Contribution to the control of systems with time-varying and state-dependent sampling

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par

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ECOLE CENTRALE DE LILLE

THESIS

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Speciality : Control theory, computer science, signal processing and image

by

Christophe Fiter

Engineer from École Centrale de Lille

PhD awarded by École Centrale de Lille

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Defended on September 25th, 2012 in presence of the committee:

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Acronyms

- ISS	=	Input-to-State Stability.
- LKF	=	Lyapunov-Krasovskii Functional.
- LMI	=	Linear Matrix Inequality.
- LTI	=	Linear Time-Invariant.
- LRF	=	Lyapunov-Razumikhin Function.
- NCS	=	Networked Control System.

Notations

Notations concerning sets:

- $\mathbb{R}_+$ is the set $\{\lambda \in \mathbb{R}, \lambda \geq 0\}$.
- $\mathbb{R}^*$ is the set $\{\lambda \in \mathbb{R}, \lambda \neq 0\}$.
- $\mathcal{M}_{n,m}(\mathbb{R})$ denotes the set of real $n \times m$ matrices.
- $\mathcal{M}_n(\mathbb{R})$ denotes the set of real $n \times n$ matrices.
- S_n denotes the set of symmetric matrices in $\mathcal{M}_n(\mathbb{R})$.
- S_n^+ (resp. S_n^{+*}) denotes the set of positive (resp. positive definite) symmetric matrices in $\mathcal{M}_n(\mathbb{R})$.
- $\text{Co}\{F_i\}_{i\in\mathcal{I}}$, for given matrices $F_i \in \mathcal{M}_{n,m}(\mathbb{R})$ and a finite set of indexes $\mathcal{I}$, denotes the convex polytope in $\mathcal{M}_{n,m}(\mathbb{R})$ formed by the vertices $F_i,\ i \in \mathcal{I}$.
- $\mathcal{C}^0(X \to Y)$, for two metric spaces X and Y, is the set of continuous functions from X to Y.
- $\mathcal{L}_2$ is the space of square-integrable functions from $\mathbb{R}^+$ to $\mathbb{R}^n$.
- λX, for a scalar $\lambda \in \mathbb{R}$ and an $\mathbb{R}$ vector space X, represents the set $\{\lambda x,\ x \in X\}$.
- $\mathbb{R}^* x$, with $x \in \mathbb{R}^n$, is the set defined as $\{y \in \mathbb{R}^n, \exists \lambda \neq 0, y = \lambda x\}$.
- $|X|$, is the cardinality of the finite set X.
- $\mathcal{P}(X)$ denotes the power set of a set X (i.e. the set of all subsets of X).

Notations concerning matrices:

- M^T stands for the transpose of $M \in \mathcal{M}_{n,m}(\mathbb{R})$.
- M^+ is the pseudoinverse of $M \in \mathcal{M}_{n,m}(\mathbb{R})$.
- $A \succeq B$ (resp. $A \succ B$) for matrices $A,\ B \in \mathcal{M}_n(\mathbb{R})$ means that $A - B$ is a positive (resp. definite positive) matrix.
- I is the identity matrix (of appropriate dimension).
- $*$, in a matrix, denotes the symmetric elements of a symmetric matrix.
- $\text{diag}(A_1, \cdots, A_m)$ is the block diagonal matrix designed by the square matrices $A_i, i \in \{1, \cdots, m\}$, of any dimension.
- $\text{rank}(M)$ is the rank of the matrix $M \in \mathcal{M}_{n,m}(\mathbb{R})$.

- $\lambda_{\max}(M)$ (resp. $\lambda_{\min}(M)$) denotes the largest (resp. lowest) eigenvalue of a symmetric matrix $M \in \mathcal{M}_n(\mathbb{R})$.
- $\rho(M)$ denotes spectral radius of $M \in \mathcal{M}_n(\mathbb{R})$.
- $|||.|||_2$ stands for the operator norm on $\mathcal{M}_n(\mathbb{R})$ associated to the norm $\|.\|_2$ on $\mathbb{R}^n$: for a matrix $M \in \mathcal{M}_n(\mathbb{R})$, $|||M|||_2 = \sup_{\|x\|_2=1} \|Mx\|_2 = \sqrt{\rho(M^T M)}$.

Notations concerning vectors:

- x^T stands for the transpose of $x \in \mathbb{R}^n$.
- $\|.\|_2$ stands for the Euclidean norm on $\mathbb{R}^n$: for a vector $x \in \mathbb{R}^n$, $\|x\|_2 = \sqrt{x^T x}$.

Notations concerning scalars:

- $\lfloor x \rfloor$ is the floor of $x \in \mathbb{R}$: the largest integer not greater than x: $x - 1 < \lfloor x \rfloor \leq x$.
- $\lceil x \rceil$ is the ceiling of $x \in \mathbb{R}$: the smallest integer not less than x: $x \leq \lceil x \rceil < x + 1$.
- $\text{sgn}(x)$ denotes the sign of the scalar x.
- $\text{sat}(x)$ denotes a scalar that is equal to -1 if the scalar $x \leq -1$, 1 if $x \geq 1$, and x otherwise.

Notations concerning functions:

- x_t (resp. $\dot{x}_t$) denotes the function in $\mathcal{C}^0([-\bar{h}, 0] \to \mathbb{R}^n)$, for a given maximal delay $\bar{h}$ such that $x_t(\theta) = x(t + \theta),\ \forall \theta \in [-\bar{h}, 0]$ (resp. $\dot{x}_t(\theta) = \dot{x}(t + \theta),\ \forall \theta \in [-\bar{h}, 0]$).
- $\|.\|_{\mathcal{L}_2}$ is the $\mathcal{L}_2$-norm on $\mathcal{L}_2$: for a function $f \in \mathcal{L}_2$, $\|f\|_{\mathcal{L}_2} = \left(\int_0^\infty \|f(t)\|_2^2 dt\right)^{\frac{1}{2}}$.
- $\|.\|_{H_\infty}$ is the H_∞-norm on $\mathcal{L}_2 \to \mathcal{L}_2$: for an operator $\Delta : u \in \mathcal{L}_2 \mapsto v \in \mathcal{L}_2$, $\|\Delta\|_{H_\infty} = \sup_{w \in \mathbb{R}_+} \|\Delta(jw)\|$, with $\|\Delta(jw)\| = \max\limits_{\|z\|_2=1,\ z \in \mathbb{C}^n} \|\Delta(jw) z\|_2$. It is equal to the $\mathcal{L}_2$-to-$\mathcal{L}_2$ norm: $\|\Delta\|_{H_\infty} = \|\Delta\|_{\mathcal{L}_2 \to \mathcal{L}_2} = \sup\limits_{u \neq 0} \dfrac{\|v\|_{\mathcal{L}_2}}{\|u\|_{\mathcal{L}_2}}$.
- A class $\mathcal{K}$ function is a function $\varphi : [0, a) \to [0, +\infty)$ that is strictly increasing, and such that $\varphi(0) = 0$.
- A class $\mathcal{K}_\infty$ function is a class $\mathcal{K}$ function such that $a = +\infty$ and $\lim_{t \to +\infty} \varphi(t) = +\infty$.
- A C^∞ function is a function that is infinitely differentiable.
- $f(n) = O(g(n))$ means that the growth-rate of the sequence $f(n)$, $n \in \mathbb{N}$, is dominated by the sequence $g(n)$, *i.e.* there exist $N \in \mathbb{N}$ and $K \in \mathbb{R}_+^*$ such that for all $n \geq N$, $|f(n)| \leq K|g(n)|$.

Notations concerning logic:

- $\wedge$ defines the "AND" logic gate.
- $\vee$ defines the "OR" logic gate.

Other notations:

- $x \equiv y$ means that the term x is denoted as y, or that the term y is denoted as x.

General introduction

Until the 50s, most systems were controlled using analogical controllers. However, the fast development of computers led to an increasing use of digital controllers. This is especially due to their computational power and flexibility. Nowadays, digital controllers have become omnipresent, and enabled the explosion of embedded systems and networked control systems. They offer several advantages: low cost installation and maintenance, increased flexibility and re-usability, reduced wiring cost, and ease of programming. Furthermore, they offer the possibility to control more than one process at a time.

Unlike analogical controllers, digital controllers, due to their nature, introduce discrete-time signals and discrete-time dynamics, via sample and hold devices [Aström 1996].

First, the information sent from the sensors to the controller is sampled, by means of an analog-to-digital (A/D) converter. Such a conversion of an input signal $x(t)$ into a sampled signal $x(s_k)$, at sampling instants $s_k,\ k \in \mathbb{N}$ is shown in Figure 1.

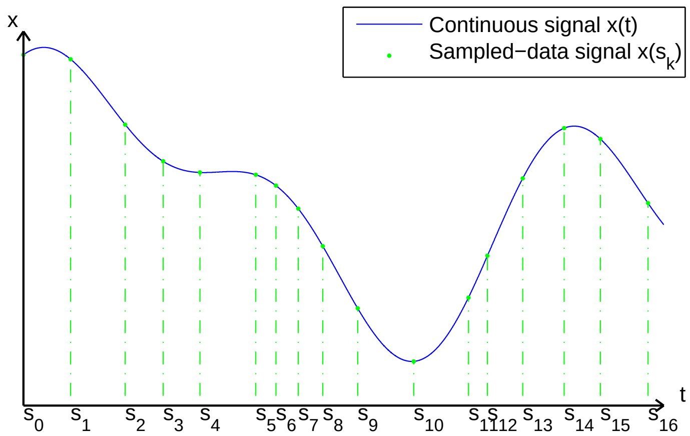

Figure 1: Analog-to-digital conversion

Moreover, since the control is computed only at discrete instants, it is necessary to use a digital-to-analog (D/A) converter (a zero-order-hold), so as to hold the control value that is sent to the actuators. The conversion of a sampled input signal $u(s_k)$ into a piecewise-constant signal $u(t)$, is shown in Figure 2.

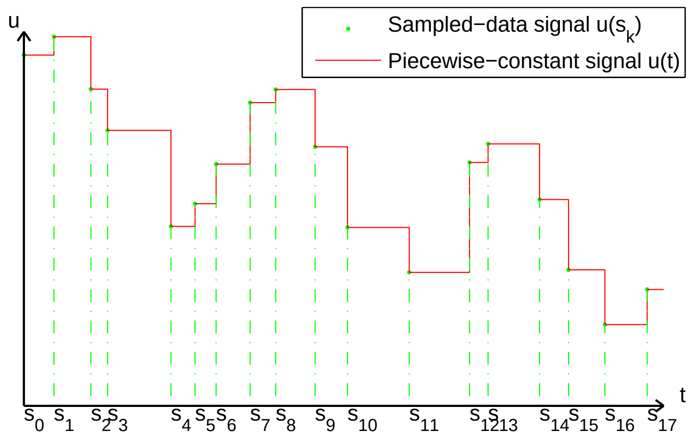

Figure 2: Digital-to-analog conversion

In embedded control applications however, a discrete-time implementation may produce undesired effects such as delays or aperiodic control executions, due to the interaction between control tasks and real-time scheduler mechanisms [Hristu-Varsakelis 2005]. The effects of these discrete-time dynamics brought up new challenges regarding the stability and stabilization, and new theories and tools have been developed for these sampled-data systems. In particular, in the last few years, two main problems have been of a great importance for control theorists:

P1) the stability of sampled-data systems with time-varying sampling;

P2) the dynamic control of the sampling events.

The new trend is to control dynamically the sampling so as to enlarge the sampling intervals and reduce the computational and energetic costs.

Goals

The work presented in this thesis is concerned with these two problems P1) and P2). The main objective is to design a sampling law that allows for reducing the sampling frequency of state-feedback control for linear sampled-data systems while ensuring the system stability.

In order avoid possible scheduling issues, the robustness with respect to time-varying sampling will also be included. The robustness aspect with respect to exogenous perturbations or delays in the control loop will be considered, so to take into account phenomena occuring in the real-time control of physical systems. Finally, a co-design of the controller and sampling law is proposed. Here, in order to reduce the conservatism, the control gains and the sampling instants will be computed jointly.

Throughout the thesis, different designs of sampling control laws will be presented. They can be used to compute a simple upper-bound for time-varying samplings, or to dynamically control the sampling intervals, using online or offline algorithms.

Structure of the thesis

The thesis is organized as follows:

Chapter 1

The first chapter is a literature survey which presents an overview of problems, challenges, and recent research directions in the domain of sampled-data systems in control theory. First, the notion of sampled-data systems is defined, and the main open problems in the literature are presented. Then, some general stability concepts necessary to the comprehension are recalled. Finally, several research directions, theories, and results are presented concerning the stability analysis of sampled-data systems with constant or time-varying sampling, or concerning the dynamic control of the sampling. The strengths and weaknesses of the different approaches are analyzed, so as to highlight which problems have already been solved, and what still remains to be done or improved.

Chapter 2

In the second chapter, a state-dependent sampling control is designed for ideal LTI systems with sampled-data. The goal is to design a sampling law that will take into account the system's state, so as to enlarge the sampling intervals, or in other terms, to generate the sampling events as sparsely as possible. The proposed state-dependent sampling function takes advantage of an offline design based on LMIs obtained thanks to a mapping of the state space, polytopic embeddings, and Lyapunov-Razumikhin stability conditions.

Chapter 3

In the third chapter, the robustness aspect with respect to exogenous disturbances is considered for the design of a state-dependent sampling law. As in the second chapter, the approach is based on Lyapunov-Razumikhin stability conditions and polytopic embeddings. After presenting the main stability results, four different applications are addressed. The first one concerns the robust stability analysis with respect to time-varying sampling. The other three applications propose different approaches to the dynamic control of the sampling with the objective to enlarge the sampling interval. Event-triggered control, self-triggered control, and the newly introduced state-dependent sampling schemes are then presented.

Chapter 4

In the fourth and last chapter, an extension to the stability analysis of perturbed time-delay linear systems is tackled, and the stabilization issue is considered. The objective here is to design a controller along with the state-dependent sampling law, so as to stabilize the considered perturbed LTI sampled-data system, and enlarge even further the allowable sampling intervals. First, the case of a classic linear state-feedback controller is considered. Then, a new controller is proposed, the gains of which are switching according to the system's state. The co-design of both the controller and the state-dependent sampling function is based on LMIs obtained thanks to the mapping of the state-space presented in the previous chapters, and thanks to a new class of Lyapunov-Krasovskii functionals with matrices switching with respect to the system's state.

Personal publications

The research exposed in this thesis can be found in the following publications:

Journals

- C. Fiter, L. Hetel, W. Perruquetti, and J.-P Richard - *A State Dependent Sampling for Linear State Feedback* - Automatica, Volume 48, Number 8, Pages 1860-1867, August 2012. doi:10.1016/j.automatica.2012.05.063

- C. Fiter, L. Hetel, W. Perruquetti, and J.-P Richard - *A Novel Stabilization Approach for State-Dependent Sampling* - International Journal of Control, provisionally accepted.

- C. Fiter, L. Hetel, W. Perruquetti, and J.-P Richard - *A Robust Stability Framework for Time-Varying Sampling* - Automatica, submitted.

International conferences

- C. Fiter, L. Hetel, W. Perruquetti, and J.-P Richard - *State Dependent Sampling: an LMI Based Mapping Approach* - 18th IFAC World Congress, Milan, Italy, September 2011.
- C. Fiter, L. Hetel, W. Perruquetti, and J.-P Richard - *State-Dependent Sampling for Perturbed Time-Delay Systems* - 51st IEEE Conference on Decision and Control, Maui, Hawaii, USA, December 2012.
- C. Fiter, L. Hetel, W. Perruquetti, and J.-P Richard - *A Robust Polytopic Approach for State-Dependent Sampling* - 12th European Control Conference, Zurich, Switzerland, July 2013 - submitted.

National conferences

- C. Fiter - *Echantillonnage Dépendant de l'Etat: une Approche par Cartographie Basée sur des LMIs* - 4èmes Journées Doctorales MACS, Marseille, France, June 2011.
- C. Fiter, L. Hetel, W. Perruquetti, and J.-P Richard - *Échantillonnage Dépendant de l'État pour les Systèmes avec Perturbations et Retards* - 8ème Colloque Francophone sur la Modélisation des Systèmes Réactifs, Villeneuve d'Ascq, France, November 2011. Journal Européen des Systèmes Automatisés, Volume 45, Number 1-2-3, Pages 189-203, 2011. doi:10.3166/jesa.45.189-203. *Best young researcher article award.*

Chapter 1

Sampled-data systems: an overview of recent research directions

In this chapter, we intend to present several basic concepts and some recent research directions about sampled-data systems. First, a short introduction of sampled-data systems will be given, along with the main mathematical definitions and problematics. Then, some general concepts of stability will be recalled, and the sampled-data systems stability and stabilizability problems will be formulated. Finally, the main recent research directions and results from the literature will be presented. They will be classified into three main categories according to their sampling type: constant sampling, time-varying sampling, and dynamic sampling control.

1.1 Introduction to sampled-data systems

1.1.1 General sampled-data systems

Sampled-data systems are dynamic systems that involve both a continuous-time dynamics and a discrete-time control. They are mathematically as follows:

Definition 1.1 (Sampled-data system)

$$\begin{aligned} &\dot{x}(t) = f(t, x(t), u(t)),\ \forall t \geq 0, \\ &u(t) = g(x(s_k), s_k),\ \forall t \in [s_k, s_{k+1}),\ k \in \mathbb{N}, \end{aligned} \tag{1.1}$$

where t is the time-variable, $x : \mathbb{R}_+ \to \mathbb{R}^n$ the "state-trajectory", $u : \mathbb{R}_+ \to \mathbb{R}^{n_u}$ the "input", or "control signal", and the scalars s_k, for $k \in \mathbb{N}$, are the sampling instants

which satisfy $0 = s_0 < s_1 < \cdots < s_k < \cdots$ *and* $\lim_{k\to+\infty} s_k = +\infty$.
The sampling law is defined as

$$s_{k+1} = s_k + \tau_k, \tag{1.2}$$

where τ_k *represents the* k^{th} *sampling interval.*

Such systems can be represented by the block diagram in Figure 1.1, in which the blocks A/D and D/A correspond to an analog-to-digital converter (a sampler) and a digital-to-analog converter (a zero-order hold) respectively.

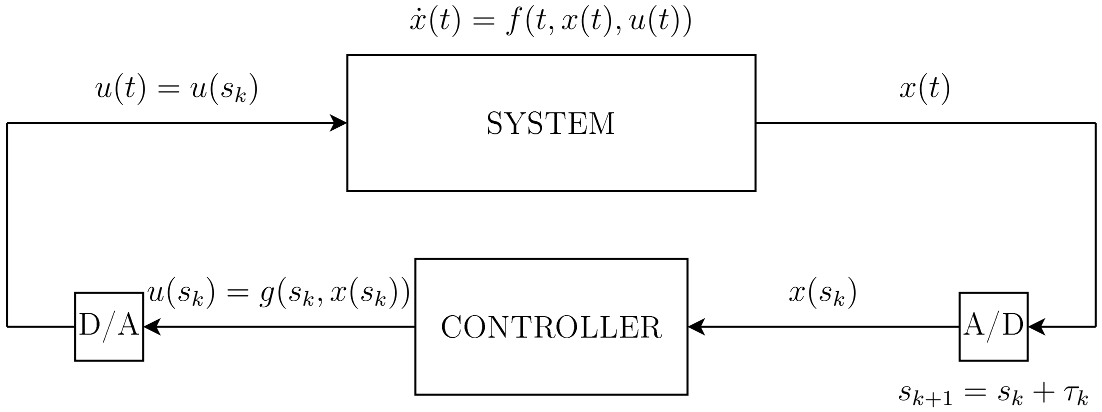

Figure 1.1: Sampled-data system

It is important to note that with these systems, the discrete-time dynamics introduced by the (digital) controller implies that during the time between two sampling instants the system is controlled in open-loop (*i.e.* without updating the feedback information). Therefore, the sampling period plays an important role in the stability of the system, and adapted tools have to be used.

1.1.2 Sampled-data linear time-invariant systems

The model of sampled-data systems provided in Definition 1.1 is very general. In this thesis, we will focus mainly on linear time-invariant sampled-data systems with state-feedback, which are defined as follows:

Definition 1.2 (Sampled-data linear time-invariant system)

$$\begin{aligned} &\dot{x}(t) = Ax(t) + Bu(t), \ \forall t \geq 0, \\ &u(t) = -Kx(s_k), \ \forall t \in [s_k, s_{k+1}), \ k \in \mathbb{N}, \end{aligned} \tag{1.3}$$

where t is the time-variable, $x : \mathbb{R}_+ \to \mathbb{R}^n$ the "state-trajectory", $u : \mathbb{R}_+ \to \mathbb{R}^{n_u}$ the "input", or "control signal", and the scalars s_k, for $k \in \mathbb{N}$, are the sampling instants which satisfy $0 = s_0 < s_1 < \cdots < s_k < \cdots$ and $\lim_{k\to+\infty} s_k = +\infty$. $A \in \mathcal{M}_n(\mathbb{R})$ is the "state matrix", $B \in \mathcal{M}_{n,n_u}(\mathbb{R})$ is the "input gain matrix", and $K \in \mathcal{M}_{n_u,n}(\mathbb{R})$ is the "control gain matrix". The sampling law is defined as

$$s_{k+1} = s_k + \tau_k, \tag{1.4}$$

where τ_k represents the k^{th} sampling interval.

This definition presents the case of "ideal" sampled-data LTI systems, in which no disturbance nor any other phenomenon is taken into account. Throughout this thesis however, additional phenomena will be considered like exogenous perturbations or delays in the feedback control-loop for example. In that case, when these classes of systems are considered, the associated system equations will be provided.

In the absence of perturbations, the evolution of the system's state between two consecutive sampling instants s_k and s_{k+1} is given by

$$\begin{aligned} x(t) &= \mathrm{e}^{A(t-s_k)}x(s_k) + \int_0^{t-s_k} \mathrm{e}^{As}dsBu(s_k) \\ &= A_d(t-s_k)x(s_k) + B_d(t-s_k)u(s_k) \\ &= [A_d(t-s_k) - B_d(t-s_k)K]\, x(s_k) \\ &= \Lambda(t-s_k)x(s_k),\ \forall t \in [s_k, s_{k+1}],\ k \in \mathbb{N}, \end{aligned} \tag{1.5}$$

with the matrix functions A_d, B_d, and Λ defined on $\mathbb{R}_+$ as

$$A_d(\sigma) = \mathrm{e}^{A\sigma},\ B_d(\sigma) = \int_0^\sigma \mathrm{e}^{As}dsB. \tag{1.6}$$

and

$$\Lambda(\sigma) = A_d(\sigma) - B_d(\sigma)K = \mathrm{e}^{A\sigma} - \int_0^\sigma \mathrm{e}^{As}dsBK. \tag{1.7}$$

Using the notation τ_k in equation (1.4), for the sampling intervals, it is then possible to obtain the following associated discrete-time model of the linear sampled-data system at instants s_k:

$$x_{k+1} = A_d(\tau_k)x_k + B_d(\tau_k)u_k = \Lambda(\tau_k)x_k,\ \forall k \in \mathbb{N}, \tag{1.8}$$

with $x_k \equiv x(s_k)$ and $u_k \equiv u(s_k)$. $A_d(\tau_k)$ and $B_d(\tau_k)$ are called the "state matrix" and the "input matrix" of the discrete-time model respectively, and $\Lambda(\tau_k)$ is called the discrete-time "transition matrix".

1.1.3 Problematics

From the control theory point of view, due to the existence of both a continuous and a discrete dynamics, sampled-data systems bring up new challenges. As in the more general frameworks of delayed-systems [Richard 2003], [Gu 2003], hybrid systems [der Schaft 2000], [Zaytoon 2001], [Goebel 2009], [Prieur 2011], or reset systems [Nesic 2008], [Beker 2004], some problems are raised.

- PROBLEM A: Determine if a sampled-data system is stable for any constant sampling interval $\tau_k \equiv \tau$ with values in a bounded subset $\Omega \subseteq \mathbb{R}_+$?

- PROBLEM B: Determine if the sampled-data system is stable for any time-varying sampling interval τ_k with values in a bounded subset $\Omega \subseteq \mathbb{R}_+$?

Lately, an additional issue has been brought up to light. With the emergence of embedded and networked systems particularly [Zhang 2001c], [Hespanha 2007], [Richard 2007], [Chen 2011], control scientists realised that computing the next control at each sampling time has a cost [Buttazzo 2002], [Cervin 2002], [Brockett 2000], [Nair 2000]. Indeed, the computations for a new control reduces the limited processor resources, in the case of embedded systems for example. In the case of networked control systems, the transmission of the sampled-data requires bandwidth, which is also limited. Therefore, a new problem arose:

- PROBLEM C: Design a sampling law $\tau_k = \tau(t, s_k, x(s_k), \cdots)$ that enlarges the sampling intervals while making the sampled-data system stable?

In this thesis, we will mainly focus on finding solutions to this last particular problem which concerns the reduction of the number of sampling instants (*i.e.* for particular systems with periodic sampling, the reduction of the sampling frequency). We will also adapt the proposed tools in order to further derive solutions to the other two problems. During this study, some stability performances will be taken into account, such as the speed of convergence of the system's state, or the robustness with respect to possible exogenous perturbations or delays.

1.2 Classical stability concepts

Before providing an overview of some works from the literature about sampled-data systems, we recall some fundamental concepts about stability, and some classic stability tools that will be used throughout the thesis.

1.2.1 Some stability definitions

Intuitively, stability is a property that corresponds to staying close to an equilibrium position, when the state is punctually disturbed. Originally, stability is analyzed for systems that are time-invariant and autonomous (i.e. for which there is no control, or for a closed-loop system with a given control). Such systems are defined as follows:

Definition 1.3 (Autonomous system) *The ordinary differential equation:*

$$\dot{x}(t) = f(x(t)), \ \forall t \geq 0, \tag{1.9}$$

with $f : \mathbb{R}^n \to \mathbb{R}^n$ Lipschitz continuous[1]*, is said to be autonomous if $f(x(t))$ does not depend explicitely on the free variable t (often regarded as time).*

An "equilibrium point" x_e represents a real solution of the equation $f(x) = 0$.

Definition 1.4 ([Khalil 2002]) *An equilibrium point x_e of the system (1.9) is*

- *stable (in the sense of Lyapunov) if $\forall \epsilon > 0, \ \exists \delta = \delta(\epsilon) > 0$ such that*

$$\|x(0) - x_e\| < \delta \Rightarrow \|x(t) - x_e\| < \epsilon, \ \forall t \geq 0;$$

- *attractive if $\exists \rho > 0$ such that*

$$\|x(0) - x_e\| < \rho \Rightarrow \lim_{t \to +\infty} \|x(t) - x_e\| = 0;$$

- *asymptotically stable if it is stable and attractive;*
- *exponentially stable if there exist three scalars $\alpha, \ \beta, \ \delta > 0$ such that*

$$\|x(0) - x_e\| < \delta \Rightarrow \|x(t) - x_e\| \leq \alpha \|x(0) - x_e\| e^{-\beta t}.$$

  For such a scalar β, called (exponential) "decay-rate", the equilibrium point is also said to be "β-stable";

- *globally asymptotically stable if it is stable and $\forall x(0) \in \mathbb{R}^n$,*

$$\lim_{t \to +\infty} \|x(t) - x_e\| = 0$$

[1] Given two metric spaces (X, d_X) and (Y, d_Y), where d_X denotes the metric on the set X and d_Y is the metric on set Y, a function $f : X \to Y$ is called Lipschitz continuous (or simply Lipschitz) if there exists a real constant $K \geq 0$ such that for all $x_1, \ x_2 \in X$, $d_Y(f(x_1), f(x_2)) \leq K d_X(x_1, x_2)$.

Note that by using a translation of the origin, it is always possible to reformulate the problem as a stability analysis around $x_e = 0$. Therefore, all the results and stability properties will now be written while taking $x_e = 0$ as the studied equilibrium point.

1.2.2 Second Lyapunov method

The most common stability tool is the Lyapunov stability approach. It is based on the fact that a system which trajectory approaches the origin, loses its energy. The Lyapunov stability approach makes use of a function $V : \mathbb{R}^n \to \mathbb{R}_+$, called "candidate Lyapunov function", which depends on the system's state, and symbolizes some sort of potential energy of the system, with respect to the origin. Very often, this function is chosen as a norm or a distance. The Lyapunov stability theory is described as follows [Khalil 2002].

Theorem 1.5 *Consider the autonomous system (1.9) with an isolated equilibrium point ($x_e = 0 \in \Omega \subseteq \mathbb{R}^n$, with Ω a neighborhood of x_e). If there exist a locally Lipschitz function $V : \mathbb{R}^n \to \mathbb{R}_+$ with continuous partial derivatives and two class $\mathcal{K}$ functions*[2] *α and β such that*

$$\alpha(\|x\|) \leq V(x) \leq \beta(\|x\|),\ \forall x \in \Omega,$$

then the origin $x = 0$ of the system is

- *stable (in the sense of Lyapunov) if*

$$\frac{dV(x)}{dt} \leq 0,\ \forall x \in \Omega,\ x \neq 0;$$

- *asymptotically stable if there exists a class $\mathcal{K}$ function φ such that*

$$\frac{dV(x)}{dt} \leq -\varphi(\|x\|),\ \forall x \in \Omega,\ x \neq 0;$$

- *exponentially stable if, moreover, there exist four scalars $\bar{\alpha},\ \bar{\beta},\ \gamma,\ p > 0$ such that*

$$\alpha(\|x\|) = \bar{\alpha}\|x\|^p,\ \beta(\|x\|) = \bar{\beta}\|x\|^p,\ \varphi(\|x\|) = \gamma\|x\|.$$

  In such a case, the equilibrium point x_e allows a decay-rate equal to $\frac{\gamma}{p}$.

There also exists a discrete-time version of the Lyapunov stability theory.

[2] A class $\mathcal{K}$ function is a function $\varphi : [0, a) \to [0, +\infty)$ that is strictly increasing, and such that $\varphi(0) = 0$.

Theorem 1.6 *Consider the discrete-time autonomous system*

$$x_{k+1} = f(x_k), \tag{1.10}$$

with an isolated equilibrium point ($x_e = 0 \in \Omega \subseteq \mathbb{R}^n$, with Ω a neighborhood of x_e). If there exist a locally Lipschitz function $V : \mathbb{R}^n \to \mathbb{R}_+$ with continuous partial derivatives and two class $\mathcal{K}$ functions α and β such that

$$\alpha(\|x\|) \leq V(x) \leq \beta(\|x\|),\ \forall x \in \Omega,$$

then the origin $x = 0$ of the system is

- *stable (in the sense of Lyapunov) if*

$$\Delta V(x_k) \leq 0,\ \forall x_k \in \Omega,\ x_k \neq 0$$

where

$$\begin{aligned} \Delta V(x_k) &= V(x_{k+1}) - V(x_k) \\ &= V(f(x_k)) - V(x_k); \end{aligned}$$

- *asymptotically stable if there exists a class $\mathcal{K}$ function φ such that*

$$\Delta V(x_k) \leq -\varphi(\|x_k\|),\ \forall x_k \in \Omega,\ x_k \neq 0;$$

- *exponentially stable if there exist four scalars $\bar{\alpha},\ \bar{\beta},\ \gamma,\ p > 0$ such that*

$$\alpha(\|x\|) = \bar{\alpha}\|x\|^p,\ \beta(\|x\|) = \bar{\beta}\|x\|^p,\ \varphi(\|x\|) = \gamma\|x\|.$$

Remark 1.7 *The local definitions of the above two theorems are globally valid if the given functions are class $\mathcal{K}_\infty$ functions*[3] *and $\Omega = \mathbb{R}^n$.*

The function $V : \mathbb{R}^n \to \mathbb{R}_+$ that verifies the properties in the previous theorems is called a "Lyapunov function". By abuse of language, especially in the case of linear systems, a system with a stable and unique equilibrium point is often called a "stable system". Furthermore, if a system is not stable, we will say that it is "unstable".

[3] A class $\mathcal{K}_\infty$ function is a class $\mathcal{K}$ function such that $a = +\infty$ and $\lim_{t \to +\infty} \varphi(t) = +\infty$.

1.2.3 Properties of linear time-invariant systems with sampled-data control

Very interesting properties arise in the context of sampled-data LTI systems, concerning continuous and discrete-time analysis approaches. One of the first concerns the equilibrium's attractivity, and is formulated as follows:

Theorem 1.8 (From [Fujioka 2009b]) *For a given sampled-data LTI system (1.3) with bounded sampling intervals and a given initial state $x(0)$, the following conditions are equivalent:*

(i) $\lim_{t\to+\infty} x(t) = 0$,

(ii) $\lim_{k\to+\infty} x(s_k) = 0$.

This property means that the attractivity of the continuous-time system (1.3) is equivalent to the attractivity of the discrete-time system (1.8).

Further analysis [Hetel 2011a] allows for proving that the continuous-time system's (asymptotic) stability is equivalent to the discrete-time system's (asymptotic) stability, in the more general case of reset control systems ([Nesic 2008], [Beker 2004] [Tarbouriech 2011], [Zaccarian 2005]).

Therefore, it is possible to use both a continuous or a discrete-time approach in order to study the stability of sampled-data systems.

In the following, we will present an overview of some results from the litterature regarding the three main studies concerning sampled-data systems:

- the stability analysis regarding a constant sampling (Problem A);
- the stability analysis regarding time-varying sampling (Problem B);
- the dynamic control of the sampling (Problem C).

1.3 Stability analysis under constant sampling

The first and easiest way to study sampled-data systems is to consider the case when the sampling interval is constant, for a given value T (see Figure 1.2).

In this case, the system's stability is usually analysed using the discrete-time LTI model of the system:

$$x_{k+1} = \Lambda(T)x_k. \tag{1.11}$$

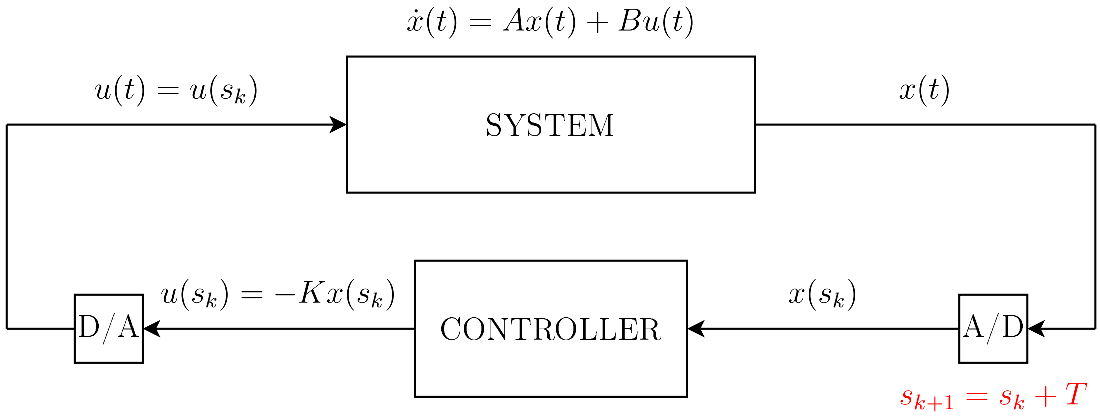

Figure 1.2: Sampled-data system with a constant sampling rate

For a given sampling period T, the most common approach to analyse the stability (the so-called "Schur method") consists in studying the eigenvalues of the transition matrix $\Lambda(T)$. We call $\lambda_{\max}(T)$ the eigenvalue of $\Lambda(T)$ with the largest modulus. We then have the following properties [Aström 1996].

Theorem 1.9 *The equilibrium $x_e = 0$ of (1.11) is*

- *Schur-stable (globally asymptotically stable) if and only if $|\lambda_{max}(T)| < 1$. In that case, $\Lambda(T)$ is called a Schur matrix;*
- *exponentially stable (globally) with a decay-rate $\alpha > 0$ if and only if $|\lambda_{max}(T)| \leq e^{-\alpha T}$.*

Equivalent Linear Matrix Inequality (LMI) stability conditions can also be obtained using the Lyapunov stability theory for discrete-time systems.

Theorem 1.10 *The considered system (1.11) is*

- *stable (globally) if and only if there exists a matrix $P \in S_n^{+*}$ such that*
$$\Lambda(T)^T P \Lambda(T) - P \preceq 0;$$
- *Schur-stable (globally asymptotically stable) if and only if there exists a matrix $P \in S_n^{+*}$ such that*
$$\Lambda(T)^T P \Lambda(T) - P \prec 0;$$

- *exponentially stable (globally) with a decay-rate $\alpha > 0$ if and only if there exists a matrix $P \in S_n^{+*}$ such that*

$$\Lambda(T)^T P \Lambda(T) - e^{-\alpha T} P \preceq 0.$$

The discrete-time analysis of sampled-data systems with a given constant sampling has since long been solved. However, some problems still remain open, since the proposed solutions remain conservative regarding the continuous-time analysis of such systems, or regarding the robustness with respect to exogenous perturbations. For more results regarding robust stability and optimal control of sampled-data systems both in continuous-time and discrete-time, we point to the handbooks [Chen 1991] and [Aström 1996]. In the following section, we will consider the robustness aspect with respect to variations in the sampling interval.

1.4 Stability analysis under time-varying sampling

In the literature, there exist numerous studies about sampled-data systems with a constant sampling interval. In practice however, it may actually be impossible to maintain a constant sampling rate during the real-time control of physical systems. Embedded and networked systems for example are often required to share a limited amount of computational and transmission resources between different applications. This may lead to fluctuations of the sampling interval, because of delays that could appear during the computation of the control, during the transmission of the information, or because of scheduling issues [Zhang 2001c], [Bushnell 2001], [Mounier 2003a]. Such systems are represented by the block diagram in Figure 1.3.

1.4.1 Difficulties and challenges

From the control theory point of view, these variations in the sampling interval bring up new challenges since they may have a destabilizing effect if they are not properly taken into account [Wittenmark 1995], [Zhang 2001b], [Li 2010].

Consider for example the system [Zhang 2001b]:

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0.6 \end{bmatrix} u(t), \ \forall t \geq 0, \\ u(t) &= -\begin{bmatrix} 1 & 6 \end{bmatrix} x(s_k), \ \forall t \in [s_k, s_{k+1}), \ k \in \mathbb{N}. \end{aligned} \tag{1.12}$$

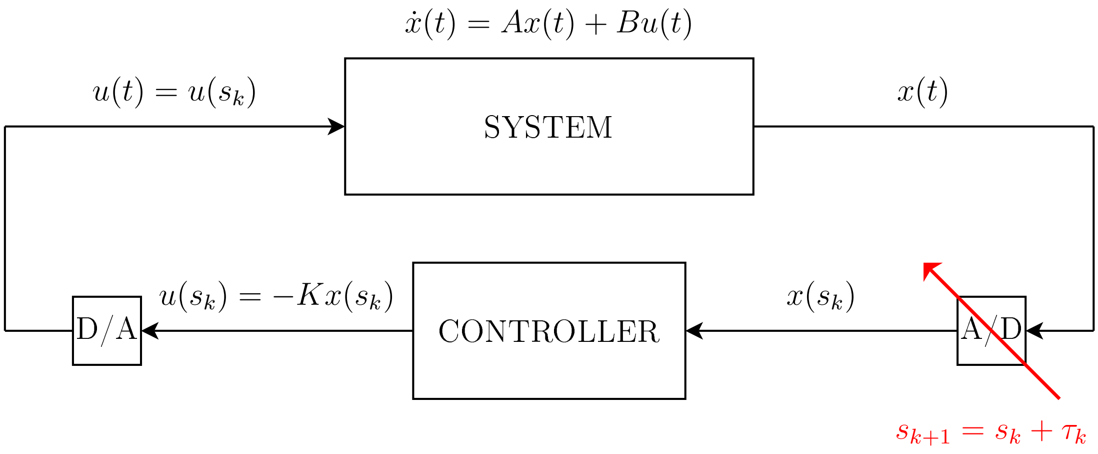

Figure 1.3: Sampled-data system with a time-varying sampling

In the case of a constant sampling rate, one can use a gridding on the sampling step T and the stability conditions from Theorem 1.9, as shown in Figure 1.4, to find that the origin of the system is Schur-stable if $T \in [0s, T^{\max}_{\text{const}} = 0.5937s]$, and unstable for $T \in [T^{\max}_{\text{const}}, 0.9s]$ (as well as for higher values).

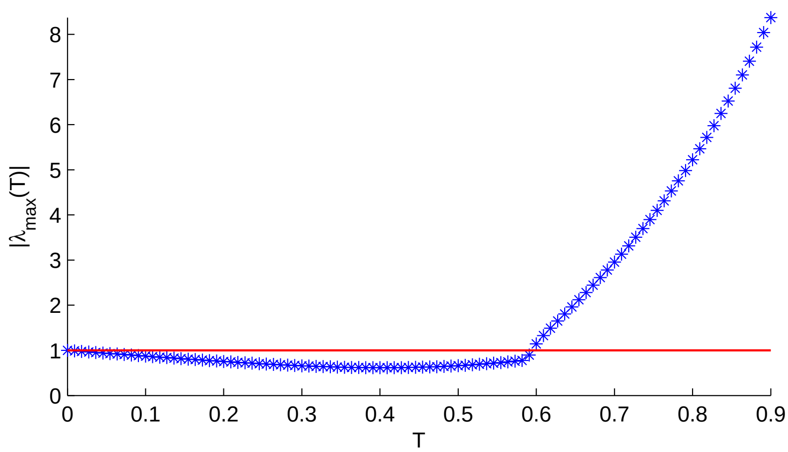

Figure 1.4: Evolution of the modulus $|\lambda_{\max}(T)|$ of the maximum eigenvalue of the transition matrix $\Lambda(T)$, depending on the sampling period T

Therefore, for constant sampling intervals $T_1 = 0.18s$ or $T_2 = 0.54s$ for example, the system is Schur-stable, as illustrated by Figure 1.5.

However, if we sample using a sequence of sampling intervals $T_1 \to T_2 \to T_1 \to T_2 \to \cdots$, the system becomes unstable, as we can see in Figure 1.6.

This is due to the fact that the Schur property of matrices is not preserved under

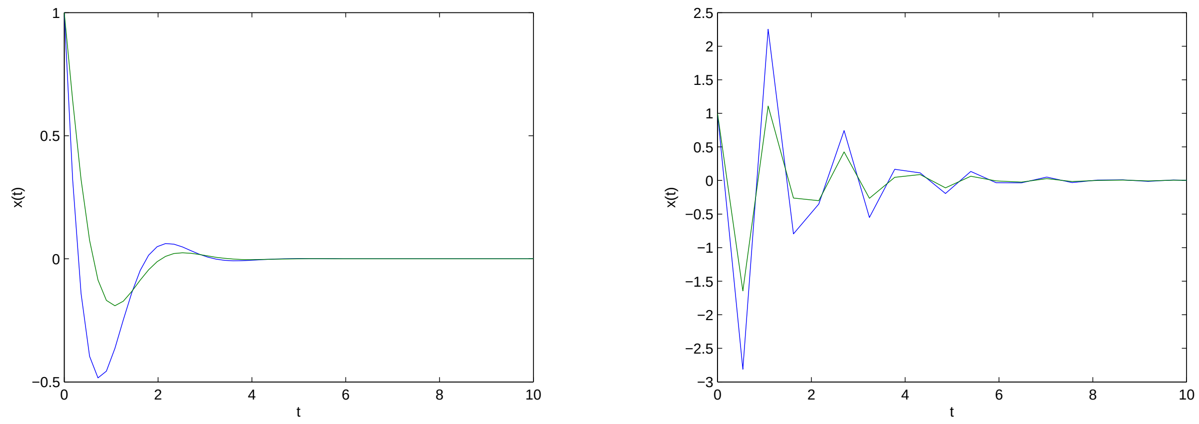

Figure 1.5: Constant sampling rate with $T_1 = 0.18s$ (left) and $T_2 = 0.54s$ (right) - Stable

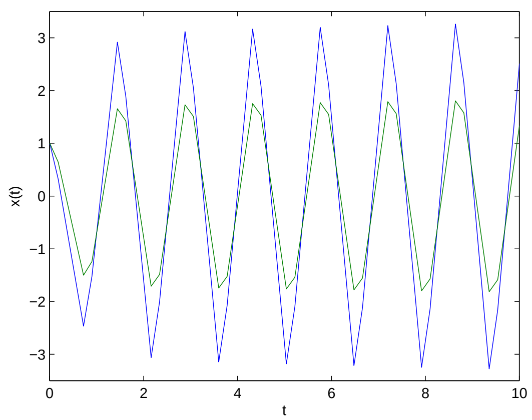

Figure 1.6: Variable sampling intervals $T_1 = 0.18s \to T_2 = 0.54s \to T_1 \to T_2 \to \cdots$ - Unstable

matrix product (*i.e.* the product of two Schur matrices is not necessarily Schur). Indeed, in this case, the discrete-time equivalent system over two sampling instants can be written as

$$x_{k+2} = \Lambda(T_2)\Lambda(T_1)x_k, \;\; \forall k \in 2\mathbb{N},$$

which can also be written as

$$x_{h+1} = \Lambda(T_1, T_2)x_h, \;\; \forall h \in \mathbb{N},$$

with h representing the $2k^{\text{th}}$ sampling, and the transition matrix

$$\Lambda(T_1, T_2) \equiv \Lambda(T_2)\Lambda(T_1) = \begin{bmatrix} 0.8069 & -3.2721 \\ 0.6133 & -2.1125 \end{bmatrix}$$

over two sampling intervals T_1 and T_2, which is not Schur in this example.

In the case of sampled-data systems with a periodic sequence of sampling intervals, it is possible to design a stability domain that depends on the sampling sequence. For instance, Figure 1.7 presents the stability domain (in blue) obtained by using a gridding on the values of T_1 and T_2, in the case of a periodic sequence of two sampling intervals, for the sampled-data system (1.43).

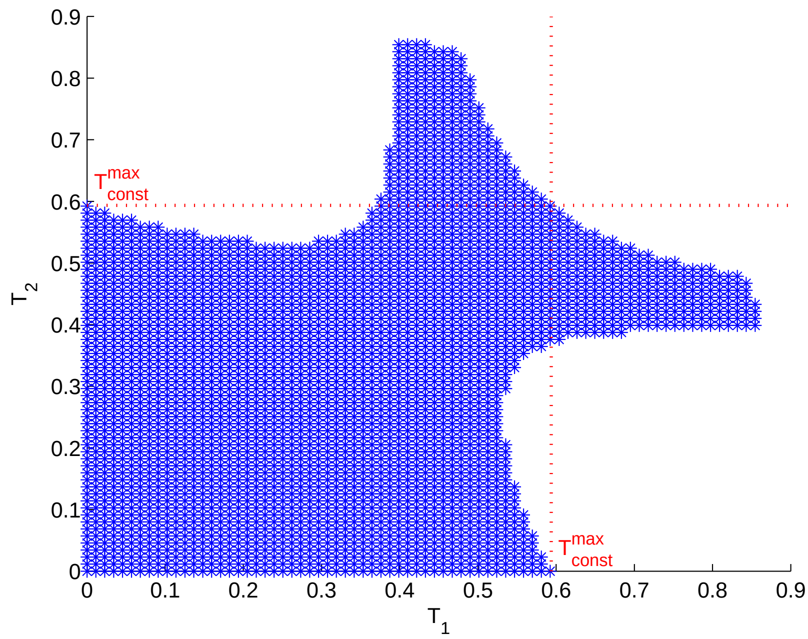

Figure 1.7: Stability domain (allowable sampling interval) for a periodic sampling sequence $T_1 \to T_2 \to T_1 \to T_2 \to \cdots$ - first example

In this figure, one can see that there exist unstable sampling sequences made of stable sampling intervals[4], which confirms our earlier remark. Also, one can see that there exist stable sampling sequences made of both stable and unstable sampling intervals (with $T_1 = 0.46s$ and $T_2 = 0.8s$ for example).

[4]by "stable sampling interval", we mean that the transition matrix of the associated sampling interval is Schur.

Consider now the example

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} 0 & 1 \\ -2 & 0.1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t), \ \forall t \geq 0, \\ u(t) &= -\begin{bmatrix} -1 & 0 \end{bmatrix} x(s_k), \ \forall t \in [s_k, s_{k+1}), \ k \in \mathbb{N}, \end{aligned} \tag{1.13}$$

and its associated stability domain (see Figure 1.8). Here, one can see that there also exist stable sampling sequences which are composed solely of unstable sampling intervals.

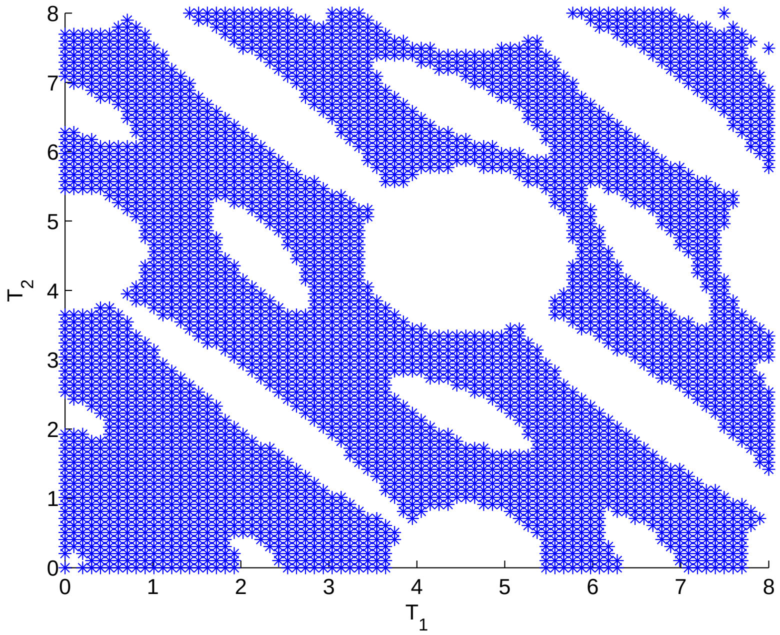

Figure 1.8: Stability domain (allowable sampling interval) for a periodic sampling sequence $T_1 \rightarrow T_2 \rightarrow T_1 \rightarrow T_2 \rightarrow \cdots$ - second example

Let us look at the sampling values $T_1 = 2.126s$ and $T_2 = 3.950s$ for example. The sampled-data system (1.13) is unstable with both constant samplings T_1 and T_2. However, as it is shown in Figure 1.9, the system's transition matrix $\Lambda(T_1, T_2)$ is Schur-stable under the periodic sampling $T_1 \rightarrow T_2 \rightarrow T_1 \rightarrow T_2 \rightarrow \cdots$.

According to the previous observations, it is clear that the existing stability tools for sampled-data systems with a constant sampling will not provide any guarantee of stability for sampled-data systems with unknown time-varying sampling that arises in real-time control conditions. For this reason, considering the difficulty of the problem, several works in the last decades have been concerned with the stability analysis of sampled-data systems

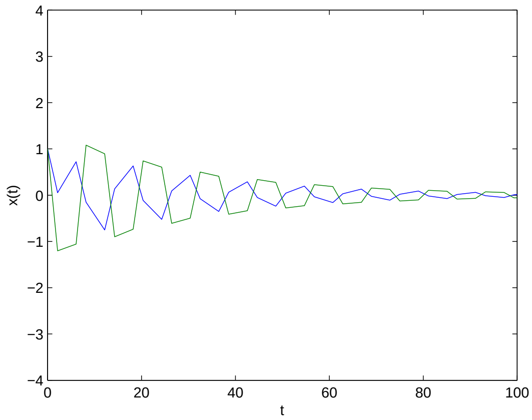

Figure 1.9: Variable sampling $T_1 = 2.126s \to T_2 = 3.950s \to T_1 \to T_2 \to \cdots$ - Stable

with time-varying samplings with bounded values [Mirkin 2007], [Naghshtabrizi 2008], [Hetel 2007], [Fujioka 2009b], [Seuret 2009], [Fridman 2010], and [Hetel 2011b]. Very often, the sampling intervals that are considered can take any value in a bounded set $[\underline{\tau}, \bar{\tau}]$. In the rest of this section, we propose a short overview of various notable methods regarding this issue.

1.4.2 Time-delay approach with Lyapunov techniques

One of the approaches to deal with time-varying sampling was initiated in [Mikheev 1988], and consists in considering the discrete-time dynamics induced by the digital controller as a piecewise continuous delay (see Figure 1.10):

$$s_k = t - (t - s_k) = t - h(t),\ \forall t \in [s_k, s_{k+1}),\ k \in \mathbb{N},$$

where $h(t) \equiv t - s_k$ is the induced delay. The LTI system with sampled-data (1.3) is then re-modeled as an LTI system with time-varying delay

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t),\ \forall t \geq 0, \\ u(t) &= -Kx(t - h(t)),\ \forall t \geq 0, \end{aligned} \tag{1.14}$$

and is studied with classical tools designed for time-delay systems [Richard 2003], [Fridman 2003], [Zhong 2006], [Mounier 2003b] which are defined by retarded functional differential equations as follows:

Definition 1.11 (Time-delay system) *A time-delay system is described by the following functional differential equation:*

$$\begin{aligned} \dot{x}(t) &= f(t, x_t), \ \forall t \geq 0, \\ x_{s_0}(\theta) &= \phi(s_0 + \theta), \ \forall \theta \in [s_0 - \bar{h}, s_0] \end{aligned} \tag{1.15}$$

where $f : \mathbb{R}_+ \times \mathcal{C}^0([-\bar{h}, 0] \to \mathbb{R}^n) \to \mathbb{R}^n$, $\phi \in \mathcal{C}^0([-\bar{h}, 0] \to \mathbb{R}^n)$, *with* $\bar{h} \geq 0$ *the maximal delay, and* $x_t \in \mathcal{C}^0([-\bar{h}, 0] \to \mathbb{R}^n)$, *which represents the state function*[5] *and is defined by:*

$$x_t(\theta) = x(t + \theta), \ \forall \theta \in [-\bar{h}, 0]. \tag{1.16}$$

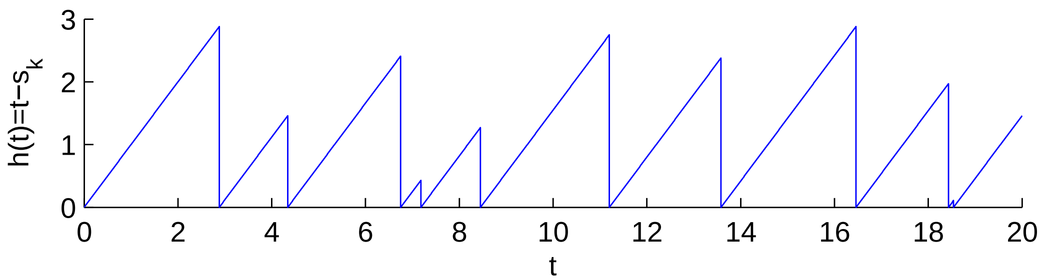

Figure 1.10: Sampling seen as a piecewise-continuous time-delay

It is assumed that there exists a unique solution to the above differential equation (some Lipschitz conditions for the existence and unicity of solutions for such systems are provided in [Gu 2003]), and that there is a unique equilibrium point[6]: $x_e = 0$ (as in the delay-free case, if the equilibrium point is not 0, we can come down to it by using a simple change of coordinates).

In the general case of time-delay systems, it is difficult to apply the classical Lyapunov stability theory from Theorem 1.5, because the derivative $\frac{dV(x)}{dt}$ will depend on the past values of the state: x_t. To overcome this issue, two different stability approaches, better suited to time-delay systems, have been developped. Both of them make use of a wider class of functions or functionals as Lyapunov candidates. The first approach is called Lyapunov-Razumikhin [Gu 2003], and makes use of a time-dependent "energy" function $V \equiv V(t, x(t))$. The second approach, called Lyapunov-Krasovskii [Gu 2003], makes use of a functional $V \equiv V(t, x_t)$ instead.

[5]Note that $x(t)$ is the value of the state at $\theta = 0$: $x(t) = x_t(0)$.

[6]Under existence and unicity of the solution, it can be shown [Dambrine 1994] that the equilibrium state defined by $\dot{x}(t) = 0$ is a constant function $x_t(\theta) \equiv x_e$, thus the expression "equilibrium point" is justified.

1.4.2.1 Lyapunov-Razumikhin approach

In this approach, it is considered a function $V \equiv V(t, x(t))$. The originality is to show that it is not necessary to check the condition $\dot{V}(t, x(t)) \leq 0$ along all the trajectories of the system. Indeed, it is possible to limit this test to solutions which tend to leave a neighbourhood $V(t, x(t)) \leq c$ of the equilibrium point. The approach is formulated as follows.

Theorem 1.12 (Lyapunov-Razumikhin (from [Gu 2003])) *Consider three continuous non-decreasing functions $\alpha,\ \beta,\ \gamma : \mathbb{R}_+ \to \mathbb{R}_+$, β strictly increasing, such that $\alpha(\theta)$ and $\beta(\theta)$ are strictly positive for all $\theta > 0$, and $\alpha(0) = \beta(0) = 0$. Assume that the vector field f from (1.15) is bounded for bounded values of its arguments.*

If there exists a continuously differentiable function $V : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$ such that:

$$\alpha(\|x\|) \leq V(t, x) \leq \beta(\|x\|),\ \forall t \in \mathbb{R}_+,\ \forall x \in \mathbb{R}^n, \tag{1.17}$$

with $\|.\|$ any norm on $\mathbb{R}^n$, and if the derivative of V along the solutions of (1.15) satisfies

$$\dot{V}(t, x(t)) \leq -\gamma(\|x(t)\|) \text{ whenever } V(t+\theta, x(t+\theta)) \leq V(t, x(t)),\ \forall \theta \in [-\bar{h}, 0], \tag{1.18}$$

then the origin of system (1.15) is uniformly stable.

If, in addition, $\gamma(\theta) > 0$ for all $\theta > 0$, and if there exists a continuous non-decreasing function $p : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $p(\theta) > \theta$ for all $\gamma > 0$, and such that condition (1.19) is strengthened to

$$\dot{V}(t, x(t)) \leq -\gamma(\|x(t)\|) \text{ whenever } V(t+\theta, x(t+\theta)) \leq p(V(t, x(t))),\ \forall \theta \in [-\bar{h}, 0], \tag{1.19}$$

then the function V is called a Lyapunov-Razumikhin function, and the origin of system (1.15) is uniformly asymptotically stable.

If in addition $\lim_{s \to +\infty} \alpha(s) = +\infty$, then the origin of system (1.15) is globally uniformly asymptotically stable.

In practice, for simplicity, most existing works about Lyapunov-Razumikhin stability use a linear function: $p(\theta) = q\theta$, with a scalar $q > 1$. Moreover, the Lyapunov-Razumikhin candidates are very often taken as quadratic and time-invariant: $V(x) = x^T P x$, where $P \in \mathcal{S}_n^{+*}$. Some works about the Lyapunov-Razumikhin approach for delayed systems include [Jankovic 2001], [Wang 2007], [Stamova 2001], [Jiao 2005], and [Yu 2004].

One of the advantages of the Lyapunov-Razumikhin stability theory is that it reduces the conservatism with respect to the classic Lyapunov stability theory, and it makes it possible to work with simple Lyapunov(-Razumikhin) functions. Its main drawback is that it may be difficult to obtain checkable delay (or sampling interval)-dependent stability conditions, since the delay (or sampling interval) is not explicitely introduced in the equations. This will be a motivation for employing Lyapunov-Krasovskii techniques to be presented now.

1.4.2.2 Lyapunov-Krasovskii approach

The Lyapunov-Krasovskii approach is an extension of the Lyapunov theory to functional differential equations. Here, we are searching for positive functionals $V \equiv V(t, x_t)$ which are decreasing along the trajectories of (1.15).

Theorem 1.13 (Lyapunov-Krasovskii (from [Gu 2003])) *Consider three continuous non-decreasing functions* $\alpha,\ \beta,\ \gamma : \mathbb{R}_+ \to \mathbb{R}_+$*, such that* $\alpha(\theta)$ *and* $\beta(\theta)$ *are strictly positive for all* $\theta > 0$*, and* $\alpha(0) = \beta(0) = 0$*. Assume that the vector field* f *from (1.15) is bounded for bounded values of its arguments.*

If there exists a continuous differentiable functional $V : \mathbb{R}_+ \times \mathcal{C}^0([-\bar{h}, 0] \to \mathbb{R}^n) \to \mathbb{R}_+$ *such that*

$$\alpha(\|\phi(0)\|) \leq V(t, \phi) \leq \beta(\|\phi\|_{\mathcal{C}}), \tag{1.20}$$

with $\|.\|$ *any norm on* $\mathbb{R}^n$*,* $\|.\|_{\mathcal{C}}$ *its associated norm on* $\mathcal{C}^0([-\bar{h}, 0] \to \mathbb{R}^n)$ *defined by* $\|\phi\|_{\mathcal{C}} = \max_{\theta \in [-\bar{h}, 0]} \|\phi(\theta)\|$*, and if*

$$\dot{V}(t, \phi) \leq -\gamma(\|\phi(0)\|), \tag{1.21}$$

then the origin of system (1.15) is uniformly stable.

If in addition $\gamma(\theta) > 0$ *for all* $\theta > 0$*, then the functional* V *is called a Lyapunov-Krasovskii functional, and the origin of system (1.15) is uniformly asymptotically stable.*

If in addition $\lim_{s \to +\infty} \alpha(s) = +\infty$*, then the origin of system (1.15) is globally uniformly asymptotically stable.*

The functionals that are being considered usually have the form [Kolmanovskii 1996]:

$$\begin{aligned} V(t, \phi) = {} & \phi^T(0) P(t) \phi(0) + \phi^T(0) \left(\int_{-\bar{h}}^0 Q(t, s) \phi(s) ds \right) + \left(\int_{-\bar{h}}^0 \phi^T(s) Q^T(t, s) ds \right) \phi(0) \\ & + \int_{-\bar{h}}^0 \int_{-\bar{h}}^0 \phi^T(s) R(t, s, p) \phi(p) ds dp + \int_{-\bar{h}}^0 \phi^T(s) S(s) \phi(s) ds, \end{aligned} \tag{1.22}$$

where P, Q, R, and $S \in \mathcal{M}_n(\mathbb{R})$. $P(t)$ and $S(s) \in S_n^{+*}$, and R satisfies $R(t,s,p) = R^T(t,p,s)$.

It was proved in [Kolmanovskii 1996] that the existence of such a Lyapunov-Krasovskii functional is necessary and sufficient to ensure the system's stability in the case of LTI systems with time-varying delay (*i.e.* when the system (1.11) is considered with $f(t,x_t) = Ax(t) + A_d x(t-h(t))$). An analytical description of fitting matrix functions Q, R and S has also been presented in [Kharitonov 2003].

In practice (see [Niculescu 2001]), these matrix terms are considered constant, and we search for functionals of the type:

$$\begin{aligned} V(t,\phi) &= \phi^T(0)P\phi(0) + 2\phi^T(0)\left(\int_{-\bar{h}}^0 Q\phi(s)ds\right) + \int_{-\bar{h}}^0 \phi^T(s)S\phi(s)ds \\ &+ \int_{-\bar{h}}^0 \int_{-\bar{h}}^0 \phi^T(s)R\phi(p)dsdp. \end{aligned} \tag{1.23}$$

Although more conservative, this form of Lyapunov-Krasovskii functionals with constant matrices allows to derive LMI stability conditions, which makes it easier to look for solutions (see [Fridman 2004] for instance).

In recent works concerning time-delay systems, the conservatism has been reduced by considering piecewise-constant matrix functions P, Q, R, and S [Fridman 2000], [Fridman 2006], [Gu 1997], [Gu 2003].

In the general case of time-delay systems, one of the drawbacks of the Lyapunov-Krasovskii approach is that the derivative $\frac{dV(t,x_t)}{dt}$ depends on the delay-derivative, which is often unknown. In the case of sampled-data systems [Fridman 2004], [Naghshtabrizi 2008], [Fridman 2010], [Seuret 2012], there is no such issue since the induced delay has a known derivative $\dot{h}(t) = 1$, for all $t \in [s_k, s_{k+1})$, $k \in \mathbb{N}$. This particularity enables to simplify the functionals that are considered and to derive less conservative stability conditions. For example, it has been shown in [Fridman 2010] that the standard time-independent term $\int_{-\bar{h}}^0 \int_{t+\theta}^0 x^T(s)Rx(s)dsd\theta$ used in [Fridman 2004] or [Park 2007] can be advantageously replaced by the term $(s_{k+1}-t)\int_{s_k}^t \dot{x}^T(s)R\dot{x}(s)ds$, which provides time-dependent stability conditions.

One of the advantages of the Lyapunov-Krasovskii approach is that it enlarges, in an essential manner, the class of Lyapunov candidates. It was shown (in [Driver 1977] for the constant delay case, and in [Kolmanovskii 1999] for the general time-varying delay case) that the existence of a Lyapunov-Razumikhin function (LRF) implies the existence of a Lyapunov-Krasovskii functional (LKF). Furthermore, it makes it possible to explicitely introduce the delay (or the sampling intervals) in the equations and to obtain delay (or sampling interval)-dependent stability conditions. Last, recent advances [Fridman 2010] are specifically tuned for sampled-data systems and re-open LKF techniques in a way that they can challenge small-gain approaches, which will be presented in the next subsection. One of the drawbacks, however, is that the design of the Lyapunov-Krasovskii functionals may not be very intuitive, which makes it difficult to identify the source of conservatism of the approach. Also, additional conservatism inherent to this technique is introduced through (sometimes heavy) upper-bounding techniques. These upper-bounds are introduced when checking the sign of the derivative $\dot{V}$, so to condition the problem in a tunable and solvable way (*e.g.* tuning nonlinear into linear matrix inequalities). However, the approach may be easily extended to control design and to the case of systems with parameter uncertainties and perturbations.

1.4.3 Small-gain approach

The idea of the small-gain approach is to consider the influence of the sampling as a perturbation with regard to the continuous control law ($w(t) = K(x(t) - x(s_k))$), and to rewrite the system (1.3) as an interconnection between the system

$$G: \begin{array}{ccc} \mathbb{R}^{n_u} & \to & \mathbb{R}^{n_u} \\ w & \mapsto & z, \end{array}$$

defined as

$$G: \begin{cases} \dot{x}(t) = A_{\mathrm{cl}}x(t) + Bw(t) \\ z(t) = Cx(t) + Dw(t) \end{cases} \tag{1.24}$$

where

$$A_{\mathrm{cl}} = A - BK,\ C = -KA, \text{ and } D = -KB,$$

and the operator

$$\Delta: \begin{array}{ccc} \mathbb{R}^{n_u} & \to & \mathbb{R}^{n_u} \\ z & \mapsto & w, \end{array}$$

defined by

$$w(t) = (\Delta z)(t) \equiv -\int_{s_k}^{t} z(\theta)d\theta,\ \forall t \in [s_k, s_{k+1}),\ k \in \mathbb{N}. \tag{1.25}$$

The stability of the obtained interconnected system (represented in Figure 1.11), can then be guaranteed by applying the small gain theorem:

Theorem 1.14 ([Khalil 2002]) *Assume that the interconnected system* $(G,\ \Delta)$ *is well-*

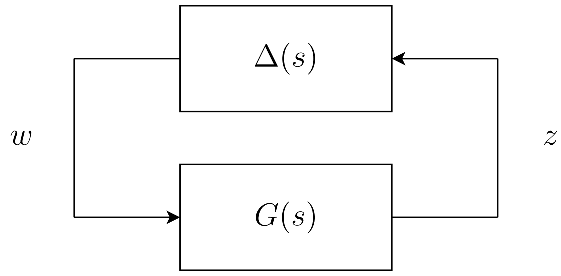

Figure 1.11: Interconnected system

posed and that $\|\Delta\|_{H_\infty}\|G\|_{H_\infty} < 1$. *Then, the closed-loop system is internally stable.*

This approach requires the analysis of the properties of the operator Δ. The first important property of Δ is that it is norm-bounded by a scalar δ_0,

$$\|\Delta\|_{H_\infty} \leq \delta_0, \tag{1.26}$$

that depends on the upper-bound on the sampling interval $\bar{\tau}$. In [Cao 1998], it was shown that $\delta_0 \leq \bar{\tau}$. In [Mirkin 2007], a better approximation of the upper-bound was found using the lifting technique: $\delta_0 \leq \frac{2}{\pi}\bar{\tau}$. This last upper-bound can be shown to be exact since it is attained for a constant sampling interval $\tau_k = s_{k+1} - s_k = \bar{\tau}$. Other properties of the operator Δ can be exploited, such as its commutativity with any linear map $W = \mathbb{R}^{n_u} \rightarrow \mathbb{R}^{n_u}$,

$$W\Delta = \Delta W. \tag{1.27}$$

With these properties, the small-gain theorem allows for writing stability conditions of the form:

$$\|WGW^{-1}\|_{H_\infty} < \frac{1}{\delta_0}, \text{ for any linear map } W = W^T \succ 0, \tag{1.28}$$

where W can be seen as a free variable. By invoking the Kalman-Yakubovich-Popov lemma [Rantzer 1996], it is then possible to derive checkable stability conditions under the form of LMIs [Cao 1998], [Mirkin 2007], [Fujioka 2009b].

Moreover, it is possible to take into account other properties of the operator Δ such as its passivity property for example:

$$< \Delta z, z > \equiv \int_0^{+\infty} z^T(\theta)(\Delta z)(\theta)d\theta \leq 0, \ \forall z \in \mathcal{L}_2, \tag{1.29}$$

as in [Fujioka 2009b]. In this case, the LMI stability conditions can be obtained through the use of Integral Quadratic Constraints [Megretski 1997]. Taking into account more properties of the operator Δ may lead to less conservative results since it may add some other free variables in the obtained LMI stability conditions. This is why in this approach, an important part of the researches are directed to finding new properties of this Δ operator.

The small-gain approach for the stability analysis of sampled-data systems with time-varying sampling is intuitive, and benefits from a large literature about the small-gain applications in robust control. The sources of conservatism from this approach are also well identified. Today however, finding new properties for the operator Δ or a better way to rewrite the sampled-data system as an interconnected system has proved to be difficult, and researches are still under progress.

Although no apparent link exists between the Lyapunov-Krasovskii approach and the small-gain approach, an interesting observation has been made in [Zhang 2001a] (in the general context of LTI systems with delay), and more recently in [Mirkin 2007] (in the particular context of LTI systems with sampled-data systems): in some cases, both approaches may lead to the same LMI stability conditions.

1.4.4 Convex-embedding approach

The convex-embedding approach [Hetel 2006], [Fujioka 2009a], [Cloosterman 2010], [Gielen 2010], is based on the property (1.5) describing the evolution of the system's state $x(t)$ with respect to the sampled-state $x(s_k)$ and the time $t - s_k$:

$$x(t) = \Lambda(t - s_k)x(s_k),\ \forall t \in [s_k, s_{k+1}),\ k \in \mathbb{N},$$

and on the study of the transition matrix operator Λ defined in (1.7). In the case of sampled-data LTI systems (1.3) with time-varying sampling intervals with values in $[\underline{\tau}, \bar{\tau}]$, $\underline{\tau} > 0$, the classic Lyapunov theory in discrete-time can be used with a simple quadratic Lyapunov function $V(x) = x^T Px$, so to obtain sufficient stability conditions under the form of parameter-dependent LMIs:

$$\Lambda(\sigma)^T P \Lambda(\sigma) - P \prec 0,\ \forall \sigma \in [\underline{\tau}, \bar{\tau}]. \tag{1.30}$$

These stability conditions involve an infinite number of LMIs, since they depend on a parameter σ that takes values in the line segment $[\underline{\tau}, \bar{\tau}]$. The idea of the convex-embedding approach is to reduce these conditions down to a finite number, by designing a polytopic over-approximation of the operator Λ. The set of matrices:

$$\mathbf{\Lambda} \equiv \{\Lambda(\tau) | \tau \in [\underline{\tau}, \bar{\tau}]\}, \tag{1.31}$$

can be over-approximated as follows:

$$\mathbf{\Lambda} \subseteq \mathrm{Co}\{F_i\}_{i \in \{1, \cdots, N\}} = \left\{ \sum_{i=1}^{N} \alpha_i F_i | \alpha = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_N \end{bmatrix} \in \mathcal{A} \right\}, \tag{1.32}$$

where $F_i \in \mathcal{M}_{n,n_u}$, $i \in \{1, \cdots, N\}$ are suitably constructed matrices, N is the number of vertices in the polytopic over-approximation, and:

$$\mathcal{A} = \left\{ \alpha \in \mathbb{R}^N | \alpha_i \geq 0, \forall i \in \{1, \cdots, N\}, \text{ and } \sum_{i=1}^{N} \alpha_i = 1 \right\}. \tag{1.33}$$

The properties of the over-approximating convex set $\mathrm{Co}\{F_i\}_{i \in \{1, \cdots, N\}}$ makes it possible to derive a finite number of sufficient stability conditions from (1.30), by writing simple LMIs over the polytope vertices:

$$F_i^T P F_i - P \prec 0, \ \forall i \in \{1, \cdots, N\}. \tag{1.34}$$

Recently, a continuous-time approach to the stability analysis of sampled-data systems based on convexification arguments has been proposed in [Hetel 2011b]. It is based on the parameter-dependent LMI:

$$\begin{bmatrix} \Lambda(\sigma) \\ I \end{bmatrix}^T \begin{bmatrix} A^T P + PA & -PBK \\ * & 0 \end{bmatrix} \begin{bmatrix} \Lambda(\sigma) \\ I \end{bmatrix} \prec 0, \ \forall \sigma \in [\underline{\tau}, \bar{\tau}], \tag{1.35}$$

and the same convexification tools.

Several over-approximation methods to design the polytope vertices F_i from (1.32) can be found in the literature. The main techniques are based on gridding and norm-bounding [Donkers 2009], [Fujioka 2009a], [Skaf 2009], Taylor series expansion [Hetel 2006], [Hetel 2011b], [Hetel 2007], real Jordan form decomposition [Olaru 2008], [de Wouw 2010], [Cloosterman 2010], or the Cayley-Hamilton theorem [Gielen 2010], [Goebel 2009]. A short comparison on numerical examples of these different approaches can be found in [Heemels 2010].

The main advantages of the convex-embedding approach for the stability analysis of sampled-data systems is that it is intuitive, and not very conservative when compared to other methods. Also, it was proved that convex embeddings allow for approaching the stability condition (1.30) as close as desired, by increasing the computational complexity of the over-approximating algorithm. The main drawback of the method is that it is complex to apply, and it may be computationally demanding, depending on the chosen numerical precision.

1.5 Dynamic control of the sampling: a short survey

In the previous section, which concerned sampled-data systems with time-varying sampling, it was shown that using a sequence of stable constant sampling intervals may destabilize the system, while using a sequence of unstable constant samplings may stabilize it. This particular behaviour [Wittenmark 1995], [Zhang 2001b], [Li 2010], very similar to the one observed in switched systems [Liberzon 1999], has been intensively studied over the past decade. In the context of embedded systems and networked control systems particularly, it raised the following problem: how to control the sampling in order to reduce the number of sampling instants while stabilizing the system?

In the last few years, an increasing attention has been brought to this question, and a number of works regarding this issue have been made. Their objective is to reduce the quantity of information sent from the sensors to the actuators, by controlling the sampling through a sampling law (see Figure 1.12):

$$s_{k+1} = s_k + \tau(t, s_k, x(s_k), \cdots), \ \forall k \in \mathbb{N}. \tag{1.36}$$

In the literature, two main approaches covering this issue can be found:

- In the first approach, the **event-triggered control** (also called **event-based control** or **event-driven control** in the literature) [Tabuada 2007], [Cogill 2007], [Heemels 2008], [Lunze 2010], [Mazo Jr. 2011], [Cervin 2007], [Velasco 2009], [Albert 2004], [Wang 2008], [Frazzoli 2012], the sampling is performed only when certain events occur. These events are usually generated when the system's state crosses a frontier in the state space. It may be generated for example when the state is leaving some neighbourhood of the origin, or when the error between the sampled-state $x(s_k)$ and the current state $x(t)$ exceeds a certain bound. A dedicated hardware is required in order to monitor the plant and generate such events.

- The second approach, the **self-triggered control** [Lemmon 2007], [Wang 2009], [Wang 2010], [Mazo Jr. 2009a], [Mazo Jr. 2010], [Anta 2010], [Anta 2009], [Anta 2012], [Dimarogonas 2010], [Araujo 2011], [Tiberi 2010], aims at emulating event-triggered control without dedicated hardware, by computing at each sampling instant a lower-bound of the next admissible sampling interval (*i.e.* an estimation of the next time an event is going to be generated).

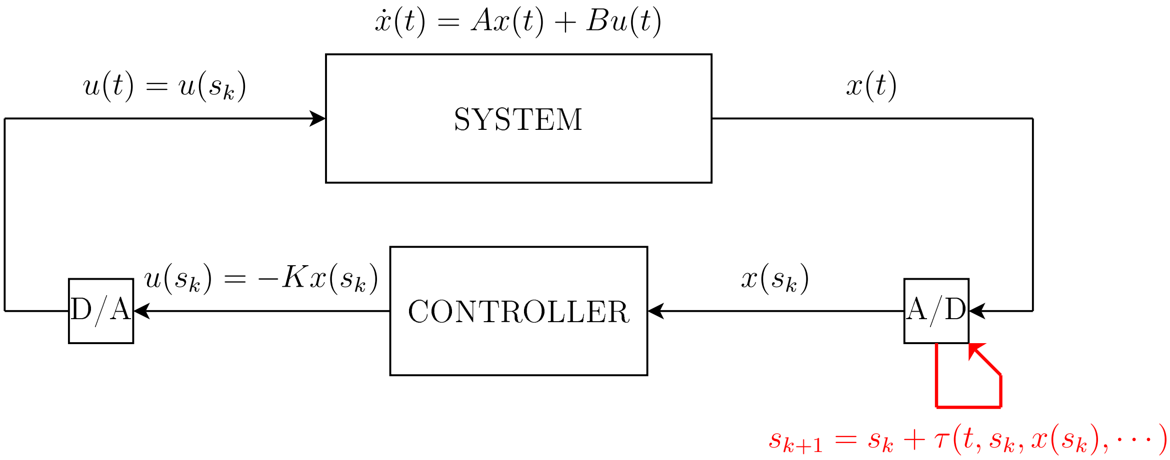

Figure 1.12: Sampled-data system with a dynamic sampling control

Although these two approaches have become very popular in the control community, it is important to note that there exist other ways of dealing with the dynamic control of the sampling, using tools from the computer science community. We can mention the approaches based on adaptive scheduling strategies such as the scheduling (m, k)-firm [Felicioni 2006], [Felicioni 2008], or the control aware computing strategy [Simon 2012] for example.

In the following, we present a brief overview of the most notable results of event-triggered control and self-triggered control from the literature. Note that although these two approaches are technically very different regarding their real-time implementation, their aims are the same: the reduction of the sampling rate. That is the reason why in the following, we will mix both approaches, and present the self-triggered control schemes (when they exist) as extensions or improvements of their associated event-triggered control schemes.

1.5.1 Deadband control approach

The main idea of the first event-triggered controllers was that it is not necessary to update the control of the system when its state is close enough to the equilibrium point. In these works (see [Otanez 2002], or [Cervin 2007] for example), the control is updated only when the state leaves (or also enters, in some works such as [Cervin 2007]) some neighbourhood of the origin. Such neighbourhood (say $\|x(t) - x_0\| \leq \bar{e}$) is called a deadband.

The system generally considered in this approach is:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + Ew(t), \ \forall t \in \mathbb{R}_+, \\ z(t) &= Cx(t), \end{aligned} \tag{1.37}$$

with $z \in \mathbb{R}$ the controlled output of the system, and u a saturated control input.(A, B) and (A, C) are assumed to be respectively stabilizable and detectable. In [Cervin 2007] for example, the authors aim at designing a controller that reduces the number of actuations, while guaranteeing that the state stays in a neighbourhood of the origin. In order to ensure that the disturbances will not make the output drift away from zero, on the one hand, the control outside the deadband is designed as follows:

$$u(t) = -\text{sgn}(z(t)), \ \forall t \geq 0. \tag{1.38}$$

On the other hand, inside the deadband, the control is:

$$u(t) = -\text{sat}(K\hat{x}(t)), \ \forall t \geq 0, \tag{1.39}$$

and is based on a simulation of the ideal evolution of the system, obtained thanks to the following reset observer (placed on the actuators):

$$\begin{aligned} \dot{\hat{x}}(t) &= A\hat{x}(t) + Bu(t), \ \forall t \in \mathbb{R}_+, \\ \hat{x}(s_k) &= x(s_k). \end{aligned} \tag{1.40}$$

In this deadband approach, the reset observer's state is updated only at time s_k when the controlled output z exceeds a certain threshold $z_{\max}$: the event $\hat{x}(t) = x(t)$ is generated when $|z(t)| = z_{\max}$.

The controller gain K is designed to assign the closed-loop matrix $A - BK$ the desired eigenvalues. Note that this observer, which resets the estimated state according to the actual state value $x(s_k)$, suggests a full state-feedback control.

Figure 1.13 presents the controlled output z and the control input u with this event-triggered control scheme for the double integrator [Cervin 2007]:

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \begin{bmatrix} 0 \\ 1 \end{bmatrix} w(t),$$
$$z(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t),$$

with a controller gain $K = \begin{bmatrix} 1 & 2 \end{bmatrix}$, a threshold $z_{\max} = 1$, and a perturbation w considered as a white noise process of intensity 0.01. Note that in the absence of exogenous disturbance, the system is locally asymptotically stable.

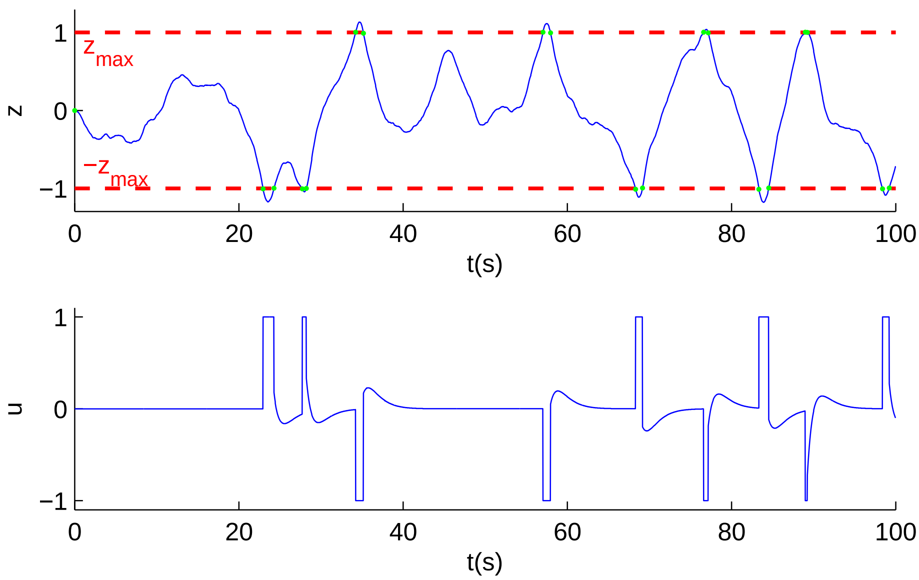

Figure 1.13: Event-triggered control from [Cervin 2007] applied on a double integrator

One of the advantages of this approach lays in its easy implementation. It guarantees the ultimate-boundedness of the system in the presence of bounded perturbations, and its asymptotic stability in their absence. Also, when the system is well known and the perturbations stay small, the number of updates to be sent to the actuator will also be very small. Indeed, in that case, the observer will take the role of a predictor. The drawbacks of the approach are that it is difficult to estimate the instants when the state leaves or enters the deadband (therefore the self-triggered implementation is not obvious), and that only ultimate-boundedness is ensured in the presence of perturbations. Also, dedicated hardware has to be used at both the actuator and sensor's sides, to compute the estimation of the state used in the control input, and to monitor the plant's state in real-time.

1.5.2 Lyapunov function levels approach

Another approach to event-triggered control consists in updating the control only when a chosen Lyapunov function crosses some predetermined energy levels: $V(x(t)) = \mathcal{V}(t, x(s_k))$.

The main idea of the approach is described in [Velasco 2009], in which it is considered a nonlinear sampled-data system:

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t)), \ \forall t \geq 0, \\ u(t) &= -g(x(s_k)), \ \forall t \in [s_k, s_{k+1}), \ k \in \mathbb{N}, \end{aligned} \tag{1.41}$$

with an event generator defined by some levels of a Lyapunov function V:

$$V(x(t)) = \eta V(x(s_k)), \tag{1.42}$$

for some given scalar $0 < \eta < 1$.

In order to ensure the stability of the system with such an event generator, it is necessary to guarantee that after each sampling instant s_k there will be a time $t > s_k$ for which the event-triggering condition (1.42) will be satisfied. Otherwise, it means that there will be no more sampling, and therefore the system will be controlled in open-loop and may become unstable (in the best case, it will be stable, but it will not be attractive). To guarantee that there will be an infinite number of sampling using the generator (1.42), the method proposed in [Velasco 2009], consists in computing an upper-bound η^* of the minimal admissible η (*i.e.* η^* is such that if η satisfies $0 < \eta^* \leq \eta < 1$, then for all $x(s_k) \in \mathbb{R}^n$, there exists $t > s_k$ such that (1.42) is satisfied). In the linear case, a method based on a gridding of the state space can be used to estimate such η^*.

To better understand the approach, consider again the double integrator [Velasco 2009]:

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t), \ \forall t \geq 0 \\ u(t) &= -\begin{bmatrix} 10 & 11 \end{bmatrix} x(s_k), \ \forall t \in [s_k, s_{k+1}), \ k \in \mathbb{N}. \end{aligned} \tag{1.43}$$

For a value of $\eta = 0.8$ (see Figure 1.14, left), one sees that the sampling sequence does not stop, and that the state $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ converges to the equilibrium point. For a value of

$\eta = 0.65$ (see Figure 1.14, right) on the contrary, the sampling sequence stops (the event-triggering condition is not satisfied anymore) and the equilibrium becomes unstable.

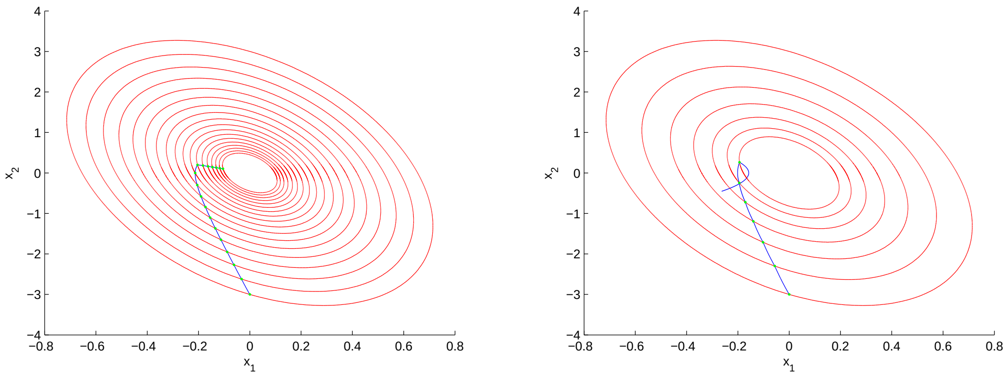

Figure 1.14: Lyapunov function levels approach to dynamic sampling control [Velasco 2009] - $\eta = 0.8 \geq \eta^*$, stable (left) and $\eta = 0.65 < \eta^*$, unstable (right)

The main advantage of such an event-triggered control scheme is that it is easy to understand the control process and to guarantee the stability of the system. Furthermore, the approach can be used in the context of nonlinear sampled-data systems, although no method is proposed to compute the value of η^* in that case. The main drawbacks are that it is still an open problem to choose or compute a suitable Lyapunov function, and that the trigger occurs when entering (and not leaving) the region $V(x(t)) \leq \eta V(x(s_k))$, which means that the sampling occurs unnecessarily.

1.5.3 Perturbation rejection approach

In the literature, one may also find event-triggered control schemes which intend to take into account exogenous perturbations in the control [Lunze 2010], [Lehmann 2011] [Stöcker 2011]. These works aim at controlling the disturbed sampled-data systems, while estimating and rejecting the perturbations, and at the same time enlarging the sampling intervals. In this case, the sensors usually need to include an observer which estimates the perturbation, and a piecewise continuous control is used. The event-generator used for this kind of controller is similar to the one used for deadband control (*i.e.* information is sent from the sensors to the actuators only when the state leaves a neighbourhood of the origin), except that, here, the error generating the trigger is computed with respect to the estimated state, instead of the equilibrium point. The events are thus generated when the measured state $x(t)$ leaves the vicinity

$$\Omega(\hat{x}(t)) = \{x | \|x - \hat{x}(t)\| \leq \bar{e}\} \tag{1.44}$$

of the estimated state $\hat{x}(t)$, for a given threshold $\bar{e}$.

In [Lunze 2010] for instance, it is considered a perturbed sampled-data LTI system:

$$\dot{x}(t) = Ax(t) + Bu(t) + Ew(t), \tag{1.45}$$

with a bounded disturbance $\|w(t)\| \leq w_{\max}$. The disturbance is estimated at each sampling instant s_k as a piecewise constant function:

$$\begin{aligned} &\hat{w}_0 = 0, \\ &\hat{w}_k = \hat{w}_{k-1} + \left(A^{-1}\left(\mathrm{e}^{A(s_k - s_{k-1})} - I\right)E\right)^{+}\left(x(s_k) - \hat{x}(s_k^-)\right), \end{aligned} \tag{1.46}$$

and an observer estimates the state while considering a continuous control feedback $u(t) = -Kx(t)$ and the estimated disturbance $\hat{w}_k$:

$$\begin{aligned} &\dot{\hat{x}}(t) = A_{\mathrm{cl}}\hat{x}(t) + E\hat{w}_k,\ \forall t \in [s_k, s_{k+1}),\ k \in \mathbb{N}, \\ &\hat{x}(s_k) = x(s_k),\ \forall k \in \mathbb{N}, \end{aligned} \tag{1.47}$$

with $A_{\mathrm{cl}} = A - BK$, the closed-loop state matrix of the system.

This estimated state serves as a reference, as the "ideal" system (i.e. continuous control and disturbance known), and is used to define the surrounding (1.44) and to generate the events.

Then, the control input is designed so as to estimate the "ideal" control input $u(t) = -Kx(t)$ (i.e. when the state is continuously available at the actuator):

$$u(t) = -K\mathrm{e}^{A_{\mathrm{cl}}(t - s_k)}x(s_k) - KA_{\mathrm{cl}}^{-1}\left(\mathrm{e}^{A_{\mathrm{cl}}(t - s_k)} - I\right)E\hat{w}_k. \tag{1.48}$$

The main advantage of this approach is that the perturbation is estimated and taken into account in the control, which means that if the system is well known and the perturbation is constant or slowly varying, the sampling will be very sparse. Also, unlike most of the event-triggered approaches which do not take into account the perturbation in the control, this scheme allows the state to converge to its equilibrium even in the case of large (slowly varying) perturbations. However, these advantages have a cost: the system requires dedicated hardware since it is needed to compute the control input in real time, it is required to monitor the plant in real time to check if the system's state does not leave the surrounding (1.44); besides, it is needed to estimate the state in real time, both at the actuator and at the sensor's sides; last, because of the complexity of the controller and the observers, no method has yet been able estimate the instants when the events are generated. Therefore, there does not exist any self-triggered control scheme adapted to this approach.

1.5.4 ISS-Lyapunov function approach

ISS Lyapunov functions constitute another popular dynamic sampling control approach in the literature, used to perform both event-triggered and self-triggered control. It was initiated by [Tabuada 2007], and further developed in [Mazo Jr. 2010], [Anta 2010]), [Anta 2009] and [Anta 2012].

In the general approach proposed in [Tabuada 2007], it is considered a nonlinear sampled-data system:

$$\dot{x}(t) = f(x(t), g(x(s_k))),\ \forall t \in [s_k, s_{k+1}),\ k \in \mathbb{N}, \tag{1.49}$$

rewritten as the reset system

$$\begin{cases} \dot{x}(t) = f(x(t), g(x(t) - e(t))),\ \forall t \geq 0, \\ e(s_k) = 0, \end{cases} \tag{1.50}$$

where $e : \mathbb{R}_+ \to \mathbb{R}^n$ is the measurement error between the current state and the last sampled state ($e(t) = x(t) - x(s_k)$). The considered system is supposed to be ISS-stable with respect to the measurement error e. The following definition is used, derived from the one of the general Input-to-State Stability from [Sontag 2004].

Definition 1.15 (Input-to-State Stability, [Tabuada 2007]) *A smooth function $V : \mathbb{R}^n \to \mathbb{R}_+$ is said to be an ISS Lyapunov function for the closed-loop system (1.50) if there exist four class $\mathcal{K}_\infty$ functions $\underline{\alpha}$, $\bar{\alpha}$, α, and γ satisfying*

$$\begin{array}{c} \underline{\alpha}(\|x\|) \leq V(x) \leq \bar{\alpha}(\|x\|), \\ \frac{\partial V}{\partial x} f(x, g(x+e)) \leq -\alpha(\|x\|) + \gamma(\|e\|), \end{array} \tag{1.51}$$

for some norm $\|.\|$ on $\mathbb{R}^n$. The closed-loop system (1.50) is said to be ISS with respect to the measurement error $e \in \mathbb{R}^n$ if there exists an ISS Lyapunov function for (1.50).

Let us assume that V is an ISS-Lyapunov function for the system (1.50). The basic idea of the approach is that if we ensure that:

$$\gamma(\|e(t)\|) \leq c\alpha(\|x(t)\|), \ \forall t \geq 0, \tag{1.52}$$

for some $0 < c < 1$, then we have:

$$\frac{\partial V}{\partial x} f(x, g(x+e)) \leq (1-c)\alpha(\|x\|), \tag{1.53}$$

which guarantees the asymptotic stability of the system. Therefore, in order to ensure this stability property, one will want to enforce inequality (1.52) by updating the control when

$$\gamma(\|e(t)\|) = c\alpha(\|x(t)\|). \tag{1.54}$$

Given some additional assumptions on the functions f, g, α and γ, one can prove that there exists a lower-bound $\underline{\tau}$ on the sampling interval such that $\gamma(\|e(s_k+\sigma)\|) < c\alpha(\|x(s_k+\sigma)\|)$, for any $\sigma \in [0, \underline{\tau}]$. In the linear case (1.3), the ISS stability conditions are:

$$\begin{array}{c} \underline{a}\|x\|_2^2 \leq V(x) \leq \bar{a}\|x\|_2^2, \\ \frac{\partial V}{\partial x}((A-BK)x - BKe) \leq -b\|x\|_2^2 + c\|e\|_2\|x\|_2, \end{array} \tag{1.55}$$

and it is possible to compute a lower-bound estimation $\underline{\tau}$ of the sampling interval such that $\gamma(\|e(s_k+\sigma)\|) < c\alpha(\|x(s_k+\sigma)\|)$, for any $\sigma \in [0, \underline{\tau}]$, by analyzing the evolution of the term $\frac{\|e\|_2}{\|x\|_2}$. The event generator in the linear case is similar to the one obtained in the nonlinear case (1.54). It is defined as:

$$b\|x(t)\|_2 = c\|e(t)\|_2. \tag{1.56}$$

[Anta 2010] proposed an extension to homogeneous systems, state-dependent homogeneous systems, and polynomial systems. The idea is that for these classes of systems, it is possible to define the sampling function $\tau : \mathbb{R}^n \to \mathbb{R}$ by using scaling laws along the homogeneous rays of the state-space. The principle for the "simple" homogeneous case is the following. Consider a system:

$$\dot{x} = f(x, u), \tag{1.57}$$

such that the control $u = g(x)$ renders the closed-loop system homegeneous of degree $d \in \mathbb{R}_+$. Then, one can show that the sampling function τ defined by the event generator

(1.56) scales according to the law:

$$\tau(\lambda^d x) = \lambda^{-d}\tau(x), \ \forall \lambda \in \mathbb{R}. \tag{1.58}$$

The procedure proposed in [Anta 2010] for designing the sampling function τ for homogeneous and polynomial control systems is based on three steps: first, one needs to design a linear system which trajectories upper-bound the trajectories of the nonlinear system around the origin; second, compute a lower-bound estimation of the maximal allowable sampling for the linear system, using the results from [Tabuada 2007] for example; third, use the proposed scaling law (1.58) for the nonlinear system.

Further developments are proposed in [Anta 2009] and [Anta 2012], where the notion of isochronous manifolds is used to design the scaling laws. In these recent works, the linear over-approximation is not designed over a ball around the system origin, but over submanifolds of the state-space containing the states for which the execution times remain constant.

One of the advantages of this ISS-Lyapunov function approach is to make it possible to compute in advance an estimation of the future maximal allowable sampling times, thus allowing to use a self-triggered control. Also, [Anta 2009], [Anta 2010], and [Anta 2012] have shown easy extensions to a wide class of systems, including linear, homogeneous, or polynomial systems. However, up to now, no method has been proposed to compute the ISS-Lyapunov function V so as to optimize the sampling intervals with this scheme, even in the linear case. Also, no perturbation is taken into account in this approach, except for potentially constant delay in [Tabuada 2007]. Finally, note that the lower-bound estimation of the maximal allowable sampling interval obtained in the linear case is constant. It does not depend on the state, which means that in the linear case, this approach will at best provide results similar to a robust analysis with respect to time-varying sampling.

1.5.5 Upper-bound on the Lyapunov function approach

In the approach presented in [Mazo Jr. 2009b] and [Mazo Jr. 2010], the sampling instants occur when a Lyapunov function crosses a predetermined boundary around the system's origin. Unlike the Lyapunov function levels approach, here the sampling occurs when the state moves away from the equilibrium point. The boundary is chosen as an exponentially decreasing function, so as to ensure the system's exponential stability. The approach aims at designing a sampling function $\tau : \mathbb{R}^n \to \mathbb{R}_+^*$ that enlarges the sampling intervals

$$s_{k+1} - s_k = \tau(x(s_k)), \tag{1.59}$$

for perturbed LTI sampled-data systems:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + w(t),\ \forall t \in \mathbb{R}_+ \\ u(t) &= -Kx(s_k),\ \forall t \in [s_k, s_{k+1}),\ k \in \mathbb{N}, \end{aligned} \tag{1.60}$$

with a disturbance $w(t)$ assumed to be essentially bounded, while ensuring the exponential input-to-state stability.

In the unperturbed case, the idea is as follows. Let V be a Lyapunov function with exponential decay-rate λ_0 for the closed-loop system with continuous feedback $\dot{x}(t) = (A - BK)x(t)$, and define the map $\delta_c(x(s_k), t) \equiv V(x(t)) - V(x(s_k))\mathrm{e}^{-\lambda(t-s_k)}$, for some $0 < \lambda < \lambda_0$. By enforcing:

$$\delta_c(x(s_k), t) \leq 0,\ \forall t \in [s_k, s_{k+1}],\ k \in \mathbb{N}, \tag{1.61}$$

the system's exponential stability is ensured in the absence of perturbation w. Therefore, the proposed event generator ideally becomes:

$$\delta_c(x(s_k), t) = 0. \tag{1.62}$$

In practice however, this condition can not be checked, and therefore the sampling map is discretized into $\delta_d(x(s_k), i) \equiv \delta_c(x(s_k), i\Delta + s_k)$, with Δ the step of discretization, and the new condition becomes:

$$\delta_d(x(s_k), i) \leq 0,\ \forall i \in \left[0, \left\lceil \frac{s_{k+1} - s_k}{\Delta} \right\rceil\right]. \tag{1.63}$$

Then, in order to predict (thanks to the new discretized map) when the event should occur, one needs to compute the maximal i such that (1.63) holds:

$$i(x) = \max_{i \in \mathbb{N}} \{i | \delta_d(x, s) \leq 0, \forall s \in \{0, \cdots, i\}\}, \tag{1.64}$$

and design the sampling function as:

$$\tau(x) = i(x)\Delta. \tag{1.65}$$

The main advantage of this approach is that it allows to perform a self-triggered control for perturbed linear systems which is less conservative than most of the self-triggered works in the literature. It is based on the discretization of the condition (1.61) which makes it unnecessary to study the Lyapunov function's derivative, through conservative upper-bounds. One of the drawbacks is that no method is proposed to choose the Lyapunov function, since it is only required to render the unperturbed closed loop system (exponentially) stable. Therefore, neither the perturbations nor the sampling is taken into account by the Lyapunov function. Furthermore, the sampling function τ is computed online, during the real-time control of the system, and, depending on the discretization step Δ, the computations may become very heavy.

1.5.6 $\mathcal{L}_2$-stability approach

One last approach, developed in [Wang 2009] and [Wang 2010], allows one to perform both event-triggered control and self-triggered control while taking into account both exogenous perturbations and delays. It is based on the notion of $\mathcal{L}_2$-stability [Khalil 2002] and involves algebraic Riccati equations.

It is considered a perturbed, delayed, sampled-data system:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + Ew(t),\ \forall t \in \mathbb{R}_+, \\ u(t) &= -B^T P x(s_k),\ \forall t \in [t_k, t_{k+1}),\ k \in \mathbb{N}, \end{aligned} \tag{1.66}$$

where s_k and t_k denote the k^{th} sampling and actuation times respectively, with a disturbance $w \in \mathcal{L}_2$, and a matrix $P \in S_n^{+*}$ satisfying the $\mathcal{H}_\infty$ algebraic Riccati equation

$$0 = PA + A^T P - PBB^T P + I + \frac{1}{\gamma^2} PEE^T P, \tag{1.67}$$

for some constant $\gamma > 0$.

The aim of the approach is to enlarge the sampling intervals while guaranteeing the $\mathcal{L}_2$-stability [Khalil 2002] of the system.

Definition 1.16 ($\mathcal{L}_2$-stability) *A linear system $\mathbf{F}$ is said to be finite-gain $\mathcal{L}_2$-stable from w to $\mathbf{F}w$ with an induced gain less than γ if $\mathbf{F}$ is a linear operator from $\mathcal{L}_2$ to $\mathcal{L}_2$ and if there exist positive real constants γ and ξ such that for all $w \in \mathcal{L}_2$,*

$$\|\mathbf{F}w\|_{\mathcal{L}_2} \le \gamma \|w\|_{\mathcal{L}_2} + \xi. \tag{1.68}$$

In order to analyze the $\mathcal{L}_2$-stability of system (1.66), we consider a positive semi-definite quadratic function $V : \mathbb{R}^n \to \mathbb{R}_+$ defined as $V(x) = x^T P x$, with the matrix P solution of the Riccati equation given in (1.67). It is possible to show that this particular function satisfies the property:

$$\dot{V}(x(t)) \leq -\beta^2 \|x(t)\|_2^2 + \gamma^2 \|w(t)\|_2^2 + e^T(t) M e(t) - x^T(s_k) N x(s_k), \tag{1.69}$$

for all $t \in [t_k, t_{k+1})$ and all $k \in \mathbb{N}$, for any scalar $\beta \in (0, 1]$, with the measurement error $e(t) = x(t) - x(s_k)$, and matrices M and N defined as:

$$\begin{aligned} M &= (1-\beta^2)I + PBB^T P, \\ N &= \tfrac{1}{2}(1-\beta^2)I + PBB^T P. \end{aligned} \tag{1.70}$$

From the inequality (1.69), we can see that if we enforce that:

$$e^T(t) M e(t) \leq x^T(s_k) N x(s_k), \ \forall t \in [t_k, t_{k+1}), \ k \in \mathbb{N}, \tag{1.71}$$

then we have:

$$\dot{V}(x(t)) \leq -\beta^2 \|x(t)\|_2^2 + \gamma^2 \|w(t)\|_2^2, \ \forall t \in \mathbb{R}_+, \tag{1.72}$$

which guarantees the $\mathcal{L}_2$-stability of the system, with an $\mathcal{L}_2$-gain less than $\frac{\gamma}{\beta}$.

Therefore, the system (1.66) with the event generator $e^T(t) M e(t) = x^T(s_k) N x(s_k)$ is $\mathcal{L}_2$-stable. Furthermore, by analyzing the evolution of the term $e^T(t) M e(t)$ for $t \geq t_k$, it is possible to compute at each sampling instant a lower-bound estimation of the next allowable sampling interval, and thus perform a self-triggered control scheme.

The main advantages of this dynamic sampling control approach are to ensure the $\mathcal{L}_2$-stability of LTI sampled-data systems in the presence of both perturbations and delays and to allow for estimating the next allowable sampling interval at each sampling instant. However, the analytical equations used to estimate the next sampling intervals are very conservative with respect to the proposed event-triggered conditions. Also, it is important to note that the Lyapunov function, which is obtained thanks to the Riccati equation (1.67), does not take into account the sampling nor the delay, and may therefore lead to conservative results, even in the case of event-triggered control.

1.6 Conclusion

This chapter has exposed some recent problems encountered in the context of sampled-data systems, and provided an overview of some important stability and stabilization results from the literature regarding time-varying sampling and dynamic control of the sampling.

The studies concerning robust stability with respect to time-varying sampling are not well fitted for the reduction of the number of sampling instants: they assume that the sampling law is undergone by the system, disregarding the information coming from the sensors. In the works about the dynamic control of the sampling, several issues also remain open. Event-triggered controllers, for instance, require a dedicated hardware to constantly monitor the plant and generate the events in real-time. In the case of self-triggered control works, which are based on Lyapunov functions, no method has been proposed yet to optimize the Lyapunov function while taking into account the effects of the sampling (nor the perturbations in most approaches in the case of perturbed systems, nor the delays in the case of delayed systems). Furthermore, the lower-bound estimations of the next maximal allowable sampling intervals are computed online, during the real-time control of the system, which often requires a heavy processor load.

In the following chapters we intend to solve these problems by proposing a novel approach to the dynamic control of the sampling, that we call "state-dependent sampling". Our point of view is to define a state-dependent sampling law (*i.e.* a map $\tau_{\max} : \mathbb{R}^n \to \mathbb{R}_+^*$) that allows for enlarging the sampling intervals, following some sampling law

$$s_{k+1} - s_k = \tau(t, x(s_k)) \in [\tau^-, \tau_{\max}(x(s_k))].$$

This map is to be computed offline so as to reduce the online computational cost. It also must ensure the stability of the sampled-data system, with some additional convergence or robustness performances. The proposed techniques will make it possible to compute the Lyapunov functions that are used so as to enlarge the lower-bound τ^* of the state-dependent sampling map (*i.e.* the maximal sampling that can be used in the worst case, which can be considered as a state-independent sampling upper-bound), just as in the works about robust stability analysis regarding time-varying sampling. The robustness aspects with respect to exogenous disturbances or delays will also be considered.

Chapter 2

A polytopic approach to dynamic sampling control for LTI systems: the unperturbed case

In this chapter, we present a new state-dependent sampling control that allows one to enlarge the sampling intervals of state-feedback control, in the case of ideal[7] LTI systems with sampled-data.

The objective of our approach is to combine the advantages of the works regarding time-varying sampling:

- maximization of an upper-bound for (state-independent) time-varying sampling;
- design of the Lyapunov function;
- offline computations;

with the advantages of the works regarding dynamic sampling control, especially self-triggered control:

- control of the sampling;
- consideration of the sampled-state in the sampling design;
- estimation of the next maximal allowable sampling interval.

Computationnally speaking, the approach we introduce here remains tractable, since it is grounded as an LMI optimization obtained thanks to:

[7] The next chapters will consider more complex classes.

- a mapping of the state-space, allowing the design of a maximal state-dependent sampling function;

- a polytopic embedding design adapted to a continuous-time stability analysis, allowing one to take into account the inter-sample behaviour;

- Lyapunov-Razumikhin-type stability conditions guaranteeing exponential stability of LTI sampled-data systems for a given decay-rate.

The chapter is organized as follows. To begin with, the next section starts by describing the system and stating the issue. Then, Section 2.2 provides some generic preliminary results, while Section 2.3 presents the main tools, and the main stability results. In Section 2.4 we describe an algorithm that allows for maximizing the state-dependent sampling function. In Section 2.5, the results are illustrated with numerical examples from the literature, and for which the number of actuations is shown to be reduced with respect to the periodic sampling case, before we conclude in Section 2.6.

The proofs of the various propositions, lemmas and theorems can be found in the Appendix A.1, while the design of the polytopic embedding and the mapping of the state-space can be found in Appendices C.2 and B respectively.

2.1 Problem statement

Consider the linear time invariant (LTI) system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \forall t \in \mathbb{R}_+ \\ x(t) &= x_0, \forall t \leq 0, \end{aligned} \tag{2.1}$$

where $x : \mathbb{R} \to \mathbb{R}^n$ and $u : \mathbb{R}_+ \to \mathbb{R}^{n_u}$ represent the system state and the control function, and the matrices A and B are constant and of appropriate dimensions. The control is a piecewise-constant state feedback

$$u(t) = -Kx(s_k), \ \forall t \in [s_k, s_{k+1}), \tag{2.2}$$

where K is fixed and such that $A - BK$ is Hurwitz[8], and where $0 = s_0 < s_1 < \cdots < s_k < \cdots$ are the sampling instants satisfying $\lim_{k\to\infty} s_k = \infty$ and defined by

$$s_{k+1} = s_k + \tau(x(s_k)), \, \forall k \in \mathbb{N}, \tag{2.3}$$

with a state-dependent sampling function $\tau : \mathbb{R}^n \to \mathbb{R}_+$. To ensure the well-posedness of the system, the function is assumed to be lower-bounded by some scalar $\delta > 0$. This guarantees that there is no Zeno phenomenon. The existence of such a lower-bound for our particular design will be proven in Remark 2.6. We denote by $\mathcal{S}$ the closed-loop system $\{(2.1), (2.2), (2.3)\}$. For a given sampling function τ, the solution of $\mathcal{S}$ with initial value x_0 is denoted by $x(t) = \varphi_\tau(t, x_0)$.

In this chapter, our main objective is to provide a way to enlarge as much as possible the state-dependent sampling function τ in (2.3) while ensuring the system's β-stability, for a chosen decay-rate β.

In order to check the system's β-stability, we use a method based on the Lyapunov-Razumikhin approach [Kolmanovskii 1992].

Proposition 2.1 *Given scalars $\alpha > 1$, $\bar{\sigma} > 0$, and $0 < \beta \leq \frac{\ln(\alpha)}{2\bar{\sigma}}$, if there exist a quadratic function $V(x) = x^T P x$, $P = P^T \succ 0 \in \mathcal{M}_n(\mathbb{R})$, and a function $\tau : \mathbb{R}^n \to \mathbb{R}_+$, $0 < \delta \leq \tau(x) \leq \bar{\sigma}$, such that*
(C1): for all $x \in \mathbb{R}^n$, for all $\sigma \in [0, \tau(x)]$, $\dot{V}(\varphi_\tau(\sigma, x)) + 2\beta V(\varphi_\tau(\sigma, x)) \leq 0$ whenever $\alpha V(\varphi_\tau(\sigma, x)) \geq V(x)$,
then the origin of $\mathcal{S}$ is globally β-stable.

Remark 2.2 *Note that α can be seen as a design parameter that can be freely chosen to fit some performances. The smaller α is, the less restrictive the stability condition will be. When α tends to the infinity, one gets a usual Lyapunov stability condition $\frac{dV}{dt} < 0$ with a quadratic Lyapunov function $V(x) = x^T P x$. When α tends to 1, the stability condition is relaxed and tends to be sufficient for ensuring stability, but not attractivity.*

Remark 2.3 *Note that if $\beta = 0$ and the inequality $\dot{V}(\varphi_\tau(\sigma, x)) \leq 0$ in (C1) is reinforced to be strict, then the classical Lyapunov-Razumikhin [Kolmanovskii 1992] theory ensures the system's asymptotic stability.*

It is also important to note that all the stability properties in this paper can be extended to state-dependent time-varying samplings $s_{k+1} = s_k + \tilde{\tau}(s_k, x(s_k))$, $\forall k \in \mathbb{N}$,

[8] a Hurwitz matrix (also called asymptotically stable matrix) is a real square matrix, each eigenvalue of which has a strictly negative real part.

with a time-varying sampling function $\tilde{\tau} : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$. The closed-loop system $\{(2.1), (2.2)\}$ with such a sampling law is denoted $\tilde{\mathcal{S}}$. Then, Proposition 2.1 becomes:

Proposition 2.4 *If there exist functions V and τ satisfying condition (C1) in Proposition 1, then the origin of $\tilde{\mathcal{S}}$ is globally β-stable for any time-varying sampling function $\tilde{\tau} : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$ satisfying $0 < \delta \leq \tilde{\tau}(t,x) \leq \tau(x)$ for all $t \in \mathbb{R}_+$ and for all $x \in \mathbb{R}^n$.*

These two propositions are proven in the Appendix A.1. Throughout this chapter, we will focus on solving two main problems. The first problem concerns the design of the sampling function and is formulated as:

Problem 1: For a given system $\{(2.1),(2.2)\}$ and a given Lyapunov-Razumikhin function (LRF) V, we denote $\tau^V_{\text{opt}}(x)$ the maximal sampling function such that (C1) holds: $\tau^V_{\text{opt}}(x) = \max \tau(x)$.
Find a lower-bound approximation of this optimal function, $\tau^V_{\text{sub}}(x) \leq \tau^V_{\text{opt}}(x)$, as large as possible.

In that formulation, the LRF is supposed to be given, which makes us wonder if there is a way to choose it. Since the objective is to sample as few times as possible, one will also want to make sure the minimal sampling interval is as large as possible by solving the following problem:

Problem 2: For a given system $\{(2.1),(2.2)\}$, we denote τ^*_{opt} the maximal lower-bound of the sampling functions satisfying (C1): $\tau^*_{\text{opt}} = \max \inf_{x\in\mathbb{R}^n} \tau(x)$.
Find an LRF V ensuring (C1) for a sampling function with a lower-bound $\tau^*_{\text{sub}} \leq \tau^*_{\text{opt}}$ as large as possible.

2.2 A generic stability property

In order to provide tractable stability conditions from Proposition 2.1, we first introduce the following Lemma:

Lemma 2.5 *Given scalars $\alpha > 1$, $\bar{\sigma} > 0$, and $0 < \beta \leq \frac{\ln(\alpha)}{2\bar{\sigma}}$, if there exist a matrix $P = P^T \succ 0 \in \mathcal{M}_n(\mathbb{R})$, a scalar $\varepsilon \geq 0$, and a function $\tau : \mathbb{R}^n \to \mathbb{R}_+$, $0 < \delta \leq \tau(x) \leq \bar{\sigma}$, such that for all $x \in \mathbb{R}^n$, for all $\sigma \in [0, \tau(x)]$,*

$$x^T \Phi(\sigma) x \leq 0, \tag{2.4}$$

with

$$\Phi(\sigma) = \begin{bmatrix} \Lambda(\sigma) \\ I \end{bmatrix}^T \Omega \begin{bmatrix} \Lambda(\sigma) \\ I \end{bmatrix}, \tag{2.5}$$

$$\Omega = \begin{bmatrix} A^TP + PA + \varepsilon\alpha P + 2\beta P & -PBK \\ * & -\varepsilon P \end{bmatrix}, \tag{2.6}$$

and

$$\Lambda(\sigma) = I + \int_0^\sigma e^{sA} ds (A - BK), \tag{2.7}$$

then the origin of $\mathcal{S}$ *is globally* β*-stable.*

Remark 2.6 *At the sampling instants,* $\Phi(0) = (A-BK)^TP + P(A-BK) + \varepsilon(\alpha-1)P + 2\beta P$*. If the matrix* P *is such that* $(A-BK)^TP + P(A-BK) \prec 0$ *(there exists such* P *since* $A - BK$ *is Hurwitz), we can find* ε *and* β *as small as needed such that* $\Phi(0) \prec 0$*. Since the function that associates the eigenvalues of* $\Phi(\sigma)$ *with each time* σ *is continuous on* $[0, \tau(x)]$*, there exists a scalar* $\delta > 0$ *such that* $\Phi(\sigma) \preceq 0$ *for all* $\sigma \in [0, \delta]$*. Therefore, with these parameters, there always exist sampling functions* τ *that satisfy Lemma 2.5 conditions, and which are lower-bounded by some scalar* $\delta > 0$*, hence avoiding any Zeno phenomenon issue.*

Remark 2.7 *The use of Lyapunov-Razumikhin type stability conditions is suggested by the delayed nature of the system, since it uses a Zero-Order-Hold control [Fridman 2004]. This method is proved to be less conservative than the usual Lyapunov theory, and the stability conditions using a quadratic function can be easily computed. Similar stability conditions can also be derived from common quadratic Lyapunov functions (see [Fiter 2011c]), input-to-state stable Lyapunov functions, or Lyapunov-Krasovskii functions. All the results that will follow in this chapter can be reformulated for such functions: all the stability conditions can be expressed in the form* $x^T\Phi(\sigma)x \leq 0$*, and only the matrix function* Φ *will change according to the type of Lyapunov function used.*

Remark 2.8 *The conditions of Lemma 2.5 are the same for a state* $x \neq 0$ *and for* λx*,* $\lambda \in \mathbb{R}^*$*. Therefore, it is sufficient to work with homogeneous state-dependent sampling functions of degree* 0 *(i.e. satisfying* $\tau(\lambda x) = \tau(x)$ *for all* $x \in \mathbb{R}^n$*,* $\lambda \in \mathbb{R}^*$*) and to check Lemma 2.5 stability conditions on the unit* n*-sphere.*

2.3 Main stability results

Lemma 2.5 gives some preliminary stability conditions for a state feedback control system with a state-dependent sampling. However, one can see that there is an infinite number of inequalities to check because of both temporal and spatial dependencies in the stability conditions.

2.3.1 Technical tools

To derive a finite number of stability conditions from Lemma 2.5, a two-step tractable methodology is proposed:

2.3.1.1 Conic covering of the state-space

First of all, the state-space is covered by a set of q conic regions

$$\mathcal{R}_s = \{x \in \mathbb{R}^n, x^T \Psi_s x \geq 0\},\ \Psi_s = \Psi_s^T \in \mathcal{M}_n(\mathbb{R}), \tag{2.8}$$

for which sampling intervals $\tau_s > 0$ are associated. We consider state-dependent sampling functions of the form

$$\tau(x) = \max_{s \in \{1,\cdots,q\} \text{ s.t. } x \in \mathcal{R}_s} \tau_s, \text{ for all } x \in \mathbb{R}^n. \tag{2.9}$$

The advantage of such a construction is that it allows to reduce the number of stability conditions from Lemma 2.5 regarding the state variable x to a finite number, by allowing to check some conditions for the finite number of regions instead of checking them for all $x \in \mathbb{R}^n$.

The choice of this conic covering is motivated by the homogeneity brought up in Remark 2.8, which says that the only caracteristic about the state that should be taken into account to design the maximal allowable sampling interval $\tau(x)$ is its direction in the state-space. An illustration of these regions in $\mathbb{R}^2$ is shown in Figure 2.1.

Two possible constructions of such regions are presented in Appendix B.

- Isotropic covering (see Appendix B.1): the first construction is based on the spherical coordinates of the state, and is called "isotropic covering" since it considers q conic regions with the same angle values. For instance, in dimension 2 (see Figure 2.1 for a graphic representation), the angles of the q conic regions have all the same value $\frac{\pi}{q}$. This covering can be designed offline, once for all, so that its online implementation

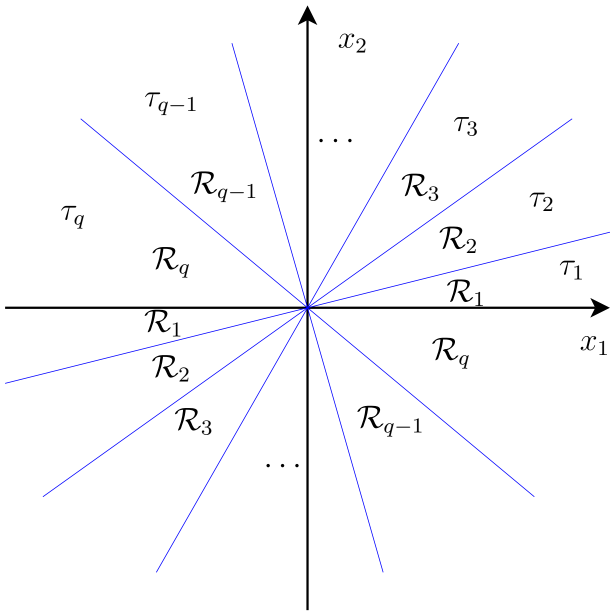

Figure 2.1: Covering the state-space of dimension 2 with q conic regions $\mathcal{R}_s$

does not need much computational power (only to compute the spherical coordinates of the state at each sampling instant).

- Anisotropic covering (see Appendix B.2): the second construction is called "anisotropic covering" and is based on the discrete-time model behaviour of the system. It is also designed offline, but its real-time implementation needs some more computations in order to evalate the position of the state position with respect to the regions.

Each of these two constructions has its own advantages and drawbacks. The advantage of the isotropic covering is that the online computations are reduced and do not depend on the number of regions which are considered. The drawback is that, for a given level of precision, the number of offline computations increases exponentially along with the system's dimension. Concerning the anisotropic covering, the situation is reversed: the offline computations for the anisotropic covering depend linearly on the system's dimension, which means that it is better suited for systems of high dimension, but the number of online computations is larger, and is linearly dependent on the number of regions.

It is important to understand that since the sampling function is defined as constant on these regions (see (2.9)), the precision of the sampling function is linked to the chosen number of conic regions. Therefore, although the system's stability is guaranteed independently of the number of regions, one needs to find a tradeoff between the offline (in the case of an isotropic covering) or online (in the case of an anisotropic covering) computational complexity, and the precision of the state-dependent sampling function.

2.3.1.2 Convex embedding according to time

Let $s \in \{1, \cdots, q\}$. The matrix function $\Phi(\sigma)$ is continuous on the compact set $[0, \tau_s]$. Therefore, it is possible to build a convex polytope defined by a finite set of vertices $\Phi_{\kappa,s}$, $\kappa \in \mathcal{K}_s$ (a finite set of indexes), such that for any $x \in \mathcal{R}_s$,

$$\left(x^T \Phi_{\kappa,s} x \leq 0,\ \forall \kappa \in \mathcal{K}_s\right) \Rightarrow \left(x^T \Phi(\sigma) x \leq 0,\ \forall \sigma \in [0, \tau_s]\right). \tag{2.10}$$

To illustrate the general idea of the approach, a 2D representation of such a convex polytope is presented in Figure 2.2. Note that in reality, this design is not performed over a 2D space, as shown in the figure, but over the space of $n \times n$ matrices: $\mathcal{M}_n(\mathbb{R})$.

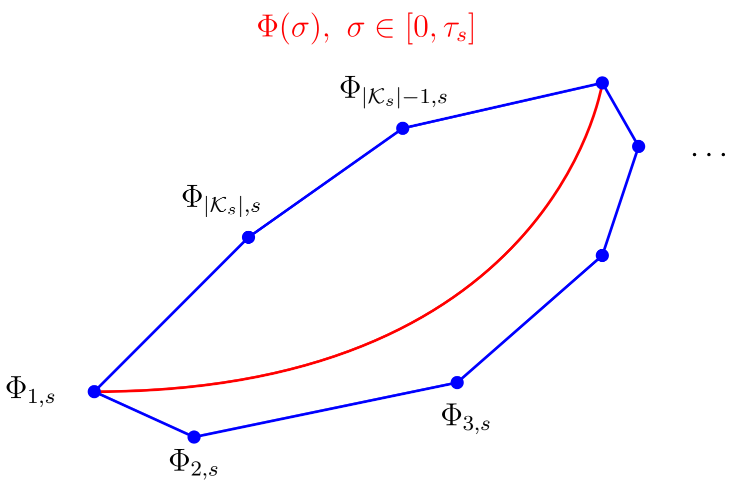

Figure 2.2: 2D representation of a convex polytope around the matrix function Φ over the time interval $\sigma \in [0, \tau_s]$

Similarly to the state-space covering, this construction allows to reduce the number of stability conditions from Lemma 2.5 regarding the time variable σ to a finite number, by allowing to check some conditions on the polytope vertices instead of for all $\sigma \in [0, \tau_s]$.

Note that the form of the matrix function Φ given by (2.5) enables to build these vertices as linearly dependent on P, and dependent on the parameters α, ε, β, and $\bar{\sigma}$, which will be very helpful to derive LMI stability conditions later on.

One possible construction of a convex polytope satisfying (2.10) is provided in Appendix C.2, Lemma C.2 (equations (C.1) to (C.7)). It makes use of the convexification technique proposed by [Hetel 2006] (presented in Appendix C.1), which allowed to build convex hulls around exponential matrix functions using Taylor polynomials. Here, the major difficulty comes from the fact that the exponential uncertainty $\Lambda(\sigma)$ appears in a bilinear manner in the stability conditions from Lemma 2.5:

$$x^T \begin{bmatrix} \Lambda(\sigma) \\ I \end{bmatrix}^T \Omega \begin{bmatrix} \Lambda(\sigma) \\ I \end{bmatrix} x \leq 0,\ \forall x \in \mathbb{R}^n \text{ and } \sigma \in [0, \tau(x)]. \tag{2.11}$$

The design of this convex polytope being quite technical and complex, it has been left to the Appendix C.2 in order to improve the readability of the manuscript, after presenting the convex polytope tools from [Hetel 2006], in Appendix C.1.

2.3.2 Stability results in the case of state-dependent sampling

Using these steps, we derive the following Theorem that guarantees the system's β-stability for a given sampling function τ.

Theorem 2.9 *Let a matrix $P = P^T \succ 0 \in \mathcal{M}_n(\mathbb{R})$, and scalars $\varepsilon \geq 0$, $\alpha > 1$, $\bar{\sigma} > 0$, and $0 < \beta \leq \frac{\ln(\alpha)}{2\bar{\sigma}}$ be given.*
Consider the conic regions (2.8), sampling intervals $\tau_1, \cdots, \tau_q$ satisfying $0 < \tau_s \leq \bar{\sigma}$, and matrices $\Phi_{\kappa,s}$ satisfying (2.10), for all $s \in \{1, \cdots, q\}$, $\kappa \in \mathcal{K}_s$. Let the sampling function $\tau : \mathbb{R}^n \to \mathbb{R}_+$ be defined as $\tau(x) = \tau_s$ for all $x \in \mathcal{R}_s$ and $s \in \{1, \cdots, q\}$.
If there exist scalars $\varepsilon_{\kappa,s} \geq 0$ such that the LMIs

$$\Phi_{\kappa,s} + \varepsilon_{\kappa,s}\Psi_s \preceq 0 \tag{2.12}$$

are satisfied for all $s \in \{1, \cdots, q\}$ and $\kappa \in \mathcal{K}_s$, then the origin of $\mathcal{S}$ is globally β-stable.

Theorem 2.9 provides sufficient conditions for Lemma 2.5, which enables to compute a lower-bound approximation of the optimal sampling function τ^V_{opt} (*i.e.* a solution to Problem 1).

Remark 2.10 *From Theorem 2.9 and Proposition 2.4, similar results can be obtained for any time-varying sampling function $\tilde{\tau} : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$ satisfying $0 < \delta \leq \tilde{\tau}(t, x) \leq \tau(x)$ for all $t \in \mathbb{R}_+$ and for all $x \in \mathbb{R}^n$.*

2.3.3 Stability results in the case of time-varying sampling

The following corollary proposes a method to analyse the stability in the case of (state-independent) time-varying sampling. It will be used so as to design the LRF, in order to optimize the lower-bound of the sampling function.

Corollary 2.11 *Consider a covering of the state-space composed of one single region $\mathcal{R} = \mathbb{R}^n$. Consider $\varepsilon \geq 0$ a tuning parameter. Let scalars $\alpha > 1$, $\bar{\sigma} > 0$, and $0 < \beta \leq \frac{\ln(\alpha)}{2\bar{\sigma}}$, and matrices Φ_κ satisfying (2.10), with $\kappa \in \mathcal{K}$ (the indexes s denoting the regions in (2.10) are dropped since we consider only one region: $\mathcal{R} = \mathbb{R}^n$). Let us assume that the sampling function $\tau : \mathbb{R}^n \to \mathbb{R}_+$ satisfies $\tau(x) = \tau^*$ for all $x \in \mathbb{R}^n$, for a given scalar $0 < \tau^* \leq \bar{\sigma}$. If there exists a matrix $P = P^T \succ 0 \in \mathcal{M}_n(\mathbb{R})$ such that the LMIs $\Phi_\kappa \preceq 0$ are satisfied for all $\kappa \in \mathcal{K}$, then the origin of system (2.1) is globally β-stable regarding the control (2.2) for any time-varying sampling bounded by τ^*.*

Remark 2.12 *For a given value of ε, one can compute the maximal τ^* (denoted τ^*_ε) for which the stability conditions from Corollary 2.11 are satisfied, by using a line search algorithm on the variable τ^* and LMI solvers. Another line search algorithm is then used on the variable ε so as to compute an estimation of the largest upper-bound for time-varying samplings: $\tau^*_{sub} = \sup_{\varepsilon \geq 0} \tau^*_\varepsilon$.*

Remark 2.13 *The state-independent Corollary 2.11 can be used to compute: an upper-bound estimation τ^*_{sub} for time-varying samplings as in the framework of robust control techniques (i.e. guaranteeing β-stability for any time-varying sampling bounded by τ^*_{sub}), which is also a lower-bound estimation of τ^*_{opt} (i.e. a solution to Problem 2); the LRF $V(x) = x^T P x$ used for the state-dependent sampling design (in Theorem 2.9).*

2.4 General algorithm to design the sampling function

Theorem 2.9 and Corollary 2.11 may be used to solve Problems 1 and 2 respectively. While Corollary 2.11 gives a way to compute the LRF parameters P and ε maximizing an estimation of the lower-bound τ^* of the sampling function τ under the stability conditions of Proposition 2.1, Theorem 2.9 gives a way to approximate the sampling function τ^V_{opt} on state regions, for given P and ε. A method to apply the proposed technique is the following:

Step 1: First, use Corollary 2.11 and the polytopic description (C.2) with $\nu = 0$. Then, the research for P is an LMI problem, and we may optimize the search of a lower-bound estimate $\hat{\tau}^*_{\text{sub}}$ of τ^*_{opt} as well as its associated ε using the technique proposed in Remark 2.12.

Step 2: Next, we compute the value ν assigned to the obtained P and ε, and we evaluate the matrix inequalities $\Phi_\kappa \preceq 0$ in Corollary 2.11 so as to obtain the value $\tau^*_{\text{sub}} \leq \hat{\tau}^*_{\text{sub}}$ which satisfies the stability conditions.

Step 3: Finally, the LMI conditions from Theorem 2.9 are used with the computed values of P, ε and ν to approximate the maximal state-dependent sampling function τ^V_{opt} (*i.e.* $\tau^V_{\text{sub}}(x) = \max \tau_s$, $\forall x \in \mathcal{R}_s$, $s \in \{1, \cdots, q\}$, such that the LMIs (2.12) hold). Note that it is possible to solve the LMIs to maximize the sampling times τ_s on each region separately.

Remark 2.14 *This algorithm provides a practical method to build a lower-bound approximation τ^V_{sub} of the optimal sampling function τ^V_{opt}. As most of the numerical methods, there is no a priori evaluation of the gap between the obtained function and the optimal function. However, the benefits of this technique are shown for some benchmarks from the literature in Section 2.5.*

2.5 Numerical examples

2.5.1 Example 1

Consider the following system from [Hetel 2011b]:

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} -0.5 & 0 \\ 0 & 3.5 \end{bmatrix} x(t) - \begin{bmatrix} 1 \\ 1 \end{bmatrix} K x(s_k), \\ K &= \begin{bmatrix} -1.02 & 5.62 \end{bmatrix}. \end{aligned}$$

After setting the polynomial approximation degree term $N = 5$, the number of polytopic subdivisions $l = 100$, and the number of equal conic regions $q = 100$ (isotropic covering on the unit sphere $x = \mathrm{e}^{\mathrm{i}\theta}$, $\theta \in [-\pi, \pi]$, see Appendix B.1), we can obtain a mapping of the state-space that gives the maximal allowable sampling interval for each state for a given decay rate $\beta > 0$ thanks to Corollary 2.11 and Theorem 2.9. For each β, after fixing $\bar{\sigma}$, we set the LRF performance parameter $\alpha > 1$ (see Remark 2.2) as small as possible and such that $\beta \leq \frac{\ln(\alpha)}{2\bar{\sigma}}$. The state-dependent sampling functions obtained offline and ensuring the β-stability of the system for different decay rates β are presented in Figure 2.3.

For a constant sampling greater than $T^{\max}_{\text{const}} = 0.469s$ the discrete-time dynamic matrix is not Schur anymore, so the system becomes unstable. However, with the proposed

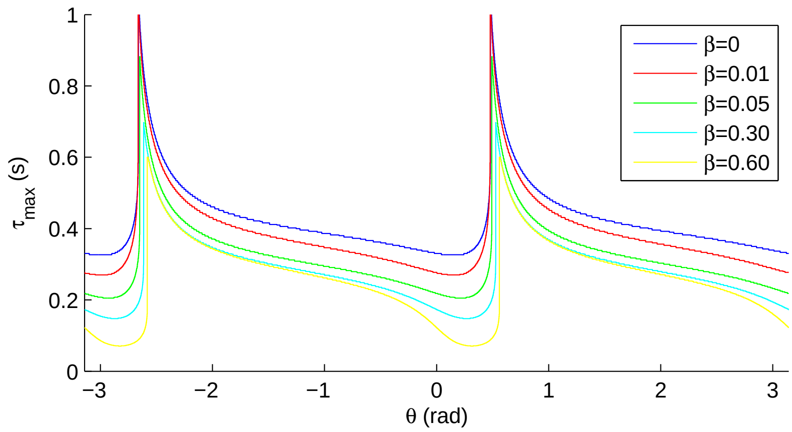

Figure 2.3: Example 1: State-angle dependent sampling function τ for different decay rates β

technique, we can go beyond the limit $T^{\max}_{\text{const}}$ for some regions of the state-space (up to $1s$ for $\beta = 0$).

Figure 2.4 (resp. Figure 2.5) shows simulation results with $\beta = 0$ (resp. $\beta = 0.05$) and a random initial state. It first shows the sampling intervals (the blue piecewise-constant curve), with the lower-bound of the offline computed state-dependent sampling function (the red horizontal line), and the limit $T^{\max}_{\text{const}}$ of the periodic case (the green horizontal line), before showing the LRF evolution. The sampling times are represented by the red dots on each graph. Note that the evolution of the LRF illustrates the conservatism of the (sufficient) stability conditions from Theorem 2.9. For $\beta = 0$, for instance (see Figure 2.4), the triggering condition from Proposition 2.1 should be $V(x(t)) = \frac{V(x(s_k))}{\alpha} \simeq V(x(s_k))$, when $\dot{V}(x(t)) > 0$ (α was set to 1.001). Thus, the gap between $V(x(s_k))$ and $V(x(t))$ at the triggering instants in the simulation represents the conservatism of the method.

In Figure 2.4 ($\beta = 0$), one can see that the number of actuations over the $20s$ time interval is 31 instead of 43 with $T^{\max}_{\text{const}}$. For any (tested) initial condition in the simulation, the average sampling time converges to $T_{\text{average}} \simeq 0.726s \simeq 155\% T^{\max}_{\text{const}}$.

For a given decay-rate $\beta > 0$, the maximal constant sampling ensuring β-stability is given by $T^{\max}_{\text{const}}(\beta) = \operatorname{argmax}\left\{T > 0, -\frac{\ln(|\lambda_{\max}|)}{T} \geq \beta\right\} < T^{\max}_{\text{const}}$, where $\lambda_{\max}$ is the eigenvalue of $A_d(T)$ with greatest modulus. In the simulation of Figure 2.5 ($\beta = 0.05$), we can observe that $T_{\text{average over 20s}}(\beta = 0.05) = 0.486s > T^{\max}_{\text{const}} = 0.469s > T^{\max}_{\text{const}}(\beta = 0.05) = 0.457s$.
This means that it is possible to sample less in average than with the maximal periodic sampling $T^{\max}_{\text{const}}$ while still ensuring asymptotic or exponential stability. Although we can

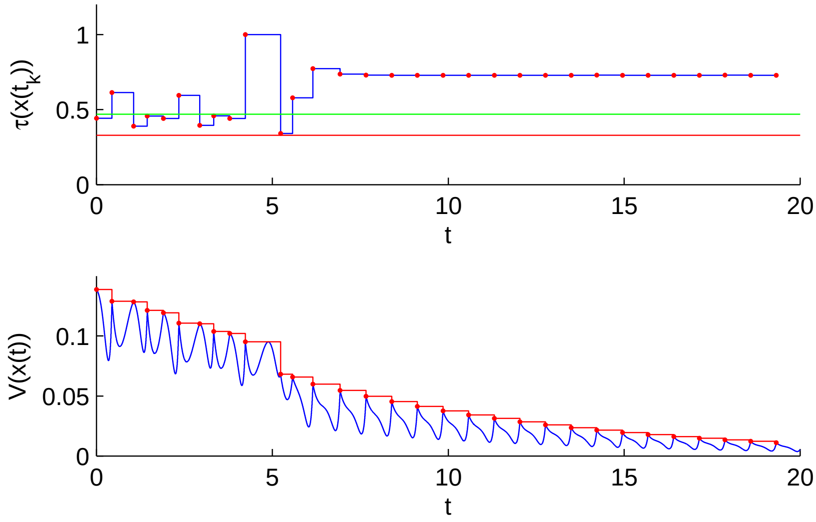

Figure 2.4: Example 1: Inter-execution times $\tau(x(s_k))$ and LRF $V(x) = x^T P x$ for a decay rate $\beta = 0$

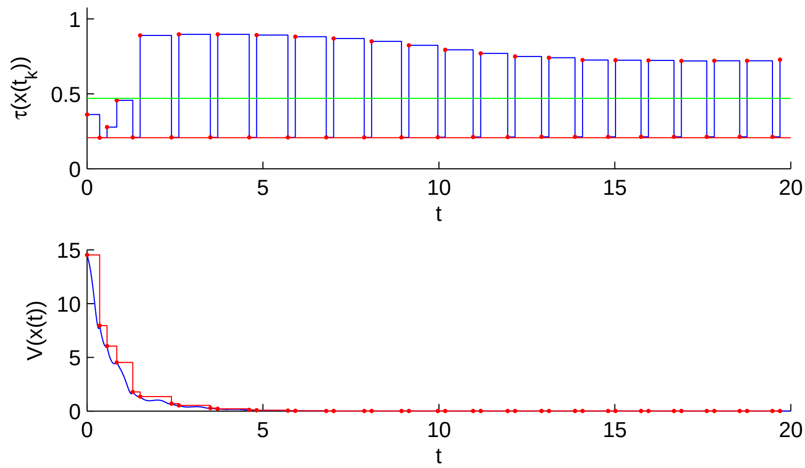

Figure 2.5: Example 1: Inter-execution times $\tau(x(s_k))$ and LRF $V(x) = x^T P x$ for a decay rate $\beta = 0.05$

not guarantee that this will always be the case, the state-dependent sampling presents some advantages compared to periodic sampling:

- It ensures some convergence performance (β-stability for a given decay-rate β, or asymptotic stability if $\beta = 0$), whereas constant sampling with $T_{\text{const}}^{\max}$ only ensures marginal stability and doesn't give any hint about the inter-sampling state behaviour.

- It guarantees robustness regarding possible fluctuations of the sampling period, which is inherent to practical applications (due to scheduling issues for example). The state-dependent sampling approach ensures the system's β-stability for any time-varying sampling period satisfying $0 < \delta \leq \tilde{\tau}(t,x) \leq \tau(x)$, for all $t \in \mathbb{R}_+$ and for all $x \in \mathbb{R}^n$ (see Remark 2.10).

Note that in many numerical examples, the lower-bound τ_{sub}^* of the sampling function is usually not far from the value of $T_{\text{const}}^{\max}$. In the worst case scenario, we can take a constant sampling interval equal to τ_{sub}^*. Also, since Remark 2.10 ensures asymptotic stability for any time-varying sampling bounded by the designed function τ with $\beta = 0$ (*i.e.* any time-varying sampling with values under the blue curve in Figure 2.3), it is also interesting to compare the lower-bound $\tau_{\text{sub}}^* = 0.329s$ (computed using Corollary 2.11) of the designed state-dependent sampling function with the maximum upper-bounds obtained in recent papers about (state-independent) time-varying sampling, as shown in Table 2.1.

[Naghshtabrizi 2008]	[Seuret 2009]	[Fujioka 2009b]	[Fridman 2010]	Corollary 2.11
$0.165s$	$0.198s$	$0.204s$	$0.259s$	$0.329s$

Table 2.1: Example 1: Maximum upper bounds τ_{sub}^* for time-varying samplings, allowable on the whole state space

2.5.2 Example 2

Consider the Batch Reactor system from [Mazo Jr. 2009a]:

$$\dot{x}(t) = \begin{bmatrix} 1.38 & -0.20 & 6.71 & -5.67 \\ -0.58 & -4.29 & 0 & 0.67 \\ 1.06 & 4.27 & -6.65 & 5.89 \\ 0.04 & 4.27 & 1.34 & -2.10 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 5.67 & 0 \\ 1.13 & -3.14 \\ 1.13 & 0 \end{bmatrix} u(t),$$

$$K = \begin{bmatrix} -0.1006 & 0.2469 & 0.0952 & 0.2447 \\ -1.4099 & 0.1966 & -0.0139 & -0.0823 \end{bmatrix}.$$

We use the same parameters $N = 5$ and $l = 100$ as in the previous example, along with $\bar{\sigma} = 1s$ and $q = 30$ conic regions built using the method proposed in the Appendix B.2, and design the mapping of the state-space for $\beta = 0$. Figure 2.6 shows a representation of this mapping with respect to the angular coordinates of the state. This state-space mapping (in dimension 3 if we consider only the angular coordinates and omit the radius of the state) provides a precise knowledge of the sampling function τ (which varies from

$\tau^*_{\text{sub}} = 0.4409$ to $0.9883 \leq \bar{\sigma}$). In comparison, the value of the maximal allowable constant sampling $T^{\max}_{\text{const}}$ is $0.5534s$. Using this mapping, we obtain the simulations shown in Figure 2.7.

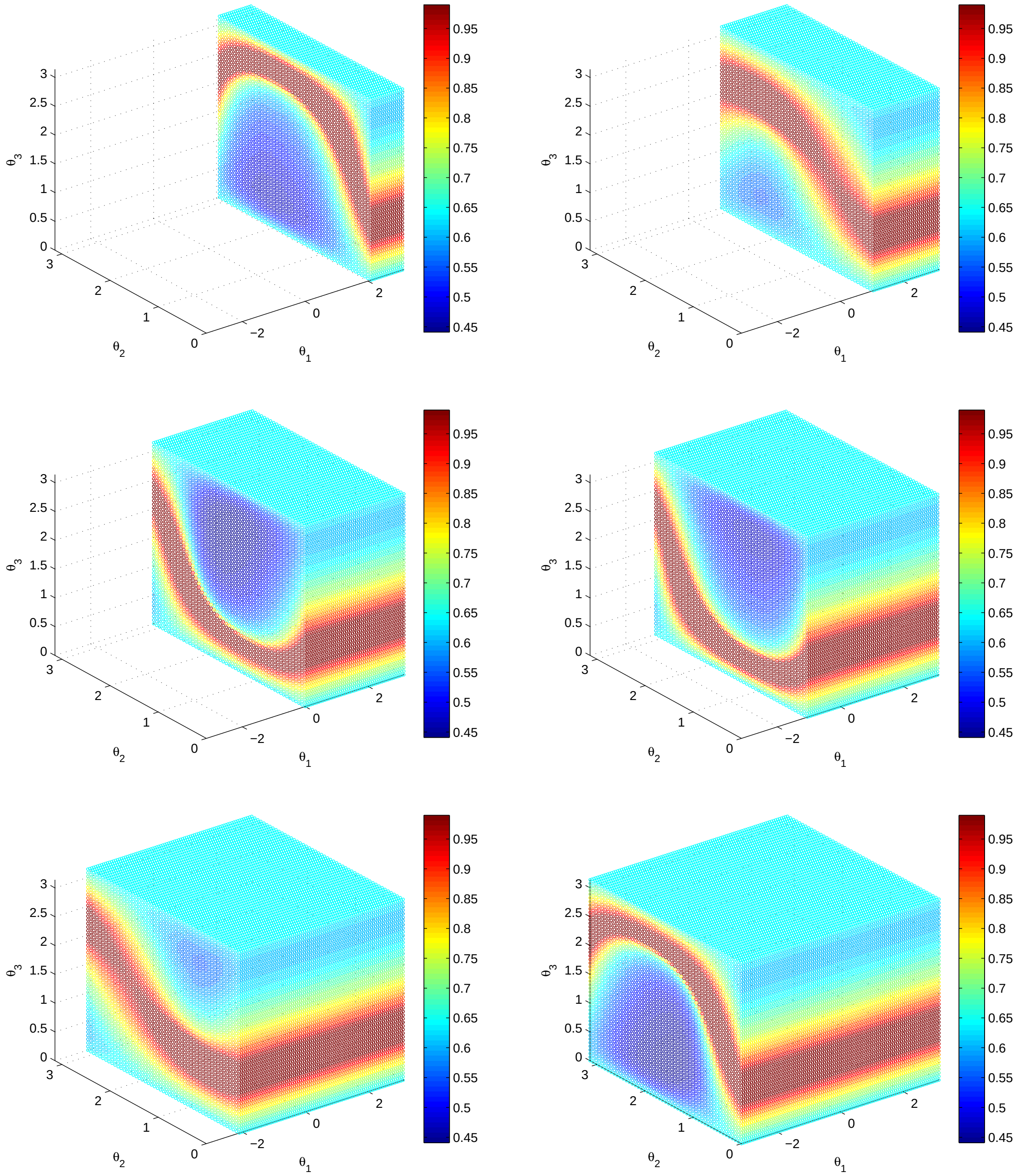

Figure 2.6: Example 2: Mapping of the state-space (regarding the angular coordinates) for $\beta = 0$ - The redder, the larger the maximal allowable sampling interval

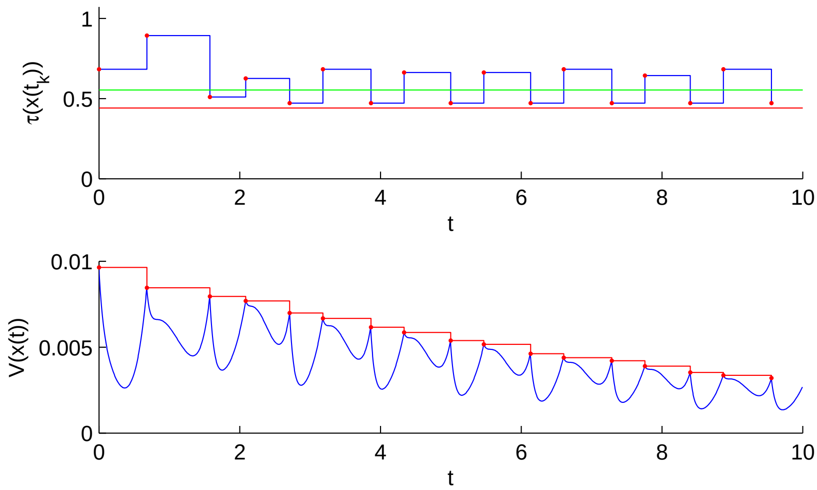

Figure 2.7: Example 2: Inter-execution times $\tau(x(s_k))$ and LRF $V(x) = x^T Px$ for a decay rate $\beta = 0$

The number of actuations over the first $10s$ time interval (see Figure 2.7) is 17, which can be compared to the number of updates presented in [Mazo Jr. 2009a] (32 in the best presented case), and the obtained average sampling time is $T_{\text{average}} = 0.5898 > T^{\max}_{\text{const}}$. This example can also be treated via the isotropic conic covering presented in Appendix B.1. With 8000 conic regions, one obtains 21 updates over the first 10s.

2.6 Conclusion

In this chapter, we have introduced an LRF based design for a state-dependent sampling function τ ensuring the exponential stability with a given decay-rate for ideal LTI sampled-data systems. The proposed method can be seen both as an offline self-triggered control scheme and as a new time-varying sampling analysis leading to a state-dependent sampling design. A lower-bound estimation of the maximal sampling function is proposed. The method presents several advantages:

- It makes it possible to maximize the lower-bound τ^* of the proposed function;
- It provides the associated LRF parameters;
- The real-time implementation takes advantage of an offline designed mapping of the next sampling interval with respect to the past sampled-state value.

Chapter 3

A polytopic approach to dynamic sampling control for LTI systems: the perturbed case

In the previous chapter, it was presented a state-dependent sampling control for ideal LTI sampled-data systems, and it was shown the benefits of the polytopic embedding approach for some benchmarks from the literature. In practice however, during the real-time control of a dynamical system, perturbations may appear: exogenous unknown inputs, parametric uncertainties, measurement noises, computation and actuation delays, unmodeled dynamics, etc. Such disturbances may destabilize the system, and thus it is necessary to analyse this robustness aspect. In this chapter, we will propose methods for robust stability with respect to perturbations in continuous-time, using convex embeddings. Note that although a large amount of works have been presented on convex embeddings [Donkers 2009], [Fujioka 2009a], [Skaf 2009], [Hetel 2006], [Hetel 2011b], [Hetel 2007], [Olaru 2008], [de Wouw 2010], [Cloosterman 2010], [Gielen 2010], [Goebel 2009], none of them has included robustness with respect to perturbations. In fact, including exogenous unknown perturbations in the stability analysis is not a simple matter.

In this chapter, we propose to include this robustness aspect with respect to unknown exogenous perturbations that are state-bounded (*i.e.* the perturbation w satisfies $\|w(t)\|^2 \leq W\|x(t_k)\|^2$, for some constant scalar W), and we provide tools to perform *robust stability analysis regarding time-varying sampling*, *event-triggered control*, *self-triggered control*, and *state-dependent sampling*. For each of these applications, we ensure the system's β-stability for a given decay-rate β, thanks to Lyapunov-Razumikhin stability conditions and convexification arguments.

The chapter is organized as follows. First, we state the problem in Section 3.1 and propose the main stability analysis in Section 3.2. Then, Sections 3.3 to 3.6 provide tools for the robust stability analysis regarding time-varying sampling and for the design of the different dynamic sampling controllers. Finally, some simulation results are shown in Section 3.7 before concluding in Section 3.8. As in the previous chapter, all the proofs are given in the Appendix A.2, and the proposed technical construction for the convex embedding can be found in Apendix C.3.

3.1 Problem statement

In this chapter, we consider the perturbed LTI system

$$\dot{x}(t) = Ax(t) + Bu(t) + Ew(t),\ \forall t \in \mathbb{R}_+, \tag{3.1}$$

where $x : \mathbb{R}_+ \to \mathbb{R}^n$, $u : \mathbb{R}_+ \to \mathbb{R}^{n_u}$, and $w : \mathbb{R}_+ \to \mathbb{R}^{n_w}$ represent respectively the system state, the control function, and the exogenous disturbances. The matrices A, B, and E are constant with appropriate dimensions.

Similarly to the case presented in Chapter 2, the control is assumed to be a piecewise-constant state feedback

$$u(t) = -Kx(s_k),\ \forall t \in [s_k, s_{k+1}),\ \forall k \in \mathbb{N}, \tag{3.2}$$

where K is fixed and such that $A - BK$ is Hurwitz[9] (*i.e.* it is assumed that the system (3.1) without perturbation is asymptotically stable with the continuous state feedback $u(t) = -Kx(t)$).

Moreover, the sampling instants $0 = s_0 < s_1 < \cdots < s_k < \cdots$ verify $\lim\limits_{k\to\infty} s_k = \infty$, and the sampling intervals are set to satisfy

$$s_{k+1} - s_k = \tau(s_k, x(s_k)) \equiv \tau_k \in [\delta, \tau_{\max}(x(s_k))],\ \forall k \in \mathbb{N}, \tag{3.3}$$

with a scalar $\delta > 0$ that ensures the well posedness of the system (no Zeno phenomenon issue), a sampling function $\tau : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$, and a maximal sampling map $\tau_{\max} : \mathbb{R}^n \to \mathbb{R}_+$. This sampling map defines the upper-bound of the sampling intervals and can be seen as a maximal time-invariant sampling function.

[9]a Hurwitz matrix (also called stable matrix) is a real square matrix for which each eigenvalue has a strictly negative real part.

The exogenous disturbance is assumed to be state-bounded in a similar way as in [Wang 2009]:

$$\exists W \geq 0, \|w(t)\|_2^2 \leq W\|x(s_k)\|_2^2, \ \forall t \in [s_k, s_{k+1}), \ \forall k \in \mathbb{N}. \tag{3.4}$$

Such a perturbation can represent model uncertainties, local nonlinearities, or measurement noise for example.

We denote by $\mathcal{S}$ the closed-loop system $\{(3.1), (3.2), (3.3), (3.4)\}$. For given sampling function τ and disturbance w, the solution of $\mathcal{S}$ with initial value x_0 is denoted by

$$x(t) = \varphi_{\tau,w}(t, x_0). \tag{3.5}$$

Our main objective is to provide a way to enlarge as much as possible the maximal sampling map $\tau_{\max}$ from (3.3) while ensuring the the system's β-stability for a chosen decay-rate β.

In order to check the β-stability of $\mathcal{S}$, as in the unperturbed case, we use a Lyapunov-Razumikhin approach [Kolmanovskii 1992] which we formulate for a wider class of perturbed systems as:

Proposition 3.1 *Consider the switched nonlinear system*

$$\dot{x}(t) = f_k(t, x(t), x(s_k), w(t)), \ \forall t \in [s_k, s_{k+1}), \ \forall k \in \mathbb{N}, \tag{3.6}$$

with switching instants s_k satisfying (3.3), and an unknown exogenous perturbation $w : \mathbb{R}_+ \to \mathbb{R}^{n_w}$ which is supposed to be locally essentially bounded [Mancilla-Aguilar 2005]. The functions $f_k : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n_w} \to \mathbb{R}^n$ are assumed to be locally Lipschitz with respect to their second variable, $x(t)$. For given sampling function τ and disturbance w, the solution of system $\{(3.3),(3.6)\}$ with initial value x_0 is denoted by $x(t) = \phi_{\tau,w}(t, x_0)$. Consider scalars $\alpha > 1$, $r > 0$, $\bar{\sigma} > 0$, and $0 < \beta \leq \frac{\ln(\alpha)}{r\bar{\sigma}}$, and a map $\tau_{max} : \mathbb{R}^n \to \mathbb{R}_+$, $0 < \delta \leq \tau_{max}(x) \leq \bar{\sigma}$. If there exist a continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}_+$, and scalars $0 < \underline{\gamma} \leq \bar{\gamma}$ such that

$$\text{For all } x \in \mathbb{R}^n, \ \underline{\gamma}\|x\|_2^r \leq V(x) \leq \bar{\gamma}\|x\|_2^r, \tag{H1}$$

and

$$\begin{gathered}\text{For all } x \in \mathbb{R}^n, \text{ for all } \sigma \in [0, \tau_{max}(x)], \\ \dot{V}(\phi_{\tau_{max},w}(\sigma, x)) + r\beta V(\phi_{\tau_{max},w}(\sigma, x)) \leq 0 \text{ whenever } \alpha V(\phi_{\tau_{max},w}(\sigma, x)) \geq V(x),\end{gathered} \tag{H2}$$

then the origin of the switched nonlinear system {(3.3),(3.6)} is globally β*-stable.*

Note that if $\beta = 0$ and the inequality $\dot{V}(\phi_{\tau_{\max},w}(\sigma, x)) \leq 0$ in (H2) is reinforced to be strict, then the classical Lyapunov-Razumikhin [Kolmanovskii 1992] theory ensures the system's asymptotic stability.

For simplicity, in this chapter, we will only consider the case of quadratic LRF $V(x) = x^T P x$, for which we derive the following stability condition:

Proposition 3.2 *Consider scalars* $\alpha > 1$, $\bar{\sigma} > 0$, $0 < \beta \leq \frac{\ln(\alpha)}{2\bar{\sigma}}$, *and* $W \geq 0$, *and a map* $\tau_{max} : \mathbb{R}^n \to \mathbb{R}_+$, $0 < \delta \leq \tau_{max}(x) \leq \bar{\sigma}$. *If there exists a quadratic function* $V(x) = x^T P x$, $P \in S_n^{+*}$ *such that*

$$\begin{gathered}\text{For all } x \in \mathbb{R}^n, \text{ for all } \sigma \in [0, \tau_{max}(x)], \\ \dot{V}(\varphi_{\tau_{max},w}(\sigma, x)) + 2\beta V(\varphi_{\tau_{max},w}(\sigma, x)) \leq 0 \text{ whenever } \alpha V(\varphi_{\tau_{max},w}(\sigma, x)) \geq V(x),\end{gathered} \tag{H3}$$

then the system $\mathcal{S}$ *is globally* β*-stable.*

As in the unperturbed case presented in the previous chapter, we will focus on solving two main problems. The first problem concerns the design of the LRF V and is formulated as:

Problem 1: Given the system {(3.1),(3.2),(3.4)}, find an LRF V such that there exists a sampling map $\tau_{\max}$ satisfying (H3) with a minimum value $\tau^* = \inf_{x \in \mathbb{R}^n} \tau_{\max}(x)$ as large as possible.

The objective of Problem 1 covers the ones in the works about robust analysis regarding time-varying sampling (see Chapter 1, Section 1.4), since it is about searching for an LRF that allows for a larger upper-bound τ^* on (state-independent) time-varying sampling. The second problem concerns the design of the sampling map $\tau_{\max}$ and is formulated as:

Problem 2: Given the system {(3.1),(3.2),(3.4)} and an LRF V, design a lower-bound approximation of the optimal sampling map $\tau_{\text{opt}}^V(x) = \max \tau_{\max}(x)$ such that (H3) holds.

This formulation covers the problems of most works about dynamic sampling control (see Chapter 1, Section 1.5).

By combining the results from these two problems (*i.e.* designing the sampling map of Problem 2 thanks to the LRF designed in Problem 1), it is possible to design a robust sampling law for which the lower bound of the sampling map (*i.e.* the maximal sampling in the worst case scenario) is optimized. Note that although the works in the literature about dynamic sampling control bring a particular attention to prove the existence of a strictly positive lower-bound on the sampling map, they do not address this issue of maximization of the lower-bound.

3.2 Main stability results

In this section, our aim is to derive sufficient stability conditions from Proposition 3.2 that depend solely on the time variable σ and on the sampled-state x.
First, we introduce the dynamics of the studied system $\mathcal{S}$ in (H3) and propose conditions that are equivalent to the ones of Proposition 3.2. It represents an extension of Lemma 2.5 to the case with perturbations.

Lemma 3.3 *Consider scalars* $\alpha > 1$, $\bar{\sigma} > 0$, $0 < \beta \leq \frac{\ln(\alpha)}{2\bar{\sigma}}$, *and* $W \geq 0$, *and a map* $\tau_{max} : \mathbb{R}^n \to \mathbb{R}_+$, $0 < \delta \leq \tau_{max}(x) \leq \bar{\sigma}$. *If there exist a matrix* $P \in S_n^{+*}$ *and a scalar* $\varepsilon \geq 0$ *such that for all* $x \in \mathbb{R}^n$, *and all* $\sigma \in [0, \tau_{max}(x)]$,

$$\begin{bmatrix} \Lambda(\sigma)x + J_w(\sigma) \\ x \\ w(\sigma) \end{bmatrix}^T \Omega \begin{bmatrix} \Lambda(\sigma)x + J_w(\sigma) \\ x \\ w(\sigma) \end{bmatrix} \leq 0, \tag{3.7}$$

with the matrices

$$\Lambda(\sigma) = I + \int_0^\sigma e^{sA} ds (A - BK), \tag{3.8}$$

$$J_w(\sigma) = \int_0^\sigma e^{A(\sigma - s)} E w(s) ds, \tag{3.9}$$

and

$$\Omega = \begin{bmatrix} A^T P + PA + \varepsilon\alpha P + 2\beta P & -PBK & PE \\ * & -\varepsilon P & 0 \\ * & * & 0 \end{bmatrix}, \tag{3.10}$$

then the system $\mathcal{S}$ *is globally* β*-stable.*

Note that in (3.7) appear the sampled state $x(s_k) \equiv x$ and the time $t - s_k \equiv \sigma$, but also other terms that result from the unknown exogenous disturbance, $w(\sigma)$ and $J_w(\sigma)$, and which need to be removed. Ideally, the aimed stability conditions have the form $x^T\Pi(\sigma)x \leq 0,\ \forall x \in \mathbb{R}^n,\ \forall \sigma \in [0, \tau_{\max}(x)]$. Such a form is adapted for the four considered techniques:

- For robust stability analysis with respect to time-varying sampling, this form allows for removing the state-dependency and derive a parameter-dependent matrix inequality $\Pi(\sigma) \preceq 0$.
- For event-triggered control, it makes it possible to derive a simple event-generator that can be checked in real-time, of the form $x(s_k)^T\Pi(t-s_k)x(s_k) = 0$.
- For self-triggered control, it allows one to compute at each sampling instant s_k a lower-bound estimation of the next maximal sampling interval, by studying the evolution of the term $x(s_k)^T\Pi(t-s_k)x(s_k)$.
- For state-dependent sampling, the form $x^T\Pi(\sigma)x \preceq 0$ is adequate to provide a stability analysis over conic regions of the state-space.

In the following theorem, we derive the central stability conditions by bounding the effects of the perturbations on the system's evolution, using the assumption (3.4).

Theorem 3.4 *Consider scalars $\alpha > 1$, $\bar{\sigma} > 0$, $0 < \beta \leq \frac{\ln(\alpha)}{2\bar{\sigma}}$, and $W \geq 0$, and a map $\tau_{max} : \mathbb{R}^n \to \mathbb{R}_+$, $0 < \delta \leq \tau_{max}(x) \leq \bar{\sigma}$.*
Then, the system $\mathcal{S}$ is globally β-stable if there exist scalars $\varepsilon \geq 0$, $\eta \geq 0$, and $\mu \geq 0$, matrices P, Φ_1, $\Phi_2 \in S_n^{+*}$, and $\Phi_3 \in S_{nw}^{+*}$, such that

$$0 \preceq M_1 + \Phi_1 + \Phi_2 \preceq \mu I, \quad \begin{bmatrix} \Phi_3 - \eta I & M_3^T \\ * & -\Phi_2 \end{bmatrix} \preceq 0, \tag{3.11}$$

and

$$x^T\Pi(\sigma)x \leq 0,\ \forall x \in \mathbb{R}^n,\ \forall \sigma \in [0, \tau_{max}(x)], \tag{3.12}$$

with

$$\begin{aligned} \Pi(\sigma) \ = \ & \Lambda(\sigma)^T M_1 \Lambda(\sigma) - \Lambda(\sigma)^T PBK - K^T B^T P \Lambda(\sigma) - \varepsilon P \\ & + M_2(\sigma)^T \Phi_1^{-1} M_2(\sigma) + M_4(\sigma)^T \Phi_3^{-1} M_4(\sigma) + W\eta I + \sigma W \mu \lambda_{max}(E^T E) f_A(\sigma) I, \end{aligned} \tag{3.13}$$

where

$$\begin{gathered} M_1 = A^T P + PA + \varepsilon\alpha P + 2\beta P,\ M_2(\sigma) = -PBK + M_1\Lambda(\sigma), \\ M_3 = PE,\ M_4(\sigma) = E^T P^T \Lambda(\sigma), \end{gathered} \tag{3.14}$$

and

$$f_A(\sigma) = \begin{cases} \frac{1}{\lambda_{max}(A+A^T)} \left(e^{\lambda_{max}(A+A^T)\sigma} - 1 \right) & \text{if } \lambda_{max}(A+A^T) \neq 0, \\ \sigma & \text{otherwise.} \end{cases} \tag{3.15}$$

The sufficient stability conditions from Theorem 3.4 will be used as a stability basis throughout the rest of the work, for robust stability analysis with respect to time-varying sampling, event-triggered control, self-triggered control, and state-dependent sampling. They involve a few LMIs (3.11) as well as the more complex set of conditions: $x^T \Pi(\sigma) x \leq 0,\ \forall x \in \mathbb{R}^n,\ \forall \sigma \in [0, \tau_{\max}(x)]$.

Remark 3.5 *In Theorem 3.4, P corresponds to the LRF matrix, ε comes from the application of the S-procedure to rewrite in a more convenient way the LRF stability conditions, and the scalars η and μ, as well as the matrices Φ_i correspond to degrees of freedom used in the majorations of the perturbations $w(\sigma)$ and $J_w(\sigma)$ from Lemma 3.3. One easy way to deal with these free matrices would be to use identity matrices. However, this would remove the degrees of freedom that were gained, and it could well result in overly conservative stability conditions. In the next section, an algorithm to efficiently compute all these parameters will be presented.*

Remark 3.6 *Similarly to the unperturbed case, for any given state $x \neq 0$, the condition (3.12) from Theorem 3.4 remains the same for any state $y = \lambda x$, $\lambda \in \mathbb{R}^*$. Therefore, it is sufficient to work with homogeneous sampling maps of degree 0 (i.e. satisfying $\tau_{max}(\lambda x) = \tau_{max}(x)$ for all $x \in \mathbb{R}^n$, $\lambda \in \mathbb{R}^*$) and to check condition (3.12) over the unit n-sphere.*

In the next sections, we show how to adapt the obtained stability conditions and how to reduce their number, so as to perform a robust analysis with respect to time-varying sampling, or a dynamic control of the sampling.

3.3 Robust stability analysis with respect to time-varying sampling - Optimization of the parameters

In this section, we study the stability for (state-independent) time-varying samplings and provide tools to compute the parameters that appear in Theorem 3.4. We consider a constant (*i.e.* state-independent) sampling map:

$$\tau_{\max}(x) = \tau_{\max}^{(\text{global})},\ \forall x \in \mathbb{R}^n, \tag{3.16}$$

and look for a stability analysis and an algorithm that allow us to compute:

- a state-independent upper-bound estimation $\tau_{\max}^{(\text{global})} = \tau^*$ for time-varying sampling as in the framework of robust control techniques (*i.e.* guaranteeing β-stability for any time-varying sampling bounded by τ^*),
- the associated LRF $V(x) = x^T P x$ (as well as additional parameters Φ_1, Φ_2, Φ_3, ε, η and μ), thus solving Problem 1.

With this aim in mind, we need to reduce the number of conditions from (3.12) in Theorem 3.4 from an infinite number in both the time σ and state x variables, to a finite number that is independent of the state x. Also, in order to compute the various parameters, we want to remove the inverse terms Φ_1^{-1} and Φ_3^{-1}, and write this finite number of conditions in the form of LMIs.

Lemma 3.7 *The condition (3.12) in Theorem 3.4, with the sampling map (3.16), is satisfied if and only if the parameter-dependent LMI*

$$\Delta(\sigma) = \begin{bmatrix} R(\sigma) & M_2(\sigma)^T & M_4(\sigma)^T \\ * & -\Phi_1 & 0 \\ * & * & -\Phi_3 \end{bmatrix} \preceq 0 \tag{3.17}$$

is satisfied for all $\sigma \in [0, \tau_{max}^{(global)}]$*, with*

$$\begin{aligned} R(\sigma) \quad = \quad & \Lambda(\sigma)^T M_1 \Lambda(\sigma) - \Lambda(\sigma)^T PBK - K^T B^T P \Lambda(\sigma) - \varepsilon P \\ & + W\eta I + \sigma W \mu \lambda_{max}(E^T E) f_A(\sigma) I. \end{aligned} \tag{3.18}$$

In order to reduce the number of conditions regarding the time-variable, we propose the following convex embedding method:

Convex embedding according to time: The matrix function Δ is continuous on the compact set $[0, \tau_{\max}^{(\text{global})}]$. Therefore, similarly to the unperturbed case, given $\tau_{\max}^{(\text{global})} \leq \bar{\sigma}$, it is possible to build a convex polytope defined by a finite set of vertices around $\Delta(\sigma)$, for $\sigma \in [0, \tau_{\max}^{(\text{global})}]$. For the sake of generality, and to define notations that can also be used in the other applications presented in this chapter, we will consider the set of vertices as a function of the maximum sampling interval considered

$$\begin{aligned} \bar{\Delta}_\kappa : \quad [0, \bar{\sigma}] \quad & \rightarrow \quad \mathcal{M}_{2n+n_w}(\mathbb{R}) \\ \tau_{\max}^{(\text{global})} \quad & \mapsto \quad \bar{\Delta}_\kappa(\tau_{\max}^{(\text{global})}). \end{aligned}$$

Similarly, we consider the set of indexes for the vertices as a function of the time

$$\mathcal{K}: \begin{array}{rcl} [0,\bar{\sigma}] & \to & \mathcal{P}(\bar{\mathcal{K}}) \\ \tau_{\max}^{(\text{global})} & \mapsto & \mathcal{K}(\tau_{\max}^{(\text{global})}), \end{array}$$

where $\bar{\mathcal{K}}$ is a finite set of indexes. In that description, $\mathcal{P}(\bar{\mathcal{K}})$ denotes the power set of $\bar{\mathcal{K}}$ and means that $\mathcal{K}(\tau_{\max}^{(\text{global})}) \subseteq \bar{\mathcal{K}}$, for all $\tau_{\max}^{(\text{global})} \in [0,\bar{\sigma}]$.

Figure 3.1 presents a 2D illustration of such a polytopic design for two different values of $\tau_{\max}^{(\text{global})}$: σ_1^* and σ_2^* $(0 < \sigma_1^* < \sigma_2^* \leq \bar{\sigma})$. Here, one can see that the number of vertices as well as their value/position changes with respect to the value of $\tau_{\max}^{(\text{global})}$. Remember that this figure shows only an intuitive representation of the convex embedding, since the function Δ evolves in the $(2n+n_w)\times(2n+n_w)$ matrices space, and thus can not be represented in a 2D space.

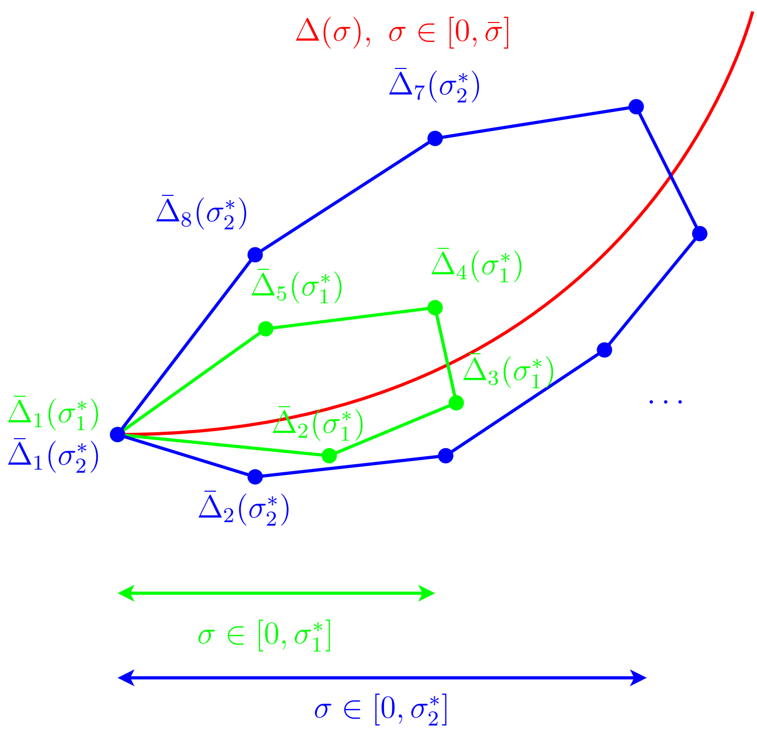

Figure 3.1: Illustration of the convex embedding design

With these notations, and for a given $\tau_{\max}^{(\text{global})}$, we can build the convex embedding such that the following property is satisfied:

$$\begin{array}{c}\left(\bar{\Delta}_\kappa(\tau_{\max}^{(\text{global})}) \preceq 0,\ \forall \kappa \in \mathcal{K}(\tau_{\max}^{(\text{global})})\right) \\ \Downarrow \\ \left(\Delta(\sigma) \preceq 0,\ \forall \sigma \in [0, \tau_{\max}^{(\text{global})}]\right).\end{array} \tag{3.19}$$

Note that the form of the matrix function Δ given by (3.17) enables to build these vertices $\bar{\Delta}_\kappa(\tau_{\max}^{(\text{global})})$ as linearly dependent on P, Φ_1, Φ_3, η, and μ, and dependent on the parameters α, ε, β, and $\bar{\sigma}$. One possible construction of a convex polytope satisfying (3.19) is provided in the Appendix C.3.

This convex embedding approach allows for obtaining the following theorem.

Theorem 3.8 *Consider $\varepsilon \geq 0$ a tuning parameter. Let a scalar $0 < \tau_{max}^{(global)} \leq \bar{\sigma}$ and the constant sampling map defined in (3.16). Let scalars $\alpha > 1$, $\bar{\sigma} > 0$, $0 < \beta \leq \frac{\ln(\alpha)}{2\bar{\sigma}}$, and $W \geq 0$, and matrices $\bar{\Delta}_\kappa(\tau_{max}^{(global)})$ satisfying (3.19), with $\kappa \in \mathcal{K}(\tau_{max}^{(global)})$.*
If there exist matrices P, Φ_1, $\Phi_2 \in S_n^{+*}$, $\Phi_3 \in S_{n_w}^{+*}$, and scalars $\eta \geq 0$ and $\mu \geq 0$, such that the LMIs (3.11) and $\bar{\Delta}_\kappa(\tau_{max}^{(global)}) \preceq 0$ are satisfied for all $\kappa \in \mathcal{K}(\tau_{max}^{(global)})$, then the system (3.1), subject to perturbations (3.4), is globally β-stable with respect to the control (3.2) for any time-varying sampling bounded by $\tau_{max}^{(global)}$.

Remark 3.9 *This theorem provides a stability analysis for systems with time-varying sampling upper-bounded by $\tau_{max}^{(global)}$. The tuning parameter ε can be optimized by using a line-search algorithm and LMI solvers. The idea is the following.*
For a given value of ε, one can compute the maximal $\tau_{max}^{(global)}$ (denoted $\tau_{max}^{(global)}(\varepsilon)$) for which the stability conditions from Theorem 3.8 are satisfied, by using a line search algorithm on the variable $\tau_{max}^{(global)}$ and LMI solvers. Then, another line search algorithm is used on the variable ε so as to compute an estimation of the largest upper-bound for time-varying sampling intervals: $\tau^* = \sup_{\varepsilon \geq 0} \tau_{max}^{(global)}(\varepsilon)$.

Using the following algorithm, it is possible to compute a lower-bound estimate of the maximal allowable sampling interval for time-varying sampling. Here, we use the polytopic description (C.14) (in Appendix C.3), which is based on Taylor series approximations. This approximation induces an estimation error which can be upper-bounded by a scalar ν, defined in (C.21).

Algorithm:

Step 1: First, we use Theorem 3.8 and the polytopic description (C.14) considering that the upper-bound on the estimation error $\nu = 0$. The search for P, Φ_1, Φ_2, Φ_3, η and μ is then an LMI problem, and we may optimize the search of the largest $\tau_{\max}^{(\text{global})}$ (denoted $\hat{\tau}^*$) and its associated parameter ε by using the method proposed in Remark 3.9.

Step 2: Then, we compute the value of the upper-bound ν that corresponds to the obtained parameters P, ε, Φ_1, Φ_2, Φ_3, η and μ. Using this value, it becomes possible to evaluate the matrix inequalities $\bar{\Delta}_\kappa(\tau_{\max}^{(\text{global})}) \preceq 0$ in Theorem 3.8 so as to obtain an estimation of the largest upper-bound for time-varying samplings $\tau^* \leq \hat{\tau}^*$ which satisfies the stability conditions.

Step 3: The maximal sampling map is then defined as

$$\tau_{\max}(x) = \tau^*, \;\forall x \in \mathbb{R}^n.$$

Remark 3.10 *Using the LRF $V(x) = x^T P x$ together with the parameters ε, Ψ_1, Ψ_2, Ψ_3, η, μ and ν obtained thanks to this algorithm allows for designing sampling maps that are lower-bounded by τ^* in the case of dynamic sampling control (i.e. event-triggered control, self-triggered control, and state-dependent sampling).*

3.4 Event-triggered control

In event-triggered control, the sampling occurs when some event is generated by the system's smart sensors. In this section three different event-triggered control schemes are presented. The first one is based on the stability conditions from Theorem 3.4, which will be used to design the self-triggered control and state-dependent sampling schemes. The last two approaches, less conservatives, allow to take into account the effects of the perturbation on the system while taking advantage of the results from the previous section, about robust stability analysis with respect to time-varying sampling.

3.4.1 Over-approximation based event-triggered control scheme

The first event-triggered control scheme is based on Theorem 3.4, which allows to define the event-generator condition for the $(k+1)^{\text{th}}$ sampling as

$$(t \geq s_k + \tau^*) \wedge \left(\left(x(s_k)^T \Pi(t - s_k) x(s_k) = 0 \right) \vee (t = s_k + \bar{\sigma}) \right),$$

with parameters P, ε, Φ_1, Φ_2, Φ_3, η, μ and ν obtained with the algorithm reported in Section 3.3. This event-generator enables to design a maximal sampling map

$$\tau_{\max}(x) = \min\left(\min\{\sigma \geq \tau^* |\ x^T \Pi(\sigma) x = 0\}, \bar{\sigma}\right)$$

during the real-time control of the system.

Note that there is no need to check the event-generator's condition during the time interval $[s_k, s_k + \tau^*]$ since Theorem 3.8 ensures that $x(s_k)^T \Pi(t - s_k) x(s_k) \leq 0$ for all $t \in [s_k, s_k + \tau^*]$. Also, note that this event-triggered control scheme does not take into account the real evolution of the perturbed system $\mathcal{S}$ since it is based on the conditions from Theorem 3.4, which are set to be satisfied for any perturbation satisfying (3.4). This scheme will be used as a referential for a comparison, in order to check the conservatism introduced in the self-triggered control and state-dependent sampling schemes.

Since event-triggered control allows to monitor the system's state at all time (and thus take into account the effect of the exogenous disturbance on the system's state evolution), we present two other approaches. One is based on the stability conditions from Lemma 3.3 (obtained before the majorations dealing with the exogenous perturbations), and the other one is based directly on the Lyapunov function for the discrete model of the system.

3.4.2 Perturbation-aware event-triggered control scheme

The second event-triggered control scheme we present is based on the following stability property, wich is derived from Lemma 3.3:

Lemma 3.11 *Consider $W \geq 0$, $\alpha > 1$, $\bar{\sigma} > 0$, $0 < \beta \leq \frac{\ln(\alpha)}{2\bar{\sigma}}$, $\varepsilon \geq 0$, and $P \in S_n^{+*}$. If*

$$\begin{bmatrix} x(t) \\ x(s_k) \end{bmatrix}^T \bar{\Omega} \begin{bmatrix} x(t) \\ x(s_k) \end{bmatrix} \leq 0,\ \forall t \in [s_k, s_{k+1}],\ k \in \mathbb{N}, \tag{3.20}$$

with

$$\bar{\Omega} = \begin{bmatrix} A^T P + PA + \varepsilon\alpha P + 2\beta P + PEE^T P & -PBK \\ * & -\varepsilon P + WI \end{bmatrix}, \tag{3.21}$$

then the sampled-data system $\{(3.1),(3.2),(3.4)\}$ with sampling intervals satisfying $s_{k+1} - s_k \in [\delta, \bar{\sigma}]$ is globally β-stable.

In order to guarantee the stability conditions from Lemma 3.11, the event-triggered generator condition for this scheme is thus defined as

$$(t \geq s_k + \tau^*) \wedge \left(\left(\begin{bmatrix} x(t) \\ x(s_k) \end{bmatrix}^T \bar{\Omega} \begin{bmatrix} x(t) \\ x(s_k) \end{bmatrix} = 0 \right) \vee (t = s_k + \bar{\sigma}) \right),$$

with parameters P and ε computed with the algorithm given in Section 3.3. Note that unlike the previous event-triggered scheme, here we use the value of the state $x(t)$, and thus take into account the effect of the perturbations on the system's state evolution from s_k to t. Also, unlike the previous scheme, this scheme does not enable to design a maximal mapping during the real-time control of the system since the event generator condition involve $x(t)$, which evolution also depends on the perturbation.

3.4.3 Discrete-time approach event-triggered control scheme

The third event-triggered control scheme fully takes into account the perturbations on the system, and thus allows to reduce the conservatism even more, with respect to the previous schemes. It is based on a discrete-time analysis of the system. The event-generator condition for the $(k+1)^{\text{th}}$ sampling is defined as

$$(t \geq s_k + \tau^*) \wedge \left(V(x(t)) = \mathrm{e}^{-2\beta(t-s_k)} V(x(s_k)) \right),$$

with the LRF V computed with the algorithm in Section 3.3. Note that just like the previous scheme, this scheme does not enable to design a maximal mapping during the real-time control of the system since the event generator condition does not depend only on the time variable and on the sampled state, but also on the perturbation and its evolution.

3.5 Self-triggered control

Self-triggered control aims at emulating event-triggered control without resorting to dedicated hardware to monitor the plant, by computing at each sampling instant a lower-bound estimation of the next maximal allowable sampling interval. In this section, we present a self-triggered control scheme derived from the stability conditions in Theorem 3.4 with the convexification arguments (3.19).

This self-triggered control scheme is based on an interesting property of the convex embedding design with polytopic subdivisions we have proposed in (C.13) and (C.14) (in the Appendix C.3) for the matrix function Δ (see equation (3.17)). We recall that in this design, which is based on Taylor series approximations, it was considered two integers N and l, which represent the order of the Taylor approximations, and the number considered of subintervals of $[0, \bar{\sigma}]$ of length $\frac{\bar{\sigma}}{l}$ respectively.

The particular property of this design is that the set of indexes in this design is expanding along with the value of the sampling interval upper-bound:

$$\forall(\sigma_1^*, \sigma_2^*) \in [0, \bar{\sigma}]^2,\ \sigma_1^* \leq \sigma_2^* \Rightarrow \mathcal{K}(\sigma_1^*) \subseteq \mathcal{K}(\sigma_2^*),$$

which means in particular that

$$\forall\sigma^* \in [0, \bar{\sigma}],\ \mathcal{K}(\sigma^*) \subseteq \mathcal{K}(\bar{\sigma}) = \{0, \cdots, N\} \times \{0, \cdots, l-1\} \equiv \bar{\mathcal{K}}.$$

Furthermore, for certain discrete values of the sampling interval upper-bound, the associated polytope vertices can be obtained directly from vertices designed for $\sigma^* = \bar{\sigma}$:

$$\begin{gathered}\forall\sigma^* = \tfrac{\bar{j}+1}{l}\bar{\sigma} \in [0, \bar{\sigma}],\ \text{for some integer } \bar{j} \in \{0, \cdots, l-1\},\\ \bar{\Delta}_\kappa(\sigma^*) = \bar{\Delta}_\kappa(\bar{\sigma}),\ \forall\kappa \in \mathcal{K}(\sigma^*) = \{0, \cdots, N\} \times \{0, \cdots, \bar{j}\} \subseteq \bar{\mathcal{K}}.\end{gathered}$$

Figure 3.2 illustrates these points. Indeed, one can see that for any integer $\bar{j} \in \{0, \cdots, l-1\}$, the convex polytope $\mathrm{Co}_{(i,j)\in\{0,\cdots,N\}\times\{0,\cdots,\bar{j}\}}\{\bar{\Delta}_{(i,j)}(\bar{\sigma})\}$ embeds the matrix function $\Delta(\sigma)$ for all values of σ in $[0, \frac{\bar{j}+1}{l}\bar{\sigma}]$.

This property is interesting because it shows that with only one set of vertices $\bar{\Delta}_\kappa(\bar{\sigma})$, with $\kappa \in \bar{\mathcal{K}} = \mathcal{K}(\bar{\sigma})$, it is possible to check the stability for different values of the upper-bound on the sampling interval (namely $\sigma^* \in \{\frac{\bar{\sigma}}{l}, \cdots, \frac{(l-1)\bar{\sigma}}{l}, \bar{\sigma}\}$), by computing the maximal index j^* for which the LMIs $\bar{\Delta}_{(i,j)}$ are satisfied for all $(i,j) \in \{0, \cdots, N\} \times \{0, \cdots, j^*\}$. If such a j^* is found, the stability is ensured for any time-varying sampling in $[0, \sigma^* = \frac{j^*+1}{l}\bar{\sigma}]$. Otherwise, if no such j^* can be computed, one can not conclude with the stability (this may be the case for small values of l, *i.e* when the considered subdivisions of the interval $[0, \bar{\sigma}]$ are too large).

It is important to note that unlike the situation of the robust stability analysis with respect to time-varying sampling in Section 3.3, which was a state-independent analysis, here we need to use equations in which the state x explicitely appears, like (3.12). To this aim, the convex embedding will be designed around the matrix function Π (equation (3.13)) instead of Δ (equation (3.17)). As it will be shown however, such a convex polytope

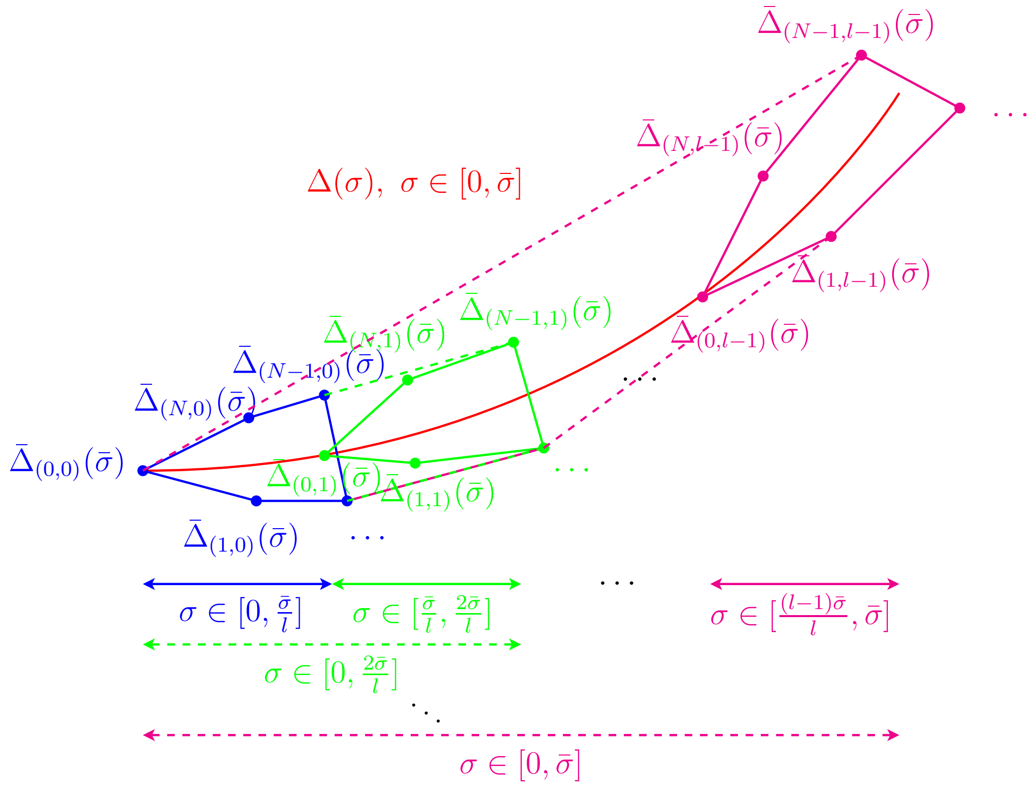

Figure 3.2: Illustration of the property of the convex embedding design with subdivisions from Appendix C.3 around the matrix function Δ

can be obtained by a simple adaptation of the one presented for Δ in the Appendix 3.3.

In a more general context than with the particular convex embedding design presented in the Appendix C.3, one may use the following property in order to design a self-triggered control scheme.

Theorem 3.12 *Consider scalars $\alpha > 1$, $\bar{\sigma} > 0$, $0 < \beta \leq \frac{\ln(\alpha)}{2\bar{\sigma}}$, $W \geq 0$, $\varepsilon \geq 0$, $\eta \geq 0$, $\mu \geq 0$, a sampling map τ_{max}, $0 < \delta \leq \tau_{max}(x) \leq \bar{\sigma}$, and matrices P, Φ_1, $\Phi_2 \in S_n^{+*}$, $\Phi_3 \in S_{n_w}^{+*}$, such that the LMIs (3.11) hold.*
Assume that there exist matrices $\Pi_\kappa \in \mathcal{M}_n(\mathbb{R})$, with $\kappa \in \bar{\mathcal{K}}$ a finite set of indexes, and a function $\tilde{\mathcal{K}} : \mathbb{R}^n \to \mathcal{P}(\bar{\mathcal{K}})$ such that for all $x \in \mathbb{R}^n$,

$$\left(x^T \Pi_\kappa x \leq 0,\ \forall \kappa \in \tilde{\mathcal{K}}(x)\right) \tag{3.22a}$$

$$\Downarrow$$

$$\left(x^T \Pi(\sigma) x \leq 0,\ \forall \sigma \in [0, \tau_{max}(x)]\right). \tag{3.22b}$$

Then, if the triggering condition (3.22a) is satisfied for all $x \in \mathbb{R}^n$, the system $\mathcal{S}$ with *the sampling map τ_{max} is globally β-stable.*

Remark 3.13 *The main advantages of such a formulation are that the triggering conditions for a given sampled state x are reduced to a finite number (see (3.22a)), and that the matrices Π_κ do not depend on x, and can be thus computed offline, for all $\kappa \in \bar{\mathcal{K}}$, once and for all.*

In the following, we provide one possible method to design the different elements involved in this theorem, and propose an adapted self-triggered control scheme. This method is based on the polytopic embedding that was proposed previously for the robust stability analysis (see Appendix C.3), and which was designed thanks to Taylor series approximations. We call N the order of the approximation, and we consider that the interval $[0, \bar{\sigma}]$ is divided into l subintervals of length $\frac{\bar{\sigma}}{l}$.

Step 1: First, in order to maximize the value of $\tau_{\max}^{(\text{global})}$ (*i.e.* the lower-bound of the sampling map) up to τ^*, we consider the parameters P, ε, Φ_1, Φ_2, Φ_3, η, μ and ν obtained using the algorithm in Section 3.3.

Step 2: Then, we compute the matrices $\bar{\Delta}_\kappa(\bar{\sigma})$, with $\kappa \in \mathcal{K}(\bar{\sigma})$ using the polytopic construction (C.13) and (C.14) (in the Appendix C.3). Note that these matrices take the form

$$\bar{\Delta}_\kappa(\bar{\sigma}) = \begin{bmatrix} \Delta_\kappa^{(1,1)} & \Delta_\kappa^{(1,2)} & \Delta_\kappa^{(1,3)} \\ * & \Delta_\kappa^{(2,2)} & 0 \\ * & * & \Delta_\kappa^{(3,3)} \end{bmatrix}.$$

Step 3: Now, we design the matrices Π_κ as

$$\Pi_\kappa = \Delta_\kappa^{(1,1)} - \Delta_\kappa^{(1,2)}[\Delta_\kappa^{(2,2)}]^{-1}[\Delta_\kappa^{(1,2)}]^T - \Delta_\kappa^{(1,3)}[\Delta_\kappa^{(3,3)}]^{-1}[\Delta_\kappa^{(1,3)}]^T, \tag{3.23}$$

and we consider a set of indexes function $\tilde{\mathcal{K}} : \mathbb{R}^n \to \mathcal{P}(\bar{\mathcal{K}})$, with $\bar{\mathcal{K}} = \mathcal{K}(\bar{\sigma}) = \{0, \cdots, N\} \times \{0, \cdots, l-1\}$, of the form:

$$\tilde{\mathcal{K}}(x) = \{0, \cdots, N\} \times \{0, \cdots, j^*(x)\}, \tag{3.24}$$

with

$$j^*(x) = \begin{cases} \max\left\{\tilde{j} \in \tilde{\mathcal{J}}(x)\right\} \text{ if } \tilde{\mathcal{J}}(x) \neq \emptyset, \\ -1 \text{ otherwise }, \end{cases}$$

and

$$\begin{aligned} \tilde{\mathcal{J}}(x) \;=\; & \left\{\tilde{j} \in \left\{\left\lfloor \tfrac{\tau^* l}{\bar{\sigma}} \right\rfloor, \cdots, l-1\right\} | x^T \Pi_{(i,j)} x \leq 0, \right. \\ & \left. \forall (i,j) \in \{0, \cdots, N\} \times \left\{\left\lfloor \tfrac{\tau^* l}{\bar{\sigma}} \right\rfloor, \cdots, \tilde{j}\right\}\right\}. \end{aligned}$$

Here, the indexes of $\tilde{\mathcal{K}}$ are composed a pair of parameters. The first parameter is linked to the Taylor approximation, whereas the second one is linked to the interval subdivision considered. For a given state x, the aim is to search for $j^*(x)$, which represents the highest subdivision for which the inequalities $x^T\Pi_{(i,j)}x \leq 0$ are satisfied for all $(i,j) \in \{0,\cdots,N\} \times \{0,\cdots,j^*(x)\}$.

Step 4: Following this construction, the sampling map for the proposed self-triggered control scheme is designed as

$$\tau_{\max}(x) = \max\left(\frac{j^*(x)+1}{l}\bar{\sigma}, \tau^*\right). \tag{3.25}$$

Using arguments similar to the ones used to prove that (3.19) is satisfied with the vertices (C.14) (see Lemma (C.3)), one can show that the matrices Π_κ defined in (3.23), with the set of indexes $\bar{\mathcal{K}} = \mathcal{K}(\bar{\sigma}) = \{0,\cdots,N\} \times \{0,\cdots,l-1\}$, and the set of indexes function $\tilde{\mathcal{K}}$ defined in (3.24) satisfy (3.22), with the sampling map (3.25).

Remark 3.14 *With this construction, the self-triggering condition during the real-time control of the system amounts to computing the value of the integer $j^*(x)$ for each sampled-state x. Note that with the parameters considered in Step 1, Theorem 3.8 ensures that if $\left\lfloor\frac{\tau^* l}{\bar{\sigma}}\right\rfloor \geq 1$, then $\Pi_{(i,j)} \leq 0$ for all $(i,j) \in \{0,\cdots,N\} \times \left\{0,\cdots,\left\lfloor\frac{\tau^* l}{\bar{\sigma}}\right\rfloor - 1\right\}$, which explains why it is not necessary to check the inequatilies $x^T\Pi_{(i,j)}x \leq 0$ for $j < \left\lfloor\frac{\tau^* l}{\bar{\sigma}}\right\rfloor$.*

Remark 3.15 *The precision of the sampling map τ_{max} is linked to the value of the integer l, which defines the number subdivisions of the time interval $[0,\bar{\sigma}]$ used in the construction of the convex polytope (C.14): the larger the integer l, the more precise the sampling map. The number of online computations required to compute $j^*(x)$ is upper-bounded by $n(n+1)(N+1)\left(l - \left\lfloor\frac{\tau^* l}{\bar{\sigma}}\right\rfloor\right)$ multiplications and $(n+1)(n-1)(N+1)\left(l - \left\lfloor\frac{\tau^* l}{\bar{\sigma}}\right\rfloor\right)$ additions. The online complexity is in $O(n^2Nl)$. It is comparable to the one obtained in the self-triggered control scheme from [Mazo Jr. 2010] for example. Here some computations are saved thanks to the optimization in Step 1 of the lower-bound τ^* of the sampling map (see Remark 3.14).*

3.6 State-dependent sampling

The state-dependent sampling aims, as introduced in the previous chapter, at emulating self-triggered control while trading online computations for offline computations, thus reducing the processor load during the real-time control of the system.

In this formulation, the sampling map is defined over regions of the state-space as

$$\tau_{\max}(x) = \tau_{\max}^{(s)},\ \forall x \in \mathcal{R}_s,\ \forall s \in \{1, \cdots, q\}. \tag{3.26}$$

Here, the homogeneity brought up in Remark 3.6, which is due to the linearity of the system, motivates us for working with conic regions of the form

$$\mathcal{R}_s = \{x \in \mathbb{R}^n, x^T \Psi_s x \geq 0\},\ \Psi_s = \Psi_s^T \in \mathcal{M}_n(\mathbb{R}). \tag{3.27}$$

Possible constructions of these conic regions using the spherical coordinates of the state or the discrete-time behaviour of the system, are presented in the Appendix B. We have the following stability property:

Theorem 3.16 *Let a matrix $P \in S_n^{+*}$, and scalars $\varepsilon \geq 0$, $\alpha > 1$, $\bar{\sigma} > 0$, $0 < \beta \leq \frac{\ln(\alpha)}{2\bar{\sigma}}$, and $W \geq 0$ be given. Let matrices Φ_1, $\Phi_2 \in S_n^{+*}$, $\Phi_3 \in S_{n_w}^{+*}$, and scalars $\eta \geq 0$ and $\mu \geq 0$, such that the LMIs (3.11) are satisfied. Consider the sampling map (3.26) defined on conic regions (3.27), with sampling intervals $\tau_{max}^{(1)}, \cdots, \tau_{max}^{(q)}$ satisfying $0 < \delta \leq \tau_{max}^{(s)} \leq \bar{\sigma}$. Assume there exist matrices $\bar{\Delta}_\kappa(\tau_{max}^{(s)})$, with $\kappa \in \mathcal{K}(\tau_{max}^{(s)})$ a finite set, satisfying for all $s \in \{1, \cdots, q\}$, and $\rho_s \geq 0$,*

$$\begin{array}{c}\left(\bar{\Delta}_\kappa(\tau_{max}^{(s)}) + \begin{bmatrix} \rho_s \Psi_s & 0 \\ * & 0 \end{bmatrix} \preceq 0,\ \forall \kappa \in \mathcal{K}(\tau_{max}^{(s)})\right) \\ \Downarrow \\ \left(\Delta(\sigma) + \begin{bmatrix} \rho_s \Psi_s & 0 \\ * & 0 \end{bmatrix} \preceq 0,\ \forall \sigma \in [0, \tau_{max}^{(s)}]\right),\end{array} \tag{3.28}$$

with $\Delta(\sigma)$ introduced in (3.17).
If there exist scalars $\rho_s \geq 0$ such that the LMIs $\bar{\Delta}_\kappa(\tau_{max}^{(s)}) + \begin{bmatrix} \rho_s \Psi_s & 0 \\ * & 0 \end{bmatrix} \preceq 0$ are satisfied for all $s \in \{1, \cdots, q\}$ and $\kappa \in \mathcal{K}(\tau_{max}^{(s)})$, then the system $\mathcal{S}$ is globally β-stable.

Theorem 3.16 provides sufficient conditions for Theorem 3.4, which enable to analyse the stability of the system for a given sampling map $\tau_{\max}$ defined on conic regions.

One possible construction for the matrices $\bar{\Delta}_\kappa(\tau_{\max}^{(s)})$, $\kappa \in \mathcal{K}(\tau_{\max}^{(s)})$, is the one proposed in (C.13) and (C.14), in the Appendix C.3. Indeed, one can show, using the same proof as the one used in Lemma C.3, that these matrices satisfy the condition (3.28) for all $s \in \{1, \cdots, q\}$ and $\rho_s \geq 0$.

A method to compute a lower-bound approximation of the optimal sampling map, solution of Problem 2, is proposed. The idea is to use the LMI conditions from Theorem 3.16 (with the values of P, ε, Φ_1, Φ_2, Φ_3, η, μ and ν computed using the algorithm in Section 3.3), in order to maximize the sampling intervals $\tau_{\max}^{(s)}$ on each region, using a line search algorithm. Then, we design a lower-bound estimation of the optimal sampling map τ_{opt}^{V} as proposed in (3.26):

$$\tau_{\max}(x) = \tau_{\max}^{(s)}, \ \forall x \in \mathcal{R}_s, \ s \in \{1, \cdots, q\}.$$

Remark 3.17 *The online complexity of the state-dependent sampling approach depends on the design of the conic covering. With the anisotropic covering proposed in Appendix B.2, the online complexity is $O(qn^2)$ (at most $(q-1)n(n+1)$ multiplications and $(q-1)(n-1)(n+1)$ additions). It can be shown that for the same precision, the number of computations in that case is divided by N compared to the self-triggered control case (Section 3.5). With the isotropic covering proposed in Appendix B.1, the online complexity becomes $O(n)$ ($9n-7$ elementary operations (additions, multiplications and divisions), 1 square-root, $n-1$ arccosine, and $n-2$ sine), which allows for saving even more computational power. Additionally, in that latter case, the online complexity does not depend on the number of regions (i.e. on the precision).*

3.7 Numerical example

Consider the system from [Tabuada 2007]:

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} x(t) - \begin{bmatrix} 0 \\ 1 \end{bmatrix} Kx(s_k) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} w(t), \\ K &= \begin{bmatrix} -1 & 4 \end{bmatrix}. \end{aligned}$$

In the following, we set the polynomial approximation degree term $N = 5$ and the number of polytopic subdivisions $l = 100$. For a given β, after fixing $\bar{\sigma}$, we set the LRF performance parameter $\alpha > 1$ (see Proposition 3.2) as small as possible and such that $\beta \leq \frac{\ln(\alpha)}{2\bar{\sigma}}$. Then, we use the algorithm proposed in Section 3.3 to perform a robust stability analysis with respect to time-varying sampling for different values of parameters β and W. The obtained upper-bounds for time-varying samplings τ^* (see the set of values provided in Figure 3.3, as well as the ones in Table 3.1, which presents a comparison with some upper-bounds obtained in the literature, without perturbation) can then be used as lower-bounds to the designed maximal sampling maps for the dynamic sampling control.

	[Naghshtabrizi 2008]	[Seuret 2009]	[Fujioka 2009b]	[Fridman 2010]	Theorem 3.8
$\beta = 0$	$0.2740s$	$0.3122s$	$0.3316s$	$0.4221s$	$0.5402s$
$\beta = 0.1$	-	$0.2795s$	-	$0.3934s$	$0.4404s$
$\beta = 0.3$	-	$0.1778s$	-	$0.3350s$	$0.3709s$

Table 3.1: Maximum upper-bounds τ^* for time-varying samplings, for different decay-rates β, without perturbation ($W = 0$)

Since the sampling maps for event-triggered control and self-triggered control are built online, we only show the ones obtained for state-dependent sampling, which are built offline. First, we set a number $q = 100$ of equal conic regions (isotropic partition on the unit sphere $x = e^{i\theta}$, $\theta \in [-\pi, \pi]$, see the design in Appendix B.1). Using the method proposed in Section 3.6, we build the maximal sampling maps for different parameters β and W, as shown in Figure 3.3. Recall that for each parameter set, β-stability is ensured for each state-dependent sampling (potentially time-varying) with values under the respective curve in Figure 3.3 (*i.e.* satisfying (3.3)). In the figure, the obtained upper-bounds τ^* for time-varying samplings (*i.e.* the lower-bounds of the sampling maps) are also provided for each parameter set.

For a constant sampling greater than $T^{\max}_{\text{const}} = 0.5947s$ the discrete-time dynamic matrix of the ideal system (without perturbation) is not Schur anymore, so the system becomes unstable. However, with the proposed sampling maps, we can go beyond the limit $T^{\max}_{\text{const}}$ for some regions of the state space (up to $1.2s$ for $\beta = 0$ and $W = 0$, or $0.9s$ for $\beta = 0.3$ and $W = 0$ for example). Figure 3.4 shows that it is even possible to sample in average less than with the constant sampling $T^{\max}_{\text{const}} = 0.5947s$ (which only ensures marginal stability), and still guarantee exponential stability. It presents simulation results obtained for a given a decay rate $\beta = 0.3$, without perturbations ($W = 0$). It first shows the sampling intervals (in blue), with the lower-bound of the offline designed sampling map (in red), and the limit $T^{\max}_{\text{const}}$ of the periodic case (in green), before showing the LRF evolution. The sampling times are represented by the red dots on the graph. The average inter-sampling time during this $20s$ simulation is $T_{\text{average}} = 0.7203s = 121\% T^{\max}_{\text{const}}$ (there are 28 updates, while there would be 34 updates with $T^{\max}_{\text{const}}$).

Finally, in Figure 3.5, we present the inter-execution times obtained in simulations for $\beta = 0.1$, $W = 0.04$ (*i.e.* with perturbations $\|w(t)\|_2 \leq 20\% \|x(s_k)\|_2$), and an initial condition $x(0) = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$, with the first event-triggered control scheme, the self-triggered

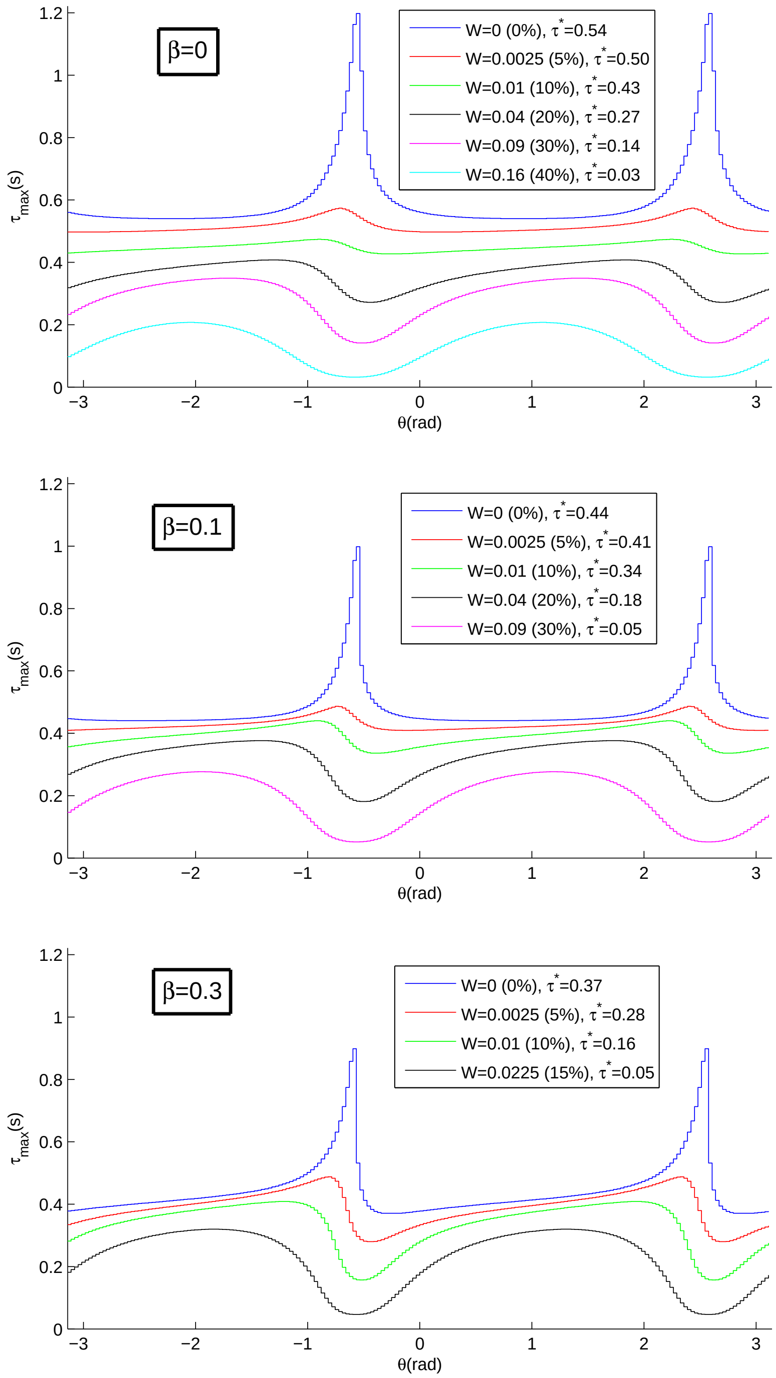

Figure 3.3: State-angle dependent sampling map $\tau_{\max}$ for different decay-rates (β) and perturbations (W)

control, and the state-dependent sampling, which are all based on stability conditions derived from Theorem 3.4.

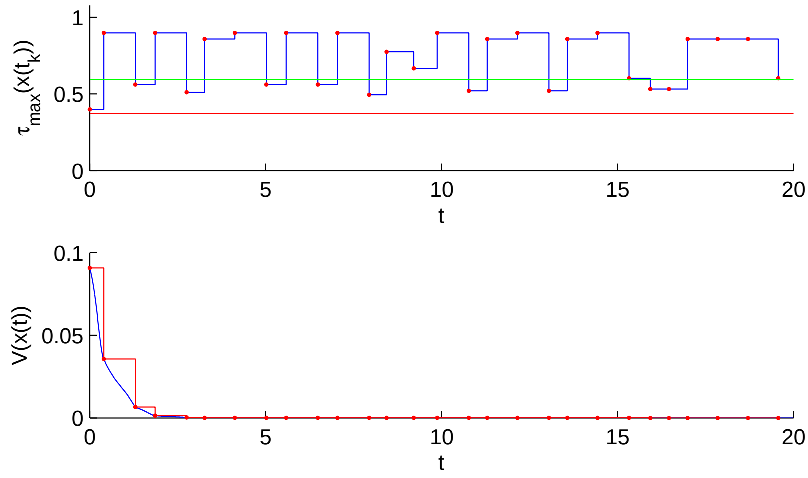

Figure 3.4: Inter-execution times $\tau_{\max}(x(s_k))$ and LRF $V(x) = x^T P x$ for a decay rate $\beta = 0.3$ and $W = 0$ - State-dependent sampling

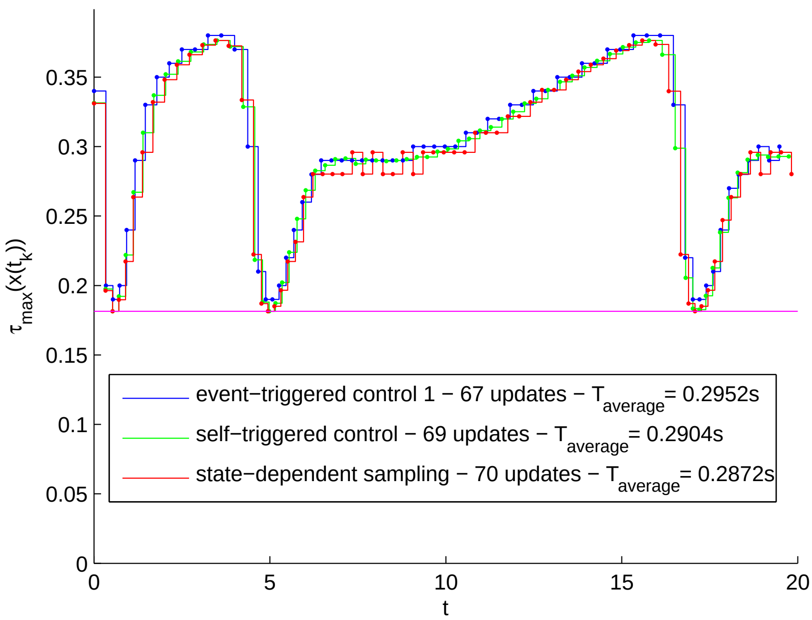

Figure 3.5: Inter-execution times $\tau_{\max}(x(s_k))$ for a decay rate $\beta = 0.1$ and $W = 0.04$ ($\|w(t)\|_2 \leq 20\%\|x(s_k)\|_2$) - First event-triggered control scheme, self-triggered control, and state-dependent sampling

These simulations show that all the proposed methods have very close results, although they are very different in their application. This illustrates the low conservatism introduced by the convex-embeddings (3.19) (used in both the self-triggered control scheme and the state-dependent sampling scheme) and the conic regions (3.27) (used in the state-dependent sampling scheme).

The second and third event-triggered control schemes, which both take into account the real value of the perturbation, provide naturally less updates (and therefore a larger average inter-execution time). With the same simulation conditions, we obtain 42 updates with the second event-triggered scheme ($T_{\text{average}} = 0.4581s$), and 33 updates with the third event-triggered scheme ($T_{\text{average}} = 0.6027s$).

3.8 Conclusion

We have introduced a Lyapunov-Razumikhin-based design for a maximal state-dependent sampling map $\tau_{\max}$ ensuring the exponential stability with a given decay-rate for perturbed linear state feedback systems. The proposed method can be used to perform:

- a robust stability analysis with respect to time-varying sampling,
- an event-triggered control scheme,
- a self-triggered control scheme,
- a state-dependent sampling scheme.

For each of these approaches, lower-bound estimation of the maximal sampling map is proposed. As in the unperturbed case, the method presents several advantages.

- It makes it possible to maximize the lower-bound τ^* of the proposed map, whatever the sampling technique.
- It provides the associated LRF parameters.
- The state-dependent map of the next maximal sampling interval with respect to the past sampled state value can be designed offline (state-dependent sampling), which helps reducing the processor load.

Chapter 4

A Lyapunov-Krasovskii approach to dynamic sampling control

In the previous two chapters, it was considered the problem of designing a sampling law that enlarges the sampling intervals while guaranteeing the stability of LTI sampled-data systems for a given controller. In this chapter, we want to go further and design at the same time a controller that will stabilize the considered LTI sampled-data system, with the objective to enlarge the sampling intervals in mind.

First of all, in the framework set in the previous chapters, we consider the stability issue, and design a state-dependent sampling function that maximizes the sampling intervals under some $\mathcal{L}_2$-stability conditions for *perturbed* linear sampled-data systems, for a given controller. An extension to systems with *delays in the feedback loop* is also proposed. Then, in the delay-free case, it will be proposed an algorithm for the design of the stabilizing feedback gain matrix either as a constant K, or as a state-dependent one $K(x_k)$.

The proposed design has the same advantages as the state-dependent sampling controls presented in the previous two chapters. Indeed, unlike the self-triggered control approach, it makes it possible:

- to reduce the number of sampling instants obtained in the worst case scenario, *i.e.* to increase the lower bound τ^+ of the largest state-independent admissible sampling interval while taking into account the perturbations and the sampling (and the delays in case of time-delay systems),
- to design *offline*, once for all, the state-dependent sampling function $\tau_{\max}(x) \geq \tau^+, x \in \mathbb{R}^n$ maximizing the sampling intervals for each state of the state space.

However, this new design also has its own advantages, since it makes it possible:

- to design a controller adapted to state-dependent sampling,
- to design a state-dependent sampling function even for some systems which are both open-loop and closed-loop (with continuous feedback) unstable,
- to adapt the controller gains depending on the state space region, to allow even larger samplings.

The stability analysis and stabilization tools in this chapter are based on a new class of Lyapunov-Krasovskii functionals (LKF) with state-dependent matrices. Just as we compute the maximal sampling $\tau_{\max}(x)$ depending on the sampled state x_k, the matrices of the LKF will switch in relation to the state space region that contains this sampled state. The obtained LMI conditions allow to compute the LKF switching matrices ensuring the largest state-dependent sampling intervals.

The *robustness study* considers an exogenous perturbation w in $\mathcal{L}_2$. Note that no other assumption is made. In particular, the perturbation is not required to be bounded or state-bounded (*i.e.* there is no need for a scalar $\delta > 0$ such that $\|w(t)\|_2 \leq \delta$ or $\|w(t)\|_2 \leq \delta \|x(t)\|_2$), as it was assumed in the previous chapter.

In the case of *systems with time-delay*, we assume that the control inputs are received by the actuator in the order they are sent (or that the packets are rearranged upon reception, as proposed in [?] for instance). However, no additional assumption regarding the actuation times is needed. In particular, the actuation times are not required to occur before the next sampling times, which means that the transmission delays can be larger than the sampling intervals, unlike in [Wang 2009] or [Wang 2010].

Concerning the *stabilization issue*, we design the state feedback gain so as to allow larger sampling intervals. An extension to the stabilization problem with a more general class of switching piecewise-linear controllers (with matrix gains that are switching according to the system's state) is also provided.

The chapter is organized as follows: First, Section 4.1 formulates the problem. Then, Section 4.2 presents the stability results, while Section 4.3 provides the stabilization results. An algorithm allowing to build off-line the adequate state-dependent sampling function is provided in both of these sections. Finally, Section 4.4 shows some simulation results, and Section 4.5 summarizes the contributions in this chapter.

4.1 Problem formulation

We consider the linear time-invariant system

$$\left.\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t) + Ew(t)\\ z(t) &= Cx(t) + Du(t)\end{aligned}\right\},\ \forall t \geq 0, \tag{4.1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $w(t) \in \mathbb{R}^{n_w}$ is an exogenous disturbance in $\mathcal{L}_2$, $u(t) \in \mathbb{R}^{n_u}$ is the control input, and $z(t) \in \mathbb{R}^{n_z}$ is the controlled output. A, B, C, D, and E are constant matrices of appropriate dimensions. The control is designed as a piecewise-constant state feedback:

$$u(t) = -Kx(s_k),\ \forall t \in [t_k, t_{k+1}), \tag{4.2}$$

with K a given feedback matrix gain, and with s_k and t_k the k^{th} sampling time and the k^{th} actuation time respectively.

For now, it is considered the case where there is no delay between the sampling and the actuation times, and thus that $t_k = s_k$. Later, in Subsection 4.2.2, the robustness aspect with respect to unknown time-varying delays $h(t)$ in the feedback control loop will be treated.

The sequence of sampling times $(s_k)_{k\geq 0}$ is assumed to satisfy $0 = s_0 < s_1 < \cdots < s_k < \cdots$ and $\lim\limits_{k\to\infty} s_k = \infty$, and the sampling law is defined as

$$s_{k+1} = s_k + \tau_k, \tag{4.3}$$

with a variable sampling step τ_k we aim to control. We denote $\mathbf{S}$, the closed-loop system $\{(4.1),(4.2),(4.3)\}$.

Due to the unknown exogenous disturbances, the system $\mathbf{S}$ is studied from the $\mathcal{L}_2$-stability point of view, which is recalled in the following definition:

Definition 4.1 *A linear system $\mathbf{F}$ is said to be finite-gain $\mathcal{L}_2$-stable from w to $\mathbf{F}w$ with an induced gain less than γ if $\mathbf{F}$ is a linear operator from $\mathcal{L}_2$ into $\mathcal{L}_2$ and there exist positive real constants γ and ξ such that for all $w \in \mathcal{L}_2$,*

$$\|\mathbf{F}w\|_{\mathcal{L}_2} \leq \gamma\|w\|_{\mathcal{L}_2} + \xi. \tag{4.4}$$

The work in the present chapter aims at designing, off-line, a state-dependent sampling function $\tau_{\max} : \mathbb{R}^n \longrightarrow \mathbb{R}^+$ which enlarges the sampling intervals

$$s_{k+1} - s_k \equiv \tau_k = \tau_{\max}(x(s_k)), \tag{4.5}$$

while ensuring the finite-gain $\mathcal{L}_2$-stability of $\mathbf{S}$ from w to z, with a gain less than a fixed $\gamma \geq 0$.

To this aim, we will use the following lemma:

Lemma 4.2 *Assume there exist a real constant $\gamma \geq 0$ and a positive definite continuous function $V : t \in \mathbb{R}^+ \rightarrow V(t) \in \mathbb{R}^+$, differentiable for all $t \neq t_k$, $k \in \mathbb{N}$, that satisfy*

$$\dot{V}(t) + z^T(t)z(t) - \gamma^2 w^T(t)w(t) \leq 0 \tag{4.6}$$

along $\mathbf{S}$. Then, $\mathbf{S}$ is $\mathcal{L}_2$-stable from w to z with a gain less than γ.

Proof: Let $t \gg 0$ and $N \in \mathbb{N}$ such that $t \in [t_N, t_{N+1})$. Integrating (4.6) over $[0, t]$ gives

$$\begin{array}{c} V(t) - V(t_N) + V(t_N^-) - V(t_{N-1}) + \cdots + V(t_0^-) \\ -V(0) + \int_0^t \left(z^T(s)z(s) - \gamma^2 w^T(s)w(s)\right) ds \leq 0. \end{array}$$

Since $V(t) \geq 0$ and $V(t_k) = V(t_k^-)$ for all $k \in \mathbb{N}$ (V is assumed to be continuous), we get:

$$\int_0^t z^T(s)z(s)ds \leq \gamma^2 \int_0^t w^T(s)w(s)ds + V(0).$$

Using the positivity of $z^T(s)z(s)$, one can show that $z = \mathbf{S}w \in \mathcal{L}_2$, and by having $t \rightarrow \infty$ one can see that the $\mathcal{L}_2$-stability condition (4.4) is satisfied, with $\xi = \sqrt{V(0)}$. ∎

As in the previous chapters, in the framework of state-dependent sampling control, we assume that the state space is covered by a set of q conic regions (not necessarily disjoint)

$$\mathcal{R}_\sigma = \{x \in \mathbb{R}^n, x^T \Psi_\sigma x \geq 0\}, \Psi_\sigma \in S_n, \sigma \in \{1, \cdots, q\}, \tag{4.7}$$

for which maximal sampling intervals τ_σ^+ will be defined. The two possible constructions presented in the previous chapters, and described in the Appendix B, may be used to design these conic regions.

Here, the sampling interval sequences $(\tau_k = s_{k+1} - s_k)_{k \geq 0}$ are set to satisfy the condition:

$$\forall k \in \mathbb{N}, \exists \sigma \in \mathcal{I}(x(s_k)), \tau^- \leq \tau_k \leq \tau_\sigma^+, \text{ with } \mathcal{I}(x) = \{\sigma \in \{1, \cdots, q\}, x \in \mathcal{R}_\sigma\}, \tag{4.8}$$

with a given minimal sampling interval $\tau^- > 0$ (this guarantees that there is no Zeno behaviour). $\mathcal{I}(x)$ denotes the set of the regions $\mathcal{R}_\sigma$ in which x belongs. Note that since the regions $\mathcal{R}_\sigma$ are not necessarily disjoint, x can belong to more than one region at a time. The condition on the samplings (4.8) ensures that at each sampling instant s_k, there is at least one region in $\sigma_k \in \mathcal{I}(x(s_k))$ for which the next sampling interval τ_k satisfies $\tau^- \leq \tau_k \leq \tau^+_{\sigma_k}$.

Our objective in this chapter is to compute the largest sampling intervals τ^+_σ for each subspace $\mathcal{R}_\sigma, \sigma \in \{1, \cdots, q\}$ which ensure the expected $\mathcal{L}_2$-stability for a fixed $\gamma \geq 0$. The state-dependent sampling function (4.5) will then be built from:

$$\tau_{\max}(x) = \max_{\sigma \in \mathcal{I}(x)} \tau^+_\sigma, \forall x \in \mathbb{R}^n. \tag{4.9}$$

We will first provide a *stability analysis* of the system for a given feedback matrix gain K and a given sampling function $\tau_{\max}$ (with an extension to systems with delays in the feedback control loop), before proposing a *stabilization method* to compute a feedback matrix gain K adapted to ensure stability for a given sampling function $\tau_{\max}$ (with an extension to the design of control laws with matrix gains that are switching according to the system's state).

In both analysis and design, we provide *algorithms* that allow to *maximize both the largest admissible state-independent sampling interval* $\tau^+ = \min_{\sigma \in \{1,\cdots,q\}} \tau^+_\sigma$ *and the state-dependent sampling function* τ_{max} *(4.9)* according to the obtained stability or stabilization conditions.

All these studies are based on a quite general class of LKF (with state-dependent matrices), which take into account the delays (in the case of delayed systems), the perturbations and the sampling. The proposed algorithms allow to compute the LKF matrices so as to optimize the state-dependent sampling function $\tau_{\max}$ (4.9).

4.2 Main $\mathcal{L}_2$-stability results

In this section, we start by proposing in Subsection 4.2.1 a stability analysis of system **S** for a given feedback matrix gain K and samplings satisfying (4.8). Then, in Subsection 4.2.2, we give an extension to systems with delays in the feedback control loop. Finally, in Subsection 4.2.3, we provide an algorithm to enlarge the sampling function $\tau_{\max}$, under the obtained stability conditions.

4.2.1 Stability analysis of the perturbed system

We consider the following LKF, which depends on the sampled-state value $x(s_k)$, the actual state $x(t)$, and the delayed derivatives of x, $\dot{x}_t$ (defined for a maximal delay $\bar{h}$ as $\dot{x}_t(\theta) = \dot{x}(t+\theta)$, $\forall\theta \in [-\bar{h}, 0]$):

$$\begin{aligned} V(t, x(t), \dot{x}_t, k) \;\; = \;\; & x^T(t)Px(t) + (s_{k+1} - t) \begin{bmatrix} x(t) \\ x(s_k) \end{bmatrix}^T \Omega_{\sigma_k} \begin{bmatrix} x(t) \\ x(s_k) \end{bmatrix} \\ & + (s_{k+1} - t) \int_{s_k}^{t} \dot{x}^T(s) U_{\sigma_k} \dot{x}(s) ds + (s_{k+1} - t)(t - s_k) x^T(s_k) S_{\sigma_k} x(s_k), \end{aligned} \tag{4.10}$$

defined for all $t \in [s_k, s_{k+1})$ and $k \in \mathbb{N}$, with the matrices $\Omega_\sigma, \sigma \in \{1, \cdots, q\}$ defined as:

$$\Omega_\sigma = \begin{bmatrix} \frac{X_\sigma + X_\sigma^T}{2} & -X_\sigma + X_{1,\sigma} \\ * & -X_{1,\sigma} - X_{1,\sigma}^T + \frac{X_\sigma + X_\sigma^T}{2} \end{bmatrix}. \tag{4.11}$$

Matrices P, U_σ, S_σ, X_σ, $X_{1,\sigma}$ are of appropriate dimensions, and parameter σ_k can be any element $\sigma \in \mathcal{I}(x(s_k))$ satisfying $\tau^- \leq \tau_k \leq \tau_\sigma^+$.

The new aspect of the LKF (4.10) compared to previous works on systems with time-varying samplings ([Fridman 2010], [Seuret 2009], [Jiang 2010b]) is the fact that *it involves elements that are switching according to the system state*. Indeed, note that the matrix term U_{σ_k} is switching at times s_k according to the region the sampled state $x(s_k)$ belongs to ($\sigma_k \in \mathcal{I}(x(s_k))$). This state-dependent switch is possible thanks to the fact that the functional V is continuous at times s_k: $V(s_k, x(s_k), \dot{x}_{s_k}, k) = \lim_{t \to s_k^-} V(t, x(t), \dot{x}_t, k-1) = x^T(s_k)Px(s_k)$.

This new type of switched LKF is well adapted to the stability analysis of systems with state-dependent sampling, but it also provides some advantages regarding the stability analysis of systems with (state-independent) time-varying sampling, as it will be shown in the Example 2 of the Numerical Examples Section 4.4.2.

In the following, as in [Fridman 2010], we denote

$$\bar{V}(t) = V(t, x(t), \dot{x}_t, k), \text{ for all } t \in [s_k, s_{k+1}),\ k \in \mathbb{N}. \tag{4.12}$$

The $\mathcal{L}_2$-stability analysis is based on Lemma 4.2 and is divided into two main steps.

- First, we prove that $\bar{V}$ is continuous over $\mathbb{R}^+$ and differentiable for all $t \in [s_k, s_{k+1})$, and provide conditions for its positive definiteness.

- Then, we differentiate $\bar{V}$, upper-bound the obtained result and derive the $\mathcal{L}_2$-

stability conditions.

4.2.1.1 Continuity, piecewise differentiability, and positivity conditions of the Lyapunov-Krasovskii Functional

To begin with, we propose the following lemma, which ensures the function's continuity, piecewise differentiability, and positivity properties.

Lemma 4.3 *The function $\bar{V}$ defined in (4.12) is continuous over $\mathbb{R}^+$ and differentiable for all $t \neq s_k$, $k \in \mathbb{N}$. If its matrix parameters satisfy $P \in S_n^{+*}$, U_σ, $S_\sigma \in S_n^+$, X_σ, $X_{1,\sigma} \in \mathcal{M}_{n,n}(\mathbb{R})$, and if there exist q scalars $\varepsilon_\sigma \geq 0$ such that, for all $\sigma \in \{1, \cdots, q\}$:*

$$\begin{bmatrix} P & 0 \\ 0 & 0 \end{bmatrix} + \tau_\sigma^+ \Omega_\sigma - \varepsilon_\sigma \begin{bmatrix} 0 & 0 \\ 0 & \Psi_\sigma \end{bmatrix} \succ 0, \tag{4.13}$$

then $\bar{V}$ is also positive definite, and there exists a scalar $\beta > 0$ such that $\bar{V}(t) \geq \beta \|x(t)\|_2^2$ for all $t \geq 0$.

Proof: $\bar{V}$, is defined on $\mathbb{R}^+$, differentiable over each time interval $[s_k, s_{k+1})$, and is designed to satisfy $\bar{V}(s_k) = \lim_{t \to s_k^-} \bar{V}(t) = x(s_k)^T P x(s_k)$ for all $k \in \mathbb{N}$. It is therefore continuous on $\mathbb{R}^+$ and differentiable over $\mathbb{R}^+ \backslash \{s_k, k \in \mathbb{N}\}$.

Now, assume that $U_\sigma, S_\sigma \in S_n^+$, with $\sigma \in \{1, \cdots, q\}$. $\bar{V}$ is positive definite if, and only if, for all $k \in \mathbb{N}$, $t \in [s_k, s_{k+1})$:

$$\begin{bmatrix} x(t) \\ x(s_k) \end{bmatrix}^T \left[\begin{bmatrix} P & 0 \\ 0 & 0 \end{bmatrix} + (s_{k+1} - t) \Omega_{\sigma_k} \right] \begin{bmatrix} x(t) \\ x(t_k) \end{bmatrix} \geq 0, \tag{4.14}$$

with equality if and only if $x(t) = x(s_k) = 0$.

Note that $0 \leq s_{k+1} - t \leq \tau_{\sigma_k}^+$. Remarking that the middle matrix term in the left part of (4.14) is linear with respect to $\lambda = s_{k+1} - t$, one can use Theorem D.8 (in the Appendix D) and show that a sufficient condition for $\bar{V}$ to be positive definite is that, for all $k \in \mathbb{N}$, $t \in [s_k, s_{k+1})$:

$$x^T(t) P x(t) > 0, \text{ for all } x(t) \neq 0, \tag{4.15}$$

and

$$\begin{bmatrix} x(t) \\ x(s_k) \end{bmatrix}^T \left[\begin{bmatrix} P & 0 \\ 0 & 0 \end{bmatrix} + \tau_{\sigma_k}^+ \Omega_{\sigma_k} \right] \begin{bmatrix} x(t) \\ x(s_k) \end{bmatrix} > 0, \text{ for all } \begin{bmatrix} x(t) \\ x(s_k) \end{bmatrix} \neq 0. \tag{4.16}$$

(4.15) is ensured by assuming that P is positive definite. Since $x(s_k) \in \mathcal{R}_{\sigma_k}$ (*i.e.* $x^T(s_k)\Psi_{\sigma_k}x(s_k) \geq 0$), the lossless version of the S-procedure [Boyd 1994] (see Theorem D.3 in Appendix D) ensures that (4.16) is fulfilled for all $k \in \mathbb{N}$ if and only if there exist q scalars $\varepsilon_\sigma \geq 0$ such that (4.13) is satisfied for all $\sigma \in \{1, \cdots, q\}$.

Furthermore, if $P \succ 0$ and the condition (4.13) is satisfied for all $\sigma \in \{1, \cdots, q\}$, then there exists q scalars $\beta_\sigma > 0, \sigma \in \{1, \cdots, q\}$, such that for all $k \in \mathbb{N}$ and $t \in [s_k, s_{k+1})$, $\begin{bmatrix} P & 0 \\ 0 & 0 \end{bmatrix} + (s_{k+1} - t)\Omega_{\sigma_k} \succ \beta_{\sigma_k} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$, and thus $\bar{V}(t) \geq \beta_{\sigma_k}\|x(t)\|_2^2$, for all $t \in [s_k, s_{k+1})$ and $k \in \mathbb{N}$. Therefore, there exists a scalar $\beta = \min_{\sigma \in \{1, \cdots, q\}} \beta_\sigma > 0$, such that $\bar{V}(t) \geq \beta\|x(t)\|_2^2$ for all $t \geq 0$, which ends the proof. ∎

4.2.1.2 $\mathcal{L}_2$-stability conditions

Conditions to ensure $\bar{V}$'s continuity, differentiability, and positivity have been proposed. In order to analyse the $\mathcal{L}_2$-stability of system **S**, we will now refer to Lemma 4.2. It is needed to provide conditions to satisfy

$$\dot{\bar{V}}(t) + z^T(t)z(t) - \gamma^2 w^T(t)w(t) \leq 0, \ \forall t \neq s_k, \ \forall k \in \mathbb{N}. \tag{4.17}$$

In order to analyse this $\mathcal{L}_2$-stability condition, we study the restriction of $\dot{\bar{V}}$ on any interval $[s_k, s_{k+1}), \ k \in \mathbb{N}$. We compute:

$$\begin{array}{rcl} \dot{\bar{V}}(t) & = & 2\dot{x}^T(t)Px(t) + ((s_{k+1} - s_k) - 2(t - s_k))x^T(s_k)S_{\sigma_k}x(s_k) \\ & & - \begin{bmatrix} x(t) \\ x(s_k) \end{bmatrix}^T \Omega_{\sigma_k} \begin{bmatrix} x(t) \\ x(s_k) \end{bmatrix} + 2(s_{k+1} - t)\dot{x}^T(t)\Omega_{1,\sigma_k} \begin{bmatrix} x(t) \\ x(s_k) \end{bmatrix} \\ & & +(s_{k+1} - t)\dot{x}^T(t)U_{\sigma_k}\dot{x}(t) - \int_{s_k}^t \dot{x}^T(s)U_{\sigma_k}\dot{x}(s)ds, \end{array} \tag{4.18}$$

with

$$\Omega_{1,\sigma_k} = \begin{bmatrix} \frac{X_{\sigma_k} + X_{\sigma_k}^T}{2} & -X_{\sigma_k} + X_{1,\sigma_k} \end{bmatrix}. \tag{4.19}$$

Using the Jensen inequality [Gu 2003] (see Theorem D.4 in Appendix D), we compute an upper bound of the integral term:

$$-\int_{s_k}^t \dot{x}^T(s)U_{\sigma_k}\dot{x}(s)ds \leq -(t - s_k)\nu^T(t)U_{\sigma_k}\nu(t), \tag{4.20}$$

with

$$\nu(t) = \frac{1}{t - s_k}\int_{s_k}^t \dot{x}(s)ds = \frac{x(t) - x(s_k)}{t - s_k}. \tag{4.21}$$

$\nu(t)$ is well defined by continuity in $t = s_k$, since when $t \to s_k$, $\nu(t) \to \dot{x}(s_k)$. Using majoration (4.20) in equation (4.18) leads to

$$\begin{aligned}
&\dot{V}(t) + z^T(t)z(t) - \gamma^2 w^T(t)w(t) \leq 2\dot{x}^T(t)Px(t) + z^T(t)z(t) - \gamma^2 w^T(t)w(t) \\
&- \begin{bmatrix} x(t) \\ x(s_k) \end{bmatrix}^T \Omega_{\sigma_k} \begin{bmatrix} x(t) \\ x(s_k) \end{bmatrix} + 2(s_{k+1} - t)\dot{x}^T(t)\Omega_{1,\sigma_k} \begin{bmatrix} x(t) \\ x(s_k) \end{bmatrix} + (s_{k+1} - t)\dot{x}^T(t)U_{\sigma_k}\dot{x}(t) \\
&-(t - s_k)\nu^T(t)U_{\sigma_k}\nu(t) + ((s_{k+1} - s_k) - 2(t - s_k))x^T(s_k)S_{\sigma_k}x(s_k).
\end{aligned} \tag{4.22}$$

Let us introduce the augmented state vector $\phi(t) \in \mathbb{R}^{3n+n_w}$:

$$\phi^T(t) = [x^T(t),\ x^T(s_k),\ \nu^T(t),\ w^T(t)]. \tag{4.23}$$

Then, there exist matrices M_i and N_j such that

$$\begin{aligned}
&\begin{bmatrix} x(t) \\ x(s_k) \end{bmatrix} = \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} \phi(t) = N_3\phi(t),\ \nu(t) = M_3\phi(t),\ w(t) = M_4\phi(t), \\
&\dot{x}(t) = (AM_1 - BKM_2 + EM_4)\phi(t) = N_1\phi(t),\ z(t) = (CM_1 - DKM_4)\phi(t) = N_2\phi(t).
\end{aligned} \tag{4.24}$$

Using these notations, we can rewrite (4.22) as

$$\begin{aligned}
&\dot{V}(t) + z^T(t)z(t) - \gamma^2 w^T(t)w(t) \leq \phi^T(t)[N_1^T PM_1 + M_1^T PN_1 + (s_{k+1} - t)N_1^T U_{\sigma_k} N_1 \\
&-(t - s_k)M_3^T U_{\sigma_k} M_3 + ((s_{k+1} - s_k) - 2(t - s_k))M_2^T S_{\sigma_k} M_2 \\
&-N_3^T \Omega_{\sigma_k} N_3 + 2(s_{k+1} - t)N_1^T \Omega_{1,\sigma_k} N_3 + N_2^T N_2 - \gamma^2 M_4^T M_4]\phi(t).
\end{aligned} \tag{4.25}$$

The relation (4.21) between $\nu(t)$, $x(t)$, and $x(s_k)$ can be written as $H(t)\phi(t) = 0$ with $H(t) = (t - s_k)M_3 - M_1 + M_2$. Therefore, by applying the Finsler's lemma [Fang 2004] (see Theorem D.2 in Appendix D) one can include this relation into (4.25) and obtain that for any matrices $Y_{\sigma_k} \in \mathcal{M}_{3n+n_w,n}(\mathbb{R})$:

$$\begin{aligned}
&\dot{V}(t) + z^T(t)z(t) - \gamma^2 w^T(t)w(t) \leq \phi^T(t)[N_1^T PM_1 + M_1^T PN_1 + (s_{k+1} - t)N_1^T U_{\sigma_k} N_1 \\
&-(t - s_k)M_3^T U_{\sigma_k} M_3 + ((s_{k+1} - s_k) - 2(t - s_k))M_2^T S_{\sigma_k} M_2 \\
&-N_3^T \Omega_{\sigma_k} N_3 + 2(s_{k+1} - t)N_1^T \Omega_{1,\sigma_k} N_3 + N_2^T N_2 - \gamma^2 M_4^T M_4 \\
&+Y_{\sigma_k}((t - s_k)M_3 - M_1 + M_2) + ((t - s_k)M_3 - M_1 + M_2)^T Y_{\sigma_k}^T]\phi(t).
\end{aligned} \tag{4.26}$$

Since equation (4.26) is linear in the variable t, it is possible to reduce the number of conditions to be checked to a finite number by applying Theorem D.8 (in the Appendix D), with the variable $\lambda = t \in [s_k, s_{k+1}]$. Then, the two obtained inequalities are both linear in the variable $s_{k+1} - s_k$. Thus we can use once again Theorem D.8 (in the Appendix D) with the variable $\lambda = s_{k+1} - s_k \in [\tau^-, \tau^+_{\sigma_k}]$ to prove that if the 4 inequalities $\xi^T \Xi_{i,j\sigma_k} \xi \leq 0$ are satisfied for all $\xi \in \mathbb{R}^{3n+n_w}$, with Ξ_{i,j,σ_k} defined as

$$\Xi_{i,1,\sigma} = \Xi_\sigma + T_{i,\sigma} \left[N_1^T U_\sigma N_1 + M_2^T S_\sigma M_2 + N_1^T \Omega_{1,\sigma} N_3 + N_3^T \Omega_{1,\sigma}^T N_1 \right], \tag{4.27}$$

$$\Xi_{i,2,\sigma} = \Xi_\sigma + T_{i,\sigma} \left[-M_3^T U_\sigma M_3 - M_2^T S_\sigma M_2 + Y_\sigma M_3 + M_3^T Y_\sigma^T \right], \tag{4.28}$$

$$\begin{aligned} \Xi_\sigma &= N_1^T P M_1 + M_1^T P N_1 + N_2^T N_2 - \gamma^2 M_4^T M_4 \\ &+ Y_\sigma(-M_1 + M_2) + (-M_1 + M_2)^T Y_\sigma^T - N_3^T \Omega_\sigma N_3, \end{aligned} \tag{4.29}$$

with

$$T_{1,\sigma} = T_1 = \tau^- \text{ and } T_{2,\sigma} = \tau^+_\sigma, \tag{4.30}$$

then $\dot{V}(t) + z^T(t)z(t) - \gamma^2 w^T(t)w(t) \leq 0$ for all $t \in [s_k, s_{k+1})$.

Note that here, we considered any sampling sequence satisfying $\tau_k = s_{k+1} - s_k \in [\tau^-, \tau^+_{\sigma_k}]$. Therefore, the $\mathcal{L}_2$-stability results we will obtain will be valid for any sampling sequence satisfying (4.8).

Eventually, since we know that $x(s_k) \in \mathcal{R}_{\sigma_k}$ (*i.e.* we have $x^T(s_k)\Psi_{\sigma_k} x(s_k) \geq 0$), we can use the lossless version of the S-procedure [Boyd 1994] (see Theorem D.3 in Appendix D) on each of the 4 obtained inequalities to show that, if there are scalars $\varepsilon_{i,j,\sigma} \geq 0$ such that the LMIs

$$\Xi_{i,j,\sigma} + \varepsilon_{i,j,\sigma} M_2^T \Psi_\sigma M_2 \preceq 0, \tag{4.31}$$

hold for $\sigma = \sigma_k$, then condition (4.17) is satisfied. Therefore, we have the following theorem:

Theorem 4.4 *Consider scalars $\gamma \geq 0$ and τ^-, and a set of q conic regions covering the state space $\mathcal{R}_\sigma = \{x, x^T \Psi_\sigma x \geq 0\}$, $\Psi_\sigma \in S_n$, $\sigma \in \{1, \cdots, q\}$, with maximal sampling intervals τ^+_σ.*
The perturbed system **S** *is finite-gain $\mathcal{L}_2$-stable from w to z with a gain less than γ for any sampling sequence satisfying (4.8) if there exist matrices $P \in S_n^{+*}$, U_σ, $S_\sigma \in S_n^+$, X_σ, $X_{1,\sigma} \in \mathcal{M}_{n,n}(\mathbb{R})$, $Y_\sigma \in \mathcal{M}_{3n+n_w,n}(\mathbb{R})$, and scalars $\varepsilon_{i,j,\sigma} \geq 0$ such that (4.13) and (4.31) are satisfied for all $\sigma \in \{1, \cdots, q\}$ and $(i,j) \in \{1,2\}^2$.*

Remark 4.5 *If w satisfies $z^T(t)z(t) - \gamma^2 w^T(t)w(t) \geq 0$, and if the LMIs (4.31) are strict, the sampled-data system* **S** *is asymptotically stable for any sampling sequence satisfying (4.8). Indeed, in such a case, $\dot{V}$ is negative definite and there is a $\beta > 0$ such that $\bar{V}(t) \geq \beta \|x(t)\|_2^2$ for all $t \geq 0$, $k \in \mathbb{N}$ and $x \in \mathbb{R}^n$. In the unperturbed case $w(t) = 0$, it is* *sufficient to verify that $\dot{V}(t) < 0$, and thus the term $z^T(t)z(t) = \phi^T(t)N_2^T N_2\phi(t)$ and the rows/columns corresponding to $w(t)$ are removed from the LMIs (4.31).*

Remark 4.6 *When searching for solutions of LMIs (4.13) and (4.31), one needs to remove the zero rows/columns that may appear in (4.31) since LMI solvers search for strict solutions. Indeed, one can see that $\Xi_{i,j,\sigma}$ is independent of ν if $j = 1$ or $T_{i,\sigma} = 0$.*

Theorem 4.4 provides stability conditions for the perturbed system $\mathbf{S}$ with a given state feedback matrix K and samplings satisfying (4.8). However, in a large variety of control implementations, delays are present in the feedback control loop. They may be induced by network communications, heavy computations, or various physical phenomena. As it has been shown in numerous works, these delays may render the system unstable. Therefore, in order to propose a state-dependent sampling law that is robust to these kind of disturbances, we propose in the following subsection an extension of the present stability results for this large class of systems.

4.2.2 Stability analysis of the perturbed system with delays

Here, we consider systems including a delay in the feedback control loop. The control law is now defined as:

$$u(t) = -Kx(s_k),\ \forall t \in [t_k, t_{k+1}), \tag{4.32}$$

with a constant feedback matrix gain K, s_k the k^{th} sampling time (when the k^{th} input is computed) and t_k the k^{th} actuation time (when the k^{th} computed input is received by the actuators).

The sampling and actuation times are linked by the relation

$$s_k = t_k - h(t_k), \tag{4.33}$$

with a delay $h(t)$ assumed to satisfy:

$$\forall t \geq 0,\ h(t) \in [h_1, h_2],\ \text{and}\ \dot{h}(t) \in [e_1, e_2], \tag{4.34}$$

for given scalars $0 \leq h_1 \leq h_2$ and $e_1 \leq e_2 < 1$.

Note that since $s_{k+1} - s_k > 0$, it implies that $t_{k+1} - t_k \geq \frac{s_{k+1}-s_k}{1-e_1} > 0$, due to $e_1 < 1$ and thus the control inputs are received by the actuator in the same order as they are sent.

The closed-loop system {(4.1),(4.32),(4.3),(4.33),(4.34)} will be denoted $\mathbf{S_d}$.

Here, we consider the LKF:

$$V_d(t, x_t, \dot{x}_t, k) = V_1(t, x_t, \dot{x}_t) + V_2(t, x_t, \dot{x}_t, k), \tag{4.35}$$

defined for all $t \in [t_k, t_{k+1})$ and $k \in \mathbb{N}$, with

$$\begin{array}{rcl} V_1(t, x_t, \dot{x}_t) & = & \int_{t-h_1}^{t} x^T(s)Q_1x(s)ds + \int_{t-h(t)}^{t-h_1} x^T(s)Q_2x(s)ds + \int_{t-h_2}^{t-h(t)} x^T(s)Q_3x(s)ds \\ & & + \int_{t-h(t)}^{t} \dot{x}^T(s)(R_1 + (h(t) - t + s)R_2)\dot{x}(s)ds \\ & & + \int_{-h_2}^{0} \int_{t+\theta}^{t} \dot{x}^T(s)R_3\dot{x}(s)dsd\theta + \int_{-h_2}^{-h_1} \int_{t+\theta}^{t} \dot{x}^T(s)R_4\dot{x}(s)dsd\theta, \end{array} \tag{4.36}$$

consisting of classical terms used for delay systems [Fridman 2001a], [Richard 2003], [Jiang 2010a], [Jiang 2010b], and an additional term

$$\begin{array}{rcl} V_2(t, x_t, \dot{x}_t, k) & = & \eta^T(t)P\eta(t) + (t_{k+1} - t) \begin{bmatrix} \eta(t) \\ \eta(t_k) \end{bmatrix}^T \Omega_{\sigma_k} \begin{bmatrix} \eta(t) \\ \eta(t_k) \end{bmatrix} \\ & & +(t_{k+1} - t) \int_{t_k}^{t} \dot{\eta}^T(s)U_{\sigma_k}\dot{\eta}(s)ds + (t_{k+1} - t)(t - t_k)\eta^T(t_k)S_{\sigma_k}\eta(t_k), \end{array} \tag{4.37}$$

similar to the switching LKF used in the non-delayed case, with the vector $\eta(t)$:

$$\eta(t) = \begin{bmatrix} x(t) \\ x(t - h(t)) \end{bmatrix}, \tag{4.38}$$

and the matrices $\Omega_\sigma, \sigma \in \{1, \cdots, q\}$ defined as:

$$\Omega_\sigma = \begin{bmatrix} \frac{X_\sigma + X_\sigma^T}{2} & -X_\sigma + X_{1,\sigma} \\ * & -X_{1,\sigma} - X_{1,\sigma}^T + \frac{X_\sigma + X_\sigma^T}{2} \end{bmatrix}. \tag{4.39}$$

The matrices P, Q_1, Q_2, Q_3, R_1, R_2, R_3, R_4, U_σ, S_σ, X_σ, $X_{1,\sigma}$ have appropriate dimensions, and the parameter σ_k can be any element $\sigma \in \mathcal{I}(x(s_k))$ satisfying $\tau^- \leq \tau_k \leq \tau_\sigma^+$ (there exists at least one, according to assumption (4.8)).

Similar to what we had with the previous simple LKF, we note that the term (4.37) is *composed of matrix terms Ω_{σ_k}, U_{σ_k}, and S_{σ_k} which are switching at times t_k according to the region $x(s_k)$ belongs to.* This state-dependent switch is possible thanks to the fact that $V_2(t_k, x_{t_k}, \dot{x}_{t_k}, k) = \lim_{t \to t_k^-} V_2(t, x_t, \dot{x}_t, k-1) = 0$, which ensures the continuity of V_2. This function with state-dependent matrices is a natural extension of the works with LKFs on systems with delays [Fridman 2001a], [Richard 2003], [Jiang 2010a], sampling [Fridman 2010], [Seuret 2009], or both delays and sampling [Jiang 2010b].

Just as we did in the delay-free case, we analyse the system's $\mathcal{L}_2$-stability by checking the conditions of Lemma 4.2 with the function

$$\bar{V}_d(t) = V_d(t, x_t, \dot{x}_t, k), \text{ for all } t \in [t_k, t_{k+1}) \text{ and } k \in \mathbb{N}, \tag{4.40}$$

with V_d defined in (4.35). Before providing the lemma ensuring this function's continuity, differentiability, and positivity, as in the non-delayed case, we introduce the following scalars

$$\begin{array}{l} T_{1,\sigma} = T_1 = \max\left\{\tau^- + h_1 - h_2, \frac{\tau^-}{1-e_1}\right\}, \\ T_{2,\sigma} = \min\left\{\tau_\sigma^+ + h_2 - h_1, \frac{\tau_\sigma^+}{1-e_2}\right\}, \end{array} \tag{4.41}$$

which are set to satisfy for any actuation step $k \in \mathbb{N}$:

$$T_{1,\sigma_k} \leq t_{k+1} - t_k \leq T_{2,\sigma_k}. \tag{4.42}$$

Indeed, since $t_{k+1} - t_k = (s_{k+1} - s_k) + (h(t_{k+1}) - h(t_k))$, one has $\tau^- + h_1 - h_2 \leq t_{k+1} - t_k \leq \tau_{\sigma_k}^+ + h_2 - h_1$. Also, since $e_1(t_{k+1} - t_k) \leq h(t_{k+1}) - h(t_k) \leq e_2(t_{k+1} - t_k)$ and $e_1 \leq e_2 < 1$, one has $\frac{\tau^-}{1-e_1} \leq t_{k+1} - t_k \leq \frac{\tau_{\sigma_k}^+}{1-e_2}$, which ends the proof.

Lemma 4.7 *The function $\bar{V}_d$ defined in (4.40) is continuous over $\mathbb{R}^+$ and differentiable for all $t \neq t_k,\ k \in \mathbb{N}$. If its matrix parameters satisfy $P \in S_{2n}^{+*}$, Q_1, Q_2, Q_3, R_1, R_2, R_3, $R_4 \in S_n^+$, U_σ, $S_\sigma \in S_{2n}^+$, X_σ, $X_{1,\sigma} \in \mathcal{M}_{2n,2n}(\mathbb{R})$, and if there exist q scalars $\varepsilon_\sigma \geq 0$ such that, for all $\sigma \in \{1, \cdots, q\}$:*

$$\begin{bmatrix} P & 0 \\ 0 & 0 \end{bmatrix} + T_{2,\sigma}\Omega_\sigma - \varepsilon_\sigma \begin{bmatrix} 0 & 0 \\ 0 & \Psi_\sigma \end{bmatrix} \succ 0, \tag{4.43}$$

then $\bar{V}_d$ is also positive definite, and there exists a scalar $\beta > 0$ such that $\bar{V}_d(t) \geq \beta \|x(t)\|_2^2$ for all $t \geq 0$.

Proof: The proof is very similar to the one in the non-delayed case. The new term $\bar{V}_1$ is obviously differentiable and positive provided that the matrix terms in the integrals are positive. ∎

We introduce the matrices $M_{i\in\{1,\cdots,11\}} \in \mathcal{M}_{n,11n+n_w}(\mathbb{R})$ and $M_{12} \in \mathcal{M}_{n_w,11n+n_w}(\mathbb{R})$:

$$\begin{bmatrix} M_1^T & \cdots & M_{12}^T \end{bmatrix} = I, \tag{4.44}$$

and define the matrices $N_{j\in\{1,\cdots,7\}}$:

$$N_1 = AM_1 - BKM_4 + EM_{12}, N_7 = CM_1 - DKM_4,$$
$$N_2 = \begin{bmatrix} M_1 \\ M_2 \end{bmatrix}, N_3 = \begin{bmatrix} N_1 \\ M_7 \end{bmatrix}, N_4 = \begin{bmatrix} M_3 \\ M_4 \end{bmatrix}, N_5 = \begin{bmatrix} N_2 \\ N_4 \end{bmatrix}, \text{ and } N_6 = \begin{bmatrix} M_8 \\ M_9 \end{bmatrix}. \tag{4.45}$$

The use of these matrices is very similar to the one in the previous simplified case and will be explained in (4.62) and (4.63).

The following theorem provides $\mathcal{L}_2$-stability conditions for the system $\mathbf{S_d}$. They take the form of $(11n+n_w)\times(11n+n_w)$ LMIs that depend on the LKF (4.35) matrices, on the conic regions description (4.7), and on some scalars (ε_σ, $\varepsilon_{i,j,l,o,\sigma}$) and matrices ($Y_{1,\sigma}$, $Y_{2,\sigma}$, $Y_{3,\sigma}$) resulting from the use of the S-procedure [Boyd 1994] (see Theorem D.3 in Appendix D) and Finsler's Lemma [Fang 2004] (see Theorem D.2 in Appendix D) respectively.

Theorem 4.8 *Consider scalars $\gamma \geq 0$, h_1, h_2, e_1, e_2, τ^-, and a set of q conic regions covering the state space $\mathcal{R}_\sigma = \{x, x^T\Psi_\sigma x \geq 0\}$, $\Psi_\sigma \in S_n$, $\sigma \in \{1,\cdots,q\}$, with maximal sampling intervals τ_σ^+. The perturbed and delayed sampled-data system $\mathbf{S_d}$ is finite-gain $\mathcal{L}_2$-stable from w to z with a gain less than γ for any sampling sequence satisfying (4.8) if there exist matrices $P \in S_{2n}^{+*}$, $Q_1, Q_2, Q_3, R_1, R_2, R_3, R_4 \in S_n^+$, $U_\sigma, S_\sigma \in S_{2n}^+$, $X_\sigma, X_{1,\sigma} \in \mathcal{M}_{2n,2n}(\mathbb{R})$, $Y_{1,\sigma} \in \mathcal{M}_{7n,2n}(\mathbb{R})$, $Y_{2,\sigma}, Y_{3,\sigma} \in \mathcal{M}_{7n,n}(\mathbb{R})$ and scalars $\varepsilon_\sigma, \varepsilon_{i,j,l,o,\sigma} \geq 0$ such that (4.43) and (4.46) are satisfied for all $\sigma \in \{1,\cdots,q\}$ and $(i,j,l,o) \in \{1,2\}^4$:*

$$\Xi_{i,j,l,o,\sigma} + \varepsilon_{i,j,l,o,\sigma} M_4^T \Psi_\sigma M_4 \preceq 0, \tag{4.46}$$

with

$$\Xi_{i,j,l,1,\sigma} = \Xi_{i,j,\sigma} + T_{l,\sigma}\left[N_4^T S_\sigma N_4 + N_3^T U_\sigma N_3 + N_3^T \Omega_{1,\sigma} N_5 + N_5^T \Omega_{1,\sigma}^T N_3\right], \tag{4.47}$$

$$\Xi_{i,j,l,2,\sigma} = \Xi_{i,j,\sigma} + T_{l,\sigma}\left[-N_4^T S_\sigma N_4 - N_6^T U_\sigma N_6 + \bar{Y}_{1,\sigma} N_6 + N_6^T \bar{Y}_{1,\sigma}^T\right], \tag{4.48}$$

$$\begin{aligned}
\Xi_{i,j,\sigma} \;=\; & N_3^T P N_2 + N_2^T P N_3 + M_1^T Q_1 M_1 + M_5^T (Q_2 - Q_1) M_5 - M_6^T Q_3 M_6 \\
& + N_1^T (R_1 + h_j R_2 + h_2 R_3 + (h_2 - h_1) R_4) N_1 - \tfrac{1}{1-e_1} M_7^T R_1 M_7 \\
& + (1 - e_i) M_2^T (Q_3 - Q_2) M_2 - \tfrac{1}{h_1} (M_1 - M_5)^T ((1 - e_i) R_2 + R_3)(M_1 - M_5) \\
& - (h_j - h_1) M_{10}^T ((1 - e_2) R_2 + R_3 + R_4) M_{10} - (h_2 - h_j) M_{11}^T (R_3 + R_4) M_{11} \\
& - N_5^T \Omega_\sigma N_5 + N_7^T N_7 - \gamma^2 M_{12}^T M_{12} + \bar{Y}_{1,\sigma}(-N_2 + N_4) + (-N_2 + N_4)^T \bar{Y}_{1,\sigma}^T \\
& + \bar{Y}_{2,\sigma}((h_j - h_1) M_{10} - M_5 + M_2) + ((h_j - h_1) M_{10} - M_5 + M_2)^T \bar{Y}_{2,\sigma}^T \\
& + \bar{Y}_{3,\sigma}((h_2 - h_j) M_{11} - M_2 + M_6) + ((h_2 - h_j) M_{11} - M_2 + M_6)^T \bar{Y}_{3,\sigma}^T,
\end{aligned} \tag{4.49}$$

$$\Omega_{1,\sigma} = \begin{bmatrix} \frac{X_\sigma + X_\sigma^T}{2} & -X_\sigma + X_{1,\sigma} \end{bmatrix}, \tag{4.50}$$

$$\bar{Y}_{1,\sigma} = \begin{bmatrix} Y_{1,\sigma} \\ 0 \end{bmatrix} \in \mathcal{M}_{11n+n_w,2n}(\mathbb{R}),\ \textit{and}\ \bar{Y}_{a,\sigma} = \begin{bmatrix} Y_{a,\sigma} \\ 0 \end{bmatrix} \in \mathcal{M}_{11n+n_w,n}(\mathbb{R}), a \in \{2,3\}. \tag{4.51}$$

Proof: Lemma 4.7 ensures that $\bar{V}_d$ is positive definite and satisfies the required continuity and differentiability properties. As in the non-delayed case, we only need to verify that the condition

$$\dot{\bar{V}}_d(t) + z^T(t)z(t) - \gamma^2 w^T(t)w(t) \leq 0,\ \forall t \neq t_k,\ \forall k \in \mathbb{N}, \tag{4.52}$$

from Lemma 4.2 is satisfied in order to ensure the system's $\mathcal{L}_2$-stability. In order to do so, we study the restriction of $\dot{\bar{V}}_d$ on any interval $[t_k, t_{k+1}), k \in \mathbb{N}$. We compute

$$\begin{aligned}
&\dot{\bar{V}}_d(t) + z^T(t)z(t) - \gamma^2 w^T(t)w(t) = I_1 + I_2 + I_3 + I_4 + z^T(t)z(t) - \gamma^2 w^T(t)w(t) \\
&+2\dot{\eta}^T(t)P\eta(t) + x^T(t)Q_1x(t) + x^T(t-h_1)(Q_2 - Q_1)x(t-h_1) - x^T(t-h_2)Q_3x(t-h_2) \\
&+(1-\dot{h}(t))x^T(t-h(t))(Q_3 - Q_2)x(t-h(t)) - (1-\dot{h}(t))\dot{x}^T(t-h(t))R_1\dot{x}(t-h(t)) \\
&+\dot{x}^T(t)(R_1 + h(t)R_2 + h_2R_3 + (h_2-h_1)R_4)\dot{x}(t) + ((t_{k+1}-t_k) - 2(t-t_k))\eta^T(t_k)S_{\sigma_k}\eta(t_k) \\
&+(t_{k+1}-t)\dot{\eta}^T(t)U_{\sigma_k}\dot{\eta}(t) - \begin{bmatrix} \eta(t) \\ \eta(t_k) \end{bmatrix}^T \Omega_{\sigma_k} \begin{bmatrix} \eta(t) \\ \eta(t_k) \end{bmatrix} + 2(t_{k+1}-t)\dot{\eta}^T(t)\Omega_{1,\sigma_k} \begin{bmatrix} \eta(t) \\ \eta(t_k) \end{bmatrix},
\end{aligned} \tag{4.53}$$

where

$$\begin{aligned}
I_1 &= -\textstyle\int_{t_k}^t \dot{\eta}^T(s)U_{\sigma_k}\dot{\eta}(s)ds, \\
I_2 &= -(1-\dot{h}(t))\textstyle\int_{t-h(t)}^t \dot{x}^T(s)R_2\dot{x}(s)ds, \\
I_3 &= -\textstyle\int_{t-h_2}^t \dot{x}^T(s)R_3\dot{x}(s)ds, \\
I_4 &= -\textstyle\int_{t-h_2}^{t-h_1} \dot{x}^T(s)R_4\dot{x}(s)ds.
\end{aligned} \tag{4.54}$$

Using the Jensen inequality [Gu 2003] (see Theorem D.4 in Appendix D), we can compute an upper bound of I_1:

$$I_1 = -\int_{t_k}^t \dot{\eta}^T(s)U_{\sigma_k}\dot{\eta}(s)ds \leq -(t-t_k)\nu_1^T(t)U_{\sigma_k}\nu_1(t), \tag{4.55}$$

with

$$\nu_1(t) = \frac{1}{t-t_k}\int_{t_k}^t \dot{\eta}(s)ds = \frac{\eta(t)-\eta(t_k)}{t-t_k}. \tag{4.56}$$

For an upper bound on the other integral terms, one writes:

$$I_2 + I_3 + I_4 = J_2 + J_3 + J_4, \tag{4.57}$$

with

$$\begin{array}{l} J_2 = -\int_{t-h_1}^{t} \dot{x}^T(s)((1-\dot{h}(t))R_2 + R_3)\dot{x}(s)ds, \\ J_3 = -\int_{t-h(t)}^{t-h_1} \dot{x}^T(s)((1-\dot{h}(t))R_2 + R_3 + R_4)\dot{x}(s)ds, \\ J_4 = -\int_{t-h_2}^{t-h(t)} \dot{x}^T(s)(R_3 + R_4)\dot{x}(s)ds. \end{array} \tag{4.58}$$

Then, using the Jensen inequality [Gu 2003] (see Theorem D.4 in Appendix D), one obtains

$$\begin{array}{ll} J_2 & \leq -\frac{1}{h_1}\nu_2^T(t)((1-\dot{h}(t))R_2 + R_3)\nu_2(t), \\ J_3 & \leq -(h(t)-h_1)\nu_3^T(t)((1-\dot{h}(t))R_2 + R_3 + R_4)\nu_3(t) \\ & \leq -(h(t)-h_1)\nu_3^T(t)((1-e_2)R_2 + R_3 + R_4)\nu_3(t), \\ J_4 & \leq -(h_2-h(t))\nu_4^T(t)(R_3+R_4)\nu_4(t), \end{array} \tag{4.59}$$

with:

$$\begin{array}{l} \nu_2(t) = x(t) - x(t-h_1), \\ \nu_3(t) = \frac{1}{h(t)-h_1}\int_{t-h(t)}^{t-h_1} \dot{x}(s)ds = \frac{x(t-h_1)-x(t-h(t))}{h(t)-h_1}, \\ \nu_4(t) = \frac{1}{h_2-h(t)}\int_{t-h_2}^{t-h(t)} \dot{x}(s)ds = \frac{x(t-h(t))-x(t-h_2)}{h_2-h(t)}. \end{array} \tag{4.60}$$

Note that $\nu_1(t)$ (respectively $\nu_3(t)$ and $\nu_4(t)$) is well defined by continuity in $t = t_k$ (respectively $h(t) = h_1$ or $h(t) = h_2$) since when $t \to t_k$ (respectively $h(t) \to h_1$ or $h(t) \to h_2$), one has $\nu_1(t) \to \dot{\eta}(t_k)$ (respectively $\nu_3(t) \to -\dot{x}(t-h_1)$ and $\nu_4(t) \to -\dot{x}(t-h_2)$).

Using majorations (4.55) and (4.59) in equation (4.53) leads to

$$\begin{array}{l} \dot{V}_d(t) + z^T(t)z(t) - \gamma^2 w^T(t)w(t) \leq z^T(t)z(t) - \gamma^2 w^T(t)w(t) \\ +2\dot{\eta}^T(t)P\eta(t) + x^T(t)Q_1x(t) + x^T(t-h_1)(Q_2-Q_1)x(t-h_1) - x^T(t-h_2)Q_3x(t-h_2) \\ +(1-\dot{h}(t))x^T(t-h(t))(Q_3-Q_2)x(t-h(t)) - (1-\dot{h}(t))\dot{x}^T(t-h(t))R_1\dot{x}(t-h(t)) \\ +\dot{x}^T(t)(R_1 + h(t)R_2 + h_2R_3 + (h_2-h_1)R_4)\dot{x}(t) + ((t_{k+1}-t_k) - 2(t-t_k))\eta^T(t_k)S_{\sigma_k}\eta(t_k) \\ +(t_{k+1}-t)\dot{\eta}^T(t)U_{\sigma_k}\dot{\eta}(t) - \begin{bmatrix}\eta(t)\\ \eta(t_k)\end{bmatrix}^T \Omega_{\sigma_k}\begin{bmatrix}\eta(t)\\ \eta(t_k)\end{bmatrix} + 2(t_{k+1}-t)\dot{\eta}^T(t)\Omega_{1,\sigma_k}\begin{bmatrix}\eta(t)\\ \eta(t_k)\end{bmatrix} \\ -(t-t_k)\nu_1^T(t)U_{\sigma_k}\nu_1(t) - \frac{1}{h_1}\nu_2^T(t)((1-\dot{h}(t))R_2+R_3)\nu_2(t) \\ -(h(t)-h_1)\nu_3^T(t)((1-e_2)R_2+R_3+R_4)\nu_3(t) - (h_2-h(t))\nu_4^T(t)(R_3+R_4)\nu_4(t). \end{array} \tag{4.61}$$

We introduce the augmented state vector $\phi(t) \in \mathbb{R}^{11n+n_w}$:

$$\begin{array}{rcl} \phi^T(t) & = & [\eta^T(t),\ \eta^T(t_k),\ x^T(t-h_1),\ x^T(t-h_2), \\ & & (1-\dot{h}(t))\dot{x}^T(t-h(t)),\ \nu_1^T(t),\ \nu_3^T(t),\ \nu_4^T(t),\ w^T(t)], \end{array} \tag{4.62}$$

and use the notations (4.44) and (4.45) to write

$$\begin{aligned}
&\eta(t) = \begin{bmatrix} x(t) \\ x(t-h(t)) \end{bmatrix} = \begin{bmatrix} M_1\phi(t) \\ M_2\phi(t) \end{bmatrix} = N_2\phi(t), \\
&\eta(t_k) = \begin{bmatrix} x(t_k) \\ x(s_k) \end{bmatrix} = \begin{bmatrix} M_3\phi(t) \\ M_4\phi(t) \end{bmatrix} = N_4\phi(t), \\
&x(t-h_1) = M_5\phi(t),\ x(t-h_2) = M_6\phi(t), \\
&(1-\dot{h}(t))\dot{x}(t-h(t)) = M_7\phi(t), \\
&\nu_1(t) = \begin{bmatrix} M_8 \\ M_9 \end{bmatrix}\phi(t) = N_6\phi(t),\ \nu_2(t) = (M_1 - M_5)\phi(t), \\
&\nu_3(t) = M_{10}\phi(t),\ \nu_4(t) = M_{11}\phi(t),\ w(t) = M_{12}\phi(t), \\
&\dot{x}(t) = (AM_1 - BKM_4 + EM_{12})\phi(t) = N_1\phi(t), \\
&\dot{\eta}(t) = \begin{bmatrix} \dot{x}(t) \\ (1-\dot{h}(t))\dot{x}(t-h(t)) \end{bmatrix} = \begin{bmatrix} N_1\phi(t) \\ M_7\phi(t) \end{bmatrix} = N_3\phi(t), \\
&\begin{bmatrix} \eta(t) \\ \eta(t_k) \end{bmatrix} = \begin{bmatrix} N_2\phi(t) \\ N_4\phi(t) \end{bmatrix} = N_5\phi(t), \\
&z(t) = (CM_1 - DKM_4)\phi(t) = N_7\phi(t).
\end{aligned} \tag{4.63}$$

Using these notations, one can see that the equations (4.56) and (4.60) about ν_1, ν_3, and ν_4 can be written as $H_i(t)\phi(t) = 0$ with $H_1(t) = (t-t_k)N_6 - N_2 + N_4$, $H_3(t) = (h(t)-h_1)M_{10} - M_5 + M_2$ and $H_4(t) = (h_2 - h(t))M_{11} - M_2 + M_6$, respectively. Therefore, by applying the Finsler's lemma [Fang 2004] (see Theorem D.2 in Appendix D) to include these relations in (4.61), one obtains that for any matrices $Y_{1,\sigma_k} \in \mathcal{M}_{7n,2n}(\mathbb{R})$, Y_{2,σ_k}, and $Y_{3,\sigma_k} \in \mathcal{M}_{7n,n}(\mathbb{R})$:

$$\begin{aligned}
&\dot{V}_d(t) + z^T(t)z(t) - \gamma^2 w^T(t)w(t) \leq \\
&\phi^T(t)[N_3^T P N_2 + N_2^T P N_3 + M_1^T Q_1 M_1 + M_5^T(Q_2 - Q_1)M_5 - M_6^T Q_3 M_6 \\
&+(1-\dot{h}(t))M_2^T(Q_3 - Q_2)M_2 + N_1^T(R_1 + h(t)R_2 + h_2R_3 + (h_2-h_1)R_4)N_1 \\
&-\tfrac{1}{1-e_1}M_7^T R_1 M_7 - N_5^T\Omega_{\sigma_k}N_5 - \tfrac{1}{h_1}(M_1-M_5)^T((1-\dot{h}(t))R_2 + R_3)(M_1 - M_5) \\
&-(h(t)-h_1)M_{10}^T((1-e_2)R_2 + R_3 + R_4)M_{10} - (h_2 - h(t))M_{11}^T(R_3+R_4)M_{11} \\
&+(t_{k+1}-t)(N_3^T U_{\sigma_k} N_3 + N_3^T\Omega_{1,\sigma_k}N_5 + N_5^T\Omega_{1,\sigma_k}^T N_3) \\
&+((t_{k+1}-t_k) - 2(t-t_k))N_4^T S_{\sigma_k} N_4 - (t-t_k)N_6^T U_{\sigma_k} N_6 + N_7^T N_7 - \gamma^2 M_{12}^T M_{12} \\
&+\bar{Y}_{1,\sigma_k}((t-t_k)N_6 - N_2 + N_4) + ((t-t_k)N_6 - N_2 + N_4)^T\bar{Y}_{1,\sigma_k}^T \\
&+\bar{Y}_{2,\sigma_k}((h(t)-h_1)M_{10} - M_5 + M_2) + ((h(t)-h_1)M_{10} - M_5 + M_2)^T\bar{Y}_{2,\sigma_k}^T \\
&+\bar{Y}_{3,\sigma_k}((h_2-h(t))M_{11} - M_2 + M_6) + ((h_2-h(t))M_{11} - M_2 + M_6)^T\bar{Y}_{3,\sigma_k}^T]\phi(t).
\end{aligned} \tag{4.64}$$

As in the delay-free case, we obtain a stability condition under the form of a parametric equation (4.64), which is linear in the variables $\dot{h}(t)$, $h(t)$, t and $t_{k+1} - t_k$. In order to reduce the number of conditions to be checked to a finite number, we use Lemma D.8 (Appendix) on (4.64) with, successively, the variables $\lambda = \dot{h}(t) \in [e_1, e_2]$, $\lambda = h(t) \in [h_1, h_2]$, $\lambda = t \in [t_k, t_{k+1}]$, and $\lambda = t_{k+1} - t_k \in [T_{1,\sigma_k}, T_{2,\sigma_k}]$, to show that if the 16 inequalities $\xi^T \Xi_{i,j,l,o,\sigma_k} \xi \leq 0$ are satisfied for all $\xi \in \mathbb{R}^{11n+n_w}$, with Ξ_{i,j,l,o,σ_k} defined in (4.47) and (4.48), then $\dot{\tilde{V}}_d(t) + z^T(t)z(t) - \gamma^2 w^T(t)w(t) \leq 0$ for all $t \in [t_k, t_{k+1})$.

Since we know that $x(s_k) \in \mathcal{R}_{\sigma_k}$ (*i.e.* $x^T(s_k)\Psi_{\sigma_k} x(s_k) \geq 0$), we use once again the lossless version of the S-procedure [Boyd 1994] (see Theorem D.3 in Appendix D) on each of the 16 obtained inequalities to show that, if there are scalars $\varepsilon_{i,j,l,o,\sigma_k} \geq 0$ such that (4.46) holds for $\sigma = \sigma_k$, then condition (4.52) is satisfied. Therefore, if (4.46) is satisfied for all $\sigma \in \{1, \cdots, q\}$, (4.52) and Lemma 4.2 allow for concluding the proof. ∎

Remark 4.9 *Setting $h_1 = 0$ reduces the size of the LMIs and the number of variables, since the state $x(t - h_1)$, the matrices Q_1 and R_4, and the integral term J_2 (in (4.57)) are not needed anymore.*

Remark 4.10 *Similar to the delay-free case, if w satisfies $z^T(t)z(t) - \gamma^2 w^T(t)w(t) \geq 0$, and if the LMIs (4.46) are strict, the delayed sampled-data system $\mathbf{S_d}$ is asymptotically stable for any sampling sequence satisfying (4.8). In the unperturbed case $w(t) = 0$, it is sufficient to verify that $\dot{\tilde{V}}(t) < 0$, and thus the term $z^T(t)z(t) = \phi^T(t)N_7^T N_7\phi(t)$ and the rows/columns corresponding to $w(t)$ are removed from the LMIs (4.46).*

Remark 4.11 *As in the non-delayed case, when searching for solutions of LMIs (4.43) and (4.46), one needs to remove the zero rows/columns that may appear in (4.46) since LMI solvers search for strict solutions. Indeed, one can see that $\Xi_{i,j,l,o,\sigma}$ is: independent of ν_1 if $o = 1$ or $T_{l,\sigma} = 0$; independent of ν_3 if $j = 1$; independent of ν_4 if $j = 2$.*

4.2.3 Algorithm to design the state-dependent sampling function $\tau_{\max}$ for a given feedback matrix gain K

Figure 4.1 provides a three-step algorithm to build a state-dependent sampling function maximizing the sampling intervals using the stability conditions from Subsections 4.2.1 and 4.2.2. This algorithm is based on a computation of the Lyapunov-Krasovskii Functional in two steps.

- First, we compute the constant ("global") LKF matrices, that allow to maximize the lower bound τ^+ of the sampling function, which leads to a classic robust analysis of perturbed (and possibly delayed) sampled-data systems with time-varying samplings.

- Then, we compute the switching ("local") LKF matrices so as to maximize the allowable sampling intervals τ_σ^+ for each region, which leads to a self-triggering algorithm except that all computations are made offline, and that the switching part of the LKF is computed at the same time as the maximal samplings τ_σ^+.

Keep in mind that all steps in the algorithm are made off-line.

STEP 1

◇ Work with a single region $\mathbb{R}^n$, or q conic regions $\mathcal{R}_\sigma$ covering the state space
◇ Fix the minimal sampling τ^-

Theorem 4.4 (resp. 4.8 with delays) LMIs + ↓ line search algorithm on τ^+

◇ Compute the largest admissible state-independent sampling τ^+ (common to every region)
◇ Compute the constant (i.e. global) LKF matrix P
(as well as the matrices Q_1, Q_2, Q_3, R_1, R_2, R_3, R_4 in the delayed case)

STEP 2

◇ Work with q conic regions $\mathcal{R}_\sigma$ covering the state space
◇ Use the same minimal sampling τ^- as in Step 1
◇ Use the LKF matrix P, (as well as the other global matrices Q_1, Q_2, Q_3, R_1, R_2, R_3, R_4 in the delayed case) computed in Step 1

Theorem 4.4 (resp. 4.8 with delays) LMIs + ↓ line search algorithm on τ_σ^+ for each region $\mathcal{R}_\sigma$

◇ Design the mapping of the largest admissible sampling $\tau_\sigma^+ \geq \tau^+$ for each region $\mathcal{R}_\sigma$
◇ Compute the switching (i.e. local) LKF matrices U_σ, S_σ, X_σ, $X_{1,\sigma}$, for each region $\mathcal{R}_\sigma$

STEP 3

Design of the state-dependent sampling function
$\tau_{\max}(x) = \max_{\sigma\in\mathcal{I}(x)} \tau_\sigma^+ \geq \tau^+, \forall x \in \mathbb{R}^n$

Figure 4.1: Algorithm to design the state-dependent sampling function $\tau_{\max}(x)$ for a given feedback matrix gain K

Note that one can compute the largest admissible state-independent sampling τ^+ (Step 1) by working with a single region ($\mathbb{R}^n$), and using the proposed LKF with constant matrices, or by working with more regions $\mathcal{R}_\sigma$, and using the LKF with state-dependent matrices switching according to the sampled state. Although the second choice is more complex, it can greatly reduce the conservatism, as it will be illustrated in Example 2. For the same reason, if the first step was proceeded with only one region $\mathbb{R}^n$, it is possible that one obtains in the second step $\min_{\sigma\in\{1,\cdots,q\}} \tau_\sigma^+ > \tau^+$, which means that even the largest state-independent sampling obtained can be increased by using the LKF with state-dependent matrices.

Also, note that very often the works are carried with a minimal sampling interval τ^- set to 0, as in [Wang 2009], [Wang 2010] or [Fujioka 2009b], or as in the polytopic approach presented in the previous two chapters. Enabling $\tau^- > 0$ allows to consider systems which are unstable with a continuous feedback control, but which are stable with sampling intervals that are lower bounded (see Example 3 for an illustration). Furthermore, enabling a larger minimal sampling makes it possible to increase the obtained maximal sampling τ_σ^+ with the proposed technique, since the stability conditions we obtained ensure stability for any sampling satisfying (4.8): $\tau^- \leq \tau_k \leq \tau_{\sigma_k}^+$.

The final step of the algorithm to design the state-dependent sampling function deals with the possible regions overlapping issue, since the regions $\mathcal{R}_\sigma$ are not necessarily disjoints.

Remark 4.12 *Unlike with the polytopic method presented in the previous two chapters, it is very difficult to design an event-triggered or a self-triggered control scheme with this LKF approach. This is due to the fact that here, the stabilty conditions are obtained through the use of an augmented state in which appear both the delays and the state derivatives. Therefore, it is difficult to isolate the sampled-state $x(s_k)$ along with the time-variable $t - t_k$ in the otained stability conditions.*

Remark 4.13 *Unlike with the polytopic approach, here it is simple to include the minimal sampling interval τ^- in the stability conditions. Although it would still be possible with the previous approach (in the delay-free case), the complexity would be very high. Indeed, one would have to design a polytope with respect to the variable $t \in [s_k, s_{k+1}]$, which would result in $N+1$ vertices with the convex embedding approach presented in the Appendix C.1 (N being the order of the Taylor series approximation), and for each of these vertices, one would need to build another polytope with respect to the variable $s_{k+1} - s_k \in [\tau^-, \tau_{max}(x)]$, which would result in a complex design of $(N+1)^2$ vertices.*

4.3 Main $\mathcal{L}_2$-stabilization results

In this section, we propose a way to design the control input, allowing to enlarge even more the state-dependent sampling function $\tau_{\max}$. Remember that the objective of this work is double: we want to maximize both the lower bound τ^+ of the state-dependent sampling function (which ensures stability for any state-independent time-varying sampling sequence in $[\tau^-, \tau^+]$), and the sampling function $\tau_{\max}$ itself.

In Subsection 4.3.1, we consider the case of a classical piecewise-constant feedback control $u(t) = -Kx(s_k)$ and provide tools to compute an adequate feedback gain K allowing to enlarge the lower bound τ^+ of the sampling function. This analysis can be seen as a robust stabilization method regarding state-independent time-varying sampling. Once this feedback gain is computed, it is possible to build its associated optimal sampling function $\tau_{\max}$ using the algorithm presented in Subsection 4.2.3.

In Subsection 4.3.2, we go a step further and design a piecewise-constant feedback control with matrices that switch according to the sampled state ($u(t) = -K_{\sigma_k}x(s_k)$), which allows to enlarge even further the sampling function $\tau_{\max}$ (4.9). Indeed, with this type of controller, one can design the feedback gains K_σ so as to enlarge the maximal allowable sampling τ_σ^+ for each region $\mathcal{R}_\sigma$.

Eventually, in Subsection 4.3.3, we provide an algorithm to be used with either obtained stabilization results, to enlarge the sampling function $\tau_{\max}$ while computing the adequate controller matrix K (or matrices K_σ).

For simplicity, in this section we assume that $D = 0$.

4.3.1 Stabilization using a piecewise-constant feedback control $u(t) = -Kx(s_k)$

Here, we want to compute the feedback matrix K that maximizes the lower-bound τ^+ of the sampling function. we provide the following stabilization theorem:

Theorem 4.14 *Consider scalars $\gamma \geq 0$, and $0 < \tau^- \leq \tau^+$. The perturbed system* $\boldsymbol{S}$ *(with $D = 0$) is finite-gain $\mathcal{L}_2$-stabilizable from w to z with a gain less than γ for any sampling sequence with values in $[\tau^-, \tau^+]$ if there exist matrices $\tilde{P} \in S_n^{+*}$, $\tilde{U}$, $\tilde{S} \in S_n^+$, $Q \in \mathcal{M}_{n,n}(\mathbb{R})$ invertible, $M \in \mathcal{M}_{n_u,n}(\mathbb{R})$, $\tilde{Y}_1$, $\tilde{Y}_2$, $\tilde{Y}_3$, $\tilde{X}$, $\tilde{X}_1 \in \mathcal{M}_{n,n}(\mathbb{R})$, and a scalar δ such that the inequalities*

$$\begin{bmatrix} \tilde{P} & 0 \\ 0 & 0 \end{bmatrix} + \tau^+ \begin{bmatrix} \frac{\tilde{X}+\tilde{X}^T}{2} & -\tilde{X} + \tilde{X}_1 \\ * & \frac{\tilde{X}+\tilde{X}^T}{2} \end{bmatrix} \succ 0 \text{ and } \hat{\Xi}_{i,j} \preceq 0 \tag{4.65}$$

are satisfied for all $(i,j) \in \{1,2\}^2$*, with*

$$\hat{\Xi}_{1,j} = \begin{bmatrix} \hat{L}_{1,1} & \hat{L}_{1,2} + T_j\tilde{Z} & \hat{L}_{1,3} & E & Q^TC^T \\ * & \hat{L}_{2,2} + T_j\tilde{U} & \hat{L}_{2,3} + T_j(-\tilde{X} + \tilde{X}_1) & \delta E & 0 \\ * & * & \hat{L}_{3,3} + T_j\tilde{S} & 0 & 0 \\ * & * & * & -\gamma^2 I & 0 \\ * & * & * & * & -I \end{bmatrix}, \tag{4.66}$$

$$\hat{\Xi}_{2,j} = \begin{bmatrix} \hat{L}_{1,1} & \hat{L}_{1,2} & \hat{L}_{1,3} & T_j\tilde{Y}_1^T & E & Q^TC^T \\ * & \hat{L}_{2,2} & \hat{L}_{2,3} & T_j\tilde{Y}_2^T & \delta E & 0 \\ * & * & \hat{L}_{3,3} - T_j\tilde{S} & T_j\tilde{Y}_3^T & 0 & 0 \\ * & * & * & -T_j\tilde{U} & 0 & 0 \\ * & * & * & * & -\gamma^2 I & 0 \\ * & * & * & * & * & -I \end{bmatrix}, \tag{4.67}$$

$$\begin{aligned} &\hat{L}_{1,1} = -\tilde{Y}_1^T - \tilde{Y}_1 + AQ + Q^TA^T - \tilde{Z},\ \hat{L}_{1,2} = \tilde{P} - \tilde{Y}_2 - Q + \delta Q^TA^T, \\ &\hat{L}_{1,3} = -BM + \tilde{Y}_1^T - \tilde{Y}_3 + \tilde{X} - \tilde{X}_1,\ \hat{L}_{2,2} = -\delta Q^T - \delta Q, \\ &\hat{L}_{2,3} = \tilde{Y}_2^T - \delta BM,\ \hat{L}_{3,3} = \tilde{Y}_3^T + \tilde{Y}_3 + \tilde{Z}_1 - \tilde{Z}, \end{aligned} \tag{4.68}$$

$$\tilde{Z} = \frac{\tilde{X} + \tilde{X}^T}{2},\ \textit{and}\ \tilde{Z}_1 = \tilde{X}_1^T + \tilde{X}_1. \tag{4.69}$$

The stabilizing feedback matrix gain is provided by $K = MQ^{-1}$*.*

Proof: We work with a single region: $\mathbb{R}^n$ and with the LKF (4.10) designed for the delay-free case. Using arguments very similar to the ones used to obtain equation (4.26), which leads to Theorem 4.4, we can show that

$$\begin{aligned} &\dot{V}(t) + z^T(t)z(t) - \gamma^2 w^T(t)w(t) \leq 2\dot{x}^T(t)Px(t) + (s_{k+1} - t)\dot{x}^T(t)U\dot{x}(t) \\ &-(t - s_k)\nu^T(t)U\nu(t) - \begin{bmatrix} x(t) \\ x(s_k) \end{bmatrix}^T \Omega \begin{bmatrix} x(t) \\ x(s_k) \end{bmatrix} + 2(s_{k+1} - t)\dot{x}^T(t)\Omega_1 \begin{bmatrix} x(t) \\ x(s_k) \end{bmatrix} \\ &+((s_{k+1} - s_k) - 2(t - s_k))x^T(s_k)Sx(s_k) + x^T(t)C^TCx(t) - \gamma^2 w^T(t)w(t) \\ &+2(x^T(t)Y_1^T + \dot{x}^T(t)Y_2^T + x^T(s_k)Y_3^T)((t - s_k)\nu(t) - x(t) + x(s_k)), \end{aligned} \tag{4.70}$$

for any matrices Y_1, Y_2, $Y_3 \in \mathcal{M}_{n,n}(\mathbb{R})$.

We use Finsler's lemma [Fang 2004] (or the descriptor method [Fridman 2001b]) to include the relation $\dot{x}(t) = Ax(t) - BKx(s_k) + Ew(t)$, (by adding the term $0 = 2(x^T(t)P_2^T + \dot{x}^T(t)P_3^T)(-\dot{x}(t) + Ax(t) - BKx(s_k) + Ew(t))$ to the previous inequality) and the augmented state vector

$$\bar{\phi}^T(t) = [x^T(t),\ \dot{x}^T(t),\ x^T(s_k),\ \nu^T(t),\ w^T(t)], \tag{4.71}$$

to show that if the matrix inequalities

$$\bar{\Xi}_{i,j} \preceq 0 \tag{4.72}$$

are satisfied, with

$$\bar{\Xi}_{1,j} = \begin{bmatrix} \bar{L}_{1,1} & \bar{L}_{1,2} + T_j Z & \bar{L}_{1,3} & P_2^T E \\ * & \bar{L}_{2,2} + T_j U & \bar{L}_{2,3} + T_j(-X + X_1) & P_3^T E \\ * & * & \bar{L}_{3,3} + T_j S & 0 \\ * & * & * & -\gamma^2 I \end{bmatrix}, \tag{4.73}$$

$$\bar{\Xi}_{2,j} = \begin{bmatrix} \bar{L}_{1,1} & \bar{L}_{1,2} & \bar{L}_{1,3} & T_j Y_1^T & P_2^T E \\ * & \bar{L}_{2,2} & \bar{L}_{2,3} & T_j Y_2^T & P_3^T E \\ * & * & \bar{L}_{3,3} - T_j S & T_j Y_3^T & 0 \\ * & * & * & -T_j U & 0 \\ * & * & * & * & -\gamma^2 I \end{bmatrix}, \tag{4.74}$$

$$\begin{array}{l} \bar{L}_{1,1} = C^T C - Y_1^T - Y_1 + P_2^T A + A^T P_2 - Z,\ \bar{L}_{1,2} = P - Y_2 - P_2^T + A^T P_3, \\ \bar{L}_{1,3} = -P_2^T BK + Y_1^T - Y_3 + X - X_1,\ \bar{L}_{2,2} = -P_3 - P_3^T, \\ \bar{L}_{2,3} = Y_2^T - P_3^T BK,\ \bar{L}_{3,3} = Y_3^T + Y_3 + Z_1 - Z, \end{array} \tag{4.75}$$

$$Z = \frac{X + X^T}{2},\ Z_1 = X_1^T + X_1, \tag{4.76}$$

and matrices P_2, P_3, Y_1, Y_2, $Y_3 \in \mathcal{M}_{n,n}(\mathbb{R})$, then the stability condition (4.6) is always satisfied.

We need to compute a feedback matrix gain K satisfying these conditions. In order to do so, we consider the case where $P_3 = \delta P_2$, with P_2 invertible, and $\delta \in \mathbb{R}$. Then, we multiply the previous matrix inequalities by $diag(P_2^{-T}, \cdots, P_2^{-T}, I)$ to the left, and $diag(P_2^{-1}, \cdots, P_2^{-1}, I)$ to the right, and use the Schur complement [Boyd 1994] (see Theorem D.1 in Appendix D) to obtain Theorem 4.14 stabilization conditions (where the notation Q denotes P_2^{-1}, and the notation $\tilde{F}$ denotes the multiplication of a matrix $F \in \mathcal{M}_n(\mathbb{R})$ from left and right by P_2^{-T} and P_2^{-1}: $\tilde{F} \equiv P_2^{-T} F P_2^{-1}$). ∎

Remark 4.15 *Note that the second inequality in (4.65) is not an LMI. It can be solved by LMI solvers however, by including a line search algorithm on the variable δ.*

Remark 4.16 *It is possible to include constraints to avoid having an unacceptable high gain $K = MQ^{-1}$. In order to do this, one can add the LMIs*

$$\begin{bmatrix} -\kappa_M I & M^T \\ * & -I \end{bmatrix} \preceq 0 \text{ and } \begin{bmatrix} Q & I \\ * & \kappa_Q I \end{bmatrix} \succeq 0, \tag{4.77}$$

which ensure that $|||K|||_2 \leq \sqrt{\kappa_M}\kappa_Q$.

Indeed, using the Schur complement, one can show that the LMIs in (4.77) imply that

$$M^T M \preceq \kappa_M I, \text{ and } \begin{cases} Q^{-1} \preceq \kappa_Q I \\ Q^{-T} \preceq \kappa_Q I \end{cases}, \tag{4.78}$$

and thus

$$|||M|||_2 = \sqrt{\rho(M^T M)} \leq \sqrt{\kappa_M}, \text{ and } |||Q^{-1}|||_2 = \sqrt{\rho(Q^{-T}Q^{-1})} \leq \sqrt{\rho(Q^{-T})\rho(Q^{-1})} \leq \kappa_Q. \tag{4.79}$$

Therefore, since $K = MQ^{-1}$ and the property of the matrix norm $|||.|||_2$ ensures that $|||K|||_2 \leq |||M|||_2|||Q^{-1}|||_2$, one has $|||K|||_2 \leq \sqrt{\kappa_M}\kappa_Q$.

Using this theorem, it is possible to compute the feedback matrix gain maximizing the lower bound τ^+ of the state-dependent sampling function. The algorithm to compute the optimal gain K along with the associated sampling function $\tau_{\max}$ is proposed in 4.3.3.

4.3.2 Stabilization using a switching piecewise-constant feedback control $u(t) = -K_{\sigma_k} x(s_k)$

The previous theorem enables to compute a feedback matrix gain K that maximizes the lower bound of the state-dependent sampling function $\tau_{\max}$. However, it seems natural to think that it may be better to adapt the control gain according to the value of the state, in order to enlarge even further the state-dependent sampling function $\tau_{\max}$. Here, therefore, we work with a more general feedback control law

$$u(t) = -K_{\sigma_k} x(s_k), \tag{4.80}$$

with feedback matrix gains K_σ that switch according to the region of the state space the sampled state is in (at each sampling step, σ_k is set to satisfy (4.8)). Using the previous stability and stabilization analysis, we can extend the obtained results to this class of systems and to show the following theorem:

Theorem 4.17 *Consider scalars $\gamma \geq 0$ and $\tau^- > 0$, matrices $P \in S_n^{+*}$, P_2, $P_3 \in \mathcal{M}_{n,n}(\mathbb{R})$, and a set of q conic regions covering the state space $\mathcal{R}_\sigma = \{x, x^T\Psi_\sigma x \geq 0\}$, $\Psi_\sigma \in S_n$, $\sigma \in \{1, \cdots, q\}$, with maximal sampling intervals τ_σ^+. The perturbed system **S** (with $D = 0$) with control input (4.80) is finite-gain $\mathcal{L}_2$-stabilizable from w to z with a gain less than γ for any sampling sequence satisfying (4.8) if there exist scalars $\varepsilon_{i,j,\sigma} \geq 0$ and matrices U_σ, $S_\sigma \in S_n^+$, $Y_{1,\sigma}$, $Y_{2,\sigma}$, $Y_{3,\sigma}$, X_σ, $X_{1,\sigma} \in \mathcal{M}_{n,n}(\mathbb{R})$, and $K_\sigma \in \mathcal{M}_{n_u,n}(\mathbb{R})$ such that the LMIs (4.13) and $\bar{\Xi}_{i,j,\sigma} \preceq 0$ are satisfied for all $(i,j) \in \{1,2\}^2$, with*

$$\bar{\Xi}_{1,j,\sigma} = \begin{bmatrix} \bar{L}_{1,1,\sigma} & \bar{L}_{1,2,\sigma} + T_{j,\sigma}Z_\sigma & \bar{L}_{1,3,\sigma} & P_2^T E \\ * & \bar{L}_{2,2,\sigma} + T_{j,\sigma}U_\sigma & \bar{L}_{2,3,\sigma} + T_{j,\sigma}(-X_\sigma + X_{1,\sigma}) & P_3^T E \\ * & * & \bar{L}_{3,3,\sigma} + T_{j,\sigma}S_\sigma + \varepsilon_{1,j,\sigma}\Psi_\sigma & 0 \\ * & * & * & -\gamma^2 I \end{bmatrix}, \tag{4.81}$$

$$\bar{\Xi}_{2,j,\sigma} = \begin{bmatrix} \bar{L}_{1,1,\sigma} & \bar{L}_{1,2,\sigma} & \bar{L}_{1,3,\sigma} & T_{j,\sigma}Y_{1,\sigma}^T & P_2^T E \\ * & \bar{L}_{2,2,\sigma} & \bar{L}_{2,3,\sigma} & T_{j,\sigma}Y_{2,\sigma}^T & P_3^T E \\ * & * & \bar{L}_{3,3,\sigma} - T_{j,\sigma}S_\sigma + \varepsilon_{2,j,\sigma}\Psi_\sigma & T_{j,\sigma}Y_{3,\sigma}^T & 0 \\ * & * & * & -T_{j,\sigma}U_\sigma & 0 \\ * & * & * & * & -\gamma^2 I \end{bmatrix}, \tag{4.82}$$

$$\begin{aligned} &\bar{L}_{1,1,\sigma} = C^TC - Y_{1,\sigma}^T - Y_{1,\sigma} + P_2^TA + A^TP_2 - Z_\sigma,\ \bar{L}_{1,2,\sigma} = P - Y_{2,\sigma} - P_2^T + A^TP_3, \\ &\bar{L}_{1,3,\sigma} = -P_2^TBK_\sigma + Y_{1,\sigma}^T - Y_{3,\sigma} + X_\sigma - X_{1,\sigma},\ \bar{L}_{2,2,\sigma} = -P_3 - P_3^T, \\ &\bar{L}_{2,3,\sigma} = Y_{2,\sigma}^T - P_3^TBK_\sigma,\ \bar{L}_{3,3,\sigma} = Y_{3,\sigma}^T + Y_{3,\sigma} + Z_{1,\sigma} - Z_\sigma, \end{aligned} \tag{4.83}$$

$$Z_\sigma = \frac{X_\sigma + X_\sigma^T}{2},\ Z_{1,\sigma} = X_{1,\sigma}^T + X_{1,\sigma}, \tag{4.84}$$

The stabilizing feedback matrix gains are directly provided as the LMI variables K_σ, with the switching law σ satisfying (4.8).

Proof: The proof is very similar to the one in the non-switching case to get the matrix inequalities (4.72), except that we are now using the LKF (4.10) with matrices switching on the conic regions defined in (4.7). ∎

Remark 4.18 *Here again, one can include constraints to avoid having unacceptable high gains K_σ by adding the LMIs*

$$\begin{bmatrix} -\kappa I & K_\sigma^T \\ * & -I \end{bmatrix} \preceq 0 \tag{4.85}$$

which ensure that $|||K_\sigma|||_2 \leq \kappa$.

Indeed, using the Schur complement, one can show that the LMI (4.85) imply that

$$\begin{cases} K_\sigma \preceq \kappa I \\ K_\sigma^T \preceq \kappa I \end{cases}, \tag{4.86}$$

and thus

$$|||K_\sigma|||_2 = \sqrt{\rho(K_\sigma^T K_\sigma)} \leq \sqrt{\rho(K_\sigma^T)\rho(K_\sigma)} \leq \kappa. \tag{4.87}$$

Remark 4.19 *Here, the matrice P from the LKF and the matrices P_2 and P_3 introduced by the application of Finsler's Lemma (or the descriptor method) in the proof of Theorem 4.14 are supposed to be given, (computed using Theorem 4.14 for example, which would give $P_2 = Q^{-1}$, $P_3 = \delta P_2$, and $P = P_2^T \tilde{P} P_2$). It is necessary to use a two-steps algorithm (one to compute P_2, P_3, and one to compute the matrices K_σ) because one can not search for P_2 and P_3 at the same time as the matrices K_σ. Indeed, the inequalities $\bar{\Xi}_{i,j,\sigma} \preceq 0$ from Theorem 4.17 would then result in BMIs, and the tricks used in the proof of Theorem 4.14 would leave us with a term $Q^T \Psi_\sigma Q$ which can not be removed using a Schur complement for example, because of the non-positivity and non-negativity of Ψ_σ.*

4.3.3 Algorithm to design the state-dependent sampling function $\tau_{\max}$ and its associated feedback matrix gain K (or gains K_σ)

Figure 4.2 provides a four-step algorithm to build a state-dependent sampling function enlarging the sampling intervals using the stabilization conditions from Subsections 4.3.1 and 4.3.2 while computing the adequate LKF function (4.10) and controller gain K (or gains K_σ). Keep in mind that all steps are made off-line.

Remark 4.20 *Step 2 in the case of switching matrix gains provides less conservative results than Step 1, since the condition $P_3 = \delta P_2$ assumed in theorem 4.14 is not valid anymore. Indeed, in Step 2, the obtained P_2 and P_3 can be any matrices in $\mathcal{M}_{n,n}(\mathbb{R})$.*

Remark 4.21 *Similar stabilization tools and algorithm can be obtained in the case of delayed systems such as the ones presented in Subsection 4.2.2. We chose not to present this study however because it concludes with LMIs of size $(13n + n_w) \times (13n + n_w)$, which are too large.*

STEP 1

◇ Work with a single region $\mathbb{R}^n$
◇ Fix the minimal sampling τ^-

↓ Theorem 4.14 LMIs + line search algorithm on δ line search algorithm on τ^+

◇ Compute the largest admissible state-independent sampling τ^+
◇ Compute the associated (global) feedback matrix gain K

STEP 2

→ Case $u(t) = -Kx(s_k)$: use Algorithm from Subsection 4.2.3 with the computed K and stop
→ Case $u(t) = -K_{\sigma_k}x(s_k)$:
◇ Work with a single region $\mathbb{R}^n$, or q conic regions $\mathcal{R}_\sigma$ covering the state space
◇ Use the same minimal sampling τ^- as in Step 1
◇ Use the feedback matrix gain K computed in Step 1

↓ Theorem 4.17 LMIs + line search algorithm on τ^+

◇ Compute a better value for τ^+
◇ Compute the associated (global) matrices P, P_2 and P_3

STEP 3

◇ Work with q conic regions $\mathcal{R}_\sigma$
◇ Use the same minimal sampling τ^- as in Step 1
◇ Use the matrices P, P_2, P_3 computed in Step 2

↓ Theorem 4.17 LMIs + line search algorithm on τ_σ^+

◇ Design the mapping of the largest admissible sampling $\tau_\sigma^+ \geq \tau^+$ for each region $\mathcal{R}_\sigma$
◇ Compute the switching (i.e. local) LKF matrices U_σ, S_σ, X_σ, $X_{1,\sigma}$, for each region $\mathcal{R}_\sigma$
◇ Compute the optimal matrix gain K_σ for each region $\mathcal{R}_\sigma$

STEP 4

Design of the controller's switching law
$\sigma_k = \text{argmax}_{\sigma \in \mathcal{I}(x(s_k))} \tau_\sigma^+$
and of the state-dependent sampling function
$\tau_{\max}(x) = \max_{\sigma \in \mathcal{I}(x)} \tau_\sigma^+ \geq \tau^+, \forall x \in \mathbb{R}^n$ $(\tau_{\max}(x(s_k)) = \tau_{\sigma_k}^+)$

Figure 4.2: Algorithm to design the state-dependent sampling function $\tau_{\max}(x)$ and its associated feedback matrix gain K (or gains K_σ)

4.4 Numerical examples

4.4.1 Example 1 - State dependent sampling for systems with perturbations and delays

We consider the system:

$$\dot{x}(t) = \begin{bmatrix} -3 & 0 \\ 0 & 1 \end{bmatrix} x(t) - \begin{bmatrix} 1 \\ 1 \end{bmatrix} Kx(s_k) + w(t), \text{ for } t \in [t_k, t_{k+1}), \text{ with } K = \begin{bmatrix} -1 & 4 \end{bmatrix},$$
$$z(t) = x(t).$$

The state-dependent sampling function (4.9) will be designed in four successive cases:

1. no delay nor perturbations ($w = 0, h = 0$, asymptotic stability);
2. perturbations on the delay-free system ($w \neq 0$, $h = 0$, $\mathcal{L}_2$-stability with $\gamma = \sqrt{10}$);
3. unperturbed system with delays $h(t) \in [10^{-4}, 10^{-1}]$ and $\dot{h}(t) \in [-0.2, 0.6]$ ($w = 0$, $h \neq 0$, asymptotic);
4. perturbed system with the same class of delays ($w \neq 0$, $h \neq 0$, $\gamma = \sqrt{10}$).

We set a number of $q = 100$ conic regions $\mathcal{R}_\sigma$, take $\tau^- \simeq 0$, and use the algorithm of Subsection 4.2.3 to build the mapping that maximizes the sampling interval for each state. We work with the isotropic covering described in the Appendix B.1, and design the conic regions using the polar coordinates (ρ, θ) of the state $x = \rho e^{i\theta}$, for the particular value $\rho = 1$ (the unit circle). Computed off-line in each of the 4 cases, Figure 4.3 presents the admissible sampling interval as a function of the state angle $\theta \in [-\pi, \pi)$. The longest state-independent sampling intervals (the lower bound of the state-dependent sampling function) we found in the four cases are presented in Table 4.1.

$w = 0, h = 0$	$w \neq 0$, $h = 0$	$w = 0$, $h \neq 0$	$w \neq 0$, $h \neq 0$
$0.535s$	$0.445s$	$0.169s$	$0.145s$

Table 4.1: Example 1: Lower-bound τ^+ of the state-dependent sampling function

Note that since τ^- has been fixed near to zero, the system $\mathcal{L}_2$-stability (or asymptotic stability) is guaranteed for any sampling intervals less than τ^+. This result corresponds to that of a classic robust stability analysis regarding (state-independent) time-varying sampling. Thanks to the mapping we built (in each of the four cases), we can extend that

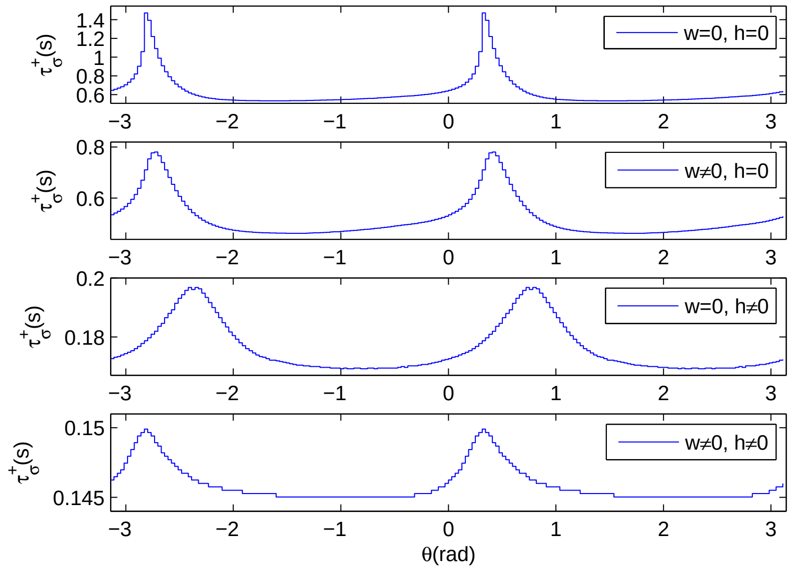

Figure 4.3: Example 1: Mapping of the maximal admissible sampling intervals τ_σ^+ with or without perturbations w and/or delays h

stability result to any state-dependent time-varying sampling with values bounded by the obtained sampling function $\tau_{\max}$ (i.e. with values below the curve presented in Figure 4.3), which allows for larger sampling intervals.

Figure 4.4 shows simulation results for a system with or without time-varying delays, with a sinusoidal disturbance $w(t)$ set to satisfy $\|w(t)\|_2 = \frac{1}{\gamma}\|z(t)\|_2 \simeq 32\%\|z(t)\|_2$. It presents the state $x(t)$, the sampling intervals $\tau_k = \tau_{\max}(x(s_k))$ and the delays $h_k = h(t_k)$ (in the delayed case).

4.4.2 Example 2 - Conservatism reduction thanks to the switched LKF

To show the conservatism reduction brought by the LKF with state-dependent matrices, we consider the system from [Hetel 2011b]:

$$\dot{x}(t) = \begin{bmatrix} -0.5 & 0 \\ 0 & 3.5 \end{bmatrix} x(t) - \begin{bmatrix} 1 \\ 1 \end{bmatrix} Kx(s_k), \quad \text{for } t \in [s_k, s_{k+1}), \text{ with } K = \begin{bmatrix} -1.02 & 5.62 \end{bmatrix},$$
$$z(t) = x(t).$$

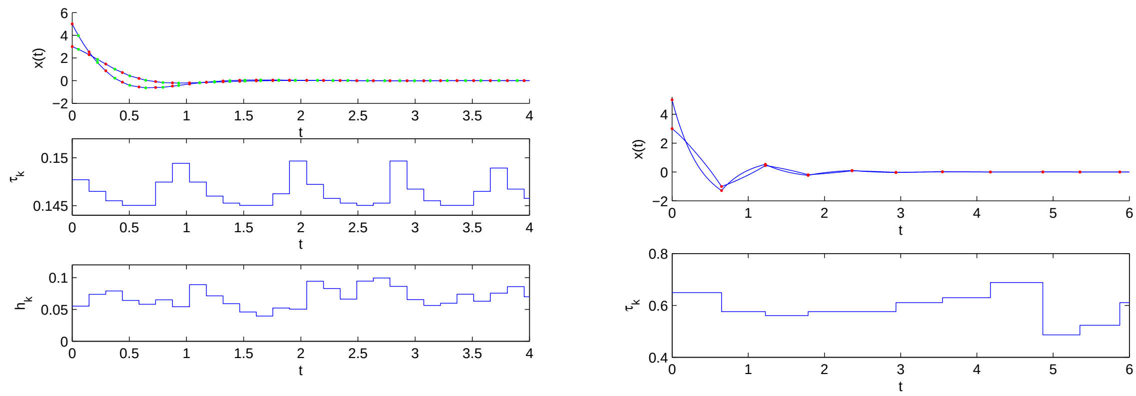

Figure 4.4: Example 1: Left side: delayed case (delays up to $0.1s$). Right side: delay-free case. In both sides, the perturbation satisfies $\|w(t)\|_2 = \frac{1}{\gamma}\|z(t)\|_2 \simeq 32\%\|z(t)\|_2$

We set $\tau^- \simeq 0$. Considering the results given by the step 1 of the algorithm described in Subsection 4.2.3 and taking only one region $\mathbb{R}^n$, the longest state-independent sampling interval τ^+ (*i.e.* admissible no matter the state) obtained is equal to $0.267s$, whereas we obtain $0.309s$ with $q = 100$ regions $\mathcal{R}_\sigma$. This corresponds to a robust stability bound that can be compared to the ones obtained in the literature, as shown in Table 4.2.

[Naghshtabrizi 2008]	[Seuret 2009]	[Fujioka 2009b]	[Fridman 2010]	Algorithm Section 4.2.3
$0.165s$	$0.198s$	$0.204s$	$0.259s$	$0.309s$

Table 4.2: Example 2: Maximum upper bounds τ^+ for time-varying samplings, allowable on the whole state space

4.4.3 Example 3 - State-dependent sampling for systems which are both open-loop and closed-loop (with a continuous feedback control) unstable

Here, we consider a system from [Gu 2003]:

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -2 & 0.1 \end{bmatrix} x(t) - \begin{bmatrix} 0 \\ 1 \end{bmatrix} Kx(s_k), \text{ for } t \in [s_k, s_{k+1}), \text{ with } K = \begin{bmatrix} -1 & 0 \end{bmatrix},$$
$$z(t) = x(t).$$

This system is asymptotically stable for a constant sampling step $\tau_{\text{const}}^{\max} = 0.25s$. However, it is unstable in open-loop, and unstable in closed-loop with a continuous state feedback (A and $A - BK$ are both not Hurwitz).

The stability tools proposed in Section 4.2 can not provide any solution for a minimal sampling interval τ^- that is too small, since the system is unstable for small sampling intervals. However, for larger values of τ^-, the proposed algorithms find solutions and allow to build state-dependent sampling functions, which is not possible in classical self-triggered works for this class of continuous closed-loop unstable sytems. The sampling functions presented in Figure 4.5 (on the left) have been obtained with a number $q = 100$ conic regions and different values for the minimal sampling inverval τ^-. On the right of the figure are shown simulation results obtained for $\tau^- = 0.25$.

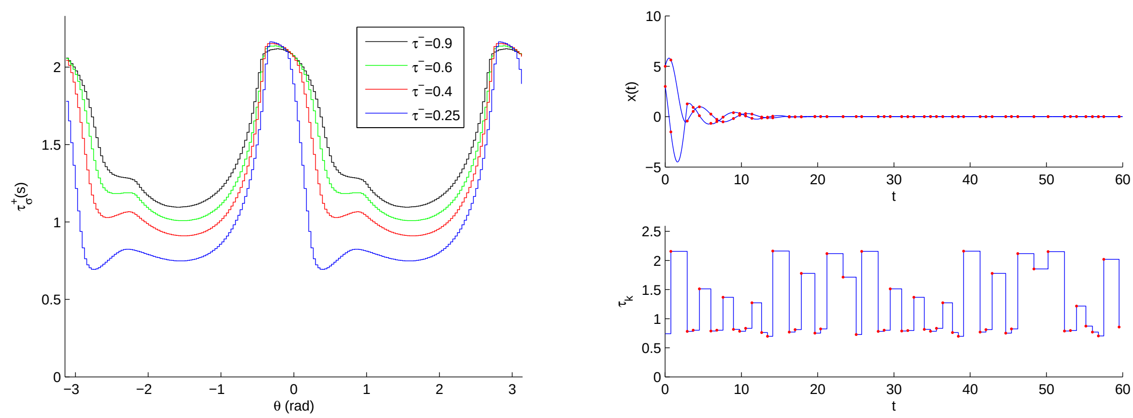

Figure 4.5: Example 3: Mapping of the maximal admissible sampling intervals for different minimal sampling intervals τ^- (on the left) and simulation results using the sampling function obtained with $\tau^- = 0.25$ (on the right)

4.4.4 Example 4 - State-dependent sampling controller for perturbed systems

Here, we consider a system from [Tabuada 2007] to which we added a switching controller:

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} x(t) - \begin{bmatrix} 0 \\ 1 \end{bmatrix} K_{\sigma_k} x(s_k) + w(t), \text{ with } z(t) = x(t), \text{ for } t \in [s_k, s_{k+1}).$$

The feedback gain matrices K_σ are computed along with the maximal admissible sampling intervals τ_σ^+ for every conic region of the state space using the algorithm proposed in

Subsection 4.3.3. Figure 4.6 presents the sampling functions obtained for various $\mathcal{L}_2$ gains γ with the proposed controller with switching matrices K_σ, and with a classic controller with a constant matrix gain K. It shows the advantages of the switching rule on the controller. These results have been obtained with $q = 100$ conic regions (isotropic design from Appendix B.1), and a lower bound on the samplings $\tau^- \simeq 0$.

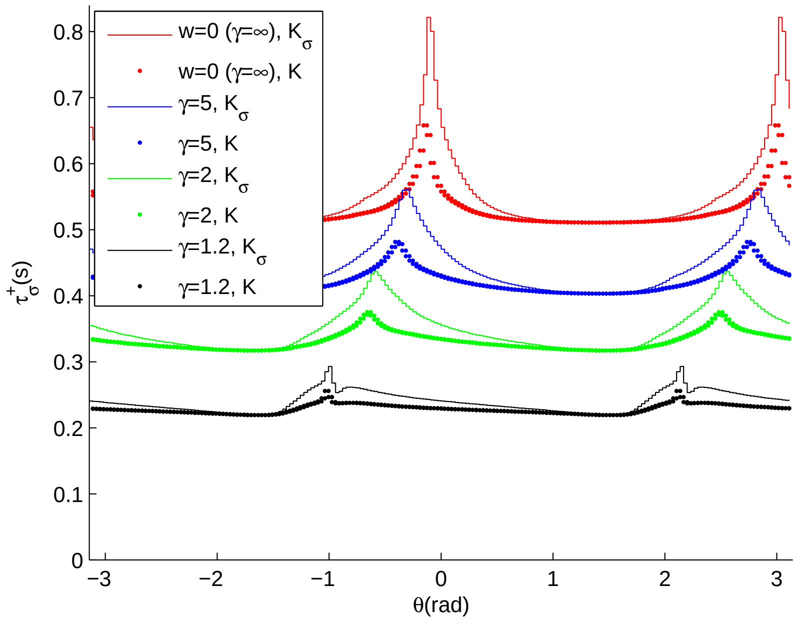

Figure 4.6: Example 4: Mapping of the maximal admissible sampling intervals for different $\mathcal{L}_2$ gains γ, with or without switching controller

Using the mapping we designed for both the maximal sampling intervals τ_σ^+ and the feedback gain matrices K_σ, we can run the simulations presented in Figure 4.7.

4.5 Conclusion

This chapter has proposed both a stability and a stabilization analysis allowing to design a state-dependent sampling that reduces the number of actuations, while ensuring the $\mathcal{L}_2$-stability for perturbed linear state feedback systems. Extensions to the stability analysis of delayed systems, and to the stabilization analysis for systems with switching feedback matrix gains are also provided.

The study is based on a new class of Lyapunov-Krasovskii functionals with state-dependent matrices that reduce the conservatism for both state-dependent sampling and

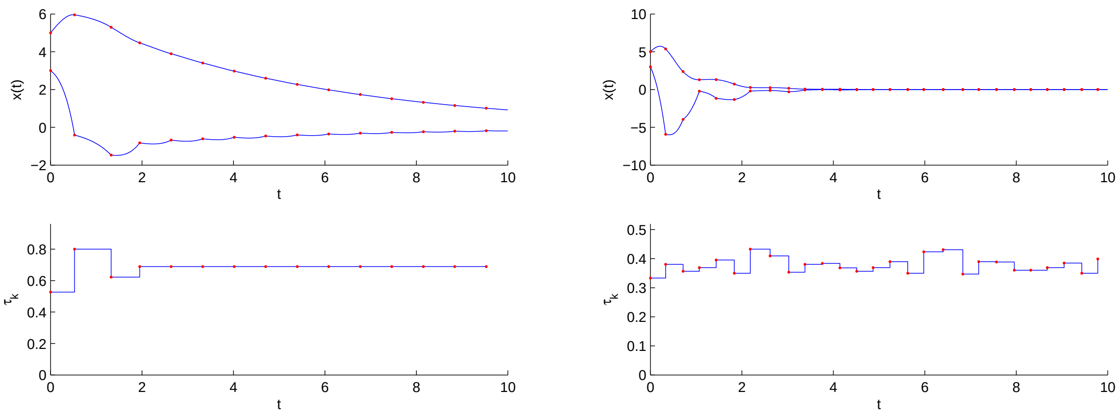

Figure 4.7: Example 4: State $x(t)$ and sampling intervals $\tau_k = \tau_{\max}(x(s_k))$ for the controlled system without perturbation (on the left) and with a perturbation satisfying $\|w(t)\|_2 = \frac{1}{\gamma}\|z(t)\|$, $\gamma = 2$ (on the right)

time-varying (state-independent) sampling.

The proposed method can be used as a self-triggered control, as a new time-varying sampling analysis leading to a state-dependent sampling design, and as a stabilizing tool. We think it presents three main advantages, since it makes it possible:

- to compute the matrix gain K (or matrix gains K_σ along with the switching rule σ in the case of switching matrix gains) adapted to the system and sampling;
- to maximize the minimal sampling interval $\tau^+ = \inf_{x\in\mathbb{R}^n} \tau_{\max}(x)$ of the state-dependent sampling function, and to compute the associated Lyapunov-Krasovskii function matrices that ensure the system $\mathcal{L}_2$-stability;
- to design off-line a mapping of the state space with a maximum allowable sampling time for each subspace. Therefore, as in most contributions in this thesis, no additional computation is required online during the control of the system.

General conclusion

This PhD thesis was dedicated to the robust stability analysis and stabilization of systems with time-varying sampling. A particular attention was given to the dynamic control of the sampling events. Its main objective was to design sampling laws that allow for reducing the number of sampling instants of state-feedback control for LTI sampled-data systems.

In this work, we have provided the foundations to a novel approach for the dynamic control of the sampling, which we called *"state-dependent sampling"*. It consists in an *offline* design of a state-dependent sampling function enlarging the sampling intervals of state-feedback control, thanks to LMIs based on a *mapping of the state-space*. One of the main advantages of this offline design is that it allows for *reducing the number of on-line computations* required to estimate in real-time the next maximal allowable sampling interval. Furthermore, it makes it possible to *optimize the lower-bound of the sampling function* by *computing the optimal Lyapunov parameters*, meaning that the maximal sampling allowed in the worst case will be optimized. This lower-bound of the state-dependent sampling function can be used as an upper-bound for the classical problem concerning the robust stability with arbitrary time-varying sampling.

First, the case of ideal LTI sampled-data systems was considered. In this context, an *extension of the common Lyapunov-Razumikhin theory to guarantee the exponential stability of sampled-data systems* was proposed. A *convex embedding design adapted to the continuous-time stability analysis* was then applied to derive the LMIs used in the design of the state-dependent sampling function. The approach was illustrated by numerical examples from the literature for which the number of actuations is shown to be reduced with respect to the periodic sampling case. This shows that our state-dependent sampling combines the robustness property (since shorter time intervals also stabilize the system) with some realism (remember that periodic sampling constitutes an idealistic assumption in real-time control situations).

Second, the robustness aspect with respect to exogenous disturbances was introduced. In this context, the method was developed so as to allow the use of the *convex-embedding* *approach in the presence of unknown perturbations.* Several possible cases of sampling functions were proposed, each of which was leading to a different kind of application. The first application concerned the *robust stability analysis with respect to time-varying sampling*, which allows one to compute an *estimation of the maximal allowable upper-bound of time-varying sampling, while taking into account both sampling and perturbations.* The other three applications proposed different approaches to the dynamic control of the sampling with the objective to enlarge the sampling interval: *event-triggered control*, *self-triggered control*, and the newly introduced *state-dependent sampling. Each of the proposed dynamic sampling control schemes takes advantage of the results about the robust stability analysis with respect to time-varying sampling*, since it allows to optimize the lower-bound of the sampling function in each case.

Finally, an extension to the stability analysis of perturbed time-delay linear systems was proposed, and the stabilization issue was considered. In this context, we developed several tools to *design a controller along with the state-dependent sampling law*, so as to stabilize the considered perturbed LTI sampled-data system, and enlarge even further the allowable sampling intervals. Two different controllers were proposed: a classic linear state-feedback controller, and a *new controller for which the gains are switching according to the system's state.* The co-design of both the controller and the state-dependent sampling function was based on LMIs obtained thanks to a mapping of the state-space, in the framework of state-dependent sampling, and thanks to a *new class of Lyapunov-Krasovskii functionals with matrices switching with respect to the system's state.* This state-dependent switch on the functional matrices allows for adapting the Lyapunov-Krasovskii functional to each region of the state-space, and thus enables to reduce the conservatism in the design of the state-dependent sampling function. Moreover, this new class of Lyapunov-Krasovskii functionals may also reduce the conservatism even in the case of state-independent time-varying sampling, as it is shown with a numerical example.

We are convinced that the perspectives that emerge from the works presented in this thesis are multiple.

First of all, an interesting research direction would be the extension of the proposed results to a larger class of sampled-data systems, like homogeneous systems or polynomial systems for example. In that case, the dynamic sampling control would then take advantage of both the state-dependent sampling approach presented in the linear case in this thesis, and of scaling properties for the sampling function like the ones expressed in [Anta 2010] for example, which are particular to the classes of systems considered.

Another research direction would be to extend the results obtained with the proposed state-dependent sampling approach to a larger class of control types and control performances. For instance, extensions to output-feedback control or observer-based control would be very useful for a wide variety of systems for which using a state-feedback controller is not physically possible. As well, the design of a perturbation-rejection control would also be interesting, so as to allow the convergence of the system state toward the equilibrium point in the case of systems with constant or slowly varying perturbations.

Finally, it would be interesting to extend the stability and stabilization results presented for systems with a state-dependent sampling to the case of systems with state-dependent delays. The study of such systems is mainly motivated by applications that may arise in Networked Controled Systems (see [Briat 2010] for the modeling of internet congestion, and [Donkers 2009] for the interaction between control tasks and scheduling algorithms for example). In this context, it would be interesting to design stability tools with respect to a known state-dependent delay $\tau(x)$, or even to propose scheduling tools that would allow for controlling this state-dependent delay so as to obtain the stability.

Résumé étendu en français

Introduction générale

Jusqu'au milieu des années 50, la plupart des systèmes étaient commandés au moyen de contrôleurs analogiques. Cependant, le développement rapide des ordinateurs à cette période poussa à une utilisation de plus en plus importante des contrôleurs numériques. Ce nouvel essor était dû notamment à la puissance de calculs de ces derniers, ainsi qu'à leur flexibilité, et leur facilité de mise en œuvre. De nos jours, les contrôleurs numériques sont devenus omniprésents, et ont permis la naissance et le développement de nouveaux systèmes de commande, tels les systèmes embarqués et les systèmes commandés par réseaux. Ils offrent de nombreux avantages: faible coût d'installation et de maintenance, flexibilité accrue, possibilité d'utilisation pour différents types d'applications, coût de câblage réduit, et facilité de programmation. Ils offrent de plus la possibilité de commander plusieurs systèmes à la fois.

Contrairement aux contrôleurs analogiques, les contrôleurs numériques introduisent naturellement des signaux et des dynamiques en temps discret, de par la présence de mécanismes tels que des échantillonneurs-bloqueurs (sample and hold devices) [Aström 1996]. Ainsi, durant la commande de systèmes en temps-réel, de nouveaux phénomènes font leur apparition.

Tout d'abord, l'information transmise par les capteurs au contrôleur est échantillonnée, à l'aide d'un convertisseur analogique numérique (A/N). Une telle conversion d'un signal d'entrée $x(t)$ en un signal échantillonné $x(s_k)$, aux instants d'échantillonnage $s_k,\ k \in \mathbb{N}$, est montrée dans la Figure 1. De plus, puisque la commande est calculée seulement à des instants discrets, il est nécessaire d'utiliser un convertisseur numérique analogique (N/A) (un bloqueur d'ordre zéro), de sorte que la valeur de la commande qui est envoyée aux actionneurs reste constante entre deux échantillonnages. La conversion d'un signal d'entrée échantillonné $u(s_k)$ en un signal constant par morceaux $u(t)$, est montrée dans la Figure 2.

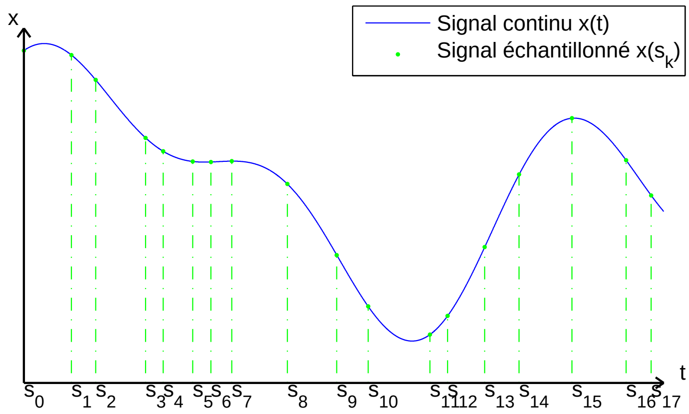

Figure 1: Conversion analogique numérique

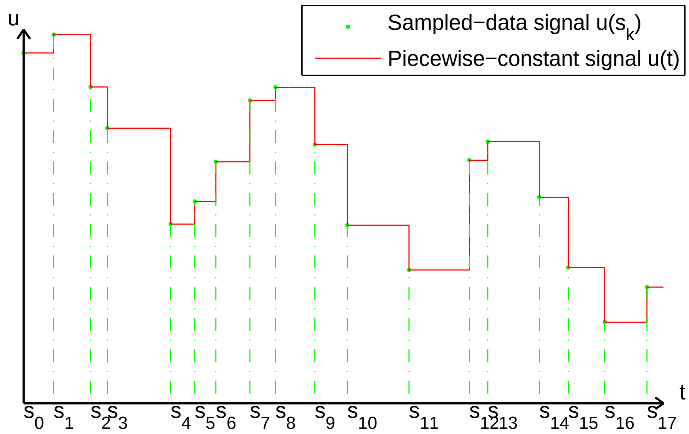

Figure 2: Conversion numérique analogique

Dans les applications de commande embarquée cependant, une implémentation en temps discret peut provoquer l'apparition d'effets indésirables tels que des retards, ou une exécution apériodique de la commande, dûs à l'interaction entre les tâches de commandes, et les mécanismes d'ordonnancement temps-réel [Hristu-Varsakelis 2005]. Les effets de ces dynamiques en temps-discret ont donné naissance à de nouveaux défis en ce qui concerne la stabilité et la stabilisation de tels systèmes, et de nouvelles théories ainsi que de nouveaux outils ont été développés spécialement pour ces systèmes dits échantillonnés. En particulier, ces dernières années, deux problèmes principaux ont retenu l'attention des automaticiens:

P1) la stabilité des systèmes échantillonnés avec un pas d'échantillonnage variable;

P2) le contrôle dynamique des instants d'échantillonnage.

La dernière tendance concerne le contrôle dynamique de l'échantillonnage dans le but d'élargir les intervalles d'échantillonnage, et ainsi réduire les coûts en termes de charge de calcul, de bande passante de réseau, ou de consommation d'énergie.

Objectifs

Le travail présenté dans cette thèse se concentre sur la résolution de ces deux problèmes P1) et P2). L'objectif principal est de modéliser une loi d'échantillonnage qui permette de réduire la fréquence d'échantillonnage pour les systèmes linéaires à temps invariant dans le temps (LTI) commandés par retour d'état, tout en assurant leur stabilité, et certains critères de performance.

Pour éviter tout problème d'ordonnancement, la robustesse vis-à-vis de la variation du pas d'échantillonnage sera également considérée. Les aspects de robustesse vis-à-vis de perturbations extérieures ou de retards dans la boucle de commande seront de même considérés, de sorte à prendre en compte des phénomènes qui apparaissent lors du contrôle en temps-réel de systèmes physiques. Enfin, un co-design du contrôleur et de la loi d'échantillonnage sera proposé. Ici, pour réduire le conservatisme et offrir des pas d'échantillonnage encore plus longs, les gains du contrôleur et les instants d'échantillonnage seront calculés en même temps.

Tout au long de cette thèse, différentes lois de contrôle de l'échantillonnage seront proposées. Elles peuvent être utilisées pour calculer une simple borne supérieure de l'échantillonnage, dans le cas d'un échantillonnage variable dans le temps, ou pour contrôler dynamiquement l'échantillonnage, au moyen d'algorithmes pouvant être mis en place soit hors-ligne, soit en-ligne.

Structure de la thèse

Le document est organisé comme suit:

Chapitre 1

Le premier chapitre présente une vue d'ensemble des différents problèmes, défis, et récents axes de recherche dans le domaine des systèmes échantillonnés en automatique. Tout d'abord, la notion de système échantillonné est définie, et les principaux problèmes ouverts dans la littérature sont présentés. Ensuite, quelques concepts de stabilité généraux nécessaires à la compréhension du travail sont rappelés. Enfin, de nombreux axes de recherche, théories, et résultats sont présentés concernant l'analyse de stabilité des systèmes échantillonnés avec échantillonnage à pas constant ou variable dans le temps, ou concernant le contrôle dynamique de l'échantillonnage. Les forces et faiblesses des différentes approches sont analysées, de façon à mettre en lumière les problèmes qui ont déjà été résolus, et ceux qu'il reste encore à résoudre, ou encore les points qu'il reste à améliorer.

Chapitre 2

Dans le deuxième chapitre, un contrôle par échantillonnage dépendant de l'état est présenté pour le cas de systèmes LTI définis par

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \forall t \in \mathbb{R}_+, \\ u(t) &= -Kx(s_k),\ \forall t \in [s_k, s_{k+1}). \end{aligned}$$

L'objectif est de concevoir une loi d'échantillonnage qui va prendre en compte l'état $x(s_k)$ du système, de manière à élargir les intervalles d'échantillonnage, ou en d'autres termes, de générer les évènements d'échantillonnage aussi peu fréquemment que possible. Pour celà, on considère la loi d'échantillonnage

$$s_{k+1} - s_k = \tau(s_k, x(s_k)) \equiv \tau_k \in (0, \tau_{\max}(x(s_k))],\ \ \forall k \in \mathbb{N},$$

où $\tau_{\max}(x)$ représente l'échantillonnage maximal associé à l'état x, avec une fonction d'échantillonnage dépendant de l'état $\tau_{\max} : \mathbb{R}^n \to \mathbb{R}_+^*$ que l'on va chercher à maximiser. L'intérêt de cette formulation est qu'ici, l'échantillonnage considéré peut être contrôlé (il dépend de l'état), mais il peut aussi varier en fonction du temps. Ainsi, la stabilité est garantie pour tout pas d'échantillonnage variable dans le temps, et borné par la fonction d'échantillonnage $\tau_{\max}$. Notons que dans le cas particulier où la fonction d'échantillonnage est constante ($\tau_{\max}(x(s_k)) = \tau^*$), l'étude se résume à une analyse de stabilité robuste classique vis-à-vis d'un échantillonnage variable.

L'objectif est alors double: nous allons chercher à maximiser la borne inférieure de la fonction d'échantillonnage maximal $\tau_{\max}$, qui correspond à une borne supérieure de stabilité robuste dans le cas d'échantillonnage variable, mais non dépendant de l'état, et nous allons aussi chercher à maximiser la fonction d'échantillonnage pour toute valeur de l'état $x(s_k)$.

La fonction d'échantillonnage dépendant de l'état que nous proposons bénéficie d'une construction hors-ligne basée sur des LMIs obtenues grâce à une cartographie de l'espace d'état réalisée par un recouvrement de régions coniques $\mathcal{R}_s = \{x \in \mathbb{R}^n, x^T \Psi_s x \geq 0\}$ (voir Figure 3).

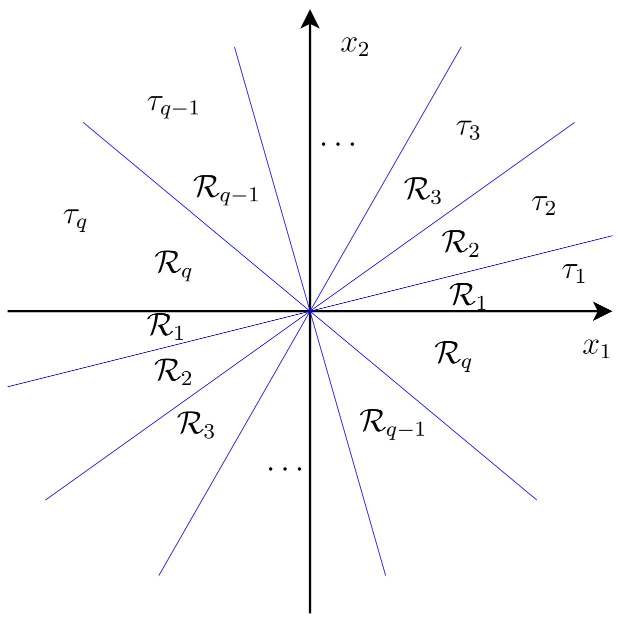

Figure 3: Recouvrement de l'espace d'état de dimension 2 par q régions coniques $\mathcal{R}_s$

La fonction d'échantillonnage est alors construite sur chacune des régions, de par la loi

$$\tau_{\max}(x) = \tau_s,\ \forall x \in \mathcal{R}_s,\ s \in \{1, \cdots, q\}.$$

Les outils utilisés dans la conception de cette fonction d'échantillonnage dépendant de l'état sont l'approche par polytopes convexes [Hetel 2006] adaptée pour permettre l'analyse de stabilité du système en temps continu, et la théorie de stabilité de Lyapunov-Razumikhin adaptée pour garantir la stabilité exponentielle pour le cas de systèmes échantillonnés, avec un taux de convergence donné.

Grâce à des exemples classiques de la littérature, nous montrons qu'il est possible avec cette nouvelle approche d'échantillonnage dépendant de l'état d'échantillonner moins souvent en moyenne qu'avec un échantillonnage périodique, tout en garantissant des performances supplémentaires de stabilité, ou de rapidité de convergence.

Chapitre 3

Dans le troisième chapitre, l'aspect de robustesse vis-à-vis de perturbations extérieures est considéré pour la conception de la loi d'échantillonnage dépendant de l'état. On considère alors le système

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t) + Ew(t), \forall t \in \mathbb{R}_+,\\ u(t) &= -Kx(s_k),\ \forall t \in [s_k, s_{k+1}),\end{aligned}$$

avec une perturbation w supposée bornée en norme par rapport à l'état du système:

$$\|w(t)\|_2^2 \leq W\|x(s_k)\|_2^2,\ \forall t \in [s_k, s_{k+1}).$$

Comme dans le deuxième chapitre, l'approche est basée sur des conditions de stabilité exponentielle de type Lyapunov-Razumikhin et des polytopes convexes.

Après avoir présenté les résultats de stabilité principaux, quatre applications différentes sont proposées.

- La première concerne l'analyse de stabilité robuste vis-à-vis des variations du pas d'échantillonnage.

- Les trois autres applications proposent différentes approches de contrôle dynamique de l'échantillonnage, avec pour objectif l'élargissement du pas d'échantillonnage. Ces approches sont présentées avec un degré de conservatisme croissant.

  – La moins conservative, mais la plus coûteuse en pratique en terme de calcul en ligne est la technique dite d'event-triggered control. Dans cette approche, les instants d'échantillonnage ont lieu lorsqu'une certaine condition analytique n'est plus satisfaite. Pour assurer la stabilité du système cependant, il est nécessaire de vérifier cette condition en temps réel, ce qui nécessite un matériel dédié pour analyser l'état du système en temps quasi-continu.

  – La deuxième approche de contrôle dynamique de l'échantillonnage que l'on propose est l'approche dite de self-triggered control, dans laquelle on essaie d'estimer en ligne à chaque pas d'échantillonnage le prochain pas maximal admissible.

– Enfin, le troisième et dernier algorithme proposé est le nouveau contrôle par échantillonnage dépendant de l'état, dans lequel une fonction estimant le prochain pas d'échantillonnage maximal admissible en fonction de l'état du système est construite hors ligne, grâce à des LMIs, pour réduire le nombre de calculs en ligne.

Chacune de ces applications de contrôle dynamique de l'échantillonnage bénéficie des résultats de l'analyse de stabilité robuste vis-à-vis des variations du pas d'échantillonnage, puisque les pas d'échantillonnage obtenus dans chacune de ces trois approches sont minorés par la borne supérieure de stabilité robuste calculée dans le cas de système avec échantillonnage variable mais non dépendant de l'état.

Il est montré grâce à des exemples de la littérature que la méthode proposée réduit le conservatisme par rapport aux travaux les plus récents, et que les résultats obtenus par les contrôles de type event-triggered, self-triggered, et d'échantillonnage dépendant de l'état que nous proposons, sont très proches, bien qu'ils soient de degrés de conservatisme croissant.

Chapitre 4

Dans le quatrième et dernier chapitre, une extension à l'analyse de stabilité pour les systèmes perturbés avec des retards variables est traitée. Le système considéré (présenté dans la Figure 4) est défini par

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + Ew(t), \\ z(t) &= Cx(t) + Du(t) \ \forall t \in \mathbb{R}_+, \end{aligned}$$

avec un contrôle échantillonné retardé

$$u(t) = -Kx(s_k), \ \forall t \in [t_k, t_{k+1}).$$

Les instants d'échantillonnage s_k et d'actuation t_k sont liés par la loi

$$s_k = t_k - h(t_k),$$

avec un retard $h(t)$ borné, et à dérivées bornées.

Tout d'abord, une loi d'échantillonnage dépendant de l'état assurant la stabilité $\mathcal{L}_2$ du système échantillonné perturbé et retardé est construite, grâce à des LMIs, de la même

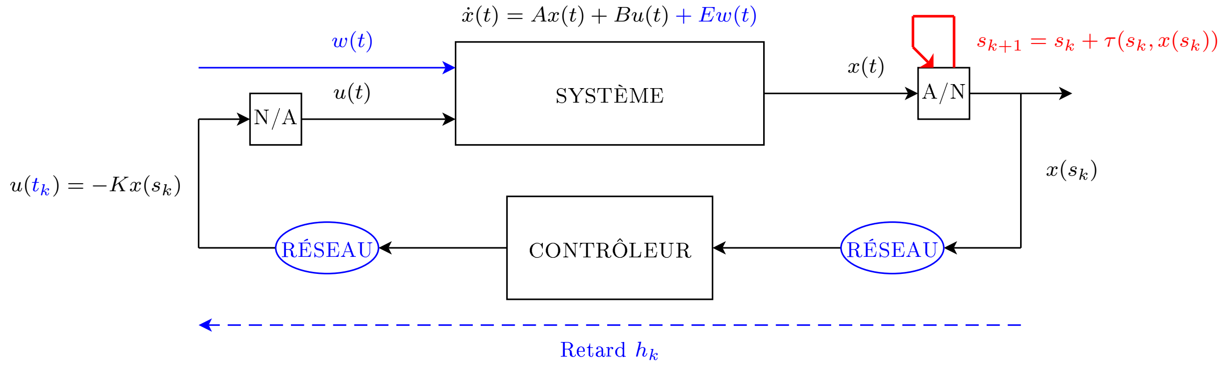

Figure 4: Système LTI échantillonné avec perturbations et retards

manière que dans les deux précédents chapitres.

Ensuite, le problème de stabilisation dans le cas non retardé est considéré. L'objectif ici est de concevoir un contrôleur en parallèle avec la loi d'échantillonnage dépendant de l'état, de sorte à stabiliser le système LTI échantillonné perturbé, et élargir encore plus les pas d'échantillonnage admissibles. Tout d'abord, le cas d'un contrôle par retour d'état linéaire classique est envisagé:

$$u(t) = -Kx(s_k), \ \forall t \in [s_k, s_{k+1}).$$

Puis, un nouveau contrôleur dont les gains vont commuter en fonction de l'état du système est proposé:

$$u(t) = -K_{\sigma(x(s_k))}x(s_k), \ \forall t \in [s_k, s_{k+1}).$$

Le co-design du contrôleur et de la fonction d'échantillonnage dépendant de l'état est basé sur des LMIs obtenues grâce à la cartographie de l'espace d'état présentée dans les précédents chapitres, et grâce à une nouvelle classe de fonctionnelles de Lyapunov-Krasovskii dont les matrices commutent en fonction de l'état du système:

$$V_{\sigma_k}(t, x_t, \dot{x}_t) = x^T(t)Px(t) + V_1(t, x_t, \dot{x}_t) + V_{2,\sigma(x(s_k))}(t, x_t, \dot{x}_t),$$

avec un terme prenant en compte le retard,

$$V_1(t, x_t, \dot{x}_t) = \int_{t-h(t)}^{t} \dot{x}^T(s)R\dot{x}(s)ds + \cdots ,$$

et un terme prenant en compte l'échantillonnage,

$$V_{2,\sigma(x(s_k))}(t, x_t, \dot{x}_t) = (t_{k+1} - t) \int_{t_k}^{t} \dot{x}^T(s) U_{\sigma(x(s_k))} \dot{x}(s) ds + \cdots ,$$

avec des matrices dépendantes de l'état.

Il est important de noter que cette nouvelle classe de fonctionnelles de Lyapunov-Krasovskii réduit le conservatisme introduit pour le cas d'échantillonnage dépendant de l'état, mais aussi dans le cas d'échantillonnage variable mais non dépendant de l'état, comme il est montré dans un des exemples traités.

Conclusions et perspectives

Cette thèse a été dédiée à l'analyse de stabilité robuste et à la stabilisation de systèmes avec des pas d'échantillonnage variables. Une attention particulière a été donnée au contrôle dynamique du pas d'échantillonnage. L'objectif principal était de construire des lois d'échantillonnage permettant de réduire le nombre d'instants d'échantillonnage pour les systèmes LTI contrôlés par retour d'état linéaire.

Dans ce travail, nous avons proposé une toute nouvelle approche de contrôle dynamique de l'échantillonnage, que nous avons appelée *"échantillonnage dépendant de l'état"*. Elle consiste en la construction *hors-ligne* d'une fonction d'échantillonnage dépendant de l'état qui permet d'élargir les pas d'échantillonnage de la commande par retour d'état, grâce à des LMIs basées sur une *cartographie de l'espace d'état*. Un des avantages majeurs de cette construction hors-ligne est qu'elle permet de *réduire le nombre de calculs en-ligne* nécessaires pour estimer en temps-réel le prochain pas d'échantillonnage maximal admissible. De plus, cette approche permet d'*optimiser la borne inférieure de la fonction d'échantillonnage* en *calculant les paramètres de Lyapunov optimaux*, ce qui signifie que le pas d'échantillonnage maximal calculé dans le pire des cas sera optimisé. Cette borne inférieure de la fonction d'échantillonnage dépendant de l'état peut aussi être utilisée comme une borne supérieure pour le problème classique de stabilité robuste de systèmes échantillonnés avec un pas d'échantillonnage variant dans le temps.

Tout d'abord, le cas de système échantillonné LTI idéal (sans aucune forme de perturbations ni d'incertitudes) a été considéré. Dans ce contexte, une *extension de la théorie classique de Lyapunov-Razumikhin pour garantir la stabilité exponentielle des systèmes échantillonnés* a été proposée. Une *construction d'enveloppe convexe adaptée pour l'analyse de stabilité en temps continu* a ensuite été appliquée afin d'obtenir les LMIs utilisées dans la construction de la fonction d'échantillonnage dépendant de l'état. L'approche a été illustrée par des exemples numériques tirés de la littérature pour lesquels il a été montré que le nombre de mises à jour de la commande est réduit par rapport au cas d'échantillonnage périodique. Un autre avantage est que l'échantillonnage dépendant de l'état que nous proposons associe des propriétés de robustesse (puisque des intervalles d'échantillonnage plus courts stabiliseraient également le système) avec du réalisme (nous rappelons que l'échantillonnage périodique est une hypothèse idéaliste, et impossible à réaliser dans les situations de contrôle temps-réel).

Ensuite, l'aspect de robustesse vis-à-vis de perturbations externes a été introduit. Dans ce contexte, la précédente méthode a été améliorée et développée de façon à permettre l'utilisation d'une *approche par polytopes convexes en présence de perturbations*. Plusieurs fonctions d'échantillonnage ont alors été proposées, chacune étant associée à un type d'application particulière. La première application consiste en une *analyse robuste de stabilité vis-à-vis d'un échantillonnage à pas variable dans le temps*, qui permet de *calculer une estimation de la borne maximale admissible de l'échantillonnage dans le cas d'un pas échantillonnage aléatoire variant dans le temps, tout en prenant en compte la présence de perturbations*. Les trois autres applications proposent différentes approches de contrôle dynamique de l'échantillonnage, avec pour objectif d'élargir les intervalles d'échantillonnage: *event-triggered control*, *self-triggered control*, et le nouvel *échantillonnage dépendant de l'état. Chacune de ces approches de contrôle dynamique de l'échantillonnage profite des résultats obtenus grâce à l'analyse de stabilité robuste vis-à-vis d'un échantillonnage à pas variable dans le temps*, puisque ces derniers permettent d'optimiser la borne inférieure de la fonction d'échantillonnage dans chacune des trois applications proposées.

Enfin, une extension à l'analyse de stabilité des systèmes LTI avec perturbations et retards a été proposée, et la question de la stabilisation a été traitée. Dans ce contexte, nous avons développé plusieurs outils permettant de *construire un contrôleur en parallèle avec la fonction d'échantillonnage dépendant de l'état*, de manière à stabiliser le système LTI perturbé et à retard considéré, et élargir encore plus les intervalles d'échantillonnage admissibles. Deux contrôleurs différents ont été proposés: un contrôleur classique par retour d'état linéaire, et un *nouveau contrôleur dont les gains commutent en fonction de l'état du système*. Le co-design du contrôleur et de la fonction d'échantillonnage dépendant de l'état est basé sur des LMIs obtenues grâce à une cartographie de l'espace d'état, dans le cadre de la méthode d'"échantillonnage dépendant de l'état" proposée, et grâce à une *nouvelle classe de fonctionnelles de Lyapunov-Krasovskii (LKF) dont les matrices commutent en fonction de l'état du système*. Cette commutation sur les matrices de la fonctionnelle permet d'adapter la LKF à chaque région de l'espace d'état. De plus, cette nouvelle classe de LKF permet également de réduire le conservatisme même dans le cas d'une analyse de stabilité robuste vis-à-vis d'un échantillonnage à pas variable, mais ne dépendant pas de l'état, comme cela est montré à l'aide un exemple numérique.

Pour conclure, nous sommes convaincus que les perspectives qui émergent des travaux présentés dans cette thèse sont multiples.

Tout d'abord, un axe de recherche intéressant serait l'extension des résultats proposés à une classe plus large de systèmes échantillonnés, comme les systèmes homogènes ou polynomiaux par exemple. Dans ce cas, le contrôle dynamique de l'échantillonnage pourrait bénéficier à la fois des avantages de l'approche par échantillonnage dépendant de l'état présentée dans le cas linéaire dans cette thèse, et des avantages des propriétés d'homogénéité (ou de mise à l'échelle, suivant la classe de système considérée) des fonctions d'échantillonnage dévoilées dans [Anta 2010] par exemple.

Un autre axe de recherche qu'il serait intéressant d'étudier serait l'extension des résultats proposés sur l'échantillonnage dépendant de l'état à une plus large classe de contrôleurs, ou en incluant d'autres types de performances de commande. Par exemple, des extensions aux cas de contrôle par retour de sortie ou de contrôle basé observateur seraient très utiles pour une large variété de systèmes pour lesquels un contrôle par retour d'état n'est physiquement pas possible. De même, la mise en place d'un contrôle avec rejet de perturbations serait très intéressante, pour permettre la stabilisation de l'état d'un système vers le point d'équilibre dans le cas de systèmes avec des perturbations constantes ou à variation lente.

Enfin, il serait intéressant d'étendre les résultats de stabilité et de stabilisation présentés pour les systèmes avec un échantillonnage dépendant de l'état aux systèmes avec retards dépendant de l'état. L'analyse de tels systèmes est principalement motivée par les applications qui apparaissent dans le cadre des systèmes commandés par réseaux (voir [Briat 2010] pour la modélisation de la congestion sur internet, et [Donkers 2009] pour l'interaction entre les tâches de commande et les algorithmes d'ordonnancement par exemple). Dans ce contexte, il serait intéressant de construire des outils pour analyser la stabilité vis-à-vis d'un retard dépendant de l'état $\tau(x)$, ou même de proposer des algorithmes d'ordonnancement qui permettraient de contrôler ce retard dépendant de l'état, de façon à obtenir la stabilité.

Appendix A

Proofs

A.1 Proofs from Chapter 2

Proof of Propositions 2.1 and 2.4: Let $\alpha > 1$, $\bar{\sigma} > 0$ and $\beta > 0$ be given. If there exist a quadratic function $V(x) = x^T P x$, $P = P^T \succ 0 \in \mathcal{M}_n(\mathbb{R})$ and a function $\tau : \mathbb{R}^n \to \mathbb{R}_+$, $0 < \delta \leq \tau(x) \leq \bar{\sigma}$, satisfying the conditions of Proposition 2.1, then the usual LRF theory [Kolmanovskii 1992] adapted to sampled data systems ensures the asymptotic stability of the system origin for both Propositions 2.1 and 2.4.

Let us take such parameters satisfying (C1), and consider a time-varying sampling function $\tilde{\tau} : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$ defining sampling instant sequences by the law $s_{k+1} = s_k + \tilde{\tau}(s_k, x(s_k)), k \in \mathbb{N}$ and satisfying $0 < \delta \leq \tilde{\tau}(t, x) \leq \tau(x)$ for all $t \in \mathbb{R}_+$ and $x \in \mathbb{R}^n$. During a sampling interval $[0, \tilde{\tau}(0, x))$ with initial state x, two cases may occur.

- The first case is that during that time interval, $V(\varphi_\tau(\sigma, x))$ never goes below $\frac{V(x)}{\alpha}$. Then, the differential inequality $\dot{V}(\varphi_\tau(\sigma, x)) + 2\beta V(\varphi_\tau(\sigma, x)) \leq 0$ is satisfied for all $\sigma \in [0, \tilde{\tau}(0, x))$ according to (C1) and therefore $V(\varphi_\tau(\tilde{\tau}(0, x), x)) \leq \mathrm{e}^{-2\beta\tilde{\tau}(0,x)} V(x)$.

- In the other case, $V(\varphi_\tau(\sigma, x))$ manages to go below $\frac{V(x)}{\alpha}$ during that time interval. According to (C1), $\dot{V}(\varphi_\tau(\sigma, x)) \leq 0$ over the set $\Upsilon_x = \{y \in \mathbb{R}^n, V(y) \geq \frac{V(x)}{\alpha}\}$, and one can show as in the framework of [Blanchini 1999] that the set $\bar{\Upsilon}_x = \{y \in \mathbb{R}^n, V(y) \leq \frac{V(x)}{\alpha}\}$ is positive invariant. Therefore, if $V(\varphi_\tau(\sigma, x))$ goes below $\frac{V(x)}{\alpha}$, one will have $V(\varphi_\tau(\tilde{\tau}(0, x), x)) \leq \frac{V(x)}{\alpha}$. Moreover, if β satisfies $\beta \leq \frac{\ln(\alpha)}{2\bar{\sigma}}$, then we get $V(\varphi_\tau(\tilde{\tau}(0, x), x)) \leq \mathrm{e}^{-2\beta\bar{\sigma}} V(x) \leq \mathrm{e}^{-2\beta\tilde{\tau}(0,x)} V(x)$.

Therefore, for any initial state x_0, for any $t \in \mathbb{R}_+$, $t \in [s_k, s_{k+1})$ for some $k \in \mathbb{N}$, one has $V(x(t)) \leq V(x(s_k)) \leq \mathrm{e}^{-2\beta \sum_{i=0}^{k-1} \tilde{\tau}(s_i, x(s_i))} V(x_0) = \mathrm{e}^{-2\beta s_k} V(x_0) \leq \mathrm{e}^{-2\beta(t-\bar{\sigma})} V(x_0)$. As a consequence, one can show that $\|x(t)\|_2 \leq \left(\sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}\mathrm{e}^{\beta\bar{\sigma}}\right)\mathrm{e}^{-\beta t}\|x_0\|_2$, which proves the β-stability of both Propositions 2.1 (with $\tilde{\tau}(t,x) = \tau(x)$ for all $t \in \mathbb{R}_+$ and $x \in \mathbb{R}^n$) and 2.4. ∎

Proof of Lemma 2.5: Let us take a quadratic function $V(x) = x^T P x$, $P = P^T \succ 0 \in \mathcal{M}_n(\mathbb{R})$, scalars $\alpha > 1$, $\bar{\sigma} > 0$, and $0 < \beta \leq \frac{\ln(\alpha)}{2\bar{\sigma}}$, and a function $\tau : \mathbb{R}^n \to \mathbb{R}_+$ upper-bounded by $\bar{\sigma}$, and let us rewrite the propositions used in the stability condition of Proposition 2.1.

Rewrite $\alpha V(\varphi_\tau(\sigma,x)) \geq V(x)$ as $\begin{bmatrix}\varphi_\tau(\sigma,x)\\ x\end{bmatrix}^T \begin{bmatrix}-\alpha P & 0\\ 0 & P\end{bmatrix}\begin{bmatrix}\varphi_\tau(\sigma,x)\\ x\end{bmatrix} \leq 0$, and $\dot{V}(\varphi_\tau(\sigma,x)) + 2\beta V(\varphi_\tau(\sigma,x)) \leq 0$ as $\begin{bmatrix}\varphi_\tau(\sigma,x)\\ x\end{bmatrix}^T \begin{bmatrix}A^TP + PA + 2\beta P & -PBK\\ -K^TB^TP & 0\end{bmatrix}\begin{bmatrix}\varphi_\tau(\sigma,x)\\ x\end{bmatrix} \leq 0$. Using the lossless version of the S-procedure [Boyd 1994] (see Theorem D.3), the stability condition from Proposition 2.1 is satisfied if and only if there exists $\varepsilon \geq 0$ such that $\begin{bmatrix}\varphi_\tau(\sigma,x)\\ x\end{bmatrix}^T \Omega \begin{bmatrix}\varphi_\tau(\sigma,x)\\ x\end{bmatrix} \leq 0$, with Ω given in (2.6). One can finally derive Lemma 2.5 stability conditions after expressing the evolution of the system state: $\varphi_\tau(\sigma,x) = \left(I + \int_0^\sigma \mathrm{e}^{sA}ds(A - BK)\right)x = \Lambda(\sigma)x$. ∎

Proof of Theorem 2.9: Let x be in $\mathbb{R}^n$. There exists a region $\mathcal{R}_s$ as in (2.8) such that $x \in \mathcal{R}_s$ and $\tau(x) = \tau_s$. Using the lossless version of the S-procedure [Boyd 1994] (see Theorem D.3), one can see that for any $\kappa \in \mathcal{K}_s$ the condition $x^T\Phi_{\kappa,s}x \leq 0, x \in \mathcal{R}_s$ is satisfied if and only if there exists a scalar $\varepsilon_{\kappa,s} \geq 0$ such that $\Phi_{\kappa,s} + \varepsilon_{\kappa,s}\Psi_s \preceq 0$. Therefore, if the condition $\Phi_{\kappa,s} + \varepsilon_{\kappa,s}\Psi_s \preceq 0$ is satisfied for all $s \in \{1, \cdots, q\}$ and $\kappa \in \mathcal{K}_s$, then for all $x \in \mathbb{R}^n$, for all $\sigma \in [0, \tau(x)]$, $x^T\Phi(\sigma)x \leq 0$, according to (2.10), and the stability conditions from Lemma 2.5 are satisfied. ∎

Proof of Corollary 2.11: This comes naturally from Theorem 2.9 and Proposition 2.4 when working with a single region: $\mathbb{R}^n$ itself. ∎

A.2 Proofs from Chapter 3

Proof of Proposition 3.1: Consider scalars $\alpha > 1$, $r > 0$, $\bar{\sigma} > 0$ and $0 < \beta \leq \frac{\ln(\alpha)}{r\bar{\sigma}}$, a map $\tau_{\max} : \mathbb{R}^n \to \mathbb{R}_+$, $0 < \delta \leq \tau_{\max}(x) \leq \bar{\sigma}$, and a sampling function $\tau : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$ satisfying (3.3). Consider a continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}_+$ and scalars $0 < \underline{\gamma} \leq \bar{\gamma}$ satisfying (H1) and (H2). Assuming that the functions f_k are locally Lipschitz in their second variable and that the perturbation w is locally essentially bounded guarantees the existence and uniqueness of solution for the differential equation (3.6) (see the framework of [Mancilla-Aguilar 2005]). The usual LRF theory [Kolmanovskii 1992] then ensures the asymptotic stability of the considered switched nonlinear system.

In order to analyse the convergence rate of the system's state, we analyse the evolution of $V(x(t))$ over each time interval $[s_k, s_{k+1})$ between two consecutive switches. During such a time interval, one has $x(t) = \phi_{\tau,w}(t, x_0) = \phi_{\tau,w}(t-s_k, x(s_k)) = \phi_{\tau,w}(\sigma, x) = \phi_{\tau_{\max},w}(\sigma, x)$, with the notations $\sigma = t - s_k$ and $x = x(s_k)$. With these notations, studying $V(x(t))$ for $t \in [s_k, s_{k+1})$ amounts to studying $V(\phi_{\tau_{\max},w}(\sigma, x))$ for $\sigma \in [0, \tau(s_k, x))$. During that time interval, two cases may occur.

- In the first case, $\alpha V(\phi_{\tau_{\max},w}(\sigma, x)) > V(x)$ for all $\sigma \in [0, \tau(s_k, x))$. According to (H2), since $\tau(s_k, x) \leq \tau_{\max}(x)$, the differential inequality $\dot{V}(\phi_{\tau_{\max},w}(\sigma, x)) + r\beta V(\phi_{\tau_{\max},w}(\sigma, x)) \leq 0$ is then satisfied for all $\sigma \in [0, \tau(s_k, x))$, and thus, one will have $V(\phi_{\tau_{\max},w}(\sigma, x)) \leq \mathrm{e}^{-r\beta\sigma} V(x)$, for all $\sigma \in [0, \tau(s_k, x))$.

- In the second case, there exists $\sigma \in [0, \tau(s_k, x))$ such that $\alpha V(\phi_{\tau_{\max},w}(\sigma, x)) \leq V(x)$. Let us denote $\sigma^* = \inf\{\sigma \in [0, \tau(s_k, x)) | \alpha V(\phi_{\tau_{\max},w}(\sigma, x)) \leq V(x)\}$. For $\sigma \in [0, \sigma^*)$, using the same arguments as in the previous case allows for proving that $V(\phi_{\tau_{\max},w}(\sigma, x)) \leq \mathrm{e}^{-r\beta\sigma} V(x)$. Let us now see what happens for $\sigma \in [\sigma^*, \tau(s_k, x))$. According to (H2), $\dot{V}(\phi_{\tau_{\max},w}(\sigma, x)) \leq 0$ over the set $\Upsilon_x = \{y \in \mathbb{R}^n, \alpha V(y) \geq V(x)\}$, and one can show as in the framework of [Blanchini 1999] that the set $\bar{\Upsilon}_x = \{y \in \mathbb{R}^n, \alpha V(y) \leq V(x)\}$ is positively invariant. Therefore, for $\sigma \in [\sigma^*, \tau(s_k, x))$, one has $\alpha V(\phi_{\tau_{\max},w}(\sigma, x)) \leq V(x)$. Then, since $\beta \leq \frac{\ln(\alpha)}{r\bar{\sigma}}$, and with the assumption that the sampling map is upper-bounded by $\bar{\sigma}$ (and thus $\sigma \leq \bar{\sigma}$), one can show that $V(\phi_{\tau_{\max},w}(\sigma, x)) \leq \mathrm{e}^{-r\beta\bar{\sigma}} V(x) \leq \mathrm{e}^{-r\beta\sigma} V(x)$.

Therefore, for any initial state x_0, for any $t \in \mathbb{R}_+$ ($t \in [s_k, s_{k+1})$ for some $k \in \mathbb{N}$), one has $V(x(t)) \leq \mathrm{e}^{-r\beta\left[\left(\sum_{i=0}^{k-1} \tau(s_i, x(s_i))\right)+(t-s_k)\right]} V(x_0) = \mathrm{e}^{-r\beta t} V(x_0)$. As a consequence, using (H1), one can show that $\|x(t)\|_2 \leq \left(\frac{\bar{\gamma}}{\underline{\gamma}}\right)^{\frac{1}{r}} \mathrm{e}^{-\beta t} \|x_0\|_2$, which proves the β-stability. ∎

Proof of Proposition 3.2: This is a particular case of Proposition 3.1, with the sampled-data system $\mathcal{S}$ which can be seen as a subclass of the switched nonlinear system $\{(3.3),(3.6)\}$, with the assumption (3.4) which ensures the perturbation w is locally essentially bounded, and with $V(x) = x^T P x$, $P \in S_n^{+*}$, $r = 2$, $\underline{\gamma} = \lambda_{\min}(P)$, and $\bar{\gamma} = \lambda_{\max}(P)$. ∎

Proof of Lemma 3.3: Consider a quadratic function $V(x) = x^T Px$, $P \in S_n^{+*}$, scalars $\alpha > 1$, $\bar{\sigma} > 0$, and $0 < \beta \leq \frac{\ln(\alpha)}{2\bar{\sigma}}$, and a sampling map $\tau_{\max} : \mathbb{R}^n \to \mathbb{R}_+$ upper-bounded by $\bar{\sigma}$. Let us rewrite the propositions from (H3) using the dynamics of the system $\mathcal{S}$.

Rewrite $\alpha V(\varphi_{\tau_{\max},w}(\sigma, x)) \geq V(x)$ as $\begin{bmatrix} \varphi_{\tau_{\max},w}(\sigma,x) \\ x \end{bmatrix}^T \begin{bmatrix} -\alpha P & 0 \\ * & P \end{bmatrix} \begin{bmatrix} \varphi_{\tau_{\max},w}(\sigma,x) \\ x \end{bmatrix} \leq 0$,

and $\dot{V}(\varphi_{\tau_{\max},w}(\sigma,x)) + 2\beta V(\varphi_{\tau_{\max},w}(\sigma,x)) \leq 0$ as $\begin{bmatrix} \varphi_{\tau_{\max},w}(\sigma,x) \\ x \\ w(\sigma) \end{bmatrix}^T \tilde{\Omega} \begin{bmatrix} \varphi_{\tau_{\max},w}(\sigma,x) \\ x \\ w(\sigma) \end{bmatrix} \leq 0$,

with $\tilde{\Omega} = \begin{bmatrix} A^T P + PA + 2\beta P & -PBK & PE \\ * & 0 & 0 \\ * & * & 0 \end{bmatrix}$. Now note that the evolution of the state satisfies

$$\varphi_{\tau_{\max},w}(\sigma, x) = \varphi_{\tau_{\max},0}(\sigma, x) + J_w(\sigma), \tag{A.1}$$

where the term $\varphi_{\tau_{\max},0}(\sigma,x) = \Lambda(\sigma)x$, with $\Lambda(\sigma)$ defined in (3.8), corresponds to the evolution of the state without perturbations, and where the term $J_w(\sigma)$, defined in (3.9), represents the effect of the disturbance on the system's evolution.

Using these notations, one can use the lossless version of the S-procedure [Boyd 1994] (see Theorem D.3) to show that the stability condition (H3) from Proposition 3.2 is satisfied if and only if there exists $\varepsilon \geq 0$ such that (3.7) is satisfied for all $x \in \mathbb{R}^n$ and $\sigma \in [0, \tau_{\max}(x)]$.

∎

Proof of Theorem 3.4: Consider a quadratic function $V(x) = x^T Px$, $P \in S_n^{+*}$, scalars $\alpha > 1$, $\bar{\sigma} > 0$, $0 < \beta \leq \frac{\ln(\alpha)}{2\bar{\sigma}}$, and $W \geq 0$, and a sampling map $\tau_{\max} : \mathbb{R}^n \to \mathbb{R}_+$ upper-bounded by $\bar{\sigma}$.

The idea of the proof is to find an upper-bound independent of the perturbation w for the left part of equation (3.7). The left part of equation (3.7) is equal to

$$\begin{aligned} G_w(\sigma,x) \equiv\; & x^T(\Lambda(\sigma)^T M_1 \Lambda(\sigma) - \Lambda(\sigma)^T PBK - K^T B^T P \Lambda(\sigma) - \varepsilon P)x \\ & + J_w(\sigma)^T M_1 J_w(\sigma) + J_w(\sigma)^T M_2 x + x^T M_2^T J_w(\sigma) \\ & + J_w(\sigma)^T M_3 w(\sigma) + w(\sigma)^T M_3^T J_w(\sigma) + w(\sigma)^T M_4 x + x^T M_4^T w(\sigma). \end{aligned} \tag{A.2}$$

In order to upper bound this term independently of the perturbation, we use the inequality in Theorem D.5, which shows that for any matrices Φ_1, $\Phi_2 \in S_n^{+*}$, and $\Phi_3 \in S_{n_w}^{+*}$, we have

$$\begin{aligned} G_w(\sigma,x) \leq\ & x^T \left[\Lambda(\sigma)^T M_1 \Lambda(\sigma) - \Lambda(\sigma)^T PBK - K^T B^T P \Lambda(\sigma) \right. \\ & \left. -\varepsilon P + M_2(\sigma)^T \Phi_1^{-1} M_2(\sigma) + M_4(\sigma)^T \Phi_3^{-1} M_4(\sigma)\right] x \\ & + w(\sigma)^T \left[M_3^T \Phi_2^{-1} M_3 + \Phi_3\right] w(\sigma) + J_w(\sigma)^T \left[M_1 + \Phi_1 + \Phi_2\right] J_w(\sigma). \end{aligned} \tag{A.3}$$

Using a classic inequality and assumption (3.4), the term $w(\sigma)^T \left[M_3^T \Phi_2^{-1} M_3 + \Phi_3\right] w(\sigma)$ from equation (A.3) can be bounded as follows:

$$\begin{aligned} w(\sigma)^T \left[M_3^T \Phi_2^{-1} M_3 + \Phi_3\right] w(\sigma) & \leq \lambda_{\max}(M_3^T \Phi_2^{-1} M_3 + \Phi_3) w(\sigma)^T w(\sigma) \\ & \leq W \lambda_{\max}(M_3^T \Phi_2^{-1} M_3 + \Phi_3) x^T x. \end{aligned} \tag{A.4}$$

Let a scalar $\eta \geq 0$ be such that $\begin{bmatrix} \Phi_3 - \eta I & M_3^T \\ * & -\Phi_2 \end{bmatrix} \preceq 0$, as assumed in (3.11). Using the Schur complement, one can show that this is equivalent to $M_3^T \Phi_2^{-1} M_3 + \Phi_3 \preceq \eta I$. Therefore, (A.4) leads to

$$w(\sigma)^T \left[M_3^T \Phi_2^{-1} M_3 + \Phi_3\right] w(\sigma) \leq W \eta x^T x. \tag{A.5}$$

We denote $Q = M_1 + \Phi_1 + \Phi_2$. The other term from (A.3), $J_w(\sigma)^T Q J_w(\sigma)$, can be written as

$$J_w(\sigma)^T Q J_w(\sigma) = \left(\int_0^\sigma \mathrm{e}^{A(\sigma-s)} E w(s) ds\right)^T Q \left(\int_0^\sigma \mathrm{e}^{A(\sigma-s)} E w(s) ds\right).$$

Let us assume that $Q \succeq 0$ (we can choose Φ_1 and Φ_2 so as to satisfy this condition. Using Jensen's inequality (Theorem D.4), one gets

$$J_w(\sigma)^T Q J_w(\sigma) \leq \sigma \int_0^\sigma w(s)^T E^T \left(\mathrm{e}^{A(\sigma-s)}\right)^T Q \left(\mathrm{e}^{A(\sigma-s)}\right) E w(s) ds.$$

Then, using the inequality in Theorem D.6 along with some other classic inequalities, as well as assumption (3.4), one gets

$$\begin{aligned} J_w(\sigma)^T Q J_w(\sigma) & \leq \sigma \lambda_{\max}(Q) \textstyle\int_0^\sigma w(s)^T E^T \left(\mathrm{e}^{A(\sigma-s)}\right)^T \left(\mathrm{e}^{A(\sigma-s)}\right) E w(s) ds \\ & \leq \sigma \lambda_{\max}(Q) \textstyle\int_0^\sigma \mathrm{e}^{(\sigma-s)\lambda_{\max}(A+A^T)} w(s)^T E^T E w(s) ds \\ & \leq \sigma \lambda_{\max}(Q) \lambda_{\max}(E^T E) \textstyle\int_0^\sigma \mathrm{e}^{(\sigma-s)\lambda_{\max}(A+A^T)} \|w(s)\|_2^2 ds \\ & \leq \sigma W \lambda_{\max}(Q) \lambda_{\max}(E^T E) \left(\textstyle\int_0^\sigma \mathrm{e}^{\lambda_{\max}(A+A^T)s} ds\right) \|x\|_2^2 \\ & = \sigma W \lambda_{\max}(Q) \lambda_{\max}(E^T E) f_A(\sigma) x^T x, \end{aligned}$$

with $f_A(\sigma)$ defined in (3.15). If Q also satisfies $Q \preceq \mu I$, for a certain $\mu \geq 0$, then one has:

$$J_w(\sigma)^T Q J_w(\sigma) \leq \sigma W \mu \lambda_{\max}(E^T E) f_A(\sigma) x^T x. \tag{A.6}$$

Implementing inequalities (A.5) and (A.6) in (A.3), it is clear that $G_w(\sigma, x) \leq x^T \Pi(\sigma) x$, with $\Pi(\sigma)$ defined in (3.13), and therefore, if $x^T \Pi(\sigma) x \leq 0$ for all $x \in \mathbb{R}^n$ and for all $\sigma \in [0, \tau_{\max}(x)]$, then the stability conditions from Lemma 3.3 are satisfied, which ends the proof. ∎

Proof of Lemma 3.7: Since the sampling map is state-independent, one can remove the state-dependency in (3.12) by rewriting the inequality under the form of a parameter-dependent LMI: $\Pi(\sigma) \preceq 0,\ \forall \sigma \in [0, \tau_{\max}^{(\text{global})}]$. Then, applying the extended version of the Schur complement allows to remove the inverse terms Φ_1^{-1} and Φ_3^{-1} that appear in the equation (3.13) of $\Pi(\sigma)$ and ensures the equivalence between $\Pi(\sigma) \preceq 0$ and (3.17). ∎

Proof of Theorem 3.8: If the condition $\bar{\Delta}_\kappa(\tau_{\max}^{(\text{global})}) \preceq 0$ is satisfied for all $\kappa \in \mathcal{K}(\tau_{\max}^{(\text{global})})$, (3.19) ensures that $\Delta(\sigma) \preceq 0$ for all $\sigma \in [0, \tau_{\max}(x)]$. Therefore, by using the result from Lemma 3.7, we show that the stability conditions from Theorem 3.4 are satisfied, and thus the system $\mathcal{S}$ is globally β-stable. ∎

Proof of Lemma 3.11: It is clear that for the sampled-data system {(3.1),(3.2),(3.4)} with sampling intervals satisfying $s_{k+1} - s_k \in [\delta, \bar{\sigma}]$, the stability conditions from Lemma 3.3 can be adapted by replacing in their statement x by $x(s_k)$, $\varphi_{\tau_{\max},w}(\sigma, x)$ by $x(t)$, and σ by $t - s_k$, and by verifying the conditions for all $t \in [s_k, s_{k+1}]$ and $k \in \mathbb{N}$ instead of veryfying them for all $x \in \mathbb{R}^n$ and $\sigma \in [0, \tau_{\max}(x)]$. From this, by rewriting the inequality (3.7) from Lemma 3.3, one can see that the studied system is globally β-stable if for all $t \in [s_k, s_{k+1}],\ k \in \mathbb{N}$,

$$\begin{bmatrix} x(t) \\ x(s_k) \end{bmatrix}^T \begin{bmatrix} A^T P + PA + \varepsilon\alpha P + 2\beta P & PBK \\ * & \varepsilon P \end{bmatrix} \begin{bmatrix} x(t) \\ x(s_k) \end{bmatrix} + x(t)^T P E w(t) + w(t)^T E^T P x(t) \leq 0.$$

Using the same tools (Theorem D.5) as in the proof of Theorem 3.4, it is possible to upper bound the crossed term in the left part of this expression as follows:

$$\begin{aligned} x(t)^T PEw(t) + w(t)^T E^T Px(t) &\leq x(t)^T PE\Phi^{-1}E^T Px(t) + w^T(t)\Phi w(t) \\ &\leq x(t)^T PE\Phi^{-1}E^T Px(t) + W\lambda_{\max}(\Phi)x^T(s_k)x(s_k), \end{aligned}$$

with any matrix $\Phi \in S^{++*}_{n_w}$. Setting $\Phi = I$ and using this majoration shows that if (3.20) is satisfied, then the system is globally β-stable. ∎

Proof of Theorem 3.12: If the sampling map $\tau_{\max}$ and the function $\tilde{\mathcal{K}} : \mathbb{R}^n \to \mathcal{P}(\bar{\mathcal{K}})$ are such that he assertion (3.22) and the triggering condition (3.22a) are satisfied for all $x \in \mathbb{R}^n$, it is clear that the condition (3.12) from Theorem 3.4 is satisfied for all $x \in \mathbb{R}^n$ and all $\sigma \in [0, \tau_{\max}(x)]$. Then, the other assumptions and conditions guarantee that all the stability conditions from Theorem 3.4 are satisfied. ∎

Proof of Theorem 3.16: Consider scalars $\rho_s \geq 0$ such that the LMIs $\bar{\Delta}_\kappa(\tau^{(s)}_{\max}) + \begin{bmatrix} \rho_s\Psi_s & 0 \\ * & 0 \end{bmatrix} \preceq 0$ are satisfied for all $s \in \{1, \cdots, q\}$ and $\kappa \in \mathcal{K}(\tau^{(s)}_{\max})$. Let $x \in \mathbb{R}^n$. There exists $s \in \{1, \cdots, q\}$ such that $x \in \mathcal{R}_s$. According to (3.28), one has $\Delta(\sigma) + \begin{bmatrix} \rho_s\Psi_s & 0 \\ * & 0 \end{bmatrix} \preceq 0$ for all $\sigma \in [0, \tau^{(s)}_{\max}]$. Thus, using the construction of Δ (equation (3.17)) and the Schur complement, we get that $\Pi(\sigma) + \rho_s\Psi_s \preceq 0$ for all $\sigma \in [0, \tau^{(s)}_{\max}]$, with Π defined in (3.13). Since $x \in \mathcal{R}_s = \{x \in \mathbb{R}^n, x^T\Psi_s x \geq 0\}$, the S-procedure [Boyd 1994] (see Theorem D.3) then ensures that $x^T\Pi(\sigma)x \preceq 0$ for all $\sigma \in [0, \tau^{(s)}_{\max} = \tau_{\max}(x)]$. Therefore, one can see that the conditions from Theorem 3.4 are satisfied, which ends the proof. ∎

Appendix B

Construction of the conic regions covering

B.1 Isotropic state-covering: using the spherical coordinates of the state

The first conic covering consists in designing sectors that describe entirely the unit n-sphere. Here, the parametrization we propose uses the generalized spherical coordinates of the state x in $\mathbb{R}^n$: $(r, \theta_1, \cdots, \theta_{n-1})$, provided by the relations

$$\begin{array}{rcl} r & = & \|x\|_2, \\ x_1 & = & r\text{cos}\theta_1, \\ x_2 & = & r\text{sin}\theta_1\text{cos}\theta_2, \\ & \vdots & \\ x_{n-1} & = & r\text{sin}\theta_1 \ldots \text{sin}\theta_{n-2}\text{cos}\theta_{n-1}, \\ x_n & = & r\text{sin}\theta_1 \ldots \text{sin}\theta_{n-2}\text{sin}\theta_{n-1}, \end{array}$$

with $\theta_1, \theta_2, \ldots, \theta_{n-2} \in [0, \pi]$, and $\theta_{n-1} \in [-\pi, \pi]$. Each region $\mathcal{R}_s$ of the covering is associated to some range of the $(n-1)$ angular coordinates θ_i:

$$(x \in \mathcal{R}_s) \Leftrightarrow \left(\forall i \in \{1, \cdots, n-1\}, \theta_i \in [\theta_{i,s}^-, \theta_{i,s}^+]\right).$$

An illustration of such conic regions in $\mathbb{R}^2$ is shown in Figure B.1.

Then, in order to build the matrices Ψ_s defining these regions $\mathcal{R}_s$ (2.8), one can use

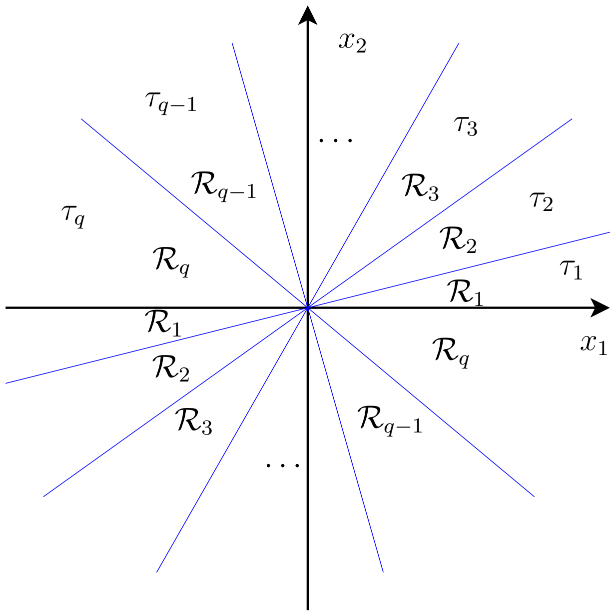

Figure B.1: Covering the state-space of dimension 2 with q conic regions $\mathcal{R}_s$

some geometric arguments: if $x \in \mathcal{R}_s$, then for $\theta_{i,s}^{-+} \in [0, \frac{\pi}{2})$,

$$\forall i \in \{1, \cdots, n-2\}, \quad \begin{cases} x_i^2 \tan^2 \theta_{i,s}^- \leq x_{i+1}^2 + \cdots + x_n^2 \\ x_i^2 \tan^2 \theta_{i,s}^+ \geq x_{i+1}^2 + \cdots + x_n^2 \end{cases}, \tag{B.1}$$

and

$$\begin{cases} x_{n-1} \geq \tan \theta_{n-1,s}^- x_n \\ x_{n-1} \leq \tan \theta_{n-1,s}^+ x_n \end{cases}. \tag{B.2}$$

Similar conditions can be obtained for $\theta_{i,s}^{-+} \in (\frac{\pi}{2}, \pi]$. The design of the conic forms Ψ_s from (B.1) and (B.2) is then trivial.

Note that with this covering, the state position is characterized by its only $n-1$ angular coordinates $\theta_1, \cdots, \theta_{n-1}$. Thus, situating $x \in \mathbb{R}^n$ in this conic covering is easy, which is important since it has to be done in real-time. The computational complexity to calculate the angular coordinates and find the right region is linear in the system's dimension ($O(n)$), and does not depend on the number of regions. More precisely, one can show that $9n-7$ elementary operations are required (additions, multiplications and divisions), added to 1 square-root, $n-1$ arccosine, and $n-2$ sine. Also, note that the smaller the ranges $[\theta_{i,s}^-, \theta_{i,s}^+]$ of each conic region, the closer the obtained state-dependent sampling function will be from the optimal sampling function.

A drawback of this covering technique is that the number of regions to be considered exponentially increases with the dimension n of the system. If one divides each angular coordinate range in m equal sectors (what we call "isotropic covering"), this provides a precision of $\frac{\pi}{m} rad$ for each angle and one needs m^{n-1} conic regions. This means that a tradeoff between the offline computational complexity and the accuracy of the approximation has to be achieved. Furthermore, there is a link between the conservatism of the proposed solution and the accuracy of approximation.

B.2 Anisotropic state-covering: using the discrete-time behaviour of the system

A second covering technique involves the dynamics of the discrete-time system. Assume that the conditions from Corollary 2.11 are satisfied for a given $\tau^* = \tau^*_{\text{sub}}$. Then, there exists a matrix $P = P^T \succ 0$ such that

$$x^T(\Lambda^T(\tau^*)P\Lambda(\tau^*) - \mathrm{e}^{-2\beta\tau^*}P)x \leq 0 \tag{B.3}$$

is satisfied for all $x \in \mathbb{R}^n$, with Λ the transition matrix function defined in (2.7).

The conic regions will be obtained by using the regions described by (B.3) for values of τ larger than τ^*. For a given scalar $\bar{\sigma} > \tau^*$, consider the following set of sampling times $T_s = \tau^* + (s-1)\frac{\bar{\sigma}-\tau^*}{q-1}$, $s \in \{1, \cdots, q\}$ ($\tau^* \leq T_s \leq \bar{\sigma}$), and design the conic regions as:

$$\mathcal{R}_s = \{x \in \mathbb{R}^n, x^T(\Lambda^T(T_s)P\Lambda(T_s) - \mathrm{e}^{-2\beta T_s}P)x \leq 0\}.$$

Such regions ensure that the function $V(x) = x^T P x$ is decreasing at sampling times along the solutions of the discrete-time model

$$x_{k+1} = \Lambda(\tau(x_k))x_k, \; s_{k+1} = s_k + \tau(x_k),$$

when $\tau(x) = \max_{s\in\{1,\cdots,q\} \text{ s.t. } x\in\mathcal{R}_s} T_s, \; \forall x \in \mathbb{R}^n$.

Using Theorem 2.9 allows us to guarantee the decay of the Lyapunov-Razumikhin function such as in Proposition 2.1 for the solution of the continuous-time model $\mathcal{S}$. Note that the case $s = 1$ corresponds to $\mathcal{R}_1 = \mathbb{R}^n$.

In this construction, the division is achieved on the time-variable T_s rather than on angular coordinates. The advantage is that the number of regions does not depend on the dimension of the system and is proportional to the numerical precision, whereas in the previous covering construction, it was an exponential function. The drawback is that more online computation is needed for situating the sampled state in its corresponding conic region: the inequalities $x^T(\Lambda^T(T_s)P\Lambda(T_s) - \mathrm{e}^{-2\beta T_s}P)x = x^T(-\Psi_s)x \leq 0$ have to be checked. Thus, with this second construction, the tradeoff moves to offline/online computational effort. At each sampling instant, the number of additions required to find the region is at most $(q-1)(n-1)(n+1)$, and the number of multiplications is at most $(q-1)n(n+1)$. The computational complexity is in $O(qn^2)$.

Appendix C

Contruction of a polytopic embedding based on Taylor polynomials

C.1 General contruction for polynomial matrix functions

The polytopic embedding approach used in this thesis is based on the results from [Hetel 2007] and [Hetel 2006], for which convex polytopes are designed around matrix exponentials using their Taylor polynomial approximation. The construction for polynomial matrix functions is based on the following property:

Theorem C.1 ([Hetel 2007]) *Consider the matrix polynomial function*

$$L(\sigma) = L_0 + L_1\sigma + \cdots + L_N\sigma^N$$

such that the variable σ is bounded and positive: $0 \leq \underline{\sigma} \leq \sigma \leq \bar{\sigma}$.
Then we can find a convex polytope formed by $N+1$ vertices which enveloppes the matrix polynomial function $L(\sigma)$, i.e. there exists an indexed family of scalars $\mu_i(\sigma) > 0$, $i \in \{0, \cdots, N\}$, verifying $\sum_{i=1}^{N} \mu_i(\sigma) = 1$, and such that

$$L(\sigma) = \sum_{i=1}^{N} \mu_i(\sigma) U_i$$

where the matrices U_i represent the vertices of the polytope and are given by

$$\begin{aligned} U_0 &= L_0 + \underline{\sigma} L_1 + \underline{\sigma}^2 L_2 + \cdots + \underline{\sigma}^N L_N \\ U_1 &= L_0 + \bar{\sigma} L_1 + \underline{\sigma}^2 L_2 + \cdots + \underline{\sigma}^N L_N \\ U_1 &= L_0 + \bar{\sigma} L_1 + \bar{\sigma}^2 L_2 + \cdots + \underline{\sigma}^N L_N \\ &\vdots \\ U_N &= L_0 + \bar{\sigma} L_1 + \bar{\sigma}^2 L_2 + \cdots + \bar{\sigma}^N L_N \end{aligned}$$

C.2 Case of unperturbed LTI systems (Chapter 2)

Here, we propose a construction of the convex polytope satisfying (2.10) for the ideal LTI sampled-data system (2.1). Let $s \in \{1, \cdots, q\}$ be the index of the considered region of the state-space. The polytope design we propose is based on a Taylor series approximation of order N performed on l subdivision intervals of $[0, \bar{\sigma}]$. The idea behind these subdivisions is to build small convex polytopes locally for each time interval subdivision, in order to refine the precision of the convex embedding. A 2D representation of the proposed convex polytope design is shown in Figure C.1. Note that each local polytope subdivision is composed of $N+1$ vertices, since each of them is designed using a Taylor series approximation of order N.

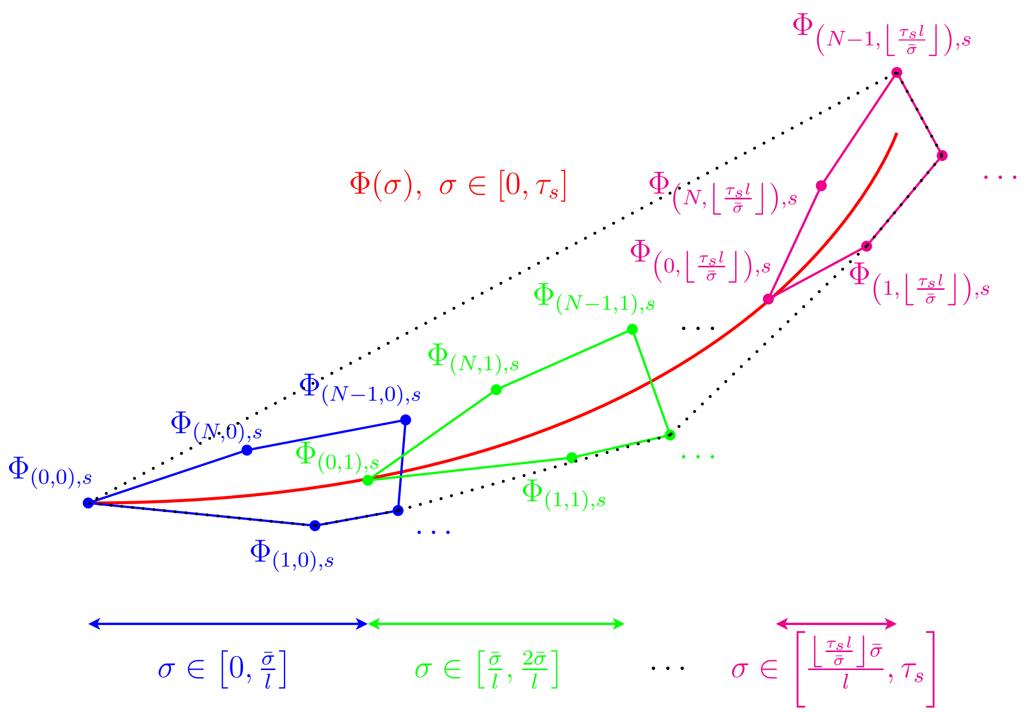

Figure C.1: 2D representation of the convex polytope design using polytopic subdivisions around the matrix function Φ over the time interval $\sigma \in [0, \tau_s]$

In this construction, we define the set of vertex indexes

$$\mathcal{K}_s = \{0, \cdots, N\} \times \left\{0, \cdots, \left\lfloor \frac{\tau_s l}{\bar{\sigma}} \right\rfloor \right\}, \tag{C.1}$$

with integers $N \geq 0$ and $l \geq 1$, and design the vertices $\Phi_{(i,j),s}$ as

$$\Phi_{(i,j),s} = \hat{\Phi}_{(i,j),s} + \nu I, \tag{C.2}$$

with

$$\begin{cases} \hat{\Phi}_{(i,j),s} = \left(\sum_{k=0}^{i} L_{k,j} \left(\frac{\bar{\sigma}}{l}\right)^k\right) \text{ if } j < \left\lfloor \frac{\tau_s l}{\bar{\sigma}} \right\rfloor, \\ \hat{\Phi}_{(i,j),s} = \left(\sum_{k=0}^{i} L_{k,j} \left(\tau_s - \frac{j\bar{\sigma}}{l}\right)^k\right) \text{ otherwise,} \end{cases} \tag{C.3}$$

$$\begin{aligned} L_{0,j} &= \Pi_{3,j}^T \Pi_1 \Pi_{3,j} - \varepsilon P - \Pi_{3,j}^T \Pi_2 - \Pi_2^T \Pi_{3,j}, \\ L_{1,j} &= \Pi_{4,j}^T (\Pi_1 \Pi_{3,j} - \Pi_2) + (\Pi_{3,j}^T \Pi_1^T - \Pi_2^T) \Pi_{4,j}, \\ L_{k\geq 2,j} &= \Pi_{4,j}^T \frac{(A^{k-1})^T}{k!} (\Pi_1 \Pi_{3,j} - \Pi_2) + (\Pi_{3,j}^T \Pi_1^T - \Pi_2^T) \frac{A^{k-1}}{k!} \Pi_{4,j} \\ &\quad + \Pi_{4,j}^T \left(\sum_{i=1}^{k-1} \frac{(A^{i-1})^T}{i!} \Pi_1 \frac{A^{k-i-1}}{(k-i)!}\right) \Pi_{4,j}, \end{aligned} \tag{C.4}$$

$$\begin{aligned} \Pi_1 &= A^T P + PA + \varepsilon \alpha P + 2\beta P, \ \Pi_2 = PBK, \\ \Pi_{3,j} &= I + M_j (A - BK), \ \Pi_{4,j} = N_j (A - BK), \end{aligned} \tag{C.5}$$

$$M_j = \int_0^{j\frac{\bar{\sigma}}{l}} \mathrm{e}^{As} ds, \ N_j = AM_j + I, \tag{C.6}$$

and an upper-bound of the approximation error

$$\nu \geq \max_{\substack{\sigma' \in [0, \frac{\bar{\sigma}}{l}], \\ r \in \{0, \cdots, l-1\}}} \lambda_{\max} \left(\Phi\left(\sigma' + r\frac{\bar{\sigma}}{l}\right) - \bar{\Phi}_{N,r}(\sigma')\right), \tag{C.7}$$

defined with the function

$$\bar{\Phi}_{N,j}(\sigma') = \sum_{k=0}^{N} L_{k,j} \sigma'^k, \sigma' \in [0, \frac{\bar{\sigma}}{l}]. \tag{C.8}$$

Lemma C.2 *Consider a vector $x \in \mathbb{R}^n$, a scalar $\bar{\sigma} > 0$, integers $N \geq 0$ and $l \geq 1$, parameters $P = P^T \succ 0 \in \mathcal{M}_n(\mathbb{R})$, $\alpha > 1$, $0 < \beta \leq \frac{\ln(\alpha)}{2\bar{\sigma}}$, and $\varepsilon \geq 0$, and a sampling interval $\tau_s > 0$. If the condition $x^T \Phi_{(i,j),s} x \leq 0$ is satisfied for all $(i,j) \in \mathcal{K}_s$ (with $\Phi_{(i,j),s}$ and $\mathcal{K}_s$ defined in (C.2) and (C.1) respectively), then for all $\sigma \in [0, \tau_s]$, one has $x^T \Phi(\sigma) x \leq 0$, with Φ defined in (2.5).*

Proof:

1. First, we divide the time interval $[0, \bar{\sigma}]$ into l subdivisions and take a time $\sigma \leq \tau_s$ into one of these subdivisions. The aim of this step is to make preparations to compute a precise estimation of the matrix function Φ by building l small convex embeddings around it instead of building one big one, as shown in Figure C.1.

2. Then, we compute a polynomial approximation of Φ for the chosen time interval subdivision.

3. Afterwards, we bound the error term from this polynomial approximation with a constant term.

4. Finally, we build a convex polytope around the polynomial approximation and the error term bound, using the method proposed in [Hetel 2006] (described in the Appendix C.1), to obtain the desired finite number of conditions.

Step (1): Let us divide the time interval $[0, \bar{\sigma}]$ into l subdivisions $[j\frac{\bar{\sigma}}{l}, (j+1)\frac{\bar{\sigma}}{l}]$, with $j \in \{0, \cdots, l-1\}$. Let $\sigma \in [0, \tau_s]$. There exists $j \in \{0, \cdots, \lfloor \frac{\tau_s l}{\bar{\sigma}} \rfloor\}$ such that $j\frac{\bar{\sigma}}{l} \leq \sigma \leq (j+1)\frac{\bar{\sigma}}{l}$. Then define $\sigma' = \sigma - j\frac{\bar{\sigma}}{l}$ ($\sigma' \in [0, \chi]$, with $\chi = \frac{\bar{\sigma}}{l}$ if $j < \lfloor \frac{\tau_s l}{\bar{\sigma}} \rfloor$, and $\chi = \tau_s - \frac{j\bar{\sigma}}{l}$ otherwise).

Step (2): We define $\Pi_1 = A^T P + PA + \varepsilon\alpha P + 2\beta P$ and $\Pi_2 = PBK$. From equations (2.5) and (2.6), we deduce that

$$\Phi(\sigma) = \Lambda(\sigma)^T \Pi_1 \Lambda(\sigma) - \Lambda^T(\sigma)\Pi_2 - \Pi_2^T \Lambda(\sigma) - \varepsilon P. \tag{C.9}$$

In order to derive a useful expression of $\Lambda(\sigma)$ (defined in (2.7)) as a function of σ', we use the property expressed in Theorem D.7

$$\int_0^{a+b} \mathrm{e}^{As} ds = \int_0^a \mathrm{e}^{As} ds + \int_0^b \mathrm{e}^{As} ds \left(A \int_0^a \mathrm{e}^{As} ds + I \right),$$

which is satisfied for any scalars a and b, to obtain

$$\begin{aligned} \Lambda(\sigma) &= I + \left(M_j + \textstyle\int_0^{\sigma'} \mathrm{e}^{As} ds N_j \right)(A - BK) \\ &= \Pi_{3,j} + \textstyle\int_0^{\sigma'} \mathrm{e}^{As} ds \Pi_{4,j}, \end{aligned} \tag{C.10}$$

with $M_j = \int_0^{j\frac{\bar{\sigma}}{l}} \mathrm{e}^{As} ds$, $N_j = AM_j + I$, $\Pi_{3,j} = I + M_j(A - BK)$, and $\Pi_{4,j} = N_j(A - BK)$. Then, note that

$$\int_0^{\sigma'} \mathrm{e}^{As} ds = \sum_{i=1}^{\infty} \frac{A^{i-1}}{i!} \sigma'^i. \tag{C.11}$$

Combining equations (C.9), (C.10) and (C.11), one gets $\Phi(\sigma) = \sum_{k=0}^{\infty} L_{k,j}\sigma'^k$, with the matrices $L_{k,j}$ defined in (C.4). It is then possible to express a polynomial approximation of order N of Φ on the interval $[j\frac{\bar{\sigma}}{l}, (j+1)\frac{\bar{\sigma}}{l}]$ as

$$\bar{\Phi}_{N,j}(\sigma') = \sum_{k=0}^{N} L_{k,j}\sigma'^k, \sigma' \in [0, \frac{\bar{\sigma}}{l}]. \tag{C.12}$$

Step (3): Let us denote the approximation error term $R_{N,j}(\sigma') = \Phi(\sigma) - \bar{\Phi}_{N,j}(\sigma')$. If we can compute a bound with a scalar ν independent of σ' such that $R_{N,j}(\sigma') \preceq \nu I$ then the condition $x^T(\bar{\Phi}_{N,j}(\sigma') + \nu I)x \leq 0$ will imply that $x^T\Phi(\sigma)x \leq 0$. Since $R_{N,j}(\sigma') = \Phi(\sigma) - \bar{\Phi}_{N,j}(\sigma')$ is symmetric, then if we denote $\lambda_{\sigma'}$ the maximal eigenvalue of $R_{N,j}(\sigma')$, we have $R_{N,j}(\sigma') \preceq \lambda_{\sigma'}I$. As a consequence, $R_{N,j}(\sigma') \preceq \nu I$ with ν a constant defined in (C.7).

Step (4): Since the function $\bar{\Phi}_{N,j}(.) + \nu I : [0, \chi] \to \mathcal{M}_n(\mathbb{R})$ is polynomial, we can use the convex polytope envelope given in [Hetel 2006] (described in the Appendix C.1), to prove that if $x^T\Phi_{(i,j),s}x \leq 0$ for all $i \in \{1, \cdots, n\}$, with $\Phi_{(i,j),s} = \left(\sum_{k=0}^{i} L_{k,j}\chi^k\right) + \nu I$, then $x^T(\bar{\Phi}_{N,j}(\sigma') + \nu I)x \leq 0$ and therefore $x^T\Phi(\sigma)x \leq 0$. ∎

C.3 Case of perturbed LTI systems (Chapter 3)

Here, we propose a construction of the convex embedding satisfying (3.19) that is based on the results from [Hetel 2007], for the perturbed LTI system (3.1).

Consider a scalar $0 \leq \sigma^* \leq \bar{\sigma}$. In this construction, we define the set of vertex indexes

$$\mathcal{K}(\sigma^*) = \{0, \cdots, N\} \times \left\{0, \cdots, \left\lfloor \frac{\sigma^* l}{\bar{\sigma}} \right\rfloor \right\}, \tag{C.13}$$

with integers $N \geq 0$ and $l \geq 1$, and design the vertices $\bar{\Delta}_{(i,j)}(\sigma^*)$ for all $(i,j) \in \mathcal{K}(\sigma^*)$, as:

$$\bar{\Delta}_{(i,j)}(\sigma^*) = \hat{\Delta}_{(i,j)}(\sigma^*) + \nu I, \tag{C.14}$$

with

$$\begin{cases} \hat{\Delta}_{(i,j)}(\sigma^*) = \left(\sum_{k=0}^{i} \tilde{\Delta}_{(k,j)} \left(\frac{\bar{\sigma}}{l}\right)^k\right) \text{ if } j < \left\lfloor \frac{\sigma^* l}{\bar{\sigma}} \right\rfloor, \\ \hat{\Delta}_{(i,j)}(\sigma^*) = \left(\sum_{k=0}^{i} \tilde{\Delta}_{(k,j)} \left(\sigma^* - \frac{j\bar{\sigma}}{l}\right)^k\right) \text{ otherwise,} \end{cases} \tag{C.15}$$

$$\tilde{\Delta}_{(0,j)} = \begin{bmatrix} L_{0,j} & -K^T B^T P + \Gamma_{1,j}^T M_1^T & \Gamma_{1,j}^T PE \\ * & -\Phi_1 & 0 \\ * & * & -\Phi_3 \end{bmatrix},$$
$$\tilde{\Delta}_{(k\geq 1,j)} = \begin{bmatrix} L_{k,j} & \Gamma_{2,j}^T \frac{(A^{k-1})^T}{k!} M_1^T & \Gamma_{2,j}^T \frac{(A^{k-1})^T}{k!} PE \\ * & 0 & 0 \\ * & * & 0 \end{bmatrix}, \tag{C.16}$$

$$\begin{aligned} \Gamma_{1,j} = I + N_j(A - BK),\ \Gamma_{2,j} = N_j'(A - BK), \\ N_j = \int_0^{j\frac{\bar{\sigma}}{l}} \mathrm{e}^{As} ds,\ N_j' = AN_j + I, \end{aligned} \tag{C.17}$$

and

$$\begin{aligned} L_{0,j} = &\ \Gamma_{1,j}^T M_1 \Gamma_{1,j} - \varepsilon P + W\eta I - \Gamma_{1,j}^T PBK - K^T B^T P \Gamma_{1,j} + \tilde{L}_{0,j}, \\ L_{1,j} = &\ \Gamma_{2,j}^T (M_1 \Gamma_{1,j} - PBK) + (\Gamma_{1,j}^T M_1^T - K^T B^T P)\Gamma_{2,j} + \tilde{L}_{1,j}, \\ L_{k\geq 2,j} = &\ \Gamma_{2,j}^T \frac{(A^{k-1})^T}{k!} (M_1 \Gamma_{1,j} - PBK) + (\Gamma_{1,j}^T M_1^T - K^T B^T P) \frac{A^{k-1}}{k!} \Gamma_{2,j} \\ & + \Gamma_{2,j}^T \left(\sum_{i=1}^{k-1} \frac{(A^{i-1})^T}{i!} M_1 \frac{A^{k-i-1}}{(k-i)!} \right) \Gamma_{2,j} + \tilde{L}_{k,j}. \end{aligned} \tag{C.18}$$

If $\lambda_{\max}(A + A^T) = 0$, the matrices $\tilde{L}_{k,j}$ are defined as

$$\begin{aligned} \tilde{L}_{0,j} &= W\mu\lambda_{\max}(E^T E) \left(j\tfrac{\bar{\sigma}}{l}\right)^2 I, \\ \tilde{L}_{1,j} &= 2W\mu\lambda_{\max}(E^T E) j\tfrac{\bar{\sigma}}{l} I, \\ \tilde{L}_{2,j} &= W\mu\lambda_{\max}(E^T E) I, \\ \tilde{L}_{k\geq 3,j} &= 0. \end{aligned} \tag{C.19}$$

Otherwise, if $\lambda_{\max}(A + A^T) \neq 0$, they are defined as

$$\begin{aligned} \tilde{L}_{0,j} = &\ W\mu \tfrac{\lambda_{\max}(E^T E)}{\lambda_{\max}(A+A^T)} j\tfrac{\bar{\sigma}}{l} \left(\mathrm{e}^{\lambda_{\max}(A+A^T) j \frac{\bar{\sigma}}{l}} - 1 \right) I, \\ \tilde{L}_{1,j} = &\ W\mu \tfrac{\lambda_{\max}(E^T E)}{\lambda_{\max}(A+A^T)} \left(\mathrm{e}^{\lambda_{\max}(A+A^T) j \frac{\bar{\sigma}}{l}} \left(1 + j\tfrac{\bar{\sigma}}{l} \lambda_{\max}(A + A^T)\right) - 1 \right) I, \\ \tilde{L}_{k\geq 2,j} = &\ W\mu \tfrac{\lambda_{\max}(E^T E)}{\lambda_{\max}(A+A^T)} \mathrm{e}^{\lambda_{\max}(A+A^T) j \frac{\bar{\sigma}}{l}} \left(j\tfrac{\bar{\sigma}}{l} \tfrac{(\lambda_{\max}(A+A^T))^k}{k!} + \tfrac{(\lambda_{\max}(A+A^T))^{k-1}}{(k-1)!} \right) I. \end{aligned} \tag{C.20}$$

Finally,

$$\nu \geq \max_{\substack{\sigma' \in [0, \frac{\bar{\sigma}}{l}], \\ r \in \{0, \cdots, l-1\}}} \lambda_{\max} \left(\Delta\left(\sigma' + r\frac{\bar{\sigma}}{l}\right) - \sum_{k=0}^{N} \tilde{\Delta}_{(k,r)} \sigma'^k \right). \tag{C.21}$$

Lemma C.3 *Consider a scalar $0 \leq \sigma^* \leq \bar{\sigma}$. The vertices $\bar{\Delta}_{(i,j)}(\sigma^*)$ defined in (C.14) satisfy the property (3.19): if the condition $\bar{\Delta}_{(i,j)}(\sigma^*) \preceq 0$ is satisfied for all $(i,j) \in \mathcal{K}(\sigma^*) = \{0, \cdots, N\} \times \left\{0, \cdots, \left\lfloor \frac{\sigma^* l}{\bar{\sigma}} \right\rfloor \right\}$, then $\Delta(\sigma) \preceq 0$ for all $\sigma \in [0, \sigma^*]$.*

Proof: The idea of the proof is similar to the one used in the construction of the convex polytopes in the unperturbed case, in the Appendix C.2. It follows the following steps:

1. First, we divide the time interval $[0, \bar{\sigma}]$ into l subdivisions and take a time $\sigma \leq \sigma^*$ into one of these subdivisions. The aim of this step is to make preparations to compute a precise estimation of the matrix function Δ by building l small convex embeddings around it instead of building one big one.

2. Then, we compute a polynomial approximation of Δ for the chosen time interval subdivision.

3. Afterwards, we bound the error term from this polynomial approximation with a constant term.

4. Finally, we build a convex polytope around the polynomial approximation and the error term bound, using the method proposed in [Hetel 2006] (see Appendx C.1), to obtain the desired finite number of conditions.

Step (1): Let us divide the time interval $[0, \bar{\sigma}]$ into l subdivisions $[j\frac{\bar{\sigma}}{l}, (j+1)\frac{\bar{\sigma}}{l}]$, with $j \in \{0, \cdots, l-1\}$. Let $\sigma \in [0, \sigma^*]$. There exists $j \in \{0, \cdots, \lfloor \frac{\sigma^* l}{\bar{\sigma}} \rfloor\}$ such that $j\frac{\bar{\sigma}}{l} \leq \sigma \leq (j+1)\frac{\bar{\sigma}}{l}$. Then define $\sigma' = \sigma - j\frac{\bar{\sigma}}{l}$ ($\sigma' \in [0, \chi]$, with $\chi = \frac{\bar{\sigma}}{l}$ if $j < \lfloor \frac{\sigma^* l}{\bar{\sigma}} \rfloor$, and $\chi = \sigma^* - \frac{j\bar{\sigma}}{l}$ otherwise).

Step (2): In this step, as in the unperturbed case, we want to compute the Taylor expansion of the matrix function $\Delta(\sigma)$ defined in (3.17) and (3.18) in order to design the convex polytope. Note that it is possible compute the Taylor approximation around the matrix function Δ bloc by bloc, by using the following property: for any C^∞ matrix functions of appropriate dimensions F, G, H and L, the Taylor expansion of the matrix function $\begin{bmatrix} F & G \\ H & L \end{bmatrix}$ can be written as

$$\begin{bmatrix} F(\sigma) & G(\sigma) \\ H(\sigma) & L(\sigma) \end{bmatrix} = \sum_{k=0}^{\infty} \begin{bmatrix} F_k(\sigma) & G_k(\sigma) \\ H_k(\sigma) & L_k(\sigma) \end{bmatrix} \sigma^k = \begin{bmatrix} \sum_{k=0}^{\infty} F_k \sigma^k & \sum_{k=0}^{\infty} G_k \sigma^k \\ \sum_{k=0}^{\infty} H_k \sigma^k & \sum_{k=0}^{\infty} L_k \sigma^k \end{bmatrix}. \tag{C.22}$$

Therefore, in order to compute the Taylor expansion of Δ defined in (3.17), one needs to compute the Taylor expansions of R defined in (3.18), as well as the ones of M_2^T and M_4^T, defined in (3.14).

All three functions involve the term $\Lambda(\sigma)$ defined in (3.8). As in the unperturbed case, we use the property expressed in Theorem D.7 to rewrite this term as a function of σ':

$$\begin{aligned}\Lambda(\sigma) &= I + \left(N_j + \int_0^{\sigma'} \mathrm{e}^{As} ds N'_j\right)(A - BK) \\ &= \Gamma_{1,j} + \int_0^{\sigma'} \mathrm{e}^{As} ds \Gamma_{2,j} \\ &= \Gamma_{1,j} + \sum_{i=1}^{\infty} \frac{A^{i-1}}{i!} \sigma'^i \Gamma_{2,j},\end{aligned} \tag{C.23}$$

with $N_j = \int_0^{j\frac{\bar{\sigma}}{l}} \mathrm{e}^{As} ds$, $N'_j = AM_j + I$, $\Gamma_{1,j} = I + N_j(A - BK)$, and $\Gamma_{2,j} = N'_j(A - BK)$.

Therefore, one has:

$$\begin{aligned}M_2(\sigma)^T &= -K^T B^T P + \Lambda(\sigma)^T M_1 \\ &= -K^T B^T P + \Gamma_{1,j}^T M_1 + \sum_{i=1}^{\infty} \Gamma_{2,j}^T \frac{(A^{i-1})^T}{i!} M_1 \sigma'^i,\end{aligned} \tag{C.24}$$

$$\begin{aligned}M_4(\sigma)^T &= \Lambda(\sigma)^T PE \\ &= \Gamma_{1,j}^T PE + \sum_{i=1}^{\infty} \Gamma_{2,j}^T \frac{(A^{i-1})^T}{i!} PE \sigma'^i,\end{aligned} \tag{C.25}$$

and

$$\begin{aligned}R(\sigma) &= \Lambda(\sigma)^T M_1 \Lambda(\sigma) - \Lambda(\sigma)^T PBK - K^T B^T P \Lambda(\sigma) - \varepsilon P \\ &\quad + W\eta I + \sigma W \mu \lambda_{\max}(E^T E) f_A(\sigma) I \\ &= \sum_{k=0}^{\infty} L_{k,j} \sigma'^k,\end{aligned} \tag{C.26}$$

with the matrices $L_{k,j}$ defined as

$$L_{0,j} = \Gamma_{1,j}^T M_1 \Gamma_{1,j} - \varepsilon P + W\eta I - \Gamma_{1,j}^T PBK - K^T B^T P \Gamma_{1,j} + \tilde{L}_{0,j}, \tag{C.27}$$

$$L_{1,j} = \Gamma_{2,j}^T (M_1 \Gamma_{1,j} - PBK) + (\Gamma_{1,j}^T M_1^T - K^T B^T P) \Gamma_{2,j} + \tilde{L}_{1,j}, \tag{C.28}$$

and

$$\begin{aligned}L_{k\geq 2,j} &= \Gamma_{2,j}^T \frac{(A^{k-1})^T}{k!} (M_1 \Gamma_{1,j} - PBK) + (\Gamma_{1,j}^T M_1^T - K^T B^T P) \frac{A^{k-1}}{k!} \Gamma_{2,j} \\ &\quad + \Gamma_{2,j}^T \left(\sum_{i=1}^{k-1} \frac{(A^{i-1})^T}{i!} M_1 \frac{A^{k-i-1}}{(k-i)!}\right) \Gamma_{2,j} + \tilde{L}_{k,j}.\end{aligned} \tag{C.29}$$

The matrices $\tilde{L}_{k,j}$ that appear in the previous equations come from the Taylor expansion of the term $\sigma W \mu \lambda_{\max}(E^T E) f_A(\sigma) I$. Two cases may occur.

In the first case, $\lambda_{\max}(A + A^T) = 0$, and thus

$$\begin{aligned}\sigma W \mu \lambda_{\max}(E^T E) f_A(\sigma) I &= W \mu \lambda_{\max}(E^T E) I \left(j\frac{\bar{\sigma}}{l} + \sigma'\right)^2 \\ &= W \mu \lambda_{\max}(E^T E) I \left(j\frac{\bar{\sigma}}{l}\right)^2 + 2W \mu \lambda_{\max}(E^T E) I j \frac{\bar{\sigma}}{l} \sigma' \\ &\quad + W \mu \lambda_{\max}(E^T E) I \sigma'^2.\end{aligned} \tag{C.30}$$

Therefore, one has

$$\begin{aligned}
\tilde{L}_{0,j} &= W\mu\lambda_{\max}(E^TE)I\left(j\tfrac{\bar{\sigma}}{l}\right)^2,\\
\tilde{L}_{1,j} &= 2W\mu\lambda_{\max}(E^TE)Ij\tfrac{\bar{\sigma}}{l},\\
\tilde{L}_{2,j} &= W\mu\lambda_{\max}(E^TE)I,\\
\tilde{L}_{k\geq 3,j} &= 0.
\end{aligned} \tag{C.31}$$

In the second case, $\lambda_{\max}(A+A^T)\neq 0$, and thus

$$\begin{aligned}
&\sigma W\mu\lambda_{\max}(E^TE)f_A(\sigma)I\\
&\quad= W\mu\tfrac{\lambda_{\max}(E^TE)}{\lambda_{\max}(A+A^T)}I\left(j\tfrac{\bar{\sigma}}{l}+\sigma'\right)\left(\mathrm{e}^{\lambda_{\max}(A+A^T)j\frac{\bar{\sigma}}{l}}\mathrm{e}^{\lambda_{\max}(A+A^T)\sigma'}-1\right)\\
&\quad= W\mu\tfrac{\lambda_{\max}(E^TE)}{\lambda_{\max}(A+A^T)}I\left[j\tfrac{\bar{\sigma}}{l}\left(\mathrm{e}^{\lambda_{\max}(A+A^T)j\frac{\bar{\sigma}}{l}}-1\right)\right.\\
&\quad+\left(\mathrm{e}^{\lambda_{\max}(A+A^T)j\frac{\bar{\sigma}}{l}}\left(1+j\tfrac{\bar{\sigma}}{l}\lambda_{\max}(A+A^T)\right)-1\right)\sigma'\\
&\quad\left.+\textstyle\sum_{k=2}^{\infty}\mathrm{e}^{\lambda_{\max}(A+A^T)j\frac{\bar{\sigma}}{l}}\left(j\tfrac{\bar{\sigma}}{l}\tfrac{(\lambda_{\max}(A+A^T))^k}{k!}+\tfrac{(\lambda_{\max}(A+A^T))^{k-1}}{(k-1)!}\right)\sigma'^k\right].
\end{aligned} \tag{C.32}$$

Therefore, one has

$$\begin{aligned}
\tilde{L}_{0,j} &= W\mu\tfrac{\lambda_{\max}(E^TE)}{\lambda_{\max}(A+A^T)}Ij\tfrac{\bar{\sigma}}{l}\left(\mathrm{e}^{\lambda_{\max}(A+A^T)j\frac{\bar{\sigma}}{l}}-1\right),\\
\tilde{L}_{1,j} &= W\mu\tfrac{\lambda_{\max}(E^TE)}{\lambda_{\max}(A+A^T)}I\left(\mathrm{e}^{\lambda_{\max}(A+A^T)j\frac{\bar{\sigma}}{l}}\left(1+j\tfrac{\bar{\sigma}}{l}\lambda_{\max}(A+A^T)\right)-1\right),\\
\tilde{L}_{k\geq 2,j} &= W\mu\tfrac{\lambda_{\max}(E^TE)}{\lambda_{\max}(A+A^T)}I\mathrm{e}^{\lambda_{\max}(A+A^T)j\frac{\bar{\sigma}}{l}}\left(j\tfrac{\bar{\sigma}}{l}\tfrac{(\lambda_{\max}(A+A^T))^k}{k!}+\tfrac{(\lambda_{\max}(A+A^T))^{k-1}}{(k-1)!}\right).
\end{aligned} \tag{C.33}$$

Using the obtained equations, one can write that $\Delta(\sigma)=\sum_{k=0}^{\infty}\tilde{\Delta}_{(k,j)}\sigma'^k$, with the matrices $\tilde{\Delta}_{(k,j)}$ defined in (C.16).

With this, a polynomial approximation of order N of Δ on the interval $[j\frac{\bar{\sigma}}{l},(j+1)\frac{\bar{\sigma}}{l}]$ can be expressed as

$$\Xi_{(N,j)}(\sigma')=\sum_{k=0}^{N}\tilde{\Delta}_{(k,j)}\sigma'^k,\ \forall\sigma'\in\left[0,\frac{\bar{\sigma}}{l}\right]. \tag{C.34}$$

Step (3): The approximation error term $R_{(N,j)}(\sigma')=\Delta(\sigma)-\Xi_{(N,j)}(\sigma')$ can be bounded using the relation $R_{(N,j)}(\sigma')\preceq\nu I$, with ν a constant scalar defined in (C.21). With this majoration, it is clear that if $\Xi_{(N,j)}(\sigma')+\nu I\preceq 0$, then $\Delta(\sigma)\preceq 0$.

Step (4): Since the function $\Xi_{(N,j)}(.)+\nu I:[0,\chi]\rightarrow\mathcal{M}_n(\mathbb{R})$ is polynomial, we can use the convex embedding design from [Hetel 2006] (see Appendx C.1), to prove that if $\bar{\Delta}_{(i,j)}(\sigma^*)\preceq 0$ for all $i\in\{1,\cdots,n\}$, with $\bar{\Delta}_{(i,j)}(\sigma^*)=\left(\sum_{k=0}^{i}\tilde{\Delta}_{(k,j)}\chi^k\right)+\nu I$, then $\Xi_{(N,j)}(\sigma')+\nu I\preceq 0$, and therefore $\Delta(\sigma)\preceq 0$. ∎

Appendix D

Some useful matrix properties

Theorem D.1 (Schur complement [Boyd 1994]) *Let Q and R be symmetric matrices. Then, the following are equivalent:*

(i) $\begin{bmatrix} Q & S \\ * & R \end{bmatrix} \succ 0$ *(resp.* $\begin{bmatrix} Q & S \\ * & R \end{bmatrix} \succeq 0$*),*

(ii) $R \succ 0,\ Q - SR^{-1}S^T \succ 0$ *(resp.* $R \succeq 0,\ Q - SR^{+}S^T \succeq 0,\ S(I - RR^{+}) = 0$*),*

where R^+ is the pseudo-inverse of R.

Theorem D.2 (Finsler's Lemma [Fang 2004]) *Let $x \in \mathbb{R}^n$, $Q \in S_n(\mathbb{R})$, and $B \in \mathcal{M}_{n,m}(\mathbb{R})$ such that rank$(B) < n$. The following statements are equivalent.*

(i) $x^TQx < 0$ *(resp.* $x^TQx \leq 0$*) for all* $Bx = 0$, $x \neq 0$,

(ii) ${B^{\perp}}^T Q B^{\perp} \prec 0$ *(resp.* ${B^{\perp}}^T Q B^{\perp} \preceq 0$*),*

(iii) there exists a scalar $\mu \in \mathbb{R}$ such that $Q - \mu B^T B \prec 0$ (resp. $Q - \mu B^T B \preceq 0$),

(iv) there exists a matrix $\mathcal{X} \in \mathcal{M}_{n,m}(\mathbb{R})$ such that $Q + \mathcal{X}B + B^T\mathcal{X}^T \prec 0$ (resp. $Q + \mathcal{X}B + B^T\mathcal{X}^T \preceq 0$),

where $B^{\perp}$ is a basis for the null space of B (i.e. all $x \neq 0$ such that $Bx = 0$ is generated by some $z \neq 0$ in the form $x = B^{\perp}z$).

Theorem D.3 (S-procedure [Yakubovich 1977], [Boyd 1994]) *Let $F_i \in \mathcal{M}_n(\mathbb{R})$, $i \in \{0, \cdots, p\}$. Then, if* *(i) there exist scalars* $\varepsilon_i \geq 0$, $i \in \{1, \cdots, p\}$, *such that* $F_0 - \sum_{i=1}^{p} \varepsilon_i F_i > 0$ *(resp.* $F_0 - \sum_{i=1}^{p} \varepsilon_i F_i \geq 0$*),*

then

(ii) $\xi^T F_0 \xi > 0$ *(resp.* $\xi^T F_0 \xi \geq 0$*) for any* $\xi \in \mathbb{R}^n$ *satisfying* $\xi^T F_i \xi \geq 0$ *for all* $i \in \{1, \cdots, p\}$.

For $p = 1$, *these two statements are equivalent.*

Theorem D.4 (Jensen's Inequality [Gu 2003]) *For any matrix* $R \in S_n^{+*}$, *scalar* $r > 0$ *and vector function* $\omega : [0, r] \to \mathbb{R}^n$ *such that the concerned inequalities are well defined, one has*

$$\left(\int_0^r \omega(s)ds\right)^T R \left(\int_0^r \omega(s)ds\right) \leq r \left(\int_0^r \omega(s)^T R \omega(s)ds\right). \tag{D.1}$$

Theorem D.5 ([Cao 1998]) *For any matrix* $R \in S_n^{+*}$ *and any scalars* $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$,

$$x^T y + y^T x \leq x^T R^{-1} x + y^T R y. \tag{D.2}$$

Theorem D.6 ([Loan 1977]) *For any matrix* $R \in \mathcal{M}_n(\mathbb{R})$ *and scalar* $t \geq 0$, *one has*

$$|||e^{Rt}|||_2 \leq e^{\lambda_{max}\left(\frac{(R+R^T)}{2}\right)t}. \tag{D.3}$$

Theorem D.7 ([Fujioka 2008]) *Consider scalars* a, $b \in \mathbb{R}^n$, *and a matrix* $A \in \mathcal{M}_n(\mathbb{R})$. *Then, the following equality holds:*

$$\int_0^{a+b} e^{As}ds = \int_0^a e^{As}ds + \int_0^b e^{As}ds \left(A \int_0^a e^{As}ds + I\right). \tag{D.4}$$

Theorem D.8 (Adapted from [Boyd 1994]) *Consider* $x \in \mathbb{R}^n$, *two matrices* Γ_1 *and* Γ_2 *in* S_n *and two scalars* $\lambda^- < \lambda^+$. *The following statements are equivalent:*
(i) $\forall \lambda \in [\lambda^-, \lambda^+], x^T(\Gamma_1 + \lambda\Gamma_2)x \leq 0$,
(ii) $x^T(\Gamma_1 + \lambda^-\Gamma_2)x \leq 0$ *and* $x^T(\Gamma_1 + \lambda^+\Gamma_2)x \leq 0$.

Proof: Let $x \in \mathbb{R}^n$ and $\lambda \in [\lambda^-, \lambda^+]$. Remarking that $\Gamma_1 + \lambda\Gamma_2 = \frac{\lambda^+ - \lambda}{\lambda^+ - \lambda^-}(\Gamma_1 + \lambda^-\Gamma_2) + \frac{\lambda - \lambda^-}{\lambda^+ - \lambda^-}(\Gamma_1 + \lambda^+\Gamma_2)$ achieves the proof since $\frac{\lambda^+ - \lambda}{\lambda^+ - \lambda^-}$ and $\frac{\lambda - \lambda^-}{\lambda^+ - \lambda^-}$ are positive. ∎

Bibliography

[Albert 2004] A. Albert. *Comparison of event-triggered and time-triggered concepts with regard to distributed control systems.* Embedded World, vol. 55, no. 9, pages 235–252, 2004.

[Anta 2009] A. Anta and P. Tabuada. *Isochronous manifolds in self-triggered control.* In 48th IEEE Conference on Decision and Control, pages 3194–3199, Shanghai, China, 2009.

[Anta 2010] A. Anta and P. Tabuada. *To sample or not to sample: self-triggered control for nonlinear systems.* IEEE Transactions on Automatic Control, vol. 55, no. 9, pages 2030–2042, 2010.

[Anta 2012] A. Anta and P. Tabuada. *Exploiting isochrony in self-triggered control.* IEEE Transactions on Automatic Control, vol. 57, no. 4, pages 950–962, 2012.

[Araujo 2011] J. Araujo, A. Anta, M. Mazo Jr., J. Faria, A. Hernandez, P. Tabuada and K.-H. Johansson. *Self-triggered control over wireless sensor and actuator networks.* In International Conference on Distributed Computing in Sensor Systems, Barcelona, Spain, 2011.

[Aström 1996] K.-J. Aström and B. Wittenmark. Computer controlled systems: theory and design. Prentice Hall, 1996.

[Beker 2004] O. Beker, C. Hollot, Y. Chait and H. Han. *Fundamental properties of reset control systems.* Automatica, vol. 40, no. 6, pages 905–915, 2004.

[Blanchini 1999] F. Blanchini. *Set invariance in control.* Automatica, vol. 35, no. 11, pages 1747–1767, 1999.

[Boyd 1994] S. Boyd, L. El-Ghaoui, E. Feron and V. Balakrishnan. Linear matrix inequalities in system and control theory, volume 15. SIAM, Studies in Applied Mathematics, 1994.

[Briat 2010] C. Briat, H. Hjalmarsson, K.-H. Johansson, U.-T. Jönsson, G. Karlsson and H. Sandberg. *Nonlinear state-dependent delay modeling and stability analysis of internet congestion control.* In 49th IEEE Conference on Decision and Control, pages 1484–1491, Atlanta, Georgia, USA, 2010.

[Brockett 2000] R. Brockett and D. Liberzon. *Quantized feedback stabilization of linear systems.* IEEE Transactions on Automatic Control, vol. 45, no. 7, pages 1279–1289, 2000.

[Bushnell 2001] L. Bushnell. *Networks and control.* IEEE Transactions on Control Systems Magazine, vol. 22, no. 1, pages 22–29, 2001.

[Buttazzo 2002] G. Buttazzo and L. Abeni. *Adaptive workload management through elastic scheduling.* Real-Time Systems, vol. 23, no. 1, pages 7–24, 2002.

[Cao 1998] Y. Cao, Y. Sun and C. Cheng. *Delay dependent robust stabilization of uncertain systems with multiple state delays.* IEEE Transactions on Automatic Control, vol. 43, no. 11, pages 1608–1612, 1998.

[Cervin 2002] A. Cervin, J. Eker, B. Bernhardsson and K.-E. Arzen. *Feedback-feedforward scheduling of control tasks.* Real-Time Systems, vol. 23, no. 1, pages 25–53, 2002.

[Cervin 2007] A. Cervin and K.-J. Aström. *On limit cycles in event-based control systems.* In 46th IEEE Conference on Decision and Control, pages 3190–3195, New Orleans, Louisiana, USA, 2007.

[Chen 1991] T. Chen and B. Francis. Optimal sampled-data control systems. Springer, 1991.

[Chen 2011] J. Chen, K.-H. Johansson, S. Olariu, I.-C. Paschalidis and I. Stojmenovic. *Guest editorial special issue on wireless sensor and actuator networks.* IEEE Transactions on Automatic Control, vol. 56, no. 10, pages 2244–2246, 2011.

[Cloosterman 2010] M.B.G. Cloosterman, L. Hetel, N. van de Wouw, W.P.M.H. Heemels, J. Daafouz and H. Nijmeijer. *Controller synthesis for networked control systems.* Automatica, vol. 46, no. 10, pages 1584–1594, 2010.

[Cogill 2007] R. Cogill, S. Lall and J. Hespanha. *A constant factor approximation algorithm for event-based sampling.* In American Control Conference, pages 305–311, New York City, New York, USA, 2007.

[Dambrine 1994] M. Dambrine. *Contribution à l'étude de la stabilité des systèmes à retard.* PhD thesis, Université de Lille, Villeneuve d'Ascq, France, 1994.

[de Wouw 2010] N. Van de Wouw, P. Naghshtabrizi, M. Cloosterman and J. Hespanha. *Tracking control for sampled-data systems with uncertain time-varying sampling intervals and delays.* International Journal of Robust and Nonlinear Control, vol. 20, no. 4, pages 1680–1685, 2010.

[der Schaft 2000] A. Van der Schaft and H. Schumacher. An introduction to hybrid dynamical systems. Springer, 2000.

[Dimarogonas 2010] D. Dimarogonas, E. Frazzoli and K.-H. Johansson. *Distributed self-triggered control for multi-agent systems.* In 49th IEEE Conference on Decision and Control, pages 6716–6721, Atlanta, Georgia, USA, 2010.

[Donkers 2009] M.C.F. Donkers, L. Hetel, W.P.M.H. Heemels, N. van de Wouw and M. Steinbuch. *Stability analysis of Networked Control Systems using a switched linear systems approach.* In 12th International Conference on Hybrid Systems: Computation and Control, pages 150–164, San Francisco, California, USA, 2009.

[Driver 1977] D.-R. Driver. Ordinary and delay differential equations. Springer-Verlag, 1977.

[Fang 2004] L. Fang, H. Lin and P.-J. Antsaklis. *Stabilization and performance analysis for a class of switched systems.* In 43rd IEEE Conference on Decision and Control, Atlantis, Paradise Island, Bahamas, 2004.

[Felicioni 2006] F. Felicioni, N. Jia, Y.-Q. Song and F. Simonot-Lion. *Impact of a (m, k)-firm data dropouts policy on the quality of control.* In 6th IEEE International Workshop on Factory Communication Systems, Torino, Italy, 2006.

[Felicioni 2008] F. Felicioni, N. Jia, F. Simonot-Lion and Y.-Q. Song. *Optimal on-line (m, k)-firm constraint assignment for real-time control tasks based on plant state information.* In 13th IEEE International Conference on Emerging Technologies and Factory Automation, Hamburg, Germany, 2008.

[Fiter 2011a] C. Fiter. *Échantillonnage dépendant de l'état: une approche par cartographie basée sur des LMIs.* In 4th Journées Doctorales MACS, Marseille, France, 2011.

[Fiter 2011b] C. Fiter, L. Hetel, W. Perruquetti and J.-P. Richard. *Échantillonnage dépendant de l'état pour les systèmes avec perturbations et retards.* In 8th Colloque Francophone sur la Modélisation des Systèmes Réactifs, volume 45, pages 189–203, Villeneuve d'Ascq, France, 2011. Journal Européen des Systèmes Automatisés.

[Fiter 2011c] C. Fiter, L. Hetel, W. Perruquetti and J.-P. Richard. *State dependent sampling: an LMI based mapping approach.* In 18th IFAC World Congress, Milan, Italy, 2011.

[Fiter 2012a] C. Fiter, L. Hetel, W. Perruquetti and J.-P. Richard. *A state dependent sampling for linear state feedback.* Automatica, vol. 48, no. 8, pages 1860–1867, 2012.

[Fiter 2012b] C. Fiter, L. Hetel, W. Perruquetti and J.-P. Richard. *State-dependent sampling for perturbed time-delay systems.* In 51st IEEE Conference on Decision and Control, Maui, Hawaii, USA, 2012.

[Fiter xa] C. Fiter, L. Hetel, W. Perruquetti and J.-P. Richard. *A novel stabilization approach for state-dependent sampling.* International Journal of Control, provisionally accepted

[Fiter xb] C. Fiter, L. Hetel, W. Perruquetti and J.-P. Richard. *A robust polytopic approach for state-dependent sampling.* In 12th European Control Conference, Zurich, Switzerland, submitted

[Fiter xc] C. Fiter, L. Hetel, W. Perruquetti and J.-P. Richard. *A robust stability framework for time-varying sampling.* Automatica, submitted

[Frazzoli 2012] E. Frazzoli and K.-H. Johansson. *Distributed event-triggered control for multi-agent systems.* IEEE Transactions on Automatic Control, vol. 57, no. 5, pages 1291–1297, 2012.

[Fridman 2000] E. Fridman and U. Shaked. *Stability and guaranteed cost control of uncertain discrete delay systems.* International Journal of Control, vol. 45, no. 12, pages 2406–2411, 2000.

[Fridman 2001a] E. Fridman. *New Lyapunov-Krasovskii functionals for stability of linear retarded and neutral type systems.* Systems and Control Letters, vol. 43, no. 4, pages 309–319, 2001.

[Fridman 2001b] E. Fridman and U Shaked. *New bounded real lemma representations for time-delay systems and their applications.* IEEE Transactions on Automatic Control, vol. 46, no. 12, pages 1973–1979, 2001.

[Fridman 2003] E. Fridman and U. Shaked. *Special issue on time-delay systems.* International Journal of Robust and Nonlinear Control, vol. 13, pages 791–792, 2003.

[Fridman 2004] E. Fridman, A. Seuret and J.-P Richard. *Robust sampled-data stabilization of linear systems: An input delay approach.* Automatica, vol. 40, no. 8, pages 1441–1446, 2004.

[Fridman 2006] E. Fridman. *Stability of systems with uncertain delays: a new "complete" Lyapunov-Krasovskii functional.* IEEE Transactions on Automatic Control, vol. 51, no. 5, pages 885–890, 2006.

[Fridman 2010] E. Fridman. *A refined input delay approach to sampled-data control.* Automatica, vol. 46, no. 2, pages 421–427, 2010.

[Fujioka 2008] H. Fujioka. *Stability analysis for a class of networked/embedded control systems: a discrete-time approach.* In American Control Conference, Seattle, Washington, USA, 2008.

[Fujioka 2009a] H. Fujioka. *A discrete-time approach to stability analysis of systems with aperiodic sample-and-hold devices.* IEEE Transactions on Automatic Control, vol. 54, no. 10, pages 2440–2445, 2009.

[Fujioka 2009b] H. Fujioka. *Stability analysis of systems with aperiodic sample-and-hold devices.* Automatica, vol. 45, no. 3, pages 771–775, 2009.

[Gielen 2010] R. Gielen, S. Olaru, M. Lazar, W. Heemels, N. Van de Wouw and S.-I. Niculescu. *On polytopic inclusions as a modeling framework for systems with time-varying delays.* Automatica, vol. 46, no. 3, pages 615–619, 2010.

[Goebel 2009] R. Goebel, R.-G. Sanfelice and A.-R. Teel. *Hybrid dynamical systems.* IEEE Control Systems Magazine, vol. 29, no. 2, pages 28–93, 2009.

[Gu 1997] K. Gu. *Discretized LMI set in the stability problem of linear uncertain time-delay systems.* International Journal of Control, vol. 68, no. 4, pages 923–934, 1997.

[Gu 2003] K. Gu, V. Kharitonov and J. Chen. Stability of time-delay systems. Boston: Birkhauser, 2003.

[Heemels 2008] W.P.M.H Heemels, J.-H. Sandee and P.P.J. Van Bosch. *Analysis of event-driven controllers for linear systems.* International Journal of Control, vol. 81, no. 4, pages 571–590, 2008.

[Heemels 2010] W.P.M.H. Heemels, N. van de Wouw, R.H. Gielen, M.C.F. Donkers, L. Hetel, S. Olaru, M. Lazar, J. Daafouz and S. Niculescu. *Comparison of overapproximation methods for stability analysis of networked control systems.* In 13th International Conference on Hybrid Systems: Computation and Control, New York City, New York, USA, 2010.

[Hespanha 2007] J.-P. Hespanha, P. Naghshtabrizi and Y. Xu. *A survey of recent results in networked control systems.* IEEE Special Issue on Technology of Networked Control Systems, vol. 95, no. 1, pages 138–162, 2007.

[Hetel 2006] L. Hetel, J. Daafouz and C. Iung. *Stabilization of Arbitrary Switched Linear Systems With Unknown Time-Varying Delays.* IEEE Transactions on Automatic Control, vol. 51, no. 10, pages 1668–1674, 2006.

[Hetel 2007] L. Hetel, J. Daafouz and C. Iung. *LMI control design for a class of exponential uncertain systems with application to network controlled switched systems.* In American Control Conference, pages 1401–1406, New York City, New York, USA, 2007.

[Hetel 2011a] L. Hetel, J. Daafouz, S. Tabouriech and C. Prieur. *Reset control systems - stabilization by nearly-periodic reset.* In 18th IFAC World Congress, Milan, Italy, 2011.

[Hetel 2011b] L. Hetel, A. Kruszewski, W. Perruquetti and J.-P Richard. *Discrete and intersample analysis of systems with aperiodic sampling.* IEEE Transactions on Automatic Control, vol. 56, no. 7, pages 1696–1701, 2011.

[Hristu-Varsakelis 2005] D. Hristu-Varsakelis and W. Levine (Editors). Handbook of networked and embedded control systems. Boston: Birkhauser, 2005.

[Jankovic 2001] M. Jankovic. *Control Lyapunov-Razumikhin functions and robust stabilization of time delay systems.* IEEE Transactions on Automatic Control, vol. 46, no. 7, pages 1048–1060, 2001.

[Jiang 2010a] W. Jiang, A. Kruszewski, E. Fridman and J.-P Richard. *Delay dependent stability analysis of interval time-delay systems.* In 9th IFAC Workshop on Time Delay Systems, Prague, Czech Republic, 2010.

[Jiang 2010b] W. Jiang and A. Seuret. *Improved stability analysis of networked control systems under asynchronous sampling and input delay.* In 2nd IFAC Workshop on Distributed Estimation and Control in Networked Systems, Annecy, France, 2010.

[Jiao 2005] X. Jiao. *Adaptive feedback control of nonlinear time-delay systems: the LaSalle-Razumikhin-based approach.* IEEE Transactions on Automatic Control, vol. 50, no. 11, pages 1909–1913, 2005.

[Khalil 2002] H. Khalil. Nonlinear systems. Prentice Hall, 2002.

[Kharitonov 2003] V.-L. Kharitonov and A.-P. Zhabko. *Lyapunov-Krasovskii approach to the robust stability analysis of time-delay systems.* Automatica, vol. 39, no. 1, pages 15–20, 2003.

[Kolmanovskii 1992] V. Kolmanovskii and A. Myshkis. Applied theory of functional differential equations. Springer, 1992.

[Kolmanovskii 1996] V. Kolmanovskii and L.-E. Shaikhet. Control of systems with aftereffect, volume 157 of *Translations of mathematical monographs.* American Mathematical Society, 1996.

[Kolmanovskii 1999] V. Kolmanovskii and A. Myshkis. Introduction to the theory and applications of functional differential equations. Springer, 1999.

[Lehmann 2011] D. Lehmann and J. Lunze. *Event-based control with communication delays.* In 18th IFAC World Congress, Milan, Italy, 2011.

[Lemmon 2007] M.-D. Lemmon, T. Chantem, X.-S. Hu and M. Zyskowski. *On self-triggered full-information H-infinity controllers.* In 10th International Conference on Hybrid Systems: Computation and Control, Pisa, Italy, 2007.

[Li 2010] X.-G. Li, A. Cela, S. Niculescu and A. Reama. *Some problems in the stability of networked-control systems with periodic scheduling.* International Journal of Control, vol. 83, no. 5, pages 996–1008, 2010.

[Liberzon 1999] D. Liberzon, J.-P. Hespanha and A.-S. Morse. *Stability of switched systems : a lie-algebraic condition.* Systems and Control Letters, vol. 37, no. 3, pages 117–122, 1999.

[Loan 1977] C.-V Loan. *The sensitivity of the matrix exponential.* SIAM Journal on Numerical Analysis, vol. 14, no. 6, pages 971–981, 1977.

[Lunze 2010] J. Lunze and D. Lehmann. *A state-feedback approach to event-based control.* Automatica, vol. 46, no. 1, pages 211–215, 2010.

[Maalej 2012a] S. Maalej, C. Fiter, L. Hetel and J.-P. Richard. *Echantillonnage dépendant de l'état: une approche discrète.* In 7th IEEE Conférence Internationale Francophone d'Automatique, Grenoble, France, 2012.

[Maalej 2012b] S. Maalej, C. Fiter, L. Hetel and J.-P. Richard. *State-dependent sampling for linear time invariant systems: a discrete time analysis.* In 20th IEEE Mediterranean Conference on Control and Automation, Barcelona, Spain, 2012.

[Mancilla-Aguilar 2005] J.-L. Mancilla-Aguilar, R. Garcia, E. Sontag and Y. Wang. *On the representation of switched systems with inputs by perturbed control systems.* Nonlinear Analysis: Theory, Methods and Applications, vol. 60, no. 6, pages 1111–1150, 2005.

[Mazo Jr. 2009a] M. Mazo Jr., A. Anta and P. Tabuada. *On self-triggered control for linear systems: guarantees and complexity.* In European Control Conference, Budapest, Hungary, 2009.

[Mazo Jr. 2009b] M. Mazo Jr. and P. Tabuada. *Input-to-state stability of self-triggered control systems.* In 48th IEEE Conference on Decision and Control, pages 928–933, Shanghai, China, 2009.

[Mazo Jr. 2010] M. Mazo Jr., A. Anta and P. Tabuada. *An ISS Self-Triggered Implementation of Linear Controllers.* Automatica, vol. 46, no. 8, pages 1310–1314, 2010.

[Mazo Jr. 2011] M. Mazo Jr. and P. Tabuada. *Decentralized event-triggered control over wireless sensor/actuator networks.* IEEE Transactions on Automatic Control, vol. 56, no. 10, pages 2456–2461, 2011.

[Megretski 1997] A. Megretski and A. Rantzer. *System analysis via integral quadratic constraints.* IEEE Transactions on Automatic Control, vol. 42, no. 6, pages 819–830, 1997.

[Mikheev 1988] Y. Mikheev, V. Sobolev and E. Fridman. *Asymptotic analysis of digital control systems.* Automation and Remote Control, vol. 49, no. 9, pages 1175–1180, 1988.

[Mirkin 2007] L. Mirkin. *Some remarks on the use of time-varying delay to model sample-and-hold circuits.* IEEE Transactions on Automatic Control, vol. 52, no. 6, pages 1109–1112, 2007.

[Mounier 2003a] H. Mounier and J.-P. Richard. *Special issue on time-delay systems for communication networks.* International Journal of Systems Science, vol. 34, no. 10–11, pages 561–561, 2003.

[Mounier 2003b] H. Mounier and J. Rudolph. *Time delay systems.* Encyclopaedia of Life and Support Systems, 6.43.19.4, 2003.

[Naghshtabrizi 2008] P. Naghshtabrizi, J.-P. Hespanha and A.-R. Teel. *Exponential stability of impulsive systems with application to uncertain sampled-data systems.* Systems and Control Letters, vol. 57, no. 5, pages 378–385, 2008.

[Nair 2000] G.-N. Nair and R.-J. Evans. *Stabilization with data-rate-limited feedback: Tightest attainable bounds.* Systems and Control Letters, vol. 41, no. 1, pages 49–56, 2000.

[Nesic 2008] D. Nesic, L. Zaccarian and A. Teel. *Stability properties of reset systems.* Automatica, vol. 44, no. 8, pages 2019–2026, 2008.

[Niculescu 2001] S.-I Niculescu. Delay effects on stability - a robust control approach, volume 269 of *Lecture Notes in Control and Information Sciences.* Springer, 2001.

[Olaru 2008] S. Olaru and S. Niculescu. *Predictive control for linear systems with delayed input subject to constraints.* In IFAC World Congress, Seoul, Korea, 2008.

[Otanez 2002] P.-G. Otanez, J.-R. Moyne and D.-M. Tilbury. *Using deadbands to reduce communication in networked control systems.* In American Control Conference, pages 3015–3020, Anchorage, Alaska, USA, 2002.

[Park 2007] P.-G. Park and J.-N. Ko. *Stability and robust stability for systems with a time-varying delay.* Automatica, vol. 43, no. 10, pages 1855–1858, 2007.

[Prieur 2011] C. Prieur and S. Tarbouriech. *New directions on hybrid control systems.* International Journal of Robust and Nonlinear Control - Special Issue, vol. 21, no. 10, pages 1063–1236, 2011.

[Rantzer 1996] A. Rantzer. *On the Kalman-Yakubovich-Popov lemma.* Systems and Control Letters, vol. 28, no. 1, pages 7–10, 1996.

[Richard 2003] J.-P Richard. *Time delay systems: an overview of some recent advances and open problems.* Automatica, vol. 39, no. 10, pages 1667–1694, 2003.

[Richard 2007] J.-P. Richard and T. Divoux. Systèmes commandés en réseau. IC2 Systèmes Automatisés. Hermès-Lavoisier, 2007.

[Seuret 2009] A. Seuret. *Stability analysis for sampled-data systems with a time-varying period.* In 48th IEEE Conference on Decision and Control, pages 8130–8135, Shanghai, China, 2009.

[Seuret 2012] A. Seuret. *A novel stability analysis of linear systems under asynchronous samplings.* Automatica, vol. 48, no. 1, pages 177–182, 2012.

[Simon 2012] D. Simon, A. Seuret and O. Sename. *On real-time feedback control systems: requirements, achievements and perspectives.* In 1st International Conference on Systems and Computer Science, Villeneuve d'Ascq, France, 2012.

[Skaf 2009] J. Skaf and S. Boyd. *Analysis and synthesis of state-feedback controllers with timing jitter.* IEEE Transactions on Automatic Control, vol. 54, no. 3, pages 652–657, 2009.

[Sontag 2004] E.-D. Sontag. *Input to state stability: basic concepts and results.* In CIME Summer Course on Nonlinear and Optimal Control Theory, pages 462–488, Cetraro, Italy, 2004. Springer.

[Stamova 2001] I.-M. Stamova and G.-T. Stamov. *Lyapunov-Razumikhin method for impulsive functional differential equations and applications to the population dynamics.* Journal of Computational and Applied Mathematics, vol. 130, no. 1–2, pages 163–171, 2001.

[Stöcker 2011] C. Stöcker and J. Lunze. *Event-based control of input-output linearizable systems.* In 18th IFAC World Congress, Milan, Italy, 2011.

[Tabuada 2007] P. Tabuada. *Event-triggered real-time scheduling of stabilizing control tasks.* IEEE Transactions on Automatic Control, vol. 52, no. 9, pages 1680–1685, 2007.

[Tarbouriech 2011] S. Tarbouriech, T. Loquen and C. Prieur. *Anti-windup strategy for reset control systems.* International Journal of Robust and Nonlinear Control, vol. 21, no. 10, pages 1159–1177, 2011.

[Tiberi 2010] U. Tiberi, C. Fischione, K.-H. Johansson and M.-D. Di Benedetto. *Adaptive self-triggered control over IEEE 802.15.4 networks.* In 49th IEEE Conference on Decision and Control, pages 2099–2104, Atlanta, Georgia, USA, 2010.

[Velasco 2009] M. Velasco, P. Marti and E. Bini. *On Lyapunov sampling for event-driven controllers.* In 48th IEEE Conference on Decision and Control, pages 6238–6243, Shanghai, China, 2009.

[Wang 2007] Q. Wang and X. Liu. *Impulsive stabilization of delay differential systems via the Lyapunov-Razumikhin method.* Applied Mathematics Letters, vol. 20, no. 8, pages 839–845, 2007.

[Wang 2008] X. Wang and M.-D. Lemmon. *Event design in event-triggered feedback control systems.* In 47th IEEE Conference on Decision and Control, pages 2105–2110, Cancun, Mexico, 2008.

[Wang 2009] X. Wang and M.-D. Lemmon. *Self-triggered feedback control systems with finite-gain $\mathcal{L}_2$ stability.* IEEE Transactions on Automatic Control, vol. 54, no. 3, pages 452–467, 2009.

[Wang 2010] X. Wang and M.-D. Lemmon. *Self-triggering under state-independent disturbances.* IEEE Transactions on Automatic Control, vol. 55, no. 6, pages 1494–1500, 2010.

[Wittenmark 1995] B. Wittenmark, J. Nilsson and M. Torngren. *Timing problems in real-time control systems.* In American Control Conference, pages 2000–2004, Seattle, Washington, USA, 1995.

[Yakubovich 1977] V.-A. Yakubovich. *S-procedure in nonlinear control theory.* Vestnik Leningrad University, vol. 4, pages 73–93, 1977.

[Yu 2004] M. Yu, L. Wang, T. Chu and F. Hao. *An LMI approach to networked control systems with data packet dropout and transmission delays.* In 43rd IEEE Conference on Decision and Control, pages 3545–3550, Atlantis, Paradise Island, Bahamas, 2004.

[Zaccarian 2005] L. Zaccarian, D. Nesic and A.-R. Teel. *First order reset elements and the Clegg integrator revisited.* In American Control Conference, volume 1, pages 563–568, Portland, Oregon, USA, 2005.

[Zaytoon 2001] J. Zaytoon. Systèmes dynamiques hybrides. Hermès-Lavoisier, 2001.

[Zhang 2001a] J. Zhang, C.-R. Knospe and P. Tsiotras. *Stability of time-delay systems: Equivalence between Lyapunov and scaled small gain conditions.* IEEE Transactions on Automatic Control, vol. 46, no. 3, pages 482–486, 2001.

[Zhang 2001b] W. Zhang. *Stability analysis of networked control systems.* PhD thesis, Department of Electrical Engineering and Computer Science, CASE Western Reserve University, USA, 2001.

[Zhang 2001c] W. Zhang, M.S. Branicky and S.M. Phillips. *Stability of networked control systems.* IEEE Control Systems Magazine, vol. 21, no. 1, pages 84–99, 2001.

[Zhong 2006] Q.-C. Zhong. Robust control of time-delay systems. Springer, 2006.

Résumé

Cette thèse est dédiée à l'analyse de stabilité des systèmes à pas d'échantillonnage variable et à la commande dynamique de l'échantillonnage. L'objectif est de concevoir des lois d'échantillonnage permettant de réduire la fréquence d'actualisation de la commande par retour d'état, tout en assurant la stabilité du système.
Tout d'abord, un aperçu des récents défis et axes de recherche sur les systèmes échantillonnés est présenté. Ensuite, une nouvelle approche de contrôle dynamique de l'échantillonnage, "échantillonnage dépendant de l'état", est proposée. Elle permet de concevoir hors-ligne un échantillonnage maximal dépendant de l'état défini sur des régions coniques de l'espace d'état, grâce à des LMIs.
Plusieurs types de systèmes sont étudiés. Tout d'abord, le cas de système LTI idéal est considéré. La fonction d'échantillonnage est construite au moyen de polytopes convexes et de conditions de stabilité exponentielle de type Lyapunov-Razumikhin. Ensuite, la robustesse vis-à-vis des perturbations est incluse. Plusieurs applications sont proposées: analyse de stabilité robuste vis-à-vis des variations du pas d'échantillonnage, contrôles event-triggered et self-triggered, et échantillonnage dépendant de l'état. Enfin, le cas de système LTI perturbé à retard est traité. La construction de la fonction d'échantillonnage est basée sur des conditions de stabilité $\mathcal{L}_2$ et sur un nouveau type de fonctionnelles de Lyapunov-Krasovskii avec des matrices dépendant de l'état. Pour finir, le problème de stabilisation est traité, avec un nouveau contrôleur dont les gains commutent en fonction de l'état du système. Un co-design contrôleur/fonction d'échantillonnage est alors proposé.

Mots-clés: Système commandé par réseau, système échantillonné, système à retard, échantillonnage variable, échantillonnage dépendant de l'état, self-triggered control, stabilité/stabilisation, inégalité matricielle linéaire

Abstract

This PhD thesis is dedicated to the stability analysis of sampled-data systems with time-varying sampling, and to the dynamic control of the sampling instants. The main objective is to design sampling laws that allow for reducing the sampling frequency of state-feedback control for linear systems while ensuring the system's stability.
First, an overview of the recent problems, challenges, and research directions regarding sampled-data systems is presented. Then, a novel dynamic sampling control approach, "state-dependent sampling", is proposed. It allows for designing offline a maximal state-dependent sampling map over conic regions of the state space, thanks to LMIs.
Various classes of systems are considered throughout the thesis. First, we consider the case of ideal LTI systems, and propose a sampling map design based on the use of polytopic embeddings and Lyapunov-Razumikhin exponential stability conditions. Then, the robustness with respect to exogenous perturbations is included. Different applications are proposed: robust stability analysis with respect to time-varying sampling, as well as event-triggered, self-triggered, and state-dependent sampling control schemes. Finally, a sampling map design is proposed in the case of LTI systems with perturbations and delays. It is based on $\mathcal{L}_2$-stability conditions and a novel type of Lyapunov-Krasovskii functionals with state-dependent matrices. Here, the stabilization issue is considered, and a new controller with gains that switch according to the system's state is presented. A co-design controller/sampling map is then proposed.

Keywords: Networked control system, sampled-data system, time-delay system, time-varying sampling, state-dependent sampling, self-triggered control, stability/stabilization, linear matrix inequality