Geometric Littelmann path model, Whittaker functions on Lie groups and Brownian motion

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24 January 2013

Sommaire

Preamble

Theorem (Discrete Pitman(1975))

Let B_n a standard random walk on $\mathbb{Z}$. Then:

$$X_n = B_n - 2 \inf_{0 \leq k \leq n} B_k$$

is Markov with transition kernel:

$$Q(x, x+1) = \frac{1}{2} \frac{x+2}{x+1} \quad Q(x, x-1) = \frac{1}{2} \frac{x}{x+1}$$

Moreover (intertwining measure):

$$\mathbb{P}\left(B_n = b | X_n = x\right) = \mathcal{U}\left(\{-x, \dots, x-4, x-2, x\}\right)$$

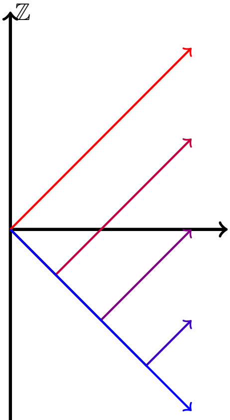
Comments

- Strange as $-\inf_{0 \leq k \leq n} B_k$ is a typical example of non-Markovian behavior.
- Very strong rigidity.

Representation theoretic explanation (1):

There is a representation-theoretic story to give here ($2 = \alpha(\alpha^\vee)$).
Consider the Lie algebra $\mathfrak{sl}_2$. For any $n \in \mathbb{N}$, highest weight, there is an irreducible representation $V(n)$ of dimension $n+1$.
$V(n) \rightsquigarrow \mathcal{B}(n)$ a crystal = a combinatorial object that can be realized as paths thanks to the Littelmann path model.

Figure: $\mathfrak{sl}_2$ path crystal with highest weight $n = 4$



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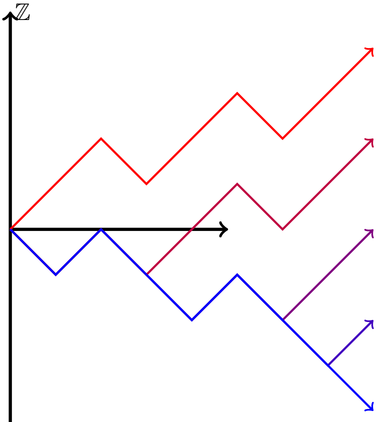

Representation theoretic explanation (2)

The Pitman transform

$$\mathcal{P} : \pi \mapsto \pi(t) - 2 \inf_{0 \leq s \leq t} \pi(s)$$

has a special interpretation in the context of the Littelmann path model: It gives the dominant path in a crystal.

Let $V(1) = \mathbb{C}^2$ be the standard representation of $\mathfrak{sl}_2$.

- Looking at the standard random walk B_n can be seen as following a weight vector in $V(1)^{\otimes n}$.

- Looking at its Pitman transform X_n means following a highest weight in a decomposition of $V(1)^{\otimes n}$ into irreducibles. The transition probabilities are given by the Clebsch-Gordan rule:

$$V(n) \otimes V(1) \approx V(n+1) \oplus V(n-1)$$

Conclusion: Pitman's theorem is about the Markov property of **a highest weight process**.

Notations

Let:

- $\mathfrak{g}$ be a complex semi-simple Lie algebra (E.g $\mathfrak{g} = \mathfrak{sl}_n$)
- $\mathfrak{h}$ a maximal abelian subalgebra - the Cartan subalgebra (typically diagonal matrices).
- (Root space decomposition) There is a root system $\Phi = \Phi^+ \sqcup -\Phi^+ \subset \mathfrak{h}^*$ such that:

  $$\mathfrak{g} = \mathfrak{h} \bigoplus_{\alpha \in \Phi^+} (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha})$$

  where:

  $$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid \forall h \in \mathfrak{h}, [h, x] = \alpha(h)x\}$$
- Define the real Cartan subalgebra as the subspace of $\mathfrak{h}$ where the roots are real.

  $$\mathfrak{h} = \mathfrak{a} + i\mathfrak{a}$$
- (Coroots) When identifying $\mathfrak{h}^*$ and $\mathfrak{h}$, coroots are defined as:

  $$\alpha^\vee = \frac{2\alpha}{(\alpha, \alpha)} \in \mathfrak{h}$$
- On every representation V, $\mathfrak{h}$ acts by simultaneously diagonalizable endomorphisms. Hence a weight space decomposition:

  $$V = \bigoplus_{\mu \in P} V_\mu$$

  where

  $$V_\mu = \{v \in V \mid \forall h \in \mathfrak{h}, h \cdot v = \mu(h)v\}$$

A group generalization by BBO

- There is a continuous version of the Littelmann path model, based on paths on $\mathfrak{a} \approx \mathbb{R}^n$ the real part of a Cartan subalgebra ($\mathfrak{a} \subset \mathfrak{g}$ a semi-simple Lie algebra).
- Consider simple roots $\Delta = \{\alpha\}_{\alpha \in \Delta}$ and define Pitman transforms for paths on $\mathfrak{a}$ for every direction:

$$\mathcal{P}_\alpha \pi(t) = \pi(t) - \inf_{0 \leq s \leq t} \alpha\left(\pi(s)\right) \alpha^\vee$$

- For $\mathbf{i} = (i_1, \ldots, i_m) \in R(w_0)$, a reduced expression for the longest element w_0 in the Weyl group:

$$\mathcal{P}_{w_0} = \mathcal{P}_{\alpha_{i_1}} \circ \mathcal{P}_{\alpha_{i_2}} \circ \cdots \circ \mathcal{P}_{\alpha_{i_m}}$$

  is well defined.
- $\mathcal{P}_{w_0}$ applied to an (irreducible) path crystal gives the highest weight element.
- Analogue of Pitman's theorem: If W is a Brownian motion then $\mathcal{P}_{w_0} W$ is Markov.

The starting point of this thesis

Theorem (Matsumoto-Yor(2000): "Geometric SL_2" Pitman)

Let $B^{(\mu)}$ a Brownian motion with drift μ. Then:

- $X_t^{(\mu)} = B_t^{(\mu)} + \log\left(\int_0^t e^{-2B_s^{(\mu)}} ds\right)$ *is Markov with inf. generator*

$$K_\mu^{-1}\left(\frac{1}{2}\frac{d^2}{dx^2} - e^{-2x} - \frac{\mu^2}{2}\right)K_\mu$$

- $K_\mu(x)$ *satisfies the eigenfunction equation:*

$$\frac{1}{2}\frac{d^2}{dx^2}K_\mu - e^{-2x}K_\mu = \frac{\mu^2}{2}K_\mu$$

- *Intertwining measure: Conditionnally of $X_t^{(\mu)} = x$, $B_t^{(\mu)}$ is distributed as a Generalized Inverse Gaussian law.*

And a generalization to the GL_n case, by O'Connell (2009).

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Question

Representation theoretic explanations of these theorems? Generalization to other groups in the spirit of BBO?

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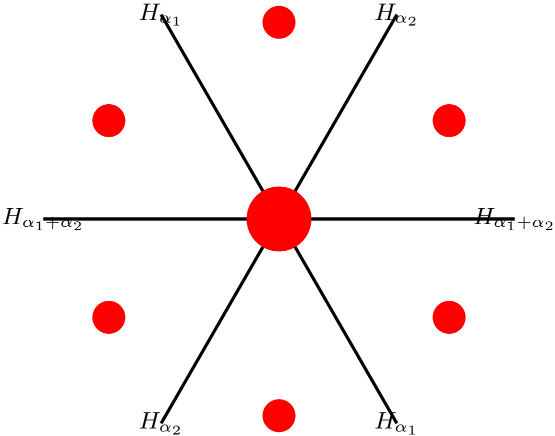
Example: The adjoint representation of $\mathfrak{sl}_3$

Let us start by the weight diagram of the adjoint representation

Figure: Weight diagram for adjoint representation of $\mathfrak{sl}_3$ (dim = 8)



Crystal = "Shadow" of a perfect basis, adapted to the weight space decomposition.

Crystal example:

The crystal graph: Oriented graph with vertices being crystal elements, and arrows being the root operators.

Figure: Crystal of tableaux associated to adjoint representation for $\mathfrak{sl}_3$

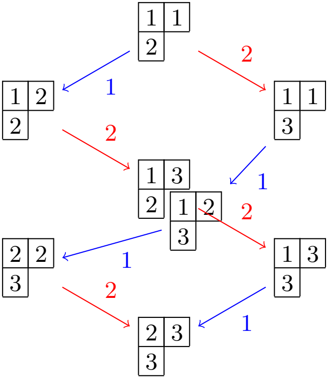
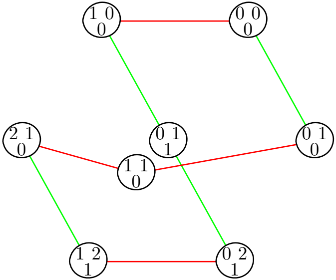


Figure: Crystal graph associated to adjoint representation for $\mathfrak{sl}_3$



Definition of combinatorial crystals

Let V be a representation of $\mathfrak{g}$. $V \rightsquigarrow \mathcal{B}$, with $\mathcal{B}$ a combinatorial object with structural maps:

- A weight map: $\gamma : \mathcal{B} \to \mathfrak{h}^*$
- Distance to edges: $\varepsilon_\alpha, \varphi_\alpha : \mathcal{B} \to \mathbb{N}$
- Root operators $e_\alpha, f_\alpha : \mathcal{B} \to \mathcal{B} \bigcup \emptyset$

It is known that irreducible representations are highest weight modules $V(\lambda)$. And:

$$V(\lambda) \rightsquigarrow \mathcal{B}(\lambda)$$

Additional properties:

- Correspondence between connected components of the crystal graph and irreducible subrepresentation of V.
- Tensor product operation on crystals such that: $V \otimes V' \rightsquigarrow \mathcal{B} \otimes \mathcal{B}'$

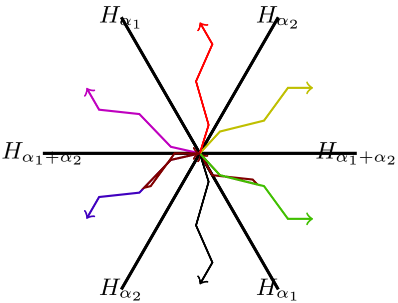
Littelmann's path model: Definitions

Littelmann realized crystals as a set of paths on $\mathfrak{a}^*$ endowed with the same operations.
Notation: $\langle \pi \rangle$ connected crystal generated by the path π.

Moreover, we have following properties:

- The weight is the endpoint.
- $$\langle \pi \rangle \otimes \langle \pi' \rangle \approx \langle \pi \star \pi' \rangle$$
- Every connected component can be written as $\langle \pi_\lambda \rangle$ where π_λ is a dominant path.
- Littelmann's independence theorem: The crystal structure of a connected crystal $\langle \pi_\lambda \rangle$ depends only on λ.

$$\langle \pi_\lambda \rangle \approx \mathcal{B}(\lambda)$$



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Introduction

Berenstein and Kazhdan introduced "geometric" crystals made out of totally positive matrices in the Borel (lower triangular) subgroup $B \subset G$. Highest weight geometric crystals are also denoted:

$$\mathcal{B}(\lambda) \subset B$$

These degenerate to the classical combinatorial crystals, and give geometric intuitions.

Our contribution:

- Identifying the natural charts for $\mathcal{B}(\lambda)$.
- Constructing a geometric path model using paths in $\mathfrak{a}$.
- Brownian motion endows geometric crystals with natural measures.

Group picture and its parametrizations

Consider $(e_\alpha)_{\alpha\in\Delta}$ to be the Chevalley generators of $\mathfrak{u}\subset\mathfrak{g}$.
Define:

$$x_\alpha(t) = e^{te_\alpha} = \phi_\alpha\left(\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix}\right)$$

Theorem (Lusztig, Fomin and Zelevinsky)

Every reduced word $\mathbf{i} = (i_1,\dots,i_m)\in R(w_0)$ *gives rise to a bijection onto its image:*

$$\begin{array}{cccc} x_{\mathbf{i}}: & \mathbb{R}^m_{>0} & \to & U^{w_0}_{>0} \\ & (t_1,\dots,t_m) & \mapsto & x_{\alpha_{i_1}}(t_1)\dots x_{\alpha_{i_m}}(t_m)\end{array}$$

The image is called the totally positive part of U*.*

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The image is called the totally positive part of U.

Definition

The geometric crystal with highest weight $\lambda \in \mathfrak{a}$ is defined as:

$$\mathcal{B}(\lambda) := \left\{ x \in B \mid x = z \bar{w}_0 e^\lambda u, z \in U_{>0}^{w_0}, u \in U_{>0}^{w_0} \right\}$$

Writing $z = x_{\mathbf{i}}(t_1, \dots, t_m)$ defines $x \in \mathcal{B}(\lambda)$ completely and is called its **Lusztig parametrization**.

Geometric crystal structure on paths

We defined a geometric crystal structure on the set $C_0([0,T];\mathfrak{a})$ of paths by endowing it with the following maps:

- A 'weight' map $\gamma : C_0([0,T];\mathfrak{a}) \to \mathfrak{a}$ defined as

$$\gamma(\pi) = \pi(T)$$

- For every $\alpha \in \Delta$, maps $\varepsilon_\alpha, \varphi_\alpha$ defined as:

$$\varepsilon_\alpha(\pi) := \log\left(\int_0^T e^{-\alpha(\pi(s))} ds\right)$$

$$\varphi_\alpha(\pi) := \alpha(\pi(T)) + \log\left(\int_0^T e^{-\alpha(\pi(s))} ds\right)$$

- The actions $(e_\alpha^\cdot)_{\alpha\in\Delta}$ defined as:

$$e_\alpha^c \cdot \pi(t) := \pi(t) + \log\left(1 + \frac{e^c - 1}{e^{\varepsilon_\alpha(\pi)}} \int_0^t e^{-\alpha(\pi(s))} ds\right) \alpha^\vee$$

Remark

Rescaling/tropicalization gives the continuous Littelmann model introduced by BBO.

Flows on B

Let $X \in C([0,T];\mathfrak{a})$ be a continuous path valued in $\mathfrak{a}$.

Definition (BBO)

Define $(B_t(X))_{t\geq 0}$ as the solution to the left-invariant equation:

$$\begin{cases} \frac{dB(X)}{dt}(t) = B_t(X)\left(\sum_{\alpha\in\Delta} f_\alpha + \frac{dX}{dt}(t)\right) \\ B_0(X) = \exp(X_0) \end{cases} \tag{1}$$

Example (A_1-type)

In the case of SL_2:

$$dB_t(X) = B_t(X)\begin{pmatrix} dX_t & 0 \\ dt & -dX_t \end{pmatrix}$$

Solving the differential equation leads to:

$$B_t(X) = \begin{pmatrix} e^{X_t} & 0 \\ e^{X_t}\int_0^t e^{-2X_s}ds & e^{-X_t} \end{pmatrix}$$

Geometric Pitman transform (1)

Definition

Define the path transform $\mathcal{T}_\alpha$ for a path π as:

$$\mathcal{T}_\alpha \pi(t) = \pi(t) + \log\left(\int_0^t \exp\left(-\alpha(\pi)\right)\right)\alpha^\vee$$

Theorem (BBO)

The path transforms $\mathcal{T}_\alpha$ satisfy the braid relations. Therefore allowing to unambiguously define:

$$\mathcal{T}_{w_0} = \mathcal{T}_{\alpha_{i_1}} \circ \cdots \circ \mathcal{T}_{\alpha_{i_m}}$$

Geometric Pitman transform (2)

Theorem (Geometric Littelmann independence theorem)

For a path crystal $\langle \pi \rangle \subset C_0\left([0,T]; \mathfrak{a}\right)$*, the crystal structure depends only on* $\lambda = \mathcal{T}_{w_0}\pi(T)$*. And:*

$$\langle \pi \rangle \approx \mathcal{B}(\lambda)$$

the isomorphism to the group picture being given by $p : \pi \mapsto B_T(\pi)$*.*

Conclusion: The transform $\mathcal{T}_{w_0}$ really gives a **highest path**, **in the geometric path model**.

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Dynamics of highest weight

Theorem (“Geometric Pitman for G semi-simple” (ADE case))

- *The process* $\Lambda_t := \mathcal{T}_{w_0} W_t^{(\mu)}$ *is Markov with inf. generator:*

$$\psi_\mu^{-1} \left(\frac{1}{2}\Delta - \sum_{\alpha \in \Delta} \frac{1}{2} \langle \alpha, \alpha \rangle e^{-\alpha(x)} - \frac{1}{2} \langle \mu, \mu \rangle \right) \psi_\mu$$

- *The function* ψ_μ *is a Whittaker function.*
- *Intertwining measure:*

$$\left(W_t^{(\mu)} | \Lambda_t = \lambda \right) \stackrel{\mathcal{L}}{=} \frac{1}{\psi_\mu(\lambda)} \exp\left(\langle \mu, x \rangle \right) DH^\lambda(dx)$$

where DH^λ *is the geometric Duistermaat-Heckman measure.*

Remark

For $G = SL_2$*, one recovers the Matsumoto-Yor theorem. And* $G = GL_n$*, a result due to O'Connell.*

Canonical measure on geometric crystals (1)

In every setting, the canonical measure is obtained by considering random walks in the path model.

Basic idea: The canonical measure = Distribution of a random crystal element conditionally to its highest weight.

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Two ingredients in the geometric setting:

- The toric reference measure on $\mathcal{B}(\lambda)$. In Lusztig coordinates:

$$\omega(dx) = \prod_{j=1}^{m} \frac{dt_j}{t_j}$$

- The superpotential map:

$$f_B(x) = \chi(z) + \chi(u)$$

  where χ is the standard character on U.

Canonical measure on geometric crystals (2)

Theorem (" Canonical measure on geometric crystals" (ADE case))

There is a canonical measure on $\mathcal{B}(\lambda)$:

$$\exp\left(-f_B(x)\right)\omega(dx)$$

in the sense that if $\Lambda_t := \mathcal{T}_{w_0} W^{(\mu)}$:

$$\left(B_t\left(W^{(\mu)}\right)|\mathcal{F}_t^{\Lambda}, \Lambda_t = \lambda\right) \stackrel{\mathcal{L}}{=} \frac{1}{\psi_\mu(\lambda)} \exp\left(\langle \mu, x\rangle - f_B(x)\right)\omega(dx)$$

Definition

Define the geometric Duistermaat-Heckman measure DH^λ on $\mathfrak{a}$ as the image measure of the canonical measure on $\mathcal{B}(\lambda)$ through the weight map.

Remark

f_B was introduced by Berenstein and Kazhdan, as an object that does the trick. In our case, Brownian motion finds it for us. It is referred as the 'superpotential' by K. Rietsch while investigating the quantum cohomology of flag manifolds and mirror symmetry.

Whittaker functions

Theorem

The Whittaker function

$$\psi_\mu(\lambda) = \int_{\mathcal{B}(\lambda)} e^{\langle \mu, \gamma(x) \rangle - f_B(x)} \omega(dx) = \int_{\mathfrak{a}} e^{\langle \mu, x \rangle} DH^\lambda(dx)$$

satisfies the following:

(i) $\psi_\mu(\lambda)$ *is an entire function in* $\mu \in \mathfrak{h} = \mathfrak{a} \otimes \mathbb{C} \approx \mathbb{C}^n$.

(ii) ψ_μ *is invariant in* μ *under the Weyl group's action.*

(iii) *For* $\mu \in C$, *the Weyl chamber, it is the unique solution (with certain growth conditions) to the quantum Toda eigenequation:*

$$\frac{1}{2} \Delta \psi_\mu(x) - \sum_{\alpha \in \Delta} \frac{1}{2} \langle \alpha, \alpha \rangle e^{-\alpha(x)} \psi_\mu(x) = \frac{1}{2} \langle \mu, \mu \rangle \psi_\mu(x)$$

Comments

- Absolutely convergent expressions ($\neq$ Jacquet).
- Representation-theoretic expression. Laplace transform of the geometric Duistermaat-Heckman measure: **"Geometric character"**

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Conclusion

What we did in this thesis:

- Give a path model for geometric crystals that is a natural deformation of Littelmann's. Description of different parametrizations and natural maps.
- By taking a random path, we endow geometric crystals with the appropriate measures.
- Markov property of highest weight and description of canonical measure.
- Describe Whittaker functions as "characters".
- (Not presented now) Describe involutions (Kashiwara and Schutzenberger) on geometric crystals.
- (Not presented now) Littlewood-Richardson rule, product formula for Whittaker functions.

Ouverture

What we could do afterwards:

- Is there a notion of geometric Demazure crystals?
- Can we make sense of branching rules?
- Explore the O'Connell-Yor semi-discrete polymer for other geometries using the previous results.
- Explore the affine case.
- Explore q-Whittaker functions (links to Kirillov-Reshetikhin crystals and q-TASEP).