Constrained control for time-delay systems.

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Constrained Control for Time-Delay Systems

by

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A thesis submitted in partial fulfillment for the
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and
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I dedicate this thesis to my parents, family and friends…

Contents

"*You cannot stay on the summit forever; you have to come down again. So why bother in the first place? Just this: What is above knows what is below, but what is below does not know what is above. One climbs, one sees. One descends, one sees no longer, but one has seen. There is an art of conducting oneself in the lower regions by the memory of what one saw higher up. When one can no longer see, one can at least still know.*"

René Daumal

"*For me, climbing is a form of exploration that inspires me to confront my own inner nature within nature. It's a means of experiencing a state of consciousness where there are no distractions or expectations. This intuitive state of being is what allows me to experience moments of true freedom and harmony.*"

Lynn Hill

Abstract

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The main interest of the present thesis is the *constrained control of time-delay system*, more specifically taking into consideration the discretization problem (due to, for example, a communication network) and the presence of constraints in the system's trajectories and control inputs.

Time-delay systems are largely used to describe some physical phenomena, where the reaction of the system with respect to exogenous signals is not instantaneous. For continuous-time systems controlled by a network (called *networked control systems*), the problem becomes interesting due to the *network-induced delays* and to the *discretization problem* (the delays can vary between the sampling instants).

The effects of data-sampling and modeling problem are studied in detail, where an uncertainty is added into the system due to additional effect of the discretization and delay. The delay variation with respect to the sampling instants is characterized by a polytopic supra-approximation of the discretization/delay induced uncertainty. The discretized system can be expressed in terms of an *augmented (or extended) state-space system.* Three cases are studied, depending on the structure of the eigenvalues of the system matrices. In order to decrease the conservativeness of these basic results, some methods and algorithms are proposed in order to obtain an improved polytopic model.

Some stabilizing techniques, based on Lyapunov's theory, are then derived for the unconstrained case. In an augmented state-space case, by using simple quadratic Lyapunov candidates, a extended state-feedback gain is obtained by using LMI techniques. Lyapunov-Krasovskii candidates were also used to obtain LMI conditions for a state-feedback, in the "original" state-space of the system.

For the *constrained control* purposes, the *set invariance theory* is used intensively, in order to obtain a region where the system is "well-behaviored", despite the presence of constraints and (time-varying) delay. Due to the *high complexity* of the *maximal delayed-state admissible set* obtained in the augmented state-space approach, in the present manuscript we proposed the concept of *set invariance* in the "original" state-space of the system with respect not just to the present states, but also with the past history of the system's trajectories. This concept is called $\mathcal{D}$-invariance, and is defined in terms of the Minkowski addition of the set supposed to be invariant and the linear set operations of the dynamics. Algebraic conditions are derived in order to obtain constructive tests (and controller synthesis), based on a LP problem. In order to reduce the conservatism of this definition, the concept of Cyclic $\mathcal{D}$-invariance is also developed, where the inclusion condition should hold for all the dynamics of the system "shifted" one-by-one. Finally, in the las part of the thesis, the MPC scheme is presented, in order to take into account the constraints and the optimality of the control solution. In order to tune the controller parameters (more precisely the weighting matrices), an inverse optimality procedure is employed. The MPC optimal control policy is obtained "on line" by the feasibility of multiparametric programming, that can be further expressed in terms of a QP problem, using as constraints the trajectories constraints and the obtained invariant set as terminal set of the optimization routine.

Résumé

Ecole Superieure d'Eletricité
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1 Introduction

Le thème principal de ce mémoire est la commande sous contraintes pour des systèmes à retard, en tenant compte de la problématique d'échantillonnage (où les informations concernant le système en temps continu sont, par exemple, envoyées par un réseau de communication) et de la présence de contraintes sur les trajectoires du système et sur l'entrée de commande.

1.1 Systèmes à retard

Les systèmes à retard sont utilisés pour décrire la classe de systèmes où la réaction n'est pas immédiate vis-à-vis de l'entrée exogène du système. Cette classe de systèmes est aussi appelé systèmes *héréditaires*, *avec mémoire*, *à temps-mort*, *aux arguments déviés*, et représente des systèmes de dimension infinie, utilisés surtout pour modéliser des phénomènes physiques, comme la dynamique des populations (reproduction, croissance, extinction, etc), les phénomènes de transport, les systèmes économiques (investissements, évolution des marchés), les procédés industriels chimiques et la production d'énergie. La principale caractéristique de ce type de systèmes est qu'ils peuvent être décrits par des équations différentielles qui incluent non seulement le présent mais aussi l'histoire passée des trajectoires des états.

La question de la stabilité des systèmes à retard est un problème toujours ouvert. Dans la boucle fermée, la présence du retard peut *dégrader la performance*, *induire des oscillations et des instabilités*, ou encore *stabiliser des systèmes instables*.

Une façon de stabiliser des systèmes malgré la présence de retard, est l'utilisation des compensateurs de retard tel que le prédicteur de Smith. La stratégie est d'éliminer le retard de la boucle de commande, si la valeur du retard est parfaitement connue. Le désavantage de cette technique est la robustesse, car à la moindre variation du retard, le système peut devenir instable.

L'analyse et la synthèse qui s'appuient sur la théorie de Lyapunov montrent aussi leur importance dans le domaine. En effet, on peut interpréter la stabilité d'un système à retard selon la théorie de Lyapunov de deux façons : soit comme étant *l'évolution du système dans un espace de fonction* (Fonctionnelles de Lyapunov-Krasovskii ou LKF) soit comme étant *l'évolution du système dans un espace Euclidien* (Fonctions de Lyapunov-Razumikhin).

Dans un premier temps, les *fonctionnelles de Lyapunov-Krasovskii* ne prennent pas en considération la taille du retard lors de la dérivation des conditions de stabilité. En revanche, en utilisant des transformations de modèle, le retard peut être pris en compte dans la analyse de stabilité et aussi dans la synthèse de correcteurs. Le concept de stabilité selon Krasovskii est lié à l'approche classique de Lyapunov, où la dérivée (ou différence en temps discret) de la candidate (définie positive) doit être négative sur les trajectoires du système. La différence par rapport aux candidates de Lyapunov-Razumikhin est que la dérivée (ou différence) doit être négative pour certaines valeurs critiques dans l'intervalle de la taille du retard. Des conditions constructives peuvent être dérivées pour obtenir des *Inégalités Linéaires Matricielles* (LMIs).

1.2 Commande par réseau

Du point de vue du problème d'échantillonnage, la question de la commande des systèmes en temps continu contrôlés par réseau (de l'anglais NCS ou « Networked Control Systems ») devient intéressante grâce aux retards induits par le réseau (les retards peuvent varier entre les périodes d'échantillonnage). La *commande par réseau* est définie pour des systèmes où la boucle est fermée au travers d'un réseau de communication. Les informations sont échangées entre les composantes du système (capteurs, actionneurs, contrôleurs, etc) en utilisant, par exemple, un réseau d'ordinateurs. Parmi les applications, l'exemple classique est celui de la téléopération des robots manipulateurs. Dans ce cas, un opérateur humain contrôle un système esclave éloigné d'un site principal, et reçoit un retour de force de l'esclave, en ayant la sensation d'être présent et en contact direct avec celui-ci.

Les informations acquises par les *capteurs* doivent être échantillonnées et envoyées au *contrôleur* par un réseau. Ainsi, la loi de commande calculée (par un ordinateur ou micro-processeur) est envoyée aux actionneurs. Les avantages sont : la réduction de la quantité de fils et de câbles (les données peuvent être envoyées via l'internet, par exemple), la facilité de diagnostique pour identifier des défauts et l'entretien. Le principal inconvénient est l'ajout des *retards* dans la boucle de commande dûs aux processus de communication et le temps de calcul de la loi de commande. Les retards introduits dans la boucle sont variables dans le temps car les données envoyées par le réseau dépendent de plusieurs paramètres, tel que le flux de données, le type de réseau, le protocole de communication, etc. Les théories conventionnelles de commande doivent être réévaluées avant d'être appliquées à la NCSs. Une autre observation importante est que les réseaux de communication ne sont pas fiables : la fiabilité est fonction du protocole, de la distance que l'information doit parcourir et des composantes physiques (réseaux sans-fils ou câblés par exemple). En effet, les paquets de données peuvent ne pas arriver à la destination finale.

1.2.1 Paradigmes de la commande par réseau

La commande par réseau est gouvernée par quelques paradigmes, de telle façon à garantir un comportement prévisible :

- *NCS basée sur le temps* : l'action est réalisée de façon périodique, à un temps pré-spécifié. Le contrôleur doit calculer la loi de commande reposant sur la dernière information reçue, avec une période T_e gérée par une horloge.
- *NCS basée sur les événements* : l'action réalisée est appuyée sur un événement, c'est à dire, seulement quand une information envoyée par les capteurs est reçue par l'ordinateur, il calcule et envoie la loi de commande, indépendamment de l'horloge.
- *D'autres paradigmes* : les paradigmes de communication basés sur la priorité et sur l'échéance ne sont pas utilisés en NCS car l'information perd l'ordre d'émission et d'arrivée, et par conséquent la dynamique du système.

Si la NCS est basée sur le temps, les paquets de donnée peuvent alors contenir *le temps où l'information a été crée*, appelé « timestamping » et nous informant sur « l'âge » du paquet. Un système, pour vérifier si le paquet est arrivé, peut aussi être utilisé, où une confirmation est envoyée dès que le paquet est arrivé à sa destination, important pour éviter la perte de données. La contre-partie est l'augmentation du retard. Dans la NCS essentiellement trois types de retard peuvent être rencontrés :

- τ^{sc}, le retard de communication entre le capteur et le contrôleur.
- τ^{cc}, le retard dû au *temps de calcul*.
- τ^{ca}, le retard de communication entre le contrôleur et les actionneurs.

Pour la commande par réseau, le *retard total* τ_k est le temps que l'actionneur prend pour utiliser un signal après l'échantillonnage et la transmission d'un état du système :

$$\tau_k = \tau^{sc} + \tau^{cc} + \tau^{ca}.$$

Ce retard peut être considéré *constant* ou *aléatoire*, en dépendant, par exemple, du *protocole de communication*. Des astuces peuvent être utilisées pour éliminer le retard variable dans la boucle, où les paquets de données peuvent être « mémorisés » puis renvoyés avec un retard spécifié et constant. Dans certains protocoles de réseau, comme *CAN* et *DeviceNet*, les retards peuvent être « considérés » constants dû à leur système de priorité d'information. En revanche, dans la plupart des schémas de communication les retards ont soit une distribution probabiliste soit aucune corrélation statistique. Certains travaux ont montré que les retards ne peuvent pas être décris pour un modèle invariant dans le temps pour une longue période de temps. D'autres protocoles de communication par réseau sont : *Fieldbus Foundation, Factory Instrumentation Protocol (FIP), PROFIBUS, et Ethernet.*

Pour garantir la cohérence du modèle utilisé dans ce manuscrit, quelques hypothèses doivent être mises en place :

Hypothèse 1) *Les contrôleurs, capteurs et actionneurs sont déclanchés par le temps et sont échantillonnés périodiquement.* Dans tous les cas il est pratiquement impossible de garantir la synchronisation entre les périodes d'échantillonnage des états du système, de l'échantillonneur et du contrôleur.

Hypothèse 2) *Pas de perte de paquet entre les capteurs et le contrôleur*, afin de garder la structure vecteur d'état augmenté.

Hypothèse 3) *« Timestamping »*, c'est à dire que les entrées de commande sont appliquées dans l'ordre où elles sont générées (une entrée de commande plus ancienne ne peut être appliquée après une entrée plus recente).

Au niveau des applications pratiques, il est « impossible » de synchroniser l'horloge des capteurs et des actionneurs. Même s'ils ont été démarrés en même temps et s'ils ont « théoriquement » la même période, on ne peut pas garantir qu'ils garderont cette périodicité. Pour modéliser ce phénomène, l'élément de variation du retard ϵ_k est introduit. De plus, pour garantir la synchronisation entre les capteurs, ils doivent être placés physiquement proches. Il en est de même pour les actionneurs.

1.3 Commande sous contraintes

Pour le cas où les contraintes ne sont pas prises en compte, des techniques de stabilisation sont proposées. Des techniques classiques de commande robuste peuvent être utilisées dans le cas du système augmenté, au travers des candidates quadratiques de Lyapunov et des techniques LMI (inégalités matricielles affines), en obtenant un retour d'état augmenté. Ensuite, l'utilisation de candidates de Lyapunov-Krasovskii pour obtenir des conditions LMI seront aussi regardées en détail, avec l'objectif d'obtenir un retour d'état dans la « taille originale » du système.

La théorie des *ensembles invariants* est largement utilisée dans ce manuscrit, car il est souhaitable d'obtenir une région de « sûreté », où le comportement du système est connu, en dépit de la présence du retard (variable) et des contraintes sur les trajectoires du système. Lorsqu'ils sont obtenus dans l'espace d'état augmenté, les ensembles invariants sont très complexes, car la dimension de l'espace Euclidien sera proportionnelle à la taille du système mais aussi à la taille du retard. Le concept de $\mathcal{D}$-invariance est ainsi proposé, où non seulement les trajectoires présentes du systèmes doivent appartenir à un ensemble, mais l'histoire des trajectoires passées doivent également respecter cette inclusion. Pour les systèmes linéaires, la $\mathcal{D}$-invariance peut être définie comme l'addition de Minkowski de l'ensemble supposé invariant et les opérations linéaires de la dynamique présente et passée. Les conditions algébriques peuvent aussi être dérivées pour l'obtention de tests constructifs et pour la synthèse de contrôleurs, basés sur des problèmes de faisabilité LP (Programmation Linéaire). Afin de réduire le conservatisme de cette approche, le concept de $\mathcal{D}$-invariance cyclique est étudié, où la condition d'inclusion des ensembles doit être respectée pour toutes les dynamiques du système déplacées une par une.

1.4 Commande prédictive

Dans la dernière partie du manuscrit, la *commande prédictive* (en anglais MPC) est présentée, pour tenir compte des contraintes sur les trajectoires et appliquer une commande optimale à l'entrée du système. Pour régler les paramètres du contrôleur (plus précisément les matrices de pondération sur les états et de la commande), un problème d'optimalité inverse est utilisé, basé sur la résolution d'une équation de Ricatti. La commande MPC est appliquée « en ligne » en résolvant un problème d'optimisation multiparamétrique, aussi exprimé au niveau d'un problème PQ (Programmation Quadratique), en utilisant comme contraintes du problème d'optimisation les contraintes de trajectoire, et l'ensemble invariant comme ensemble terminal de la procédure de prédiction à horizon glissant.

2 Modélisation

Ici, la modélisation de systèmes linéaires échantillonnés est discutée. Pendant le processus d'échantillonnage (et par conséquent l'intégration des entrées), le retard variable dans le temps peut être traité comme une incertitude en fonction de cette variation. Le but est de confiner cette fonction dans un polytope, de façon à couvrir toutes les variations possibles du retard. Cette technique doit être valable malgré l'absence d'informations statistiques concernant la variation.

Les variations du retard peuvent être bornées dans une période d'échantillonnage (Chapitre 3) ou dans plusieurs périodes (Chapitre 5), appelées dans cette thèse *variation intra-échantillon* et *multi-échantillon*. Le modèle peut être donc exprimé en utilisant un vecteur d'état augmenté contenant les états actuels et les entrés de commande passées. Un modèle polytopique augmenté est obtenu, c'est à dire que le système continu à retard variable est transformé en un système échantillonné avec des incertitudes polytopiques. L'avantage est la possibilité d'utiliser des techniques classiques de commande robuste.

Le terme en fonction de la variation du retard peut être catégorisé en fonction du niveau des valeurs propres de la matrice d'état du système en temps continu, notamment *matrice non-défective avec des valeurs propres réels*, *matrice non-défective avec des valeurs propres complexes conjugués* et finalement *matrice défective avec des valeurs propres réels*. En utilisant ces trois cas de base, toutes les autres combinaisons possibles de systèmes peuvent être aussi calculées. Il est montré que la *fonction d'incertitude* peut être enveloppée par un polytope en utilisant des calculs simples, reposant sur la *décomposition de Jordan*. Un des inconvénients de cette technique est qu'elle est considérée conservative. Un algorithme ayant pour finalité de diminuer le volume des polytopes autour de la fonction d'incertitude est donc proposé.

2.1 La problématique de l'échantillonnage et du réseau

L'échantillonnage de systèmes est un problème intéressant du point de vue de la commande par réseau, où la transmission des données (états et commande) va induire et/ou ajouter un retard au système. Pendant l'échantillonnage, le retard variable peut être traité comme une fonction d'incertitude du système.

Dans ce manuscrit, ce problème est détaillé dans la Section 3.1 du Chapitre 3 ainsi que dans le Chapitre 5, en fonction des bornes de la variation du retard. Dans tous les cas, le modèle peut être obtenu en utilisant un vecteur d'état augmenté en fonction des entrées de commande passées et des états du système. Le modèle polytopique est obtenu dans l'espace d'état augmenté et le système à retard est exprimé comme étant un système augmenté avec des incertitudes polytopiques. Le principal avantage est que ce type de système peut être facilement stabilisé en utilisant des techniques classiques de commande robuste pour des systèmes incertains.

En tenant compte des hypothèses décrites ci-dessus, deux cas d'échantillonnage peuvent être décrits selon la variation du retard : dans le premier, la variation du retard est plus petite qu'une période d'échantillonnage et dans le deuxième, plus grande. L'entrée de commande est constante par morceau. Ceci en considérant $u(t) = u(a_k)$ pour $a_{k-d_k} \in \mathbb{R}_{[t_k,t_{k+1})}$, avec une période d'échantillonnage T_e et $t_k = kT_e$ le k-ième instant, et en considérant aussi les instants d'activation de la commande $a_{k-d_k} = \tau_k + t_{k-d_k}$, où τ_k est le retard et $d_k = \frac{\tau_k}{T_e}$. La variation du retard est $\epsilon_k \in \mathbb{Z}_{[0,T_e]}$.

Nous pouvons facilement voir que :

$$\tau_k = a_{k-d_k} - t_{k-d_k}.$$

Considérons un système linéaire en temps continu avec un retard variable dans l'entrée :

$$\dot{x}(t) = A_c x(t) + B_c u(t - \tau^c(t)),$$

pour tout $t > 0$, avec $A_c \in \mathbb{R}^{n\times n}$, $B_c \in \mathbb{R}^{n\times m}$, sous les conditions initiales appropriées. Considérons que le système est échantillonné de façon périodique, dont la période est T_e et notons $t_k = kT_e$ le k-ième instant. Considérons la séquence d'activation des entrées de commande $\{a_k\}_{k=0}^{\infty}$ définies par $a_k = \tau_k + t_k$ où $\tau_k > 0$ sans aucune information statistique à propos de cette variation. Le retard évolue $\tau^c(t)$ impulsivement :

$$\dot{\tau}^c(t) = 1, \, \forall t \neq a_k$$

et

$$\tau^c(a_k^+) = a_{k-d_k} - t_{k-d_k} = \tau_k,$$

où

$$a_k^+ = \lim_{\substack{t \to a_k \\ t > a_k}} t.$$

2.2 Variation du retard inférieure à une période d'échantillonnage

Dans ce cas, le retard est borné par les valeurs extrêmes $\tau_{min} = d_k T_e$ et $\tau_{max} = (d_k+1)T_e$ tel que $0 \leq \tau_{min} \leq \tau_k \leq \tau_{max}$. Le système échantillonné est le suivant :

$$x_{k+1} = Ax_k + Bu_{k-d_k} - \Delta(\epsilon_k)(u_{k-d_k} - u_{k-d_k+1}),$$

où les matrices $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}$ et la fonction $\Delta(\epsilon) : \mathbb{R}_{(0,\bar{\epsilon}]} \to \mathbb{R}^{n\times m}$ sont données par :

$$\begin{array}{rcl} A & = & e^{A_c T_e}; \\ B & = & \displaystyle\int_0^{T_e} e^{A_c(T_e-\theta)} B_c d\theta; \\ \Delta(\epsilon_k) & = & \displaystyle\int_{T_e-\epsilon_k}^{T_e} e^{A_c(T_e-\theta)} B_c d\theta = \int_{-\epsilon_k}^{0} e^{-A_c\tau} B_c d\tau. \end{array}$$

La fonction $\Delta(\epsilon_k)$ est la fonction d'incertitude vis à vis de la variation du retard.

Les équations ci-dessus peuvent être réécrites sous une forme linéaire compacte tel que le système augmenté obtenu soit :

$$\xi_{k+1} = A_\Delta \xi_k + B_\Delta u_k,$$

avec le vecteur d'état augmenté suivant :

$$\xi_k^\top = \begin{bmatrix} x_k^\top & u_{k-d}^\top & \dots & u_{k-1}^\top & \end{bmatrix}^\top \in \mathbb{R}^{n\times n+md}$$

et les matrices d'état augmentées :

$$A_\Delta = \begin{bmatrix} A & B-\Delta(\epsilon) & \Delta(\epsilon) & \dots & 0 \\ 0 & 0 & I_m & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & I_m \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}; \; B_\Delta = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ I_m \end{bmatrix},$$

où $A_\Delta \in \mathbb{R}^{(n+md)\times(n+md)}$ et $B_\Delta \in \mathbb{R}^{(n+md)\times m}$.

L'idée est d'envelopper $\Delta(\epsilon_k)$ par un ensemble polytopique de façon à couvrir $\epsilon_k \in \mathbb{R}_{(0,\bar{\epsilon}_k]}$, tout en considérant $\bar{\epsilon} = T_e$, et en rendant le système indépendant de ϵ_k. Le modèle polytopique de $n+1$ sommets $\Delta(\epsilon_k) \in Co\{\Delta_0, \dots, \Delta_n\}$ peut être obtenu pour un modèle dans l'espace d'état augmenté :

$$\begin{array}{rcl} \xi_{k+1} & = & A_\Delta \xi_k + B_\Delta u_k \\ A_\Delta & \in & \mathrm{Co}\{A_{\Delta_0},\ A_{\Delta_1},\ \ldots A_{\Delta_n}\}. \end{array}$$

Pour obtenir les sommets, la décomposition de Jordan de la matrices A_c peut être utilisée : $A_c = V\Sigma V^{-1}$, où $\Sigma \in \mathbb{R}^{n\times n}$ est une matrice bloc diagonale comme suit :

$$\Sigma = \begin{bmatrix} \Sigma_{1,m_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \Sigma_{p,m_p} \end{bmatrix},$$

pour tout $i \in \mathbb{Z}[1,p]$ et $\Sigma_{i,m_i} \in \mathbb{R}^{m_i\times m_i}$ sont les blocs de Jordan, avec :

$$\Sigma_{i,m_i} = \begin{bmatrix} \sigma_i & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \sigma_i & 1 \\ 0 & 0 & \cdots & 0 & \sigma_i \end{bmatrix} = \sigma_i \underbrace{\begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}}_{\Lambda_i} + \underbrace{\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}}_{\Gamma_i}.$$

Selon la *multiplicité algébrique* et *géométrique* de la matrice A_c, trois classes de systèmes peuvent être obtenues, notamment :

- Blocs de Jordan Σ_{i,m_i} avec $m_i = 1$ correspondant aux valeurs propres *réelles* avec la multiplicité algébrique égale à la multiplicité géométrique.
- Blocs de Jordan Σ_{i,m_i} avec $m_i = 1$ correspondant aux valeurs propres *complexes-conjuguées* avec la multiplicité algébrique égale à la multiplicité géométrique.
- Blocs de Jordan Σ_{i,m_i} avec $m_i > 1$ correspondant aux valeurs propres *réels* avec la multiplicité algébrique inférieure à la multiplicité géométrique.

Pour chaque cas décrit ci-dessus nous avons développé une technique afin d'envelopper la fonction $\Delta(\epsilon_k)$ dans un polytope au niveau de sommets/hyperplans ($\mathcal{V}$-représentation et $\mathcal{H}$-représentation). Du point de vue du calcul informatique, il est intéressant d'obtenir le polytope le moins complexe possible dans l'espace Euclidien, soit un *simplex* : l'enveloppe convexe de $(n+1)$ points et indépendants dans l'espace Euclidien de dimension n.

L'avantage principal de l'approche dans l'espace d'état augmenté est la possibilité d'utilisation des techniques de commande robuste classiques basées sur des LMIs. Le problème devant être résolu sera proportionnel à la taille du système ($\mathbb{R}^{(n+md)\times(n+md)}$) et au nombre de sommets du polytope $(n+1)$.

2.2.1 Matrice du système non-défective avec des valeurs propres réelles

Pour les valeurs limites de $\bar{\epsilon}_k$, la fonction $\Delta(\epsilon_k)$ peut être enveloppée dans la $\mathcal{V}$-représentation par les équations :

$$\Delta_0 = \mathbf{0}_{n\times m},$$
$$\Delta_i = V\int_{-\bar{\epsilon}_k}^{0} e^{-\Sigma_{i,1}\tau} d\tau V^{-1}B_c,\ \forall i = 1,\dots,n,$$

où $\Sigma_{i,1}$ est la matrice Σ seulement avec le i-ème terme (qui n'est qu'un bloc de Jordan de taille $m_i = 1$ pour $i \in \mathbb{Z}_{[1,p]}$, indice omit par simplification), en considérant la décomposition :

$$\Sigma = \underbrace{\begin{bmatrix} \tilde{\Sigma}_1 & 0 & \dots & 0 \\ 0 & 0 & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \dots & \dots & 0 \end{bmatrix}}_{\Sigma_{1,1}} + \underbrace{\begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & \tilde{\Sigma}_2 & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \dots & \dots & 0 \end{bmatrix}}_{\Sigma_{2,1}} + \dots + \underbrace{\begin{bmatrix} 0 & \dots & \dots & 0 \\ \vdots & 0 & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \dots & \dots & \tilde{\Sigma}_n \end{bmatrix}}_{\Sigma_{n,1}},$$

où $\tilde{\Sigma}_i$ sont les blocs de Jordan.

2.2.2 Matrice du système non-défective avec des valeurs propres complexes-conjugués

Pour cette classe de systèmes la fonction $\Delta(\epsilon_k)$ est non-monotone, et représente plus précisément une spirale. L'enveloppement par un simplex est une tâche difficile. Les méthodes basées sur les hypercubes sont une solution possible. Par contre le résultat est conservatif. De plus, un hypercube a 2^n sommets, nous avons donc une augmentation de la complexité du polytope.

À partir de la décomposition de Jordan de $A_c = V\Sigma V^{-1}$:

$$\begin{aligned} \Delta(\epsilon) &= A_c^{-1}\left(e^{A_c\bar{\epsilon}} - 1\right)B_c \\ &= V\Sigma^{-1}e^{\Sigma\bar{\epsilon}}V^{-1}Bc - V\Sigma^{-1}V^{-1}B_c. \end{aligned}$$

Deux cas indépendants doivent être traités :

(i) A_c stable : l'hypercube peut être représenté dans la $\mathcal{H}$-représentation par :

$$F = \begin{bmatrix} I_n \\ -I_n \end{bmatrix};\ w = \begin{bmatrix} \left|V\Sigma^{-1}\right|\left|V^{-1}B_c\right| + S \\ \left|V\Sigma^{-1}\right|\left|V^{-1}B_c\right| - S \end{bmatrix}$$

(ii) A_c instable : l'hypercube peut être representé dans la $\mathcal{H}$-représentation par :

$$F = \begin{bmatrix} I_n \\ -I_n \end{bmatrix}; \; w = \begin{bmatrix} \left|V\Sigma^{-1}\right| e^{\Sigma\bar{\epsilon}} \left|V^{-1}B_c\right| + S \\ \left|V\Sigma^{-1}\right| e^{\Sigma\bar{\epsilon}} \left|V^{-1}B_c\right| - S \end{bmatrix}$$

Les sommets peuvent être facilement obtenus par des transformations polyédrales entre la $\mathcal{H}$-représentation, donnée par $\mathcal{P} = \{x \in \mathbb{R}^n \mid Fx \leq w\}$, par $\mathcal{V}$-représentation, $\mathcal{P} = Co\{\Delta_1, \Delta_2, \ldots, \Delta_{2^n}\}$.

2.2.3 Matrice du système défective avec des valeurs propres réelles

Notons la décomposition de Jordan : $A_c = V\Sigma V^{-1}$ et la matrice bloc diagonale $\Sigma \in \mathbb{R}^{n\times n}$, nous avons :

$$\Sigma = \begin{bmatrix} \Sigma_{1,m_1} & 0 & \cdots & 0 \\ 0 & \Sigma_{2,m_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Sigma_{p,m_p} \end{bmatrix},$$

pour tout $i \in [1, ..., p]$ en résultant $\Sigma_{i,m_i} \in \mathbb{R}^{m_i\times m_i}$ avec :

$$\Sigma_{i,m_i} = \begin{bmatrix} \sigma_i & 1 & \cdots & 0 & 0 \\ 0 & \sigma_i & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \sigma_i & 1 \\ 0 & 0 & \cdots & 0 & \sigma_i \end{bmatrix} = \sigma_i \underbrace{\begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}}_{\Lambda_i} + \underbrace{\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}}_{\Gamma_i}.$$

En exploitant le fait que la matrice Γ_i soit nilpotente, nous obtenons :

$$\Sigma = \begin{bmatrix} \sigma_1\Lambda_1 + \Gamma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2\Lambda_2 + \Gamma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_p\Lambda_p + \Gamma_p \end{bmatrix} =$$

$$= \sum_{i=1}^{p} \sigma_i \underbrace{\begin{bmatrix} 0 & & & \\ & \ddots & & \\ & & \Lambda_i & \\ & & & \ddots & \\ & & & & 0 \end{bmatrix}}_{L_i} + \sum_{i=1}^{p} \underbrace{\begin{bmatrix} 0 & & & \\ & \ddots & & \\ & & \Gamma_i & \\ & & & \ddots & \\ & & & & 0 \end{bmatrix}}_{G_i},$$

qui peut être écrite comme suit :

$$\Sigma = \sum_{i=1}^{p} \sigma_i L_i + G_i,$$

où L_i sont des matrices diagonales de même taille que Σ avec des valeurs « un » dans la diagonale principale, dans la cellule correspondante au bloc de Jordan Σ_{i,m_i}. De façon similaire, G_i sont des matrices diagonales de la même taille que Σ avec des valeurs « un » dans la supra-diagonale principale à la cellule correspondante au bloc de Jordan Σ_{i,m_i}.

Pour enfermer $\Delta(\epsilon)$, nous pouvons réécrire :

$$\begin{aligned}\Delta(\epsilon) &= \int_0^{\bar{\epsilon}} e^{A_c\tau} B_c d\tau = \int_0^{\bar{\epsilon}} e^{V\Sigma V^{-1}\tau} d\tau \ B_c \\ &= \int_0^{\bar{\epsilon}} V e^{\Sigma\tau} V^{-1} d\tau \ B_c = V \int_0^{\bar{\epsilon}} e^{\Sigma\tau} d\tau \ V^{-1} B_c.\end{aligned}$$

En exploitant la structure des blocs de Jordan, nous avons :

$$e^{\Sigma_{i,m_i}\tau} = e^{\sigma_i \Lambda_i \tau} e^{\Gamma_i \tau}.$$

En développant l'expansion de Taylor de $e^{\Sigma_{i,m_i}\tau}$ jusqu'au $(m_i - 1)$-ième élément, (puisque Σ_i est nilpotente) nous obtenons :

$$e^{\Sigma_{i,m_i}\tau} = e^{\sigma_i \Lambda_i \tau} \left(I + \frac{1}{1!}\Gamma_i \tau + \frac{1}{2!}\Gamma_i^2 \tau^2 + ... + \frac{1}{(m_i - 1)!}\Gamma_j^{m_i - 1} \tau^{m_i - 1} \right).$$

Deux cas peuvent être distingués :

1. Blocs de Jordan Σ_{i,m_i} avec $m_i = 1$. Dans ce cas, G_i n'existe pas et l'approche décrite auparavant peut être implémentée :

$$\Delta_{j,1}(\epsilon) = V \int_0^{\bar{\epsilon}} e^{\Sigma\tau} d\tau \ V^{-1} B_c = V L_j \int_0^{\bar{\epsilon}} e^{\sigma_j \tau} d\tau \ V^{-1} B_c.$$

2. Le bloc de Jordan Σ_{i,m_i} correspond a une valeur propre répétée avec multiplicité algébrique $m_i \geq 2$. Le problème de l'enveloppement peut être découplé dans ce cas :

$$\begin{aligned}&\Delta_j(\epsilon) = V L_j \int_0^{\bar{\epsilon}} e^{\sigma_j \tau} d\tau \ V^{-1} B_c; \\ &\Delta_{j,1}(\epsilon) = V \tfrac{1}{1!} G_j \int_0^{\bar{\epsilon}} \tau e^{\sigma_j \tau} d\tau \ V^{-1} B_c; \\ &\qquad \vdots \\ &\Delta_{j,m_j-1}(\epsilon) = V \tfrac{1}{(m_j-1)!} G_j^{m_j-1} \int_0^{\bar{\epsilon}} \tau^{m_j-1} e^{\sigma_j \tau} d\tau \ V^{-1} B_c.\end{aligned}$$

En développant, nous pouvons voir que :

- $e^{\sigma_i \tau}$, avec $i \in [1, ..., p]$ sont des fonctions monotones et positives en $\tau \in [0, \bar{\epsilon}]$.
- $\tau_p^k e^{\sigma_i \tau}$ avec $i \in [1, ..., p]$, $k \in [1, ..., n-p]$ sont des fonctions monotones et positives en $\tau \in [0, \bar{\epsilon}]$.

Globalement, pour un $\bar{\epsilon}$ donné, nous avons « p » fonctions monotones et positives en $\tau \in [0, \bar{\epsilon}]$. En notant ces fonctions comme $g_1(\tau), ..., g_n(\tau)$, il y a une séquence ordonnée $i_1, ..., i_n$ avec $i_k \in [1, ..., n]$ telle que :

$$g_{i_1}(\bar{\epsilon}) \geq g_{i_2}(\bar{\epsilon}) \geq \ldots \geq g_{i_n}(\bar{\epsilon}).$$

Pour tout $g_i(\epsilon)$, il existe $\Delta(\epsilon)$, et pour tout $g_{i_k}(\epsilon)$ il existe $\Delta_{g_{i_k}}(\epsilon)$.

Pour avoir une approche moins conservative, nous avons :

$$\begin{aligned}
&\Delta_j(\epsilon) = V\left(I + \tfrac{1}{1!}\Gamma + ... + \tfrac{1}{(p-1)!}\Gamma^{p-1}\right)\int_0^{\bar{\epsilon}} \tau^{p-1} e^{\sigma_j \tau} d\tau \; V^{-1} B_c \\
&\Delta_{j,1}(\epsilon) = V\left(I + ... + \tfrac{1}{(p-2)!}\Gamma^{p-2}\right)\int_0^{\bar{\epsilon}} \left(-\frac{1}{(p-1)!}\tau^{p-1} + \tau^{p-2}\right) e^{\sigma_j \tau} d\tau \; V^{-1} B_c \\
&\Delta_{j,2}(\epsilon) = V\left(I + ... + \tfrac{1}{(p-3)!}\Gamma^{p-3}\right)\int_0^{\bar{\epsilon}} \left(-\frac{1}{(p-2)!}\tau^{p-2} + \tau^{p-3}\right) e^{\sigma_j \tau} d\tau \; V^{-1} B_c \\
&\vdots \\
&\Delta_{j,m_j-1} = V\int_0^{\bar{\epsilon}} (-\tau + 1)\, e^{\sigma_j \tau} d\tau \; V^{-1} B_c
\end{aligned}$$

Les autres éléments Δ_{j,m_k}, $k \in [1, ..., n-p]$ sont les permutations :

$$\begin{aligned}
\Delta_{j,m_j} &= \Delta_j(\epsilon) + \Delta_{j,1}(\epsilon) \\
\Delta_{j,m_j+1} &= \Delta_j(\epsilon) + \Delta_{j,2}(\epsilon) \\
&\vdots \\
\Delta_{j,m_k} &= \Delta_j(\epsilon) + \Delta_{j,1}(\epsilon) + \ldots + \Delta_{j,k-1}(\epsilon)
\end{aligned}$$

2.3 Optimisation sur une classe de polytopes

Nous pouvons obtenir des polytopes moins conservatifs avec $n+1$ sommets à partir de l'utilisation d'un problème d'optimisation approprié. Le principe est de diviser la fonction $\Delta(\epsilon_k)$ en sous-intervalles et de construire une enveloppe convexe autour de ces sous-intervalles. Le but est ensuite de minimiser un simplex autour de ces sous-ensembles, en utilisant les sommets comme contraintes. Cette procédure est exécutée itérativement, jusqu'à ce que la précision désirée soit obtenue.

Étant donné la fonction :

$$D(\epsilon_k) = \int_0^{\epsilon_k} e^{\Sigma\tau} d\tau.$$

La fonction D peut aussi être exprimée dans un vecteur de dimension n.

Le polytope avec $n+1$ sommets est décrit par :

$$\mathcal{E} = Co\left\{D_0, D_1, \ldots, D_n\right\},$$

où $D_i = diag(v_{i,1}, \ldots, v_{i,n})$ est représentée par les vecteurs :

$$v_0 = \begin{bmatrix} v_{0,1} \\ v_{0,2} \\ \vdots \\ v_{0,n} \end{bmatrix};\ v_1 = \begin{bmatrix} v_{1,1} \\ v_{1,2} \\ \vdots \\ v_{1,n} \end{bmatrix};\ \ldots;\ v_n = \begin{bmatrix} v_{n,1} \\ v_{n,2} \\ \vdots \\ v_{n,n} \end{bmatrix}.$$

Le simplex le moins conservatif est obtenu grâce à la solution du *problème d'optimisation* suivant :

$$\begin{array}{ll} \min\limits_{v_0,\ldots,v_n} & Vol(Co\left\{v_0,\ldots,v_n\right\}) = \frac{1}{n!}\left|det\begin{bmatrix} v_0 & \ldots & v_n \\ 1 & \ldots & 1\end{bmatrix}\right|. \\ \text{subject to} & v(\epsilon) \in Co\left\{v_0,\ldots,v_n\right\} \end{array}$$

Ce problème minimise le volume d'un simplex tout en utilisant un ensemble de points comme contraintes (dans ce cas, les sommets de $\mathcal{E}$). L'idée maintenant est de prendre les ensembles locaux, obtenus à partir de la sous-division de $\Delta(\epsilon_k)$ dans les intervalles $[0, \bar{\epsilon}_k]$ dans ℓ sous-intervalles $[0, \epsilon_1] \cup [\epsilon_1, \epsilon_2] \cup \cdots \cup [\epsilon_{\ell-1}, \bar{\epsilon}]$. Le volume minimal sera obtenu par la solution de l'optimisation pour l'ensemble de sommets des ensembles locaux $D(\epsilon) \in \bigcup\limits_{i=0}^{\ell-1} \mathcal{E}_{\epsilon_i,\epsilon_{i+1}}$.

$$\begin{array}{ll} \min\limits_{v_0,\ldots,v_n} & \frac{1}{n!}\left|det\begin{bmatrix} v_0 & \ldots & v_n \\ 1 & \ldots & 1\end{bmatrix}\right|. \\ \text{subject to} & d_i^{[\epsilon_k,\epsilon_{k+1}]} \in Co\left\{v_0,\ldots,v_n\right\},\ k \in \mathbb{Z}_{[0,\ell]} \end{array}$$

2.4 Variation du retard supérieure à une période d'échantillonnage

L'échantillonnage de systèmes, quand la variation du retard est supérieure à une période d'échantillonnage, est un problème difficile à traiter. Nous pouvons imaginer cette difficulté en tenant compte du fait que les entrées de commande peuvent s'appuyer sur les états à t_k, t_{k-1}, ou même t_{k-d_k} (un instant plus « ancien » que t_{k-2}). Si nous acceptions les hypothèses initiales (« timestamping », capteurs et actionneurs basés sur le temps, absence de perte de paquets de communication, etc), le système peut être échantillonnée en considérant les bornes *connues* du retard, d_1 et d_2 tel que $d_1 > d_k > d_2$. Le système peut être décrit ainsi :

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_{k-d} - \Delta(\epsilon_k)(u_{k-d} - u_{k-d+1}); \\ & \quad d \in \{d_{min}, d_{min}+1, d_{min}+2, \ldots, d_{max}-1, d_{max}\}, \end{aligned}$$

où la variation du retard dans une période d'échantillonnage peut être décrite par la fonction $\Delta(\epsilon_k)$ enveloppée par le polytope.

En utilisant le même vecteur d'état augmenté ξ_k, le problème est l'incertitude introduite par la variation du retard $d \in \mathbb{Z}_{[d_{min},d_{max}]}$ d'un côté, et l'incertitude paramétrique introduite par le terme $\epsilon_k \in \mathbb{R}_{(0,\bar{\epsilon}]}$ de l'autre. Une façon de traiter ce type de modèles est de définir un modèle linéaire du type commuté affecté par une incertitude paramétrique. Pour un vecteur donné $\delta \in \{0,1\}^{d_{max}-d_{min}+1}$, avec les composantes $\delta_i, i \in \mathbb{Z}_{[1,n]}$ les matrices suivantes sont introduites :

$$F(\delta_1,\delta_2,\Delta) = \left[\begin{array}{c|c|c|c|c|c|c|c} A & \Psi_{d_{max}}(\delta_1,\delta_2,\epsilon) & \ldots & \cdots & \Psi_{d_i}(\delta_1,\delta_2,\epsilon) & \cdots & \cdots & \Psi_1(\delta_1,\delta_2,\epsilon) \\ 0 & 0 & I_m & \cdots & 0 & \cdots & \cdots & \vdots \\ \vdots & \ddots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & I_m & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 & I_m & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & I_m \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \end{array}\right];$$

$$G(\delta_1,\delta_2,\Delta) = \begin{bmatrix} \Psi_0(\delta_1,\delta_2,\epsilon) \\ 0 \\ \vdots \\ \vdots \\ 0 \\ I_m \end{bmatrix},$$

où :

$$\Psi_i(\delta_1,\delta_2,\Delta) = \begin{cases} \mathbf{0}_{n\times m} & \text{if} \quad i < d_{min}, \\ (B - \Delta(\epsilon))\,\delta_1(i) + \Delta(\epsilon)\delta_2(i) & \text{if} \quad i \in \mathbb{Z}_{[d_{min},d_{max}]}, \end{cases}$$

et la notation :

$$\delta_\ell = \begin{bmatrix} \delta_\ell(d_{min}) & \ldots & \delta_\ell(i) & \ldots & \delta_\ell(d_{max}) \end{bmatrix},$$

pour $\ell = 1, 2$ et $i \in \mathbb{Z}_{[d_{min}, d_{max}]}$ est utilisée.

Du point de vue de la commutation, il existe plusieurs restrictions qui peuvent être résumées par les contraintes linéaires sous les variables binaires :

$$\left\{\begin{array}{l} \sum_{i=d_{min}}^{d_{max}} \delta_\ell(i) = 1, \quad \text{for} \quad \ell = 1, 2 \\ \sum_{i=d_{min}}^{d_{max}} i\delta_\ell(i) > \sum_{j=d_{min}}^{d_{max}} j\delta_\ell(j). \end{array}\right.$$

En utilisant la même représentation en espace d'état augmenté, nous pouvons obtenir le modèle le suivant, avec (A, B) et $\bar{d} = \left\lfloor \frac{\bar{\tau}}{T_e} \right\rfloor$: $\xi_{k+1} = \bar{F}\xi_k + \bar{G}u_k$, avec $\bar{F}, \bar{G}$ donné par les équations :

$$\bar{F} = \left[\begin{array}{c|c|c|c} A & \overbrace{0\ \ 0\ \ \dots\ \ 0}^{d_{max}-\bar{d}} & B & \overbrace{0\ \ \dots\ \ \dots\ \ 0}^{\bar{d}-1} \\ \hline d_{max}-\bar{d}\left\{\begin{matrix}0\\ \vdots\\ \vdots\\ 0\end{matrix}\right. & \begin{matrix}0 & I_m & \ddots & \\ & \ddots & \ddots & 0 \\ & & \ddots & I_m \\ & & & 0\end{matrix} & \begin{matrix}0\\ \ddots\\ 0\\ I_m\end{matrix} & \vdots \\ \hline \vdots & \ddots & 0 & I_m\ \ 0\ \ \ddots\ \ \ddots \\ \hline \bar{d}-1\left\{\begin{matrix}\vdots\\ \vdots\\ \vdots\\ 0\end{matrix}\right. & \dots & \dots & \begin{matrix}0 & I_m & \ddots & \\ & \ddots & \ddots & 0\\ & & \ddots & I_m \\ & & & 0\end{matrix}\end{array}\right];$$

$$\bar{G} = \left[\begin{array}{c|c|c|c} 0 & 0\ \ \dots\ \ \dots\ \ \dots & 0 & \dots\ \ \dots\ \ 0\ \ I_m \end{array}\right]^T.$$

3 Commande

Dans la partie *Commande* du manuscrit, nous discutons des techniques de stabilisation basées sur la théorie de Lyapunov, comme des ensembles invariants pour des systèmes à retard et de la commande prédictive.

3.1 Stabilisation des systèmes à retard

Au cours de cette thèse, pour stabiliser des systèmes à retard, nous nous sommes basés sur la théorie de Lyapunov. Les candidates de Lyapunov sont généralement très simples, comme des fonctions quadratiques. En utilisant cette méthode, nous pouvons trouver un retour d'état qui stabilise le système malgré la présence du retard variable dans la boucle. Comme le polytope a $n+1$ sommets, le problème est de trouver un retour d'état qui satisfait $n+1$ LMIs pour les deux cas décrits de variation du retard. Par contre, la taille des matrices du système est proportionnelle au retard et la dimension de l'espace Euclidien utilisé (des sommets du polytope) est fonction de la taille des matrices d'état du système continu, augmentant la complexité du problème. Cet ensemble de facteurs va influencer la complexité et le domaine de feasabilité du problème LMI.

La flexibilité des procédures LMI est bien connue dans la littérature. Cela signifie que ce type de problème peut être résolu dans un temps polynomial et que les algorithmes utilisés pour le résoudre sont considérés efficaces.

Une autre possibilité est l'utilisation des candidates de Lyapunov-Krasovskii. Ce type de technique stabilise le système dans l'espace Euclidien original. L'inconvénient est la complexité de la candidate, considérée importante.

3.1.1 Approche de commande robuste classique

L'idée est d'obtenir une loi de commande reposant sur un retour d'état pour les systèmes discrétisés décrits ci-dessous :

$$u_k = K\xi_k,$$

où $K \in \mathbb{R}^{m\times(n+dm)}$.

Considérons, donc, le système polytopique décrit dans les étapes précédentes. La matrice de retour d'état augmenté $K \in \mathbb{R}^{m\times(n+dm)}$ qui stabilise le système est donnée par :

$$K = YS^{-1},$$

où $Y \in \mathbb{R}^{m\times(n+md)}$, $S \in \mathbb{R}^{(n+md)\times(n+md)}$, $S = S^\top > I_{m+dm}$ sont obtenus par la solution du problème d'optimisation suivant :

$$\min_{\gamma,S,Y} \gamma$$
$$\text{soumis à :} \quad ,$$

$$\begin{bmatrix} S & SA_{\Delta_i}^\top + Y^\top B_\Delta^\top & SQ^{1/2} & Y^\top R^{1/2} \\ A_{\Delta_i} S + B_\Delta Y & S & 0 & 0 \\ Q^{1/2} S & 0 & \gamma I & 0 \\ R^{1/2} Y & 0 & 0 & \gamma I \end{bmatrix} \geq 0,$$

$$S \succeq 0;$$

$$\text{pour tout} \quad i = 0, \ldots, n,$$

où $Q \in \mathbb{R}^{(n+md)\times(n+md)}$, $R \in \mathbb{R}^{m\times m}$ et $\gamma \in \mathbb{R}$.

3.1.2 Approche de Lyapunov-Krasovskii

L'idée est à présent d'obtenir une loi de commande du type retour d'état :

$$u_k = Kx_k,$$

où $K \in \mathbb{R}^{m\times n}$.

Une possibilité est l'utilisation des *fonctionnelles de Lyapunov-Krasovskii* (ou simplement LKF) et *Fonctions de Lyapunov-Razumikhin*. Dans le temps discret, les candidates sont, en fait, des *fonctions* (contrairement au cas continu, où elles sont des fonctionnelles).

À partir de transformations de modèle, deux types d'approche LMI peuvent être obtenues pour des candidates de *Lyapunov-Krasovskii* : *indépendante du retard* et *dépendante du retard*. La différence est l'inclusion de l'information du retard dans le résultat de stabilisation. Les caractéristiques de ces deux approches sont :

- **Indépendante du retard:** la propriété de stabilité est respectée pour toutes les valeur positives (et finies) du retard.
- **Dépendante du retard:** la stabilité est garantie pour quelques valeurs du retard. Pour les autres valeurs le système peut être instable.

En observant ces affirmations, il est clair que l´approche indépendante du retard est la plus conservative.

Maintenant nous allons montrer un résultat indépendant du retard, où le retour d'état suivant doit être obtenu :

$$u_k = Kx_k,$$

où $K \in \mathbb{R}^{m\times n}$.

Considérons le modèle en temps discret :

$$x_{k+1} = Ax_k + \Delta(\epsilon_k)Kx_{k-d+1} + (B - \Delta(\epsilon_k))Kx_{k-d}.$$

La matrice qui stabilise le système ci-dessus est donné par :

$$K = YG^{-1},$$

où $G \in \mathbb{R}^{n\times n}$, $G = G^\top > 0$ et $Y \in \mathbb{R}^{m\times n}$ sont obtenus par la solution du problème d'optimisation suivant :

$$\min_{\gamma,\ G,\ G_x,\ G_y,\ Y} \quad \gamma$$

soumis à :

$$\begin{bmatrix} G & 0 & 0 & G_x & 0 & GA^\top & GQ^{\frac{1}{2}} & 0 & 0 \\ 0 & G_x & 0 & 0 & G_y & Y^\top \Delta_i^\top & 0 & Y^\top R_1^{\frac{1}{2}} & 0 \\ 0 & 0 & G_y & 0 & 0 & Y^\top (B-\Delta_i)^\top & 0 & 0 & Y^\top R_2^{\frac{1}{2}} \\ G_x & 0 & 0 & G_x & 0 & 0 & 0 & 0 & 0 \\ 0 & G_y & 0 & 0 & G_y & 0 & 0 & 0 & 0 \\ AG & \Delta_i Y & (B-\Delta_i)Y & 0 & 0 & G & 0 & 0 & 0 \\ Q^{\frac{1}{2}}G & 0 & 0 & 0 & 0 & 0 & \gamma I & 0 & 0 \\ 0 & R_1^{\frac{1}{2}}Y & 0 & 0 & 0 & 0 & 0 & \gamma I & 0 \\ 0 & 0 & R_2^{\frac{1}{2}}Y & 0 & 0 & 0 & 0 & 0 & \gamma I \end{bmatrix} \geq 0;$$

$$i = 1, \ldots, n_\Delta,$$

où $Q \in \mathbb{R}^{n\times n}$, $R_1 \in \mathbb{R}^{m\times m}$, $R_2 \in \mathbb{R}^{m\times m}$, $G_x \in \mathbb{R}^{n\times n}$, $G_y \in \mathbb{R}^{n\times n}$ et γ est un scalaire.

Pour dériver un résultat indépendant du retard, considérons le modèle en temps discret suivant :

$$x_{k+1} = Ax_k + Bu_{k-d},$$

où $x_k \in \mathbb{R}^n$ est le vecteur d'états, $A \in \mathbb{R}^{n\times n}$ et $B \in \mathbb{R}^{n\times m}$ et $d \in \mathbb{Z}_+^*$ est le retard discret. La loi de commande est sous la forme :

$$u_k = Kx_k,$$

où $K \in \mathbb{R}^{m\times n}$. Donc la boucle fermé peut être écrite comme suit :

$$x_{k+1} = Ax_k + BKx_{k-d}.$$

La relation suivante peut être dérivée :

$$x_k - x_{k-d} = \sum_{i=-d}^{-1} (x_{k+i+1} - x_{k+i}) .$$

Le système à retard, après la transformation de modèle imposé par la relation ci-dessus, devient :

$$x_{k+1} = (A + BK)x_k - BK \sum_{i=-d}^{-1} (x_{k+i+1} - x_{k+i}) .$$

La matrice de retour d'état qui stabilise ce système est donnée par :

$$K = LG^{-1},$$

où $G = G^\top$ et L sont obtenus par la solution du problème d'optimisation suivant :

$$\min_{\gamma, G, L, H, T, J, W} \gamma,$$
$$\text{soumis à : .}$$

$$\begin{bmatrix} T + T^\top + dH + J - G & -T & GA^\top & dG(A - T') & GQ_1^{1/2} & 0 \\ -T^\top & -J & L^\top B^\top & dL^\top B^\top & 0 & L^\top R_d^{1/2} \\ AG & BL & -G & 0 & 0 & 0 \\ d(A - I)G & dBL & 0 & -dG & 0 & 0 \\ Q_1^{1/2}G & 0 & 0 & 0 & -\gamma I & 0 \\ 0 & R_d^{1/2}L & 0 & 0 & 0 & -\gamma I \end{bmatrix} \leq 0;$$

$$\begin{bmatrix} H & T \\ T^\top & G \end{bmatrix} \geq 0, \; G > 0, \; J > 0.$$

3.2 Les ensembles invariants

Les techniques de stabilisation présentées auparavant ne prennent en compte ni les contraintes de trajectoire du système ni de l'entrée de commande, et peuvent alors causer des instabilités, saturations, endommagement des composantes physiques, etc. Une alternative, afin d'éviter ces conséquences et de surmonter la présence de contraintes, est l'utilisation de la théorie des ensembles invariants.

Du point de vue de la construction effective, deux types d'ensembles invariants sont en évidence : les *ellipsoïdaux* et les *polyédraux*. Au cours de cette thèse, nous utilisons plutôt les ensembles polyédraux grâce à la possibilité de les exprimer en $\mathcal{V}$-représentation et $\mathcal{H}$-représentation, plus adaptées à l'utilisation dans le cadre de la commande prédictive.

Le concept de *l'ensemble maximal des états à retard* sera également défini dans l'espace d'état augmenté en utilisant les techniques de modélisation déjà décrites. Le point faible est la complexité très importante de ce type d'ensemble : le système original étant d'ordre "n", l'ensemble obtenu est d'ordre $n + dm$. Pour surmonter cet inconvénient, le concept de $\mathcal{D}$-invariance est introduit. La $\mathcal{D}$-invariance peut être comprise comme l'invariance des états actuels du système mais aussi des états passés. Ceci étant, tous les états, quelque soit le moment de l'histoire de l'évolution du système, doivent être inclus dans un ensemble. Les propriétés associées à la $\mathcal{D}$-invariance sont également présentées. Des conditions nécessaires et suffisantes sont dérivées, comme les conditions d'existence algébriques, en ayant comme résultat des tests constructifs. Pour réduire le conservatisme, le concept de *$\mathcal{D}$-invariance cyclique* est alors proposé. Un algorithme permettant d'obtenir des ensembles $\mathcal{D}$-invariant de manière itérative est présenté.

3.2.1 Invariance des ensembles dans l'espace d'état augmenté

Dans le cadre de la commande sous contraintes pour des systèmes à retard, le but est de développer une loi de commande qui stabilise le système tout en respectant un ensemble de contraintes. Un tel ensemble peut être décrit en fonction de l'état augmenté ξ, comme suit :

$$\Gamma\xi_k + Du_k \leq W.$$

En utilisant la loi de commande stabilisante trouvée par les techniques LMI, le domaine polyédral peut être défini dans l'espace d'état augmenté, notamment :

$$P = \left\{\xi \in \mathbb{R}^{(n+md)} \mid \ (\Gamma + DK)\xi \leq W\right\}.$$

Pour un système soumis à des incertitudes polytopiques nous aurons :

$$\begin{aligned} \xi_{k+1} &= \Phi\xi_k; \\ \Phi &\in \Omega_K; \\ \Omega_K &= \mathrm{Co}\{(A_{\Delta_0} + B_{\Delta_n}K);\ \ldots;\ (A_{\Delta_n} + B_{\Delta_n}K)\} \end{aligned}$$

et un ensemble pré-défini P, *l'ensemble maximal des états admissibles à retard* O_∞^Ω est défini comme la collection de tous les états initiaux ξ_0 pour lesquels les trajectoires du système restent à l'intérieur de P pour tous les instant futurs $k \geq 0$. Cet ensemble peut être décrit comme suit :

$$O_\infty^\Omega = \left\{\xi_0 \mid \prod_{i=0}^{k} \Phi_i\xi_0 \in P, \forall\Phi_i \in \Omega_K, \forall k \in \mathbb{Z}\right\}.$$

Prenons les hypothèses suivantes :

1. Il existe une fonction de Lyapunov en commun qui stabilise asymptotiquement les systèmes $\xi_{k+1} = \Phi\xi_k, \Phi \in \Omega_K$.
2. L'ensemble P est borné.
3. $0 \in \text{int } P$.

Donc l'algorithme pour obtenir O_∞^Ω s'arrête en temps fini, si et seulement si $O_N^\Omega = O_{N+1}^\Omega$, où :

$$O_N^\Omega = \left\{ \xi_0 \mid \prod_{i=0}^{N} \Phi_i \xi_0 \in P, \forall \Phi_i \in \Omega_K, \forall i \in \mathbb{Z}_{[1,N]} \right\}$$

Cet ensemble peut être réécrit ainsi :

$$O_{N+1}^\Omega = \{\xi \in O_N \mid \Phi\xi \in P, \forall \Phi \in \Omega_K\}$$

et à partir de la notation $\Phi_i = A_{\Delta_i} + B_\Delta K$, $i \in \mathbb{Z}_{[0,n]}$ nous pouvons obtenir la forme algorithmique suivante :

$$O_{N+1}^\Omega = \{\xi \in O_N \mid \Phi_i\xi \in P, \forall \Phi_i \in \{\Phi_0, \ldots, \Phi_n\}\},$$

à utiliser récursivement pour obtenir l'ensemble maximal des états admissibles à retard.

3.2.2 Invariance des ensembles dans l'espace d'état « originel »

Pour définir le concept de $\mathcal{D}$-invariance et $\mathcal{D}$-contractivité, l'équation à différences à retard suivante est considérée :

$$x_{k+1} = \sum_{i=0}^{d} A_i x_{k-i},$$

où $x_k \in \mathbb{R}^n$ est le vecteur d'état dans le temps $k \in \mathbb{Z}_+$. $A_i \in \mathbb{R}^{n\times n}$, pour tout $i \in \mathbb{Z}_{[0,d]}$ et les conditions initiales sont données par la séquence $x_{-i} \in \mathbb{R}^n$, $i \in \mathbb{Z}_{[0,d]}$. Ce type de système peut être trouvé dans plusieurs applications pratiques, décrites dans le manuscrit.

Le concept de $\mathcal{D}$-invariance, défini ultérieurement, sera utilisé extensivement pendant la thèse et est important pour caractériser l'invariance pour des systèmes à retard.

Considérons $\varepsilon \in \mathbb{R}_{[0,1]}$. Un ensemble $\mathcal{P} \subseteq \mathbb{R}^n$ qui contient l'origine est dit $\mathcal{D}$-contractif par rapport à l'équation à différences à retard si l'inclusion suivante est respectée :

$$\bigoplus_{i=0}^{d} A_i \mathcal{P} \subseteq \varepsilon \mathcal{P}.$$

Quand $\varepsilon = 1$, $\mathcal{P}$ est considéré comme un ensemble $\mathcal{D}$-invariant.

À partir de cette définition quelques propriétés intéressantes peuvent être exploitées :

- Les affirmatives suivantes sont équivalentes :
  i) Un ensemble $\mathcal{P} \subseteq \mathbb{R}^n$ est $\mathcal{D}$-invariant pour l'équation à différences à retard.
  ii) $\bigoplus_{i=0}^{d} A_i \mathcal{P} \subseteq \mathcal{P}$.
- Considérons un ensemble convexe $\mathcal{P} \subseteq \mathbb{R}^n$ qui contient l'origine. Si $\mathcal{P}$ est $\mathcal{D}$-invariant par rapport au système alors $\mathcal{P}$ est invariant par rapport a n'importe quelle dynamique LTI (Linéaire et invariante dans le temps) :

$$x_{k+1} = A_i x_k, \quad \forall i \in \mathbb{Z}_{[0,d]}.$$

  De façon équivalente :

$$A_0 \mathcal{P} \subseteq \mathcal{P},\ A_1 \mathcal{P} \subseteq \mathcal{P},\ \ldots,\ A_d \mathcal{P} \subseteq \mathcal{P}.$$

- Si $\mathcal{P} \subseteq \mathbb{R}^n$ est $\mathcal{D}$-invariant par rapport au système alors $\alpha\mathcal{P}$ est $\mathcal{D}$-invariant par rapport à la même dynamique pour n'importe quel $\alpha \in \mathbb{R}_+^*$.
- Considérons $\mathcal{P}_1, \mathcal{P}_2 \subseteq \mathbb{R}^n$ comme étant deux ensembles $\mathcal{D}$-invariants par rapport à la même dynamique. Donc $\mathcal{P}_1 \cap \mathcal{P}_2$ est un ensemble $\mathcal{D}$-invariant pour le même système.
- Considérons l'ensemble $\mathcal{P} \subseteq \mathbb{R}^n$ qui contient l'origine. Si $\mathcal{P}$ est $\mathcal{D}$-invariant par rapport à la dynamique alors $\mathcal{P}$ est positif invariant par rapport à la même dynamique linéaire :

$$\begin{aligned} x_{k+1} &= A_0 x_k, \\ &\vdots \\ x_{k+1} &= A_d x_k. \end{aligned}$$

  De façon équivalente, $A_i \mathcal{P} \subseteq \mathcal{P}$, avec $i \in \mathbb{Z}_{[0,d]}$.

- Etant donné un ensemble $\mathcal{D}$-invariant $\mathcal{P} \subseteq \mathbb{R}^n$ par rapport à la dynamique, alors $\mathcal{P}$ est $\mathcal{D}$-invariant pour :

$$x_{k+1} = A_d x_k + \cdots + A_0 x_{k-d}.$$

- Pour un $d \in \mathbb{Z}_+^*$, considérons un ensemble $\mathcal{D}$-invariant $\mathcal{P} \subseteq \mathbb{R}^n$ par rapport à la dynamique. Ainsi $\mathcal{P}$ est $\mathcal{D}$-invariant par rapport à la même dynamique pour n'importe quel $d \in \mathbb{Z}_+^*$.

Conditions nécessaires pour la $\mathcal{D}$-invariance :

- Les rayons spectraux des matrices A_i sont sous-unitaires :

$$\rho(A_i) \leq 1, \ \forall i \in \{0, d\};$$

- Le rayon spectral de la matrice $\sum_{i=0}^{d}(A_i)$ est sous-unitaire :

$$\rho\left(\sum_{i=0}^{d}(A_i)\right) \leq 1;$$

- Le rayon spectral de la matrice d'état étendu est sous-unitaire :

$$\rho\left(\begin{bmatrix} A_0 & A_1 & \dots & A_{d-1} & A_d \\ I & 0 & \dots & 0 & 0 \\ 0 & I & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & I & 0 \end{bmatrix}\right) < 1.$$

Conditions suffisantes pour la $\mathcal{D}$-invariance :

- L'addition des rayons spectraux de chaque A_i est sous-unitaire :

$$\rho(A_0) + \cdots + \rho(A_d) < 1.$$

- Dans le cas de la matrice non-singulière A_0 (ou A_d) :

$$\begin{gathered}(1 + \sigma(A_0^{-1} \dots A_d))\sigma(A_0) \leq 1, \\ \vdots \\ (1 + \sigma(A_d^{-1} \dots A_{d-1}))\sigma(A_d) \leq 1.\end{gathered}$$

Nous pouvons dériver les conditions algébriques pour la $\mathcal{D}$-invariance d'un ensemble : considérons un ensemble polyédrale $\mathcal{P}$ en $\mathbb{R}^n$ qui contient l'origine, alors il existe $F \in \mathbb{R}^{r\times n}$ tel que :

$$\mathcal{P} = \{x \in \mathbb{R}^n \mid Fx \leq \mathbb{1}\}$$

est $\mathcal{D}$-contractif par rapport au système, si et seulement si, les matrices $H_i \in \mathbb{R}^{r\times r}$ pour $i \in \mathbb{Z}_{[0,d]}$ avec des éléments non-négatifs telles que :

$$FA_i = H_i F$$

et

$$\left(\sum_{i=0}^{d} H_i\right)\mathbb{1} \leq \varepsilon\mathbb{1}.$$

Quand $\varepsilon = 1$, $\mathcal{P}$ est considéré un ensemble $\mathcal{D}$-invariant.

Deux types de vérification de la $\mathcal{D}$-invariance sont ainsi proposés en tenant compte les définitions ci-dessus :

- Test fondé sur l'addition de Minkowski : en appliquant directement la définition de $\mathcal{D}$-invariance, si l'inclusion est respectée, l'ensemble peut être dit $\mathcal{D}$-invariant par rapport à la dynamique.
- Test fondé sur la faisabilité : en utilisant les conditions algébriques, trois types de test peuvent être dérivés, tels que :
  - Application directe des conditions algébriques sous la forme d'un PL (Programmation Linéaire) : L'ensemble polyédral $\mathcal{P}$ est $\mathcal{D}$-invariant par rapport à la dynamique s'il existe un vecteur $\bar{h} \in \mathbb{R}^{dr^2}$ avec des éléments non-négatifs obtenus comme solutions de :

$$\min_{\bar{h}} \quad \varepsilon$$
$$\text{soumis à :} \quad \begin{cases} A_{eq}\bar{h} = b_{eq} \\ A_{in}\bar{h} \leq b_{in} \\ \bar{h} \geq 0 \\ 0 \leq \varepsilon \leq 1 \end{cases}$$

    Si $\varepsilon < 1$ l'ensemble $\mathcal{P}$ est $\mathcal{D}$-contractif.
  - La $\mathcal{D}$-invariance est vérifiée en utilisant la forme duale suivante :

$$\mathcal{P} = \left\{x \in \mathbb{R}^n \middle| \left\{f_{j,[1,n]}\right\} x \leq 1,\, j \in \mathbb{Z}_{[1,r]},\, \forall f \in \tilde{\mathcal{P}}\right\},$$
$$\tilde{\mathcal{P}} = \left\{f_j \in \mathbb{R}^n \middle| \left\{f_{j,[1,n]}\right\} x \leq 1,\, j \in \mathbb{Z}_{[1,r]},\, \forall x \in \mathcal{P}\right\},$$

où $\left\{f_{j,[1,n]}\right\}$ est la j^{eme} ligne de la matrice F. L'ensemble $\tilde{\mathcal{P}}$ est $\mathcal{D}$-invariant s'il existe un vecteur $\tilde{h} \in \mathbb{R}^{ndr}$ avec des éléments non-négatifs obtenus comme solutions de :

$$\min_{\tilde{h}} \quad \varepsilon$$
$$\text{soumis à :} \quad \begin{cases} \tilde{A}_{eq}\tilde{h} = \tilde{b}_{eq} \\ 0 \leq \varepsilon \leq 1 \\ \tilde{h} \geq 0 \end{cases}$$

où $\tilde{A}_{eq} \in \mathbb{R}^{drn \times dr^2+1}$, $\tilde{b}_{eq} \in \mathbb{R}^{drn}$, $\tilde{A}_{in} \in \mathbb{R}^{dr^2+r+2 \times dr^2+1}$, $\tilde{b}_{in} \in \mathbb{R}^{dr^2+r+2}$ et $\tilde{h} \in \mathbb{R}^{dr^2+1}$. Si $\varepsilon < 1$ l'ensemble $\mathcal{P}$ est $\mathcal{D}$-contractif.

– La $\mathcal{D}$-invariance est vérifiée en utilisant la fonction de support. Dans ce cas, les ensembles convexes sont partiellement ordonnés par une inclusion et la fonction de support préserve cette structure. Considérons deux ensembles $\mathcal{P}_1$ et $\mathcal{P}_2$, la relation entre leurs fonction de support $S(\mathcal{P}_1, u) \leq S(\mathcal{P}_2, u)$ est préservée pour tous $u \in \mathbb{R}^n$ si et seulement si $\mathcal{P}_1 \subset \mathcal{P}_2$. Pour utiliser ce fait dans un test de $\mathcal{D}$-invariance, nous avons besoin d'exploiter deux propriétés de la fonction de support pour un vecteur $u \in \mathbb{R}^n$:
  * $S(\bigoplus_{i=0}^{d} \mathcal{P}_i, u) = \sum_{i=0}^{d} S(\mathcal{P}_i, u)$.
  * $S(A\mathcal{P}, u) = S(\mathcal{P}, A^T u)$.

  Le test de $\mathcal{D}$-invariance peut être implémenté au niveau d'un problème PL en tenant compte du nombre de lignes de la matrice F qui décrit l'ensemble $P = \{x \in \mathbb{R}^n | Fx \leq \mathbb{1}\}$.

$$\begin{aligned} S(\textstyle\bigoplus_{i=0}^{d} A_i\mathcal{P}, f_{j,[1:n]}^T) = \quad & \min_{x_0,\dots,x_d} -\gamma \\ \text{soumis à :} \quad & \gamma \leq f_{j,[1:n]}(A_0x_0 + \cdots + A_dx_d) \\ & Fx_i \leq \mathbb{1}, i = 0, \dots, d \end{aligned}$$

  Finalement, si $S(\bigoplus_{i=0}^{d} A_i\mathcal{P}, f_{j,[1:n]}^T) \leq \varepsilon, \forall j$, l'ensemble $\mathcal{P}$ est $\mathcal{D}$-invariant.

En utilisant aussi les conditions algébriques, nous pouvons dériver la synthèse d'un correcteur, de telle façon à respecter la propriété de $\mathcal{D}$-invariance. Considérons un système sous la forme :

$$x_{k+1} = \sum_{i=0}^{d} A_i x_{k-i} + \sum_{i=0}^{d} B_i u_{k-i},$$

où $x_k \in \mathbb{R}^n$ est le vecteur d'état, $u_k \in \mathbb{R}^r$ est l'entrée de commande à l'instant $k \in \mathbb{Z}_+$, $A_i \in \mathbb{R}^{n \times n}$, $B_i \in \mathbb{R}^{n \times m}$, pour tout $i \in \mathbb{Z}_{[0,d]}$ et les conditions initiales sont données par la séquence $x_{-i}, u_{-i} \in \mathbb{R}^n$, $i \in \mathbb{Z}_{[0,d]}$.

Nous pouvons supposer que le système peut être stabilisé par un retour d'état :

$$u_k = Kx_k,$$

avec $K \in \mathbb{R}^{m\times n}$, en ayant la boucle fermée :

$$x_{k+1} = \sum_{i=0}^{d} (A_i + B_i K)x_{k-i}.$$

Considérons l'ensemble polyédral de contraintes suivant :

$$\mathcal{P} = \{x \in \mathbb{R}^n \mid Fx \leq \mathbb{1}\},$$

avec $F \in \mathbb{R}^{r\times n}$, est $\mathcal{D}$-invariant par rapport au système et la loi de commande est la solution stabilisante, si et seulement s'il existent des matrices $H_i \in \mathbb{R}^{r\times r}$ pour $i \in \mathbb{Z}_{[0,d]}$, $K \in \mathbb{R}^{m\times n}$ et $\lambda \in \mathbb{R}_{[0,1]}$ tel que :

$$F(A_i + B_i K) = H_i F$$

et

$$\left(\sum_{i=0}^{d} H_i\right)\mathbb{1} \leq \lambda\mathbb{1}.$$

La définition de $\mathcal{D}$-invariance cyclique est importante pour réduire le conservatisme de la définition stricte de $\mathcal{D}$-invariance. Les ensembles $\mathcal{P}_i \subseteq \mathbb{R}^n$ qui contiennent l'origine sont considérés $\mathcal{D}$-invariant cycliques par rapport à la dynamique si les inclusions suivantes sont respectées :

$$\begin{gathered} A_0\mathcal{P}_0 \oplus A_1\mathcal{P}_1 \oplus \cdots \oplus A_d\mathcal{P}_d \subseteq \mathcal{P}_0; \\ A_0\mathcal{P}_d \oplus A_1\mathcal{P}_0 \oplus \cdots \oplus A_d\mathcal{P}_{d-1} \subseteq \mathcal{P}_d; \\ \vdots \\ A_0\mathcal{P}_1 \oplus A_1\mathcal{P}_2 \oplus \cdots \oplus A_d\mathcal{P}_0 \subseteq \mathcal{P}_1. \end{gathered}$$

Après « d » cycles, la séquence revient à l'ordre initial et nous fournit une formulation indépendante du temps au niveau de l'invariance des ensembles. En même temps les conditions algébriques pour la $\mathcal{D}$-invariance cyclique peuvent être dérivées. Considérons les ensembles polyédraux $\mathcal{P}_i \subseteq \mathbb{R}^n$, pour $i \in \mathbb{Z}_{[0,d]}$ qui contiennent l'origine, il existe $F_i \in \mathbb{R}^{r\times n}$ tel que :

$$\mathcal{P}_i = \{x \in \mathbb{R}^n \mid F_i x \leq \mathbb{1}\}$$

La séquence $\{P_0, \ldots, P_d\}$ est $\mathcal{D}$-invariante cyclique par rapport à la dynamique, si et seulement si, il exist les matrices $H_{ij} \in \mathbb{R}^{r\times r}$ for $i,j \in \mathbb{Z}_{[0,d]}$ avec des éléments non-négatifs tel que :

$$F_i A_j = H_{ij} F_i$$

et

$$\left(\sum_{j=0}^{d} H_{0j}\right)\mathbb{1} \leq \mathbb{1}; \quad \cdots ; \quad \left(\sum_{j=0}^{d} H_{dj}\right)\mathbb{1} \leq \mathbb{1}.$$

Des ensembles $\mathcal{D}$-invariants peuvent être obtenus par des algorithmes, dans un cadre d'itérations des ensembles, en considérant l'ensemble de contraintes comme étant l'ensemble initial. Considérons les matrices $A_i \in \mathbb{R}^{n\times n}$, pour $i \in \mathbb{Z}_{[0,d]}$, les itérations suivantes $\Phi : ComC(\mathbb{R}^n) \to ComC(\mathbb{R}^n)$ et $\Psi : ComC(\mathbb{R}^n) \to ComC(\mathbb{R}^n)$ sont définies par :

$$\Phi(\mathcal{P}) = \bigoplus_{i=0}^{d} A_i\mathcal{P};$$

$$\Psi(\mathcal{P}) = Co(\mathcal{P}, \bigoplus_{i=0}^{d} A_i\mathcal{P}) = Co(\mathcal{P}, \Phi(\mathcal{P})).$$

Les itérations ci-dessus peuvent être utilisées pour définir k-itérations :

$$\Phi^k(\mathcal{P}) = \Phi(\Phi^{k-1}(\mathcal{P})), k \geq 0 \text{ with } \Phi^0(\mathcal{P}) = \mathcal{P},$$
$$\Psi^k(\mathcal{P}) = \Psi(\Psi^{k-1}(\mathcal{P})), k \geq 0 \text{ with } \Psi^0(\mathcal{P}) = \mathcal{P}.$$

Quelques propriétés concernant les itérations des ensembles peuvent être dérivées, par exemple une séquence des ensembles décroissante/croissante. Considérons l'itération $\Pi(.)$ comme une fonction de $ComC(\mathbb{R}^n)$ sur lui même. Si $\mathcal{P} \subseteq \Pi(\mathcal{P})$ pour un ensemble $\mathcal{P} \subseteq \mathbb{R}^n$ alors la séquence d'itérations $\Pi(\ldots(\Pi(\mathcal{P})))$ est non-décroissante. De façon similaire, si $\Pi(\mathcal{P}) \subseteq \mathcal{P}$ alors la séquence est non-croissante. Donc les propriétés suivantes sont vraies :

1. $\Psi^k(\mathcal{P})$ est non-décroissante :

   $$\Psi^{k-1}(\mathcal{P}) \subseteq \Psi^k(\mathcal{P}),$$

   pour tout $k \in \mathbb{Z}_+^*$ et pour n'importe quel $\mathcal{P} \in ComC(\mathbb{R}^n)$.

2. Si $\Phi(\mathcal{P}) \subseteq \mathcal{P}$ alors $\Phi^k(\mathcal{P})$ avec $k \in \mathbb{Z}_+$ est non-croissante :

   $$\Phi^k(\mathcal{P}) \subseteq \Phi^{k-1}(\mathcal{P}),$$

   pour tout $k \in \mathbb{Z}_+^*$.

3. Si $\mathcal{P}$ est un ensemble convexe et $\mathcal{D}$-invariant alors $\mathcal{P}$ est un *point fixe* pour $\Psi(.)$, tel que :

$$\Psi(\mathcal{P}) = \mathcal{P}.$$

Étant donné un ensemble $X \subseteq \mathbb{R}^n$, si $\mathcal{P} \subseteq X$ est $\mathcal{D}$-invariant et pour tous les ensembles $\mathcal{D}$-invariants $\mathcal{R} \subseteq X$, alors l'ensemble $\mathcal{R} \subseteq \mathcal{P}$, $\mathcal{P}$ est appelé l'ensemble $\mathcal{D}$-invariant maximal en X.

L'itération $\Phi(\cdot)$ en $\Pi(.)$ est une *contraction*, s'il existe un facteur $\alpha \in \mathbb{R}_{[0,1)}$ tel que, pour deux ensembles arbitraires $\mathcal{P}_1 \subseteq \mathbb{R}^n$ et $\mathcal{P}_2 \subseteq \mathbb{R}^n$ la *distance de Hausdorff* est vérifiée :

$$d_H(\Phi(\mathcal{P}_1), \Phi(\mathcal{P}_2)) \leq \alpha d_H(\mathcal{P}_1, \mathcal{P}_2).$$

Le comportement contractif de $\Psi(\cdot)$ peut être défini de façon similaire.

L'algorithme pour la construction des ensembles $\mathcal{D}$-invariants utilise les bases théoriques développées ci-dessus, de telle sorte à obtenir l'ensemble $\mathcal{D}$-invariant maximal par rapport à la dynamique. La procédure est fondée sur les propriétés de contraction des ensembles et converge vers l'ensemble $\mathcal{D}$-invariant maximal. Quelques définitions auxiliaires doivent être définies avant d'introduire l'algorithme :

- Si $\mathcal{P} \in ComC(\mathbb{R}^n)$ est un ensemble $\mathcal{D}$-invariant pour la dynamique alors pour n'importe quel sous-ensemble $\mathcal{S} \subseteq \mathcal{P}$ il est vrai que :

$$\Phi^k(\mathcal{S}) \subseteq \mathcal{P}, \forall k \in \mathbb{Z}_+^*.$$

- Etant donné $\mathcal{P} \subset \mathcal{X}$, la suite est vraie :

$$\hat{\mathcal{P}} \subseteq \mathcal{X};$$
$$\hat{\mathcal{P}} = \{\mathcal{D} \subseteq \mathcal{X} | \mathcal{P} \subseteq \mathcal{D},\ \mathcal{D} \text{ is } \mathcal{D}\text{-invariant}\}.$$

- $$d^* = \inf_{\mathcal{D} \in \hat{\mathcal{P}}} d_H(\mathcal{D}, \mathcal{P}) \geq 0$$

- $$\hat{\mathcal{P}}^* = \left\{\mathcal{D} \subseteq \hat{\mathcal{P}} | d_H(\mathcal{D}, \mathcal{P}) = d^*\right\}.$$

- Considérons la suite :

$$\lim_{k \to \infty} \Psi^k(\mathcal{P}) \in \hat{\mathcal{P}}^*.$$

  Alors la limite est un des ensembles $\mathcal{D}$-invariants avec la distance de Hausdorff minimale.

- Étant donné un ensemble $\mathcal{P} \in ComC(\mathbb{R}^n)$, la séquence $\Psi^k(\mathcal{P}), k \geq 0$ converge vers le « plus proche » (dans un sens de la distance de Hausdorff) ensemble $\mathcal{D}$-invariant.

Finalement, à partir de l'itération des ensembles, un ensemble convexe et $\mathcal{D}$-invariant peut être obtenu. Le but est d'obtenir une approximation de l'ensemble $\mathcal{D}$-invariant maximal dans une région pré-définie $\mathcal{X} \subseteq \mathbb{R}^n$. Étant donnée une équation à différences à retard et un ensemble compact et convexe $\mathcal{X} \subseteq \mathbb{R}^n$, l'ensemble initial est noté $\mathcal{R}_0 = \mathcal{X}_0$. À partir de l'opération $\mathcal{T} = Co(\mathcal{R}_j, x)$ avec $x \in int(\mathcal{X} \setminus \mathcal{R}_j)$, l'ensemble limite suivant est obtenu : $\mathcal{D} = \lim_{k\to\infty} \Phi^k(\mathcal{T})$. Si cet ensemble est $\mathcal{D}$-invariant alors l'algorithme continue en ajoutant des régions tel que $x \in int(\mathcal{X} \setminus \mathcal{R}_j)$. Sinon une autre région doit être choisie.

3.3 La commande prédictive

La commande prédictive à base de modèle (MPC) est une des techniques les plus importantes dans la recherche et dans l'industrie. Grâce à sa performance au niveau des applications industrielles (aussi appelée *Commande par horizon glissant*). La MPC peut être appliquée au cas monovariable (SISO), multivariable (MIMO) et même au cas non-linéaire. L'avantage est l'application d'une loi de commande optimale, tout en considérant les contraintes physiques du systèmes et le retard durant la synthèse de la loi de commande.

Dans les techniques de commande sous contraintes classiques, le but est d´éviter l'activation des contraintes, de telle façon à éviter la saturation des actionneurs et l'endommagement des composantes physiques, en ayant comme résultat des approches conservatives. En commande prédictive, grâce à la théorie des ensembles invariants, les trajectoires du système peuvent être ramenées près des bornes des contraintes. De plus, une loi de commande optimale est appliquée, en garantissant une meilleure performance par rapport aux approches de commande classique.

En parlant de la commande prédictive par modèle, quelques caractéristiques peuvent être mises en évidence :

- Comme son propre nom l'indique, le *modèle* du système doit être utilisé pour « prévoir » les trajectoires futures du système dans une « fenêtre » de taille « N » périodes d'échantillonnage dans le futur, appelé horizon de prédiction.

- La loi de commande est donnée par la minimisation d'une fonction de coût, calculée à chaque pas d'échantillonnage.

- Le principe de l'horizon glissant signifie le « déplacement » de l'horizon de prédiction dans le futur.

- Le premier échantillon de la séquence de commande optimale est appliqué et recalculé à chaque pas.
- La taille de l'horizon de prédiction doit être plus grande que la taille du retard en temps discret, toutefois, pour prendre en compte les effets du retard dans le schéma prédictif.

À chaque nouvel instant de temps t_k, les trajectoires du système sont prédites ($\hat{x}_{k+j|k}$, pour $j \in \mathbb{Z}_{[1,N]}$) en utilisant le modèle du système. Il est évident que les prédictions dépendent aussi des entrées de commande futures du système ($u_{k+j|k}$, pour $j \in \mathbb{Z}_{[1,N-1]}$). La fonction de coût est minimisée par rapport à un critère de façon à garder les trajectoires du système les plus proches possible de la référence, en pondérant les états et la commande. Généralement, ce critère est une fonction quadratique de l'erreur entre la sortie prédite et les trajectoires futures du système. Pour finir, l'entrée de commande u_k est envoyée lors que le reste de la séquence ($j \in \mathbb{Z}_{[1,N-1]}$) est rejetée.

La commande prédictive construit à chaque pas « k » la séquence de commande optimale :

$$\mathbf{k}_u^* = \{u_{k|k}, \ldots, u_{k+N-d-1|k}\},$$

par rapport a un critère de performance vis à vis des trajectoires du système dans un horizon fini $k+1, \ldots, k+N$. En effet, la prédiction repose sur le modèle nominal du système, mais le système réel est affecté par une retard discret d_{max}. Pour prendre en compte toutes les variations possibles du retard, nous devons considérer $N \geq d_{max}$.

Comme nous avons mentionné ci-dessus, la première composante $\mathbf{k}_u^*$ est appliquée au système :

$$u_k = \mathbf{k}_u^*(1) = u_{k|k},$$

lors que le reste est rejeté. En utilisant les nouvelles sorties du système, la procédure d'optimisation recommence et ainsi la boucle est fermée.

Si les contraintes physiques du système sont décrites sous la forme polyédrale du type $Cx_k \leq W$, l'implémentation de la commande prédictive sera donnée par un problème d'optimisation de la forme suivante :

$$\mathbf{k}_u^* = \underset{\{u_{k|k},\ldots,u_{k+N-d-1|k}\}}{\arg\min} \left\{ x_{k+N|k}{}^{\top} \bar{P} x_{k+N|k} + \sum_{j=1}^{N} x_{k+j|k}{}^{\top} \bar{Q} x_{k+j|k} + \sum_{j=0}^{N-d-1} u_{k+j|k}^{\top} \bar{R} u_{k+j|k} \right\}$$

soumis à :

$$\begin{cases} x_{k+j+1|k} = \bar{F}x_{k+j|k} + \bar{G}u_{k+j|k} \\ Cx_{k+j|k} \leq W;\ j = 1, \ldots, N-1 \\ u_{k+i|k} = 0, \quad i = N-d, \ldots, N-1 \\ x_{k+N} \in O_N; \end{cases}$$

où O_N est l'ensemble terminal (dans le cadre de cette thèse, un ensemble maximal des états à retard), décrit par le polyèdre :

$$O_N = \left\{ \xi \in \mathbb{R}^{n+dm} \mid H_{O_N}\xi \leq W_{O_n} \right\},$$

où $H_{O_N} \in \mathbb{R}^{r\times(n+dm)}$ (pour le cas où nous travaillons dans l'espace d'état augmenté). Pour le cas $\mathcal{D}$-invariant, nous avons l'ensemble O_N :

$$O_N = \{x \in \mathbb{R}^n \mid H_{O_N}x \leq W_{O_n}\},$$

où $H_{O_N} \in \mathbb{R}^{r\times n}$.

La loi de commande prédictive sera influencée par le choix de l'horizon de prédiction « N », de la matrice de pondération des états $\bar{Q} = \bar{Q}^\top \geq 0$ et de la commande $\bar{R} = \bar{R}^\top \geq 0$. Pour la « pénalisation » de l'ensemble terminal, la matrice $\bar{P}$ est normalement construite telle que l'horizon de prédiction soit étendu à l'infini, par l'introduction du terme $x_{k+N|k}{}^\top \bar{P}x_{k+N|k}$. Par contre, $x_{k+N|k}$ doit satisfaire des conditions imposées par l'ensemble terminal, décrit par l'ensemble invariant, qui « force » les états à atteindre l'ensemble X_N dans l'horizon de prédiction. Un choix habituel est *l'ensemble maximal des états à retard* (ou maximal de sortie admissible) $X_N = O_\infty$, construit pour notre système à retard, avec la loi de commande qui satisfait l'équation de Riccati :

$$\begin{aligned} \bar{P} &= \bar{Q} + \bar{F}^\top \bar{P}\bar{F} - \bar{K}^\top \left(\bar{R} + \bar{G}^\top \bar{P}G\right)\bar{K} \\ \bar{K} &= -\left(\bar{R} + \bar{G}^\top \bar{P}G\right)^{-1} \bar{G}^\top F \end{aligned}$$

Le problème décrit ci-dessus peut être modifié de sorte à obtenir une version dans un format plus adapté aux applications pratiques. En faisant la notation $\mathbf{x} = \left\{x_{k+1|k}^\top, \ldots, x_{k+N|k}^\top\right\}$, les états peuvent être mises sous la forme compacte, et la fonction de coût peut être récrite ainsi :

$$\begin{aligned} &\arg\min_{\mathbf{k}_u} x_k^\top Qx_k + \mathbf{x}^\top \bar{Q}\mathbf{x} + \mathbf{k}_u^\top \bar{R}\mathbf{k}_u; \\ &\text{soumis a : } \left\{ \bar{C}\mathbf{k}_u + \tilde{E}x + \bar{E}\mathbf{x} \leq \bar{W} \right., \end{aligned}$$

où :

$$\bar{Q} = \text{diag}(\text{diag}(Q, N), P); \quad \bar{R} = \text{diag}(R, N); \quad \bar{C} = \text{diag}(C, N);$$

$$\bar{E} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ E & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & E & 0 \end{bmatrix}; \quad \bar{W} = \begin{bmatrix} W \\ \vdots \\ W \end{bmatrix}; \quad \tilde{E} = \begin{bmatrix} E \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

où P est donnée, par exemple, par la solution d'une équation de Riccati ou une fonction de Lyapunov.

Dans le cadre de cette thèse, les dimensions de matrices dépendent de la modélisation considérée et de la technique de stabilisation choisie. Par exemple, si le modèle est « nominal » et la technique de stabilisation choisie et celle de Lyapunov-Krasovskii, les dimensions des matrices du système seront $A \in \mathbb{R}^{n\times n}$ et $B \in \mathbb{R}^{m\times n}$ et ont une influence directe sur les matrices et vecteurs suivants : $\bar{Q} \in \mathbb{R}^{Nn\times Nn}$, $\bar{R} \in \mathbb{R}^{Nm\times Nm}$, $\bar{C} \in \mathbb{R}^{Nr\times Nn}$, $\bar{E} \in \mathbb{R}^{Nr\times Nn}$, $\bar{W} \in \mathbb{R}^{Nr\times 1}$, $\tilde{E} \in \mathbb{R}^{Nr\times n}$.

Si le modèle utilisé est « augmenté », le modèle nominal suivant peut être exprimé :

$$\xi_{k+1} = A_\Theta \xi_k + B_\Theta u_k,$$

avec le vecteur d'état augmenté suivant :

$$\xi_k^\top = \begin{bmatrix} x_k^\top & u_{k-d}^\top & \ldots & u_{k-1}^\top & \end{bmatrix}^\top$$

et les matrices du système augmenté :

$$A_\Theta = \begin{bmatrix} A & B & 0 & \ldots & 0 \\ 0 & 0 & I_m & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & I_m \\ 0 & 0 & 0 & \ldots & 0 \end{bmatrix}; \; B_\Theta = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ I_m \end{bmatrix}^\top,$$

où $A_\Theta \in \mathbb{R}^{(n+md)\times(n+md)}$ et $B_\Theta \in \mathbb{R}^{(n+md)\times m}$, et par conséquent, les dimensions seront : $\bar{Q} \in \mathbb{R}^{N(n+dm)\times N(n+dm)}$, $\bar{R} \in \mathbb{R}^{Nm\times Nm}$, $\bar{D} \in \mathbb{R}^{Nr\times N(n+dm)}$, $\bar{E} \in \mathbb{R}^{Nr\times N(n+dm)}$, $\bar{W} \in \mathbb{R}^{Nr\times 1}$, $\tilde{E} \in \mathbb{R}^{Nr\times(n+dm)}$.

Le système de prédiction peut être décrit par :

$$\mathbf{x} = \Theta x_k + \Upsilon \mathbf{k}_u,$$

où :

$$\Theta = \begin{bmatrix} A \\ A^2 \\ \vdots \\ A^N \end{bmatrix}; \quad \Upsilon = \begin{bmatrix} B & 0 & \cdots & 0 \\ AB & B & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A^{N-1}B & A^{N-2}B & \cdots & B \end{bmatrix},$$

quand nous travaillons dans l'espace d'état originel. Sinon, pour le cas augmenté, la substitution de A et de B pour A_Θ et B_Θ doit être faite.

Le problème d'optimisation peut maintenant être reformulé comme un problème quadratique multiobjectif :

$$\begin{aligned} &\mathbf{k}_u^*(\xi_k) = \arg\min_{\mathbf{k}_u} \left\{0.5\mathbf{k}_u^\top H \mathbf{k}_u + \mathbf{k}_u^\top G x_k\right\} \\ &\text{soumis a : } A_{in}\mathbf{k}_u \leq b_{in} + B_{in}x_k \end{aligned}$$

où $H = \bar{R} + \Upsilon^\top \bar{Q}\Upsilon$, $G = \Upsilon\bar{Q}\Theta$, $A_{in} = \bar{C} + \bar{E}\Upsilon$, $B_{in} = \tilde{E} + \bar{E}\Theta$, $b_{in} = \bar{W}$ et le vecteur x_k a le rôle de paramètre. Pour rajouter des contraintes terminales à partir de l'ensemble O_N, les inégalités peuvent être décrites par :

$$H_{O_N}\left(A^N x_k + \begin{bmatrix} A^{N-1}B & A^{N-2}B & \ldots & B \end{bmatrix} \mathbf{k}_u\right) \leq W_{O_N},$$

ou encore :

$$H_{O_N}\begin{bmatrix} A^{N-1}B & A^{N-2}B & \ldots & B \end{bmatrix} \mathbf{k}_u \leq W_{O_N} - H_{O_N}A^N x_k.$$

Donc, l'ensemble polyédral de contraintes est le suivant :

$$A_{in} = \begin{bmatrix} \bar{C} + \bar{E}\Upsilon \\ H_{O_N}\begin{bmatrix} A^{N-1}B & A^{N-2}B & \ldots & B \end{bmatrix} \end{bmatrix}; \quad B_{in} = \begin{bmatrix} \tilde{E} + \bar{E}\Theta \\ H_{O_N}A^N \end{bmatrix}; \quad b_{in} = \begin{bmatrix} \bar{W} \\ W_{O_N} \end{bmatrix}.$$

Considérez un problème de commande du type min-max pour le système à retard à commander :

$$\min_{u_k, u_{k+1}, u_{k+2}\cdots} \quad \max_{F \in \Omega_\xi} J_\infty$$

avec :

$$\begin{aligned} J_\infty &= \sum_{i=0}^{\infty} \xi_{k+i}^\top Q \xi_{k+i} + u_{k+i}^\top R u_{k+i}, \\ u_k &= K\xi_k, \end{aligned}$$

où $Q > 0$, $R > 0$ sont des matrices de pondération déterminés *a priori* et K est la matrice de retour d'état.

Par les techniques présentées précédemment, un retour d'état peut être obtenu dans l'espace d'état augmenté ou originel du système, qui n'est qu'une solution de stabilisation pour le système dans le cas sans contraintes. L'idée est d'utiliser par la suite cette information pour régler les paramètres Q, R et P de façon plus optimale. L'obtention de ce critère de performance correspond à la solution d'un problème du type LQ ($K = YS^{-1} \leftrightarrow K_{LQ}$) et peut être vu comment un problème d'optimalité inverse. Étant données les matrices $\bar{F}, \bar{G}$ et Y, S, les matrices $\bar{Q} \geq 0$ et $\bar{R} > 0$ (et indirectement $\bar{P} \geq 0$) sont reconstruites de telle façon que la solution optimale pour le cas sans contrainte soit :

$$\mathbf{k}_u^* = \begin{bmatrix} YS^{-1} \\ YS^{-1}(\bar{F} + \bar{G}YS^{-1}) \\ \vdots \\ YS^{-1}(\bar{F} + \bar{G}YS^{-1})^{N-1} \end{bmatrix} \xi_k$$

La paire (non-unique) $(\bar{Q}, \bar{R})$ doit satisfaire les relations :

$$\bar{Q} = \bar{P} - \bar{F}^\top \bar{P}\bar{F} + \left\{YS^{-1}\right\}^\top (\bar{R} + \bar{G}^\top \bar{P}\bar{G})YS^{-1}$$

$$\bar{R}YS^{-1} + \bar{G}^\top \bar{P}\bar{G}YS^{-1} + \bar{G}^\top \bar{P}\bar{F} = 0$$

Le problème ci-dessus peut être résolu dans le cas général à partir de la formulation LMI suivante :

$$\begin{gathered} \min \alpha \\ \bar{P} - \bar{F}^\top \bar{P}\bar{F} + \left\{YS^{-1}\right\}^\top (\bar{R} + \bar{G}^\top \bar{P}\bar{G})YS^{-1} \geq 0 \\ \begin{bmatrix} Z & \bar{R}YS^{-1} + \bar{G}^\top \bar{P}\bar{G}YS^{-1} + \bar{G}^\top \bar{P}\bar{F} \\ * & I \end{bmatrix} \geq 0 \\ Z \prec \alpha I, \quad \bar{P} \succ 0 \end{gathered}$$

4 Conclusion

4.1 Conclusions

Le principal intérêt de cette thèse est la *commande sous contraintes pour des systèmes à retard* avec la prise en compte du problème de l'échantillonnage dans la boucle.

Le problème de la modélisation a été étudié en détail, lorsqu'une incertitude est ajoutée au système échantillonné à cause du retard variable. Cette variation peut être caractérisée comme une fonction contenue par un enveloppe convexe. Le système échantillonné peut être décrit au niveau d'un *système d'état augmenté*. Dans ce cadre, trois cas on été étudiés en fonction des valeurs propres des matrices du système. Devant le conservatisme des résultats, une méthode algorithmique a été proposée.

Des techniques de stabilisation on été proposées pour le cas sans contraintes, appuyés par la théorie de Lyapunov. Dans le cas de la description du système augmenté, un retour d'état augmenté a été obtenu en utilisant des techniques LMI et des candidates quadratiques de Lyapunov. Des candidates de Lyapunov-Krasovskii ont aussi été utilisées pour stabiliser le système dans l'espace d'état « originel ».

La théorie des *ensembles invariants* est extensivement utilisée dans le cadre de la commande sous contraintes, où le but est d'obtenir une région dans laquelle le comportement du système est connu, malgré la présence des contraintes et du retard variable dans le temps. La complexité des *ensembles maximaux des états admissibles à retard* obtenus dans le cadre du système augmenté est trop importante. Pour faire face à ce problème, nous avons proposé le concept de l'invariance des ensembles dans l'espace originel de systèmes à retard, appelé ici $\mathcal{D}$-invariance, où l'invariance doit être respectée non seulement pour les états présents, mais aussi pour l'historique passé des trajectoires du système. La $\mathcal{D}$-invariance peut être définie au niveau de l'inclusion de l'addition de Minkowski de cet ensemble et les opérations linéaires des ensembles dans la dynamique. Les conditions algébriques sont dérivées de façon à obtenir des tests constructifs, basés sur des problèmes d'optimisation. Le concept de $\mathcal{D}$-invariance cyclique est aussi présenté comme solution pour le conservatisme de cette définition, où la condition d'inclusion doit être respectée pour chaque dynamique, de manière cyclique, une par une.

Dans la dernière partie du manuscrit, la commande prédictive est présentée, en tenant compte des contraintes et de l'optimalité de la solution de la commande. Pour régler les paramètres du contrôleur, plus précisément des matrices de pondération, un problème d'optimalité inverse est proposé. La loi de commande prédictive est obtenue « en ligne » par la faisabilité d'un problème d'optimisation multiparamétrique, qui peut être exprimé comme un problème PQ, en utilisant comme contraintes les trajectoires du système et l'ensemble invariant obtenu comme ensemble terminal de la routine d'optimisation.

4.2 Perspectives

Malgré l'utilisation extensive de la théorie des ensembles invariants dans la littérature, il existe de nombreuses opportunités au niveau des systèmes à retard. Au cours de cette thèse de nouveaux concepts ont été introduits, notamment concernant le problème de l'invariance des ensembles. Malgré la classe de systèmes pour lesquels les conditions d'invariance fonctionnent, le but principal fut d'introduire le sujet et donner les premiers pas dans cette direction. Des améliorations et des opportunités peuvent être observées, notamment :

- Généralisation du concept de $\mathcal{D}$-invariance pour d'autres classes de système.
- Obtention de conditions dépendantes du retard pour la $\mathcal{D}$-invariance.
- Existence d'un algorithme exact pour obtenir des ensembles $\mathcal{D}$-invariants.
- Caractérisation des ensembles maximaux/minimaux pour des systèmes à retard.

Resumo

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Doutor em Ciências Fisicas

por Warody Claudinei Lombardi

O foco desta tese é o controle sob restrições de sistemas com atraso, mais precisamente considerando em detalha a discretização (devido à, por exemplo, uma rede de comunicação de computadores) e a presença de restrições nas trajetorias do sistema bem como nas entradas de controle.

Entende-se por sistemas com atraso a classe de sistemas utilisada para descrever fenômenos fisicos onde a reação do sistema com relação à entrada de sinais externos não é estantânea. Para sistemas em tempo continuo *controlador via rede* (NCS em inglês), o problema torna-se interessante devido aos atrasos induzidos pela rede e pela discretização (os sensores captam os sinais e enviam pela rede), o atraso podendo assim variar entre os periodos de amostragem.

Os efeitos da discretização e da modelagem são estudados detalhadamente, onde uma incerteza é adicionada ao sistema devido aos efeitos da discretização do atraso. A variação do atraso com relação aos instantes de amostragem pode ser caracterizada por um politopo da incerteza induzida pelo atraso/discretização. O sistema discretizado pode ser descrito em termos de um *vetor aumentado do espaço de estados*. Três casos são estudados, em função dos autovalores da matriz do sistema em tempo continuo. Um algoritmo é proposto para diminuir o conservatismo dos resultados, minimizando o volume do politopo.

Durante a tese algumas técnicas de estabilização, baseadas na teoria de Lyapunov, são obtidas para o caso sem restrições. No caso do sistema aumentado, a partir de candidatas de Lyapunov simples, tal como funções quadraticas, um ganho de realimentação de estados é obtido em termos de condições LMI. Candidatas de Lyapunov-Krasovskii também são obtidas em termos de LMIs para a estabilização a partir de uma realimentação de estados, no espaço de estados "original" do sistema.

Para o controle sob restrições, a teoria dos conjuntos invariantes é extensivamente utilisada, de tal forma a obter uma região onde o sistema tem o comportamento conhecido, mesmo diante da presença de restrições e atraso variavel. Devido à alta complexidade do conjunto invariante obtido na abordagem de sistema aumentado, o autor propôs o conceito de conjunto invariante no espaço de estados original do sistema, com relação não apenas aos estados presentes do sistema, mas também com a historia passada do mesmo. Nesta tese, este conceito é chamado de $\mathcal{D}$-invariância ($\mathcal{D}$ de *delay*), cuja definição é feita em termos da soma de Minkowski do conjunto que supõe-se invariante e as operações lineares das dinâmicas do sistemas pelo conjunto devem respeitar a inclusão do conjunto. Condições algébricas são derivadas de forma a obter testes construtivos, baseados em um problema de otimização LP. Para reduzir o conservatismo da definição de $\mathcal{D}$-invariância, o conceito de $\mathcal{D}$-invariância ciclica é introduzieo, onde a condição de inclusão deve ser respeitada para as dinâmicas do sistema deslocas "uma por uma".

Finalmente, na ultima parte da tese, o control preditivo (MPC) é apresentado, permitindo que as restrições sejam levadas em conta com a aplicação de uma lei de controle otima. Para regular os parâmetros do controlados (mais precisamente as matrizes de poderação), um problema de otimização inversa é utilizado. A lei de controle preditivo é obtida "on line" a partir de um problema de otimização multiparâmetros, que pode ser escrito como um problema QP, utilizando como restrições de otimização as restrições nos estados e o conjunto invariante como conjunto terminal da rotina de otimização.

List of Figures

List of Tables

Abbreviations

LKF	**L**yapunov-**K**rasovskii **F**unction or **F**unctional
LMI	**L**inear **M**atrix **I**nequality
LP	**L**inear **P**rogramming
LTI	**L**inear **T**ime **I**nvariant
MIMO	**M**ulti-**I**nput-**M**ulti-**O**utput
MPC	**M**odel **P**redictive **C**ontrol
SISO	**S**ingle-**I**nput-**S**ingle-**O**utput
QP	**Q**uadratic **P**rogramming

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Part I

Introduction

Chapter 1

Introduction

This thesis deals with dynamical systems affected by time-delays in presence of constraints, covering set of problems ranging from the analysis to constrained controller design. Time-delay systems are systems for which the dynamics is determined not only by the present states, but also by past states. Time-delay system emerge as an important research topic due to its variety of applications when modeling some physical, biological, communication, etc, processes. Lately, an important example of application is represented by the so-called *Networked Control Systems*(NCS), where the information are exchanged by means of a computer network, and presence of a communication delay is intrinsic. The NCS problem becomes even more interesting when the system to be controlled is in continuous-time and needs to be sampled, in order to send the discrete-time information by the network. The control action can be calculated, for example, by a computer far away from the plant, with all the information being exchanged over the communication network. In this context, *Model Predictive Control* rises as an attractive design methodology for the control of these systems, by its capacity of dealing with delays, even with the presence of physical constraints (displacement, velocity, acceleration, voltage, current, etc). The main focuses of the present thesis are the stability and the robustness issues of sampled systems with delayed inputs. The entire exposition will concern linear dynamics and the design problems will be generally treated in a discrete-time framework which is concordant with the classical Model Predictive Control framework.

1.1 Time-delay systems

Time-delay systems, or just *delay systems* (as [136], called also *hereditary* or with *memory*, *deviating arguments*, *aftereffects*, *post actions*, *dead-time*, or *time-lag*) represent a class of infinite-dimensional systems largely used to describe some physical phenomena (see [88] for some detailed examples), such as population dynamics (reproduction, growing, extinction, etc), transport phenomena, economic systems (investments policy, commodity market evolution), chemical industrial processes and energy production [142], etc. The main characteristic of the systems described above is that their dynamics can be described by differential equations including the present and the past history of the system's evolution. Some examples can be found in [71, 88, 136, 142, 170].

One of the important points about time-delay systems as for any dynamical systems is the *stability problem* [66, 88, 128, 136]. In the closed-loop schemes, delays can induce complex behaviors, as oscillations, instabilities, performance degeneration, etc, but in some configurations can help in process of stabilization of unstable systems [139, 172]. The surveys [83, 158] and more recently [170] discuss about some existing techniques and open problems.

One way to overcome the undesirable effect of delays is the use of *time-delay compensators* [171], where the main strategy is to "eliminate" the delay of the control loop (if the delay value if *perfectly known* and fixed), so called *Smith predictors*. Historically, this celebrated scheme is known to suffer from the robustness point of view and some recent works as [142, 163–165] points towards the robustness of the *Smith predictors* by using filters, even with unstable plants.

Another important research topic on time-delay systems is the synthesis/analysis based on the Lyapunov's theory. Actually, when considering time-delay systems, there are two different ways of interpreting the stability of the considered system: as an *evolution in a function space* (Lyapunov-Krasovskii functionals of LKF) [92–94] or as an *evolution in the Euclidean space* (Lyapunov-Razumikhin functions) [154]. Actually, these two methods differ in their essential features.

Lyapunov-Krasovskii functionals, at a first sight, do not take into account the delay size information to develop stability conditions. However, by means of *model transformations*, the delay size can be considered on the stability synthesis/analysis. The idea of this kind of candidate is somehow linked to the classical Lyapunov approach, where the derivative (or difference in discrete-time) of the candidate must decrease along all the system's trajectories to obtain stability. On the other hand, in order to characterize stability, the derivative of *Lyapunov-Razumikhin candidate* should be negative just for some "critical" values between the delay interval, and not for all trajectories. Further discussions and interpretations can be found in [136].

In this context, the LKF combined with an appropriated model transformation provide constructive conditions for the stability analysis and state feedback design. Some results concerning the stability conditions for time-delay systems via LKF are presented in [48, 49, 59, 84, 89]. Some works, as [19, 53, 54, 67, 68, 103, 109, 110, 174], used a discretization of a continuous-time LKF to obtain a Lyapunov function candidate, called by convention discrete-time LKF. The Lyapunov-Razumikhin-based stability conditions [71, 154] have been applied in [56, 104, 108, 138, 147, 186]. Although largely explored in the literature, the connections between Lyapunov-Krasovskii and Lyapunov-Razumikhin, to the best of the author's knowledge, have never been studied.

In discrete-time, the past control inputs or states affected by time-delays can be stored in an extended state-space vector, as addressed by [51, 57, 77, 114, 115]. Although these augmented state-space systems can be easily stabilized (by using, for example, quadratic Lyapunov candidates) in [78] the equivalence between a Lyapunov candidate in the augmented state-space and a LKF in the original state-space is shown.

1.2 Networked control systems

Networked control systems are systems wherein the control loops are closed through a real-time communication network [9, 187, 194, 195]. Actually, all the information among the system components (sensors, controller and actuators) are exchanged by means of, for example, a computer network [42, 43, 72, 135, 153]. An example of NCS is *teleoperation*, where the human operator (master) commands a robotic system (slave) from a remote place and receives a force feedback, studied in [4, 26, 27, 73, 100, 191] and many other applications. A comparative study on teleoperation controllers can be found in [7].

Figure 1.1 shows a schematic bloc of a networked control system. The information collected by the *sensors* have to be sampled and are sent to the *controller* by means of the *communication network*, the same as for the calculated control law (by a computer or micro-processor). The advantages are reduced system wiring (all the information can be sent, for example, by internet), easy of diagnosis and maintenance, etc. The main inconvenient is the introduction of *time-delays* (varying or fixed, under certain assumptions) in the *control loop*, due to the communication process and the time spent to compute the control law. Moreover, these introduced delays are *time-varying* by conception, once the information sent by the network depends on some parameters, as data flow, network type, communication protocol, etc. One can observe that conventional control theories (where most of the assumptions are ideal) must be reevaluated before being applied to NCS. Another observation to be taken into account is that networks are not reliable (depending on the protocol and the distance the information should travel) and the physical components involved (wireless or cable, for example). Package dropouts can occur, i.e. information is sent but cannot achieve its destination.

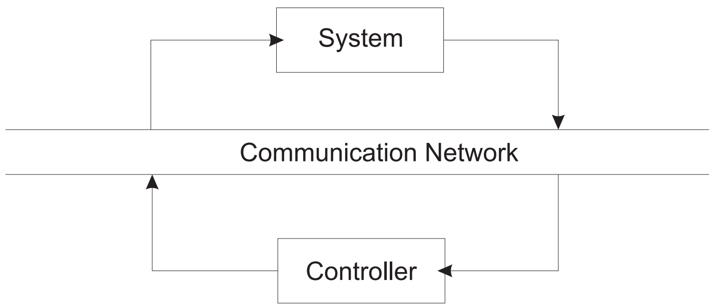


FIGURE 1.1: Schematic bloc of a NCS.

To the completeness, additional information is provided subsequently.

1.2.1 Networked control systems paradigms

According to the synchronization between the control scheme elements, networked control systems can be governed by some paradigms:

- *Time-triggered networked control system*: the action is taken at a pre-specified time (periodically). For example, the controller computes the control law with the last information received, with a period T_e.

- *Event-based networked control system*: the action is taken when an event occurs, i.e., when an information is received by another element. For example, the controller computes the control law just after being received the information of the sensors.

- *Other paradigms*: the priority-based and deadline-based communication paradigms are not used in networked control system because the information *queue* is not respected and the dynamics of the system can be lost.

In time-triggered networked control system, the data packets can contain the information about the *time it was generated*, called *timestamping*, telling "how old" the packet is. Moreover, an "according" system can also be employed, i.e., a confirmation that a sent information packet has achieved its destination, useful in order to avoid information loss. If an information achieves its destination, the receiver sends a confirmation to the sender; otherwise, the sender should "re-send" the information. The counter effect is the increase of the delays.

In Fig. 1.2 a more detailed schematic bloc of a NCS is shown. Essentially, in a NCS scheme there are three kinds of delay:

- τ^{sc}, communication delay between the sensor and the controller.

- τ^{cc}, the *computation time* delay.

- τ^{ca}, communication delay between the controller and the actuator.

To control design purposes, the *control delay* [141] is the time taken to use a control signal in the actuator after a state is sampled and transmitted:

$$\tau_k = \tau^{sc} + \tau^{cc} + \tau^{ca}.$$

In NCS, delays are essentially considered *constant* or *random*, depending, for example, on the *communication protocol*. Anyway, time-varying delays can be eliminated from the control loop. This concept is used in [90, 120], where the data packets are first stored and after the data stream is reconstructed with a specified and constant delay. In *CAN* and *DeviceNet* network protocols, delays can be considered constant (with minimal or no fluctuation) by its information priority system. However, in most of the communication schemes, delays might depend on a probabilistic distribution, or just have no statistical relationship [134]. Actually, as stated in [134], network delays cannot be described with a single time-invariant mathematical model over a long period of time. Other network protocols are: *Fieldbus Foundation, Factory Instrumentation Protocol (FIP), PROFIBUS, and Ethernet.* The details about communication protocols and computer networks can be found in [95, 148, 177].

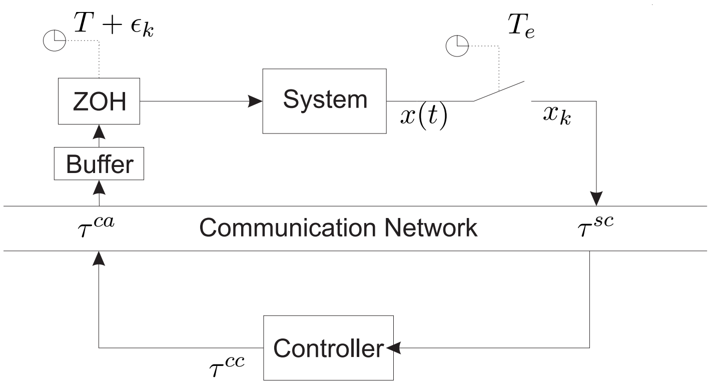


FIGURE 1.2: A more detailed schematic bloc of a NCS.

In order to guarantee the coherence of the modeling framework used during the thesis, some assumptions have to be made:

- *The controller, sensors and actuators are time-triggered, i.e. sampled periodically.* It is obvious that it is almost impossible to guarantee the synchronization between the sampling period of the system states (T_e, shown in Fig. 1.2), the period of the zero-order hold ($T + \epsilon_k$) and a cycle of the controller. This topic will be treated in detail later, when the discretization problem is posed.

- *No package loss from sensor to controller*, in order to respect the augmented state-space structure (More information is provided on Chapter 3). This task is made by the buffer (see Fig. 1.2).

- *Timestamping*, i.e., the control inputs are applied in order of generation (an older control input cannot be applied after a newer one). This assumption is also validated by the presence of a buffer.

One has to consider that in practical application it is impossible to synchronize the *sensor's clock* (T_e) and the *zero-order-hold clock* ($T + \epsilon_k$) [1]. Even if both of them are started at the same time and have "theoretically" the same periods, i.e. $T_e = T$, one cannot guarantee that their periods will remain exactly equal. To model this phenomenon, the variation element ϵ_k is introduced, as one can see in Fig. 1.2. Moreover, in order to guarantee synchronization between the control inputs, all the actuators must be physically closer and respect the same clock $T + \epsilon_k$. The same can be said to the sensors, with respect to the clock T_e.

1.2.2 Discretization problem over NCS

The problem of *data-sampling* over a network is interesting on networked control systems point of view (see, for instance, [29–32, 50, 79, 141, 195]). In this thesis, the problem is detailed in Section 3.1 of the Chapter 3 and also in Chapter 5, according to the delay variation.

Under the above assumptions, two discretization cases are characterized according to the delay variation: in the first one, the delay variation is smaller than a sampling interval; in the second one, the delay variation can be larger than an interval. Figure 1.3 shows, schematically, a discretization scheme, where the control input is piecewise constant, i.e. $u(t) = u(a_k)$ for $a_{k-d_k} \in \mathbb{R}_{[t_k, t_{k+1})}$, with a sampling period T_e and $t_k = kT_e$ is the k^{th} sampling instant, and by considering also the control actuation times $a_{k-d_k} = \tau_k + t_{k-d_k}$, where τ_k is the delay and $d_k = \frac{\tau_k}{T_e}$. The delay variation is $\epsilon_k \in \mathbb{R}_{[0,T_e]}$.

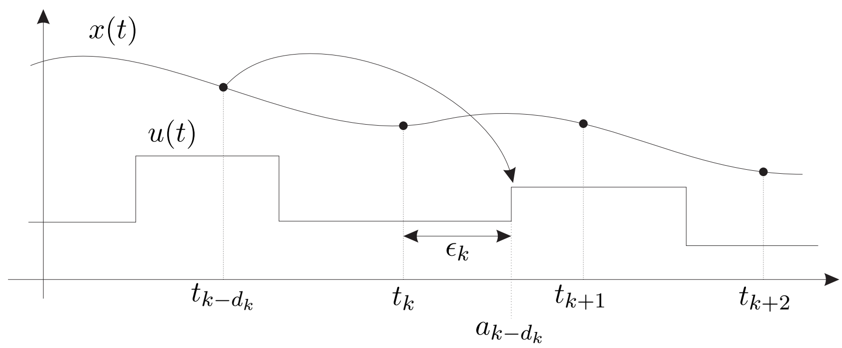


FIGURE 1.3: Primary discretization scheme of a time-delay system.

One can easily see that:

$$\tau_k = a_{k-d_k} - t_{k-d_k}. \tag{1.1}$$

These elements will be used in the study of the discretization problem in the following.

1.2.2.1 Delay variation smaller than a sampling period

Consider a nominal linear continuous-time system affected by variable input delay:

$$\dot{x}(t) = A_c x(t) + B_c u(t - \tau^c(t)), \tag{1.2}$$

for all $t > 0$, with $A_c \in \mathbb{R}^{n\times n}$, $B_c \in \mathbb{R}^{n\times m}$ under appropriate initial conditions, evolving as in Fig. 1.4. Assume that the system state is sampled periodically with the period T_e and we denote $t_k = kT_e$ the k^{th} sampling instant. Consider the sequence of control actuations times $\{a_k\}_{k=0}^{\infty}$ defined as $a_k = \tau_k + t_k$ where $\tau_k > 0$ is bounded by the extreme values $\tau_{min} = d_k T_e$ and $\tau_{max} = (d_k+1)T_e$ such that $0 \le \tau_{min} \le \tau_k \le \tau_{max}$ without other statistical information of this variation. The delay $\tau^c(t)$ is evolving in an impulsive manner, i.e.

$$\dot{\tau}^c(t) = 1, \ \forall t \neq a_k$$

and

$$\tau^c(a_k^+) = a_{k-d_k} - t_{k-d_k} = \tau_k,$$

where

$$a_k^+ = \lim_{\substack{t \to a_k \\ t > a_k}} t.$$

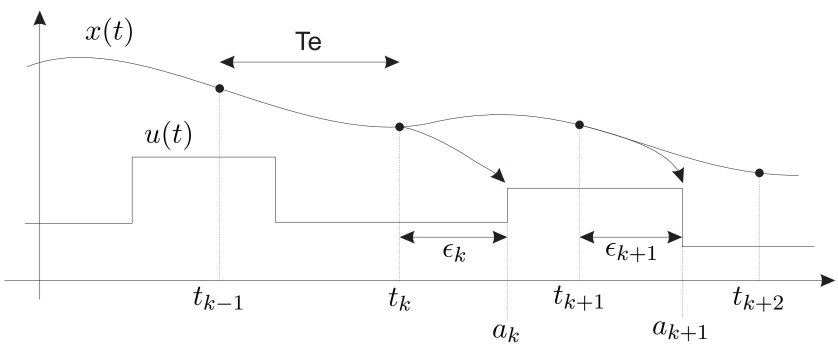


FIGURE 1.4: Detailed discretization scheme when the delay is smaller than a sampling period.

The system can be discretized taking into account the effect of the delay variation:

$$x_{k+1} = e^{A_c T_e} x_k + \int_0^{T_e - \epsilon_k} e^{A_c \theta} B_c d\theta u_{a_k} + \int_{T_e - \epsilon_k}^{T_e} e^{A_c \theta} B_c d\theta u_{a_{k+1}}. \tag{1.3}$$

The discretization is obtained by assuming that the control action u is piecewise constant between sampling instants.

1.2.2.2 Delay variation larger than a sampling period

The discretization problem for delay variation larger than a sampling period is slightly more complicated. One can imagine that the control actuation times can be based on the states t_k, as in Fig. 1.5, t_{k-1} (Fig. 1.6) or t_{k-d_k} (a time instant "older" than t_{k-2}), Fig. 1.7. By taking into account the initial assumptions (timestamping, time-triggered sensors and actuators, no package loss, etc), one can discretize the system by considering the *known* delay bounds d_1 and d_2 such that $d_1 > d_k > d_2$. The details about this problem are presented in Chapter 5.

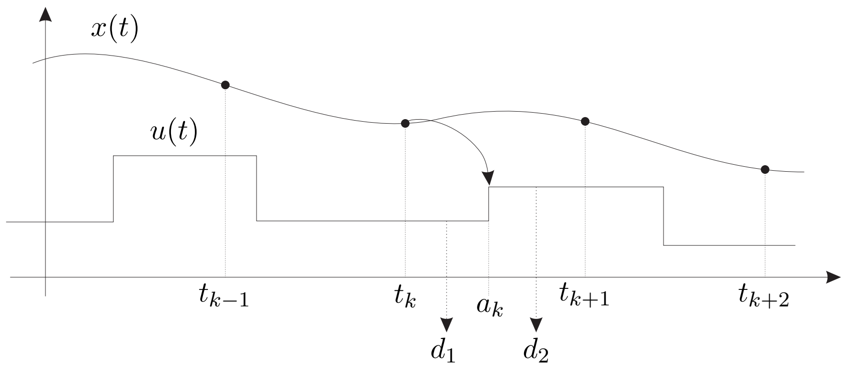


FIGURE 1.5: Primary discretization scheme when the delay variation is smaller than a sampling period.

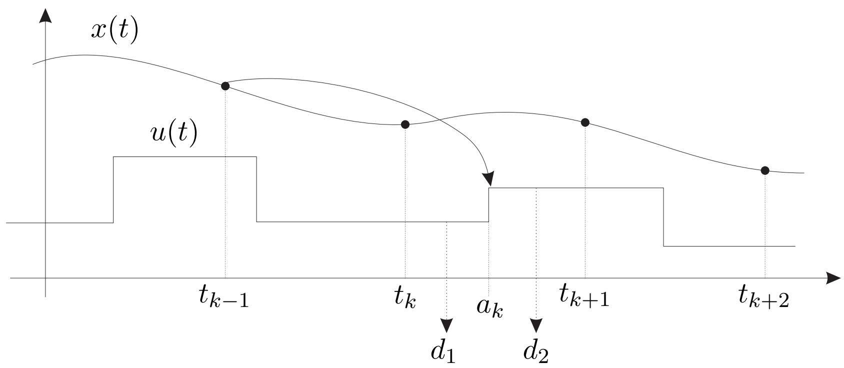


FIGURE 1.6: Primary discretization scheme when the delay variation is larger than one sampling period.

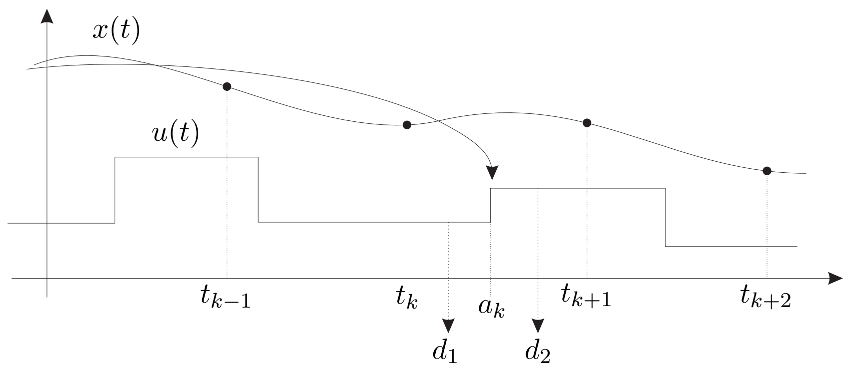


FIGURE 1.7: Primary discretization scheme when the delay variation is larger than several sampling periods.

1.3 Constrained control and positive invariance

The presence of constraints leads to high complexity control problems [61, 179], not just on control and dynamical systems theory, but also in practical frameworks. Basically, there are two categories of constraints in real-world [127] : physical and operational.

Physical constraints represent the physical limitations of the system itself. The objective here is not *just* the stability of the system, but the *stability of the system in the presence of a set of constraints* on the system's *states* and *control inputs* [107, 161]. Once these constraints are violated, physical components can be damaged (for example, the components of an electronic circuit) or the system's actuators saturation [22, 25, 183] can occur, leading to instabilities, oscillations and degrading performance (for example, the motors of a robot, or the servo-motors of the navigation system of an airplane). It can become even more challenging with the presence of delays [156, 178, 181, 182], uncertainties [117, 180] and disturbances [87] on the feedback control loop, and for switched and piecewise affine systems [3, 70, 101, 129]. *Operational constraints* are introduced in order to guarantee performance requirements, equipment longevity and environmental problems (for example, the temperature of biochemical processes with enzymes, leading to contamination and low quality products).

The first works that considered constraints and their effects were in an *optimal control framework* as in [80, 81]. Even if it has been shown by [52, 175] that *linear* control laws cannot be used for *global stability* purposes of systems subjected to input saturation/constraints, some works point out the synthesis of simple controllers [188] and semi-global stability [106] under certain conditions.

The first steps to stabilize such a kind of system were in the sense of non-linear feedback control law [185], and to consider and adaptation of the control law with respect to input saturation, called "anti-windup", more recently treated by [60, 184]. These methods are popular in practical control applications [159].

Recently, the research on *constrained control topics* emerges [61, 167], where the constraints are taken into account during the synthesis [8, 13, 14, 24, 190, 192]. By its principle *positive invariance theory* [16, 17] shows its importance on dealing with constraints. A *certainty* region is obtained, i.e. a region or a set where the *stability* or the *constraints satisfaction* are respected. This region can be expressed, for example, in terms of the set of initial conditions [58]. The minimal invariant region concept is introduced in [150].

Invariant sets can also be used in different frameworks, as controller design under polyhedral constraints [37, 41, 130] by using geometrical approaches [40], systems subject to additive disturbances [131], nonlinear systems by using differential inclusions [46], by using viability theory for polytopic systems [47], to estimate the domain of attraction of Lur'e systems [2], self-bounded systems [35, 38, 39] and multiple outputs [36].

For time-delay systems, set invariance has not been largely explored. In discrete-time, the subject is addressed in [110, 114, 115, 145], in an augmented state-space framework. Algebraic conditions are derived in [33, 62, 75, 76, 112], in the original state-space of the system, inspired from the results of [13, 14]. In [113] an algorithm to construct invariant sets for time-delay systems is proposed.

In the present thesis the interest is on polyhedral invariant sets. Even if the complexity of this kind of representation is higher than in the ellipsoidal case [96], polyhedral sets have the advantage to follow accurately the shape of the limit (maximal/minimal) invariant sets in different frameworks [8].

1.4 Model predictive control

Model predictive control (MPC) is one of the recently emerging control techniques by its significant impact on industrial process control [142]. Because of of its characteristics, MPC can be applied to single-input-single-output (SISO) and multi-input-multi-output

(MIMO) systems, nonlinear systems [6, 45] and hybrid systems [11, 101]. One of the main advantages of MPC is that an optimal control signal can be applied taking constraints [123, 125, 151] and delay into account during the synthesis of the controllers. In addition, the prediction (understood as anticipation) is the natural counteract of the delay. See, for instance, [5, 23, 121, 142, 152, 155, 160, 193]. By its functioning principles, it is also called *receding horizon control* [168].

In classical constrained control, it is desirable to avoid constraints, in order to keep the system far from actuators saturation and components damage, leading to conservative control approaches. In MPC, it is possible to operate the system close the constraints boundary, by using, for example, set-invariance theory. Moreover, an optimal control policy is applied, guaranteeing better performance when compared with classical constrained control.

The path to MPC development passed by several control techniques and theories, such as optimal control [34, 102] by using optimization techniques [176], constrained control [82], etc. In the 70's MPC started to be used in the process industry [157]. *Generalized predictive control* was formulated by [28], where an optimal control policy is applied in a determined prediction window, by choosing the appropriate design parameters, and more recently by [143] for the multivariable case, based on a Smith predictor scheme.

More recently [126, 149], the *tube* approach [98, 105, 162] is used, motivated by the presence of disturbances, by using set-invariance theory. Within this approach, the *initial state* of the model is a decision variable in addition with the usual control sequence. As main advantages, one can easily prove the robust stability and attractivity, the tube of trajectories originated by the initial state can handle uncertainties and disturbances, and the considered invariant set is the terminal set of the optimization procedure [126].

Some recent works pointed towards the MPC robustness issues, where the problematics to guarantee constraints satisfaction and robust closed-loop stability. In [119], a linear estimator of the output and of the disturbances is developed in order to guarantee robust stability under a certain criterion (determined off-line by LMI techniques), based on a QP, despite the presence of constraints. It is shown in [118] the relationship between a min-max problem with MPC, in terms of inverse optimality problems, in order to "obtain" the robustness characteristics of the system. In [117], a robust constrained MPC is developed for systems under unstructured uncertainties, in terms of a linear state estimator. In the case time-varying system with disturbances is developed in [124], by using "tubes" and a time-varying estimator.

One of the open problems, either for the linear or for the nonlinear framework, is the "on-line" solution of non-convex control problems [122]. This affects mainly the practical implementation, as the optimal control law should be calculated and sent at a pre-defined time interval. It is well known that MPC of linear systems under convex constraints yield convex control problems [122]. It is also well known that quadratic cost function under linear constraints yield quadratic control problems. In both of the cases, the optimization algorithms are efficient and the time to find the *global minima* is satisfactory in the practical application point of view. However, if the optimization problem is non-convex, there exist probably just *local minima*, and the optimization routines are not efficient for application purposes, as the control law can not be applied on line. This can happen in the nonlinear context, or even under the presence of non-convex constraints.

Some features have to be emphasized when dealing with MPC:

- As its name says, in MPC one has to use the *model* of the system in order to predict the future states trajectories in a "window" of "N" future sampling instants, called prediction horizon, and N_u, called control horizon. In the present thesis, we consider $N = N_u$ for simplicity.
- The control law sequence is done by the minimization of a cost function, between each sampling period.
- The *receding horizon control* principle means the displacement of the prediction horizon towards the future.
- The first control signal of the optimal control sequence is applied and recalculated at each step.
- When dealing with delays, the size of the prediction horizon should be larger than the discrete-time delay, in order to take the delay effects into account in the prediction scheme.

The principle of working is characterized in Fig. 1.8. At each time instant t_k, the system's trajectories are predicted ($\hat{x}_{k+j|k}$, for $j \in \mathbb{Z}_{[1,N]}$) using the system's model. Of course

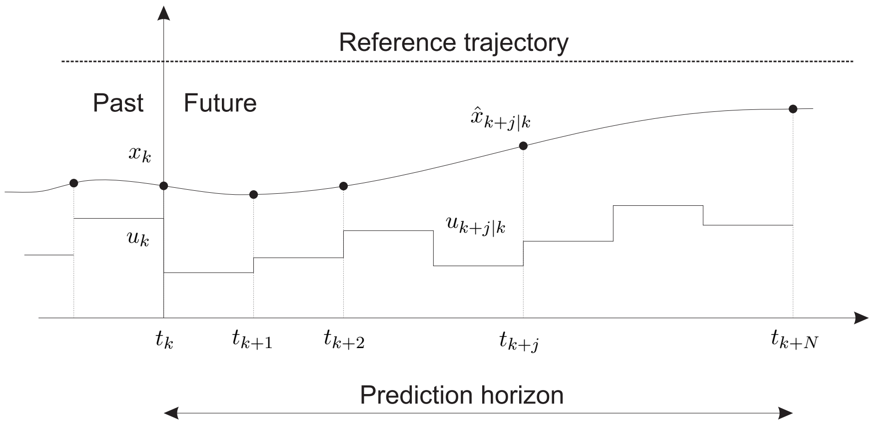


FIGURE 1.8: MPC scheme.

these predictions depend also of the future control signals ($u_{k+j|k}$, for $j \in \mathbb{Z}_{[1,N-1]}$). As it was mentioned above, a cost function is minimized with respect to some criteria, in order to keep the system as close as possible to the reference, by weighting the states and control inputs. Generally, this criteria is a quadratic cost function of the error [142] between the predicted output signal and the predicted reference trajectories. Finally, the control signal u_k is sent to the system and the "tail" ($j \in \mathbb{Z}_{[1,N-1]}$) is discarded.

1.5 Outline of the thesis

The present thesis is organized in three parts:

- Part I: Introduction
- Part II: Modeling
- Part III: Control
- Part IV: Conclusions

Part I contains two chapters, *Introduction* (Chapter 1) and *Mathematical definitions and notations* (Chapter 2). Chapter 2 gives the reader the mathematical background and related definitions to be used during the thesis, namely:

- Mathematical definitions and notations:

  - Sets and operations with sets
  - Vectors, matrices and norms
  - Linear set iteration
  - Polyhedra and polyhedral representation

- Basic stability definitions:

  - Delay-free difference equations
  - Delay-difference equations

- Preliminaries on set invariance:

  - Maximal output admissible set

In Part II the modeling of time-delay systems is developed, where a system of the type:

$$\dot{x}(t) = A_c x(t) + B_c u(t - \tau^c(t)),$$

is sampled (by, for example, a set of sensors and the information is sent via a communication network). The delay $\tau^c(t)$ (induced by the network or not) is considered, by hypothesis, time-varying. More precisely, in Chapter 3 the delay can vary inside a sampling period. All the problematics of the discretization of this class of system is presented in Section 3.1 in terms of the delay variation according to the sampling period. As it was mentioned before, the objective is to model the system above as a linear system under polytopic uncertainties. During the discretization we have an exponential integral, function of the delay variation, to be contained by a polytope. In Section 3.2, the polytopic models are obtained based on the Jordan decomposition of the continuous-time system matrix, according to its eigenvalues. Once the system is sampled and the polytope is obtained, one can set an augmented polytopic system (called *augmented or extended state-space framework* during the thesis), to be used further for stabilization purposes.

Due to the conservativeness of such an approach (the polytope is large with respect to the uncertainty function), in Chapter 4, an algorithm is derived in order to minimize the volume of the polytope around the uncertainty function. The basic principle is to make "sub-divisions" of the uncertainty function and by the application of the techniques presented in Chapter 3, build local embeddings of each "sub-function". The polytope is minimized iteratively, subjected to the local embeddings extreme realizations, until a desired precision is achieved.

In Chapter 5, the last one of the Part II, the delay variation is larger than a sampling period. Differently of the Chapter 3, where the polytope was a simplex ($n+1$ vertices), in this chapter the polytope is also function of the *intersampled delay variation*, where the delay can vary through all the sampling instants. Basically, the models developed in this part of the thesis are said in *augmented state-space*.

Part III concerns the control of time-delay systems. Once the modeling is established, several stabilizing techniques can be derived, based on the Lyapunov's method, as in Chapter 6. Classical robust control techniques based on LMIs are developed for the extended state-space framework in Section 6.1. The objective is to obtain a static state-feedback in order to stabilize the polytopic system. If the objective is to stabilize the system in its *orignal state-space*, one can use Lyapunov-Krasovskii candidates, developed in Section 6.2. This kind of candidate can be applied in a delay-independent (Section 6.2.1) or a delay dependent frameworks (Section 6.2.2), where the delay information can be either taken into account or ignored, during the controller synthesis.

In Chapter 7, the set-invariance theory is developed for time-delay systems. In the extended state-space framework (Section 7.1), the concept of *maximal delayed-states admissible set* is defined inspired by the classical concept of *maximal output admissible set*. The novelty is the obtention of an invariant set for a polytopic system in the extended state-space framework. Due to the high complexity of these kinds of sets, it is worth preserving the original state space, in order to obtain sets in the same dimensional space of the original system. The concept of $\mathcal{D}$-invariance, i.e. the set invariance with respect to the present and past history of the system formulation, presented in Section 7.2, rises as alternative method for dealing with the invariance constraints of time-delay systems. In this same chapter, the properties concerning $\mathcal{D}$-invariance properties are presented in Section 7.2.1, while the necessary and sufficient conditions are defined in Section 7.2.2. The algebraic conditions for $\mathcal{D}$-invariance are presented in Section 7.2.3, and further, based on these algebraic conditions, some constructive $\mathcal{D}$-invariant tests, based on feasibility problems, are presented in Section 7.2.4:

- Tests based on the $\mathcal{D}$-invariance definition, based on the Minkowski addition, presented in Section 7.2.4.1.

- Direct application of the algebraic conditions, obtaining a LP, in Section 7.2.4.2.
- Dual of the algebraic conditions, in Section 7.2.4.3.
- Test based on the support function of the candidate set, in Section 7.2.4.4.

A controller can be obtained by the same conditions, where a static feedback gain is obtained in order to keep the set $\mathcal{D}$-invariant with respect to the dynamics, called in this thesis *controlled $\mathcal{D}$-invariance*, developed in Section 7.2.5. In Section7.3, the concept of cyclic $\mathcal{D}$-invariance is defined in order to reduce the conservatism of the previous approaches, with its algebraic conditions (Section 7.3.1). The relation between $\mathcal{D}$-invariance and cyclic $\mathcal{D}$-invariance is established in Section 7.3.2, in terms of Lyapunov-Razumikhin stability. In order to obtain $\mathcal{D}$-invariant constraints, in Section 7.4, algorithms based on set iterates are proposed, within several properties (Section 7.4.1) and auxiliary algorithmic routines (Section B.2).

Chapter 8 is the last one of the Part III of the thesis and talks about MPC for the constrained control of time-delay systems purposes. In Section 8.1, the general framework of optimization-based MPC for time-delay systems is considered, in terms of a multiparametric programming (Section 8.1.1). To tune optimally the parameters of the presented MPC scheme (the weights on the states and control), Section 8.1.2 exposes an inverse optimization problem based on the solution of a Riccati equation.

Part IV contains the concluding remarks and future perspectives of this work (Chapter 9) and the appendices, where in Appendix A the publication list of the authors is given, while in the Appendix B is reserved to detail the algorithms used during the thesis and Appendix C contains parts of lengthly proofs.

Chapter 2

Mathematical definitions and notations

2.1 Mathematical definitions and Notations

Sets and operations with sets

Let $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{Z}$ and $\mathbb{Z}_+$ denote the field of real numbers, the set of non-negative reals, the set of integer numbers and the set of non-negative integers, respectively.

We denote $(\mathbb{R}^n)^d := \underbrace{\mathbb{R}^n \times \mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{d \text{ times}}$.

For every subset Π of $\mathbb{R}$ we define $\mathbb{R}_\Pi := \{k \in \mathbb{R} \mid k \in \Pi\}$ and $\mathbb{Z}_\Pi := \{k \in \mathbb{Z} \mid k \in \Pi\}$.

For the Euclidean space $\mathbb{R}^n$, we denote by $Com(\mathbb{R}^n)$ the space of compact subsets of $\mathbb{R}^n$.

A convex and compact set in $\mathbb{R}^n$ that contains the origin in its interior is called a $\mathcal{C}$-set. $ComC(\mathbb{R}^n)$ denotes the space of $\mathcal{C}$-subsets of $\mathbb{R}^n$ containing the origin. For an arbitrary set $\mathcal{A} \subseteq \mathbb{R}^n$, $int(\mathcal{A})$ denotes the interior of $\mathcal{A}$.

For two arbitrary sets $\mathcal{A} \subseteq \mathbb{R}^n$ and $\mathcal{B} \subseteq \mathbb{R}^n$:

- $\mathcal{A} \cap \mathcal{B} := \{x \in \mathbb{R}^n \mid x \in \mathcal{A} \text{ and } x \in \mathcal{B}\}$ denotes their intersection.
- $\mathcal{A} \cup \mathcal{B} := \{x \in \mathbb{R}^n \mid x \in \mathcal{A} \text{ or } x \in \mathcal{B}\}$ denotes their union.

- $\mathcal{B} \setminus \mathcal{A} := \{x \in \mathcal{B} \mid x \notin \mathcal{A}\}$ denotes their set difference.
- $\mathcal{A} \oplus \mathcal{B} := \{x + y \mid x \in \mathcal{A}, y \in \mathcal{B}\}$ denotes their Minkowski sum.
- $\mathcal{A} \sim \mathcal{B} := \{x \in \mathbb{R}^n \mid x + \mathcal{B} \subseteq \mathcal{A}\}$ denotes their Pontryagin difference.
- $Co(\mathcal{A}, \mathcal{B})$ denotes the convex hull between $\mathcal{A}$ and $\mathcal{B}$.
- $d_H(\mathcal{A}, \mathcal{B}) := \max\left(\max_{x\in\mathcal{A}} \min_{y\in\mathcal{B}} d(x, y),\ \max_{x\in\mathcal{B}} \min_{y\in\mathcal{A}} d(x, y)\right)$ is the Hausdorff distance between $\mathcal{A}$ and $\mathcal{B}$, where $d(x, y)$ is the Euclidean distance between the points x and y in $\mathbb{R}^n$, $d(x, y) := \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$, where x_i and y_i are the Cartesian coordinates of the points x and y.

For a given set $\mathcal{A} \subset \mathcal{B} \times \mathcal{C}$ the projection of $\mathcal{A}$ onto $\mathcal{B}$ is defined as

$$Proj_{\mathcal{B}}\mathcal{A} := \{b \in \mathcal{B} \mid \exists c \in \mathcal{C} \text{ such that } (b, c) \in \mathcal{A}\}.$$

For an arbitrary set $\mathcal{A} \subseteq \mathbb{R}^n$ and for any $\alpha \in \mathbb{R}_+$, we define:

$$\alpha\mathcal{A} := \{\alpha x \mid x \in \mathcal{A}\}.$$

Given a sequence of subsets of $\mathbb{R}^n$, i.e. $\{\mathcal{A}_i\}_{i\in\mathbb{Z}_{[a,b]}}$ with $a \in \mathbb{Z}_+$ and $b \in \mathbb{Z}_{\geq a}$, we define:

$$\bigoplus_{i=a}^{b} A_i := A_a \oplus \cdots \oplus A_b.$$

For a non-empty closed convex set $\mathcal{P} \in \mathbb{R}^n$, the support function $S(\mathcal{P}, .) : \mathbb{R}^n \to \mathbb{R}$ is defined by:

$$S(\mathcal{P}, u) = \sup_{x\in\mathcal{P}} \langle x, u\rangle \tag{2.1}$$

where $\langle ., . \rangle$ denotes the inner product on $\mathbb{R}^n$. In the Euclidean space $\mathbb{R}^n$, the inner product $\langle (x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n)\rangle$ is $x_1y_1 + x_2y_2 + \cdots + x_ny_n$.

$\mathbb{B}^n_{0,r}$ denotes the ball of radius r in Euclidean norm, centered in the origin of $\mathbb{R}^n$.

Vectors, matrices and norms

For a real number $a \in \mathbb{R}$, $|a|$ denotes its absolute value, $\lceil a \rceil$ denote the smallest integer larger than a (or floor) and $\lfloor a \rfloor$ denotes the biggest integer smaller than a (or ceil).

For a vector $x \in \mathbb{R}^n$ let $\|x\|$ denote its Euclidean norm.

For a matrix $A \in \mathbb{R}^{n\times m}$, $A^\top$ denotes its transpose, A^{-1} denotes its inverse (if it exists) and $det(A)$ denotes its determinant.

For a matrix $A \in \mathbb{R}^{n\times n}$, the minor $M_{i,j}$ of $a_{i,j}$ (where $a_{i,j}$ is the element of A in the i^{th} row and the j^{th} column) is the determinant calculated by omitting the i^{th} row and the j^{th} column of A. The cofactor $C_{i,j}$ of $a_{i,j}$ is $C_{i,j} = (-1)^{i+j} M_{i,j}$.

The symbol $*$ in the matrix description has to be interpreted as the transposed of the symmetric element with respect to the main diagonal.

For a matrix $A \in \mathbb{R}^{n\times m}$, $A > 0$ and $A \succ 0$ denotes that A is positive definite. For all $x \in \mathbb{R}^n - \{0\}$ it holds that $x^\top A x > 0$ and $A = A^\top$.

For a matrix $A \in \mathbb{R}^{r\times s}$, $\{a_{ij}\} \in \mathbb{R}$ denotes the i^{th} row and j^{th} column element, for $i \in \mathbb{Z}_{[1,r]}$ and $j \in \mathbb{Z}_{[1,s]}$.

For the matrices A and B, $\text{diag}(A, k)$ denotes a diagonal matrix with k matrices A on the main diagonal, or $\text{diag}(A, B)$ denotes:

$$\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}.$$

I_n denotes the identity matrix of dimension $n \times n$, $\mathbb{1}$ denotes a vector of appropriated dimensions containing exclusively "1"s and $\mathbf{0}$ denotes a matrix or a vector of appropriated dimensions containing exclusively "0"s.

A matrix $A \in \mathbb{R}^{n\times m}$ is called *nilpotent* if there exists some k such that $A^k = \mathbf{0}_{n\times m}$.

For a matrix $A \in \mathbb{R}^{n\times n}$, $\lambda(A)$ denotes the eigenvalues of A, called also *spectrum* of A. Its spectral radius is defined as:

$$\rho(A) := \max_{\lambda \in \lambda(A)} (|\lambda|).$$

The spectrum of a matrix pencil (or matrix pair) (A, B) is the collection of complex numbers λ for which $det(A - \lambda B) = 0$ and is denoted $\lambda(A, B)$. With this notation we will use: $\bar{\lambda}(A, B) := \max_{\lambda \in \lambda(A,B)}(|\lambda|)$. The spectral norm will be denoted $\sigma(A)$ and is defined as:

$$\sigma(A) := \sqrt{\rho(A^T A)}.$$

For a matrix $A \in \mathbb{R}^{n \times n}$, $s_n(A)$ (so-called *singular values* of A) are the square roots of A^*A, where A^* is the conjugate transpose of A.

For a matrix $A \in \mathbb{R}^{n \times n}$ and eigenvalues $\lambda(A)$, the *algebraic multiplicity* of an eigenvalue $\lambda_i(A)$ is defined as the multiplicity of the corresponding root of the characteristic polynomial. The *geometric multiplicity* of an eigenvalue is defined as the dimension of the associated eigenspace, i.e., the number of independent eigenvectors with that eigenvalue.

Proposition 2.1 ([21] Schur complement)**.** *Consider the matrices* $Q(x) = Q(x)^\top$, $R(x) = R(x)^\top$ *and* $S(x)$ *be affinely dependent on* x*. So the Linear Matrix Inequality (LMI):*

$$\begin{bmatrix} Q(x) & S(x) \\ S(x)^\top & R(x) \end{bmatrix} > 0,$$

is equivalent to the set of nonlinear inequalities:

$$R(x) > 0, \; Q(x) - S(x)R(x)^{-1}S(x)^\top > 0,$$

or equivalently:

$$Q(x) > 0, \; R(x) - S(x)^\top Q(x)^{-1}S(x) > 0.$$

Theorem 2.2 (Cayley-Hamilton Theorem)**.** *If* $p(\lambda) := det(\lambda I_n - A)$ *is the characteristic polynomial of a matrix* $A \in \mathbb{R}^{n \times n}$ *then* $p(A) = 0$*.* □

Definition 2.3 (Non-defective Matrix)**.** An arbitrary square matrix $A \in \mathbb{R}^{n \times n}$ is called non-defective if and only if it has n linearly independent eigenvectors, i.e., all the eigenvalues have the algebraic multiplicity equal to the geometric multiplicity.

Definition 2.4 (Defective Matrix)**.** An arbitrary square matrix $A \in \mathbb{R}^{n \times n}$ is called defective if and only if it does not have n linearly independent eigenvectors, i.e., at least one of the eigenvalues has the algebraic multiplicity larger than the geometric multiplicity.

A defective matrix always has fewer than n distinct eigenvalues, since distinct eigenvalues always have linearly independent eigenvectors.

Linear set mapping

For an arbitrary matrix $A \in \mathbb{R}^{m\times n}$ and a set $\mathcal{S} \subseteq \mathbb{R}^n$, we define:

$$A\mathcal{S} := \{y \in \mathbb{R}^m | y = Ax,\, x \in \mathcal{S}\}.$$

Polyhedra and polyhedral representation

A polyhedron (or a polyhedral set) in $\mathbb{R}^n$ is a convex set obtained as the intersection of a finite number of open and/or closed half-spaces. The vertices of a polyhedron $\mathcal{A} \subseteq \mathbb{R}^n$ are denoted $\mathcal{V}(\mathcal{A})$.

There are two main polyhedral representations, namely:

- The $\mathcal{V}$-representation of a polyhedron $\mathcal{P}$ is also called *vertex representation*, i.e. the convex hull of a finite set: $n, m \in \mathbb{Z}_+$ with $m > n \geq 1$ and m points $v_1, v_2, \ldots, v_m$ in $\mathbb{R}^n$ such that $\mathcal{P} = Co\{v_1, v_2, \ldots, v_m\}$. Consequently, $v_1, v_2, \ldots, v_m$ are $\mathcal{V}(\mathcal{P})$.
- The $\mathcal{H}$-representation of a polyhedron $\mathcal{P}$ is also called *halfspace representation*, i.e. $\mathcal{P}$ is the intersection of halfspaces: $n, m \in \mathbb{Z}_+$ with $m > n \geq 1$, a matrix $F \in \mathbb{R}^{m\times n}$ and a vector $w \in \mathbb{R}^m$ such that $P = \{x \in \mathbb{R}^n | Fx \leq w\}$.

A *polytope* is a bounded and convex polyhedron.

The volume of a simplex $\mathcal{P} \in \mathbb{R}^n$ in $\mathcal{V}$-representation is done by:

$$Vol(\mathcal{P}) = \frac{1}{n!} \det \begin{bmatrix} v_1 - v_0 & v_2 - v_0 & \ldots & v_{n-1} - v_0 & v_n - v_0 \end{bmatrix},$$

where v_i for $i \in \mathbb{Z}_{[0,n]}$ are the vertices of $\mathcal{P}$.

2.2 Basic stability definitions

2.2.1 Delay-free difference equations

Consider the discrete-time autonomous system:

$$x_{k+1} = f(x_k), \tag{2.2}$$

where $x_k \in \mathbb{R}^n$ is the state vector at the time $k \in \mathbb{Z}_+$ and f is a mapping $f : \mathbb{R}^n \to \mathbb{R}^n$.

Theorem 2.5 ([69] Lyapunov Stability Theorem)**.** *Consider the Lyapunov function $V(x) : \mathbb{R}^n \to \mathbb{R}_+$. Suppose there exist radially unbounded functions $\phi(\cdot), \omega(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ continuous and non-decreasing with $\phi(0) = \omega(0) = 0$ and $\lambda \in \mathbb{R}_{[0,1)}$ such that:*

(i) $\phi(\|x\|) \leq V(x) \leq \omega(\|x\|),\ \forall x \in \mathbb{R}^n$,

(ii) $V(f(x)) - \lambda V(x) \leq 0,\ \forall x \in \mathbb{R}^n$.

Then the system (2.2) is asymptotically stable.

If $\lambda = 1$ the function $V(x)$ is a weak *Lyapunov function and the system is stable but not asymptotically stable.* □

Definition 2.6 ([15] Minkowski functions)**.** Consider a convex and compact polyhedral set containing the origin:

$$\mathcal{P} = \{x \in \mathbb{R}^n | Fx \leq \mathbb{1}\},$$

with $F \in \mathbb{R}^{r\times n}$. A Minkowski polyhedral function $V_{\mathcal{P}}(x) : \mathbb{R}^n \to \mathbb{R}_+$ associated to $\mathcal{P}$ is defined as:

$$V_{\mathcal{P}}(x) = \max_{j\in\mathbb{Z}_{[1,r]}} \{\max\{(Fx)_j, 0\}\}. \tag{2.3}$$

where $(Fx)_j$ denotes the j^{th} element of the vector Fx. □

Remark 2.7. The Minkowski function (2.3) can be interpreted as an infinity-norm for an appropriate vector ([85, 116]):

$$V_{\mathcal{P}}(x) = \|\max\{Fx, 0\}\|_\infty. \tag{2.4}$$

For *symmetric* polyhedral sets with respect to the origin, (2.3)-(2.4) can be replaced by:

$$V_{\mathcal{P}}(x) = \|Fx\|_{\infty}. \tag{2.5}$$

Remark 2.8. The Minkowski polyhedral function (2.3) can be used as a polyhedral Lyapunov function.

2.2.2 Delay-difference equations

Consider a delay-difference equation of the form:

$$x_{k+1} = \sum_{i=0}^{d} A_i x_{k-i}, \tag{2.6}$$

where $x_k \in \mathbb{R}^n$ is the state vector at the time $k \in \mathbb{Z}_+$. $A_i \in \mathbb{R}^{n\times n}$, for all $i \in \mathbb{Z}_{[0,d]}$ and the initial conditions are given by a sequence $x_{-i} \in \mathbb{R}^n$, $i \in \mathbb{Z}_{[0,d]}$.

Theorem 2.9 (Lyapunov-Razumikhin Stability Theorem)**.** *Consider the Lyapunov-Razumikhin function $V : \mathbb{R}^n \to \mathbb{R}_+$ such that there exist the radially unbounded functions $\phi(\cdot), \omega(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ continuous and non-decreasing with $\phi(0) = \omega(0) = 0$ and $\varepsilon \in \mathbb{R}_{[0,1)}$.*

Denote $\mathbf{x}_k^\top = \begin{bmatrix} x_k^\top & x_{k-1}^\top & \dots & x_{k-d}^\top \end{bmatrix}^\top \in (\mathbb{R}^n)^{d+1}$.

Consider the function $\tilde{V} : (\mathbb{R}^n)^{d+1} \to \mathbb{R}^n$ *with* $\tilde{V}(\mathbf{x}_k) \triangleq \max_{i\in\mathbb{Z}_{[0,d]}} \{V(x_{k-i})\}$.

If the following hold:

(i) $\phi(\|x\|) \leq V(x) \leq \omega(\|x\|)$, $\forall x \in \mathbb{R}^n$,

(ii) $V(x_{k+1}) - \varepsilon\tilde{V}(\mathbf{x}_k) \leq 0$, $\forall k \in \mathbb{Z}_+$, $\forall \mathbf{x}(0) \in (\mathbb{R}^n)^{d+1}$

then the system (2.6) is globally asymptotically stable. If $\varepsilon = 1$ the function $V(x_k)$ is called a weak Lyapunov-Razumikhin function.

Theorem 2.10 (Lyapunov-Krasovskii Stability Theorem)**.** *Consider the Lyapunov-Krasovskii function $\tilde{V} : (\mathbb{R}^n)^{d+1} \to \mathbb{R}^n_+$ defined as $\tilde{V}(\mathbf{x}_k) \triangleq \sum_{i\in\mathbb{Z}_{[0,d]}} V(x_{k-i})$ with $V : \mathbb{R}^n \to \mathbb{R}_+$. If there exist the radially unbounded functions $\phi(\cdot), \omega(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ continuous and non-decreasing with $\phi(0) = \omega(0) = 0$ and $\varepsilon \in \mathbb{R}_{[0,1)}$ such that the following hold:*

(i) $\phi(\|x\|) \leq V(x) \leq \omega(\|x\|)$, $\forall x \in \mathbb{R}^n$,

(ii) $\tilde{V}(\mathbf{x}_{k+1}) - \varepsilon \tilde{V}(\mathbf{x}_k) \leq 0$, $\forall k \in \mathbb{Z}_+$.

then the system (2.6) *is globally asymptotically stable. If* $\varepsilon = 1$ *the function* $V(x_k)$ *is called a* weak Lyapunov-Krasovskii function.

Remark 2.11. Although the existence of a weak Lyapunov-Razumikhin (Theorem 2.9) and Lyapunov-Krasovskii (Theorem 2.10) functions does not imply global asymptotic stability, it induces invariant sets.

2.3 Preliminaries On Set Invariance

Here we present some classical set invariance properties and results, to be used as basic concepts during the paper.

Definition 2.12 (Set invariance)**.** Let $\varepsilon \in \mathbb{R}_{[0,1)}$. A set $\mathcal{P} \subseteq \mathbb{R}^n$ is called *contractive* with respect to system (2.2) if:

$$f(\mathcal{P}) \subseteq \varepsilon \mathcal{P}. \tag{2.7}$$

For $\varepsilon = 1$, $\mathcal{P}$ is called an *invariant* set with respect to (2.2). □

The next result shows that invariance property is linked to the classical notion of Lyapunov stability [69].

Proposition 2.13. *If* $V(x)$ *is a Lyapunov function for the dynamical system* (2.2)*, then the set* $\mathcal{N}(V, c) = \{x : V(x) \leq c\}$ *is an invariant set with respect to the same dynamics.*

Proposition 2.14 (Algebraic conditions for set invariance)**.** *[13] The convex polyhedral set:*

$$\mathcal{P} = \{x \in \mathbb{R}^n | Fx \leq w\},$$

with $F \in \mathbb{R}^{r \times n}$*,* $w \in \mathbb{R}^r$*, is an invariant set with respect to*

$$x_{k+1} = Ax_k \tag{2.8}$$

if and only if there exists a matrix $H \in \mathbb{R}^{r \times r}$ *with nonnegative elements such that:*

$$FA - HF = 0 \tag{2.9}$$

and

$$(H - \mathbb{1})w \leq 0. \tag{2.10}$$

□

2.3.1 Maximal Output Admissible Set

The concept of *Maximal Output Admissible Set* is done by [58]. For a discrete-time linear system of the type:

$$x_{k+1} = Ax_k + Bu_k, \tag{2.11}$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$. The control is done by a feedback control law $u_k = Kx_k$, where $K \in \mathbb{R}^{m \times n}$. The state and control constraints are summarized, by an appropriated choice of F and G, by the set inclusion:

$$Fx_k + Gu_k \in \mathcal{X}, \tag{2.12}$$

where $Y \subseteq \mathbb{R}^n$ is convex and contains the origin.

Definition 2.15 (Maximal State Admissible set)**.** Consider the system (2.11) in closed-loop with the stabilizing feedback law $u_k = Kx_k$ and the state constraints (2.12). A *state admissible set* is a safe set of initial conditions, defined as a subset $\mathcal{Z} \subseteq \mathbb{R}^n$ for which $x_0 \in \mathcal{Z}$ implies (2.12) for all $k \in \mathbb{Z}_+$. The *maximal state admissible set* is the reunion of all the *state admissible sets* for the given closed-loop dynamics and the associated constraints. □

The same concept can be applied to systems with output constraints, where the discrete dynamics:

$$\begin{aligned} x_{k+1} &= Ax_k; \\ y_k &= Cx_k. \end{aligned} \tag{2.13}$$

satisfies:

$$y_k \in \mathcal{X}, \tag{2.14}$$

for all $k \in \mathbb{Z}_+$.

Definition 2.16 (Maximal Output Admissible set)**.** Consider the system (2.13) and the output constraints (2.14). The *maximal output admissible set* is a safe set of initial conditions, i.e. a set $\mathcal{Z} \subseteq \mathbb{R}^n$ such that $x_0 \in \mathcal{Z}$ implies (2.14) for all $k \in \mathbb{Z}_+$. □

The algorithms to determine *maximal output admissible sets* are done by [58].

Part II

Modeling

Chapter 3

Intersampled delay-variation

In this chapter the modeling of linear time-delay system is discussed. During the discretization process the variable input delay can be treated as an uncertainty function. The goal is to confine this function of the uncertainty inside a polytopic description, in order to convex cover all the possible realizations of the variation. It is worth mentioning that this technique can be applied even if there is no statistical information about the delay variation (in fact from the statistical point of view the authors consider a continuous uniform distribution inside the extreme limits of variations).

The delay variation bounds can be smaller (Section 3.1.1) or larger (Chapter 5) than a sampling period, this difference being marked in the manuscript under the terminology of *intersampled* and *multisampled delay variation*. In both cases the model can be obtained by using an extended vector of the states and control inputs. In order to model the effects of the delay variation, a polytopic supra-approximation of the uncertainty generated by the delay during the discretization process is constructed. A polytopic model is then obtained in an extended state-space framework, i.e. the original continuous-time time-delay system is expressed as a discretized polytopic system. By using classical robust control techniques, the stabilizing control law synthesis for the associated polytopic system is straightforward.

The delay variation uncertainty function can be classified in terms of the original continuous-time system eigenvalues:

- non-defective system matrix with real eigenvalues,

- non-defective system matrix with complex-conjugated eigenvalues
- defective system matrix with real eigenvalues.

By using these three main classes, all the other combinations of a general n-dimensional system can be covered. It is shown that the uncertainty function can be contained by polytopic embeddings, by using simple calculations. The inconvenient is that these techniques are considered conservative. To solve this problem an algorithm based on the minimization of the volume of the polytopes is proposed. In this case, the goal is to minimize the volume of the polytopic embedding containing the uncertainty function.

3.1 Time-delay system discretization - problem formulation

Consider a nominal linear continuous-time system affected by variable input delay:

$$\dot{x}(t) = A_c x(t) + B_c u(t - \tau^c(t)), \forall\ t > 0 \tag{3.1}$$

with $A_c \in \mathbb{R}^{n\times n}$, $B_c \in \mathbb{R}^{n\times m}$ under appropriate initial conditions. We assume that the system state is sampled periodically with the period T_e and we denote $t_k = kT_e$ the k^{th} sampling instant. Consider the sequence of control actuations times $\{a_k\}_{k=0}^{\infty}$ defined as $a_k = t_k + \tau_k$ where $\tau_k > 0$ is bounded by the extreme values τ_{min} and τ_{max} such that $0 \leq \tau_{min} \leq \tau_k \leq \tau_{max}$ without other statistical information of this variation. The delay $\tau^c(t)$ is evolving in an impulsive manner, i.e. $\dot{\tau}^c(t) = 1, \forall t \neq a_k$ and $\tau^c(a_k^+) = \tau_k$, where $a_k^+ = \lim_{e\to 0, e>0} a_k + e$.

In the presence of a discrete-time control signal function at a chosen sampling period T_e such that $u(t) = u(kT_e)$, for $t \in \mathbb{R}_{[kT_e,(k+1)T_e)}$, a corresponding discrete-time model will be constructed considering the time instants $t_k = kT_e$. By noting the state at discrete-time instants $x_k = x(t_k)$, one can represent a discrete-time dynamical model correspondent for (3.1):

$$x_{k+1} = Ax_k + Bu_{k-d}, \tag{3.2}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$. For $d_{min}, d_{max} \in \mathbb{Z}$ and $d \in \mathbb{Z}_{[d_{min},d_{max}]}$, the limits of variation of the discrete-time delay are $d_{min} = \left\lfloor \frac{\tau_{min}}{T_e} \right\rfloor \in \mathbb{Z}_+$ and $d_{max} = \left\lceil \frac{\tau_{max}}{T_e} \right\rceil \in \mathbb{Z}_+$.

It can be observed that this model fails to retain the intersampled behavior of the original continuous system, i.e., the delay variation can lay between two sampling instants. The consequence is as uncertainty function of the delay variation. This essential feature (see for example the latest results in networked control systems discussed by [74]) will be the treated in detail in the following by means of a polytopic supra-approximations, where the linear continuous-time delay systems are modeled as linear discrete-time systems with polytopic uncertainty.

3.1.1 Intersampled delay variation

If the delay variation is intersampled, i.e., inside a sampling period, one can consider $d = d_{max}$ for the simplicity of notation and define the delay variation interval as follows:

$$\tau_k \in \mathbb{R}_{(dT_e,(d+1)T_e]}. \tag{3.3}$$

As discussed in the Introduction of the present thesis, Chapter 1, the delay variation is defined by $\epsilon_k = \tau_k - dT_e$, with $\epsilon_k \in \mathbb{R}_{[0,\bar{\epsilon}]}$ and $\bar{\epsilon} = T_e$, as long as the maximal variation cannot overlap the sampling period.

Under the assumption (3.3), the discrete time model which takes into account the effect of the continuous time delay variation will be:

$$x_{k+1} = Ax_k + Bu_{k-d} - \Delta(\epsilon_k)(u_{k-d} - u_{k-d+1}), \tag{3.4}$$

where the matrices $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}$ and the function $\Delta(\epsilon_k) : \mathbb{R}_{(0,\bar{\epsilon}]} \to \mathbb{R}^{n\times m}$ are given by:

$$\begin{aligned} A &= e^{A_c T_e}; & (3.5)\\ B &= \int_0^{T_e} e^{A_c(T_e-\theta)} B_c d\theta; & (3.6)\\ \Delta(\epsilon_k) &= \int_{T_e-\epsilon_k}^{T_e} e^{A_c(T_e-\theta)} B_c d\theta = \int_{-\epsilon_k}^{0} e^{-A_c\tau} B_c d\tau. & (3.7) \end{aligned}$$

These relations reflect the non-existence of an exact correspondence between the delay in continuous-time ($\tau^c(t)$) and discrete-time ("d"). This issue impose the definition of the uncertainty function (3.7) in the discrete-time model, in order to compensate the continuous-time and the discrete-time model mismatch and the delay variation bounds inside a sampling period (ϵ).

The general problem addressed in this chapter and in the next two chapter is the *construction of a polytopic overapproximation of $\Delta(\epsilon_k)$ given by the relation (3.7).*

Exploiting (3.7), the extreme realizations of the discrete-time model are obtained for $\epsilon = 0$:

$$x_{k+1} = Ax_k + Bu_{k-d} \tag{3.8}$$

and $\bar{\epsilon} = T_e$ respectively:

$$x_{k+1} = Ax_k + (B - \Delta(\bar{\epsilon}))u_{k-d} + \Delta(\bar{\epsilon})u_{k-d+1} \tag{3.9}$$

but all the values of $\epsilon_k \in \mathbb{R}_{(0,\bar{\epsilon}]}$, with $\bar{\epsilon} = T_e$, have to be considered.

3.2 Jordan Decomposition-based polytopic models

In this section the idea is to confine $\Delta(\epsilon_k)$ in a polytopic set which covers $\epsilon_k \in \mathbb{R}_{(0,\bar{\epsilon}]}$, by considering the worst variation maximal bound $\bar{\epsilon} = T_e$, leading the system independent of ϵ_k.

Equation (3.4) can be rewritten on the compact linear form such that an extended (or augmented) state-space dynamics is obtained:

$$\xi_{k+1} = A_\Delta \xi_k + B_\Delta u_k, \tag{3.10}$$

with the extended state vector:

$$\xi_k^\top = \begin{bmatrix} x_k^\top & u_{k-d}^\top & \dots & u_{k-1}^\top \end{bmatrix}^\top \tag{3.11}$$

and the augmented system matrices:

$$A_\Delta = \begin{bmatrix} A & B-\Delta(\epsilon_k) & \Delta(\epsilon_k) & \dots & 0 \\ 0 & 0 & I_m & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & I_m \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix};\ B_\Delta = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ I_m \end{bmatrix}^\top, \tag{3.12}$$

where $A_\Delta \in \mathbb{R}^{(n+md)\times(n+md)}$ and $B_\Delta \in \mathbb{R}^{(n+md)\times m}$.

Remark 3.1. The matrix A_Δ in Eq. (3.12) is an *uncertain extended state-space matrix* in function of the intersample delay variation ϵ, to be covered by a polytope (Section 3.2).

By considering the embedding (or containment) of the matrix $\Delta(\epsilon_k)$ in a polytopic model with $n+1$ extreme realizations $\Delta(\epsilon) \in Co\{\Delta_0, \dots, \Delta_n\}$, one obtains the following global polytopic model in an extended state space:

$$\begin{aligned} \xi_{k+1} &= A_\Delta \xi_k + B_\Delta u_k \\ A_\Delta &\in \text{Co}\{A_{\Delta_0},\ A_{\Delta_1},\ \dots A_{\Delta_n}\}, \end{aligned} \tag{3.13}$$

where the extended state vector is done by (3.11). Note the particularity of this embedding which used only $n+1$ extreme realizations. It is worth mentioning that from the computational point of view, it is interesting to obtain the lowest complexity admissible polytope. The simplest polytope to be defined in an Euclidean space $\mathbb{R}^n$ is a *simplex*:

Definition 3.2 (Simplex or n-simplex)**.** Is the convex hull of a set of $(n+1)$ affinely independent points in an Euclidean Space of n dimension. □

Equation (3.13) can be understood as a *simplex* type of geometrical construction and has the important advantage of being "the less" complex in an n-dimensional space.

In order to obtain the vertices of this polytopic embedding, the Jordan canonical form of the matrix A_c can be used: $A_c = V\Sigma V^{-1}$, where $\Sigma \in \mathbb{R}^{n\times n}$ is a block diagonal matrix as follows:

$$\Sigma = \begin{bmatrix} \Sigma_{1,m_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \Sigma_{p,m_p} \end{bmatrix}, \tag{3.14}$$

for all $i \in \mathbb{Z}[1,p]$ and $\Sigma_{i,m_i} \in \mathbb{R}^{m_i \times m_i}$ are the Jordan blocks, with:

$$\Sigma_{i,m_i} = \begin{bmatrix} \sigma_i & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \sigma_i & 1 \\ 0 & 0 & \cdots & 0 & \sigma_i \end{bmatrix} = \sigma_i \underbrace{\begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}}_{\Lambda_i} + \underbrace{\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}}_{\Gamma_i}.$$

According to the *algebraic* and *geometrical multiplicity* of the matrix A_c, three classes of systems can be identified:

- Jordan blocks Σ_{i,m_i} with $m_i = 1$ corresponding to *real* eigenvalues with algebraic multiplicity equal to the geometric multiplicity (Section 3.2.1).
- Jordan blocks Σ_{i,m_i} with $m_i = 1$ corresponding to *complex-conjugated* eigenvalues with algebraic multiplicity equal to the geometric multiplicity (Section 3.2.2).
- Jordan blocks Σ_{i,m_i} with $m_i > 1$ corresponding to *real* eigenvalues with algebraic multiplicity larger than the geometric multiplicity (Section 3.2.3).

Remark 3.3. The notions of *algebraic* and *geometrical multiplicity* are briefly recalled in Chapter 2.

For each class presented above, a corresponding technique is derived in order to enclose the function $\Delta(\epsilon)$ inside a polytope in terms of polytope vertices/hyperplanes ($\mathcal{V}$ and $\mathcal{H}$-representation).

Remark 3.4. As it is mentioned above, one of the advantages of the extended state-space framework is the possibility of using classical LMI-based robust stabilization techniques. The LMI problem complexity is proportional to the system's size ($\mathbb{R}^{(n+md)\times(n+md)}$) and polytope vertices $(n+1)$.

3.2.1 Non-defective system matrix with real eigenvalues

Equation (3.7) can be written as:

$$\Delta(\epsilon_k) = \int_0^{\epsilon_k} e^{A_c\tau} B_c d\tau, \tag{3.15}$$

Remark 3.5. Actually, one has the exponential term of the matrix Σ, by Eq. (3.15) and the Jordan decomposition of $A_c = V\Sigma V^{-1}$:

$$e^{\Sigma} = e^{\begin{bmatrix} \bar{\Sigma}_1 & & \\ & \ddots & \\ & & \bar{\Sigma}_n \end{bmatrix}} = \begin{bmatrix} e^{\bar{\Sigma}_1} & & \\ & \ddots & \\ & & e^{\bar{\Sigma}_n} \end{bmatrix}.$$

In order to obtain the vertices of the polytopic embedding, the exponential needs to be decomposed in terms of each Jordan bloc. By considering the decomposition:

$$e^{\Sigma} = \begin{bmatrix} e^{\bar{\Sigma}_1} & & \\ & \ddots & \\ & & 0 \end{bmatrix} + \cdots + \begin{bmatrix} 0 & & \\ & \ddots & \\ & & e^{\bar{\Sigma}_n} \end{bmatrix},$$

one can see the following:

$$e^{\Sigma} \neq e^{\begin{bmatrix} \bar{\Sigma}_1 & & \\ & \ddots & \\ & & 0 \end{bmatrix}} + \cdots + e^{\begin{bmatrix} 0 & & \\ & \ddots & \\ & & \bar{\Sigma}_n \end{bmatrix}}, \tag{3.16}$$

because:

$$e^{\begin{bmatrix} \bar{\Sigma}_1 & & \\ & \ddots & \\ & & 0 \end{bmatrix}} = \begin{bmatrix} e^{\bar{\Sigma}_1} & & \\ & \ddots & \\ & & I \end{bmatrix}.$$

In order to obtain such a kind of structure and for the further theoretical developments of this chapter, the notation should be taken into account: for a matrix $X \in \mathbb{R}^{n\times m}$ we denote $exp(X)$ the matrix $Y = exp(X) \in \mathbb{R}^{n\times m}$ which satisfies

$$Y_{i,j} = \begin{cases} 0 & \text{if} \quad X_{i,j} = 0,\ i \in \mathbb{Z}_{[1,n]},\ j \in \mathbb{Z}_{[1,m]} \\ e^{X_{i,j}} & \text{if} \quad X_{i,j} \neq 0,\ i \in \mathbb{Z}_{[1,n]},\ j \in \mathbb{Z}_{[1,m]} \end{cases}$$

Thus stressing the fact exposed in Eq. (3.16). □

Then, for the limit values of $\bar{\epsilon}$, the simplex vertices can be obtained in the $\mathcal{V}$-representation:

$$\Delta_0 = \mathbf{0}_{n\times m}, \tag{3.17}$$

$$\Delta_i = V\int_{-\epsilon_k}^{0} exp(-\Sigma_{i,1}\tau)d\tau V^{-1}B_c,\ \forall i = 1,\dots,n, \tag{3.18}$$

where $\Sigma_{i,1}$ is the Σ matrix just with the i^{th} Jordan block (which is a Jordan block of size $m_i = 1$ for $i \in \mathbb{Z}_{[1,p]}$, that can be suppressed of the notation for the sake of simplicity), all the other ones being replaced by "zero", by considering the decomposition:

$$\Sigma = \underbrace{\begin{bmatrix} \tilde{\Sigma}_1 & 0 & \dots & 0 \\ 0 & 0 & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \dots & \dots & 0 \end{bmatrix}}_{\Sigma_{1,1}} + \underbrace{\begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & \tilde{\Sigma}_2 & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \dots & \dots & 0 \end{bmatrix}}_{\Sigma_{2,1}} + \dots + \underbrace{\begin{bmatrix} 0 & \dots & \dots & 0 \\ \vdots & 0 & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \dots & \dots & \tilde{\Sigma}_n \end{bmatrix}}_{\Sigma_{n,1}},$$

where $\tilde{\Sigma}_i$ are the Jordan blocks.

The advantage of this technique in comparison with the one derived in [145] is that the matrix A_c does not need to be invertible.

The next theorem uses the convex embedding of the function $\Delta(\epsilon_k)$ for $\epsilon_k \in \mathbb{R}_{(0,\bar{\epsilon}]}$, to describe the system dynamics in terms of linear difference inclusion (discrete-time linear system with polytopic uncertainty).

Theorem 3.6. *For any $\epsilon_k \in \mathbb{R}_{(0,\bar{\epsilon}]}$ the state matrix A_Δ satisfies:*

$$A_\Delta \in Co\{A_{\Delta_0}, A_{\Delta_1}, \dots, A_{\Delta_n}\} \tag{3.19}$$

[1] *and vertices A_{Δ_i} are given by:*

$$A_{\Delta_i} = \begin{bmatrix} A & B - n\Delta_i & n\Delta_i & \dots & 0 \\ 0 & 0 & I_m & \dots & 0 \\ \dots & \dots & \dots & \ddots & \dots \\ 0 & 0 & 0 & \dots & I_m \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}, \; i = 0, \dots, n. \tag{3.20}$$

Proof. For any $0 \leq \epsilon_k \leq \bar{\epsilon}$ and $i = 1, \dots, n$ there exists $0 \leq \beta_i \leq 1$ such that:

$$\begin{aligned}
\Delta(\epsilon_k) &= \int_{-\epsilon_k}^{0} e^{-A_c\tau} B_c d\tau = V \int_{-\epsilon_k}^{0} e^{-\Sigma\tau} d\tau V^{-1} B_c = \\
&= \sum_{i=1}^{n} V \int_{-\epsilon_k}^{0} exp(-\Sigma_i\tau) d\tau V^{-1} B_c = \\
&= \sum_{i=1}^{n} V \left[(1-\beta_i)\mathbf{0}_{n\times n} + \beta_i \int_{-\epsilon_k}^{0} exp(-\Sigma_i\tau) d\tau \right] V^{-1} B_c = \\
&= (n - \sum_{i=1}^{n} \beta_i) \underbrace{\mathbf{0}_{n\times m}}_{\Delta_0} + \sum_{i=1}^{n} \beta_i \underbrace{V \int_{-\epsilon_k}^{0} exp(-\Sigma_i\tau) d\tau V^{-1} B_c}_{\Delta_i} \\
&= \underbrace{\frac{(n - \sum_{i=1}^{n} \beta_i)}{n}}_{\alpha_0} n\Delta_0 + \sum_{i=1}^{n} \underbrace{\frac{\beta_i}{n}}_{\alpha_i} n\Delta_i.
\end{aligned}$$

The matrix $\Delta(\epsilon_k)$ appears in a linear manner in the structure of A_Δ as it can be seen in (3.12) and by using the scalars $\alpha_i \geq 0, i = 0, \dots, n$ found before, one can write $A_\Delta = \sum_{i=0}^{n} \alpha_i A_{\Delta_i}$. By noting that $\sum_{i=0}^{n} \alpha_i = 1$ the proof is completed. ■

Example 3.1. *Consider the same example as the used in [145], an unstable system with delay:*

$$\dot{x}(t) = \begin{bmatrix} 1.1 & -0.1 \\ 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t-h)$$

with $h \in (0.3, 0.4)$. Sampling period $T_e = 0.1$ a discrete model is obtained with a fixed discrete delay of $d = 3$ sampling and the uncertainty $0 < \epsilon_k \leq 0.1$.

[1] For some nonnegative scalars $\alpha_0, \alpha_1, \alpha_2, \dots, \alpha_n$ summing to one

$$A_\Delta = \sum_{i=0}^{n} \alpha_i A_{\Delta_i}$$

By using the relations (3.17-3.18) to confine (3.7), the following vertices were obtained:

$$\Delta_0(\epsilon_k) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \; ; \quad \Delta_1(\epsilon_k) = \begin{bmatrix} 0.1057 \\ 0.0057 \end{bmatrix} \; ; \quad \Delta_2(\epsilon_k) = \begin{bmatrix} 0.0999 \\ -0.0006 \end{bmatrix} . \qquad (3.21)$$

The embedding result is shown in the Fig. 3.1.

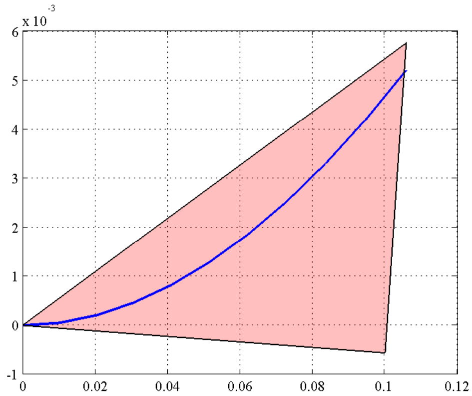


FIGURE 3.1: Illustration of the containment of $\Delta(\epsilon_k)$ by a simplex: non-defective system matrix with real eigenvalues.

The uncertain model can now be expressed as a polytopic model, by using the vertices (3.21) and the relation (3.20):

$$A_{\Delta_0} = \begin{bmatrix} 1.1157 & -0.0106 & 0.1057 & 0 & 0 \\ 0.1057 & 0.9995 & 0.0052 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} ;$$

$$A_{\Delta_1} = \begin{bmatrix} 1.1157 & -0.0106 & -0.1058 & 0.2115 & 0 \\ 0.1057 & 0.9995 & -0.0063 & 0.0115 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} ;$$

$$A_{\Delta_2} = \begin{bmatrix} 1.1157 & -0.0106 & -0.0942 & 0.1999 & 0 \\ 0.1057 & 0.9995 & 0.0063 & -0.0011 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

3.2.2 Non-defective system matrix with complex-conjugated eigenvalues

For this class of systems the function $\Delta(\epsilon_k)$ is non-monotone, i.e. is of spiral-type, as shown in the example below. The containment of $\Delta(\epsilon_k)$ by a simplex is a difficult task, and in a first stage the difficulty will obviated by using direct methods based on hypercubes (elementwise bounds) which provide conservative embeddings [114]. Moreover, an hypercube has 2^n vertices, augmenting the complexity of the polytope in higher dimensions. We have however conscience about these limitations and will get back to this issues in the Chapter 4 where is shown how generic simplex embeddings can be obtained using these basic constructions for each type of Jordan blocks.

Example 3.2. *Consider the uncertainty function* (3.7)*:*

$$\Delta(\epsilon_k) = \int_0^{\epsilon_k} e^{A_c \tau} B_c d\tau,$$

where ϵ_k variation is $0 \leq \epsilon_k \leq 50$. Two cases are studied here, whose evolution of $\Delta(\epsilon_k)$ are shown in Fig. 3.2 and 3.3. For the first one, the system matrices are:

$$A_c = \begin{bmatrix} -0.3 & -0.05 & -0.0500 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}; \; B_c = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

whose eigenvalues $\lambda(A_c)$ are:

$$\lambda(A_c) = \begin{bmatrix} -0.4424 \\ 0.0712 + 0.3285i \\ 0.0712 - 0.3285i \end{bmatrix}.$$

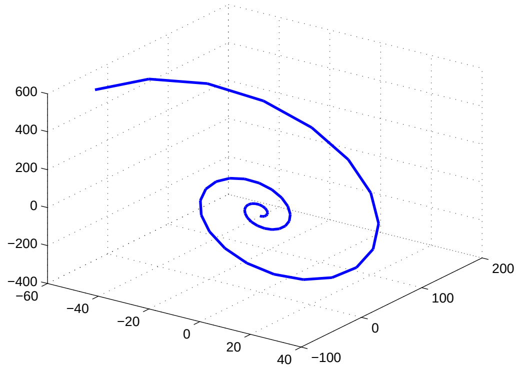


FIGURE 3.2: Illustration of $\Delta(\epsilon_k)$: system matrix with complex-conjugated eigenvalues.

Example 3.3. *For the second case the system matrices are:*

$$A_c = \begin{bmatrix} -0.09 & -0.0915 & 0.0009 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}; \; B_c = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

whose eigenvalues $\lambda(A_c)$ *are:*

$$\lambda(A_c) = \begin{bmatrix} 0.01 \\ -0.05 + 0.3i \\ -0.05 + 0.3i \end{bmatrix}.$$

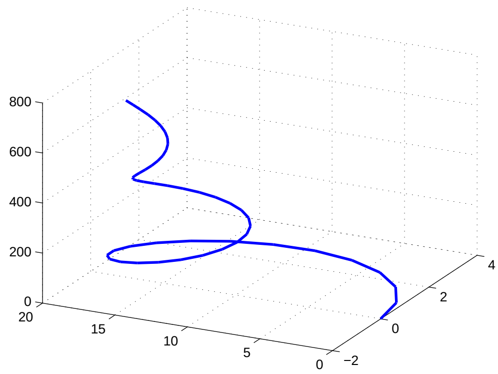


FIGURE 3.3: Illustration of $\Delta(\epsilon_k)$: system matrix with complex-conjugated eigenvalues.

Exploiting the Jordan decomposition of $A_c = V\Sigma V^{-1}$ (A_c must be nonsingular):

$$\begin{aligned}\Delta(\epsilon_k) &= A_c^{-1}\left(e^{A_c\epsilon_k}-1\right)B_c \\ &= \underbrace{V\Sigma^{-1}e^{\Sigma\epsilon_k}V^{-1}B_c}_{\theta} - V\Sigma^{-1}V^{-1}B_c\end{aligned} \tag{3.22}$$

Two independent cases need to be treated:

(i) A_c stable - in this case, the bounds corresponding to each Jordan block can be obtained by considering the forward dynamic for the autonomous system $\dot{\Sigma} = A_c(\Sigma - S) + B_c$. The coordinates $S = -A_c^{-1}B_c$ represent the *convergence point* of the $\Delta(\epsilon_k)$ function when $\epsilon_k \to \infty$. As proposed in [76], one can say that $|\Sigma - S| \leq \left|V\Sigma^{-1}\right|\left|V^{-1}B_c\right|$ (the notation $|\theta|$ represents the elementwise absolute value of the vector θ). The hypercube obtained can be represented in the $\mathcal{H}$-representation by:

$$F = \begin{bmatrix} I_n \\ -I_n \end{bmatrix};\ w = \begin{bmatrix} \left|V\Sigma^{-1}\right|\left|V^{-1}B_c\right| + S \\ \left|V\Sigma^{-1}\right|\left|V^{-1}B_c\right| - S \end{bmatrix} \tag{3.23}$$

(ii) A_c unstable - in this case, ϵ_k will "travel" along $[0, \bar{\epsilon}]$ from $\bar{\epsilon}$ to 0. The non-monotone behavior can be considered to be driven in the reverse time by a transfer matrix $-A_c$ which is stable. The point $S = -A_c^{-1}B_c$ are the *convergence point coordinates.* The hypercube can be constructed in the $\mathcal{H}$-representation with:

$$F = \begin{bmatrix} I_n \\ -I_n \end{bmatrix};\ w = \begin{bmatrix} \left|V\Sigma^{-1}\right| e^{\Sigma\bar{\epsilon}}\left|V^{-1}B_c\right| + S \\ \left|V\Sigma^{-1}\right| e^{\Sigma\bar{\epsilon}}\left|V^{-1}B_c\right| - S \end{bmatrix} \tag{3.24}$$

The vertices can be obtained by transforming the polyhedron in $\mathcal{H}$-representation, done by $\mathcal{P} = \{x \in \mathbb{R}^n \mid Fx \leq w\}$, to $\mathcal{V}$-representation, $\mathcal{P} = Co\{\Delta_1, \Delta_2, \ldots, \Delta_{2^n}\}$.

Example 3.4. *Consider the uncertainty function* (3.7) *as in [114]:*

$$\Delta(\epsilon_k) = \int_0^{\epsilon_k} e^{A_c\tau}B_c d\tau,$$

where the variation of ϵ_k is $0 \leq \epsilon_k \leq 100$. Two cases are studied here, whose evolution of $\Delta(\epsilon_k)$ and its containment by hypercubes by using the techniques presented above are *shown in Fig. 3.4 and 3.5. For the first one, the system matrix A_c is stable:*

$$A_c = \begin{bmatrix} -0.1 & 0.21 \\ -0.21 & -0.1 \end{bmatrix} ; \; B_c = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

whose eigenvalues $\lambda(A_c)$ are:

$$\lambda(A_c) = \begin{bmatrix} -0.1 + 0.21i \\ -0.1 - 0.21i \end{bmatrix}.$$

By the direct application of (3.23), the settling point is:

$$S = \begin{bmatrix} 5.7301 \\ -2.0333 \end{bmatrix} ; \; F = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} ; \; w = \begin{bmatrix} 11.8103 \\ 4.04691 \\ 0.35005 \\ 8.11345 \end{bmatrix}$$

while the polytopic embedding is described by the half planes description:

$$F = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} ; \; w = \begin{bmatrix} 11.8103 \\ 4.04691 \\ 0.35005 \\ 8.11345 \end{bmatrix}.$$

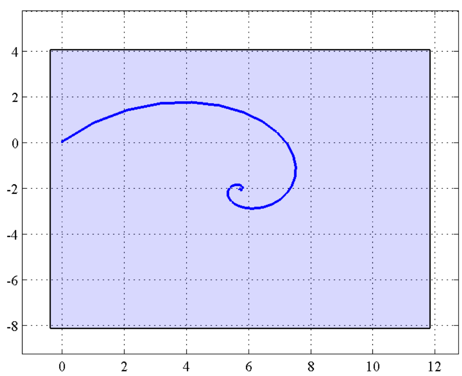


FIGURE 3.4: $\Delta(\epsilon_k)$ containment by hypercubes: Non-defective stable system matrix with complex-conjugated eigenvalues.

In Fig. 3.4 the conservatism can be observed by the fact that the "box" does not encloses the function optimally.

Example 3.5. *In the second case, the system matrix A_c is unstable, as in [114]:*

$$A_c = \begin{bmatrix} 0.01 & 0.21 \\ -0.21 & 0.01 \end{bmatrix};\ B_c = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

whose eigenvalues $\lambda(A_c)$ are:

$$\lambda(A_c) = \begin{bmatrix} 0.01 + 0.21i \\ 0.01 - 0.21i \end{bmatrix}.$$

By the direct application of (3.23), the settling point is :

$$S = \begin{bmatrix} 4.5249 \\ -4.9774 \end{bmatrix}$$

while the polytopic embedding is described by the half planes description:

$$F = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix};\ w = \begin{bmatrix} 22.81 \\ 13.3078 \\ 13.7603 \\ 23.2625 \end{bmatrix}.$$

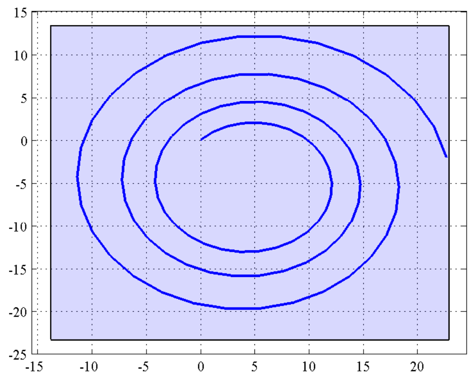


FIGURE 3.5: $\Delta(\epsilon_k)$ containment by hypercubes: Non-defective unstable system matrix with complex-conjugated eigenvalues.

In Fig. 3.5 the conservatism is reduced by the fact that the “box” is closer to the function (in the convex covering in the case of the symmetrical set).

3.2.3 Defective system matrix with real eigenvalues

Here the case of continuous models with repeated eigenvalues is considered, with the Jordan decomposition of $A_c = V\Sigma V^{-1}$ and the block diagonal matrix $\Sigma \in \mathbb{R}^{n\times n}$. As developed by [115], one has:

$$\Sigma = \begin{bmatrix} \Sigma_{1,m_1} & 0 & \cdots & 0 \\ 0 & \Sigma_{2,m_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Sigma_{p,m_p} \end{bmatrix}, \tag{3.25}$$

for all $i \in [1, ..., p]$ which leads $\Sigma_{i,m_i} \in \mathbb{R}^{m_i \times m_i}$ with:

$$\Sigma_{i,m_i} = \begin{bmatrix} \sigma_i & 1 & \cdots & 0 & 0 \\ 0 & \sigma_i & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \sigma_i & 1 \\ 0 & 0 & \cdots & 0 & \sigma_i \end{bmatrix} = \sigma_i \underbrace{\begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}}_{\Lambda_i} + \underbrace{\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}}_{\Gamma_i}.$$

One should note that the matrix Γ_i is nilpotent. As it is briefly described in Chapter 2, the matrix Γ_i is nilpotent if there is some k such that $\Gamma_i^k = \mathbf{0}$. This fact will be exploited below.

Define the explicit matrices:

$$\Sigma = \begin{bmatrix} \sigma_1\Lambda_1 + \Gamma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2\Lambda_2 + \Gamma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_p\Lambda_p + \Gamma_p \end{bmatrix} =$$

$$= \sum_{i=1}^{p} \sigma_i \underbrace{\begin{bmatrix} 0 & & & \\ & \ddots & & \\ & & \Lambda_i & \\ & & & \ddots & \\ & & & & 0 \end{bmatrix}}_{L_i} + \sum_{i=1}^{p} \underbrace{\begin{bmatrix} 0 & & & \\ & \ddots & & \\ & & \Gamma_i & \\ & & & \ddots & \\ & & & & 0 \end{bmatrix}}_{G_i}, \tag{3.26}$$

which leads to a compact form:

$$\Sigma = \sum_{i=1}^{p} \sigma_i L_i + G_i, \tag{3.27}$$

where L_i are diagonal matrices of the size of Σ with "1"s on the diagonal of the cell corresponding to the Jordan block Σ_{i,m_i}. Similarly G_i are diagonal matrices of the size of Σ with "1"s on the upper-diagonal of the cell corresponding to the Jordan block Σ_{i,m_i}.

The convex cover of the function $\Delta(\epsilon_k)$, Eq. (3.7) can be rewritten:

$$\begin{aligned} \Delta(\epsilon_k) &= \int_0^{\epsilon_k} e^{A_c\tau} B_c d\tau = \int_0^{\epsilon_k} e^{V\Sigma V^{-1}\tau} d\tau \ B_c \\ &= \int_0^{\epsilon_k} V e^{\Sigma\tau} V^{-1} d\tau \ B_c = V \int_0^{\epsilon_k} e^{\Sigma\tau} d\tau \ V^{-1} B_c. \end{aligned} \tag{3.28}$$

Exploiting the structure of the Jordan blocks one can rewrite:

$$e^{\Sigma_{i,m_i}\tau} = e^{\sigma_i \Lambda_i \tau} e^{\Gamma_i \tau}. \tag{3.29}$$

Using the Taylor expansion of $e^{\Sigma_{i,m_i}\tau}$ up to the $(m_i-1)^{th}$ term (as the m_i^{th} term will be identically zero matrix by the fact that Γ_i is nilpotent), Eq. (3.20) becomes:

$$e^{\Sigma_{i,m_i}\tau} = e^{\sigma_i \Lambda_i \tau} \left(I + \frac{1}{1!}\Gamma_i \tau + \frac{1}{2!}\Gamma_i^2 \tau^2 + ... + \frac{1}{(m_i-1)!}\Gamma_j^{m_i-1}\tau^{m_i-1} \right). \tag{3.30}$$

This expansion is finite by exploiting the nilpotent characteristics of Γ_i, i.e. $\Gamma_i^m = \mathbf{0}$.

Two cases can be distinguished:

1. Jordan blocks Σ_{i,m_i} with $m_i = 1$. In this case G_i does not exist and (3.19) can be handled upon the techniques developed in Section 3.2.1, by the vertices:

$$\Delta_{j,1}(\epsilon_k) = V \int_0^{\epsilon_k} exp(\Sigma\tau) d\tau \ V^{-1} B_c = V L_j \int_0^{\epsilon_k} exp(\sigma_j \tau) d\tau \ V^{-1} B_c. \tag{3.31}$$

2. Jordan block Σ_{i,m_i} corresponds to a repeated eigenvalue with geometrical multiplicity $m_i \geq 2$. The embedding problem can be decoupled in this case on the embedding of the blocks:

$$\begin{aligned} \Delta_j(\epsilon_k) &= VL_j \int_0^{\epsilon_k} exp(\sigma_j \tau)d\tau\ V^{-1}B_c; \\ \Delta_{j,1}(\epsilon_k) &= V\tfrac{1}{1!}G_j \int_0^{\epsilon_k} \tau exp(\sigma_j \tau)d\tau\ V^{-1}B_c; \\ &\vdots \\ \Delta_{j,m_j-1}(\epsilon_k) &= V\tfrac{1}{(m_j-1)!}G_j^{m_j-1} \int_0^{\epsilon_k} \tau^{m_j-1} exp(\sigma_j \tau)d\tau\ V^{-1}B_c. \end{aligned} \tag{3.32}$$

Using (3.25) one can see that:

- $exp(\sigma_i \tau)$, with $i \in [1, ..., p]$ are monotones and positive functions on $\tau \in [0, \bar{\epsilon}]$.
- $\tau_p^k exp(\sigma_i \tau)$ with $i \in [1, ..., p]$, $k \in [1, ..., n-p]$ are monotones and positive functions on $\tau \in [0, \bar{\epsilon}]$.

Globally, for a given $\bar{\epsilon}$ having n monotone and positive functions on $\tau \in [0, \bar{\epsilon}]$. Noting these functions as $g(\tau), ..., g_n(\tau)$, there exists an ordered sequence $i_1, ..., i_n$ with $i_k \in [1, ..., n]$ such that:

$$g_{i_1}(\bar{\epsilon}) \geq g_{i_2}(\bar{\epsilon}) \geq \ldots \geq g_{i_n}(\bar{\epsilon}). \tag{3.33}$$

For all $g_i(\epsilon_k)$ there exists a $\Delta(\epsilon_k)$ as in (3.23) and by its correspondences there exists $\Delta_{g_{i_k}}(\epsilon_k)$ for all $g_{i_k}(\epsilon_k)$.

Theorem 3.7. *For any $0 \leq \epsilon_k \leq \bar{\epsilon}$ there exists Δ_i satisfying:*

$$\Delta(\epsilon_k) \in Co\{\Delta_0, \Delta_{g_1}(\bar{\epsilon}), \ldots, \Delta_{g_n}(\bar{\epsilon})\}.$$

Proof. One can have $\forall \epsilon_k \in [0, \bar{\epsilon}]$, $\exists\ 0 \leq \beta_i \leq 1$ such that:

$$\Delta_{g_i}(\epsilon_k) = \beta_i \Delta_{g_i}(\bar{\epsilon}),$$

where Δ_{g_i} are the matrices formed by non-zero blocks at the position g_{i_k}, which can be proved by the monotonicity of Δ_{g_i}:

$$\begin{aligned}
\Delta(\epsilon_k) &= \sum_{i=1}^{n}\{\beta_i\Delta_{g_i} + (1-\beta_i)\Delta_0\} \\
&= \sum_{i=1}^{n}\beta_1\Delta_{g_i} + \sum_{i=2}^{n}(\beta_i-\beta_1)\Delta_{g_i} \\
&= \beta_1\sum_{i=1}^{n}\Delta_{g_i} + \sum_{i=2}^{n}(\beta_2-\beta_1)\Delta_{g_i} + \sum_{i=3}^{n}(\beta_i-\beta_2)\Delta_{g_i} \\
&\vdots \\
&= \underbrace{\beta_1}_{\alpha_1}\sum_{i=1}^{n}\Delta_{g_i} + \underbrace{(\beta_2-\beta_1)}_{\alpha_2}\sum_{i=2}^{n}\Delta_{g_i} + \cdots + \underbrace{(\beta_n-\beta_{n-1})}_{\alpha_n}\Delta_{g_n} \\
&= \alpha_1\sum_{i=1}^{n}\Delta_{g_i} + \alpha_2\sum_{i=2}^{n}\Delta_{g_i} + \cdots + \alpha_n\Delta_{g_n} + \underbrace{(1-\beta_n)}_{\alpha_0}\Delta_0
\end{aligned}$$

One can easily check that $\alpha_i \leq 1, \forall i = 1,\ldots,n$ and $\sum_{i=0}^{n}\alpha_i = 1$.

Then for a given $\epsilon_k \in [0,\ \bar{\epsilon}]$ by ordering the convex combination coefficients β_i:

$$\beta_1 \leq \beta_2 \leq \cdots \leq \beta_n$$

wich implies $0 \leq \alpha_i$ and concludes the proof. ■

Example 3.6. *Consider the system* (3.1) *with matrices:*

$$A_c = \begin{bmatrix} 5 & -8 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix};\ B_c = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

with sampling period $T_e = 0.1s$ and the structure of the Jordan decomposition $A_c = V\Sigma V^{-1}$:

$$V = \begin{bmatrix} 1 & 4 & 0 \\ 1 & 2 & -1 \\ 1 & 1 & -1 \end{bmatrix};\quad \Sigma = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix},$$

one can remark that there are two eigenvalues in Σ one of them with algebraic multiplicity 2 and geometrical multiplicity 1. Following the procedure presented above:

$$\Delta_1(\epsilon_k) = V \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \int_0^{\epsilon_k} e^{\tau} d\tau \; V^{-1} B_c; \; \Delta_2(\epsilon_k) = V \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \int_0^{\epsilon_k} e^{2\tau} d\tau \; V^{-1} B_c;$$

$$\Delta_3(\epsilon_k) = V \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \int_0^{\epsilon_k} \tau e^{2\tau} d\tau \; V^{-1} B_c.$$

The considered delay variation is of one sampling period, i.e. $\epsilon_k = T_e$. The simplex volume is $3 * 10^{-4}$, and $\Delta(\epsilon_k)$ values are:

$$\Delta_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}; \quad \Delta_1(T_e) = \begin{bmatrix} 0.105 \\ 0.105 \\ 0.105 \end{bmatrix}; \quad \Delta_2(T_e) = \begin{bmatrix} 0 \\ -0.110 \\ -0.110 \end{bmatrix}; \quad \Delta_3(T_e) = \begin{bmatrix} 0.223 \\ 0.011 \\ 0.005 \end{bmatrix}$$

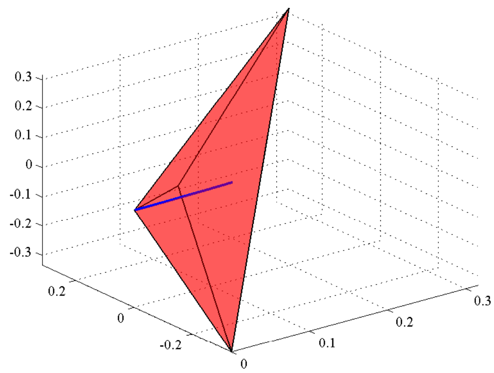


FIGURE 3.6: Containment of $\Delta(\epsilon_k)$ by a simplex: Defective system matrix with real eigenvalues.

3.2.3.1 Exploiting the monotonicity

The previous approach provides a (simple) automatic construction of the embedding. However, it can be considered conservative (as it is the case for the previous example due to the fact that the extreme value of the function $\Delta(\epsilon_k)$ in not placed on a vertex of the embedding). One can exploit the monotonicity characteristics of the exponential functions, to a better comprehension of the structure of $\Delta(\epsilon_k)$ and further obtain a less conservative embedding as sketched by the following example.

Example: Consider the system (3.1) of order 4 with algebraic multiplicity 4 and geometrical multiplicity 1. One can rearrange the composition of the block exponential and instead of scalar embedding of the functions on the upper diagonal. Deal with the difference between them on a partition of the interval of variation $\epsilon_k \in [0, T_e)$ after an appropriate ordering of the function one can obtain:

$$
\begin{aligned}
\Delta_{1max} &= V \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \max\left\{ \int_0^{\epsilon_k} \frac{1}{3!}\tau^3 e^{\tau} d\tau \right\} V^{-1} B_c \\
\Delta_{2max} &= V \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \max\left\{ \int_0^{\epsilon_k} \left(-\frac{1}{3!}\tau^3 + \frac{1}{2!}\tau^2 \right) e^{\tau} d\tau \right\} V^{-1} B_c \\
\Delta_{3max} &= V \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \max\left\{ \int_0^{\epsilon_k} \left(-\frac{1}{2!}\tau^2 + \tau \right) e^{\tau} d\tau \right\} V^{-1} B_c \\
\Delta_{4max} &= V \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \max\left\{ \int_0^{\epsilon_k} (-\tau + 1)\, e^{\tau} d\tau \right\} V^{-1} B_c
\end{aligned}
\tag{3.34}
$$

Exploiting the monotonicity of the exponential terms, one should order the elements of the Eqs. (3.34), by analyzing the values of σ and τ as in the figure 3.7.

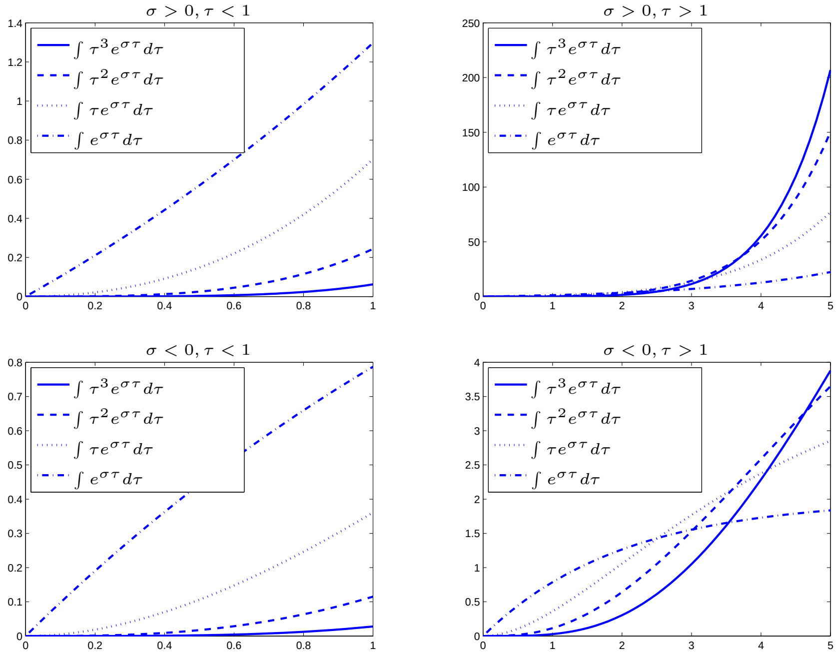


FIGURE 3.7: Illustration of the exponential plots of (3.34) in order to exploit the monotonicity to order the terms.

3.2.3.2 Formal Statements

Looking toward a less conservative approach of (3.34), one can have:

$$
\begin{aligned}
&\Delta_j(\epsilon_k) = V\left(I + \tfrac{1}{1!}\Gamma + ... + \tfrac{1}{(p-1)!}\Gamma^{p-1}\right)\int_0^{\epsilon_k} \tau^{p-1} exp(\sigma_j\tau)d\tau \ V^{-1}B_c \\
&\Delta_{j,1}(\epsilon_k) = V\left(I + ... + \tfrac{1}{(p-2)!}\Gamma^{p-2}\right)\int_0^{\epsilon_k}\left(-\frac{1}{(p-1)!}\tau^{p-1} + \tau^{p-2}\right) exp(\sigma_j\tau)d\tau \ V^{-1}B_c \\
&\Delta_{j,2}(\epsilon_k) = V\left(I + ... + \tfrac{1}{(p-3)!}\Gamma^{p-3}\right)\int_0^{\epsilon_k}\left(-\frac{1}{(p-2)!}\tau^{p-2} + \tau^{p-3}\right) exp(\sigma_j\tau)d\tau \ V^{-1}B_c \\
&\vdots \\
&\Delta_{j,m_j-1} = V\int_0^{\epsilon_k} (-\tau + 1)\, exp(\sigma_j\tau)d\tau \ V^{-1}B_c
\end{aligned}
\tag{3.35}
$$

The other Δ_{j,m_k}, $k \in [1, ..., n-p]$ elements are the permutation between the elements in (3.35):

$$
\begin{aligned}
&\Delta_{j,m_j} = \Delta_j(\epsilon_k) + \Delta_{j,1}(\epsilon_k) \\
&\Delta_{j,m_j+1} = \Delta_j(\epsilon_k) + \Delta_{j,2}(\epsilon_k) \\
&\vdots \\
&\Delta_{j,m_k} = \Delta_j(\epsilon_k) + \Delta_{j,1}(\epsilon_k) + \ldots + \Delta_{j,k-1}(\epsilon_k)
\end{aligned}
\tag{3.36}
$$

Example 3.7. *Consider the system* (3.1) *with the same matrices as the previous example. As result, by exploiting the monotonicity as described above, one has the case of ordered nonlinear function. One can obtain the embedding with a volume* $6.3 * 10^{-5}$ *for the extreme values of* $\Delta(\epsilon_k)$*:*

$$\Delta_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}; \quad \Delta_1(T_e) = \begin{bmatrix} 0.105 \\ 0.105 \\ 0.105 \end{bmatrix}; \quad \Delta_2(T_e) = \begin{bmatrix} 0.023 \\ 0.005 \\ 0 \end{bmatrix}; \quad \Delta_3(T_e) = \begin{bmatrix} 0 \\ -0.105 \\ -0.105 \end{bmatrix}$$
$$\Delta_4(T_e) = \begin{bmatrix} 0.023 \\ -0.099 \\ -0.105 \end{bmatrix}; \quad \Delta_5(T_e) = \begin{bmatrix} 0.128 \\ 0.111 \\ 0.105 \end{bmatrix}; \quad \Delta_6(T_e) = \begin{bmatrix} 0.105 \\ 0.0002 \\ 0.0002 \end{bmatrix}; \quad \Delta_7(T_e) = \begin{bmatrix} 0.128 \\ 0.006 \\ 0.0002 \end{bmatrix}$$

obtained upon all possible arrangements of sums of terms Δ_{imax}.

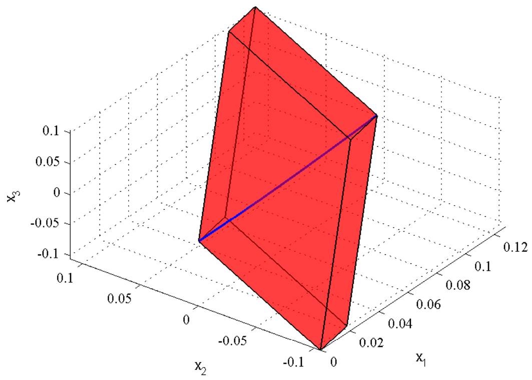


FIGURE 3.8: Containment of $\Delta(\epsilon_k)$ by a polyhedron: Defective system matrix with real eigenvalues.

3.3 Illustrative Example

The next example will be used during the thesis as a possible application of the modeling, analysis and control design techniques developed. The objective is the position control of a DC motor as developed in [173]. The practical system is shown in Fig. 3.9(a) and a more detailed picture of the position system is given in Fig. 3.9(b).

The bloc representation in open loop is presented in Fig. 3.10.

(a)

(b)

FIGURE 3.9: Position control plant.

The control objective is the positioning of a linear cursor upon a reference signal u_{ref}. The linear cursor is attached to a belt moved by a DC motor, while a pulley (or reducer) transforms the rotation of the motor (θ) into linear movement (x) in terms of the electrical signal applied to the motor. This system has two sensors: a position sensor, who measures the linear position of the cursor and a tachometric sensor, which gives an electric signal proportional to the rotation speed of the motor.

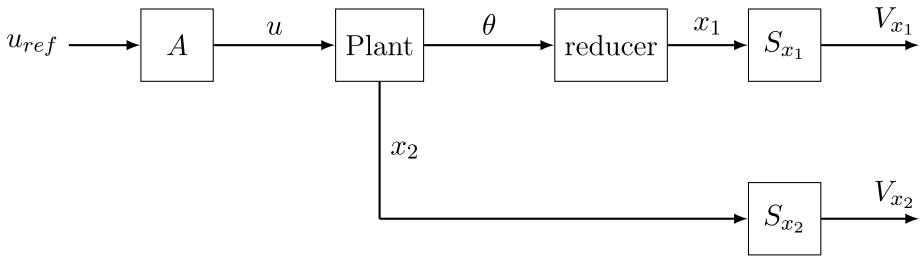


FIGURE 3.10: General scheme for the plant to be controlled.

The position sensor S_{x_1} is physically limited as $-6\text{cm} \leq x_1 \leq 6\text{cm}$, that can be translated in an electrical limitation of $-10\text{V} \leq V_{x_1} \leq 10\text{V}$. The electrical tension of the tachometric sensor can deliver is $-3\text{V} \leq V_{x_2} \leq 3\text{V}$, that corresponds to $-8.5\text{m/s} \leq x_1 \leq 8.5\text{m/s}$. Both of the sensors (position and tachometric), S_{x_i}, for $i = 2$ are considered simple gains who transform the physical entries into electric signals.

The block A represents a power amplifier, necessary to feed the motor with an adequate control input u.

Example 3.8 (Model of a DC motor with time-varying delays - intersampled delay variation)**.** *The simplified system equation can be defined as:*

$$\beta\ddot{x}(t) + \dot{x}(t) = K_s u(t - \tau),$$

where β is the mechanical time constant of the motor, in $[s]$, given by $\beta \approx \frac{RJ}{\alpha R + \phi^2}$, where ϕ is the electrical flux and coupling constant, R is the electrical resistance of the motor, J is the motor inertia and α is the internal viscosity constant. K_s is the system speed constant in $[rad/s/V]$, given by $K_s = \frac{\phi_0}{\alpha R + \phi^2}$. Actually, the states should be in terms of V_{x_i}, for $i = 1, 2$, i.e. in terms of the electrical tensions given by the sensors. But in order to respect the notation of the thesis, we consider the states $x_i = V_{x_i}$, for $i = 1, 2$. The state-space representation is given by:

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & \frac{-1}{\beta} \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ \frac{K_s}{\beta} \end{bmatrix} u(t - \tau).$$

The system parameters identified by least squares method are $K_s = 210$ and $\tau = 0.017$. The sampling period is $T_e = 0.003s$, the delay is $\tau \in \mathbb{R}_{[0.006s, 0.009s]}$. It must be noted that for analysis purposes, such a variable time-delay can be induced by the use of a Matlab

Simulink real-time environment. Practically, for the NCS framework, this will cover the asynchronous phenomena as described in the Introduction (Chapter 1).

The discrete-time model is given by:

$$x_{k+1} = \begin{bmatrix} 1 & 0.0028 \\ 0 & 0.8382 \end{bmatrix} x_k + \begin{bmatrix} 0.0525 \\ 33.9731 \end{bmatrix} u_{k-d} - \Delta(u_{k-d} - u_{k-d+1}), \text{ with d} = 3. \tag{3.37}$$

In the first instance, the embedding of the intersampled uncertainty matrix Δ have to be obtained. Due to the fact that in the original representation, we deal with a 2-dimensional state vector x_k and a scalar input u_k, the poytopic uncertainty will be given by a simplex in a 2-dimensional (Fig. 3.11) space due to the fact that $\Delta \in \mathbb{R}^{2\times 1}$, by following the procedure presented in Section 3.1.1:

$$\Delta \in Co\left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 37.0588 \end{bmatrix}, \begin{bmatrix} 0.0525 \\ 33.9731 \end{bmatrix} \right\}, \tag{3.38}$$

which volume is $Vol = 0.9720$.

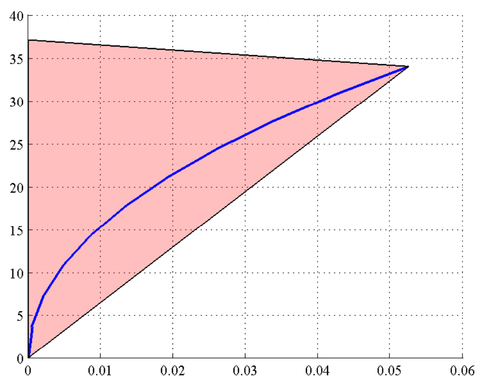


FIGURE 3.11: Δ function containment by the simplex.

The uncertain model can now be expressed as a polytopic model, by using the vertices (3.38) *and the relation* (3.20)*:*

$$A_{\Delta_0} = \begin{bmatrix} 1 & 0.0028 & 0.0525 & 0 & 0 \\ 0 & 0.8382 & 33.9731 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}; \quad A_{\Delta_1} = \begin{bmatrix} 1 & 0.0028 & 0.0525 & 0 & 0 \\ 0 & 0.8382 & -3.0857 & 37.0588 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix};$$

$$A_{\Delta_2} = \begin{bmatrix} 1 & 0.0028 & 0 & 0.0525 & 0 \\ 0 & 0.8382 & 0 & 33.9731 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Chapter 4

Optimization over a class of polytopes

Less conservative polytopic embeddings with the fixed complexity (of $n+1$ vertices in the case of a simplex) can be obtained by defining an appropriate constrained optimization problem, where the restrictions are generated by the inclusion of the envelope of the curve ($\Delta(\epsilon_k)$ in the case of the variable delay we are interested in) by an over-approximation, as developed in two papers, [114] and [115], and will be explained here in a generic form. We have seen in the previous chapter that the delay uncertainty function can be contained by a polytopic region. One of the basic ideas in the present chapter is that the same principle can be used, if the uncertainty function will be divided in sub-intervals (of variations for the parameter ϵ_k) and corresponding local embeddings are generated. The second basic idea is the use of a cost function to be minimized upon the volume (in fact the "hypervolume" as long as the space is in n-dimension) of a simplex by using the vertices. The link between these two ingredients is represented by the fact that the local embeddings are included in containment type of constraints. The ultimate goal is to obtain a less conservative polytopic representation of the time-delay system.

The general problem to be addressed in this chapter is the *overapproximation of* $\Delta(\epsilon)$ *as in* (3.7) *be a simplex with an optimized volume.*

4.1 Toward less conservative polytopes

For low complexity embeddings, the Jordan normal form is the main ingredient, where the key element is the exponential of a block-diagonal matrix:

$$D(\epsilon_k) = \int_0^{\epsilon_k} e^{\Sigma\tau} d\tau.$$

By its diagonal structure, this function will impose the size of the set of generators. By the diagonal structure of D one can map the diagonal elements in a n-dimensional vector and thus the minimization of the embedding to the optimization of the volume of a n-dimensional polytope. Specifically, we will be interested by the polytopes with the lowest complexity - the simplices.

In the following we will discuss about polytopic embeddings of $D(\epsilon_k)$ in terms of $n+1$ vertices in a n dimensional space:

$$\mathcal{E} = Co\{D_0, D_1, \ldots, D_n\},$$

where $D_i = diag(v_{i,1}, \ldots, v_{i,n})$ will be represented by the following vectors:

$$v_0 = \begin{bmatrix} v_{0,1} \\ v_{0,2} \\ \vdots \\ v_{0,n} \end{bmatrix};\ v_1 = \begin{bmatrix} v_{1,1} \\ v_{1,2} \\ \vdots \\ v_{1,n} \end{bmatrix};\ \ldots;\ v_n = \begin{bmatrix} v_{n,1} \\ v_{n,2} \\ \vdots \\ v_{n,n} \end{bmatrix}. \tag{4.1}$$

A less conservative simplex embedding is obtained as solution of the following *optimization problem*:

$$\begin{aligned} \min_{v_0,\ldots,v_n} \quad & Vol(Co\{v_0, \ldots, v_n\}), \\ \text{subject to} \quad & v(\epsilon_k) \in Co\{v_0, \ldots, v_n\} \end{aligned} \tag{4.2}$$

The cost function can be explicitly expressed in the form:

$$Vol(Co\{v_0, \ldots, v_n\}) = \frac{1}{n!} \left| det \begin{bmatrix} v_0 & \ldots & v_n \\ 1 & \ldots & 1 \end{bmatrix} \right| \tag{4.3}$$

This problem minimizes the volume of a simplicial polytope by using a set of points as constraints (in the case of (4.2) the constraints are the vertices of $\mathcal{E}$). The nonlinear dependence in ϵ_k (especially in the constraints expression) of (4.2) will influence the convergence of the optimization problem, making the global approach not very interesting (at least for larger values of "n"). The idea is to take local embeddings, to be constructed by splitting of $\Delta(\epsilon_k)$ evaluated in the interval $[0, \bar{\epsilon}]$ into ℓ subintervals $[0, \epsilon_1] \cup [\epsilon_1, \epsilon_2] \cup \dots \cup [\epsilon_{\ell-1}, \bar{\epsilon}]$.

Then the minimum volume simplex problem is to be solved for a set of extreme points generated by the local embeddings $D(\epsilon_k) \in \bigcup_{i=0}^{\ell-1} \mathcal{E}_{\epsilon_i,\epsilon_{i+1}}$ instead of the conservative non-linear expression of $D(\epsilon_k)$.

$$\begin{aligned} \min_{v_0,\dots,v_n} \quad & \frac{1}{n!}\left| det \begin{bmatrix} v_0 & \dots & v_n \\ 1 & \dots & 1 \end{bmatrix} \right|. \\ \text{subject to} \quad & d_i^{[\epsilon_k,\epsilon_{k+1}]} \in Co\{v_0,\dots,v_n\}, \, k \in \mathbb{Z}_{[0,\ell]} \end{aligned} \tag{4.4}$$

For this problem, efficient optimization routines can be obtained by formulating it as an instance of polynomial programming with linear constraints. A constructive way of expressing the embedding of a set of points is to use the $\mathcal{H}$-representation, which express a simplex in $\mathbb{R}^n$ by the $n+1$ bounding hyperplanes:

$$h_{i0} + \sum_{j=0}^{n} h_{ij}x_j = 0, \, \forall i \in \mathbb{Z}[0, n]. \tag{4.5}$$

By noting $H \in \mathbb{R}^{(n+1)\times n}$ and $H_0 \in \mathbb{R}^{n+1}$ the matrix with elements h_{ij}, $i, j \in \mathbb{Z}_{[0,n]}$ and the vector with h_{i0}, $i \in \mathbb{Z}_{[0,n]}$, the constraints in (4.4) are translated in:

$$Hd_i^{[\epsilon_k,\epsilon_{k+1}]} \leq H_0, \, \forall k \in \mathbb{Z}_{[0,\ell]}. \tag{4.6}$$

The cost function can be adapted using the next result.

Theorem 4.1 ([86]Simplex volume in $\mathcal{H}$-representation)**.** *The volume of a simplex $\mathcal{S} = \{x \in \mathbb{R}^n \mid Hx \leq k\}$, in $\mathcal{H}$-representation, is given by:*

$$Vol(\mathcal{S}) = \frac{|det(H)|^n}{n!\Pi_{i=0}^{n} H_{i0}}, \tag{4.7}$$

where H_{i0} is the cofactor of h_{i0} in "H".

Sketch of Proof.(See [86] for a more detailed proof.) Consider the transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ defined by:

$$T : \mathbf{x}' = H\mathbf{x} + h_0$$

The idea is to take a simplex $\mathcal{S}' \in \mathbb{R}^n$ with vertices in the coordinate axis, described by:

$$\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}; \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}; \cdots; \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix},$$

and make the transformation of $\mathcal{S}$ into $\mathcal{S}'$ which is bounded by the hyperplanes S'_i for $i \in \mathbb{Z}_{[0,n]}$. On has:

$$Vol(\mathcal{S}) = det(J)Vol(\mathcal{S}'),$$

where J is the Jacobian matrix of the transformation. It is also known that $Vol(S') = \frac{1}{n!}$. By denoting the intercept of S_0 with each x'_i-axis by c_i, by a simple coordinates transformation $y_i = \frac{x'_i}{c_i}$, one has:

$$Vol(\mathcal{S}') = \int_{\mathcal{S}'} 1dx'_1 \cdots dx'_{n+1} = \prod_{i=1}^{n+1} \frac{c_i}{n!}.$$

To find the c_i's, let $P_i = \cap_{k \neq i} S_k$, $i \in \mathbb{Z}_{[0,n]}$ be the vertex which lies in all the hyperplanes except S_i. If the position vector of P_i is designated $\mathbf{x}_i$, with $\mathbf{x}_i = [x_{i1}, \ldots, x_{in}]$, then by Cramer's rule:

$$x_{ij} = \frac{H^0_{ij}}{H^0_{i0}}, \tag{4.8}$$

where H^0_{ij} is the co-factor h_{ij} in H^0. Now T maps P_i into P'_i. Since $P'_i = \cap_{j \neq i} S'_j$ it follows that:

$$x'_{ij} = 0, \; for j \neq i$$

and

$$x'_{ii} = \sum_{k=1}^{n} h_{ik} x_{ik} + h_{i0}.$$

By substituting from (4.8), one has:

$$c_{ii} = x'_{ii} = \sum_{k=1}^{n} h_{ik} \frac{H^0_{ik}}{H^0_{i0}} + h_{i0} = \frac{det(H^0)}{H^0_{i0}}. \tag{4.9}$$

Because of the Jacobian of the transformation T is $det(H)^{-1} = (H^0_{00})^{-1}$, one can see from (4.8) and (4.9) that (4.7) holds.

■

Remark 4.2. The articles [63, 64, 86] provide a tutorial on polytope, polyhedra and simplex calculations and computational aspects in direct relationship with the present problem of embedding optimization and have been used as references for the geometrical concepts vehiculated in the present modeling approach.

With these elements, one can state the tractable optimization problem for the reduced complexity embedding of $v(\epsilon)$ [1]

$$\begin{array}{ll} \min\limits_{H} & \frac{|det(H)|^n}{n!\Pi_{i=0}^{n} H_{i0}}. \\ \text{subject to} & Hd_i^{[\epsilon_k,\epsilon_{k+1}]} \leq H_0, \ \forall k \in \mathbb{Z}_{[0,\ell]};\ \forall i \in \mathbb{Z}_{[1,n]} \end{array} \tag{4.10}$$

The volume minimization problem will be used in an interactive scheme for embedding optimization. The principle is the following:

- Firstly one should divide the interval $[0, \bar{\epsilon}]$ in sub-intervals, for example, $[0,\ \bar{\epsilon}/2]$, and $[\bar{\epsilon}/2,\ \bar{\epsilon}]$, construct local embeddings for each sub-interval and collect all the vertices.
- The next step is to minimize a simplex by using the obtained vertices as constraints and calculate its volume, by using (4.10).
- By comparing the calculated volume with the preceding simplex volume, one has the precision of the algorithmic step. If the precision is smaller than the given by the user, the algorithm stops. Otherwise the local embedding which "touches" the simplex and has the biggest volume will be re-sampled and the algorithm restarts.

The formal statements of this algorithm can be found in the Appendix B, Algorithm 1.

The vertices of the embedding will be obtained from a classical transformation from the $\mathcal{H}$-representation to the $\mathcal{V}$-representation. Although this procedure is expensive in the

[1] Further improvements can be obtained by exploiting the *centroid property* which states that the centroid of each simplex facet touches the convex hull of points to be enclosed. Zhou and Suri [196] proposed an algorithm with linear complexity for the construction of the ϵ_k approximation for minimum volume ellipsoid in $\mathbb{R}^3$.

computational point of view [55], we remark that we are in one of the most optimistic case as long as we are handling simplex bodies ($n + 1$ vertices). The last points to be clarified is the convergence of the algorithm and the degree of approximation of the solution found by solving (4.10) with respect to the minimum volume ellipsoid sought by (4.2).

Proposition 4.3. *Let $\mathcal{E}_\ell^a$ and $\mathcal{E}_{\ell+1}^a$ be the approximations obtained at two consecutive iterations of the Algorithm 1.*

$$Vol(\mathcal{E}_\ell^a) \geq Vol(\mathcal{E}_{\ell+1}^a) \tag{4.11}$$

Proof. These two simplex approximations are built upon the decomposition of the interval $\epsilon \in \mathbb{R}_{(0,\bar{\epsilon}]}$ in:

$$[0, \epsilon_1^\ell] \cup [\epsilon_1^\ell, \epsilon_2^\ell] \cup \cdots \cup [\epsilon_{i-1}^\ell, \epsilon_i^\ell] \cup [\epsilon_{\ell-1}^\ell, \bar{\epsilon}]$$

and

$$[0, \epsilon_1^{\ell+1}] \cup [\epsilon_1^{\ell+1}, \epsilon_2^{\ell+1}] \cup \cdots \cup [\epsilon_{i-1}^{\ell+1}, \epsilon_i^{\ell+1}] \cup [\epsilon_i^{\ell+1}, \epsilon_{i+1}^{\ell+1}] \cup [\epsilon_\ell^{\ell+1}, \bar{\epsilon}].$$

The subintervals are identical, excepting the ones who are re-sampled: $[\epsilon_{i-1}^\ell, \epsilon_i^\ell]$ is decomposed in $[\epsilon_{i-1}^{\ell+1}, \epsilon_i^{\ell+1}] \cup [\epsilon_i^{\ell+1}, \epsilon_{i+1}^{\ell+1}]$ at the second iteration.

The local embedding for

$$\Delta(\epsilon_k) = \int_{\epsilon_i}^{\epsilon} e^{A_c\tau} B_c d\tau,$$

for all $\epsilon \in [\epsilon_{i-1}, \epsilon_i]$ is denoted $\mathcal{E}_\ell^i$ and the ones corresponding to $[\epsilon_{i-1}^{\ell+1}, \epsilon_i^{\ell+1}] \cup [\epsilon_i^{\ell+1}, \epsilon_{i+1}^{\ell+1}]$ by $\mathcal{E}_{\ell+1}^i \cup \mathcal{E}_{\ell+1}^{i+1}$.

By the fact that:

$$\left\{\mathcal{E}_{\ell+1}^i \cup \mathcal{E}_{\ell+1}^{i+1}\right\} \subset \mathcal{E}_\ell^i$$

and recalling (4.10), which can be rewritten compactly as the volume minimization of $Vol(\mathcal{E}^a)$ subject to $Co\left\{\mathcal{E}_\ell^0, \ldots, \mathcal{E}_\ell^\ell\right\} \subset \mathcal{E}^a$, it becomes obvious that the set of constraints for $\mathcal{E}_{\ell+1}^a$ is redundant with respect to the one for $\mathcal{E}_\ell^a$. By consequence the corresponding optimum satisfies $Vol(\mathcal{E}_\ell^a) \geq Vol(\mathcal{E}_{\ell+1}^a)$, as long as the cost function is the same and the set of constraints is less restrictive at step $\ell + 1$.. ∎

Definition 4.4. Consider the vertex v_i of a local embedding $\mathcal{E}_\ell^i$ that is placed on the border of $\mathcal{E}_\ell^a$. By noting v_i^c the most distanced vertex from v_i inside $\mathcal{E}_\ell^i$, and $v_i^{\mathcal{E}} \in$

$int\{Co(\Delta(\epsilon))\}$ an arbitrary point in the interior of $Co\{\Delta(\epsilon_k)\}$ for all $\epsilon \in \mathbb{R}_{(0,\bar{\epsilon}]}$. One can define c_i as the ratio:

$$c_i = \frac{d(v_i, v_i^c)}{d(v_i, v_i^{\mathcal{E}})} \tag{4.12}$$

where $d(.,.)$ is understood as the Euclidean distance as defined in Chapter 2.

Theorem 4.5. *Let the minimum volume simplex embedding for $\Delta(\epsilon)$ for all $\epsilon \in \mathbb{R}_{(0,\bar{\epsilon}]}$ be $\mathcal{E}$ and the simplex resulting from the solution of (4.10) with respect to the local embeddings on $[0,\epsilon_1] \cup [\epsilon_1,\epsilon_2] \cup \cdots \cup [\epsilon_{\ell-1},\bar{\epsilon}]$ be $\mathcal{E}^a$. Then, by considering the constant $c^* = \max c_j \in \mathbb{R}_+^*$, for $j \in \mathbb{Z}_{[1,\ell]}$, over all the scalars defined by (4.12), the following relationship is verified:*

$$Vol(c^*\mathcal{E}^a) \geq \left[(Vol(\mathcal{E}^a)^{\frac{1}{n}}) - (Vol(\mathcal{E})^{\frac{1}{n}})\right]^n. \tag{4.13}$$

Proof. Consider $\mathcal{C} = Co\left\{\Delta(\epsilon), \forall \epsilon \in \mathbb{R}_{(0,\bar{\epsilon}]}\right\}$. Then there exist a homothetic transformation to cover all the local embeddings:

$$Co\{\mathcal{E}_i, \forall i = 1, \ldots, \ell\} \subseteq \mathcal{C} \oplus c^*\mathcal{C}.$$

The ideal simplex embedding $\mathcal{C} \subset \mathcal{E}$ implies that $\{\mathcal{C} \oplus c^*\mathcal{C}\} \subset \{\mathcal{E} \oplus c^*\mathcal{E}\}$ and by consequence:

$$Vol\{\mathcal{E} \oplus c^*\mathcal{E}\} \geq Vol\{\mathcal{E}^a\}. \tag{4.14}$$

By using the Brunn-Minkowski theorem [166], with equality instead of inequality due to the homotetic transformation, and the fact that $\mathcal{E}^a$ is minimum-volume for the approximated containment problem, one can write Eq. (4.14) with the power terms $1/n$ such that:

$$\begin{aligned}
&Vol\{\mathcal{E} \oplus c^*\mathcal{E}\}^{\frac{1}{n}} \geq Vol\{\mathcal{E}^a\}^{\frac{1}{n}}, \\
&Vol\{\mathcal{E}\}^{\frac{1}{n}} + Vol\{c^*\mathcal{E}\}^{\frac{1}{n}} \geq Vol\{\mathcal{E}^a\}^{\frac{1}{n}}, \\
&Vol\{c^*\mathcal{E}\} \geq \left[Vol\{\mathcal{E}^a\}^{\frac{1}{n}} - Vol\{\mathcal{E}\}^{\frac{1}{n}}\right]^n.
\end{aligned}$$

By the fact that $Vol\{c^*\mathcal{E}^a\} \geq Vol\{c^*\mathcal{E}\}$, the proof is complete. ■

The previous result provides an effective way of measuring the degree of approximation between two volume measurements of Algorithm 1. It should be however noticed that because of the nonlinear characteristics of the optimization problem (4.10) (and specifically the cost as a function of the optimization arguments), c^* can increase between two iterations, but this phenomenon will not affect the global convergence of the algorithm as long as the proposition 4.3 assures the decreasing behavior of the volume.

The next example shows the principle of the algorithm step-by-step.

Example 4.1 (Model of a DC motor with time-varying delays - a less conservative approach)**.** *Consider the same example as the one used in the Example 3.8 of the Chapter 3. In this example, the volume of the simplex which contains the uncertainty function* $\Delta(\epsilon)$ *is* $Vol = 0.9720$*. The objective here is to reduce the conservatism of the polytopic model, by the reduction of its volume. The sampling period is* $T_e = 0.003s$*, the maximal delay bounds are* $0.009s \leq \tau \leq 0.012s$*, i.e.* $\epsilon = 0.003s$*.*

The first step is to re-sample the interval of variation for the argument of the function $\Delta(\epsilon_k)$ *in three parts* $\mathbb{R}_{(0,\, \bar{\epsilon}/3]}$, $\mathbb{R}_{[\bar{\epsilon}/3,\, 2\bar{\epsilon}/3]}$ *and* $\mathbb{R}_{[2\bar{\epsilon}/3,\, \bar{\epsilon}]}$*, and build a local embedding (based on* (3.17) *and* (3.18)*) for each interval, as it is shown in Fig 4.1:*

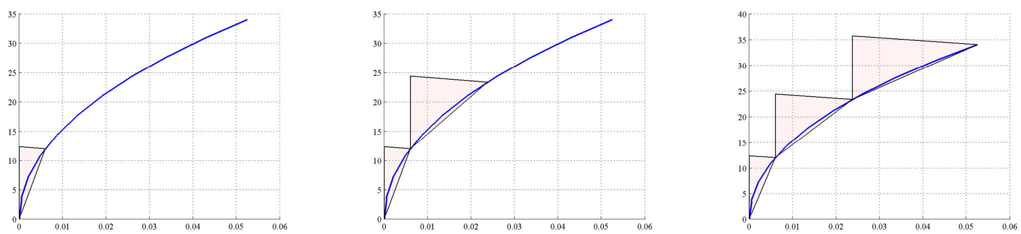


FIGURE 4.1: Construction of the local embeddings for $\Delta(\epsilon)$.

The next step is to optimize the volume of a simplex by using the vertices of the local embeddings as constraints. The result, shown in the following (Fig. 4.2), has the volume $Vol = 0.8796$.

In the iterative refinement stage one need to find the vertices which "touch" the simplex (i.e. saturate the constraints of the optimization routine), as shown in Figure 4.3(a), the red points *in the vertices. Once a local embedding contains one of these vertices is identified, then its volume is found. Finally, the local embedding who has the biggest volume among all (Fig. 4.3(b)) belongs to the interval to be re-sampled (Fig. 4.3(c)).*

The result, shown in the following (Fig. 4.4(a)), has the volume $Vol = 0.8649$*. As a result, the incremental improvement in the tightness of the embedding between two algorithmic iterations is* $\delta = 0.0147$*. The algorithm stops when the precision is smaller than a given precision. Figure 4.4(b) shows the final result with a precision* $\delta = 10^{-7}$.

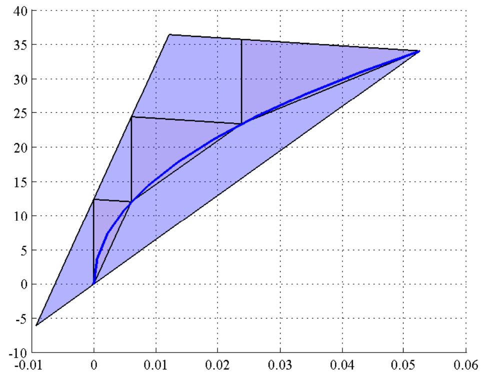
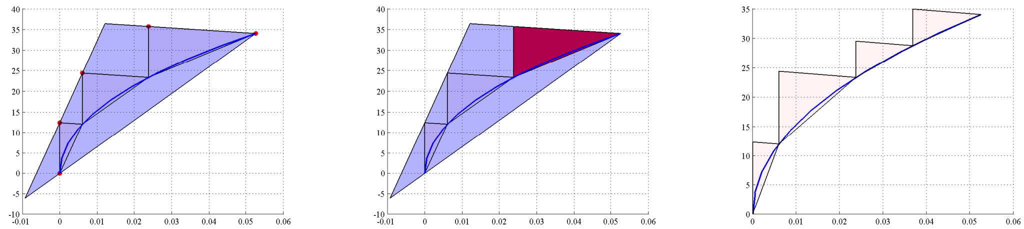


FIGURE 4.2: Optimization of the simplex for the example 4.1.

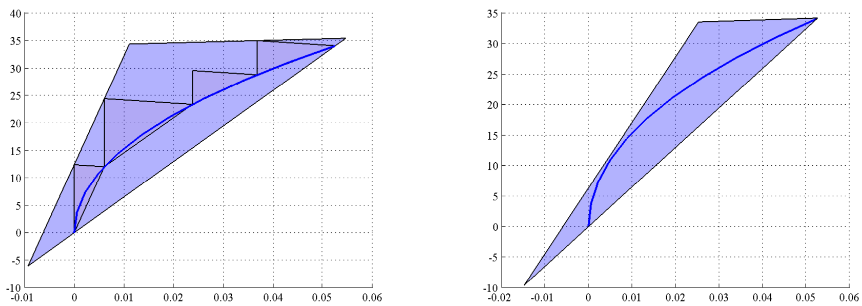


(a) (b) (c)

FIGURE 4.3: Construction of the local embeddings for $\Delta(\epsilon)$.

The resulting volume is $Vol = 0.5756$ *and the polytope is described by:*

$$\Delta \in Co \left\{ \begin{bmatrix} -0.0148 \\ -9.5824 \end{bmatrix}, \begin{bmatrix} 0.0526 \\ 34.0630 \end{bmatrix}, \begin{bmatrix} 0.0252 \\ 33.4281 \end{bmatrix} \right\}.$$



(a) (b)

FIGURE 4.4: Final optimized simplex embedding.

4.2 Comparative study with existing techniques

In this section is presented a comparative study between the techniques developed above (Sections 3.2.1, 3.2.2, 3.2.3 and 4.1) and some existing polytopic techniques, namely *Cayley-Hamilton Decomposition* (Section 4.2.1) developed by [57] and *Taylor Expansion Approximation* (Section 4.2.2), according to [77].

4.2.1 Cayley-Hamilton Decomposition

This method is based on [57], where the uncertainty is covered by a polytope generated by the Cayley-Hamilton Theorem (see 2.2 on Chapter 2).

Theorem 2.2, from the above mentioned reference, is recalled:

Lemma 4.6. *Let:*

$$g_j(\tau_k) = \int_0^{\epsilon} f_j(T_s - \theta) d\theta,$$

for some $c_{i,j} \in \mathbb{R}$ *and* $f_j(T_s - \theta)$ *expressed by:*

$$f_j(T_s - \theta) = \sum_{i=0}^{\infty} \alpha_{i,j} (T_s - \theta)^i,$$

where $a_{i,j} = \frac{c_{i,j}}{i!}$. *Then:*

$$\Delta_k = (g_0(\epsilon) I_n + g_1(\epsilon) A_c + \cdots + g_{n-1}(\epsilon) A_c^{n-1}) B_c.$$

The expression of the vertices, in $\mathcal{V}$-representation is obtained by the following theorem:

Theorem 4.7. *For any* $\tau_k \in [0, \epsilon_{max}]$, Δ_k *satisfies:*

$$\Delta_k \in Co \left\{ n\Delta_0, \ldots, n\Delta_{2n-1} \right\},$$

where:

$$\Delta_j = g_{j,l} A_c^j; \; \Delta_{j+n} = g_{j,u} A_c^j.$$

□

Sketch of Proof. By taking the integral expression of $\Delta(\epsilon)$ and by expressing the characteristics polynomial of A_c, one can see that for any $\tau_k \in [0, \bar{\tau}]$ and $g_j(\tau_k)$ there exists a $\nu_j \in \mathbb{R}_{[0,1]}$ and $\mu_j = 1 - \nu_j$ such that Δ_k is convex covered:

$$\begin{aligned}\Delta_k &= (g_0(\epsilon)I_n + g_1(\epsilon)A_c + \cdots + g_{n-1}(\epsilon)A_c^{n-1})B_c \\ &= (\delta_0 n g_{1,l} I + \delta_n n g_{1,u} I + \cdots + \delta_{2n-1} n g_{n-1,u} A_c^{n-1})B_c.\end{aligned}$$

With $\delta_i = \frac{\nu_j}{n}$, $\delta_{i+n} = \frac{\mu_j}{n}$ and $\sum_{i=0}^{2n-1} \delta_i = 1$ the proof is complete.

The proof details can be found in [57]. ■

4.2.2 Taylor Expansion

This method is based on [77], where the uncertainty is covered by a polytope generated by a Taylor series expansion of (3.7).

Proposition 4.8. *The elements of* (3.7) *can be expressed as:*

$$\Delta_k = \left(- \sum_{q=1}^{\infty} \frac{(-\epsilon)^i}{i!} A_c^{i-1} e^{A_c T_e} \right) B_c.$$

The expression of the vertices, in $\mathcal{V}$-representation is obtained by the following theorem:

Theorem 4.9. *For any* $\tau_k \in [0, \epsilon_{max}]$, Δ_k *satisfies:*

$$\Delta_k \in Co\left\{\Delta_0, \ldots, \Delta_{h-n}\right\},$$

where:

$$\Delta_k = \left(- \sum_{q=1}^{h} \frac{(-\epsilon)^i}{i!} A_c^{i-1} e^{A_c T_e} \right) B_c,$$

where the Taylor expansion is truncated on the h *first elements.*

Sketch of Proof. By taking the integral expression of $\Delta(\epsilon)$ and making its Taylor series expansion, one obtains:

$$\Delta(\epsilon) = - \sum_{q=1}^{\infty} \frac{(-x)^q}{q!} A_c^{q-1} e^{A_c T_e},$$

i.e., $x = \epsilon$. By truncating the series on the h first elements, the proof is complete.

The proof details can be found in [77]. ∎

4.2.3 Illustrative examples

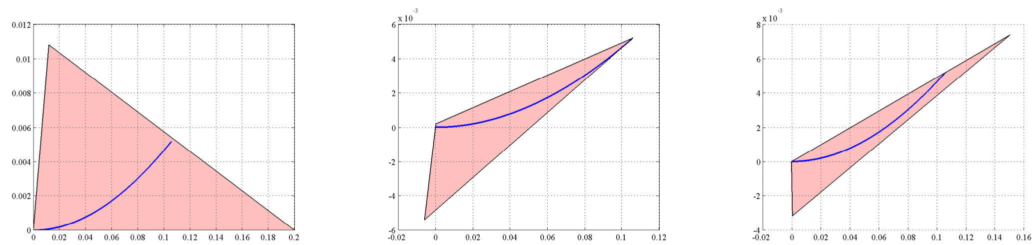
Example 4.2 (Non-defective system matrix with real eigenvalues, delay variation $\epsilon \in \mathbb{R}_{(0,0.1]}$). *Consider the same example as the used in [145], an unstable system with delay:*

$$\dot{x}(t) = \begin{bmatrix} 1.1 & -0.1 \\ 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t-h)$$

with sampling period is $T_e = 0.1$. *To perform the algorithm presented in Section 4.1, the precision is* $\delta = 10^{-7}$.

Figure 4.5 shows the containment of $\Delta(\epsilon)$ *by a polytope for the case of non-defective system matrix with real eigenvalues using the three methods presented in this section: Jordan decomposition based polytope + optimization (Section 3.2 and 4.1), Fig. 4.5(c), Cayley-Hamilton decomposition based polytope (Section 4.2.1), Fig. 4.5(a), and Taylor expansion based polytope (Section 4.2.2), Fig. 4.5(b). The delay variation* ϵ *is smaller or equal a sampling period.*

The volumes of the polytopes are employed as comparison between the methods. For Fig. 4.5(a) the computed volume is 0.0011. *In Fig. 4.5(b) the measured volume is* 0.0000279 *and for 4.5(c) the volume is* 0.0000238.



(a) Cayley-Hamilton (b) Taylor (c) Jordan + Optimization

FIGURE 4.5: Containment of $\Delta(\epsilon)$ by a polyhedron for non-defective system matrix with real eigenvalues: intersample delay variation.

Example 4.3 (Non-defective system matrix with complex-conjugated eigenvalues, delay variation $\epsilon \in \mathbb{R}_{(0,0.1]}$). *Consider an unstable system with delay:*

$$\dot{x}(t) = \begin{bmatrix} 0.2 & -0.26 \\ 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t-h),$$

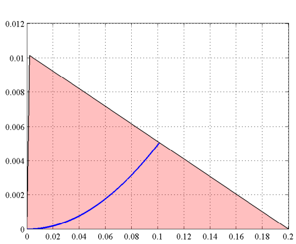
whose eigenvalues $\lambda(A_c)$ *are:*

$$\lambda(A_c) = \begin{bmatrix} 0.1 + 0.5i \\ 0.1 - 0.5i \end{bmatrix},$$

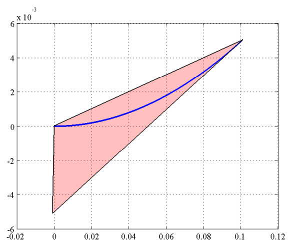
with a sampling period $T_e = 0.1$*. To perform the algorithm presented in Section 4.1, the precision is* $\delta = 10^{-7}$*.*

Figure 4.6 shows the containment of $\Delta(\epsilon)$ *by a polytope for the case of defective system matrix with real eigenvalues using the three methods presented in this section: Jordan decomposition based polytope + optimization (Section 3.2 and 4.1), Fig. 4.6(c), Cayley-Hamilton decomposition based polytope (Section 4.2.1), Fig. 4.6(a), and Taylor expansion based polytope (Section 4.2.2), Fig. 4.6(b). The delay variation* ϵ *is smaller or equal a sampling period.*

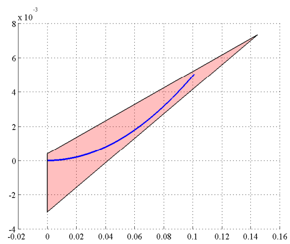
For Fig. 4.6(a) the volume is 0.0010*. In Fig. 4.6(b) the computed volume is* 0.0000255 *and for 4.6(c) the volume is* 0.0000243*.*



(a) Cayley-Hamilton



(b) Taylor



(c) Jordan + Optimization

Figure 4.6: Containment of $\Delta(\epsilon)$ by a polyhedron for non-defective system matrix with complex-conjugated eigenvalues: intersample delay variation.

Example 4.4 (Defective system matrix with real eigenvalues, delay variation $\epsilon \in \mathbb{R}_{(0,0.1]}$). *Consider the example, an unstable system with delay:*

$$\dot{x}(t) = \begin{bmatrix} 4 & -4 \\ 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t-h)$$

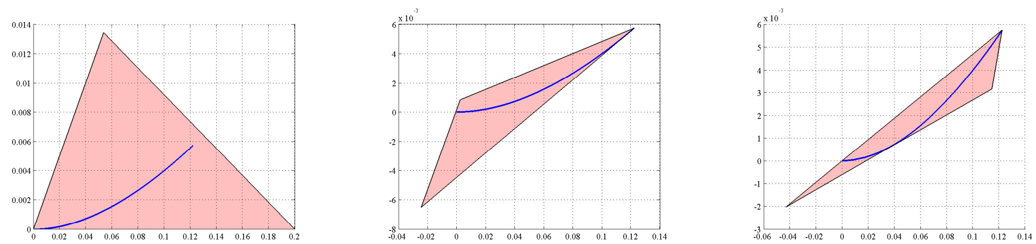
with sampling period is $T_e = 0.1$*. By the Jordan decomposition* $A_c = V\Sigma V^{-1}$*:*

$$V = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}; \quad \Sigma = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$$

one can remark that there is Jordan block formed by the eigenvalue 2*. To perform the algorithm presented in Section 4.1, the precision is* $\delta = 10^{-7}$*.*

Figure 4.7 shows the containment of $\Delta(\epsilon)$ *by a polytope for the case of defective system matrix with real eigenvalues using the three methods presented in this section: Jordan decomposition based polytope (Section 3.2), Fig. 4.7(c), Cayley-Hamilton decomposition based polytope (Section 4.2.1), Fig. 4.7(a), and Taylor expansion based polytope (Section 4.2.2), Fig. 4.7(b). The delay variation* ϵ *is smaller or equal a sampling period.*

To compare the performance between the methods, the volumes measurements are employed. For Fig. 4.7(a) the volume is 0.0013*. In Fig. 4.7(b) the measured volume is* 0.0000372 *and for 4.7(c) the volume is* 0.0000183*.*



(a) Cayley-Hamilton (b) Taylor (c) Jordan + Optimization

FIGURE 4.7: Containment of $\Delta(\epsilon)$ by a polyhedron for defective system matrix with real eigenvalues: intersample delay variation.

The Jordan decomposition offers a simplistic way to obtain rough embeddings for the parameters variations originated by time varying delays. It is comparable with the alternative methods in the literature as those based on Cayley-Hamilton and Taylor series approximations. Nevertheless, it was shown that the embedding can be tighten using an optimization based algorithm and re-sampling of the original interval of variation.

4.2.4 An assessment of overapproximation methods

The *overapproximation* methods used to cover (3.7), as a result of the sampling process, can be viewed into three perspectives, the "complexity" of the obtained polytope, in function of the number of vertices, the "quality" of the approximation, where the polytope describes (3.7) in a less conservative perspective and "calculation complexity", where the method used to obtain the vertices is in evidence.

When we want to talk about "complexity", there are mainly three categories of polytopes and methods presented in the literature, for $\mathbb{R}^n$:

- Simplex ($n + 1$ extreme points).
- Hypercubes (2^n extreme points).
- Others (number of extreme points does not depend on n of are widely larger than n).

The two first methods are presented in this thesis, in Chapter 3, where the calculation are based on the Jordan Canonical form. These approximations are the simplest ones, and therefore, the more conservative ones. It is obvious by the compromise: *the simpler* is the polytope, *the worse* the approximation of the function will be.

The "quality" means "how closed" the polytope is of $\Delta(\epsilon_k)$, how its description corresponds to $\Delta(\epsilon_k)$. In [74], the authors tried to evaluate the methods presented in the literature according to the "volume" of the polytopes with respect to $\Delta(\epsilon_k)$. [74] is "inconclusive" in the sense that *there is not a better overapproximation method*; which of them has its particularities, advantages and disadvantages.

In the present manuscript we do not intend to compare methods, but to show the existence of different frameworks. Moreover, in this Chapter 4 we presented an optimization procedure that gives the "best" overapproximation (the one with the smallest volume). This method can be applied to any technique described in the literature: the polytopes based on the sub-intervals can be obtained by Jordan Decomposition [114, 115, 145], Taylor Expansion [77] and Cayley-Hamilton Decomposition [57], and the extreme points can be used as constraint in the optimization. This composes a *complete* framework of the over approximation methods for time-delay systems.

The "calculation complexity" means "how hard" the calculations involved are, in order to obtain the polytopes. Although this aspect is not deeply analyzed in the literature, we consider the three main methods (Jordan, Taylor and Cayley-Hamilton) of low complexity.

Chapter 5

Multisampled delay variation

Chapter 3 deals with the modeling of time-delay systems as uncertain systems with polytopic uncertainty for an inter-sampled delay variation, i.e. the delay variation bounds remains inside a sampling period. If these variation bounds are larger than a sampling period, an extended model can be obtained for each sampling period variation and for each vertex of the polytope (given by Chapter 3 and 4). In this case the system can be considered a switched linear system.

5.1 Multisampled delay variation

In the case where the continuous-time delay variation bounds are larger than a sampling period, the discrete-time delay variation $d \in \mathbb{Z}_{[d_{min}, d_{max}]}$ has to be considered:

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_{k-d_1} - \Delta(\epsilon_k)(u_{k-d_1} - u_{k-d_2}) \\ \Delta(\epsilon_k) &\in Co\{\Delta_0, \Delta_1, \ldots, \Delta_n\} \\ \forall d_1, d_2 &\in \{d_{min}, d_{min}+1, d_{min}+2, \ldots, d_{max}-1, d_{max}\} \text{ such that } \quad d_1 > d_2, \end{aligned} \tag{5.1}$$

where d_1 and d_2 are time varying and n is the Euclidean space dimension $\mathbb{R}^n$. The intersample variations were considered to be represented by the function $\Delta(\epsilon_k)$, done by (3.7):

$$\Delta(\epsilon_k) = \int_{-\epsilon_k}^{0} e^{-A_c\tau} B_c d\tau,$$

and practically covered by a polytopic uncertainty with the extreme realizations (see the polytopic constructions in Chapter 3). The delay variation between sampling periods is described by $(d_i+1)T_e = d_i T_e + \epsilon_k$, where $i \in \mathbb{Z}_{[min,max]}$.

By preserving the same augmented state vector ξ_k as in Eq. (3.11), the problem is in fact a structural uncertainty introduced by the variation of the discrete-time delay $d \in \mathbb{Z}_{[d_{min},d_{max}]}$ on one side, and the parametric uncertainty induced by $\epsilon_k \in \mathbb{R}_{(0,\bar{\epsilon}]}$, where $\bar{\epsilon} = T_e$, appearing in the linear dynamics. In order to deal with all these characteristics, a switching linear model affected by polytopic uncertainty is proposed. For a given vector $\delta \in \{0,1\}^{d_{max}-d_{min}+1}$, with components $\delta_i, i \in \mathbb{Z}_{[1,n]}$ the following matrices are introduced:

$$F(\delta_1,\delta_2,\Delta(\epsilon_k)) = \left[\begin{array}{c|c|c|c|c|c|c|c} A & \Psi_{d_{max}}(\delta_1,\delta_2,\epsilon_k) & \dots & \cdots & \Psi_{d_i}(\delta_1,\delta_2,\epsilon_k) & \cdots & \cdots & \Psi_1(\delta_1,\delta_2,\epsilon_k) \\ 0 & 0 & I_m & \cdots & 0 & \cdots & \cdots & \vdots \\ \vdots & \ddots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & I_m & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 & I_m & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & I_m \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \end{array}\right]; \tag{5.2}$$

$$G(\delta_1,\delta_2,\Delta(\epsilon_k)) = \begin{bmatrix} \Psi_0(\delta_1,\delta_2,\epsilon_k) \\ 0 \\ \vdots \\ \vdots \\ 0 \\ I_m \end{bmatrix}, \tag{5.3}$$

where:

$$\Psi_i(\delta_1,\delta_2,\epsilon_k) = \begin{cases} \mathbf{0}_{n\times m} & \text{if} \quad i \in \mathbb{Z}_{<d_{min}}, \\ (B-\Delta(\epsilon_k))\,\delta_1(i) + \Delta(\epsilon_k)\delta_2(i) & \text{if} \quad i \in \mathbb{Z}_{[d_{min},d_{max}]}, \end{cases} \tag{5.4}$$

and the notation:

$$\delta_\ell = \begin{bmatrix} \delta_\ell(d_{min}) & \dots & \delta_\ell(i) & \dots & \delta_\ell(d_{max}) \end{bmatrix}, \tag{5.5}$$

for $\ell = 1, 2$ and $i \in \mathbb{Z}_{[d_{min}, d_{max}]}$ has been used.

We note that the particularity of this case is the block switch at the extended matrices (5.2)-(5.3). This can be covered by a polytopic overapproximation plus the introduction of the binary variables (5.4). We complete the study of this case with a formal result for the polytopic model construction in this case.

From the switching point of view there are several restrictions resumed by the linear constraints on the binary variables:

$$\begin{cases} \sum_{i=d_{min}}^{d_{max}} \delta_\ell(i) = 1, \quad \text{for} \quad \ell = 1, 2 \\ \sum_{i=d_{min}}^{d_{max}} i\delta_\ell(i) > \sum_{j=d_{min}}^{d_{max}} j\delta_\ell(j). \end{cases} \tag{5.6}$$

The next theorem states the existence of a polytopic over-approximation for discretized linear systems affected by a variable input delay larger than the sampling period.

Theorem 5.1. *For any linear continuous time dynamics* (3.1) *with a piecewise constant control signal and affected by variable input delay* $\tau \in \mathbb{R}_{[\tau_{min}, \tau_{max}]}$*, there exist a linear discrete time model with polytopic uncertainty:*

$$\begin{aligned} \xi_{k+1} &\in F\xi_k + Gu_k; \\ (F, G) &= Co\,\{(F(\delta_1, \delta_2, \Delta_i), G(\delta_1, \delta_2, \Delta_i))\,, \\ &\qquad (\delta_1, \delta_2) \in \{0,1\}^{d_{max}-d_{min}+1} \times \{0,1\}^{d_{max}-d_{min}+1}, i \in \mathbb{Z}_{[0,n]}\}\,, \end{aligned} \tag{5.7}$$

where Δ_i *are the vertices of the polytope that covers* $\Delta(\epsilon_k)$ *(see Chapters 3 and 4), such that for a given state* $x(t) = x(kT_e)$ *and the piecewise constant control function* $u(t); t \in \mathbb{R}_{(t-\tau_{min}, 0]}$*:*

$$x(t + T_e) \in Proj_x\,\{F\xi_k + Gu_k\}\,. \tag{5.8}$$

Proof. By using the piecewise constant property of the control signal and the given continuous-time initial conditions, an extended state vector ξ_k can be constructed as in (3.11). This represents a set of well defined initial conditions for the discrete-time system (5.7). To prove that the continuous time trajectories sampled at $t = t - kT_e$, in the presence of varying input delay, are covered by the polyhedral approximation of the discrete-time dynamics, one has to think about the fact that the pair (F, G) contains a discrete value "d" such that the continuous-time variation will be placed between two discrete-time instants $(k - d_1)T_e\tau \leq kT_e - \tau \leq (k - d_1 + 1)T_e$. The intersample variation is covered by $\Delta(\epsilon_k) \in Co\{\Delta_1, \ldots, \Delta_n\}$ and corrected in discrete model by the term $\Delta(u_{k-d_1} - u_{k-d_2})$ and the polytopic methods presented in Chapters 3 and 4.

Formally, a given pair of discrete delays (d_1, d_2) corresponds to one and only one pair of vectors $(\delta_1, \delta_2) \in \{0, 1\}^{d_{max}-d_{min}+1} \times \{0, 1\}^{d_{max}-d_{min}+1}$, and thus

$$(F, G) \in Co\left\{(F_i(\delta), G_i(\delta)), i \in \mathbb{Z}_{[0,n]}\right\}.$$

Further, by allowing the discrete-time delays to have values in the interval $d_{min} \leq d_1$ and $d_2 \leq d_{max}$, all the possible realizations are covered by the polytopic model in (5.7). ∎

Remark 5.2. It can be observed from (5.2) that the delay variation will affect the block G only if $d_{min} \leq T_e$. For the case when $d_{min} > T_e$, Eq. (5.7) can be simplified and described in a compact form by exploiting the fact that $G_0(\delta)$ is constant (as long as $\Delta_0 = \mathbf{0}_{m\times n}$, for the polytopic covering described in Chapter 3), in terms of the polytope vertices (variable i, corresponding to $n + 1$ vertices) and switching modes (variable j, corresponding to $d_{max} - d_{min}$ switching modes):

$$\begin{aligned} \xi_{k+1} &\in F\xi_k + G_0 u_k; \\ F &= Co\left\{F_{i,j}, i \in \mathbb{Z}_{[0,n]}, j \in \mathbb{Z}_{[d_{min},d_{max}]}\right\}. \end{aligned} \tag{5.9}$$

Remark 5.3. The number of vertices (or extreme realizations) is $s = (d_{max} - d_{min} + 1)(d_{max} - d_{min})(n + 1)/2$, due to the fact that there are $2(d_{max} - d_{min} + 1)$ binary variables which are combined under the restrictions (5.6). For the polytopic covering of $\Delta(\epsilon_k)$ described in Chapter 3, indeed, $n - 1$ of the binary variables combinations, corresponding to $\Delta = \mathbf{0}_{m\times n}$, are in fact spanned by the neighbor combinations of logic variables such that the overall complexity of the polytopic model is given by $s = (d_{max} - d_{min} + 1)(d_{max} - d_{min})n/2 + 1$.

Remark 5.4. The linear structure of the continuous-time model and the piecewise constant nature of the control signal make the delay variation be modeled by a vector $\|\delta\| = 1$

and thus be described by (5.7). Unfortunately, the convex description in (5.7) will relax the discrete-time nature of the vector $\delta \in \{0,1\}^{d_{max}-d_{min}+1}$ to $\delta \in \mathbb{R}^{d_{max}-d_{min}+1}$. The result is the conservatism in the control synthesis stage, where the polytopic uncertainty is convenient for LMI-based approaches. It is conservative by the fact that other possible dynamical models with distributed delays are introduced inside a sampling period, between the subintervals $\mathbb{R}[\tau_{min}, \tau_{max}]$, corresponding to a decomposition of $\delta_{d_{min}} + \delta_{d_{min}+1} + \cdots + \delta_{d_{max}} = 1$, $\delta_i \in \mathbb{Z}_{[0,1)}$ and $i \in \mathbb{Z}_{[d_{min}, d_{max}]}$.

Using the same extended state-space representation, based on the equation (3.4), one can obtain, by using (A, B) and $\bar{d} = \left\lfloor \frac{\bar{\tau}}{T_e} \right\rfloor$, the model:

$$\xi_{k+1} = \bar{F}\xi_k + \bar{G}u_k, \tag{5.10}$$

with $\bar{F}, \bar{G}$ given by the equations:

$$\bar{F} = \left[\begin{array}{c|c|c|c}
A & \overbrace{\begin{matrix} 0 & 0 & \dots & 0 \end{matrix}}^{d_{max}-\bar{d}} & B & \overbrace{\begin{matrix} 0 & \dots & \dots & 0 \end{matrix}}^{\bar{d}-1} \\ \hline
d_{max}-\bar{d}\left\{\begin{matrix} 0 \\ \vdots \\ \vdots \\ 0 \end{matrix}\right. & \begin{matrix} 0 & I_m & \ddots & \\ & \ddots & \ddots & 0 \\ & & \ddots & I_m \\ & & & 0 \end{matrix} & \begin{matrix} 0 \\ \ddots \\ 0 \\ I_m \end{matrix} & \vdots \\ \hline
\vdots & \ddots & 0 & \begin{matrix} I_m & 0 & \ddots & \ddots \end{matrix} \\ \hline
\bar{d}-1\left\{\begin{matrix} \vdots \\ \vdots \\ \vdots \\ 0 \end{matrix}\right. & \dots & \dots & \begin{matrix} 0 & I_m & \ddots & \\ & \ddots & \ddots & 0 \\ & & \ddots & I_m \\ & & & 0 \end{matrix}
\end{array}\right]; \tag{5.11}$$

$$\bar{G} = \left[\begin{array}{c|ccccc|c|cccc} 0 & 0 & \dots & \dots & \dots & & 0 & \dots & \dots & 0 & I_m \end{array}\right]^T. \tag{5.12}$$

Consider the polytopic systems (3.10) and the extended model (3.12), affected by polytopic uncertainty (5.7); these will be used for the construction of a robust stabilizing control law, based on classical LMI techniques.

The next example shows the construction of the same DC-motor as the one used in the Chapter 3, Example 3.8, but for a multisampled delay variation.

Example 5.1 (Model of a DC motor with time-varying delays - multisampled delay variation)**.** *Consider the same example as the one used in the Example 3.8 of the Chapter 3, where sampling period is $T_e = 0.003s$ and the maximal delay bounds are $\tau \in \mathbb{R}_{[0s,0.06s]}$. So $d \in \mathbb{Z}_{[d_{min},d_{max}]}$ where $d_{min} = 0$ and $d_{max} = 2$.*

By using the techniques developed in Chapter 3 the polytope and its vertices are:

$$\Delta \in Co\left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 37.0588 \end{bmatrix}, \begin{bmatrix} 0.0525 \\ 33.9731 \end{bmatrix} \right\}.$$

Thus by using the Eqs. (5.2) and (5.3) for all the vertices of Δ and for all the delay variation, one can build the model:

$$F_1 = \begin{bmatrix} 1 & 0.0028 & 0.0525 & 0 \\ 0 & 0.8382 & 33.9731 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}; \quad G_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}; \quad F_2 = \begin{bmatrix} 1 & 0.0028 & 0.0525 & 0 \\ 0 & 0.8382 & -3.0857 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}; \quad G_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix};$$

$$F_3 = \begin{bmatrix} 1 & 0.0028 & 0 & 0.0525 \\ 0 & 0.8382 & 0 & 33.9731 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}; \quad G_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}; \quad F_4 = \begin{bmatrix} 1 & 0.0028 & 0.0525 & 0 \\ 0 & 0.8382 & 33.9731 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}; \quad G_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix};$$

$$F_5 = \begin{bmatrix} 1 & 0.0028 & 0 & 0.0525 \\ 0 & 0.8382 & 0 & 33.9731 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}; \quad G_5 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}; \quad F_6 = \begin{bmatrix} 1 & 0.0028 & 0.0525 & 0 \\ 0 & 0.8382 & -3.0857 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}; \quad G_6 = \begin{bmatrix} 0 \\ 37.0588 \\ 0 \\ 1 \end{bmatrix};$$

$$F_7 = \begin{bmatrix} 1 & 0.0028 & 0 & 0.0525 \\ 0 & 0.8382 & 0 & 33.9731 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}; \quad G_7 = \begin{bmatrix} 0 \\ 37.0588 \\ 0 \\ 1 \end{bmatrix}; \quad F_8 = \begin{bmatrix} 1 & 0.0028 & 0 & 0 \\ 0 & 0.8382 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}; \quad G_8 = \begin{bmatrix} 0.0525 \\ 37.0588 \\ 0 \\ 1 \end{bmatrix};$$

$$F_9 = \begin{bmatrix} 1 & 0.0028 & 0 & 0 \\ 0 & 0.8382 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}; \quad G_9 = \begin{bmatrix} 0.0525 \\ 37.0588 \\ 0 \\ 1 \end{bmatrix}.$$

If a less conservative approach is desired, as presented in Chapter 4, the polytope and its vertices are:

$$\Delta \in Co\left\{ \begin{bmatrix} -0.0148 \\ -9.5824 \end{bmatrix}, \begin{bmatrix} 0.0526 \\ 34.0630 \end{bmatrix}, \begin{bmatrix} 0.0252 \\ 33.4281 \end{bmatrix} \right\}.$$

Thus by using the Eqs. (5.2) *and* (5.3) *for all the vertices of* Δ *and for all the delay variation, one can build the model:*

$$F_1 = \begin{bmatrix} 1 & 0.0028 & 0.0673 & -0.0148 \\ 0 & 0.8382 & 43.5555 & -9.5824 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}; \quad G_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}; \quad F_2 = \begin{bmatrix} 1 & 0.0028 & -0.0001 & 0.0526 \\ 0 & 0.8382 & -0.0899 & 34.0630 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}; \quad G_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix};$$

$$F_3 = \begin{bmatrix} 1 & 0.0028 & 0.0272 & 0.0252 \\ 0 & 0.8382 & 0.5450 & 33.4281 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}; \quad G_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}; \quad F_4 = \begin{bmatrix} 1 & 0.0028 & 0.0673 & 0 \\ 0 & 0.8382 & 43.5555 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}; \quad G_4 = \begin{bmatrix} -0.0148 \\ -9.5824 \\ 0 \\ 1 \end{bmatrix};$$

$$F_5 = \begin{bmatrix} 1 & 0.0028 & 0 & 0.0673 \\ 0 & 0.8382 & 0 & 43.5555 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}; \quad G_5 = \begin{bmatrix} -0.0148 \\ -9.5824 \\ 0 \\ 1 \end{bmatrix}; \quad F_6 = \begin{bmatrix} 1 & 0.0028 & -0.0001 & 0 \\ 0 & 0.8382 & -0.0899 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}; \quad G_6 = \begin{bmatrix} 0.0526 \\ 34.0630 \\ 0 \\ 1 \end{bmatrix};$$

$$F_7 = \begin{bmatrix} 1 & 0.0028 & 0 & -0.0001 \\ 0 & 0.8382 & 0 & -0.0899 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}; \quad G_7 = \begin{bmatrix} 0.0526 \\ 34.0630 \\ 0 \\ 1 \end{bmatrix}; \quad F_8 = \begin{bmatrix} 1 & 0.0028 & 0.0272 & 0 \\ 0 & 0.8382 & 0.5450 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}; \quad G_8 = \begin{bmatrix} 0.0252 \\ 33.4281 \\ 0 \\ 1 \end{bmatrix};$$

$$F_9 = \begin{bmatrix} 1 & 0.0028 & 0 & 0.0272 \\ 0 & 0.8382 & 0 & 0.5450 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}; \quad G_9 = \begin{bmatrix} 0.0252 \\ 33.4281 \\ 0 \\ 1 \end{bmatrix}.$$

Part III

Analysis and Control Design

Chapter 6

Stability of linear discrete-time time-delay systems

In view of stability analysis and control design of the systems modeled in the previous chapter, several techniques can be proposed, based on the Lyapunov theory. The advantage of the extended state-space framework (Section 6.1) is that time-delay systems can be stabilized by using classical robust control methods for linear systems affected by polytopic uncertainty, such as those based on Linear Matrix Inequalities (LMIs) (see, for instance, [20]). Moreover, simple Lyapunov candidates can be considered, as long as quadratic forms are used in the LMI constructions. Within this method, one can find a state feedback gain that stabilizes the polytopic system, i.e., a static gain that stabilizes in closed loop the extreme realizations of the system dynamics and by consequence (and convexity) all the realizations inside the polytopic model.

The size of the extended system is function of the state dimension for the continuous-time and the size of discrete-time delay. As the simplicial polytopes have $n+1$ vertices, the problem will be to find a feedback gain that satisfies $n+1$ LMIs for the *intersampled delay variation case* or $(d_{max} - d_{min} + 1)(d_{max} - d_{min})(n+1)/2$ for the *multisampled delay variation case*. All these elements are important to establish the complexity of the feasibility domain of the LMI problem.

The main advantage of the LMI-based procedure is *tractability* [20], which means that they can be solved in polynomial time [55]. In other words, these problems can be considered of *low computational complexity* and from the practical point of view, the algorithms to solve them are considered effective, even for an on-line implementation [91]. More information about LMI optimization problems can be found in [20].

Another approach for the stabilization problem of discrete-time delay systems is based on Lyapunov-Krasovskii candidates (Section 6.2). Altough these techniques stabilize the system in its original state-space, the Lyapunov candidates remain highly complex.

6.1 Extended state-space framework

In this section the models developed in the Part II of the present thesis will be used as starting point for the design of an extended state-feedback (unconstrained) control law which stabilizes the system (3.1) in the presence of time-varying delay.

The idea is to design an extended state-feedback control law by using the discretized systems (3.10) and (5.2)-(5.3) as model:

$$u_k = K\xi_k, \tag{6.1}$$

where $K \in \mathbb{R}^{m\times(n+dm)}$ and the extended state vector is:

$$\xi_k^\top = \begin{bmatrix} x_k^\top & u_{k-d}^\top & \dots & u_{k-1}^\top \end{bmatrix}^\top$$

Next section states the LMI procedure to obtain (6.1).

6.1.1 Linear Matrix Inequalities for polytopic systems

It is well known that uncertain systems can be modeled as polytopic systems [20]. For polytopic systems, as developed in the Part II of the present thesis, the set $\Omega \in \mathbb{R}^{n+md}$ is the polytope:

$$\begin{aligned} \xi_{k+1} &= A_\Delta \xi_k + B_\Delta u_k; \\ \begin{bmatrix} A_\Delta & B_\Delta \end{bmatrix} &\in \Omega; \\ \Omega &= Co\left\{ \begin{bmatrix} A_{\Delta_0} & B_{\Delta_0} \end{bmatrix}, \begin{bmatrix} A_{\Delta_1} & B_{\Delta_1} \end{bmatrix}, \dots, \begin{bmatrix} A_{\Delta_n} & B_{\Delta_n} \end{bmatrix} \right\}. \end{aligned}$$

This means that for some $\alpha_i \geq 0$ for $i \in \mathbb{Z}_{[0,n]}$ satisfies:

$$\sum_{i=0}^{n} \alpha_i = 1,$$

we have

$$\begin{bmatrix} A_\Delta & B_\Delta \end{bmatrix} = \sum_{i=0}^{n} \alpha_i \begin{bmatrix} A_{\Delta_i} & B_{\Delta_i} \end{bmatrix}.$$

As defined in [20], a LMI is a matrix inequality of the form:

$$F(x) = F_0 + \sum_{j=0}^{\ell} x_j F_j > 0,$$

where x_j for all $j \in \mathbb{Z}_{[0,\ell]}$ are the variables and $F_j = F_j^\top \in \mathbb{R}^{(n+dm)\times(n+dm)}$ are given. The advantage is that for the polytopic case, multiple LMIs $F_i(x) > 0$, for $i \in \mathbb{Z}_{[0,n]}$, can be expressed as a single LMI:

$$\text{diag}(F_i(x)) > 0, \forall i \in \mathbb{Z}_{[0,n]}.$$

So actually, the set of LMIs $F_i(x) > 0$ means $\text{diag}(F_i(x)) > 0$, but no distinction is made between them in terms of notation. It is also well known that a LMI-based problem is the minimization of a linear function subject to LMI constraints:

$$\begin{gathered} \min c^\top x; \\ \text{subject to: } F(x) > 0, \end{gathered}$$

where F is symmetric and depends affinely on the optimization variable x.

Remark 6.1. In [20] there are several details about LMIs and the algorithms to solve them efficiently.

6.1.2 Classical robust stabilization approach

Theorem 6.2. *Consider the polytopic systems* (3.10) *and* (5.2)-(5.3)*. An extended state-feedback matrix* $K \in \mathbb{R}^{m\times(n+dm)}$ *that stabilizes the system is given by:*

$$K = YS^{-1},$$

where $Y \in \mathbb{R}^{m\times(n+md)}$, $S \in \mathbb{R}^{(n+md)\times(n+md)}$, $S = S^\top > I_{m+dm}$ *are obtained from the solution of the following minimization problem:*

$$\begin{aligned} &\min_{\gamma,S,Y} \gamma \\ &\textit{subject to:} \quad (6.3), \end{aligned} \tag{6.2}$$

with:

$$\begin{aligned} &\begin{bmatrix} S & SA_{\Delta_i}^\top + Y^\top B_\Delta^\top & SQ^{1/2} & Y^\top R^{1/2} \\ A_{\Delta_i}S + B_\Delta Y & S & 0 & 0 \\ Q^{1/2}S & 0 & \gamma I & 0 \\ R^{1/2}Y & 0 & 0 & \gamma I \end{bmatrix} \geq 0, \\ &S \geq I; \\ &\textit{for all} \quad i = 0, \ldots, n, \end{aligned} \tag{6.3}$$

where $Q \in \mathbb{R}^{(n+md)\times(n+md)}$, $R \in \mathbb{R}^{m\times m}$ *and* $\gamma \in \mathbb{R}$.

Proof. For the minimization of the Lyapunov function $V(\xi)$ for the polytopic system (3.10) and (5.2)-(5.3), the quadratic candidate is considered:

$$V(\xi) = \xi_k^\top P \xi_k; \; P = P^\top > 0, \tag{6.4}$$

The minimization problem is equivalent to:

$$\begin{aligned} &\min_{\gamma,P} \gamma \\ &\text{subject to:} \; \begin{cases} \xi_k^\top P \xi_k \leq \gamma; \\ \gamma > 0; \\ P = P^\top > 0. \end{cases} \end{aligned}$$

Defining $S = \gamma P^{-1} > 0$, with $S = S^\top$, and by using the Schur complement formula (as stated in Chapter 2) one has:

$$\begin{aligned} &\min_{\gamma,S} \gamma \\ &\text{subject to:} \; \begin{cases} \begin{bmatrix} I & \xi_k^\top \\ \xi_k & S \end{bmatrix} \geq 0; \\ \gamma > 0; \\ S = S^\top > 0. \end{cases} \end{aligned}$$

In order to assure asymptotic stability with a performance criterion, the following inequality has to be equally satisfied:

$$\Delta V = V(\xi_{k+1}) - V(\xi_k) \leq -\left[\xi_{k+i}^\top Q \xi_{k+i} + u_{k+i}^\top R u_{k+i}\right]. \tag{6.5}$$

By using (6.1) to rewrite (6.5), one has:

$$\xi_{k+i}^\top \left[(A_\Delta + B_\Delta K)^\top P (A_\Delta + B_\Delta K) - P + K^\top R K + Q\right] \xi_{k+i} \leq 0 \tag{6.6}$$

By noting $P = \gamma S^{-1}$ and by the fact that $S \geq I_{n+dm}$ and $K = YS^{-1}$, the Schur complement formula can be applied two times [20] and the following LMI is obtained:

$$\begin{bmatrix} S & SA_{\Delta_i}^\top + Y^\top B_\Delta^\top & SQ^{1/2} & Y^\top R^{1/2} \\ A_{\Delta_i} S + B_\Delta Y & S & 0 & 0 \\ Q^{1/2} S & 0 & \gamma I & 0 \\ R^{1/2} Y & 0 & 0 & \gamma \end{bmatrix} \geq 0. \tag{6.7}$$

This inequality is satisfied for all $A_{\Delta_i} = Co\{A_{\Delta_0}, \ldots, A_{\Delta_n}\}$ because (6.7) is affine in A_{Δ_i}. ■

Remark 6.3. As this is a classical result, the proof details were suppressed. Further details can be obtained in [20, 91].

Remark 6.4. The number of vertices of the polytopic covering A_{Δ_i}, as seen in the Part II of the thesis can be a function of the Euclidean space $\mathbb{R}^n$ (Chapter 3) but also of the multisampled delay variation behavior (Chapter 5), but has been omitted in the proof for the sake of simplicity of notation.

Example 6.1 (Position control of a DC motor with time-varying delays - intersample delay variation)**.** *Consider the model developed in the Examples 3.8 and 4.1 of the Chapter 3 and 4, with the less conservative model as proposed by Chapter 4. The sampling period is* $T_e = 0.003s$*, the delay is* $\tau \in \mathbb{R}_{[0.006s, 0.009s]}$*. The discrete-time model is given by:*

$$x_{k+1} = \begin{bmatrix} 1 & 0.0028 \\ 0 & 0.8382 \end{bmatrix} x_k + \begin{bmatrix} 0.0525 \\ 33.9731 \end{bmatrix} u_{k-d} - \Delta(u_{k-d} - u_{k-d+1}), \text{ with d} = 3.$$

The polytopic embedding is described by:

$$\Delta \in Co\left\{\begin{bmatrix} -0.0148 \\ -9.5824 \end{bmatrix}, \begin{bmatrix} 0.0526 \\ 34.0630 \end{bmatrix}, \begin{bmatrix} 0.0252 \\ 33.4281 \end{bmatrix}\right\}.$$

The polytopic model, by using the vertices expressed above, is given by:

$$A_{\Delta_0} = \begin{bmatrix} 1 & 0.0028 & 0.0673 & -0.0148 & 0 \\ 0 & 0.8382 & 43.5555 & -9.5824 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}; \quad A_{\Delta_1} = \begin{bmatrix} 1 & 0.0028 & -0.0001 & 0.0526 & 0 \\ 0 & 0.8382 & -0.0899 & 34.0630 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix};$$

$$A_{\Delta_2} = \begin{bmatrix} 1 & 0.0028 & 0.0272 & 0.0252 & 0 \\ 0 & 0.8382 & 0.5450 & 33.4281 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The extended-state feedback gain obtained by Theorem 6.2 with the matrices $Q = 10^{-5} I_{n+md}$ *and* $R = 1$ *is:*

$$K = \begin{bmatrix} -0.0677 & -0.0018 & -0.0430 & -0.0696 & 0.1424 \end{bmatrix}.$$

In order to study the dynamical behavior of the system, a simulation in Matlab Simulink is taken in place. The initial conditions are $x_0 = \begin{bmatrix} 1 & -1 \end{bmatrix}^\top$ *and* $x_i = 0$ *for* $i \in \mathbb{Z}_{[-d,-1]}$.

We obtain a state feedback control law but the extended state realization used in the design implies the storage of the delayed control actions in a buffer. This implies a certain technological complexity of the real-time control scheme which has to contain memory components. These can be important if the delay is important and it increases by the augmentation of the sampling period.

Actually, in the Example 6.1 the control law should be applied in the original system (which size is smaller). In order to “resize” the control vector to be applied, one can do

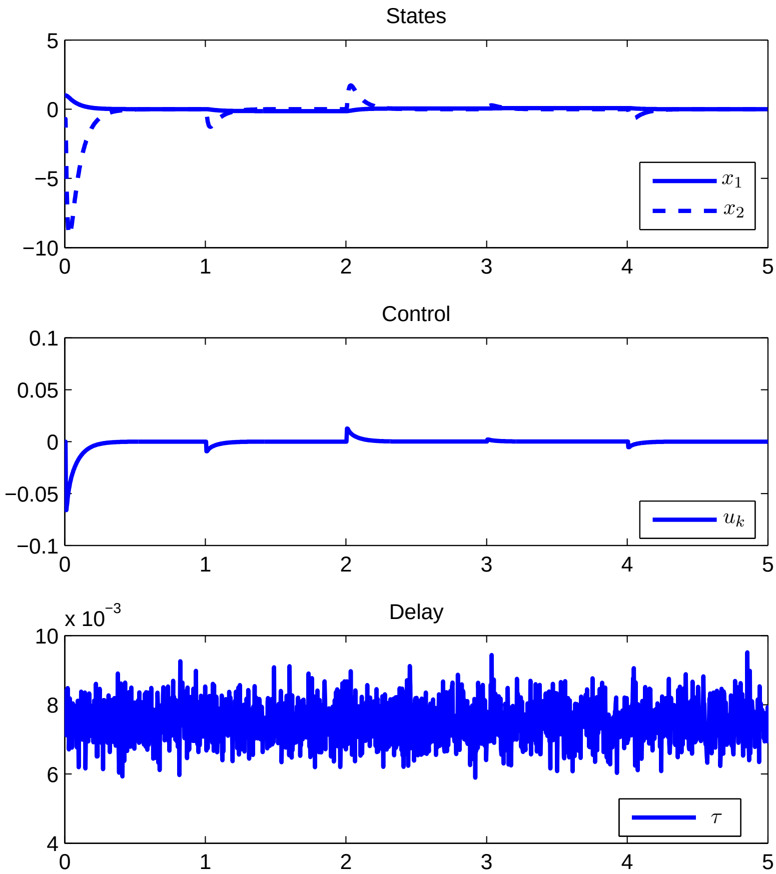


FIGURE 6.1: Temporal behavior of the system. The x axis is the time in seconds [x] of simulation. The y axis are [cm] for the states, [v] for the control and seconds [s].

the following :

$$u_k = K_x x_k - 0.043u_{k-3} - 0.0696u_{k-2} + 0.1424u_{k-1},$$

where K_x is the block part of the feedback gain K corresponding to the states.

Example 6.2 (Position control of a DC motor with time-varying delays - multisample delay variation)**.** *The system to be controlled here is the same as the one developed in Example 5.1 of the Chapter 5. Sampling period is* $T_e = 0.003s$ *and the maximal delay bounds are* $\tau \in \mathbb{R}_{[0s,0.06s]}$*. So* $d \in \mathbb{Z}_{[d_{min},d_{max}]}$ *where* $d_{min} = 0$ *and* $d_{max} = 2$*. The* *polytopic embedding is described by:*

$$\Delta \in Co\left\{\begin{bmatrix} -0.0148 \\ -9.5824 \end{bmatrix}, \begin{bmatrix} 0.0526 \\ 34.0630 \end{bmatrix}, \begin{bmatrix} 0.0252 \\ 33.4281 \end{bmatrix}\right\}.$$

The extreme realizations are the same as the ones obtained in the Example 5.1 in Chapter 5.

The extended-state feedback gain obtained by Theorem 6.2 with the matrices $Q = 10^{-5} I_{n+md}$ *and* $R = 1$ *is:*

$$K = \begin{bmatrix} -0.0159 & -0.0007 & 0.0822 & 0.2924 \end{bmatrix}.$$

In order to study the dynamical behavior of the system, a simulation in Matlab Simulink is taken in place. The initial conditions are $x_0 = \begin{bmatrix} 1 & -1 \end{bmatrix}^\top$ *and* $x_i = 0$ *for* $i \in \mathbb{Z}_{[-d,-1]}$.

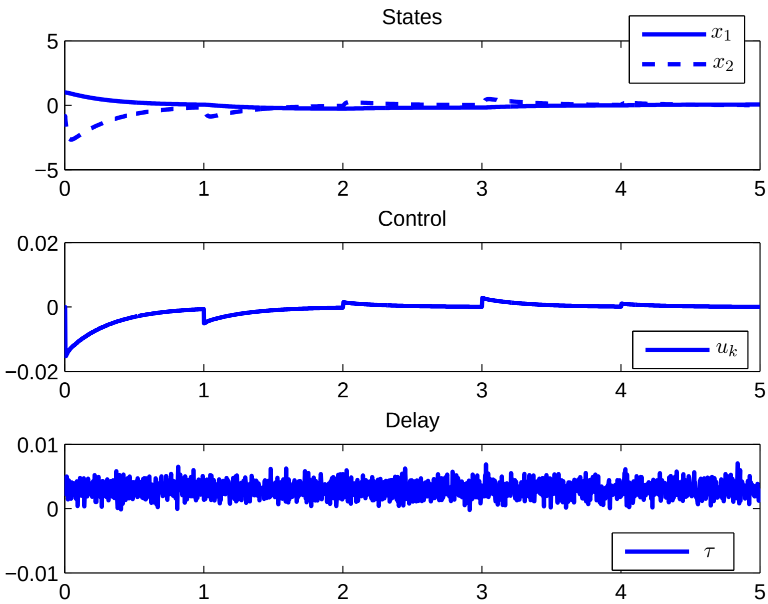


FIGURE 6.2: Temporal behavior of the system. The x axis is the time in seconds [x] of simulation. The y axis are [cm] for the states, [v] for the control and seconds [s].

6.2 Original state-space framework

Although the stabilization technique presented in Section 6.1 is simple in terms of Lyapunov candidate, the size of the augmented system rises as an important issue. As seen in the Part II of the thesis, the polytopes have $n+1$ vertices and the problem is to find a feedback gain that satisfies $n+1$ LMIs for the *intersampled delay variation case* or $(d_{max} - d_{min} + 1)(d_{max} - d_{min})(n+1)/2$ for the *multisampled delay variation case*, and moreover, the extended-state system's size is function of the delay size ($n + md$, where $m, m, d \in \mathbb{Z}_+^*$) and continuous system's size, affecting the feasibility domain of the LMI problem.

So the idea is to design a state-feedback gain which avoids in the same time the technical difficulties related to the storage of past control/state values, of the type:

$$u_k = Kx_k, \tag{6.8}$$

where $K \in \mathbb{R}^{m \times n}$.

One of the possible solutions is to use the discrete time correspondent of *Lyapunov-Krasovskii functionals* (also called LKF) and *Lyapunov-Razumikhin functionals* [136]. In discrete-time, both of these classes of Lyapunov candidates are indeed *Lyapunov-Krasovskii functions* and respectively *Lyapunov-Razumikhin functions*. The simplicity of the control law (6.8) comes with the inconvenient of a high complexity of this kind of candidate.

In Sections 6.2.1 and 6.2.2, by using model transformations, the state-feedback design is based on *Lyapunov-Krasovskii candidates* leading on two LMI-based procedures: the first one is *delay-independent* (Section 6.2.1) and the second is *delay-dependent* (Section 6.2.2). The difference is the inclusion of the delay information on the delay size in the corresponding stability result (see for instance [137, 140] and the references therein). According to [133], the characteristics of these two approaches are:

- **Delay-independent:** the stability property holds for all positive (and finite) delay values.
- **Delay-dependent:** the stability is guaranteed for some values of delays. For the other values the system is unstable.

By observing the affirmatives above, it is obvious that the delay-independent approach is more conservative. For more information, see, for instance, the survey [140].

From the delay variation point of view, the results presented in Sections 6.2.1 and 6.2.2 are more restrictive than the one presented in Section 6.1.2. In Section 6.2.1, the delay-independent approach is designed to cover the intersampled delay variation, i.e. $\tau \in \mathbb{R}_{(dT_e,(d+1)T_e]}$. By using the polytopic constructions presented in Chapters 3 and 4, the stability is guaranteed for the system (3.1) despite the uncertainty function $\Delta(\epsilon)$, described by (3.7). On the other hand, in Section 6.2.2 a discrete-time time-delay system is directly considered. The discrete-time delay "d" is time-varying and uncertain in the bounds $d \in \mathbb{Z}_{(0,d_{max}]}$.

6.2.1 Delay-independent Lyapunov-Krasovskii approach

The idea [111] is to design a control law as $u_k = Kx_k$, where $K \in \mathbb{R}^{m\times n}$ is the static state feedback gain. Based on (3.4), one can have the final discrete-time model:

$$x_{k+1} = Ax_k + \Delta(\epsilon_k)Kx_{k-d+1} + (B - \Delta(\epsilon_k))Kx_{k-d}. \tag{6.9}$$

To derive the stability conditions for (6.9), the following theorem is stated:

Theorem 6.5. *Consider the discrete-time system (6.9). A state feedback matrix $K \in \mathbb{R}^{m\times n}$ that solves a robust performance criteria and stabilizes the system is given by:*

$$K = YG^{-1}, \tag{6.10}$$

where the decision variables $G \in \mathbb{R}^{n\times n}$, $G = G^\top > 0$ and $Y \in \mathbb{R}^{m\times n}$ are obtained from the solution of the following linear objective minimization problem:

$$\begin{aligned} &\min_{\gamma,\ G,\ G_x,\ G_y,\ Y} && \gamma \\ &\textit{subject to:} && (6.12) \end{aligned} \tag{6.11}$$

where the decision variables are $Q \in \mathbb{R}^{n\times n}$, $R_1 \in \mathbb{R}^{m\times m}$, $R_2 \in \mathbb{R}^{m\times m}$, $G_x \in \mathbb{R}^{n\times n}$, $G_x = G_x^\top > 0$, $G_y \in \mathbb{R}^{n\times n}$, $G_y = G_y^\top > 0$, and γ is a scalar.

Proof. For the system (6.9) consider the LKF candidate:

$$V(x) = x_k^\top P_0 x_k + \sum_{i=1}^{d-1}\left\{x_{k-i}^\top P_1 x_{k-i}\right\} + x_{k-d}^\top P_2 x_{k-d} > 0, \tag{6.13}$$

$$\begin{bmatrix} G & 0 & 0 & G_x & 0 & GA^\top & GQ^{\frac{1}{2}} & 0 & 0 \\ 0 & G_x & 0 & 0 & G_y & Y^\top\Delta_i^\top & 0 & Y^\top R_1^{\frac{1}{2}} & 0 \\ 0 & 0 & G_y & 0 & 0 & Y^\top(B-\Delta_i)^\top & 0 & 0 & Y^\top R_2^{\frac{1}{2}} \\ G_x & 0 & 0 & G_x & 0 & 0 & 0 & 0 & 0 \\ 0 & G_y & 0 & 0 & G_y & 0 & 0 & 0 & 0 \\ AG & \Delta_i Y & (B-\Delta_i)Y & 0 & 0 & G & 0 & 0 & 0 \\ Q^{\frac{1}{2}}G & 0 & 0 & 0 & 0 & 0 & \gamma I & 0 & 0 \\ 0 & R_1^{\frac{1}{2}}Y & 0 & 0 & 0 & 0 & 0 & \gamma I & 0 \\ 0 & 0 & R_2^{\frac{1}{2}}Y & 0 & 0 & 0 & 0 & 0 & \gamma I \end{bmatrix} \geq 0; \quad i = 1,\ldots,n_\Delta, \tag{6.12}$$

where the matrices $P_0 \in \mathbb{R}^{n\times n}$, $P_1 \in \mathbb{R}^{n\times n}$ and $P_2 \in \mathbb{R}^{n\times n}$ are symmetric positive definite.

In order to assure asymptotic stability the LKF has to satisfy:

$$\Delta V = V(x_{k+1}) - V(x_k) \leq 0. \tag{6.14}$$

This imposes a decrease condition of the LKF function on every possible system trajectory.

A robust performance criterion can further be imposed:

$$\begin{aligned} \Delta V &\leq -\tfrac{1}{\gamma}\left(x_k^\top Q x_k + u_{k-d+1}^\top R_1 u_{k-d+1} + u_{k-d}^\top R_2 u_{k-d}\right) \\ &= -\tfrac{1}{\gamma}\left(x_k^\top Q x_k + x_{k-d+1}^\top K^\top R_1 K x_{k-d+1} + x_{k-d}^\top K^\top R_2 K x_{k-d}\right), \end{aligned} \tag{6.15}$$

where the weighting matrices $R_1 \in \mathbb{R}^{m\times m}$, $R_2 \in \mathbb{R}^{m\times m}$ and $Q \in \mathbb{R}^{n\times n}$ are positive definite.

Exploiting the systems dynamics (6.9):

$$\begin{aligned} \Delta V = \; & \left[Ax_k + (B-\Delta(\epsilon))Kx_{k-d} + \Delta(\epsilon)Kx_{k-d+1}\right]^\top P_0 \\ & \left[Ax_k + (B-\Delta(\epsilon))Kx_{k-d} + \Delta(\epsilon)Kx_{k-d+1}\right] + \\ & + \sum_{i=1}^{d-1}\left\{x_{k-i+1}^\top P_1 x_{k-i+1}\right\} + x_{k-d+1}^\top P_2 x_{k-d+1} - x_k^\top P_0 x_k - \\ & - \sum_{i=1}^{d-1}\left\{x_{k-i}^\top P_1 x_{k-i}\right\} - x_{k-d}^\top P_2 x_{k-d}. \end{aligned} \tag{6.16}$$

It can be seen that some terms of (6.16) can be simplified:

$$\sum_{i=1}^{d-1} x_{k-i+1}^\top P_1 x_{k-i+1} - \sum_{i=1}^{d-1} x_{k-i}^\top P_1 x_{k-i} = x_k^\top P_1 x_k - x_{k-d+1}^\top P_1 x_{k-d+1}. \tag{6.17}$$

Equation (6.16) becomes:

$$\begin{aligned}\Delta V = \; & x_k^\top \left(A^\top P_0 A - P_0 + P_1\right) x_k + x_{k-d+1}^\top \left(K^\top \Delta(\epsilon)^\top P_0 \Delta K - P_1 + P_2\right) x_{k-d+1} + \\ & + x_{k-d}^\top \left[K^\top (B-\Delta(\epsilon))^\top P_0 (B-\Delta) K - P_2\right] x_{k-d} + x_k^\top A^\top P_0 (B-\Delta(\epsilon)) K x_{k-d} + \\ & + x_k^\top A^\top P_0 \Delta(\epsilon) K x_{k-d+1} + x_{k-d+1}^\top K^\top \Delta(\epsilon)^\top P_0 A x_k + \\ & x_{k-d+1}^\top K^\top \Delta(\epsilon)^\top P_0 (B-\Delta) K x_{k-d} + x_{k-d}^\top K^\top (B-\Delta(\epsilon))^\top P_0 A x_k + \\ & x_{k-d}^\top K^\top (B-\Delta(\epsilon))^\top P_0 \Delta(\epsilon) K x_{k-d+1}.\end{aligned} \tag{6.18}$$

By noting:

$$\zeta_k^\top = \begin{bmatrix} x_k^\top & x_{k-d+1}^\top & x_{k-d}^\top \end{bmatrix}^\top, \tag{6.19}$$

equation (6.16) can be written in the following block matrix form:

$$\Delta V = \zeta_k^\top A_\Delta \zeta_k, \tag{6.20}$$

where:

$$A_\Delta = \begin{bmatrix} A^\top P_0 A - P_0 + P_1 & A^\top P_0 \Delta(\epsilon) K & A^\top P_0 (B-\Delta(\epsilon)) K \\ K^\top \Delta(\epsilon)^\top P_0 A & K^\top \Delta(\epsilon)^\top P_0 \Delta(\epsilon) K - P_1 + P_2 & K^\top \Delta(\epsilon)^\top P_0 (B-\Delta(\epsilon)) K \\ K^\top (B-\Delta(\epsilon))^\top P_0 A & K^\top (B-\Delta(\epsilon))^\top P_0 \Delta(\epsilon) K & K^\top (B-\Delta(\epsilon))^\top P_0 (B-\Delta(\epsilon)) K - P_2 \end{bmatrix} \tag{6.21}$$

Now the condition (6.15) can be written in the form:

$$\Delta V = -\zeta_k^\top A_\Delta \zeta_k \geq -\zeta_k^\top Q_\Delta \zeta_k, \tag{6.22}$$

where:

$$Q_\Delta = \begin{bmatrix} \frac{1}{\gamma} Q & 0 & 0 \\ 0 & \frac{1}{\gamma} K^\top R_1 K & 0 \\ 0 & 0 & \frac{1}{\gamma} K^\top R_2 K \end{bmatrix}. \tag{6.23}$$

and A_Δ is done by (6.21).

So (6.22) can be now considered as:

$$\zeta_k^\top \left(-A_\Delta - Q_\Delta\right) \zeta_k \geq 0. \tag{6.24}$$

The inequality (6.24) can be written in the form $M - N^\top R^{-1} N \geq 0$:

$$\begin{bmatrix} P_0 - \frac{1}{\gamma}Q & 0 & 0 \\ 0 & P_1 - \frac{1}{\gamma}K^\top R_1 K & 0 \\ 0 & 0 & P_2 - \frac{1}{\gamma}K^\top R_2 K \end{bmatrix} - \begin{bmatrix} P_1 & 0 & 0 \\ 0 & P_2 & 0 \\ P_0 A & P_0\Delta(\epsilon)K & P_0(B-\Delta(\epsilon))K \end{bmatrix}^\top \begin{bmatrix} P_1^{-1} & 0 & 0 \\ 0 & P_2^{-1} & 0 \\ 0 & 0 & P_0^{-1} \end{bmatrix} \begin{bmatrix} P_1 & 0 & 0 \\ 0 & P_2 & 0 \\ P_0 A & P_0\Delta(\epsilon)K & P_0(B-\Delta(\epsilon))K \end{bmatrix} \geq 0. \tag{6.25}$$

Schur complement formula can now be applied. In the obtained equation the substitutions $P_0 = G^{-1}$, $P_1 = G_1^{-1}$ and $P_2 = G_2^{-1}$ were made. Now by congruence with diag$(G, 6)$ and with the notations $Y = KG$, $G_x = GG_1^{-1}G$, $G_y = GG_2^{-1}G$ one obtains:

$$\begin{bmatrix} G & 0 & 0 & G_x & 0 & GA^\top \\ 0 & G_x & 0 & 0 & G_y & Y^\top\Delta(\epsilon)^\top \\ 0 & 0 & G_y & 0 & 0 & Y^\top(B-\Delta(\epsilon))^\top \\ G_x & 0 & 0 & G_x & 0 & 0 \\ 0 & G_y & 0 & 0 & G_y & 0 \\ AG & \Delta Y & (B-\Delta(\epsilon))Y & 0 & 0 & G \end{bmatrix} - \begin{bmatrix} GQ^{\frac{1}{2}} & 0 & 0 \\ 0 & Y^\top R_1^{\frac{1}{2}} & 0 \\ 0 & 0 & Y^\top R_2^{\frac{1}{2}} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\gamma}I & 0 & 0 \\ 0 & \frac{1}{\gamma}I & 0 \\ 0 & 0 & \frac{1}{\gamma}I \end{bmatrix} \begin{bmatrix} Q^{\frac{1}{2}}G & 0 & 0 & 0 & 0 & 0 \\ 0 & R_1^{\frac{1}{2}}Y & 0 & 0 & 0 & 0 \\ 0 & 0 & R_2^{\frac{1}{2}}Y & 0 & 0 & 0 \end{bmatrix} \geq 0. \tag{6.26}$$

By applying Schur complement formula one obtains the final LMI (6.12) with the decision variables G, G_x, G_y, γ and Y and Δ being replaced by the generators of the polytopic over-approximation. By saying that the state feedback is given by $K = YG^{-1}$ the proof is complete. ■

6.2.2 Delay-dependent Lyapunov-Krasovskii approach

Consider a linear discrete-time system with input delay, as in [110]:

$$x_{k+1} = Ax_k + Bu_{k-d}, \tag{6.27}$$

where $x_k \in \mathbb{R}^n$ is the state, $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ are the state space representation and $d \in \mathbb{Z}_+^*$, is the input delay. The sought control law has the form:

$$u_k = Kx_k, \tag{6.28}$$

where $K \in \mathbb{R}^{m\times n}$ is the state feedback gain matrix. Therefore, the system (6.27) can be written as:

$$x_{k+1} = Ax_k + BKx_{k-d}. \tag{6.29}$$

In the sequel, the stability conditions for the closed-loop system (6.29) are stated. These conditions are the starting point for a stabilization design proposed in the next section.

Lemma 6.6. *Consider the linear discrete-time system (6.29) with an uncertain input time-delay $d \in \mathbb{Z}_{(0,d_{max}]}$. If there exists the matrices $P > 0$, $Q > 0$, R, Y and Z such that the following inequalities hold:*

$$\begin{bmatrix} \Gamma_1 & \Gamma_2 \\ \Gamma_2^\top & \Gamma_3 \end{bmatrix} \leq 0, \tag{6.30}$$

$$\begin{bmatrix} R & Y \\ Y^\top & Z \end{bmatrix} \geq 0, \tag{6.31}$$

where:

$$\begin{aligned}\Gamma_1 &= A^\top PA - P + Y + Y^\top + dR + Q + d(A-I)^\top Z(A-I),\\ \Gamma_2 &= A^\top PBK - Y + d(A-I)^\top ZBK,\\ \Gamma_3 &= K^\top B^\top PBK + dK^\top B^\top ZBK - Q,\end{aligned} \tag{6.32}$$

then the system is asymptotically stable.

Proof. Consider the following discrete-time Lyapunov candidate:

$$V_k = V_{1k} + V_{2k} + V_{3k} \geq 0, \tag{6.33}$$

where V_{1k}, V_{2k} and V_{3k} are, respectively:

$$\begin{gathered}V_{1k} = x_k^\top P x_k,\\ V_{2k} = \sum_{i=-d}^{-1}\sum_{j=i}^{-1}(x_{k+j+1}^\top - x_{k+j}^\top)Z(x_{k+j+1} - x_{k+j}),\\ V_{3k} = \sum_{i=-d}^{-1}(x_{k+i}^\top Q x_{k+i}).\end{gathered}$$

Since it holds that:

$$x_k - x_{k-d} = \sum_{i=-d}^{-1}\left(x_{k+i+1} - x_{k+i}\right), \tag{6.34}$$

the system (6.29) becomes:

$$x_{k+1} = (A+BK)x_k - BK\sum_{i=-d}^{-1}\left(x_{k+i+1} - x_{k+i}\right). \tag{6.35}$$

Stability is assured by the condition:

$$\Delta V = V_{k+1} - V_k = \Delta V_1 + \Delta V_2 + \Delta V_3 \leq 0. \tag{6.36}$$

The term ΔV_1 satisfies the relation:

$$\begin{aligned}
\Delta V_1 &= V_{1(k+1)} - V_{1k} = x_{k+1}^\top P x_{k+1} - x_k^\top P x_k = \\
&= [(A+BK)x_k - BK\sum_{i=-d}^{-1}\{x_{k+i+1} - x_{k+i}\}]^\top P \\
&\quad [(A+BK)x_k - BK\sum_{i=-d}^{-1}\{x_{k+i+1} - x_{k+i}\}] - x_k^\top P x_k \\
&= x_k^\top[(A+BK)^\top P(A+BK) - P]x_k - \\
&\quad -2x_k^\top(A+BK)^\top PBK\sum_{i=-d}^{-1}\{x_{k+i+1} - x_{k+i}\} + \\
&+ \sum_{i=-d}^{-1}\{x_{k+i+1} - x_{k+i}\}^\top K^\top B^\top PBK\sum_{i=-d}^{-1}\{x_{k+i+1} - x_{k+i}\}.
\end{aligned}$$

In the stability results presented here, the upper bound for the inner product of two vectors plays an important role. Taking advantage of the ideas stated in [132] one has that for $a(\cdot) \in \mathbb{R}^{n_a}$, $b(\cdot) \in \mathbb{R}^{n_b}$ and $N(\cdot) \in \mathbb{R}^{n_a \times n_b}$ defined on some interval φ, and for any matrices $R \in \mathbb{R}^{n_a \times n_a}$, $Y \in \mathbb{R}^{n_a \times n_b}$ and $Z \in \mathbb{R}^{n_b \times n_b}$, the following condition holds:

$$-2\sum_{i\in\varphi} a(i)^\top N b(i) \leq \sum_{i\in\varphi} \begin{bmatrix} a(i) \\ b(i) \end{bmatrix}^\top \begin{bmatrix} R & Y - N \\ Y^\top - N^\top & Z \end{bmatrix} \begin{bmatrix} a(i) \\ b(i) \end{bmatrix}, \tag{6.37}$$

where

$$\begin{bmatrix} R & Y \\ Y^\top & Z \end{bmatrix} \geq 0. \tag{6.38}$$

By choosing $a = x_k$, $b = (x_{k+i+1} - x_{k+i})$ and $N = (A+BK)^\top PBK$ one has:

$$\begin{aligned}
&-2x_k^\top(A+BK)^\top PBK\sum_{i=-d}^{-1}\{x_{k+i+1} - x_{k+i}\} \leq \\
&\sum_{i=-d}^{-1}\Big\{(x_{k+i+1} - x_{k+i})^\top Z(x_{k+i+1} - x_{k+i})\Big\} + dx_k^\top R x_k + \\
&+2x_k^\top\left[Y - (A+BK)^\top PBK\right]\sum_{i=-d}^{-1}\{x_{k+i+1} - x_{k+i}\}.
\end{aligned} \tag{6.39}$$

Considering (6.39) and (6.34) one obtains:

$$\begin{aligned}\Delta V_1 &\leq \sum_{i=-d}^{-1}\Big\{(x_{k+i+1}-x_{k+i})^\top Z\,(x_{k+i+1}-x_{k+i})\Big\}+\\ &+ x_k^\top(A^\top PA-P+Y+Y^\top+dR)x_k+\\ &+ x_{k-d}^\top K^\top B^\top PBKx_{k-d}+2x_k^\top[A^\top PBK-Y]x_{k-d}.\end{aligned}$$

Furthermore for ΔV_2:

$$\begin{aligned}\Delta V_2 &= V_{2(k+1)}-V_{2k}=d(x_{k+1}-x_k)^\top Z(x_{k+1}-x_k)-\\ &- \sum_{i=-d}^{-1}\Big\{(x_{k+i+1}^\top-x_{k+i}^\top)Z(x_{k+i+1}-x_{k+i})\Big\}.\end{aligned}$$

Now by replacing x_{k+1} with (6.29), one gets:

$$\begin{aligned}\Delta V_2 &= dx_k^\top(A-I)^\top Z(A-I)x_k+2dx_k^\top(A-I)^\top ZBKx_{k-d}\\ &- \sum_{i=-d}^{-1}\Big\{(x_{k+i+1}-x_{k+i})^\top Z(x_{k+i+1}-x_{k+i})\Big\}+\\ &+ dx_{k-d}^\top K^\top B^\top ZBKx_{k-d}.\end{aligned}$$

Finally, ΔV_3 is equal with:

$$\Delta V_3 = V_{3(k+1)}-V_{3k}=x_k^\top Qx_k-x_{k-d}^\top Qx_{k-d}.$$

Therefore (6.36) can be rewritten as:

$$\begin{aligned}\Delta V \leq\; & x_k^\top\left[A^\top PA-P+Y+Y^\top+dR+Q+d(A-I)^\top Z(A-I)\right]x_k+\\ & 2x_k^\top\left[A^\top PBK-Y+d(A-I)^\top ZBK\right]x_{k-d}+\\ & +x_{k-d}^\top\left[BK^\top PBK+dK^\top B^\top ZBK-Q\right]x_{k-d}\leq 0.\end{aligned} \tag{6.40}$$

with R, Y and N satisfying (6.38). By rewriting in a block matrix form, these relations are equivalent with the ones presented in Lemma 6.6. ■

Remark 6.7. There exist several alternative methods to derive delay-dependent stability conditions for the closed-loop system (6.29) in the literature. Most of the approaches take advantage on the system's dynamics over some delay "intervals" (model transformation) (see for instance [48, 49, 84, 89, 104]) or on some constructive procedures for deriving appropriated upper bounds for the inner product of state vectors (see for instance [54, 132, 146, 174]). From the best of the authors knowledge, the present approach has never been exploited and has the advantage of a Lyapunov construction based on three components in (6.36).

Theorem 6.8. *Consider the linear discrete-time system* (6.29) *with an uncertain input time-delay* $d \in [0, d_{max})$*. If there exists the matrices* $G = G^\top > 0$*,* $J > 0$*,* H*,* T*,* W *and* L *such that the following inequalities hold:*

$$\begin{bmatrix} T + T^\top + dH + J - G & -T & GA^\top & d_{max}G(A-I)^\top \\ -T^\top & -J & L^\top B^\top & d_{max}L^\top B^\top \\ AG & BL & -G & 0 \\ d_{max}(A-I)G & d_{max}LB & 0 & -d_{max}W \end{bmatrix} \leq 0, \tag{6.41}$$

$$\begin{bmatrix} H & T \\ T^\top & GW^{-1}G \end{bmatrix} \geq 0, \tag{6.42}$$

where $0 < d < d_{max}$*, then the system is asymptotically stable and a state feedback matrix* K *that stabilizes the system is given by:*

$$K = LG^{-1}. \tag{6.43}$$

Proof. The inequality (6.30) can be rewritten as:

$$\begin{bmatrix} A^\top P & d(A-I)^\top Z \\ K^\top B^\top P & dK^\top B^\top Z \end{bmatrix} \begin{bmatrix} P^{-1} & 0 \\ 0 & \frac{1}{d}Z^{-1} \end{bmatrix} \begin{bmatrix} PA & PBK \\ dZ(A-I) & dZBK \end{bmatrix} + \begin{bmatrix} Y + Y^\top + dR + Q - P & -Y \\ -Y^\top & -Q \end{bmatrix} \leq 0. \tag{6.44}$$

Now by applying Schur complement, pre- and post-multiplying the resulting inequality with $\mathrm{diag}(P^{-1}, P^{-1}, P^{-1}, Z^{-1})$ and making the notations $P^{-1} = G$, $Z^{-1} = W$, $P^{-1}YP^{-1} = T$, $P^{-1}QP^{-1} = J$, $P^{-1}RP^{-1} = H$, $KP^{-1} = L$ the equation (6.41) is obtained. By pre- and post-multiplying equation (6.31) with $\mathrm{diag}(P^{-1}, P^{-1})$, equation (6.42) yields. Finally, the stabilizing gain will be given by the relation $K = LG^{-1}$. ∎

6.2.2.1 Robust performance conditions

In order to achieve robust performance, one can impose restrictions on the Lyapunov candidate most related to a particular decreasing rate. In a general form, this constraint takes the form:

$$\Delta V \leq -\frac{1}{\gamma} \begin{bmatrix} x_k \\ u_{k-1} \\ \vdots \\ u_{k-d} \end{bmatrix}^\top \left[\begin{array}{c|ccc} Q_1 & & & \\ \hline & R_d & L_{i,j} & \\ & L_{i,j}^\top & \ddots & \\ & & & R_d \end{array} \right] \begin{bmatrix} x_k \\ u_{k-1} \\ \vdots \\ u_{k-d} \end{bmatrix}, \tag{6.45}$$

with the convention that some $R_i = 0$, for some (but not all) indexes $i \in \mathbb{Z}_{[1,d_{max}]}$. By suppressing the corresponding lines and columns the remaining weighting matrix should be positive definite.

A particular construction in this sense considers $R_i = 0$ for $i \in \mathbb{Z}_{[2,d_{max}]}$ and thus enforces:

$$\Delta V \leq -\frac{1}{\gamma} \left[x_k^\top Q_1 x_k + u_{k-d}^\top R_d u_{k-d} \right]. \tag{6.46}$$

Theorem 6.9. *Consider the linear discrete-time system (6.29) with an uncertain input time-delay $d \in \mathbb{Z}_{(0,d_{max}]}$. A stabilizing state feedback matrix K for the system (6.27) subject to robust performance constraints (6.46) is given by:*

$$K = LG^{-1}, \tag{6.47}$$

where $G = G^\top$ and L are obtained as the solution of the following linear objective minimization problem:

$$\begin{gathered} \min_{\gamma, G, L, H, T, J, W} \gamma, \\ \textit{subject to: } (6.49) \textit{ and } (6.50). \end{gathered} \tag{6.48}$$

where:

$$\begin{bmatrix} T+T^\top+d_{max}H+J-G & -T & GA^\top & d_{max}G(A-T') & GQ_1^{1/2} & 0 \\ -T^\top & -J & L^\top B^\top & d_{max}L^\top B^\top & 0 & L^\top R_d^{1/2} \\ AG & BL & -G & 0 & 0 & 0 \\ d_{max}(A-I)G & d_{max}BL & 0 & -d_{max}G & 0 & 0 \\ Q_1^{1/2}G & 0 & 0 & 0 & -\gamma I & 0 \\ 0 & R_d^{1/2}L & 0 & 0 & 0 & -\gamma I \end{bmatrix} \leq 0; \tag{6.49}$$

where $0 < d < d_{max}$*, and*

$$\begin{bmatrix} H & T \\ T^\top & G \end{bmatrix} \geq 0,\ G > 0,\ J > 0. \tag{6.50}$$

Proof. The inequality (6.46) can be rewritten as:

$$\begin{aligned} &\begin{bmatrix} A^\top P & d(A-I)^\top Z \\ K^\top B^\top P & dK^\top B^\top Z \end{bmatrix} \begin{bmatrix} P^{-1} & 0 \\ 0 & \frac{1}{d}Z^{-1} \end{bmatrix} \begin{bmatrix} PA & PBK \\ dZ(A-I) & dZBK \end{bmatrix} \\ &+ \begin{bmatrix} Y+Y^\top+dR+Q-P+\frac{1}{\gamma}Q_1 & -Y \\ -Y^\top & -Q+\frac{1}{\gamma}K^\top R_d K \end{bmatrix} \leq 0. \end{aligned} \tag{6.51}$$

By applying Schur complement formula:

$$\begin{aligned} &\begin{bmatrix} Y+Y^\top+dR+Q-P & -Y & A^\top P & d(A-I)^\top Z \\ -Y^\top & -Q & K^\top B^\top P & dK^\top B^\top Z \\ PA & PBK & -P & 0 \\ dZ(A-I) & dZBK & 0 & -dZ \end{bmatrix} + \\ &\begin{bmatrix} Q_1^{\frac{1}{2}} & 0 \\ 0 & K^\top R_d^{\frac{1}{2}} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\gamma}I & 0 \\ 0 & \frac{1}{\gamma}I \end{bmatrix} \begin{bmatrix} Q_1^{\frac{1}{2}} & 0 & 0 & 0 \\ 0 & R_d^{\frac{1}{2}}K & 0 & 0 \end{bmatrix} \leq 0. \end{aligned}$$

By applying again Schur complement, pre- and post-multiplying the obtained inequality with $\mathrm{diag}(P^{-1}, P^{-1}, P^{-1}, Z^{-1}, I, I)$, making the notations $P^{-1} = G$, $Z^{-1} = W$, $P^{-1}YP^{-1} = T$, $P^{-1}QP^{-1} = J$, $P^{-1}RP^{-1} = H$, $KP^{-1} = L$ and considering $W = G$ the equation (6.49) follows.

For $W = G$, pre- and post-multiplying equation (6.31) with $\text{diag}(P^{-1}, P^{-1})$, gives the inequality (6.50).

By observing that the feedback K is obtained by $K = LG^{-1}$ the proof is complete. ■

Example 6.3. *Consider the position control of a DC-motor as described in Section 3.3. The sampling period is $T_e = 0.003s$ and the discrete-time delay bounds are $d \in \mathbb{Z}_{[0,3]}$. The discrete-time model is given by:*

$$x_{k+1} = \begin{bmatrix} 1 & 0.0028 \\ 0 & 0.8382 \end{bmatrix} x_k + \begin{bmatrix} 0.0525 \\ 33.9731 \end{bmatrix} u_{k-d}.$$

The state-feedback gain obtained by Theorem 6.9 with the weighting matrices $Q_1 = 10^{-4} I_{n+md}$, $R_1 = 10^{-4}$ and $d = 3$ is:

$$K = \begin{bmatrix} -0.0018 & -0.0031 \end{bmatrix}.$$

In order to study the dynamical behavior of the system, a simulation in Matlab Simulink is taken in place. The initial conditions are $x_0 = \begin{bmatrix} 1 & -1 \end{bmatrix}^\top$ and $x_i = 0$ for $i \in \mathbb{Z}_{[-d,-1]}$.

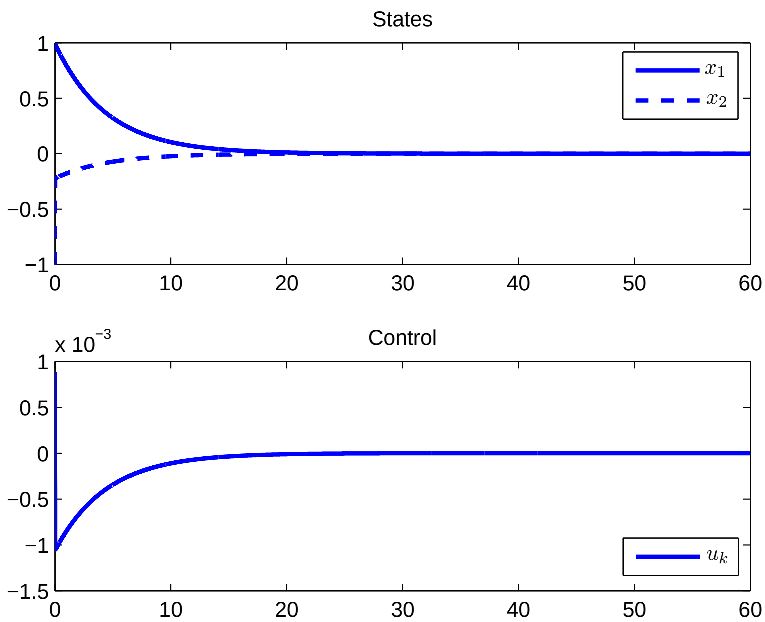


FIGURE 6.3: Temporal behavior of the system. The x axis is the time in seconds [x] of simulation. The y axis are [cm] for the states, [v] for the control and seconds [s].

Remark 6.10. In order to draw a conclusion on the models introduced in Chapters 3 and 5 we can say that:

- Delay-independent results in Section 6.2.1 can be applied directly without the extended state-space implications.
- Delay-independent approach in Section 6.2.1 covers the same class of systems by avoiding the state augmentation.
- Delay-dependent approach in Section 6.2.2, on the other hand, provides a less conservative approach but the results are based on the model 6.27 and 3.8, which is a restricted class of time-delay systems over the ones presented in Chapters 3 and 5.

Chapter 7

Set invariance for time-delay systems

The stabilization techniques presented in the previous chapter do not consider neither state nor control input constraints. When applied to real systems it can lead to instabilities, saturation, damage of the physical components, among other drastic consequences. The invariant set (control) theory provides the appropriated framework for the characterization of the dynamical systems behavior in presence of such physical or mathematical constraints.

The two main types of invariant sets used in practice are the *polyhedral* and the *ellipsoidal* ones, due to their computational advantages (we restrict ourselves to convex invariant sets, the case of non-convex constructions being detailed where appropriate by collections of convex sets). Here the polyhedral sets are developed because they can be expressed in the $\mathcal{V}$-representation and $\mathcal{H}$-representation, more convenient to predictive control purposes by preserving the linear description of the constraints. On the other hand, from a geometrical point of view, the polyhedral regions can provide approximations for any convex body at the price of the computational complexity [63, 64].

In Section 7.1 the concept of *maximal state admissible sets* (introduced by [58]) is recalled for the extended state-space framework, by using the modeling techniques presented in the Chapter 2 and the stabilization techniques developed in Section 6.1. The disadvantage is the high complexity of the obtained set: the original system's dimension is n while the obtained set is described a $n + dm$ dimensional space.

To overcome this inconvenient, the concept of $\mathcal{D}$-invariance is introduced in Section 7.2. The $\mathcal{D}$-invariance can be understood as the state invariance in both actual and delayed states, i.e., all of them should belong to the same set, which is invariant. In the same section some properties concerning the $\mathcal{D}$-invariance are provided. Necessary and sufficient conditions are presented in Section 7.2.2, and the algebraic existence conditions [75, 76] are discussed in Section 7.2.3, with some constructive $\mathcal{D}$-invariant tests. If the maximal admissible set suffers from the complexity point of view, the $\mathcal{D}$-invariant sets may prove to be simple but conservative alternatives. Section 7.3 tries to cope with this conservativeness by deriving the concept of Cyclic $\mathcal{D}$-invariance. A very important step in the invariant set characterization is the constructive procedures. This is known to be a challenging issue even in the LTI case. An algorithm to obtain $\mathcal{D}$-invariant sets is presented in Section 7.4, by means of set iterates [113], and represent an effective way to produce invariant sets for further predictive control constructions, for example.

7.1 Extended state-space framework

The objective in *constrained control for time-delay systems* is to design a control law which regulates the system state while robustly satisfying a set of constraints. A mixed state-input set of constraints will be described in a linear formulation by a collection of inequalities:

$$Cx_k + Du_k \leq W \tag{7.1}$$

7.1.1 Maximal delayed-state admissible set

In order to deal with the constraints, the first step is to rewrite (7.1) in terms of the augmented state variable ξ:

$$\Gamma\xi_k + Du_k \leq W. \tag{7.2}$$

Using the stabilizing control law $u_k = K\xi_k = YS^{-1}\xi_k$ found by solving (6.3) the following polyhedral domain can be defined in the augmented state space:

$$P = \left\{\xi \in \mathbb{R}^{(n+md)} \mid \ (\Gamma + DK)\xi \leq W\right\}. \tag{7.3}$$

The interest is in the extension of the concept of *maximal state admissible sets* [58] to polytopic models, obtained by the closed loop connection of (3.13) with $u_k = K\xi_k$. The following definition provides the necessary details in this direction.

Definition 7.1. For a system with polytopic uncertainty:

$$\begin{aligned} \xi_{k+1} &= \Phi\xi_k; \\ \Phi &\in \Omega_K; \\ \Omega_K &= \mathrm{Co}\{(A_{\Delta_0} + B_{\Delta_n}K);\ \ldots;\ (A_{\Delta_n} + B_{\Delta_n}K)\} \end{aligned} \tag{7.4}$$

and a predefined set P, the *maximal delayed-state admissible set* O_∞^Ω is defined as the collection of all the initial states ξ_0, for which the state trajectory remains in the interior of P, for all future instants $k \geq 0$. □

In other words, the maximal admissible set is described readily as:

$$O_\infty^\Omega = \left\{ \xi_0 \in \mathbb{R}^{n+dm} \mid \prod_{i=0}^{k} \Phi_i \xi_0 \in P, \forall \Phi_i \in \Omega_K, \forall k \in \mathbb{Z} \right\}. \tag{7.5}$$

By the relationship:

$$O_\ell^\Omega = \left\{ \xi_0 \in \mathbb{R}^{n+dm} \mid \prod_{i=0}^{k} \Phi_i \xi_0 \in P, \forall \Phi_i \in \Omega_K, \forall k \in \mathbb{Z},\ k < \ell \right\},$$

one can construct the sequence O_1^Ω, O_2^Ω, $\ldots$ and we can expect that $O_i^\Omega = O_{i+1}^\Omega$. This is what we call in the following "finite determiness".

Theorem 7.2. *If the following assumptions are fulfilled:*

1. *There is a common quadratic Lyapunov function that assures the asymptotic stability of the systems $\xi_{k+1} = \Phi\xi_k, \Phi \in \Omega_K$.*

2. *The polytope P is bounded.*

3. $0 \in$ *int* P.

Then O_∞^Ω is finitely determined.

Proof. Assuming the feasibility of the LMI-based stabilization in the Theorem 6.2, the system is asymptotic stable and thus:

$$\lambda\left(\prod_{i=0}^{k}\Phi_i\right)\to 0$$

as $k\to\infty$. Using the fact that the set P is bounded, there exists a finite "k" such that any vertex of

$$\prod_{i=0}^{k}\Phi_i \text{ with } \Phi_i\in \mathrm{Co}\{(A_{\Delta_0}+B_\Delta K);\ \ldots;\ (A_{\Delta_n}+B_\Delta K)\}$$

assures a contraction factor satisfying the strict inclusion:

$$\prod_{i=0}^{k+1}\Phi_i P\subset P,$$

by observing that the initial set P is bounded and contains the origin in its strict interior.

■

It should be noticed that the feasibility of the Theorem 6.2 assures the satisfaction of the first hypothesis of the previous theorem.

Similarly to the linear case, the construction algorithm can exploit the fact that O_∞^Ω is finitely determined if and only if $O_N=O_{N+1}$ where:

$$O_N^\Omega=\left\{\xi_0 \mid \prod_{i=0}^{N}\Phi_i\xi_0\in P, \forall\Phi_i\in\Omega_K, \forall i\in\mathbb{Z}_{[1,N]}\right\} \tag{7.6}$$

Observing that the same set can be rewritten as:

$$O_{N+1}^\Omega=\left\{\xi\in O_N^\Omega \mid \Phi\xi\in P, \forall\Phi\in\Omega_K\right\} \tag{7.7}$$

and further by noting $\Phi_i=A_{\Delta_i}+B_\Delta K$, $i\in\mathbb{Z}_{[0,n]}$ one can obtain a direct computable expression:

$$O_{N+1}^\Omega=\left\{\xi\in O_N^\Omega \mid \Phi_i\xi\in P, \forall\Phi_i\in\{\Phi_0,\ \ldots,\ \Phi_n\}\right\}, \tag{7.8}$$

which can be used in a recursive manner to obtain the maximal robust output admissible set.

The set O_∞^Ω enjoys by definition (7.5) robust positively invariance properties [16] and thus it can be used in several control design strategies, as for example, the predictive control scheme in which the invariance of the terminal set is instrumental in stability analysis [61].

Example 7.1 (Position control of a DC motor with time-varying delays - intersample delay variation - invariant set construction)**.** *Consider the model developed in the Examples 3.8 and 4.1 of the Chapter 3 and 4, with the less conservative model as proposed by Chapter 4 with the control law obtained in Example 6.1.*

The maximal delayed-state admissible set obtained is shown in Fig. 7.1. The system being described in an extended state-space $\mathbb{R}^5$, in order to plot the result in $\mathbb{R}^3$, one has to cut or project the higher dimensions in the thirtieth one. In this case, the cut is done in $x_4 = 0$ and $x_5 = 0$.

The chosen sampling period is $T_e = 0.003s$, the delay is $\tau \in \mathbb{R}_{[0.006s, 0.009s]}$. The discrete-time model is given by:

$$x_{k+1} = \begin{bmatrix} 1 & 0.0028 \\ 0 & 0.8382 \end{bmatrix} x_k + \begin{bmatrix} 0.0525 \\ 33.9731 \end{bmatrix} u_{k-d} - \Delta(u_{k-d} - u_{k-d+1}), \text{ with d} = 3.$$

The polytopic embedding is the same as the one obtained in Example 4.1 of the Chapter 4.

The extended-state feedback gain obtained by Theorem 6.2 with the matrices $Q = 10^{-5} I_{n+md}$ and $R = 1$ is, (the same as the one obtained in Example 6.1 of the Chapter 6):

$$K = \begin{bmatrix} -0.0677 & -0.0018 & -0.0430 & -0.0696 & 0.1424 \end{bmatrix}.$$

7.2 Original state-space framework - The $\mathcal{D}$-invariance

To define the concept of $\mathcal{D}$-invariance and $\mathcal{D}$-contractiveness, a linear delay-difference equation in the general form is considered:

$$x_{k+1} = \sum_{i=0}^{d} A_i x_{k-i}, \tag{7.9}$$

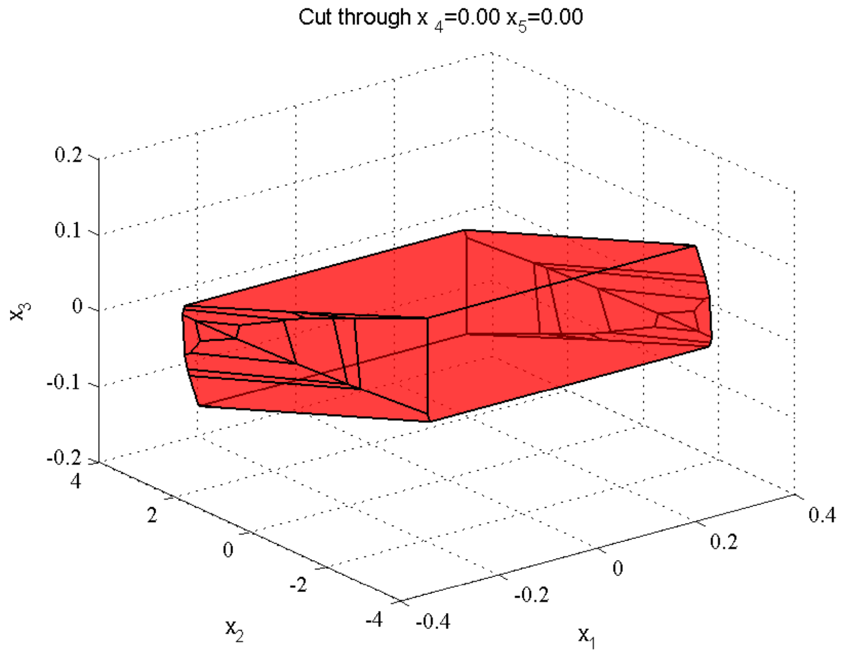


FIGURE 7.1: Maximal delayed-state admissible set, cutting through the coordinates $x_4 = 0$ and $x_5 = 0$.

where $x_k \in \mathbb{R}^n$ is the state vector at the time $k \in \mathbb{Z}_+$. $A_i \in \mathbb{R}^{n\times n}$, for all $i \in \mathbb{Z}_{[0,d]}$ and the initial conditions are given by a sequence $x_{-i} \in \mathbb{R}^n$, $i \in \mathbb{Z}_{[0,d]}$.

It is obvious that the discrete-time version of the input-delay systems is a special case of this class. However we would like to stress that the next results can be related to more general physical systems as long as several systems described by this equation can be found in practice. In [10], with suitable assumptions, the same type of dynamics can be related to the system:

$$\dot{x}(t) = \sum_{i=0}^{N} A_i x(t - T_i)$$

as the equation for a mixing tank of a dye from a central tank, as dyed water circulates through a number of pipes (discussed in Example 7.2). The same equation is considered by [169] for the circulation model for the distribution of ^{131}I-albumin in man.

Example 7.2. *The modeling example shown in Fig. 7.2, is studied by [10], where "A" is a tank of unity volume filled with water which circulates through three pipes with volumetric flow rates* A_1*,* A_2 *and* A_3*, in* $[cm^3/s]$*, and flow times* T_1*,* T_2 *and* T_3*, in* $[s]$ *(this is the time required for a particle to traverse the given pipe). At* $t = 0$*, the quantity of* $1g$ *of dye is introduced into the tank "A", and it is assumed that the dye in "A" is always mixed* *instantaneously. Thus the initial dye concentration in the tank is* unitary *and the initial dye concentration in the pipes is* zero. *Dyed water leaves the system at "C" through a leak pipe with volumetric flow rate "L" and pure water enters the system at "B", with the same flow rate. This is equivalent to assuming a pipe with infinite flow time and with flow rate "L". It is assumed that no mixing takes place in the pipes, that is, diffusion is neglected. The flow is assumed constant over a cross section of any pipe.*

If $x(t)$ denotes the dye concentration in the tank at time "t", then the concentration variation can be described by the differential-difference equation:

$$\dot{x}(t) = -(A_1 + A_2 + A_3 + L)x(t) + A_1 x(t - T_1) + A_2 x(t - T_2) + A_3 x(t - T_3),$$

with $t \in \mathbb{R}_+$. The initial condition describes the dye concentration in the tank and pipes at $t = 0$. For the system described above the appropriate initial conditions are:

$$x(t) = 0;\ -T_3 \leq t < 0;\ x(0) = 1.$$

If "N" pipes are considered rather than three, then the circulation model is:

$$\dot{x}(t) = \sum_{i=0}^{N} A_i x(t - T_i),$$

with initial conditions

$$x(t) = 0;\ -T_N \leq t < 0;\ x(0) = 1.$$

Example 7.3. *In [18], the equation:*

$$\frac{\partial f(x)}{\partial x} = -\sum_{i=1}^{N} A_i(x) f(x - i)$$

describes the transport theory of neutrons which are slowed-down in an infinite medium containing a mixture of "N" elements, where "x" is the relative lethargy of a neutron (the logarithmic difference of the neutron energy before and after a collision) and the matrices A_i are functions of the probability of scattering of a neutron which has the collision with the i^{th} element of the mixture. It is interesting to notice that this equation is not *a function of* time *and cannot be seen as a classical dynamic system.*

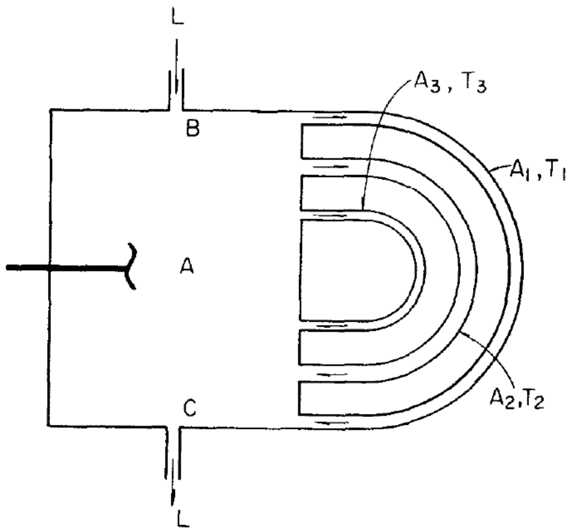


FIGURE 7.2: Circulation tank [10]

The $\mathcal{D}$-invariance and $\mathcal{D}$-contractiveness concepts [112], defined below, will be used extensively throughout the chapter. These definitions are important and useful to characterize *set invariance* and contractive sets for time-delay systems. $\mathcal{D}$-invariance can be understood as the set invariance in both the current and delayed states, as follows:

Definition 7.3 ($\mathcal{D}$-invariance and $\mathcal{D}$-contractiveness [112])**.** Let $\varepsilon \in \mathbb{R}_{[0,1]}$. A set $\mathcal{P} \subseteq \mathbb{R}^n$ containing the origin is called $\mathcal{D}$-contractive set with respect to (7.9) if for all $x_{k-i} \in \mathcal{P}$, for $i \in \mathbb{Z}_{[0,d]}$, we have $x_{k+i} \in \varepsilon\mathcal{P}$. When $\varepsilon = 1$, $\mathcal{P}$ is called a $\mathcal{D}$-invariant set with respect to (7.9) (see related definition (2.7)). □

Example 7.4. *An example is presented here in order to illustrate the theoretical results above. Consider a delay-difference equation, where $n = 2$ and $d = 1$, with the matrices:*

$$A_0 = \begin{bmatrix} 0.2 & -0.34 \\ 0.34 & 0.2 \end{bmatrix}; A_1 = \begin{bmatrix} 0.24 & -0.17 \\ 0.17 & 0.24 \end{bmatrix}.$$

Consider also the set $\mathcal{P}$ as the 1-norm unit circle in $\mathbb{R}^2$. Figure 7.3 shows the set iterations $A_0\mathcal{P}$ and $A_1\mathcal{P}$ and the set $\mathcal{P}$, respectively. As we can see, $\mathcal{P}$ is invariant with the individual dynamics, as $A_i\mathcal{P} \subset \mathcal{P}$, for $i = 0, 1$.

The Minkowski addition $A_0\mathcal{P} \oplus A_1\mathcal{P}$ is shown graphically in Fig. 7.4. This result is also $\mathcal{D}$-invariant, as $A_0\mathcal{P} \oplus A_1\mathcal{P} \subseteq \mathcal{P}$, as shown in in Fig. 7.4.

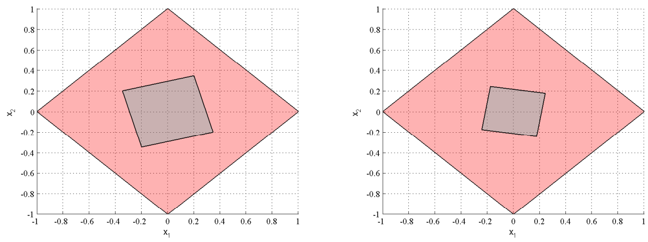


FIGURE 7.3: Set $\mathcal{P}$ and set iterates $A_0\mathcal{P}$ and $A_1\mathcal{P}$.

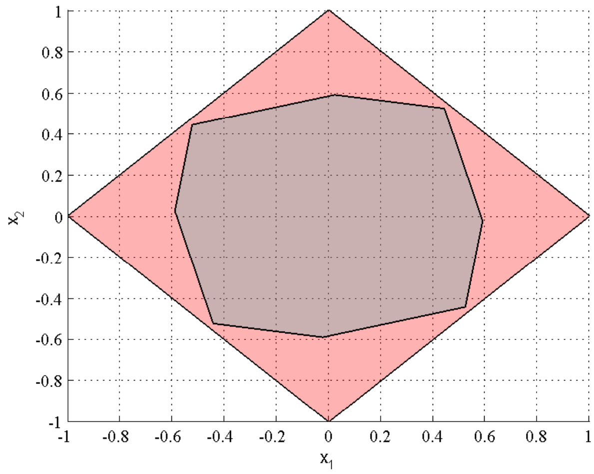


FIGURE 7.4: Set $\mathcal{P}$ and set iterate $A_0\mathcal{P} \oplus A_1\mathcal{P}$.

Example 7.5. *Another example is presented here, where the set $\mathcal{P} \subset \mathbb{R}^3$. The delay-difference equation has the matrices:*

$$A_0 = \begin{bmatrix} 0.1 & 0.23 & -0.03 \\ 0.14 & 0.22 & -0.07 \\ 0.01 & 0.07 & 0.01 \end{bmatrix};\ A_1 = \begin{bmatrix} 0.01 & 0.26 & -0.11 \\ 0.08 & 0.1257 & 0.10 \\ 0.05 & -0.02 & 0.06 \end{bmatrix}.$$

The matrix F and the vector w of the polyhedral set $\mathcal{P} = \{x \in \mathbb{R}^n | Fx \leq w\}$ are presented in the Appendix C, Section C.5. Figure 7.5 presents the Minkowski addition $A_0\mathcal{P} \oplus A_1\mathcal{P}$. As $A_0\mathcal{P} \oplus A_1\mathcal{P} \subset \mathcal{P}$, one can say that $\mathcal{P}$ is $\mathcal{D}$-contractive.

7.2.1 $\mathcal{D}$-invariance properties

By using the Definition 7.3, one can exploit some consequent properties [113], to be developed trough this section. The next theorem states the relationship between the

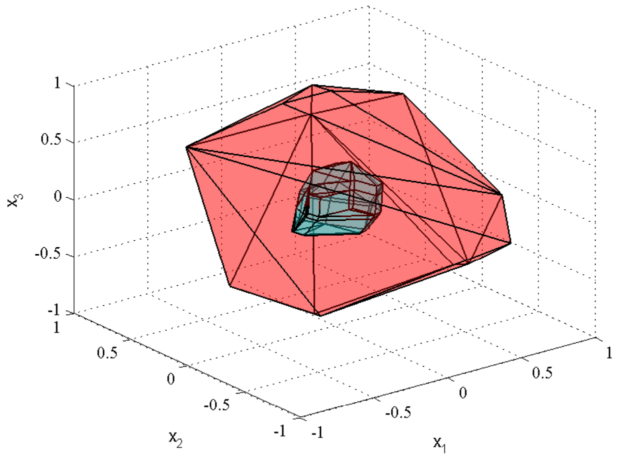


FIGURE 7.5: Set $\mathcal{P}$ and set iterate $A_0\mathcal{P} \oplus A_1\mathcal{P}$.

inclusion (ii) and $\mathcal{D}$-invariance.

Theorem 7.4. *The following affirmations are equivalent:*

i) A set $\mathcal{P} \subseteq \mathbb{R}^n$ is $\mathcal{D}$-invariant for system (7.9).

ii) $\bigoplus_{i=0}^{d} A_i\mathcal{P} \subseteq \mathcal{P}$.

Proof. i $\rightarrow$ ii: The $\mathcal{D}$-invariance of $\mathcal{P}$ implies that $x_1 \in \mathcal{P}$ for all the sequence $x_i \in \mathcal{P}$, for $i \in \mathbb{Z}_{[0,d]}$. It is equivalent to

$$\sum_{i=0}^{d} A_i x_{k-i} \in \mathcal{P},$$

for all $x_i \in \mathcal{P}$.

ii $\rightarrow$ i: For all $x_i \in \mathcal{P}$, with $i \in \mathbb{Z}_{[0,d]}$, the following holds:

$$x_1 = \sum_{i=0}^{d} A_i x_{k-i} \in \bigoplus_{i=0}^{d} A_i\mathcal{P} \subseteq P.$$

Then, by induction, $x_k \in \mathcal{P}$ for all $k \in \mathbb{Z}_+^*$. ■

Proposition 7.5. *Consider a convex set $\mathcal{P} \subseteq \mathbb{R}^n$ containing the origin. If $\mathcal{P}$ is $\mathcal{D}$-invariant with respect to (7.9) then $\mathcal{P}$ is positive invariant with respect to any LTI dynamics:*

$$x_{k+1} = A_i x_k, \quad \forall i \in \mathbb{Z}_{[0,d]}. \tag{7.10}$$

Equivalently:

$$A_0\mathcal{P} \subseteq \mathcal{P},\ A_1\mathcal{P} \subseteq \mathcal{P},\ \ldots,\ A_d\mathcal{P} \subseteq \mathcal{P}.$$

Proof. Using the fact that $\{0\} \in \mathcal{P}$ and the $\mathcal{D}$-invariance property with respect to (7.9) then for any $i \in \mathbb{Z}_{[0,d]}$ the following set inclusions hold:

$$A_i\mathcal{P} = \bigoplus_{l=0}^{i-1} A_l\{0\} \bigoplus A_i\mathcal{P} \bigoplus_{l=i+1}^{d} A_l\{0\} \subset \bigoplus_{l=0}^{d} A_l\mathcal{P} \subset \mathcal{P},$$

which corresponds to the definition of a positive invariant set $\mathcal{P}$ with respect to the dynamics in (7.10). ■

Proposition 7.6. *If $\mathcal{P} \in \mathbb{R}^n$ is $\mathcal{D}$-invariant with respect to (7.9) then $\alpha\mathcal{P}$ is $\mathcal{D}$-invariant with respect to the same dynamics for any $\alpha \in \mathbb{R}^*_+$.*

Proof. By using basic properties of the Minkowski addition (see [166]) for all $\mathcal{P} \subseteq ComC(\mathbb{R}^n)$ and Theorem 7.4 one obtains:

$$\bigoplus_{i=0}^{d} \alpha A_i\mathcal{P} = \alpha\left(\bigoplus_{i=0}^{d} A_i\mathcal{P}\right) \subseteq \alpha\mathcal{P}.$$

■

Proposition 7.7. *Let $\mathcal{P}_1, \mathcal{P}_2 \subseteq \mathbb{R}^n$ be two $\mathcal{D}$-invariant sets for the dynamics (7.9). Then $\mathcal{P}_1 \cap \mathcal{P}_2$ is a $\mathcal{D}$-invariant set for the same dynamical system.*

Proof. Consider a sequence of points $x_i \in \mathbb{R}^n$ for $i \in \mathbb{Z}_{[0,d]}$ such that $x_1 \in \mathcal{P}_1 \cap \mathcal{P}_2$, $x_2 \in \mathcal{P}_1 \cap \mathcal{P}_2$, $\ldots$, $x_d \in \mathcal{P}_1 \cap \mathcal{P}_2$. The set $\mathcal{P}_1$ is $\mathcal{D}$-invariant and $x_i \in \mathcal{P}_1$, for $i \in \mathbb{Z}_{[0,d]}$, thus:

$$\sum_{i=0}^{d} A_i x_i \in \mathcal{P}_1.$$

Similarly $\mathcal{P}_2$ is $\mathcal{D}$-invariant and $x_i \in \mathcal{P}_1$, for $i \in \mathbb{Z}_{[0,d]}$, imply:

$$\sum_{i=0}^{d} A_i x_i \in \mathcal{P}_2$$

and thus observing that x_i were chosen arbitrarily:

$$\sum_{i=0}^{d} A_i x_i \in \mathcal{P}_1 \cap \mathcal{P}_2,$$

that completes the proof. ■

Proposition 7.8. *Given a $\mathcal{D}$-invariant set $\mathcal{P} \in \mathbb{R}^n$ with respect to the system (7.9), then $\mathcal{P}$ is $\mathcal{D}$-invariant for:*

$$x_{k+1} = A_d x_k + \cdots + A_0 x_{k-d}. \tag{7.11}$$

Proof. This represents a direct implication of Theorem 7.4 since the set theoretic description of the $\mathcal{D}$-invariance with respect to (7.9) and (7.11) is identical due to commutativity of the Minkowski addition. ■

Remark 7.9. The results in Proposition 7.8 can be generalized to any permutation of the pair of matrices $(A_0, \ldots, A_d)$ over the delay interval.

Proposition 7.10. *For some $d \in \mathbb{Z}_{>0}$, consider a $\mathcal{D}$-invariant set $\mathcal{P} \in \mathbb{R}^n$ for the system (7.9). Then $\mathcal{P}$ is $\mathcal{D}$-invariant for (7.9) for any $d \in \mathbb{Z}_{>0}$.*

Proof. The set $\mathcal{P}$ is $\mathcal{D}$-invariant for the system (7.9) with $d = 1$ if $A_0\mathcal{P} \oplus A_1\mathcal{P} \subset \mathcal{P}$. When $d = 2$ the set $\mathcal{P}$ is $\mathcal{D}$-invariant if $A_0\mathcal{P} \oplus A_1\mathcal{P} \oplus A_2\mathcal{P} \subset \mathcal{P}$ and by induction, the set $\mathcal{P}$ is $\mathcal{D}$-invariant if $\bigoplus_{i=0}^{d} A_i\mathcal{P} \subset \mathcal{P}$ for all $d \in \mathbb{Z}_{>0}$. ■

The properties 7.8 and 7.10 underline the fact that $\mathcal{D}$-invariance is related to the algebraic properties of A_i, for $i \in \mathbb{Z}_{[0,d]}$, and thus is delay-independent from a structural point of view. The effective link with the specificity of dynamical systems affected by delays is resumed by the conditions $x_{-i} \in \mathcal{P}, i \in \mathbb{Z}_{[1,d]}$ (with their dimensional and complexity implications).

7.2.2 On Necessary and Sufficient Conditions for $\mathcal{D}$-invariance

It is obvious that the existence of a non-degenerated and bounded $\mathcal{D}$-invariant set[1] is related to the stability of the discrete-time dynamical system affected by delay (7.9).

[1]It is easy to observe that sets like $\{0\}$ or $\mathbb{R}^n$ are $\mathcal{D}$-invariant but they do not satisfy the non-degenerate or boundedness conditions.

Let us introduce the following notation for the extended state-space matrix:

$$A_\xi = \begin{bmatrix} A_0 & A_1 & \dots & A_{d-1} & A_d \\ I & 0 & \dots & 0 & 0 \\ 0 & I & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & I & 0 \end{bmatrix} \tag{7.12}$$

7.2.2.1 Necessary conditions

Considering the system (7.9), the existence of a $\mathcal{D}$-invariant $\mathcal{C}$-set P implies:

N1. The spectral radii of the matrices A_i for $i \in \mathbb{Z}_{[0,d]}$ are sub-unitary: $\rho(A_i) \leq 1,\ \forall i \in \mathbb{Z}_{[0,d]}$;

N2. The spectral radius of the matrix $\rho\left(\sum_{i=0}^{d} A_i\right)$ is sub-unitary: $\rho\left(\sum_{i=0}^{d} A_i\right) \leq 1$;

N3. The spectral radius of the extended state-space matrix is subunitary:

$$\rho\left(A_\xi\right) \leq 1.$$

In order to support our claims we provide here the sketch of proof for the case $x_{k+1} = A_0 x_k + A_d x_{k-d}$. The general forms 1, 2 and 3 follow similarly and we omit them here for the sake of brevity.

Proof. Before entering into the details of each case we recall that a set $P \subset \mathbb{R}^n$ is positive invariant for a linear time-invariant dynamic $x_{k+1} = Ax_k$ only if $\rho(A) \leq 1$.

N1. Let an arbitrary vector $x_{k-i} \in P$ and $x_{k-j} = 0$ for $i \neq j$ with $i, j \in \mathbb{Z}_{[0,d]}$. For this specific choice of initial conditions, the equation (7.9) becomes:

$$x_{k+1} = A_i x_{k-i} \tag{7.13}$$

The fact that x_{k-i} is chosen arbitrarily in P assures that $x_{k+1} \in P$ and thus $\mathcal{D}$*-invariance of* P *w.r.t.* (7.9) *implies the positive invariance of* P *with respect to* (7.13).

In the same time

the positive invariance of P *with respect to* (7.13) $\Rightarrow \rho(A_i) \leq 1$

and the necessity of the condition on the spectral radii of the matrices A_0 and A_d is proved.

N2. Let an arbitrary vector $x_{k-i} \in P$ and $x_{k-j} = x_{k-i}$ for $j \neq i$ with $i, j \in \mathbb{Z}_{[0,d]}$. For this specific choice of initial conditions, the equation (7.9) becomes:

$$x_{k+1} = (\sum_{j=0}^{d} A_j) x_{k-i} \tag{7.14}$$

The fact that x_{k-i} is chosen arbitrarily in P assures that $x_{k+1} \in P$ and thus $\mathcal{D}$-*invariance of* P *with respect to* (7.9) *implies the positive invariance of* P *with respect to* (7.14).

In the same time *positive invariance of* P *with respect to* (7.14) $\Rightarrow \rho(A_i) \leq 1$

and the necessity of the condition on the spectral radius of the matrix $\sum_{i=0}^{d} A_i$ is proved.

N3. Let the extended state vector:

$$\xi_k^\top = \left[x_k^\top, \ldots, x_{k-d}^\top \right]$$

and the associated dynamics

$$\xi_{k+1} = A_\xi \xi_k \tag{7.15}$$

Using the set $P_\xi = \underbrace{P \times \cdots \times P}_{d+1 \text{ times}}$ we have the implications:

$\mathcal{D}$-*invariance of* P *with respect to* (7.9) $\Rightarrow$ *pos. inv. of* P_ξ *with respect to* (7.15).

In the same time

the positive invariance of P_ξ *with respect to* (7.15) *implies* $\rho(A_\xi) \leq 1$ and the necessity of the condition on the spectral radius of the extended state matrix is proved.

■

7.2.2.2 Sufficient conditions

Considering the system (7.9), the existence of a bounded, non-degenerate $\mathcal{D}$-invariant is guaranteed by the satisfaction of one of the following conditions:

S1. The sum of the largest singular values of A_i, for $i \in \mathbb{Z}_{[0,d]}$, is sub-unitary:

$$\sum_{i=0}^{d} (\bar{s}_n(A_i)) < 1,$$

where $\bar{s}_n(A_i)$ is the largest singular value of A_i, for $i \in \mathbb{Z}_{[0,d]}$.

S2. In the case of nonsingular matrix A_0 (or A_i, for $i \in \mathbb{Z}_{[1,d]}$)

$$\begin{gathered}(1 + \sigma(A_0^{-1}A_1) + \cdots + \sigma(A_0^{-1}A_d))\sigma(A_0) \leq 1,\\ \vdots \\ (1 + \sigma(A_d^{-1}A_0) + \cdots + \sigma(A_d^{-1}A_{d-1}))\sigma(A_d) \leq 1.\end{gathered}$$

Proof. S1. Consider the candidate set $\mathbb{B}^n_{0,r}$ for some positive $r \in \mathbb{R}$. By applying the linear transformations A_0 and A_d one can obtain:

$$A_0\mathbb{B}^n_{0,r} \subset \sigma(A_0)\mathbb{B}^n_{0,r} \tag{7.16}$$

$$\vdots$$

$$A_d\mathbb{B}^n_{0,r} \subset \sigma(A_d)\mathbb{B}^n_{0,r} \tag{7.17}$$

Using properties of the Minkowski addition:

$$\mathbb{B}^n_{0,r} \supset (\sum_{i=0}^{d} \sigma(A_i))\mathbb{B}^n_{0,r} \supset$$

$$\sigma(A_0)\mathbb{B}^n_{0,r} \oplus \cdots \oplus \sigma(A_d)\mathbb{B}^n_{0,r} \supset A_0\mathbb{B}^n_{0,r} \oplus \cdots \oplus A_d\mathbb{B}^n_{0,r} \tag{7.18}$$

the $\mathcal{D}-$invariance of the candidate set is proved.

S2. The dynamics (7.9) can be equivalently expressed (under the hypothesis of non-singular matrices A_0, A_d) as:

$$x_{k+1} = A_0 x_k + A_d x_{k-d} \tag{7.19}$$
$$= A_0(x(k) + A_0^{-1} A_d x_{k-d}) \tag{7.20}$$
$$= A_d(A_d^{-1} A_0 x_k + x_{k-d}). \tag{7.21}$$

This leads by exploiting (7.20) to:

$$\mathbb{B}^n_{0,r} \supset \sigma(A_0)(1 + \sigma(A_0^{-1} A_d))\mathbb{B}^n_{0,r} \supset$$

$$A_0(\mathbb{B}^n_{0,r} \oplus (A_0^{-1} A_d)\mathbb{B}^n_{0,r}) \supset A_0\mathbb{B}^n_{0,r} \oplus A_d\mathbb{B}^n_{0,r} \tag{7.22}$$

to the proof of $\mathcal{D}-$invariance for $\mathbb{B}^n_{0,r}$. The same inference can be made starting from (7.21) and the proof is complete.

■

7.2.3 $\mathcal{D}$-invariance algebraic conditions for Polyhedral Sets

Lemma 7.11 presents the algebraic existence conditions of the matrices H_i, i.e. the condition (7.23) has a non-empty set of solutions. But this is not enough to establish invariance. As it will be stated in Theorem 7.12, matrices H_i with non-negative elements are required, within the satisfaction of conditions (7.26) and (7.27).

Lemma 7.11. *Let $F \in \mathbb{R}^{r\times n}$, $r \geq n$ and* $\text{rank}(F) \leq n$. *If $Fx = 0$ implies $FA_i x = 0$ for all $i \in \mathbb{Z}_{[0,d]}$ then there exist $H_i \in \mathbb{R}^{r\times r}$ such that:*

$$FA_i = H_i F. \tag{7.23}$$

Proof. Since $Fx = 0$ implies $FA_i x = 0$ for $i \in \mathbb{Z}_{[0,d]}$ the row vectors $\{f_{j,[1,n]}\}$ and $\{(FA_i)_{j,[1,n]}\}$ of the matrices F and FA_i belong to the same $rank(F)$-dimensional subspace of $\mathbb{R}^n$, that is, orthogonal to the $(n-rank(F))$-dimensional null space of the matrix F. As the $rank(F) \leq n$, from the rows of F one can obtain a basis for this subspace. Therefore, for any index $j \in \mathbb{Z}_{[1,m]}$ and $i \in \mathbb{Z}_{[0,d]}$ there exist $\left\{h^i_{j,1}\right\}, \left\{h^i_{j,2}\right\}, \dots, \left\{h^i_{j,r}\right\}$

such that

$$\begin{aligned}\left\{(FA_i)_{j,[1,n]}\right\} = \left\{h^i_{j1}\right\}\left\{f_{1,[1,n]}\right\} + \left\{h^i_{j2}\right\}\left\{f_{2,[1,n]}\right\} + \dots \\ + \left\{h^i_{jn}\right\}\left\{f_{m,[1,n]}\right\},\end{aligned} \tag{7.24}$$

which leads to the matrix $H_i = \{h^i_{j,k}\}$, $H_i \in \mathbb{R}^{r\times r}$ in (7.23). ■

The next theorem provides the algebraic $\mathcal{D}$-invariance conditions of a polyhedral set for a given time-delay system (7.9).

Theorem 7.12 ([76] Algebraic conditions for $\mathcal{D}$-invariance)**.** *Let $\mathcal{P}$ be a polyhedral set in $\mathbb{R}^n$ containing the origin in its interior, i.e. there exists a $F \in \mathbb{R}^{r\times n}$ such that:*

$$\mathcal{P} = \{x \in \mathbb{R}^n \mid Fx \leq \mathbb{1}\} \tag{7.25}$$

is $\mathcal{D}$-contractive with respect to the system (7.9) if and only if there exist the matrices $H_i \in \mathbb{R}^{r\times r}$ for $i \in \mathbb{Z}_{[0,d]}$ with non-negative elements such that:

$$FA_i = H_iF \tag{7.26}$$

and

$$\left(\sum_{i=0}^{d} H_i\right)\mathbb{1} \leq \varepsilon\mathbb{1}. \tag{7.27}$$

When $\varepsilon = 1$, $\mathcal{P}$ is called a $\mathcal{D}$-invariant set.

Remark 7.13. For symmetrical polyhedral sets containing the origin in their interior, $\mathcal{P}$ is defined as:

$$\mathcal{P} = \{x \in \mathbb{R}^n \mid Fx \leq \mathbb{1},\ -Fx \leq \mathbb{1}\}. \tag{7.28}$$

Condition (7.26) holds and condition (7.27) becomes:

$$\left(\sum_{i=0}^{d} |H_i|\right)\mathbb{1} \leq \varepsilon\mathbb{1}. \tag{7.29}$$

Proof. **(of the Theorem 7.12).** Consider the Lyapunov-Razumikhin candidate:

$$\tilde{V}(\mathbf{x}_k) = \max_{i\in\mathbb{Z}_{[0,d]}} \{V(x_{k-i})\}, \tag{7.30}$$

with $V : \mathbb{R}^n \to \mathbb{R}_+$ defined as a Minkowski function for the set $\mathcal{P}$ i.e.:

$$V(x_k) = \left\| \max\{Fx, 0\} \right\|_\infty , \tag{7.31}$$

where the maximum is taken componentwise.

Note that $\tilde{V}(\mathbf{x}_k) \leq \rho$, with $\rho \in \mathbb{R}_+$ implies $V(x_{k-i}) \leq \rho$ for all $i \in \mathbb{Z}_{[0,d]}$ and consequently $x_{k-i} \in \rho\mathcal{P}$ for all $i \in \mathbb{Z}_{[0,d]}$.

Sufficient conditions: One way to prove the sufficiency is to show that $\Delta\tilde{V} \leq 0$, which qualifies $\tilde{V}$ as a weak Lyapunov-Razumikhin function for the system (7.9) and implicitly proves the $\mathcal{D}$-invariance of its sublevel sets with respect to the same dynamics.

For the difference $\Delta\tilde{V}$ one can write explicitly:

$$\Delta\tilde{V} = \tilde{V}(\mathbf{x}_{k+1}) - \varepsilon\tilde{V}(\mathbf{x}_k) = \max_{i_1 \in \mathbb{Z}_{[-1,d-1]}} \{V(x_{k-i_1})\} - \varepsilon \max_{i_2 \in \mathbb{Z}_{[0,d]}} \{V(x_{k-i_2})\} . \tag{7.32}$$

If the optimal argument i_1^* for the first maximization problem in (7.32) satisfies $i_1^* \in \mathbb{Z}_{[0,d-1]}$ then the inequality $\Delta\tilde{V} \leq 0$ is satisfied trivially as a consequence of the inclusion relation between the feasible domains:

$$\mathbb{Z}_{[0,d-1]} \subset \mathbb{Z}_{[0,d]}.$$

The case that deserves attention is $i_1^* = -1$ and $i_2^* \in \mathbb{Z}_{[0,d-1]}$ for which:

$$V(x_{k+1}) - \varepsilon \max_{i \in \mathbb{Z}_{[0,d]}} \{V(x_{k-i})\} = \left\| \max\left\{ F \sum_{i=0}^{d} A_i x_{k-i}, 0 \right\} \right\|_\infty - \varepsilon\tilde{V}(\mathbf{x}_k) =$$
$$\left\| \max\left\{ \sum_{i=0}^{d} F A_i x_{k-i}, 0 \right\} \right\|_\infty - \varepsilon\tilde{V}(\mathbf{x}_k) \leq \left\| \sum_{i=0}^{d} \max\{F A_i x_{k-i}, 0\} \right\|_\infty - \varepsilon\tilde{V}(\mathbf{x}_k).$$

Using (7.26), one obtains:

$$\left\| \sum_{i=0}^{d} \max\{F A_i x_{k-i}, 0\} \right\|_\infty - \varepsilon\tilde{V}(\mathbf{x}_k) = \left\| \sum_{i=0}^{d} \max\{H_i F x_{k-i}, 0\} \right\|_\infty - \varepsilon\tilde{V}(\mathbf{x}_k)$$
$$\leq \left\| \sum_{i=0}^{d} H_i \max\{F x_{k-i}, 0\} \right\|_\infty - \varepsilon\tilde{V}(\mathbf{x}_k),$$

which is equal to:

$$\left\|\sum_{i=0}^{d} H_i \max\{Fx_{k-i},0\}\right\|_{\infty} - \varepsilon \max_{i\in\mathbb{Z}_{[0,d]}} \{\|\max\{Fx_{k-i},0\}\|_{\infty}\} = \left\|\sum_{i=0}^{d} H_i \max\{Fx_{k-i},0\}\right\|_{\infty} - \varepsilon\rho \leq \left\|\sum_{i=0}^{d} H_i \rho \mathbb{1}\right\|_{\infty} - \varepsilon\rho \leq 0,$$

where the last inequality is a direct consequence of (7.27).

Necessary conditions: If $\mathcal{P}$ is $\mathcal{D}$-invariant with respect to (7.9) then using Proposition 7.5 one can exploit invariance with respect to the linear dynamics $x_{k+1} = A_i x_k$ for any $i \in \mathbb{Z}_{[0,d]}$. This is equivalent to the existence of matrices U_i with non-negative elements satisfying:

$$FA_i = U_i F. \tag{7.33}$$

The $\mathcal{D}$-invariance implies weak Lyapunov-Razumikhin stability with $\Delta\tilde{V} \leq 0$. Then for all $x_{k-i} \in \mathcal{P}$ for $i \in \mathbb{Z}_{[0,d]}$:

$$\max_{i_1\in\mathbb{Z}_{[-1,d-1]}} \{V(x_{k-i_1})\} - \max_{i_2\in\mathbb{Z}_{[0,d]}} \{V(x_{k-i_2})\} \leq 0. \tag{7.34}$$

This guarantees that:

$$V(x_{k+1}) - \max_{i\in\mathbb{Z}_{[0,d]}} \{V(x_{k-i})\} \leq 0, \tag{7.35}$$

which is equivalent to:

$$\|\max\{Fx_{k+1},0\}\|_{\infty} - \max_{i\in\mathbb{Z}_{[0,d]}} \{\|\max\{Fx_{k-i},0\}\|_{\infty}\} \leq 0. \tag{7.36}$$

By substituting x_{k+1} according to the system dynamics in (7.9), one obtains:

$$\left\|\max\left\{F\sum_{i=0}^{d} A_i x_{k-i},0\right\}\right\|_{\infty} - \max_{i\in\mathbb{Z}_{[0,d]}} \{\|\max\{Fx_{k-i},0\}\|_{\infty}\} = \left\|\max\left\{\sum_{i=0}^{d} FA_i x_{k-i},0\right\}\right\|_{\infty} - \max_{i\in\mathbb{Z}_{[0,d]}} \{\|\max\{Fx_{k-i},0\}\|_{\infty}\} \leq 0.$$

One can exploit the relation (7.33) and rewrite this inequalities in terms of $U_i, i \in \mathbb{Z}_{[0,d]}$:

$$\left\|\max\left\{\sum_{i=0}^{d} U_i F x_{k-i},0\right\}\right\|_{\infty} \leq \max_{i\in\mathbb{Z}_{[0,d]}} \{\|\max\{Fx_{k-i},0\}\|_{\infty}\}. \tag{7.37}$$

By considering the upper and lower bounds, one obtains:

$$\left\| \sum_{i=0}^{d} U_i F x_{k-i} \right\|_\infty \leq \left\| \max \left\{ \sum_{i=0}^{d} U_i F x_{k-i}, 0 \right\} \right\|_\infty \leq 1. \tag{7.38}$$

By considering each row of U_i independently, $j \in \mathbb{Z}_{[1,r]}$:

$$\sum_{i=0}^{d} \left\{ u^i_{j,[1,r]} \right\} F x_{k-i} \leq 1, \tag{7.39}$$

where $\left\{ u^i_{j,k} \right\}$ is the j^{th} row and k^{th} column of U_i, for $j, k \in \mathbb{Z}_{[1,r]}$. This inequalities hold for all $x_{k-i} \in \mathcal{P}$, $i \in \mathbb{Z}_{[0,d]}$ and by consequence hold also for the result of the optimization problem:

$$\begin{array}{ll} \max\limits_{x_{k-i}, \forall i \in \mathbb{Z}_{[0,d]}} & \sum\limits_{i=0}^{d} \left\{ u^i_{j,[1,r]} \right\} F x_{k-i} \\ \text{subject to:} & F x_{k-i} \leq \mathbb{1} \end{array}$$

as well as for its dual:

$$\begin{array}{ll} \min\limits_{\left\{ h^i_{j,[1,r]} \right\}, \forall i \in \mathbb{Z}_{[0,d]}} & \sum\limits_{i=0}^{d} \left\{ h^i_{j,[1,r]} \right\} \mathbb{1} \\ \text{subject to:} & \left\{ \begin{array}{l} \left\{ h^i_{j,[1,r]} \right\} F = \left\{ u^i_{j,[1,r]} \right\} F \\ \left\{ h^i_{j,l} \right\} \geq 0, \forall l \in \mathbb{Z}_{[1,r]}, \forall i \in \mathbb{Z}_{[0,d]}. \end{array} \right. \end{array} \tag{7.40}$$

Putting together the optimal arguments for the LP problems (7.40) the matrices $H_i^*, i \in \mathbb{Z}_{[0,d]}$ are obtained and one concludes that:

$$\begin{aligned} \sum_{i=0}^{d} H_i^* \mathbb{1} &\leq \mathbb{1}; \\ H_i^* F &= U_i F; \\ H_i^* &\geq 0, \forall i \in \mathbb{Z}_{[0,d]} \end{aligned}$$

and finally, using (7.33):

$$\begin{aligned} \sum_{i=0}^{d} H_i^* \mathbb{1} &\leq \mathbb{1}; \\ F A_i &= H_i^* F; \\ H_i^* &\geq 0, \forall i \in \mathbb{Z}_{[0,d]}. \end{aligned}$$

■

Remark 7.14. The proof of Proposition 7.12 is stated for a general polyhedral set $\mathcal{P}$ containing the origin in its interior. The result holds similarly for a symmetric polyhedral set containing the origin in its interior if:

$$V(x_k) = \|Fx\|_\infty . \quad (7.41)$$

Remark 7.15. Similar results have been obtained in the context of positive invariance of polyhedral sets with respect to multivariable discrete-time systems described by ARMA models in [189]. Note also that the proof of Proposition 7.12 can be derived alternatively by exploiting the extended Farka's Lemma (see [76]).

Theorem 7.12 was presented in [112]. In the meantime we become aware that a similar result proved in [76] based on the Farka's Lemma. We produce here our original proof as long as it provides an interesting insight on the relationship between $\mathcal{D}$-contractiveness and Lyapunov-Razumikhin stability proofs.

7.2.4 $\mathcal{D}$-Invariance verification methods

An important aspect is the possibility of testing the $\mathcal{D}$-invariance of a set with respect to a given delay-difference equation (7.9). In this section two types of tests are presented: *Minkowski addition based* tests and *feasibility-based* tests.

7.2.4.1 Minkowski addition based methods

The tests based on the Minkowski addition are a direct application of the relation (ii). So if the set dynamics $\bigoplus_{i=0}^{d} A_i\mathcal{P}$, for $A_i \in \mathbb{R}^{n\times n}$, is an inclusion of $\mathcal{P} \subseteq \mathbb{R}^n$, the set is called $\mathcal{D}$-invariant for the dynamics. The MatLab MPT-Multiparametric Toolbox [97] provides useful tools to calculate the Minkowski addition of polytopes.

7.2.4.2 Feasibility-based $\mathcal{D}$-Invariance verification

The approach presented in Section 7.2.4.1 has the inconvenience of being computationally expensive [112], as the Minkowski addition for high dimensional polytopic sets is a non-trivial operation, usually based on the vertex enumeration and the limitation of application to a restrict class of systems. To avoid this inconvenient, one should keep the polyhedral operations in terms of the half-space representation. One way of posing this problem is by stating a feasibility problem.

Theorem 7.16. *The polyhedral set (7.25), described by* $\mathcal{P} = \{x \in \mathbb{R}^n \mid Fx \leq w\}$ *is* $\mathcal{D}$*-invariant with respect to the system (7.9) if there exists the vector* $\bar{h} \in \mathbb{R}^{dr^2}$ *with nonnegative elements obtained as the feasible solution of:*

$$\min_{\bar{h}} \quad \varepsilon$$
$$\textit{subject to:} \quad \begin{cases} A_{eq}\bar{h} = b_{eq} \\ A_{in}\bar{h} \leq b_{in} \\ \bar{h} \geq 0 \\ 0 \leq \varepsilon \leq 1 \end{cases} \tag{7.42}$$

where $A_{eq} \in \mathbb{R}^{drn \times dr^2}$ *and* $b_{eq} \in \mathbb{R}^{drn}$ *are done by:*

$$A_{eq} = \mathrm{diag}(F^\top, r \times n \times d);$$
$$b_{eq}^\top = \begin{bmatrix} \{\bar{f}_{1,1}^1\} & \{\bar{f}_{2,1}^1\} & \cdots & \{\bar{f}_{r,n}^1\} & \{\bar{f}_{1,1}^2\} & \cdots & \cdots & \{\bar{f}_{r,n}^d\} \end{bmatrix}^\top,$$

where $\bar{f}_{pq}^i$ *are the elements of* FA_i*, for* $p \in \mathbb{Z}_{[1,r]}$*,* $q \in \mathbb{Z}_{[1,n]}$ *and* $i \in \mathbb{Z}_{[0,d]}$*, with* $F \in \mathbb{R}^{r \times n}$*. The matrices* $A_{in} \in \mathbb{R}^{dr^2+r \times dr^2}$ *and* $b_{in} \in \mathbb{R}^{dr^2+r}$ *are done by:*

$$A_{in} = \begin{bmatrix} \mathrm{diag}(w^\top, r) & \cdots & \mathrm{diag}(w^\top, r) \\ -1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & -1 \end{bmatrix}; \quad b_{in} = \begin{bmatrix} w \\ 0 \\ \vdots \\ 0 \end{bmatrix};$$

and the vector $\bar{h} \in \mathbb{R}^{dr^2}$ *is done by:*

$$\bar{h}^\top = \begin{bmatrix} \{h_{1,1}^1\} & \{h_{1,2}^1\} & \cdots & \{h_{1,r}^1\} & \{h_{2,1}^2\} & \cdots & \cdots & \{h_{r,r}^d\} \end{bmatrix}^\top,$$

where h_{pz}^i *are the elements of* H_i*. See the construction details in Appendix C.*

If $\varepsilon < 1$ *the set* $\mathcal{P}$ *is* $\mathcal{D}$*-contractive.*

Proof. The Theorem 7.12 proves that $\mathcal{P}$ is $\mathcal{D}$-invariant with respect to (7.9). Appendix C provides the construction of the matrices $A_{eq} \in \mathbb{R}^{drn \times dr^2}$, $b_{eq} \in \mathbb{R}^{drn}$, $A_{in} \in \mathbb{R}^{dr^2+r \times dr^2}$

and the vectors $b_{in} \in \mathbb{R}^{dr^2+r}$ and $\bar{h} \in \mathbb{R}^{dr^2}$. ■

7.2.4.3 Duality-based feasibility verification

Within this approach, the $\mathcal{D}$-invariance is verified by using the same ideas as in [192], using the dual of (7.26) and (7.27), where the $\mathcal{D}$-invariance of $\mathcal{P}$ can be verified by using the dual of (7.25):

$$\begin{aligned} \mathcal{P} &= \left\{ x \in \mathbb{R}^n | \left\{ f_{j,[1,n]} \right\} x \leq 1,\ j \in \mathbb{Z}_{[1,r]},\ \forall f \in \tilde{\mathcal{P}} \right\}, \\ \tilde{\mathcal{P}} &= \left\{ f_j \in \mathbb{R}^n | \left\{ f_{j,[1,n]} \right\} x \leq 1,\ j \in \mathbb{Z}_{[1,r]},\ \forall x \in \mathcal{P} \right\}, \end{aligned} \tag{7.43}$$

where $\left\{ f_{j,[1,n]} \right\}$ denotes the j^{th} row of the matrix F.

To verify the conditions above, the following theorem can be stated:

Corollary 7.17. *The polyhedral set* (7.25) *is $\mathcal{D}$-invariant with respect to the system* (7.9) *if there exists the vector $\tilde{h} \in \mathbb{R}^{ndr}$ with nonnegative elements obtained as the feasible solution of:*

$$\begin{aligned} &\min_{\tilde{h}} \quad \varepsilon \\ &\textit{subject to:} \quad \begin{cases} \tilde{A}_{eq}\tilde{h} = \tilde{b}_{eq} \\ 0 \leq \varepsilon \leq 1 \\ \tilde{h} \geq 0 \end{cases} \end{aligned} \tag{7.44}$$

where $\tilde{A}_{eq} \in \mathbb{R}^{drn \times dr^2+1}$, $\tilde{b}_{eq} \in \mathbb{R}^{drn}$ are done by:

$$\underbrace{\begin{bmatrix} \Lambda & & & \\ & \Lambda & & \\ & & \ddots & \\ & & & \Lambda \end{bmatrix}}_{\tilde{A}_{eq}} \underbrace{\begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_r \end{bmatrix}}_{\tilde{h}} = \underbrace{\begin{bmatrix} \Gamma_1 \\ \Gamma_2 \\ \vdots \\ \Gamma_r \end{bmatrix}}_{\tilde{b}_{eq}};$$

and

$$\underbrace{\begin{bmatrix} F^\top & & & \\ & F^\top & & \\ & & \ddots & \\ & & & F^\top \\ \mathbb{1} & \mathbb{1} & \dots & \mathbb{1} \end{bmatrix}}_{\Lambda} \underbrace{\begin{bmatrix} \{y_j^1\} \\ \{y_j^2\} \\ \vdots \\ \{y_j^d\} \end{bmatrix}}_{Y_j} = \underbrace{\begin{bmatrix} A_1^\top \{f_{j,[1,n]}\}^\top \\ A_2^\top \{f_{j,[1,n]}\}^\top \\ \vdots \\ A_d^\top \{f_{j,[1,n]}\}^\top \\ \varepsilon \end{bmatrix}}_{\Gamma_j},$$

where

$$\left\{y^i_{j,[1,r]}\right\} = \left[\left\{h^i_{j,1}\right\} \quad \left\{h^i_{j,2}\right\} \quad \dots \quad \left\{h^i_{j,r}\right\}\right],$$

for $i \in \mathbb{Z}_{[0,d]}$ *and* $j \in \mathbb{Z}_{[1,r]}$.

If $\varepsilon < 1$ *the set* $\mathcal{P}$ *is* $\mathcal{D}$*-contractive.*

Proof. By using (7.43) one can write (ii) in its dual form:

$$\bigoplus_{i=0}^{d} A_i^\top \tilde{\mathcal{P}} \subseteq \varepsilon \tilde{\mathcal{P}}. \tag{7.45}$$

By the condition (7.25), the Eqs. (7.26) can be written as:

$$\begin{aligned} \sum_{i=0}^{d} A_i^\top \{f_{j,[1,n]}\}^\top &= \sum_{k=1}^{r} \left\{h^i_{j,k}\right\} \{f_{k,[1,n]}\}^\top,\ j \in \mathbb{Z}_{[1,r]}; \\ \sum_{k=1}^{r} \left\{h^i_{j,k}\right\} &= \varepsilon; \\ \left\{h^i_{j,k}\right\} &\geq 0, \end{aligned} \tag{7.46}$$

where $\{f_{j,k}\}$ means the j^{th} row and the k^{th} column of the matrix F. By denoting

$$\left\{y^i_{j,[1,r]}\right\} = \left[\left\{h^i_{j,1}\right\} \quad \left\{h^i_{j,2}\right\} \quad \dots \quad \left\{h^i_{j,r}\right\}\right],$$

for $i \in \mathbb{Z}_{[0,d]}$ and $j \in \mathbb{Z}_{[1,r]}$, the following hold:

$$\underbrace{\begin{bmatrix} F^\top & & & \\ & F^\top & & \\ & & \ddots & \\ & & & F^\top \\ \mathbb{1} & \mathbb{1} & \dots & \mathbb{1} \end{bmatrix}}_{\Lambda} \underbrace{\begin{bmatrix} \{y_j^1\} \\ \{y_j^2\} \\ \vdots \\ \{y_j^d\} \end{bmatrix}}_{Y_j} = \underbrace{\begin{bmatrix} A_1^\top \{f_{j,[1,n]}\}^\top \\ A_2^\top \{f_{j,[1,n]}\}^\top \\ \vdots \\ A_d^\top \{f_{j,[1,n]}\}^\top \\ \varepsilon \end{bmatrix}}_{\Gamma_j}, \tag{7.47}$$

for $j \in \mathbb{Z}_{[1,r]}$. By expanding the j terms:

$$\underbrace{\begin{bmatrix} \Lambda & & & \\ & \Lambda & & \\ & & \ddots & \\ & & & \Lambda \end{bmatrix}}_{\tilde{A}_{eq}} \underbrace{\begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_r \end{bmatrix}}_{\tilde{h}} = \underbrace{\begin{bmatrix} \Gamma_1 \\ \Gamma_2 \\ \vdots \\ \Gamma_r \end{bmatrix}}_{\tilde{b}_{eq}}; \qquad \tilde{h} \geq 0. \tag{7.48}$$

■

The computational aspects concerning the $\mathcal{D}$-invariance tests are presented in [112] and in the Section C.4 of the Appendix C.

7.2.4.4 Feasibility test based on support functions

The convex sets are partially ordered by inclusion and the support functions preserve this order structure. Given two sets $\mathcal{P}_1$ and $\mathcal{P}_2$, the relationship between their support function $S(\mathcal{P}_1, u) \leq S(\mathcal{P}_2, u)$ holds $\forall u \in \mathbb{R}^n$ if and only if $\mathcal{P}_1 \subset \mathcal{P}_2$. In order to use this in a $\mathcal{D}$-invariance test, one needs to exploit two properties of the support functions for a vector $u \in \mathbb{R}^n$:

- $S(\bigoplus_{i=0}^{d} \mathcal{P}_i, u) = \sum\limits_{i=0}^{d} S(\mathcal{P}_i, u)$.
- $S(A\mathcal{P}, u) = S(\mathcal{P}, A^\top u)$.

Using these facts and the definition of the support function in (2.1), one can implement the $\mathcal{D}$-invariance test in terms of a set of LP problems according to the number of rows of the matrix F, describing the set $\mathcal{P} = \{x \in \mathbb{R}^n | Fx \leq \mathbb{1}\}$.

$$\begin{aligned} S(\textstyle\bigoplus_{i=0}^{d} A_i\mathcal{P}, f_{j,[1:n]}^\top) = \quad & \min_{x_0,\dots,x_d} -\gamma \\ \text{subject to:} \quad & \gamma \leq f_{j,[1:n]}(A_0x_0 + \cdots + A_dx_d) \\ & Fx_i \leq \mathbb{1}, i = 0, \dots, d \end{aligned} \tag{7.49}$$

Finally, if $S(\bigoplus_{i=0}^{d} A_i\mathcal{P}, f_{j,[1:n]}^\top) \leq \varepsilon, \forall j$ then the set $\mathcal{P}$ is $\mathcal{D}$-invariant.

7.2.5 Controlled $\mathcal{D}$-invariance

This section considers the constrained stabilization problem by means of the obtention of a state feedback control law $u_k = Kx_k$ using the $\mathcal{D}$-contractiveness ($\mathcal{D}$-invariance) conditions for the polyhedral set $\mathcal{P} \subseteq \mathbb{R}^n$.

Consider a discrete time-delay system of the form:

$$x_{k+1} = \sum_{i=0}^{d} A_i x_{k-i} + \sum_{i=0}^{d} B_i u_{k-i}, \tag{7.50}$$

where $x_k \in \mathbb{R}^n$ is the state vector, $u_k \in \mathbb{R}^r$ is the control input at the time $k \in \mathbb{Z}_+$, $A_i \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n\times m}$, for all $i \in \mathbb{Z}_{[0,d]}$ and the initial conditions are given by a sequence $x_{-i}, u_{-i} \in \mathbb{R}^n$, $i \in \mathbb{Z}_{[0,d]}$..

Assume that the system (7.50) can be stabilized using a state feedback control law:

$$u_k = Kx_k, \tag{7.51}$$

with $K \in \mathbb{R}^{m\times n}$. By closing the loop in (7.50) with (7.51):

$$x_{k+1} = \sum_{i=0}^{d} (A_i + B_i K) x_{k-i}. \tag{7.52}$$

Proposition 7.18. *The polyhedral set of constraints:*

$$\mathcal{P} = \{x \in \mathbb{R}^n \mid Fx \leq \mathbb{1}\}, \tag{7.53}$$

with $F \in \mathbb{R}^{r\times n}$, is $\mathcal{D}$-invariant with respect to the system (7.50) and the control law (7.51) is a solution to the stabilization of (7.50) if and only if there exist the matrices $H_i \in \mathbb{R}^{r\times r}$ for $i \in \mathbb{Z}_{[0,d]}$, $K \in \mathbb{R}^{m\times n}$ and $\lambda \in \mathbb{R}_{[0,1]}$ such that:

$$F(A_i + B_i K) = H_i F \tag{7.54}$$

and

$$\left(\sum_{i=0}^{d} H_i\right) \mathbb{1} \leq \lambda \mathbb{1}. \tag{7.55}$$

Proof. Analogously to the Section 7.2.4.2, the proof can derived using the Theorem 7.12 and substituting the transition matrices A_i by $(A_i + B_i K)$ for all $i \in \mathbb{Z}_{[0,d]}$. Appendix B provides the construction of the matrices $A_{eq} \in \mathbb{R}^{drn \times dr^2}$, $b_{eq} \in \mathbb{R}^{drn}$, $A_{in} \in \mathbb{R}^{dr^2+r \times dr^2}$ and the vectors $b_{in} \in \mathbb{R}^{dr^2+r}$ and $\bar{h} \in \mathbb{R}^{dr^2}$. ■

Remark 7.19. The synthesis problem is equivalent to a LP (linear program) and λ can play the role of cost function which can be optimized in order to obtain a contraction of the polyhedral set if $\lambda < 1$. See the details in Appendix B.

7.3 The Cyclic $\mathcal{D}$-invariance

Consider the following extended system:

$$\mathbf{x}(k+1) = \Phi(A_i)\mathbf{x}(k), \tag{7.56}$$

$\mathbf{x}(k) = \begin{bmatrix} x(k)^T & x(k-1)^T & \dots & x(k-d)^T \end{bmatrix}^T \in (\mathbb{R}^n)^{d+1}$ and the transition matrix

$$\Phi(A_0, A_1, \cdots, A_d) = \begin{bmatrix} A_0 & A_1 & \cdots & A_{d-1} & A_d \\ I_n & 0 & \cdots & 0 & 0 \\ 0 & I_n & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I_n & 0 \end{bmatrix}. \tag{7.57}$$

The notion of $\mathcal{D}$-invariance inherits structural conservativeness with respect to the invariant sets corresponding to the extended system (7.56). The existence of a $\mathcal{D}$-invariant set $\mathcal{P} \subset \mathbb{R}^n$ with respect to (7.9) implies the invariance of the set $\mathcal{P}^{d+1} \subset (\mathbb{R}^n)^{d+1}$ with respect to (7.56). The degenerate set $\mathcal{P} = \{0\} \subset \mathbb{R}^n$ represents a $\mathcal{D}$-invariant set, representing in fact a fix point for (7.9) but in general terms the existence of invariant set in $(\mathbb{R}^n)^{d+1}$ with respect to (7.56) represents only a necessary condition for the existence of a non-degenerate $\mathcal{D}$-invariant set in $\mathbb{R}^n$ with respect to (7.9). For convex sets is interesting to find if the $\mathcal{D}$-invariance holds when operation that preserve convexity (like *intersection* or *convex hull*) are used.

In order to fix the ideas, let us consider another numerical example with the dynamics:

Example 7.6.

$$x(k+1) = \begin{bmatrix} 0.25 & 0.43 \\ -0.43 & 0.25 \end{bmatrix} x(k) + \begin{bmatrix} 0.2 & 0.35 \\ -0.26 & 0.15 \end{bmatrix} x(k-1),$$

Let the polyhedral sets P_0 and P_1 as those described graphically in Figure 7.6 and Figure 7.7. In the same figures it is illustrated that the $\mathcal{D}$-invariance properties are not verified, neither for P_0, nor for P_1.

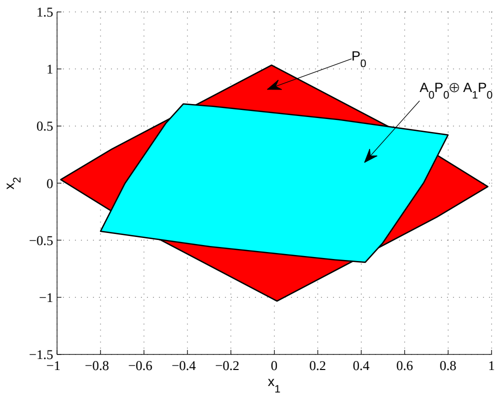


FIGURE 7.6: The set P_0 and the Minkowski sum $A_0P_0 \oplus A_1P_0$.

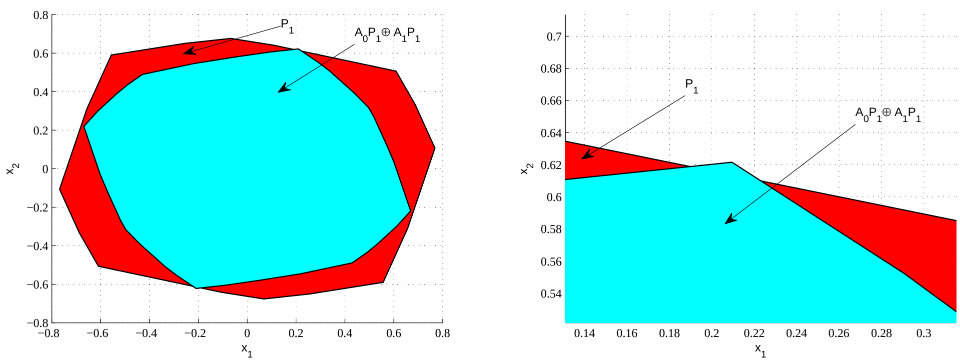


FIGURE 7.7: The set P_1 and the Minkowski sum $A_0P_1 \oplus A_1P_1$. In the right hand side a zoom of the same plot which proves $P1 \not\subset A_0P_1 \oplus A_1P_1$.

However, by using the sequence of sets $\{P_0, P_1\}$ for which the one step circular shift leads to the sequence $\{P_1, P_0\}$, we can verify the inclusions

$$P_1 \subset A_0P_0 \oplus A_1P_1; \; P_0 \subset A_0P_1 \oplus A_1P_0,$$

as illustrated in Figure 7.8.

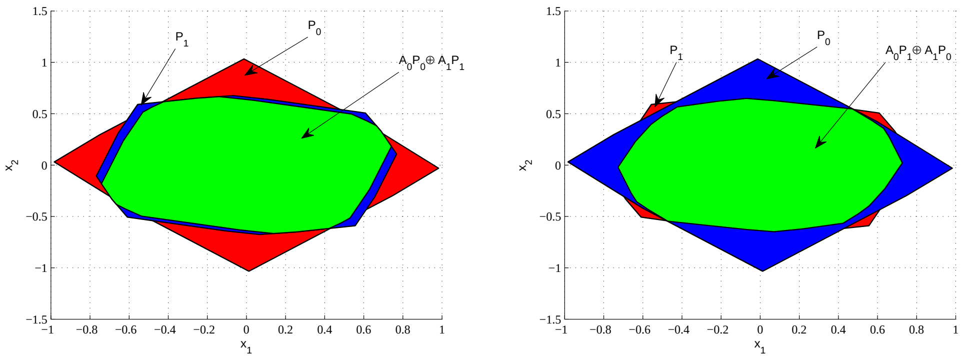


FIGURE 7.8: $\mathcal{D}$-invariance with different topology of the sets on the shifted sequence.

This construction can be generalized for any linear discret-time dynamics affected by delay. If the $\mathcal{D}$-invariance definition could be illustrated by a graph of length "d", similar to the one depicted in Fig. 7.9. Similarly, the schematic representation of the cyclic $\mathcal{D}$-invariance (contractiveness) concept we want to introduce in the present section is given in Figure 7.10.

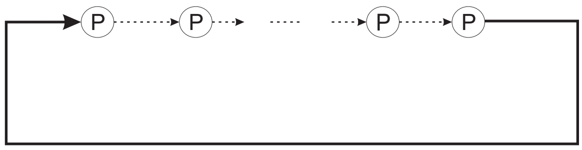


FIGURE 7.9: $\mathcal{D}$-invariant sequence

Each one-step cyclic shift performs a rotation of sets in the sequence with the previous last set inserted in the first position. Mathematically this insertion in the first position is allowed by the inclusion properties as resumed by the next definition.

Definition 7.20 (Cyclic $\mathcal{D}$-invariance)**.** The sets $\mathcal{P}_i \subseteq \mathbb{R}^n$ containing the origin are called cyclic $\mathcal{D}$-invariant sets with respect to (7.9) if:

$$\begin{aligned} A_0\mathcal{P}_0 \oplus A_1\mathcal{P}_1 \oplus \cdots \oplus A_d\mathcal{P}_d &\subseteq \mathcal{P}_0; \\ A_0\mathcal{P}_d \oplus A_1\mathcal{P}_0 \oplus \cdots \oplus A_d\mathcal{P}_{d-1} &\subseteq \mathcal{P}_d; \\ &\vdots \\ A_0\mathcal{P}_1 \oplus A_1\mathcal{P}_2 \oplus \cdots \oplus A_d\mathcal{P}_0 &\subseteq \mathcal{P}_1. \end{aligned} \tag{7.58}$$

□

After "d" cycles, the sequence comes back to the initial order and thus provides a time-independent formulation of the invariance concept.

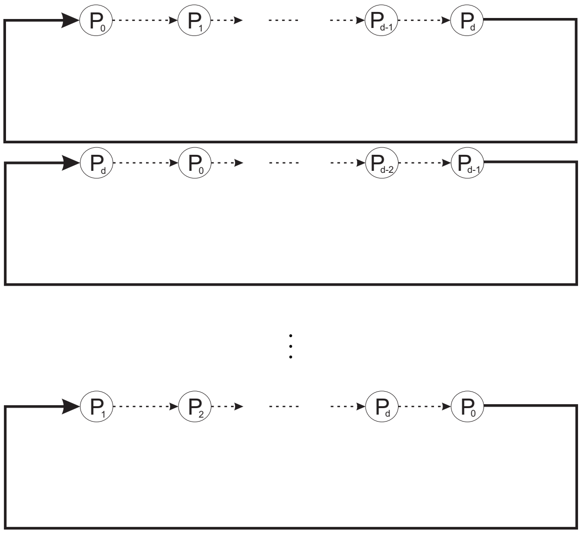


FIGURE 7.10: Cyclic $\mathcal{D}$-invariant sequence

7.3.1 Algebraic conditions for Cyclic $\mathcal{D}$-invariance

In order to introduce the algebraic conditions for cyclic $\mathcal{D}$-invariance consider the following constructions based on the given matrices:

$$\Psi = diag(F_0, \ldots, F_d); \ \Theta = diag(A_1, \ldots, A_d) \tag{7.59}$$

$$\Gamma_i = \underbrace{\begin{bmatrix} F_i & F_i & \ldots & F_i \end{bmatrix}}_{d+1}, \ i \in \mathbb{Z}_{[0,d]}$$

and the permutation matrix:

$$\Pi = \begin{bmatrix} 0 & 0 & \ldots & I_n \\ I_n & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \ldots & I_n & 0 \end{bmatrix} \in (\mathbb{R}^n)^{d+1}$$

The next result resumes the invariance properties which can be used for *Linear Programming* - based analysis.

Theorem 7.21. *The sequence of sets* $\{\mathcal{P}_0, \ldots, \mathcal{P}_d\} \subseteq (\mathbb{R}^n)^{d+1}$ *as in* $\mathcal{P}_i = \{x \in \mathbb{R}^n \mid F_i x \leq \mathbb{1}\}$ *is cyclic* $\mathcal{D}$*-invariant with respect to* (7.9) *if and only if there exist a sequence of matrices* $\Omega_i \in \mathbb{R}^{r \times r(d+1)}, i \in \mathbb{Z}_{[0,d]}$ *with non-negative elements and* $\|\Omega_i\|_\infty \leq 1$ *such that:*

$$\Omega_i \Psi \Pi^i = \Gamma_i \Theta$$

Proof. The series of $d+1$ inclusions defining the $\mathcal{D}$-invariance in (7.20) can be rewritten explicitly using the half-space representation of the sets P_i in $\mathcal{P}_i = \{x \in \mathbb{R}^n \mid F_i x \leq \mathbb{1}\}$ and the permutation matrix:

$$\Psi\Pi^i \mathbf{x} \leq \mathbb{1} \Longrightarrow \Gamma_i \Theta \mathbf{x} \leq \mathbb{1},\ \forall i \in \mathbb{Z}_{[0,d]} \tag{7.60}$$

But this inclusion implication is equivalent upon the Extended Farkas Lemma to the existence of matrices Ω_i with non-negative elements satisfying

$$\begin{aligned} \Omega_i \Psi \Pi^i &= \Gamma_i \Theta \\ \Omega_i \mathbb{1} &\leq \mathbb{1} \end{aligned} \tag{7.61}$$

Taking into account the elementwise non-negativeness of the matrices Ω_i, the second inequality can be interpreted in terms of matrix norm as $\|\Omega_i\|_\infty \leq 1$ which completes the proof. ■

7.3.2 Relationship between cyclic $\mathcal{D}$-invariance and $\mathcal{D}$-invariance

In order to link the basic $\mathcal{D}$-invariance property to the introduced cyclic $\mathcal{D}$-invariance, one can observe that for any $\mathcal{D}$-invariant set, it exists a trivial cyclic $\mathcal{D}$-invariant sequence $\left\{\underbrace{P, \ldots, P}_{\text{“}d\text{'' times}}\right\}$.

The next result points out the relationship which can be resumed as follows: *cyclic $\mathcal{D}$-invariance induces $\mathcal{D}$-invariance.*

Proposition 7.22. *Consider the polyhedral sets $\mathcal{P}_i \subseteq \mathbb{R}^n$, for $i \in \mathbb{Z}_{[0,d]}$, containing the origin in its interior. If the sequence $\{P_0, \ldots, P_d\}$ is cyclic $\mathcal{D}$-invariant with respect to (7.9) then the intersection:*

$$\mathcal{I} = \bigcap_{i=0}^{d} P_i \tag{7.62}$$

is $\mathcal{D}$-invariant.

Proof. Without loss of generality, consider the cyclic $\mathcal{D}$-invariant sets $\mathcal{P}_0, \mathcal{P}_1 \subseteq \mathbb{R}^n$ with respect to the system:

$$x(k+1) = A_0 x(k) + A_1 x(k-1),$$

So the cyclic $\mathcal{D}$-invariance:

$$A_0\mathcal{P}_0 \oplus A_1\mathcal{P}_1 \subseteq \mathcal{P}_0; \quad A_0\mathcal{P}_1 \oplus A_1\mathcal{P}_0 \subseteq \mathcal{P}_1,$$

implies $\mathcal{D}$-invariance with respect to $\mathcal{I}$:

$$A_0\mathcal{I} \oplus A_1\mathcal{I} \subseteq \mathcal{P}_0; \quad A_0\mathcal{I} \oplus A_1\mathcal{I} \subseteq \mathcal{P}_1,$$

by consequence:

$$A_0\mathcal{I} \oplus A_1\mathcal{I} \subseteq \mathcal{P}_0 \cap \mathcal{P}_1 \quad = \quad A_0\mathcal{I} \oplus A_1\mathcal{I} \subseteq \mathcal{I}.$$

■

These relationships prove that the cyclic $\mathcal{D}$-invariance is a less conservative formulation of the same concept.

7.4 $\mathcal{D}$-invariant constraints based on set-iterates

Once the inclusions presented in Definitions 7.20 and 7.3 do not hold, i.e. the set $\mathcal{P} \subseteq \mathbb{R}^n$ is not a $\mathcal{D}$-invariant set with respect to (7.9), the question one can ask is the following: *Suppose the existence of $\mathcal{D}$-invariant sets is assured. How to practically construct $\mathcal{D}$-invariant set by taking a $\mathcal{D}$-invariant candidate (which accessorily is a subset), with respect to a given dynamics?* By using the equivalence between the $\mathcal{D}$-invariance as stated in Definition 7.3 and the set-theoretic counterpart evidenced in Theorem 7.4, one can derive constructive algorithms for obtaining $\mathcal{D}$-invariant sets in a *set-theoretic* framework.

By considering the matrices $A_i \in \mathbb{R}^{n\times n}$, for $i \in \mathbb{Z}_{[0,d]}$, as in (7.9), the following mappings $\Phi : ComC(\mathbb{R}^n) \to ComC(\mathbb{R}^n)$ and $\Psi : ComC(\mathbb{R}^n) \to ComC(\mathbb{R}^n)$ are defined:

$$\Phi(\mathcal{P}) = \bigoplus_{i=0}^{d} A_i\mathcal{P}; \tag{7.63}$$

$$\Psi(\mathcal{P}) = Co(\mathcal{P}, \bigoplus_{i=0}^{d} A_i\mathcal{P}) = Co(\mathcal{P}, \Phi(\mathcal{P})). \tag{7.64}$$

Remark 7.23. By using (7.63) and by a direct consequence of the Theorem 7.4, a given compact set $\mathcal{P}$ is $\mathcal{D}$-invariant if $\Phi(\mathcal{P}) \subseteq \mathcal{P}$.

The mappings (7.63)-(7.64) can be further used to define k-iterates over the family of $\mathcal{C}$-sets:

$$\begin{aligned} \Phi^k(\mathcal{P}) &= \Phi(\Phi^{k-1}(\mathcal{P})), k \geq 0 \text{ with } \Phi^0(\mathcal{P}) = \mathcal{P}, \\ \Psi^k(\mathcal{P}) &= \Psi(\Psi^{k-1}(\mathcal{P})), k \geq 0 \text{ with } \Psi^0(\mathcal{P}) = \mathcal{P}. \end{aligned} \tag{7.65}$$

7.4.1 Properties

Several properties about the iterative set mappings can be pointed out, in order to state the theoretical framework, to be used in the numerical and algorithmic constructions in the sequel.

In this context, the following definition, similar to the one in [8], is characterizing the non-increasing/non-decreasing sequence of iterates.

Definition 7.24. Consider the mapping $\Pi(.)$ as a map from $ComC(\mathbb{R}^n)$ to itself. If $\mathcal{P} \subseteq \Pi(\mathcal{P})$ for some set $\mathcal{P} \subseteq \mathbb{R}^n$ then the sequence of iterates $\Pi(\ldots(\Pi(\mathcal{P})))$ is set-wise non-decreasing. Likewise, if $\Pi(\mathcal{P}) \subseteq \mathcal{P}$ then the set iterates are set-wise non-increasing. □

For the mappings defined in (7.63)-(7.64), the following properties hold:

1. $\Psi^k(\mathcal{P})$ is set-wise non-decreasing, i.e.

$$\Psi^{k-1}(\mathcal{P}) \subseteq \Psi^k(\mathcal{P}),$$

for all $k \in \mathbb{Z}_+^*$ and for any $\mathcal{P} \in ComC(\mathbb{R}^n)$.

2. If $\Phi(\mathcal{P}) \subseteq \mathcal{P}$ then $\Phi^k(\mathcal{P})$ with $k \in \mathbb{Z}_+$ is set-wise non-increasing, i.e.:

$$\Phi^k(\mathcal{P}) \subseteq \Phi^{k-1}(\mathcal{P}),$$

for all $k \in \mathbb{Z}_+^*$.

3. If $\mathcal{P}$ is $\mathcal{D}$-invariant and convex then $\mathcal{P}$ is a *fixed point* for $\Psi(.)$, i.e.:

$$\Psi(\mathcal{P}) = \mathcal{P}. \tag{7.66}$$

Definition 7.25. Given a subset X of $\mathbb{R}^n$, if $\mathcal{P} \subseteq X$ is $\mathcal{D}$-invariant and for all $\mathcal{D}$-invariant sets $\mathcal{R} \subseteq X$ it holds that $\mathcal{R} \subseteq \mathcal{P}$, $\mathcal{P}$ is called the maximal $\mathcal{D}$-invariant set in X. □

The property 3 points out that a $\mathcal{D}$-invariant set is related to a *fixed point* for the set-dynamics described by (7.64). By starting with a set $\mathcal{P} \in ComC(\mathbb{R}^n)$, the convergence of the iterates $\Psi^k(\mathcal{P})$ to a *fixed point* is directly related to the convergence of the *Hausdorff distance* between the *fixed point* and the iterates to zero.

The mapping $\Phi(\cdot)$ in (7.63) is a *contraction* if there exists an $\alpha \in \mathbb{R}_{[0,1)}$, such that for two arbitrary sets $\mathcal{P}_1 \subseteq \mathbb{R}^n$ and $\mathcal{P}_2 \subseteq \mathbb{R}^n$ the following holds:

$$d_H(\Phi(\mathcal{P}_1), \Phi(\mathcal{P}_2)) \leq \alpha d_H(\mathcal{P}_1, \mathcal{P}_2),$$

The contractive behavior of the mapping $\Psi(\cdot)$ in (7.64) is defined similarly.

Next, by using the theoretical results shown above, an algorithmic procedure for the computation of maximal $\mathcal{D}$-invariant set approximations for the system (7.9) is proposed. This procedure considers as input argument a predefined region in the state space which confines the $\mathcal{D}$-invariant candidates and exploits the contractive properties of the mappings (7.63)-(7.64) to assure the convergence of the sequence of candidates towards the maximal $\mathcal{D}$-invariant set.

Example 7.7. *Consider a dynamical system as in (7.9), with:*

$$A_0 = \begin{bmatrix} 0.0809 & -0.0588 \\ 0.0588 & 0.0809 \end{bmatrix}, \quad A_d = \begin{bmatrix} 0.8257 & -0.1308 \\ 0.1308 & 0.8257 \end{bmatrix}. \tag{7.67}$$

Consider a set $\mathcal{X}_0$ which is the 1-norm unit circle in $\mathbb{R}^2$. By applying Algorithm 5, in Appendix B the inner-outer approximation of the limit $\mathcal{D}$-invariant set is obtained iteratively. Fig. 7.11 presents the inner-outer approximation for each iteration (indexed on the z axis).

Figure 7.12 presents the evolution of the Hausdorff distance between the inner and the outer approximation as a function of the number of iterations.

The test of $\mathcal{D}$-invariance is depicted in Figure 7.13.

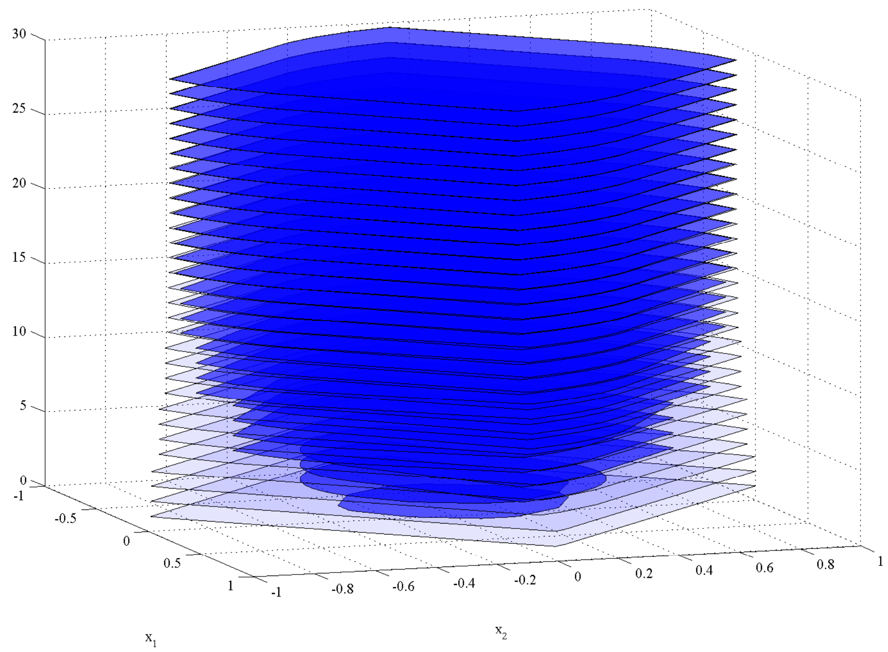


FIGURE 7.11: Sequences of inner and outer approximations $(\mathcal{R}_i, \mathcal{P}_i)$ indexed on the x_3 axis according to the number of iterations in the numerical procedure.

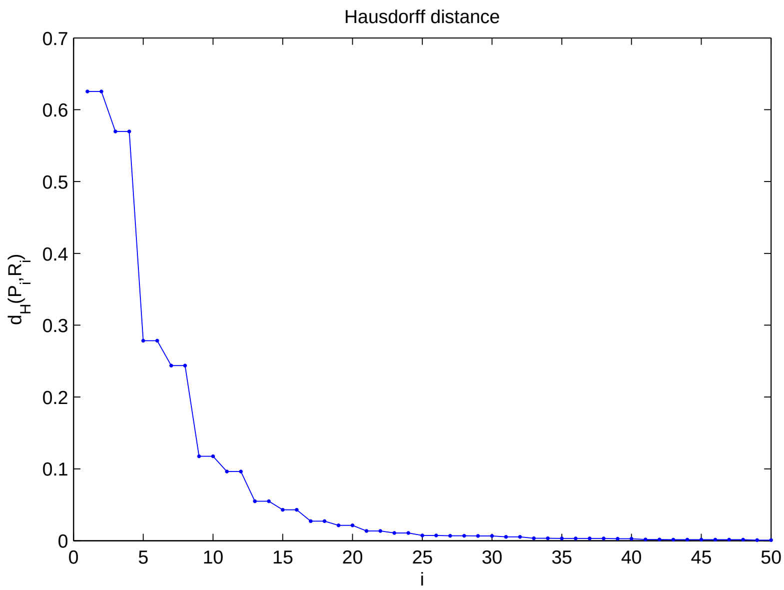


FIGURE 7.12: $d_H(\mathcal{R}_i, \mathcal{P}_i)$ vs. the iteration index i.

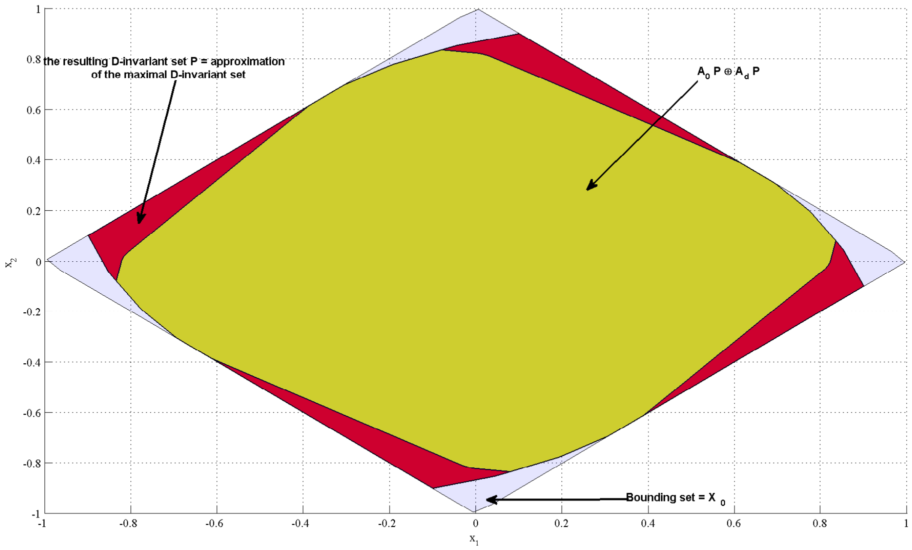


FIGURE 7.13: Graphical illustration of $\mathcal{D}$-invariance $A_0\mathcal{P} \oplus A_d\mathcal{P} \subseteq \mathcal{P}$. In blue, the bounding box - $\mathcal{X}_0$; in red, the $\mathcal{D}$-invariant set; in yellow, the set $A_0\mathcal{P} \oplus A_d\mathcal{P}$.

Chapter 8

Predictive control

In this chapter a general *Model Predictive Control* (MPC) procedure is presented. MPC algorithms solve at each sampling period a finite-time optimal control problem over a receding prediction horizon. The objective is to implement a predictive control technique for time-delay systems, by using the models obtained in Chapter 3, 4 and 5. In order to impose stability in the MPC synthesis, the invariant sets obtained in Chapter 7 are used as terminal set of constraints on the on-line optimization problem (to be stabilized by using the techniques presented in Chapter 6). The states are forced to attain the *maximal delayed-state admissible set* in a finite horizon, even if the initial states are outside this set. The system is stabilized while robustly satisfying the constraints in the states and control input. We restrict ourselves to this well established formulation, but we point towards the potential use of $\mathcal{D}$-invariance principle in MPC.

In Section 8.1 a MPC procedure is described: in Section 8.1.1 the optimization problem is detailed and in Section 8.1.2 the weighting matrices are found by an inverse optimality problem.

8.1 MPC Strategy

A standard MPC strategy will construct at each sampling instant "k" an optimal control sequence:

$$\mathbf{k}_u^* = \{u_{k|k}, \ldots, u_{k+N-d-1|k}\}, \tag{8.1}$$

with respect to a *performance index* which evaluates the system dynamics over a finite horizon $k+1, \ldots, k+N$. As a basic remark, for time-delay systems, the prediction horizon has to be larger than the delay, i.e. $N \geq d_{max}$, in order to have an effective measure of how the delays affect the system dynamics. The prediction is constructed upon the nominal model but the real system may be affected by delays up to d_{max} samples. In order to cope with all the possible delay variations, it will be considered that $N \geq d_{max}$.

The first component of $\mathbf{k}_u^*$ is effectively applied as control action to the system:

$$u_k = \mathbf{k}_u^*(1) = u_{k|k}, \tag{8.2}$$

while the tail is discarded. Using the new measurements, the optimization procedure is restarted. Thus a closed-loop control scheme can be obtained.

A *classical performance index* has a quadratic form and commensurate the state (tracking error) trajectory and the associated control effort. If the admissible trajectories are described by a polyhedral set of constraints of the type $Cx_k \leq W$, the MPC implementation passes by the resolution of the optimization problem of the form:

$$\mathbf{k}_u^* = \underset{\{u_{k|k},\ldots,u_{k+N-d-1|k}\}}{\arg\min} \left\{ {x_{k+N|k}}^\top \bar{P} x_{k+N|k} + \sum_{j=1}^{N} {x_{k+j|k}}^\top \bar{Q} x_{k+j|k} + \sum_{j=0}^{N-d-1} u_{k+j|k}^\top \bar{R} u_{k+j|k} \right\} \tag{8.3}$$

subject to:

$$\begin{cases} x_{k+j+1|k} = \bar{F} x_{k+j|k} + \bar{G} u_{k+j|k} \\ C x_{k+j|k} \leq W; \; j = 1, \ldots, N-1 \\ u_{k+i|k} = 0, \quad i = N-d, \ldots, N-1 \\ x_{k+N} \in O_N; \end{cases} \tag{8.4}$$

where O_N is the *terminal set* (a maximal delayed-state admissible set or a $\mathcal{D}$-invariant set), described by the polyhedron:

$$O_N = \left\{ \xi \in \mathbb{R}^{n+dm} \mid H_{O_N} \xi \leq W_{O_n} \right\},$$

where $H_{O_N} \in \mathbb{R}^{r \times (n+dm)}$ (for the extended case), obtained in the Section 7.1 of the Chapter 7. For the original state-space dimension, O_N is a $\mathcal{D}$-invariant set (obtained in

Section 7.4 of the Chapter 7):

$$O_N = \{x \in \mathbb{R}^n \mid H_{O_N} x \leq W_{O_n}\},$$

where $H_{O_N} \in \mathbb{R}^{r\times n}$. The prediction model in (8.4) can be used as the nominal model in a polytopic framework (see, for instance, (5.10)).

The construction of the predictive control law will be influenced by the choice of the prediction horizon "N", the weighting factors on the state trajectory $\bar{Q} = \bar{Q}^\top \geq 0$ and the control effort, $\bar{R} = \bar{R}^\top \geq 0$. For the "penalty" on the terminal state, the matrix $\bar{P}$ is usually constructed such that the prediction horizon is extended to infinity, by the introduction of the term $x_{k+N|k}{}^\top \bar{P} x_{k+N|k}$, as in (8.3). However, $x_{k+N|k}$ has to satisfy some mild conditions materialized by the terminal constraint set, which force this prediction to reach the predefined invariant set X_N. The usual choice in this sense [58] is the *maximal delayed-state admissible set* (or maximal output admissible set) $X_N = O_\infty$, constructed for the delay system (see Chapter 7), with the optimal control law satisfying the discrete algebraic Riccati equation:

$$\begin{aligned} \bar{P} &= \bar{Q} + \bar{F}^\top \bar{P}\bar{F} - K^\top \left(\bar{R} + \bar{G}^\top \bar{P} G\right) K \\ K &= -\left(\bar{R} + \bar{G}^\top \bar{P} G\right)^{-1} \bar{G}^\top F \end{aligned} \tag{8.5}$$

In the Section 8.1.2, the choice of the performance index is discussed (in particular how to obtain the matrices $\bar{Q}, \bar{R}$ and indirectly $\bar{P}$) such that the resulting control law to present a certain degree of robustness with respect to the variable delay.

8.1.1 Multiparametric programming

In order to make the optimization problem "user friendly", Eq. (8.3) can be translated in a modified version of the constrained infinite horizon linear quadratic problem, where both the cost index and the inequality constraints are defined over the prediction horizon "N". By noting $\mathbf{x} = \left\{x_{k+1|k}^\top, \ldots, x_{k+N|k}^\top\right\}$, the states of (8.3) can be compacted and the cost function can be rewritten as:

$$\begin{aligned} &\arg\min_{\mathbf{k}_u} x_k^\top Q x_k + \mathbf{x}^\top \bar{Q}\mathbf{x} + \mathbf{k}_u^\top \bar{R}\mathbf{k}_u; \\ &\text{subject to}: \left\{ \bar{C}\mathbf{k}_u + \tilde{E}x + \bar{E}\mathbf{x} \leq \bar{W} \right. , \end{aligned} \tag{8.6}$$

where:

$$\bar{Q} = \mathrm{diag}(\mathrm{diag}(Q, N), P); \quad \bar{R} = \mathrm{diag}(R, N); \quad \bar{C} = \mathrm{diag}(C, N);$$

$$\bar{E} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ E & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & E & 0 \end{bmatrix}; \qquad \bar{W} = \begin{bmatrix} W \\ \vdots \\ W \end{bmatrix}; \qquad \tilde{E} = \begin{bmatrix} E \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

where P is done by, for example, Eq. (8.5).

One can note that all the matrix dimensions depend on the modeling of the system and on the used stabilization technique. If the model is the nominal one (and by consequence the stabilization technique is the Lyapunov-Krasovskii ones, for example), the system matrix dimensions $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{m\times n}$ will directly influence all the matrices and vectors above. The dimensions are: $\bar{Q} \in \mathbb{R}^{Nn\times Nn}$, $\bar{R} \in \mathbb{R}^{Nm\times Nm}$, $\bar{C} \in \mathbb{R}^{Nr\times Nn}$, $\bar{E} \in \mathbb{R}^{Nr\times Nn}$, $\bar{W} \in \mathbb{R}^{Nr\times 1}$, $\tilde{E} \in \mathbb{R}^{Nr\times n}$.

If the used model is the extended one, with respect to the modeling framework presented in Chapters 3-5, then the following nominal extended model can be expressed:

$$\xi_{k+1} = A_\Theta \xi_k + B_\Theta u_k,$$

with the extended state vector:

$$\xi_k^\top = \begin{bmatrix} x_k^\top & u_{k-d}^\top & \dots & u_{k-1}^\top & \end{bmatrix}^\top$$

and the augmented system matrices:

$$A_\Theta = \begin{bmatrix} A & B & 0 & \dots & 0 \\ 0 & 0 & I_m & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & I_m \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}; \; B_\Theta = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ I_m \end{bmatrix}^\top, \tag{8.7}$$

where $A_\Theta \in \mathbb{R}^{(n+md)\times(n+md)}$ and $B_\Theta \in \mathbb{R}^{(n+md)\times m}$, and by consequence the dimensions are: $\bar{Q} \in \mathbb{R}^{N(n+dm)\times N(n+dm)}$, $\bar{R} \in \mathbb{R}^{Nm\times Nm}$, $\bar{D} \in \mathbb{R}^{Nr\times N(n+dm)}$, $\bar{E} \in \mathbb{R}^{Nr\times N(n+dm)}$, $\bar{W} \in \mathbb{R}^{Nr\times 1}$, $\tilde{E} \in \mathbb{R}^{Nr\times(n+dm)}$.

The following prediction system can be described:

$$\mathbf{x} = \Theta x_k + \Upsilon \mathbf{k}_u,$$

where:

$$\Theta = \begin{bmatrix} A \\ A^2 \\ \vdots \\ A^N \end{bmatrix}; \quad \Upsilon = \begin{bmatrix} B & 0 & \cdots & 0 \\ AB & B & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A^{N-1}B & A^{N-2}B & \cdots & B \end{bmatrix},$$

for the original state-space model. For the extended state-space approach, one should just substitute A and B by A_Θ and B_Θ.

The optimization problem in (8.3) can now be reformulated as a multiparametric quadratic problem [12, 44, 61, 144]:

$$\begin{aligned} &\mathbf{k}_u^*(\xi_k) = \arg\min_{\mathbf{k}_u} \left\{0.5\mathbf{k}_u^\top H \mathbf{k}_u + \mathbf{k}_u^\top G x_k\right\} \\ &\text{subject to}: \; A_{in}\mathbf{k}_u \leq b_{in} + B_{in}x_k \end{aligned} \tag{8.8}$$

where $H = \bar{R} + \Upsilon^\top \bar{Q} \Upsilon$, $G = \Upsilon \bar{Q} \Theta$, $A_{in} = \bar{C} + \bar{E}\Upsilon$, $B_{in} = \tilde{E} + \bar{E}\Theta$, $b_{in} = \bar{W}$ and the vector x_k plays the role of parameter. To add the terminal set of constraints O_N on (8.8) (where O_N is a maximal delayed-state admissible set in the augmented state-space framework or a $\mathcal{D}$-invariant set in the original state-space framework, developed in Chapter 7), one can state this set inequalities as:

$$H_{O_N}\left(A^N x_k + \begin{bmatrix} A^{N-1}B & A^{N-2}B & \ldots & B \end{bmatrix} \mathbf{k}_u\right) \leq W_{O_N},$$

or else:

$$H_{O_N}\begin{bmatrix} A^{N-1}B & A^{N-2}B & \ldots & B \end{bmatrix} \mathbf{k}_u \leq W_{O_N} - H_{O_N}A^N x_k.$$

Thus, the final constraints of (8.8) are:

$$A_{in} = \begin{bmatrix} \bar{C} + \bar{E}\Upsilon \\ H_{O_N}\begin{bmatrix} A^{N-1}B & A^{N-2}B & \ldots & B \end{bmatrix} \end{bmatrix}; \quad B_{in} = \begin{bmatrix} \tilde{E} + \bar{E}\Theta \\ H_{O_N}A^N \end{bmatrix}; \quad b_{in} = \begin{bmatrix} \bar{W} \\ W_{O_N} \end{bmatrix}.$$

Example 8.1 (Predictive position control of a DC motor with time-varying delays - intersampled delay variation)**.** *Consider the model developed in the Examples 3.8 and 4.1 of the Chapters 3 and 4, with the less conservative model as proposed by Chapter*

4, the control law as the one obtained in Example 6.1 and the maximal delayed-state admissible set as in Example 7.1.

The sampling period is $T_e = 0.003s$*, the delay is* $\tau \in \mathbb{R}_{[0.006s, 0.009s]}$*, without any statistical information about the variation in this interval. The discrete-time model is given by:*

$$x_{k+1} = \begin{bmatrix} 1 & 0.0028 \\ 0 & 0.8382 \end{bmatrix} x_k + \begin{bmatrix} 0.0525 \\ 33.9731 \end{bmatrix} u_{k-d} - \Delta(u_{k-d} - u_{k-d+1}), \text{ with d} = 3.$$

The polytopic embedding is described by:

$$\Delta \in Co\left\{ \begin{bmatrix} -0.0148 \\ -9.5824 \end{bmatrix}, \begin{bmatrix} 0.0526 \\ 34.0630 \end{bmatrix}, \begin{bmatrix} 0.0252 \\ 33.4281 \end{bmatrix} \right\}.$$

The polytopic model, by using the vertices expressed above, is given by:

$$A_{\Delta_0} = \begin{bmatrix} 1 & 0.0028 & 0.0673 & -0.0148 & 0 \\ 0 & 0.8382 & 43.5555 & -9.5824 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}; \; A_{\Delta_1} = \begin{bmatrix} 1 & 0.0028 & -0.0001 & 0.0526 & 0 \\ 0 & 0.8382 & -0.0899 & 34.0630 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix};$$

$$A_{\Delta_2} = \begin{bmatrix} 1 & 0.0028 & 0.0272 & 0.0252 & 0 \\ 0 & 0.8382 & 0.5450 & 33.4281 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The extended-state feedback gain obtained by Theorem 6.2 with the matrices $Q = 10^{-5} I_{n+md}$ *and* $R = 1$ *is:*

$$K = \begin{bmatrix} -0.0677 & -0.0018 & -0.0430 & -0.0696 & 0.1424 \end{bmatrix}.$$

The prediction is made for an horizon $N = 5$*. The terminal set (a maximal delayed-states admissible set)* O_N^{Ω} *obtained is the same as the one in Example 7.1. This set can be viewed in Fig. 8.2, with the states evolution shown in detail. The initial conditions* *are* $x_0 = \begin{bmatrix} -0.34 & 2.95 & 0 & 0 & 0 \end{bmatrix}^\top$ *and* $x_i = 0$ *for* $i \in \mathbb{Z}_{[-d,-1]}$. *The temporal evolution of the system can be noticed in Fig. 8.1. The system remains stable even when the constraints are activated, assuring the robustness of the results.*

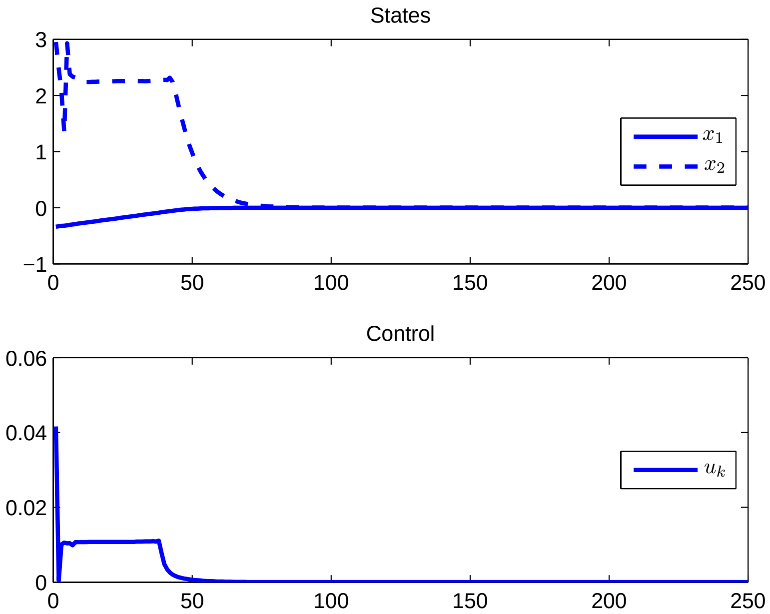


FIGURE 8.1: States evolution and temporal behavior of the system. The x axis is the number of simulation steps. The y axis are [cm] for the states and [v] for the control.

It is worth mentioning that the initial conditions are outside of the maximal delayed-states admissible set. This set is attained by the states in a number of iteration steps inside the prediction horizon.

8.1.2 Tuning MPC for robustness

Globally the optimum $\mathbf{k}_u^*(\xi_k)$ is a piecewise affine function over a polyhedral partition of the state space.

Explicit solutions for the MPC law can be obtained by retaining the first component of $\mathbf{k}_u^*(\xi_k)$ and thus expressing the predictive control in terms of a piecewise affine feedback law:

$$u_k = K_i^{MPC}\xi + \kappa_i^{MPC}, \quad \text{with} \quad i \text{ s.t. } x \in D_i\,, \tag{8.9}$$

for D_i, polyhedral regions in $\mathbb{R}^{n+dm}$.

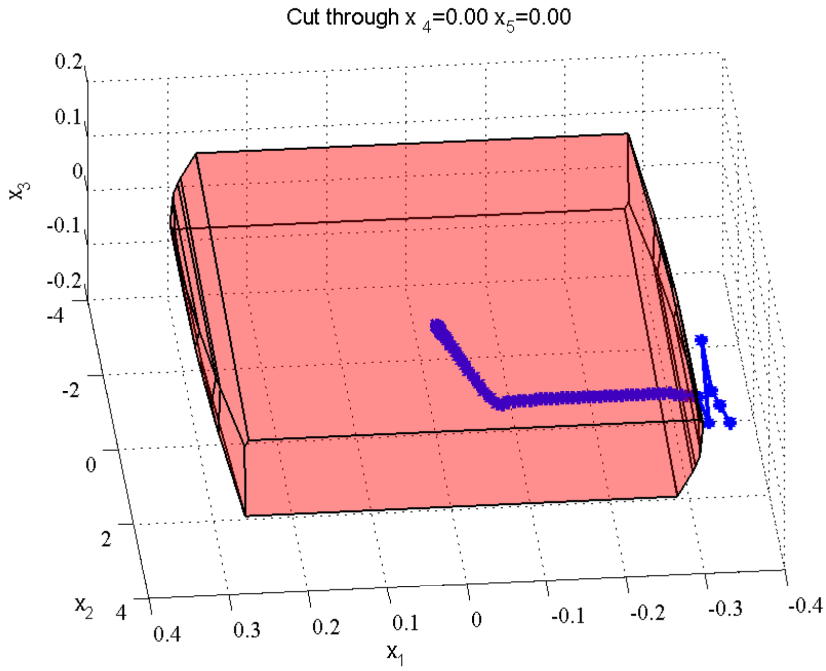


FIGURE 8.2: States trajectories with respect to the maximal delayed-states admissible set.

Remark 8.1. The prediction model is linear; the origin is a feasible point (in the most cases placed on the interior of the feasible domain) and thus represents an equilibrium point for the polytopic system. The problem (8.3), and further (8.8), are feasible, and more than that, the associated optimum will be unconstrained in a neighborhood of this equilibrium point. The consequence is that the affine control law corresponding to the region D_{i_0} containing the origin ($0 \in D_{i_0}$) is in fact a linear feedback ($\kappa_{i_0}^{MPC} = 0$) and it corresponds to the unconstrained optimal control law ($K_{i_0}^{MPC} = K_{LG}$ if $\bar{P}$ is build upon (8.5)). If the constraints are symmetric, the region containing the origin will be the central region of the partition.

Consider an infinite-horizon min-max control problem for the time-delay system to be controlled:

$$\min_{u_k, u_{k+1}, u_{k+2} \cdots} \quad \max_{F \in \Omega_\xi} J_\infty \tag{8.10}$$

with:

$$J_\infty = \sum_{i=0}^{\infty} \xi_{k+i}^\top Q \xi_{k+i} + u_{k+i}^\top R u_{k+i}, \tag{8.11}$$

$$u_k = K\xi_k, \tag{8.12}$$

where $Q > 0$, $R > 0$ are suitable weighting matrices fixed *a priori* and K is the feedback gain (playing in fact the role of the optimization argument).

By using the techniques presented in Chapter 6, a state-feedback matrix is obtained, in the extended or the original state-space of the time-delay system. The resulting law $u_k = Kx_k$ represents a robust stabilizing control law for the unconstrained case. The idea in the following is to use this information when tuning the nominal MPC parameters in (8.3), namely Q, R and P. One can start with the remark that the MPC law is a piecewise affine function of the state and the central region (or the region containing the origin, if the constraints are not symmetric) is characterized by the unconstrained optimum for the chosen performance index in (8.3). Constructing this performance index such that the optimal solution corresponds to the LQ solution ($K = YS^{-1} \leftrightarrow K_{LQ}$) can be seen as an inverse optimality problem [81].

In short the tuning procedure is the following: given the matrices $\bar{F}, \bar{G}$ and Y, S from Chapter 3, the matrices $\bar{Q} \geq 0$ and $\bar{R} > 0$ (and indirectly $\bar{P} \geq 0$) will be constructed such that the optimal solution to the unconstrained problem (8.3) to be:

$$\mathbf{k}_u^* = \begin{bmatrix} YS^{-1} \\ YS^{-1}(\bar{F} + \bar{G}YS^{-1}) \\ \vdots \\ YS^{-1}(\bar{F} + \bar{G}YS^{-1})^{N-1} \end{bmatrix} \xi_k \tag{8.13}$$

The (not unique) pair $(\bar{Q}, \bar{R})$ has to satisfy:

$$\bar{Q} = \bar{P} - \bar{F}^\top \bar{P} \bar{F} + \left\{ YS^{-1} \right\}^\top (\bar{R} + \bar{G}^\top \bar{P} \bar{G}) YS^{-1} \tag{8.14}$$

$$\bar{R}YS^{-1} + \bar{G}^\top \bar{P} \bar{G} YS^{-1} + \bar{G}^\top \bar{P} \bar{F} = 0 \tag{8.15}$$

This problem can be solved in the general case by employing an LMI formulation [99]:

$$
\begin{gathered}
\min_{\alpha,\bar{R}} \alpha \\
\text{subject to:} \\
\bar{P} - \bar{F}^\top \bar{P}\bar{F} + \left\{YS^{-1}\right\}^\top (\bar{R} + \bar{G}^\top \bar{P}\bar{G})YS^{-1} \geq 0 \\
\begin{bmatrix} Z & \bar{R}YS^{-1} + \bar{G}^\top \bar{P}\bar{G}YS^{-1} + \bar{G}^\top \bar{P}\bar{F} \\ * & I \end{bmatrix} \geq 0 \\
Z < \alpha I, \quad \bar{P} > 0
\end{gathered}
\tag{8.16}
$$

Theorem 8.2. *The nominal MPC control law, designed upon a performance index obtained by inverse optimality with respect to an unconstrained robust linear feedback, is robustly stabilizing the extended system given by Chapters 3, 4 and 5, despite the constraints on a non-degenerated neighborhood of the origin V.*

Proof. The proof is constructive and follows the arguments described in this section. Using the LMI formulation a robustly stabilizing control law is obtained for the unconstrained polytopic of nominal systems affected by delays. The corresponding gains done by Chapter 6 $\bar{K} = YS^{-1}$ (Section 6.1.2), $K = YG^{-1}$ (Section 6.2.1) and $K = YG^{-1}$ (Section 6.2.2) will be used together with the nominal model for the resolution of the LMI problem (8.16) which provides by inverse optimality the matrix $\bar{R}$. The matrix $\bar{Q}$ is obtained with a simple evaluation of (8.14) and the structure of the performance index in (8.3) is completed. The prediction horizon of the same performance index can be chosen according with the desired performances and complexity of the explicit solution. Independently of this choice, if the matrix $\bar{P}$ satisfies (8.5), then the nominal MPC leads to a piecewise affine control law and for the region D_{i_0} with $0 \in Int(D_{i_0})$ the explicit control law will be:

$$
u_k = K_{i_0}^{MPC}\xi_k + \kappa_{i_0}^{MPC} = YS^{-1}\xi_k. \tag{8.17}
$$

This region is polyhedral and the robust stabilizing properties are verified for an invariant subset with respect to the closed loop dynamics. If the general form of the invariant set given by the level set is considered:

$$
E(\sigma) = \left\{\xi | \xi^\top \bar{P}\xi \leq \sigma\right\} \tag{8.18}
$$

then one can find $\sigma > 0$ satisfying $V = E(\sigma) \subset D_{i_0}$. ∎

This LMI-based tuning procedure can be used in conjunction with the classical MPC design 8.3 to improve the overall robustness, by avoiding the complex min – max procedure, tube-based MPC and all other alternative MPC schemes available in the literature [105, 117, 126].

Part IV

Conclusions and Perspectives

Chapter 9

Conclusions

This chapter draws the concluding remarks and the future perspectives about the thesis.

9.1 Conclusions

The main interest of the present thesis is the *constrained control of time-delay system*, more specifically taking into consideration the discretization problem (due to, for example, a communication network) and the presence of constraints in the system's trajectories and control inputs.

The modeling problem is studied in detail, where an uncertainty is added into the system due to the discretization and delay. The delay variation with respect to the sampling instants is characterized by a polytopic covering of the discretization/delay induced uncertainty. The discretized system can be expressed in terms of an *augmented (or extended) state-space system.* Three cases are studied, depending on the eigenvalues of the system matrices. Despite the conservativeness of these results, some methods and algorithms are proposed in order to obtain a less conservative approach, by employing optimization-based techniques, all by preserving a fixed complexity.

Once the modeling issues are clarified we concentrated on the stabilizing techniques, based on Lyapunov's theory. The approach is concentrated on a first stage to the unconstrained stabilization. In the augmented state-space case, by using simple quadratic

Lyapunov candidates, an extended state-feedback gain is obtained by using LMI techniques. Lyapunov-Krasovskii candidates were also used to obtain LMI conditions for a state-feedback, in the "original" state-space of the system.

For the *constrained control* purposes, the use of the *set invariance theory* is used intensively, in order to obtain a region where the system is "well-behaviored", despite the presence of constraints and (time-varying) delay. Due to the *high complexity* of the *maximal delayed-state admissible set* obtained in the augmented state-space approach, the concept of *set invariance* is defined in the "original" state-space of the system with respect not just to the present states, but also with the past history of the system's trajectories. In this manuscript, this concept is called $\mathcal{D}$-invariance, and is defined in terms of the Minkowski addition of the set supposed to be invariant and the linear set operations of the dynamics. The algebraic conditions are also derived in order to obtain constructive tests (and controller synthesis), based on a LP problem. In order to reduce the conservatism of this definition, the concept of Cyclic $\mathcal{D}$-invariance is also developed, where the inclusion condition should hold for all the dynamics of the system "shifted" one-by-one. In the last part of the thesis, the MPC scheme is presented, in order to take into account the constraints and the optimality of the control solution. In order to tune the controller parameters (more precisely the weighting matrices), an inverse optimality synthesis is used. The MPC optimal control policy is obtained "on line" by the feasibility of multiparametric programming, that be further expressed in terms of a QP problem, using as constraints the trajectories constraints and the obtained invariant set as terminal set of the optimization routine.

9.2 Perspectives

Even if the set-invariance theory is extensively exploited in the literature, there are several opportunities in terms of set invariance for time-delay systems. In the present thesis, we introduced some new concepts, specially linked to the invariance problem. Although the class of systems is not large and the conditions are quite restrictive, the aim is to formulate the problem and underline the interest of research of the subject and give some constructive solutions. Several improvements in this theory can be made, namely:

- Generalize the concept of $\mathcal{D}$-invariance to other classes of systems.
- Obtain less conservative conditions for $\mathcal{D}$-invariance.
- Characterize the "maximal/minimal" invariant sets for time-delay systems.
- Apply the concept of $\mathcal{D}$-invariance and generalize the results on the MPC scheme.
- Generalize the $\mathcal{D}$-invariance by using the *comparison principle*.

Appendices

Appendix A

List of Publications

A.1 Accepted

A.1.1 Chapters

- W. Lombardi, S. Olaru, S.-I. Niculescu. Polytopic discrete-time models for systems with time-varying delays. In 'Time Delay Systems - Methods, Applications and New Trends', LNCIS Springer, 2011

A.1.2 Conferences

- W. Lombardi, S. Olaru, and S.-I. Niculescu. Invariant sets for a class of linear systems with variable time-delay. In Proceedings of the European Control Conference, 2009.

- W. Lombardi, S. Olaru, and S.-I. Niculescu. Robust invariance for a class of time-delay systems with repeated eigenvalues. In Proceedings of the IFAC TDS, 2009.

- W. Lombardi, A. Luca, S. Olaru, and S.-I. Niculescu. State admissible sets for discrete systems under delay constraints. In Proceedings of the American Control Conference, 2010.

- W. Lombardi, A. Luca, S. Olaru, S.-I. Niculescu, and J. Cheong. Feedback stabilization and motion synchronization of systems with time-delay in the communication network. In Proceedings of the IFAC TDS, 2010.

- W. Lombardi, A. Luca, S. Olaru, S.-I. Niculescu, J. Cheong and P. Boucher. Synchronisation de mouvements sous contraintes pour des systèmes à retard. CIFA 2010

- W. Lombardi, S. Olaru, M. Lazar, and S.-I. Niculescu. On positive invariance for delay difference equations. In Proceedings of the American Control Conference, 2011.

- W. Lombardi, S. Olaru, M. Lazar, G. Bitsoris, and S.-I. Niculescu. On the polyhedral set-invariance conditions for time-delay systems. In Proceedings of IFAC World Congress, 2011.

A.2 Submited

A.2.1 Journals

- W. Lombardi, S. Olaru, S.-I. Niculescu and L. Hetel. A predictive control scheme for systems with variable time-delay. Submitted on the International Journal of Control.

- W. Lombardi, S. Olaru, G. Bitsoris, S.-I. Niculescu. Cyclic invariance for discrete time-delay systems. Submitted on Automatica.

Appendix B

Algorithms

B.1 Approximation of the minimal-volume embedding

Algorithm 1 describes the recursive procedure which provides an approximation of the minimal-volume embedding for the function $\Delta(\epsilon)$.

Algorithm 1 Computes $\mathcal{E}^a(\delta)$, a δ approximation of the minimum-volume simplex embedding with $n+1$ generators for $\Delta(\epsilon) = \int_0^\epsilon e^{A_c\tau} B_c d\tau,$ with $\forall\epsilon \in [0,\ \bar{\epsilon}]$.

Require: A_c, B_c, $\bar{\epsilon}$, δ
Ensure: $\{\Delta_0, \Delta_1, \ldots, \Delta_n\}$
1: Compute an embedding based on Section 3.2
2: Sample the intervals $[0,\ \bar{\epsilon}/2]$ and $[\bar{\epsilon}/2,\ \bar{\epsilon}]$
3: For each pair of consecutive points in $[0,\ \bar{\epsilon}]$ construct a local embedding and collect all the resulting extreme points
4: Solve (4.10) with the constraints given by the points obtained at step 3 using as initial conditions the existing embedding or the one obtained in step 1
5: **if** $\delta < \mathit{precision}$ **then**
6:     **return** Polytope Δ_i
7: **else**
8:     Take the local embedding with maximum volume and with an extreme point placed on the boundary of the global simplex embedding $\mathcal{E}^a$
9:     Divide the i-th interval $[\epsilon_i, \epsilon_{i+1}]$
10:     Return to step 2
11: **end if**

B.2 A generic procedure for $\mathcal{D}$-invariant set construction

In this subsection, the main steps of an iterative construction of $\mathcal{D}$-invariant sets is described. One of the main concepts to be exploited is the upper boundedness of the set-wise non-decreasing sequence of sets $\Psi(\mathcal{P})$ in case of the existence of a bounded $\mathcal{D}$-invariant set. The following results are instrumental in this sense.

Lemma B.1. *If $\mathcal{P} \in ComC(\mathbb{R}^n)$ is a $\mathcal{D}$-invariant set for (7.9) then for any subset $\mathcal{S} \subseteq \mathcal{P}$ one has:*

$$\Phi^k(\mathcal{S}) \subseteq \mathcal{P}, \forall k \in \mathbb{Z}_+^*.$$

Proof. By assumption, the set $\mathcal{P}$ is $\mathcal{D}$-invariant which assures $\Phi(\mathcal{P}) \subseteq \mathcal{P}$. Since $\mathcal{S} \subseteq \mathcal{P}$ one has $\Phi(\mathcal{S}) \subseteq \Phi(\mathcal{P}) \subseteq \mathcal{P}$. The result holds for all $k \geq 0$ by induction. ■

Definition B.2. Given a $\mathcal{D}$-invariant set $\mathcal{P} \subset \mathcal{X}$ we denote $\hat{\mathcal{P}}$ the class of $\mathcal{D}$-invariant sets contained in $\mathcal{X}$ and containing $\mathcal{P}$:

$$\hat{\mathcal{P}} = \{\mathcal{D} \subseteq \mathcal{X} | \mathcal{P} \subseteq \mathcal{D}, \, \mathcal{D} \text{ is } \mathcal{D}\text{-invariant}\}. \tag{B.1}$$

□

Definition B.3.

$$d^* = \inf_{\mathcal{D} \in \hat{\mathcal{P}}} d_H(\mathcal{D}, \mathcal{P}) \geq 0 \tag{B.2}$$

□

Definition B.4.

$$\hat{\mathcal{P}}^* = \left\{ \mathcal{D} \subseteq \hat{\mathcal{P}} | d_H(\mathcal{D}, \mathcal{P}) = d^* \right\}. \tag{B.3}$$

□

Theorem B.5. *Consider:*

$$\lim_{k \to \infty} \Psi^k(\mathcal{P}) \in \hat{\mathcal{P}}^*. \tag{B.4}$$

So, the limit is one of the $\mathcal{D}$-invariant supersets with minimal Hausdorff distance.

Theorem B.6. *Given a convex set $\mathcal{P} \in ComC(\mathbb{R}^n)$, the sequence $\Psi^k(\mathcal{P}), k \geq 0$ converges toward the closest (in Hausdorff distance sense) $\mathcal{D}$-invariant convex superset.*

Proof. Let $\mathcal{D} \subseteq \mathbb{R}^n$ be a convex $\mathcal{D}$-invariant superset of $\mathcal{P}$ (a set which contains $\mathcal{P}$). Using Lemma B.1, $\mathcal{P} \subseteq \mathcal{D}$ implies $\Phi(\mathcal{P}) \subseteq \mathcal{D}$. Exploiting the fact that all the sets are convex by hypothesis, one has $Co(\mathcal{P}, \Phi(\mathcal{P})) \subseteq \mathcal{D}$ which further implies $\Psi(\mathcal{P}) \subseteq \mathcal{D}$. By induction $\Psi^k(\mathcal{P}) \subseteq \mathcal{D}$ and in the same time $\Psi^k(\mathcal{P})$ is a non-decreasing sequence of sets. Thus the following inclusion holds:

$$\mathcal{P} \subseteq \Psi(\mathcal{P}) \subseteq \Psi^2(\mathcal{P}) \subseteq \cdots \subseteq \Psi^k(\mathcal{P}) \subseteq \cdots \subseteq \mathcal{D}. \tag{B.5}$$

The relationship (B.5) holds for any $\mathcal{D}$-invariant $\mathcal{C}$-set containing 0 in its interior and thus for any $\mathcal{D}$-invariant $\mathcal{C}$-set with the minimal Hausdorff distance with respect to $\mathcal{P}$ (note the property 7.7 in Subsection 7.2.1).

At this point, one proves the existence of a non-decreasing sequence of sets upperbounded by a $\mathcal{D}$-invariant set. In order to complete the proof, the limit set $\bar{\mathcal{D}} = \lim_{k\to\infty} \Phi^k(\mathcal{P})$ should be a $\mathcal{D}$-invariant set. This is immediate by the fact that $\Phi(\Psi^k(\mathcal{P})) \subseteq \Psi^{k+1}(\mathcal{P})$ and thus:

$$\lim_{k\to\infty} \Phi(\Psi^k(\mathcal{P})) \subseteq \lim_{k\to\infty} \Psi^{k+1}(\mathcal{P}) \Rightarrow \Phi(\bar{\mathcal{D}}) \subseteq \bar{\mathcal{D}} \tag{B.6}$$

■

This main result offers a basis for a generic construction of convex $\mathcal{D}$-invariant sets based on set-iterations. The main objective is to obtain an enlarged $\mathcal{D}$-invariant set in a predefined region $\mathcal{X} \subseteq \mathbb{R}^n$. Given a delay-difference equation as (7.9) and a compact and convex set $\mathcal{X} \subseteq \mathbb{R}^n$, by denoting the initial set $\mathcal{R}_0 = \mathcal{X}_0$ and by applying the operation $\mathcal{T} = Co(\mathcal{R}_j, x)$ with $x \in int(\mathcal{X} \setminus \mathcal{R}_j)$, compute the limit set $\mathcal{D} = \lim_{k\to\infty} \Phi^k(\mathcal{T})$ and test if it is $\mathcal{D}$-invariant. If it is $\mathcal{D}$-invariant, then this is the new continue including new x, with $x \in int(\mathcal{X} \setminus \mathcal{R}_j)$, otherwise one has to chose an alternative x. The algorithm details are presented in Algorithm 2, Appendix B.

An advantage of this algorithm is that at each intermediary step one can dispose of a $\mathcal{D}$-invariant set and the procedure will produce a sequence of mathematical increasing $\mathcal{D}$-invariant sets in $\mathcal{X}$.

In order to have an effective implementation of this algorithm, there are two issues that have to be clarified :

- How to estimate the Hausdorff distance between the current $\mathcal{D}$-invariant set and the maximal $\mathcal{D}$-invariant set (the former one being à priori unknown). In fact, this estimation will be used in the evaluation of a stopping criteria for the set-iteration.
- How to chose points in $\mathcal{X} \setminus \mathcal{R}_i$ and how to use them in the approximation of the limit $\mathcal{D}$-invariant set.

In order to answer to these questions, in the following some auxiliary routines are proposed.

B.2.1 Auxiliary routines

Several operations which are instrumental for the proposed practical $\mathcal{D}$-invariant set construction are here defined in an algorithmic form .

The first in this enumeration is the routine providing a $\mathcal{D}$-invariant set by set-iterations from an initial arbitrary set $\mathcal{S}$.

- $\mathcal{R} = get\_Dinv(\mathcal{S})$: an algorithmic construction of the limit set:

$$\lim_{k\to\infty} \Psi^k(\mathcal{S}).$$

  The idea is: given a delay-difference equation as (7.9) and a compact and convex set $\mathcal{S} \subseteq \mathbb{R}^n$, compute the mapping $\mathcal{R}_{i+1} = \Psi(\mathcal{R}_j)$, given by (7.65), with the initial set $\mathcal{R}_0 = \mathcal{S}$, while $\mathcal{R}_i \neq \mathcal{R}_{j-1}$. The details are presented in Algorithm 3 in Appendix B.

In order to evaluate the speed of converge of the maximal $\mathcal{D}$-invariant set approximation, a pair of inner-outer approximations has to be used. If the inner approximation is mainly based on the numerical exploitation of the Theorem B.6, for the outer approximation it direct the use of the complementary set for a $\mathcal{D}$-invariant set $\mathcal{R}$ with respect to a given subspace of the state space $\mathcal{X}_0$. This will be defined as the collection of points in $\mathcal{X}_0$, which mapped by all A_i, for $i \in \mathbb{Z}_{[0,d]}$, can be summed up (in the Minkowski sense) with any point in $A_i\mathcal{R}$ and remain in the collection.

- $\mathcal{P} = outer(\mathcal{R}, \mathcal{X}_0, A_0, \ldots, A_d)$: An iterative algorithm which uses a $\mathcal{D}$-invariant set $\mathcal{R}$ to construct the largest set $\mathcal{P} \subseteq \mathcal{X}_0 \subseteq \mathbb{R}^n$ that verifies:

$$\begin{aligned} A_0\mathcal{P} \oplus \bigoplus_{i=1}^{d} \{A_i\mathcal{R}\} &\subseteq \mathcal{P}; \\ A_0\mathcal{R} \oplus A_1\mathcal{P} \oplus \bigoplus_{i=2}^{d} \{A_i\mathcal{R}\} &\subseteq \mathcal{P}; \\ &\vdots \\ \bigoplus_{i=0}^{d-1} \{A_i\mathcal{P}\} \oplus A_d\mathcal{R} &\subseteq \mathcal{P}. \end{aligned}$$

  This algorithm takes a $\mathcal{D}$-invariant set $\mathcal{R} \subseteq \mathbb{R}^n$ with respect to (7.9), with $\mathcal{R}$ as initial set, i.e. $\mathcal{P}_0 = \mathcal{R}$ and apply the mapping:

$$\mathcal{P}_{j+1} = \{x \in \mathcal{P}_0 \mid \quad \begin{aligned} A_0 x \oplus \textstyle\bigoplus_{i=1}^{d} A_i\mathcal{R} &\subseteq \mathcal{P}_0; \\ A_0\mathcal{R} \oplus A_1 x \oplus \textstyle\bigoplus_{i=2}^{d} A_i\mathcal{R} &\subseteq \mathcal{P}_0; \\ &\vdots \\ \textstyle\bigoplus_{i=0}^{d-1} A_i\mathcal{R} \oplus A_d x &\subseteq \mathcal{P}_0\}, \end{aligned}$$

  while $\mathcal{P}_j \neq \mathcal{P}_{j-1}$.

  The formal statement about this algorithm are presented in Algorithm 4.

In order to adapt the outer approximation of the maximal $\mathcal{D}$-invariant set after each refinement of the inner approximation one has to introduce new points in the step 2) of Algorithm 2. In our implementation, at each iteration, the vertice of the outer approximation corresponding to the point which generated the Hausdorff distance between the inner and the outer approximation is used.

- $[dH, v_H] = d_H(\mathcal{P}_1, \mathcal{P}_2)$: is a function which returns the Hausdorff distance between $\mathcal{P}_1 \supseteq \mathcal{P}_2$. The second output argument is the point $v_H \in \mathcal{P}_1$ for which $d_H(\mathcal{P}_1, \mathcal{P}_2) = d_H(v_H, \mathcal{P}_2)$ [1].

[1] It is well known the fact that the Hausdorff distance ban be found by solving a quadratic program for each vertex of the set $\mathcal{A}$.

B.2.2 Effective approximation of the maximal $\mathcal{D}$-invariant set

Once all the generic steps are defined and the specific routines are available, one can bring together all the elements and describe an effective algorithm for the ϵ-approximation of the maximal $\mathcal{D}$-invariant set.

The first step is to obtain a $\mathcal{D}$-invariant set $\mathcal{S} \subseteq \mathbb{R}^n$ with respect to (7.9) by using $\mathcal{S} = get\_Dinv(\mathcal{X}_0)$, such that $\mathcal{S} \subseteq \mathcal{X}_0$. Do $\mathcal{P}_0 = outer(\mathcal{X}_0, \mathcal{R}, A_0, \ldots, A_d)$. Calculate the Hausdorff distance $[dH, v_H] = d_H(\mathcal{P}_0, \mathcal{R}_0)$, the convergence metric between the iterations. While $dH \leq \mathcal{E}$, i.e. the Hausdorff distance is bigger than the given precision, choose a point (λv_H) between the sets $\mathcal{P}_j$ and $\mathcal{R}_j$ and do $\mathcal{T} = Co(\lambda v_H, \mathcal{R}_j)$. Do $\mathcal{R}_{j+1} = get\_Dinv(\mathcal{T})$ and $\mathcal{P}_{j+1} = outer(\mathcal{P}_j, \mathcal{R}_{j+1}, A_0, \ldots, A_d)$. If the set $\mathcal{R}_{j+1}$ is not equal or a subset of $\mathcal{P}_{j+1}$, then $\mathcal{P}_{j+1} = Co(\mathcal{R}_j, \lambda v_H, \mathcal{V}(\mathcal{P}_j) \setminus \{v_H\})$.

The details are expressed in Algorithm 5.

In order to obtain the maximal $\mathcal{D}$-invariant set, Algorithm 5 will construct a sequence of sets which satisfies the following property:

$$\mathcal{X}_0 = \mathcal{P}_0 \supseteq \mathcal{P}_1 \supseteq \cdots \supseteq \mathcal{P}_j \supseteq \cdots \supseteq \mathcal{R}_j \supseteq \mathcal{R}_1 \supseteq \mathcal{R}_0. \tag{B.7}$$

In terms of Hausdorff distance:

$$d_H(\mathcal{P}_0, \mathcal{R}_0) \geq d_H(\mathcal{P}_1, \mathcal{R}_1) \geq \cdots \geq d_H(\mathcal{P}_j, \mathcal{R}_j) \geq \ldots \tag{B.8}$$

and due to the finite number of vertices and the the fact the points λv_H are in the strict interior of $\mathcal{P}_j \setminus \mathcal{R}_j$ one has:

$$\lim_{j \to \infty} d_H(\mathcal{P}_j, \mathcal{R}_j) = 0. \tag{B.9}$$

Thus Algorithm 5 has an effective stopping criteria in the condition $d_H < \mathcal{E}$. A transparent tuning parameter for the proposed routine is the scalar λ which place the points to be treated closer to the inner or to the outer approximation of the maximal $\mathcal{D}$-invariant set. It is interesting to remark the fact that λ can be adapted at each iteration.

Algorithm 2 describes a generic procedure to obtain $\mathcal{D}$-invariant sets in $\mathbb{R}^n$ based on set iterations.

Algorithm 2 Approximation of the maximal $\mathcal{D}$-invariant set in a predefined region $\mathcal{X} \subseteq \mathbb{R}^n$.

Require: A convex set $\mathcal{X} \in ComC(\mathbb{R}^n)$, $A_i \in \mathbb{R}^{n \times n}$ for $i \in \mathbb{Z}_{[0,d]}$
Ensure: A inner approximation of the maximal $\mathcal{D}$-invariant set in $\mathcal{X}$
1: Obtain $\mathcal{R}_0$, a initial $\mathcal{D}$-invariant set in $\mathcal{X}_0$. Set $j = 0$
2: Compute a set $\mathcal{T} = Co(\mathcal{R}_j, x)$ with $x \in int(\mathcal{X} \setminus \mathcal{R}_j)$
3: Compute the limit set $\mathcal{D} = \lim_{k \to \infty} \Phi^k(\mathcal{T})$
4: **if** $\mathcal{D} \subseteq \mathcal{X}$ **then**
5: $\quad \mathcal{R}_{j+1} = \mathcal{D}$
6: $\quad$ **if** $\mathcal{R}_j$ is not a acceptable approximation **then**
7: $\quad\quad$ Start over from step 2 with $j = j + 1$
8: $\quad$ **else**
9: $\quad\quad$ **return** $\mathcal{R}_j$
10: $\quad$ **end if**
11: **else**
12: $\quad$ Start over from step 2 with a new point x and note that the previous selected point is not contained in the maximal convex $\mathcal{D}$-invariant set
13: **end if**

This algorithm provides a $\mathcal{D}$-invariant set by set-iterations from an initial arbitrary set $\mathcal{S}$.

Algorithm 3 $\mathcal{R} = get\_Dinv(\mathcal{S})$: an algorithmic construction of the limit set $\lim\limits_{k \to \infty} \Psi^k(\mathcal{S})$.

Require: A convex set $\mathcal{S} \in ComC(\mathbb{R}^n)$, $A_i \in \mathbb{R}^{n \times n}$ for $i \in \mathbb{Z}_{[0,d]}$
Ensure: $\mathcal{R}$
1: $\mathcal{R}_0 = \mathcal{S}$
2: $\mathcal{R}_1 = \Psi(\mathcal{R}_0)$
3: **while** $\mathcal{R}_i \neq \mathcal{R}_{j-1}$ **do**
4: $\quad \mathcal{R}_{i+1} = \Psi(\mathcal{R}_j)$
5: $\quad j = j + 1$
6: **end while**
7: **return** $\mathcal{R} = \mathcal{R}_j$

This algorithm uses a $\mathcal{D}$-invariant set $\mathcal{R}$ to construct the largest set $\mathcal{P} \subseteq \mathcal{X}_0 \subseteq \mathbb{R}^n$ that verifies:

$$\begin{aligned}
A_0\mathcal{P} \oplus \bigoplus_{i=1}^{d} \{A_i\mathcal{R}\} &\subseteq \mathcal{P}; \\
A_0\mathcal{R} \oplus A_1\mathcal{P} \oplus \bigoplus_{i=2}^{d} \{A_i\mathcal{R}\} &\subseteq \mathcal{P}; \\
&\vdots \\
\bigoplus_{i=0}^{d-1} \{A_i\mathcal{P}\} \oplus A_d\mathcal{R} &\subseteq \mathcal{P}.
\end{aligned}$$

This algorithms provides a constructive routine to obtain the maximal $\mathcal{D}$-invariant set.

Algorithm 4 $\mathcal{P} = outer(\mathcal{R}, \mathcal{X}_0, A_0, \ldots, A_d)$: An iterative algorithm which use a $\mathcal{D}$-invariant set $\mathcal{R}$ to construct the largest set $\mathcal{P} \subseteq \mathcal{X}_0 \subseteq \mathbb{R}^n$

Require: Convex sets $\mathcal{R} \subseteq \mathcal{X}_0 \in Com\mathbb{R}^n$, $A_i \in \mathbb{R}^{n\times n}$ for $i \in \mathbb{Z}_{[0,d]}$
Ensure: $\mathcal{P}$

$$\mathcal{P}_1 = \{x \in \mathcal{P}_0 \mid \begin{array}{c} A_0 x \oplus \bigoplus_{i=1}^{d} A_i\mathcal{R} \subseteq \mathcal{P}_0; \\ A_0\mathcal{R} \oplus A_1 x \oplus \bigoplus_{i=2}^{d} A_i\mathcal{R} \subseteq \mathcal{P}_0; \\ \vdots \\ \bigoplus_{i=0}^{d-1} A_i\mathcal{R} \oplus A_d x \subseteq \mathcal{P}_0\} \end{array}$$

while $\mathcal{P}_j \neq \mathcal{P}_{j-1}$ **do**

$$\mathcal{P}_{j+1} = \{x \in \mathcal{P}_0 \mid \begin{array}{c} A_0 x \oplus \bigoplus_{i=1}^{d} A_i\mathcal{R} \subseteq \mathcal{P}_0; \\ A_0\mathcal{R} \oplus A_1 x \oplus \bigoplus_{i=2}^{d} A_i\mathcal{R} \subseteq \mathcal{P}_0; \\ \vdots \\ \bigoplus_{i=0}^{d-1} A_i\mathcal{R} \oplus A_d x \subseteq \mathcal{P}_0\} \end{array}$$

  $j = j + 1$
end while
return $\mathcal{P} = \mathcal{P}_j$

Algorithm 5 ϵ-approximation of the maximal $\mathcal{D}$-invariant set.

Require: Convex set $\mathcal{X}_0 \in Com\mathbb{R}^n$, $A_i \in \mathbb{R}^{n\times n}$, for $i \in \mathbb{Z}_{[0,d]}$, and a $\mathcal{E} \in \mathbb{R}$
Ensure: $\mathcal{R}$, a convex $\mathcal{D}$-invariant set representing a ϵ-approximation of the maximal convex $\mathcal{D}$-invariant set in $\mathcal{X}_0$

```
S = get_Dinv(X_0)
Find γ such that γS ⊆ X_0
R_0 = γS
P_0 = outer(X_0, R, A_0, ..., A_d)
j = 1
[dH, v_H] = d_H(P_0, R_0)
while dH ≤ E do
   Choose λ ∈ R_(0,1)] such that λv_H ∈ int(P_j \ R_j)
   T = Co(λv_H, R_j)
   R_{j+1} = get_Dinv(T)
   P_{j+1} = outer(P_j, R_{j+1}, A_0, ..., A_d)
   if R_{j+1} ⊈ P_{j+1} then
      R_{j+1} = R_j
      P_{j+1} = Co(R_j, λv_H, V(P_j) \ {v_H})
   end if
   j = j + 1
   [dH, v_H] = d_H(P_j, R_j)
end while
return R_j
```

Appendix C

Proofs, constructions and computational aspects

C.1 Matrices construction for the Theorem 7.16 and 7.17

Let us define $\bar{F}_i = FA_i$, for $i \in \mathbb{Z}_{[0,d]}$. Substituting $\bar{F}_i$ in Eq. (7.23) we have $\bar{F}_i^\top = H_i^\top F^\top$. The matrices are formed by the elements: $\bar{F}_i = \{\bar{f}^i_{pq}\}$, $A_i = \{a^i_{qs}\}$, $H_i = \{h^i_{pz}\}$, $F = \{f_{pq}\}$, $w = \{w_p\}$ for $p, z \in \mathbb{Z}_{[1,r]}$, $q, s \in \mathbb{Z}_{[1,n]}$ and $i \in \mathbb{Z}_{[0,d]}$.

The matrices $\bar{F}_i$ are:

$$\bar{F}_i = \begin{bmatrix} \sum_{j=1}^{n} \{f_{1,j}\}\{a^i_{j,1}\} & \sum_{j=1}^{n} \{f_{1,j}\}\{a^i_{j,2}\} & \cdots & \sum_{j=1}^{n} \{f_{1,j}\}\{a^i_{j,n}\} \\ \sum_{j=1}^{n} \{f_{2,j}\}\{a^i_{j,1}\} & \sum_{j=1}^{n} \{f_{2,j}\}\{a^i_{j,2}\} & \cdots & \sum_{j=1}^{n} \{f_{2,j}\}\{a^i_{j,n}\} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{j=1}^{n} \{f_{r,j}\}\{a^i_{j,1}\} & \sum_{j=1}^{n} \{f_{r,j}\}\{a^i_{j,2}\} & \cdots & \sum_{j=1}^{n} \{f_{r,j}\}\{a^i_{j,n}\} \end{bmatrix} \tag{C.1}$$

Developing also $\bar{F}_i = H_iF$, we have:

$$\begin{gathered} \begin{bmatrix} \{\bar{f}^i_{1,1}\} & \{\bar{f}^i_{2,1}\} & \dots & \{\bar{f}^i_{m,n}\} \end{bmatrix}^\top = \\ \begin{bmatrix} \sum_{j=1}^{r} \{f^i_{j,1}\}\{h^i_{1j}\} & \sum_{j=1}^{r} \{f^i_{j,1}\}\{h^i_{2,j}\} & \dots & \sum_{j=1}^{r} \{f^i_{j,n}\}\{h^i_{m,j}\} \end{bmatrix}^\top \end{gathered} \tag{C.2}$$

Developing $(\sum_{i=0}^{d} H_i)w \leq w$, we have:

$$\begin{bmatrix} \sum_{i=1}^{d}\sum_{j=1}^{r}\{h_{1,j}^{i}\}\{w_j\} \\ \sum_{i=1}^{d}\sum_{j=1}^{r}\{h_{2,j}^{i}\}\{w_j\} \\ \vdots \\ \sum_{i=1}^{d}\sum_{j=1}^{r}\{h_{r,j}^{i}\}\{w_j\} \end{bmatrix} \leq \begin{bmatrix} \{w_1\} \\ \{w_2\} \\ \vdots \\ \{w_r\} \end{bmatrix}. \tag{C.3}$$

To guarantee the positivity of the elements of H_i, the constraint $-\bar{h} \leq 0$ must be added.

The matrices $A_{eq} \in \mathbb{R}^{drn \times dr^2}$ and $b_{eq} \in \mathbb{R}^{drn}$, done by (C.4), are obtained by using (C.2):

$$\begin{aligned} A_{eq} &= \operatorname{diag}(F^\top, r \times n \times d); \\ b_{eq}^\top &= \begin{bmatrix} \{\bar{f}_{1,1}^{1}\} & \{\bar{f}_{2,1}^{1}\} & \cdots & \{\bar{f}_{r,n}^{1}\} & \{\bar{f}_{1,1}^{2}\} & \cdots & \cdots & \{\bar{f}_{r,n}^{d}\} \end{bmatrix}^\top. \end{aligned} \tag{C.4}$$

The matrices $A_{in} \in \mathbb{R}^{dr^2+r \times dr^2}$ and $b_{in} \in \mathbb{R}^{dr^2+r}$, done by (C.5), are obtained by using (C.3):

$$A_{in} = \begin{bmatrix} \operatorname{diag}(w^\top, r) & \cdots & \operatorname{diag}(w^\top, r) \\ -1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & -1 \end{bmatrix}; \quad b_{in} = \begin{bmatrix} w \\ 0 \\ \vdots \\ 0 \end{bmatrix}; \tag{C.5}$$

and the vector $\bar{h} \in \mathbb{R}^{dr^2}$ is done by (C.6):

$$\bar{h}^\top = \begin{bmatrix} \{h_{1,1}^{1}\} & \{h_{1,2}^{1}\} & \cdots & \{h_{1,r}^{1}\} & \{h_{2,1}^{2}\} & \cdots & \cdots & \{h_{r,r}^{d}\} \end{bmatrix}^\top. \tag{C.6}$$

C.2 Details about the Section 7.2.5

In order to obtain the control law (7.51), the following theorem can be stated:

Theorem C.1. *The polyhedral set* (7.53) *is* $\mathcal{D}$*-invariant with respect to the closed loop system* (7.52) *if there exists the state feedback* $K \in \mathbb{R}^{m \times n}$*, the vector* $\bar{h} \in \mathbb{R}^{dr^2}$ *and the* *contraction factor* $\lambda \in \mathbb{R}_{[0,1]}$ *done as the feasible solution of:*

$$\begin{aligned} \min_{\bar{h}} \quad & \lambda \\ \textit{subject to:} \quad & \begin{cases} \bar{A}_{eq}\bar{h} = \bar{b}_{eq} \\ \bar{A}_{in}\bar{h} \leq \bar{b}_{in} \\ 0 \leq \lambda \leq 1 \\ \bar{h} \geq 0. \end{cases} \end{aligned} \tag{C.7}$$

where $\bar{A}_{in}$*,* $\bar{h}$ *and* $\bar{b}_{in}$ *are done by:*

$$\bar{A}_{in} = \begin{bmatrix} A_{in} & 0 & 0 \\ -I_{dr^2} & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix} \quad \bar{b}_{in} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad \bar{h} = \begin{bmatrix} \tilde{h} \\ \bar{K}^{\top} \\ \lambda \end{bmatrix} \tag{C.8}$$

where $\bar{K}^{\top}$ *is done by:*

$$\bar{K}^{\top} = \begin{bmatrix} \{k_{1,1}\} & \{k_{1,2}\} & \dots & \{k_{m,n}\} \end{bmatrix}^{\top}. \tag{C.9}$$

Proof. The matrices A_{in}, b_{in} and $\tilde{h}$ are presented in the Appendix A and $\bar{A}_{in}$, $\bar{b}_{in}$, $\bar{A}_{eq}$, $\bar{b}_{eq}$ and $\bar{h}$ are presented in the Section C.3. ■

C.3 Obtention of the Matrices of Section 7.2.5

Let us define $\bar{F}_i = F(A_i + B_i K)$, for $i \in \mathbb{Z}_{[0,d]}$. Substituting $\bar{F}_i$ in Eq. (7.23) one has $\bar{F}_i = H_i F$. The matrices are formed by the elements: $\bar{F}_i = \{\bar{f}^i_{p,q}\}$, $A_i = \{a^i_{q,y}\}$, $B_i = \{b^i_{q,s}\}$, $K = \{k_{s,q}\}$ $H_i = \{h^i_{p,z}\}$, $F = \{f_{p,q}\}$, $w = \{w_p\}$ for $p, z \in \mathbb{Z}_{[1,r]}$, $q, y \in \mathbb{Z}_{[1,n]}$, $s \in \mathbb{Z}_{[1,m]}$ and $i \in \mathbb{Z}_{[0,d]}$.

The matrices $\bar{F}_i$ are:

$$\bar{F}_i = F(A_i + B_i K) =$$
$$\begin{bmatrix} \sum_{g=1}^{n} \{f_{1,g}\} \left(\{a^i_{g,1}\} + \sum_{j=1}^{m} \{b^i_{g,j}\} \{k_{j,1}\}\right) & \ldots \\ \sum_{g=1}^{n} \{f_{2,g}\} \left(\{a^i_{g,1}\} + \sum_{j=1}^{m} \{b^i_{g,j}\} \{k_{j,1}\}\right) & \ldots \\ \vdots & \ddots \\ \sum_{g=1}^{n} \{f_{r,g}\} \left(\{a^i_{g,1}\} + \sum_{j=1}^{m} \{b^i_{g,j}\} \{k_{j,1}\}\right) & \ldots \end{bmatrix} \tag{C.10}$$

One can let the terms $\{h^i_{pp}\}$ and $\{k_{sq}\}$ on the right part side by manipulating (C.10):

$$\begin{bmatrix} \sum_{j=1}^{n} \{f_{1,j}\} \{a^i_{j,1}\} & \cdots \\ \sum_{j=1}^{n} \{f_{2,j}\} \{a^i_{j,1}\} & \cdots \\ \vdots & \ddots \\ \sum_{j=1}^{n} \{f_{r,j}\} \{a^i_{j,1}\} & \cdots \end{bmatrix} =$$
$$\begin{bmatrix} \sum_{j=1}^{r} \{f^i_{j,1}\} \{h^i_{1,j}\} - \sum_{g=1}^{n} \{f_{1,g}\} \sum_{j=1}^{m} \{b^i_{g,j}\} \{k_{j,1}\} & \cdots \\ \sum_{j=1}^{r} \{f^i_{j,1}\} \{h^i_{2,j}\} - \sum_{g=1}^{n} \{f_{2,g}\} \sum_{j=1}^{m} \{b^i_{g,j}\} \{k_{j,1}\} & \cdots \\ \vdots & \ddots \\ \sum_{j=1}^{r} \{f^i_{j,1}\} \{h^i_{r,j}\} - \sum_{g=1}^{n} \{f_{r,g}\} \sum_{j=1}^{m} \{b^i_{g,j}\} \{k_{j,1}\} & \cdots \end{bmatrix} \tag{C.11}$$

The matrices $\bar{A}_{eq} \in \mathbb{R}^{drn \times dr^2}$ and $\bar{b}_{eq} \in \mathbb{R}^{drn}$, done by (C.12) and (C.13), are obtained by using (C.11).

$$\bar{A}_{eq} = \left[\begin{array}{c|cc} & \text{diag}(\sum_{g=1}^{m} \{b_{1,g}\} \{f_{g,1}^1\}, n) & \cdots \\ & \text{diag}(\sum_{g=1}^{m} \{b_{2,g}\} \{f_{g,1}^1\}, n) & \cdots \\ A_{eq} & \vdots & \ddots \\ & \text{diag}(\sum_{g=1}^{m} \{b_{r,g}\} \left\{f_{g,1}^d\right\}, n) & \cdots \end{array}\right] \tag{C.12}$$

$$\bar{b}_{eq}^\top = \left[\{\bar{f}_{1,1}^1\} \quad \{\bar{f}_{2,1}^1\} \quad \cdots \quad \{\bar{f}_{r,n}^1\} \quad \{\bar{f}_{1,1}^2\} \quad \cdots \quad \cdots \quad \{\bar{f}_{r,n}^d\}\right]^\top \tag{C.13}$$

Developing $(\sum_{i=0}^{d} H_i)w \leq \lambda w$, one has:

$$\begin{bmatrix} \sum_{i=1}^{d}\sum_{j=1}^{r} \{h_{1,j}^i\} \{w_j\} - \lambda \{w_1\} \\ \sum_{i=1}^{d}\sum_{j=1}^{r} \{h_{2,j}^i\} \{w_j\} - \lambda \{w_2\} \\ \vdots \\ \sum_{i=1}^{d}\sum_{j=1}^{r} \{h_{r,j}^i\} \{w_j\} - \lambda \{w_r\} \end{bmatrix} \leq 0. \tag{C.14}$$

To guarantee the positivity of the elements of H_i, the constraint $-\bar{h} \leq 0$ must be added.

The matrices $\bar{A}_{in} \in \mathbb{R}^{dr^2+r+2 \times dr^2}$ and $\bar{b}_{in} \in \mathbb{R}^{dr^2+r+2}$, done by (C.15), are obtained by using (C.14) and (C.5):

$$\bar{A}_{in} = \begin{bmatrix} A_{in} & 0 & 0 \\ -I_{dr^2} & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}; \quad b_{in} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}; \tag{C.15}$$

and the vector $\bar{h} \in \mathbb{R}^{dr^2}$ is done by (C.16):

$$\bar{h}^\top = \begin{bmatrix} \tilde{h}^\top & \bar{K}^\top & \lambda \end{bmatrix}^\top, \tag{C.16}$$

where $\bar{K}^\top$ is done by:

$$\bar{K}^\top = \begin{bmatrix} \{k_{1,1}\} & \{k_{1,2}\} & \dots & \{k_{1,n}\} & \{k_{2,1}\} & \dots & \dots & \{k_{m,n}\} \end{bmatrix}^\top. \tag{C.17}$$

C.4 Computational aspects of the $\mathcal{D}$-invariance tests

The *time* to solve a $\mathcal{D}$*-invariance verification problem* of a set in $\mathbb{R}^n$ for a delay-system is studied using two different methods: Minkowski-based tests (Section 7.2.4.1) and feasibility-based tests (Sections 7.2.4.2 and 7.2.4.3). All the simulations were run on a Intel Core2 (32 bi=ts) 2.13 GHz CPU, with a 2 Gb RAM memory, Windows Vista Enterprise and MatLab R2007b. The Minkowski additions were calculated by using the Multi-Parametric Toolbox Version 2.6.2 platform [97].

The simulated system is of the same type as (7.9). Two series of tests are proposed. In the first type of test, presented in Section C.4.0.1, the delay size is fixed as $d = 1$, and the variable is the Euclidean space dimension n and the number of vertices j of the polyhedral set. A second series of tests is performed (Section C.4.0.2), by fixing the Euclidean space dimension $n = 2$, number of vertices $j = 3$ and the varying parameter is the delay size d. The matrices $A_i \in \mathbb{R}^{n\times n}$, for $i \in \mathbb{Z}_{[0,1]}$ are randomly generated by using the Jordan decomposition of $A_i = LML^{-1}$.

C.4.0.1 Fixed d and varying n and r

For each state-space dimension n and extreme point j, 30 tests are performed, in order to obtain an average solving time. At each new test, when the dimension of the state-space changes, a new system is found, by generating randomly a new generalized eigenvectors matrix L and a new Jordan canonical form matrix M.

Before performing the tests, the set $\mathcal{P}$ was provided in the half-space representation, i.e. $\mathcal{P} = \{x \in \mathbb{R}^n | Fx \leq w\}$.

The Tables C.1, C.2 and C.3 show the time, in seconds, spent to solve the Minkowski-based and feasibility-based $\mathcal{D}$-invariance tests, in its original and dual versions, respectively. In the columns the state-space size $\mathbb{R}^n$ are shown, where $n \in \mathbb{Z}_{[2,7]}$; in the rows one has the number of extreme points $j \in \mathbb{Z}_{[3,10]}$ of the set $\mathcal{P} \subset \mathbb{R}^n$.

j \ n	2	3	4	5	6	7
3	0.150					
4	0.140	0.140				
5	0.141	0.143	0.148			
6	0.144	0.150	0.175	0.249		
7	0.147	0.157	0.206	0.408	2.036	
8	0.146	0.165	0.249	0.835	4.136	22.748
9	0.152	0.171	0.312	1.606	22.908	68.940
10	0.155	0.182	0.415	3.871	34.727	353.054

TABLE C.1: Computation time in seconds for the Minkowski addition procedure, presented in Section 7.2.4.1

j \ n	2	3	4	5	6	7
3	0.252					
4	0.236	0.231				
5	0.237	0.241	0.235			
6	0.239	0.247	0.246	0.243		
7	0.243	0.256	0.271	0.264	0.249	
8	0.245	0.268	0.326	0.382	0.319	0.264
9	0.251	0.281	0.429	0.694	0.717	0.411
10	0.249	0.310	0.617	1.900	3.195	2.181

TABLE C.2: Computation time in seconds for the feasibility procedure, presented in Section 7.2.4.2

C.4.0.2 Fixed n and r and varying d

The set $\mathcal{P}$ is provided in the half-space representation, i.e. $\mathcal{P} = \{x \in \mathbb{R}^n | Fx \leq w\}$. At each new test, when the size of the delay d changes, once again a new system is found, by

j \ n	2	3	4	5	6	7
3	0.230					
4	0.233	0.231				
5	0.239	0.231	0.230			
6	0.234	0.243	0.242	0.236		
7	0.237	0.251	0.272	0.263	0.238	
8	0.243	0.266	0.329	0.380	0.316	0.247
9	0.242	0.285	0.433	0.734	0.728	0.414
10	0.248	0.304	0.627	1.891	2.920	2.122

TABLE C.3: Computation time in seconds for the dual feasibility procedure, presented in Section 7.2.4.3

generating randomly a new generalized eigenvectors matrix L and a new Jordan canonical form matrix M. The delay variation is $d \in \mathbb{Z}_{[0,50]}$. Figure C.1 shows the time in seconds to solve $\mathcal{D}$-invariance verification problem in function of the delay size d.

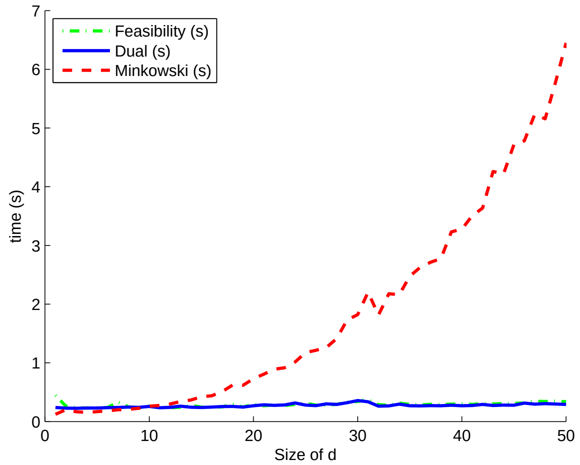


FIGURE C.1: $\mathcal{D}$-invariant tests time in function of the delay size

C.4.0.3 Discussion

Analyzing the time complexity of the tests reported in Sections C.4.0.1 and C.4.0.2, the feasibility-based tests have a better performance. According to [65], there is no polynomial time for computing the Minkowski addition in an $\mathbb{R}^n$ when n is a variable and the number o vertices j is fixed. This can be seen on the last line of the table C.1.

When n and j are fixed but d is variable, the Minkowski addition time complexity increases exponentially with time. Although [65] say that this problem *can* have a polynomial time complexity, it is not what happens here, as one can observe in the numerical results. Moreover a LP-problem has also an exponential time complexity, but the practical algorithms used to solve it have a good performance [55].

For the feasibility tests, MatLab chose the Interior-Point algorithm, where problems are solvable in polynomial time. In this case, an average of 72% was dedicated to built-in and manipulate the matrices constraints. As the size of the constraints matrices are almost the same in the feasibility and it dual, there is no important difference between the calculation times.

Analyzing the Minkowski addition tests in detail, an average of 88% of the computation time was spent on the *vertex enumeration* problem, in other words, to transform the half-space representation into a vertex representation. This procedure is computationally expensive and complex, and this fact explains the *time complexity* difference between the tests.

C.5 Matrix and vector of the Example 7.5

$$F = \begin{bmatrix} 0.4848 & 0.6989 & -0.5258 \\ 0.4536 & 0.2618 & -0.8519 \\ -0.0438 & -0.6158 & 0.7867 \\ -0.2009 & -0.6549 & 0.7285 \\ -0.6652 & -0.7039 & 0.2489 \\ 0.3335 & 0.3103 & 0.8902 \\ -0.5356 & -0.0108 & -0.8444 \\ -0.8044 & -0.5416 & -0.2443 \\ -0.2026 & 0.6379 & 0.7430 \\ -0.0178 & 0.4864 & 0.8735 \\ -0.3266 & 0.9448 & -0.0257 \\ 0.4701 & 0.7091 & -0.5256 \\ -0.2113 & 0.9023 & 0.3756 \\ 0.8566 & 0.2634 & 0.4438 \\ -0.7566 & -0.6196 & -0.2092 \\ -0.7158 & -0.6983 & 0.0065 \\ 0.9710 & -0.2202 & 0.0931 \\ 0.0531 & -0.6612 & -0.7483 \\ 0.9778 & -0.0394 & 0.2058 \\ 0.9776 & 0.0860 & 0.1923 \\ 0.9733 & 0.2267 & 0.0371 \\ 0.2635 & -0.3633 & -0.8936 \end{bmatrix}, \quad w = \begin{bmatrix} 0.6427 \\ 0.8849 \\ 0.7149 \\ 0.6450 \\ 0.2872 \\ 0.9337 \\ 0.8289 \\ 0.1572 \\ 0.8593 \\ 0.9174 \\ 0.6180 \\ 0.6443 \\ 0.7617 \\ 0.7589 \\ 0.1119 \\ 0.1788 \\ 0.7121 \\ 0.6480 \\ 0.6089 \\ 0.5916 \\ 0.6576 \\ 0.8361 \end{bmatrix}$$

Bibliography

[1] J. M. Akre, J. Juillard, S. Olaru, D. Galayko, and E. Colinet. Determination of the behaviour of self-sampled digital phase-locked loops. In *Circuits and Systems (MWSCAS), 2010 53rd IEEE International Midwest Symposium on*, pages 1089–1092. IEEE, 2010.

[2] T. Alamo, A. Cepeda, M. Fiacchini, and E. F. Camacho. Convex invariant sets for discrete-time lur'e systems. *Automatica*, 45(4):1066 – 1071, 2009.

[3] T. Alamo, A. Cepeda, D. Limon, J. M. Bravo, M. Fiacchini, and E. F. Camacho. On the computation of robust control invariant sets for piecewise affine systems. In *International Workshop on Assessment and Future Directions of Nonlinear Model Predictive Control*, August 2005.

[4] A. Alfi and M. Farrokhi. Force reflecting bilateral control of master—slave systems in teleoperation. *J. Intell. Robotics Syst.*, 52(2):209–232, 2008.

[5] F. Allgöwer and A. Zhen, editors. *Nonlinear Model Predictive Control*, volume 26 of *Progress in Systems and Control Theory*. Birkhauser, 2000.

[6] F. Allgöwer and A. Zheng. *Nonlinear model predictive control*. Birkhäuser, 2000.

[7] P. Arcara and C. Melchiorri. Control schemes for teleoperation with time delay: A comparative study. *Robotics and Autonomous Systems*, 38(1):49–64, 2002.

[8] Z. Artstein and S. V. Rakovic. Feedback and invariance under uncertainty via set-iterates. *Automatica*, 44(2):520–525, 2008.

[9] K. E. Årzén and A. Cervin. Control and embedded computing: Survey of research directions. In *16th IFAC World Congress, Prague, Czech Republic*, 2005.

[10] H. R. Bailey and M. Z. Williams. Some results on the differential-difference equation $\dot{x}(t) = \sum_{i=0}^{n} A_i x(t - T_i)$. *Journal of Mathematical Analysis and Applications*, 15:569–587, 1966.

[11] A. Bemporad and M. Morari. Predictive control of constrained hybrid systems. *Nonlinear model predictive control*, page 71, 2000.

[12] A. Bemporad, M. Morari, V. Dua, and E. Pistikopoulos. The explicit linear quadratic regulator for constrained systems. *Automatica*, 38:3–20, 2002.

[13] G. Bitsoris. On the positive invariance of polyhedral sets for discrete-time systems. *System & control Letters*, 11(3):243–248, 1988.

[14] G. Bitsoris. Positively invariant polyhedral sets of discrete-time linear systems. *International Journal of Control*, 47(6):1713–1726, 1988.

[15] F. Blanchini. Nonquadratic Lyapunov functions for robust control. *Automatica*, 31(3):451–461, 1995.

[16] F. Blanchini. Set invariance in control. *Automatica*, 35:1747–1467, 1999.

[17] F. Blanchini and S. Miani. *Set-theoretic methods in control*. Springer, 2008.

[18] V. Boffi and R. Scozzafava. A first-order linear differential-difference equation with n delays. *Journal of mathematical analysis and applications*, 17:577–589, 1967.

[19] E.-K. Boukas. Discrete-time systems with time-varying time delay: Stability and stabilizability. *Mathematical Problems in Engineering*, 2006:42489, 2006.

[20] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan. *Linear Matrix Inequalities in System and Control Theory*. Studies in Applied Mathematics. SIAM, Philadelphia, PA, June 1994.

[21] S. Boyd and L. Vandenberghe. *Convex optimization*. Cambridge Univ Pr, 2004.

[22] C. Burgat and S. Tarbouriech. Stability and control of saturated linear systems. *Nonlinear Systems: Stability and Stabilization*, page 113, 1995.

[23] E. F. Camacho and C. Bordons. *Model Predictive Control*. Springer-Verlag, 2003.

[24] E. B. Castelan and S. Tarbouriech. Positively invariant polyhedral sets for discrete-time singular systems with additive perturbations. In *Decision and Control, 1996., Proceedings of the 35th IEEE*, volume 1, pages 992–993. IEEE, 1996.

[25] E. B. Castelan, S. Tarbouriech, J. M. Gomes da Silva Jr, and I. Queinnec. L2-Stabilization of continuous-time linear systems with saturating actuators. *International Journal of Robust and Nonlinear Control*, 16(18):935–944, 2006.

[26] J. Cheong and S.-I. Niculescu. Robust motion synchronization control for interconnected systems with human interaction. In *Proceedings of the 17th IFAC World Congress*, 2008.

[27] J. Cheong, S.-I. Niculescu, and C. Kim. Motion synchronization control of distributed multisubsystems with invariant local natural dynamics. *IEEE Transactions on Robotics*, 25(2):382–398, 2009.

[28] D. W. Clarke, C. Mohtadi, and P. S. Tuffs. Generalized predictive control - part i and ii. *Automatica*, 23(2):137–160, 1987.

[29] M. B. G. Cloosterman. *Control over communication networks: modeling, analysis, and synthesis*. PhD thesis, Technische Universiteit Eindhoven, Eindhoven, the Netherlands, 2008.

[30] M. B. G. Cloosterman, L. Hetel, N. van de Wouw, W. Heemels, J. Daafouz, and H. Nijmeijer. Controller synthesis for networked control systems. *Automatica*, 2010.

[31] M. B. G. Cloosterman, N. van de Wouw, W. M. P. H. Heemels, and H. Nijmeijer. Robust stability of networked control systems with time-varying network-induced delays. In *45th IEEE Conference on Decision and Control*, pages 4980–4985, 2006.

[32] M. B. G. Cloosterman, N. van de Wouw, W. M. P. H. Heemels, and H. Nijmeijer. Stability of networked control systems with large delays. In *46th IEEE Conference on Decision and Control*, pages 5017–5022, 2007.

[33] M. Dambrine, J.-P. Richard, and P. Borne. Feedback control of time-delay systems with bounded control and state. *Mathematical Problems on Engineering*, 1(1):77–87, 1995.

[34] R. Datko. Foundations of Optimal Control Theory., 1969.

[35] C. E. T. Dorea. Self-bounded (a, b)-invariant polyhedra and constant reference tracking in constrained linear systems. *Sba: Controle & Automação Sociedade Brasileira de Automatica*, 15(1):10–16, 2004.

[36] C. E. T. Dorea. Set-invariant estimators for multiple-output discrete-time systems. In *Decision and Control, 2006 45th IEEE Conference on*, pages 4538–4543. IEEE, 2006.

[37] C. E. T. Dorea. Output-feedback controlled-invariant polyhedra for constrained linear systems. In *Decision and Control, 2009 held jointly with the 2009 28th Chinese Control Conference. CDC/CCC 2009. Proceedings of the 48th IEEE Conference on*, pages 5317–5322. IEEE, 2009.

[38] C. E. T. Dorea and J.-C. Hennet. Self-bounded (a, b)-invariant polyhedra of discrete-time systems. In *Decision and Control, 2000. Proceedings of the 39th IEEE Conference on*, volume 4, pages 3163–3168. IEEE, 2000.

[39] C. E. T. Dorea, J.-C. Hennet, F. Blanchini, and S. Miani. Discussion on:'(a, b)-invariance conditions of polyhedral domains for continuous-time systems'. *European Journal of Control*, 5(1):82–86, 1999.

[40] C. E. T. Dorea and J.C. Hennet. A geometric approach to the l1 linear control problem. In *Decision and Control, 1997., Proceedings of the 36th IEEE Conference on*, volume 2, pages 1552–1557. IEEE, 1997.

[41] C. E. T Dorea and B. E. A Milani. Design of lq regulators for state constrained continuous-time systems. *IEEE Transactions on Automatic Control*, 40(3):544–548, 1995.

[42] L. Dritsas and A. Tzes. Robust stability analysis of networked systems with varying delays. *International Journal of Control*, 82(12):2347–2355, 2009.

[43] L. Dritsas and A. Tzes. Robust guaranteed-cost controllers for networked systems with varying delays. In *Proceedings of the 18th Mediterranean Conference on Control & Automation (MED)*, 2010.

[44] V. Dua, E. N. Pistikopoulos, and M. C. Georgiadis. *Multi-Parametric Model-Based Control: Theory and Applications.* Wiley-VCH Verlag, Weinheim, Germany, 2007.

[45] R. M. Esfanjani, M. Reble, U. Munz, S. K. Y. Nikravesh, and F. Allgöwer. Model predictive control of constrained nonlinear time-delay systems. In *Proceedings of the Control and Decision Conference*, Shangai, China, 2009.

[46] M. Fiacchini, T. Alamo, and E. F. Camacho. On the computation of convex robust control invariant sets for nonlinear systems. *Automatica*, 46(8):1334 – 1338, 2010.

[47] M. Fiacchini, S. Tarbouriech, and C. Prieur. Polytopic control invariant sets for continuous-time systems: a viability theory approach. 2011.

[48] E. Fridman. Stability of systems with uncertain delays: A new "complete" Lyapunov-Krasovskii functional. *IEEE Transactions on Automatic Control*, 51(5):885–890, 2006.

[49] E. Fridman and S.-I. Niculescu. On complete Lyapunov-Krasovskii functional techniques for uncertain systems with fast-varying delays. *International Journal of Robust and Nonlinear Control*, 18(3):364–374, 2008.

[50] E. Fridman, A. Seuret, and J.-P. Richard. Robust sampled-data stabilization of linear systems: an input delay approach. *Automatica*, 40(8):1441–1446, 2004.

[51] H. Fujioka, T. Nakai, and L. Hetel. A switched Lyapunov function approach to stability analysis of non-uniformly sampled-data systems. In *American Control Conference*, Baltimore, Maryland, 2010.

[52] A. T. Fuller. In-the-large stability of relay and saturating control systems with linear controllers. *International Journal of Control*, 10(4):457–480, 1969.

[53] H. Gao and T. Chen. New results on stability of discrete-time systems with time-varying state delay. *IEEE Transactions on Automatic Control*, 52(2):328–334, 2007.

[54] H. Gao, C. Wang J. Lam, and Y. Wang. Delay-dependent output-feedback stabilization of discrete-time systems with time-varying state delay. *IEEE Proceedings on Control Theory and Applications*, 151(6):691–698, 2004.

[55] M. R. Garey and D. S. Johnson. *Computers and Intractability; A Guide to the Theory of NP-Completeness.* W. H. Freeman & Co., New York, NY, USA, 1990.

[56] R. H. Gielen, M. Lazar, and I. V. Kolmanovsky. On Lyapunov theory for delay difference inclusions. In *American Control Conference*, Baltimore, Maryland, 2010.

[57] R. H. Gielen, S. Olaru, and M. Lazar. On polytopic approximations as a modelling framework for networked control systems. In *NMPC Workshop*, Pavia, Italy, 2008.

[58] E. G. Gilbert and K. T. Tan. Linear systems with state and control constraints, the theory and application of maximal output admissible sets. *IEEE Transaction on Automatic Control*, 36(9):1008–1020, 1991.

[59] G. Goebel, U. Münz, and F. Allgöwer. Stabilization of linear systems with distributed input delay. In *Proc. American Control Conf. (ACC)*, pages 5800–5806, Baltimore, Maryland, USA, 2010.

[60] J. M. Gomes da Silva Jr and S. Tarbouriech. Antiwindup design with guaranteed regions of stability: an LMI-based approach. *IEEE Transactions on Automatic Control*, 50(1):106–111, 2005.

[61] G. C. Goodwin, M. M. Seron, and J. A. De Dona. *Constrained control and estimation: an optimization approach.* Springer, 2004.

[62] A. Goubet-Bartholomeus, M. Dambrine, and J.-P. Richard. Bounded domains and constrained control of linear time-delay systems. *Journal europeen des systemes automatises*, 31(6):1001–1014, 1997.

[63] P. Gritzmann and V. Klee. On the complexity of some basic problems in computational convexity: I. containment problems. *Discrete Mathematics*, 136(1-3):129–174, 1994.

[64] P. Gritzmann and V. Klee. On the complexity of some basic problems in computational convexity: Ii. volume and mixed volumes. *Universitat Trier, Mathematik/Informatik*, 07, 1994.

[65] P. Gritzmann and B. Sturmfels. Minkowski addition of polytopes: Computational complexity and applications to Gröbner basis. *SIAM Journal on Discrete Mathematics*, 6(2):246–269, 1993.

[66] K. Gu, J. Chen, and V. Kharitonov. *Stability of Time-Delay Systems.* Birkhauser Boston, 2003.

[67] K. Gu and Q.-L. Han. Controller design for time-delay systems using discretized Lyapunov functional approach. In *Proceedings of the Control and Decision Conference*, Sidney, Australia, 2000.

[68] K. Gu and Q.-L. Han. Discretized Lyapunov functional for linear uncertain systems with time-varying delay. In *Proceedings of the American Control Conference*, Chig, Australia, 2000.

[69] W. Hahn. *Stabilty of motion*. Grundlehren Math. Wiss. Springer, Berlin, 1967. Transl. from the German.

[70] H. Haimovich and M. M. Seron. Componentwise ultimate bound and invariant set computation for switched linear systems. *Automatica*, 2010.

[71] J. K. Hale and S. M. V. Lunel. *Introduction to functional differential equations*. Springer, 1993.

[72] Y. Halevi and A. Ray. Integrated communication and control systems: Part I-Analysis. *Journal of Dynamic Systems, Measurement and Control*, 110(4):367–373, 1988.

[73] B. Hannaford. A design framework for teleoperators with kinesthetic feedback. *IEEE Transactions on Robotics and Automation*, 5(4):426–434, 1989.

[74] M. Heemels, N. Van De Wouw, R. Gielen, T. Donkers, L. Hetel, S. Olaru, M. Lazar, J. Daafouz, and S.-I. Niculescu. A comparison of overapproximation methods for stability analysis of networked control systems. In *13th International Conference on Hybrid Systems: Computation and Control, HSCC 2010*, Stockholm Sweden, 2010.

[75] J.-C. Hennet and S. Tarbouriech. Stability and stabilization of delay differential systems. *Automatica*, 33(3):347–354, 1997.

[76] J.-C. Hennet and S. Tarbouriech. Stability conditions of constrained delay systems via positive invariance. *International Journal of Robust and Nonlinear Control*, 8(3):265–278, 1998.

[77] L. Hetel, J. Daafouz, and C. Iung. Stabilization of arbitrary switched linear systems with unknown time varying delays. *IEEE Transactions on Automatic Control*, 51(10):1668–1674, 2006.

[78] L. Hetel, J. Daafouz, and C. Iung. Equivalence between the Lyapunov-Krasovskii functionals approach for discrete delay systems and that of the stability conditions for switched systems. *Nonlinear Analysis: Hybrid Systems*, 2(3):697–705, 2008.

[79] L. Hetel, J. Daafouz, J.-P. Richard, and M. Jungers. Delay-dependent sampled-data control based on delay estimates. *Systems & Control Letters*, 60:146–150, 2011.

[80] R. E. Kalman. Contributions to the theory of optimal control. *Bol. Soc. Mat. Mexicana*, 5(2):102–119, 1960.

[81] R. E. Kalman. When is a linear control system optimal? *Trans. ASME, Journal of Basic Engineering, Series D*, 86:81–90, 1964.

[82] S. S. Keerthi and E. G. Gilbert. Optimal infinite-horizon feedback laws for a general class of constrained discrete-time systems: Stability and moving-horizon approximations. *Journal of Optimization Theory and Applications*, 57(2):265–293, 1988.

[83] V. Kharitonov. Robust stability analysis of time delay systems: A survey. *Annual Reviews in Control*, 23:185–196, 1999.

[84] V. Kharitonov. Complete type Lyapunov-Krasovskii functionals. In S.-I. Niculescu and K. Gu, editors, *Advances in Time-Delay Systems*. Springer-Verlag, 2004.

[85] H. Kiendl, J. Adamy, and P. Stelzner. Vector norms as Lyapunov functions for linear systems. *IEEE Transactions on Automatic Control*, 37(6):839–842, 1992.

[86] F. C. Klebaner, A. Sudbury, and G. A. Watterson. The volumes of simplices, or, find the penguin. *Journal of the Australian Mathematical Society*, 47(02):263–268, 1989.

[87] E. Kofman, H. Haimovich, and M. M. Seron. A systematic method to obtain ultimate bounds for perturbed systems. *International Journal of Control*, 80:167 – 178, 2007.

[88] V. B. Kolmanovskii and A. D. Myshkis. *Introduction to the theory and applications of functional differential equations*. Springer Netherlands, 1999.

[89] V. B. Kolmanovskii, S.-I. Niculescu, and J.-P. Richard. On the Liapunov-Krasovskii functionals for stability analysis of linear delay systems. *International Journal of Control*, 72(4):374–384, 1999.

[90] K. Kosuge, H. Murayama, and K. Takeo. Bilateral feedback control of telemanipulators via computer network. In *Proceedings of International Conference on the Intelligent Robots and Systems '96, IROS 96, Proceedings of the 1996 IEEE/RSJ*, 2008.

[91] M. V. Kothare, V. Balakrishnan, and M. Morari. Robust constrained model predictive control using linear matrix inequalities. *Automatica*, 32(10):1361–1379, 1996.

[92] N. N. Krasovskii. Stability of motion. *SIAM Review*, 6(4):471–472, 1960.

[93] N. N. Krasovskii. On the analytic construction of an optimal control in a system with time lags. *Journal of Applied Mathematics and Mechanics*, 26(1):50–67, 1962.

[94] N. N. Krasovskii. *Stability of motion: applications of Lyapunov's second method to differential systems and equations with delay.* Stanford Univ Pr, 1963.

[95] J. F. Kurose and K. W. Ross. *Computer networking*, volume 75. Addison-Wesley Boston, 2001.

[96] A. B. Kurzhanski and I. Valyi. Ellipsoidal Calculus for Dynamics and Control. *Birkhauser, Boston*, 1998.

[97] M. Kvasnica, P. Grieder, and M. Baotić. Multi-parametric toolbox (mpt), 2004.

[98] W. Langson, I. Chryssochoos, S. V. Rakovic, and D. Q. Mayne. Robust model predictive control using tubes. *Automatica*, 40(1):125–133, 2004.

[99] V. Larin. About the inverse problem of optimal control. *Journal of Applied and Computational Mathematics*, 2:90–97, 2003.

[100] D. A. Lawrence. Stability and transparency in bilateral teleoperation. *IEEE Transactions on Robotics and Automation*, 9(5):624–637, 1993.

[101] M. Lazar. *Model predictive control of hybrid systems: Stability and robustness.* PhD thesis, Technische Universiteit Eindhoven, Eindhoven, the Netherlands, 2006.

[102] E. B. Lee. Foundations of optimal control theory. Technical report, Minnesota University Minneapolis Center for Control Sciences, 1967.

[103] V. J. S. Leite, S. Tarbouriech, and P. L. D. Peres. A convex approach for robust state feedback control of discrete-time systems with state delay. In *Proceedings of the American Control Conference*, Boston, Massachusetts, 2004.

[104] X. Li and C.E. De Souza. Criteria for robust stability and stabilization of uncertain linear systems with state delay. *Automatica*, 33(9):1657–1662, 1997.

[105] D. Limon, I. Alvarado, T. Alamo, and E. F. Camacho. Robust tube-based MPC for tracking of constrained linear systems with additive disturbances. *Journal of Process Control*, 20(3):248–260, 2010.

[106] Z. Lin and A. Saberi. Semi-global exponential stabilization of linear systems subject to. *Systems & Control Letters*, 21(3):225–239, 1993.

[107] Z. Lin, A. A. Stoorvogel, and A. Saberi. Output regulation for linear systems subject to input saturation. *Automatica*, 32(1):29–47, 1996.

[108] B. Liu and Horacio J. Marquez. Razumikhin-type stability theorems for discrete delay systems. *Automatica*, 43:1219–1225, 2007.

[109] X. G. Liu, R. R. Martin, M. Wu, and M. L. Tang. Delay-dependent robust stabilisation of discrete-time systems with time-varying delay. *IEEE Proceedings on Control Theory and Applications*, 153(6):689–702, 2006.

[110] W. Lombardi, A. Luca, S. Olaru, and S.-I. Niculescu. State admissible sets for discrete systems under delay constraints. In *Proceedings of the American Control Conference*, 2010.

[111] W. Lombardi, A. Luca, S. Olaru, S.-I. Niculescu, and J. Cheong. Feedback stabilization and motion synchronization of systems with time-delay in the communication network. In *Proceedings of the IFAC TDS*, 2010.

[112] W. Lombardi, S. Olaru, M. Lazar, G. Bitsoris, and S.-I. Niculescu. On the polyhedral set-invariance conditions for time-delay systems. In *Proceedings of IFAC World Congress*, 2011.

[113] W. Lombardi, S. Olaru, M. Lazar, and S.-I. Niculescu. On positive invariance for delay difference equations. In *Proceedings of the American Control Conference*, 2011.

[114] W. Lombardi, S. Olaru, and S.-I. Niculescu. Invariant sets for a class of linear systems with variable time-delay. In *Proceedings of the European Control Conference*, 2009.

[115] W. Lombardi, S. Olaru, and S.-I. Niculescu. Robust invariance for a class of time-delay systems with repeated eigenvalues. In *Proceedings of the IFAC TDS*, 2009.

[116] K. Loskot, A. Polanski, and R. Rudnicki. Further comments on “Vector norms as Lyapunov functions for linear systems". *IEEE Transactions on Automatic Control*, 43(2):289–291, 1998.

[117] C. Lovaas, M. M. Seron, and G. C. Goodwin. Robust output-feedback model predictive control for systems with unstructured uncertainty. *Automatica*, 44(8):1933–1943, 2008.

[118] C. Lovaas, M. M. Seron, and G. C. Goodwin. Inverse minimax optimality of model predictive control policies. *Systems & Control Letters*, 58(1):31–38, 2009.

[119] C. Lovaas, M. M. Seron, and G. C. Goodwin. Robust output-feedback mpc with integral action. *IEEE Transactions on Automatic Control*, 55(7):1531 –1543, 2010.

[120] R. Luck and A. Ray. An observer-based compensator for distributed delays* 1. *Automatica*, 26(5):903–908, 1990.

[121] J. M. Maciejowski. *Predictive control: with constraints*. Pearson education, 2002.

[122] D. Q. Mayne. Nonlinear model predictive control: Challenges and opportunities. *Nonlinear model predictive control*, pages 23–44, 2000.

[123] D. Q. Mayne, S. V. Rakovic, R. Findeisen, and F. Allgöwer. Robust output feedback model predictive control of constrained linear systems. *Automatica*, 42(7):1217–1222, 2006.

[124] D. Q. Mayne, S. V. Rakovic, R. Findeisen, and F. Allgöwer. Robust output feedback model predictive control of constrained linear systems: Time varying case. *Automatica*, 45(9):2082–2087, 2009.

[125] D. Q. Mayne, J. B. Rawlings, C. V. Rao, and P. O. M. Scokaert. Constrained model predictive control: Stability and optimality. *Automatica*, 36:789–814, 2000.

[126] D. Q. Mayne, M. M. Seron, and S. V. Rakovic. Robust model predictive control of constrained linear systems with bounded disturbances. *Automatica*, 41(2):219 – 224, 2005.

[127] A. M. Medioli. *Maximal controllability via reduced complexity model predictive control*. PhD thesis, University of Newcastle, Newcastle, Australia, 2008.

[128] W. Michiels and S.-I. Niculescu. *Stability and stabilization of time-delay systems*. SIAM, 2007.

[129] B. E. A. Milani. Piecewise-affine lyapunov functions for discrete-time linear systems with saturating controls. *Automatica*, 38(12):2177 – 2184, 2002.

[130] B. E. A. Milani and A. N. Carvalho. Robust linear regulator design for discrete-time systems under polyhedral constraints. *Automatica*, 31(10):1489 – 1493, 1995.

[131] B. E. A. Milani and C. E. T. Dorea. On invariant polyhedra of continuous-time systems subject to additive disturbances. *Automatica*, 32(5):785 – 789, 1996.

[132] Y. S. Moon, P. Park, W. H. Kwon, and Y. S. Lee. Delay-dependent robust stabilization of uncertain state-delayed systems. *International Journal of Control*, 74(14):1366–5820, 2001.

[133] T. Mori. Criteria for asymptotic stability of linear time-delay systems. *IEEE Transactions on Automatic Control*, 30(2):158–161, 1985.

[134] A. Mukherjee. On the dynamics and significance of low frequency components of Internet load. *Internetworking-Research and Experience*, 5(4):163–206, 1994.

[135] P. Naghshtabrizi, J. P. Hespanha, and A. R. Teel. Exponential stability of impulsive systems with application to uncertain sampled-data systems. *Systems & Control Letters*, 57(5):378–385, 2008.

[136] S.-I. Niculescu. *Delay Effects on Stability: A Robust Control Approach*. Springer: Heidelberg, LNCIS, 2001.

[137] S.-I. Niculescu. On delay-dependent stability under model transformations of some neutral linear systems. *International Journal of Control*, 74(6):609–617, 2001.

[138] S.-I. Niculescu, J.-M. Dion, and L. Dugard. Mixed delay-independent/delay-dependent stability of linear systems with delayed state: a lmi approach. In *Proceedings of IFAC TDS*, 1998.

[139] S.-I. Niculescu, W. Michiels, K. Gu, and C. Abdallah. Delay effects on output feedback control of dynamical systems. *Complex Time-Delay Systems*, pages 63–84, 2010.

[140] S.-I. Niculescu, E. Verriest, L. Dugard, and J. M. Dion. Stability and robust stability of time-delay systems: A guided tour. *Stability and control of time-delay systems*, pages 1–71, 1998.

[141] J. Nilsson. *Real-time control systems with delays.* PhD thesis, Lund Institute of Technology, 1998.

[142] J. Normey-Rico and E. F. Camacho. *Control of dead-time processes.* Springer, 2007.

[143] J. E. Normey-Rico and E. F. Camacho. Multivariable generalised predictive controller based on the Smith predictor. In *Proceedings of the Control Theory and Applications*, 2000.

[144] S. Olaru and D. Dumur. Avoiding constraints redundancy in predictive control optimization routines. *IEEE Transactions on Automatic Control*, 50(9):1459–1466, 2005.

[145] S. Olaru and S.-I. Niculescu. Predictive control for systems with delay and constraints. In *Proceedings of IFAC World Congress*, 2008.

[146] P. Park. A delay-dependent stability criterion for systems with uncertain time-invariant delays. *IEEE Transactions on Automatic Control*, 44(4):876–877, 1999.

[147] S. Peng and Y. Zhang. Razumikhin-Type Theorems on p th Moment Exponential Stability of Impulsive Stochastic Delay Differential Equations. *IEEE Transactions on Automatic Control*, 55(8):1917–1922, 2010.

[148] L. L. Peterson and B. S. Davie. *Computer networks: a systems approach.* Morgan Kaufmann Pub, 2007.

[149] S. V. Rakovic, P. Grieder, M. Kvasnica, D. Q. Mayne, and M. Morari. Computation of invariant sets for piecewise affine discrete time systems subject to bounded disturbances. In *In Proceeding 43th IEEE Conference on Decision and Control*, December 2004.

[150] S. V. Rakovic, E. C. Kerrigan, K. I. Kouramas, and D. Q. Mayne. Invariant approximations of the minimal robust positively invariant set. *IEEE Transactions on Automatic Control*, 50(3):406–410, 2005.

[151] S. V. Rakovic, E. C. Kerrigan, D. Q. Mayne, and J. Lygeros;. Reachability analysis of discrete-time systems with disturbances. *IEEE Transactions on Automatic Control*, 51:546 – 561, 2006.

[152] J. B. Rawlings and D. Q. Mayne. *Model predictive control: Theory and design.* Nob Hill Pulications, 2009.

[153] A. Ray and Y. Halevi. Integrated communication and control systems: Part II-design considerations. *Journal of Dynamic Systems, Measurement and Control*, 110(4):374–381, 1988.

[154] B. S. Razumikhin. On the stability of systems with a delay. *Prikl. Math. Meh.*, 20:500–512, 1956.

[155] M. Reble and F. Allgöwer. Stabilizing design parameters for model predictive control of constrained nonlinear time-delay systems. In *Proc. 9th IFAC Workshop on Time Delay Systems*, 2010.

[156] M. Reble, R. M. Esfanjani, S. K. Y. Nikravesh, and F. Allgöwer. Model predictive control of constrained nonlinear time-delay systems. *IMA Journal of Mathematical Control and Information*, 2010.

[157] J. Richalet, A. Rault, J.L. Testud, and J. Papon. Model predictive heuristic control: Applications to industrial processes. *Automatica*, 14(5):413 – 428, 1978.

[158] J.-P. Richard. Time-delay systems:an overview of some recent advances and open problems. *Automatica*, 39(10):1667–1694, 2003.

[159] C. Roos, J.-M. Biannic, S. Tarbouriech, C. Prieur, and M. Jeanneau. On-ground aircraft control design using a parameter-varying anti-windup approach. *Aerospace Science and Technology*, 14(7):459 – 471, 2010.

[160] J. A. Rossiter. *Model-Based Predictive Control.* CRC-Press, 2003.

[161] A. Saberi, A. Stoorvogel, and P. Sannuti. *Control of linear systems with regulation and input constraints.* Springer, 2000.

[162] T. L. M. Santos, D. Limon, T. Alamo, and J. E. Normey-Rico. Robust tube based model predictive control for constrained systems with dead-time. In *Proceedings of the UKACC International Conference on Control*, Coventry - Inglaterra, Setembro 2010.

[163] T. L. M. Santos and J. E. Normey-Rico. Robust model predictive controller with terminal weighting for multivariable dead-time processes. In *Proceedings of the 8th Workshop on Time Delay Systems*, Sinaia - Romania, 2009.

[164] T. L. M. Santos, J. E. Normey-Rico, and D. Limon. Explicit input-delay compensation for robustness improvement in MPC. In *Proceedings of the 9th Workshop on Time Delay Systems*, Prague - Czec Republic, Junho 2010.

[165] T. L. M. Santos, L. Roca, J. L. Guzman, J. E. Normey-Rico, and M. Berenguel. Practical mpc with robust dead-time compensation applied to a solar desalination plant. In *18th IFAC World Congress*, Milano, September 2011.

[166] R. Schneider. *Convex bodies: The Brunn-Minkowski Theory*. Cambridge University Press, Cambridge, 1993.

[167] M. M. Seron, G. C. Goodwin, and J. H. Braslavsky. *Fundamental Limitations in Filtering and Control*. Springer, 1997.

[168] M. M. Seron, G. C. Goodwin, and J. A. De Dona. Characterisation of receding horizon control for constrained linear systems. *Asian Journal of Control*, 5(2):271–286, 2003.

[169] A. S. C. Sinha. Analysis of a mathematical model for transport of ^{1}31I-albumin. *Journal of Theoretical Biology*, 50(2):259–267, 1975.

[170] R. Sipahi, S.-I. Niculescu, C. T. Abdallah, W. Michiels, and K. Gu. Stability and stabilization of systems with time delay - limitations and opportunities. *IEEE Control Systems Magazine*, 31(1):38 – 65, 2011.

[171] O. J. M. Smith. A controller to overcome dead time. *IsA Jvurnal*, 6:28–33, 1959.

[172] G. Stépán. *Retarded dynamical systems: stability and characteristic functions.* Longman Scientific & Technical, 1989.

[173] F. Stoican, S. Olaru, M. Nesic, and S. Marinkovic. Control design of a positioning system upon a fault tolerant multisensor scheme. In *Proceedings of the 17th Telecommunications forum TELFOR*, Belgrade, Serbia, 2009.

[174] X.-M. Sun, G.-P. Liu, D. Rees, and W. Wanga. Delay-dependent stability for discrete systems with large delay sequence based on switching techniques. *Automatica*, 44(11):2902–2908, 2008.

[175] H. J. Sussmann and Y. Yang. On the stabilizability of multiple integrators by means of bounded feedback controls. In *Decision and Control, 1991., Proceedings of the 30th IEEE Conference on*, pages 70–72. IEEE, 1991.

[176] D. Tabak. Applications of mathematical programming techniques in optimal control: a survey. *IEEE Transactions on Automatic Control*, 15(6):688–690, 1970.

[177] A. S. Tanenbaum. Network protocols. *ACM Computing Surveys (CSUR)*, 13(4):453–489, 1981.

[178] S. Tarbouriech. Local stabilization of continuous-time delay systems with bounded inputs. *Stability and control of time-delay systems*, pages 302–317, 1998.

[179] S. Tarbouriech, G. Garcia, and A. H. Glattfelder. *Advanced strategies in control systems with input and output constraints*. Springer Verlag, 2007.

[180] S. Tarbouriech, G. Garcia, and J. M. Gomes da Silva Jr. Robust stability of uncertain polytopic linear time-delay systems with saturating inputs: an LMI approach. *Computers & Electrical Engineering*, 28(3):157–169, 2002.

[181] S. Tarbouriech, J. M. Gomes da Silva Jr, and G. Garcia. Delay-dependent anti-windup strategy for linear systems with saturating inputs and delayed outputs. *International Journal of Robust and Nonlinear Control*, 14(7):665–682, 2004.

[182] S. Tarbouriech, P. L. D. Peres, G. Garcia, and I. Queinnec. Delay-dependent stabilisation and disturbance tolerance for time-delay systems subject to actuator saturation. In *Proceedings of Control Theory and Applications*, 2002.

[183] S. Tarbouriech, I. Queinnec, T. Alamo, M. Fiacchini, and E. F. Camacho. Ultimate bounded stability and stabilization of linear systems interconnected with generalized saturated functions. *Automatica*, 47(7):1473 – 1481, 2011.

[184] S. Tarbouriech, I. Queinnec, and G. Garcia. Stability region enlargement through anti-windup strategy for linear systems with dynamics restricted actuator. *International journal of systems science*, 37(2):79–90, 2006.

[185] A. R. Teel. Global stabilization and restricted tracking for multiple integrators with bounded controls. *Systems & Control Letters*, 18(3):165–171, 1992.

[186] A. R. Teel. Connections between Razumikhin-type theorems and the ISS nonlinear small gain theorem. *IEEE Transactions on Automatic Control*, 43(7):960–964, 1998.

[187] Y. Tipsuwan and M. Y. Chow. Control methodologies in networked control systems. *Control engineering practice*, 11(10):1099–1111, 2003.

[188] M. C. Turner and I. Postlethwaite. A new perspective on static and low order anti-windup synthesis. *International Journal of Control*, 77(1):27 – 44, January 2004.

[189] M. Vassilaki and G. Bitsoris. Constrained feedback control of discrete-time systems decribed by arma models. In *Proceedings of the European Control Conference*, 1999.

[190] R. Vidal, S. Schaffert, J. Lygeros, and S. Sastry. *Controlled Invariance of Discrete Time Systems*. HSCC LNCS 1790. Springer, 2000.

[191] Y. Yokokohji and T. Yoshikawa. Bilateral control of master-slave manipulators for ideal kinesthetic coupling. *IEEE Transactions on Robotics and Automation*, 10(5):605–620, 1994.

[192] K. Yoshida, H. Kawabe, and Y. Nishimura. Simple LP-type criteria for positively invariant polyhedral sets. *IEEE Transactions on Automatic Control*, 45(1):98–101, 2000.

[193] S. Yu, C. Böhm, H. Chen, and F. Allgöwer. Robust model predictive control with disturbance invariant sets. In *Proc. American Control Conf. (ACC)*, pages 6262–6267, Baltimore, Maryland, USA, 2010.

[194] W. Zhang. *Stability analysis of networked control systems*. PhD thesis, Case Western Reserve University, 2001.

[195] W. Zhang, M. S. Branicky, and S. M. Phillips. Stability of networked control systems. *IEEE control systems magazine*, 21(1):84–99, 2001.

[196] Y. Zhou and S. Suri. Algorithms for a minimum volume enclosing simplex in three dimensions. *SIAM J. Comput*, 31:2002, 2002.

Le thème principal de ce mémoire est la commande sous contraintes pour des systèmes à retard, en tenant compte de la problématique d'échantillonnage (où les informations concernant le système en temps continu sont, par exemple, envoyées par un réseau de communication) et de la présence de contraintes sur les trajectoires du système et sur l'entrée de commande.

Pendant le processus d'échantillonnage (et par conséquent l'intégration des entrées), le retard variable dans le temps peut être traite comme une incertitude en fonction de cette variation. Le but est de confiner cette fonction dans un polytope, de façon a couvrir toutes les variations possibles du retard. Cette technique doit être valable malgré l'absence d'informations statistiques concernant la variation. Les variations du retard peuvent être bornées dans une période d'échantillonnage ou dans plusieurs périodes, appelées dans cette thèse variation intra-échantillon et multi-échantillon. Le modèle peut être donc exprimé en utilisant un vecteur d'état augmenté contenant les états actuels et les entres de commande passées. Un modèle polytopique augmente est obtenu, c'est a dire que le système continu a retard variable est transforme en un système échantillonne avec des incertitudes polytopiques. L'avantage est la possibilité d'utiliser des techniques classiques de commande robuste. Le terme en fonction de la variation du retard peut être catégorise en fonction du niveau des valeurs propres de la matrice du système en temps continu. Il est montré que la fonction d'incertitude peut être enveloppée par un polytope en utilisant des calculs simples, reposant sur la décomposition de Jordan. Un des inconvénients de que cette technique est qu'elle est considérée conservative. Un algorithme ayant pour finalité de diminuer le volume des polytopes autour de la fonction est donc propose.

Au cours de cette thèse, pour stabiliser des systèmes a retard, nous nous sommes bases sur la théorie de Lyapunov. En utilisant cette méthode, nous pouvons trouver un retour d'état qui stabilise le système malgré la présence du retard variable dans la boucle. Dans l'espace d'état augmenté, la taille des matrices du système est proportionnelle au retard et la dimension de l'espace Euclidien utilise (des sommets du polytope) est fonction de la taille des matrices du système continu. Cet ensemble de facteurs va influencer la complexité et le domaine de faisabilité du problème LMI. Une autre possibilité est l'utilisation des candidates de Lyapunov-Krasovskii. Ce type de technique stabilise le système dans l'espace Euclidien original. L'inconvénient est la complexité de la candidate, considérée importante.

La théorie des ensembles invariants est largement utilisée durant ce manuscrit, car il est souhaitable d'obtenir une région de ≪ sureté ≫, ou le comportement du système est connu, en dépit de la présence du retard (variable) et des contraintes sur les trajectoires du système. Lorsqu'ils sont obtenus dans l'espace d'état augmente, les ensembles invariants sont très complexes, car la dimension de l'espace Euclidien sera proportionnelle a la taille du système mais aussi a la taille du retard. Le concept de D-invariance est ainsi propose, ou non seulement les trajectoires présentes du système doivent appartenir a un ensemble, mais l'histoire des trajectoires passées doivent également respecter cette inclusion. Pour les systèmes linéaires, la D-invariance peut être définie comme l'addition de Minkowski de l'ensemble suppose invariant et les opérations linéaires de la dynamique présente et passée. Les conditions algébriques peuvent aussi être dérivées pour l'obtention de tests constructifs et pour la synthèse de contrôleurs, bases sur des problèmes de faisabilité LP (Programmation Linéaire). Afin de réduire le conservatisme de cette approche, le concept de D-invariance cyclique est étudie, ou la condition d'inclusion des ensembles doit être respectée pour toutes les dynamiques du système déplacées une par une.

Dans la dernière partie du manuscrit, la commande prédictive (en anglais MPC) est présentée, pour tenir compte des contraintes sur les trajectoires et appliquer une commande optimale a l'entrée du système. Pour régler les paramètres du contrôleur (plus précisément les matrices de pondération des états et de la commande), un problème d'optimalité inverse est utilise, base sur la résolution d'une équation de Ricatti. La commande MPC est appliquée « en ligne » en résolvant un problème d'optimisation multiparamétrique, aussi exprime au niveau d'un problème PQ (Programmation Quadratique), en utilisant comme contraintes du problème d'optimisation les contraintes de trajectoire, et l'ensemble invariant comme ensemble terminal de la procédure de prédiction à horizon glissant.