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Hochschild cohomology VS Chevalley-Eilenberg cohomology

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Introduction

This text is an abridged english version (no proofs) of the my original thesis manuscript which is written in french.

Let $\mathfrak{g}$ be a Lie algebra on a commutative ring $\mathbb{K}$, $U\mathfrak{g}$ it's universal enveloping algebra, and consider a $U\mathfrak{g}$-bimodule M to which a right "adjoint" $\mathfrak{g}$-module M^{ad} is canonically associated. It is well known, since the work of Cartan and Eilenberg, that there exists an antisymmetrisation map

$$\begin{array}{rcl} F_* : C_*(\mathfrak{g}; M^{ad}) & \to & CH_*(U\mathfrak{g}; M) \\ m \otimes g_1 \wedge \cdots \wedge g_n & \mapsto & \displaystyle\sum_{\sigma \in \Sigma_n} \mathrm{sgn}(\sigma) m \otimes g_{\sigma(1)} \otimes \cdots \otimes g_{\sigma(n)} \end{array} \tag{0.0.1}$$

from the Chevalley-Eilenberg complex of $\mathfrak{g}$ with coefficients in M^{ad} to the Hochschild complex of $U\mathfrak{g}$ with coefficients in M, that induces an isomorphism

$$H_*(F_*) : H_*(\mathfrak{g}; M^{ad}) \cong HH_*(U\mathfrak{g}; M) \tag{0.0.2}$$

between the Lie algebra homology of $\mathfrak{g}$ and the Hochschild homology of its universal enveloping algebra, denoted by $HH_*(U\mathfrak{g}; M)$. An analogous statement stands in cohomology. The goal of this manuscript is to answer the following question :

Question 0.0.0.1. *Is there a quasi-inverse map*

$$G_* : CH_*(U\mathfrak{g}; M) \to C_*(\mathfrak{g}; M^{ad})$$

, defined at the chains level, such that the induced map $HH_*(U\mathfrak{g}; M) \to H_*(\mathfrak{g}; M^{ad})$ provides the inverse of the preceeding antisymmetrisation map $H_*(F_*)$?
In case the answer is positive, can such a quasi-inverse be given by an explicit formula, as Cartan and Eilenberg did for F_* ?

In the case when $\mathbb{K}$ is a field, the existence of such a G_* is clear since one can choose adapted bases of the kernel and image of the differentials. However, it is not obvious whether G_* can be defined by an intrinsic and natural formula that doesn't rely on choices of bases.

For a finite dimensionnal abelian Lie algebra $\mathfrak{g}$, $U\mathfrak{g}$ can be identified (as an **algebra**) with the symmetric algebra $S\mathfrak{g}$ and the isomorphism

$$H_*(F) : H_*(\mathfrak{g}; S\mathfrak{g}) = S\mathfrak{g} \otimes \Lambda^n \mathfrak{g} \to HH_*(S\mathfrak{g}; S\mathfrak{g})$$

can be seen as a polynomial version of the Hochschild-Kostant-Rosenberg theorem ([Lod98], [Hal01]) which identifies Kähler forms of a smooth commutative algebra with its Hochschild homology.

Moreover, Hochschild homology admits an interpretation in terms of derived functor, and the quasi-isomorphism (0.0.1) comes from the choice of two different resolutions to compute it: the bar-resolution and the Koszul resolution. In [Con85], A. Connes shows how to build a Koszul-type resolution $CK_*(A)$ of the A-bimodule A, when $A := \mathcal{C}^\infty(V;\mathbb{C})$ is the algebra of complex valued functions on a compact smooth manifold V. $CK_n(A)$ is defined as the space of smooth sections of the pullback of the complexification of the n-th exterior power of the cotangent bundle of V through the second projection $\mathrm{pr}_2 : V \times V \to V$ i.e.

$$CK_n(A) := \Gamma^\infty(V \times V; E_n)$$

with $E_n := \mathrm{pr}_2^*(\Lambda^n T^*_{\mathbb{C}} V)$. The differential $d^K : CK_n(A) \to CK_{n-1}(A)$, which has degree -1, is the inner product ι_X by a Euler vector field (vanishing on the diagonal) $X : V \times V \to \mathrm{pr}_2^* T_{\mathbb{C}} V$ defined thanks to the choice of a connection on V. To prove the contractibility of the complex $(CK_*(A), d^K)$ the author exhibits a contracting homotopy $s : CK_*(A) \to CK_{*+1}(A)$ which is defined by a formula similar to the one defining the usual contraction of the so called Poincaré lemma, and uses this homotopy to define an isomorphism

$$F^* : HH^*_c(A; A^\vee) \overset{\cong}{\to} D^*(V)$$

between the continuous Hochschild cohomology of A with values in its dual $A^\vee$ computed with the usual continuous Hochschild complex, and $D^*(V)$, the graded vector space of De Rham currents on V. This isomorphism can be seen as a continuous cohomological version of the classical antisymmetrization isomorphism of the H-K-R theorem.

Notice that when V is a Lie group, there exists a canonical connection on V given by left translations. The case $V = (\mathbb{R}^m, +)$ has been treated in [BGH$^+$05] where M. Bordemann, G. Ginot, G. Halbout, H-C Herbig et S. Waldmann apply Connes' construction to get a contraction h^K of the continuous Koszul resolution $CK^c_*(\mathcal{C}^\infty(\mathbb{R}^m)) := \mathcal{C}^\infty(\mathbb{R}^{2m}) \otimes \Lambda^*(\mathbb{R}^m)^\vee$ from wich they deduce a morphism of resolutions

$$G^B_* : (B^c_*(\mathcal{C}^\infty(\mathbb{R}^m)), d^B) \to (CK^c_*(\mathcal{C}^\infty(\mathbb{R}^m)), d^K)$$

over the identity of $\mathcal{C}^\infty(\mathbb{R}^m)$. Here, $(B^c_*(\mathcal{C}^\infty(\mathbb{R}^m)), d^B)$ denotes the usual continuous bar resolution of the topological algebra $\mathcal{C}^\infty(\mathbb{R}^m)$. More precisely, G^B_* is given in degree n by

$$G^B_n(\phi)(a,b) = \sum_{i_1,\cdots,i_n=1}^{m} e_{i_1}\wedge\cdots\wedge e_{i_n} \int_0^1 dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n \frac{\partial^n \phi}{\partial x^1_{i_1}\cdots\partial x^n_{i_n}}(a, t_1 a+(1-t_1)b, \cdots, t_n a+(1-t_n)b, b)$$

for every n-chain

$$\begin{array}{rcl} \phi : (\mathbb{R}^m)^{n+2} & \to & \mathbb{R} \\ (a, x^1, \cdots, x^n, b) & \mapsto & \phi(a, x^1, \cdots, x^n, b) \end{array}$$

in $B_n^c(\mathcal{C}^\infty(\mathbb{R}^m)) := \mathcal{C}^\infty(\mathbb{R}^{\times(n+2)m})$ and for all a and b in $\mathbb{R}^m$. In the above formula, e_1, ..., e_m stands for the dual base associated to the canonical base of $\mathbb{R}^m$. In fact, G_*^B is the unique morphism of $\mathcal{C}^\infty(\mathbb{R}^m)$-bimodules satifying the defining condition

$$G_{n+1}^B(1 \otimes f_1 \otimes \cdots f_n \otimes 1) = h^K \circ G_n \circ d^B(1 \otimes f_1 \otimes \cdots f_n \otimes 1) \tag{0.0.3}$$

for all f_1, ..., f_n in $\mathcal{C}^\infty(\mathbb{R}^m)$, once G_0^B is fixed. The authors then show that the application induced in homology by G_*^B after tensorization with the identity of $\mathcal{C}^\infty(\mathbb{R}^m)$ over $\mathcal{C}^\infty(\mathbb{R}^{2m})$ gives an inverse for the homological continuous H-K-R antisymmetrization map, realizing the complex of differential forms on $\mathbb{R}^m$ with zero differential as a deformation retract of the continuous Hochschild chain complex of $\mathcal{C}^\infty(\mathbb{R}^m)$ with coefficients in itself.

In the more general case 0.0.0.1 we are interested in, the map G_* we are looking for should also come from a morphism of resolutions G_*^B between the bar and Koszul resolutions, no longer of the algebra of functions on some Lie group V, but of the universal envelopping algebra $U\mathfrak{g}$. Recall that when $\mathfrak{g}$ is a finite dimensionnal Lie algebra over $\mathbb{K} = \mathbb{R}$, $U\mathfrak{g}$ can be identified with the bialgebra of distributions supported at the neutral element of any Lie group whose associated Lie algebra is $\mathfrak{g}$ (see for example[Ser06]). Thus, to apply the method of [BGH+05]and use an analogue of formula (0.0.3) to define G_*^B, we need a contracting homotopy h_K which may be obtained by dualizing and localizing at the neutral element the geometric interpretation of the Koszul resolution given by Connes.

Let's now give a brief description of each of the three chapters of this manuscript.

Chapter 1

The chapter begins with the definitions of the various chain complexes involved in 0.0.0.1, followed by a sketch of the proof given in [CE56] that the antisymmetrization map (0.0.2) is an isomorphism. The first step consists in the comparison of projective resolutions of the ground ring $\mathbb{K}$ in the category of $U\mathfrak{g}$-modules and projective resolutions of $U\mathfrak{g}$ in the category of $U\mathfrak{g}$-bimodules, and is given by the first point of theorem 1.1.2.2 which is itsel a consequence of a general principle called "change of ring" by the authors. Then, one has to see that antisymmetrisation map $F_* : C_*(\mathfrak{g}, M^{ad}) \to CH_*(U\mathfrak{g}; M)$ given by (0.0.1) indeed comes from a morphism of resolutions $F^K : CK(U\mathfrak{g}) \to B_*(U\mathfrak{g})$, from the Koszul resolution $CK_*(U\mathfrak{g})$ of $U\mathfrak{g}$ obtained by tensoring the Chevalley-Eilenberg resolution of $\mathbb{K}$ by $U\mathfrak{g}^e := U\mathfrak{g} \otimes U\mathfrak{g}^{op}$ over $U\mathfrak{g}$, to the usual bar resolution of $U\mathfrak{g}$, denoted by $B_*(U\mathfrak{g})$. Finally, the fundamental lemma of the calculus of derived functors 1.2.1.1 ensures not only that the antisymmetrisation map F_* is a quasi-isomorphism, but also that any morphism of resolutions $G_*^B : B_*(U\mathfrak{g}) \to CK_*(U\mathfrak{g})$ will induce a quasi-inverse of F_*. In the last section, we explain how to apply the method of [BGH+05] described above to build such a G_*^B from a contraction h of the Koszul resolution. Note that the same method has been used by Suslin and Wodzicki in

Chapter 2

This chapter is devoted to the construction of a contraction h of the Koszul resolution. First, we produce a geometric interpretation of $C_*(\mathfrak{g})$, the Chevalley-Eilenberg *resolution* of the ground field $\mathbb{K} = \mathbb{R}$ seen as a trivial $\mathfrak{g}$ module, when $\mathfrak{g}$ is a finite dimensionnal Lie algebra. $C_*(\mathfrak{g})$ appears as a subcomplex of the complex of currents supported at the neutral element of the connected and simply connected Lie group G associated to $\mathfrak{g}$ by Lie's third theorem. Then, we prove that the dual $s^\vee$ of the contraction given by the Poincaré lemma existing on the De Rham complex of germs of differential forms at the neutral element of G restricts to a contraction $s : C_*(\mathfrak{g}) \to C_{*+1}(\mathfrak{g})$ of the Chevalley-Eilenberg resolution. These to facts correspond respectively to propositions 2.1.1.8 and 2.1.2.13. Moreover, proposition 2.1.2.13 gives an explicit formula (2.1.5) for s, involving the canonical contraction ϕ_t of $U\mathfrak{g}$ defined thanks to the Hopf algebra structure of $U\mathfrak{g}$. Section 2.2 is divided in two subsections : the first explains how to transfer the contracting homotopy s to a contraction $h : CK_*(U\mathfrak{g}) \to CK_{*+1}(U\mathfrak{g})$ of the Koszul resolution $CK_*(U\mathfrak{g})$, and the second shows that formula (2.2.3) still makes sense for an arbitrary dimensionnal Lie algebra, providing an explicit contraction of $C!K_*(U\mathfrak{g})$ without any assumption on dimension.

Chapter 3

This last chapter is divided in two distinct but linked sections. In the first one, we give explicit computations, in degree 1 and 2, of the morphism of resolutions $G_*^B : B_*(U\mathfrak{g}) \to CK_*(U\mathfrak{g})$ deduced from the contraction h obtained in chapter 2 by applying the strategy developped in 1.2.3, i.e. formulae (1.2.5) and (0.0.3).

The second section begins with the study of a commutative diagram of the form

$$\begin{array}{ccccc} & & C^*_{loc}(G;\mathbb{R}) & & \\ & \overset{I}{\nearrow} & \downarrow{\scriptstyle T'} & \overset{T}{\searrow} & \\ C_*(\mathfrak{g};\mathbb{R}) & \xrightarrow[G^*]{} & CH_*(U\mathfrak{g};\mathbb{R}) & \xrightarrow[F^*]{} & C_*(\mathfrak{g};\mathbb{R}) \end{array} \tag{0.0.4}$$

where, again, G is a Lie group with tangent space at its neutral element e equal to $\mathfrak{g}$, $C_{loc}(G;\mathbb{R})$ is the complex of group cochains on G which are smooth in a neighbourhood of e, and $T : C^*_{loc}(G;\mathbb{R}) \to C_*(\mathfrak{g};\mathbb{R})$ (resp. $I : C_*(\mathfrak{g};\mathbb{R}) \to C^*_{loc}(G;\mathbb{R})$) is the morphism of cochain complexes that can be seen as a "derivation" (resp. integration) transformation of locally smooth global group cochains (resp. Lie algebra cochains of $\mathfrak{g}$) in Lie algebra cochains (resp. locally smooth group cochains). Theses maps have been studied in [Nee04] in the framework of infinite dimensionnal Lie groups, for $n = 2$. The morphism $G^* : C^*(\mathfrak{g};\mathbb{R}) \to CH_*(U\mathfrak{g};\mathbb{R})$ induced by G_*^B, and the antisymmetrisation map F^* then appear, via T', as algebraic analogues of geometric maps, respectively I and T. This allows us, thanks to the cubical integration of Lie cochains formula given in 3.2.1.5, to guess an explicit and closed formula for the morphisms G_*^B and G^* : that's the content of proposition 3.2.1.8. The end of the chapter is an attempt to mimick the construction of diagram (0.0.4) in entirely algebraic terms. The Lie group G is here replaced by $\hat{G}$, the Malcev group consisting of grouplike elements of the I-adic completion of $U\mathfrak{g}$ with respect to the augmentation ideal I. The continuity and smoothness conditions defined in 3.2.2, although very restricive, are automatically satisfied when the Lie algebra $\mathfrak{g}$ is nilpotent.

The author wants to thank M. Bordemann for his numerous enlightening advices, especially concerning the use of the eulerian idempotent in the construction of the quasi-inverse G^*.

Chapter 1

Precise statement of the problem and strategy of resolution

1.1 The antisymmetrisation map of Cartan-Eilenberg

This section start with a quick recollection on Lie algebra and associative algebra (co)homology theories, and then moves to a sketch of the proof of the bijectivity of the antisymmetrization map following [CE56].

1.1.1 Hochschild and Chevalley-Eilenberg's complexes

No details nor proofs are given here : the reader may found more information on the suject in the litterature, for instance in [Wei95], [Lod98], [CE56] or [Lan75]. The cohomological versions of the differents complex appearing here are defined in appendix B.

Hochschild homology

In all this subsection, A is an $\mathbb{K}$-projective associative and unital algebra over some commutative ring $\mathbb{K}$, and M is an A-bimodule. The notation A^{op} stands for the opposite algebra of A and $A^e := A \otimes A^{op}$ denotes its envelopping algebra so that the category of A bimodules is identified with the one of left A^e-modules.

Definition 1.1.1.1. *The* ***homological Hochschild complex*** *of A with coefficients in M, is the graded $\mathbb{K}$-module $CH_*(A; M)$ defined by*

$$CH_n(A; M) := M \otimes A^{\otimes n}$$

for all integer n, with differential d^H of degree -1, defined on n chains by

$$d^H(m \otimes a_1 \otimes \cdots a_n) := ma_1 \otimes a_2 \otimes \cdots a_n + \sum_{i=1}^{n-1} (-1)^i m \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n + (-1)^n a_n m \otimes a_1 \otimes \cdots \otimes a_{n-1}$$

for all m in M and a_1, ..., a_n in A. The Hochschild homology $HH_*(A;M)$ of A with coefficients in M is the homolgy of this complex i.e.

$$HH_*(A;M) := H_*(CH_*(A;M), d^H)$$

In the same fashion (see Appendix B or [Lod98]), on can define the cohomological Hochschild complex of A with values in M, denoted by $CH^*(A;M)$, with differential d_H, whose homology $HH^*(A;M)$ is called Hochschild cohomology of A with values in M.

Hochschild homology is a derived functor and to see why we need to introduce a particular resolution of the A bimodule A :

Definition 1.1.1.2. *The* ***bar-resolution of*** *A is the complex of A-bimodules $B_*(A)$ defined by*

$$B_n(A) := A^{\otimes n+2}$$

for all integer n, with differential d^B defined by

$$d^B(a_0 \otimes a_1 \otimes \cdots \otimes a_n \otimes a_{n+1}) := \sum_{i=0}^{n} (-1)^i a_0 \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes a_{i+1} \otimes \cdots \otimes a_n$$

for all a_0, ..., a_{n+1} in A. The A-bimodule structure on $B_n(A) = A^{\otimes n}$ is given by

$$a(a_0 \otimes a_1 \otimes \cdots \otimes a_n \otimes a_{n+1})b := (aa_0) \otimes a_1 \otimes \cdots \otimes a_n \otimes (a_{n+1}b)$$

for all a, b, a_0, ..., a_{n+1} in A.

Proposition 1.1.1.3. *The bar-resolution of A est is a projective resolution (see [CE56] for a definition of projective resolution) of the left A^e-module A and the obvious isomorphism of graded modules*

$$M \otimes_{A^e} B_*(A) \cong CH_*(A;M)$$

sends $Id_M \otimes d^B$ on d^H. Thus

$$HH_*(A;M) = \mathrm{Tor}_*^{A^e}(A;M)$$

and similarly

$$HH^*(A;M) = \mathrm{Ext}^*_{A^e}(A;M)$$

Homology of Lie algebras

In this subsection, $\mathfrak{g}$ is a Lie algebra over some commutative ring $\mathbb{K}$ and N (resp. N') is left (resp. right) $\mathfrak{g}$-module.

Definition 1.1.1.4. ***The universal enveloping algebra of*** $\mathfrak{g}$ *(often abridged in* ***enveloping algebra*** *of $\mathfrak{g}$ in the following), denoted by $U\mathfrak{g}$, is the quotient of the tensor algebra*

$T\mathfrak{g} := \oplus_{n\geq 0}\mathfrak{g}^{\otimes n}$ by the idéal I generated by elements of the form $g\otimes g' - g'\otimes g - [g,g']$ when g and g' run over $\mathfrak{g}$, i.e.

$$U\mathfrak{g} := T\mathfrak{g}/_{I} \quad , \quad I := < g\otimes g' - g'\otimes g - [g,g'] \,/\, g,g' \in \mathfrak{g} >$$

The product on $U\mathfrak{g}$, denoted by μ, is induced by the concatenation product of $T\mathfrak{g}$, and we will write $xy := \mu(x\otimes y)$ the product of two elements x and y of $U\mathfrak{g}$. L'unité $\eta : \mathbb{K} \to U\mathfrak{g}$ provient de l'inclusion canonique de $g^{\otimes 0} = \mathbb{K}$ dans $T\mathfrak{g}$.

***The augmentation** of $U\mathfrak{g}$, is the map $\epsilon : U\mathfrak{g} \to \mathbb{K}$ induced by the projection of $T\mathfrak{g}$ on its term $\mathbb{K}$ along $\oplus_{n\geq 1} g^{\otimes n}$. This algebra morphism makes $\mathbb{K}$ into a $U\mathfrak{g}$-module.*

The category of left $\mathfrak{g}$-modules is canonically isomorphic to the one of left $U\mathfrak{g}$-modules.

Definition 1.1.1.5. *The **homological Chevalley-Eilenberg complex** of $\mathfrak{g}$ with coefficients in N is the graded module $C_*(\mathfrak{g};N)$ defined by*

$$C_*(\mathfrak{g};N) := N \otimes \Lambda^*\mathfrak{g}$$

for all integer n, with differential d^{CE} defined on n-chains by

$$\begin{aligned} d^{CE}(m\otimes g_1\wedge g_2\wedge\cdots\wedge g_n) := &\sum_{i=1}^{n}(-1)^{i+1} m\cdot g_i \otimes g_1\wedge g_2\wedge\cdots\wedge\hat{g}_i\wedge\cdots\wedge g_n \\ &+ \sum_{1\leq i<j\leq n}(-1)^{j+1} m\otimes g_1\wedge\cdots\wedge[g_i,g_j]\wedge\cdots\wedge\hat{g}_j\cdots\wedge g_n\,, \end{aligned}$$

*for all m in N and for all g_1, ... g_n in $\mathfrak{g}$. Here, $\Lambda^*\mathfrak{g}$ stands for the graded commutative algebra generated by $\mathfrak{g}$ seen as a graded module concentrated in degree 1, and notation $\hat{g}_i$ means "ommit g_i". **The homology of $\mathfrak{g}$ with coefficients in** N, denoted by $H_*(\mathfrak{g};N)$, is the graded $\mathbb{K}$-module*

$$H_*(\mathfrak{g};N) := H_*(C_*(\mathfrak{g};N), d^{CE})$$

The cohomological version will be written $H^*(\mathfrak{g};N')$ (See appendix B. or [Wei95] for more details).

When the module M is chosen to be $U\mathfrak{g}$ itself for the action given by multiplication, the Chevalley-Eilenberg $C_*(\mathfrak{g};U\mathfrak{g})$ will be denoted by $C_*(\mathfrak{g})$ and refered to as the **Chevalley-Eilenberg resolution** for the following reason :

Proposition 1.1.1.6. *The complex $C_*(\mathfrak{g}) := C_*(\mathfrak{g};U\mathfrak{g})$ is acyclic and its degree 0 homology is equal to $\mathbb{K}$. Thus, it is a resolution of $\mathbb{K}$ in the category of left $U\mathfrak{g}$-modules.*

Corollary 1.1.1.7. *If $\mathfrak{g}$ is projective[1] over $\mathbb{K}$, then*

$$H_*(\mathfrak{g};N) = \mathrm{Tor}_*^{U\mathfrak{g}}(\mathbb{K};N)$$

et

$$H^*(\mathfrak{g};N') = \mathrm{Ext}^*_{U\mathfrak{g}}(\mathbb{K};N')$$

[1] $\mathbb{K}$-flat is enough for homology.

1.1.2 The antisymmetrisation map is a quasi-isomorphism

In this subsection, the Lie algebra $\mathfrak{g}$ is free over $\mathbb{K}$ and M is a $U\mathfrak{g}$-bimodule.

Definition 1.1.2.1. *The right $\mathfrak{g}$-module* ***adjoint*** *to M is the right $\mathfrak{g}$-module M^{ad} whose underlying $\mathbb{K}$-module is M, on which $\mathfrak{g}$ acts on the right by*

$$m \cdot g := mg - gm$$

for all M^{ad} and g in $\mathfrak{g}$.

We now have two homological invariants associated to the pair $(\mathfrak{g}, M)$: $H_*(\mathfrak{g}; M^{ad})$ and $HH_*(U\mathfrak{g}; M)$. It turns out that these two graded modules are the same, and this fact is part of the content of theorem 5.1, chapter XIII, of [CE56]. Let us recall this theorem in a special form, adapted to our context:

Theorem 1.1.2.2 ([CE56])**.** *Let $\mathfrak{g}$ be a $\mathbb{K}$ Lie algebra, supposed to be free as a $\mathbb{K}$-module.*

1. *If X_* is a projective resolution of $\mathbb{K}$ as a left $U\mathfrak{g}$-module, then $U\mathfrak{g}^e \otimes_{U\mathfrak{g}} X_*$ is a projective resolution of $U\mathfrak{g}$ as a left $U\mathfrak{g}^e$-module.*

2. *As a consequence, the antisymmetrization map $F_* : C_*(\mathfrak{g}; M^{ad}) \to CH_*(U\mathfrak{g}; M)$ defined in degree n by*

$$F_n(m \otimes g_1 \wedge g_2 \wedge \cdots \wedge g_n) := \sum_{\sigma \in \Sigma_n} \mathrm{sgn}(\sigma) m \otimes g_{\sigma(1)} \otimes g_{\sigma(2)} \otimes \cdots \otimes g_{\sigma(n)}$$

   for all m in M and g_1, ..., g_n in $\mathfrak{g}$, is a quasi-isomorphism of chain complexes. .

The proof of this statement given in [CE56] relies on a general principle named "change of rings" by the authors, which allows the comparison of the derived functors $\mathrm{Tor}^A_*(-; Q_A)$ (resp. $\mathrm{Ext}^*_A(Q_A; -)$) and $\mathrm{Tor}^B_*(-; Q_B)$ (resp. $\mathrm{Ext}^*_B(Q_B; -)$), where A and B are two rings linked by a commutative diagram of the form

$$\begin{array}{ccc} A & \xrightarrow{\epsilon_A} & Q_A \\ \downarrow E & & \downarrow \\ B & \xrightarrow{\epsilon_B} & Q_B \end{array} \tag{1.1.1}$$

Here, Q_A (resp. Q_B) is a left A-module (resp. B-module) and ϵ_A (resp. ϵ_B) is a morphism of left A-modules (resp. B-modules). In particular, B is endowed with an A-module structure via the map E which is supposed to be a ring isomorphism. The next section recalls some key facts appearing in the proof of 1.1.2.2 and explains why they imply the existence of quasi-inverse to the antisymmetrization map, and how they can be used to elaborate a strategy for the construction of an explicit one.

1.2 Going backward using a contraction

This section begins with a short overview of things that have been developed in great details by S. Eilenberg and H. Cartan in [CE56]. Here, $\mathfrak{g}$ is supposed to be free as a $\mathbb{K}$-module.

1.2.1 Derived functors and projective resolutions

In this subsection, A is a ring, $\mathcal{M}$ is the category of left A-modules, and $\mathcal{A}$ denotes the category of abelian groups. The computation of derived functors associated to additive functors between categories of modules (and more generally between abelian categories) relies on the following well known fondamental lemma:

Lemma 1.2.1.1 (Lemme fondamental.)**.** *Let P and Q be two left A-modules, $X_* \to P$ be a complex of projective left A-modules over P and $Y_* \to Q$ be a (non necessarily projective) resolution of Q. Then, any A-linear map $f : P \to Q$ can be lifted to a morphism of complexes of A-modules $F_* : X_* \to Y_*$ over $f : P \to Q$. Moreover, F_* is unique up to homotopy.*

To any additive and right exact functor $T : \mathcal{M} \to \mathcal{A}$ is associated its n-th left derived functor L^nT which, evaluated on an A-module Q is defined to be ne n-th homology group of the complex of abelian groups $(T(X_*), T(d^X))$ image by T of any projective resolution (X_*, d^X) of Q. The fact that any A-module admits a projective resolution is established in [CE56]. The fact that $L^nT(Q)$ is well defined is a consequence of lemma 1.2.1.1: if $(X_*, d^X) \to Q$ and $(Y_*, d^Y) \to Q$ are two projective resolutions of Q, the fundamental lemma asserts the existence of two lifts $F_*^X : X_* \to Y_*$ and $G_*^Y : Y_* \to X_*$ of the identity map of Q whose composition in both possible orders as to be homotopic to the identity. This implies that $T(F_*^X)$ and $T(F_*^Y)$ are quasi-isomorphisms, each one being a quasi-inverse of the other and shows that the value of L^nT on Q doesn't depend on the choice of a projective resolution of Q.

Definition 1.2.1.2. *When the functor T is of the form $Q \mapsto P \otimes_A Q$, for a given right A-module P, le n-th derived functor of T evaluated on an A-module Q is denoted by*

$$\mathrm{Tor}_n^A(P, Q)$$

Similarly, if P is a left A-module, the right derived functors of $Q \mapsto \mathrm{Hom}_A(P, Q)$ evaluated at Q are commonly written $\mathrm{Ext}_A^*(P, Q)$.

1.2.2 Application to the Hochschild/Chevalley-Eilenberg case

The goal of this subsection is to explain why the first part of theorem 1.1.2.2 implies the second one. Let's first precise the right $U\mathfrak{g}$-module structure on $U\mathfrak{g}^e$ involved in the statement of 1.1.2.2.

The Hopf algebra structure on $U\mathfrak{g}$

For a complete exposition of the facts developped below, the reader may consult [Kas95]. Appendix A. is a quick recollection of the main properties of Hopf algebras that are used in this document.

We have already defined the augmentation morphism $\epsilon : U\mathfrak{g} \to \mathbb{K}$ and the product $\mu : U\mathfrak{g}^{\otimes 2} \to U\mathfrak{g}$ in section 1.1.1.

Definition 1.2.2.1. *The coproduct of* $U\mathfrak{g}$ *is the unique morphism of algebras* $\Delta : U\mathfrak{g} \to U\mathfrak{g} \otimes U\mathfrak{g}$ *satisfying*

$$\Delta(g) := g \otimes 1 + 1 \otimes g$$

for all g *in* $\mathfrak{g} \subset U\mathfrak{g}$. *The antipode* $S : U\mathfrak{g} \to U\mathfrak{g}^{op}$ *is the unique algebra isomorphism satisfying*

$$S(g) = -g$$

for all g *in* $\mathfrak{g}$. *The augmenation of the algebra* $U\mathfrak{g}^e$ *is the morphism of left* $U\mathfrak{g}^e$*-modules* $\rho : U\mathfrak{g}^e \to U\mathfrak{g}$ *defined by* $\rho(x \otimes y) := \mu(x \otimes y)$ *for all* x *in* $U\mathfrak{g}$ *and* y *in* $U\mathfrak{g}^{op}$.

Notation 1.2.2.2. *In the following, iterated coproducts are written using* ***Sweedler's notation****: the* k*-times iterated coproduct of an element* x *of* $U\mathfrak{g}$ *reads*

$$\sum_{(x)} x^{(1)} \otimes x^{(2)} \otimes \cdots \otimes x^{(k+1)} := (\Delta \otimes \mathrm{Id}^{\otimes k-1}) \circ (\Delta \otimes \mathrm{Id}^{\otimes k-3}) \circ \cdots \circ (\Delta \otimes \mathrm{Id}) \circ \Delta(x) \in U\mathfrak{g}^{\otimes k+1}$$

In particular,

$$\Delta(x) = \sum_{(x)} x^{(1)} \otimes x^{(2)}$$

Proposition 1.2.2.3. *The six-tuple* $(U\mathfrak{g}, \mu, \eta, \Delta, \epsilon, S)$ *is a cocommutative connected Hopf algebra*[2].

We are now ready to introduce the change of rings morphsim:

Definition 1.2.2.4. *The morphism of algebras* $E : U\mathfrak{g} \to U\mathfrak{g}^e$ *is defined by*

$$E := (Id \otimes S) \circ \Delta$$

The change of resolutions

The commutative square

$$\begin{array}{ccc} U\mathfrak{g} & \xrightarrow{\epsilon} & \mathbb{K} \\ \downarrow E & & \downarrow \eta \\ U\mathfrak{g}^e & \xrightarrow{\rho} & U\mathfrak{g} \end{array} \qquad (1.2.1)$$

satisfies technical conditions E.1) et E.2) of the "change of ring" theorem 6.1 of Cartan-Eilenberg and this implies point 1) of theorem 1.1.2.2. In particular, the complex of $U\mathfrak{g}$-bimodules $(U\mathfrak{g}^e \otimes_{U\mathfrak{g}} C_*(\mathfrak{g}), Id \otimes d^{CE})$ is a projective resolution of $U\mathfrak{g}$.

[2] See [Qui69] for a definition of connected.

Definition-Proposition 1.2.2.5. *The projective resolution* $(U\mathfrak{g}^e \otimes_{U\mathfrak{g}} C_*(\mathfrak{g}), Id \otimes d^{CE})$ *is isomorphic to the complex of* $U\mathfrak{g}$*-bimodules* $(CK_*(U\mathfrak{g}), d^K)$ *defined in degree* n *by*

$$CK_n(U\mathfrak{g}) := U\mathfrak{g} \otimes \Lambda^n \mathfrak{g} \otimes U\mathfrak{g}$$

and

$$\begin{aligned} d^K(x \otimes g_1 \wedge \cdots \wedge g_n \otimes y) := & \sum_{i=1}^{n} (-1)^{i+1} (x g_i \otimes g_1 \wedge \cdots \wedge \hat{g}_i \wedge \cdots \wedge g_n \otimes y - x \otimes g_1 \wedge \cdots \wedge \hat{g}_i \wedge \cdots \wedge g_n \otimes g_i y) \\ & + \sum_{1 \le i < j \le n} (-1)^{j+1} x \otimes g_1 \wedge \cdots \wedge [g_i, g_j] \wedge g_{i+1} \wedge \cdots \wedge \hat{g}_j \wedge \cdots \wedge g_n \otimes y \end{aligned}$$

for all x*,* y *in* $U\mathfrak{g}$ *and* g_1*, ...,* g_n *in* $\mathfrak{g}$*. This acyclic complex is called the* ***Koszul resolution*** *of* $U\mathfrak{g}$*.*

Thus, we have a "smaller" complex than the usual one $CH_*(U\mathfrak{g}; M)$ to compute the Hochschild homology of $U\mathfrak{g}$ with coefficients in M, which is obtained by tensoring the Koszul resolution by M over $U\mathfrak{g}^e$. One easily checks that this smaller complex is itsel isomorphic to $C_*(\mathfrak{g}; M^{ad})$, the Chevalley-Eilenberg complex of $\mathfrak{g}$ with coefficients in the adjoint module M^{ad}. Moreover, we have seen that lemma 1.2.1.1 implies the existence of morphisms of complexes $F_*^K : CK_*(U\mathfrak{g}) \to B_*(U\mathfrak{g})$ et $G_*^B : B_*(U\mathfrak{g}) \to CK_*(U\mathfrak{g})$ lifting the identity map of $U\mathfrak{g}$, and inducing quasi-isomorphisms $F_* : C_*(\mathfrak{g}; M^{ad}) \to CH_*(U\mathfrak{g}; M)$ and $G_* : CH_*(U\mathfrak{g}; M) \to C_*(\mathfrak{g}; M^{ad})$ inverse of each other at the homology level. In fact, an explicit example of such a F_*^K is given by the antisymmetrization map :

Proposition 1.2.2.6. *The map* $F_*^K : CK_*(U\mathfrak{g}) \to B_*(U\mathfrak{g})$ *defined on* n*-chains by*

$$F_*^K(x \otimes g_1 \wedge g_2 \wedge \cdots \wedge g_n \otimes y) := \sum_{\sigma \in \Sigma_n} \mathrm{sgn}(\sigma) x \otimes g_{\sigma(1)} \otimes g_{\sigma(2)} \otimes \cdots \otimes g_{\sigma(n)} \otimes y$$

for all x*,* y *in* $U\mathfrak{g}$ *and* g_1*, ...,* g_n *in* $\mathfrak{g}$ *is a morphism of complexes of* $U\mathfrak{g}$*-bimodules over* $U\mathfrak{g}$ *lifting the identity map of* $U\mathfrak{g} \to U\mathfrak{g}$*. In addition, the quasi-isomorphism* $F_* : C_*(\mathfrak{g}; M^{ad}) \to CH_*(U\mathfrak{g}; M)$ *induced by* F_*^K *is exactly the antisymmetrization map of theorem 1.1.2.2.*

The fact that the antisymmetrization map F_* of theorem 1.1.2.2 comes from a morphism of resolutions as two immediate consequences, according to lemma 1.2.1.1 : F_* is necessarily a quasi-isomorphism, and any explicit lift $G_*^B : B_*(U\mathfrak{g}) \to CK_*(U\mathfrak{g})$ of the identity map of $U\mathfrak{g}$ will induce one of its explicit quasi-inverses. The following subsection explains how to build such a lift assuming that an explicit contracting homotopy of the Koszul resolution is known.

1.2.3 How to define a quasi-inverse ?

The Lie algebra $\mathfrak{g}$ is still assumed to be free over $\mathbb{K}$ so that the previous considerations apply. Suppose that a contraction $h : CK_*(U\mathfrak{g}) \to CK_{*+1}(U\mathfrak{g})$ is given, that is a degree +1 graded map satisfying

$$hd^K + d^K h = Id_{CK_*(U\mathfrak{g})} \tag{1.2.2}$$

Define isomorphisms of chain complexes θ and θ' by

$$\begin{array}{rcl} \theta : C_*(\mathfrak{g}; M^{ad}) & \overset{\cong}{\to} & M \otimes_{U\mathfrak{g}^e} CK_*(U\mathfrak{g}) \\ m \otimes g_1 \wedge g_2 \wedge \cdots \wedge g_n & \mapsto & m \otimes 1 \otimes g_1 \wedge \cdots \wedge g_n \otimes 1 \end{array} \tag{1.2.3}$$

and

$$\begin{array}{rcl} \theta' : CH_*(U\mathfrak{g}; M) & \to & M \otimes_{U\mathfrak{g}^e} B_*(U\mathfrak{g}) \\ m \otimes x_1 \otimes \cdots \otimes x_n & \mapsto & m \otimes 1 \otimes x_1 \otimes \cdots \otimes x_n \otimes 1 \end{array} \tag{1.2.4}$$

The following proposition shows how to build morphsim of resolutions $G_*^B : B_*(U\mathfrak{g}) \to CK_*(U\mathfrak{g})$:

Proposition 1.2.3.1. *The graded liner map $G_*^B : B_*(U\mathfrak{g}) \to CK_*(U\mathfrak{g})$ defined by induction on the homological degree n by*

$$G_0^B := \mathrm{Id} : U\mathfrak{g} \otimes U\mathfrak{g} \to U\mathfrak{g} \otimes U\mathfrak{g}$$

and

$$G_n^B(x \otimes x_1 \otimes \cdots \otimes x_n \otimes y) := x \left(hG_{n-1}d^B(1 \otimes x_1 \otimes \cdots \otimes x_n \otimes 1 \right) y \quad , \quad \forall n > 1 \tag{1.2.5}$$

for all x, y, x_1, ..., x_n in $U\mathfrak{g}$, is a morphism of complexes of $U\mathfrak{g}$-bimodules inducing, when tensored by the identity map of M over $U\mathfrak{g}^e$ and via the identifications θ and θ' given by (1.2.3) et (1.2.4), a quasi-isomorphism

$$G_* : CH_*(U\mathfrak{g}; M) \to C_*(\mathfrak{g}; M^{ad})$$

inverse of F_* in homology. Similarly, applying the functor $\mathrm{Hom}_{U\mathfrak{g}^e}(-, M)$ to G_*^B brings a quasi-inverse to the cohomological antisymmetrization map F^* in the cohomological framework:

$$G^* : C^*(\mathfrak{g}; M^{ad}) \to CH^*(U\mathfrak{g}; M)$$

It is now clear that all we need to define an explicit quasi-inverse to the antisymmetrization map is an explicit contracting homotopy h of the Koszul resolution of $U\mathfrak{g}$. The next chapter is devoted to the construction of such an h.

Chapter 2

A contracting homotopy for the Koszul complex

In this chapter, we restrict to the case $\mathbb{K} = \mathbb{R}$ for which the existence of a quasi-inverse G_* of the antisymmetrization map F_* is garanted by the fact that $\mathbb{R}$ is a field. However, the second part of question 0.0.0.1 is still non trivial.

2.1 Contraction of the Chevalley-Eilenberg resolution

In the whole section, $\mathfrak{g}$ is a finite dimensionnal Lie algebra over $\mathbb{R}$ with $m := \dim \mathfrak{g}$. G is a Lie group with neutral element e, product $\mu_G : G \times G \to G$, and is tangent space T_eG is supposed equal to $\mathfrak{g}$. Such a Lie group exists by Lie's third theorem. The diagonal G is the map $\Delta_G : G \to G \times G$ that sends every element z of G to (z, z) in $G \times G$.

The contracting homotopy of the Koszul resolution h we are looking for will be built from a contracting homotopy s of the Chevalley-Eilenberg resolution $C_*(\mathfrak{g})$, and to define this last, we need to understand it geometrically.

2.1.1 Geometric interpretation of $C_*(\mathfrak{g})$

It is well known (see [Lod98], [FOT08] and [Nee04] for arbitrary coefficients) that the cohomological Chevalley-Eilenberg *complex* with trivial coefficients $C^*(\mathfrak{g}; \mathbb{R})$ is isomorphic to the De Rhamc omplex of left invariant differential forms on the Lie group$G$. To give an analogous interpretation of the Chevalley-Eilenberg resolution $C_*(\mathfrak{g})$, one may observe that $U\mathfrak{g}$ can be thought of as the continuous dual of the space of germs of functions at e on G, that is as the dual of germs 0-differential forms on G. Let's get this more precise.

Punctual distributions and differential operators

The universal enveloping algebra $U\mathfrak{g}$ is a Hopf algebra that admits at least two geometric interpretations, the first in terms of punctual distributions on G supported at the neutral element e, the second in terms of left invariant differential operators on G. Denote by $\mathcal{C}^\infty_e(G)$

the algebra of germs of functions at e on G obtained by quotienting the algebra of real valued smooth functions on G by the equivalence relation for which two function are said equivalent when their restrictions to some open set of G containing e are equal. $\mathcal{C}_e^\infty(G)$ has a unique maximal ideal which consists of all the functions vanishing at e. The diffeomorphism of G corresponding to left multiplication by some fixed element z will be written $L_z : G \to G$.

Definition 2.1.1.1. *A **punctual distribution** at e on G is a linear form D on $\mathcal{C}_e^\infty(G)$, continuous for the $\mathfrak{m}$-adic topology. This means that there exists an integer r such that D is zero on $\mathfrak{m}^r$. The set of puntctual distributions at e, temporarily denoted by U, is a bialgebra with product $\cdot$ given by the **convolution** of distributions:*

$$D \cdot D'(\bar{f}) = D \otimes D'(f \circ \mu_G) := D(z \mapsto D'(f \circ L_z))$$

*for all germ $\bar{f}$ at e, and for all distributions D and D'. The **coproduct** $\delta : U \to U \otimes U$ is defined by*

$$\delta(D)(\bar{f}) = D(f \circ \Delta_G)$$

for all germ f at (e,e) on the group $G \times G$ and for all punctual distributrion D on G. Note that in the preceeding formula, $U \otimes U$ has been identified with the algebra of punctual distributions at (e,e) on $G \times G$.

Proposition 2.1.1.2. *The bialgebras $U\mathfrak{g}$ and U are isomorphic.*

Definition 2.1.1.3. *Let g be a vector in $\mathfrak{g}$. Le **left invariant vector field generated by** g, denoted by X_g, is the vector field $X_g : G \to TG$ defined by*

$$X_g(a) := T_e L_a(g)$$

for all a in G.

Let U' be the unital subalgebra of $\operatorname{End}_{\mathbb{R}}(\mathcal{C}^\infty(G))$ (whose product is given by the composition of linear endomorphisms of $\mathcal{C}^\infty(G)$, the vector space of real valued smooth funtions on G) generated by left invariant vector fields, that is vector fields of the form X_g for some g in $\mathfrak{g}$.

*A **differential operator** is sum of functions and of compositions of vector fields. Thus U' is the algebra of left invariant differential operators on G (scalars correspond to left invariant, i.e. constant, functions on G)*

Remark 2.1.1.4. *See for example [God04] for a more precise definition of differential operators.*

Proposition 2.1.1.5. *The map $U\mathfrak{g} \to U'$ assigning to each monomial $g_1 g_2 \cdots g_n$ in $U\mathfrak{g}$ the left invariant vector field $X_{g_1} \circ X_{g_2} \circ \cdots \circ X_{g_n}$ is an isomorphism of algebras.*

The following technical lemma is a bridge betwenn the two geometric interpretations of the product of $U\mathfrak{g}$:

Lemma 2.1.1.6. *Let x be an element of $U\mathfrak{g}$, g an element of $\mathfrak{g}$ and f a germ of function at e on G. Then, by propsition 2.1.1.2:*

$$x(X_g f) = xg(f)$$

Inclusion dans le complexe des courants

Denote by $\Omega_e^n(G)$ the vector space of germs of n-differential forms at e on G.

Definition 2.1.1.7. *A n-**current supported at** e is a linear form on $\Omega_e^n(G)$. The **complex of currents (supported at e) on** G is the graded vector space $\Lambda_e^* G := \Omega_e^*(G)^\vee := \mathrm{Hom}_{\mathbb{R}}(\Omega_e^*(G), \mathbb{R})$ endowed with the differential $d_{DR}^\vee$, dual of the usual De Rham differential on forms.*

The Chevalley-Eilenberg resolution $C_*(\mathfrak{g})$ can be seen as a subomplex of $\Lambda_e^* G$ in the following manner: Let $x \otimes g_1 \wedge \cdots \wedge g_n$ be an elementary tensor in $U\mathfrak{g} \otimes \Lambda^n \mathfrak{g}$. Define

$$R(x \otimes g_1 \wedge \cdots \wedge g_n)(\bar{\omega}) := x\left(z \mapsto \omega_z(X_{g_1}(z), \cdots X_{g_n}(z))\right)$$

for all germ of n-form $\bar{\omega}$.

Proposition 2.1.1.8. *The map*

$$\begin{array}{rcl} R : C_*(\mathfrak{g}) & \to & \Lambda_*(G) \\ x \otimes g_1 \wedge \cdots \wedge g_n & \mapsto & \big(\bar{\omega} \mapsto R(x \otimes g_1 \wedge \cdots \wedge g_n)(w)\big) \end{array}$$

is an injective morphism of chain complexes.

Remark 2.1.1.9. [D. Calaque]

Recall that $\mathfrak{m}$ is the maximal ideal of $\mathcal{C}_e^\infty(G)$ consisting of germs of functions vanishing at e.

*Since $\Omega_e^*(G)$ is a complex of left $\mathcal{C}_e^\infty(G)$-modules, the $\mathfrak{m}$-adic topology on $\mathcal{C}_e^\infty(G)$ induces a $\mathfrak{m}$-adic topology on each $\Omega_e^*(G)$ by declaring that the family of submodules $(\mathfrak{m}^n \Omega_e^*(G))_{n \geq 0}$ forms a basis of neighbourhoods of 0. The map R identifies the Chevalley-Eilenberg resolution with the subcomplex of **continuous** punctual currents on G that is the subcomplex of linear forms on $\Omega_e^*(G)$ that are continuous for the $\mathfrak{m}$-adic topology.*

2.1.2 Transfer of the Poincaré contraction

In this subsection, we show that the contracting homotopy of $\Omega_e^*(G)$ given by the Poincaré lemma induces, by dualization and restriction, a contracting homotopy of the Chevalley-Eilenberg resolution $C_*(\mathfrak{g})$ that can be rewritten using only the Hopf algebra structure on $U\mathfrak{g}$.

Exactness of $\Lambda_e^* G$

The complex of germs of differential forms at a point p of a smooth manifold M is always exact: this is the so called "Poincaré lemma".

Lemma 2.1.2.1. *Let M be a smooth manifold and U be an open subset of M. Suppose that U can be contracted to on of its points p, which means that there exists a smooth homotopy (fixing p) $\varphi : [0, 1] \times U \to U$ between the identity map on U and the constant map which sends every* *element of U to p. For all n-differential form ω defined on an open subset V of G containing U, define the $(n-1)$-form $s\omega$ on U by*

$$s\omega = \int_0^1 dt\, \iota_{\frac{\partial}{\partial t}} \varphi^* \omega$$

where ι_X stands for the inner product with a vector field X. Then, the degree -1 map

$$\begin{array}{rcl} s : \Omega_p^*(M) & \to & \Omega_p^*(M) \\ \bar{\omega} & \mapsto & \overline{s\omega} \end{array} \tag{2.1.1}$$

is a contracting homotopy of the complex $\Omega_p^*(M)$, i.e.

$$sd_{DR} + d_{DR}s = Id_{\Omega_p^*(M)}$$

Since any point of a smooth manifold admits a contractible neighbourhood we have the following corollary:

Corollary 2.1.2.2. *For all point p of a smooth manifold M, the complex $\Omega_p^*(M)$ is contractible. A contracting homotopy is given by the map s of lemma 2.1.2.1. Dually, the homotopy s induces a contracting homotopy $s^\vee$ of the complex of punctual currents $\Lambda_e^* G$.*

When $M = G$, a special contraction is given by the exponential map. Let V be an open subset of $\mathfrak{g}$ containing 0 on which the exponential map $\exp : V \to U$ is a diffeomorphism on its image $U := \exp(V)$. Denote by $\ln : U \to V$ its inverse.

Definition 2.1.2.3. *The* ***canonical contraction*** *associated to G is the smooth map $\varphi : [0,1] \times U \to U$ defined by*

$$\varphi(t, a) := \exp(t \ln a)$$

for all t in $[0,1]$ and a in U.

An algebraic contracting homotopy for $C_*(\mathfrak{g})$

To be able to get an algebraic expression of the contracting homotopy induced on $C_*(\mathfrak{g})$ by $s^\vee$, we need to extend the dictionary between geometry and algebra outlined in the previous section.

Definition 2.1.2.4. *Let $f : V \to W$ be a smooth map between two open subsets V and W of two smooth manifolds M and N. If p is a given point in V, the linear map $f_* : \mathcal{C}_p(M)^\vee \to \mathcal{C}_{f(p)}(N)^\vee$* ***induced by*** *$f$ at p is defined by*

$$f_* D(\bar{h}) := D(\overline{h \circ f})$$

for all linear form D on $\mathcal{C}_p^\infty(M)$ and for all germ of function $\bar{h}$ in $\mathcal{C}_{f(p)}(N)$. In the case when $M = N = G$ and $p = e = f(e)$, f_* restricts to an endomorphism of the coalgebra $U\mathfrak{g}$, still denoted by f_*.

Another feature we'll need in the following is the notion of **convolution** of linear endomorphisms of $U\mathfrak{g}$:

Definition 2.1.2.5. *Let f and g be two $\mathbb{R}$-linear endomorphisms of $U\mathfrak{g}$. Their **convolution prodct** is the linear endomorphism $f \star g$ defined by*

$$f \star g := \mu(f \otimes g)\Delta$$

*Clearly, $(\mathrm{End}_{\mathbb{R}}(U\mathfrak{g}), +, \star)$ is an associative algebra with unit given by $\eta\epsilon$. The **canonical projection** of $U\mathfrak{g}$ is the linear map $\mathrm{pr} : U\mathfrak{g} \to U\mathfrak{g}$ defined by*

$$\mathrm{pr} := \sum_{k\geq 1} \frac{(-1)^{k+1}}{k}(Id - \eta\epsilon)^{\star k}$$

Proposition 2.1.2.6. *The canonical projection takes its values in $\mathfrak{g} \subset U\mathfrak{g}$ and satisfies the following identities*

$$\mathrm{pr}^{\star p} \circ \mathrm{pr}^{\star q} = \begin{cases} \mathrm{pr}^{\star p} & si\ p = q \\ 0 & sinon \end{cases} \tag{2.1.2}$$

for all integers p and q, and

$$\mathrm{pr}_{|\mathfrak{g}} = Id_{\mathfrak{g}}$$

Remark 2.1.2.7. *When p runs over all integers, the $\mathrm{pr}^{\star p}$'s form a family of orthogonal idempotents named **eulerian idempotents** in [Lod98] and [Reu93].*

Proposition 2.1.2.8. *Let t be a real number in $[-1, 1]$*

1. *The endomorphism of coalgebra $(\varphi_t)_* : U\mathfrak{g} \to U\mathfrak{g}$ induced by the differential map*

$$\begin{array}{rcl} \varphi_t : U & \to & U \\ a & \mapsto & \varphi(t, a) \end{array}$$

where φ is the canonical contraction defined in 2.1.2.3, is equal to the endomorphism of coalgebra ϕ_t defined by

$$\phi_t = \sum_{n\geq 0} \frac{t^n}{n!}\mathrm{pr}^{\star n}$$

2. *For all s in $[-1, 1]$,*

$$\phi_s \circ \phi_t = \phi_{st}$$

In particular, the antipode $S = \phi_{-1}$ satisfies

$$S \circ \phi_t = \phi_{-t}$$

3. *For all s in $[-1, 1]$,*

$$\phi_t \star \phi_s = \phi_{s+t}$$

The proof of the preceeding proposition is based on the two following lemma which give the translations of $\ln : U \to V$ and $\exp : V \to U$ in the algebraic framework. Recall that when one forgets the Lie bracket on $\mathfrak{g}$, one gets an abelian Lie algebra, still denoted by $\mathfrak{g}$, which is the tangent space at 0 of the Lie group $(\mathfrak{g}, +)$., and whose universal envelopping algebra is the symmetric algebra $S\mathfrak{g}$.

Lemma 2.1.2.9. *The isomorphism of coalgebra* $\ln_* : U\mathfrak{g} \to S\mathfrak{g}$ *induced by the logarithm is given by*

$$\ln_* = \sum_{k \geq 0} \frac{1}{k!} \mathrm{pr}^{\star k} \tag{2.1.3}$$

where the convolution $\star$ *is taken in* $\mathrm{Hom}_{\mathbb{R}}(U\mathfrak{g}, S\mathfrak{g})$.

Lemma 2.1.2.10. *The isomorphism of coalgebras* $\exp_* : S\mathfrak{g} \to U\mathfrak{g}$ *induced by the exponential map* $V \subset \mathfrak{g} \to U \subset G$ *is the symmetrization map of Poincaré-Birkhoff-Witt* $\beta : S\mathfrak{g} \to U\mathfrak{g}$ *: its value on every monomial* $g_1 g_2 \cdots g_p$ *of lengh* p *is given by*

$$\exp_*(g_1 g_2 \cdots g_p) = \sum_{\sigma \in \Sigma_p} g_{\sigma(1)} g_{\sigma(2)} \cdots g_{\sigma(p)} =: \beta(g_1 g_2 \cdots g_p) \tag{2.1.4}$$

where Σ_p *stands for the group of permutations of the set* $\{1, 2, \cdots, p\}$.

Definition 2.1.2.11. *Let* t *be a real number.* $A_t : U\mathfrak{g}^{\otimes 2} \to U\mathfrak{g}$ *is the bilinear map defined by*

$$A_t(x, y) := A_t(x \otimes y) := \sum_{(x)} \phi_{-t}(x^{(1)}) \phi_t(x^{(2)} y)$$

for all x *and* y *in* $U\mathfrak{g}$.

Proposition 2.1.2.12. *For all* x *in* $U\mathfrak{g}$, g *in* $\mathfrak{g}$, *and for all real number* t*:*

$$A_t(x, g) \in \mathfrak{g}$$

Theorem 2.1.2.13. *Let* $\mathfrak{g}$ *be a finite dimensionnal Lie algebra over* $\mathbb{R}$ *and* G *be a Lie group with Lie algebra* $\mathfrak{g}$. *The contracting homotopy* $s^\vee$ *of* $\Lambda_e^* G$ *defined in 2.1.2.2 thanks to te canonical contraction* φ *of* G *restricts, via the injection* $R : C_*(\mathfrak{g})\Lambda_e^* G$ *defined at the begining of the chapter, to a contracting homotopy* s *of the Chevalley-Eilenberg resolution. Moreover,* s *is given on* p*-chains by*

$$s(x \otimes g_1 \wedge g_2 \wedge \cdots \wedge g_p) = \sum_{(x)} \int_0^1 dt\, \phi_t(x^{(1)}) \otimes \mathrm{pr}(x^{(2)}) \wedge A_t(x^{(3)}, g_1) \wedge \cdots \wedge A_t(x^{(p+2)}, g_p) \tag{2.1.5}$$

for all x *in* $U\mathfrak{g}$ *and* g_1, ..., g_p *in* $\mathfrak{g}$.

2.2 Contraction of the Koszul resolution

The first part of this section shows how to use the contraction s of the Chevalley-Eilenberg resolution defined in (2.1.5) to obtain another contracting homotopy h, but this time for the Koszul resolution $CK(U\mathfrak{g}$. The second part deals with the suppression of the finiteness hypothesis on the dimension $\mathfrak{g}$ that has been assumed until now, arguing that formula (2.1.5) still makes sens in arbitrary dimension

2.2.1 From s to h

Notations are the same as those of section 1.2.2. In particular, the map $E : U\mathfrak{g} \to U\mathfrak{g}^e$ defined by $E := (Id \otimes S)\Delta$ makes the diagram (1.2.1) commute, and fully determines the right $U\mathfrak{g}$-module on $U\mathfrak{g}^e$. Moreover, the map

$$\begin{array}{rcl} \theta'' : CK_*(U\mathfrak{g}) & \to & U\mathfrak{g}^e \otimes_{U\mathfrak{g}} C_*(\mathfrak{g}) \\ x \otimes g_1 \wedge \cdots \wedge g_n \otimes y & \mapsto & x \otimes y \otimes 1 \otimes g_1 \wedge \cdots \wedge g_n \end{array} \tag{2.2.1}$$

is an isomorphism of complexes of $U\mathfrak{g}$-bimodules, and it could seem natural to define the contracting homotopy h we are looking for by transporting the map $Id_{U\mathfrak{g}^e} \otimes s$ to $CK_*(U\mathfrak{g})$ via θ''. Unfortunately, $Id_{U\mathfrak{g}^e} \otimes s$ is not well defined on $U\mathfrak{g}^e \otimes_{U\mathfrak{g}} C_*(\mathfrak{g})$ since s is not $U\mathfrak{g}$-linear. To fix this, we can proceed in the same way that we did in 1.2.3 to force the $U\mathfrak{g}^e$-linearity of the map G^B_*. Define a degree +1 linear map $\tilde{h} : U\mathfrak{g}^e \otimes_{U\mathfrak{g}} C_*(\mathfrak{g}) \to U\mathfrak{g}^e \otimes_{U\mathfrak{g}} C_*(\mathfrak{g})$ by

$$\tilde{h}(x \otimes y \otimes z \otimes g_1 \wedge \cdots \wedge g_n) := \sum_{(x)} 1 \otimes x^{(1)} y \otimes s(x^{(2)} z \otimes g_1 \wedge \cdots \wedge g_n) \tag{2.2.2}$$

for all x, y, z, w in $U\mathfrak{g}$ and g_1, ..., g_n in $\mathfrak{g}$.

Proposition 2.2.1.1. *The map $\tilde{h}$ is a well defined contracting homotopy of the complex $U\mathfrak{g}^e \otimes_{U\mathfrak{g}} C_*(\mathfrak{g})$.*

Corollary 2.2.1.2. *Let $\mathfrak{g}$ be a finite dimensionnal Lie algebra over $\mathbb{R}$. The degree +1 graded linear map $h : CK_*(U\mathfrak{g}) \to CK_*(U\mathfrak{g})$ defined in degree n by*

$$h(x \otimes g_1 \wedge \cdots \wedge g_n \otimes y) := \int_0^1 dt\, \phi_t(x^{(1)}) \otimes \mathrm{pr}\, x^{(2)} \wedge A_t(x^{(3)}, g_1) \wedge \cdots \wedge A_t(x^{(n+2)}, g_n) \otimes \phi_{1-t}(x^{(n+3)}) y \tag{2.2.3}$$

for all x, y in $U\mathfrak{g}$ and g_1, ..., g_n in $\mathfrak{g}$ is a contracting homotpoy of the Koszul resolution $C_*(\mathfrak{g})$.

2.2.2 Removing the finiteness condition

In all the above discussion, the dimension Lie algebra is always assumed to be finite in order use the geometric interpretation of $C_*(\mathfrak{g})$, that is its inclusion R in the complex of currents $\Lambda^*_e G$, to induce a contracting homotopy s on it by applying the Poincaré lemma to the canonical contraction φ. Note that we never had to show that s satisfies $sd + ds = \mathrm{Id} - \eta\epsilon$ since the dual identity was given by the usual Poincaré lemma. However, formula (2.1.5) makes sense in arbitrary dimension and a natural question one could ask is whether it still defines a contracting homotopy of $C_*(\mathfrak{g})$. The answer is given by the following theorem: 2.1.2.13 :

Theorem 2.2.2.1. *Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{R}$. A contracting homotopy $s : C_*(\mathfrak{g}) \to C_*(\mathfrak{g})$ of the Chevalley-Eilenberg resolution $C_*(\mathfrak{g})$ is given by*

$$s(x \otimes g_1 \wedge \cdots \wedge g_n) := \sum_{(x)} \int_0^1 dt\, \phi_t(x^{(1)}) \otimes \mathrm{pr}\, x^{(2)} \wedge A_t(x^{(3)}, g_1) \wedge \cdots \wedge A_t(x^{(n+2)}, g_n)$$

for all x in $U\mathfrak{g}$ and g_1, ..., g_n in $\mathfrak{g}$.

which as corollary

Corollary 2.2.2.2. *Let $\mathfrak{g}$ be a Lie algbra over $\mathbb{R}$. Formula (2.2.3) defines a contracting homotopy h of the Koszul resolution $CK_*(U\mathfrak{g})$.*

Remark 2.2.2.3. *The preceding corollary implies the acyclicity of the Koszul resolution with no need of point* 1. *of theorem 1.1.2.2.*

The proof of theorem is based on the following lemmas, which are algebraic analogues of usual identities in ordinary differential calculus.

Lemma 2.2.2.4. *Let t belong to $[-1, 1]$, g be in $\mathfrak{g}$ and x in $U\mathfrak{g}$. Then*

1.
$$\frac{d}{dt}\phi_t = \phi_t \star \mathrm{pr} = \mathrm{pr} \star \phi_t$$

2.
$$\frac{d}{dt}A_t(x, g) = \mathrm{pr}(xg) + \sum_{(x)} [A_t(x^{(1)}, g), \mathrm{pr}\, x^{(2)}]$$

Lemma 2.2.2.5. *Let g, h be two elements of $\mathfrak{g}$, and x dansbe an element of $U\mathfrak{g}$. Then, for all real number t:*

$$-\sum_{(x)} [A_t(x^{(1)}, g), A_t(x^{(2)}, h)] = A_t(xg, h) - A_t(xh, g) - A_t(x, [g, h])$$

Thus, when the ground ring is $\mathbb{R}$, we have built an explicit contracting homotopy h of the Koszul resolution that allows us to apply the strategy developped in 1.2.3 to get a quasi-inverse to the anisymmetrization map of Cartan and Eilenberg.

Chapter 3

Inverting the Cartan-Eilenberg isomorphism

This chapter is divided in two sections. In the first one, we give explicit computations, in degree $* = 1$ and $* = 2$, of the morphisms of complexes $G_* : CH_*(U\mathfrak{g}; M) \to C_*(\mathfrak{g}; M^{ad})$ and $G^* : C^*(\mathfrak{g}; M^{ad}) \to CH^*(U\mathfrak{g}; M)$, quasi-inverses of the antisymmetrization map respectively in the homological and cohomological framework (see 1.2.3), and both induced by the morphism of resolution G_*^B, itself obtained by applying the strategy developed in 1.2.3.1 and whose definition involves the contractiong homotopy h of chapter 2. In section 2, we show that when $M = \mathbb{R}$, the map G^* can be see as an analogue of the integration of Lie algebra 2-cocycles in locally-smooth group group cochain map described in [Nee04] and [Cov10]. In what follows, $\mathfrak{g}$ is a Lie algebra over $\mathbb{R}$ and M is a $U\mathfrak{g}$-bimodule.

3.1 Construction of the quasi-inverse in degrees 1 and 2

To obtain $G_* : CH_*(U\mathfrak{g}; M) \to C_*(\mathfrak{g}; M^{ad})$, we need to make the morphism of resolutions $G_*^B : B_*(U\mathfrak{g}) \to CK_*(U\mathfrak{g})$ of proposition 1.2.3.1 more explicit.

3.1.1 Computation of G_*^B in low degree

G_*^B and G_* are defined in 1.2.3 with the contraction $h : CK_*(U\mathfrak{g}) \to CK_*(U\mathfrak{g})$ defined by formula (2.2.3). Recall that $G_0^B := \mathrm{Id}_{U\mathfrak{g}\otimes U\mathfrak{g}}$. Let x, y, z and w be elements of $U\mathfrak{g}$. The image of $x \otimes y \otimes z$ by G_1^B can be computed via (1.2.5):

$$\begin{aligned} G_1^B(x \otimes y \otimes x) :=& x(hG_0 d^B(1 \otimes y \otimes 1))z \\ =& x(h(y \otimes 1 - 1 \otimes y))z \\ =& \int_0^1 dt\; x\phi_t(y^{(1)}) \otimes \mathrm{pr}\; y^{(2)} \otimes \phi_{1-t}(y^{(3)})z - x \otimes \mathrm{pr}\; 1 \otimes yz \end{aligned}$$

which implies, since $\mathrm{pr}\,1 = 0$:

$$G_1^B(x \otimes y \otimes z) = \int_0^1 dt\; x\phi_t(y^{(1)}) \otimes \mathrm{pr}\, y^{(2)} \otimes \phi_{1-t}(y^{(3)})z \tag{3.1.1}$$

Similarly, using denitions and formula (3.1.1), we get:

$$\begin{aligned} G_2^B(x \otimes y \otimes z \otimes w) =& x \left(h G_1^B d^B (1 \otimes y \otimes z \otimes 1) \right) w \\ =& x \left(h G_1^B \left(y \otimes z \otimes 1 - 1 \otimes yz \otimes 1 + 1 \otimes y \otimes z \right) \right) w \\ =& \int_0^1 dt\; x \Bigg(h\big(y\phi_t(z^{(1)}) \otimes \mathrm{pr}\, z^{(2)} \otimes \phi_{1-t}(z^{(3)})\big) \\ & - \overbrace{h\big(\phi_t((yz)^{(1)}) \otimes \mathrm{pr}((yz)^{(2)}) \otimes \phi_{1-t}((yz)^{(3)})\big)}^{A} \\ & + \overbrace{h\big(\phi_t(y^{(1)}) \otimes \mathrm{pr}\, y^{(2)} \otimes \phi_{1-t}(y^{(3)})z\big)}^{B} \Bigg) w \end{aligned}$$

Let's show that A et B are zero. First, notice that relation (2.1.2) implies

$$\phi_t \circ \mathrm{pr} = \mathrm{pr} \circ \phi_t = t\, \mathrm{pr}$$

which, combined with point 1. of lemma 2.2.2.4, points 2. and 3. of proposition 2.1.2.8, and using the fact that ϕ_t is a coalgebra endomorphism, gives:

$$\begin{aligned} A &= t \int_0^1 ds\; \phi_{st}((yz)^{(1)}) \otimes \mathrm{pr}((yz)^{(2)}) \wedge A_s \left(\phi_t((yz)^{(3)}), \mathrm{pr}((yz)^{(4)}\right) \otimes \phi_{1-t}((yz)^{(5)}) \\ &= t \int_0^1 ds\; \phi_{st}((yz)^{(1)}) \otimes \mathrm{pr}((yz)^{(2)}) \wedge \phi_{-st}((yz)^{(3)}) \phi_s(\frac{d}{dt}\phi_t((yz)^{(4)})) \otimes \phi_{1-t}((yz)^{(5)}) \\ &= t \int_0^1 ds\; \phi_{st}((yz)^{(1)}) \otimes \mathrm{pr}((yz)^{(2)}) \wedge \frac{d}{du}\phi_{s(u-t)}((yz)^{(3)})_{|u=t} \otimes \phi_{1-t}((yz)^{(4)}) \\ &= t \int_0^1 s ds\; \phi_{st}((yz)^{(1)}) \otimes \mathrm{pr}((yz)^{(2)}) \wedge \mathrm{pr}((yz)^{(3)}) \otimes \phi_{1-t}((yz)^{(4)}) \end{aligned}$$

As Δ is cocommutative, A is invariant under the action of the transposition $(1\,2) \in \Sigma_2$, thus

$$-A = \mathrm{sgn}\left((1\,2)\right) A = (1\,2) A = A$$

i.e.

$$A = 0$$

$B = 0$ can be shown in the same way, replacing yz by y. Finally, we have

$$G_2^B(x \otimes y \otimes z \otimes w) = \int_0^1 dt\, x\Big(h\big(y\phi_t(z^{(1)}) \otimes \mathrm{pr}\, z^{(2)} \otimes \phi_{1-t}(z^{(3)})\big)\Big)w$$
$$= \int_0^1 \int_0^1 ds\, dt\, x\phi_s(y^{(1)}\phi_t(z^{(1)})) \otimes \mathrm{pr}(y^{(2)}\phi_t(z^{(2)})) \wedge A_s(y^{(3)}\phi_t(z^{(3)}), \mathrm{pr}\, z^{(6)}) \otimes$$
$$\otimes\, \phi_{1-s}(y^{(4)}\phi_t(z^{(4)}))\phi_{1-t}(z^{(5)})w \qquad (3.1.2)$$

3.1.2 G_* and G^* in low degrees

The quasi-isomorphismG_* : $CH_*(\mathfrak{g}; M) \to C_*(\mathfrak{g}; M^{ad})$ corresponds to $\mathrm{Id} \otimes G^B : M \otimes_{U\mathfrak{g}^e} B_*(U\mathfrak{g}) \to M \otimes_{U\mathfrak{g}^e} CK_*(U\mathfrak{g})$ via the isomorphisms θ and θ' defined in (1.2.3) and (1.2.4). A direct computation using (3.1.1) and (3.1.2) gives the expression of G_1 and G_2 on elementary tensorss :

Proposition 3.1.2.1. *The quasi-inverse $G_* : CH_*(U\mathfrak{g}; M) \to C_*(\mathfrak{g}; M^{ad})$ of the antisymmetrization map F_* of theorem 1.1.2.2 induced by the contracting homotopy h of corollary 2.2.1.2 satisfies*

$$G_1(m \otimes x) = \int_0^1 dt\, \phi_{1-t}(x^{(1)})m\phi_t(x^{(2)}) \otimes \mathrm{pr}\, x^{(3)}$$

and

$$G_2(m \otimes x \otimes y) = \int_0^1 \int_0^1 ds dt \phi_{1-s}(x^{(1)}\phi_t(y^{(1)}))\phi_{1-t}(y^{(5)})m\phi_s(x^{(2)}\phi_t(y^{(2)})) \otimes \mathrm{pr}(x^{(3)}\phi_t(y^{(3)})) \wedge A_s(x^{(4)}\phi_t(y^{(4)}), \mathrm{pr} y^{(6)})$$

for all m in M, x and y in $U\mathfrak{g}$.

The cohomological version $G^* : C^*(\mathfrak{g}; M^{ad}) \to CH^*(U\mathfrak{g}; M)$, for which M^{ad} is endowed with its **left** $U\mathfrak{g}$-module structure, is obtained by transporting the quasi-isomorphism

$$\begin{array}{ccc} \mathrm{Hom}^*_{U\mathfrak{g}^e}(CK_*(U\mathfrak{g}), M) & \overset{\simeq}{\to} & \mathrm{Hom}^*_{U\mathfrak{g}^e}(B_*(U\mathfrak{g}), M) \\ f & \mapsto & f \circ G^B_* \end{array}$$

via the isomorphisms of complexes of $U\mathfrak{g}$-bimodules

$$\begin{array}{ccc} \mathrm{Hom}^*_{U\mathfrak{g}^e}(B_*(U\mathfrak{g}), M) & \overset{\cong}{\to} & CH^*(U\mathfrak{g}; M) := \{\mathrm{Hom}(U\mathfrak{g}^{\otimes n}, M)\}_{n \geq 0} \\ f & \mapsto & \big(x_1 \otimes \cdots \otimes x_n \mapsto f(1 \otimes x_1 \otimes \cdots \otimes x_n \otimes 1)\big) \end{array}$$

and

$$\begin{array}{ccc} C^*(\mathfrak{g}; M^{ad}) := \{\mathrm{Hom}(U\mathfrak{g}^{\otimes n}, M)\}_{n \geq 0} & \overset{\cong}{\to} & \mathrm{Hom}^*_{U\mathfrak{g}^e}(CK_*(U\mathfrak{g}), M) \\ f & \mapsto & \big(x \otimes x_1 \wedge \cdots \wedge x_n \otimes y \mapsto x f(x_1 \wedge \cdots \wedge x_n) y\big) \end{array}$$

Thus G^* can also be given explicitely in low degrees, still thanks to relations (3.1.1) and (3.1.2):

Proposition 3.1.2.2. *The morphisms of cochain complexes $G^* : C^*(\mathfrak{g}; M^{ad}) \to CH^*(U\mathfrak{g}; M)$, quasi-inverse of the cohomological antisymmetrization map F^* induced by G_*^B, satisfies*

$$G^1(f_1)(x) := \int_0^1 dt\, \phi_t(x^{(1)}) f_1(x^{(2)}) \phi_{1-t}(x^{(3)})$$

and

$$G^2(f_2)(x \otimes y) = \int_0^1 \int_0^1 ds dt \phi_s(x^{(1)} \phi_t(y^{(1)})) f\left(\mathrm{pr}(x^{(2)} \phi_t(y^{(2)})) \wedge A_s(x^{(3)} \phi_t(y^{(3)}), \mathrm{pr}\, y^{(6)})\right) \phi_{1-s}(x^{(4)} \phi_t(y^{(4)})) \phi_{1-t}(y^{(5)})$$

for all 1*-cochain f_1 in $C^1(\mathfrak{g}; M^{ad})$, for all* 2*-cochain f_2 in $C^2(\mathfrak{g}; M^{ad})$, and for all x, y in $U\mathfrak{g}$.*

Remark 3.1.2.3. *Let $f : \mathfrak{g} \to M^{ad}$ be a* 1*-Lie algebra cocycle. In [Dix74], J. Dixmier explains how to associate a* 1*-cocycle de Hochschild $\hat{f} : U\mathfrak{g} \to M$ to f. $\hat{f}$ satisfies the following defining conditions:*

$$\hat{f}(xy) = x f(y) + f(x) y \quad , \quad \forall\, x,\, y \in U\mathfrak{g} \tag{3.1.3}$$

and

$$\hat{f}(g) = f(g) \quad , \quad \forall g \in \mathfrak{g}. \tag{3.1.4}$$

Since G^1 is a morphism of cochain complexes, $G^1(f)$ is a Hochschild cocycle i.e. satisfies (3.1.3), and thus

$$\hat{f} = G^1(f)$$

3.1.3 Example: G^2 in the abelian case

In this subsection, $\mathfrak{g}$ is assumed to be abelian ($[x, y] = 0$ for all g and h in $\mathfrak{g}$). In that case, the eneloping algebra $U\mathfrak{g}$ is the symmetric algebra $S\mathfrak{g}$ and is graded by the lengh of monomials. Moreover, the canonical projection pr is exactly the projection $\mathrm{proj} : S\mathfrak{g} \to \mathfrak{g}$ of 2.1.2.9, which is a derivation of the algebra $U\mathfrak{g}$ along ϵ that is:

$$\mathrm{pr}(xy) = \epsilon(x)\, \mathrm{pr}\, y + \epsilon(y)\, \mathrm{pr}\, x$$

for all x and y in $U\mathfrak{g}$. Denote by $|x|$ the degree of an element x of $U\mathfrak{g} = S\mathfrak{g}$. One easily checks that the operators pr and ϕ_t satisfy

$$\mathrm{pr} x = \begin{cases} x & \text{si } |x| = 1 \\ 0 & \text{sinon} \end{cases}$$

and

$$\phi_t(x) = t^{|x|} x$$

for all real number t. In particular, thanks to point 3. of proposition 2.1.2.8:

$$t^{|x^{(1)}|} s^{|x^{(2)}|} x^{(1)} x^{(2)} = \phi_s \star \phi_t(x) = \phi_{s+t}(x) = (s+t)^{|x|} x$$

By cocommutativity of the coproduct and since pr is a derivation along ϵ, this gives

$$\mathrm{pr}(u\,\phi_t(v^{(1)}))\wedge \mathrm{pr}(v^{(2)}) = \epsilon(u)\mathrm{pr}(\phi_t(v^{(1)}))\wedge \mathrm{pr}(v^{(2)}) + \epsilon(\phi_t(v^{(1)}))\mathrm{pr}(u)\wedge \mathrm{pr}(v^{(2)}) = \mathrm{pr}\, u\wedge \mathrm{pr}\, v$$

for all u an v in$U\mathfrak{g}$. Thus, by 3.1.2.2,

$$\begin{aligned} G^2(f_2)(x\otimes y) =& \int_0^1\int_0^1 t^{|y^{(1)}|+|y^{(3)}|+|y^{(4)}|+|y^{(5)}|}(1-t)^{|y^{(6)}|}s^{|x^{(1)}|+|y^{(1)}|+|x^{(4)}|+|y^{(4)}|+1}(-s)^{|x^{(3)}|+|y^{(3)}|}(1-s)^{|x^{(5)}|+|y^{(5)}|} \\ & x^{(1)}y^{(1)}f\left(\mathrm{pr}(x^{(2)}\phi_t(y^{(2)}))\wedge x^{(3)}y^{(3)}x^{(4)}y^{(4)}\mathrm{pr}\,(y^{(7)})\right)x^{(5)}y^{(5)}y^{(6)}ds\,dt \\ =& \int_0^1\int_0^1 (st)^{|y^{(1)}|}s^{|x^{(1)}|+1}(1-s)^{|x^{(3)}|}(1-st)^{|y^{(3)}|}\,x^{(1)}y^{(1)}f\left(\mathrm{pr}(x^{(2)})\wedge \mathrm{pr}\,(y^{(2)})\right)x^{(3)}y^{(3)}ds\,dt \end{aligned}$$

for every 2-Lie algebra cochain $f:\Lambda^2\mathfrak{g}\to M^{ad}$ and for all x, y in $S\mathfrak{g}$. Using the change of variable $u := st$, we get

$$G^2(f)(x\otimes y) = \int_0^1 ds\int_0^s du\, u^{|y^{(1)}|}s^{|x^{(1)}|}(1-s)^{|x^{(3)}|}(1-u)^{|y^{(3)}|}\,x^{(1)}y^{(1)}f\left(\mathrm{pr}(x^{(2)})\wedge \mathrm{pr}\,(y^{(2)})\right)x^{(3)}y^{(3)}$$

As, for all integers p and q

$$\int_0^s du u^p(1-u)^q = \frac{p!q!}{(p+q+1)!} - \sum_{i=0}^{p}\frac{p!q!}{(p-i)!(q+i+1)!}s^{p-i}(1-s)^{q+i+1}$$

we have

$$\begin{aligned} G^2(f)(x\otimes y) =& \int_0^1 ds\Bigg(\frac{|y^{(1)}|!\,|y^{(3)}|!}{(|y|+1-|y^{(2)}|)!}\,s^{|x^{(1)}|}(1-s)^{|x^{(3)}|}\,x^{(1)}y^{(1)}f\left(\mathrm{pr}(x^{(2)})\wedge \mathrm{pr}\,(y^{(2)})\right)x^{(3)}y^{(3)} \\ & -\sum_{i=0}^{|y^{(1)}|}\frac{|y^{(1)}|!\,|y^{(3)}|!}{(|y^{(1)}|-i)!(|y^{(3)}|+1+i)!}s^{|x^{(1)}|+|y^{(1)}|-i}(1-s)^{|x^{(3)}|+|y^{(3)}|+1+i}x^{(1)}y^{(1)}f\left(\mathrm{pr}(x^{(2)})\wedge \mathrm{pr}\,(y^{(2)})\right)x^{(3)}y^{(3)}\Bigg) \\ =& \frac{|y^{(1)}|!\,|y^{(3)}|!\,|x^{(1)}|!\,|x^{(3)}|!}{(|y|+1-|y^{(2)}|)!(|x|+1-|x^{(2)}|)!}\,x^{(1)}y^{(1)}f\left(\mathrm{pr}(x^{(2)})\wedge \mathrm{pr}\,(y^{(2)})\right)x^{(3)}y^{(3)} \\ & -\sum_{i=0}^{|y^{(1)}|}\frac{|y^{(1)}|!\,|y^{(3)}|!(|x^{(1)}|+|y^{(1)}|-i)!(|x^{(3)}|+|y^{(3)}|+1+i)!}{(|y^{(1)}|-i)!(|y^{(3)}|+1+i)!(|x|+|y|+2-|x^{(2)}|-|y^{(2)}|)!}\,x^{(1)}y^{(1)}f\left(\mathrm{pr}(x^{(2)})\wedge \mathrm{pr}\,(y^{(2)})\right)x^{(3)}y^{(3)} \\ =& \left(\frac{1}{(|y|+1-|y^{(2)}|)!(|x|+1-|x^{(2)}|)!} - \sum_{i=0}^{|y^{(1)}|}\frac{\binom{|x^{(1)}|+|y^{(1)}|-i}{|y^{(1)}|-i}\binom{|x^{(3)}|+|y^{(3)}|+1+i}{|y^{(3)}|+1+i}}{(|x|+|y|+2-|x^{(2)}|-|y^{(2)}|)!}\right) \\ & |y^{(1)}|!\,|y^{(3)}|!\,|x^{(1)}|!\,|x^{(3)}|!\,x^{(1)}y^{(1)}f\left(\mathrm{pr}(x^{(2)})\wedge \mathrm{pr}\,(y^{(2)})\right)x^{(3)}y^{(3)} \end{aligned}$$

Since pr vanishes on every element of lengh non equal to 1, we can assume that $|x^{(2)}| = |y^{(2)}| = 1$ in the preceeding equality, which leads to

$$G^2(f)(x\otimes y) = \left(\frac{1}{|y|!\,|x|!} - \sum_{i=0}^{|y^{(1)}|} \frac{\binom{|x^{(1)}|+|y^{(1)}|-i}{|y^{(1)}|-i}\binom{|x^{(3)}|+|y^{(3)}|+1+i}{|y^{(3)}|+1+i}}{(|x|+|y|)!}\right)|y^{(1)}|!\,|y^{(3)}|!\,|x^{(1)}|!\,|x^{(3)}|!\; x^{(1)}y^{(1)}f\left(\mathrm{pr}\,x^{(2)}\wedge \mathrm{pr}\,y^{(2)}\right)x^{(3)}y^{(3)}$$

Since $|y^{(1)}| + |y^{(3)}| + 1 = |y|$ and $|x^{(1)}| + |x^{(3)}| + 1 = |x|$, the change of variable $j := |y^{(1)}| - i$ gives :

$$G^2(f)(x\otimes y) = \left(\frac{1}{|y|!\,|x|!} - \sum_{j=0}^{|y^{(1)}|} \frac{\binom{|x^{(1)}|+j}{j}\binom{|x^{(3)}|+|y|-j}{|y|-j}}{(|x|+|y|)!}\right)|y^{(1)}|!\,|y^{(3)}|!\,|x^{(1)}|!\,|x^{(3)}|!\; x^{(1)}y^{(1)}f\left(\mathrm{pr}\,x^{(2)}\wedge \mathrm{pr}y^{(2)}\right)x^{(3)}y^{(3)}$$

Thus:

Proposition 3.1.3.1. *If $\mathfrak{g}$ is an abelian Lie algebra over $\mathbb{R}$, M a $U\mathfrak{g}$-bimodule, and $f : \Lambda^2\mathfrak{g} \to M^{ad}$ a 2-cochain of Lie algebra with values in M^{ad}, then*

$$G^2(f)(x\otimes y) = \left(\frac{1}{|y|!\,|x|!} - \sum_{j=0}^{|y^{(1)}|} \frac{\binom{|x^{(1)}|+j}{j}\binom{|x^{(3)}|+|y|-j}{|y|-j}}{(|x|+|y|)!}\right)|y^{(1)}|!\,|y^{(3)}|!\,|x^{(1)}|!\,|x^{(3)}|!\; x^{(1)}y^{(1)}f\left(\mathrm{pr}\,x^{(2)}\wedge \mathrm{pr}\,y^{(2)}\right)x^{(3)}y^{(3)}$$

for all x and y in $U\mathfrak{g} = S\mathfrak{g}$.

Corollary 3.1.3.2. *Under the hypothesis of previous proposition, if we ask in addition for the bimodule structure on M to be trivial ($xm = mx = \epsilon(x)m$ for all x in $S\mathfrak{g}$ and m in M), then*

$$G^2(f)(x\otimes y) = \begin{cases} \frac{1}{2}f(x\wedge y) & si\ |x| = |y| = 1 \\ 0 & sinon. \end{cases} \tag{3.1.5}$$

for all x and y in $U\mathfrak{g} = S\mathfrak{g}$.

The case of the Heisenberg algebra: It is well known ([Wei95], [Lod98], [CE56]) that to every 2-cocycle of Lie algebra $f : \Lambda^2 \to M^{ad}$ can be associated an abelian extension of Lie algebras

$$0 \to M^{ad} \to \mathfrak{h} \to \mathfrak{g} \to 0$$

such that the Lie bracket on the vector space $\mathfrak{h} \cong M^{ad} \oplus \mathfrak{g}$ is given by

$$[(0,X),(0,Y)] := (f(X\wedge Y),[X,Y])$$

for all X and Y in $\mathfrak{g}$. Similarly (see [Lan75]), the second cohomology group $HH^2(U\mathfrak{g}; M)$ classifies singular extensions (always split over the field $\mathbb{R}$) of the form

$$0 \to M \to H \to U\mathfrak{g} \to 0$$

We are now going to describe the the extension encoded by the Hochschild 2-cocycle $G^2(f) : U\mathfrak{g}^{\otimes 2} \to M$ associated to a special Lie cocycle: the Heisenberg cocycle. Let $\mathfrak{g} :=< X, Y >$ be the abelian Lie algebra of dimension 2 generated by X and Y, and denote by $M = \mathbb{R}Z$ the trivial bimodule of dimension 1 with generator Z.

Definition 3.1.3.3. *The* ***Heisenberg cocycle*** *is the Lie algebra cocycle $c_H : \Lambda^2\mathfrak{g} \to M = \mathbb{R}Z$ defined by*

$$c_H(X \wedge Y) = Z$$

The corresponding central extension is

$$0 \to \mathbb{R}Z \to \mathfrak{h} \to \mathfrak{g} \to 0$$

whose middle term $\mathfrak{h}$ is called ***the Heisenberg algebra of dimension*** 3*).*

Let $H := \mathbb{R}Z \oplus S\mathfrak{g}$ be the extension of associative algebra $S\mathfrak{g} = U\mathfrak{g}$ with kernel $\mathbb{R}Z$ corresponding to $G^2(c_H)$. The family of monomials $(X^\alpha Y^\beta)_{\alpha,\beta\geq 0}$ is a base of the $\mathbb{R}$-vector space $S\mathfrak{g}$, so that the associative product on H is fully determined by the values of $G^2(c_H)$ on such monomials. Thanks to (3.1.5), we see that

$$G^2(c_H)(X^\alpha Y^\beta \otimes X^\gamma Y^\delta) = \begin{cases} \frac{1}{2}Z & \text{si } (\alpha,\beta) = (1,0) \text{ et } (\gamma,\delta) = (0,1) \\ -\frac{1}{2}Z & \text{si } (\alpha,\beta) = (0,1) \text{ et } (\gamma,\delta) = (1,0) \\ 0 & \text{sinon.} \end{cases}$$

for all α, β, γ and δ in $\mathbb{N}$. For λ and μ in $\mathbb{R}$, denote by $\lambda Z + \mu X^\alpha Y^\beta$ the element $(Z, X^\alpha Y^\beta)$ of $H = \mathbb{R}Z \oplus S\mathfrak{g}$ and by $\cdot$ the product on H. Then

$$(\lambda Z + \mu X^\alpha Y^\beta) \cdot (X^\gamma Y^\delta) = \begin{cases} \lambda Z + \mu X^\alpha Y^\beta & \text{si } \gamma + \delta = 0 \\ \frac{\lambda}{2}Z + \mu XY & \text{si } (\alpha,\beta,\gamma,\delta) = (1,0,0,1) \\ \frac{\lambda}{2}Z + \mu XY & \text{si } (\alpha,\beta,\gamma,\delta) = (0,1,1,0) \\ \mu X^{\alpha+\gamma}Y^{\beta+\delta} & \text{dans les autres cas.} \end{cases} \tag{3.1.6}$$

A direct inspection shows that the $\mathbb{R}$-linear injective map $i : \mathfrak{h} \to H$ defined by

$$i(X) = X \quad , \quad i(Y) = Y \quad , \text{et} \quad i(Z) = Z$$

is a Lie algebra morphism (the bracket on H being the commutator associated to the product $\cdot$). By universal property of the universal enveloping algebra, i induces a surjective morphism of associative algebras

$$p : U\mathfrak{h} \twoheadrightarrow H$$

Let's determine the kernel of p. To do so, notice that by the Poincaré-Birkhoff-Witt theorem, the family $(Z^\alpha X^\beta Y^\gamma)_{\alpha,\beta,\gamma\geq 0}$ is a base of the real vector space $U\mathfrak{h}$. Any element B of $U\mathfrak{g}$ can be written uniquely as a linear combination

$$B := \sum_{\alpha,\beta,\gamma\geq 0} B_{\alpha,\beta,\gamma} Z^\alpha X^\beta Y^\gamma ,$$

where $(B_{\alpha,\beta,\gamma})_{\alpha,\beta,\gamma\geq 0}$ is a family of real numbers, a finite number of them being non zero . Suppose that $p(B) = 0$. Then

$$p(B) = \sum_{\alpha,\beta,\gamma\geq 0} B_{\alpha,\beta,\gamma} p(Z^\alpha X^\beta Y^\gamma) = \sum_{\alpha,\beta,\gamma\geq 0} B_{\alpha,\beta,\gamma} \overbrace{Z\cdot Z\cdots Z}^{\alpha \text{ fois}} \cdot \overbrace{X\cdot X\cdots X}^{\beta \text{ fois}} \cdot \overbrace{Y\cdot Y\cdots Y}^{\gamma \text{ fois}} = 0$$

But according to (3.1.6), $Z \cdot A = 0$ for all A in the augmentation ideal $\bar{S}\mathfrak{g} \subset H$ of $S\mathfrak{g}$. Thus,

$$\sum_{\beta,\gamma\geq 0} B_{0,\beta,\gamma} \overbrace{X\cdot X\cdots X}^{\beta \text{ fois}} \cdot \overbrace{Y\cdot Y\cdots Y}^{\gamma \text{ fois}} + \sum_{\alpha\geq 1} B_{\alpha,0,0} \overbrace{Z\cdot Z\cdots Z}^{\alpha \text{ fois}} = 0$$

Applying (3.1.6) again, we get

$$\sum_{\beta,\gamma\geq 0} B_{0,\beta,\gamma} X^\beta Y^\gamma + \frac{1}{2}(B_{0,2,0} + B_{0,1,1} + B_{0,0,2} + 2B_{1,0,0})Z = 0$$

which implies that $B_{1,0,0} = 0$ and

$$B_{0,\beta,\gamma} = 0 \quad , \quad \forall \beta,\gamma \geq 0 \, .$$

Thus, $p(B) = 0$ if and only if B belongs to the ideal of $U\mathfrak{h}$ generated by ZX and ZY, denoted by $< ZX, ZY >$ i.e. $\operatorname{Ker} p =< ZX, ZY >$. We have proven

Proposition 3.1.3.4. *The singular extension H of the associative algebra $S\mathfrak{g}$ associated to the Hochschild cocycle $G^2(c_H)$, where c_H is the Heisenberg cocycle defined in 3.1.3.3, is isomorphic to the quotient of $U\mathfrak{h}$ by the ideal generated by ZX and ZY :*

$$H \cong U\mathfrak{h}/_{< ZX, ZY >}$$

3.2 Interpretation as an algebraic integration process

In this section, we explain how the quasi-isomorphism of cochain complexes $G^* : C^*(\mathfrak{g}; M^{ad}) \to CH^2(U\mathfrak{g}; M)$ of proposition 3.1.2.2 can be seen as an infinitesimal version of the integration map desribed in degree 2 in [Nee04] and [Cov10], which assign to every 2-cocycle of Lie algebra $\omega : \Lambda^2\mathfrak{g} \to \mathbb{R}$, a 2-cocycle of group $I_s(\omega) : G^{\times 2} \to \mathbb{R}$, smooth in a neighbourhood of the neutral element e of G. Here, G is a Lie (or Fréchet-Lie) group with Lie algebra $\mathfrak{g}$.

To simplify formulas, we restrict to trivial bimodule $M = \mathbb{R}$ on which the action of $U\mathfrak{g}$ is given by

$$xm = mx = \epsilon(x)m$$

for all x in $U\mathfrak{g}$ and m in $\mathbb{R}$. The left $\mathfrak{g}$-module on $\mathbb{R}^{ad}$ then satisfies

$$g \cdot m = 0$$

for all g in $\mathfrak{g}$ and m in $\mathbb{R}$.

Let's now recall the simplicial integration of Lie 2-cocycles "à la van Est" when $\mathfrak{g}$ is finite dimensionnal.

3.2.1 Integration of Lie algebra cocycles in group cocycles

Let G be a Lie group with neutral element e and Lie algebra $\mathfrak{g}$. For $n \geq 1$, denote by $C^n_{loc}(\mathfrak{g};\mathbb{R})$ the $\mathbb{R}$-vector space of smooth group n-cochains, defined in a neighbourhood of e, with values in the trivial G module $\mathbb{R}$. We refer to [Nee04] and appendix B. for a definition of $C^n_{loc}(\mathfrak{g};\mathbb{R})$ and of the associated differential $d_G : C^n_{loc}(\mathfrak{g};\mathbb{R}) \to C^{n+1}_{loc}(\mathfrak{g};\mathbb{R})$. Let $T : C^n_s(G;\mathbb{R}) \to C^n(\mathfrak{g};\mathbb{R})$ be the derivation map defined by

$$T(f)(g_1 \wedge \cdots \wedge g_n) := \sum_{\sigma \in \Sigma_n} \mathrm{sgn}(\sigma)(g_{\sigma(1)} \otimes \cdots \otimes g_{\sigma(n)})(f)$$

for all local smooth group n-cochain $f : G^{\times n} \to \mathbb{R}$, an for all g_1, ..., g_n in $\mathfrak{g} \subset U\mathfrak{g}$. Here $(g_1 \otimes g_2 \otimes \cdots g_n)$ is considered as a punctual distribution supported at $(e, \cdots, e)$ on $G^{\times n}$ via the isomorphism:

$$U(\mathfrak{g}^{\oplus n}) \cong (U\mathfrak{g})^{\otimes n}$$

For example, when f is 2-cochain in $C^2(\mathfrak{g};\mathbb{R})$ we have

$$T(f)(g_1 \wedge g_2) = (g_1 \otimes g_2 - g_2 \otimes g_1)(f) = d^2_{(e,e)}f((g_1, 0), (0, g_2)) - d^2_{(e,e)}f((g_2, 0), (0, g_1))$$

Proposition 3.2.1.1. *The map* $T : (C^n_{loc}(G;\mathbb{R}), d_G)_{n>0} \to (C^n(\mathfrak{g};\mathbb{R}), d_{CE})_{n>0}$ *is a morphism of cochain complexes.*

We now want to describe the image of T : Given a Chevalley-Eilenberg cochain $f : \Lambda^n\mathfrak{g} \to \mathbb{R}$ is it possible to build a local smooth group n-cochain $\tilde{G}^n(f) : G^{\times n} \to \mathbb{R}$ whose image under T is f ? When $n = 2$, the answer is given by van Est's method.

The dual/left-invariant version of the injective morphism $R : C_*(\mathfrak{g}) \to \Lambda^* G$ of proposition 2.1.1.8, studied for example in [Nee04] and [FOT08], is the isomorphism of cochain complexes $R' : C^*(\mathfrak{g};\mathbb{R}) \overset{\cong}{\to} \Omega^*_{inv}(G)$ defined by

$$R'(\omega)_z(X_1(z), X_2(z), \cdots, X_n(z)) := \omega^{inv}_z(X_1(z), X_2(z), \cdots, X_n(z)) := \omega(T_z L_{z^{-1}} X_1(z), \cdots, T_z L_{z^{-1}} X_n(z))$$

for all z in U, for all X_1, ..., X_n vector fields on G, and for all Chevalley-Eilenberg 2-cochain $\omega : \mathfrak{g} \wedge \mathfrak{g} \to \mathbb{R}$. Here, $\Omega^*_{inv}(G)$ is the complex of left invariant differential forms[1]. Note that the

[1] i.e forms ω such that $L^*_z\omega = \omega$ for all z in G.

inverse of R' is nothing but the evaluation map ev_e at e. Now, let V be a convex open subset of $\mathfrak{g} = \mathbb{R}^n$ containing 0, equipped with a diffeomorphism $\varphi : U \to V$ sending e to 0, where U is an open subset of G containing e. The following lemma gives a section of T :

Lemma 3.2.1.2. [Lemme V.2 de [Nee04], version simpliciale] *If U' is an open subset of U such that $U'U' \subset U$ and $\varphi^{-1}(U)$ is a convex open subset of $\mathfrak{g}$ containing 0, and if $T_0\varphi^{-1} = \mathrm{Id}_{\mathfrak{g}}$, then the map*

$$\begin{array}{rcl} I_s : C^2(\mathfrak{g};\mathbb{R}) & \to & C^2_{loc}(G;\mathbb{R}) \\ \omega & \mapsto & \left((z_1,z_2) \mapsto \int_{\Delta^2} \gamma^*_{z_1,z_2}\omega^{inv}\right) \end{array}$$

where for every (z_1, z_2) in $U' \times U'$, $\gamma_{z_1,z_2} : \Delta^2 \to U \subset G$ is the smooth 2-simplex defined by

$$\gamma_{z_1,z_2}(s,t) := \varphi\left(s\,\varphi^{-1}\left(z_1\varphi(t\varphi^{-1}(z_2))\right) \;+\; t\,\varphi^{-1}\left(z_1\varphi((1-s)\varphi^{-1}(z_2))\right)\right)$$

is left inverse to T i.e.

$$T \circ I_s = \mathrm{Id}_{C^*(\mathfrak{g};\mathbb{R})}$$

Moreover, I_s sends Lie algebra 2-cocycles (resp. 2-cocoundaries) to group 2-cocyles (resp.group coboudaries).

We now provide a cubical analogue of this lemma, based on the choice of particular n-cubes in G :

Definition 3.2.1.3. *Let $U_n \subset U$ be an open subset whose preimage by φ is a star shapen open of $\mathfrak{g}$ centered in 0 and such that $\overbrace{U_nU_n\cdots U_n}^{n \text{ times}} \subset U$. For all $(z_1,\cdots,z_n)$ in $U_n^{\times n}$, $\gamma^n_{z_1,\cdots,z_n} : [0,1]^n \to U \subset G$ is the smooth n-cube defined by unduction on n by*

$$\gamma^n_{z_1,\cdots,z_n}(t_1,\cdots,t_n) := \varphi_{t_1}\left(z_1\gamma^{n-1}_{z_1,\cdots,z_{n-1}\varphi_{t_n}(z_n)}(t_1,\cdots,t_{n-1}\right) \quad n>1 \tag{3.2.1}$$

and

$$\gamma^1_z(t) := \varphi_t(z) \tag{3.2.2}$$

where, for all $0 \le t \le 1$, $\varphi_t : U \to U$ is the smooth map defined by

$$\varphi_t(z) := \varphi^{-1}(t\varphi(z)) \tag{3.2.3}$$

for all z in U.

Remark 3.2.1.4. *The map φ_t defined in (3.2.3) coincides with the map φ_t of proposition 2.1.2.8 when one chooses φ to be the logarithm.*

Lemma 3.2.1.5. [Lemme V.2 de [Nee04], cubical version] *The map*

$$\begin{array}{rcl} I_c : C^*(\mathfrak{g};\mathbb{R}) & \to & C^*_{loc}(G;\mathbb{R}) \\ \omega \in C^n(\mathfrak{g};\mathbb{R}) & \mapsto & \left((z_1,\cdots,z_n) \mapsto \int_{[0,1]^n} (\gamma^n_{z_1,\cdots,z_n})^*\omega^{inv}\right) \in C^n_{loc}(G;\mathbb{R}) \end{array}$$

is a morphism of cochain complexes, right inverse to T i.e.

$$I_c \circ d_{CE} = d_G \circ I_c \tag{3.2.4}$$

and

$$T \circ I_c = \mathrm{Id}_{C^*(\mathfrak{g};\mathbb{R})} \tag{3.2.5}$$

To make the link between the integration map I_c and the morphism of complexes G^* defined in 3.1.2.2 more precise, let's introduce operators Γ et B defined as follows:

Definition 3.2.1.6. *Let n be an integer, j be an element of $\{1,\cdots,n\}$, and t_1, ..., t_n ve real numbers in $[0,1]$. The operators $\Gamma_{t_1,\cdots,t_n} : U\mathfrak{g}^{\otimes n} \to U\mathfrak{g}$ and $B^j_{t_1,\cdots,t_n} : U\mathfrak{g}^{\otimes n} \to U\mathfrak{g}$ are defined by*

$$\Gamma_{t_1,\cdots,t_n}(x_1,\cdots,x_n) := \Gamma_{t_1,\cdots,t_n}(x_1\otimes\cdots\otimes x_n) := \phi_{t_1}\left(x_1\phi_{t_2}\big(x_2\phi_{t_3}(\cdots\phi_{t_{n-1}}(x_{n-1}\phi_{t_n}(x_n))\cdots)\big)\right) \tag{3.2.6}$$

for all x_1, ..., x_n in $U\mathfrak{g}$, and

$$B^j_{t_1,\cdots,t_n} := \Gamma_{-t_1,t_2,\cdots,t_n} \star \frac{\partial}{\partial t_j}\Gamma_{t_1,\cdots,t_n} \tag{3.2.7}$$

Here, the convolution product $\star$ on $\mathrm{Hom}_{\mathbb{R}}(U\mathfrak{g}^{\otimes n},, U\mathfrak{g})$ is the one associated to the coproduct $x_1\otimes\cdots\otimes x_n \mapsto (x_1^{(1)}\otimes\cdots x_n^{(1)})\otimes(x_1^{(2)}\otimes\cdots\otimes x_n^{(2)})$ on $U\mathfrak{g}^{\otimes n}$.

Remark 3.2.1.7. *$\Gamma_{t_1,\cdots,t_n}$ is the morphism of punctual distributions coalgebras $(\gamma^n(t_1,\cdots,t_n))_* : U\mathfrak{g}^{\otimes n} \to U\mathfrak{g}$ induced (in the sens of 2.1.2.4) by the smooth map $\gamma^n(t_1,\cdots,t_n) : U'^{\times n} \to U$ that sends each $(z_1,\cdots,z_n)$ to $\gamma^n_{z_1,\cdots,z_n}(t_1,\cdots,t_n)$.*

The next proposition, linking T', I_c and G^*, generalizes formulae given for G^1 and G^2 in 3.1.2 and provides a closed formula for G^n for arbitrary n.

Proposition 3.2.1.8. *If $\varphi : U \to V$ is the logarithm map defined on an openU of G with values a convex open V of $\mathfrak{g}$ containing 0, the the morphism of cochain complexes $G^* : C^*(\mathfrak{g};\mathbb{R}) \to CH^*(U\mathfrak{g};\mathbb{R})$ satisfies*

$$G^* = T' \circ I_c \tag{3.2.8}$$

i.e. the image by G^n of a n-cochain ω in $C^n(\mathfrak{g};\mathbb{R})$ satisfies

$$G^n(\omega)(x_1\otimes\cdots\otimes x_n) = \int_{[0,1]^n} dt_1\cdots dt_n\, \omega\left(B^1_{t_1,\cdots,t_n}(x_1^{(1)},\cdots,x_n^{(1)}),\cdots,B^n_{t_1,\cdots,t_n}(x_1^{(n)},\cdots,x_n^{(n)})\right) \tag{3.2.9}$$

The proof of the preceeding proposition is a direct consequence of the following lemma, which provides a **general** closed formula to compute G^* for for an arbitrary bimodule M:

Lemma 3.2.1.9. *Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{R}$ and M be a $U\mathfrak{g}$-bimodule. Then, 3.2.1.8, for all integer n and for all x, y, x_1, ..., x_n in $U\mathfrak{g}$,*

$$\begin{aligned} G^B_n(x\otimes x_1\otimes\cdots\otimes x_n\otimes y) = \int_{[0,1]^n} dt_1\cdots dt_n\, x\Gamma_{t_1,\cdots,t_n}(x_1^{(1)},\cdots,x_n^{(1)})\otimes B^1_{t_1,\cdots,t_n}(x_1^{(2)},\cdots,x_n^{(2)})\wedge\cdots \\ \cdots\wedge B^n_{t_1,\cdots,t_n}(x_1^{(n+1)},\cdots,x_n^{(n+1)})\otimes\Gamma_{-t_1,t_2,\cdots,t_n}(x_1^{(n+2)},\cdots,x_n^{(n+2)})x_1^{(n+3)}x_2^{(n+3)}\cdots x_n^{(n+3)}y \end{aligned} \tag{3.2.10}$$

Concluons cette sous-section en en résumant les principaux résultats établis à la commutativité du diagramme Finally, we can summarize the results of this subsection in the commutativity of the following diagram

$$\begin{array}{ccccc} & & \xrightarrow{\quad\quad\mathrm{Id}\quad\quad} & & \\ C^*(\mathfrak{g};\mathbb{R}) & \xrightarrow{I_c} & C^*_{loc}(G;\mathbb{R}) & \xrightarrow{T} & C^*(\mathfrak{g};\mathbb{R}) \\ & \searrow^{G^*} & \downarrow T' & \nearrow_{F^*} & \\ & & CH^*(U\mathfrak{g};\mathbb{R}) & & \end{array} \tag{3.2.11}$$

in the category of cochain complexes.

3.2.2 Algebraic version of the integration map

Malcev completion of $U\mathfrak{g}$

Denote by $I := \mathrm{Ker}\epsilon$ the augmentation ideal of $U\mathfrak{g}$, I^n its n-th power, and consider the inverse system of projections

$$\mathbb{R} = U\mathfrak{g}/_I \twoheadleftarrow U\mathfrak{g}/_{I^2} \twoheadleftarrow U\mathfrak{g}/_{I^3} \twoheadleftarrow \cdots \twoheadleftarrow U\mathfrak{g}/_{I^n} \twoheadleftarrow U\mathfrak{g}/_{I^{n+1}} \twoheadleftarrow \cdots$$

Definition 3.2.2.1. *The* ***completion*** *of $U\mathfrak{g}$, denoted by $\hat{U\mathfrak{g}}$, is defined by*

$$\hat{U\mathfrak{g}} := \lim_{\substack{\leftarrow \\ n}} U\mathfrak{g}/_{I^n}$$

Proposition 3.2.2.2. *The connected cocommutative Hopf algebra structure $(\mu, \eta, \Delta, \epsilon, S)$ on $U\mathfrak{g}$ induces a onnected cocommutative Hopf algebra structure on $\hat{U\mathfrak{g}}$ denoted in the same way. The primitive elements of $\hat{U\mathfrak{g}}$ assemble in a Lie subalgebra $\hat{\mathfrak{g}}$ of $\hat{U\mathfrak{g}}$ which contains $\mathfrak{g}$.*

Definition 3.2.2.3. *The* ***Malcev group*** *associated to $\mathfrak{g}$, denoted by $\hat{G}$, is the subset of $1+\hat{I} \subset \hat{U\mathfrak{g}}$ consisting of* ***grouplike*** *elements x that is elements verifing:*

$$\Delta x = x \otimes x \in \hat{U\mathfrak{g}} \hat{\otimes} \hat{U\mathfrak{g}}$$

Proposition 3.2.2.4. *The exponential map*

$$\begin{array}{rcl} \exp := \hat{\mathfrak{g}} & \to & \hat{G} \\ g & \mapsto & \exp(g) := e^g := \sum_{n\geq 0} \frac{1}{n!} g^n \end{array}$$

is a bijection.

The inclusion

$$\hat{G} \hookrightarrow \hat{U\mathfrak{g}}$$

induces, by universal property of the group algebra , a morphism of associative algebras

$$\mathbb{R}\hat{G} \to \hat{U\mathfrak{g}}$$

and thus a morphism of cochain complexes

$$Q : CH^*(\hat{U\mathfrak{g}};\mathbb{R}) \to C^*(\hat{G};\mathbb{R}) = CH^*(\mathbb{R}\hat{G};\mathbb{R})$$

The obvious map $U\mathfrak{g} \to \hat{U\mathfrak{g}}$ induces

$$P : CH^*(\hat{U\mathfrak{g}};\mathbb{R}) \to CH^*(U\mathfrak{g};\mathbb{R})$$

The situation is then

$$\begin{array}{ccc} C^*(\hat{G};\mathbb{R}) & & \\ {\scriptstyle Q}\uparrow & & \\ CH^*(\hat{U\mathfrak{g}};\mathbb{R}) & & C^*(\mathfrak{g};\mathbb{R}) \\ {\scriptstyle P}\downarrow & \overset{F^*}{\nearrow}\ \underset{G^*}{\swarrow} & \\ CH^*(U\mathfrak{g};\mathbb{R}) & & \end{array} \tag{3.2.12}$$

To define the maps T and I_c of the preceeding subsection, we had to restrict to local smooth group cochains. The goal of the following subsection is to give an analogue of smoothness in the algebraic framework.

Continuous cochains

One can define a topology on $\mathfrak{g}$ and $U\mathfrak{g}$ making the inclusion $\mathfrak{g} \hookrightarrow U\mathfrak{g}$ continuous thanks to the augmentation ideal I. A basis of neighbourhood of 0 in $U\mathfrak{g}$ is given by the powers I^k of I et l'on obtient une base de voisinages de tout autre point par translation. The tensor product $U\mathfrak{g}^{\otimes n}$ can be equipped with a product topology for which a basis of neigbourhoods of O is given by the powers J^k , $k \geq 0$ of the augmentation ideal[2] J defined as follows

$$J := I \otimes U\mathfrak{g}^{\otimes(n-1)} + U\mathfrak{g} \otimes I \otimes U\mathfrak{g}^{\otimes(n-2)} + \cdots + U\mathfrak{g}^{(n-1)} \otimes I \subset U\mathfrak{g}^{\otimes n} \tag{3.2.13}$$

This topology on $U\mathfrak{g}$ and its tensor powers is called **I-adic topology**.

Proposition 3.2.2.5. *Suppose that $\mathbb{R}$ is given the usual topology. Then, a Hochschild n-cochain $\omega : U\mathfrak{g}^{\otimes n} \to \mathbb{R}$ is said**continuous** with respect to the I-adic topology if there exists an integer r such that*

$$\omega(J^k) = \{0_{\mathbb{R}}\}$$

In the case of $\mathfrak{g}$, there is also a natural filtration:

Definition 3.2.2.6. *The **lower central filtration** associated to $\mathfrak{g}$ is the decreasing sequence of ideals*

$$D_1(\mathfrak{g}) := [\mathfrak{g},\mathfrak{g}] \subset \cdots \subset D_k(\mathfrak{g}) \subset D_{k+1}(\mathfrak{g}) \subset \cdots$$

[2] Where the augmentation is $\epsilon^{\otimes n} : U\mathfrak{g}^{\otimes n} \to \mathbb{R}^{\otimes n} \cong \mathbb{R}$.

defined by

$$D_k(\mathfrak{g}) := [D_{k-1}(\mathfrak{g}), \mathfrak{g}]$$

for all $k \geq 2$.

$\mathfrak{g}$ *is said to be* ***nilpotent****when* $D_k(\mathfrak{g}) = \{0\}$ *for some integer* k.

Declaring this sequence to be a basis of neignbourhoods of 0 *defines a topology on* $\mathfrak{g}$ *called the* D*-adic topology. One can extend this topology to every* $\Lambda^n\mathfrak{g}$ *to define the graded vector subspace* $C_c^n(\mathfrak{g};\mathbb{R})$ *of the Chevalley-Eilenberg complex of* $\mathfrak{g}$ *consisting of Lie algbra cochains that are continous with respect to the* D*-adic topology.*

Proposition 3.2.2.7. *The differentials* d_H *et* d_{CE} *restricts to the subspaces defined above to give* $(CH_c^*(U\mathfrak{g};\mathbb{R}), d_H)$ *et* $(C_c^*(\mathfrak{g};\mathbb{R}), d_{CE})$ *which subcomplexes respectively of* $(CH^*(U\mathfrak{g};\mathbb{R}), d_H)$ *and of* $(C^*(\mathfrak{g},\mathbb{R}), d_{CE})$.

The link between I and D adic topologies is given by the following proposition:

Proposition 3.2.2.8. *The* D*-adic topology is the one induced by the* I*-adic topology on* $U\mathfrak{g}$ *via the inclusion* $\mathfrak{g} \hookrightarrow U\mathfrak{g}$.

The previous proposition relies on the following lemma

Lemma 3.2.2.9. *The canonical projection* $\mathrm{pr} : U\mathfrak{g} \to \mathfrak{g}$ *is continuous. As a consequence, the morphism of coalgebras* $\phi_t : U\mathfrak{g} \to U\mathfrak{g}$ *and all its derivatives* $\frac{d^n}{dt^n}\phi_t$ *are also continous.*

Proposition 3.2.2.10. *Les morphisms* $F^* : CH^*(U\mathfrak{g};\mathbb{R}) \to C^*(\mathfrak{g};\mathbb{R})$ *and* $G^* : C^*(\mathfrak{g};\mathbb{R}) \to CH^*(U\mathfrak{g};\mathbb{R})$ *restrict to the subcomplexes of continuous cochains to give an equivalence*

$$CH_c^*(U\mathfrak{g};\mathbb{R}) \underset{G^*}{\overset{F^*}{\rightleftarrows}} C_c^*(\mathfrak{g};\mathbb{R})$$

Let's now precise the topology on the completed algebra $\hat{U\mathfrak{g}}$.

Definition 3.2.2.11. *The*$\hat{I}$*-adic topology on* $\hat{U\mathfrak{g}}$ *is the one generated by the basis of neighbourhood of zero given by the powers of the augmentation ideal* $\hat{I}$ *with*

$$\hat{I} := \lim_{\substack{\leftarrow \\ n}} I/_{I^n} \subset \hat{U\mathfrak{g}}$$

The completed tensor powers[3] $\hat{U\mathfrak{g}}^{\hat{\otimes} n}$ *of* $\hat{U\mathfrak{g}}$ *can be topologized in the same way as above. Continuous Hochschild cochains* $\omega : \hat{U\mathfrak{g}}^{\hat{\otimes} n} \to \mathbb{R}$ *form a cochain complex* $CH_c^*(\hat{U\mathfrak{g}};\mathbb{R})$ *with differential* d_H *obtained by completion of the usual* d_H.

Proposition 3.2.2.12. *The restriciton morphism* $P : CH^*(\hat{U\mathfrak{g}};\mathbb{R}) \to CH^*(U\mathfrak{g};\mathbb{R})$ *induces an isomorphism of cochain complexes*

$$P : CH_c^*(\hat{U\mathfrak{g}};\mathbb{R}) \overset{\cong}{\to} C_c^*(U\mathfrak{g};\mathbb{R})$$

[3]See [Qui69]

Thus, (3.2.12) can be enriched in

$$\begin{array}{ccc}
C^*(\hat{G};\mathbb{R}) & & \\
\uparrow{\scriptstyle Q} & & \\
CH^*_c(\hat{U}\mathfrak{g};\mathbb{R}) & & C^*_c(\mathfrak{g};\mathbb{R}) \\
{\scriptstyle P^{-1}}\uparrow\downarrow{\scriptstyle P} & \overset{F^*}{\nearrow}\ \underset{G^*}{\swarrow} & \\
CH^*_c(U\mathfrak{g};\mathbb{R}) & &
\end{array} \tag{3.2.14}$$

The last step deals with the notion of smoothness at $1 \in \hat{G}$:

Definition 3.2.2.13. *Let p be an integer. A n-cochain of group $f : G^{\otimes n} \to \mathbb{R}$ is said* ***p-smooth at*** *1 when there exist p multilinear and symmetric maps*

$$D^i f : (\hat{\mathfrak{g}}^{\oplus n})^{\hat{\otimes} i} \to \mathbb{R} \quad , i \in \{1, \cdots , p\} ,$$

and a continuous map $O : \hat{\mathfrak{g}}^{\oplus n} \to \mathbb{R}$, satisfying conditions

1. $$f(e^{g_1}, \cdots , e^{g_n}) = f(1, \cdots , 1) + \sum_{i=1}^{n} D^i f\big((g_1, \cdots , g_n) \otimes \cdots \otimes (g_1, \cdots , g_n)\big) + O(g_1, \cdots , g_n) \tag{3.2.15}$$
   for all $(g_1, \cdots , g_n)$ in $\hat{\mathfrak{g}}^{\oplus n}$,

2. *and*
   $$\lim_{t \to 0} \frac{1}{t^p} O(g(t)) = 0 \tag{3.2.16}$$
   for all polynomial arc $g : [-1, 1] \to \hat{\mathfrak{g}}^{\oplus n}$ such that $g(0) = 0$.

The map $D^i f$ is called ***the i-th différential*** *of f. A* ***smooth*** *cochain is a cochain which is p-smooth at 1 for all p.*

Proposition 3.2.2.14. *Smooth cochains at 1 on $\hat{G}$ form a subcomplex of $C^*(\hat{G};\mathbb{R})$, denoted by $C^*_s(\hat{G};\mathbb{R})$. Moreover, the restriction map $Q : CH^*(\hat{U}\mathfrak{g};\mathbb{R}) \to C^*(\hat{G};\mathbb{R})$ restricts to*

$$Q : CH^*_c(\hat{U}\mathfrak{g};\mathbb{R}) \to C^*_s(\hat{G};\mathbb{R})$$

Definition 3.2.2.15. *La version algébrique de l'application d'intégration est le morphisme de complexes de cochaînes $I_c : C^*_c(\mathfrak{g};\mathbb{R}) \to C^*_s(\hat{G};\mathbb{R})$ défini par*

$$I_c := Q \circ P^{-1} \circ G^*$$

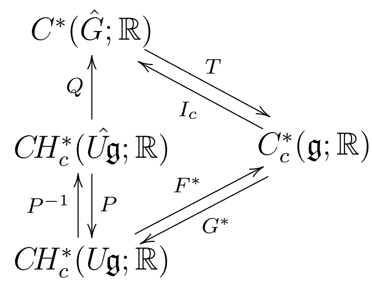
Proposition 3.2.2.16. *The map $T'' : C_s^*(\hat{G};\mathbb{R}) \to C_c^*(\hat{\mathfrak{g}};\mathbb{R})$, defined in degree n by*

$$T''(f)(g_1 \wedge \cdots \wedge g_n) := n! \sum_{\sigma \in \Sigma_n} \mathrm{sgn}(\sigma) D^n f((g_{\sigma(1)}, 0, \cdots, 0) \hat{\otimes} \cdots \hat{\otimes} (0, \cdots, 0, g_{\sigma(n)}))$$

for all g_1, ..., g_n in $\hat{\mathfrak{g}}$, is a morphism of cochain complexes inducing, via the canonical map $\mathfrak{g} \to \hat{\mathfrak{g}}$, a morphism of cochain complexes $T : C_s^*(\hat{G};\mathbb{R}) \to C_c^*(\mathfrak{g};\mathbb{R})$. Moreover

$$T \circ I_c = \mathrm{Id}_{C_c^*(\hat{\mathfrak{g}};\mathbb{R})}$$

We finally established the existence of morphism of complexes T et I_c such that



is commutative.

The nilpotent case

When $\mathfrak{g}$ is nilpotent, we have the following result, due to P.F Pickel ([Pic78]):

Proposition 3.2.2.17. *If $\mathfrak{g}$ is a nilpotent Lie algebra over $\mathbb{Q}$, then the morphisms*

$$P : CH^*(\hat{U}\mathfrak{g};\mathbb{R}) \to CH^*(U\mathfrak{g};\mathbb{R})$$

and

$$Q : CH^*(\hat{U}\mathfrak{g};\mathbb{R}) \to C^*(\hat{G};\mathbb{R})$$

are quasi-isomorphisms.

Remark 3.2.2.18. *When $\mathfrak{g}$ is nilpotent,*

1. *(a) $\hat{\mathfrak{g}} = \mathfrak{g}$ and $\hat{G} = G = \{e^g\ ,\ g \in \mathfrak{g}\}$ ([Qui69]),*

   (b) Every Chevalley-Eilenberg cochain is continuous.

2. *In [Tam03], D. Tamarkin uses the fact that when $H^*(\mathfrak{g};\mathbb{R})$ has finite type, the restriction map $P : CH^*(\hat{U}\mathfrak{g};\mathbb{R}) \to CH^*(U\mathfrak{g};\mathbb{R})$ is necessarily a quasi-isomorphism.*

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Résumé : Le but de ce travail est d'expliquer en quoi l'application de d'antisymétrisation de Cartan-Eilenberg F^*, qui permet d'identifier la cohomologie de Chevalley-Eilenberg d'une algèbre de Lie $\mathfrak{g}$ à la cohomologie de Hochschild de son algèbre enveloppante $U\mathfrak{g}$, est l'analogue algébrique de l'application usuelle de dérivation de cochaînes de groupe lisses au voisinage de l'élément neutre d'un groupe de Lie, et comment un de ses quasi-inverses peut être construit et compris comme une application d'intégration de cocycles de Lie. De plus, nous montrons qu'un tel quasi-inverse, bien que provenant d'une contraction d'origine géométrique, peut s'écrire de manière totalement intinsèque, en n'utilisant que la structure d'algèbre de Hopf cocommutative connexe sur $U\mathfrak{g}$.

Mots clés : Algèbre de Hopf - (co)homologie de Hochschild - algèbre de Lie - (co)homologie de Chevalley-Eilenberg - (co)homologie de groupe - homotopie - complétion I-adique - exponentielle

Summary : This thesis aims at explaining why Cartan and Eilenberg's antisymmetrisation map F^*, which provides an explicit identification between the Chevalley-Eilenberg cohomology of a free lie algebra $\mathfrak{g}$ and the Hochschild cohomology of its universal enveloping algebra $U\mathfrak{g}$, can be seen as an algebraic analogue of the well-known derivation map from the complex of locally smooth group cochains to the one of Lie algebra cochains, and how one of its quasi-inverses can be built and thought of as an integration of Lie algebra cochains in Lie group cochains process. Moreover, we show that such a quasi-inverse, even if it is defined thanks to a Poincaré contraction coming from geometry, can be written using a totally intrinsic formula that involves only the connex cocommutative Hopf algebra structure on $U\mathfrak{g}$.

Key words : Hopf algebra - Hochschild (co)homology - Lie algebra - Chevalley-Eilenberg (co)homology - group (co)homology - homotopie - I-adic completion - exponential map