$I^*P^*(\delta_{(0,1)}^1) - \delta_{(0,1)}^1 = A \circ I - \delta_{(0,-1)}^1 - \delta_{(0,1)}^1 = (A \circ I)' = b_{(0,1)} + b_{(0,-1)}$$

$$I^*P^*(\delta_{(0,-1)}^1) - \delta_{(0,-1)}^1 = \delta_{(0,-1)}^2 - \delta_{(0,-1)}^1 = -e_{(0,-1)}^1$$

So for $F \in PL(S)$ such that $F(x, y) = 0$ for $x > 0$, we denote $d_1 = d(F, (0, 1))$, $d_2 = d(F, (0, -1))$. Then $F' = \sum_\alpha m_\alpha b_\alpha + d_1 b_{(0,1)} + d_2 b_{(0,-1)}$, and so we have:

$$I^*P^*(F') = I^*P^*(F - d_1\delta_{(0,1)}^1 - d_2\delta_{(0,-1)}^1) =$$

$$= F \circ \mu_{(0,1)}' - d_1(A \circ I)' + d_2 e_{(0,-1)}^1 =$$

$$= \sum_\alpha m_\alpha b_{\mu_{(0,1)}(\alpha)} - d_1 b_{(0,-1)} + d_2(b_{(0,-1)} + e_{(0,-1)}) = \mu_{(0,1)}(F')$$

Similarly the action of I^*P^* and $\mu_{(0,1)}$ coincides on the functions F' such that $F \in PL(S)$ and $F(x, y) = 0$ for $x > 0$. So to complete the proof of the lemma, we just need to verify that the action coincide on $A' = b_{(1,0)} + b_{(-1,0)}$.

$$I^*P^*(A') = I^*P^*(A - \delta_{(1,0)}^1 - \delta_{(-1,0)}^1) =$$

$$= A \circ \mu_{(0,1)} - \delta_{(0,-1)}^1 - \delta_{(-1,1)}^1 - \delta_{(1,0)}^1 =$$

$$= (A \circ \mu_{(0,1)})' = b_{(-1,1)} + b_{(0,-1)} + b_{(1,0)} = \mu_{(0,1)}(b_{(1,0)} + b_{(0,-1)})$$

So the lemma is proved. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

The action of the element $\mu_{(0,1)} = I^* P^*$ may be called the mutation applied to the vector $v = (0,1)$. By analogy we would say, that $\gamma \circ \mu \circ \gamma^{-1}$ is the mutation applied to the vector v if $v = \gamma((0,1))$.

Actually the action of the mutation on $B_0 \oplus E$ admits even further simplification. Namely make the following notations:

$$\Theta = \mathbf{Z}^2 \setminus (0,0)$$

$$W = \bigoplus_{\alpha \in \Theta} \mathbf{Z} p_\alpha$$

Θ is a set of points, W is a free abelian group.
Remind that the for any $v \in S \subset \Theta$ the action of the mutation μ_v on Θ is given by:

$$\mu_v : w \mapsto \begin{cases} w + (v \wedge w)v & \text{if } v \wedge w \geq 0 \\ w & \text{if } v \wedge w < 0. \end{cases}$$

We define the action of μ_v on W as follows:

$$\mu_v : p_\alpha \mapsto \begin{cases} p_{\alpha + (v \wedge \alpha)v} + (v \wedge \alpha)p_{-v} & \text{if } v \wedge \alpha > 0 \\ p_\alpha & \text{if } v \wedge \alpha < 0 \\ p_{\alpha - v} - p_{-v} & \text{if } v \wedge \alpha = 0 \end{cases}$$

Remark that μ_v maps p_v to $-p_{-v}$.
In particular $\mu_{(0,1)}$ acts on W as follows:

$$p_{(x,y)} \mapsto p_{(x,y)} \text{ if } x > 0$$

$$p_{(x,y)} \mapsto p_{(x,y-x)} - x p_{(-1,0)} \text{ if } x < 0$$

$$p_{(0,y)} \mapsto p_{(0,y-1)} - p_{(0,-1)}$$

There is a group morphism $W \to \mathbf{Z}^2$, $p_{(x,y)} \mapsto (x,y)$, and denote it's kernel by V_1. So we have an exact sequence of abelian groups:

$$0 \to V_1 \to W \to \mathbf{Z}^2 \to 0$$

For an element $\gamma \in SL(2,\mathbf{Z})$ and $\alpha \in \mathbf{Z}^2$ we define:

$$\gamma(p_\alpha) := p_{\gamma(\alpha)}$$

As a consequence of previous lemmas we can prove the following theorem:

Theorem 3.3.1. *There is an isomorphism between $V \subset Pic(X_H)$ and $V_1 \subset W$ and it is equivariant under the action of $H = <P, SL(2, \mathbf{Z})>$.*

Proof. We use an isomorphism $\Theta : V \rightarrow B_0 \oplus E$ and the lemma (3.3.5), which claims that the induced action of I^*P^* on $B_0 \oplus E$ coincides with the action of $\mu_{(0,1)}$. Let us now construct an isomorphism $\Phi : V_1 \rightarrow B_0 \oplus E$, which for any $v \in S$ and k positive integer is given by the formula:

$$\Phi : p_{kv} \rightarrow e_v^{k-1} + 2e_v^{k-2} + ... + (k-1)e_v^1 + kb_v$$

It is clear that Φ defines an isomorphism between W and $B \oplus E$, but it is also consistent with the projections of W and B to $\mathbf{Z}^2$, so the kernels of this projections V_1 and $B_0 \oplus E$ respectively are isomorphic.

For any $\gamma \in SL(2, \mathbf{Z})$ we have $\Phi(p_{\gamma(\alpha)}) = \gamma(\Phi(p_\alpha))$. So we just need to verify that the action of $\mu_{(0,1)}$ commute with Φ, because $\mu_{0,1}$ and $SL(2, \mathbf{Z})$. But actually Φ commute with μ_v for any $v \in S$.

For $s \in S$ and $k > 0$ we denote $\alpha = ks$ and we consider four cases using the formulas for the action of μ_v on $B \oplus E$:

1. If $s \wedge v > 0$ then

$$\mu_v(\Phi(p_\alpha)) = \mu_v(e_s^k + 2e_s^{k-1} + .. + (k-1)e_s^1 + kb_s) = \Phi(p_\alpha)$$

2. If $s \wedge v < 0$ then

$$\mu_v(\Phi(p_\alpha)) = \mu_v(e_s^k + 2e_s^{k-1} + .. + (k-1)e_s^1 + kb_s) =$$

$$= e_{\mu_v(s)}^k + 2e_{\mu_v(s)}^{k-1} + .. + (k-1)e_{\mu_v(s)}^1 + kb_{\mu_v(s)} + k(s \wedge v)b_{-v} =$$

$$= \Phi(p_{\mu_v(\alpha)} + (\alpha \wedge v)p_{-v})$$

3. If $\alpha = kv$ then

$$\mu_v(\Phi(p_\alpha)) = \mu_v(e_v^k + 2e_v^{k-1} + .. + (k-1)e_v^1 + kb_v) =$$

$$= e_v^{k-1} + 2e_v^{k-2} + .. + (k-2)e_v^1 + (k-1)(b_v + b_{-v}) + kb_{-v} = \Phi(p_{(k-1)v} - p_{-v})$$

4. If $\alpha = -kv$ then

$$\mu_v(\Phi(p_\alpha)) = \mu_v(e_{-v}^k + 2e_{-v}^{k-1} + .. + (k-1)e_{-v}^1 + kb_{-v}) =$$

$$= e_{-v}^{k+1} + 2e_{-v}^k + .. + (k-1)e_{-v}^2 + k(b_{-v} + e_{-v}^1) = \Phi(p_{-(k+1)v} - p_{-v})$$

This concludes the proof of the theorem. $\square$

3.3.2 Cluster mutations

By calling a piecewise linear transformation μ mutation, we made an allusion to a cluster mutation, so let us try to justify this similarity.

To a given cluster $\underline{t}$ we associate a vector space $V(\underline{t}) = \bigoplus_{\lambda \in \Lambda(\underline{t})} \mathbf{C} e_\lambda$, where $\Lambda(\underline{t})$ is the set of indices for cluster variables at this cluster, we suppose $b_{ij}(\underline{t})$ is an antisymmetric matrix of exponents. Suppose a mutation μ_i at variable $i \in \Lambda(\underline{t})$ relates a cluster $\underline{t}$ to $\underline{t}'$, then we have a bijection $\sigma_i : \Lambda(\underline{t}) \to \Lambda(\underline{t}')$. Let us define a map $\mu_i : V(\underline{t}') \to V(\underline{t})$ in the following way:

$$e_{\sigma(k)}(\underline{t}') \mapsto e_k + max(b_{ik}(\underline{t}), 0) e_i$$

$$e_{\sigma(i)(\underline{t}')} \mapsto -e_i$$

It is easy to check that $b(\underline{t})$ which may be considered as an element of $\bigwedge^2 V^*(\underline{t})$ taking value b_{ij} at $e_i \wedge e_j$ transforms in a way, consistent with the transformation of mutation matrix.

Let us now set $\Lambda = \mathbf{Z}^2$ and let us allow mutations only at the primitive vectors $S \subset \Lambda$. Take all the seeds to be isomorphic and set $b_{ij} = i \wedge j$, so rather than having different seeds we would act by mutations on a seed data. First of all σ_v for $v \in S$ acts on Λ as follows: $\sigma_v : w \mapsto w - min(v \wedge w, 0) v$ if $w \wedge v \neq 0$

$\sigma_v : w \mapsto w - v$ if $w = kv$

It is an automorphism of Λ. It's easy to see that $\sigma_{\gamma(v)} = \gamma \circ \sigma_v \circ \gamma^{-1}$ for any $\gamma \in \Gamma$. We may also denote $\mu := \sigma_{(1,0)}$.

In this notations we deduce the following presentation of T:

Theorem 3.3.2. *T is generated by Γ and μ. The full set of relations is:*

$$(I^2 \mu)^2 = U^{-1}$$

$$(I^{-1} \mu)^5 = 1$$

$$(I\mu)^7 = 1$$

Proof. Recall the alternative description of the Thompson group T given in the Theorem(2.2.1):

$$T = \langle L, C' | I = LC'L, C'^3 = I'^4 = L^5 = (C'I'L)^7 = 1, I'^2 C' = C'I'^2 \rangle$$

We put dashes to distinguish these elements from generators of $SL(2, \mathbf{Z})$. Let us invert this presentation, i.e. put $P = L^{-1}$, $C = C'^{-1}$, $I = I'^{-1}$. Then we have:

$$T = \langle P, C | I = PCP, C^3 = I^4 = P^5 = (PIC)^7 = 1, I^2 C = CI^2 \rangle$$

Now we can send the group $\Gamma = SL(2, \mathbf{Z})$ to T by mapping $\begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$ to C, and $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ to I. Denote $U = CI = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\mu = IP$. Then we can express U in terms of μ and I using relation $PCP = I$:

$$U^{-1} = I^{-1}C^{-1} = I^{-1}(P^{-1}IP^{-1})^{-1} = (I^*P^*)^2 = (I^2\mu)^2$$

Relation $P^5 = 1$ is equivalent to $(I^{-1}\mu)^5 = 1$, because $P = I^{-1}\mu$. The relation $(PIC)^7 = 1$ may be replaced by $(CPI)^{-7} = 1$, but using $CP = P^{-1}I$ we have $(CPI)^{-1} = (P^{-1}II)^{-1} = I^2P = I\mu$, so we obtained the last relation $(I\mu)^7 = 1$. $\qquad\square$

Now the vector space V_0 associated to the cluster as explained above is $V_0 = \bigoplus_{s \in \Lambda} \mathbf{C}e_s$, and the action of μ_v is:

$$e_w \mapsto e_{\sigma_v(w)} + max(v \wedge w, 0)e_{-v}$$

$$e_v \mapsto -e_{-v}$$

And this is the formula for the action of μ in the representation of H, except for e_w when w is collinear with v.

Chapter 4

Cremona group

4.1 Introduction

The Cremona group is the group of automorphisms of the field $K = \mathbf{C}(x,y)$ of two independent commuting variables. Alternatively it is the group of birational automorphisms of $\mathbf{CP^2}$ or any other rational surface, because K is it's field of rational functions.

We construct a non-commutative associative algebra A, which is a consecutive localization of a free non-commutative algebra of two generators at the set of elements outside a commutator ideal. The main result is that the Cremona group acts on this non-commutative algebra by outer automorphisms. First proof of this statement is a direct computation: we provide non-commutative analogs of generators of Cremona group and verify that the relations known in commutative setting also hold in a non-commutative up to inner automorphisms.

Then we prove the same fact more abstractly, namely for a smooth proper algebraic variety X we construct a triangulated category $\widetilde{C}(X)$, which is a birational invariant of X. By this we mean that if X and Y are birationally equivalent, then $\widetilde{C}(X)$ and $\widetilde{C}(Y)$ are equivalent as triangulated categories. The category $\widetilde{C}(X)$ is a quotient of derived category of coherent sheaves that are left orthogonal to the structure sheaf of O_X by the full subcategory of complexes of sheaves with the support of codimension at least 1. Then we study the structure of $\widetilde{C}(\mathbf{CP^2})$ and prove that it is generated by one object P which is the image of $O(1)$. We also check that this object is preserved under equivalences, so the Cremona group will act by outer automorphisms on the ring of endomorphisms of this object. Then we compute that $RHom^0_{\widetilde{C}(\mathbf{CP^2})}(P,P) = A$, so we get an action of the Cremona group on the same non-commutative algebra as previous. Then we verify that two actions coincide.

4.2 Construction of the noncommutative algebra A

In what follows we will construct a non-commutative algebra A together with algebra homomorphism $\phi : A \to K$ such that:

1. $R = \mathbf{C} < x,y > \subset A$, R is an algebra of polynomials in two non-commuting variables.

2. $I = A[A,A]A$ is the kernel of ϕ

3. All the elements in $A \setminus I$ are invertible

Let us construct such an algebra by induction. Put $A_0 := R = \mathbf{C}<x,y>$, $I_0 := R[R,R]R = R(xy-yx)R$(just notice that for associative noncommutative algebras B the following holds $B[B,B]B = [B,B]B = B[B,B]$ and we call this submodule a commutator of B). We successively construct A_{i+1} as a localization of A_i at $A_i \setminus I_i$. I_i is just a commutator of A_i, denote $S_i = A_i \setminus I_i$. First let us explain what do we mean by localization: consider $B_i := A_i < u_s | s \in S_i >$ and let P_i be a bi-ideal in B_i generated by $u_s s - 1$ and $s u_s - 1$. We put $A_{i+1} := B_i / P_i$. We also have induced maps $A_i \to A_{i+1}$, so we may take an inductive limit $A := \varinjlim A_i$.
We verify the following:

Lemma 4.2.1. (1). $\phi_i : A_i \to K$ *is well defined and consistent with the maps* $A_i \to A_{i+1}$.
(2). $ker(\phi_i) = I_i$.

Corollary. $\phi : A \to K$ is well defined and $ker(\phi) = I$.
We can define ϕ on the inductive limit of A_k's which is A and the statement $ker(\phi) = I$ follows from the corresponding statement about ϕ_k.

Proof. For $i = 0$ we define an obvious map $\phi_0 : R \to \mathbf{C}[x,y] \subset K$. It has the commutator I_0 as the kernel.
By induction we suppose that $ker(\phi_i) = I_i$ for $i = k$. Then ϕ_k maps S_k to non-zero thus invertible elements of K, so we may extend the map $\phi_k(u_s) := \phi_k(s)^{-1}$ to B_k and it is easily seen that P_k maps to 0, so we may define ϕ_{k+1} in a consistent way on $A_{k+1} = B_k / P_k$.
So S_k may be characterized as the set of elements that maps by ϕ_k not to zero, so it is a multiplicative system. We may verify the following equality in A_{k+1}:
$u_s u_t = u_{ts}$, because $ts(u_{ts} - u_s u_t) = (ts u_{ts} - 1) + t(s u_s - 1) u_t + (t u_t - 1) \in P_k$. From the other side $(u_{ts} ts - 1)(u_{ts} - u_s u_t) \in P_k$. It follows that $(u_{ts} - u_s u_t) \in P_k$.
Let us prove now that $ker(\phi_{k+1}) = I_{k+1}$ by induction on k. Commutator of A_{k+1} is contained in the kernel of ϕ_{k+1}, because K is commutative, so we have to prove another inclusion. For any element $b \in B_k$ there exist $a, s \in A_k$ such that $b - a u_s \in B_k[B_k, B_k] + P_k$, because

$$a_1 u_{s_1} a_2 u_{s_2} .. a_n - a_1 a_2 ... a_n u_{s_1} u_{s_2} .. u_{s_{n-1}} \in B_k[B_k, B_k]$$

$$a u_{s_1} u_{s_2} .. u_{s_k} - a u_{s_k ... s_2 s_1} \in P_k$$
$$a u_s + b u_p - (ap + bs) u_{ps} \in B_k[B_k, B_k] + P_k$$

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In particular for any $b' \in A_{k+1} = B_k/P_k$ there exist $a', s' \in A_k$, such that $b' - a'u'_s \in A_{k+1}[A_{k+1}, A_{k+1}] = I_{k+1}$. So if $\phi_{k+1}(b') = 0$ then $\phi_{k+1}(a'u_{s'}) = \phi_k(a')\phi_k(s')^{-1} = 0$, so $\phi_k(a') = 0$ and by induction hypothesis $a' \in A_k[A_k, A_k] = I_k$. This proves that $b' \in I_{k+1}$. $\qquad\square$

To define a morphism from an algebra A to any algebra B we should prescribe images of x, y. Then a map from A_0 to B will be defined. If the image of elements not in commutator of A_k is invertible in B, then we can extend it to the map from A_{k+1} to B.

Lemma 4.2.2. *Suppose $a, b \in A$ such that the field morphism $i : K \to K$ given by $i : x \mapsto \phi(a)$, $i : y \mapsto \phi(b)$ is injective. Then there is a well defined algebra homomorphism $f : A \to A$ with $f(x) = a$, $f(y) = b$.*

Proof. Given $a, b \in A$ we uniquely define $f_0 : A_0 \to A$ by prescribing $f_0(x) = a$, $f_0(y) = b$. Now by induction suppose that the map $f_k : A_k \to A$ is well defined and $f_k(x) = a$, $f_k(y) = b$. Now we use $\phi \circ f_k = i \circ \phi_k$. It follows that $\phi(f(S_k))$ is non-zero, because $\phi_k(S_k)$ is non-zero and i is injective. So $f_k(S_k)$ is invertible and we can uniquely extend f_k to $f_{k+1} : A_{k+1} \to A$. $\quad\square$

To see that A is indeed non-commutative and has non-trivial elements(it's not obvious from the construction that A doesn't coincide with K) let us construct algebra morphisms from A to the following algebra $M = M_{k\times k}(K[[\epsilon]])$ - algebra of matrices over $K[[\epsilon]]$ series with coefficients in K. Let us choose two matrices of the form $X = xId + \epsilon X_1$, $Y = yId + \epsilon Y_1$, where X_1, Y_1 are some matrices in M. We can invert element of M if and only if we can invert it's reduction modulo ϵ. This justifies that the map $R \to M$ given by $x \mapsto X$, $y \mapsto Y$ may be extended to the morphism $A \to M$.
For example we see that $XY - YX = \epsilon^2(X_1Y_1 - Y_1X_1)$ may be non-zero, implying that $xy - yx$ is nonzero in A, so A is indeed non-commutative.

Lemma 4.2.3. *The natural map $i : A_0 = \mathbf{C} < x, y > \to A$ is an injection.*

Proof. Let us consider an algebra $M = M_{k\times k}(K[[\epsilon]])$ of matrices of big enough size k with entries in the ring of formal power series $K[[\epsilon]]$. For any pair of complex matrices $S, T \in M_{k\times k}(\mathbf{C})$ it exists a ring homomorphism $f : A \to M$ given by $x \mapsto xId + \epsilon S$, $y \mapsto yId + \epsilon T$. So if some non-commutative polynomial $a \in A_0$ was in the kernel of the map $i : A_0 \to A$, then $f \circ i(a) = 0$ and it's a polynomial in ϵ with coefficients in matrices, in particular for the highest power of ϵ we would have $a(S, T) = 0$. So we would have that some non-commutative polynomial is identically 0 if we put

as arguments any complex matrices of any size. This is false, so we proved the lemma. $\qquad\square$

4.3 Direct extension of the Cremona group to non-commutative variables

Suppose $a = \begin{pmatrix} P(x) & Q(x) \\ R(x) & S(x) \end{pmatrix} \in PGL(2, k(x))$. It gives an automorphism of K over $\mathbf{C}$:

$$a : (x, y) \mapsto (x, \frac{yP(x) + Q(x)}{yR(x) + S(x)})$$

Define also a map $\tau : (x, y) \mapsto (y, x)$.

Let us summarize the results of [9]:
The Cremona group Cr is generated by τ and $PGL(2, k(x))$ with the described action.
Consider an exact sequence of groups $1 \to PGL(2, k(x)) \to B \to PGL(2, k) \to 1$, where B preserves the subfield $k(x) \subset k(x, y)$. $D = PGL(2, k) \times PGL(2, k) = D_1 \times D_2 \subset B$ preserves both $k(x)$ and $k(y)$, where the action of $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in D_1$ is given by $(x, y) \mapsto (\frac{ax+b}{cx+d}, y)$. And the action of $N = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in D_2$ is given by $(x, y) \mapsto (x, \frac{ay+b}{cy+d})$.

The Cremona group is generated by τ, B and by [9] all the relations are the consequences of the following:
1. $\tau^2 = 1$
2. $\tau D \tau = D$
3. $(\tau e)^3 = a$
in the last relation $e : (x, y) \mapsto (x, \frac{x}{y}) \in B$ and $a : (x, y) \mapsto (\frac{1}{x}, \frac{1}{y}) \in D$

Now let us reinterpret these relations: we denote $D_1 = D_2 = PGL(2, k)$ and $G = PGL(2, k(x))$. Field inclusion $k \subset k(x)$ induces group inclusion $D_2 = G(k) \subset G = G(k(x))$. D_1 acts on $k(x)$, so we have also the action of D_1 on coefficients of $PGL(2, k(x)) = G$ which leaves D_2 stable.
Consider the group generated by G and $\mathbf{Z}/2 =< 1, \tau >$, on which we put the following relations:
1. $D_1 = \tau D_2 \tau$ normalizes G and the induced action of D_1 on G is induced from the action of $PGL(2, k) = D_1$ on $k(x)$.

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2. $(\tau e)^3 = a$, where $e = \begin{pmatrix} 0 & x \\ 1 & 0 \end{pmatrix} \in G$ and $a = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \times \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in D_1 \times D_2$.

The group B from previous notations is included in the exact sequence: $1 \to G \to B \to D_1 \to 1$. So the the group that we described is the Cremona group, as this set of relations is equivalent to the previous one.

Let us now introduce an action on the previously constructed ring A. $a \in GL(2, k(x))$ is a matrix $\begin{pmatrix} P(x) & Q(x) \\ R(x) & S(x) \end{pmatrix}$. It gives automorphisms of A:

$$t_a : (x, y) \mapsto (x, (yR(x) + S(x))^{-1}(yP(x) + Q(x)))$$

$$p_a : (x, y) \mapsto (x, (P(x)y + Q(x))(R(x)y + S(x))^{-1})$$

Before proceeding further, let us show how those automorphisms are related: the dual algebra for $R = \mathbf{C} < x, y >$ would be $R^{op} = \mathbf{C} < x, y >$. They are isomorphic as algebras:

$$\rho : \qquad\qquad R \longrightarrow \qquad\qquad R^{op}$$
$$(x, y) \longmapsto \qquad\qquad (x, y)$$

In particular $\rho(xy) = yx$ and $\rho(x^2 y^3 x^7) = x^7 y^3 x^2$.

We can extend ρ as an isomorphism between A and A^{op}, so we would have $\rho(P^{-1}Q) = \rho(Q)\rho(P)^{-1}$. Now we observe that formulas for t_a and p_a are dual:

$$\rho \circ t_a = p_a \circ \rho$$

So the result we would prove for t_a would also be true for p_a.

Define a map $\tau : (x, y) \mapsto (y, x)$, and consider a group Cr^{nc} generated by τ and t_a for all $a \in GL(2, k(x))$. Commutator $A[A, A]A$ is preserved by any automorphism, so Cr^{nc} also acts on $A/A[A, A]A = \mathbf{C}(x, y)$ i.e. maps to commutative Cremona group Cr. Let us denote Cr^{in} the subgroup of Cr^{nc} generated by inner automorphisms, i.e. of the type $(x, y) \mapsto (rxr^{-1}, ryr^{-1})$ for some invertible $r \in A^*$. We want to prove that Cr^{in} is the kernel of the map $Cr^{nc} \to Cr$.

Theorem 4.3.1. *The kernel of the map $Cr^{nc} \to Cr$ consists of inner automorphisms of A.*

Proof. For this we need to check that when we lift a relation for generators of Cr to Cr^{nc} it belongs to Cr^{in}.

Lemma 4.3.1. $t_a t_b = t_{ab}$ *for any* $a, b \in GL(2, k(x))$

Proof. Let $b = \begin{pmatrix} P(x) & Q(x) \\ R(x) & S(x) \end{pmatrix}$, $a = \begin{pmatrix} P_1(x) & Q_1(x) \\ R_1(x) & S_1(x) \end{pmatrix}$. The action of a on A is given by

$$t_b : (x, y) \mapsto (x, (yR(x) + S(x))^{-1}(yP(x) + Q(x))) =: (x, y_1)$$

$$t_a : (x, y_1) \mapsto (x, (y_1 R_1(x) + S_1(x))^{-1}(y_1 P_1(x) + Q_1(x))) =: (x, y_2)$$

$$y_1 R_1(x) + S_1(x) = (yR + S)^{-1}(y(PR_1 + RS_1) + (QR_1 + SS_1))$$

$$y_1 P_1(x) + Q_1(x) = (yR + S)^{-1}(y(PP_1 + RQ_1) + (QP_1 + SQ_1))$$

$$t_a t_b : (x, y) \mapsto (x, (y(PR_1 + RS_1) + (QR_1 + SS_1))^{-1}(y(PP_1 + RQ_1) + (QP_1 + SQ_1)))$$

So $t_a t_b = t_{ab}$, because

$$ab = \begin{pmatrix} P_1(x)P(x) + Q_1(x)R(x) & P_1(x)Q(x) + Q_1(x)S(x) \\ R_1(x)P(x) + S_1(x)R(x) & R_1(x)Q(x) + S_1(x)S(x) \end{pmatrix}$$

$\square$

When $GL(2, k(x)) \subset Cr^{nc}$ maps to Cr it has a center in the kernel, so suppose $d = \begin{pmatrix} d(x) & 0 \\ 0 & d(x) \end{pmatrix}$ is a diagonal matrix, then $t_d : (x, y) \mapsto (x, d(x)^{-1}yd(x))$ is a conjugation by $d(x)$, so $t_d \in Cr^{in}$.

Let us verify a first relation: suppose $m = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in D_2 = GL(2, k)$. Then $m_1 = \tau t_m \tau : (x, y) \mapsto ((cx + d)^{-1}(ax + b), y)$. From this formula we easily see that m_1 normalizes a subgroup $GL(2, k(x)) \subset Cr^{nc}$ and it's action is induced from it's action on $k(x)$, so the first relation is verified.

Let us now verify the second relation: $(\tau e)^3 = a$. Remind that

$$\tau : (x, y) \mapsto (y, x)$$

$$e : (x, y) \mapsto (x, y^{-1}x)$$

$$a : (x, y) \mapsto (x^{-1}, y^{-1})$$

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Then we calculate

$$\tau e : (x, y) \mapsto (y^{-1}x, x)$$

$$(\tau e)^2 : (x, y) \mapsto (x^{-1}y^{-1}x, y^{-1}x)$$

$$a^{-1}(\tau e)^3 : (x, y) \mapsto (x^{-1}yxy^{-1}x, x^{-1}yx)$$

The last automorphism is a conjugation by $x^{-1}y$, so $a^{-1}(\tau e^3) \in Cr^{in}$. $\quad\square$

4.4 The proof using derived categories

In this section we would construct a triangulated category $\widetilde{C}$ with a specific isomorhism class of objects P, and it would be a birational invariant of rational surfaces. As a consequence, the Cremona group will act on this category by equivalences and will preserve P, so it will act by outer automorphisms on the graded ring $H^*(End_C(P))$, in particular it's degree 0 part coincides with the non-commutative algebra A considered previously. We would then verify that the action of Cr on A coincides with the one given by explicit formulas.

4.4.1 Notations

First let us make some conventional notations:

D - bounded derived category of coherent sheaves on $\mathbf{CP}^2$

$\pi : X \to \mathbf{CP}^2$ - sequence of blow-ups of $\mathbf{CP}^2$ or equivalently a regular map invertible outside a finite set of points.

$D(X)$ - bounded derived category of coherent sheaves on X. By omitting X we would mean the corresponding category for $\mathbf{CP}^2$.

$O, O(1), O(2)$ - line bundles $O_{\mathbf{CP}^2}, O_{\mathbf{CP}^2}(1), O_{\mathbf{CP}^2}(2)$, we would also see them as objects of D placed at degree 0. Depending on the context we would denote by the same symbols the objects $L\pi^*O, L\pi^*O(1), L\pi^*O(2) \in D(X)$ for the functor $L\pi^* : D \to D(X)$.

E - exceptional curve of a blow-up.

$\widetilde{D}(X)$ - full subcategory of $D(X)$ consisting o complexes of sheaves left orthogonal to O_X.

$D^1(X)$ - full subcategory of $D(X)$ of complexes of sheaves with the dimension of support ≤ 1.

$\widetilde{D}^1(X)$ - full subcategory of $D(X)$ of objects which are both in $\widetilde{D}(X)$ and in $D^1(X)$.

$RHom(F, G)$ would usually be a derived functor in category $D(X)$. If both

F, G belong to D, we consider $RHom$ in D, because $L\pi^*$ is a fully faithful functor.

$\widetilde{C}(X) = \widetilde{D}(X)/\widetilde{D}^1(X)$ - quotient triangulated category.

$P \in \widetilde{C}(X)$ - isomorphism class of object $O(1)$.

$a, b, e \in Hom(O, O(1))$ - sections corresponding to X, Y, Z.

$R = \mathbf{C} < x, y >$ - ring of polynomials in non-commutative variables.

A - non-commutative algebra, constructed in previous sections.

$K = \mathbf{C}(x, y)$

$A^\bullet[1]$ a shift of a complex $A^\bullet$ to the left

4.4.2   Strategy of the proof

First we prove that the quotient category $\widetilde{C} = \widetilde{D}/\widetilde{D}^1$ is split generated by one object P and $RHom_{\widetilde{C}}(P, P) \cong A^\bullet$ is some dg-algebra. We prove that this dg-algebra is concentrated in non-positive degrees and $H^0(A^\bullet) = A$ is a non-commutative algebra that we constructed earlier.

Next we prove that the functor $L\pi^* : D \to D(X)$ induces an equivalence of categories $\widetilde{C}$ and $\widetilde{C}(X)$ for any sequence of blow-ups $\pi : X \to \mathbf{CP}^2$. From this we construct an action of the Cremona group on the category $\widetilde{C}$ by equivalences of category.

Then we verify, that this action preserves the isomorphism class of object P, so the Cremona group would act by outer automorphisms on A.

At last we compute this action on elements $x, y \in A$ and verify that it coincides with our explicit formulas.

To start with, we show how this construction works for the category D without taking left orthogonal to the structure sheaf, so we get an action of the Cremona group on the category $C = D/D^1$. In this case we just recover the action of Cremona group on the field $K = \mathbf{C}(\mathbf{CP}^2)$. The functor $\widetilde{C} \to C$ induced from inclusion $\widetilde{D} \subset D$ corresponds to the "commutativization" morphism $comm : A \to K$.

4.4.3   Action of the Cremona group on K through derived categories

$\mathbf{C}(X)$ the field of rational functions on X. $D^b(\mathbf{C}(X) - mod)$ - bounded derived category of finitely generated modules over it.

Lemma 4.4.1. *The functor of restriction to a generic point induces an*

equivalence of triangulated categories $C(X) = D(X)/D^1(X)$ and $D^b(\mathbf{C}(X)-mod)$.

Proof. Let $\eta(F)$ be a restriction of a coherent sheaf F from X to a generic point. Clearly $\eta(F)$ is a finite rank vector space over $\mathbf{C}(X)$. It induces a functor $\eta : D(X) \to D^b(C(X) - mod)$. Complexes of sheaves with the support of codimension ≥ 1 are exactly the ones that become acyclic after applying this functor. So we have a functor from the quotient category $C(X) = D(X)/D^1(X)$ to $D^b(\mathbf{C}(X) - mod)$. All the rank 1 sheaves are isomorphic in $C(X)$, let us denote by O it's isomorphism class. All the vector bundles on X are obtained by extensions from rank 1 sheaves, so the triangulated subcategory of $C(X)$ generated by cones and shifts of O contains images of vector bundles and thus is equivalent to $C(X)$.

If Z is a closed sub-scheme of X and $i : U \hookrightarrow X$ is a complement of Z, then $D(X)/D_Z(X)$ is equivalent to $D(U)$, where the equivalence is induced by a restriction functor $i^* : D(X) \to D(U)$. It's a well-known fact. In particular if U is affine then there is a commutative algebra $Q = O_U(U)$ and $D(U)$ is equivalent to $D^b(Q - mod)$. In particular $Hom^i_{D(X)/D_Z(X)}(O_U, O_U) = 0$ for $i \neq 0$ and $Hom^0_{D(X)/D_Z(X)}(O, O) = O_U(U)$.

By definition of the quotient category

$$Hom^i C(X)(A, B) = \varinjlim_f Hom^i_{D(X)}(A, F)$$

The limit in the right hand side is taken over morphisms $f : B \to F$ such that $Cone(f) \in D^1(X)$. For a given f let Z_1 be a support of $Cone(f)$. It is a closed proper sub-scheme of X. We choose a closed sub-scheme Z, such that $Z_1 \subset Z \subset X$ and the complement of Z in X is affine. So we may rewrite the formula as follows:

$$Hom^i C(X)(A, B) = \varinjlim_Z \varinjlim_g Hom^i_{D(X)} D(A, F)$$

First limit is over closed sub-schemes Z with affine complement, second limit is over g such that $Cone(g)$ is supported in Z. But now the second limit equals to $Hom^i_{D(X)/D_Z X}(F, G) = Hom^i_{D(U)}(F, G)$. In particular $Hom^i_{C(X)}(O, O) = 0$ for $i \neq 0$ and $Hom^0_{C(X)}(O, O) = \varinjlim_Z O_U(U) = \mathbf{C}(X)$. So the functor induced from restriction to generic point $C(X) = D(X)/D^1(X) \to D^b(\mathbf{C}(X) - mod)$ is fully faithful on O. And because every object of $C(X)$ is a direct sum of shifts of O, it is fully faithful, thus an equivalence. $\qquad\square$

So we recover a field $K = \mathbf{C}(X)$ of rational functions from derived category of coherent sheaves. Let us now recover the action of the group of

birational automorphisms on it. Suppose $\pi : X \to Y$ is a blow-up with a smooth center.

Lemma 4.4.2. . *The functor between quotient categories $\Psi : C(Y) \to C(X)$ induced by $L\pi^*$ is an equivalence and is a composition of equivalences $C(X) \xleftarrow{\eta_X^*} D^b(K - mod) \xrightarrow{\eta_Y^*} C(Y)$.*

Proof. If η_X is an embedding of a generic point in X and η_Y in Y, then clearly $\pi \circ \eta_X = \eta_Y$, so in the derived categories of coherent sheaves we have a functor isomorphism $\eta_X^* \circ L\pi^* = \eta_Y^*$

Let E be an exceptional curve of this blow-up, then we have a semi-orthogonal decomposition $D(X) = < L\pi^* D(Y), O_{|E} >$ and also $D^1(X) = < L\pi^* D^1(Y), O_{|E} >$. So we can use that the quotient triangulated category $< A, B > /B$ is equivalent to A. It follows that $D(X)/ < O_{|E} > \simeq L\pi^* D(Y)$, so

$$D(X)/D^1(X) = (D(X)/ < O_{|E} >)/L\pi^* D(Y) \simeq L\pi^* D(Y)/L\pi^* D^1(Y).$$

So $L\pi^*$ induces an equivalence between $C(Y)$ and $C(X)$. $\qquad\square$

It means that the quotient categories are independent of the rational surface, thus implying that the Cremona group acts on both. All line bundles placed in degree 0 become isomorphic in this category $C(X)$, so the Cremona group preserves the isomorphism class of line bundles and acts in a natural way on the endomorphism ring of this class, which is K. It's easy to verify that this action coincides with the natural action of $Cr = Aut(\mathbf{C}(x, y)/\mathbf{C})$ on $K = \mathbf{C}(x, y)$.

4.4.4   Construction of the birational invariant

Here we make a general construction, which to a smooth proper algebraic manifold X associates a triangulated category $\widetilde{C}$ invariant under birational isomorphisms.
Let $\widetilde{D}(X)$ be a full subcategory of $D^b(Coh(X))$ consisting of objects left orthogonal to O_X, i.e. complexes of sheaves F^*, such that $RHom_{D(X)}(F^*, O_X)$ is acyclic. It is a triangulated category and it may be empty for some manifolds.
Consider it's full triangulated subcategory $\widetilde{D}^1(X)$ consisting of the complexes of sheaves that are left orthogonal to O_X and have a positive codimension of support, it means the cohomology of the complex have support on the

subvarieties of dimension strictly less than the dimension of X.

Define $\widetilde{C}(X)$ as a quotient of triangulated categories $\widetilde{C}(X) = \widetilde{D}(X)/\widetilde{D}^1(X)$ in a sense of Verdier[22].

So for any object Y in $\widetilde{D}(X)$ let Q_Y be a category of morphisms $f : Y \to Z$, such that $Cone(f)$ is isomorphic to an object of $\widetilde{D}^1(X)$. Then by results of Verdier that we reproduce from formula (12.1) of [8] we compute the morphisms in a quotient category this way:

$$Ext^i_{\widetilde{D}(X)/\widetilde{D}^1(X)}(U, Y) = \varinjlim_{(Y \to Z) \in Q_Y} Ext^i_{\widetilde{D}(X)}(U, Z) \qquad (4.4.1)$$

Theorem 4.4.1. *If smooth X and Z are birationally equivalent, then $\widetilde{C}(X)$ and $\widetilde{C}(Z)$ are equivalent as triangulated categories.*

Proof. In virtue of the results of [7](weak factorization theorem) any birational automorphism may be decomposed as a sequence of blow-ups and blow-downs with smooth centers. It means that it is enough to prove our theorem for such kind of morphisms.

Let $\pi : Z \to X$ be a blow-up of X with a smooth center Y. We have a pair of adjoint functors $(L\pi^*, R\pi_*$ on $D(X), D(Z)$ and moreover $L\pi^*O_X = O_Z$ and $R\pi_*O_Z = O_X$.

Let us denote $D_0(Z)$ a full subcategory of $D(Z)$ consisting of objects F such that $R\pi_*F$ is acyclic, and let $L\pi^*D(X)$ a full subcategory of $D(Z)$ consisting of images of $L\pi^*$.

Lemma 4.4.3. (1). *$D_0(Z)$ and $L\pi^*D(X)$ are triangulated categories.*

(2). *The natural transformation $F \to R\pi_*L\pi^*F$ in $D(X)$ is an isomorphism.*

(3). *$L\pi^*$ induces an equivalence of triangulated categories $D(X)$ and $L\pi^*D(X)$.*

(4). *$L\pi^*D(X)$ is left orthogonal to $D_0(Z)$, i.e. $RHom_{D(Z)}(L\pi^*F, G)$ is acyclic.*

(5). *For any object $F \in D(X)$ there is an exact triangle $F_1 \to F \to F_0$, where $F_0 \in D_0(Z), F_1 \in L\pi^*D(X)$, and moreover such a triangle is unique up to isomorphism. F_1 is isomorphic to $L\pi^*R\pi_*F$.*

(6). *$D_0(Z)$ is supported on exceptional divisor of the blow-up.*

Proof. (1) trivial. (2) is a local statement, but locally on X any object of $D(X)$ has a free bounded resolution by a structure sheaf, for which the map $O_X \to R\pi_*L\pi^*O_X$ is an isomorphism. (3) follows from projection formula $RHom_{D(Z)}(L\pi^*F, L\pi^*G) = RHom_{D(X)}(F, R\pi_*L\pi^*G)$ and (2). (4) also follows from the projection formula. The triangle in (5) is constructed

by completing a natural map $L\pi^* R\pi_* F \to F$ to an exact triangle, map coming from the adjunction of identity morphism $R\pi_* F \to R\pi_* F$. Actually F_1 represents a functor $X \mapsto Hom_{L\pi^* D(X)}(X, F)$ and F_0 represents a functor $X \mapsto Hom_{D_0(Z)}(F, X)^*$. (6) follows from the definition $R\pi_*$, because π is an isomorphism outside an exceptional divisor. $\qquad \square$

A concise way to summarize lemma is the statement that $D(Z)$ admits a semi-orthogonal decomposition $D(Z) = < D_0(Z), L\pi^* D(X) >$. Actually we have a stronger result proved at Proposition 3.4 of [17]: there is a semi-orthogonal decomposition $D(Z) = < D(Y)_{-r+1}, .., D(Y)_1, L\pi^* D(X) >$, where r is a codimension of Y in X, but we would not use it here.
We may also introduce the full triangulated subcategory $D_1(Z) \subset D$, which consists of objects $K_Z^{-1} \otimes^L F$ for $F \in D_0(Z)$. Because of Serre duality it is a left orthogonal to $L\pi^* D(X)$ and we have a semi-orthogonal decomposition $D(Z) = < L\pi^* D(X), D_1(Z) >$.

Lemma 4.4.4. (1). $D_1(Z)$ is full triangulated subcategory of $\widetilde{D}^1(Z)$.
(2). There are semi-orthogonal decompositions

$$\widetilde{D}(Z) = < L\pi^* \widetilde{D}(X), D_1(Z) >$$

$$\widetilde{D}^1(Z) = < L\pi^* \widetilde{D}^1(X), D_1(Z) >$$

Proof. Objects is $D_1(Z)$ are left orthogonal to $L\pi^* D(X)$ and in particular to $O_Z = L\pi^* O_X$. The support of an object of $D(Z)$ doesn't change when we tensor it with a linear bundle, so the elements of $D_1(Z)$ are supported on an exceptional divisor of blow-up, because the elements of $D_0(Z)$ are, as we see from the last statement of the previous lemma. $\qquad \square$

This lemma implies the statement of the theorem, because to obtain

$$\widetilde{C}(Z) = \widetilde{D}(Z)/\widetilde{D}^1(Z) = < L\pi^* \widetilde{D}(X), D_1(Z) > / < L\pi^* \widetilde{D}^1(X), D_1(Z) >$$

we may first quotient $\widetilde{D}(Z)$ by it's full subcategory $D_1(Z)$, so we will have an equivalence

$$\widetilde{D}(Z)/\widetilde{D}^1(Z) \to L\pi^* \widetilde{D}(X)/L\pi^* \widetilde{D}^1(X)$$

And the latter is equivalent to $\widetilde{C}(X)$, because $L\pi^*$ is fully faithful functor.
$\qquad \square$

4.4.5 General statements about quotients of dg-categories

For convenience of the reader, let us summarize few facts concerning quotients of dg-categories. Our main reference would be [8].

Suppose we have a dg-algebra $A^\bullet$ over a field k with a unit. We can see it as a dg-category $\mathcal{A}$ with one object P and the endomorphism ring $Hom_\mathcal{A}(P,P) = A^\bullet$. Then from this dg-category with one object we construct a pre-triangulated category $\mathcal{A}^{pre-tr}$ (cf.[D, 2.4]), which has the formal sums $B = (\oplus_{j=1}^n P^{\oplus m_j}[k_j], q)$ as objects. Here q stands for a homogeneous degree one matrix

$$q_{ij} \in Hom_\mathcal{A}^1(P^{\oplus m_j}, P^{\oplus m_i})[k_i - k_j] = Mat_{m_i \times m_j}(A^{k_i - k_j + 1})$$

We ask that $q_{ij} = 0$ for $i \geq j$ and $dq + q^2 = 0$. The homomorphism between the formal sums $B = (\oplus_{j=1}^{j=n} P^{\oplus m_j}[k_j], q)$ and $B' = (\oplus_{j=1}^{j=n'} P^{\oplus m_j'}[k_j'], q')$ is given by matrices $f = (f_{ij})$ where

$$f_{ij} \in Hom(P^{\oplus m_j}, P^{\oplus m_i'})[k_i' - k_j] = Mat_{m_i' \times m_j}(A^{k_i' - k_j})$$

The differential on homomorphism groups is $df = (df_{ij}) + q'f + (-1)^l fq$, where $l = deg(f_{ij})$. Define the triangulated category $\mathcal{A}^{tr}$ as the homotopy category of $\mathcal{A}^{pre-tr}$, which means $Hom_{\mathcal{A}^{tr}}(X,Y) = H^0(Hom_{\mathcal{A}^{pre-tr}}(X,Y))$. The category $\mathcal{A}^{pre-tr}$ contains cones. Let us consider $M \in Hom_\mathcal{A}(P^m, P^m)$ a $m \times m$-matrix with coefficients in A of degree 0. An element $Cone(M)$ would be an object $(P^m \oplus P^m[1], q)$, where $q_{21} = M[-1]$ and $q_{11} = q_{12} = q_{22} = 0$. Naturally this element goes to the cone of a morphism M in the triangulated category. In the set up of [8] if $\mathcal{B}$ is a full dg-subcategory of $\mathcal{A}$, then the quotient dg-category $\mathcal{A}/\mathcal{B}$ has the same objects as $\mathcal{A}$ and for every object $X \in \mathcal{B}$ it has an additional morphism $u_X \in Hom_{\mathcal{A}/\mathcal{B}}^{-1}(X,X)$ with differential $d(u_X) = id_X$. The resulting homotopy category $(\mathcal{A}/\mathcal{B})^{tr}$ is equivalent to the quotient $\mathcal{A}^{tr}/\mathcal{B}^{tr}$.

4.4.6 Localization in dg-sense

Suppose we have a closed morphism in dg-category $\mathcal{A}$. It will give some morphism in the homotopy category $\mathcal{A}^{tr}$ and we want to understand what does it mean to invert this morphism. It corresponds to taking a quotient $\mathcal{A}/Cone(a)$, so that $(\mathcal{A}/Cone(a))^tr$ is a localization of $\mathcal{A}^{tr}$ at a.

Let us now consider a dg-category $\mathcal{A}$ with one object P, where $Hom_\mathcal{A}(P,P) = A^\bullet$ is a dg-algebra concentrated in non-positive degrees. Let $Q = Cone(a) \in$

$\mathcal{A}^{pre-tr}$ for some element $a \in A^0$ of degree 0. We want to consider a quotient of dg-category $D_1 =< P, Q >$ which has two objects P, Q by dg-subcategory $D_2 =< Q >$ with one object Q. Both D_1 and D_2 are seen as full subcategories of $\mathcal{A}^{pre-tr}$.

Lemma 4.4.5. *Quotient dg-category $D_1/D_2 =< P, Q > / < Q >$ is equivalent to a dg-category with one object P and $End_{D_1/D_2}(P) = B$ - a dg-algebra, concentrated in non-positive degrees. Moreover $H^0(B)$ is a localization of $H^0(A)$ at a.*

Proof. First let us write homomorphism groups in the category D_1:

$End_{D_1}(P) = A^\bullet$

$Hom_{D_1}(P, Q) = A^\bullet \oplus A^\bullet[1] = A^\bullet e_0 \oplus A^\bullet e_{-1}$

$Hom_{D_1}(Q, P) = A^\bullet \oplus A^\bullet[-1] = A^\bullet f_0 \oplus A^\bullet f_1$

$End_{D_1}(Q) = A^\bullet[1] \oplus A^\bullet \oplus A^\bullet \oplus A^\bullet[-1] = A^\bullet e_0 f_1 \oplus A^\bullet e_0 f_0 \oplus A^\bullet e_{-1} f_1 \oplus A^\bullet e_{-1} f_0$

We would also have the following relations: $f_1 e_0 = f_0 e_{-1} = 0$, $f_1 e_{-1} = f_0 e_0 = 1$, $id_Q = e_0 f_0 + e_{-1} f_1$. Lower indices indicate the degrees of homomorphisms. Also all e_{-1}, e_0, f_0, f_1 commute with $A^\bullet$.

On these spaces there would be a differential: $de_{-1} = ae_0$, $df_0 = -f_1 a$.

The procedure described in [8] suggests that the quotient category would have one more homomorphism $u \in End(Q)$ of degree -1, such that $du = id_Q$. So as an $A^\bullet$-algebra $B = End_{D_1/D_2}(P)$ would be freely generated by four elements: $t = f_1 u e_0$, $x = f_1 u e_{-1}$, $y = f_0 u e_0$, $z = -f_0 u e_{-1}$. Of them t would be of degree 0, x, y - of degree -1, and z of degree -2. One verifies the following formulas: $dt = 0$, $dx = ta - 1$, $dy = 1 - at$, $dz = ax + ya$. The elements of B of degree 0 are $A^0 < z >$ - non-commutative polynomials in z with coefficients of degree 0 in A, the elements of degree -1 are $a_1 x a_2 + a_3 y a_4 + a_5$, where a_i are of degree 0 and a_5 is of degree -1. So $H^0(B) = H^0(A^\bullet)[z]/(az - 1, za - 1)$ which is a localization of $H^0(A^\bullet)$ at a, and the lemma is proved. □

Comment. At this point it is not clear, whether the algebras that we obtain have non-trivial cohomologies in negative degrees.

For further use we should improve this lemma for cones of more general type. As before we consider $D_1 =< P, Q >$, $D_2 =< Q >$ full subcategories of $\mathcal{A}^{pre-tr}$, but now $Q = Cone(M)$ where M is a matrix over $A^\bullet$ of homogeneous degree 0. We would prove:

Lemma 4.4.6. *Suppose that* $M = \begin{pmatrix} 1_{k \times k} & 0 \\ 0 & a \end{pmatrix} + dM_{-1}$, $a \in A^\bullet$ *is a degree 0 element. Then* $End_{D_1/D_2}(P) = B$ *- a dg-algebra, concentrated in non-positive degrees and* $H^0(B)$ *is a localization of* $H^0(A^\bullet)$ *at* a.

Proof. Remember that $Cone(M) = (P^{\oplus m} \oplus P^{\oplus m}[1], q = \begin{pmatrix} 0 & M \\ 0 & 0 \end{pmatrix})$. So we would have the basis e_0^i, e_{-1}^i, $i = 1..k+1$ of $Hom_{D_1}(P, Q)$ as an $A^\bullet$-module. We would for simplicity denote by the column vectors $e_0 = \begin{pmatrix} e_0^1 \\ .. \\ e_0^{k+1} \end{pmatrix}$, $e_{-1} = \begin{pmatrix} e_{-1}^1 \\ .. \\ e_{-1}^{k+1} \end{pmatrix}$. Moreover we can multiply these elements by $A^\bullet$ from the left as well as from the right and these multiplications commute. $de_{-1} = Me_0$. Then we can pick a dual basis of $Hom_{D_1}(Q, P)$: f_0^i, f_1^i, $i = 1..k+1$, and similarly denote row vectors $f_0 = (f_0^1, .., f_0^{k+1})$, $f_1 = (f_1^1, .., f_1^{k+1})$. Then $df_0 = -f_1 M$. We would have $f_1^i e_0^j = f_0^i e_{-1}^j = 0$, $f_1^i e_{-1}^j = f_0^i e_0^j = \delta_{ij}$, $id_Q = \Sigma_{i=0}^{k+1}(e_0^i f_0^i + e_{-1}^i f_1^i)$.

Again the quotient category is obtained adding element u of degree -1 to the space $End(Q)$ such that $du = id_Q$. So $B = End_{D_1/D_2}(P)$ has no elements in positive degree. In degree 0 it is freely generated by $f_1^i u e_0^j$ as A^0-algebra. So B^0 as an algebra is isomorphic to $A^0 < u_{ij} >$ - non-commutative polynomials over A^0 and u_{ij} maps through isomorphism to $f_1^i u e_0^j$. Further notice that $d(f_0^i u e_0^j) = (df_0^i) u e_0^j + f_0^i (du) e_0^j = -(\Sigma_k m^{ik} f_1^k u e_0^j) + f_0^i Id_Q e_0^j = -(\Sigma_k dm_{-1}^{ik}(f_1^k u e_0^j)) - (m^{ii} - dm_{-1}^{ii}) f_1^i u e_0^j + \delta_{ij}$, where m_{-1}^{ij} stand for entries of the matrix M_{-1}. We know that $d(f_1^k u e_0^j) = 0$, so modulo the border $\delta_{ij} = (m^{ii} - dm_{-1}^{ii}) f_1^i u e_0^j$. But $(m^{ii} - dm_{-1}^{ii}) = 1$ for $i \neq k+1$, so in $H^0(B)$ we have $\delta_{ij} = f_1^i u e_0^j$ for $i \neq k+1$. The similar calculation of $d(f_0^i u e_{-1}^j)$ will show that $\delta_{ij} = f_1^i u e_0^j$ for $j \neq k+1$. So we deduce that $H^0(B)$ is generated by $t = f_1^{k+1} u e_0^{k+1}$ over $H^0(A^\bullet)$. The border will be generated by $d(f_0^{k+1} u e_0^{k+1}) = -at + 1$ and $d(f_1^{k+1} u e_{-1}^{k+1}) = -ta + 1$. So $H^0(B)$ is a localization of $H^0(A^\bullet)$ at a, which proves the lemma. $\square$

4.4.7 Computation of the quotient category for a rational surface

Let us calculate explicitly $\widetilde{C}(\mathbf{CP}^2) = \widetilde{D}/\widetilde{D}^1$. It is known that a derived category D of coherent sheaves on $\mathbf{CP}^2$ is generated by $O, O(1), O(2)$. Con-

sider a subcategory $\widetilde{D}$ which is left orthogonal to O, it is generated by $O(1)$ and $O(2)$. Denote $V = RHom_D(O(1), O(2)) = Hom_{\mathbf{CP^2}}(O(1), O(2))$ - a 3-dimensional vector space.

Denote $P_1 = O(1), P_2 = O(2)$, then $Hom_D(P_1, P_2) = V$ and all the other higher hom's between these objects vanish. $End(P_1) = End(P_2) = \mathbf{C}$. Any object in D may be written as a direct sum of shifts of the complexes $P_1^m \to P_2^n$ and the arrow is given by an $n \times m$ matrix with elements in V.

D^1 is a full subcategory of D, consisting of complexes of coherent sheaves with the dimension of support at most 1. Then $\widetilde{D}^1$ is the intersection of D^1 and $\widetilde{D}$, more precisely it is full subcategory of D, consisting of objects left orthogonal to O and with a dimension of support at most 1. The condition of left orthogonality to O means that objects in $\widetilde{D}^1$ are direct sums of shifts of $Cone(M)$, where $M : O(1)^{\oplus m} \to O(2)^{\oplus n}$. The condition to have support of dimension at most 1 means, that the map given by M is generically invertible, which is exactly the case when $m = n$ and the commutative determinant of M is non-trivial.

Choose a non-zero element $e \in V$. First we would prove that $\widetilde{D}/Cone(e) \simeq D^b(R-mod)$, where $D^b(R-mod)$ is the bounded derived category of complexes R-modules of finite type.
We would proceed in a dg-setting, so we consider a dg-category Q with two objects P_1, P_2 with $Hom(P_1, P_2) = V$ - 3-dimensional of degree 0, $Hom(P_2, P_1) = 0$, endomorphism rings of both objects are one-dimensional and are generated by identity morphisms. This category corresponds to a Kronecker quiver with three arrows. The associated derived category $\mathcal{Q}^{tr}$ is equivalent to $\widetilde{D}$. First we would quotient it by an object $Cone(e : P_1 \to P_2)$, where e is some non-trivial morphism. Denote by Q_1 a full dg-subcategory of $\mathcal{Q}^{pre-tr}$ consisting of objects $P_1, P_2, Cone(e)$. Let us also denote by $\mathcal{R}$ a dg-category with one object P and $Hom_{\mathcal{R}}(P, P) = R = \mathbf{C} < x, y >$ - an algebra of non-commutative polynomials on x, y of degree 0.

Lemma 4.4.7. *Dg-categories $\mathcal{R}$ and $Q_1/Cone(e)$ are quasi-equivalent.*

Corollary. Proposition 2.5[8] then imply that $\mathcal{R}^{pre-tr}$ and $(Q_1/Cone(e))^{pre-tr}$ are quasi-equivalent, which means that $D^b(R-mod)$ and $(Q_1/Cone(e))^{tr}$ are equivalent as triangulated categories.

Proof. We understand $Cone(e)$ as a $P_2 \oplus P_1[1]$ with a differential $e : P_1 \to P_2$ and symbolically write it as a column $\begin{pmatrix} P_2 \\ P_1[1] \end{pmatrix}$, so the endomorphism dg-algebra $End_{Q_1}(Cone(e))$ looks like

$$\begin{pmatrix} Hom(P_2, P_2) & Hom(P_1, P_2)[-1] \\ Hom(P_2, P_1)[1] & Hom(P_1, P_1) \end{pmatrix} = \begin{pmatrix} k & V[-1] \\ 0 & k \end{pmatrix}$$

k means 1-dimensional vector spaces of degree 0, and $V = Hom(P_1, P_2)$. The differential is the following: $d : \begin{pmatrix} x & v \\ 0 & y \end{pmatrix} \mapsto \begin{pmatrix} 0 & (x-y)e \\ 0 & 0 \end{pmatrix}$. The multiplication is coming from the matrix multiplication. Denote this dg-algebra by $R^\bullet$.

Choose a 2-dimensional vector subspace $V' \subset V$, such that $V' \oplus ke = V$ and let a, b be it's basis. By construction of [8] dg-category $Q_1/Cone(e)$ is obtained from Q_1 by adding a morphism $\eta \in Hom^{-1}(Cone(e), Cone(e))$ such that $d\eta = id_{Cone(e)}$. We also have

$$Hom_{Q_1}(P_2, Cone(e)) = \begin{pmatrix} k \\ 0 \end{pmatrix}$$

$$Hom_{Q_1}(Cone(e), P_2) = \begin{pmatrix} k & V[-1] \end{pmatrix}$$

They are respectively left and right $R^\bullet$-dg-modules.

$$Hom_{Q_1/Cone(e)}(P_2, P_2) = \bigoplus_{n=0}^{\infty} Hom_{(n)}(P_2, P_2)$$

$$Hom_{(n)}(P_2, P_2) = Hom_{Q_1}(Cone(e), P_2) \otimes \eta \otimes R^\bullet \otimes \eta \otimes ... R^\bullet \otimes \eta \otimes Hom_{Q_1}(P_2, Cone(e))$$

n is a number of times that η shows up. In particular we see that this is a dg-algebra concentrated in non positive degrees. The degree 0 part is:

$$k \oplus V\eta k \oplus V\eta V\eta k \oplus ... = T(V)$$

$T(V)$ is a tensor algebra of V.
This allows us to construct a DG functor $F : \mathcal{R} \to Q_1/Cone(e)$. We define $F(P) := P_2$, and $F(x) = a\eta i$, $F(y) = b\eta i$, where i is a generator of $Hom_{Q_1}(P_2, Cone(e))$. Remind that $Hom_{\mathcal{R}}(P, P) = R = \mathbf{C} < x, y >= T(W)$, where W is a vector space spanned by x, y. So F sends $T(W)$ to

$T(V') \subset T(V) = Hom_{Q_1/Cone(e)}(P_2, P_2)$.

In the dg-category $Q_1/Cone(e)$ objects P_1 and P_2 are homotopy equivalent, because $e \in Hom^0_{Q_1/Cone(e)}(P_2, P_1)$ and $j\eta i \in Hom^0_{Q_1/Cone(e)}(P_1, P_2)$ are inverse of each other up to a homotopy(j is a generator of $Hom_{Q_1}(Cone(e), P_1)$). So the functor $Ho(F)$ is essentially surjective and to conclude the lemma we need just to prove that F induces a quasi-isomorphism between $End_{\mathcal{R}}(P)$ and $End_{Q_1/Cone(e)}(P_2)$, in other words we have to prove that the cohomology of the latter complex is exactly $T(V')$.

Introduce a dg-subalgebra $R_0^\bullet \subset R^\bullet$ consisting of $\begin{pmatrix} k & ke \\ 0 & k \end{pmatrix}$ and let $R' \subset R^\bullet$ be a submodule $\begin{pmatrix} 0 & V' \\ 0 & 0 \end{pmatrix}$, so that we have a decomposition $R^\bullet = R_0^\bullet \oplus R'$.

Introduce a dg-subalgebra $T_\eta(R_0^\bullet) \subset Hom_{Q_1/Cone(e)}(P_2, P_2)$ by the following formula:

$$T_\eta(R_0^\bullet) = \bigoplus_{n=0}^{\infty} \eta \otimes R_0^\bullet \otimes \eta ... R_0^\bullet \otimes \eta$$

n is the the number of times the $R_0^\bullet$ shows up.

The cohomology of $R_0^\bullet$ is one dimensional and generated by the identity matrix Id. Let us introduce a subspace $T_\eta(Id) \subset T_\eta(R_0^\bullet)$ defined as follows:

$$T_\eta(Id) = k\eta \oplus k\eta \otimes Id \otimes \eta \oplus k\eta \otimes Id \otimes \eta \otimes Id \otimes \eta \oplus ...$$

Let $C_0 = \begin{pmatrix} k & ke \end{pmatrix}$, so that we have decompositions:

$$Hom_{Q_1}(Cone(e), P_2) = C_0 \oplus R'$$

$$R^\bullet = R_0^\bullet \oplus R'$$

Substituting this to the formula(4.4.7) we get

$$Hom_{Q_1/Cone(e)}(P_2, P_2) = \bigoplus_{m=0}^{\infty} Hom'_m(P_2, P_2)$$

$$Hom'_m(P_2, P_2) = (R' \oplus C_0) \otimes T_\eta(R_0^\bullet) \otimes R' \otimes T_\eta(R_0^\bullet) \otimes ... \otimes R' \otimes T_\eta(R_0^\bullet) \otimes ki$$

Or in a more concise way

$$Hom_{Q_1/Cone(e)}(P_2, P_2) = (R' \oplus C_0) \otimes T_\eta(R_0^\bullet) \otimes T(R' \otimes T_\eta(R_0^\bullet)) \otimes ki$$

Observe here that as a complex it is a direct sum of tensor products of direct summands of complexes of the type $C_0 \otimes T_\eta(R_0^\bullet) \otimes R'$ and $R' \otimes T_\eta(R_0^\bullet) \otimes R'$. There is a well-defined filtration on the number of appearances of η, so we

can compute the cohomology of these complexes using the spectral sequence with respect to these filtration. First term of the first spectral sequence is $H^\bullet(C_0) \otimes T(H^*(R^\bullet)) \otimes R'$, H^* - is cohomology. C_0 is acyclic, so $H^*(C_0) = 0$ and $C_0 \otimes T_\eta(R_0^\bullet) \otimes R'$ is acyclic. First term of the second spectral sequence is $R' \otimes T_\eta Id \otimes R'$ and the differential is given by $d(\eta) = 1$. η is odd, so the cohomology of this complex is $R'\eta R'$. We deduce from here that the cohomology of $Hom_{Q_1/Cone(e)}(P_2, P_2)$ is equal to $\bigoplus_{k=0}^{\infty} R'\eta R'\eta...R'\eta k i = T(R') = T(V')$. It's exactly what we needed to prove.

$\square$

Next we have to quotient the triangulated category $D^b(R - mod)$ by the elements $Cone(M)$, where $M : R^{\oplus k} \to R^{\oplus k}$ is given by the matrices with entries in vector space $V = < 1, x, y >$ and non-zero commutative determinant, because this objects generate the image of category $\widetilde{D}^1$ in the quotient $\widetilde{D}/Cone(e)$.

Let us proceed by induction on the size of matrices M: denote Mat_k a set of $k \times k$ matrices with coefficients in a 3-dimensional vector space $V = < 1, x, y >$ with non-negative commutative determinant. Actually we want to invert these matrices. On each step we would obtain a dg-algebra $A_k^\bullet$ concentrated in non-positive degrees, such there the cohomology of this algebra would be the endomorphisms of the object P in a quotient category $\mathcal{A}^{tr}/ < Cone(M)|M \in Mat_k >$.

Theorem 4.4.2. . *There exist dg-algebras $A_k^\bullet$, $k = 0, 1, 2, ..$ concentrated in non-positive degrees, such that:*
1. $A_0^\bullet = R = \mathbf{C} < x, y >$ in degree 0;
2. For a triangulated category $D_k = D^b(R - mod)/Cone(M)|M \in Mat_k$ we have $RHom_{D_k}(P, P) \cong A_k^\bullet$;
3. $H^0(A_{k+1})$ is a localization of $H^0(A_k)$ at some set outside the commutator ideal;
4. All matrices in Mat_k are invertible over $H^0(A_k)$

Corollary 1. This procedure gives us a dg-algebra $A^\bullet = \varinjlim A_k$, such that for the triangulated category $\widetilde{C} = \widetilde{D}/\widetilde{D^1}$ we have an equivalence $RHom_{\widetilde{C}}(P, P) \cong A^\bullet$.

Corollary 2. $H = \varinjlim H^0(A_k)$ is a non-commutative algebra and may be described as a consecutive localization of R at some sets, which are the elements not lying in the commutator ideal.

Proof. We proceed by induction: let's put $A_0^\bullet = R$ concentrated in degree 0. For $k = 0$ there is no statement to prove. Let us prove the theorem for $k+1$: take a matrix $M \in Mat_{k+1}$, as it has non-trivial commutative determinant, we can find sub-matrix M' of size $k \times k$ that would also have non-trivial determinant. This matrix would be invertible over $H^0(A_k^\bullet)$, so let M_1' be the lift of it's inverse over $A_k^\bullet$. We have $M'M_1' = 1 + dM_2$ over $A_k^\bullet$. So if matrix M looks like $\begin{pmatrix} M' & b \\ a & c \end{pmatrix}$, then we would multiply it from the left by matrix $\begin{pmatrix} M_1' & 0 \\ -aM_1' & 1 \end{pmatrix}$ and from the right by matrix $\begin{pmatrix} 1 & -M_1'b \\ 0 & 1 \end{pmatrix}$. So the resulting matrix would be $M_0 = \begin{pmatrix} 1_{k \times k} & 0 \\ 0 & q \end{pmatrix} + dM_3$, where $q = -aM_1'b + c \in A_k^\bullet$ and M_3 is some $(k+1) \times (k+1)$ matrix of degree -1 defined over $A_k^\bullet$. Now we can apply the previous lemma, which would say that the quotient of $(\mathcal{A}_k^\bullet)^{pre-tr}$ by $Cone(M)$ is dg-equivalent to $\mathcal{B}_{\mathcal{M}}^{pre-tr}$, where B_M is some dg-algebra, concentrated in non-positive degrees and $H^0(B)$ is a localization of $H^0(A)$ at q. When we quotient by a set of matrices, the procedure of [8] takes the direct limit of dg-algebras, so we would obtain $A_{k+1}^\bullet = \varinjlim_M B_M$ and $H^0(A_{k+1}^\bullet)$ would be just a localization of $H^0(A_k^\bullet)$ at the set of elements obtained as q in the beginning of the proof. After the commutativization q goes to the determinant of the matrix M divided by the determinant of the matrix M', so definitely it is non-zero, so it lies outside the commutator ideal. $\square$

As a consequence of the theorem we obtain a non-commutative algebra $H = H^0(A^\bullet) = RHom^0_{C(\mathbf{CP}^2)}(P, P)$ and it is a consecutive localization of $R = \mathbf{C} < x, y >$ at some sets outside the commutator ideal. Although that is enough to get the action of the Cremona group on the algebra A, because we have an algebra morphism $H \to A$, we can prove the more precise result:

Lemma 4.4.8. *The map $H = H^0(A^\bullet) \to A$ is an isomorphism.*

Proof. We can summarize the procedure of the theorem as follows: consider a matrix $M \in M_k(V)$ with values in a vector space $V = < 1, x, y >$ and non-zero commutative determinant. Remind that M is invertible over H_k, or more precisely up to permuting the columns of M we can represent it as a product $M = UT$, where U is a lower triangular matrix with coefficients in H_{k-1} and it's diagonal elements are invertible over H_{k-1} and the last diagonal element is 1, T is an upper triangular matrix with coefficients in H_{k-1}. The elements on the diagonal of T are $(1, 1, .., 1, \Delta)$. As proved in

the theorem, Δ is invertible in H_k. Let S denote a subset of elements of H, consisting of images of elements Δ in H_k for all matrices M with non-zero commutative determinant and of any size k. In particular S contains non-zero elements of V for $k = 1$.

Let $comm : H \to K = \mathbf{C}(x, y)$ be a natural morphism induced from the map $R = \mathbf{C} < x, y > \to K$. We will prove that S consists of all the elements of H that are not in the kernel of $comm$.

Lemma 4.4.9. *1. $\Delta \in S$ implies $\Delta^{-1} \in S$.*
2. $\Delta_1, \Delta_2 \in S$ implies $\Delta_1\Delta_2 \in S$.
3. If $\Delta_1, \Delta_2 \in S$ and $comm(\Delta_1 + \Delta_2) \neq 0$ then $\Delta_1 + \Delta_2 \in S$.

Proof. Notice first that if we have a matrix $M \in Mat_{k \times k}(V)$ and an equality $UM = T$, where U is lower triangular and T is upper triangular with diagonal elements $(1, .., 1, \Delta)$, then for any $\lambda \in \mathbf{C}^*$ let us consider a diagonal matrix $D = diag(1, .., 1, \lambda)$. Then MD has also coefficients in V and non-zero determinant and U(MD)=(TD), where (TD) has elements $(1, .., 1, \lambda\Delta)$ on diagonal. This implies that $\lambda\Delta \in S$.

Given M and decomposition $UM = T$, let us consider a matrix $M' = \begin{pmatrix} M & e(k) \\ e(k)^t & 1 \end{pmatrix}$, where $e(k)$ is a $k \times 1$ matrix with $e(k)_{i1} = 0$ for all i except $e(k)_{k1} = 1$. Clearly M' has all coefficients in V. Let Δ_M be the last diagonal element of T and consider the lower-triangular matrix U' such that it's non-zero entries are $u'_{ii} = 1$ for $i = 1..k - 1$, $u'_{kk} = \Delta_M^{-1}$, $u'_{k+1k} = -1$, $u'_{k+1k+1} = 1$. Then we have a decomposition $(U'U)M' = T'$, here U is extended to a $(k + 1) \times (k + 1)$ matrix by putting $u_{k+1k+1} = 1$. So we have that T' has all diagonal elements equal to 1 except $t'_{k+1k+1} = \Delta_M^{-1}$. This proves that if $\Delta \in S$ then $\Delta^{-1} \in S$, which is the assertion (1) of the lemma. Suppose we have $UM = T$ and $WN = Q$, and $t_{kk} = \Delta_1$, $q_{nn} = \Delta_2$. Let M_0 be the first $k - 1$ columns of M and M_1 be the last column of M, N_0 - first $(l - 1)$ columns of N and N_1 the last column of N. Let $O(k, l)$ be a $k \times l$ matrix consisting of zero's and $e(l)$ be a $l \times 1$ vector which is zero except for $e(l)_{l1} = 1$. Introduce matrices

$$P = \begin{pmatrix} M_0 & e(k) & O(k, l-1) & M_1 \\ O(l, k-1) & N_1 & N_0 & O(l, 1) \end{pmatrix}$$

$$U' = \begin{pmatrix} U & O(k, l) \\ O(l, k) & W \end{pmatrix}$$

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Then we have

$$U'P = \begin{pmatrix} UM_0 & Ue(k) & O(k,l-1) & UM_1 \\ O(l,k-1) & WN_1 & WN_0 & O(l,1) \end{pmatrix}$$

Notice that $Ue(k) = e(k)$ because U is lower triangular with the last diagonal entry equal to 1. Moreover the bottom entry of WN_1 equals to Δ_2 and the bottom entry of UM_1 is Δ_1. Consider now a $l \times k$ matrix $W' = \begin{pmatrix} O(l,k-1) & -WN_1 \end{pmatrix}$ and a $(k+l) \times (k+l)$ matrix

$$U'' = \begin{pmatrix} 1_{k \times k} & O(k,l) \\ W' & 1_{l \times l} \end{pmatrix}$$

$W'UM_0 = 0$ and the bottom entry of $W'UM_1$ is $-\Delta_2 \Delta_1$. So we see that $U''U'P$ is an upper-triangular matrix with all the diagonal entries equal to 1 except for the last one which is $-\Delta_2 \Delta_1$. The matrix P has coefficients in vector space $V = < 1, x, y >$ and $U''U'$ is lower triangular with 1's on the diagonal. So it implies that $-\Delta_2 \Delta_1 \in S$, which proves the second statement of the lemma.

To prove the last statement, for a $\Delta_1 \in S$ we find the decomposition $UM = T$ where $t_{kk} = -\Delta_1 \in S$. Let also $WN = Q$ with $q_{ll} = \Delta_2$. The notations for $M_0, M_1, N_0, N_1, O(k,l), e(k)$ are as previously. Consider matrices

$$P = \begin{pmatrix} M_0 & e(k) & O(k,l-1) & M_1 \\ O(l,k-1) & e(l) & N_0 & N_1 \end{pmatrix}$$

$$U' = \begin{pmatrix} U & O(k,l) \\ O(l,k) & W \end{pmatrix}$$

$$U'P = \begin{pmatrix} UM_0 & e(k) & O(k,l-1) & UM_1 \\ O(l,k-1) & e(l) & WN_0 & WN_1 \end{pmatrix}$$

The matrix $U'P$ is upper-triangular except for one entry $(U'P)_{k+l,k} = 1$. Let us multiply this matrix to the left by U'' such that $u''_{ii} = 1$ for all i, $u''_{k+l,k} = -1$ and other entries are zero. Then $U''U'P$ would be upper triangular with the last diagonal entry equal to $\Delta_1 + \Delta_2$. $\square$

Remind that H is constructed as the consecutive localization at the elements coming from determinants. It follows from lemma(4.4.9) that any element outside of the kernel of the map $comm : H \to K$ is invertible. So we invert all the elements outside of the commutator ideal, which is a construction of A. So we proved the lemma. $\square$

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4.4.8 Action of the Cremona group on the triangulated category

We study the action of the Cremona group on $\widetilde{C}$. Suppose that $f \in Cr$ is an element of the Cremona group. There exists a diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\;g\;} & Y \\
{\scriptstyle \pi_X}\downarrow & & \downarrow{\scriptstyle \pi_Y} \\
\mathbf{CP}^2 & \xrightarrow{\;f\;} & \mathbf{CP}^2
\end{array}
$$

π_X, π_Y - are sequences of blow-ups and g is a regular iso-morphism. By Theorem4.4.1 $(L\pi_X^*)^{-1} \circ Lg^* \circ L\pi_Y^*$ is a composition of equivalences of triangulated categories, so it defines an auto-equivalence ψ_f of $\widetilde{C} = \widetilde{D}/\widetilde{D}^1$. Moreover the composition of such auto-equivalences $\psi_f \circ \psi_g$ is non-canonically isomorphic to $\psi_{f \circ g}$, so we have a well-defined <u>weak</u> action of the Cremona group on $\widetilde{C}$.

Notice that $Lg^* \circ L\pi_Y^*(O_{\mathbf{CP}^2}(1)) = g^*(\pi_Y(O_{\mathbf{CP}^2}(1)))$ is a line bundle on X left orthogonal to O_X.

Lemma 4.4.10. *The isomorphism class of an object $P = O(1) \in \widetilde{C}$ is preserved by the action of the Cremona group.*

Proof. As previously we suppose that our birational automorphism is decomposed into blow-up $\pi : X \to \mathbf{CP}^2$ and blow-down $f : X \to \mathbf{CP}^2$ and it is given by $f \circ \pi^{-1}$. So $L = f^*O(1) = Lf^*O(1)$ is a line bundle on X, left orthogonal to O. If an automorphism is given by the element of $PGL(3, \mathbf{C})$ then we don't need blowing up and f is this regular morphism, so $Lf^*O(1) = O(1)$ and the isomorphism class is preserved. If automorphism is a quadratic transformation, then π is blow-up of $\mathbf{CP}^2$ at three points and E is an exceptional curve. Then $Lf^*O(1) = O(2 - E)$ is a line bundle and moreover an embedding $O(2 - E) \hookrightarrow O(2)$ induces isomorphism of $Lf^*O(1)$ and $O(2)$ which is in turn isomorphic to $O(1)$ in $\widetilde{C}(X)$. $\qquad\square$

4.4.9 The coherence of two actions

Cremona group acts on $\widetilde{C}$ and fixes P, so it acts by outer automorphisms on $H^k(End(P))$. And as we calculated in Lemma(4.4.8) $H^0(End(P)) = A$ - an

algebra constructed previously(4.2). This equality is also specified up to inner conjugation. So we obtain an action of Cr on A by outer automorphisms.

Lemma 4.4.11. . *The action of the Cremona group on $A = H^0(End_{\widetilde{D}/\widetilde{D}^1}(P))$ coincides with the one constructed by explicit formulas(4.3).*

Proof. Our isomorphism between A and $H^0(End_{\widetilde{D}/\widetilde{D}^1}(P))$ depended on the choice of the basis of vector space $H^0(\mathbf{CP}^2, O(1)) = V = < e, a, b >$. We used e to identify $O(1)$ and $O(2)$ and we denoted $e^{-1}a, e^{-1}b$ by $x, y \in End(P)$. Let e', a', b' be another basis, they are some linear combinations of e, a, b, so $\gamma = e^{-1}e', \alpha = e^{-1}a', \beta = e^{-1}b'$ are some linear combinations of $1, x, y$. So $x' = e'^{-1}a' = (e^{-1}e')^{-1}(e^{-1}a') = \gamma^{-1}\alpha$, $y' = \gamma^{-1}\beta$.
It follows that if we have an automorphism of $\mathbf{CP}^2$ given by $(x : y : 1) \mapsto (\alpha, \beta, \gamma)$, then it would induce an automorphism of $H^0(End_{\widetilde{D}/\widetilde{D}^1}(P)) = A$ given by $(x, y) \mapsto (\gamma^{-1}\alpha, \gamma^{-1}\beta)$. We should verify that this map is conjugate to the one, given by explicit formulas.
If $\alpha = ax + by + c$, $\beta = dy + e$, $\gamma = 1$, then both maps are written as $(x, y) \mapsto (ax + by + c, dy + e)$.
If $\alpha = y$, $\beta = x$, $\gamma = 1$, then both maps are given by $(x, y) \mapsto (y, x)$.
If $\alpha = x$, $\beta = 1$, $\gamma = y$, then map coming from derived category is $(x, y) \mapsto (y^{-1}x, y^{-1})$. A map defined by explicit formulas is a composition of $(x, y) \mapsto (y^{-1}x, y)$ and $(x, y) \mapsto (x, y^{-1})$, which is $(x, y) \mapsto (y^{-1}x, y^{-1})$, so both maps in question coincide.
This three types of maps generate the action of the group $GL(3, \mathbf{C})$ on A by outer automorphisms, and we've just proved that the two actions coincide. The Cremona group is known to be generated by linear automorphisms and a quadratic transformation, so we are left to verify that actions are conjugate for quadratic transformation $a : (x, y) \mapsto (\frac{1}{x}, \frac{1}{y})$.
The explicit formula have the form $a : (x, y) \mapsto (x^{-1}, y^{-1})$, because $a \in D_1 \times D_2$.

As we noticed before, if $f : X \to Y$ is a blow-up at one point, then $D(X)$ has a semi-orthogonal decomposition $D(X) = < Lf^*D(Y), O_E >$, where E is an exceptional curve and O_E is it's structure sheaf.

Let us now consider surface X which is a blow-up of $\mathbf{CP}^2$ at 3 points $(0 : 0 : 1), (0 : 1 : 0), (1 : 0 : 0)$ and denote this map by π. Let E be an exceptional curve, so it is a union of three non-intersecting rational curves. Quadratic transformation $f : (x : y : z) \mapsto (yz, xz, xy)$ is a composition

$g \circ \pi^{-1}$, where g is a regular morphism $g : X \to \mathbf{CP}^2$.
$Lg^*O(1) = O(2 - E)$, $Lg^*O(2) = O(4 - 2E)$. So the basis elements $e, a, b \in Hom(O(1), O(2))$ maps by Lg^* to some elements $e', a', b' \in Hom_X(O(2 - E), O(4 - 2E))$, so to understand the action of this quadratic transformation on non-commutative algebra A we need to understand the elements $e'^{-1}a', e'^{-1}b' \in End_{\widetilde{D}/\widetilde{D}^1}(P)$.

Consider a sheaf $O(1-E)$ on X. Let us prove that it is left orthogonal to O_X, $RHom(O(1-E), O) = R\Gamma(O(E-1))$ and we have an exact sequence of sheaves $0 \to O(-1) \to O(-1+E) \to O(-1+E)_{|E} \to 0$. Actually $O(-1+E)$ restricted to any component E_i of E is isomorphic to $O(E_i)_{|E_i} \cong O(-1)_{|\mathbf{CP}^1}$, so it's acyclic. $O(-1)_X$ is also acyclic, so $R\Gamma(O(-1+E)) = 0$ which proves that $O(1 - E) \in \widetilde{D}(X)$.
Now in $\widetilde{D}(X)$ we have the following maps:

$$u_x, u_y, u_z \in Hom(O(1 - E), O(2 - E)) = Hom(O(1), O(2))$$
$$f = xyz \in Hom(O(1), O(4 - 2E))$$
$$a', b', e' \in Hom(O(2 - E), O(4 - 2E))$$
$$i : O(1 - E) \hookrightarrow O(1), j : O(2 - E) \hookrightarrow O(2)$$

By definition: $j \circ u_x = x \circ i$, $j \circ u_y = y \circ i$, $j \circ u_z = z \circ i$. Also we have the following relations: $f \circ i = a' \circ u_x = b' \circ u_y = e' \circ u_z$, just because $a' = yz, b' = xz, c' = xy$ as elements of $Hom(O(2 - E), O(4 - 2E)) = \Gamma(O(2 - E))$.
Now we can pass to a quotient category $\widetilde{C}(X) = \widetilde{D}(X)/\widetilde{D}^1(X)$, where all the previously described maps become invertible, because they define non-trivial maps between line bundles, so their cones have support of dimension 1. So in the quotient category we have: $a' \circ j^{-1} \circ x \circ i = e' \circ j^{-1} \circ z \circ i$ which implies $e'^{-1}a' = j^{-1} \circ zx^{-1} \circ j$ and $e'^{-1}b' = j^{-1} \circ zy^{-1} \circ j$, so this map is conjugate to the map $(x, y) \mapsto (x^{-1}, y^{-1})$, which concludes the lemma. $\qquad \square$

Chapter 5

Non-commutative cluster mutations

5.1 Quantization

Let us apply the result (4.3.1) to the group H. In the ring A we consider an element $q = x^{-1}y^{-1}xy$. Introduce the group $H^{nc} \subset Cr^{nc}$ as the group of automorphisms of A generated by the following elements :

$$\widetilde{P} : (x, y) \mapsto (x^{-1}yx, x^{-1}(1+y))$$

$$\widetilde{C} : (x, y) \mapsto (x^{-2}yx, x^{-1})$$

$$\widetilde{q} : (x, y) \mapsto (qxq^{-1}, qyq^{-1})$$

The group morphism $Cr^{nc} \to Cr$ clearly induces a group morphism $H^{nc} \to H$, such that $\widetilde{P} \mapsto P$, $\widetilde{C} \mapsto C$, $\widetilde{q} \mapsto 1$. Let $\mathbf{C}_1(q) \subset \mathbf{C}(q)$ be a sub-ring of rational functions on q, which are well defined at $q = 1$, i.e. denominator doesn't contain $(q-1)$. Now we can prove the following result:

Lemma 5.1.1. *The kernel of the group morphism $H^{nc} \to H$ consists of conjugations by the elements of $\mathbf{C}_1(q)$.*

Proof. The theorem 4.3.1 implies that the kernel of this map consists of conjugations by the elements $a \in A$. Moreover this conjugation preserves q, which means that a commutes with q, which is only possible if a is some rational function depending on q. Notice also that $q - 1 = x^{-1}y^{-1}xy - 1 = x^{-1}y^{-1}(xy - yx)$ belongs to the ideal of A generated by commutator, so it is not an invertible element of A. We conclude that $a \in \mathbf{C}_1(q)$. $\qquad\square$

We expect actually that the kernel of this map would be a free abelian group generated by q. To support this expectation let us verify that the basic relations in H from the section(3.2.1) hold also in H^{nc} up to conjugation by q. We may consider for convenience the following element of Cr^{nc} that also preserves q:

$$\widetilde{I} : (x, y) \mapsto (y^{-1}, y^{-1}xy)$$

Lemma 5.1.2. *We have the following relations on $\widetilde{P}, \widetilde{C}, \widetilde{I}, \widetilde{q}$:*

$$[\widetilde{q}, \widetilde{C}] = [\widetilde{q}, \widetilde{P}] = 1$$

$$\widetilde{C}^3 = \widetilde{q}$$

$$\widetilde{P}\widetilde{C}\widetilde{P} = \widetilde{q}\widetilde{I}$$

$$\widetilde{I}^4 = \widetilde{q}^{-1}$$

$$[\widetilde{C}, \widetilde{I}^2] = 1$$

$$\widetilde{P}^5 = \widetilde{q}$$

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Proof. 1. $\widetilde{q}$ is preserved by $\widetilde{P}$ and $\widetilde{C}$. It follows that $\widetilde{q}$ is the central element of H^{nc} because $\widetilde{P}, \widetilde{C}$ are the generators of H^{nc}.

2. Using the notation $q = x^{-1}y^{-1}xy$ and the property that $\widetilde{C}(q) = q$ we rewrite $\widetilde{C}(x,y) = (x^{-1}yq^{-1}, x^{-1})$, so we have that

$$\widetilde{C}^2(x,y) = (qy^{-1}xx^{-1}q^{-1}, qy^{-1}x) = (qy^{-1}q^{-1}, qy^{-1}x)$$

$$\widetilde{C}^3(x,y) = (qxq^{-1}, qxx^{-1}yq^{-1}) = (qxq^{-1}, qyq^{-1}) = \widetilde{q}(x,y)$$

3. Let us compute $\widetilde{P}\widetilde{C}\widetilde{P}$:

$$\widetilde{C}\widetilde{P} : (x,y) \mapsto ((x^{-1}y^{-1}x)x^{-1}(1+y)q^{-1}, x^{-1}y^{-1}x) = (x^{-1}(1+y^{-1})q^{-1}, qy^{-1})$$

$$\widetilde{P}\widetilde{C}\widetilde{P} : (x,y) \mapsto ((q(1+y^{-1})^{-1}x)x^{-1}y^{-1}x(x^{-1}(1+y^{-1})q^{-1},$$
$$(q(1+y^{-1})^{-1}x)(1+qy^{-1})) = (qy^{-1}q^{-1}, qx) = \widetilde{q}(y^{-1}, xq) = \widetilde{q}\widetilde{I}(x,y)$$

In particular this formula proves that $\widetilde{I} \in H^{nc}$.

4. First we compute $\widetilde{I}^2$:

$$\widetilde{I}^2(x,y) \mapsto (y^{-1}x^{-1}y, (y^{-1}x^{-1}y)y^{-1}(y^{-1}xy)) = (q^{-1}x^{-1}, y^{-1}q$$

$$\widetilde{I}^4(x,y) \mapsto (q^{-1}xq, q^{-1}yq) = \widetilde{q}^{-1}(x,y)$$

5. Here we use the previous calculation for $\widetilde{I}^2$:

$$\widetilde{C}\widetilde{I}^2(x,y) \mapsto (xqy^{-1}qq^{-1}, xq) = (y^{-1}x, xq)$$

$$\widetilde{I}^2\widetilde{C}(x,y) \mapsto (q^{-1}qy^{-1}x, xq) = (y^{-1}x, xq) = \widetilde{C}\widetilde{I}^2(x,y)$$

6. Let us make the following computations:

$$\widetilde{P}^2 : (x,y) \mapsto ((x^{-1}y^{-1}x)(x^{-1}(1+y))(x^{-1}yx), (x^{-1}y^{-1}x)(1+x^{-1}(1+y))) =$$
$$= (x^{-1}(1+y)q^{-1}, x^{-1}y^{-1}(1+x+y))$$

$$\widetilde{P}^3 : (x,y) \mapsto ((q(1+y)^{-1}x)x^{-1}y^{-1}(1+x+y)(x^{-1}(1+y)q^{-1}),$$
$$(q(1+y)^{-1}x)(1+x^{-1}y^{-1}(1+x+y))) = (qy^{-1}(x^{-1}+(1+y)^{-1})(1+y)q^{-1},$$
$$q(1+y)^{-1}(1+y^{-1})(1+x)) = (x^{-1}y^{-1}(1+x+y)q^{-1}, qy^{-1}(1+x))$$

$$\widetilde{P}^4 : (x,y) \mapsto ((q(1+x+y)^{-1}yx)x^{-1}y^{-1}x(1+x)(x^{-1}y^{-1}(1+x+y)q^{-1}),$$
$$(q(1+x+y)^{-1}yx)(1+qy^{-1}(1+x))) = (q(1+x+y)^{-1}(1+x)y^{-1}(1+x+y)q^{-1},$$
$$q(1+x+y)^{-1}(yx+x(1+x))) = (qy^{-1}(1+x)y^{-1}yq^{-1}, qx) =$$
$$= (qy^{-1}(1+x)q^{-1}, qx)$$

$$\widetilde{P}^5 : (x,y) \mapsto ((q(1+x)^{-1}yq^{-1})qx(qy^{-1}(1+x)q^{-1}), q(1+x)^{-1}yq^{-1}(1+qx)) =$$
$$= (qxq^{-1}, qyq^{-1}) = \widetilde{q}(x,y)$$

$\square$

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5.2 Integrable dynamical system on the pair of matrices.

We consider a discrete dynamical system on the pairs of complex matrices and prove that it's integrable.

Now let X, Y be two $n \times n$ complex matrices. Consider the following transformation:

$$T : (X, Y) \mapsto (Y, (Y + Y^{-1})X^{-1}Y)$$

Denote by M_n the space of pairs of complex $n \times n$ matrices, then T defines the birational automorphism of this space. It's inverse is given by the map:

$$T^{-1} : (X, Y) \mapsto (XY^{-1}(X + X^{-1})^{-1}, X)$$

The following matrices are invariant and the action of T:

$$F = Y^{-1}X + YX^{-1} + Y^{-1}X^{-1}$$

$$G = Y^{-1}XYX^{-1}$$

Let us consider a $2n \times 3n$ matrix

$$M = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}$$

a, b, c, d, e, f - $n \times n$ matrices.
Also make the following notations

$$U_1 = \begin{pmatrix} a & b \\ d & e \end{pmatrix}$$

$$U_2 = \begin{pmatrix} b & c \\ e & f \end{pmatrix}$$

U_1, U_2 are $2n \times 2n$ matrices.
Consider a space S_n of matrices M that verify the following property:

$$U_1^2 = U_2^2 = -1$$

Lemma 5.2.1. *A map $\Psi : M \mapsto (a, b)$ defines a birational isomorphism of spaces $\Psi : S_n \to M_n$.*

Proof. Let us attempt to construct a map inverse to Ψ. Given matrices X, Y, with the condition $U_1^2 = -1$ we have the unique choice for d, e. Now if we know b, e then the condition $U_2^2 = -1$ gives unique values for c, f. Namely the inverse of Ψ is given by the following formula:

$$\Psi^{-1}(X, Y) = \begin{pmatrix} X & Y & (Y + Y^{-1})X^{-1}Y \\ -Y^{-1}(1 + X^2) & -Y^{-1}XY & -Y^{-1}XYX^{-1}Y \end{pmatrix}$$

$\square$

Observe that with this formulas

$$U = U_2 U_1 = \begin{pmatrix} -F & -1 \\ G & 0 \end{pmatrix}$$

Moreover $U^{-1} = U_1 U_2$, so U and U^{-1} are conjugate. In particular if λ is an eigenvalue of U, then so is λ^{-1}.

Define a dynamical system on S_n by the formula:

$$P : M \mapsto -UM$$

By this map $P(U_1) = -UU_1 = -U_2 U_1 U_1 = U_2$, $P(U_2) = -UU_2 = -U_2 U_1 U_2$. Again we have that $P(U_1)^2 = P(U_2)^2 = -1$, so $P(M)$ belongs to S_n. Moreover $P(U) = P(U_2)P(U_1) = (-U_2 U_1 U_2)U_2 = U$, so U is invariant under P. U is invertible, because U_1, U_2 are invertible, and so P is invertible. Now we can summarize the previous discussion in the following theorem:

Theorem 5.2.1. *The integrable system (S_n, P) is equivalent to (M_n, T), namely there is a birational isomorphism $\Psi : S_n \to M_n$, such that $\Psi^{-1} \circ T \circ \Psi = P$.*

Proof. We need to prove that:$\Psi P \Psi^{-1} = T$. Because Ψ takes only first two entries of the first row of the matrix $P(M)$, it's enough to know $P(U_1)$ which equals to U_2, so $\Psi P \Psi^{-1}(X, Y)$ equals to the upper entries of U_2, which is $(Y, (Y + Y^{-1})X^{-1}Y)$ and it equals to $T(X, Y)$, so we are done. $\square$

Let $R(t)$ be a rational function of variable t with the following property:

$$R(t)R(\frac{1}{t}) = 1$$

Lemma 5.2.2. *The map $M \mapsto R(U)M$ is a well defined birational automorphism of S_n and it preserves $U = U_2 U_1$.*

Proof. This map sends U_1 to $R(U)U_1$, and U_2 to $R(U)U_2$. We should see that $(R(U)U_i)^2 = -1$. But we have:

$$R(U)U_1R(U)U_1 = -R(U)U_1R(U_2U_1)U_1^{-1} =$$

$$= -R(U)R(U_1U_2) = -R(U)R(U^{-1}) = -1$$

The proof of the equality $(R(U)U_2)^2 = -1$ is analogous. $\qquad\square$

Let B_n be a set of $2n \times 2n$ matrices $U = \begin{pmatrix} -F & -1 \\ G & 0 \end{pmatrix}$ such that U is conjugate to U^{-1}. The dimension of B_n is $(2n^2 - n)$, because the arbitrary matrices (F, G) form a $2n^2$ dimensional manifold, and the condition that U is conjugate to U^{-1} implies that the characteristic polynomial $Ch(t)$ of U satisfies the following property:

$$Ch(\frac{1}{t})t^{2n} = Ch(t)$$

This condition gives n additional conditions on B_n and later we will prove that the fiber of π is at least n-dimensional, which would imply that $dim_{\mathbf{C}}(B_n) = 2n^2 - n$.

We have a well-defined map:

$$\pi : S_n \to B_n$$

$$\pi : M \mapsto U = U_2U_1$$

Theorem 5.2.2. *The fiber over generic U has an open orbit of the free action of the n-dimensional algebraic torus.*

Proof. For a function $R(t)$ as in the lemma(5.2)we see that $M \mapsto R(U)M$ preserves the fibers of π. For the fixed U let us introduce the multiplicative group $G(U)$ of matrices $R(U)$ for all possible R such that $R(U)$ is non-degenerate. Then $G(U)$ is an abelian algebraic group and it acts freely on the fibers of π. If we take $R(t) = -t$, then the corresponding map $M \mapsto -UM$ is exactly P, so P preserves the fibers of π as well.

Let us understand the structure of $G(U)$. For a generic $U \in B_n$ the eigenvalues of U are different, then U is conjugate to a diagonal matrix $D = diag(x_1, x_1^{-1}, ..., x_n, x_n^{-1})$. Then $R(U)$ is conjugate to

$$R(D) = diag(R(x_1), R(x_1^{-1}), .., R(x_n), R(x_n^{-1})) =$$

$$= diag(R(x_1), R(x_1)^{-1}, .., R(x_n), R(x_n)^{-1})$$

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We can take $R(t)$ to be the product of functions $\frac{t+c}{ct+1}$ for different c. So we can prove that the images of $R(U)$ generate the group of matrices $diag(a_1, a_1^{-1}, .., a_n, a_n^{-1})$ which is isomorphic to an algebraic torus $(\mathbf{C}^*)^n$. So we have proved that for generic U, the group $G(U)$ is isomorphic to $(\mathbf{C}^*)^n$. $\qquad\square$

5.3 Non-commutative cluster mutations.

Let us consider the following birational automorphism of $\mathbf{CP}^2$:

$$(x, y) \mapsto (y, \frac{1 + y^2}{x})$$

It is one of the simplest cluster mutations of cluster variables (x, y). This automorphism preserves the function $\frac{1}{xy} + \frac{x}{y} + \frac{y}{x}$, so it preserves the fibers of this function, which are elliptic curves.

For a positive integer k we define a mutation on non-commutative variables by the formula:

$$\mu_k : (X, Y) \mapsto (X^{-1}YX, X^{-1}(1 + Y^k))$$

We can assume that X, Y belong a non-commutative ring A and this formula defines an automorphism of A. We can verify that the element $q = X^{-1}Y^{-1}XY$ is preserved by μ_k.

As it was suggested by M.Kontsevich, the non-commutative Laurent phenomenon should hold for a mutation defined in this way. Namely, if $L = \mathbf{C}[F_2] = \mathbf{C} < X, X^{-1}, Y, Y^{-1} >$ is a group ring of a free group on two generators X, Y, then $\mu^n(X), \mu^n(Y) \in L \subset A$ for any n. We examine more in detail the case $k = 2$. From now on we denote:

$$\mu : (X, Y) \mapsto (X^{-1}YX, X^{-1}(1 + Y^2))$$

Let us define a recursion as follows:

$$X_0 := X$$

$$X_1 := \mu(X) = X^{-1}YX$$

$$X_n = \mu^n(X)$$

Lemma 5.3.1.
$$T(X_{n-1}, X_n) = (X_n, X_{n+1})$$

Proof. It is enough to prove it for $n = 1$. Remind that $T : (X, Y) \mapsto (Y, (Y + Y^{-1})X^{-1}Y)$, so we just compute directly:

$$\mu(X, Y) = (X^{-1}YX, X^{-1}(1 + Y^2))$$

$$X_2 = (X^{-1}YX)^{-1}(X^{-1}(1 + Y^2))(X^{-1}YX) = X^{-1}(Y + Y^{-1})X^{-1}YX$$

$$T(X_0, X_1) = (X_1, (X_1 + X_1^{-1})X_0 X_1) = (X_1, X^{-1}(Y + Y^{-1})XX^{-1}(X^{-1}YX)) =$$
$$= (X_1, X_2)$$

$\square$

Now we deduce the following theorem:

Theorem 5.3.1. *The non-commutative Laurent phenomenon holds for* $k = 2$, *i.e.* $\mu^n(X), \mu^n(Y) \in L$.

Proof. We have $X_n = \mu^n(X) = \mu^{n-1}(X^{-1}YX) = \mu^{n-1}(Yq^{-1})$, where $q = X^{-1}Y^{-1}XY$ and it is preserved by μ. So $X_n = \mu^{n-1}(Y)q^{-1}$ and if we prove that $X_n \in L$ it would follow that $\mu^{n-1}(Y) \in L$, because $q \in L$.
The consequence of the lemma(5.3.1) and of the theorem(5.2.1) is that

$$(X_n, X_{n+1}) = T^n(X_0, X_1) = (\Psi P \Psi^{-1})^n(X, X^{-1}YX) = \Psi \circ (-U)^n \circ \Psi^{-1}(X, X^{-1}YX)$$

We can compute the latter expression by passing on the variety S_n through Ψ:

$$\Psi^{-1}(X, X^{-1}YX) =$$

$$= \begin{pmatrix} X & X^{-1}YX & X^{-1}(Y + Y^{-1})X^{-1}YX \\ -X^{-1}Y^{-1}X(1 + X^2) & -X^{-1}Y^{-1}XYX & -X^{-1}Y^{-1}XYX^{-1}YX \end{pmatrix}$$

Notice that all the entries of this matrix are Laurent polynomials. So both U_1 and U_2, which are made of this entries, have coefficients in L. The same holds true for $U = U_2 U_1$ and then for $(-U)^n$, which proves that $(X_n, X_{n+1}) = \Psi \circ (-U)^n \circ \Psi^{-1}(X, X^{-1}YX)$ belong to L. $\square$

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