Deformation and construction of minimal surfaces

Antoine Coutant

Laboratoire d'Analyse et de Mathématiques Appliquées — UMR 8050
École Doctorale Mathématiques et Sciences et Technologies de l'Information et de la Communication — MSTIC

Thèse présentée en vue de l'obtention du diplôme
de docteur de l'université de Paris-Est
en mathématiques

Déformation et construction de surfaces minimales

présentée par

Antoine Coutant

le 5 décembre 2012 devant le jury composé de

Gérard Besson	Université de Grenoble 1	Examinateur
Benoît Daniel	Université de Lorraine	Examinateur
Laurent Hauswirth	Université Paris-Est Marne-la-Vallée	Examinateur
Emmanuel Hebey	Université de Cergy-Pontoise	Rapporteur
Frank Pacard	École Polytechnique	Directeur de thèse
Martin Traizet	Université François Rabelais	Rapporteur

Résumé

L'objet de cette thèse consiste en la construction de nouveaux exemples de surfaces (ou hypersurfaces) minimales dans les espaces euclidiens $\mathbb{R}^3$, $\mathbb{R}^n \times \mathbb{R}$ avec $n \geqslant 3$ ou dans l'espace homogène $\mathbb{S}^2 \times \mathbb{R}$.

Nous prouvons dans le chapitre I l'existence de surfaces minimales dans $\mathbb{R}^3$ arbitrairement proches d'un polygone convexe.

Dans le chapitre II, nous prouvons l'existence d'hypersurfaces minimales de type Riemann dans $\mathbb{R}^n \times \mathbb{R}$, $n \geqslant 3$. Celles-ci peuvent être interprétées comme étant une famille d'hyperplans horizontaux (des bouts) reliés les uns aux autres par des morceaux de caténoïdes déformés (des cous). Nous donnons un résultat général pour ce type d'objet quand il est périodique ou bien quand il a un nombre fini de bouts horizontaux. Cela se fait sous certaines hypothèses de contraintes sur les forces intervenant dans la construction. Nous finissons en donnant plusieurs exemples, notamment l'existence d'une hypersurface de type Wei verticale qui n'existe pas en dimension 3.

Dans le chapitre III, nous prouvons l'existence d'une surface minimale de type Riemann dans $\mathbb{S}^2 \times \mathbb{R}$ telle que deux sphères sont reliés entre elles alternativement par 1 cou et 2 cous. Là aussi, nous mettons en évidence le rôle joué par les forces lors de la construction. De même que dans le chapitre précédent, la méthode repose sur un processus de recollement.

Dans le chapitre IV, nous donnons une description très précise de la caténoïde et la surface de Riemann dans $\mathbb{S}^2 \times \mathbb{R}$.

Enfin, dans le chapitre V, nous établissons l'existence dans $\mathbb{R}^n \times \mathbb{R}$ d'hypersurfaces de type Scherk lorsque $n \geqslant 3$.

Mots-clés : surfaces minimales, surfaces minimales de Riemann, surfaces de Scherk, méthode de recollement, espaces à poids.

Abstract

This thesis is devoted to the construction of numerous examples of minimal surfaces (or hypersurfaces) in the Euclidean 3-space, $\mathbb{R}^n \times \mathbb{R}$ with $n \geqslant 3$ or in the homogeneous space $\mathbb{S}^2 \times \mathbb{R}$.

We prove in the chapter I the existence of minimal surfaces in $\mathbb{R}^3$ as close as we want to a convex polygon.

In the chapter II, we prove the existence of minimal hypersurfaces in $\mathbb{R}^n \times \mathbb{R}$, $n \geqslant 3$, that have Riemann's type. These ones could be considered as a family of horizontal hyperplanes (the ends) which are linked to each other by pieces of deformed catenoids (the necks). We provide a general result in the simply-periodic case together with the case of a finite number of hyperplanar ends. Our construction lies on some conditions associates with the forces that characterize the different configurations. We end with giving some examples ; in particular, we exhibit the existence of vertical Wei example that does not exists in the 3-dimensional case.

In the chapter III, we prove the existence of the analogous of the Wei example in $\mathbb{S}^2 \times \mathbb{R}$. The surface is such that two spheres are linked by 1 neck and 2 necks alternatively. Here again, we highlight the role that the forces play in the construction. Moreover, like in the previous chapter, the method lies on a gluing process.

In the chapter IV, We give an accurate description of the catenoid and the Riemann's minimal example in $\mathbb{S}^2 \times \mathbb{R}$.

Finally, in the chapter V, we demonstrate the existence of Scherk type hypersurfaces in $\mathbb{R}^n \times \mathbb{R}$ when $n \geqslant 3$.

Keywords : minimal surfaces, Riemann minimal examples, Scherk surfaces, gluing method, weighted spaces.

À Camille.

Table des matières

Remerciements

Je tiens tout d'abord à exprimer ma reconnaissance envers mon directeur de thèse, Frank Pacard. Je le remercie d'avoir accepté de superviser mes travaux pendant ces trois années et de m'avoir accordé sa confiance. De par ses remarques, ses intuitions et ses questions, son encadrement a toujours été pertinent.

Je suis de même très reconnaissant envers Emmanuel Hebey et Martin Traizet pour avoir rapporté cette thèse. Je remercie aussi Gérard Besson, Benoît Daniel et Laurent Hauswirth pour avoir accepté d'être membres du jury.

Je remercie également les chercheurs qui ont toujours pris plaisir à répondre à mes questions. En particulier, je tiens à témoigner ma gratitude envers Laurent Mazet, Yuxin Ge, Benoît Daniel, Laurent Hauswirth, Sébastien Cartier, Éric Toubiana et Rabah Souam. J'ai une pensée toute particulière pour Harold Rosenberg et Frédéric Hélein dont les cours exemplaires m'ont fait découvrir la géométrie riemannienne, et plus particulièrement la théorie des surfaces minimales.

Je n'oublie pas de remercier l'ensemble des membres du LAMA dans lequel j'ai eu la chance de travailler, et en particulier, les autres doctorants avec qui j'ai partagé mon quotien au sein du laboratoire : Eduardo *the chessman*, Francesco *l'italien*, Johann *le mailer*, David, Jean-Maxime, Xian, Peng, Cecilia, Salwa et Nour. J'en profite pour adresser un grand merci à Bertrand *the questionman* qui est co-organisateur du séminaire des doctorants.

Je remercie les personnes avec qui j'ai partagé des moments fort sympathiques en-dehors du cadre de la thèse. Tout d'abord, mes pensées se dirigent vers les irréductibles Cachannais que sont Olivier, Florent, Julien, Xavier, Grégory, Corentin et Guillaume. Ensuite, un immense merci aux membres de l'association musicale *Not So Small Band* avec qui j'ai eu le plaisir de jouer, notamment à Stéphane, Sven, Nicolas, Vincent, Ghislain et Guillaume. Je remercie également les membres des *Planches à Musique* grâce à qui je me suis bien *amusiqué* : entre autres Mathias, Émilien, Tom, Jérémy, Cécile et Mélanie.

Je remercie aussi mes amis Benjamin, Mathieu, Pierre-Antoine, Louis-Joseph et Pascal. Leurs rires, leurs histoires et leur personnalité n'ont pas été sans importance.

Mes parents, ma sœur et l'ensemble de ma famille ont su montrer leur soutien constant pendant toute la durée des mes études et je leur en suis infiniment reconnaissant.

J'ai également une pensée pour toutes les personnes que j'aurais pu oublier dans ces remerciements. J'espère qu'elles ne m'en tiendront pas rigueur !

Enfin, c'est avec une profonde gratitude et grande humilité que je remercie ma femme, Camille. Sa présence au quotidien est le plus merveilleux des présents. Elle est pour moi une source inépuisable de motivation et de bonheur. Pendant cette thèse, elle a su épauler, aimer, faire rire, rendre heureux, surprendre, faire réfléchir, initier à la GMM, expliquer, embrasser, rendre perplexe, écrire, faire découvrir, encourager, créer, accompagner, discuter, faire de moi quelqu'un de meilleur et surtout, me faire le plus *génialissime* des cadeaux : Alma.

Introduction

Les surfaces minimales constituent une branche de la géométrie riemannienne qui a connu de vastes progrès lors des dernières décennies. Celles-ci sont des points critiques pour la fonctionnelle aire, c'est-à-dire qu'une surface $\mathcal{S}$ d'une variété M de dimension 3 est dite *minimale* si pour toute courbe fermée simple γ contenue dans cette surface, la partie de $\mathcal{S}$ située à l'intérieur de γ est point critique de la fonctionnelle aire parmi toutes les surfaces qui ont pour bord γ. On peut prouver que cette notion est équivalente au fait que la courbure moyenne de $\mathcal{S}$ est nulle, en d'autres termes, que la moyenne de ses courbures principales s'annule. Bien entendu, ce genre d'objet se généralise aux hypersurfaces incluses dans une variété de dimension n où n est un entier supérieur ou égal à 4.

Parmi les différentes questions que pose la théorie des surfaces minimales, il semble primordial de pouvoir exhiber des exemples et c'est là l'objet de ma thèse.

Bref historique

La première surface minimale non triviale a été découverte par Leonhard Euler en 1744 : il s'agit de la caténoïde. Mis à part le plan, c'est l'unique surface minimale de révolution de l'espace euclidien $\mathbb{R}^3$. Il s'agit de la surface obtenue par rotation d'une chaînette autour d'un axe vertical. Nous verrons dans le chapitre II que la généralisation de cet objet dans l'espace euclidien de dimension supérieure existe et qu'il en est de même dans $\mathbb{S}^2 \times \mathbb{R}$ (voir les chapitres III et IV). Nous renvoyons à la figure 1 pour une illustration.

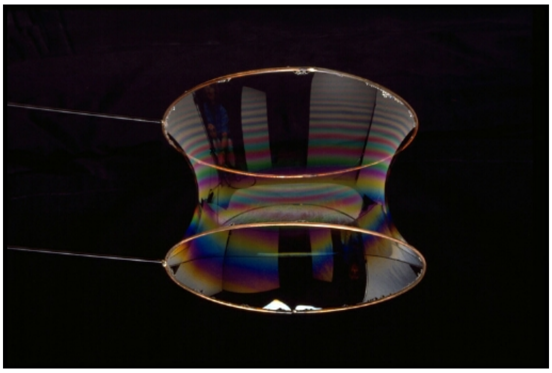


FIGURE 1 – La caténoïde.

En 1755, grâce au calcul des variations, Joseph-Louis Lagrange formule une équation aux dérivées partielles qui correspond à une condition nécessaire et suffisante pour qu'une fonction u d'un ouvert de $\mathbb{R}^2$ dans $\mathbb{R}$ décrive un graphe minimal, à savoir

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0, \tag{0.0.1}$$

où $|\cdot|$ désigne la norme euclidienne de $\mathbb{R}^2$. Une équation semblable existe pour les surfaces (ou hypersurfaces) de $M \times \mathbb{R}$ qui sont des graphes au-dessus d'un ouvert de la variété M. Nous verrons que cette équation nous a été utile pour tous les travaux présentés dans cette thèse, excepté pour le chapitre V

En 1834, Heinrich Scherk démontre l'existence des surfaces minimales qui portent maintenant son nom. Celles-ci peuvent être considérées comme étant des graphes minimaux définis au-dessus d'un carré et qui prennent alternativement les valeurs $+\infty$ et $-\infty$ sur ses côtés. En dimension supérieure, des objets similaires existent : c'est l'objet du chapitre V.

L'un des outils les plus puissants concernant la théorie des surfaces minimales réside en le théorème de représentation de Karl Weirstrass (1866) qui relie profondément ces objets à la la théorie des fonctions complexes. Malheureusement, les techniques développées grâce à ce théorème sont propres à la dimension 2 — par exemple, elles ne peuvent pas se généraliser aux hypersurfaces de $\mathbb{R}^n$ avec n quelconque.

Dans un article publié à titre posthume [Rie98], Bernhard Riemann découvre les surfaces minimales de Riemann. Celles-ci sont simplement périodiques, plongées dans $\mathbb{R}^3$ et admettent une infinité de bouts planaires horizontaux reliés entre eux par des *cous* ; par ailleurs, elles sont feuilletées par des cercles horizontaux. Nous renvoyons à la figure 2 pour une représentation. Nous étudions ce type d'objets dans les chapitres II, III et IV.

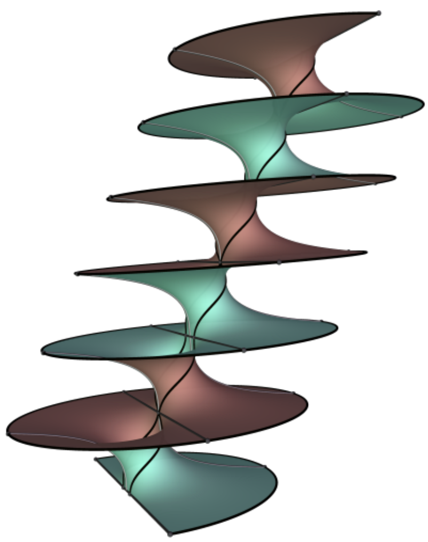


FIGURE 2 – La surface minimale de Riemann.

En 1873, Joseph Plateau, de par son étude du comportement des films de savon — qui sont en réalité des surfaces minimales — énonce les lois de Plateau qui régissent la géométrie de ces interfaces physiques. Le problème de Plateau, quant à lui, consiste à déterminer si une courbe fermée simple est le contour d'une surface minimale. Ce problème sera résolu dans les années 30 de deux façons différentes : d'une part par Jesse Douglas qui utilise les intégrales qui portent dorénavant son nom et pour lesquelles il reçoit la médaille Fields en 1936 et d'autre part par Tibor Radó [Rad32] qui utilise l'existence d'un minimiseur de l'énergie. Ce type de résultat sera notamment utilisé dans le chapitre V.

En 1984, Celso J. Costa a relancé la recherche dans le domaine des surfaces minimales en prouvant [Cos84] l'existence d'une surface minimale, complète, de genre 1 avec trois bouts. L'année d'après, David Hoffman et William H. Meeks démontrent [HM85] que c'est en fait une surface plongée. Jusqu'alors, il était conjecturé que les seules surfaces minimales complètes, plongées et de topologie finie étaient le plan, l'hélicoïde et la caténoïde. Dès lors, de nombreux exemples ont été construits. Parmi eux, citons la surface de Costa-Hoffman-Meeks qui est la généralisation d'une surface de Costa de genre quelconque [HM90] ou bien l'hélicoïde de genre 1 [DHW93] qui est en fait une hélicoïde à laquelle on a rajouté une poignée. La preuve de l'existence de ces théorèmes se fait essentiellement à l'aide de leur représentation de Weirstrass. Pour cela, la procédure est la suivante : on se donne une image assez correcte de la surface que l'on souhaite construire, on en déduit un bon candidat pour sa structure de surface de Riemann sous-jacente et une application méromorphe de Gauss associée (à certains paramètres près), on utilise les différentes symétries et les différents bouts que nous voulons prescrire à la surface pour trouver une famille raisonnable de représentations généralisées de Weirstrass ; reste ensuite à déterminer les derniers paramètres — c'est le fameux *period problem* — de telle sorte que l'on obtienne finalement la surface que l'on veut.

En parallèle à ces techniques, des démonstrations d'analyse linéaire et non linéaire reposant essentiellement sur les propriétés de l'opérateur de Jacobi (qui doit être considéré comme le linéarisé de la courbure moyenne) associé aux surfaces ont permis de construire de nouvelles surfaces minimales par des méthodes de recollement. L'exemple sans doute le plus parlant est en fait celui dû à Riemann. En effet, on peut considérer la surface de Riemann comme étant la somme connexe d'une infinité de caténoïdes ; c'est d'ailleurs cette approche que nous développons dans les chapitres II et III. Généralement, l'idée essentielle repose sur un argument de recollement. On considère deux surfaces minimales Σ_1 et Σ_2 auxquelles on enlève un petit disque D_i de rayon r_ϵ avec $i \in \{1,2\}$, où r_ϵ est un paramètre qui tend vers 0 quand ϵ tend vers 0. On prouve ensuite que l'on peut déformer les surfaces $\Sigma_i \setminus D_i$ alors obtenues en gardant le critère minimal et en assignant une certaine condition de bord sur ∂D_i. On prouve ensuite que l'on peut trouver des conditions de bord de telle sorte que l'on puisse recoller les surfaces alors déformées. De nombreux exemples ont été élaborés par cette méthode et nous reviendrons sur ceux-ci lors de la présentation des chapitres II et III. Cette approche a le mérite de fournir généralement une description assez précise de la surface alors obtenue, notamment aux points de recollement. Toutefois, ces techniques ne procurent des exemples qu'avec un paramètre ϵ petit ; en d'autres termes, la surface construite est presque singulière, ce qui n'est pas nécessairement le cas pour la méthode utilisant la représentation de Weirstrass.

Initiation aux outils utiles à la thèse

Notre approche concernant la théorie des surfaces (ou hypersurfaces) minimales se base essentiellement sur des méthodes d'analyses propres aux différentes équations aux dérivées partielles qui peuvent intervenir. Ces dernières sont des EDP de même nature que l'équation développée par Lagrange (0.0.1).

En particulier, comme cela est le cas dans les chapitres I, II et III, nous séparons l'analyse en deux parties. En effet, d'une part, nous étudions les propriétés d'injectivité ou de surjectivité des linéarisés des opérateurs que nous rencontrons. D'autre part, on traite les termes restants (que nous qualifions de terme d'erreur) qui regroupent les termes quadratiques et supérieurs à l'aide de théorèmes d'inversion locale, de fonctions implicites ou de points fixes, l'idée consistant à dire que ces termes sont négligeables devant le terme linéaire et que l'on peut bouger un peu ce dernier de telle sorte qu'il compense l'erreur commise.

Il s'avère que pour les exemples que nous voulons construire, les opérateurs ci-nommés sont en général définis sur des espaces de fonctions qui sont elles-mêmes définies soit au-dessus d'un domaine non régulier comme c'est le cas dans le chapitre I (c'est un triangle), soit sur des domaines non compacts comme c'est le cas pour les trois chapitres suivants. C'est la raison pour laquelle nous privilégions le travail dans des *espaces à poids*. Ces derniers sont des espaces de fonctions dont le comportement est en quelque sorte prescrit par un terme de croissance ou de décroissance : par exemple, pour des triangles, on impose que le comportement près des sommets soit au pire d'ordre r^δ pour un certain réel δ où r représente la distance à un sommet ; pour des cylindres du type $M \times \mathbb{R}$, on impose un comportement du type $e^{-\delta t}$ où t est la seconde coordonnée dans $M \times \mathbb{R}$. Il est à noter que dans le cas du triangle, si on effectue un changement de variable $r = e^{-t}$, les espaces à poids associés peuvent être considérés comme des espaces à poids sur des fonctions définies sur des domaines non compacts. Nous donnons une définition pour illustrer cette notion dans un cadre L^2 mais en pratique, nous utiliserons plutôt des cadres de régularité hölderienne.

Définition 1 — Si $C = M \times \mathbb{R}_+$ est un demi-cylindre et δ est un nombre réel, on définit l'espace à poids $L^2_\delta(C)$ comme étant

$$L^2_\delta(C) \quad := \quad e^{\delta t} L^2(C),$$

où t désigne la seconde coordonnée de $M \times \mathbb{R}$.

De façon générale, l'analyse d'un opérateur linéaire $\mathcal{L}$ d'ordre deux sur un cylindre $C = M \times \mathbb{R}$ se fait tout d'abord par l'étude de ses *racines indicielles* qui sont définies comme suit.

Définition 2 — Un nombre réel δ est appelé racine indicielle en $+\infty$ de l'opérateur linéaire $\mathcal{L}$ s'il existe une fonction v de classe $\mathcal{C}^2$ sur C et un nombre $\delta' < \delta$ tels que les deux assertions suivantes soient vérifiées :

$$\liminf \|v\|_{L^\infty(M\times\{t\})} > 0 \qquad \text{et} \qquad \lim \left(e^{-\delta' t}\mathcal{L}\left(e^{\delta t}v\right)\right) = 0$$

quand t tend vers $+\infty$.

En particulier, on en déduit que la fonction $\mathcal{L}\left(e^{\delta t}v\right)$ est négligeable devant $e^{\delta t}$. En pratique, nous déterminerons certaines racines indicielles en utilisant des éléments du noyau de $\mathcal{L}$. Le rôle des racines indicielles est primordiales. On peut montrer que sous certaines hypothèse sur l'opérateur $\mathcal{L}$, ce dernier a de très bonnes propriétés si on le considère comme un opérateur de $L^2_\delta(C)$ dans lui-même du moment que δ n'est pas une racine indicielle : des propriétés de type Fredholm peuvent être prouvées et des arguments de dualité peuvent être utilisés pour relier la surjectivité à l'injectivité, etc...

Dans nombre d'articles est utilisée la théorie des opérateurs sur des espaces à poids. Nous donnerons de nombreuses références par la suite. Toutefois, un exposé complet de celle-ci pourra être trouvée dans les lectures [Pac09] qui traitent de l'analyse sur des variétés ayant un nombre fini de bouts de type cylindrique.

Le chapitre V se distingue des autres chapitres de par la nature de la construction que nous faisons. Celle-ci repose essentiellement sur le principe du maximum et le principe de réflection. Nous en rappelons les énoncés — se reporter à [GT01, Theorem 3.3] ou [CM05, 1.3] pour le principe du maximum, à [1] [ST, Lemma 3.1] pour le principe de réflexion.

Théorème 1 (Principe du maximum)
Soit Σ_1 et Σ_2 deux hypersurfaces minimales dans $\mathbb{R}^n \times \mathbb{R}$ telles que Σ_i est le graphe d'une fonction $u_i : D \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ où $\mathcal{D}$ est un compact. Alors :

1) si le bord de Σ_1 est au dessus du bord de Σ_2, la surface Σ_1 est au-dessus de Σ_2 ;

2) si Σ_1 et Σ_2 sont tangentes en un point p qui appartient à l'intérieur de ces deux surfaces et que l'une est au-dessus de l'autre, alors Σ_1 et Σ_2 sont égales.

Théorème 2 (Principe de réflexion)
Soit $\Omega \subset \mathbb{R}^n$ un domaine dont la frontière contient un ouvert V d'un hyperplan H et supposons que Ω est situé de l'un des deux côtés de H et que $\partial\Omega \cap H = \overline{V}$.

Soit s la réflexion orthogonale de $\mathbb{R}^n$ par rapport à H et $u : \Omega \longrightarrow \mathbb{R}$ une solution à l'équation des graphes minimaux qui est continue sur $\Omega \cup V$ telle qu'elle s'annule sur V. Alors u peut être analytiquement prolongée à travers V en une

1. Dans cet article, le principe de réflexion est écrit pour des graphes dans $\mathbb{H}^n \times \mathbb{R}$ mais la preuve donnée fonctionne églement dans $\mathbb{R}^n \times \mathbb{R}$.

> *fonction* $\widetilde{u} : \Omega \cup V \cup s(\Omega) \longrightarrow \mathbb{R}$ *qui satisfait également l'équation des graphes minimaux en posant*
>
> $$\widetilde{u}(p) = \begin{cases} u(p) & when \quad p \in \Omega \cup V, \\ -u(s(p)) & when \quad p \in s(\Omega). \end{cases}$$

Présentation des travaux effectués

Le parti pris est de formuler les différents résultats obtenus de façon indépendante. En particulier, les chapitres II et III, même s'ils sont légèrement redondants, doivent être considérés comme deux prépublications différentes.

Chapitre I.
Construction de polygones minimaux dans $\mathbb{R}^3$

Bien que ce ne soit pas le résultat essentiel de ma thèse, j'ai choisi de présenter ce chapitre en premier lieu car il est à mon sens un moyen efficace de se familiariser avec la théorie des espaces à poids et d'illustrer les propriétés énoncées dans [Pac09].

Nous présentons donc ici la construction de surfaces minimales dans l'espace euclidien $\mathbb{R}^3$ de type polygonal. Par polygone, nous voulons signifier que la surface construite est compacte et sa frontière est de classe $\mathcal{C}^{2,\alpha}$ par morceaux et ne présente qu'un nombre fini de points singuliers.

Pour cela, nous déformons un polygone plat $\mathcal{P}$ que l'on suppose inclus dans le plan horizontal $\{x_3 = 0\}$ de $\mathbb{R}^3$. Nous supposons qu'il a m sommets. Il est à noter que des résultats de déformation avaient déjà été obtenus par Brian White [Whi87] pour des surfaces à bord suffisamment régulier. L'idée de la preuve dans les deux cas consiste à appliquer un théorème des fonctions implicites qui utilise les propriétés de l'opérateur de Jacobi J associé. Toutefois, dans notre cas, celui-ci n'est autre que le Laplacien classique car nous avons choisi un polygone plat.

Comme nous l'avons déjà dit plus haut, le rôle des espaces à poids est primordial pour traiter les singularités que constituent les sommets du polygone $\mathcal{P}$. D'une certaine manière, ce changement de point de vue transforme ce dernier en une variété de dimension 2 qui a m bouts de type cylindrique. Cela consiste à dire que la croissance d'une fonction u définie sur $\mathcal{P}$ est typiquement r_i^δ pour un certain nombre réel δ où r_i désigne la distance au i-ème sommet. Le lecteur remarquera que, puisque l'on travaille *a priori* avec des petites déformations, le paramètre de poids δ sera choisi positif de telle façon à ce que r^δ tende vers 0 quand on s'approche de l'un des sommets.

Par exemple, pour tout m-uplet $\overline{\delta} = (\delta_i)$, on peut définir l'espace à poids $L^\infty_{\overline{\delta}}(\mathcal{P})$ comme l'espace des fonctions u de $L^\infty_{loc}(\mathcal{P})$ telles que la norme

$$\|u\|_{L^\infty_{\overline{\delta}}(\mathcal{P})} = \sum_i \left\| r_i^{-\delta_i} u \right\|_{L^\infty(\mathcal{P})}$$

soit finie. Pour la définition un peu plus technique des espaces à poids $\mathcal{C}^{2,\alpha}_{\overline{\delta}}(\mathcal{P})$ de type Hölder, nous renvoyons à la section 3 du chapitre dont il est question.

Le théorème que nous prouvons peut être résumé comme suit.

Théorème 3
Notons $0 < \omega_i < 2\pi$ l'angle au i-ème sommet. Alors pour tout m-uplet $\overline{\delta}$ qui satisfait

$$\forall i, \qquad \delta_i \quad \in \quad [1,2] \cap \left(0, \frac{\pi}{\omega_i}\right),$$

si $Z : \mathcal{P} \longrightarrow \mathbb{R}^3$ est un champ de vecteur de classe $\mathcal{C}^3$ suffisamment petit, il existe une unique fonction u dans l'espace à poids $\mathcal{C}^{2,\alpha}_{\overline{\delta}}(\mathcal{P})$ qui s'annule au bord telle que la surface $\mathcal{P}_{u,Z}$ dont le graphe est donné par le graphe

$$p \in \mathcal{P} \quad \longmapsto \quad p + Z(p) + u(p)\,\mathbf{e}_3$$

soit minimale, où $\mathbf{e}_3$ est le vecteur unitaire vertical qui pointe vers le haut de $\mathbb{R}^3$.

Il est intéressant de noter au passage que le bord du polygone minimal $\mathcal{P}_{u,Z}$ alors construit est donné par le graphe de

$$p \quad \longmapsto \quad p + Z(p).$$

La condition sur les poids vient essentiellement de deux arguments différents.
- L'inégalité $\delta \leqslant 2$ provient du fait que lors de notre démonstration, un point légèrement technique consiste à vérifier qu'un certain opérateur $\widehat{H}$ doit envoyer l'espace à poids de paramètre $\overline{\delta}$ dans l'espace à poids $\overline{\delta} - \overline{2}$ où $\overline{2}$ est le m-uplet $(2, \ldots, 2)$.
- D'autre part, l'inégalité $\delta_i \geqslant 1$ est à considérer comme étant plutôt une condition sur l'angle ω_i qui doit, pour qu'un δ_i convenable existe, être plus petit que π. La raison en est que dans ce cas, on ne peut plus s'attendre à ce que la surface minimale construite soit un graphe au-dessus d'un domaine que l'on souhaite prescrire — nous renvoyons pour cela à la dernière remarque du chapitre I.

Chapitre II.
Construction d'hypersurfaces minimales de type Riemann dans $\mathbb{R}^n \times \mathbb{R}$

Ce chapitre constitue la partie la plus importante de mon travail et correspond à la prépublication [CP11].

Les surfaces minimales découvertes par Riemann forment une famille à 1 paramètre de surfaces qui sont feuilletées par des cercles horizontaux. Si des caractérisations de telles surfaces ont été développées dans [HKR91] et [MPR], nous nous intéressons plutôt à la généralisation de l'existence de ce type de surfaces. Elles sont essentiellement de deux natures : d'une part l'existence avec un nombre arbitraire de cous entre les différents niveaux, d'autre part l'existence dans des espaces homogènes $M \times \mathbb{R}$ où M est une variété de dimension au moins égale à 2. Avant d'aller plus loin, nous donnons la définition de ce que nous appelons (hyper)surfaces de type Riemann.

Définition 3 — Nous disons qu'une surface Σ d'une variété homogène $M \times \mathbb{R}$ est de *type Riemann* si elle est connexe, complète, plongée et qu'elle présente des bouts asymptotiques à $M \times \{s_i\}$ pour des réels s_i (en nombre fini ou infini). Nous appelons $M \times \{s_i\}$ le i-ème niveau. Nous appelons *cous* les parties de Σ situées entre deux niveaux consécutifs. De façon générale, les surfaces construites constituent une famille à 1 paramètre (ou plus) ϵ arbitrairement petit. Nous appelons *point de recollement* le point correspondant à la limite des cous quand le paramètre ϵ tend vers 0.

Il est à noter que dans cette définition, nous ne supposons pas de critère de périodicité. Nous nous concentrons sur le cas particulier où $M = \mathbb{S}^2$ dans le chapitre III.

Le premier exemple autre que celui de Riemann a été produit par F. Wei [Wei94] dans $\mathbb{R}^2 \times \mathbb{R}$. Celui-ci consiste en une surface périodique, présentant de nombreuses symétries, à savoir une symétrie centrale et une symétrie par rapport au plan vertical $\{x_1 = 0\}$, telle que les niveaux $2l$ et $2l+1$ sont reliés entre eux par 1 cou tandis que les niveaux $2l+1$ et $2l+2$ sont reliés entre eux par 2 cous. Le lecteur pourra en trouver une illustration — se reporter à la figure III.1 page 98.

Dans $\mathbb{R}^2 \times \mathbb{R}$, des cas très généraux ont été prouvés par M. Traizet [Tra02a] et [Tra02b]. Dans le premier article, il prouve que sous certaines contraintes de configuration géométrique des points où sont placés les cous, à savoir les conditions *équilibrée* et *non dégénérée*, on peut produire des exemples de surfaces de type Riemann périodiques avec un nombre arbitraire de cous. Dans le second article, il prouve que l'on peut produire, sous des contraintes similaires, des surfaces de type Riemann non périodiques avec un nombre fini de bouts et également un nombre arbitraire de cous. Des exemples numériques ont été fournis et le lecteur pourra trouver des illustrations dans [Tra] Encore plus récemment, en collaboration avec F. Morabito [MT11], il a prouvé l'existence de ce type de surface non périodique ayant un nombre infini de bouts. La méthode utilisée repose en grande partie sur les représentations de Weirstrass et ne peuvent donc pas s'appliquer au cas de $\mathbb{R}^n \times \mathbb{R}$.

Toutefois, par des méthodes de recollement issues de l'analyse d'opérateurs, S. Fakhi et mon directeur de thèse F. Pacard [FP00] ont prouvé l'existence d'hypersurfaces de type Riemann dans $\mathbb{R}^n \times \mathbb{R}$, avec n supérieur ou égal à 3, ayant un nombre fini de bouts, telles que deux niveaux sont reliés entre eux par 1 cou. Plus récemment a été prouvée dans [KP07] l'existence de l'exemple de Riemann en plus grande dimension : la surface est périodique avec une infinité de bouts reliés deux à deux par 1 cou. En revanche, contrairement au cas $n = 2$, elles ne sont pas feuilletées par des sphères horizontales.

Mon travail a consisté à obtenir des résultats similaires à ceux de M. Traizet en dimension plus grande. La méthode utilisée fait en quelque sorte la jonction entre ses travaux et ceux de F. Pacard. En effet, bien que la méthode que nous utilisons soit différente en ce qui concerne l'analyse, nous avons obtenu des résultats tout à fait comparables à ceux de M. Traizet en ce qui concerne les configurations de points — et tout particulièrement les conditions *équilibrée* et *non dégénérée* — et l'idée de la preuve qui consiste à procéder à divers recollements.

Il est à noter que nous obtenons des configurations pour lesquelles nous avons un degré de liberté supplémentaire. Cela est essentiellement dû au fait que les caténoïdes de $\mathbb{R}^2 \times \mathbb{R}$ et ceux de $\mathbb{R}^n \times \mathbb{R}$ avec $n > 2$ n'ont pas le même type de comportement. En effet, si le caténoïde classique présente une croissance logarithmique, le caténoïde en dimension supérieure est en revanche asymptote à deux hyperplans horizontaux. Ainsi, si l'on cherche dans $\mathbb{R}^2 \times \mathbb{R}$ à obtenir des surfaces à bouts horizontaux, il faut que la contribution des caténoïdes, qui est du type

$$\sum_i a_i \ln(|x|)$$

sur un bout $\mathbb{R}^2 \times \{s_i\}$, où les a_i correspondent aux tailles des caténoïdes que l'on placera au i-ème point du i-ème niveau, soit telle que la limite quand $|x|$ tend vers l'infini soit finie. En d'autres termes, il faut nécessairement imposer la condition

$$\sum_i a_i \quad = \quad 0$$

à tous les niveaux, ce qui n'est pas le cas en dimension supérieure. Cela explique par exemple que pour la surface de Wei, la distance entre deux niveaux reliés par un seul cou soit deux fois plus grande que la distance entre deux niveaux reliés par deux cous.

Comme je l'ai dit ci-dessus, nous pouvons produire des surfaces uniquement sous certaines conditions. Ces dernières sont en réalité formulées en terme de *forces*. Il faut comprendre la force en un point de recollement comme l'interaction qu'il y a entre ce point et les autres points de recollement. Notez au passage que cette force apparait de façon naturelle comme le terme linéaire du développement limité des fonctions de Green. Ces dernières sont des fonctions harmoniques dont le Laplacien est une somme de masses de Dirac. Nous prouvons que leur existence et leur comportement sont étroitement liés à l'existence des caténoïdes. Plus de détails à ce sujet seront donnés dans le chapitre II.

En revanche, il semble essentiel dans cette introduction de donner plus d'informations concernant les forces. On se donne, pour chaque niveau k, un nombre fini n_k de points $p_{k,j}$ de $\mathbb{R}^n$ avec $1 \leqslant j \leqslant n_k$. Géométriquement, ces points sont les lieux où l'on procède aux recollements. Pour illustrer, l'exemple dû à Riemann est tel que n vaut 2, qu'il y a une infinité de niveaux $k \in \mathbb{Z}$, que n_k vaut 1 et que l'on obtient $p_{k+1,1}$ à partir de $p_{k,1}$ grâce à une translation $\mathbf{t}_{\text{hor}}$ de $\mathbb{R}^2$ qui ne dépend pas du niveau auquel on se situe. Pour chaque niveau, on se donne un paramètre de poids a_k qui est un nombre réel strictement positif. Géométriquement, ce paramètre est relié à la distance entre les niveaux k et $k+1$; c'est la même chose que de dire que a_k détermine la taille des caténoïdes que l'on recolle entre ces deux niveaux. On prouve que celle-ci est de l'ordre de

$$(a_k\,\epsilon)^{\frac{1}{n-1}}$$

dans $\mathbb{R}^n \times \mathbb{R}$ avec $n \geqslant 3$. Dans ce cas, contrairement à ce qui se passe en dimension inférieure, on peut choisir les poids de façon indépendante, ce qui procure un degré de liberté supplémentaire comme annoncé. J'invite le lecteur à se reporter à la figure 3 pour avoir une idée de l'allure de l'objet que l'on souhaite construire.

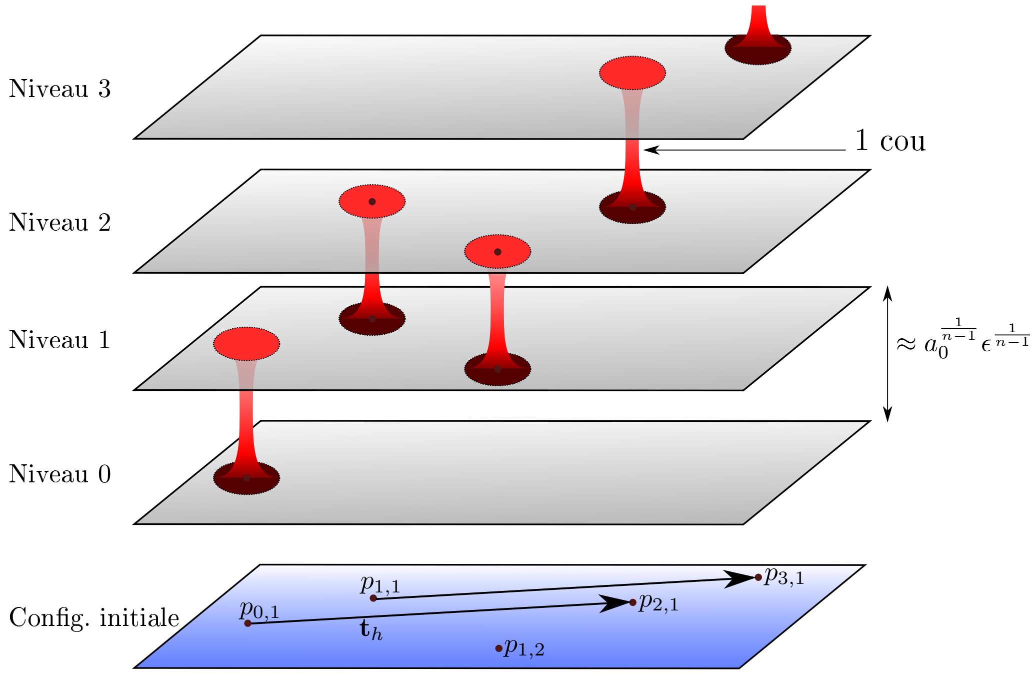


FIGURE 3 – Un exemple de configuration périodique avec consécutivement 1 cou et 2 cous entre deux niveaux consécutifs.

On définit la force $f(p,q)$ d'interaction entre deux points comme étant le vecteur

$$f(p,q) \quad := \quad (n-2)\,\frac{p-q}{|p-q|^n}$$

et la force totale $F_{k,j}$ qu'exercent tous les autres points sur le point $p_{k,j}$ comme étant le vecteur

$$F_{k,j} \quad := \quad 2\sum_{\substack{i=1\\i\neq j}}^{n_k} a_k f(p_{k,j},p_{k,i}) - \sum_{i=1}^{n_{k-1}} a_{k-1} f(p_{k,j},p_{k-1,i}) - \sum_{i=1}^{n_{k+1}} a_{k+1} f(p_{k,j},p_{k+1,i})\,.$$

Ainsi, celle-ci dépend uniquement de l'interaction avec les points du même niveau (avec un facteur 2) et de celle avec les niveaux juste au-dessus et juste en-dessous.

Définition 4 — On dit que la configuration de points pondérés $\{(a_k, p_{k,j})\}$ est *équilibrée* si toutes les forces sont nulles.

Le sens géométrique de ces forces pour les surfaces de type de Riemann peut être interprété comme étant la façon dont on *penche* les cous lors du recollement. En effet, si on fait un zoom sur l'un des cous de l'exemple classique de Riemann, on obtient la figure 4. On constate effectivement dans ce cas que les cous ne sont pas verticaux mais légèrement penchés dans une direction privilégiée, à savoir la direction opposée à celle donnée par le terme horizontal de la période.

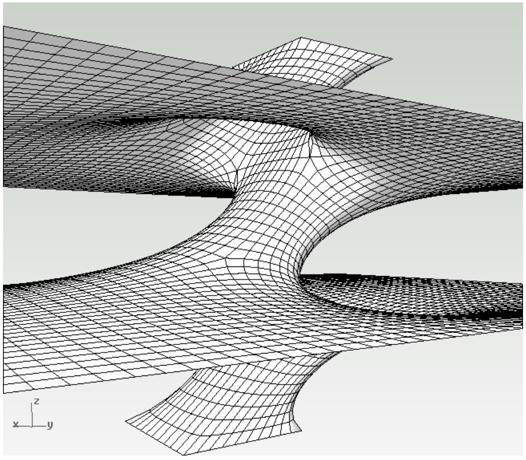


FIGURE 4 – Dans l'exemple de Riemann, les cous sont orientés selon une certaine direction.

Le fait que la configuration soit balancée correspond géométriquement au fait que les cous ne peuvent pas être « tordus », c'est-à-dire que leur axe n'est pas courbé.

La condition de non dégénérescence, quant à elle, est légèrement plus technique. On la résume dans la définition suivante.

Définition 5 — Une configuration initiale de points est dite *non dégénérée* si l'application $\mathcal{F}$ qui associe, à une configuration, l'ensemble des forces totales $F_{k,j}$ est de rang maximal en cette configuration.

Nous explicitons le rang maximal dont il est question dans l'introduction du chapitre. Essentiellement, le fait que $\mathcal{F}$ ne soit pas de rang plein provient du groupe de symétries $\mathfrak{G}$ inhérent aux surfaces que l'on veut construire : parmi celles-ci citons notamment les rotations, les dilatations et les translations. Géométriquement, le fait que la condition soit non dégénérée se traduit par la capacité à pouvoir bouger légèrement les points et les poids de la configuration initiale de façon à prescrire n'importe quelle force arbitrairement petite.

Le théorème d'existence que nous prouvons est le suivant.

Théorème 4
Étant donnée une configuration balancée et non dégénérée, on peut construire une famille à 1 paramètre d'hypersurfaces minimales $(\mathcal{S}_\epsilon)_\epsilon$ de type Riemann dans $\mathbb{R}^n \times \mathbb{R}$ telles que les cous entre deux niveaux consécutifs sont placés dans un voisinage des points de la configuration initiale. De plus, la distance entre deux niveaux est de l'ordre de $a_k^{1/(n-1)}\epsilon^{1/(n-1)}$.

Nous terminons le chapitre en donnant quelques exemples.

Chapitre III. Construction de surfaces minimales de type Riemann-Wei dans $\mathbb{S}^2 \times \mathbb{R}$

Ce chapitre se consacre à nouveau aux surfaces minimales de type Riemann, mais cette fois-ci dans $\mathbb{S}^2 \times \mathbb{R}$. C'est l'objet de la prépublication [CP12]. Nous démontrons le théorème suivant.

> **Théorème 5**
> *Il existe une famille de surfaces minimales de type Riemann dans $\mathbb{S}^2 \times \mathbb{R}$ qui correspond à l'analogue de la surface de Wei.*

Tout d'abord, précisons ce que l'on entend par « analogue de la surface de Wei ». Dans un article non publié [Wei94], F. Wei établit l'existence dans $\mathbb{R}^2 \times \mathbb{R}$ d'une surface minimale simplement périodique avec une infinité de bouts plans qui sont reliés entre eux par alternativement 1 cou et 2 cous — nous renvoyons à la figure III.1 page 98. Le théorème prouvé pendant cette thèse stipule donc qu'une telle surface existe dans $\mathbb{S}^2 \times \mathbb{R}$: on peut la voir comme une surface périodique (dans un sens que l'on précise dans le chapitre III) qui relie une infinité de sphères par alternativement 1 cou et 2 cous.

La raison pour laquelle nous n'avons pas démontré un théorème plus global de type Traizet comme nous l'avons effectué dans le chapitre II est que la géométrie de $\mathbb{S}^2$ complique techniquement le problème.

Ceci dit, nous utilisons à nouveau des arguments faisant appel à des conditions balancées et non dégénérées, même si cette dernière condition n'est pas explicitée car elle est cachée dans la preuve de la dernière proposition du chapitre.

Chapitre IV. Paramétrisation des surfaces minimales de Riemann dans $\mathbb{S}^2 \times \mathbb{R}$

Ce court chapitre se consacre à l'étude de la paramétrisation de l'analogue de l'exemple de Riemann dans $\mathbb{S}^2 \times \mathbb{R}$.

Son existence a déjà été démontrée par L. Hauswirth [Hau06]. L'intérêt du chapitre réside essentiellement dans le fait qu'il permet d'obtenir une description relativement précise de ladite surface. De plus, elle a le mérite d'être très parlante s'agissant de sa représentation géométrique.

En particulier, nous en déduisons l'existence de l'analogue de la caténoïde dans $\mathbb{S}^2 \times \mathbb{R}$. De plus, nous démontrons que les cous reliant deux niveaux de la surface de Riemann ressemblent à des caténoïdes penchés comme cela est prouvé dans [HP07].

Chapitre V.

Construction d'hypersurfaces de Scherk dans $\mathbb{R}^n \times \mathbb{R}$

Ce dernier chapitre a été placé en dernier car les techniques utilisées pendant les preuves diffèrent complètement des précédentes. Nous y démontrons l'existence d'hypersurfaces de type Scherk dans $\mathbb{R}^n \times \mathbb{R}$ avec $n \geqslant 3$.

Je tiens à souligner que ce chapitre n'est pas complet au sens où j'ai arrêté de m'y consacrer au bout d'un an. En effet, il s'agit de mon premier travail effectué pendant la thèse. Malheureusement, nous nous sommes rendu compte en fin de première année qu'un article d'E. Tomaini [Tom86] utilisant surtout des techniques de théorie géométrique de la mesure résolvait toutes les questions que nous nous posions et ce, dans un cadre beaucoup plus général que le nôtre. Malgré tout, j'ai tenu à le mettre en raison des techniques utilisées qui diffèrent des outils classiques permettant d'étudier les surfaces de type Scherk. En particulier, l'essentiel de nos raisonnements se font *à la main* et reposent sur des arguments très visuels.

La surface de Scherk est une surface minimale qui a 4 bouts plans parallèles deux à deux. Dans ce chapitre, nous construisons d'autres hypersurfaces minimales de type Scherk — nous en donnons la définition.

Définition 6 — Une hypersurface minimale Σ de $\mathbb{R}^n \times \mathbb{R}$ est de *type Scherk* si elle peut être représentée comme le graphe d'une fonction u au-dessus d'un domaine D de $\mathbb{R}^n$ qui vaille $+\infty$ ou $-\infty$ sur une partie du bord.

Mon directeur de thèse a déjà prouvé dans [Pac02] qu'il existe une hypersurface analogue dans l'espace euclidien de dimension plus grande. Par ailleurs, R. Sa Earp et É. Toubiana [ST] ont prouvé l'existence d'hypersurfaces de type Scherk dans $\mathbb{H}^n \times \mathbb{R}$ et $\mathbb{R}^n \times \mathbb{R}$ qui sont des graphes au-dessus de domaines ayant de nombreuses symétries, à savoir sur des polytopes réguliers.

H. Jenkins et J. Serrin [JS65] ont prouvé qu'il était possible d'exhiber de tels objets modulo certaines contraintes sur le domaine D dans l'espace euclidien de dimension 3. En effet, ils prouvent que les parties du bord sur lesquelles u prend une valeur infinie doivent nécessairement être des segments droits. De plus, des conditions concernant leur longueur doivent être satisfaites : par exemple, si l'on veut prescrire sur le bord de D des valeurs toutes infinies, la longueur ℓ_- des parties des ∂D sur laquelle u vaut $+\infty$ doit être égale à la longueur ℓ_- des parties de ∂D sur laquelle u vaut $-\infty$. Cela est dû à une condition de flux sur la surface ; géométriquement, si par exemple $\ell_+ > \ell_-$, on trouve un graphe u qui vaut $+\infty$ partout, ce qui n'est pas d'un vif intérêt. Des résultats semblables ont été établis par B. Nelli et H. Rosenberg [NR02] dans $\mathbb{H}^2 \times \mathbb{R}$ et par A. Pinheiro [Pin09] ou encore L. Mazet, M. Rodríguez et H. Rosenberg [MRR11] dans le cas de $M \times \mathbb{R}$, où M est une variété de dimension 2. Dans ces derniers cas, les segments sont remplacés par des géodésiques.

Ainsi, il est tout à fait raisonnable de s'attendre à avoir des contraintes géométriques sur D. En particulier, les parties de ∂D sur lesquelles u prend une valeur infinie sont choisies comme étant elles-mêmes des hypersurfaces minimales de $\mathbb{R}^n$. Nous ne prouvons pas que c'est une condition nécessaire mais le lecteur pourra en trouver l'explication dans [Tom86]. Nous démontrons différents résultats (parfois partiels).

Théorème 6
Supposons que le bord du domaine D est exactement $\mathcal{S} \cup \Gamma$, où $\mathcal{S}$ (resp. Γ) est le graphe d'une fonction s (resp. γ) au-dessus d'un même ouvert pseudoconvexe de $\mathbb{R}^{n-1}$ tel que $\mathcal{S}$ est minimale et Γ est convexe. Alors il existe une surface de type Scherk qui soit le graphe d'une fonction u au-dessus de D qui s'annule sur l'intérieur de Γ et qui prenne la valeur $+\infty$ sur l'intérieur de $\mathcal{S}$.

Ce théorème permet ensuite de retrouver les exemples de [ST]. Nous construisons également un exemple dans $\mathbb{R}^3 \times \mathbb{R}$ au dessus d'un octaèdre dont les faces sont des triangles minimaux.

Enfin, nous terminons en donnant des conditions sur la géométrie de D de type Jenkins-Serrin en ce qui concerne le $(n-1)$-volume des parties du bord sur lesquelles u est continue ou infinie.

Chapitre I

Construction de polygones minimaux dans $\mathbb{R}^3$

Introduction

The theory of minimal surfaces in the 3-dimensional Euclidean space has been specifically developed for the last thirty years. In particular, numerous examples of perturbations of minimal surfaces have been produced. For example, in [Whi87], B. White proved that if a compact minimal surface has smooth boundary, then one can perturb its boundary keeping the surface minimal. One of the most useful application of this kind of perturbation is to construct new minimal surfaces by performing a connected sum of two different surfaces Σ_1 and Σ_2 : the idea is to take off a small disk in Σ_1 and Σ_2 and to deform the punctured minimal surfaces we obtain in order to match their boundary data. For more details, we refer to [MP01], [MPP01] or [Pac98].

In this chapter, we are interested in the perturbation of specific minimal surfaces, namely the polygons. The type of result we obtain could be expected to be similar to the one of B. White, but the fact there are vertices modifies its proof.

The main idea of the method lies in applying a well chosen implicit function theorem. It amounts to study the Laplacian operator — which is the linearization of the mean curvature — about domains with singularities. P. Grisvard in [Gri92] or M. Dauge in [Dau88] already studied this operator around polygons. Nevertheless, it seems that the theory of weighted spaces proves its efficiency in our case. We refer to the lectures [Pac09] for main results.

In the following, we only consider the case of a triangle but it can be easily extended to the case of polygons. Let $\mathcal{T}$ be a triangle in $\mathbb{R}^3$ such that its interior $\mathring{\mathcal{T}}$ is non empty and $\mathcal{T}$ is closed, that is to say $\overline{\mathcal{T}} = \mathcal{T}$. Without loss of generality, we assume that $\mathcal{T}$ is horizontal, in other words, we assume that $\mathcal{T}$ belongs to the plane $\{x_3 = 0\}$. Of course, since $\mathcal{T}$ is flat, it is a minimal surface. To perform a perturbation, we introduce two types of terms, namely :

- normal perturbation given by

$$p \in \mathcal{T} \quad \longmapsto \quad p + u\,(p)\,\mathbf{e}_3,$$

where $\mathbf{e}_3 = (0,0,1)$ is the unit normal vector of $\mathcal{T}$ that points upwards and u is a regular enough function which vanishes on the boundary $\partial\mathcal{T}$ of the triangle ;
- any perturbation given by

$$p \in \mathcal{T} \quad \longmapsto \quad p + Z(p),$$

where $Z : \mathcal{T} \longrightarrow \mathbb{R}^3$ is a regular enough vector field.

The reader may wonder why we do not have directly chosen a vector field Z that would hold the component $u(p)\,\mathbf{e}_3$. The reason for this is that u corresponds to the classical perturbation parameter (usually, we consider normal perturbations) while Z has to be understood as a parameter that transforms the boundary of the triangle. We then denote by $\mathcal{T}_{u,Z}$ the surface in $\mathbb{R}^3$ which is the graph of $t_{u,Z}$ whose definition is

$$t_{u,Z} : p \in \mathcal{T} \longmapsto p + u(p)\,\mathbf{e}_3 + Z(p) \in \mathbb{R}^3.$$

Notice that for u and Z small enough, $\mathcal{T}_{u,Z}$ is an embedded surface with 3 vertices. We now state the main theorem [1].

> **Theorem 0.0.1**
> *For all $\overline{\delta} = (\delta_1, \delta_2, \delta_3)$ such that*
>
> $$\forall i \in \{1,2,3\}, \qquad \delta_i \in [1,2] \cap \left(-\frac{\pi}{\omega_i}, \frac{\pi}{\omega_i}\right)$$
>
> *where the ω_i's denote the angles of $\mathcal{T}$, there exists a neighbourhood $\mathcal{U}$ of 0 in $\mathcal{C}^{2,\alpha}_{\overline{\delta},0}(\mathcal{T},\mathbb{R})$, a neighborhood $\mathcal{V}$ of 0 in $\mathcal{C}^{2,\alpha}(\mathcal{T},\mathbb{R}^3)$ and an application $\varphi : \mathcal{V} \longrightarrow \mathcal{U}$ such that $\mathcal{T}_{u,Z}$ is a minimal surface with $(u,Z) \in \mathcal{U}\times\mathcal{V}$ if and only if $u = \varphi(Z)$.*

1 The minimal surface equation

In this section, we establish the formula which ensures that the deformed triangle $\mathcal{T}_{u,Z}$ is minimal.

The Jacobi operator $J_{\mathcal{T}}$ about $\mathcal{T}$ is the linearization of the mean curvature operator. It reads

$$J_{\mathcal{T}} \quad = \quad \Delta_{g_{\mathcal{T}}} + |A_{g_{\mathcal{T}}}|^2$$

where $\Delta_{g_{\mathcal{T}}}$ is the Laplace Beltrami operator on $\mathcal{T}$ endowed with metric $g_{\mathcal{T}}$ and $A_{g_{\mathcal{T}}}$ is the shape operator of $\mathcal{T}$. But the metric on $\mathcal{T}$ is nothing but the one induced by the Euclidean metric and since $\mathcal{T}$ is flat, we find that

$$\Delta_{g_{\mathcal{T}}} = \Delta_{\mathbb{R}^2} \qquad \text{and} \qquad A_{g_{\mathcal{T}}} = 0.$$

1. See the definition of the weighted spaces $\mathcal{C}^{2,\alpha}_{\overline{\delta}}$ in section 2.1.

According to [BdC84], the mean curvature of the surface $\mathcal{T}_{u,0}$ obtained performing only normal perturbations is given by

$$H\left(\mathcal{T}_{u,0}\right) = \Delta u + Q\left(u\right),$$

where Q is a non linear term that collects all terms of order larger or equal to 2. When there is no confusion, we note $\Delta = \Delta_{\mathbb{R}^2}$.

Lemma 1.0.2 — *Let u and Z be small enough. Then the mean curvature of $\mathcal{T}_{u,Z}$ is given by*

$$H\left(\mathcal{T}_{u,Z}\right) = \Delta u + \Delta \left\langle Z, \mathbf{e}_3\right\rangle_{\mathbb{R}^3} + Q\left(u, Z\right). \tag{1.0.1}$$

where Q is a non linear expression.

Proof

We give the proof in our case because it is very simple. Notice that it is enough to prove that the linear term in Z is exactly $\Delta\left\langle Z, \mathbf{e}_3\right\rangle$.

Assume in the first place that Z is always vertical, that is to say that $Z\left(p\right) = Z^3\left(p\right)\mathbf{e}_3$. Then Z is a normal perturbation term and the linear term is ΔZ^3 which is equal to $\left\langle Z, \mathbf{e}_3\right\rangle$.

Assume in the second place Z is tangent to the triangle. Then the surface $\mathcal{T}_{0,Z}$ lies in the horizontal plane $x_3 = 0$, thus it is minimal. It is the same to say $H\left(\mathcal{T}_{0,Z}\right)$ vanishes. □

Corollary 1.0.3 — *The surface $\mathcal{T}_{u,Z}$ is minimal if and only if the relation*

$$\Delta u + \Delta \left\langle Z, \mathbf{e}_3\right\rangle_{\mathbb{R}^3} = -Q\left(u, Z\right) \tag{1.0.2}$$

holds true.

2 Analysis in weighted spaces

The weighted function spaces provide a powerful theory of analysis on non-compact domains or on domains with singularities. The idea is to reduce to analysis on an infinite cylinder with prescribed asymptotic behaviour at infinity.

Denote by S_1, S_2 and S_3 the vertices of $\mathcal{T}$, chosen in counterclockwise. For all i in $\{1, 2, 3\}$, we define ω_i to be the oriented angle in S_i and $\mathcal{B}_i$ to be the open set of $\mathcal{T}$ such that

$$\mathcal{B}_i := \left\{P \in \mathcal{T} : \left\|P - S_i\right\|_2 < 1\right\}.$$

Without loss of generality, we can assume that the sides of $\mathcal{T}$ are big enough to ensure the $\mathcal{B}_i$ do not intersect themselves. Finally, we define $\mathcal{K}$ to be the compact set

$$\mathcal{K} \quad := \quad \mathcal{T} \setminus \cup_i \mathcal{B}_i.$$

Denote by (x, y) Cartesian coordinates on $\mathcal{T}$. If $P \in \mathcal{B}_i$, we can choose to work with polar coordinates (see the figure I.1) (r_i, θ_i) or cylindrical coordinates (t, θ) with the help of following identifications :

$$\begin{aligned} (x, y) \in \mathcal{B}_i \iff & (x, y) = (r_i \cos\theta_i, r_i \sin\theta_i) && \text{with} \quad (r_i, \theta_i) \in \mathcal{B}_{i,\text{pol}} \\ \iff & (x, y) = (e^{-t_i}\cos\theta_i, e^{-t_i}\sin\theta_i) && \text{with} \quad (t_i, \theta_i) \in \mathcal{C}_i \end{aligned}$$

where $\mathcal{B}_{i,\text{pol}}$ is the open set $(0, 1) \times (0, \omega_i)$ and $\mathcal{C}_i$ is the half-cylinder $(0, +\infty) \times (0, \omega_i)$. Note that we have assumed — without loss of generality — that the line $(S_i S_{i+1})$ coincides with the x-axis. By using cylindrical coordinates, we notice that one can consider the triangle as a manifold with three cylindrical ends : it is exactly the framework used in [Pac09]. In addition to that, the cylindrical coordinates are conformal.

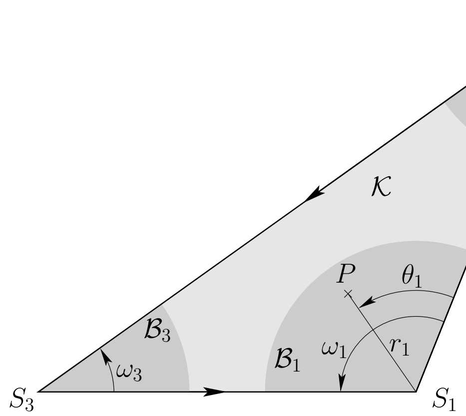


Figure I.1: Notations for the angles in $\mathcal{T}$.

We work with different points of view on $\mathcal{T}$: with cartesian, polar or cylindrical coordinates. More precisely, let $u : \mathcal{T} \longrightarrow \mathbb{R}$. We collect the different coordinates we use in the following table.

coordinates	set	function
cartesian	$(x, y) \in \mathcal{T}$	$u_{\text{car}} : (x, y) \longmapsto u(x, y)$
polar	$(r, \theta) \in \mathcal{B}_{\text{pol}}$	$u_{\text{pol}} : (r, \theta) \longmapsto u(r\cos\theta, r\sin\theta)$
cylindrical	$(t, \theta) \in \mathcal{C}$	$u_{\text{cyl}} : (t, \theta) \longmapsto u(e^{-t}\cos\theta, e^{-t}\sin\theta)$

Then the expression of the metric or the Laplace-Beltrami operator defined by

$$\Delta_g \quad = \quad \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x^j} \left(\sqrt{\det g}\, g^{ij} \frac{\partial}{\partial x^i} \right),$$

where $\left(g^{ij}\right)_{i,j}$ denotes the inverse of the matrix $\left(g_{ij}\right)_{i,j}$, depends of our choice of coordinates. We sum up this in the following table :

coordinates	metric	Laplace-Beltrami operator
Cartesian	$g_{\text{car}} = \mathrm{d}x^2 + \mathrm{d}y^2$	$\Delta_{\text{car}} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$
polar	$g_{\text{pol}} = \mathrm{d}r^2 + r^2\mathrm{d}\theta^2$	$\Delta_{\text{pol}} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial \theta^2}$
cylindrical	$g_{\text{cyl}} = e^{-2t}\left(\mathrm{d}t^2 + \mathrm{d}\theta^2\right)$	$\Delta_{\text{cyl}} = \frac{1}{e^{-2t}}\left(\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial \theta^2}\right)$

The most useful expression is undoubtedly the last one because it gives us a way to highlight the asymptotic behaviour when one approaches one of the vertices.

2.1 Weighted spaces

As announced, the theory of weighted spaces is very useful in our study because they provide an efficient tool to study analysis on manifolds with cylindrical ends.

Definition 2.1.1 — Recall that the cylinder $\mathcal{C}$ is defined to be $(0,+\infty)\times(0,\omega)$. Let $\delta\in\mathbb{R}$, $k\in\mathbb{N}$ and $\alpha\in(0,1)$. We define the *weighted Hölder space* $\mathcal{C}^{k,\alpha}_{\delta}(\mathcal{C})$ with weight δ over $\mathcal{C}$ by

$$\mathcal{C}^{k,\alpha}_{\delta}(\mathcal{C}) \quad := \quad e^{\delta t}\mathcal{C}^{k,\alpha}(\mathcal{C}),$$

endowed with the norm

$$\|u_{\text{cyl}}\|_{\mathcal{C}^{k,\alpha}_{\delta}(\mathcal{C})} \quad := \quad \left\|e^{-\delta t}u_{\text{cyl}}\right\|_{\mathcal{C}^{k,\alpha}(\mathcal{C})},$$

where the classical Hölder norm is defined to be

$$\|v\|_{\mathcal{C}^{k,\alpha}(D)} \quad := \quad \sum_{i=0}^{k}\sum_{|\beta|=i}\left\|\frac{\partial^\beta v_{\text{cyl}}}{\partial z^\beta}\right\|_{L^\infty(D)} + \sum_{|\beta|=k}\sup_{p\neq q\in D}\frac{\left|\frac{\partial^\beta v_{\text{cyl}}}{\partial z^\beta}(p) - \frac{\partial^\beta v_{\text{cyl}}}{\partial z^\beta}(q)\right|}{\|p-q\|^\alpha}.$$

for any domain $D\subset\mathbb{R}^2$ and any $v : D \longrightarrow \mathbb{R}$.

Remark 2.1.2 — Note that we can also define weighted Hölder spaces on $\mathcal{B}_{\text{pol}}$ by writing

$$u_{\text{pol}}\in\mathcal{C}^{k,\alpha}_{\delta}(\mathcal{B}_{\text{pol}}) \quad\Longleftrightarrow\quad u_{\text{cyl}}\in\mathcal{C}^{k,\alpha}_{-\delta}(\mathcal{C}).$$

The idea is that a function u belongs to the weighted space if u and its derivatives $\frac{\partial^k u}{\partial r^k}$ are bounded by a constant times $r^{\delta-k}$.

Typically, we want to deal with bounded perturbations to build a new minimal surface arbitrarily close to $\mathcal{T}$, so we do not want that u explodes near a vertice S. Therefore we choose $\delta>0$ to ensure $\lim_{r\to 0} r^\delta = 0$.

With the help of the above definition, it is natural to define weighted Hölder spaces on the triangle.

Definition 2.1.3 — Let $\overline{\delta} = (\delta_1,\delta_2,\delta_3)\in\mathbb{R}^3$. We define *the weighted Hölder space* $\mathcal{C}^{k,\alpha}_{\overline{\delta}}(\mathcal{T})$ to be the set

$$\mathcal{C}^{k,\alpha}_{\overline{\delta}}(\mathcal{T}) \quad := \quad \left\{u\in\mathcal{C}^{k,\alpha}(\mathcal{T}) : \forall i\in\{1,2,3\}, u_{\text{cyl},i}\in\mathcal{C}^{k,\alpha}_{\delta_i}(\mathcal{C}_i)\right\},$$

endowed with the norm

$$\|u\|_{\mathcal{C}^{k,\alpha}_{\overline{\delta}}(\mathcal{T})} \quad := \quad \left\|u_{|\mathcal{K}}\right\|_{\mathcal{C}^{k,\alpha}(\mathcal{K})} + \sum_{i=1}^{3}\|u_{\text{cyl},i}\|_{\mathcal{C}^{k,\alpha}_{\delta_i}(\mathcal{C}_i)}.$$

Remark 2.1.4 — For our study, it is also relevant to define the weighted spaces with vanishing data boundary

$$\mathcal{C}^{k,\alpha}_{\delta,0}(\mathcal{C}) \quad := \quad \left\{u \in \mathcal{C}^{k,\alpha}_{\delta}(\mathcal{C}) : u_{|\partial\mathcal{C}} = 0\right\}$$

together with

$$\mathcal{C}^{k,\alpha}_{\overline{\delta},0}(\mathcal{T}) \quad := \quad \left\{u \in \mathcal{C}^{k,\alpha}_{\overline{\delta}}(\mathcal{T}) : u_{|\partial\mathcal{T}} = 0\right\}.$$

2.2 The Laplace-Beltrami operator in Hölder weighted spaces over a cylinder

In order to become familiar with weighted spaces, we introduce the following property.

Property 2.2.1 — *Let k be a positive integer such that $k \geqslant 2$. Then if u_{cyl} belongs to $\mathcal{C}^{k,\alpha}_{-\delta}(\mathcal{C})$, the function $\Delta_{cyl} u_{cyl}$ belongs to $\mathcal{C}^{k-2,\alpha}_{-\delta+2}(\mathcal{C})$.*

Proof
It does not represent any difficulty, but it is a way to have a better understanding of these spaces. Therefore assume that u_{cyl} belongs to $\mathcal{C}^{k,\alpha}_{-\delta}(\mathcal{C})$. There exists $v_{\text{cyl}} \in \mathcal{C}^{k,\alpha}(\mathcal{C})$ such that

$$u_{\text{cyl}} \quad = \quad e^{-\delta t} v_{\text{cyl}}.$$

We deduce from this relation that

$$\frac{\partial^2 u_{\text{cyl}}}{\partial t^2} \;=\; e^{-\delta t}\left(\delta^2 v_{\text{cyl}} - 2\delta \frac{\partial v_{\text{cyl}}}{\partial t} + \frac{\partial^2 v_{\text{cyl}}}{\partial t^2}\right) \qquad \text{and} \qquad \frac{\partial^2 u_{\text{cyl}}}{\partial \theta^2} \;=\; e^{-\delta t}\frac{\partial^2 v_{\text{cyl}}}{\partial \theta^2}.$$

Consequently, there exists exists a function $w_{\text{cyl}} \in \mathcal{C}^{k-2,\alpha}(\mathcal{C})$ such that

$$\Delta_{\text{cyl}} u_{\text{cyl}} \quad = \quad \frac{1}{e^{-2t}}\left(\frac{\partial^2 u_{\text{cyl}}}{\partial t^2} + \frac{\partial^2 u_{\text{cyl}}}{\partial \theta^2}\right) \quad = \quad e^{(-\delta+2)t} w_{\text{cyl}} \tag{2.2.3}$$

and the conclusion follows. □

We are interested in the study of the following operator

$$\Delta_{\text{cyl},-\delta} : \begin{array}{ccc} \mathcal{C}^{k,\alpha}_{-\delta,0}(\mathcal{C}) & \longrightarrow & \mathcal{C}^{k-2,\alpha}_{-\delta+2}(\mathcal{C}) \\ u_{\text{cyl}} & \longmapsto & \dfrac{1}{e^{-2t}}\left(\dfrac{\partial^2 u_{\text{cyl}}}{\partial t^2} + \dfrac{\partial^2 u_{\text{cyl}}}{\partial \theta^2}\right). \end{array}$$

Instead of giving results about $\Delta_{\text{cyl},-\delta}$, we rather study the elliptic operator $\mathcal{L}_{\text{cyl},-\delta}$ defined by

$$\mathcal{L}_{\text{cyl},-\delta} : u_{\text{cyl}} \in \mathcal{C}^{k,\alpha}_{-\delta,0}(\mathcal{C}) \longmapsto e^{-2t}\Delta_{\text{cyl},-\delta} u_{\text{cyl}} \in \mathcal{C}^{k-2,\alpha}_{-\delta}(\mathcal{C}).$$

Note that $\mathcal{L}_{\text{cyl}}$ is nothing but the Laplace-Beltrami operator on $\mathcal{C}$ endowed with metric $\mathrm{d}t^2 + \mathrm{d}\theta^2$ which satisfies conditions that appear in [Pac09].

The role of indicial roots is fundamental in studying the mapping properties of an elliptic operator. They are deeply linked to mapping properties of the operator that acts on weighted spaces.

Definition 2.2.2 — A real number δ is an *indicial root* of $\mathcal{L}_{\text{cyl}}$ if there exists a non-zero function $u_{\text{cyl}} \in \mathcal{C}_0^2(\mathcal{C}) \setminus \{0\}$ and $\delta' < \delta$ such that

$$\lim_{t\to+\infty} \inf \|u_{\text{cyl}}\|_{L^\infty(\{t\}\times(0,\omega))} \quad > \quad 0$$

and

$$e^{-\delta' t}\mathcal{L}_{\text{cyl}}\left(e^{\delta t}u\right) \xrightarrow[t\to+\infty]{} 0.$$

We denote by $\mathrm{Ind}(\mathcal{L}_{\text{cyl}})$ the associated set and by $\mathrm{Isom}(\mathcal{L}_{\text{cyl}})$ the set defined to be

$$\mathrm{Isom}(\mathcal{L}_{\text{cyl}}) := \left\{\delta : \forall \delta' \in \mathrm{Ind}(\mathcal{L}_{\text{cyl}}) \cap \mathbb{R}^+, \delta \in (\delta', -\delta')\right\}.$$

The set $\mathrm{Isom}(\mathcal{L}_{\text{cyl}})$ is essential in our study : we make use of its definition in order to establish that some operators are isomorphisms (see [Pac09, Proposition 12.4.3]).

Proposition 2.2.3 — *For all δ such that $-\frac{\pi}{\omega} < \delta < \frac{\pi}{\omega}$, the operator $\mathcal{L}_{-\delta}$ is an isomorphism.*

Of course, we directly deduce from the above proposition an analysis result for the Laplace-Beltrami operator.

Corollary 2.2.4 — *For all δ such that $-\frac{\pi}{\omega} < \delta < \frac{\pi}{\omega}$, the operator $\Delta_{cyl,-\delta}$ is an isomorphism.*

Proof

The proof is organized as follows : we determine the indicial roots of $\mathcal{L}_{\text{cyl}}$, we prove that the operator $\mathcal{L}_\delta$ is injective when δ is negative and we conclude by using duality results.

First step — the spectrum of the Laplacian over $(0,\omega)$. Analysis about the cylinder $\mathcal{C} = \mathbb{R}_+ \times (0,\omega)$ can be done with the help of a Fourier type decomposition of $(0,\omega)$. We thus introduce the operator l_ω defined to be

$$l_\omega : \begin{array}{ccc} \mathcal{C}_0^2((0,\omega)) & \longrightarrow & \mathcal{C}^0((0,\omega)) \\ \varphi & \longmapsto & -\frac{\mathrm{d}^2\varphi}{\mathrm{d}\theta^2}, \end{array}$$

where $\mathcal{C}_0^2\left((0,\omega)\right)$ is the set of $\mathcal{C}^2$ functions which vanish on the boundary $\{0,\omega\}$. Then it is easy to check that the spectrum of l_ω is given by

$$\operatorname{Sp}\left(l_\omega\right) \quad = \quad \left\{\frac{m^2\pi^2}{\omega^2} : m \in \mathbb{N}^*\right\},$$

with associated eigenfunctions

$$\varphi_{m,\omega} : \theta \longmapsto \sin\left(\frac{m\pi}{\omega}\theta\right).$$

Second step — indicial roots. Let $m \in \mathbb{N}^*$. We use the spectrum of l_ω in order to study the ODE

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} - \frac{m^2\pi^2}{\omega^2} = 0.$$

Its solutions is the set spanned by the function $e^{\pm\frac{m\pi}{\omega}t}$, therefore, according to [Pac09, proposition 5.2.1.], the indicial roots of $\mathcal{L}_{\mathrm{cyl}}$ are

$$\operatorname{Ind}\left(\mathcal{L}_{\mathrm{cyl}}\right) \quad = \quad \left\{\pm\frac{m\pi}{\omega} : m \in \mathbb{N}^*\right\}. \tag{2.2.4}$$

Therefore, we obtain

$$\operatorname{Isom}\left(\mathcal{L}_{\mathrm{cyl}}\right) = \left(-\frac{\pi}{\omega}, \frac{\pi}{\omega}\right).$$

Third step — injectivity. Let us fix some $\delta_0 > 0$ and $u_{\mathrm{cyl}} \in \ker\left(\mathcal{L}_{-\delta_0}\right)$. Then u_{cyl} is harmonic over $\mathcal{C}$ and there exists $v_{\mathrm{cyl}} \in \mathcal{C}^{k,\alpha}\left(\mathcal{C}\right)$ such that

$$u_{\mathrm{cyl}} \quad = \quad e^{-\delta_0 t} v_{\mathrm{cyl}}.$$

The idea is to apply the maximum principle to infinity : let $M > 0$ and define the bounded cylinder $\mathcal{C}^M$ to be

$$\mathcal{C}^M \quad := \quad (0, M) \times (0, \omega) \quad \subset \quad \mathcal{C}.$$

Then the restriction $u_{\mathrm{cyl}|\mathcal{C}^M}$ of u_{cyl} to $\mathcal{C}^M$ is harmonic and its boundary data satisfies

$$\begin{cases} u_{\mathrm{cyl}|\partial\mathcal{C}^M} & = \quad 0 & \text{over} \quad \partial\mathcal{C}^M \setminus \left(\{M\} \times (0,\omega)\right), \\ \left|u_{cyl_{|\partial\mathcal{C}^M}}\right| & \leqslant \quad e^{-\delta_0 M} \left\|v_{\mathrm{cyl}}\right\|_{L^\infty(C)} & \text{over} \quad \{M\} \times (0,\omega). \end{cases}$$

According to the classical maximum principle, we then obtain

$$\sup_{\mathcal{C}^M} \left|u_{\mathrm{cyl}}\right| \quad = \quad \sup_{\mathcal{C}^M} \left|u_{cyl_{|\partial\mathcal{C}^M}}\right| \quad \leqslant \quad e^{-\delta_0 M} \left\|v_{\mathrm{cyl}}\right\|_{L^\infty(C)}.$$

Since it is true for all $M > 0$, we end up with $u_{\mathrm{cyl}} \equiv 0$ and thus, the operator $\mathcal{L}_{-\delta_0}$ is injective.

Conclusion. According to second step, there exists $\delta_0 > 0$ such that

$$-\delta_0 \in \operatorname{Isom}\left(\mathcal{L}\right).$$

But we know by third step that $\mathcal{L}_{-\delta_0}$ is injective. Application of [Pac09, Proposition 12.4.3] then implies the result. $\square$

2.3 The Laplace-Beltrami operator in Hölder weighted spaces over a triangle

We make use of the above results to deduce properties of the Laplacian about the triangle.

Definition 2.3.1 — We construct a partial order $\prec$ in $\mathbb{R}^3$ by

$$\overline{\delta} \prec \overline{\delta'} \quad \Longleftrightarrow \quad \forall i \in \{1,2,3\}, \, \delta_i < \delta'_i.$$

We can define the same way $\preceq$, $\succ$ or $\succeq$.

We now extend the notion of indicial roots to the case of the triangle.

Definition 2.3.2 — A 3-tuple $\overline{\delta} = (\delta_1, \delta_2, \delta_3)$ is an *indicial root* of Δ_{car} if for all $i \in \{1,2,3\}$, the real number δ_i is an indicial root of

$$\Delta_{\text{cyl},i} : \mathcal{C}^k(\mathcal{C}_i) \longrightarrow \mathcal{C}^{k-2}(\mathcal{C}_i)$$

where $\mathcal{C}_i = (0,+\infty) \times (0,\omega_i)$. We denote their set by $\operatorname{Ind}(\Delta_{\text{car}})$ and we define $\operatorname{Isom}(\Delta_{\text{car}})$ to be the set

$$\operatorname{Isom}(\Delta_{\text{car}}) \quad := \quad \left\{ -\overline{\delta} : \forall -\overline{\delta}' \in \operatorname{Ind}(\Delta_{\text{car}}), \; -\overline{\delta}' \succeq \overline{0} \Longrightarrow \overline{\delta}' \prec -\overline{\delta} \prec -\overline{\delta}' \right\},$$

where $\overline{0} := (0,0,0)$.

Let $-\overline{\delta} \in \mathbb{R}^3$. Like in the previous section, we deal with the following operator :

$$\Delta_{\text{car},-\overline{\delta}} : \mathcal{C}^{2,\alpha}_{-\overline{\delta},0}(\mathcal{T}) \longrightarrow \mathcal{C}^{0,\alpha}_{-\overline{\delta}+\overline{2}}(\mathcal{T}),$$

where $\overline{2} = (2,2,2)$. It is well defined according to (2.2.3). Besides, by (2.2.4) in the second step of the proof of property 2.2.3, we easily check that

$$\operatorname{Ind}(\Delta_{\text{car}}) \quad = \quad \left\{ \left(\pm\frac{m_1\pi}{\omega_1}, \pm\frac{m_2\pi}{\omega_2}, \pm\frac{m_3\pi}{\omega_3} \right) : \forall i, m_i \in \mathbb{N}^* \right\}, \tag{2.3.5}$$

from what we deduce

$$\operatorname{Isom}(\Delta_{\text{car}}) \quad = \quad \prod_{i=1}^{3} \left(-\frac{\pi}{\omega_i}, \frac{\pi}{\omega_i} \right).$$

To deal with injectivity, let $\overline{\delta}_0 \succ 0$ and let

$$u_{\text{car}} \in \ker\left(\Delta_{\text{car},-\overline{\delta}_0}\right).$$

Then $u_{\text{car}} \in \mathcal{C}^{2,\alpha}(\mathcal{T})$, is harmonic over $\mathcal{T}$, continuous[2] over $\overline{\mathcal{T}}$ and vanishes over $\partial\mathcal{T}$. By the maximum principle, u vanishes on $\mathcal{T}$, ie $\Delta_{car,-\overline{\delta}_0}$ is injective. Since there exists $-\overline{\delta}_0 \in \operatorname{Isom}(\Delta_{\text{car}})$ such that $\overline{\delta}_0 \succ 0$, the proposition 12.4.3. in [Pac09] implies the following one :

2. To see u_{car} is continuous in a neighborhood of a vertex, note that for all i, $u_{cyl,i} \in \mathcal{C}^{k,\alpha}_{-\delta_i}(\mathcal{B}_i)$, therefore $\lim_{P\to S_i} u_{\text{car}}(P) = 0$ since $\delta_i > 0$.

Proposition 2.3.3 — *For all $-\overline{\delta} \in Isom\,(\Delta_{car})$, $\Delta_{car,-\overline{\delta}}$ is an isomorphism.*

3 Perturbing triangles

Recall that $\mathcal{T}_{u,Z}$ is minimal if and only if equation (1.0.2) is satisfied. As announced, our aim is to apply an implicit function theorem. But first, we have to define which spaces u and Z belong to. Of course, we would like to work in weighted spaces.

In this purpose, we define the mean curvature operator $\widehat{H}_{-\overline{\delta}}$ in the following proposition.

Proposition 3.0.4 — *Let $\overline{\delta} \in \mathbb{R}^3$ a weight parameter. Then there exists a positive real number $\epsilon_0 > 0$ such that if $\widetilde{\mathcal{U}}_{-\overline{\delta}}$ and $\widetilde{\mathcal{V}}$ denote the open sets defined to be*

$$\left\{ \begin{array}{lll} \widetilde{\mathcal{U}}_{-\overline{\delta}} & := & \left\{ u_{car} \in \mathcal{C}^{2,\alpha}_{-\overline{\delta},0}\left(\mathcal{T}\right) : \|u\|_{\mathcal{C}^{2,\alpha}_{-\overline{\delta},0}(\mathcal{T})} < \epsilon_0 \right\}, \\ \widetilde{\mathcal{V}} & := & \left\{ Z_{car} \in \mathcal{C}^3\left(\mathcal{T}, \mathbb{R}^3\right) : \|Z\|_{\mathcal{C}^3(\mathcal{T},\mathbb{R}^3)} < \epsilon_0 \right\}, \end{array} \right.$$

then the mean curvature operator

$$\widehat{H}_{-\overline{\delta}} : \begin{array}{ccc} \widetilde{\mathcal{U}}_{-\overline{\delta}} \times \widetilde{\mathcal{V}} & \longrightarrow & \mathcal{C}^{0,\alpha}_{-\overline{\delta}+\overline{2}}\left(\mathcal{T}\right) \\ (u, Z) & \longmapsto & H\left(\mathcal{T}_{u,Z}\right) \end{array}$$

is well defined for all $\overline{\delta}$ such that

$$\overline{1} \preceq \overline{\delta} \preceq \overline{2}.$$

Remark 3.0.5 — The choice of ϵ_0 is a technical condition under which the perturbed surface $\mathcal{T}_{u,Z}$ is a well defined embedded surface in $\mathbb{R}^3$.

Proof
It is enough to work near one of the vertices. We omit the index i to relieve notations.

The Hölder's condition : Observe that

$$\mathcal{C}^3\left(\mathcal{T}, \mathbb{R}^3\right) \subset \mathcal{C}^{2,\alpha}\left(\mathcal{T}, \mathbb{R}^3\right).$$

Thus it is clear that when $\mathcal{T}_{u,Z}$ is a well defined surface,

$$(u, Z) \in \mathcal{C}^{2,\alpha}_{-\overline{\delta},0}\left(\mathcal{T}\right) \times \mathcal{C}^3\left(\mathcal{T}, \mathbb{R}^3\right) \quad \Longrightarrow \quad H\left(\mathcal{T}_{u,Z}\right) \in \mathcal{C}^{0,\alpha}\left(\mathcal{T}\right).$$

The weight's condition : It remains to see why there is the weight $-\overline{\delta}+\overline{2}$. For example, according to [BG92, 10.6.5], the mean curvature of a surface Σ whose parametrization is given by some function $f:(r,\theta)\longmapsto f(x,y)\in\mathbb{R}^3$ satisfies the relation

$$H \quad = \quad \frac{1}{2}\frac{E\,D''+G\,D+2F\,D'}{(E\,G-F^2)^{3/2}},$$

where E, F and G are the coefficients of the metric on Σ, that is to say

$$E=\left\|\frac{\partial f}{\partial x}\right\|^2,\quad F=\left\langle\frac{\partial f}{\partial x},\frac{\partial f}{\partial y}\right\rangle,\quad \text{and}\quad G=\left\|\frac{\partial f}{\partial y}\right\|^2,$$

and the other coefficients are defined to be

$$D=\det\left(\frac{\partial^2 f}{\partial x^2},\frac{\partial f}{\partial x},\frac{\partial f}{\partial y}\right),\quad D'=\det\left(\frac{\partial^2 f}{\partial x\partial y},\frac{\partial f}{\partial x},\frac{\partial f}{\partial y}\right)$$

together with

$$D''=\det\left(\frac{\partial^2 f}{\partial y^2},\frac{\partial f}{\partial x},\frac{\partial f}{\partial y}\right).$$

In our case, we work with the parametrization $t_{u,Z}$ given by

$$t_{u,Z}:\begin{array}{ccc}\mathcal{B}_i & \longrightarrow & \mathbb{R}^3\\ (x,y) & \longmapsto & \begin{pmatrix} x & + & Z^1(x,y)\\ y & + & Z^2(x,y)\\ & & A(x,y)\end{pmatrix}\end{array}\quad\text{where}\quad A:=Z^3+u.$$

Then it is easy to compute the first derivatives

$$\partial_x t_{u,Z}=\begin{pmatrix}1 & + & \partial_x Z^1\\ & & \partial_x Z^2\\ & & \partial_x A\end{pmatrix}\quad\text{and}\quad \partial_y t_{u,Z}=\begin{pmatrix} & & \partial_y Z^1\\ 1 & + & \partial_y Z^2\\ & & \partial_y A\end{pmatrix}$$

together with the second derivatives

$$\partial_{xx}t_{u,Z}=\begin{pmatrix}\partial_{xx}Z^1\\ \partial_{xx}Z^2\\ \partial_{xx}A\end{pmatrix},\quad \partial_{yy}t_{u,Z}=\begin{pmatrix}\partial_{yy}Z^1\\ \partial_{yy}Z^2\\ \partial_{yy}A\end{pmatrix}\quad\text{and}\quad \partial_{xy}t_{u,Z}=\begin{pmatrix}\partial_{xy}Z^1\\ \partial_{xy}Z^2\\ \partial_{xy}A\end{pmatrix}.$$

From now on, it is useful to consider the asumptions we made about Z and u. More precisely, one checks that

$$\partial_w Z=\mathcal{O}(\epsilon_0),\quad \partial_w u=\mathcal{O}\left(r^{\delta-1}\epsilon_0\right),\quad \partial_{wz}Z=\mathcal{O}(\epsilon_0)\quad\text{and}\quad \partial_{wz}u=\mathcal{O}\left(r^{\delta-2}\epsilon_0\right),$$

where the index w or z denote an element of $\{x,y\}$. It follows that, since he weight parameter δ is chosen so that $\delta\in[1,2]$,

$$\partial_w A=\mathcal{O}(\epsilon_0)\quad\text{and}\quad \partial_{wz}A=\mathcal{O}\left(r^{\delta-2}\epsilon_0\right).$$

We deduce from the above estimates that the induced metric is such that

$$E = 1 + \mathcal{O}(\epsilon_0), \quad F = \mathcal{O}(\epsilon_0) \quad \text{and} \quad G = 1 + \mathcal{O}(\epsilon_0)$$

while the determinants D, D' and D'' can be written as

$$\begin{vmatrix} \mathcal{O}(\epsilon_0) & 1+\mathcal{O}(\epsilon_0) & \mathcal{O}(\epsilon_0) \\ \mathcal{O}(\epsilon_0) & \mathcal{O}(\epsilon_0) & 1+\mathcal{O}(\epsilon_0) \\ \mathcal{O}\left(r^{\delta-2}\epsilon_0\right) & \mathcal{O}(\epsilon_0) & \mathcal{O}(\epsilon_0) \end{vmatrix} = \mathcal{O}\left(r^{\delta-2}\epsilon_0\right)$$

for ϵ_0 small enough. Therefore, the mean curvature satisfies

$$H = \frac{1}{2}\frac{\mathcal{O}\left(r^{\delta-2}\epsilon_0\right)}{\left(1+\mathcal{O}(\epsilon_0)\right)^{\frac{3}{2}}} = \mathcal{O}\left(r^{\delta-2}\epsilon_0\right),$$

and the result follows. □

Remark 3.0.6 — Indeed, the condition $\overline{\delta} \succeq \overline{1}$ is necessary. If $\widehat{H}$ is well-defined, we can take $Z \equiv 0$ and hence :

$$\forall u_{\text{car}} \in \mathcal{C}^{k,\alpha}_{-\overline{\delta},0}(\mathcal{T}), \quad H\left(\mathcal{T}_{u_{\text{car}},0}\right) \in \mathcal{C}^{k-2,\alpha}_{-\overline{\delta}+\overline{2}}(\mathcal{T}).$$

But, recall that for graphs, the mean curvature is given by the following equation [3] is

$$H\left(\mathcal{T}_{u_{\text{car}},0}\right) = \frac{1}{2}\operatorname{div}\left(\frac{\nabla u_{\text{car}}}{\sqrt{1+|\nabla u_{\text{car}}|^2}}\right).$$

It is useful to rewrite the above equation as follows :

$$H\left(\mathcal{T}_{u_{\text{car}},0}\right) = \frac{1}{2\sqrt{1+|\nabla u_{\text{car}}|^2}}\left[\Delta u_{\text{car}} - \frac{\operatorname{Hess}(u_{\text{car}})(\nabla u_{\text{car}}, \nabla u_{\text{car}})}{1+|\nabla u_{\text{car}}|^2}\right].$$

The Laplacian term can be estimated as

$$\Delta u_{\text{car}} = \mathcal{O}\left(r^{\delta-2}\right)$$

while the second term satisfies

$$\operatorname{Hess}(u_{\text{car}})(\nabla u_{\text{car}}, \nabla u_{\text{car}}) = \mathcal{O}\left(r^{3\delta-4}\right).$$

Consequently, if we want $H\left(\mathcal{T}_{u_{\text{car}}}\right)$ to belong to the weighted space $\mathcal{C}^{0,\alpha}_{-\overline{\delta}+\overline{2}}(\mathcal{T})$, it is necessary to ensure

$$r^{3\delta-4} \leqslant r^{\delta-2}$$

for small r. Thus it is necessary that the following inequality

$$\delta \geqslant 1$$

holds true.

3. See for example [CM99].

3.1 Proof of theorem 0.0.1

First of all, observe that there always exists a 3-tuple $\overline{\delta}$ such that

$$\overline{1} \preceq \overline{\delta} \preceq \overline{2} \qquad \text{together with} \qquad -\overline{\delta} \in \operatorname{Isom}\left(\Delta_{\mathrm{car}}\right) \tag{3.1.6}$$

because for all i in $\{1,2,3\}$, the angle ω_i is less than π and the set $\operatorname{Isom}\left(\Delta_{\mathrm{car}}\right)$ is described in (2.3). More precisely, it is easy to check that $\overline{\delta}$ satisfies (3.1.6) if and only if for all $i \in \{1,2,3\}$, δ_i belongs to $[1,2] \cap \left(-\frac{\pi}{\omega_i}, \frac{\pi}{\omega_i}\right)$; this is why it is this condition which appears in the theorem.

The idea lies in applying an implicit function theorem to the operator $\widehat{H}_{-\overline{\delta}}$ we have defined in proposition 3.0.4.

First, observe that since $\mathcal{T}$ is minimal, $\widehat{H}_{-\overline{\delta}}$ satisfies

$$\widehat{H}_{-\overline{\delta}}\left(0,0\right) \quad = \quad 0.$$

Next, according to the equation (1.0.1), we can calculate the differential $D_1\widehat{H}_{-\overline{\delta}}\left(0,0\right)$ of $\widehat{H}_{-\overline{\delta}}$ in comparison with the first variable u in $(0,0)$. We find

$$D_1\widehat{H}_{-\overline{\delta}}\left(0,0\right) : \begin{array}{rcl} \mathcal{C}^{2,\alpha}_{-\overline{\delta},0}\left(\mathcal{T},\mathbb{R}\right) & \longrightarrow & \mathcal{C}^{0,\alpha}_{-\overline{\delta}+\overline{2}}\left(\mathcal{T},\mathbb{R}\right) \\ u_{\mathrm{car}} & \longmapsto & \Delta_{\mathrm{car}}(u_{\mathrm{car}}). \end{array}$$

Consequently, $D_1\widehat{H}_{-\overline{\delta}}\left(0,0\right)$ is nothing but the operator $\Delta_{car,-\overline{\delta}}$. Therefore, according to the proposition 2.3.3, it is an isomorphism.

We can then apply the implicit function theorem to $\widehat{H}_{-\overline{\delta}}$ with which we build $\mathcal{U}$ in $\widetilde{\mathcal{U}}_{-\overline{\delta}}$, $\mathcal{V}$ in $\widetilde{\mathcal{V}}$ and suitable φ, QED.

3.2 Perturbation of polygons

Our work naturally extends to some polygons $\mathcal{P}$ — with j vertices — which are included in the plane $\{z=0\}$. We define in the same way the angles ω_i, the sets $\mathcal{B}_i$ or $\mathcal{C}_i$, the weighted spaces, and so on... This gives a theorem similar to theorem 0.0.1.

Theorem 3.2.1
For all $-\overline{\delta}$ such that

$$\overline{1} \preceq \overline{\delta} \preceq \overline{2} \qquad \textit{and} \qquad -\overline{\delta} \in Isom\left(\Delta_{car}\right),$$

there exists a neighbourhood $\mathcal{U}$ of 0 in $\widetilde{\mathcal{U}}_{-\overline{\delta}} \subset \mathcal{C}^{k,\alpha}_{\overline{\delta},0}\left(\mathcal{P},\mathbb{R}\right)$, a neighbourhood $\mathcal{V}$ of 0 in $\mathcal{C}^{k,\alpha}\left(\mathcal{P},\mathbb{R}^3\right)$ and an application $\varphi : \mathcal{V} \longrightarrow \mathcal{U}$ such that $\mathcal{P}_{u,Z}$ is a minimal surface with $(u,Z) \in \mathcal{U} \times \mathcal{V}$ if and only if $u = \varphi(Z)$.

Remark 3.2.2 — Note that such a $\overline{\delta}$ does not exist in general case : if one of the angles ω_i is more that π — in other words, if $\mathcal{P}$ is not convex — then we can not use this above theorem. Heuristically, what happens is that if ω_1 is more than π, then we lose the property that the map is a vertical : even if the solution to the Plateau's problem exists, the surface is not a vertical graph over $\mathcal{P}$.

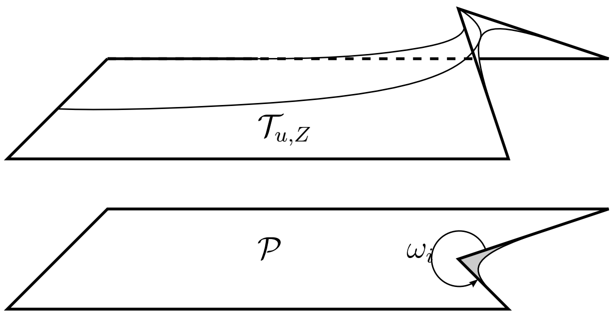


Figure I.2: $\mathcal{T}_{u,Z}$ is not a graph over $\mathcal{P}$.

Chapter II

Construction d'hypersurfaces minimales de type Riemann dans $\mathbb{R}^n \times \mathbb{R}$

1 Introduction and statements of the results

The *Riemann minimal surfaces* — or hypersurfaces — form a subject that has been studied over last years. They are minimal surfaces with planar ends which are simply-periodic, embedded and complete. Usually, a Riemann minimal surface belongs to a one parameter family.

In [Rie98], Bernhard Riemann discovered one family of such surfaces foliated by horizontal circles in 3-dimensional Euclidean-space. Such a surface could be seen as planes which are linked to each other by one catenoid (or one *neck*). Enneper in 1869 and Shiffman [Shi56] in 1956 gave a characterization of these : a minimal annulus which holds two circles in parallel planes is either a part of catenoid or a part of the Riemann example. More general characterizations have been given the last twenty years, especially by Hoffman, Karcher and Rosenberg [HKR91] or Meeks, Pérez and Ros [MPR]. L. Hauswirth [Hau06] proved the existence and classify the minimal surfaces foliated by horizontal constant curvature curves in $\mathbb{R}^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$ and $\mathbb{S}^2 \times \mathbb{R}$.

In an unpublished paper [Wei94], F. Wei builds a more general Riemann example with alternatively one neck and two necks between horizontal planes. Recently, Martin Traizet proved the existence of such families with an arbitrary number of necks and planes in [Tra02a] and [Tra02b] (in this last paper, there is a finite number of planar ends) in the Euclidean space $\mathbb{R}^3$. With this aim, he introduces a points configuration which satisfies different hypotheses, namely the *balanced* and the *non-degenerate* conditions. The first is necessary for existence while the second is a condition under which he can produce examples by using implicit function theorem. The Weirstrass representation of minimal surface plays a significant role in the construction and the same method cannot be applied in a higher dimensional case. Besides, it will be shown that in $\mathbb{R}^{n+1}$ with $n \geqslant 3$, there are more degrees of freedom because the catenoid lies between two horizontal hyperplanes while the catenoid in $\mathbb{R}^3$ is not bounded in any direction ; equivalently, to enforce the surfaces to be embedded, the flux at infinity has to vanish. More recently, F. Morabito and M. Traizet [MT11] proved the existence of non-periodic minimal surfaces with an infinite number of parallel ends. In [HP07], L. Hauswirth and F. Pacard add genus to the Riemann example ($1 \leqslant genus \leqslant 37$) : these surfaces have two ends which are asymptotic to halves of Riemann's surface.

In papers by S. Fakhi and F. Pacard [FP00] or by S. Kaabachi and F. Pacard [KP07], the existence of some examples of such hypersurfaces is proved when $n \geqslant 3$: one with a finite number of planar ends (non periodic example) and one which generalises the Riemann example, namely horizontal hyperplanes with only one neck between two of them. However, the tools are very different from the Weirstrass representation theorem since they come from non linear analysis. Note that this kind of method provides us with an accurate description of the surface.

In this paper, we give more general examples in $\mathbb{R}^{n+1}$ with $n \geqslant 3$ for points configurations that satisfy similar conditions to those of M. Traizet. First of all, we need to give some definitions.

Let N be a positive natural number and $\mathbf{t}_h$ be a vector in $\mathbb{R}^n$. For all k in $\mathbb{Z}$, we assume we are given

(i) n_k a positive natural number,

(ii) for all $j \in [\![1, n_k]\!]$, a point $p_{k,j}$ of $\mathbb{R}^n$ with a weight a_k, where a_k denotes a positive real number,

(iii) for all $j \neq j' \in [\![1, n_k]\!]$, for all $j'_+ \in [\![1, n_{k+1}]\!]$, for all $j'_- \in [\![1, n_{k-1}]\!]$,

$$p_{k,j} \neq p_{k,j'}, \qquad p_{k,j} \neq p_{k+1,j'_+} \qquad \text{and} \qquad p_{k,j} \neq p_{k-1,j'_-},$$

(iv) for all k and for all $j \in [\![1, n_k]\!]$,

$$a_{k+N} = a_k \qquad \text{and} \qquad p_{k+N,j} = p_{k,j} + \mathbf{t}_h.$$

We then say the family $\{(a_k, p_{k,j})\}$ is a *$\mathbf{t}_h$-periodic weighted points configuration.*

The interpretation is the following : k is the index of the k-th horizontal hyperplane, n_k is the number of necks we want to put between the k-th hyperplane and the $(k+1)$-th one, $p_{k,j}$ is the emplacement of those necks while the weight a_k is their size — notice that the distance between two consecutive hyperplanes has to be independent of the different necks we glue, thus it is why a_k is chosen so that it does not depend on $p_{k,j}$ but only on k. The hypothesis (iii) is necessary since we do not want to glue a neck to another. Finally, the hypothesis (iv) is the periodicity of the configuration and N is the number of hyperplanes we want to consider modulo to this condition.

We define in the same way a *non-periodic weight points configuration* with a finite number $N+1$ of hyperplanes : we take $k \in [\![0, N-1]\!]$ rather than $k \in \mathbb{Z}$ and we omit the periodicity condition (iv).

Remark 1.0.3 — In what follows, it will also be convenient to consider weighted configurations $\{(a_{k,j}, p_{k,j})\}$ with $a_{k,j}$ chosen in a small neighbourhood of a_k.

In all cases, we denote by Ne the total number of necks we consider, that is to say

$$\mathrm{Ne} \quad := \quad \sum_{k=0}^{N-1} n_k.$$

The *force* $f(p,q)$ between two distinct points is defined to be the vector in $\mathbb{R}^n$ such that

$$f(p,q) \quad := \quad (n-2)\frac{p-q}{|p-q|^n}.$$

The total force $F_{k,j}$ that all the points exert on $p_{k,j}$ is

$$F_{k,j} \quad := \quad 2\sum_{\substack{i=1\\ i\neq j}}^{n_k} a_k f(p_{k,j},p_{k,i}) \;-\; \sum_{i=1}^{n_{k-1}} a_{k-1} f(p_{k,j},p_{k-1,i}) \\ -\; \sum_{i=1}^{n_{k+1}} a_{k+1} f(p_{k,j},p_{k+1,i}), \quad (1.0.1)$$

in other words, we consider the interaction between $p_{k,j}$ the points of the same level with a factor 2 and the interaction between $p_{k,j}$ and the points of level $k-1$ and $k+1$ with a factor -1. This definition also makes sense for non-periodic configurations if we omit the terms that do not exist in this case (for example, in $F_{0,j}$, we replace the contribution at level -1 by 0). Note that the forces $F_{k,j}$ depend on the emplacement of the points together with the weights. Moreover, note it also makes sense to define these forces with weights $a_{k,j}$: it is enough to replace a_k (resp. a_{k-1}, a_{k+1}) by $a_{k,i}$ (resp. $a_{k-1,i}$, $a_{k+1,i}$).

Definition 1.0.4 — We say the configuration $\{(a_k, p_{k,j})\}$ is *balanced* if all forces vanish, i.e. if

$$\forall (k,j), \quad F_{k,j} = 0.$$

This condition could be interpreted as as geometrical one. As a matter of fact, the forces are deeply linked to the way we *bend* the necks we glue between two consecutive levels. To say the configuration is balanced is the same to say the axis of each neck is straight.

Those forces are quite similar to the ones M. Traizet develops in his papers concerning the construction of minimal surfaces. However, let us remark that in our case, there are more freedom degrees, namely the family of weights $\{a_{k,j}\}$, while in the 2-dimensional case, a_k is prescribed by $a_k = \frac{1}{n_k}$. The reason for this is the catenoid in $\mathbb{R}^3$ goes to infinity, it is not asymptotic to any plane. The flux at infinity has to vanish and then, the weight is prescribed.

It turns out that our construction lies on the inverse function theorem, thus we have to determine under which conditions we could prescribe the forces. Assume we are given a $\mathbf{t}_h$-periodic weighted configuration. For convenience, let us define the linear subspace W of $\mathbb{R}^n$ spanned by the points, that is to say

$$W \quad = \quad \operatorname{Span}\{q-p \ : \ p,q \in \{p_{k,j}\}\}.$$

Since the problem is invariant under the group of translations, we can assume, without loss of generality, that all $p_{k,j}$ are in W — in other words, that the affine space $W_{\text{aff}} = p_{0,1} + W$ passes through 0. Thus we identify a point $p_{k,j}$ of the affine space with a vector in W. Note that the forces are in W. We denote by m the dimension of W and by $(\mathbf{e}_i)_{1\leqslant i\leqslant n}$ an orthonormal basis of $\mathbb{R}^n$ such that $(\mathbf{e}_i)_{1\leqslant i\leqslant m}$ is an orthonormal basis of W. We define $\mathfrak{G}_W = id_W \otimes O\left(W^\perp\right)$ to be the subgroup of the isometry group of $\mathbb{R}^n$ whose elements u can be written

$$u = \begin{pmatrix} I_m & 0 \\ 0 & v \end{pmatrix} \quad \text{with} \quad v \in O_{n-m}(\mathbb{R}).$$

Notice that an element u of $\mathfrak{G}_W$ is such that $u(p_{k,j}) = p_{k,j}$ for all k and for all j. We also define the subgroup $\mathfrak{G}_{W^\perp} = O(W) \otimes id_{W^\perp}$ the same way. From now on, the vectors $\mathbf{e}_i$ is either considered as a vector of $\mathbb{R}^n$ or as a vector of $\mathbb{R}^n \times \mathbb{R}$, in which case $\mathbf{e}_i$ is said to be *horizontal*. We note $\mathbf{e}_{n+1} = (0, \cdots, 0, 1)$ a unit *vertical* vector in $\mathbb{R}^n \times \mathbb{R}$.

With the help of these definitions, we notice that the forces satisfy different relations under the action of translations, rotations and dilation.

- If $u_{\mathbf{t}}$ is a translation, then $F_{k,j} \circ u_{\mathbf{t}} = F_{k,j}$: the forces are invariant under the group of translations.
- If r is a rotation of $\mathfrak{G}_{W^\perp}$, then $F_{k,j} \circ r = r \circ F_{k,j}$. Besides, in the $\mathbf{t}_h$-periodic case, if we enforce the new configuration $\{r(p_{k,j})\}$ to be $\mathbf{t}_h$-periodic, r has to be chosen so that $r(\mathbf{t}_h) = \mathbf{t}_h$. It is relevant only when $\mathbf{t}_h \neq 0$.
- In the non-periodic case or in the 0-periodic case, if $\lambda \cdot id_{\mathbb{R}^n}$ is a dilation (or a contraction) with scale factor $\lambda > 0$, then $F_{k,j} \circ \lambda \cdot id_{\mathbb{R}^n} = \lambda^{1-n} F_{k,j}$. Here, we do not consider the $\mathbf{t}_h$-periodic case with $\mathbf{t}_h \neq 0$ since a dilation with scale factor $\lambda \neq 1$ changes the period into $\lambda \mathbf{t}_h \neq \mathbf{t}_h$.

It follows that, when the configuration $\{(\mathring{a}_k, \mathring{p}_{k,j})\}$ is *balanced*, the points force function

$$\mathscr{F} : \{p_{k,j}\} \in W^{Ne} \longmapsto \{F_{k,j}\} \in W^{Ne} \tag{1.0.2}$$

can not be a diffeomorphism near the initial configuration $\{\mathring{p}_{k,j}\}$ since the kernel of its differential holds the three following linear subspaces :

$$\begin{cases} V_t \ := \ \operatorname{Span}\left\{(\mathbf{v}, \cdots, \mathbf{v}) \in (\mathbb{R}^n)^{Ne} : \mathbf{v} \in W\right\} & \text{for translations,} \\ V_r \ := \ \left\{\left(\mathcal{R}\mathring{p}_{0,1}, \cdots, \mathcal{R}\mathring{p}_{N-1,n_{N-1}}\right) : \mathcal{R} \in \operatorname{Skew}_{\mathbf{t}_h}\right\} & \text{for rotations,} \\ V_d \ := \ \left\{\lambda\left(\mathring{p}_{0,1}, \cdots, \mathring{p}_{N-1,n_{N-1}}\right) \in W^{Ne} : \lambda \in \mathbb{R}\right\} & \text{for dilation,} \end{cases}$$

where $\operatorname{Skew}_{\mathbf{t}_h}$ is the set of the skew-symmetric matrices which span, with the help of the exponential mapping, the rotations r of $\mathfrak{G}_W$ such that $r(\mathbf{t}_h) = \mathbf{t}_h$, that is to say

$$\operatorname{Skew}_{\mathbf{t}_h} = \left\{\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{M}_n(\mathbb{R}) : A \in \mathcal{M}_m(\mathbb{R}) \text{ is skew-symmetric and } A\mathbf{t}_h = 0\right\}.$$

Furthermore, notice that the sum $V_t + V_r + V_d$ is direct and we do not consider V_d in the non-vanishing periodic case.

As a matter of fact, we also can collect information regarding the image of the force function. For any configuration $\{(a_{k,j}, p_{k,j})\}$, since $f(p,q) = -f(q,p)$, the relation

$$\sum_{k=0}^{N-1} \sum_{j=1}^{n_k} a_{k,j} F_{k,j} \quad = \quad 0 \tag{1.0.3}$$

holds true. Moreover, for all skew-symmetric matrix A, the scalar product $\langle f(p,q), Ap\rangle$ is equal to $-(n-2)\frac{\langle q,Ap\rangle}{|p-q|^n}$ and $\langle q, Ap\rangle = -\langle Aq, p\rangle$. Consequently, we obtain the additional property

$$\forall \mathcal{R} \in \mathrm{Skew}_{\mathbf{t}_h}, \quad \sum_{k=0}^{N-1} \sum_{j=1}^{n_k} a_{k,j} \langle F_{k,j}, \mathcal{R} p_{k,j}\rangle \quad = \quad 0. \tag{1.0.4}$$

Thus, $\mathscr{F}$ does not have full rank. That is why we introduce the *non-degenerate* condition.

Definition 1.0.5 — An initial weighted points balanced configuration

$$\mathring{C} \quad := \quad \{(\mathring{a}_k, \mathring{p}_{k,j})\}$$

is said to be *non-degenerate* if the differential of the force function $\mathscr{F}$ defined to be so that

$$\mathscr{F} : \{(a_k, p_{k,j})\} \longmapsto \{F_{k,j}\}$$

has maximum rank on account of the invariants at the initial configuration $\mathring{C}$, that is to say

$$\dim \mathrm{Im}\left(\mathrm{d}\mathscr{F}_{\mathring{C}}\right) \quad = \begin{cases} (\mathrm{Ne}-1)\,m - \frac{(m-1)(m-2)}{2} & \text{when } \mathbf{t}_h \neq 0, \\ (\mathrm{Ne}-1)\,m - \frac{m(m-1)}{2} & \begin{cases}\text{when } \mathbf{t}_h = 0, \\ \text{in the non-periodic case.}\end{cases} \end{cases}$$

The reader should pay attention to the fact that in this definition, we enforce the $a_{k,j}$ to be the same at level k.

Remark 1.0.6 — If we write $\mathrm{d}\mathscr{F} = \mathrm{d}_a\mathscr{F} + \mathrm{d}_p\mathscr{F}$ where the index denotes the derivative parameter (the weights or the points), then

$$\dim\left(\ker \mathrm{d}_p\mathscr{F}\right) \geqslant \begin{cases} m + \frac{(m-1)(m-2)}{2} & \text{when } \mathbf{t}_h \neq 0, \\ m + \frac{m(m-1)}{2} + 1 & \begin{cases}\text{when } \mathbf{t}_h = 0 \\ \text{in the non-periodic case.}\end{cases} \end{cases}$$

Heuristically, when we can dilate the configuration, we assume that a perturbation of the weight parameters offsets the part of the kernel that comes from the dilation. It is the same kind of hypothesis as the one in the theorem 1 of [Tra02b].

We are now in a position to state the main result of the paper.

Theorem 1.0.7
Let $\{(a_k, p_{k,j})\}$ be a balanced and non-degenerate configuration. Then there exists a 1-parameters family of embedded, complete and minimal hypersurfaces $(\mathscr{S}_\epsilon)_{\epsilon\in(0,\epsilon_0)}$ such that the following assertions hold true :

symmetries : *$\mathscr{S}_\epsilon$ is invariant under the action of the subgroup $\mathfrak{G}_{W\times\mathbb{R}} = \mathfrak{G}_W \otimes id_{\mathbb{R}}$ of the isometry group of $\mathbb{R}^n \times \mathbb{R}$ whose elements u can be written*

$$u = \begin{pmatrix} u' & 0 \\ 0 & 1 \end{pmatrix} \quad \textit{with} \quad u' \in \mathfrak{G}_W;$$

the $\mathbf{t}_h$-periodic case : *(i) $\mathscr{S}_\epsilon$ is simply $\mathbf{t}$-periodic, with $\mathbf{t} = \mathbf{t}_h + \mathbf{t}_v$ where the vertical vector $\mathbf{t}_v = \mathcal{O}_{\epsilon\to 0}\left(\epsilon^{\frac{1}{n-1}}\right)\mathbf{e}_{n+1}$;*

(ii) $\mathscr{S}_\epsilon/\mathbf{t}\mathbb{Z}$ has N horizontal hyperplanar ends ;

(iii) from a topological point of view, $\mathscr{S}_\epsilon/\mathbf{t}\mathbb{Z}$ is the connected sum of N horizontal hyperplanes H_k with $n_k + n_{k-1}$ punctures at $p_{k,j}$ and $p_{k-1,j}$;

the non-periodic case : *(i) $\mathscr{S}_\epsilon$ has $N+1$ horizontal hyperplanar ends and the distance between the two extremal hyperplanar ends is $\mathcal{O}_{\epsilon\to 0}\left(\epsilon^{\frac{1}{n-1}}\right)$;*

(ii) from a topological point of view, $\mathscr{S}_\epsilon$ is the connected sum of $N+1$ horizontal hyperplanes H_k with $n_k + n_{k-1}$ punctures at $p_{k,j}$ and $p_{k-1,j}$.

2 Adding "necks" to hyperplane

The goal of the sections concerning horizontal hyperplanes is to build minimal surfaces close enough to these (it will correspond to the hyperplanar ends of the Riemann surfaces we want to construct) such that they have necks (or catenoidal shape) at each gluing point $p_{k,j}$.

As announced in the introduction, we work with a slightly perturbed weighted configuration $\{(a_{k,j}, p_{k,j})\}$ of $\{(a_k, p_{k,j})\}$. Let us fix $l \in \mathbb{Z}$ (it is the index of the l-th end) and consider the n_l weighted points

$$((a_{l,1}, p_{l,1}), \cdots, (a_{l,1}, p_{l,n_l}))$$

together with the n_{l-1} weighted points

$$\left((a_{l-1,1}, p_{l-1,1}), \cdots, \left(a_{l-1,n_{l-1}}, p_{l-1,n_{l-1}}\right)\right).$$

We then define the l-th Green function by

$$\Gamma : \begin{array}{ccc} \mathbb{R}^n & \longrightarrow & \mathbb{R} \\ x & \longmapsto & -\sum_{i=1}^{n_{l-1}} a_{l-1,i} \left|x - p_{l-1,i}\right|^{2-n} \quad + \quad \sum_{i=1}^{n_l} a_{l,i} \left|x - p_{l,i}\right|^{2-n}. \end{array}$$

It is well known that Γ is a harmonic function over the set $\mathbb{R}^n_l := \mathbb{R}^n \backslash \{p_{l-1,1}, \cdots, p_{l,n_l}\}$. More precisely, Γ satisfies following equation :

$$\Delta_{\mathbb{R}^n}\Gamma \quad = \quad c(n)\left[-\sum_{i=1}^{n_{l-1}} a_{l-1,i}\delta_{p_{l-1,i}} \quad + \quad \sum_{i=1}^{n_l} a_{l,i}\delta_{p_{l,i}}\right],$$

where $c(n) = -\frac{1}{\mathrm{Vol}(\mathbb{S}^{n-1})}$ is a constant that depends on n.

Besides, note that this Green function is chosen so that it point upwards near the points $p_{l,i}$ and downwards near the other points $p_{l-1,i}$. As a matter of fact, we will use Γ at the l-th level to glue this one to the $(l+1)$-th level near $p_{l,i}$ and to the $(l-1)$-th level near $p_{l-1,i}$. Note that Γ and its derivatives continuously depend on the choice of weighted points.

Remark 2.0.8 — All the results we prove for the above Green's function could be applied in the case of a finite number $N+1$ of horizontal ends, i.e. when we give a finite number of weighted points $(a_{k,j}, p_{k,j})_{j\in[\![1,n_k]\!]}$ with $k = 0, 1, \cdots, N$. We define the 0-th Green's function and the N-th Green's function by

$$\Gamma_0 := \sum_{i=1}^{n_0} a_{0,i}\left|\cdot - P_{0,i}\right|^{2-n} \qquad \text{and} \qquad \Gamma_N := -\sum_{i=1}^{n_{N-1}} a_{N-1,i}\left|\cdot - p_{N-1,i}\right|^{2-n}.$$

At level 0, we only add necks that point upwards since there is not any lower level while at level N, we only add necks that point downwards since there is not any upper level.

2.1 Behaviour near singularities

We have in mind to conduct a gluing process to build minimal hypersurfaces. This kind of method requires a thorough description of local behaviour near the gluing points. Thus it is useful to give the Taylor expansion of Green's function near its singularities. In this purpose, we give the typical Taylor expansion

$$|x-p|^{2-n} \quad = \quad |p|^{2-n} + (n-2)\,|p|^{-n}\langle x, p\rangle + \mathring{\mathcal{O}}_{x\to 0}\left(|x|^2\right),$$

where we write $f(x) = \mathring{\mathcal{O}}_{x\to 0}(|x|^m)$ with $m \in \mathbb{Z}$ if for all $k \in \mathbb{N}$, the equality $\nabla^k f(x) = \mathcal{O}_{x\to 0}(|x|^{m-k})$ is satisfied. We will see later that this kind of equality proves to be very efficient in weighted spaces theory.

Behaviour near $p_{l-1,j}$ with $1 \leqslant j \leqslant n_{l-1}$

Without difficulty, one finds

$$\begin{aligned}\Gamma(x) \quad = \quad & -a_{l-1,j}\left|x - p_{l-1,j}\right|^{2-n} \; + \; C_{l-1,j,+} \\ & + \; \langle x - p_{l-1,j}\;,\; F_{l-1,j,+}\rangle \; + \; \mathring{\mathcal{O}}_{x\to p_{l-1,j}}\left(\left|x - p_{l-1,j}\right|^2\right), \quad (2.1.5)\end{aligned}$$

where the real constant $C_{l-1,j,+}$ is given by

$$C_{l-1,j,+} \quad := \quad -\sum_{\substack{i=1\\ i\neq j}}^{n_{l-1}} a_{l-1,i} \left|p_{l-1,j} - p_{l-1,i}\right|^{2-n} \;+\; \sum_{i=1}^{n_l} a_{l,i} \left|p_{l-1,j} - p_{l,i}\right|^{2-n}, \tag{2.1.6}$$

and the partial force $F_{l-1,j,+}$ of $\mathbb{R}^n$ is defined by

$$F_{l-1,j,+} \quad := \quad -\sum_{\substack{i=1\\ i\neq j}}^{n_{l-1}} a_{l-1,i} f\left(p_{l-1,j}, p_{l-1,i}\right) \;+\; \sum_{i=1}^{n_l} a_{l,i} f\left(p_{l-1,j}, p_{l,i}\right), \tag{2.1.7}$$

Behaviour near $p_{l,j}$ with $1 \leqslant j \leqslant n_l$

Similar calculus leads us to

$$\begin{aligned}\Gamma(x) \quad = \quad & a_{l,j} \left|x - p_{l,j}\right|^{2-n} \;+\; C_{l,j,-} \\ & + \;\langle x - p_{l,j} \;,\; F_{l,j,-}\rangle \;+\; \underset{x\to p_{l,j}}{\mathring{\mathcal{O}}} \left(\left|x - p_{l,j}\right|^2\right),\end{aligned} \tag{2.1.8}$$

where the real constant $C_{l,j,-}$ is given by

$$C_{l,j,-} \quad := \quad -\sum_{i=1}^{n_{l-1}} a_{l-1,i} \left|p_{l,j} - p_{l-1,i}\right|^{2-n} \;+\; \sum_{\substack{i=1\\ i\neq j}}^{n_l} a_{l,i} \left|p_{l,j} - p_{l,i}\right|^{2-n}, \tag{2.1.9}$$

and the partial force $F_{l,j,-}$ of $\mathbb{R}^n$ is defined by

$$F_{l,j,-} \quad := \quad -\sum_{i=1}^{n_{l-1}} a_{l-1,i} f\left(p_{l,j}, p_{l-1,i}\right) \;+\; \sum_{\substack{i=1\\ i\neq j}}^{n_l} a_{l,i} f\left(p_{l,j}, p_{l,i}\right), \tag{2.1.10}$$

Remark 2.1.1 —
- It is relevant to note that the partial forces $F_{k,j,+}$ (resp. $F_{k,j,-}$) corresponds to the interaction between the point $p_{k,j}$ and all other points of same level k together with the the level $k-1$ (resp. $k+1$).
- Geometrically, the force term in $p_{k,j}$ corresponds to leaning a neck in some special direction during the gluing process.

2.2 The first error term and its correction

Let $\epsilon > 0$ and $r_\epsilon := \epsilon^{\frac{2}{3(n-1)}}$ — we'll see later why we have chosen this radius r_ϵ (cf. remark 5.3.1). Let us note $\mathbb{R}^n_{l,\epsilon}$ the set $\mathbb{R}^n$ without small balls of radius r_ϵ centred in the singularities $p_{k,i}$, namely

$$\mathbb{R}^n_{l,\epsilon} \quad := \quad \mathbb{R}^n \setminus \left(\bigcup_{p=p_{l-1,1},\cdots,p_{l-1,n_{l-1}}} B\left(p, r_\epsilon\right) \bigcup_{p=p_{l,1},\cdots,p_{l,n_l}} B\left(p, r_\epsilon\right) \right).$$

From now on, we suppose the parameter ϵ small enough in comparison with the distance between the points $p_{k,j}$ for the purpose of ensuring that the balls $B\left(p, r_\epsilon\right)$ do not intersect.

Unfortunately, although the hyperplane $\{x_{n+1} = 0\}$ is clearly a minimal hypersurface in $\mathbb{R}^n \times \mathbb{R}$, it is not the case with regard to the graph of Green's function. On the other hand, this one performs a relatively good approximation to the minimal surface equation. More precisely, a function f over an open set of $\mathbb{R}^n$ defines a minimal hypersurface if it satisfies the equation

$$\operatorname{div}\left(\frac{\nabla f}{\sqrt{1+|\nabla f|^2}}\right) = 0.$$

In this paper, it is more convenient to translate this equation into

$$\Delta f - \mathcal{G}(f) = 0 \qquad \text{with} \qquad \mathcal{G}(f) = \frac{\nabla^2 f(\nabla f, \nabla f)}{1+|\nabla f|^2},$$

where $\nabla^2 f$ is the symmetric bilinear form defined by the Hessian of f. This writing makes it possible to put the role of harmonic functions (especially Green's function) forward. From a heuristics point of view, if a function f has small $\mathcal{C}^2$ norm, then the main term of previous equation is given by the linear part, in other words the Laplacian, whereas the remainder is cubic type and so is small in comparison to the Laplacian. More exactly, in our case, let us search for the error we make if we consider the function Γ over $\mathbb{R}^n_{l,\epsilon}$. We multiply it by a small parameter ϵ in order to ensure the small $\mathcal{C}^2$ norm of $\epsilon\Gamma$. The question is to estimate the term $\mathcal{G}(\epsilon\Gamma)$ because the Laplacian vanishes. Since Green's function explodes near the singularities, it is enough to work in a neighbourhood of one of them, let's say $p_{l,j}$ for example. We write

$$\epsilon\Gamma(x) = \epsilon\, a_{l,j}\,|x-p_{l,j}|^{2-n} + \epsilon\, C_{l,j,-} + \underset{\substack{x\to p_{l,j}\\ x\in\mathbb{R}^n_{l,\epsilon}}}{\mathring{\mathcal{O}}}\left(\epsilon\,|x-p_{l,j}|\right)$$

In this case, easy computation shows that we can write

$$\mathcal{G}(\epsilon\Gamma) - \Delta(\epsilon\Gamma) = (2-n)^3(n-1)\,{a_{l,j}}^3\epsilon^3|x-p_{l,j}|^{2-3n} + \mathring{\mathcal{O}}\left(\epsilon^3\,|x-p_{l,j}|^{1-2n}\right).$$

Note that the main term is radial, which is in agreement with the construction of Green's function : if we are close to the singularity $p_{l,j}$, the contribution of $|x-p_{k,i}|^{2-n}$ for $p_{k,i} \neq p_{l,j}$ is bounded whereas the radial term $|x-p_{l,j}|^{2-n}$ explodes. Besides, the rough estimate of the error is $\epsilon^3 r_\epsilon^{2-3n} = \epsilon r_\epsilon^{-1}$, the one for the next term being $\epsilon r_\epsilon^{n-2}$.

For the gluing method we will conduct later, it is useful to introduce a new function Cor such that Cor corrects the main term of the above error. In a neighbourhood of the singularity $p_{l,j}$, the function $x \mapsto \frac{(n-2)^3}{2(3n-4)} a_{l,j}^3\epsilon^3\,|x-p_{l,j}|^{4-3n}$ is such that its Laplacian is equal to this one. Since there are other singularities, we rather introduce the function Cor defined over $\mathbb{R}^n_{l,\epsilon}$ by

$$Cor(x) := \frac{(n-2)^3}{2(3n-4)}\epsilon^3\left[-\sum_{i=1}^{n_{l-1}} a_{l-1,i}^3\,|x-p_{l-1,j}|^{4-3n} + \sum_{i=1}^{n_l} a_{l,i}^3\,|x-p_{l,j}|^{4-3n}\right].$$

The choice of the quantity r_ϵ leads to equality $\epsilon^3 r_\epsilon^{4-3n} = \epsilon r_\epsilon$. In other words, this correcting function has the same rough estimate than the second term in the Taylor expansion of Green's function when one approaches the boundary of the ball $B(p, r_\epsilon)$, although they are different types : its main term is radial and thus does not favour any directions.

By construction, it follows that we can approach a minimal hypersurface with much more precision. It is the object of the following lemma.

Lemma 2.2.1 — *Let $\Gamma_{cor,\epsilon}$ the function defined by $\Gamma_{cor,\epsilon} := \epsilon\Gamma + Cor$. Then for all $k \in \mathbb{N}$, there exists a constant $c_k := c(n,k)$ such that for all $x \in \mathbb{R}^n_{l,\epsilon}$, we have the inequality*

$$\left|\nabla^k \left(\Delta\Gamma_{cor,\epsilon} - \mathcal{G}\left(\Gamma_{cor,\epsilon}\right)\right)(x)\right| \quad \leqslant \quad c_k \epsilon r_\epsilon^{3n-3} r^{1-2n-k}, \tag{2.2.11}$$

where r is chosen to be the distance between x and the set of points $\left\{p_{l-1,i}\right\}_{i\in[\![1,n_l]\!]} \cup \left\{p_{l,i}\right\}_{i\in[\![1,n_l]\!]}$. It amounts to writing

$$\left(\Delta\Gamma_{cor,\epsilon} - \mathcal{G}\left(\Gamma_{cor,\epsilon}\right)\right)(x) = \epsilon r_\epsilon^{3n-3} \underset{r\to 0}{\overset{\circ}{\mathcal{O}}}\left(r^{1-2n}\right).$$

In other words, we have improved the approximation of a solution to the minimal hypersurface equation over $\mathbb{R}^n_{l,\epsilon}$ by a factor $\frac{\epsilon r_\epsilon^{n-2}}{\epsilon r_\epsilon^{-1}} = r_\epsilon^{n-1}$ compared with the approximation in the case where we only consider Green's function $\epsilon\Gamma$.

3 Analysis in weighted spaces

We have in mind to glue the "necks" of the graph of $\Gamma_{\mathrm{cor},\epsilon}$ with small truncated catenoids. Besides, there are two types of terms of order 1 in the Taylor expansion of $\Gamma_{\mathrm{cor},\epsilon}$, namely a radial one that comes from the correcting function Cor, and a "force term" with one direction (*a priori*, it does not vanish) that comes from Green's function. Their rough estimate is ϵr_ϵ. Henceforth, our aim is to build a minimal graph over $\mathbb{R}^n_{l,\epsilon}$ whose boundary data is $\Gamma_{\mathrm{cor},\epsilon} + \Phi$ where $\|\Phi\| < \kappa\epsilon r_\epsilon$ for some positive constant κ that does not depend on ϵ and that we will determine later.

For this, we are looking for a small perturbation $\Gamma_{\mathrm{cor},\epsilon} + v$ of $\Gamma_{\mathrm{cor},\epsilon}$ where v is a small function such that we are able to solve the following problem :

$$\left\{\begin{array}{rcll} \Delta\left(\Gamma_{\mathrm{cor},\epsilon} + v\right) - \mathcal{G}\left(\Gamma_{\mathrm{cor},\epsilon} + v\right) & = & 0 & \text{in } \mathbb{R}^n_{l,\epsilon}; \\ \Gamma_{\mathrm{cor},\epsilon} + v & = & \Gamma_{\mathrm{cor},\epsilon} + \Phi & \text{on } \partial\mathbb{R}^n_{l,\epsilon}. \end{array}\right. \tag{3.0.12}$$

In other words, given an arbitrary element Φ, can we find a minimal graph over the non compact domain $\mathbb{R}^n_{l,\epsilon}$ that takes boundary value $\left(\Gamma_{\mathrm{cor},\epsilon}\right)_{|\partial\mathbb{R}^n_{l,\epsilon}} + \Phi$?

For practical use, we write $\Phi_p : \mathbb{S} \longrightarrow \mathbb{R}$ such that for all $z \in \mathbb{S}$,

$$\Phi_p(z) \quad := \quad \Phi\left(p + \frac{z}{r_\epsilon}\right)$$

for each $p = p_{l-1,1}, \cdots, p_{l-1,n_{l-1}}, p_{l,1}, \cdots, p_{l,n_l}$.

Let us very briefly expose the heuristics to solve this problem :

- if v_0 and ϵ are small enough, then

$$\Delta \left(\Gamma_{\text{cor},\epsilon} + v \right) - \mathcal{G} \left(\Gamma_{\text{cor},\epsilon} + v \right) \approx \Delta v,$$

  thus we first solve $\Delta v_0 = 0$ with $v_0 \approx \Phi$ on the boundary ;
- by a fixed point theorem argument, we search for a solution v that takes the form $\omega = \epsilon\Gamma_{\text{cor},\epsilon} + v_0 + v$ with $\|v\| << \|v_0\|$.

3.1 Laplacian and weighted spaces

We are interested in the following Dirichlet problem :

$$\begin{cases} \Delta_{\mathbb{R}^n} v &= f \quad \text{over } \mathbb{R}^n_{l,\epsilon}; \\ v &= 0 \quad \text{over } \partial\mathbb{R}^n_{l,\epsilon}. \end{cases} \tag{3.1.13}$$

It is a partial differential equation over an unbounded domain. To deal with it in a suitable theory, we introduce the weighted spaces. These have already proved to be useful, especially for gluing process.

We have seen in the previous section that the Laplacian plays a prominent role in the resolution to the minimal graph equation. Indeed, it is its inverse that gives us the possibility to apply a well chosen fixed point theorem.

We introduce two quantities

$$\widetilde{\rho}_0 \quad := \quad \min_{p \neq q \in \left\{p_{l-1,i}\right\}_{i \in [\![1,n_{l-1}]\!]} \cup \left\{p_{l,i}\right\}_{i \in [\![1,n_l]\!]}} \left\{ \frac{\text{dist}\,(p,q)}{3} \right\}$$

and

$$\rho_* \quad := \quad \max_{p \neq q \in \left\{p_{l-1,i}\right\}_{i \in [\![1,n_{l-1}]\!]} \cup \left\{p_{l,i}\right\}_{i \in [\![1,n_l]\!]}} \left\{ \text{dist}\,(0_{\mathbb{R}^n}, p) + \text{dist}\,(p,q) \right\} + 1.$$

We then define the sets :

$$\begin{cases} \mathbb{R}^n_{l,*} &:= \mathbb{R}^n \setminus \{p_{l-1,1}, \cdots, p_{l,n_l}\}, & \text{not bounded open set;} \\ B^p &:= B\left(p, \widetilde{\rho}_0\right), p = p_{l-1,1}, \cdots, p_{l,n_l}, & \text{relatively compact open set;} \\ B_r &:= B\left(0, r\right) \text{ for } r > 0,; & \text{relatively compact open set;} \\ \mathcal{A}_r &:= B_{2r} \setminus \overline{B_r} \text{ for } r > 0, & \text{relatively compact open annulus;} \\ K &:= \overline{B_{\rho_*}} \setminus \left(\bigcup_{p = p_{l-1,1}, \cdots, p_{l,n_l}} B^p \right), & \text{compact set;} \\ \Omega &:= \mathbb{R}^n \setminus \overline{B_{\rho_*}}, & \text{not bounded open set.} \end{cases}$$

Note that we have chosen those sets and $\widetilde{\rho}_0$ together with ρ^* such that the balls B^p do not intersect and B_{ρ_*} is a large ball centred in 0 which contains all the B^p. Besides, none of those depend on the parameter ϵ.

Definition 3.1.1 — Let μ and ν be real numbers. We define the *weighted space* $L^\infty_{\mu,\nu}\left(\mathbb{R}^n_{l,*}\right)$ as the set of functions $f \in L^\infty_{loc}\left(\mathbb{R}^n_{l,*}\right)$ such that

$$\|f\|_{L^\infty_{\mu,\nu}\left(\mathbb{R}^n_{l,*}\right)} := \sum_{p=p_{l-1,1},\cdots,p_{l,n_l}} \left\| |x-p|^{-\mu} f \right\|_{L^\infty(B^p)} + \|f\|_{L^\infty(K)} + \left\| |x|^{-\nu} f \right\|_{L^\infty(\Omega)} < +\infty.$$

Remark 3.1.2 — In particular, if f is an element of $L^\infty_{\mu,\nu}\left(\mathbb{R}^n_{l,*}\right)$, then we control f near the possible singularities $p = p_{l-1,1},\cdots,p_{l,n_l}$: f does not increase faster than r^μ when r tends to 0. Likewise, the behavior of f near infinity is at the most r^ν when r is large and f is bounded for all compact set included in $\mathbb{R}^n_{l,*}$.

We also give the definition of such weighted spaces for more regular Hölder functions.

Definition 3.1.3 — Let $\mu,\nu \in \mathbb{R}$, $k \in \mathbb{N}$ and $\alpha \in (0,1)$. We define the *Hölder weighted space* $\mathcal{C}^{k,\alpha}_{\mu,\nu}\left(\mathbb{R}^n_{l,*}\right)$ as the set of functions $f \in \mathcal{C}^{k,\alpha}_{loc}\left(\mathbb{R}^n_{l,*}\right)$ such that the following norm is finite :

$$\begin{aligned}\|f\|_{\mathcal{C}^{k,\alpha}_{\mu,\nu}\left(\mathbb{R}^n_{l,*}\right)} := \sum_{p=p_{l-1,1},\cdots,p_{l,n_l}} & \left(\sum_{i=0}^{k} \left\| |x-p|^{i-\mu} \nabla^i f \right\|_{L^\infty(B^p\setminus\{p\})} \right. \\ & \left. + \sup_{0<2r<\widetilde{\rho}_0} \left\{ r^{k+\alpha-\mu} \sup_{x\neq y \in P+A_r} \frac{\left|\nabla^k f(x) - \nabla^k f(y)\right|}{|x-y|^\alpha} \right\} \right) \\ & + \|f\|_{\mathcal{C}^{k,\alpha}(K)} \\ & + \sum_{i=0}^{k} \left\| |x|^{i-\nu} \nabla^i f \right\|_{L^\infty(\Omega)} \\ & + \sup_{r>\rho_*^2} \left\{ r^{k+\alpha-\nu} \sup_{x\neq y\in A_r} \frac{\left|\nabla^k f(x) - \nabla^k f(y)\right|}{|x-y|^\alpha} \right\}.\end{aligned}$$

Remark 3.1.4 — For practical purposes, a function $f \in \mathcal{C}^{k,\alpha}_{\mu,\nu}\left(\mathbb{R}^n_{l,*}\right)$ (resp. its derivatives) is bounded by $c\ r^\mu$ (resp. the derivatives of r^μ) near the points $p = p_{l-1,1},\cdots,p_{l,n_l}$, by r^ν (resp. the derivatives of r^ν) near infinity, where c is a constant ($c = \|f\|_{\mathcal{C}^{2,\alpha}_{\mu,\nu}\left(\mathbb{R}^n_{l,*}\right)}$). Moreover,

$$\left|\nabla^k f(x) - \nabla^k f(y)\right| \quad < \quad c\ r^{\mu-k-\alpha} |x-y|^\alpha$$

(resp. $< c\ r^{\nu-k-\alpha} |x-y|^\alpha$).

Definition 3.1.5 — Let U be an open subset of $\mathbb{R}^n_{l,*}$. Then we define in the same way the *weighted spaces* $L^\infty_{\mu,\nu}(U)$ (resp. $\mathcal{C}^{k,\alpha}_{\mu,\nu}(U)$) endowed with the norm $\|\cdot\|_{L^\infty_{\mu,\nu}(U)}$ (resp. $\|\cdot\|_{\mathcal{C}^{k,\alpha}_{\mu,\nu}(U)}$) induced by $\|\cdot\|_{L^\infty_{\mu,\nu}\left(\mathbb{R}^n_{l,*}\right)}$ (resp. $\|\cdot\|_{\mathcal{C}^{k,\alpha}_{\mu,\nu}\left(\mathbb{R}^n_{l,*}\right)}$).

Remark 3.1.6 — The spaces we have defined are Banach spaces. In particular, classical fixed point theorems can be used.

Remark 3.1.7 — Suppose $k \geqslant 2$. The definition of weighted spaces implies that if $f \in \mathcal{C}^{k,\alpha}_{\mu,\nu}\left(\mathbb{R}^n_{l,*}\right)$, then $\Delta_{\mathbb{R}^n} f \in \mathcal{C}^{k-2,\alpha}_{\mu-2,\nu-2}\left(\mathbb{R}^n_{l,*}\right)$.

Proposition 3.1.8 — *Assume that $\mu, \nu \in (2-n, 0)$. Then there exists some constant $c := c(\mu,\nu,\alpha,n,l)$ such that for all $f \in \mathcal{C}^{k-2,\alpha}_{\mu-2,\nu-2}\left(\mathbb{R}^n_{l,*}\right)$, there exists one and only one function v such that :*

$$\left\{\begin{array}{rcll} v & \in & \mathcal{C}^{k,\alpha}_{\mu,\nu}\left(\mathbb{R}^n_{l,*}\right); & \\ \Delta_{\mathbb{R}^n} v & = & f & \text{over } \mathbb{R}^n_{l,*}. \end{array}\right. \tag{3.1.14}$$

Besides, for such a solution v, we have the following estimate :

$$\|v\|_{\mathcal{C}^{2,\alpha}_{\mu,\nu}\left(\mathbb{R}^n_{l,*}\right)} \leqslant c\,\|f\|_{\mathcal{C}^{0,\alpha}_{\mu-2,\nu-2}\left(\mathbb{R}^n_{l,*}\right)}. \tag{3.1.15}$$

We note $\Delta^{-1}_{\mu,\nu,*} : \mathcal{C}^{k-2,\alpha}_{\mu-2,\nu-2}\left(\mathbb{R}^n_{l,*}\right) \longrightarrow \mathcal{C}^{k,\alpha}_{\mu,\nu}\left(\mathbb{R}^n_{l,*}\right)$ the right inverse.

Proof

We use the same approach than the study of similar problem in L^∞ with only one singularity in [Pac09, Chapter 4]. Essentially, we divide up our case into two parts : one for the singularities and one for infinity. Estimates are obtained by well chosen barrier functions and the maximum principle.

The Laplacian for well chosen radial functions : Observe, by using the expression of Laplacian in polar coordinates, that

$$\Delta_{\mathbb{R}^n} |x|^\beta = \beta(n+\beta-2)|x|^{\beta-2}.$$

Moreover, note that the constant $\beta(n+\beta-2)$ is negative if, and only if, $\beta \in (2-n, 0)$. This simple property will be very useful to solve our Dirichlet problem since it is the first brick in the construction of a barrier function.

Decomposition of f : We write $f = f_{\widetilde{K}} + f_\infty$ where $f_{\widetilde{K}}$ is supported in the relatively compact set $\widetilde{K} := B_{2\rho_*} \setminus \{p_{l-1,1}, \cdots, p_{l,n_l}\}$ ($\widetilde{K}$ is chosen so that $\overline{\widetilde{K}}$ is a ball large enough to hold all singularities), and f_∞ is supported in the non compact domain Ω. We may assume $f_{\widetilde{K}}$ and f_∞ have the same regularity as f [1].

The $f_{\widetilde{K}}$-part : We want to build a solution $v_{\widetilde{K}}$ such that $\Delta_{\mathbb{R}^n} v_{\widetilde{K}} = f_{\widetilde{K}}$ over $\mathbb{R}^n_{l,*}$. The domain is not compact, thus we cannot apply directly classical existence results. To make up for this problem, we solve it by using a sequence of solutions on open sets $(U_i)_i$ that converges to $\mathbb{R}^n_{l,*}$ defined by

$$U_i := B(0, i+2\rho_*) \setminus \bigcup_{p=p_{l-1,1},\cdots,p_{l,n_l}} \overline{B\left(p, \frac{\rho_0}{i}\right)}.$$

1. For example, with the help of a well chosen cut-off function $\mathcal{C}^{\infty,\alpha}$-function $\chi : \mathbb{R}^n_{l,*} \longrightarrow [0,1]$ such that $\chi = 1$ over $B_{\rho_*^2}$ and $\chi = 0$ over $\mathbb{R}^n \setminus B_{2\rho_*^2}$: then put $f_{\widetilde{K}} := \chi \cdot f$ and $f_\infty := f - f_{\widetilde{K}}$.

By [GT01, Theorem 4.3.], there exists a unique solution $v_{i,\widetilde{K}} \in \mathcal{C}^{2,\alpha}(U_i)$ to the following Dirichlet problem

$$\begin{cases} \Delta_{\mathbb{R}^n} v_{i,\widetilde{K}} & = f_{\widetilde{K}} \quad \text{in } U_i; \\ v_{i,\widetilde{K}} & = 0 \quad \text{on } \partial U_i. \end{cases}$$

Before doing $i \to \infty$, it is necessary to obtain estimates for this solution. For this purpose, we build a barrier function. Let

$$w \quad := \quad C \frac{\left\| f_{\widetilde{K}} \right\|_{\mathcal{C}^{0,\alpha}_{\mu-2,\nu-2}\left(\mathbb{R}^n_{l,*}\right)}}{\mu(2-n-\mu)} \sum_{p=p_{l-1,1},\cdots,p_{l,n_l}} |\cdot - p|^{\mu} - v_{i,\widetilde{K}},$$

where C is a positive constant. Thus the Laplacian of w is determined by

$$\Delta_{\mathbb{R}^n}(w)(x) \quad = \quad -C \left\| f_{\widetilde{K}} \right\|_{\mathcal{C}^{0,\alpha}_{\mu-2,\nu-2}\left(\mathbb{R}^n_{l,*}\right)} \sum_{p=p_{l-1,1},\cdots,p_{l,n_l}} |x-p|^{\mu-2} - f_{\widetilde{K}}(x).$$

Then one proves that we can take $C = C(\mu, \nu)$ big enough (which does not depend on i) to ensure that for all $x \in U_i$, the inequality $\Delta_{\mathbb{R}^n}(w)(x) \leqslant 0$ holds true. Moreover, $w \geqslant 0$ over ∂U_i. Consequently, we can apply the maximum principle (cf. [GT01, Theorem 3.3]) to obtain the following pointwise bound :

$$\forall x \in U_i, \qquad v_{i,C}(x) \quad \leqslant \quad C \left\| f_{\widetilde{K}} \right\|_{\mathcal{C}^{0,\alpha}_{\mu-2,\nu-2}\left(\mathbb{R}^n_{l,*}\right)} \sum_{p=p_{l-1,1},\cdots,p_{l,n_l}} |x-p|^{\mu}.$$

Working in a similar way with $-v_{i,\widetilde{K}}$, we then obtain the uniform bound :

$$\forall x \in U_i, \qquad \left| v_{i,\widetilde{K}}(x) \right| \quad \leqslant \quad C \left\| f_{\widetilde{K}} \right\|_{\mathcal{C}^{0,\alpha}_{\mu-2,\nu-2}\left(\mathbb{R}^n_{l,*}\right)} \sum_{p=p_{l-1,1},\cdots,p_{l,n_l}} |x-p|^{\mu}.$$

According to [GT01, Corollary 4.7.] [2], the previous inequality and a diagonal extraction argument, we conclude that there exists $v_{\widetilde{K}} \in \mathcal{C}^{2,\alpha}_{loc}\left(\mathbb{R}^n_{l,*}\right)$ such that $\Delta_{\mathbb{R}^n} v_{\widetilde{K}} = f_{\widetilde{K}}$ over $\mathbb{R}^n_{l,*}$. Besides, since the convergence is uniform, with the help of same inequality, we have :

$$\forall x \in \mathbb{R}^n_{l,*}, \qquad \left| v_{\widetilde{K}}(x) \right| \quad \leqslant \quad C \left\| f_{\widetilde{K}} \right\|_{\mathcal{C}^{0,\alpha}_{\mu-2,\nu-2}\left(\mathbb{R}^n_{l,*}\right)} \sum_{p=p_{l-1,1},\cdots,p_{l,n_l}} |x-p|^{\mu}, \tag{3.1.16}$$

in other words, we already know that $v_{\widetilde{K}}$ belongs to a space of the form $L^{\infty}_{\mu,*}\left(\mathbb{R}^n_{l,*}\right)$; $v_{\widetilde{K}}$ has the right behaviour near the singularity points p.
To deal with the case $x \to \infty$, we use another barrier function. More precisely, since $f_{\widetilde{K}}$ vanishes over $\mathbb{R}^n \setminus \widetilde{K}$, the function $v_{\widetilde{K}}$ is harmonic over $\mathbb{R}^n \setminus \widetilde{K}$. Note

2. *"Any bounded sequence of solutions of Poisson's equation $\Delta_{\mathbb{R}^n} v = f$ in a domain U with $f \in \mathcal{C}^{0,\alpha}(U)$ contains a subsequence converging uniformly on compact subdomains to a solution."*

$\widetilde{v}_{\widetilde{K}}$ the function induced by $v_{\widetilde{K}}$ on $\mathbb{R}^n \setminus \widetilde{K}$. But we know that Green's function $|\cdot|^{2-n}$ is also harmonic. Let $K\left[\widetilde{v}_{\widetilde{K}}\right]$ be the Kelvin transform of $\widetilde{v}_{\widetilde{K}}$, ie :

$$K\left[\widetilde{v}_{\widetilde{K}}\right] : \begin{array}{rcl} B_{1/\rho_*^2} \setminus \{0\} & \longrightarrow & \mathbb{R} \\ y & \longmapsto & |y|^{2-n}\, \widetilde{v}_{\widetilde{K}}\left(\frac{y}{|y|^2}\right). \end{array}$$

According to [ABR01, Theorem 4.7], this function is harmonic. According to the pointwise estimate (3.1.16), the limit $\lim_{x\to\infty} \widetilde{v}_{\widetilde{K}}$ vanishes. Therefore, applying [ABR01, Theorem 4.8], $K\left[\widetilde{v}_{\widetilde{K}}\right]$ has a removable singularity at the origin and the maximum principle leads us to :

$$\sup_{y\in B_{1/\rho_*^2}} \left|K\left[\widetilde{v}_{\widetilde{K}}\right](y)\right| \quad = \quad \sup_{|y|=1/\rho_*^2} \left|K\left[\widetilde{v}_{\widetilde{K}}\right](y)\right|.$$

In other words, we have obtained a new pointwise bound for $v_{\widetilde{K}}$:

$$\forall x \in \mathbb{R}^n_{l,*} \setminus \widetilde{K}, \qquad \left|v_{\widetilde{K}}(x)\right| \quad \leqslant \quad C\ \left\|f_{\widetilde{K}}\right\|_{\mathcal{C}^{0,\alpha}_{\mu,\nu}\left(\mathbb{R}^n_{l,*}\right)} |x|^{2-n}.$$

Note that this inequality is stronger than the previous one for x large enough since $2-n<\mu$. Indeed, we conclude that $v_{\widetilde{K}}$ belongs to $L^\infty_{\mu,\nu}\left(\mathbb{R}^n_{l,*}\right)$.

The f_∞-part : by similar arguments, we prove that there exists $v_\infty \in \mathcal{C}^{2,\alpha}_{loc}\left(\mathbb{R}^n_{l,*}\right)$ a solution to the problem $\Delta_{\mathbb{R}^n} v_\infty = f_\infty$ on $\mathbb{R}^n_{l,*}$ with the estimate

$$\forall x \in \mathbb{R}^n_{l,*}, \qquad |v_\infty(x)| \quad \leqslant \quad C\ \left\|f_{\widetilde{K}}\right\|_{\mathcal{C}^{0,\alpha}_{\mu,\nu}\left(\mathbb{R}^n_{l,*}\right)} |x|^\nu, \tag{3.1.17}$$

where $C = C(n,\mu,\nu)$.

To deal with the case $x \to p$ with $p = p_{l-1,1}, \cdots, p_{l,n_l}$, we use classic results about isolated singularity for harmonic functions(Cf. [ABR01, theorem 10.5]) to show the singularities at P are removable since $\nu > 2-n$. By maximum principle, we find

$$\sup_{|x|\leqslant\rho_*^2} |v_\infty(x)| \quad \leqslant \quad C\left\|f_{\widetilde{K}}\right\|_{\mathcal{C}^{0,\alpha}_{\mu,\nu}\left(\mathbb{R}^n_{l,*}\right)} \sum_{p=p_{l-1,1},\cdots,p_{l,n_l}} |x-p|^\mu,$$

for some positive constant $C = C(n,\mu,\nu)$.

Collecting previous cases : we are now able to solve the problem (3.1.14). Let $v := v_{\widetilde{K}} + v_\infty$. By linearity of the Laplacian operator, it is clear that $\Delta_{\mathbb{R}^n} v = f$. It remains to see why v belongs to the Hölder weighted space $\mathcal{C}^{2,\alpha}_{\mu,\nu}\left(\mathbb{R}^n_{l,*}\right)$. Collecting all previous estimates, we see that

$$v \in L^\infty_{\mu,\nu}\left(\mathbb{R}^n_{l,*}\right) \cap \mathcal{C}^{2,\alpha}_{loc}\left(\mathbb{R}^n_{l,*}\right)$$

and there exists $c = c(n,\mu,\nu) > 0$ such that

$$\|v\|_{L^\infty_{\mu,\nu}\left(\mathbb{R}^n_{l,*}\right)} \quad \leqslant \quad c\,\|f\|_{\mathcal{C}^{0,\alpha}_{\mu,\nu}\left(\mathbb{R}^n_{l,*}\right)}.$$

We now have in mind to obtain the same type estimate of for Hölder spaces as in L^∞. We only prove that the estimate for case $|x|$ very large ; we can prove in the same way that the estimate in a neighbourhood of the singularities. The key of the proof is the Schauder's interior estimates (cf. [GT01, Theorem 6.2.]). More precisely, for $|x|$ large enough, we work on an annulus $\mathcal{A}_R$ with R big enough and we boil down to the case $\mathcal{A}_1$ after a contraction ; we apply the estimate to this new function and we perform a dilatation to return to the case $|x|$ large. First of all, since for $R \geqslant \rho_*$, the set $B^p \cap \mathcal{A}_R$ is empty, we have the estimate

$$\begin{aligned} \|v\|_{\mathcal{C}^{2,\alpha}_{\mu,\nu}(\mathcal{A}_R)} &\leqslant (2R)^{-\nu} \sum_{i=0}^{2} \left\| |x|^i \nabla^i v \right\|_{L^\infty(\mathcal{A}_R)} \\ &\quad + R^{-\nu} \sup_{R<2r<2R} \left(r^{k+\alpha} \sup_{x \neq y \in \mathcal{A}_r} \frac{|\nabla^2 v(x) - \nabla^2 v(y)|}{|x-y|^\alpha} \right), \end{aligned}$$

therefore, applying Schauder's interior estimates, there exists $C = C(n, \alpha, \nu, l)$ that does not depend on R such that

$$\|v\|_{\mathcal{C}^{2,\alpha}_{\mu,\nu}(\mathcal{A}_R)} \leqslant C \|f\|_{\mathcal{C}^{0,\alpha}_{\mu,\nu}(\mathcal{A}_R)},$$

from what we conclude that

$$\|v\|_{\mathcal{C}^{2,\alpha}_{\mu,\nu}(\Omega)} \leqslant C \|f\|_{\mathcal{C}^{0,\alpha}_{\mu,\nu}(\Omega)}.$$

By using exactly the same method in the B^p and K, we finally end up with

$$\|v\|_{\mathcal{C}^{2,\alpha}_{\mu,\nu}\left(\mathbb{R}^n_{l,*}\right)} \leqslant c \|f\|_{\mathcal{C}^{0,\alpha}_{\mu,\nu}\left(\mathbb{R}^n_{l,*}\right)},$$

where $c = c(\mu, \nu, \alpha, n, l)$.

Uniqueness : It is again an application of the maximum principle. Assume that we have two solutions v_1 and v_2 to the problem (3.1.14). Then $v_1 - v_2$ is harmonic on $\mathbb{R}^n_{l,*}$ and belongs to $\mathcal{C}^{2,\alpha}_{\mu,\nu}\left(\mathbb{R}^n_{l,*}\right)$. So, if $p = p_{l-1,1}, \cdots, p_{l,n_l}$, p is a removable singularity since $\mu > 2-n$. Therefore $v_1 - v_2$ is bounded over all compact sets. Moreover, since $\nu < 0$,

$$(v_1 - v_2)(x) \underset{|x|\to\infty}{\longrightarrow} 0,$$

then, $v_1 - v_2$ is bounded over $\mathbb{R}^n$. According to Liouville's theorem [ABR01, Theorem 2.1], $v_1 - v_2$ is constant ; but the behavior at infinity implies this constant is necessarily 0, in other words $v_1 = v_2$.

Using exactly the same arguments, one can prove the following :

Proposition 3.1.9 — *Assume that $\mu, \nu \in (2-n, 0)$. Then there exists some constant $c := c(\mu, \nu, \alpha, n, l)$ such that for all $\epsilon > 0$ and for all $f \in \mathcal{C}^{k-2,\alpha}_{\mu-2,\nu-2}\left(\mathbb{R}^n_{l,*}\right)$, there exists one and only one function v such that :*

$$v \in \mathcal{C}^{k,\alpha}_{\mu,\nu}\left(\mathbb{R}^n_{l,\epsilon}\right) \quad \text{and} \quad \left\{ \begin{array}{rl} \Delta_{\mathbb{R}^n} v &= f \quad \text{over } \mathbb{R}^n_{l,\epsilon}; \\ v &= 0 \quad \text{on } \partial\mathbb{R}^n_{l,\epsilon}. \end{array} \right. \tag{3.1.18}$$

Besides, for such a solution v, we have the following estimate :

$$\|v\|_{\mathcal{C}^{2,\alpha}_{\mu,\nu}\left(\mathbb{R}^n_{l,\epsilon}\right)} \leqslant c\,\|f\|_{\mathcal{C}^{0,\alpha}_{\mu-2,\nu-2}\left(\mathbb{R}^n_{l,\epsilon}\right)}. \tag{3.1.19}$$

We note $\Delta^{-1}_{\mu,\nu,\epsilon} : \mathcal{C}^{k-2,\alpha}_{\mu-2,\nu-2}\left(\mathbb{R}^n_{l,\epsilon}\right) \longrightarrow \mathcal{C}^{k,\alpha}_{\mu,\nu}\left(\mathbb{R}^n_{l,\epsilon}\right)$ the right inverse.

Note that the important point is that the constant c does not depend on ϵ.

3.2 Harmonic function

3.2 — (a) The harmonic extension on $\mathbb{R}^n \setminus B_1$

In the previous proposition, we have studied the problem of a prescribed Laplacian with boundary data equal to 0 over $\mathbb{R}^n_{l,\epsilon}$ and the result corresponds to the part "fixed point theorem" we will apply later to build minimal hypersurface in a neighbourhood of the graph of $\Gamma_{\text{cor},\epsilon}$. However, we also must consider the problem of prescribed boundary data that will be the key of the gluing process.

We look for an operator W^e that is equivalent to an *exterior harmonic extension.* Indeed, we are interested in the following Dirichlet problem :

$$\begin{cases} \Delta_{\mathbb{R}^n} W^e(\Phi) &= 0 \quad \text{over } \mathbb{R}^n \setminus B_1; \\ W^e(\Phi) &= \Phi \quad \text{over } \partial B_1 = \mathbb{S}; \end{cases} \tag{3.2.20}$$

where Φ belongs to $L^2(\mathbb{S})$. The role of Fourier decomposition is fundamental. According to [GHL93, Corollary 4.49 and Lemma 4.50], the eigenvalues of $-\Delta_{\mathbb{S}^2}$ are the $\lambda_j := j(n-2+j)$ for $j \in \mathbb{N}$. Moreover, by [Hö68], there exists some constant $c = c(n)$ such that for all j, for all eigenfunction Φ^j that belongs to the j^{th} eigenspace E^j, we have the following Hörmander's estimate :

$$\left\|\Phi^j\right\|_{L^\infty(\mathbb{S})} \leqslant c\,|\lambda_j|^{\frac{n-2}{4}} \left\|\Phi^j\right\|_{L^2(\mathbb{S})}, \tag{3.2.21}$$

and according to [Don06], the dimension of E^j is bounded by

$$c\,|\lambda_j|^{\frac{n-2}{2}}. \tag{3.2.22}$$

Furthermore, the L^2-orthogonal basis of the E^j span $L^2(\mathbb{S})$, thus we can write $\Phi = \sum_{j=0}^{+\infty} \Phi^j$ with $\Delta_{\mathbb{S}^2}\Phi^j = -\lambda_j \Phi^j$.

Notation 3.2.1 — It will be convenient to write π^j the orthogonal projection on the eigenspace E^j and $\pi^\perp$ the projection on the modes $j \geqslant 2$; thus we could decompose Φ as $\Phi^0 + \Phi^1 + \Phi^\perp$. Besides, recall that the mode 1 is spanned by the $\langle \cdot\,, \mathbf{e}_i\rangle$ for $i \in [\![1, n]\!]$; we write $\pi^{1,i}$ the orthogonal projection on the eigenfunction $\langle \cdot\,, \mathbf{e}_j\rangle$ and we could decompose Φ^1 as $\sum_{i=1}^n \Phi^{1,i}\langle \cdot\,, \mathbf{e}_i\rangle$. Note that Φ^0 and the $\Phi^{1,i}$ are real numbers.

Let I be a real interval, bounded or not. We note $\mathcal{E}^\perp(I \times \mathbb{S})$ (resp. $\mathcal{E}^0(I \times \mathbb{S})$ and $\mathcal{E}^1(I \times \mathbb{S})$) the set of function f on $I \times \mathbb{R}$ such that for all $s \in I$, the function $f(s, \cdot) : \mathbb{S} \longrightarrow \mathbb{R}$ belongs to $E^\perp$ (resp. E^0 and E^1).

Proposition 3.2.2 — *There exists some constant $c = c(n, \alpha)$ and one and only one linear operator $W^e : \mathcal{C}^{2,\alpha}(\mathbb{S}) \longrightarrow \mathcal{C}^{2,\alpha}_{0,2-n}(\mathbb{R}^n \setminus B_1)$ such that for all $\Phi \in \mathcal{C}^{2,\alpha}(\mathbb{S})$, $W^e(\Phi)$ is a solution to Dirichlet problem (3.2.20) that belongs to the Hölder weighted space $\mathcal{C}^{2,\alpha}_{0,2-n}(\mathbb{R}^n \setminus B_1)$. Besides, we have the following estimate :*

$$\|W^e(\Phi)\|_{\mathcal{C}^{2,\alpha}_{0,2-n}(\mathbb{R}^n \setminus B_1)} \leqslant c\, \|\Phi\|_{\mathcal{C}^{2,\alpha}(\mathbb{S})} ; \tag{3.2.23}$$

More precisely, one can refine this kind of estimate : if $\Phi \not\equiv 0$ and $j_0 := \min\{j : \Phi^j \not\equiv 0\}$, then $W^e(\Phi)$ is an element of $\mathcal{C}^{2,\alpha}_{0,2-n-j_0}(\mathbb{R}^n \setminus B_1)$ and

$$\|W^e(\Phi)\|_{\mathcal{C}^{2,\alpha}_{0,2-n-j_0}(\mathbb{R}^n \setminus B_1)} \leqslant c\, \|\Phi\|_{\mathcal{C}^{2,\alpha}(\mathbb{S})} . \tag{3.2.24}$$

Note that c does not depend on j_0.

Proof

Using similar method to the proof of proposition 3.1.8, one can demonstrate the existence and the uniqueness of such a solution, as well as the estimate (3.2.23). However, there exists another proof (by explicit construction of the solution) that leads to better estimate (3.2.24).

Formal solution : if $\Phi \equiv 0$, then it is clear that $W^e(\Phi) \equiv 0$ is the solution. Otherwise, we formally define

$$W^e(\Phi) : x \in \mathbb{R}^n \setminus B_1 \longmapsto \sum_{j \geqslant j_0} |x|^{2-n-j}\, \Phi^j\left(\frac{x}{|x|}\right).$$

It is an easy calculus to check that if this series has good properties (uniform convergence, ...), then it defines a solution. So, it is enough to prove the convergence of the formal series to conclude.

$W^e(\Phi)$ has the right weight in L^∞ : using Hörmander's estimate (3.2.21), we have :

$$|x|^{2-n-j} \left|\Phi^j\left(\frac{x}{|x|}\right)\right| \leqslant c\, |x|^{2-n-j_0}\, |x|^{j_0-j}\, |\lambda_j|^{\frac{n-2}{4}} \left\|\Phi^j\right\|_{L^2(\mathbb{S})} .$$

But, since Φ^j is the L^2-orthogonal projection on the eigenspace E^j,

$$\left\|\Phi^j\right\|^2_{L^2(\mathbb{S})} \leqslant \|\Phi\|^2_{L^2(\mathbb{S})} \leqslant \mathrm{Vol}(\mathbb{S})\ \|\Phi\|^2_{L^\infty(\mathbb{S})} ,$$

and thus there exists some constant $c = c(n)$ such that for all $j \in \mathbb{N}$, following inequality holds :

$$\left\|\Phi^j\right\|_{L^2(\mathbb{S})} \leqslant c\ \|\Phi\|_{L^\infty(\mathbb{S})} \leqslant c\ \|\Phi\|_{\mathcal{C}^{2,\alpha}(\mathbb{S})} .$$

We can deduce

$$|W^e(\Phi)(x)| \leqslant c\ |x|^{2-n-j_0}\, \|\Phi\|_{\mathcal{C}^{2,\alpha}(\mathbb{S})} \sum_{j \geqslant j_0} |x|^{j_0-j}\, |\lambda_j|^{\frac{n-2}{4}} . \tag{3.2.25}$$

Observe that for all $i \in \mathbb{N}^*$, since $\lambda_j \sim_{j\to\infty} j^2$, this series uniformly converges on $\mathbb{R}^n \setminus B_{1+1/i}$. So we just have proved that $W^e(\Phi)$ is well defined for $|x| > 1 + 1/i$ and is an element of $L^\infty_{0,2-n-j_0}\left(\mathbb{R}^n \setminus B_{1+1/i}\right)$. It remains to prove that it also defines an element of $L^\infty_{0,2-n-j_0}(\mathbb{R}^n \setminus B_1) \cap \mathcal{C}^{2,\alpha}_{loc}(\mathbb{R}^n \setminus B_1)$.

$W^e(\Phi)$ has the right regularity : From standard estimates for elliptic operators ([Aub82, Theorem 3.58]) together with an induction argument, one can demonstrate that for all $k \in \mathbb{N}$, there exists some constant $c = c(n,k)$ such that for all $j \geqslant j_0$, we have :

$$\left\|\Phi^j\right\|_{W^{2k,2}(\mathbb{S})} \quad \leqslant \quad c\ \|\Phi\|_{\mathcal{C}^{2,\alpha}(\mathbb{S})} \sum_{\ell=0}^{k} |\lambda_j|^l .$$

We apply Sobolev imbedding theorem for compact manifold ([Aub82, Theorem 2.20]) with $k_0 := \left\lceil \frac{n-1}{4} + 1 + \frac{\alpha}{2} \right\rceil$ to see that Φ^j belongs to $\mathcal{C}^{2,\alpha}(\mathbb{S})$ and there exists some constant $c = c(n,\alpha)$ such that

$$\left\|\Phi^j\right\|_{\mathcal{C}^{2,\alpha}(\mathbb{S})} \quad \leqslant \quad c\ P_n(j)\, \|\Phi\|_{\mathcal{C}^{2,\alpha}(\mathbb{S})} ,$$

where P_n denotes a polynomial expression of degree $2k_0$. Let us fix $i > 0$ and let $\mathcal{A}_i$ be the annulus $B_{1+i} \setminus B_{1+1/i}$ which tends to $\mathbb{R}^n \setminus B_1$. Then collecting previous inequalities, there exists some constant $c = c(n,\alpha,i)$ such that

$$\left\|r^{2-n-j}\Phi^j\right\|_{\mathcal{C}^{2,\alpha}(\mathcal{A}_i)} \quad \leqslant \quad c\,\|\Phi\|_{\mathcal{C}^{2,\alpha}(\mathbb{S})}\, P_n(j) \left(1 + \frac{1}{i}\right)^{2-n-j} ,$$

which is the general term of a convergent series (with index j). Therefore, for all i, $W^e(\Phi)$ belongs to $\mathcal{C}^{2,\alpha}(\mathcal{A}_i)$, thus belongs to $\mathcal{C}^{2,\alpha}_{loc}(\mathbb{R}^n \setminus B_1)$. Now we know that $W^e(\Phi)$ is a well defined harmonic function in $\mathbb{R}^n \setminus B_1$, thus by maximum principle and inequality (3.2.25), for all $1 < |x| < 2$:

$$\begin{aligned} |W^e(\Phi)(x)| \quad &\leqslant \quad \max\left\{ \|\Phi\|_{L^\infty(\mathbb{S})}\ ,\ c\ \|\Phi\|_{\mathcal{C}^{2,\alpha}(\mathbb{S})} \sum_{j \geqslant j_0} 2^{2-n-j} |\lambda_j|^{\frac{n-2}{4}} \right\} \\ &< \quad +\infty, \end{aligned}$$

so $W^e(\Phi) \in L^\infty_{0,2-n-j_0}(\mathbb{R}^n \setminus B_1) \cap \mathcal{C}^{2,\alpha}_{loc}(\mathbb{R}^n \setminus B_1)$.

In weighted Hölder space : To conclude with the derivatives of $W^e(\Phi)$, it is enough to apply Schauder's estimates : it is exactly the same argument as in the proof of proposition 3.1.8. □

3.2 — (b) The harmonic extension on $\mathbb{R}^n_{l,\epsilon}$

Let $\Phi : \partial\mathbb{R}^n_{l,\epsilon} \longrightarrow \mathbb{R}$ be a $\mathcal{C}^{2,\alpha}$ function. We use the construction of harmonic functions on $\mathbb{R}^n \setminus B_1$ in proposition 3.2.2 to define the following harmonic extension on $\mathbb{R}^n_{l,\epsilon}$:

$$h_\Phi : x \in \mathbb{R}^n_{l,\epsilon} \longmapsto \sum_{p = p_{l-1,1},\cdots,p_{l-1,n_{l-1}}} W^e(\Phi_p)\left(\frac{x-p}{r_\epsilon}\right) + \sum_{p = p_{l,1},\cdots,p_{l,n_l}} W^e(\Phi_p)\left(\frac{x-p}{r_\epsilon}\right).$$

The reader will pay attention to the boundary data of h_Φ. Indeed, in each singularity $p_{k,j}$, the terms that come from the $p_{k',j'}$ for $(k',j') \neq (k,j)$ create *small* perturbations of the boundary function Φ we want to prescribe, namely a contribution whose rough estimate is r_ϵ^{n-2}. It is the object of the following proposition whose result arises from the previous one 3.2.2.

Proposition 3.2.3 — *The function h_Φ belongs to $\mathcal{C}^{2,\alpha}_{2-n,2-n}\left(\mathbb{R}^n_{l,\epsilon}\right)$ and there exists $c = c(n,\alpha,l)$ such that*

$$\|h_\Phi\|_{\mathcal{C}^{2,\alpha}_{2-n,2-n}\left(\mathbb{R}^n_{l,\epsilon}\right)} \leqslant c\, r_\epsilon^{n-2} \|\Phi\|_{\mathcal{C}^{2,\alpha}(\partial\mathbb{R}^n_\epsilon)}, \tag{3.2.26}$$

where we have defined

$$\|\Phi\|_{\mathcal{C}^{2,\alpha}(\partial\mathbb{R}^n_\epsilon)} := \max_{\substack{p=p_{l-1,1},\cdots,p_{l,n_{l-1}} \\ p=p_{l,1},\cdots,p_{l,n_l}}} \left\{\|\Phi_p\|_{\mathcal{C}^{2,\alpha}(\mathbb{S})}\right\}.$$

Moreover, near the boundary $\partial\mathbb{R}^n_{l,\epsilon}$, for all $p = p_{l-1,1},\cdots,p_{l,n_l}$, we have

$$\left\| h_\Phi(x) - W^e(\Phi_p)\left(\frac{x-p}{r_\epsilon}\right)\right\|_{\mathcal{C}^{2,\alpha}(p+\mathcal{A}_{r_\epsilon})} \leqslant c \cdot r_\epsilon^{n-2} \|\Phi\|_{\mathcal{C}^{2,\alpha}(\partial\mathbb{R}^n_\epsilon)}.$$

4 Deforming Green's function to find a minimal graph

In the previous sections, we have developed two essential points :

- How to build small catenoidal "necks" by using the harmonic Green's function $\Gamma_{\mathrm{cor},\epsilon}$ and how to deal with the main part of the error to the minimal graph equation.
- A thorough analysis of the Laplacian operator in $\mathbb{R}^n$. It should be noted that in our case, we rather consider it like the Jacobi operator over the hyperplane $\{x_{n+1} = 0\}$, in other words, the differential of the mean curvature. Combining surjectivity with vanishing boundary data and the harmonic extensions with prescribed boundary data, we are in a position to solve a more general problem $\Delta_{\mathbb{R}^n} f = g$ on $\mathbb{R}^n_{l,\epsilon}$ with boundary data $\Gamma_{\mathrm{cor},\epsilon} + \Phi +$ (*small term in comparison with* Φ).

For all function Φ over $\partial\mathbb{R}^n_{l,\epsilon}$, let define ω_Φ (that continuously depends on the weighted points) by $\omega_\Phi := \Gamma_{\mathrm{cor},\epsilon} + h_\Phi$. Let $\mu,\nu \in (2-n,0)$. From the writing of minimal graph problem (3.0.12), if v is a fixed point of the operator $\mathcal{F}$ defined by

$$\mathcal{F}: \begin{array}{rcl} \mathcal{C}^{2,\alpha}_{\mu,\nu}(\mathbb{R}^n_\epsilon) & \longrightarrow & \mathcal{C}^{2,\alpha}_{\mu,\nu}\left(\mathbb{R}^n_{l,\epsilon}\right) \\ v & \longmapsto & \Delta^{-1}_{\mu,\nu,\epsilon}\left(\mathcal{G}(\omega_\Phi + v) - \Delta(Cor)\right), \end{array}$$

then $\omega_\Phi + v$ satisfies the minimal graph equation.

4.1 A fixed point theorem

We propose to prove a fixed point theorem for the operator $\mathcal{F}$. The reader will keep in mind that we will perform similar method for the deformation of a truncated catenoid. We briefly expose the reasoning.

- We analyze the image of the function 0, especially its rough estimate R, in order to find one suitable radius $2R$ for a closed ball centered whose image by the functional $\mathcal{F}$ is inside itself.
- We show that the functional is $\frac{1}{2}$-lipschitz over a closed ball with radius $2R$. Thus we could demonstrate $\mathcal{F}$ has a fixed point. Note that even if it is enough in our case, one could prove that for all $0 < k < 1$, $\mathcal{F}$ is k-contracting over a ball whose radius is $R + a_k$ where $(a_k)_k$ tends to 0 when k tends to 0.

4.1 — (a) Definition of the operator $\mathcal{F}$

First and foremost, we have to justify that $\mathcal{F}$ is a well defined operator for some suitable weight parameters. By proposition 3.2.3, together with the definition of $\Gamma_{\mathrm{cor},\epsilon}$, the functions $\Gamma_{\mathrm{cor},\epsilon}$ and h_Φ belong to the weighted Hölder space $\mathcal{C}^{2,\alpha}_{2-n}\left(\mathbb{R}^n_{l,\epsilon}\right)$, which is included in $\mathcal{C}^{2,\alpha}_{\mu,\nu}\left(\mathbb{R}^n_{l,\epsilon}\right)$ for all $\mu, \nu \in (2-n, 0)$, from what we conclude that $\mathcal{G}\left(\omega_\Phi + v\right) \in \mathcal{C}^{2,\alpha}_{\mu-2,\nu-2}\left(\mathbb{R}^n_{l,\epsilon}\right)$, and the conclusion holds.

4.1 — (b) The ω_Φ-part

Here, we want to estimate the contribution of the term ω_Φ in the definition of the operator $\mathcal{F}$; in other words, we are interested in the study of $\mathcal{F}(0)$. According to the construction of Green's function, the corrective term Cor and the harmonic extension h_Φ (3.2.26), there exists $c = c(\rho, n, \alpha, l)$ such that

$$\epsilon\Gamma(x) = \mathring{\mathcal{O}}\left(\epsilon r^{2-n}\right), \qquad Cor(x) = \mathring{\mathcal{O}}\left(\epsilon^3 \epsilon r^{4-3n}\right)$$

and

$$h_\Phi(x) = \mathring{\mathcal{O}}\left(r_\epsilon^{n-2} r^{2-n} \left\|\Phi\right\|\right).$$

Let us write

$$\left|\mathcal{G}\left(\omega_\Phi\right) - \Delta Cor\right|(x) \quad \leqslant \quad \left|\mathcal{G}\left(\Gamma_{\mathrm{cor},\epsilon}\right) - \Delta Cor\right|(x) + \left|\mathcal{G}\left(\omega_\Phi\right) - \mathcal{G}\left(\Gamma_{\mathrm{cor},\epsilon}\right)\right|(x).$$

First term : We already have an estimate that comes from the construction of the corrected Green's function $\Gamma_{\mathrm{cor},\epsilon}$, namely the estimation (2.2.11) in lemma 2.2.1 : there exists $c = c(n, \alpha)$ such that

$$\left\|\mathcal{G}\left(\Gamma_{\mathrm{cor},\epsilon}\right) - \Delta Cor\right\|_{\mathcal{C}^{0,\alpha}_{\mu-2,\nu-2}\left(\mathbb{R}^n_{l,\epsilon}\right)} \quad \leqslant \quad c\epsilon r_\epsilon^{3n-3} r_\epsilon^{1-2n-(\mu-2)} = c\epsilon r_\epsilon^{n+1-\mu}.$$

Second term : Here we use a Taylor formula. More precisely, computation proves that the differential of $\mathcal{G}$ is such that for all f and g, we have

$$\mathrm{d}\mathcal{G}_f(g) \quad = \quad \frac{\nabla^2 g\left(\nabla f, \nabla f\right) + 2\nabla^2 f\left(\nabla f, \nabla g\right)}{1 + \left|\nabla f\right|^2} - 2\frac{\left\langle \nabla f, \nabla g\right\rangle \mathcal{G}(f)}{1 + \left|\nabla f\right|^2}.$$

We have in mind to apply the mean value theorem [3]

$$\|\mathcal{G}(\omega_\Phi) - \mathcal{G}(\Gamma_{\text{cor},\epsilon})\|_{\mathcal{C}^{0,\alpha}_{\mu-2,\nu-2}\left(\mathbb{R}^n_{l,\epsilon}\right)} \leqslant \sup_{f\in[\omega_\Phi,\Gamma_{\text{cor},\epsilon}]} \|\mathrm{d}\mathcal{G}_f\| \, \|\omega_\Phi - \Gamma_{\text{cor},\epsilon}\|_{\mathcal{C}^{2,\alpha}_{\mu,\nu}\left(\mathbb{R}^n_{l,\epsilon}\right)}.$$

If we choose Φ such that its norm is smaller than $\kappa\epsilon r_\epsilon$ in the space $\mathcal{C}^{2,\alpha}\left(\mathbb{R}^n_{l,\epsilon}\right)$, then Cor and h_Φ have same rough estimate, namely

$$|Cor|(x) \leqslant c\epsilon r_\epsilon \qquad \text{and} \qquad |h_\Phi|(x) \leqslant \kappa c\epsilon r_\epsilon.$$

Thus, for all function f in the set $[\omega_\Phi, \Gamma_{\text{cor},\epsilon}]$, it is possible to decompose f as

$$f = \epsilon\Gamma + \widetilde{f}, \qquad \text{where} \qquad \left|\widetilde{f}\right|(x) \leqslant (1+\kappa)\, c\epsilon r_\epsilon,$$

that is to say $\widetilde{f}$ is very small in comparison with Green's function $\epsilon\Gamma$. Then we check that for all function g in $\mathcal{C}^{2,\alpha}_{\mu,\nu}\left(\mathbb{R}^n_{l,\epsilon}\right)$, for all x in $\mathbb{R}^n_{l,\epsilon}$,

$$\mathrm{d}\mathcal{G}_f(g)(x) \leqslant c\left[\epsilon^2 r^{2-2n} + \epsilon^2 r^{2-2n} + \left(\epsilon^2 r^{2-2n}\right)^2\right] r^{\mu-2} \|g\|_{L^\infty_{\mu-2,\nu-2}\left(\mathbb{R}^n_{l,\epsilon}\right)}.$$

So, for a parameter ϵ small enough,

$$\|\mathrm{d}\mathcal{G}_f(g)\|_{L^\infty_{\mu-2,\nu-2}\left(\mathbb{R}^n_{l,\epsilon}\right)} \leqslant c\epsilon^2 r_\epsilon^{2-2n} \|g\|_{L^\infty_{\mu-2,\nu-2}\left(\mathbb{R}^n_{l,\epsilon}\right)}.$$

By similar calculus, we finally end up with

$$\|\mathrm{d}\mathcal{G}_f\| \leqslant c\epsilon^2 r_\epsilon^{2-2n}.$$

Besides, since $\omega_\Phi - \Gamma_{\text{cor},\epsilon} = h_\Phi$, the estimate (3.2.26) implies

$$\|\omega_\Phi - \Gamma_{\text{cor},\epsilon}\|_{\mathcal{C}^{2,\alpha}_{\mu,\nu}\left(\mathbb{R}^n_{l,\epsilon}\right)} \leqslant c r_\epsilon^{-\mu} \|\Phi\|_{\mathcal{C}^{2,\alpha}\left(\partial\mathbb{R}^n_{l,\epsilon}\right)}.$$

Using the two previous inequalities together with $\|\Phi\|_{\mathcal{C}^{2,\alpha}\left(\partial\mathbb{R}^n_{l,\epsilon}\right)} \leqslant \kappa\epsilon r_\epsilon$, we get

$$\|\mathcal{G}(\omega_\Phi) - \mathcal{G}(\Gamma_{\text{cor},\epsilon})\|_{\mathcal{C}^{0,\alpha}_{\mu-2,\nu-2}\left(\mathbb{R}^n_{l,\epsilon}\right)} \leqslant c\kappa\epsilon^3 r_\epsilon^{3-2n-\mu} = c\kappa\epsilon r_\epsilon^{n-\mu}.$$

Conclusion : For the parameter ϵ small enough,

$$\|\mathcal{G}(\Gamma_{\text{cor},\epsilon}) - \Delta Cor\|_{\mathcal{C}^{0,\alpha}_{\mu-2,\nu-2}\left(\mathbb{R}^n_{l,\epsilon}\right)} \leqslant c_\kappa \epsilon r_\epsilon^{n-\mu}. \tag{4.1.27}$$

4.1 — (c) The contraction mapping

Proposition 4.1.1 — *For all $\mu, \nu \in (2-n, 0)$, there exists some constant $c = c(n,\mu,\nu,\alpha) > 0$ such that for all $\kappa > 0$, there exists $\epsilon_\kappa > 0$ such that :*

for all $\epsilon \in (0,\epsilon_\kappa)$, for all Φ which satisfies $\|\Phi\|_{\mathcal{C}^{2,\alpha}(\partial\mathbb{R}^n_\epsilon)} < \kappa\,\epsilon\, r_\epsilon$, $\mathcal{F}$ maps the ball $\mathscr{B}$ of radius $2c\,\epsilon\, r_\epsilon^{n-\mu}$, centred in the function 0, which is contained in $\mathcal{C}^{2,\alpha}_{\mu,\nu}\left(\mathbb{R}^n_{l,\epsilon}\right)$, into itself and is a $\frac{1}{2}$-Lipschitz operator in the ball $\mathscr{B}$: for all $v_1, v_2 \in \mathscr{B}$,

$$\|\mathcal{F}(v_1) - \mathcal{F}(v_2)\|_{\mathcal{C}^{2,\alpha}_{\mu,\nu}\left(\mathbb{R}^n_{l,\epsilon}\right)} \leqslant \frac{1}{2}\, \|v_1 - v_2\|_{\mathcal{C}^{2,\alpha}_{\mu,\nu}\left(\mathbb{R}^n_{l,\epsilon}\right)}. \tag{4.1.28}$$

3. In this case, we work with the operator $\mathcal{G} : \mathcal{C}^{2,\alpha}_{\mu,\nu}\left(\mathbb{R}^n_{l,\epsilon}\right) \longrightarrow \mathcal{C}^{0,\alpha}_{\mu-2,\nu-2}\left(\mathbb{R}^n_{l,\epsilon}\right)$.

PROOF

The image of 0 **:** According to previous estimate (4.1.27) together with the estimate of the norm of operator $\Delta^{-1}_{\mu,\nu_\epsilon}$ (3.1.19), there exists

$$c = c\,(n, \mu, \nu, \alpha) > 0$$

such that

$$\|\mathcal{F}\,(0)\|_{\mathcal{C}^{0,\alpha}_{\mu,\nu}\left(\mathbb{R}^n_{l,\epsilon}\right)} \quad \leqslant \quad c_\kappa \epsilon r_\epsilon^{n-\mu}$$

Note that it is this estimate which conducts us to the choice of the radius of $\mathscr{B}$. Besides, $\mu \in (2-n, 0)$ implies that for small ϵ (and thus for small r_ϵ),

$$\epsilon r_\epsilon^{n-\mu} << \epsilon r_\epsilon^{1-\mu},$$

thus $F\,(0)$ is small in comparison with the corrective term Cor, the harmonic extension h_Φ and the first order term of $\epsilon\Gamma$.

The Contracting part : let $v_1, v_2 \in \mathscr{B}$. To deal with the difference of the $\mathcal{F}\,(v_i)$, we make use of the PDE they satisfy. Thus we obtain

$$\Delta\left(\mathcal{F}\,(v_1) - \mathcal{F}\,(v_2)\right) \quad = \quad \mathcal{G}\,(\omega_\Phi + v_1) - \mathcal{G}\,(\omega_\Phi + v_2)\,.$$

According to Taylor's theorem, we get

$$\begin{aligned} &\|\mathcal{G}\,(\omega_\Phi + v_1) - \mathcal{G}\,(\omega_\Phi + v_2)\|_{\mathcal{C}^{2,\alpha}_{\mu-2,\nu-2}\left(\mathbb{R}^n_{l,\epsilon}\right)} \\ &\qquad\qquad \leqslant \quad \sup_{f\in[\omega_\Phi+v_1,\omega_\Phi+v_2]} \|\mathrm{d}\mathcal{G}_f\|\;\|v_1 - v_2\|_{\mathcal{C}^{2,\alpha}_{\mu,\nu}\left(\mathbb{R}^n_{l,\epsilon}\right)}\,, \end{aligned}$$

therefore it is enough to estimate the norm of the differential of $\mathcal{G}$. It is the same kind of computation than the one we used previously, except f belongs to $[\omega_\Phi + v_1, \omega_\Phi + v_2]$ and not to $[\omega_\Phi, \Gamma_{\mathrm{cor},\epsilon}]$. However, since v_i is small in comparison with ω_Φ, if f is an element in $[\omega_\Phi, \Gamma_{\mathrm{cor},\epsilon}]$, its main part is given by $\epsilon\Gamma$ and it follows that

$$\sup_{f\in[\omega_\Phi+v_1,\omega_\Phi+v_2]} \|\mathrm{d}\mathcal{G}_f\| \quad \leqslant \quad c\,\epsilon^2 r_\epsilon^{2-2n}\,,$$

and conclusion holds since $\Delta^{-1}_{\mu,\nu,\epsilon}$ is a continuous linear operator. □

4.1 — (d) A theorem of existence

If we apply a fixed point theorem with parameters

$$\left\{\left(a_{l-1,1}, p_{l-1,1}\right), \cdots, \left(a_{l,n_l}, p_{l,n_l}\right)\right\}$$

for the operator $\mathcal{F}$, one deduces the next theorem.

Theorem 4.1.2
For all $\mu, \nu \in (2-n, 0)$, there exists some constant $c = c(n, \mu, \nu, \alpha) > 0$ such that for all $\kappa > 0$, there exists $\epsilon_\kappa > 0$ such that :

for all $\epsilon \in (0, \epsilon_\kappa)$, for all Φ which satisfies $\|\Phi\|_{\mathcal{C}^{2,\alpha}(\partial \mathbb{R}^n_\epsilon)} < \kappa\, \epsilon\, r_\epsilon$, there exists v_Φ satisfying following assertions :

(i) $\Gamma_{cor,\epsilon} + h_\Phi + v_\Phi$ *satisfies the minimal graph equation on* $\mathbb{R}^n_{l,\epsilon}$.

(ii) v_Φ *belongs to* $\mathcal{C}^{2,\alpha}_{\mu,\nu}\left(\mathbb{R}^n_{l,\epsilon}\right)$ *with*

$$\|v_\Phi\|_{\mathcal{C}^{2,\alpha}_{\mu,\nu}\left(\mathbb{R}^n_{l,\epsilon}\right)} \quad \leqslant \quad 2c\, \epsilon\, r_\epsilon^{n-\mu};$$

(iii) The solution v_Φ continuously depends on the weighted points.

4.2 Description of the solution near its boundaries

We have seen that there exists v_Φ such that the surface whose parametrization is given by $\epsilon\Gamma + h_\Phi + v_\Phi$ is minimal, but we do not know much about the boundary data of this minimal hypersurface ; by construction, we only can say that $h_\Phi + v_\Phi$ *looks like* $\Gamma_{\text{cor},\epsilon} + \Phi$ on its boundary. But we have to give a more accurate description for gluing process. Instead of looking this problem on $\partial\mathbb{R}^n_{l,\epsilon}$, it is enough to consider the case near one of the balls $\partial B\left(p, r_\epsilon\right)$ for some $p = p_{k,j}$, the other cases can be deduced from this one. We dilate the function near a neighbourhood of $p + \partial B_{r_\epsilon}$ into

$$u_{\Phi,p,\pm} : \begin{array}{rcl} \mathcal{A}_1 & \longrightarrow & \mathbb{R} \\ x & \longmapsto & \left(\Gamma_{\text{cor},\epsilon} + h_\Phi + v_\Phi\right)\left(p + r_\epsilon\, x\right), \end{array}$$

where $\mathcal{A}_1$ is the open annulus $B_2 \setminus \overline{B_1}$ and the index $\pm$ is $-$ (resp. $+$) when $k = l$ (resp. $k = l-1$). We omit it until the end of this section to relieve notations.

Theorem 4.2.1
(i) $u_{\Phi,p}$ *is an element of* $\mathcal{C}^{2,\alpha}\left(\mathcal{A}_1\right)$ *and*

$$\left\|u_{\Phi,p} - \Gamma_{cor,\epsilon}\left(r_\epsilon \cdot\right) - W^e\left(\Phi_p\right)\right\|_{\mathcal{C}^{2,\alpha}\left(\mathcal{A}_1\right)} \quad \leqslant \quad c\ \kappa\ \epsilon\ r_\epsilon^{n-1}; \qquad (4.2.29)$$

in particular, if $a_p = a_{k,j}$ (resp. $a_p = -a_{k,j}$) when $k = l$ (resp. $k = l-1$) and $\pm$ is $-$ (resp. $+$) when $k = l$ (resp. $k = l-1$), the difference function $\mathfrak{d}_{\Phi,p}$ defined by

$$\mathfrak{d}_{\Phi,p}\left(x\right) \quad := \quad u_{\Phi,p}\left(x\right) - \left(a_p \epsilon r_\epsilon^{2-n}\left|x\right|^{2-n} + \frac{\left(n-2\right)^3}{2\left(3n-4\right)} a_p^3 \epsilon r_\epsilon \left|x\right|^{4-3n} \right.$$
$$\left. + C_{k,j,\pm}\epsilon + \epsilon r_\epsilon \left\langle x, F_{k,j,\pm}\right\rangle + W^e\left(\Phi_p\right)\left(x\right)\right),$$

is such that

$$\|\mathfrak{d}_{\Phi,p}\|_{\mathcal{C}^{2,\alpha}(\mathcal{A}_1)} \leqslant c\,\epsilon\, r_\epsilon^2 \tag{4.2.30}$$

and it depends continuously on the weighted points $\left((a_{l-1,1},p_{l-1,1}),\cdots,(a_{l-1,n_{l-1}},p_{l-1,n_{l-1}})\right)$.

(ii) If Φ *and* $\overline{\Phi}$ *are smaller than* $\kappa\,\epsilon\, r_\epsilon$*, then*

$$\left\|\mathfrak{d}_{\Phi,p} - \mathfrak{d}_{\overline{\Phi},p}\right\|_{\mathcal{C}^{2,\alpha}(\mathcal{A}_1)} \leqslant c\,\epsilon\, r_\epsilon^{n-2}\left\|\Phi - \overline{\Phi}\right\|_{\mathcal{C}^{2,\alpha}(\partial R^n_\epsilon)}; \tag{4.2.31}$$

it follows that the map $\Phi \mapsto \mathfrak{d}_{\Phi,p}$ *is a contraction for* ϵ *small enough.*

Remark 4.2.2 — As a matter of fact, inequality (4.2.30) will be useful to describe the behaviour of the minimal hypersurface near one of its boundaries $p + \partial B_{r_\epsilon}$ and thus for the gluing procedure. The second inequality (4.2.31) is established because we have in mind to apply an other fixed point theorem in the gluing process.

Proof

(i) First, note that

$$\begin{aligned}&\|u_\Phi - \epsilon\Gamma(r_\epsilon\cdot) - W^e(\Phi_p)\|_{\mathcal{C}^{2,\alpha}(\mathcal{A}_1)}\\ &\qquad\leqslant \|h_\Phi - W^e(\Phi_p)\|_{\mathcal{C}^{2,\alpha}(\mathcal{A}_1)} + \|v_\Phi(r_\epsilon\cdot)\|_{\mathcal{C}^{2,\alpha}(\mathcal{A}_1)}.\end{aligned}$$

Using the proposition 3.2.3 for the harmonic exterior extension part together with the existence theorem 4.1.2 for the v_Φ part, one then checks that

$$\begin{aligned}&\|u_\Phi - \epsilon\Gamma(r_\epsilon\cdot) - W^e(\Phi_p)\|_{\mathcal{C}^{2,\alpha}(\mathcal{A}_1)}\\ &\qquad\leqslant c r_\epsilon^{n-2}\|\Phi\|_{\mathcal{C}^{2,\alpha}\left(\partial\mathbb{R}^n_{l,\epsilon}\right)} + 2c\epsilon r_\epsilon^{n-\mu} r_\epsilon^{\mu},\end{aligned}$$

and inequality (4.2.29) holds for ϵ small enough.

Concerning the difference function $\mathfrak{d}_{\Phi,p_0}$, it directly follows from the expansion of Green's function Γ in (2.1.5).

(ii) Notice that

$$\begin{aligned}\left\|\mathfrak{d}_{\Phi,p_0} - \mathfrak{d}_{\overline{\Phi},P_0}\right\|_{\mathcal{C}^{2,\alpha}(\mathcal{A}_1)} &\leqslant \|v_\Phi(r_\epsilon\cdot) - v_{\overline{\Phi}}(r_\epsilon\cdot)\|_{\mathcal{C}^{2,\alpha}(\mathcal{A}_1)}\\ &\quad+\left\|h_\Phi(r_\epsilon\cdot) - W^e(\Phi_p) - \left(h_{\overline{\Phi}}(r_\epsilon\cdot) - W^e\left(\overline{\Phi}_p\right)\right)\right\|_{\mathcal{C}^{2,\alpha}(\mathcal{A}_1)}.\end{aligned}$$

- To deal with the first term, we use linearity of the exterior harmonic extension operator together with the result of proposition 3.2.3 to obtain

$$\begin{aligned}&\left\|h_\Phi(r_\epsilon\cdot) - W^e(\Phi_p) - \left(h_{\overline{\Phi}}(r_\epsilon\cdot) - W^e\left(\overline{\Phi}_p\right)\right)\right\|_{\mathcal{C}^{2,\alpha}(\mathcal{A}_1)}\\ &\qquad\leqslant c\cdot r_\epsilon^{n-2}\left\|\Phi - \overline{\Phi}\right\|_{\mathcal{C}^{2,\alpha}\left(\partial\mathbb{R}^n_{l,\epsilon}\right)}.\end{aligned}$$

- To deal with the other term, we use similar tools to the proof of proposition 4.1.2. By construction, the difference $v_\Phi - v_{\overline{\Phi}}$ satisfies the PDE
$$\Delta\left(v_\Phi - v_{\overline{\Phi}}\right) \quad = \quad \mathcal{G}\left(\omega_\Phi + v_\Phi\right) - \mathcal{G}\left(\omega_{\overline{\Phi}} + v_{\overline{\Phi}}\right),$$
thus, if we use the same method than for the estimate of $\left\|\mathcal{G}\left(\omega_\Phi\right) - \mathcal{G}\left(\Gamma_{\mathrm{cor},\epsilon}\right)\right\|$ and the continuity of the linear operator $\Delta^{-1}_{\mu,\nu,\epsilon}$ we end up with
$$\begin{aligned}&\left\|v_\Phi - v_{\overline{\Phi}}\right\|_{\mathcal{C}^{2,\alpha}_{\mu,\nu}\left(\mathbb{R}^n_{l,\epsilon}\right)} \\ &\qquad\leqslant \quad c\, r_\epsilon^{n-1}\left[\left\|h_\Phi - h_{\overline{\Phi}}\right\|_{\mathcal{C}^{2,\alpha}_{\mu,\nu}\left(\mathbb{R}^n_{l,\epsilon}\right)} + \left\|v_\Phi - v_{\overline{\Phi}}\right\|_{\mathcal{C}^{2,\alpha}_{\mu,\nu}\left(\mathbb{R}^n_{l,\epsilon}\right)}\right],\end{aligned}$$
from what we deduce, with the help of proposition 3.2.3, that
$$\left(1 - c\, r_\epsilon^{n-1}\right)\left\|v_\Phi - v_{\overline{\Phi}}\right\|_{\mathcal{C}^{0,\alpha}_{\mu,\nu}\left(\mathbb{R}^n_{l,\epsilon}\right)} \quad \leqslant \quad c\, r_\epsilon^{n-1-\mu}\left\|\Phi - \overline{\Phi}\right\|_{\mathcal{C}^{2,\alpha}\left(\partial\mathbb{R}^n_{l,\epsilon}\right)},$$
where c is a universal constant that does not depend on the parameter κ. Consequently, for ϵ small enough, since $c r_\epsilon^{n-1} << 1$,
$$\left\|v_\Phi - v_{\overline{\Phi}}\right\|_{\mathcal{C}^{2,\alpha}_{\mu,\nu}\left(\mathbb{R}^n_{l,\epsilon}\right)} \leqslant c\, r_\epsilon^{n-1-\mu}\left\|\Phi - \overline{\Phi}\right\|_{\mathcal{C}^{2,\alpha}\left(\partial\mathbb{R}^n_{l,\epsilon}\right)},$$
and the conclusion holds. □

5 The n-catenoid

5.1 Some properties of the n-catenoid

The catenoid is a well known minimal surface in $\mathbb{R}^3$: it is the minimal surface with rotational invariance whose boundary is given by two parallel circles. There exists a generalization of such a surface in higher dimension : it is what we call the *n-catenoid*. Since we need different explicit formula for our study, we briefly recall the construction of this minimal hypersurface in $\mathbb{R}^{n+1}$.

We are looking for a hypersurface of revolution around the $\mathbf{e_{n+1}}$-axis — the "vertical" one — thus we choose a parametrization whose form is the following :
$$X : \begin{array}{rcl} \mathbb{S}^{n-1}\times\mathbb{R} & \longrightarrow & \mathbb{R}^{n+1} \\ (z,s) & \longmapsto & \left(\varphi(s)z, \psi(s)\right), \end{array}$$
where φ and ψ are unknown. The function φ corresponds to the radius while the function ψ corresponds to the height. Then the metric g_c of such a hypersurface is given by :
$$g_c = \left(\dot\varphi^2 + \dot\psi^2\right)\mathrm{d}s^2 + \varphi^2 g_{\mathbb{S}^{n-1}}.$$
We enforce the equality $\dot\varphi^2 + \dot\psi^2 = \varphi^2$ in order to ensure a conform parametrization $g_c = \varphi^2\left(\mathrm{d}s^2 + g_{\mathbb{S}^{n-1}}\right)$. Therefore, the second fundamental form and the mean curvature satisfy the relations
$$\mathrm{II}_c = \frac{\dot\varphi\ddot\psi - \ddot\varphi\dot\psi}{\varphi}\mathrm{d}s^2 + \dot\psi g_{\mathbb{S}^{n-1}} \qquad \text{and} \qquad H_c = \frac{\dot\varphi\ddot\psi - \ddot\varphi\dot\psi}{\varphi^3} + (n-1)\,\frac{\dot\psi}{\varphi^2}.$$

Thus, if we are looking a solution of $H_c = 0$ for which we enforce $\dot{\psi}$ to be a polynomial expression in φ, we find that the couple of functions (φ, ψ) has to be a solution to the ODE system given by

$$\left\{ \begin{array}{rcl} \dot{\psi} & = & \varphi^{2-n}; \\ \dot{\varphi}^2 + \varphi^{4-2n} & = & \varphi^2. \end{array} \right. \tag{5.1.32}$$

If we want the size of the neck to be 1, an explicit solution is chosen so that

$$\varphi(s) = \cosh\left((n-1)\, s\right)^{\frac{1}{n-1}} \qquad \text{and} \qquad \psi(s) = \int_0^s \varphi^{2-n}(t)\mathrm{d}t.$$

It will be convenient to use different formula to study deformations of the n-catenoid. We collect them into the next lemma.

Lemma 5.1.1 — *The unit normal is chosen to be*

$$N_c = \left(-\varphi^{1-n} z, \dot{\varphi}/\varphi\right);$$

the second fundamental form is

$$II_c = \varphi^{2-n}\left((1-n)\,\mathrm{d}s^2 + g_{\mathbb{S}^{n-1}}\right);$$

the radius function is a solution to the ordinary differential equation

$$\ddot{\varphi} = \varphi\left(1 + (n-2)\,\varphi^{2-2n}\right). \tag{5.1.33}$$

Besides, Taylor expansions provide us the behaviour of the radius even function φ near infinity

$$\varphi\,(s) = 2^{-\frac{1}{n-1}}\, e^s + \frac{2^{-\frac{1}{n-1}}}{n-1}\, e^{(3-2n)s} + \underset{s\to+\infty}{\mathcal{O}}\left(e^{(5-4n)s}\right) \tag{5.1.34}$$

and the height odd function ψ

$$\psi\,(s) = \frac{H}{2} - \frac{2^{\frac{n-2}{n-1}}}{n-2}\, e^{(2-n)s} + \frac{2^{\frac{n-2}{n-1}}\,(n-2)}{(n-1)\,(3n-4)}\, e^{(4-3n)s} + \underset{s\to+\infty}{\mathcal{O}}\left(e^{(6-5n)s}\right), \tag{5.1.35}$$

where $\frac{H}{2}$ denotes the half height of the n-catenoid, that is to say

$$H = \int_{-\infty}^{+\infty} \varphi^{2-n}(t)\mathrm{d}t = -2\sqrt{\pi}\frac{\Gamma\left(\frac{n-2}{2(n-1)}\right)}{\Gamma\left(-\frac{1}{2(n-1)}\right)}.$$

Remark 5.1.2 — The unit normal is almost vertical for large $|s|$; it points up when s tends to $+\infty$ and points down when s tends to $-\infty$. Moreover, unlike the case $n = 2$ where catenoids have infinite height, catenoids in greater dimension have finite height H and admits two horizontal asymptotic hyperplanes.

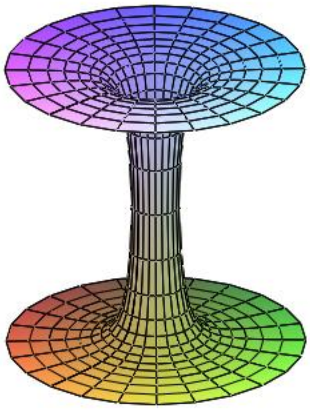


Figure II.1: The catenoid in $\mathbb{R}^n \times \mathbb{R}$ has two asymptotic hyperplanar ends.

PROOF
Here, we only give the proof for the half height of catenoid. For convenient purpose, we consider the catenoid of $\mathbb{R}^{n+1} \times \mathbb{R}$. First of all, integrating by substitution $u = \cosh(nt)^{-1}$, we find

$$\int_0^{+\infty} (\cosh(nt))^{\frac{1-n}{n}}\, \mathrm{d}t \;=\; \frac{1}{n}\int_0^1 u^{-\frac{n+1}{n}} u \left(1-u^2\right)^{-\frac{1}{2}} \mathrm{d}u,$$

then integrating by substitution $v = \sqrt{1-u^2}$,

$$\int_0^{+\infty} (\cosh(nt))^{\frac{1-n}{n}}\, \mathrm{d}t = \frac{1}{n}\int_0^1 \left(1-v^2\right)^{-\frac{n+1}{2n}} \mathrm{d}v.$$

If y is defined to be $y = \frac{1}{2} + \frac{v}{2}$, then

$$\int_0^{+\infty} (\cosh(nt))^{\frac{1-n}{n}}\, \mathrm{d}t = \frac{2^{-\frac{1+n}{n}}}{n}\int_0^1 (y(1-y))^{-\frac{n+1}{2n}} \mathrm{d}y = \frac{2^{-\frac{1+n}{n}}}{n} B\left(\tfrac{n-1}{2n}, \tfrac{n-1}{2n}\right),$$

where B denotes the Bêta function. Therefore,

$$\int_0^{+\infty} (\cosh(nt))^{\frac{1-n}{n}}\, \mathrm{d}t = \frac{2^{-\frac{1+n}{n}}}{n} \frac{\Gamma\left(\frac{n-1}{2n}\right)^2}{\Gamma\left(\frac{n-1}{n}\right)}.$$

The result directly follows from the application of the duplication Legendre formula together with the analytic continuation formula

$$\frac{2^{\frac{1}{n}}\sqrt{\pi}}{\Gamma\left(\frac{n-1}{2n}+\frac{1}{2}\right)} = \frac{\Gamma\left(\frac{n-1}{2n}\right)}{\Gamma\left(\frac{n-1}{n}\right)} \qquad \text{and} \qquad \Gamma\left(-\frac{1}{2n}\right) = -2n\Gamma\left(\frac{n-1}{2n}+\frac{1}{2}\right). \qquad \square$$

5.2 Local description of a truncated n-catenoid near its boundaries

We have in mind to glue a truncated catenoid with the graph of perturbed Green functions over hyperplanes. The catenoid has two ends, namely the horizontal hyperplanes $\left\{x_{n+1} = \pm\frac{H}{2}\right\}$. It is useful to write the parametrization of this minimal hypersurface in a neighbourhood of those hyperplanes as a vertical graph, which amounts to consider the height function over some annulus $B\left(0, x_\epsilon\right) \setminus B\left(0, x_\epsilon/2\right)$ where x_ϵ is large when ϵ is small. Let us make the change of variables defined by $x := \varphi\left(s\right) z$. By Taylor expansion of the radius function (5.1.34), calculus leads us to the equality

$$e^s = 2^{\frac{1}{n-1}} \left|x\right| \left(1 - \frac{1}{4\left(n-1\right)} \left|x\right|^{2-2n} + \mathcal{O}\left(\left|x\right|^{4-4n}\right)\right),$$

which could be written

$$s = \ln\left(2^{\frac{1}{n-1}} \left|x\right|\right) - \frac{1}{4\left(n-1\right)} \left|x\right|^{2-2n} + \mathcal{O}\left(\left|x\right|^{4-4n}\right)$$

as well. Injecting this relation into the Taylor expansion of the height function (5.1.35), we find

$$\psi\left(\varphi^{-1}\left(\left|x\right|\right)\right) = \frac{H}{2} - \frac{1}{n-2} \left|x\right|^{2-n} - \frac{1}{2\left(3n-4\right)} \left|x\right|^{4-3n} + \mathcal{O}\left(\left|x\right|^{6-5n}\right). \qquad (5.2.36)$$

for the upper part of the catenoid. As regards the lower part, it is enough to multiply this equality by a factor -1.

Remark 5.2.1 — The first non constant term corresponds to Green's function whose singularity is at the origin. This highlights once again the essential role that Green function plays in the theory of deformation of minimal hypersurfaces. Furthermore, we could have computed the second term with the help of the minimal hypersurface graph equation in $\mathbb{R}^n$: it exactly matches the correcting term Cor we have introduced for Γ. More accurately, one easily checks that

$$\begin{aligned}\Delta_{\mathbb{R}^2}\left(\frac{1}{2\left(3n-4\right)} \left|\cdot\right|^{4-3n}\right) &= -\left(n-1\right)\left|x\right|^{2-3n} \\ &= \nabla^2\left(-\frac{1}{n-2}\left|\cdot\right|^{2-n}\right)\left(\nabla\left(-\frac{1}{n-2}\left|\cdot\right|^{2-n}\right), \nabla\left(-\frac{1}{n-2}\left|\cdot\right|^{2-n}\right)\right).\end{aligned}$$

5.3 Rescaling of the n-catenoid

The upper part of the catenoid can be parametrised by

$$x \in \mathbb{R}^n \setminus B_1 \longmapsto \left(x, u\left(x\right)\right),$$

where $u\left(x\right) = \psi\left(s\right) = \psi \circ \varphi^{-1}\left(\left|x\right|\right)$; remark that the choice of B_1 is done in such a way that $\min_{\mathbb{R}} \varphi = 1$. Besides, the behaviour of u when s is large (or $\left|x\right|$) is given by (5.2.36).

Let $\eta > 0$ be a small dilation factor ; we rescale the n-catenoid by η to find following parametrization with $y := \eta x$:

$$y \longmapsto \left(y, \eta \frac{H}{2} - \frac{1}{n-2} \eta^{n-1} |y|^{2-n} - \frac{1}{2(3n-4)} \eta^{3(n-1)} |y|^{4-3n} + \underset{y \to 0}{\mathring{\mathcal{O}}} \left(\eta^{5(n-1)} |y|^{6-5n} \right) \right).$$

Since we have in mind to glue a catenoid with the minimal surface we have constructed for the hyperplane-case, at the point $p_{k,j}$, we choose η to be

$$\eta_{k,j} \quad := \quad (n-2)^{\frac{1}{n-1}} a_{k,j}^{\frac{1}{n-1}} \epsilon^{\frac{1}{n-1}}. \tag{5.3.37}$$

It is essential for a good understanding of the situation to notice that for an arbitrary choice of parameters $a_{k,j}$, the catenoid $C_{p_{k,j}}$ has — *a priori* — a different height from the other catenoids $C_{P_{k',j'}}$. However, we will see in gluing process that for fixed k, the $(a_{k,j})_{j \in [\![1, n_k]\!]}$ are almost equal ; in other words, the catenoids we glue between levels k and $k+1$ have almost the same height.

Remark 5.3.1 — For such a choice of η, the coefficients in front of $|x|^{2-n}$ are the same for the catenoid and for Green's function $\epsilon\Gamma$. Besides, if we want to enforce the coefficient $|x|^{4-3n}$ to have the same rough estimate than the term $\langle \cdot, F \rangle$ of $\Gamma_{\mathrm{cor},\epsilon}$, we find $\epsilon^3 r_\epsilon^{4-3n} \approx \epsilon r_\epsilon$: this is why we define the radius r_ϵ to be equal to $\epsilon^{\frac{2}{3(n-1)}}$.

In terms of r_ϵ, we have following relation :

$$r_\epsilon \quad = \quad ((n-2)\, a_{k,j})^{-\frac{2}{3(n-1)}} \eta_{k,j}^{\frac{2}{3}}.$$

Therefore, we note that the coefficients of Taylor expansion are given by

$$\begin{array}{lll} -\epsilon a_{k,j} & \text{for} & |x|^{2-n}, \\ -\frac{1}{2(3n-4)} (n-2)^3 a_{k,j}^3 \epsilon^3 & \text{for} & |x|^{4-3n}, \end{array}$$

in other words, they coincide with the expansion of the function $\Gamma_{cor,\epsilon}$ we have introduced in the hyperplane case near $p_{k,j}$ — see the section 2.2.

Since $|x| = \varphi(s)$, it will be convenient to define a large real number $s_{\epsilon,k,j}$ as follows

$$\varphi(s_{\epsilon,k,j}) = \frac{1}{\eta_{k,j}} r_\epsilon, \qquad \text{i.e.} \qquad s_{\epsilon,k,j} := \varphi^{-1}\left(\frac{r_\epsilon}{\eta_{k,j}}\right).$$

To relieve notations, we omit the indices k and j for all sections concerning the catenoid. To clarify the context, it is very useful to have estimates of the different quantities based on ϵ we use in this paper :

$$s_\epsilon \underset{\epsilon \to 0}{\sim} \ln\left(\frac{r_\epsilon}{\eta}\right) \underset{\epsilon \to 0}{\sim} \epsilon^{-\frac{1}{3(n-1)}}.$$

Notice that the equivalence does not depend on the point $p_{k,j}$.

5.4 Some operators on the n-catenoid

5.4 — (a) The Jacobi operator

The Jacobi operator is interpreted as the differential of the mean curvature operator. Like in the hyperplane case in which we have given an accurate description of the Laplacian, we have to develop similar propositions for the catenoid.

By construction, the metric g_c on the n-catenoid can we written as $g_c = \varphi^2 \left(\mathrm{d}s^2 + g_{\mathbb{S}^{n-1}}\right)$. Thus, the Laplace-Beltrami operator is given by

$$\Delta_c \quad = \quad \frac{1}{\varphi^n}\,\partial_s\left(\varphi^{n-2}\partial_s\cdot\right) + \frac{1}{\varphi^2}\,\Delta_{\mathbb{S}^{n-1}}.$$

The Jacobi operator on the n-catenoid and its conjugate. Recall that we can write

$$\mathrm{II}_c \quad = \quad \begin{pmatrix} (1-n)\,\varphi^{2-n} & 0 \\ 0 & \varphi^{2-n} g_{\mathbb{S}^{n-1}} \end{pmatrix},$$

so we obtain the expression of the Jacobi operator $J_c = \Delta_c + |A_c|^2$, where A_c denotes the shape operator :

$$J_c \quad = \quad \frac{1}{\varphi^n}\,\partial_s\left(\varphi^{n-2}\partial_s\cdot\right) + \frac{1}{\varphi^2}\,\Delta_{\mathbb{S}^{n-1}} + n\,(n-1)\,\varphi^{-2n}.$$

The conjugate operator on the n-catenoid. Since the above expression is relatively inconvenient, we rather choose to study the conjugate operator L_c defined by

$$L_c \quad := \quad \varphi^{\frac{n+2}{2}}\,J_c\left(\varphi^{\frac{2-n}{2}}\,\cdot\right).$$

Then a direct computation shows that

$$L_c \quad = \quad \partial_s^2 + \frac{2-n}{2}\,\ddot{\varphi}\,\varphi^{-1} - \frac{n^2-6n+8}{4}\,\dot{\varphi}^2\,\varphi^{-2} + \Delta_{\mathbb{S}^{n-1}} + n\,(n-1)\,\varphi^{2-2n}.$$

Using the differential equation (5.1.32) for φ together with the expression of the second derivative (5.1.33), we end up with

$$L_c \quad = \quad \partial_s^2 + \Delta_{\mathbb{S}^{n-1}} - \left(\frac{n-2}{2}\right)^2 + \frac{n\,(3n-2)}{4}\,\varphi^{2-2n}, \tag{5.4.38}$$

whose formula is relatively simpler than the one for the Jacobi operator. Note that the potential term is bounded and is almost equal to $\left(\frac{n-2}{2}\right)^2$ when s is large. We will use this fact in the study of the Fredholm properties of this operator.

5.4 — (b) Jacobi fields and conjugate Jacobi fields

Definition 5.4.1 — A function f is a Jacobi (resp. conjugate Jacobi) field if $J_c(f)$ (resp. $L_c(f)$) vanishes everywhere.

An efficient way to produce such fields is to consider the space of transformations which leave invariant the mean curvature. These transformations provide elements which belong to the kernel of the Jacobi operator since modify the vanishing mean curvature. For the n-catenoid, there are three classes of transformations, namely the translations, the dilatations and the rotations around the vertical axis $\mathrm{Span}\left(\mathbf{e}_{n+1}\right)$.

Jacobi field associated to dilation. In this case, $Y = X_c$, in other words, the vectorfield associated with dilation is nothing but the position vector. Then we get

$$\langle Y, N_c\rangle = -\varphi^{2-n} + \frac{\psi \cdot \dot{\varphi}}{\varphi}.$$

The conjugate Jacobi field for the conjugate operator L_c is given by :

$$\phi^0_- = -\varphi^{\frac{2-n}{2}} + \varphi^{\frac{n-2}{2}} \ \frac{\dot{\varphi}}{\varphi} \ \psi. \tag{5.4.39}$$

Moreover, ϕ^0_- admits the following Taylor expansion :

$$\phi^0_-(s) = a_0^- e^{\frac{n-2}{2}\, s} + b_0^- e^{-\frac{n-2}{n}\, s} + \mathcal{O}\left(e^{\frac{2-3n}{2}\, s}\right), \tag{5.4.40}$$

with $a_0^- = 2^{-\frac{n-2}{2(n-1)}}\frac{H}{2}$ and $b_0^- = \frac{1}{n-2} - 2^{\frac{n-2}{2(n-1)}}$.

Jacobi field associated with the vertical translation. In this case, $Y = \mathbf{e}_{n+1}$, the Jacobi field is given by $\langle Y, N_c\rangle = \frac{\dot{\varphi}}{\varphi}$ and the conjugate Jacobi field is

$$\phi^0_+ = \varphi^{\frac{n-2}{2}} \ \frac{\dot{\varphi}}{\varphi} \ . \tag{5.4.41}$$

Moreover, ϕ^0_+ admits the following Taylor expansion :

$$\phi^0_+(s) = a_0^+ e^{\frac{n-2}{2}\, s} + b_0^+ e^{-\frac{3n-2}{2}\, s} + \mathcal{O}\left(e^{\frac{2-3n}{2}\, s}\right), \tag{5.4.42}$$

with $a_0^+ = 2^{-\frac{n-2}{2(n-1)}}$ and $b_0^+ = \frac{2-3n}{2(n-1)}\, 2^{-\frac{n-2}{2(n-1)}}$.

Jacobi fields associated with the horizontal translations. Since the $(e_j)_{1\leqslant j \leqslant n}$ span the horizontal translations, it is enough to consider the case $Y = \mathbf{e}_j$ for some $j \in [\![1, n]\!]$. Then the Jacobi field is $\varphi^{1-n}\langle z\ ,\ \mathbf{e}_j\rangle$ and the conjugate Jacobi field is given by

$$\phi^{1,j}_+(s,z) = \phi^1_+(s)\langle z\ ,\ \mathbf{e}_j\rangle \qquad \text{where} \qquad \phi^1_+ = \varphi^{-\frac{n}{2}}. \tag{5.4.43}$$

Moreover, ϕ^1_+ has the following asymptotic behaviour :

$$\phi^1_+(s) = 2^{\frac{n}{2(n-1)}} e^{-\frac{n}{2}s} + \mathcal{O}\left(e^{\frac{4-5n}{2}s}\right).$$

Jacobi fields associated with horizontal rotations. In this case, we chose Y to be such that the Jacobi field is

$$\langle Y, N_c\rangle = \left(\varphi^{1-n}\psi + \dot{\varphi}\right)\langle z\ ,\ \mathbf{e}_j\rangle,$$

and the conjugate Jacobi field is given by

$$\phi^{1,j}_-(s,z) = \phi^1_-(s)\langle z\,, \mathbf{e}_j\rangle \qquad \text{where} \qquad \phi^1_- = \varphi^{-\frac{n}{2}}\psi + \varphi^{\frac{n}{2}}\frac{\dot{\varphi}}{\varphi}.$$

Furthermore, ϕ_-^1 admits the following asymptotic behaviour :

$$\phi_-^1(s) \quad = \quad a_1^- e^{\frac{n}{2}s} + b_1^- e^{-\frac{n}{2}s} + \mathcal{O}\left(e^{\frac{4-3n}{2}s}\right), \tag{5.4.44}$$

where $a_1^- = 2^{-\frac{n}{2(n-1)}}$ and $b_1^- = 2^{\frac{n}{2(n-1)}}\frac{H}{2}$.

Remark 5.4.2 — $\phi_+^{1,j}$ is an even function while $\phi_-^{1,j}$ is an odd one. Besides, ϕ_+^1 exponentially decreases when $|s|$ tends to ∞ while ϕ_-^1 exponentially increases. We'll see it is a central point for deforming a catenoid : we won't be able to deform it anyhow since we'll have to enforce some conditions about the data boundary Ψ — see the remark 6.3.7.

6 Fredholm properties of the Laplace Beltrami and the Jacobi operators

6.1 Indicial roots

The indicial roots of a second order elliptic operator play an essential role in studying its mapping properties. They provide a relatively simple method to check injectivity, surjectivity, together with asymptotic behaviour of functions. The reader could find more details in the lectures [Pac09].

Definition 6.1.1 — If $\mathcal{L}$ is an elliptic operator on a cylinder $M \times \mathbb{R}$, we say that a real number δ is an *indicial root* of $\mathcal{L}$ in $+\infty$ if there exists a a $\mathcal{C}^2$-function v on $M \times \mathbb{R}$ and a real number δ' such that following assertions hold :

(i) $\delta' < \delta$;

(ii) $\underline{\lim}_{s\to+\infty} \|v\|_{L^\infty(M\times\{s\})} > 0$;

(iii) $e^{-\delta' s}\mathcal{L}\left(e^{\delta s}v\right) \underset{s\to+\infty}{\longrightarrow} 0$.

6.1 — (a) Decomposition on eigenspaces associated with the sphere $\mathbb{S}^{n-1}$.

We use here the tools of Fourier analysis we have given in section 3.2 — (a).

Let us write, for all function $w \in \mathcal{C}^{2,\alpha}(\mathbb{S}^{n-1}\times\mathbb{R})$ the formula $w = \sum_{j\in\mathbb{N}} w^j$, where for all $s \in \mathbb{R}$, the function $w^j(s,\cdot)$ belongs to the eigenspace E^j. It is the same to write $w^j = \pi^j(w)$, where the projection π^j is defined in 3.2.1. More precisely, if $\left(\varepsilon^{j,i}\right)_i$ denotes an orthonormal basis of E^j, there exists $w^{j,i} : \mathbb{R} \longrightarrow \mathbb{R}$ such that for all s and for all z,

$$w(s,z) \quad = \quad \sum_j \left(\sum_i w^{j,i}(s)\,\varepsilon^{j,i}(z)\right).$$

Then one checks that $L_c(w)$ vanishes if, and only if, for all $j \in \mathbb{N}$, for all $i \in [\![1, \dim E_j]\!]$, $L_j\left(w^{j,i}\right) = 0$ where

$$L_j \quad := \quad \partial_s^2 + \frac{n(3n-2)}{4}\varphi^{2-2n} - \left(\frac{n-2}{2} + j\right)^2$$

defines an ordinary differential equation.

Remark 6.1.2 — • We recall that if P is a homogeneous harmonic polynomial of degree j defined in $\mathbb{R}^n$, then its restriction to the sphere $\mathbb{S}^{n-1}$ is an element of the eigenspace E^j. Thus constant functions belong to E^0 and functions of form $\langle z, \mathbf{e}_j \rangle$ are in E^1. Therefore, the jacobi fields associated with dilation and vertical translation $\phi_0^{\pm}$ satisfy $L_0\left(\phi_0^{\pm}\right) = 0$ and the jacobi fields $\phi_{1,j}^{\pm}$ associated with horizontal translations or rotations satisfy $L_1\left(\phi_{1,j}^{\pm}\right) = 0$. In particular, the kernels of L_0 and L_1 are non empty.

- Moreover, $\phi_{\pm}^i$ (with $i \in \{0,1\}$) satisfy an ODE whose form is given by $y''+by = 0$ and we notice that the coefficient in front of y' vanishes. Consequently, the wronskian w^i of $\left(\phi_-^i, \phi_+^i\right)$ is constant. More precisely, $w^0 = 1-n$ and $w^1 = -n$. For practical purpose, we multiply $\phi_{\pm}^i$ by real constants $c_{\pm}^i$ such that the wronskian of the family $\left(c_+^i \phi_+^i, c_-^i \phi_-^i\right)$ is equal to 1.

6.1 — (b) Indicial roots of the operator L_j

Let us note $\delta_j := \frac{n-2}{2} + j$. We easily check that

$$L_j\left(e^{\pm\delta_j s}\right) = \frac{n(3n-2)}{4}\varphi^{2-2n}(s)\, e^{\pm\delta_j s}.$$

But the term $\varphi^{2-2n}(s)$ exponentially tends to 0 when $|s|$ tends to infinity. Indeed, the contribution of the term $\frac{n(3n-2)}{4}\varphi^{2-2n}(s)$ in the operator L_j (or L_c) near infinity can be "neglected". More precisely, we the following lemma.

Lemma 6.1.3 — *The indicial roots of L_j are $\pm\delta_j$.*

Corollary 6.1.4 — *The indicial roots of L_c are the same than the operator $\partial_s^2 + \Delta_{\mathbb{S}^{n-1}} - \left(\frac{n-2}{2}\right)^2$, namely the $\pm\delta_j$ for $j \in \mathbb{N}$.*

Proof (Of lemma 6.1.3)
Note that

$$e^{-\delta' s} L_j\left(e^{\pm\delta_j s}\right) \underset{s\to\infty}{\sim} \frac{n(3n-2)}{4} e^{(2-2n\pm\delta_j-\delta')s} \underset{s\to\infty}{\longrightarrow} 0$$

provided $\delta' \in \left(\pm\delta_j + (2-2n), \pm\delta_j\right)$, which is possible since $2-2n<0$. Therefore $\pm\delta_j$ is an indicial root of L_j.

It remains to prove that $\pm\delta_j$ are the only ones. By [Pac09, 5.2.1], the indicial roots of $\partial_s^2 - \delta_j^2$ are $\pm\delta_j$. Therefore, if δ is an indicial root of L_j and $\delta' < \delta$ together with v are the associated real number and function with δ (see definition 6.1.1), then

$$\begin{aligned} e^{-\delta' s}\left(\partial_{ss}^2 - \delta_j^2\right)\left(e^{\delta s} v\right) &= e^{-\delta' s} L_j\left(e^{\delta s} v\right) + \frac{n(3n-2)}{4}\varphi^{2-2n}(s)\, e^{(\delta-\delta')s} v \\ &\underset{s\to\infty}{\longrightarrow} 0, \end{aligned}$$

from what we conclude that δ is an indicial root of $\partial_s^2 - \delta_j^2$ and conclusion holds. □

6.1 — (c) Injectivity of L_c

Proposition 6.1.5 — *Let $\delta < -\frac{n}{2}$. Suppose there exists some function w and some constant $c > 0$ such that*

$$\left\{ \begin{array}{rcl} L_c(w) & = & 0\,; \\ \forall z \in \mathbb{S}^{n-1}, |w(s,z)| & \leqslant & c\,(\cosh s)^{\delta}\,. \end{array} \right.$$

Then $w = 0$.

Proof

We write $w(s,z) = \sum_{j,i} w^{j,i}(s)\,\varepsilon^{j,i}(z)$. Then, according to paragraph 6.1 — (a), we get $L_j(w^{j,i}) = 0$. Besides, for all s, $w^{j,i}(s)$ is the orthogonal projection of $w(s,\cdot)$ with respect to the space $\mathrm{Span}(\varepsilon^{j,i})$, thus $|w^{j,i}|(s) \leqslant \|w(s,\cdot)\|_{L^2(\mathbb{S}^{n-1})}$; therefore, for all $s \in \mathbb{R}$,

$$\left|w^{j,i}\right|(s) \quad \leqslant \quad \widetilde{c}\cdot(\cosh s)^{\delta}\,.$$

We claim that $w^{j,i} \equiv 0$.

The case $j = 0, 1$: it is enough to remark that the conjugate Jacobi fields we have described in 5.4 — (b) span all solutions of $L_0(w) = 0$ and $L_1(w) = 0$. More precisely, by Cauchy-Lipschitz theory, the dimension of the solutions to the ordinary differential equation $L_j w = 0$ is 2. Moreover, E_0 has dimension 1 and E_1 has dimension n. Since $\left(\phi^0_-, \phi^0_+\right)$ is a linearly independent family, it spans $\ker L_0$, just as the family $(\phi^{\pm}_{1,j})_{j\in[\![1,n]\!]}$ span $\ker L_1$. But none of these elements decrease as quickly than $(\cosh s)^{\delta}$. Thus $w^{j,i} \equiv 0$.

The case $j \geqslant 2$: the idea is to apply a maximum principle. Indeed, we compare the function $w^{j,i}$ with one of the conjugate Jacobi fields we have studied previously.

Since $L_j(w^{j,i}) = 0$ and $\phi^+_{1,1}$ belongs to $\ker(L_1)$, we use the two differential equations they satisfy to define

$$w_t : s \longmapsto \varphi^{-\frac{n}{2}}(s) - t\,w^{j,i}(s)\,,$$

where t denotes a real number. Then w_t satisfies

$$L_j(w_t) \quad = \quad -(j-1)(n-1+j)\,\varphi^{-\frac{n}{2}} \quad < \quad 0 \tag{6.1.45}$$

because $j > 1$.

Moreover, we know the asymptotic behaviour of φ (see (5.1.34)) and $w^{j,i}$ near infinity :

$$w^{j,i}(s) \;=\; \underset{|s|\to\infty}{\mathcal{O}}\left(\cosh(s)^{\delta}\right) \qquad \left(= \underset{|s|\to\infty}{o}\left(\varphi^{-\frac{n}{2}}(s)\right)\right).$$

It follows that w_t is non negative near infinity.

Reductio ad absurdum, suppose $w^{j,i}$ does not vanish everywhere. We then choose t in such a way that w_t is positive on $\mathbb{R}$ and vanishes in at least one point s_0. Therefore w_t reaches its minimum 0 in s_0 and $\ddot{w}_t$ must be positive : it contradicts the inequality (6.1.45). □

One proves with similar method the following proposition.

Proposition 6.1.6 — *Let $s_1 < s_2$. Suppose there exists some bounded function $w : [s_1, s_2] \times \mathbb{S}^{n-1} \longrightarrow \mathbb{R}$ such that*

$$\left\{\begin{array}{rcl} & L_c(w) & = & 0\,; \\ \forall s \in (s_1, s_2)\,, & w(s,\cdot) & \in & E_{0,1}^{\perp}\,; \\ \forall j \in \{1,2\}\,, & w(s_j,\cdot) & = & 0. \end{array}\right.$$

Then $w = 0$.

6.2 Analysis in weighted spaces

We have seen in the proposition 6.1.5 that when we enforce a certain behaviour of a function near infinity, we can give some properties about the action of operator L_c on such a function. This is why we consider our problem in well chosen function spaces : the weighted spaces. The reason is quite similar to the hyperplane case.

Definition 6.2.1 — Let δ be a real number.

- Let $p \in [1, +\infty]$. We define the *weighted space* L^p_δ on $\mathbb{R} \times \mathbb{S}^{n-1}$ as follows :

$$L^p_\delta\left(\mathbb{R} \times \mathbb{S}^{n-1}\right) \quad := \quad (\cosh s)^{\delta}\, L^p\left(\mathbb{R} \times \mathbb{S}^{n-1}\right),$$

  endowed with the norm $\|\cdot\|_{L^p_\delta(\mathbb{R}\times\mathbb{S}^{n-1})}$ defined by

$$\|f\|_{L^p_\delta(\mathbb{R}\times\mathbb{S}^{n-1})} \quad := \quad \left\|(\cosh s)^{-\delta} f\right\|_{L^p(\mathbb{R}\times\mathbb{S}^{n-1})}.$$

- Let $k \in \mathbb{N}$ and $\alpha \in (0,1)$. We define the *weighted space* $\mathcal{C}^{k,\alpha}_\delta$ on $\mathbb{R} \times \mathbb{S}^{n-1}$ as follows :

$$\mathcal{C}^{k,\alpha}_\delta\left(\mathbb{R} \times \mathbb{S}^{n-1}\right) \quad := \quad (\cosh s)^{\delta}\, \mathcal{C}^{k,\alpha}\left(\mathbb{R} \times \mathbb{S}^{n-1}\right),$$

  endowed with the norm $\|\cdot\|_{L^p_\delta(\mathbb{R}\times\mathbb{S}^{n-1})}$ defined by

$$\|f\|_{\mathcal{C}^{k,\alpha}_\delta(\mathbb{R}\times\mathbb{S}^{n-1})} \quad := \quad \left\|(\cosh s)^{-\delta} f\right\|_{\mathcal{C}^{k,\alpha}(L^p(\mathbb{R}\times\mathbb{S}^{n-1}))}.$$

Heuristically, a function f which belongs to some weighted space is bounded by a constant times $(\cosh s)^{\delta}$: it prevents functions from exploding too fast. Differential operators on weighted functions have a kernel and an image that depend on the choice of the weight parameter δ.

Remark 6.2.2 — One can define in the same way the weighted spaces $L^p_\delta\left(I \times \mathbb{S}^{n-1}\right)$ (resp. $\mathcal{C}^{k,\alpha}_\delta\left(I \times \mathbb{S}^{n-1}\right)$) where I is an interval of $\mathbb{R}$.

Remark 6.2.3 — Unlike the case of weighted spaces on hyperplanes, for all $k \geqslant 2$, the operator L_c satisfies

$$L_c : \mathcal{C}_\delta^{k,\alpha}\left(\mathbb{R} \times \mathbb{S}^{n-1}\right) \longrightarrow \mathcal{C}_\delta^{k-2,\alpha}\left(\mathbb{R} \times \mathbb{S}^{n-1}\right),$$

and the weight parameter δ does not change. Mainly, it can be explained by the fact that the derivative of an exponential $e^{\delta s}$ is also $e^{\delta s}$ while the derivative of r^δ is $r^{\delta-1}$.

According to the proposition 6.1.5, we obtain Fredholm properties for L_c on weighted spaces.

Proposition 6.2.4 — *Let* $\delta < -\frac{n}{2}$. *Then* $L_c : \mathcal{C}_\delta^{2,\alpha}\left(\mathbb{R} \times \mathbb{S}^{n-1}\right) \longrightarrow \mathcal{C}_\delta^{0,\alpha}\left(\mathbb{R} \times \mathbb{S}^{n-1}\right)$ *is an injective operator.*

Previous proposition is an injectivity property, but we do not know yet if there is surjectivity or isomorphism. It is the object of the following proposition.

Proposition 6.2.5 — *Let* $\delta \in \left(\frac{n}{2}, \frac{n+2}{2}\right)$. *Then there exists a constant* $c = c(\delta, \alpha, n)$ *such that for all* $f \in \mathcal{C}_\delta^{0,\alpha}\left(\mathbb{R} \times \mathbb{S}^{n-1}\right) \cap \mathcal{E}^\perp\left(\mathbb{R} \times \mathbb{S}^{n-1}\right)$ *(recall* $\mathcal{E}$ *is defined in notation 3.2.1), there exists one and only one function* v *such that :*

$$\left\{\begin{array}{rcl} v & \in & \mathcal{C}_\delta^{2,\alpha}\left(\mathbb{R} \times \mathbb{S}^{n-1}\right); \\ L_c(v) & = & f \quad \text{over} \quad \mathbb{R} \times \mathbb{S}^{n-1}. \end{array}\right. \tag{6.2.46}$$

Besides, for such a solution, we have the following estimate :

$$\|v\|_{\mathcal{C}_\delta^{2,\alpha}(\mathbb{R}\times\mathbb{S}^{n-1})} \leqslant c\,\|f\|_{\mathcal{C}_\delta^{0,\alpha}(\mathbb{R}\times\mathbb{S}^{n-1})}. \tag{6.2.47}$$

We note $L_{c_\delta,\perp}^{-1} : \mathcal{C}_\delta^{0,\alpha}\left(\mathbb{R} \times \mathbb{S}^{n-1}\right) \cap \mathcal{E}^\perp\left(\mathbb{R} \times \mathbb{S}^{n-1}\right) \longrightarrow \mathcal{C}_\delta^{2,\alpha}\left(\mathbb{R} \times \mathbb{S}^{n-1}\right)$ *the right inverse.*

Proof

Before giving the details of the proof, we briefly explain the main ideas. First, we first study the compact case : we solve the PDE on a compact set $(s_0, s_1) \times \mathbb{S}^{n-1}$ and we give an estimate which depends *a priori* on the choice (s_0, s_1). Next, we prove by contradiction that, *a fortiori*, the estimate does not depend on these parameters. In this purpose, we use the fact that

- a function which vanishes at a point x and takes value 1 at a point y with y close to x has a gradient that explodes ;
- the Arzela-Peano theorem [GT01, theorem 4.6] could be used to study a limit PDE.

Bounded case : We show that we are able to solve the problem over an interval of the form $[s_0, s_1]$, together with a right estimate, namely :
for all $s_0 < s_1$, there exists a solution to

$$\begin{cases} v & \in & \mathcal{C}^{2,\alpha}_{\delta}\left((s_0, s_1) \times \mathbb{S}^{n-1}\right); \\ L_c(v) & = f & \text{over } \ (s_0, s_1) \times \mathbb{S}^{n-1}; \\ v & = 0 & \text{on } \ \{s_0, s_1\} \times \mathbb{S}^{n-1}. \end{cases}$$

Besides, there exists some constant c such that we have the following estimate :

$$\|v\|_{\mathcal{C}^{2,\alpha}_{\delta}((s_0,s_1)\times\mathbb{S}^{n-1})} \quad \leqslant \quad c\,\|f\|_{\mathcal{C}^{0,\alpha}_{\delta}((s_0,s_1)\times\mathbb{S}^{n-1})}.$$

The existence follows from the injectivity of L_c for bounded functions (proposition 6.1.6) together with the compactness of $[s_0, s_1] \times \mathbb{R}$ and the self-adjoint property of the operator L_c.

The constant depends neither on s_0 nor on s_1 : We only prove that it is the case for s_1 and we conclude by symmetry. Reductio ad absurdum, suppose it is not the case, i.e. suppose there exists a sequence $(s_m)_{m\in\mathbb{N}^*}$ of $(s_0, +\infty)$ such that for all $m > 0$, there exists f_m and v_m which satisfy

$$\begin{cases} Lc\,(v_m) & = & f_m; \\ \|f_m\|_{L^{\infty}_{\delta}((s_0,s_m)\times\mathbb{S}^{n-1})} & \xrightarrow[m\to\infty]{} & 0; \\ \|v_m\|_{L^{\infty}_{\delta}((s_0,s_m)\times\mathbb{S}^{n-1})} & = & 1. \end{cases}$$

We then define a sequence $\left((s_m^*, z_m^*)\right)_m$ such that

$$(\cosh s_m^*)^{-\delta}\,|v_m(s_m^*, z_m^*)| \quad = \quad \|v_m\|_{L^{\infty}_{\delta}((s_0,s_m)\times\mathbb{S}^{n-1})} \quad = \quad 1.$$

We then compute a new function v_m^* which is chosen to be

$$v_m^* : \begin{array}{rcl} [s_0 - s_m^*, s_m - s_m^*] \times \mathbb{S}^{n-1} & \longrightarrow & \mathbb{R} \\ (s, z) & \longmapsto & (\cosh s_m^*)^{-\delta}\, v_m(s + s_m^*, z). \end{array}$$

Remark that v_m^* is nothing but a translation and a dilation of v_m such that $|v_m^*|$takes value 1 at $(0, z_m^*)$ and $v_m^*(s_m - s_m^*, \cdot) = 0$.

We claim that $(s_m - s_m^*)_m$ does not tend to 0 : indeed, if it is the case, the gradient of v_m^* would degenerate near $(0, z_m^*)$ and it is impossible. More precisely, note that for all s and z,

$$|v_m(s, z)| \leqslant (\cosh s)^{\delta} \qquad \text{and} \qquad |f_m(s, z)| \leqslant (\cosh s)^{\delta}\,\|f_m\|_{L^{\infty}_{\delta}([s_0,s_m]\times\mathbb{S}^{n-1})}.$$

Moreover, using the PDE that v_m satisfies and after noticing that φ^{2-2n} is a bounded function, simple calculation leads us to

$$\left|\left(\partial_s^2 + \Delta_{\mathbb{S}^{n-1}}\right)(v_m)\right| \quad \leqslant \quad c\ (\cosh s_m)^{\delta},$$

where $c = c(n)$. By [GT01, Theorem 4.11], there exists $c = c(n)$ such that for all s in $[s_m - 1, s_m]$ and for all z in $\mathbb{S}^{n-1}$,

$$\|\nabla v_m(s,z)\| \quad \leqslant \quad c\ (\cosh s_m)^{\delta}.$$

By Taylor's theorem between (s_m, z_m) and (s_m^*, z_m), we then deduce that

$$1 \quad \leqslant \quad c(\cosh s_m)^{\delta} |s_m - s_m^*|,$$

thus

$$|s_m - s_m^*| \quad \geqslant \quad c(\cosh s_m)^{-\delta}$$

and this quantity does not tend to 0 since $s_m > s_0$ and $-\delta > 0$.

Even if it means extracting a subsequence, one can assume that the sequence $(s_m^*)_m$ converges in $\overline{\mathbb{R}}$.
Now, let us write the partial differential equation the function v_m^* satisfies.

$$\begin{aligned}&\left(\partial_s^2 + \Delta_{\mathbb{S}^{n-1}} - \left(\frac{n-2}{2}\right)^2\right) v_m^*(s,z) \\ &\qquad = \quad -\frac{n(3n-2)}{4}\varphi^{2-2n}(s+s_m^*)\, v_m^*(s,z) + f_m(s+s_m,z) \quad (6.2.48)\end{aligned}$$

First case — $s_m \to +\infty$: We prove that we can extract a subsequence of $(v_m^*)_m$ which converges to a solution v_∞^* to the following equation :

$$\left(\partial_s^2 + \Delta_{\mathbb{S}^{n-1}} - \left(\frac{n-2}{2}\right)^2\right) v_\infty^*(s,z) \quad = \quad 0. \qquad (6.2.49)$$

In other words, the contribution of the terms in the second member of (6.2.48) can be neglected when m is large since φ^{2-2n} tends to 0 when s tends to ∞ and f_m converges to 0.
We first assume $s_m - s_m^* \to c_0$. The number c_0 is not equal to 0 by previous claim. If a is a fixed real number, we have a uniform bound for the quantity

$$\|v_m\|_{\mathcal{C}^{2,\alpha}([s_m^*-a, s_m^*+a]\times\mathbb{S}^{n-1})}.$$

By Arzela-Peano theorem, one can extract a subsequence which uniformly converges in $\mathcal{C}^{2,\alpha}\left([s_m^* - a, s_m^* + a]\times\mathbb{S}^{n-1}\right)$ to v_∞^*. Therefore, equation (6.2.49) is satisfied. By an diagonal argument, one checks that it is true on $(-\infty, c_0)\times\mathbb{S}^{n-1}$.
Besides, if we want the boundary data of v_∞^*, a simple argument proves it is 0 :

$$\begin{aligned}&|v_m^*(s_m - s_m^*, \cdot) - v_\infty^*(c_0,\cdot)| \\ &\quad \leqslant \quad |(v_m^* - v_\infty^*)(s_m - s_m^*,\cdot)| + |v_\infty^*(s_m - s_m^*,\cdot) - v_\infty^*(c_0,\cdot)| \\ &\quad \xrightarrow[m\to\infty]{} 0,\end{aligned}$$

and conclusion holds since $v_m^*(s_m - s_m^*, \cdot) = 0$.
Moreover, if we study the norm of v_∞^*, one finds

$$\left|(\cosh s)^{-\delta} v_m^*(s,z)\right| = \left|(\cosh s)^{-\delta} (\cosh s_m^*)^{-\delta} w_m(s + s_m^*, z)\right|,$$

thus

$$\|v_m^*\|_{L_\delta^\infty((-\infty,c_0)\times\mathbb{S}^{n-1})} \underset{m\to\infty}{\sim} c\,\|v_m\|_{L_\delta^\infty((s_0,s_m)\times\mathbb{S}^{n-1})} \leqslant c$$

and v_∞^* belongs to the weighted space $L_\delta^\infty((-\infty,c_0)\times\mathbb{S}^{n-1})$.
But, according to proposition 6.2.4, $v_\infty^* \equiv 0$ and it is a contradiction with $\|v_\infty^*(0,\cdot)\|_{L^\infty(\mathbb{S}^{n-1})} = 1$.
With the help of very similar arguments, one also finds a contradiction in the case $s_m - s_m^* \to +\infty$.

Second case — $s_m \to s_\infty \in \mathbb{R}$ **:** like in the previous case, even if it means extracting a subsequence, one shows that we can assume $(v_m^*)_m$ uniformly converges (on all compact sets) to a solution v_∞^* of the following partial differential equation

$$L_c(v_\infty^*) = 0,$$

and we conclude in the same way as previous one.

Construction of a solution on $\mathbb{R}\times\mathbb{S}^{n-1}$ **:** we conclude in the same way of the study of the hyperplane case. More precisely, one considers a sequence of solutions on $(-m,m)\times\mathbb{S}^{n-1}$ with boundary data 0 and we let $m\to+\infty$. We use the universal constant c to conclude we have the right estimate (6.2.47).

Uniqueness : if v_1 and v_2 are two solutions, then $L_c(v_1 - v_2) = 0$. But $v_1 - v_2$ belongs to $\mathcal{C}_\delta^{2,\alpha}(\mathbb{R}\times\mathbb{S}^{n-1})$ and is orthogonal to the modes 0 and 1. Moreover, recall these modes are deeply linked to the indicial roots $\frac{n-2}{2}$ and $\frac{n}{2}$ while the weight parameter δ is chosen so that $\delta > \frac{n}{2}$. According to [Pac09, proposition 12.4.1], $v_1 - v_2 \equiv 0$. □

Proposition 6.2.6 — *Let* $\delta \in \left(\frac{n}{2}, \frac{n+2}{2}\right)$. *Then there exists a constant* $c = c(\delta,\alpha,n)$ *such that for all* $f \in \mathcal{C}_\delta^{0,\alpha}(\mathbb{R}\times\mathbb{S}^{n-1}) \cap \mathcal{E}^{0,1}(\mathbb{R}\times\mathbb{S}^{n-1})$, *there exists one function* v *such that :*

$$\begin{cases} v \in \mathcal{C}_\delta^{2,\alpha}(\mathbb{R}\times\mathbb{S}^{n-1}); \\ L_c(v) = f \quad \text{over } \mathbb{R}\times\mathbb{S}^{n-1}. \end{cases} \tag{6.2.50}$$

Besides, for such a solution, we have the following estimate :

$$\|v\|_{\mathcal{C}_\delta^{2,\alpha}(\mathbb{R}\times\mathbb{S}^{n-1})} \leqslant c\,\|f\|_{\mathcal{C}_\delta^{0,\alpha}(\mathbb{R}\times\mathbb{S}^{n-1})}. \tag{6.2.51}$$

We note $L_{c_\delta}^{-1} : \mathcal{C}_\delta^{0,\alpha}(\mathbb{R}\times\mathbb{S}^{n-1}) \cap \mathcal{E}^{0,1}(\mathbb{R}\times\mathbb{S}^{n-1}) \longrightarrow \mathcal{C}_\delta^{2,\alpha}(\mathbb{R}\times\mathbb{S}^{n-1})$ *the right inverse.*

PROOF
Let us write $f = f^0 + f^1$ where f^j belongs to $\mathcal{E}^j$ for $j = 0, 1$. By linearity, it is enough to solve the problem with $v = v^0 + v^1$ where $L_j\left(v^j\right) = f^j$.

Explicit construction of solutions : we use the variation of constants to define solutions to $L_j\left(v^j\right) = f^j$ together with the help of the construction of conjugate Jacobi fields that belong to $\ker L_0$ and $\ker L_1$. Recall that the wronskian of $\left(\phi_-^i, \phi_+^i\right)$ with $i \in \{0, 1\}$ is chosen so that it is equal to 1 — see remark 6.1.2. We check that

$$v^0(s) := \phi_+^0(s)\int_0^s \phi_-^0(t)\, f^0(t)\,\mathrm{d}t - \phi_-^0(s)\int_0^s \phi_+^0(t)\, f^0(t)\,\mathrm{d}t$$

and

$$v^1(s,z) := \sum_{i=1}^n \left(\phi_+^{1,i}(s,z)\int_0^s \phi_-^{1,i}(t,z)\, f^{1,i}(t,z)\,\mathrm{d}t - \phi_-^{1,i}(s,z)\int_0^s \phi_+^{1,i}(t,z)\, f^{1,i}(t,z)\,\mathrm{d}t\right)$$

are solutions.

Right weighted space and estimate : we have not proved that our solutions v_j belong to the space $\mathcal{C}_\delta^{2,\alpha}\left(\mathbb{R}\times\mathbb{S}^{n-1}\right)$. By classical results on elliptic partial differential equations, v_j belongs to $\mathcal{C}_{loc}^{2,\alpha}\left(\mathbb{R}\times\mathbb{S}^{n-1}\right)$. It remains to prove the weight part. According to Schauder's estimates, it is enough to demonstrate that v_j belongs to $L_\delta^\infty\left(\mathbb{R}\times\mathbb{S}^{n-1}\right)$.

For example, we deal with the first term of v_0 as follows : we use the definition of the conjugate Jacobi fields together with the estimate

$$\left\|f^0\right\|_{L_\delta^\infty(\mathbb{R}\times\mathbb{S}^{n-1})} \leqslant c\left\|f\right\|_{L_\delta^\infty(\mathbb{R}\times\mathbb{S}^{n-1})}$$

to get

$$\begin{aligned}&\left|\phi_+^0(s,z)\int_0^s \phi_-^0(t,z)\, f^{0,1}(t)\,\mathrm{d}t\right| \\ &\qquad\qquad \leqslant \quad c\left((\cosh s)^\delta + (\cosh s)^{\frac{2-n}{2}}\right)\|f\|_{L_\delta^\infty(\mathbb{R}\times\mathbb{S}^{n-1})},\end{aligned}$$

where $c = c(\delta, n)$. Doing similar computation for the other term, we finally obtain

$$|v_0(s,z)| \quad \leqslant \quad c\,\|f\|_{L_\delta^\infty(\mathbb{R}\times\mathbb{S}^{n-1})}\left((\cosh s)^{\frac{n-2}{2}} + (\cosh s)^{-\frac{n-2}{2}} + (\cosh s)^\delta\right),$$

thus

$$\|v_0\|_{L_\delta^\infty(\mathbb{R}\times\mathbb{S}^{n-1})} \quad \leqslant \quad c\,\|f\|_{L_\delta^\infty(\mathbb{R}\times\mathbb{S}^{n-1})},$$

provided $\delta \geqslant \frac{n-2}{2}$ (and it is the case), where $c = c(\delta, n)$. $\square$

6.3 Harmonic extensions

Like in the case of the study of minimal hypersurfaces over punctured hyperplanes, we make use of the harmonic function theory. As a matter of fact, the previous analysis provides us a way to solve the problem $L_c(v) = f$ but we have not yet prescribed a Dirichlet data. It is the object of this section.

Let us define the *harmonic operator* H_c as follows :

$$H_c \quad := \quad \partial_s^2 + \Delta_{\mathbb{S}^{n-1}} - \left(\frac{n-2}{2}\right)^2.$$

First of all, note that H_c is also equal to $L_c - \frac{n(3n-2)}{4}\varphi^{2-2n}$, thus it is very close to the operator L_c when $|s|$ is large. From a heuristic point of view, the solutions to Dirichlet problems

$$\begin{cases} H_c(h) &= 0 \\ h(\pm s_\epsilon, z) &= \Psi \end{cases} \quad \text{and} \quad \begin{cases} L_c(h) &= 0 \\ h(\pm s_\epsilon, z) &= \Psi \end{cases}$$

are similar and have same kinds of estimate when s_ϵ is sufficiently large for a boundary data Ψ on $\{\pm s_\epsilon\} \times \mathbb{S}^{n-1}$.

The reason for which we have called H_c the harmonic extension is that there is a deep link between H_c on a cylinder $\mathbb{R} \times \mathbb{S}^{n-1}$ and the *interior harmonic extension* W^i on the ball B_1. Indeed, it is the same object in different coordinates.

The interior harmonic extension has the same type than W^e, except it defines an harmonic function on the interior of the ball B_1 with prescribed boundary data while W^e defines an harmonic function on the exterior of the ball $\overline{B}_1$. Like in the proof of proposition 3.2.2, one checks that if Ψ is a function on $\mathbb{S}^{n-1}$, then the explicit formula for $W^i(\Psi)$ is given by

$$\forall t \in B_1, \qquad W^i(\Psi)(t) \quad = \quad \sum_{j \geqslant 0} |t|^j \, \Psi^j \left(\frac{t}{|t|}\right).$$

Moreover, if $h_\Psi : (s,z) \in [0,+\infty) \times \mathbb{S}^{n-1} \longmapsto f_c(s,z) \in \mathbb{R}$ is a solution to the problem

$$\begin{cases} H_c(h_\Psi) &= 0 & \text{in} & [0,+\infty) \times \mathbb{S}^{n-1}, \\ h_\Psi(0,\cdot) &= \Psi(\cdot) & \text{on} & \mathbb{S}^{n-1}, \end{cases} \tag{6.3.52}$$

then calculus shows that for all $t \in B_1 \setminus \{0\}$,

$$W^i(\Psi)(t) \quad = \quad |t|^{\frac{2-n}{2}} h_\Psi \left(-\log|t|, \frac{t}{|t|}\right).$$

Given an arbitrary boundary data Ψ, we decompose each function $\Psi_\pm(\cdot) = \Psi(\pm s_\epsilon, \cdot)$ defined on the sphere $\mathbb{S}^{n-1}$ by using eigenmodes of the Laplacian. We distinguish the modes 2, 3, 4 and so on from the other ones because we will use the conjugate Jacobi fields to study the problem with modes 0 and 1.

6.3 — (a) The modes 2, 3, 4, etc

Proposition 6.3.1 — *There exists some constant $c := c(\alpha, n)$ such that for all $\Psi \in \mathcal{C}^{2,\alpha}(\mathbb{S}^{n-1}) \cap \mathcal{E}^{\perp}(\mathbb{S}^{n-1})$, there exists one and only one function h_Ψ in $\mathcal{C}^{2,\alpha}_{loc}([0,+\infty) \times \mathbb{S}^{n-1})$ which is a solution to the problem (6.3.52). Besides, h_Ψ belongs to $\mathcal{C}^{2,\alpha}_{\frac{n+2}{2}}([0,+\infty) \times \mathbb{S}^{n-1})$ and following estimate holds true :*

$$\|h_\Psi\|_{\mathcal{C}^{2,\alpha}_{\frac{n+2}{2}}([0,+\infty)\times\mathbb{S}^{n-1})} \leqslant c\,\|\Psi\|_{\mathcal{C}^{2,\alpha}(\mathbb{S}^{n-1})}. \tag{6.3.53}$$

Remark 6.3.2 — The weight parameter could be improved in some special cases : if $\Psi \neq 0$ and $j_0 := \min\{j \leqslant 2 : \Psi^j \neq 0\}$, then we have same result with weight parameter δ_{j_0}.

Before giving the proof, we deduce from the above proposition the

Corollary 6.3.3 — *There exists some constant $c := c(\alpha, n)$ such that for all $s_0 \in \mathbb{R}_+$ and $\Psi \in \mathcal{C}^{2,\alpha}(\{\pm s_0\} \times \mathbb{S}^{n-1}) \cap \mathcal{E}^{\perp}$, there exists one function h_Ψ in $\mathcal{C}^{2,\alpha}_{loc}([-s_0,+s_0] \times \mathbb{S}^{n-1})$ which satisfies the assertions :*

(i) h_Ψ is harmonic, i.e.

$$H_c(h_\Psi) \quad = \quad 0 \qquad in\ [-s_0,+s_0] \times \mathbb{S}^{n-1};$$

(ii) h_Ψ belongs to $\mathcal{C}^{2,\alpha}_{\frac{n+2}{2}}([-s_0,+s_0] \times \mathbb{S}^{n-1})$ and we have the estimate

$$\|h_\Psi\|_{\mathcal{C}^{2,\alpha}_{\frac{n+2}{2}}([-s_0,+s_0]\times\mathbb{S}^{n-1})} \quad \leqslant \quad c\cosh(s_0)^{-\frac{n+2}{2}}\,\|\Psi\|_{\mathcal{C}^{2,\alpha}(\mathbb{S}^{n-1})}; \tag{6.3.54}$$

(iii) we have an accurate description of the solution in a neighbourhood of the boundary, namely :

$$\begin{aligned}\left\|h_\Psi(s,z) - e^{\frac{n-2}{2}(s-s_0)}W^i(\Psi_+)\left(e^{s-s_\epsilon}z\right)\right\|_{\mathcal{C}^{2,\alpha}([s_0-2,s_0]\times\mathbb{S}^{n-1})}& \\ \leqslant \quad c\cosh(s_0)^{-(n+2)}\,\|\Psi\|_{\mathcal{C}^{2,\alpha}(\mathbb{S}^{n-1})}.&\end{aligned} \tag{6.3.55}$$

Remark 6.3.4 — Of course, there exists a similar result for the lower part of the catenoid, namely the inequality

$$\begin{aligned}\left\|h_\Psi(s,z) - e^{\frac{n-2}{2}(-s+s_0)}W^i(-\Psi_-)\left(e^{s-s_\epsilon}z\right)\right\|_{\mathcal{C}^{2,\alpha}([-s_0,-s_0+2]\times\mathbb{S}^{n-1})}& \\ \leqslant \quad c\cosh(s_0)^{-(n+2)}\,\|\Psi\|_{\mathcal{C}^{2,\alpha}(\mathbb{S}^{n-1})}.&\end{aligned}$$

The reader will mind the change of signs in the interior harmonic extension.

Sketch of the proof (Of proposition 6.3.1)
Either we use the link between W^i and H_c or one checks, like in the hyperplane case, that

$$h_\Psi (s, z) \quad := \quad \sum_{j=2}^{\infty} \sum_{i=1}^{\dim E^j} e^{-\delta_j s} \Psi^{j,i} (z)$$

is a solution with weight parameter $-\delta_2 = -\frac{n+2}{2}$. □

Proof (of the corollary)
It is enough to apply previous proposition. Let $\widetilde{h}_{\Psi_+}$ (resp. $\widetilde{h}_{\Psi_-}$) be the solution given by the problem (6.3.52) with boundary data $\Psi (s_0, \cdot)$ (resp. $\Psi (-s_0, \cdot)$). We define h_Ψ by

$$h_\Psi : (s, z) \longmapsto \widetilde{h}_{\Psi_+} (s_0 - s, z) + \widetilde{h}_{\Psi_-} (s_0 + s, z) \,.$$

We easily check that this sum of translated functions satisfies $H_c (h_\Psi) = 0$.

We deal with the estimate (6.3.54) as follows : we prove it in the $L^\infty_{\frac{n+2}{2}}$ sense, then Schauder's estimates give the result in $\mathcal{C}^{2,\alpha}_{\frac{n+2}{2}}$. The boundary estimate (6.3.55) can be proved with same method by noting the equality

$$\widetilde{h}_{\Psi_+} (s_0 - s, z) \quad = \quad e^{\frac{2-n}{2}(s-s_0)} W^i (\Psi_+) \left(e^{s-s_0}, z\right) .$$ □

Remark 6.3.5 — Note that since $\delta_{j_0} \geqslant \frac{n+2}{2}$, the harmonic extension h_Ψ always belongs to $\mathcal{C}^{2,\alpha}_\delta \left((s_0, +\infty) \times \mathbb{S}^{n-1}\right)$ for all $\delta \in \left(\frac{n}{2}, \frac{n+2}{2}\right)$.

6.3 — (b) The mode 0 and the s-odd mode 1

Unlike the method described in particular in the articles [FP00] or [KP07, Part 4] in which the gluing process is conducted by leaning the deformed catenoid in some direction (in our case, we would need to look in the direction given by the force F), we will treat terms of order 0 and 1 by using the conjugate Jacobi fields because they span the kernel of L_c for weight parameter smaller than $\frac{n}{2}$. However, we will also prove that this approach is very close to that of harmonic extensions.

Definition 6.3.6 — For all function Ψ over $\{\pm s_\epsilon\} \times \mathbb{S}^{n-1}$, we decompose its mode 1, namely Ψ^1 as

$$\Psi^1 \quad = \quad \Psi^1_{odd} + \Psi^1_{even}$$

where Ψ^1_{odd} (resp. Ψ^1_{even}) is s-odd (resp. s-even). By misuse of notation, we also identify the function Ψ^1_{odd} (resp. Ψ^1_{even}) with the vector Ψ^1_{odd} such that $\Psi^1_{\mathrm{odd}} (\pm s_\epsilon, z) = \langle \pm \Psi^1_{\mathrm{odd}}, z \rangle$ (resp. $\Psi^1_{even} (\pm s_\epsilon, z) = \langle \Psi^1_{even}, z \rangle$).

Remark 6.3.7 — We say that en element Ψ is s-odd when $\Psi (s_0, z) = -\Psi (-s_0, z)$, in other words, when

$$\Psi_+ = -\Psi_- \quad \text{where} \quad \Psi_\pm := \Psi (\pm s_0, \cdot) \,.$$

In this section, we cannot use the s-even Jacobi fields $\phi^{+}_{1,j}$. This particular choice is a fundamental point on which we have to enlarge upon. The nature of Jacobi fields associated with rotation or dilation turns out to be very different. Indeed, a rough asymptotic behaviour of ϕ^{1}_{-} is given by $(\cosh s)^{\frac{n}{2}}$ which explodes when s is large while the inverse phenomena occurs for ϕ^{1}_{+} which decreases like $(\cosh s)^{-\frac{n}{2}}$. If we chose any function Ψ with no symmetry, we would observe that we have to make use a term like ϕ^{1}_{+} and the solution we want to construct in the following proposition would not be bounded in $s = 0$: its rough estimate would be $(\cosh (s_0))^{\frac{n-2}{2}}$. In other words, a generic boundary data implies we do not control the norm of the solution and we no longer are able to perform the fixed point theorem to deform a truncated catenoid. Geometrically, this condition states that the axis of the deformed truncated catenoid is straight.

Proposition 6.3.8 — *There exists some constant $c := c(\alpha, n)$ such that for all $s_0 \in \mathbb{R}_+$ and $\Psi \in \mathcal{C}^{2,\alpha}(\{\pm s_0\} \times \mathbb{S}^{n-1}) \cap \mathcal{E}^0$ (resp. $\mathcal{C}^{2,\alpha}(\{\pm s_0\} \times \mathbb{S}^{n-1}) \cap \mathcal{E}^1_{odd}$, where $\mathcal{E}^1_{odd}$ denotes the the s-odd functions of $\mathcal{E}^1$), there exists one function $\ell_\Psi \in \mathcal{C}^{2,\alpha}_{loc}([-s_0, s_0] \times \mathbb{S}^{n-1})$ which is a solution to the following problem :*

$$\left\{ \begin{array}{rcll} L_c(\ell_\Psi) & = & 0 & \textit{in} \quad [-s_0, s_0] \times \mathbb{S}^{n-1}; \\ \ell_\Psi & = & \Psi & \textit{on} \quad \{\pm s_0\} \times \mathbb{S}^{n-1}. \end{array} \right. \tag{6.3.56}$$

Besides, ℓ_Ψ belongs to $\mathcal{C}^{2,\alpha}_{\frac{n-2}{2}}([-s_0, s_0] \times \mathbb{S}^{n-1})$ (resp. $\mathcal{C}^{2,\alpha}_{\frac{n}{2}}[-s_0, s_0] \times \mathbb{S}^{n-1}$) and following estimates hold :

$$\begin{array}{rcll} \|\ell_\Psi\|_{\mathcal{C}^{2,\alpha}_{\frac{n-2}{2}}([-s_0,s_0]\times\mathbb{S}^{n-1})} & \leqslant & c(\cosh s_0)^{-\frac{n-2}{2}} \|\Psi\|_{\mathcal{C}^{2,\alpha}(\mathbb{S}^{n-1})} & \textit{if } \Psi \in \mathcal{E}^0 \\ (\textit{resp.} & \leqslant & c(\cosh s_0)^{-\frac{n}{2}} \|\Psi\|_{\mathcal{C}^{2,\alpha}(\mathbb{S}^{n-1})} & \textit{if } \Psi \in \mathcal{E}^{1^-}). \end{array}$$

Moreover, it is possible to have more efficient estimate near the boundary, namely :

$$\begin{array}{l} \left\| \ell_\Psi(s,z) - e^{\frac{n-2}{2}(s-s_0)} W^i(\Psi_+)\left(e^{s-s_0}, z\right) \right\|_{\mathcal{C}^{2,\alpha}([s_0-2,s_\epsilon]\times\mathbb{S}^{n-1})} \\ \qquad \leqslant \left\{ \begin{array}{ll} c\cosh(s_0)^{-n+2} \|\Psi\|_{\mathcal{C}^{2,\alpha}(\mathbb{S}^{n-1})} & \textit{if} \quad \Psi \in \mathcal{E}^0; \\ c\cosh(s_0)^{-n} \|\Psi\|_{\mathcal{C}^{2,\alpha}(\mathbb{S}^{n-1})} & \textit{if} \quad \Psi \in \mathcal{E}^{1^-}. \end{array} \right. \end{array} \tag{6.3.57}$$

Remark 6.3.9 — Concerning the lower part, the same kind can be proved with a change of signs in interior harmonic extensions, just like in remark 6.3.4.

Proof

Here, it us very useful to explicit the solutions since an accurate description of the solution highlights the role of the harmonic extensions.

The mode 0 **:** Suppose $\Psi \in E^0$. We already know that the family $\left(\phi_-^0, \phi_+^0\right)$ spans the Jacobi fields associated with this mode. Thus we look for a solution ℓ_Ψ whose form is given by $\lambda\phi_-^0 + \mu\phi_+^0$. The boundary conditions for $s = \pm s_0$ imply the following linear system

$$\begin{pmatrix} \phi_-^0(s_0) & \phi_+^0(s_0) \\ \phi_-^0(s_0) & -\phi_+^0(s_0) \end{pmatrix} \begin{pmatrix} \lambda \\ \mu \end{pmatrix} = \begin{pmatrix} \Psi_+ \\ \Psi_- \end{pmatrix},$$

from what we deduce

$$\ell_\Psi = \Psi_+ \left(\frac{\phi_-^0}{2\phi_-^0(s_0)} + \frac{\phi_+^0}{2\phi_+^0(s_0)} \right) + \Psi_- \left(\frac{\phi_-^0}{2\phi_-^0(s_0)} - \frac{\phi_+^0}{2\phi_+^0(s_0)} \right).$$

It is an easy computation to see it is really a solution to the problem (6.3.56). It remains to check this function ℓ_Ψ begins to some weighted space and to estimate its norm. As a matter of fact, this directly derives from the construction of Jacobi field (cf. (5.4.39) et (5.4.41)). First of all, observe those fields acts as $\varphi^{\frac{n-2}{2}}$, thus exponentially increases as $(\cosh s)^{\frac{n-2}{2}}$, so ℓ_Ψ is an element of the weighted space $L^\infty_{\frac{n-2}{2}}\left([-s_0, s_0] \times \mathbb{S}^{n-1}\right)$.

To estimate its norm, we use the Taylor expansion (5.4.40) : there exists a constant $c = c(n)$ which does not depend on s_0, such that for all $s \in \mathbb{R}_+$, we can decompose φ_0^- into

$$\varphi_0^-(s) = a_0^- e^{\frac{n-2}{2}s} + r_0^-(s)$$

where the function r_0^- is much smaller than the first term, namely $\left|r_0^-(s)\right| \leqslant ce^{-\frac{n-2}{2}s}$. Therefore, the inequality

$$\left| \frac{\phi_-^0(s)}{\phi_-^0(s_0)} - e^{\frac{n-2}{2}(s-s_0)} \right| \leqslant c\, e^{\frac{n-2}{2}(s-s_0)} \left| e^{(2-n)s} - e^{(2-n)s_0} \right|,$$

provides a way to prove

$$\left| e^{-\frac{n-2}{2}s} \frac{\phi_-^0(s)}{\phi_-^0(s_0)} \right| \leqslant ce^{-\frac{n-2}{2}s_0}.$$

By doing likewise for the case $s \in \mathbb{R}_-$, we finally end up with

$$\left\| \frac{\phi_-^0(\cdot)}{\phi_-^0(s_0)} \right\|_{L^\infty_{\frac{n-2}{2}}([-s_0,s_0]\times\mathbb{S}^{n-1})} \leqslant c\,(\cosh s_0)^{-\frac{n-2}{2}}.$$

The calculus for the ϕ_+^0 part is quite similar. We find

$$\|\ell_\Psi\|_{L^\infty_{\frac{n-2}{2}}([-s_0,s_0]\times\mathbb{S}^{n-1})} \leqslant c\,(\cosh s_0)^{-\frac{n-2}{2}} \|\Psi\|_{\mathcal{C}^{2,\alpha}(\{\pm s_0\}\times\mathbb{S}^{n-1})}.$$

the same kind of computation regarding the norm $\mathcal{C}^{2,\alpha}_{\frac{n-2}{2}}\left([-s_0, s_0] \times \mathbb{S}^{n-1}\right)$ together with the Schauder's estimates leads to the estimate (6.3.8).

Furthermore, in order to compare the solution with the harmonic extension, the estimate (6.3.57) can be easily checked by noting that $W^i(\Psi_+) = \Psi_+$ — it is the constant term of the interior harmonic extension — and by applying above inequalities.

The mode 1 **:** It is an argument quite similar to the previous one. Indeed, if $\Psi_+(z) = -\Psi_-(z) = \sum_i \Psi^{1,i}\langle z, \mathbf{e}_i\rangle$, then the solution is given by

$$\ell_\Psi(s,z) = \sum_i \Psi^{1,i}\frac{\phi_1^-(s)}{\phi_1^-(s_0)}\langle z, \mathbf{e}_i\rangle .$$

The estimate (6.3.57) can be deduced from the equality

$$e^{\frac{n}{2}(s-s_0)}\Psi^+(z) = e^{\frac{n-2}{2}(s-s_0)}W^i(\Psi_+)\left(e^{s-s_0}, z\right). \qquad \square$$

6.3 — (c) To deal with the s-even mode 1

In the above proposition, we enforce the boundary data Ψ^1 to be s-odd. However, for a general case, there is *a priori* no symmetry in the weighted configuration $\{a_{k,j}, p_{k,j}\}$, thus no reason why the truncated catenoids of the gluing method could have s-odd mode 1 boundary data. It turns out that for any function Ψ^1, we are not able to solve the minimal surface equation with the method we use. To make up for this problem, we rather solve a problem whose type is

$$H_\omega = \varphi^*\langle V, \text{Jacobi fields}\rangle$$

where $*$ is a well chosen real number. We explain the choice of this equation and the choice of $*$ in section 7.2.

Proposition 6.3.10 — *Let $|\chi_\epsilon|$ be a smooth cutoff function with values in $[0,1]$ that takes value 1 for $|s| \geqslant s_\epsilon - 1$, vanishes for $|s| \leqslant s_\epsilon - 2$. We suppose its $\mathcal{C}^\infty$-norm does not depend on ϵ. There exists a constant $c = c(n,\alpha)$ such that for all $\Psi^1_{even} \in \mathcal{C}^{2,\alpha}(\{\pm s_\epsilon\}\times\mathbb{S}^{n-1})$, there exists one function $\widetilde{\ell}_{\Psi^1_{even}}$ which is a solution to the problem*

$$\begin{cases} L_c\left(\widetilde{\ell}_{\Psi^1_{even}}\right)(s,z) = -\frac{1}{M_\epsilon}\left(1 - |\chi_\epsilon(s)|\right)\phi^1_+(s)\langle\Psi^1_{even}, z\rangle & \text{over } [-s_\epsilon, s_\epsilon]\times\mathbb{S}^{n-1}, \\ \widetilde{\ell}_{\Psi^1_{even}}(\pm s_\epsilon, z) = \langle\Psi^1_{even}, z\rangle & \text{over } \{\pm s_\epsilon\}\times\mathbb{S}^{n-1}, \end{cases} \tag{6.3.58}$$

where M_ϵ is a large positive constant we determine during the proof. Besides, $\widetilde{\ell}_{\Psi^1_{even}}$ belongs to the Hölder weighted space $\mathcal{C}^{2,\alpha}_{\frac{n}{2}}([-s_\epsilon, s_\epsilon]\times\mathbb{S}^{n-1})$ and

$$\left\|\widetilde{\ell}_{\Psi^1_{even}}\right\|_{\mathcal{C}^{2,\alpha}_{\frac{n}{2}}([-s_\epsilon,s_\epsilon]\times\mathbb{S}^{n-1})} \leqslant c\cosh(s_\epsilon)^{-\frac{n}{2}}\left\|\Psi^1_{even}\right\|_{\mathcal{C}^{2,\alpha}(\{\pm s_\epsilon\}\times\mathbb{S}^{n-1})}. \tag{6.3.59}$$

Moreover, it is possible to obtain more efficient estimate near the boundary with the help of harmonic extensions, namely :

$$\begin{aligned} &\left\|\widetilde{\ell}_{\Psi^1_{even}}(s,z) - e^{\frac{n-2}{2}(s-s_\epsilon)}W^i\left(\langle\Psi^1_{even}, z\rangle\right)\left(e^{s-s_\epsilon}z\right)\right\|_{\mathcal{C}^{2,\alpha}([s_\epsilon-2,s_\epsilon]\times\mathbb{S}^{n-1})} \\ &\qquad \leqslant c\cosh(s_\epsilon)^{-n}\left\|\Psi^1_{even}\right\|_{\mathcal{C}^{2,\alpha}(\{\pm s_\epsilon\}\times\mathbb{S}^{n-1})}. \end{aligned} \tag{6.3.60}$$

Proof
By standard variation of constants for the operator L_1 together with the remark 6.1.2 about the wronskian of $\left(\phi_-^1, \phi_+^1\right)$, the function $\widetilde{\ell}$ defined by

$$\begin{aligned}\widetilde{\ell}(s) \quad &:= \quad (1 - |\chi_\epsilon(s)|) \\ &\cdot \left(\phi_+^1(s) \int_0^s \varphi(t)^{-\frac{n}{2}} \phi_-^1(t)\, \mathrm{d}t - \phi_-^1(s) \int_0^s \varphi(t)^{-\frac{n}{2}} \phi_+^1(t)\, \mathrm{d}t\right)\end{aligned}$$

satisfies $L_1\left(\widetilde{\ell}\right) = (1 - |\chi_\epsilon|)\, \varphi^{-\frac{n}{2}}$. Besides, according to the construction of the Jacobi fields $\phi_\pm^1$, we check that there exists a constant $c = c(n)$ such that for all s, $\left|\widetilde{\ell}(s)\right| \leqslant c \cosh(s_\epsilon)^{\frac{n}{2}}$.

We choose $M_\epsilon := \widetilde{\ell}(s_\epsilon)$ and

$$\widetilde{\ell}_{\Psi^1_{\text{even}}}(s, z) \quad := \quad -\frac{1}{\widetilde{\ell}(s_\epsilon)} \widetilde{\ell}(s) \left\langle \Psi^1_{\text{even}}, z\right\rangle.$$

Then $\widetilde{\ell}_{\Psi^1_{\text{even}}}$ is a solution to the problem (8.2.88). Besides, the above inequality together with Schauder's estimates prove the estimate (6.3.59).

To conclude with (6.3.60), it is enough to obtain Taylor expansion for $\widetilde{\ell}(s)$ when $|s|$ is large. For example, note that

$$\begin{aligned}\int_0^s \phi_+^1(t)\, \varphi(t)^{-\frac{n}{2}}\, \mathrm{d}t &= \int_0^{+\infty} \varphi(t)^{-n}\, \mathrm{d}t - \int_s^{+\infty} \varphi(t)^{-n}\, \mathrm{d}t \\ &= \int_0^{+\infty} \varphi(t)^{-n}\, \mathrm{d}t + \underset{|s| \to +\infty}{\mathcal{O}} \left(\cosh(s)^{-n}\right).\end{aligned}$$

We deal with the other terms of $\widetilde{\ell}$ in the same way. □

Remark 6.3.11 — It is a result analogous in shape to the one we prove for the s-odd part Ψ^1_{odd}. Besides, in a neighbourhood of $\pm s_\epsilon$, the solution $\widetilde{\ell}_{\Psi^1_{\text{even}}}$ looks like the solution to the one of the problem $L_c(\widetilde{\ell}_{\Psi^1_{\text{even}}}) = 0$ with boundary data Ψ^1_{even} ; it can be fundamentally explained with the inequality

$$\left|\frac{1 - |\chi_\epsilon|}{\widetilde{\ell}(s_\epsilon)} \varphi^{-\frac{n}{2}} \Psi^1_{\text{even}}\right| \quad \leqslant \quad c \cosh(s_\epsilon)^{-n} \left\|\Psi^1_{\text{even}}\right\| \quad << \quad \left\|\Psi^1_{\text{even}}\right\|.$$

7 Hypersurface near the n-catenoid

7.1 Mean curvature and minimal hypersurfaces

We have in mind to deform a truncated n-catenoid whose boundary data is prescribed. In this purpose, we consider normal deformations. For a function ω on $\mathbb{S}^{n-1} \times \mathbb{R}$, let

$$X_\omega := X_c + \omega N_c = X_c + \widetilde{\omega} \widetilde{N}_c$$

where

$$\widetilde{\omega} = \frac{\omega}{\varphi} \ , \quad \widetilde{N}_c = \varphi N_c$$

and denote by Σ_ω the associated hypersurface.

It is well konwn (cf. [BdC84] for example) that the Jaboci operator is the linearization of the mean curvature. Nevertheless, for our purposes, it is necessary to give an accurate description of the minimal surface equation we want to solve : we compute the first variation of area to obtain asymptotic behaviour of the non linear terms.

7.1 — (a) The normal deformations case

Metric on Σ_ω. Calculus demonstrates that the metric on the surface Σ_ω is given by the matrix

$$g_\omega = \left(\langle \partial_{x^i} X_\omega \ , \ \partial_{x^j} X_\omega \rangle_{\mathbb{R}^{n+1}}\right)_{i,j}$$

which satisfies equalities

$$(g_\omega)_{s,s} = \varphi^2 \left[1 + 2\left(n-1\right)\widetilde{\omega}\varphi^{1-n} + \widetilde{\omega}^2\left(1 + n\left(n-2\right)\varphi^{2-2n}\right) + \dot{\widetilde{\omega}}^2 + 2\widetilde{\omega}\dot{\widetilde{\omega}}\frac{\dot{\varphi}}{\varphi}\right],$$

together with

$$(g_\omega)_{z^i,z^j} = \varphi^2 \left[\left(1 - \widetilde{\omega}\varphi^{1-n}\right)^2 \left(g_{\mathbb{S}^{n-1}}\right)_{i,j} + \partial_{z^j}\widetilde{\omega}\partial_{z^i}\widetilde{\omega}\right]$$

and

$$(g_\omega)_{s,z^i} = \varphi^2 \partial_{z^i}\widetilde{\omega}\left(\dot{\widetilde{\omega}} + \widetilde{\omega}\frac{\dot{\varphi}}{\varphi}\right).$$

Let $g_{\omega,\mathbb{S}^{n-1}}$ be the extracted $(n-1)$-matrix from g_ω given by the spherical coordinates. For small ω, we can write

$$g_{\omega,\mathbb{S}^{n-1}} = \varphi^2\left(1 - \widetilde{\omega}\varphi^{1-n}\right)^2 \cdot g_{\mathbb{S}^{n-1}} \cdot \left(I_{n-1} + \frac{1}{\left(1 - \widetilde{\omega}\varphi^{1-n}\right)^2} g_{\mathbb{S}^{n-1}}^{-1}\left(\partial_{z^i}\widetilde{\omega}\partial_{z^j}\widetilde{\omega}\right)_{i,j}\right).$$

Note the matrix equality $g_{\mathbb{S}^{n-1}}^{-1}\left(\partial_{z^i}\widetilde{\omega}\partial_{z^j}\widetilde{\omega}\right)_{i,j} = \left(\partial_{z^l}\widetilde{\omega}\nabla^k_{\mathbb{S}^{n-1}}\widetilde{\omega}\right)_{k,l}$. For practical purpose, we give the following definition.

Definition 7.1.1 — For a function f, we write

$$f\left(\widetilde{\omega}\right) = Q_i\left(\widetilde{\omega}\right) = Q_i\left(s, \widetilde{\omega}, \nabla\widetilde{\omega}, \nabla^2\widetilde{\omega}\right)$$

for $0 \leqslant i \leqslant 3$ (resp. $i = 4$) if the following assertions are satisfied :

(i) $Q_i\left(\omega\right)$ is an expression of order i (resp. collects all the terms whose order is larger than 3) in $\widetilde{\omega}$ and its derivatives ;

(ii) the coefficients of Q_i and their derivatives are s-uniformly bounded functions.

In particular, there exists universal constant $c_i = c_i\,(n, \alpha)$ such that

$$|Q_i\,(\widetilde{\omega})| \quad \leqslant \quad c\,\|\widetilde{\omega}\|^i_{\mathcal{C}^{2,\alpha}}$$

and

$$|Q_i\,(\widetilde{\omega}_1 - \widetilde{\omega}_2)| \quad \leqslant \quad c\,\|\widetilde{\omega}_1 - \widetilde{\omega}_2\|_{\mathcal{C}^{2,\alpha}} \max\left\{\|\widetilde{\omega}_1\|_{\mathcal{C}^{2,\alpha}}, \|\widetilde{\omega}_2\|_{\mathcal{C}^{2,\alpha}}\right\}^{i-1}.$$

Thus, the determinant of $g_{\omega,\mathbb{S}^{n-1}}$ satisfies :

$$\begin{aligned}
&\det g_{\omega,\mathbb{S}^{n-1}} = \varphi^{2(n-1)} \cdot \det g_{\mathbb{S}^{n-1}} \\
&\left(1 - 2\,(n-1)\,\widetilde{\omega}\varphi^{1-n} + (n-1)\,(2n-3)\,\widetilde{\omega}^2\varphi^{2(1-n)} + \varphi^{3(1-n)}Q_3\,(\widetilde{\omega}) + \varphi^{4(1-n)}Q_4\,(\widetilde{\omega})\right) \\
&\qquad \cdot \det\Big(I_{n-1} + \left(1 + \varphi^{1-n}Q_1\,(\widetilde{\omega}) + \varphi^{2(1-n)}Q_2\,(\widetilde{\omega})\right)\left(\partial_{z^l}\widetilde{\omega}\nabla^k_{\mathbb{S}^{n-1}}\widetilde{\omega}\right)_{k,l}\Big).
\end{aligned}$$

We use the classical asymptotic formula for the last term

$$\det\,(I_{n-1} + H) = 1 + \mathrm{tr}\,(H) + \frac{1}{2}\left((\mathrm{tr}\,(H))^2 - \mathrm{tr}\left(H^2\right)\right) + q_3\,(H) + q_4\,(H)$$

where q_3 collects all the terms of degree 3 and q_4 collects the higher terms in order to obtain, according to the identity $|\nabla_{\mathbb{S}^{n-1}}\widetilde{\omega}|^2_{\mathbb{S}^{n-1}} = \sum_i \partial_{z^i}\widetilde{\omega}\nabla^i_{\mathbb{S}^{n-1}}\widetilde{\omega}$,

$$\begin{aligned}
&\det\Big(I_{n-1} + \left(1 + \varphi^{1-n}Q_1\,(\widetilde{\omega}) + \varphi^{2(1-n)}Q_2\,(\widetilde{\omega})\right)\left(\partial_{z^l}\widetilde{\omega}\nabla^k_{\mathbb{S}^{n-1}}\widetilde{\omega}\right)_{k,l}\Big) \\
&\qquad\qquad = 1 + |\nabla_{\mathbb{S}^{n-1}}\widetilde{\omega}|^2_{\mathbb{S}^{n-1}} + \varphi^{1-n}Q_3\,(\widetilde{\omega}) + Q_4\,(\widetilde{\omega}).
\end{aligned}$$

The reader will mind the different coefficients in front of Q_3 and Q_4. The terms whose higher is larger than 4 come from two kinds of objects : on one hand, those we obtain with quantity $\varphi^{2(1-n)}Q_2\,(\widetilde{\omega}) \cdot \partial_{z^l}\widetilde{\omega}\nabla^k_{\mathbb{S}^{n-1}}\widetilde{\omega}$, on the other hand those we obtain with quantity $\partial_{z^l}\widetilde{\omega}\nabla^k_{\mathbb{S}^{n-1}}\widetilde{\omega} \cdot \partial_{z^{l'}}\widetilde{\omega}\nabla^{k'}_{\mathbb{S}^{n-1}}\widetilde{\omega}$. Therefore, the determinant of the metric g_ω is given by

$$\begin{aligned}
\det g_\omega \;=\; \varphi^{2n}\det g_{\mathbb{S}^{n-1}}\Big[&1 + \widetilde{\omega}^2 + \widetilde{\omega}^2\left(-n^2 + n - 1\right)\varphi^{2-2n} + \dot{\widetilde{\omega}}^2 + 2\widetilde{\omega}\dot{\widetilde{\omega}}\frac{\dot{\varphi}}{\varphi} \\
&+ \|\nabla\widetilde{\omega}\|^2_{\mathbb{S}^{n-1}} + \varphi^{1-n}Q_3\,(\widetilde{\omega}) + Q_4\,(\widetilde{\omega})\Big].
\end{aligned}$$

In terms of ω, we get

$$\begin{aligned}
\det g_\omega \;=\; \varphi^{2n}\det g_{\mathbb{S}^{n-1}}\Big[&1 + \varphi^{-2}\left(\dot{\omega}^2 + \|\nabla\omega\|^2_{\mathbb{S}^{n-1}}\right) - n\,(n-1)\,\omega^2\varphi^{-2n} \\
&+ \varphi^{1-n}Q_3\left(\varphi^{-1}\omega\right) + Q_4\left(\varphi^{-1}\omega\right)\Big],
\end{aligned}$$

thus

$$\begin{aligned}
\sqrt{\det g_\omega} \;=\; \varphi^n\sqrt{\det g_{\mathbb{S}^{n-1}}}\Big[&1 + \frac{\varphi^{-2}}{2}\left(\dot{\omega}^2 + \|\nabla\omega\|^2_{\mathbb{S}^{n-1}}\right) - \frac{n\,(n-1)}{2}\omega^2\varphi^{-2n} \\
&+ \varphi^{1-n}Q_3\left(\varphi^{-1}\omega\right) + Q_4\left(\varphi^{-1}\omega\right)\Big]. \qquad (7.1.61)
\end{aligned}$$

Minimal hypersurface equation. Here, we want to obtain the description of the link between the first variation of area and the Jacobi operator which is the differential of mean curvature. By definition, the area of X_ω is given by $\mathcal{A}_\omega = \int_{\mathbb{R}\times\mathbb{S}^{n-1}} \sqrt{\det g_\omega}\ \mathrm{d}s\ \mathrm{d}z$. According to the equation (7.1.61), one proves the differential of the area functionnal at ω is given by

$$\begin{aligned}\mathrm{d}\mathcal{A}_\omega(h) &= \int_{\mathbb{R}\times\mathbb{S}^{n-1}} \sqrt{\det g_{\mathbb{S}^{n-1}}}\ \Big(\varphi^{n-2}\dot\omega\dot h + \varphi^{n-2}\left\langle \nabla_{\mathbb{S}^{n-1}}\omega, \nabla_{\mathbb{S}^{n-1}}h\right\rangle_{\mathbb{S}^{n-1}} \\ &\qquad - n(n-1)\omega h\varphi^{-n} + \widetilde{Q}_2\left(\varphi^{-1}\omega\right)(h) + \varphi^{n-1}\widetilde{Q}_3\left(\varphi^{-1}\omega\right)(h)\Big)\ \mathrm{d}s\ \mathrm{d}z,\end{aligned}$$

where $\widetilde{Q}_i\left(\widetilde{\omega}\right)(h)$ is the differential of Q_{i+1} at point $\widetilde{\omega}$ in h. Integration by parts leads us to

$$\mathrm{d}\mathcal{A}_\omega(h) = -\int_{\mathbb{R}\times\mathbb{S}^{n-1}} \sqrt{\det g_{\mathbb{S}^{n-1}}}\ \Big(\varphi^n J_c(\omega) + \widetilde{Q}_2\left(\varphi^{-1}\omega\right) + \varphi^{n-1}\widetilde{Q}_3\left(\varphi^{-1}\omega\right)\Big)\ h\ \mathrm{d}s\ \mathrm{d}z. \tag{7.1.62}$$

It follows that Σ_ω is minimal if this differential vanishes, i.e. when

$$L_c\left(\varphi^{\frac{n-2}{2}}\omega\right) = \varphi^{\frac{2-n}{2}}\widetilde{Q}_2\left(\varphi^{-1}\omega\right) + \varphi^{\frac{n}{2}}\widetilde{Q}_3\left(\varphi^{-1}\omega\right).$$

Instead of considering the parametrization $X_\omega = X_c + \omega N_c$, it will be convenient to use another *conjugate* parametrization $X_\omega := X_c + \varphi^{\frac{2-n}{2}}\omega N_c$. Then the previous minimal surface equation turns into

$$L_c(\omega) = \varphi^{\frac{2-n}{2}}\widetilde{Q}_2\left(\varphi^{\frac{n}{2}}\omega\right) + \varphi^{\frac{n}{2}}\widetilde{Q}_3\left(\varphi^{\frac{n}{2}}\omega\right). \tag{7.1.63}$$

7.1 — (b) The mean curvature equation

In the above paragraph, we develop an accurate description of the minimal graph equation. Nevertheless, as said in section 6.3 — (c), for a problem with no symmetries, we are not able to solve it. We rather solve equation whose type is "mean curvature = sum of Jacobi fields". Consequently, we have to explicit the mean curvature equation. In this purpose, we use the classical formula for normal deformations $X_\omega = X_c + \omega N_c$

$$\mathrm{d}\mathcal{A}_\omega(h) = \int_{\Sigma_\omega} H_\omega \left\langle N_\omega, N_c\right\rangle\ h\ \mathrm{d}vol_{\Sigma_\omega},$$

where H_ω is the mean curvature of Σ_ω, N_ω is its unit normal and $\mathrm{d}vol_{\Sigma_\omega}$ is its volume form. According to this equation together with (7.1.62), we find

$$H_\omega = -\frac{\varphi^n J_c(\omega) + \widetilde{Q}_2\left(\varphi^{-1}\omega\right) + \varphi^{n-1}\widetilde{Q}_3\left(\varphi^{-1}\omega\right)}{\left\langle N_\omega, N_c\right\rangle\sqrt{\det g_\omega}}\sqrt{\det g_{\mathbb{S}^{n-1}}}.$$

The only data we do not have described yet is the unit normal N_ω. We give a broad outline of its tedious computation. We look for a representation such that

$$N_\omega = \frac{1}{\left(\alpha^2 + \sum_i \beta_i^2 + \gamma^2\right)^{1/2}}\left(\alpha z + \sum_i \beta_i \partial_{z^i} z, \gamma\right) \in \mathbb{R}^n \times \mathbb{R}.$$

Relations $\langle N_\omega, \partial_s X_\omega\rangle = 0$ and $\langle N_\omega, \partial_{z^i} X_\omega\rangle = 0$ show we can choose

$$\begin{array}{rcl}\alpha & = & -\left(\varphi^{3-n} + \dot{\varphi}\dot{\omega} + (1-n)\,\omega\varphi^{3-2n}\right),\\ \beta_i & = & -\varphi\nabla^i_{\mathbb{S}^{n-1}}\omega + \varphi^{3-n}Q_2\left(\varphi^{-1}\omega\right) + \varphi^{4-2n}Q_3\left(\varphi^{-1}\omega\right),\\ \gamma & = & \dot{\varphi}\varphi - \dot{\omega}\varphi^{2-n} + (n-1)\,\omega\dot{\varphi}\varphi^{1-n}.\end{array}$$

Consequently, in order to normalize the vector N_ω, it remains to explicit

$$\begin{aligned}&\left(\alpha^2 + \sum_i \beta_i^2 + \gamma^2\right)^{-1/2}\\ &\quad = \varphi^{-2}\left(1 - (n-1)\,\varphi^{-1}\omega\left(\varphi^{1-n} - 2\varphi^{3(1-n)}\right) + Q_2\left(\varphi^{-1}\omega\right) + \varphi^{1-n}Q_3\left(\varphi^{-1}\omega\right)\right).\end{aligned}$$

It follows that

$$\langle N_\omega, N_c\rangle^{-1} \quad = \quad 1 + \varphi^{1-n}Q_1\left(\varphi^{-1}\omega\right) + Q_2\left(\varphi^{-1}\omega\right),$$

from what we conclude

$$H_\omega \quad = \quad -J_c\left(\omega\right) + \varphi^{-n}Q_2\left(\varphi^{-1}\omega\right) + \varphi^{-1}Q_3\left(\varphi^{-1}\omega\right). \tag{7.1.64}$$

As well as in the minimal hypersurface equation (7.1.63), if we use the *conjugate* parametrization $X_\omega := X_c + \varphi^{\frac{2-n}{2}}\omega N_c$, the above equality turns into

$$H_\omega \quad = \quad \varphi^{-\frac{n+2}{2}}\left[-L_c\left(\omega\right) + \varphi^{\frac{2-n}{2}}Q_2\left(\varphi^{\frac{n}{2}}\omega\right) + \varphi^{\frac{n}{2}}Q_3\left(\varphi^{\frac{n}{2}}\omega\right)\right]. \tag{7.1.65}$$

7.1 — (c) A more convenient deformation

As said previously, when $|s|$ is very large, the catenoid is "almost flat", asymptotic to a horizontal hyperplane and its unit normal is quasi vertical. For the gluing process, it will be more convenient to consider that in this case, we can write the piece of catenoid as a graph over an open set of an horizontal hyperplane. We choose an annulus whose radius is large enough to ensure we can write the piece of catenoid as a vertical graph. To avoid this annulus depends on the choice of the small perturbation ω we perform, we change the unit normal N_c as a vector N_ϵ such that N_ϵ is equal to N_c for $|s|$ small and N_ϵ is a vertical vector when $|s|$ is large. We then work with the small perturbation $X_{\omega,\epsilon} = X_c + \varphi^{\frac{2-n}{n}}\omega N_\epsilon$.

Let χ_ϵ a smooth increasing cutoff function over $\mathbb{R}$ with values in $[0,1]$ such that

$$\chi_\epsilon\left(s\right) \quad = \quad \left\{\begin{array}{rlrcl} 0 & \text{if} & |s| & \in & [0, s_\epsilon - 2],\\ 1 & \text{if} & s & \in & [s_\epsilon - 1, +\infty],\\ -1 & \text{if} & -s & \in & [s_\epsilon - 1, +\infty],\end{array}\right.$$

and such that its $\mathcal{C}^\infty$ norm does not depend on the choice of parameter ϵ. We then define $N_\epsilon := (1 - |\chi_\epsilon|)\,N_c + \chi_\epsilon \mathbf{e}_{n+1}$. In this case, we can write

$$\begin{array}{rcl}\langle N_\epsilon, N_c\rangle\left(s,z\right) - 1 & = & -\left|\chi_\epsilon\right| + \chi_\epsilon\left\langle \mathbf{e}_{n+1}, N_c\right\rangle\\ & = & \chi_\epsilon\left(s\right)\cdot\left(-\mathrm{sgn}(s) + \tanh\left(\left(n-1\right)s\right)\right),\end{array}$$

and we check that for all $k \in \mathbb{N}$, there exists a universal constant $c_k = c(n,k)$ such that for all $|s| \in [s_\epsilon - 2, s_\epsilon]$,

$$\left|\nabla^k \left(\langle N_\epsilon, N_c\rangle (s,z) - 1\right)\right| \quad \leqslant \quad c_k \left(\cosh s_\epsilon\right)^{2-2n}.$$

Therefore, N_ϵ corresponds to a very small perturbation of the unit normal N_c. According to the minimal graph equation for the case of normal deformations (7.1.63) together with an argument whose source is the inverse function theorem [4], we prove that the mean curvature $H_{\omega,\epsilon}$ of $X_{\omega,\epsilon}$ satisfies the following PDE :

$$H_{\omega,\epsilon} \quad = \quad \varphi^{-\frac{n+2}{2}} \left[-L_c(\omega) - L_\epsilon(\omega) + \varphi^{\frac{2-n}{2}} Q_{2,\epsilon}\left(\varphi^{\frac{n}{2}}\omega\right) + \varphi^{\frac{n}{2}} Q_{3,\epsilon}\left(\varphi^{\frac{n}{2}}\omega\right)\right], \tag{7.1.66}$$

where

- for all $|s| \in [0, s_\epsilon - 2]$, the different quantities are such that

  $$L_\epsilon = 0, \quad Q_{2,\epsilon} = Q_2 \quad \text{and} \quad Q_{3,\epsilon} = Q_3$$

  because for small $|s|$, $N_\epsilon = N_c$;
- L_ϵ can be interpreted as a linear error term whose coefficients are very small — bounded by $c_k \left(\cosh s_\epsilon\right)^{2-2n}$ for all k in norm $\mathcal{C}^{k,\alpha}\left([-s_\epsilon, s_\epsilon] \times \mathbb{S}^{n-1}\right)$. Indeed, L_ϵ is equal, modulo the conjugate operation, to the difference between the Jacobi operator and the first variation of area operator for an hypersurface parametrised by $X_{\omega,\epsilon}$. We then check in this case that $L_\epsilon = L_c\left(\left(1 - \langle N_c, N_\epsilon\rangle\right)\cdot\right)$.
- $Q_{2,\epsilon}$ (resp. $Q_{3,\epsilon}$) is a quadratic term (resp. a term which includes the higher ones) whose coefficients and their derivatives do not depend on ϵ.

7.2 Resolution of the mean curvature equation

As announced, we want to solve problem

$$\left\{\begin{array}{rcll} H_{\omega,\epsilon} & = & \frac{1}{\widetilde{\ell}(s_\epsilon)}\left(1 - |\chi_\epsilon|\right)\varphi^{*}\phi_1^{+}\langle \Psi^1_{even}, z\rangle & \text{over} \quad [-s_\epsilon, s_\epsilon] \times \mathbb{S}^{n-1}, \\ \omega & = & \Psi + \text{smaller terms} & \text{over} \quad \{\pm s_\epsilon\} \times \mathbb{S}^{n-1}, \end{array}\right. \tag{7.2.67}$$

for well chosen parameters $\epsilon > 0$, $\Psi \in \mathcal{C}^{2,\alpha}\left(\{\pm s_\epsilon\} \times \mathbb{S}^{n-1}\right)$ and $* \in \mathbb{R}$ — recall the definition of $\widetilde{\ell}$ in the proof of proposition 6.3.10. First of all, note that like in the hyperplane case, we do not solve exactly the boundary condition $\omega = \Psi$. According to the equation (7.1.66), if ω is a solution, then

$$-L_c(\omega) - L_\epsilon(\omega) + \varphi^{\frac{2-n}{2}} Q_2\left(\varphi^{\frac{n}{2}}\omega\right) + \varphi^{\frac{n}{2}} Q_3\left(\varphi^{\frac{n}{2}}\omega\right) \quad = \quad \frac{1 - |\chi_\epsilon|}{\widetilde{\ell}\left(s_\epsilon\right)} \varphi^{\frac{n+2}{2} + *} \phi_+^1 \left\langle \Psi^1_{even}, z\right\rangle.$$

That is why we choose $* := -\frac{n+2}{2}$. Indeed, in first approximation, the linear term satisfies $L_c(\omega) = -\frac{1-|\chi_\epsilon|}{\widetilde{\ell}(s_\epsilon)}\phi_+^1 \langle \Psi^1_{even}, z\rangle$, and this is why we have previously considered the PDE problem (8.2.88).

4. The reader could refer to the section 3.3 and the annex of the article by R. Mazzeo, F. Pacard and D. Pollack [MPP01]

From now on, we suppose

$$\|\Psi\|_{\mathcal{C}^{2,\alpha}(\{\pm s_\epsilon\}\times\mathbb{S}^{n-1})} \leqslant \kappa\epsilon r_\epsilon \left(\cosh s_\epsilon\right)^{2+\frac{n}{2}}$$

where κ is a constant. We explain this choice in remark 7.2.6.

Then we define ω_Ψ to be

$$\omega_\Psi : (s,z) \in [-s_\epsilon, s_\epsilon] \times \mathbb{S}^{n-1} \longmapsto \ell_{\Psi^0} + \ell_{\Psi^1_{odd}} + \widetilde{\ell}_{\Psi^1_{even}} + h_{\Psi^\perp}.$$

As a result, we remark that

$$L_c\left(\omega_\Psi\right) \quad = \quad -\frac{1-|\chi_\epsilon|}{\widetilde{\ell}\left(s_\epsilon\right)}\phi^1_+ \left\langle \Psi^1_{\text{even}}, z\right\rangle + \frac{n\left(3n-2\right)}{4}\ \varphi^{2-2n} h_{\Psi^\perp}$$

and we check that it is almost a solution.

We perform a small perturbation of ω_Ψ : we look for a small function v such that the graph of $X_{\omega,\epsilon}$ with $\omega = \omega_\Psi + v$ is a solution. In terms of v, the equation (7.1.66) turns into

$$\begin{aligned} L_c\left(v\right) \quad = \quad & L_\epsilon\left(\omega_\Psi + v\right) + \varphi^{\frac{2-n}{2}} Q_{2,\epsilon}\left(\varphi^{\frac{n}{2}}\left(\omega_\Psi + v\right)\right) \\ & + \varphi^{\frac{n}{2}} Q_{3,\epsilon}\left(\varphi^{\frac{n}{2}}\left(\omega_\Psi + v\right)\right) - \frac{n\left(3n-2\right)}{4}\ \varphi^{2-2n} h_{\Psi_\perp}. \end{aligned} \tag{7.2.68}$$

To solve this equation, we have in mind to prove a fixed point theorem for a well chosen operator $\mathcal{F}_c$: the approach is quite similar to the steps we used in the case of the hyperplane.

However, we have to pay attention to the definition of operator L_{c_δ} which is defined for functions over $\mathbb{R}\times\mathbb{S}^{n-1}$ whereas ω_Ψ is defined only over $[-s_\epsilon, s_\epsilon]\times\mathbb{S}^{n-1}$. It is the reason why we build an operator extension $\mathcal{E}$ and an injection operator $\mathcal{I}$. This last is nothing but the canonical one $\mathcal{I} : f \in \mathcal{C}^{2,\alpha}_\delta\left(\mathbb{R}\times\mathbb{S}^{n-1}\right) \longmapsto f_{|[-s_\epsilon,s_\epsilon]\times\mathbb{S}^{n-1}} \in \mathcal{C}^{2,\alpha}_\delta\left([-s_\epsilon, s_\epsilon]\times\mathbb{S}^{n-1}\right)$, which is linear and continuous. Moreover, its linear norm does not depend on the choice of ϵ. Regarding the extension one, we define a cut-off function χ (independent of ϵ), smooth, increasing, such that $\chi\left(\mathbb{R}_-\right) = \{1\}$ and $\chi\left(1,+\infty\right) = \{0\}$, from what we set

$$\mathcal{E} : \begin{array}{ccc} \mathcal{C}^{2,\alpha}_\delta\left([-s_\epsilon, s_\epsilon]\times\mathbb{S}^{n-1}\right) & \longrightarrow & \mathcal{C}^{2,\alpha}_\delta\left(\mathbb{R}\times\mathbb{S}^{n-1}\right) \\ f & \longmapsto & \mathcal{E}\left(f\right) \end{array}$$

where

$$\mathcal{E}\left(f\right) : (s,z) \longmapsto \begin{cases} f\left(s,z\right) & \text{if} \quad s \in [-s_\epsilon, s_\epsilon]\,; \\ \chi\left(s - s_\epsilon\right) f\left(s_\epsilon, z\right) & \text{if} \quad s \geqslant s_\epsilon; \\ \chi\left(-s - s_\epsilon\right) f\left(s_\epsilon, z\right) & \text{if} \quad s \leqslant -s_\epsilon. \end{cases}$$

By construction, the restriction of the function $\mathcal{E}\left(f\right)$ to $[-s_\epsilon, s_\epsilon]\times\mathbb{S}^{n-1}$ is f and its support is included in the compact set $[-s_\epsilon - 1, s_\epsilon + 1]\times\mathbb{S}^{n-1}$. Furthermore, $\mathcal{E}$ is linear, continuous and its linear norm does not depend on ϵ.

Henceforth, we can rewrite the mean curvature equation (7.2.68) as a fixed point for the functional

$$\mathcal{F}_c : \mathcal{C}^{2,\alpha}_{\delta}\left([-s_\epsilon, s_\epsilon] \times \mathbb{S}^{n-1}\right) \longmapsto \mathcal{C}^{2,\alpha}_{\delta}\left([-s_\epsilon, s_\epsilon] \times \mathbb{S}^{n-1}\right)$$

defined by

$$\mathcal{F}_c(v) = \mathcal{I} \circ L^{-1}_{c_\delta} \circ \mathcal{E}\Big[L_\epsilon(\omega_\Psi + v) + \varphi^{\frac{2-n}{2}} Q_{2,\epsilon}\left(\varphi^{\frac{n}{2}}(\omega_\Psi + v)\right) + \varphi^{\frac{n}{2}} Q_{3,\epsilon}\left(\varphi^{\frac{n}{2}}(\omega_\Psi + v)\right) - \frac{n(3n-2)}{4}\,\varphi^{2-2n} h_{\Psi^\perp}\Big].$$

The ω_Ψ-part

We are interested in the study of $\mathcal{F}_c(0)$. According to inequalities from propositions 6.3.3, 6.3.8, 6.3.10, together with the properties of L_ϵ, $Q_{2,\epsilon}$ and $Q_{3,\epsilon}$, the main part of the quantity

$$\left| L_\epsilon(\omega_\Psi) + \varphi^{\frac{2-n}{2}} Q_{2,\epsilon}\left(\varphi^{\frac{n}{2}}\omega_\Psi\right) + \varphi^{\frac{n}{2}} Q_{3,\epsilon}\left(\varphi^{\frac{n}{2}}\omega_\Psi\right) - \frac{n(3n-2)}{4}\,\varphi^{2-2n} h_{\Psi^\perp}\right|(s,z)$$

concentrates into the last term $\varphi^{2-2n}h_{\Psi^\perp}$ for $\epsilon < \epsilon_\kappa$ small enough. Therefore, for such ϵ,

$$\left\| L_\epsilon(\omega_\Psi) + \varphi^{\frac{2-n}{2}} Q_{2,\epsilon}\left(\varphi^{\frac{n}{2}}\omega_\Psi\right) + \varphi^{\frac{n}{2}} Q_{3,\epsilon}\left(\varphi^{\frac{n}{2}}\omega_\Psi\right) - \frac{n(3n-2)}{4}\,\varphi^{2-2n} h_{\Psi^\perp}\right\|_{L^\infty_\delta([-s_\epsilon,s_\epsilon]\times\mathbb{S}^{n-1})} \leqslant c(\cosh s_\epsilon)^{-\frac{n+2}{2}}\|\Psi\|,$$

where c does not depend on κ and $\|\Psi\| := \|\Psi\|_{\mathcal{C}^{2,\alpha}(\{\pm s_\epsilon\}\times\mathbb{S}^{n-1})}$. We prove the same kind of estimate in the weighted space $\mathcal{C}^{0,\alpha}_\delta([-s_\epsilon, s_\epsilon]\times\mathbb{S}^{n-1})$. Using propositions 6.2.5 and 6.2.6, we conclude that

$$\|\mathcal{F}_c(0)\|_{\mathcal{C}^{2,\alpha}_\delta([-s_\epsilon,s_\epsilon]\times\mathbb{S}^{n-1})} \quad\leqslant\quad c(\cosh s_\epsilon)^{-\frac{n+2}{2}}\|\Psi\|. \tag{7.2.69}$$

Remark 7.2.1 — Note that this previous estimate shows that in a neighbourhood of $|s| = s_\epsilon$, $F_c(0)(s,z)$ is small as compared with the term $h_{\Psi^\perp}(s,z)$. It is the first step to prove that the image of a small ball by $\mathcal{F}$ stays in the same ball.

The contracting part

In this paragraph, we demonstrate that $\mathcal{F}_c$ is a contracting operator in a small ball around 0. Let v_1 and v_2 be two functions of $\mathcal{C}^{2,\alpha}_\delta([-s_\epsilon, s_\epsilon]\times\mathbb{S}^{n-1})$ such that their $\mathcal{C}^{2,\alpha}_\delta-$ norms are less than $c(\cosh s_\epsilon)^{-\frac{n+2}{2}}\|\Psi\|$. We are interested in an estimate of the quantity $\mathcal{F}_c(v_1) - \mathcal{F}_c(v_2)$. By linearity, we note that it is enough to estimate the three quantities

$$L_\epsilon(v_1 - v_2), \qquad \varphi^{\frac{2-n}{2}}\left[Q_{2,\epsilon}\left(\varphi^{\frac{n}{2}}(\omega_\Psi + v_1)\right) - Q_{2,\epsilon}\left(\varphi^{\frac{n}{2}}(\omega_\Psi + v_2)\right)\right]$$

and

$$\varphi^{\frac{n}{2}}\left[Q_{3,\epsilon}\left(\varphi^{\frac{n}{2}}(\omega_\Psi + v_1)\right) - Q_{3,\epsilon}\left(\varphi^{\frac{n}{2}}(\omega_\Psi + v_2)\right)\right].$$

First term : According to the properties of L_ϵ, one checks that

$$\|L_\epsilon (v_1 - v_2)\|_{\mathcal{C}^{0,\alpha}_\delta([-s_\epsilon,s_\epsilon]\times\mathbb{S}^{n-1})} \leqslant \underbrace{c\,(\cosh s_\epsilon)^{2-2n}}_{\xrightarrow{\epsilon\to 0} 0} \|v_1 - v_2\|_{\mathcal{C}^{2,\alpha}_\delta([-s_\epsilon,s_\epsilon]\times\mathbb{S}^{n-1})}.$$

Second and third terms : We only deal with the second one since the third one follows from similar arguments. Using the property

$$|Q_{2,\epsilon}(f) - Q_{2,\epsilon}(g)| \leqslant c\max\{|f|,|g|\}\,|f-g|$$

together with the fact that the main part of $\omega_\Psi + v_i$ concentrates into ω_Ψ, we end up with

$$\begin{aligned}
&(\cosh s)^{-\delta}\varphi(s)^{\frac{2-n}{2}}\left|Q_{2,\epsilon}\left(\varphi^{\frac{n}{2}}(\omega_\Psi + v_1)\right) - Q_{2,\epsilon}\left(\varphi^{\frac{n}{2}}(\omega_\Psi + v_2)\right)\right|(s,z)\\
&\qquad\leqslant c\,(\cosh s)^{\frac{2+n}{2}}\left(\frac{\cosh s}{\cosh s_\epsilon}\right)^{\frac{n-2}{2}} \|\Psi\|\,\|v_1 - v_2\|_{\mathcal{C}^{2,\alpha}_\delta([-s_\epsilon,s_\epsilon]\times\mathbb{S}^{n-1})}\\
&\qquad\leqslant \underbrace{c\kappa\,(\cosh s_\epsilon)^{4-2n}}_{\xrightarrow{\epsilon\to 0} 0}\|v_1 - v_2\|_{\mathcal{C}^{2,\alpha}_\delta([-s_\epsilon,s_\epsilon]\times\mathbb{S}^{n-1})}.
\end{aligned}$$

Conclusion : for $\epsilon < \epsilon_\kappa$ small enough,

$$\|\mathcal{F}_c(v_1) - \mathcal{F}_c(v_2)\|_{\mathcal{C}^{2,\alpha}_\delta([-s_\epsilon,s_\epsilon]\times\mathbb{S}^{n-1})} \leqslant \frac{1}{2}\,\|v_1 - v_2\|_{\mathcal{C}^{2,\alpha}_\delta([-s_\epsilon,s_\epsilon]\times\mathbb{S}^{n-1})} \qquad (7.2.70)$$

7.2 — (a) Construction of hypersurface near the n-catenoid with prescribed curvature

According to the previous computations, we can deduce an important corollary by the fixed point theorem, namely :

Theorem 7.2.2
For all $\delta \in \left(\frac{n}{2}, \frac{n+2}{2}\right)$, there exists some constant $c := c(n,\alpha,\delta) > 0$ such that for all $\kappa > 0$, there exists $\epsilon_\kappa > 0$ such that :

for all $\epsilon \in (0,\epsilon_\kappa)$, for all Ψ which belongs to the space $\mathcal{C}^{2,\alpha}(\{\pm s_\epsilon\}\times\mathbb{S}^{n-1})$ and whose norm is less than $\kappa\epsilon r_\epsilon(\cosh s_\epsilon)^{2+\frac{n}{2}}$, there exists v_Ψ satisfying following assertions :

(i) the function $X_c + \varphi^{\frac{2-n}{2}}(\omega_\Psi + v_\Psi)N_\epsilon$ defines a hypersurface C_Ψ on $[-s_\epsilon, s_\epsilon]\times\mathbb{S}^{n-1}$ whose mean curvature is exaclty $\frac{1-|\chi_\epsilon(s)|}{\tilde{\ell}(s_\epsilon)}\varphi(s)^{-\frac{n+2}{2}}\phi^1_+(s)\langle\Psi^1_{even}, z\rangle$;

(ii) v_Ψ belongs to $\mathcal{C}^{2,\alpha}_\delta([-s_\epsilon,s_\epsilon]\times\mathbb{S}^{n-1})$ with

$$\|v_\Psi\|_{\mathcal{C}^{2,\alpha}_\delta([-s_\epsilon,s_\epsilon]\times\mathbb{S}^{n-1})} \leqslant 2c\,(\cosh s_\epsilon)^{-\frac{n+2}{2}}\|\Psi\|.$$

Remark 7.2.3 — It should be noted that for $|s| \in [s_\epsilon - 1, s_\epsilon]$, the mean curvature of Σ_ω vanishes. Therefore, it is minimal in a neighbourhood of its boundary.

7.2 — (b) The link with harmonic extensions

In the case of hyperplane, we already have seen that the role of harmonic extensions over $\mathbb{R}^n \backslash B_1$ is really of importance. Moreover, we also have seen that harmonic extensions on B_1 come naturally considering the resolution of the prescribed mean curvature equation : it is the second method we use in the proof of proposition 6.3.1.

Like in the deformation of hyperplane, it is essential to describe the behaviour of ω_Ψ near the boundary. In short, it amounts to prove that when s_ϵ is large enough, the term in Ψ_- is from very small contribution at $+s_\epsilon$ and reciprocally, the function behaves like harmonic extension. Heuristically, near the neighbourhood, the operator L_c looks like H_c up to a factor φ^{2-2n}. But this quantity is negligible in comparison with the constant $\left(\frac{n-2}{2}\right)^2$ when $|s|$ is large enough. It is the object of the following proposition.

Proposition 7.2.4 — *There exists a constant $c = c(n, \alpha)$ such that for all $\Psi \in \mathcal{C}^{2,\alpha}\left(\{\pm s_\epsilon\} \times \mathbb{S}^{n-1}\right)$,*

$$\left\| \varphi^{\frac{2-n}{2}}(s)\, \omega_\Psi(s,z) - x_\epsilon^{\frac{2-n}{2}} W^i(\Psi_+)\left(\frac{x}{x_\epsilon}\right) \right\|_{\mathcal{C}^{2,\alpha}([s_\epsilon-2, s_\epsilon]\times\mathbb{S}^{n-1})} \leqslant c \cosh(s_\epsilon)^{-\frac{3(n-2)}{2}} \|\Psi\|_{\mathcal{C}^{2,\alpha}(\{\pm s_\epsilon\}\times\mathbb{S}^{n-1})}, \quad (7.2.71)$$

where $x := \varphi(s)\, z$, $x_\epsilon := \varphi(s_\epsilon) = \frac{r_\epsilon}{\eta}$ and W^i is the interior harmonic extension operator defined in the beginning of the section 6.3. There is also the same kind of result regarding the lower part of the catenoid, but care must be taken to replace Ψ_+ by $-\Psi_-$.

Proof

Here, we only give a L^∞ estimate. Schauder's theory provides the $\mathcal{C}^{2,\alpha}$ case.

First of all, we decompose the problem into three parts by writing

$$\begin{aligned}
&\varphi^{\frac{2-n}{2}}(s)\, \omega_\Psi(s,z) - x_\epsilon^{\frac{2-n}{2}} W^i(\Psi_+)\left(\frac{x}{x_\epsilon}\right) \\
&= \varphi^{\frac{2-n}{2}}(s)\left[\omega_\Psi(s,z) - e^{\frac{n-2}{2}(s-s_\epsilon)} W^i(\Psi_+)\left(e^{s-s_\epsilon} z\right)\right] \\
&\quad + \varphi^{\frac{2-n}{2}}(s)\, e^{\frac{n-2}{2}(s-s_\epsilon)}\left[W^i(\Psi_+)\left(e^{s-s_\epsilon} z\right) - W^i(\Psi_+)\left(\frac{x}{x_\epsilon}\right)\right] \\
&\quad + \left[\varphi^{\frac{2-n}{2}}(s)\, e^{\frac{n-2}{2}(s-s_\epsilon)} - x_\epsilon^{\frac{2-n}{2}}\right] W^i(\Psi_+)\left(\frac{x}{x_\epsilon}\right).
\end{aligned}$$

First term : It is enough to collect the results of corollary 6.3.3 for the $h_{\Psi_\perp}$ part, those of propositions 6.3.8 and 6.3.10 for ℓ_Ψ and $\widetilde{\ell}_\Psi$ parts. We end up with

$$\left\| \varphi^{\frac{2-n}{2}}(s)\left[\omega_\Psi(s,z) - e^{\frac{n-2}{2}(s-s_\epsilon)} W^i(\Psi_+)\left(e^{s-s_\epsilon, z}\right)\right] \right\|_{L^\infty([s_\epsilon-2, s_\epsilon]\times\mathbb{S}^{n-1})} \leqslant c \cosh(s_\epsilon)^{-\frac{3(n-2)}{2}} \|\Psi\|_{\mathcal{C}^{2,\alpha}(\mathbb{S}^{n-1})}.$$

Second term : Here, the use the regularity of $W^i(\Psi_+)$. Indeed, after noticing the inequality

$$\left|\frac{e^s}{e^{s_\epsilon}} - \frac{|x|}{x_\epsilon}\right| \quad \leqslant \quad c\cosh(s_\epsilon)^{-2(n-1)},$$

we find

$$\left\|\varphi^{\frac{2-n}{2}}(s)\, e^{\frac{n-2}{2}(s-s_\epsilon)}\left[W^i(\Psi_+)\left(e^{s-s_\epsilon}z\right) - W^i(\Psi_+)\left(\frac{x}{x_\epsilon}\right)\right]\right\|_{L^\infty([s_\epsilon-2,s_\epsilon]\times\mathbb{S}^{n-1})}$$
$$\leqslant \quad c\cosh(s_\epsilon)^{\frac{-5n+6}{2}}\|\Psi\|_{\mathcal{C}^{2,\alpha}(\mathbb{S}^{n-1})}.$$

Last term : The inequality

$$\left\|\left[\varphi^{\frac{2-n}{2}}(s)\, e^{\frac{n-2}{2}(s-s_\epsilon)} - x_\epsilon^{\frac{2-n}{2}}\right] W^i(\Psi_+)\left(\frac{x}{x_\epsilon}\right)\right\|_{L^\infty([s_\epsilon-2,s_\epsilon]\times\mathbb{S}^{n-1})}$$
$$\leqslant \quad c\cosh(s_\epsilon)^{\frac{-5n+6}{2}}\|\Psi\|_{\mathcal{C}^{2,\alpha}(\mathbb{S}^{n-1})}$$

holds true because the Taylor expansion of φ in (5.1.34) implies

$$\varphi^{\frac{2-n}{2}s} \quad = \quad 2^{\frac{n-2}{2(n-1)}} e^{\frac{2-n}{2}s} + \mathcal{O}\left(e^{\frac{-5n+6}{2}s}\right)$$

Therefore, the main term is the first one since $-5n+6 < -3n+6$ and conclusion holds. $\square$

7.2 — (c) Description of the solution near its boundaries

We have in mind to obtain the behaviour of the solution near its boundaries. For that purpose, let us rescale the deformed n-catenoid by[5] η, ie $x = \frac{y}{\eta}$. Then, if C_Ψ denotes the perturbed catenoid we have constructed in theorem 7.2.2, the upper part of the associated rescaled hypersurface is the graph of the function

$$\overline{u}_{\Psi_+} : \begin{array}{rcl} \mathcal{A}_{1/2} & \longrightarrow & \mathbb{R} \\ t & \longmapsto & \eta\left(\psi + \varphi^{\frac{2-n}{2}}(\omega_\Psi + v_\Psi)\right)\left(\varphi^{-1}\left(\frac{y_\epsilon|t|}{\eta}\right), \frac{t}{|t|}\right) \end{array}$$

where $y = r_\epsilon t$ (recall that $r_\epsilon = \eta x_\epsilon$). The point in this dilatation lies in having a function defined over a normalised annulus $\mathcal{A}_{1/2}$. It will simplify the gluing in $\mathbb{S}^{n-1}$ with the function defined over $\mathcal{A}_1$ in the hyperplane case.

Theorem 7.2.5
(i) $\overline{u}_{\Psi_+}$ *is an element of* $\mathcal{C}^{2,\alpha}\left(\mathcal{A}_{1/2}\right)$ *and*

$$\left\|\overline{u}_{\Psi_+} - \eta\psi\left(\varphi^{-1}\left(\frac{r_\epsilon|\cdot|}{\eta}\right)\right) - W^i\left(\eta^{\frac{n}{2}} r_\epsilon^{\frac{2-n}{2}}\Psi_+\right)\right\|_{\mathcal{C}^{2,\alpha}\left(\mathcal{A}_{1/2}\right)}$$
$$\leqslant \quad 2c\epsilon r_\epsilon(\cosh s_\epsilon)^{\left(\delta-\frac{n+2}{2}\right)}. \quad (7.2.72)$$

5. We have introduced the dilatation factor η in (5.3.37).

In particular, if $\eta = \eta_a$ and $\overline{\mathfrak{d}}_{\Psi_+}$ denotes the difference defined by

$$\overline{\mathfrak{d}}_{\Psi_+} := \overline{u}_{\Psi_+} - \left(\eta\frac{H}{2} - a\epsilon r_\epsilon^{2-n}\left|\cdot\right|^{2-n} - \frac{(n-2)^3}{2(3n-4)}a^3\epsilon r_\epsilon\left|\cdot\right|^{4-3n}\right.$$
$$\left. + W^i\left(\eta^{\frac{n}{2}}r_\epsilon^{\frac{2-n}{2}}\Psi_+\right)\right),$$

then

$$\left\|\overline{\mathfrak{d}}_{\Psi_+}\right\|_{\mathcal{C}^{2,\alpha}\left(\mathcal{A}_{1/2}\right)} \leqslant 2c\epsilon r_\epsilon\left(\cosh s_\epsilon\right)^{\left(\delta-\frac{n+2}{2}\right)}. \tag{7.2.73}$$

(ii) If Ψ and $\overline{\Psi}$ are smaller than $\kappa\epsilon r_\epsilon\left(\cosh s_\epsilon\right)^{2+\frac{n}{2}}$, then

$$\left\|\overline{\mathfrak{d}}_{\Psi_+} - \overline{\mathfrak{d}}_{\overline{\Psi}_+}\right\|_{\mathcal{C}^{2,\alpha}\left(\mathcal{A}_{1/2}\right)} \leqslant c\left(\cosh s_\epsilon\right)^{2-n}\left\|\eta^{\frac{n}{2}}r_\epsilon^{\frac{2-n}{2}}\left(\Psi - \overline{\Psi}\right)\right\|_{\mathcal{C}^{2,\alpha}\left(\mathbb{S}^{n-1}\right)}. \tag{7.2.74}$$

Remark 7.2.6 — • Here, we can justify the choice of the norm of Ψ. As a matter of fact, the quantity $\eta^{\frac{n}{2}}r_\epsilon^{\frac{2-n}{2}}$ has rough estimate $\epsilon^{\frac{4+n}{6(n-1)}}$, i.e. $\cosh\left(s_\epsilon\right)^{-\left(2+\frac{n}{2}\right)}$. Thus, the rough estimate of $\eta^{\frac{n}{2}}r_\epsilon^{\frac{2-n}{2}}\Psi$ is $\kappa\epsilon r_\epsilon$, that is to say, the same than Φ we use in the hyperplane case. It is consistent for the gluing process.

- Besides, note that $\delta - \frac{n+2}{2}$ is negative, thus $\overline{\mathfrak{d}}_{\Psi_+}$ is small in comparison with ϵr_ϵ.
- Similar result holds for the lower part of the catenoid, but we have to mind the signs. More precisely, $\overline{u}_{\Psi_-}$ and $\overline{\mathfrak{d}}_{\Psi_-}$ are defined to be

$$\overline{u}_{\Psi_-} : t \in \mathcal{A}_{1/2} \longmapsto \eta\left(\psi + \varphi^{\frac{2-n}{2}}\left(\omega_\Psi + v_\Psi\right)\frac{\dot{\varphi}}{\varphi}\right)\left(\varphi^{-1}\left(-\frac{y_\epsilon\left|t\right|}{\eta}\right), \frac{t}{\left|t\right|}\right) \in \mathbb{R}$$

and

$$\overline{\mathfrak{d}}_{\Psi_-} := \overline{u}_{\Psi_-} + \left(\eta\frac{H}{2} - a\epsilon r_\epsilon^{2-n}\left|\cdot\right|^{2-n} - \frac{(n-2)^3}{2(3n-4)}a^3\epsilon r_\epsilon\left|\cdot\right|^{4-3n}\right.$$
$$\left. + W^i\left(\eta^{\frac{n}{2}}r_\epsilon^{\frac{2-n}{2}}\Psi_-\right)\right).$$

Proof

(i) Recall that $\varphi(s)z = \frac{r_\epsilon}{\eta}t$ and W^i is a linear operator. It follows that

$$\begin{aligned}
&\overline{u}_{\Psi_+}(t) - \eta\psi\varphi^{-1}\left(\frac{r_\epsilon\left|t\right|}{\eta}\right) - W^i\left(\eta^{\frac{n}{2}}r_\epsilon^{\frac{2-n}{2}}\Psi_+\right)(t) \\
= \;& \eta^{\frac{n}{2}}r_\epsilon^{\frac{2-n}{2}}\left|t\right|^{\frac{2-n}{2}}v_\Psi(s,z) + \eta^{\frac{n}{2}}r_\epsilon^{\frac{2-n}{2}}\left|t\right|^{\frac{2-n}{2}}\left(\omega_\Psi(s,z) - \left|t\right|^{\frac{n-2}{2}}W^i\left(\Psi_+\right)(t)\right).
\end{aligned}$$

Now, we must give an upper bound for the two above terms. By existence theorem 7.2.2, we can estimate v_Ψ in order to obtain the inequality, for all $s \in (s_\epsilon - 2, s_\epsilon)$

$$|v_\Psi(s,z)| \quad \leqslant \quad c(\cosh s_\epsilon)^{-\frac{n+2}{2}+\delta} \|\Psi\|_{\mathcal{C}^{2,\alpha}(\{\pm s_\epsilon\}\times\mathbb{S}^{n-1})}.$$

Regarding the second term, we use proposition 7.2.4 to find

$$\left|\omega_\Psi(s,z) - |t|^{\frac{n-2}{2}} W^i(\Psi_+)(t)\right| \quad \leqslant \quad c(\cosh s_\epsilon)^{-(n-2)} \|\Psi\|_{\mathcal{C}^{2,\alpha}(\{\pm s_\epsilon\}\times\mathbb{S}^{n-1})}.$$

This last one is negligible in comparison with the first one. Consequently, we end up with the estimate (7.2.72) by noticing

$$\eta^{\frac{n}{2}} r_\epsilon^{\frac{2-n}{2}} |t|^{\frac{2-n}{2}} (\cosh s_\epsilon)^{-\frac{n+2}{2}+\delta} \|\Psi\|_{\mathcal{C}^{2,\alpha}(\{\pm s_\epsilon\}\times\mathbb{S}^{n-1})} \approx \kappa\epsilon r_\epsilon (\cosh s_\epsilon)^{\delta-\frac{n+2}{2}}$$

and by choosing κ large enough ($\kappa \approx 2c$).
The estimate (7.2.73) comes from the above one, together with the work of section 5.3. Note that $\mathcal{O}\left(\eta^{5(n-1)} r_\epsilon^{6-5n}\right) = o(\epsilon r_\epsilon)$.

(ii) We study difference of two solutions with different boundary data. According to the linearity of $\Psi \mapsto \omega_\Psi$ and the linearity of W^i, we find

$$\begin{aligned}\mathfrak{d}_{\Psi_+}(t) - \mathfrak{d}_{\overline{\Psi}_+}(t) \quad &= \quad \eta^{\frac{n}{2}} r_\epsilon^{\frac{2-n}{2}} |t|^{\frac{2-n}{2}} (v_\Psi - v_{\overline{\Psi}})(s,z) \\ &\quad + \eta^{\frac{n}{2}} r_\epsilon^{\frac{2-n}{2}} |t|^{\frac{2-n}{2}} \left(\omega_{\Psi-\overline{\Psi}}(s,z) - |t|^{\frac{n-2}{2}} W^i\left(\Psi_+ - \overline{\Psi}_+\right)(t)\right).\end{aligned}$$

The second term is already handled during the analysis of (i) and we get :

$$\begin{aligned}&\left\|\omega_{\Psi-\overline{\Psi}}(s,z) - |t|^{\frac{n-2}{2}} W^i\left(\Psi_+ - \overline{\Psi}_+\right)(t)\right\|_{\mathcal{C}^{2,\alpha}((s_\epsilon-2,s_\epsilon)\times\mathbb{S}^{n-1})} \\ &\qquad \leqslant c(\cosh s_\epsilon)^{-(n-2)} \left\|\Psi - \overline{\Psi}\right\|_{\mathcal{C}^{2,\alpha}(\{\pm s_\epsilon\}\times\mathbb{S}^{n-1})}. \qquad (7.2.75)\end{aligned}$$

The difficulty lies in the estimation of the difference $v_\Psi - v_{\overline{\Psi}}$. The method we use is the same kind of the one we use for the hyperplane case, namely the PDE that this difference satisfies — see the proof of theorem 4.2.1. Indeed, we can write

$$L_c(v_\Psi - v_{\overline{\Psi}}) \quad = \quad L_\epsilon(v_\Psi - v_{\overline{\Psi}}) + L_\epsilon\left(\omega_{\Psi-\overline{\Psi}}\right) + e + f + g,$$

where

$$\begin{aligned} e \quad &:= \quad \varphi^{\frac{2-n}{2}}\left[Q_{2,\epsilon}\left(\varphi^{\frac{n}{2}}(\omega_\Psi + v_\Psi)\right) - Q_{2,\epsilon}\left(\varphi^{\frac{n}{2}}(\omega_{\overline{\Psi}} + v_{\overline{\Psi}})\right)\right], \\ f \quad &:= \quad \varphi^{\frac{n}{2}}\left[Q_{3,\epsilon}\left(\varphi^{\frac{n}{2}}(\omega_\Psi + v_\Psi)\right) - Q_{3,\epsilon}\left(\varphi^{\frac{n}{2}}(\omega_{\overline{\Psi}} + v_{\overline{\Psi}})\right)\right],\end{aligned}$$

and

$$g \quad := \quad -\frac{n(3n-2)}{4}\varphi^{2-2n}\left(h_{\Psi_\perp} - h_{\overline{\Psi}_\perp}\right).$$

It remains to give estimates for the five quantities.

- First of all, according to the definition of L_ϵ, we check

$$\begin{aligned}\|L_\epsilon (v_\Psi - v_{\overline{\Psi}})\|_{\mathcal{C}^{0,\alpha}((s_\epsilon-2,s_\epsilon)\times\mathbb{S}^{n-1})} \\ \leqslant \quad c(\cosh s_\epsilon)^{2-2n} \|v_\Psi - v_{\overline{\Psi}}\|_{\mathcal{C}^{2,\alpha}((s_\epsilon-2,s_\epsilon)\times\mathbb{S}^{n-1})} .\end{aligned}$$

- Likewise, since

$$\left\|\omega_{\Psi-\overline{\Psi}}\right\|_{\mathcal{C}^{2,\alpha}((s_\epsilon-2,s_\epsilon)\times\mathbb{S}^{n-1})} \quad \leqslant \quad c\left\|\Psi-\overline{\Psi}\right\|_{\mathcal{C}^{2,\alpha}(\{\pm s_\epsilon\}\times\mathbb{S}^{n-1})} ,$$

  we get

$$\left\|L_\epsilon\left(\omega_{\Psi-\overline{\Psi}}\right)\right\|_{\mathcal{C}^{0,\alpha}((s_\epsilon-2,s_\epsilon)\times\mathbb{S}^{n-1})} \quad \leqslant \quad c(\cosh s_\epsilon)^{2-2n}\left\|\Psi-\overline{\Psi}\right\|_{\mathcal{C}^{2,\alpha}(\{\pm s_\epsilon\}\times\mathbb{S}^{n-1})} .$$

- For all $(s,z)\in(s_\epsilon-2,s_\epsilon)\times\mathbb{S}^{n-1}$, using properties of $Q_{2,\epsilon}$ like we have done to prove that $\mathcal{F}_c$ is a contracting operator, we demonstrate that

$$\begin{aligned}|e(s,z)| \quad \leqslant \quad & c\kappa(\cosh s_\epsilon)^{4-2n}\left\|\Psi-\overline{\Psi}\right\|_{\mathcal{C}^{2,\alpha}(\{\pm s_\epsilon\}\times\mathbb{S}^{n-1})} \\ & + c\kappa(\cosh s_\epsilon)^{4-2n}\|v_\Psi - v_{\overline{\Psi}}\|_{\mathcal{C}^{2,\alpha}((s_\epsilon-2,s_\epsilon)\times\mathbb{S}^{n-1})} .\end{aligned}$$

  This estimate holds in L^∞, but Schauder's estimates prove that there is the same kind of estimate in Hölder space.
- With similar method, we get

$$\begin{aligned}\|f\|_{\mathcal{C}^{0,\alpha}((s_\epsilon-2,s_\epsilon)\times\mathbb{S}^{n-1})} \quad \leqslant \quad & c\kappa^2(\cosh s_\epsilon)^{6-3n}\left\|\Psi-\overline{\Psi}\right\|_{\mathcal{C}^{2,\alpha}(\{\pm s_\epsilon\}\times\mathbb{S}^{n-1})} \\ & + c\kappa^2(\cosh s_\epsilon)^{6-3n}\|v_\Psi - v_{\overline{\Psi}}\|_{\mathcal{C}^{2,\alpha}((s_\epsilon-2,s_\epsilon)\times\mathbb{S}^{n-1})} .\end{aligned}$$

- Finally, following inequality holds :

$$\|g\|_{\mathcal{C}^{0,\alpha}((s_\epsilon-2,s_\epsilon)\times\mathbb{S}^{n-1})} \quad \leqslant \quad c(\cosh s_\epsilon)^{2-2n}\left\|\Psi-\overline{\Psi}\right\|_{\mathcal{C}^{2,\alpha}(\{\pm s_\epsilon\}\times\mathbb{S}^{n-1})} .$$

We deduce from above calculus that

$$\begin{aligned}&\|L_c(v_\Psi - v_{\overline{\Psi}})\|_{\mathcal{C}^{2,\alpha}((s_\epsilon-2,s_\epsilon))} \\ &\quad \leqslant \quad c\,a_\kappa\left(\left\|\Psi-\overline{\Psi}\right\|_{\mathcal{C}^{2,\alpha}(\{\pm s_\epsilon\}\times\mathbb{S}^{n-1})} + \|v_\Psi - v_{\overline{\Psi}}\|_{\mathcal{C}^{2,\alpha}((s_\epsilon-2,s_\epsilon)\times\mathbb{S}^{n-1})}\right),\end{aligned}$$

where $a_\kappa = \left[(\cosh s_\epsilon)^{2-2n} + \kappa(\cosh s_\epsilon)^{4-2n} + \kappa^2(\cosh s_\epsilon)^{6-3n}\right]$ tends to 0 when ϵ tends to 0 and whose main term is $\kappa(\cosh s_\epsilon)^{4-2n}$. Consequently,

$$(1-ca_\kappa)\|v_\Psi - v_{\overline{\Psi}}\|_{\mathcal{C}^{2,\alpha}((s_\epsilon-2,s_\epsilon)\times\mathbb{S}^{n-1})} \quad \leqslant \quad ca_\kappa\left\|\Psi-\overline{\Psi}\right\|_{\mathcal{C}^{2,\alpha}(\{\pm s_\epsilon\}\times\mathbb{S}^{n-1})} ,$$

from what we deduce that at fixed κ, for $\epsilon\leqslant\epsilon_\kappa$, we find

$$\begin{aligned}&\|v_\Psi - v_{\overline{\Psi}}\|_{\mathcal{C}^{2,\alpha}((s_\epsilon-2,s_\epsilon)\times\mathbb{S}^{n-1})} \\ &\quad \leqslant \quad c\kappa(\cosh s_\epsilon)^{4-2n}\left\|\Psi-\overline{\Psi}\right\|_{\mathcal{C}^{2,\alpha}(\{\pm s_\epsilon\}\times\mathbb{S}^{n-1})} . \qquad (7.2.76)\end{aligned}$$

According to inequalities (7.2.75) and (7.2.76) together with $4-2n\leqslant 2-n$, the conclusion holds, up to reduce the parameter ϵ_κ. $\square$

8 The gluing process : proof of theorem 1.0.7

In previous sections, we have developed a way to build minimal hypersurfaces near hyperplanes with small catenoidal necks and hypersurfaces with prescribed small mean curvature near truncated catenoids. In this section, we explain how to glue these different hypersurfaces in order to obtain more complex ones that look like a family of horizontal hyperplanes which are linked together by small truncated catenoids.

First of all, according to the regularity theory for minimal hypersurfaces (cf. [CM05]), if a minimal hypersurface is $\mathcal{C}^1$, then it is $\mathcal{C}^\infty$. Therefore, it is enough to perform a $\mathcal{C}^1$ gluing to ensure the connected sum has the mean curvature we want to prescribe.

Besides, as said before, since the quantity $\eta^{\frac{n}{2}} r_\epsilon^{\frac{2-n}{2}}$ behaves like $(\cosh s_\epsilon)^{-2-\frac{n}{2}}$, it will be shrewder to work with a boundary data $\Upsilon := \eta^{\frac{n}{2}} r_\epsilon^{\frac{2-n}{2}} \Psi$ regarding the catenoid. Note in this case that Υ and Φ have the same rough estimate $\kappa\epsilon r_\epsilon$.

Furthermore, a small perturbation of one of the weighted points generates a small continuous perturbation of η, $\Gamma_{\text{cor},\epsilon}$ and $u_{\Phi,P}$. We will allow ourselves to perturb the parameters in a small neighbourhood.

8.1 The gluing equations

The point is to glue the hypersurfaces we build near the singularities $p_{k,j}$. In this purpose, we use the description we gave for the deformed horizontal hyperplane $u_{\Phi,P}$ over $\mathcal{A}_1$ together with the one we gave for the truncated catenoid $\overline{u}_{\Upsilon_\pm}$ over $\mathcal{A}_{1/2}$. Then the point lies in $\mathcal{C}^1$-matching the boundary data over $\partial B_1 = \mathbb{S}^{n-1}$. This is why we have performed changes in scales.

As done before, it is wise to distinguish the modes 0 and 1 (especially the s-even and the s-odd ones) from the others. So we write the gluing equations we obtain by projection on those different modes.

8.1 — (a) The choice of boundary data Φ for hyperplanes and Υ for catenoids

Because of the different symmetries of the problem (especially the rotations and the translations which preserve the mean curvature), we can't choose any boundary data.

For example, in the N-periodic case, or in the case of a non periodic hypersurface with $N+1$ horizontal ends, we consider the following boundary data

$$(\Phi, \Upsilon) \quad = \quad \Big(\big(\Phi_{p_{k,j,\pm}}, \Upsilon_{p_{k,j,\pm}} \big)_{j \in [\![1,n_k]\!]} \Big)_{k \in [\![0,N-1]\!]}$$

for which we enforce the vectors [6] $\Upsilon^1_{k,j,\text{even}}$ to be equal to $(\Upsilon^1_{\text{even}} + \mathcal{R} p_{k,j})$, where Υ^1_{even}

6. Recall we identify the first eigenmode of $Upsilon^1_{k,j}$ with a vector in $\mathbb{R}^n$ — see definition 6.3.6.

is a vector of W and $\mathcal{R}$ is an element of $\mathrm{Skew}_{\mathbf{t}_h}$ such that

$$\max\left(\left\|\Upsilon^1_{\text{even}}\right\|_\infty, \|\mathcal{R}\|_\infty\right) \quad \leqslant \quad \kappa\epsilon r_\epsilon.$$

Notice that Υ^1_{even} and $\mathcal{R}$ do not depend on the points $p_{k,j}$. We have made this choice to ensure that for well chosen parameter ν, the mean curvature of the hypersurface we construct vanishes — see the proof of proposition 8.3.1.

Therefore, there are $2 \cdot 2 \cdot \mathrm{Ne}$ boundary data functions over $\mathbb{S}^n$. The first factor 2 comes from the gluing with upper or lower level ; the second 2 corresponds to different boundary data for the hyperplane or the catenoid.

For the N-periodic case, this above definition holds because

$$(a_{k+N,j}, p_{k+N,j}) = (a_{k,j}, p_{k,j} + \mathbf{t}_h)$$

for all $k \in \mathbb{N}$ and $j \in [\![1, n_k]\!]$. Thus it is enough to perform the gluing on a period. The data boundary for other levels are obtained by

$$\left(\Phi_{p_{k+N\mathbb{Z},j,\pm}}, \Upsilon_{p_{k+N\mathbb{Z},j,\pm}}\right) \quad := \quad \left(\Phi_{p_{k,j,\pm}}, \Upsilon_{p_{k,j,\pm}}\right).$$

In all cases, $\Phi_{k,j,-}$ (resp. $\Phi_{k,j,+}$) is the boundary data we want to enforce at the hyperplane k (resp. $k+1$) at point $p_{k,j}$ while $\Psi_{k,j,+}$ (resp. $\Psi_{k,j,-}$) is the boundary data we want to enforce for the upper (resp. lower) part of the catenoid at point $p_{k,j}$ at the hyperplane $k+1$ (resp. k). Besides, we define the norm of Φ as the maximum of the $\mathcal{C}^{2,\alpha}(\mathbb{S})$-norms of its different elements, in other words,

$$\|\Phi\|_{\mathcal{C}^{2,\alpha}} \quad := \quad \max_{k,j,\pm} \left\|\Phi_{p_{k,j,\pm}}\right\|_{\mathcal{C}^{2,\alpha}(\mathbb{S})},$$

and we have the same kind of definition for $\|\Upsilon\|_{\mathcal{C}^{2,\alpha}}$.

8.1 — (b) Shape of the gluing equation

We consider one of the gluing points, say $p_{k,j}$ for example. We have in mind to $\mathcal{C}^1$-match the boundary data of the minimal hypersurface we obtain for the k-th hyperplane at $p_{k,j}$ (which points upwards) to that of the lower part of the catenoid we obtain with parameter $\eta = \eta_{k,j}$. Besides, we want to proceed likewise regarding the $k+1$-th deformed hyperplane at $p_{k,j}$ (which points downwards) to that of the upper part of the above catenoid.

Before writing equations, let us note that it is more convenient to translate the different hypersurfaces. Let $h_{k,-}$ be the height of the k-th horizontal hyperplane, $h_{k,+}$ the the height of the $k+1$-th horizontal hyperplane and δh_k be the height difference between the $k+1$-th and k-th ones, that is to say $\delta h_k = h_{k+1,-} - h_{k,-} = h_{k,+} - h_{k-1,+}$. We will determine this quantity when we perform the gluing of modes 0 in proposition 8.2.1. Let $\mathbf{t}_{k,j,v} = t_{k,j,v}\mathbf{e}_{n+1}$ be a vertical vector where $t_{k,j,v}$ is a positive number we will determine in section 8.2 — (a). Instead of considering the catenoid $C_{p_{k,j}}$ centred in 0, we rather consider this one after a translation of $\mathbf{t}_{k,j,v} + p_{k,j}$.

It follows that the gluing equation at point $p_{k,j}$ can be written

$$\forall z \in \mathbb{S}^{n-1}, \left\{ \begin{array}{rcl} u_{\Phi,p_{k,j,\pm}}(z) + h_{k,\pm} & = & \overline{u}_{\Upsilon,p_{k,j,\pm}}(z) + t_{k,j,v}, \\ \partial_r u_{\Phi,p_{k,j,\pm}}(z) & = & \partial_r \overline{u}_{\Upsilon,p_{k,j,\pm}}(z), \end{array} \right. \tag{8.1.77}$$

where the index $\pm$ is $-$ at level k and is $+$ at level $k+1$.

Here, it is relevant to use the description of local behaviour near the boundaries that we have given in previous description theorems 7.2.5 and 4.2.1 in order to rewrite the above system. Recall that

$$\begin{aligned} u_{\Phi,p_{k,j,\pm}} = \mathfrak{d}_{\Phi,p_{k,j,\pm}} \mp a_{k,j}\epsilon r_\epsilon^{2-n} \left|\cdot\right|^{2-n} \mp \frac{(n-2)^3}{2(3n-4)} a_{k,j}^3 \epsilon r_\epsilon \left|\cdot\right|^{4-3n} \\ + \epsilon C_{k,j,\pm} + \epsilon r_\epsilon \left\langle \cdot, F_{k,j,\pm} \right\rangle + W^e\left(\Phi_{p_{k,j,\pm}}\right) \end{aligned}$$

and

$$\begin{aligned} \overline{u}_{\Upsilon,p_{k,j,\pm}} = \overline{\mathfrak{d}}_{\Phi,p_{k,j,\pm}} \mp a_{k,j}\epsilon r_\epsilon^{2-n} \left|\cdot\right|^{2-n} \mp \frac{(n-2)^3}{2(3n-4)} a_{k,j}^3 \epsilon r_\epsilon \left|\cdot\right|^{4-3n} \\ \pm \eta_{k,j}\frac{H}{2} \pm W^i\left(\Upsilon_{p_{k,j,\pm}}\right), \end{aligned}$$

where the sign $\mp$ is the opposite of $\pm$.

Remark 8.1.1 — According to the choice of $\eta_{k,j}$ we have explained in section 5.3, the terms $|x|^{2-n}$ and $|x|^{4-3n}$ are the same ones in the right and left members.

Consequently, the system (8.1.77) is written

$$\forall z \in \mathbb{S}^{n-1}, \left\{ \begin{array}{rl} \mathfrak{d}_{\Phi,p_{k,j,\pm}}(z) + \epsilon C_{k,j,\pm} + \epsilon r_\epsilon \left\langle \cdot, F_{k,j,\pm}\right\rangle + W^e\left(\Phi_{p_{k,j,\pm}}\right)(z) + h_{k,\pm} & \\ = \overline{\mathfrak{d}}_{\Upsilon,p_{k,j,\pm}}(z) \pm \eta_{k,j}\frac{H}{2} \pm W^i\left(\Upsilon_{p_{k,j,\pm}}\right)(z) + t_{k,j,v}, & \\ \partial_r\left(\mathfrak{d}_{\Phi,p_{k,j,\pm}} + \epsilon r_\epsilon \left\langle \cdot, F_{k,j,\pm}\right\rangle + W^e\left(\Phi_{p_{k,j,\pm}}\right)\right)(z) & \\ = \partial_r\left(\overline{\mathfrak{d}}_{\Upsilon,p_{k,j,\pm}} \pm W^i\left(\Upsilon_{p_{k,j,\pm}}\right)\right)(z). & \end{array}\right. \tag{8.1.78}$$

8.1 — (c) Gluing equations after orthogonal projections on eigenmodes

We perform the projection of the solutions on the different modes since they do not behave in the same way.

The mode 0. We find following equations at $p_{k,j}$:

$$\left\{ \begin{array}{rcl} h_{k,\pm} + C_{k,j,\pm}\,\epsilon + \Phi^0_{p_{k,j,\pm}} + \pi^0\left(\mathfrak{d}_{\Phi,p_{k,j,\pm}}\right) & = & \pm\eta_{k,j}\frac{H}{2} \pm \Upsilon^0_{p_{k,j,\pm}} + \pi^0\left(\overline{\mathfrak{d}}_{\Upsilon,p_{k,j,\pm}}\right) + t_{k,j,v}, \\ (2-n)\,\Phi^0_{p_{k,j,\pm}} + \pi^0\left(\partial_r \mathfrak{d}_{\Phi,p_{k,j,\pm}}\right) & = & \pi^0\left(\partial_r \overline{\mathfrak{d}}_{\Upsilon,p_{k,j,\pm}}\right), \end{array}\right. \tag{8.1.79}$$

where the sign $\pm$ is $-$ at level k and is $+$ at level $k+1$. Mind the change of signs in front the boundary data Υ and $\eta_{k,j}\frac{H}{2}$.

The mode 1. When f is a function over $\mathbb{S}$, we write $\pi^1(f) = F^1$ the vector such that for all $z \in \mathbb{S}$, $f^1(z) = \left\langle F^1, z\right\rangle$. Then equations can be written at $p_{k,j}$:

$$\left\{ \begin{array}{rcl} F_{k,j,\pm}\epsilon r_\epsilon + \Phi^1_{p_{k,j,\pm}} + \pi^1\left(\mathfrak{d}_{\Phi,p_{k,j,\pm}}\right) & = & \Upsilon^1_{p_{k,j},\text{odd}} \pm \left(\Upsilon^1_{\text{even}} + \mathcal{R}p_{k,j}\right) + \pi^1\left(\overline{\mathfrak{d}}_{\Upsilon,p_{k,j,\pm}}\right), \\ F_{k,j,\pm}\epsilon r_\epsilon + (1-n)\,\Phi^1_{p_{k,j,\pm}} + \pi^1\left(\partial_r \mathfrak{d}_{\Phi,p_{k,j,\pm}}\right) & = & \Upsilon^1_{p_{k,j},\text{odd}} \pm \left(\Upsilon^1_{\text{even}} + \mathcal{R}p_{k,j}\right) + \pi^1\left(\partial_r \overline{\mathfrak{d}}_{\Upsilon,p_{k,j,\pm}}\right). \end{array}\right. \tag{8.1.80}$$

The mode $\perp$. Regarding the modes 2, 3, etc., we get

$$\left\{\begin{array}{rcl} \Phi^{\perp}_{p_{k,j,\pm}} + \pi^{\perp}\left(\mathfrak{d}_{\Phi,p_{k,j,\pm}}\right) & = & \pm\Upsilon^{\perp}_{p_{k,j,\pm}} + \pi^{\perp}\left(\overline{\mathfrak{d}}_{\Upsilon,p_{k,j,\pm}}\right), \\ \partial_r W^e\left(\Phi^{\perp}_{p_{k,j,\pm}}\right) + \pi^{\perp}\left(\partial_r \mathfrak{d}_{\Phi,p_{k,j,\pm}}\right) & = & \partial_r W^i\left(\pm\Upsilon^{\perp}_{p_{k,j,\pm}}\right) + \pi^{\perp}\left(\partial_r \mathfrak{d}_{\Upsilon,p_{k,j,\pm}}\right). \end{array}\right.$$

Indeed, it is more ingenious to rewrite the above system by highlighting the contracting part. In this purpose, we define the operator $\mathcal{H}$ by

$$\mathcal{H} : f \in \mathcal{C}^{2,\alpha}\left(\mathbb{S}^{n-1}\right) \longmapsto \partial_r\left(\left(W^e - W^i\right)(f)\right)_{|\mathbb{S}^{n-1}} \in \mathcal{C}^{1,\alpha}\left(\mathbb{S}^{n-1}\right).$$

It is known, according to the article [MP99] by R. Mazzeo et F. Pacard, that $\mathcal{H}$ is an isomorphism. Besides, the space $E^{\perp}$ is $\mathcal{H}$-stable. Therefore the system turns into :

$$\left\{\begin{array}{rcl} \Phi^{\perp}_{p_{k,j,\pm}} & = & \mathcal{H}^{-1}\circ\pi^{\perp}\left[\partial_r\left(\left(Id - W^i\right)\left(\overline{\mathfrak{d}}_{\Upsilon,p_{k,j,\pm}} - \mathfrak{d}_{\Phi,p_{k,j,\pm}}\right)\right)\right], \\ \pm\Upsilon^{\perp}_{p_{k,j,\pm}} & = & \Phi^{\perp}_{p_{k,j,\pm}} - \pi^{\perp}\left(\overline{\mathfrak{d}}_{\Upsilon,p_{k,j,\pm}} - \mathfrak{d}_{\Phi,p_{k,j,\pm}}\right). \end{array}\right. \tag{8.1.81}$$

8.2 Resolution of gluing equations

In this section, we explain how to solve all gluing equations. One of the key arguments is the fixed point theorem. Of course, we decompose the resolution into three parts : one for each mode we consider.

We briefly expose the ideas to solve these different equations. The mode 1 is the most complex : it is the last we solve.

- Regarding the mode 0, the main point lies in choosing suitable vertical translations parameters — it is the object of proposition 8.2.1 — then we use a fixed point theorem by using the contracting properties we have demonstrated for both cases.
- For the mode 2, 3, etc, the same kind of fixed point holds true.
- For the mode 1, we highlight the balanced and non degenerate conditions and we make use of a Brouwer fixed point theorem by changing parameters of the construction.

8.2 — (a) The mode 0, the height of horizontal hyperplanes and the height of catenoids

In the resolution of this mode, very different terms take place. More exactly, when k is fixed, there are :

- The height of k-th level $h_{k,-} = h_{k-1,+}$. I has to be the same for all $p_{k,j}$ with $j \in [\![1, n_k]\!]$.
- The vertical translation term $\mathbf{t}_{k,j,v}$ for the catenoid we glue at $p_{k,j}$ between the levels k and $k+1$. It does not depend on level k or level $k+1$.
- The constant term $\pm\eta_{k,j}\frac{H}{2}$ which we could interpret as the height of the catenoid at $p_{k,j}$. It has to be the same at levels k and $k+1$.
- The constant term $C_{k,j,\pm}\,\epsilon$ that comes from Green function.
- The constant terms that come from the orthogonal projection of Φ and Υ on mode 0.

- Very small constant terms that come from perturbation of Green function (the $\mathfrak{d}$-part) or catenoid (the $\overline{\mathfrak{d}}$-part).

For all weighted configuration $\{a_{k,j}, p_{k,j}\}$, we define the constant $C_{k,j}$ by

$$\begin{aligned} C_{k,j} &= C_{k,j,-} - C_{k,j,+} \\ &= 2\sum_{\substack{i=1\\ i\neq j}}^{n_k} a_{k,j} \left|p_{k,j} - p_{k,i}\right|^{2-n} - \sum_{i=1}^{n_{k-1}} a_{k-1,j} \left|p_{k,j} - p_{k-1,i}\right|^{2-n} \\ &\qquad - \sum_{i=1}^{n_{k+1}} a_{k+1,j} \left|p_{k,j} - p_{k+1,i}\right|^{2-n}. \end{aligned}$$

Note that this constant only depends on the configuration but not depends on the small parameter ϵ. Besides, this definition is similar to the one of the force $F_{k,j}$.

Determining the heights of hyperplanes, the translation vectors and a well chosen weighted configuration. In this paragraph, we suppose we are given a configuration $\{a_k, p_{k,j}\}$ together with the height $h_{0,-} \in \mathbb{R}$ of level 0.

According to the second equation of the system (8.1.79), $\Phi^0_{p_{k,j,\pm}}$ is such that

$$\Phi^0_{p_{k,j,\pm}} \quad = \quad \frac{1}{n-2}\pi^0 \left[\partial r \left(\mathfrak{d}_{\Phi,p_{k,j,\pm}} - \overline{\mathfrak{d}}_{\Upsilon,p_{k,j,\pm}}\right)\right], \tag{8.2.82}$$

therefore its rough estimate is given by $c\epsilon r_\epsilon \cosh\left(s_\epsilon\right)^{\delta-\frac{n+2}{2}}$. Thus Φ^0 is necessarily small in comparison with $\kappa\epsilon r_\epsilon$.

Most of the information lies in the first equation of system (8.1.79) whose main terms are the heights $h_{k,\pm}$ of hyperplanes, the half height of catenoids $\pm\eta_{k,j}\frac{H}{2}$ and the constant term $C_{k,j,\pm}\epsilon$. Moreover, if we subtract the equation at level k from the one at level $k+1$, we get

$$h_{k,+} - h_{k,-} \quad = \quad \eta_k H + C_{k,j}\left(\{(a_k, p_{k,j})\}\right)\epsilon + E'_{k,j}\left(\epsilon, \{a_k, p_{k,j}\}, \Phi, \Upsilon\right) \tag{8.2.83}$$

where $\eta_k := \left((n-2)\, a_k \epsilon\right)^{\frac{1}{n-1}}$, the error function $E'_{k,j}$ is continuous and its L^∞ norm is smaller than $\kappa\epsilon r_\epsilon$, thus is very small in comparison with the other terms. Therefore, if $h_{0,-}$ denotes a fixed real number, we define the height of other levels $h_{k,\pm}$ and the vertical vector $\mathbf{t}_{k,j,v}$ to be

$$h_{k,+} = h_{k+1,-} \quad := \quad h_{k,-} + \eta_k H + C_{k,1}\left(\{a_k, p_{k,j}\}\right)\epsilon, \tag{8.2.84}$$

$$\mathbf{t}_{k,j,v} \quad := \quad \left[h_{k,+} - \eta_k \frac{H}{2} + C_{k,j,+}\left(\{a_k, p_{k,j}\}\right)\epsilon\right]\mathbf{e}_{n+1}. \tag{8.2.85}$$

However, the difference of heights between the hyperplanes $k+1$ and k has to be independent of the choice of the gluing point $p_{k,j}$ for $j \in [\![1, n_k]\!]$. This kind of problem occurs when there are several catenoids to glue between those hyperplanes, i.e. when $n_k \geqslant 2$. This is why we introduce the following proposition. The reader could notice that the $h_{k,\pm}$ and $\mathbf{t}_{k,j,v}$ continuously depend on the weighted configuration $\{a_k, p_{k,j}\}$.

Proposition 8.2.1 — *Given a configuration $\{a_k, p_{k,j}\}$ and the real number $h_{0,-}$, we define the $h_{k,\pm}$ and $\mathbf{t}_{k,j,v}$ as above. Then there exists a $\mathcal{C}^1$ "weight" mapping*

$$\mathcal{W} : \left(\mathbb{R}_+^*\right)^N \times W^{Ne} \longmapsto \left(\mathbb{R}_+^* \times W\right)^{Ne}$$

such that the weighted configuration $\{\{a_{k,j}, p_{k,j}\}\} = \mathcal{W}(\{a_k, p_{k,j}\})$ satisfies following assertions :

(i) for all $j \in [\![1, n_k]\!]$, we can write $a_{k,j} = a_k\left(1 + \epsilon^{\frac{n-2}{n-1}} \alpha_{k,j}\right)$ with $\alpha_{k,1} = 0$ and $|\alpha_{k,j}| \leqslant M$ for some positive constant M ;

(ii) for all k such that $n_k \geqslant 2$, for all $j \neq j' \in [\![1, n_k]\!]$,

$$(\eta_{k,j} - \eta_{k,j'})\, H + \epsilon\,(C_{k,j} - C_{k,j'})\,(\{a_{l,i}, p_{l,i}\}) \quad = \quad \underset{\epsilon \to 0}{\mathcal{O}}\left(\epsilon^{\frac{2n-3}{n-1}}\right). \qquad (8.2.86)$$

Besides, $\mathcal{W}$ does not depend on ϵ and does not change the placement of the points.

Proof

It turns out that solving the equation (8.2.86) at order 1 is enough. Moreover, note that the problem can be reduced to the case $j = 1$ and any j'. More exactly, when $\alpha_{l,i}$ is bounded, we rewrite it as

$$1 - \left(1 + \epsilon^{\frac{n-2}{n-1}} \alpha_{k,j'}\right)^{\frac{1}{n-1}} \quad = \quad -\epsilon^{\frac{n-2}{n-1}} c_k\,(C_{k,1} - C_{k,j'})\,(\{a_l, p_{l,i}\}) + \underset{\epsilon \to 0}{\mathcal{O}}\left(\epsilon^{2\frac{n-2}{n-1}}\right),$$

where $c_k := ((n-2)\, a_k)^{-\frac{1}{n-1}} H^{-1}$. This equation justifies we look for parameters $a_{k,j'}$ whose form is given by $a_k\left(1 + \epsilon^{\frac{n-2}{n-1}} \alpha_{k,j'}\right)$. The main part of the right member does not depend on the family $(\alpha_{l,i})_{l,i}$. A Taylor expansion when ϵ is small gives

$$\frac{1}{n-1} \alpha_{k,j'} \quad = \quad c_k\,(C_{k,1} - C_{k,j'})\,(\{a_l, p_{l,i}\}) + \underset{\epsilon \to 0}{\mathcal{O}}\left(\epsilon^{\frac{n-2}{n-1}}\right),$$

therefore $\alpha_{k,j'} := (n-1)\, c_k\,(C_{k,1} - C_{k,j'})\,(\{a_l, p_{l,i}\})$ suits to the problem. □

Determining the boundary data Φ^0 and Υ^0. From now on, given initial weights $\{a_k\}$, we build from any configuration $\{p_{k,j}\}$ a new configuration $\{a_{k,j}, p_{k,j}\}$ given by the mapping $\mathcal{W}$.

Proposition 8.2.2 — *Let (Φ^i, Υ^i) be elements whose mode is i for $i = 1, \perp$. Assume their $\mathcal{C}^{2,\alpha}$ norm is smaller than $\kappa \epsilon r_\epsilon$. Then there exists (Φ^0, Υ^0) such that if $\Phi = \Phi^0 + \Phi^1 + \Phi^\perp$ and $\Upsilon = \Upsilon^0 + \Upsilon^1 + \Upsilon^\perp$, then following assertions hold :*

(i) the gluing equation (8.1.79) of mode 0 is satisfied ;

(ii) $\max\{|\Phi^0|, |\Upsilon^0|\} \leqslant c\epsilon r_\epsilon \cosh(s_\epsilon)^{\delta - \frac{n-2}{2}}$.

Furthermore, (Φ^0, Υ^0) continuously depends on the parameters ϵ, $\{a_k\}$, $\{p_{k,j}\}$, $(\Phi^i, \Upsilon^i)_{i \in \{2,\perp\}}$ and is a contraction mapping on the variables $(\Phi^i, \Upsilon^i)_{i \in \{2,\perp\}}$.

Proof
As said before, Φ^0 has to satisfy equation (8.2.82). Therefore, according to the theorems 4.2.1 and 7.2.5, it describes the $\Phi^0_{p_{k,j,\pm}}$ as functions which depend, in a contracting way, on $(\Phi^i, \Upsilon^i)_{i\in\{1,\perp\}}$ and on Υ^0. Besides, these functions are continuous on ϵ and $\{p_{l,i}\}_{l,i}$. By a standard fixed point with parameters, we can solve this equation.

It remains to determine Υ^0. For example, at level k, we use the definition of $h_{k,\pm}$ and $\mathbf{t}_{k,j,v}$ to rewrite the first equation of the system (8.1.79) as

$$\begin{aligned}
\Upsilon^0_{p_{k,j,-}} \quad = \quad & (\eta_{k,1} - \eta_{k,j})\,\frac{H}{2} + \epsilon\left[C_{k,1}\left(\{a_{l,i}, p_{l,i}\}\right) - C_{k,j}\left(\{a_{l,i}, p_{l,i}\}\right)\right] \\
& + \epsilon\left[C_{k,1}\left(\{a_l, p_{l,i}\}\right) - C_{k,1}\left(\{a_{l,i}, p_{l,i}\}\right)\right] \\
& + \epsilon\left[C_{k,j,+}\left(\{a_l, p_{l,i}\}\right) - C_{k,j,+}\left(\{a_{l,i}, p_{l,i}\}\right)\right] \\
& \quad - \Phi^0_{p_{k,j,-}} + \pi^0\left(\overline{\mathfrak{d}}_{\Upsilon,p_{k,j,-}} - \mathfrak{d}_{\Phi,p_{k,j,-}}\right).
\end{aligned}$$

According to the weighted configuration proposition 8.2.1, the first three lines of the right member do not depend on the choice of (Φ, Υ) and their norms have the rough estimate

$$\epsilon^{\frac{2n-3}{n-1}} \quad = \quad \underset{\epsilon\to 0}{o}\left(\epsilon r_\epsilon \cosh\left(s_\epsilon\right)^{\delta - \frac{n-2}{2}}\right).$$

With similar arguments than of the Φ^0 case, the conclusions follows. ☐

8.2 — (b) The mode $\perp$

This mode is by far the easiest to solve since it is a straightforward application of the contracting properties. Once again, we suppose we are given the slightly perturbed configuration $\{a_{k,j}, p_{k,j}\}$. As a matter of fact, the system (8.1.81) is "almost" solved. More precisely, we can check the two equations describe $\Phi^\perp_{p_{k,j,\pm}}$ and $\Upsilon^\perp_{p_{k,j,\pm}}$ as functions which depend, in a contracting way, on Φ and Υ : according to the description theorems 4.2.1 and 7.2.5 together with properties of $\mathcal{H}$, we prove that for $\epsilon \leqslant \epsilon_\kappa$,

$$\max\left\{\left\|\Phi^\perp_{p_{k,j,\pm}}\right\|_{\mathcal{C}^{2,\alpha}(\mathbb{S})}, \left\|\Upsilon^\perp_{p_{k,j,\pm}}\right\|_{\mathcal{C}^{2,\alpha}(\mathbb{S})}\right\} \quad \leqslant \quad \frac{1}{2}\max\left\{\|\Phi\|_{\mathcal{C}^{2,\alpha}}, \|\Upsilon\|_{\mathcal{C}^{2,\alpha}}\right\}.$$

Note also that the right members of system (8.1.81) continuously depend on weighted points, Φ and Υ and their $\mathcal{C}^{2,\alpha}$-norms are less than $c\epsilon r_\epsilon \cosh\left(s_\epsilon\right)^{\delta-\frac{n+2}{2}}$. By a standard fixed point theorem with parameters together with the previous proposition, we get the following :

Proposition 8.2.3 — *Let (Φ^1, Υ^1) be elements whose mode is 1. Assume its $\mathcal{C}^{2,\alpha}$ norm is smaller than $\kappa\epsilon r_\epsilon$. Then there exists $\left(\Phi^0, \Phi^\perp, \Upsilon^0, \Upsilon^\perp\right)$ such that if $\Phi = \Phi^0 + \Phi^1 + \Phi^\perp$ and $\Upsilon = \Upsilon^0 + \Upsilon^1 + \Upsilon^\perp$, then the following assertions hold true :*

(i) the gluing equations (8.1.79) *and* (8.1.81) *of modes* 0 *and* $\perp$ *are satisfied ;*

(ii) $\max\left\{|\Phi^0|,|\Upsilon^0|,\left\|\Phi^\perp\right\|_{\mathcal{C}^{2,\alpha}},\left\|\Upsilon^\perp\right\|_{\mathcal{C}^{2,\alpha}}\right\} \leqslant c\epsilon r_\epsilon \cosh(s_\epsilon)^{\delta-\frac{n-2}{2}}$.

Furthermore, $\left(\Phi^0,\Phi^\perp,\Upsilon^0,\Upsilon^\perp\right)$ *continuously depends on the parameters* ϵ*,* $\{a_k\}$*,* $\{p_{k,j}\}$*,* (Φ^1,Υ^1) *and is a contraction mapping on the variables* (Φ^1,Υ^1)*.*

8.2 — (c) The mode 1 : balanced and non-degenerate configurations in the periodic case

This mode is the trickiest one to solve. Its resolution implies the use of hypothesis regarding the *balanced* and *non-degenerate* conditions. First of all, we have to rewrite the system (8.1.80) in order to highlight the contracting part. If we subtract the second equation to the first one, we find

$$\Phi^1_{p_{k,j,\pm}} = \frac{1}{n}\pi^1\left[(Id-\partial_r)\left(\bar{\mathfrak{d}}_{\Upsilon,p_{k,j,\pm}}-\bar{\mathfrak{d}}_{\Phi,p_{k,j,\pm}}\right)\right],$$

therefore a similar fixed point theorem with parameters to the one we have used for the mode $\perp$ holds. Besides, the norm of $\Phi^1_{p_{k,j,\pm}}$ admits the rough estimate $2c\epsilon r_\epsilon\cosh(s_\epsilon)^{\delta-\frac{n+2}{2}}$.

It remains to solve the first equation of the system (8.1.80). The main problem lies in the fact that $\Upsilon^1_{k,j,\text{even}}$ is enforced to be equal to $\Upsilon^1_{\text{even}}+\mathcal{R}p_{k,j}$. Therefore, there are *a priori* more equations than unknowns. Let us write the equations at levels k and $k+1$:

$$\begin{cases} F_{k,j,-}\epsilon r_\epsilon+\Phi^1_{p_{k,j,-}}+\pi^1\left(\mathfrak{d}_{\Phi,p_{k,j,-}}\right) = \Upsilon^1_{p_{k,j},\text{odd}}-\left(\Upsilon^1_{\text{even}}+\mathcal{R}p_{k,j}\right)+\pi^1\left(\bar{\mathfrak{d}}_{\Upsilon,p_{k,j,-}}\right), \\ F_{k,j,+}\epsilon r_\epsilon+\Phi^1_{p_{k,j,+}}+\pi^1\left(\mathfrak{d}_{\Phi,p_{k,j,+}}\right) = \Upsilon^1_{p_{k,j},\text{odd}}+\left(\Upsilon^1_{\text{even}}+\mathcal{R}p_{k,j}\right)+\pi^1\left(\bar{\mathfrak{d}}_{\Upsilon,p_{k,j,+}}\right), \end{cases}$$

from what we deduce that

$$\begin{aligned} 2\Upsilon^1_{p_{k,j},\text{odd}} &= \left(F_{k,j,-}+F_{k,j,+}\right)\epsilon r_\epsilon+\left(\Phi^1_{p_{k,j,-}}+\Phi^1_{p_{k,j,+}}\right) \\ &\quad+\pi^1\left(\mathfrak{d}_{\Phi,p_{k,j,-}}-\bar{\mathfrak{d}}_{\Upsilon,p_{k,j,-}}+\mathfrak{d}_{\Phi,p_{k,j,+}}-\bar{\mathfrak{d}}_{\Upsilon,p_{k,j,+}}\right) \end{aligned} \tag{8.2.87}$$

together with

$$\begin{aligned} 2\left(\Upsilon^1_{\text{even}}+\mathcal{R}p_{k,j}\right) &= F_{k,j}\epsilon r_\epsilon+\left(-\Phi^1_{p_{k,j,-}}+\Phi^1_{p_{k,j,+}}\right) \\ &\quad+\pi^1\left(-\mathfrak{d}_{\Phi,p_{k,j,-}}+\bar{\mathfrak{d}}_{\Upsilon,p_{k,j,-}}+\mathfrak{d}_{\Phi,p_{k,j,+}}-\bar{\mathfrak{d}}_{\Upsilon,p_{k,j,+}}\right), \end{aligned} \tag{8.2.88}$$

where $F_{k,j}$ is the force at $p_{k,j}$ defined in definition 1.0.1 :

$$F_{k,j} := \left(-F_{k,j,-}+F_{k,j,+}\right).$$

As before, the equation (8.2.87) can be solved by similar arguments to the previous ones. To deal with the equation (8.2.88) is quite more difficult because of the term $\Upsilon^1_{\text{even}}+\mathcal{R}p_{k,j}$. It is the object of the next proposition.

Proposition 8.2.4 — *Let $\mathring{C} = \{\mathring{a}_k, \mathring{p}_{k,j}\}$ be a balanced and non-degenerate configuration. Then for all $\delta \in \left(\frac{n}{2}, \frac{n+2}{2}\right)$, for all $\epsilon < \epsilon_0$ small enough and for all $\beta \in \left(0, \frac{1}{3(n-1)}\left(\frac{n+2}{2} - \delta\right)\right)$, there exists a weighted points configuration $\{a_{k,j}, p_{k,j}\}$ such that :*

(i) all the gluing equations are solved ;

(ii) for all k, j, $|p_{k,j} - \mathring{p}_{k,j}| \leqslant \epsilon^\beta$.

(iii) **The $\mathbf{t}_h$-periodic case with $\mathbf{t}_h \neq 0$:** *the configuration $\{a_{k,j}, p_{k,j}\}$ is $\mathbf{t}_h$-periodic and the weights $a_{k,j}$ are associated to the configuration $\{\mathring{a}_k, p_{k,j}\}$. In particular, $a_{k,j} - \mathring{a}_k = \mathcal{O}\left(\epsilon^{\frac{n-2}{n-1}}\right)$.*

The $\mathbf{t}_h$-periodic case with $\mathbf{t}_h = 0$: *the configuration $\{a_{k,j}, p_{k,j}\}$ is 0-periodic and the weights $a_{k,j}$ are associated to the configuration $\{a_k, p_{k,j}\}$ where $|a_k - \mathring{a}_k| \leqslant \epsilon^\beta$. In particular, $a_{k,j} - \mathring{a}_k = \mathcal{O}\left(\epsilon^\beta\right)$.*

The non-periodic case : *the weights $a_{k,j}$ are associated to the configuration $\{a_k, p_{k,j}\}$ where $|a_k - \mathring{a}_k| \leqslant \epsilon^\beta$. In particular, $a_{k,j} - \mathring{a}_k = \mathcal{O}\left(\epsilon^\beta\right)$.*

Besides, in the periodic case, the hypersurface $\mathscr{S}$ we obtain is $\mathbf{t}$-periodic, where $\mathbf{t}$ can be decomposed as $\mathbf{t} = \mathbf{t}_h + \mathbf{t}_v$ and its vertical component $\mathbf{t}_v$ is

$$\mathbf{t}_v = \sum_{k=0}^{N-1} \left[\eta_k H + \epsilon C_{k,1}\left(\{a_l, p_{l,i}\}\right)\right] \mathbf{e}_{n+1}.$$

In the non-periodic case, $|\mathbf{t}_v|$ is the distance between the two extremal horizontal hyperplanes.

Remark 8.2.5 — Note that $\epsilon^{\frac{n-2}{n-1}} = o_{\epsilon\to 0}\left(\epsilon^\beta\right)$. In the periodic case with non-vanishing period, we can solve the equations without changing the weights $\mathring{a}_k$ while it is not the same for the other cases in which we have to deal with the influence of the dilation.

Proof

The idea lies in applying a well chosen Brouwer fixed point : we prove that if we slightly deform the initial configuration of points $\{\mathring{a}_k, \mathring{p}_{k,j}\}$, then we can solve the problem.

To any configuration $\{a_k, p_{k,j}\}$ such that

$$|p_{k,j} - \mathring{p}_{k,j}| \leqslant \epsilon^\beta \qquad \text{and} \qquad |a_k - \mathring{a}_k| \leqslant \epsilon^\beta$$

is continuously associated a new close weighted configuration

$$\{a_{k,j}, p_{k,j}\} = \mathcal{W}\left(\{a_k, p_{k,j}\}\right)$$

given by the proposition 8.2.1. According to previous paragraphs, for all Υ^1_{even} and $\mathcal{R}$ such that their norm is less than $\kappa\epsilon r_\epsilon$, we can find boundary data (Φ, Υ) such that all gluing equations are solved except the equation (8.2.88) together with

$$\forall k, \forall j, \qquad \Upsilon^1_{k,j,\text{even}} = \Upsilon^1_{\text{even}} + \mathcal{R} p_{k,j}$$

Besides, Φ and Υ continuously depend on Υ^1_{even}, $\mathcal{R}$ and $\{a_{k,j}, p_{k,j}\}$.

From now on, we assume the configuration is either 0-periodic or non-periodic [7]. To relieve notations, we define

$$\widetilde{\Upsilon}^1_{\text{even}} := \frac{2}{\epsilon r_\epsilon}\Upsilon^1_{\text{even}} \qquad \text{and} \qquad \widetilde{\mathcal{R}} := \frac{2}{\epsilon r_\epsilon}\mathcal{R}$$

so that their norm has rough estimate 1. Then we can rewrite all the equations (8.2.88) as

$$\mathscr{G}\left(\{a_k, p_{k,j}\}, \widetilde{\Upsilon}^1_{\text{even}}, \widetilde{\mathcal{R}}\right) \quad = \quad \mathscr{E}\left(\{a_k, p_{k,j}\}, \widetilde{\Upsilon}^1_{\text{even}}, \widetilde{\mathcal{R}}\right) \tag{8.2.89}$$

where the mapping $\mathscr{G}$ is defined to be

$$\begin{aligned}&\mathscr{G} : \left(\{a_k, p_{k,j}\}, \widetilde{\Upsilon}^1_{\text{even}}, \widetilde{\mathcal{R}}\right) \in \left(\mathbb{R}^*_+\right)^N \times W^{\text{Ne}} \times W \times \text{Skew}_{\mathbf{t}_h} \\ &\longmapsto \mathscr{F} \circ \mathcal{W}\left(\{a_k, p_{k,j}\}\right) - \left(\widetilde{\Upsilon}^1_{\text{even}}, \cdots, \widetilde{\Upsilon}^1_{\text{even}}\right) - \left(\widetilde{\mathcal{R}}p_{0,1}, \cdots, \widetilde{\mathcal{R}}p_{N-1,n_{N-1}}\right) \in W^{\text{Ne}},\end{aligned}$$

the force function $\mathscr{F} : \{a_{k,j}, p_{k,j}\} \longmapsto \{F_{k,j}\}$ is defined in (1.0.2) and the error function $\mathscr{E}$ is such that

$$\mathscr{E} : \left(\{a_k, p_{k,j}\}, \widetilde{\Upsilon}^1_{\text{even}}, \widetilde{\mathcal{R}}\right) \longmapsto \left\{E'_{k,j}\left(\{a_l, p_{l,i}\}_{l,i}, \Upsilon^1_{\text{even}}, \widetilde{\mathcal{R}}\right)\right\}_{k,j}.$$

The error functions $E'_{k,j}$ are continuous and their L^∞-norm is smaller than $2c\cosh(s_\epsilon)^{\delta-\frac{n+2}{2}}$. Necessarily, equation (8.2.89) implies the forces to be small : this is why we are close to the *balanced* configuration.

Description of $\mathscr{G}$. We claim that $\mathscr{G}$ is locally a $\mathcal{C}^1$-submersion at point $(\mathring{C}, 0, 0)$. More precisely, the rank of a matrix is an open property. Since $\mathcal{W}$ slightly changes the weights, the rank of $\mathrm{d}\mathscr{G}$ is the same than the rank of $\mathrm{d}\mathscr{G}^0$ at $(\mathring{C}, 0, 0)$, where $\mathscr{G}^0$ is defined to be

$$\mathscr{G}^0 : \left(\{a_k, p_{k,j}\}, \widetilde{\Upsilon}^1_{\text{even}}, \widetilde{\mathcal{R}}\right) \longmapsto \mathscr{F}\left(\{a_k, p_{k,j}\}\right) - \left(\widetilde{\Upsilon}^1_{\text{even}} + \widetilde{\mathcal{R}}p_{0,1}, \cdots\right).$$

Differentiation at $(\mathring{C}, 0, 0)$ then leads us to

$$\begin{aligned}&\mathrm{d}\mathscr{G}^0_{(\mathring{C},0,0)} : \left(\{w_k, \mathbf{p}_{k,j}\}, \boldsymbol{\Upsilon}^1_{\text{even}}, \mathbf{R}\right) \in \mathbb{R}^N \times W^{\text{Ne}} \times W \times \text{Skew}_{\mathbf{t}_h} \\ &\longmapsto \mathrm{d}\mathscr{F}_{\mathring{C}}\left(\{w_k, \mathbf{p}_{k,j}\}\right) - \left(\boldsymbol{\Upsilon}^1_{\text{even}} + \mathbf{R}\mathring{p}_{0,1}, \cdots, \boldsymbol{\Upsilon}^1_{\text{even}} + \mathbf{R}\mathring{p}_{N-1,n_{N-1}}\right) \in W^{\text{Ne}}.\end{aligned}$$

Because of the non-degenerate hypothesis, the first term $\mathrm{d}\mathscr{F}_{\mathring{C}}$ has full rank $(\text{Ne}-1)\,m - \dim(\text{Skew}_{\mathbf{t}_h})$. Thus the idea is to prove the contribution of $\boldsymbol{\Upsilon}^1_{\text{even}}$ and $\mathbf{R}$ makes up for the loss of $m + \dim(\text{Skew}_{\mathbf{t}_h})$ dimensions. But recall the definitions of the kernels V_t and V_d in the introduction. Then it is enough to demonstrate that

$$\text{Im}\left(\mathrm{d}\mathscr{F}_{\mathring{C}}\right) + V_t + V_r \quad = \quad W^{\text{Ne}}.$$

7. See the remark 8.2.6 for the periodic case with $\mathbf{t}_h \neq 0$

We have already established that the sum $V_t + V_r$ is direct. Besides, its dimension is $m + \dim(\mathrm{Skew}_{\mathbf{t}_h})$. Therefore, it remains to prove

$$\mathrm{Im}\left(\mathrm{d}\mathscr{F}_{\mathring{C}}\right) \cap \left(V_t \oplus V_r\right) \quad = \quad \{0\}\,.$$

Consider an element e in this intersection. Then there exists $\left(\{w_k, \mathbf{p}_{k,j}\}, \boldsymbol{\Upsilon}^1_{\text{even}}, \mathbf{R}\right)$ such that e is read as

$$e \;=\; \mathrm{d}\mathscr{F}_{\mathring{C}}\left(\{w_k, \mathbf{p}_{k,j}\}\right) \;=\; \left(\boldsymbol{\Upsilon}^1_{\text{even}} + \mathbf{R}p_{0,1}, \cdots, \boldsymbol{\Upsilon}^1_{\text{even}} + \mathbf{R}p_{N-1,n_{N-1}}\right).$$

Furthermore, recall that we have provided forces relations in (1.0.3) and (1.0.4) which are valid for any configuration. The differentiation of the first one implies

$$\sum_{k,j} \mathring{a}_k \mathrm{d}\left(F_{k,j}\right)_{\mathring{C}}\left(\{w_l, \mathbf{p}_{l,i}\}\right) \quad = \quad 0 \quad = \quad \sum_{k,j} \mathring{a}_k \left(\boldsymbol{\Upsilon}^1_{\text{even}} + \mathbf{R}\mathring{p}_{k,j}\right)$$

and since the configuration $\mathring{C}$ is balanced,

$$\sum_{k,j} \mathring{a}_k \left\langle \mathrm{d}\left(F_{k,j}\right)_{\mathring{C}}\left(\{w_l, \mathbf{p}_{l,i}\}\right), \mathbf{R}\mathring{p}_{k,j}\right\rangle = 0 = \sum_{k,j} \mathring{a}_k \left\langle \boldsymbol{\Upsilon}^1_{\text{even}} + \mathbf{R}\mathring{p}_{k,j}, \mathbf{R}\mathring{p}_{k,j}\right\rangle$$

regarding the second one. Therefore, injecting the first relation into the second one, we end with

$$\left(\sum_{k'} n_{k'}\mathring{a}_{k'}\right) \cdot \left(\sum_{k,j} \mathring{a}_k \left|\mathbf{R}\mathring{p}_{k,j}\right|^2\right) \quad = \quad \left|\sum_{k,j} \mathring{a}_k \mathbf{R}\mathring{p}_{k,j}\right|^2.$$

Consequently, by standard Cauchy-Schwarz inequality, all $\mathbf{R}\mathring{p}_{k,j}$ are aligned, thus $\mathbf{R}$ vanishes and so does $\boldsymbol{\Upsilon}^1_{\text{even}}$, from what we deduce $e = 0$.
Therefore, $\mathscr{G}^0$ is locally a submersion near initial configuration $\mathring{C}$.

Last fixed point theorem. According to the submersion theorem, there exists a neighbourhood $\mathcal{U}$ of $(\mathring{C}, 0, 0)$, a $\mathcal{C}^1$-diffeomorphism

$$\Lambda : \mathcal{U} \longmapsto \Lambda\left(\mathcal{U}\right) \subset \mathbb{R}^N \times W^{\mathrm{Ne}} \times W \times \mathrm{Skew}_{\mathbf{t}_h}$$

with

$$\Lambda\left(\mathring{C}, 0, 0\right) \quad = \quad \left(\{0, 0\}, 0, 0\right)$$

such that

$$\mathscr{G} \circ \Lambda^{-1}\left(\left\{a'_k, p'_{k,j}\right\}, \Upsilon'^1_{\text{even}}, \mathcal{R}'\right) \quad = \quad \mathscr{F} \circ \mathcal{W}\left(\mathring{C}\right) + \left(p'_{0,1}, \cdots, p'_{N-1,n_{N-1}}\right).$$

Up to reducing $\mathcal{U}$, we assume that for r small enough, $\Lambda\left(U\right)$ is the compact convex set

$$\Lambda\left(U\right) \quad = \quad [-r, r]^N \times \left(B_W\left(0, r\right)\right)^{\mathrm{Ne}} \times \overline{B}_W\left(0, r\right) \times \overline{B}_{\mathrm{Skew}_{\mathbf{t}_h}}\left(0, r\right).$$

The choice of r does not depend on $\epsilon < \epsilon_0$ where ϵ_0 is small enough. By construction, the size (or the diameter) of $\mathcal{U}$ is of same type than the size of $\Lambda(\mathcal{U})$, that is to say r since $\mathscr{G}$ behaves like $\mathscr{G}^0$ which does not depend on ϵ. Then, if we look for a solution of (8.2.89) in $\mathcal{U}$, we can rewrite it as

$$\mathscr{F} \circ \mathcal{W}\left(\mathring{C}\right) + \left(p'_{0,1}, \cdots, p'_{N-1,n_{N-1}}\right) \quad = \quad \mathscr{E} \circ \Lambda^{-1}\left(\left\{a'_k, p'_{k,j}\right\}, \Upsilon'^1_{\text{even}}, \mathcal{R}'\right).$$

To apply a classical Brouwer fixed point theorem, it remains to give rough estimates of the different quantities which appear in the above equation. We claim that for $\epsilon < \epsilon_0$ small enough and for $r := \epsilon^\beta$, the inclusion $\mathscr{E}(\mathcal{U}) \subset \mathscr{G}(\mathcal{U})$ holds true.

As a matter of fact, for all $(\{a_k, p_{k,j}\}, \widetilde{\Upsilon}^1_{\text{even}}, \widetilde{\mathcal{R}}) \in \mathcal{U}$, we have the estimate

$$\mathscr{E}\left(\{a_k, p_{k,j}\}, \widetilde{\Upsilon}^1_{\text{even}}, \widetilde{\mathcal{R}}\right) \quad = \quad \underset{\epsilon\to 0}{\mathcal{O}}\left(\cosh(s_\epsilon)^{\delta - \frac{n+2}{2}}\right) \quad = \quad \underset{\epsilon\to 0}{o}\left(\epsilon^\beta\right).$$

Note that this is why we have chosen such a definition of β. Besides, according to the choice of the $a_{k,j}$ in proposition 8.2.1 together with the *balanced* condition of the initial configuration $\mathring{C}$, we get

$$\mathscr{G}\left(\mathring{C}, 0, 0\right) = \mathscr{F} \circ \mathcal{W}\left(\mathring{C}\right) = \underset{\epsilon\to 0}{\mathcal{O}}\left(\epsilon^{\frac{n-2}{n-1}}\right) = \underset{\epsilon\to 0}{o}\left(\epsilon^\beta\right),$$

therefore $\mathscr{E}(\mathcal{U}) \subset \mathscr{G}(\mathcal{U})$ is true for ϵ small enough.

It follows that for all

$$\left(\{a'_k\}, \Upsilon'^1_{\text{even}}, \mathcal{R}'\right) \qquad \text{such that} \qquad \left\|\{a'_k\}, \Upsilon'^1_{\text{even}}, \mathcal{R}'\right\| \leqslant \epsilon^\beta,$$

we are looking for a configuration

$$\left\{p'_{k,j}\right\} \in W^{\mathrm{Ne}} \qquad \text{such that} \qquad \left|p'_{k,j}\right| \leqslant \epsilon^\beta$$

which satisfies the equation

$$\left(p'_{0,1}, \cdots, p'_{N-1,n_{N-1}}\right) \quad = \quad \mathscr{E} \circ \Lambda^{-1}\left(\left\{a'_k, p'_{k,j}\right\}, \Upsilon'^1_{\text{even}}, \mathcal{R}'\right) + \mathscr{G}\left(\mathring{C}\right)$$

is solved. Since $\mathscr{E}$ is continuous, it is also the case of $\mathscr{E} \circ \Lambda^{-1}$. The right member of the above equation is made of vectors whose norm is bounded by is much more smaller than ϵ^β while the left member is made of vectors whose norm describes $\left[0, \epsilon^\beta\right]$ by construction. besides, both of us are continuous data. According to the Brouwer fixed point, there exists a solution which depends on the choice of $(\{a'_k\}, \Upsilon'^1_{\text{even}}, \mathcal{R}')$. It follows that

$$\left(\{a_k, p_{k,j}\}, \Upsilon^1_{\text{even}}, \mathcal{R}\right) \quad := \quad \Lambda\left(\left\{P'_{k,j}\left(\{a'_k\}, \boldsymbol{\Upsilon}'^1_{\text{even}}, \mathcal{R}'\right)\right\}, \boldsymbol{\Upsilon}^1_{\text{even}}, \mathcal{R}\right)$$

is a solution to the equation (8.2.89), QED. □

Remark 8.2.6 — The proof is a similar one in the periodic case when $\mathbf{t}_h \neq 0$, except it is not necessary to change the weights. As a matter of fact, suppose the family $\{a_k\} = \{\mathring{a}_k\}$ is fixed. Since the kernel of the force function $\mathscr{F}_{\{\mathring{a}_k\}} : W^{\mathrm{Ne}} \longmapsto W^{\mathrm{Ne}}$ is exactly $V_t \oplus V_r$, to say $\mathscr{F}$ is non-degenerate at $\{\mathring{a}_k, \mathring{p}_{k,j}\}$ is the same than to say the differential of $\mathscr{F}_{\{\mathring{a}_k\}}$ at $\{\mathring{p}_{k,j}\}$ has full rank $\mathrm{Ne} - \dim(V_t \oplus V_r)$. Thus the same proof with fixed $\{a_k\}$ holds true.

8.3 The mean curvature vanishes

To end with this section, it remains to prove that the mean curvature of the smooth hypersurface $\mathscr{S}$ we have built by gluing process vanishes everywhere. We know by construction that the parts which are close to the horizontal hypersurfaces are minimal hypersurfaces, but it is not yet the case for the small catenoids. It is the object of following proposition.

Proposition 8.3.1 — *Assume the weight parameter $\nu < \frac{3-n}{2}$. Then when ϵ is small enough, $\mathscr{S}$ is minimal.*

Proof

We only give the proof in the periodic case. The non-periodic one can be demonstrated in the same way.

The result comes from the first variation formula of area by using the mean curvature on small catenoids which behaves like the Jacobi field associated to the horizontal translations.

Reductio ad absurdum, suppose $\mathscr{S}$ is not minimal. Then its mean curvature does not vanish anywhere, that is to say $(\Upsilon^1_{\text{even}}, \mathcal{R}) \neq (0,0)$, where Υ^1_{even} is a vector in W and $\mathcal{R}$ is a skew-symmetric matrix such that the boundary data satisfies

$$\Upsilon^1_{k,j,\text{even}} \quad = \quad \Upsilon^1_{\text{even}} + \mathcal{R} p_{k,j}.$$

First of all, let us introduce some notations. We denote by $\widetilde{\mathscr{S}} \subset \mathscr{S}$ an element which represents the periodic surface $\mathscr{S}$, that is to say such that $\widetilde{\mathscr{S}} + \mathbf{t}\mathbb{Z} = \mathscr{S}$. Without loss of generality, we assume $\widetilde{\mathscr{S}}$ contains the levels $0, 1, \cdots, N-1$. Finally, for large radius $R > R_0$, we denote by $\mathscr{S}_R$ the intersection of the cylinder $B(0,R) \times \mathbb{R}$ with the surface $\widetilde{\mathscr{S}}$, in other words

$$\mathscr{S}_R = (B(0,R) \times \mathbb{R}) \cap \widetilde{\mathscr{S}}.$$

The ball $B(0,R)$ denotes a subset in $\mathbb{R}^n$. Besides, we assume R_0 is large enough to ensure all the $p_{k,j}$ are in this ball, for $k = 0, \cdots, N-1$.

Let $\mathbf{w}$ be a vector of $W \times \{0\}$ and $r = \exp \mathbf{R}'$ be the rotation of $\mathbb{R}^n \times \mathbb{R}$ such that

$$\mathbf{R}' \quad := \quad \begin{pmatrix} \mathcal{R}' & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathcal{R}' \in \mathrm{Skew}_{\mathbf{t}_h}.$$

Then the area of $\mathscr{S}_R$ is the same than the translated and rotated surface $r(\mathscr{S}_R) + \mathbf{w}$. Thus, according to the first variation formula of area, if we denote by $\mathcal{A}$ the area function, we can write

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} \left[\mathcal{A}\Big(\exp(t\mathbf{R}')(\mathscr{S}_R) + t\mathbf{w} \right] - \mathcal{A}(\mathscr{S}_R) \Big) &= 0 \\ = -\int_{\mathscr{S}_R} \langle \mathbf{H}, \mathbf{w} + \mathbf{R}'\mathscr{S}_R \rangle \, \mathrm{d}vol_{\mathscr{S}_R} &+ \int_{\partial \mathscr{S}_R} \langle \mathbf{n}, \mathbf{w} + \mathbf{R}'\mathscr{S}_R \rangle \, \mathrm{d}vol_{\partial \mathscr{S}_R}, \end{aligned}$$

where $\mathbf{H}$ is the mean curvature vector of $\mathscr{S}_R$ and $\mathbf{n}$ is the outward pointing unit normal field of the boundary $\partial\mathscr{S}_R$. By construction, the mean curvature vanishes for the part of $\mathscr{S}_R$ which corresponds to the case of hyperplanes. Furthermore, the boundary $\partial\mathscr{S}_R$ is the union of N connected components $\partial\mathscr{S}_{R,k}$ which are graphs over $\partial B(0,R)$. Thus the above formula turns into

$$\sum_{k=0}^{N-1}\sum_j \int_{\mathrm{Cat}_{k,j}} \langle \mathbf{H}, \mathbf{w}+\mathbf{R}'\mathscr{S}_R\rangle \,\mathrm{d}vol_{\mathrm{Cat}_{k,j}} = \sum_{k=0}^{N-1}\int_{\partial\mathscr{S}_{R,k}} \langle \mathbf{n}_k, \mathbf{w}+\mathbf{R}'\mathscr{S}_R\rangle \,\mathrm{d}vol_{\partial\mathscr{S}_{R,k}}, \tag{8.3.90}$$

where $\mathrm{Cat}_{k,j}$ is the catenoid we glue at $p_{k,j}$ between levels k and $k+1$ and $\mathbf{n}_k$ is the outward pointing unit normal field of the boundary $\partial\mathscr{S}_{R,k}$.

The clue then lies in proving that the right member vanishes for well chosen weight parameter ν. We then consider well chosen $\mathbf{w}$ and $\mathcal{R}'$ to demonstrate that the vector Υ^1_{even} is 0 and that the skew symmetric matrix $\mathcal{R}$ vanishes, that is to say the rotation is nothing but the identity.

The integral over $\partial\mathscr{S}_{R,k}$: notice that the left member does not depend on R, thus it is also the case for the $\int_{\partial\mathscr{S}_{R,k}}$-part. The idea is to let $R\to\infty$ and to prove that the unit normal is close enough to the unit normal associated with a sphere of radius R in $\mathbb{R}^n$.

To relieve notations, we omit the index k. The key is the minimal graph over $\mathbb{R}^n_\epsilon$ is almost flat far away from the singularities. More exactly, according to theorem 4.1.2, $\mathscr{S}_R$ is parametrized by

$$X:(r,z)\in[0,R]\times\mathbb{S}\longmapsto\begin{pmatrix} rz\\ h(r,z)\end{pmatrix}\in B(0,R)\times\mathbb{R},$$

where the height function h satisfies

$$h(r,z) = h_{k,-}+\epsilon\left(\underset{r\to\infty}{\mathring{\mathcal{O}}}\left(r^{2-n}\right)+r_\epsilon^{n-\mu}\underset{r\to\infty}{\mathring{\mathcal{O}}}\left(r^\nu\right)\right) = h_{k,-}+\epsilon r_\epsilon^{n-\mu}\underset{r\to\infty}{\mathring{\mathcal{O}}}\left(r^\nu\right)$$

because $\nu>2-n$. Here, the important point is that the constants which appear on the definition of $\mathcal{O}$ do not depend on ϵ, they are universal, that is to say there exists c such that for all ϵ small enough,

$$\left|\nabla^i\left(\mathcal{O}\left(r^\nu\right)\right)\right| \leqslant cr^{\nu-i}.$$

Besides, since $\mathbf{R}'(\mathbf{e}_{n+1})=0$, we have the relation

$$\mathbf{R}'(X(r,z)) = (r\mathcal{R}'z,0).$$

Since the tangent space of $\mathscr{S}_R$ is spanned by the vectors $\partial_r X$ and $\partial_{z^i}X$, there exists a family of $n-1$ real numbers (λ_i) such that $\mathbf{n}$ is written

$$\mathbf{n} = c_{\mathbf{n}}\left(\sum_{i=1}^{n-1}\lambda_i\frac{1}{r}\partial_{z^i}X+\partial_r X\right) = c_{\mathbf{n}}\begin{pmatrix}\sum_i\lambda_i\partial_{z^i}z+z\\ \epsilon r_\epsilon^{n-\mu}\left(\sum_i\lambda_i+1\right)\underset{r\to\infty}{\mathring{\mathcal{O}}}\left(r^{\nu-1}\right)\end{pmatrix},$$

where $c_{\mathbf{n}}$ is a constant such that $\|\mathbf{n}\| = 1$. Besides, $\mathbf{n}$ must be orthogonal to the boundary $\partial \mathscr{S}_R$ which is parametrized by $z \in \mathbb{S} \mapsto X(R, z)$. Thus for all $j \in [\![1, n-1]\!]$, $\langle \mathbf{n}, \partial_{z^j} X \rangle = 0$, from what we conclude

$$\lambda_i = \epsilon^2 r_\epsilon^{2n-2\mu} \underset{r\to\infty}{\overset{\circ}{\mathcal{O}}} \left(r^{2\nu-2}\right).$$

Finally, we deduce the unit normal $\mathbf{n}$ is very close to the normal to $\partial B(0, R) \times \{0\}$, namely

$$\mathbf{n}(z, r) = \begin{pmatrix} z \\ 0 \end{pmatrix} + \begin{pmatrix} \epsilon^2 r_\epsilon^{2n-2\mu} \underset{r\to\infty}{\overset{\circ}{\mathcal{O}}} \left(r^{2\nu-2}\right) \\ \epsilon r_\epsilon^{n-\mu} \underset{r\to\infty}{\overset{\circ}{\mathcal{O}}} \left(r^{\nu-1}\right) \end{pmatrix}.$$

With similar arguments, we prove that there exists a constant c which does not depend on ϵ such that in cylindrical coordinates,

$$\left|\mathrm{d}vol_{\partial\mathscr{S}_R} - R^{n-1}\mathrm{d}vol_{\mathbb{S}}\right| \leqslant c\,\epsilon^2 r_\epsilon^{2n-2\mu} R^{n-1+2\nu-2}\mathrm{d}vol_{\mathbb{S}}.$$

Therefore, if we write $\mathbf{w} = (w, 0)$ with $w \in \mathbb{R}^n$ and if we use the equality $\int_{\mathbb{S}} z\,\mathrm{d}vol_{\mathbb{S}} = 0$, we can estimate the integral as

$$\left|\int_{\partial\mathscr{S}_R} \langle \mathbf{n}, \mathbf{w} + \mathbf{R}'\mathscr{S}_R\rangle_{\mathbb{R}^{n+1}}\,\mathrm{d}vol_{\partial\mathscr{S}_R}\right| \leqslant c\left(\|w\|_\infty + \|\mathcal{R}'\|_\infty\right)\epsilon^2 r_\epsilon^{n-\mu} R^{n-3+2\nu}.$$

This right member tends to 0 when R tends to 0 when $n - 3 + 2\nu < 0$: this is why we have chosen such a ν. Since the integral does not depend on R,

$$\left|\int_{\partial\mathscr{S}_R} \langle \mathbf{n}, \mathbf{w} + \mathbf{R}'\mathscr{S}_R\rangle_{\mathbb{R}^{n+1}}\,\mathrm{d}vol_{\partial\mathscr{S}_R}\right| = 0. \tag{8.3.91}$$

The integral over a catenoid : by construction, the mean curvature vector is $\mathbf{H}_{k,j} = H_{k,j}\,\mathbf{N}_{k,j}$ where $\mathbf{N}_{k,j}$ is the unit normal vector associated to the truncated catenoid $\mathrm{Cat}_{k,j}$ whose boundary data is prescribed. Note that $H_{k,j}$ is given by theorem 7.2.2 up to the dilation factor $\eta_{k,j}$. Besides, recall that we have enforced the relation

$$\Psi^1_{k,j,\mathrm{even}} = \eta_{k,j}^{-\frac{n}{2}} r_\epsilon^{\frac{2-n}{2}} \left(\Upsilon^1_{\mathrm{even}} + \mathcal{R}p_{k,j}\right)$$

in the gluing process. For practical use, we denote by N_c (resp. $N_{k,j}$, N_ϵ) the vector of $\mathbb{R}^n$ such that its components are the n first components of the vector $\mathbf{N}_c$ of $\mathbb{R}^{n+1}$ (resp. $\mathbf{N}_{k,j}$, $\mathbf{N}_\epsilon$). We choose $\mathbf{w}$ and $\mathcal{R}'$ such that

$$w = \eta_{0,1}^{-\frac{n}{2}} r_\epsilon^{\frac{2-n}{2}} \Upsilon^1_{\mathrm{even}} \qquad \text{and} \qquad \mathcal{R}' = \eta_{0,1}^{-\frac{n}{2}} r_\epsilon^{\frac{2-n}{2}} \mathcal{R}.$$

We check that the rotated catenoid $\mathcal{R}'\mathrm{Cat}_{k,j}$ is parametrized by

$$\mathcal{R}'\mathrm{Cat}_{k,j} = \mathcal{R}'p_{k,j} + \eta_{k,j}\mathcal{R}'\left(X_c + \varphi^{\frac{2-n}{2}}\omega_{k,j}\mathbf{N}_\epsilon\right).$$

According to the definition of the Killing field ϕ_+^1, in local coordinates, we can write

$$
\begin{aligned}
&\int_{\mathrm{Cat}_{k,j}} \langle \mathbf{H}_{k,j}, \mathbf{w} + \mathbf{R}' \mathscr{S}_R \rangle \,\mathrm{d}vol_{\mathrm{Cat}_{k,j}} \\
&\quad = \int_{-s_\epsilon}^{s_\epsilon} \int_{\mathbb{S}} \Big[f_{k,j}(s,z) \langle w + \mathcal{R}' p_{k,j}, N_c(s,z) \rangle \\
&\qquad \cdot \Big\langle N_{k,j}(s,z), w + \mathcal{R} p_{k,j} + \eta_{k,j} \mathcal{R}' \Big(X_c + \varphi^{\frac{2-n}{2}} \omega_{k,j} N_\epsilon \Big) \Big\rangle \Big] \mathrm{d}s\mathrm{d}z,
\end{aligned}
$$

where the positive function $f_{k,j}$ is defined to be

$$
f_{k,j}(s,z) \quad := \quad \left(\frac{\eta_{0,1}}{\eta_{k,j}} \right)^{\frac{n}{2}} \frac{1 - |\chi_\epsilon(s)|}{\eta_{k,j} \widetilde{\ell}(s_\epsilon)} \varphi^{-2}(s) \sqrt{|g_{k,j}(s,z)|}
$$

and $g_{k,j}$ is the metric on $\mathrm{Cat}_{k,j}$. We check that $|g_{k,j}|$ is almost $\eta_{k,j}^n |g_c|$. Calculus demonstrates there exists a constant c which does not depend on $\epsilon < \epsilon_0$ such that for all (k,j),

$$
\begin{aligned}
&\frac{1 - |\chi_\epsilon(s)|}{c} \epsilon \cosh(s_\epsilon)^{-\frac{n}{2}} \varphi^{n-2}(s) \\
&\qquad \leqslant \quad f_{k,j}(s,z) \quad \leqslant \quad c\left(1 - |\chi_\epsilon(s)|\right) \epsilon \cosh(s_\epsilon)^{-\frac{n}{2}} \varphi^{n-2}(s). \qquad (8.3.92)
\end{aligned}
$$

Consequently, the mapping

$$
\begin{aligned}
&\langle \cdot, \cdot \rangle_{W \times \mathrm{Skew}_{\mathbf{t}_h}} : \left((w_1, \mathcal{R}_1), (w_2, \mathcal{R}_2) \right) \\
&\qquad \longmapsto \sum_{k,j} \int_{-s_\epsilon}^{s_\epsilon} \int_{\mathbb{S}} f_{k,j}(s,z) \langle w_1 + \mathcal{R}_1 p_{k,j}, N_c \rangle \langle w_2 + \mathcal{R}_2 p_{k,j}, N_c \rangle \,\mathrm{d}s\mathrm{d}z
\end{aligned}
$$

defines a scalar product on $W \times \mathrm{Skew}_{\mathbf{t}_h}$.

Furthermore, according to the definition of $\mathbf{N}_c$, we can check that there exists some positive constant c such that for all $\epsilon < \epsilon_0$, for all vector w' in W, for all $\mathcal{R}'$ in $\mathrm{Skew}_{\mathbf{t}_h}$, we obtain the equality

$$
\begin{aligned}
&\frac{1}{c} \left(\varphi^{1-n} \right)^2 \| w' + \mathcal{R}' p_{k,j} \|^2 \\
&\qquad \leqslant \quad \int_{\mathbb{S}} \langle w' + \mathcal{R}' p_{k,j}, \mathbf{N}_c \rangle^2 \,\mathrm{d}z \quad \leqslant \quad c \left(\varphi^{1-n} \right)^2 \| w' + \mathcal{R}' p_{k,j} \|^2,
\end{aligned}
$$

from what we deduce the norm $\|\cdot\|_{W \times \mathrm{Skew}_{\mathbf{t}_h}}$ satisfies

$$
\begin{aligned}
&\frac{1}{c} \epsilon \cosh(s_\epsilon)^{-\frac{n}{2}} \left(\|w'\|_\infty + \|\mathcal{R}'\|_\infty \right)^2 \\
&\quad \leqslant \quad \|(w', \mathcal{R}')\|^2_{W \times \mathrm{Skew}_{\mathbf{t}_h}} \quad \leqslant \quad c \epsilon \cosh(s_\epsilon)^{-\frac{n}{2}} \left(\|w'\|_\infty + \|\mathcal{R}'\|_\infty \right)^2. \qquad (8.3.93)
\end{aligned}
$$

To end with this case, we put equation (8.3.91) in equation (8.3.90), we write $N_{k,j} = N_c + (N_{k,j} - N_c)$ in order to obtain

$$\begin{aligned} 0 &= \sum_{k,j} \int_{\mathrm{Cat}_{k,j}} \langle \mathbf{H}_{k,j}, \mathbf{w} + \mathbf{R}\mathscr{S}_R \rangle \, \mathrm{d}vol_{\mathrm{Cat}_{k,j}} \\ &= \|(w, \mathcal{R})\|^2_{W \times \mathrm{Skew}_{\mathbf{t}_h}} + I_1 + I_2, \quad (8.3.94) \end{aligned}$$

where the integrals I_1 and I_2 are defined by

$$I_1 = \sum_{k,j} \int f_{k,j} \langle N_c, w + \mathcal{R}p_{k,j} \rangle \cdot \langle N_{k,j} - N_c, w + \mathcal{R}p_{k,j} \rangle \, \mathrm{d}s\mathrm{d}z$$

and

$$I_2 = \sum_{k,j} \int f_{k,j} \langle N_c, w + \mathcal{R}p_{k,j} \rangle \cdot \left\langle N_{k,j}, \eta_{k,j} \mathcal{R} \left(X_c + \varphi^{\frac{2-n}{2}} \omega_{k,j} N_\epsilon \right) \right\rangle \mathrm{d}s\mathrm{d}z.$$

Similar arguments than those in section 7.1 — (b) prove that $N_{k,j} - N_c = \mathcal{O}_{\epsilon \to 0}(\epsilon r_\epsilon)$. Consequently, according to the Cauchy-Schwarz inequality, there exists $c > 0$ such that for all $\epsilon < \epsilon_0$,

$$|I_1| \leqslant c\epsilon r_\epsilon \|(w, \mathcal{R})\|^2_{W \times \mathrm{Skew}_{\mathbf{t}_h}}.$$

Moreover, since $\langle z, \mathcal{R}z \rangle = 0$, we can write

$$\left\langle N_{k,j}, \mathcal{R} \left(X_c + \varphi^{\frac{2-n}{2}} \omega_{k,j} N_\epsilon \right) \right\rangle = \left\langle N_{k,j} - N_c, \mathcal{R} \left(X_c + \varphi^{\frac{2-n}{2}} \omega_{k,j} N_\epsilon \right) \right\rangle,$$

consequently,

$$|I_2| \leqslant c\eta_{k,j} \epsilon r_\epsilon \|\mathcal{R}\|_\infty \|(w, \mathcal{R})\|_{W \times \mathrm{Skew}_{\mathbf{t}_h}} \cdot \left| \sum_{k,j} \int f_{k,j} \varphi^2 \right|^{\frac{1}{2}}.$$

According to the description of $f_{k,j}$ (8.3.92) together with inequality (8.3.93), we find

$$|I_2| \leqslant c r_\epsilon^{\frac{5n}{4}} \|(w, \mathcal{R})\|^2_{W \times \mathrm{Skew}_{\mathbf{t}_h}}.$$

Finally, we put the estimates for I_1 and I_2 in (8.3.94) : there exists some positive constant c which does not depend on $\epsilon < \epsilon_0$ such that

$$\|(w, \mathcal{R})\|^2_{W \times \mathrm{Skew}_{\mathbf{t}_h}} \leqslant c r_\epsilon^{\frac{5n}{4}} \|(w, \mathcal{R})\|^2_{W \times \mathrm{Skew}_{\mathbf{t}_h}},$$

thus $w = 0$ and $\mathcal{R} = 0$ for ϵ small enough. In other words, for all k, j, the quantity $\Psi^1_{k,j,\mathrm{even}}$ vanishes : the surface is minimal. □

9 Examples

We provide some examples of balanced and non-degenerate weighted points configurations. For convenience, in all this section, we omit the factor $(n-2)$ in the definition of the force $f(p,q)$ between two points p and q — it is possible since the rank of the force function $\mathscr{F}$ is invariant under the multiplication by a non-vanishing real number.

9.1 Periodic examples with non-vanishing horizontal period

9.1 — (a) The Riemann minimal hypersurface example

Here, we prove the generalization of the Riemann's example (only one neck between planes) in higher dimensional space given by S. Kaabachi and F. Pacard [KP07].

Let ρ be a positive real number and consider the $\rho\mathbf{e}_1$-periodic configuration given by

$$\{p_{k,1}\}_k \quad := \quad \{(k\rho, 0, \cdots, 0)\}_k \,.$$

We note a the weight of all points, that is to say that equality $a = a_k$ holds for all k. Then the configuration is always balanced because of the symmetry of the problem.

Moreover, $W = \mathrm{Span}\{\mathbf{e}_1\}$ is a one-dimensional linear space. Thus the configuration is non-degenerate if, and only if the rank of the jacobian matrix is 0 (ant it is its maximal rank) : it is always the case.

9.1 — (b) The Wei example

We have in mind to construct the analogue of the Wei example [Wei94] with alternatively one neck and two necks between horizontal hyperplanes. Notice that this kind of example has already been produced in $\mathbb{S}^2 \times \mathbb{R}$ [CP12].

Let ρ be a positive real number and consider the $2\mathbf{e}_1$- periodic configuration $\{P_{k,j}\}$ defined as follows :

$$\forall k \in \mathbb{Z}, \begin{cases} n_{2k} = 1 \quad \text{and} & p_{2k,1} = (2k, 0, \cdots, 0)\,, \\ n_{2k+1} = 2 \quad \text{and} & p_{2k+1,1} = (2k+1, \rho, 0, \cdots, 0)\,, \\ & p_{2k+1,2} = (2k+1, -\rho, 0, \cdots, 0)\,. \end{cases}$$

Note that this configuration is invariant under the action of the orthogonal symmetry with respect to the vertical hyperplane $\{x_2 = 0\}$. Then the linear space spanned by the points is $W = \mathrm{Span}\{\mathbf{e}_1, \mathbf{e}_2\}$. In particular, its dimension is 2 and the dimension of $\mathrm{Skew}_{\mathbf{t}_h}$ is 0. Consequently, the configuration is non degenerate when the Jacobian of $\mathscr{F}$ has rank 4.

Then the forces of the weighted configuration are such that

$$\begin{aligned} F_{0,1} &= 0, \\ F_{1,1} = -F_{1,2} &= 2\left(a_1\,(2\rho)^{1-n} - a_0\frac{\rho}{(1+\rho^2)^{\frac{n}{2}}}\right)\mathbf{e}_2, \end{aligned}$$

thus the configuration is balanced if, and only if $a_{1,1}$ and $a_{0,1}$ are related by the equation

$$a_1 (2\rho)^{1-n} \quad = \quad a_0 \frac{\rho}{(1+\rho^2)^{\frac{n}{2}}}.$$

Since $\mathbf{t}_h \neq 0$, it is enough to consider the force function $\mathscr{F}$ with fixed weights. In this case, computation leads us to Jacobian matrix $J_{\mathscr{F}} \in \mathcal{M}_6(\mathbb{R})$ of the force function $\mathscr{F}$

$$J_{\mathscr{F}} \quad = \quad \left(C^1_{0,1}, C^2_{0,1}, C^1_{1,1}, C^2_{1,1}, C^1_{1,2}, C^2_{1,2}\right),$$

where the columns $C^i_{k,j} = \frac{\partial \mathscr{F}}{\partial p^i_{k,j}}$ are such that its components $(C^i_{k,j})_l$ vanish when $l \notin \{i, i+2, i+4\}$. Besides, $C^i_{0,1} + C^i_{1,1} + C^i_{1,2} = 0$ since $V = \{\mathbf{v}, \cdots, \mathbf{v}\}$ belongs to the kernel of $J_{\mathscr{F}}$. Thus we easily check the rank of the Jacobian is the same than the rank of its 4 first columns. After calculus, we obtain

$$\left(C^1_{0,1}, C^2_{0,1}, C^1_{1,1}, C^2_{1,1}\right) \quad = \quad \begin{pmatrix} -4b\alpha & 0 & 2b\alpha & 0 \\ 0 & -4b\beta & 0 & 2b\beta \\ 2a\alpha & 0 & a - 2a\alpha & 0 \\ 0 & 2a\beta & 0 & (1-n)\,a - 2a\beta \\ 2a\alpha & 0 & -a & 0 \\ 0 & 2a\beta & 0 & -(1-n)\,a \end{pmatrix},$$

where the real numbers a, b, α and β are defined to be

$$a = \frac{a_0}{(1+\rho^2)^{\frac{n}{2}}}, \quad b = \frac{a_1}{(1+\rho^2)^{\frac{n}{2}}}, \quad \alpha = 1 - \frac{n}{1+\rho^2} \quad \text{and} \quad \beta = 1 - \frac{n\rho^2}{1+\rho^2}.$$

Therefore, the above matrix has maximal rank 4 if α and β do not vanish, i.e. when $\rho \neq (n-1)^{\pm\frac{1}{2}}$ and in this case, the configuration is non-degenerate.

The conclusion is that we construct the Wei example for parameters

$$\rho \neq (n-1)^{\pm\frac{1}{2}}, \qquad a_0 \in \mathbb{R}^*_+ \qquad \text{and} \qquad a_1 = a_0 \frac{(2)^{n-1}\,\rho^n}{(1+\rho^2)^{\frac{n}{2}}}.$$

Moreover, the period is such its horizontal component is $2\mathbf{e}_1$.

9.2 Periodic example with vanishing horizontal period

Here, we assume $\mathbf{t}_h = 0$. We give examples for which we do not suppose the weights are fixed. We prove the existence of a type of surface that does not exist in $\mathbb{R}^2 \times \mathbb{R}$, namely a kind of "degenerate Wei example" : a configuration similar to the one of section 9.1 — (b), except we enforce the horizontal period to vanish. It could be named the *vertical* Wei's example.

Let ρ be a positive real number and consider the $\mathbf{0}$-periodic configuration defined as follows :

$$\forall k \in \mathbb{Z}, \begin{cases} n_{2k} = 1 \quad \text{and} & p_{2k,1} = (0, 0, \cdots, 0), \\ n_{2k+1} = 2 \quad \text{and} & p_{2k+1,1} = (\rho, 0, \cdots, 0), \\ & p_{2k+1,2} = (-\rho, 0, \cdots, 0). \end{cases}$$

Then $W = \mathrm{Span}\{\mathbf{e}_1\}$ and the configuration is non-degenerate if the rank of the differential of the force function is 2.

9.2 — (a) Why we have to change the weights

We note $a = a_{2k}$ and $b = a_{2k+1}$. Then the forces satisfy

$$\begin{aligned} F_{0,1} &= 0, \\ F_{1,1} = -F_{1,2} &= 2\rho^{1-n}\left(2^{1-n}b - a\right)\mathbf{e}_1. \end{aligned}$$

Consequently, the configuration is balanced for $2^{1-n}b = a$. Easy calculus demonstrates that the Jacobian $J_{\mathscr{F},p}$ defined to be the matrix associated with the differential $\mathrm{d}_p\mathscr{F}$ of $\mathscr{F}$ with respect to the points (and not the weights) is given by

$$J_p\mathscr{F} \quad = \quad \left(\frac{\partial \mathscr{F}}{\partial p^1_{0,1}}, \frac{\partial \mathscr{F}}{\partial p^1_{1,1}}, \frac{\partial \mathscr{F}}{\partial p^1_{1,2}}\right) \quad = \quad \alpha \begin{pmatrix} -2b & b & b \\ a & 2^{-n}b - a & -2^{-n}b \\ a & -2^{-n}b & 2^{-n}b - a \end{pmatrix},$$

where $\alpha = 2\frac{1-n}{\rho^n}$. But the balancing condition enforces $b = 2^{n-1}a$ and then $2^{-n}b - a = -\frac{1}{2}\,a$: $J_{\mathscr{F}}$ has rank 1 and thus has not full rank 2. This is why it is necessary to change the weights, which is impossible for minimal surfaces in $\mathbb{R}^2 \times \mathbb{R}$.

9.2 — (b) If we change the weights

Heuristically, the problem of the above configuration comes from the symmetries : they are too numerous to ensure the differential of $\mathscr{F}$ to have maximal rank.

As announced, we change the weights parameters a and b. We then check that the differential $\mathrm{d}_a\mathscr{F}$ of $\mathscr{F}$ with respect to the weights is given by

$$J_a\mathscr{F} \quad = \quad \left(\frac{\partial \mathscr{F}}{\partial a}, \frac{\partial \mathscr{F}}{\partial b}\right) \quad = \quad \begin{pmatrix} 0 & 0 \\ -2\rho^{1-n} & 2^{2-n}\rho^{1-n} \\ 2\rho^{1-n} & -2^{2-n}\rho^{1-n} \end{pmatrix}.$$

Obviously, this above matrix has rank 1 and its columns don't belong to the linear space that the columns of $J_p\mathscr{F}$ span.

Therefore, $\mathrm{d}\mathscr{F}$ has rank 2 : the configuration is balanced and non-degenerate when the equality $2^{1-n}b = a$ holds true. In this case, it is possible to produce the vertical Wei's example.

9.3 Non-periodic example

We have in mind to construct a minimal hypersurface with 3 hyperplanar ends and 3 necks.

We consider the non-periodic configuration given by

$$\begin{cases} n_0 = 1 & \text{and} \quad p_{0,1} = (0, 0, \cdots, 0), \\ n_1 = 2 & \text{and} \quad p_{1,1} = (\rho, 0, \cdots, 0), \\ & \qquad\quad p_{1,2} = (-\rho, 0, \cdots, 0), \end{cases}$$

where ρ denotes a positive real number. For convenience, we note $a := a_0$ and $b := a_1$. Then the forces satisfy

$$F_{0,1} = 0, \quad F_{1,1} = \left(2^{2-n}b - a\right)\rho^{n-1}\mathbf{e}_1 \quad \text{and} \quad F_{1,2} = -\left(2^{2-n}b - a\right)\rho^{n-1}\mathbf{e}_1.$$

Therefore, the configuration is balanced when the relation

$$a = 2^{2-n}b$$

holds true.

For the non-degenerate part, since here the space W that the points span is 1-dimensional, we have to prove that the rank of $\mathrm{d}\mathscr{F}$ is equal to 2. An easy computation yields to

$$\begin{aligned}
&\left(\frac{\partial \mathscr{F}}{\partial a}, \frac{\partial \mathscr{F}}{\partial b}, \frac{\partial \mathscr{F}}{\partial p^1_{0,1}}, \frac{\partial \mathscr{F}}{\partial p^1_{1,1}}, \frac{\partial \mathscr{F}}{\partial p^1_{1,2}}\right) \\
&\qquad = \rho^{n-1}(n-1)\begin{pmatrix} 0 & 0 & 2b & -a & -a \\ -\frac{1}{n-1} & \frac{2^{2-n}}{n-1} & a & 2^{1-n}b - a & -2^{1-n}b \\ \frac{1}{n-1} & -\frac{2^{2-n}}{n-1} & a & -2^{1-n}b & 2^{1-n}b - a \end{pmatrix}
\end{aligned}$$

whose rank is 2 when the configuration is balanced. Consequently, the configuration is non-degenerate and it provides a non-periodic Riemann example when $a = 2^{2-n}b$.

Chapitre III

Construction de surfaces minimales de type Riemann-Wei dans $\mathbb{S}^2 \times \mathbb{R}$

1 Introduction

The classical Riemann minimal surfaces form a 1-parameter family of simply periodic minimal surfaces with an infinite number of planar ends which are linked to each other by one "neck". One way to study it is to consider it as the connected sum of Euclidean catenoids.

Many generalizations of this kind of minimal surfaces have been done, especially since the last twenty years. Essentially, there are three types of results that have been established, namely : characterizations in [HKR91] or in [MPR], generalizations with an arbitrary number of necks in the Euclidean 3-space in [Tra02a], [Tra02b], [MT11] or in [HP07] and generalizations in other homogeneous spaces. In this paper, we are interested in this last kind of result. In 2006, L. Hauswirth ([Hau06]) proved that the Riemann example exists in space products $\mathbb{H} \times \mathbb{R}$ and $\mathbb{S}^2 \times \mathbb{R}$. In [KP07] or [FP00], the existence of this kind of minimal hypersurface is proved in $\mathbb{R}^n \times \mathbb{R}$ with $n \geqslant 3$. In [CP11] is proved an extension of the results by M. Traizet with an arbitrary number of necks in $\mathbb{R}^n \times \mathbb{R}$.

For the time being, there are few examples of minimal surfaces in $\mathbb{S}^2 \times \mathbb{R}$. In this paper, we prove that there exists the analogue of the Wei example in $\mathbb{S}^2 \times \mathbb{R}$. The example that Wei (cf. [Wei94]) produced is a simply periodic minimal surface in $\mathbb{R}^2 \times \mathbb{R}$ with alternatively one and two necks between two consecutive planar ends — see figure III.1. An other way to describe it is to consider the classical Riemann minimal surface and to add one handle every two planar ends. Notice that the distance between two planar ends that are linked to each other by only one neck is twice larger than the distance in the case where there are two necks. The reason for that is that the logarithmic growths of the catenoids have to make up for one another.

In our case, we generate in $\mathbb{S}^2 \times \mathbb{R} \subset \mathbb{R}^3 \times \mathbb{R}$ a minimal periodic minimal surface Σ that can be seen as punctured spheres (the analogue of punctured planar ends) which

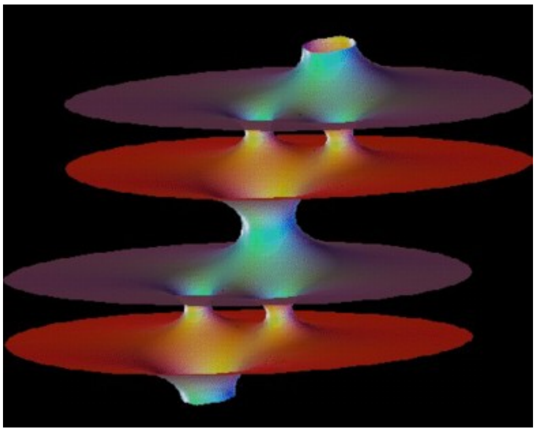


Figure III.1: The Wei's example in $\mathbb{R}^2 \times \mathbb{R}$.

are linked to each other by alternatively one and two small truncated catenoids. By *periodic*, we mean there exists a vertical vector $\mathbf{t}_{\text{ver}} \in \text{Span}(\mathbf{e}_4)$ where $\mathbf{e}_4 = (0,0,0,1)$ and a rotation $\mathcal{R}$ of $\mathbb{R}^4$ that preserves the vertical vector $\mathbf{e}_4$ such that

$$\mathcal{R}(\Sigma) + \mathbf{t}_{\text{ver}} = \Sigma.$$

We note $\mathfrak{k}$ the period transformation, i.e. $\mathfrak{k}$ maps a point (s,t) of $\mathbb{S}^2 \times \mathbb{R}$ to $\mathcal{R}(s,t) + \mathbf{t}_{ver}$ and $\mathfrak{K}$ the group spanned by $\mathfrak{k}$. As a matter of fact, $\mathcal{R}$ has to be seen as the generalization of the horizontal component of the translation that characterizes the period in $\mathbb{R}^2 \times \mathbb{R}$.

First of all, we define the gluing points, that is to say the points in which we will glue catenoids. Let s_p, s_q and s_r be three points on the sphere $\mathbb{S}^2 \subset \mathbb{R}^3$ whose geometric configuration is given by an isosceles triangle on the sphere :

$$s_p := (\sin\theta_+, 0, \cos\theta_+), \quad s_q := (0, \sin\theta_-, \cos\theta_-) \text{ and } s_r := (0, -\sin\theta_-, \cos\theta_-), \tag{1.0.1}$$

where θ_+ and θ_- are real numbers in $(0,\pi)$. Note that the North pole N is exactly the middle of the geodesic which passes through the two points s_q and s_r. We identify this sphere with the sphere at level 0, that is to say $\mathbb{S}^2 \times \{0\}$. Then the punctured sphere at upper level 1 will be given by $\mathbb{S}^2 \times \{t_1\}$ (where t_1 is a positive real number that corresponds to the size of the catenoid we glue at s_p) and the upper level 2 of this last will be given by $\mathbb{S}^2 \times \{t_{\text{ver}}\}$ where $t_{\text{ver}} = t_1 + t_2$. The points s_p^1, s_q^1 and s_r^1 on $\mathbb{S}^2 \simeq \mathbb{S}^2 \times \{t_1\}$ we consider are given by

$$s_p^1 = s_p, \qquad s_q^1 = (\cos\theta_- \sin 2\theta_+, \sin\theta_-, \cos\theta_- \cos 2\theta_+)$$

and

$$s_r^1 = (\cos\theta_- \sin 2\theta_+, -\sin\theta_-, \cos\theta_- \cos 2\theta_+)$$

while at level 2, we consider the points

$$s_p^2 = (\sin 3\theta_+, 0, \cos 3\theta_+), \qquad s_q^2 = s_q^1 \qquad \text{and} \qquad s_r^2 = s_r^1.$$

In other words, the configuration at level 2 is obtained by the configuration at level 0 after the rotation $\mathcal{R}_{\mathbb{S}^2}$ of angle $-2\theta_+$ around the $\mathbf{e}_2$-axis. We then define s_p^k for all $k \in \mathbb{Z}$ by using the periodicity condition

$$\forall g \in \{p, q, r\}, \forall k \in \mathbb{Z}, \qquad s_g^{k+2} = \mathcal{R}_{\mathbb{S}^2}\left(s_g^k\right)$$

and the level k of the sphere is

$$\mathbb{S}^2 \times \{t_1 + lt_{\mathrm{ver}}\} \quad \text{when} \quad k = 2l+1 \qquad \text{and} \qquad \mathbb{S}^2 \times \{lt_{\mathrm{ver}}\} \quad \text{when} \quad k = 2l.$$

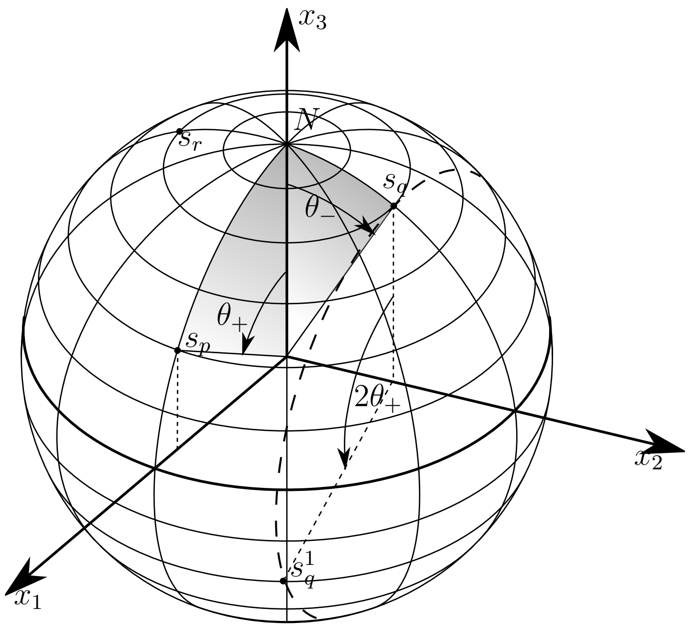


Figure III.2: The different points s_g we consider on $\mathbb{S}^2$.

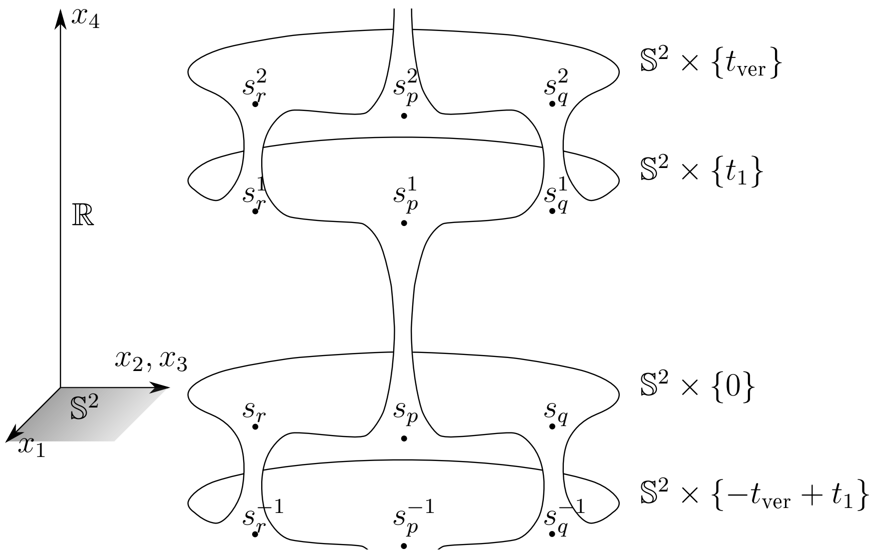


Figure III.3: The kind of surface whom we prove the existence.

Indeed, the rotation $\mathcal{R}$ associated with the period of the example we want to construct is such that its restriction on the sphere is $\mathcal{R}_{|\mathbb{S}^2} = \mathcal{R}_{\mathbb{S}^2}$.

In our construction, we glue one truncated catenoid at point s_p^{2l} and s_p^{2l+1} between levels $2l$ and $2l+1$ while we glue two truncated catenoids at points s_q^{2l+1} (resp. s_r^{2l+1}) and s_q^{2l+2} (resp. s_r^{2l+2}) so that there is alternatively one neck or two necks between two consecutive levels.

Before giving the main theorem of this paper, we have to give some definitions. First of all, at all level, one easily checks that the configuration is invariant under the action of the orthogonal symmetry $\mathfrak{s}$ with respect to the plane $\{x_2 = 0\}$ — note that it is a real plane if we consider the k-th level as a sphere $\mathbb{S}^2$ or it is a hyperplane if we consider the global product space $\mathbb{S}^2 \times \mathbb{R}$. We note $\mathfrak{G}$ the group of isometries $\{\mathrm{Id}, \mathfrak{s}\}$. According to different cases, we see this group as a group that acts on $\mathbb{S}^2$ or $\mathbb{S}^2 \times \mathbb{R}$. Notice that $\mathfrak{s}\left(s_p^i\right) = s_p^i$ and $\mathfrak{s}\left(s_q^i\right) = s_r^i$ for all $i \in \mathbb{Z}$. We also denote by $\mathfrak{c}$ the point reflection of $\mathbb{S}^2 \times \mathbb{R}$ in respect to the point $s_p \times \left\{\frac{t_1}{2}\right\}$ and by $\mathfrak{H}$ the group spanned by $\mathfrak{c}$ and $\mathfrak{s}$; by *point reflection*, we mean that if $\left(s, \frac{t_1}{2} + t\right)$ belongs to $\mathbb{S}^2 \times \left\{\frac{t_1}{2} + t\right\}$ for some real number t, then $\mathfrak{c}\left(s, \frac{t_1}{2} + t\right)$ is the point $\mathfrak{c}\left(s', \frac{t_1}{2} - t\right)$ of $\mathbb{S}^2 \times \left\{\frac{t_1}{2} - t\right\}$ where s' is chosen to be the point of $\mathbb{S}^2$ so that s_p is the middle of the geodesic which passes through the points s and s'. In particular, the isometry $\mathfrak{c}$ satisfies the relations $\mathfrak{c}\left(s_r^1\right) = s_q^0$, $\mathfrak{c}\left(s_q^1\right) = s_r^0$, $\mathfrak{c}\left(s_r^2\right) = s_q^{-1}$, $\mathfrak{c}\left(s_q^2\right) = s_r^{-1}$ and $\mathfrak{c}\left(s_p^2\right) = s_p^{-1}$. Notice that the point $s_p \times \left\{\frac{t_1}{2}\right\}$ is seen as the centre of the only neck which links the level $\mathbb{S}^2 \times \{0\}$ with the level $\mathbb{S}^2 \times \{t_1\}$.

Definition 1.0.1 — We say the configuration given by $\theta_\pm$ is *balanced* if the relation

$$\theta_- \quad = \quad \arccos\left(\frac{-1 + \sqrt{1 + 8\cos^2\theta_+}}{2\cos\theta_+}\right).$$

holds true.

Even if this definition seems technical, it is essentially a generalization of the condition associated with the force which is used in the papers of M. Traizet or in [CP11]. It comes from the Taylor expansion of Green's function. We rather explain it in the discussion we have made after the equivalent definition 2.1.7.

We now state the result of this paper.

Theorem 1.0.2
Let a be a positive number and $(\mathring{\theta}_+, \mathring{\theta}_-) \in (0, \pi)$. Then, if the point configuration $\{s_p, s_q, s_r\}$ given by (1.0.1) is balanced, there exists a 1-parameter family of surfaces $(\Sigma_\epsilon)_{\epsilon \in (0, \epsilon_0)}$ embedded in $\mathbb{S}^2 \times \mathbb{R}$ such that for all ϵ,

(i) Σ_ϵ is minimal ;

(ii) Σ_ϵ is invariant under the action of the group $\mathfrak{H}$;

(iii) Σ_ϵ is 1-periodic with parameters $\mathcal{R}$ and $\mathbf{t}_{ver}$ where

$$\mathcal{R}_{|\mathbb{S}^2} \quad = \quad \mathcal{R}_{\mathbb{S}^2, \mathring{\theta}_+} \qquad \text{and} \qquad t_{ver} \quad = \quad \frac{3}{2}\, a\, \epsilon \ln \epsilon + \underset{\epsilon \to 0}{\mathcal{O}}(\epsilon)\,.$$

In addition to that, the quotient space $\Sigma_\omega/\mathfrak{K}$ is the connected sum $\mathbb{S}^2 \times \{0\}$ with the upper level $\mathbb{S}^2 \times \{t_1\}$ at point s_p and of this last one with $\mathbb{S}^2 \times \{t_{ver}\}$ in a neighbourhood of points s_q and s_r, where

$$t_1 \quad = \quad a\ \epsilon \ln \epsilon + \mathcal{O}(\epsilon).$$

Remark 1.0.3 — For all ϵ in $(0, \epsilon_0)$, the distance between levels $2l$ and $2l+1$ is

$$t_1 \quad = \quad a\ \epsilon \ln \epsilon + \mathcal{O}(\epsilon)$$

and the distance t_2 between levels $2l+1$ and $2l$ is

$$t_2 \quad = \quad \frac{1}{2}\, a\ \epsilon \ln \epsilon + \mathcal{O}(\epsilon).$$

This is in agreement with the remark we have made about Wei's example in $\mathbb{R}^2 \times \mathbb{R}$.

2 Analysis about a sphere in $\mathbb{S}^2 \times \mathbb{R}$

2.1 Green's function

In [CP11], we have already highlighted the key role of Green's function Γ associated to the Jacobi operator in order to build minimal Riemann hypersurfaces in $\mathbb{R}^n \times \mathbb{R}$ with an infinite number of hyperplanar ends. They satisfy the PDE equation

$$\Delta_{\mathbb{R}^n} \Gamma \quad = \quad \sum_i a_g \delta_{p_i},$$

where the p_i are the points in which we perform the connected sum of hyperplanes and a_g are non vanishing real numbers that provide the size of necks between two consecutive hyperplanar ends. We have in mind to generalize it to the sphere case.

For all g in $\{p, q, r\}$, let a_g be a positive real number — more precisely, it is what we call a *weight parameter*. Green's function we have in mind to construct is such that

$$\Delta_{\mathbb{S}^2} \Gamma \quad = \quad -2\pi \left(a_p \delta_{s_p} - a_q \delta_{s_q} - a_r \frac{1}{2} \delta_{s_r} \right).$$

Notice that for the gluing process, it is essential to provide an accurate description of the solution near its singularities. In particular, near one of the s_g's, the term of order 1 in its Taylor expansion corresponds to what we call the *force* F_g which is nothing but a tangent vector that describes the interaction between s_g and the other singularities.

In this paper, we make use of the stereographic projection $\pi : \mathbb{S}^2 \longmapsto \mathbb{R}^2$ from North pole. We choose to work with this projection because it is conformal and thus, it simplifies the analysis on the sphere.

Notation 2.1.1 — For convenience, we define $x := \pi(s)$ to be the point of $\mathbb{R}^2$ which represents the point s of the sphere and for all $g \in \{p, q, r\}$, we note $x_g := \pi(s_g)$.

The metric associated with the parametrization π^{-1} is given by

$$g_{\pi,\mathbb{S}^2} = \phi\, dx^2$$

where ϕ is the conformal parameter, i.e.

$$\phi = \frac{4}{\left(1 + |\ |^2\right)^2},$$

where $|\ |$ denotes the classical Euclidean norm on $\mathbb{R}^2$. The Laplace Beltrami operator in these coordinates is

$$\Delta_{\pi,\mathbb{S}^2} = \frac{\left(1+|x|^2\right)^2}{4} \Delta_{\mathbb{R}^2}.$$

2.1 — (a) The existence part

First of all, since constant functions belong to the kernel of the Laplace-Beltrami operator, Green's functions are defined up to a constant. To ensure uniqueness, we agree that

$$\int_{\mathbb{S}^2} \Gamma = 0.$$

Next, the choice of the coefficients in the definition of Green's function has to satisfy a necessary condition. Indeed, if Γ suits the problem with points s_g and weights a_g, then for any smooth function f on $\mathbb{S}^2$,

$$\sum_g a_g f(s_g) = \int_{\mathbb{S}^2} f\ \Delta_{\mathbb{S}^2}\Gamma = \int_{\mathbb{S}^2} \Delta_{\mathbb{S}^2} f\ \Gamma.$$

Thus if we choose $f \equiv 1$ the constant function, we end up with

$$\sum_g a_g = 0.$$

Consequently, this fact together with the fact we want to construct a minimal surface which is invariant under the action of the orthogonal symmetry with respect to the vertical set $\{x_2 = 0\}$, we enforce the coefficients to have the type

$$(a_p, a_q, a_r) = a\left(1, -\frac{1}{2}, -\frac{1}{2}\right)$$

for some positive real number a.

As a matter of fact, another way to explain the above choice is to recall that our method consists in gluing small catenoids with the sphere. However, catenoids in the Euclidean space $\mathbb{R}^2 \times \mathbb{R}$ are not bounded and have logarithmic growth. The weight parameter a_g matches the size of the catenoid we want to glue. The condition $a_p = a_q + a_r$ is an equality under which the asymptotic behaviour of classical catenoids make up for each other. It is the same condition that M. Traizet uses in its construction — see [Tra02a] or [Tra02b].

Lemma 2.1.2 — *For all positive a, there exists an unique Green's function* Γ *such that*

$$\Delta_{\mathbb{S}^2}\Gamma \quad = \quad -2\pi a\left(\delta_{s_p} - \frac{1}{2}\delta_{s_q} - \frac{1}{2}\delta_{s_r}\right) \qquad \text{together with} \qquad \int_{\mathbb{S}^2}\Gamma = 0.$$

Moreover, the explicit formula for Γ *is given by*

$$\Gamma : \begin{array}{ccl} \mathbb{S}^2 & \longrightarrow & \mathbb{R} \\ s & \longmapsto & \sum_{g\in\{p,q,r\}} a_g \ln\left(\left|\pi\left(s\right) - \pi\left(s_g\right)\right|\right) + c_\pi, \end{array}$$

where the constant c_π *is chosen so that the integral of* Γ *on the sphere vanishes.*

Remark 2.1.3 —

- Since $\sum_g a_g = 0$, we check that
$$\sum_g a_g \ln\left(\left|\pi\left(s\right) - \pi\left(s_g\right)\right|\right) \xrightarrow[s\to\text{North}]{} 0.$$
Therefore, it makes sense to consider Γ as a continuous function on the punctured sphere $\mathbb{S}^2_* = \mathbb{S}^2 \setminus (s_p, s_q, s_r)$. If $\sum_g a_g \neq 0$, then Green's function would tend to infinity when one approaches the North pole. It is another way to understand why the assumption about the coefficients has to be true.
- The graph of Green's function points upwards when one approaches the singularity s_p and points downwards when one approaches the singularity s_q or s_r. Heuristically, the positive part well be used to glue some catenoid with an upper level while the negative part will be used to connect two catenoids with the lower one.
- According to the explicit formula of Γ, one easily checks that if Σ denotes the surface we obtain as the graph of Γ over $\mathbb{S}^2 \setminus \{s_p, s_q, s_r\}$, then Σ is invariant under the action of the group $\mathfrak{G}$.

Proof (Of lemma 2.1.2)

Naturally, we deal with the equation by using the conformal properties of the stereographic projection. For convenient purpose, if f is a function on the sphere, then we compute $\widetilde{f} := f \circ \pi^{-1}$ the associated function on $\mathbb{R}^2$. The problem then turns into

$$\begin{aligned} -2\pi\sum_i a_g f\left(s_g\right) = \int_{\mathbb{S}^2}\Delta_{\mathbb{S}^2}\Gamma\ f &= \int_{\mathbb{R}^2}\frac{\left(1+|x|^2\right)^2}{4}\Delta_{\mathbb{R}^2}\widetilde{\Gamma}\ \widetilde{f}\ \frac{4dx^2}{\left(1+|x|^2\right)^2} \\ &= \int_{\mathbb{R}^2}\Delta_{\mathbb{R}^2}\widetilde{\Gamma}\ \widetilde{f}dx^2 = -2\pi\sum_i a_g\widetilde{f}\left(\pi\left(s_g\right)\right). \end{aligned}$$

Furthermore, it is well known that, in $\mathbb{R}^2$, we have

$$\Delta_{\mathbb{R}^2} \left(\ln \left(|x-p|\right)\right) \quad = \quad -2\pi\delta_p.$$

Consequently, the function

$$s \longmapsto \sum_g a_g \ln \left(|\pi\left(s\right) - \pi\left(s_g\right)|\right)$$

is chosen to be a solution — up to a constant. The constant c_π is defined to ensure $\int_{\mathbb{S}} \Gamma = 0$. Note that the definition of c_π makes sense since in stereographic coordinates, we can rewrite its expression as follows :

$$c_\pi \quad = \quad \int_{\mathbb{R}^2} \sum_g a_g \ln \left(|x - x_g|\right) \frac{4dx^2}{\left(1+|x|^2\right)^2}.$$

At infinity, the integral converges absolutely because

$$\ln \left(|x-x_g|\right) \frac{4}{\left(1+|x|^2\right)^2} \quad \underset{x \to \infty}{\sim} \quad \frac{\ln |x|}{|x|^4}$$

with 4>2. Near a singularity x_i, it is enough to remark $\int_{B_{\mathbb{R}^2}(x_i,1)} \ln \left(|x-x_i|\right)$ converges absolutely since in polar coordinates,

$$\int_{B_{\mathbb{R}^2}(0,1)} \ln \left(|x|\right) \quad = \quad 2\pi \int_0^1 r \ln |r| \, \mathrm{d}r \quad = -\frac{\pi}{2}.$$

The uniqueness follows from maximum principle. □

2.1 — (b) Local description near the singularities

We have in mind to give an accurate description of Green's function in a neighbourhood of one of the points s_g. The main difficulty lies in relating the asymptotic behaviour of the term $\ln\left(|x-x_g|\right)$ which explodes when s tends to s_g with $\ln\left(r_g\right)$ where $r_g\left(s\right)$ denotes the geodesic distance between the points s and s_g on the sphere.

We define angles θ_g on small geodesic circles $\partial B\left(s_g, r_0\right)$ with $0 < r_0 << 1$ to be the oriented angle in $\mathbb{R}^3$ between a point that belongs to the plane that holds $\partial B\left(s_g, r_0\right)$ and the unit vector $\mathbf{e}_g$, based at the center of the circle (seen as an object in $\mathbb{R}^3$), which belongs to this plane and whose coordinates are $\left(e_g^1, 0, e_g^3\right)$ with e_g^3 negative. We also denote by $\mathbf{e}_g^\perp\left(0, e_g^{\perp,2}, e_g^{\perp,3}\right)$ the unit vector which is orthogonal to $\mathbf{e}_g$ and tangent to the sphere at s_g such that its component $e_g^{\perp,2}$ is positive on the Northern hemisphere and is negative on the Southern hemisphere. In other words, $\left(r_g, \theta_g\right)$ denotes the geodesic coordinates. The reader can refer to the figure III.4 for an illustration.

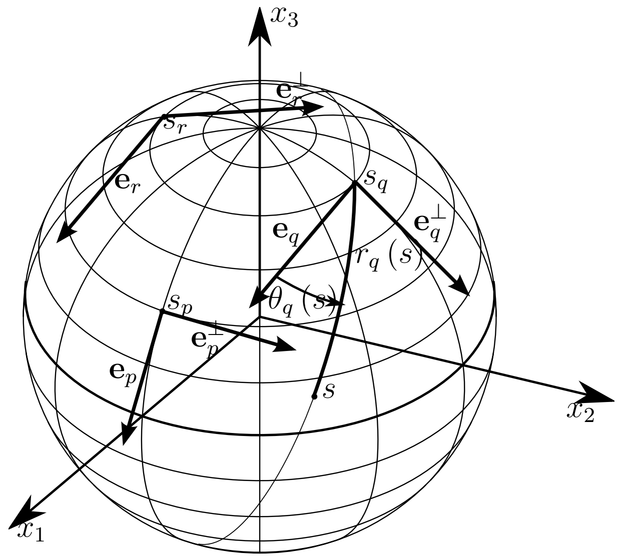


Figure III.4: The geometric meaning of r_g and θ_g.

Lemma 2.1.4 — *Recall that $x = \pi(s)$. In a neighbourhood of s_g, following expansion holds true :*

$$\ln(r_g(s)) \quad = \quad \ln(|x - x_g|) - c_{\pi,g} - \frac{\langle x - x_g, x_g \rangle}{1 + |x_g|^2} + \underset{x \to 0}{\mathcal{O}}\left(|x - x_g|^2\right), \qquad (2.1.2)$$

where $c_{\pi,g}$ is a constant that depends on s_g and the projection π we determine in the proof.

Before giving the proof of this technical lemma, we give a description of Γ near one of its singularities.

Corollary 2.1.5 — *Near s_g, Green's function has following expansion :*

$$\begin{aligned}\Gamma(s) \quad = \quad & a_g \ln r_g(s) \; + \; c_g \\ & + \; r_g(s) \; \left\langle F_g \; , \; \cos\theta_g(s)\,\mathbf{e}_\rho + \sin\theta_g(s)\,\mathbf{e}_\rho^\perp \right\rangle_{\mathbb{S}^2} + \underset{s \to s_g}{\mathcal{O}_\infty}\left(r_g^2(s)\right),\end{aligned}$$

where the force $F_g \in T_{s_g}(\mathbb{S}^2)$ is given by

$$F_g \quad := \quad -\frac{1}{2} \sum_{\overline{g} \neq g} a_{\overline{g}} \; cotan\left(\frac{r_g(s_{\overline{g}})}{2}\right)\left(\cos\theta_g(s_{\overline{g}})\,\mathbf{e}_g + \sin\theta_g(s_{\overline{g}})\,\mathbf{e}_g^\perp\right) \qquad (2.1.3)$$

and where the constant c_g is

$$c_g \quad = \quad a_g c_{\pi,g} \; + \; c_\pi \; + \; \sum_{\overline{g} \neq g} a_{\overline{g}} \ln(|x_g - x_{\overline{g}}|).$$

Remark 2.1.6 — As a matter of fact, the force F_g can be seen as the gradient at point s_g of the $\mathcal{C}^\infty$ function $\Gamma - a_g(\ln(r_g(s)))$ in a ball centred in s_g.

Proof (Of the corollary 2.1.5)
According to the lemma 2.1.2 together with asymptotic behaviour 2.1.2, we can write

$$\begin{aligned}
\Gamma(s) &= a_g \ln(|x-x_g|) + \sum_{\overline{g}\neq g} \frac{1}{2} \ln\left(|x-x_g+x_g-x_{\overline{g}}|^2\right) + c_\pi \\
&= a_g \ln(r_g) + a_g c_{\pi_i} + a_g \frac{\langle x-x_g, x_g\rangle}{1+|x_g|^2} + c_\pi + \sum_{\overline{g}\neq g} a_{\overline{g}} \ln(|x_g - x_{\overline{g}}|) \\
&\quad + \frac{1}{2}\sum_{\overline{g}\neq g} \ln\left(1 + 2\frac{\langle x-x_g, x_g-x_{\overline{g}}\rangle}{|x_g-x_{\overline{g}}|^2} + \mathcal{O}_\infty\left(|x-x_g|^2\right)\right) \\
&\quad + \mathcal{O}_\infty\left(|x-x_g|^2\right),
\end{aligned}$$

from what we deduce (because $|x-x_g| = \mathcal{O}_\infty(r_g)$) the following expansion :

$$\Gamma(s) \quad = \quad a_g \ln r_g \ + \ c_g \ + \ f_g(s) \ + \ \underset{s\to s_g}{\mathcal{O}_\infty}\left(r_g^2\right),$$

where the function f_g on the punctured sphere $\mathbb{S}^2_*$ is defined to be

$$f_g(s) \quad = \quad a_g \frac{\langle x-x_g, x_g\rangle}{1+|x_g|^2} + \sum_{\overline{g}\neq g} a_{\overline{g}} \frac{\langle x-x_g, x_g-x_{\overline{g}}\rangle}{|x_g-x_{\overline{g}}|^2}.$$

We then express this above formula in more useful coordinates $r_g e^{\imath\theta_g}$. If π denotes the stereographic projection from the antipodal point $-s_g$, then the formula of f_g becomes

$$f_g(s) \quad = \quad -\sum_{\overline{g}\neq g} a_{\overline{g}} \frac{\langle x, x_{\overline{g}}\rangle_{\mathbb{R}^2}}{|x_{\overline{g}}|^2_{\mathbb{R}^2}}.$$

But one easily transposes this formula in spherical coordinates since $|x|_{\mathbb{R}^2}$ is given by $\tan\frac{r_g}{2}$. We then get

$$\begin{aligned}
f_g(s) &= -\sum_{\overline{g}\neq g} a_{\overline{g}} \frac{\left\langle \tan\frac{r_g(s)}{2} e^{\imath\theta_g(s)} \ , \ \tan\frac{r_g\left(s_{\overline{g}}\right)}{2} e^{\imath\theta_g\left(s_{\overline{g}}\right)} \right\rangle_{\mathbb{R}^2}}{\tan^2 \frac{r_g\left(s_{\overline{g}}\right)}{2}} \\
&= -\sum_{\overline{g}\neq g} a_{\overline{g}} \frac{\tan\frac{r_g(s)}{2}}{\tan\frac{r_g\left(s_{\overline{g}}\right)}{2}} \left\langle e^{\imath\theta_g(s)}, e^{\imath\theta_g\left(s_{\overline{g}}\right)}\right\rangle_{\mathbb{R}^2}.
\end{aligned}$$

We now use the Taylor expansion of tan to obtain

$$f_g(s) \quad = \quad r_g(s) \ \left\langle F_g \ , \ \cos\theta_g(s)\,\mathbf{e}_\rho + \cos\theta_g(s)\,\mathbf{e}_\rho^\perp\right\rangle_{\mathbb{S}^2} + \mathcal{O}\left(r^3\right),$$

and the result follows. $\square$

Proof (Of the lemma 2.1.4)
The main idea is to explicit the link between $r_g(s)$ that does not depend on the parametrization of the sphere and the quantity $|x - x_g|$.

The metric $g_{\pi,\mathbb{S}^2}$ on the associated with the stereographic projection from North pole is such that

$$g_{\pi,\mathbb{S}^2}(x) = \frac{4}{\left(1+|x_g|^2\right)^2}\left(1 - \frac{4\langle x_g, x - x_g\rangle}{1+|x_g|^2} + \mathcal{O}\left(|x-x_g|^2\right)\right) dx^2.$$

Moreover, since the stereographic projection is conformal, the equation of the geodesics $\gamma : \mathbb{R} \longmapsto \mathbb{R}^2$ is given by

$$\ddot{\gamma} - \frac{1}{2\phi}|\dot{\gamma}|^2 \nabla\phi + \frac{1}{\phi}\langle\nabla\phi, \dot{\gamma}\rangle\dot{\gamma} = 0.$$

Therefore, if γ is a geodesic with unit initial speed v such that

$$\gamma(0) = x_g \qquad \text{and} \qquad |\dot{\gamma}(0)|_{\pi,\mathbb{S}^2} = |v|_{\pi,\mathbb{S}^2} = 1, \quad \text{i.e.} \quad |v| = \phi(x_g)^{-\frac{1}{2}},$$

then we get

$$\gamma(t) - x_g = tv + \frac{t^2}{2\phi(x_g)}\left(\frac{1}{2}|v|^2\nabla\phi(x_g) - \langle\nabla\phi(x_g), v\rangle v\right) + \underset{t\to 0}{\mathcal{O}}\left(t^3\right).$$

Consequently, if we denote by x the quantity $\gamma(t)$, we obtain

$$t = \frac{|x-x_g|}{|v|}\left(1 + \frac{1}{4\phi(x_g)}\langle x - x_g, \nabla\phi(x_g)\rangle\right) + \underset{x\to 0}{\mathcal{O}}\left(|x-x_g|^3\right),$$

from what we deduce

$$r_g(s) = \frac{2}{1+|x_g|^2}|x-x_g|\left(1 - \left\langle x - x_g, \frac{x_g}{1+|x_g|^2}\right\rangle + \mathcal{O}\left(|x-x_g|^2\right)\right). \tag{2.1.4}$$

Finally, we end up with expression (2.1.2) where the constant c is

$$c_{\pi,g} := -\frac{1}{2}\ln\left(\phi(x_g)\right).$$

□

2.1 — (c) Forces and balanced condition

We need to explicit the different force terms because their behaviour is not the same and play an essential role in constructing the minimal Riemann surface for which we want to prove the existence. As a matter of fact, these terms geometrically explain how to bend the small truncated catenoids in the gluing process — we will discuss about that in the proof of proposition 4.2.5.

For all g, we decompose F_g into

$$F_g = F_g^{1,1}\mathbf{e}_g + F_g^{1,2}\mathbf{e}_g^{\perp}.$$

Then for F_p, one checks that

$$F_p^{1,1} \quad = \quad \frac{a}{2\tan\frac{r_g(s_q)}{2}} \cos\left(\theta_g\left(s_q\right)\right) \qquad \text{and} \qquad F_p^{1,2} \quad = \quad 0.$$

Thus the symmetries enforce the force F_p to lie in the vertical plane $\{x_2 = 0\}$. Heuristically, the catenoid we will glue at s_p will be bent only in the direction $\mathbf{e}_g$.

Moreover, since $\theta_q\left(s_r\right) = \frac{3\pi}{2}$, the force F_q is such that

$$F_q^{1,1} = -\frac{a}{2\tan\frac{r_q(s_p)}{2}} \cos\theta_q\left(s_p\right)$$

and

$$F_q^{1,2} = \frac{a}{2}\left[\frac{1}{\tan\frac{r_q(s_p)}{2}} \sin\left(-\theta_q\left(s_p\right)\right) - \frac{1}{2\tan\frac{r_q(s_r)}{2}}\right] \tag{2.1.5}$$

where we notice that $\sin\left(-\theta_q\left(s_p\right)\right)$ is positive since $\theta_q\left(s^p\right)$ belongs to $(\pi, 2\pi)$. By symmetry, we obtain a similar formula for F_r. Unlike the case of F_p, *a priori*, the force F_q does not favour any direction. But for symmetries reasons, the catenoids we will glue at s_q and s_r can't bend in any direction. This is why we introduce the

Definition 2.1.7 — We say the configuration (s_p, s_q, s_r) is *balanced* if the component $F_q^{1,2}$ and $F_r^{1,2}$ of the forces F_q and F_r vanish. It is the same to say that for all g, the force F_g belongs to the line $\mathrm{Span}\left(\mathbf{e}_g\right)$.

Of course, the above definition is equivalent to the definition 1.0.1 of the introduction. Indeed, the distance between the points s_q and s_r is $r_q\left(s_r\right) = 2\theta_-$. To obtain the distance $r_q\left(s_p\right)$ between the points s_p and s_q, we use the formula

$$r_q\left(s_p\right) \quad = \quad \arccos\left(\cos\theta_+ \ \cos\theta_-\right)$$

and thus, according to the relation $\tan\theta = \sqrt{\frac{1-\cos\theta}{1+\cos\theta}}$ for all θ in $(0,\pi)$, we get

$$\tan\left(\frac{r_q\left(s_p\right)}{2}\right) \quad = \quad \sqrt{\frac{1-\cos\theta_+ \ \cos\theta_-}{1+\cos\theta_+ \ \cos\theta_-}}.$$

To determine the angle $\theta_q\left(s_p\right)$, we use

$$\theta_q\left(s_p\right) \quad = \quad \frac{3\pi}{2} - \arcsin\left(\frac{\sin\theta_+}{\sqrt{1-\cos^2\theta_+ \ \cos^2\theta_-}}\right)$$

from what we deduce that

$$\sin\left(-\theta_q\left(s_p\right)\right) \quad = \quad \frac{\sin\theta_- \ \cos\theta_+}{\sqrt{1-\cos^2\theta_+ \ \cos^2\theta_-}}.$$

We put these different formula in the expression of $F_q^{1,2}$ and we find

$$\begin{aligned} F_q^{1,2} &= \frac{a}{2}\left[\frac{\sin\theta_- \ \cos\theta_+}{1-\cos\theta_+ \ \cos\theta_-} - \frac{1}{2\tan\theta_-}\right] \\ &= -\frac{a}{4}\,\frac{\cos^2\theta_- \ \cos\theta_+ + \cos\theta_- - 2\cos\theta_+}{\sin\theta_- \ (1-\cos\theta_+ \ \cos\theta_-)}. \end{aligned} \tag{2.1.6}$$

Given θ_+, it is always possible to find θ_- such that we obtain a balanced configuration. One checks that it is enough to choose

$$\theta_- = \arccos\left(\frac{-1+\sqrt{1+8\cos^2\theta_+}}{2\cos\theta_+}\right),$$

which is nothing but the relation of definition 1.0.1.

2.2 Introduction of a first corrective term

Green's function is a tool that produces a graph with singularities whose type is locally radial and logarithmic. In some sense, it *looks like* to the classical expansion of the catenoid. In this section, we describe the difference between this graph and a minimal surface.

First of all, recall that for a surface Σ which is the graph of a function f defined over the sphere, its mean curvature H is given by the formula

$$H = \frac{1}{2}\operatorname{div}_{\mathbb{S}^2}\left(\frac{\nabla_{\mathbb{S}^2} f}{\sqrt{1+|\nabla_{\mathbb{S}^2} f|^2_{\mathbb{S}^2}}}\right).$$

Therefore, the graph is minimal if and only if its mean curvature vanishes, in other words if and only if f is a solution to the following PDE :

$$\Delta_{\mathbb{S}^2} f = \mathcal{G}(f) \qquad \text{where} \qquad \mathcal{G}(f) = \frac{\operatorname{Hess}_{\mathbb{S}^2} f\,(\nabla_{\mathbb{S}^2} f, \nabla_{\mathbb{S}^2} f)}{1+|\nabla_{\mathbb{S}^2} f|^2_{\mathbb{S}^2}}.$$

We have chosen to work with this expression because it highlights the essential role that harmonic functions play in the minimal surface theory.

The natural question which arises in our context is to determine the kind of error we produce when we use Green's function alone.

We define a smooth increasing function $\chi : \mathbb{R} \longmapsto [0,1]$ such that

$$\chi(-\infty,1) = \{0\} \qquad \text{and} \qquad \chi(2,+\infty) = \{1\}.$$

Proposition 2.2.1 — *Let ρ_0 be a positive real number such that the geodesic balls $B(s_g, \rho_0)$ do not intersect themselves and do not hold the North pole. For small positive parameter ϵ, let $\Gamma_{cor,\epsilon}$ be the function on the sphere such that*

$$\Gamma_{cor,\epsilon}(s) \quad = \quad \epsilon\Gamma(s) - \sum_{g \in \{p,q,r\}} \epsilon^3 \frac{a_g^3}{4} \frac{1 - \chi\left(2\frac{r_g}{\rho_0}\right)}{r_g^2}.$$

Then for all k in $\mathbb{N}$, there exists a positive constant c_k such that

$$\left|\nabla^k_{\mathbb{S}^2}\left(\Delta_{\mathbb{S}^2}\left(\Gamma_{cor,\epsilon}\right) - \mathcal{G}\left(\Gamma_{cor,\epsilon}\right)\right)\right| \quad \leqslant \quad c_k \epsilon^3 \underline{r}^{-3},$$

where the quantity $\underline{r}$ is the minimum of $\{r_p, r_q, r_s\}$.

Before giving the proof, note that this result is really similar to the one we have obtained in the case of punctured hyperplanes in $\mathbb{R}^n \times \mathbb{R}$. It is a little bit more technical since we work with graphs over the sphere, but the main ideas are the same. The corrective term only appears in the neighbourhood of the singularities of Γ.

Remark 2.2.2 — Once again, notice that if Σ denotes the surface we obtain as the graph of $\Gamma_{\mathrm{cor},\epsilon}$, then Σ is invariant under the action of $\mathfrak{G}$.

PROOF

The proof divides into two parts : in the first, we study $\mathcal{G}(\epsilon\Gamma)$ and in the second, we introduce a corrective function that makes up for the main term of $\mathcal{G}(\epsilon\Gamma)$.

First, let us give some formula which are useful to solve the problem. If (r, θ) denotes the polar coordinates in $\mathbb{R}^2$, then the stereographic projection is such that

$$\pi^{-1} : (r, \theta) \longrightarrow \left(\frac{2r\cos\theta}{1+r^2}, \frac{2r\sin\theta}{1+r^2}, \frac{-1+r^2}{1+r^2}\right) \in \mathbb{S}^2.$$

Consequently, the induced metric is given by

$$g \quad = \quad \frac{4}{\left(1+r^2\right)^2}\mathrm{d}r^2 + \frac{4r^2}{\left(1+r^2\right)^2}\mathrm{d}\theta^2$$

and the square root of its determinant is

$$\sqrt{|g|} \quad = \quad \frac{4r}{\left(1+r^2\right)^2}.$$

The Christoffel symbols satisfy following expressions :

$$\left\{\begin{array}{lcl} \Gamma^r_{rr} & = & -\frac{2r}{1+r^2}, \\ \Gamma^r_{r\theta} & = & 0, \\ \Gamma^r_{\theta\theta} & = & r\frac{-1+r^2}{1+r^2} \end{array}\right. \qquad \text{and} \qquad \left\{\begin{array}{lcl} \Gamma^\theta_{rr} & = & 0, \\ \Gamma^\theta_{r\theta} & = & \frac{-1+r^2}{r(1+r^2)}, \\ \Gamma^\theta_{\theta\theta} & = & 0. \end{array}\right.$$

The Laplace Beltrami operator is given by

$$\Delta_{\mathbb{S}^2} = \frac{\left(1+r^2\right)^2}{4}\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right),$$

the gradient satisfies

$$\nabla_{\mathbb{S}^2}(f) = \frac{\left(1+r^2\right)^2}{4}\frac{\partial f}{\partial r}\partial_r+\frac{\left(1+r^2\right)^2}{4r^2}\frac{\partial f}{\partial \theta}\partial_\theta$$

and the hessian is

$$\mathrm{Hess}_{\mathbb{S}^2} = \begin{pmatrix} \frac{\partial^2}{\partial r^2}+\frac{2r}{1+r^2}\frac{\partial}{\partial r} & \frac{\partial^2}{\partial r\partial\theta}+\frac{1-r^2}{r(1+r^2)}\frac{\partial}{\partial\theta} \\ \frac{\partial^2}{\partial r\partial\theta}+\frac{1-r^2}{r(1+r^2)}\frac{\partial}{\partial\theta} & \frac{\partial^2}{\partial\theta^2}+r\frac{1-r^2}{1+r^2}\frac{\partial}{\partial r}\end{pmatrix}.$$

We now compute the estimate of the error we perform when use only Green's function. Assume that $r_g \geqslant \rho_0$ for one of the elements $g \in \{p,q,r\}$. It is convenient to work with the stereographic projection π_{-s_g} from the antipodal point of the singularity s_g since in this case, $x_g = 0$. According to the formula (24), we can write

$$\epsilon\Gamma(s) = \epsilon a_g \ln(r_g)+\epsilon c+\underset{r_g\to 0}{\mathcal{O}_\infty}(\epsilon r_g),$$

from what we deduce [1]

$$\nabla_{\mathbb{S}^2}(\epsilon\Gamma) = \left(\epsilon\frac{a_g}{4r}+\mathcal{O}_\infty(\epsilon)\right)\partial r+\mathcal{O}_\infty\left(\epsilon r^{-1}\right)\partial_\theta$$

and

$$\mathrm{Hess}_{\mathbb{S}^2}(\epsilon\Gamma) = \begin{pmatrix} -\epsilon a_g r^{-2}+\mathcal{O}_\infty\left(\epsilon r^{-1}\right) & \mathcal{O}_\infty\left(\epsilon r^{-1}\right) \\ \mathcal{O}_\infty\left(\epsilon r^{-1}\right) & \mathcal{O}_\infty\left(\epsilon r^{-1}\right)\end{pmatrix}.$$

Consequently, we check that

$$\mathcal{G}(\epsilon\Gamma) = -\epsilon^3\frac{a_g^3}{4^2 r^4}+\mathcal{O}_\infty\left(\epsilon^3 r^{-3}\right).$$

Note that the main term is radial because the behaviour of Γ near s_g is given by the rotational invariant function $\epsilon a_g \ln r_g$.

We now make use of the above expression and we add the corrective term

$$\mathrm{Cor}(s) := -\sum_g \epsilon^3\frac{a_g^3}{4}\frac{1-\chi\left(2\frac{r_g}{\rho_0}\right)}{r_g^2}.$$

1. The reader will pay attention to the fact that $r = |x|$ is not exactly r_g.

When s belongs to the geodesic ball $B\left(s_g, \frac{\rho_0}{2}\right)$, then this function is equal to $-\epsilon^3 \frac{a_g^3}{4r_g^2}$. According to the formula (2.1.4) we have developed in the proof of lemma 2.1.4, in stereographic coordinates from the antipodal point $-s_g$, we get

$$\mathrm{Cor}\,(s) \quad = \quad -\epsilon^3 \frac{a_g^3}{4^2 r^2} + \mathcal{O}_\infty\left(\frac{\epsilon^3}{r}\right)$$

and thus, its Laplacian is

$$\Delta_{\mathbb{S}^2}\mathrm{Cor}\,(s) \quad = \quad -\epsilon^3 \frac{a_g^3}{4^2 r^4} + \mathcal{O}_\infty\left(\frac{\epsilon^3}{r^3}\right);$$

its main term is the same than the main term $\mathcal{G}\,(\epsilon\Gamma)$ and this is why we have introduced this corrective function.

This proves the result for the continuity property in a neighbourhood of s_g, but same kind of estimates hold true for the derivatives. Regarding the points which are *far* from the s_g's, it is enough to note that Γ and its derivatives are bounded by a constant times ϵ : it is small as compared with what happens near singularities. □

2.3 The weighted spaces

In this paragraph, we give some definitions in order to use PDE theory in well chosen spaces to deal with the singularities. Before giving the results, we define the weighted Hölder spaces in punctured spheres.

We note $\mathbb{S}^2_*$ the sphere without the singularities s_g for all $g \in \{p, q, r\}$. Recall ρ_0 is a positive real number such that the geodesic balls $B\,(s_g, \rho_0)$ do not intersect themselves and do not hold the North pole. Finally, let K be the compact set defined to be the sphere excised from the three balls $B\left(s_g, \frac{\rho_0}{2}\right)$ in $\mathbb{S}^2$. It is also convenient to define $r_g\,(s)$ the geodesic distance between a point s of the sphere and s_g.

Definition 2.3.1 — Let μ be a real number. We define the *weighted space* $L^\infty_\mu\left(\mathbb{S}^2_*\right)$ to be the set of all functions $f \in L^\infty_{loc}\left(\mathbb{S}^2_*\right)$ such that the quantity

$$\|f\|_{L^\infty_\mu(\mathbb{S}^2_*)} \quad := \quad \sum_{g\in\{p,q,r\}} \left\|r_g^{-\mu} f\right\|_{L^\infty(B(s_g,\rho_0)\setminus\{s_g\})} + \|f\|_{L^\infty(K)}$$

is finite, that is to say

(i) f is bounded far away the singularities ;

(ii) in a neighbourhood of s_g, f is bounded by a constant times r_g^μ.

Remark 2.3.2 — If U denotes an open space of $\mathbb{S}^2_*$, we define in the same way the weighted space $L^\infty_\mu\,(U)$ endowed with the norm $\|\cdot\|_{L^\infty_\mu(U)}$.

Definition 2.3.3 — Let μ be a real number, k be a non negative integer and $\alpha \in (0, 1)$. Then the weighted Hölder space $\mathcal{C}^{k,\alpha}_\mu\left(\mathbb{S}^2_*\right)$ is defined to be the set of functions f that belong to $\mathcal{C}^{k,\alpha}_{\mathrm{loc}}(\mathbb{S}^2_*)$ such that the following norm

$$\begin{aligned} \|f\|_{\mathcal{C}^{2,\alpha}_\mu(\mathbb{S}^2_*)} &:= \|f\|_{\mathcal{C}^{2,\alpha}(K)} + \sum_{l=0}^{k} \sum_g \left\| r_g^{-\mu+l} \nabla^l f \right\|_{L^\infty(B(s_g,\rho_0)\setminus\{s_g\})} \\ &\quad + \sum_g \sup_{\substack{r\in(0,\rho_0) \\ s_1\neq s_2 \in B(s_g,r)\setminus B\left(s_g,\frac{r}{2}\right)}} r^{k+\alpha-\mu} \frac{\left\|\nabla^k f(s_1) - \nabla^k f(s_2)\right\|}{\left|r_g(s_1) - r_g(s_2)\right|^\alpha} \end{aligned}$$

is finite.

Remark 2.3.4 — It is convenient to remark that this definition allows us to consider the punctured sphere $\mathbb{S}^2_*$ as a manifold with three ends. In fact, we could identify the ball $B(s_g, \rho_0)$ with a cylinder $[a, +\infty) \times \mathbb{S}^1$ by writing $r_g = e^{-t}$ with $t \in [a, +\infty)$. We will use PDE theory developed in lectures [Pac09].

The properties of an operator defined on this kind of manifolds come from the study of its indicial roots : these are a tool that yields to estimates of solutions when one approaches the singularities. It is an easy check to see the indicial roots of the Laplace-Beltrami operator on the punctured sphere are the elements of $\mathbb{Z}$ (it can be seen with the help of classical Fourier series on the sphere $\mathbb{S}^1$ for which the eigenfunctions are the elements $e^{\imath j\theta}$ for $j \in \mathbb{Z}$).

2.4 The harmonic extension

We have already highlighted the essential role that Green's function plays in the theory of minimal surfaces. In this section, we build harmonic extensions on a punctured sphere in order to prescribe local behaviour near singularities.

For all $\epsilon > 0$, let r_ϵ be the radius $r_\epsilon := \epsilon^{\frac{2}{3}}$ — we will explain this choice in remark 3.1. Let $\mathbb{S}^2_\epsilon$ be the sphere punctured by geodesic balls of radius r_ϵ around the singularities s_g :

$$\mathbb{S}^2_\epsilon \quad := \quad \mathbb{S}^2 \setminus \bigcup_g B(s_g, r_\epsilon).$$

From now on, we identify a point z_g of the circle $\partial B(s_g, r\epsilon)$ with its angle θ_g. Let $\Phi := (\Phi_g)_g$ be a family of 3 functions defined on the circle $\mathbb{S}^1$. Our goal is to build an harmonic extension h_Φ on $\mathbb{S}^2_\epsilon$ such that

$$\Delta_{\mathbb{S}^2} h_\Phi \quad = \quad 0$$

in one hand and

$$\forall g\,\{p,q,r\}, \quad \forall z_g \in \partial B(s_g, r_\epsilon), \qquad h_\Phi(z_g) \quad = \quad \Phi_g(\theta_g)$$

in other hand. Indeed, we do not solve exactly this problem : in what follows, the data boundary is almost equal to Φ_g modulo an error term whose rough estimate is small in comparison with Φ.

Definition 2.4.1 — By misuse of language, we say that Φ is $\mathfrak{G}$-invariant if

$$\Phi_p(\theta_p) = \Phi_p(-\theta_p) \qquad \text{and} \qquad \phi_q(\theta_q) = \Phi_r(-\theta_r).$$

This definition makes sense because is we consider the graph $\mathcal{L}$ in $\mathbb{S}^2 \times \mathbb{R}$ of Φ over $\cup_g \partial B(s_g, r_\epsilon)$, then $\mathcal{L}$ is invariant under the action of $\mathfrak{G}$.

Notation 2.4.2 —
- For all Φ defined on the circle, we use classical Fourier analysis to write

$$\Phi = \sum_{k \geqslant 0} \Phi^k,$$

where Φ^k belongs to the k-th eigenmode of the Laplacian on the circle, that is to say $\Delta_{\mathbb{S}^1}\left(\Phi^k\right) = -k^2 \Phi^k$. In other words, Φ^k is a linear combination of $\cos(k\theta)$ and $\sin(k\theta)$. For i in $\{0,1\}$ We denote by π^i (resp. $\pi^\perp$) the linear function which maps a L^2 function f on f^i (resp. on $\sum_{j \geqslant 2} f^j$).
- We note $\mathcal{C}^{2,\alpha}\left(\partial \mathbb{S}^2_\epsilon\right)^{\perp_0}$ the set whose elements Φ have vanishing 0-th eigenmode, in other words such that for all g, Φ^0_g vanishes. It is the same to say that for all g, the average of Φ_g is equal to 0.
- We define the Hölder norm of an element Φ as the maximum of the norms of Φ_g, in other words :

$$\|\Phi\|_{\mathcal{C}^{2,\alpha}(\partial \mathbb{S}^2_\epsilon)} := \max_{g \in \{p,q,r\}} \left\{ \|\Phi_g\|_{\mathcal{C}^{2,\alpha}(\mathbb{S}^1)} \right\}.$$

Before giving the result for harmonic extensions on $\mathbb{S}^2$, we first give a result concerning the harmonic extension on $\mathbb{R}^2 \setminus B(0,1)$. It is this one we use to build our solution for $\mathbb{S}^2_\epsilon$.

2.4 — (a) Harmonic extension on $\mathbb{R}^2 \setminus B(0,1)$

Definition 2.4.3 — Let μ be a real number, $k \in \mathbb{N}$ and α an element of $(0,1)$. We define the *weighted Hölder space* $\mathcal{C}^{k,\alpha}_\mu\left(\mathbb{R}^2 \setminus B(0,1)\right)$ as the set of functions $f \in \mathcal{C}^{k,\alpha}_{loc}\left(\mathbb{R}^2 \setminus B(0,1)\right)$ such that the following norm

$$\begin{aligned} \|f\|_{\mathcal{C}^{2,\alpha}_\mu(\mathbb{R}^2 \setminus B(0,1))} := & \sum_{i=0}^{k} \left\| |\ |^{-\mu} \nabla^i f \right\|_{L^\infty(\mathbb{R}^2 \setminus B(0,1))} \\ & + \sup_{r \geqslant 1} \left\{ r^{k+\alpha-\mu} \sup_{x \neq y \in B(0,2r) \setminus B(0,r)} \frac{\left|\nabla^k f(x) - \nabla^k f(y)\right|}{|x-y|^\alpha} \right\} \end{aligned}$$

is finite.

Proposition 2.4.4 — *There exists an universal positive constant c such that there exists a continuous linear operator*

$$W^e : \Phi \in \mathcal{C}^{2,\alpha}\left(\mathbb{S}^1\right)^{\perp_0} \longmapsto W^e(\Phi) \in \mathcal{C}^{2,\alpha}_{-1}\left(\mathbb{R}^2 \setminus B(0,1)\right)$$

such that $W^e(\Phi)$ is harmonic and $W^e(\Phi) = \Phi$ on $\mathbb{S}^1$. Moreover, the following estimate holds true

$$\|W^e(\Phi)\|_{\mathcal{C}^{2,\alpha}_{-1}(\mathbb{R}^2\setminus B(0,1))} \leqslant c\,\|\Phi\|_{\mathcal{C}^{2,\alpha}(\mathbb{S}^1)}. \tag{2.4.7}$$

We call W^e the *exterior harmonic extension*. We will see that when we deform a truncated catenoid, the *interior harmonic extension* W^i plays a similar role.

Remark 2.4.5 — We choose to work with vanishing 0-eigenmode. The reason is that harmonic extensions of constants are either constants or logarithmic functions. In both cases, it implies estimates we cannot use for the construction of the minimal surface.

Proof

We only give the explicit formula that will be used in the gluing process :

$$W^e(\Phi)\left(re^{\imath\theta}\right) = \sum_{j\geqslant 1} r^{-j}\Phi^j(\theta).$$

For precise proof, it is a slightly modified method than the proposition 3.1 in [PR] in which is proved same kind of result for the interior harmonic extension. □

2.4 — (b) Harmonic extensions on punctured sphere

We make use of the above proposition in order to construct harmonic extensions on $\mathbb{S}^2_\epsilon$.

Proposition 2.4.6 — *There exists an universal positive constant c such that for all $\epsilon > 0$, there exists a continuous linear operator*

$$h : \Phi \in \mathcal{C}^{2,\alpha}\left(\partial\mathbb{S}^2_\epsilon\right)^{\perp_0} \longmapsto h_\Phi \in \mathcal{C}^{2,\alpha}_{-1}\left(\mathbb{S}^2_\epsilon\right)$$

such that h_Φ is harmonic and satisfies following assertions :

(i) the estimate

$$\|h_\Phi\|_{\mathcal{C}^{2,\alpha}_{-1}(\mathbb{S}^2_\epsilon)} \leqslant c\;\|\Phi\|_{\mathcal{C}^{2,\alpha}(\partial\mathbb{S}^2_\epsilon)} \tag{2.4.8}$$

holds true ;

(ii) for all g, we have the estimate

$$\|h_\Phi(r_\epsilon r,\theta_g) - W^e(\Phi_g)(r,\theta_g)\|_{\mathcal{C}^{2,\alpha}\left(B_{\mathbb{R}^2}(0,2)\setminus B_{\mathbb{R}^2}(0,1)\right)} \leqslant c\,r_\epsilon\,\|\Phi\|_{\mathcal{C}^{2,\alpha}(\partial\mathbb{S}^2_\epsilon)}; \tag{2.4.9}$$

(iii) furthermore, if Φ is $\mathfrak{G}$-invariant, then the surface we obtain as the graph of h_Φ is invariant under the action of $\mathfrak{G}$.

Remark 2.4.7 — The point *(ii)* of the proposition specifies that in a small annulus aroud s_g, the harmonic extension h_Φ is equal to Φ modulo an error term whose rough estimate is $r_\epsilon \|\Phi\|$: it is small in comparison with the data boundary we want to prescribe.

Proof

With the help of proposition 2.4.4, it is relatively simple. As a matter of fact, we perform the sum of the harmonic extensions of each Φ_g.

Let π_{-s_g} be the stereographic projection from the antipodal point $-s_g$ of s_g. The reader will pay attention to the orientation we choose concerning the plane on which we perform this projection. As a bases of this plane, we choose $\left(\mathbf{e}_g, \mathbf{e}_g^\perp\right)$ where $\mathbf{e}_g^\perp$ is the unit vector we obtain after a rotation of angle $\theta_g = \frac{\pi}{2}$.

More precisely, if we define $h^{\mathcal{P}}_{\Phi,g}$ as

$$h^{\mathcal{P}}_{\Phi,g} : re^{\imath\theta_g} \in \mathbb{R}^2 \setminus B_{\mathbb{R}^2}\left(0, \tan\frac{r_\epsilon}{2}\right) \longrightarrow W^e\left(\Phi_g\right)\left(\frac{r}{\tan\frac{r_\epsilon}{2}}\, e^{\imath\theta_g}\right)$$

where we have noted the exponant $\mathcal{P}$ to describe functions on planes we identify with $\mathbb{R}^2$, then the induced function $h_{\Phi,g} = h^{\mathcal{P}}_{\Phi,g} \circ \pi_{-s_g}$ is harmonic on $\mathbb{S}^2 \setminus B_{\mathbb{S}^2}\left(s_g, r_\epsilon\right)$ (because the stereographic projection is conform) and is equal to $\Phi_g\left(\theta_g\right)$ on $\partial B_{\mathbb{S}^2}\left(s_g, r_\epsilon\right)$. Notice that the quantity $\tan\frac{r_\epsilon}{2}$ is nothing but the radius of the projected circle $\partial B\left(s_g, r_\epsilon\right)$ of $\mathbb{S}^2$ on the plane after the stereographic projection. In particular, one easily checks that in spherical coordinates $\left(r_g, \theta_g\right)$, we have the relation

$$h_{\Phi,g}\left(r_g, \theta_g\right) \quad = \quad W^e\left(\frac{\tan\frac{r_g}{2}}{\frac{r_\epsilon}{2}}, \theta_g\right).$$

We then claim that

$$h_\Phi \quad := \quad \sum_g h_{\Phi,g}$$

suits to the problem.

It is an harmonic function by construction. Besides, according to the estimate (2.4.7), together with the formula $\tan\frac{r_\epsilon}{2} = \frac{r_\epsilon}{2} + \mathcal{O}\left(r_\epsilon^3\right)$, we obtain the estimate

$$\left|h^{\mathcal{P}}_{\Phi,g}\left(re^{\imath\theta_g}\right)\right| \quad \leqslant \quad c\,\frac{r_\epsilon}{r}\,\|\Phi\|_{\mathcal{C}^{2,\alpha}\left(\partial\mathbb{S}^2_\epsilon\right)}.$$

It gives locally that $h_{\Phi,g}$ belongs to the weighted space $L^\infty_{-1}\left(B\left(s_g, \rho_0\right) \setminus \{s_g\}\right)$.

Concerning the second estimate (2.4.9), if s belongs to the small annulus $B_{\mathbb{S}^2}\left(s_g, 2r_\epsilon\right) \setminus B_{\mathbb{S}^2}\left(s_g, r_\epsilon\right)$, we write $r_g = r_\epsilon r$ with $r \in (1,2)$ and

$$\begin{aligned} & h_\Phi\left(r_\epsilon r, \theta_g\right) - W^e\left(\Phi_g\right)\left(r, \theta_g\right) \\ & \qquad = \quad W^e\left(\Phi_g\right)\left(\frac{\tan\frac{r_\epsilon r}{2}}{\tan\frac{r_\epsilon}{2}}, \theta_g\right) - W^e\left(\Phi_g\right)\left(r, \theta_g\right) + \sum_{\overline{g} \neq g} h_{\Phi,\overline{g}}\left(s\right). \end{aligned}$$

If we focus on the influence of $h_{\Phi,\overline{g}}$ near the point s_g with $\overline{g} \neq g$, then one finds that for all s in the annulus $B_{\mathbb{S}^2}(s_g, 2r_\epsilon) \setminus B_{\mathbb{S}^2}(s_g, r_\epsilon)$, we get the estimate

$$|h_{\Phi,\overline{g}}(s)| \quad \leqslant \quad c\, r_\epsilon \, \|\Phi\|_{\mathcal{C}^{2,\alpha}(\partial \mathbb{S}^2_\epsilon)} .$$

If we focus on the difference on the harmonic extensions, we note that

$$\frac{\tan \frac{r_g}{2}}{\tan \frac{r_\epsilon}{2}} \quad = \quad 1 + \mathcal{O}_\infty\left(r_\epsilon^2\right),$$

therefore we obtain

$$\left| W^e(\Phi_g)\left(\frac{\tan \frac{r_\epsilon r}{2}}{\tan \frac{r_\epsilon}{2}}, \theta_g\right) - W^e(\Phi_g)(r, \theta_g) \right| \quad \leqslant \quad c\, r_\epsilon^2 \, \|\Phi\|_{\mathcal{C}^{2,\alpha}(\partial \mathbb{S}^2_\epsilon)},$$

which is less than the contribution of $h_{\Phi,\overline{g}}$ for small ϵ.

Consequently, the estimates are proved in the L^∞ sense. We obtain the result in $\mathcal{C}^{2,\alpha}$ with the help of Schauder's estimates for the derivatives.

To end up with the proof, it is enough to observe that if Φ is $\mathfrak{G}$-invariant, then, using explicit formula of $h_{\Phi,g}$, one finds

$$h^{\mathcal{P}}_{\Phi,p}(r,\theta) = h_{\Phi,p}(r,-\theta) \qquad \text{and} \qquad h^{\mathcal{P}}_{\Phi,q}(r,\theta) = h^{\mathcal{P}}_{\Phi,r}(r,-\theta),$$

thus the associated surface Σ is invariant under the action of the group $\mathfrak{G}$. □

2.5 Analysis of the Laplacian on the punctured sphere

In what follows, we go on studying properties of the Laplace-Beltrami operator over a punctured sphere. In the previous section, we have turned our attention to find harmonic functions with prescribed data boundary. In this section, we focus on mapping properties of this operator.

In the above proposition, we give a result about the injectivity of the operator.

Proposition 2.5.1 — *Assume that μ belongs to $(0,1)$. Then if a function f satisfies*

$$\Delta_{\mathbb{S}^2} f = 0 \quad \text{on} \quad \mathbb{S}^2_* \qquad \text{together with} \qquad f \in \mathcal{C}^{2,\alpha}_\mu\left(\mathbb{S}^2_*\right),$$

then f is the function 0.

Proof

The proof is relatively simple in this case. Indeed, if f is a harmonic function over $\mathbb{S}^2_*$ such that in a neighbourhood of the singularities,

$$f(s) \quad \leqslant \quad r_g^\mu,$$

we observe that f tends to 0 as one approaches the point s_g. By standard harmonic function theory, the singularities of f are removable and thus, f is harmonic on $\mathbb{S}^2_*$.

In particular, f is continuous and, since the sphere is compact, f is bounded. By Liouville theorem, we conclude from this fact that f is constant. But this constant value is necessary equal to the limit of $f(s)$ when s tends to s_g, thus this constant is nothing but 0. □

We now apply duality theory in manifolds with ends together with estimates ([Pac09, Chapter 10 and 12]) in order to obtain the following :

Proposition 2.5.2 — *For all $\mu \in (-1,0)$, there exists an universal constant c such that for all ϵ, there exists a continuous operator*

$$\Delta_\mu^{-1} : \mathcal{C}^{0,\alpha}_{\mu-2}\left(\mathbb{S}^2_*\right) \longrightarrow \mathcal{C}^{2,\alpha}_{\mu}\left(\mathbb{S}^2_*\right)$$

such that for all f which belongs to the weighted space $\mathcal{C}^{0,\alpha}_{\mu-2}\left(\mathbb{S}^2_*\right)$,

$$\Delta_{\mathbb{S}^2} \circ \Delta_\mu^{-1}(f) \quad = \quad f$$

together with the estimate

$$\left\|\Delta_\mu^{-1}(f)\right\|_{\mathcal{C}^{2,\alpha}_{\mu}(\mathbb{S}^2_*)} \quad \leqslant \quad c\ \|f\|_{\mathcal{C}^{0,\alpha}_{\mu-2}(\mathbb{S}^2_*)}. \tag{2.5.10}$$

Moreover, we can can choose Δ_μ^{-1} so that if the graph Σ_f of f is invariant under the action of $\mathfrak{G}$, then it is also the case of the graph of $\Delta_\mu^{-1}(f)$.

Remark 2.5.3 —

- There is not uniqueness of such an operator. The reason for this lack of uniqueness is that the Laplace-Beltrami operator over the punctured sphere $\mathbb{S}^2_*$ has a 3-dimensional kernel on the weighted spaces $\mathcal{C}^{0,\alpha}_{\mu}\left(\mathbb{S}^2_*\right)$ when μ belongs to $(-1,0)$. More exactly, it is spanned by constant functions and two Green's functions ω_{pq} and ω_{pr} which satisfy

$$\Delta_{\mathbb{S}^2}\left(\omega_{pq}\right) = \delta_{s_p} - \delta_{s_q} \qquad \text{and} \qquad \Delta_{\mathbb{S}^2}\left(\omega_{pr}\right) = \delta_{s_p} - \delta_{s_r}.$$

  The construction of these two functions could be performed by a slightly modified proof of the construction of Γ (see lemma 2.1.2). We then check that near singularities, we have logarithmic growth : it is what ensures these functions to belong to the weighted space with parameter $\mu \in (-1,0)$ because $\ln r = o\left(r^\mu\right)$. In what follows, we always consider the operator Δ_μ^{-1} such that for all f, $\Delta_\mu^{-1}(f)$ is L^2-orthogonal to the constants, w_{pq} and w_{pr}.
- Similar result holds true for functions defined over $\mathbb{S}^2_\epsilon$ and the constant does not depend on the small parameter ϵ. The continuous operator is noted $\Delta_{\mu,\epsilon}^{-1}$.

Proof

The only point that does not come from [Pac09] is the geometric invariance under the action of $\mathfrak{G}$. Indeed, if we have an operator $\overline{\Delta}_\mu^{-1}$ which is a right inverse of $\Delta_{\mathbb{S}^2}$ on $\mathbb{S}^2_*$, then it is an easy check to see that the operator Δ_μ^{-1} defined by

$$\forall s = (s_1, s_2, s_3) \in \mathbb{S}^2, \qquad \Delta_\mu^{-1}(f)(s) := \frac{1}{2}\left[\overline{\Delta}_\mu^{-1}(f)(s) + \overline{\Delta}_\mu^{-1}(f)(\overline{s})\right],$$

where $\overline{s} = (s_1, -s_2, s_3)$, suits to the problem. □

2.6 Resolution of the minimal graph equation

Green's function with its corrective term transforms the sphere into a sphere with necks whose type is catenoidal. The harmonic extensions are a tool in order to prescribe data boundary. We perform a small perturbation of these two objects to construct a minimal graph over $\mathbb{S}^2_\epsilon$.

Let μ be a real number in $(-1, 0)$. Assume we are given a data boundary Φ whose norm is smaller than $\kappa \epsilon r_\epsilon$ and whose first eigenmode vanishes. Then if v is a function over $\mathbb{S}^2_\epsilon$, we know that the surface described by the graph of $\omega_{\Phi,v}$ where $\omega_{\Phi,v}$ is the function

$$\omega_{\Phi,v} : s \in \mathbb{S}^2_\epsilon \longmapsto \Gamma_{\mathrm{cor},\epsilon}(s) + h_\Phi(s) + v(s)$$

is minimal of and only if v is a fixed point of the operator $\mathcal{F}$ defined by

$$\mathcal{F} : \begin{array}{rcl} \mathcal{C}^{2,\alpha}_\mu(\mathbb{S}^2_\epsilon) & \longrightarrow & \mathcal{C}^{2,\alpha}_\mu(\mathbb{S}^2_\epsilon) \\ v & \longmapsto & \Delta^{-1}_{\mu,\epsilon}\left(\mathcal{G}(\omega_{\Phi,v}) - \Delta_{\mathbb{S}^2}(\mathrm{Cor})\right). \end{array}$$

Therefore, we use classical arguments to prove that $\mathcal{F}$ admits a fixed point : first, we compute the image of 0 in order to find a well chosen radius for a ball of $\mathcal{C}^{2,\alpha}_\mu(\mathbb{S}^2_\epsilon)$ centred in 0, next we prove that $\mathcal{F}$ is a contracting operator.

The image of 0 by the operator $\mathcal{F}$. The first step consists in establishing an estimate for the difference $\mathcal{G}(\omega_{\Phi,0}) - \Delta_{\mathbb{S}^2}(\mathrm{Cor})$. We then use properties of the Laplacian ant its inverse.

We divide the above difference into two parts as follows :

$$\mathcal{G}(\omega_{\Phi,0}) - \Delta_{\mathbb{S}^2}(\mathrm{Cor}) = \left(\mathcal{G}(\omega_{\Phi,0}) - \mathcal{G}(\Gamma_{\mathrm{cor},\epsilon})\right) + \left(\mathcal{G}(\Gamma_{\mathrm{cor},\epsilon}) - \Delta_{\mathbb{S}^2}(\mathrm{Cor})\right).$$

The last term has already been studied in proposition 2.2.1 from which we deduce

$$\left\|\mathcal{G}(\Gamma_{\mathrm{cor},\epsilon}) - \Delta_{\mathbb{S}^2}(\mathrm{Cor})\right\|_{\mathcal{C}^{2,\alpha}_{\mu-2}(\mathbb{S}^2_\epsilon)} \leqslant c\,\epsilon\, r_\epsilon^{2-\mu}.$$

The first term is a little bit more technical. We only give a broad outline of the situation because the main ideas are the same than those we have used for the hyperplane case in $\mathbb{R}^n \times \mathbb{R}$. We write

$$\mathcal{G}(\omega_{\Phi,0}) - \mathcal{G}(\Gamma_{\mathrm{cor},\epsilon}) = \mathrm{d}\mathcal{G}_{\Gamma_{\mathrm{cor},\epsilon}}(h_\Phi) + \mathcal{O}\left(h_\Phi^2\right),$$

where [2]

$$\mathrm{d}\mathcal{G}_{\Gamma_{\mathrm{cor},\epsilon}}(f) = \frac{\mathrm{Hess}(f)(\nabla\Gamma_{\mathrm{cor},\epsilon}, \nabla\Gamma_{\mathrm{cor},\epsilon}) + 2\mathrm{Hess}(\Gamma_{\mathrm{cor},\epsilon})(\nabla f, \nabla\Gamma_{\mathrm{cor},\epsilon})}{1 + |\nabla\Gamma_{\mathrm{cor},\epsilon}|^2} - 2\frac{\langle\nabla f, \nabla\Gamma_{\mathrm{cor},\epsilon}\rangle\,\mathcal{G}(\Gamma_{\mathrm{cor},\epsilon})}{1 + |\nabla\Gamma_{\mathrm{cor},\epsilon}|^2}.$$

2. In this formula, we have relieved notations by omitting index $\mathbb{S}^2$, but the hessian, the norm and the scalar product have to be considered on $\mathbb{S}^2$.

We use the formula

$$\Gamma_{\mathrm{cor},\epsilon}(s) = \mathcal{O}(\epsilon \ln \underline{r})$$

together with the estimate (2.4.8) to obtain

$$\|\mathcal{G}(\omega_{\Phi,0}) - \mathcal{G}(\Gamma_{\mathrm{cor},\epsilon})\|_{\mathcal{C}^{2,\alpha}_{\mu-2}(\mathbb{S}^2_\epsilon)} \leqslant c\left(r_\epsilon^{-2-\mu}\|\Phi\|_{\mathcal{C}^{2,\alpha}(\partial\mathbb{S}^2_\epsilon)} + r_\epsilon^2\|\Phi\|^2_{\mathcal{C}^{2,\alpha}(\partial\mathbb{S}^2_\epsilon)}\right)$$

where c does not depend on ϵ or κ.

Collecting previous estimates, there exists $c > 0$ such that for all $\kappa > 0$, there exists ϵ_κ such that for all $0 < \epsilon < \epsilon_\kappa$,

$$\|\mathcal{G}(\omega_{\Phi,0}) - \Delta_{\mathbb{S}^2}(\mathrm{Cor})\|_{\mathcal{C}^{2,\alpha}_{\mu-2}(\mathbb{S}^2_\epsilon)} \leqslant c\,\epsilon^{\frac{7-2\mu}{3}}.$$

Finally, we end up with

$$\|\mathcal{F}(0)\|_{\mathcal{C}^{2,\alpha}_{\mu}(\mathbb{S}^2_\epsilon)} \leqslant c\,\epsilon^{\frac{7-2\mu}{3}}. \tag{2.6.11}$$

$\mathcal{F}$ is a contracting mapping. Assume we are given two functions v_1 and v_2 in the weighted space $\mathcal{C}^{2,\alpha}_\mu(\mathbb{S}^2_\epsilon)$ such that their norm is smaller than $2c\epsilon^{\frac{7-2\mu}{3}}$ — as announced, we choose a radius on this function space that depends on the image of 0 by the operator $\mathcal{F}$. We then claim that, up to reducing ϵ_κ

$$\|\mathcal{F}(v_1) - \mathcal{F}(v_2)\|_{\mathcal{C}^{2,\alpha}_{\mu}(\mathbb{S}^2_\epsilon)} \leqslant \frac{1}{2}\|v_1 - v_2\|_{\mathcal{C}^{2,\alpha}_{\mu}(\mathbb{S}^2_\epsilon)}, \tag{2.6.12}$$

which is obviously a contracting criterion.

The proof of this claim has same kind than the work we did in the above paragraph to compare $\mathcal{G}(\omega_{\Phi,0})$ with $\mathcal{G}(\Gamma_{\mathrm{cor},\epsilon})$. As a matter of fact, we make use of the PDE

$$\Delta_{\mathbb{S}^2}(\mathcal{F}(v_1) - \mathcal{F}(v_2)) = \mathcal{G}(\omega_{\Phi,v_1}) - \mathcal{G}(\omega_{\Phi,v_2}).$$

As done previously, we use the linearization of $\mathcal{G}$ in order to estimate this quantity. We prove that

$$\|\mathcal{G}(\omega_{\Phi,v_1}) - \mathcal{G}(\omega_{\Phi,v_2})\|_{\mathcal{C}^{2,\alpha}_{\mu-2}(\mathbb{S}^2_\epsilon)} \leqslant c\,\epsilon^{\frac{2}{3}}\|v_1 - v_2\|_{\mathcal{C}^{2,\alpha}_{\mu}(\mathbb{S}^2_\epsilon)},$$

from what we deduce the coefficient belongs to $\left(0, \frac{1}{2}\right)$ for ϵ small enough and the conclusion holds true.

A first fixed point theorem. According to previous paragraphs, we can use a fixed point with parameters in order to construct a minimal surface. We have proved the following :

Theorem 2.6.1
For all μ that belongs to $(0,1)$, there exists a universal positive constant c such that for all positive parameter κ, there exists $\epsilon_\kappa > 0$ such that for all $\epsilon \in (0, \epsilon_\kappa)$, for all data boundary Φ whose $\mathcal{C}^{2,\alpha}$ norm is smaller than $\kappa\epsilon r_\epsilon$ and whose 0-eigenmode vanishes, then there exists a function v_Φ which satisfies the following assertions :

(i) the surface given by the graph of $\Gamma_{cor,\epsilon} + h_\Phi + v_\Phi$ over $\mathbb{S}^2_\epsilon$ is minimal ;

(ii) the function v_Φ belongs to the weighted space $\mathcal{C}^{2,\alpha}_\mu(\mathbb{S}^2_\epsilon)$ and its norm is such that

$$\|v_\Phi\|_{\mathcal{C}^{2,\alpha}_\mu(\mathbb{S}^2_\epsilon)} \leqslant 2c\,\epsilon^{\frac{7-2\mu}{3}}; \tag{2.6.13}$$

(iii) the function v_Φ continuously depends on parameters a, θ_p and θ_q ;

(iv) if Φ is $\mathfrak{G}$-invariant, then the associated surface Σ_Φ is invariant under the action of $\mathfrak{G}$.

Notice that the estimate (2.6.13) provides us an idea of the description of the minimal surface near its boundaries : the main terms come from the Γ function, the corrective term, the harmonic extension while inequality

$$|v_\Phi(s)| \leqslant 2c\,\epsilon^{\frac{7-2\mu}{3}}\underline{r}^\mu$$

implies v_Φ has rough estimate $\epsilon^{\frac{7}{3}} = \epsilon r_\epsilon^2$ near one singularity, which is very small in comparison with the quantity ϵr_ϵ that comes from

1) the force term that $\epsilon\Gamma$ yields ;
2) the radial term that Cor produces ;
3) the choice of the rough estimate for the boundary data Φ.

2.7 Local description of the minimal surface near its boundaries

In the above theorem, we have established the existence part : we can build a minimal surface under some conditions. The goal of this section is to compute an accurate description of the solution in a neighbourhood of s_g for some $g \in \{p, q, r\}$. Essentially, there are two reasons to explicit that : in one hand, it produces a way to have a good geometric idea of the surface we have built, in the other hand, it will be necessary when we perform the gluing process.

Among main problems we have encountered, there is the fact that in $\mathbb{S}^2$, different local coordinates can be used to describe the same phenomenon. Until now, we have used different stereographic projections. However, we will see that for the catenoid we want to glue, it is suitable to consider spherical coordinates $(r, \theta) \mapsto (\sin r \cos\theta, \sin \sin\theta, \cos r)$. For our description, we choose these last coordinates.

In spherical coordinates with origin in g, we note $s_g + r_g e^{\imath\theta_g}$ the point of the sphere whose distance from s_g is given by r_g and whose angle with unit vector $\mathbf{e}_g$ is exactly θ_g. We then define a function $u_{\Phi,g}$ on the annulus $\mathcal{A}_1 = B_{\mathbb{R}^2}(0,2) \setminus B_{\mathbb{R}^2}(0,1)$ as follows :

$$u_{\Phi,g} : re^{\imath\theta_g} \longmapsto (\Gamma_{\mathrm{cor},\epsilon} + h_\Phi + v_\Phi)\left(s_g + r_\epsilon r e^{\imath\theta_g}\right).$$

Note that this function yields the behaviour of our solution in a small annulus a radii $2r_\epsilon$ and r_ϵ near s_g. We have performed a change of scales to help along the gluing method.

Theorem 2.7.1
The function $u_{\Phi,g}$ satisfies following assertions :

(i) it is an element of $\mathcal{C}^{2,\alpha}(\mathcal{A}_1)$;

(ii) if we denote by $\mathfrak{d}_{\Phi,g}$ the difference

$$\begin{aligned}\mathfrak{d}_{\Phi,g}\left(re^{\imath\theta_g}\right) \quad := \quad u_{\Phi,g} - \Big(& a_g\epsilon\ln(r_\epsilon r) + \epsilon c_{\Gamma,g} - \epsilon^3 r_\epsilon^{-2}\frac{a_g^3}{4r^2} \\ & + \epsilon r_\epsilon F_g\left(re^{\imath\theta_g}\right) + W^e(\Phi_g)\left(re^{\imath\theta_g}\right)\Big),\end{aligned}$$

then we get the estimate

$$\left\|\mathfrak{d}_{\Phi,g}\right\|_{\mathcal{C}^{2,\alpha}(\mathcal{A}_1)} \quad \leqslant \quad c\ \epsilon^{\frac{7}{3}}. \tag{2.7.14}$$

(iii) up to reducing ϵ_κ, the contracting property

$$\left\|\mathfrak{d}_{\overline{\Phi},g} - \mathfrak{d}_{\overline{\Phi},g}\right\|_{\mathcal{C}^{2,\alpha}(\mathcal{A}_1)} \quad \leqslant \quad \frac{1}{2}\left\|\Phi - \overline{\Phi}\right\|_{\mathcal{C}^{2,\alpha}(\mathcal{A}_1)}. \tag{2.7.15}$$

holds true.

Proof
We use the formula of $u_{\Phi,g}$. According to the corollary 2.1.5 together with the proposition 2.2.1, the corrected Green's function $\Gamma_{\mathrm{cor},\epsilon}$ provides the Taylor expansion

$$\begin{aligned}\Gamma_{\mathrm{cor},\epsilon}\left(s_g + r_\epsilon re^{\imath\theta_g}\right) \quad = \quad & a_g\epsilon\ln(r_\epsilon r) + \epsilon c_{\Gamma,g} - \epsilon^3 r_\epsilon^{-2}\frac{a_g^3}{4r^2} \\ & + \epsilon r_\epsilon r\left\langle F_g\ ,\ e^{\imath\theta_g}\right\rangle + \mathcal{O}\left(\epsilon r_\epsilon^2\right).\end{aligned}$$

We then end up with the estimate (2.7.14) by using proposition 2.4.6 and in particular the inequality (2.4.9).

In order to obtain the contracting inequality, we use similar method than the one we have used to prove that the operator $\mathcal{F}$ is contracting (cf. inequality (2.6.12)).□

Remark 2.7.2 — This last theorem is undoubtedly one of the keys for the gluing process. The point *(ii)* will be very useful to establish gluing equations wile the point *(iii)* will be useful to solve them by a fixed point argument.

3 The small truncated catenoid in $\mathbb{S}^2 \times \mathbb{R}$

In this section, we prove the existence of small truncated catenoids in $\mathbb{S}^2 \times \mathbb{R}$ together with the existence of minimal surfaces in a neighbourhood of these catenoids we obtain with the help of a small perturbation argument.

We briefly expose the method we use. First, we recall some well-known about the classical Euclidean catenoid in $\mathbb{R}^3$; in particular, we explain our choice of the parameter r_ϵ after a dilation of this surface.

Next, we make use of normal coordinates on the sphere near the singularities s_g in order to obtain a metric on the sphere that can be locally written

$$g_{\mathbb{S}^2} \quad = \quad g_{\text{eucl}} + \mathcal{O}\left(r_g^2\right),$$

in other words, if we zoom in one point of the sphere, then we can consider locally the sphere is *almost flat*. We then inject the coordinates of a small truncated Euclidean catenoid — which is not minimal in $\mathbb{S}^2 \times \mathbb{R}$ — and we compute the mean curvature equation for small perturbations of this surface. We solve the equation by a fixed point method.

3.1 Small catenoid in $\mathbb{R}^3$ and choice of parameters

The well known catenoid of $\mathbb{R}^3$, found by Euler, is the minimal surface of revolution whose parametrization is (up to a dilation)

$$X : (t, \theta) \in \mathbb{R} \times \mathbb{S}^1 \longmapsto \begin{pmatrix} \cosh(t) \;\; e^{\imath\theta} \\ t \end{pmatrix} \in \mathbb{R}^3$$

and we note the associated surface Σ_c. We choose the normal vector to this surface so that it points upwards for the upper part of the catenoid, in other words :

$$N(t, \theta) \quad = \quad \frac{1}{\cosh t} \begin{pmatrix} -e^{\imath\theta} \\ \sinh t \end{pmatrix}.$$

Notice that when t is large, this normal is almost vertical. More precisely, one checks the :

Lemma 3.1.1 — *Let $N_v := (0, 0, 1)$ be the vertical unit vector of $\mathbb{R}^2 \times \mathbb{R}$. Then for all non negative integer k, there exists an universal constant c_k such that the following estimate holds true :*

$$\forall t > 0, \qquad \left|\nabla^k \left(\langle N, N_v\rangle - 1\right)\right| \;\leqslant\; c_k \left(\cosh t\right)^{-2}.$$

Remark 3.1.2 — Obviously, same kind of estimate can be obtained with negative t, i.e. for the lower part of the catenoid.

An other way to parametrize the catenoid consists in writing its upper part (i.e. the part with positive height t) as a graph over $\mathbb{R}^2 \setminus B(0,1)$. More precisely, let $x := \cosh(t)\, e^{\imath\theta}$. Then

$$t \quad = \quad \operatorname{arccosh}|x| \quad = \quad \ln\left(|x| + \sqrt{|x|^2 - 1}\right).$$

An easy calculus leads us to the asymptotic behaviour :

$$t \quad = \quad \ln|x| + \ln 2 - \frac{1}{4|x|^2} + \underset{x\to\infty}{\mathcal{O}}\left(\frac{1}{|x|^4}\right).$$

Let $\eta > 0$ be a small dilation factor and consider the change of variables $y := \eta x$. Then the corresponding catenoid is parametrized by

$$y \in \mathbb{R}^2 \setminus B(0,\eta) \quad \longmapsto \quad \begin{pmatrix} y \\ \eta\ln|y| - \eta\ln\eta + \eta\ln 2 - \eta^3 \frac{1}{4|y|^2} + \underset{|y|\to+\infty}{\mathcal{O}}\left(\eta^5 \frac{1}{|y|^4}\right) \end{pmatrix}.$$

In order to glue the necks over the sphere with catenoids, we want the main terms to have same rough estimates. For the catenoid, it is given by $\eta \ln|y|$ while for the punctured sphere, it is given by Green's function, more exactly, by $a_g \epsilon \ln r_g$ near s_g. In other words, we enforce relation [3]

$$\eta \quad \sim \quad a_g \epsilon.$$

Obviously, η depends on g. For the catenoid part, we note η alone but in the gluing process, in order to avoid ambiguities, we note η_g to precise that at point s_g, we glue a catenoid whose size is determined by η_g.

In the gluing process, we deal with the constants by using suitable vertical translations. The next non-constant term has rough estimate

$$\epsilon r_\epsilon \ \text{ for the punctured sphere} \qquad \text{and} \qquad \eta^3 r_\epsilon^{-2} \ \text{ for the small catenoid.}$$

This is why we choose boundary the radius r_ϵ to be such that $\eta^3 r_\epsilon^{-2} \sim \epsilon r_\epsilon$ and the boundary data so that $\|\Phi\| \sim \epsilon r_\epsilon$.

For practical purpose, we also define the large parameter t_ϵ to be such that

$$\cosh t_\epsilon \quad := \quad \frac{r_\epsilon}{\eta}.$$

Here again, t_ϵ depends on the weight parameter a_g at point s_g. Notice that $t_\epsilon \sim \ln \epsilon^{-\frac{1}{3}}$.

We briefly recall some well known operators on the catenoid. First, its Jacobi operator J_c, which is the linearization of the mean curvature operator for normal deformations

$$(t,z) \in \mathbb{R} \times \mathbb{S}^1 \quad \longmapsto \quad X(t,z) + \omega(t,z)\, N(t,z)$$

3. As a matter of fact, in the gluing process, we prove that we can choose $\eta = a_g \epsilon + o(\epsilon)$.

is given by the formula

$$J_c(\omega) = \frac{1}{\varphi^2}\Delta_{\mathbb{R}\times\mathbb{S}^1}\omega + \frac{2}{\varphi^4}\omega,$$

where $\varphi := \cosh$. Moreover, since φ^{-4} is very small in comparison with φ^{-2} when t is large, it is natural to introduce the operator

$$H_c := \frac{1}{\varphi^2}\Delta_{\mathbb{R}\times\mathbb{S}^1}.$$

Jacobi fields. The Jacobi fields are functions that belong to the kernel of J_c. Using some isometries which preserve the mean curvature, we can explicit different ones : these that come from the dilation, the translations and the horizontal rotations.

Of course, one does not modify the mean curvature is we perform a dilation of the catenoid. Then the associated deformation can be seen as

$$p \in \Sigma_c \longmapsto p + c\, X_c$$

for a constant c. Then one finds the Jacobi field

$$k(t,\theta) = \phi^0_-(t) = \langle N, X_c\rangle = 1 - t\tanh t.$$

If we perform a vertical translation, the transformation is given by

$$p \in \Sigma_c \longmapsto p + c\,\mathbf{e}_3$$

and we find the Jacobi field

$$k(t,\theta) = \phi^0_+(t,\theta) = \tanh t.$$

We do same kind of calculus for the horizontal translations to obtain killings fields that are given by

$$k(t,\theta) = \phi^1_+(t)\,(A\cos\theta + B\sin\theta)$$

where A and B are any constants and

$$\phi^1_+(t) = \frac{1}{\cosh t}.$$

Finally, for the rotations that preserve the vertical unit vectors $\mathbf{e}_1$ or $\mathbf{e}_2$, we end up with

$$k(t,\theta) = \phi^1_-(t)\,(A\cos\theta + B\sin\theta)$$

where A and B are any constants and

$$\phi^1_-(t) = \frac{t}{\cosh t} + \sinh t.$$

3.2 A small catenoid in $\mathbb{S}^2 \times \mathbb{R}$

In this section, we describe how to put a small deformed truncated catenoid in $\mathbb{S}^2 \times \mathbb{R}$. To do this, we use the classical catenoid of $\mathbb{R}^3$ we dilate by a small factor $\eta > 0$ and we compute its mean curvature in $\mathbb{S}^2 \times \mathbb{R}$.

Let Σ_0 be the classical truncated Euclidean catenoid we put in $\mathbb{S}^2 \times \mathbb{R}$, that is to say that the upper part of Σ_0 is parametrized by

$$X_0 \; : \; (r, \theta) \in (\eta, r_\epsilon) \times \mathbb{S}^1 \longmapsto \begin{pmatrix} \sin r \cos\theta \\ \sin r \sin\theta \\ \cos r \\ f_0\,(t) \end{pmatrix} \in \mathbb{S}^2 \times \mathbb{R},$$

where

$$t \;:=\; \operatorname{arccosh}\frac{r}{\eta} \qquad \text{and} \qquad f_0\,(t) \;=\; \eta \operatorname{arccosh}(t)\,.$$

The lower part of Σ_0 has same parametrization with $-f_0\,(t)$ instead of f_0. As announced, its mean curvature is very small : its rough estimate is given by η — see the formula (3.3.17) with $\omega = 0$.

Our method to produce minimal surfaces which are close to Σ_0 lies in performing a small perturbation of this surface. Let ω be a function which is defined over the cylinder $[-t_\epsilon, t_\epsilon] \times \mathbb{S}^1$ and $\widetilde{N}$ be a unit vectorfield. We then consider the surface Σ_ω which is given by

$$p = X_0\,(r, \theta) \in \Sigma_0 \quad \longmapsto \quad p + \eta\; \omega\,(t, \theta)\, \widetilde{N}\,(p)\,.$$

Here, it is necessary to discuss the choice of $\widetilde{N}$. In general case, when we perform deformations of a surface, we consider normal deformations. It turns out that in our case, we proceed to a slightly modified transformation. The reason for this is that the normal vector to Σ_0 is almost vertical when r is close to r_ϵ and that it is more useful to describe the surface we construct near its boundaries as the graph of a function over an annulus, like we have done for the deformation of the punctured sphere. Consequently, if N_0 denotes the normal vector of Σ_0 that points upwards for its upper part and $N_v = \mathbf{e}_4$ be the vertical unit vector in $\mathbb{S}^2 \times \mathbb{R}$, we choose $\widetilde{N}$ so that near the boundary $r = r_\epsilon$, we have the inequality $N = N_v$. However, then one approaches the neck of Σ_0 (i.e. when r tends to η), then Σ_0 is not a vertical graph since N_0 is horizontal. We thus enforce $\widetilde{N} = N_0$ near $r = \eta$. To sum up, we choose

$$\widetilde{N} \quad = \quad (1 - |\widetilde{\chi}|)\, N_0 + \chi_\epsilon \mathbf{e}_4$$

where $\widetilde{\chi}$ is a function we construct with the help of the cut-off function χ we have introduced for the proposition 2.2.1 so that

$$\widetilde{\chi}\,(t) \quad = \quad \chi\,(t) - \chi\,(-t)\,.$$

In other words, $\widetilde{\chi}$ is a smooth increasing function whose Hölder norm does not depend on t_ϵ such that

$$\widetilde{\chi}\,((2, +\infty)) = \{1\}\,, \quad \widetilde{\chi}\,((-\infty, -2)) = \{-1\} \quad \text{and} \quad \widetilde{\chi}\,((-1, 1)) = \{0\}\,.$$

3.3 The mean curvature equation

We have in mind to compute the mean curvature of Σ_ω. In this purpose, we first calculate the mean curvature for vertical graphs (i.e. we consider the case $\widetilde{N} = \mathbf{e}_4$), then we give the more general formula for the $\widetilde{N}$ defined in the above paragraph. The reason for what we have chosen to work with vertical graphs is it is more convenient for the calculus and it highlights the fact that the sphere is almost flat if we zoom in enough.

We thus define a graph over a piece of the sphere $\mathbb{S}^2$ as follows :

$$(r,\theta) \in (\eta, r_\epsilon) \times \mathbb{S}^1 \longmapsto \begin{pmatrix} \sin r \cos\theta \\ \sin r \sin\theta \\ \cos r \\ f_\omega(r,\theta) \end{pmatrix} \in \mathbb{S}^2 \times \mathbb{R},$$

where f_ω is the function

$$f_\omega : \begin{array}{ccc} B(0,r_\epsilon) \setminus B(0,\eta) & \longrightarrow & \mathbb{R} \\ (r\cos\theta, r\sin\theta) & \longmapsto & \eta \operatorname{arccosh}\frac{r}{\eta} + \eta\,\omega\left(\operatorname{arccosh}\frac{r}{\eta}, \theta\right). \end{array}$$

Before giving the formula for the mean curvature of this graph, we introduce the following notation.

Definition 3.3.1 — For all $i \in \{0,1,2,3\}$, we note Q_i a function

$$Q_i : \mathcal{C}^{2,\alpha}\left(-(t_\epsilon, t_\epsilon) \times \mathbb{S}^1\right) \longrightarrow \mathcal{C}^{0,\alpha}\left(-(t_\epsilon, t_\epsilon) \times \mathbb{S}^1\right)$$

such that there exists a positive constant c which satisfies

- Q_0 only depends on t, Q_1 is linear, Q_2 is quadratic and Q_3 collects all terms of higher order ;
- for all f, $Q_i(f)$ depends on t, f, ∇f and $\operatorname{Hess}(f)$;
- for all f,

$$|Q_i(f)(t)| \quad \leqslant \quad c\,|f(t)|^i$$

and for all f_1 and f_2, for $i \geqslant 1$,

$$|Q_i(f_1) - Q_i(f_2)|(t) \quad \leqslant \quad c\ |f_1(t) - f_2(t)|\ \left(|f_1(t)| + |f_2(t)|\right)^{i-1}.$$

Lemma 3.3.2 — *The mean curvature H_ω of the surface Σ_ω that we obtain by vertical deformations satisfies following equation :*

$$\begin{aligned} H_\omega \quad = \quad \frac{1}{2}\Bigg[& \eta\, \frac{r\cos r - \sin r}{r^2 \sin r} \\ & + \eta\, \frac{(r^2-\eta^2)^{\frac{1}{2}}}{r}\left(\frac{1}{r^2}\ddot{\omega} + \frac{1}{\sin^2 r}\omega_{\theta\theta}\right) \\ & + \eta\, \frac{r\cos r\,(r^2-\eta^2) - \sin r\,(r^2 - 3\eta^2)}{r^4 \sin r}\dot{\omega} \\ & + \eta r^{-2} Q_2\left(\varphi^{-1}\omega\right) + r^{-1} Q_3\left(\varphi^{-1}\omega\right)\Bigg]. \end{aligned} \qquad (3.3.16)$$

Before giving a more suitable formula for H_ω with the help of the Jacobi operator, let us discuss the different terms very quickly. The first comes from the catenoid we put in $\mathbb{S}^2 \times \mathbb{R}$. More precisely, since we have used the parametrization of the Euclidean truncated catenoid of $\mathbb{R}^3$, there is an error term caused by the curvature of the sphere. The second term is very close to the Laplacian of ω. Besides, note that there is no linear term in which ω appears. It can be easily explained by the fact that a vertical translation does not change the mean curvature.

Corollary 3.3.3 — *There exists some universal constant c such that there exists $\epsilon_0 > 0$ such that for all $\epsilon \in (0, \epsilon_0)$, the mean curvature of the surface Σ_ω that we obtain via $\widetilde{N}$-deformations can be rewritten as follows :*

$$\begin{aligned} H_\omega &= \frac{1}{2\eta}\Bigg(J_c(\omega) + J_\epsilon(\omega) - \frac{\eta^2}{3} + \eta^4\varphi^2 Q_{0,\epsilon} + \eta^2\varphi Q_{1,\epsilon}\left(\varphi^{-1}\omega\right) + \\ &\qquad \varphi^{-2}Q_{2,\epsilon}\left(\varphi^{-1}\omega\right) + \varphi^{-1}Q_{3,\epsilon}\left(\varphi^{-1}\omega\right)\Bigg), \end{aligned} \tag{3.3.17}$$

where J_ϵ is a second order operator whose coefficients are bounded by $c\varphi(t_\epsilon)^{-2}$ and the $Q_{i,\epsilon}$'s enjoy similar properties than the Q_i's except they depend also on ϵ but the constant that appears in the definition of the $Q_{i,\epsilon}$ is c.

This above formula is more useful than the previous one since it highlights the role that the Jacobi operator of an Euclidean catenoid plays here ; besides, its mapping properties are well known. The definition of the constant ϵ_0 is such that the mean curvature is very close to the mean curvature of a catenoid with vertical deformation in $\mathbb{R}^3$ provided the area in which we perform the calculus is small enough.

The proof of the lemma is a little bit technical, but the proof of the corollary is more interesting.

PROOF (OF LEMMA 3.3.2)
First of all, we recall some riemannian geometry formula associated with the local parametrization of the sphere near its North pole :

$$X_{\mathbb{S}^2} : (r,\theta) \longmapsto (\sin r\cos\theta, \sin r\sin\theta, \cos r) \in \mathbb{S}^2 \subset \mathbb{R}^3.$$

We note ∂_r (resp. ∂_θ) the tangent vector $\frac{\partial X_{\mathbb{S}^2}}{\partial r}$ (resp. $\frac{\partial X_{\mathbb{S}^2}}{\partial \theta}$). Then the metric satisfies

$$g = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2 r \end{pmatrix}$$

thus the Christoffel symbols are given by

$$\left\{\begin{aligned} \Gamma^r_{rr} &= 0, \\ \Gamma^r_{\theta\theta} &= -\cos r\sin r, \\ \Gamma^r_{r\theta} &= 0 \end{aligned}\right. \qquad \text{and} \qquad \left\{\begin{aligned} \Gamma^\theta_{rr} &= 0, \\ \Gamma^\theta_{\theta\theta} &= 0, \\ \Gamma^\theta_{r\theta} &= \tan^{-1} r. \end{aligned}\right.$$

For convenience, we define $\widetilde{\omega} := \varphi^{-1}\omega$ and $t = \text{argcosh}\frac{r}{\eta}$ so that

$$f_\omega(r,\theta) = \eta\ \text{argcosh}\frac{r}{\eta} + r\ \widetilde{\omega}(t,\theta).$$

We note

$$\dot{\widetilde{\omega}} = \frac{\partial\widetilde{\omega}}{\partial t} \qquad \text{and} \qquad \widetilde{\omega}_\theta = \frac{\partial\widetilde{\omega}}{\partial\theta}.$$

We make use of the mean curvature formula for graphs :

$$H_\omega = \frac{1}{2}\,\text{div}\frac{\nabla f_\omega}{\sqrt{1+|\nabla f_\omega|^2}}$$

where all terms are considered endowed with the metric $g_{\mathbb{S}^2}$. To compute this formula for f_ω, we give the main steps of the calculus. First of all, the gradient is

$$\nabla f_\omega = \left(\frac{\eta}{\sqrt{r^2-\eta^2}} + \widetilde{\omega} + \frac{r}{\sqrt{r^2-\eta^2}}\dot{\widetilde{\omega}}\right)\partial_r + \frac{r}{\sin^2 r}\widetilde{\omega}_\theta\partial_\theta.$$

The reader could notice that, since r is small, the quantity $\frac{r}{\sin^2 r}$ has rough estimate $\frac{1}{r}$: it is the term which appears in the gradient if we consider polar coordinates in the plane $\mathbb{R}^2$. According to the Taylor's expansion

$$\frac{r^2}{\sin^2 r} = 1 + \mathcal{O}_\infty\left(r^2\right),$$

the above expression yields to

$$1+|\nabla f_\omega|^2 = \frac{r^2}{r^2-\eta^2}\left(1 + 2\eta\frac{\left(r^2-\eta^2\right)^{\frac{1}{2}}}{r^2}\widetilde{\omega} + 2\eta\frac{1}{r}\dot{\widetilde{\omega}} + Q_2\left(\widetilde{\omega}\right)\right).$$

Note that the contribution of terms which come specifically from the sphere and not from the plane are quadratic.

The Laplacian of f_ω is given by the expression

$$\begin{aligned}\Delta f_\omega = {} & \eta\frac{-r\sin r + \cos r\left(r^2-\eta^2\right)}{\sin r\left(r^2-\eta^2\right)^{\frac{3}{2}}} + \tan^{-1}\widetilde{\omega} \\ & + \frac{\sin r\left(r^2-\eta^2\right) + r\cos r\left(r^2-\eta^2\right)}{\sin r\left(r^2-\eta^2\right)^{\frac{3}{2}}} + \frac{r}{r^2-\eta^2}\ddot{\widetilde{\omega}} + \frac{r}{\sin^2 r}\widetilde{\omega}_{\theta\theta}.\end{aligned}$$

Moreover, the Hessian is given by

$$\begin{cases} \text{Hess}\left(f_\omega\right)_{rr} = -\eta\frac{r}{\left(r^2-\eta^2\right)^{\frac{3}{2}}} + \frac{r}{r^2-\eta^2}\ddot{\widetilde{\omega}} + \frac{r^2-2\eta^2}{\left(r^2-\eta^2\right)^{\frac{3}{2}}}\dot{\widetilde{\omega}}, \\ \text{Hess}\left(f_\omega\right)_{r\theta} = Q_1\left(\widetilde{\omega}\right), \\ \text{Hess}\left(f_\omega\right)_{\theta\theta} = r\widetilde{\omega}_{\theta\theta} + \cos r\sin r\left(\eta\frac{1}{\left(r^2-\eta^2\right)^{\frac{1}{2}}} + \widetilde{\omega} + \frac{r}{\left(r^2-\eta^2\right)^{\frac{1}{2}}}\dot{\widetilde{\omega}}\right), \end{cases}$$

from what we deduce that

$$\begin{aligned}\mathrm{Hess}\left(f_\omega\right)\left(\nabla f_\omega, \nabla f_\omega\right) &= -\eta^3 \frac{r}{\left(r^2-\eta^2\right)^{\frac{5}{2}}} \\ &\quad + \eta^2 \frac{r}{\left(r^2-\eta^2\right)^2}\left(-2\widetilde{\omega}+\ddot{\widetilde{\omega}}\right) - \eta^2 \frac{r^2+2\eta^2}{\left(r^2-\eta^2\right)^{\frac{5}{2}}}\dot{\widetilde{\omega}} \\ &\quad + \eta r^{-2} Q_2\left(\widetilde{\omega}\right) + r^{-1} Q_3\left(\widetilde{\omega}\right).\end{aligned}$$

We make use of the formula

$$H_\omega = \frac{1}{2}\left(\frac{\Delta f_\omega}{\left(1+\left|\nabla f_\omega\right|^2\right)^{\frac{1}{2}}} - \frac{\mathrm{Hess}\left(f_\omega\right)\left(\nabla f_\omega, \nabla f_\omega\right)}{\left(1+\left|\nabla f_\omega\right|^2\right)^{\frac{3}{2}}}\right)$$

in order to obtain, after a tedious calculus, the expression

$$\begin{aligned}H_\omega = \frac{1}{2}\Bigg[&\eta\,\frac{r\cos r-\sin r}{r^2\sin r} \\ &+ \frac{r^2\cos r+\eta^2\left(3r^2\sin r-2r^3\cos r\right)+\eta^4\left(r\cos r-3\sin r\right)}{r^4\sin r\left(r^2-\eta^2\right)^{\frac{1}{2}}}\widetilde{\omega} \\ &+ \frac{r^4\sin r+r^5\cos r-2\eta^2 r^3\cos r+\eta^4\left(r\cos r-\sin r\right)}{r^3\sin r\left(r^2-\eta^2\right)}\dot{\widetilde{\omega}} \\ &+ \frac{\left(r^2-\epsilon^2\right)^{\frac{1}{2}}}{r^2}\ddot{\widetilde{\omega}} + \frac{\left(r^2-\epsilon^2\right)^{\frac{1}{2}}}{\sin^2 r}\widetilde{\omega}_{\theta\theta} + \eta r^{-2}Q_2\left(\widetilde{\omega}\right) + r^{-1}Q_3\left(\widetilde{\omega}\right).\Bigg]\end{aligned} \tag{3.3.18}$$

We express the above expression in terms of ω. In particular, we use relations

$$\dot{\widetilde{\omega}} = \eta\frac{1}{r}\dot{\omega} - \eta\frac{\left(r^2-\eta^2\right)^{\frac{1}{2}}}{r^2}\omega, \qquad \widetilde{\omega}_{\theta\theta} = \eta\frac{1}{r}\omega_{\theta\theta}$$

and

$$\ddot{\widetilde{\omega}} = \eta\frac{1}{r}\ddot{\omega} - 2\eta\frac{\left(r^2-\eta^2\right)^{\frac{1}{2}}}{r^2}\dot{\omega} + \eta\frac{r^2-2\eta^2}{r^3}\omega.$$

Injecting these relations into equation 3.3.18, we get the result. □

Proof (of the corollary 3.3.3)
There are two kinds of terms to evaluate : those that come from the curvature of the sphere and those that can be associated with the Jacobi operator.

We first perform the formula for vertical deformations.

We begin with the expression which does not depend on ω. Classical Taylor's expansion provides

$$\eta\,\frac{r\cos r-\sin r}{r^2\sin r} = -\eta\frac{1}{3} - \eta\frac{1}{45}r^2 + \mathcal{O}_\infty\left(\eta r^4\right).$$

Same method leads us to the term $\dot{\omega}$:

$$\eta \, \frac{r \cos r \left(r^2 - \eta^2\right) - \sin r \left(r^2 - 3\eta^2\right)}{r^4 \sin r} \quad = \quad \eta^3 \frac{2}{r^4} - \eta \frac{r^2 - \eta^2}{3r^2} + \mathcal{O}_\infty \left(\epsilon r^2\right).$$

We then rewrite the equation (3.3.16) in terms of φ rather than in terms of r. We obtain :

$$\begin{aligned} H_\omega \quad = \quad & \frac{1}{2\eta} \Bigg[-\eta^2 \frac{1}{3} + \frac{\sqrt{\varphi^2 - 1}}{\varphi^3} \left(\ddot{\omega} + \omega_{\theta\theta}\right) + \frac{2}{\varphi^4} \dot{\omega} \\ & \qquad + \eta^4 \varphi^2 Q_0 + \eta^2 \varphi Q_1 \left(\varphi^{-1} \omega\right) + \varphi^{-2} Q_2 \left(\varphi^{-1} \omega\right) + \varphi^{-1} Q_3 \left(\varphi^{-1} \omega\right) \Bigg]. \end{aligned}$$

Note that the linear term Q_1 is small in comparison with the other linear terms.

To establish the link with the Jacobi operator, we could use the formula of [PR, Proposition 2.2] that provides the linearization of the mean curvature operator for any deformation of a surface. In our case, we rather perform the calculus explicitly. If $N_0 = (0, 0, 1)$ is the vertical unit vector in $\mathbb{R}^2 \times \mathbb{R}$ and N is the unit vector of the Euclidean catenoid, then we obtain

$$J_c \left(\langle N_0, N \rangle_{\mathbb{R}^3} \, \omega \right) \quad = \quad \frac{\sqrt{\varphi^2 - 1}}{\varphi^3} \left(\ddot{\omega} + \omega_{\theta\theta}\right) + \frac{2}{\varphi^4} \dot{\omega}$$

and the same equation of the proposition for vertical deformations holds true follows by taking

$$J_\epsilon \left(\omega\right) \quad := \quad J_c \Big[\big(\langle N, N_v \rangle_{\mathbb{R}^3} - 1 \big) \omega \Big],$$

where N is the normal to the Euclidean catenoid and N_v is the vertical vector in $\mathbb{R}^3$, and by using the lemma 3.1.1.

For the general case with deformations which make use of $\widetilde{N}$, it is same kind of method except that one finds

$$J_\epsilon \left(\omega\right) \quad := \quad J_c \left[\left(\left\langle \widetilde{N}_{\mathbb{R}^3}, N \right\rangle - \operatorname{sgn} \left(\left\langle \widetilde{N}_{\mathbb{R}^3}, N \right\rangle \right) \right) \omega \right],$$

where the vector $\widetilde{N}_{\mathbb{R}^3}$ has a similar definition than the definition of $\widetilde{N}$, namely

$$\widetilde{N}_{\mathbb{R}^3} \quad := \quad \left(1 - |\widetilde{\chi}|\right) N + \chi N_v \quad \in \mathbb{R}^3.$$

□

3.4 Construction of a minimal surface near the catenoid

3.4 — (a) Analysis results

The Jacobi operator J_c over the catenoid is a well known operator. We recall the main results. First, we define the weighted spaces on cylinder $\mathbb{R} \times \mathbb{S}^1$.

Definition 3.4.1 — Let δ be a real number, k a non negative integer and α an element of $(0,1)$. We define the Hölder weighted space $\mathcal{C}^{k,\alpha}_{\delta}\left(\mathbb{R} \times \mathbb{S}^1\right)$ to be the set of functions f that belong to $\mathcal{C}^{k,\alpha}_{\mathrm{loc}}\left(\mathbb{R} \times \mathbb{S}^1\right)$ such that the norm

$$\|f\|_{\mathcal{C}^{k,\alpha}_{\delta}(\mathbb{R}\times\mathbb{S}^1)} \quad := \quad \sup_{t\in\mathbb{R}} \left((\cosh t)^{-\delta} \|f\|_{\mathcal{C}^{2,\alpha}((t-1,t+1)\times\mathbb{S}^1)} \right)$$

is finite.

Remark 3.4.2 — We define in the same way the weighted Hôlder spaces $\mathcal{C}^{k,\alpha}_{\delta}\left((t_1,t_2) \times \mathbb{S}^1\right)$ where $t_1 < t_2$.

We recall the result from proposition 3 in [Pac98] or proposition 4 in [MPP01] for the existence of a right inverse for the Jacobi operator J_c.

Proposition 3.4.3 — *For all weight parameter δ that belongs to $(1,2)$, there exists an universal constant $c = c(\alpha,\delta)$ such that for all positive real number t_0, there exists a continuous linear operator*

$$J^{-1}_{\delta} : f \in \mathcal{C}^{2,\alpha}_{\delta-2}\left((-t_0,t_0)\times\mathbb{S}^1\right) \longmapsto J^{-1}_{\delta}(f) \in \mathcal{C}^{2,\alpha}_{\delta}\left((-t_0,t_0)\times\mathbb{S}^1\right)$$

such that

$$\begin{cases} J_c\left(J^{-1}_{\delta}(f)\right) &= f \quad on \quad (-t_0,t_0)\times\mathbb{S}^1, \\ \pi^{\perp}\left(J^{-1}_{\delta}(f)\right) &= 0 \quad on \quad \{\pm t_0\}\times\mathbb{S}^1. \end{cases}$$

Moreover, the estimate

$$\left\|J^{-1}_{\delta}(f)\right\|_{\mathcal{C}^{2,\alpha}_{\delta}(\mathbb{R}\times\mathbb{S}^1)} \quad \leqslant \quad c\ \|f\|_{\mathcal{C}^{2,\alpha}_{\delta-2}(\mathbb{R}\times\mathbb{S}^1)}$$

holds true. Furthermore, if f is $\mathfrak{G}$-invariant (that is to say $f(t,\theta) = f(t,-\theta)$, then we can choose $J^{1}_{\delta}(f)$ also $\mathfrak{G}$-invariant.

Remark 3.4.4 — The proof in the references does not hold the $\mathfrak{G}$-invariant part. However, to prove this last, it is enough to use similar arguments than in the proposition 2.5.2.

The above proposition is useful to prescribe the quantity $J_c(v)$. We have the counterpart in order to obtain a result to enforce data boundary — see the proposition 5 in [MPP01].

Before giving the proposition, we introduce W^i to be the linear operator that maps a boundary data Υ on $\mathbb{S}^1$ to its interior harmonic extension $W^i(\Upsilon)$ defined on $B_{\mathbb{R}^2}(0,1)$. This operator satisfies similar properties than the exterior harmonic extension W^e — see for example the proposition 2.4.4 or paper [CP11].

Proposition 3.4.5 — *There exists an universal constant $c = c(\alpha)$ such that for all real number $t_0 \geqslant 2$, then there exists a continuous linear operator*

$$h : \Psi^{\perp} = \sum_{j \geqslant 2} \Psi^j \in \mathcal{C}^{2,\alpha}\left(\{\pm t_0\} \times \mathbb{S}^1\right) \longmapsto h_{\Psi^{\perp}}$$

such that

$$H_c\left(h_{\Psi^{\perp}}\right) = 0 \quad on \quad (-t_0, t_0) \times \mathbb{S}^1$$

together with

$$\begin{aligned} &\left\| h_{\Psi^{\perp}}(t,\theta) - W^i\left(\Psi^{\perp}_{+}\right)\left(e^{t-t_0},\theta\right)\right\|_{\mathcal{C}^{2,\alpha}([t_0-2,t_0]\times\mathbb{S}^1)} \\ &\qquad\qquad \leqslant c\left(\varphi(t_0)\right)^{-4}\left\|\Psi^{\perp}\right\|_{\mathcal{C}^{2,\alpha}(\{\pm t_0\}\times\mathbb{S}^1)}. \end{aligned} \tag{3.4.19}$$

Moreover, $h_{\Psi^{\perp}}$ belongs to the weighted space $\mathcal{C}^{2,\alpha}_2\left((-t_0,t_0)\times\mathbb{S}^1\right)$ and the estimate

$$\left\|h_{\Psi^{\perp}}\right\|_{\mathcal{C}^{2,\alpha}_2((-t_0,t_0)\times\mathbb{S}^1)} \leqslant c\,\frac{1}{\varphi(t_0)^2}\left\|\Psi^{\perp}\right\|_{\mathcal{C}^{2,\alpha}(\{\pm t_0\}\times\mathbb{S}^1)}$$

holds true. Furthermore, if $\Psi^{\perp}$ is $\mathfrak{G}$-invariant, then so is $h_{\Psi^{\perp}}$.

Remark 3.4.6 — Of course, same type of estimate (3.4.19) takes place for the lower part of the catenoid, i.e. when $t \in (-t_0, -t_0+2)$.

To deal with the first eigenmode is slightly different. Indeed, we already know the functions associated with the first eigenmode, namely the Jacobi fields $\phi^1_{\pm}$. As said previously, ϕ^1_{-} and ϕ^1_{+} do not behave in the same way : the Jacobi field ϕ^1_{+} associated with the horizontal translation is a function that exponentially decreases like $(\cosh t)^{-1}$ while the Jacobi field ϕ^1_{-} associated with the rotation exponentially increases like $\cosh t$. This is why we decompose data the eigenmode Ψ^1 of a boundary Ψ on $\{\pm t_0\} \times \mathbb{S}^1$ into

$$\Psi^1 = \Psi^1_{\text{odd}} + \Psi^1_{\text{even}}$$

where

$$\Psi^1_{\text{odd}}(t_0,\cdot) = -\Psi^1_{\text{odd}}(-t_0,\cdot) \qquad \text{and} \qquad \Psi^1_{\text{even}}(t_0,\cdot) = \Psi^1_{\text{even}}(-t_0,\cdot).$$

We then check, by using the odd Jacobi field ϕ^1_{-} :

Proposition 3.4.7 — *There exists an universal constant $c = c(\alpha)$ such that for all real number $t_0 \geqslant 2$, then there exists a continuous linear operator*

$$\ell : \Psi^1_{odd} \in \mathcal{C}^{2,\alpha}\left(\{\pm t_0\} \times \mathbb{S}^1\right) \longmapsto \ell_{\Psi^1_{odd}}$$

such that

$$\begin{cases} J_c\left(\ell_{\Psi^1_{odd}}\right) &= 0 \quad on \quad (-t_0, t_0) \times \mathbb{S}^1, \\ \ell_{\Psi^1_{odd}} &= \Psi^1_{odd} \quad on \quad \{\pm t_0\} \times \mathbb{S}^1. \end{cases}$$

Moreover, $\ell_{\Psi^1_{odd}}$ *belongs to the weighted space* $\mathcal{C}^{2,\alpha}_1((-t_0,t_0) \times \mathbb{S}^1)$ *and the two estimates*

$$\left\|\ell_{\Psi^1_{odd}}\right\|_{\mathcal{C}^{2,\alpha}_1((-t_0,t_0)\times\mathbb{S}^1)} \leqslant c\,\frac{1}{\varphi(t_0)}\left\|\Psi^1_{odd}\right\|_{\mathcal{C}^{2,\alpha}(\{\pm t_0\}\times\mathbb{S}^1)}$$

and

$$\begin{aligned} &\left\|\ell_{\Psi^1_{odd}}(t,\theta) - W^i\left(\Psi^1_{odd,+}\right)\left(e^{t-t_0},\theta\right)\right\|_{\mathcal{C}^{2,\alpha}([t_0-2,t_0]\times\mathbb{S}^1)} \\ &\qquad\qquad \leqslant c\, t_0\,\left(\varphi(t_0)\right)^{-2}\left\|\Psi^\perp\right\|_{\mathcal{C}^{2,\alpha}(\{\pm t_0\}\times\mathbb{S}^1)}. \end{aligned} \tag{3.4.20}$$

hold true. Furthermore, if Ψ^1_{odd} *is* $\mathfrak{G}$*-invariant, then so is* $\ell_{\Psi^1_{odd}}$*.*

Remark 3.4.8 — Of course, same type of estimate (3.4.20) takes place for the lower part of the catenoid, i.e. when $t \in (-t_0, -t_0 + 2)$.

Proof
We easily check that

$$\ell_\Psi(t,\theta) := \frac{\phi^1_-}{\phi^1_-(t_0)}\,\Psi^1_{\text{odd},+}(\theta)$$

suits to the problem. The estimate (3.4.20) is obtained by noticing that for all $t \in (t_0 - 2, t_0)$,

$$\left|\frac{\phi^1_-(t)}{\phi^1_-(t_0)} - e^{t-t_0}\right| \leqslant c\, t_0 e^{-2t_0}. \qquad \square$$

3.4 — (b) Dealing with the curvature of the sphere

In the mean curvature equation appears the term $-\frac{\eta^2}{3}$ which comes form the fact that the metric on the sphere is not exactly flat near the North pole. For the construction we have in mind, it is primordial to explain how to deal with this term with the help of a corrective function.

We would like to solve

$$J_c(f) = -\frac{\eta^2}{3}.$$

However, we do not solve exactly the above equation but a more simpler one, namely

$$H_c(f) \quad = \quad -\frac{\eta^2}{3}.$$

The reason is that the difference between H_c and J_c is very small when $|t|$ is large and that we can produce explicit solutions for the operator H_c. Thus, we define Cor to be a solution of the ordinary differential equation

$$\frac{1}{\varphi^2}\ddot{\mathrm{Cor}} \quad = -\frac{\eta^2}{3}$$

and we easily checks that

$$\mathrm{Cor}\,(t) \quad := \quad -\frac{\eta^2}{12}\left(\varphi^2(t) + t^2\right)$$

suits to the equation for H_c. More exactly, we obtain the

Lemma 3.4.9 — *For all no negative integer k, there exists a universal constant $c = c(k)$ such that there exists $\epsilon_k > 0$ such that for all ϵ that belongs to $(0, \epsilon_k)$, we have inequality*

$$\left|\nabla^k_{\mathbb{R}^2}\left(J_c\,(Cor) + \frac{\eta^2}{3}\right)\right| \quad \leqslant \quad c_k\frac{\eta^2}{\varphi^2} \qquad on \qquad [-t_\epsilon, t_\epsilon] \times \mathbb{S}^1.$$

3.4 — (c) An application of a fixed point theorem

In the above sections, we have described a method to produce minimal surfaces with catenoidal necks over the punctured sphere. Here, we explain how to build minimal surface in $\mathbb{S}^2 \times \mathbb{R}$ close to a small truncated catenoid. Like before, we first prove the existence and then we give an accurate description of the solution near its boundaries in order to use it for the gluing process.

Let Ψ be a boundary data on $\{\pm t_\epsilon\} \times \mathbb{S}^1$ such that

$$\Psi^0_+ = \Psi^0_- = 0 \qquad \text{and} \qquad \Psi^1_{\mathrm{even}} = 0.$$

We then define ω_Ψ by

$$\omega_\Psi \quad := \quad \mathrm{Cor} + h_{\Psi^\perp} + \ell_{\Psi^1_{\mathrm{odd}}}.$$

Our method relies on a fixed point theorem. Indeed, the idea is to perform a small perturbation of ω_Ψ in order to solve the equation $H_\omega = 0$. If v is some function in the weighted space $\mathcal{C}^{2,\alpha}_\delta([-t_\epsilon, t_\epsilon] \times \mathbb{S}^1)$ with $\delta \in (1, 2)$, we then define

$$\omega \quad := \quad \omega_\Psi + v.$$

According to lemma 3.4.9, we see that

$$J_c(\omega) \quad = \quad J_c(v) + \frac{\eta^2}{3} + \eta^2\varphi^{-2}Q_0 + \frac{2}{\varphi^4}h_{\Psi^\perp}.$$

We remark that the constant term in front of Q_0 is $\eta^2\varphi^{-2}$ which is much larger from $\eta^4\varphi^2$ (the term in front of Q_0 in the equation of H_ω). Therefore, injecting this relation in the equation (3.3.17) from corollary 3.3.3, we obtain that the surface Σ_ω is minimal if and only if v is a fixed point for the operator $\mathcal{F}$ whose definition is the following :

$$\begin{aligned}\mathcal{F}(v) \quad := \quad J_\delta^{-1}\Bigg[& -\frac{2}{\varphi^4}h_{\Psi^\perp} - J_\epsilon(\omega) + \eta^2\varphi^{-2}Q_0 + \eta^2\varphi Q_1\left(\varphi^{-1}\omega\right) \\ & + \varphi^{-2}Q_2\left(\varphi^{-1}\omega\right) + \varphi^{-1}Q_3\left(\varphi^{-1}\omega\right)\Bigg].\end{aligned}$$

Theorem 3.4.10
For all $\delta \in (1,2)$, there exists an universal positive constant $c = c(\alpha,\delta)$ such that for all positive κ, there exists $\epsilon_\kappa > 0$ such that for all $\epsilon \in (0,\epsilon_\kappa)$, for all boundary data $\Psi \in \mathcal{C}^{2,\alpha}(\{\pm t_\epsilon\} \times \mathbb{S}^1)$ which satisfies

(i) its eigenmode associated with eigenvalue 0 vanishes,

(ii) its eigenmode associated with the eigenvalue 1 is odd and

(iii) its norm is smaller than κr_ϵ,

then there exists v_Ψ in the weighted space $\mathcal{C}^{2,\alpha}_\delta([-t_\epsilon,t_\epsilon] \times \mathbb{S}^1)$ such that following assertions hold true :

(i) the surface Σ_ω is minimal and

(ii) we have the estimate

$$\|v_\Psi\|_{\mathcal{C}^{2,\alpha}_\delta([-t_\epsilon,t_\epsilon]\times\mathbb{S}^1)} \quad \leqslant \quad 2\,c\,\epsilon^{\frac{4}{3}}.$$

In addition to that, if Ψ is $\mathfrak{G}$-invariant (resp. $\mathfrak{H}$-invariant), then the minimal surface Σ_ω is invariant under the action of $\mathfrak{G}$ (resp. $\mathfrak{H}$).

Proof

We use a fixed point argument with parameters : first, we evaluate the image of 0 by $\mathcal{F}$, then we prove that $\mathcal{F}$ is a contraction mapping on a small ball centred in 0. We prove all estimates in L^∞_δ — the result for the derivatives comes from Schauder's theory.

First of all, we estimate ω_Ψ. According to the definition of Cor together with the propositions 3.4.5 and 3.4.7, we find

$$\begin{aligned}\forall (t,\theta) \in [-t_\epsilon,t_\epsilon] \times \mathbb{S}^1, \qquad |\omega_\Psi(t,\theta)| \quad & \leqslant \quad \eta^2\varphi^2(t) + \frac{\varphi(t)}{\varphi(t_\epsilon)}\|\Psi\|_{\mathcal{C}^{2,\alpha}(\{\pm t_\epsilon\}\times\mathbb{S}^1)} \\ & \leqslant \quad c\,\kappa\,\eta\,\varphi(t).\end{aligned}$$

We then claim that the main term in the definition of $\mathcal{F}$ is given by the harmonic extension $h_{\Psi^\perp}$. As a matter of fact, we check the following estimates :

$$\begin{array}{lll}
\left|\varphi^{2-\delta}\varphi^{-4}h_{\Psi^\perp}\right| & \leqslant & c\kappa\epsilon^{\frac{4}{3}}, \\
\left|\varphi^{2-\delta}J_\epsilon\left(\omega_\Psi\right)\right| & \leqslant & c\kappa\epsilon^{\frac{5}{3}}, \\
\left|\varphi^{2-\delta}\eta^2\varphi^{-2}\ Q_0\right| & \leqslant & c\epsilon^2, \\
\left|\varphi^{2-\delta}\eta^2\varphi\ Q_1\left(\varphi^{-1}\omega_\Psi\right)\right| & \leqslant & c\kappa\epsilon^{\frac{7+\delta}{3}}, \\
\left|\varphi^{2-\delta}\varphi^{-2}\ Q_2\left(\varphi^{-1}\omega_\Psi\right)\right| & \leqslant & c\kappa^2\epsilon^2, \\
\left|\varphi^{2-\delta}\varphi^{-1}\ Q_3\left(\varphi^{-1}\omega_\Psi\right)\right| & \leqslant & c\kappa^3\epsilon^3.
\end{array}$$

It implies that there exists $\epsilon_\kappa > 0$ such that for all $\epsilon \in (0, \epsilon_\kappa)$,

$$\left\|\mathcal{F}\left(0\right)\right\|_{\mathcal{C}^{2,\alpha}_\delta([-t_\epsilon,t_\epsilon]\times\mathbb{S}^1)} \quad \leqslant \quad c\ \kappa\ \epsilon^{\frac{4}{3}}.$$

This above estimate provides us the choice of a suitable radius for a small ball centred in 0. Let v_1 and v_2 be two functions such that

$$\text{for } i = 1, 2, \qquad \left\|v_i\right\|_{\mathcal{C}^{2,\alpha}_\delta([-t_\epsilon,t_\epsilon]\times\mathbb{S}^1)} \quad \leqslant \quad 2c\ \kappa\ \epsilon^{\frac{4}{3}}.$$

We then claim that

$$\left\|\mathcal{F}\left(v_1\right) - \mathcal{F}\left(v_2\right)\right\|_{\mathcal{C}^{2,\alpha}_\delta([-t_\epsilon,t_\epsilon]\times\mathbb{S}^1)} \quad \leqslant \quad \frac{1}{2}\ \left\|v_1 - v_2\right\|_{\mathcal{C}^{2,\alpha}_\delta([-t_\epsilon,t_\epsilon]\times\mathbb{S}^1)}$$

for all $\epsilon \in (0, \epsilon_\kappa)$ and thus, a fixed point with parameters gives the result. Note that

$$\begin{array}{rcl}
J_c\left(\mathcal{F}\left(v_1\right) - \mathcal{F}\left(v_2\right)\right) & = & -J_\epsilon\left(\omega_1 - \omega_2\right) + \eta^2\varphi\ \ \left(Q_1\left(\omega_1\right) - Q_1\left(\omega_2\right)\right) \\
& & +\,\varphi^{-2}\left(Q_2\left(\omega_1\right) - Q_2\left(\omega_2\right)\right) + \varphi^{-1}\left(Q_3\left(\omega_1\right) - Q_3\left(\omega_2\right)\right).
\end{array}$$

and that

$$\omega_1 - \omega_2 \quad = \quad v_1 - v_2.$$

We give the main estimates to establish $\mathcal{F}$ is $\frac{1}{2}$-contraction mapping. According to the definition of the linear operator J_ϵ — see corollary 3.3.3 — we find

$$\begin{array}{rcl}
\left|\varphi^{2-\delta}J_\epsilon\left(\omega_1 - \omega_2\right)\right| & = & \left|\varphi^{2-\delta}J_\epsilon\left(v_1 - v_2\right)\right| \\
& & \qquad\qquad \leqslant \quad c\ \epsilon^{\frac{2}{3}}\left\|v_1 - v_2\right\|_{\mathcal{C}^{2,\alpha}_\delta([-t_\epsilon,t_\epsilon]\times\mathbb{S}^1)}.
\end{array}$$

We deal with the terms Q_1 (we use the fact that Q_1 is linear) to obtain

$$\left|\varphi^{2-\delta}\eta^2\varphi\left(Q_1\left(\omega_1\right) - Q_1\left(\omega_2\right)\right)\right| \quad \leqslant \quad c\ \epsilon^{\frac{4}{3}}\left\|v_1 - v_2\right\|_{\mathcal{C}^{2,\alpha}_\delta([-t_\epsilon,t_\epsilon]\times\mathbb{S}^1)}.$$

By definition of Q_2 and Q_3, we get

$$\left|\varphi^{2-\delta}\varphi^{-2}\left(Q_2\left(\omega_1\right) - Q_2\left(\omega_2\right)\right)\right| \quad \leqslant \quad c\ \kappa\ \epsilon\left\|v_1 - v_2\right\|_{\mathcal{C}^{2,\alpha}_\delta([-t_\epsilon,t_\epsilon]\times\mathbb{S}^1)}$$

and

$$\left|\varphi^{2-\delta}\varphi^{-1}\left(Q_3\left(\omega_1\right) - Q_3\left(\omega_2\right)\right)\right| \quad \leqslant \quad c\ \kappa^2\epsilon^2\left\|v_1 - v_2\right\|_{\mathcal{C}^{2,\alpha}_\delta([-t_\epsilon,t_\epsilon]\times\mathbb{S}^1)}.$$

Collecting the above inequalities, we end up with

$$\left\|\mathcal{F}\left(v_1\right) - \mathcal{F}\left(v_2\right)\right\|_{\mathcal{C}^{2,\alpha}_\delta([-t_\epsilon,t_\epsilon]\times\mathbb{S}^1)} \quad \leqslant \quad c\left(\epsilon^{\frac{2}{3}} + \kappa\ \epsilon + \kappa^2\epsilon^2\right)$$

and the conclusion holds — up to reducing ϵ_κ. □

3.4 — (d) Local description of the minimal surface near its boundaries

To perform the gluing, we need an accurate description of the surface we have constructed in a neighbourhood of its boundaries. For all boundary data Ψ which satisfies the hypothesis of theorem 3.4.10, $\Sigma_{\omega_\Psi+v_\Psi}$ is a vertical graph on an small annulus $B(N, r_\epsilon) \setminus B\left(N, \frac{r_\epsilon}{2}\right)$ — here, N refers to the North pole of $\mathbb{S}^2$.

To describe the upper part of the small catenoid, it is useful to perform a change of scales : we define $\overline{u}_{\Psi_+}$ on $B_{\mathbb{R}^2}(0,1) \setminus B_{\mathbb{R}^2}\left(0, \frac{1}{2}\right)$ to be

$$\overline{u}_{\Psi_+}(y) \quad := \quad f_{\omega_\Psi+v_\Psi}(r_\epsilon y).$$

Theorem 3.4.11
Let $\delta \in (1,2)$, Ψ a boundary data which satisfies the previous hypothesis. Then the function $\overline{\mathfrak{d}}_{\Psi_+}$ defined on the annulus $B_{\mathbb{R}^2}(0,1) \setminus B_{\mathbb{R}^2}\left(0, \frac{1}{2}\right)$ by

$$\overline{\mathfrak{d}}_{\Psi_+}(y) \quad := \quad \overline{u}_{\Psi_+}(y) - \eta \ln\left(2\frac{r_\epsilon}{\eta}|y|\right) + \frac{1}{4}\eta^3 r_\epsilon^{-2}|y|^{-2} - W^i(\eta\Psi_+)(y)$$

satisfies following assertions :

(i) we have the estimate

$$\left\|\overline{\mathfrak{d}}_{\Psi_+}\right\|_{\mathcal{C}^{2,\alpha}\left(B_{\mathbb{R}^2}(0,1)\setminus B_{\mathbb{R}^2}\left(0,\frac{1}{2}\right)\right)} \quad \leqslant \quad c\epsilon^{\frac{5}{3}}\epsilon^{\frac{2-\delta}{3}}; \tag{3.4.21}$$

(ii) the mapping $\Psi \longmapsto \overline{\mathfrak{d}}_{\Psi_+}$ is $\frac{1}{2}$-contracting.

Remark 3.4.12 —
- The estimate (3.4.21) demonstrates that the terms that come from v_Ψ are very small in comparison with the data boundary.
- Obviously, same kind of result holds true for the lower part of the catenoid. In this case, the reader will pay attention to the fact that there is a change of signs : $\overline{\mathfrak{d}}_{\Psi_-}$ is defined by

$$\overline{\mathfrak{d}}_{\Psi_-}(y) \quad := \quad \overline{u}_{\Psi_-}(y) + \eta \ln\left(2\frac{r_\epsilon}{\eta}|y|\right) - \frac{1}{4}\eta^3 r_\epsilon^{-2}|y|^{-2} + W^i(\eta\Psi_-)(y).$$

PROOF
For convenience, we write $t = \operatorname{argcosh}\left(\frac{r_\epsilon}{\eta}|y|\right)$ and $\theta = \frac{y}{|y|}$.

(i) We recall the expansion

$$\operatorname{argcosh} x \quad = \quad \ln(2x) - \frac{1}{4x^2} + \underset{x\to\infty}{\mathcal{O}}\left(\frac{1}{x^4}\right)$$

in order to obtain

$$\eta \operatorname{argcosh}\left(\frac{r_\epsilon}{\eta}|y|\right) \quad = \quad \eta \ln\left(2\frac{r_\epsilon}{\eta}|y|\right) - \frac{1}{4}\eta^3 r_\epsilon^{-2}|y|^{-2} + \underset{\epsilon\to 0}{\mathcal{O}}\left(\epsilon^{\frac{7}{3}}\right).$$

Moreover, the contribution of the corrective function Cor is such that

$$|\eta \operatorname{Cor}(|t|)| \quad \leqslant \quad c\,\epsilon^{\frac{7}{3}},$$

thus it is very small in comparison with $\epsilon^{\frac{5}{3}}\epsilon^{\frac{2-\delta}{3}}$.
Furthermore, according to the construction of the term v_Ψ in theorem 3.4.10, we get

$$\eta\,|v_\Psi(t,\theta)| \quad \leqslant \quad 2c\,\epsilon^{\frac{5}{3}}\epsilon^{\frac{2-\delta}{3}}.$$

It is this term that yields to estimate (3.4.21).
It remains to deal with the part that establishes the link with the interior harmonic extensions. In this purpose, we make use of inequalities (3.4.19) and (3.4.20) to finally obtain

$$\left|\eta\left(h_{\Psi^\perp}+\ell_{\Psi^1_{\text{odd}}}\right)(t,\theta)-W^i\left(\eta\Psi_+\right)(y)\right| \quad \leqslant \quad c\,\kappa\,\epsilon^{\frac{7}{3}}\,t_\epsilon.$$

Notice that this quantity in also small in comparison with the contribution of v_Ψ. The estimate follows from these different estimates.

(ii) Let Ψ and $\overline{\Psi}$ be two boundary data on $\{\pm t_\epsilon\}\times\mathbb{S}^1$ such that their $\mathcal{C}^{2,\alpha}$-norm is smaller than κr_ϵ. Then we can write

$$\begin{aligned}\mathfrak{d}_{\Psi_+}(t,\theta)-\mathfrak{d}_{\overline{\Psi}_+}(t,\theta) \quad &= \quad \eta\left(\omega_\Psi-\omega_{\overline{\Psi}}\right)(t,\theta)\\ &\qquad -W^i\left(\eta\left(\Psi-\overline{\Psi}\right)\right)(y)+\eta\left(v_\Psi-v_{\overline{\Psi}}\right)(t,\theta).\end{aligned}$$

Like in the previous inequality, we check that

$$\begin{aligned}&\left|\eta\left(\omega_\Psi-\omega_{\overline{\Psi}}\right)(t,\theta)-W^i\left(\eta\left(\Psi-\overline{\Psi}\right)\right)(y)\right|\\ &\qquad\qquad \leqslant \quad c\,\eta^{\frac{5}{3}}t_\epsilon\left\|\Psi-\overline{\Psi}\right\|_{\mathcal{C}^{2,\alpha}(\{\pm t_\epsilon\}\times\mathbb{S}^1)}.\end{aligned}$$

Concerning the part with $v_\Psi - v_{\overline{\Psi}}$, we use a method similar to the one we have used in the proof of theorem 3.4.10 after writing

$$\begin{aligned}J_c\left(v_\Psi-v_{\overline{\Psi}}\right) \quad = \quad & -\frac{2}{\varphi^4}h_{\Psi^\perp-\overline{\Psi}^\perp}-J_\epsilon\left(\omega_\Psi-\omega_{\overline{\Psi}}+v_\Psi-v_{\overline{\Psi}}\right)\\ &+\eta^2\varphi Q_1\left(\frac{1}{\varphi}\left(\omega_\Psi-\omega_{\overline{\Psi}}+v_\Psi-v_{\overline{\Psi}}\right)\right)\\ &+\varphi^{-2}\left(Q_2\left(\omega_\Psi+v_\Psi\right)-Q_2\left(\omega_{\overline{\Psi}}+v_{\overline{\Psi}}\right)\right)\\ &+\varphi^{-1}\left(Q_3\left(\omega_\Psi+v_\Psi\right)-Q_3\left(\omega_{\overline{\Psi}}+v_{\overline{\Psi}}\right)\right).\end{aligned}$$

This implies that

$$\begin{aligned}\left|\varphi^{2-\delta}J_c\left(v_\Psi-v_{\overline{\Psi}}\right)\right| \quad \leqslant \quad & c\epsilon^{\frac{2}{3}}\left\|\Psi-\overline{\Psi}\right\|_{\mathcal{C}^{2,\alpha}(\{\pm t_\epsilon\}\times\mathbb{S}^1)}\\ &\qquad + c\epsilon^{\frac{2}{3}}\left\|v_\Psi-v_{\overline{\Psi}}\right\|_{\mathcal{C}^{2,\alpha}_\delta([-t_\epsilon,t_\epsilon]\times\mathbb{S}^1)},\end{aligned}$$

from which we deduce

$$\left(1 - c\epsilon^{\frac{2}{3}}\right) \|v_\Psi - v_{\overline{\Psi}}\|_{\mathcal{C}^{2,\alpha}_\delta([-t_\epsilon,t_\epsilon]\times\mathbb{S}^1)} \quad \leqslant \quad c\epsilon^{\frac{2}{3}} \left\|\Psi - \overline{\Psi}\right\|_{\mathcal{C}^{2,\alpha}(\{\pm t_\epsilon\}\times\mathbb{S}^1)} .$$

Thus for ϵ small enough,

$$|v_\Psi - v_{\overline{\Psi}}| (t, \theta) \quad \leqslant \quad c\, \epsilon^{\frac{2}{3}} \varphi(t)^\delta \left\|\Psi - \overline{\Psi}\right\|_{\mathcal{C}^{2,\alpha}(\{\pm t_\epsilon\}\times\mathbb{S}^1)} ,$$

for all $t \in [t_\epsilon - 2, t_\epsilon]$ and the result follows since $\epsilon^{\frac{2}{3}}\varphi(t_\epsilon)$ tends to 0 as ϵ tends to 0. □

4 The gluing

In the previous sections, we have produced a way to build two types of minimal surfaces in the homogeneous space $\mathbb{S}^2\times\mathbb{R}$: one on a punctured sphere with catenoidal necks and one that looks like to an Euclidean truncated catenoid. We now perform the gluing, i.e. we prove that we can glue these two kinds of minimal surfaces by choosing suitable boundary data. By minimal surface theory, it is enough to proceed to a $\mathcal{C}^1$ gluing on the boundaries. For convenience, we define the boundary data $\Upsilon := \eta\Psi$ so that Υ and Φ have the same rough estimate $\kappa\epsilon r_\epsilon$.

To this aim, the local description (see theorems 3.4.11 and 2.7.1) of the surfaces we have constructed plays an essential role. In particular, there are different kinds of terms in this :

- for both constructions, the main term that is given by the logarithmic growth of the form $a_g\epsilon \ln(r_\epsilon |y|)$ — it is this comparison that yields to the choice of η ;
- the following radial term whose form is $a_g^3\epsilon^3 r_\epsilon^{-2} |y|^{-2}$ up to a constant ;
- a constant — we will deal with it by using vertical translations ;
- the force that comes from Green's function whose rough estimate is $\kappa\epsilon r_\epsilon$;
- the harmonic extensions of boundary data with same rough estimate than the force — the properties of the linear operator $W^i - W^e$ will be necessary for the gluing ;
- smaller terms.

The idea is again to apply a fixed point theorem by using the contracting properties of the operators $\mathfrak{d}$ and $\overline{\mathfrak{d}}$.

4.1 The choice of boundary data

Here, we explain the shape of boundary data Υ and Φ that the object we want to construct enforces. Like the reader could suspect, we highlight the role of the different eigenmodes by using orthogonal projection.

First of all, recall that in our construction, the constant projections Φ^0 and Υ^0 vanish. The reason for this is that harmonic extensions of constants are either logarithmic (which explodes) or constants. In all cases, it implies that the estimates we have made are not valid.

Moreover, the Riemann minimal surface we have in mind to construct is invariant under the action of the group of isometries $\mathfrak{G}$. Consequently, on one hand, for Φ, we enforce the relations

$$\Phi_p(\cos\theta,\sin\theta) = \Phi_p(\cos\theta,-\sin\theta)$$

and

$$\Phi_q(\cos\theta,\sin\theta) = \Phi_r(\cos\theta,-\sin\theta).$$

These equalities hold because of the orthogonal symmetry $\mathfrak{s}$ with respect to the vertical set $\{x_1 = 0\}$. In other hand, for Υ, we enforce relations

$$\Upsilon_{p,+}(\cos\theta,\sin\theta) = \Upsilon_{p,+}(\cos\theta,-\sin\theta) = \Upsilon_{p,-}(-\cos\theta,-\sin\theta)$$

together with

$$\begin{aligned}\Upsilon_{q,+}(\cos\theta,\sin\theta) &= \Upsilon_{q,-}(-\cos\theta,\sin\theta)\\ &= \Upsilon_{r,+}(\cos\theta,-\sin\theta) = \Upsilon_{r,-}(-\cos\theta,-\sin\theta).\end{aligned} \quad (4.1.22)$$

In addition to that, recall that in our construction of a deformed truncated catenoid, we have enforced the mode 1 to be odd, that is to say

$$\Upsilon^1_{g,+} = -\Upsilon^1_{g,-}. \quad (4.1.23)$$

Here again, me make use of the above orthogonal symmetry. Regarding the link between the upper data and the lower data Υ_+ and Υ_-, we use the point reflection $\mathfrak{c}$ with respect to the center of the catenoid we glue at s_p. Therefore, to give a boundary data is the same to give a 9-tuple

$$\left(\Phi^{1,1}_p, \Phi^{\perp}_p, \Phi^{1,1}_q, \Phi^{1,2}_q, \Phi^{\perp}_q, \Upsilon^{1,1}_{p,+}, \Upsilon^{\perp}_{p,+}, \Upsilon^{1,1}_{q,+}, \Upsilon^{\perp}_{q,+}\right)$$

where $\Phi^{1,i}_g$ and $\Upsilon^{1,i}_g$ are real numbers such that [4]

$$\Phi^1_g(\cos\theta,\sin\theta) = \Phi^{1,1}_g\cos\theta + \Phi^{1,2}_g\sin\theta,$$

$\Phi^{\perp}_g$ and $\Upsilon^{\perp}_g$ are functions that satisfy the symmetries hypothesis.

Remark 4.1.1 — It could be interesting to develop the case of Υ_q in order to check that the equations (4.1.22) and (4.1.23) are in agreement when $\Upsilon^{1,2} = 0$. First of all, let us write

$$\Upsilon_{q,+}(\cos\theta,\sin\theta) = \underbrace{0}_{\text{mode } 0} + \underbrace{\Upsilon^{1,1}_{q,+}\cos\theta + \Upsilon^{1,2}_{q,+}\sin\theta}_{\text{mode } 1} + \Upsilon^{\perp}_{q,+}(\cos\theta,\sin\theta).$$

According to the equation (4.1.22), the function $\Upsilon_{q,-}$ then satisfies

$$\Upsilon_{q,-}(\cos\theta,\sin\theta) = \Upsilon_{q,+}(-\cos\theta,\sin\theta) = -\Upsilon^{1,1}_{q,+}\cos\theta + \Upsilon^{1,2}_{q,+}\sin\theta + \Upsilon^{\perp}_{q,+}(-\cos\theta,\sin\theta),$$

It follows that the equation (4.1.23) implies

$$\Upsilon^{1,1}_{q,+}\cos\theta + \Upsilon^{1,2}_{q,+}\sin\theta = -\left(-\Upsilon^{1,1}_{q,+}\cos\theta + \Upsilon^{1,2}_{q,+}\sin\theta\right) = \Upsilon^{1,1}_{q,+}\cos\theta - \Upsilon^{1,2}_{q,+}\sin\theta.$$

Consequently, $\Upsilon^{1,2}_{q,+}$ has to be chosen so that it vanishes. Geometrically, it means that the necks we glue at s_q and s_r are sloped away from the direction given by $\pm\mathbf{e}_q$ and $\pm\mathbf{e}_r$.

4. Same definition holds for Υ^1_g.

4.2 The gluing equations

4.2 — (a) Shape

Because of the symmetries of the problem, it is enough to conduct the gluing in s_p and s_q : what happens in s_r is obtained by the orthogonal symmetry with respect to the plane $\{x_1 = 1\}$ while what happens at upper and lower levels can be deduced with the help of a suitable translation and the point reflection.

Since we do not change the mean curvature under the action of the vertical translations, we introduce $\mathbf{t}_{g,\text{ver}} := t_{g,\text{ver}}\mathbf{e}_4$ a vertical vector we determine in the gluing of the mode constant. Rather than directly using $\overline{u}_\Psi$, we allow ourselves to perform a vertical translation of the catenoid Σ_{ω_g}. We also allow ourselves to use a suitable rotation so that s_g is the North pole. Then the general shape of the gluing equation in s_g can be written

$$\forall \theta \in \mathbb{S}^1, \qquad \left\{ \begin{array}{lcl} u_{\Phi,g}(\cos\theta, \sin\theta) & = & \overline{u}_{\Psi_{\pm},g}(\cos\theta, \sin\theta) + t_{g,\text{ver}}, \\ \partial_r u_{\Phi,g}(\cos\theta, \sin\theta) & = & \partial_r \overline{u}_{\Psi_{\pm},g}(\cos\theta, \sin\theta), \end{array} \right. \tag{4.2.24}$$

where the index $\pm$ for the catenoid part is "$-$" when $g = p$ (we glue the lower part of a small truncated catenoid with the catenoidal neck that points upwards whose main term is given by $\epsilon a_p \ln(|y|)$) and is "$+$" when $g = q$ (we glue the upper part of a small truncated catenoid with the catenoidal neck that points downwards whose main term is given by $-\epsilon a_q \ln(|y|)$).

4.2 — (b) Resolution : proof of theorem 1.0.2

As announced, we project the equations of the system (4.2.24) on the different eigenmodes 0, 1 and the others. We begin with the constant mode by using a Brouwer fixed point theorem in order to determine η which lies in a neighbourhood of $a_g\epsilon$, then we determine the vertical translation vector. After, we solve the modes associated with 2, 3, etc. A fixed point with parameters will be useful by applying the contracting properties for $\mathfrak{d}$ and $\overline{\mathfrak{d}}$. We end up with the mode 1 which is the more difficult to solve since we have to deal with the force term and we explain in what the balanced condition is satisfied.

The mode 0. Here, it is not necessary to distinguish in which point we perform the gluing because the method can be applied of all s_g. We project on the constant mode the equations of the system 4.2.24 with the help of the description theorems 3.4.11 and 2.7.1. We obtain

$$\left\{ \begin{array}{lcl} a_g\epsilon \ln r_\epsilon + \epsilon c_{\Gamma,g} - \frac{1}{4}a_g^3\epsilon^3 r_\epsilon^{-2} + \pi^0(\mathfrak{d}_{\Phi,g}) & = & \eta_g \ln\left(2\frac{r_\epsilon}{\eta_g}\right) - \frac{1}{4}\eta_g^3 r_\epsilon^2 \\ & & +\pi^0\left(\overline{\mathfrak{d}}_{\Psi,\pm,g}\right) + t_{g,\text{ver}}, \\ a_g\epsilon + \frac{1}{8}\epsilon^3 r_\epsilon^{-2} a_g^3 + \pi^0(\partial_r \mathfrak{d}_{\Phi,g}) & = & \eta_g + \frac{1}{8}\eta_g^3 r_\epsilon^{-2} + \pi^0\left(\partial_r \overline{\mathfrak{d}}_{\Psi,\pm,g}\right). \end{array} \right. \tag{4.2.25}$$

These above equations do not depend on the choice of the boundary data Υ and Φ. Besides, note that if we can solve the second equation, then we can also solve the first one by using a suitable vertical translation parameter. Moreover, the second equation implies that

$$(a_g\epsilon-\eta_g)\left(1+\frac{1}{8}r_\epsilon^{-2}\left(a_g^2\epsilon^2+\eta_g^2+a_g\epsilon\eta_g\right)\right) \quad = \quad \pi^0\left(\partial_r\overline{\mathfrak{d}}_{\Psi,\pm,g}-\partial_r\mathfrak{d}_{\Phi,g}\right), \quad (4.2.26)$$

what we could rewrite, if we choose η_g with rough estimate is ϵ, as

$$(a_g\epsilon-\eta_g)\left(1+\mathcal{O}\left(\epsilon^{\frac{2}{3}}\right)\right) \quad = \quad \mathcal{O}\left(\epsilon^{\frac{5}{3}}\epsilon^{\frac{2-\delta}{3}}\right).$$

That is why we have to choose η_g close to $a_g\epsilon$. More precisely, we have the

Proposition 4.2.1 — *Assume that Φ and Υ have the same rough estimate $\kappa\epsilon r_\epsilon$. Then for all, g, there exists $\mathbf{t}_{g,ver}$ and η_g such that the system* (4.2.25) *is solved. Moreover, following equations hold true :*

$$\eta=a_g\epsilon+\mathcal{O}\left(\epsilon^{\frac{5}{3}}\epsilon^{\frac{2-\delta}{3}}\right) \qquad \textit{and} \qquad t_{g,ver}=\eta_g\ln(\eta_g)+\mathcal{O}(\epsilon).$$

The main term of the translation vector is $\eta_g\ln(\eta_g)$: the vertical period of the surface we want to build has rough estimate $\epsilon\ln(\epsilon)$.

Proof

As announced previously, it is enough to solve the second equation and we use a Brouwer fixed point theorem. By construction, $\partial_r\overline{\mathfrak{d}}_{\Psi,\pm,g}$ continuously depends on η_g for $\eta_g\sim a_g r_\epsilon$. To fix ideas, assume that

$$a_g\epsilon-\epsilon^{\frac{5}{3}} \quad \leqslant \quad \eta_g \quad \leqslant \quad a_g\epsilon+\epsilon^{\frac{5}{3}}.$$

Then equation (4.2.26) can be rewritten

$$\eta_g \quad = \quad a_g\epsilon\left(1+\mathcal{O}\left(\epsilon^{\frac{2}{3}}\epsilon^{\frac{2-\delta}{3}}\right)\right)$$

where $\mathcal{O}\left(\epsilon^{\frac{2}{3}}\epsilon^{\frac{2-\delta}{3}}\right)$ denotes a function that continuously depends on η_g and which is bounded by a constant times $\epsilon^{\frac{2}{3}}\epsilon^{\frac{2-\delta}{3}}$ where the constant does not depend on η_g. By Brouwer fixed point theorem, there exists a solution η_g. □

Remark 4.2.2 — Indeed, one can prove that $|\eta_g-a_g\epsilon|$ belongs to $(0,\epsilon^*)$ where $*$ is a power such that $\epsilon^* >> \epsilon^{\frac{2}{3}}\epsilon^{\frac{2-\delta}{3}}$.

The mode $\perp$. From now on, we suppose ϵ and η_g are fixed such that they satisfy the conditions of the above proposition.

This mode is maybe the easiest to solve since it is nothing but an application of a fixed point theorem for contracting mappings. Like for the constant mode, the resolution of the projected system

$$\text{on } \mathbb{S}^1, \quad \left\{\begin{array}{lclclcl} \Phi_g^\perp & + & \pi^\perp(\mathfrak{d}_{\Phi,g}) & = & \Upsilon^\perp & + & \pi^\perp\left(\overline{\mathfrak{d}}_{\Psi,\pm,g}\right), \\ \partial_r W^e\left(\Phi_g^\perp\right) & + & \pi^\perp(\partial_r\mathfrak{d}_{\Phi,g}) & = & \partial_r W^i\left(\Upsilon^\perp\right) & + & \pi^\perp\left(\partial_r\overline{\mathfrak{d}}_{\Psi,\pm,g}\right) \end{array}\right.$$

can be generally done without considering a specific point s_g : the method works for the three points s_p, s_q and s_r. It is more useful to rewrite the above system in order to highlight the contracting part. In this purpose, we recall the following lemma (cf. section 11 in [MP01]).

Lemma 4.2.3 — *The linear mapping*

$$\mathcal{H} : \Upsilon^{\perp} \in \pi^{\perp}\left(\mathcal{C}^{2,\alpha}\left(\mathbb{S}^1\right)\right) \longmapsto \partial_r\Big(\left(W^e - W^i\right)\left(\Upsilon^{\perp}\right)\Big)_{|\mathbb{S}^1} \in \pi^{\perp}\left(\mathcal{C}^{1,\alpha}\left(\mathbb{S}^1\right)\right)$$

is an isomorphism.

We deduce from it we can rewrite the above system as :

$$\text{on } \mathbb{S}^1, \qquad \begin{cases} \Phi_g^{\perp} - \Upsilon_{\pm,g}^{\perp} &= \pi^{\perp}\left(\overline{\mathfrak{d}}_{\Psi,\pm,g} - \mathfrak{d}_{\Phi,g}\right), \\ \Upsilon_{\pm,g}^{\perp} &= \mathcal{H}^{-1}\Big[\pi^{\perp}\left(\partial_r\left(\overline{\mathfrak{d}}_{\Psi,\pm,g} - \mathfrak{d}_{\Phi,g}\right)\right) \\ & \qquad -\partial_r W^e\left(\pi^{\perp}\left(\overline{\mathfrak{d}}_{\Psi,\pm,g} - \mathfrak{d}_{\Phi,g}\right)\right)\Big]. \end{cases} \tag{4.2.27}$$

By contracting properties of the mappings $\mathfrak{d}$ and $\overline{\mathfrak{d}}$, one checks that we can apply a fixed point theorem with parameters to obtain the

Proposition 4.2.4 — *Assume that for all boundary data Υ, Φ whose norm is less than $\kappa\epsilon r_\epsilon$ and whose mode 1 is prescribed. Then we can solve the system (4.2.27) and the solution continuously depends on the parameters Φ^1 and $\Upsilon_{\pm}^1$.*

The forces and the mode 1. From now on, we assume for all boundary data with fixed eigenmode 1, we have solved the gluing equations of modes 0 and $\perp$. It remains to match the mode 1. We perform the projection of system (4.2.24) to obtain :

$$\text{on } \mathbb{S}^1, \quad \begin{cases} \epsilon r_\epsilon F_g\left(1, e^{\imath\theta_g}\right) + \Phi_g^1\left(e^{\imath\theta_g}\right) + \pi^1\left(\mathfrak{d}_{\Phi,g}\right) &= \Upsilon_{g,\pm}^1 + \pi^1\left(\overline{\mathfrak{d}}_{\Upsilon,\pm,g}\right), \\ \epsilon r_\epsilon F_g\left(1, e^{\imath\theta_g}\right) - \Phi_g^1\left(e^{\imath\theta_g}\right) + \pi^1\left(\partial_r\mathfrak{d}_{\Phi,g}\right) &= \Upsilon_{g,\pm}^1 + \pi^1\left(\partial_r\overline{\mathfrak{d}}_{\Upsilon,\pm,g}\right), \end{cases} \tag{4.2.28}$$

where we have used the relations for the harmonic extensions

$$W^e\left(\Phi\right)\left(re^{\imath\theta}\right) = \sum_{j\geqslant 1} r^{-j}\Phi^j\left(\theta\right), \qquad \partial_r W^e\left(\Phi\right)\left(re^{\imath\theta}\right) = \sum_{j\geqslant 1} -j\ r^{-j-1}\Phi^j\left(\theta\right),$$

together with

$$W^i\left(\Upsilon_{\pm}\right)\left(re^{\imath\theta}\right) = \sum_{j\geqslant 1} r^j\Phi^j\left(\theta\right) \quad \text{and} \quad \partial_r W^i\left(\Upsilon_{\pm}\right)\left(re^{\imath\theta}\right) = \sum_{j\geqslant 1} j\ r^{j-1}\Upsilon_{\pm}^j\left(\theta\right).$$

Proposition 4.2.5 — *Assume that the configuration associated with parameters* $(\mathring{\theta}_+, \mathring{\theta}_-)$ *is balanced. Then there exists* θ_- *such that*

(i) θ_- *is close to* $\mathring{\theta}_-$ *in the sense that for all parameter* $\frac{1}{3} \leqslant * \leqslant \frac{2}{3}$,

$$\theta_- - \mathring{\theta}_- = \mathcal{O}\left(\epsilon^*\right)$$

and

(ii) there exists boundary data Υ *and* Φ *such that all gluing equations are solved.*

Proof

Here, we can't go on without being more precise because the resolution at point s_p differs from the resolution at s_q. For all eigenfunction f^1 associated with the eigenmode 1, we write

$$f^1\left(e^{\imath\theta}\right) = f^{1,1}\cos\theta + f^{1,2}\sin\theta$$

and $\pi^{1,i}$ the associated projections. Note that because of the symmetries, the component $\Upsilon^{1,2}_{g,\pm}$ always vanishes and thus, according to the construction of the truncated deformed catenoid,

$$\pi^{1,2}\left(\overline{\mathfrak{d}}_{\Psi,\pm,g}\right) = \pi^{1,2}\left(\partial_r\overline{\mathfrak{d}}_{\Psi,\pm,g}\right) = 0.$$

Then, in one hand, at s_p, the above system (4.2.28) has components on $\cos\theta_p$ only and it is written

$$\left\{\begin{array}{lcl} \epsilon r_\epsilon F^{1,1}_p + \Phi^{1,1}_p + \pi^{1,1}\left(\mathfrak{d}_{\Phi,p}\right) & = & \Upsilon^{1,1}_{p,-} + \pi^{1,1}\left(\overline{\mathfrak{d}}_{\Upsilon,-,p}\right), \\ \epsilon r_\epsilon F^{1,1}_p - \Phi^{1,1}_p + \pi^{1,1}\left(\partial_r\mathfrak{d}_{\Phi,p}\right) & = & \Upsilon^{1,1}_{p,-} + \pi^{1,1}\left(\partial_r\overline{\mathfrak{d}}_{\Upsilon,-,p}\right), \end{array}\right.$$

from what we deduce

$$\left\{\begin{array}{lcl} 2\Phi^{1,1}_p & = & \pi^{1,1}\left[(\mathrm{Id}-\partial_r)\left(\overline{\mathfrak{d}}_{\Upsilon,-,p} - \mathfrak{d}_{\Phi,p}\right)\right], \\ \Upsilon^{1,1}_{p,-} & = & \epsilon r_\epsilon F^{1,1}_p + \Phi^{1,1}_p + \pi^{1,1}\left(\mathfrak{d}_{\Phi,p} - \overline{\mathfrak{d}}_{\Upsilon,-,p}\right). \end{array}\right. \tag{4.2.29}$$

On the other hand, at s_q we have to deal with the projections $\pi^{1,1}$ and $\pi^{1,2}$. We get, after rewriting equations as above,

$$\left\{\begin{array}{lcl} 2\Phi^{1,1}_q & = & \pi^{1,1}\left[(\mathrm{Id}-\partial_r)\left(\overline{\mathfrak{d}}_{\Upsilon,+,q} - \mathfrak{d}_{\Phi,q}\right)\right], \\ \Upsilon^{1,1}_{q,+} & = & \epsilon r_\epsilon F^{1,1}_q + \Phi^{1,1}_q + \pi^{1,1}\left(\mathfrak{d}_{\Phi,q} - \overline{\mathfrak{d}}_{\Upsilon,+,q}\right). \end{array}\right. \tag{4.2.30}$$

The shape for after projection $\pi^{1,2}$ is completely different because the small catenoids do not have bend on the associated direction. We get the system :

$$\left\{\begin{array}{lcl} \epsilon r_\epsilon F^{1,2}_q + \Phi^{1,2}_q + \pi^{1,2}\left(\mathfrak{d}_{\Phi,q}\right) & = & 0, \\ \epsilon r_\epsilon F^{1,2}_q - \Phi^{1,2}_q + \pi^{1,2}\left(\partial_r\mathfrak{d}_{\Phi,q}\right) & = & 0. \end{array}\right.$$

It is this last system in which the balanced condition appears : we can rewrite it as

$$\left\{ \begin{array}{lcl} 2\Phi_q^{1,2} & = & \pi^{1,2}\left[\left(\mathrm{Id} - \partial_r\right)\left(\mathfrak{d}_{\Phi,q}\right)\right], \\ 2\epsilon r_\epsilon F_q^{1,2} & = & \pi^{1,2}\left[\left(\mathrm{Id} + \partial_r\right)\left(\mathfrak{d}_{\Phi,q}\right)\right]. \end{array} \right. \tag{4.2.31}$$

Therefore $F_q^{1,2}$ has rough estimate $\epsilon^{\frac{2}{3}}$. This is why we assume $F_q^{1,2}$ vanishes at the initial condition. We then claim that a Brouwer fixed point theorem yields to the result.

More exactly, we first consider the five equations given by

- the systems (4.2.29) and (4.2.30) together with
- the first equation of the system (4.2.31).

Using the contracting properties of $\mathfrak{d}$ and $\overline{\mathfrak{d}}$, it is easy to apply a fixed point theorem with parameters to prove that we can solve these five equations — and the solution continuously depends on θ_- and θ_+.

It remains to solve

$$F_q^{1,2} \quad = \quad \frac{1}{2\epsilon r_\epsilon}\pi^{1,2}\left[\left(\mathrm{Id} + \partial_r\right)\left(\mathfrak{d}_{\Phi,q}\right)\right]. \tag{4.2.32}$$

The right member continuously depends on θ_+ and θ_- by construction. Moreover, there exists a constant c that does not depend on $\theta_\pm$ such that for all constructed solution (Φ, Υ),

$$-c\ \epsilon^{\frac{2}{3}} \quad \leqslant \quad \frac{1}{2\epsilon r_\epsilon}\pi^{1,2}\left[\left(\mathrm{Id} + \partial_r\right)\left(\mathfrak{d}_{\Phi,q}\right)\right] \quad \leqslant \quad c\ \epsilon^{\frac{2}{3}}.$$

But, according to the expression of $F_q^{1,2}$ (cf. (2.1.6)), one proves that for all $*$ in $\left(\frac{1}{3}, \frac{2}{3}\right)$, for all θ_- that belongs to $[\mathring{\theta}_- - \epsilon^*, \mathring{\theta}_- + \epsilon^*]$, $F_q^{1,2}$ describes an interval whose form is $[-c_1\epsilon^*,\ c_2\epsilon^*]$ where the constants c_i have rough estimate 1. Since ϵ^* is much more larger than $\epsilon^{\frac{2}{3}}$, a Brouwer fixed point gives the result. $\square$

Chapitre IV

Paramétrisation des surfaces minimales de Riemann dans $\mathbb{S}^2 \times \mathbb{R}$

Introduction

In this chapter, we provide a thorough local description of the Riemann's minimal example in the homogeneous space $\mathbb{S}^2 \times \mathbb{R}$. In particular, we prove that its necks behave like small truncated catenoids.

Similar work was done in [HP07] in $\mathbb{R}^2 \times \mathbb{R}$. The authors used it to study the Jacobi operator on a half Riemann's example in order to construct new minimal surfaces. These examples could be considered as two half Riemann's surfaces connected to each other by a number k of catenoidal necks with $1 \leqslant k \leqslant 37$.

Although we do not provide such examples in $\mathbb{S}^2 \times \mathbb{R}$, it is a first step to a better understanding of the Riemann's minimal example in this space. In particular, we highlight its different symmetries and provide the description of a catenoid in $\mathbb{S}^2 \times \mathbb{R}$.

1 Riemann minimal surface in $\mathbb{R}^3$

As a warm-up, we briefly expose the computation of the classical Riemann example. Recall that it is foliated by circles whose center describes a straight line.

Let t denote the height parameter, $r(t)$ be the radius of the horizontal circle whose center is $(a(t), 0, t)$. We then look for functions a and r such that the parametrization

$$X : \begin{array}{ccc} \mathbb{R}^2 & \longrightarrow & \mathbb{R}^3 \\ (t, \theta) & \longmapsto & \begin{pmatrix} a(t) \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} r(t)\cos\theta \\ r(t)\sin\theta \\ t \end{pmatrix} \end{array} \tag{1.0.1}$$

describes a minimal surface. The first term can be interpreted as the action of an horizontal translation. The function t could be interpreted as the *height funcion.* Notice that when $a \equiv 0$, then this problem is nothing but the construction of the catenoid.

Furthermore, the surface that X describes is invariant under the action of the orthogonal reflection with recpect to the plane $\{x_2 = 0\}$.

The induced metric is given by

$$g = \begin{pmatrix} \dot r^2 + \dot a^2 + 2\dot r\dot a\cos\theta + 1 & -r\dot a\sin\theta \\ -r\dot a\sin\theta & r^2 \end{pmatrix},$$

where $\dot f$ denotes $\frac{\mathrm{d}}{\mathrm{d}t}f$. Then one checks that the unit normal is such that

$$N = \frac{1}{\sqrt{1 + \dot r^2 + \dot a^2\cos^2\theta + 2\dot r\dot a\cos\theta}} \begin{pmatrix} -\cos\theta \\ -\sin\theta \\ \dot r + \dot a\cos\theta \end{pmatrix}$$

and the second fundamental form is

$$\mathrm{II} = \frac{1}{\sqrt{1 + \dot r^2 + \dot a^2\cos^2\theta + 2\dot r\dot a\cos\theta}} \begin{pmatrix} -\ddot r - \ddot a\cos\theta & 0 \\ 0 & r \end{pmatrix}.$$

Consequently, X describes a minimal surface if its mean curvature H vanishes, that is to say if the ordinary differential equation

$$1 + \dot r^2 + \dot a^2 - r\ddot r + (2\dot r\dot a - r\ddot a)\cos\theta = 0$$

is satisfied. Since it has to be true for any angle θ, this above equality can be reduced to the system

$$\begin{cases} r\ddot r = 1 + \dot r^2 + \dot a^2, \\ 2\dot r\dot a = r\ddot a. \end{cases} \tag{1.0.2}$$

The second equation can be integrated and we find $\dot a = Cr^2$, where C is a positive constant. Thus the first one can be rewritten in order to obtain

$$1 + \dot r^2 = r\ddot r - C^2r^4.$$

Thus, if r is a solution, we easily check that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\dot r^2}{r^2}\right) = 2\dot r r^{-3} + 2C^2 r\dot r$$

and integration leads us to the system

$$\begin{cases} 1 + \dot r^2 = Ar^2 + C^2r^4, \\ \dot a = Cr^2, \end{cases} \tag{1.0.3}$$

where A and C are constants.

2 Riemann minimal surface in $\mathbb{S}^2 \times \mathbb{R}$

We look for the analogue of the previous section in the geometry of the homogeneous space $\mathbb{S}^2 \times \mathbb{R}$. From now on, we consider $\mathbb{S}^2$ as a submanifold of $\mathbb{R}^3$ and we denote by $(\mathbf{e}_i)_{1 \leqslant i \leqslant 4}$ an orthonormal basis of $\mathbb{R}^3 \times \mathbb{R}$.

Like in the Riemann example, we enforce this surface to be foliated by circles. However, we can't work with an horizontal translation term as before. In order to make up for this problem, it is enough to consider the horizontal translation in $\mathbb{R}^2 \times \mathbb{R}$ as an "horizontal" isometry. If we transpose this in $\mathbb{S}^2 \times \mathbb{R}$, it is natural to consider "horizontal" rotations. In addition to that, recall that in the Euclidean case, the translation term lies in the line Span $(\mathbf{e}_1)$; in $\mathbb{S}^2 \times \mathbb{R}$, we choose a rotation R so that $\mathbf{e}_2$ is invariant under its action. More exactly, we consider the following parametrization :

$$X : \begin{array}{ccc} \mathbb{R}^2 & \longrightarrow & \mathbb{S}^2 \times \mathbb{R} \\ (t, \theta) & \longmapsto & \begin{pmatrix} R(t) & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} X_s(t, \theta) \\ t \end{pmatrix}, \end{array}$$

where

$$X_s(t, \theta) = \begin{pmatrix} \cos b(t) \cos \theta \\ \cos b(t) \sin \theta \\ \sin b(t) \end{pmatrix} \in \mathbb{S}^2 \quad \text{and} \quad R(t) = \begin{pmatrix} \cos a(t) & 0 & -\sin a(t) \\ 0 & 1 & 0 \\ \sin a(t) & 0 & \cos a(t) \end{pmatrix}$$

denotes the rotation of angle $a(t)$ which preserves the direction $\mathbf{e}_2$. This choice can be explained in the same way that the choice of translations in previous case in which the center of the circles covers straight line. We refer to the following figure to illustrate this kind of parametrization.

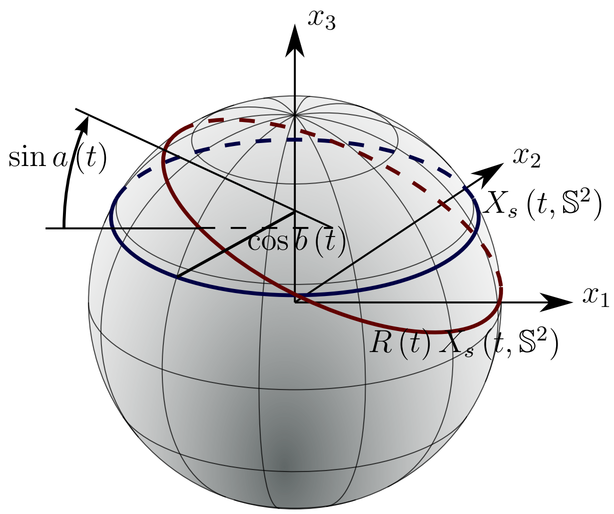


Figure IV.1: Geometric meaning of the parametrization.

The parameter b corresponds to the radius r of the $\mathbb{R}^2 \times \mathbb{R}$-case : the surface associated with the parameter is foliated by circles chose radius is $\cos b$. Moreover, the parameter a corresponds to the translation term of the $\mathbb{R}^2 \times \mathbb{R}$-case, except that in $\mathbb{S}^2 \times \mathbb{R}$, we replace the translation with rotations. Obviously, the surface we construct with the help of X is invariant under the action of the orthogonal reflection with respect to the linear space $\{x_2 = 0\}$ of $\mathbb{R}^3 \times \mathbb{R}$.

From now on, we assume that b is not a constant function. This hypothesis makes sense because of the Euclidean case : regarding the classical catenoid or the Riemann minimal exammple, the radius function is not constant and there does not exist minimal surfaces foliated by circles of constant radius in $\mathbb{R}^2 \times \mathbb{R}$.

2.1 The minimal surface equation

Our aim is to obtain a similar ODE system than the one we have explicited in the Euclidean case — see (1.0.3). Thus we compute the mean curvature of a surface in $\mathbb{S}^2 \times \mathbb{R}$ whose parametrization is given by X.

When f is a function which depends on the variable t, we note $\dot{f}(t) = \frac{\mathrm{d}f}{\mathrm{d}t}(t)$. Thus the tangent vectors of the surface are given by

$$\dot{X}(t,\theta) = \begin{pmatrix} \dot{R}(t)\, X_s(t,\theta) + R(t)\, \dot{X}_s(t,\theta) \\ t \end{pmatrix} \quad \text{and} \quad \partial_\theta X(t,\theta) = \begin{pmatrix} R(t)\, \partial_\theta X_s(t,\theta) \\ 0 \end{pmatrix}.$$

In order to compute the induced metric, we check the relations

$$\left\| \dot{R} X_s \right\|^2 = \dot{a}^2 \left(\cos^2 b \cos^2\theta + \sin^2 b \right) \qquad \text{and} \qquad \left\langle R\dot{X}_s, \dot{R}X_s \right\rangle = \dot{a}\dot{b}\cos\theta.$$

Besides, since R is an isometry of $\mathbb{R}^3$, we obtain

$$\left\| R\dot{X}_s \right\|^2 = \left\| \dot{X}_s \right\|^2 = \dot{b}^2.$$

Therefore, the metric induced by the parametrization is given by

$$g = \begin{pmatrix} 1 + \dot{a}^2\cos^2 b\cos^2\theta + \dot{a}^2\sin^2 b + 2\dot{a}\dot{b}\cos\theta + \dot{b}^2 & \dot{a}\cos b\sin b\sin\theta \\ \dot{a}\cos b\sin b\sin\theta & \cos^2 b \end{pmatrix}.$$

In order to describe an unit normal N, we remark that it has to be orthogonal to $\dot{X}$, $\partial_\theta X$ and also to $(X_s, 0)$. The last vector takes place because we have chosen to work with the sphere $\mathbb{S}^2$ we suppose embedded in $\mathbb{R}^3$. Thus if we choose N so that [1]

$$N = c \begin{pmatrix} \alpha\dot{R}X_s + \beta R\dot{X}_s \\ \gamma \end{pmatrix},$$

where α, β, γ are real numbers and c is a positive real number such that N is a unit vector, we find that N can be written

$$N = c \begin{pmatrix} R\dot{X}_s \\ -\dot{b}^2 - \dot{a}\dot{b}\cos\theta \end{pmatrix}.$$

1. Recall that b is not constant, thus $\dot{b} \neq 0$ and N does not vanish.

Note that we do not have to explicit the positive real number c because we want to build a minimal surface ; it would not be the case if we want to build a non-vanishing constant mean curvature surface.

Regarding the second fundamental form, we make use of relations

$$\|X_s\|^2 = 1, \qquad \left\langle X_s, \dot{X}_s \right\rangle = 0 \qquad \text{and} \qquad \left\langle \dot{X}_s, \ddot{X}_s \right\rangle = \dot{b}\ddot{b},$$

together with the isometry properties of the rotation R and

$$\dot{a}\ddot{R} \quad = \quad \ddot{a}\dot{R} - \dot{a}^3 \left(R - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right).$$

Then calculus demonstrates that the second fundamental form is given by the following matrix :

$$\mathrm{II} = c \begin{pmatrix} \dot{b}\ddot{b} + \ddot{a}\dot{b}\cos\theta - \dot{a}^2\dot{b}\cos b \sin b \sin^2\theta & -\dot{a}\dot{b}\cos^2 b \sin\theta \\ -\dot{a}\dot{b}\cos^2 b \sin\theta & \dot{b}\cos b \sin b \end{pmatrix}.$$

Since the mean curvature is given by the formula

$$H \quad = \quad \frac{\mathrm{II}_{11} g_{11} - 2\mathrm{II}_{12} g_{12} + \mathrm{II}_{22} g_{22}}{\det g}$$

it is enough that the quantity $\mathrm{II}_{11} g_{11} - 2\mathrm{II}_{12} g_{12} + \mathrm{II}_{22} g_{22}$ vanishes to obtain a minimal surface. Consequently, X describes a minimal surface if a and b are solutions of the following equation :

$$0 \quad = \quad \ddot{b}\cos b + \left(1 + \dot{a}^2 + \dot{b}^2\right)\sin b + \left(\ddot{a}\cos b + 2\dot{a}\dot{b}\sin b\right)\cos\theta.$$

The hypothesis about $\dot{b}$ is reasonable : the quantity b is the analogue of the radius r in the $\mathbb{R}^2 \times \mathbb{R}$ case and this last never is a constant.

Since the above equation has to be true for any angle θ, it is equivalent to the system

$$\left\{ \begin{array}{rcl} \ddot{b}\cos b + \left(1 + \dot{a}^2 + \dot{b}^2\right)\sin b & = & 0, \\ \ddot{a}\cos b + 2\dot{a}\dot{b}\sin b & = & 0. \end{array} \right. \tag{2.1.4}$$

Notice the similarity of this system with the classical Riemann example (1.0.2). As done in $\mathbb{R}^2 \times \mathbb{R}$, we can integrate the second equation in order to obtain

$$\dot{a} \quad = \quad C\cos^2 b,$$

where C is a constant. Then the first one can be rewritten

$$\ddot{b}\cos b + \left(1 + C^2\cos^4 b + \dot{b}^2\right)\sin b \quad = \quad 0.$$

However, if b is a solution, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\dot{b}^2}{\cos^2 b}\right) \quad = \quad -2\dot{b}\sin b\left(\cos^{-3} b + C^2\cos b\right).$$

Thus the system (2.1.4) turns into

$$\begin{cases} 1+\dot{b}^2 &= A\cos^2 b + C^2\cos^4 b, \\ \dot{a} &= C\cos^2 b, \end{cases} \tag{2.1.5}$$

where A and C are constants. Once again, the reader will pay attention to the likeness between this case and the $\mathbb{R}^2 \times \mathbb{R}$ case.

2.2 The analogue of the catenoid in $\mathbb{S}^2 \times \mathbb{R}$

The catenoid and the Riemann minimal surface of $\mathbb{R}^2 \times \mathbb{R}$ are both foliated by circles. However, we could consider the catenoid as a specific Riemann example : a *vertical* example, that is to say when the center of circles don't move ; it is the case when the function a is a constant. In terms of equation (1.0.3), the catenoid matches with the case in which the constant C vanishes. Therefore, it is natural to explain the similar case in $\mathbb{S}^2 \times \mathbb{R}$: it is what we call the *catenoid*.

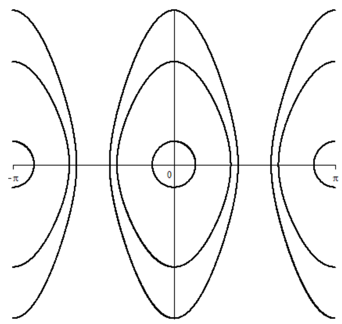
Therefore, let us assume $C = 0$. Then $a(t) = a_0$ is constant. Without loss of generality, we assume $a_0 = 0$, up to applying a suitable rotation. The ordinary differential equation of b turns into

$$1+\dot{b}^2 \quad = \quad A\cos^2 b. \tag{2.2.6}$$

Consequently, the constant A has to be chosen so that $A > 1$. The case $A = 1$ is left out since X no longer parametrizes a surface. Furthermore, this equation implies that $|\cos b| \geqslant \frac{1}{\sqrt{A}}$, thus we introduce the critical angle b_A such that

$$b_A := \arccos\left(\frac{1}{\sqrt{A}}\right) \qquad \text{and} \qquad \forall t,\ b(t) \in [-b_A, b_A].$$

To interpret this, we should remark that it enforces b to be different from the angle $\frac{\pi}{2}$. This condition then says that the radius $\cos b$ of the horizontal circle never vanishes.



On the left are represented phase portraits associated to the ODE (2.2.6) that b satisfies. The variable b can be read on the abscissa while its derivative $\dot{b}$ is on the ordinate. Moreover, we clockwise cover the curves. Here, we have represented the cases $A = 1.2$, $A = 5$ and $A = 10$. The size of A matches the size of curves.

Lemma 2.2.1 — *The ordinary differential equation* (2.2.6) *has a periodic solution. Besides, its period is such that*

- *when A tends to 1,*

$$T = 2\pi - \frac{\pi}{2}(A-1) + \mathcal{O}\left((A-1)^2\right);$$

- *when A tends to ∞,*

$$T = 2\frac{\ln(A)}{\sqrt{A}} + \mathcal{O}\left(A^{-1}\right).$$

Furthermore, if $\dot{b}(0) = 0$, then for all $|t| \leqslant t_A$ with $t_A = \frac{\ln A - 1}{2\sqrt{A}}$,

$$b(t) = -\frac{\pi}{2} + \frac{1}{\sqrt{A}}\cosh\left(\sqrt{A}t\right) + \underset{A\to\infty}{\mathcal{O}_\infty}\left(\frac{1}{A^{\frac{3}{2}}}\cosh\left(\sqrt{A}t\right)\sinh^2\left(\sqrt{A}t\right)\right), \tag{2.2.7}$$

where $\mathcal{O}_\infty(f(t))$ is a function that is bounded by a constant times f (and its derivatives are bounded by a constant times the derivatives of f).

Remark 2.2.2 —

- The expansion (2.2.7) at first order is equivalent to the classical vertical catenoid in $\mathbb{R}^3$ whose minimal radius is $\frac{1}{\sqrt{A}}$. The reason is relatively simple. As a matter of fact, when A is large, $b(0)$ is close to $-\frac{\pi}{2}$, thus the radius $\cos(b(0))$ is arbitrary small. Moreover, locally, when the radius is small, in a small area around the initial configuration, $\mathbb{S}^2$ behaves like $\mathbb{R}^2$ and its metric is almost flat. It is in agreement with the catenoidal shape of the surface near this point.
- Unlike the Euclidean case, the periodicity condition implies that the catenoid of $\mathbb{S}^2 \times \mathbb{R}$ is also periodic, that is to say that the surface $\mathscr{S}$ we obtain is invariant under the action of the vertical translation $\mathbf{t} = T\mathbf{e}_4$. In addition to that, if $b(0) = -b_A$, then we check that $\mathscr{S}$ is invariant under the action of point reflection with respect to the origin. Finally, notice that since a is chosen so that it vanishes everywhere, $\mathscr{S}$ is a surface of revolution around the vertical axis.

Proof

Assume $b(0) = -b_A$. Then, a solution b is necessarily order-preserving in a neighbourhood of 0^+. It follows that, for positive height t small enough, we can write

$$\mathrm{d}t = \frac{\mathrm{d}b}{\sqrt{A\cos^2 b - 1}} \qquad \text{and thus} \qquad t(b) = \int_{-b_A}^{b} \frac{\mathrm{d}u}{\sqrt{A\cos^2 u - 1}}.$$

Therefore, it is enough to prove that the height does not explode when b increases. For example, near $b = -b_A$, a Taylor's expansion provides the expression

$$\left(A\cos^2(-b_A + h) - 1\right)^{-1/2} \underset{h\to 0}{\sim} \left(2\sqrt{A-1}\,h\right)^{-\frac{1}{2}}$$

and this is the term of an improper integral that converges when h tends to 0. Same kind of result holds when b lies in a neighbourhood of $+b_A$.

Therefore, the height is finite for all angle b : the solution is defined on $\mathbb{R}$. Thus the solution is periodic and its period is

$$T \quad := \quad 4\int_0^{b_A} \frac{\mathrm{d}u}{\sqrt{A\cos^2 u - 1}} \quad = \quad 4\int_{\frac{1}{\sqrt{A}}}^{1} \frac{\mathrm{d}v}{\sqrt{(1-v^2)(Av^2-1)}}.$$

The behaviour of T when A tends to 1 or ∞ follows from the positivity of the integrand together with its asymptotic expansion — in the ∞-case, we make use of formula

$$\int_{\frac{1}{\sqrt{A}}}^{1} \frac{\mathrm{d}v}{\sqrt{Av^2-1}} \quad = \quad \frac{\ln(A)}{2\sqrt{A}} + \mathcal{O}\left(A^{-1}\right).$$

In the case A very large, it is more convenient to define β such that $b(t) = -\frac{\pi}{2} + \beta(t)$ and we define the critical angle β_A to be

$$\beta_A := \arcsin\left(\frac{1}{\sqrt{A}}\right) = \beta(0).$$

Thus, $\beta(t)$ is very small in a neighbourhood of $t = 0$. In first approximation, equation (2.2.6) turns into

$$1 + \dot{\beta}^2 \quad = A\beta^2,$$

whose solution is

$$\beta_A \cosh\left(\sqrt{A}t\right) + \mathcal{O}\left(\frac{1}{A}\right)\sinh\left(\sqrt{A}t\right).$$

Therefore, it makes sense to look for a solution whose form is given by

$$\beta(t) \quad = \quad \beta_A \cosh\left(\sqrt{A}v(t)\right)$$

for some function v such that $v(0) = 0$. In terms of v, (2.2.6) gives equation

$$\begin{aligned} \dot{v}^2 \quad &= \quad \frac{\sin^2\left(\beta_A \cosh\left(\sqrt{A}v\right)\right) - \sin^2(\beta_A)}{\beta_A^2 \sinh^2\left(\sqrt{A}v\right)} \\ &= \quad \sum_{k=1}^{\infty} (-1)^{m+1} \frac{2^{2m-1}}{(2m)!} \beta_A^{2m-2} \frac{\cosh^{2m}\left(\sqrt{A}v\right) - 1}{\cosh^2\left(\sqrt{A}v\right) - 1}. \end{aligned}$$

Assume $t(1-c) \leqslant v(t) \leqslant t$ for some positive real number $c \in (0,1)$. Then previous expansion implies

$$1 - c\beta_A^2 \cosh^2\left(\sqrt{A}t\right) \quad \leqslant \quad \dot{v} \quad \leqslant \quad 1,$$

therefore

$$v(t) \quad = \quad t + \underset{A\to\infty}{\mathcal{O}_\infty}\left(\beta_A^2 \frac{\sinh\left(2\sqrt{A}t\right)}{4\sqrt{A}}\right).$$

It follows that the assumption is correct for $|t| \leqslant t_A$. We end by using the Taylor's expansion of β_A

$$\beta_A \quad = \quad \frac{1}{\sqrt{A}}\left(1 + \underset{A\to\infty}{\mathcal{O}}\left(\frac{1}{A}\right)\right). \qquad \square$$

2.3 Analysis in the case C very large

Instead of considering the functions a and b, we rescale the problem by introducing $\overline{a}$ and $\overline{b}$ such that

$$\overline{a}(t) := a\left(\frac{t}{C}\right) \qquad \text{and} \qquad \overline{b}(t) := b\left(\frac{t}{C}\right).$$

Then the minimal surface system equations (2.1.5) turns into

$$\left\{\begin{array}{rcl} \dot{\overline{a}} & = & \cos^2 \overline{b}, \\ \dot{\overline{b}}^2 & = & \mu \cos^2 \overline{b} + \cos^4 \overline{b} - \lambda^2 \end{array}\right. \tag{2.3.8}$$

where

$$\lambda := \frac{1}{C} \qquad \text{and} \qquad \mu := \frac{A}{C^2}$$

From now on, we omit the overline on functions to relieve notations and we assume μ is a constant real number.

2.3 — (a) A first approach to reduce the problem

In order to describe solutions of the above system when C is very large (that is to say when λ is very small), it makes sense to analyse it when $\lambda = 0$: it will provide us different data to study the general case. Therefore, we are interested in the system in which we replace λ in the second equation with 0 and we find

$$\left\{\begin{array}{rcl} \dot{a} & = & \cos^2 b, \\ \dot{b}^2 & = & \cos^2 b\left(\mu + \cos^2 b\right). \end{array}\right. \tag{2.3.9}$$

Notice that the last equation implies that μ has to be chosen so that$\mu \geqslant -1$.

Lemma 2.3.1 — *Assume $b(0) > 0$ and $\dot{b}(0) > 0$. Then solutions of the above system are :*

- *such that* $\dot{a}$ *and* b *are periodic functions when* μ *belongs to* $(-1, 0)$ *and for all* t, $b(t) \in [-b_\lambda, b_\lambda]$ *where*

$$b_\lambda \quad := \quad \arccos\left(\sqrt{-\mu}\right);$$

- *non-periodic when* $\mu \geqslant 0$, *defined for all height* $t \in \mathbb{R}$ *and*

$$b(t) \xrightarrow[t\to\pm\infty]{} \pm\frac{\pi}{2}.$$

Furthermore, if $a(0) = 0$, *then*

$$a(t) \xrightarrow[t\to\pm\infty]{} \pm a_\infty \qquad \textit{where} \qquad a_\infty := \arcsin\left(\frac{1}{\sqrt{1+\mu}}\right).$$

Remark 2.3.2 — The lack of periodicity can be explained by the approximation we do when we assume $\lambda = 0$.

PROOF
The second equation implies that for t small enough,

$$\mathrm{d}t \quad = \quad \frac{\mathrm{d}b}{\cos b\sqrt{\mu + \cos^2 b}}.$$

On one hand, when $\mu \in (-1, 0)$, as done in the proof of lemma 2.2.1, we check in this case the solution is periodic and its period T is

$$T \quad = \quad 4\int_0^{b_\lambda} \frac{\mathrm{d}u}{\cos u\sqrt{\mu + \cos^2 u}} \quad = \quad 4\int_{\sqrt{-\mu}}^{1} \frac{\mathrm{d}v}{v\sqrt{(1-v^2)(\mu+v^2)}}.$$

Besides, the limit radius $\cos b_\lambda$ corresponds to the case in which $\mu + \cos^2 b_\lambda$ vanishes, that is to say $b_\lambda = \arccos\left(\sqrt{-\mu}\right)$.

On the other hand, when μ is positive or vanishes, the only problem which might occur is when b approaches $\frac{\pi}{2}$. Since $b \mapsto \frac{1}{\cos b}$ is not integrable in $\frac{\pi}{2}$, it follows that infinite height is necessary to reach this critical angle. Besides, b reaches all angles in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Moreover, we check that

$$\frac{\mathrm{d}a}{\mathrm{d}b} \quad = \quad \frac{\cos b}{\sqrt{\mu + \cos^2 b}}.$$

To integrate this relation between the heights $\pm\infty$ is the same to integrate it between angles $\pm\frac{\pi}{2}$ for b. We then find

$$\lim_{t\to\infty}\left(a(t) - a(-t)\right) \quad = \quad \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos b}{\sqrt{\mu + \cos^2 b}}\mathrm{d}b.$$

If we integrate it by substitution $u = \sin b$, we end up with the explicit formula of a_∞. □

2.3 — (b) What happens when we also consider the contribution of λ

We now consider the problem with additional term λ^2, like in system (2.3.8). Here we assume $b(0)$ is the minimal value of b and μ is a positive real number ; in particular, $\dot{b}(0) = 0$. Up to using a suitable rotation, we also assume $a(0) = 0$.

We introduce the angle b_m such that $b(0) = -\frac{\pi}{2} + b_m$. Since

$$\lambda^2 = \sin^2 b_m \left(\mu + \sin^2 b_m\right),$$

$\sin b_m$ has to be small and so does b_m. Easy calculus demonstrates that

$$b_m = \frac{\lambda}{\sqrt{\mu}} + \underset{\lambda \to 0}{\mathcal{O}}\left(\lambda^3\right). \tag{2.3.10}$$

Lemma 2.3.3 — *Assume $\lambda \in (0,1)$. Then solutions to the minimal surface system equations* (2.3.8) *are such that $\dot{a}$ and b are periodic functions. Moreover, the period T_λ is such that*

$$T_\lambda = -4\frac{\ln \lambda}{\sqrt{\mu}} + \underset{\lambda \to 0}{o}(1)$$

and if $a(0) = 0$, a describes $[-a_\infty, a_\infty]$ where

$$a_\infty := \arcsin\left(\frac{\cos b_m}{\sqrt{1+\mu}}\right) = \arcsin\left(\frac{1}{\sqrt{1+\mu}}\right) - \frac{\lambda^2}{2\mu^{\frac{3}{2}}} + \underset{\lambda \to 0}{\mathcal{O}}\left(\lambda^3\right).$$

The rough estimate of the period makes sense regarding the case $\lambda = 0$ in lemma 2.3.1 since it tends to infinity.

Remark 2.3.4 — The periodicity condition implies that the minimal surface $\mathscr{S}$ also admits a period. More precisely, $\mathscr{S}$ is invariant under the action of the isometry wich maps a point (s,t) of $\mathbb{S}^2 \times \mathbb{R}$ to the point $(s', t + T_\lambda)$ where $s' = \begin{pmatrix} \cos(2a_\infty) & 1 & -\sin(2a_\infty) \\ 0 & 1 & 0 \\ \sin(2a_\infty) & 0 & \cos(2a_\infty) \end{pmatrix}(s)$.

Proof

As done in previous proofs, we use relation between height and angle, namely

$$\mathrm{d}t = \frac{\mathrm{d}b}{\sqrt{\cos^2 b\left(\mu + \cos^2 b\right) - \lambda^2}}.$$

Thus it is enough to prove that it is integrable in $b(0)$. Let β be defined so that $b(t) = -\frac{\pi}{2} + \beta(t)$. According to the equality

$$\sin^2\beta\left(\mu + \sin^2\beta\right) - \lambda^2 = \left(\sin\beta - \sin b_m\right)\left(\sin\beta + \sin b_m\right) \cdot \left(1 + \sin^2\beta + \sin^2 b_m\right),$$

it is enough to prove that $\beta \mapsto (\sin \beta - \sin b_m)^{-\frac{1}{2}}$ is integrable in b_m. It is the case since

$$\sin(b_m + h) - \sin(b_m) \quad = \quad h \cos b_m + \underset{h \to 0}{\mathcal{O}}\left(h^2\right).$$

Therefore, solutions are periodic and the period is

$$T \quad = \quad 4 \int_0^{\frac{\pi}{2} - b_m} \frac{\mathrm{d}b}{\sqrt{\cos^2 b\,(\mu + \cos^2 b) - \lambda^2}}.$$

We perform a change of variables and we define ϵ is defined to be

$$\epsilon := \sin(b_m) = \left(\frac{\sqrt{\mu^2 + 4\lambda^2} - \mu}{2}\right)^{\frac{1}{2}} = \frac{\lambda}{\sqrt{\mu}} + \underset{\lambda \to 0}{\mathcal{O}}\left(\lambda^3\right).$$

Then the period is given by the formula

$$T_\lambda \quad = \quad 4 \int_\epsilon^1 \frac{\mathrm{d}u}{\sqrt{(1 - u^2)(\mu + u^2 + \epsilon^2)(u^2 - \epsilon^2)}}.$$

In order to estimate T when ϵ tends to 0, we first consider the integral in which we neglect the contribution at $u = 1$ and we obtain

$$4 \int_\epsilon^1 \frac{\mathrm{d}u}{\sqrt{\mu(u^2 - \epsilon^2)}} \quad = \quad -4 \frac{\ln \epsilon}{\sqrt{\mu}} + \underset{\epsilon \to 0}{\mathcal{O}}(1).$$

Then we prove that the difference between this above term and the period T is bounded. More precisely, we prove that there exists a constant c which only depends on μ such that

$$\left| T - 4 \int_\epsilon^1 \frac{\mathrm{d}u}{\sqrt{\mu(u^2 - \epsilon^2)}} \right| \quad \leqslant \quad c \left(1 + \int_\epsilon^{\frac{1}{2}} \frac{1 - \sqrt{1 - u^2}}{\sqrt{u^2 - \epsilon^2}} \mathrm{d}u\right).$$

According to the inequality

$$1 - \sqrt{1 - u^2} \quad \leqslant \quad u^2,$$

we end up with

$$\left| T - 4 \int_\epsilon^1 \frac{\mathrm{d}u}{\sqrt{\mu(u^2 - \epsilon^2)}} \right| \quad \leqslant \quad c \left(1 + \int_\epsilon^1 \frac{u^2\,\mathrm{d}u}{\sqrt{u^2 - \epsilon^2}}\right) = c\left(1 + \frac{1}{8} + \underset{\epsilon \to 0}{o}(1)\right).$$

The result follows. □

Local description of the solution

Here, we prove that a similar description to the one in [HP07, Lemma 3.1] holds true.

Lemma 2.3.5 — *The solution* $(a(t), b(t))$ *of system* (2.3.8) *has the following expansion :*

$$\forall t \in [-T_\lambda + 1, T_\lambda - 1],$$
$$\left\{ \begin{array}{rcl} b(t) & = & -\frac{\pi}{2} + \rho(t) + \underset{\lambda \to 0}{\mathcal{O}_\infty} \left(\lambda^3 \cosh\left(\sqrt{\mu}t\right) \sinh^2\left(\sqrt{\mu}t\right)\right), \\ a(t) & = & m(t) + \underset{\lambda \to 0}{\mathcal{O}_\infty} \left(\lambda^4 \cosh^3\left(\sqrt{\mu}t\right) \sinh\left(\sqrt{\mu}t\right)\right), \end{array} \right. \tag{2.3.11}$$

where the functions ρ *and* m *are defined to be*

$$\rho(t) := \frac{\lambda}{\sqrt{\mu}} \cosh\left(\sqrt{\mu}t\right) \qquad \textit{and} \qquad m(t) := \frac{\lambda^2}{2\mu} t + \frac{\lambda^2}{4\mu^{\frac{3}{2}}} \sinh\left(2\sqrt{\mu}t\right)$$

and $\mathcal{O}_\infty(f(t))$ *denotes a function that is bounded by a constant times* f *(and its derivatives are bounded by a constant times the derivatives of* f*).*

Proof

To simplify the analysis, we define ω by

$$b(t) \quad = \quad -\frac{\pi}{2} + b_m \cosh(\omega(t))$$

with $\omega(0) = 0$ to ensure $b(0) = -\frac{\pi}{2} + b_m$. According to the definition of b_m, in terms of ω, the ordinary differential equation that b satisfies changes into

$$\dot{\omega}^2 \quad = \quad \left(\mu + \sin^2 b_m + \sin^2(b_m \cosh\omega)\right) \frac{\sin^2(b_m \cosh\omega) - \sin^2 b_m}{b_m^2\left(\cosh^2\omega - 1\right)}.$$

Assume that $\left|\omega(t) - \sqrt{\mu}t\right| < c$ for some positive constant for all t in $[-t_*, t_*]$ with $t_* > 0$. Then the solution ω is such that

$$\forall t \in [-t_*, t_*], \qquad \omega(t) \quad = \quad \sqrt{\mu}t + \underset{\lambda \to 0}{\mathcal{O}_\infty}\left(\lambda^2 \cosh^2\left(\sqrt{\mu}t\right)\right),$$

and we check the estimate about ω holds true provided c is chosen big enough for $t_* = T_\lambda - 1$. The result for b then follows and the estimate for a is established by using Taylor's expansion e of $\cos^2 b$ together with the integration $\dot{a} = e$. $\square$

It is very useful to describe the solution as a vertical graph upon a small annulus. In this purpose, we perform the change of coordinates

$$\left(r\cos z, r\sin z, -\sqrt{1-r^2}, t\right) \qquad = \qquad X(t, \theta).$$

According to the above lemma, we demonstrate the following corollary.

Corollary 2.3.6 — *Let r_ϵ be the small radius defined by $r_\lambda := \sqrt{\frac{\lambda}{\mu}}$. Then following expansions holds true :*

$$t(r,z) = \frac{1}{\sqrt{\mu}} \ln\left(\frac{r}{\lambda}\right) + \frac{1}{2\sqrt{\mu}} \ln(4\mu) - \frac{1}{2\sqrt{\mu}} r \cos z + \underset{\lambda \to 0}{\mathcal{O}_\infty}(\lambda),$$

where $\mathcal{O}_\infty(f(r))$ denotes a function that is bounded by a constant times f on the annulus $\mathcal{A}_{r_\lambda} := B(0, 2r_\lambda) \backslash \overline{B}\left(0, \frac{r_\lambda}{2}\right)$ (and its derivatives are bounded by a constant times the derivatives of f).

Proof

It is similar to the proof of lemma 3.2 in [HP07]. We briefly recall the main steps. Let us define $t_\lambda := -\frac{\ln(\lambda)}{2\sqrt{\mu}}$ so that $t_\lambda \sim \frac{T_\lambda}{8}$ and $r_\lambda = \frac{\lambda}{\sqrt{\mu}} e^{\sqrt{\mu} t_\lambda}$. From the lemma 2.3.5, ρ and m satisfy the expansions

$$\rho(t) = \frac{\lambda}{2\sqrt{\mu}} e^{\sqrt{\mu} t} + \mathcal{O}_\infty\left(\lambda^{\frac{3}{2}}\right) \qquad \text{and} \qquad m(t) = \frac{\lambda^2}{8\mu^{\frac{3}{2}}} e^{2\sqrt{\mu} t} + \mathcal{O}_\infty\left(\lambda^2 \ln \lambda\right),$$

where $\mathcal{O}_\infty(f(t))$ denotes a function that is bounded by a constant times f on the set $[-T_\lambda + 1, T_\lambda - 1]$. Then

$$m(t) - \frac{\rho^2(t)}{2} = \mathcal{O}_\infty\left(\lambda^2 \ln \lambda\right).$$

Moreover, by the choice of new coordinates,

$$\rho(t)^2 = r(t)^2 - 2r(t)\, m(t) \cos z + \mathcal{O}_\infty\left(\lambda^2\right).$$

Consequently, if we inject this relation into the previous one, we get

$$m(r) = \frac{r^2}{2} + \mathcal{O}_\infty\left(\lambda^{\frac{3}{2}}\right),$$

where $\mathcal{O}_\infty(f(r))$ affects functions defined on the annulus $\mathcal{A}_{r_\lambda}$. The expansion of t follows. □

Chapitre V

Construction d'hypersurfaces de Scherk dans $\mathbb{R}^n \times \mathbb{R}$

Introduction

The Scherk surface, discovered in 1834 by Heinrich Scherk, is an unbounded minimal surface that can be seen as the graph over a square which takes alternatively $+\infty$ value and $-\infty$ value on its boundaries. More precisely, it is the graph of u over the domain $D = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ defined by

$$u : (x, y) \in D \longmapsto \log\cos x - \log\cos y.$$

Then u is $+\infty$ on the sides $\left\{y = \pm\frac{\pi}{2}\right\}$ and is $-\infty$ on the other sides $\left\{x = \pm\frac{\pi}{2}\right\}$. The goal of this paper is to prove the existence of such objects in $\mathbb{R}^n \times \mathbb{R}$.

We can except to have some restrictions about the geometry of domain D we choose. Indeed, H. Jenkins and J. Serrin [JS65] proved that in $\mathbb{R}^2 \times \mathbb{R}$, if a minimal graph u takes infinite value on a part B of the boundary of the domain, then B must be a geodesic. Moreover, in this paper is proved that the lengths of the boundary on which u takes infinite value have to satisfy some conditions — for example, when there are only infinite values on ∂D, the length of the boundary part in which u takes $+\infty$ value has to be equal to the length of the boundary part in which u takes $-\infty$ value. This kind of result has been extended by B. Nelli and H. Rosenberg in $\mathbb{H}^2 \times \mathbb{R}$ [NR02], then by A. L. Pinheiro [Pin09] or L. Mazet, M. M. Rodríguez and H. Rosenberg [MRR11] in $M^2 \times \mathbb{R}$. That is why we choose to work with domain whose par of boundary is a minimal surface. Moreover, the existence of Scherk type hypersurface has been proved by F. Pacard in [Pac02].

We have in mind to generalize the existence of such u for domains with many symmetries. For example, we would like to construct a Scherk type hypersurface over a regular octahedron which takes alternatively infinite values $\pm\infty$ over the faces. The existence and uniquiness have been proven by R. Sa Earp and É. Toubiana in [ST] for some polyedra in $\mathbb{H}^n$ or $\mathbb{R}^n$. In our case, we will use different approach and build Sherk type hypersurfaces over an ocathedron whose faces are minimal surfaces (it is the analogous of the condition under which in dimension 2, the part B of the boundary is a geodesic).

The general case together with a Jenkins-Serrin theorem has been solved by E. Tomaini [Tom86]. However, it seems relevant to explain our method which is almost completely self-contained.

Definition 0.3.7 — Let D be a bounded domain in $\mathbb{R}^n$ with continuous boundary ∂D. We say $\Sigma \subset \mathbb{R}^{n+1}$ is a *Scherk type hypersurface* over D if there exists $u : D \longrightarrow \mathbb{R}$ such that following conditions hold true:

(i) Σ is the graph of u ;

(ii) Σ is a minimal hypersurface, that is to say that u is a solution of the minimal graph equation

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) \quad = \quad 0;$$

(iii) there exists B a subset of ∂D such that its interior $\mathring{B}$ is non empty and u takes infinite value over B : $u_{|B} = +\infty$ (or $-\infty$).

We now state the main result.

Theorem 0.3.8
Let D a compact domain of $\mathbb{R}^n$ such that its boundary is $\Gamma \cup \mathcal{S}$ where Γ and $\mathcal{S}$ are respectively a convex graph and a minimal graph over a pseudoconvex[1] set $\Omega \subset \mathbb{R}^{n-1}$. Then there exists a Scherk type hypersurface Σ over D in $\mathbb{R}^n \times \mathbb{R}$ such that it is the graph of some function u with

$$u_{|\mathring{\Gamma}} = 0 \qquad \textit{and} \qquad u_{|\mathring{\mathcal{S}}} = +\infty. \tag{0.3.1}$$

Besides, the boundary of Σ is given by

$$\partial\Sigma \quad = \quad (\Gamma \times \{0\}) \cup \left(\partial\mathcal{S} \times \mathbb{R}^+\right). \tag{0.3.2}$$

By using reflection principle (see [ST, Lemma 3.1]), we then obtain the existence of Scherk type hypersurfaces over deformed octahedron.

Corollary 0.3.9 — *Let $\mathcal{O} \subset \mathbb{R}^3$ be a simply connected domain such that*

(i) the origin $0_{\mathbb{R}^3}$ belongs to the interior $\mathring{\mathcal{O}}$ of $\mathcal{O}$,

(ii) $\mathcal{O}$ is invariant under the action of the orthogonal reflections s_i of R^3 ($i \in \{1,2,3\}$) with respect to the plane $\{x_i = 0\}$,

(iii) its boundary $\mathcal{O}$ is the collection of 8 faces $\mathcal{F}_j$ which are minimal surfaces of $\mathbb{R}^3$.

Then there exists a function $u : \mathcal{O} \longrightarrow \mathbb{R}$ which is a solution to the minimal graph equation and which takes alternatively $+\infty$ value and $-\infty$ value on its faces.

We then discuss about the regularity of such hypersurfaces near their boundary, also with the help of reflection principle.

Lemma 0.3.10 — *Let Σ denote the minimal hypersurface of $\mathbb{R}^3 \times \mathbb{R}$ associated with the above corollary when $\mathcal{O}$ is a regular octahedron. Then Σ is smooth everywhere except on the vertices of $\mathcal{O}$ in which the surface is continuous but not differentiable.*

Finally, we give in the last section a result whose type is the same than the Jenkins-theorem.

1 Building Scherk type hypersurfaces in $\mathbb{R}^n \times \mathbb{R}$

1.1 Geometry of domains

As announced, geometric restrictions about the geometry of the domain D are expected in view of the Jenkins-Serrin theorem. We already know that the Dirichlet problem with continuous data over the boudary of a domain $\mathcal{D}$ is solvable if $\mathcal{D}$ is a $\mathcal{C}^2$ bounded domain in $\mathbb{R}^n$ whose boundary has nonnegative mean curvature — see theorem 16.8 in [GT01] or the article [JS68]. There also exists more sophisticated existence theorem in [Mir71] for locally pseudoconvexs sets in $\mathbb{R}^n$, generalized by R.C. Bassanezi and U. Massari in [BM78] for pseudoconvex sets.

Definition 1.1.1 — A subset $\mathcal{D} \subset \mathbb{R}^n$ is *pseudoconvex* if for all open bounded subset A of $\mathbb{R}^n$ and for all E such that $\overline{E}$ is a compact of A, following inequality holds true :

$$\int_A |D_{\mathbb{1}_\mathcal{D}}| \quad \leqslant \quad \int_A |D_{\mathbb{1}_{\mathcal{D}\cup E}}|\,,$$

where $\int_A |D_{\mathbb{1}_\mathcal{D}}|$ denotes total variation of the function $\mathbb{1}_\mathcal{D}$ over A, that is to say

$$\int_A |D_{\mathbb{1}_\mathcal{D}}| \quad = \quad \sup_{\left\{g\in\mathcal{C}_0^1(A,\mathbb{R}^n):\|g\|_{L^\infty(A)}\leqslant 1\right\}} \left(\int_A \mathbb{1}_\mathcal{D}\,\operatorname{div} g\right).$$

Note that if $\mathcal{D}$ has Lipshitz continuous boundary, we are in the case of [Mir71] ; if $\mathcal{D}$ has $\mathcal{C}^2$ boundary, then we are in the case of [JS68].

First and last, let us fix some notation and hypothesis for the rest of our study. We assume we are given a pseudoconvex set Ω of $\mathbb{R}^{n-1}$ such that its interior $\mathring{\Omega}$ is non empty. Moreover, we also assume that there exists a minimal hypersurface $\mathcal{S}$ of $\mathbb{R}^n$ which is the graph of some function s defined on Ω and a convex hypersurface Γ which is the graph of some convex function γ defined over Ω such that $\partial\Gamma = \partial\mathcal{S}$, in other words, such that the functions s and γ line up on the boundary $\partial\Omega$ of Ω. We also assume that $\gamma(p) \neq s(p)$ for all $p \in \mathring{\Omega}$. Notice that this assumption makes sense because of the convex hull property for minimal hypersurfaces ([CM99, proposition

1.7]) that specifies a minimal graph lies in the convex hull of its boundary. In particular, we can suppose without loss of generality that the surface $\mathcal{S}$ is below Γ and Γ is not minimal. We also denote by $\mathcal{C}$ a convex set of $\mathbb{R}^n$ which holds the domain D of $\mathbb{R}^n$ whose boundary is exactly $\Gamma \cup \mathcal{S}$ — for example, we could choose D so that it would be the convex hull of $\Gamma \cup \mathcal{S}$. We refer to the figure V.1 for an illustration.

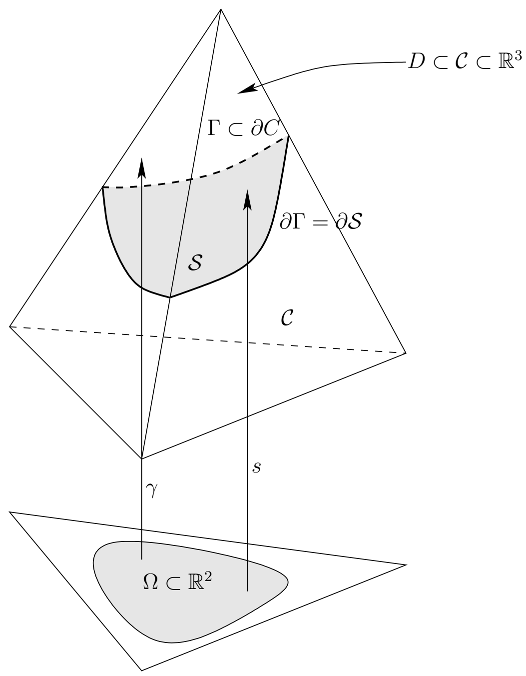


Figure V.1: Example of domain.

Notice that if $\partial\Gamma$ is continuous, then the existence of a minimal graph $\mathcal{S}$ over the pseudoconvex set Ω is provided by the result of R. Bassanezi and U. Masseri [BM78].

Example 1.1.2 — As an illustration of the above conditions, the classical Scherk surface provides a nice example. In this case, we choose Ω to be $(-1, 1)$. Then the minimal surface is in this case a geodesic : to fix ideas, we choose $\mathcal{S} = \Omega \times \{0\}$, in other words, $s \equiv 0$. Concerning Γ, we choose

$$\gamma(x) \quad = \quad 1 - |x| .$$

Of course, one check the boundary condition

$$\gamma(-1) = s(-1) = 0 \qquad \text{and} \qquad \gamma(+1) = s(+1) = 0.$$

We give an illustration in figure V.2.

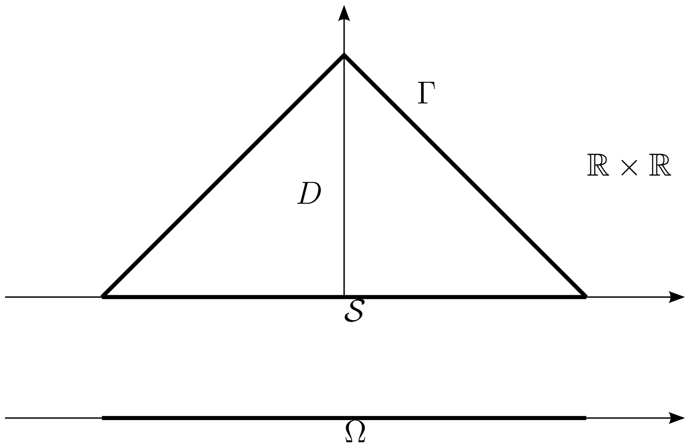


Figure V.2: The Scherk surface takes $+\infty$ value on $\mathring{S}$ and vanishes on $\mathring{\Gamma}$ as a graph on D.

Remark 1.1.3 — We do not give directly the result for deformed polyhedra because we use this above theorem to construct this new hypersurface with reflections. For example, note the classical Sherk surface can be obtained by both the existence of minimal surface over a triangle which vanishes over two sides et takes infinite data over the other side and reflections — see figure V.3.

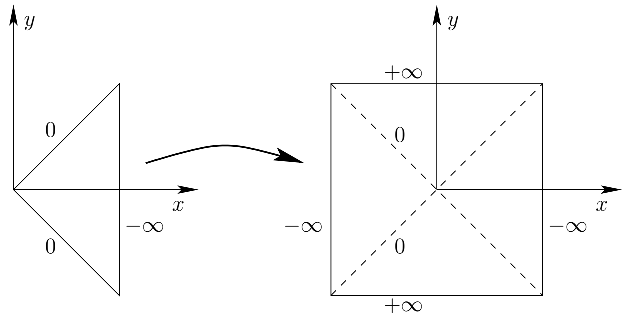


Figure V.3: Scherk surface over a triangle then over a square.

1.2 "Horizontal" problem

There are two main ideas to prove the above theorem :

1) we solve a bounded problem by prescribing boundary data $+m$ for some positive integer m rather than $+\infty$ then we let m to tend to $+\infty$;

2) we change what we call the "vertical" problem into the "horizontal" problem — it is more useful to prove that the sequence of solutions has a limit when m tends to $+\infty$.

1.2 — (a) Bounded "horizontal" problem

As announced, we introduce the following "vertical" Dirichlet problem for $m \in \mathbb{N}^*$:

$$\begin{cases} u_m^{\mathrm{ver}} & \text{is a minimal graph over } D; \\ \forall x \in \partial D, & u_m^{\mathrm{ver}}(x) = 0 \quad \text{if} \quad x \in \Gamma, \quad \text{else} \quad m. \end{cases} \tag{1.2.3}$$

We denote by Σ_m^{ver} the graph of such a solution. Indeed, we have in mind to use compactness results for minimal graphs. It is known that in the case $n = 2$, we can obtain a solution of Dirichlet problem (0.3.1) by letting $m \to \infty$ — see [JS65]. We use a different approach with the introduction to the following "horizontal" Dirichlet problem :

$$\begin{cases} u_m^{\mathrm{hor}} & \text{is a minimal graph over } \Omega_0^m := \Omega \times [0, m]\,; \\ \forall p \in \partial\Omega, & \forall x \in [0, m]\,, \quad u_m^{\mathrm{hor}}(p, x) = \gamma(p) = s(p); \\ \forall p \in \Omega, & u_m^{\mathrm{hor}}(p, 0) = \gamma(p); \\ \forall p \in \Omega, & u_m^{\mathrm{hor}}(p, m) = s(p). \end{cases} \tag{1.2.4}$$

We denote by Σ_m^{hor} the graph of such a solution u_m^{hor}. Note that, by construction, we have

$$\partial\Sigma_m^{\mathrm{hor}} = (\Gamma \times \{0\}) \cup (\partial\mathcal{S} \times [0, m]) \cup (\mathcal{S} \times \{m\}) = \partial\Sigma_m^{\mathrm{ver}}. \tag{1.2.5}$$

The idea of the transformation lies in the figure V.4. The point is that in the horizontal problem, the boundary data is continuous and we are able to give an uniform bound for the solutions.

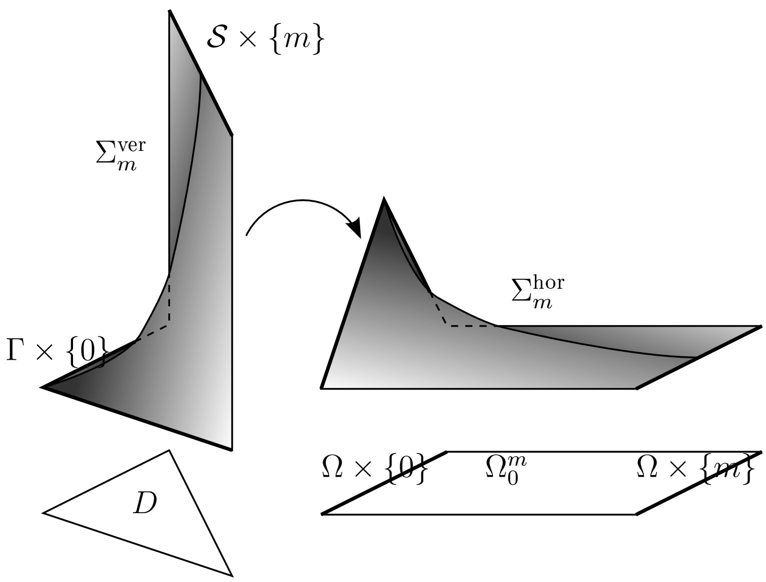


Figure V.4: The horizontal problem and the vertical problem.

Notice that according to [BM78], since Ω_0^m is pseudoconvex (because Ω is pseudoconvex), the Dirichelt problem (1.2.4) has a solution. Besides, according to the maximum principle, the solution is unique. Therefore, it makes sense to consider u_m^{hor}.

As announced, we prove the existence of an uniform bound, which allows us to consider the case $m \to +\infty$.

Proposition 1.2.1 — *The sequence $\left(u_m^{hor}\right)_m$ is uniformly bounded. More precisely, for all $m \in \mathbb{N}^*$, for all $(p,t) \in \Omega_0^m$, we have the following inequalities :*

$$\gamma(p) \quad \geqslant \quad u_m^{hor}(p,t) \quad \geqslant \quad s(p). \tag{1.2.6}$$

PROOF
There are two types of arguments : we deal with the lower bound with the help of the maximum principle while we deal with the upper bound by using the convex hull property.

First, since $\mathcal{S}$ is a minimal graph over Ω, it is clear that $\mathcal{S} \times [0,m]$ is also a minimal graph over $\Omega \times [0,m]$. Moreover, its boundary is given by

$$\partial\left(\mathcal{S} \times [0,m]\right) \quad = \quad \left(\mathcal{S} \times \{0\}\right) \cup \left(\partial\mathcal{S} \times [0,m]\right) \cup \left(\mathcal{S} \times \{m\}\right).$$

Therefore, the boundary of the minimal surface $\mathcal{S} \times [0,m]$ is below the boundary of Σ_m^{hor} because Γ is above $\mathcal{S}$. According to the maximum principle (see [GT01] for example), we deduce from this fact that Σ_m^{hor} is above $\mathcal{S} \times [0,m]$. In other words, we obtain the uniform lower bound

$$\forall (p,t) \in \Omega_0^m, \qquad u_m^{\text{hor}}(p,t) \quad \geqslant \quad s(p).$$

Next, by the convex hull property, it is an easy check to see that Σ_m^{hor} lies in the set $D \times [0,m]$. In particular, Σ_m^{hor} is below $\Gamma \times [0,m]$, from what we deduce the uniform upper bound. $\square$

The above proposition yields to the convergence of the sequence of solutions $\left(u_m^{\text{hor}}\right)_m$.

Corollary 1.2.2 — *For all $t_0 \in \mathbb{R}^+$, for all $p \in \Omega$, the sequence $(u_m^{hor}(p,t_0))_{m \geqslant t_0}$ converges simply to a limit, denoted by $u^{hor}(p,t_0)$.*

PROOF
Here again, it is a well chosen of the maximum principle.

The key lies in proving the increasing property

$$\forall t_0 \in \mathbb{R}^+, \forall p \in \Omega, \forall m \geqslant t_0, \qquad u_m^{\text{hor}}(p,t_0) \quad \leqslant \quad u_{m+1}^{\text{hor}}(p,t_0). \tag{1.2.7}$$

The above inequality together with the uniform upper bound we have demonstrated in the proposition 1.2.1 then yields to the conclusion.

Consequently, let us prove the claim (1.2.7). We refer to the figure V.5 for a better understanding. We observe that u_m^{hor} and the restriction $u_{m+1|\Omega_0^m}^{\text{hor}}$ of u_{m+1}^{hor} to Ω_0^m are minimal graphs. Moreover, by construction, these functions coincide on the part $\partial\Omega_0^m \setminus (\Omega \times \{m\})$ of the boundary of Ω_0^m. In addition to that, by construction, we also know that for all $p \in \Omega$,

$$u_m^{\text{hor}}(p, m) \quad = \quad s(p)$$

and using property 1.2.1,

$$u_{m+1}^{\text{hor}}(p, m) \quad \geqslant \quad s(p).$$

In other words, u_{m+1}^{hor} is greater than u_m^{hor} on the part $\Omega \times \{m\}$ of the boundary $\partial\Omega_0^m$. Then, by maximum principle, we end up with

$$u_m^{\text{hor}} \quad \leqslant \quad u_{m+1|\Omega_0^m}^{\text{hor}} \qquad \text{over} \qquad \Omega_0^m,$$

so inequality (1.2.7) holds true.

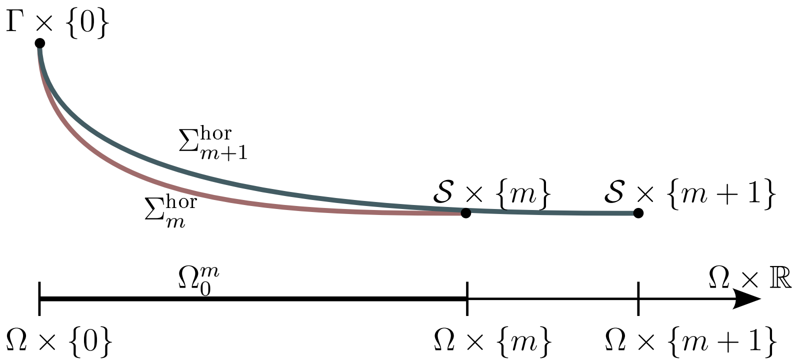


Figure V.5: Comparison of Σ_m^{hor} and $\Sigma_{m+1}^{\text{hor}}$.

Proposition 1.2.3 — *Let $m \in \mathbb{N}^*$, p be a point of $\Omega \setminus \partial\Omega$ and $t < t' \in [0, m]$. Then*

$$u_m^{hor}(p, t) \quad > \quad u_m^{hor}(p, t'). \tag{1.2.8}$$

In other words, u_m^{hor} is decreasing along the direction t. Note that the case $p \in \partial\Omega$ is more easy since for all $t \in [0, m]$, $u_m^{\text{hor}}(p, t) = s(p)$. This property is central to show that we can consider a solution of (1.2.4) as a solution of the other Dirichlet problem (1.2.3) — see corallary 1.2.4.

Proof
The main idea of this proof is to compare Σ_m^{hor} with another minimal surface that is nothing but a translated of Σ_m^{hor} and to use a maximum principle.

More precisely, let $0 \leqslant t < t' \leqslant m$, and we consider the translated $\Omega_{-(t'-t)}^{m-(t'-t)}$ of Ω_0^m defined by

$$\Omega_{-(t'-t)}^{m-(t'-t)} \quad := \quad \Omega \times [-(t'-t), m - (t'-t)]$$

together with the translated function w_m of u_m^{hor} defined by

$$w_m : (p, \tau) \in \Omega_{-(t'-t)}^{m-(t'-t)} \longmapsto u_m^{\text{hor}}(p, \tau + (t'-t)) \in \mathbb{R}.$$

Then w_m satisfies the minimal graph equation over $\Omega_{-(t'-t)}^{m-(t'-t)}$ since we just have performed a translation. We note $\Sigma_{t',t}$ the associated surface. We define $\widetilde{\Omega}$ to be the intersection of the two domains we consider, that is to say

$$\widetilde{\Omega} \quad := \quad \Omega \times [0, m-(t'-t)] \quad = \quad \Omega_0^m \cap \Omega_{-(t'-t)}^{m-(t'-t)}.$$

We refer to the figure V.6 for the idea of the proof and the notations.

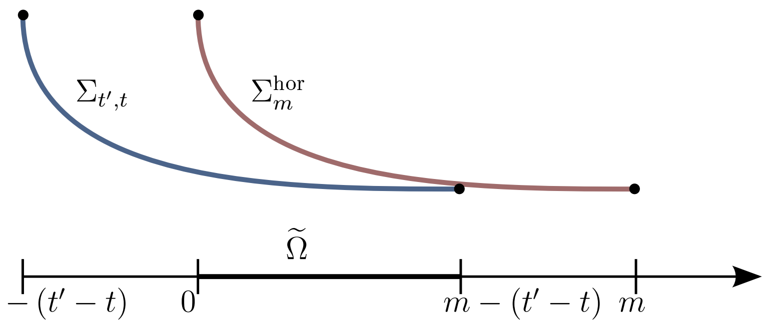


Figure V.6: The surface Σ_m^{hor} and its translated $\Sigma_{t',t}^{\mathrm{hor}}$.

Large inequality : It is once again an application of the maximum principle, in other words, we compare the boundaries of the surface defined over the same same domain $\widetilde{\Omega}$.

Applying inequalities (1.2.6) of property 1.2.1, we obtain for all $(p, 0)$ which belongs to the part $\Omega \times \{0\}$ of the boundary $\partial\widetilde{\Omega}$,

$$u_m^{\mathrm{hor}}(p,0) \quad = \quad \gamma(p) \quad \geqslant \quad u_m^{\mathrm{hor}}(p, t'-t) \quad = \quad w_m(p,0).$$

With similar method, for all $(p, m-(t'-t))$ which belongs to the part $\Omega \times \{p, m-(t'-t)\}$ of the boundary $\partial\widetilde{\Omega}$, we get

$$u_m^{\mathrm{hor}}(p, m-(t'-t)) \quad \geqslant \quad s(p) \quad = \quad u_m^{\mathrm{hor}}(p,m) \quad = \quad w_{m_{|\widetilde{\Omega}}}(p,(t'-t)).$$

It remains to consider the points $(p,\tau) \in \partial\Omega \times [0, m-(t'-t)]$ of $\partial\widetilde{\omega}$ for which the equality

$$u_m^{\mathrm{hor}}(p,\tau) \quad = \quad s(p) \quad = \quad w_{m_{|\widetilde{\Omega}}}(p,\tau)$$

holds true. According to the maximum principle, since

$$w_{m_{|\widetilde{\Omega}}} \quad \leqslant \quad u_{m_{|\widetilde{\Omega}}}^{\mathrm{hor}}$$

on $\partial\widetilde{\Omega}$, we end up with

$$w_{m_{|\widetilde{\Omega}}} \quad \leqslant \quad u_{m_{|\widetilde{\Omega}}}^{\mathrm{hor}},$$

on $\widetilde{\Omega}$. Consequently, for all $p \in \Omega$, we get the inequality

$$u_m^{\mathrm{hor}}(p,t) \quad \geqslant \quad w_m(p,t) \quad = \quad u_m^{\mathrm{hor}}(p,t'),$$

hence decreasing property is proved.

Strict inequality : It remains to prove that for there is no equality in the above inequality when p belongs to $\Omega \setminus \partial\Omega$. By the maximum principle, it is enough to prove that there exists a point p_0 of Ω such that

$$u_m^{\text{hor}}(p_0, t') \quad < \quad u_m^{\text{hor}}(p_0, t).$$

Reductio ad absurdum, assume it is not the case. Then for all $p \in \Omega$, the relation

$$u_m^{\text{hor}}(p, t') \quad = \quad u_m^{\text{hor}}(p, t)$$

holds true. Since u_m^{hor} is decreasing along t, we then obtain

$$\forall (p, \tau) \in \Omega \times [t, t'], \qquad u_m^{\text{hor}}(p, \tau) \quad = \quad u_m^{\text{hor}}(p, t).$$

In other words, the part of Σ_m^{hor} which is the graph of u_m^{hor} over $\Omega \times [t, t']$ has cylindrical type $\overline{\mathcal{S}} \times [t, t']$ where $\overline{\mathcal{S}}$ is the graph of

$$p \in \Omega \quad \longmapsto \quad u_m^{\text{hor}}(p, t).$$

Refer to the following figure for an illustration.

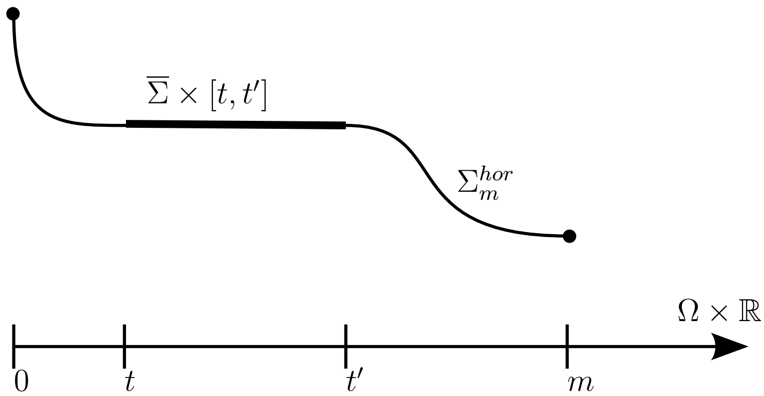


Figure V.7: The part of the surface over $\Omega \times [t, t']$ does not depend on $\tau \in [t, t']$.

Consequently, $\overline{\Sigma}$ has to be a minimal graph over Ω. Moreover, on $\partial\Omega$, $\overline{\Sigma}$ and $\mathcal{S}$ are equal. According to the maximum principle, $\overline{\Sigma}$ is nothing but $\mathcal{S}$. We deduce from this fact that Σ_m^{hor} and $\mathcal{S} \times [0, m]$ are tangent minimal surfaces in a point which belongs to their interior, thus they are equal. That contradicts $\Gamma \times \{0\}$ belongs to Σ_m^{hor}. □

Corollary 1.2.4 — *For all $m \in \mathbb{N}^*$, we can consider a solution u_m^{hor} of Dirichlet problem (1.2.4) as a solution u_m^{ver} of problem (1.2.3).*

Proof
From property 1.2.3, for all $p \in \Omega \setminus \partial\Omega$,

$$u_m^{\text{hor},(p)} : t \longmapsto u_m^{\text{hor}}(p, t)$$

is a bijective function which maps $[0,m]$ to $[s(p), \gamma(p)]$. Let $((p,\tau),0) \in D \times \mathbb{R}$ — in other words, τ is such that $\tau \in [s(p), \gamma(p)]$. Then if $p \in \Omega \setminus \partial\Omega$, there exists only one $t \in [0,m]$ such that $u_m^{\mathrm{hor},(p)}(t) = \tau$. We deduce from these facts that the function u_m^{ver} chosen so that

$$u_m^{\mathrm{ver}} : (p,\tau) \in D \subset \mathbb{R}^n \longmapsto \begin{cases} \left(u_m^{\mathrm{hor},(p)}\right)^{-1}(\tau) & \text{if} \quad (p,\tau) \in D \setminus (\Gamma \cap \mathcal{S}), \\ m & \text{if} \quad (p,\tau) \in \partial\Omega \times \{0\}, \end{cases}$$

defines a solution to Dirichlet problem (1.2.3) since the graph of $u_m^{\mathrm{hor},(p)}$ over D coincides with Σ_m^{hor} — which is a minimal hypersurface — and boundary conditions are satisfied by construction. □

1.2 — (b) Not bounded "horizontal" problem

We now prove that the limit u^{hor} we have defined in corollary 1.2.2 is actually the Scherk type surface we want to construct.

Theorem 1.2.5
The graph of u^{hor} is minimal over the interior $\mathring{\Omega}_{\mathbb{R}^+}$ where the set $\Omega_{\mathbb{R}^+}$ is defined to be

$$\Omega_{\mathbb{R}^+} \quad := \quad \Omega \times \mathbb{R}^+.$$

Moreover, if Σ^{hor} denotes the associated surface, then its boundary satisfies

$$\partial\Sigma^{hor} \quad = \quad (\Gamma \times \{0\}) \cup \left(\partial\mathcal{S} \times \mathbb{R}^+\right), \tag{1.2.9}$$

and the hypersurface $\mathcal{S} \times \mathbb{R}^+$ is asymptotic to Σ^{hor} when t tends to $+\infty$.

Before giving the proof, the reader should note the last point of the above theorem is the reason for which we have chosen a domain D whose boundary is $\Gamma \cup \mathcal{S}$ with $\mathcal{S}$ minimal. It is nothing but the generalization of the condition of Jenkins and Serrin that states if a minimal graph takes $+\infty$ value on a part of the boundary, then this part must be a geodesic.

Proof
The proof turns on four points. We briefly expose them before giving all the details :

1) we prove that the surface is minimal — it comes from a compactness principle together with the uniform bound we have proved ;
2) we prove the boundary conditions are satisfies ;
3) we demonstrate that $u^{\mathrm{hor}}(p,t)$ converges when t tends to $+\infty$ — it comes from a decreasing property like the one we have used in the proof of proposition 1.2.3 ;
4) we end up with proving that the limit is exactly s that parametrizes the minimal surface $\mathcal{S}$.

First step. We claim u^{hor} is a minimal graph over $\mathring{\Omega}_{\mathbb{R}^+}$. To see that, for positive integer k large enough, let us define $\widetilde{\Omega}^k$ to be the compact subset of $\mathring{\Omega}_{\mathbb{R}^+}$ such that

$$\widetilde{\Omega}^k \quad := \quad \left\{ (p,x) \in \Omega_0^k : d\left((p,x), \partial\Omega_0^k\right) \geqslant \frac{1}{2^k} \right\}.$$

Notice that the sequence of sets $(\widetilde{\Omega}^k)_k$ converges to the set $\mathring{\Omega}_{\mathbb{R}^+}$. Besides, according to propostion 1.2.1, the sequence $(u^{\text{hor}}_{m|\widetilde{\Omega}^k})_{m\geqslant k}$ of minimal graphs over $\widetilde{\Omega}^k$ is uniformly bounded. From compactness principle (cf. [GT01]), there exists a subsequence that uniformly converges to a minimal graph v_k over $\widetilde{\Omega}^k$. But corollary 1.2.2 implies that

$$u^{\text{hor}}_{m_{|\widetilde{\Omega}^k}} \xrightarrow[k\to+\infty]{} u^{\text{hor}}_{|\widetilde{\Omega}^k}.$$

By uniqueness of the limit,

$$u^{\text{hor}}_{|\widetilde{\Omega}^k} \quad = \quad v_k,$$

hence u^{hor} is a minimal graph other $\widetilde{\Omega}^k$ for all k and the claim follows.

Second step. Let us show that boundary conditions (1.2.9) are satisfied and that u is continuous over $\Omega_{\mathbb{R}^+}$. It follows directly from applications of property 1.2.1 together with the construction of u^{hor}_m which is continuous, like γ and s. Concerning the part $\Omega \times \{0\}$ of the boundary $\partial\Omega_{\mathbb{R}^+}$, we check that the inequalities

$$\forall\,(p,x) \in \Omega_0^1, \qquad u_1^{\text{hor}}(p,x) \leqslant u^{\text{hor}}(p,x) \leqslant \gamma(p)$$

hold true. But, by construction, we get the limits

$$u_1^{\text{hor}}(p,x) \xrightarrow[(p,x)\to(p_0,0)]{} \gamma(p_0) \qquad \text{and} \qquad \gamma(p) \xrightarrow[p\to p_0]{} \gamma(p_0).$$

Injecting these relations into the previous inequalities, we finally find that

$$u^{\text{hor}}(p,x) \xrightarrow[(p,x)\to(p_0,0)]{} \gamma(p_0),$$

in other words, $\Gamma \times \{0\}$ belongs to the boundary of $\partial\Sigma^{\text{hor}}$.

With similar method, we prove that $\partial\mathcal{S} \times \mathbb{R}^+$ also belongs to the boundary of this surface.

Third step. We claim that $u^{\text{hor}}(p,t)$ converges when t tends to $+\infty$. We check that for all $p \in \Omega$, the function $u^{\text{hor},(p)}$ defined to be

$$u^{\text{hor},(p)} : t \in \mathbb{R} \longmapsto u^{\text{hor}}(p,t) \in \mathbb{R}$$

is decreasing — it follows from proposition 1.2.3. But proposition 1.2.1 implies that $u^{\text{hor},(p)}$ is bounded. Then it converges. We denote the limit by $v : \Omega \longrightarrow \mathbb{R}$.

Fourth step. To complete the proof, it remains to demonstrate that the limit function v is exactly s. The idea of the proof looks like the proof of property 1.2.3 : we make use of a suitable translation of Σ^{hor} along the t-axis.

(i) Let $\omega_m^{+\infty}$ be the set

$$\Omega_{-m}^{+\infty} \quad := \quad \Omega \times [-m, +\infty[$$

and we define the translated function z_m to be

$$z_m : \begin{array}{ccc} \Omega_{-m}^{+\infty} & \longrightarrow & \mathbb{R} \\ (p,t) & \longmapsto & u^{\text{hor}}(p, t+m) \,. \end{array}$$

It is clear that z_m satisfies the minimal graph equation. Moreover, its boundary data is determined by

$$\forall p \in \Omega, \quad z_m(p, -m) = s(p) \qquad \text{and} \qquad \forall (p,t) \in \partial\Omega \times \mathbb{R}^+, \quad z_m(p,t) = s(p).$$

Furthermore, according to the third step, z_m has a limit when m tends to $+\infty$:

$$z_m(p,t) \quad \xrightarrow[m \to \infty]{} \quad v(p).$$

(ii) Let us define z to be the t-invariant function

$$z : \begin{array}{ccc} \Omega_{\mathbb{R}} & \longrightarrow & \mathbb{R} \\ (p,t) & \longmapsto & v(p). \end{array}$$

We claim that z satisfies the minimal graph equation. For that, we consider an increasing sequence of compact subsets of $\mathring{\Omega}_{\mathbb{R}}$ which converges to $\mathring{\Omega}_{\mathbb{R}}$. Then analogous arguments to first step enable us to conclude.

(iii) Now, let us show that $v = s$. First, note that similar methods to second step show that z in an element of $\mathcal{C}^0(\Omega_{\mathbb{R}}, \mathbb{R})$. Then Σ_z is a minimal hypersurface in $\mathbb{R}^{n+1}$. But it is also invariant under translations along the t-axis. Consequently, v is a minimal graph over Ω. But $v_{\partial\Omega} = s_{\partial\Omega}$ because of both the continuity of v and

$$\forall ((p,t), m) \in (\partial\Omega \times \mathbb{R}) \times \mathbb{N}^*, z_m(p,t) = s(p).$$

The maximum principle for minimal graphs over Ω leads to the conclusion. $\square$

1.3 Proof of theorem 0.3.8

Like we have done to prove that to solve the bounded horizontal problem is the same than to solve the bounded vertical problem, we prove that the solution u^{hor} determines a way to compute a solution u^{ver} of the vertical version of the problem.

First step. First, we claim that for all $p \in \mathring{\Omega}$, the function

$$u^{\mathrm{hor},(p)} : (p,t) \in \Omega_{\mathbb{R}^+} \longmapsto u^{\mathrm{hor}}(p,x) \in \mathbb{R}$$

is strictly decreasing — it is constant equal to $s(p)$ if $p \in \partial\Omega$.
We already know that $u^{\mathrm{hor},(p)}$ is decreasing according to the third step in previous proof. Let $t < t' \in \mathbb{R}^+$ and $m \in \mathbb{N}^*$ such that $m \geqslant t'$. Then we consider the restriction of u^{hor} to $\Omega_0^{m-t'+t}$ and the following translation of u^{hor} :

$$w : \begin{array}{ccc} \Omega_0^{m-(t'-t)} & \longrightarrow & \mathbb{R} \\ (p,\tau) & \longmapsto & u^{\mathrm{hor}}\left(p, \tau + (t'-t)\right). \end{array}$$

By using exactly the same method than in the proof of proposition 1.2.3, the claim follows.

Second step. Σ^{hor} can be seen as the graph of some function u^{ver} which is a solution to Dirichlet problem (0.3.1) with boundary conditions (0.3.2). The mechanisms are the same than in the proof of corollary 1.2.4. Since for all p in $\Omega \setminus \partial\Omega$, the function $u^{\mathrm{hor},(p)}$ is bijective and maps $\mathbb{R}^+$ to $[s(p), \gamma(p)]$, we can define u^{ver} to be :

$$u^{\mathrm{ver}} : \begin{array}{ccc} D \subset \mathbb{R}^n & \longrightarrow & \mathbb{R}^+ \cup \{+\infty\} \\ (p,\tau) & \longmapsto & \left\{ \begin{array}{rll} \left(u^{\mathrm{hor},(p)}\right)^{-1}(\tau) & \text{if} & (p,\tau) \in D \setminus \mathcal{S}; \\ +\infty & \text{if} & (p,\tau) \in \mathcal{S}. \end{array} \right. \end{array}$$

By construction, we check that

$$\begin{aligned} &\left\{ \left(p, u^{\mathrm{ver}}(p,\tau), \tau\right) : (p,\tau) \in D \right\} \\ &\qquad = \quad \left\{ \left(p, t, u^{\mathrm{hor}}(p,t)\right) : (p,t) \in \Omega_{\mathbb{R}^+ \cup \{+\infty\}} \setminus \left(\partial\Omega \times \mathbb{R}^+\right) \right\}, \end{aligned}$$

where we have defined

$$u^{\mathrm{hor}}(p,+\infty) := s(p) = \lim_{t\to\infty} u^{\mathrm{hor}}(p,x) \qquad \text{and} \qquad \Omega_{\mathbb{R}^+ \cup \{+\infty\}} := \Omega \times \left(\mathbb{R}^+ \cup \{+\infty\}\right).$$

We deduce from this equality that

$$\begin{aligned} &\left\{ \left(p, u^{\mathrm{ver}}(p,\tau), \tau\right) : (p,\tau) \in D \right\} \\ &\qquad = \quad \left(\Sigma \cup \{(p,+\infty,s(p)) : p \in \Omega\}\right) \setminus \left\{ (p,t,\gamma(p)) : (p,t) \in \partial\Omega \times \mathbb{R}^+ \right\} \end{aligned}$$

which is a minimal hypersurface of $\mathbb{R}^{n+1}$. Then u^{ver} is a solution of first Dirichlet problem and satisfies theorem 0.3.8.

2 Scherk type hypersurface over a deformed octahedron and regularity

We have in mind to study some properties of such minimal surfaces. Note the Sherk surface is nothing but the case $n = 2$; this surface is $\mathcal{C}^2$ everywhere —

including on its boundary. Besides, we can obtain this surface from a Scherk type surface over a triangle [2]. In this section, we work with $n = 3$ and our goal is to prove the corollary 0.3.9 together with the lemma 0.3.10.

2.1 Example : regular octahedron

Let us consider an octahedron $\mathcal{O} \subset \mathbb{R}^3$ with (open) faces $\mathcal{F}_1, \ldots, \mathcal{F}_8$. We would like to build a minimal hypersurface in $\mathbb{R}^3 \times \mathbb{R}$ over $\mathcal{O}$ such that it takes alternately $\pm\infty$ data value over faces. It is already known that such a hypersurface exists — see the article of R. Sa Earp and É. Toubiana [ST]. We briefly recall the idea of the construction.

Only two results are necessary : on the one hand, the existence of Sherk type hypersurface over a tetrahedron which takes value 0 on three faces and $+\infty$ on the last one, on the other hand, the reflection principle — we refer to [ST, Lemma 3.1]. The result in this last paper is proved for minimal graphs in $\mathbb{H}^n \times \mathbb{R}$ but similar arguments work in $\mathbb{R}^n \times \mathbb{R}$.

Denote by $OABC$ the open tetrahedron $\mathcal{T}$ such that (OC), (OB) and (OA) are orthogonal with

$$\text{length}\,(OA) = \text{length}\,(OB) = \text{length}\,(OC)\,,$$

and by $\mathcal{F}_1$ the (open) face ABC. According to theorem 0.3.8, there exists a minimal graph $u : \mathcal{T} \longrightarrow \mathbb{R}$ such that (see figure V.8) :

$$\begin{cases} u &= & 0 & \text{over} & \mathring{\partial T} \setminus \mathcal{F}_1; \\ u &= & +\infty & \text{over} & \mathcal{F}_1. \end{cases}$$

By reflection principle, we extend u to some new minimal graph v over the domain $\mathcal{T} \cup (OAB) \cup \mathcal{T}'$ where $\mathcal{T}'$ is the reflection of $\mathcal{T}$ with respect to the plane $\{z = 0\}$ as follows :

$$v(x,y,z) \quad := \quad \begin{cases} u(x,y,z) & \text{if} & (x,y,z) & \in & \mathcal{T}; \\ -u(x,y,-z) & \text{if} & (x,y,-z) & \in & \mathcal{T}; \\ 0 & \text{if} & (x,y,z) & \in & (OAB)\,. \end{cases}$$

Then v is a function which satisfies the minimal graph equation together with (see figure V.9) :

$$\begin{cases} v &= & 0 & \text{over} & \operatorname{Int}\left(\partial\left(\mathcal{T} \cup (OAB) \cup \mathcal{T}'\right)\right) \setminus (\mathcal{F}_1 \cup \mathcal{F}_5)\,; \\ v &= & +\infty & \text{over} & \mathcal{F}_1; \\ v &= & -\infty & \text{over} & \mathcal{F}_5. \end{cases}$$

We can go on with reflections and build a minimal hypersurface over the octahedron $\mathcal{O}$ — see figure V.10 : we use the reflection with respect to the plane $\{y = 0\}$, then with respect to the plane $\{x = 0\}$.

2. See remark 1.1.3 page 165 and figure V.3 page 165.

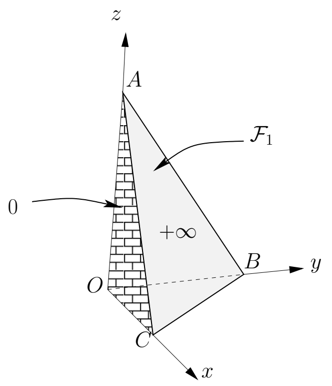


Figure V.8: Scherk type hypersurface over a tetrahedron $\mathcal{T}$ and the function u.

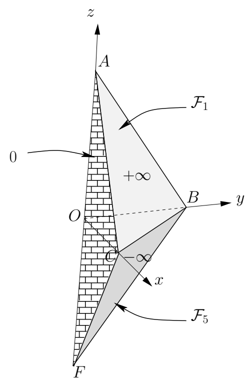


Figure V.9: Scherk hypersurface after a reflection and the function v.

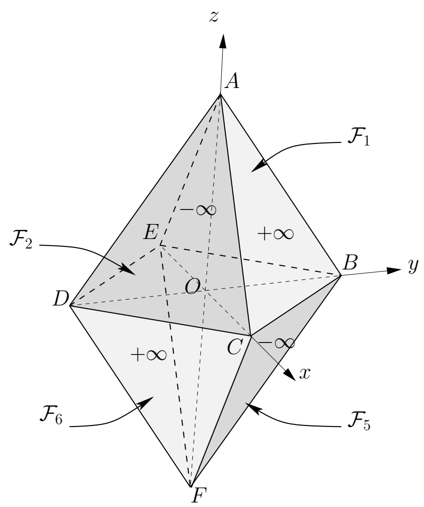


Figure V.10: Scherk type hypersurface over an octahedron.

It is interesting to observe that this construction proves that the function u is C^2 over $\overline{\partial T} \setminus \overline{\mathcal{F}_1}$.

So we can wonder what is the regularity of the Scherk type hypersurface on its boundary.

Lemma 2.1.1 — *Let Σ the Scherk type hypersurface over $\mathcal{O} \subset \mathbb{R}^3$ in $\mathbb{R}^3 \times \mathbb{R}$. Recall that its boundary is given by*

$$\partial\Sigma = (\partial(\partial\mathcal{O}) \times \mathbb{R}) \subset \left(\mathbb{R}^3 \times \mathbb{R}\right), \tag{2.1.10}$$

where $\partial(\partial\mathcal{O})$ is the collection of the sides of the octahedron. Then Σ is $\mathcal{C}^2$ everywhere, except in the vertices of $\mathcal{O}$ in which Σ is continuous but is not $\mathcal{C}^1$.

Consequently, we can't expect to have the same regularity than in the 2-dimensional case.

PROOF
We first note that it is enough to consider the case in which Σ is a minimal graph over a tetrahedron $\mathcal{T}$.

Regularity over (open side) $\times \mathbb{R}_+^*$ **:** By construction of Scherk type hypersuface Σ over $\mathcal{T}$, Σ can be seen as the graph of u^{hor} over the domain $(ABC) \times \mathbb{R}^+$ where (ABC) denotes the (open) triangle whose vertices are A, B and C. Moreover, on the boundary of $(ABC) \times \mathbb{R}^+$, u^{hor} is constructed so that

$$u^{\mathrm{hor}} = 0 \quad \text{over} \quad \partial\,(ABC) \times \mathbb{R}^+;$$

over the other part $(ABC) \times \{0\}$ of the boundary, u^{hor} is the graph of the upper piece of the tetrahedron, in other words, it is the graph of $\partial\mathcal{T} \setminus (ABC)$. Without loss of generality, we may assume the open segment (AB) belongs to the plane $\{x = 0\}$. We then construct a new hypersurface over $(ACBC') \times \mathbb{R}^+$ by reflection principle — we refer to figure V.11 to illustrate this. Then this new graph is $\mathcal{C}^2$ over the interior of this domain. In particular, it is $\mathcal{C}^2$ over $(AB) \times \mathbb{R}_+^*$, so is Σ.

Figure V.11: The first symmetry.

Regularity over (open side $\times \{0\}$**) :** It is the most important point in the proof of the lemma. We need several steps to conclude.

First step : We work with the solution u_m^{hor} of the bounded horizontal problem.

First of all, we introduce notations — see figure V.12. Let us denote by G the orthogonal projection of the vertex O of the tetrahedron $\mathcal{T}$ on the plane that holds A, B and C — in other words, G is the centre of mass of the triangle (ABC). Let $P \in (AB)$ and denote by Q the point of (ABC) such that $Q \in (AG) \cup (BG)$ and the straight lines (PQ) and (BC) are parallel. We note, for all $t \in [0, m]$, the point $P_t := (P, t)$ (we define in the same way A_t, B_t,...). Then we claim that u_m^{hor} is monotonic over $[P_t, Q_t]$.

To prove this, let $R_t, T_t \in [P_t, Q_t]$ such that

$$\|R_t - P_t\| \quad < \quad \|T_t - P_t\|\,.$$

We want to demonstrate that the inequality

$$u_m^{\mathrm{hor}}\,(R_t) \leqslant u_m^{\mathrm{hor}}\,(T_t)\,. \tag{2.1.11}$$

holds true. Let M_t be the middle of $[R_t, T_t]$ and $\mathcal{P}$ the plane such that $M_t \in \mathcal{P}$ and $\mathcal{P}$ is parallel to the plane that holds B,G,B_m and G_m. We denote by M^1 and M^2 the points given by the intersection of $\mathcal{P}$ with ∂ABC. Finally, let $\Omega' \in \mathbb{R}^3$ the domain whose boundary is

$$\left(\partial\left(M^1M^2A\right) \times [0,m]\right) \cup \left(\left(M^1M^2A\right) \times \{0,m\}\right)$$

and Ω'' (resp. A^1) be the symmetric of Ω' (resp. A) with respect to the plane $\mathcal{P}$.

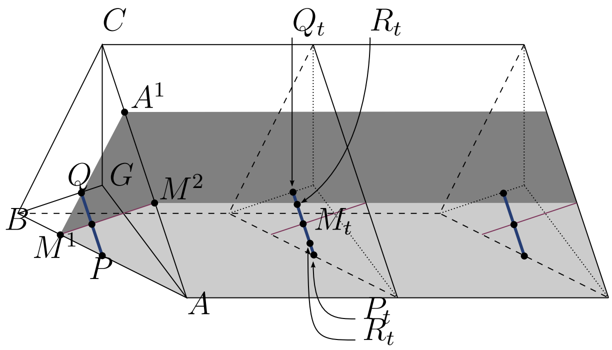


Figure V.12: Notations. Ω' is the light gray piece ; Ω'' is the dark gray piece.

Now, denote by s^1 the restriction of the function u_m^{hor} to Ω'' and s^2 the symmetry of the restriction of u_m^{hor} to Ω' with respect to the plane $\mathcal{P}$. By construction, s^1 et s^2 satisfy the minimal graph equation over Ω'' and their regularity is such that they belong to $\mathcal{C}^2(\Omega'') \cup \mathcal{C}^0\left(\overline{\Omega''}\right)$. Moreover, we can explicit the boundary conditions as follows

$$\begin{cases} \forall P \in \mathcal{P} \cap \partial\left(\Omega''\right), & s^1(P) = s^2(P) = u_m^{\text{hor}}(P); \\ \forall P \in \overline{(M^1A^1A_m^1M_m^1)}, & s^1(P) = s^2(P) = 0; \\ \forall P \in \overline{(M^1M^2M_m^2M_m^1)}, & s^1(P) = s^2(P) = 0; \\ \forall P \in \left(M^1M^2A^1\right), & s^2(P) \leqslant s^1(P). \end{cases}$$

Note that the last inequality holds true because of the geometry of a regular tetrahedron $\mathcal{T}$. Maximum principle then implies that $s^2 \leqslant s^1$ over Ω'', thus

$$u_m^{\text{hor}}(R_t) = s^2(Q_t) \leqslant s^1(Q_t) = u_m^{\text{hor}}(Q_t).$$

Second step : By letting $m \to +\infty$ in inequality (2.1.11), one can find similar property for u^{hor} :

$$\forall t \in \mathbb{R}_+, \qquad u^{\text{hor}}(R_t) \leqslant u^{\text{hor}}(T_t). \tag{2.1.12}$$

Third step : The second step gives us an indication of the geometry of Σ. We denote by Σ' the graph of u^{hor} over $(ABG) \times \mathbb{R}_+$. We claim that the orthogonal projection π of Σ' onto the sloping hyperplane H of $\mathbb{R}^4$ that holds A, B, O and the t-axis is injective. Denote by $H^{\text{s}} \subset H$ its image.

Notice that it is enough to prove that

$$\pi\left(R_t, u^{\text{hor}}\left(R_t\right)\right) \quad \neq \quad \pi\left(T_t, u^{\text{hor}}\left(T_t\right)\right)$$

because of the inclusion

$$\pi\left(\left[P_t, Q_t\right]\right) \quad \subset \quad \mathcal{D}_{t,P},$$

where $\mathcal{D}_{t,P}$ is the line that holds the points $\left(P, t, u^{\text{hor}}\left(P, t\right)\right)$ and $\left(Q, t, u^{\text{hor}}\left(Q, 0\right)\right)$. The reader can observe the configuration of the problem in figure V.13.

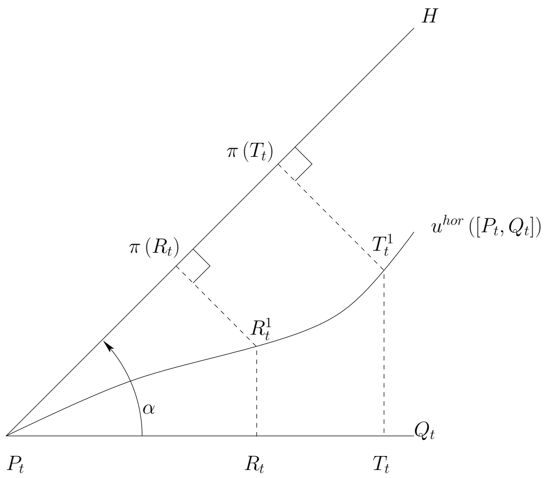


Figure V.13: Configuration to prove the injectivity of π. Note that convex hull property already shows that the graph of u^{hor} is below H.

Trigonometric calculus leads us to [3] :

$$\begin{aligned} \left\|P - \pi\left(R\right)\right\| &= \cos\left(\alpha - \arctan\left(\frac{u\left(R\right)}{d\left(P, R\right)}\right)\right)\sqrt{\left\|P - R\right\|^2 + u\left(R\right)^2} \\ &= \|P - R\|\cos\alpha + u(R)\sin\alpha. \end{aligned}$$

Since u^{hor} is an increasing function on $[P_t, Q_t]$ together with the fact that the distance $\|P - R\|$ is less than the distance $\|P - T\|$, we end up with

$$\left\|P - \pi\left(R\right)\right\| \quad < \quad \|P - T\|\cos\alpha + u(T)\sin\alpha \quad = \quad d\left(P, \pi\left(Q\right)\right),$$

from what we deduce that

$$\pi\left(Q_t\right) \quad \neq \quad \pi\left(R_t\right)$$

and thus, π is injective.

Last step : according to previous step, Σ^{hor} is then the graph of some function u^{s} over H^{s} such that u^{s} vanishes over $([A, B] \times \mathbb{R}_+) \cup (ABO)$. We use twice the reflection principle : first, the reflection with respect to the plane, included in H, which holds A, B and the direction t-axis, next with respect to the plane $\{t = 0\} \subset H$. We obtain a new minimal surface over a domain such that $(AB) \times \{0\}$ belongs to its interior : Σ is $\mathcal{C}^2$ over $(AB) \times \{0\}$.

3. We relieve notations by omitting index t and exponent *hor*.

Regularity over (vertex) $\times \mathbb{R}_+^*$ **:** It directly follows from other reflections and from our choice of $\mathcal{T}$: since the angle

$$\widehat{BAC} = \frac{\pi}{3},$$

we can build a hypersurface over $(ACA'C'A''C'') \times \mathbb{R}_+$ — see figure V.14.

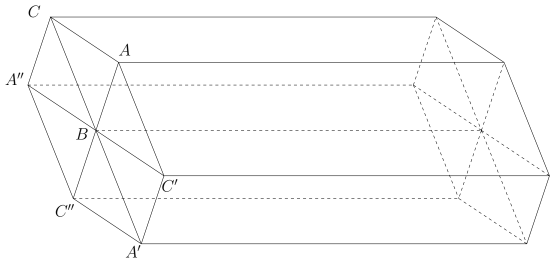


Figure V.14: hexagon $\times \mathbb{R}_+$.

We conclude like before that Σ is $\mathcal{C}^2$ over $\{B\} \times \mathbb{R}_+^*$.

Σ **is not** $\mathcal{C}^1$ **over (vertex)** $\times \{0\}$ **:** We give a proof by contradiction. Suppose Σ in $\mathcal{C}^1$ on : $(A, 0)$. Then its the tangent space $T_{(A,0)}\Sigma$ is well defined. However, the t-axis belongs to $T_{(A,0)}\Sigma$ since we know the boundary of Σ (cf. (2.1.10)), and so do (AO), (BO) and (CO) : we have four independent directions in the 3-dimensional space $T_{(A,0)}\Sigma$ and it cannot be the case. □

Remark 2.1.2 — Similar results hold true in higher dimension : if $n > 3$ and Σ is a Scherk type hypersurface over a regular polytope $\mathcal{O} \subset \mathbb{R}^n$ with 2^n hyperfaces [4], then Σ is $\mathcal{C}^2$ everywhere, except over $\partial\left(\partial\left(\partial\mathcal{O}\right)\right) \times \{0\}$.

2.2 Sherk type hypersurfaces over deformed tetrahedron

We keep notations of the study of Scherk hypersurface over a regular octahedron $\mathcal{T} = (OABC)$. Let Ω be a convex subset of (ABC) such that G belongs to the interior of Ω. We are typically in the case of figure V.1 page 164. According to the theorem 0.3.8, there exists a minimal graph u over D such that u vanishes over Γ and takes infinite value $+\infty$ over $\mathcal{S}_1 := \mathcal{S}$; the surface $\mathcal{S}_1$ will play the role of the face $\mathcal{F}_1$ of the regular octahedron $\mathcal{T}$. By reflections, like in the previous section, we can build a new minimal surface Σ over a deformed octahedron that takes alternatively $\pm\infty$ value over its boundary. Note that its boundary is the union of eight minimal surfaces $\mathcal{S}_i$ and all $\mathcal{S}_i$ are obtained from $\mathcal{S}$ by symmetries — cf. figure V.15.

Note that this construction shows that u is $\mathcal{C}^2$ over $\partial\mathcal{T}\backslash\overline{\mathcal{S}}$, but we cannot expect to have regularity results for Σ near $\mathcal{S} \times \mathbb{R}$ by using symmetries.

Of course, there are similar constructions in higher dimensions.

4. Each hyperface is a regular simplex.

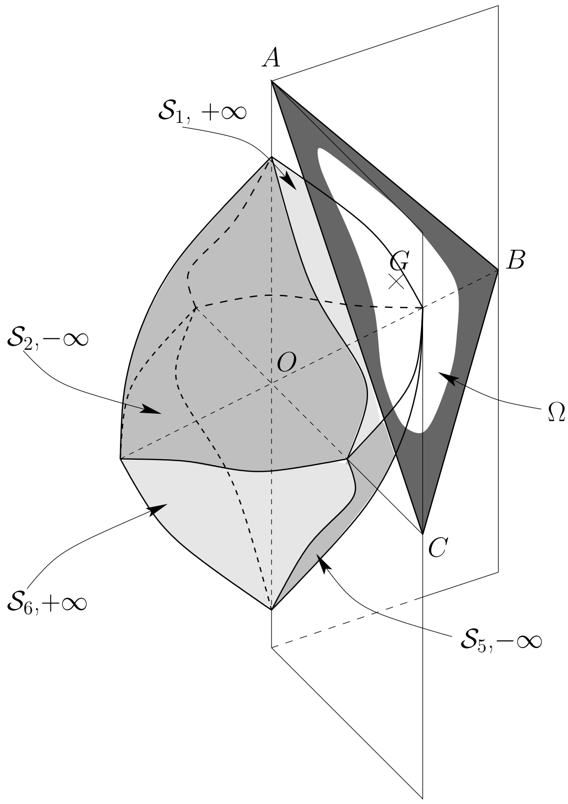


Figure V.15: Scherk hypersurface over a deformed tetrahedron.

3 A Jenkins-Serrin type condition with $3 \leqslant n \leqslant 7$

In this section, we give the proof of a Jenkins-Serrin type condition which is necessary for the existence of some Scherk type hypersurfaces for $3 \leqslant n \leqslant 7$.

Let $\mathcal{D}$ on open bounded pseudoconvex simply connected domain of $\mathbb{R}^n$ whose boundary is piecewise $\mathcal{C}^1$ and satisfies

$$\partial\mathcal{D} \quad = \quad \left(\cup_{i=1}^{k}\mathcal{S}_i^{+}\right)\bigcup\left(\cup_{j=1}^{l}\mathcal{S}_j^{-}\right)\bigcup\left(\cup_{s=1}^{h}\mathcal{S}_s^{0}\right),$$

where $\mathcal{S}_i^{+}$, $\mathcal{S}_j^{-}$ and $\mathcal{S}_s^{0}$ are minimal hypersurfaces of $\mathbb{R}^n$. Let $\mathcal{P}$ be a piecewise $\mathcal{C}^1$ subset of $\mathcal{D}$ and define the non negative real numbers α, β and γ to be

$$\alpha := \sum_{i\in[\![1,k]\!]} \left\|\mathcal{S}_i^{+}\cap\partial\mathcal{P}\right\|, \quad \beta := \sum_{j\in[\![1,l]\!]} \left\|\mathcal{S}_j^{-}\cap\partial\mathcal{P}\right\| \quad \text{and} \quad \gamma := \|\partial\mathcal{P}\|,$$

where $\|.\|$ denotes the $(n-1)$-volume.

Theorem 3.0.1
Suppose there exists a minimal graph u over $\mathcal{D}$ such that u takes $+\infty$ (resp. $-\infty$) value over $\mathcal{S}_i^{+}$ (resp. $\mathcal{S}_j^{-}$) and is continuous over $\mathcal{S}_s^{0}$. Then

$$\begin{array}{lll} \textit{if } \gamma = 0, & & \textit{then } \alpha = \beta; \\ \textit{else} & 2\alpha < \gamma & \textit{and } 2\beta < \gamma. \end{array} \tag{3.0.13}$$

Sketch of the proof

We use smilar arguments to the articles [JS65], [NR02], [Pin09] and [MRR11]. The proof is done as follows : first, we formulate the flow condition, then we prove that N is vertical only on the minimal surfaces $\mathcal{S}_i+$ and $\mathcal{S}_j^-$. We conclude with differentiating the cases $\gamma = 0$ and $\gamma \neq 0$.

Let

$$N \quad := \quad \frac{1}{\sqrt{1+\|\nabla u\|^2}} \begin{pmatrix} -\nabla u \\ 1 \end{pmatrix}$$

be the normal to the minimal hyperface, let $\nu_{\mathbb{R}^n}$ be the outward conormal to the boundary of $\mathcal{P}$ and let $\nu_{\mathbb{R}^{n+1}} := (\nu_{\mathbb{R}^n}, 0)$.

First step. Let $\mathcal{D}'$ be a piecewise $\mathcal{C}^1$ compact subset of $\mathcal{D}$. Since u satisfies the minimal graph equation, we have the equality

$$\int_{\mathcal{D}'} \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+\|\nabla u\|^2}}\right) \quad = \quad 0.$$

According to Stoke's theorem, we then rewrite the above relation as

$$0 \quad = \quad \int_{\partial\mathcal{D}'} \left\langle \frac{\nabla u}{\sqrt{1+\|\nabla u\|^2}}, \nu'_{\mathbb{R}^n} \right\rangle \quad = \quad -\int_{\partial\mathcal{D}'} \left\langle N, \nu'_{\mathbb{R}^{n+1}} \right\rangle,$$

where $\nu'_{\mathbb{R}^{n+1}}$ is the outward conormal to the boundary of $\mathcal{D}'$. This above equation is nothing but a flow condition. It also holds tre for $\mathcal{P}$ which, in general case, is not a compact subset of $\mathcal{D}$: it is enough to consider a sequence of compact piecewise $\mathcal{C}^1$ subdomains of $\mathcal{P}$ which converges to $\mathcal{P}$ — for example

$$(\mathcal{D}'_m)_m \quad := \quad \left(\left\{ P \in \mathcal{P} : d(P, \partial\mathcal{P}) \geqslant \frac{1}{m} \right\}\right)_m$$

for m large enough. Thus, we obtain

$$\int_{\partial P} \left\langle N, \nu_{\mathbb{R}^{n+1}} \right\rangle \quad = \quad 0. \tag{3.0.14}$$

Second step. We claim that if p belongs to the interior of $\mathcal{S}_s^0$, then

$$\left|\left\langle N(p), \nu_{\mathbb{R}^{n+1}}(p) \right\rangle\right| \quad < \quad 1.$$

The proof is similar to the one of [Pin09, assertion 5.2] : suppose it is not the case. Then the hypersurfaces Σ and $\mathcal{S}_s^0 \times \mathbb{R}$ have the same vertical tangent space in $(p, u(p))$ and Σ is in the same side of $\mathcal{S}_s^0 \times \mathbb{R}$. Then the boundary maximum principle implies that those two minimal surfaces coincide in a neighbourhood of $(p, u(p))$: it is not possible.

Third step. If p belongs to one of the $\operatorname{Int}\left(\mathcal{S}_i^+\right)$ (resp. $\operatorname{Int}\left(\mathcal{S}_i^-\right)$), then we claim that

$$\left\langle N(p), \nu_{\mathbb{R}^{n+1}}(p) \right\rangle \quad = \quad -1 \qquad (\text{resp.} = 1). \tag{3.0.15}$$

The proof is similar to the proof of [NR02, Lemma 1] : suppose it is not the case. We only consider the problem for $\mathcal{S}_i^+$. Then there exists $\varepsilon > 0$ and a sequence $(p_m)_m$ of $\mathcal{D}$ which converges to p as m tends to $+\infty$ such that

$$\langle N(p_m), \nu_{\mathbb{R}^{n+1}}(p)\rangle \quad > \quad -1+\varepsilon. \tag{3.0.16}$$

Besides, there exists $r > 0$ such that, if $\mathcal{B}(q,r)$ denotes the disk contained in Σ centered at q of intrinsic radius r, for all m, $\mathcal{B}((p_m, u(p_m)), r) \subset \Sigma$ because u takes $+\infty$ value over $\mathcal{S}_i^+$. According to [SSY75, Theorem 3] for $3 \leqslant n \leqslant 5$ together with [Sim] for $n = 6$ or $n = 7$, we can use curvatures estimates, namely there exists an absolute constant $C = C(r)$ such that :

$$\forall m, \forall q \in \mathcal{B}\left((p_m, u(p_m)), \frac{r}{2}\right), \qquad |\mathcal{A}(q)| \quad \leqslant \quad C,$$

where $\mathcal{A}$ denotes the second funamental form of Σ. Hence Σ is a bounded graph over a disk $B((p_m, u(p_m)), r')$ contained in the tangent space $T_{(p_m, u(p_m))}\Sigma$. Notice that r' can be chosen so that it does not depend on m. But the horizontal projection of those disks is not contained in $\mathcal{D}$ when p_m approaches p because of inequality (3.0.16) : it is a contradiction.

Last step : Finally, collecting the equations (3.0.14), (3.0.16) and (3.0.15) give, we end up with

$$\begin{aligned} 0 \quad = \quad & \sum_{i=1}^{k} \int_{\mathcal{S}_i^+ \cap \partial\mathcal{P}} \langle N, \nu_{\mathbb{R}^{n+1}}\rangle + \sum_{j=1}^{l} \int_{\mathcal{S}_j^- \cap \partial\mathcal{P}} \langle N, \nu_{\mathbb{R}^{n+1}}\rangle \\ & + \int_{\partial P \setminus \left(\left(\cup_i \mathcal{S}_i^+\right) \bigcup \left(\cup_j \mathcal{S}_j^-\right)\right)} \langle N, \nu_{\mathbb{R}^{n+1}}\rangle, \end{aligned}$$

and thus, we find the flow equation

$$\alpha - \beta \quad = \quad \int_{\partial P \setminus \left(\left(\cup_i \mathcal{S}_i^+\right) \bigcup \left(\cup_j \mathcal{S}_j^-\right)\right)} \langle N, \nu_{\mathbb{R}^{n+1}}\rangle. \tag{3.0.17}$$

It remains to separate the cases.

First case : $\gamma = 0$ and we consider $\mathcal{P} := \mathcal{D}$. Then the equation (3.0.17) yields to

$$\alpha - \beta \quad = \quad 0. \tag{3.0.18}$$

Second case : $\gamma \neq 0$. If p is a point of $\partial P \setminus \left(\left(\cup_i \mathcal{S}_i^+\right) \bigcup \left(\cup_j \mathcal{S}_j^-\right)\right)$, either p belongs to the interior of $\mathcal{D}$ and then $|\langle N, \nu_{\mathbb{R}^{n+1}}\rangle| < 1$ or p belongs to one of the $\mathrm{Int}\left(\mathcal{S}_s^0\right)$ and same inequality holds. Therefore equation (3.0.17) provides

$$\begin{aligned} \alpha \quad & < \quad \beta + \left\| \partial P \setminus \left(\left(\cup_i \mathcal{S}_i^+\right) \bigcup \left(\cup_j \mathcal{S}_j^-\right)\right) \right\| \\ & < \quad \beta + (\gamma - (\alpha + \beta)), \end{aligned}$$

from what we deduce

$$2\alpha \quad < \quad \gamma. \tag{3.0.19}$$

The inequality with β can be proved in the same way. $\square$

Table des figures

Bibliographie

[ABR01] Sheldon Axler, Paul Bourdon, and Wade Ramey. *Harmonic Function Theory.* Springer, second edition, 2001.

[Aub82] Thierry Aubin. *Nonlinear Analysis on Manifolds, Monge-Ampère Equations.* Springer-Verlag, 1982.

[BdC84] J. Lucas Barbosa and Manfredo do Carmo. Stability of Hypersurfaces with Constant Mean Curvature. *Mathematische Zeitschrift*, 185(3) :339–353, 1984.

[BG92] Marcel Berger and Bernard Gostiaux. *Géométrie différentielle : variétés, courbes et surfaces.* Presses universitaires de France, second edition, 1992.

[BM78] Rodney Carlos Bassanezi and Umberto Massari. The Dirichlet Problem for the Minimal Surface Equation in Nonregular Domains. *Ann. Univ. Ferrara Sez. VII (N.S.)*, 24 :181–189, 1978.

[CM99] T. Colding and W. Minicozzi. *Minimal Surfaces — Courant Lecture Notes in Mathematics*, volume 4. Courant Institute of Mathematical Sciences, 1999.

[CM05] T. Colding and W. Minicozzi. Minimal submanifolds. *arXiv :math/0504158v2*, 2005.

[Cos84] C. J. Costa. Example of a complete minimal immersion in r3 of genus one and three embedded ends. *Bol. Soc. Brasil. Mat.*, 15(1) :47–54, 1984.

[CP11] Antoine Coutant and Frank Pacard. Riemann minimal hypersurfaces in $\mathbf{R}^n \times \mathbf{R}$. *Preprint*, 2011.

[CP12] Antoine Coutant and Frank Pacard. The Wei-Riemann minimal surface in $\mathbf{S}^2 \times \mathbf{R}$. *Preprint*, 2012.

[Dau88] Monique Dauge. *Elliptic Boundary Value Problems on Corner Domains.* Springer, 1988.

[DHW93] H. Karcher D. Hoffman and F. Wei. Adding Handles to the Helicoid. *Bulletin of The American Mathematical Society*, 29(1) :77–85, 1993.

[Don06] Harold Donnelly. Eigenfunctions of the Laplacian on compact Riemannian manifolds. *Asian Journal of Mathematics*, 10(1) :115–125, 2006.

[FP00] Saadia Fakhi and Frank Pacard. Existence of complete minimal hypersurfaces with finite total curvature. *Manuscripta Mathematica*, 103 :465–512, 2000.

[GHL93] Sylvestre Gallot, Dominique Hulin, and Jacques Lafontaine. *Riemannian Geometry.* Springer-Verlag, second edition, 1993.

[Gri92] Pierre Grisvard. *Singularities in Boundary Value Problems.* Springer, 1992.

[GT01] David Gilbarg and Neil S. Trudinger. *Elliptic Partial Differential Equations of Second Order.* Springer, third edition, 2001.

[Hö68] Lars Hörmander. The spectral function of an elliptic operator. *Acta Mathematica*, 121(1) :193–218, 1968.

[Hau06] L. Hauswirth. Minimal surfaces of Riemann type in three-dimensional product manifolds. *Pacific J. Math.*, 224(1) :91–117, 2006.

[HKR91] D. Hoffman, H. Karcher, and H. Rosenberg. Embedded minimal annuli in $\mathbf{R}^3$ bounded by a pair of straight lines. *Comment. Math. Helv.*, 66(4) :599–617, 1991.

[HM85] D. Hoffman and W. H. Meeks. A complete embedded minimal surface in $\mathbf{R}^3$ with genus one and three ends. *J. Differential Geom.*, 21(1) :109–127, 1985.

[HM90] D. Hoffman and W. H. Meeks. Embedded Minimal Surfaces of Finite Topology. *The Annals of Mathematics*, 131(1) :1–34, January 1990.

[HP07] L. Hauswirth and F. Pacard. Higher genus riemann minimal surfaces. *Invent. Math.*, 169(3) :569–620, 2007.

[JS65] Howard Jenkins and James Serrin. Variational Problems for Minimal Surface Type, II. Boundary Value Problems for the Minimal Surface Equation. *Archive for Rational Mechanics and Analysis*, 21(4) :321–342, 1965.

[JS68] Howard Jenkins and James Serrin. The Dirichlet Problem for the Minimal Surface Equation in Higher Dimension. *Journal für die Reine und Angewandte Mathematik*, 229 :170–187, 1968.

[KP07] S. Kaabachi and F. Pacard. Riemann minimal surfaces in higher dimensions. *Journal of the Institute of Mathematics of Jussieu*, 4(4) :613–637, 2007.

[Mir71] M. Miranda. Un principio di Massimo Forte per le frontiere minimali e una sua applicazione alla risoluzione del problema al contorno per l'equazione delle superfici di area minima. *Rendiconti del Seminario Mathematica della Università di Padova*, (45) :355–366, 1971.

[MP99] R. Mazzeo and F. Pacard. Constant scalar curvature metrics with isolated singularities. *Duke Math. J.*, 99(3) :353–418, 1999.

[MP01] R. Mazzeo and F. Pacard. Constant mean curvature surfaces with delaunay ends. *Comm. Analysis and Geometry*, 9(1) :169–237, 2001.

[MPP01] R. Mazzeo, F. Pacard, and D. Pollack. Connected sums of constant mean curvature surfaces in euclidean 3 space. *J. Reine Angew. Math.*, 536 :115–165, 2001.

[MPR] W. Meeks, J. Pérez, and A. Ros. Uniqueness of the Riemann minimal surfaces. *Inventiones Mathematicae*, 133(1) :107–132.

[MRR11] Laurent Mazet, M. Magdalena Rodríguez, and Harold Rosenberg. The Dirichlet Problem for the Minimal Surface Equation — with possible infinite boundary data — over domains in a riemannian surface. *Proc. London. Math. Soc.*, (102), 2011.

[MT11] F. Morabito and M. Traizet. Non-periodic Riemann examples with handles. *Advances in Math*, 2011.

[NR02] Barbara Nelli and Harold Rosenberg. Minimal Surfaces in $\mathbb{H}^2\times\mathbb{R}$. *Bulletin of the Brazilian Mathematical Society*, 33(2) :263–292, 2002.

[Pac98] Frank Pacard. Construction de surfaces à courbure moyenne constante. *Séminaire de Théorie Spectrale et Géométrie*, 17 :139–157, 1998.

[Pac02] Frank Pacard. Higher dimensional Scherk's hypersurface. *J. Math. Pures Appl.*, IX(3) :241–258, 2002.

[Pac09] Frank Pacard. *Lectures on Connected Sum Constructions in Geometry and Nonlinear Analysis.* Preprint, 2009.

[Pin09] Ana Lucia Pinheiro. A Jenkins-Serrin Theorem in $\mathbb{M}^2 \times \mathbb{R}$. *Bulletin of the Brazilian Mathematical Society*, 40(1) :117–148, March 2009.

[PR] Frank Pacard and Harold Rosenberg. Attaching handles to Delaunay nodoïds. *Preprint.*

[Rad32] Tibor Radò. *The Problem of Plateau.* Springer, 1932.

[Rie98] Bernhard Riemann. *Œuvres mathématiques de Riemann.* Gauthier-Villars et fils, Paris, 1898.

[Shi56] Max Shiffman. On surfaces of stationary area bounded by two circles, or convex curves, in parallel planes. *Ann. of Math.*, 63(2) :77–90, 1956.

[Sim] Leon Simon. Remarks on curvature estimates for minimal hypersurfaces. *Duke Math. J.*, 43(3) :545–553.

[SSY75] Richard Schoen, Leon Simon, and Shing-Tung Yau. Curvature estimates for minimal hypersurfaces. *Acta Mathematica*, 134(3–4) :275–288, 1975.

[ST] R. Sa Earp and É. Toubiana. Minimal Graphs in $\mathbb{H}^n \times \mathbb{R}$ and $\mathbb{R}^{n+1}$. *Annales de l'Institut Fourier*, 60(7) :2373–2402.

[Tom86] Ermanna Tomaini. Infinite boundary value problems for surfaces of prescribed mean curvature. *Rendiconti del. seminario matematico della Università di Padova*, 76 :59–74, 1986.

[Tra] Martin Traizet. Embedded minimal surfaces of finite total curvature. `http://www.lmpt.univ-tours.fr/~traizet/minimal.html`.

[Tra02a] Martin Traizet. Adding handles to riemann minimal examples. *Journal Institut Mathématique de Jussieu*, 1(1) :145–174, 2002.

[Tra02b] Martin Traizet. An embedded minimal surface with no symmetries. *Journal of Differential Geometry*, 60 :103–153, 2002.

[Wei94] F. Wei. Adding handles to the Riemann examples. *Preprint*, 1994.

[Whi87] Brian White. The Space of m-dimensional surfaces that are stationary for a parametric elliptic functional. *Indiana University Mathematics Journal*, 36 :567–603, 1987.