Jeux différentiels stochastiques à information incomplète

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Introduction

1 Résumé

L'objectif de cette thèse est l'étude des jeux différentiels stochastiques à information incomplète. Nous considérons un jeu à deux joueurs adverses qui contrôlent une diffusion afin de minimiser, respectivement de maximiser un paiement spécifique. Pour modéliser l'incomplétude des informations, nous suivrons la célèbre approche d'Aumann et Maschler [3]. Nous supposons qu'il existe des états de la nature différents dans laquelle le jeu peut avoir lieu. Avant que le jeu commence, l'état est choisi au hasard. L'information est ensuite transmise à un joueur alors que le second ne connaît que les probabilités respectives pour chaque état.

L'observation frappante pour le modèle Aumann-Maschler est ce qu'on appelle le théorème Cav u auquel nous nous référons comme la représentation duale dans ce qui suit. Elle dit que l'on peut envisager un jeu à information incomplète comme un jeu à information complète, où le joueur informé, en addition à son contrôle habituel, peut contrôler la dynamique du nouveau jeu avec l'aide de certaines mesures martingale. Cette représentation peut alors être utilisée pour étudier des jeux à information incomplète, avec l'aide de jeux à information complète. En particulier, elle permet de déterminer des stratégies optimales pour le joueur informé.

Les idées célèbres d'Aumann et Maschler, qui remontent aux années 1960, ont été largement étudiées dans le cadre de jeux répétés dans les dernières décennies. Cependant, ce n'est que récemment que les jeux différentiels à information incomplète furent considérés, en premier lieu par Cardaliaguet dans [23] et [24]. L'existence et l'unicité d'une fonction valeur pour les jeux différentiels stochastiques à information incomplète ont ensuite été données par Cardaliaguet et Rainer dans [28], en utilisant des solutions de viscosité d'une certaine équation aux dérivées partielles (EDP) complètement non-linéaire. Dans un travail ultérieur, Cardaliaguet et Rainer [27] établissent dans un exemple déterministe simple, une représentation duale pour ces jeux en temps continue.

Dans le chapitre 3, nous étendons les résultats de Cardaliaguet et Rainer [27] et établissons une représentation duale pour les jeux différentiels stochastiques à information incomplète. Ici, nous utilisons largement la théorie des équations différentielles stochastiques rétrogrades (EDSRs), qui se révèle être un outil indispensable dans cette étude. En outre, nous montrons comment, sous certaines restrictions, cette représentation permet

de construire des stratégies optimales pour le joueur informé. Ces résultats sont basés sur l'article :

1. *A BSDE approach to stochastic differential games with incomplete information*, Stochastic Processes and their Applications, vol. 122, no. 4, pp. 1917 - 1946, (2012).

Dans le chapitre 4, nous donnons, en utilisant la représentation duale, une preuve particulièrement simple de la semiconvexité de la fonction valeur des jeux différentiels à information incomplète.

2. *A note on regularity for a fully non-linear PDE arising in game theory*, (2011), Preprint.

Ce résultat, uniquement basé sur des techniques probabilistes, est nouveau et serait probablement beaucoup plus difficile à établir du point de vue de la théorie des EDP.

Le chapitre 5 est consacré à des schémas numériques pour les jeux différentiels stochastiques à information incomplète. Nous nous intéressons à la construction explicite d'une approximation de la fonction valeur. À cette fin, nous donnons un schéma qui est entièrement discrétisé en temps avec l'inconvénient que, comme dans les jeux différentiels stochastiques ordinaires, seul la valeur, et non les stratégies optimales, peut être approchée. Les résultats présentés dans ce chapitre peuvent être trouvés dans :

3. *A probabilistic numerical scheme for stochastic differential games with incomplete information*, arXiv :1111.4136v1, (2011), soumis.

Dans le chapitre 6, nous étudions des jeux d'arrêt optimal en temps continue, appelés jeux de Dynkin, à information incomplète. Nous montrons que ces jeux ont une valeur et une caractérisation unique par des EDP complètement non-linéaires avec obstacles pour lesquelles nous fournissons un principe de comparaison. Aussi, nous établissons une représentation duale pour les jeux de Dynkin à information incomplète. Ce chapitre est basé sur l'article :

4. *On a continuous time Dynkin game with incomplete information*, en cours de préparation.

2 La boîte à outils mathématiques : EDSR et EDP

2.1 Équations différentielles stochastiques rétrogrades

Bien que déjà évoqué dans un travail de Bismut [12] en 1973 l'étude des équations différentielles stochastiques rétrogrades (EDSRs) commence vraiement en 1990 avec l'article pionnier de Pardoux et Peng [88]. Dans une série de travaux ultérieurs [89], [91], [92], [93] et [94] Pardoux et Peng ont posé les bases de l'étude des EDSR et de leurs connexions à d'autres domaines des mathématiques comme le contrôle optimal et les équations aux dérivées partielles. Les années suivantes la théorie des EDSR a connu un développement considérable et s'est avérée être un outil très précieux pour diverses applications, notamment en finance mathématique. Pour cet aspect, nous nous référons à l'étude d'El Karoui, Peng et Quenez [45].

La base de la théorie des EDSR est le théorème de représentation des martingales. En effet soit $(\Omega, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P}_0)$ un espace de probabilité filtré, avec les hypothèses usuelles, muni d'un mouvement Brownien B. Si $(\mathcal{F}_t) = \sigma(B_s, s \leq t)$, nous avons par le théorème de représentation des martingales pour toute variable aléatoire ξ, $\mathcal{F}_T$-mesurable et de carré intégrable une décomposition de la forme

$$\xi = \mathbb{E}[\xi|\mathcal{F}_0] + \int_0^T Z_s dB_s, \tag{2.1}$$

où Z est un processus adapté de carré intégrable. En appliquant le théorème de représentation des martingales à la variable aléatoire $\mathcal{F}_s$-mesurable $Y_s := \mathbb{E}[\xi|\mathcal{F}_s]$ nous obtenons avec (2.1) l'équation suivante :

$$Y_s = \xi - \int_s^T Z_r dB_r. \tag{2.2}$$

L'équation (2.2) est appelé EDSR linéaire et le couple de processus adaptés (Y, Z) est appelé la solution de (2.2).

Plus géneralement, on appelle EDSR une équation de la forme

$$Y_s = \xi + \int_s^T f(r, Y_r, Z_r)dr - \int_s^T Z_r dB_r, \tag{2.3}$$

où le générateur f est une fonction aléatoire donnée, c'est à dire $f = f_\omega(\cdot)$. Nous remarquons que, si f est indépendante de Y, (2.3) se lit comme suit

$$Y_s = \mathbb{E}\left[\xi + \int_s^T f(r, Z_r)dr\big|\mathcal{F}_s\right]. \tag{2.4}$$

Sous l'hypothèse que f est uniformément Lipschitz, l'existence d'une solution (Y, Z) à (2.3) a d'abord été démontrée par Pardoux et Peng [88] via un argument de point fixe. En outre, l'unicité a été établie par un principe de comparaison. Depuis lors, de nombreux auteurs ont contribué à affaiblir les hypothèses sur le générateur f et sur la donnée finale ξ (voir par exemple Briand and Hu [16], Delbaen, Hu et Bao [37], Delbaen, Hu et Richou [38], Kobylansky [67], Lepeltier et San Martin [80]).

Nous tenons également à mentionner que, si la filtration $(\mathcal{F}_t)$ est plus grande que $\sigma(B_s, s \leq t)$, le théorème de représentation des martingales ne s'applique pas. Au lieu de cela on peut utiliser la décomposition de Galtchouk-Kunita-Watanabe (voir par exemple Ansel and Stricker [2]). Elle implique que, pour toute variable aléatoire ξ, $\mathcal{F}_T$-mesurable de carré intégrable, nous avons la décomposition

$$\xi = \mathbb{E}_{\mathbb{P}}[\xi|\mathcal{F}_0] + \int_0^T Z_s dB_s + N_T, \tag{2.5}$$

où Z est un processus adapté de carré intégrable et N est une martingale de carré intégrable, avec $N_0 = 0$, qui est fortement orthogonal à B. Comme dans El Karoui et Huang [43] on peut aussi considérer des EDSR de la forme suivante

$$Y_s = \xi + \int_s^T f(r, Y_r, Z_r)dr - \int_s^T Z_r dB_r - (N_T - N_s) \tag{2.6}$$

avec un triplet (Y, Z, N) comme solution.

2.2 EDS progressives-rétrogrades et leur lien avec les EDP

Un cas très important, d'abord examiné la première fois par Peng dans [91], est quand le générateur f et la donnée finale ξ de l'EDSR dépendent d'une équation différentielle stochastique (EDS)

$$dX_s^{t,x} = b(s, X_s^{t,x})ds + \sigma(s, X_s^{t,x})dB_s \qquad X_t^{t,x} = x. \tag{2.7}$$

Pour des fonctions déterministes données f et g, on considère alors l'équation suivante :

$$Y_s^{t,x} = g(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})dr - \int_s^T \sigma^*(s, X_s^{t,x})Z_r^{t,x}dB_r, \tag{2.8}$$

où nous avons légèrement modifié la dernière intégrale pour des raisons de notation. Le couple (2.7), (2.8) est appelé EDS progressive-rétrograde (EDSPR).

En supposant une régularité suffisante sur les coefficients, l'existence et l'unicité de la solution de (2.8) peuvent être établies grâce à la théorie des EDP semi-linéaire. Pour celles-ci on pourra par exemple consulter le livre de Ladyženskaja, Solonnikov et Uralceva [73]. En particulier, si u est une solution lisse de l'EDP

$$\begin{aligned} \tfrac{\partial u}{\partial t} + \tfrac{1}{2}\mathrm{tr}(\sigma\sigma^*(t,x)D_x^2 u) + \langle b(t,x), D_x u\rangle + f(t,x,u,D_x u) &= 0 \\ u(T,x) &= g(x), \end{aligned} \tag{2.9}$$

alors d'après la formule de Itô, le couple

$$\begin{aligned} Y_s^{t,x} &:= u(s, X_s^{t,x}) \\ Z_s^{t,x} &:= D_x u(s, X_s^{t,x}) \end{aligned} \tag{2.10}$$

est une solution de (2.8).

Cette connexion a été d'abord établie par Peng dans [91]. C'est en outre l'idée centrale de ce qu'on appelle le "four step scheme" de Ma, Protter et Yong [82]. Il y est montré qu'on peut aussi appliquer cette méthode pour trouver des solutions d'EDSPR entièrement couplées, c'est à dire où les coefficients de l'EDS (2.7) dépendent aussi de $(Y^{t,x}, Z^{t,x})$. L'existence et l'unicité de solutions pour des EDSPR entièrement couplées, c'est à dire du triplet $(X^{t,x}, Y^{t,x}, Z^{t,x})$, sont étudiées dans de nombreux articles au-delà du cadre Markovien (par exemple Hu et Peng [62], Hu et Yong [63], Pardoux et Tang [90], Peng et Wu [95]). Pour plus de références nous conseillons le livre de Ma et Yong [83].

D'autre part une question naturelle est de savoir si la solution de l'EDSR (2.8) fournit une solution à l'EDP semi-linéaire parabolique (2.9). Peng établit dans [91] cette généralisation de la célèbre formule de Feynman-Kac pour le cas semi-linéaire. En fait, la fonction u définie par

$$u(t,x) := Y_t^{t,x} \tag{2.11}$$

est - sous des hypothèses de régularité sur les coefficients - lisse et une solution classique de (2.9). Ainsi Peng donne dans [91] une preuve entièrement probabiliste de l'existence d'une solution d'une EDP semi-linéaire. Pour des raisons ultérieures nous remarquons qu'un pas modeste mais important dans la preuve est de montrer que $u(t, x)$ est déterministe. Ici, c'est une conséquence facile de la loi du 0-1 de Blumenthal.

Sous des hypothèses uniquement Lipschitz sur les coefficients, Peng [92] a montré que $u(t, x)$ résout l'EDP dans un sens plus faible ; nommément dans le sens des solutions de viscosité. Cette notion a été introduite pour l'étude des problèmes de contrôle dans le début des années 1980 par Crandall et Lions [30]. La référence principale pour la théorie des solutions de viscosité est la monographie de Crandall, Ishii et Lions [29]. Nous allons donner dans les chapitres suivants une définition précise des solutions de viscosité pour les différents cas qui nous intéressent.
Comme dans le cas lisse, la théorie des EDSR donne une preuve probabiliste de l'existence de solutions de viscosité des EDPs semi-linéaires. Cependant ces solutions de viscosité ne sont en général pas assez régulières pour construire comme dans (2.10) des solutions à des EDSR.

3 Jeux différentiels stochastiques

3.1 Le problème

Un jeu différentiel stochastique à somme nulle est en général un jeu, où deux joueurs adverses contrôlent une quantité diffusive, tandis qu'ils s'observent mutuellement. Nous allons donner ici la forme standard de ce problème et fixer les notations pour les sections suivantes.

Pour la description mathématique, il est commode de considérer la dynamique stochastique sur l'espace canonique $\mathcal{C}([0, T]; \mathbb{R}^d)$ muni de la mesure de Wiener $\mathbb{P}_0$. Dans la suite, nous désignons par $B_s(\omega_B) = \omega_B(s)$ l'application coordonnées sur $\mathcal{C}([0, T]; \mathbb{R}^d)$, par $\mathcal{H} = (\mathcal{H}_s)$ la filtration engendrée par $s \mapsto B_s$ et par $(\mathcal{H}_{t,s})_{s\in[t,T]}$ la filtration engendrée par $s \mapsto B_s - B_t$.

Pour toutes les données initiales $t \in [0, T]$ et $x \in \mathbb{R}^d$, les joueurs contrôlent une diffusion donnée par

$$\begin{aligned} dX_s^{t,x,u,v} &= b(s, X_s^{t,x,u,v}, u_s, v_s)ds + \sigma(s, X_s^{t,x,u,v}, u_s, v_s)dB_s \\ X_t^{t,x,u,v} &= x, \end{aligned} \tag{3.1}$$

où nous supposons que les contrôles des joueurs u, v prennent leurs valeurs dans certains ensembles U, V, respectivement, où U, V sont des sous-ensembles compacts de certains espaces de dimension finie. L'objectif des joueurs est respectivement de minimiser ou maximiser l'espérance du gain

$$J(t, x, u, v) = \mathbb{E}\left[\int_t^T l(s, X_s^{t,x,u,v}, u_s, v_s)ds + g(X_T^{t,x,u,v})\right], \tag{3.2}$$

où l désigne le paiement courant du jeu et g le paiement final. Nous notons que, en général, les coefficients pourraient être des fonctions aléatoires.

3.2 Jeux différentiels et jeux différentiels stochastiques via EDP

Les premières études des jeux différentiels déterministes (ce qui correspond à la situation, où $\sigma = 0$ et les coefficients sont des fonctions déterministes) remontent au début des années 1940 avec les oeuvres d'Isaacs [64] et Pontryagin [96], [97]. Le principal problème pour l'investigation des jeux en temps continu est de spécifier comment les joueurs peuvent jouer. D'une part ils doivent avoir la possibilité de réagir aux actions de leur adversaire alors que d'autre part une définition appropriée doit éviter des changements instantanés. Pour contourner les difficultés qui sont posées par le temps continu, une approche habituelle consiste à discrétiser le jeu dans le temps (voir par exemple Fleming [49], [50], Friedman [54], Krasovskiĭ and Subbotin [69], Subbotina, Subbotin and Tretjakov [105],....). Le résultat du jeu en temps continu est alors la limite du résultat du jeu en temps discret.

Une approche différente pour l'étude des jeux différentiels déterministes est donnée par Evans et Souganidis [47] en utilisant la notion de stratégies non-anticipatives mises en place par Elliot et Kalton [46]. Leur preuve s'appuie largement sur la technique des solutions de viscosité introduites par Crandall et Lions [30]. Les résultats d' Evans et Souganidis [47] ont été généralisés par Fleming et Souganidis dans [52] pour le cas des jeux différentiels stochastiques, où le système est markovien, c'est à dire les coefficients sont des fonctions déterministes.

Afin d'éviter les changements instantanés, Fleming et Souganidis [52] laissent les joueurs jouer des contrôles contre des stratégies, en utilisant les définitions suivantes :

Définition 3.1. *Pour tout $t \in [0,T]$ un contrôle admissible $u = (u_s)_{s\in[t,T]}$ pour le joueur 1 est un processus progressivement mesurable par rapport à la filtration $(\mathcal{H}_{t,s})_{s\in[t,T]}$ à valeurs dans U. L'ensemble des contrôles admissibles pour le joueur 1 est désigné par $\mathcal{U}(t)$. La définition des contrôles admissibles $v = (v_s)_{s\in[t,T]}$ pour le joueur 2 est similaire. L'ensemble des contrôles admissibles pour le joueur 2 est désigné par $\mathcal{V}(t)$.*

Définition 3.2. *Une stratégie pour le joueur 1 à l'instant $t \in [0,T]$ est une application non-anticipative $\alpha : [t,T] \times \mathcal{V}(t) \to \mathcal{U}(t)$, c'est-à-dire que pour tout $v, v' \in \mathcal{V}(t)$, $s \in [t,T]$*

$$v = v' \ \text{ sur } [t,s] \to \alpha(v) = \alpha(v') \text{ sur } [t,s].$$

L'ensemble des stratégies pour le joueur 2 est désigné par $\mathcal{A}(t)$.
La définition de stratégies $\beta : [t,T]\times\mathcal{U}(t) \to \mathcal{V}(t)$ pour le joueur 2 est similaire. L'ensemble des stratégies pour le joueur 2 est désigné par $\mathcal{B}(t)$.

La valeur inférieure du jeu est ici

$$V^-(t,x) = \inf_{\alpha\in\mathcal{A}(t)} \sup_{v\in\mathcal{V}(t)} J(t,x,\alpha,v), \tag{3.3}$$

où $J(t,x,\alpha,v)$ est associé au couple de contrôles $(\alpha(\cdot,v),v) \in \mathcal{U}(t) \times \mathcal{V}(t)$. De même, la valeur supérieure est

$$V^+(t,x) = \sup_{\beta\in\mathcal{B}(t)} \inf_{u\in\mathcal{U}(t)} J(t,x,u,\beta), \tag{3.4}$$

où $J(t,x,\alpha,v)$ est associé au couple de contrôles $(u,\beta(\cdot,u)) \in \mathcal{U}(t) \times \mathcal{V}(t)$.

Définition 3.3. *On dit que le jeu a une valeur si*

$$V^-(t,x) = V^+(t,x). \tag{3.5}$$

$V(t,x) := V^-(t,x) = V^+(t,x)$ *est appelé la valeur du jeu.*

Pour montrer que le jeu différentiel stochastique a une valeur, Fleming et Souganidis [52] utilisent la théorie des solutions de viscosité. En effet, sous des hypothèses adéquates, on peut montrer que V^+ est la solution de viscosité de l'équation de Hamilton-Jacobi-Isaacs (HJI),

$$\frac{\partial w}{\partial t} + H^+(t, x, D_x w, D_x^2 w) = 0, \tag{3.6}$$

où, pour chaque $t \in [0, T]$, $x \in \mathbb{R}^d$, $\xi \in \mathbb{R}^d$, $A \in \mathcal{S}^d$

$$H^+(t, x, \xi, A) = \inf_{u \in U} \sup_{v \in V} \left\{ \langle b(t, x, u, v), \xi \rangle + \frac{1}{2}\mathrm{tr}(\sigma\sigma^*(t, x, u, v)A) + l(t, x, u, v) \right\}. \tag{3.7}$$

En outre, par les mêmes méthodes, V^- est une solution de viscosité de

$$\frac{\partial w}{\partial t} + H^-(t, x, D_x w, D_x^2 w) = 0 \tag{3.8}$$

avec

$$H^-(t, x, \xi, A) = \sup_{v \in V} \inf_{u \in U} \left\{ \langle b(t, x, u, v), \xi \rangle + \frac{1}{2}\mathrm{tr}(\sigma\sigma^*(t, x, u, v)A) + l(t, x, u, v) \right\}. \tag{3.9}$$

Si on suppose maintenant la condition d'Isaacs

$$H^+(t, x, \xi, A) = H^-(t, x, \xi, A) =: H(t, x, \xi, A), \tag{3.10}$$

la propriété de solution de viscosité donne, avec un principe de comparaison pour l'équation HJI (voir par exemple Crandall, Isshii and Lions [29]), le résultat de Fleming et Souganidis [52] :

Théorème 3.4. *Pour chaque $(t, x) \in [0, T] \times \mathbb{R}^d$ le jeu différentiel stochastique a une valeur $V(t, x)$ et la fonction $(t, x) \mapsto V(t, x)$ est la solution de viscosité unique de*

$$\begin{array}{rcl} \frac{\partial w}{\partial t} + H(t, x, D_x w, D_x^2 w) & = & 0 \\ w(T, x) & = & g(x). \end{array} \tag{3.11}$$

3.3 Jeux différentiels stochastiques via EDSR

L'étude des jeux différentiels stochastiques via la théorie des EDSR a été initiée par Hamadène et Lepeltier dans [58], [59]. La contribution principale de la théorie des EDSR consiste en la possibilité de considérer les systèmes non-markoviens où les arguments des EDP de Fleming et Souganidis [52] ne peuvent pas être appliqués. Ces idées ont ensuite été généralisées à d'autres situations dans Hamadène, Lepeltier et Peng [61], El Karoui et Hamadène [42], Hamadène et Lepeltier [60] et Hamadène et Hassani [57].

En effet, d'après la définition (3.2), le paiement pour tout $(u, v) \in \mathcal{U}(t) \times \mathcal{V}(t)$ peut être écrit comme $J(t, x, u, v) = \mathbb{E}\left[Y_t^{t,x,u,v}\right]$, où $Y^{t,x,u,v}$ est la solution de l'EDSR

$$Y_s^{t,x,u,v} \quad = \quad g(X_T^{t,x,u,v}) + \int_s^T l(r, X_r^{t,x,u,v}, u_r, v_r)dr - \int_s^T \sigma^*(r, X_r^{t,x})Z_r^{t,x,u,v}dB_r \tag{3.12}$$

avec $X^{t,x,u,v}$ défini comme

$$\begin{array}{rcl} dX_s^{t,x,u,v} & = & b(s, X_s^{t,x,u,v}, u_s, v_s)ds + \sigma(s, X_s^{t,x,u,v}, u_s, v_s)dB_s \\ X_t^{t,x,u,v} & = & x. \end{array} \tag{3.13}$$

Dans Hamadène et Lepeltier [59] un jeu différentiel stochastique de somme nulle est considéré, où le coefficient de diffusion ne peut pas être contrôlé par les joueurs, c'est à dire

$$\sigma(t,x,u,v)=\sigma(t,x) \tag{3.14}$$

et σ est supposé être non dégénéré. L'idée de Hamadène et Lepeltier [59] est de considérer le jeu sous une transformation de Girsanov pour découpler la dynamique progressive du contrôle. Ensuite, il est possible de construire un couple de contrôles optimaux pour les joueurs en utilisant le principe de comparaison pour les EDSR. En inversant la transformation de Girsanov on obtient alors un point selle pour le jeu.

En fait, de manière équivalente à l' EDS (3.13) on peut considérer pour chaque $(u,v)\in\mathcal{U}(t)\times\mathcal{V}(t)$

$$\begin{aligned} dX_s^{t,x} &= \sigma(s,X^{t,x})dB_s \\ X_t^{t,x} &= x \end{aligned} \tag{3.15}$$

sous $d\mathbb{P}^{u,v}=\Gamma_T^{u,v}d\mathbb{P}$ avec

$$\Gamma_s^{u,v}=\mathcal{E}\left(\int_t^s b(r,X_r^{t,x},u_r,v_r)\sigma^*(r,X_r^{t,x})^{-1}dB_r\right), \tag{3.16}$$

où $\mathcal{E}$ est l'exponentielle de Doléans-Dade. Une fonctionnelle du paiement comme (3.2) peut alors être exprimée par

$$J(t,x,u,v)=\mathbb{E}_{\mathbb{P}^{u,v}}\left[Y_t^{t,x,u,v}\right] \tag{3.17}$$

où $Y_s^{t,x,u,v}$ résout l' EDSR

$$\begin{aligned} Y_s^{t,x,u,v} &= g(X_T^{t,x})+\int_s^T\left(l(r,X_r^{t,x},u_r,v_r)+b(r,X_r^{t,x},u_r,v_r)Z_r^{t,x,u,v}\right)dr \\ &\quad -\int_s^T\sigma^*(r,X_r^{t,x})Z_r^{t,x,u,v}dB_r \end{aligned} \tag{3.18}$$

avec le $\mathbb{P}$-mouvement Brownien B.

Comme dans Fleming et Souganidis [52], la condition d'Isaacs, ici trajectorielle, est supposée :

$$\begin{aligned} &\sup_{v\in V}\inf_{u\in U}\{\langle b(t,x,u,v),\xi\rangle+l(t,x,u,v)\} \\ &=\inf_{u\in U}\sup_{v\in V}\{\langle b(t,x,u,v),\xi\rangle+l(t,x,u,v)\}:=H(t,x,\xi) \end{aligned} \tag{3.19}$$

et il est possible de définir pour tout $t\in[0,T]$, $x\in\mathbb{R}^d$, $\xi\in\mathbb{R}^d$ les fonctions (aléatoires) $u^*(t,x,\xi)$, $v^*(t,x,\xi)$ respectivement, telles que :

$$\begin{aligned} &H(t,x,\xi)\geq\langle b(t,x,u^*(t,x,\xi),v),\xi\rangle+l(t,x,u^*(t,x,\xi),v) \quad \text{for all } v\in V \\ &H(t,x,\xi)\leq\langle b(t,x,u,v^*(t,x,\xi)),\xi\rangle+l(t,x,u,v^*(t,x,\xi)) \quad \text{for all } u\in U. \end{aligned} \tag{3.20}$$

On peut maintenant définir les processus

$$\begin{aligned} \bar{u}_s &= u^*(s,X_s^{t,x},Z_s^{t,x}) \\ \bar{v}_s &= v^*(s,X_s^{t,x},Z_s^{t,x}), \end{aligned} \tag{3.21}$$

où $Z^{t,x}$ est donnée comme une solution de

$$Y_s^{t,x} \quad = \quad g(X_T^{t,x}) + \int_s^T H(r, X_r^{t,x}, Z_r^{t,x})dr - \int_s^T \sigma^*(r, X_r^{t,x})Z_r^{t,x}dB_r. \tag{3.22}$$

En outre, il suit du principe de comparaison pour les EDSR que pour tout $u \in \mathcal{U}(t)$, $v \in \mathcal{V}(t)$, on a

$$Y_t^{t,x,\bar{u},v} \leq Y_t^{t,x} \leq Y_t^{t,x,u,\bar{v}} \qquad \mathbb{P}\text{-p.s.} \tag{3.23}$$

et depuis $\mathbb{E}\left[Y_t^{t,x}\right] = J(t, x, \bar{u}, \bar{v})$, on obtient un point selle, dans le sens suivant :

Théorème 3.5. *Pour chaque* $(u, v) \in \mathcal{U}(t) \times \mathcal{V}(t)$

$$J(t, x, \bar{u}, v) \leq J(t, x, \bar{u}, \bar{v}) \leq J(t, x, u, \bar{v}) \tag{3.24}$$

et on peut définir $V(t, x) = \mathbb{E}\left[Y_t^{t,x}\right]$ *comme la valeur du jeu différentiel stochastique.*

Il mérite d'être mentionné, que le couple de contrôles optimaux $(\bar{u}, \bar{v})$ est adapté, mais n'est pas, en général, sous une forme de "feed-back" comme la formule (3.21) pourrait l'indiquer, c'est à dire il n'est en général pas donné en fonction du temps t et de l'état du système à l'instant t. En effet, on définit $(\bar{u}, \bar{v})$ sous $\mathbb{P}$. Mais la vraie dynamique du jeu est donnée sous $\mathbb{P}^{\bar{u},\bar{v}}$. Ainsi, afin de calculer les stratégies optimales, on doit effectuer un changement de mesure, et en général $Z^{t,x}$ sous le changement de mesure peut tout aussi bien dépendre de toute l'histoire de $\bar{u}$ et $\bar{v}$. Pour une discussion détaillée de ce problème assez profond nous nous référons à l'article de Rainer [98].

D'un autre côté, si les coefficients sont des fonctions déterministes il est évident que la valeur du jeu défini par Fleming et Souganidis [52] coïncide avec celle trouvée dans Hamadène et Lepeltier, puisque la solution de l'EDSR (3.22) donne une solution de viscosité de l'équation HJI (3.11). En outre, si $V \in \mathcal{C}^{1,2}([t, T], \mathbb{R}^d)$, alors par unicité de la solution de l'EDSR (3.22) et (2.8)

$$Z_s^{t,x} \quad = \quad D_x V(s, X_s^{t,x}) \tag{3.25}$$

et les contrôles optimaux en "feed-back" sont donnés par

$$\begin{aligned} \bar{u}_s &= u^*(s, X_s^{t,x}, D_x V(s, X_s^{t,x})) \\ \bar{v}_s &= v^*(s, X_s^{t,x}, D_x V(s, X_s^{t,x})) \end{aligned} \tag{3.26}$$

sous $\mathbb{P}^{\bar{u},\bar{v}}$ et donc sous la dynamique du monde réel. Afin de jouer de façon optimale il faut mettre à chaque temps s la valeur réelle du système, qui est précisément $X^{t,x}$ sous $\mathbb{P}^{\bar{u},\bar{v}}$, dans (3.26).

3.4 Fleming Souganidis revisité

Nous avons vu dans la section précédente que la théorie de EDSR peut être utilisée pour étudier les jeux non-markoviens en établissant l'existence d'un point selle. Buckdahn et Li ont montré dans [19] qu'elle peut également être utilisée pour simplifier la preuve du Fleming et Souganidis [52], qui est techniquement assez lourde. En fait, ce dernier travail a pour désavantage que les contrôles de $\mathcal{U}(t)$ et de $\mathcal{V}(t)$ sont respectivement limités à ne pas dépendre des chemins du mouvement brownien avant le temps t. Cette restriction implique l'utilisation de techniques difficiles dans la preuve de Fleming et Souganidis [52]. Buckdahn et Li affaiblissent cette condition en imposant que les contrôles admissibles soient des processus mesurables par rapport à la plus grande filtration $(\mathcal{H}_s)_{s\in[t,T]}$.

De plus dans Buckdahn et Li [19] des fonctionnelles de coûts plus générales sont considérées

$$J(t,x,u,v) = Y_t^{t,x,u,v}, \tag{3.27}$$

où $Y_t^{t,x,u,v}$ est définie comme la solution à une EDSPR, $(X_t^{t,x,u,v}, Y_t^{t,x,u,v}, Z_t^{t,x,u,v})$ avec des coefficients qui sont, comme dans Fleming et Souganidis [52], des fonctions déterministes. Cependant pour u, v censés être $\mathcal{H}_s$-mesurable, contrairement à (2.11) la loi du 0-1 de Blumenthal ne s'applique pas, donc $Y_t^{t,x,u,v}$ n'est en général pas déterministe.

Une étape centrale dans le travail de Buckdahn et Li [19] est de montrer que la fonction valeur inférieure

$$V^-(t,x) = \text{essinf}_{\alpha\in\mathcal{A}(t)}\text{esssup}_{v\in\mathcal{V}(t)} Y_t^{t,x,\alpha,v}, \tag{3.28}$$

et la fonction valeur supérieure

$$V^+(t,x) = \text{esssup}_{\beta\in\mathcal{B}(t)}\text{essinf}_{u\in\mathcal{U}(t)} Y_t^{t,x,u,\beta}, \tag{3.29}$$

sont déterministes. Cela est accompli grâce une idée élégante, à savoir, en montrant que $V^+(t,x)$ et $V^-(t,x)$ sont invariantes par rapport aux variations dans l'espace de Cameron-Martin.

Le choix plus général sur (u,v) permet alors une preuve directe de la propriété de solution de viscosité pour les fonctions de valeur supérieure et inférieure de certaines équations HJI qui sont, sous une condition d'Isaacs, à nouveau égales. Les résultats de Buckdahn et Li [19] sont étendus à des cas plus généraux dans les travaux de Buckdahn et Li [20], Buckdahn, Hu et Li [18] et Lin [81].

4 Jeux différentiels stochastiques à information incomplète

4.1 Jeux à information incomplète

Le formalisme introduit par Aumann et Maschler [3] en 1968 considère des jeux à somme nulle et à information incomplète de structure suivante :

- Il y a I états de la nature différents dans lesquels le jeu peut avoir lieu. Avant le début du jeu un état est choisi avec une probabilité p, qui est connue.
- L'information est transmise au joueur 1, tandis que le joueur 2 ne sait que p.
- Le joueur 1 veut minimiser le gain, tandis que le joueur 2 veut le maximiser.
- Nous supposons que les deux joueurs observent le contrôle de leur adversaire.

Alors que dans les jeux à une étape la dernière hypothèse est redondante, dans les jeux répétés elle devient cruciale. Comme son adversaire observe le joueur informé, il est important pour lui de trouver à chaque étape le bon équilibre entre l'utilisation de l'information ou sa dissimulation en agissant de manière moins optimale afin d'être capable de l'utiliser à un stade ultérieur. En effet, il se trouve qu'il est optimal pour les joueurs de jouer au hasard, selon un aléatoire supplémentaire.

L'idée célèbre d'Aumann et Maschler [3] est alors de considérer un jeu à information incomplète comme un jeu aléatoire à information complète, où les deux joueurs ne connaissent pas l'état de la nature. A chaque étape tous les états de la nature sont joués simultanément avec une certaine probabilité. Cette probabilité reflète l'opinion du joueur non informé sur quel état de la nature a été choisi en fonction de ses informations actuelles et donne donc lieu à une martingale discrète. Puisque les croyances sont contrôlées par les actions du joueur informé, il agira tel que cette martingale donne un gain minimal des jeux qui sont joués simultanément.

Les jeux répétés à information incomplète d'un ou des deux côtés ont été largement étudiés depuis les travaux fondateurs d'Aumann et Maschler [3]. Il est resté jusqu'à aujourd'hui un champ de recherche très actif. Pour une analyse et de nombreuses références sur l'étude des jeux répétés à informations incomplètes, nous nous référons à l'ouvrage de Sorin [99]. Pour des études récentes dans ce domaine, nous pouvons citer les travaux de De Meyer and Rosenberg [34], De Meyer, Lehrer and Rosenberg [33], Gensbittel [55], Laraki [74], [75] et Sorin [100], [101]. En outre, une application pour les marchés boursiers peut être trouvés dans le travail de De Meyer [32].

4.2 Jeux différentiels stochastiques à information incomplète

Ce n'est que récemment que le formalisme d'Aumann et Maschler a été généralisé pour les jeux différentiels déterministes dans Cardaliaguet [23], [24] et pour les jeux différentiels stochastiques dans Cardaliaguet et Rainer [28]. Là, la dynamique du jeu est donnée par une diffusion contrôlée : pour $t \in [0,T], x \in \mathbb{R}^d$,

$$dX_s^{t,x,u,v} = b(s, X_s^{t,x,u,v}, u_s, v_s)ds + \sigma(s, X_s^{t,x,u,v}, u_s, v_s)dB_s, \qquad X_t^{t,x} = x. \tag{4.1}$$

Comme dans le modèle d'Aumann et Maschler il y a $I \in \mathbb{N}^*$ différents états de la nature correspondant à I différents

(i) coûts courants : $(l_i)_{i\in\{1,\dots,I\}} : [0,T] \times \mathbb{R}^d \times U \times V \to \mathbb{R}$ et

(ii) coûts finals : $(g_i)_{i\in\{1,\dots,I\}} : \mathbb{R}^d \to \mathbb{R}$.

Avant que le jeu commence, un de ces états est choisi selon une probabilité $p \in \Delta(I)$, où $\Delta(I)$ désigne le simplexe de $\mathbb{R}^I$. L'information est transmise au joueur 1 uniquement. Le joueur 1 cherche à minimiser le profit espéré, le joueur 2 à le maximiser. Nous supposons que les deux joueurs observent le contrôle de leur adversaire.

Comme dans le cas de jeux différentiels à information complète, il est supposé qu' une condition d'Isaacs est satisfaite. Dans le cas d'information incomplète, elle s'écrit comme suit :

$$\begin{aligned}&\sup_{v\in V} \inf_{u\in U} \left\{\langle b(t,x,u,v),\xi\rangle + \tfrac{1}{2}\mathrm{tr}(\sigma\sigma^*(t,x,u,v)A) + \langle p, l(t,x,u,v)\rangle\right\}\\ &\quad= \inf_{u\in U} \sup_{v\in V} \left\{\langle b(t,x,u,v),\xi\rangle + \tfrac{1}{2}\mathrm{tr}(\sigma\sigma^*(t,x,u,v)A) + \langle p, l(t,x,u,v)\rangle\right\} \qquad (4.2)\\ &\quad:= H(t,x,p,\xi,A)\end{aligned}$$

La définition des stratégies admissibles diffère légèrement de Fleming et Souganidis [52] :

Définition 4.1. *Pour tout $t \in [0,T]$ un contrôle admissible $u = (u_s)_{s\in[t,T]}$ pour le joueur 1 est un processus càdlàg progressivement mesurable par rapport à la filtration $(\mathcal{H}_{t,s})_{s\in[t,T]}$ à valeurs dans U. L'ensemble des contrôles admissibles pour le joueur 1 est désigné par $\mathcal{U}(t)$.*
La définition des contrôles admissible $v = (v_s)_{s\in[t,T]}$ pour le joueur 2 est similaire. L'ensemble des contrôles admissibles pour le joueur 2 est désigné par $\mathcal{V}(t)$.

Comme dans le cas des jeux répétés les joueurs apprennent et s'adaptent à l'information qu'ils apprennent, et donc une valeur définie en jouant stratégie contre contrôle comme dans Fleming et Souganidis [52] n'est pas suffisante. Afin de permettre une interaction, donc de jouer stratégie contre stratégie avec une valeur qui est toujours définie, Cardaliaguet et Rainer adaptent dans [28] la notion de stratégies non-anticipatives avec délai introduite dans Buckdahn, Cardaliaguet, Rainer [17] au cadre stochastique.
En outre on doit prendre en compte le fait que le joueur informé essaie de cacher ses informations. Pour ce faire, il doit être autorisé à ajouter de l'aléatoire à son comportement. Nous constatons qu'il est aussi raisonnable de permettre au jouer non informé d'utiliser des stratégies aléatoires. Comme montré pour les jeux déterministes par Souquière dans [103], le joueur non informé joue aussi de manière aléatoire, afin de se rendre moins vulnérable à la manipulation.
Ces deux caractéristiques nécessaires pour les stratégies dans les jeux à information incomplète sont incorporées dans la définition suivante de Cardaliaguet et Rainer [28] :

Nous désignons U_t, respectivement V_t, l'ensemble des fonctions càdlàg de $[t,T]$ dans U, respectivement V. Soit $\mathcal{I}$ un ensemble fixe d' espaces de probabilité qui est non trivial et stable par produit fini.

Définition 4.2. *Une stratégie aléatoire pour le joueur 1 à l'instant $t \in [0,T]$ est une paire $((\Omega_\alpha, \mathcal{G}_\alpha, \mathbb{P}_\alpha), \alpha)$, où $(\Omega_\alpha, \mathcal{G}_\alpha, \mathbb{P}_\alpha)$ est un espace de probabilité de $\mathcal{I}$ et où $\alpha : [t,T] \times \Omega_\alpha \times \mathcal{C}([t,T];\mathbb{R}^d) \times V_t \to U_t$ satisfait*

(i) α est une fonction mesurable, où Ω_α est muni de la tribu $\mathcal{G}_\alpha$,

(ii) il existe $\delta > 0$ tel que pour tout $s \in [t,T]$ et pour tout $\omega, \omega' \in \mathcal{C}([t,T];\mathbb{R}^d)$ et $v, v' \in V_t$ on a :

$$\omega = \omega' \text{ et } v = v' \text{ presque partout sur } [t,s]$$
$$\Rightarrow \alpha(\cdot, \omega, v) = \alpha(\cdot, \omega', v') \text{ presque partout sur } [t, s+\delta] \text{ pout tout } \omega_\alpha \in \Omega_\alpha.$$

L'ensemble des stratégies aléatoires pour le joueur 1 est désigné par $\mathcal{A}^r(t)$.
La définition de stratégies aléatoires pour le joueur 2 $((\Omega_\beta, \mathcal{G}_\beta, \mathbb{P}_\beta), \beta)$, où $\beta : [t,T] \times \Omega_\beta \times \mathcal{C}([t,T];\mathbb{R}^d) \times U_t \to V_t$, est similaire.

Nous notons que la définition 4.2. est, contrairement à la définition 3.2 de Fleming et Souganidis [52], une définition trajectorielle. Par conséquent, pour s'assurer que l' EDS (4.1.) est bien posée, nous devons assumer plus de régularité sur les contrôles afin de construire une intégrale stochastique trajectorielle dans (4.1). Pour une étude concise des constructions trajectorielles nous nous référons à Karandikar [66].

Dans Cardaliaguet et Rainer [28] Lemma 2.1. il est démontré que, grâce à ce délai, il est possible d'associer pour tout $(\omega_\alpha, \omega_\beta) \in \Omega_\alpha \times \Omega_\beta$ à chaque couple de stratégies aléatoires $(\alpha, \beta) \in \mathcal{A}^r(t) \times \mathcal{B}^r(t)$ un couple unique de stratégies admissibles $(u^{\omega_\alpha,\omega_\beta}, v^{\omega_\alpha,\omega_\beta}) \in \mathcal{U}(t) \times \mathcal{V}(t)$, tel que pour tout $\omega \in \mathcal{C}([t,T];\mathbb{R}^d)$, $s \in [t,T]$,

$$\alpha(s, \omega_\alpha, \omega, v^{\omega_\alpha,\omega_\beta}(\omega)) = u_s^{\omega_\alpha,\omega_\beta}(\omega) \quad \text{ et } \quad \beta(s, \omega_\beta, \omega, u^{\omega_\alpha,\omega_\beta}(\omega)) = v_s^{\omega_\alpha,\omega_\beta}(\omega)\ .$$

Donc pour tout $(t,x,p) \in [0,T] \times \mathbb{R}^d \times \Delta(I)$, $(\bar{\alpha}_1, \ldots, \bar{\alpha}_I) \in (\mathcal{A}^r(t))^I$, $\beta \in \mathcal{B}^r(t)$ le gain

$$J(t,x,p,\bar{\alpha},\beta) = \sum_{i=1}^{I} p_i\ \mathbb{E}_{\bar{\alpha}_i,\beta}\left[\int_t^T l_i(s, X_s^{t,x,u_i,v_i}, (u_i)_s, (v_i)_s)ds + g_i(X_T^{t,x,u_i,v_i})\right] \tag{4.3}$$

avec (u_i, v_i), tel que $u_i = \alpha_i(v_i), v_i = \beta(u_i)$, est bien défini. Nous remarquons que l'avantage d'informations du joueur 1 est traduit dans (4.3) par la possibilité de choisir une stratégie $\bar{\alpha}_i$ pour chaque état de nature $i \in \{1, \dots, I\}$.

Comme dans Fleming et Souganidis [52] on peut maintenant définir la valeur inférieure, respectivement la valeur supérieure, d'un jeu différentiel stochastique à information incomplète comme

$$\begin{aligned} V^-(t,x,p) &= \sup_{\beta \in \mathcal{B}^r(t)} \inf_{\bar{\alpha} \in (\mathcal{A}^r(t))^I} J(t,x,p,\bar{\alpha},\beta) \\ V^+(t,x,p) &= \inf_{\bar{\alpha} \in (\mathcal{A}^r(t))^I} \sup_{\beta \in \mathcal{B}^r(t)} J(t,x,p,\bar{\alpha},\beta). \end{aligned} \tag{4.4}$$

Par la définition même il suit que $V^-(t,x,p) \leq V^+(t,x,p)$. Pour montrer que le jeu a une valeur, l'inégalité inverse est établie par Cardaliaguet et Rainer dans [28] en utilisant la théorie des solutions de viscosité. Cependant, la fonction valeur satisfait - contrairement à l'équation de HJI (3.11) pour les jeux différentiels stochastiques à information complète - une équation de HJI avec une contrainte de convexité dans la variable p. Le résultat de Cardaliaguet et Rainer dans [28] ainsi que la caractérisation grâce aux EDP par Cardaliaguet [25] sont résumés dans :

Théorème 4.3. *Pour tout $(t,x,p) \in [0,T] \times \mathbb{R}^d \times \Delta(I)$ le jeu différentiel stochastique à information incomplète a une valeur $V(t,x,p)$. La fonction $V : [0,T] \times \mathbb{R}^d \times \Delta(I) \to \mathbb{R}$ est l'unique solution de viscosité de*

$$\begin{aligned} \min\left\{ \frac{\partial w}{\partial t} + H(t,x,D_x w, D_x^2 w, p), \lambda_{\min}\left(p, \frac{\partial^2 w}{\partial p^2}\right)\right\} &= 0 \\ w(T,x,p) &= \textstyle\sum_i p_i g_i(x), \end{aligned} \tag{4.5}$$

où pour tout $p \in \Delta(I)$, $A \in \mathcal{S}^I$

$$\lambda_{\min}(p,A) := \min_{z \in T_{\Delta(I)(p)} \setminus \{0\}} \frac{\langle Az, z\rangle}{|z|^2}$$

et $T_{\Delta(I)(p)}$ désigne le cône tangent à $\Delta(I)$ en p, soit $T_{\Delta(I)(p)} = \overline{\cup_{\lambda>0}(\Delta(I)-p)/\lambda}$.

4.3 Représentation duale pour les jeux différentiels stochastiques

L'objectif est maintenant d'établir un analogue à la représentation duale d'Aumann et Maschler [3] pour le cas des jeux différentiels stochastiques. Un exemple de jeu déterministe en temps continu est considéré dans Cardaliaguet et Rainer [27] en utilisant une minimisation sur des mesures de martingales. Une technique similaire est introduite dans De Meyer [32] dans le cadre de marchés financièrs avec des agents informés.
Dans le chapitre 3, nous généralisons ce dernier résultat au cas où la dynamique est donnée, comme dans Hamadène et Lepeltier [59], par une diffusion contrôlée, c'est à dire pour $t \in [0,T], x \in \mathbb{R}^d$, on a :

$$dX_s^{t,x,u,v} = b(s, X_s^{t,x,u,v}, u_s, v_s)ds + \sigma(s, X_s^{t,x,u,v})dB_s \qquad X_t^{t,x} = x. \tag{4.6}$$

Comme dans Hamadène et Lepeltier [59], il s'avère essentiel d'assumer la condition de non-dégénérescence pour $\sigma(t,x)$.

Suivant les notations de Cardaliaguet et Rainer [28] nous avons pour chaque donnée initiale $(t, x, p) \in [0, T] \times \mathbb{R}^d \times \Delta(I)$ et chaque stratégie aléatoire du joueur informé $\bar{\alpha} \in (\mathcal{A}^r(t))^I$ et du joueur non informé $\beta \in \mathcal{B}^r(t)$ un gain espéré

$$J(t, x, p, \bar{\alpha}, \beta) = \sum_{i=1}^{I} p_i \ \mathbb{E}_{\bar{\alpha}_i, \beta} \left[\int_t^T l_i(s, X_s^{t,x,\bar{\alpha}_i,\beta}, (\bar{\alpha}_i)_s, \beta_s) ds + g_i(X_T^{t,x,\bar{\alpha}_i,\beta}) \right], \tag{4.7}$$

tandis que la condition d'Isaacs s'écrit ici :

$$\begin{aligned} &\inf_{u \in U} \sup_{v \in V} \{ \langle b(t, x, u, v), \xi \rangle + \langle p, l(t, x, u, v) \rangle \} \\ &= \sup_{v \in V} \inf_{u \in U} \{ \langle b(t, x, u, v), \xi \rangle + \langle p, l(t, x, u, v) \rangle \} \\ &=: H(t, x, \xi, p). \end{aligned} \tag{4.8}$$

Pour donner une représentation duale, il faut grossir l'espace canonique $\mathcal{C}([0, T]; \mathbb{R}^d)$ de la dynamique brownienne, en ajoutant $\mathcal{D}([0, T]; \Delta(I))$, où $\mathcal{D}([0, T]; \Delta(I))$ désigne l'ensemble des fonctions càdlàg de $\mathbb{R}$ dans $\Delta(I)$, qui sont constantes sur $(-\infty, 0)$ et sur $[T, +\infty)$. Nous désignons par $\mathbf{p}_s(\omega_p) = \omega_p(s)$ l'application coordonnée sur $\mathcal{D}([0, T]; \Delta(I))$ et par $\mathcal{G} = (\mathcal{G}_s)$ la filtration engendrée par $s \mapsto \mathbf{p}_s$. Nous munissons l'espace produit $\Omega := \mathcal{D}([0, T]; \Delta(I)) \times \mathcal{C}([0, T]; \mathbb{R}^d)$ de la filtration $\mathcal{F} = \mathcal{G} \otimes \mathcal{H}$, où $\mathcal{F}_t = \cap_{s>t} \mathcal{F}_s^0$ avec $(\mathcal{F}_s^0) = (\mathcal{G}_s \otimes \mathcal{H}_s)$. (Nous rappellons que $(\mathcal{H}_s)$ ètè definie comme la filtration engendrée par le processus canonique B.) Dans ce qui suit, chaque fois que nous travaillons avec une probabilité fixe $\mathbb{P}$ sur Ω, nous complétons la filtration $\mathcal{F}$ par rapport des ensembles de mesure nulle sans changer la notation.

Nous allons munir l'espace filtré Ω de mesures $\mathbb{P}$ permettant de modéliser les croyances du joueur non informé par le processus ajouté $\mathbf{p}$. Avant que le jeu commence, l'information du joueur non informé est juste la distribution initiale p. À la fin du jeu, l'information est révélée donc $\mathbf{p}_T \in \{e_i, i = 1, \ldots, I\}$. Comme l'état est choisi avant que le jeu commence, $\mathbf{p}_T$ est indépendante de $(B_s)_{s \in (-\infty, T]}$. Enfin, la propriété de martingale, $\mathbf{p}_t = \mathbb{E}_{\mathbb{P}}[\mathbf{p}_T | \mathcal{F}_t]$, est satisfaite à cause de la meilleure estimation sur l'état réel de la nature du joueur non informé. Ces caractéristiques sont intégrées dans la définition suivante :

Définition 4.4. *Soit $p \in \Delta(I)$, $t \in [0, T]$. Nous désignons par $\mathcal{P}(t, p)$ l'ensemble des mesures de probabilité $\mathbb{P}$ sur Ω telles que, sous $\mathbb{P}$,*
- *(i) $\mathbf{p}$ est un martingale, telle que*
  - *(a) $\mathbf{p}_s = p \ \forall s < t$,*
  - *(b) $\mathbf{p}_s \in \{e_i, i = 1, \ldots, I\} \ \forall s \geq T$ $\mathbb{P}$-p.s. et*
  - *(c) $\mathbf{p}_T$ est indépendant de $(B_s)_{s \in (-\infty, T]}$,*
- *(ii) $(B_s)_{s \in [0, T]}$ est un mouvement brownien.*

Vu que dans notre cas, l'Hamiltonien $H(t, x, \xi, p)$ définie par (4.8) dépend d'un paramètre supplémentaire $\xi \in \mathbb{R}^d$, une représentation duale directe utilisant l' Hamiltonien comme dans Cardaliaguet et Rainer [27] est impossible. Inspiré par Hamadène et Lepeltier [59], nous utilisons la théorie des EDSR pour résoudre ce problème. Pour tout $t \in [0, T]$, $x \in \mathbb{R}^d$ on définit le processus $X^{t,x}$ par

$$X_s^{t,x} = x \quad s < t, \qquad X_s^{t,x} = x + \int_t^s \sigma(r, X_r^{t,x}) dB_r \quad s \geq t. \tag{4.9}$$

Soit $p \in \Delta(I)$. Nous considérons pour chaque $\mathbb{P} \in \mathcal{P}(t,p)$ l'EDSR

$$\begin{aligned} Y_s^{t,x,\mathbb{P}} &= \langle \mathbf{p}_T, g(X_T^{t,x})\rangle + \int_s^T H(r, X_r^{t,x}, Z_r^{t,x,\mathbb{P}}, \mathbf{p}_r)dr \\ &\quad - \int_s^T \sigma^*(r, X_r^{t,x}) Z_r^{t,x,\mathbb{P}} dB_r - N_T + N_s, \end{aligned} \tag{4.10}$$

où N est une martingale de carré intégrable qui est fortement orthogonale à B.

En particulier, on a

$$Y_{t-}^{t,x,\mathbb{P}} = \mathbb{E}_{\mathbb{P}}\left[\int_t^T H(r, X_r^{t,x}, Z_r^{t,x,\mathbb{P}}, \mathbf{p}_r)dr + \langle \mathbf{p}_T, g(X_T^{t,x})\rangle \big| \mathcal{F}_{t-}\right]. \tag{4.11}$$

Comme dans Hamadène et Lepeltier [59], on peut voir que $Y_{t-}^{t,x,\mathbb{P}}$ est la valeur (aléatoire) d'un jeu différentiel stochastique à information complète avec une dynamique progressive supplémentaire $\mathbf{p}$.

Dans le chapitre 3 de cette thèse, la représentation duale suivante est établie :

Théorème 4.5. *Pour tout $(t,x,p) \in [0,T] \times \mathbb{R}^d \times \Delta(I)$ la valeur du jeu à information incomplète $V(t,x,p)$ peut être caractérisée comme*

$$V(t,x,p) = \text{essinf}_{\mathbb{P} \in \mathcal{P}(t,p)} Y_{t-}^{t,x,\mathbb{P}}. \tag{4.12}$$

Nous remarquons que nous pouvons identifier chaque $\mathbb{P} \in \mathcal{P}(t,p)$ sur $\mathcal{F}_{t-}$ à une mesure de probabilité commune $\mathbb{Q} = \delta(p) \otimes \mathbb{P}_0$, où $\delta(p)$ est la mesure sous laquelle $\mathbf{p}$ est constante et égale à p et $\mathbb{P}_0$ est la mesure de Wiener. Donc le terme de droite de l'équation (4.12) est défini $\mathbb{Q}$ -p.s. mais, a priori, n'est pas déterministe. Pour établir que $\text{essinf}_{\mathbb{P} \in \mathcal{P}(t,p)} Y_{t-}^{t,x,\mathbb{P}}$ est déterministe, une étape essentielle dans la preuve du théorème 4.5. est d'adapter les idées de Buckdahn et Li [19] à notre cas.

4.4 "Stratégies" optimales dans le cas stochastique

Nous avons avec le théorème 4.5. une représentation pour le jeu différentiel stochastique à information incomplète, mais - comme dans Hamadène et Lepeltier [59] dans le cas d'informations complète - avec une dynamique qui est donnée sous une transformation de Girsanov du monde réel. Ainsi, afin d'utiliser la représentation (4.12) pour étudier le jeu et décrire le comportement optimal du joueur informé, comme dans l'exemple de Cardaliaguet et Rainer [27], nous devons inverser cette transformation.

Ceci est en effet possible : nous fournissons dans le chapitre 3 de cette thèse un résultat sous l'hypothèse supplémentaire que $V \in \mathcal{C}^{1,2,2}([t,T) \times \mathbb{R}^d \times \Delta(I); \mathbb{R})$ et qu'il existe un $\bar{\mathbb{P}} \in \mathcal{P}(t,p)$, tel que

$$V(t,x,p) = Y_{t-}^{t,x,\bar{\mathbb{P}}}. \tag{4.13}$$

Grâce à la condition d'Isaacs, on peut définir la fonction $u^*(t,x,p,\xi)$ comme une sélection Borel mesurable de $\text{argmin}_{u \in U} \max_{v \in V} \{\langle b(t,x,u,v), \xi\rangle + \langle p, l(t,x,u,v)\rangle\}$, donc

$$H(t,x,\xi,p) = \max_{v \in V} \left\{\langle b(t,x,u^*(t,x,p,\xi),v), \xi\rangle + \langle p, l(t,x,u^*(t,x,p,\xi),v)\rangle\right\}. \tag{4.14}$$

Comme dans (3.26), nous définissons le processus

$$\bar{u}_s = u^*(s, X_s^{t,x}, D_x V(s, X_s^{t,x}, \mathbf{p}_s), \mathbf{p}_s), \tag{4.15}$$

et pour tout $\beta \in \mathcal{B}(t)$ la mesure équivalente $\bar{\mathbb{P}}^{\bar{u},\beta} = (\Gamma_T^{\bar{u},\beta})\bar{\mathbb{P}}$ avec

$$\Gamma_s^{\bar{u},\beta} = \mathcal{E}\left(\int_t^s b(r, X_r^{t,x}, \bar{u}_r, \beta(\bar{u})_r)\sigma^*(r, X_r^{t,x})^{-1} dB_r\right),$$

si $s \geq t$, et $\Gamma_s^{\bar{u},\beta} = 1$, si $s < t$.
Puisque le joueur informé connaît l'état de la nature, il jouera en sachant le résultat du choix de l'état de la nature au début du jeu : pour tout $A \in \mathcal{F}$ on pose

$$\bar{\mathbb{P}}_i^{\bar{u},\beta(\bar{u})}[A] = \bar{\mathbb{P}}^{\bar{u},\beta(\bar{u})}[A|\mathbf{p}_T = e_i] = \frac{1}{p_i}\bar{\mathbb{P}}^{\bar{u},\beta(\bar{u})}[A \cap \{\mathbf{p}_T = e_i\}], \quad \text{si } p_i > 0,$$

et $\bar{\mathbb{P}}_i^{\bar{u},\beta(\bar{u})}[A] = \bar{\mathbb{P}}^{\bar{u},\beta(\bar{u})}[A]$ sinon.

Dans le chapitre 3 nous démontrons :

Théorème 4.6. *Pour chaque état de la nature $i = 1, \ldots, I$ et toute stratégie $\beta \in \mathcal{B}(t)$ du joueur non informé, il est optimal pour le joueur informé de jouer*

$$\bar{u}_s = u^*(s, X_s^{t,x}, D_x V(s, X_s^{t,x}, \mathbf{p}_s), \mathbf{p}_s) \quad \textit{avec la probabilité } \bar{\mathbb{P}}_i^{\bar{u},\beta(\bar{u})}. \tag{4.16}$$

Nous voudrions cependant mentionner que le résultat du théorème 4.6. a quelques subtilités. Contrairement au cas à information complète, où (3.26) donne des contrôles “feed-back” optimaux, (4.16) n'en donne en général pas. En fait $\bar{u}$ dépend de l'état du système, c'est à dire de $X^{t,x}$ sous $\bar{\mathbb{P}}_i^{\bar{u},\beta(\bar{u})}$ et de la randomisation $\mathbf{p}$ qui est transformée sous la mesure optimale $\bar{\mathbb{P}}_i^{\bar{u},\beta(\bar{u})}$. Puisque ce changement dépend de la stratégie β du joueur non-informé, nous ne trouvons pas un contrôle de type “feed-back” aléatoire, mais une sorte de stratégie aléatoire pour le joueur informé, ce qui n'est pas compatible avec les stratégies aléatoires exprimées dans la Définition 4.2. Pour obtenir une telle stratégie aléatoire, il serait nécessaire de démontrer certaines propriétés de la mesure optimale $\bar{\mathbb{P}}$.

4.5 Un résultat de régularité

Un fait assez remarquable, est que la représentation dans le théorème 4.5., via des solutions d' EDSR, nous donne la possibilité d'obtenir, avec des outils probabilistes, un résultat de régularité pour la fonction valeur V. En effet en supposant une régularité supplémentaire pour les coefficients :
(i) $(g_i)_{i\in\{1,\ldots,I\}} : \mathbb{R}^d \to \mathbb{R}$ est différentiable par rapport à x à dérivée bornée, continue Lipschitz
(ii) $\sigma : [0,T] \times \mathbb{R}^d \to \mathbb{R}^{d\times d}$ est pour tout $t \in [0,T]$ différentiable par rapport à x à dérivée bornée, continue uniformément Lipschitz
(iii) $H : [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \times \Delta(I) \to \mathbb{R}$ est pour tout $t \in [0,T]$ différentiable par rapport à x et z à dérivées bornées, continues uniformément Lipschitz
nous démontrons dans le chapitre 4 :

Théorème 4.7. *La fonction valeur V est semiconcave en x avec un module linéaire.*

La preuve ressemble beaucoup aux preuves de régularité des EDP semi-linéaires par des techniques d' EDSR dans Pardoux et Peng [89]. Cependant, compte tenu de (4.12), on ne peut probablement pas s'attendre à ce que la fonction valeur soit lisse, en particulier aux points où il pourrait y avoir plusieurs mesures minimisantes. Dans ce contexte, la notion naturelle de régularité est la semiconcavité, qui est une propriété précieuse si on considère les problèmes de contrôle (voir par exemple le livre de Cannarsa et Sinestrari [21]). Notamment par le théorème de Alexandroff on peut on en conclure :

Corollaire 4.8. *V est deux fois différentiable p.p. en x, soit pour tout $t \in [0,T], p \in \Delta(I)$ presque tout $x_0 \in \mathbb{R}^d$ il existe $\xi \in \mathbb{R}^d$, $A \in \mathcal{S}^I$, tel que*

$$\lim_{x \to x_0} \frac{V(t,x,p) - V(t,x_0,p) - \langle \xi, x - x_0 \rangle + \langle A(x-x_0), x - x_0 \rangle}{|x - x_0|^2} = 0. \tag{4.17}$$

En outre, le gradient $D_x V(t,x,p)$ est définie p.p. et appartient à la classe des fonctions à variation localement bornée.

5 Approximation de jeux différentiels stochastiques à information incomplète

5.1 Approximation de jeux différentiels stochastiques et des équations d'HJI associées

L'approximation des jeux différentiels déterministes à information complète remonte aux années 1960. Comme nous l'avons déjà mentionné dans la section 3, celle-ci était effectivement utilisée pour définir des valeurs des jeux en temps continu par une approximation avec des jeux en temps discret. Avec cette approche, il est même possible de dériver des contrôles "feed-back" ϵ-optimaux pour des jeux déterministes en temps continu. Là, il suffit que le joueur agisse uniquement sur une grille en temps discret suffisamment fine. Les résultats et de nombreuses références peuvent être trouvés dans le livre de Krasovskiĭ et Subbotin [69].

Pour l'approximation numérique de jeux différentiels stochastiques, une méthode d'approximation par des chaînes de Markov est largement utilisée, comme décrit dans le livre de Kushner et Dupuis [72]. La preuve de convergence utilise généralement des techniques d' EDP. Une preuve purement probabiliste de la convergence se trouve dans Kushner [70]. Cependant le calcul des contrôles "feed-back" ϵ-optimaux pour des jeux différentiels stochastiques avec un schéma numérique est plus délicat en raison de la nature stochastique des jeux. Limiter le premier joueur à jouer sur une grille de temps tandis que l'autre peut encore agir et s'adapter au bruit brownien sur les intervalles, pourrait offrir au deuxième joueur la possibilité de faire des profits de manière disproportionnée, à moins qu'il n'y ait d'autres hypothèses restrictives remplies.

Dans l'esprit de l'approche par des EDSR pour des jeux différentiels stochastiques de Hamadène and Lepeltier in [59], Bally dérive dans [4] une méthode pour approcher la fonction valeur d'un jeu différentiel stochastique par l'approximation des solutions d' EDSR. Dans l'article de Bally [4] l'approximation est - contrairement à l'approximation par des chaînes de Markov - sous une transformation de Girsanov du système, c'est-à-dire l'approximation d'une EDSR avec l' Hamiltonien comme générateur. Des contrôles "feed-back" ϵ-optimaux pour les deux joueurs sont ensuite dérivés en inversant la transformation. Cependant les hypothèses de Bally [4] pour établir l' ϵ-optimalité sont encore plutôt restrictives.

Outre l'approche de Bally dans [4], il existe de nombreuses autres méthodes d'approximation pour les EDSR. Parmi les premiers à poursuivre le développement de la théorie étaient Bouchard et Touzi [15] et Zhang [108] en 2001. Des travaux ultérieurs sur des approximations pour les EDSR et leurs relations avec les EDP sont donnés par Bender et Denk [8], Bender et Zhang [9], Delarue et Menozzi [36], [35]. En fait, compte tenu du lien étroit des EDP avec les EDSR - l'approximation d'une EDSR peut être considérée comme une approximation complètement probabiliste pour les solutions d' EDP semi-linéaires, y compris l'équation de HJI (3.11) qui caractérise les jeux différentiels stochastiques.

D'autre part, on peut considérer directement une approximation des solutions de viscosité d'EDP semi-linéaires et complètement non-linéaires. Les conditions pour la convergence des schémas d'approximation, notamment une condition de monotonie, sont étudiées par Barles et Souganidis dans [7]. D'une manière très naturelle de telles approximations donnent lieu à des schémas d'approximation pour les EDSR. Des schémas d'approximation monotones sont également appliqués dans le travail récent de Fahim, Touzi and Warin [48], où des EDP paraboliques complètement non-linéaires sont considérées.

5.2 Schéma numérique pour des jeux différentiels stochastiques à information incomplète

L' approximation des jeux différentiels déterministes à information incomplète a été étudié par Cardaliaguet dans [26]. Le cas déterministe à information incomplète des deux côtés a été examiné par Souquière dans [102]. Dans le chapitre 5, nous étendons l'approximation au cas des jeux différentiels stochastiques, où nous considérons comme dans la section 4.2 une diffusion avec une dérive contrôlée, mais une volatilité incontrôlée :

$$dX_s^{t,x,u,v} = b(s, X_s^{t,x,u,v}, u_s, v_s)ds + \sigma(s, X_s^{t,x,u,v})dB_s \qquad X_t^{t,x} = x. \tag{5.1}$$

Là encore, pour chaque donnée initiale $(t, x, p) \in [0, T) \times \mathbb{R}^d \times \Delta(I)$ et chaque stratégie aléatoire du joueur informé $\bar{\alpha} \in (\mathcal{A}^r(t))^I$ et du joueur non informé $\beta \in \mathcal{B}^r(t)$, nous avons un gain espéré

$$J(t, x, p, \bar{\alpha}, \beta) = \sum_{i=1}^{I} p_i \ \mathbb{E}_{\bar{\alpha}_i,\beta}\left[\int_t^T l_i(s, X_s^{t,x,\bar{\alpha}_i,\beta}, (\bar{\alpha}_i)_s, \beta_s)ds + g_i(X_T^{t,x,\bar{\alpha}_i,\beta})\right], \tag{5.2}$$

où la condition d'Isaacs est supposée

$$\begin{aligned} &\inf_{u\in U} \sup_{v\in V} \{\langle b(t, x, u, v), \xi\rangle + \langle p, l(t, x, u, v)\rangle\} \\ &= \sup_{v\in V} \inf_{u\in U} \{\langle b(t, x, u, v), \xi\rangle + \langle p, l(t, x, u, v)\rangle\} \\ &=: H(t, x, \xi, p) \end{aligned} \tag{5.3}$$

ainsi que la non-dégénérescence de $\sigma(t, x)$.

En contraste avec Cardaliaguet [26] et Souquière [102], nous pouvons utiliser cette dernière hypothèse pour travailler, comme Bally dans [4], sur le problème sous une transformation de Girsanov. Nous considérons ensuite un algorithme stochastique, qui est très proche de celui étudié dans Fahim, Touzi et Warin [48], pour une EDP semi-linéaire avec une convexification en p à chaque étape de temps. L'algorithme est construit comme suit :

Pour $L \in \mathbb{N}$, on définit une partition de $[0,T]$ $\Pi^\tau = \{0 = t_0, t_1, \ldots, t_L = T\}$ avec le pas de discrétisation $\tau = \frac{T}{L}$. Nous approchons la fonction valeur en reculant dans le temps. Pour cela nous posons pour tout $x \in \mathbb{R}^d$, $p \in \Delta(I)$

$$V^\tau(t_L, x, p) = \langle p, g(x) \rangle \tag{5.4}$$

et nous définissons de façon récursive pour $k = L-1, \ldots, 0$

$$\begin{aligned} V^\tau(t_k, x, p) = \mathrm{Vex}_p\big(\mathbb{E}\left[V^\tau(t_{k+1}, x + \sigma(t_k, x)\Delta B^k, p)\right] \\ + \tau H(t_k, x, \bar{z}_k(x,p), p)\big), \end{aligned} \tag{5.5}$$

où $\Delta B^j = B_{t_{j+1}} - B_{t_j}$ et $\bar{z}_k(x,p)$ est donné par

$$\bar{z}_k(x,p) = \frac{1}{\tau}\mathbb{E}\left[V^\tau(t_{k+1}, x + \sigma(t_k, x)\Delta B^k, p)(\sigma^*)^{-1}(t_k, x)\Delta B^k\right] \tag{5.6}$$

et Vex_p désigne l'enveloppe convexe par rapport à la variable p, c'est-à-dire la plus grande fonction qui est convexe dans la variable p, et ne dépasse pas la fonction donnée.

Comme dans Barles et Souganidis [7], nous montrons dans le chapitre 5 de cette thèse la convergence du schéma vers la valeur du jeu :

Théorème 5.1. *V^τ converge uniformément sur les compacts de $[0,T] \times \mathbb{R}^d \times \Delta(I)$ vers $V(t,x,p)$, en ce sens que*

$$\lim_{\tau \downarrow 0,\ t_k \to t,\ x' \to x,\ p' \to p} V^\tau(t_k, x', p') = V(t,x,p). \tag{5.7}$$

6 Jeux d'arrêt optimal à information incomplète

6.1 Jeux de Dynkin : histoire et résultats généraux

Les jeux de Dynkin ont été introduits par Dynkin en 1969 dans [39] comme un problème de jeu d'arrêt optimal. Le jeu est joué par deux joueurs adverses qui veulent respectivement minimiser ou maximiser un certain profit. Contrairement aux jeux que nous avons décrits dans les sections précédentes, les joueurs ont la possibilité d'arrêter le jeu à tout moment en se soumettant à une certaine pénalité. Ce problème a beaucoup attiré l'attention des scientifiques travaillant dans le domaine des probabilités, ainsi que dans la théorie des EDP. Parmi ces derniers, les travaux de Bensoussan et Lions [11], Bensoussan et Friedmann [10], Friedman [53] étaient les premiers à considérer les jeux d'arrêt optimal en temps continu en établissant une relation avec les EDP variationnelles.

Outre l'approche analytique, diverses méthodes purement probabilistes sont appliquées pour étudier les jeux de Dynkin (voir par exemple Alario-Nazaret, Lepeltier et Marchal [1], Bismut [13], Ekström et Peskir [40], Eckström et Villeneuve [41], Lepeltier et Maingueneau [79], Morimoto [85], Stettner [104] et le travail très récent de Kobylanski, Quenez et de Campagnolle [68]). En combinaison avec des diffusions contrôlées, des méthodes d' EDSR ont également été appliquées par Cvitanic et Karatzas [31], Hamadène et Lepeltier [60]. Là, le jeu d'arrêt optimal mène à l'étude des EDSR réfléchies introduites dans El Karoui, Kapoudjian, Pardoux, Peng, Quenez [44]. La plupart des travaux ont en commun la célèbre condition de Mokobodski, qui est en quelque sorte un équivalent à la condition d'Isaacs. Elle peut être complètement supprimée en introduisant des temps d'arrêt aléatoires. Ceci fut d'abord démontré dans Touzi et Vieille dans [106] puis généralisé par Laraki et Solan dans [77].

6.2 Un exemle simple pour l'approche analytique

Nous tenons à répéter ici ce qui est bien connu pour les jeux de Dynkin à information complète dans un cadre markovien assez simple, c'est à dire lorsque la dynamique est donnée comme dans la section 3.1 sur un espace de Wiener $(\mathcal{C}([0,T];\mathbb{R}^d),(\mathcal{H}_t),\mathbb{P}_0)$ par

$$dX_s^{t,x} = b(s,X_s^{t,x})ds + a(s,X_s^{t,x})dB_s \qquad X_t^{t,x} = x. \tag{6.1}$$

Nous considérons un jeu où il y a deux joueurs adverses qui veulent minimiser, respectivement maximiser, un profit $g(X_T^{t,x})$, où

$$g : \mathbb{R}^d \to \mathbb{R}.$$

Cependant - en contraste avec les jeux que nous avons étudiés dans la section 3 - les joueurs ont la possibilité d'arrêter le jeu à tout moment tout en subissant une certaine pénalité . Le joueur 1 choisit $\tau \in [0,T]$ afin de minimiser, le joueur 2 choisit $\sigma \in [0,T]$ afin de maximiser le gain espéré

$$J(t,x,\tau,\sigma) = \mathbb{E}\left[f(\sigma,X_\sigma^{t,x})1_{\sigma<\tau\leq T} + h(\tau,X_\tau^{t,x})1_{\tau\leq\sigma,\tau<T} + g(X_T^{t,x})1_{\sigma=\tau=T}\right], \tag{6.2}$$

où il y a

(i) gain d'exercice anticipé du joueur 2 : $f : [0,T]\times\mathbb{R}^d \to \mathbb{R}$

(ii) gain d'exercice anticipé du joueur 1 : $h : [0,T]\times\mathbb{R}^d \to \mathbb{R}$.

Comme d'habitude pour l'approche par des EDP (voir par exemple Bensoussan and Friedmann [10]), on suppose que pour tout $t \in [0,T]$, $x \in \mathbb{R}^d$

$$f(t,x) \leq h(t,x) \text{ et } f(T,x) \leq g(x) \leq h(T,x). \tag{6.3}$$

Définition 6.1. *Au temps $t \in [0,T]$, un temps d'arrêt admissible pour les deux joueurs est un temps d'arrêt par rapport à la filtration $(\mathcal{H}_{t,s})_{s\in[t,T]}$ à valeurs dans $[t,T]$. On désigne l'ensemble des temps d'arrêt admissibles par $\mathcal{T}(t)$.*

On peut alors définir la fonction valeur inférieure par

$$V^-(t,x) = \sup_{\sigma\in\mathcal{T}(t)} \inf_{\tau\in\mathcal{T}(t)} J(t,x,\tau,\sigma) \tag{6.4}$$

et la fonction valeur supérieure par

$$V^+(t,x) = \inf_{\tau\in\mathcal{T}(t)} \sup_{\sigma\in\mathcal{T}(t)} J(t,x,\tau,\sigma). \tag{6.5}$$

Là encore, on peut utiliser des méthodes d' EDP pour montrer que le jeu a une valeur, c'est-à-dire V $^-(t,x) = V^+(t,x) = V(t,x)$. La caractérisation suivante remonte à Bensoussan et Friedmann [10] pour un cas lisse et peut être trouvée dans le livre de Barles [6].

Théorème 6.2. *Pour toute $(t, x) \in [0, T] \times \mathbb{R}^d$ le jeu de Dynkin a une valeur. La fonction $V : [0, T] \times \mathbb{R}^d \to \mathbb{R}$ est la solution de viscosité unique de*

$$\begin{aligned} \max\{\min\{(-\frac{\partial}{\partial t} - \mathcal{L})[w], w - f(t,x)\}, w - h(t,x)\} &= 0 \\ w(T,x) &= g(x), \end{aligned} \tag{6.6}$$

où

$$\mathcal{L}[w](t,x) := \frac{1}{2}\mathrm{tr}(aa^*(t,x)D_x^2 w(t,x)) + b(t,x)D_x w(t,x).$$

6.3 Jeux de Dynkin à information incomplète

Dans le chapitre 6, nous considérons un jeu d'arrêt optimal à information incomplète. Comme dans l'approche par Aumann et Maschler [3], nous supposons qu'il existe $I \in \mathbb{N}^*$ différents états de la nature pour le jeu. Avant que le jeu commence l'état de la nature est choisi selon une probabilité $p \in \Delta(I)$. Le jeu est joué par deux joueurs adverses, qui veulent minimiser, respectivement maximiser, un certain profit $g_i(X_T^{t,x})$ en fonction de l'état de la nature $i \in \{1, \ldots, I\}$ et de la diffusion $X_T^{t,x}$ donnée par (6.1), où

$$(g_i) : \mathbb{R}^d \to \mathbb{R}, i \in \{1, \ldots, I\}$$

Comme dans l'exemple précédent, les joueurs ont la possibilité supplémentaire d'arrêter le jeu à tout moment tout en subissant une certaine pénalité, i.e. pour $i \in \{1, \ldots, I\}$

(i) gain d'exercice anticipé du joueur 2 : $f_i : [0, T] \times \mathbb{R}^d \to \mathbb{R}$,

(ii) gain d'exercice anticipé du joueur 1 : $h_i : [0, T] \times \mathbb{R}^d \to \mathbb{R}$.

Pour tous les états de la nature $i \in \{1, \ldots, I\}$, nous supposons

$$f_i(t,x) \leq h_i(t,x) \qquad \text{et} \qquad f_i(T,x) \leq g_i(x) \leq h_i(T,x). \tag{6.7}$$

Lorsque le jeu commence, le joueur 1 est informé de l'état de la nature $i \in \{1, \ldots, I\}$, le joueur 2 ne connaît que les probabilités marginales p_i. Nous supposons que les deux joueurs observent le contrôle de leur adversaire. Cela signifie qu'ils savent tout de suite, quand le jeu est arrêté et le gain révélé.

Comme dans Cardaliaguet et Rainer [28], nous permettons aux joueurs de jouer au hasard pour cacher leur information ou pour se rendre moins vulnérables aux manipulations. Ce qui signifie qu'ils peuvent prendre la décision d'arrêter à l'aide d'un générateur de hasard supplémentaire. En utilisant la définition dans Laraki et Solan [77] nous définissons :

Définition 6.3. *Un temps d'arrêt randomisé après le temps $t \in [0, T]$ est une fonction mesurable $\mu : [0, 1] \times \mathcal{C}([t, T]; \mathbb{R}^d) \to [t, T]$, tel que pour tout $r \in [0, 1]$*

$$\tau^r(\omega) := \mu(r, \omega) \in \mathcal{T}(t)$$

Nous désignons l'ensemble des temps d'arrêt randomisés par $\mathcal{T}^r(t)$.

Pour tout $(t, x, p) \in [0, T] \times \mathbb{R}^d \times \Delta(I)$, $(\mu_1, \ldots, \mu_I) \in (\mathcal{T}^r(t))^I, \nu \in \mathcal{T}^r(t)$ on définit

$$\begin{aligned} J(t,x,p,\mu,\nu) = \textstyle\sum_{i=1}^I p_i \mathbb{E}_{\mathbb{P}_0 \otimes \lambda \otimes \lambda} \Big[& f_i(\nu, X_\nu^{t,x}) 1_{\nu < \mu_i \leq T} \\ & + h_i(\mu_i, X_{\mu_i}^{t,x}) 1_{\mu_i \leq \nu, \mu_i < T} + g_i(X_T^{t,x}) 1_{\mu_i = \nu = T} \Big], \end{aligned} \tag{6.8}$$

où λ est la mesure de Lebesgue sur $[0,1]$. (Dans la suite nous omettrons l'indice $\mathbb{P}_0 \otimes \lambda \otimes \lambda$.) Nous définissons la fonction valeur inférieure par

$$V^-(t,x,p) = \sup_{\nu \in \mathcal{T}^r(t)} \inf_{\mu \in (\mathcal{T}^r(t))^I} J(t,x,p,\mu,\nu) \tag{6.9}$$

et la fonction valeur supérieure par

$$V^+(t,x,p) = \inf_{\mu \in (\mathcal{T}^r(t))^I} \sup_{\nu \in \mathcal{T}^r(t)} J(t,x,p,\mu,\nu). \tag{6.10}$$

Pour montrer que le jeu a une valeur, nous établissons le

Théorème 6.4. *Pour tout $(t,x,p) \in [0,T] \times \mathbb{R}^d \times \Delta(I)$, le jeu a une valeur $V(t,x,p)$. La fonction $V : [0,T] \times \mathbb{R}^d \times \Delta(I) \to \mathbb{R}$ est l'unique solution de viscosité de l'équation*

$$\begin{aligned} \max\Big\{ \max\{\min\{(-\tfrac{\partial}{\partial t} - \mathcal{L})[w], w - \langle f(t,x),p\rangle\}, \\ w - \langle h(t,x),p\rangle\}, -\lambda_{\min}\Big(p, \tfrac{\partial^2 w}{\partial p^2}\Big) \Big\} &= 0 \\ w(T,x,p) &= \textstyle\sum_{i=1,\ldots,I} p_i g_i(x), \end{aligned} \tag{6.11}$$

où

$$\mathcal{L}[w](t,x,p) := \frac{1}{2}\mathrm{tr}(aa^*(t,x)D_x^2 w(t,x,p)) + b(t,x)D_x w(t,x,p)$$

et pour tout $p \in \Delta(I)$, $A \in \mathcal{S}^I$

$$\lambda_{\min}(p,A) := \min_{z \in T_{\Delta(I)(p)} \setminus \{0\}} \frac{\langle Az,z\rangle}{|z|^2}$$

avec $T_{\Delta(I)(p)} = \overline{\cup_{\lambda>0}(\Delta(I)-p)/\lambda}$.

6.4 Représentation duale pour les jeux de Dynkin à information incomplète

Dans la deuxième partie du chapitre 6, nous utilisons la caractérisation par des EDP du théorème 6.4. pour établir une représentation duale de la fonction valeur. Comme dans la section 4.1 on élargit l'espace canonique de Wiener à l'espace $(\mathcal{D}([0,T];\Delta(I)) \times \mathcal{C}([0,T];\mathbb{R}^d), \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ avec $\mathbb{P} \in \mathcal{P}(t,p)$. Ainsi, l'espace élargi supporte en plus du mouvement brownien B, des croyances du joueur non informé $\mathbf{p}$ comme dynamique supplémentaire. Pour chaque $\mathbb{P} \in \mathcal{P}(t,p)$, nous considérons un jeu d'arrêt avec cette dynamique supplémentaire. Cependant, nous devons modifier la définition de temps d'arrêt admissible :

Définition 6.5. *Au temps $t \in [0,T]$, un temps d'arrêt admissible pour les deux joueurs est un temps d'arrêt par rapport à la filtration $(\mathcal{F}_s)_{s\in[t,T]}$ à valeurs dans $[t,T]$. On désigne l'ensemble des temps d'arrêt admissibles par $\bar{\mathcal{T}}(t)$.*

Nous notons, que contrairement à la définition (6.1) les temps d'arrêt admissibles au temps t pourrait également dépendre désormais des chemins du mouvement brownien avant le temps t.

Pour chaque $\mathbb{P} \in \mathcal{P}(t,p)$, nous considérons les jeux d'arrêt avec un gain

$$\begin{aligned} J(t,x,\tau,\sigma,\mathbb{P})_{t-} \quad := \quad & \mathbb{E}_{\mathbb{P}}\Big[\langle \mathbf{p}_\sigma, f(\sigma, X_\sigma^{t,x})\rangle 1_{\sigma<\tau<T} + \langle \mathbf{p}_\tau, h(\tau, X_\tau^{t,x})\rangle 1_{\tau\leq\sigma,\tau<T} \\ & \qquad + \langle \mathbf{p}_T, g(X_T^{t,x})\rangle 1_{\sigma=\tau=T} | \mathcal{F}_{t-}\Big], \end{aligned} \tag{6.12}$$

où $\tau \in \bar{\mathcal{T}}(t)$ désigne le temps d'arrêt choisi par le joueur et $\sigma \in \bar{\mathcal{T}}(t)$ désigne le temps d'arrêt choisi par le joueur 2. En contraste avec la considération dans la section précédente nous ne travaillons ici seulement avec des temps d'arrêt non randomisés.

Nous tenons à mentionner que les résultats connus dans la littérature n'impliquent pas que les jeux avec des fonctionnelles de gain (6.12) ont une valeur pour tout $\mathbb{P} \in \mathcal{P}(t,p)$ fixe. Cependant, notre intérêt porte sur la valeur du jeu où les croyances du joueur non informé $\mathbf{p}$ sont manipulés de manière optimale. À cette fin, nous définissons la fonction valeur inférieure par

$$W^-(t,x,p) = \text{essinf}_{\mathbb{P}\in\mathcal{P}(t,p)} \text{esssup}_{\sigma\in\bar{\mathcal{T}}(t)} \text{essinf}_{\tau\in\bar{\mathcal{T}}(t)} J(t,x,\tau,\sigma,\mathbb{P})_{t-} \tag{6.13}$$

et la fonction valeur supérieure par

$$W^+(t,x,p) = \text{essinf}_{\mathbb{P}\in\mathcal{P}(t,p)} \text{essinf}_{\tau\in\bar{\mathcal{T}}(t)} \text{esssup}_{\sigma\in\bar{\mathcal{T}}(t)} J(t,x,\tau,\sigma,\mathbb{P})_{t-}, \tag{6.14}$$

et en utilisant la caractérisation EDP du théorème 6.3. nous établissons :

Théorème 6.6. *Pour tout $(t,x,p) \in [0,T] \times \mathbb{R}^d \times \Delta(I)$ la valeur du jeu peut être exprimée par*

$$V(t,x,p) = W^+(t,x,p) = W^-(t,x,p). \tag{6.15}$$

7 Conclusion et perspectives

Dans ce travail, nous avons contribué à l'étude des jeux différentiels stochastiques à information incomplète. Nous avons établi une représentation duale pour les jeux différentiels stochastiques à l'aide d'une procédure de minimisation des solutions d'EDSR $Y_{t-}^{t,x,\mathbb{P}}$. Ces EDSR peuvent être associées à des jeux différentiels avec les croyances du joueur non informé comme dynamique progressive supplémentaire. Cette représentation nous permet d'établir, par une preuve remarquablement simple, un résultat de régularité pour la fonction valeur avec les méthodes d'EDSR. En outre, sous l'hypothèse que la fonction valeur V est suffisamment lisse et qu' il y a un $\bar{\mathbb{P}}$ tel que la valeur du jeu est donnée par une solution d'EDSR $Y_{t-}^{t,x,\bar{\mathbb{P}}}$, nous dérivons des stratégies optimales. Toutefois, les conditions sous lesquelles cette dernière hypothèse est satisfaite sont loin d'être évidentes. En effet l'existence et la structure d'un tel $\bar{\mathbb{P}}$ optimal reste un problème difficile et ouvert à des recherches plus poussées.

Dans une partie suivante de cette thèse nous considérons des approximations numériques pour les jeux différentiels stochastiques à information incomplète. Là on peut se faire une idée de comment des mesures de martingales approchées peuvent être construites sur une grille à temps discret. Cependant passer à la limite pose également ici un problème délicat. Un autre défi consiste à déterminer quelles sont les hypothèses minimales qu'on doit imposer pour la construction, par des méthodes numériques, de stratégies ϵ-optimales pour le joueur informé. Les problèmes que l'on rencontre en raison de la nature stochastique du jeu sont encore pour les jeux différentiels stochastiques à information complète très difficile à étudier.

La dernière partie de cette thèse est consacrée à l'étude d'un type différent de jeu à information incomplète, à savoir des jeux d'arrêt optimal. De nouveau, nous montrons que ces jeux ont une valeur qui peut être caractérisée comme une solution de viscosité à un EDP variationnelle. En outre, nous obtenons une représentation duale de la fonction valeur en termes d'une procédure de minimisation. En contraste avec la première partie, il n'est pas clair ici si les jeux sur lesquels nous minimisons ont effectivement une valeur. Pourtant, nous montrons qu' asymptotiquement la valeur supérieure et inférieure de la représentation duale coïncide en utilisant la caractérisation par les EDP de la fonction valeur. Une extension naturelle serait de considérer les jeux différentiels stochastiques d'arrêt optimal, que l'on appelle les jeux mixtes comme dans Hamadène et Lepeltier [60]. Toutefois, puisque les croyances du joueur non informé ne sont pas continues mais seulement càdlàg, les théorèmes de Hamadène et Lepeltier [60] ne s'appliquent pas directement et doivent être généralisés.

Chapter 2

Introduction (English version)

1 Summary

The objective of this thesis is the study of stochastic differential games with incomplete information. We consider a game with two opponent players who control a diffusion in order to minimize, respectively maximize a certain payoff. To model the information incompleteness we will follow the famous ansatz of Aumann and Maschler [3]. We assume that there are different states of nature in which the game can take place. Before the game starts the state is chosen randomly. The information is then transmitted to one player while the second one only knows the respective probabilities for each state.

The striking observation for the Aumann-Maschler-model is the so called Cav u theorem to which we refer in the following as dual representation. It says that one can consider a game with incomplete information as a game with information completeness where the informed player - additional to his usual control - can control the dynamics of the new game with the help of some martingale measures. This representation can then be used to investigate games with incomplete information with the help of a game with complete information. In particular, it allows to derive optimal strategies for the informed player.

The celebrated ideas of Aumann and Maschler, which date back to the late 1960s, have been studied extensively for repeated games in the last decades. However it was only recently that continuous time differential games with incomplete information were first investigated by Cardaliaguet in [23],[24]. The existence and uniqueness of a value function for stochastic differential games with incomplete information were then given by Cardaliaguet and Rainer in [28] using viscosity solutions to some fully non-linear partial differential equation (PDE). In a subsequent work Cardaliaguet and Rainer in [27] establish in a simple deterministic setting a dual representation for these games.

In chapter 3 we extend the results of Cardaliaguet and Rainer [27] and establish a dual representation for stochastic differential games with incomplete information. Therein we make a vast use of the theory of backward stochastic differential equations (BSDEs), which turns out to be an indispensable tool in this study. Moreover we show how under some restrictions that this representation allows to construct optimal strategies for the informed player. The results are based on the main paper:

1. *A BSDE approach to stochastic differential games with incomplete information*, Stochastic Processes and their Applications, vol. 122, no. 4, pp. 1917 - 1946, (2012).

In a subsequent note in chapter 4 we give - using the dual representation - a strikingly simple proof for semiconvexity of the value function of differential games with incomplete information.

2. *A note on regularity for a fully non-linear PDE arising in game theory*, (2011), Preprint.

This result - merely based on probabilistic techniques - is new and would be from the point of view of PDE theory much harder to establish.

Chaper 5 is devoted to numerical schemes for stochastic differential games with incomplete information. We are here concerned with how to explicitly construct an approximation for the value function. To that end we give a scheme which is fully discretized in time with the drawback that - as in ordinary stochastic differential games - the value but not the optimal strategies can be approximated. The results presented in this chapter can be found in:

3. *A probabilistic numerical scheme for stochastic differential games with incomplete information*, arXiv:1111.4136v1, (2011), submitted.

In Chaper 4 we investigate continuous time optimal stopping games, so called Dynkin games, with information incompleteness. We show that these games have a value and a unique characterization by a fully non-linear variational PDE for which we provide a comparison principle. Also we establish a dual representation for Dynkin games with incomplete information. This chapter is based on:

4. *On a continuous time Dynkin game with incomplete information*, in progress.

2 The mathematical toolbox: BSDE and PDE

2.1 Backward stochastic differential equations

Though first noted already in a work of Bismut [12] in 1973 the study of backward stochastic differential equations (BSDEs) has its real starting point in 1990 with the pioneering paper of Pardoux and Peng [88]. In a series of subsequent works Peng [91], [92], [93] et [94] and Pardoux and Peng [89] laid the basis for the investigation of BSDEs and their connection to other fields of mathematics as optimal control and partial differential equations. The following years the theory of BSDE theory has seen a tremenduous development and proved to be a most valuable tool for various applications, notably in mathematical finance. For the latter we refer to the survey of El Karoui, Peng and Quenez [45].

The very basis of BSDE theory is the classical martingale representation theorem. Indeed let $(\Omega, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P}_0)$ be a filtered probability space with the usual assumptions carrying a Brownian motion B. If $(\mathcal{F}_t) = \sigma(B_s, s \leq t)$, then we have by the martingale representation theorem for any square integrable $\mathcal{F}_T$ measurable random variable ξ a decomposition

$$\xi = \mathbb{E}[\xi|\mathcal{F}_0] + \int_0^T Z_s dB_s, \tag{2.1}$$

where Z is an adapted square integrable process. Applying the martingale representation theorem to the $\mathcal{F}_s$ measurable random variable $Y_s := \mathbb{E}[\xi|\mathcal{F}_s]$ we get with (2.1) the following equation:

$$Y_s = \xi - \int_s^T Z_r dB_r. \tag{2.2}$$

(2.2) is called linear BSDE and the couple of adapted processes (Y, Z) is called the solution to (2.2).

More generally equations of the following form are denoted BSDE:

$$Y_s = \xi + \int_s^T f(r, Y_r, Z_r) dr - \int_s^T Z_r dB_r, \tag{2.3}$$

where the driver f is a given random function, i.e. $f = f_\omega(\cdot)$. We note that, if f is independent of Y, (2.3) reads

$$Y_s = \mathbb{E}\left[\xi + \int_s^T f(r, Z_r) dr \Big| \mathcal{F}_s\right]. \tag{2.4}$$

Under a uniform Lipschitz assumption on the driver, the existence of a solution (Y, Z) to (2.3) has first been shown by Pardoux and Peng [88] via a fixed point argument. Furthermore the uniqueness has been established via a comparison principle. Since then many authors contributed to weaken the assumptions on f and on the terminal condition ξ (see e.g. Briand and Hu [16], Delbaen, Hu and Bao [37], Delbaen, Hu and Richou [38], Kobylansky [67], Lepeltier and San Martin [80]).

For later purposes we would also like to mention that, if the filtration $(\mathcal{F}_t)$ is larger than $\sigma(B_s, s \leq t)$, the martingale representation theorem does not apply. Instead one can use the Galtchouk-Kunita-Watanabe decomposition (see e.g. Ansel and Stricker [2]). It implies that for any square integrable $\mathcal{F}_T$ measurable random variable ξ we have a decomposition

$$\xi = \mathbb{E}_\mathbb{P}[\xi|\mathcal{F}_0] + \int_0^T Z_s dB_s + N_T, \tag{2.5}$$

with an adapted square integrable process Z and a square integrable martingale N with $N_0 = 0$, which is strongly orthogonal to B. As in El Karoui and Huang [43] one can consider BSDE of the following form

$$Y_s = \xi + \int_s^T f(r, Y_r, Z_r) dr - \int_s^T Z_r dB_r - (N_T - N_s). \tag{2.6}$$

with a triple (Y, Z, N) as solution.

2.2 Forward BSDE and their connection with PDE

A very important case - first considered by Peng in [91] - is when the driver and the terminal condition of the BSDE depend on a stochastic differential equation (SDE)

$$dX_s^{t,x} = b(s, X_s^{t,x}) ds + \sigma(s, X_s^{t,x}) dB_s \qquad X_t^{t,x} = x. \tag{2.7}$$

For given are deterministic functions g and f one considers

$$Y_s^{t,x} = g(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})dr - \int_s^T \sigma^*(s, X_s^{t,x}) Z_r^{t,x} dB_r, \tag{2.8}$$

where we slightly changed the last integral for notational reasons. The couple of equations (2.7), (2.8) is called forward-backward stochastic differential equation (FBSDE).

Assuming sufficient regularity on the coefficients the existence and uniqueness for the solution of (2.8) can be established by the theory of semi-liner PDE as can be found in the book of Ladyženskaja, Solonnikov and Uralceva [73]. Indeed, if u denotes a smooth solution to PDE

$$\begin{aligned} \frac{\partial u}{\partial t} + \frac{1}{2}\mathrm{tr}(\sigma\sigma^*(t,x)D_x^2 u) + \langle b(t,x), D_x u\rangle + f(t,x,u,D_x u) &= 0 \\ u(T,x) &= g(x), \end{aligned} \tag{2.9}$$

then by Itô s formula

$$\begin{aligned} Y_s^{t,x} &:= u(s, X_s^{t,x}) \\ Z_s^{t,x} &:= D_x u(s, X_s^{t,x}) \end{aligned} \tag{2.10}$$

solves (2.8).

This connection has been first established by Peng in [91]. It is furthermore the central idea of the so called four step scheme of Ma, Protter and Yong [82]. Therein it is shown that this method applies also to find solutions for fully coupled FBSDE, i.e. the coefficient of the forward part (2.7) depend also on $(Y^{t,x}, Z^{t,x})$. The existence and uniqueness for solutions for fully coupled FBSDE, namely the triplet $(X^{t,x}, Y^{t,x}, Z^{t,x})$, are investigated in numerous works also going beyond the Markovian framework (e.g. Hu and Peng [62], Hu and Yong [63], Pardoux and Tang [90], Peng and Wu [95]). For a survey and further references we would also like to refer to the textbook of Ma and Yong [83].

On the other hand a natural question is, whether the solution of the BSDE (2.8) provides a solution to the semilinear parabolic PDE (2.9). Peng established in [91] this generalization of the famous Feynman-Kac formula to the semilinear case. Indeed, the function u defined by

$$u(t,x) := Y_t^{t,x} \tag{2.11}$$

is under regularity assumptions on the coefficient smooth and a classical solution to (2.9). Hence Peng gives in [91] a completely probabilistic proof for the existence of a solution of a semi-linear PDE. For later purposes we note that one small but important step in the proof is to show that $u(t,x)$ is deterministic. Here it is an easy consequence of the Blumenthal zero-one law.

With merely Lipschitz assumptions on the coefficients Peng [92] showed that $u(t,x)$ solves the PDE in a weaker sense namely in the sense of viscosity solutions. This notion was introduced for the investigation of control problems in the beginning of the 1980s by Crandall and Lions [30]. The main reference for the theory of viscosity solutions is the survey of Crandall, Ishii and Lions [29]. We will give in the following chapters a concise definition for viscosity solutions in our cases of interest.

As in the smooth case the theory of BSDEs gives a probabilistic proof of the existence of viscosity solutions of semi-linear PDEs. However viscosity solutions are in general not smooth enough to construct as in (2.10) solutions to BSDE.

3 Stochastic differential games

3.1 The problem

A zero-sum stochastic differential game is in general a game, where two opponent players while observing each other control a diffusive quantity. We shall give here the standard form of this problem and fix the notation for the subsequent sections.

For the mathematical description it is convenient to consider the stochastic dynamics on the canonical space $\mathcal{C}([0,T];\mathbb{R}^d)$ equipped with the Wiener measure $\mathbb{P}_0$. For the reminder of the introduction we denote by $B_s(\omega_B) = \omega_B(s)$ the coordinate mapping on $\mathcal{C}([0,T];\mathbb{R}^d)$. By $\mathcal{H} = (\mathcal{H}_s)$ the filtration generated by $s \mapsto B_s$ and by $(\mathcal{H}_{t,s})_{s\in[t,T]}$ the filtration generated by $s \mapsto B_s - B_t$.

For all initial data $t \in [0,T]$, $x \in \mathbb{R}^d$ the players control a diffusion given by

$$\begin{aligned} dX_s^{t,x,u,v} &= b(s, X_s^{t,x,u,v}, u_s, v_s)ds + \sigma(s, X_s^{t,x,u,v}, u_s, v_s)dB_s \\ X_t^{t,x,u,v} &= x, \end{aligned} \tag{3.1}$$

where we assume that the controls of the players u, v can only take their values in some sets U, V respectively, where U, V are compact subsets of some finite dimensional spaces. The aim of the players is to minimize, respectively maximize the expected outcome

$$J(t,x,u,v) = \mathbb{E}\left[\int_t^T l(s, X_s^{t,x,u,v}, u_s, v_s)ds + g(X_T^{t,x,u,v})\right], \tag{3.2}$$

where l denotes the running costs of the game and g the terminal payoff. We note that in general the coefficients might be random functions.

3.2 Differential games and stochastic differential games via PDE

The first studies of deterministic differential games (corresponding to the situation, where $\sigma = 0$ and the coefficients are deterministic functions) date back to the early 1940s with the works of Isaacs [64] and Pontryagin [96], [97]. The main problem for the investigation of games in continuous time is to specify how the players can play. On one hand they have to be given the possibility to react on the actions of their adversary while on the other hand a proper definition has to avoid instantaneous switches. To circumvent the difficulties the continuous time poses a common ansatz is to discretize the game in time (see e.g. Fleming [49], [50], Friedman [54], Krasovskiĭ and Subbotin [69], Subbotina, Subbotin and Tretjakov [105],....). The outcome of the continuous time game is then the limit of the outcome of the discrete time one.

A different ansatz for the investigation of deterministic differential games is given by Evans and Souganidis [47] using the notion of non-anticipative strategies introduced by Elliot and Kalton [46]. Their proof relies heavily on the technique of viscosity solutions introduced by Crandall and Lions [30]. The results of Evans and Souganidis [47] were

generalized by Fleming and Souganidis in [52] to the case of stochastic differential games, where the system is Markovian, i.e. the coefficients are deterministic functions.
In order to avoid instantaneous switches Fleming and Souganidis [52] let the players play control against strategy using the following definitions:

Definition 3.1. *For any $t \in [0,T]$ an admissible control $u = (u_s)_{s\in[t,T]}$ for Player 1 is a progressively measurable process with respect to the filtration $(\mathcal{H}_{t,s})_{s\in[t,T]}$ with values in U. The set of admissible controls for Player 1 is denoted by $\mathcal{U}(t)$.*
The definition for admissible controls $v = (v_s)_{s\in[t,T]}$ for Player 2 is similar. The set of admissible controls for Player 2 is denoted by $\mathcal{V}(t)$.

Definition 3.2. *A strategy for Player 1 at time $t \in [0,T]$ is a non-anticipative map $\alpha : [t,T] \times \mathcal{V}(t) \to \mathcal{U}(t)$, i.e. for any $v, v' \in \mathcal{V}(t)$, $s \in [t,T]$*

$$v = v' \quad \text{on} \quad [t,s] \Rightarrow \alpha(v) = \alpha(v') \quad \text{on } [t,s].$$

The set of strategies for Player 2 is denoted by $\mathcal{A}(t)$.
The definition of strategies $\beta : [t,T] \times \mathcal{U}(t) \to \mathcal{V}(t)$ for Player 2 is similar. The set of strategies for Player 2 is denoted by $\mathcal{B}(t)$.

The lower value of the game is then defined as

$$V^-(t,x) = \inf_{\alpha\in\mathcal{A}(t)} \sup_{v\in\mathcal{V}(t)} J(t,x,\alpha,v), \tag{3.3}$$

where $J(t,x,\alpha,v)$ is associated with the couple of controls $(\alpha(\cdot,v),v) \in \mathcal{U}(t) \times \mathcal{V}(t)$, and similarly the upper value is defined as

$$V^+(t,x) = \sup_{\beta\in\mathcal{B}(t)} \inf_{u\in\mathcal{U}(t)} J(t,x,u,\beta), \tag{3.4}$$

where $J(t,x,\alpha,v)$ is associated with the couple of controls $(u,\beta(\cdot,u)) \in \mathcal{U}(t) \times \mathcal{V}(t)$.

Definition 3.3. *One says that the game has a value if*

$$V^-(t,x) = V^+(t,x) \tag{3.5}$$

and $V(t,x) := V^-(t,x) = V^+(t,x)$ is called the value of the game.

To show that the stochastic differential game has a value Fleming and Souganidis [52] use the theory of viscosity solutions. Indeed, under suitable assumptions one can show that V^+ is a viscosity solution to the Hamilton-Jacobi-Isaacs (HJI) equation

$$\frac{\partial w}{\partial t} + H^+(t,x,D_x w, D_x^2 w) = 0, \tag{3.6}$$

where for each $t \in [0,T]$, $x \in \mathbb{R}^d$, $\xi \in \mathbb{R}^d$, $A \in \mathcal{S}^d$

$$H^+(t,x,\xi,A) = \inf_{u\in U} \sup_{v\in V} \left\{ \langle b(t,x,u,v),\xi\rangle + \frac{1}{2}\mathrm{tr}(\sigma\sigma^*(t,x,u,v)A) + l(t,x,u,v) \right\}. \tag{3.7}$$

Furthermore by the very same methods V^- is a viscosity solution to

$$\frac{\partial w}{\partial t} + H^-(t,x,D_x w, D_x^2 w) = 0 \tag{3.8}$$

with

$$H^-(t,x,\xi,A) = \sup_{v\in V}\inf_{u\in U}\left\{\langle b(t,x,u,v),\xi\rangle + \frac{1}{2}\mathrm{tr}(\sigma\sigma^*(t,x,u,v)A) + l(t,x,u,v)\right\}. \quad (3.9)$$

If one now assumes Isaacs' condition

$$H^+(t,x,\xi,A) = H^-(t,x,\xi,A) =: H(t,x,\xi,A), \quad (3.10)$$

the viscosity solution property yields thoghether with a comparison principle for the HJI equation (see e.g. Crandall, Isshii and Lions [29]) the result of Fleming and Souganidis [52]:

Theorem 3.4. *For any $(t,x)\in[0,T]\times\mathbb{R}^d$ the stochastic differential game has a value $V(t,x)$ and the function $(t,x)\mapsto V(t,x)$ is the unique viscosity solution to*

$$\begin{array}{rcl} \frac{\partial w}{\partial t} + H(t,x,D_x w, D_x^2 w) & = & 0 \\ w(T,x) & = & g(x). \end{array} \quad (3.11)$$

3.3 Stochastic differential games via BSDE

The study of stochastic differential games via the theory of BSDE was initiated by Hamadène and Lepeltier in [58] ,[59]. The main contribution in using BSDE consists in the possibility to consider non-Markovian systems where the PDE arguments of Fleming and Souganidis in [52] cannot be applied. The ideas were later generalized to other situations in Hamadène, Lepeltier and Peng [61], El Karoui and Hamadène [42], Hamadène and Lepeltier [60] and Hamadène and Hassani [57].

Indeed, by the very definition (3.2) the payoff for any $(u,v)\in\mathcal{U}(t)\times\mathcal{V}(t)$ can be written as $J(t,x,u,v) = \mathbb{E}\left[Y_t^{t,x,u,v}\right]$, where $Y^{t,x,u,v}$ is the solution to the BSDE

$$Y_s^{t,x,u,v} \quad = \quad g(X_T^{t,x,u,v}) + \int_s^T l(r, X_r^{t,x,u,v}, u_r, v_r)dr - \int_s^T \sigma^*(r, X_r^{t,x})Z_r^{t,x,u,v} dB_r \quad (3.12)$$

with $X^{t,x,u,v}$ defined as

$$\begin{array}{rcl} dX_s^{t,x,u,v} & = & b(s, X_s^{t,x,u,v}, u_s, v_s)ds + \sigma(s, X_s^{t,x,u,v}, u_s, v_s)dB_s \\ X_t^{t,x,u,v} & = & x. \end{array} \quad (3.13)$$

In Hamadène and Lepeltier [59] a zero-sum stochastic differential game is considered, where the diffusion coefficient cannot be controlled by the players, i.e.

$$\sigma(t,x,u,v) = \sigma(t,x). \quad (3.14)$$

and σ is assumed to be non-degenerate. The idea in Hamadène and Lepeltier [59] is to consider the game under a Girsanov transformation to decouple the forward dynamics from the control. Then it is possible construct a couple of optimal controls for the players by using the comparison prinicle for BSDE. By inverting the Girsanov transformation this yields a saddle point equilibrium for the game.

Indeed, equivalently to the SDE (3.13) one can consider for each $(u,v)\in\mathcal{U}(t)\times\mathcal{V}(t)$

$$\begin{array}{rcl} dX_s^{t,x} & = & \sigma(s, X^{t,x})dB_s \\ X_t^{t,x} & = & x \end{array} \quad (3.15)$$

under $d\mathbb{P}^{u,v} = \Gamma_T^{u,v} d\mathbb{P}$ with

$$\Gamma_s^{u,v} = \mathcal{E}\left(\int_t^s b(r, X_r^{t,x}, u_r, v_r)\sigma^*(r, X_r^{t,x})^{-1} dB_r\right). \tag{3.16}$$

where $\mathcal{E}$ denotes the Doléans-Dade exponential.
A cost functional as in (3.2) can then be expressed as

$$J(t,x,u,v) = \mathbb{E}_{\mathbb{P}^{u,v}}\left[Y_t^{t,x,u,v}\right], \tag{3.17}$$

where $Y_s^{t,x,u,v}$ solves the BSDE

$$\begin{aligned} Y_s^{t,x,u,v} &= g(X_T^{t,x}) + \int_s^T \left(l(r, X_r^{t,x}, u_r, v_r) + b(r, X_r^{t,x}, u_r, v_r) Z_r^{t,x,u,v}\right) dr \\ &\quad - \int_s^T \sigma^*(r, X_r^{t,x}) Z_r^{t,x,u,v} dB_r \end{aligned} \tag{3.18}$$

with the $\mathbb{P}$-Brownian motion B.

As in Fleming and Souganidis [52] Isaacs' condition is assumed, which in this case is supposed to hold pathwise:

$$\begin{aligned} &\sup_{v\in V} \inf_{u\in U} \{\langle b(t,x,u,v), \xi\rangle + l(t,x,u,v)\} \\ &= \inf_{u\in U} \sup_{v\in V} \{\langle b(t,x,u,v), \xi\rangle + l(t,x,u,v)\} := H(t,x,\xi) \end{aligned} \tag{3.19}$$

and it is possible to define for all $t \in [0,T]$, $x \in \mathbb{R}^d$, $\xi \in \mathbb{R}^d$ $u^*(t,x,\xi)$, $v^*(t,x,\xi)$ respectively, such that :

$$\begin{aligned} H(t,x,\xi) &\geq \langle b(t,x,u^*(t,x,\xi),v),\xi\rangle + l(t,x,u^*(t,x,\xi),v) \quad \text{for all } v \in V \\ H(t,x,\xi) &\leq \langle b(t,x,u,v^*(t,x,\xi)),\xi\rangle + l(t,x,u,v^*(t,x,\xi)) \quad \text{for all } u \in U. \end{aligned} \tag{3.20}$$

One can now define the processes

$$\begin{aligned} \bar{u}_s &= u^*(s, X_s^{t,x}, Z_s^{t,x}) \\ \bar{v}_s &= v^*(s, X_s^{t,x}, Z_s^{t,x}), \end{aligned} \tag{3.21}$$

where $Z^{t,x}$ is given by a solution to the BSDE

$$Y_s^{t,x} = g(X_T^{t,x}) + \int_s^T H(r, X_r^{t,x}, Z_r^{t,x}) dr - \int_s^T \sigma^*(r, X_r^{t,x}) Z_r^{t,x} dB_r. \tag{3.22}$$

Furthermore for all $u \in \mathcal{U}(t)$, $v \in \mathcal{V}(t)$ the comparison principle for BSDEs yields

$$Y_t^{t,x,\bar{u},v} \leq Y_t^{t,x} \leq Y_t^{t,x,u,\bar{v}} \tag{3.23}$$

$\mathbb{P}$-a.s. and since $\mathbb{E}\left[Y_t^{t,x}\right] = J(t,x,\bar{u},\bar{v})$, this gives a saddle point for the game in the following sense:

Theorem 3.5. *For any* $(u,v) \in \mathcal{U}(t) \times \mathcal{V}(t)$

$$J(t,x,\bar{u},v) \leq J(t,x,\bar{u},\bar{v}) \leq J(t,x,u,\bar{v}) \tag{3.24}$$

and one can define $V(t,x) = \mathbb{E}\left[Y_t^{t,x}\right]$ *as the value of the stochastic differential game.*

It is worth to mention that the optimal pair of controls $(\bar{u}, \bar{v})$ is adapted but in general not of a feedback form as formula (3.21) might indicate, i.e. is in general not given as a function of time t and state of the system at time t. Indeed (3.21), defines $\bar{u}, \bar{v}$ under $\mathbb{P}$. But the real dynamics of the game are given under $\mathbb{P}^{\bar{u},\bar{v}}$. So in order to calculate the optimal strategies one has to perform a measure change and in general $Z^{t,x}$ might under the measure change quite as well depend on the whole history of $\bar{u}, \bar{v}$. For a detailed discussion of this rather deep issue we refer to the paper of Rainer [98].

On the other hand if the coefficients are deterministic functions it is clear that the value of the game defined by Fleming and Souganidis [52] coincides with the one found in Hamadène and Lepeltier, since the solution of the BSDE (3.22) gives a viscosity solution to the HJI equation (3.11). Furthermore if $V \in \mathcal{C}^{1,2}([t,T], \mathbb{R}^d)$ then by uniqueness of the solution of the BSDE (3.22) and (2.8)

$$Z_s^{t,x} \quad = \quad D_x V(s, X_s^{t,x}) \tag{3.25}$$

and the optimal feedback controls are given by

$$\begin{aligned} \bar{u}_s &= u^*(s, X_s^{t,x}, D_x V(s, X_s^{t,x})) \\ \bar{v}_s &= v^*(s, X_s^{t,x}, D_x V(s, X_s^{t,x})) \end{aligned} \tag{3.26}$$

under $\mathbb{P}^{\bar{u},\bar{v}}$ hence the real world dynamics. In order to play optimally one has to set at each time s the actual value of the system, which is precisely $X^{t,x}$ under $\mathbb{P}^{\bar{u},\bar{v}}$, into (3.26).

3.4 Fleming Souganidis revisited

We have seen in the previous section that the theory of BSDE can be used to study games in non-Markovian systems by establishing the existence of a saddlepoint. Buckdahn and Li showed in [19] that BSDEs can also be used to unburden the technically rather heavy proof of Fleming and Souganidis [52]. Indeed, the latter work has as disadvantage that the controls in $\mathcal{U}(t)$, $\mathcal{V}(t)$ respectively, are restricted not to depend on the paths of the Brownian motion before time t. This restriction implies heavy technicalities in the proof of Fleming and Souganidis [52]. Buckdahn and Li relax in [19] this condition by imposing that admissible controls are measurable process with respect to the whole filtration $(\mathcal{H}_s)_{s\in[t,T]}$.

Furthermore in Buckdahn and Li [19] more general cost functionals are considered, namely

$$J(t,x,u,v) = Y_t^{t,x,u,v}, \tag{3.27}$$

where $Y_t^{t,x,u,v}$ is defined as the solution to a FBSDEs $(X_t^{t,x,u,v}, Y_t^{t,x,u,v}, Z_t^{t,x,u,v})$ with coefficients that are as in Fleming and Souganidis [52] deterministic functions. However for u, v supposed to be $\mathcal{H}_s$ and not $\mathcal{H}_{t,s}$ measurable in contrast to (2.11) the Blumental zero-one law does not apply, hence $Y_t^{t,x,u,v}$ is in general not deterministic.

One central step in the work of Buckdahn and Li [19] is to show that the lower value function

$$V^-(t,x) = \text{essinf}_{\alpha\in\mathcal{A}(t)}\text{esssup}_{v\in\mathcal{V}(t)} Y_t^{t,x,\alpha,v}, \tag{3.28}$$

and the upper value function

$$V^+(t,x) = \text{esssup}_{\beta\in\mathcal{B}(t)}\text{essinf}_{u\in\mathcal{U}(t)} Y_t^{t,x,u,\beta}, \tag{3.29}$$

are deterministic. They accomplish this with an elegant idea, namely by showing that $V^+(t,x)$ and $V^-(t,x)$ are invariant under variations on the Cameron Martin space.

The more general choice of u, v enables now a direct proof of the viscosity solution property for the upper and lower value functions to some HJI equations which are under an Isaacs' condition equal again. The results of Buckdahn and Li [19] are extended to more general cases in subsequent papers by Buckdahn and Li [20], Buckdahn, Hu and Li [18] and Lin [81].

4 Stochastic differential games with incomplete information

4.1 General games with incomplete information

The formalism introduced by Aumann and Maschler in [3] in 1968 considers zero-sum games with incomplete information of the following structure:

- There are I differerent states of nature the game can take place. Before it starts one state is picked with a probability p, which is commonly known.
- The information is transmitted to Player 1, while Player 2 only knows p.
- Player 1 wants to minimize, Player 2 wants to maximize his payoff.
- We assume both players observe their opponents control.

While in one shot games the last assumption is redundant in repeated games, where the game takes place in multiple stages, it becomes crucial. Since his opponent observes the informed player, it is important for him to find at each stage the right balance between using the information and hence revealing it or hiding it in acting less optimal in order to be able to use it at a later stage. Indeed it turns out that it is optimal for the players to play randomly according to an additional random advice.

The famous idea of Aumann and Maschler [3] is now that one can consider a game with incomplete information as a random game with complete information, where both players do not know the state of nature. At each step all states of nature are played simultaneously with a certain probability. This probability reflects the belief of the uninformed player about which state of nature has been chosen according to his current information and hence gives rise to a discrete martingale. Since the beliefs are controlled by the actions of the informed player he will act such that this martingale gives a minimal outcome of the simultaneously played games. This representation is known as the famous cav u theorem (Though vex u is more correct in our case, since the informed player is the minimizer). In the following we will refer to this representation of a game with incomplete information also as dual representation.

The case of repeated games with incomplete information on one or both sides has been studied extensively since the seminal work of Aumann and Maschler [3] and is up to now an active field of research. For a survey and numerous references on the study of repeated games with incomplete information we refer to the textbook of Sorin [99]. For recent reseach in that field we like to mention the works of De Meyer and Rosenberg [34], De Meyer, Lehrer and Rosenberg [33], Gensbittel [55], Laraki [74], [75] and Sorin [100],[101]. Furthermore an application to stock markets can be found in the work of De Meyer [32].

4.2 Stochastic differential games with incomplete information

Only recently the setting of Aumann and Maschler was generalized to deterministic differential games in Cardaliaguet in [23], [24] and to stochastic differential games by Cardaliaguet and Rainer in [28]. Therein the dynamic of the game is given by a controlled diffusion, i.e. for $t \in [0,T], x \in \mathbb{R}^d$

$$dX_s^{t,x,u,v} = b(s, X_s^{t,x,u,v}, u_s, v_s)ds + \sigma(s, X_s^{t,x,u,v}, u_s, v_s)dB_s \qquad X_t^{t,x} = x. \tag{4.1}$$

As in the model of Aumann and Maschler there are $I \in \mathbb{N}^*$ different states of nature corresponding to I different

(i) running costs: $(l_i)_{i\in\{1,\dots,I\}} : [0,T] \times \mathbb{R}^d \times U \times V \to \mathbb{R}$ and

(ii) terminal payoffs: $(g_i)_{i\in\{1,\dots,I\}} : \mathbb{R}^d \to \mathbb{R}$.

Before the game starts one of these states is chosen according to a probability $p \in \Delta(I)$, where $\Delta(I)$ denotes the simplex of $\mathbb{R}^I$. The information is transmitted to Player 1 only. Player 1 chooses his control to minimize, Player 2 chooses his control to maximize the expected payoff. We assume both players observe their opponents control.

As in the case of differential games with complete information it is assumed, that an Isaacs' condition holds. In the case of incomplete information it reads:

$$\begin{aligned} &\sup_{v\in V} \inf_{u\in U} \left\{ \langle b(t,x,u,v), \xi\rangle + \tfrac{1}{2}\mathrm{tr}(\sigma\sigma^*(t,x,u,v)A) + \langle p, l(t,x,u,v)\rangle \right\} \\ &= \inf_{u\in U} \sup_{v\in V} \left\{ \langle b(t,x,u,v), \xi\rangle + \tfrac{1}{2}\mathrm{tr}(\sigma\sigma^*(t,x,u,v)A) + \langle p, l(t,x,u,v)\rangle \right\} \\ &:= H(t,x,p,\xi,A) \end{aligned} \tag{4.2}$$

For later purpose the definition of admissible strategies differs slightly from Fleming and Souganidis [52]:

Definition 4.1. *For any $t \in [0,T]$ an admissible control $u = (u_s)_{s\in[t,T]}$ for Player 1 is a progressively measurable càdlàg process with respect to the filtration $(\mathcal{H}_{t,s})_{s\in[t,T]}$ with values in U. The set of admissible controls for Player 1 is denoted by $\mathcal{U}(t)$.*
The definition for admissible controls $v = (v_s)_{s\in[t,T]}$ for Player 2 is similar. The set of admissible controls for Player 2 is denoted by $\mathcal{V}(t)$.

As in the case of repeated games the players learn and adapt to the information they learn, so a value defined by playing strategy against control as in Fleming and Souganidis [52] is not sufficient. To allow an interaction, hence playing strategy against strategy with a value that is still defined, Cardaliaguet and Rainer adapt in [28] the notion of non anticipative strategies with delay introduced in Buckdahn, Cardaliaguet and Rainer [17] to the stochastic setting.
Furthermore one has to take into account that the informed player tries to hide his information. In order to do this he has to be allowed to add an extra randomness to his behavior. Note that it is also reasonable to allow the uninformed to use random strategies. As shown for deterministic games by Souquière in [103], the uninformed player plays random as well in order to make himself less vulnerable to the manipulation.
Both features required for strategies in games with incomplete information are incorporated in the following definition of Cardaliaguet and Rainer [28]:

Let U_t, respectively V_t, denote the set of càdlàg maps from [t,T] to U, respectively V. Let $\mathcal{I}$ be a fixed set of probability spaces that is nontrivial and stable by finite product.

Definition 4.2. *A random strategy for Player 1 at time* $t \in [0,T]$ *is a a pair* $((\Omega_\alpha, \mathcal{G}_\alpha, \mathbb{P}_\alpha), \alpha)$*, where* $(\Omega_\alpha, \mathcal{G}_\alpha, \mathbb{P}_\alpha)$ *is a probability space in* $\mathcal{I}$ *and* $\alpha : [t,T] \times \Omega_\alpha \times \mathcal{C}([t,T];\mathbb{R}^d) \times V_t \to U_t$ *satisfies*

(i) α *is a measurable function, where* Ω_α *is equipped with the* σ*-field* $\mathcal{G}_\alpha$*,*

(ii) there exists $\delta > 0$ *such that for all* $s \in [t,T]$ *and for any* $\omega, \omega' \in \mathcal{C}([t,T];\mathbb{R}^d)$ *and* $v, v' \in V_t$ *we have:*

$$\omega = \omega' \text{ and } v = v' \text{ a.e. on } [t,s]$$
$$\Rightarrow \alpha(\cdot,\omega,v) = \alpha(\cdot,\omega',v') \text{ a.e. on } [t,s+\delta] \text{ for any } \omega_\alpha \in \Omega_\alpha.$$

The set of random strategies for Player 1 is denoted by $\mathcal{A}^r(t)$*.*
The definition of random strategies $((\Omega_\beta, \mathcal{G}_\beta, \mathbb{P}_\beta), \beta)$*, where* $\beta : [t,T] \times \Omega_\beta \times \mathcal{C}([t,T];\mathbb{R}^d) \times U_t \to V_t$ *for Player 2 is similar. The set of random strategies for Player 2 is denoted by* $\mathcal{B}^r(t)$*.*

Note that Definition 4.2. is in contrast to Definition 3.2 of Fleming and Souganidis [52] a pathwise one. Hence to ensure the well posedness of the SDE (4.1.) we have to assume more regularity on the controls in order perform a pathwise construction of stochastic integral in (4.1). For a concise study of this construction we refer to Karandikar [66].

In Cardaliaguet and Rainer [28] Lemma 2.1. it is shown, that thanks to the delay it is possible to associate to each couple of random strategies $(\alpha,\beta) \in \mathcal{A}^r(t) \times \mathcal{B}^r(t)$ for any $(\omega_\alpha, \omega_\beta) \in \Omega_\alpha \times \Omega_\beta$ a unique couple of admissible strategies $(u^{\omega_\alpha,\omega_\beta}, v^{\omega_\alpha,\omega_\beta}) \in \mathcal{U}(t) \times \mathcal{V}(t)$, such that for all $\omega \in \mathcal{C}([t,T];\mathbb{R}^d)$, $s \in [t,T]$

$$\alpha(s,\omega_\alpha,\omega,v^{\omega_\alpha,\omega_\beta}(\omega)) = u_s^{\omega_\alpha,\omega_\beta}(\omega) \qquad \text{and} \qquad \beta(s,\omega_\beta,\omega,u^{\omega_\alpha,\omega_\beta}(\omega)) = v_s^{\omega_\alpha,\omega_\beta}(\omega)\ .$$

Hence for any $(t,x,p) \in [0,T] \times \mathbb{R}^d \times \Delta(I)$, $(\bar{\alpha}_1,\ldots,\bar{\alpha}_I) \in (\mathcal{A}^r(t))^I$, $\beta \in \mathcal{B}^r(t)$ the payoff

$$J(t,x,p,\bar{\alpha},\beta) = \sum_{i=1}^I p_i\ \mathbb{E}_{\bar{\alpha}_i,\beta}\left[\int_t^T l_i(s,X_s^{t,x,u_i,v_i},(u_i)_s,(v_i)_s)ds + g_i(X_T^{t,x,u_i,v_i})\right] \tag{4.3}$$

with (u_i,v_i) such that $u_i = \bar{\alpha}_i(v_i)$, $v_i = \beta(u_i)$ is well defined. We note that the information advantage of Player 1 is reflected in (4.3) by having the possibility to choose a strategy $\bar{\alpha}_i$ for each state of nature $i \in \{1,\ldots,I\}$.

And as in Fleming and Souganidis [52] one can now define the lower value, respectively the upper value of a stochastic differential game with incomplete information as

$$\begin{aligned} V^-(t,x,p) &= \sup_{\beta\in\mathcal{B}^r(t)} \inf_{\bar{\alpha}\in(\mathcal{A}^r(t))^I} J(t,x,p,\bar{\alpha},\beta) \\ V^+(t,x,p) &= \inf_{\bar{\alpha}\in(\mathcal{A}^r(t))^I} \sup_{\beta\in\mathcal{B}^r(t)} J(t,x,p,\bar{\alpha},\beta). \end{aligned} \tag{4.4}$$

By the very definition we have $V^-(t,x,p) \le V^+(t,x,p)$. To show that the game has a value the reverse inequality is established by Cardaliaguet and Rainer in [28] using the theory of viscosity solutions. However in contrast to the HJI equation (3.11) for stochastic differential games with complete information the value function satisfies a HJI equation with a convexity constraint in the variable p. The result of Cardaliaguet and Rainer in [28] together with the PDE characterization by Cardaliaguet in [25] are summed up in:

Theorem 4.3. *For any* $(t,x,p) \in [0,T] \times \mathbb{R}^d \times \Delta(I)$ *the stochastic differential game with incomplete information has a value* $V(t,x,p)$*. The function* $V : [0,T] \times \mathbb{R}^d \times \Delta(I) \to \mathbb{R}$ *is* *the unique viscosity solution to*

$$\begin{aligned} \min\left\{\frac{\partial w}{\partial t}+H(t,x,D_xw,D_x^2w,p),\lambda_{\min}\left(p,\frac{\partial^2 w}{\partial p^2}\right)\right\} &= 0 \\ w(T,x,p) &= \textstyle\sum_i p_i g_i(x), \end{aligned} \tag{4.5}$$

where for all $p\in\Delta(I),\ A\in\mathcal{S}^I$

$$\lambda_{\min}(p,A):=\min_{z\in T_{\Delta(I)(p)}\setminus\{0\}}\frac{\langle Az,z\rangle}{|z|^2}$$

and $T_{\Delta(I)(p)}$ *denotes the tangent cone to* $\Delta(I)$ *at* p*, i.e.* $T_{\Delta(I)(p)}=\overline{\cup_{\lambda>0}(\Delta(I)-p)/\lambda}$ *.*

4.3 Dual representation for stochastic differential games

The aim now is to establish an analog to the dual representation of Aumann and Maschler [3] for the case of stochastic differential games. An example of a deterministic game in a continuous time setting is considered in Cardaliaguet and Rainer [27] using a minimization over martingale measures. A similar technique is introduced in De Meyer [32] in the framework of financial markets with informed agents.

In chapter 3 we generalize their result to the case where the dynamics are given as in Hamadène and Lepeltier [59] by a controlled diffusion, i.e. for $t\in[0,T],x\in\mathbb{R}^d$

$$dX_s^{t,x,u,v}=b(s,X_s^{t,x,u,v},u_s,v_s)ds+\sigma(s,X_s^{t,x,u,v})dB_s\qquad X_t^{t,x}=x. \tag{4.6}$$

As in Hamadène and Lepeltier [59] it turns out to be crucial to assume the non-degeneracy condition for $\sigma(t,x)$.

Following the notation of Cardaliaguet and Rainer [28] we have for each initial data $(t,x,p)\in[0,T]\times\mathbb{R}^d\times\Delta(I)$ and each random strategy of the informed player $\bar\alpha\in(\mathcal{A}^r(t))^I$ and the uninformed player $\beta\in\mathcal{B}^r(t)$ an expected payoff

$$J(t,x,p,\bar\alpha,\beta)=\sum_{i=1}^I p_i\ \mathbb{E}_{\bar\alpha_i,\beta}\left[\int_t^T l_i(s,X_s^{t,x,\bar\alpha_i,\beta},(\bar\alpha_i)_s,\beta_s)ds+g_i(X_T^{t,x,\bar\alpha_i,\beta})\right], \tag{4.7}$$

while Isaacs condition here reads:

$$\begin{aligned} &\inf_{u\in U}\sup_{v\in V}\left\{\langle b(t,x,u,v),\xi\rangle+\langle p,l(t,x,u,v)\rangle\right\}\\ &=\sup_{v\in V}\inf_{u\in U}\left\{\langle b(t,x,u,v),\xi\rangle+\langle p,l(t,x,u,v)\rangle\right\}\\ &=: H(t,x,\xi,p). \end{aligned} \tag{4.8}$$

To provide a dual representation we enlarge the canonical space $\mathcal{C}([0,T];\mathbb{R}^d)$ of the Brownian dynamics to the product space $\Omega:=\mathcal{D}([0,T];\Delta(I))\times\mathcal{C}([0,T];\mathbb{R}^d)$, where $\mathcal{D}([0,T];\Delta(I))$ denotes the set of càdlàg functions from $\mathbb{R}$ to $\Delta(I)$, which are constant on $(-\infty,0)$ and on $[T,+\infty)$. We denote by $\mathbf{p}_s(\omega_p)=\omega_p(s)$ the coordinate mapping on $\mathcal{D}([0,T];\Delta(I))$ and by $\mathcal{G}=(\mathcal{G}_s)$ the filtration generated by $s\mapsto\mathbf{p}_s$. We equip the space Ω with the right-continuous filtration $\mathcal{F}$, where $\mathcal{F}_t=\cap_{s>t}\mathcal{F}_s^0$ with $(\mathcal{F}_s^0)=(\mathcal{G}_s\otimes\mathcal{H}_s)$, where

$(\mathcal{H}_s)$ was defined as the filtration generated by the canonical process B on $\mathcal{C}([0,T];\mathbb{R}^d)$. In the following we shall, whenever we work under a fixed probability $\mathbb{P}$ on Ω, complete the filtration $\mathcal{F}$ with $\mathbb{P}$-nullsets without changing the notation.

We shall equip the filtered space Ω with measures $\mathbb{P}$ in order to model the beliefs of the uninformed player by the additional process $\mathbf{p}$. Before the game starts the information of the uninformed player is just the initial distribution p. At the end of the game the information is revealed hence $\mathbf{p}_T \in \{e_i, i = 1,\dots,I\}$, but this is determined before the game starts so $\mathbf{p}_T$ is independent of $(B_s)_{s\in(-\infty,T]}$. Finally, the martingale property, $\mathbf{p}_t = \mathbb{E}_{\mathbb{P}}[\mathbf{p}_T|\mathcal{F}_t]$, is due to the best guess of the uninformed player about the actual state of nature. These features are incorporated in the following definition:

Definition 4.4. *Given $p \in \Delta(I)$, $t \in [0,T]$, we denote by $\mathcal{P}(t,p)$ the set of probability measures $\mathbb{P}$ on Ω, such that under $\mathbb{P}$*

(i) $\mathbf{p}$ is a martingale, such that

(a) $\mathbf{p}_s = p\ \forall s < t$,

(b) $\mathbf{p}_s \in \{e_i, i = 1,\dots,I\}\ \forall s \geq T$ $\mathbb{P}$-a.s. and

(c) $\mathbf{p}_T$ is independent of $(B_s)_{s\in(-\infty,T]}$,

(ii) $(B_s)_{s\in[0,T]}$ is a Brownian motion.

As in our case the Hamiltonian $H(t,x,\xi,p)$ defined in (4.8) depends on an additional parameter $\xi \in \mathbb{R}^d$ a direct dual representation using the Hamiltonian as in Cardaliaguet and Rainer [27] is not possible. Inspired by Hamadène and Lepeltier [59] we use the theory of BSDE to solve this problem. For all $t \in [0,T]$, $x \in \mathbb{R}^d$ we define the process $X^{t,x}$ by

$$X_s^{t,x} = x \quad s < t, \qquad X_s^{t,x} = x + \int_t^s \sigma(r, X_r^{t,x})dB_r \quad s \geq t. \tag{4.9}$$

Let $p \in \Delta(I)$. We consider for each $\mathbb{P} \in \mathcal{P}(t,p)$ the BSDE

$$\begin{aligned} Y_s^{t,x,\mathbb{P}} &= \langle \mathbf{p}_T, g(X_T^{t,x})\rangle + \int_s^T H(r, X_r^{t,x}, Z_r^{t,x,\mathbb{P}}, \mathbf{p}_r)dr \\ &\quad - \int_s^T \sigma^*(r, X_r^{t,x}) Z_r^{t,x,\mathbb{P}} dB_r - N_T + N_s, \end{aligned} \tag{4.10}$$

where N is a square integrable martingale which is strongly orthogonal to B.
In particular we have

$$Y_{t-}^{t,x,\mathbb{P}} = \mathbb{E}_{\mathbb{P}}\left[\int_t^T H(r, X_r^{t,x}, Z_r^{t,x,\mathbb{P}}, \mathbf{p}_r)dr + \langle \mathbf{p}_T, g(X_T^{t,x})\rangle \big| \mathcal{F}_{t-}\right]. \tag{4.11}$$

As in Hamadène and Lepeltier [59] one can see that $Y_{t-}^{t,x,\mathbb{P}}$ is the (random) outcome of a stochastic differential game with complete information with an additional forward dynamic $\mathbf{p}$. Note that with the definition of $\mathcal{P}(t,p)$, $\mathbf{p}_s$ is merely $\mathcal{F}_s$ adapted hence might as well depend on the paths of Brownian motion B before the game starts. So we are in a similar situation as in Buckdahn and Li [19].

In chapter 3 of this thesis the following dual representation for the value function is established:

Theorem 4.5. *For any $(t,x,p) \in [0,T) \times \mathbb{R}^d \times \Delta(I)$ the value of the game with incomplete information $V(t,x,p)$ can be characterized as*

$$V(t,x,p) = \operatorname{essinf}_{\mathbb{P}\in\mathcal{P}(t,p)} Y_{t-}^{t,x,\mathbb{P}}. \tag{4.12}$$

Note that we can identify each $\mathbb{P} \in \mathcal{P}(t,p)$ on $\mathcal{F}_{t-}$ with a common probability measure $\mathbb{Q} = \delta(p) \otimes \mathbb{P}_0$, where $\delta(p)$ is the measure under which $\mathbf{p}$ is constant and equal to p and $\mathbb{P}_0$ is a Wiener measure. So the right hand side of (4.12) is defined $\mathbb{Q}$-a.s. but a priori not deterministic. To establish that $\operatorname{essinf}_{\mathbb{P}\in\mathcal{P}(t,p)} Y_{t-}^{t,x,\mathbb{P}}$ is deterministic an essential step in the proof of Theorem 4.5. is to adapt the ideas of Buckdahn and Li [19] to our setting.

4.4 Optimal "strategies" in the stochastic case

We have with Theorem 4.5. a representation for the stochastic differential game with incomplete information. But - as in Hamadène and Lepeltier [59] for the complete information case - with dynamics that are given under Girsanov transformation of the real world. So in order to use the representation (4.11) to investigate the game and describe the optimal behavior of the informed player as in Cardaliaguet and Rainer [27] we have to reverse this transformation.

This is indeed possible and we provide in chapter 3 of this thesis a result under the additional assumption that $V \in \mathcal{C}^{1,2,2}([t,T] \times \mathbb{R}^d \times \Delta(I); \mathbb{R})$ and there is $\bar{\mathbb{P}} \in \mathcal{P}(t,p)$, such that

$$V(t,x,p) = Y_{t-}^{t,x,\bar{\mathbb{P}}}. \tag{4.13}$$

Thanks to Isaacs condition one can define the function $u^*(t,x,p,\xi)$ as a Borel measurable selection of $\operatorname{argmin}_{u\in U} \max_{v\in V}\{\langle b(t,x,u,v),\xi\rangle + \langle p, l(t,x,u,v)\rangle\}$, hence

$$H(t,x,\xi,p) = \max_{v\in V}\left\{\langle b(t,x,u^*(t,x,p,\xi),v),\xi\rangle + \langle p, l(t,x,u^*(t,x,p,\xi),v)\rangle\right\}. \tag{4.14}$$

As in (3.26) we define the process

$$\bar{u}_s = u^*(s, X_s^{t,x}, D_x V(s, X_s^{t,x}, \mathbf{p}_s), \mathbf{p}_s) \tag{4.15}$$

and for any $\beta \in \mathcal{B}(t)$ the equivalent measure $\bar{\mathbb{P}}^{\bar{u},\beta} = (\Gamma_T^{\bar{u},\beta})\bar{\mathbb{P}}$ with

$$\Gamma_s^{\bar{u},\beta} = \mathcal{E}\left(\int_t^s b(r, X_r^{t,x}, \bar{u}_r, \beta(\bar{u})_r)\sigma^*(r, X_r^{t,x})^{-1} dB_r\right)$$

for $s \geq t$ and $\Gamma_s^{\bar{u},\beta} = 1$ for $s < t$.
Since the informed player knows the state of nature he will be playing conditional to the outcome of the choice of the state at the beginning of the game. Hence we define now for any $i \in \{1,\ldots,I\}$ and for any $\beta \in \mathcal{B}(t)$ a probability measure $\bar{\mathbb{P}}_i^{\bar{u},\beta(\bar{u})}$ by: for all $A \in \mathcal{F}$ we have

$$\bar{\mathbb{P}}_i^{\bar{u},\beta(\bar{u})}[A] = \bar{\mathbb{P}}^{\bar{u},\beta(\bar{u})}[A|\mathbf{p}_T = e_i] = \frac{1}{p_i}\bar{\mathbb{P}}^{\bar{u},\beta(\bar{u})}[A \cap \{\mathbf{p}_T = e_i\}], \quad \text{if } p_i > 0,$$

and $\bar{\mathbb{P}}_i^{\bar{u},\beta(\bar{u})}[A] = \bar{\mathbb{P}}^{\bar{u},\beta(\bar{u})}[A]$ else.

In chapter 3 we establish:

Theorem 4.6. *For each state of nature $i = 1,\ldots,I$ and any strategy of the uninformed player $\beta \in \mathcal{B}(t)$, playing*

$$\bar{u}_s = u^*(s, X_s^{t,x}, D_x V(s, X_s^{t,x}, \mathbf{p}_s), \mathbf{p}_s) \text{ with probability } \bar{\mathbb{P}}_i^{\bar{u},\beta(\bar{u})} \tag{4.16}$$

is optimal for the informed player.

We would however like to mention that the result of Theorem 4.6. has some subtleties. Different to the case of complete information where (3.26) gives optimal feedback controls, (4.16) does in general not. Indeed $\bar{u}$ depends on the state of the system, i.e. $X^{t,x}$ under $\bar{\mathbb{P}}_i^{\bar{u},\beta(\bar{u})}$ and the shifted randomization $\mathbf{p}$ under the optimal measure $\bar{\mathbb{P}}_i^{\bar{u},\beta(\bar{u})}$. Since this shift depends on the strategy β of the uninformed player, we do not find a random feedback control but a kind of random strategy for the informed player which is not consistent with the random strategies defined in Definition 4.2. To get such a random strategy it would be necessary to show a certain structure of the optimal measure $\bar{\mathbb{P}}$.

4.5 A regularity result

A rather remarkable fact is that the representation in theorem 4.5. via solutions of BSDEs gives us the possibility to derive with probabilistic tools a regularity result for the value function V . Indeed assuming additional regularity for the coefficients:

(i) $(g_i)_{i\in\{1,\dots,I\}} : \mathbb{R}^d \to \mathbb{R}$ is differentiable with bounded, Lipschitz continuous derivative

(ii) $\sigma : [0,T]\times\mathbb{R}^d \to \mathbb{R}^{d\times d}$ is for any $t\in[0,T]$ differentiable with respect to x with bounded, uniformly Lipschitz continuous derivative

(iii) $\tilde{H} : [0,T]\times\mathbb{R}^d\times\mathbb{R}^d\times\Delta(I) \to \mathbb{R}$ is for any $t\in[0,T]$ differentiable in x and z with bounded, uniformly Lipschitz continuous derivative

we show in chapter 4:

Theorem 4.7. *The value function V is semiconcave in x with linear modulus.*

The proof is very similar to the regularity proofs for semilinear PDEs via BSDE techniques in Pardoux and Peng [89]. However in view of (4.12) we probably cannot expect the value function to be smooth, in particular at points where there might be several minimizing measures. In that context the natural notion of regularity is indeed semiconcavity, which is a valuable property if one considers control problems (see e.g. the book of Cannarsa and Sinestrari [21]). In particular by Alexandroff's Theorem we can conclude:

Corollary 4.8. *V is twice differentiable a.e. in x, i.e. for all $t\in[0,T], p\in\Delta(I)$ and a.e. $x_0\in\mathbb{R}^d$ there exists $\xi\in\mathbb{R}^d$, $A\in\mathcal{S}^I$ such that*

$$\lim_{x\to x_0}\frac{V(t,x,p)-V(t,x_0,p)-\langle\xi,x-x_0\rangle+\langle A(x-x_0),x-x_0\rangle}{|x-x_0|^2}=0. \qquad (4.17)$$

Furthermore the gradient $D_xV(t,x,p)$ is defined a.e. and belongs to the class of functions with locally bounded variation.

5 Approximation of stochastic differential games with incomplete information

5.1 Approximation of stochastic differential games and associated HJI equations

The approximation of deterministic differential games with complete information dates back to the 1960s. As we already mentioned in section 3 they were actually used to define values for continuous time games by approximation with discrete time ones. With this ansatz it is even possible to derive ϵ-optimal feedback controls for deterministic continuous time games, where it is sufficient that the player just acts on a sufficiently fine discrete time grid. The results and numerous references can be found in the book of Krasovskiĭ and Subbotin [69].

For the numerical approximation of stochastic differential games a Markov chain approximation method is widely used as described in the textbook of Kushner and Dupuis [72]. The convergence proof usually uses PDE techniques. A purely probabilistic proof of convergence is found in Kushner [70]. However the derivation of ϵ-optimal feedback controls for stochastic differential games with a numerical scheme is more tricky due to the stochastic nature of the game. Indeed, reducing the first player to play on a time grid while the other one still can act and adapt to the Brownian noise during the intervals, might offer the second player a possibility to disproportionally make profit, unless there are rather restrictive additional assumptions fulfilled.

In the spirit of the BSDE ansatz for stochastic differential games of Hamadène and Lepeltier in [59] Bally derives in [4] a method to approximate of the value function of a stochastic differential game via the approximation of solutions of BSDEs. In the article of Bally [4] the approximation is - different to the Markov chain approximation - under a Girsanov transformation of the system, i.e. the approximation of a BSDE with the Hamiltonian as driver is used. ϵ-optimal feedback controls for both players are then derived by reversing the transformation. However the assumptions in Bally [4] for establishing the ϵ-optimality are again rather restrictive.

Besides the ansatz of Bally in [4] there are various other methods to approximate solutions of BSDE. Among the first to further develop the theory were Bouchard and Touzi [15] and Zhang [108] in 2001. Later works on approximations for BSDE and their relations to PDE are given by Bender and Denk [8], Bender and Zhang [9], Delarue and Menozzi [36] and [35]. Indeed, in the light of the close connection of PDEs with BSDEs approximating a BSDE can be seen as a completely probabilistic approximation for the solutions of semilinear PDE - including the HJI equation characterizing stochastic differential games as an example.

On the other hand one can directly consider an approximation of viscosity solutions to semilinear and fully non-linear PDE. The conditions for the convergence of approximation schemes, most notably a monotonicity condition, are investigated by Barles and Souganidis in [7]. In a very natural way such approximations give rise to approximation schemes for BSDE. Monotone approximation schemes were also applied in the recent work of Fahim, Touzi and Warin [48] where fully nonlinear parabolic PDEs are treated. We would also like to refer to the recent survey of Bouchard, Elie and Touzi [14] on the numerical approximation of BSDEs and related PDEs.

5.2 Numerical scheme for stochastic differential games with incomplete information

The approximation for deterministic differential games with incomplete information was given by Cardaliaguet in [26]. The deterministic case with information completeness on both sides has been investigated by Souquière in [102]. In chapter 5 we extend the approximation of [26] to the framework of stochastic differential games, where we consider as in section 4.2 a diffusion with controlled drift but uncontrolled volatility

$$dX_s^{t,x,u,v} = b(s, X_s^{t,x,u,v}, u_s, v_s)ds + \sigma(s, X_s^{t,x,u,v})dB_s \quad X_t^{t,x} = x. \tag{5.1}$$

Again, for each initial data $t \in [0,T]$, $x \in \mathbb{R}^d$, $p \in \Delta(I)$ and random strategy of the informed player $\bar{\alpha} \in (\mathcal{A}^r(t))^I$ and the uninformed player $\beta \in \mathcal{B}^r(t)$ we have an expected payoff

$$J(t,x,p,\bar{\alpha},\beta) = \sum_{i=1}^{I} p_i \, \mathbb{E}_{\bar{\alpha}_i,\beta}\left[\int_t^T l_i(s, X_s^{t,x,\bar{\alpha}_i,\beta}, (\bar{\alpha}_i)_s, \beta_s)ds + g_i(X_T^{t,x,\bar{\alpha}_i,\beta})\right], \tag{5.2}$$

where Isaacs condition is assumed

$$\begin{aligned} &\inf_{u\in U}\sup_{v\in V}\left\{\langle b(t,x,u,v),\xi\rangle + \langle p, l(t,x,u,v)\rangle\right\} \\ &= \sup_{v\in V}\inf_{u\in U}\left\{\langle b(t,x,u,v),\xi\rangle + \langle p, l(t,x,u,v)\rangle\right\} \\ &=: H(t,x,\xi,p) \end{aligned} \tag{5.3}$$

as well as the non-degeneracy of $\sigma(t,x)$.

In constrast to Cardaliaguet [26] and Souquière [102], we can use the latter assumption to work as Bally in [4] on the problem under a Girsanov transform. We then consider an stochastic algorithm, which is very close to the one investigated in Fahim, Touzi and Warin [48], for a semi-linear PDE together with a convexification in p at each time step. The algorithm is constructed as follows:

For $L \in \mathbb{N}$ we define a partition of $[0,T]$ $\Pi^\tau = \{0 = t_0, t_1, \ldots, t_L = T\}$ with stepsize $\tau = \frac{T}{L}$. We will approximate the value function backwards in time. To do so we set for all $k = 0, \ldots, L$, $x \in \mathbb{R}^d$, $p \in \Delta(I)$

$$V^\tau(t_L, x, p) = \langle p, g(x)\rangle \tag{5.4}$$

and we define recursively for $k = L-1, \ldots, 0$

$$\begin{aligned} V^\tau(t_k,x,p) = \mathrm{Vex}_p\big(&\mathbb{E}\left[V^\tau(t_{k+1}, x + \sigma(t_k,x)\Delta B^k, p)\right] \\ &+\tau H(t_k, x, \bar{z}_k(x,p), p)\big), \end{aligned} \tag{5.5}$$

where $\Delta B^j = B_{t_{j+1}} - B_{t_j}$ and $\bar{z}_k(x,p)$ is given by

$$\bar{z}_k(x,p) = \frac{1}{\tau}\mathbb{E}\left[V^\tau(t_{k+1}, x + \sigma(t_k,x)\Delta B^k, p)(\sigma^*)^{-1}(t_k,x)\Delta B^k\right] \tag{5.6}$$

and Vex_p denotes the convex hull with respect to p, i.e. the largest function that is convex in the variable p and does not exceed the given function.

As in Barles and Souganidis [7] we show in chapter 5 of this thesis the convergence to the value of the game:

Theorem 5.1. *V^τ converges uniformly on the compact subsets of $[0,T] \times \mathbb{R}^d \times \Delta(I)$ to $V(t,x,p)$, in the sense that*

$$\lim_{\tau\downarrow 0,\ t_k\to t,\ x'\to x,\ p'\to p} V^\tau(t_k, x', p') = V(t,x,p). \tag{5.7}$$

6 Games of optimal stopping under information incompleteness

6.1 Dynkin Games: history and general results

Dynkin games have been introduced by Dynkin in 1969 in [39] as a game problem of optimal stopping. The game is played by two opponent players who want to minimize resp. maximize a certain payoff. In contrast to the games we described in the previous sections the players are given the possibility to stop the game at any time by undergoing a certain penalty. The problem has attracted a lot of attention from scientists working in the field of probability as well as in the theory of PDE. Concering the latter, the works of Bensoussan and Lions [11], Bensoussan and Friedmann [10], Friedman [53] were the first to consider continuous time stopping games by establishing a relation with variational PDE.

Besides the analytic ansatz there are various purely probabilistic methods applied to study Dynkin games (see e.g. Alario-Nazaret, Lepeltier and Marchal [1], Bismut [13], Ekström and Peskir [40], Eckström and Villeneuve [41], Lepeltier and Maingueneau [79], Morimoto [85], Stettner [104] and the very recent work of Kobylanski, Quenez and de Campagnolle [68]). In combination with controlled diffusions also BSDE methods have been applied by Cvitanic and Karatzas [31], Hamadène and Lepeltier [60]. Therein the optimal stopping game leads to the study of reflected BSDEs introduced in El Karoui, Kapoudjian, Pardoux, Peng, Quenez [44]. Most of the works have the famous Mokobodski condition in common, which is in some sense an eqivalent to Isaacs' condition. It can be completely removed by introducing random stopping times as first shown in Touzi and Vieille in [106] and elaborated by Laraki and Solan in [77].

6.2 A simple example for the analytic approach

For our purpouses we would like to repeat what is well known for Dynkin games with complete information in a rather simple Markovian framework, i.e. as in section 3.1 on the canonical Wiener space $(\mathcal{C}([0,T];\mathbb{R}^d),(\mathcal{H}_t),\mathbb{P}_0)$. The dynamics are given by a diffusion

$$dX_s^{t,x} = b(s,X_s^{t,x})ds + a(s,X_s^{t,x})dB_s \qquad X_t^{t,x} = x. \tag{6.1}$$

We consider a game with two opponent players who want to minimize, respectively maximize a certain payoff $g(X_T^{t,x})$, where

$$g : \mathbb{R}^d \to \mathbb{R}.$$

However - in contrast to the games we considered in section 3 - the players have the possibility to stop the game at any time while undergoing a certain punishment. Player 1 chooses $\tau \in [0,T]$ to minimize, Player 2 chooses $\sigma \in [0,T]$ to maximize the expected payoff

$$J(t,x,\tau,\sigma) = \mathbb{E}\bigg[f(\sigma,X_\sigma^{t,x})1_{\sigma<\tau<T} + h(\tau,X_\tau^{t,x})1_{\tau\leq\sigma,\tau<T} + g(X_T^{t,x})1_{\sigma=\tau=T}\bigg] \tag{6.2}$$

with

(i) early execution payoff for Player 2: $f:[0,T]\times\mathbb{R}^d\to\mathbb{R}$,
(ii) early execution payoff for Player 1: $h:[0,T]\times\mathbb{R}^d\to\mathbb{R}$,

where as usual for the PDE ansatz (see e.g. Bensoussan and Friedmann [10]) it is assumed that for all $t \in [0,T]$, $x \in \mathbb{R}^d$

$$f(t,x) \leq h(t,x) \text{ and } f(T,x) \leq g(x) \leq h(T,x). \tag{6.3}$$

Definition 6.1. *At time $t \in [0,T]$ an admissible stopping time for either player is a $(\mathcal{H}_{t,s})_{s\in[t,T]}$ stopping time with values in $[t,T]$. We denote the set of admissible stopping times by $\mathcal{T}(t)$.*

One can then define the lower value function by

$$V^-(t,x) = \sup_{\sigma\in\mathcal{T}(t)} \inf_{\tau\in\mathcal{T}(t)} J(t,x,\tau,\sigma) \tag{6.4}$$

and the upper value function by

$$V^+(t,x) = \inf_{\tau\in\mathcal{T}(t)} \sup_{\sigma\in\mathcal{T}(t)} J(t,x,\tau,\sigma). \tag{6.5}$$

Again one can use PDE methods to show that the game has a value, i.e. $V^-(t,x) = V^+(t,x) = V(t,x)$. The following characterization dates back to Bensoussan and Friedmann [10] for a smooth case and can be found in the book of Barles [6].

Theorem 6.2. *For any $(t,x) \in [0,T]\times\mathbb{R}^d$ the game has a value $V(t,x)$. The function $V : [0,T]\times\mathbb{R}^d \to \mathbb{R}$ is the unique viscosity solution to*

$$\begin{aligned} \max\{\min\{(-\tfrac{\partial}{\partial t} - \mathcal{L})[w], w - f(t,x)\}, w - h(t,x)\} &= 0 \\ w(T,x) &= g(x), \end{aligned} \tag{6.6}$$

where

$$\mathcal{L}[w](t,x) := \frac{1}{2}\mathrm{tr}(aa^*(t,x)D_x^2 w(t,x)) + b(t,x)D_x w(t,x).$$

6.3 Stochastic Dynkin games with incomplete information

In chapter 6 we consider an optimal stopping game with incomplete information. As in the ansatz of Aumann and Maschler [3], we assume that there are I different states of nature for the game. Before the game starts the state of nature is chosen according to a probability $p \in \Delta(I)$.
The game is played by two opponent players, who want to minimize, respectively maximize a certain payoff $g_i(X_T^{t,x})$ depending on the state of nature $i \in \{1,\dots,I\}$ and on the terminal value of the diffusion $X_T^{t,x}$ given by (6.1), where

$$g_i : \mathbb{R}^d \to \mathbb{R},\ i \in \{1,\dots,I\}$$

As in the previous example the players have the possibility to stop the game at any time while undergoing a certain punishment, i.e. for $i \in \{1,\dots,I\}$

(i) early execution payoff for Player 2: $f_i : [0,T]\times\mathbb{R}^d \to \mathbb{R}$,
(ii) early execution payoff for Player 1: $h_i : [0,T]\times\mathbb{R}^d \to \mathbb{R}$.

For all states of nature $i \in I$ we assume

$$f_i(t,x) \le h_i(t,x) \qquad \text{and} \qquad f_i(T,x) \le g_i(x) \le h_i(T,x). \tag{6.7}$$

When the game starts Player 1 is informed about the state of nature $i \in \{1,\dots,I\}$, Player 2 just knows the respective probabilities p_i. We assume both players observe their opponents control. That means they know immediately, when the game is stopped and the payoff is revealed.

As in Cardaliaguet and Rainer [28] we allow the players to play randomly to hide their information or to make themselves less vulnerable to manipulation. Meaning that they can choose their stopping decision with an additional random device. Using the definition in Laraki and Solan [77] we define:

Definition 6.3. *At time $t \in [0,T]$ an admissible stopping time for either player is a $(\mathcal{H}_{t,s})_{s\in t,T}$ stopping time with values in $[t,T]$. We denote the set of admissible stopping times by $\mathcal{T}(t)$.*
A randomized stopping time after time $t \in [0,T]$ is a measurable function $\mu : [0,1] \times \mathcal{C}([t,T];\mathbb{R}^d) \to [t,T]$ such that for all $r \in [0,1]$

$$\tau^r(\omega) := \mu(r,\omega) \in \mathcal{T}(t)$$

We denote the set of randomized stopping times by $\mathcal{T}^r(t)$.

For any $(t,x,p) \in [0,T]\times\mathbb{R}^d\times\Delta(I)$, $(\mu_1,\dots,\mu_I) \in (\mathcal{T}^r(t))^I, \nu \in \mathcal{T}^r(t)$ we set

$$\begin{aligned} J(t,x,p,\mu,\nu) = \textstyle\sum_{i=1}^I p_i \mathbb{E}_{\mathbb{P}_0\otimes\lambda\otimes\lambda}\Big[& f_i(\nu, X_\nu^{t,x})1_{\nu<\mu_i\le T} \\ & + h_i(\mu_i, X_{\mu_i}^{t,x})1_{\mu_i\le\nu,\mu_i<T} + g_i(X_T^{t,x})1_{\mu_i=\nu=T}\Big], \end{aligned} \tag{6.8}$$

where λ denotes the Lebesgue measure on $[0,1]$. (In the following we will skip the subscript $\mathbb{P}_0\otimes\lambda\otimes\lambda$.)
We define the lower value function by

$$V^-(t,x,p) = \sup_{\nu\in\mathcal{T}^r(t)} \inf_{\mu\in(\mathcal{T}^r(t))^I} J(t,x,p,\mu,\nu) \tag{6.9}$$

and the upper value function by

$$V^+(t,x,p) = \inf_{\mu\in(\mathcal{T}^r(t))^I} \sup_{\nu\in\mathcal{T}^r(t)} J(t,x,p,\mu,\nu). \tag{6.10}$$

To show that the game has a value we establish:

Theorem 6.4. *For any $(t,x,p) \in [0,T]\times\mathbb{R}^d\times\Delta(I)$ the game has a value $V(t,x,p)$. The function $V : [0,T]\times\mathbb{R}^d\times\Delta(I) \to \mathbb{R}$ is the unique viscosity solution to*

$$\begin{aligned} \max\Big\{ \max\{\min\{(-\tfrac{\partial}{\partial t} - \mathcal{L})[w], w - \langle f(t,x),p\rangle\}, & \\ w - \langle h(t,x),p\rangle\}, -\lambda_{\min}\left(p, \tfrac{\partial^2 w}{\partial p^2}\right) \Big\} &= 0 \\ w(T,x,p) &= \textstyle\sum_{i=1,\dots,I} p_i g_i(x), \end{aligned} \tag{6.11}$$

where

$$\mathcal{L}[w](t,x,p) := \frac{1}{2}\mathrm{tr}(aa^*(t,x)D_x^2 w(t,x,p)) + b(t,x)D_x w(t,x,p)$$

and for all $p \in \Delta(I)$, $A \in \mathcal{S}^I$

$$\lambda_{\min}(p,A) := \min_{z \in T_{\Delta(I)(p)} \setminus \{0\}} \frac{\langle Az, z\rangle}{|z|^2}$$

with $T_{\Delta(I)(p)} = \overline{\cup_{\lambda>0}(\Delta(I)-p)/\lambda}$.

6.4 Dual representation for Dynkin games with incomplete information

In a second part in chapter 6 we use the PDE characterization to establish a dual representation of the value function. As in section 4.1 we enlarge the canonical Wiener space to the space $(\mathcal{D}([0,T];\Delta(I)) \times \mathcal{C}([0,T];\mathbb{R}^d), \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ with $\mathbb{P} \in \mathcal{P}(t,p)$. Hence the enlarged space carries besides a Brownian motion B the beliefs of the uninformed player $\mathbf{p}$ as additional dynamic. For each $\mathbb{P} \in \mathcal{P}(t,p)$ we consider a stopping game with this additional dynamic $\mathbf{p}$. However we have to modify the definition of admissible stopping times:

Definition 6.5. *At time* $t \in [0,T]$ *an admissible stopping time for either player is a* $(\mathcal{F}_s)_{s\in[t,T]}$ *stopping time with values in* $[t,T]$*. We denote the set of admissible stopping times by* $\bar{\mathcal{T}}(t)$*.*

We note that in contrast to Definition 6.1 the admissible stopping times at time t might now also depend on the paths of the Brownian motion before time t.

For each $\mathbb{P} \in \mathcal{P}(t,p)$ we consider stopping games with a payoff

$$\begin{aligned} J(t,x,\tau,\sigma,\mathbb{P})_{t-} \quad := \quad \mathbb{E}_{\mathbb{P}}\Big[&\langle \mathbf{p}_\sigma, f(\sigma, X_\sigma^{t,x})\rangle 1_{\sigma<\tau<T} + \langle \mathbf{p}_\tau, h(\tau, X_\tau^{t,x})\rangle 1_{\tau\leq\sigma,\tau<T} \\ &+\langle \mathbf{p}_T, g(X_T^{t,x})\rangle 1_{\sigma=\tau=T} | \mathcal{F}_{t-}\Big], \end{aligned} \tag{6.12}$$

where $\tau \in \bar{\mathcal{T}}(t)$ denotes the stopping time choosen by Player 1, who minimizes, and $\sigma \in \bar{\mathcal{T}}(t)$ denotes the stopping time choosen by Player 2, who maximizes the expected outcome. In contrast to the consideration in the previous section here we are only working with non randomized stopping times.

We would like to mention that the known results in the literature do not imply that the games with cost functionals (6.12) have a value for any fixed $\mathbb{P} \in \mathcal{P}(t,p)$. However our case of interest is the value of the game where the beliefs of the uninformed player $\mathbf{p}$ are manipulated in an optimal way. To that end we define the lower value function by

$$W^-(t,x,p) = \mathrm{essinf}_{\mathbb{P}\in\mathcal{P}(t,p)} \mathrm{esssup}_{\sigma\in\bar{\mathcal{T}}(t)} \mathrm{essinf}_{\tau\in\bar{\mathcal{T}}(t)} J(t,x,\tau,\sigma,\mathbb{P})_{t-} \tag{6.13}$$

and the upper value function by

$$W^+(t,x,p) = \mathrm{essinf}_{\mathbb{P}\in\mathcal{P}(t,p)} \mathrm{essinf}_{\tau\in\bar{\mathcal{T}}(t)} \mathrm{esssup}_{\sigma\in\bar{\mathcal{T}}(t)} J(t,x,\tau,\sigma,\mathbb{P})_{t-}, \tag{6.14}$$

and using the PDE characterization of theorem 6.3. we establish:

Theorem 6.6. *For any* $(t,x,p) \in [0,T] \times \mathbb{R}^d \times \Delta(I)$ *the value of the game can be written as*

$$V(t,x,p) = W^+(t,x,p) = W^-(t,x,p). \tag{6.15}$$

7 Conclusion and perspective

In this work we contributed to the study of stochastic differential games with incomplete information. We established a dual representation for stochastic differential games in terms of a minimization procedure of solutions of BSDEs $Y_{t-}^{t,x,\mathbb{P}}$. These BSDE can be associated to differential games with the beliefs of the uninformed player as additional forward dynamic. This representation allows us to establish by a strikingly simple proof a regularity result for the value function with BSDE methods. Furthermore under the assumption that the value function V is sufficiently smooth and that there is a $\bar{\mathbb{P}}$ such that the value of the game is given by a solution of a BSDE $Y_{t-}^{t,x,\bar{\mathbb{P}}}$ we derive optimal strategies. However the conditions under which the latter assumption is fulfilled is far from being obvious. Indeed the existence and the structure of such an optimal $\bar{\mathbb{P}}$ leaves a challenging open problem for further research.

In a following part of this thesis we consider numerical approximations for stochastic differential games with incomplete information. Therein one can get an idea how approximate martingale measures can be constructed on a discrete time grid. However passing to the limit poses also here a tricky problem. Another challenge is which minimal assumptions one has to impose for the determination of ϵ-optimal strategies for the informed player via numerical methods. The problems one meets due to the stochastic nature of the game are even for stochastic differential games with complete information very difficult to deal with.

The last part of this thesis is devoted to the study of a different kind of game with incomplete information, namely games of optimal stopping. Again, we show that these games have a value which can be characterized as a viscosity solution to a variational PDE. Furthermore we derive a dual representation of the value function in terms of a minimization procedure. In contrast to the first part, it is not clear here if the games over which we are minimizing actually do have a value. Yet we show that in the limit upper and lower value of the dual representation coincide using the PDE characterization of the value function. A natural extension would be to consider stochastic differential games with optimal stopping, so called mixed games as in Hamadène and Lepeltier [60]. However, since the beliefs of the uninformed player are only assumed to be càdlàg the theorems of Hamadène and Lepeltier [60] do not directly apply and need to be generalized.

Chapter 3

A BSDE approach to stochastic differential games with incomplete information

1 Introduction

In this chapter we consider a two player zero-sum game, where the underlying dynamics are given by a diffusion with controlled drift but uncontrolled (non-degenerate) volatility. The game can take place in $I \in \mathbb{N}^*$ different scenarios for the running cost and the terminal outcome as in a classical stochastic differential game. Before the game starts one scenario is picked with the probability $p \in \Delta(I)$, where $\Delta(I)$ denotes the simplex of $\mathbb{R}^I$. The information is transmitted only to Player 1. So at the beginning of the game he knows in which scenario he is playing, while Player 2 only knows the probability p. It is assumed that both players observe the actions of the other one, so Player 2 might infer from the actions of his opponent in which scenario the game is actually played.

It has been proved in Cardaliaguet and Rainer [28] that this game has a value. To investigate the game under the perspective of information transmission we establish an alternative representation of this value. We achieve this by directly modeling the amount of information the informed player reveals during the game. To that end we enlarge the canonical Wiener space to a space which carries besides a Brownain motion, càdlàg martingales with values in $\Delta(I)$. These martingales can be interpreted as possible beliefs of the uninformed player, i.e. the probability in which scenario the game is played in according to his information at time t.

The very same ansatz has been used in the case of deterministic differential games in Cardaliaguet and Rainer [27], while the original idea of the so called a posteriori martingale can already be found in the classical work of Aumann and Maschler (see [3]). Bearing in mind the ideas of Hamadène and Lepeltier [59] we show that the value of our game can be represented by minimizing the solution of a backward stochastic differential equation (BSDE) with respect to possible beliefs of the uninformed player.

A cornerstone in the investigation of stochastic differential games has been laid by Fleming and Souganidis in [52] who extend the results of Evans and Souganidis [47] to a stochastic framework. Therein it is shown that under Isaacs condition the value function of a stochastic differential game is given as the unique viscosity solution of a Hamilton-Jacobi-Isaacs (HJI) equation.

The theory of BSDE, which was originally developped by Peng [92] for stochastic control theory, has been introduced to stochastic differential games by Hamadène and Lepeltier [59] and Hamadène, Lepeltier and Peng [61]. The former results have been extended to cost functionals defined by controlled BSDEs in Buckdahn and Li [19], where the admissible control processes are allowed to depend on events occurring before the beginning of the game.

The study of games with incomplete information has its starting point in the pioneering work of Aumann and Maschler (see [3] and references given therein). The extension to stochastic differential games has been given in Cardaliaguet and Rainer [28]. The proof is accomplished introducing the notion of dual viscosity solutions to the HJI equation of a usual stochastic differential game, where the probability p just appears as an additional parameter. A different unique characterization via the viscosity solution of the HJI equation with an obstacle in the form of a convexity constraint in p is given in Cardaliaguet [25]. We use this latter characterization in order to prove our main representation result.

The outline of the chapter is as follows. In section 2 we describe the game and restate the results of [28] and [25] which build the basis for our investigation. In section 3 we give our main theorem and derive the optimal behaviour for the informed player under some smoothness condition. The whole section 4 is devoted to the proof of the main theorem, while in the appendix we summarize extensions to classical BSDE results, which are needed in our case.

2 Setup

2.1 Formal description of the game

Let $\mathcal{C}([0,T];\mathbb{R}^d)$ be the set of continuous functions from $\mathbb{R}$ to $\mathbb{R}^d$, which are constant on $(-\infty,0]$ and on $[T,+\infty)$. We denote by $B_s(\omega_B)=\omega_B(s)$ the coordinate mapping on $\mathcal{C}([0,T];\mathbb{R}^d)$ and define $\mathcal{H}=(\mathcal{H}_s)$ as the filtration generated by $s\mapsto B_s$. We denote $\Omega_t=\{\omega\in\mathcal{C}([t,T];\mathbb{R}^d)$ and $\mathcal{H}_{t,s}$ the σ-algebra generated by paths up to time s in Ω_t. Furthermore we provide $\mathcal{C}([0,T];\mathbb{R}^d)$ with the Wiener measure $\mathbb{P}_0$ and we consider the respective filtration $\mathcal{H}$ augmented by $\mathbb{P}_0$ nullsets without changing the notation.

In the following we investigate a two-player zero-sum differential game starting at a time $t\geq 0$ with terminal time T. The dynamic is given by a controlled diffusion on $(\mathcal{C}([t,T];\mathbb{R}^d),(\mathcal{H}_{t,s})_{s\in[t,T]},\mathcal{H},\mathbb{P}_0)$, i.e. for $t\in[0,T], x\in\mathbb{R}^d$

$$dX_s^{t,x,u,v}=b(s,X_s^{t,x,u,v},u_s,v_s)ds+\sigma(s,X_s^{t,x,u,v})dB_s \qquad X_t^{t,x}=x. \tag{2.1}$$

We assume that the controls of the players u,v can only take their values in some sets U, V respectively, where U,V are compact subsets of some finite dimensional spaces.

Let $I\in\mathbb{N}^*$ and $\Delta(I)$ denote the simplex of $\mathbb{R}^I$. The objective to optimize is characterized by

(i) running costs: $(l_i)_{i\in\{1,\dots,I\}}:[0,T]\times\mathbb{R}^d\times U\times V\to\mathbb{R}$

(ii) terminal payoffs: $(g_i)_{i\in\{1,\dots,I\}}:\mathbb{R}^d\to\mathbb{R}$,

which are chosen with probability $p\in\Delta(I)$ before the game starts. Player 1 chooses his control u to minimize, Player 2 chooses his control v to maximize the expected payoff. We assume both players observe their opponents control. However Player 1 knows which payoff he optimizes, Player 2 just knows the respective probabilities p_i for scenario $i\in\{1,\dots,I\}$.

The following will be the standing assumption throughout the paper.

Assumption (H)

(i) $b : [0,T] \times \mathbb{R}^d \times U \times V \to \mathbb{R}^d$ is bounded and continuous in all its variables and Lipschitz continuous with respect to (t,x) uniformly in (u,v).

(ii) For $1 \leq k,l \leq d$ the function $\sigma_{k,l} : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ is bounded and Lipschitz continuous with respect to (t,x). For any $(t,x) \in [0,T] \times \mathbb{R}^d$ the matrix $\sigma^*(t,x)$ is non-singular and $(\sigma^*(t,x))^{-1}$ is bounded and Lipschitz continuous with respect to (t,x).

(iii) $(l_i)_{i\in\{1,\dots,I\}} : [0,T] \times \mathbb{R}^d \times U \times V \to \mathbb{R}$ is bounded and continuous in all its variables and Lipschitz continuous with respect to (t,x) uniformly in (u,v). $(g_i)_{i\in\{1,\dots,I\}} : \mathbb{R}^d \to \mathbb{R}$ is bounded and Lipschitz continuous.

(iv) Isaacs condition: for all $(t,x,\xi,p) \in [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \times \Delta(I)$

$$\begin{aligned} &\inf_{u\in U} \sup_{v\in V} \left\{ \langle b(t,x,u,v), \xi \rangle + \sum_{i=1}^I p_i l_i(t,x,u,v) \right\} \\ &= \sup_{v\in V} \inf_{u\in U} \left\{ \langle b(t,x,u,v), \xi \rangle + \sum_{i=1}^I p_i l_i(t,x,u,v) \right\} \\ &=: H(t,x,\xi,p). \end{aligned} \tag{2.2}$$

By assumption (H) the Hamiltonian H is Lipschitz in (ξ,p) uniformly in (t,x) and Lipschitz in (t,x) with Lipschitz constant $c(1+|\xi|)$, i.e. for all $t,t' \in [0,T]$, $x,x' \in \mathbb{R}^d$, $\xi,\xi' \in \mathbb{R}^d$, $p,p' \in \Delta(I)$

$$|H(t,x,\xi,p)| \leq c(1+|\xi|) \tag{2.3}$$

and

$$\begin{aligned} &|H(t,x,\xi,p) - H(t',x',\xi',p')| \\ &\qquad \leq c(1+|\xi|)(|x-x'|+|t-t'|) + c|\xi-\xi'| + c|p-p'|. \end{aligned} \tag{2.4}$$

2.2 Strategies and value function

Definition 2.1. *For any $t \in [0,T]$ an admissible control $u = (u_s)_{s\in[t,T]}$ for Player 1 is a progressively measurable càdlàg process with respect to the filtration $(\mathcal{H}_{t,s})_{s\in[t,T]}$ with values in U. The set of admissible controls for Player 1 is denoted by $\mathcal{U}(t)$.*
The definition for admissible controls $v = (v_s)_{s\in[t,T]}$ for Player 2 is similar. The set of admissible controls for Player 2 is denoted by $\mathcal{V}(t)$.

In differential games with complete information as in [52] it is sufficient, that one player chooses at the beginning an admissible control and the other one chooses the optimal reaction to it. In our case the uninformed player tries to infer from the actions of his opponent in which scenario the game is played and adapts his behavior to his beliefs. Thus a permanent interaction has to be allowed. To this end it is necessary to restrict admissible strategies to have a small delay in time.

Let U_t, respectively V_t, denote the set of càdlàg maps from [t,T] to U, respectively V.

Definition 2.2. *A strategy for Player 1 at time $t \in [0,T]$ is a map $\alpha : [t,T] \times \mathcal{C}([t,T];\mathbb{R}^d) \times V_t \to U_t$ which is nonanticipative with delay, i.e. there is $\delta > 0$ such that for all $s \in [t,T]$*

for any $\omega, \omega' \in \mathcal{C}([t,T];\mathbb{R}^d)$ *and* $v, v' \in V_t$ *we have:* $\omega = \omega'$ *and* $v = v'$ *a.e. on* $[t,s] \Rightarrow \alpha(\cdot,\omega,v) = \alpha(\cdot,\omega',v')$ *a.e. on* $[t,s+\delta]$. *The set of strategies for Player 1 is denoted by* $\mathcal{A}(t)$.
The definition of strategies $\beta : [t,T] \times \mathcal{C}([t,T];\mathbb{R}^d) \times U_t \to V_t$ *for Player 2 is similar. The set of strategies for Player 2 is denoted by* $\mathcal{B}(t)$.

Furthermore it is crucial that the players are allowed to choose their strategies with a certain additional randomness. Intuitively this can be explained by the incentive of the players to hide their information, respectively to protect themselves from manipulation. Thus for the evaluation of a game with incomplete information we introduce random strategies.

Let $\mathcal{I}$ be a fixed set of probability spaces that is nontrivial and stable by finite product.

Definition 2.3. *A random strategy for Player 1 at time* $t \in [0,T]$ *is a a pair* $((\Omega_\alpha, \mathcal{G}_\alpha, \mathbb{P}_\alpha), \alpha)$, *where* $(\Omega_\alpha, \mathcal{G}_\alpha, \mathbb{P}_\alpha)$ *is a probability space in* $\mathcal{I}$ *and* $\alpha : [t,T] \times \Omega_\alpha \times \mathcal{C}([t,T];\mathbb{R}^d) \times V_t \to U_t$ *satisfies*

(i) α *is a measurable function, where* Ω_α *is equipped with the* σ*-field* $\mathcal{G}_\alpha$,

(ii) there exists $\delta > 0$ *such that for all* $s \in [t,T]$ *and for any* $\omega, \omega' \in \mathcal{C}([t,T];\mathbb{R}^d)$ *and* $v, v' \in V_t$ *we have:*

$$\omega = \omega' \text{ and } v = v' \text{ a.e. on } [t,s]$$
$$\Rightarrow \alpha(\cdot,\omega,v) = \alpha(\cdot,\omega',v') \text{ a.e. on } [t,s+\delta] \text{ for any } \omega_\alpha \in \Omega_\alpha.$$

The set of random strategies for Player 1 is denoted by $\mathcal{A}^r(t)$.
The definition of random strategies $((\Omega_\beta, \mathcal{G}_\beta, \mathbb{P}_\beta), \beta)$, *where* $\beta : [t,T] \times \Omega_\beta \times \mathcal{C}([t,T];\mathbb{R}^d) \times U_t \to V_t$ *for Player 2 is similar. The set of random strategies for Player 2 is denoted by* $\mathcal{B}^r(t)$.

Remark 2.4. *In [28] it is shown that one can now associate to each couple of random strategies* $(\alpha,\beta) \in \mathcal{A}^r(t) \times \mathcal{B}^r(t)$ *for any* $(\omega_\alpha, \omega_\beta) \in \Omega_\alpha \times \Omega_\beta$ *a unique couple of admissible strategies* $(u^{\omega_\alpha,\omega_\beta}, v^{\omega_\alpha,\omega_\beta}) \in \mathcal{U}(t) \times \mathcal{V}(t)$, *such that for all* $\omega \in \mathcal{C}([t,T];\mathbb{R}^d)$, $s \in [t,T]$

$$\alpha(s,\omega_\alpha,\omega,v^{\omega_\alpha,\omega_\beta}(\omega)) = u_s^{\omega_\alpha,\omega_\beta}(\omega) \quad \text{and} \quad \beta(s,\omega_\beta,\omega,u^{\omega_\alpha,\omega_\beta}(\omega)) = v_s^{\omega_\alpha,\omega_\beta}(\omega)\ .$$

Furthermore $(\omega_\alpha,\omega_\beta) \to (u^{\omega_\alpha,\omega_\beta}, v^{\omega_\alpha,\omega_\beta})$ *is a measurable map, from* $\Omega_\alpha \times \Omega_\beta$ *equipped with the* σ*-field* $\mathcal{G}_\alpha \otimes \mathcal{G}_\beta$ *to* $\mathcal{V}(t) \times \mathcal{U}(t)$ *equipped with the Borel* σ*-field associated to the* L^1*-distance.*

For any $(t,x,p) \in [0,T] \times \mathbb{R}^d \times \Delta(I)$, $(\bar\alpha_1,\dots,\bar\alpha_I) \in (\mathcal{A}^r(t))^I$, $\beta \in \mathcal{B}^r(t)$ we define for any $(\omega_{\bar\alpha_i},\omega_\beta)$ the process $X^{t,x,\bar\alpha_i,\beta}$ as solution to (2.1) with the associated couple of controls $(u^{\omega_{\bar\alpha_i},\omega_\beta}, v^{\omega_{\bar\alpha_i},\omega_\beta})$. Furthermore we set

$$J(t,x,p,\bar\alpha,\beta) = \sum_{i=1}^I p_i\ \mathbb{E}_{\bar\alpha_i,\beta}\left[\int_0^T l_i(s,X_s^{t,x,\bar\alpha_i,\beta},(\bar\alpha_i)_s,\beta_s)ds + g_i(X_T^{t,x,\bar\alpha_i,\beta})\right], \tag{2.5}$$

where $\mathbb{E}_{\bar\alpha_i,\beta}$ is the expectation on $\Omega_{\bar\alpha_i} \times \Omega_\beta \times \mathcal{C}([t,T];\mathbb{R}^d)$ with respect to the probability $\mathbb{P}_{\bar\alpha_i} \otimes \mathbb{P}_\beta \otimes \mathbb{P}_0$. Here $\mathbb{P}_0$ denotes the Wiener measure on $\mathcal{C}([t,T];\mathbb{R}^d)$. We note that the information advantage of Player 1 is reflected in (2.5) by having the possibility to choose a strategy $\bar\alpha_i$ for each state of nature $i \in \{1,\dots,I\}$.

Under assumption (H) the existence of the value of the game is proved in a more general setting in [28].

Theorem 2.5. *For any $(t, x, p) \in [0, T] \times \mathbb{R}^d \times \Delta(I)$ the game with incomplete information has a value $V(t, x, p)$ given by*

$$\begin{aligned} V(t, x, p) &= \inf_{\bar{\alpha} \in (\mathcal{A}^r(t))^I} \sup_{\beta \in \mathcal{B}^r(t)} J(t, x, p, \bar{\alpha}, \beta) \\ &= \sup_{\beta \in \mathcal{B}^r(t)} \inf_{\bar{\alpha} \in (\mathcal{A}^r(t))^I} J(t, x, p, \bar{\alpha}, \beta). \end{aligned} \tag{2.6}$$

Remark 2.6. *It is well known (e.g. [28] Lemma 3.1) that it suffices for the uninformed player to use admissible (non-random) strategies in the first line of (2.6). So we can use the easier expression*

$$V(t, x, p) = \inf_{\bar{\alpha} \in (\mathcal{A}^r(t))^I} \sup_{\beta \in \mathcal{B}(t)} J(t, x, p, \bar{\alpha}, \beta). \tag{2.7}$$

The existence and uniqueness of the value function $V : [0, T] \times \mathbb{R}^d \times \Delta(I) \to \mathbb{R}$ is first given [28] using the concept of dual viscosity solutions to HJI equations. Starting from this a characterization of the value function as solution of an obstacle problem is given in [25].

Theorem 2.7. *The function $V : [0, T] \times \mathbb{R}^d \times \Delta(I) \to \mathbb{R}$ is the unique viscosity solution to*

$$\min \left\{ \frac{\partial w}{\partial t} + \frac{1}{2} \mathrm{tr}(\sigma\sigma^*(t, x) D_x^2 w) + H(t, x, D_x w, p), \lambda_{\min}\left(p, \frac{\partial^2 w}{\partial p^2}\right) \right\} = 0 \tag{2.8}$$

with terminal condition $w(T, x, p) = \sum_i p_i g_i(x)$ in the class of bounded, uniformly continuous functions, which are uniformly Lipschitz continuous in p. For all $p \in \Delta(I)$, $A \in \mathcal{S}^I$ (where $\mathcal{S}^I$ denotes the set of symmetric $I \times I$ matrices) we set in (2.8)

$$\lambda_{\min}(p, A) := \min_{z \in T_{\Delta(I)(p)} \setminus \{0\}} \frac{\langle Az, z \rangle}{|z|^2}.$$

where $T_{\Delta(I)(p)}$ denotes the tangent cone to $\Delta(I)$ at p, i.e. $T_{\Delta(I)(p)} = \overline{\cup_{\lambda > 0} (\Delta(I) - p)/\lambda}$.

Remark 2.8. *Note that unlike the standard definition of viscosity solutions (see e.g. [29]) the subsolution property to (2.8) is required only on the interior of $\Delta(I)$ while the supersolution property to (2.8) is required on the whole domain $\Delta(I)$ (see [25] and [27]). This is due to the fact that we actually consider viscosity solutions with a state constraint, namely $p \in \Delta(I) \subsetneq \mathbb{R}^I$. For a concise investigation of such problems we refer to [22].*

We do not go into detail about the rather technical proof of Theorem 2.7. in [25]. However there is an easy intuitive explanation of the convexity constraint, which we give in the following remark.

Remark 2.9. *Let $(t, x) \in [0, T] \times \mathbb{R}^d$ be fixed. For any $p_0 \in \Delta(I)$ let $\lambda \in (0, 1)$, $p_1, p_2 \in \Delta(I)$, such that $p_0 = (1 - \lambda)p_1 + \lambda p_2$.*
We consider the game in two steps. First the initial distribution for the game with incomplete information p_1, p_2 is picked with probability $(1 - \lambda), \lambda$. If the outcome is transmitted only to Player 1, the value of this game is $V(t, x, (1 - \lambda)p_1 + \lambda p_2) = V(t, x, p_0)$.
On the other hand we consider the game in which both players are told the outcome of the pick of the initial distribution p_1, p_2. The expected outcome of this game is $(1 - \lambda)V(t, x, p_1) + \lambda V(t, x, p_2)$.
In the first game the informed player knows more, hence, if we make the rather reasonable assumption that the value of information is positive, we have $V(t, x, p_0) \leq (1 - \lambda)V(t, x, p_1) + \lambda V(t, x, p_2)$.

3 Alternative representation of the value function

3.1 Enlargement of the canonical space

In the following we establish a representation of the value function by enlarging the canonical Wiener space to a space which will carry besides a Brownian motion a new dynamic. We use this additional dynamic to model the incorporation of the private information into the game. More precisely we model the probability in which scenario the game is played in according to the information of the uninformed Player 2.

To that end let us denote by $\mathcal{D}([0,T];\Delta(I))$ the set of càdlàg functions from $\mathbb{R}$ to $\Delta(I)$, which are constant on $(-\infty,0)$ and on $[T,+\infty)$. We denote by $\mathbf{p}_s(\omega_p)=\omega_p(s)$ the coordinate mapping on $\mathcal{D}([0,T];\Delta(I))$ and by $\mathcal{G}=(\mathcal{G}_s)$ the filtration generated by $s\mapsto\mathbf{p}_s$. Furthermore we recall that $\mathcal{C}([0,T];\mathbb{R}^d)$ denotes the set of continuous functions from $\mathbb{R}$ to $\mathbb{R}^d$, which are constant on $(-\infty,0]$ and on $[T,+\infty)$. We denote by $B_s(\omega_B)=\omega_B(s)$ the coordinate mapping on $\mathcal{C}([0,T];\mathbb{R}^d)$ and by $\mathcal{H}=(\mathcal{H}_s)$ the filtration generated by $s\mapsto B_s$. We equip the product space $\Omega:=\mathcal{D}([0,T];\Delta(I))\times\mathcal{C}([0,T];\mathbb{R}^d)$ with the right-continuous filtration $\mathcal{F}$, where $\mathcal{F}_t=\cap_{s>t}\mathcal{F}^0_s$ with $(\mathcal{F}^0_s)=(\mathcal{G}_s\otimes\mathcal{H}_s)$. In the following we shall, whenever we work under a fixed probability $\mathbb{P}$ on Ω, complete the filtration $\mathcal{F}$ with $\mathbb{P}$-nullsets without changing the notation.

For $0\le t\le T$ we denote $\Omega_t=\{\omega\in\mathcal{D}([t,T];\Delta(I))\times\mathcal{C}([t,T];\mathbb{R}^d)\}$ and $\mathcal{F}_{t,s}$ the (right-continuous) σ-algebra generated by paths up to time $s\ge t$ in Ω_t. Furthermore we define the space

$$\Omega_{t,s}=\{\omega\in\mathcal{D}([t,s];\Delta(I))\times\mathcal{C}([t,s];\mathbb{R}^d)\}$$

for $0\le t\le s\le T$. If $r\in(t,T]$ and $\omega\in\Omega_t$ then let

$$\omega_1=1_{[-\infty,r)}\omega \qquad \omega_2=1_{[r,+\infty]}(\omega-\omega_{r-})$$

and denote $\pi\omega=(\omega_1,\omega_2)$. The map $\pi:\Omega_t\to\Omega_{t,r}\times\Omega_r$ induces the identification $\Omega_t=\Omega_{t,r}\times\Omega_r$ moreover $\omega=\pi^{-1}(\omega_1,\omega_2)$, where the inverse is defined in an evident way.

For any measure $\mathbb{P}$ on Ω, we denote by $\mathbb{E}_{\mathbb{P}}[\cdot]$ the expectation with respect to $\mathbb{P}$. We equip Ω with a certain class of measures.

Definition 3.1. *Given $p\in\Delta(I)$, $t\in[0,T]$, we denote by $\mathcal{P}(t,p)$ the set of probability measures $\mathbb{P}$ on Ω such that, under $\mathbb{P}$*

(i) $\mathbf{p}$ is a martingale, such that $\mathbf{p}_s=p$ $\forall s<t$, $\mathbf{p}_s\in\{e_i,i=1,\dots,I\}$ $\forall s\ge T$ $\mathbb{P}$-a.s., where e_i denotes the i-th coordinate vector in $\mathbb{R}^I$, and $\mathbf{p}_T$ is independent of $(B_s)_{s\in(-\infty,T]}$,

(ii) $(B_s)_{s\in[0,T]}$ is a Brownian motion.

Remark 3.2. *Assumption (ii) is naturally given by the Brownian structure of the game. Assumption (i) is motivated as follows. Before the game starts the information of the uninformed player is just the initial distribution p. The martingale property, implying $\mathbf{p}_t=\mathbb{E}_{\mathbb{P}}[\mathbf{p}_T|\mathcal{F}_t]$, is due to the best guess of the uninformed player about the scenario he is in. Finally, at the end of the game the information is revealed hence $\mathbf{p}_T\in\{e_i,i=1,\dots,I\}$ and since the scenario is picked before the game starts the outcome $\mathbf{p}_T$ is independent of the Brownian motion.*

3.2 BSDEs for stochastic differential games with incomplete information

The value of a game with incomplete information is studied in [27] in a simpler setting, namely where $X^{t,x,u,v}$ is constant. An alternative representation of the value is given by directly minimizing the expectation of the Hamiltonian over a similar class of martingale measures as in Definition 3.1. (i). In our case $X^{t,x,u,v}$ is a diffusion where the drift is controlled by the players, hence the Hamiltonian (2.2) depends on the first derivative of the value function and a "direct" representation is not possible.
To solve this problem we use the theory of BSDE and extend the result of [59], where the value of an ordinary stochastic differential game is expressed by the solution of a BSDE with the Hamiltonian as driver. In the case of incomplete information we will have additional to the diffusion an extra forward dynamic, namely the beliefs $\mathbf{p}$ of the uninformed player, which are manipulated by the actions of the informed one. He chooses his control - hence indirectly the dynamics of $\mathbf{p}$ - in order to minimize the expected outcome. In this paper we show that we can represent the value function over a direct minimization of the solutions of BSDEs which can be interpreted as the outcome of a stochastic differential game with information completeness and an additional forward dynamic $\mathbf{p}$ (see (3.2)).

First we introduce the following spaces. For any $p \in \Delta(I)$, $t \in [0,T]$ and fixed $\mathbb{P} \in \mathcal{P}(t,p)$ we denote by $\mathcal{L}^2_T(\mathbb{P})$ the set of a square integrable $\mathcal{F}_T$-measurable random variables. We define by $\mathcal{S}^2(\mathbb{P})$ the set of real-valued adapted càdlàg processes ϑ, such that $\mathbb{E}\left[\sup_{s\in[0,T]} \vartheta_s^2\right] < \infty$, furthermore by $\mathcal{H}^2(\mathbb{P})$ the space of $\mathbb{R}^d$-valued progressively measurable processes, such that $\int_0^\cdot \theta_s dB_s$ is a square integrable martingale, i.e. $\mathbb{E}\left[\int_0^T |\theta_s|^2 ds\right] < \infty$. We denote by $\mathcal{M}_0^2(\mathbb{P})$ the space of square integrable martingales null at zero. In the following we shall identify any $N \in \mathcal{M}_0^2(\mathbb{P})$ with its càdlàg modification.
For all $t \in [0,T]$, $x \in \mathbb{R}^d$ we define the process $X^{t,x}$ by

$$X_s^{t,x} = x \quad s < t, \qquad X_s^{t,x} = x + \int_t^s \sigma(r, X_r^{t,x}) dB_r \quad s \geq t. \tag{3.1}$$

Let $p \in \Delta(I)$. We consider for each $\mathbb{P} \in \mathcal{P}(t,p)$ the BSDE

$$\begin{aligned} Y_s^{t,x,\mathbb{P}} &= \langle \mathbf{p}_T, g(X_T^{t,x})\rangle + \textstyle\int_s^T H(r, X_r^{t,x}, Z_r^{t,x,\mathbb{P}}, \mathbf{p}_r) dr \\ &\quad - \textstyle\int_s^T \sigma^*(r, X_r^{t,x}) Z_r^{t,x,\mathbb{P}} dB_r - N_T^{\mathbb{P}} + N_s^{\mathbb{P}}, \end{aligned} \tag{3.2}$$

where $N^{\mathbb{P}} \in \mathcal{M}_0^2(\mathbb{P})$ is strongly orthogonal to the Brownian motion B.

Existence and uniqueness results for the BSDE (3.2) can be found in more generality in [43]. Our case is much simpler, since the driver does not depend on the jump parts. We mention the results which will be relevant for us in the Appendix. Note in particular that as in the standard case one can establish a comparison principle (Theorem 6.3.), which will be crucial in our further calculations.

Theorem 3.3. *Under the assumption (H) the BSDE (3.2) has a unique solution* $(Y^{t,x,\mathbb{P}}, Z^{t,x,\mathbb{P}}, N^{\mathbb{P}}) \in \mathcal{S}^2(\mathbb{P}) \times \mathcal{H}^2(\mathbb{P}) \times \mathcal{M}_0^2(\mathbb{P})$ *and for any* $s \leq T$

$$Y_s^{t,x,\mathbb{P}} = \mathbb{E}_{\mathbb{P}}\left[\int_s^T H(r, X_r^{t,x}, Z_r^{t,x,\mathbb{P}}, \mathbf{p}_r) dr + \langle \mathbf{p}_T, g(X_T^{t,x})\rangle \big| \mathcal{F}_s\right].$$

In particular we get, that

$$Y_{t-}^{t,x,\mathbb{P}} = \mathbb{E}_{\mathbb{P}}\left[\int_t^T H(r, X_r^{t,x}, Z_r^{t,x,\mathbb{P}}, \mathbf{p}_r)dr + \langle \mathbf{p}_T, g(X_T^{t,x})\rangle \big| \mathcal{F}_{t-}\right]. \tag{3.3}$$

Fix $t \in [0,T]$, $x \in \mathbb{R}^d$, $p \in \Delta(I)$. Note that all $\mathbb{P} \in \mathcal{P}(t,p)$ are equal on $\mathcal{F}_{t-}$, i.e. the distribution of $(B_s, \mathbf{p}_s)$ $s \in [0,t)$ is given by $\delta(p) \otimes \mathbb{P}_0$, where $\delta(p)$ is the measure under which $\mathbf{p}$ is constant and equal to p and $\mathbb{P}_0$ is a Wiener measure. So we can identify each $\mathbb{P} \in \mathcal{P}(t,p)$ on $\mathcal{F}_{t-}$ with a common probability measure $\mathbb{Q}$ and define

$$W(t,x,p) = \operatorname{essinf}_{\mathbb{P}\in\mathcal{P}(t,p)} Y_{t-}^{t,x,\mathbb{P}} \qquad \mathbb{Q}\text{-a.s.} \tag{3.4}$$

The aim of this paper is to show the following alternative representation for the value function.

Theorem 3.4. *For any $(t,x,p) \in [0,T]\times\mathbb{R}^d\times\Delta(I)$ the value of the game with incomplete information $V(t,x,p)$ can be characterized as*

$$V(t,x,p) = \operatorname{essinf}_{\mathbb{P}\in\mathcal{P}(t,p)} Y_{t-}^{t,x,\mathbb{P}}. \tag{3.5}$$

We give the proof in the section 4, where we first show that $W(t,x,p)$ is a deterministic function. Then we establish a Dynamic Programming Principle and show that $W(t,x,p)$ is a viscosity solution to (2.8). Since $V(t,x,p)$ is by Theorem 2.7. uniquely defined as the viscosity solution to (2.8), the equality (3.5) is immediate. Before, let us first investigate under smoothness assumptions a possible behavior of an optimal measure and show how the representation is related to the original game.

3.3 A sufficient condition for optimality

We give a sufficient condition for a $\mathbb{P} \in \mathcal{P}(t,p)$ to be optimal in (3.5). We assume, that $V \in \mathcal{C}^{1,2,2}([t,T)\times\mathbb{R}^d\times\Delta(I);\mathbb{R})$ and set

$$\mathcal{H} \;=\; \left\{(t,x,p) \in [0,T)\times\mathbb{R}^d\times\Delta(I) : \frac{\partial V}{\partial t} + \frac{1}{2}\operatorname{tr}(\sigma\sigma^*(t,x)D_x^2V) + H(t,x,D_xV,p) = 0\right\}$$

and

$$\mathcal{H}(t,x) = \{p \in \Delta(I) :\; (t,x,p) \in \mathcal{H}\}.$$

In the theory of games with incomplete information the set $\mathcal{H}$ is usually called the non-revealing set. This is due to the fact that on $\mathcal{H}$ the value function fullfills the standard HJI equation, hence the informed player is not "actively" using his information because the belief of the uninformed player stays unchanged.

Theorem 3.5. *Let $(t,x,p) \in [0,T)\times\mathbb{R}^d\times\Delta(I)$. We assume, that $V \in \mathcal{C}^{1,2,2}([t,T)\times\mathbb{R}^d\times\Delta(I);\mathbb{R})$. Let $\bar{\mathbb{P}} \in \mathcal{P}(t,p)$, such that*

(i) $\mathbf{p}_s \in \mathcal{H}(s, X_s^{t,x})$ $\forall s \in [t,T]$ $\bar{\mathbb{P}}$-a.s.,

(ii) $\bar{\mathbb{P}}$-a.s. we have $\forall s \in [t,T]$, that

$$V(s, X_s^{t,x}, \mathbf{p}_s) - V(s, X_s^{t,x}, \mathbf{p}_{s-}) - \langle \frac{\partial}{\partial p} V(s, X_s^{t,x}, \mathbf{p}_{s-}), \mathbf{p}_s - \mathbf{p}_{s-}\rangle = 0,$$

(iii) $\mathbf{p}$ *is under* $\bar{\mathbb{P}} \in \mathcal{P}(t,p)$ *a purely discontinuous martingale.*
Then $\bar{\mathbb{P}}$ *is optimal for* $V(t,x,p)$ *and* $Z_s^{t,x,\bar{\mathbb{P}}} = D_x V(s, X_s^{t,x}, \mathbf{p}_s)$.

Remark 3.6. *The analysis of the deterministic case in [27] indicates that the conditions (i) and (ii) might also be necessary even in the non-smooth case. In fact under certain assumptions the conditions (i)-(iii) of Theorem 3.5. can expected to be necessary and sufficient. (See [27] Example 4.4.)*

Proof. By definition $V(T,x,p) = \langle g(x), p\rangle$. Since $V \in \mathcal{C}^{1,2,2}$ and $\mathbf{p}$ is purely discontinuous we have by Itô's formula and the assumptions (i)-(iii)

$$\begin{aligned}
\langle g(X_T^{t,x}), \mathbf{p}_T\rangle &= V(T, X_T^{t,x}, \mathbf{p}_T) \\
&= V(s, X_s^{t,x}, \mathbf{p}_s) \\
&\quad + \int_s^T \left(\frac{\partial}{\partial t} V(r, X_r^{t,x}, \mathbf{p}_r) + \frac{1}{2} tr(\sigma\sigma^*(r, X_r^{t,x}) D_x^2 V(s, X_r^{t,x}, \mathbf{p}_r)\right) dr \\
&\quad + \int_s^T \sigma^*(r, X_r^{t,x}) D_x V(r, X_r^{t,x}, \mathbf{p}_r) dB_r \qquad (3.6)\\
&\quad + \sum_{s\leq r\leq T} V(r, X_r^{t,x}, \mathbf{p}_r) - V(r, X_r^{t,x}, \mathbf{p}_{r-}) - \langle \frac{\partial}{\partial p} V(r, X_r^{t,x}, \mathbf{p}_{r-}), \mathbf{p}_r - \mathbf{p}_{r-}\rangle \\
&= V(s, X_s^{t,x}, \mathbf{p}_s) - \int_s^T H(r, X_r^{t,x}, D_x V(r, X_r^{t,x}, \mathbf{p}_r), \mathbf{p}_r) dr \\
&\quad + \int_s^T \sigma^*(r, X_r^{t,x}) D_x V(r, X_r^{t,x}, \mathbf{p}_r) dB_r.
\end{aligned}$$

So by comparison (Theorem 6.3.) the triplet $(Y_s^{t,x,\bar{\mathbb{P}}}, Z_s^{t,x,\bar{\mathbb{P}}}, N_s^{t,x,\bar{\mathbb{P}}}) := (V(s, X_s^{t,x}, \mathbf{p}_s), D_x V(s, X_s^{t,x}, \mathbf{p}_s), 0)$ is the unique solution to the BSDE (3.2).
We have in particular

$$V(t,x,p) = \langle g(X_T^{t,x}), \mathbf{p}_T\rangle - \int_t^T H(s, X_s^{t,x}, Z_s^{t,x,\bar{\mathbb{P}}}, \mathbf{p}_s) ds + \int_t^T \sigma^*(s, X_s^{t,x}) Z_s^{t,x,\bar{\mathbb{P}}} dB_s,$$

hence the result follows from taking conditional expectation and the representation in Theorem 3.3. □

3.4 Optimal "strategies" for the informed player

Our aim is to quantify the amount of information the informed player has to reveal in order to play optimally. Note that in the representation we consider as in [59] the original game under a Girsanov transformation. Hence an optimal measure in (3.5) gives an information structure of the game only up to a Girsanov transformation, which we have to reverse to get back to our original problem.

We assume that $V \in \mathcal{C}^{1,2,2}([t,T) \times \mathbb{R}^d \times \Delta(I); \mathbb{R})$. Let $\bar{\mathbb{P}} \in \mathcal{P}(t,p)$, such that the conditions of Theorem 3.5. are fulfilled, hence $Z_s^{t,x,\bar{\mathbb{P}}} = D_x V(s, X_s^{t,x}, \mathbf{p}_s)$.

Thanks to Isaacs condition, assumption (H) (iv), one can define the function $u^*(t,x,p,\xi)$ as a Borel measurable selection of $\operatorname{argmin}_{u\in U}\{\max_{v\in V}\langle b(t,x,u,v), \xi\rangle + \sum_{i=1}^I p_i l_i(t,x,u,v)\}$,

hence

$$H(t,x,\xi,p) = \max_{v\in V}\big\{\langle b(t,x,u^*(t,x,p,\xi),v),\xi\rangle + \sum_{i=1}^{I} p_i l_i(t,x,u^*(t,x,p,\xi),v)\big\}. \tag{3.7}$$

We define the process

$$\bar{u}_s = u^*(s, X_s^{t,x}, D_x V(s, X_s^{t,x}, \mathbf{p}_s), \mathbf{p}_s), \tag{3.8}$$

where by definition $\bar{u}$ is progressively measurable with respect to the filtration $(\mathcal{F}_s)_{s\in[t,T]}$ with values in U. In the following we will denote the set of such processes the set of controls $\bar{\mathcal{U}}(t)$ and the set of progressively measurable processes with respect to the filtration $(\mathcal{F}_s)_{s\in[t,T]}$ with values in V the set of controls $\bar{\mathcal{V}}(t)$.

We consider for each control $v \in \bar{\mathcal{V}}(t)$ the (F)BSDE

$$\begin{aligned}
X_s^{t,x} &= x + \int_t^s \sigma(r, X_r^{t,x}) dB_r \\
Y_s^{t,x,\bar{u},v} &= \langle \mathbf{p}_T, g(X_T^{t,x})\rangle + \int_s^T \big(\langle \mathbf{p}_r, l(r, X_r^{t,x}, \bar{u}_r, v_r)\rangle \\
&\qquad + \langle b(r, X_r^{t,x}, \bar{u}_r, v_r), D_x V(r, X_r^{t,x}, \mathbf{p}_r)\rangle\big) dr \\
&\qquad - \int_s^T \sigma^*(r, X_r^{t,x}) D_x V(r, X_r^{t,x}, \mathbf{p}_r) dB_r - (N_T - N_s).
\end{aligned} \tag{3.9}$$

Theorem 3.7. *For any $v \in \bar{\mathcal{V}}(t)$ we have, that*

$$Y_{t-}^{t,x,\bar{u},v} \leq Y_{t-}^{t,x,\bar{\mathbb{P}}} = V(t,x,p) \qquad \bar{\mathbb{P}}\text{-a.s..} \tag{3.10}$$

Proof. Since

$$\begin{aligned}
&H(r, X_r^{t,x}, D_x V(r, X_r^{t,x}, \mathbf{p}_r), \mathbf{p}_r) \\
&= \min_{u\in U}\max_{v\in V}\Big\{\langle b(r, X_r^{t,x}, u, v), D_x V(r, X_r^{t,x}, \mathbf{p}_r)\rangle + \langle \mathbf{p}_r, l(r, X_r^{t,x}, u_r, v)\rangle\Big\} \\
&= \max_{v\in V}\Big\{\langle b(r, X_r^{t,x}, \bar{u}_r, v), D_x V(r, X_r^{t,x}, \mathbf{p}_r)\rangle + \langle \mathbf{p}_r, l(r, X_r^{t,x}, \bar{u}_r, v)\rangle\Big\} \\
&\geq \langle b(r, X_r^{t,x}, \bar{u}_r, v_r), D_x V(r, X_r^{t,x}, \mathbf{p}_r)\rangle + \langle \mathbf{p}_r, l(r, X_r^{t,x}, \bar{u}_r, v_r)\rangle,
\end{aligned}$$

(3.10) follows from the comparison Theorem 6.3. □

As in [59] we define now for any $v \in \bar{\mathcal{V}}(t)$ the equivalent measure $\bar{\mathbb{P}}^{\bar{u},v} = (\Gamma_T^{\bar{u},v})\bar{\mathbb{P}}$ with

$$\Gamma_s^{\bar{u},v} = \mathcal{E}\left(\int_t^s b(r, X_r^{t,x}, \bar{u}_r, v_r)\sigma^*(r, X_r^{t,x})^{-1} dB_r\right).$$

for $s \geq t$ and $\Gamma_s^{\bar{u},v} = 1$ for $s < t$. By Girsanov (see e.g. Theorem III.3.24 [65]) we have the following Lemma.

Lemma 3.8. *For any $p \in \Delta(I)$, $t \in [0,T]$, $v \in \bar{\mathcal{V}}(t)$,* *(i)* $X^{t,x}$ *is under* $\bar{\mathbb{P}}^{\bar{u},v}$ *a solution to*

$$X_s^{t,x} = x + \int_t^s b(r, X_r^{t,x}, \bar{u}_r, v_r)dr + \int_t^s \sigma(r, X_r^{t,x})d\bar{B}_r, \tag{3.11}$$

where $\bar{B}$ *is a* $\bar{\mathbb{P}}^{\bar{u},v}$*-Brownian motion.*

(ii) $\mathbf{p}$ *is a* $\bar{\mathbb{P}}^{\bar{u},v}$ *martingale, such that* $\mathbf{p}_s = p \ \forall s < t$, $\mathbf{p}_s \in \{e_i, i = 1, ..., I\} \ \forall s \geq T$ $\bar{\mathbb{P}}^{\bar{u},v}$*-a.s. and* $\mathbf{p}_T$ *is independent of* $(B_s)_{s\in(-\infty,T]}$.

For any $\beta \in \mathcal{B}(t)$ (nonanticipative with delay), we can define the process $\beta(\bar{u})_s = \beta(s, \cdot, \bar{u}_s)$. By definition $\beta(\bar{u}) \in \bar{\mathcal{V}}(t)$. So we can define for any $\beta \in \mathcal{B}(t)$ the measure $\bar{\mathbb{P}}^{\bar{u},\beta(\bar{u})}$.

To take into account that the informed player knows the scenario, we define now for any scenario $i \in \{1, \dots, I\}$ and for any $\beta \in \mathcal{B}(t)$ a probability measure $\bar{\mathbb{P}}_i^{\bar{u},\beta(\bar{u})}$ by: for all $A \in \mathcal{F}$ we have, that

$$\bar{\mathbb{P}}_i^{\bar{u},\beta(\bar{u})}[A] = \bar{\mathbb{P}}^{\bar{u},\beta(\bar{u})}[A|\mathbf{p}_T = e_i] = \frac{1}{p_i}\bar{\mathbb{P}}^{\bar{u},\beta(\bar{u})}[A \cap \{\mathbf{p}_T = e_i\}], \quad \text{if } p_i > 0,$$

and $\bar{\mathbb{P}}_i^{\bar{u},\beta(\bar{u})}[A] = \bar{\mathbb{P}}^{\bar{u},\beta(\bar{u})}[A]$ else. Note that by Lemma 3.8. (ii) $\bar{B}$ is a $\bar{\mathbb{P}}_i^{\bar{u},\beta(\bar{u})}$-Brownian motion, hence $X^{t,x}$ is under $\bar{\mathbb{P}}_i^{\bar{u},\beta(\bar{u})}$ a solution of the SDE (3.11) with $v = \beta(\bar{u})$.

Theorem 3.9. *For any scenario* $i = 1, \dots, I$ *and any strategy of the uninformed player* $\beta \in \mathcal{B}(t)$ *the information transmission* $\bar{\mathbb{P}}_i^{\bar{u},\beta(\bar{u})}$ *is optimal for the informed player in the sense that for any* $\beta \in \mathcal{B}(t)$

$$\sum_{i=1}^{I} p_i \mathbb{E}_{\bar{\mathbb{P}}_i^{\bar{u},\beta(\bar{u})}} \left[\int_t^T l_i(s, X_s^{t,x}, \bar{u}_s, \beta(\bar{u})_s)ds + g_i(X_T^{t,x})\right] \leq V(t, x, p). \tag{3.12}$$

Proof. By definition we have

$$\begin{aligned}
&\sum_{i=1}^{I} p_i \mathbb{E}_{\bar{\mathbb{P}}_i^{\bar{u},\beta(\bar{u})}} \left[\int_t^T l_i(s, X_s^{t,x}, \bar{u}_s, \beta(\bar{u})_s)ds + g_i(X_T^{t,x})\right] \\
&= \sum_{i=1}^{I} p_i \mathbb{E}_{\bar{\mathbb{P}}^{\bar{u},\beta(\bar{u})}} \left[\int_t^T l_i(s, X_s^{t,x}, \bar{u}_s, \beta(\bar{u})_s)ds + g_i(X_T^{t,x})\big|\mathbf{p}_T = e_i\right] \\
&= \sum_{i=1}^{I} \bar{\mathbb{P}}^{\bar{u},\beta(\bar{u})}[\mathbf{p}_T = e_i]\mathbb{E}_{\bar{\mathbb{P}}^{\bar{u},\beta(\bar{u})}} \left[\int_t^T l_i(s, X_s^{t,x}, \bar{u}_s, \beta(\bar{u})_s)ds + g_i(X_T^{t,x})\big|\mathbf{p}_T = e_i\right] \\
&= \sum_{i=1}^{I} \mathbb{E}_{\bar{\mathbb{P}}^{\bar{u},\beta(\bar{u})}} \left[1_{\{\mathbf{p}_T = e_i\}} \int_t^T l_i(s, X_s^{t,x}, \bar{u}_s, \beta(\bar{u})_s)ds + g_i(X_T^{t,x})\right] \\
&= \mathbb{E}_{\bar{\mathbb{P}}^{\bar{u},\beta(\bar{u})}} \left[\langle \mathbf{p}_T, \int_t^T l(s, X_s^{t,x}, \bar{u}_s, \beta(\bar{u})_s)ds\rangle + \langle \mathbf{p}_T, g(X_T^{t,x})\rangle\right] \\
&= \mathbb{E}_{\bar{\mathbb{P}}^{\bar{u},\beta(\bar{u})}} \left[\int_t^T \langle \mathbf{p}_s, l(s, X_s^{t,x}, \bar{u}_s, \beta(\bar{u})_s)\rangle ds + \langle \mathbf{p}_T, g(X_T^{t,x})\rangle\right],
\end{aligned}$$

where in the last step we used the product rule for the $\bar{\mathbb{P}}^{\bar{u},\beta(\bar{u})}$-martingale $\mathbf{p}$ and the adapted finite variation process $\int_t^{\cdot} l(s, X_s^{t,x}, \bar{u}_s, \beta(\bar{u})_s)ds$.
Furthermore we have

$$\mathbb{E}_{\bar{\mathbb{P}}^{\bar{u},\beta(\bar{u})}} \left[\int_t^T \langle \mathbf{p}_s, l(s, X_s^{t,x}, \bar{u}_s, \beta(\bar{u})_s)\rangle ds + \langle \mathbf{p}_T, g(X_T^{t,x})\rangle\right] = \mathbb{E}_{\bar{\mathbb{P}}^{\bar{u},\beta(\bar{u})}} \left[Y_{t-}^{t,x,\bar{u},\beta(\bar{u})}\right],$$

since by Girsanov $Y_s^{t,x,\bar{u},\beta(\bar{u})}$ is under $\bar{\mathbb{P}}^{\bar{u},\beta(\bar{u})}$ given by

$$\begin{aligned} Y_s^{t,x,\bar{u},\beta(\bar{u})} &= \langle \mathbf{p}_T, g(X_T^{t,x})\rangle + \int_s^T \langle \mathbf{p}_r, l(r, X_r^{t,x}, \bar{u}_r, \beta(\bar{u})_r)\rangle dr \\ &\quad - \int_s^T \sigma^*(r, X_r^{t,x}) D_x V(r, X_r^{t,x}, \mathbf{p}_r) d\bar{B}_r - (N_T - N_s). \end{aligned} \tag{3.13}$$

So since by Theorem 3.7. $Y_{t-}^{t,x,\bar{u},\beta(\bar{u})} \leq V(t,x,p)$ $\bar{\mathbb{P}}$-a.s. and $\bar{\mathbb{P}}$ is equivalent to $\bar{\mathbb{P}}^{\bar{u},\beta(\bar{u})}$, we have

$$\mathbb{E}_{\bar{\mathbb{P}}^{\bar{u},\beta(\bar{u})}}\left[Y_{t-}^{t,x,\bar{u},\beta(\bar{u})}\right] \leq V(t,x,p).$$

□

Remark 3.10. *In the simpler case of [27] the representation (3.5) allowed to derive an optimal random control for the informed player in a direct feedback form. Here however there are significant differences. By the Girsanov transformation we have for each $\beta \in \mathcal{B}(t)$ at each time $s \in [t,T]$ an optimal reaction $\bar{u}_s = u^*(s, X_s^{t,x}, D_x V(s, X_s^{t,x}, \mathbf{p}_s), \mathbf{p}_s)$ of the informed player. It depends on the state of the system, i.e. $X^{t,x}$ under $\bar{\mathbb{P}}_i^{\bar{u},\beta(\bar{u})}$ and the shifted randomization $\mathbf{p}$ under the optimal measure $\bar{\mathbb{P}}_i^{\bar{u},\beta(\bar{u})}$. Since this shift depends on the strategy β of the uninformed player, we do not find a random control but a kind of random strategy for the informed player. Note that this "strategy" - none of the less giving us a recipe how the informed player can generate the optimal information flow - is in general not of the form required in Definition 2.4. To get a classical random strategy it would be necessary to show a certain structure of the optimal measure $\bar{\mathbb{P}}$. In a subsequent paper we show how this can be established for ϵ-optimal measures leading to ϵ-optimal strategies in the sense of Definition 2.4.*

4 Proof of Theorem 3.4.

4.1 The function $W(t,x,p)$ and ϵ-optimal strategies

Recall that we defined $W(t,x,p)$ $\mathbb{Q}$-a.s. as $\text{essinf}_{\mathbb{P}\in\mathcal{P}(t,p)} Y_{t-}^{t,x,\mathbb{P}}$, where by definition a random variable ξ is called $\text{essinf}_{\mathbb{P}\in\mathcal{P}(t,p)} Y_{t-}^{t,x,\mathbb{P}}$, if

(i) $\xi \leq Y_{t-}^{t,x,\mathbb{P}}$, $\mathbb{Q}$-a.s., for any $\mathbb{P} \in \mathcal{P}(t,p)$

(ii) if there is another random variable η such that $\eta \leq Y_{t-}^{t,x,\mathbb{P}}$, $\mathbb{Q}$-a.s., for any $\mathbb{P} \in \mathcal{P}(t,p)$, then $\eta \leq \xi$, $\mathbb{Q}$-a.s.

So by its very definition $W(t,x,p)$ is merely a $\mathcal{F}_{t-}$ measurable random variable. However we show that it is deterministic and hence a good candidate to represent the deterministic value function $V(t,x,p)$. Our proof is mainly based on the methods in [19].

Proposition 4.1. *For any $t \in [0,T]$, $x \in \mathbb{R}^d$, $p \in \Delta(I)$*

$$W(t,x,p) = \mathbb{E}_{\mathbb{Q}}[W(t,x,p)] \qquad \mathbb{Q}\text{-a.s.} \tag{4.1}$$

Hence identifying $W(t,x,p)$ with its deterministic version $\mathbb{E}_{\mathbb{Q}}[W(t,x,p)]$ we can consider $W : [0,T] \times \mathbb{R}^d \times \Delta(I) \to \mathbb{R}$ as a deterministic function.

To prove that $W(t,x,p)$ is deterministic it suffices to show that it is independent of the σ-algebra $\sigma(B_s, s \in [0,t])$. Since $\mathbf{p}$ is on $[0,t)$ $\mathbb{Q}$-a.s. a constant Proposition 4.1. follows with Lemma 4.1. in [19].

To show the independence of $\sigma(B_s, s \in [0,t])$ we will use as in [19] a perturbation of $\mathcal{C}([0,T];\mathbb{R}^d)$ with certain elements of the Cameron-Martin space. Let H denote the Cameron-Martin space of all absolutely continuous elements $h \in \mathcal{C}([0,T];\mathbb{R}^d)$, whose Radon-Nikodym derivative $\dot{h}$ belongs to $L^2([0,T];\mathbb{R}^d)$. Denote $H_t = \{h \in H : h(\cdot) = h(\cdot \wedge t)\}$. For any $h \in H_t$, we define for all $(\omega_p, \omega_B) \in \mathcal{D}([0,T];\Delta(I)) \times \mathcal{C}([0,T];\mathbb{R}^d)$ the mapping $\tau_h(\omega_p, \omega_B) := (\omega_p, \omega_B + h)$. Then $\tau_h : \mathcal{D}([0,T];\Delta(I)) \times \mathcal{C}([0,T];\mathbb{R}^d) \to \mathcal{D}([0,T];\Delta(I)) \times \mathcal{C}([0,T];\mathbb{R}^d)$ is a $\mathcal{F} - \mathcal{F}$ measurable bijection with $[\tau_h]^{-1} = \tau_{-h}$.

Lemma 4.2. *For any $h \in H_t$*

$$W(t,x,p) \circ \tau_h = W(t,x,p). \tag{4.2}$$

Proof. Obviously $\tau_h, \tau_h^{-1} : \mathcal{D}([0,T];\Delta(I)) \times \mathcal{C}([0,T];\mathbb{R}^d) \to \mathcal{D}([0,T];\Delta(I)) \times \mathcal{C}([0,T];\mathbb{R}^d)$ are $\mathcal{F}_t - \mathcal{F}_t$ measurable and $(B_s - B_t) \circ \tau_h = (B_s - B_t)$ for all $s \in [t,T]$.

Step 1: Observe that $X_s^{t,x} \circ \tau_h = X_s^{t,x}$ for all $s \in [t,T]$. Then $Y^{t,x,\mathbb{P}} \circ \tau_h$ is the solution to the BSDE

$$\begin{aligned}(Y^{t,x,\mathbb{P}} \circ \tau_h)_s \;&=\; \langle \mathbf{p}_T, g(X_T^{t,x})\rangle + \textstyle\int_s^T H(r, X_r^{t,x}, (Z^{t,x,\mathbb{P}} \circ \tau_h)_r, \mathbf{p}_r) ds \\ &\quad - \textstyle\int_s^T \sigma^*(r, X_r^{t,x})(Z^{t,x,\mathbb{P}} \circ \tau_h)_r dB_r - (N \circ \tau_h)_T + (N \circ \tau_h)_s\end{aligned} \tag{4.3}$$

which is the original BSDE (3.2) however under the different $\mathbb{P} \circ [\tau_h]^{-1}$ dynamics for $\mathbf{p}$. Furthermore $X_{s\in[t,T]}^{t,x}$ under $\mathbb{P}$ and under $\mathbb{P} \circ [\tau_h]^{-1}$ are by Girsanov $\mathbb{P}$-a.s. equal. So under $\mathbb{P} \circ [\tau_h]^{-1}$ the process $Y^{t,x,\mathbb{P}\circ[\tau_h]^{-1}}$ solves (4.3). Since the solution of (4.3) is unique we have in particular

$$Y_{t-}^{t,x,\mathbb{P}} \circ \tau_h = Y_{t-}^{t,x,\mathbb{P}\circ[\tau_h]^{-1}}. \tag{4.4}$$

Step 2: We claim that

$$\left(\mathrm{essinf}_{\mathbb{P}\in\mathcal{P}(t,p)} Y_{t-}^{t,x,\mathbb{P}}\right) \circ \tau_h = \mathrm{essinf}_{\mathbb{P}\in\mathcal{P}(t,p)} \left(Y_{t-}^{t,x,\mathbb{P}} \circ \tau_h\right) \qquad \mathbb{Q}\text{-a.s.} \tag{4.5}$$

Observe that

$$\mathbb{P} \circ [\tau_h]^{-1} = \exp\left(\int_0^t \dot{h}_s dB_s - \frac{1}{2}\int_0^t |\dot{h}_s|^2 ds\right) \mathbb{P} \tag{4.6}$$

for all probability measures $\mathbb{P}$ on Ω. Define $I(t,x,p) = \mathrm{essinf}_{\mathbb{P}\in\mathcal{P}(t,p)} Y_{t-}^{t,x,\mathbb{P}}$. Then $I(t,x,p) \leq Y_{t-}^{t,x,\mathbb{P}}$ $\mathbb{Q}$ a.s.. Since $\mathbb{Q} \circ [\tau_h]^{-1}$ is equivalent to $\mathbb{Q}$ on $\mathcal{F}_{t-}$, we have $I(t,x,p) \circ \tau_h \leq Y_{t-}^{t,x,\mathbb{P}} \circ \tau_h$ $\mathbb{Q}$-a.s.
Furthermore let ξ be a $\mathcal{F}_{t-}$-measurable random variable, such that $\xi \leq Y_{t-}^{t,x,\mathbb{P}} \circ \tau_h$ $\mathbb{Q}$-a.s. Then $\xi \circ [\tau_h]^{-1} \leq Y_{t-}^{t,x,\mathbb{P}}$ $\mathbb{Q}$-a.s.. hence $\xi \circ [\tau_h]^{-1} \leq I(t,x,p)$, so $\xi \leq I(t,x,p) \circ \tau_h$.
Consequently we have

$$I(t,x,p) \circ \tau_h = \mathrm{essinf}_{\mathbb{P}\in\mathcal{P}(t,p)} (Y_{t-}^{t,x,\mathbb{P}} \circ \tau_h).$$

Step 3: Using (4.4) and (4.5) we have $\mathbb{Q}$-a.s.

$$W(t,x,p) \circ \tau_h \;=\; (\mathrm{essinf}_{\mathbb{P}\in\mathcal{P}(t,p)} Y_{t-}^{t,x,\mathbb{P}}) \circ \tau_h$$

$$= \text{essinf}_{\mathbb{P}\in\mathcal{P}(t,p)}(Y_{t-}^{t,x,\mathbb{P}} \circ \tau_h)$$
$$= \text{essinf}_{\mathbb{P}\in\mathcal{P}(t,p)} Y_{t-}^{t,x,\mathbb{P}\circ[\tau_h]^{-1}}.$$

Note that in general $\mathbb{P} \circ [\tau_h]^{-1} \notin \mathcal{P}(t,p)$, since under $\mathbb{P} \circ [\tau_h]^{-1}$ the process B is no longer a Brownian motion on $[0,t]$. We define $\mathbb{P}^h$ on $\Omega = \Omega_{0,t} \times \Omega_t$, such that

$$\mathbb{P}^h = (\delta(p) \otimes \mathbb{P}_0) \otimes (\mathbb{P} \circ [\tau_h]^{-1}|_{\Omega_t}),$$

where $\delta(p)$ is the measure under which $\mathbf{p}$ is constant and equal to p and $\mathbb{P}_0$ is the Wiener measure on $\Omega_{0,t}$.
So by definition $(B_s)_{s\in[t,T]}$ is a Brownian motion under $\mathbb{P}^h$. Also $(\mathbf{p}_s)_{s\in[t,T]}$ is still a martingale under $\mathbb{P}^h$. We can see this immediately, since for all $t \leq s \leq r \leq T$ by (4.6)

$$\mathbb{E}_{\mathbb{P}^h}[\mathbf{p}_r|\mathcal{F}_s] = \mathbb{E}_{\mathbb{P}\circ[\tau_h]^{-1}}[\mathbf{p}_r|\mathcal{F}_s] = \mathbb{E}_{\mathbb{P}}[\mathbf{p}_r|\mathcal{F}_s].$$

Furthermore the remaining conditions of Definition 3.1. are obviously met. Hence $\mathbb{P}^h \in \mathcal{P}(t,p)$ and, since $Y^{t,x,\mathbb{P}\circ[\tau_h]^{-1}}$ is a solution of a BSDE, we have

$$Y_{t-}^{t,x,\mathbb{P}\circ[\tau_h]^{-1}} = Y_{t-}^{t,x,\mathbb{P}^h}.$$

On the other hand by considering $\mathbb{P}\circ\tau_h$ one can associate to any $\mathbb{P} \in \mathcal{P}(t,p)$ a $\mathbb{P}^{-h} \in \mathcal{P}(t,p)$, such that

$$Y_{t-}^{t,x,\mathbb{P}^{-h}\circ[\tau_h]^{-1}} = Y_{t-}^{t,x,\mathbb{P}}.$$

Hence $\left\{Y_{t-}^{t,x,\mathbb{P}\circ[\tau_h]^{-1}} : \mathbb{P} \in \mathcal{P}(t,p)\right\} = \left\{Y_{t-}^{t,x,\mathbb{P}} : \mathbb{P} \in \mathcal{P}(t,p)\right\}$ and

$$\text{essinf}_{\mathbb{P}\in\mathcal{P}(t,p)} Y_{t-}^{t,x,\mathbb{P}\circ[\tau_h]^{-1}} = \text{essinf}_{\mathbb{P}\in\mathcal{P}(t,p)} Y_{t-}^{t,x,\mathbb{P}} = W(t,x,p).$$

□

In the following section we establish some regularity results and a dynamic programming principle. To this end we work with ϵ-optimal measures. Note that since we are taking the essential infimum over a family of random variables, existence of an ϵ-optimal $\mathbb{P}^\epsilon \in \mathcal{P}(t,p)$ is not standard. Therefore we provide a technical lemma, the proof of which is also strongly inspired by [19].

Lemma 4.3. *For any $(t,x,p) \in [0,T] \times \mathbb{R}^d \times \Delta(I)$ there is an ϵ-optimal $\mathbb{P}^\epsilon \in \mathcal{P}(t,p)$ in the sense that*

$$Y_{t-}^{t,x,\mathbb{P}^\epsilon} \leq W(t,x,p) + \epsilon \qquad \mathbb{Q}\text{-a.s.}$$

Proof. Note that there exists a sequence $(\mathbb{P}^n)_{n\in\mathbb{N}}$, $\mathbb{P}^n \in \mathcal{P}(t,p)$, such that

$$W(t,x,p) = \text{essinf}_{\mathbb{P}\in\mathcal{P}(t,p)} Y_{t-}^{t,x,\mathbb{P}} = \inf_{n\in\mathbb{N}} Y_{t-}^{t,x,\mathbb{P}^n}.$$

For an $\epsilon > 0$ set $\Gamma_n := \{W(t,x,p) + \epsilon \geq Y_{t-}^{t,x,\mathbb{P}^n}\} \in \mathcal{F}_{t-}$ for any $n \in \mathbb{N}$. Then $\bar{\Gamma}_1 := \Gamma_1$, $\bar{\Gamma}_n := \Gamma_n \setminus (\cup_{m=1,\ldots,n-1}\bar{\Gamma}_m)$ for $n \geq 2$ form a $\mathcal{F}_{t-}$ measurable partition of Ω.
We define $\mathbb{P}^\epsilon$, such that on $\Omega = \Omega_{0,t} \times \Omega_t$

$$\mathbb{P}^\epsilon = (\delta(p) \otimes \mathbb{P}_0) \otimes \hat{\mathbb{P}},$$

where $\delta(p)$ denotes the measure under which $\mathbf{p}$ is constant and equal to p, $\mathbb{P}_0$ is the Wiener measure on $\Omega_{0,t}$ and $\hat{\mathbb{P}}$ is the measure on Ω_t defined by: for all $A \in \mathcal{B}(\Omega_t)$

$$\hat{\mathbb{P}}[A] = \sum_{n\in\mathbb{N}} (\delta(p) \otimes \mathbb{P}_0)[\bar{\Gamma}_n]\mathbb{P}^n[A|\bar{\Gamma}_n].$$

So by definition $(B_s)_{s\in[t,T]}$ is a Brownian motion under $\mathbb{P}^\epsilon$ and $(\mathbf{p}_s)_{s\in[t,T]}$ is still a martingale under $\mathbb{P}^\epsilon$, since for all $t \leq s \leq r \leq T$

$$\mathbb{E}_{\mathbb{P}^\epsilon}[\mathbf{p}_r|\mathcal{F}_s] = \sum_{n\in\mathbb{N}} \mathbb{E}_{\mathbb{P}^n}[1_{\bar{\Gamma}_n}\mathbf{p}_r|\mathcal{F}_s] = \sum_{n\in\mathbb{N}} 1_{\bar{\Gamma}_n}\mathbb{E}_{\mathbb{P}^n}[\mathbf{p}_r|\mathcal{F}_s] = \sum_{n\in\mathbb{N}} 1_{\bar{\Gamma}_n}\mathbf{p}_s = \mathbf{p}_s.$$

Again the remaining conditions of Definition 3.1. are obviously met. Thus $\mathbb{P}^\epsilon \in \mathcal{P}(t,p)$ and

$$W(t,x,p) + \epsilon \geq \sum_{n\in\mathbb{N}} 1_{\bar{\Gamma}_n} Y_{t-}^{t,x,\mathbb{P}^n} = Y_{t-}^{t,x,\mathbb{P}^\epsilon}.$$

□

Furthermore for technical reasons we introduce the set $\mathcal{P}^f(t,p)$ as the set of all measures $\mathbb{P} \in \mathcal{P}(t,p)$, such that there exists a finite set $S \subset \Delta(I)$ with $\mathbf{p}_s \in S$ $\mathbb{P}$-a.s. for all $s \in [t,T]$.

Remark 4.4. *Note that for any $(t,x,p) \in [0,T] \times \mathbb{R}^d \times \Delta(I)$ $\epsilon > 0$ we can choose an ϵ-optimal $\mathbb{P}^\epsilon$ in the smaller class $\mathcal{P}^f(t,p)$. The idea of the proof is as follows: first choose $\frac{\epsilon}{2}$-optimal measure $\mathbb{P}^\epsilon \in \mathcal{P}(t,p)$. Since $\mathbf{p}$ progressively measurable we can approximate it by an elementary processes $\bar{\mathbf{p}}^\epsilon$, such that with BSDE estimates one has*

$$|Y_{t-}^{t,x,\mathbb{P}^\epsilon} - Y_{t-}^{t,x,\bar{\mathbb{P}}^\epsilon}| \leq \frac{\epsilon}{2},$$

where $\bar{\mathbb{P}}^\epsilon$ is the distribution of $(B, \bar{\mathbf{p}}^\epsilon)$.

4.2 Some regularity results

For technical reasons we will consider the BSDE (3.2) with a slightly different notation. For any $t \in [0,T]$, $x \in \mathbb{R}^d$, $\mathbb{P} \in \mathcal{P}(t,p)$ let

$$\begin{aligned} Y_s^{t,x,\mathbb{P}} &= \langle \mathbf{p}_T, g(X_T^{t,x})\rangle + \int_s^T \tilde{H}(r, X_r^{t,x}, z_r^{t,x,\mathbb{P}}, \mathbf{p}_r)dr \\ &\quad - \int_s^T z_r^{t,x,\mathbb{P}} dB_r - N_T^{\mathbb{P}} + N_s^{\mathbb{P}}, \end{aligned} \tag{4.7}$$

where $\tilde{H}(t,x,p,\xi) = H(t,x,p,(\sigma^*(t,x))^{-1}\xi)$. Setting $Z_s^{t,x,\mathbb{P}} = (\sigma^*(s,X_s^{t,x}))^{-1} z_s^{t,x,\mathbb{P}}$ then gives the solution to (3.2).

In the following we will use the notation $Y_s^{t,x,\mathbb{P}} = Y_s^{t,x}$, $z^{t,x,\mathbb{P}} = z^{t,x}$, $N^{\mathbb{P}} = N$, whenever we work under a fixed $\mathbb{P} \in \mathcal{P}(t,p)$.

Remark 4.5. *Observe that by (H) we have that $\tilde{H}$ is uniformly Lipschitz continuous in (ξ,p) uniformly in (t,x) and Lipschitz continuous in (t,x) with Lipschitz constant $c(1+|\xi|)$, i.e. for all $t,t' \in [0,T]$, $x,x' \in \mathbb{R}^d$, $\xi,\xi' \in \mathbb{R}^d$, $p,p' \in \Delta(I)$*

$$|\tilde{H}(t,x,\xi,p)| \leq c(1+|\xi|) \tag{4.8}$$

and

$$\begin{aligned} &|\tilde{H}(t,x,\xi,p) - \tilde{H}(t',x',\xi',p')| \\ &\leq c(1+|\xi|)(|x-x'| + |t-t'|) + c|\xi-\xi'| + c|p-p'|. \end{aligned} \tag{4.9}$$

Proposition 4.6. *$W(t,x,p)$ is uniformly Lipschitz continuous in x and uniformly Hölder continuous in t.*

Proof. The Lipschitz continuity can be shown by straightforward calculation using Proposition 6.2. For the Hölder continuity in time, let $t,t' \in [0,T]$ such that $t' \le t$ and assume $W(t',x,p) > W(t,x,p)$. Let $\mathbb{P}^\epsilon \in \mathcal{P}(t,p)$ be ϵ-optimal for $W(t,x,p)$ for a sufficiently small ϵ. Note that since $t' \le t$ $\mathbb{P}^\epsilon \in \mathcal{P}(t',p)$. Then we have with Hölder's inequality and Proposition 6.2.

$$\begin{aligned}
0 &\le W(t',x,p) - W(t,x,p) - \epsilon \\
&\le \mathbb{E}_{\mathbb{P}^\epsilon}\left[\int_{t'}^{t} \tilde{H}(s, X_s^{t',x}, z_s^{t',x}, \mathbf{p}_s)ds\right] \\
&\quad + \mathbb{E}_{\mathbb{P}^\epsilon}\left[\int_{t}^{T} \left(\tilde{H}(s, X_s^{t',x}, z_s^{t',x}, \mathbf{p}_s) - \tilde{H}(s, X_s^{t,x}, z_s^{t,x}, \mathbf{p}_s)\right) ds\right] \\
&\quad + \mathbb{E}_{\mathbb{P}^\epsilon}\left[\langle \mathbf{p}_T, g(X_T^{t',x}) - g(X_T^{t,x})\rangle\right] \\
&\le c\, \mathbb{E}_{\mathbb{P}^\epsilon}\left[\int_{t'}^{t} (1+|z_s^{t',x}|)ds\right] \\
&\quad + c\, \mathbb{E}_{\mathbb{P}^\epsilon}\left[\int_{t}^{T} \left((1+|z_s^{t',x}|)|X_s^{t',x} - X_s^{t,x}| + |z_s^{t',x} - z_s^{t,x}|\right) ds\right] \\
&\quad + \mathbb{E}_{\mathbb{P}^\epsilon}\left[|X_T^{t',x} - X_T^{t,x}|\right] \\
&\le c|t'-t|^{\frac{1}{2}} + c\left(\mathbb{E}_{\mathbb{P}^\epsilon}\left[\int_{t}^{T} |X_s^{t',x} - X_s^{t,x}|^2 ds + |X_T^{t',x} - X_T^{t,x}|^2\right]\right)^{\frac{1}{2}} \\
&\le c|t'-t|^{\frac{1}{2}}.
\end{aligned}$$

For the case $t' \le t$, $W(t',x,p) < W(t,x,p)$ choose a $\mathbb{P}^\epsilon \in \mathcal{P}(t',p)$, which is ϵ-optimal for $W(t',x,p)$ for a sufficiently small ϵ. We define then the probability measure $\bar{\mathbb{P}}^\epsilon$, such that on $\Omega = \Omega_{0,t} \times \Omega_t$

$$\bar{\mathbb{P}}^\epsilon = (\delta(p) \otimes \mathbb{P}_0) \otimes \mathbb{P}^\epsilon|_{\Omega_t},$$

where $\delta(p)$ denotes the measure under which $\mathbf{p}$ is constant and equal to p and $\mathbb{P}_0$ is a Wiener measure on $\Omega_{0,t}$. So by definition $(B_s)_{s\in[t,T]}$ is a Brownian motion under $\bar{\mathbb{P}}^\epsilon$. Furthermore the remaining conditions of Definition 3.1. are met, hence $\bar{\mathbb{P}}^\epsilon \in \mathcal{P}(t,p)$ and the same argument as above applies in that case. □

Proposition 4.7. *$W(t,x,p)$ is convex and uniformly Lipschitz continuous with respect to p.*

Proof. 1. To show the convexity in p let $p_1, p_2 \in \Delta(I)$ and let $\mathbb{P}^1 \in \mathcal{P}(t,p_1)$, $\mathbb{P}^2 \in \mathcal{P}(t,p_2)$ be ϵ-optimal for $W(t,x,p_1)$, $W(t,x,p_2)$ respectively. For $\lambda \in [0,1]$ define a martingale measure $\mathbb{P}^\lambda \in \mathcal{P}(t,p_\lambda)$, such that for all measurable $\phi : \mathcal{D}([0,T];\Delta(I)) \times \mathcal{C}([0,T];\mathbb{R}^d) \to \mathbb{R}_+$

$$\mathbb{E}_{\mathbb{P}^\lambda}[\phi(\mathbf{p},B)] = \lambda \mathbb{E}_{\mathbb{P}^1}[\phi(\mathbf{p},B)] + (1-\lambda)\mathbb{E}_{\mathbb{P}^2}[\phi(\mathbf{p},B)].$$

Observe that this can be understood as identifying Ω with $\Omega \times \{1,2\}$ with weights λ and $(1-\lambda)$ for $\Omega \times \{1\}$ and $\Omega \times \{2\}$, respectively. So

$$Y^{t,x,\mathbb{P}^\lambda} = 1_{\Omega\times\{1\}} Y^{t,x,\mathbb{P}^1} + 1_{\Omega\times\{2\}} Y^{t,x,\mathbb{P}^2}.$$

Hence

$$\begin{aligned} W(t,x,p_\lambda) &\leq Y_{t-}^{t,x,\mathbb{P}^\lambda} = 1_{\Omega\times\{1\}} Y_{t-}^{t,x,\mathbb{P}^1} + 1_{\Omega\times\{2\}} Y_{t-}^{t,x,\mathbb{P}^2} \\ &\leq 1_{\Omega\times\{1\}} W(t,x,p_1) + 1_{\Omega\times\{2\}} W(t,x,p_2) + 2\epsilon \end{aligned}$$

and the convexity follows by taking expectation, since ϵ can be chosen arbitrarily small.

2. It remains to prove the uniform Lipschitz continuity in p. Since we have convexity in p, it is sufficient to establish the Lipschitz continuity with respect to p on the extreme points e_i. Observe that $\mathcal{P}(t,e_i)$ consists in the single probability measure $\delta(e_i)\otimes\mathbb{P}_0$, where $\delta(e_i)$ is the measure under which $\mathbf{p}$ is constant and equal to e_i and $\mathbb{P}_0$ is a Wiener measure. The case $W(t,x,e_i)-W(t,x,p)<0$ is immediate. Assume $W(t,x,e_i)-W(t,x,p)>0$. For $\epsilon>0$ let $\mathbb{P}^\epsilon\in\mathcal{P}(t,p)$ be ϵ-optimal for $W(t,x,p)$. Then

$$\begin{aligned} & W(t,x,e_i) - W(t,x,p) - \epsilon \\ & \leq \mathbb{E}_{\mathbb{P}^\epsilon}\left[\int_t^T \left(\tilde{H}(s,X_s^{x,t},z_s^{t,x,e_i},e_i) - \tilde{H}(s,X_s^{t,x},z_s^{t,x},\mathbf{p}_s)\right) ds + \langle \mathbf{p}_T - e_i, g(X_T^{x,t})\rangle \big| \mathcal{F}_{t-}\right] \\ & \leq Y_{t-}^{t,x,e_i} - Y_{t-}^{t,x}. \end{aligned}$$

The uniform Lipschitz continuity of $\tilde{H}$ in ξ and p yields

$$\begin{aligned} Y_{t-}^{t,x,e_i} - Y_{t-}^{t,x} &\leq \langle e_i - \mathbf{p}_T, g(X_T^{x,t})\rangle + c\int_t^T \left(|z_s^{t,x} - z_s^{t,x,e_i}| + |\mathbf{p}_s - e_i|\right) ds \\ &\quad - \int_t^T (z_s^{t,x,e_i} - z_s^{t,x}) dB_s - (N - N^{e_i})_T + (N - N^{e_i})_{t-} \\ &\leq c\left(\int_t^T (1-(\mathbf{p}_s)_i) ds + 1 - (\mathbf{p}_T)_i\right) + c\int_t^T |z_s^{t,x} - z_s^{t,x,e_i}| ds \\ &\quad - \int_t^T (z_s^{t,x,e_i} - z_s^{t,x}) dB_s - (N - N^{e_i})_T + (N - N^{e_i})_{t-}, \end{aligned}$$

where we used that for all $p\in\Delta(I)$ $0\leq|p-e_i|\leq c(1-p_i)$. We define $(\hat{Y},\hat{z},\hat{N})$ as the unique solution to the BSDE

$$\hat{Y}_s = c\left(\int_s^T (1-(\mathbf{p}_r)_i) dr + 1 - (\mathbf{p}_T)_i\right) + c\int_s^T |\hat{z}_r| dr - \int_s^T \hat{z}_r dB_r - (\hat{N}_T - \hat{N}_s).$$

Then by comparison (Theorem 6.3.) we have

$$Y_{t-}^{t,x,e_i} - Y_{t-}^{t,x} \leq \hat{Y}_{t-}.$$

We claim that $\hat{Y}_s = (1-(\mathbf{p}_s)_i)\,\tilde{Y}_s$, where $(\tilde{Y},\tilde{z},\tilde{N})$ is on $[t,T]$ the solution to

$$\tilde{Y}_s = c + c\,(T-s) + \int_s^T |\tilde{z}_r| dr - \int_s^T \tilde{z}_r dB_r. \tag{4.10}$$

This follows directly by applying the Itô folmula

$$(1-(\mathbf{p}_s)_i)\,\tilde{Y}_s \;=\; c\,\left(1-(\mathbf{p}_T)_i\right) + c\int_s^T (1-(\mathbf{p}_r)_i)\,dr + \int_s^T |(1-(\mathbf{p}_r)_i)\,\tilde{z}_r|\,ds$$

$$-\int_s^T (1-(\mathbf{p}_r)_i)\,\tilde{z}_r dB_r + \int_s^T \tilde{Y}_r d(\mathbf{p}_r)_i$$

and identifying $\hat{z}_s = (1-(\mathbf{p}_s)_i)\,\tilde{z}_s$ and $\tilde{N}_s = \int_0^s \tilde{Y}_r d(\mathbf{p}_r)_i$ which is by the definition of $\mathcal{P}^f(t,p)$ purely discontinuous, hence strongly orthogonal to B. Furthermore

$$1-(\mathbf{p}_{t-})_i = 1-p_i \le c\sum_j |(p)_j - \delta_{ij}| \le c\sqrt{I}|p-e_i|,$$

hence

$$Y_{t-}^{t,x,e_i} - Y_{t-}^{t,x} \le \hat{Y}_{t-} = (1-(\mathbf{p}_{t-})_i)\tilde{Y}_{t-} \le c\sqrt{I}|p-e_i|\tilde{Y}_{t-}.$$

It is well known (see e.g. [45]) that, the solution $\tilde{Y}$ to (4.10) is continuous, bounded in $\mathcal{L}^1$ and $\tilde{Y}_t$ is deterministic. So $\tilde{Y}_{t-} = \tilde{Y}_t \le c$ and we have

$$Y_{t-}^{t,x,e_i} - Y_{t-}^{t,x} \le c\sqrt{I}|p-e_i|.$$

□

4.3 Essential Lemmas

In this section we will show two Lemmas which will be essential for the proof of the viscosity solution property.

Lemma 4.8. *For all $\mathbb{P} \in \mathcal{P}^f(t,p)$, $t' \in [t,T]$*

$$Y_{t'-}^{t,x,\mathbb{P}} \ge W(t', X_{t'}^{t,x}, \mathbf{p}_{t'-}) \qquad \mathbb{P}\text{-a.s.}. \tag{4.11}$$

Proof. Fix $\mathbb{P} \in \mathcal{P}^f(t,p)$ and $t' \in [t,T]$. Let $(A_l)_{l\in\mathbb{N}}$ be a partition of $\mathbb{R}^d$ in Borel sets, such that $\mathrm{diam}(A_l) \le \epsilon$ and choose for any $l \in \mathbb{N}$ some $y^l \in A_l$. Let z^{t',y^l} denote the z term of the solution of BSDE (4.7) with forward dynamics X^{t',y^l} instead of $X^{t,x}$. First observe that

$$\begin{aligned}
Y_{t'-}^{t,x} &= \mathbb{E}_{\mathbb{P}}\left[\int_{t'}^T \tilde{H}(s, X_s^{t,x}, z_s^{t,x}, \mathbf{p}_s)ds + \langle \mathbf{p}_T, g(X_T^{t,x})\rangle \big| \mathcal{F}_{t'-}\right] \\
&\ge \sum_{l=1}^{\infty} \mathbb{E}_{\mathbb{P}}\left[\int_{t'}^T \tilde{H}(s, X_s^{t',y^l}, z_s^{t',y^l}, \mathbf{p}_s)ds + \langle \mathbf{p}_T, g(X_T^{t',y^l})\rangle \big| \mathcal{F}_{t'-}\right] 1_{\{X_{t'}^{t,x}\in A^l\}} \\
&\quad -c\sum_{l=1}^{\infty} \mathbb{E}_{\mathbb{P}}\Bigg[\int_{t'}^T \left(|z_s^{t',y^l} - z_s^{x,t}| + (1+|z_s^{x,t}|)|X_s^{t',y^l} - X_s^{t,x}|\right) ds \\
&\qquad\qquad + |X_T^{t',y^l} - X_T^{t,x}| \big| \mathcal{F}_{t'-}\Bigg] 1_{\{X_{t'}^{t,x}\in A^l\}},
\end{aligned}$$

where by Hölder's inequality, Proposition 6.2. and Gronwall's inequality

$$\begin{aligned}
&\sum_{l=1}^{\infty} \mathbb{E}_{\mathbb{P}}\Bigg[\int_{t'}^T \left(|z_s^{t',y^l} - z_s^{x,t}| + (1+|z_s^{x,t}|)|X_s^{t',y^l} - X_s^{t,x}|\right) ds \\
&\qquad\qquad + |X_T^{t',y^l} - X_T^{t,x}| \big| \mathcal{F}_{t'-}\Bigg] 1_{\{X_{t'}^{t,x}\in A^l\}}
\end{aligned}$$

$$\le c \sum_{l=1}^{\infty} \left(\mathbb{E}_{\mathbb{P}} \left[\int_{t'}^{T} |X_s^{t',y^l} - X_s^{t,x}|^2 ds \right. \right.$$
$$\left. \left. + |X_T^{t',y^l} - X_T^{t,x}|^2 \middle| \mathcal{F}_{t'-} \right] \right)^{\frac{1}{2}} 1_{\{X_{t'}^{t,x} \in A^l\}}$$
$$\le c \sum_{l=1}^{\infty} |X_{t'}^{t,x} - y^l| 1_{\{X_{t'}^{t,x} \in A^l\}} \le c\epsilon.$$

Hence

$$\sum_{l=1}^{\infty} Y_{t'-}^{t',y^l} 1_{\{X_{t'}^{t,x} \in A^l\}} - c\epsilon \le Y_{t'-}^{t,x} \le \sum_{l=1}^{\infty} Y_{t'-}^{t',y^l} 1_{\{X_{t'}^{t,x} \in A^l\}} + c\epsilon, \tag{4.12}$$

where the upper bound is given by a similar argument. Furthermore by assumption there exist $S = \{p^1, \ldots, p^k\}$, such that $\mathbb{P}[\mathbf{p}_{t'-} \in S] = 1$. We define for $m = 1, \ldots, k$ the probablility measures $\mathbb{P}^m$, such that on $\Omega = \Omega_{0,t'} \times \Omega_{t'}$

$$\mathbb{P}^m = (\delta(p^m) \otimes \mathbb{P}_0) \otimes \hat{\mathbb{P}}^m,$$

where $\delta(p^m)$ denotes the measure under which $\mathbf{p}$ is constant and equal to p^m, $\mathbb{P}_0$ is the Wiener measure on $\Omega_{0,t'}$ and $\hat{\mathbb{P}}^m$ is the measure on $\Omega_{t'}$ defined by: for all $A \in \mathcal{B}(\Omega_{t'})$

$$\hat{\mathbb{P}}^m[A] = \mathbb{P}[\mathbf{p}_{t'-} = p_m] \mathbb{P}[A | \mathbf{p}_{t'-} = p_m].$$

So by definition $(B_s)_{s \in [t',T]}$ is a Brownian motion under $\mathbb{P}^m$ and $(\mathbf{p}_s)_{s \in [t',T]}$ is a martingale. We see this, since for $t' \le s \le T$

$$\mathbb{E}_{\mathbb{P}^m}[\mathbf{p}_s | \mathcal{F}_{t'-}] = \mathbb{E}_{\mathbb{P}}[1_{\{\mathbf{p}_{t'-} = p_m\}} \mathbf{p}_s | \mathcal{F}_{t'-}] = 1_{\{\mathbf{p}_{t'-} = p^m\}} \mathbf{p}_{t'-} = p^m.$$

Furthermore the remaining conditions of Definition 3.1. are met, hence $\mathbb{P}^m \in \mathcal{P}^f(t', p)$ for $m = 1, \ldots, k$ and

$$Y_{t'-}^{t',y^l,\mathbb{P}^m} 1_{\{\mathbf{p}_{t'-} = p_m\}} \ge W(t', y^l, p^m) 1_{\mathbf{p}_{t'-} = p^m}.$$

So we have, that

$$Y_{t'-}^{t',y^l,\mathbb{P}} = \sum_{m=1}^{k} Y_{t'-}^{t',y^l,\mathbb{P}^m} 1_{\{\mathbf{p}_{t'-} = p_m\}} \ge \sum_{m=1}^{k} W(t', y^l, p^m) 1_{\{\mathbf{p}_{t'-} = p_m\}} = W(t', y^l, \mathbf{p}_{t'-}).$$

Since W is uniformly Lipschitz continuous in x, we have with (4.12)

$$Y_{t'-}^{t,x,\mathbb{P}} \ge W(t', X_{t'}^{x,t}, \mathbf{p}_{t'-}) - c\epsilon.$$

for an arbitrarily small $\epsilon > 0$. □

Lemma 4.9. *For any $\epsilon > 0$, $t' \in [t, T]$ and $\mathbb{P} \in \mathcal{P}^f(t, p)$ one can choose a $\mathbb{P}^\epsilon \in \mathcal{P}^f(t, p)$, such that*

(i) $\mathbb{P}^\epsilon = \mathbb{P}$ *on* $\mathcal{F}_{t'-}$

(ii)

$$Y_{t'-}^{t,x,\mathbb{P}^\epsilon} \leq W(t', X_{t'}^{t,x}, \mathbf{p}_{t'-}) + \epsilon. \tag{4.13}$$

Proof. Fix a $\mathbb{P} \in \mathcal{P}^f(t,p)$. Let $t' \in [t,T]$. By assumption there exist $S = \{p^1, \ldots, p^k\}$, such that $\mathbb{P}[\mathbf{p}_{t'-} \in S] = 1$. Furthermore let $(A_l)_{l\in\mathbb{N}}$ be a partition of $\mathbb{R}^d$ by Borel sets, such that $\operatorname{diam}(A_l) \leq \bar{\epsilon}$ and choose for any $l \in \mathbb{N}$ some $y^l \in A_l$.
Define for any l, m a measure $\mathbb{P}^{l,m} \in \mathcal{P}^f(t', p^m)$, such that

$$\begin{aligned}
&\mathbb{E}_{\mathbb{P}^{l,m}}\left[\int_{t'}^T \tilde{H}(s, X_s^{t',y^l}, z_s^{t',y^l,\mathbb{P}^{l,m}}, \mathbf{p}_s)ds + \langle \mathbf{p}_T, g(X_T^{t',y^l})\rangle \big| \mathcal{F}_{t'-}\right] \\
&\leq \inf_{\mathbb{P}\in\mathcal{P}^f(t',p^m)} \mathbb{E}_{\mathbb{P}}\left[\int_{t'}^T \tilde{H}(s, X_s^{t',y^l}, z_s^{t',y^l,\mathbb{P}}, \mathbf{p}_s)ds + \langle \mathbf{p}_T, g(X_T^{t',y^l})\rangle \big| \mathcal{F}_{t'-}\right] + \epsilon \\
&= W(t', y^l, p^m) + \epsilon.
\end{aligned}$$

We define the probablility measures $\mathbb{P}^\epsilon$, such that on $\Omega = \Omega_{0,t'} \times \Omega_{t'}$

$$\mathbb{P}^\epsilon = (\mathbb{P}|_{\Omega_{0,t'}}) \otimes \hat{\mathbb{P}}.$$

where $\hat{\mathbb{P}}$ is the measure on $\Omega_{t'}$ defined by: for all $A \in \mathcal{B}(\Omega_{t'})$

$$\hat{\mathbb{P}}[A] = \sum_{m=1}^k \sum_{l=1}^\infty \mathbb{P}[X_{t'}^{t,x} \in A^l, \mathbf{p}_{t'-} = p_m]\mathbb{P}^{l,m}[A].$$

So by definition $(B_s)_{s\in[t,T]}$ is a Brownian motion under $\mathbb{P}^\epsilon$. Also $(\mathbf{p}_s)_{s\in[t,T]}$ is a martingale, since for $t' \leq r \leq s \leq T$

$$\mathbb{E}_{\mathbb{P}^\epsilon}[\mathbf{p}_s|\mathcal{F}_r] = \sum_{m=1}^k \sum_{l=1}^\infty 1_{\{X_{t'}^{t,x}\in A^l, \mathbf{p}_{t'-}=p_m\}} \mathbb{E}_{\mathbb{P}^{l,m}}[\mathbf{p}_s|\mathcal{F}_r] = \sum_{m=1}^k \sum_{l=1}^\infty 1_{\{X_{t'}^{t,x}\in A^l, \mathbf{p}_{t'-}=p_m\}} \mathbf{p}_r = \mathbf{p}_r.$$

Furthermore the remaining conditions of Definition 3.1. are obviously met, hence $\mathbb{P}^\epsilon \in \mathcal{P}^f(t,p)$.
Note that by the uniform Lipschitz continuity of $\tilde{H}$ and Proposition 6.2. we have as in (4.12)

$$\begin{aligned}
Y_{t'-}^{t,x,\mathbb{P}^\epsilon} &= \sum_{m=1}^k \sum_{l=1}^\infty 1_{\{X_{t'}^{t,x}\in A^l, \mathbf{p}_{t'-}=p_m\}} Y_{t'-}^{t,x,\mathbb{P}^\epsilon} \\
&\leq \sum_{m=1}^k \sum_{l=1}^\infty 1_{\{X_{t'}^{t,x}\in A^l, \mathbf{p}_{t'-}=p_m\}} Y_{t'-}^{t',y^l,\mathbb{P}^\epsilon} \\
&\qquad + c\sum_{m=1}^k \sum_{l=1}^\infty \mathbb{E}_{\mathbb{P}}\left[1_{\{X_{t'}^{t,x}\in A^l, \mathbf{p}_{t'-}=p_m\}} |X_{t'}^{t,x} - y^l| \big| \mathcal{F}_{t'-}\right] \\
&\leq \sum_{m=1}^k \sum_{l=1}^\infty 1_{\{X_{t'}^{t,x}\in A^l, \mathbf{p}_{t'-}=p_m\}} Y_{t'-}^{t',y^l,\mathbb{P}^\epsilon} + c\bar{\epsilon}.
\end{aligned}$$

So it follows by the definition of $\mathbb{P}^\epsilon$, that

$$Y_{t'-}^{t,x,\mathbb{P}^\epsilon} \leq \sum_{m=1}^k \sum_{l=1}^\infty 1_{\{X_{t'}^{t,x}\in A^l, \mathbf{p}_{t'-}=p_m\}} Y_{t'-}^{t',y^l,\mathbb{P}^{l,m}} + c\bar{\epsilon}$$

$$\leq \sum_{m=1}^{k}\sum_{l=1}^{\infty} 1_{\{X_{t'}^{t,x}\in A^l, \mathbf{p}_{t'-}=p_m\}} W(t',y^l,p^m) + \epsilon + c\bar{\epsilon}$$
$$\leq W(t', X_{t'}^{t,x}, \mathbf{p}_{t'-}) + \epsilon + c\bar{\epsilon}$$

and the result follows, since $\bar{\epsilon}$ can be chosen arbitrarily small. □

4.4 Viscosity solution property

To prove that W is a viscosity solution to (2.8) we first show the subsolution property.

Proposition 4.10. *W is a viscosity subsolution to (2.8).*

Proof. Let $\phi : [0,T] \times \mathbb{R}^d \times \Delta(I) \to \mathbb{R}$ be a test function such that $W - \phi$ has a strict global maximum at $(\bar{t}, \bar{x}, \bar{p}) \in [0,T) \times \mathbb{R}^d \times \mathrm{Int}(\Delta(I))$ with $W(\bar{t},\bar{x},\bar{p}) - \phi(\bar{t},\bar{x},\bar{p}) = 0$. We have to show, that

$$\min\left\{\frac{\partial\phi}{\partial t} + \frac{1}{2}\mathrm{tr}(\sigma\sigma^*(t,x)D_x^2\phi) + H(t,x,D_x\phi,p), \lambda_{\min}\left(\frac{\partial^2\phi}{\partial p^2}\right)\right\} \geq 0 \qquad (4.14)$$

holds at $(\bar{t},\bar{x},\bar{p})$.
By Proposition 4.7. W is convex in p. So since $\bar{p} \in \mathrm{Int}(\Delta(I))$, we already get

$$\lambda_{\min}\left(\frac{\partial^2\phi}{\partial p^2}(\bar{t},\bar{x},\bar{p})\right) \geq 0.$$

Set $\mathbb{P} := \delta(\bar{p}) \otimes \mathbb{P}_0$. Then by Lemma 4.9 we can choose for any $\epsilon > 0$ and any $t \in (\bar{t}, T]$ a measure $\mathbb{P}^\epsilon \in \mathcal{P}^f(\bar{t},\bar{p})$, such that $\mathbb{P} = \mathbb{P}^\epsilon$ on $\mathcal{F}_{t-}$ and we have the estimate

$$Y_{t-}^{\bar{t},\bar{x},\mathbb{P}^\epsilon} \leq W(t, X_t^{\bar{t},\bar{x}}, \bar{p}) + \epsilon(t-\bar{t}). \qquad (4.15)$$

Then since $\mathbb{P} = \mathbb{P}^\epsilon$ on $\mathcal{F}_{t-}$

$$Y_{t-}^{\bar{t},\bar{x},\mathbb{P}^\epsilon} = \mathbb{E}_{\mathbb{P}}\left[\int_{\bar{t}}^{t} \tilde{H}(s, X_s^{\bar{t},\bar{x}}, z_s^{\bar{t},\bar{x},\mathbb{P}^\epsilon}, \bar{p})ds + Y_{t-}^{\bar{t},\bar{x},\mathbb{P}^\epsilon} \big| \mathcal{F}_{t-}\right]. \qquad (4.16)$$

So by comparison

$$Y_{\bar{t}-}^{\bar{t},\bar{x},\mathbb{P}^\epsilon} \leq \bar{Y}_{\bar{t}-}^{\bar{t},\bar{x}} + \epsilon(t-\bar{t}), \qquad (4.17)$$

where $\bar{Y}_{\bar{t}-}^{\bar{t},\bar{x}}$ is given by the solution to

$$\bar{Y}_s^{\bar{t},\bar{x}} = W(t, X_t^{\bar{t},\bar{x}}, \bar{p}) + \int_s^t \tilde{H}(s, X_s^{\bar{t},\bar{x}}, \bar{z}_s^{\bar{t},\bar{x}}, \bar{p})ds + \int_s^t \bar{z}_s^{\bar{t},\bar{x}} dB_s.$$

Hence by (4.17)

$$W(\bar{t},\bar{x},\bar{p}) \leq Y_{\bar{t}-}^{\bar{t},\bar{x},\mathbb{P}^\epsilon} \leq \mathbb{E}_{\mathbb{P}}\left[\int_{\bar{t}}^{t} \tilde{H}(s, X_s^{\bar{t},\bar{x}}, \bar{z}_s^{\bar{t},\bar{x}}, \bar{p})ds + W(t, X_t^{\bar{t},\bar{x}}, \bar{p})\big|\mathcal{F}_{\bar{t}-}\right] + \epsilon(t-\bar{t}).$$

Since by standard Markov arguments $\mathbb{E}\left[\int_{\bar{t}}^{t} \tilde{H}(s, X_s^{\bar{t},\bar{x}}, \bar{z}_s^{\bar{t},\bar{x}}, \bar{p})ds + W(t, X_t^{\bar{t},\bar{x}}, \bar{p})\big|\mathcal{F}_{\bar{t}-}\right]$ is deterministic and $\phi(\bar{t},\bar{x},\bar{p}) = W(\bar{t},\bar{x},\bar{p})$ and $W \leq \phi$ by construction, this yields

$$\phi(\bar{t},\bar{x},\bar{p}) \leq \mathbb{E}_{\mathbb{P}}\left[\int_{\bar{t}}^{t} \tilde{H}(s, X_s^{\bar{t},\bar{x}}, z_s^{\bar{t},\bar{x}}, \bar{p})ds + \phi(t, X_t^{\bar{t},\bar{x}}, \bar{p})\right] + \epsilon(t-\bar{t}), \qquad (4.18)$$

which implies (4.14) as $t \downarrow \bar{t}$ by standard results (see e.g. [45]) since ϵ can be chosen arbitrarily small. □

Proposition 4.11. *W is a viscosity supersolution to (2.8).*

Proof. Let $\phi : [0,T] \times \mathbb{R}^d \times \Delta(I) \to \mathbb{R}$ be a smooth test function, such that $W - \phi$ has a strict global minimum at $(\bar{t}, \bar{x}, \bar{p}) \in [0,T) \times \mathbb{R}^d \times \Delta(I)$ with $W(\bar{t}, \bar{x}, \bar{p}) - \phi(\bar{t}, \bar{x}, \bar{p}) = 0$ and such that its derivatives are uniformly Lipschitz in p.
We have to show, that

$$\min\left\{\frac{\partial \phi}{\partial t} + \frac{1}{2}\mathrm{tr}(\sigma\sigma^*(t,x)D_x^2\phi) + H(t,x,D\phi,p), \lambda_{\min}\left(\frac{\partial^2 \phi}{\partial p^2}\right)\right\} \leq 0 \qquad (4.19)$$

holds at $(\bar{t}, \bar{x}, \bar{p})$. Observe that, if $\lambda_{\min}\left(\frac{\partial^2\phi}{\partial p^2}\right) \leq 0$ at $(\bar{t}, \bar{x}, \bar{p})$, then (4.19) follows immediately.
We assume in the subsequent steps strict convexity of ϕ in p at $(\bar{t}, \bar{x}, \bar{p})$, i.e. there exist $\delta, \eta > 0$ such that for all $z \in T_{\Delta(I)(\bar{p})}$

$$\langle \frac{\partial^2 \phi}{\partial p^2}(t,x,p)z, z\rangle > 4\delta|z|^2 \qquad \forall (t,x,p) \in B_\eta(\bar{t}, \bar{x}, \bar{p}). \qquad (4.20)$$

Since ϕ is a test function for a purely local viscosity notion, one can modify it outside a neighborhood of $(\bar{t}, \bar{x}, \bar{p})$ such that for all $(s,x) \in [\bar{t}, T] \times \mathbb{R}^d$ the function $\phi(s,x,\cdot)$ is convex on the whole convex domain $\Delta(I)$. Thus for any $p \in \Delta(I)$

$$W(t,x,p) \geq \phi(t,x,p) \geq \phi(t,x,\bar{p}) + \langle \frac{\partial \phi}{\partial p}(t,x,p), p - \bar{p}\rangle. \qquad (4.21)$$

We divide the proof in several steps. First we apply an estimate which is stronger than (4.21) basing on assumption (4.20). In the second step we use the stronger estimate and the dynamic programming to establish estimates for $\mathbf{p}$. The subsequent steps are rather close to the standard case. We reduce the problem by considering a BSDE on a smaller time interval. Then we establish estimates for the auxiliary BSDE, which we use in the last step to show the viscosity supersolution property.

Step 1: As in [27] one can show that there exist $\eta, \delta > 0$, such that for all $(t,x) \in [\bar{t}, \bar{t}+\eta] \times B_\eta(\bar{x})$, $p \in \Delta(I)$

$$W(t,x,p) \geq \phi(t,x,\bar{p}) + \langle \frac{\partial \phi}{\partial p}(t,x,p), p - \bar{p}\rangle + 2\delta|p - \bar{p}|^2. \qquad (4.22)$$

By (4.22) we have for any $t > \bar{t}$ such that $(t - \bar{t}) < \eta$

$$\begin{aligned}
&W(t, X_t^{\bar{t},\bar{x}}, \mathbf{p}_{t-}) \\
&\geq 1_{\{|X_t^{\bar{t},\bar{x}} - \bar{x}| < \eta\}} \left(\phi(t, X_t^{\bar{t},\bar{x}}, \bar{p}) + \langle \frac{\partial \phi}{\partial p}(t, X_t^{\bar{t},\bar{x}}, \bar{p}), \mathbf{p}_{t-} - \bar{p}\rangle + \delta|\mathbf{p}_{t-} - \bar{p}|^2\right) \\
&\quad + 1_{\{|X_t^{\bar{t},\bar{x}} - \bar{x}| \geq \eta\}} \phi(t, X_t^{\bar{t},\bar{x}}, \mathbf{p}_{t-}) \\
&\geq \phi(t, X_t^{\bar{t},\bar{x}}, \bar{p}) + \langle \frac{\partial \phi}{\partial p}(t, X_t^{\bar{t},\bar{x}}, \bar{p}), \mathbf{p}_{t-} - \bar{p}\rangle + 1_{\{|X_t^{\bar{t},\bar{x}} - \bar{x}| < \eta\}} \delta|\mathbf{p}_{t-} - \bar{p}|^2 \\
&\quad + 1_{\{|X_t^{\bar{t},\bar{x}} - \bar{x}| \geq \eta\}} \left(\phi(t, X_t^{\bar{t},\bar{x}}, \mathbf{p}_{t-}) - \phi(t, X_t^{\bar{t},\bar{x}}, \bar{p}) - \langle \frac{\partial \phi}{\partial p}(t, X_t^{\bar{t},\bar{x}}, \bar{p}), \mathbf{p}_{t-} - \bar{p}\rangle\right).
\end{aligned}$$

Recalling that ϕ is convex with respect to p, we get

$$\begin{aligned} &W(t, X_t^{\bar{t},\bar{x}}, \mathbf{p}_{t-}) \\ &\geq \phi(t, X_t^{\bar{t},\bar{x}}, \bar{p}) + \langle \tfrac{\partial \phi}{\partial p}(t, X_t^{\bar{t},\bar{x}}, \bar{p}), \mathbf{p}_{t-} - \bar{p}\rangle + \delta 1_{\{|X_t^{\bar{t},\bar{x}} - \bar{x}| < \eta\}} |\mathbf{p}_{t-} - \bar{p}|^2. \end{aligned} \tag{4.23}$$

Step 2: Next we establish with the help of (4.23) an estimate for $\mathbf{p}$. First we choose for any $\epsilon > 0$, $t > \bar{t}$ a $\mathbb{P}^\epsilon \in \mathcal{P}^f(\bar{t}, \bar{p})$ such that we have

$$\begin{aligned} &W(\bar{t}, \bar{x}, \bar{p}) + \epsilon(t - \bar{t}) \\ &\geq \mathbb{E}_{\mathbb{P}^\epsilon}\left[\int_{\bar{t}}^t \tilde{H}(s, X_s^{\bar{t},\bar{x}}, z_s^{\bar{t},\bar{x}}, \mathbf{p}_s)ds + Y_{t-}^{\bar{t},\bar{x}} \big| \mathcal{F}_{t-}\right], \end{aligned} \tag{4.24}$$

where used (and will use in the following) the notation $z_s^{\bar{t},\bar{x},\mathbb{P}^\epsilon} = z_s^{\bar{t},\bar{x}}$, $Y^{\bar{t},\bar{x},\mathbb{P}^\epsilon} = Y^{\bar{t},\bar{x}}$. Note that Lemma 4.8. implies that $\mathbb{P}^\epsilon$-a.s.

$$Y_{t'-}^{\bar{t},\bar{x}} \geq W(t, X_t^{\bar{t},\bar{x}}, \mathbf{p}_{t-}). \tag{4.25}$$

So using (4.25) in (4.24) we get

$$\epsilon(t - \bar{t}) \geq \mathbb{E}_{\mathbb{P}^\epsilon}\left[\int_{\bar{t}}^t \tilde{H}(s, X_s^{\bar{t},\bar{x}}, z_s^{\bar{t},\bar{x}}, \mathbf{p}_s)ds + W(t, X_t^{\bar{t},\bar{x}}, \mathbf{p}_{t-}) - W(\bar{t}, \bar{x}, \bar{p}) \big| \mathcal{F}_{\bar{t}-}\right]. \tag{4.26}$$

Hence with (4.23) it follows for all $t > \bar{t}$, such that $(t - \bar{t}) < \eta$,

$$\begin{aligned} \epsilon(t - \bar{t}) \;\geq\; \mathbb{E}_{\mathbb{P}^\epsilon}\Bigg[&\int_{\bar{t}}^t \tilde{H}(s, X_s^{\bar{t},\bar{x}}, z_s^{\bar{t},\bar{x}}, \mathbf{p}_s)ds + \phi(t, X_t^{\bar{t},\bar{x}}, \bar{p}) - \phi(\bar{t}, \bar{x}, \bar{p}) \\ &+ \langle \tfrac{\partial \phi}{\partial p}(t, X_t^{\bar{t},\bar{x}}, \bar{p}), \mathbf{p}_{t-} - \bar{p}\rangle + \delta 1_{\{|X_t^{\bar{t},\bar{x}} - \bar{x}| < \eta\}} |\mathbf{p}_{t-} - \bar{p}|^2 \big| \mathcal{F}_{\bar{t}-}\Bigg]. \end{aligned} \tag{4.27}$$

With (4.8) we have for a generic constant c

$$\left|\mathbb{E}_{\mathbb{P}^\epsilon}\left[\int_{\bar{t}}^t \tilde{H}(s, X_s^{\bar{t},\bar{x}}, z_s^{\bar{t},\bar{x}}, \mathbf{p}_s)ds \big| \mathcal{F}_{\bar{t}-}\right]\right| \leq c\mathbb{E}_{\mathbb{P}^\epsilon}\left[\int_{\bar{t}}^t (1 + |z_s^{\bar{t},\bar{x}}|)ds \big| \mathcal{F}_{\bar{t}-}\right] \leq c(t - \bar{t})^{\frac{1}{2}}, \tag{4.28}$$

since by Proposition 6.2.

$$\mathbb{E}_{\mathbb{P}^\epsilon}\left[\int_{\bar{t}}^t |z_s^{\bar{t},\bar{x}}|^2 ds \big| \mathcal{F}_{\bar{t}-}\right] \leq c\mathbb{E}_{\mathbb{P}^\epsilon}\left[\int_{\bar{t}}^t |X_s^{\bar{t},\bar{x}}|^2 ds \big| \mathcal{F}_{\bar{t}-}\right] \leq c.$$

Furthermore by Itô's formula and the assumptions on ϕ it follows, that

$$\mathbb{E}_{\mathbb{P}^\epsilon}\left[|\phi(t, X_t^{\bar{t},\bar{x}}, \bar{p}) - \phi(\bar{t}, \bar{x}, \bar{p})| \big| \mathcal{F}_{\bar{t}-}\right] \leq c(t - \bar{t})^{\frac{1}{2}}. \tag{4.29}$$

Next, let $f : [\bar{t}, t] \times \mathbb{R}^n \to \mathbb{R}^n$ be a smooth bounded function, with bounded derivatives. Recall that under any $\mathbb{P} \in \mathcal{P}^f(\bar{t}, \bar{p})$ the process $\mathbf{p}$ is strongly orthogonal to B. So since under $\mathbb{P}^\epsilon$ the process $\mathbf{p}$ is a martingale with $\mathbb{E}_{\mathbb{P}^\epsilon}[\mathbf{p}_{t-}|\mathcal{F}_{\bar{t}-}] = \bar{p}$, we have by Itô's formula, that

$$\mathbb{E}_{\mathbb{P}^\epsilon}\left[f_i(t, X_t^{\bar{t},\bar{x}})(\mathbf{p}_{t-} - \bar{p})_i \big| \mathcal{F}_{\bar{t}-}\right]$$

$$
\begin{aligned}
&= \mathbb{E}_{\mathbb{P}^\epsilon}\left[\int_{\bar t}^{t} f_i(s,X_s^{\bar t,\bar x})d(\mathbf{p}_s)_i + \int_{\bar t}^{t}(\mathbf{p}_s-\bar p)_i df_i(s,X_s^{\bar t,\bar x})\big|\mathcal{F}_{\bar t-}\right]\\
&= \mathbb{E}_{\mathbb{P}^\epsilon}\Bigg[\int_{\bar t}^{t}\Bigg(\frac{\partial}{\partial t}f_i(s,X_s^{\bar t,\bar x}) + \langle D_x f_i(s,X_s^{\bar t,\bar x}), b(X_s^{\bar t,\bar x})\rangle\\
&\qquad + \frac{1}{2}\mathrm{tr}(\sigma\sigma^*(s,X_s^{\bar t,\bar x})D_x^2 f_i(s,X_s^{\bar t,\bar x}))\Bigg)(\mathbf{p}_s-\bar p)_i ds\big|\mathcal{F}_{\bar t-}\Bigg].
\end{aligned}
$$

Hence by (H)

$$
\begin{aligned}
\left|\mathbb{E}_{\mathbb{P}^\epsilon}\left[\langle \tfrac{\partial\phi}{\partial p}(t,X_t^{\bar t,\bar x},\bar p),\mathbf{p}_{t-}-\bar p\rangle\big|\mathcal{F}_{\bar t-}\right]\right| &\le c\mathbb{E}_{\mathbb{P}^\epsilon}\left[\textstyle\int_{\bar t}^{t}|\mathbf{p}_s-\bar p|ds\big|\mathcal{F}_{\bar t-}\right]\\
&\le c(t-\bar t).
\end{aligned}
\tag{4.30}
$$

Furthermore observe that, since $|\mathbf{p}_{t-}-\bar p|\le 1$, we have for $\epsilon'>0$, that by Young's and Hölder's inequality

$$
\begin{aligned}
&\mathbb{E}_{\mathbb{P}^\epsilon}\left[1_{\{|X_t^{\bar t,\bar x}-\bar x|<\eta\}}|\mathbf{p}_{t-}-\bar p|^2\big|\mathcal{F}_{\bar t-}\right]\\
&\ge \mathbb{E}_{\mathbb{P}^\epsilon}\left[|\mathbf{p}_{t-}-\bar p|^2\big|\mathcal{F}_{\bar t-}\right] - \frac{1}{\eta}\mathbb{E}_{\mathbb{P}^\epsilon}\left[|X_t^{\bar t,\bar x}-\bar x||\mathbf{p}_{t-}-\bar p|^2\big|\mathcal{F}_{\bar t-}\right]\\
&\ge (1-\frac{\epsilon'}{\eta})\mathbb{E}_{\mathbb{P}^\epsilon}\left[|\mathbf{p}_{t-}-\bar p|^2\big|\mathcal{F}_{\bar t-}\right] - \frac{1}{4\eta\epsilon'}\mathbb{E}_{\mathbb{P}^\epsilon}\left[|X_t^{\bar t,\bar x}-\bar x|^2\big|\mathcal{F}_{\bar t-}\right],
\end{aligned}
$$

hence

$$
\begin{aligned}
&\mathbb{E}_{\mathbb{P}^\epsilon}\left[1_{\{|X_t^{\bar t,\bar x}-\bar x|<\eta\}}|\mathbf{p}_{t-}-\bar p|^2\big|\mathcal{F}_{\bar t-}\right]\\
&\ge (1-\tfrac{\epsilon'}{\eta})\mathbb{E}_{\mathbb{P}^\epsilon}\left[|\mathbf{p}_{t-}-\bar p|^2\big|\mathcal{F}_{\bar t-}\right] - \tfrac{1}{4\eta\epsilon'}(t-\bar t).
\end{aligned}
\tag{4.31}
$$

Choosing $0<\epsilon'<\eta$ and combining (4.27) with the estimates (4.28)-(4.31) there exists a constant c, such that

$$
\mathbb{E}_{\mathbb{P}^\epsilon}\left[|\mathbf{p}_{t-}-\bar p|^2\big|\mathcal{F}_{\bar t-}\right] \le c(t-\bar t)^{\frac{1}{2}}.
\tag{4.32}
$$

Since $\mathbf{p}$ is a martingale, it follows that for all $s\in[\bar t,t)$

$$
\mathbb{E}_{\mathbb{P}^\epsilon}\left[|\mathbf{p}_s-\bar p|^2\big|\mathcal{F}_{\bar t-}\right] \le c(t-\bar t)^{\frac{1}{2}},
$$

hence

$$
\mathbb{E}_{\mathbb{P}^\epsilon}\left[\int_{\bar t}^{t}|\mathbf{p}_s-\bar p|ds\big|\mathcal{F}_{\bar t-}\right] \le (t-\bar t)^{\frac{1}{2}}\left(\mathbb{E}_{\mathbb{P}^\epsilon}\left[\int_{\bar t}^{t}|\mathbf{p}_s-\bar p|^2ds\big|\mathcal{F}_{\bar t-}\right]\right)^{\frac{1}{2}} \le c(t-\bar t)^{\frac{5}{4}}.
\tag{4.33}
$$

Step 3: Note that under $\mathbb{P}^\epsilon\in\mathcal{P}^f(\bar t,\bar p)$ the triplet $(Y_s^{\bar t,\bar x}, z_s^{\bar t,\bar x}, N_s)_{s\in[\bar t,T]}$ is the unique solution of the BSDE

$$
Y_s^{\bar t,\bar x} = \langle \mathbf{p}_T, g(X_T^{\bar t,\bar x})\rangle + \int_s^T \tilde H(r,X_r^{\bar t,\bar x}, z_r^{\bar t,\bar x},\mathbf{p}_r)dr - \int_s^T z_r^{\bar t,\bar x}dB_r - N_T + N_s.
$$

To consider an auxiliary BSDE with terminal time t we define as in the standard case (see e.g. [45])

$$
G(s,x,p) = \frac{\partial\phi}{\partial t}(s,x,p) + \frac{1}{2}\mathrm{tr}(\sigma\sigma^*(s,x)D^2\phi(s,x,p)) + \tilde H(s,x,\sigma^*(s,x)D\phi(s,x,p),p)
$$

$$= \frac{\partial\phi}{\partial t}(s,x,p) + \frac{1}{2}\mathrm{tr}(\sigma\sigma^*(s,x)D^2\phi(s,x,p)) + H(s,x,D\phi(s,x,p),p)$$

and set

$$\begin{aligned}\tilde{Y}_s^{\bar{t},\bar{x}} &= Y_s^{\bar{t},\bar{x}} - \phi(s,X_s^{\bar{t},\bar{x}},\mathbf{p}_s) - \int_s^t G(r,\bar{x},\bar{p})dr \\ &\quad + \sum_{\bar{t}\le r\le s}\left(\phi(r,X_r^{\bar{t},\bar{x}},\mathbf{p}_r) - \phi(r,X_r^{\bar{t},\bar{x}},\mathbf{p}_{r-}) - \langle\frac{\partial}{\partial p}\phi(r,X_r^{\bar{t},\bar{x}},\mathbf{p}_{r-}),\mathbf{p}_r - \mathbf{p}_{r-}\rangle\right) \\ \tilde{z}_s^{\bar{t},\bar{x}} &= z_s^{\bar{t},\bar{x}} - \sigma^*(s,X_s^{\bar{t},\bar{x}})D_x\phi(s,X_s^{\bar{t},\bar{x}},\mathbf{p}_s).\end{aligned}$$

Then by Itô's formula $(\tilde{Y}^{\bar{t},\bar{x}},\tilde{z}^{\bar{t},\bar{x}},N)$ is on $[\bar{t},t)$ the solution to the BSDE

$$\begin{aligned}\tilde{Y}_s^{\bar{t},\bar{x}} = \xi + \int_s^t &\Big(\tilde{H}(r,X_r^{\bar{t},\bar{x}},\tilde{z}_r^{\bar{t},\bar{x}} + \sigma^*(r,X_r^{\bar{t},\bar{x}})D_x\phi(r,X_r^{\bar{t},\bar{x}},\mathbf{p}_r),\mathbf{p}_r) \\ &+\frac{\partial\phi}{\partial t}(r,X_r^{\bar{t},\bar{x}},\mathbf{p}_r) + \frac{1}{2}\mathrm{tr}(\sigma\sigma^*(r,X_r^{\bar{t},\bar{x}})D^2\phi(r,X_r^{\bar{t},\bar{x}},\mathbf{p}_r)) - G(r,\bar{x},\bar{p})\Big)dr \\ -\int_s^t &\tilde{z}_r^{\bar{t},\bar{x}}dB_r - N_{t-} + N_s\end{aligned}$$

with the terminal value

$$\begin{aligned}\xi &= Y_{t-}^{\bar{t},\bar{x}} - \phi(t,X_t^{\bar{t},\bar{x}},\mathbf{p}_{t-}) \\ &\quad + \textstyle\sum_{\bar{t}\le r<t}\left(\phi(r,X_r^{\bar{t},\bar{x}},\mathbf{p}_r) - \phi(r,X_r^{\bar{t},\bar{x}},\mathbf{p}_{r-}) - \langle\frac{\partial}{\partial p}\phi(r,X_r^{\bar{t},\bar{x}},\mathbf{p}_{r-}),\mathbf{p}_r - \mathbf{p}_{r-}\rangle\right).\end{aligned}$$

Note that by the (strict) convexity assumption on ϕ we get, that $\mathbb{P}^\epsilon$-a.s.

$$\sum_{\bar{t}\le r<t}\left(\phi(r,X_r^{\bar{t},\bar{x}},\mathbf{p}_r) - \phi(r,X_r^{\bar{t},\bar{x}},\mathbf{p}_{r-}) - \langle\frac{\partial}{\partial p}\phi(r,X_r^{\bar{t},\bar{x}},\mathbf{p}_{r-}),\mathbf{p}_r - \mathbf{p}_{r-}\rangle\right) \ge 0. \tag{4.34}$$

Furthermore by Lemma 4.8. and the choice of ϕ we have $Y_{t-}^{\bar{t},\bar{x}} \ge W(t,X_t^{\bar{t},\bar{x}},\mathbf{p}_{t-}) \ge \phi(t,X_t^{\bar{t},\bar{x}},\mathbf{p}_{t-})$, hence $\xi \ge 0$.
Consider now the solution to the BSDE with the same driver but with the smaller target 0, i.e.

$$\begin{aligned}\bar{Y}_s^{\bar{t},\bar{x}} &= \textstyle\int_s^t \big(\tilde{H}(r,X_r^{\bar{t},\bar{x}},\bar{z}_r^{\bar{t},\bar{x}} + \sigma^*(r,X_r^{\bar{t},\bar{x}})D_x\phi(r,X_r^{\bar{t},\bar{x}},\mathbf{p}_r),\mathbf{p}_r) \\ &\quad +\tfrac{\partial\phi}{\partial t}(r,X_r^{\bar{t},\bar{x}},\mathbf{p}_r) + \tfrac{1}{2}\mathrm{tr}(\sigma\sigma^*(r,X_r^{\bar{t},\bar{x}})D^2\phi(r,X_r^{\bar{t},\bar{x}},\mathbf{p}_r)) - G(r,\bar{x},\bar{p})\big)dr \\ &\quad -\textstyle\int_s^t \bar{z}_r^{\bar{t},\bar{x}}dB_r - \bar{N}_{t-} + \bar{N}_s.\end{aligned} \tag{4.35}$$

Proposition 6.2. yields

$$\mathbb{E}_{\mathbb{P}^\epsilon}\left[\int_{\bar{t}}^t |\bar{z}_s^{\bar{t},\bar{x}}|^2 ds\big|\mathcal{F}_{\bar{t}-}\right] \le c\mathbb{E}_{\mathbb{P}^\epsilon}\left[\int_{\bar{t}}^t |\bar{f}_s|^2 ds\big|\mathcal{F}_{\bar{t}-}\right] \tag{4.36}$$

with

$$\bar{f}_s := \tilde{H}(s,X_s^{\bar{t},\bar{x}},\sigma^*(s,X_s^{\bar{t},\bar{x}})D_x\phi(s,X_s^{\bar{t},\bar{x}},\mathbf{p}_s),\mathbf{p}_s)$$

$$+\frac{\partial\phi}{\partial t}(s,X_s^{\bar t,\bar x},\mathbf{p}_s)+\frac{1}{2}\mathrm{tr}(\sigma\sigma^*(s,X_s^{\bar t,\bar x})D^2\phi(s,X_s^{\bar t,\bar x},\mathbf{p}_s))-G(s,\bar x,\bar p).$$

Because $\tilde H$ is uniformly Lipschitz continuous in p and the derivatives of ϕ with respect to p are uniformly bounded, we have

$$\begin{aligned}|\bar f_s| \;\leq\; & \Big|\tilde H(s,X_s^{\bar t,\bar x},\sigma^*(s,X_s^{\bar t,\bar x})D_x\phi(s,X_s^{\bar t,\bar x},\bar p),\bar p)\\ &\quad +\frac{\partial\phi}{\partial t}(s,X_s^{\bar t,\bar x},\bar p)+\frac{1}{2}\mathrm{tr}(\sigma\sigma^*(s,X_s^{\bar t,\bar x})D^2\phi(s,X_s^{\bar t,\bar x},\bar p))-G(s,\bar x,\bar p)\Big|\\ & +c\,|\mathbf{p}_s-\bar p|\end{aligned}$$

and we get as in [45] by the estimate (4.33), that for all $\epsilon'>0$

$$\begin{aligned}\mathbb{E}_{\mathbb{P}^\epsilon}\left[\int_{\bar t}^t|\bar f_s|^2ds\big|\mathcal{F}_{\bar t-}\right] &\leq \frac{1}{4\epsilon'}(t-\bar t)O(t-\bar t)+\epsilon' c\;\;\mathbb{E}_{\mathbb{P}^\epsilon}\left[\int_{\bar t}^t|\mathbf{p}_s-\bar p|^2\,ds\big|\mathcal{F}_{\bar t-}\right]\\ &\leq \frac{1}{4\epsilon'}(t-\bar t)O(t-\bar t)+\epsilon' c(t-\bar t)^{\frac{3}{2}},\end{aligned}$$

where $O(t-\bar t)\to 0$ as $t\to\bar t$. Hence we have by (4.36) and Cauchy inequality

$$\mathbb{E}_{\mathbb{P}^\epsilon}\left[\int_{\bar t}^t|\bar z_s^{\bar t,\bar x}|ds\big|\mathcal{F}_{\bar t-}\right]\leq c\left((t-\bar t)O(t-\bar t)+(t-\bar t)^{\frac{5}{4}}\right) \tag{4.37}$$

and

$$\begin{aligned}\bar Y_{\bar t-}^{\bar t,\bar x} &\geq -c\mathbb{E}_{\mathbb{P}^\epsilon}\left[\int_{\bar t}^t|\bar f_s|ds\big|\mathcal{F}_{\bar t-}\right]-c\mathbb{E}_{\mathbb{P}^\epsilon}\left[\int_{\bar t}^t|\bar z_s^{\bar t,\bar x}|ds\big|\mathcal{F}_{\bar t-}\right]\\ &\geq -c\left((t-\bar t)O(t-\bar t)+(t-\bar t)^{\frac{5}{4}}\right).\end{aligned}$$

Since by the comparison theorem (Theorem 6.3.) we have $\tilde Y_{\bar t-}^{\bar t,\bar x}\geq\bar Y_{\bar t-}^{\bar t,\bar x}$, this yields the following estimate for the auxiliary BSDE

$$\tilde Y_{\bar t-}^{\bar t,\bar x}\geq -c\left((t-\bar t)O(t-\bar t)+(t-\bar t)^{\frac{5}{4}}\right). \tag{4.38}$$

Step 4: The theorem is proved, if we show $G(\bar t,\bar x,\bar p)\leq 0$. Note that by definition of $\tilde Y^{\bar t,\bar x}$

$$\tilde Y_{\bar t-}^{\bar t,\bar x} \;=\; Y_{\bar t-}^{\bar t,\bar x}-\phi(\bar t,\bar x,\bar p)-\textstyle\int_{\bar t}^t G(r,\bar x,\bar p)dr, \tag{4.39}$$

while we have by the choice of $\mathbb{P}^\epsilon$ (4.24)

$$\begin{aligned}& Y_{\bar t-}^{\bar t,\bar x}-W(\bar t,\bar x,\bar p)\\ \leq\;& Y_{\bar t-}^{\bar t,\bar x}-\mathbb{E}_{\mathbb{P}^\epsilon}\left[\int_{\bar t}^t\tilde H(s,X_s^{\bar t,\bar x},z_s^{\bar t,\bar x},\mathbf{p}_s)ds+Y_{t-}^{\bar t,\bar x}\big|\mathcal{F}_{\bar t-}\right]+\epsilon(t-\bar t)\\ =\;&\epsilon(t-\bar t).\end{aligned}$$

Since $\phi(\bar t,\bar x,\bar p)=W(\bar t,\bar x,\bar p)$,

$$Y_{\bar t-}^{\bar t,\bar x}-\phi(\bar t,\bar x,\bar p)\leq c\epsilon(t-\bar t). \tag{4.40}$$

Thus with (4.39) we have

$$\tilde{Y}^{\bar{t},\bar{x}}_{\bar{t}-} + \int_{\bar{t}}^{t} G(r,\bar{x},\bar{p})dr \leq c\epsilon(t-\bar{t})$$

and finally by the estimate (4.38)

$$-c\left((t-\bar{t})O(t-\bar{t}) + (t-\bar{t})^{\frac{5}{4}}\right) + \int_{\bar{t}}^{t} G(r,\bar{x},\bar{p})dr \leq c\epsilon(t-\bar{t}),$$

hence

$$\frac{1}{(t-\bar{t})}\int_{\bar{t}}^{t} G(s,\bar{x},\bar{p})ds \leq c\ \left(O(t-\bar{t}) + (t-\bar{t})^{\frac{1}{4}}\right) + c\epsilon,$$

which implies (4.19) as $t \downarrow \bar{t}$ since $\epsilon > 0$ can be chosen arbitrary small. □

Thus by Proposition 4.10., 4.11. and comparison theorem for (2.8) (see [27], [25]) we now have the following result.

Theorem 4.12. *W is the unique viscosity solution to (2.8) in the class of bounded uniformly continuous functions, which are uniformly Lipschitz in p.*

Theorem 3.4. follows directly from Theorem 4.12. and the characterization of the value function in Theorem 2.8.

5 Concluding remarks

In this paper we have shown an alternative representation of the value function in terms of a minimization of solutions of certain BSDEs over some specific martingale measures. These BSDEs correspond to the dynamics of a stochastic differential game with the beliefs of the uninformed player (modulo a Girsanov transformation) as an additional forward dynamic. We used this to show how to explicitly determine the optimal reaction of the informed player under some rather restrictive assumptions. To have a representation like

$$V(t,x,p) \quad = \quad Y^{t,x,\bar{\mathbb{P}}}_{t-}$$

in a more general case a careful analysis of the optimal measure is necessary. In the simpler framework of [27] the existence of a weak limit $\mathbb{P}^*$ for a minimizing sequence is straightforward using [84]. In our case any limiting procedure needs to take into account the BSDE structure. The question of existence of an optimal measure under which there is a representation by a solution to a BSDE poses therefore a rather delicate problem, which shall be addressed in a subsequent work.

6 Appendix: Results for BSDE on $\mathcal{D}([0,T];\Delta(I))\times\mathcal{C}([0,T];\mathbb{R}^d)$

Here we restate existence and uniqueness results of for BSDEs adapted to our setting. More general results can be found in [43].

Let $\Omega := \mathcal{D}([0,T];\Delta(I)) \times \mathcal{C}([0,T];\mathbb{R}^d)$ and $(\Omega, \mathcal{F}, (\mathcal{F}_s)_{s\in[t,T]})$ be defined as in section 3.1. We fix a $\mathbb{P} \in \mathcal{P}(t,p)$ and denote $\mathbb{E}_{\mathbb{P}}[\cdot] = \mathbb{E}[\cdot]$. Let $\xi \in \mathcal{L}^2_T(\mathbb{P})$, i.e. ξ is a square integrable $\mathcal{F}_T$-measurable random variable. Let $f : \Omega \times [0,T] \times \mathbb{R}^d \to \mathbb{R}$ be $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d)$ measurable, such that $f(\cdot,0) \in \mathcal{H}^2(\mathbb{P})$ and such that, there exists a constant c, such that $\mathbb{P} \otimes dt$ a.e.

$$|f(\omega,s,z^1) - f(\omega,s,z^2)| \le c|z^1 - z^2| \quad \forall z^1, z^2 \in \mathbb{R}^d. \tag{6.1}$$

We consider on $\mathcal{D}([0,T];\Delta(I)) \times \mathcal{C}([0,T];\mathbb{R}^d)$ the BSDE

$$Y_s \;=\; \xi + \int_s^T f(r,z_r)ds + \int_s^T z_r dB_r - (N_T - N_s). \tag{6.2}$$

The existence and uniqueness can be shown by a combination of the proof for the solvability of BSDE via a fixed point argument as in [45] and the Galtchouk-Kunita-Watanabe decomposition (see e.g. [2]).

Theorem 6.1. *For any fixed $\mathbb{P} \in \mathcal{P}(t,x)$ there exists a solution $(Y,z,N) \in \mathcal{S}^2(\mathbb{P}) \times \mathcal{H}^2(\mathbb{P}) \times \mathcal{M}^2_0(\mathbb{P})$ to (6.2), such that N is strongly orthogonal to the Brownian motion B. Furthermore (Y,z) are unique in $\mathcal{S}^2(\mathbb{P}) \times \mathcal{H}^2(\mathbb{P})$ and $N \in \mathcal{M}^2_0(\mathbb{P})$ is unique up to indistinguability.*

Furthermore we note that we have the following dependence on the data.

Proposition 6.2. *For $i = 1,2$, let $\xi^i \in \mathcal{L}^2_T(\mathbb{P})$. Let $f^i : \Omega \times [0,T] \times \mathbb{R}^d \to \mathbb{R}$ be two generators for the BSDE (6.2), i.e. $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d)$ measurable, $f^i(\cdot,0) \in \mathcal{H}^2(\mathbb{P})$ and f^i are uniformly Lipschitz continuous in z.*
Let $(Y^i,z^i,N^i) \in \mathcal{S}^2(\mathbb{P}) \times \mathcal{H}^2(\mathbb{P}) \times \mathcal{M}^2_0(\mathbb{P})$ be the respective solutions. Set $\delta z = z^1 - z^2$ and $\delta\xi = \xi^1 - \xi^2$, $\delta f = f^1(\cdot, z^2_\cdot) - f^2(\cdot, z^2_\cdot)$. Then we have that for any $s \in [0,T]$

$$\begin{aligned}&\mathbb{E}\left[\sup_{r\in[s,T]} |\delta Y_r|^2 \middle| \mathcal{F}_s\right] + \mathbb{E}\left[\int_s^T |\delta z_r|^2 dr|\mathcal{F}_s\right]\\ &\qquad \le c\left(\mathbb{E}\left[|\delta\xi|^2|\mathcal{F}_s\right] + \mathbb{E}\left[\int_s^T |\delta f_r|^2 dr\middle|\mathcal{F}_s\right]\right)\end{aligned} \tag{6.3}$$

Also we have a comparison principle which can be established like as in [45].

Theorem 6.3. *For $i = 1,2$, let $\xi^i \in \mathcal{L}^2_T(\mathbb{P})$. Let $f^i : \Omega \times [0,T] \times \mathbb{R}^d \to \mathbb{R}$ be two generators for the BSDE (6.2), i.e. f^i is $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d)$ measurable, uniformly Lipschitz continuous in z and $f^i(\cdot,0) \in \mathcal{H}^2(\mathbb{P})$.*
Let $(Y^i,z^i,N^i) \in \mathcal{S}^2(\mathbb{P}) \times \mathcal{H}^2(\mathbb{P}) \times \mathcal{M}^2_0(\mathbb{P})$ be the respective solutions. Assume
(i) $\delta\xi = \xi^1 - \xi^2 \ge 0$ $\mathbb{P}$-a.s.
(ii) $\delta f = f^1(\cdot, z^2_\cdot) - f^2(\cdot, z^2_\cdot) \ge 0$ $\mathbb{P} \otimes dt$-a.s.
Then for any time $s \in [0,T]$ we have $Y^1_s - Y^2_s \ge 0$ $\mathbb{P}$-a.s.

Chapter 4

A note on regularity for a fully non-linear partial differential equation arising in game theory

1 Introduction

We consider a quasilinear partial differential equation (PDE) with a convexity constraint. More precisely we shall investigate the following type of equation: Let $I \in \mathbb{N}^*$, $\Delta(I)$ denote the simplex of $\mathbb{R}^I$ and $\mathcal{S}^I$ the set of symmetric matrices of size $I \times I$. We consider

$$\min \left\{ \frac{\partial w}{\partial t} + \frac{1}{2} \mathrm{tr}(\sigma\sigma^*(t,x) D_x^2 w) + H(t,x,D_x w, p), \lambda_{\min}\left(p, \frac{\partial^2 w}{\partial p^2}\right) \right\} = 0 \tag{1.1}$$

with terminal condition $w(T,x,p) = \sum_{i=1,\ldots,I} p_i g_i(x)$, where for all $p \in \Delta(I)$ and for all $A \in \mathcal{S}^I$,

$$\lambda_{\min}(p,A) := \min_{z \in T_{\Delta(I)(p)} \setminus \{0\}} \frac{\langle Az, z\rangle}{|z|^2}$$

with $T_{\Delta(I)(p)}$ denoting the tangent cone to $\Delta(I)$ at p, i.e. $T_{\Delta(I)(p)} = \overline{\cup_{\lambda>0}(\Delta(I)-p)/\lambda}$. This kind of PDE was used in [28] to investigate some stochastic differential games with incomplete information in the spirit of the celebrated model of Aumann and Maschler (see [3]). Since we are only dealing with the PDE itself we do not go into detail about the game and refer the reader to [28] and the references given therein.
The equation (1.1) is highly degenerate, thus it is especially difficult to deal with. A study of obstacle problems as (1.1) with a convexity constraint but no additional dynamic in x can be found in [86], [87]. In this paper we give an easy proof for semiconcavity in x for the solution of the obstacle problem (1.1) which gives us with Alexandrov's theorem weak differentiability in x. The method we use is purely probabilistic and uses the representation of the solution via a specific minimum over solutions of backward stochastic differential equations (BSDE), which was given in [56].
It is well known that the study of BSDE initiated by Peng in [92] also gives insight to existence and regularity properties for solutions to quasilinear PDE. Notably, we would like to mention [89] whose methods we shall adapt to our case.

2 Main assumption and known results

The following will be the standing assumption throughout the paper.

Assumption (H)

(i) For $1 \leq k,l \leq d$ the function $\sigma_{k,l} : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ is bounded and Lipschitz continuous with respect to (t,x). For any $(t,x) \in [0,T] \times \mathbb{R}^d$ the matrix $\sigma^*(t,x)$ is non-singular and $(\sigma^*(t,x))^{-1}$ is bounded and Lipschitz continuous with respect to (t,x).

(ii) $(g_i)_{i\in\{1,\dots,I\}} : \mathbb{R}^d \to \mathbb{R}$ are bounded and uniformly Lipschitz continuous.

(iii) $H : [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \times \Delta(I) \to \mathbb{R}$ is uniformly Lipschitz continuous in (ξ,p) uniformly in (t,x) and Lipschitz continuous in (t,x) with Lipschitz constant $c(1+|\xi|)$, i.e. for all $t,t' \in [0,T]$, $x,x' \in \mathbb{R}^d$, $\xi,\xi' \in \mathbb{R}^d$, $p,p' \in \Delta(I)$

$$|H(t,x,\xi,p)| \leq c(1+|\xi|) \tag{2.1}$$

and

$$\begin{aligned}&|H(t,x,\xi,p) - H(t',x',\xi',p')| \\ &\qquad \leq c(1+|\xi|)(|x-x'|+|t-t'|) + c|\xi-\xi'| + c|p-p'|.\end{aligned} \tag{2.2}$$

It has been shown in [28], that:

Theorem 2.1. *There is a unique viscosity solution $V : [0,T] \times \mathbb{R}^d \times \Delta(I) \to \mathbb{R}$ to (1.1) in the class of bounded, uniformly continuous functions, which are uniformly Lipschitz continuous in p.*

In [56] we give a representation of the viscosity solution to (1.1) in terms of BSDE. To that end, we enlarge the canonical Wiener space to a space which carries besides a Brownian motion, càdlàg martingales with values in $\Delta(I)$. More precisely we define for any $p \in \Delta(I)$, $t \in [0,T]$ on the canonical space $\Omega := \mathcal{D}([0,T];\Delta(I)) \times \mathcal{C}([0,T];\mathbb{R}^d)$ with canonical process $(\mathbf{p},B)$ the set $\mathcal{P}(t,p)$ as the set of probability measures $\mathbb{P}$ such that, under $\mathbb{P}$,

(i) $\mathbf{p}$ is a martingale, such that $\mathbf{p}_s = p \ \forall s < t$, $\mathbf{p}_s \in \{e_i, i=1,\dots,I\} \ \forall s \geq T$ $\mathbb{P}$-a.s., where e_i denotes the i-th coordinate vector in $\mathbb{R}^I$, and $\mathbf{p}_T$ is independent of $(B_s)_{s\in(-\infty,T]}$,

(ii) $(B_s)_{s\in[0,T]}$ is a Brownian motion.

We show in [56], that V can be represented by minimizing the solutions of a backward stochastic differential equation (BSDE) with respect to the measures $\mathbb{P} \in \mathcal{P}(t,p)$. More precisely we consider for each $\mathbb{P} \in \mathcal{P}(t,p)$

$$X_s^{t,x} = x \quad s<t, \qquad X_s^{t,x} = x + \int_t^s \sigma(r,X_r^{t,x})dB_r \quad s \geq t \tag{2.3}$$

and the associated BSDE

$$\begin{aligned}Y_s^{t,x,\mathbb{P}} &= \langle \mathbf{p}_T, g(X_T^{t,x})\rangle + \int_s^T H(r,X_r^{t,x},Z_r^{t,x,\mathbb{P}},\mathbf{p}_r)dr \\ &\qquad - \int_s^T \sigma^*(r,X_r^{t,x})Z_r^{t,x,\mathbb{P}}dB_r - N_T^{\mathbb{P}} + N_s^{\mathbb{P}},\end{aligned} \tag{2.4}$$

where $N^{\mathbb{P}}$ is a square integrable martingale which is strongly orthogonal to the Brownian motion B. In [56] we establish:

Theorem 2.2. *For any $(t,x,p) \in [0,T] \times \mathbb{R}^d \times \Delta(I)$ the solution $V = V(t,x,p)$ of (1.1) can be characterized as*

$$V(t,x,p) = \operatorname{essinf}_{\mathbb{P} \in \mathcal{P}(t,p)} Y_{t-}^{t,x,\mathbb{P}}. \tag{2.5}$$

Furthermore we have (with the help of BSDE techniques) the following regularity.

Proposition 2.3. *$V(t,x,p)$ is uniformly Lipschitz continuous in x, uniformly Hölder continuous in t, convex and uniformly Lipschitz continuous with respect to p.*

3 Regularity

3.1 BSDE results

As we are again using BSDE techniques for providing a regularity proof, we repeat some results which can be found in more generality in [43]. For technical reasons we will consider the BSDE (2.4) with a slightly different notation. For any $t \in [0,T]$, $x \in \mathbb{R}^d$, $\mathbb{P} \in \mathcal{P}(t,p)$ let

$$\begin{aligned} Y_s^{t,x,\mathbb{P}} \quad = \quad & \langle \mathbf{p}_T, g(X_T^{t,x}) \rangle + \int_s^T \tilde{H}(r, X_r^{t,x}, z_r^{t,x,\mathbb{P}}, \mathbf{p}_r) dr \\ & - \int_s^T z_r^{t,x,\mathbb{P}} dB_r - N_T^{\mathbb{P}} + N_s^{\mathbb{P}}, \end{aligned} \tag{3.1}$$

where $\tilde{H}(t,x,p,\xi) = H(t,x,p,(\sigma^*(t,x))^{-1}\xi)$. Setting $Z_s^{t,x,\mathbb{P}} = (\sigma^*(s,X_s^{t,x}))^{-1} z_s^{t,x,\mathbb{P}}$ then gives the solution $(Y^{t,x,\mathbb{P}}, Z^{t,x,\mathbb{P}})$ to (2.4).

In the following we will use the notation $Y_s^{t,x,\mathbb{P}} = Y_s^{t,x}$, $z^{t,x,\mathbb{P}} = z^{t,x}$, $N^{\mathbb{P}} = N$, whenever we work under a fixed $\mathbb{P} \in \mathcal{P}(t,p)$.

Remark 3.1. *Observe that by (H) we have that $\tilde{H}$ is uniformly Lipschitz continuous in (ξ,p) uniformly in (t,x) and Lipschitz continuous in (t,x) with Lipschitz constant $c(1+|\xi|)$, i.e. for all $t,t' \in [0,T]$, $x,x' \in \mathbb{R}^d$, $\xi,\xi' \in \mathbb{R}^d$, $p,p' \in \Delta(I)$*

$$|\tilde{H}(t,x,\xi,p)| \leq c(1+|\xi|) \tag{3.2}$$

and

$$\begin{aligned} & |\tilde{H}(t,x,\xi,p) - \tilde{H}(t',x',\xi',p')| \\ & \qquad \leq c(1+|\xi|)(|x-x'|+|t-t'|) + c|\xi-\xi'| + c|p-p'|. \end{aligned} \tag{3.3}$$

First we introduce the following spaces. For any $p \in \Delta(I)$, $t \in [0,T]$ and fixed $\mathbb{P} \in \mathcal{P}(t,p)$ we denote by $\mathcal{L}_T^2(\mathbb{P})$ the set of a square integrable $\mathcal{F}_T$-measurable random variables. We define by $\mathcal{S}^2(\mathbb{P})$ the set of real-valued adapted càdlàg processes ϑ, such that $\mathbb{E}\left[\sup_{s\in[0,T]} \vartheta_s^2\right] < \infty$, furthermore by $\mathcal{H}^2(\mathbb{P})$ the space of $\mathbb{R}^d$-valued progressively measurable processes, such that $\int_0^\cdot \theta_s dB_s$ is a square integrable martingale, i.e. $\mathbb{E}\left[\int_0^T |\theta_s|^2 ds\right] < \infty$. We denote by $\mathcal{M}_0^2(\mathbb{P})$ the space of square integrable martingales null at zero. In the following we shall identify any $N \in \mathcal{M}_0^2(\mathbb{P})$ with its càdlàg modification.

Theorem 3.2. *Under the assumption (H) the BSDE (3.1) has a unique solution* $(Y^{t,x,\mathbb{P}}, z^{t,x,\mathbb{P}}, N^{t,x,\mathbb{P}}) \in \mathcal{S}^2(\mathbb{P}) \times \mathcal{H}^2(\mathbb{P}) \times \mathcal{M}_0^2(\mathbb{P})$ *and for any* $s \leq T$

$$Y_s^{t,x,\mathbb{P}} = \mathbb{E}_{\mathbb{P}}\left[\int_s^T \tilde{H}(r, X_r^{t,x}, z_r^{t,x,\mathbb{P}}, \mathbf{p}_r)dr + \langle \mathbf{p}_T, g(X_T^{t,x})\rangle \big| \mathcal{F}_s\right].$$

In particular

$$Y_{t-}^{t,x,\mathbb{P}} = \mathbb{E}_{\mathbb{P}}\left[\int_t^T \tilde{H}(r, X_r^{t,x}, z_r^{t,x,\mathbb{P}}, \mathbf{p}_r)dr + \langle \mathbf{p}_T, g(X_T^{t,x})\rangle \big| \mathcal{F}_{t-}\right]. \tag{3.4}$$

Furthermore we have the general result:

Proposition 3.3. *For* $i = 1, 2$, *let* $\xi^i \in \mathcal{L}_T^2(\mathbb{P})$ *be two terminal values. Let* $f^i : \Omega \times [0,T] \times \mathbb{R}^d \to \mathbb{R}$ *be two generators for the BSDE (3.1), i.e.* $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d)$ *measurable,* $f^i(\cdot, 0) \in \mathcal{H}^2(\mathbb{P})$ *and* f^i *are uniformly Lipschitz continuous in* z.
Let $(Y^i, z^i, N^i) \in \mathcal{S}^2(\mathbb{P}) \times \mathcal{H}^2(\mathbb{P}) \times \mathcal{M}_0^2(\mathbb{P})$ *be the respective solutions. Set* $\delta z = z^1 - z^2$ *and* $\delta\xi = \xi^1 - \xi^2$, $\delta f = f^1(\cdot, z_\cdot^2) - f^2(\cdot, z_\cdot^2)$. *Then for any* $s \in [0,T]$

$$\mathbb{E}\left[\int_s^T |\delta z_r|^2 dr \big| \mathcal{F}_s\right] \leq c\left(\mathbb{E}\left[|\delta\xi|^2 | \mathcal{F}_s\right] + \mathbb{E}\left[\int_s^T |\delta f_r|^2 dr \big| \mathcal{F}_s\right]\right) \tag{3.5}$$

Also we have a comparison principle.

Theorem 3.4. *For* $i = 1, 2$, *let* $\xi^i \in \mathcal{L}_T^2(\mathbb{P})$. *Let* $f^i : \Omega \times [0,T] \times \mathbb{R}^d \to \mathbb{R}$ *be two generators for the BSDE (3.1). Let* $(Y^i, z^i, N^i) \in \mathcal{S}^2(\mathbb{P}) \times \mathcal{H}^2(\mathbb{P}) \times \mathcal{M}_0^2(\mathbb{P})$ *be the respective solutions. Assume*
(i) $\delta\xi = \xi^1 - \xi^2 \geq 0$ $\mathbb{P}$*-a.s.*
(ii) $\delta f = f^1(\cdot, z_\cdot^2) - f^2(\cdot, z_\cdot^2) \geq 0$ $\mathbb{P} \otimes dt$*-a.s.*
Then for any time $s \in [0,T]$ *we have* $Y_s^1 - Y_s^2 \geq 0$ $\mathbb{P}$*-a.s.*

3.2 Semiconcavity in x

In order to show semiconcavity we need some stronger assumptions:
Assumption (H')
(i) $(g_i)_{i \in \{1,\ldots,I\}} : \mathbb{R}^d \to \mathbb{R}$ is differentiable with bounded, uniformly Lipschitz continuous derivative.
(ii) $\sigma : [0,T] \times \mathbb{R}^d \to \mathbb{R}^{d \times d}$ is for any $t \in [0,T]$ differentiable with respect to x with bounded, uniformly Lipschitz continuous derivative.
(iii) $\tilde{H} : [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \times \Delta(I) \to \mathbb{R}$ is for any $t \in [0,T]$ differentiable in x and z with bounded, uniformly Lipschitz continuous derivative.

Proposition 3.5. *Under (H), (H') the value function* V *is semiconcave in* x *with linear modulus.*

The proof is very similar to [89] which provides strong regularity for solutions of BSDEs. Since here we are only dealing with the essential infimum of solutions to BSDEs our result is of course weaker.

Proof. Since V is continuous in x, it remains to show that there is a constant $c > 0$, such that for all $h > 0$, $\nu \in S^{d-1}$

$$V(t, x + h\nu, p) - 2V(t, x, p) + V(t, x - h\nu, p) \leq ch^2 \tag{3.6}$$

For $h > 0$ let $\mathbb{P}^\epsilon$ be ϵ-optimal for $V(t,x,p)$ for an $\epsilon = \mathcal{O}(h^2)$. By the representation (2.5) we have under $\mathbb{P}^\epsilon$

$$\begin{aligned} &V(t,x+h\nu,p) - 2V(t,x,p) + V(t,x-h\nu,p) \\ &\leq Y_{t-}^{t,x+h\nu} - 2Y_t^{t,x} + Y_{t-}^{t,x-h\nu} + \epsilon. \end{aligned} \tag{3.7}$$

We set for $s \in [t,T]$

$$\Delta^h Y_s^{t,x} = Y_s^{t,x+h\nu} - 2Y_s^{t,x} + Y_s^{t,x-h\nu} \tag{3.8}$$

and

$$\Delta^h z_s^{t,x} = z_s^{t,x+h\nu} - 2z_s^{t,x} + z_s^{t,x-h\nu}. \tag{3.9}$$

Then $\Delta^h Y_s^{t,x}$ is given by

$$\begin{aligned} \Delta^h Y_s^{t,x} \quad = \quad & \langle \mathbf{p}_T, g(X_T^{t,x+h\nu}) - 2g(X_T^{t,x}) + g(X_T^{t,x-h\nu}) \rangle \\ & + \int_s^T \Big(\tilde{H}(r, X_r^{t,x+h\nu}, z_r^{t,x+h\nu}, \mathbf{p}_r) - 2\tilde{H}(r, X_r^{t,x}, z_r^{t,x}, \mathbf{p}_r) \\ & \qquad + \tilde{H}(r, X_r^{t,x-h\nu}, z_r^{t,x-h\nu}, \mathbf{p}_r) \Big) dr \\ & - \int_s^T \Delta^h z_r^{t,x} dB_r - \tilde{N}_T + \tilde{N}_s, \end{aligned} \tag{3.10}$$

where $\tilde{N} = N^{t,x+h\nu} - 2N^{t,x} + N^{t,x-h\nu} \in \mathcal{M}_0^2(\mathbb{P})$ is strongly orthogonal to B.

Let $\chi := \langle \mathbf{p}_T, g(X_T^{t,x+h\nu}) - 2g(X_T^{t,x}) + g(X_T^{t,x-h\nu}) \rangle$. We claim that

$$\mathbb{E}\left[|\chi|^2 \right] \leq ch^4 \tag{3.11}$$

with a constant c independent of h. First: Since g is differentiable in x, we have for $\Theta^1 = \int_0^1 D_x g(X_T^{t,x+h\nu} + r(X_T^{t,x} - X_T^{t,x+h\nu}))dr$, $\Theta^2 = \int_0^1 D_x g(X_T^{t,x-h\nu} + r(X_T^{t,x} - X_T^{t,x-h\nu}))dr$

$$\begin{aligned} & \mathbb{E}\left[|g(X_T^{t,x+h\nu}) - 2g(X_T^{t,x}) + g(X_T^{t,x-h\nu})|^2 \right] \\ & \quad = \mathbb{E}\left[|\Theta^1(X_T^{t,x+h\nu} - X_T^{t,x}) + \Theta^2(X_T^{t,x-h\nu} - X_T^{t,x})|^2 \right] \\ & \quad \leq c\left(\mathbb{E}\left[|\Theta^1(X_T^{t,x+h\nu} - 2X_T^{t,x} + X_T^{t,x-h\nu})|^2 \right] + \mathbb{E}\left[|(\Theta^2 - \Theta^1)(X_T^{t,x-h\nu} - X_T^{t,x})|^2 \right] \right). \end{aligned}$$

Furthermore since $D_x g$ is uniformly bounded we have

$$\mathbb{E}\left[|\Theta^1(X_T^{t,x+h\nu} - 2X_T^{t,x} + X_T^{t,x-h\nu})|^2 \right] \leq c\mathbb{E}\left[|X_T^{t,x+h\nu} - 2X_T^{t,x} + X_T^{t,x-h\nu}|^2 \right].$$

And since by assumption (H') $D_x\sigma$ is uniformly Lipschitz continuous it is well known (see e.g. [89]) that for all $s \in [t,T]$

$$\mathbb{E}\left[|X_s^{t,x+h\nu} - 2X_s^{t,x} + X_s^{t,x-h\nu}|^2 \right] \leq ch^4. \tag{3.12}$$

On the other hand, since $D_x g$ is by (H') Lipschitz continuous, we have

$$\mathbb{E}\left[|(\Theta^2 - \Theta^1)(X_T^{t,x-h\nu} - X_T^{t,x})|^2 \right] \leq c\mathbb{E}\left[|X_T^{t,x+h\nu} - X_T^{t,x-h}|^2 |X_T^{t,x-h\nu} - X_T^{t,x}|^2 \right] \leq ch^4.$$

Hence (3.11) follows since $|\mathbf{p}_T| = 1$.
Next, we provide estimates for the driver of (3.10)

$$\tilde{H}(s, X_s^{t,x+h\nu}, z_s^{t,x+h\nu}, \mathbf{p}_s) - 2\tilde{H}(s, X_s^{t,x}, z_s^{t,x}, \mathbf{p}_s) + \tilde{H}(s, X_s^{t,x-h\nu}, z_s^{t,x-h\nu}, \mathbf{p}_s). \tag{3.13}$$

Since $\tilde{H}$ is by (H') differentiable with respect to x and ξ, we have with

$$\begin{aligned}
\Lambda_s^1 &:= \int_0^1 D_x\tilde{H}(s, X_s^{t,x} + r(X_s^{t,x+h\nu} - X_s^{t,x}), z_s^{t,x+h\nu}, \mathbf{p}_s)dr \\
\Lambda_s^2 &:= \int_0^1 D_x\tilde{H}(s, X_s^{t,x-h\nu} + r(X_s^{t,x} - X_s^{t,x-h\nu}), z_r^{t,x}, \mathbf{p}_s)dr \\
\Gamma_s^1 &:= \int_0^1 D_z\tilde{H}(s, X_s^{t,x}, z_s^{t,x+h\nu} + r(z_s^{t,x} - z_s^{t,x+h\nu}), \mathbf{p}_s)dr \\
\Gamma_s^2 &:= \int_0^1 D_z\tilde{H}(s, X_s^{t,x-h\nu}, z_s^{t,x} + r(z_s^{t,x-h\nu} - z_s^{t,x}), \mathbf{p}_s)dr,
\end{aligned}$$

that $\mathbb{P}^\epsilon$ a.s.

$$\begin{aligned}
&\tilde{H}(s, X_s^{t,x+h\nu}, z_s^{t,x+h\nu}, \mathbf{p}_s) - \tilde{H}(s, X_s^{t,x}, z_s^{t,x}, \mathbf{p}_s) \\
&= \Lambda_s^1(X_s^{t,x+h\nu} - X_s^{t,x}) + \Gamma_s^1(z_s^{t,x+h\nu} - z_s^{t,x})
\end{aligned}$$

and

$$\begin{aligned}
&\tilde{H}(s, X_s^{t,x}, z_s^{t,x}, \mathbf{p}_s) - \tilde{H}(s, X_s^{t,x-h}, z_s^{t,x-h}, \mathbf{p}_s) \\
&= \Lambda_s^2(X_s^{t,x} - X_s^{t,x-h}) + \Gamma_s^2(z_s^{t,x} - z_s^{t,x-h}).
\end{aligned}$$

So (3.13) can be written as

$$\begin{aligned}
&\tilde{H}(s, X_s^{t,x+h\nu}, z_s^{t,x+h\nu}, \mathbf{p}_s) - 2\tilde{H}(s, X_s^{t,x}, z_s^{t,x}, \mathbf{p}_s) + \tilde{H}(s, X_s^{t,x-h\nu}, z_s^{t,x-h\nu}, \mathbf{p}_s) \\
&= \Lambda_s^1(X_s^{t,x+h\nu} - 2X_s^{t,x} + X_s^{t,x-h\nu}) + \Gamma_s^1(z_s^{t,x+h\nu} - 2z_s^{t,x} + z_s^{t,x-h\nu}) \\
&\quad +(\Lambda_s^1 - \Lambda_s^2)(X_s^{t,x} - X_s^{t,x-h\nu}) + (\Gamma_s^1 - \Gamma_s^2)(z_s^{t,x} - z_s^{t,x-h}).
\end{aligned} \tag{3.14}$$

Since $D_x\tilde{H}$, $D_z\tilde{H}$ are unifomly bounded and Lipschitz continuous by (H'), we have as in the estimate for the terminal value, that the driver in (3.10) can be estimated from above by

$$\begin{aligned}
&F_s(\cdot, \Delta z_s^{t,x}) \\
&:= c|\Delta z_s^{t,x}| + c\Big(|X_s^{t,x+h\nu} - 2X_s^{t,x} + X_s^{t,x-h\nu}| + |X_s^{t,x+h\nu} - X_s^{t,x-h\nu}|^2 \\
&\quad +|X_s^{t,x+h\nu} - X_s^{t,x}|^2 + |z_s^{t,x} - z_s^{t,x-h\nu}|^2 + |z_s^{t,x+h\nu} - z_s^{t,x-h\nu}|^2\Big).
\end{aligned} \tag{3.15}$$

So by the Comparison Theorem 3.4. $\Delta^h Y^{t,x}$ can be estimated from above by the solution of the BSDE

$$\bar{Y}_s = \chi + \int_s^T F_r(\bar{z}_r)dr - \int_s^T \bar{z}_r dB_r - \bar{N}_T + \bar{N}_s \tag{3.16}$$

and from below by the solution of the BSDE

$$\tilde{Y}_s \quad = \quad \chi - \int_s^T F_r(\tilde{z}_r)dr - \int_s^T \tilde{z}_r dB_r - \tilde{N}_T + \tilde{N}_s. \tag{3.17}$$

By Theorem 3.4. we have

$$\tilde{Y}_{t-} \leq \Delta^h Y_{t-}^{t,x} \leq \bar{Y}_{t-}, \tag{3.18}$$

hence

$$\mathbb{E}\left[|\Delta^h Y_{t-}^{t,x}|^2\right] \leq \mathbb{E}\left[|\bar{Y}_{t-}|^2 + |\tilde{Y}_{t-}|^2\right] \leq c\mathbb{E}\left[|\chi|^2 + \int_0^T |F_s(\cdot,0)|^2 ds\right], \tag{3.19}$$

while by (3.11) and Proposition 3.3.

$$\mathbb{E}\left[|\chi|^2 + \int_0^T |F_s(\cdot,0)|^2 ds\right] \leq ch^4.$$

So by (3.7) and Hölder, since the left hand side is deterministic

$$\begin{aligned} &V(t,x+h,p) - 2V(t,x,p) + V(t,x-h,p) \\ &\leq c\left(\mathbb{E}\left[|\Delta^h Y_{t-}^{t,x}|^2\right]\right)^{\frac{1}{2}} + \mathcal{O}(h^2) \leq ch^2 \end{aligned}$$

and the desired result follows. □

By Alexandroff's Theorem (see, e.g., [21]) we have

Corollary 3.6. *V is twice differentiable a.e. in x, i.e. for all $t \in [0,T], p \in \Delta(I)$ and a.e. $x_0 \in \mathbb{R}^d$ there exists $\xi \in \mathbb{R}^d$, $A \in \mathcal{S}^d$ (denoting the set of symmetric matrices of size $d \times d$), such that*

$$\lim_{x \to x_0} \frac{V(t,x,p) - V(t,x_0,p) - \langle \xi, x - x_0 \rangle + \langle A(x-x_0), x - x_0 \rangle}{|x - x_0|^2} = 0. \tag{3.20}$$

Furthermore the gradient $D_x V(t,x,p)$ is defined a.e. and belongs to the class of functions with locally bounded variation.

Chapter 5

A probabilistic-numerical approximation for an obstacle problem arising in game theory

1 Introduction

In 1967 Aumann and Maschler presented their celebrated model for games with incomplete information, see [3] and references therein. The game they consider consists in a set of, say I, standard discrete time two person zero-sum games. At the beginning of the game one of these zero-sum games is picked at random according to a commonly known probability p. The information which game was picked is transmitted to Player 1 only, while Player 2 just knows p. It is assumed that both players observe the actions of the other one, so Player 2 might infer from the actions of his opponent which game is actually played. It turns out that it is optimal for the informed player to play with an additional randomness. Namely in a such a way, that he optimally manipulates the beliefs of the uninformed player.

The extension to two-player zero-sum stochastic differential games has recently been given by Cardaliaguet and Rainer in [28], [25], where the value function is characterized as the unique viscosity solution of a Hamilton Jacobi Isaacs (HJI) equation with an obstacle in the form of a convexity constraint in p. The HJI equation without obstacle is the one which is also found to characterize stochastic differential games in the classical work of Fleming and Souganidis [52]. The probability p appears as an additional parameter in which the value function has to be convex.

In Cardaliaguet [26] an approximation scheme for the value function of deterministic differential games with incomplete information is introduced. An extension of [26] to deterministic games with information incompleteness on both sides is given in Souquière [102]. We consider the case where the underlying dynamic is given by a diffusion with controlled drift but uncontrolled non-degenerate volatility. In constrast to [26] and [102] we can use a Girsanov transform. This transform is a well known tool to consider stochastic games with complete information in the context of backward stochastic differential equations (BSDEs) (see Hamadène and Lepeltier [59]). An approximation of the value function of a stochastic differential game via BSDEs has been discussed in Bally [4]. In contrast with [4] our algorithm is closely related to the work of Barles and Souganidis [7] who consider monotone approximation schemes for fully nonlinear second order partial differential equations. The latter was also applied in the recent work of Fahim, Touzi and Warin [48] where fully non-linear parabolic PDEs are treated. As in [48] we use a kind of finite difference scheme for the HIJ backwards in time and and capture the effect of the information incompleteness by taking the convex envelope in p of the resulting expression. Note that this rather direct ansatz using a probabilistic PDE scheme also significantly differs from the Makov chain approximation method for stochastic differential games described in Kushner [71].

From the very beginning of the investigation of BSDEs initiated by Peng in [92] the close relationship with optimal control problems and quasilinear PDEs has been exploited. Consequently, also the approximation of solutions to BSDEs and to quasilinear PDEs are closely related. For a survey on BSDEs we refer to El Karoui, Peng and Quenez [45], while a survey on the numerical approximation of BSDEs can be found in Bouchard, Elie and Touzi [14]. In this sense our result can also be interpreted as an approximation of the solutions to the BSDEs which appear in the BSDE representation of the value function for stochastic differential games with incomplete information in [56].

It is natural to ask whether this approximation might be used to determine optimal feedback strategies for the informed player. In the deterministic games with complete information it is well known that the answer is positive (see the step by step motions associated with feedbacks in Krasovskii and Subbotin [69]). The case of deterministic games with incomplete information has been treated in [26].

The approximation of optimal strategies for stochastic differential games is a more delicate topic even in the case with complete information. Bally - also considering the game under a Girsanov transform - gives in [4] a partwise answer under a weak Lipschitz assumption of the feedback control. The result is shown by using approximations of BSDEs however not in a completely discrete framework. In the very recent paper [51] of Fleming and Hernández-Hernández approximately Markov strategies are constructed with an approximation that in contrast to ours takes into account the actions of the other player during the time intervals. This however makes the approximation much harder to implement.

In fact, if we use the approximation for the construction of optimal strategies for the informed player we are in the same situation as Kushner [71]. For the approximation of the value function in [71] nearly optimal policies are constructed which possess a certain optimality in the approximative discrete time games instead of the continuous time one. To the authors knowledge the problem of finding an efficient approximation of optimal strategies in stochastic differential games (with or without incomplete information) is open and poses an interesting problem for further research.

The outline of the paper is as follows. In section 2 we describe the game and restate the results of [28] and [25] which build the basis for our investigation. In section 3 we present the approximation scheme and give some regularity proofs. Section 4 is devoted to the convergence proof.

2 Setup

2.1 Formal description of the game

Let $\mathcal{C}([0,T];\mathbb{R}^d)$ be the set of continuous functions from $\mathbb{R}$ to $\mathbb{R}^d$, which are constant on $(-\infty,0]$ and on $[T,+\infty)$. We denote by $B_s(\omega_B)=\omega_B(s)$ the coordinate mapping on $\mathcal{C}([0,T];\mathbb{R}^d)$ and define $\mathcal{H}=(\mathcal{H}_s)$ as the filtration generated by $s\mapsto B_s$. We denote $\Omega_t=\{\omega\in\mathcal{C}([t,T];\mathbb{R}^d)\}$ and $\mathcal{H}_{t,s}$ the σ-algebra generated by paths up to time s in Ω_t.

Furthermore we provide $\mathcal{C}([0,T];\mathbb{R}^d)$ with the Wiener measure $\mathbb{P}_0$ on $(\mathcal{H}_s)$ and complete the respective filtration with respect to $\mathbb{P}_0$-nullsets without changing notation.
In the following we investigate a two-player zero-sum differential game starting at a time $t \geq 0$ with terminal time T. For any fixed initial data $t \in [0,T], x \in \mathbb{R}^d$ the two players control a diffusion on $(\mathcal{C}([t,T];\mathbb{R}^d), (\mathcal{H}_{t,s})_{s\in[t,T]}, \mathcal{H}, \mathbb{P}_0)$ given by

$$dX_s^{t,x,u,v} = b(s, X_s^{t,x,u,v}, u_s, v_s)ds + \sigma(s, X_s^{t,x,u,v})dB_s \quad X_t^{t,x} = x, \tag{2.1}$$

where we assume that the controls of the players u, v can only take their values in some compact subsets of some finite dimensional spaces, denoted by U, V respectively.
Let $I \in \mathbb{N}^*$ and $\Delta(I)$ denote the simplex of $\mathbb{R}^I$. The game is characterized by

(i) running costs: $(l_i)_{i\in\{1,\dots,I\}} : [0,T] \times \mathbb{R}^d \times U \times V \to \mathbb{R}$

(ii) terminal payoffs: $(g_i)_{i\in\{1,\dots,I\}} : \mathbb{R}^d \to \mathbb{R}$,

which are chosen according to a probability $p \in \Delta(I)$ before the game starts. At the beginning of the game this information is transmitted only to Player 1. We assume that Player 1 chooses his control to minimize, Player 2 chooses his control to maximize the expected payoff. Furthermore we assume both players observe their opponent's control. So Player 2, knowing only the probability p_i for scenario $i \in \{1,\dots,I\}$ at the beginning of the game, will try to guess the missing information from the behavior of his opponent. The following will be the standing assumption throughout the paper.

Assumption (A)

(i) $b : [0,T] \times \mathbb{R}^d \times U \times V \to \mathbb{R}^d$ is bounded and continuous in all its variables and Lipschitz continuous with respect to (t,x) uniformly in (u,v).

(ii) For $1 \leq k,l \leq d$ the function $\sigma_{k,l} : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ is bounded and Lipschitz continuous with respect to (t,x). For any $(t,x) \in [0,T] \times \mathbb{R}^d$ the matrix $\sigma^*(t,x)$ is non-singular and $(\sigma^*)^{-1}(t,x)$ is bounded and Lipschitz continuous with respect to (t,x).

(iii) $(l_i)_{i\in\{1,\dots,I\}} : [0,T]\times\mathbb{R}^d\times U\times V \to \mathbb{R}$ is bounded and continuous in all its variables and Lipschitz continuous with respect to (t,x) uniformly in (u,v). $(g_i)_{i\in\{1,\dots,I\}} : \mathbb{R}^d \to \mathbb{R}$ is bounded and Lipschitz continuous.

(iv) Isaacs condition: for all $(t,x,\xi,p) \in [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \times \Delta(I)$

$$\begin{aligned} &\inf_{u\in U}\sup_{v\in V}\left\{\langle b(t,x,u,v),\xi\rangle + \textstyle\sum_{i=1}^I p_i l_i(t,x,u,v)\right\} \\ &= \sup_{v\in V}\inf_{u\in U}\left\{\langle b(t,x,u,v),\xi\rangle + \textstyle\sum_{i=1}^I p_i l_i(t,x,u,v)\right\} \\ &=: H(t,x,\xi,p). \end{aligned} \tag{2.2}$$

By assumption (A) the Hamiltonian H is Lipschitz continuous in (ξ,p) uniformly in (t,x) and Lipschitz continuous in (t,x) with Lipschitz constant $c(1+|\xi|)$, i.e. for all $t,t' \in [0,T]$, $x,x' \in \mathbb{R}^d$, $\xi,\xi' \in \mathbb{R}^d$, $p,p' \in \Delta(I)$

$$|H(t,x,\xi,p)| \leq c(1+|\xi|) \tag{2.3}$$

and

$$|H(t,x,\xi,p) - H(t',x',\xi',p')| \leq c(1+|\xi|)(|x-x'|+|t-t'|) + c|\xi-\xi'| + c|p-p'|. \tag{2.4}$$

2.2 Strategies and value function

We now give some definitions and results of [25] and [28] which are the starting point of our investigation.

Definition 2.1. *For any $t \in [0,T]$ an admissible control $u = (u_s)_{s\in[t,T]}$ for Player 1 is a progressively measurable càdlàg process with respect to the filtration $(\mathcal{H}_{t,s})_{s\in[t,T]}$ with values in U. The set of admissible controls for Player 1 is denoted by $\mathcal{U}(t)$.*
The definition for admissible controls $v = (v_s)_{s\in[t,T]}$ for Player 2 is similar. The set of admissible controls for Player 2 is denoted by $\mathcal{V}(t)$.

Let U_t, respectively V_t, denote the set of càdlàg maps from [t,T] to U, respectively V.

Definition 2.2. *A strategy for Player 1 at time $t \in [0,T]$ is a map $\alpha : [t,T]\times\mathcal{C}([t,T];\mathbb{R}^d)\times V_t \to U_t$ which is nonanticipative with delay, i.e. there is $\delta > 0$ such that for all $s \in [t,T]$ for any $\omega, \omega' \in \mathcal{C}([t,T];\mathbb{R}^d)$ and $v, v' \in V_t$ we have: $\omega = \omega'$ and $v = v'$ a.e. on $[t,s] \Rightarrow \alpha(\cdot,\omega,v) = \alpha(\cdot,\omega',v')$ a.e. on $[t,s+\delta]$. The set of strategies for Player 1 is denoted by $\mathcal{A}(t)$.*
The definition of strategies $\beta : [t,T]\times\mathcal{C}([t,T];\mathbb{R}^d)\times U_t \to V_t$ for Player 2 is similar. The set of strategies for Player 2 is denoted by $\mathcal{B}(t)$.

With Definition 2.2. it is possible to prove via a fixed point argument the following Lemma, which is a slight modification of Lemma 5.1. in [28].

Lemma 2.3. *To each pair of strategies $(\alpha,\beta) \in \mathcal{A}(t)\times\mathcal{B}(t)$ one can associate a unique couple of admissible controls $(u,v) \in \mathcal{U}(t)\times\mathcal{V}(t)$, such that for all $\omega \in \mathcal{C}([t,T];\mathbb{R}^d)$*

$$\alpha(s,\omega,v(\omega)) = u_s(\omega) \qquad \text{and} \qquad \beta(s,\omega,u(\omega)) = v_s(\omega) \ .$$

A characteristic feature of games with incomplete or asymmetric information is that the players have to find a balance between acting optimally according to their information and hiding it. To this end it turns out that the informed player will give his behavior a certain additional randomness (see [28] or [27], [56]). Note that it is also reasonable to allow the uninformed to use random strategies. As shown in the approximation for deterministic games in [26], the uninformed player plays random as well in order to make himself less vulnerable to the manipulation. This effect is captured in the following definition of [28]:

Let $\mathcal{I}$ be a fixed set of probability spaces that is nontrivial and stable by finite product.

Definition 2.4. *A random strategy for Player 1 at time $t \in [0,T]$ is a a pair $((\Omega^\alpha,\mathcal{G}_\alpha,\mathbb{P}_\alpha),\alpha)$, where $(\Omega^\alpha,\mathcal{G}_\alpha,\mathbb{P}_\alpha)$ is a probability space in $\mathcal{I}$ and $\alpha : [t,T]\times\Omega^\alpha\times\mathcal{C}([t,T];\mathbb{R}^d)\times V_t \to U_t$ satisfies*

(i) α is a measurable function, where Ω^α is equipped with the σ-field $\mathcal{G}_\alpha$,

(ii) there exists $\delta > 0$ such that for all $s \in [t,T]$ and for any $\omega,\omega' \in \mathcal{C}([t,T];\mathbb{R}^d)$ and $v, v' \in V_t$ we have:

$\omega = \omega'$ and $v = v'$ a.e. on $[t,s] \Rightarrow \alpha(\cdot,\omega,v) = \alpha(\cdot,\omega',v')$ a.e. on $[t,s+\delta]$ for any $\omega_\alpha \in \Omega_\alpha$.

The set of random strategies for Player 1 is denoted by $\mathcal{A}^r(t)$.
The definition of random strategies $((\Omega^\beta,\mathcal{G}_\beta,\mathbb{P}_\beta),\beta)$, where $\beta : [t,T]\times\Omega^\beta\times\mathcal{C}([t,T];\mathbb{R}^d)\times U_t \to V_t$ for Player 2 is similar. The set of random strategies for Player 2 is denoted by $\mathcal{B}^r(t)$.

Remark 2.5. *Again one can associate to each couple of random strategies* $(\alpha, \beta) \in \mathcal{A}^r(t) \times \mathcal{B}^r(t)$ *for any* $(\omega_\alpha, \omega_\beta) \in \Omega^\alpha \times \Omega^\beta$ *a unique couple of admissible strategies* $(u^{\omega_\alpha,\omega_\beta}, v^{\omega_\alpha,\omega_\beta}) \in \mathcal{U}(t) \times \mathcal{V}(t)$*, such that for all* $\omega \in \mathcal{C}([t,T];\mathbb{R}^d)$*,* $s \in [t,T]$

$$\alpha(s, \omega_\alpha, \omega, v^{\omega_\alpha,\omega_\beta}(\omega)) = u_s^{\omega_\alpha,\omega_\beta}(\omega) \qquad \text{and} \qquad \beta(s, \omega_\beta, \omega, u^{\omega_\alpha,\omega_\beta}(\omega)) = v_s^{\omega_\alpha,\omega_\beta}(\omega)\ .$$

Furthermore $(\omega_\alpha, \omega_\beta) \to (u^{\omega_\alpha,\omega_\beta}, v^{\omega_\alpha,\omega_\beta})$ *is a measurable map, from* $\Omega^\alpha \times \Omega^\beta$ *equipped with the* σ*-field* $\mathcal{G}_\alpha \otimes \mathcal{G}_\beta$ *to* $\mathcal{V}(t) \times \mathcal{U}(t)$ *equipped with the Borel* σ*-field associated to the* L^1*-distance.*

For any $(t,x,p) \in [0,T] \times \mathbb{R}^d \times \Delta(I)$, $(\bar{\alpha}_1, \ldots, \bar{\alpha}_I) \in (\mathcal{A}^r(t))^I$, $\beta \in \mathcal{B}^r(t)$ we define for any $(\omega_{\bar{\alpha}_i}, \omega_\beta)$ the process $X^{t,x,\bar{\alpha}_i,\beta}$ as solution to (2.1) with the associated couple of controls $(u^{\omega_{\bar{\alpha}_i},\omega_\beta}, v^{\omega_{\bar{\alpha}_i},\omega_\beta})$. Furthermore we set

$$J(t,x,p,\bar{\alpha},\beta) = \sum_{i=1}^{I} p_i\ \mathbb{E}_{\bar{\alpha}_i,\beta}\left[\int_0^T l_i(s, X_s^{t,x,\bar{\alpha}_i,\beta}, (\bar{\alpha}_i)_s, \beta_s)ds + g_i(X_T^{t,x,\bar{\alpha}_i,\beta})\right], \tag{2.5}$$

where $\mathbb{E}_{\bar{\alpha}_i,\beta}$ is the expectation on $\Omega_{\bar{\alpha}_i} \times \Omega_\beta \times \mathcal{C}([t,T];\mathbb{R}^d)$ with respect to the probability $\mathbb{P}_{\bar{\alpha}_i} \otimes \mathbb{P}_\beta \otimes \mathbb{P}_0$. Here $\mathbb{P}_0$ denotes the Wiener measure on $\mathcal{C}([t,T];\mathbb{R}^d)$. We note that the information advantage of Player 1 is reflected in (2.5) by having the possibility to choose a strategy $\bar{\alpha}_i$ for each state of nature $i \in \{1, \ldots, I\}$.

Under assumption (A) the existence of the value of the game and its characterization as a viscosity solution to an obstacle problem is shown in [25],[28].

Theorem 2.6. *For any* $(t,x,p) \in [0,T] \times \mathbb{R}^d \times \Delta(I)$ *the game with incomplete information has a value* $V(t,x,p)$ *given by*

$$\begin{aligned} V(t,x,p) &= \inf_{\bar{\alpha}\in(\mathcal{A}^r(t))^I} \sup_{\beta\in\mathcal{B}^r(t)} J(t,x,p,\bar{\alpha},\beta) \\ &= \sup_{\beta\in\mathcal{B}^r(t)} \inf_{\bar{\alpha}\in(\mathcal{A}^r(t))^I} J(t,x,p,\bar{\alpha},\beta). \end{aligned} \tag{2.6}$$

Furthermore the function $V : [0,T] \times \mathbb{R}^d \times \Delta(I) \to \mathbb{R}$ *is the unique viscosity solution to*

$$\begin{aligned} \min\left\{\frac{\partial w}{\partial t} + \frac{1}{2}\mathrm{tr}(\sigma\sigma^*(t,x)D_x^2 w) + H(t,x,D_x w,p), \lambda_{\min}\left(p, \frac{\partial^2 w}{\partial p^2}\right)\right\} &= 0 \\ w(T,x,p) &= \langle p, g(x)\rangle, \end{aligned} \tag{2.7}$$

in the class of bounded, uniformly continuous functions, which are uniformly Lipschitz continuous in p*. For all* $p \in \Delta(I)$*,* $A \in \mathcal{S}^I$ *(where* $\mathcal{S}^I$ *denotes the set of symmetric* $I \times I$ *matrices) we set in (2.7)*

$$\lambda_{\min}(p,A) := \min_{z\in T_{\Delta(I)(p)}\setminus\{0\}} \frac{\langle Az, z\rangle}{|z|^2}. \tag{2.8}$$

and $T_{\Delta(I)(p)}$ *denotes the tangent cone to* $\Delta(I)$ *at* p*, i.e.* $T_{\Delta(I)(p)} = \overline{\cup_{\lambda>0}(\Delta(I)-p)/\lambda}$ *.*

Remark 2.7. *Unlike the standard definition of viscosity solutions (see e.g. [29]) the subsolution property to (2.7) is required only on the interior of* $\Delta(I)$ *while the supersolution property to (2.7) is required on the whole domain* $\Delta(I)$ *(see [25] and [28]). This is due to the fact that we actually consider viscosity solutions with a state constraint, namely* $p \in \Delta(I) \subsetneq \mathbb{R}^I$*. For more details we refer to [22].*

3 Approximation of the value function

3.1 Numerical scheme

Our numerical scheme for the value function basically amounts to approximate the solution of the obstacle problem (2.7). In order to do so it is convenient to consider the real dynamics of the game (2.1) under a Girsanov transform. This technique - first applied to stochastic differential games by [59] - enables us to decouple the forward dynamics (2.1) from the controls of the players. As in [4] where this transformation is applied in the context of numerical approximation for stochastic differential games via BSDE we will use the following approximation for the forward dynamics.
For $L \in \mathbb{N}$ we define a partition of $[0,T]$ with stepsize $\tau = \frac{T}{L}$ by $\Pi^\tau = \{0 = t_0, t_1, \ldots, t_L = T\}$. Then for all $k = 0, \ldots, L$, $x \in \mathbb{R}^d$, $p \in \Delta(I)$ let $(X_s^{t_k,x})_{s \in [t_k,T]}$ denote the diffusion

$$X_s^{t_k,x} = x + \int_{t_k}^s \sigma(r, X_r^{t,x}) dB_r. \tag{3.1}$$

Furthermore we define the discrete process $(\bar{X}_n^{k,x})_{n=k,\ldots,L}$ as the standard Euler scheme approximation for (3.1) on Π^τ

$$\bar{X}_n^{k,x} = x + \sum_{j=k}^{n-1} \sigma(t_j, \bar{X}_j^{k,x}) \Delta B^j, \tag{3.2}$$

where $\Delta B^j = B_{t_{j+1}} - B_{t_j}$.
We will approximate the value function (2.6) backwards in time. To do so we set for all $x \in \mathbb{R}^d$, $p \in \Delta(I)$

$$V^\tau(t_L, x, p) = \langle p, g(x) \rangle \tag{3.3}$$

and we define recursively for $k = L-1, \ldots, 0$

$$V^\tau(t_{k-1}, x, p) \quad = \quad \mathrm{Vex}_p \left(\mathbb{E} \left[V^\tau(t_k, \bar{X}_k^{k-1,x}, p) \right] + \tau H(t_{k-1}, x, \bar{z}_{k-1}(x,p), p) \right), \tag{3.4}$$

where $\bar{z}_{k-1}(x,p)$ is given by

$$\bar{z}_{k-1}(x,p) = \frac{1}{\tau} \mathbb{E} \left[V^\tau(t_k, \bar{X}_k^{k-1,x}, p) (\sigma^*)^{-1}(t_{k-1}, x) \Delta B^{k-1} \right] \tag{3.5}$$

and Vex_p denotes the convex envelope with respect to p, i.e. the largest function that is convex in the variable p and does not exceed the given function. Furthermore for $t \in (t_{k-1}, t_k)$ we define $V^\tau(t,x,p)$ by linear interpolation.

3.2 Some regularity properties

3.2.1 Monotonicity

First we show that our scheme fulfills a monotonicity condition which corresponds to the one in [7] (2.2). It is well known that this criteria is crucial for the convergence of general finite difference schemes.

Lemma 3.1. *Let $\phi : \mathbb{R}^d \to \mathbb{R}$ be a Lipschitz continuous function with Lipschitz constant M. Then there exists for all $x, x' \in \mathbb{R}^d$ a $\theta \in \mathbb{R}^d$ with $|\theta| \leq M$*

$$\phi(x) - \phi(x') = \langle \theta, x - x' \rangle$$

Proof. For $\phi \in C^1$ the result follows from partial integration with $\theta = \in 0^1 D_x\phi(x + r(x' - x))dr$. For the case of general Lipschitz continuous function ϕ one chooses a sequence of $\mathcal{C}^1$ functions $(\phi^\epsilon)_{\epsilon>0}$ which converges uniformly to ϕ. Since ϕ is uniformly Lipschitz continuous, we may assume that the absolute value of $D_x\phi^\epsilon$ and hence the corresponding θ^ϵ are uniformly bounded by the constant M. Consequently, possibly passing though a subsequence, there exists a $\theta \in \mathbb{R}^d$ with $|\theta| \leq M$ such that the lemma is valid. □

With the help of Lemma 3.1 we now establish:

Lemma 3.2. *Let $k \in \{0, \ldots, L-1\}$ and $\phi, \psi : \mathbb{R}^d \to \mathbb{R}$ be two Lipschitz continuous functions with $0 \leq \phi(x) - \psi(x) \leq c$. Then for any $x \in \mathbb{R}$, $p \in \Delta(I)$*

$$\begin{aligned}
&\mathbb{E}\left[\phi(\bar{X}^{k,x}_{k+1})\right] + \tau H(t_k, x, \frac{1}{\tau}\mathbb{E}\left[\phi(\bar{X}^{k,x}_{k+1})(\sigma^*)^{-1}(t_k,x)\Delta B^k\right], p) \\
&\quad \geq \mathbb{E}\left[\psi(\bar{X}^{k,x}_{k+1})\right] + \tau H(t_k, x, \frac{1}{\tau}\mathbb{E}\left[\psi(\bar{X}^{k,x}_{k+1})(\sigma^*)^{-1}(t_k,x)\Delta B^k\right], p) - \tau\mathcal{O}(\tau),
\end{aligned}$$

where $\mathcal{O}(\tau)$ is independent of p.

Proof. By (2.4) H is uniformly Lipschitz continuous in ξ. So by Lemma 3.1. there exists a $\theta \in \mathbb{R}^d$ with $|\theta| \leq M$, where M denotes the Lipschitz constant of H, such that

$$\begin{aligned}
&\mathbb{E}\left[(\phi-\psi)(\bar{X}^{k,x}_{k+1})\right] + \tau\Bigg(H(t_k, x, \frac{1}{\tau}\mathbb{E}\left[\phi(\bar{X}^{k,x}_{k+1})(\sigma^*)^{-1}(t_k,x)\Delta B^k\right], p) \\
&\qquad -H(t_k, x, \frac{1}{\tau}\mathbb{E}\left[\psi(\bar{X}^{k,x}_{k+1})(\sigma^*)^{-1}(t_k,x)\Delta B^k\right], p))\Bigg) \\
&= \mathbb{E}\left[(\phi-\psi)(\bar{X}^{k,x}_{k+1})\right] + \Bigg\langle \tau\theta, \Bigg(\frac{1}{\tau}\mathbb{E}\left[\phi(\bar{X}^{k,x}_{k+1})(\sigma^*)^{-1}(t_k,x)\Delta B^k\right] \\
&\qquad\qquad -\frac{1}{\tau}\mathbb{E}\left[\psi(\bar{X}^{k,x}_{k+1})(\sigma^*)^{-1}(t_k,x)\Delta B^k\right]\Bigg)\Bigg\rangle \\
&= \mathbb{E}\left[(\phi-\psi)(\bar{X}^{k,x}_{k+1})\right] + \left\langle \theta, \mathbb{E}\left[(\phi-\psi)(\bar{X}^{k,x}_{k+1})(\sigma^*)^{-1}(t_k,x)\Delta B^k\right]\right\rangle \\
&= \mathbb{E}\left[(\phi-\psi)(\bar{X}^{k,x}_{k+1})\left(1 + \langle\theta, (\sigma^*)^{-1}(t_k,x)\Delta B^k\rangle\right)\right].
\end{aligned}$$

Since $0 \leq \phi(x) - \psi(x) \leq c$ for any $x \in \mathbb{R}$, we have

$$\begin{aligned}
&\mathbb{E}\left[(\phi-\psi)(\bar{X}^{k,x}_{k+1})\left(1 + \langle\theta, (\sigma^*)^{-1}(t_k,x)\Delta B^k\rangle\right)\right] \\
\geq\ &\mathbb{E}\left[(\phi-\psi)(\bar{X}^{k,x}_{t_{k+1}})1_{|\Delta B^k| \geq \|\theta\sigma^{-1}\|_\infty^{-1}}\langle\theta, (\sigma^*)^{-1}(t_k,x)\Delta B^k\rangle\right] \\
\geq\ &-C\mathbb{E}\left[1_{|\Delta B^k| \geq \frac{1}{C}}|\Delta B^k|\right]
\end{aligned}$$

with $C := \|M\sigma^{-1}\|_\infty$ independent of (t_k, x, p) and τ. Furthermore we can explicitely calculate

$$\mathbb{E}\left[1_{|\Delta B^k| \geq \frac{1}{C}}|\Delta B^k|\right] = \frac{1}{(2\pi)^{\frac{d}{2}}(\tau)^{\frac{1}{2}}}\int_{|x| \geq \frac{1}{C}}^{\infty} |x| e^{-\frac{x^2}{2\tau}}dx = \frac{1}{2^{\frac{d}{2}-1}\Gamma(\frac{d}{2})}\tau^{\frac{1}{2}}e^{-\frac{1}{2C^2\tau}},$$

where Γ denotes the gamma function. □

3.2.2 Lipschitz continuity in x

To show that the Lipschitz continuity in x is preserved under the scheme, we establish the following lemma.

Lemma 3.3. *Let $k \in \{0, \dots, L-1\}$ and $\phi : \mathbb{R}^d \to \mathbb{R}$ be a uniformly Lipschitz continuous function with Lipschitz constant M. Then for any $k \in \{0, \dots, L-1\}$, $x, x' \in \mathbb{R}$, $p \in \Delta(I)$*

$$\begin{aligned}\Bigg|\mathbb{E}\left[\phi(\bar{X}^{k,x}_{k+1})\right] &+ \tau H(t_k, x, \frac{1}{\tau}\mathbb{E}\left[\phi(\bar{X}^{k,x}_{k+1})(\sigma^*)^{-1}(t_k,x)\Delta B^k\right], p) \\ &-\mathbb{E}\left[\phi(\bar{X}^{k,x'}_{k+1})\right] - \tau H(t_k, x', \frac{1}{\tau}\mathbb{E}\left[\phi(\bar{X}^{k,x'}_{k+1})(\sigma^*)^{-1}(t_k,x')\Delta B^k\right], p)\Bigg| \\ &\leq C^{M,\tau}|x - x'|,\end{aligned}$$

where $C^{M,\tau} = M(1 + c\tau) + c\tau$ with c independent of p.

Proof. We fix $k \in \{0, \dots, L-1\}$, $x, x' \in \mathbb{R}$, $p \in \Delta(I)$ and write

$$\begin{aligned}&\mathbb{E}\left[\phi(\bar{X}^{k,x}_{k+1}) - \phi(\bar{X}^{k,x'}_{k+1})\right] + \tau\Bigg(H(t_k, x, \tfrac{1}{\tau}\mathbb{E}\left[\phi(\bar{X}^{k,x}_{k+1})(\sigma^*)^{-1}(t_k,x)\Delta B^k\right], p) \\ &\quad -H(t_k, x', \tfrac{1}{\tau}\mathbb{E}\left[\phi(\bar{X}^{k,x'}_{k+1})(\sigma^*)^{-1}(t_k,x')\Delta B^k\right], p)\Bigg) \\ &= \mathbb{E}\left[\phi(\bar{X}^{k,x}_{k+1}) - \phi(\bar{X}^{k,x'}_{k+1})\right] \\ &\quad +\tau\Bigg(H(t_k, x, \tfrac{1}{\tau}\mathbb{E}\left[\phi(\bar{X}^{k,x}_{k+1})(\sigma^*)^{-1}(t_k,x)\Delta B^k\right], p) \\ &\qquad\qquad -H(t_k, x, \tfrac{1}{\tau}\mathbb{E}\left[\phi(\bar{X}^{k,x'}_{k+1})(\sigma^*)^{-1}(t_k,x')\Delta B^k\right], p)\Bigg) \\ &\quad +\tau\Bigg(H(t_k, x, \tfrac{1}{\tau}\mathbb{E}\left[\phi(\bar{X}^{k,x'}_{k+1})(\sigma^*)^{-1}(t_k,x')\Delta B^k\right], p) \\ &\qquad\qquad -H(t_k, x', \tfrac{1}{\tau}\mathbb{E}\left[\phi(\bar{X}^{k,x'}_{k+1})(\sigma^*)^{-1}(t_k,x')\Delta B^k\right], p)\Bigg).\end{aligned} \tag{3.6}$$

Assume that $\phi \in \mathcal{C}^1$ with $|D_x\phi| \leq M$. First we consider the last term of (3.6). We have for $\Theta^1 := \int_0^1 D_x\phi(x' + r\sigma(t_k,x')\Delta B^k)dr$ that $|\Theta^1| \leq M$ and

$$\begin{aligned}\left|\frac{1}{\tau}\mathbb{E}\left[\phi(\bar{X}^{k,x'}_{k+1})(\sigma^*)^{-1}(t_k,x')\Delta B^k\right]\right| &= \frac{1}{\tau}\left|\mathbb{E}\left[\phi(x')(\sigma^*)^{-1}(t_k,x')\Delta B^k + \Theta^1|\Delta B^k|^2\right]\right| \\ &\leq M.\end{aligned}$$

Since by (2.4) the Hamiltonian H is uniformly Lipschitz continuous in x with Lipschitz constant $c(1 + |\xi|)$ we get, that

$$\begin{aligned}&\tau\Bigg(H(t_k, x, \frac{1}{\tau}\mathbb{E}\left[\phi(\bar{X}^{k,x'}_{k+1})(\sigma^*)^{-1}(t_k,x')\Delta B^k\right], p) \\ &\qquad\qquad -H(t_k, x', \frac{1}{\tau}\mathbb{E}\left[\phi(\bar{X}^{k,x'}_{k+1})(\sigma^*)^{-1}(t_k,x')\Delta B^k\right], p)\Bigg) \\ &\leq \tau c(1 + M)|x - x'|.\end{aligned}$$

For the remaining terms in (3.6) we note that by (2.4) the Hamiltonian H is uniformly Lipschitz continuous. So there exists as in Lemma 3.1. a $\theta^1 \in \mathbb{R}^d$ with $|\theta^1| \le c$, such that

$$
\begin{aligned}
&\mathbb{E}\left[\phi(\bar X^{k,x}_{k+1}) - \phi(\bar X^{k,x'}_{k+1})\right] \\
&\quad + \tau\Bigg(H(t_k, x, \tfrac{1}{\tau}\mathbb{E}\left[\phi(\bar X^{k,x}_{k+1})(\sigma^*)^{-1}(t_k,x)\Delta B^k\right], p) \\
&\qquad\qquad - H(t_k, x, \tfrac{1}{\tau}\mathbb{E}\left[\phi(\bar X^{k,x'}_{k+1})(\sigma^*)^{-1}(t_k,x')\Delta B^k\right], p)\Bigg) \\
&= \mathbb{E}\left[\phi(\bar X^{k,x}_{k+1}) - \phi(\bar X^{k,x'}_{k+1})\right] \\
&\quad + \left\langle \theta^1, \left(\mathbb{E}\left[\phi(\bar X^{k,x}_{k+1})(\sigma^*)^{-1}(t_k,x)\Delta B^k\right] - \mathbb{E}\left[\phi(\bar X^{k,x'}_{k+1})(\sigma^*)^{-1}(t_k,x')\Delta B^k\right]\right)\right\rangle \\
&= \mathbb{E}\left[(\phi(\bar X^{k,x}_{k+1}) - \phi(\bar X^{k,x'}_{k+1}))(1 + \langle \theta^1, (\sigma^*)^{-1}(t_k,x)\Delta B^k\rangle)\right] \\
&\quad + \mathbb{E}\left[\langle \theta^1, \phi(\bar X^{k,x'}_{k+1})((\sigma^*)^{-1}(t_k,x) - (\sigma^*)^{-1}(t_k,x'))\Delta B^k\rangle\right].
\end{aligned}
\tag{3.7}
$$

For the first term of (3.7) we have with $\Theta^2 := \int_0^1 D_x\phi(\bar X^{k,x}_{k+1} + r(\bar X^{k,x'}_{k+1} - \bar X^{k,x}_{k+1}))dr$

$$
\begin{aligned}
&\mathbb{E}\left[(\phi(\bar X^{k,x}_{k+1}) - \phi(\bar X^{k,x'}_{k+1}))(1 + \langle \theta^1, (\sigma^*)^{-1}(t_k,x)\Delta B^k\rangle)\right] \\
&= \mathbb{E}\left[\left\langle \Theta^2, \bar X^{k,x}_{k+1} - \bar X^{k,x'}_{k+1}\right\rangle (1 + \langle \theta^1, (\sigma^*)^{-1}(t_k,x)\Delta B^k\rangle)\right] \\
&\le \mathbb{E}\left[\left\langle \Theta^2, (1 + \langle \theta^1, (\sigma^*)^{-1}(t_k,x)\Delta B^k\rangle)(x - x') + (\sigma(t_k,x) - \sigma(t_k,x'))\Delta B^k\right\rangle\right] \\
&\qquad + c\tau|x - x'|.
\end{aligned}
$$

We finally use Cauchy-Schwarz (note that in the expansion of the square the ΔB^k parts vanish when taking expectation), $|\Theta^2| \le M$ and the Lipschitz continuity of σ to get

$$
\begin{aligned}
&\mathbb{E}\left[\left\langle \Theta^2, (1 + \langle \theta^1, (\sigma^*)^{-1}(t_k,x)\Delta B^k\rangle)(x - x') + (\sigma(t_k,x) - \sigma(t_k,x'))\Delta B^k\right\rangle\right] \\
&\le M\mathbb{E}\left[\left\|(1 + \langle \theta^1, (\sigma^*)^{-1}(t_k,x)\Delta B^k\rangle)(x - x') + (\sigma(t_k,x) - \sigma(t_k,x'))\Delta B^k\right\|^2\right]^{\frac{1}{2}} \\
&\le M|x - x'|\left(\mathbb{E}\left[1 + c|\Delta B^k|^2\right]\right)^{\frac{1}{2}} = M|x - x'|(1 + c\tau)^{\frac{1}{2}} \le M|x - x'|(1 + \frac{c}{2}\tau).
\end{aligned}
$$

For the second term of (3.7) we use the uniform Lipschitz continuity of $(\sigma^*)^{-1}$ (by assumption (A)) to have with the $\mathbb{R}^d$-valued random variable $\Theta^3 := \int_0^1 D_x\phi(\bar X^{k,x'}_{k+1} + r(\bar X^{k,x'}_{k+1} - x'))dr$

$$
\begin{aligned}
&\mathbb{E}\left[\langle \theta^1, \phi(\bar X^{k,x'}_{k+1})((\sigma^*)^{-1}(t_k,x) - (\sigma^*)^{-1}(t_k,x'))\Delta B^k\rangle\right] \\
&= \mathbb{E}\left[\langle \theta^1, (\phi(x') + \langle \Theta^3, \sigma(t,x')\Delta B^k\rangle)((\sigma^*)^{-1}(t_k,x) - (\sigma^*)^{-1}(t_k,x'))\Delta B^k\rangle\right] \\
&\le cM\tau|x - x'|.
\end{aligned}
$$

The case of Lipschitz continuous ϕ follows by approximation with a sequence of $\mathcal{C}^1$ functions $(\phi^\epsilon)_{\epsilon>0}$ which converges uniformly to ϕ. Since ϕ is uniformly Lipschitz continuous with constant M, we may assume that $|D_x\phi^\epsilon| \le M$ for all $\epsilon > 0$. □

With the previous Lemma it is easy to show the Lipschitz continuity of $V^\tau(t.,x,p)$ in x.

Proposition 3.4. *$V^\tau(t.,x,p)$ is uniformly Lipschitz continuous in x with a Lipschitz constant that depends only on the constants of assumption (A).*

Proof. We will show Proposition 3.4. by induction. With (A) we have that $V^\tau(t_L,x,p)$ is Lipschitz continuous in x with a constant M_L that depends only on the constants of assumption (A). Let M_k be the Lipschitz constant for $V^\tau(t_k,\cdot,p)$ then by (3.4) and Lemma 3.3. and since Vex is monotonic, we have

$$|V^\tau(t_{k-1},x,p) - V^\tau(t_{k-1},x',p)| \le (M_k(1+c\tau)+c\tau)|x-x'|.$$

Hence $M_{k-1} := M_k(1+c\tau)+c\tau$ is a Lipschitz constant for $V^\tau(t_{k-1},\cdot,p)$ and $M := M_L C e^{CT}$ for a C independent of τ, x, p is a constant dominating the recursively defined Lipschitz constants $(M_k)_{k=0,\dots,L}$. □

With the uniform Lipschitz continuity of V^τ in x it follows that the value function is uniformly bounded.

Proposition 3.5. *$V^\tau(t.,x,p)$ is uniformly bounded by a constant only depending on the constants of assumption (A).*

Proof. Fix $k \in \{0, L-1\}$, $x \in \mathbb{R}^d$, $p \in \Delta(I)$. Assume first that V^τ is at t_{k+1} continuously differentiable in the second variable with $|D_x V^\tau| \le M$. Then with $\Theta := \int_0^1 D_x V^\tau(t_{k+1}, x + r\sigma(t_k,x)\Delta B^k, p)dr$

$$\begin{aligned} |\bar{z}_k(x,p)| &= \tfrac{1}{\tau}\left|\mathbb{E}\left[V^\tau(t_{k+1}, x+\sigma(t_k,x)\Delta B^k, p)(\sigma^*)^{-1}(t_k,x)\Delta B^k\right]\right| \\ &= \tfrac{1}{\tau}\left|\mathbb{E}\left[V^\tau(t_{k+1},x,p)(\sigma^*)^{-1}(t_k,x)\Delta B^k + \Theta|\Delta B^k|^2\right]\right| \qquad (3.8) \\ &\le M. \end{aligned}$$

Since V^τ is by Lemma 3.3. uniformly Lipschitz continuous in x one has (3.8) in the general case again by regularization.
By (A) $V^\tau(t_L,x,p)$ is bounded by a constant M_L that depends only on the constants of assumption (A). Let M_k be a bound for $|V^\tau(t_k,\cdot,p)|$ then by (2.3) the definition (3.4) and (3.8) we have

$$\mathbb{E}\left[V^\tau(t_k, \bar{X}_k^{k-1,x}, p)\right] + \tau H(t_{k-1}, x, \bar{z}_{k-1}(x,p), p) \le M_k + c\tau(1+M)$$

and $M_L + cT(1+M)$ is a constant dominating the recursively defined constants $(M_k)_{k=0,\dots,L}$. □

3.2.3 Lipschitz continuity in p

The Lipschitz continuity of $V^\tau(t.,x,p)$ in p can be shown with similar methods.

Lemma 3.6. *Let $k \in \{0, \dots, L-1\}$ and $\phi : \mathbb{R}^d \times \Delta(I) \to \mathbb{R}$ be a uniformly Lipschitz continuous function with Lipschitz constant M. Then for any $k \in \{0, \dots, L-1\}$, $x \in \mathbb{R}^d$, $p, p' \in \Delta(I)$*

$$\begin{aligned}
&\Big| \mathbb{E}\left[\phi(\bar{X}^{k,x}_{k+1}, p)\right] + \tau H(t_k, x, \frac{1}{\tau}\mathbb{E}\left[\phi(\bar{X}^{k,x}_{k+1}, p)(\sigma^*)^{-1}(t_k, x)\Delta B^k\right], p) \\
&\quad - \mathbb{E}\left[\phi(\bar{X}^{k,x}_{k+1}, p')\right] - \tau H(t_k, x, \frac{1}{\tau}\mathbb{E}\left[\phi(\bar{X}^{k,x}_{k+1}, p')(\sigma^*)^{-1}(t_k, x)\Delta B^k\right], p') \Big| \\
&\quad \le \bar{C}^{M,\tau}|p - p'|,
\end{aligned}$$

where $\bar{C}^{M,\tau} = M(1 + c\tau) + c\tau$.

Proof. We fix $k \in \{0, \dots, L-1\}$, $x \in \mathbb{R}^d$, $p, p' \in \Delta(I)$. First note that by (2.4) the Hamiltonian is uniformly Lipschitz in p. Hence

$$\begin{aligned}
&\mathbb{E}\left[\phi(\bar{X}^{k,x}_{k+1}, p) - \phi(\bar{X}^{k,x}_{k+1}, p')\right] + \tau\Big(H(t_k, x, \frac{1}{\tau}\mathbb{E}\left[\phi(\bar{X}^{k,x}_{k+1}, p)(\sigma^*)^{-1}(t_k, x)\Delta B^k\right], p) \\
&\quad - H(t_k, x, \frac{1}{\tau}\mathbb{E}\left[\phi(\bar{X}^{k,x}_{k+1}, p')(\sigma^*)^{-1}(t_k, x)\Delta B^k\right], p') \Big) \\
&\le \mathbb{E}\left[\phi(\bar{X}^{k,x}_{k+1}, p) - \phi(\bar{X}^{k,x}_{k+1}, p')\right] + \tau\Big(H(t_k, x, \frac{1}{\tau}\mathbb{E}\left[\phi(\bar{X}^{k,x}_{k+1}, p)(\sigma^*)^{-1}(t_k, x)\Delta B^k\right], p) \\
&\quad - H(t_k, x, \frac{1}{\tau}\mathbb{E}\left[\phi(\bar{X}^{k,x}_{k+1}, p')(\sigma^*)^{-1}(t_k, x)\Delta B^k\right], p) \Big) + c\tau|p - p'|.
\end{aligned}$$

By (2.4) the Hamiltonian H is uniformly Lipschitz continuous in ξ with a constant c. So by Lemma 3.1.

$$\begin{aligned}
&\mathbb{E}\left[\phi(\bar{X}^{k,x}_{k+1}, p) - \phi(\bar{X}^{k,x}_{k+1}, p')\right] + \tau\Big(H(t_k, x, \frac{1}{\tau}\mathbb{E}\left[\phi(\bar{X}^{k,x}_{k+1}, p)(\sigma^*)^{-1}(t_k, x)\Delta B^k\right], p) \\
&\quad - H(t_k, x, \frac{1}{\tau}\mathbb{E}\left[\phi(\bar{X}^{k,x}_{k+1}, p')(\sigma^*)^{-1}(t_k, x)\Delta B^k\right], p) \Big) \\
&= \mathbb{E}\left[\phi(\bar{X}^{k,x}_{k+1}, p) - \phi(\bar{X}^{k,x}_{k+1}, p')\right] \\
&\quad + \Big\langle \theta, \mathbb{E}\left[\phi(\bar{X}^{k,x}_{k+1}, p)(\sigma^*)^{-1}(t_k, x)\Delta B^k\right] - \mathbb{E}\left[\phi(\bar{X}^{k,x}_{k+1}, p')(\sigma^*)^{-1}(t_k, x)\Delta B^k\right] \Big\rangle \\
&= \mathbb{E}\left[(\phi(\bar{X}^{k,x}_{k+1}, p) - \phi(\bar{X}^{k,x}_{k+1}, p'))(1 + \langle \theta, (\sigma^*)^{-1}(t_k, x)\Delta B^k\rangle)\right].
\end{aligned}$$

Assume for a while that ϕ is differentiable in p with $|D_p\phi| \le M$. Then with $\Theta := \int_0^1 D_p\phi(\bar{X}^{k,x}_{k+1}, p + r(p - p'))dr$ we have

$$\begin{aligned}
&\mathbb{E}\left[(\phi(\bar{X}^{k,x}_{k+1}, p) - \phi(\bar{X}^{k,x}_{k+1}, p'))(1 + \langle \theta, (\sigma^*)^{-1}(t_k, x)\Delta B^k\rangle)\right] \\
&= \mathbb{E}\left[\langle \Theta, (1 + \langle \theta, (\sigma^*)^{-1}(t_k, x)\Delta B^k\rangle)(p - p')\rangle\right] \\
&\le M|p - p'| \left(\mathbb{E}\left[1 + c|\Delta B^k|^2\right]\right)^{\frac{1}{2}} = M|p - p'|(1 + c\tau)^{\frac{1}{2}} \le M|p - p'|(1 + \frac{c}{2}\tau),
\end{aligned}$$

where for the first estimate in the last line we used again Cauchy Schwartz as in the previous Lemma. The general case follows again by regularization.

□

It is now easy to show the Lipschitz continuity of $V^\tau(t_\cdot, x, p)$ in p as in Proposition 3.4. by using the fact that the convex hull of a Lipschitz continuous function on $\Delta(I)$ is still Lipschitz continuous with the same Lipschitz constant. The latter result is due to [76].

Proposition 3.7. *$V^\tau(t_\cdot, x, p)$ is uniformly Lipschitz continuous in p with a Lipschitz constant only depending on the constants of assumption (A).*

3.2.4 Hölder continuity in t.

Finally we use the Lipschitz continuity of V^τ in x to establish a Hölder continuity in time on the grid points.

Proposition 3.8. *For all $L \in \mathbb{N}$, $x \in \mathbb{R}^d$, $p \in \Delta(I)$ we have that $(t_\cdot, x, p) \to V^\tau(t_\cdot, x, p)$ is Hölder continuous in $t_\cdot$, in the sense that for all $k \in \{1, \dots, L-1\}, l \in \{1, \dots L-k\}$, there exists a constant c only depending on the constants of assumption (A), such that*

$$|V^\tau(t_{k+l}, x, p) - V^\tau(t_k, x, p)| \le c|t_{k+l} - t_k|^{\frac{1}{2}}.$$

Proof. We fix $(x, p) \in \mathbb{R}^d \times \Delta(I)$. By (3.4), (2.3) and the convexity of V^τ in p we have

$$\begin{aligned}
&|V^\tau(t_{k+l}, x, p) - V^\tau(t_k, x, p)| \\
&\quad = \left|V^\tau(t_{k+l}, x, p) - \mathrm{Vex}_p\left(\mathbb{E}\left[V^\tau(t_{k+1}, \bar{X}^{k,x}_{k+1}, p)\right] + \tau H(t_k, x, \bar{z}_k(x, p), p)\right)\right| \\
&\quad \le \left|\mathbb{E}\left[V^\tau(t_{k+l}, x, p) - V^\tau(t_{k+1}, \bar{X}^{k,x}_{k+1}, p)\right]\right| + c\tau(1 + M),
\end{aligned}$$

where we used that by (3.8) $|\bar{z}_k(x, p)|$ is bounded uniformly in $p \in \Delta(I)$ by the Lipschitz constant of V^τ in x. Note that by definition (3.4)

$$V^\tau(t_{k+1}, \bar{X}^{k,x}_{k+1}, p) = \mathrm{Vex}_p\left(\mathbb{E}\left[V^\tau(t_{k+2}, \bar{X}^{k+1,x'}_{k+2}, p)\right] + \tau H(r, x', \bar{z}_{k+1}(x', p), p)\right)\Big|_{x'=\bar{X}^{k,x}_{k+1}}.$$

Hence by (A) and the fact that V^τ is convex in p we have

$$\begin{aligned}
&\left|V^\tau(t_{k+l}, x, p) - \mathbb{E}\left[V^\tau(t_{k+1}, \bar{X}^{k,x}_{k+1}, p)\right]\right| \\
&\quad = \Bigg|V^\tau(t_{k+l}, x, p) \\
&\qquad\quad - \mathbb{E}\left[\mathrm{Vex}_p\left(\mathbb{E}\left[V^\tau(t_{k+2}, \bar{X}^{k+1,x'}_{k+2}, p)\right] + \tau H(r, x', \bar{z}_{k+1}(x', p), p)\right)\Big|_{x'=\bar{X}^{k,x}_{k+1}}\right]\Bigg| \\
&\quad \le \left|V^\tau(t_{k+l}, x, p) - \mathbb{E}\left[V^\tau(t_{k+2}, \bar{X}^{k+1,\bar{X}^{k,x}_{k+1}}_{k+2}, p)\right]\right| + c\tau(1 + M) \\
&\quad = \left|V^\tau(t_{k+l}, x, p) - \mathbb{E}\left[V^\tau(t_{k+2}, \bar{X}^{k,x}_{k+2}, p)\right]\right| + c\tau(1 + M).
\end{aligned}$$

Since $l\tau = |t_{k+l} - t_k|$ repeating this now $l - 2$ times gives

$$\begin{aligned}
&|V^\tau(t_{k+l}, x, p) - V^\tau(t_k, x, p)| \\
&\quad \le \left|V^\tau(t_{k+l}, x, p) - \mathbb{E}\left[V^\tau(t_{k+l}, \bar{X}^{k,x}_{k+l}, p)\right]\right| + c(1 + M)|t_{k+l} - t_k|.
\end{aligned}$$

Furthermore the Lipschitz continutity of V^τ in x and (A) yields

$$\left|V^\tau(t_{k+l}, x, p) - \mathbb{E}\left[V^\tau(t_{k+l}, \bar{X}^{k,x}_{k+l}, p)\right]\right| \le M\mathbb{E}\left[|\bar{X}^{k,x}_{k+l} - x|\right] \le c|t_{k+l} - t_k|^{\frac{1}{2}},$$

hence

$$|V^\tau(t_{k+l},x,p) - V^\tau(t_k,x,p)| \le M|t_{k+l}-t_k|^{\frac{1}{2}} + c(1+M)|t_{k+l}-t_k|.$$

□

3.3 One step a posteriori martingales and DPP

Since V^τ is bounded we have that at each time step t_k for any $x \in \mathbb{R}^d$ and $p \in \Delta(I)$ the convex envelop with respect to p in (3.4) can be written as a minimization problem

$$\begin{aligned}
&\mathrm{Vex}_p \left(\mathbb{E}\left[V^\tau(t_k, \bar{X}_k^{k-1,x}, p)\right] + \tau H(t_{k-1}, x, \bar{z}_{k-1}(x,p), p)\right) \\
&= \inf_{A(p)} \Big\{ \lambda^1 \left(\mathbb{E}\left[V^\tau(t_k, \bar{X}_k^{k-1,x}, p^1)\right] + \tau H(t_{k-1}, x, \bar{z}_{k-1}(x,p^1), p^1)\right) \\
&\quad + \ldots + \lambda^{I+1} \left(\mathbb{E}\left[V^\tau(t_k, \bar{X}_k^{k-1,x}, p^{I+1})\right] + \tau H(t_{k-1}, x, \bar{z}_{k-1}(x,p^{I+1}), p^{I+1})\right) \Big\}
\end{aligned}$$

where $A(p)$ is the set of $(\lambda^1,\ldots,\lambda^{I+1},\pi^1,\ldots,\pi^{I+1}) \in \Delta(I+1)\times(\Delta(I))^{I+1}$, such that

$$\sum_{l=1}^{I+1} \lambda_l \pi^l = p \quad \text{and} \quad \sum_{l=1}^{I+1} \lambda_l = 1.$$

So by a standard measurable selection theorem there exists at each time step t_k for any $x \in \mathbb{R}^d$ and $p \in \Delta(I)$ a linear combination of $\pi^{k,1}(x,p),\ldots,\pi^{k,I+1}(x,p) \in \Delta(I)$, such that

$$\sum_{l=1}^{I+1} \lambda_l^k(x,p)\pi^{k,l}(x,p) = p \qquad \sum_{l=1}^{I+1} \lambda_l^k(x,p) = 1 \tag{3.9}$$

and

$$\begin{aligned}
&V^\tau(t_k,x,p) \\
&= \textstyle\sum_{l=1}^{I} \lambda_l^k(x,p) \Big(\mathbb{E}\left[V^\tau(t_{k+1}, \bar{X}_{k+1}^{k,x}, \pi^{k,l}(x,p))\right] \\
&\quad + \tau H(t_k, x, \bar{z}_k(x, \pi^{k,l}(x,p)), \pi^{k,l}(x,p)) \Big)
\end{aligned} \tag{3.10}$$

with

$$\bar{z}_k(x,\pi^{k,l}(x,p)) = \frac{1}{\tau}\mathbb{E}\left[V^\tau(t_{k+1}, \bar{X}_{k+1}^{k,x}, \pi^{k,l}(x,p))(\sigma^*)^{-1}(t_k,x)\Delta B^k \right], \tag{3.11}$$

where $(x,p) \to \lambda^k(x,p) \in \Delta(I+1)$ and $(x,p) \to \pi^k(x,p) \in (\Delta(I))^{I+1}$ are Borel measurable.

With the help of weight $(\lambda_1^k(x,p),\ldots,\lambda_{I+1}^k(x,p)) \in \Delta(I+1)$ and the points $\pi^{k,1}(x,p)$, $\ldots$, $\pi^{k,I+1}(x,p) \in \Delta(I)$ it is now possible to construct as in [27], so called one-step a posteriori martingales, which start in p and jump then to one of the support points of the convex hull $\pi^{k,1}(x,p),\ldots,\pi^{k,I+1}(x,p) \in \Delta(I)$.

Definition 3.9. *For all $i = 1,\ldots,I$, $k = 0,\ldots,L$, $x \in \mathbb{R}^n$ and $p \in \Delta(I)$ we define the one step feedbacks $\mathbf{p}_{k+1}^{i,x,p}$ as $\Delta(I)$-valued random variables which are independent of $\sigma(B_s)_{s\in\mathbb{R}}$, such that* *(i) for* $k = 0, \dots, L-1$

(a) if $p_i = 0$ *set* $\mathbf{p}^{i,x,p}_{k+1} = p$

(b) if $p_i > 0$*:* $\mathbf{p}^{i,x,p}_{k+1} \in \{\pi^{k,1}(x,p), \dots, \pi^{k,I+1}(x,p)\}$ *with probability*

$$\begin{aligned}\mathbb{P}\left[\mathbf{p}^{i,x,p}_{k+1} = \pi^{k,l}(x,p) | (\mathbf{p}^{j,x',p'}_m)_{j\in\{1,\dots,I\},x'\in\mathbb{R},p'\in\Delta I, m\in\{1,\dots,k\}}\right] \\ = \lambda^k_l(x,p)\frac{(\pi^{k,l}(x,p))_i}{p_i}\end{aligned}$$

(ii) for $k = L$ *set* $\mathbf{p}^{i,x,p}_{L+1} = e^i$.

Furthermore we define $\mathbf{p}^{x,p}_{k+1} = \mathbf{p}^{\mathbf{i},x,p}_{k+1}$*, where the index* $\mathbf{i}$ *is a random variable with law* p*, independent of* $\sigma(B_s)_{s\in[0,T]}$ *and* $(\mathbf{p}^{j,x',p'}_m)_{j\in\{1,\dots,I\},x'\in\mathbb{R},p'\in\Delta I,m\in\{1,\dots,L\}}$.

Lemma 3.10. *For all* $k = 0, \dots, L$, $x \in \mathbb{R}^n$ *and* $p \in \Delta(I)$ $\mathbf{p}^{x,p}_{k+1}$ *is a one step martingale.*

The martingale property is a direct consequence of Definition 3.9. It can be shown along the lines of the proof of the one step dynamic programming given in the Lemma below.

Lemma 3.11. *For all* $k = 0, \dots, L-1$, $x \in \mathbb{R}^d$, $p \in \Delta(I)$ *we have*

$$V^\tau(t_k, x, p) = \mathbb{E}\left[V^\tau(t_{k+1}, \bar{X}^{k,x}_{k+1}, \mathbf{p}^{x,p}_{k+1}) + \tau H(t_k, x, \bar{z}_k(x, \mathbf{p}^{x,p}_{k+1}), \mathbf{p}^{x,p}_{k+1})\right] \tag{3.12}$$

with

$$\bar{z}_k(x, \mathbf{p}^{x,p}_{k+1}) = \frac{1}{\tau}\mathbb{E}\left[V^\tau(t_{k+1}, \bar{X}^{k,x}_{k+1}, p)(\sigma^*)^{-1}(t_k, x)\Delta B^k\right]\Bigg|_{p=\mathbf{p}^{x,p}_{k+1}}. \tag{3.13}$$

Proof. Assume $(p)_i > 0$ for all $i = 1, \dots, I$. By Definition 3.9. we have that for all suitable functions $f : \Delta(I) \to \mathbb{R}$

$$\begin{aligned}\mathbb{E}[f(\mathbf{p}^{x,p}_{k+1})] &= \sum_{i=1}^I \mathbb{E}\left[1_{\{\mathbf{i}=i\}} f(\mathbf{p}^{i,x,p}_{k+1})\right] = \sum_{i=1}^I \mathbb{E}\left[1_{\{\mathbf{i}=i\}}\right]\mathbb{E}\left[f(\mathbf{p}^{i,x,p}_{k+1})\right] \\ &= \sum_{i=1}^I p_i \sum_{l=1}^I \lambda^k_l(x,p)\frac{(\pi^{k,l}(x,p))_i}{p_i} f((\pi^{k,l}(x,p))) \\ &= \sum_{l=1}^I \lambda^k_l(x,p) f(\pi^{k,l}(x,p))\end{aligned}$$

and the Lemma follows with (3.10). □

4 Convergence

4.1 Main Theorem

Our aim is to establish:

Theorem 4.1. *Under (A) we have uniform convergence of* $(V^\tau, \tau > 0)$ *on the compact subsets of* $[0,T] \times \mathbb{R}^d \times \Delta(I)$*, i.e.*

$$\lim_{\tau\downarrow 0, t'\to t, x'\to x, p'\to p} V^\tau(t', x', p') = V(t, x, p). \tag{4.1}$$

First we note the following:

Proposition 4.2. *$(V^\tau, \tau > 0)$ is compact for the topology of uniform convergence.*

Proof. Note that by Proposition 3.5. the family $(V^\tau, \tau > 0)$ is uniformly bounded. Furthermore by Proposition 3.4. and 3.7., since V^τ is defined by linear interpolation on the time grid, for all $\tau > 0$ the functions V^τ are Lipschitz in x and p with a common Lipschitz constant independent of τ. Proposition 3.8. implies with the linear interpolation that there exists a constant c independent of τ, such that for all $t, s \in [0, T]$

$$|V^\tau(t, x, p) - V^\tau(s, x, p)| \leq c|s - t|^{\frac{1}{2}} + c\tau^{\frac{1}{2}}. \tag{4.2}$$

Hence we have Hölder continuity with an error of the order $\tau^{\frac{1}{2}}$. As it vanishes as $\tau \downarrow 0$ we have with a small modification of the Arzela Ascoli lemma (see e.g. [107]) compactness for the topology of uniform convergence. □

We note that Proposition 4.2. induces a priori several candidates for the limit in Theorem 4.1., where any candidate for the limit is bounded, continuous and in particular Lipschitz continuous in p. Furthermore any candidate is convex in p as a limit of convex functions.
Let $w : [0, T] \times \mathbb{R}^d \times \Delta(I) \to \mathbb{R}$ be one of the candidates for the limit. Then with the results of the subsequent section we have.

Proposition 4.3. *w is a viscosity solution to (2.7).*

Proof. Theorem 4.1. Since we have by Theorem 5.1. in [25] uniqueness of the viscosity solution in the class of bounded, continuous functions from $[0, T] \times \mathbb{R}^d \times \Delta(I)$ to $\mathbb{R}$ which are Lipschitz continuous in the last variable p, with Proposition 4.3. all candidates for the limit coincide. Furthermore since the viscosity solution property uniquely characterizes the value function V the convergence (4.1) follows. □

4.2 Viscosity solution property

4.2.1 Viscosity subsolution property of w

Proposition 4.4. *w is a viscosity subsolution to (2.7).*

Proof. Let $\phi : [0, T] \times \mathbb{R} \times \Delta(I) \to \mathbb{R}$ be a test function such that $w - \phi$ has a strict global maximum at $(\bar{t}, \bar{x}, \bar{p}) \in [0, T) \times \mathbb{R}^d \times \mathrm{Int}(\Delta(I))$. We have to show, that

$$\min \left\{ \frac{\partial \phi}{\partial t} + \frac{1}{2}\mathrm{tr}(\sigma\sigma^*(t, x) D_x^2 \phi) + H(t, x, D_x\phi, p), \lambda_{\min}\left(p, \frac{\partial^2 \phi}{\partial p^2}\right) \right\} \geq 0 \tag{4.3}$$

holds at $(\bar{t}, \bar{x}, \bar{p})$. As a limit of convex functions w is convex in p and we have since $\bar{p} \in \mathrm{Int}(\Delta(I))$

$$\lambda_{\min}\left(\bar{p}, \frac{\partial^2 \phi}{\partial p^2}(\bar{t}, \bar{x}, \bar{p})\right) \geq 0.$$

So it remains to show

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}\mathrm{tr}(\sigma\sigma^*(t, x) D_x^2 \phi) + H(t, x, D_x\phi, p) \geq 0. \tag{4.4}$$

Note that by standard arguments (e.g. [5]) there exists a sequence $(\bar{t}_k, \bar{x}_k, \bar{p}_k)_{k\in\mathbb{N}}$ such that $\bar{t}_k = l_k \frac{T}{k} = l_k\tau \in \Pi^\tau$ converges to $\bar{t}$ and $(\bar{x}_k, \bar{p}_k)$ converge to $(\bar{x}, \bar{p})$ and such that $V^\tau - \phi$ has a global maximum at $(\bar{t}_k, \bar{x}_k, \bar{p}_k)$.
Define $\phi^\tau = \phi + \Delta_\tau$, where $\Delta_\tau = V^\tau(\bar{t}_k, \bar{x}_k, \bar{p}_k) - \phi(\bar{t}_k, \bar{x}_k, \bar{p}_k))$. Then for all $x \in \mathbb{R}, p \in \Delta(I)$

$$V^\tau(\bar{t}_k + \tau, x, p) - \phi^\tau(\bar{t}_k + \tau, x, p) \leq V^\tau(\bar{t}_k, \bar{x}_k, \bar{p}_k) - \phi^\tau(\bar{t}_k, \bar{x}_k, \bar{p}_k) = 0.$$

Set

$$\bar{X}_{k+1} = \bar{x}_k + \sigma(\bar{t}_k, \bar{x}_k)\Delta B^{l_k}$$

and

$$\bar{z}_k = \frac{1}{\tau}\mathbb{E}\left[V^\tau(\bar{t}_k + \tau, \bar{X}_{k+1}, \bar{p}_k)(\sigma^*)^{-1}(\bar{t}_k, \bar{x}_k)\Delta B^{l_k}\right].$$

By the definition of V^τ (3.4) we get that

$$\begin{aligned} 0 &= \operatorname{Vex}_p\left(\mathbb{E}\left[V^\tau(\bar{t}_k + \tau, \bar{X}_{k+1}, \bar{p}_k) + \tau H(\bar{t}_k, \bar{x}_k, \bar{z}_k, \bar{p}_k)ds\right]\right) - V^\tau(\bar{t}_k, \bar{x}_k, \bar{p}_k) \\ &\leq \mathbb{E}\left[V^\tau(\bar{t}_k + \tau, \bar{X}_{k+1}, \bar{p}_k)\right] + \tau H(\bar{t}_k, \bar{x}_k, \bar{z}_k, \bar{p}_k) - V^\tau(\bar{t}_k, \bar{x}_k, \bar{p}_k). \end{aligned}$$

Hence by the monotonicity Lemma 3.2. we have for all $\tau > 0$

$$\begin{aligned} 0 \leq\ & \mathbb{E}\Bigg[V^\tau(\bar{t}_k + \tau, \bar{X}_{k+1}, \bar{p}_k) - V^\tau(\bar{t}_k, \bar{x}_k, \bar{p}_k) \\ & +\tau H(\bar{t}_k, \bar{x}_k, \frac{1}{\tau}\mathbb{E}\left[V^\tau(\bar{t}_k + \tau, \bar{X}_{k+1}, \bar{p}_k)(\sigma^*)^{-1}(\bar{t}_k, \bar{x}_k)\Delta B^{l_k}\right], \bar{p}_k)\Bigg] \\ \leq\ & \mathbb{E}\Bigg[\phi^\tau(\bar{t}_k + \tau, \bar{X}_{k+1}, \bar{p}_k) - \phi^\tau(\bar{t}_k, \bar{x}_k, \bar{p}_k) \\ & +\tau H(\bar{t}_k, \bar{x}_k, \frac{1}{\tau}\mathbb{E}\left[\phi^\tau(\bar{t}_k + \tau, \bar{X}_{k+1}, \bar{p}_k)(\sigma^*)^{-1}(\bar{t}_k, \bar{x}_k)\Delta B^{l_k}\right], \bar{p}_k)\Bigg] \\ & +\tau\mathcal{O}(\tau). \end{aligned}$$

We obtain inequality (4.4) by expanding ϕ^τ, which is just equal to ϕ up to the constant Δ_τ.

□

4.2.2 Viscosity supersolution property of w

Proposition 4.5. *w is a viscosity supersolution to (2.7).*

Proof. To show that $w(t, x, p)$ is a viscosity supersolution of (2.7) let $\phi : [0, T] \times \mathbb{R} \times \Delta(I)$ be a test function, such that $w - \phi$ has a strict global minimum at $(\bar{t}, \bar{x}, \bar{p}) \in [0, T) \times \mathbb{R}^d \times \Delta(I)$ with $w(\bar{t}, \bar{x}, \bar{p}) - \phi(\bar{t}, \bar{x}, \bar{p}) = 0$ and such that its derivatives are uniformly Lipschitz continuous in p.
We have to show, that

$$\min\left\{\frac{\partial\phi}{\partial t} + \frac{1}{2}\operatorname{tr}(\sigma\sigma^T(t, x)D_x^2\phi) + b(t, x)D_x\phi + H(t, x, D_x\phi, p), \lambda_{\min}\left(p, \frac{\partial^2\phi}{\partial p^2}\right)\right\} \leq 0 \quad (4.5)$$

holds at $(\bar{t}, \bar{x}, \bar{p})$. Observe that, if $\lambda_{\min}\left(\frac{\partial^2\phi}{\partial p^2}\right) \leq 0$ at $(\bar{t}, \bar{x}, \bar{p})$, then (4.5) follows immediately. So we now assume $\lambda_{\min}\left(\frac{\partial^2\phi}{\partial p^2}\right) > 0$ at $(\bar{t}, \bar{x}, \bar{p})$.

By standard arguments (e.g. [5]) there exists a sequence $(\bar{t}_k, \bar{x}_k, \bar{p}_k)_{k\in\mathbb{N}}$ such that $\bar{t}_k = l_k\tau \in \Pi^\tau$ converges to $\bar{t}$ and $(\bar{x}_k, \bar{p}_k)$ converge to $(\bar{x}, \bar{p})$ and such that $V^\tau - \phi$ has a global minimum at $(\bar{t}_k, \bar{x}_k, \bar{p}_k)$.
Define $\phi^\tau = \phi + (V^\tau(\bar{t}_k, \bar{x}_k, \bar{p}_k) - \phi(\bar{t}_k, \bar{x}_k, \bar{p}_k)) = \phi + \Delta_\tau$. Since the minimum is global, we have

$$V^\tau(\bar{t}_k + \tau, x, p) - \phi^\tau(\bar{t}_k + \tau, x, p) \geq V^\tau(\bar{t}_k, \bar{x}_k, \bar{p}_k) - \phi(\bar{t}_k, \bar{x}_k, \bar{p}_k) = 0.$$

Note that by the assumption $\lambda_{\min}\left(\frac{\partial^2\phi}{\partial p^2}\right) > 0$ there exists $\delta, \eta > 0$ such that for all k great enough we have

$$\langle \frac{\partial^2 \phi^\tau}{\partial p^2}(t, x, p)z, z\rangle > 4\delta|z|^2 \qquad \forall (x, p) \in B_\eta(\bar{x}_k, \bar{p}_k),\ t \in [\bar{t}_k, \bar{t}_k + \tau],\ z \in T_{\Delta(I)(\bar{p}_k)}. \quad (4.6)$$

Since ϕ^τ is a test function for a purely local viscosity notion, one can modify it outside a neighborhood of $(\bar{t}_k, \bar{x}_k, \bar{p}_k)$, such that for all $(s, x) \in [\bar{t}_k, T] \times \mathbb{R}^d$ the function $\phi^\tau(s, x, \cdot)$ is convex on the whole convex domain $\Delta(I)$. Thus for any $p \in \Delta(I)$

$$V^\tau(s, x, p) \geq \phi^\tau(s, x, p) \geq \phi^\tau(s, x, \bar{p}_k) + \langle \frac{\partial \phi^\tau}{\partial p}(s, x, p), p - \bar{p}_k\rangle. \quad (4.7)$$

We proceed in several steps.
(1) First we show a local estimate which is stronger than (4.7) using (4.6).
(2) In the second step we establish estimates for $\mathbf{p}_{k+1} := \mathbf{p}^{\bar{p}_k, \bar{x}_k}_{l_k+1}$ where $\mathbf{p}^{\bar{p}_k, \bar{x}_k}_{l_k+1}$ is defined as one step martingale with initial data $(\bar{t}_k, \bar{x}_k, \bar{p}_k)$ as in Definition 4.2.
(3) Then we use the estimates of the second step together with the monotonicity in Lemma 3.3. to conclude the viscosity supersolution property.

Step 1: We claim that there exist $\eta, \delta > 0$, such that for all $\tau > 0$ small enough (meaning k great enough)

$$V^\tau(\bar{t}_k + \tau, x, p) \geq \phi^\tau(\bar{t}_k + \tau, x, \bar{p}_k) + \langle \frac{\partial \phi^\tau}{\partial p}(\bar{t}_k + \tau, x, \bar{p}_k), p - \bar{p}_k\rangle + \delta|p - \bar{p}_k|^2 \quad (4.8)$$

for all $x \in B_\eta(\bar{x}_k)$, $p \in \Delta(I)$. By Taylor expansion in p

$$\phi^\tau(t, x, p) \geq \phi^\tau(t, x, \bar{p}_k) + \langle \frac{\partial \phi^\tau}{\partial p}(t, x, p), p - \bar{p}_k\rangle + 2\delta|p - \bar{p}_k|^2 \quad (4.9)$$

holds for $(x, p) \in B_\eta(\bar{x}_k, \bar{p}_k)$, $t \in [\bar{t}_k, \bar{t}_k + \tau]$. Hence (4.8) is true locally in p. To establish (4.8) for all $p \in \Delta(I)$ we set for $p \in \Delta(I) \setminus \mathrm{Int}(B_\eta(\bar{p}_k))$

$$\tilde{p} = \bar{p}_k + \frac{p - \bar{p}_k}{|p - \bar{p}_k|}\eta.$$

So by the convexity of V^τ in p and (4.9) we have for a $\hat{p} \in \partial V^{\tau -}(\bar{t}_k, \bar{x}_k, \tilde{p})$

$$\begin{aligned} V^\tau(\bar{t}_k, \bar{x}_k, p) &\geq V^\tau(\bar{t}_k, \bar{x}_k, \tilde{p}) + \langle \hat{p}, p - \tilde{p}\rangle \\ &\geq \phi^\tau(\bar{t}_k, \bar{x}_k, \bar{p}_k) + \langle \frac{\partial \phi^\tau}{\partial p}(\bar{t}_k, \bar{x}_k, \bar{p}_k), \tilde{p} - \bar{p}_k\rangle + 2\delta\eta^2 + \langle \hat{p}, p - \tilde{p}\rangle \\ &\geq \phi^\tau(\bar{t}_k, \bar{x}_k, \bar{p}_k) + \langle \frac{\partial \phi^\tau}{\partial p}(\bar{t}_k, \bar{x}_k, \bar{p}_k), p - \bar{p}_k\rangle + 2\delta\eta^2 \end{aligned}$$

$$+\langle \hat{p} - \frac{\partial \phi^\tau}{\partial p}(\bar{t}_k, \bar{x}_k, \bar{p}_k), p - \tilde{p}\rangle.$$

Since $\frac{\partial \phi^\tau}{\partial p}(\bar{t}_k, \bar{x}_k, \bar{p}_k) \in \partial V^{\tau -}(\bar{t}_k, \bar{x}_k, \bar{p}_k)$ and $p - \tilde{p} = c(p - \bar{p}_k)$ $(c > 0)$ and V^τ is convex in p we get that

$$\langle \hat{p} - \frac{\partial \phi^\tau}{\partial p}(\bar{t}_k, \bar{x}_k, \bar{p}_k), p - \tilde{p}\rangle \geq 0.$$

So we have for all $p \in \Delta(I) \setminus \mathrm{Int}(B_\eta(\bar{p}_k))$

$$V^\tau(\bar{t}_k, \bar{x}_k, p) \geq \phi^\tau(\bar{t}_k, \bar{x}_k, \bar{p}_k) + \langle \frac{\partial \phi^\tau}{\partial p}(\bar{t}_k, \bar{x}_k, \bar{p}_k), p - \bar{p}_k\rangle + 2\delta\eta^2 \tag{4.10}$$

which gives in the limit for all $p \in \Delta(I) \setminus \mathrm{Int}(B_\eta(\bar{p}))$

$$w(\bar{t}, \bar{x}, p) \geq \phi(\bar{t}, \bar{x}, \bar{p}) + \langle \frac{\partial \phi}{\partial p}(\bar{t}, \bar{x}, \bar{p}), p - \bar{p}\rangle + 2\delta\eta^2. \tag{4.11}$$

Assume now that (4.8) does not hold for a $p \in \Delta(I)$. Hence there exists a sequence $(\tau, x_{k_n}, p_{k_n}) \to (0, 0, p)$ with $\tau = \frac{T}{n}$, $p_{k_n} \in \Delta(I) \setminus B_\eta(\bar{p}_{k_n})$, such that

$$\begin{aligned}
&V^\tau(\bar{t}_{k_n} + \tau, \bar{x}_{k_n} + x_{k_n}, p_{k_n}) \\
&\quad < \phi^\tau(\bar{t}_{k_n} + \tau, \bar{x}_{k_n} + x_{k_n}, \bar{p}_{k_n}) + \langle \frac{\partial \phi^\tau}{\partial p}(\bar{t}_{k_n} + \tau, \bar{x}_{k_n} + x_{k_n}, p_{k_n}), p_{k_n} - \bar{p}_{k_n}\rangle \\
&\qquad + \delta |p_{k_n} - \bar{p}_{k_n}|^2.
\end{aligned}$$

Thus for $n \to \infty$, $p \in \Delta(I) \setminus \mathrm{Int}(B_\eta(\bar{p}))$ and

$$w(\bar{t}, \bar{x}, p) < \phi(\bar{t}, \bar{x}, \bar{p}) + \langle \frac{\partial \phi}{\partial p}(\bar{t}, \bar{x}, \bar{p}), p - \bar{p}\rangle + \delta\eta^2 \tag{4.12}$$

which contradicts (4.11).

In the following we denote

$$\bar{X}_{k+1} = \bar{x}_k + \sigma(\bar{t}_k, \bar{x}_k)\Delta B^{l_k}.$$

where $\Delta B^{l_k} = B_{\bar{t}_k + \tau} - B_{\bar{t}_k}$. With the estimate (4.8) we have for τ small enough for all $p \in \Delta(I)$

$$\begin{aligned}
&\mathbb{E}\left[V^\tau(\bar{t}_k + \tau, \bar{X}_{k+1}, p)\right] \\
&= \mathbb{E}\left[V^\tau(\bar{t}_k + \tau, \bar{X}_{k+1}, p) 1_{|\bar{X}_{k+1} - \bar{x}_k| < \eta}\right] + \mathbb{E}\left[V^\tau(\bar{t}_k + \tau, \bar{X}_{k+1}, p) 1_{|\bar{X}_{k+1} - \bar{x}_k| \geq \eta}\right] \\
&\geq \mathbb{E}\Bigg[\Bigg(\phi^\tau(\bar{t}_k + \tau, \bar{X}_{k+1}, \bar{p}_k) + \langle \frac{\partial}{\partial p}\phi^\tau(\bar{t}_k + \tau, \bar{X}_{k+1}, \bar{p}_k), p - \bar{p}_k\rangle \\
&\qquad\qquad + \delta |p - \bar{p}_k|^2\Bigg) 1_{|\bar{X}_{k+1} - \bar{x}_k| < \eta}\Bigg] \\
&\quad + \mathbb{E}\left[\phi^\tau(\bar{t}_k + \tau, \bar{X}_{k+1}, p) 1_{|\bar{X}_{k+1} - \bar{x}_k| \geq \eta}\right] \\
&= \mathbb{E}\Bigg[\phi^\tau(\bar{t}_k + \tau, \bar{X}_{k+1}, \bar{p}_k) + \langle \frac{\partial}{\partial p}\phi^\tau(\bar{t}_k + \tau, \bar{X}_{k+1}, \bar{p}_k), p - \bar{p}_k\rangle \\
&\qquad + \delta 1_{|\bar{X}_{k+1} - \bar{x}_k| < \eta} |p - \bar{p}_k|^2)\Bigg]
\end{aligned}$$

$$+\mathbb{E}\Bigg[1_{|\bar{X}_{k+1}-\bar{x}_k|\geq\eta}\Bigg(\phi^\tau(\bar{t}_k+\tau,\bar{X}_{k+1},p)-\phi^\tau(\bar{t}_k+\tau,\bar{X}_{k+1},\bar{p}_k) -\langle\frac{\partial}{\partial p}\phi^\tau(\bar{t}_k+\tau,\bar{X}_{k+1},\bar{p}_k),p-\bar{p}_k\rangle\Bigg)\Bigg].$$

Recalling that ϕ^τ is convex with respect to p, we get for all $p\in\Delta(I)$

$$\begin{aligned}&\mathbb{E}\left[V^\tau(\bar{t}_k+\tau,\bar{X}_{k+1},p)\right]\\ &\geq\mathbb{E}\Bigg[\phi^\tau(\bar{t}_k+\tau,\bar{X}_{k+1},\bar{p}_k)+\langle\tfrac{\partial}{\partial p}\phi^\tau(\bar{t}_k+\tau,\bar{X}_{k+1},\bar{p}_k),p-\bar{p}_k\rangle\\ &\qquad+\delta 1_{|\bar{X}_{k+1}-\bar{x}_k|<\eta}|p-\bar{p}_k|^2)\Bigg].\end{aligned}\tag{4.13}$$

Step 2: Next we establish an estimate for $\mathbf{p}_{k+1}:=\mathbf{p}^{\bar{p}_k,\bar{x}_k}_{l_k+1}$ where $\mathbf{p}^{\bar{p}_k,\bar{x}_k}_{l_k+1}$ is defined as one step martingale as in Definition 3.9. with initial data $(\bar{t}_k,\bar{x}_k,\bar{p}_k)$.
Note that by the one step dynamic programming in Lemma 3.11.

$$V^\tau(\bar{t}_k,\bar{x}_k,\bar{p}_k)=\mathbb{E}\left[V^\tau(\bar{t}_k+\tau,\bar{X}_{k+1},\mathbf{p}_{k+1})+\tau H(\bar{t}_k,\bar{x}_k,\bar{z}_k(\bar{x}_k,\mathbf{p}_{k+1}),\mathbf{p}_{k+1})\right].\tag{4.14}$$

Together with $V^\tau(\bar{t}_k,\bar{x}_k,\bar{p}_k)=\phi^\tau(\bar{t}_k,\bar{x}_k,\bar{p}_k)$ and the estimate (4.13) we have, for all $\tau>0$ small enough,

$$\begin{aligned}\phi^\tau(\bar{t}_k,\bar{x}_k,\bar{p}_k)\ \geq\ &\mathbb{E}\Bigg[\phi^\tau(\bar{t}_k+\tau,\bar{X}_{k+1},\bar{p}_k)+\tau H(\bar{t}_k,\bar{x}_k,\bar{z}_k(\bar{x}_k,\mathbf{p}_{k+1}),\mathbf{p}_{k+1})\\ &+\langle\frac{\partial}{\partial p}\phi^\tau(\bar{t}_k+\tau,\bar{X}_{k+1},\bar{p}_k),\mathbf{p}_{k+1}-\bar{p}_k\rangle+\delta 1_{|\bar{X}_{k+1}-\bar{x}_k|<\eta}|\bar{p}_k-\mathbf{p}_{k+1}|^2\Bigg].\end{aligned}$$

Since $\mathbf{p}_{k+1}$ and ΔB^{l_k} are independent, ϕ^τ has bounded derivatives and $\mathbf{p}_{k+1}$ is a one step martingale, we get

$$\begin{aligned}&\mathbb{E}\left[\langle\frac{\partial}{\partial p}\phi^\tau(\bar{t}_k+\tau,\bar{X}_{k+1},\bar{p}_k),\mathbf{p}_{k+1}-\bar{p}_k\rangle\right]\\ &\quad=\mathbb{E}\left[\langle\frac{\partial}{\partial p}\phi^\tau(\bar{t}_k+\tau,\bar{x}_k+\sigma(\bar{t}_k,\bar{x}_k)\Delta B^{l_k},\bar{p}_k),\mathbf{p}_{k+1}-\bar{p}_k\rangle\right]=0.\end{aligned}$$

Furthermore by the Markovian inequality and assumption (A) we have

$$\begin{aligned}&\mathbb{E}\left[1_{|\bar{X}_{k+1}-\bar{x}_k|<\eta}|\mathbf{p}_{k+1}-\bar{p}_k|^2\right]\\ &\quad=\mathbb{E}\left[1_{|\sigma(\bar{t}_k,\bar{x}_k)\Delta B^{l_k}|<\eta}|\mathbf{p}_{k+1}-\bar{p}_k|^2\right]\geq c(1-\tau^{\frac{1}{2}})\mathbb{E}\left[|\mathbf{p}_{k+1}-\bar{p}_k|^2\right]\end{aligned}$$

with a sufficiently small constant c independent of k. Thus

$$\begin{aligned}0\ \geq\ &\mathbb{E}\Bigg[\phi^\tau(\bar{t}_k+\tau,\bar{X}_{k+1},\bar{p}_k)-\phi^\tau(\bar{t}_k,\bar{x}_k,\bar{p}_k)+\tau H(\bar{t}_k,\bar{x}_k,\bar{z}_k(\bar{x}_k,\mathbf{p}_{k+1}),\mathbf{p}_{k+1})\\ &+c\delta(1-\tau^{\frac{1}{2}})|\mathbf{p}_{k+1}-\bar{p}_k|^2\Bigg].\end{aligned}\tag{4.15}$$

Since ϕ^τ has bounded derivatives we get with assumption (A) that

$$\left|\mathbb{E}\left[\phi^\tau(\bar{t}_k+\tau,\bar{X}_{k+1},\bar{p}_k)-\phi^\tau(\bar{t}_k,\bar{x}_k,\bar{p}_k)\right]\right| \quad \leq \quad c\tau \tag{4.16}$$

and since $\mathbb{E}\left[|\bar{z}_k(\bar{x}_k,\mathbf{p}_{k+1})|\right]\leq c$ by (3.8), (A) and Hölder's inequality yields

$$\mathbb{E}\left[\tau H(\bar{t}_k,\bar{x}_k,\bar{z}_k(\bar{x}_k,\mathbf{p}_{k+1}),\mathbf{p}_{k+1})\right] \quad \leq \quad c\tau. \tag{4.17}$$

Combining (4.15)-(4.17) we have for small enough $\tau>0$ and a generic constant $c'>0$

$$\mathbb{E}[|\mathbf{p}_{k+1}-\bar{p}_k|^2]\leq \frac{c'}{c\delta(1-\tau^{\frac{1}{2}})}\tau,$$

hence for small enough τ and a constant $c''>0$

$$\mathbb{E}[|\mathbf{p}_{k+1}-\bar{p}_k|^2]\leq c''\tau. \tag{4.18}$$

Step 3:

Furthermore we have with (4.13), (4.14) and the monotonicity Lemma 3.2., since $V^\tau(\bar{t}_k,\bar{x}_k,\bar{p}_k)=\phi^\tau(\bar{t}_k,\bar{x}_k,\bar{p}_k)$

$$\begin{aligned}0\geq \mathbb{E}\Big[&\phi^\tau(\bar{t}_k+\tau,\bar{X}_{k+1},\mathbf{p}_{k+1})-\phi^\tau(\bar{t}_k,\bar{x}_k,\bar{p}_k)+\tau H(\bar{t}_k,\bar{x}_k,\tilde{z}_k(\bar{x}_k,\mathbf{p}_{k+1}),\mathbf{p}_{k+1})\Big]\\ &-\tau\mathcal{O}(\tau),\end{aligned} \tag{4.19}$$

where

$$\tilde{z}_k(\bar{x}_k,\mathbf{p}_{k+1})=\frac{1}{\tau}\mathbb{E}\left[\phi^\tau(\bar{t}_k+\tau,\bar{X}_{k+1},p)(\sigma^*)^{-1}(\bar{t}_k,\bar{x}_k)\Delta B^{l_k}\right]\big|_{p=\mathbf{p}_{k+1}}.$$

From the construction of $\mathbf{p}_{k+1}$ and the fact that ϕ^τ is convex we get with (4.11) that

$$\begin{aligned}&\mathbb{E}\left[\phi^\tau(\bar{t}_k+\tau,\bar{X}_{k+1},\mathbf{p}_{k+1})\right]\\ &\quad\geq \mathbb{E}\left[\phi^\tau(\bar{t}_k+\tau,\bar{X}_{k+1},\bar{p}_k)+\langle\tfrac{\partial}{\partial p}\phi^\tau(\bar{t}_k+\tau,\bar{X}_{k+1},\bar{p}_k),\mathbf{p}_{k+1}-\bar{p}_k\rangle\right]\\ &\quad=\mathbb{E}\left[\phi^\tau(\bar{t}_k+\tau,\bar{X}_{k+1},\bar{p}_k)\right].\end{aligned} \tag{4.20}$$

It remains to get a suitable estimate for $\tilde{z}_k(\bar{x}_k,\mathbf{p}_{k+1})$. Since ϕ^τ is uniformly Lipschitz continuous in x, we get by Taylor expansion in x that

$$\begin{aligned}\tilde{z}_k(\bar{x}_k,\mathbf{p}_{k+1}) &= \frac{1}{\tau}\mathbb{E}\left[\phi^\tau(\bar{t}_k+\tau,\bar{X}_{k+1},p)(\sigma^*)^{-1}(\bar{t}_k,\bar{x}_k)\Delta B^{l_k}\right]\big|_{p=\mathbf{p}_{k+1}}\\ &= \frac{1}{\tau}\mathbb{E}\left[\phi^\tau(\bar{t}_k+\tau,\bar{x}_k,p)(\sigma^*)^{-1}(\bar{t}_k,\bar{x}_k)\Delta B^{l_k}\right]\big|_{p=\mathbf{p}_{k+1}}\\ &\qquad+\frac{1}{\tau}\mathbb{E}\left[D_x\phi^\tau(\bar{t}_k+\tau,\bar{x}_k,p)|\Delta B^{l_k}|^2\right]\big|_{p=\mathbf{p}_{k+1}}+\mathcal{O}(\tau)\\ &= \frac{1}{\tau}\mathbb{E}\left[D_x\phi^\tau(\bar{t}_k+\tau,\bar{x}_k,p)|\Delta B^{l_k}|^2\right]\big|_{p=\mathbf{p}_{k+1}}+\mathcal{O}(\tau).\end{aligned}$$

Furthermore since $D_x\phi^\tau$ is Lipschitz continuous in p, it follows with (4.18) that

$$\mathbb{E}\left[\left|D_x\phi^\tau(\bar{t}_k+\tau,\bar{x}_k,\mathbf{p}_{k+1})|\Delta B^{l_k}|^2-D_x\phi^\tau(\bar{t}_k+\tau,\bar{x}_k,\bar{p}_k)|\Delta B^{l_k}|^2\right|\right]$$

$$\leq c\mathbb{E}\left[|\mathbf{p}_{k+1} - p_k||\Delta B^{l_k}|^2\right] \leq c\tau^{\frac{3}{2}}.$$

So from (4.20) we have, that

$$0 \geq \mathbb{E}\left[\phi^\tau(\bar{t}_k + \tau, \bar{X}_{k+1}, \bar{p}_k) - \phi^\tau(\bar{t}_k, \bar{x}_k, \bar{p}_k) + \tau H(\bar{t}_k, \bar{x}_k, D_x\phi^\tau(\bar{t}_k + \tau, \bar{x}_k, \bar{p}_k), \bar{p}_k)\right] - c(\tau^{\frac{3}{2}} + \tau\mathcal{O}(\tau)),$$

which implies (4.5) since ϕ^τ is equal to ϕ up to a linear shift.

□

Proposition 4.3. follows immediately by Proposition 4.4. and 4.5.

Chapter 6

On Dynkin games with incomplete information

1 Introduction

In this paper we consider a Dynkin game with incomplete information. The game starts at time 0 and ends at time T paying off a certain terminal payoff. In between the players can choose to stop the game and recieve a certain payment dependent on who stopped the game first. However with regard to the payoffs stopping might be less favourable for them than waiting for the other one to stop the game or the game to terminate. We assume that the game is played by two players. One player is informed about the payoffs, while the other one only knows them with a certain probability $(p_i)_{i\in\{1,\dots,I\}}$. Furthermore we assume that the players observe each other during the game so the uninformed player will try to guess his missing information.

Games with this kind of information incompleteness have been introduced by Aumann and Maschler (see [3]) in discrete time setting. Differential games and stochastic differential games with incomplete information in their spirit have been considered in Cardaliaguet and Rainer [28], who give a characterization of the value function in terms of a fully non linear partial differential equation. As in the case of stochastic differential games with incomplete information studied by Cardaliaguet and Rainer [28], we allow the players to use an additional randomization device. We note that randomized stopping times have already been used in Touzi and Vieille [106] and Laraki and Solan [77] in a different context. As a result even if the informed player knows the exact state of nature he might not stop when it is optimal to stop for him in order to preserve his information advantage.

It turns out that as in the discrete time setting of Aumann and Maschler the randomization device can be interpreted as a certain minimal martingale with a state space in the probability measures on $\{1,\dots I\}$. With the optimal measure this representation then allows to determine optimal strategies for the informed agent. This result has been generalized to differential games by Cardaliaguet and Rainer in [27] and to stochastic differential games by the author in [56]. A similar technique of minimization over martingale measures is introduced in De Meyer [32] to determine optimal strategies for informed agents in a financial market.

In this paper we extend the previous results to the framework of Dynkin games. We show that the value function of Dynkin games with information incompleteness exists and is determined by a solution to a fully non-linear second order variational partial differential

equation. We use the latter characterization in order to establish a dual representation of the value via a minimization procedure over some martingale measures. This representation then allows - under some additional assumptions - to derive optimal strategies for the informed player.
Dynkin games were introduced by E. Dynkin in [39] as a gametheoretical version of an optimal stopping problem. Ever since there has been a vast variety of results obtained by using analytical or purely probabilistic tools. As we are considering continuous time Dynkin games with a diffusion as underlying dynamic we would notably like to mention the works of Bensoussan and Friedman [10] and Friedman [53] who were the first to connect Dynkin games to solutions of second order variational partial differential equations. For for a probabilistic approach we refer to Alario-Nazaret, Lepeltier and Marchal [1], Bismut [13], Ekström and Peskir [40], Eckström and Villeneuve [41], Lepeltier and Maingueneau [79], Morimoto [85], Stettner [104] and the recent work of Kobylanski, Quenez et de Campagnolle [68]. In combination with controlled diffusions also BSDE methods were applied by Cvitanic and Karatzas [31] and Hamadène and Lepeltier [60]. Though the extension of the current paper to Dynkin games, where also the drift of the diffusion is controlled, might seem rather straight forward there are some subtleties to consider. Especially when generalizing the BSDE approach of [56] to an approach with reflected BSDE we have to take into account that for the well-posedness of reflected BSDE as in Hamadène and Lepeltier [60] or Hamadène and Hassani [57] one basically needs that $\mathbf{p}$ is continuous. This however implies a severe restriction on the set of martingale measures $\mathcal{P}(t,p)$, making it impossible to just follow the proofs in [56].
Of course our way to consider information incompleteness is rather specific and far from being the only way to model Dynkin games with incomplete information. A very interesting paper with a completely different ansatz is the recent work of Lempa and Matomäki [78].

2 Description of the game

2.1 Canonical setup and standing assumptions

Let $\mathcal{C}([0,T];\mathbb{R}^d)$ be the set of continuous functions from $\mathbb{R}$ to $\mathbb{R}^d$, which are constant on $(-\infty,0]$ and on $[T,+\infty)$. We denote by $B_s(\omega_B)=\omega_B(s)$ the coordinate mapping on $\mathcal{C}([0,T];\mathbb{R}^d)$ and define $\mathcal{H}=(\mathcal{H}_s)$ as the filtration generated by $s\mapsto B_s$. We denote $\mathcal{H}_{t,s}$ the σ-algebra generated by paths up to time s in $\mathcal{C}([t,T];\mathbb{R}^d)$. Furthermore we provide $\mathcal{C}([0,T];\mathbb{R}^d)$ with the Wiener measure $\mathbb{P}_0$ on $(\mathcal{H}_s)$ and we consider the respective filtration augmented by $\mathbb{P}_0$ nullsets without changing the notation.

In the following we investigate a two-player zero-sum differential game starting at a time $t\geq 0$ with terminal time T. The dynamic is given by an uncontrolled diffusion on $(\mathcal{C}([0,T];\mathbb{R}^d)$, $(\mathcal{H}_{t,s})_{s\in[t,T]},\mathcal{H},\mathbb{P}_0)$, i.e. for $t\in[0,T], x\in\mathbb{R}^d$

$$dX_s^{t,x}=b(s,X_s^{t,x})ds+a(s,X_s^{t,x})dB_s \qquad X_t^{t,x}=x. \tag{2.1}$$

Let $I\in\mathbb{N}^*$ and $\Delta(I)$ denote the simplex of $\mathbb{R}^I$. The objective to optimize is characterized by

(i) terminal payoffs: $(g_i)_{i\in\{1,\ldots,I\}}:\mathbb{R}^d\to\mathbb{R}$,
(ii) early execution payoffs for Player 2: $(f_i)_{i\in\{1,\ldots,I\}}:[0,T]\times\mathbb{R}^d\to\mathbb{R}$,

(iii) early execution payoffs for Player 1: $(h_i)_{i\in\{1,\dots,I\}} : [0,T]\times\mathbb{R}^d \to \mathbb{R}$,
which are chosen with probability $p = (p_i)_{i\in\{1,\dots,I\}} \in \Delta(I)$ before the game starts. Player 1 chooses $\tau \in [0,T]$ to minimize, Player 2 chooses $\sigma \in [0,T]$ to maximize the expected payoff:

$$J_i(t,x,\tau,\sigma) = \mathbb{E}\Bigg[f_i(\sigma, X^{t,x}_\sigma) 1_{\sigma<\tau,\sigma<T} + h_i(\tau, X^{t,x}_\tau) 1_{\tau\le\sigma,\tau<T} + g_i(X^{t,x}_T) 1_{\sigma=\tau=T} \Bigg]. \quad (2.2)$$

We assume that both players observe their opponents control. However Player 1 knows which payoff he minimizes, Player 2 just knows the respective probabilities p_i for scenario $i \in \{1,\dots,I\}$.

The following will be the standing assumption throughout the paper.

Assumption (A)
(i) $b : [0,T]\times\mathbb{R}^d \to \mathbb{R}^d$ is bounded and Lipschitz continuous with respect to (t,x). For $1 \le k,l \le d$ the function $\sigma_{k,l} : [0,T]\times\mathbb{R}^d \to \mathbb{R}$ is bounded and Lipschitz continuous with respect to (t,x).
(ii) $(g_i)_{i\in\{1,\dots,I\}} : \mathbb{R}^d \to \mathbb{R}$, $(f_i)_{i\in\{1,\dots,I\}} : [0,T]\times\mathbb{R}^d \to \mathbb{R}$ and $(h_i)_{i\in\{1,\dots,I\}} : [0,T]\times\mathbb{R}^d \to \mathbb{R}$ are bounded and Lipschitz continuous. For all $i \in \{1,\dots,I\}$, $t\in[0,T]$, $x \in \mathbb{R}^d$ we have that

$$f_i(t,x) \le h_i(t,x) \quad (2.3)$$

and

$$f_i(T,x) \le g_i(x) \le h_i(T,x). \quad (2.4)$$

Remark 2.1. *Note that (A) (ii) implies: for all $t\in[0,T]$, $x\in\mathbb{R}^d$, $p\in\Delta(I)$*

$$\langle p, f(t,x)\rangle \le \langle p, h(t,x)\rangle \quad (2.5)$$

and

$$\langle p, f(T,x)\rangle \le \langle p, g(x)\rangle \le \langle p, h(T,x)\rangle. \quad (2.6)$$

2.2 Random stopping times

In Dynkin games both players have the possibility to stop the game with undergoing a certain punishment (early execution payment), so strategies in this case consist of a stopping decision.

Definition 2.2. *At time $t\in[0,T]$ an admissible stopping time for either player for the game terminating at time T is a $(\mathcal{H}_{t,s})_{s\in[t,T]}$ stopping time with values in $[t,T]$. We denote the set of admissible stopping times by $\mathcal{T}(t,T)$. In the following we shall omit T in the notation whenever it is obvious.*

As in [77], [106] we allow the players to choose their stopping decision randomly

Definition 2.3. *A randomized stopping time after time $t\in[0,T]$ is a measurable function $\mu : [0,1]\times\mathcal{C}([t,T];\mathbb{R}^d) \to [t,T]$ such that for all $r\in[0,1]$*

$$\tau^r(\omega) := \mu(r,\omega) \in \mathcal{T}(t).$$

We denote the set of randomized stopping times by $\mathcal{T}^r(t)$.

For any $(t,x,p) \in [0,T] \times \mathbb{R}^d \times \Delta(I)$, $\mu = (\mu_i)_{i\in\{1,\dots,I\}} \in (\mathcal{T}^r(t))^I, \nu \in \mathcal{T}^r(t)$ we set for $i \in \{1,\dots,I\}$

$$\begin{aligned} J_i(t,x,\mu_i,\nu) \quad = \quad & \mathbb{E}_{\mathbb{P}_0\otimes\lambda\otimes\lambda}\Big[f_i(\nu, X_\nu^{t,x})1_{\nu<\mu_i,\nu<T} \\ & \qquad\qquad + h_i(\mu_i, X_{\mu_i}^{t,x})1_{\mu_i\le\nu,\mu_i<T} + g_i(X_T^{t,x})1_{\mu_i=\nu=T}\Big], \end{aligned} \tag{2.7}$$

where λ denotes the Lebesgue measure on $[0,1]$. (In the following we will skip the subscript $\mathbb{P}_0 \otimes \lambda \otimes \lambda$.) Furthermore we set

$$J(t,x,p,\mu,\nu) = \sum_{i=1}^{I} p_i J_i(t,x,\mu_i,\nu). \tag{2.8}$$

We note that the information advantage of Player 1 is reflected in (2.8) by having the possibility to choose a randomized stopping time μ_i for each state of nature $i \in \{1,\dots,I\}$.

2.3 An example

To illustrate the importance of not immediately revealing the information advantage we would like to conclude this section with a basic deterministic example. Assume that the game takes place between times $t = 0$ and $T = 1$. There are two possible states of nature $i = 1,2$ picked with probability $(p, 1-p)$ before the game starts. They are associated to the two payoff functionals

$$J_1(\tau,\sigma) = (2\tau+1)1_{\tau<\sigma,\tau<1} + (2\sigma-1)1_{\sigma\le\tau,\sigma<1} + 2\ 1_{\sigma=\tau=1} \tag{2.9}$$

and

$$J_2(\tau,\sigma) = (3-\tau)1_{\tau<\sigma,\tau<1} + (2-\sigma)1_{\sigma\le\tau,\sigma<1} + \frac{3}{2}\ 1_{\sigma=\tau=1}. \tag{2.10}$$

Player 1, who is informed about the actual state of nature, chooses $\tau \in [0,1]$ to minimize and Player 2 chooses $\sigma \in [0,1]$ to maximize the payoff functional. However Player 2 is not informed whether it is J_1 or J_2 he has to optimize.
Now if the informed player plays a revealing strategy: he immediately stops the game i.e. $\tau = 0$, if $i = 1$ is picked, and the payoff is $J_1(0,\sigma) = 1$. In case $i = 2$ he does not stop, i.e. $\tau = 1$, for $i = 2$. Player 2 does not know i a priori, but if he sees that the revealing Player 1 does not stop he can be sure $i = 2$, hence the information advantage is lost. In this case it is optimal for Player 2 to stop immediately which yields the payoff $J_2(\tau,0) = 2$. So the overall payoff for a revealing strategy of Player 1 would be $pJ_1(0,\sigma)+(1-p)J_2(\tau,0) = 2-p$. On the other hand if Player 1 plays non-revealing, that means acting as if he does not know i, both player face a stopping game with payoff

$$\begin{aligned} &((3-2p)+(3p-1)\tau)1_{\tau<\sigma,\tau<1} + ((2-3p)+(3p-1)\sigma)1_{\sigma\le\tau,\sigma<1} \\ &\qquad\qquad +(\tfrac{3}{2}+\tfrac{1}{2}p)\ 1_{\sigma=\tau=1}, \end{aligned} \tag{2.11}$$

where only $p \in [0,1]$ is known to both players. For $p < \frac{1}{7}$ the uninformed player in his turn will stop immediately. Hence in this case, we have an overall payoff of $pJ_1(\tau,0) + (1-p)J_2(\tau,0) = 2-3p$, which is indeed smaller than the revealing case. As we see later in section 6.3. in general a mixing of randomly revealing and non-revealing strategies will be optimal for the informed player.

3 Value of the game

For any $(t, x, p) \in [0, T] \times \mathbb{R}^d \times \Delta(I)$ we define the lower value function by

$$V^-(t, x, p) = \sup_{\nu \in \mathcal{T}^r(t)} \inf_{\mu \in (\mathcal{T}^r(t))^I} J(t, x, p, \mu, \nu) \tag{3.1}$$

and the upper value function by

$$V^+(t, x, p) = \inf_{\mu \in (\mathcal{T}^r(t))^I} \sup_{\nu \in \mathcal{T}^r(t)} J(t, x, p, \mu, \nu). \tag{3.2}$$

Remark 3.1. *It is well known (e.g. [28] Lemma 3.1) that it suffices for the uninformed player to use admissible non-random strategies in (3.2). So we can use the easier expression*

$$V^+(t, x, p) = \inf_{\mu \in (\mathcal{T}^r(t))^I} \sup_{\sigma \in \mathcal{T}(t)} J(t, x, p, \mu, \sigma). \tag{3.3}$$

To show that the game has a value we establish:

Theorem 3.2. *For any $(t, x, p) \in [0, T] \times \mathbb{R}^d \times \Delta(I)$ the value of the game is given by*

$$V(t, x, p) := V^+(t, x, p) = V^-(t, x, p). \tag{3.4}$$

Remark 3.3. *Note that by definition $V^+(t, x, p) \geq V^-(t, x, p)$.*

To establish $V^+(t, x, p) \leq V^-(t, x, p)$ we will show that V^+ is a viscosity subsolution and V^- a viscosity supersolution to a nonlinear obstacle problem. More precisely we define the differential operator $\mathcal{L}[w](t, x, p) := \frac{1}{2}\mathrm{tr}(aa^*(t, x)D_x^2 w(t, x, p)) + b(t, x)D_x w(t, x, p)$ and consider

$$\max\Big\{ \max\{\min\{(-\tfrac{\partial}{\partial t} - \mathcal{L})[w], w - \langle f(t, x), p\rangle\}, \\ w - \langle h(t, x), p\rangle\}, -\lambda_{\min}\Big(p, \tfrac{\partial^2 w}{\partial p^2}\Big) \Big\} = 0 \tag{3.5}$$

with terminal condition $w(T, x, p) = \sum_{i=1,\ldots,I} p_i g_i(x)$, where for all $p \in \Delta(I)$, $A \in \mathcal{S}^I$ (where $\mathcal{S}^I$ denotes the set of symmetric $I \times I$ matrices)

$$\lambda_{\min}(p, A) := \min_{z \in T_{\Delta(I)(p)} \setminus \{0\}} \frac{\langle Az, z\rangle}{|z|^2}.$$

and $T_{\Delta(I)(p)}$ denotes the tangent cone to $\Delta(I)$ at p, i.e. $T_{\Delta(I)(p)} = \overline{\cup_{\lambda > 0}(\Delta(I) - p)/\lambda}$.

Remark 3.4. *Note that since by (2.5), (2.6) the obstacles are separated, we one can consider as in the classical case (3.5) as*

$$\max\Big\{ \min\{\max\{(-\tfrac{\partial}{\partial t} - \mathcal{L})[w], w - \langle h(t, x), p\rangle\}, \\ w - \langle f(t, x), p\rangle\}, -\lambda_{\min}\Big(p, \tfrac{\partial^2 w}{\partial p^2}\Big) \Big\} = 0. \tag{3.6}$$

Definition 3.5. *A function $w : [0,T] \times \mathbb{R}^d \times \Delta(I) \to \mathbb{R}$ is a viscosity subsolution to (3.5) if and only if for all $(\bar{t},\bar{x},\bar{p}) \in [0,T) \times \mathbb{R}^d \times \mathrm{Int}(\Delta(I))$ and any test function $\phi : [0,T] \times \mathbb{R}^d \times \Delta(I) \to \mathbb{R}$ such that $w - \phi$ has a (strict) maximum at $(\bar{t},\bar{x},\bar{p})$ with $w(\bar{t},\bar{x},\bar{p}) - \phi(\bar{t},\bar{x},\bar{p}) = 0$ we have, that*

$$\max\left\{ \max\{\min\{(-\tfrac{\partial}{\partial t} - \mathcal{L})[\phi], \phi - \langle f(t,x),p\rangle\}, \phi - \langle h(t,x),p\rangle\}, -\lambda_{\min}\left(p, \tfrac{\partial^2\phi}{\partial p^2}\right) \right\} \leq 0$$

at $(\bar{t},\bar{x},\bar{p})$. This is equivalent to:
(i) $\lambda_{\min}\left(p, \frac{\partial^2\phi}{\partial p^2}\right) \geq 0$
(ii) $w(\bar{t},\bar{x},\bar{p}) = \phi(\bar{t},\bar{x},\bar{p}) \leq \langle h(\bar{t},\bar{x}),\bar{p}\rangle$
(iii) If $w(\bar{t},\bar{x},\bar{p}) = \phi(\bar{t},\bar{x},\bar{p}) > \langle f(\bar{t},\bar{x}),\bar{p}\rangle$*, then* $(\frac{\partial}{\partial t} + \mathcal{L})[\phi](\bar{t},\bar{x},\bar{p}) \geq 0$.

Definition 3.6. *A function $w : [0,T] \times \mathbb{R}^d \times \Delta(I) \to \mathbb{R}$ is a viscosity supersolution to (3.5) if and only if for all $(\bar{t},\bar{x},\bar{p}) \in [0,T) \times \mathbb{R}^d \times \Delta(I)$ and any test function $\phi : [0,T] \times \mathbb{R}^d \times \Delta(I) \to \mathbb{R}$ such that $w - \phi$ has a (strict) minimum at $(\bar{t},\bar{x},\bar{p})$ with $w(\bar{t},\bar{x},\bar{p}) - \phi(\bar{t},\bar{x},\bar{p}) = 0$ we have, that*

$$\max\left\{ \min\{\max\{(-\tfrac{\partial}{\partial t} - \mathcal{L})[\phi], \phi - \langle f(t,x),p\rangle\}, \phi - \langle h(t,x),p\rangle\}, -\lambda_{\min}\left(p, \tfrac{\partial^2\phi}{\partial p^2}\right) \right\} \geq 0$$

at $(\bar{t},\bar{x},\bar{p})$. This is equivalent to: if

$$\lambda_{\min}\left(p, \frac{\partial^2\phi}{\partial p^2}\right) > 0,$$

we have, that
(i) $w(\bar{t},\bar{x},\bar{p}) = \phi(\bar{t},\bar{x},\bar{p}) \geq \langle f(\bar{t},\bar{x}),\bar{p}\rangle$
(ii) If $w(\bar{t},\bar{x},\bar{p}) = \phi(\bar{t},\bar{x},\bar{p}) < \langle h(\bar{t},\bar{x}),\bar{p}\rangle$*, then* $(\frac{\partial}{\partial t} + \mathcal{L})[\phi](\bar{t},\bar{x},\bar{p}) \leq 0$.

An essential part of the proof of Theorem 3.2. is given by the following comparison result. We postpone the proof to the appendix.

Theorem 3.7. *Let $w_1 : [0,T] \times \mathbb{R}^d \times \Delta(I) \to \mathbb{R}$ be a bounded, continuous viscosity subsolution to (3.5), which is uniformly Lipschitz continuous in p, and $w_2 : [0,T] \times \mathbb{R}^d \times \Delta(I) \to \mathbb{R}$ be a bounded, continuous viscosity supersolution to (3.5), which is uniformly Lipschitz continuous in p. Assume that*

$$w_1(T,x,p) \leq w_2(T,x,p) \tag{3.7}$$

for all $x \in \mathbb{R}^d, p \in \Delta(I)$. Then

$$w_1(t,x,p) \leq w_2(t,x,p) \tag{3.8}$$

for all $(t,x,p) \in [0,T] \times \mathbb{R}^d \times \Delta(I)$.

4 Dynamic programming

4.1 Regularity properties

Proposition 4.1. *$V^+(t,x,p)$ and $V^-(t,x,p)$ are uniformly Lipschitz continuous in x and p and Hölder continuous in t.*

Proof. The proof of the Lipschitz continuity in x and p is straightforward and omitted here. For the Hölder continuity in time let $t,t' \in [0,T]$ with $t \le t'$. Assume $V^+(t,x,p) > V^+(t',x,p)$. Then

$$\begin{aligned} 0 &< V^+(t,x,p) - V^+(t',x,p) \\ &= \inf_{\mu\in(\mathcal{T}^r(t))^I} \sup_{\nu\in\mathcal{T}^r(t)} J(t,x,p,\mu,\nu) - \inf_{\mu\in(\mathcal{T}^r(t'))^I} \sup_{\nu\in\mathcal{T}^r(t')} J(t',x,p,\mu,\nu). \end{aligned}$$

Now for $\epsilon > 0$ choose $\bar\mu \in (\mathcal{T}^r(t'))^I$ ϵ-optimal for $V^+(t',x,p)$. Since $t \le t'$ we have $\bar\mu \in (\mathcal{T}^r(t))^I$. Furthermore choose $\bar\nu \in \mathcal{T}^r(t)$ ϵ-optimal for $\sup_{\nu\in\mathcal{T}^r(t)} J(t,x,p,\bar\mu,\nu)$ and define $\hat\nu \in \mathcal{T}^r(t')$

$$\hat\nu = \begin{cases} t' & \text{on } \{\bar\nu < t'\} \\ \bar\nu & \text{on } \{\bar\nu \ge t'\}. \end{cases} \tag{4.1}$$

Then we have

$$V^+(t,x,p) - V^+(t',x,p) - 2\epsilon \le J(t,x,p,\bar\mu,\bar\nu) - J(t',x,p,\bar\mu,\hat\nu). \tag{4.2}$$

Since

$$\begin{aligned} &J(t,x,p,\bar\mu,\bar\nu) - J(t',x,p,\bar\mu,\hat\nu) \\ &= \mathbb{E}[(f_i(\bar\nu, X^{t,x}_{\bar\nu}) - f_i(t',x))1_{\bar\nu<t'}] \\ &\quad + \mathbb{E}\Big[f_i(\bar\nu, X^{t,x}_{\bar\nu}) - f_i(\bar\nu, X^{t',x}_{\bar\nu})1_{t'\le\bar\nu<\bar\mu_i,\bar\nu<T} + h_i(\bar\mu_i, X^{t,x}_{\bar\mu_i}) - h_i(\bar\mu_i, X^{t',x}_{\bar\mu_i})1_{t'\le\bar\mu_i\le\bar\nu,\bar\mu_i<T} \\ &\qquad + (g_i(X^{t,x}_T) - g_i(X^{t',x}_T))1_{\bar\mu_i=\bar\nu=T}\Big], \end{aligned}$$

the claim follows with assumption (A) by standard estimates, since ϵ can be chosen arbitrarily small. The case $V^+(t,x,p) < V^+(t',x,p)$ follows by similar arguments. □

The following is a key property in games with incomplete information (see [3]). Our proof follows closely [28].

Proposition 4.2. *For all $(t,x) \in [0,T] \times \mathbb{R}^d$ $V^+(t,x,p)$ and $V^-(t,x,p)$ are convex in p.*

Proof. That $V^-(t,x,p)$ is convex in p can be easily seen by the following reformulation

$$\begin{aligned} V^-(t,x,p) &= \sup_{\nu\in\mathcal{T}^r(t)} \inf_{\mu\in(\mathcal{T}^r(t))^I} J(t,x,p,\mu,\nu) \\ &= \sup_{\nu\in\mathcal{T}^r(t)} \sum p_i \inf_{\mu\in\mathcal{T}^r(t)} J_i(t,x,p,\mu,\nu). \end{aligned} \tag{4.3}$$

To show that $V^+(t,x,p)$ is convex in p: fix $(t,x) \in [0,T] \times \mathbb{R}^d$ and let $p,p^1,p^2 \in \Delta(I)$, $\lambda \in [0,1]$ such that $p = \lambda p^1 + (1-\lambda)p^2$.

Furthermore choose $\mu^1 \in (\mathcal{T}^r(t))^I$, $\mu^2 \in (\mathcal{T}^r(t))^I$ ϵ-optimal for $V^+(t,x,p^1)$, $V^+(t,x,p^2)$ respectively. Then as in [28] Proposition 2.1. one can construct a $\hat{\mu} \in (\mathcal{T}^r(t))^I$, such that for any $\nu \in \mathcal{T}^r(t)$ we have that

$$\sum_{i=1}^I p_i J_i(t,x,\hat{\mu}_i,\nu) = \lambda \sum_{i=1}^I p_i^1 J_i(t,x,\mu_i^1,\nu) + (1-\lambda)\sum_{i=1}^I p_i^2 J_i(t,x,\mu_i^2,\nu). \tag{4.4}$$

Maximizing over $\nu \in \mathcal{T}^r(t)$ (4.4) yields then

$$V^+(t,x,p) \leq \lambda V^+(t,x,p^1) + (1-\lambda)V^+(t,x,p^2) + 2\epsilon$$

and the result follows since ϵ can be chosen arbitrarily small. □

Furthermore from the very definition of V^+, V^- we have the following:

Proposition 4.3. *For all $(t,x,p) \in [0,T] \times \mathbb{R}^d \times \Delta(I)$ we have that*

$$\langle f(t,x),p\rangle \leq V^+(t,x,p) \leq \langle h(t,x),p\rangle \tag{4.5}$$

and

$$\langle f(t,x),p\rangle \leq V^-(t,x,p) \leq \langle h(t,x),p\rangle. \tag{4.6}$$

4.2 Subdynamic programming principle for V^+

Theorem 4.4. *Let $(\bar{t},\bar{x},\bar{p}) \in [0,T] \times \mathbb{R}^d \times \Delta(I)$. Then for any $t \in [\bar{t},T]$*

$$\begin{aligned} V^+(\bar{t},\bar{x},\bar{p}) \leq \inf_{\tau\in\mathcal{T}(\bar{t},t)} \sup_{\sigma\in\mathcal{T}(\bar{t},t)} \mathbb{E}\Big[\Big\langle \bar{p}, f(\sigma, X_\sigma^{\bar{t},\bar{x}}) 1_{\sigma<\tau,\sigma<t} \Big\rangle \\ + \Big\langle \bar{p}, h(\tau, X_\tau^{\bar{t},\bar{x}}) 1_{\tau\leq\sigma,\tau<t} \Big\rangle + V^+(t, X_t^{\bar{t},\bar{x}}, \bar{p}) 1_{\tau=\sigma=t} \Big]. \end{aligned} \tag{4.7}$$

Proof. Fix $(\bar{t},\bar{x},\bar{p}) \in [0,T] \times \mathbb{R}^d \times \Delta(I)$. Let A^j be a partition of $\mathbb{R}^d$ with $diam(A^j) \leq \delta$ for a $\delta > 0$. For any $j \in \mathbb{N}$, choose a $y^j \in A^j$ and $\mu^j \in (\mathcal{T}^r(t))^I$ ϵ-optimal for $V^+(t,y^j,\bar{p})$. Furthermore choose $\bar{\mu} \in (\mathcal{T}^r(\bar{t},t))^I$ to be ϵ optimal for

$$\begin{aligned} \inf_{\mu\in(\mathcal{T}^r(\bar{t}))^I} \sup_{\sigma\in\mathcal{T}(\bar{t})} \sum_{i=1}^I \bar{p}_i \mathbb{E}\Big[f_i(X_\nu^{\bar{t},\bar{x}}) 1_{\nu<\mu_i,\nu<t} + h_i(X_{\mu_i}^{\bar{t},\bar{x}}) 1_{\mu_i\leq\nu,\mu_i<t} \\ + V^+(t, X_t^{\bar{t},\bar{x}}, \bar{p}) 1_{\mu_i=\nu=t} \Big]. \end{aligned} \tag{4.8}$$

We shall build with $\bar{\mu}$ and $(\mu^j)_{j\in\mathbb{N}}$ a randomized stopping time $\hat{\mu} \in (\mathcal{T}^r(\bar{t}))^I$ in the following way

$$\hat{\mu} = \begin{cases} \bar{\mu} & \text{on } \{\bar{\mu} < t\} \\ (\mu^j)_{j\in\mathbb{N}} & \text{on } \{\bar{\mu} = t, X_t^{\bar{t},\bar{x}} \in A^j\}. \end{cases} \tag{4.9}$$

First note that for any $\sigma \in \mathcal{T}(\bar{t})$.

$$\begin{aligned}
&\sum_{i=1}^I \bar{p}_i \mathbb{E}\Big[f_i(X_\sigma^{\bar{t},\bar{x}}) 1_{\sigma<\hat{\mu}_i, \sigma<T} + h_i(X_{\mu_i}^{\bar{t},\bar{x}}) 1_{\hat{\mu}_i \leq \sigma, \hat{\mu}_i<T} + g_i(X_T^{\bar{t},\bar{x}}) 1_{\hat{\mu}_i=\sigma=T}\Big] \\
&= \sum_{i=1}^I \bar{p}_i \mathbb{E}\Big[f_i(X_\sigma^{\bar{t},\bar{x}}) 1_{\sigma<\hat{\mu}_i, \sigma<t} + h_i(X_{\hat{\mu}_i}^{\bar{t},\bar{x}}) 1_{\hat{\mu}_i \leq \sigma, \hat{\mu}_i<t}\Big] \\
&\quad + \sum_{i=1}^I \bar{p}_i \mathbb{E}\Big[f_i(X_\sigma^{\bar{t},\bar{x}}) 1_{t \leq \sigma<\hat{\mu}_i, \sigma<T} + h_i(X_{\hat{\mu}_i}^{\bar{t},\bar{x}}) 1_{t \leq \hat{\mu}_i \leq \sigma, \hat{\mu}_i<T} \\
&\qquad\qquad + g_i(X_T^{\bar{t},\bar{x}}) 1_{\hat{\mu}_i=\sigma=T}\Big],
\end{aligned} \tag{4.10}$$

while by the uniform Lipschitz continuity of the coefficients by (A) and of V^+ by Proposition 4.1. we have for a generic constant $c > 0$

$$\begin{aligned}
&\sum_{i=1}^I \bar{p}_i \mathbb{E}\Big[f_i(X_\sigma^{\bar{t},\bar{x}}) 1_{t \leq \sigma<\hat{\mu}_i, \sigma<T} + h_i(X_{\mu_i}^{\bar{t},\bar{x}}) 1_{t \leq \hat{\mu}_i \leq \sigma, \hat{\mu}_i<T} + g_i(X_T^{\bar{t},\bar{x}}) 1_{\hat{\mu}_i=\sigma=T}\Big] \\
&\leq \sum_{j\in\mathbb{N}} \sum_{i=1}^I \bar{p}_i \mathbb{E}\Big[(f_i(X_\sigma^{t,y^j}) 1_{t \leq \sigma<\hat{\mu}_i, \sigma<T} + h_i(X_{\mu_i}^{t,y^j}) 1_{t \leq \hat{\mu}_i \leq \sigma, \hat{\mu}_i<T} \\
&\qquad\qquad + g_i(X_T^{t,y^j}) 1_{\hat{\mu}_i=\sigma=T}) 1_{X_t^{\bar{t},\bar{x}} \in A^j}\Big] + c\delta \\
&\leq \mathbb{E}\Big[V^+(t, X_t^{\bar{t},\bar{x}}, \bar{p}) 1_{\hat{\mu}_i \geq t, \sigma \geq t}\Big] + c\delta + \epsilon.
\end{aligned} \tag{4.11}$$

Hence combining (4.8) with (4.10) and (4.11) and choosing $\hat{\sigma} \in \mathcal{T}(\bar{t})$ to be ϵ-optimal for $V^+(\bar{t}, \bar{x}, \bar{p})$ (3.3) we get

$$\begin{aligned}
&V^+(\bar{t}, \bar{x}, \bar{p}) \\
&\leq \inf_{\mu \in (\mathcal{T}^r(\bar{t},t))^I} \sup_{\sigma \in \mathcal{T}(\bar{t},t)} \sum_{i=1}^I \bar{p}_i \mathbb{E}\Big[f_i(X_\sigma^{\bar{t},\bar{x}}) 1_{\sigma<\mu_i, \sigma<t} + h_i(X_\mu^{\bar{t},\bar{x}}) 1_{\mu_i \leq \sigma, \hat{\mu}_i<t} \\
&\qquad\qquad + V^+(t, X_t^{\bar{t},\bar{x}}, \bar{p}) 1_{\mu_i=\sigma=t}\Big] + c\delta + 2\epsilon \\
&\leq \inf_{\tau \in \mathcal{T}(\bar{t},t)} \sup_{\sigma \in \mathcal{T}(\bar{t},t)} \mathbb{E}\Big[\langle \bar{p}, f(X_\sigma^{\bar{t},\bar{x}}) 1_{\sigma<\tau, \sigma<t} + h(X_\tau^{\bar{t},\bar{x}}) 1_{\tau \leq \sigma, \tau<t} \rangle \\
&\qquad\qquad + V^+(t, X_t^{\bar{t},\bar{x}}, \bar{p}) 1_{\tau=\sigma=t}\Big] + c\delta + 2\epsilon.
\end{aligned}$$

The claim follows since ϵ and δ can be chosen arbitrarily small. □

In contrast to the subdynamic programming for V^+ a superdynamic programming principle for V^- can not be derived directly. As in [28] we are led to consider the convex conjugate.

4.3 Convex conjugate of V^- and implications

For $V^- : [0,T] \times \mathbb{R}^d \times \Delta(I) \to \mathbb{R}$ we define the convex conjugate $(V^-)^* : [0,T] \times \mathbb{R}^d \times \mathbb{R}^I \to \mathbb{R}$ as

$$(V^-)^*(t,x,\hat{p}) = \sup_{p \in \Delta(I)} \{\langle \hat{p}, p\rangle - V^-(t,x,p)\}. \tag{4.12}$$

Let $\phi : [0,T] \times \mathbb{R}^d \times \Delta(I) \to \mathbb{R}$ such that $V^- - \phi$ has a strict global minimum at $(\bar{t},\bar{x},\bar{p}) \in [0,T) \times \mathbb{R}^d \times \Delta(I)$ with $V^-(\bar{t},\bar{x},\bar{p}) - \phi(\bar{t},\bar{x},\bar{p}) = 0$ and

$$\lambda_{\min}\left(p, \frac{\partial^2 \phi}{\partial p^2}\right) > 0. \tag{4.13}$$

Then by [25] there exists a $\delta, \eta > 0$ such that for all $p \in \Delta(I)$, $(t,x) \in [\bar{t}, \bar{t}+\eta] \times B_\eta(\bar{x})$

$$V^-(t,x,p) \geq \phi(t,x,\bar{p}) + \langle \frac{\partial \phi}{\partial p}(t,x,\bar{p}), p - \bar{p}\rangle + \delta |p-\bar{p}|^2. \tag{4.14}$$

Consequently, for any $\hat{p} \in \mathbb{R}^I$

$$\begin{aligned}(V^-)^*(t,x,\hat{p}) &= \sup_{p \in \Delta(I)} \{\langle \hat{p}, p\rangle - V^-(t,x,p)\}\\ &\leq -\phi(t,x,\bar{p}) + \sup_{p\in\Delta(I)}\{\langle \hat{p}, p\rangle - \langle \tfrac{\partial \phi}{\partial p}(t,x,\bar{p}), p-\bar{p}\rangle + \delta|p-\bar{p}|^2\} \\ &\leq -\phi(t,x,\bar{p}) + \langle \hat{p}, \bar{p}\rangle + \tfrac{1}{4\delta}|\tfrac{\partial\phi}{\partial p}(t,x,\bar{p}) - \hat{p}|^2,\end{aligned} \tag{4.15}$$

which implies by choosing $\hat{p} = \frac{\partial \phi}{\partial p}(t,x,\bar{p})$

$$(V^-)^*(t,x,\tfrac{\partial\phi}{\partial p}(t,x,\bar{p})) \leq -\phi(t,x,\bar{p}) + \langle \tfrac{\partial \phi}{\partial p}(t,x,\bar{p}), \bar{p}\rangle$$

and for $(t,x) = (\bar{t},\bar{x})$ with (4.15)

$$\begin{aligned}(V^-)^*(\bar{t},\bar{x},\tfrac{\partial\phi}{\partial p}(\bar{t},\bar{x},\bar{p})) &= -V(\bar{t},\bar{x},\bar{p}) + \langle \tfrac{\partial \phi}{\partial p}(\bar{t},\bar{x},\bar{p}), \bar{p}\rangle \\ &= -\phi(\bar{t},\bar{x},\bar{p}) + \langle \tfrac{\partial \phi}{\partial p}(\bar{t},\bar{x},\bar{p}), \bar{p}\rangle.\end{aligned} \tag{4.16}$$

Note that (4.15) and (4.16) imply in particular:

Lemma 4.5. *If there is a test function $\phi : [0,T] \times \mathbb{R}^d \times \Delta(I) \to \mathbb{R}$ such that $V^- - \phi$ has a strict global minimum at $(\bar{t},\bar{x},\bar{p}) \in (0,T) \times \mathbb{R}^d \times \Delta(I)$ with $V^-(\bar{t},\bar{x},\bar{p}) - \phi(\bar{t},\bar{x},\bar{p}) = 0$ and*

$$\lambda_{\min}\left(p, \frac{\partial^2 \phi}{\partial p^2}\right) > 0, \tag{4.17}$$

then $\frac{\partial (V^-)^*}{\partial p}$ exists at $(\bar{t},\bar{x},\hat{p})$ and is equal to $\bar{p}$.

4.4 Subdynamic programming principle for $(V^-)^*$

Instead of a superdynamic programming principle for V^- we can with regard to (4.16) show a subdynamic programming principle for $(V^-)^*$. To that end the following reformulation of $(V^-)^*$ will be useful.

Proposition 4.6. *For any $(t,x,\hat{p}) \in [0,T] \times \mathbb{R}^d \times \mathbb{R}^I$ we have that*

$$(V^-)^*(t,x,\hat{p}) = \inf_{\nu \in \mathcal{T}^r(t)} \sup_{\mu \in \mathcal{T}^r(t)} \max_{i \in \{1,\dots,I\}} \{\hat{p}_i - J_i(t,x,\mu,\nu)\}. \tag{4.18}$$

We recall:

$$J_i(t,x,\mu,\nu) \quad = \quad \mathbb{E}\Big[f_i(\nu, X_\nu^{t,x})1_{\nu<\mu,\nu<T} + h_i(\mu, X_\mu^{t,x})1_{\mu\leq\nu,\mu<T} + g_i(X_T^{t,x})1_{\mu=\nu=T}\Big].$$

Remark 4.7. *Again as in Remark 2.1. we can rewrite (4.18) as*

$$(V^-)^*(t,x,\hat{p}) \quad = \quad \inf_{\nu\in\mathcal{T}^r(t)} \sup_{\tau\in\mathcal{T}(t)} \max_{i\in\{1,\dots,I\}} \{\hat{p}_i - J_i(t,x,\tau,\nu)\}. \tag{4.19}$$

Proof. Denote $w(t,x,\hat{p})$ the right hand side of (4.18). Since V^- is convex in p we have that $((V^-)^*)^* = V^-$. Hence it suffices to prove $w^* = V^-$.
First we show convexity of w in $\hat{p}$. To that end let $\hat{p}, \hat{p}^1, \hat{p}^2 \in \mathbb{R}^I$, $\lambda \in (0,1)$ such that $\hat{p} = \lambda\hat{p}^1 + (1-\lambda)\hat{p}^2$. Choose $\hat{\nu}^1, \hat{\nu}^2$ ϵ-optimal for $w(t,x,\hat{p}^1)$, $w(t,x,\hat{p}^2)$ respectively. Furthermore define as in [28] a $\hat{\nu} \in \mathcal{T}^r(t)$ such that for all $\mu \in \mathcal{T}^r(t)$

$$J_i(t,x,\mu,\hat{\nu}) = \lambda J_i(t,x,\mu,\hat{\nu}^1) + (1-\lambda)J_i(t,x,\mu,\hat{\nu}^2). \tag{4.20}$$

Then for all $\mu \in \mathcal{T}^r(t)$

$$\begin{aligned}
&\max_{i\in\{1,\dots,I\}} \{\hat{p}_i - J_i(t,x,\mu,\hat{\nu})\}\\
&= \max_{i\in\{1,\dots,I\}} \left\{\lambda(\hat{p}_i - J_i(t,x,\mu,\hat{\nu}^1)) + (1-\lambda)(\hat{p}_i - J_i(t,x,\mu,\hat{\nu}^2))\right\}\\
&\leq \lambda \max_{i\in\{1,\dots,I\}} \left\{\hat{p}_i - J_i(t,x,\mu,\hat{\nu}^1)\right\} + (1-\lambda) \max_{i\in\{1,\dots,I\}} \left\{\hat{p}_i - J_i(t,x,\mu,\hat{\nu}^2)\right\}\\
&\leq \lambda w(t,x,\hat{p}^1) + (1-\lambda)w(t,x,\hat{p}^2).
\end{aligned}$$

The convexity follows then by choosing $\hat{\mu}$ ϵ-optimal for $w(t,x,\hat{p})$.
Next we calculate w^*. By definition of the convex conjugate we have

$$\begin{aligned}
&w^*(t,x,p)\\
&= \sup_{\hat{p}\in\mathbb{R}^I} \left\{\langle\hat{p},p\rangle + \sup_{\nu\in\mathcal{T}^r(t)} \inf_{\mu\in\mathcal{T}^r(t)} \min_{j\in\{1,\dots,I\}} \{J_j(t,x,\mu,\nu) - \hat{p}_j\}\right\}\\
&= \sup_{\nu\in\mathcal{T}^r(t)} \sup_{\hat{p}\in\mathbb{R}^I} \left\{\textstyle\sum_{i=1}^I p_i \min_{j\in\{1,\dots,I\}} \inf_{\mu\in\mathcal{T}^r(t)} \{J_j(t,x,\mu,\nu) + \hat{p}_i - \hat{p}_j\}\right\},
\end{aligned}$$

where the supremum is attained for $\hat{p}_j = \inf_{\mu\in\mathcal{T}^r(t)} J_j(t,x,\mu,\nu)$. Hence

$$w^*(t,x,p) = \sup_{\nu\in\mathcal{T}^r(t)} \left\{\sum_{i=1}^I p_i \inf_{\mu\in\mathcal{T}^r(t)} J_i(t,x,\mu,\nu)\right\} = \sup_{\nu\in\mathcal{T}^r(t)} \inf_{\mu\in(\mathcal{T}^r(t))^I} \sum_{i=1}^I p_i J_i(t,x,\mu,\nu).$$

□

As a direct consequence of (4.18) we have:

Proposition 4.8. *For any $(t,x,\hat{p}) \in [0,T] \times \mathbb{R}^d \times \mathbb{R}^I$ we have that*

$$\max_{i\in\{1,\dots,I\}} \{\hat{p}_i - h_i(x)\} \leq (V^-)^*(t,x,\hat{p}) \leq \max_{i\in\{1,\dots,I\}} \{\hat{p}_i - f_i(x)\}. \tag{4.21}$$

Furthermore we have with (4.18) as in Proposition 4.1.:

Proposition 4.9. *$(V^-)^*(t,x,\hat{p})$ is uniformly Lipschitz continuous in x and $\hat{p}$ and Hölder continuous in t.*

Now we can establish a subdynamic programming principle.

Theorem 4.10. *Let $(\bar{t},\bar{x},\hat{p}) \in [0,T]\times\mathbb{R}^d\times\mathbb{R}^I$. Then for all $t\in[\bar{t},T]$*

$$\begin{aligned}(V^-)^*(\bar{t},\bar{x},\hat{p})\\ \leq \inf_{\sigma\in\mathcal{T}(\bar{t},t)}\sup_{\tau\in\mathcal{T}(\bar{t},t)}\mathbb{E}\Big[\max_{i\in\{1,\dots,I\}}\{\hat{p}_i - f_i(X_\nu^{\bar{t},\bar{x}})\}1_{\sigma<\tau,\sigma<t}\\ +\max_{i\in\{1,\dots,I\}}\{\hat{p}_i - h_i(X_\tau^{\bar{t},\bar{x}})\}1_{\tau\leq\sigma,\tau<t} + (V^-)^*(t,X_t^{\bar{t},\bar{x}},\hat{p})1_{\tau=\sigma=t}\Big].\end{aligned} \tag{4.22}$$

Proof. Fix $(\bar{t},\bar{x},\hat{p})\in[0,T]\times\mathbb{R}^d\times\Delta(I)$. Let A^j be a partition of $\mathbb{R}^d$ with $diam(A^j)\leq\delta$ for a $\delta>0$. For any $j\in\mathbb{N}$, choose a $y^j\in A^j$ and $\nu^j\in\mathcal{T}^r(t)$ ϵ-optimal for $(V^-)^*(t,y^j,\hat{p})$. Furthermore fix some $\bar{\sigma}\in\mathcal{T}(\bar{t},t)$ ϵ-optimal for the right hand side of (4.22).
We shall build with $\bar{\sigma}$ and $(\nu^j)_{j\in\mathbb{N}}$ a randomized stopping time $\hat{\nu}\in\mathcal{T}^r(\bar{t})$ in the following way:

$$\hat{\nu} = \begin{cases}\bar{\nu} & \text{on } \{\bar{\nu}<t\}\\ (\nu^j)_{j\in\mathbb{N}} & \text{on } \{\bar{\nu}=t, X_t^{\bar{t},\bar{x}}\in A^j\}.\end{cases} \tag{4.23}$$

First note that for any $\tau\in\mathcal{T}(\bar{t})$

$$\begin{aligned}&\max_{i\in\{1,\dots,I\}}\Big\{\hat{p}_i - \mathbb{E}\Big[f_i(X_{\hat{\nu}}^{\bar{t},\bar{x}})1_{\hat{\nu}<\tau,\hat{\nu}<T} + h_i(X_\tau^{\bar{t},\bar{x}})1_{\tau\leq\hat{\nu},\tau<T} + g_i(X_T^{\bar{t},\bar{x}})1_{\tau=\hat{\nu}=T}\Big]\Big\}\\ &= \max_{i\in\{1,\dots,I\}}\Big\{\hat{p}_i - \mathbb{E}\Big[f_i(X_{\hat{\nu}}^{\bar{t},\bar{x}})1_{\hat{\nu}<\tau,\hat{\nu}<t} + h_i(X_\tau^{\bar{t},\bar{x}})1_{\tau\leq\hat{\nu},\tau<t}\Big]\\ &\qquad -\mathbb{E}\Big[f_i(X_{\hat{\nu}}^{\bar{t},\bar{x}})1_{t\leq\hat{\nu}<\tau,\hat{\nu}<T} + h_i(X_\tau^{\bar{t},\bar{x}})1_{t\leq\tau\leq\hat{\nu},\tau<T} + g_i(X_T^{\bar{t},\bar{x}})1_{\tau=\hat{\nu}=T}\Big]\Big\}\\ &\leq \max_{i\in\{1,\dots,I\}}\Big\{\mathbb{E}\Big[(\hat{p}_i - f_i(X_{\hat{\nu}}^{\bar{t},\bar{x}}))1_{\hat{\nu}<\tau,\hat{\nu}<t} + (\hat{p}_i - h_i(X_\tau^{\bar{t},\bar{x}}))1_{\tau\leq\hat{\nu},\tau<t}\Big]\Big\}\\ &\qquad +\max_{i\in\{1,\dots,I\}}\Big\{\mathbb{E}\Big[\hat{p}_i 1_{t\leq\tau,t\leq\hat{\nu}} - f_i(X_{\hat{\nu}}^{\bar{t},\bar{x}})1_{t\leq\hat{\nu}<\hat{\mu},\hat{\nu}<T}\\ &\qquad\qquad -h_i(X_\tau^{\bar{t},\bar{x}})1_{t\leq\tau\leq\hat{\nu},\tau<T} - g_i(X_T^{\bar{t},\bar{x}})1_{\tau=\hat{\nu}=T}\Big]\Big\}.\end{aligned} \tag{4.24}$$

Furthermore by the uniform Lipschitz continuity of the coefficients by (A) we have for a generic constant $c>0$

$$\begin{aligned}&\Big|\mathbb{E}\Big[f_i(X_{\hat{\nu}}^{\bar{t},\bar{x}})1_{t\leq\hat{\nu}<\tau,\hat{\nu}<T} + h_i(X_\tau^{\bar{t},\bar{x}})1_{t\leq\tau\leq\hat{\nu},\tau<T} + g_i(X_T^{\bar{t},\bar{x}})1_{\tau=\hat{\nu}=T}\Big]\\ &-\textstyle\sum_{j\in\mathbb{N}}\mathbb{E}\Big[f_i(X_{\hat{\nu}}^{t,y^j})1_{t\leq\hat{\nu}<\tau,\hat{\nu}<T} + h_i(X_{\tilde{\mu}}^{t,y^j})1_{t\leq\tau\leq\hat{\nu},\tau<T} + g_i(X_T^{t,y^j})1_{\tau=\hat{\nu}=T}1_{X_t^{\bar{t},\bar{x}}\in A^j}\Big]\Big|\\ &\leq c\delta.\end{aligned}$$

And since $v \mapsto \max_{i\in\{1,\dots,I\}} v_i$ is convex, we have by taking conditional expectation, the fact that $X^{\bar{t},\bar{x}}$ is Markovian and the choice of $\hat{\nu}$ in (4.24)

$$\max_{i\in\{1,\dots,I\}}\Bigg\{\mathbb{E}\Bigg[\hat{p}_i 1_{\tau\geq t,\hat{\nu}\geq t} - f_i(X^{\bar{t},\bar{x}}_{\hat{\nu}})1_{t\leq\hat{\nu}<\tau,\hat{\nu}<T}$$

$$-h_i(X^{\bar{t},\bar{x}}_{\tau})1_{t\leq\tau\leq\hat{\nu},\tau<T} - g_i(X^{\bar{t},\bar{x}}_{T})1_{\tau=\hat{\nu}=T}\Bigg]\Bigg\}$$

$$\leq \sum_{j\in\mathbb{N}} \mathbb{E}\Bigg[\max_{i\in\{1,\dots,I\}}\Bigg\{\hat{p}_i - \mathbb{E}\Bigg[f_i(X^{t,y^j}_{\nu^j})1_{\nu^j<\tau,\nu_j<T} + h_i(X^{t,y^j}_{\tau})1_{\tau\leq\nu^j,\tau<T}$$

$$+g_i(X^{t,y^j}_{T})1_{\tau=\nu^j=T}\Bigg]\Bigg\}1_{X^{\bar{t},\bar{x}}_t\in A^j}1_{\tau\geq t,\hat{\nu}\geq t}\Bigg] + c\delta$$

$$\leq \sum_{j\in\mathbb{N}} \mathbb{E}\Bigg[(V^-)^*(t,y^j,\hat{p})1_{X^{\bar{t},\bar{x}}_t\in A^j}1_{\tau\geq t,\hat{\nu}\geq t}\Bigg] + c\delta + \epsilon,$$

which yields with the Lipschitz property of $(V^-)^*$ in x by Proposition 4.6.

$$\begin{aligned}&\max_{i\in\{1,\dots,I\}}\Bigg\{\mathbb{E}\Bigg[\hat{p}_i 1_{\tau\geq t,\hat{\nu}\geq t} - f_i(X^{\bar{t},\bar{x}}_{\hat{\nu}})1_{t\leq\hat{\nu}<\tau,\hat{\nu}<T}\\ &\qquad -h_i(X^{\bar{t},\bar{x}}_{\tilde{\mu}})1_{t\leq\tau\leq\hat{\nu},\tau<T} - g_i(X^{\bar{t},\bar{x}}_{T})1_{\tau=\hat{\nu}=T}\Bigg]\Bigg\}\\ &\leq \mathbb{E}\Bigg[(V^-)^*(t,X^{\bar{t},\bar{x}}_t,\hat{p})1_{\tau\geq t,\hat{\nu}\geq t}\Bigg] + 2c\delta + \epsilon.\end{aligned} \tag{4.25}$$

Let $\hat{\tau}\in\mathcal{T}(\bar{t})$ be ϵ-optimal for $(V^-)^*(\bar{t},\bar{x},\hat{p})$ (4.18) then combining (4.24) with (4.25) we get

$$(V^-)^*(\bar{t},\bar{x},\bar{p})$$

$$\leq \max_{i\in\{1,\dots,I\}}\Bigg\{\mathbb{E}\Bigg[(\hat{p}_i - f_i(X^{\bar{t},\bar{x}}_{\hat{\nu}}))1_{\hat{\nu}<\hat{\tau},\hat{\nu}<t} + (\hat{p}_i - h_i(X^{\bar{t},\bar{x}}_{\hat{\tau}}))1_{\hat{\tau}\leq\hat{\nu},\tau<t}$$

$$+(V^-)^*(t,X^{\bar{t},\bar{x}}_t,\hat{p})1_{\hat{\tau}\geq t,\hat{\nu}\geq t}\Bigg]\Bigg\} + \epsilon + 2c\delta.$$

$$\leq \inf_{\sigma\in\mathcal{T}(\bar{t},t)}\sup_{\tau\in\mathcal{T}(\bar{t},t)}\mathbb{E}\Bigg[\max_{i\in\{1,\dots,I\}}\{\hat{p}_i - f_i(X^{\bar{t},\bar{x}}_{\sigma})\}1_{\sigma<\tau,\sigma<t}$$

$$+\max_{i\in\{1,\dots,I\}}\{\hat{p}_i - h_i(X^{\bar{t},\bar{x}}_{\tau})\}1_{\tau\leq\sigma,\tau<t} + (V^-)^*(t,X^{\bar{t},\bar{x}}_t,\hat{p})1_{\sigma=\tau=t}\Bigg]$$

$$+2\epsilon + 2c\delta.$$

The claim follows since ϵ and δ can be chosen arbitrarily small.

□

5 Viscosity solution property

5.1 Subsolution property for V^+

Theorem 5.1. *V^+ is a viscosity subsolution to (3.5).*

Proof. Let $(\bar{t},\bar{x},\bar{p}) \in [0,T) \times \mathbb{R}^d \times \mathrm{Int}(\Delta(I))$ and $\phi : [0,T] \times \mathbb{R}^d \times \Delta(I) \to \mathbb{R}$ a test function such that $V^+ - \phi$ has a strict global maximum at $(\bar{t},\bar{x},\bar{p})$ with $V^+(\bar{t},\bar{x},\bar{p}) - \phi(\bar{t},\bar{x},\bar{p}) = 0$. Because of the convexity of V^+ by Proposition 4.2. and since $\bar{p} \in \mathrm{Int}(\Delta(I))$ we have

$$\lambda_{\min}\left(p, \frac{\partial^2 \phi}{\partial p^2}\right) \geq 0. \tag{5.1}$$

So it remains to show

$$\max\{\min\{(-\tfrac{\partial}{\partial t} - \mathcal{L})(\phi), \phi - \langle f(t,x),p\rangle\}, \phi - \langle h(t,x),p\rangle\} \leq 0 \tag{5.2}$$

at $(\bar{t},\bar{x},\bar{p})$.
Note that by Proposition 4.3. we already have

$$\phi(\bar{t},\bar{x},\bar{p}) - \langle h(\bar{t},\bar{x}),\bar{p}\rangle \leq 0. \tag{5.3}$$

So it remains to show that for $V^+(\bar{t},\bar{x},\bar{p}) - \langle f(\bar{t},\bar{x}),\bar{p}\rangle = \phi(\bar{t},\bar{x},\bar{p}) - \langle f(\bar{t},\bar{x}),\bar{p}\rangle > 0$ we have that $(-\frac{\partial \phi}{\partial t} - \mathcal{L})[\phi](\bar{t},\bar{x},\bar{p}) \leq 0$, which is just a classical consequence of the subdynamic programming principle for V^+. Indeed if we set $\tau = t$ in the dynamic programming (4.22) we have for an $\epsilon(t-\bar{t})$ optimal $\sigma_\epsilon \in \mathcal{T}(\bar{t})$

$$\begin{aligned} \phi(\bar{t},\bar{x},\bar{p}) &= V^+(\bar{t},\bar{x},\bar{p}) \\ &\leq \mathbb{E}\left[\langle \bar{p}, f(X^{\bar{t},\bar{x}}_{\sigma^\epsilon})\rangle 1_{\sigma^\epsilon < t} + V^+(t, X^{\bar{t},\bar{x}}_t, \bar{p}) 1_{\sigma^\epsilon = t}\right] - \epsilon(t-\bar{t}) \\ &\leq \mathbb{E}\left[\langle \bar{p}, f(X^{\bar{t},\bar{x}}_{\sigma^\epsilon})\rangle 1_{\sigma^\epsilon < t} + \phi(t, X^{\bar{t},\bar{x}}_t, \bar{p}) 1_{\sigma^\epsilon = t}\right] - \epsilon(t-\bar{t}). \end{aligned} \tag{5.4}$$

If we now assume

$$V^+(\bar{t},\bar{x},\bar{p}) - \langle f(\bar{t},\bar{x}),\bar{p}\rangle = \phi(\bar{t},\bar{x},\bar{p}) - \langle f(\bar{t},\bar{x}),\bar{p}\rangle > 0 \tag{5.5}$$

and

$$(-\frac{\partial \phi}{\partial t} - \mathcal{L})[\phi](\bar{t},\bar{x},\bar{p}) > 0, \tag{5.6}$$

then there exists $h, \delta > 0$ such that for all $(s,x) \in [\bar{t}, \bar{t}+h] \times B_h(\bar{x})$

$$\phi(s,x,\bar{p}) - \langle f(s,x),\bar{p}\rangle \geq \delta \qquad \text{and} \qquad (-\tfrac{\partial \phi}{\partial t} - \mathcal{L})[\phi](s,x,\bar{p}) \geq \delta.$$

Define $A := \{\inf_{s\in[\bar{t},t]} |X^{\bar{t},\bar{x}}_s - \bar{x}| > h\}$ and note that there exists a constant c depending only on the parameters of $X^{\bar{t}.\bar{x}}$ such that $\mathbb{P}[A] \leq \frac{c(t-\bar{t})^2}{h^4}$. By the Itô formula we have since the coefficients ϕ and all its derivatives are bounded

$$\begin{aligned}
\phi(\bar{t},\bar{x},\bar{p}) &= \mathbb{E}\left[\phi(\sigma^\epsilon, X^{\bar{t},\bar{x}}_{\sigma^\epsilon},\bar{p}) + \textstyle\int_{\bar{t}}^{\sigma^\epsilon}(-\frac{\partial}{\partial t}-\mathcal{L})(s,X^{\bar{t},\bar{x}}_s,\bar{p})ds\right] \\
&\geq \mathbb{E}\left[1_{A^c}\left(\phi(\sigma^\epsilon, X^{\bar{t},\bar{x}}_{\sigma^\epsilon},\bar{p}) + \textstyle\int_{\bar{t}}^{\sigma^\epsilon}(-\frac{\partial}{\partial t}-\mathcal{L})(s,X^{\bar{t},\bar{x}}_s,\bar{p})ds\right)\right] - c\tfrac{(t-\bar{t})^2}{h^4} \\
&\geq \mathbb{E}\Bigg[1_{A^c}\Bigg((\langle f(\sigma^\epsilon, X^{\bar{t},\bar{x}}_{\sigma^\epsilon}),\bar{p}\rangle + \delta)1_{\sigma^\epsilon<t} + \phi(\sigma^\epsilon, X^{\bar{t},\bar{x}}_{\sigma^\epsilon},\bar{p})1_{\sigma^\epsilon=t} \\
&\qquad +\delta(\sigma^\epsilon-\bar{t})\Bigg)\Bigg] - c\tfrac{(t-\bar{t})^2}{h^4} \\
&\geq \mathbb{E}\left[\langle f(\sigma^\epsilon, X^{\bar{t},\bar{x}}_{\sigma^\epsilon}),\bar{p}\rangle 1_{\sigma^\epsilon<t} + \phi(\sigma^\epsilon, X^{\bar{t},\bar{x}}_{\sigma^\epsilon},\bar{p})1_{\sigma^\epsilon=t}\right] \\
&\qquad +\delta\mathbb{E}\left[1_{\sigma^\epsilon<t} + (\sigma^\epsilon-\bar{t})\right] - 2c\tfrac{(t-\bar{t})^2}{h^4}.
\end{aligned}$$

Furthermore note that for $1 \geq (t-\bar{t})$ we have that

$$\mathbb{E}\left[1_{\sigma^\epsilon<t} + (\sigma^\epsilon-\bar{t})\right] = \mathbb{E}\left[(1+\sigma^\epsilon-\bar{t})1_{\sigma^\epsilon<t} + (t-\bar{t})1_{\sigma^\epsilon=t}\right] \geq (t-\bar{t}). \tag{5.7}$$

So

$$\begin{aligned}
\phi(\bar{t},\bar{x},\bar{p}) &\geq \mathbb{E}\left[\langle f(\sigma^\epsilon, X^{\bar{t},\bar{x}}_{\sigma^\epsilon}),\bar{p}\rangle 1_{\sigma^\epsilon<t} + \phi(\sigma^\epsilon, X^{\bar{t},\bar{x}}_{\sigma^\epsilon},\bar{p})1_{\sigma^\epsilon=t}\right] \\
&\qquad +\delta(t-\bar{t}) - 2c\tfrac{(t-\bar{t})^2}{h^4},
\end{aligned}$$

which gives with (5.4)

$$\delta(t-\bar{t}) - 2c\frac{(t-\bar{t})^2}{h^4} - \epsilon(\bar{t}-t) \leq 0.$$

Hence

$$\delta - 2c\frac{(t-\bar{t})}{h^4} - \epsilon \leq 0, \tag{5.8}$$

which yields a contradiction, since $(t-\bar{t})$ and ϵ can be choosen arbitrarily small.

□

5.2 Supersolution property of V^-

With the subdynamic programming principle for $(V^-)^*$ Theorem 4.10. and the estimate in Proposition 4.9. we can now as in Theorem 5.1. establish:

Theorem 5.2. $(V^-)^*$ *is convex and is a viscosity subsolution to the obstacle problem*

$$\begin{aligned}
\max\Big\{ &\min\left\{(-\tfrac{\partial}{\partial t}-\mathcal{L})[w], w - \max_{i\in\{1,\dots,I\}}\{\hat{p}_i - h_i(x)\}\right\}, \\
&\qquad w - \max_{i\in\{1,\dots,I\}}\{\hat{p}_i - f_i(x)\}\Big\} = 0
\end{aligned} \tag{5.9}$$

with terminal condition $w(T,x,p) = \max_{i\in\{1,\dots,I\}}\{\hat{p}_i - g_i(x)\}$.

We are now using Theorem 4.2 to conclude the supersolution property for V^-.

Theorem 5.3. *V^- is a viscosity supersolution to (3.5).*

Proof. Assume that $p = e_i$ for an $i \in \{1, \ldots, I\}$, where e_i denotes the i-th coordinate vector in $\mathbb{R}^I$. Then (5.9) reduces to the PDE for a game with complete information, i.e.

$$\max\{\min\{(-\tfrac{\partial}{\partial t} - \mathcal{L})[w], w - f_i(t,x)\}, w - h_i(t,x)\} = 0 \tag{5.10}$$

with terminal condition $w(T,x,p) = g_i(x)$ and the result is standard.

Let $\bar{p} \notin \{e_i,\ i = 1, \ldots, I\}$ and $\phi : [0,T] \times \mathbb{R}^d \times \Delta(I) \to \mathbb{R}$ such that $V^- - \phi$ has a strict global minimum at $(\bar{t}, \bar{x}, \bar{p}) \in [0,T) \times \mathbb{R}^d \times \Delta(I)$ with $V^-(\bar{t},\bar{x},\bar{p}) - \phi(\bar{t},\bar{x},\bar{p}) = 0$. We have to show

$$\begin{aligned} \max\Big\{ & \max\{\min\{(-\tfrac{\partial}{\partial t} - \mathcal{L})[\phi], \phi - \langle f(t,x), p\rangle\}, \\ & \qquad \phi - \langle h(t,x), p\rangle\}, -\lambda_{\min}\left(p, \tfrac{\partial^2 \phi}{\partial p^2}\right) \Big\} \geq 0 \end{aligned} \tag{5.11}$$

at $(\bar{t},\bar{x},\bar{p})$. If

$$\lambda_{\min}\left(p, \frac{\partial^2 \phi}{\partial p^2}\right) \leq 0$$

at $(\bar{t},\bar{x},\bar{p})$ (5.11) obviously holds. So assume

$$\lambda_{\min}\left(p, \frac{\partial^2 \phi}{\partial p^2}\right) > 0. \tag{5.12}$$

Note that by Proposition 4.3. we have that $V^+(\bar{t},\bar{x},\bar{p}) - \langle f(\bar{t},\bar{x}), p\rangle = \phi(\bar{t},\bar{x},\bar{p}) - \langle f(\bar{t},\bar{x}), \bar{p}\rangle \geq 0$. So to show (5.11) it remains to show, that for $\phi(\bar{t},\bar{x},\bar{p}) < \langle h(t,x), p\rangle$, we have that

$$(-\frac{\partial}{\partial t} - \mathcal{L})[\phi](\bar{t},\bar{x},\bar{p}) \geq 0. \tag{5.13}$$

Recall that (5.12) implies by Lemma 4.5. that $(V^-)^*(\bar{t},\bar{x},\hat{p})$ is differentiable at $\hat{p} := \frac{\partial \phi}{\partial p}(\bar{t},\bar{x},\bar{p})$ with a derivative equal to $\frac{\partial (V^-)^*(\bar{t},\bar{x},\hat{p})}{\partial \hat{p}} = \bar{p}$.
From Proposition 4.8. we have

$$(V^-)^*(\bar{t},\bar{x},\hat{p}) \quad \geq \quad \max_{i \in \{1,\ldots,I\}}\{\hat{p}_i - h_i(\bar{t},\bar{x})\}. \tag{5.14}$$

Indeed we have strict inequality in (5.14) for $\bar{p} \notin \{e_i,\ i = 1, \ldots, I\}$. Assume that

$$(V^-)^*(\bar{t},\bar{x},\hat{p}) \quad = \quad \max_{i \in \{1,\ldots,I\}}\{\hat{p}_i - h_i(\bar{t},\bar{x})\}. \tag{5.15}$$

Since $\max_{i \in \{1,\ldots,I\}}\{\hat{p}_i - h_i(\bar{t},\bar{x})\}$ is convex in $\hat{p}$, we would have that $\max_{i \in \{1,\ldots,I\}}\{\hat{p}_i - h_i(\bar{t},\bar{x})\}$ is also differentiable at $\hat{p}$ with a derivative equal to $\frac{\partial (V^-)^*(\bar{t},\bar{x},\hat{p})}{\partial \hat{p}} = \bar{p}$.
However the map $\hat{p}' \to \max_{i \in \{1,\ldots,I\}}\{\hat{p}'_i - h_i(\bar{t},\bar{x})\}$ is only differentiable at points for which there is a unique $i_0 \in \{1, \ldots, I\}$ such that $\max_{i \in \{1,\ldots,I\}}\{\hat{p}'_i - h_i(\bar{t},\bar{x})\} = \hat{p}'_{i_0} - h_{i_0}(\bar{t},\bar{x})$ and in this case its derivative is given by e_{i_0}. This is impossible since $\bar{p} \neq e_{i_0}$. Therefore

$$(V^-)^*(\bar{t},\bar{x},\hat{p}) \quad > \quad \max_{i \in \{1,\ldots,I\}}\{\hat{p}_i - h_i(\bar{t},\bar{x})\} \tag{5.16}$$

holds, which implies with (4.16)

$$\begin{aligned} V^-(\bar{t},\bar{x},\bar{p}) \quad & < \quad \langle \hat{p}, \bar{p}\rangle - \max_{i \in \{1,\ldots,I\}}\{\hat{p}_i - h_i(\bar{t},\bar{x})\} \\ & = \quad \langle \hat{p}, \bar{p}\rangle + \min_{i \in \{1,\ldots,I\}}\{-\hat{p}_i + h_i(\bar{t},\bar{x})\} \\ & \leq \quad \langle h(\bar{t},\bar{x}), \bar{p}\rangle. \end{aligned} \tag{5.17}$$

If we now recall the dynamic programming for $(V^-)^*$ with setting $\sigma = t$, i.e.

$$(V^-)^*(\bar{t},\bar{x},\hat{p}) \leq \sup_{\tau\in\mathcal{T}(\bar{t},t)} \mathbb{E}\Big[\max_{i\in\{1,\ldots,I\}}\{\hat{p}_i - h_i(X_\tau^{\bar{t},\bar{x}})\}1_{\tau<t} + (V^-)^*(t, X_t^{\bar{t},\bar{x}}, \hat{p})1_{\tau=t}\Big], \tag{5.18}$$

we have with the upper bound of $(V^-)^*$ (5.16) that $(V^-)^*$ has the viscosity subsolution property to

$$(-\frac{\partial}{\partial t} - \mathcal{L})[w] = 0 \tag{5.19}$$

at $(\bar{t},\bar{x},\hat{p})$. And as in [25] V^- has the viscosity supersolution property to (5.19) at $(\bar{t},\bar{x},\bar{p})$, hence (5.13) holds. □

5.3 Viscosity solution property of the value function

To establish Theorem 3.2. it remains with Remark 3.3. to show that $V^- \geq V^+$. This is however a direct consequence of Theorem 5.1. and Theorem 5.2. together with the comparison Theorem 3.7.. We then have the following characterization of the value.

Corollary 5.4. *The value function $V : [0,T]\times\mathbb{R}^d\times\Delta(I) \to \mathbb{R}$ is the unique viscosity solution to (3.5) in the class of bounded, uniformly continuous functions, which are uniformly Lipschitz continuous in p.*

6 Alternative representation

In a second part we use the PDE characterization to establish a representation of the value function via a minimization procedure over certain martingale measures. To do so we enlarge the canonical Wiener space to a space which carries besides a Brownain motion B a new dynamic $\mathbf{p}$. We use this additional dynamic to model the incorporation of the private information into the game. More precisely we model the probability in which scenario the game is played in according to the information of the uninformed Player 2.

6.1 Enlargement of the canonical space

To that end let us denote by $\mathcal{D}([0,T];\Delta(I))$ the set of càdlàg functions from $\mathbb{R}$ to $\Delta(I)$, which are constant on $(-\infty,0)$ and on $[T,+\infty)$. We denote by $\mathbf{p}_s(\omega_p) = \omega_p(s)$ the coordinate mapping on $\mathcal{D}([0,T];\Delta(I))$ and by $\mathcal{G} = (\mathcal{G}_s)$ the filtration generated by $s \mapsto \mathbf{p}_s$. Furthermore we recall that $\mathcal{C}([0,T];\mathbb{R}^d)$ denotes the set of continuous functions from $\mathbb{R}$ to $\mathbb{R}^d$, which are constant on $(-\infty,0]$ and on $[T,+\infty)$. We denote by $B_s(\omega_B) = \omega_B(s)$ the coordinate mapping on $\mathcal{C}([0,T];\mathbb{R}^d)$ and by $\mathcal{H} = (\mathcal{H}_s)$ the filtration generated by $s \mapsto B_s$. We equip the product space $\Omega := \mathcal{D}([0,T];\Delta(I)) \times \mathcal{C}([0,T];\mathbb{R}^d)$ with the right-continuous filtration $\mathcal{F}$, where $\mathcal{F}_t = \cap_{s>t}\mathcal{F}_t^0$ with $(\mathcal{F}_s^0) = (\mathcal{G}_s)\otimes(\mathcal{H}_s)$. In the following we shall, whenever we work under a fixed probability $\mathbb{P}$ on Ω, complete the filtration $\mathcal{F}$ with $\mathbb{P}$-nullsets without changing the notation.

For $0 \leq t \leq T$ we denote $\Omega_t = \mathcal{D}([t,T];\Delta(I)) \times \mathcal{C}([t,T];\mathbb{R}^d)$ and $\mathcal{F}_{t,s}$ the (right-continuous) σ-algebra generated by paths up to time $s \geq t$ in Ω_t. Furthermore we define the space

$$\Omega_{t,s} = \mathcal{D}([t,s];\Delta(I)) \times \mathcal{C}([t,s];\mathbb{R}^d)$$

for $0 \leq t \leq s \leq T$. If $r \in (t,T]$ and $\omega \in \Omega_t$ then let

$$\omega_1 = 1_{[-\infty,r)}\omega \qquad \omega_2 = 1_{[r,+\infty]}(\omega - \omega_{r-})$$

and denote $\pi\omega = (\omega_1, \omega_2)$. The map $\pi : \Omega_t \to \Omega_{t,r} \times \Omega_r$ induces the identification $\Omega_t = \Omega_{t,r} \times \Omega_r$ moreover $\omega = \pi^{-1}(\omega_1, \omega_2)$, where the inverse is defined in an evident way.

For any measure $\mathbb{P}$ on Ω, we denote by $\mathbb{E}_{\mathbb{P}}[\cdot]$ the expectation with respect to $\mathbb{P}$. We equip Ω with a certain class of measures.

Definition 6.1. *Given $p \in \Delta(I)$, $t \in [0,T]$, we denote by $\mathcal{P}(t,p)$ the set of probability measures $\mathbb{P}$ on Ω such that, under $\mathbb{P}$*

(i) $\mathbf{p}$ is a martingale, such that $\mathbf{p}_s = p$ $\forall s < t$, $\mathbf{p}_s \in \{e_i, i = 1,\dots,I\}$ $\forall s \geq T$ $\mathbb{P}$-a.s., where e_i denotes the i-th coordinate vector in $\mathbb{R}^I$, and $\mathbf{p}_T$ is independent of $(B_s)_{s\in(-\infty,T]}$,

(ii) $(B_s)_{s\in[0,T]}$ is a Brownian motion.

Remark 6.2. *Assumption (ii) is naturally given by the Brownian structure of the game. Assumption (i) is motivated as follows. Before the game starts the information of the uninformed player is just the initial distribution p. The martingale property, implying $\mathbf{p}_t = \mathbb{E}_{\mathbb{P}}[\mathbf{p}_T|\mathcal{F}_t]$, is due to the best guess of the uninformed player about the scenario he is in. Finally, at the end of the game the information is revealed hence $\mathbf{p}_T \in \{e_i, i = 1,\dots,I\}$ and since the scenario is picked before the game starts the outcome $\mathbf{p}_T$ is independent of the Brownian motion.*

6.2 Auxiliary games and representation

From now on we will consider stopping times on the enlarged space $\Omega = \mathcal{D}([0,T];\Delta(I)) \times \mathcal{C}([0,T];\mathbb{R}^d)$.

Definition 6.3. *At time $t \in [0,T]$ an admissible stopping time for either player is a $(\mathcal{F}_s)_{s\in[t,T]}$ stopping time with values in $[t,T]$. We denote the set of admissible stopping times by $\bar{\mathcal{T}}(t,T)$. In the following we shall omit T in the notation whenever it is obvious.*

We note that in contrast to Definition 2.2. the admissible stopping times at time t might now also depend on the paths of the Brownian motion before time t.

One can now consider a stopping game with this additional dynamic, namely with a payoff given by

$$\begin{aligned} J(t,x,\tau,\sigma,\mathbb{P})_{t-} \quad := \quad \mathbb{E}_{\mathbb{P}}\Big[& \langle \mathbf{p}_\sigma, f(\sigma, X^{t,x}_\sigma)\rangle 1_{\sigma<\tau,\sigma<T} + \langle \mathbf{p}_\tau, h(\tau, X^{t,x}_\tau)\rangle 1_{\tau\leq\sigma,\tau<T} \\ & + \langle \mathbf{p}_T, g(X^{t,x}_T)\rangle 1_{\sigma=\tau=T} | \mathcal{F}_{t-}\Big], \end{aligned} \tag{6.1}$$

where $\tau \in \bar{\mathcal{T}}(t)$ denotes the stopping time choosen by Player 1, who minimizes, and $\sigma \in \bar{\mathcal{T}}(t)$ denotes the stopping time choosen by Player 2, who maximizes the expected outcome. In contrast to the previous consideration here we are only working with non randomized stopping times. Indeed the randomization is in some sense shifted to the additional dynamic $\mathbf{p}$.

Note that the known results in literature do not imply that these games have a value for any fixed $\mathbb{P} \in \mathcal{P}(t,p)$, i.e.

$$\begin{aligned} &\operatorname{esssup}_{\sigma\in\bar{\mathcal{T}}(t)}\operatorname{essinf}_{\tau\in\bar{\mathcal{T}}(t)} J(t,x,\tau,\sigma,\mathbb{P})_{t-} \\ &\qquad = \operatorname{essinf}_{\tau\in\bar{\mathcal{T}}(t)}\operatorname{esssup}_{\sigma\in\bar{\mathcal{T}}(t)} J(t,x,\tau,\sigma,\mathbb{P})_{t-}. \end{aligned} \tag{6.2}$$

Indeed since $\mathbf{p}$ is only assumed to be càdlàg the theorems of [60] or [57] requiring basically the continuity of $\mathbf{p}$ do not apply. We would like however mention that the very recent result of [68], where there is only a continuity in expectation supposed, seems to be applicable.

For us however it is for now not important since our first goal is an alternative representation of the value function, for which we have a PDE representation. Since $\mathbf{p}$ can be interpreted as a manipulation of the uninformed player by the informed one the outcome of the game should be some minimum in this manipulation.

Fix $t \in [0,T]$, $x \in \mathbb{R}^d$, $p \in \Delta(I)$. Note that all $\mathbb{P} \in \mathcal{P}(t,p)$ are equal on $\mathcal{F}_{t-}$, i.e. the distribution of $(B_s, \mathbf{p}_s)$ on $[0,t)$ is given by $\delta(p) \otimes \mathbb{P}_0$, where $\delta(p)$ is the measure under which $\mathbf{p}$ is constant and equal to p and $\mathbb{P}_0$ is the Wiener measure on $\Omega_{0,t}$. So we can identify each $\mathbb{P} \in \mathcal{P}(t,p)$ on $\mathcal{F}_{t-}$ with a common probability measure $\mathbb{Q}$ and define $\mathbb{Q}$-a.s. the lower value function

$$W^-(t,x,p) = \operatorname{essinf}_{\mathbb{P}\in\mathcal{P}(t,p)}\operatorname{esssup}_{\sigma\in\bar{\mathcal{T}}(t)}\operatorname{essinf}_{\tau\in\bar{\mathcal{T}}(t)} J(t,x,\tau,\sigma,\mathbb{P})_{t-} \tag{6.3}$$

and the upper value function

$$W^+(t,x,p) = \operatorname{essinf}_{\mathbb{P}\in\mathcal{P}(t,p)}\operatorname{essinf}_{\tau\in\bar{\mathcal{T}}(t)}\operatorname{esssup}_{\sigma\in\bar{\mathcal{T}}(t)} J(t,x,\tau,\sigma,\mathbb{P})_{t-}, \tag{6.4}$$

where by definition we have $W^-(t,x,p) \le W^+(t,x,p)$.

Theorem 6.4. *For any $(t,x,p) \in [0,T] \times \mathbb{R}^d \times \Delta(I)$ we have that*

$$W(t,x,p) := W^+(t,x,p) = W^-(t,x,p). \tag{6.5}$$

Furthermore the value of the Dynkin game with incomplete information can be written as

$$V(t,x,p) = W(t,x,p). \tag{6.6}$$

To prove the theorem we establish a subdynamic programming for W^+ and a superdynamic programming principle for W^-. Then we show that W^+ is a subsolution and W^- a supersolution to the PDE (3.5). After establishing that W^+ and W^- are bounded, uniformly continuous functions, which are uniformly Lipschitz continuous in p, the comparison result Theorem 3.7. gives us the equalities (6.5) and (6.6).

6.3 Optimal strategies for the informed player

The motivation for the alternative representation is that, as in [27], [56] it allows to determine optimal strategies for the informed player. Indeed, if we assume that there exists a $\bar{\mathbb{P}} \in \mathcal{P}(t,p)$, such that

$$V(t,x,p) \quad = \quad \operatorname{essinf}_{\tau\in\bar{\mathcal{T}}(t)}\operatorname{esssup}_{\sigma\in\bar{\mathcal{T}}(t)} J(t,x,\tau,\sigma,\bar{\mathbb{P}})_{t-}, \tag{6.7}$$

then we can define for any scenario $i \in \{1, \ldots, I\}$ a probability measure $\bar{\mathbb{P}}_i$ by: for all $A \in \mathcal{F}$ we have that

$$\bar{\mathbb{P}}_i[A] = \bar{\mathbb{P}}[A|\mathbf{p}_T = e_i] = \frac{1}{p_i}\bar{\mathbb{P}}[A \cap \{\mathbf{p}_T = e_i\}], \quad \text{if } p_i > 0,$$

and $\bar{\mathbb{P}}_i[A] = \bar{\mathbb{P}}[A]$ else. It is clear by Definition 6.1. that B is still a Brownian motion under $\mathbb{P}^i$.
We note that the right-continuity of $\mathbf{p}$ allows to define the stopping time $\tau^* = \inf\{s \in [0,T], (s, X_s^{t,x}, \mathbf{p}_s) \in D\}$, where $D = \{(t,x,p) \in [0,T] \times \mathbb{R}^d \times \Delta(I) : V(t,x,p) \geq \langle h(t,x), p\rangle\}$ is a closed set by the continuity of V and g.
The couple $(\tau^*, \bar{\mathbb{P}}_i)$ then defines a randomized stopping time for the first player. Indeed, for each state of nature $i \in \{1, \ldots, I\}$ the informed player stops when $(s, X_s^{t,x}, \mathbf{p}_s)$ enters D under $\bar{\mathbb{P}}_i$, where $X^{t,x}$ is the diffusion both players observe and $\mathbf{p}$ under $\bar{\mathbb{P}}_i$ represents his own randomization device.

Theorem 6.5. *For any scenario $i = 1, \ldots, I$ and any stopping time of the uninformed player $\sigma \in \bar{\mathcal{T}}(t)$ playing $(\tau^*, \bar{\mathbb{P}}_i)$ is optimal for the informed player in the sense that*

$$\begin{aligned} \textstyle\sum_{i=1}^I p_i \mathbb{E}_{\bar{\mathbb{P}}_i}\Big[& f_i(\sigma, X_\sigma^{t,x}) 1_{\sigma<\tau^*, \sigma<T} \\ & + h_i(\tau^*, X_{\tau^*}^{t,x}) 1_{\tau^* \leq \sigma, \tau^*<T} + g_i(X_T^{t,x}) 1_{\tau^*=\sigma=T}\Big] \leq V(t,x,p). \end{aligned} \tag{6.8}$$

Proof. By definition of $\bar{\mathbb{P}}_i$ we have

$$\begin{aligned} & \sum_{i=1}^I p_i \mathbb{E}_{\bar{\mathbb{P}}_i}\Big[f_i(\sigma, X_\sigma^{t,x}) 1_{\sigma<\tau^*, \sigma<T} + h_i(\tau^*, X_{\tau^*}^{t,x}) 1_{\tau^* \leq \sigma, \tau^*<T} + g_i(X_T^{t,x}) 1_{\tau^*=\sigma=T}\Big] \\ & = \sum_{i=1}^I \bar{\mathbb{P}}[\mathbf{p}_T = e_i] \mathbb{E}_{\bar{\mathbb{P}}}\Big[f_i(\sigma, X_\sigma^{t,x}) 1_{\sigma<\tau^*, \sigma<T} \\ & \qquad + h_i(\tau^*, X_{\tau^*}^{t,x}) 1_{\tau^* \leq \sigma, \tau^*<T} + g_i(X_T^{t,x}) 1_{\tau^*=\sigma=T} | \mathbf{p}_T = e_i\Big] \\ & = \sum_{i=1}^I \mathbb{E}_{\bar{\mathbb{P}}}\Big[1_{\{\mathbf{p}_T = e_i\}} \Big(f_i(\sigma, X_\sigma^{t,x}) 1_{\sigma<\tau^*, \sigma<T} \\ & \qquad + h_i(\tau^*, X_{\tau^*}^{t,x}) 1_{\tau^* \leq \sigma, \tau^*<T} + g_i(X_T^{t,x}) 1_{\tau^*=\sigma=T}\Big)\Big] \\ & = \mathbb{E}_{\bar{\mathbb{P}}}\Big[\langle \mathbf{p}_T, f(\sigma, X_\sigma^{t,x})\rangle 1_{\sigma<\tau^*, \sigma<T} \\ & \qquad + \langle \mathbf{p}_T, h(\tau^*, X_{\tau^*}^{t,x})\rangle 1_{\tau^* \leq \sigma, \tau^*<T} + \langle \mathbf{p}_T, g(X_T^{t,x})\rangle 1_{\tau^*=\sigma=T}\Big], \end{aligned}$$

while, since $\mathbf{p}$ is a martingale, we have by conditioning

$$\begin{aligned} \mathbb{E}_{\bar{\mathbb{P}}}\Big[& \langle \mathbf{p}_T, f(\sigma, X_\sigma^{t,x})\rangle 1_{\sigma<\tau^*, \sigma<T} \\ & + \langle \mathbf{p}_T, h(\tau^*, X_{\tau^*}^{t,x})\rangle 1_{\tau^* \leq \sigma, \tau^*<T} + \langle \mathbf{p}_T, g(X_T^{t,x})\rangle 1_{\tau^*=\sigma=T}\Big] \end{aligned}$$

$$= \mathbb{E}_{\bar{\mathbb{P}}}\Big[\langle \mathbf{p}_\sigma, f(\sigma, X_\sigma^{t,x})\rangle 1_{\sigma<\tau^*,\sigma<T}$$
$$+\langle \mathbf{p}_{\tau^*}, h(\tau^*, X_{\tau^*}^{t,x})\rangle 1_{\tau^*\leq\sigma,\tau^*<T} + \langle \mathbf{p}_T, g(X_T^{t,x})\rangle 1_{\tau^*=\sigma=T}\Big].$$

(6.8) follows then with (6.7) by standard results. □

6.4 The functions W^+, W^- and ϵ-optimal martingale measures

We conclude this section with some important technical remarks. Note that by its very definition $W^+(t,x,p)$ and $W^-(t,x,p)$ are merely $\mathcal{F}_{t-}$ measurable random fields. However we can show that they are deterministic and hence a good candidate to represent the deterministic value function $V(t,x,p)$. The proof is mainly based on the methods in [19] using perturbation of $\mathcal{C}([0,T];\mathbb{R}^d)$ with certain elements of the Cameron-Martin space. We already adapted these arguments to the framework of games with incomplete information in [56]. The proof is very similar here and thus omitted.

Proposition 6.6. *For any $t \in [0,T]$, $x \in \mathbb{R}^d$, $p \in \Delta(I)$ we have that*

$$\begin{aligned} W^+(t,x,p) &= \mathbb{E}_{\mathbb{Q}}[W^+(t,x,p)] \qquad \mathbb{Q}\text{-a.s.} \\ W^-(t,x,p) &= \mathbb{E}_{\mathbb{Q}}[W^-(t,x,p)] \qquad \mathbb{Q}\text{-a.s.} \end{aligned}$$

Hence identifying W^+, W^- respectively with its deterministic version we can consider $W^+ : [0,T]\times\mathbb{R}^d\times\Delta(I)\to\mathbb{R}$ and $W^- : [0,T)\times\mathbb{R}^d\times\Delta(I)\to\mathbb{R}$ as deterministic functions.

In the following section we establish some regularity results and a dynamic programming principle. To this end we work with ϵ-optimal measures. Note that since we are taking the essential infimum over a family of random variables, existence of an ϵ-optimal $\mathbb{P}^\epsilon \in \mathcal{P}(t,p)$ is as in [56] not standard. Therefore we provide a technical lemma, the proof of which can be provided along the lines of [19], [56] respectively.

Lemma 6.7. *For any $(t,x,p)\in[0,T]\times\mathbb{R}^d\times\Delta(I)$ there is an ϵ-optimal $\mathbb{P}^\epsilon\in\mathcal{P}(t,p)$ in the sense that* $\mathbb{Q}$-a.s.

$$W^-(t,x,p)+\epsilon \quad\geq\quad \mathrm{esssup}_{\sigma\in\bar{\mathcal{T}}(t)}\mathrm{essinf}_{\tau\in\bar{\mathcal{T}}(t)}J(t,x,\tau,\sigma,\mathbb{P}^\epsilon)_{t-}.$$

Furthermore for any $(t,x,p)\in[0,T]\times\mathbb{R}^d\times\Delta(I)$ there is an ϵ-optimal $\mathbb{P}^\epsilon\in\mathcal{P}(t,p)$ in the sense that $\mathbb{Q}$-a.s.

$$W^+(t,x,p)+\epsilon \quad\geq\quad \mathrm{essinf}_{\tau\in\bar{\mathcal{T}}(t)}\mathrm{esssup}_{\sigma\in\bar{\mathcal{T}}(t)}J(t,x,\tau,\sigma,\mathbb{P}^\epsilon)_{t-}.$$

For technical reasons we furthermore introduce the set $\mathcal{P}^f(t,p)$ as the set of all measures $\mathbb{P}\in\mathcal{P}(t,p)$, such that there exists a finite set $S\subset\Delta(I)$ with $\mathbf{p}_s\in S$ $\mathbb{P}$-a.s. for all $s\in[t,T]$.

Remark 6.8. *Note that for any $(t,x,p)\in[0,T]\times\mathbb{R}^d\times\Delta(I)$ $\epsilon>0$ we can choose an ϵ-optimal $\mathbb{P}^\epsilon$ in the smaller class $\mathcal{P}^f(t,p)$. The idea of the proof is as follows: first choose $\frac{\epsilon}{2}$-optimal measure $\mathbb{P}^\epsilon\in\mathcal{P}(t,p)$ for $W^-(t,x,p)$. Since $\mathbf{p}$ progressively measurable we can approximate it by an elementary processes $\bar{\mathbf{p}}^\epsilon$, such that one has*

$$|\mathrm{esssup}_{\sigma\in\bar{\mathcal{T}}(t)}\mathrm{essinf}_{\tau\in\bar{\mathcal{T}}(t)}J(t,x,\tau,\sigma,\mathbb{P}^\epsilon)_{t-}-\mathrm{esssup}_{\sigma\in\bar{\mathcal{T}}(t)}\mathrm{essinf}_{\tau\in\bar{\mathcal{T}}(t)}J(t,x,\tau,\sigma,\bar{\mathbb{P}}^\epsilon)_{t-}|\leq\frac{\epsilon}{2},$$

where $\bar{\mathbb{P}}^\epsilon$ distribution of $(B,\bar{\mathbf{p}}^\epsilon)$. The same argument works for W^+.

7 Dynamic programming for W^+, W^-

7.1 Regularity properties

Proposition 7.1. *For all $(t,x) \in [0,T] \times \mathbb{R}^d$ $W^+(t,x,p)$ and $W^-(t,x,p)$ are convex in p.*

Proof. Let $(t,x) \in [0,T] \times \mathbb{R}^d$ and $p_1, p_2 \in \Delta(I)$. Let $\mathbb{P}^1 \in \mathcal{P}(t,p_1)$, $\mathbb{P}^2 \in \mathcal{P}(t,p_2)$ be ϵ-optimal for $W^+(t,x,p_1)$, $W^+(t,x,p_2)$ respectively. For $\lambda \in [0,1]$ define a martingale measure $\mathbb{P}^\lambda \in \mathcal{P}(t,p_\lambda)$, such that for all measurable $\phi : \mathcal{D}([0,T];\Delta(I)) \times \mathcal{C}([0,T];\mathbb{R}^d) \to \mathbb{R}_+$

$$\mathbb{E}_{\mathbb{P}^\lambda}[\phi(\mathbf{p},B)] = \lambda \mathbb{E}_{\mathbb{P}^1}[\phi(\mathbf{p},B)] + (1-\lambda)\mathbb{E}_{\mathbb{P}^2}[\phi(\mathbf{p},B)].$$

Observe that this can be understood as identifying Ω with $\Omega \times \{1,2\}$ with weights λ and $(1-\lambda)$ for $\Omega \times \{1\}$ and $\Omega \times \{2\}$, respectively. So

$$\begin{aligned} W^+(t,x,p_\lambda) &\leq \operatorname{essinf}_{\tau \in \bar{\mathcal{T}}(t)} \operatorname{esssup}_{\sigma \in \bar{\mathcal{T}}(t)} J(t,x,\tau,\sigma,\mathbb{P}^\lambda)_{t-} \\ &= 1_{\Omega \times \{1\}} \operatorname{essinf}_{\tau \in \bar{\mathcal{T}}(t)} \operatorname{esssup}_{\sigma \in \bar{\mathcal{T}}(t)} J(t,x,\tau,\sigma,\mathbb{P}^1)_{t-} \\ &\qquad + 1_{\Omega \times \{2\}} \operatorname{essinf}_{\tau \in \bar{\mathcal{T}}(t)} \operatorname{esssup}_{\sigma \in \bar{\mathcal{T}}(t)} J(t,x,\tau,\sigma,\mathbb{P}^2)_{t-} \\ &\leq 1_{\Omega \times \{1\}} W^+(t,x,p_1) + 1_{\Omega \times \{2\}} W^+(t,x,p_2) + 2\epsilon \end{aligned}$$

and the convexity follows by taking expectation, since ϵ can be chosen arbitrarily small. The proof for W^- follows by similar arguments. □

Proposition 7.2. *$W^+(t,x,p)$ and $W^-(t,x,p)$ are uniformly Lipschitz continuous in x and p and Hölder continuous in t.*

Proof. The proof of Lipschitz continuity in x is straightforward, while the Hölder continuity in t can be shown as in Proposition 4.1. and Proposition 4.6. in [56].
It remains to prove the uniform Lipschitz continuity in p. Since we have convexity in p, it is sufficient to establish the Lipschitz continuity with respect to p on the extreme points e_i. Observe that $\mathcal{P}(t,e_i)$ consists in the single probability measure $\delta(e_i) \otimes \mathbb{P}_0$, where $\delta(e_i)$ is the measure under which $\mathbf{p}$ is constant and equal to e_i and $\mathbb{P}_0$ is a Wiener measure.
Assume $W^+(t,x,e_i) - W^+(t,x,p) > 0$. For $\epsilon > 0$ let $\mathbb{P}^\epsilon \in \mathcal{P}(t,p)$ be ϵ-optimal for $W^+(t,x,p)$. Then

$$\begin{aligned} &W^+(t,x,e_i) - W^+(t,x,p) - 3\epsilon \\ &\leq \operatorname{essinf}_{\tau \in \bar{\mathcal{T}}(t)} \operatorname{esssup}_{\sigma \in \bar{\mathcal{T}}(t)} J(t,x,\tau,\sigma,\delta(e_i) \otimes \mathbb{P}_0)_{t-} \\ &\qquad - \operatorname{essinf}_{\tau \in \bar{\mathcal{T}}(t)} \operatorname{esssup}_{\sigma \in \bar{\mathcal{T}}(t)} J(t,x,\tau,\sigma,\mathbb{P}^\epsilon)_{t-} - 2\epsilon. \end{aligned} \tag{7.1}$$

Choose now $\bar{\tau} \in \bar{\mathcal{T}}(t)$ to be ϵ-optimal for $\operatorname{essinf}_{\tau \in \bar{\mathcal{T}}(t)} \operatorname{esssup}_{\sigma \in \bar{\mathcal{T}}(t)} J(t,x,\tau,\sigma,\mathbb{P}^\epsilon)_{t-}$ and $\bar{\sigma} \in \bar{\mathcal{T}}(t)$ to be ϵ-optimal for $\operatorname{esssup}_{\sigma \in \bar{\mathcal{T}}(t)} J(t,x,\bar{\tau},\sigma,,\delta(e_i) \otimes \mathbb{P}_0))_{t-}$. Then we have with (7.1)

$$\begin{aligned} &W^+(t,x,e_i) - W^+(t,x,p) - 3\epsilon \\ &\leq \mathbb{E}_{\mathbb{P}^\epsilon}\Big[\langle e_i - \mathbf{p}_{\bar{\sigma}}, f(\bar{\sigma}, X^{t,x}_{\bar{\sigma}})\rangle 1_{\bar{\sigma}<\bar{\tau}\leq T} + \langle e_i - \mathbf{p}_{\bar{\tau}}, h(\bar{\tau}, X^{t,x}_{\bar{\tau}})\rangle 1_{\bar{\tau}\leq\bar{\sigma},\bar{\tau}<T} \\ &\qquad + \langle e_i - \mathbf{p}_T, g(X^{t,x}_T)\rangle 1_{\bar{\sigma}=\bar{\tau}=T} | \mathcal{F}_{t-}\Big]. \end{aligned} \tag{7.2}$$

Since for all $p \in \Delta(I)$ $0 \le |p - e_i| \le c(1-p_i)$ we have by the boundedness of the coefficients with (7.2) and the fact that $\mathbf{p}$ is a $\mathbb{P}^\epsilon$-martingale with mean p

$$\begin{aligned} & W^+(t,x,e_i) - W^+(t,x,p) - 3\epsilon \\ \le\ & c\left(1 - \mathbb{E}_{\mathbb{P}^\epsilon}\left[(\mathbf{p}_{\bar\sigma})_i 1_{\bar\sigma < \bar\tau \le T} + (\mathbf{p}_{\bar\tau})_i 1_{\bar\tau \le \bar\sigma, \bar\tau < T} + (\mathbf{p}_T)_i 1_{\bar\sigma = \bar\tau = T} | \mathcal{F}_{t-}\right]\right) \\ \le\ & c(1-p_i). \end{aligned}$$

Using now

$$1 - p_i \le c \sum_j |(p)_j - \delta_{ij}| \le c\sqrt{I}|p - e_i|,$$

the claim follows since ϵ can be chosen arbirarily small. The case $W^+(t,x,p) - W^+(t,x,e_i) > 0$ is immediate.
The Lipschitz continuity of W^- in p can be established by similar arguments. □

7.2 Subdynamic programming for W^+

Theorem 7.3. *Let $(\bar t, \bar x, \bar p) \in [0,T] \times \mathbb{R}^d \times \Delta(I)$. Then for all $t \in [\bar t, T]$*

$$\begin{aligned} & W^+(\bar t, \bar x, \bar p) \\ & \le \operatorname{essinf}_{\mathbb{P} \in \mathcal{P}(\bar t, \bar p)} \operatorname{essinf}_{\tau \in \bar{\mathcal{T}}(\bar t, t)} \operatorname{esssup}_{\sigma \in \bar{\mathcal{T}}(\bar t, t)} \mathbb{E}_{\mathbb{P}}\Big[\langle \mathbf{p}_\sigma, f(\sigma, X_\sigma^{\bar t, \bar x}) 1_{\sigma < \tau, \sigma < t}\rangle \\ & \qquad + \langle \mathbf{p}_\tau, h(\tau, X_\tau^{\bar t, \bar x}) 1_{\tau \le \sigma, \tau < t}\rangle + W^+(t, X_t^{\bar t, \bar x}, \mathbf{p}_{t-}) 1_{\tau = \sigma = t} | \mathcal{F}_{\bar t -}\Big]. \end{aligned} \tag{7.3}$$

Proof. Let $\mathbb{P} \in \mathcal{P}^f(t,p)$, $t \in [\bar t, T]$. By assumption there exist $S = \{p^1, \ldots, p^k\}$, such that $\mathbb{P}[\mathbf{p}_{t-} \in S] = 1$. Furthermore let $(A_l)_{l \in \mathbb{N}}$ be a partition of $\mathbb{R}^d$ by Borel sets, such that $\operatorname{diam}(A_l) \le \bar\epsilon$ and choose for any $l \in \mathbb{N}$ some $y^l \in A_l$.

Define for any l, m measures $\mathbb{P}^{l,m} \in \mathcal{P}^f(t, p^m)$, such that they are ϵ-optimal for $W^+(t, p^m, y^l)$ and ϵ-optimal stopping times $\tau^{l,m}$. We define the probablility measure $\mathbb{P}^\epsilon$, such that on $\Omega = \Omega_{0,t} \times \Omega_t$

$$\mathbb{P}^\epsilon = (\mathbb{P}|_{\Omega_{0,t}}) \otimes \hat{\mathbb{P}}, \tag{7.4}$$

where for all $A \in \mathcal{B}(\Omega_t)$:

$$\hat{\mathbb{P}}[A] = \sum_{m=1}^k \sum_{l=1}^\infty \mathbb{P}[X_t^{\bar t, \bar x} \in A^l, \mathbf{p}_{t-} = p_m] \mathbb{P}^{l,m}[A],$$

and the stopping time

$$\hat\tau = \begin{cases} \tau & \text{on } \{\tau < t\} \\ \tau^{l,m} & \text{on } \{\bar\tau \ge t, X_t^{\bar t, \bar x} \in A^l, \mathbf{p}_{t-} = p^m\}. \end{cases} \tag{7.5}$$

Note that by definition $(B_s)_{s \in [\bar t, T]}$ is a Brownian motion under $\mathbb{P}^\epsilon$. Also $(\mathbf{p}_s)_{s \in [\bar t, T]}$ is a martingale, since for $t \le r \le s \le T$

$$\mathbb{E}_{\mathbb{P}^\epsilon}[\mathbf{p}_s | \mathcal{F}_r] = \sum_{m=1}^k \sum_{l=1}^\infty 1_{\{X_t^{t,x} \in A^l, \mathbf{p}_{t-} = p^m\}} \mathbb{E}_{\mathbb{P}^{l,m}}[\mathbf{p}_s | \mathcal{F}_r] = \sum_{m=1}^k \sum_{l=1}^\infty 1_{\{X_t^{\bar t, \bar x} \in A^l, \mathbf{p}_{t-} = p^m\}} \mathbf{p}_r = \mathbf{p}_r.$$

Furthermore the remaining conditions of Definition 6.1. are obviously met, hence $\mathbb{P}^\epsilon \in \mathcal{P}^f(t,p)$. By the definition of W^+ we have

$$\begin{aligned} &W^+(\bar t,\bar x,\bar p)\\ &\leq \text{esssup}_{\sigma\in\bar{\mathcal{T}}(\bar t)}\mathbb{E}_{\mathbb{P}^\epsilon}\Big[\langle \mathbf{p}_\sigma, f(\sigma,X_\sigma^{\bar t,\bar x})\rangle 1_{\sigma<\hat\tau,\sigma<T} + \langle \mathbf{p}_{\hat\tau}, h(\hat\tau,X_{\hat\tau}^{\bar t,\bar x})\rangle 1_{\hat\tau\leq\sigma,\hat\tau<T} \\ &\qquad +\langle \mathbf{p}_T, g(X_T^{t,x})\rangle 1_{\sigma=\hat\tau=T}|\mathcal{F}_{t-}\Big]. \end{aligned} \tag{7.6}$$

Note that using the Lipschitz continuity of W^+ we have for any $\sigma\in\bar{\mathcal{T}}(\bar t)$

$$\begin{aligned} &\mathbb{E}_{\mathbb{P}^\epsilon}\Big[\langle \mathbf{p}_\sigma, f(\sigma,X_\sigma^{\bar t,\bar x})\rangle 1_{t\leq\sigma<\hat\tau,\sigma<T} + \langle \mathbf{p}_{\hat\tau}, h(\hat\tau,X_{\hat\tau}^{\bar t,\bar x})\rangle 1_{t\leq\hat\tau\leq\sigma,\hat\tau<T} + \langle \mathbf{p}_T, g(X_T^{\bar t,\bar x})\rangle 1_{\sigma=\tau=T}|\mathcal{F}_{t-}\Big]\\ &\leq \mathbb{E}_{\mathbb{P}^\epsilon}\Big[W^+(t,y^l,p^m)1_{\{X_t^{\bar t,\bar x}\in A^l,\mathbf{p}_{t-}=p^m\}}1_{\{\sigma\geq t,\hat\tau\geq t\}}|\mathcal{F}_{t-}\Big]+c\delta+2\epsilon\\ &\leq \mathbb{E}_{\mathbb{P}^\epsilon}\Big[W^+(t,X_t^{\bar t,\bar x},\mathbf{p}_{t-})1_{\{\sigma\geq t,\hat\tau\geq t\}}|\mathcal{F}_{t-}\Big]+2c\delta+2\epsilon. \end{aligned}$$

Hence we have with (7.6)

$$\begin{aligned} &W^+(\bar t,\bar x,\bar p)\\ &\leq \text{esssup}_{\sigma\in\bar{\mathcal{T}}(\bar t,t)}\mathbb{E}_{\mathbb{P}^\epsilon}\Big[\langle \mathbf{p}_\sigma, f(\sigma,X_\sigma^{\bar t,\bar x})\rangle 1_{\sigma<\tau,\sigma<t} + \langle \mathbf{p}_\tau, h(\tau,X_\tau^{\bar t,\bar x})\rangle 1_{\tau\leq\sigma,\tau<t}\\ &\qquad +W^+(t,X_t^{\bar t,\bar x},\mathbf{p}_{t-})1_{\{\sigma=\tau=t\}}|\mathcal{F}_{t-}\Big]+2c\delta+2\epsilon. \end{aligned}$$

Now choosing $\mathbb{P}$, $\tau\in\bar{\mathcal{T}}(\bar t,t)$ such that they are ϵ optimal for the right hand side of (7.3) gives the desired result. □

7.3 Superdynamic programming for W^-

Theorem 7.4. *Let $(\bar t,\bar x,\bar p)\in[0,T]\times\mathbb{R}^d\times\Delta(I)$. Then for all $t\in[\bar t,T]$*

$$\begin{aligned} &W^-(\bar t,\bar x,\bar p)\\ &\geq \text{essinf}_{\mathbb{P}\in\mathcal{P}(\bar t,\bar p)}\text{esssup}_{\sigma\in\bar{\mathcal{T}}(\bar t,t)}\text{essinf}_{\tau\in\bar{\mathcal{T}}(\bar t,t)}\mathbb{E}_{\mathbb{P}}\Big[\langle \mathbf{p}_\sigma, f(\sigma,X_\sigma^{\bar t,\bar x})\rangle 1_{\sigma<\tau,\sigma<t}\\ &\qquad +\langle \mathbf{p}_\tau, h(\tau,X_\tau^{\bar t,\bar x})1_{\tau\leq\sigma,\tau<t}\rangle + W^-(t,X_t^{\bar t,\bar x},\mathbf{p}_{t-})1_{\tau=\sigma=t}|\mathcal{F}_{\bar t-}\Big]. \end{aligned} \tag{7.7}$$

Proof. We choose a $\mathbb{P}^\epsilon\in\mathcal{P}^f(\bar t,\bar p)$ to be ϵ-optimal for $W^-(\bar t,\bar x,\bar p)$,

$$\begin{aligned} &W^-(\bar t,\bar x,\bar p)\\ &\geq \text{essup}_{\sigma\in\bar{\mathcal{T}}(\bar t)}\text{essinf}_{\tau\in\bar{\mathcal{T}}(\bar t)}\mathbb{E}_{\mathbb{P}^\epsilon}\Big[\langle \mathbf{p}_\sigma, f(\sigma,X_\sigma^{t,x})\rangle 1_{\sigma<\tau,\sigma<T} + \langle \mathbf{p}_\tau, h(\tau,X_\tau^{t,x})\rangle 1_{\tau\leq\sigma,\tau<T}\\ &\qquad +\langle \mathbf{p}_T, g(X_T^{t,x})\rangle 1_{\sigma=\tau=T}|\mathcal{F}_{\bar t-}\Big]-\epsilon. \end{aligned} \tag{7.8}$$

By assumption there exist $S = \{p^1, \dots, p^k\}$, such that $\mathbb{P}^\epsilon[\mathbf{p}_{t-} \in S] = 1$. Furthermore let $(A_l)_{l\in\mathbb{N}}$ be a partition of $\mathbb{R}^d$ by Borel sets, such that $\mathrm{diam}(A_l) \le \bar{\epsilon}$ and choose for any $l \in \mathbb{N}$ some $y^l \in A_l$.
With the help of $\mathbb{P}^\epsilon$ define $\mathbb{P}^{l,m}$ as

$$\mathbb{P}^{l,m} = (\mathbb{P}_0 \otimes \delta(p^m)) \otimes \hat{\mathbb{P}}^{l,m}, \tag{7.9}$$

where $\delta(p^m)$ denotes the measure under which $\mathbf{p}$ is constant and equal to p^m, $\mathbb{P}_0$ is a Wiener measure on $\Omega_{0,t}$ and for all $A \in \mathcal{B}(\Omega_t)$

$$\hat{\mathbb{P}}^{l,m} = \mathbb{P}^\epsilon[\mathbf{p}_{t-} = p^m, X_t^{\bar{t},\bar{x}} \in A^l]\mathbb{P}^\epsilon[A|\mathbf{p}_{t-} = p^m, X_t^{\bar{t},\bar{x}} \in A^l].$$

Furthermore define stopping times $\sigma^{l,m} \in \bar{\mathcal{T}}(t)$ which are ϵ-optimal for

$$\begin{aligned}&\mathrm{esssup}_{\sigma\in\bar{\mathcal{T}}(t)}\mathrm{essinf}_{\tau\in\bar{\mathcal{T}}(t)}\mathbb{E}_{\mathbb{P}^{l,m}}\Big[\langle \mathbf{p}_\sigma, f(\sigma, X_\sigma^{t,y^l})\rangle 1_{\sigma<\tau,\sigma<T}\\ &\qquad +\langle \mathbf{p}_\tau, h(\tau, X_\tau^{t,y^l})\rangle 1_{\tau\le\sigma,\tau<T} + \langle \mathbf{p}_T, g(X_T^{\bar{t},\bar{x}})\rangle 1_{\sigma=\tau=T}|\mathcal{F}_{t-}\Big],\end{aligned} \tag{7.10}$$

which implies that for all $\tau \in \bar{\mathcal{T}}(t)$

$$\begin{aligned}&\mathbb{E}_{\mathbb{P}^{l,m}}\Big[\langle \mathbf{p}_{\sigma^{l,m}}, f(\sigma^{l,m}, X_{\sigma^{l,m}}^{t,y^l})\rangle 1_{\sigma^{l,m}<\tau,\sigma^{l,m}<T} + \langle \mathbf{p}_\tau, h(\tau, X_\tau^{\bar{t},\bar{x}})\rangle 1_{\tau\le\sigma^{l,m},\tau<T}\\ &\qquad +\langle \mathbf{p}_T, g(X_T^{\bar{t},\bar{x}})\rangle 1_{\sigma^{l,m}=\tau=T}|\mathcal{F}_{t-}\Big]\\ &\ge \mathrm{esssup}_{\sigma\in\bar{\mathcal{T}}(t)}\mathrm{essinf}_{\tau\in\bar{\mathcal{T}}(t)}\mathbb{E}_{\mathbb{P}^{l,m}}\Big[\langle \mathbf{p}_\sigma, f(\sigma, X_\sigma^{t,y^l})\rangle 1_{\sigma<\tau,\sigma<T}\\ &\qquad +\langle \mathbf{p}_\tau, h(\tau, X_\tau^{t,y^l})\rangle 1_{\tau\le\sigma,\tau<T} + \langle \mathbf{p}_T, g(X_T^{\bar{t},\bar{x}})\rangle 1_{\sigma=\tau=T}|\mathcal{F}_{t-}\Big] - \epsilon\end{aligned} \tag{7.11}$$

and

$$\begin{aligned}&\mathbb{E}_{\mathbb{P}^{l,m}}\Big[\langle \mathbf{p}_{\sigma^{l,m}}, f(\sigma^{l,m}, X_{\sigma^{l,m}}^{t,y^l})\rangle 1_{\sigma^{l,m}<\tau,\sigma^{l,m}<T} + \langle \mathbf{p}_\tau, h(\tau, X_\tau^{\bar{t},\bar{x}})\rangle 1_{\tau\le\sigma^{l,m},\tau<T}\\ &\qquad +\langle \mathbf{p}_T, g(X_T^{\bar{t},\bar{x}})\rangle 1_{\sigma^{l,m}=\tau=T}|\mathcal{F}_{t-}\Big]\\ &\ge \mathrm{essinf}_{\mathbb{P}\in\mathcal{P}(t,p^m)}\mathrm{esssup}_{\sigma\in\bar{\mathcal{T}}(t)}\mathrm{essinf}_{\tau\in\bar{\mathcal{T}}(t)}\mathbb{E}_P\Big[\langle \mathbf{p}_\sigma, f(\sigma, X_\sigma^{t,y^l})\rangle 1_{\sigma<\tau,\sigma<T}\\ &\qquad +\langle \mathbf{p}_\tau, h(\tau, X_\tau^{t,y^l})\rangle 1_{\tau\le\sigma,\tau<T} + \langle \mathbf{p}_T, g(X_T^{\bar{t},\bar{x}})\rangle 1_{\sigma=\tau=T}|\mathcal{F}_{t-}\Big] - \epsilon\\ &= W^-(t, p^m, y^l) - \epsilon.\end{aligned} \tag{7.12}$$

For any $\sigma \in \bar{\mathcal{T}}(\bar{t})$ define

$$\hat{\sigma} = \begin{cases}\sigma & \text{on } \{\sigma < t\}\\ \sigma^{l,m} & \text{on } \{\sigma \ge t, X_t^{\bar{t},\bar{x}} \in A^l, \mathbf{p}_{t-} = p^m\}.\end{cases} \tag{7.13}$$

Note that using the Lipschitz continuity of the coefficients and W^- and the definition of $\hat\sigma$ and $\mathbb{P}^{l,m}$ we have for any $\tau \in \bar{\mathcal{T}}(\bar t)$

$$\begin{aligned}
&\mathbb{E}_{\mathbb{P}^\epsilon}\Big[\langle \mathbf{p}_{\hat\sigma}, f(\hat\sigma, X^{t,x}_{\hat\sigma})\rangle 1_{t\le\hat\sigma<\tau,\hat\sigma<T} + \langle \mathbf{p}_\tau, h(\tau, X^{t,x}_\tau)\rangle 1_{t\le\tau\le\hat\sigma,\tau<T} \\
&\qquad\qquad + \langle \mathbf{p}_T, g(X^{t,x}_T)\rangle 1_{\hat\sigma=\tau=T,\hat\sigma,\tau\ge t}|\mathcal{F}_{\bar t-}\Big] \\
&= \mathbb{E}_{\mathbb{P}^\epsilon}\Big[\mathbb{E}_{\mathbb{P}^\epsilon}\Big[\langle \mathbf{p}_{\hat\sigma}, f(\hat\sigma, X^{t,x}_{\hat\sigma})\rangle 1_{t\le\hat\sigma<\tau,\hat\sigma<T} + \langle \mathbf{p}_\tau, h(\tau, X^{t,x}_\tau)\rangle 1_{t\le\tau\le\hat\sigma,\tau<T} \\
&\qquad\qquad + \langle \mathbf{p}_T, g(X^{t,x}_T)\rangle 1_{\hat\sigma=\tau=T,\hat\sigma,\tau\ge t}|\mathcal{F}_{t-}\Big]|\mathcal{F}_{\bar t-}\Big] \\
&\ge \mathbb{E}_{\mathbb{P}^\epsilon}\Big[1_{\sigma,\tau\ge t, X^{\bar t,\bar x}_t\in A^l, \mathbf{p}_{t-}=p^m}\mathbb{E}_{\mathbb{P}^{l,m}}\Big[\langle \mathbf{p}_{\sigma^{l,m}}, f(\sigma^{l,m}, X^{t,y^l}_{\sigma^{l,m}})\rangle 1_{\sigma^{l,m}<\tau,\sigma^{l,m}<T} \\
&\qquad + \langle \mathbf{p}_\tau, h(\tau, X^{\bar t,\bar x}_\tau)\rangle 1_{\tau\le\sigma^{l,m},\tau<T} + \langle \mathbf{p}_T, g(X^{\bar t,\bar x}_T)\rangle 1_{\sigma^{l,m}=\tau=T}|\mathcal{F}_{t-}\Big]|\mathcal{F}_{\bar t-}\Big] \\
&\quad -c\delta \\
&\ge \mathbb{E}_{\mathbb{P}^\epsilon}\Big[1_{\sigma,\tau\ge t, X^{\bar t,\bar x}_t\in A^l, \mathbf{p}_{t-}=p^m} W(t, y^l, p^m)|\mathcal{F}_{\bar t-}\Big] - c\delta - \epsilon \\
&\ge \mathbb{E}_{\mathbb{P}^\epsilon}\Big[1_{\sigma,\tau\ge t} W(t, X^{\bar t,\bar x}_t, \mathbf{p}_{t-})|\mathcal{F}_{\bar t-}\Big] - 2c\delta - \epsilon.
\end{aligned}$$

This gives with (7.12) for any $\sigma \in \bar{\mathcal{T}}(\bar t, t)$

$$\begin{aligned}
&W^-(\bar t, \bar x, \bar p) \\
&\quad\ge \operatorname{essinf}_{\tau\in\bar{\mathcal{T}}(\bar t,t)}\mathbb{E}_{\mathbb{P}^\epsilon}\Big[\langle \mathbf{p}_\sigma, f(\sigma, X^{t,x}_\sigma)\rangle 1_{\sigma<\tau,\sigma<t} + \langle \mathbf{p}_\tau, h(\tau, X^{t,x}_\tau)\rangle 1_{\tau\le\sigma,\tau<t} \\
&\qquad\qquad + W(t, X^{\bar t,\bar x}_t, \mathbf{p}_{t-}) 1_{\sigma=\tau=t}|\mathcal{F}_{\bar t-}\Big] - 2c\delta - \epsilon.
\end{aligned} \tag{7.14}$$

So in particular when choosing $\bar\sigma$ ϵ-optimal for

$$\begin{aligned}
&\operatorname{esssup}_{\sigma\in\bar{\mathcal{T}}(\bar t,t)}\operatorname{essinf}_{\tau\in\bar{\mathcal{T}}(\bar t,t)}\mathbb{E}_{\mathbb{P}^\epsilon}\Big[\langle \mathbf{p}_\sigma, f(\sigma, X^{t,x}_\sigma)\rangle 1_{\sigma<\tau,\sigma<t} \\
&\qquad + \langle \mathbf{p}_\tau, h(\tau, X^{t,x}_\tau)\rangle 1_{\tau\le\sigma,\tau<t} + W(t, X^{\bar t,\bar x}_t, \mathbf{p}_{t-}) 1_{\sigma=\tau=t}|\mathcal{F}_{\bar t-}\Big]
\end{aligned} \tag{7.15}$$

we get

$$\begin{aligned}
&W^-(\bar t, \bar x, \bar p) \\
&\quad\ge \operatorname{esssup}_{\sigma\in\bar{\mathcal{T}}(\bar t,t)}\operatorname{essinf}_{\tau\in\bar{\mathcal{T}}(\bar t,t)}\mathbb{E}_{\mathbb{P}^\epsilon}\Big[\langle \mathbf{p}_\sigma, f(\sigma, X^{t,x}_\sigma)\rangle 1_{\sigma<\tau,\sigma<t} + \langle \mathbf{p}_\tau, h(\tau, X^{t,x}_\tau)\rangle 1_{\tau\le\sigma,\tau<t} \\
&\qquad + \langle \mathbf{p}_\tau, h(\tau, X^{t,x}_\tau)\rangle 1_{t>\sigma\ge\tau} + W(t, X^{\bar t,\bar x}_t, \mathbf{p}_{t-}) 1_{\sigma=\tau=t}|\mathcal{F}_{\bar t-}\Big] - 2c\delta - 2\epsilon
\end{aligned}$$

and the claim follows by taking the essential infimum in $\mathbb{P} \in \mathcal{P}(\bar t, \bar p)$ since δ and ϵ can be chosen arbitrarily small.

□

8 Viscosity solution property W^+, W^-

8.1 Subsolution property of W^+

Theorem 8.1. *W^+ is a viscosity subsolution to the obstacle problem*

$$\max\Big\{\max\{\min\{(-\tfrac{\partial}{\partial t}-\mathcal{L})[w], w-\langle f(t,x),p\rangle\}, \\ w-\langle h(t,x),p\rangle\}, -\lambda_{\min}\Big(p,\tfrac{\partial^2 w}{\partial p^2}\Big)\Big\}=0 \tag{8.1}$$

with terminal condition $w(T,x,p)=\sum_{i=1,\dots,I} p_i g_i(x)$.

Proof. Let $\phi:[0,T]\times\mathbb{R}^d\times\Delta(I)\to\mathbb{R}$ be a test function such that $W^+-\phi$ has a strict global maximum at $(\bar t,\bar x,\bar p)\in[0,T)\times\mathbb{R}^d\times\mathrm{Int}(\Delta(I))$ with $W(\bar t,\bar x,\bar p)-\phi(\bar t,\bar x,\bar p)=0$. We have to show, that

$$\max\Big\{\max\{\min\{(-\tfrac{\partial}{\partial t}-\mathcal{L})[\phi], \phi-\langle f(t,x),p\rangle\}, \\ \phi-\langle h(t,x),p\rangle\}, -\lambda_{\min}\Big(p,\tfrac{\partial^2 w}{\partial p^2}\Big)\Big\}\le 0 \tag{8.2}$$

at $(\bar t,\bar x,\bar p)$.
By Proposition 7.2 W^+ is convex in p. So since $\bar p\in\mathrm{Int}(\Delta(I))$, we have that

$$-\lambda_{\min}\left(\frac{\partial^2\phi}{\partial p^2}(\bar t,\bar x,\bar p)\right)\le 0.$$

So it remains to show, that

$$\max\{\min\{(-\tfrac{\partial}{\partial t}-\mathcal{L})[\phi], \phi-\langle f(t,x),p\rangle\}, \phi-\langle h(t,x),p\rangle\}\le 0 \tag{8.3}$$

at $(\bar t,\bar x,\bar p)$. Note that the subdynamic programming for W^+ implies for $\mathbb{P}=\mathbb{P}_0\otimes\delta(\bar p)$ in particular

$$\begin{aligned} &W^+(\bar t,\bar x,\bar p)\\ &\le \mathrm{essinf}_{\tau\in\bar{\mathcal{T}}(\bar t,t)}\mathrm{esssup}_{\sigma\in\bar{\mathcal{T}}(\bar t,t)}\mathbb{E}_{\mathbb{P}}\Big[\langle\bar p, f(\sigma,X_\sigma^{\bar t,\bar x})1_{\sigma<\tau,\sigma<t}\rangle\\ &\qquad +\langle\bar p,h(\tau,X_\tau^{\bar t,\bar x})1_{\tau\le\sigma,\tau<t}\rangle+W^+(t,X_t^{\bar t,\bar x},\bar p)1_{\tau=\sigma=t}|\mathcal{F}_{\bar t-}\Big].\end{aligned}$$

So (8.2) follows by the standard arguments we mentioned already in the proof of Theorem 5.1.

□

8.2 Supersolution property of W^-

Theorem 8.2. *W^- is a viscosity supersolution to the obstacle problem*

$$\max\Big\{\max\{\min\{(-\tfrac{\partial}{\partial t}-\mathcal{L})[w], w-\langle f(t,x),p\rangle\}, \\ w-\langle h(t,x),p\rangle\}, -\lambda_{\min}\Big(p,\tfrac{\partial^2 w}{\partial p^2}\Big)\Big\}=0 \tag{8.4}$$

with terminal condition $w(T,x,p)=\sum_{i=1,\dots,I} p_i g_i(x)$.

Proof. Let $\phi : [0,T] \times \mathbb{R}^d \times \Delta(I) \to \mathbb{R}$ be a smooth test function with uniformly bounded derivatives such that $W^- - \phi$ has a strict global minimum at $(\bar{t}, \bar{x}, \bar{p}) \in [0,T) \times \mathbb{R}^d \times \Delta(I)$ with $W^-(\bar{t}, \bar{x}, \bar{p}) - \phi(\bar{t}, \bar{x}, \bar{p}) = 0$. We have to show

$$\max\Big\{ \max\{\min\{(-\tfrac{\partial}{\partial t} - \mathcal{L})[\phi], \phi - \langle f(t,x), p\rangle\}, \\ \phi - \langle h(t,x), p\rangle\}, -\lambda_{\min}\Big(p, \tfrac{\partial^2 \phi}{\partial p^2}\Big) \Big\} \geq 0 \tag{8.5}$$

at $(\bar{t}, \bar{x}, \bar{p})$. If

$$\lambda_{\min}\left(p, \frac{\partial^2 \phi}{\partial p^2}\right) \leq 0$$

at $(\bar{t}, \bar{x}, \bar{p})$ (8.5) obviously holds. So we assume in the subsequent steps strict convexity of ϕ in p at $(\bar{t}, \bar{x}, \bar{p})$, i.e. there exist $\delta, \eta > 0$ such that for all $z \in T_{\Delta(I)(\bar{p})}$

$$\langle \frac{\partial^2 \phi}{\partial p^2}(t,x,p) z, z\rangle > 4\delta |z|^2 \qquad \forall (t,x,p) \in B_\eta(\bar{t}, \bar{x}, \bar{p}). \tag{8.6}$$

Since ϕ is a test function for a purely local viscosity notion, one can modify it outside a neighborhood of $(\bar{t}, \bar{x}, \bar{p})$ such that for all $(s,x) \in [\bar{t}, T] \times \mathbb{R}^d$ the function $\phi(s,x,\cdot)$ is convex on the whole convex domain $\Delta(I)$. Thus for any $p \in \Delta(I)$ we have that

$$W^-(t,x,p) \geq \phi(t,x,p) \geq \phi(t,x,\bar{p}) + \langle \frac{\partial \phi}{\partial p}(t,x,\bar{p}), p - \bar{p}\rangle. \tag{8.7}$$

Step 1: Estimate for $\mathbf{p}$.
As in (4.14) we have with (8.6) a stronger estimate, namely there exist $\delta, \eta > 0$ such that for all $p \in \Delta(I)$, $t \in [\bar{t}, \bar{t} + \eta]$, $x \in B_\eta(\bar{x})$

$$W^-(t,x,p) \geq \phi(t,x,\bar{p}) + \langle \frac{\partial \phi}{\partial p}(t,x,\bar{p}), p - \bar{p}\rangle + \delta |p - \bar{p}|^2. \tag{8.8}$$

As in the proof of Theorem 4.1. we can set in the dynamic programming for W^- $\sigma = t$ to get

$$W^-(\bar{t}, \bar{x}, \bar{p}) \\ \geq \operatorname{essinf}_{\mathbb{P} \in \mathcal{P}(\bar{t}, \bar{p})} \operatorname{essinf}_{\tau \in \bar{\mathcal{T}}(\bar{t}, t)} \mathbb{E}_{\mathbb{P}}\Big[\langle \mathbf{p}_\tau, h(\tau, X_\tau^{\bar{t}, \bar{x}}) 1_{\tau < t}\rangle + W^-(t, X_t^{\bar{t}, \bar{x}}, \mathbf{p}_{t-}) 1_{\tau = t} | \mathcal{F}_{\bar{t}-}\Big]. \tag{8.9}$$

So for $\epsilon(t - \bar{t})$-optimal $\mathbb{P}^\epsilon \in \mathcal{P}^f(t,p)$ and a $\epsilon(t - \bar{t})$-optimal stopping time τ^ϵ we have

$$\begin{aligned} &W^-(\bar{t}, \bar{x}, \bar{p}) \\ &\geq \mathbb{E}_{\mathbb{P}^\epsilon}\Big[\langle \mathbf{p}_{\tau^\epsilon}, h(\tau^\epsilon, X_{\tau^\epsilon}^{\bar{t}, \bar{x}})\rangle 1_{\tau^\epsilon < t} + W^-(t, X_t^{\bar{t}, \bar{x}}, \mathbf{p}_{t-}) 1_{\tau^\epsilon = t} | \mathcal{F}_{\bar{t}-}\Big] - 2\epsilon(t - \bar{t}) \\ &= \mathbb{E}_{\mathbb{P}^\epsilon}\Big[\langle \mathbf{p}_{\tau^\epsilon -}, h(\tau^\epsilon, X_{\tau^\epsilon}^{\bar{t}, \bar{x}})\rangle 1_{\tau^\epsilon < t} + W^-(t, X_t^{\bar{t}, \bar{x}}, \mathbf{p}_{t-}) 1_{\tau^\epsilon = t} | \mathcal{F}_{\bar{t}-}\Big] - 2\epsilon(t - \bar{t}) \\ &\geq \mathbb{E}_{\mathbb{P}^\epsilon}\Big[W(\tau^\epsilon, X_{\tau^\epsilon}^{\bar{t}, \bar{x}}, \mathbf{p}_{\tau^\epsilon -}) | \mathcal{F}_{\bar{t}-}\Big] - 2\epsilon(t - \bar{t}), \end{aligned} \tag{8.10}$$

since $\langle p, h(t,x)\rangle \geq W^-(t,x,p)$ for all $(t,x,p) \in [0,T] \times \mathbb{R}^d \times \Delta(I)$. Using (8.7) and (8.8) we get since $W^-(\bar{t},\bar{x},\bar{p}) = \phi(\bar{t},\bar{x},\bar{p})$

$$\begin{aligned} 0 \geq \mathbb{E}_{\mathbb{P}^\epsilon}\Big[&\phi(\tau^\epsilon, X^{\bar{t},\bar{x}}_{\tau^\epsilon},\bar{p}) - \phi(\bar{t},\bar{x},\bar{p}) - \langle \tfrac{\partial \phi}{\partial p}(\tau^\epsilon, X^{\bar{t},\bar{x}}_{\tau^\epsilon},\bar{p}), \mathbf{p}_{\tau^\epsilon -} - \bar{p}\rangle \\ &+\delta 1_{\{|X^{\bar{t},\bar{x}}-x|<\eta\}} |\mathbf{p}_{\tau^\epsilon -} - \bar{p}| \big| \mathcal{F}_{\bar{t}-}\Big] - 2\epsilon(t-\bar{t}). \end{aligned} \tag{8.11}$$

Now by Itô's formula and since the derivatives of ϕ are uniformly bounded we have that

$$\Big|\mathbb{E}_{\mathbb{P}^\epsilon}\left[\phi(\tau_\epsilon, X^{\bar{t},\bar{x}}_{\tau^\epsilon},\bar{p}) - \phi(\bar{t},\bar{x},\bar{p})\big|\mathcal{F}_{\bar{t}-}\right]\Big| \leq c\mathbb{E}_{\mathbb{P}^\epsilon}[(\tau^\epsilon - \bar{t})\;|\mathcal{F}_{\bar{t}-}] \leq c(t-\bar{t}). \tag{8.12}$$

Next, let $f : [\bar{t},t] \times \mathbb{R}^n \to \mathbb{R}^n$ be a smooth bounded function, with bounded derivatives. Recall that under any $\mathbb{P} \in \mathcal{P}^f(\bar{t},\bar{p})$ the process $\mathbf{p}$ is strongly orthogonal to B. So since under $\mathbb{P}^\epsilon$ the process $\mathbf{p}$ is a martingale with $\mathbb{E}_{\mathbb{P}^\epsilon}[\mathbf{p}_{\tau^\epsilon -}|\mathcal{F}_{\bar{t}-}] = \bar{p}$, we have by Itô's formula that

$$\mathbb{E}_{\mathbb{P}^\epsilon}\left[f_i(\tau^\epsilon, X^{\bar{t},\bar{x}}_{\tau^\epsilon})(\mathbf{p}_{\tau^\epsilon -} - \bar{p})_i\big|\mathcal{F}_{\bar{t}-}\right] = \mathbb{E}_{\mathbb{P}^\epsilon}\left[\int_{\bar{t}}^{\tau^\epsilon}\left((\frac{\partial}{\partial t} + \mathcal{L})f_i(s, X^{\bar{t},\bar{x}}_s)\right)(\mathbf{p}_s - \bar{p})_i ds\big|\mathcal{F}_{\bar{t}-}\right].$$

Hence by the assumption on the coefficients of the diffusion (A)(i)

$$\left|\mathbb{E}_{\mathbb{P}^\epsilon}\left[\langle \frac{\partial \phi}{\partial p}(\tau^\epsilon, X^{\bar{t},\bar{x}}_{\tau^\epsilon},\bar{p}), \mathbf{p}_{\tau^\epsilon -} - \bar{p}\rangle\big|\mathcal{F}_{\bar{t}-}\right]\right| \leq c\mathbb{E}_{\mathbb{P}^\epsilon}[(\tau^\epsilon - \bar{t})\;|\mathcal{F}_{\bar{t}-}] \leq c(t-\bar{t}). \tag{8.13}$$

Furthermore observe that, since $|\mathbf{p}_{\tau^\epsilon -} - \bar{p}| \leq 1$, we have, that for $\epsilon' > 0$ by Young and Hölder inequality

$$\begin{aligned} &\mathbb{E}_{\mathbb{P}^\epsilon}\left[1_{\{|X^{\bar{t},\bar{x}}_{\tau^\epsilon}-\bar{x}|<\eta\}}|\mathbf{p}_{\tau^\epsilon -} - \bar{p}|^2\big|\mathcal{F}_{\bar{t}-}\right] \\ &\geq \mathbb{E}_{\mathbb{P}^\epsilon}\left[|\mathbf{p}_{\tau^\epsilon -} - \bar{p}|^2\big|\mathcal{F}_{\bar{t}-}\right] - \tfrac{1}{\eta}\mathbb{E}_{\mathbb{P}^\epsilon}\left[|X^{\bar{t},\bar{x}}_{\tau^\epsilon} - \bar{x}||\mathbf{p}_{\tau^\epsilon -} - \bar{p}|^2\big|\mathcal{F}_{\bar{t}-}\right] \\ &\geq (1-\tfrac{\epsilon'}{\eta})\mathbb{E}_{\mathbb{P}^\epsilon}\left[|\mathbf{p}_{\tau^\epsilon -} - \bar{p}|^2\big|\mathcal{F}_{\bar{t}-}\right] - \tfrac{1}{4\eta\epsilon'}\mathbb{E}_{\mathbb{P}^\epsilon}\left[|X^{\bar{t},\bar{x}}_{\tau^\epsilon} - \bar{x}|^2\big|\mathcal{F}_{\bar{t}-}\right] \\ &\geq (1-\tfrac{\epsilon'}{\eta})\mathbb{E}_{\mathbb{P}^\epsilon}\left[|\mathbf{p}_{\tau^\epsilon -} - \bar{p}|^2\big|\mathcal{F}_{\bar{t}-}\right] - \tfrac{c}{4\eta\epsilon'}\mathbb{E}_{\mathbb{P}^\epsilon}[(\tau^\epsilon - \bar{t})\;|\mathcal{F}_{\bar{t}-}], \end{aligned}$$

hence

$$\begin{aligned} &\mathbb{E}_{\mathbb{P}^\epsilon}\left[1_{\{|X^{\bar{t},\bar{x}}_{\tau^\epsilon}-\bar{x}|<\eta\}}|\mathbf{p}_{\tau^\epsilon -} - \bar{p}|^2\big|\mathcal{F}_{\bar{t}-}\right] \\ &\geq (1-\tfrac{\epsilon'}{\eta})\mathbb{E}_{\mathbb{P}^\epsilon}\left[|\mathbf{p}_{\tau^\epsilon -} - \bar{p}|^2\big|\mathcal{F}_{\bar{t}-}\right] - \tfrac{c}{4\eta\epsilon'}(t-\bar{t}). \end{aligned} \tag{8.14}$$

Choosing $0 < \epsilon' < \eta$ and combining (8.11) with the estimates (8.12)-(8.14) there exists a constant c, such that

$$\mathbb{E}_{\mathbb{P}^\epsilon}\left[|\mathbf{p}_{\tau^\epsilon -} - \bar{p}|^2\big|\mathcal{F}_{\bar{t}-}\right] \leq c(t-\bar{t}). \tag{8.15}$$

This implies in particular for $h > 0$ by Doob's inequality

$$\mathbb{P}^\epsilon\left[\sup_{s\in[\bar{t},\tau^\epsilon[}|\mathbf{p}_s - \bar{p}| > h\right] \leq c\frac{\mathbb{E}_{\mathbb{P}^\epsilon}\left[|\mathbf{p}_{\tau^\epsilon -} - \bar{p}|^2\right]}{h^2} \leq c\frac{(t-\bar{t})}{h^2}. \tag{8.16}$$

Step 2: Viscosity supersolution property
To show the viscosity supersolution property we have to show that

$$W^-(\bar{t},\bar{x},\bar{p}) - \langle h(\bar{t},\bar{x}),\bar{p}\rangle = \phi(\bar{t},\bar{x},\bar{p}) - \langle h(\bar{t},\bar{x}),\bar{p}\rangle < 0$$

implies

$$(\frac{\partial \phi}{\partial t} + \mathcal{L})[\phi](\bar{t},\bar{x},\bar{p}) \leq 0.$$

We will argue by contradiction. Assume that

$$\phi(\bar{t},\bar{x},\bar{p}) - \langle h(\bar{t},\bar{x}),\bar{p}\rangle < 0 \qquad \text{and} \qquad (\frac{\partial \phi}{\partial t} + \mathcal{L})[\phi](\bar{t},\bar{x},\bar{p}) > 0. \tag{8.17}$$

Then there exist $h,\delta > 0$ such that for all $(s,x,p) \in [\bar{t},\bar{t}+h] \times B(\bar{x},\bar{p})$

$$\langle h(s,x),\bar{p}\rangle - \phi(s,x,\bar{p}) \geq \delta \qquad \text{and} \qquad (\tfrac{\partial \phi}{\partial t} + \mathcal{L})[\phi](s,x,\bar{p}) \geq \delta. \tag{8.18}$$

By the Itô formula we have, that

$$\phi(\tau^\epsilon, X^{\bar{t},\bar{x}}_{\tau^\epsilon}, \mathbf{p}_{\tau^\epsilon}) \geq \phi(\bar{t},\bar{x},\bar{p}) + \textstyle\int_{\bar{t}}^{\tau^\epsilon} (\frac{\partial}{\partial t} + \mathcal{L})[\phi](s, X^{\bar{t},\bar{x}}_s, \mathbf{p}_s)ds, \tag{8.19}$$

where we used the fact that by the convexity of ϕ we have $\mathbb{P}^\epsilon$-a.s., that

$$\sum_{\bar{t}\leq r<\tau^\epsilon} \left(\phi(r, X^{\bar{t},\bar{x}}_r, \mathbf{p}_r) - \phi(r, X^{\bar{t},\bar{x}}_r, \mathbf{p}_{r-}) - \langle \frac{\partial}{\partial p}\phi(r, X^{\bar{t},\bar{x}}_r, \mathbf{p}_{r-}), \mathbf{p}_r - \mathbf{p}_{r-}\rangle\right) \geq 0.$$

Define $A := \{\inf_{s\in[\bar{t},t]} |\mathbf{p}_{s-} - \bar{p}| > h\}$ and $B := \{\inf_{s\in[\bar{t},t]} |X^{\bar{t},\bar{x}}_t - \bar{x}| > h\}$.
Note that by (8.16) and since $\mathbb{P}^\epsilon[B] \leq \frac{c(t-\bar{t})^2}{h^4}$ we have that

$$\begin{aligned} \mathbb{E}_{\mathbb{P}^\epsilon}[1_A 1_B] &\leq c(\mathbb{E}_{\mathbb{P}^\epsilon}[1_A])^{\frac{1}{2}}(\mathbb{E}_{\mathbb{P}^\epsilon}[1_B])^{\frac{1}{2}} \\ &\leq c(\tfrac{(t-\bar{t})}{h^2})^{\frac{1}{2}}(\tfrac{(t-\bar{t})^2}{h^4})^{\frac{1}{2}} = c\tfrac{(t-\bar{t})^{\frac{3}{2}}}{h^3}. \end{aligned} \tag{8.20}$$

Now we can continue as in the proof of Theorem 5.1. By using (8.19) we get

$$\begin{aligned} \phi(\bar{t},\bar{x},\bar{p}) &\leq \mathbb{E}_{\mathbb{P}^\epsilon}\left[1_{A^c}1_{B^c}\left(\phi(\tau^\epsilon, X^{\bar{t},\bar{x}}_{\tau^\epsilon}, \mathbf{p}_{\tau^\epsilon-}) - \textstyle\int_{\bar{t}}^{\tau^\epsilon}(\frac{\partial}{\partial t} + \mathcal{L})[\phi](s, X^{\bar{t},\bar{x}}_s, \mathbf{p}_s)ds\right)\right] \\ &\quad + c\tfrac{(t-\bar{t})^{\frac{3}{2}}}{h^3} \\ &\leq \mathbb{E}_{\mathbb{P}^\epsilon}\left[\langle h(\tau^\epsilon, X^{\bar{t},\bar{x}}_{\tau^\epsilon}), \mathbf{p}_{\tau^\epsilon}\rangle 1_{\tau^\epsilon<t} + \phi(t, X^{\bar{t},\bar{x}}_t, \mathbf{p}_{t-})1_{\tau^\epsilon=t}\right] \\ &\quad - \delta\mathbb{E}_{\mathbb{P}^\epsilon}\left[1_{\tau^\epsilon<t}\right] - \delta\mathbb{E}_{\mathbb{P}^\epsilon}\left[(\tau^\epsilon - \bar{t})\right] + 2c\tfrac{(t-\bar{t})^{\frac{3}{2}}}{h^3}. \end{aligned}$$

As in (5.7) we have that for $1 \geq (t-\bar{t})$

$$(t-\bar{t}) \leq \mathbb{E}_{\mathbb{P}^\epsilon}\left[1_{\tau^\epsilon<t}\right] + \mathbb{E}_{\mathbb{P}^\epsilon}\left[(\tau^\epsilon - \bar{t})\middle|\mathcal{F}_{\bar{t}-}\right] \tag{8.21}$$

so

$$\phi(\bar{t},\bar{x},\bar{p}) \leq \mathbb{E}_{\mathbb{P}^\epsilon}\left[\langle h(\tau^\epsilon, X^{\bar{t},\bar{x}}_{\tau^\epsilon}), \mathbf{p}_{\tau^\epsilon}\rangle 1_{\tau^\epsilon<t} + \phi(t, X^{\bar{t},\bar{x}}_t, \mathbf{p}_{t-})1_{\tau^\epsilon=t}\right]$$

$$-\delta(t-\bar{t}) + 2c\frac{(t-\bar{t})^{\frac{3}{2}}}{h^3},$$

which gives with (8.19)

$$-\delta(t-\bar{t}) + 2c\frac{(t-\bar{t})^{\frac{3}{2}}}{h^3} + 2\epsilon(t-\bar{t}) \geq 0.$$

Dividing by $(t-\bar{t})$ we have

$$-\delta + 2c\frac{(t-\bar{t})^{\frac{1}{2}}}{h^3} + 2\epsilon \geq 0. \tag{8.22}$$

However (8.22) contradicts $\delta > 0$, since ϵ and $t-\bar{t}$ can be chosen arbitrarily small. □

The proof of Theorem 6.4 is now straightforward using the subsolution property of W^+, the supersolution property of W^- and the comparison result of Theorem 3.7.

9 Appendix: Comparison

In this section we provide the proof of the comparison result Theorem 3.7. for the fully non linear variational PDE (3.5)

$$\max\left\{\max\{\min\{(-\tfrac{\partial}{\partial t} - \mathcal{L})[w], w - \langle f(t,x),p\rangle\}, \right.$$
$$\left. w - \langle h(t,x),p\rangle\}, -\lambda_{\min}\left(p, \tfrac{\partial^2 w}{\partial p^2}\right)\right\} = 0$$

with terminal condition $w(T,x,p) = \sum_{i=1,\dots,I} p_i g_i(x)$. The proof is more or less a straight forward adaption of the results in [25].

9.1 Reduction to the faces

Let $\tilde{I} \subset \{1,\dots,I\}$ and we define the set $\Delta(\tilde{I})$ by

$$\Delta(\tilde{I}) = \{p \in \Delta(I) : p_i = 0 \text{ if } i \notin \tilde{I}\}. \tag{9.1}$$

Note that by Definition 3.3. the supersolution property is obviously preserved under restriction. We just state

Proposition 9.1. *Let $w : [0,T]\times\mathbb{R}^d\times\Delta(I) \to \mathbb{R}$ be a bounded, continuous viscosity supersolution to (3.5), which is uniformly Lipschitz continuous in p. Then the restriction of w to $\Delta(\tilde{I})$ is a supersolution to (3.5) on $[0,T]\times\mathbb{R}^d\times\Delta(\tilde{I})$.*

The subsolution property is however not immediate, since $\mathrm{Int}(\Delta(\tilde{I})) \not\subseteq \mathrm{Int}(\Delta(I))$.

Proposition 9.2. *Let $w : [0,T]\times\mathbb{R}^d\times\Delta(I) \to \mathbb{R}$ be a bounded, continuous viscosity subsolution to (3.5), which is uniformly Lipschitz continuous in p. Then the restriction of w to $\Delta(\tilde{I})$ is a subsolution to (3.5) on $[0,T]\times\mathbb{R}^d\times\Delta(\tilde{I})$.*

Proof. Set $\tilde{w} = w\big|_{\Delta(\tilde{I})}$. Let $(\bar{t},\bar{x},\bar{p}) \in (0,T)\times\mathbb{R}^d\times\mathrm{Int}(\Delta(\tilde{I}))$ and $\phi : [0,T]\times\mathbb{R}^d\times\Delta(\tilde{I}) \to \mathbb{R}$ a test function such that $\tilde{w}-\phi$ has a strict minimum at $(\bar{t},\bar{x},\tilde{p})$ with $\tilde{w}(\bar{t},\bar{x},\tilde{p})-\phi(\bar{t},\bar{x},\tilde{p}) = 0$. By using the viscosity subsolution property of w on $[0,T]\times\mathbb{R}^d\times\Delta(I)$ we have to show:

(i) $\lambda_{\min}\left(\tilde{p},\frac{\partial^2\phi}{\partial p^2}\right)\geq 0$

(ii)
$$\max\{\min\{(-\tfrac{\partial}{\partial t}-\mathcal{L})[\phi],\phi-\langle f(t,x),p\rangle\},\phi-\langle h(t,x),p\rangle\}\leq 0$$
at $(\bar{t},\bar{x},\tilde{p})$.

However $\tilde{p}\notin\mathrm{Int}(\Delta(I))$ so we have to use an appropriate approximation. Let $\mu\in\mathbb{R}^I$ such that $\mu_i=0$ if $i\in\tilde{I}$ and $\mu_i=1$ else. Furthermore we define for $p\in\Delta(I)$ the projection Π onto $\Delta(\tilde{I})$ by
$$\Pi(p)_i=\begin{cases}p_i+\left(\sum_{j\notin\tilde{I}}p_i\right)/|\tilde{I}| & \text{if } i\in\tilde{I},\\ 0 & \text{else.}\end{cases}$$

Since w is uniformly Lipschitz continuous with Lipschitz constant k with respect to p, we have
$$w(t,x,p)\leq\tilde{w}(t,x,\Pi(p))+(k+1)|\Pi(p)-p|$$
with an equality for $p\in\Delta(\tilde{I})$, hence
$$w(t,x,p)\leq\phi(t,x,\Pi(p))+2(k+1)\langle\mu,p\rangle$$
with an equality only at $(\bar{t},\bar{x},\tilde{p})$, where we used
$$|\Pi(p)-p|\leq\sum_{j\in\tilde{I}}|\Pi(p)_j-p_j|+\sum_{j\notin\tilde{I}}p_j=(1+1/|\tilde{I}|)\sum_{j\notin\tilde{I}}p_j=2\langle\mu,p\rangle.$$

For $\epsilon>0$ small we now consider
$$\max_{(t,x,p)\in[0,T]\times\mathbb{R}^d\times\Delta(I)} w(t,x,p)-\phi_\epsilon(t,x,p) \tag{9.2}$$
with
$$\phi_\epsilon(t,x,p)=\phi(t,x,\Pi(p))+2(k+1)\langle\mu,p\rangle-\epsilon\sigma(p)$$
and $\sigma(p)=\sum_{j\notin\tilde{I}}\ln(p_i(1-p_i))$. For ϵ sufficiently small this problem has a maximum $(t_\epsilon,x_\epsilon,p_\epsilon)$ which converges to $(\bar{t},\bar{x},\tilde{p})$ as $\epsilon\downarrow 0$. By the definition of σ and the fact that $\tilde{p}\in\mathrm{Int}(\Delta(\tilde{I}))$ we have that $p_\epsilon\in\mathrm{Int}(\Delta(I))$. Hence by the subsolution property of w we have, that $\lambda_{\min}\left(p_\epsilon,\frac{\partial^2\phi_\epsilon}{\partial p^2}\right)(t_\epsilon,x_\epsilon,p_\epsilon)\geq 0$. Note that since Π is affine, $\Pi|_{\Delta(\tilde{I})}=\mathrm{id}$ and σ does not depend on p_i for $i\in\tilde{I}$, we have

$$\liminf_{\epsilon\downarrow 0}\lambda_{\min}\left(p_\epsilon,\tfrac{\partial^2\phi_\epsilon}{\partial p^2}\right)(t_\epsilon,x_\epsilon,p_\epsilon)$$
$$\leq\liminf_{\epsilon\downarrow 0}\min_{z\in T_{\Delta(\tilde{I})(\tilde{p})}\setminus\{0\}}\tfrac{\langle\partial^2\phi_\epsilon(t_\epsilon,x_\epsilon,p_\epsilon)z,z\rangle}{|z|^2}$$
$$\leq\liminf_{\epsilon\downarrow 0}\min_{z\in T_{\Delta(\tilde{I})(\tilde{p})}\setminus\{0\}}\tfrac{\langle\partial^2\phi_\epsilon(t_\epsilon,x_\epsilon,\Pi(p_\epsilon))z,z\rangle}{|z|^2}=\lambda_{\min}\left(\tilde{p},\tfrac{\partial^2\phi}{\partial p^2}\right)(\bar{t},\bar{x},\tilde{p}).$$

And since $\lambda_{\min}\left(p_\epsilon,\frac{\partial^2\phi_\epsilon}{\partial p^2}\right)(t_\epsilon,x_\epsilon,p_\epsilon)\geq 0$, we have
$$\lambda_{\min}\left(\tilde{p},\frac{\partial^2\phi}{\partial p^2}\right)(\bar{t},\bar{x},\tilde{p})\geq 0. \tag{9.3}$$

(ii) follows then by the subsolution property of w, i.e.

$$\max\Big\{\min\big\{(-\tfrac{\partial}{\partial t}-\mathcal{L})[\phi_\epsilon](t_\epsilon,x_\epsilon,p_\epsilon)),\\ \phi(t_\epsilon,x_\epsilon,p_\epsilon)-\langle f(t_\epsilon,x_\epsilon),p_\epsilon\rangle\big\},\phi(t_\epsilon,x_\epsilon,p_\epsilon)-\langle h(t_\epsilon,x_\epsilon),p_\epsilon\rangle\Big\}\leq 0 \tag{9.4}$$

by letting $\epsilon\downarrow 0$. □

9.2 Proof of Theorem 3.7

Let $w_1:[0,T]\times\mathbb{R}^d\times\Delta(I)\to\mathbb{R}$ be a bounded, continuous viscosity subsolution to (3.5), which is uniformly Lipschitz continuous in p, and $w_2:[0,T]\times\mathbb{R}^d\times\Delta(I)\to\mathbb{R}$ be a bounded, continuous viscosity supersolution to (3.5), which is uniformly Lipschitz continuous in p. Assume that

$$w_1(T,x,p)\leq w_2(T,x,p) \tag{9.5}$$

for all $x\in\mathbb{R}^d, p\in\Delta(I)$. We want to show that

$$w_1(t,x,p)\leq w_2(t,x,p) \tag{9.6}$$

for all $(t,x,p)\in[0,T]\times\mathbb{R}^d\times\Delta(I)$. As in [25] we prove (9.6) by induction over I. Indeed if $I=1$, (3.5) reduces to

$$\max\Big\{\min\big\{(-\tfrac{\partial}{\partial t}-\mathcal{L})[w],w-f_1(t,x)\big\},w-h_1(t,x)\Big\}=0, \tag{9.7}$$

where comparison is a classical result, see e.g. [57]. Assume that Theorem 3.7. holds for $I\in\mathbb{N}^*$. That means for $w_1,w_2:[0,T]\times\mathbb{R}^d\times\Delta(I+1)$ we have by Proposition 9.1. and 9.2. that

$$w_1(t,x,p)\leq w_2(t,x,p)\qquad\forall(t,x,p)\in[0,T]\times\mathbb{R}^d\times\partial(\Delta(I)). \tag{9.8}$$

We will show (9.6) by contradiction. Assume

$$M:=\sup_{(t,x,p)\in[0,T]\times\mathbb{R}^d\times\Delta(I))}(w_1-w_2)>0. \tag{9.9}$$

Since w_1 and w_2 are bounded we have for $\epsilon,\alpha,\eta>0$ that

$$M_{\epsilon,\alpha,\eta}:=\max_{(t,s,x,y,p)\in[0,T]^2\times\mathbb{R}^{2d}\times\Delta(I)}\Big\{w_1(t,x,p)-w_2(s,y,p)\\ -\tfrac{|t-s|^2+|x-y|^2}{2\epsilon}-\tfrac{\alpha}{2}(|x|^2+|y|^2)+\eta t\Big\} \tag{9.10}$$

is finite and achieved at a point $(\bar{t},\bar{s},\bar{x},\bar{y},\bar{p})$ (dependent on ϵ,α,η). Furthermore we have for the limit

$$\lim_{\epsilon,\alpha,\eta\downarrow 0}M_{\epsilon,\alpha,\eta}=\sup_{(t,x,p)\in[0,T]\times\mathbb{R}^d\times\Delta(I))}(w_1-w_2)=M>0. \tag{9.11}$$

With (9.5) and the Hölder continuity of w_1 and w_2 we have with (9.11) that $\bar{t},\bar{s}<T$ for ϵ,α,η small enough. Also note that $\bar{p}\in\mathrm{Int}(\Delta(I))$ as soon as $M_{\epsilon,\alpha,\eta}>0$.
We now consider a new penalization: For $\beta,\delta>0$ small

$$M_{\epsilon,\alpha,\eta,\delta,\beta}:=\max_{(t,s,x,y,p,q)\in[0,T]^2\times\mathbb{R}^{2d}\times\Delta(I)^2}\Big\{w_1(t,x,p)-w_2(s,y,p)\\ -\tfrac{|t-s|^2+|x-y|^2}{2\epsilon}-\tfrac{|p-q|}{2\delta}-\tfrac{\alpha}{2}(|x|^2+|y|^2)+\eta t+\tfrac{\beta}{2}(|p|^2+|q|^2)\Big\} \tag{9.12}$$

is attained at a point $(\tilde{t}, \tilde{s}, \tilde{x}, \tilde{y}, \tilde{p}, \tilde{q})$ (dependent on $\epsilon, \alpha, \eta, \delta, \beta$), where

$$\frac{|\tilde{t}-\tilde{s}|^2 + |\tilde{x}-\tilde{y}|^2}{2\epsilon}, \frac{|\tilde{p}-\tilde{q}|}{2\delta}, \alpha|\tilde{x}|^2, \alpha|\tilde{y}|^2, \beta|\tilde{p}|^2, \beta|\tilde{q}|^2 \leq 2(|w_1|_\infty + |w_2|_\infty). \tag{9.13}$$

Furthermore we have with (9.11)

$$w_1(\tilde{t}, \tilde{x}, \tilde{p}) - w_2(\tilde{s}, \tilde{y}, \tilde{q}) > 0. \tag{9.14}$$

So for $\beta, \delta \downarrow 0$ $(\tilde{t}, \tilde{s}, \tilde{x}, \tilde{y}, \tilde{p}, \tilde{q})$ converges (up to subsequences) to some $(\bar{t}, \bar{s}, \bar{x}, \bar{y}, \bar{p}, \bar{p})$, where $(\bar{t}, \bar{s}, \bar{x}, \bar{y}, \bar{p})$ is a maximum point of (9.12). Hence for β, δ sufficiently small we have that $\tilde{p}, \tilde{q} \in \text{Int}(\Delta(I))$.

From the usual maximum principle (see e.g. [29]) we have that:
for all $\sigma \in (0,1)$ there exist $X_1, X_2 \in S^d$, $P_1, P_2 \in S^I$ such that on $[0,T]^2 \times \mathbb{R}^{2d} \times T_I$ with $T_I = \{z \in \mathbb{R}^I : \sum_i z_i = 0\}$ we have

$$\left(\frac{\tilde{t}-\tilde{s}}{\epsilon} - \eta,\ \frac{\tilde{x}-\tilde{y}}{\epsilon} + \alpha\tilde{x},\ \frac{\tilde{p}-\tilde{q}}{\delta} - \beta\tilde{p},\ X_1,\ P_1|_{T_I}\right) \in \bar{\mathcal{D}}^{1,2,2,-} w_1(\tilde{t}, \tilde{x}, \tilde{p})$$

and

$$\left(\frac{\tilde{t}-\tilde{s}}{\epsilon},\ \frac{\tilde{x}-\tilde{y}}{\epsilon} - \alpha\tilde{y},\ \frac{\tilde{p}-\tilde{q}}{\delta} + \beta\tilde{q},\ X_2,\ P_2|_{T_I}\right) \in \bar{\mathcal{D}}^{1,2,2,+} w_2(\tilde{s}, \tilde{y}, \tilde{q})$$

with

$$\text{diag}\left(\begin{pmatrix} X_1 & 0 \\ 0 & -X_2 \end{pmatrix}, \begin{pmatrix} P_1|_{T_I} & 0 \\ 0 & -P_2|_{T_I} \end{pmatrix}\right) \leq A + \sigma A^2,$$

where

$$A = \text{diag}\left\{\frac{1}{\epsilon}\begin{pmatrix} \text{id}_d & -\text{id}_d \\ -\text{id}_d & \text{id}_d \end{pmatrix} + \alpha \text{id}_{2d}, \frac{1}{\delta}\begin{pmatrix} \text{id}_I & -\text{id}_I \\ -\text{id}_I & \text{id}_I \end{pmatrix} - \beta \text{id}_{2I}\right\}.$$

Note that

$$\begin{pmatrix} X_1 & 0 \\ 0 & -X_2 \end{pmatrix} \leq \left(\frac{1}{\epsilon} + 2\frac{\sigma}{\epsilon^2} + 2\frac{\alpha\sigma}{\epsilon}\right)\begin{pmatrix} \text{id}_d & -\text{id}_d \\ -\text{id}_d & \text{id}_d \end{pmatrix} + (\alpha + \alpha^2\sigma)\text{id}_{2d} \tag{9.15}$$

and

$$(P_1 - P_2)|_{T_I} \leq (-\beta + \sigma\beta^2)\text{id}_{2I}. \tag{9.16}$$

Since w_1 is a viscosity subsolution to (3.5) we have

$$\lambda_{\min}(\tilde{p}, P_1) \geq 0. \tag{9.17}$$

And since $\tilde{p} \in \text{Int}(\Delta(I))$, this yields with (9.16) to

$$\lambda_{\min}(\tilde{q}, P_2) > 0. \tag{9.18}$$

Furthermore since w_1 is a viscosity subsolution and w_2 is a viscosity supersolution we have

$$\begin{aligned} w_1(\tilde{t}, \tilde{x}, \tilde{p}) &\leq \langle h(\tilde{t}, \tilde{x}), \tilde{p}\rangle \\ w_2(\tilde{s}, \tilde{y}, \tilde{q}) &\geq \langle f(\tilde{s}, \tilde{y}), \tilde{q}\rangle, \end{aligned} \tag{9.19}$$

which yields for $\epsilon, \alpha, \eta, \delta, \beta$ small enough with (9.11)

$$\begin{aligned} w_1(\tilde{t}, \tilde{x}, \tilde{p}) &> \langle f(\tilde{t}, \tilde{x}), \tilde{p}\rangle \\ w_2(\tilde{s}, \tilde{y}, \tilde{q}) &< \langle h(\tilde{s}, \tilde{y}), \tilde{q}\rangle. \end{aligned} \tag{9.20}$$

So again using the subsolution property of w_1 and the supersolution property of w_2 we have with (9.20)

$$\begin{aligned} \tfrac{\tilde{t}-\tilde{s}}{\epsilon} - \eta + \tfrac{1}{2}\mathrm{tr}(aa^*(\tilde{t}, \tilde{x})X_1) + b(\tilde{t}, \tilde{x})\left(\tfrac{\tilde{x}-\tilde{y}}{\epsilon} + \alpha\tilde{x}\right) &\geq 0 \\ \tfrac{\tilde{t}-\tilde{s}}{\epsilon} + \tfrac{1}{2}\mathrm{tr}(aa^*(\tilde{s}, \tilde{y})X_2) + b(\tilde{t}, \tilde{x})\left(\tfrac{\tilde{x}-\tilde{y}}{\epsilon} - \alpha\tilde{y}\right) &\leq 0. \end{aligned} \tag{9.21}$$

Now using (9.15) and (9.16) in (9.21) yields a contradiction for ϵ, α, η sufficiently small as in the standard case (see [29]).

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