Construction of rational maps with prescribed dynamics

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Construction de fractions rationnelles à dynamique prescrite

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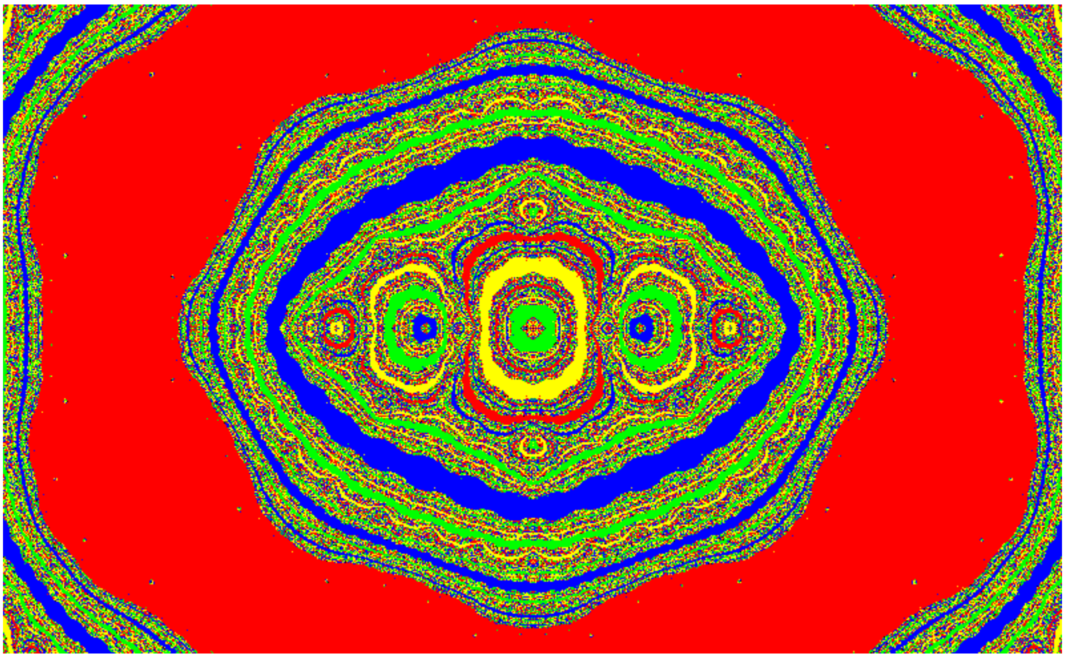
"Une activité intense, que ce soit à l'école ou à l'université, à l'église ou au marché, est le symptôme d'un manque d'énergie alors que la faculté d'être oisif est la marque d'un large appétit et d'une conscience aiguë de sa propre identité."

R. L. Stevenson


A Persian carpet

Construction de fractions rationnelles à dynamique prescrite

Résumé

Dans cette thèse, nous nous intéressons aux critères d'existence et à la construction effective de fractions rationnelles à dynamique prescrite. Nous commençons par étudier le même problème pour certains revêtements ramifiés post-critiquement finis et nous donnons une méthode de construction à partir de dynamiques d'arbres. Puis nous présentons un théorème de Thurston qui fournit une caractérisation combinatoire pour passer du cadre topologique au cadre analytique. En particulier, nous généralisons aux applications non post-critiquement finies un résultat de Levy qui simplifie le critère de Thurston dans le cas polynomial. Nous illustrons cette généralisation par une condition suffisante d'existence de polynômes ayant un disque de Siegel fixe de type borné. Ensuite nous détaillons la construction par chirurgie quasiconforme d'un exemple de fraction rationnelle non post-critiquement finie dont la dynamique est décrite par un arbre. Plus généralement, nous montrons qu'un résultat de Cui Guizhen et Tan Lei permet de construire une famille de fractions rationnelles à ensemble de Julia disconnexe à partir de certains arbres de Hubbard pondérés.

Construction of rational maps with prescribed dynamics

Abstract

In this thesis, we are interested in the existence criterions and the effective construction of rational maps with prescribed dynamics. We start by studying the same problem for some post-critically finite ramified coverings and we give a construction method from dynamical trees. Then we present a Thurston's theorem which provides a combinatorial characterization to go from the topological point of view to the analytical one. In particular, we generalize to non-post-critically finite maps a Levy's result which simplifies the Thurston's criterion in the polynomial case. We illustrate this generalization by a sufficient condition for existence of polynomials with a fixed Siegel disk of bounded type. Next we detail the construction by quasiconformal surgery of an example of non-post-critically finite rational map whose dynamics is described by a tree. More generally, we show that a result of Cui Guizhen and Tan Lei allows to construct a family of rational maps with disconnected Julia sets from some weighted Hubbard trees.

Contents

Chapter 1

Résumé français

1.2 Introduction

Nous commençons par introduire la théorie des systèmes dynamiques holomorphes et en particulier la définition des ensembles de Fatou et de Julia (définition 2.1). Le but n'est pas de refaire une étude systémique des concepts de base mais plutôt de citer quelques résultats essentiels qui motivent cette thèse. En particulier nous rappelons que si l'ensemble de Julia d'une fraction rationnelle de degré plus grand que deux n'est pas connexe alors il possède une infinité non dénombrable de composantes connexes (théorème 2.2). Nous nous intéressons ici en particulier à la dynamique d'échange des composantes de Julia. Par exemple nous rappelons un résultat classique (théorème 2.3) qui affirme que sous l'hypothèse que tous les points critiques sont capturés par le bassin immédiat d'attraction d'un même point fixe attractif alors l'ensemble de Julia est un ensemble de Cantor et la dynamique est celle du shift. Un des objectifs de cette thèse est d'obtenir un résultat similaire pour des ensembles de Julia plus compliqués. Nous présentons ainsi un exemple dû à C. T. McMullen (figure 2.2) où l'ensemble de Julia est homéomorphe au produit de l'ensemble triadique de Cantor par des courbes de Jordan et la dynamique d'échange des composantes de Julia est celle du shift (proposition 2.4). Enfin nous énonçons un résultat que nous démontrerons dans le chapitre 6 fournissant un autre exemple concret d'une fraction rationnelle à ensemble Julia disconnexe (figure 2.4) dont la dynamique d'échange d'une partie des composantes de Julia (qui sont toutes des courbes de Jordan sauf un nombre au plus dénombrable de préimages d'une composante fixe quasiconforme à l'ensemble de Julia connexe d'une autre fraction rationnelle) est conjuguée à l'action d'un polynôme quadratique sur un ensemble de Cantor formé par l'intersection de son arbre de Hubbard associé et de son ensemble de Julia.

1.3 Dynamique prescrite

L'objectif de ce chapitre est d'introduire les portraits de ramifications (définition 3.1) qui donne un cadre précis à la notion de dynamique prescrite. L'idée est de conserver l'information dynamique d'un revêtement ramifié (ou plus généralement d'une fraction rationnelle) sur son ensemble post-critique. Nous rappelons ensuite des propriétés et des définitions classiques autour de cette notion. Le concept le plus important est celui de la réalisation d'un portrait de ramifications (section 3.2) à l'aide de la relation d'équivalence de similarité (définition 3.9). Une dynamique prescrite sera donc vu comme la donnée d'une classe d'équivalence pour cette relation. Dans le cas des ensembles post-critiques infinis, nous introduisons aussi le concept de réalisation asymptotique (définition 3.12) qui nous permettra de ne conserver qu'un nombre fini d'informations dynamiques (celles portées par l'ensemble d'accumulation supposé fini de l'ensemble post-critique). Enfin nous terminons ce chapitre par la présentation de deux exemples représentatifs des deux sortes de problèmes (ou d'obstructions) qui peuvent survenir dans la question de savoir si un portrait de ramification est réalisé par une fraction rationnelle. Le premier (exemple 3.14) illustre les restrictions topologiques imposées par la formule de Riemann-Hurwitz (théorème A.12 en appendice) à l'aide d'un portrait de ramification pour lequel il n'existe même pas de revêtement ramifié qui le réalise. En particulier cet exemple motive le chapitre suivant où nous discuterons de la réalisation de portraits de ramifications par des revêtements ramifiés (un point de vue purement topologique). Le second exemple (exemple 3.15) présente un portrait de ramification que nous réaliserons par un revêtement ramifié noté f_{ana} au chapitre suivant. Nous ne présentons dans ce chapitre que la restriction à l'axe réel d'une telle application continue. D'autre part, nous verrons au chapitre 5 que f_{ana} ne pourra être "pertubée" afin d'obtenir une fraction rationnelle réalisant le même portrait de ramification. En particulier cet exemple motive le chapitre 5 où nous discuterons de la réalisation par des fractions rationnelles de portraits de ramifications déjà réalisés par des revêtements ramifiés (un point de vue purement analytique).

1.4 Arbres topologiques

L'objectif de ce chapitre est double : introduire les arbres dynamiques et démontrer le théorème de réalisation 4.12. Nous commençons donc par définir les arbres planaires (définition 4.1) que nous équipons ensuite de dynamiques (définition 4.3). Dans la section suivante, nous allons utiliser ces arbres afin de construire des revêtements ramifiés réalisant des portraits de ramifications particuliers : tous les points critiques sont périodiques et un point critique fixe joue le rôle du point à l'infini pour les polynômes (définition 4.9). Nous commençons (lemme 4.10) par réaliser les portraits de ramifications n'ayant qu'un seul cycle de points critiques (autre que le point à l'infini). L'idée est de partir d'un arbre étoilé dont la dynamique est la rotation autour de la racine puis de l'étendre en un graphe sur la sphère (le point à l'infini étant un sommet) dont les composantes connexes du complémentaire sont des disques topologiques. Nous pouvons alors définir des homéomorphismes au bord de ces composantes qui prolongent la dynamique de l'arbre initial. Ces homéomorphismes se prolongent à l'intérieur de chacune de ces composantes à l'aide du théorème de Schönflies (théorème A.3 en appendice). Il suffit ensuite de vérifier le dégré à l'infini du revêtement ramifié obtenu afin de prouver que le portrait de ramification initial est bien réalisé. Le lemme 4.11 est un rafinement du résultat précédent où nous augmentons l'arbre étoilé afin de construire en plus un point fixe du revêtement ramifié obtenu. Enfin le théorème 4.12 prouve le résultat pour un nombre quelconque de cycle de points critiques. La preuve consiste à recoller par leur point fixe les arbres étoilés de chacun des cycles de points critiques. Nous concluons ce chapitre en discutant d'une possible généralisation de cette méthode pour d'autres portraits de ramifications. Nous illustrons cette discussion par la construction d'un revêtement ramifié f_{ana} réalisant le portrait de ramification du dernier exemple du chapitre précédent (exemple 4.13).

1.5 Obstructions analytiques

Dans ce chapitre nous discutons de la réalisation par des fractions rationnelles de portraits de ramifications déjà réalisés par des revêtements ramifiés. L'outil principal est un théorème de Thurston qui charactérise les fractions rationnelles post-critiquement finies. Tout d'abord nous rapellons la définition de l'équivalence de Thurston (définition 5.2) qui donne le bon cadre pour la suite puis celle des obstructions de Thurston (définition 5.8). Nous énonçons ensuite le théorème de Thurston (théorème 5.9) qui charactérise les revêtements ramifiés post-critiquement finis dont la classe d'équivalence de Thurston contient une fraction rationnelle qui réalise donc le même portrait de ramifications. Nous discutons aussi de la difficulté de vérifier ce critère combinatoire malgré quelques tentatives de simplifications (proposition 5.12). Nous nous intéressons dans la section suivante (section 5.3) à la simplification de ce critère dans le cas polynomial grâce aux cycles de Levy. Nous démontrons ainsi une généralisation au cas non post-critiquement fini d'un résultat de S. V. F. Levy (théorème 5.17) qui nous permet en particulier d'énoncer un résultat positif à propos de la réalisation par des polynômes des portraits de ramifications considérés dans le chapitre précédent (corollaire 5.21). De plus cette simplification nous permet de montrer que le revêtement ramifié f_{ana} construit au chapitre précédent n'est pas équivalent au sens de Thurston à un polynôme (exemple 5.19). Nous concluons ce chapitre par une section indépendante du fil conducteur de cette thèse mais qui illustre l'intérêt du théorème 5.17. En combinant ce théorème avec un résultat de Zhang Gaofei (théorème 5.25) nous donnons un critère simple d'existence de polynômes ayant un disque de Siegel fixe de type borné (théorème 5.26) et dont la dynamique peut être prescrite par des dynamiques d'arbres comme au chapitre précédent (exemple 5.27).

1.6 D'un arbre à un tapis persan

Les deux chapitres suivants sont les plus novateurs de cette thèse. Nous commençons par poursuivre la conversation entamée au chapitre 4 à propos des arbres dynamiques afin d'introduire la notion d'arbres de Hubbard (définition 6.5 et exemple 6.6) un outil combinatoire capturant plus d'informations dynamiques que les portraits de ramifications. Nous définissons ensuite des arbres de Hubbards pondérés (définition 6.8) qui nous permettrons d'encoder la dynamique d'échange des composantes de Julia de certaines fractions rationnelles non post-critiquement finies sous l'hypothèse que ces arbres vérifient une condition combinatoire similaire à celle du critère de Thurston (définition 6.10). Dans la section suivante (section 6.2), nous détaillons minutieusement la construction par chirurgie quasiconforme d'une telle fraction rationnelle f dont la dynamique est "encodée" par un arbre de Hubbard pondéré $(\mathcal{H}, w)$. L'idée est de partir d'une fraction rationnelle post-critiquement finie $\widehat{f}$ dont la dynamique respecte celle d'un arbre $(\widehat{\mathcal{T}}, \widehat{w})$ déduit de $(\mathcal{H}, w)$ en supprimant son point de "pliage". Ensuite nous choisissons avec soin des equipotentielles dans le bassin immédiat d'attraction de $\widehat{f}$ (lemme 6.12) afin de découper la sphère en plusieurs morceaux sur lesquels nous allons définir une application quasirégulière F. Nous procédons pas à pas à cette définition. Le point le plus crucial est certainement la réalisation du "pliage" (étape 4) qui s'effectue à l'aide d'une application envoyant un anneau sur un disque topologique (figure 6.10), entraînant l'apparition d'un nouveau point critique. Nous prenons soin à ce que l'orbite de ce nouveau point critique accumule le cycle super-attractif issu de $(\mathcal{H}, w)$ afin de ne pas créer d'autre phénomène dynamique. Finalement (étape finale) nous montrons que la construction de l'application quasirégulière F suit un principe de chirurgie quasiconforme (théorème C.13 en appendice) et par conséquent F est quasiconformément conjuguée à une fraction rationnelle f comme nous le souhaitons. L'objectif de la section suivante (section 6.3) est de fournir une formule algébrique de l'application f_p construite précédemment vue comme une famille de fractions rationnelles dépendante d'un paramètre complexe p correspondant à la position d'un des points du cycle super-attractif (le point de "pliage"). Nous donnons également l'expression du point critique p' dont l'orbite accumule le cycle super-attractif. Nous pouvons ainsi produire plusieurs images numériques : tout d'abord le plan des paramètres (figure 6.14) afin de choisir p dans une petite composante hyperbolique réalisant la dynamique souhaitée (figure 6.15) et ensuite l'ensemble de Julia correspondant (figure 6.16) appelé un tapis persan. Nous en profitons pour citer un résultat de Tan Lei et K. Pilgrim (théorème 6.14) qui décrit la géométrie d'un grand nombre des composantes de Julia. Dans la section suivante (section 6.4) nous justifions la condition combinatoire (similaire à celle du critère de Thurston) vérifiée par l'arbre de Hubbard pondéré $(\mathcal{H}, w)$ en produisant un contre-exemple qui ne vérifie pas cette condition. Enfin nous concluons ce chapitre en décrivant comment la dynamique d'échange des composantes de Julia induite par notre application f construite précédemment est encodé par $(\mathcal{H}, w)$. Nous démontrons en particulier le théorème 6.19 énoncé en introduction.

1.7 Une collection de tapis persans

Dans ce chapitre nous discutons d'une généralisation possible de la construction du chapitre précédent. Nous commençons par définir les arbres de Hubbard pondérés dont nous allons réaliser le portrait de ramifications associé (définition 7.1). Ensuite nous construisons de nouveau pas à pas une application F. En particulier le lemme 7.2 nous permet de généraliser l'application réalisant le "pliage" du chapitre précédent. Cependant nous serons moins exigeant ici qu'au chapitre précédent quant à la régularité de F. En effet, au lieu de conclure par un principe de chirurgie quasiconforme, nous allons utiliser un résultat de Cui Guizhen et Tan Lei (théorème 7.5) qui généralise le théorème de Thurston à certains revêtements ramifiés non post-critiquement finis. Finalement nous obtenons un théorème de réalisation asymptotique de certains portraits de ramifications par des fractions rationnelles à ensemble de Julia disconnexe (théorème 7.6). Nous illustrons ce résultat dans la section suivante (section 7.2) à l'aide de la formule algébrique d'une famille de fractions rationnelles h_e à un paramètre complexe e correspondant à une extrémité de l'arbre de Hubbard pondéré considéré. En choisissant ce paramètre au centre des composantes hyperboliques d'une copie de l'ensemble de Mandelbrot (figure 7.5) nous produisons plusieurs exemples d'ensembles de Julia dont la dynamique d'échange des composante est encodée par des arbres pondérés (figure 7.4, figure 7.6 et figure 7.7). Le choix d'un paramètre correspondant à un point de Misiurewicz de la copie de l'ensemble de Mandelbrot permet même d'obtenir d'autres exemples qui ne sont pas couverts par notre résultat (figure 7.8).

1.8 Conclusion

Finalement nous concluons cette thèse par quelques questions soulevées par ces travaux et par des projets futurs. Tout d'abord nous envisageons un raffinement du théorème 6.18 afin d'étendre continuement la conjugaison sur les composantes de Julia à toute la sphère de Riemann. Nous proposons une méthode à l'aide du résultat de Cui Guizhen et Tan Lei (théorème 7.5) discuté au chapitre précédent. Cette méthode permet aussi d'espérer un encodage pour les applications produites par le théorème 7.6. Ensuite nous discutons d'une généralisation possible de la construction du chapitre précédent à des arbres de Hubbard pondérés plus généraux que ceux de la définition 7.1. Enfin nous soulevons des questions à plus long terme à propos de l'unicité (à conjugaison quasiconforme près) de fractions rationnelles encodées par un même arbre de Hubbard pondéré et également à propos du problème inverse, à savoir déduire la structure d'arbre derrière la dynamique d'échange des composantes de Julia d'une fraction rationnelle donnée.

Chapter 2

Introduction

Let $\widehat{\mathbb{C}}$ be the Riemann sphere. We will use $\mathbb{S}^2$ when we wish to think topologically (that is the one-point compactification of the complex plane $\mathbb{C}$) and $\widehat{\mathbb{C}}$ when we wish to think analytically (emphasizing the complex structure as one-dimensional complex manifold). Recall that the set of holomorphic maps on $\widehat{\mathbb{C}}$ is equal to the set of rational maps, that is the set of ratios of polynomials with complex coefficients.

Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational map. The aim of holomorphic dynamical systems theory is to study the forward orbit under iterations by f of every starting point $z_0 \in \widehat{\mathbb{C}}$:

$$z_0 \overset{f}{\mapsto} z_1 \overset{f}{\mapsto} z_2 \overset{f}{\mapsto} z_3 \overset{f}{\mapsto} \dots$$

To do so, the Riemann sphere is divided in two sets of starting points which lead to two different kinds of behavior. These sets are named after two mathematicians whose works flowered the global study of holomorphic dynamical systems during the early 20^{th} century.

Definition 2.1 (Fatou and Julia sets)**.** Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational map. The **Fatou set** of f, denoted by $\boldsymbol{\mathcal{F}(f)}$, is the domain of normality for the collection of iterates $\{f^{\circ n} \,/\, n \geqslant 1\}$, that is the set of $z \in \widehat{\mathbb{C}}$ which admits a neighbourhood $V_z \subset \widehat{\mathbb{C}}$ such that the sequence $(f^{\circ n}_{|V_z})_{n \geqslant 1}$ of holomorphic maps has subsequence which converges uniformly on compact subsets of V_z. The **Julia set** of f, denoted by $\boldsymbol{\mathcal{J}(f)}$, is the complement in $\widehat{\mathbb{C}}$ of the Fatou set.

$$\mathcal{J}(f) = \widehat{\mathbb{C}} - \mathcal{F}(f)$$

Many authors present the basics to study the geometrical and dynamical properties of those two sets, see in particular [Bea91], [CG93], [BM01] or [Mil06]. We would like here to focus on some results which motivate this thesis.

Theorem 2.2. *For any rational map $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ of degree two or more, the Julia set $\mathcal{J}(f)$ is a nonempty fully invariant compact set without isolated point. Furthermore*

- *either $\mathcal{J}(f)$ is connected,*
- *or else $\mathcal{J}(f)$ has uncountably many connected components.*

Figure 2.1 illustrates this dichotomy.

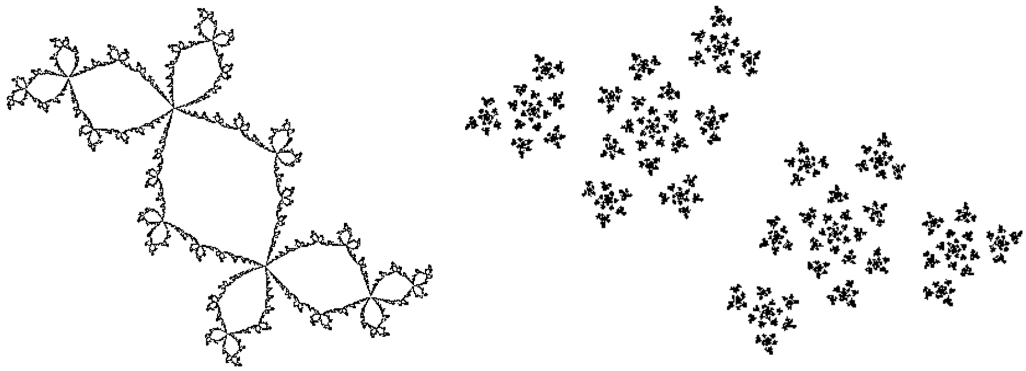


Figure 2.1: Two Julia sets of quadratic polynomials: one connected called the Douady's rabbit and one totally disconnected

We are going to discuss the second case and particularly the exchanging dynamics of Julia components (that is the induced dynamical system on the set of every connected components of the Julia set).

In this way, we are going to state at first a classical result (see [Bea91]). Recall that a fixed point $z_0 \in \widehat{\mathbb{C}}$ of a rational map $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is said attracting if it satisfies $|f'(z_0)| < 1$ (or $|\lim_{z\to\infty} \frac{1}{f'(z)}| < 1$ if $z_0 = \infty$). In that case we define the immediate attracting bassin of z_0 to be the connected component containing z_0 of the set of points whose forward orbits accumulate z_0. Recall also that a Cantor set is a nonempty compact set which is perfect (without isolated point) and totally disconnected (each connected component is a single point). For instance the second Julia set in Figure 2.1 is a Cantor set (actually it is a consequence of the following result).

Theorem 2.3. *Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational map of degree $d \geqslant 2$. If there exists an attracting fixed point z_0 of f such that every critical point of f lies in the immediate attracting bassin of z_0 then $\mathcal{J}(f)$ is a Cantor set. More precisely there exists a homeomorphism $\varphi : \mathcal{J}(f) \to \Sigma_d$ such that the following diagram commutes*

$$\begin{array}{ccc} \mathcal{J}(f) & \xrightarrow{\ f\ } & \mathcal{J}(f) \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\varphi} \\ \Sigma_d & \xrightarrow[\sigma]{} & \Sigma_d \end{array}$$

where

- $\Sigma_d = \{1, 2, \dots, d\}^{\mathbb{N}}$ *is a Cantor set for the metric*

$$\forall \varepsilon, \varepsilon' \in \Sigma_d, \quad d_{\Sigma_d}(\varepsilon, \varepsilon') = \sum_{i=0}^{+\infty} \frac{|\varepsilon_i - \varepsilon'_i|}{(d+1)^i}$$

- $\sigma : \Sigma_d \to \Sigma_d$ *is the shift map that is* $\forall \varepsilon \in \Sigma_d$, $\sigma(\varepsilon_0\varepsilon_1\varepsilon_2 \dots) = \varepsilon_1\varepsilon_2\varepsilon_3 \dots$

Notice that the assumption is satisfied for quadratic polynomials of the form $z \mapsto z^2 + c$ with a sufficiently large value of c (like the second Julia set in Figure 2.1).

This result allows to understand the entire exchanging dynamics of Julia components in that case. For instance, if follows easily that the periodic points are dense in the Julia set or that there is a dense set of points in the Julia set whose forward orbits are dense in the Julia set.

We would like to obtain a similar result for Julia sets with more complicated components than single points. Unfortunatly we do not know several examples of such Julia sets. Indeed B. Branner and J. H. Hubbard proved in [BH92] that in polynomial case, all but countably many Julia components must be single points. This is certainly not true for arbitrary rational maps but we will see later (see Theorem 6.14) that Tan Lei and K. Pilgrim proved in [PT00] that under a certain condition, all but countably many Julia components must be either a point or a Jordan curve.

The example in Figure 2.2, due to C. T. McMullen, has uncountably many connected components which are Jordan curves (see [Bea91] or [Mil06]). Roughly speaking, this Julia set is a Cantor set of circles (denoted by “CoC” in shorter) that is like the set $\bigcup_{r \in \mathcal{C}} \{z \in \mathbb{C} \,/\, |z| = r\}$ where $\mathcal{C}$ is some Cantor set on the positive real line.

Actually we can describe the entire exchanging dynamics of Julia components for this example (see [Bea91]). We denote by f_{CoC} the rational map

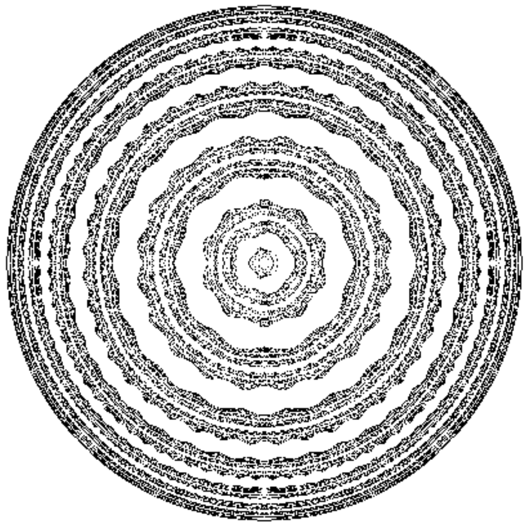


Figure 2.2: Example of Julia set which is a Cantor set of circles

whose Julia set is drawn in Figure 2.2 and by $\mathcal{J}_{\mathrm{CoC}}$ the set of Julia components equipped with the Hausdorff topology coming from the following Hausdorff metric:

$$\forall J, J' \in \mathcal{J}_{\mathrm{CoC}},\ d_{\mathrm{H}}(J, J') = \max\left\{\sup_{z\in J}\inf_{z'\in J'}|z-z'|, \sup_{z'\in J'}\inf_{z\in J}|z-z'|\right\}$$

Then f_{CoC} induces on $\mathcal{J}_{\mathrm{CoC}}$ a continuous dynamical system denoted also by $f_{\mathrm{CoC}} : \mathcal{J}_{\mathrm{CoC}} \to \mathcal{J}_{\mathrm{CoC}}$.

Proposition 2.4. *There exists a homeomorphism $\phi : \mathcal{J}_{CoC} \to \Sigma_2$ such that the following diagram commutes*

$$\begin{array}{ccc} \mathcal{J}_{CoC} & \xrightarrow{f_{CoC}} & \mathcal{J}_{CoC} \\ \phi\downarrow & & \downarrow\phi \\ \Sigma_2 & \xrightarrow[\sigma]{} & \Sigma_2 \end{array}$$

Recall that the Cantor ternary set Σ_2 may be seen as the non-escaping set of a continuous dynamical system on the unit segment $[0, 1]$:

$$\begin{array}{rcl} \tau_{\mathrm{C}} : [0,1] & \to & [0,1] \\ x & \mapsto & \begin{cases} 3x & \text{if } x \in [0, \frac{1}{2}] \\ 3(1-x) & \text{if } x \in [\frac{1}{2}, 1] \end{cases} \end{array}$$

Σ_2 is homeomorphic to the following non-escaping set

$$\mathcal{J}_{\mathrm{C}} = \left\{ x \in [0,1] \,/\, \forall n \geqslant 0,\ \tau_{\mathrm{C}}^{\circ n}(x) \in \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right] \right\}$$

We may thus reformulate Proposition 2.4 as follows:

Proposition 2.5. *There exists a homeomorphism* $\varphi : \mathcal{J}_{CoC} \to \mathcal{J}_C$ *such that the following diagram commutes*

$$\begin{array}{ccc} \mathcal{J}_{CoC} & \xrightarrow{f_{CoC}} & \mathcal{J}_{CoC} \\ \varphi \downarrow & & \downarrow \varphi \\ \mathcal{J}_C & \xrightarrow[\tau_C]{} & \mathcal{J}_C \end{array}$$

Any Julia component $J \in \mathcal{J}_{CoC}$*, that is any any preimage by* φ *of a point in* $\mathcal{J}_C$*, is a Jordan curve.*

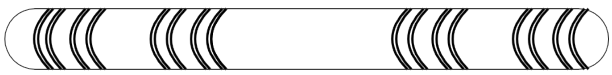


Figure 2.3: A "thickening" of $\mathcal{J}_{\mathrm{C}} \subset [0,1] \hookrightarrow \mathbb{R}^3$
(compare with Figure 2.2)

Heuristically speaking, we may think of the action of f_{CoC} as that one of τ_{C} on the boundary (homeomorphic to the sphere $\mathbb{S}^2$) of a small "thickening" of the unit segment $[0,1]$ embedded in $\mathbb{R}^3$ as it is suggested in Figure 2.3. Indeed f_{CoC} is of the form $z \mapsto z^2 + \lambda/z^3$ where λ is sufficiently small and a study of Fatou components (see [Bea91]) shows that the immediate attracting bassin F_∞ of ∞ is simply connected (that corresponds to a neighbourhood of 0 for τ_{C}), the preimage F_0 of F_∞ contains the other critical points and is also simply connected (that corresponds to a neighbourhood of 1 for τ_{C}) and all other Fatou components are doubly connected.

In this thesis, we would like to construct some other examples of rational maps with complicated Julia sets and whose dynamics is encoded by some combinatorial data. To do so, we will discuss the construction of rational maps with prescribed dynamics. The two main tools in this discussion will be the quasiconformal surgery method and the Thurston's characterization of post-critically finite rational maps.

In particular, we will get the following concrete example (see Chapter 6).

Theorem 2.6. *There exists a rational map $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ whose Julia set is drawn in Figure 2.4 and a homeomorphism φ from a subset $\mathcal{J}_{\mathcal{H}}(f)$ of Julia components of f equipped with a certain topology such that the following diagram commutes*

$$\begin{array}{ccc} \mathcal{J}_{\mathcal{H}}(f) & \xrightarrow{\ f\ } & \mathcal{J}_{\mathcal{H}}(f) \\ \varphi \downarrow & & \downarrow \varphi \\ \mathcal{J}_{\mathcal{H}} & \xrightarrow[P_c]{} & \mathcal{J}_{\mathcal{H}} \end{array}$$

where $\mathcal{J}_{\mathcal{H}}$ is a Cantor set which is the intersection between the Julia set of a quadratic polynomial $P_c : z \mapsto z^2 + c$ (actually $c \approx$-0.157+1.032i) and the associated Hubbard tree (see Example 6.6).

Moreover there exists only one fixed Julia component in $\mathcal{J}_{\mathcal{H}}(f)$ denoted by J_α (which is quasiconformally mapped onto the connected Julia set of an other rational map) and any Julia component $J \in \mathcal{J}_{\mathcal{H}}(f)$ which is not mapped after some iterations of f onto J_α is a Jordan curve.

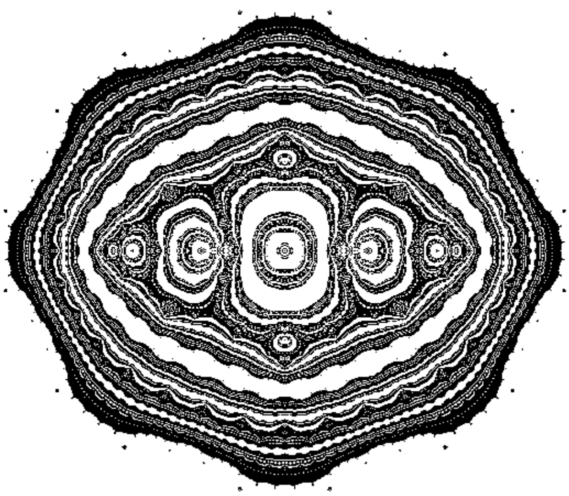


Figure 2.4: A disconnected Julia set

In particular that proves there are some Julia components of f which does not meet the boundary of any Fatou components (for instance J_α and all of their preimages).

Chapter 3

Prescribed dynamics

We are going to give some meanings behind what is prescribed dynamics. The aim is to impose the action of a rational map, or more generally of a ramified covering, on its critical and post-critical sets. To do so, we will introduce the ramification portraits and some related definitions (following [Koc07] and according to works in [BFH92]). Then we will begin to discuss the question of finding rational maps of prescribed dynamics. The arising problems should lead to the two next chapters.

3.1 Ramification portraits

We first establish some standard notations and definitions. To each ramified covering $f : \mathbb{S}^2 \to \mathbb{S}^2$ we denote by $\Omega_f = \{\text{critical points of } f\}$ its critical set and by $P_f = \bigcup_{n\geqslant 1} f^{\circ n}(\Omega_f)$ its post-critical set. The action of f on $\Omega_f \cup P_f$ is totally encoded by the surjective restriction $\sigma_f = f_{|\Omega_f\cup P_f} : \Omega_f \cup P_f \to P_f$ and the local degree $\nu_f = \deg_{\text{loc}}(f)_{|\Omega_f\cup P_f} : \Omega_f \cup P_f \to \mathbb{N} - \{0\}$. That leads to the following definition (according to [Koc07]).

Definition 3.1 (ramification portrait)**.** A **ramification portrait** is the data of

- a finite set $\Omega \subset \mathbb{S}^2$
- a set (not necessarily finite or disjoint from Ω) $P \subset \mathbb{S}^2$
- a surjective map $\sigma : \Omega \cup P \to P$
- a function $\nu : \Omega \cup P \to \mathbb{N} - \{0\}$ such that $\nu^{-1}(\{n \in \mathbb{N}/n \geqslant 2\}) = \Omega$

We denote by $\boldsymbol{\mathcal{R} = (\Omega, P, \sigma, \nu)}$ such a ramification portrait.

Remark that the critical set of a ramified covering $f : \mathbb{S}^2 \to \mathbb{S}^2$ is well finite since the critical points of f are isolated points of the compact set $\mathbb{S}^2$. In particular, f is of finite degree $\deg(f)$ and it has exactly $2\deg(f) - 2$ critical points, counted with multiplicity, by the Riemann-Hurwitz formula (see Theorem A.12 in appendix).

Definition 3.2 (degree of a ramification portrait)**.** The **degree** of a ramification portrait $\mathcal{R} = (\Omega, P, \sigma, \nu)$, denoted by $\boldsymbol{\deg(\mathcal{R})}$, is the number

$$\deg(\mathcal{R}) = 1 + \frac{1}{2}\sum_{\omega\in\Omega} (\nu(\omega) - 1)$$

Obviously we have:

Proposition 3.3. *If f is a ramified covering of critical set Ω_f and post-critical set P_f then $(\Omega_f, P_f, \sigma_f = f_{|\Omega_f\cup P_f}, \nu_f = \deg_{loc}(f)_{|\Omega_f\cup P_f})$ is a ramification portrait and* $\deg(\mathcal{R}_f)$ *is equal to the degree* $\deg(f)$ *of f.*

Definition 3.4 (ramification portrait of a ramified covering)**.** We denote by $\boldsymbol{\mathcal{R}_f = (\Omega_f, P_f, \sigma_f = f_{|\Omega_f\cup P_f}, \nu_f = \deg_{\mathrm{loc}}(f)_{|\Omega_f\cup P_f})}$ the ramification portrait associated to a ramified covering $f : \mathbb{S}^2 \to \mathbb{S}^2$. In case $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ would be a rational map, we identify $\mathbb{S}^2$ with $\widehat{\mathbb{C}}$ as topological manifolds in order to define the associated ramification portrait of f as well.

Notice that the degree of a ramification portrait $\mathcal{R} = (\Omega, P, \sigma, \nu)$ is not necessarily an integer, indeed $\deg(\mathcal{R}) \in \frac{1}{2}\mathbb{Z}$. Moreover the number of preimages by σ of a point in P may be larger than the degree $\deg(\mathcal{R})$. So we will restrict our discussion to a special subset of ramification portraits defined below.

Definition 3.5 (branch compatible ramification portrait)**.** A ramification portrait $\mathcal{R} = (\Omega, P, \sigma, \nu)$ is said **branch compatible** if

$$\forall\omega \in P, \sum_{\substack{x\in\Omega\cup P\\ \sigma(x)=\omega}} \nu(x) \leqslant \deg(\mathcal{R})$$

Definition 3.6 (ramification portrait of polynomial type)**.** $\mathcal{R} = (\Omega, P, \sigma, \nu)$ is a ramification portrait of **polynomial type** if

(i) $\mathcal{R}$ is branch compatible

(ii) $\exists\omega \in \Omega \cup P/\, \sigma(\omega) = \omega$ and $\nu(\omega) = \deg(\mathcal{R})$

A point which satisfies the second condition is called an **infinity point** of the ramification portrait $\mathcal{R}$.

Remark that an infinity point is necessarily in Ω as soon as $\deg(\mathcal{R}) \geqslant 2$. We enlarge the definition of ramification portrait of polynomial type from [Koc07] to allow ramification portraits similar to those of polynomials $z \mapsto z^d$, $d \geqslant 2$.

Proposition 3.7. *Let $\mathcal{R} = (\Omega, P, \sigma, \nu)$ be a ramification portrait of polynomial type. We have the following properties.*

1. *If c_∞ denotes an infinity point then*

$$\deg(\mathcal{R}) = 1 + \sum_{\omega \in \Omega - \{c_\infty\}} (\nu(\omega) - 1)$$

   In particular the degree of $\mathcal{R}$ is a positive integer.

2. *If $\deg(\mathcal{R}) \geqslant 2$ then there exist at most two infinity points $c_\infty \in \Omega$. These points also satisfy $\sigma^{-1}(c_\infty) = \{c_\infty\}$.*

Proof of Proposition 3.7. The first statement is obvious since $\nu(c_\infty) = \deg(\mathcal{R})$. If there exist at least three points $c \in \Omega$ such that $\nu(c) = \deg(\mathcal{R})$ then

$$\deg(\mathcal{R}) = 1 + \frac{1}{2} \sum_{\omega \in \Omega} (\nu(\omega) - 1) \geqslant 1 + \frac{3}{2} (\deg(\mathcal{R}) - 1)$$

and this inequality is equivalent to $\deg(\mathcal{R}) \leqslant 1$. Assume now that there exists $x_\infty \in \Omega \cup P - \{c_\infty\}$ such that $\sigma(x_\infty) = c_\infty$ where $c_\infty \in \Omega$ is an infinity point. We get

$$\sum_{\substack{x \in \Omega \cup P \\ \sigma(x) = c_\infty}} \nu(x) \geqslant \nu(x_\infty) + \nu(c_\infty) \geqslant 1 + \deg(\mathcal{R})$$

contradicting the fact that $\mathcal{R}$ is branch compatible. □

3.2 Realization

Example 3.8. Let $\mathcal{R}$ be a ramification portrait of polynomial type displayed below.

$$c_0 \xrightarrow{2} c_1 \xrightarrow{1} c_2 \circlearrowleft 1 \qquad\qquad c_\infty \circlearrowleft 2$$

In this example, $\Omega = \{c_0, c_\infty\}$ and $P = \{c_1, c_2, c_\infty\}$. The arrows above depict the map $\sigma : \Omega \cup P \to P$. To each $\omega \in \Omega \cup P$ the integer $\nu(\omega)$ is assigned to the arrow leaving from ω: $\nu(c_0) = \nu(c_\infty) = 2$ and $\nu(c_1) = \nu(c_2) = 1$.

To find a polynomial f (f acts on $\widehat{\mathbb{C}}$ but we identify $\widehat{\mathbb{C}}$ to $\mathbb{S}^2$) such that $\mathcal{R}_f = \mathcal{R}$ implies at first that $\deg(f) = \deg(\mathcal{R}) = 2$ and $c_\infty = \infty$. Hence f is a quadratic polynomial. From the informations $f'(c_0) = 0$, $f(c_0) = c_1$ and $f(c_1) = c_2$ we get that the form of f is $f(z) = \frac{(c_2-c_1)}{(c_1-c_0)^2}(z - c_0)^2 + c_1$. The last information $f(c_2) = c_2$ gives an algebraic relation linking c_0, c_1 and c_2 that is $c_0 = \frac{c_1+c_2}{2}$. Finally there exists a polynomial of associated ramification portrait $\mathcal{R}$ if and only if $c_\infty = \infty$ and $c_0 = \frac{c_1+c_2}{2}$. For instance we get $z \mapsto z^2 - 2$ for $c_0 = 0$, $c_1 = -2$ and $c_2 = 2$.

As it is shown in the example above, to find a polynomial of given associated ramification portrait may be not possible. Fixing the position of critical and post-critical points on the sphere $\mathbb{S}^2$ is too restrictive.

Definition 3.9 (similar ramification portraits)**.** Two ramification portraits $\mathcal{R} = (\Omega, P, \sigma, \nu)$ and $\mathcal{R}' = (\Omega', P', \sigma', \nu')$ are said **similar** if there exists a bijection $\beta : \Omega \cup P \to \Omega' \cup P'$ such that the following two diagrams commute:

$$\begin{array}{ccc} \Omega \cup P & \xrightarrow{\beta} & \Omega' \cup P' \\ \sigma \downarrow & & \downarrow \sigma' \\ P & \xrightarrow{\beta_{|P}} & P' \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \Omega \cup P & \xrightarrow{\beta} & \Omega' \cup P' \\ & \nu \searrow \quad \swarrow \nu' & \\ & \mathbb{N} - \{0\} & \end{array}$$

We will write $\boldsymbol{\mathcal{R} \sim_{\mathrm{sim}} \mathcal{R}'}$ in this case.

Proposition 3.10. *$\sim_{sim}$ is an equivalence relation on the set of all ramification portraits.*

Proof of Proposition 3.10. Only the symmetry is not trivial. It's enough to check that the map $\beta_{|P_1} : P_1 \to P_2$ in Definition 3.9 is a bijection. At first $\beta_{|P_1}$ is an injective map as a restriction of a bijection. Surjectivity comes from the commutativity of the diagram: $\beta_{|P_1} \circ \sigma_1 = \sigma_2 \circ \beta$ where σ_2 and β are surjective maps. $\square$

Definition 3.11 (realization of a ramification portrait)**.** We say that a ramified covering $f : \mathbb{S}^2 \to \mathbb{S}^2$ (or a rational map $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$) **realizes** a ramification portrait $\mathcal{R}$ if $\mathcal{R}_f \sim_{\text{sim}} \mathcal{R}$.

For instance, the polynomial $z \mapsto z^2 - 2$ realizes the ramification portrait of Example 3.8 for any choice of points c_0, c_1, c_2 and c_∞ on $\mathbb{S}^2$.

So we may consider a prescribed dynamics as an equivalence class of $\sim_{\text{sim}}$. That does not provide a combinatorial tool which captures the whole dynamical information about a ramified covering and we will discuss later some attempts to make the meaning behind prescribed dynamics sharper. But for the moment, to find a rational map with prescribed dynamics means to realize a ramification portrait by a rational map.

However we will need a less restrictive definition in case of infinite ramification portrait in order to deal with only a finite number of informations.

Definition 3.12 (asymptotic realization)**.** Let $f : \mathbb{S}^2 \to \mathbb{S}^2$ be a ramified covering (or more generally let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational map). We denote by Ω'_f the set of critical points with finite forward orbit under iteration of f and by $\mathcal{R}'_f$ the induced ramification portrait:

- $\Omega'_f = \{\omega \in \Omega_f \,/\, |\bigcup_{n\geqslant 1} f^{\circ n}(\omega)| < \infty\}$
- $\mathcal{R}'_f = (\Omega'_f, P'_f = \bigcup_{n\geqslant 1} f^{\circ n}(\Omega'_f), \sigma'_f = f_{|\Omega'_f \cup P'_f}, \nu'_f = \deg_{\text{loc}}(f)_{|\Omega'_f \cup P'_f})$

Then we say that f **realizes asymptotically** a finite ramification portrait $\mathcal{R}$ if the following conditions hold:

(i) $\mathcal{R}'_f \sim_{\text{sim}} \mathcal{R}$

(ii) the accumulation set of the post-critical set of f is contained in $\Omega'_f \cup P'_f$, in other words the orbit $\bigcup_{n\geqslant 1} f^{\circ n}(\omega)$ of every critical point $\omega \in \Omega_f - \Omega'_f$ accumulates a periodic cycle in $\Omega'_f \cup P'_f$

Example 3.13. Consider a quadratic polynomial P_c of the form $z \mapsto z^2 + c$ with a sufficiently large value of c. Then P_c realizes asymptotically the ramification portrait below

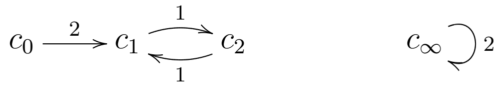
$$\infty \circlearrowright 2 \qquad\qquad 0 \overset{2}{\leadsto} c$$

where the waved arrow means that the critical point 0 is mapped in the immediate attracting bassin of ∞.

3.3 Examples

Unfortunately not every ramification portrait can be realized by rational maps. We will see two examples of such ramification portraits which are characteristic of the two kinds of problems (or obstructions) that could arise.

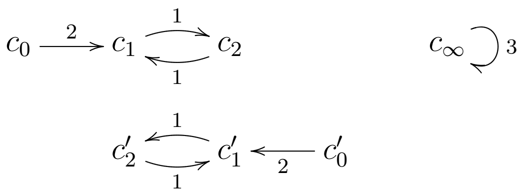
Example 3.14 (Topological obstruction)**.** Let $\mathcal{R}_{\mathrm{top}}$ be the ramification portrait below.



If a rational map realizes this ramification portrait, this map would be of degree $\deg(\mathcal{R}_{\mathrm{top}}) = 2$. But there are at least three preimages of c_1, counted with multiplicity (c_2 of multiplicity 1 and c_0 of multiplicity 2), which is impossible for a map of degree two. In fact $\mathcal{R}_{\mathrm{top}}$ is not branch compatible.

Indeed the Riemann-Hurwitz formula (see Theorem A.12 in appendix) provides many restrictions on the ramification portrait of a ramified covering. If a ramification portrait violates these restrictions, it cannot be realized by any ramified covering (and in particular by any rational map). Consequently our first objective will be to discuss the realization of ramification portraits by ramified coverings (that is a purely topological discussion).

Example 3.15 (Analytical obstruction)**.** Let $\mathcal{R}_{\mathrm{ana}}$ be the ramification portrait of polynomial type below.



This example seems to have no topological obstruction as the previous example. Each point of the post-critical set has less than three preimages, counted with multiplicity, and $\deg(\mathcal{R}_{\mathrm{ana}}) = 3$. Indeed we will see in the next chapter that $\mathcal{R}_{\mathrm{ana}}$ can be realized by a ramified covering f_{ana}. For the moment, identifying $\mathbb{S}^2 - \{c_\infty\}$ with the complex plane $\mathbb{C}$ and assuming that the critical and post-critical sets belong to the real line $\mathbb{R}$, we can at least construct the restriction $f_{\mathrm{ana}|\mathbb{R}}$ on the real axis (see Figure 3.1): c_0 and c_0' are the only points where $f_{\mathrm{ana}|\mathbb{R}}$ is not locally injective (the "critical points"), even where $f_{\mathrm{ana}|\mathbb{R}}$ is locally two-to-one (of "local degree two") and the action of $f_{\mathrm{ana}|\mathbb{R}}$ on the forward orbits of c_0 and c_0' is the same, up to composition by a bijection, as the one of σ_{ana}.

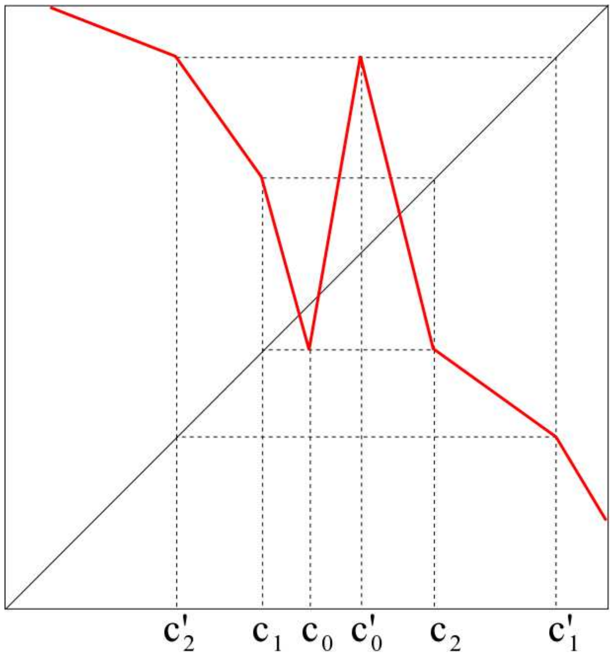


Figure 3.1: A continuous and piecewise affine map $f_{\text{ana}|\mathbb{R}}$ which realizes $\mathcal{R}_{\text{ana}}$

But we will see (in Chapter 5) that f_{ana} cannot be "perturbed" in order to get a polynomial! In that case the obstruction is not topological (since $\mathcal{R}_{\text{ana}}$ can be realized by a ramified covering) but it comes from the analytical structure we equip the sphere $\mathbb{S}^2$ with. Therefore our second objective will be to discuss the realization by some rational maps of ramification portraits already realized by some ramified coverings. We will forget the topological obstructions to concentrate on the analytical part of the problem.

Chapter 4

Topological trees

In this chapter, we would like to concentrate on the realization of finite ramification portraits by some ramified coverings. To do so, we introduce dynamical trees and some related definitions (following [Poi93]) to be also useful for discussions coming later (Chapter 6). Then we will construct ramified coverings realizing some particular ramification portraits and we will discuss the general case with an example.

4.1 Dynamical trees

Definition 4.1 (planar tree)**.** A **planar tree** is a finite connected acyclic planar graph $\boldsymbol{T = (V, E)}$, that is the data of

- a finite set $V \subset \mathbb{C}$ of vertices
- a finite set E of pairwise disjoint (except possibly at the ends) Jordan arcs e_{v_1,v_2}, called edges, between some unordered pairs of distinct vertices $v_1, v_2 \in V$

such that for any pair of distinct vertices $v, v' \in V$ there exists a unique Jordan arc linking v and v' as the union of distinct edges:

$$[v, v']_T = e_{v_0=v,v_1} \cup e_{v_1,v_2} \cup \cdots \cup e_{v_{n-1},v_n=v'}$$

We will not distinguish between the tree T and its planar pattern $\bigcup_{e\in E} e \subset \mathbb{C}$.

More precisely planar graph in definition above means a simplicial complex of dimension 1 embedded in the complex plane, and it is a planar tree if its complement in the complex plane is connected.

Definition 4.2 (valency)**.** For every vertex $v \in V$ of a planar tree $T = (V, E)$, let $\boldsymbol{E_v} \subset E$ be the set of edges of T with v as a common endpoint. We call **valency** the number of edges in E_v which is equal to the number of connected components of $T - \{v\}$. We say that v is a branching point if $|E_v| > 2$ and an end if $|E_v| = 1$.

Remark that every E_v comes with a cyclic order induced by the usual counterclockwise orientation on $\mathbb{C}$.

For any integer $n \in \mathbb{N}$, denote by $\mathbb{U}_n$ the cyclicly ordered group of n^{th} roots of unity.

Definition 4.3 (dynamical tree)**.** A **dynamical tree** is the data of

- an underlying planar tree $T = (V, E)$
- a map of vertices dynamics $\tau : V \to V$
- a local degree function $\delta : V \to \mathbb{N} - \{0\}$

such that

(i) For any edge $e_{v,v'} \in E$ with endpoints $v, v' \in V$, the two images $\tau(v), \tau(v') \in V$ must be distinct.

The assumption above allows us to extend τ continuously to T as follows: for any edge $e_{v,v'} \in E$ with endpoints $v, v' \in V$, define $\tau : e_{v,v'} \to [\tau(v), \tau(v')]_T$ to be a homeomorphism where $[\tau(v), \tau(v')]_T$ is the unique Jordan arc in T linking $\tau(v)$ to $\tau(v')$. This extension of τ induces naturally a well defined map $\tau_v : E_v \to E_{\tau(v)}$ for any vertex $v \in V$.

(ii) For any vertex $v \in V$, there exist an order preserving bijection $\beta_v : E_{\tau(v)} \to \mathbb{U}_n$, where $n = |E_{\tau(v)}|$ is the valency of $\tau(v)$, together with an order preserving injection $\iota_v : E_v \to \mathbb{U}_{\delta(v)n}$ where $\delta(v)$ is the assigned degree at v, such that the following diagram commutes

$$\begin{array}{ccc} E_v & \xhookrightarrow{\quad \iota_v \quad} & \mathbb{U}_{\delta(v)n} \\ {\scriptstyle \tau_v}\big\downarrow & & \big\downarrow {\scriptstyle e^{2i\pi\theta} \mapsto e^{2i\pi\delta(v)\theta}} \\ E_{\tau(v)} & \xrightarrow[\beta_v]{\quad \sim \quad} & \mathbb{U}_n \end{array}$$

We denote by $\boldsymbol{\mathcal{T} = (T, \tau, \delta)}$ such a dynamical tree.

Figure 4.1 shows an example illustrating the second condition. One way to interpret this condition is that we lift by $z \mapsto z^{\delta(v)}$ a star of n edges to get a star of $\delta(v)n$ edges, and our map τ near v should behave like $z \mapsto z^{\delta(v)}$ on a sub-star of the lifted star. This condition implies in particular

$$\forall v \in V,\ |E_v| \leqslant \delta(v)|E_{\tau(v)}|$$

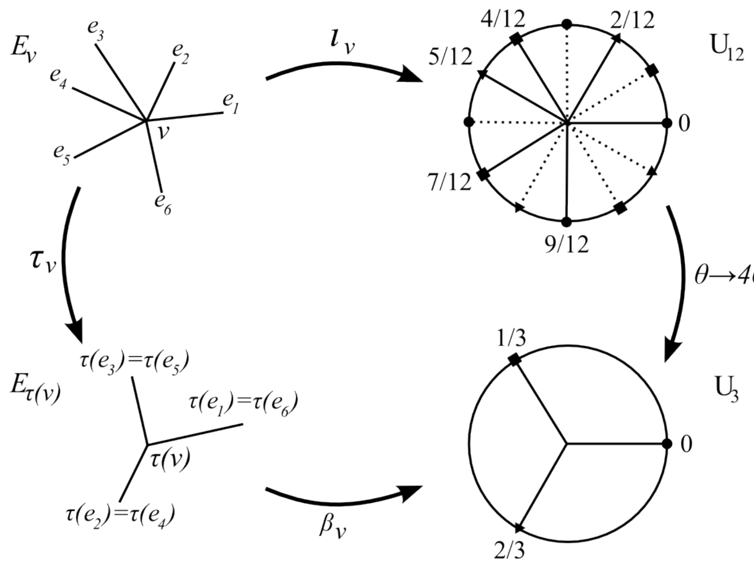


Figure 4.1: Example of vertex in dynamical tree with local degree 4

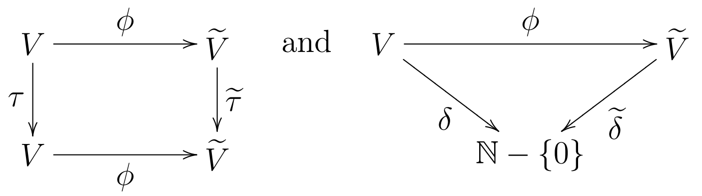
Definition 4.4 (critical and post-critical points of a dynamical tree)**.** We say that a vertex $v \in V$ of a dynamical tree $\mathcal{T} = (T, \tau, \delta)$ is a **critical point** if $\delta(v) \geqslant 2$ and a **post-critical point** if there exist an integer $n \geqslant 1$ and a critical point $w \in V$ such that $v = \tau^{\circ n}(w)$. We denote by $\boldsymbol{\Omega_{\mathcal{T}}}=$ {critical points of $\mathcal{T}$} the critical set of $\mathcal{T}$ and by $\boldsymbol{P_{\mathcal{T}}}= \bigcup_{n\geqslant 1} \tau^{\circ n}(\Omega_{\mathcal{T}})$ the post-critical set.

Definition 4.5 (degree of a dynamical tree)**.** The **degree** of a dynamical tree $\mathcal{T} = (T, \tau, \delta)$, denoted by $\mathbf{deg}(\boldsymbol{\mathcal{T}})$, is the number

$$\deg(\mathcal{T}) = 1 + \sum_{v \in V} (\delta(v) - 1)$$

Definition 4.6 (extension of a dynamical tree)**.** Let $\mathcal{T} = (T, \tau, \delta)$ and $\widetilde{\mathcal{T}} = (\widetilde{T}, \widetilde{\tau}, \widetilde{\delta})$ be two dynamical trees of same degree $\deg(\mathcal{T}) = \deg(\widetilde{\mathcal{T}})$. We say that $\widetilde{\mathcal{T}}$ is an **extension** of $\mathcal{T}$ if there exists an injective map $\phi : \begin{cases} V \to \widetilde{V} \\ E \to \widetilde{E} \end{cases}$ such that

(i) the following two diagrams commute



(ii) for any vertex $v \in V$, ϕ induces a cyclic order preserving injection of E_v into $E_{\phi(v)}$

We will write $\boldsymbol{\mathcal{T} \preceq \widetilde{\mathcal{T}}}$ in this case.

To obtain $\widetilde{T}$ from T we may add extra non-critical vertices on the edges of T and/or extra edges linking vertices of T to extra non-critical vertices outside of T. Obviously we have:

Proposition 4.7. *$\preceq$ is a partial order relation on the set of all dynamical trees.*

Therefore the following definition determines an equivalence relation.

Definition 4.8 (equivalent dynamical trees)**.** Two dynamical trees $\mathcal{T} = (T, \tau, \delta)$ and $\mathcal{T}' = (T', \tau', \delta')$ of same degree are said **equivalent** if $\mathcal{T} \preceq \mathcal{T}'$ and $\mathcal{T}' \preceq \mathcal{T}$. We will write $\boldsymbol{\mathcal{T} \simeq \mathcal{T}'}$ in this case.

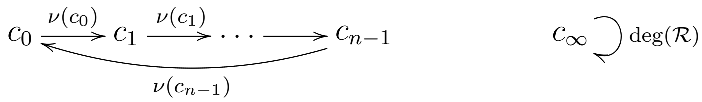
4.2 Stars and surgery

We would like to prove that we may extend (using "topological surgery") some dynamical trees to the whole complex plane $\mathbb{C}$ by ramified coverings in order to realize some particular ramification portraits of polynomial type.

Definition 4.9 (N-cyclic ramification portrait of polynomial type)**.** Let $\mathcal{R} = (\Omega, P, \sigma, \nu)$ be a ramification portrait of polynomial type and c_∞ be an infinity point. $\mathcal{R}$ is said ***N*-cyclic** for a positive integer N if every critical point $\omega \in \Omega - \{c_\infty\}$ is periodic under iteration of σ (i.e. $\Omega \subset P$) and $P - \{c_\infty\}$ is the union of exactly N disjoint periodic cycles.

Remark that this definition does not depend on the choice of infinity point in case there exist two of them. Moreover if a ramification portrait of polynomial type $\mathcal{R}$ is N-cyclic for a positive integer N then $\deg(\mathcal{R}) \geqslant 2$.

Lemma 4.10. *Let* $\mathcal{R} = (\Omega, P, \sigma, \nu)$ *be a 1-cyclic ramification portrait of polynomial type denoted by*



We identify $\mathbb{S}^2 - \{c_\infty\}$ *with* $\mathbb{C}$ *as topological manifolds. Then there exist a planar tree* $T = (V, E)$ *and a ramified covering* $f : \mathbb{S}^2 \to \mathbb{S}^2$ *such that*

(i) T *is a starlike tree that is a planar tree formed by the set of vertices* $V = \{\alpha, c_0, c_1, \dots, c_{n-1}\}$ *where* $\alpha \in \mathbb{C} - \{c_0, c_1, \dots, c_{n-1}\}$ *is the unique branching point and by the cyclicly ordered set of edges* $E = E_\alpha = \{e_{\alpha,c_0}, e_{\alpha,c_1}, \dots, e_{\alpha,c_{n-1}}\}$

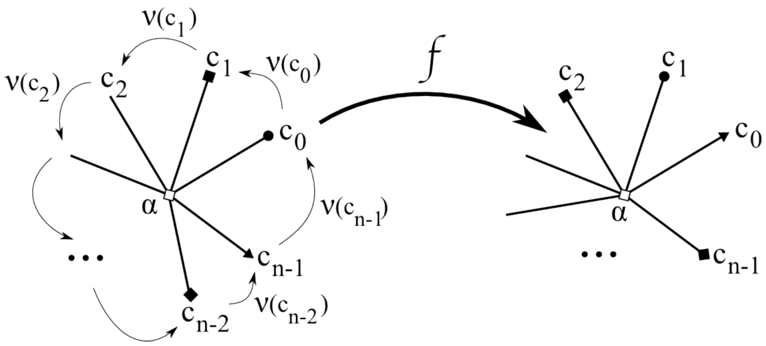


Figure 4.2: A starlike tree associated to a 1-cyclic ramification portrait of polynomial type

(ii) T *is invariant by* f*, that is* $f(T) \subset T$

(iii) f *induces on* T *a dynamical tree* $\mathcal{T} = (T, \tau = f_{|T}, \delta = \deg_{loc}(f)_{|V})$ *which fixes the branching point* α *and acts as the counterclockwise circular shift on the ends* $\{c_0, c_1, \dots, c_{n-1}\}$

(iv) $\mathcal{R}$ *is the ramification portrait associated to* f

Proof of Lemma 4.10. From the fact that the complex plane minus a finite number of points is arcwise connected, we can easily find a planar tree $T = (V, E)$ which satisfies the condition *(i)* for any choice of branching point $\alpha \in \mathbb{C} - \{c_0, c_1, \ldots, c_{n-1}\}$. It is also possible to construct carefully the edges of T in such a way they are locally connected. Let $\tau : V \to V$ be the map which fixes the branching point α and acts as the counterclockwise circular shift on the ends $\{c_0, c_1, \ldots, c_{n-1}\}$. Extend continuously this map to T as in Definition 4.3 and denote by $\tau = f_{|T}$ this extension. Remark that any extension of $f_{|T}$ to $\mathbb{S}^2$ satisfies the condition *(ii)*. Recall that if $\mathcal{R} = (\Omega, P, \sigma, \nu)$ is the ramification portrait associated to such an extension f (as in condition *(iv)*), then $\deg_{\text{loc}}(f)_{|V} = \nu$ (with the extra definition $\nu(\alpha) = 1$). Moreover $\mathcal{T} = (T, \tau = f_{|T}, \delta = \nu)$ is a well defined dynamical tree as in condition *(iii)*. Therefore it is sufficient to prove that $f_{|T}$ extends to $\mathbb{S}^2$ as a ramified covering f whose associated ramification portrait is $\mathcal{R}$.

Now consider the extension $\widetilde{\mathcal{T}} = (\widetilde{T}, \widetilde{\tau}, \widetilde{\delta})$ of $\mathcal{T}$ as follows (see Figure 4.3)

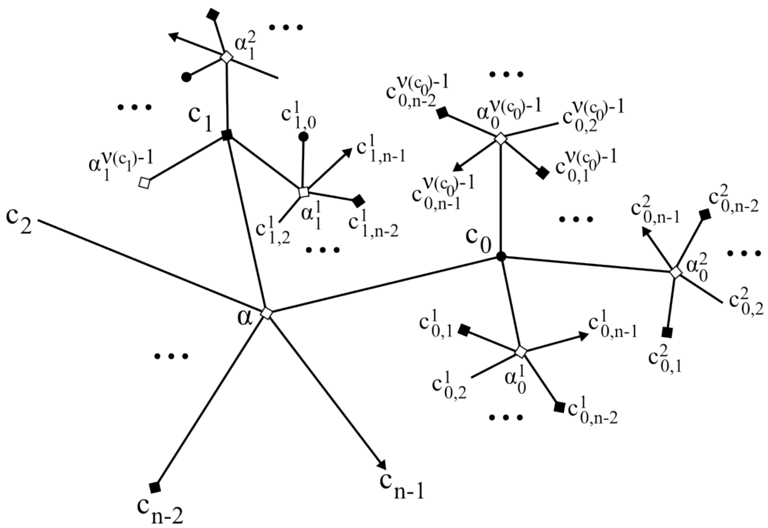


Figure 4.3: The extension $\widetilde{\mathcal{T}}$ in proof of Lemma 4.10

- for each end c_k where $k \in \{0, 1, \ldots, n-1\}$, add $\nu(c_k) - 1$ extra edges linking c_k to extra vertices denoted by $\alpha_k^1, \alpha_k^2, \ldots, \alpha_k^{\nu(c_k)-1}$ such that $E_{c_k} = \{e_{c_k,\alpha}, e_{c_k,\alpha_k^1}, e_{c_k,\alpha_k^2}, \ldots, e_{c_k,\alpha_k^{\nu(c_k)-1}}\}$ is cyclicly ordered
- for each vertex α_k^j where $j \in \{1, 2, \ldots, \nu(c_k)-1\}$, add $n-1$ extra edges linking α_k^j to extra vertices denoted by $c_{k,0}^j, \ldots, c_{k,k-1}^j, c_{k,k+1}^j, \ldots, c_{k,n-1}^j$ such that $E_{\alpha_k^j} = \{e_{\alpha_k^j,c_{k,0}^j}, \ldots, e_{\alpha_k^j,c_{k,k-1}^j}, e_{\alpha_k^j,c_{k,k+1}^j}, \ldots, e_{\alpha_k^j,c_{k,n-1}^j}\}$ is cyclicly ordered

- define $\widetilde{\tau} : \widetilde{V} \to \widetilde{V}$ as follows

$$\begin{cases} \widetilde{\tau}(\alpha) = \alpha & \\ \widetilde{\tau}(c_k) = c_{k+1} & \text{where } k \in \{0, 1, \ldots, n-1\} \\ \widetilde{\tau}(\alpha_k^j) = \alpha & \text{where } k \in \{0, 1, \ldots, n-1\},\ j \in \{1, 2, \ldots, \nu(c_k) - 1\} \\ \widetilde{\tau}(c_{k,\ell}^j) = c_{\ell+1} & \text{where } k \neq \ell \in \{0, 1, \ldots, n-1\},\ j \in \{1, 2, \ldots, \nu(c_k) - 1\} \end{cases}$$

  with the notation $c_n = c_0$

- extend continuously $\widetilde{\tau}$ to $\widetilde{T}$ by τ on T and by homeomorphisms on every extra edge

Remark that we may construct carefully each extra edge in such a way they are locally connected. Remark also that $\widetilde{\tau}(\widetilde{T}) = T$ and the valency of each c_k where $k \in \{0, 1, \ldots, n-1\}$ is now $\widetilde{\delta}(c_k) = \nu(c_k)$ in $\widetilde{T}$. Therefore

$$\forall k \in \{0, 1, \ldots, n-1\},\ |\widetilde{E}_{c_k}| = \nu(c_k) |\widetilde{E}_{\widetilde{\tau}(c_k)}| \tag{4.1}$$

Consider $\widetilde{T}$ and its image $\widetilde{\tau}(\widetilde{T}) = T$ as if they belong to two different copies of $\mathbb{S}^2$ (recall that we identify $\mathbb{C}$ with $\mathbb{S}^2 - \{c_\infty\}$), say respectively S_1 and S_2 (see Figure 4.4). We are going to construct a ramified covering f which extends continuously $\widetilde{\tau}$ by surgery.

Since T is simply connected, we may easily define a cyclicly ordered set A_2 of $|A_2| = n$ disjoint (except at c_∞) Jordan arcs in $S_2 - T$ (except at c_k) linking c_∞ to each c_k where $k \in \{0, 1, \ldots, n-1\}$ (see Figure 4.4).

Since T is connected, Riemann's mapping theorem provides a biholomorphic map $\varphi_1 : \mathbb{D} \to S_1 - \widetilde{T}$ such that $\varphi_1(0) = c_\infty$. Moreover we have constructed $\widetilde{T}$ to be locally connected. By Carathéodory's theorem we may thus extend continuously φ_1 to $\overline{\mathbb{D}}$. Remark that

- each end $c_{k,\ell}^j$ where $k \neq \ell \in \{0, 1, \ldots, n-1\}$ and $j \in \{1, 2, \ldots, \nu(c_k) - 1\}$ has exactly one preimage in $\partial\mathbb{D}$ by φ_1

- each branching point c_k where $k \in \{0, 1, \ldots, n-1\}$ has exactly $\nu(c_k)$ preimages in $\partial\mathbb{D}$ by φ_1 (since $\nu(c_k)$ corresponds to the valency of c_k)

Then define a cyclic ordered set A_1 of Jordan arcs (see Figure 4.4) as images by φ_1 of straight rays in $\overline{\mathbb{D}}$ linking 0 to those preimages in $\partial\mathbb{D}$. Using

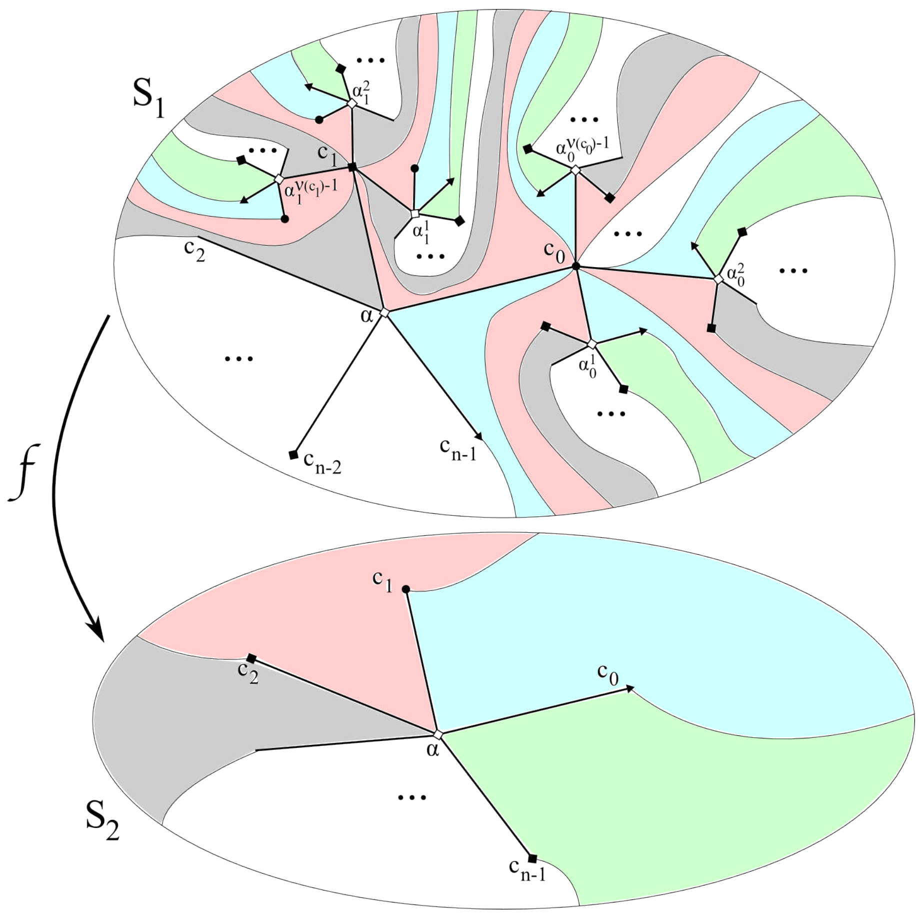


Figure 4.4: The piecewise definition of the ramified covering f in Lemma 4.10

Proposition 3.7, we get:

$$\begin{aligned} |A_1| &= (n-1)\sum_{k=0}^{n-1}(\nu(c_k)-1)+\sum_{k=0}^{n-1}\nu(c_k) \\ &= n\sum_{k=0}^{n-1}(\nu(c_k)-1)+n \\ &= n(\deg(\mathcal{R})-1)+n \\ &= \deg(\mathcal{R})|A_2| \end{aligned} \tag{4.2}$$

Now define $f_{|\widetilde{T}\cup A_1} : \widetilde{T}\cup A_1 \to T\cup A_2$ to be equal to $\widetilde{\tau}$ on $\widetilde{T}$ and and to be continuously extended to each Jordan arc in A_1 linking $a \in \widetilde{T}$ to c_∞ by homeomorphism onto the Jordan arc in A_2 linking $\widetilde{\tau}(a)$ to c_∞. It remains $\deg(\mathcal{R}) \times n$ connected components of $S_1 - (\widetilde{T} \cup A_1)$ which are topological disks whose boundaries are Jordan curves formed by the union of exactly two edges of $\widetilde{T}$ and two Jordan arcs in A_1 (see Figure 4.4). Remark that $f_{|\widetilde{T}\cup A_1}$ is defined as a homeomorphism on each of those Jordan curves. Therefore we may extend homeomorphically $f_{|\widetilde{T}\cup A_1}$ to every connected component of $S_1 - (\widetilde{T} \cup A_1)$ onto a connected component of $S_2 - (T \cup A_2)$ by Schönflies' theorem (see Theorem A.3 in appendix). Finally the equalities (4.1) and (4.2) together with the cyclic orders on A_1, A_2 and $\widetilde{E}_{c_k}$ for every $k \in \{0, 1, \dots, n-1\}$ ensure that we get a ramified covering $f : (S_1 = \mathbb{S}^2) \to (S_2 = \mathbb{S}^2)$ whose associated ramification portrait is $\mathcal{R}$ as required. □

Before doing as well for any N-cyclic ramification portrait of polynomial type, we need the following sharpening.

Lemma 4.11. *Use notations from Lemma 4.10 and assume in addition that c_0 is a critical point. Then we can find a planar tree $T = (V, E)$ and a ramified covering $f : \mathbb{S}^2 \to \mathbb{S}^2$ which satisfy conditions (i), (ii), (iii), (iv) together with*

(v) there exists an extension $\mathcal{T}' = (T', \tau' = f_{|T'}, \delta' = \deg_{loc}(f)_{|V'})$ of $\mathcal{T}$ which consists in adding one extra vertex $\beta \in \mathbb{C} - \{\alpha, c_0, c_1, \dots, c_{n-1}\}$ and one extra edge $e_{c_0,\beta}$ such that $f(\beta) = \beta$ and $f(e_{c_0,\beta}) \subset T'$

Notice that with a suitable indexing of points in $\Omega \cup P$, we may always assume that c_0 is a critical point. The condition *(v)* implies in particular (see Figure 4.5)

$$f(e_{c_0,\beta}) = [c_1, \beta]_{T'} = e_{c_1,\alpha} \cup e_{\alpha,c_0} \cup e_{c_0,\beta}$$

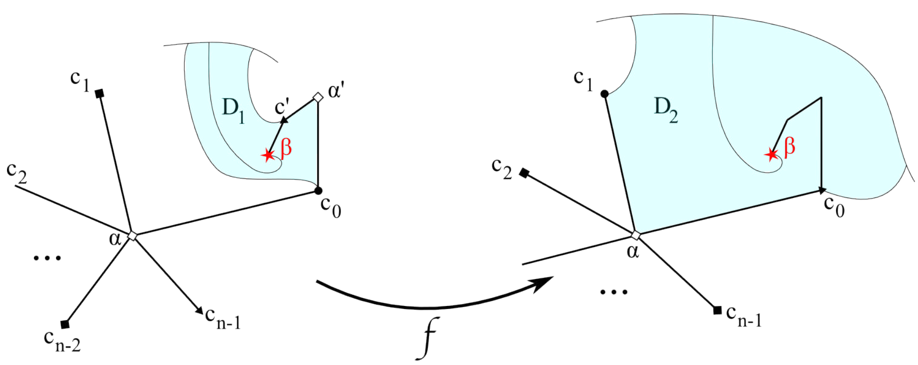


Figure 4.5: The beta point

Proof of Lemma 4.11. Let us come back to the proof of Lemma 4.10. Denote by D_1 the connected component of $S_1 - (\widetilde{T} \cup A_1)$ whose boundary is formed by the union of two edges of $\widetilde{T}$ and two Jordan arcs in A_1 linking $c' = c_{0,n-1}^{\nu(c_0)-1}$, $\alpha' = \alpha_0^{\nu(c_0)-1}$, c_0 and c_∞ (it exists since $\nu(c_0) \geqslant 2$). Remark that the proof of Lemma 4.10 holds for any choice of cyclicly ordered set A_2 of n disjoint (except at c_∞) Jordan arcs in $S_2 - T$ (except at c_k) linking c_∞ to each c_k where $k \in \{0, 1, \dots, n-1\}$. For such a set A_2, denote by D_2 the image of D_1 by f that is the connected component of $S_2 - (T \cup A_2)$ whose boundary is formed by the union of two edges of T and two Jordan arcs in A_2 linking c_0, α, c_1 and c_∞. See Figure 4.5 and compare with Figure 4.3 and Figure 4.4.

Remark that $\overline{D_1}$ contains no other points in $V = \{\alpha, c_0, c_1, \dots, c_{n-1}\}$ than c_0. Therefore there exists a cyclicly ordered set A_2 of n disjoint (except at c_∞) Jordan arcs in $S_2 - T$ (except at c_k) linking c_∞ to each c_k where $k \in \{0, 1, \dots, n-1\}$, such that D_1 is contained in D_2. Carry on the proof of Lemma 4.10 for this choice of A_2. Now pick a point $\beta \in D_1$. Let $e_{c',\beta}$ be a Jordan arc in D_1 (except at c') linking c' to β and e_{β,c_∞} be a Jordan arc in $D_1 - e_{c',\beta}$ (except for its endpoints) linking β to c_∞. Redefine f on D_1 as follows

- $f(\beta) = \beta$
- let $f_{|e_{c',\beta}} : e_{c',\beta} \to e_{c_0,\alpha'} \cup e_{\alpha',c'} \cup e_{c',\beta}$ be an homeomorphism
- let $f_{|e_{\beta,c_\infty}} : e_{\beta,c_\infty} \to e_{\beta,c_\infty}$ be an homeomorphism
- extend homeomorphically f to the two connected components of $D_1 - (e_{c',\beta} \cup e_{\beta,c_\infty})$ by Schönflies' theorem

The result follows with $e_{c_0,\beta} = e_{c_0,\alpha'} \cup e_{\alpha',c'} \cup e_{c',\beta}$. □

Theorem 4.12. *Let $\mathcal{R} = (\Omega, P, \sigma, \nu)$ be a N-cyclic ramification portrait of polynomial type where N is a positive integer. Then there exists a ramified covering $f : \mathbb{S}^2 \to \mathbb{S}^2$ whose associated ramification portrait is $\mathcal{R}$.*

Clearly the ramified covering f is not unique. Indeed the construction suggested here depends strongly on the choice of the starlike tree in Lemma 4.10. A different choice of shape for trees would lead to a different ramified covering f with same associated ramification portrait. In particular the condition starlike is not even necessary.

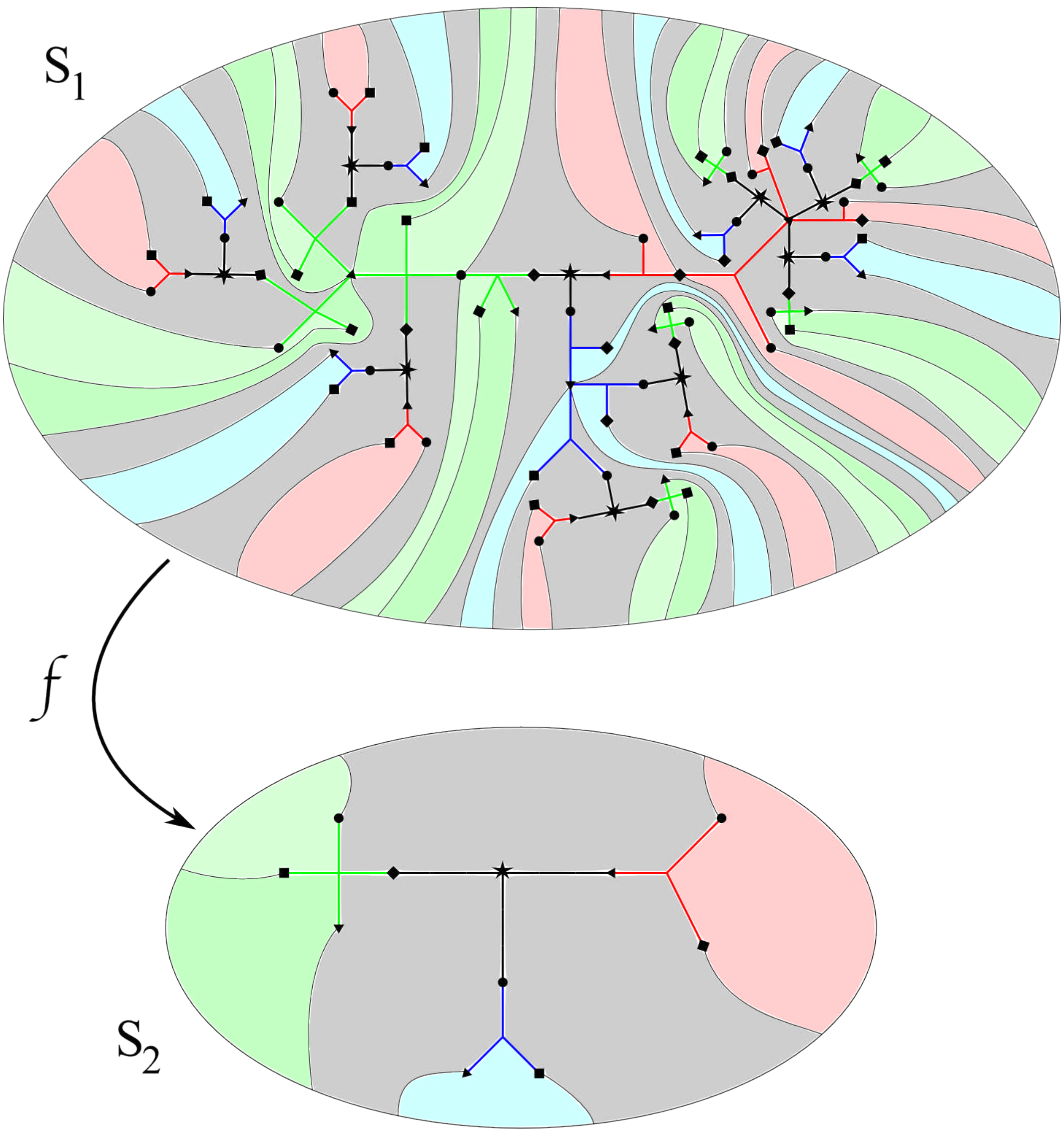


Figure 4.6: Example of a 3-cyclic ramification portrait of polynomial type realized by a ramified covering of degree 9

Proof of Theorem 4.12. The proof is similar to that one of Lemma 4.10 with another shape of tree. Denote by $P - \{c_\infty\} = \bigcup_{i=1}^{N} P_i$ the union of the disjoint periodic cycles of $\mathcal{R}$. For every $i \in \{1, 2, \ldots, N\}$, let $\mathcal{R}_i$ be the 1-cyclic ramification portrait of polynomial type induced by the infinity point c_∞ together with P_i and the restrictions $\sigma_{|P_i}$ and $\nu_{|P_i}$. Denote by n_i the number of points in P_i.

Let $\mathcal{T}_i = (T_i, \tau_i, \delta_i)$ be a starlike dynamical tree associated to $\mathcal{R}_i$ as in Lemma 4.10. Construct carefully the edges of T_i in such a way they are locally connected.

Now consider the extension $\widetilde{\mathcal{T}_i}$ of $\mathcal{T}_i$ defined in the proof of Lemma 4.10. Briefly $\widetilde{\mathcal{T}_i}$ consists of adding to each end of T_i some little copies of T_i (see Figure 4.3).

According to Lemma 4.11, extend $\widetilde{\mathcal{T}_i}$ by adding an extra locally connected edge linking a critical point to a picked fixed point $\beta \in \mathbb{S}^2 - P$ which is the same for every $i \in \{1, 2, \dots, N\}$. Therefore we get a big dynamical tree $\widetilde{\mathcal{T}}$ formed by the union of planar trees $\widetilde{T_i}$ where $i \in \{1, 2, \dots, N\}$ together with the extra edges linking them to the fixed point β. Do likewise for the planar trees T_i to get a big dynamical tree $\mathcal{T}$ such that $\widetilde{\mathcal{T}}$ is an extension of $\mathcal{T}$ and $\widetilde{\tau}(\widetilde{T}) = T$.

Finally, using notations of the proof of Lemma 4.10, add to $\widetilde{\mathcal{T}_i}$ some extra edges linking the preimages of c_0 (that is the critical point of $\mathcal{T}_i$ which is linked to β by an extra edge) to a little copy of $T - T_i$. More precisely:

- at a vertex c^j_{k,n_i-1} where $k \in \{0, 1, \dots, n_i - 2\}$, $j \in \{1, 2, \dots, \nu(c_k) - 1\}$ add one extra edge linking c^j_{k,n_i-1} to the corresponding β point of a little copy of $T - T_i$

- at a vertex c_{n_i-1}, add $\nu(c_{n_i-1})$ extra edges inserted between the edges of $\widetilde{T}$ with c_{n_i-1} at a common endpoint, linking c_{n_i-1} to the corresponding β points of $\nu(c_{n_i-1})$ little copies of $T - T_i$

- extend $\widetilde{\tau}$ to each little copy of $T - T_i$ according to the definition of $\widetilde{\tau}$ on $T - T_i$

An example of such a construction is drawn in Figure 4.6. We call again $\widetilde{\mathcal{T}}$ this extension. Notice that we still have $\widetilde{\tau}(\widetilde{T}) = T$. Consider $\widetilde{T}$ and its image T as if they belong to two different copies of $\mathbb{S}^2$, say respectively S_1 and S_2.

We may carry on the construction of a ramified covering f which extends continuously $\widetilde{\tau}$ as in Lemma 4.10. Define a cyclicly ordered set A_2 of disjoint (except at c_∞) Jordan arcs in $S_2 - T$ (except at endpoints) linking c_∞ to each endpoint in T (see Figure 4.6). Furthermore, by Riemann's mapping theorem and Carathéodory's theorem, define a cyclicly ordered set A_1 of disjoint (except at c_∞) Jordan arcs in $S_1 - \widetilde{T}$ (except at endpoints) linking c_∞ to each preimage by $\widetilde{\tau}$ of endpoint in T (see Figure 4.6). Now extend $\widetilde{\tau}$ to each Jordan arc in A_1 by homeomorphism, and then to each connected component of $S_1 - (\widetilde{T} \cup A_1)$ (which is a topological disk) by Schönflies' theorem.

The cyclic orders on A_1, A_2 and around each branching point ensure that we get a ramified covering $f : (S_1 = \mathbb{S}^2) \to (S_2 = \mathbb{S}^2)$. Moreover the associated ramification portrait of f is $\mathcal{R}$ except possibly for the local degree at c_∞. Actually it remains to prove that $|A_1| = \deg(\mathcal{R})|A_2|$.

Since the number of Jordan arcs in A_2 is equal to the number of endpoints in T and every endpoint in T_i except one (the point c_0) is an endpoint of T we have:

$$\begin{aligned} |A_2| &= (n_1 - 1) + (n_2 - 1) + \cdots + (n_N - 1) \\ &= n - N \end{aligned} \tag{4.3}$$

where $n = n_1 + n_2 + \cdots + n_N$ is the number of points in $P - \{c_\infty\}$ (compare with Figure 4.6 where $N = 3$, $n = 3 + 4 + 3$ and $|A_2| = 7$).

Now for every $i \in \{1, 2, \ldots, N\}$, consider the subset A_1^i of Jordan arcs in A_1 linking c_∞ to a vertex in T_i or a vertex in a little copy of $T - T_i$. Remark that for every such a little copy of $T - T_i$, there are exactly $|A_2| - (n_i - 1) = n - N - n_i + 1$ Jordan arcs linking c_∞ to a vertex of this little copy. Use the notations from the proof of Lemma 4.10 to denote the vertices of T_i.

(a) a vertex $c^j_{0,\ell}$ where $\ell \in \{1, 2, \ldots, n_i - 1\}$ and $j \in \{1, 2, \ldots, \nu(c_0) - 1\}$ is an endpoint of exactly one Jordan arc in A_1 if and only if $\ell \neq n_i - 1$ (since $\widetilde{\tau}(c^j_{0,\ell}) = c_{\ell+1}$ with the notation $c_{n_i} = c_0$)

(a') a vertex c^j_{0,n_i-1} where $j \in \{1, 2, \ldots, \nu(c_0) - 1\}$ is an endpoint of exactly one little copy of $T - T_i$ if and only if $j \neq \nu(c_0) - 1$ (since $c^{\nu(c_0)-1}_{0,n_i-1}$ is an endpoint of the original $T - T_i$)

(b) a vertex $c^j_{k,\ell}$ where $k \in \{1, 2, \ldots, n_i - 2\}$, $\ell \in \{0, 1, \ldots, n_i - 1\} - \{k\}$ and $j \in \{1, 2, \ldots, \nu(c_k) - 1\}$ is an endpoint of exactly one Jordan arc in A_1 if and only if $\ell \neq n_i - 1$

(b') a vertex c^j_{k,n_i-1} where $k \in \{1, 2, \ldots, n_i - 2\}$ and $j \in \{1, 2, \ldots, \nu(c_0) - 1\}$ is an endpoint of exactly one little copy of $T - T_i$

(c) a vertex $c^j_{n_i-1,\ell}$ where $\ell \in \{0, 1, \ldots, n_i - 2\}$, $j \in \{1, 2, \ldots, \nu(c_{n_i-1}) - 1\}$ is an endpoint of exactly one Jordan arc in A_1

(d) a vertex c_k where $k \in \{0, 1, \ldots, n_i - 2\}$ is an endpoint of exactly $\nu(c_k)$ Jordan arcs in A_1

(e) the vertex c_{n_i-1} is an endpoint of exactly $\nu(c_{n_i-1})$ little copies of $T - T_i$

Using Proposition 3.7, we deduces

$$\begin{aligned}
|A_1^i| &= \underbrace{(n_i-2)(\nu(c_0)-1)}_{\text{(a)}} + \underbrace{(\nu(c_0)-2)(n-N-n_i+1)}_{\text{(a')}} \\
&\quad + \sum_{k=1}^{n_i-2}\left(\underbrace{(n_i-2)(\nu(c_k)-1)}_{\text{(b)}} + \underbrace{(\nu(c_k)-1)(n-N-n_i+1)}_{\text{(b')}}\right) \\
&\quad + \underbrace{(n_i-1)(\nu(c_{n_i-1})-1)}_{\text{(c)}} \\
&\quad + \sum_{k=0}^{n_i-2}\underbrace{\nu(c_k)}_{\text{(d)}} + \underbrace{\nu(c_{n_i-1})(n-N-n_i+1)}_{\text{(e)}} \\
&= (n_i-1)\sum_{k=0}^{n_i-1}(\nu(c_k)-1) + \sum_{k=0}^{n_i-1}(\nu(c_k)-1)(n-N-n_i+1) \\
&\quad +(n_i-1) \\
&= (n-N)(\deg(\mathcal{R}_i)-1) + (n_i-1)
\end{aligned}$$

Remark that Proposition 3.7 implies

$$\sum_{i=1}^{N}(\deg(\mathcal{R}_i)-1) = \deg(\mathcal{R})-1$$

It follows from (4.3):

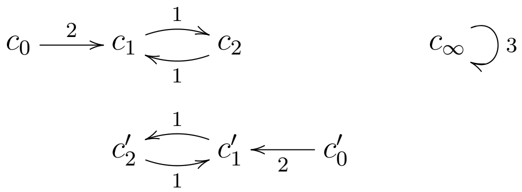
$$\begin{aligned}
|A_1| &= \sum_{i=1}^{N}|A_1^i| \\
&= (n-N)\sum_{i=1}^{N}(\deg(\mathcal{R}_i)-1) + \sum_{i=1}^{N}(n_i-1) \\
&= (n-N)(\deg(\mathcal{R})-1) + (n-N) \\
&= \deg(\mathcal{R})|A_2|
\end{aligned}$$

Consequently the local degree of the ramified covering f at c_∞ is $\deg(\mathcal{R})$ as required (compare with Figure 4.6 where $|A_1| = 63 = 9 \times 7 = \deg(\mathcal{R})|A_2|$). □

4.3 Generalization

We will not need a result more general than Theorem 4.12 in this thesis. However we may easily adapt our construction to ramified covering realizing more complicated ramification portrait as it is suggested in the following example.

Example 4.13. Consider the continuous and piecewise affine map $f_{\text{ana}|\mathbb{R}}$ defined in Example 3.15. Let T_{ana} be the planar tree formed by the set of vertices $V_{\text{ana}} = \{c'_2, c_1, c_0, c'_0, c_2, c'_1\}$ and the real segments linking them together as edges. $f_{\text{ana}|\mathbb{R}}$ induces on T_{ana} a dynamical tree $\mathcal{T}_{\text{ana}}$ whose local degree function on vertices is given by that one of $\mathcal{R}_{\text{ana}}$. Recall this ramification portrait:



Notice that the post-critical points c_1 and c'_1 each have three preimages counted with multiplicity whereas c_2 and c'_2 each have only one. So we consider the extension $\widetilde{\mathcal{T}}_{\text{ana}} = (\widetilde{T}_{\text{ana}}, \widetilde{\tau}_{\text{ana}}, \widetilde{\delta}_{\text{ana}})$ of $\mathcal{T}_{\text{ana}}$ defined as follows (see Figure 4.7)

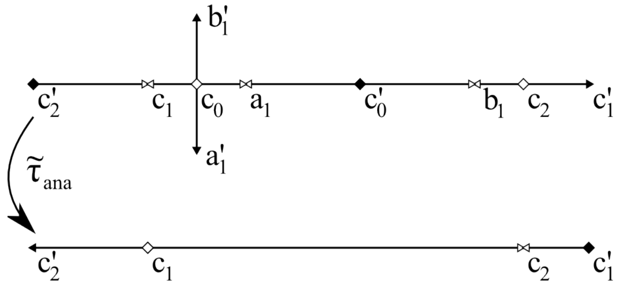


Figure 4.7: The extension $\widetilde{\mathcal{T}}_{\text{ana}}$

- add two extra vertices $a_1 \in [c_0, c'_0]$ and $b_1 \in [c'_0, b_2]$ corresponding to two preimages of c_2 by f_{ana} (see Figure 3.1)
- add two extra edges e_{c_0,a'_1} and e_{c_0,b'_1} linking c_0 to two extra vertices a'_1 in the lower half plane and b'_1 in the upper half plane
- define $\widetilde{\tau}_{\text{ana}} : e_{c_0,a'_1} \to e_{c_1,c'_2} = [c'_2, c_1]$ and $\widetilde{\tau}_{\text{ana}} : e_{c_0,b'_1} \to e_{c_1,c'_2} = [c'_2, c_1]$ to be homeomorphisms, in particular $\widetilde{\tau}_{\text{ana}}(a'_1) = \widetilde{\tau}_{\text{ana}}(b'_1) = c'_2$

Now consider $\widetilde{T}_{\rm ana}$ and its image by $\widetilde{\tau}_{\rm ana}$ as if they belong to two different copies of $\mathbb{S}^2$ (recall that we identify $\mathbb{C}$ with $\mathbb{S}^2 - \{c_\infty\}$), say respectively S_1 and S_2 (see Figure 4.8).

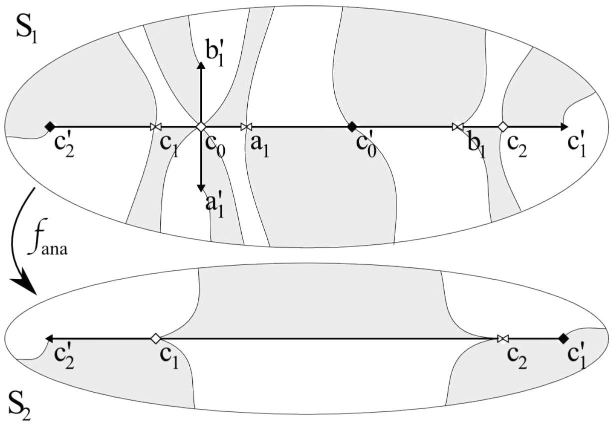


Figure 4.8: The construction of $f_{\rm ana}$

We may define six Jordan arcs in $S_2 - \widetilde{\tau}_{\rm ana}(\widetilde{T}_{\rm ana})$ (except for the endpoints) linking c_∞ to each post-critical points c_2', c_1, c_2 and c_1' as in the proof of Lemma 4.10 (by Riemann's mapping theorem and Carathéodory's theorem). In this way, S_2 is divided into six topological disks. We may do likewise in $S_1 - \widetilde{T}_{\rm ana}$ in order to get eighteen $(\deg(\mathcal{R}_{\rm ana}) \times 6)$ topological disks (see Figure 4.8). Now define $f_{\rm ana}$ to be equal to $\widetilde{\tau}_{\rm ana}$ on $\widetilde{T}_{\rm ana}$ and to be an homeomorphism from each Jordan arc in S_1 linking c_∞ to $v \in \widetilde{V}_{\rm ana}$ to the corresponding Jordan arc in S_2 linking c_∞ to $\widetilde{\tau}_{\rm ana}(v) = f_{{\rm ana}|\mathbb{R}}(v)$ (do it carefully in order to respect the cyclic order of Jordan arcs around c_∞). Finally extend homeomorphically $f_{\rm ana}$ on each topological disk by Schönflies' theorem.

Such a construction provides a ramified covering $f_{\rm ana}$ which realizes the ramification portrait $\mathcal{R}_{\rm ana}$ as required.

Chapter 5

Analytical obstructions

Now we would like to discuss the realization by some rational maps of ramification portrait already realized by some ramified coverings. The main tool is the Thurston's characterization of post-critically finite rational maps stated by W. P. Thurston in 1982. We will present this very powerful theorem in holomorphic dynamical systems after some required definitions and the readers are referred to [DH93] for a proof. Then we will discuss how we may simplify this criterion in polynomial case with Levy cycles (according to works in [Lev85]). In particular we will extend a result from [Lev85] to non-post-critically finite rational maps providing to simplify many Thurson-like characterization. For instance we will give a criterion about polynomials with one fixed bounded-type Siegel disk by using a result from [Zha08].

5.1 Thurston equivalence

At first we establish some standard notations to write up the aim of this chapter.

Definition 5.1 (Thurston map)**.** A **Thurston map** is an orientation-preserving ramified covering $f : \mathbb{S}^2 \to \mathbb{S}^2$ whose post-critical set P_f is finite: $|P_f| < \infty$.

Remark that every ramified covering considered in the previous chapter is actually a Thurston map.

Definition 5.2 (Thurston equivalence)**.** Two Thurston maps f and g are said **Thurston equivalent**, or **combinatorially equivalent**, if there exist two orientation-preserving homeomorphisms φ_0 and φ_1 of $\mathbb{S}^2$ such that

(i) the following diagram commutes

$$\begin{array}{ccc} \mathbb{S}^2 & \xrightarrow{\varphi_1} & \mathbb{S}^2 \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ \mathbb{S}^2 & \xrightarrow[\varphi_0]{} & \mathbb{S}^2 \end{array}$$

(ii) $\varphi_0(P_f) = \varphi_1(P_f) = P_g$

(ii) φ_0 is isotopic to φ_1 relative to P_f, that is there exists an isotopy $\Phi : [0,1] \times \mathbb{S}^2 \to \mathbb{S}^2$, $(t,.) \mapsto \Phi(t,.) = \varphi_t$ from φ_0 to φ_1 such that its restriction on P_f is constant with respect to t (in particular, $\varphi_{0|P_f} = \varphi_{1|P_f}$)

We will write $\boldsymbol{f \sim_T g}$ in this case.

Proposition 5.3. *We have the following properties*

1. $\sim_T$ *is an equivalence relation on the set of Thurston maps.*
2. *If two Thurston maps are Thurston equivalent then their associated ramification portraits are similar.*

In particular if a Thurston equivalence class of a Thurston map contains a rational map then the associated ramification portrait of the given Thurston map is realized by the rational map as required. The Thurston theorem (Theorem 5.9) characterizes these classes by a topological criterion.

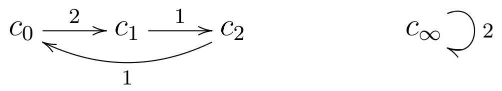
Proof of Proposition 5.3. There is no difficulty for the first statement. For the second one, let f and g be two Thurston maps and assume they are Thurston equivalent. Using the notations of Definition 5.2, call β the restriction of the homeomorphism φ_1 to the set $\Omega_f \cup P_f$, that is $\beta = \varphi_{1|\Omega_f \cup P_f}$. Then β is a bijection starting from $\Omega_f \cup P_f$ and $\beta(P_f) = P_g$. Since $\varphi_0 \circ f = g \circ \varphi_1$ and φ_0, φ_1 are homeomorphisms we get

$$\forall x \in \mathbb{S}^2, \quad \deg_{\text{loc}}(f)(x) = \deg_{\text{loc}}(g)(\varphi_1(x))$$

As a consequence $\beta(\Omega_f) = \varphi_1(\Omega_f) = \Omega_g$ proving that β is a bijection from $\Omega_f \cup P_f$ to $\Omega_g \cup P_g$. Finally, β satisfies exactly Definition 3.9 for the associated ramification portraits $\mathcal{R}_f$ and $\mathcal{R}_g$ since $\beta_{|P_f} = \varphi_{1|P_f} = \varphi_{0|P_f}$. □

Nevertheless the converse of the second statement of Proposition 5.3 is false as we will see in the following example.

Example 5.4. Consider the ramification portrait of polynomial type below.



If this ramification portrait is associated to a polynomial of the form $z \mapsto z^2 + c$ (we fix $c_\infty = \infty$, $c_0 = 0$ and $c_1 = c$) then the complex parameter c is a root of the equation $(c^2+c)^2+c = 0$. This equation has four roots: one trivial root $c = 0$ which does not correspond to the ramification portrait above (otherwise $c_0 = c_1 = c_2$), one real negative root called $c_{airplane} \approx -1.755$ and two complex conjugated roots called $c_{rabbit} \approx -0.123+0.745i$ and $c_{corabbit} = \overline{c_{rabbit}}$. We denote by $f_{airplane}$, f_{rabbit} and $f_{corabbit}$ the corresponding quadratic polynomials. Hence we get three ramification portraits associated, all similar to the ramification portrait above. But we will see from the unicity part of Thurston's theorem 5.9 that those three quadratic polynomials cannot be Thurston equivalent to each other since they are not conjugated by Möbius transformations (every quadratic polynomial is conjugated by Möbius transformations to a unique map of the form $z \mapsto z^2 + c$).

In fact, $f_{airplane}$, f_{rabbit} and $f_{corabbit}$ are the only polynomials, up to conjugation by Möbius transformations, which realize the ramification portrait above. Moreover applying the Levy's theorem 5.17, any Thurston map which realizes the ramification portrait above is Thurston equivalent to one of those three quadratic polynomials. For instance taking a Dehn twist T around the two non-critical points c_1 and c_2 of f_{rabbit} (i.e. a homeomorphism which is the identity map outside an annulus A surrounding c_1 and c_2 and is conjugated on A to the map $(r, z) \mapsto (r, ze^{2i\pi r})$ on $[0, 1] \times \mathbb{S}^1$), we get a Thurston map $T^m \circ f$ for every $m \in \mathbb{Z}$ which is Thurston equivalent to one of $f_{airplane}$, f_{rabbit} and $f_{corabbit}$. The question of which one was asked by J. H. Hubbard (see the Hubbard's twisted rabbit problem in [Pil03]) and was answered by L. Bartholdi and V. Nekrashevych in [BN06] using iterated monodromy groups.

In [Kam01] the author uses an orientation-preserving argument in order to give another example of two polynomials with the same associated ramification portrait but which are not Thurston equivalent.

Those examples lead to discuss how to make ramification portraits more restrictive in order to precise the meaning behind prescribed dynamics. We postpone this discussion to the next chapter (see Definition 6.5 and following remarks) where we will define combinatorial data which catch more information about dynamical properties.

5.2 Thurston obstructions

We follow the definitions and notations from [DH93]. For every ramified covering, we denote by $\mathcal{P}_f = \overline{P_f}$ the closure of its post-critical set but we keep the notation P_f in case f is a Thurston map.

Definition 5.5 (multi-curve)**.** Let f be a ramified covering. A Jordan curve γ is called non-peripheral if each connected component of $\mathbb{S}^2 - \gamma$ contains at least two points of $\mathcal{P}_f$. A **multi-curve** $\Gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_n\}$ is a finite set of disjoint, non-homotopic and non-peripheral Jordan curves in $\mathbb{S}^2 - \mathcal{P}_f$.

Notice that there exist an infinite number of multi-curves as soon as $|P_f| \geqslant 4$ but:

Lemma 5.6. *Any multi-curve of a Thurston map f contains at most $|P_f| - 3$ curves.*

Proof of Lemma 5.6. The result is clearly true for small values of $|P_f|$. Assume by induction that it is true for every post-critical set of cardinality smaller than a fixed integer $p \geqslant 6$. Let Γ be a multi-curve of a Thurston map f satisfying $|P_f| = p$. We may assume that there exists a curve $\gamma_0 \in \Gamma$ such that each connected component of $\mathbb{S}^2 - \gamma_0$ contains at least three points of P_f (adding such a curve in Γ if necessary). We denote by D_1 and D_2 the two distinct connected components of $\mathbb{S}^2 - \gamma_0$ and by x_2, y_2, z_2 three points in $P_f \cap D_2$. Γ is the union of $\{\gamma_0\}$ together with two disjoint multi-curves $\Gamma_1 = \{\gamma \in \Gamma \,/\, \gamma \subset D_1\}$ and $\Gamma_2 = \{\gamma \in \Gamma \,/\, \gamma \subset D_2\}$. Remark now that $(P_f \cap D_1) \cup \{x_2, y_2\}$ may be seen as a new post-critical set P_g of a Thurston map g (existence of g is ensured by discussions in Chapter 4). Moreover $|P_g| < |P_f|$ (since $z_2 \in P_f - P_g$) and $\{\gamma_0\} \cup \Gamma_1$ is a multi-curve associated to the post-critical set P_g. By induction hypothesis we get $1 + |\Gamma_1| \leqslant (|P_f \cap D_1| + 2) - 3$. We may likewise prove that $1 + |\Gamma_2| \leqslant (|P_f \cap D_2| + 2) - 3$. Finally

$$|\Gamma| = 1 + |\Gamma_1| + |\Gamma_2| \leqslant |P_f \cap D_1| + |P_f \cap D_2| - 3 = |P_f| - 3$$

□

Since every ramified covering $f : \mathbb{S}^2 \to \mathbb{S}^2$ is of finite degree and $f(\mathcal{P}_f) \subset \mathcal{P}_f$, each connected component δ of the preimage of a Jordan curve γ in $\mathbb{S}^2 - \mathcal{P}_f$ is still a Jordan curve in $S^2 - \mathcal{P}_f$ and the degree of the map $f_{|\delta} : \delta \to \gamma$ is finite. That justifies the following definition.

Definition 5.7 (Thurston linear transformation)**.** Let f be a ramified covering and $\Gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_n\}$ be a multi-curve. For every pair of integers $i, j \in \{1, 2, \ldots, n\}$ denote by $\delta^\alpha_{i,j}$ the connected components of $f^{-1}(\gamma_j)$ homotopic to γ_i in $\mathbb{S}^2 - \mathcal{P}_f$ (where we index the components by α) and $d^\alpha_{i,j}$ the degree of the map $f_{|\delta^\alpha_{i,j}} : \delta^\alpha_{i,j} \to \gamma_j$. The **Thurston linear transformation** $\boldsymbol{f_\Gamma} : \mathbb{R}^\Gamma \to \mathbb{R}^\Gamma$ is defined as follows

$$f_\Gamma(\gamma_j) = \sum_{i,\alpha} \frac{1}{d^\alpha_{i,j}} \gamma_i$$

with the convention that the value of the empty sum is zero. We denote by $\boldsymbol{F_\Gamma} = (\sum_\alpha \frac{1}{d^\alpha_{i,j}})$ its associated n-square matrix called the **transition matrix**.

Notice that there is only a finite number of possible transition matrices for a Thurston map f of given number of post-critical points $|P_f|$ and given degree d since the order of a transition matrix is less than $|P_f|-3$ (see Lemma 5.6) and the number of terms in the sum for each entry and every degree $d^\alpha_{i,j}$ are less than d.

Since transition matrix has non-negative entries, there is an eigenvalue of largest modulus which is real and non-negative (see Perron-Frobenius theorem B.7 and Corollary B.8 in appendix).

Definition 5.8 (Thurston obstruction)**.** Let f be a ramified covering. For every multi-curve Γ, we denote by $\boldsymbol{\lambda(f_\Gamma)}$ the largest non-negative eigenvalue of the associated Thurston linear transformation. Any multi-curve Γ with $\lambda(f_\Gamma) \geqslant 1$ is called a **Thurston obstruction**.

The Thurston's topological characterization of rational maps is the following.

Theorem 5.9 (Thurston's topological characterization)**.** *A Thurston map with hyperbolic orbifold is Thurston equivalent to a rational map if and only if it has no Thurston obstruction. In that case, the rational map is unique up to conjugation by a Möbius transformation.*

Some remarks:

- The notion of orbifold can be found in [DH93]. We just mention that if a Thurston map f has a non-hyperbolic orbifold then $|P_f| \leqslant 4$ and every example in this thesis has a hyperbolic orbifold.

- We refer the readers to [DH93] for a proof.

- Initially Thurston obstructions are defined for stable multi-curves (see Definition 5.10). But we will see in Proposition 5.12 that the two definitions are equivalent.

- The Thurston's criterion does not give an algorithm to decide if a Thurston map is equivalent to a rational function, nor to construct the rational map. Many attempts were made in this direction, see in particular [Tan92], [ST00], [Kam01], [Pil03] and [BN06].

- Even if there is only a finite number of computations to do and conditions to check, the Thurston's criterion is difficult to implement numerically. Indeed the action of the Thurston map f on a subset of the fundamental group of $\mathbb{S}^2 - P_f$ (the free group of $|P_f| - 1$ generators) is needed.

- The Thurston's criterion deals only with the post-critically finite case but we refer the readers to [CT07] where the authors extend the Thurston's theorem to the sub-hyperbolic semi-rational maps (see Theorem 7.5).

We are going to show that we may restrict the criterion to special subsets of obstructions. Recall that a n-square matrix M is reducible if there exists a permutation matrix P (i.e. a n-square matrix that has exactly one entry 1 in each row and each column and 0's elsewhere) such that

$$P^{-1}MP = \begin{pmatrix} A & 0 \\ * & B \end{pmatrix}$$

where A is a k-square block with $1 \leqslant k < n$.

Definition 5.10 (irreducible and stable multi-curves)**.** Let Γ be a multi-curve associated to a ramified covering f.

- Γ is said **irreducible** if the transition matrix F_Γ of its associated Thurston linear transformation is not reducible.

- Γ is said **stable** if for each curve $\gamma \in \Gamma$, every non-peripheral connected component of $f^{-1}(\gamma)$ is homotopic in $\mathbb{S}^2 - \mathcal{P}_f$ to a curve in Γ.

The following combinatorial characterization will be useful later.

Lemma 5.11. *A multi-curve $\Gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_n\}$ associated to a ramified covering f is irreducible if and only if*

$$\begin{array}{l} \forall i, j \in \{1, 2, \ldots, n\},\ \exists r \geqslant 1 \textit{ and } i_0 = i, i_1, i_2, \ldots, i_{r-1}, i_r = j \in \{1, 2, \ldots, n\}/ \\ \quad \forall k \in \{1, 2, \ldots, r\},\ \exists \delta_{i_{k-1}, i_k} \textit{ connected component of } f^{-1}(\gamma_{i_k}) \\ \quad\quad \textit{homotopic to } \gamma_{i_{k-1}} \textit{ in } \mathbb{S}^2 - \mathcal{P}_f \end{array}$$

As a consequence for any curve γ in an irreducible multi-curve Γ there exists at least one connected component of the preimage $f^{-1}(\Gamma)$ which is homotopic to γ in $\mathbb{S}^2 - \mathcal{P}_f$.

Proof of Lemma 5.11. By definition of the Thurston linear transformation f_Γ (Definition 5.7), the existence of a connected component δ_{i_{k-1},i_k} is equivalent to the fact that the $(i_{k-1}, i_k)^{\text{th}}$ entry of the transition matrix F_Γ, say m_{i_{k-1},i_k}, is not zero. Therefore the contrapositive claims that F_Γ is reducible if and only if there exists a pair of indices $i_0, j_0 \in \{1, 2, \dots, n\}$ such that for every integer $r \geqslant 1$ the $(i_0, j_0)^{\text{th}}$ entry of the matrix F_Γ^r, say $m^r_{i_0,j_0}$, is zero. Necessity follows from

$$P^{-1} F_\Gamma^r P = \begin{pmatrix} A^r & 0 \\ * & B^r \end{pmatrix}$$

where A is a k-square block with $1 \leqslant k < n$ and P is a transition matrix associated with a permutation $\sigma \in \mathfrak{S}_n$. Thus for every integer $r \geqslant 1$, the entry $m^r_{\sigma(1),\sigma(n)}$ is zero. To prove sufficiency, let I be the set of integers $i \in \{1, 2, \dots, n\}$ such that there exists $r \geqslant 1$ with $m^r_{i_0,i} > 0$. We may assume that I is not empty, otherwise each entry of the i_0^{th} row of F_Γ is zero and then F_Γ is reducible (choosing P as the permutation matrix of the transposition exchanging 1 and i_0). Moreover the complement of I in $\{1, 2, \dots, n\}$ is not empty since $j \notin I$. Hence with a suitable choice of a permutation matrix P, we may assume that $I = \{1, 2, \dots, k\}$ where $1 \leqslant k < n$. Now for any $i \in I$, that is $m^r_{i_0,i} > 0$ for $r \geqslant 1$, and any $j \notin I$, if $m^s_{i,j} > 0$ for $s \geqslant 1$ then $m^{(r+s)}_{i_0,j} \geqslant m^r_{i_0,i} m^s_{i,j} > 0$ (because F_Γ has non-negative entries) contradicting $j \notin I$. In particular we get $m_{i,j} = 0$ for every $i \in \{1, 2, \dots, k\}$ and $j \in \{k+1, k+2, \dots, n\}$ proving that F_Γ is reducible. For the consequence, just take $\gamma_i = \gamma_j = \gamma$ then δ_{i_0,i_1} is a suitable connected component. □

The following result was stated in [ST00].

Proposition 5.12. *The following holds.*

1. *If f is a ramified covering then any Thurston obstruction contains an irreducible Thurston obstruction.*

2. *If f is a Thurston map, then any irreducible Thurston obstruction is homotopically (in $\mathbb{S}^2 - P_f$) contained in a stable Thurston obstruction.*

In particular a Thurston map with hyperbolic orbifold is Thurston equivalent to a rational map if and only if it has no stable Thurston obstruction.

Proof of Proposition 5.12. 1. Let F_Γ be the transition matrix of a Thurston obstruction $\Gamma = \{\gamma_1, \gamma_2, \dots, \gamma_n\}$. By induction we can find a permutation matrix P such that $P^{-1} F_\Gamma P$ is a lower block triangular matrix whose all blocks are irreducible. Remark that $P^{-1}F_\Gamma P$ is the transition matrix associated to the multi-curve $\Gamma_\sigma = \{\gamma_{\sigma(1)}, \gamma_{\sigma(2)}, \dots, \gamma_{\sigma(n)}\}$ for a certain permutation $\sigma \in \mathfrak{S}_n$. Therefore Γ can be considered as a disjoint union of sub-multi-curves (Γ_k) such that each associated transition matrix F_{Γ_k} is an irreducible block of $P^{-1}F_\Gamma P$. Moreover $\lambda(f_\Gamma) = \max_k\{\lambda(f_{\Gamma_k})\}$, thus any Γ_k with $\lambda(f_{\Gamma_k}) \geqslant 1$ is an irreducible Thurston obstruction.

2. Assume that Γ_0 is an irreducible Thurston obstruction. Let Γ_1 be the set of all non-peripheral connected components in $f^{-1}(\Gamma_0)$ quotiented by the equivalent relation of homotopy in $\mathbb{S}^2 - P_f$ (keeping only non-homotopic and non-peripheral Jordan curves). Since the curves in Γ_0 are disjoint from each other, so are the curves in $f^{-1}(\Gamma_0)$. Hence Γ_1 is a multi-curve. Moreover the consequence of Lemma 5.11 proves that Γ_0 is homotopically contained in Γ_1. By induction we can construct for every integer $k \geqslant 1$ a multi-curve (Γ_k) as the set of non-peripheral curves in $f^{-1}(\Gamma_{k-1})$ up to homotopy in $\mathbb{S}^2 - P_f$ such that Γ_{k-1} is homotopically contained in Γ_k. In particular $(|\Gamma_k|)$ is an increasing sequence of integers which is bounded by $|P_f| - 3$ as we noticed in Lemma 5.6. Thus there exists an integer $k \geqslant 1$ such that $|\Gamma_k| = |\Gamma_{k-1}|$ proving that Γ_{k-1} is a stable multi-curve which contains homotopically the Thurston obstruction Γ_0. The conclusion follows with Proposition B.5 and Proposition B.6 in appendix.

□

5.3 Levy cycles

Fortunately the Thurston's criterion may be simplified in polynomial case as we will see in this section.

Definition 5.13 (topological polynomial)**.** A ramified covering f is a **topological polynomial** if there exists a critical point $\omega \in \Omega_f$, called **infinity point**, such that $f^{-1}(\omega) = \{\omega\}$.

The infinity point is not necessarily unique but we can prove that there exist at most two infinity points as soon as the degree of the ramified covering is greater than two (the proof is similar as that one of Proposition 3.7).

Remark that the ramification portrait associated to a topological polynomial is of polynomial type.

Lemma 5.14. *Let f be a topological polynomial. For any Jordan curve γ in $\mathbb{S}^2 - \mathcal{P}_f$, if D denote the connected component of $\mathbb{S}^2 - \gamma$ which does not contain the infinity point of f then every connected component of $f^{-1}(D)$ is a topological disk.*

In order to simplify the Thurston's criterion in polynomial case, the lemma above is a key result. It will allow us to study preimages of topological disks instead of Jordan curves.

Proof of Lemma 5.14. Denote by ω the infinity point of f. At first notice that D is well defined since $\omega \in \mathcal{P}_f$. Let W be a connected component of $f^{-1}(\mathbb{S}^2 - D)$. f is a ramified covering of finite degree and $\mathbb{S}^2 - D$ is a connected compact set, therefore $f(W) = \mathbb{S}^2 - D$ (see Proposition A.11 in appendix). Furthermore $\omega \in \mathbb{S}^2 - D$ and $f^{-1}(\omega) = \{\omega\}$ thus $\omega \in W$. Consequently $f^{-1}(\mathbb{S}^2 - D)$ has only one connected component and every connected component of the set $f^{-1}(D)$ which is equal to $\mathbb{S}^2 - f^{-1}(\mathbb{S}^2 - D)$ (because of surjectivity of f) is simply connected. □

Definition 5.15 (Levy cycle)**.** Let f be a ramified covering. A multi-curve $\Gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_n\}$ is called a **Levy cycle** if for every integer $j \in \{1, 2, \ldots, n\}$, $f^{-1}(\gamma_j)$ has a connected component $\delta_{j-1,j}$ homotopic to γ_{j-1} in $\mathbb{S}^2 - \mathcal{P}_f$ (with the notation $\gamma_0 = \gamma_n$) and the map $f_{|\delta_{j-1,j}} : \delta_{j-1,j} \to \gamma_j$ is of degree one.

Proposition 5.16. *Any Levy cycle is an irreducible Thurston obstruction.*

Proof of Proposition 5.16. At first remark that a Levy cycle satisfies easily the characterization of irreducible multi-curve from Lemma 5.11 (choose by induction $i_k = i_{k-1} + 1$). Let F_Γ be the transition matrix associated to a Levy cycle Γ. By definition of Levy cycle, we have $F_\Gamma \geqslant M$ where M is the following non-negative permutation matrix

$$M = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ \vdots & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \ldots & 0 & 1 \\ 1 & 0 & \ldots & \ldots & 0 \end{pmatrix}$$

The conclusion follows from the fact that the spectral radius of M is 1 and from the monotonicity of spectral radius on the set of non-negative matrices (see Proposition B.5 in appendix). □

Initially Levy cycles are introduced to reduce the Thurston's criterion for quadratic ramified coverings (see [Lev85]). But they may also used to simplify the polynomial case as we are going to see now.

Theorem 5.17. *If a topological polynomial f has a Thurston obstruction then*

1. *f has a Levy cycle Γ contained in the Thurston obstruction*

2. *Denote by $\mathcal{D}$ the union over every $\gamma \in \Gamma$ of connected components of $\mathbb{S}^2 - \gamma$ which do not contain the infinity point of f. The following holds*

   (a) $\mathcal{D} \cap \mathcal{P}_f \neq \emptyset$

   (b) $f(\mathcal{D} \cap \mathcal{P}_f) \subset \mathcal{D} \cap \mathcal{P}_f$

   (c) $\mathcal{D} \cap \mathcal{P}_f$ *does not contain any critical point of f*

   As a consequence, there exist some post-critical points of f whose iterations do not accumulate a critical point.

The following proof was inspired by discussions in [ST00] in order to make that one from [Lev85] sharper and shorter. Actually we prove in addition that the result holds for non-post-critically finite maps whereas the works in [Lev85] deal only with post-critically finite maps.

Proof of Theorem 5.17. Let $\Gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_n\}$ be a Thurston obstruction associated to a topological polynomial f. By Proposition 5.12, we may assume that Γ is irreducible. Let ω be the infinity point of f. For every integer $i \in \{1, 2, \ldots, n\}$ denote by D_i the connected component of $\mathbb{S}^2 - \gamma_i$ which does not contain ω. Our first goal is to show that every topological disk D_i is disjoint from each other. For that we use the following key lemma.

Lemma 5.18. *We say that a topological disk D_i for a certain $i \in \{1, 2, \ldots, n\}$ is innermost if it contains no other topological disk $D_{i'}$ where $i' \neq i$.*

1. *Assume there exist a pair of integers $i, j \in \{1, 2, \ldots, n\}$ such that*

   (i) *D_j is innermost*

   (ii) *there exists a connected component $\delta_{i,j}$ of $f^{-1}(\gamma_j)$ homotopic to γ_i in $\mathbb{S}^2 - \mathcal{P}_f$*

   Then D_i is innermost

2. *For every $i \in \{1, 2, \ldots, n\}$, D_i is innermost. As a consequence the topological disks $D_1, D_2, \ldots, D_n$ are pairwise disjoint.*

Proof of Lemma 5.18. 1. By contradiction assume there exists $i' \neq i$ such that $D_{i'} \subset D_i$. The consequence of Lemma 5.11 proves the existence of an integer $j' \in \{1, 2, \dots, n\}$ and a connected component $\delta_{i',j'}$ of $f^{-1}(\gamma_{j'})$ which is homotopic to $\gamma_{i'}$ in $\mathbb{S}^2 - \mathcal{P}_f$. By Lemma 5.14, $\delta_{i,j}$ (respectively $\delta_{i',j'}$) is boundary of a topological disk $B_{i,j}$ (respectively $B_{i',j'}$) such that $f(B_{i,j}) = D_j$ (respectively $f(B_{i',j'}) = D_{j'}$). Now we are going to use some results from planar topology (see Appendix A.1) as it is suggested in Figure 5.1.

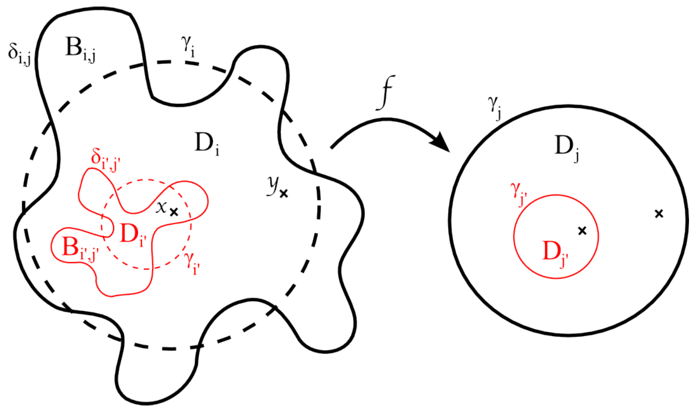


Figure 5.1: Nested Jordan curves in proof of Lemma 5.18

Since $\gamma_{i'}$ is non-peripheral and homotopic to $\delta_{i',j'}$ in $\mathbb{S}^2 - \mathcal{P}_f$, there exists at least one point $x \in \mathcal{P}_f$ which belongs to $D_{i'} \cap B_{i',j'}$ (see Lemma A.6). But $D_{i'} \subset D_i$ and γ_i is homotopic to $\delta_{i,j}$, therefore x is also in $B_{i,j}$. In particular we get $j \neq j'$ (otherwise $B_{i,j}$ and $B_{i',j}$ would be disjoint as connected components of $f^{-1}(D_j)$). So $\delta_{i,j}$ and $\delta_{i',j'}$ are disjoint since their images γ_j and $\gamma_{j'}$ are disjoint. Applying Lemma A.4, we get either $B_{i,j} \subset B_{i',j'}$ or $B_{i',j'} \subset B_{i,j}$. Furthermore γ_i and $\gamma_{i'}$ are not homotopic in $\mathbb{S}^2 - \mathcal{P}_f$ so there exists a point $y \in \mathcal{P}_f$ which belongs to $D_i - D_{i'}$. By Lemma A.6, y belongs also to $B_{i,j} - B_{i',j'}$. Consequently $B_{i',j'} \subset B_{i,j}$ and, pulling forward by f, $D_{j'} \subset D_j$ that is a contradiction.

2. Since Γ is a finite set (here is the key argument so that the proof still holds for the non-post-critically finite case), we may prove by induction that there exists an innermost disk, say D_j. Fix an integer $i \in \{1, 2, \dots, n\}$. The irreducibility of Γ implies the existence of a path of $r+1$ curves $\gamma_{i_0} = \gamma_i, \gamma_{i_1}, \gamma_{i_2}, \dots, \gamma_{i_{r-1}}$ and $\gamma_{i_r} = \gamma_j$ such that for every integer $k \in \{1, 2, \dots, r\}$ there exists a connected component δ_{i_{k-1},i_k} of $f^{-1}(\gamma_{i_k})$ homotopic to $\gamma_{i_{k-1}}$ in $\mathbb{S}^2 - \mathcal{P}_f$ (see Lemma 5.11). Apply-

ing successively the first point, we get that D_i is innermost. Finally it follows from Lemma A.4 that every D_i is disjoint from each other.

□

Let us come back to the proof of Theorem 5.17. Now we will discuss the form of the transition matrix $F_\Gamma = (\sum_\alpha \frac{1}{d^\alpha_{i,j}})$.

1. Let $\delta^\alpha_{i,j}$ and $\delta^\beta_{i,j}$ be two distinct connected components of $f^{-1}(\gamma_j)$ homotopic to γ_i in $\mathbb{S}^2 - \mathcal{P}_f$. By Lemma 5.14, $\delta^\alpha_{i,j}$ (respectively $\delta^\beta_{i,j}$) is boundary of a topological disk $B^\alpha_{i,j}$ (respectively $B^\beta_{i,j}$) where $B^\alpha_{i,j}$ and $B^\beta_{i,j}$ are two disjoint connected components of $f^{-1}(D_j)$. Let x be a point of $\mathcal{P}_f \cap D_i$ (it exists because γ_i is non-peripheral). Since $\delta^\alpha_{i,j}$ and $\delta^\beta_{i,j}$ are homotopic to γ_i, x is also in the intersection $B^\alpha_{i,j} \cap B^\beta_{i,j}$ (see Lemma A.6) that is impossible. Consequently, for every pair of integers $i, j \in \{1, 2, \ldots, n\}$, there exists at most one connected component of $f^{-1}(\gamma_j)$ homotopic to γ_i in $\mathbb{S}^2 - \mathcal{P}_f$. In other words, the sum in each entry of F_Γ contains at most one term.

2. Let $\delta_{i,j}$ be a connected component of $f^{-1}(\gamma_j)$ and $\delta_{i,k}$ be a connected component of $f^{-1}(\gamma_k)$ such that $j \neq k$ and they are both homotopic to γ_i in $\mathbb{S}^2 - \mathcal{P}_f$. By Lemma 5.14, $\delta_{i,j}$ (respectively $\delta_{i,k}$) is boundary of a topological disk $B_{i,j}$ (respectively $B_{i,k}$) where $B_{i,j}$ (respectively $B_{i,k}$) is a connected component of $f^{-1}(D_j)$ (respectively $f^{-1}(D_k)$). Let x be a point of $\mathcal{P}_f \cap D_i$ (it exists because γ_i is non-peripheral). Since $\delta_{i,j}$ and $\delta_{i,k}$ are homotopic to γ_i, x is also in the intersection $B_{i,j} \cap B_{i,k}$ (see Lemma A.6). Hence $f(x) \in D_j \cap D_k$ but recall that D_j and D_k are disjoints (by Lemma 5.18). Consequently, for every integer $i \in \{1, 2, \ldots, n\}$, there exists at most one curve γ_j such that $f^{-1}(\gamma_j)$ has a connected component homotopic to γ_i in $\mathbb{S}^2 - \mathcal{P}_f$. In other words, each row of F_Γ contains at most one non-zero entry.

3. Furthermore it follows from the consequence of Lemma 5.11 that each column of F_Γ contains at least one non-zero entry.

Finally, up to conjugation by a permutation matrix (that is with a suitable choice of an order for curves in Γ), we get from the results above:

$$F_\Gamma = \begin{pmatrix} 0 & \frac{1}{d_{1,2}} & 0 & \ldots & 0 \\ \vdots & 0 & \frac{1}{d_{2,3}} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \ldots & 0 & \frac{1}{d_{n-1,n}} \\ \frac{1}{d_{n,1}} & 0 & \ldots & \ldots & 0 \end{pmatrix}$$

Therefore $\lambda(f_\Gamma) = (d_{1,2}d_{2,3}\dots d_{n-1,n}d_{n,1})^{-1} \geqslant 1$. Since each $d_{k-1,k}$ is an integer, they are all equal to 1 concluding the proof of the first point.

For the second one, it follows from Lemma A.6 that

$$\forall j \in \{1,2,\dots,n\},\ f(D_{j-1}\cap\mathcal{P}_f) = f(B_{j-1,j}\cap\mathcal{P}_f) \subset D_j\cap\mathcal{P}_f$$

with the notations $D_0 = D_n$ and $B_{0,1} = B_{n,1}$. Since each $D_j\cap\mathcal{P}_f$ is not empty (because γ_j is non-peripheral), the union of these sets is a nonempty set of post-critical points which is stable after iteration of f. If one element of this set is a critical point, say $c \in D_{j_0-1}$, then the map $f_{|B_{j_0-1,j_0}} : B_{j_0-1,j_0} \to D_{j_0}$ is of degree at least two (e.g. by Riemann-Hurwitz formula, see Theorem A.12). That is a contradiction since the map $f_{|\delta_{j_0-1,j_0}} : \delta_{j_0-1,j_0} \to \gamma_{j_0}$ is of degree one. □

Example 5.19. Come back to f_{ana} defined in Example 4.13. Now we are able to prove what we claimed in Example 3.15, that is f_{ana} is not Thurston equivalent to a polynomial. Let γ_1 be a Jordan curve surrounding the edge of T_{ana} with endpoints c_2', c_1 and γ_2 be a Jordan curve surrounding the edge of T_{ana} with endpoints c_2, c_1' (see Figure 5.2).

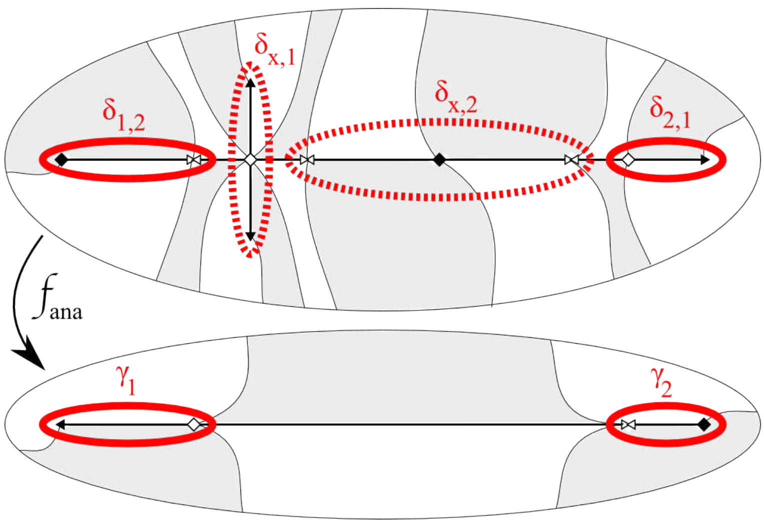


Figure 5.2: Example of a Levy cycle

If we pullback γ_1, we get two preimages: one, say $\delta_{2,1}$, is homotopic to γ_2 and the other one, say $\delta_{\times,1}$ is not homotopic to either γ_2 or γ_1. We get likewise for preimages of γ_2: $\delta_{1,2}$ which is homotopic to γ_1 and $\delta_{\times,2}$. Furthermore, if γ_1 and γ_2 are chosen small enough then they do not surround any critical points of f_{ana} (i.e. neither c_0 nor c_0'). It follows that the restriction maps

$f_{|\delta_{1,2}} : \delta_{1,2} \to \gamma_2$ and $f_{|\delta_{2,1}} : \delta_{2,1} \to \gamma_1$ are of degree one. Finally, the multicurve $\{\gamma_1, \gamma_2\}$ is a Levy cycle for the ramified covering f_{ana} and Theorem 5.17 and Theorem 5.9 imply that f_{ana} is not Thurston equivalent to a polynomial.

The following result due to [Lev85] simplifies the Thurston's criterion in polynomial case. In particular that allows us to show that the maps constructed in Theorem 4.12 are Thurston equivalent to polynomials.

Theorem 5.20 (Levy)**.** *Let f be Thurston map. Assume that f is a topological polynomial and that every critical point of f falls after some iterations into a periodic cycle containing a critical point. Then f is Thurston equivalent to a polynomial.*

Of course this condition is easier to check than to find a Thurston obstruction.

Proof of Theorem 5.20. The result follows from the consequence of the second point of Theorem 5.17 in the post-critically finite case and from Theorem 5.9. □

Corollary 5.21. *Any N-cyclic ramification portrait of polynomial type where N is a positive integer is realized by a polynomial.*

Proof of Corollary 5.21. Apply Theorem 4.12 and then Theorem 5.20. □

For a given N-cyclic ramification portrait of polynomial type $\mathcal{R}$, the Thurston class of a polynomial which realizes $\mathcal{R}$ is not necessarily unique. The same remark as for Theorem 4.12 holds: two different shapes for trees which extend as ramified coverings with associated ramification portrait $\mathcal{R}$ would lead to two polynomials which are not necessarily Thurston equivalent.

5.4 Siegel rational maps

We conclude this chapter by illustrating how Levy cycles are useful in order to give an easy to check criterion from a Thurston-like characterization. More precisely we present a result from [Zha08] and we would like to merge it with Theorem 5.17 to get a simple condition ensuring the existence of Siegel rational maps with prescribed dynamics.

We follow the notations from [Zha08].

Definition 5.22 (Siegel rational map)**.** A **Siegel rational map** is a rational map $g : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that

(i) g has a Siegel disk D_g with rotation number $\theta \in \mathbb{R}$, that is there exist a topological disk D_g and a biholomorphic map $\phi : D_g \to \mathbb{D}$ such that the following diagram is commutative

$$\begin{array}{ccc} D_g & \xrightarrow{\ \phi\ } & \mathbb{D} \\ {\scriptstyle g}\downarrow & & \downarrow {\scriptstyle z \mapsto e^{2i\pi\theta}z} \\ D_g & \xrightarrow{\ \phi\ } & \mathbb{D} \end{array}$$

(ii) ∂D_g is a quasicircle (i.e. the image of $\partial\mathbb{D}$ by a quasiconformal map)

(iii) $\mathcal{P}_g - \overline{D_g}$ is a finite set

If a Siegel rational map g has a rotation number θ of bounded type (i.e. whose associated sequence of its infinite continued fraction representation is bounded) then ∂D_g must contain at least one critical point of g (by a result of J. Graczyck and G. Swiatek, see [Zha08]). That justifies the second condition in definition below.

Definition 5.23 (Siegel ramified covering of bounded type)**.** A **Siegel ramified covering of bounded type** is an orientation-preserving ramified covering $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that

(i) $f_{|\Delta_f} : z \mapsto e^{2i\pi\theta}z$ where Δ_f is the unit disk and $\theta \in \mathbb{R}$, called the rotation number of f, is of bounded type

(ii) $\partial\Delta_f \cap \Omega_f \neq \emptyset$

(iii) $\mathcal{P}_f - \overline{\Delta_f}$ is a finite set

Notice that a Siegel rational map with rotation number of bounded type is a Siegel ramified covering of bounded type.

Definition 5.24 (combinatorially equivalence)**.** Two Siegel ramified coverings of bounded type f and g are said **combinatorially equivalent** if there exist two orientation-preserving homeomorphisms φ_0 and φ_1 of $\widehat{\mathbb{C}}$ such that

(i) the following diagram commutes

$$\begin{array}{ccc} \widehat{\mathbb{C}} & \xrightarrow{\varphi_1} & \widehat{\mathbb{C}} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ \widehat{\mathbb{C}} & \xrightarrow{\varphi_0} & \widehat{\mathbb{C}} \end{array}$$

(ii) $\varphi_0(\mathcal{P}_f) = \varphi_1(\mathcal{P}_f) = \mathcal{P}_g$

(iii) φ_0 is isotopic to φ_1 relative to $\mathcal{P}_f$

(iv) $\varphi_{0|\Delta_f} = \varphi_{1|\Delta_f} : \Delta_f \to \Delta_g$ is holomorphic

Compare the definition above with Definition 5.2. As Thurston theorem for the Thurston equivalence classes (see Theorem 5.9), the following result characterizes the combinatorially equivalence classes containing a Siegel rational map.

Theorem 5.25 (Zhang Gaofei)**.** *A Siegel ramified covering of bounded type $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is combinatorially equivalent to a Siegel rational map if and only if it has no Thurston obstruction on the outside of the rotation disk, that is no Thurston obstruction Γ such that every curve $\gamma \in \Gamma$ lies in $\widehat{\mathbb{C}} - (\overline{\Delta_f} \cup \mathcal{P}_f)$.*

Unfortunatly this Thurston-like characterization is difficult to check in pratical since we already remarked that we have no algorithm to decide if there exists a Thurston obstruction.

However Theorem 5.17 allows to simplify the criterion as follows.

Theorem 5.26. *Let f be a Siegel topological polynomial of bounded type. Assume that every periodic cycles in $\mathcal{P}_f - \overline{\Delta_f}$ contains a critical point. Then f is combinatorially equivalent to a Siegel polynomial.*

Notice that we need the statement of Theorem 5.17 in the non-post-critically finite case since the post-critical set of a Siegel rational map is not finite. Actually if f is a Siegel ramified covering of bounded type then it follows from conditions *(i)* and *(ii)* of Definition 5.23 that $\mathcal{P}_f$ contains $\partial\Delta_f$.

Proof of Theorem 5.26. Assume by contradiction that there exists a Thurston obstruction on the outside of the rotation disk of f. It follows from the first point of Theorem 5.17 that f has a Levy cycle Γ on the outside of the rotation disk of f. For every $\gamma_i \in \Gamma$, denote by D_i the connected component of $\widehat{\mathbb{C}} - \gamma_i$ which does not contain the infinity point of f.

Assume that there exists a curve $\gamma_i \in \Gamma$ such that D_i contains the rotation disk $\overline{\Delta_f}$. Recall that $\partial\Delta_f$ is contained in $\mathcal{P}_f$ and $\partial\Delta_f$ contains a critical point. Therefore $D_i \cap \mathcal{P}_f$ contains a critical point which is not possible by the second point of Theorem 5.17.

Consequently the union $\mathcal{D}$ of topological disks D_i over every curve $\gamma_i \in \Gamma$ is contained in $\widehat{\mathbb{C}} - \overline{\Delta_f}$. Since $\mathcal{P}_f - \overline{\Delta_f}$ is finite, every post-critical point in $\mathcal{D}$ falls after some iterations into a periodic cycle which contains a critical point by assumption. That contredicts the second point of Theorem 5.17.

Finally there is no Thurston obstruction on the outside of the rotation disk of f. The conclusion follows with Theorem 5.25. □

Example 5.27. In order to produce Siegel topological polynomials of bounded type which satisfy the assumption of Theorem 5.26, we may use a construction similar to that one explained in Chapter 4. For instance, Figure 5.3 shows the Julia set of a cubic Siegel polynomial coming from the gluing at the beta point (see Lemma 4.11) of a starlike tree (see Lemma 4.10) corresponding to the ramification portrait of the quadratic polynomial f_{rabbit} (see Example 5.4) together with another combinatorial data associated to a quadratic polynomial with one fixed Siegel disk Δ which has a rotation number equals to the golden ratio (we may take the union of a dynamical tree with the rotation disk as combinatorial data).

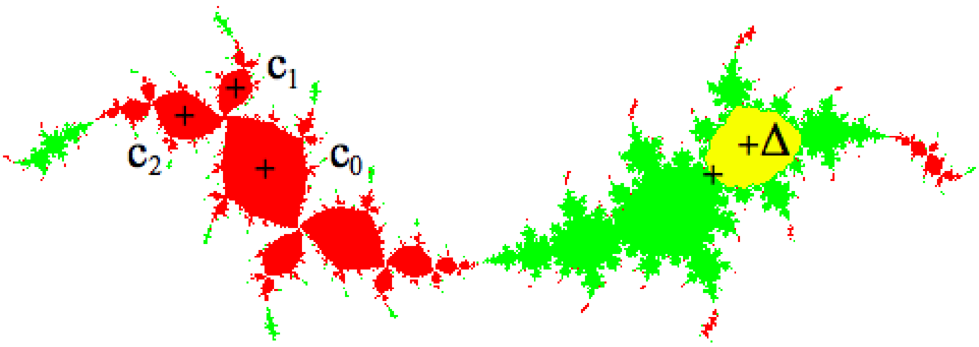


Figure 5.3: Example of cubic Siegel polynomial

Chapter 6

From a tree to a Persian carpet

We have presented different kind of obstructions occuring in realization of prescribed dynamics by post-critically rational map. In accordance with previous discussions, we would like now to construct a concrete example of non-post-critically rational map whose dynamics is encoded by a sharpening of a dynamical tree.

6.1 Weighted Hubbard trees

We will push further the discussion began in Section 4.1.

Definition 6.1 (restriction of a planar tree)**.** Let $T = (V, E)$ be a planar tree and let $W \subset V$ be a subset of vertices. The **restriction** of T determined by W is the planar tree $\boldsymbol{T[W] = (V[W], E[W])}$ where

- $V[W] \subset V$ is the union of vertices in W together with the vertices of V which are in the intersection of at least two Jordan arcs $[v, v']_T$ between vertices in W
- $E[W]$ is the set of Jordan arcs $[v, v']_T$ between vertices in $V[W]$

In some sense, the restriction $T[W]$ is the convex hull of W in T. Recall that for every dynamical tree $\mathcal{T} = (T, \tau, \delta)$, the dynamical map $\tau : V \to V$ can be extended continuously to T (see Definition 4.3).

Proposition 6.2. *Let $\mathcal{T} = (T, \tau, \delta)$ be a dynamical tree. We have:*

1. *For any Jordan arc $[v, v']_T$ between vertices $v, v' \in V$ containing no critical point of $\mathcal{T}$ except possibly for its endpoints,*

$$\tau([v, v']_T) = [\tau(v), \tau(v')]_T$$

2. *For any subset of vertices $W \subset V$,*

$$T[\tau(W)] \subset \tau(T[W]) \subset T[\tau(W \cup \Omega_{\mathcal{T}})]$$

Proof of Proposition 6.2. 1. Clearly $[\tau(v), \tau(v')]_T \subset \tau([v, v']_T)$. Now assume by contradiction that the previous inclusion is strict. Therefore there exists an edge outside of $[\tau(v), \tau(v')]_T$ which belongs to the images of at least two edges in $[v, v']_T$, thus $\tau_{|[v,v']_T}$ is not injective. Consider the nonempty set $K = \{(z_1, z_2) \in ([v, v']_T)^2 \,/\, z_1 \neq z_2 \text{ and } \tau(z_1) = \tau(z_2)\}$. Since $\tau_{|[v,v']_T}$ is locally injective by assumption (see Definition 4.3), K is a compact set. Take $(z_1, z_2) \in K$ such that $|z_1 - z_2|$ is minimal and let z be a point in the Jordan arc $[z_1, z_2]_T$ distinct from z_1 and z_2. Notice that $\tau(z)$ is necessarily distinct from $\tau(z_1) = \tau(z_2)$ since $|z_1 - z_2|$ is minimal. Moreover

$$[\tau(z_1), \tau(z)]_T = [\tau(z), \tau(z_2)]_T \subset \tau([z_1, z]_T) \cap \tau([z, z_2]_T)$$

Hence we can find $z_1' \in [z_1, z]_T$ distinct from z_1 and $z_2' \in [z, z_2]_T$ distinct from z_2 such that $\tau(z_1') = \tau(z_2')$ contradicting the minimality of $|z_1 - z_2|$.

2. Since τ is a continuous map, $\tau(T[W])$ is a convex subset of T containing $\tau(W)$, and so the convex hull $T[\tau(W)]$. Furthermore $T[W]$ is the union of Jordan arcs of the form $[v, v']_T$ between endpoints $v, v' \in W \cup \Omega_{\mathcal{T}}$ containing no critical point of $\mathcal{T}$ except possibly for its endpoints. Therefore the second inclusion follows from the first part.

□

The proposition above justifies the following definition.

Definition 6.3 (restriction of a dynamical tree)**.** Let $\mathcal{T} = (T, \tau, \delta)$ be a dynamical tree and let $W \subset V$ such that $\tau(W) \cup \Omega_{\mathcal{T}} \subset W$. The **restriction** of $\mathcal{T}$ determined by W is the dynamical tree $\boldsymbol{\mathcal{T}[W] = (T[W], \tau_{|T[W]}, \delta_{|V[W]})}$ where

- $T[W] = (V[W], E[W])$ is the restriction of the planar tree $T = (V, E)$ determined by W
- $\tau_{|T[W]} : T[W] \to \tau(T[W]) = T[\tau(W)] \subset T[W]$ is the restriction of the map τ on $T[W]$
- $\delta_{|V[W]} : V[W] \to \mathbb{N} - \{0\}$ is the restriction of the map δ on $V[W]$

Actually the restriction may be seen as the inverse operation of extension (see Definition 4.6) in the sense that $\widetilde{\mathcal{T}}$ is an extension of $\mathcal{T} = (T, \tau, \delta)$ where $T = (V, E)$ if and only if $\mathcal{T} \simeq \widetilde{\mathcal{T}}[V]$.

Recall that the set of all dynamical trees is partially ordered by $\preceq$ (Proposition 4.7)

Theorem 6.4. *Any dynamical tree has a unique minimal restriction. Moreover this dynamical tree is the restriction determined by the union of the critical and post-critical points.*

Proof of Theorem 6.4. Clearly $\mathcal{T}[\Omega_{\mathcal{T}} \cup P_{\mathcal{T}}]$ is the least element in the set of all restrictions of $\mathcal{T}$. □

Definition 6.5 (abstract Hubbard tree)**.** An **abstract Hubbard tree** is a minimal restriction $\boldsymbol{\mathcal{H}} = \boldsymbol{\mathcal{T}}[\boldsymbol{\Omega_{\mathcal{H}}} \cup \boldsymbol{P_{\mathcal{H}}}]$ of a dynamical tree $\mathcal{T}$ of degree $\deg(\mathcal{T}) = \deg(\mathcal{H}) \geqslant 2$.

Example 6.6. A. Douady and J. H. Hubbard introduced Hubbard tree associated to post-critically finite polynomial in [DH84] as follows. In the filled Julia set $K(P)$ of a post-critically finite polynomial P, consider Jordan arcs such that their intersections with any Fatou component (which is necessary associated to a super-attracting cycle) consist of the union of finitely many internal rays. Then the Hubbard tree associated to P is the smallest closed connected infinite union of those particular Jordan arcs which contains the union of critical and post-critical sets of P. A. Douady an J. H. Hubbard proved that this construction is unique and defines a topological tree, called the Hubbard tree of P. The action of any post-critically finite polynomial on its associated Hubbard tree provides a family of examples of abstract Hubbard trees.

Observe that Hubbard trees of post-critically finite polynomials encode more informations about dynamical properties than ramification portraits. Actually A. Douady and J. H. Hubbard showed in [DH84] that two non-conformally conjugate post-critically finite polynomials provide two different tree structures (a Hubbard tree, a dynamics on it coming from the polynomial and a bit of extra information) but they did not give criterion for realization. That was done by A. Poirier in [Poi93] who considered abstract Hubbard trees close to those of Definition 6.5 but with two more assumptions: one "angled" dynamical property around a certain type of vertices and one expanding condition. See also [AF00] where the authors study how much of dynamical information about a quadratic Misiurewicz polynomial (whose critical point is preperiodic) is captured by the Hubbard tree.

The following particular points play important role in dynamics of abstract Hubbard trees.

Definition 6.7 (Misiurewicz points of an abstract Hubbard tree)**.** Let $\mathcal{H} = (T, \tau, \delta)$ be an abstract Hubbard tree. A point in $T = \bigcup_{e \in E} e$ is said of **Misiurewicz type** if it is mapped after a finite number of iterations of τ to a periodic vertex whose corresponding cycle does not contain any critical point.

Now we are going to equip abstract Hubbard trees with more informations in order to encode the dynamics of some rational maps.

Definition 6.8 (weighted Hubbard tree)**.** A **weighted Hubbard tree** is the data of

- an abstract Hubbard tree $\mathcal{H} = (T, \tau, \delta)$ where $T = (V, E)$
- a weight function $w : E \to \mathbb{N} - \{0\}$

We denote by $\boldsymbol{(\mathcal{H}, w)}$ such a weighted Hubbard tree.

In the following, the tree T of a weighted Hubbard tree $\mathcal{H} = (T, \tau, \delta)$ will be considered to be embedded in $\mathbb{R}^3$. So we may forget the cyclic order of edges at a common endpoint.

Definition 6.9 (transition matrix of a weighted Hubbard tree)**.** Let $(\mathcal{H}, w)$ be a weighted Hubbard tree and $E = \{e_1, e_2, \ldots, e_n\}$ be the set of edges of the associated planar tree. For every pair of integers $i, j \in \{1, 2, \ldots, n\}$ define the following non-negative entries

$$h_{i,j} = \begin{cases} \frac{1}{w_{\mathcal{H}}(e_i)} & \text{if } e_j \subset \tau_{\mathcal{H}}(e_i) \\ 0 & \text{otherwise} \end{cases}$$

The n-square matrix $M = (h_{i,j})$ is called a **transition matrix** of $(\mathcal{H}, w)$.

Compare this definition with Definition 5.7. Remark that two distinct ordering of the edges in E lead to two transition matrices that are conjugated by a permutation matrix. Therefore the largest non-negative eigenvalue (see Corollary B.8 in appendix) of the transition matrix is well defined and is independent on the edge ordering. We may thus define:

Definition 6.10 (unobstructed weighted Hubbard tree)**.** Let $(\mathcal{H}, w)$ be a weighted Hubbard tree. We denote by $\boldsymbol{\lambda(\mathcal{H})}$ the largest non-negative eigenvalue of an associated transition matrix. If $\lambda(\mathcal{H}) < 1$, we say that $(\mathcal{H}, w)$ is **unobstructed**.

Example 6.11. Consider the weighted Hubbard tree $(\mathcal{H}, w)$ displayed in Figure 6.1. The arrows depict the dynamics of the map τ, and the numbers depict the weight function w. Hence

$$\begin{cases} \tau(e_{\alpha,c_1}) = e_{\alpha,c_2} \\ \tau(e_{\alpha,c_2}) = e_{\alpha,c_3} \\ \tau(e_{\alpha,c_3}) = e_{\alpha,c_1} \end{cases} \text{and} \begin{cases} w(e_{\alpha,c_1}) = 2 \\ w(e_{\alpha,c_2}) = 2 \\ w(e_{\alpha,c_3}) = 1 \end{cases}$$

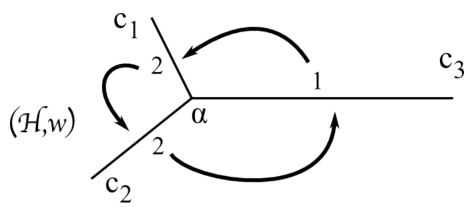


Figure 6.1: Example of unobstructed weighted Hubbard tree

Using the same order of the edges as above, we get the following transition matrix of $(\mathcal{H}, w)$

$$M = \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \\ 1 & 0 & 0 \end{pmatrix}$$

It follows that $\lambda(\mathcal{H})$ is the largest non-negative root of $4X^3 - 1$, that is $2^{-2/3} < 1$. Therefore $(\mathcal{H}, w)$ is unobstructed.

6.2 Weaving by quasiconformal surgery

Consider the weighted Hubbard tree $(\mathcal{H}, w)$ displayed in Figure 6.2.

The abstract Hubbard tree $\mathcal{H} = (T, \tau, \delta)$ has one fixed branching point α of Misiurewicz type and one periodic cycle of four vertices $\{c_0, c_1, c_2, c_3\}$ containing a critical point c_0 of local degree $\delta(c_0) = 2$. Actually $\mathcal{H}$ is the Hubbard tree associated to a quadratic polynomial of the form $z \mapsto z^2 + c$ where $c \approx -0.157 + 1.032i$ (see Example 6.6). The Julia set of this quadratic polynomial is drawn in Figure 6.3.

We have

$$\begin{cases} \tau(e_{\alpha,c_0}) = e_{\alpha,c_1} \\ \tau(e_{\alpha,c_1}) = e_{\alpha,c_2} \\ \tau(e_{\alpha,c_2}) = e_{\alpha,c_0} \cup e_{c_0,c_3} \\ \tau(e_{c_0,c_3}) = e_{\alpha,c_1} \cup e_{\alpha,c_0} \end{cases} \text{and} \begin{cases} w(e_{\alpha,c_0}) = 1 \\ w(e_{\alpha,c_1}) = 2 \\ w(e_{\alpha,c_2}) = 2 \\ w(e_{c_0,c_3}) = 1 \end{cases}$$

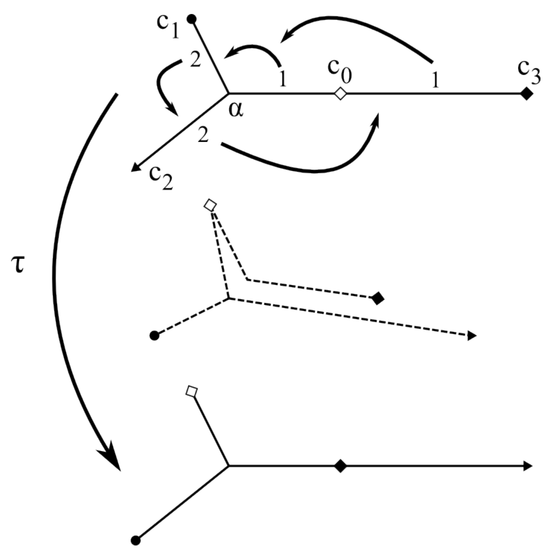


Figure 6.2: The weighted Hubbard tree $(\mathcal{H}, w)$

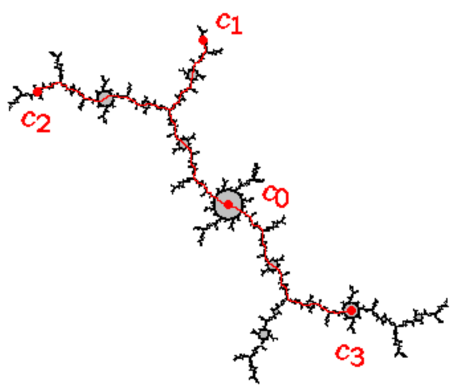


Figure 6.3: Hubbard tree of $z \mapsto z^2 - 0.157 + 1.032i$

Using the same order for the edges of $\mathcal{H}$ as above, we get the following transition matrix of $(\mathcal{H}, w)$

$$M = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

An easy computation shows that $\lambda(\mathcal{H})$ is the largest non-negative root of $4X^4 - 2X - 1$ that is $\lambda(\mathcal{H}) \approx 0.918 < 1$. Thus the weighted Hubbard tree $(\mathcal{H}, w)$ is unobstructed. We will discuss in Section 6.4 what it happens for a different choice of weight function w in order that $(\mathcal{H}, w)$ is no longer unobstructed.

Now consider the dynamical tree $\widehat{\mathcal{T}} = (\widehat{T}, \widehat{\tau}, \widehat{\delta})$ deduced from $\mathcal{H}$ by removing the critical point c_0 and by sending c_3 to c_1 instead of c_0, as displayed in Figure 6.4. In some sense, we have deleted the folding point of $\mathcal{H}$.

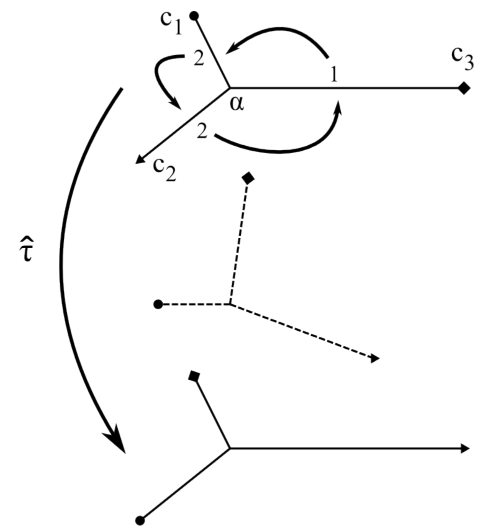


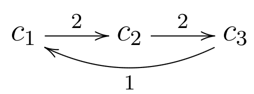
Figure 6.4: The dynamical tree $\widehat{\mathcal{T}}$ deduced from $\mathcal{H}$

We equip $\widehat{\mathcal{T}}$ with a weight function $\widehat{w} : \widehat{E} \to \mathbb{N} - \{0\}$ deduced from w by removing the weight on e_{c_0,c_3} and by keeping that one on e_{α,c_0}

$$\begin{cases} \widehat{w}(e_{\alpha,c_3}) = 1 \\ \widehat{w}(e_{\alpha,c_1}) = 2 \\ \widehat{w}(e_{\alpha,c_2}) = 2 \end{cases}$$

In this way, we get the weighted Hubbard tree of Example 6.11. In particular $(\widehat{\mathcal{H}}, \widehat{w})$ is still unobstructed.

We can easily find a rational map "encoded" by $(\widehat{\mathcal{T}}, \widehat{w})$, that is a rational mal which realizes the ramification portrait $\widehat{\mathcal{R}}$ below where the degrees come from the weight function $\widehat{w}$ on the corresponding edges.



Indeed choosing $c_1 = 1$, $c_2 = \infty$ and $c_3 = 0$, this ramification portrait is realized by the map $z \mapsto z^2$ precomposed with a Möbius transformation which acts as a circular permutation on $\{0, 1, \infty\}$:

$$\widehat{f} = \left(z \mapsto z^2\right) \circ \left(z \mapsto \frac{1}{1-z}\right) = \left(z \mapsto \frac{1}{(1-z)^2}\right)$$

The Julia set of $\widehat{f}$ is drawn in Figure 6.5. This Julia set plays an important role in the Hubbard's twisted rabbit problem disccussed in Example 5.4.

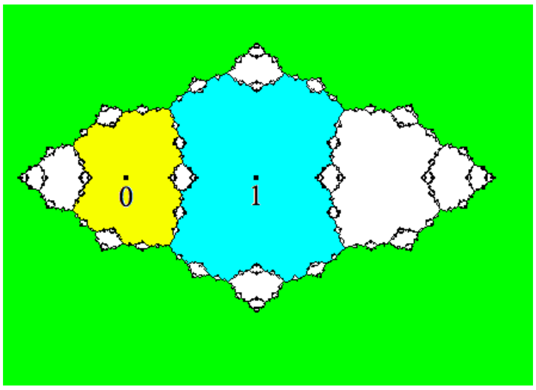


Figure 6.5: The Julia set of $\widehat{f}$

Now we are going to explain more precisely how $\widehat{f}$ is "encoded" by $(\widehat{\mathcal{T}}, \widehat{w})$. To do so, for any point c in the attracting cycle $\{0, 1, \infty\}$ of $\widehat{f}$, consider $B(c)$ the connected component containing c of the immediate attracting bassin (i.e. the set of points whose forward orbits accumulate the super-attracting cycle containing c). A classical result in holomorphic dynamical systems claims that $\widehat{f}^{\circ 3}_{|B(c)} : B(c) \to B(c)$ is conjugated to $(z \mapsto z^4) : \mathbb{D} \to \mathbb{D}$ by a biholomorphic map ϕ_c, called the Böttcher coordinates (the ϕ_c are unique up to multiplication by a third root of unity), with $\phi_c(c) = 0$. Furthermore the ϕ_c together for $c \in \{0, 1, \infty\}$ make the following diagram commutative:

$$\begin{array}{ccc}
B(0) & \xrightarrow{\phi_0} & \mathbb{D} \\
\Big\downarrow \widehat{f} & & \Big\downarrow z \mapsto z \\
B(1) & \xrightarrow{\phi_1} & \mathbb{D} \\
\Big\downarrow \widehat{f} & & \Big\downarrow z \mapsto z^2 \\
B(\infty) & \xrightarrow{\phi_\infty} & \mathbb{D} \\
\Big\downarrow \widehat{f} & & \Big\downarrow z \mapsto z^2 \\
B(0) & \xrightarrow{\phi_0} & \mathbb{D}
\end{array}$$

Then define a map $\widehat{\pi} : \widehat{\mathbb{C}} \to \widehat{T} = e_{\alpha,c_3} \cup e_{\alpha,c_1} \cup e_{\alpha,c_2} \subset \mathbb{C}$ as follows

$$\widehat{\pi} : z \mapsto \begin{cases} |\phi_0(z)|\alpha + (1 - |\phi_0(z)|)c_3 \in e_{\alpha,c_3} & \text{if } z \in B(0) \\ |\phi_1(z)|\alpha + (1 - |\phi_1(z)|)c_1 \in e_{\alpha,c_1} & \text{if } z \in B(1) \\ |\phi_\infty(z)|\alpha + (1 - |\phi_\infty(z)|)c_2 \in e_{\alpha,c_2} & \text{if } z \in B(\infty) \\ \alpha & \text{otherwise} \end{cases}$$

In this way, $\widehat{\pi}$ is a continuous and surjective map such that the preimage of each point of $\widehat{T} - \{\alpha\}$ is an equipotential (i.e. a preimage of a circle centered at the origin by the Böttcher coordinates). Moreover we may find a suitable continuous extension of $\widehat{\tau} : \widehat{V} \to \widehat{V}$ to $\widehat{T}$ in Definition 4.3 in order to make the following diagram commutative (see Figure 6.6)

$$\begin{array}{ccc} \overline{\bigcup B(c)} & \xrightarrow{\widehat{\pi}} & \widehat{T} \\ \widehat{f}\downarrow & & \downarrow\widehat{\tau} \\ \overline{\bigcup B(c)} & \xrightarrow[\widehat{\pi}]{} & \widehat{T} \end{array}$$

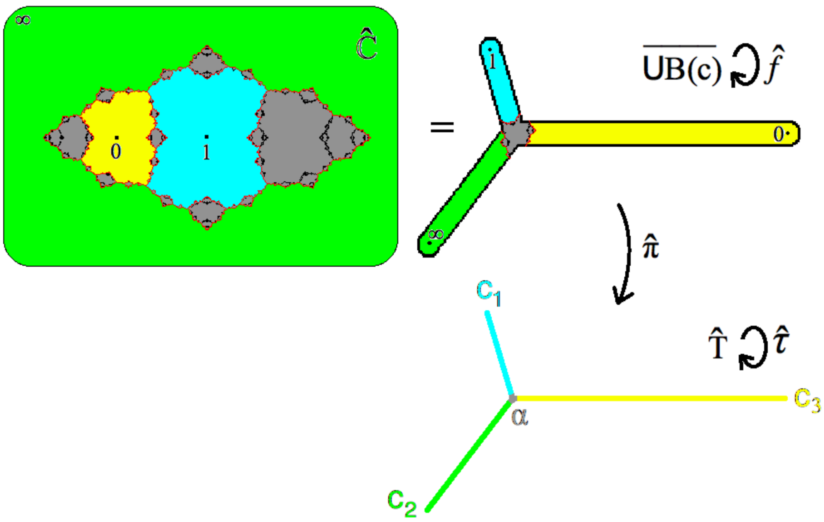


Figure 6.6: $\widehat{f}$ is encoded by $(\widehat{\mathcal{T}}, \widehat{w})$

Actually define $\widehat{\tau}$ as follows

$$\widehat{\tau} : x \mapsto \begin{cases} \mu\alpha + (1-\mu)c_1 \in e_{\alpha,c_1} & \text{if } x = \mu\alpha + (1-\mu)c_3 \in e_{\alpha,c_3} \\ \mu^2\alpha + (1-\mu^2)c_2 \in e_{\alpha,c_2} & \text{if } x = \mu\alpha + (1-\mu)c_1 \in e_{\alpha,c_1} \\ \mu^2\alpha + (1-\mu^2)c_3 \in e_{\alpha,c_3} & \text{if } x = \mu\alpha + (1-\mu)c_2 \in e_{\alpha,c_2} \\ \alpha & \text{otherwise} \end{cases}$$

Heuristically speaking, we may think of the action of $\widehat{f}$ on $\overline{\bigcup B(c)}$ as that one of $\widehat{\tau}$ on the boundary of a small "thickening" of the tree $\widehat{T}$ embedded in $\mathbb{R}^3$ as it is suggested in Figure 6.6.

Now we would like to construct a rational map f whose dynamics is encoded by the weighted Hubbard tree $(\mathcal{H}, w)$. To do so, we start with the rational map $\widehat{f}$ and we create a kind of "folding" point c_0 inside $B(0)$ (which corresponds to the edge e_{α,c_3}) by a surgery process. More precisely, we are going to divide the Riemann sphere $\widehat{\mathbb{C}}$ into several pieces whose boundaries are equipotentials, and then we will define a continuous and sufficiently regular map $F : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ which is encoded by $(\mathcal{H}, w)$ as a piecewise map. Finally the quasiconformal surgery principle (see Theorem C.13 in appendix) will provide us a rational map f as required.

Step 1 - Cutting off For every $c \in \{0, 1, \infty\}$, denote by α_c the boundary of $B(c)$ (recall that $B(c)$ is a topological disk since it is biholomorphically mapped onto the unit disk $\mathbb{D}$).

We follow the notations of Definition C.4 in appendix: for any pair of disjoint continua γ and γ', we denote by $A(\gamma, \gamma')$ the unique doubly connected component of $\widehat{\mathbb{C}} - (\gamma \cup \gamma')$ and by $\mathrm{mod}(\gamma, \gamma') > 0$ its modulus. In particular if γ and γ' are two equipotentials of levels $|\phi_c(\gamma')| < |\phi_c(\gamma)|$ in $B(c)$ for any $c \in \{0, 1, \infty\}$, then

$$\mathrm{mod}(\gamma, \gamma') = \frac{1}{2\pi} \log\left(\frac{|\phi_c(\gamma)|}{|\phi_c(\gamma')|}\right)$$

Lemma 6.12. *Given any positive constant $C > 0$, there exist five equipotentials β_0, β_1, β_2, γ_{-3} and γ_{+3} such that*

(i) $\beta_0 \subset B(0)$, $\beta_1 \subset B(1)$ and $\beta_2 \subset B(2)$

(ii) $\gamma_{-3}, \gamma_{+3} \subset B(0)$ and $|\phi_0(\beta_0)| > |\phi_0(\gamma_{-3})| > |\phi_0(\gamma_{+3})|$

(iii) the following inequalities hold

$$\left\{ \begin{array}{rcl} \mathrm{mod}(\alpha_1, \beta_1) & < & \mathrm{mod}(\alpha_0, \beta_0) \\ \frac{1}{2}\,\mathrm{mod}(\alpha_2, \beta_2) & < & \mathrm{mod}(\alpha_1, \beta_1) \\ \frac{1}{2}\,\mathrm{mod}(\alpha_0, \beta_0) + \frac{1}{2}\,\mathrm{mod}(\gamma_{-3}, \gamma_{+3}) & < & \mathrm{mod}(\alpha_2, \beta_2) \\ \mathrm{mod}(\alpha_0, \beta_0) + \mathrm{mod}(\alpha_1, \beta_1) + C & < & \mathrm{mod}(\gamma_{-3}, \gamma_{+3}) \end{array} \right. \tag{6.1}$$

$$\tfrac{1}{2}\,\mathrm{mod}(\alpha_0, \gamma_{+3}) < \mathrm{mod}(\alpha_2, \beta_2) \tag{6.2}$$

If $\beta'_{2,0}$, $\beta'_{0,1}$, $\beta'_{1,2}$, $\gamma'_{2,-3}$ and $\gamma'_{2,+3}$ denote the equipotentials which are the respective preimages by g of β_0, β_1, β_2, γ_{-3} and γ_{+3} then the previous inequalities implies (see Proposition C.5 in appendix)

$$\begin{cases} |\phi_0(\beta'_{0,1})| > |\phi_0(\beta_0)| > |\phi_0(\gamma_{-3})| > |\phi_0(\gamma_{+3})| \\ |\phi_1(\beta'_{1,2})| > |\phi_1(\beta_1)| \\ |\phi_\infty(\beta'_{2,0})| > |\phi_\infty(\gamma'_{2,-3})| > |\phi_\infty(\gamma'_{2,+3})| > |\phi_\infty(\beta_2)| \end{cases}$$

The pattern of those equipotentials on $\widehat{\mathbb{C}}$ are displayed in Figure 6.7.

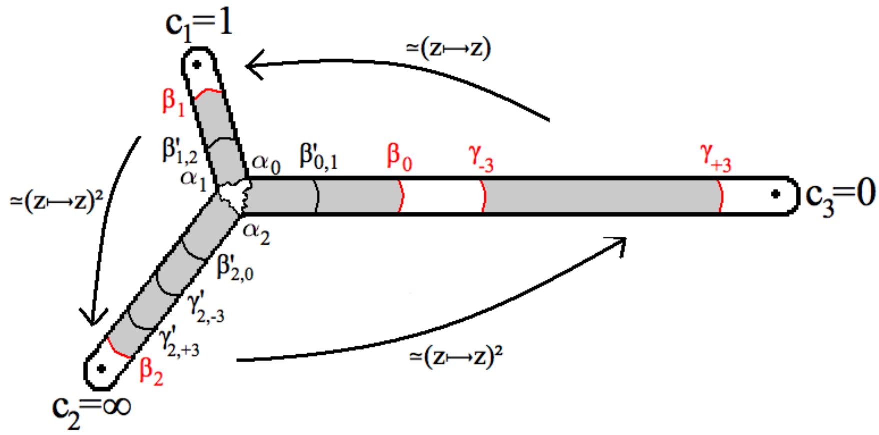


Figure 6.7: Equipotentials in Lemma 6.12

Proof of Lemma 6.12. Compare the inequalities (6.1) with the transition matrix of the weighted Hubbard tree $(\mathcal{H}, w)$:

$$M = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

By Corollary B.8 in appendix and since $\lambda(\mathcal{H}) < 1$, there exists a vector $x \in \mathbb{R}^4$ with positive entries such that $Mx < x$. Let $\mu > 0$ be large enough such that

$$M\mu x + \begin{pmatrix} 0 \\ 0 \\ 0 \\ C \end{pmatrix} < \mu x \tag{6.3}$$

Let $\beta_0 \subset B(0)$, $\beta_1 \subset B(1)$ and $\beta_2 \subset B(2)$ be three equipotentials such that

$$\begin{pmatrix} \mathrm{mod}(\alpha_0,\beta_0) \\ \mathrm{mod}(\alpha_1,\beta_1) \\ \mathrm{mod}(\alpha_2,\beta_2) \\ * \end{pmatrix} = \mu x$$

In this way, the equipotentials β_0, β_1 and β_2 are uniquely defined. Denote by $m > 0$ the last entry of μx. The third row of the linear system of simultaneous inequations (6.3) is $\frac{1}{2}\,\mathrm{mod}(\alpha_0,\beta_0) + \frac{1}{2}m < \mathrm{mod}(\alpha_2,\beta_2)$. So, let $\gamma_{-3} \subset B(0)$ be an equipotential such that $|\phi_0(\beta_0)| > |\phi_0(\gamma_{-3})|$ and

$$\frac{1}{2}\,\mathrm{mod}(\beta_0,\gamma_{-3}) < \mathrm{mod}(\alpha_2,\beta_2) - \left(\frac{1}{2}\,\mathrm{mod}(\alpha_0,\beta_0) + \frac{1}{2}m\right) \qquad (6.4)$$

Then $\mathrm{mod}(\gamma_{-3},\gamma_{+3}) = m$ defines uniquely an equipotential $\gamma_{+3} \subset B(0)$ such that $|\phi_0(\gamma_{-3})| > |\phi_0(\gamma_{+3})|$. Now (6.3) is exactly the linear system of simultaneous inequations (6.1). It remains to prove (6.2) which follows from (6.4):

$$\begin{aligned} \frac{1}{2}\,\mathrm{mod}(\alpha_0,\gamma_{+3}) &= \frac{1}{2}\big(\mathrm{mod}(\alpha_0,\beta_0) + \mathrm{mod}(\beta_0,\gamma_{-3}) + \mathrm{mod}(\gamma_{-3},\gamma_{+3})\big) \\ &< \mathrm{mod}(\alpha_2,\beta_2) \end{aligned}$$

□

Step 2 - The branching piece Let $D(0,\beta'_{0,1})$ be the topological disk bounded by $\beta'_{0,1}$ which contains 0 (see Figure 6.8).

Then define F as a holomorphic map on $\widehat{\mathbb{C}} - D(0,\beta'_{0,1})$ by

$$F_{|\widehat{\mathbb{C}}-D(0,\beta'_{0,1})} = \widehat{f}_{|\widehat{\mathbb{C}}-D(0,\beta'_{0,1})}$$

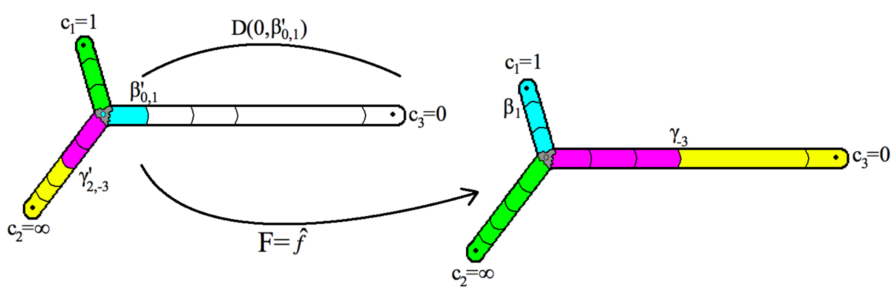


Figure 6.8: Definition of F on the branching piece

Step 3 - Preimage of the branching piece Recall the last inequality of the system of simultaneous inequations (6.1)

$$\mathrm{mod}(\alpha_0, \beta_0) + \mathrm{mod}(\alpha_1, \beta_1) + C < \mathrm{mod}(\gamma_{-3}, \gamma_{+3})$$

We have not fixed the value of the positive constant C yet. By a result of Cui Guizhen and Tan Lei in [CT07] (see Lemma C.7 in appendix) there exists $C > 0$ such that for every pair of equipotentials $\beta_0 \subset B(0)$ and $\beta_1 \subset B(1)$,

$$\mathrm{mod}(\beta_1, \beta_0) < \mathrm{mod}(\alpha_0, \beta_0) + \mathrm{mod}(\alpha_1, \beta_1) + C$$

For this C, apply the Lemma 6.12. We get as a consequence

$$\mathrm{mod}(\beta_1, \beta_0) < \mathrm{mod}(\gamma_{-3}, \gamma_{+3})$$

Therefore we can find two equipotentials $\beta'_{-3,1}$ and $\beta'_{+3,0}$ in $B(0)$ such that

$$\begin{cases} |\phi_0(\gamma_{-3})| > |\phi_0(\beta'_{-3,1})| > |\phi_0(\beta'_{+3,0})| > |\phi_0(\gamma_{+3})| \\ \\ \mathrm{mod}(\beta'_{-3,1}, \beta'_{+3,0}) = \mathrm{mod}(\beta_1, \beta_0) \end{cases}$$

These two equipotentials are displayed in Figure 6.9.

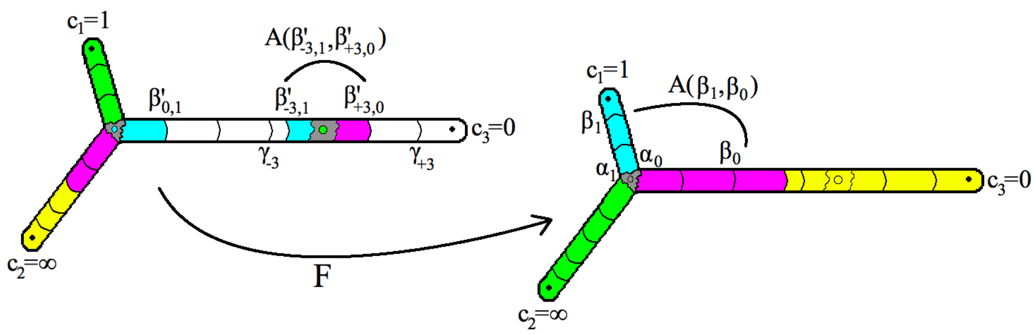


Figure 6.9: Realization of a preimage of the branching piece

Now define F on $A(\beta'_{-3,1}, \beta'_{+3,0})$ to be any biholomorphic map such that

- F maps $A(\beta'_{-3,1}, \beta'_{+3,0})$ onto $A(\beta_1, \beta_0)$
- F extends diffeomorphically to $\overline{A(\beta'_{-3,1}, \beta'_{+3,0})}$ mapping $\beta'_{-3,1}$ onto β_1 and $\beta'_{+3,0}$ onto β_0

Step 4 - Folding We are going to realize the folding point c_0 of $\mathcal{H}$ by creating some critical points of F in $A(\beta'_{0,1}, \beta'_{-3,1})$.

Lemma 6.13. *There exists a biholomorphic map φ mapping $A(\beta_0, \gamma_{-3})$ onto an annulus of the form $A_r = \{z \in \mathbb{C} \,/\, r < |z| < \frac{1}{r}\}$ where $r \in]0,1[$ and having a diffeomorphical extension to $\overline{A(\beta_0, \gamma_{-3})}$.*

Proof of Lemma 6.13. Consider the composition map below

$$\widetilde{\varphi} = (z \mapsto \mu z) \circ \phi_0 : A(\beta'_{0,1}, \beta'_{-3,1}) \to \{z \in \mathbb{D} \,/\, \mu|\phi_0(\beta'_{-3,1})| < |z| < \mu|\phi_0(\beta'_{0,1})|\}$$

with $\mu = 1/\sqrt{|\phi_0(\beta_0)||\phi_0(\gamma_{-3})|}$ and take $\varphi = \widetilde{\varphi}_{|A(\beta_0, \gamma_{-3})}$. □

The aim is now to map the annulus A_r onto a topological disk.

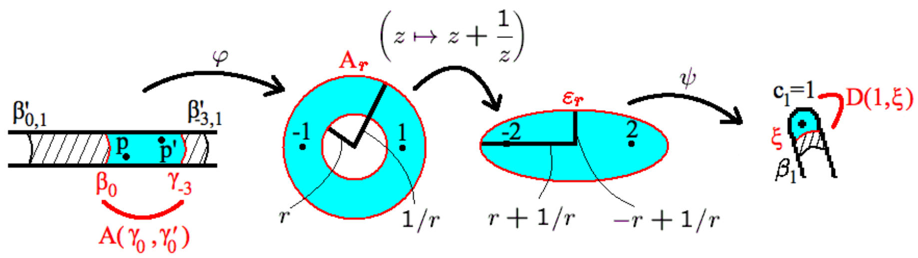


Figure 6.10: Creation of some critical points

Consider the rational map $(z \mapsto z + \frac{1}{z})$ on A_r. It has two simply critical points 1 and -1 and it maps A_r onto an ellipse denoted by ε_r. Denote by p (respectively p') the unique preimage of the critical point 1 (respectively -1) by φ.

Now let ξ be an equipotential in $B(1)$ such that $|\phi_1(\beta_1)| > |\phi_1(\xi)|$ and denote $D(1, \xi)$ the topological disk bounded by ξ which contains 1. Let ψ be any biholomorphic map such that $\psi(\varepsilon_r) = D(1, \xi)$, $\psi(2) = 1$ and ψ extends diffeomorphically to $\overline{\varepsilon_r}$. All the process is resumed in Figure 6.10.

Finally define F as a holomorphic map on $A(\beta_0, \gamma_{-3})$ by

$$F_{|A(\beta_0, \gamma_{-3})} = \psi \circ (z \mapsto z + \frac{1}{z}) \circ \varphi$$

Notice that F extends diffeomorphically to the boundary $\partial A(\beta_0, \gamma_{-3})$. Moreover F has now two new critical points p and p'. The first one $c_0 = p$ is sent exactly to $c_1 = 1$ (in the future super-attracting cycle) and the second one p' is sent near 1 (more precisely in $D(1, \xi) \subset B(1)$).

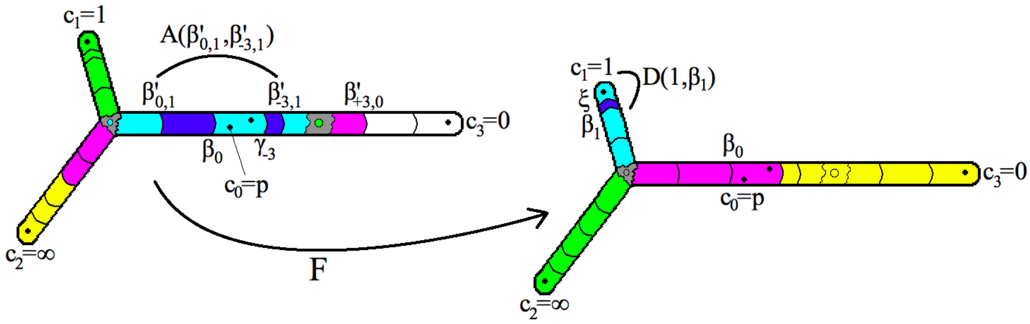


Figure 6.11: Realization of the folding

On the remaining annuli $\overline{A(\beta'_{0,1},\beta_0)}$ and $\overline{A(\gamma_{-3},\beta'_{-3,1})}$, just extend quasiregularly F mapping $\overline{A(\beta'_{0,1},\beta_0)}$ and $\overline{A(\gamma_{-3},\beta'_{-3,1})}$ onto $\overline{A(\beta_1,\xi)}$ (see Proposition C.12 in appendix). Figure 6.11 illustrates how F realizes a folding point at the new critical point $c_0 = p$.

Step 5 - End with an end It remains to define F on the topological disk $D(0,\beta'_{+3,0})$ (see Figure 6.12). The main difficulty is that this domain must contain a preimage of itself in order for F to be encoded by $(\mathcal{H}, w)$ as required.

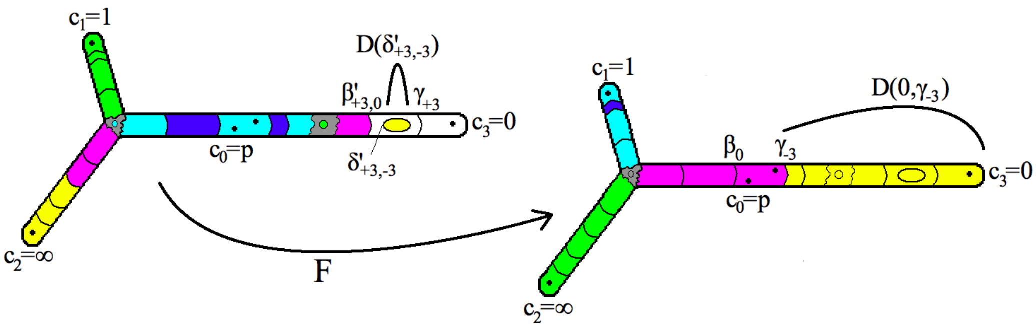


Figure 6.12: Realization of a preimage of an end

Let $\delta'_{+3,-3}$ be a smooth curve (i.e. the image of the unit circle $\mathbb{S}^1$ by a diffeomorphism) in $A(\beta'_{+3,0},\gamma_{+3})$ which does not separate $\beta'_{+3,0}$ and γ_{+3} (i.e. they are both in the same connected component of $\widehat{\mathbb{C}} - \delta'_{+3,-3}$). Denote by $D(\delta'_{+3,-3})$ the connected component of $\widehat{\mathbb{C}} - \delta'_{+3,-3}$ disjoint with $\beta'_{+3,0} \cup \gamma_{+3}$.

Define F on $D(\delta'_{+3,-3})$ to be any biholomorphic map such that

- F maps $D(\delta'_{+3,-3})$ onto $D(0,\gamma_{-3})$
- F extends diffeomorphically to $\overline{D(\delta'_{+3,-3})}$ mapping $\delta'_{+3,-3}$ onto γ_{-3}

In this way, we have realized a preimage of $D(0,\gamma_{-3})$ (see Figure 6.12). Now we are going to realize the mapping of the end $c_3 = 0$ to the folding point $c_0 = p$.

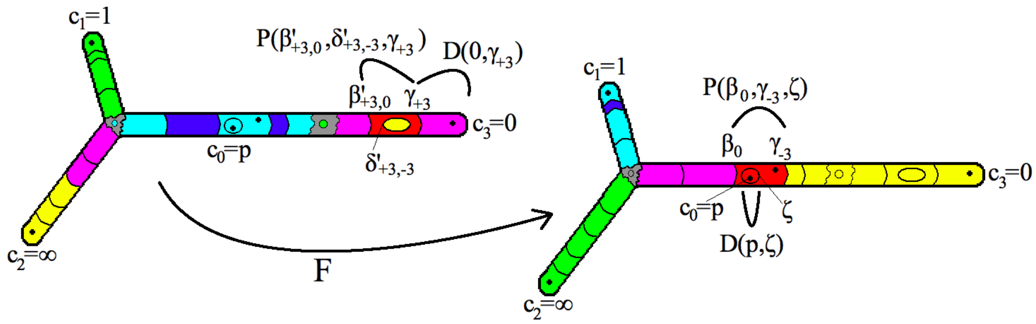


Figure 6.13: The map F on the whole Riemann sphere

Let ζ be a smooth curve in $A(\beta_0,\gamma_{-3})$ which does not separate β_0 and γ_{-3} such that the connected component of $\widehat{\mathbb{C}} - \zeta$ disjoint with $\beta_0 \cup \gamma_{-3}$, denoted by $D(p,\zeta)$, contain the critical point p but not the critical point p' (see Figure 6.13).

Define F on $D(0,\gamma_{+3})$ to be any biholomorphic map such that

- F maps $D(0,\gamma_{+3})$ onto $D(p,\zeta)$
- $F(0) = p$
- F extends diffeomorphically to $\overline{D(0,\gamma_{+3})}$ mapping γ_{+3} onto ζ

Finally let $P(\beta'_{+3,0},\delta'_{+3,-3},\gamma_{+3})$ be the remaining pair of pants bounded by $\beta'_{+3,0}$, $\delta'_{+3,-3}$ and γ_{+3} and let $P(\beta_0,\gamma_{-3},\zeta)$ be as well. Extend quasiregularly F to $\overline{P(\beta'_{+3,0},\delta'_{+3,-3},\gamma_{+3})}$ mapping $\overline{P(\beta'_{+3,0},\delta'_{+3,-3},\gamma_{+3})}$ onto $\overline{P(\beta_0,\gamma_{-3},\zeta)}$ (see Proposition C.12).

Figure 6.13 shows the entire piecewise definition of F on $\widehat{\mathbb{C}}$.

Final Step To sum up, we have defined piecewisely a quasiregular map $F : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that

- F is holomorphic on an open subset $H \subset \widehat{\mathbb{C}}$

$$H = \underbrace{\left(\widehat{\mathbb{C}} - \overline{D(0, \beta'_{0,1})}\right)}_{\text{Step 2}} \bigcup \underbrace{A(\beta'_{-3,1}, \beta'_{+3,0})}_{\text{Step 3}} \bigcup \underbrace{A(\beta_0, \gamma_{-3})}_{\text{Step 4}} \bigcup \underbrace{D(\delta'_{+3,-3}) \cup D(0, \gamma_{+3})}_{\text{Step 5}}$$

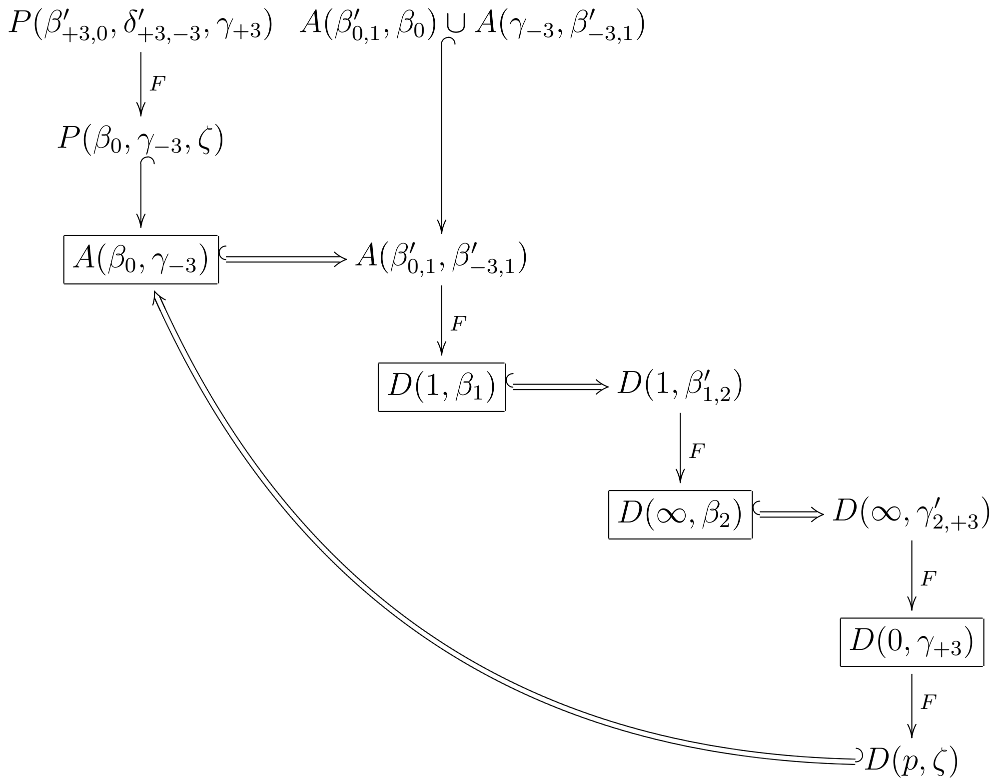
- F extends quasiregularly to the complement $Q = \widehat{\mathbb{C}} - H$

$$Q = \underbrace{\overline{A(\beta'_{0,1}, \beta_0)} \cup \overline{A(\gamma_{-3}, \beta'_{-3,1})}}_{\text{Step 4}} \bigcup \underbrace{\overline{P(\beta'_{+3,0}, \delta'_{+3,-3}, \gamma_{+3})}}_{\text{Step 5}}$$

- There exists an open set $A \subset H$ such that $F(A) \subset A$ and $F^{\circ 2}(Q) \subset A$

$$A = A(\beta_0, \gamma_{-3}) \cup D(1, \beta_1) \cup D(\infty, \beta_2) \cup D(0, \gamma_{+3})$$

The last point is justified by the following diagram



where the first row is the interior of Q, the framed sets are in A, fletched arrows depict inclusions and doubly fletched arrows depict compact inclusions.

We may thus apply the Shishikura's principle on quasiconformal surgery (see Theorem C.13 in appendix) to get a F-invariant almost complex structure. Therefore F is conjugated by a quasiconformal map (coming from the measurable Riemann's mapping theorem C.10) to a rational map f on $\widehat{\mathbb{C}}$. In particular f is still "encoded" by the weighted Hubbard tree $(\mathcal{H}, w)$ as required. Notice that f is non-post-critically finite since the forward orbit of the new critical point p' is infinite.

6.3 Pictures

In general such a construction does not provide an algebraic formula for the rational map we obtain. Since we add critical points by quasiconformal surgery to some rational map, the degree increases quickly as soon as we require a sufficiently interesting tree structure. So the algebraic relations behind such examples are complicated to study. However the particular weighted Hubbard tree $(\mathcal{H}, w)$ of the previous section is simple enough to provide an algebraic formula for such a rational map f_p depending on the critical point $p \in \mathbb{C}$.

Recall at first that f_p realizes asymptotically (see Definition 3.12) the ramification portrait below.

$$p \xrightarrow{2} 1 \xrightarrow{2} \infty \xrightarrow{2} 0 \qquad\qquad p' \overset{2}{\rightsquigarrow} 1$$

$$0 \xrightarrow{1} p$$

The last arrow means that the critical point p' is mapped in the connected component containing 1 of the immediate attracting bassin of f_p. Remark that f_p has four critical points of multiplicity one and then $\deg(f_p) = 3$ (by Riemann-Hurwitz formula). In particular f_p is of the form

$$f_p : z \mapsto \frac{az^3 + bz^2 + cz + d}{Az^3 + Bz^2 + Cz + D}$$

Since 1 is mapped to ∞ with a local degree two, the denominator may factor as

$$f_p : z \mapsto \frac{az^3 + bz^2 + cz + d}{(z-1)^2(C'z + D')}$$

We do likewise for ∞ which is mapped to 0 with a local degree two

$$f_p : z \mapsto \frac{cz + d}{(z-1)^2(C'z + D')}$$

Now use the fact that $f_p(0) = p$ to get

$$f_p : z \mapsto \frac{cz + p}{(z-1)^2(C'z+1)}$$

It remains two informations $f_p(p) = 1$ and $f_p'(p) = 0$ which lead to two equations satisfied by c and C'

$$\begin{cases} (1-p)^2(pC'+1) &= p(c+1) \\ \\ c(1-p)^2(pC'+1) &= p(c+1)((1-4p+3p^2)C' - 2 + 2p) \end{cases}$$

Remark that we may easily simplify the second equation by using the first one (luckily). Then we get the following linear system of two equations

$$\begin{cases} pc - p(1-p)^2C' &= 1 - 3p + p^2 \\ \\ c - (1-p)(1-3p)C' &= -2 + 2p \end{cases}$$

Finally we obtain

$$\begin{cases} c = \dfrac{-1+4p-6p^2+p^3}{2p^2} \\ \\ C' = \dfrac{-1+p+p^2}{2p^2(1-p)} \end{cases}$$

and

$$f_p : z \mapsto \frac{(1-p)\Big[(-1+4p-6p^2+p^3)z + 2p^3\Big]}{(z-1)^2\Big[(-1+p+p^2)z + 2p^2(1-p)\Big]}$$

Some more computations provide an algebraic formula for the last critical point depending on the position of the first one

$$p' = \frac{-p\big(5p^4 - 10p^3 + 11p^2 - 6p + 1\big)}{\big(p^2+p-1\big)\big(p^3 - 6p^2 + 4p - 1\big)}$$

We have proved in the previous section that there exist some choices of p (in order to make $f_p(p')$ close to 1) such that f_p is “encoded” by the weighted Hubbard tree $(\mathcal{H}, w)$. Indeed for $p \approx -0.0005$ we get the bifurcation locus in Figure 6.14.

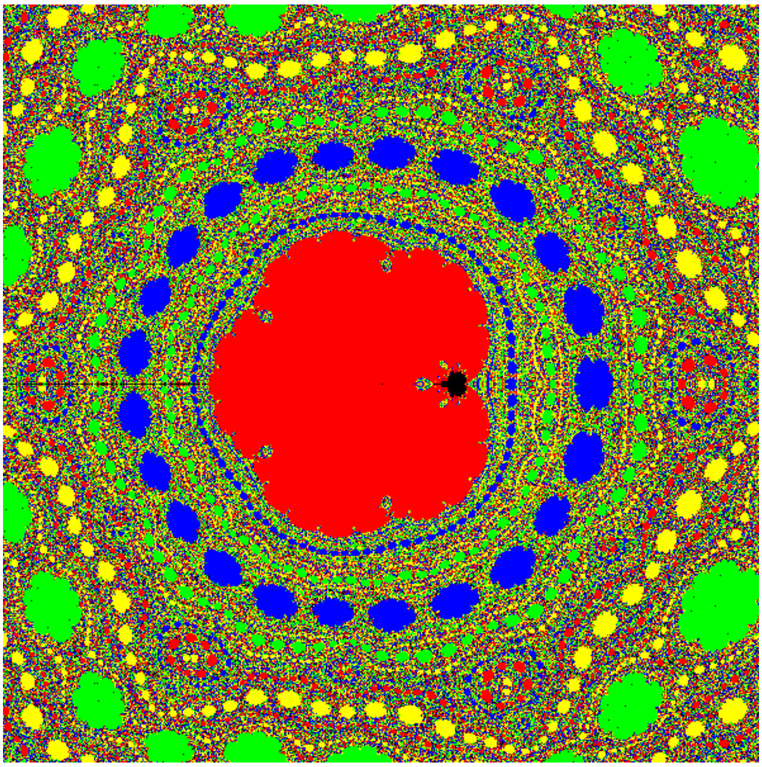


Figure 6.14: The hyperbolic component of the family $(f_p)_{p\in\mathbb{C}}$ inside a disk of center -0.0005 and radius 0.005

Picking a parameter p inside the big hyperbolic component, we obtain the Julia set $\mathcal{J}(f_p)$ in Figure 6.16 called a Persian carpet.

Observe that the Julia set of $\widehat{f}$ is displayed as a "watermark" or more precisely as a buried Julia component inside $\mathcal{J}(f_p)$ (compare with Figure 6.5). Actually this Julia component is fixed, and it's the only one periodic Julia component with a complicated topology. More precisely Tan Lei and K. Pilgrim proved in [PT00] that except for this fixed Julia component and its countable collection of preimages, every Julia component of $\mathcal{J}(f_p)$ is a point or a Jordan curve (see Theorem 6.14).

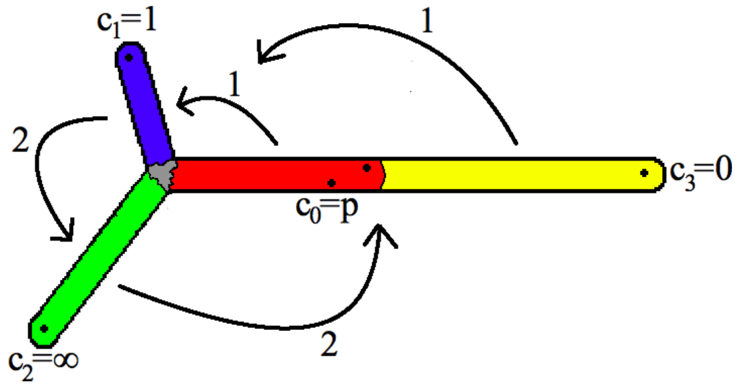


Figure 6.15: The weighted Hubbard tree $(\mathcal{H}, w)$ which encodes f_p

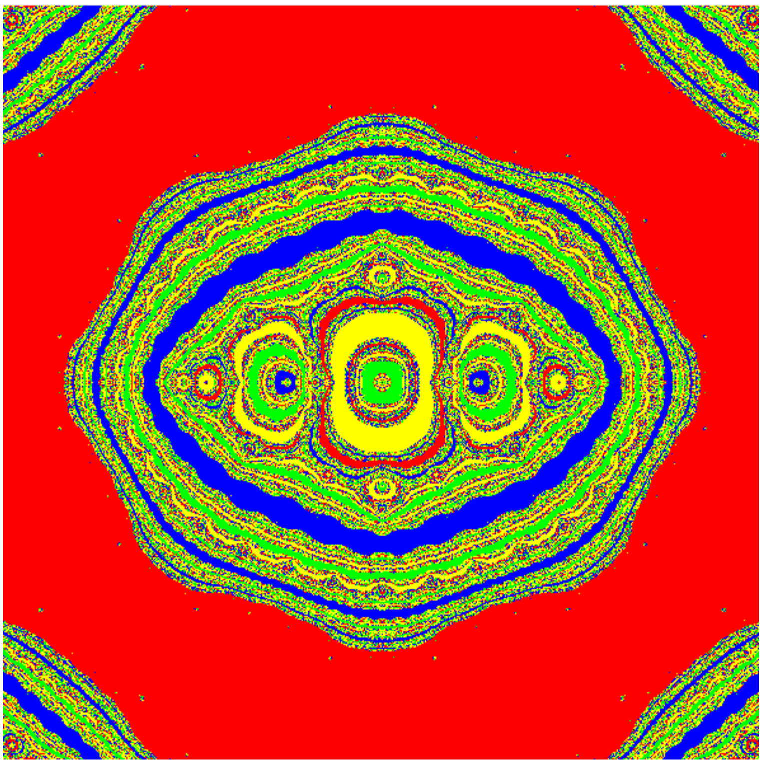


Figure 6.16: The Julia set of $f_{-0.0005}$ which is encoded by $(\mathcal{H}, w)$

Theorem 6.14 (Tan Lei-Pilgrim)**.** *Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational map with disconnected Julia set $\mathcal{J}(f)$. Denote by $\mathcal{P}'_f$ the accumulation set of its post-critical set. The following holds.*

1. *If $\mathcal{P}'_f \cap \mathcal{J}(f)$ is finite then every wandering Julia component is a point or a Jordan curve.*

2. *If $\mathcal{P}'_f \cap \mathcal{J}(f)$ is contained in finitely many Julia components then every wandering Julia component is either simply or doubly connected.*

See also [CPT09] for a sharpening of this result.

With regards to the Fatou components, they are preimages of the immediate attracting bassin of f_p shown in Figure 6.17.

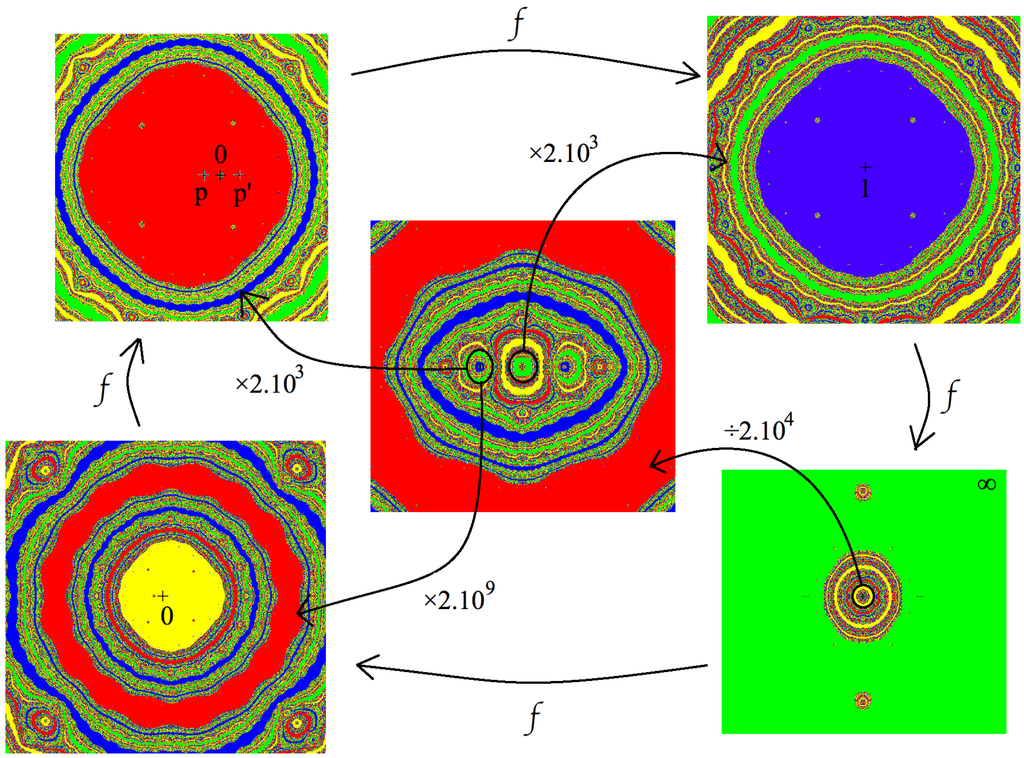


Figure 6.17: Some details of a Persian carpet

6.4 Counterexample

Let us come back to the beginning of the construction in Section 6.2. Now equip the abstract Hubbard tree $\mathcal{H}$ with the following weight function:

$$\begin{cases} w'(e_{\alpha,c_0}) = 1 \\ w'(e_{\alpha,c_1}) = 1 \\ w'(e_{\alpha,c_2}) = 2 \\ w'(e_{c_0,c_3}) = 1 \end{cases}$$

We deduce the following transition matrix of $(\mathcal{H}, w')$

$$M' = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

$\lambda'(\mathcal{H})$ is the largest non-negative root of $2X^4 - 2X - 1$ that is $\lambda'(\mathcal{H}) \approx 1.130$. Thus the new weighted Hubbard tree $(\mathcal{H}, w')$ is no longer unobstructed.

However we may try to carry out the same construction. The first difficulty comes from Lemma 6.12 but we may overcome it. For instance, define five equipotentials which satisfy the conditions *(i)*, *(ii)* together with the linear system of simultaneous inequations (6.1) but not the inequation (6.2) (see Proof of Lemma 6.12).

After that all is right until the final step. There the big diagram is no longer true. Indeed the compact inclusion $D(\infty, \zeta_2) \subset D(\infty, \delta_2)$ which came from the inequation (6.2) does not hold any more. So we cannot apply the Shishikura's principle on quasiconformal surgery and the construction provides only a quasiregular map which is not necessarily quasiconformally conjugated to a rational map.

Actually a result of C. T. McMullen from [McM94] implies that such a rational map does not exist.

Theorem 6.15 (McMullen). *Let f be a rational map which is not a Lattès example (i.e. whose Fatou set $\mathcal{F}(f)$ is nonempty). Assume that there exists a Thurston obstruction Γ. Then $\lambda(f_\Gamma) = 1$ and at least one curve in the multi-curve Γ is contained in the union of Fatou components where f is biholomorphically conjugated to a rotation.*

In our case, every critical points accumulate a super-attracting cycle and consequently every Fatou component is a preimage of a connected component of the immediate attracting bassin. In particular there is no rotation domain in the Fatou set.

Therefore the assumption of unobstructed weighted Hubbard tree is required in order to complete the construction of Section 6.2 successfully.

In particular, remark that the choice of weight function w in Section 6.2 is the simplest possible to make the abstract Hubbard tree $\mathcal{H}$ unobstructed.

6.5 Encoding

We conclude this section by showing how the exchanging dynamics of Julia components induced by the rational map f constructed in Section 6.2 is "encoded" by the weighted Hubbard tree $(\mathcal{H}, w)$.

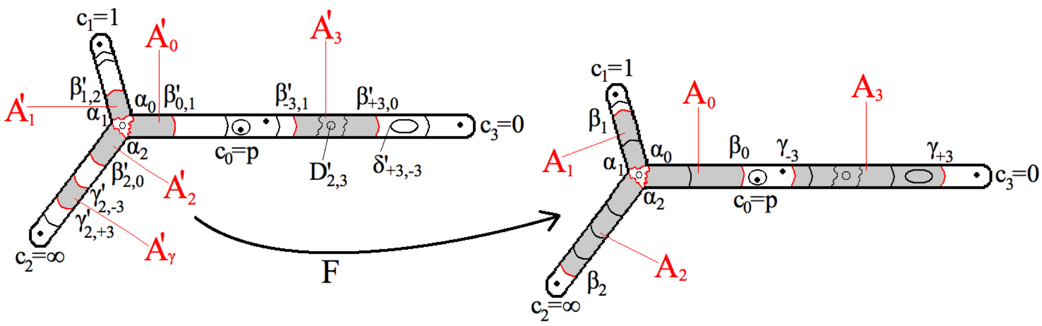


Figure 6.18: Some annuli

We follow the notations from Section 6.2. We consider the following annuli (see Figure 6.18):

- For every $k \in \{0,1,2\}$, $A'_k = A(\alpha_k, \beta'_{k,k+1})$ (with the notation $\beta'_{2,3} = \beta'_{2,0}$) and $A_k = A(\alpha_k, \beta_k)$
- $A'_3 = A(\beta'_{-3,1}, \beta'_{+3,0})$ and $A_3 = A(\gamma_{-3}, \gamma_{+3})$
- $A'_\gamma = A(\gamma'_{2,-3}, \gamma'_{2,+3})$

We have:

1. *(topology)* For every $k \in \{0,1,2,3\}$, $A'_k \subset A_k$ and $A'_\gamma \subset A_2$.
2. *(dynamics)* The following restrictions are unbranched coverings:

$$F_{|A'_0} : A'_0 \to A_1 \quad , \quad F_{|A'_1} : A'_1 \to A_2 \quad , \quad F_{|A'_2} : A'_2 \to A_0$$

$$F_{|A'_\gamma} : A'_\gamma \to A_3 \quad , \quad F_{|A'_3} : A'_3 \to A(\beta_1, \beta_0) \supset A_1 \cup A_0$$

Now we consider as well open sets on the associated Hubbard tree $\mathcal{H} = (T, \tau, \delta)$. Recall that T is the union of four edges which are isometric to four segments denoted by $e_{\alpha,c_0} = [\alpha, c_0]$, $e_{\alpha,c_1} = [\alpha, c_1]$, $e_{\alpha,c_2} = [\alpha, c_2]$ and $e_{c_0,c_3} = [c_0, c_3]$ (see Figure 6.3).

Lemma 6.16. *There exist open segments in T of the form*

- *For every $k \in \{0,1,2\}$, $I'_k =]\alpha, b'_{k,k+1}[\subset e_{\alpha,c_k}$ (with the notation $b'_{2,3} = b'_{2,0}$) and $I_k =]\alpha, b_k[\subset e_{\alpha,c_k}$*
- *$I'_3 =]b'_{-3,1}, c_0] \cup [c_0, b'_{+3,0}[\subset e_{\alpha,c_0} \cup e_{c_0,c_3}$ and $I_3 =]g_{-3}, c_0] \cup [c_0, g_{+3}[\subset e_{\alpha,c_0} \cup e_{c_0,c_3}$*
- *$I'_\gamma =]g_{2,-3}, g_{2,+3}[\subset]b'_{2,0}, c_2[\subset e_{\alpha,c_2} - I'_2$*

and a suitable continuous extension of $\tau : V \to V$ to T in Definition 4.3 such that

1. *(topology) For every $k \in \{0,1,2,3\}$, $I'_k \subset I_k$ and $I'_\gamma \subset I_2$.*
2. *(dynamics) The following restrictions are surjective affine maps:*
$$\tau_{|I'_0} : I'_0 \to I_1 \quad , \quad \tau_{|I'_1} : I'_1 \to I_2 \quad , \quad \tau_{|I'_2} : I'_2 \to I_0$$
$$\tau_{|I'_\gamma} : I'_\gamma \to I_3 \quad , \quad \tau_{|I'_3} : I'_3 \to I_1 \cup \{\alpha\} \cup I_0$$
3. *(expansion) the set*
$$\mathcal{J}_{\mathcal{H}} = \{x \in T \,/\, (\tau^{\circ n}(x))_{n \geqslant 1} \text{ does not accumulate } \{c_0, c_1, c_2, c_3\}\}$$
is a Cantor set which is equal to
$$\mathcal{J}_{\mathcal{H}} = \{x \in T \,/\, \forall n \geqslant 1,\, \tau^{\circ n}(x) \in \{\alpha\} \cup I'_0 \cup I'_1 \cup I'_2 \cup I'_\gamma \cup I'_3\}$$

Remark that the intersection between the Hubbard tree and the Julia set of the quadratic polynomial in Figure 6.3 is well a Cantor set. Actually this intersection is homeomorphic to $\mathcal{J}_{\mathcal{H}}$ and the dynamics is conjugated to τ.

Proof of Lemma 6.16. At first, we may easily consider open segments satisfying the assumptions together with the first condition *(topology)*. Moreover we may require that every new picked points on boundaries of the segments are not in $V = \{\alpha, c_0, c_1, c_2, c_3\}$ and every inclusions in the first condition are compact. Define τ on the new picked points by

$$\begin{cases} \tau(b'_{0,1}) = b_1,\ \tau(b'_{1,2}) = b_2,\ \tau(b'_{2,0}) = b_0 \\ \tau(b'_{-3,1}) = b_1,\ \tau(b'_{+3,0}) = b_0 \\ \tau(g_{2,-3}) = g_{-3},\ \tau(g_{2,+3}) = g_{+3} \end{cases}$$

Recall the definition of τ on V

$$\tau(c_0) = c_1,\ \tau(c_1) = c_2,\ \tau(c_2) = c_3,\ \tau(c_3) = c_0 \text{ and } \tau(\alpha) = \alpha$$

Now extend continuously τ on T by linear interpolation. We get easily the second condition *(dynamics)* and we may classically check the third one *(expansion)* (see for instance [Bea91] or [Mil06]). $\square$

Given a point $x \in \mathcal{J}_{\mathcal{H}}$, we may look at its itinerary that is the iterative address relative to the set of intervals $\{I'_0, I'_1, I'_2, I'_\gamma, I'_3\}$. This motivates the association of the following symbolic dynamical system (Σ, σ):

- $\Sigma = \{\varepsilon \in \{0, 1, 2, \gamma, 3\}^{\mathbb{N}} \,/\, \forall i \in \mathbb{N},\ \varepsilon_i\varepsilon_{i+1} \in \{01, 12, 20, \gamma3, 30, 31\}\}$
- $\sigma : \Sigma \to \Sigma$ is the shift map that is $\forall \varepsilon \in \Sigma,\ \sigma(\varepsilon_0\varepsilon_1\varepsilon_2 \dots) = \varepsilon_1\varepsilon_2\varepsilon_3 \dots$

More precisely (Σ, σ) is a subshift of finite type. Classically we may equip Σ with a metric d_Σ so that Σ is a Cantor set and σ is continuous (see [Bea91] or Chapter 2).

Lemma 6.17. *There is a continuous semi-conjugacy $\psi : (\Sigma, \sigma) \to (\mathcal{J}_{\mathcal{H}}, \tau)$ that is a continuous and surjective map making the following diagram commutative*

$$\begin{array}{ccc} \Sigma & \xrightarrow{\ \sigma\ } & \Sigma \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle\psi} \\ \mathcal{J}_{\mathcal{H}} & \xrightarrow[\ \tau\ ]{} & \mathcal{J}_{\mathcal{H}} \end{array}$$

such that

(i) for every $\varepsilon = \varepsilon_0\varepsilon_1\varepsilon_2 \dots \in \Sigma$ we have

$$\begin{cases} \psi(\varepsilon) \in I'_{\varepsilon_0} \cup \{\alpha\} & \text{if } \varepsilon_0 \in \{0, 1, 2\} \\ \psi(\varepsilon) \in I'_\gamma & \text{if } \varepsilon_0 = \gamma \\ \psi(\varepsilon) \in I'_3 & \text{if } \varepsilon_0 = 3 \end{cases}$$

(ii) $\psi^{-1}(\alpha) = \{\overline{012}, \overline{120}, \overline{201}\}$

(iii) for any inverse image $x \in \bigcup_{n \geqslant 1} \tau^{-n}(\alpha)$,

$$\psi^{-1}(x) = \{\mu3\overline{012}, \mu3\overline{120}, \mu3\overline{201}\}$$

where $\mu \in \{0, 1, 2, \gamma, 3\}^N$ is a finite word of length $N \geqslant 0$

(iv) for any other $x \in \mathcal{J}_{\mathcal{H}} - \bigcup_{n \geqslant 0} \tau^{-n}(\alpha)$, $\psi^{-1}(x)$ is a unique point in Σ

Proof of Lemma 6.17. We give only the main ideas of the proof. Consider the following inverse branches (by Lemma 6.16):

$$(\tau_{|\overline{I'_0}})^{-1} : \overline{I_1} \to \overline{I'_0} \quad , \quad (\tau_{|\overline{I'_1}})^{-1} : \overline{I_2} \to \overline{I'_1} \quad , \quad (\tau_{|\overline{I'_2}})^{-1} : \overline{I_0} \to \overline{I'_2}$$

$$(\tau_{|\overline{I'_\gamma}})^{-1} : \overline{I_3} \to \overline{I'_\gamma} \quad , \quad (\tau_{|\overline{I'_3}})^{-1} : \overline{I_1 \cup I_0} \to \overline{I'_3}$$

Then given any $\varepsilon \in \Sigma$ the following intersection

$$\bigcap_{i=0}^{+\infty} \left[(\tau_{|\overline{I'_{\varepsilon_0}}})^{-1} \circ (\tau_{|\overline{I'_{\varepsilon_1}}})^{-1} \circ \cdots \circ (\tau_{|\overline{I'_{\varepsilon_{i-1}}}})^{-1} \right] (\overline{I'_{\varepsilon_i}})$$

is reduced to exactly one point x in $\mathcal{J}_{\mathcal{H}}$ as decreasing sequence of nonempty compact subsets with diameters tending to zero. Now define $\psi(\varepsilon) = x$ and the remaining follows. □

Similarly define $\mathcal{J}_{\mathcal{H}}(F)$ to be the set of connected components of the following non-escaping set

$$\{z \in \widehat{\mathbb{C}} \,/\, \forall n \geqslant 0,\ F^{\circ n}(x) \in J_\alpha \cup A'_0 \cup A'_1 \cup A'_2 \cup A'_\gamma \cup A'_3 - D'_{3,2}\}$$

where J_α is the Julia set of $\widehat{f}$ (see Figure 6.5) and $D'_{3,2}$ is the unique topological disk which is mapped by F onto the connected component of the complement of J_α containing $c_2 = \infty$ (denoted by $D(\infty, \alpha_2)$ according to our notations). $D'_{3,2}$ is shown in Figure 6.18.

We equip $\mathcal{J}_{\mathcal{H}}(F) - \bigcup_{n\geqslant 0} F^{-n}(J_\alpha)$ with the Hausdorff topology coming from the following Hausdorff metric:

$$\forall J, J' \in \mathcal{J}_{\mathcal{H}}(F) - \bigcup_{n\geqslant 0} F^{-n}(J_\alpha),$$

$$d_{\mathrm{H}}(J, J') = \max\left\{ \sup_{z\in J} \inf_{z'\in J'} |z - z'|, \sup_{z'\in J'} \inf_{z\in J} |z - z'| \right\}$$

We add to this topology some neighbourhoods of each component $J \in \mathcal{J}_{\mathcal{H}}(F)$ mapped onto J_α after some iterations of F defined as follows

$$\forall k \geqslant 1,\ \mathcal{V}_{1/k}(J) = \left\{ J' \in \mathcal{J}_{\mathcal{H}}(F)\,,\ \inf_{z\in J, z'\in J'} |z - z'| < \frac{1}{k} \right\}$$

In this way, $\mathcal{J}_{\mathcal{H}}(F)$ is a topological space on where F induces a continuous dynamical system denoted also by $F : \mathcal{J}_{\mathcal{H}}(F) \to \mathcal{J}_{\mathcal{H}}(F)$.

Remark that the forward orbit of every component $J \in \mathcal{J}_{\mathcal{H}}(F)$ does not accumulate the attracting cycle $\{c_0 = p, c_1 = 1, c_2 = \infty, c_3 = 0\}$. But there exist some more points in $\widehat{\mathbb{C}}$ with this property. For instance, some components in $\mathcal{J}_{\mathcal{H}}(F)$ have preimages by F in the forgotten topological disk $D'_{3,2}$ or in the topological disk $D(\delta'_{+3,-3})$ (see Step 5 in Section 6.2). We will not discuss the dynamics of such points in $D'_{3,2} \cup D(\delta'_{+3,-3})$ and all of their preimages.

Nevertheless we have the following result whose proof is similar to that one of Lemma 6.17.

Lemma 6.18. *There is a continuous semi-conjugacy* $\phi : (\Sigma, \sigma) \to (\mathcal{J}_{\mathcal{H}}(F), F)$ *such that*

(i) for every $\varepsilon = \varepsilon_0\varepsilon_1\varepsilon_2 \cdots \in \Sigma$ *we have*

$$\begin{cases} \phi(\varepsilon) \subset A'_{\varepsilon_0} \cup J_\alpha & \text{if } \varepsilon_0 \in \{0,1,2\} \\ \phi(\varepsilon) \subset A'_\gamma & \text{if } \varepsilon_0 = \gamma \\ \phi(\varepsilon) \subset A'_3 & \text{if } \varepsilon_0 = 3 \end{cases}$$

(ii) $\phi^{-1}(J_\alpha) = \{\overline{012}, \overline{120}, \overline{201}\}$

(iii) for any component $J \in \mathcal{J}_{\mathcal{H}}(F)$ *mapped onto* J_α *after some iterations of* F*,*

$$\phi^{-1}(J) = \{\mu 3\overline{012}, \mu 3\overline{120}, \mu 3\overline{201}\}$$

where $\mu \in \{0,1,2,\gamma,3\}^N$ *is a finite word of length* $N \geqslant 0$

(iv) for any other component $J \in \mathcal{J}_{\mathcal{H}}(F)$*,* $\phi^{-1}(J)$ *is a unique point in* Σ

Recall that F is conjugated by a quasiconformal map to the rational map f on $\widehat{\mathbb{C}}$. Therefore the same result holds for the subset $\mathcal{J}_{\mathcal{H}}(f)$ of Julia components of f which correspond quasiconformally to the components in $\mathcal{J}_{\mathcal{H}}(F)$. $\mathcal{J}_{\mathcal{H}}(f)$ is equipped with a topology induced from that one of $\mathcal{J}_{\mathcal{H}}(F)$. Hence we get a semi-conjugacy $\varphi : \Sigma \to \mathcal{J}_{\mathcal{H}}(f)$ which is quasiconformally conjugated to ϕ. That allows to encode every Julia components in $\mathcal{J}_{\mathcal{H}}(f)$ as in Lemma 6.18.

Finally the following result about the encoding of a subset of Julia components of f folllows easily from Lemma 6.17 and Lemma 6.18 together with Theorem 6.14.

Theorem 6.19. *There is a continuous semi-conjugacy $\varphi : (\Sigma, \sigma) \to (\mathcal{J}_{\mathcal{H}}(f), f)$ such that*

(i) $\psi \circ \varphi^{-1} : (\mathcal{J}_{\mathcal{H}}(f), f) \to (\mathcal{J}_{\mathcal{H}}, \tau)$ *is a conjugacy that is a continuous and bijective map making the following diagram commutative*

$$\begin{array}{ccc} \mathcal{J}_{\mathcal{H}}(f) & \xrightarrow{\ f\ } & \mathcal{J}_{\mathcal{H}}(f) \\ {\scriptstyle \psi \circ \varphi^{-1}} \downarrow & & \downarrow {\scriptstyle \psi \circ \varphi^{-1}} \\ \mathcal{J}_{\mathcal{H}} & \xrightarrow[\ \tau\ ]{} & \mathcal{J}_{\mathcal{H}} \end{array}$$

(ii) for any Julia component $J \in \mathcal{J}_{\mathcal{H}}(f)$, it is a Jordan curve if and only if $\varphi^{-1}(J)$ is a unique point in Σ, in other words if and only if it is not mapped after some iterations of f onto the fixed Julia component which corresponds quasiconformally to J_α

Chapter 7

A collection of Persian carpets

Now we would like to generalize the construction of the previous chapter in order to get a large family of non-post-critically rational maps encoded by weighted Hubbard trees. As a consequence, we obtain a result about realization of some particular infinite ramification portraits. We will conclude by some computable examples.

7.1 General construction

As the particular construction in Section 6.2, we would like to construct piecewisely a map F. But we will use a result of Cui Guizhen and Tan Lei instead of a quasiconformal surgery method to conclude. That allows us to construct a map F which is required to be less regular than a quasiregular map. Nevertheless we keep in head that the aim (the final rational map f) is similar to that one in Section 6.2.

Step 0 - The starting data At first we need a post-critically map $\widehat{f}$ "encoded" by a particular weighted Hubbard tree $(\widehat{\mathcal{T}}, \widehat{w})$ deduced from a weighted Hubbard tree $(\mathcal{H}, w)$ by removing some critical points as in the beginning of the construction in Section 6.2. In other words, we would like to start the construction by a rational map $\widehat{f}$ which realizes a given finite ramification portrait $\widehat{\mathcal{R}}$. But it is not a so easy task. Actually we have already discussed in Chapter 4 and Chapter 5 the obstructions that could arise. In particular Corollary 5.21 ensures that we may find such a post-critically rational map $\widehat{f}$ if the ramification portrait $\widehat{\mathcal{R}}$ is N-cyclic where N is a positive integer.

This assumption is not necessary. Although we have seen in Section 4.3 that we may find another sufficient condition for a larger class of ramification portraits which are realized by ramified coverings, the discussions in Chapter 5 showed that the analytical part of the problem does not provide an easy to check criterion in general. Therefore we restrict here the issue to N-cyclic ramification portraits. That will lead to a large enough family of rational maps.

Let N be a positive integer and let $\widehat{\mathcal{R}}$ be a N-cyclic ramification portrait denoted by

$$\begin{array}{l}
c_1^1 \xrightarrow{\widehat{\nu}(c_1^1)} c_2^1 \xrightarrow{\widehat{\nu}(c_2^1)} \cdots \xrightarrow{\widehat{\nu}(c_{n_1-1}^1)} c_{n_1}^1, \quad c_{n_1}^1 \xrightarrow{\widehat{\nu}(c_{n_1}^1)} c_1^1 \\
c_1^2 \xrightarrow{\widehat{\nu}(c_1^2)} c_2^2 \xrightarrow{\widehat{\nu}(c_2^2)} \cdots \xrightarrow{\widehat{\nu}(c_{n_2-1}^2)} c_{n_2}^2, \quad c_{n_2}^2 \xrightarrow{\widehat{\nu}(c_{n_2}^2)} c_1^2 \\
\vdots \\
c_1^N \xrightarrow{\widehat{\nu}(c_1^N)} c_2^N \xrightarrow{\widehat{\nu}(c_2^N)} \cdots \xrightarrow{\widehat{\nu}(c_{n_N-1}^N)} c_{n_N}^N, \quad c_{n_N}^N \xrightarrow{\widehat{\nu}(c_{n_N}^N)} c_1^N \\
c_\infty \circlearrowright \deg(\widehat{\mathcal{R}})
\end{array}$$

Without loss of generality, we may write the last cycle like the previous ones with notations $n_{N+1} = 1$, $c_1^{N+1} = c_\infty$ and $\widehat{\nu}(c_1^{N+1}) = \deg(\widehat{\mathcal{R}})$. Notice that

(1) $\forall i \in \{1, 2, \dots, N\}$, $\exists k \in \{1, 2, \dots, n_i\} \,/\, \widehat{\nu}(c_k^i) \geqslant 2$

(2) $\widehat{\nu}(c_1^{N+1}) = 1 + \sum_{i\in\{1,2,\dots,N\}} \left(\sum_{k\in\{1,2,\dots,n_i\}} (\widehat{\nu}(c_k^i) - 1)\right)$

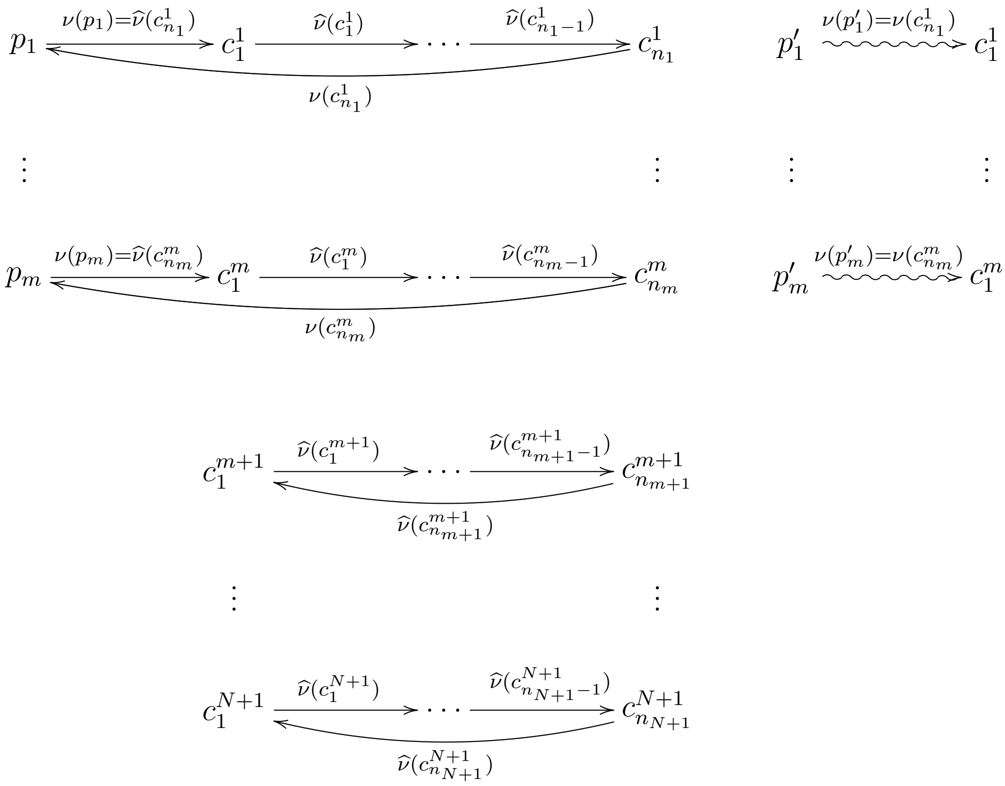
Let $\widehat{f}$ be a rational map realizing the ramification portrait $\widehat{\mathcal{R}}$. We follow notations from Section 6.2: for every integers $i \in \{1, 2, \dots, N+1\}$ and $k \in \{1, 2, \dots, n_i\}$ denote by $B(c_k^i)$ the connected component containing c_k^i of the immediate attracting bassin of the i^{th} cycle and by $\phi_{c_k^i}$ the associated Böttcher coordinates. We may show as in Section 6.2 that the action of $\widehat{f}$ is encoded by a weighted Hubbard tree $(\widehat{\mathcal{T}}, \widehat{w})$, that is there exist a continuous and surjective map $\widehat{\pi} : \widehat{\mathbb{C}} \to \widehat{T}$ and a suitable continuous extension of $\widehat{\tau} : \widehat{V} \to \widehat{V}$ to $\widehat{T}$ in Definition 4.3 such that the following diagram is commutative

$$\begin{array}{ccc}
\overline{\bigcup B(c_k^i)} & \xrightarrow{\widehat{\pi}} & \widehat{T} \\
\widehat{f}\big\downarrow & & \big\downarrow\widehat{\tau} \\
\overline{\bigcup B(c_k^i)} & \xrightarrow[\widehat{\pi}]{} & \widehat{T}
\end{array}$$

Indeed the weighted Hubbard tree $(\widehat{\mathcal{T}}, \widehat{w})$ is a starlike tree, that is a tree formed by the set of vertices $\widehat{V} = \{\alpha\} \cup (\bigcup\{c_k^i\})$ where α is the unique branching point and by the set of edges $E = E_\alpha = \bigcup\{e_{\alpha, c_k^i}\}$ (recall that a weighted Hubbard tree is considered to be embedded in $\mathbb{R}^3$ and thus without cyclic order of edges at a common endpoint), whose dynamics $\widehat{\tau}$ fixes the branching point α and acts as the dynamics $\widehat{\sigma}$ of $\widehat{\mathcal{R}}$ on the ends $\{c_k^i\}$ and whose weight function $\widehat{w}$ is deduced from the local degrees function $\widehat{\nu}$ of $\widehat{\mathcal{R}}$:

$$\forall i \in \{1, 2, \dots, N+1\}, \forall k \in \{1, 2, \dots, n_i\},\ \widehat{w}(e_{\alpha, c_k^i}) = \widehat{\nu}(c_k^i)$$

Recall that we would like to construct a rational map f from $\widehat{f}$ which has two new critical points inside some $B(c_{n_i}^i)$, say p_i and p_i'. In other words, up to relabelling the cycles in $\widehat{\mathcal{R}}$, we would like to construct a rational map f which realizes asymptotically the big ramification portrait $\mathcal{R}$ below



where m is an integer in $\{1, 2, \dots, N+1\}$ and the waved arrows means that the critical points p_i' are mapped near c_1^i (in the connected component containing c_1^i of the immediate attracting bassin of the i^{th} cycle). We will discuss later the new local degrees $\nu(p_i)$ and $\nu(p_i')$. We should think of the ramification portrait $\mathcal{R}$ as it comes from a starting weighted Hubbard tree $(\mathcal{H}, w)$ as in the beginning of the construction in Section 6.2.

This discussion leads naturally to the following definition.

Definition 7.1 (admissible weighted Hubbard tree)**.** A weighted Hubbard tree $(\mathcal{H}, w)$ is said **admissible** if the following conditions are satisfied.

(i) **(tree shape condition)** the tree T is formed by the set of vertices

$$V = \{\alpha\} \cup \left(\bigcup_{\substack{i \in \{1,2,\ldots,N+1\} \\ k \in \{1,2,\ldots,n_i\}}} \{c_k^i\} \right) \cup \left(\bigcup_{i \in \{1,2,\ldots,m\}} \{p_i\} \right)$$

and by the set of edges

$$\begin{aligned} E &= \left(\bigcup_{i \in \{1,2,\ldots,N+1\}} E_\alpha^i \right) \cup \left(\bigcup_{i \in \{1,2,\ldots,m\}} \{e_{p_i, c_{n_i}^i}\} \right) \\ &\text{where} \quad E_\alpha^i = \left(\bigcup_{k \in \{1,2,\ldots,n_i - 1\}} \{e_{\alpha, c_k^i}\} \right) \cup \{e_{\alpha, x_i}\} \\ &\text{with the notation} \quad x_i = \begin{cases} p_i & \text{if } i \in \{1,2,\ldots,m\} \\ c_{n_i}^i & \text{otherwise} \end{cases} \end{aligned}$$

moreover the dynamics τ on T is given by

$$\begin{cases} \tau(\alpha) = \alpha \\ \tau(c_k^i) = c_{k+1}^i & \text{if } i \in \{1,2,\ldots,N+1\}, k \in \{1,2,\ldots,n_i - 1\} \\ \tau(c_{n_i}^i) = p_i & \text{if } i \in \{1,2,\ldots,m\} \\ \tau(x_i) = c_1^i & \text{if } i \in \{1,2,\ldots,N+1\} \end{cases}$$

(ii) **(realization condition)** there exists an integer $i_\infty \in \{1,2,\ldots,N+1\}$ such that

(0) $n_{i_\infty} = 1$

(1) $\forall i \in \{1,2,\ldots,N+1\} - \{i_\infty\}$, $\exists e \in E_\alpha^i \,/\, w(e) \geqslant 2$

(2) $w(e_{\alpha, x_{i_\infty}}) = 1 + \sum_{i \in \{1,2,\ldots,N+1\} - \{i_\infty\}} \left(\sum_{e \in E_\alpha^i} (w(e) - 1) \right)$

(iii) **(Thurston condition)** $(\mathcal{H}, w)$ is unobstructed

Notice that we have already discussed the necessity of the Thurston condition *(iii)* in Section 6.4.

Finally let $(\mathcal{H}, w)$ be an admissible weighted Hubbard tree. We consider another weighted Hubbard tree $(\widehat{\mathcal{H}}, \widehat{w})$ deduced from $(\mathcal{H}, w)$ by removing the critical points p_i where $i \in \{1, 2, \dots, m\}$ according to the previous discussions. We associate likewise a ramification portrait $\widehat{\mathcal{R}}$ whose local degrees function $\widehat{\nu}$ is deduced from the weight function $\widehat{w}$. Remark that the realization condition *(ii)* implies that $\widehat{\mathcal{R}}$ is N-cyclic. As a consequence, Corollary 5.21 provides a rational map $\widehat{f}$ realizing the ramification portrait $\widehat{\mathcal{R}}$ as required.

Step 1 - The preexisting post-critical points At first we define a neighbourhood of the post-critical set $P_{\widehat{f}} = \bigcup \{c_k^i\}$ of $\widehat{f}$ as follows. For every $i \in \{1, 2, \dots, N+1\}$ and every $k \in \{1, 2, \dots, n_i\}$, let $U(c_k^i)$ be a small topological disk containing c_k^i in the immediate attracting bassin of $\widehat{f}$ such that

- $\overline{U(c_k^i)}$ are pairwise disjoint
- $\forall i \in \{1, 2, \dots, N+1\}, \forall k \in \{1, 2, \dots, n_i\}, \widehat{f}\left(\overline{U(c_k^i)}\right) \subset U(c_{k+1}^i)$

with the notation $c_{n_i+1}^i = c_1^i$. Such a collection of neighbourhoods exists since every periodic cycle in $P_{\widehat{f}}$ is super-attracting (it contains at least one critical point because $\widehat{\mathcal{R}}$ is N-cyclic).

Now for every $i \in \{1, 2, \dots, m\}$ and every $k \in \{1, 2, \dots, n_i\}$, define F on $U(c_k^i)$ as a holomorphic map by the restriction map $\widehat{f}_{|U(c_k^i)}$. Remark that F maps c_k^i to c_{k+1}^i with a local degree

$$\widehat{\nu}(c_k^i) = \widehat{w}(e_{\alpha, c_k^i}) = w(e_{\alpha, c_k^i})$$

Do as well for every $i \in \{m+1, \dots, N, N+1\}$ and every $k \in \{1, 2, \dots, n_i\}$. There are hence two kinds of remaining post-critical points:

- the folding points p_i where $i \in \{1, 2, \dots, m\}$
- the ending points $c_{n_i}^i$ where $i \in \{1, 2, \dots, m\}$

Step 2 - Foldings Now we would like to realize the folding points of $\mathcal{H}$ by creating some critical points p_i and p_i' whose multiplicities are deduced from the weight function w. The aim is to map an annulus onto a topological disk (compare with Step 4 in Section 6.2). However the map $(z \mapsto z + \frac{1}{z})$ does no longer realize what it is required since the degrees on the boundaries are not necessary equal to one any more. So we need some sharpenings of that map.

Lemma 7.2. *For every pair of positive integers $d, d' \in \mathbb{N} - \{0\}$, there exists a ramified covering G of degree $d + d'$ such that*

(i) G is defined on a neighbourhood of a closed annulus $\overline{A(\eta, \eta')}$ between two Jordan curves η and η'

(ii) G maps $A(\eta, \eta')$ onto a topological disk $D(\xi)$ with a Jordan curve ξ as boundary and $\eta \cup \eta'$ onto ξ

(iii) the restriction maps $G_{|\eta} : \eta \to \xi$ and $G_{|\eta'} : \eta' \to \xi$ are respectively of degree d and d'

(iv) G has exactly two critical points, say q, q', which belong to $A(\eta, \eta')$ and which are respectively of multiplicity d and d'

(v) G is holomorphic on two disjoint small topological disks $U(q), U(q') \subset A(\eta, \eta')$ containing respectively q and q'

Notice that we may then produce a rational map g which is quasiconformally conjugated to G (see Theorem C.13) whereas it is not so easy to find a required algebraic formula generalizing $(z \mapsto z + \frac{1}{z})$ with classical methods of calculus. Furthermore this process provides in addition a better understanding of the action of such a rational map g.

Proof of Lemma 7.2. We do not give a detailed proof, we only sketch the construction with Figure 7.1 illustrating the piecewise definition of the map G in the particular case where $d = 3$ and $d' = 2$. There is no difficulty to guess a generalization of this example for any pair of degrees.

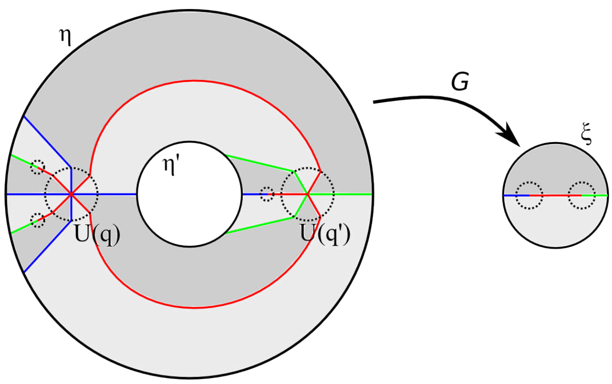


Figure 7.1: A map G satisfying Lemma 7.2 for $d = 3$ and $d' = 2$

□

Now we may define F on a neighbourhood of every "folding" as a ramified covering given by Lemma 7.2. More precisely, for every $i \in \{1, 2, \ldots, m\}$, let $A(\gamma_i, \gamma_i') \subset \widehat{\mathbb{C}}$ be a small annulus between two Jordan curves γ_i and γ_i' such that

- $A(\gamma_i, \gamma_i')$ is disjoint from the post-critical set $P_{\widehat{f}} = \bigcup \{c_k^i\}$ of $\widehat{f}$
- if $D(\gamma_i')$ denote the topological disks in $\widehat{\mathbb{C}} - A(\gamma_i, \gamma_i')$ with γ_i' as boundary, then $\overline{U(c_{n_i}^i)} \subset D(\gamma_i')$
- $A(\gamma_i, \gamma_i')$ is biholomorphically conjugated to an annulus $A(\eta_i, \eta_i')$ associated to a ramified covering G_i given by Lemma 7.2 with

$$\begin{cases} d_i &= w(e_{\alpha,p_i}) = \widehat{\nu}(c_{n_i}^i) = \widehat{w}(e_{\alpha,c_{n_i}^i}) \\ \\ d_i' &= w(e_{p_i,c_{n_i}^i}) = \nu(c_{n_i}^i) \end{cases} \tag{7.1}$$

  We denote by φ_i a biholomorphic map from $A(\gamma_i, \gamma_{i'})$ onto $A(\eta_i, \eta_{i'})$ and by p_i (respectively p_i') the preimage by φ_i of the critical point q_i (respectively q_i') of G_i.

Let ψ_1^i be any biholomorphic map from the topological disk $U(c_1^i)$ onto $D(\xi_i)$ such that $\psi_1^i(c_1^i) = G_i(q_i)$. Finally define F on $A(\gamma_i, \gamma_i')$ as a ramified covering by the following commutative diagram

$$\begin{array}{ccc} A(\gamma_i, \gamma_i') & \xrightarrow{\varphi_i} & A(\eta_i, \eta_i') \\ {\scriptstyle F_{|A(\gamma_i,\gamma_i')}}\downarrow & & \downarrow{\scriptstyle G_i} \\ U(c_1^i) & \xrightarrow[\psi_1^i]{} & D(\xi_i) \end{array}$$

In particular, remark that F maps p_i to c_1^i with a local degree

$$d_i = w(e_{\alpha,p_i}) = \widehat{\nu}(c_{n_i}^i) = \widehat{w}(e_{\alpha,c_{n_i}^i})$$

and F is holomorphic in the neighbourhood of p_i and p_i'.

Step 3 - Endings For every $i \in \{1, 2, \ldots, m\}$, consider the small topological disk $U(p_i) = \varphi_i^{-1}(U(q_i)) \subset A(\gamma_i, \gamma_i') - \{p_i'\}$ containing p_i and let $\phi_i : U(p_i) \to \mathbb{D}$ be any biholomorphic map such that $\phi_i(p_i) = 0$. Likewise, let $\psi_{n_i}^i : U(c_{n_i}^i) \to \mathbb{D}$ be a biholomorphic map such that $\psi_{n_i}^i(c_{n_i}^i) = 0$.

Then define F on $U(c^i_{n_i})$ by the following commutative diagram

$$\begin{array}{ccc} U(c^i_{n_i}) & \xrightarrow{\psi^i_{n_i}} & \mathbb{D} \\ F_{|U(c^i_{n_i})}\downarrow & & \downarrow z\mapsto z^{d'_i} \\ U(p_i) & \xrightarrow{\phi_i} & \mathbb{D} \end{array}$$

In particular, remark that F maps $c^i_{n_i}$ to p_i with a local degree

$$d'_i = w(e_{p_i,c^i_{n_i}}) = \nu(c^i_{n_i})$$

Step 4 - Extension Observe that we may easily improve Step 1 and Step 3 in order to extend F to the complement of a neighbourhood of $\bigcup \overline{A(\gamma_i,\gamma'_i)}$. It remains to define F on small annuli of the form $A(\beta_i,\gamma_i)$ and $A(\gamma'_i,\beta'_i)$ where $i\in\{1,2,\ldots,m\}$ and β_i,β'_i are two Jordan curves. The equalities (7.1) imply that

$$\begin{cases} \deg(F_{|\beta_i}) = \widehat{\nu}(c^i_{n_i}) &= d_i = \deg(F_{|\gamma_i}) \\ \\ \deg(F_{|\gamma'_i}) &= d'_i = \deg(F_{|\beta'_i}) \end{cases}$$

Therefore F may be extended onto every $A(\beta_i,\gamma_i)$ and every $A(\gamma'_i,\beta'_i)$ as a ramified covering without extra critical point.

To sum up, we finally obtain a ramified covering $F:\widehat{\mathbb{C}}\to\widehat{\mathbb{C}}$ such that

- F is holomorphic on

$$U = \left(\bigcup_{\substack{i\in\{1,2,\ldots,N+1\}\\ k\in\{1,2,\ldots,n_i\}}} U(c^i_k)\right) \cup \left(\bigcup_{i\in\{1,2,\ldots,m\}} U(p_i)\cup U(p'_i)\right)$$

- the closure of the post-critical set of F is

$$\begin{aligned}\mathcal{P}_F &= \left(\bigcup_{\substack{i\in\{1,2,\ldots,N+1\}\\ k\in\{1,2,\ldots,n_i\}}} \{c^i_k\}\right) \cup \left(\bigcup_{i\in\{1,2,\ldots,m\}} \{p_i\}\right) \\ &\cup \left(\bigcup_{i\in\{1,2,\ldots,m\}} \overline{\{F^{\circ n}(p'_i)\,/\,n\geqslant 1\}}\right)\end{aligned}$$

- $\mathcal{P}_F\subset U$

The third point follows from the construction since the forward orbit of each critical point p'_i where $i\in\{1,2,\ldots,m\}$ accumulates the super-attracting periodic cycle $p_i,c^i_1,p^i_2,\ldots,p^i_{n_i}$.

Step 5- A powerful theorem We are going to conclude the construction (that is to get a rational map f from the ramified covering F) by using a Thurston-like characterization which generalizes Theorem 5.9 for a larger set of ramified coverings which contains F. We follow the notations from [CT07].

Definition 7.3 (sub-hyperbolic semi-rational map)**.** Let $F : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be an orientation-preserving ramified covering. Denote by $\mathcal{P}'_F$ the accumulation set of its post-critical set. We say that F is a **sub-hyperbolic semi-rational map** if the following conditions hold

(i) $\mathcal{P}'_F$ is finite

(ii) F is holomorphic in a neighbourhood of $\mathcal{P}'_F$

(iii) every periodic points in $\mathcal{P}'_F$ are either attracting or super-attracting

Definition 7.4 (combinatorially equivalence)**.** Two sub-hyperbolic semi-rational maps F and G are said **combinatorially equivalent** if there exist two orientation-preserving homeomorphisms φ_0 and φ_1 of $\widehat{\mathbb{C}}$ such that

(i) the following diagram commutes

$$\begin{array}{ccc} \widehat{\mathbb{C}} & \xrightarrow{\varphi_1} & \widehat{\mathbb{C}} \\ F \downarrow & & \downarrow G \\ \widehat{\mathbb{C}} & \xrightarrow{\varphi_0} & \widehat{\mathbb{C}} \end{array}$$

(ii) $\varphi_0(\mathcal{P}_F) = \varphi_1(\mathcal{P}_F) = \mathcal{P}_G$

(iii) φ_0 is isotopic to φ_1 relative to $\mathcal{P}_F$

(iv) φ_0 is holomorphic on a neighbourhood of $\mathcal{P}'_F$

Compare the definition above with Definition 5.2. Notice that φ_0 and φ_1 are actually equal on a neighbourhood of $\mathcal{P}'_F$ (by the isolated zero theorem) and then realize a local conformal conjugacy.

The Thurston-like characterization for sub-hyperbolic semi-rational map is the following.

Theorem 7.5 (Cui Guizhen-Tan Lei)**.** *A sub-hyperbolic semi-rational map F with $\mathcal{P}'_F \neq \emptyset$ is combinatorially equivalent to a rational map if and only if it has no Thurston obstruction. In that case, the rational map is unique up to conjugation by a Möbius transformation.*

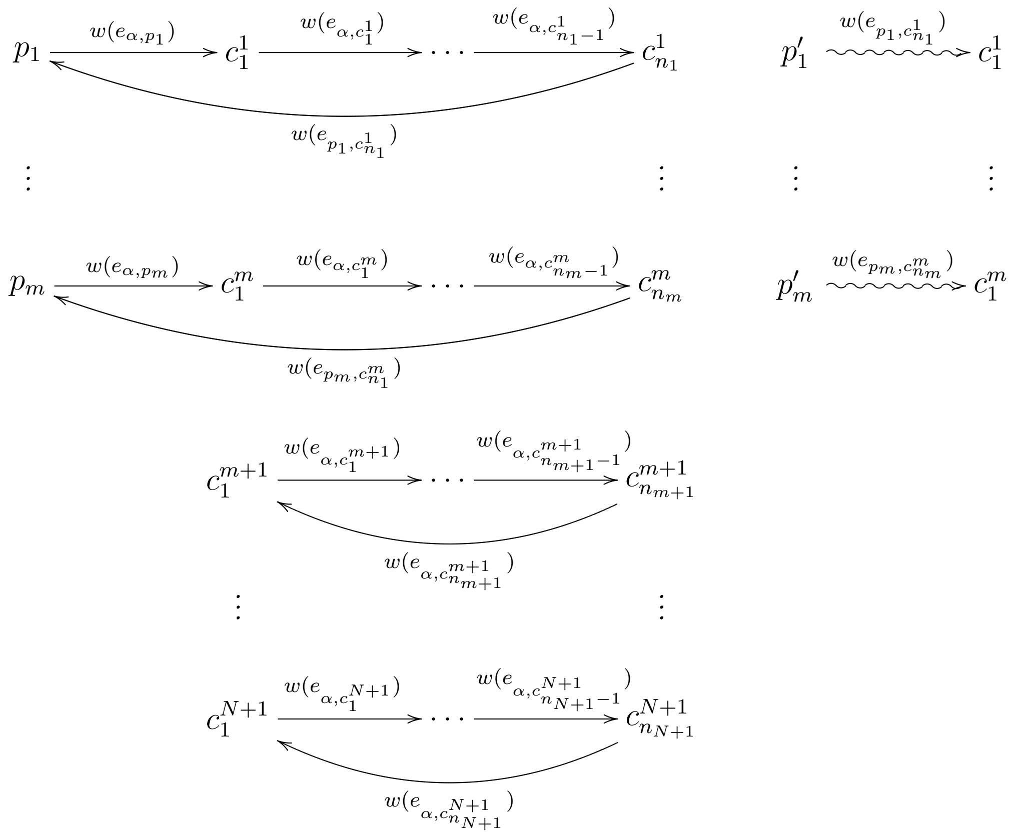
Compare this result with Theorem 5.9. We refer the readers to [CT07] for a proof.

Final Step Let us come back to our construction. Remark at first that the map F we have constructed is well a sub-hyperbolic semi-rational map as we noticed in Step 4. By Theorem 7.5, it is thus enough to check that F has no Thurston obstruction in order to prove the existence of a rational map as required. Indeed if follows from a Corollary of Theorem 7.5 in [CT07] that the Thurston condition *(iii)* in Definition 7.1 ensures that F has no Thurston obstruction.

Finally, we have proved in particular:

Theorem 7.6. *For every admissible weighted Hubbard tree* $(\mathcal{H}, w)$ *there exists a rational map* f *such that*

(i) f *realizes asymptotically the big ramification portrait below*



where the waved arrows mean that the critical points p'_i *are mapped near* c_1^i *(in the connected component containing* c_1^i *of the immediate attracting bassin of the* i^{th} *cycle)*

(ii) the Julia set $\mathcal{J}(f)$ *is disconnected*

7.2 Mandelbrotesque carpets

Consider the abstract Hubbard tree $\mathcal{H} = (T, \tau, \delta)$ displayed in Figure 7.2.

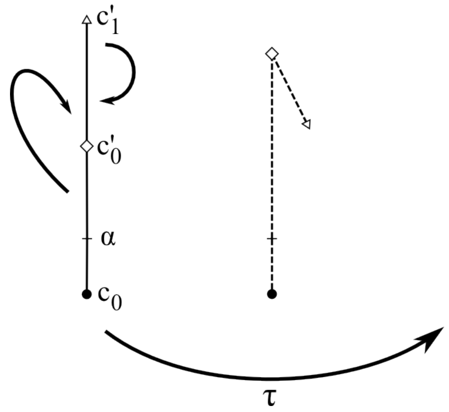


Figure 7.2: The abstract Hubbard tree $\mathcal{H}$

It has one fixed branching point α of Misiurewicz type and two periodic cycle of vertices: one fixed end c_0 and one periodic cycle $\{c_0', c_1'\}$ of period two containing a critical point c_0'. We have

$$\begin{cases} \tau(e_{\alpha,c_0}) = e_{\alpha,c_0} \\ \tau(e_{\alpha,c_0'}) = e_{\alpha,c_0'} \cup e_{c_0',c_1'} \\ \tau(e_{c_0',c_1'}) = e_{c_0',c_1'} \end{cases}$$

Equip $\mathcal{H}$ with a weight function w. Using the same order for the edges of $\mathcal{H}$ as above, we get the following transition matrix of $(\mathcal{H}, w)$

$$M = \begin{pmatrix} w(e_{\alpha,c_0})^{-1} & 0 & 0 \\ 0 & w(e_{\alpha,c_0'})^{-1} & w(e_{\alpha,c_0'})^{-1} \\ 0 & 0 & w(e_{c_0',c_1'})^{-1} \end{pmatrix}$$

In particular $\lambda(\mathcal{H})$ is the largest reciprocal of the weights. Therefore the weighted Hubbard tree $(\mathcal{H}, w)$ is unobstructed if and only if every weight is at least two.

For instance consider the weight function w which is equal to two on each edge. Then the weighted Hubbard tree $(\mathcal{H}, w)$ is admissible and consequently there exist a rational map h_p depending on the parameter $p \in \mathbb{C}$ realizing asymptotically the ramification portrait below.

$$0 \circlearrowright 2 \qquad 1 \underset{2}{\overset{4}{\rightleftarrows}} \infty \qquad p \overset{2}{\rightsquigarrow} \infty$$

The last arrow means that the critical point p is mapped in the connected component containing ∞ of the immediate attracting bassin of h_p. Remark that h_p has three critical points of multiplicity one and one critical point of multiplicity three, so $\deg(h_p) = 4$ (by Riemann-Hurwitz formula). In particular h_p is of the form

$$h_p : z \mapsto \frac{az^4 + bz^3 + cz^2 + dz + e}{Az^4 + Bz^3 + Cz^2 + Dz + E}$$

Since 1 is mapped to ∞ with a local degree four, the denominator may factor as

$$h_p : z \mapsto \frac{az^4 + bz^3 + cz^2 + dz + e}{(z-1)^4}$$

In particular

$$h_p : \frac{1}{z} \mapsto \frac{(a-1) + (b+4)z + (c-6)z^2 + (d+4)z^3 + ez^4}{(1-z)^4} + 1$$

Since ∞ is mapped to 1 with a local degree two, it follows $a = 1$ and $b = -4$

$$h_p : z \mapsto \frac{z^4 - 4z^3 + cz^2 + dz + e}{(z-1)^4}$$

An easy computation gives the derivative function of h_p

$$f_p' : z \mapsto -\frac{2(c-6)z^2 + (2c+3d)z + (d+4e)}{(z-1)^5}$$

Since 0 and p are critical points of h_p we get the following two equations

$$\begin{cases} d + 4e &= 0 \\ -\dfrac{2c+3d}{2(c-6)} &= p \end{cases} \Rightarrow \begin{cases} d &= -4e \\ c &= \dfrac{12p - 3d}{2p+2} = 6\left(\dfrac{p+e}{p+1}\right) \end{cases}$$

$$h_p : z \mapsto \frac{z^4 - 4z^3 + 6\left(\frac{p+e}{p+1}\right)z^2 - 4ez + e}{(z-1)^4} \tag{7.2}$$

It remains the information $h_p(0) = 0$ which lead to $e = 0$

$$h_p : z \mapsto \frac{z^2\left[z^2 - 4z + 6\left(\frac{p}{p+1}\right)\right]}{(z-1)^4}$$

We know that there exist some choices of p (in order to make $h_p(p)$ close to ∞) such that h_p is encoded by the weighted Hubbard tree $(\mathcal{H}, w)$. Indeed for $p \approx -1$ we get the bifurcation locus in Figure 7.3. Picking a parameter p inside the big hyperbolic component, we obtain the Julia set $\mathcal{J}(h_p)$ in Figure 7.4.

Observe that every Julia component of $\mathcal{J}(h_p)$ is a point or a Jordan curve (by the result of Tan Lei and K. Pilgrim, see Theorem 6.14).

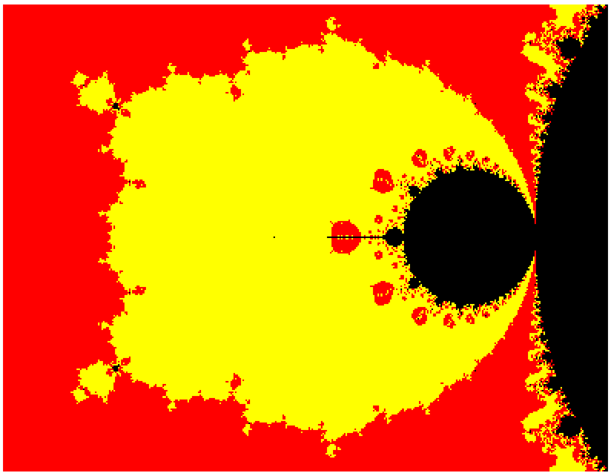


Figure 7.3: The hyperbolic component of the family $(h_p)_{p\in\mathbb{C}}$ inside a disk of center -1 and radius 0.15

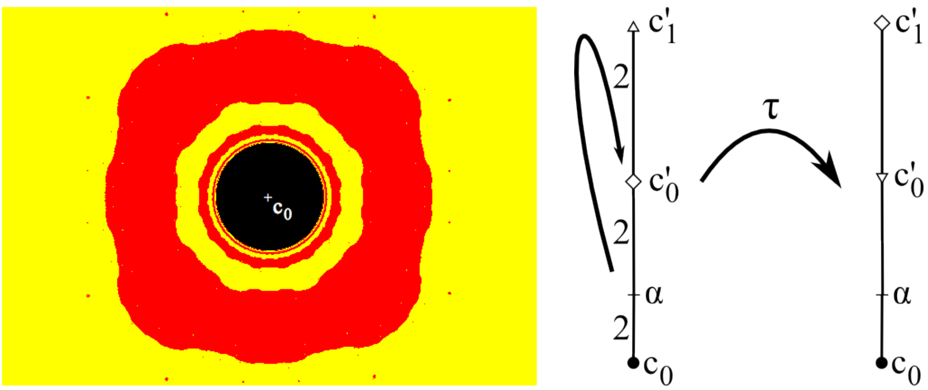


Figure 7.4: A Persian carpet with a disk motif

Now fixing such a value for the critical point p, we may vary the image of the critical point 0 in order that 0 is no longer fixed. Denote by e the image of 0. The asymptotic ramification portrait is now

$$0 \xrightarrow{2} e \qquad\qquad 1 \underset{2}{\overset{4}{\rightleftarrows}} \infty \qquad\qquad p \overset{2}{\rightsquigarrow} \infty$$

Actually we have already computed an algebraic formula for the rational map h_e seen now as depending on the parameter $e \in \mathbb{C}$ (see (7.2)):

$$h_e : z \mapsto \frac{z^4 - 4z^3 + 6\left(\frac{p+e}{p+1}\right) z^2 - 4ez + e}{(z-1)^4}$$

For $e \approx 0$ we get the bifurcation locus in Figure 7.5. The big hyperbolic component containing the parameter $e = 0$ is the main cardioid of a copy of the Mandelbrot set. Then this copy provides several different kind of rational maps with disconnected Julia sets, encoded by different admissible weighted Hubbard tree (see Figure 7.6 and Figure 7.7). Actually we may find parameter such that the associated rational map is still "encoded" by a weighted Hubbard tree which is no longer admissible (see Figure 7.8).

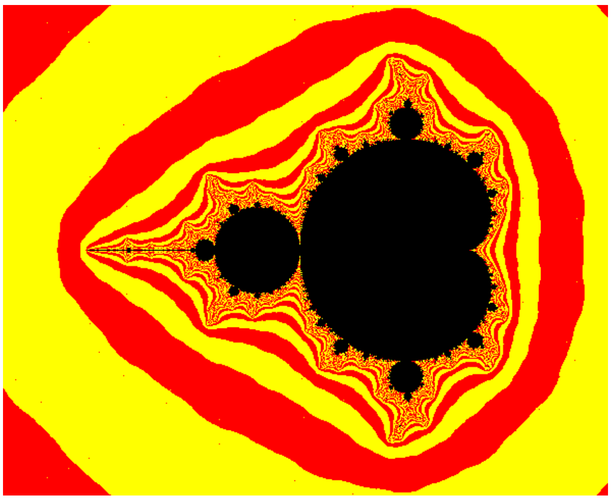


Figure 7.5: The hyperbolic component of the family $(h_e)_{e\in\mathbb{C}}$ ($p = -1.001$) inside a disk of center 0 and radius 0.0005

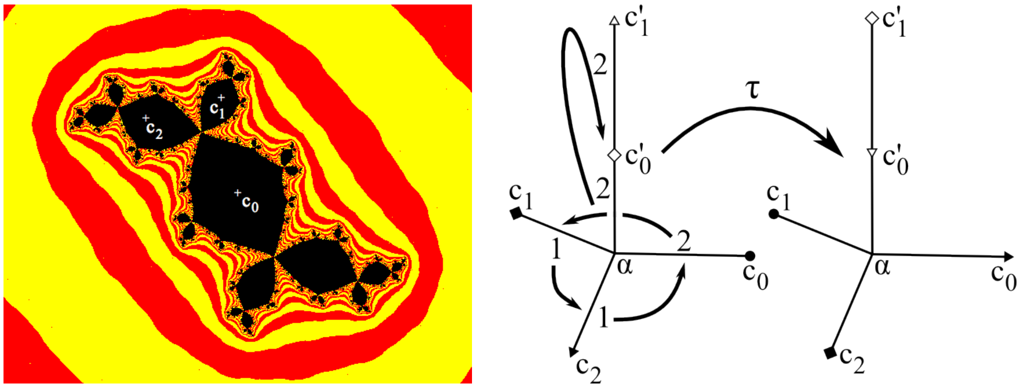


Figure 7.6: A Persian carpet with a rabbit motif

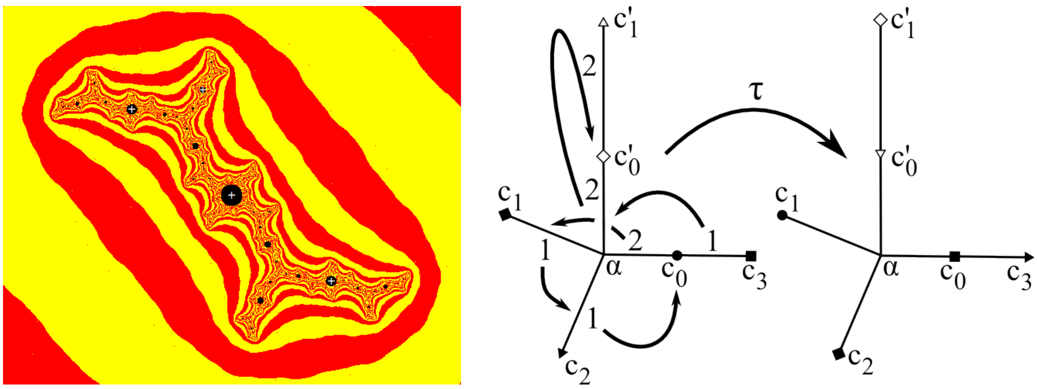


Figure 7.7: Another Persian carpet with a quadratic motif
(compare with Figure 6.3)

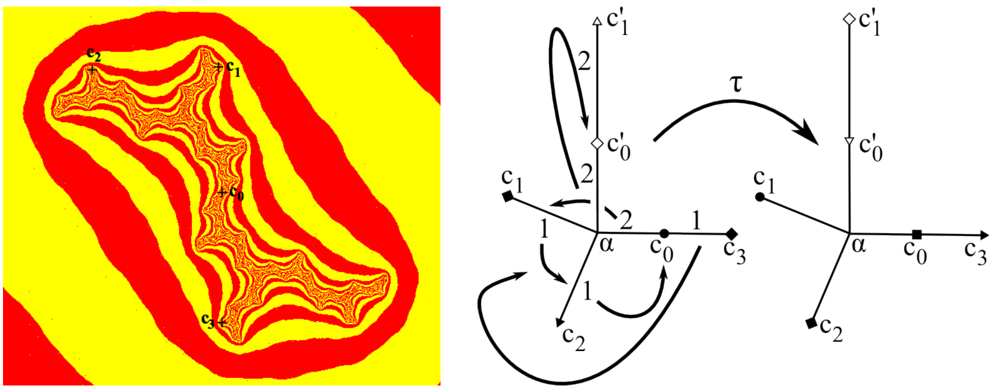


Figure 7.8: A Persian carpet with a dendrite motif

Chapter 8

Concluding remarks: future works

Theorem 6.18 gives a conjugacy of the rational map f on a subset $\mathcal{J}_{\mathcal{H}}(f)$ of Julia components. But it seems that the induced continuous and surjective map, say π, from $\bigcup_{J\in\mathcal{J}_{\mathcal{H}}(f)} J$ onto the Hubbard tree $\mathcal{H}$ may be continuously extended to the whole Riemann sphere $\widehat{\mathbb{C}}$ (compare with the map $\widehat{\pi}$ defined in Section 6.2). Unfortunatly, it is difficult to define precisely π on each piece constructed in Section 6.2, especially on $D'_{3,2}$ and $D(\delta'_{+3,-3})$ which contains some Julia components as we explained in Section 6.5. Another possible way to show this is to begin a new construction in a abstract way in order to get a sub-hyperbolic semi-rational map which is already "encoded" by the weighted Hubbard tree $(\mathcal{H}, w)$. Nevertheless, Theorem 7.5 provides only a combinatorially equivalence, not a topological equivalence as required. Therefore that will remain some works to conclude. This issue is linked to a question occurring naturally at the end of Chapter 7: how to encode precisely the dynamics of the constructed rational maps by the weighted Hubbard tree behind ?

We have also to generalize the construction from Chapter 7 to more complicated tree than those of Definition 7.1. More precisely, we may at first try to enlarge the realization condition *(ii)* by extending Theorem 5.21. To do so, we have to generalize Theorem 4.12 as we already discussed in Section 4.3 (topological part) and to consider only the ramified coverings without Thurston obstruction (analytical part). Such a generalization seems reasonable and it is suggested by Figure 7.8. We may also try to enlarge the tree shape condition *(i)*. That would be done by using Theorem 7.5 in a more general situation.

Furthermore several question occurs naturally in the way of studying the dynamics of rational maps with disconnected Julia set. In order to know how much of dynamical informations is captured by the weighted Hubbard tree, here is a list of issues:

- What about the unicity ? More precisely, if two rational maps are encoded by a same weighted Hubbard tree, are they quasiconformally conjugated ?
- What about the converse problem ? If a rational map has a disconnected Julia set, is it always possible to find a weighted Hubbard tree structure behind ?

The second guess seems to be true at least for sub-hyperbolic semi-rational map.

Appendix A

Topological tools

This appendix gathers all the topological facts required.

A.1 Plane Topology

We would like to prove two lemmas which were used several times in this thesis. As many other results in plane topology, both are consequence of the powerful Jordan's theorem. We will also recall some classical definitions and results.

Definition A.1 (Jordan arc and Jordan curve)**.** Let X be a topological space. A **Jordan arc** between two points $a, b \in X$ is the image of the unit segment $[0,1]$ by a homeomorphism $\ell : [0,1] \to X$ such that $\ell(0) = a$ and $\ell(1) = b$. A **Jordan curve** in X is the image of the unit circle $\mathbb{S}^1$ by a homeomorphism $\gamma : \mathbb{S}^1 \to X$. We will write only ℓ (respectively γ) to denote a Jordan arc (respectively a Jordan curve) when there is no ambiguity.

Theorem A.2 (Jordan's curve theorem)**.** *The complement of any Jordan curve γ in $\mathbb{S}^2$ consists of two distinct connected components and γ is the boundary of each component.*

There exist several proofs of this fundamental result in plane topology (using complex analysis, or Brouwer's fixed point theorem, or graph theory,...). The following classical sharpening discuss the topological nature of connected components in the complement of a Jordan curve.

Theorem A.3 (Schönflies' theorem)**.** *Let $\gamma : \mathbb{S}^1 \to \mathbb{S}^2$ be a Jordan curve in $\mathbb{S}^2$ and denote by D a connected component of $\mathbb{S}^2 - \gamma$. Then there is a homeomorphism $\widetilde{\gamma} : \mathbb{D} \to D$ which extends γ on the whole unit disk.*

Lemma A.4. *Fix a point ω in $\mathbb{S}^2$. Let γ (respectively γ') be a Jordan curve in $\mathbb{S}^2$ and denote by D (respectively by D') the connected component of $\mathbb{S}^2 - \gamma$ (respectively $\mathbb{S}^2 - \gamma'$) which does not contain ω. Then*

$$\begin{cases} \gamma \cap \gamma' = \emptyset \\ D \cap D' \neq \emptyset \end{cases} \Rightarrow D \subset D' \text{ or } D' \subset D$$

Proof of Lemma A.4. Let x be in the intersection $D \cap D'$. Since γ is connected and disjoint from γ', the Jordan's curve theorem applied to γ' implies either $\gamma \subset D'$ or $\gamma \subset \mathbb{S}^2 - \overline{D'}$. In the second case, we get $\gamma \cap D' = \emptyset$ and then the connected set D' is necessarily in the connected component of $\mathbb{S}^2 - \gamma$ which contains x, that is $D' \subset D$. So we may assume $\gamma \subset D'$ and $\gamma' \subset D$ by symmetry. In particular γ and γ' are not contained in the boundary of $D \cup D'$. Furthermore this boundary is contained in $\partial D \cup \partial D'$ which is equal to $\gamma \cup \gamma'$ by the Jordan's curve theorem. Finally $D \cup D' = \mathbb{S}^2$, that is a contradiction with the existence of the point ω. □

Definition A.5 (homotopic Jordan curves)**.** Let $P \subset \mathbb{S}^2$ be a finite set. Two Jordan curves γ_0 and γ_1 in $\mathbb{S}^2 - P$ are said **homotopic** in $\mathbb{S}^2 - P$ if there exists a homotopy $\varphi : [0,1] \times \mathbb{S}^1 \to \mathbb{S}^2 - P$, $(t,.) \mapsto \varphi(t,.)$ from $\varphi(0, \mathbb{S}^1) = \gamma_0$ to $\varphi(1, \mathbb{S}^1) = \gamma_1$ where $\varphi(0,.)$ and $\varphi(1,.)$ are homeomorphisms.

Notice that the distinction between a homeomorphism and its image to define a Jordan curve is important in definition above. For instance, the homeomorphisms $e^{2i\pi\theta} \mapsto e^{2i\pi\theta}$ and $e^{2i\pi\theta} \mapsto e^{-2i\pi\theta}$ are not homotopic in $\widehat{\mathbb{C}} - \{0, \infty\}$ (with identification of $\mathbb{S}^2$ and $\widehat{\mathbb{C}}$ as topologic manifolds) but they define the same Jordan curve.

Lemma A.6. *Let P be a subset of $\mathbb{S}^2$. Let γ (respectively γ') be a Jordan curve in $\mathbb{S}^2 - P$ and let D (respectively D') be a connected component of $\mathbb{S}^2 - \gamma$ (respectively $\mathbb{S}^2 - \gamma'$). If γ and γ' are homotopic in $\mathbb{S}^2 - P$ then $D \cap P$ is equal to either $D' \cap P$ or $(\mathbb{S}^2 - \overline{D'}) \cap P$.*

In order to prove the lemma above, we will need the following useful tool.

Definition A.7 (winding number)**.** Let $\phi : \mathbb{S}^1 \to \mathbb{C}$ be a continuous function and a be a point in $\mathbb{C} - \phi(\mathbb{S}^1)$. We can rewrite $\phi : e^{2i\pi t} \mapsto a + r(t)e^{2i\pi\theta(t)}$ where $r : [0,1] \to]0, +\infty[$ and $\theta : [0,1] \to \mathbb{R}$ are required to be continuous. The **winding number** of ϕ around a, denoted by $\mathbf{Ind}_{\boldsymbol{\phi}}\boldsymbol{(a)}$, is the integer

$$\mathrm{Ind}_\phi(a) = \theta(1) - \theta(0)$$

and it does not depend on choice of continuous function $\theta : [0,1] \to \mathbb{R}$.

Recall some results about the winding number.

Proposition A.8. *We have the following properties.*

1. *The winding number is continuous: for every continuous function* $\phi : \mathbb{S}^1 \to \mathbb{C}$*,* Ind_ϕ *is constant on each connected component of* $\mathbb{C} - \phi(\mathbb{S}^1)$*.*

2. *The winding number is constant on the homotopy classes: for every continuous functions* $\phi_0, \phi_1 : \mathbb{S}^1 \to \mathbb{C}$*, and every point* $a \in \mathbb{C} - (\phi_0(\mathbb{S}^1) \cup \phi_1(\mathbb{S}^1))$*, if there exists a homotopy* $\varphi : [0,1] \times \mathbb{S}^1 \to \mathbb{C} - \{a\}$*,* $(t,.) \mapsto \varphi(t,.)$ *from* $\varphi(0,.) = \phi_0$ *to* $\varphi(1,.) = \phi_1$ *then* $\mathrm{Ind}_{\phi_0}(a) = \mathrm{Ind}_{\phi_1}(a)$*.*

3. *The winding number of a Jordan curve* γ *on* $\mathbb{C}$ *is equal to 1 or -1 on the bounded connected component of* $\mathbb{C} - \gamma$ *and 0 on the unbounded connected component.*

Let us come back to the Lemma A.6.

Proof of Lemma A.6. We may assume that $P \neq \emptyset$ and, by Jordan's curve theorem, $(\mathbb{S}^2 - \overline{D}) \cap (\mathbb{S}^2 - \overline{D'}) \cap P \neq \emptyset$ (exchanging D with $\mathbb{S}^2 - \overline{D}$ and D' with $\mathbb{S}^2 - \overline{D'}$ if necessary). We have thus to prove that $D \cap P = D' \cap P$. Let ω be a point in the intersection $(\mathbb{S}^2 - \overline{D}) \cap (\mathbb{S}^2 - \overline{D'}) \cap P$ and identify $\mathbb{S}^2 - \{\omega\}$ with $\mathbb{C}$ as topological manifolds. Now compare Ind_γ and $\mathrm{Ind}_{\gamma'}$ at every point $a \in P - \{\omega\}$. Since γ and γ' are homotopic in $\mathbb{S}^2 - \{a\}$, if follows $\mathrm{Ind}_\gamma(a) = \mathrm{Ind}_{\gamma'}(a)$ from the second point of Proposition A.8. Moreover $\mathrm{Ind}_\gamma(a)$ (respectively $\mathrm{Ind}_{\gamma'}(a)$) is not equal to 0 if and only if $a \in D$ (respectively $a \in D'$) by the third point of Proposition A.8. The conclusion follows. □

A.2 Ramified coverings

We would like to recall the definition of ramified coverings together with some classical results. We refer the readers to [Dou05] for deeper discussions.

Definition A.9 (ramified covering)**.** Let U_1 and U_2 be two toplogical manifolds of real dimension two (not necessarily equipped with a Riemann surface structure). A continuous function $f : U_1 \to U_2$ is a **ramified covering** if for every $y \in U_2$ there exists a connected neighbourhood $D(y)$ such that

- $f^{-1}(D(y))$ is a nonempty union (possibly finite) of pairwise disjoints connected open sets

$$f^{-1}(D(y)) = \bigcup_{k \geqslant 1} B(x_k)$$

- for every integer $k \geqslant 1$, there exist a positive integer $d_k \geqslant 1$ and two charts (i.e. homeomorphisms) $\varphi_k : B(x_k) \to \mathbb{D}$ and $\varphi : D(y) \to \mathbb{D}$ such that $\varphi_k(x_k) = 0$, $\varphi(y) = 0$ and the following diagram commutes

$$\begin{array}{ccc} B(x_k) & \xrightarrow{\varphi_k} & \mathbb{D} \\ f \downarrow & & \downarrow z \mapsto z^{d_k} \\ D(y) & \xrightarrow{\varphi} & \mathbb{D} \end{array}$$

The integer d_k is called the **local degree** of f at x_k and it is denoted by $\mathbf{deg_{loc}(f)(x_k)}$. Furthermore if $\deg_{\mathrm{loc}}(f)(x_k) > 1$, we say that x_k is a **critical point** of f of **multiplicity** $\deg_{\mathrm{loc}}(f)(x_k) - 1$. Finally an **unbranched covering** is a ramified covering without critical point.

Definition A.10 (orientation-preserving ramified covering)**.** Let $f : U_1 \to U_2$ be a ramified covering between two orientable topological manifolds of real dimension two. f is said **orientation-preserving** if for every pair of charts $\varphi_1 : V_1 \to \mathbb{D}$ and $\varphi_2 : V_2 \to \mathbb{D}$ in Definition A.9 where $V_1 \cap V_2$ is nonempty, the transition map

$$\varphi_{1,2} = (\varphi_2 \circ \varphi_1^{-1})_{|\varphi_1(V_1 \cap V_2)} : \varphi_1(V_1 \cap V_2) \to \varphi_2(V_1 \cap V_2)$$

is orientation-preserving (where $\mathbb{D} \subset \mathbb{C}$ comes with the usual counterclockwise orientation).

Proposition A.11. *Let $f : U_1 \to U_2$ be a ramified covering between two topological manifolds of real dimension two. The following holds.*

1. *The critical points of f are isolated points in U_1.*

2. *There exists $d \in \mathbb{N} \cup \{\infty\}$, called the degree of f, such that*

$$\forall y \in U_2, \sum_{\substack{x \in U_1 \\ f(x)=y}} \deg_{loc}(f)(x) = d$$

   In particular f is a surjective map.

3. *For every connected open set $V_2 \subset U_2$ and every V_1 connected component of $f^{-1}(V_2)$, $f_{|V_1} : V_1 \to V_2$ is a ramified covering.*

4. *In case $d < \infty$, for every connected compact set $K_2 \subset U_2$ and every K_1 connected component of $f^{-1}(K_2)$, $f(K_1) = K_2$.*

The following powerful result makes a link between ramification and algebraic topology.

Theorem A.12 (Riemann-Hurwitz formula)**.** *Let V_1 and V_2 be two connected open sets in $\mathbb{S}^2$. Denote by m_1 (respectively m_2) the number of connected components in $\mathbb{S}^2 - V_1$ (respectively $\mathbb{S}^2 - U_2$). If $f : U_1 \to U_2$ is a ramified covering of degree $d \in \mathbb{N}$ then*

$$m_1 - 2 = d(m_2 - 2) + r \quad \textit{where} \quad r = \sum_{x \in V_1} (\deg_{loc}(f)(x) - 1)$$

As a consequence, a rational map $R : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ of degree $d \geqslant 1$ has exactly $2d - 2$ critical points counted with multiplicity.

Appendix B

Linear algebra

We would like to recall in this appendix some classical definitions and results in linear algebra, notably the Perron-Frobenius theorem. All these results are needed in the discussion around the Thurston's theorem (Chapter 5). We will not give proofs, except for the last corollary which justifies crucial Definition 5.8 and Definition 6.10.

Definition B.1 (spectral radius)**.** The **spectral radius** of a matrix $M \in \mathcal{M}_n(\mathbb{C})$, denoted by $\boldsymbol{\rho(M)}$ is the largest modulus of its eigenvalues.

$$\rho(M) = \max\{|\lambda| \,/\, \lambda \in \mathrm{Sp}(M)\}$$

In other words, the spectral radius is the largest modulus of roots of the characteristic polynomial: $\chi_M(X) = \det(M - XI_n) \in \mathbb{C}_n[X]$. From the fact that polynomial roots depend continuously on coefficients, we get:

Proposition B.2. *For any matrix norm $\|.\|$ on $\mathcal{M}_n(\mathbb{C})$, the spectral radius is a continuous function from $(\mathcal{M}_n(\mathbb{C}), \|.\|)$ to $(\mathbb{R}, |.|)$.*

Theorem B.3 (Gelfand's formula)**.** *For any matrix norm $\|.\|$ on $\mathcal{M}_n(\mathbb{C})$, we have:*

$$\forall M \in \mathcal{M}_n(\mathbb{C}),\ \rho(M) = \lim_{k\to\infty} \|M^k\|^{\frac{1}{k}}$$

Definition B.4 (positive or non-negative matrices)**.** A matrix is said **positive**, respectively **non-negative**, if all its coefficients are positive, respectively non-negative. For two matrices $A, B \in \mathcal{M}_n(\mathbb{C})$ we write $\boldsymbol{A \leqslant B}$ if the matrix $B - A$ is non-negative.

Proposition B.5. *The spectral radius is increasing on the set of non-negative matrices.*

$$\forall A, B \in \mathcal{M}_n(\mathbb{R}_+),\ A \leqslant B \Rightarrow \rho(A) \leqslant \rho(B)$$

Proposition B.6. *Let $M \in \mathcal{M}_n(\mathbb{R}_+)$ be a non-negative matrix. The spectral radius of any square sub-block $M_{i,j} \in \mathcal{M}_k(\mathbb{R}_+)$ is less than the spectral radius of M.*

$$\rho(M_{i,j}) \leqslant \rho(M) \text{ where } M = \begin{pmatrix} * & * & * \\ * & M_{i,j} & * \\ * & * & * \end{pmatrix}$$

Theorem B.7 (Perron-Frobenius theorem)**.** *Let M be a positive matrix. Then $\rho(M)$ is an eigenvalue of M and it is the only one with the largest modulus. Moreover there exists an associated eigenvector with positive entries.*

Corollary B.8. *Let M be a non-negative matrix. Then $\rho(M)$ is an eigenvalue of M. Moreover for every $\varepsilon > 0$ there exists a vector y with positive entries such that $My \leqslant (\rho(M) + \varepsilon)y$.*

Proof of Corollary B.8. Denote by $(m_{i,j})$ the entries of M. For every integer $k \geqslant 1$, let M_k be the positive matrix of coefficients $(m_{i,j} + \frac{1}{k})$. By the Perron-Frobenius theorem, there exists a positive eigenvector v_k associated to the eigenvalue $\rho(M_k)$. Denote by $x_k = \frac{v_k}{\|v_k\|}$ the normalized vector for any vector norm $\|.\|$ on $\mathbb{R}^n$. Since the set $S = \{x \in \mathbb{R}^n \,/\, \|x\| = 1\}$ is compact, we can extract a sub-sequence $(x_{\varphi(k)})$ which tends to a non-negative vector $x \in S$. Moreover $\lim_{k\to\infty} M_k = M$ and $\lim_{k\to\infty} \rho(M_k) = \rho(M)$ because of continuity of spectral radius (Proposition B.2). Finally we get

$$Mx = \lim_{k\to\infty} M_{\varphi(k)} x_{\varphi(k)} = \lim_{k\to\infty} \rho(M_{\varphi(k)}) x_{\varphi(k)} = \rho(M)x$$

Now let $k \geqslant 1$ be large enough such that the inequality $\rho(M_{\varphi(k)}) \leqslant \rho(M) + \varepsilon$ holds. The result follows with $y = x_{\varphi(x)}$. □

Appendix C

Complex analysis

This last appendix gathers some classical results in complex analysis which were often used in this thesis.

C.1 Riemann's mapping theorem

Theorem C.1 (Riemann's mapping theorem). *If U is a topological disk in the Riemann sphere, that is a nonempty simply connected open subset of $\widehat{\mathbb{C}}$ whose complement contains at least two points, then there exists a biholomorphic map $\varphi : \mathbb{D} \to U$ from the unit disk onto U.*

We refer the readers to [Ahl79] or any other lecture about complex analysis for a proof. Notice that up to precomposition by an automorphism of $\mathbb{D}$, that is a map of the form

$$\mathbb{D} \to \mathbb{D},\ z \mapsto e^{i\theta}\frac{z-a}{1-|a|z} \quad \text{where } e^{i\theta} \in \partial\mathbb{D} \text{ and } a \in \mathbb{D}$$

we may always prescribe the image of 0 and the argument of $\varphi'(0)$ (actually the map φ is unique for given $(\varphi(0), \frac{\varphi'(0)}{|\varphi'(0)|}) \in U \times \partial\mathbb{D}$).

This powerful result has several consequences in many fields of mathematics. In particular it provides a classification of all the simply connected open subsets of $\widehat{\mathbb{C}}$ (and likewise for any Riemann surface). There are some kinds of generalization. We are going to present two of them. At first the question of extension on the boundary of the unit disk as in the following result.

Theorem C.2 (Carathéodory's theorem). *Let U be a topological disk in $\widehat{\mathbb{C}}$ with a locally connected boundary ∂U. Then any biholomorphic map $\varphi : \mathbb{D} \to U$ extends continuously to the unit circle: $\widetilde{\varphi} : \overline{\mathbb{D}} \to \overline{U}$.*

Secondly consider the same problem for doubly connected domains. We have the following classical results.

Theorem C.3. *Let U be a topological disk in $\widehat{\mathbb{C}}$ and let K be a connected and relatively compact subset of U which contains at least two points. Then there exists a unique $r \in]0,1[$ such that $U - K$ is biholomorphically conjugated to the annulus $\{z \in \mathbb{C} \,/\, r < |z| < 1\}$.*

That justifies the following definition.

Definition C.4 (modulus of annulus)**.** Let γ and γ' be two disjoint continua in the Riemann sphere, that is two nonempty compact connected subsets of $\widehat{\mathbb{C}}$. We denote by $\boldsymbol{A(\gamma,\gamma')}$ the unique doubly connected component of $\widehat{\mathbb{C}} - (\gamma \cup \gamma')$. The **modulus** of the annulus $A(\gamma,\gamma')$ is the unique positive number $\mathbf{mod}(\boldsymbol{\gamma},\boldsymbol{\gamma'})> 0$ such that $A(\gamma,\gamma')$ is biholomorphically conjugated to the annulus

$$\{z \in \mathbb{C} \,/\, r < |z| < 1\} \quad \text{where} \quad r = e^{-2\pi \operatorname{mod}(\gamma,\gamma')} \Leftrightarrow \operatorname{mod}(\gamma,\gamma') = \frac{1}{2\pi}\log\left(\frac{1}{r}\right)$$

By extension we write $\mathbf{mod}(\boldsymbol{A})$ for every annulus A.

Proposition C.5. *Let $f : A_1 \to A_2$ be a conformal map of degree $d \geqslant 1$ between two annuli $A_1, A_2 \subset \widehat{\mathbb{C}}$. Then*

$$\operatorname{mod}(A_1) = \frac{1}{d}\operatorname{mod}(A_2)$$

Theorem C.6 (Grötzsch's inequality)**.** *Let $A \subset \widehat{\mathbb{C}}$ be an annulus and $A_1, A_2 \subset A$ be two disjoint annulli such that the two connected components of the boundary of A are neither in the same connected component of $\widehat{\mathbb{C}} - A_1$ nor in the same connected component of $\widehat{\mathbb{C}} - A_2$. Then*

$$\operatorname{mod}(A_1) + \operatorname{mod}(A_2) \leqslant \operatorname{mod}(A)$$

The following useful result is due to Cui Guizhen and Tan Lei. An equipotential in a topological disk $U \subset \widehat{\mathbb{C}}$ is the image of a round circle $\{z \in \mathbb{D} \,/\, |z| = r\}$ where $r \in]0,1[$ by any biholomorphic map $\varphi : \mathbb{D} \to U$.

Lemma C.7 (Inverse Grötzsch's inequality)**.** *Let U_1 and U_2 be two disjoint topological disks in $\widehat{\mathbb{C}}$ whose boundary are respectively denoted by α_1 and α_2. Then there exists a positive constant $C > 0$ such that for every pair of equipotentials $\gamma_1 \subset U_1$ and $\gamma_2 \subset U_2$ the following inequalities hold*

$$\operatorname{mod}(\alpha_1,\gamma_1) + \operatorname{mod}(\alpha_2,\gamma_2) \leqslant \operatorname{mod}(\gamma_1,\gamma_2) \leqslant \operatorname{mod}(\alpha_1,\gamma_1) + \operatorname{mod}(\alpha_2,\gamma_2) + C$$

The left hand side is the classical Grötzsch's inequality. The right hand side is a consequence of Koebe 1/4-theorem. We refer the readers to [CT07] for a complete proof.

C.2 Quasiconformal surgery

A. Douady, J. H. Hubbard and D. Sullivan were the first to understand the power of quasiconformal maps in holomorphic dynamical systems. These maps allowed in particular to prove the Sullivan's non-wandering domains theorem or the sharpest value of the number of non-repelling cycles by M. Shishikura. It is now a standard tool. We would like to present a brief overview of this theory, especially the quasiconformal surgery method. The readers are refered to [Ahl06] for more overall presentation of quasiconformal maps and to [Bea91], [CG93] or [BM01] for proofs of classical results in holomorphic dynamical systems using the quasiconformal surgery method.

If a map $\varphi : \mathbb{C} \to \mathbb{C}$ is differentiable, we denote by φ_x and φ_y its partial derivatives and we use the following standard differential operators:

$$\begin{cases} \varphi_z = \frac{1}{2}(\varphi_x - i\varphi_y) \\ \varphi_{\overline{z}} = \frac{1}{2}(\varphi_x + i\varphi_y) \end{cases}$$

Observe that if φ is holomorphic then

$$\begin{cases} \varphi_z = \varphi'(z) \\ \varphi_{\overline{z}} = 0 \end{cases}$$

The generalization of these equations is the Beltrami equation:

$$\varphi_{\overline{z}} = \mu(z)\varphi_z \tag{C.1}$$

where μ is some suitable complex function.

Definition C.8 (almost complex structure)**.** Let U be a domain in the Riemann sphere, that is a nonempty open connected subset of $\widehat{\mathbb{C}}$. An **almost complex structure** on U is a Lebesgue measurable function $\boldsymbol{\mu}$: $U \to \mathbb{C}$ such that $\|\mu\|_\infty < 1$.

Definition C.9 (quasiconformal map)**.** Let U be a domain in $\widehat{\mathbb{C}}$. A homeomorphism $\varphi : U \to \widehat{\mathbb{C}}$ is a **quasiconformal map** if φ is a solution of the Beltrami equation (C.1) for an almost complex structure μ on U, called the **dilatation** or the **Beltrami coefficient** of φ. Moreover if

$$\|\mu\|_\infty \leqslant k < 1$$

then φ is said to be a ***k*-quasiconformal map**.

If φ is differentiable at $z \in U$, the preimage of a circle by the differential $d\varphi(z) = \varphi_z dz + \varphi_{\overline{z}} d\overline{z}$ is an ellipse whose the argument of the major axis is

$\frac{1}{2}(\pi + \arg(\mu))$ and the ratio of major to minor axis is $\frac{1+|\mu|}{1-|\mu|}$. So the dilatation μ of a quasiconformal map φ may be seen as an infinitesimal ellipse field. Moreover φ is k-quasiconformal means that the "stretching" (i.e. the ratio of major to minor axis of infinitesimal ellipse) of φ is uniformally bounded:

$$1 \leqslant \frac{1+|\mu|}{1-|\mu|} \leqslant \frac{1+k}{1-k}$$

The following result provides a positive answer for the question of the existence of solutions for Beltrami equation (C.1).

Theorem C.10 (measurable Riemann's mapping theorem)**.** *Let μ be an almost complex structure on a domain U in $\widehat{\mathbb{C}}$. Then there exists a quasiconformal map $\varphi : U \to \widehat{\mathbb{C}}$ with dilatation μ.*

Moreover it is shown that there is uniqueness up to postcomposing by a biholomorphic map. Notice that for U simply connected (whose complement contains at least two points) and μ identically zero we get the classical Riemann's mapping theorem (see Theorem C.1).

Definition C.11 (quasiregular map)**.** Let U be a domain in $\widehat{\mathbb{C}}$. A map $F : U \to \widehat{\mathbb{C}}$ is said **quasiregular** on $V \subset U$ if for every $z \in V$, F is holomorphic or quasiconformal on a neighbourhood of z.

Notice that if F is quasiregular on a compact set K then there exists a positive number $k < 1$ such that for every $z \in K$, F is holomorphic or k-quasiconformal on a neighbourhood of z.

Proposition C.12. *Let U be a domain in $\widehat{\mathbb{C}}$ and let K be a connected and relatively compact subset of U whose boundary is the union of a finite number of disjoint smooth curves (i.e. the image of the unit circle $\mathbb{S}^1$ by a diffeomorphism). Let $f : U - K \to \widehat{\mathbb{C}}$ be a holomorphic map which extends diffeomorphically to the boundary ∂K. Then f extends to U by a quasiregular map without critical points in K.*

The proposition above was used in this thesis in order to define piecewisely some quasiregular maps. That is not the most general statement, but we did not require more.

The very useful following result was stated by M. Shishikura in [Shi87] and it is the basic principle of the quasiconformal surgery method.

Theorem C.13 (Shishikura's principle on quasiconformal surgery)**.** *Assume that a map $F : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ satisfies the following points:*

- $F_{|H} : H \to \widehat{\mathbb{C}}$ *is holomorphic on an open set* $H \subset \widehat{\mathbb{C}}$
- F *extends quasiregularly to the complement* $Q = \widehat{\mathbb{C}} - H$
- *there exists an open set* $A \subset H$ *such that* $F(A) \subset A$ *and*

$$\exists N \in \mathbb{N} \,/\, F^{\circ N}(Q) \subset A$$

Then there exists an almost complex structure μ *on* $\widehat{\mathbb{C}}$ *which is* F*-invariant (i.e. such that every infinitesimal ellipse associated to* μ *at* z *is mapped by* $dF(z)$ *to the infinitesimal ellipse associated to* μ *at* $F(z)$ *for almost every* $z \in \widehat{\mathbb{C}}$*).*

In particular, if $\varphi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ *is the quasiconformal map of dilatation* μ *given by measurable Riemann's mapping theorem then the map* $f = \varphi \circ F \circ \varphi^{-1}$ *is holomorphic.*

The main idea of the proof of this result is to pullback infinite number of times an infinitesimal circle field by F. Then we get an F-invariant infinitesimal ellipse field which is well defined since we go only a finite number of times through Q where the "stretching" of F is uniformally bounded (it is null elsewhere since F is holomorphic).

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