UNIVERSITÉ DE NICE - SOPHIA ANTIPOLIS

ÉCOLE DOCTORALE STIC

SCIENCES ET TECHNOLOGIES DE L'INFORMATION ET DE LA COMMUNICATION

T H È S E

pour obtenir le titre de

Docteur en Sciences

de l'Université de Nice - Sophia Antipolis

Mention : INFORMATIQUE

Présentée et soutenue par

Leonardo SAMPAIO ROCHA

Algorithmic aspects of graph colouring heuristics

Thèse dirigée par Frédéric HAVET

préparée au sein du projet Mascotte, I3S(CNRS et UNS) et INRIA

soutenue le 19 novembre 2012

Jury :

Rapporteurs :	Stéphane BESSY	-	Univeristé Montpellier II (LIRMM)
	Éric SOPENA	-	Université Bordeaux I (LaBRI)
Directeur :	Frédéric HAVET	-	Projet Mascotte, I3S(CNRS et UNS) et INRIA
Examinateurs :	David COUDERT	-	Projet Mascotte, I3S(CNRS et UNS) et INRIA
	Mario VALENCIA-PABON	-	Université Paris-Nord (LIPN)
	Yann VAXÈS	-	Université de la Méditerranée (LIF)

Algorithmic aspects of graph colouring heuristics

Leonardo Sampaio

19 november 2012

Aspects algorithmiques d'heuristiques de coloration de graphes

Résumé : Une coloration propre d'un graphe est une fonction qui attribue une couleur à chaque sommet du graphe avec la restriction que deux sommets voisins ont des couleurs distinctes. Les colorations propres permettent entre autres de modéliser des problèmes d'ordonnancement, d'allocation de fréquences ou de registres. Le problème de trouver une coloration propre d'un graphe qui minimise le nombre de couleurs est un problème NP-difficile très connu.

Dans cette thèse, nous étudions le nombre de Grundy et le nombre b-chromatique des graphes, deux paramètres qui permettent d'évaluer la performance de quelques heuristiques pour le problème de la coloration propre. Nous commençons par dresser un état de l'art des résultats sur ces deux paramètres. Puis, nous montrons que déterminer le nombre de Grundy est NP-difficile sur les graphes bipartis ou cordaux.

Ensuite, nous montrons que déterminer le nombre b-chromatique est NP-difficile pour un graphe cordal et distance-héréditaire, et nous donnons des algorithmes polynomiaux pour certaines sous-classes de graphes: graphes des blocs, complémentaires des graphes bipartis et P_4-sparses.

Nous considérons également la complexité à paramètre fixé de déterminer le nombre de Grundy (resp. nombre b-chromatique) et en particulier, nous montrons que décider si le nombre de Grundy (ou le nombre b-chromatique) d'un graphe G est au moins $|V(G)| - k$ admet un algorithme FPT lorsque k est le paramètre.

Enfin, nous considérons la complexité de nombreux problèmes liés à la comparaison du nombre de Grundy et du nombre b-chromatique avec divers autres paramètres d'un graphe.

Mots clés : Coloration de graphes, coloration gloutonne, b-coloration, NP-complétude, Complexité à Paramètre Fixé

Algorithmic aspects of graph colouring heuristics

Abstract: A proper colouring of a graph is a function that assigns a colour to each vertex with the restriction that adjacent vertices are assigned with distinct colours. Proper colourings are a natural model for many problems, like scheduling, frequency assignment and register allocation. The problem of finding a proper colouring of a graph with the minimum number of colours is a well-known NP-hard problem.

In this thesis we study the Grundy number and the b-chromatic number of graphs, two parameters that evaluate some heuristics for finding proper colourings. We start by giving the state of the art of the results about these parameters. Then, we show that the problem of determining the Grundy number of bipartite or chordal graphs is NP-hard.

After, we show that the problem of determining the b-chromatic number of a chordal distance-hereditary graph is NP-hard, and we give polynomial-time algorithms for subclasses of block graphs, complement of bipartite graphs and P_4-sparse graphs.

We also consider the fixed-parameter tractability of determining the Grundy number and the b-chromatic number, and in particular we show that deciding if the Grundy number (or the b-chromatic number) of a graph G is at least $|V(G)| - k$ admits an FPT algorithm when k is the parameter.

Finally, we consider the computational complexity of many problems related to comparing the b-chromatic number and the Grundy number with various other related parameters of a graph.

Keywords: Graph colouring, greedy colouring, b-colouring, NP-completeness, Fixed Parameter Complexity

Contents

Chapter 1

Outline of this thesis and our contributions

The most important concept of this thesis is that of a *graph colouring*. A *graph colouring* is a partition of the vertices of a graph into sets that are called *colour classes*. They are a natural model for problems in which a set of objects is to be partitioned according to some prescribed rules. The objects and the relations between them are modelled by a graph and one seeks to find a partition of the vertices into classes that satisfy the particular constraints of the problem.

Depending on the constraints that are imposed to the partitions, one obtains different kinds of colouring. Of particular interest are colourings in which every colour class corresponds to a *stable set* of the graph, that is a set of pairwise non-adjacent vertices. Such colourings are called *proper colourings*, and they model a number of practical and theoretical applications. For example, problems of *scheduling* [120], *frequency assignment* [45] and *register allocation* [22, 23], besides of the *finite element method* [112], are naturally modelled by proper colourings. While it is easy to find a proper colouring when no bound is imposed on the number of colour classes, a challenging problem consists in finding one that minimizes the number of colours. The minimum number of colours of a proper colouring of a graph G is its *chromatic number*, that is denoted $\chi(G)$.

The problem of finding a proper colouring with the minimum number of colours was one of the first problems proved to be NP-hard: to decide if a graph admits a proper colouring with k colours is an NP-complete problem, even if k is fixed [67]. Moreover, the chromatic number is hard to approximate: for all $\epsilon > 0$, there is no algorithm that approximates the chromatic number within a factor of $n^{1-\epsilon}$ unless P = NP [123]. On the other hand, a number of algorithms, exacts or not, were proposed. As examples we refer to [96, 49, 59].

Although determining the chromatic number of a graph is a hard problem, there are heuristics that quickly produce proper colourings. Actually many of the upper bounds that are known for the chromatic number are obtained from colouring heuristics.

A very simple and widespread heuristic is the *greedy colouring algorithm*, that takes in turn each vertex of the graph and assigns to it the smallest colour that does not appears in one of its neighbours. A colouring obtained this way is called a *greedy colouring*. The *Grundy number* of a graph is defined as the maximum number of colours of a greedy colouring of the graph, and is denoted $\Gamma(G)$. The Grundy number and its ratio with the chromatic number is a measure of how bad the greedy colouring algorithm may perform on a graph. Early works on the Grundy number go back to the 30's in an article of Grundy [53], who first defined it in the context of directed graphs corresponding to games. Christen and Selkow [24] were the first to define it in the context of graph colourings. After that the Grundy number was investigated by many authors [121, 4, 36, 10, 116]. Since it is very easy to obtain a greedy colouring of a graph, and since any greedy colouring provides an upper bound on the chromatic number, a natural application of the greedy colouring is to evaluate the perfomance of any graph coloring heuristics. For example, the greedy colouring algorithm applied to interval graphs was used as a subroutine of the algorithm *Buddy Decreasing Size*, introduced by Chrobak and Slusarek [25], to solve the *Dynamic Storage Allocation problem*. Motivated by this, a number of authors [81, 82, 104, 108, 109] worked at giving upper bounds on the Grundy number of an interval graph as a function of its chromatic number, since these bounds give a performance guarantee to the above-mentioned subroutine.

Another heuristic to find a proper colouring of a graph can be as follows. Start with some arbitrary colouring of the graph (for example the trivial one in which every vertex has a distinct colour) and choose one colour class. Recolour each vertex in this colour class by a colour that does not appear in the neighbourhood of the vertex under consideration. This heuristic can be used to reduce the number of colours until a colouring is found in which every colour class has at least one vertex with neighbours from every other colour class. Such a colouring is called a *b-colouring* and the special vertex in each colour class is called a *b-vertex*. The *b-chromatic number*, denoted $\chi_b(G)$, is then the maximum number of colours of a b-colouring. In the same way as the Grundy number, the b-chromatic number and its ratio with the chromatic number measure how bad the b-colouring algorithm may perform on a graph. The b-colouring was first defined in 1999 by Irving and Manlove [70], and since that there are many papers dealing with it [90, 12, 118, 19, 99]. As it happens in the case of the Grundy number, a natural application of the b-coloring is to evaluate the perfomance of any graph coloring heuristics, since any b-coloring provides an upper bound on the chromatic number, and a b-coloring of a graph can be easily obtained. Moreover, the concept of b-coloring was used in databases clustering [32] and in automatic recognition of documents [43].

The objective of this thesis is to study the previously defined colourings and the parameters associated with them from an algorithmic point of view. It is organized as follows.

- Chapter 2 contains the necessary background for this thesis. The main graph theory concepts and notations that are used in the following chapters are introduced, as well as the colouring problems that are investigated. Moreover, the state of the art of these colouring problems is presented.

- In Chapter 3 we investigate the complexity of computing the Grundy number of a graph. Some of the results in this chapter were published in [62]. We prove that, given a bipartite graph G, deciding if it admits a greedy colouring with $\Delta(G) + 1$ colours is an NP-complete problem. As a consequence, a characterization of the graphs with Grundy number equal to $\Delta(G) + 1$ that is checkable in polynomial time is unlikely to exists, under the classical complexity assumptions. We then prove that the previous problem remains NP-complete when restricted to chordal graphs instead of bipartite graphs. We end the chapter with open problems concerning the complexity of computing the Grundy number on other graph classes.

- In Chapter 4 the b-chromatic number of graphs is studied. The results presented in this chapter were published in [61]. A well-known upper bound for the b-chromatic number of a graph G is the *m-degree*, denoted $m(G)$, which is defined as the maximum k such that there are k vertices of degree at least $k - 1$. Motivated by this upper bound on the b-chromatic number, we define the *tight graphs*, which are graphs with exactly $m(G)$ vertices of degree exactly $m(G) - 1$. We show that determining if $\chi_b(G) = m(G)$, for a given tight connected chordal distance-hereditary graph G, is an NP-complete problem. After, we investigate this problem for other classes of tight graphs. In order to do so, two operations that may be applied to a tight graph are defined, that are able to relate the b-colouring problem to the classical colouring problem and the precolouring extension problem. By using previously known results about these two problems, polynomial-time algorithms are given to decide if $\chi_b(G) = m(G)$ when G belongs to particular classes of tight graphs, such as complement of bipartite graphs, P_4-sparse graphs and block graphs.

- In Chapter 5 the parameterized complexity of problems involving the b-chromatic number and the Grundy number is investigated. Some of the results in this chapter were published in [62]. In particular, we consider the problem of deciding, given a graph G and k being the parameter, if $\Gamma(G) \geq |V(G)| - k$. We show that this problem is Fixed-Parameter Tractable(FPT). The analog problem in which the question is if $\chi_b(G) \geq |V(G)| - k$ is considered, and we prove that it is also FPT. Finally, we consider the problems of deciding if $\Gamma(G) = \Delta(G) + 1$ and $\chi_b(G) = \Delta(G) + 1$, and we show that these problems are FPT when $\Delta(G)$ is the parameter.

- In Chapter 6 we consider the computational complexity of comparing between the colouring parameters. More specifically, for a fixed integer $c \geq 1$, we determine the complexity of the problem of deciding if $\phi(G) \leq c\psi(G)$, where $\phi(G), \psi(G) \in\{\omega(G)$, $\chi(G)$, $\chi_b(G)$, $\Gamma(G)$, $\partial\Gamma(G)$, $\zeta(G)$, $\Delta(G) + 1\}$ (See the nomenclature at page 121 for the definition of all these parameters). In Table 1, the cell with row $\phi(G)$ and column $\psi(G)$ gives the complexity of the problem of deciding if $\phi(G) \leq c\psi(G)$. The meaning of the abbreviations are described in what follows.

  (=): $\phi(G) = \psi(G)$.

  (≤): the problem is trivial because $\phi(G) \leq \psi(G)$;

  (NPC): for every $c \geq 1$, the problem is NP-complete;

  (=-NPC): for $c = 1$ the problem is NP-complete;

  (>-NPC): for $c > 1$ the problem is NP-complete;

  (coNPC): for every $c \geq 1$, the problem is co-NP-complete;

  (=-coNPC): for $c = 1$ the problem is co-NP-complete;

  (>-coNPC): for $c > 1$ the problem is co-NP-complete;

  (NPH): for every $c \geq 1$, the problem is NP-hard;

  (=-NPH): for $c = 1$ the problem is NP-hard;

  (>-NPH): for $c > 1$ the problem is NP-hard;

  (Poly): the problem is polynomial-time solvable.

  We end Chapter 6 by considering analogues of a well-known conjecture from Reed [110]. The conjecture states that $\chi(G) \leq \left\lceil \frac{\omega(G) + \Delta(G) + 1}{2} \right\rceil$, where $\omega(G)$ is the maximum size of a clique in G, and $\Delta(G)$ is its maximum degree. We investigate variations of that conjecture involving $\chi_b(G)$, $\Gamma(G)$ and $\partial\Gamma(G)$, and their upper and lower bounds.

- Finally, Chapter 7 contains the conclusion of this thesis.

	$\omega(G)$	$\chi(G)$	$\chi_b(G)$	$\Gamma(G)$	$\partial\Gamma(G)$	$\zeta(G)$	$\Delta(G)+1$
$\omega(G)$	=	$\leq$	$\leq$	$\leq$	$\leq$	$\leq$	$\leq$
$\chi(G)$	NPC	=	$\leq$	$\leq$	$\leq$	$\leq$	$\leq$
$\chi_b(G)$	NPH	coNPC	=	NPH	$\leq$	$\leq$	$\leq$
$\Gamma(G)$	NPH	coNPC	NPH	=	$\leq$	$\leq$	$\leq$
$\partial\Gamma(G)$	Unknown	$>$-NPH	=-NPH	NPH	=	$\leq$	$\leq$
$\zeta(G)$	Unknown	$>$-NPH	=-NPC	NPC	=-NPC	=	$\leq$
$\Delta(G)+1$	Unknown	$>$-coNPC [1]	NPC	NPC	NPC	Poly	=

Table 1.1: Complexity of deciding equality between the parameters.

Other works. During this thesis I had the opportunity to work in other graph problems other than the graph colouring one. In the appendices, two research reports about graph convexities and the hull number problem are presented. Some of the results on the first report, entitled "On the Hull number of some graph classes", where presented at the Sixth European Conference on Combinatorics, Graph Theory and Applications (EuroComb 2011) [1] and accepted for publication at the journal Theoretical Computer Science. The second report, entitled "Hull number: P_5-free graphs and reduction rules" was accepted to be presented at the 2nd Bordeaux Graph Workshop (BGW 2012).

[1] For $c = 1$ the problem can be solved in polynomial time.

Chapter 2

Introduction

In Section 2.1 we give the preliminaries on graph theory. A reader familiar with the standard graph theory concepts and notations may skip this section and consult the remissive index or the nomenclature chapter if necessary. In Section 2.2 we give a short introduction to proper colouring of graphs and in Section 2.3 we present the b-colourings and the Grundy colourings as well as the main results about these colourings. Finally, the partial Grundy number is presented in Section 2.4, and we briefly analyze the relations between the many colouring parameters that are introduced in this chapter. These relations will be studied in more detail in Chapter 6.

2.1 Preliminaries

Basic graph definitions. A *graph* G is a pair (V, E), where V is a set of *vertices* and $E \subset [V]^2$ a set of *edges*, where $[V]^2$ denotes all the 2-element subsets of V. The cardinality of $V(G)$ is the *order* of G and the cardinality of $E(G)$ is its *size*. The letters n and m are used to denote $|V(G)|$ and $|E(G)|$ respectively. We use the notation uv to indicate the edge $\{u, v\}$. If $uv \in E(G)$, we say that vertices u and v are its *endvertices*.

Let $G = (V, E)$ be a graph. If $uv \in E(G)$, then u and v are *neighbours* in G. The set of all neighbours of vertex $v \in V(G)$ is the *neighbourhood* of v, and is denoted $N_G(v)$. This notation is extended to a set of vertices $S \subseteq V(G)$, in case $N_G(S) = \bigcup_{v \in S} N(v)$. The *degree* $d_G(v)$ of vertex $v \in V(G)$ is the cardinality of $N_G(v)$. We may omit the subscript in the last notations, when the graph we are talking about is clear from the context. The *minimum degree* of G, denoted $\delta(G)$, is the smallest degree of a vertex of G, while its *maximum degree* $\Delta(G)$ is the largest one. A graph is *k-regular* if all its vertices have degree k. The *complement* of G, denoted by $\overline{G}$, is the graph with $V(\overline{G}) = V(G)$ and $E(\overline{G}) = \{uv \in [V]^2 \mid uv \notin E(G)\}$.

The graph H is a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq \{uv \in E(G) \mid u, v \in V(H)\}$. If $E(H) = \{uv \in E(G) \mid u, v \in V(H)\}$, H is an *induced subgraph* of G. The subgraph of G induced by $S \subset V(G)$ is denoted $G[S]$. A *homomorphism* ϕ from H to G is an injective function $\phi : V(H) \to V(G)$ such that if $uv \in E(H)$ then $\phi(u)\phi(v) \in E(G)$. The function ϕ is an *isomorphism* if it is a bijection $\phi : V(H) \to V(G)$ such that $uv \in E(H)$ if and only if $\phi(u)\phi(v) \in E(G)$. If G has a subgraph which is isomorphic to H, G is said to *contain* H as a subgraph. If G does not contain H as a induced subgraph, G is said to be *H-free*.

A *path* is a graph P with vertex set $V(P) = \{v_1, v_2, \ldots, v_k\}$ and edge set $E(P) = \{v_i v_{i+1} \mid 1 \leq i \leq k-1\}$. The *length* of a path is the cardinality of its edge set. The vertices v_1 and v_k are the *endvertices* of P and $\{v_2, \ldots, v_{k-1}\}$ are its *inner vertices*. A path with endvertices u and v is called a (u, v)-*path*. The path with k vertices is denoted P_k. A graph G is *connected* if there is a path between any pair of distinct vertices. A *connected component* of G is a maximal connected subgraph. We denote by $dist(u, v)$ the *distance* between u and v, that is the length of the smallest (u, v)-path. A *vertex cut* is a set $S \subset V(G)$ such that $G[V(G) \backslash S]$ is disconnected. The *connectivity* $\kappa(G)$ is the size of a smallest vertex cut. A graph is called k-*connected* if its connectivity is k or more. A maximal 2-connected subgraph of a graph is called a *block*.

A *cycle* is a graph C with vertex set $V(C) = \{v_1, v_2, \ldots, v_k\}$ and edge set $E(C) = \{v_i v_{i+1} \mid 1 \leq i \leq k-1\} \cup \{(v_k, v_1)\}$. The *length* of a cycle is the cardinality of its edge set. The cycle of length k is denoted C_k. A graph is *acyclic* if it does not contains any cycle as a subgraph. An acyclic graph is also called a *forest*. A *tree* is a connected forest.

The *complete graph of order* n, denoted K_n, is the graph in which every pair of vertices is joined by an edge. A *clique* of cardinality k in a graph G is a set of k pairwise adjacent vertices of G. The size of the largest clique of G is denoted by $\omega(G)$. A *stable set* of order k in G is a set of pairwise non-adjacent vertices of G. We denote by S_n the graph consisting of a stable set of order n. The cardinality of the maximum stable set is denoted $\alpha(G)$. A graph is called *bipartite* if its vertex set can be partitioned into two stable sets. The notation $G = (A \cup B, E)$ is used to indicate that A and B are the parts of G. It is well-known that a graph is bipartite if and only if contains no cycle of odd length as a subgraph. Given integers $p, q \geq 1$, the *complete bipartite graph* with parts of size p and q, denoted $K_{p,q}$, is such that there is one edge between every pair of vertices from different parts. For $p \geq 2$, the graph $K_{1,p-1}$ is called the *star of order* p.

Vertex covers, stable sets, matchings and edge covers. A *vertex cover* of a graph G is a set $C \subseteq V(G)$ such that every edge in $E(G)$ has at least one of its endvertices in C. The *vertex cover number* $\tau(G)$ is the size of a minimum vertex cover of G. An *edge cover* of G is a set $L \subseteq E(G)$ such that any vertex in $V(G)$ is the endvertex of at least one edge in L. The *edge cover number* $\rho(G)$ is the minimum size of an edge cover of G. A *matching* is a set of edges $M \subseteq E(G)$ such that no pair of edges from M have common endvertices. A vertex is said to be *saturated* by M if it is an endvertex of some edge of M, while M is a *perfect matching* if it saturates all vertices from $V(G)$. The *matching number* $\mu(G)$ is the size of a maximum matching of G. The Gallai Identities relate the parameters that were just defined.

Theorem 2.1.1 (Gallai Identities [44])**.** *For any graph G,*

(i) $\tau(G) + \alpha(G) = |V(G)|$.

(ii) $\mu(G) + \rho(G) = |V(G)|$, *if G has no vertices of degree* 0.

Proof. (i) It is clear from the definitions of stable set and vertex cover that S is a stable set if and only if $V(G) \backslash S$ is a vertex cover.

(ii) Consider an edge cover L of minimum size $\rho(G)$. Since L is minimal it has to be a union of s stars. The number of vertices in each star is one more than the number of edges in the star. By taking an edge from each star one obtains a matching, thus implying $\mu(G) \geq s$. This means $n = s + \rho(G) \leq \mu(G) + \rho(G)$. Conversely, consider a maximum matching M of size $\mu(G)$. The set $U = V(G) - V(M)$ is a stable set. For each vertex in U pick an edge incident to it. There will always be such an edge, as there are no vertices of degree 0. Call this set of edges E'. Clearly $E' \cup E(M)$ form an edge cover of G, so $\rho(G) \leq (n - 2\mu(G)) + \mu(G)$.

□

There is also a relation between $\mu(G)$ and $\tau(G)$ in any graph G: $\mu(G) \leq \tau(G)$. To see that, consider a vertex cover C and a matching M of G. Since C is a vertex cover, it must contain at least one vertex of each edge from M. Therefore, for any matching M and vertex cover C, it is valid that $|M| \leq |C|$, and so $\mu(G) \leq \tau(G)$. Another inequality that is true for any graph G is $\tau(G) \leq 2\mu(G)$. It follows from the fact that given a is a maximal matching M of G, the set of endvertices of the edges in M forms a vertex cover of G.

In the case of bipartite graphs an even stronger fact is true about $\mu(G)$ and $\tau(G)$.

Theorem 2.1.2 (König's Minimax Theorem ([83], 1931))**.** *If G is bipartite, then $\tau(G) = \mu(G)$.*

Proof. We present here a proof due to Lovász [95]. It was shown in the last paragraph that $\mu(G) \leq \tau(G)$. Therefore it suffices to show that $\tau(G) \leq \mu(G)$. Now consider a subgraph G' of G such that $\tau(G') = \tau(G)$ and for any $e \in E(G')$, $\tau(G' \backslash \{e\}) < \tau(G)$. If no two edges of G' have a vertex in common, then clearly $\tau(G) = \tau(G') = \mu(G') \leq \mu(G)$.

So assume there are two edges $x, y \in E(G')$ that are both incident to a same vertex v. Consider the graph $G' \backslash \{x\}$. Because of the minimality of G', $\tau(G' \backslash \{x\}) = \tau(G') - 1$. Moreover, in a minimum vertex cover S_x of $G' \backslash \{x\}$, no endvertex of x belongs to S_x. Similarly, there is a minimum vertex cover S_y of $G' \backslash \{y\}$ such that no endvertex of y is in S_y, and $|S_x| = |S_y|$.

Let $G'' = G'[\{v\} \cup (S_x \Delta S_y)]$, where Δ denotes the symmetric difference between S_x and S_y, $S_x \Delta S_y = (S_x \backslash (S_x \cap S_y)) \cup (S_y \backslash (S_x \cap S_y))$. Let $t = |S_x \cap S_y|$. Then $|V(G'')| = 2(\tau(G') - 1 - t) + 1$. The graph G'' is bipartite, since it is a subgraph of G. Let T be the smaller of the two parts of G''. Then T is a vertex cover of G'' and $|T| \leq \tau(G') - 1 - t$. □

König's theorem is a central theorem in graph theory. It provides a very useful characterization of the matching number of a bipartite graph. A matching of size k can be used to show that $\mu(G) \geq k$. If one wants to prove that no larger matching exists, it is sufficent to give a vertex cover of size k, since it would imply that $\mu(G) \leq k$. Moreover, given a set of k edges (resp. set of k vertices), one can verify in polynomial time if it consists of a matching (resp. vertex cover) of the graph. König's Theorem is a particular case of many other theorems, like the max-flow min-cut theorem, the total unimodularity theorem of linear programming and the weak perfect graph theorem (The first two may be found in classical combinatorial optimization books like the one from Nemhauser and Wolsey [105] while the third may be found in Diestel's Graph Theory book [29]).

Another very important result from graph theory is the so called Hall's Theorem.

Theorem 2.1.3 (Hall's Theorem ([55], 1935))**.** *Let $G = (A \cup B, E)$ be a bipartite graph. Then G has a matching that saturates A if and only if $|N(X)| \geq |X|$ for all $X \subseteq A$.*

It is easy to see that a bipartite graph $G = (A \cup B, E)$ has a matching that saturates A if and only if $\mu(G) = |A|$. As a consequence, Hall's theorem may be seen as a corollary of König's Theorem. Actually, König's and Hall's theorem, together with the following result from Frobenius can all be shown to be equivalent.

Theorem 2.1.4 (Frobenius Marriage Theorem ([41], 1917))**.** *A bipartite graph $G = (A \cup B, E)$ has a perfect matching if and only if $|A| = |B|$ and for each $X \subseteq A$, $|X| \leq |N(X)|$.*

For other results on matching theory we refer to the excellent book of Lovász and Plummer [94].

Planar Graphs. A graph is *planar* if it can be drawn in the plane in such a way that no edges cross each other. A graph H is said to be a *subdivision* of a graph G if H may be obtained by replacing the edges of G by paths. The paths replacing distinct edges from G are vertex-disjoint, except possibly for their endvertices. The well-known Kuratowski's Theorem states that a graph is planar if and only if it does not contain a subgraph that is a subdivision of the complete graph K_5 or the complete bipartite graph $K_{3,3}$. A graph is *outerplanar* if it is planar and can be drawn in the plane in such a way that all the vertices belong to the unbounded face of the drawing. In other words, no vertex is totally surrounded by edges. Outerplanar graphs have a forbidden graph characterization anologous to Kuratowski's Theorem. A graph is outerplanar if and only if it does not contain a subgraph that is a subdivision of the complete graph K_4 or the complete bipartite graph $K_{2,3}$.

Chordal graphs and tree decomposition. A graph is *chordal* if it does not contains induced cycles of size greater than 3. A vertex v is said to be *simplicial* if v and its neighbourhood form a clique. A *perfect elimination order* of a graph G is an ordering of the vertices such that vertex v is simplicial in the subgraph induced by v and the succeding vertices in the ordering. A graph G is chordal if and only if it has a perfect elimination order [42]. Some examples of chordal graphs are trees and *split graphs*, which are the graphs such that their vertex set can be partitioned into a stable set and a clique. Another subclass of chordal graphs that is going to be considered in this thesis is the one of *block graphs*, which are the graphs such that every block is a clique.

A *tree decomposition* of G is a pair $D = (X, T)$ such that $T = (I, F)$ is a tree and X is a family of subsets of $V(G)$ satisfying:

1. $\bigcup_{i \in I} X_i = V(G)$;
2. for every edge $uv \in E(G)$, there is $i \in I$ such that $\{u, v\} \subseteq X_i$;
3. for i, j, k in I, if j is in the (i, k)-path of T, then $X_i \cap X_k \subseteq X_j$.

The *size of the decomposition* is given by $max_{i \in I}(|X_i| - 1)$. The *treewidth* of a graph is the minimum size of a tree decomposition of it, and is denoted $tw(G)$. The treewidth measure how close the graph is from a tree. It is easy to see that $tw(T) = 1$, if T is a tree. A graph G has $tw(G) \leq 2$ if and only if it is a subgraph of a *series-parallel graph*, that is a graph that does not contains K_4 as a subdvision. Many problems that are NP-complete in their general formulation are polynomial given that the input graph is of bounded treewidth.

The class of *k-trees* is defined recursively as follows:

- The complete graph on k vertices is a k-tree.
- A k-tree G of $n+1$ vertices ($n \geq k$) can be constructed from a k-tree H of n vertices by adding a vertex and making it adjacent to k vertices corresponding to a k-clique of H.

It is easy to see that a k-tree is a chordal graph. A graph is a *partial k-tree* if it is a subgraph of a partial k-tree. Partial k-trees are precisely the graphs of treewidth at most k.

Graphs with few induced P_4's. A *cograph* is a P_4-free graph. Any cograph can be obtained by using the following rules [26]:

- The graph consisting of a single vertex is a cograph.
- Given cographs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, the *disjoint union* $G_1 + G_2 = (V_1 \cup V_2, E_1 \cup E_2)$ is a cograph.
- Given cographs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, the *join* $G_1 \oplus G_2 = (V_1 \cup V_2, E_1 \cup E_2 \cup V_1 \times V_2)$ is a cograph.

A graph is P_4-*sparse* if any 5 vertices of G induces at most one copy of P_4. Clearly, the class of the P_4-sparse graphs strictly contains the one of the cographs. A *spider* is a graph whose vertex set can be partitioned into S, C and R, where $S = \{s_1, \ldots, s_k\}$ is a stable set, $k \geq 2$; $C = \{c_1, c_2, \ldots, c_k\}$ is a clique; s_i is adjacent to c_j if and only $i = j$ (*thin spider*), or s_i is adjacent to c_j if and only if $i \neq j$ (a *thick spider*); R is allowed to be empty and if it is not, then all the vertices in R are adjacent to all the vertices in C and non-adjacent to all the vertices in S. P_4-sparse graphs, like the cographs, also admit a nice decomposition theorem. Hoàng [63] proved that if G is a non-trivial P_4-sparse graph, then either G or $\overline{G}$ is not connected, or G is a spider.

Also the P_5-free graphs admits a good characterization. A connected graph G is P_5-free if and only if for every induced subgraph $H \subseteq G$ either H has a dominating clique or a dominating cycle on five vertices [6].

Graph products. We end this section by defining some graph operations that are mentioned in this thesis. Given two graphs G and H, their *disjoint union* is the graph $G + H = (V(G) \cup V(H), E(G) \cup E(H))$, while their *join* is $G \oplus H = (V(G) \cup V(H), E(G) \cup E(H) \cup V(G) \times V(H))$. The *direct product* $G \times H$, the *lexicographic product* $G[H]$, the *cartesian product* $G \Box H$ and the *strong product* $G \boxtimes H$ all have vertex set $V(G) \times V(H)$ and the following edge sets:

- $E(G \times H) = \{(a, x)(b, y) \mid ab \in E(G)$ and $xy \in E(H)\}$.
- $E(G[H]) = \{(a, x)(b, y) \mid$ either $ab \in E(G)$ or $a = b$ and $xy \in E(H)\}$.
- $E(G \Box H) = \{(a, x)(b, y) \mid$ either $a = b$ and $xy \in E(H)$ or $ab \in E(G)$ and $x = y\}$.
- $E(G \boxtimes H) = E(G \times H) \cup E(G \Box H)$.

2.2 Graph colourings

A *graph colouring* is a partition of the vertices of a graph into sets that are called *colour classes*. If k colour classes are used in the partition and we want to make that explicit, we call it a k-*colouring*. Alternatively, a k-colouring of a graph $G = (V, E)$ is simply a function $c : V \to \{1, 2, \ldots, k\}$.

Often it is useful to consider colourings in which some condition is imposed on the colour classes. A *proper colouring* is one in which the colour classes are stable sets of the graph. In other words, in a proper colouring adjacent vertices are assigned distinct colours. A trivial proper colouring is obtained by putting each vertex in a separate colour class. The interest is in finding proper colourings with as few colours as possible. The *chromatic number* is precisely the smallest value k such that the graph in question admits a proper k-colouring. The chromatic number of graph G is denoted $\chi(G)$.

The problem of determining the chromatic number is one of Karp's twenty one NP-complete problems [80] from 1972. Deciding if a given graph admits a colouring with 3 colours is an NP-complete problem even if the graph is a planar 4-regular graph [28].

Lund and Yannakakis were the first to prove a negative result concerning the existence of approximation algorithms for the chromatic number. They proved that it is NP-hard to approximate the chromatic number of a graph to within a factor n^c for some constant $c > 0$ [97]. Later, Zuckerman [123] proved that for all $\epsilon > 0$, there is no algorithm that approximates the chromatic number within a factor of $n^{1-\epsilon}$, assuming P $\neq$ NP [123]. Currently the best known approximation algorithm for the chromatic number is the one of Halldórsson, that has an approximation factor $O\left(\frac{n(\log\log n)^2}{(\log n)^3}\right)$ [58]. He conjectured that the best possible approximation factor for graph colouring is $\Theta\left(\frac{n}{(\log n)^c}\right)$.

There is a simple heuristic to produce proper colourings of a graph, the greedy algorithm, that will be studied in more detail in Section 2.3.

GREEDY ALGORITHM

INPUT: G and a vertex ordering $\sigma = v_1 < v_2 < \cdots < v_n$ of $V(G)$
OUTPUT: a proper colouring c

---

1. For i from 1 to n:

   1.1 Assign to $c(v_i)$ the smallest positive integer that does not appear in $c(N(v_i))$.

2. return c.

See Figure 2.2 for two colourings of the Petersen graph obtained by the greedy algorithm. Clearly, distinct vertex orderings may give distinct colourings, that may also use a different number of colours.

A graph is said to be k-*degenerate* if each of its subgraphs has a vertex of degree at most k. The *degeneracy* of graph G is the smallest k such that it is k-degenerate. We denote the degeneracy by $\delta^*(G)$. A *degenerate ordering* of the vertices is an ordering $v_1, v_2, \ldots, v_n$ such that v_i has at most $\delta^*(G)$ neighbours in $G\backslash\{v_1, v_2, \ldots, v_{i-1}\}$. Observe that such an ordering can be obtained by iteratively taking a vertex v_i of smallest degree in the graph $G\backslash\{v_1, v_2, \ldots, v_{i-1}\}$ and putting it in the end of the of the ordering $\{v_1, v_2, \ldots, v_{i-1}\}$. An implication of this observation is that the degeneracy of a graph can be computed in polynomial time. Obviously, the degeneracy of a graph cannot exceed its maximum degree, therefore

$$\delta^*(G) \leq \Delta(G).$$

Now consider the colouring obtained by the greedy algorithm applied to a degenerate ordering of the vertices of the graph. At the moment the algorithm colours vertex v_i, it has at most $\delta^*(G)$ neighbours that are already coloured, since they are precisely the ones preceding it in the ordering. But then there is always a colour in $\{1, 2, \ldots, \delta^*(G) + 1\}$ that can be given to v_1. As a consequence, the greedy algorithm uses at most $\delta^*(G) + 1$ colours when applied to that ordering, and therefore

$$\chi(G) \leq \delta^*(G) + 1.$$

Combining the last two inequalities we obtain that

$$\chi(G) \leq \delta^*(G) + 1 \leq \Delta(G) + 1.$$

This bound is tight, as shown by the following Proposition.

Proposition 2.2.1 (folklore). *Let G be a graph and $n \geq 1$.*

(i) If $G \cong C_n$ and n odd, then $\chi(G) = \Delta(G) + 1 = 3$.

(ii) If $G \cong K_n$, then $\chi(G) = \Delta(G) + 1 = n$.

Proof. First let $G \cong C_n$, where n is odd. Clearly $\chi(G) > 1$, so suppose c is a proper colouring with two colours and let $v_1, v_2, \ldots, v_n$ be the vertices as they appear in the cycle. We may assume without loss of generality that $c(v_1) = 1$, and since it is adjacent to v_2 and the colouring is proper, then $c(v_2) =$

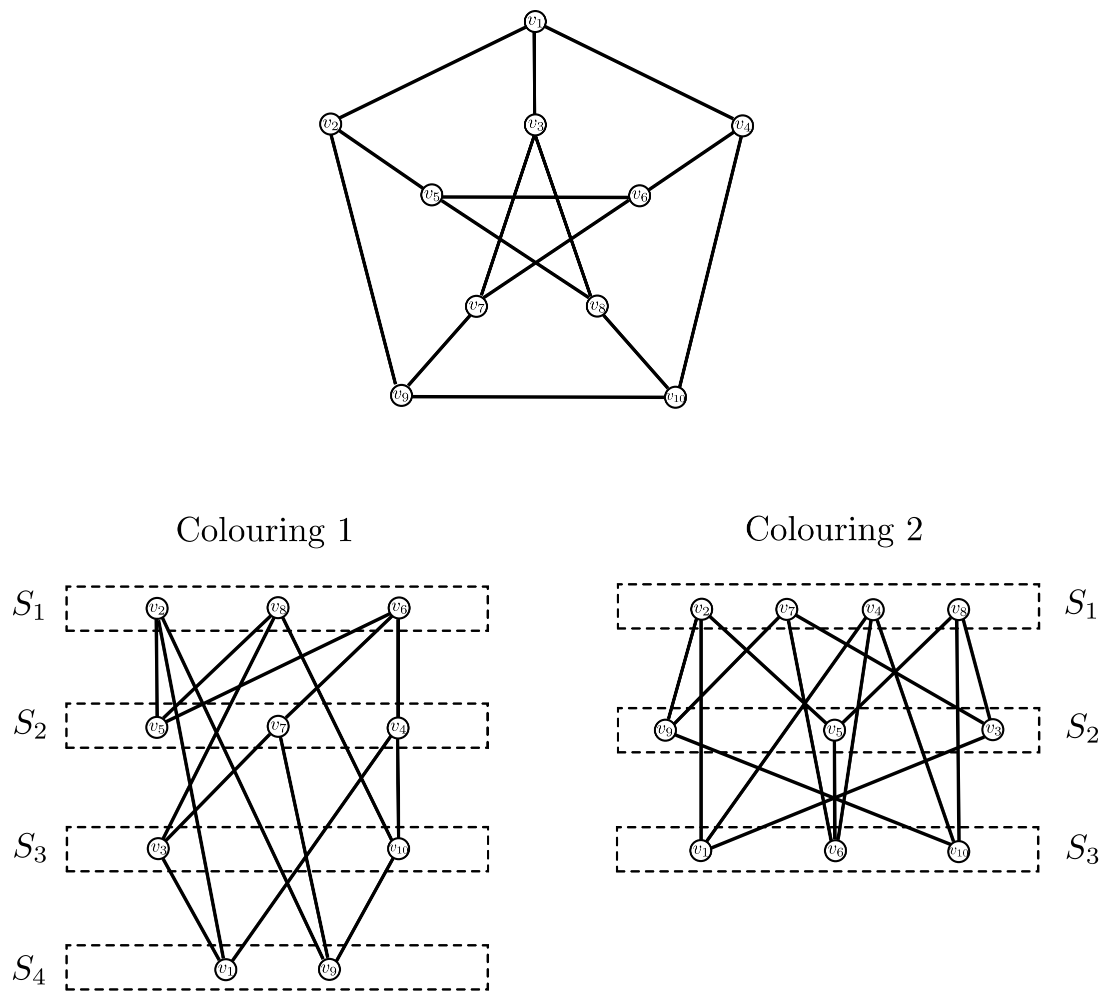


Figure 2.1: The Petersen graph and two colourings of it obtained by the greedy algorithm. The colourings correspond to an application of the greedy algorithm to any vertex ordering in which the vertices of S_i preceed the ones in S_j, whenever $i < j$.

2. By repeatedly using this argument, we get that $c(v_i) = 1$, in case i is odd, and $c(v_i) = 2$ otherwise. Since v_1 and v_n are adjacent, and n is odd, we get a contradiction. So, $\chi(G) = \Delta(G) + 1 = 3$.

Now let $G \cong K_n$. Since every pair of vertices of G is adjacent, in any proper colouring of G all vertices should receive distinct colours. So, $\chi(G) = n = \Delta(G) + 1$. □

But the graphs in Proposition 2.2.1 are essentialy the only ones for which $\chi(G) = \Delta(G) + 1$, as shown in the following theorem of Brooks. The proof presented here uses the greedy algorithm and is due to Lovász [95].

Theorem 2.2.1 (Brooks [15])**.** *Let G be a connected graph. Then, $\chi(G) \leq \Delta(G)$, unless G is an odd cycle or a complete graph, in which case $\chi(G) = \Delta(G) + 1$.*

Proof. We already shown that for any G, $\chi(G) \leq \Delta(G) + 1$. Suppose, by contradiction, that G is neither an odd cycle nor a complete graph and that $\chi(G) = \Delta(G) + 1$. Moreover assume G is a graph with the minimum number of vertices satisfying that.

We claim that G is 2-connected. If this is not the case, let v be a cut vertex, $C_1, C_2, \dots, C_p$ the vertices in the connected components of $G\backslash\{v\}$, and let $k = \max_{1\leq i\leq p} \chi(G_i)$, where $G_i = G[C_i \cup \{v\}]$. It is clear that $k \leq \chi(G)$. If $k < \chi(G)$ then every G_i may be coloured with less than k colours. We can change the colouring of each G_i so that v is coloured with the same colour in all the colourings. By doing so we obtain a proper colouring of G that uses less than $\chi(G)$ colours, a contradiction. Therefore, $k = \chi(G)$. Let G_j be such that $\chi(G_j) = k$. Since we assumed that $\chi(G) = \Delta(G) + 1$ and G is a counterexample with the minimum number of vertices, G_j has to be a complete graph with $\Delta(G) + 1$ vertices. But then G_j should be a connected component of G, since otherwise v has degree greater than $\Delta(G)$ in G. Therefore G has to be 2-connected.

Now we show how to colour G with $\Delta(G)$ colours, by giving an ordering of its vertices for which the greedy algorithm uses no more than $\Delta(G)$ colours. Since G is not complete, there is at least a pair of vertices, say x, y, such that $xy \notin E(G)$. If x and y have a common neighbour z, make $u = x$, $v = y$ and $w = z$. Otherwise, x and y are at distance more than two from each other. In this case take u, v and w as three consecutive vertices in a shortest path between x and y. The vertices u, v, w are such that $uv, vw \in E(G)$ and $uw \notin E(G)$.

Now, we order the vertices of the graph by making $v_1 = u, v_2 = v$ and ordering the vertices in $V(G)\backslash\{v_1, v_2\}$ in such a way that if $3 \leq i \leq j$ then $dist(v, v_i) \geq dist(v, v_j)$. Observe that this ordering can obtained by doing a breadth-first search in $G\backslash\{v_1\}$ starting at v and then reversing the ordering. This ordering is such that for every $i < n$, there is $j > i$ such that $v_iv_j \in E(G)$. As a consequence, every v_i, $i < n$, has at most $\Delta(G) - 1$ neighbours preceeding him, and so the greedy algorithm applied to this ordering uses at most $\Delta(G)$ colours to colour $\{v_1, v_2, \dots, v_{n-1}\}$. Finally, vertex v_n is also assigned a colour that is at most $\Delta(G)$, since it is adjacent to v_1 and v_2 that have both the same colour, namely colour 1.

□

As a consequence to Brooks' Theorem, it can be decided in polynomial time if a graph G has chromatic number equal to $\Delta(G) + 1$.

Extensions of Brooks' Theorem have been considered in the literature. Borodin and Kostochka [13] conjectured that every graph of maximal degree $\Delta \geq 9$ and chromatic number at least Δ has a Δ-clique. Reed [111] proved that this is true when Δ is sufficiently large, thus settling a conjecture of Beutelspacher and Herring [9]. More information about this problem can be found in the monograph of Jensen and Toft [78, Problem 4.8]. A generalization of this problem has also been studied by Farzad, Molloy and Reed [40] and Molloy and Reed [102]. In particular, it is proved in [102] that determining whether a graph with large constant maximum degree Δ is $(\Delta - q)$-colourable can be done in linear time if $(q + 1)(q + 2) \leq \Delta$. This threshold is optimal by a result of Emden-Weinert, Hougardy and Kreuter [33], since they proved that for any two constants Δ and $q \leq \Delta - 3$ such that $(q + 1)(q + 2) > \Delta$, determining whether a graph of maximum degree Δ is $(\Delta - q)$-colourable is NP-complete.

Brooks' theorem has been adapted and proved for more general versions of the colouring problem. In the following we show a generalization of this result to *list colouring*, in which the set of colours that are available to each vertex is restricted.

A *list assignment* L is an assignment of a list of colours $L(v)$, for every vertex v. If each list is of size at least k, we say that L is a *k-list-assignment*. An *L-colouring* c is a proper colouring such that the colours are assigned according to the list assignment L, that is for every $v \in V(G)$, $c(v) \in L(v)$. A graph is *L-colourable* if it admits a proper L-colouring. One of the main concepts in list colouring is that of a *k-choosable graph*. A graph is *k-choosable* if it is L-colourable for every k-list-assignment L. The *choice number* of a graph G is the least k such that G is k-choosable, and is denoted $ch(G)$. Given a list assignment L where all lists are identical and of size k, it is easy to see that G is L-colourable if and only if it is k-colourable. Then clearly

$$ch(G) \geq \chi(G).$$

The choice number of a graph can be arbitrarily far apart from the chromatic number. For example, the complete bipartite graph $K_{k,k}$ is such that its choice number tends to infinity with k, as shown by Erdös, Rubin and Taylor [107].

Consider a list assignment L of size $\delta^*(G) + 1$. In a degenerate ordering $v_1, v_2, \dots v_n$, one have that vertex v_i has only at most $\delta^*(G)$ neighbours in $\{v_1, v_2, \dots, v_{i-1}\}$. Since $|L(v_i)| \geq \delta^*(G)+1$, we can always greedily assign v_i a colour from $L(v_i)$ that was not assigned to any of $\{v_1, v_2, \dots, v_{i-1}\}$. Therefore,

$$ch(G) \leq \delta^*(G) + 1.$$

Since the degeneracy of a graph cannot exceed its maximum degree, we get that:

$$ch(G) \leq \delta^*(G) + 1 \leq \Delta(G) + 1.$$

Before generalizing Brooks' theorem for list colouring we just need the following Lemma.

Lemma 2.2.1. *Let G be a connected graph and a list assignment L such that for all $v \in V(G)$ $|L(v)| \geq d(v)$ and there is a vertex u such that $|L(u)| \geq d(u) + 1$. Then G is L-colourable.*

Proof. Take a breadth-first ordering of the vertices starting from u and reverse it. The obtained ordering $v_1, v_2, \dots, v_n$ of the vertices is such that for every $i < n$, vertex v_i has neighbour v_j, $j > i$. Moreover, $v_n = u$. We will greedily colour the vertices according to this ordering. When we are about to colour vertex v_i, $i < n$, at most $d(v_i) - 1$ of its neighbours are already coloured, and since $|L(v_i)| \geq d(v_i)$, there is always a colour from $L(v_i)$ that can be assigned to v_i. In the end all vertices are coloured except for $v_n = u$. But $|L(u)| \geq d(u) + 1$, and so there is a colour from $L(u)$ that can be assigned to u that does not appear in any of its neighbours. □

Theorem 2.2.2. *Let G be a connected graph. Then, $ch(G) \leq \Delta(G)$, unless G is an odd cycle or a complete graph, in which case $ch(G) = \Delta(G) + 1$.*

Proof. We already showed that for any G, $ch(G) \leq \Delta(G) + 1$. If G is an odd cycle or a complete graph, $\chi(G) = \Delta(G) + 1$, and since $ch(G) \geq \chi(G)$ we get that $ch(G) = \Delta(G) + 1$. Suppose, by contradiction, that G is neither an odd cycle nor a complete graph and that $ch(G) = \Delta(G) + 1$.

Let L be a $\Delta(G)$-list assignment.

Assume first that G is not 2-connected. Let v be a cut vertex, C_1 be a connected component of $G \backslash \{v\}$, $C_2 = V(G) \backslash (C_1 \cup \{v\})$ and $G_i = G[C_i \cup \{v\}]$, $i \in \{1, 2\}$. Then G_1 and G_2 are connected and there is no edge between vertices of C_1 and C_2. Moreover $d_G(v) = d_{G_1}(v) + d_{G_2}(v)$. Let $L'(v)$ be the set of colours α such that there exists an L-colouring of G_2 with v coloured α.

Because of Lemma 2.2.1, for any subset S of $L(v)$ of size $d_{G_2}(v) + 1$, there is an L-colouring of G_2 with v coloured in S. Therefore, $|L'(v)| \geq \Delta(G) - d_{G_2}(v) \geq d_{G_1}(v)$. But then, because of the minimality of G, G_1 is $\Delta(G_1)$-choosable, and so there is an L-colouring c_1 of G_1 such that $c_1(v) \in L'(v)$. Since $c_1(v) \in L'(v)$, there is an L-colouring c_2 of G_2 such that $c_2(v) = c_1(v)$. The union of c_1 and c_2 is an L-colouring of G.

It remains to consider the case when G is 2-connected. In case L is such that the lists of all vertices are equal, we may assume without loss of generality that these lists are $\{1, 2, \dots, \Delta(G)\}$ and an L-colouring is simply a $\Delta(G)$-colouring of the graph. Then by Brooks' theorem, G is L-colourable, as it is neither an odd cycle nor a complete graph. So we may assume that there are vertices u and v such that $L(u) \neq L(v)$. Moreover, as G is connected, we may assume $uv \in E(G)$.

Let $v_1 = u, \ldots, v_n = v$ be an ordering of $V(G)$ obtained by doing a breadth-first search on $G - u$ starting from v and then reversing it and putting $v_1 = u$ in the beginning of the ordering. This ordering always exists, since G is 2-connected, and it is such that for every $1 \leq i < n$, v_i has a neighbour in $\{v_{i+1}, \ldots, v_n\}$. First, colour v_1 with a colour in $L(v_1)\backslash L(v_n)$. Then we colour greedily the vertices $\{v_2, \ldots, v_{n-1}\}$ according to the ordering. When we consider vertex v_i, $1 < i < n$, it has at most $d(v_i) - 1$ neighbours that are already coloured, and since $|L(v_i)| = \Delta(G)$, there is always one colour that may be assigned to v_i. It remains to colour v_n. But then, it has at most $d(v_i) - 1$ neighbours that may have been coloured with a colour from $L(v_n)$, since $v_1 v_n \in E(G)$ and we coloured v_1 with a colour in $L(v_1)\backslash L(v_n)$. Therefore, there is always a colour from $L(v_n)$ that may be assigned to v_n. □

Precolouring extension. We end this section by introducing the precolouring extension problem, as this problem and some results about its computational complexity are going to be used in the next chapters.

A *precolouring* of a graph G is a function $p_c : W \to \{1, 2, ..., k\}$, where $W \subset V(G)$ is the set of *precoloured vertices* of G and $V(G)\backslash W$ are the *precolourless vertices*. It is possible that $W = \emptyset$. Given a precolouring p_c of G, we want to decide if it can be *extended* to a proper colouring of the entire graph.

PRECOLOURING EXTENSION
INPUT : Graph G and precolouring $p_c : W \to \{1, 2, ..., k\}$, $W \subset V(G)$.
OUTPUT : Is there a proper colouring $c : V(G) \to \{1, 2, ..., k\}$ such that $p_c(v) = c(v)$, if $v \in W$?

Now, we mention some known results about the computational complexity of PRECOLOURING EXTENSION. We will denote by $W_1, W_2, \ldots, W_k$ the subsets of W that are precoloured with colours $1, 2, \ldots, k$, respectively. Moreover, the ℓ-PRECOLOURING EXTENSION is the same problem as PRECOLOURING EXTENSION but with the additional constraint that $|W_i| \leq \ell$, $1 \leq i \leq k$.

In case $W = \emptyset$, PRECOLOURING EXTENSION corresponds to the problem of deciding if G admits a proper colouring with k colours, that we already mentioned to be NP-complete. The following results can be found in [117]. PRECOLOURING EXTENSION is NP-complete in bipartite graphs, even if $|W| = 3$. It is NP-complete even for a planar bipartite graph and $k = 3$. The problem is solvable in polynomial time for perfect graphs if $W_i = \emptyset$, $i \geq 3$, and is NP-complete otherwise. PRECOLOURING EXTENSION is polynomial for cographs, split graphs, and complement of bipartite graphs.

The complexity of 1-PRECOLOURING EXTENSION is of particular interest to us, since this problem is related to the method we present in Chapter 4. From what was mentioned in the last paragraph, we get that 1-PRECOLOURING EXTENSION is polynomial for cographs, split graphs, and complement of bipartite graphs. Moreover, it is polynomial for P_4-sparse graphs [69] and chordal graphs [100]. On the other hand, 1-PRECOLOURING EXTENSION is NP-complete for perfect graphs [117].

2.3 Colouring heuristics and their worst-case behaviour

As mentioned in the last section, the problem of finding a proper colouring of a graph with the minimum number of colours cannot be solved in polynomial time, unless P = NP. We also mentioned that under the same assumption not even a constant factor approximation algorithm exists for the chromatic number. In this section we present two simple heuristics to that problem. These heuristics are going to motivate the study of two special kinds of colourings that we study in detail in the next chapters of this thesis.

2.3.1 Greedy colourings and the Grundy number

The colourings produced by the greedy algorithm, presented in the last section, satisfy the following property:

For every (i, j) with $j < i$, every vertex v in S_i has a neighbour in S_j. $\quad (P)$

Otherwise v would have been coloured by an integer not greater than j. The converse also holds. A colouring satisfying Property (P) can be seen as the colouring obtained by the greedy algorithm applied to any vertex ordering in which the vertices of S_i precede those of S_j when $i < j$. We call a colouring satisfying (P) a *greedy colouring*.

The *Grundy number* of G is the largest k such that G has a greedy k-colouring. We denote this number by $\Gamma(G)$. The Grundy number and its ratio with the chromatic number measure how bad the greedy algorithm may perform on a graph.

A greedy colouring is a proper colouring, therefore

$$\chi(G) \leq \Gamma(G).$$

Moreover, starting with a colouring that uses $\chi(G)$ colours and ordering the vertices according to their colour classes, the greedy algorithm can be used to produce a greedy colouring that uses $\chi(G)$ colours. Consequently, there is always a proper colouring that uses the minimum number of colours that is a greedy colouring.

Finally, Property (P) implies that if a vertex is coloured i in a greedy colouring, it has at least $i-1$ neighbours, so

$$\Gamma(G) \leq \Delta(G) + 1.$$

The *Grundy function* of a directed acyclic graph was defined in 1939 by Grundy [53] as the unique function assigning to each vertex v the smallest nonnegative integer $g(v)$ assigned to no vertex u such that there is a directed edge from u to v. If the vertices of a graph are ordered $\sigma = v_1 < v_2 < \cdots < v_n$ and we direct each edge uv from u to v whenever $u < v$, then clearly the Grundy function g of the resulting acyclic digraph is such that $g(u) + 1$ is the colour assigned by the greedy algorithm applied to the graph with the order σ. Grundy gave this definition motivated by the study of directed graphs corresponding to games.

Christen and Selkow [24], in 1979, were the first to define it in the context of graph colourings. They characterized the $\Gamma\chi$-perfect graphs, which are the graphs such that for any induced subgraph the Grundy number is equal to the chromatic number. They also characterized the $\Gamma\omega$-perfect graphs, which are the graphs such that for any induced subgraph the Grundy number is equal to the clique number. In [36], Erdös et al. proved that the Grundy number was in fact the same as the ordered chromatic number that was studied independently by Simmons [115].

Computational complexity of computing the Grundy number of a graph. In their article from 1982, Hedetniemi, Hedetniemi, and Beyer [10] proved that the Grundy number of a tree can be computed in linear time and asked about the complexity of the problem in the general case. They observed that the Grundy number of a tree T equals the largest order of a *binomial tree* that is contained in T, a binomial tree being recursively defined as follows:

- $\mathcal{B}_1 = K_1$.
- $\mathcal{B}_2 = K_2$.
- The binomial tree of order k, denoted $\mathcal{B}_k$, is constructed from a copy H of $\mathcal{B}_{k-1}$ by adding $|V(H)|$ vertices and matching them with the vertices in $V(H)$.

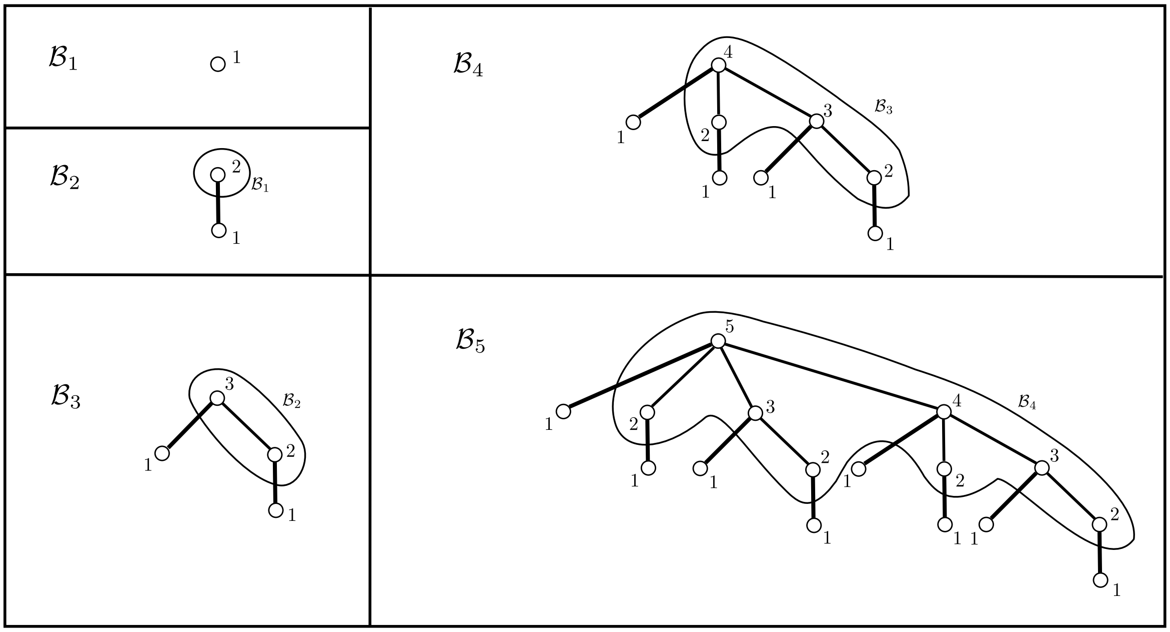


Figure 2.2: The binomial tree $\mathcal{B}_k$, $k = 1, 2, \ldots, 5$ together with a greedy colouring with $\Gamma(\mathcal{B}_k)$ colours. The bold edges indicate the matching as in the definition.

See Figure 2.3.1. Clearly, $\Gamma(\mathcal{B}_1) = 1$ and $\Gamma(\mathcal{B}_2) = 2$. Given that $\Gamma(\mathcal{B}_{k-1}) = k - 1$ one may obtain a greedy colouring of $\mathcal{B}_k$ with k colours by giving colour 1 to the vertices that are added in the construction of $\mathcal{B}_k$ and then greedily colouring the rest of the graph using the same ordering of the vertices of $\mathcal{B}_{k-1}$ that gives a greedy colouring with $\Gamma(\mathcal{B}_{k-1})$ colours.

The algorithm in [10] computes, for every vertex of the tree, the maximum k such that the graph contains a binomial tree of order k in which the vertex being considered is a vertex with maximum degree (therefore the one that is coloured k in the greedy colouring with k colours from $\mathcal{B}_k$). This yields a linear time algorithm for computing the Grundy number of a tree. Telle and Proskurowski [116] gave a linear time algorithm for partial k-trees, that correspond to the graphs of tree-width at most k.

The complexity of the problem in the general case was still open in 1995, appearing as Problem 10.4 in the monograph of Jensen and Toft [78]. It was in 1997 that Goyal and Vishvanathan [50] proved the NP-completeness of the problem for a general graph. Later, Zaker [121] proved that the problem is NP-complete even for the complement of a bipartite graph. In Chapter 3, we complement the later result by showing that the problem is NP-complete for bipartite graphs. An implication of our reductions is that any characterization of the graphs whose Grundy number equals to $\Delta(G) + 1$ is unlikely to be checkable in polynomial time, under the classical complexity assumptions. In other words, no Brooks' type theorem shall exist for the Grundy number.

Approximation algorithms. Currently, there is no known constant-factor approximation algorithm for the Grundy number of a general graph. Then only result concerning the approximability of the Grundy number for general graphs was obtained by Kortsarz [84]. He proved that there is a constant $c > 1$ such that approximating the Grundy number within c is not possible, unless $NP \subseteq RP$, where RP ("Randomized Polynomial time") is the class of decision problems for which there exists a probabilistic Turing machine that runs in polynomial time in the size of the input that may reject a correct input with probability at most $\frac{1}{2}$ but that only accepts correct inputs. The inclusion $NP \subseteq RP$ is trivial, and $NP \neq RP$ assuming the commonly believed conjectures $P \neq NP$ and $P = BPP$, where BPP ("Bounded-error Probabilistic Polynomial time") is the class of decision problems admiting a probabilistic turing machine that runs in polynomial time in the size of the input but that may reject (resp. accept) a correct (resp. incorrect) input with probability at most $\frac{1}{3}$. We refer to [3] for more detailed information about these complexity classes.

For some specific classes of graphs, the Grundy number can be approximated within a constant factor. It is the case of interval graphs, complement of bipartite graphs and complement of chordal graphs. A graph G is said to be *perfect* if $\chi(H) = \omega(H)$, for every induced subgraph H of G. The chromatic number of a perfect graph can be determined in polynomial time [52]. The existence of constant factor approximation algorithms for the above mentioned classes is a consequence of these graphs being perfect and their Grundy number being bounded from above by a constant factor of their clique number. These bounds are going to be proved later in this section. To conclude the discussion about approximation algorithms, we mention that Kortsarz [84] observed that for a general graph not even an $o(n)$-approximative algorithm is known for the Grundy number, leaving the problem of finding such algorithm as an open problem.

The greedy algorithm and online colouring algorithms. The greedy algorithm can be seen as an *online colouring algorithm*. In an *online colouring algorithm*, the vertices of the graph are coloured in some order $v_1, v_2, \dots, v_n$. The colour of v_i is assigned by only looking at the subgraph of G induced by the set $\{v_1, \dots, v_i\}$ and the assigned colour of v_i is never changed.

One way to analyze and compare online colouring algorithms is to consider their *performance ratio*. The *performance ratio* of an online colouring algorithm is the maximum ratio of the number of colours used by the algorithm to the chromatic number of the graph, ranging over all input graphs. In a formal way, let $\Gamma_A(G)$ be the maximum number of colours used by the online colouring algorithm A over all possible vertex orderings of graph G; the *performance ratio* of A is the value $max_G \left\{ \frac{\Gamma_A(G)}{\chi(G)} \right\}$.

An online colouring algorithm with sublinear performance ratio was given by Lovázs, Saks and Trotter [92], the performance ratio being $O(\frac{n}{\log^* n})$. Vishwanathan [119] gave a randomized algorithm with performance ratio $O(\frac{n}{\sqrt{\log n}})$. His algorithm was modified in [56] to improve the performance ratio to $O(\frac{n}{\log n})$. Halldorsson and Szegedy [57] gave lower bounds on the performance ratio of *any* online colouring algorithm:

Theorem 2.3.1 ([57]). *The performance ratio of any deterministic online colouring algorithm is at least* $\frac{2n}{log^2 n}$.

Theorem 2.3.2 ([57]). *The performance ratio of any randomized online colouring algorithm is at least* $\frac{n}{16 log^2 n}$.

The greedy algorithm has a good performance ratio on certain classes of graphs. Let the *performance ratio of an online colouring algorithm A on the class of graphs C* be the value $max_{G \in C} \left\{ \frac{\Gamma_A(G)}{\chi(G)} \right\}$.

The performance ratio of the greedy algorithm on the class of interval graphs received particular attention. In 1988, Chrobak and Slusarek [25] introduced an algorithm, called *Buddy Decreasing Size* to solve the *Dynamic Storage Allocation problem*, and this algorithm used the greedy colouring algorithm on interval graphs as a subroutine. Motivated by that application, Kierstead [81] proved that the performance ratio of the greedy algorithm on interval graphs is at most 40, what implies that $\Gamma(G) \leq 40\omega(G)$ for an interval graph G. In 1995, Kierstead and Qin [82] improved this bound to 26. Pemmaraju, Raman and Varadarajan [108] improved this bound to 10 and mentioned that they believed it could be reduced to 8. The proof that $\Gamma(G) \leq 8\omega(G)$ appears in Raman's thesis [109], another proof also being given by Narayanaswamy and Babu [104].

Gyárfás and Lehel [54] proved many results concerning the performance ratio of the greedy algorithm on other particular graph classes:

Proposition 2.3.1 ([54]). *If G is a split graph, then* $\Gamma(G) \leq \omega(G) + 1$.

Proof. Let $X \cup Y$ be the parts of G, where X is complete with $|X| = \omega(G)$ and Y is a stable set. Clearly, no vertex of Y can be coloured greater than $\omega(G)$ in a greedy colouring, as it has degree at most $\omega(G) - 1$. Suppose there is a greedy colouring c in which a vertex $v \in X$ has $c(v) = \omega(G) + 2$. It cannot be the case where all the neighbours of v with a colour in $\{1, 2, \ldots, \omega(G) + 1\}$ are all in X, since $|X| = \omega(G)$. Let k be the maximum value from $\{1, 2, \ldots, \omega(G) + 1\}$ such that the only neighbours of v coloured k are in Y, and let $u \in Y$ be one such neighbour. Since all the neighbours of u are in X, the colours $\{1, 2, \ldots, k - 1\}$ must all appear in X. Then the colours $\{1, 2, \ldots, k - 1\} \cup \{k + 1, k + 2, \ldots, \omega(G) + 1\} \cup \{\omega(G) + 2\}$ all appear in X, a contradiction to the fact that $|X| = \omega(G)$. □

Proposition 2.3.2 ([54]). *If G is the complement of a bipartite graph, then* $\Gamma(G) \leq \frac{3}{2}\omega(G)$.

Proof. Let $G = (A \cup B, E)$ and consider a greedy colouring c with $\Gamma(G)$ colours. Since G is the complement of a bipartite graph, $\alpha(G) \leq 2$. Therefore any colour class is of size at most 2. Let X be the set of pairs (a, b) such that $a \in A$, $b \in B$ and a and b form a colour class, and let Y the set of vertices that are alone in their colour classes. Clearly, $|X| + |A \cap Y| \leq \omega(G)$ and $|X| + |B \cap Y| \leq \omega(G)$. Moreover, since c is a greedy colouring Y is a clique, and therefore $|Y| \leq \omega(G)$. From these inequalities, $\Gamma(G) = |X| + |Y| \leq \frac{3}{2}\omega(G)$. □

Proposition 2.3.3 ([54]). *If G is the complement of a chordal graph, then* $\Gamma(G) \leq 2\omega(G) - 1$.

Proof. The proof is by induction on $|V(G)|$. The proposition is true if G is a complete graph. Let $C_1, C_2, \ldots, C_k$ be the colour classes of a greedy colouring of G with k colours. This corresponds to a clique partition $C_1, C_2, \ldots, C_k$ of $V(\overline{G})$ in which C_i is a maximal clique on the graph $\overline{G} - (C_1 \cup \ldots \cup C_{i-1})$. To obtain the desired result, we shall show that $\alpha(\overline{G}) \geq \frac{k+1}{2}$, which implies $\omega(G) \geq \frac{k+1}{2}$.

Let $\overline{G}' = \overline{G} - V(C_1)$. If $\overline{G}'$ has more components than $\overline{G}$, then the claim easily follows from the fact that $C_2, C_3, \ldots, C_k$ is a clique partition of $\overline{G}'$ and the inductive hypothesis. Otherwise, C_1 intersects at most one other maximal clique in $\overline{G}$, and therefore it should contains a simplicial vertex. This vertex can be added to any of the maximal cliques $C_2, C_3, \ldots, C_k$ of $\overline{G}'$ and induction can be used to show that:

$$\alpha(\overline{G}) \geq \alpha(\overline{G}') + 1 \geq \frac{(k-1)+1}{2} + 1 > \frac{k+1}{2}.$$

□

Proposition 2.3.4 ([54]). *If G is a cograph, then* $\Gamma(G) = \chi(G)$.

Proof. It is well known that a graph is a cograph if and only if it consists of a single vertex or it is obtained from graphs G_1 and G_2 by applying either the join or the disjoint union operation. The result follows by induction on $|V(G)|$, the base case being that of a single vertex that is trivial.

First, suppose that $G = G_1 \times G_2$. Then, $\Gamma(G_1 \times G_2) = \Gamma(G_1) \times \Gamma(G_2) = \chi(G_1) \times \chi(G_2)$, applying the inductive hypothesis.

In case $G = G_1 \oplus G_2$ we have that $\Gamma(G_1 + G_2) = \max\{\Gamma(G_1), \Gamma(G_2)\} = \max\{\chi(G_1), \chi(G_2)\}$, again by the inductive hypothesis. □

The graphs in the previous propositions are all subclasses of perfect graphs, which implies that they satisfy $\chi(G) = \omega(G)$. Therefore the bounds given by Gyárfás and Lehel show that the performance ratio of the greedy algorithm is constant for these classes.

For general graphs the perfomance ratio of the greedy algorithm can be rather disappointing:

Theorem 2.3.3. *The performance ratio of the greedy algorithm can be as bad as* $\frac{n}{4}$.

Proof. Let $M_{k,k}$ be the bipartite graph with $V(M_{k,k}) = \{v_1, v_2, \ldots, v_k, w_1, w_2, \ldots, w_k\}$ and $E(M_{k,k}) = \{v_i w_j \mid i \neq j\}$. Consider the colouring c obtained by the greedy algorithm applied to the ordering $v_1, w_1, v_2, w_2, \ldots, v_k, w_k$. It is easy to see that $c(v_i) = c(w_i) = i$, and therefore $\Gamma(M_{k,k}) = k$, since $\Delta(M_{k,k}) = k - 1$ and the Grundy number is at most the maximum degree plus one. As $M_{k,k}$ is a bipartite graph, $\chi(M_{k,k}) = 2$. Consequently, since $M_{k,k}$ is a graph with $|V(M_{k,k})| = 2k$ that is coloured in the worst-case by the greedy algorithm with $\frac{|V(M_{k,k})|}{2} = k$ colours, we have the desired performance ratio of $\frac{n}{4}$. □

Observe that the the family of bipartite graphs $\{M_{k,k} \mid k \geq 2\}$ that are defined in the proof of Theorem 2.3.3 is such that the difference between the Grundy number and the chromatic number is not bounded even for bipartite graphs. Also observe that bipartite graphs are perfect, so despite the performance ratio of the greedy algorithm being bounded by a constant on some subclasses of perfect graphs, this is not true for the class of all perfect graphs.

Expected behaviour of the greedy algorithm. Although there are graphs with Grundy number arbitrarily far from the chromatic number, the greedy algorithm has a good expected behaviour on the random graph $G_{n,p}$. Grimmett and McDiarmid [51] proved that for $G \in G_{n,p}$, $\mathbb{E}[\Gamma(G)] \leq (1 + o(1))\frac{n}{logn}$. In the same paper they proved that $\mathbb{E}[\chi(G)] \geq (1 - o(1))\frac{n}{2logn}$, thus implying that $\mathbb{E}[\Gamma(G)] = (2 + o(1))\,\mathbb{E}[\chi(G)]$. On the other hand, Kucera [91] considered the randomized version of the greedy colouring algorithm, in which the input vertex ordering is randomly chosen. He proved that, given constants $c_1, c_2, c_3 > 0$, for every sufficient large n there is a graph such that the probability that the randomized greedy algorithm uses less than $(1-c_2)\frac{n}{logn}$ colours when applied to G is $o(n^{-c_3})$. Thus, even a polynomial number of applications of the randomized greedy algorithm is not likely to give a good result for some graphs.

Fixed-parameter-complexity of determining the Grundy number. We now go back to algorithmic aspects of the Grundy number, and consider the problem of deciding if the Grundy number of a graph is at least a fixed value k. Zaker [121] introduced the concept of a k-*atom*, and used it to obtain a necessary and sufficient condition for a graph to have Grundy number at least k. The family of the k-atoms, which we denote by $\mathcal{A}_k$, is defined as follows.

- $\mathcal{A}_1 = \{K_1\}$.
- $\mathcal{A}_2 = \{K_2\}$.
- An element from $\mathcal{A}_k$ is constructed from an element $H \in \mathcal{A}_{k-1}$ as follows. Fix a value $1 \leq m \leq |V(H)|$ and add a copy of the stable set S_m to H. Then, construct a matching between the vertices in H and S_m that saturates the vertices in S_m. Finally, make each vertex from H

that is not saturated by the matching adjacent to exactly one vertex in S_m. In other words, that corresponds to adding a maximal (with respect to inclusion) stable dominating set to the graph H. Observe that the binomial tree $\mathcal{B}_k$ belongs to $\mathcal{A}_k$ and is obtained by adding in each recursive step a matching of the same size as the graph from the previous step..

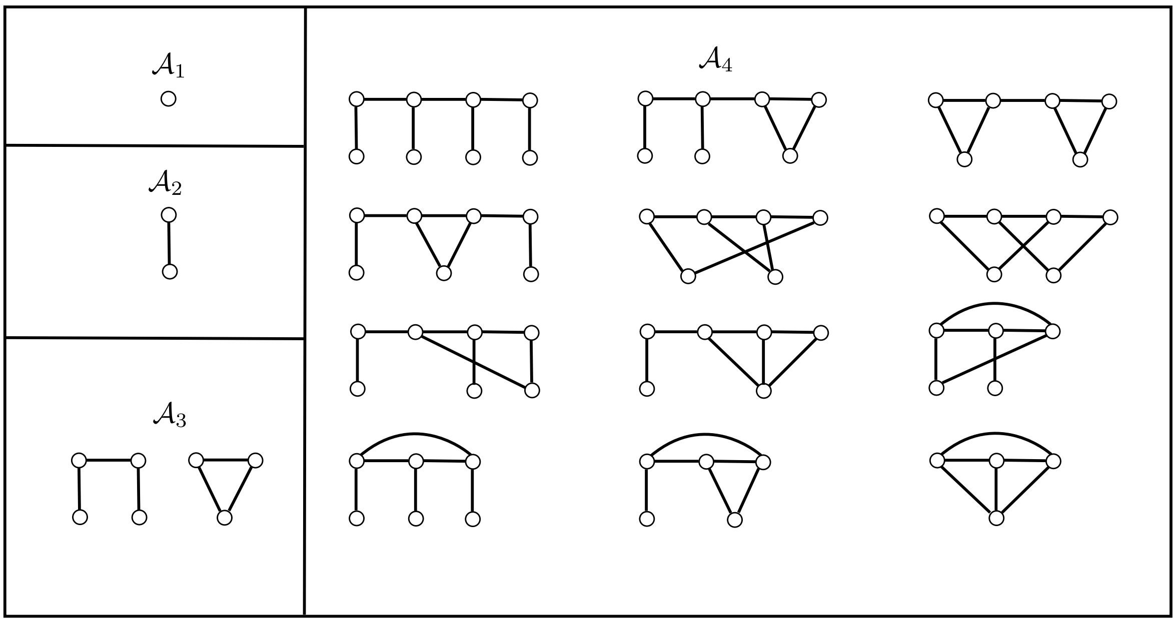


Figure 2.3: The family $\mathcal{A}_k$, $k = 1, 2, \dots, 4$.

See Figure 2.3. Clearly, the only 1-atom has Grundy number 1, while the only 2-atom has Grundy number 2. Similarly with what happens with the binomial tree $\mathcal{B}_k$, a k-atom admits a greedy colouring with k colours. To obtain a greedy colouring of $G \in \mathcal{A}_k$ with k colours, assuming that every member of $\mathcal{A}_{k-1}$ has Grundy number $k-1$, one should assign colour 1 to the vertices that are added in the constructive definition of $\mathcal{A}_k$ and then greedily colour the rest of the graph using an ordering of the vertices corresponding to a member of $\mathcal{A}_{k-1}$ that gives a greedy colouring with $\Gamma(\mathcal{A}_{k-1})$ colours.

Theorem 2.3.4 ([121])**.** *For a given graph G, $\Gamma(G) \geq k$ if and only if G contains a k-atom as an induced subgraph.*

Proof. It is easily seen by the recursive definition of the atoms that a k-atom admits a Grundy colouring with k colours. Therefore if G contains a k-atom, then $\Gamma(G) \geq k$.

Now suppose $\Gamma(G) \geq k$ and let $S_1, S_2, \dots, S_k$ be the colour classes of a Grundy colouring of G with $\Gamma(G)$ colours. We prove by induction on k that G contains a k-atom. let $H = G \backslash S_1$. Then clearly $\Gamma(H) \geq k-1$ and by induction we have that H contains a $(t-1)$-atom H'. But S_1 is a maximal stable set of G, therefore every vertex of H' is adjacent to a vertex from S_1 in G. Then it is clear by the construction of a k-atom that there is a set of vertices $S \subseteq S_1$ that together with the vertices in H' induces a k-atom in G. □

Observe that the requirement in Theorem 2.3.4 that the k-atom is contained as an induced subgraph is important. For example C_4 contains $\mathcal{B}_3$ (which is P_4) as a subgraph but $\Gamma(C_4) = 2$. This observation implies that the removal of an edge may increase the Grundy number of a graph. This is a remarkable difference between $\chi(G)$ and $\Gamma(G)$, since for any subgraph H of G, $\chi(H) \leq \chi(G)$. On the other hand, as pointed in Proposition 1 of [4], if H is an induced subgraph of G, $\Gamma(H) \leq \Gamma(G)$. Theorem 2.3.4 has algorithmic consequences on the computation of the Grundy number of graphs.

Corollary 2.3.1 ([122]). *A connected graph G satisfies $\Gamma(G) = 2$ if and only if it is a complete bipartite graph.*

Proof. Suppose G is connected and $\Gamma(G) = 2$. Since a greedy colouring is a proper colouring, G must be bipartite. Let A and B be the parts of G. If G is not complete bipartite, there exists $u \in A$ and $v \in B$ such that $uv \notin E(G)$. But then, since G is connected and bipartite, $dist(u, v) \geq 4$, and so G contains an induced P_4. Therefore Theorem 2.3.4 implies that $\Gamma(G) \geq 3$, since $P_4 = \mathcal{B}_3$.

The converse is trivial, since a complete bipartite graph does not induces neither a P_4 nor a K_3, which are the only 3-atoms. □

Corollary 2.3.2 ([121]). *Let k be a fixed integer. Determining whether $\Gamma(G) \geq k$ for a given graph G can be done in polynomial time.*

Proof. We first observe that there is a finite number $f(k)$ of k-atoms that depends only on k. Moreover, a member of $\mathcal{A}_k$ has at most 2^{k-1} vertices. Therefore we are able to verify if G contains a given k-atom in time $O(f(k)n^{2^{k-1}}) = O(n^{2^{k-1}})$. □

The algorithm given in Corollary 2.3.2 is polynomial but the order of the polynomial depends on k. Therefore a natural question is whether there is an FPT algorithm to decide if a graph G has $\Gamma(G) \geq k$, when k is the parameter. We go back to that and other related questions in Chapter 5.

To end the discussion about the existence of greedy colourings for a fixed value, we mention that Christen and Selkow [24] considered the question of whether a graph G has greedy colourings for every value k such that $\chi(G) \leq k \leq \Gamma(G)$. They answer the question in the positive. The proof we present now is different from the original and is simpler.

Theorem 2.3.5 ([24]). *Let G be a graph and k an integer such that $\chi(G) \leq k \leq \Gamma(G)$. There is a greedy colouring of G that uses exactly k colours.*

Proof. We proceed by induction on $\chi(G)$, the result holding trivially for stable sets. Let G be a graph with chromatic number $\chi(G) > 1$. Let $\chi \leq k \leq \Gamma(G)$. If $k = \chi(G)$ then trivially there is a greedy k-colouring of G, so we may assume that $k \geq \chi(G) + 1$. Let $(S_1, S_2, \dots, S_{\Gamma(G)})$ be a greedy colouring of G. Then S_1 is a dominating stable set in G. Let us consider $G' = G - S_1$. Then $\Gamma(G') = \Gamma(G) - 1$ and $\chi(G') \leq \chi(G)$. So $\chi(G') \leq k - 1 \leq \Gamma(G')$ and by the induction hypothesis, G' admits a greedy $(k-1)$-colouring $(S'_2, \dots, S'_k)$. Thus $(S_1, S'_2, \dots, S'_k)$ is a greedy k-colouring of G. □

Grundy number of graphs with few induced P_4's. In Proposition 2.3.4 we presented the proof that cographs are such that their chromatic number is equal to their Grundy number. Since a cograph is perfect, this implies that the Grundy number of a cograph can be computed in polynomial time. Campos et. al [21] gave a polynomial algorithm for determining the Grundy number of $(q, q-4)$-graphs, for a fixed $q \geq 4$, that are the graphs such that every q vertices induces at most $q - 4$ P_4's. Observe cographs and P_4-sparse graphs are the particular cases $q = 4$ and $q = 5$ respectively. Araújo and Linhares Sales [2] gave polynomial algorithms for subclasses of P_5-free graphs that strictly contain P_4-free graphs.

Grundy number of graph products. The Grundy number of graph products was investigated by some authors. Asté, Havet and Linhares-Sales [4] studied the Grundy number of the lexicographic and cartesian product of graphs. They proved that $\Gamma(G[H]) \leq 2^{\Gamma(G)-1}(\Gamma(H)-1)+\Gamma(G)$ and that in case $\Gamma(G)=\Delta(G)+1$ or G is a tree, $\Gamma(G[H])=\Gamma(G)\times\Gamma(H)$. They also prove that $\Gamma(G\Box H)$ is not upper bounded as a function of $\Gamma(G)$ and $\Gamma(H)$, but that $\Gamma(G\Box H)\leq\Delta(G)\cdot 2^{\Gamma(H)-1}+\Gamma(H)$. Campos et. al [18] show that the Grundy number of the direct product $G\times H$ and strong product $G\boxtimes H$ are also not bounded as a function of $\Gamma(G)$ and $\Gamma(H)$ and disprove some conjectures concerning the Grundy number of graph products.

2.3.2 b-colourings and the b-chromatic number

A vertex v is said to be a *b-vertex* in a colouring c if $c(v)=i$ and v has at least one neighbour in every colour class S_j, $j\neq i$. Consider the following colouring heuristic.

---

b-COLOURING ALGORITHM

INPUT: $G=(\{v_1,v_2,\ldots,v_n\},E)$
OUTPUT: a proper colouring c

---

1. For every $1\leq i\leq n$, set $c(v_i)=i$
2. While c has a colour class S_j with no b-vertices do:
   2.1 For every $v\in S_j$ do:
      2.1.1 Assign to v a colour that does not appear in $c(N(v))$.
   2.2 Remove S_j.
3. return c.

At each iteration, the b-colouring algorithm modifies the current colouring and eliminates a colour class without b-vertices by changing the colours of all its vertices. It stops when in the current colouring every colour class has a b-vertex. We call a colouring satisfying this property a *b-colouring*. See Figure 2.4. The *b-chromatic number* of G, denoted $\chi_b(G)$, is the largest integer k such that G admits a b-colouring with k colours. In the same way as the Grundy number, the b-chromatic number and its ratio with the chromatic number measure how bad the b-colouring algorithm may perform on a graph.

A vertex colouring with $\chi(G)$ colours is necessarily a b-colouring, for otherwise we could eliminate one colour class from it and obtain a colouring with less than $\chi(G)$ colours. Then,

$$\chi(G)\leq\chi_b(G).$$

But as it happens in the case of the Grundy number, the difference between the b-chromatic number and the chromatic number of a graph can be arbitrarily big.

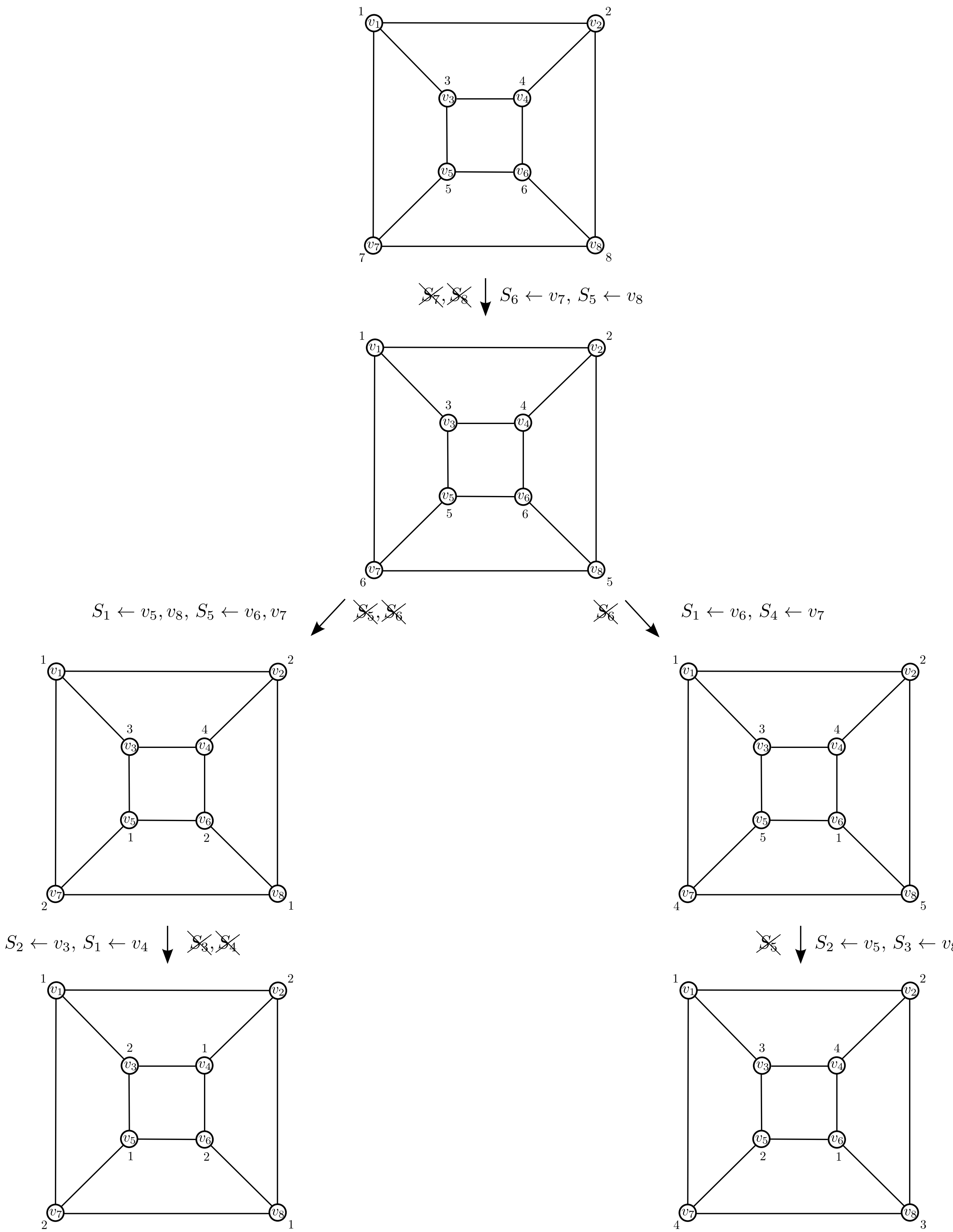


Figure 2.4: Two b-colourings of the 3-cube obtained from distinct executions of the b-colouring algorithm to the same initial colouring.

Proposition 2.3.5 ([70]). *For any positive integer k, there is a graph G such that $\chi_b(G) - \chi(G) \geq k$.*

Proof. Set $n = k + 2$ and consider the bipartite graph $M_{n,n}$ such that $V(M_{n,n}) = \{v_1, v_2, \ldots, v_n, w_1, w_2, \ldots, w_n\}$ and $E(M_{n,n}) = \{v_i w_j \mid i \neq j\}$. Now consider the colouring c such that $c(v_i) = c(w_i) = i$. Clearly, c is a b-colouring with n colours, and since $\Delta(M_{n,n}) = n - 1$, we get that $\chi_b(M_{n,n}) = n$. Since $M_{n,n}$ is a bipartite graph, $\chi(M_{n,n}) = 2$. So $\chi_b(G) - \chi(M_{n,n}) = n - 2 = k$. □

Although there are graphs with b-chromatic number far appart from the chromatic number, Kratochvíl, Tuza and Voigt proved that the b-chromatic number of the random graph $G_{n,p}$ is not much larger than the chromatic number. They proved that for a fixed edge probability $0 < p < 1$ and $q = 1 - p$, $\left(\frac{1}{2} - o(1)\right) \frac{n \log \frac{1}{q}}{\log n} \leq \mathbb{E}[\chi_b(G)] \leq \left(\frac{n \log \frac{1}{q}}{\log n}\right)$.

Now observe that a b-vertex in a b-colouring with k colours must have at least $k - 1$ neighbours, one in each other colour class. As a consequence,

$$\chi_b(G) \leq \Delta(G) + 1.$$

Irving and Manlove [70] were the first to define b-colourings, in 1999. They also introduced an upper bound for the b-chromatic number of a graph G that is better than $\Delta(G) + 1$. If G admits a b-colouring with k colours, then in that colouring each of the k colour classes must have a b-vertex, which should therefore be of degree at least $k - 1$. As a consequence, there should be k vertices of degree at least $k - 1$ in G. The *m-degree* of a graph G is the largest integer m such that G has m vertices of degree at least $m - 1$, and is denoted by $m(G)$.

Clearly,

$$\chi_b(G) \leq m(G).$$

In what follows, we will say that a vertex is *dense* if it has degree at least $m(G) - 1$. The upper bound $m(G)$ may be computed in linear time. To do so, it suffices to order the vertices $v_1, v_2, \ldots, v_n$ in a way that $d(v_i) \geq d(v_{i+1})$, $1 \leq i \leq n - 1$, and compute the maximum k such that $d(v_k) \geq k - 1$.

Already in [70], Irving and Manlove show that the b-chromatic number can be arbitrarily far apart from $m(G)$.

Proposition 2.3.6 ([70]). *For any positive integer k, there is a graph G such that $m(G) - \chi_b(G) = k$.*

Proof. Set $n = k + 1$. Consider the complete bipartite graph $K_{n,n}$. Clearly, $m(K_{n,n}) = n + 1$. Now suppose by contradiction that $\chi_b(K_{n,n}) > 2$ and that c is a b-colouring with $\chi_b(K_{n,n})$ colours. Moreover, let A and B are the parts of $K_{n,n}$. Then there are at least two b-vertices with different colours in one of the parts of $K_{n,n}$, say $u, v \in A$. Since v is adjacent to every vertex in B, $c(v)$ cannot appear in B. Since u is a b-vertex it should have a neighbour with colour $c(v)$, but all its neighbours are in B and so we get a contradiction. Therefore, $m(K_{n,n}) - \chi_b(K_{n,n}) = n - 1 = k$. □

The b-chromatic number of trees and graphs of large girth. Irving and Manlove also proved that the b-chromatic number of a tree can be computed in linear time. Their proof introduces the notion of a *pivoted tree*. A tree T is a *pivoted tree* if T has exactly $m(T)$ dense vertices and a special vertex v, that is called a *pivot*, such that:

(i) v is not dense.

(ii) Every dense vertex is adjacent to v or to a dense vertex adjacent to v.

(iii) Every dense vertex adjacent to v and another dense vertex has degree $m(T) - 1$.

See Figure 2.5.

Proposition 2.3.7 ([70])**.** *A pivoted tree is such that* $\chi_b(T) < m(T)$.

Proof. First observe that in any b-colouring of T with $m(T)$ colours, all dense vertices have to be coloured with distinct colours from $\{1, 2 \ldots, m(T)\}$, since there are only $m(T)$ dense vertices. Now, if the pivot v is coloured i, then obviously none of the dense vertices adjacent to it may be coloured i, as the colouring is proper. Let d be the dense vertex that is coloured i, and let d' be its dense neighbour that is adjacent to v, whose existance is assured by (ii). Since d' is dense, it should be a b-vertex, and because of (iii) it has exactly $m(T) - 1$ neighbours. But d' is adjacent to v and d, that are both coloured i, so it does not have enough neighbours for the other $m(T) - 1$ colours that should appear in its neighbourhood, a contradiction. □

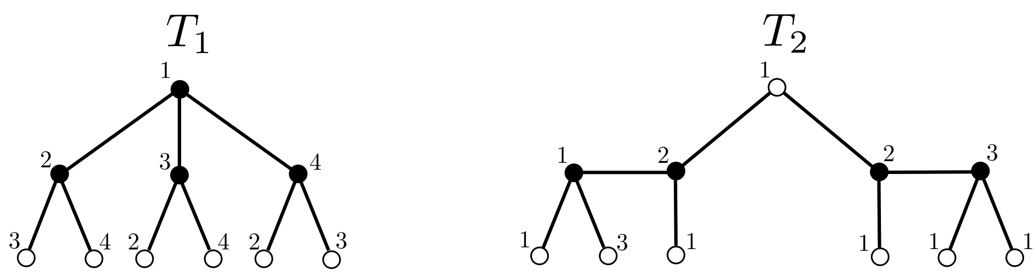


Figure 2.5: T_1 is a non-pivoted tree and T_2 is a pivoted tree. $m(T_1) = m(T_2) = 4$, but $\chi_b(T_1) = 4$ while $\chi_b(T_2) = 3$.

Irving and Manlove prove that a pivoted tree T can be coloured with $m(T) - 1$ colours, and therefore $\chi_b(T) = m(T) - 1$. Furthermore, they show that if T is not pivoted, there is always a set of $m(T)$ vertices that can play the role of b-vertices in a b-colouring with $m(T)$ colours, and therefore a non-pivoted tree T satisfies $\chi_b(T) = m(T)$. Their proofs yelds a linear time algorithm to find the b-chromatic number of a tree. In Chapter 4 we define the *tight graphs*, which are graphs such that they have exactly $m(G)$ dense vertices and such that every dense vertex has degree exactly $m(G) - 1$ neighbours. We generalize the notion of pivoted tree and give a sufficient condition for a tight graph G to satisfy $\chi_b(G) < m(G)$.

In her Phd thesis [114] Silva observed that Irving and Manlove's algorithm for trees also works on graphs with girth at least 11, and consequently these graphs also satisfy $\chi_b(G) \in \{m(G) - 1, m(G)\}$ and admit a polynomial time algorithm to compute their b-chromatic number. This result was improved by Campos, Farias and Silva [17], who proved that the b-chromatic number of a graph of girth at least 9 can also be computed in polynomial time and is also in $\{m(G) - 1, m(G)\}$.

The difference $m(G) - \chi(G)$ **for graphs with structure similar to a tree.** The results mentioned in the last paragraphs shows that the difference $m(G) - \chi_b(G)$ is at most one when G is a tree or a graph with sufficiently large girth. It is natural to ask if this is the case for other graphs that have a structure similar to a tree, like cacti, block graphs, k-trees, etc. This was proved true for cacti [19, 114] and outerplanar graphs [99, 114]. On the other hand, it was proved in Silva's Phd Thesis [114] that this difference can be arbitrarily large for block graphs, series-parallel graphs and line graphs of trees. All the constructions make use of large complete bipartite subgraphs (not necessarily induced) in order to achive this difference. Inspired by the results they obtained while investigating the cartesian product of trees by paths, cycles and stars [98] and also by other existing results concerning graphs that doesn't contain $K_{2,3}$ as a subgraph [19, 39, 70, 88, 86], Maffray and Silva conjectured that if G has no $K_{2,3}$ as subgraph, then $\chi_b(G) \geq m(G) - 1$, the only exception being the cartesian product of two triangles, that they prove to have $\chi_b(G) < m(G) - 1$.

Computational hardness results. The problem of determining the b-chromatic number of a graph was already shown to be NP-Hard in the paper of Irving and Manlove [70]. Kratochvíl, Tuza and Voigt [90] proved that, given a graph G and an integer k, deciding if G admits a b-colouring with k colours is an NP-complete problem even if G is a connected bipartite graph and $k = m(G) = \Delta(G) + 1$. As a consequence, no Brooks' like theorem is likely to exist for the b-chromatic number. In Chapter 4, we prove that deciding if a graph admits a b-colouring with k colours, k given as input, is an NP-complete problem even if the graph is a connected chordal graph.

There is no known approximation algorithm for the b-chromatic number of graphs. Corteel, Valencia-Pabon and Vera[27] proved that for all $\epsilon > 0$, the b-chromatic number cannot be approximated by a factor of $\frac{120}{133} - \epsilon$ in polynomial time, unless P = NP.

Singularities of b-colourings. b-colourings have many singularities when compared to other kinds of colourings. For example, we already mentioned that if H is an induced subgraph of G and $\Gamma(H) = k$, then $\Gamma(G) \geq k$, the analogous statement being true for the chromatic number. This property allows the definition of atoms and critical graphs, respectively in the case of the Grundy number and the chromatic number. This property does not holds for b-colourings.

Proposition 2.3.8 ([90]). *For any positive integer k, there is a graph G and an induced subgraph H of G such that $\chi_b(H) - \chi_b(G) \geq k$.*

Proof. Let $n = k + 3$ and consider the graph defined as follows. Start with the complete bipartite graph $K_{n,n}$ and remove a matching of size $n - 1$.

We claim that the resulting graph G has b-chromatic number 2. Suppose A and B are the parts of G. Since we started with a complete bipartite graph and removed only removed $n - 1$ edges, there is a pair $u \in A, v \in B$ such that $N(u) = B$ and $N(v) = A$. Now suppose c is a b-colouring of G with at least 3 colours. We may assume without loss of generality that $c(u) = 1$ and $c(v) = 2$. But since $N(u) = B$, the colour 1 cannot appear in B, and with a similar argument we have that colour 2 cannot appear in A. But a b-vertex of colour 3 should be adjacent to at least one vertex of colour 1 and colour 2, a contradiction.

Now it suffices to see that G contains an induced subgraph H which is isomorphic to $M_{n-1,n-1}$, and we already observed that $\chi_b(M_{k,k}) = k$. Then, $\chi_b(H) - \chi_b(G) = n - 1 - 2 = k$. □

Another singularity of b-colourings is presented now. Given a graph G, the set of values such that the graphs admits a proper colouring is contiguous, in the sense that for every $\chi(G) \leq k \leq |V(G)|$ there exists a k-colouring. A similar statement can be said about the greedy colourings, as shown in Theorem 2.3.5: for every $\chi(G) \leq k \leq \Gamma(G)$ there is a greedy colouring with k colours. The case of b-colourings is different.

Proposition 2.3.9 ([90]). *For every positive k, there exists a graph G that admits a b-colouring with p colours if, and only if, $p \in \{2, k\}$.*

Proof. Consider again the bipartite graph $M_{k,k}$ such that $V(M_{k,k}) = \{v_1, v_2, \ldots, v_k, w_1, w_2, \ldots, w_k\}$ and $E(M_{k,k}) = \{v_iw_j \mid i \neq j\}$. Since we already observed that $\chi_b(M_{k,k}) = k$ and since $\chi(M_{k,k}) = 2$, then there are b-colourings of $M_{k,k}$ with $p \in \{2, k\}$ colours.

Now suppose c is a b-colouring with q colours, $2 < q < k$, and let A and B be the parts of $M_{k,k}$. Since $q \geq 3$, we may assume without loss of generality that there are two b-vertices from c that belong to A. Again without loss of generality, we may assume that v_1 and v_2 are those b-vertices, and that $c(v_1) = 1$ and $c(v_2) = 2$. But then, as v_1 is a b-vertex, it should have a neighbour that is coloured 2, and this vertex has to be w_2, as all other vertices of B are adjacent to v_2. With a similar argument, we get that $c(w_1) = 1$. All the vertices in $V(M_{k,k})\backslash\{v_1, v_2, w_1, w_2\}$ should receive colour greater than 2, since they are all adjacent to a vertex with colour 1 and 2. Now we can assume again without loss of generality that either v_3 or w_3 is a b-vertex of colour 3, and apply the same argument as before to obtain that $c(v_3) = c(w_3) = 3$ and that the vertices in $V(M_{k,k})\backslash\{v_1, v_2, v_3, w_1, w_2, w_3\}$ should have colour greater than 3. By repeatedly using this argument we get a contradiction, as we assumed $q < k$. □

The b-*spectrum* of graph $G = (V, E)$ is the set $S_b(G) = \{k \mid G$ admits a b-colouring with k colours$\}$. Kratochvíl, Tuza and Voigt [90] asked for which sets of positive integers S there exists graphs G with $S_b(G) = S$. This question was answered by Barth, Cohen and Faik [8].

Theorem 2.3.6 ([8]). *For every finite non-empty set $I \subset \mathbb{N}\backslash\{0, 1\}$ there is a graph G such that $S_b(G) = I$.*

b-continuity. Proposition 2.3.9 and Theorem 2.3.6 motivate the definition of b-continuous graphs. A graph G is said to be b-*continuous* if $S_b(G) = \{k \in \mathbb{Z} \mid \chi(G) \leq k \leq \chi_b(G)\}$. To decide if a graph is b-continuous is a NP-complete problem [8]. Some graph classes have been proved to be b-continuous, like chordal graphs [38, 79], P_4-sparse graphs [12], and the Kneser graphs $K(n, 2)$, for $n \geq 17$ [76]. Other results concerning the b-continuity of graphs can be found in [38, 79, 8, 12].

b-perfect graphs. Hoáng and Kouider [64] introduced the notion of a b-*perfect graph*. A graph is b-*perfect* if $\chi_b(H) = \chi(H)$ for all induced subgraph H of G. In [64] they characterize all b-perfect bipartite graphs and P_4-sparse graphs by minimal forbidden induced subgraphs. Hoàng, Linhares Sales and Maffray [65] gave a list of 22 minimal forbidden subgraphs for b-perfect graphs and conjectured that a graph is b-perfect if and only if it does not contains any graph from this list. They proved this is true for diamond-free graphs and graphs with chromatic number at most 3. The conjecture was proved to be true by Hoàng, Maffray and Mechebbek [66].

The b-chromatic number of d-regular graphs. There are a number of results about d-regular graphs with girth at least 5 [71, 85, 86, 11, 16]. Kratochvíl, Tuza and Voigt [90] were the first to consider b-colourings of d-regular graphs, and they proved the following:

Proposition 2.3.10 ([90]). *If G is a d-regular graph with at least d^4 vertices, then $\chi_b(G) = d + 1$.*

Proof. We claim that G has a set of vertices $D = \{v_1, v_2, \ldots, v_{d+1}\}$ such that $dist(v_i, v_j) \geq 4$, for all $i \neq j$. Such a set may be found as follows. First choose an arbitrary vertex v_1 and then remove the vertices that are at distance at most 3 from it. The number of removed vertices is at most $1 + d + d(d-1) + d(d-1)(d-1) < d^3$. We then proceed in the same way on the resulting graph in order to choose v_2, and so on to choose $v_3, \ldots, v_d$. Since the number of removed vertices altogether is smaller than d^4, there remains at least one vertex that may be chosen as v_{d+1} and the claim follows.

Now it suffices to see that we may colour v_i with colour $i + 1$, for $1 \leq i \leq d + 1$, and colour its neighbours with distinct colours in $\{1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{d+1}\}$, so that v_i is a b-vertex of colour i. Because of the distance constraints on the vertices of D, this partial colouring is proper, and all the colour classes have a b-vertex. It remains to show that this $(d + 1)$-colouring can be extended to the entire graph, as some vertices may be still uncoloured. Since $\Delta(G) = d$, every uncoloured vertex has at most d distinct colours that already appear in its neighbourhood, and therefore there is always one colour that may be assigned to it, since there are $d + 1$ available colours. □

An implication of this proposition is that there is only a finite number of d-regular graphs with $\chi_b(G) < d + 1$, and as a consequence determining the b-chromatic number of a d-regular graph may be done in polynomial time. Kouider [85] proved that any d-regular graph of girth at least 6 admits a b-colouring with $d + 1$. Kouider and Sahili [86] asked if this is true for graphs with girth 5 and solved it for the case where the graphs have no C_6 as subgraph. Blidia, Maffray and Zemir [11] pointed that the Petersen graph is a counter-example to their question, as it has girth 5 and b-chromatic number equal to 3. They conjectured that the Petersen graph is the only counter-example, and show that for $d \leq 6$, every d-regular graph of girth 5 that is not the Petersen graphs satisfies $\chi_b(G) = d+1$. Cabello and Jakovac [16] showed that a d-regular graph of girth 5 satisfies $\chi_b(G) \geq \lfloor \frac{d+1}{2} \rfloor$.

b-chromatic number of graphs with few induced P_4's. The b-chromatic number of graph classes with few P_4's was considered by some authors. Bonomo et. al [12] proved that the b-chromatic number of a P_4-sparse graph G can be computed in $O(|V(G)|^3)$-time. They asked if this result could be extended to *distance-hereditary graphs*, that are graphs in which every induced path is a shortest path. We answer in the negative to this question by showing that determining the b-chromatic number of a chordal distance-hereditary graphs is NP-hard, what is done in Chapter 4. We also show that the b-chromatic number can be determined in linear-time for a subclass of the P_4-sparse graphs, namely the tight P_4-sparse graphs, which are defined in Chapter 4. Some extensions of the results in [12] were considered. Bonomo, Koch and Velasquez [118] proved that the b-chromatic number can be determined in polynomial-time for a *P_4-tidy graph*, that is a graph in which for every set A inducing a P_4 there is at most one vertex x such that the subgraph induced by $A \cup \{x\}$ has more than one induced P_4. Its easy to see that a P_4-sparse graph is P_4-tidy. Campos et. al [20] gave a polynomial-time algorithm for the class of $(q, q-4)$-graphs, for a fixed $q \geq 4$, that is the class of graphs such that every set of q vertices induces at most $q - 4$ P_4's. Observe that cographs and P_4-sparse graphs are the particular cases $q = 4$ and $q = 5$ respectively. There is no containment relationship between the classes P_4-tidy and $(q, q-4)$-graphs.

b-chromatic number of graph products. The b-chromatic number of graph products was also studied. The cartesian product of complete graphs, stars, paths and cycles was considered in [77, 88, 98]. In particular, Kouider and Mahéo [88] prove that $\chi_b(G \square H) \geq \chi_b(G) + \chi_b(H) - 1$ when both G and H have b-colourings with $\chi_b(G)$ and $\chi_b(H)$ colours respectively in which the b-vertices are a stable set. The b-chromatic number of the strong product, lexicographic product and direct product is studied in [72].

2.4 The relations between the colouring parameters

In this section we discuss the relations between the colouring parameters that we introduced before and present the *partial Grundy colourings*, that may be seen as a relaxation of greedy colourings and also of b-colourings.

Observe that in general there is no relation between $\chi_b(G)$ and $\Gamma(G)$, as shown in the following propositions.

Proposition 2.4.1. *For every positive k, there exists a graph G with $\Gamma(G) - \chi_b(G) = k$.*

Proof. Let $n = k + 1$ and consider again the graph G obtained from the complete bipartite graph $K_{n,n}$ by removing a matching of size $n - 1$.

We have already shown in Proposition 2.3.8 that G has b-chromatic number 2. G contains an induced subgraph H which is isomorphic to $M_{n-1,n-1}$, and we proved in Theorem 2.3.3 that $\Gamma(M_{p,p}) = p$, $p \geq 2$. One can start with the greedy colouring of H from Theorem 2.3.3 that uses $n-1$ colours and then assign colours n and $n+1$ to the two vertices in $V(G)\backslash V(H)$. By doing so we obtain a greedy colouring of G with $n+1$ colours and since $\Delta(G) = n$ we have that $\Gamma(G) = n+1$. Therefore $\Gamma(G) - \chi_b(G) = n + 1 - 2 = k$. □

Proposition 2.4.2. *For every positive k, there exists a graph G with $\chi_b(G) - \Gamma(G) = k$.*

Proof. Let $n = k + 2$ and let the consider the star $K_{1,p-1}$. Now let G be the disjoint union of n copies $S_1, S_2, \ldots, S_n$ of the star of order n. It is clear that $m(G) = n$, and that a b-colouring with n colours may be obtained by colouring the vertex of degree $n-1$ from the star S_i, say v_i, with colour i, and then colouring its neighbours with all the colours different from i, therefore making v_i a b-vertex. Then, $\chi_b(G) = n$.

Now, the Grundy number of G is equal to the maximum of the Grundy number of its connected components. But it is easy to see that $\Gamma(K_{1,n-1}) = 2$, therefore we have that $\chi_b(G) - \Gamma(G) = n - 2 = k$ □

Given a colouring c, a vertex v is said to be a *Grundy vertex* if it has at least one neighbour in every colour class S_j, $j < c(v)$. A greedy colouring is therefore a colouring in which every vertex is a Grundy vertex. The concept of a greedy colouring can be relaxed as follows. A *partial greedy colouring* is a proper colouring such that each colour class contains at least one Grundy vertex. The *partial Grundy number* $\partial\Gamma(G)$ is the maximum k such that G has a partial greedy k-colouring. Partial greedy colourings were introduced by Dunbar et al. [31] and studied by Erdős et al. [37], Shi et al. [113], and Balakrishnan and Kavaskar [7]. Erdös et al. [37] related partial greedy colourings to other graph properties such as the parsimonious proper colouring number and the maximum degree.

In any partial greedy colouring with k colours, the colour class S_k should have a vertex with neighbours in all other $k - 1$ colour classes, and therefore $k - 1 \leq \Delta(G)$, therefore

$$\partial\Gamma(G) \leq \Delta(G) + 1.$$

Moreover, as a partial greedy colouring is a proper colouring,

$$\chi(G) \leq \partial\Gamma(G).$$

A greedy colouring is a partial greedy colouring, consequently

$$\Gamma(G) \leq \partial\Gamma(G).$$

The same can be said about the b-chromatic number: any b-colouring is a partial greedy colouring. Therefore

$$\chi_b(G) \leq \partial\Gamma(G).$$

Therefore the partial Grundy number can be seen as an upper bound to the Grundy and b-chromatic numbers of a graph. One could expect that in contrast to what happens with the Grundy number, the partial Grundy number could be easy to compute. This is not the case, as Shi et al. [113] proved that deciding if $\partial\Gamma(G) = \Delta(G) + 1$ is a NP-complete problem even when restricted to a bipartite or a chordal graph. They also show that the $\partial\Gamma(G)$ can be computed in polynomial time, given that G has girth larger than 8, thus implying that the partial Grundy number can be computed in polynomial time for trees. It is also shown that it can be computed in polynomial time for r-regular graphs, for a fixed r. They also present an upper bound on the partial Grundy number that is presented in the next paragraph.

A sequence S of r distinct vertices $(g_1, \ldots, g_r)$ of a graph G is a *feasible Grundy sequence* if for $1 \leq i \leq r$ the degree of g_i in $G \backslash \{g_{i+1}, \ldots, g_r\}$ is at least $i-1$. The *stair factor* of G, denoted $\zeta(G)$, is the maximum cardinality of a feasible Grundy sequence of G. In [113] a linear-time algorithm is presented to compute $\zeta(G)$.

Consider a partial greedy colouring of G with k colours. Let v_i be a Grundy vertex from colour i, for every $1 \leq i \leq k$. It is easy to see that $(v_k, \ldots, v_1)$ is a feasible Grundy sequence, and therefore $\zeta(G) \geq k$. As a consequence,

$$\partial\Gamma(G) \leq \zeta(G).$$

Since we already observed that $\Gamma(G) \leq \partial\Gamma(G)$ and $\chi_b(G) \leq \partial\Gamma(G)$,

$$\Gamma(G) \leq \zeta(G)$$

and

$$\chi_b(G) \leq \zeta(G).$$

We investigate all these parameters in Chapter 6, where we consider the computational complexity of comparing between them for a given graph, and also analogue versions of the conjecture from Reed [110] which states that $\chi(G) \leq \left\lceil \dfrac{\omega(G) + \Delta(G) + 1}{2} \right\rceil$.

Chapter 3

Complexity of the Grundy number problem

3.1 Introduction

In this chapter we investigate the complexity of the following problem:

> GRUNDY NUMBER
> INPUT : A graph G and an integer k.
> OUTPUT : $\Gamma(G) \geq k$?

The NP-completeness of the problem in the general case is proved in [50]. The only specific class of graphs for which the problem was known to be NP-complete was the complements of bipartite graphs [121].

In Theorem 3.2.1 we prove that the problem is also NP-complete for bipartite graphs. Theorem 3.2.1 also implies that no Brooks' like characterization of the graphs G with $\Gamma(G) = \Delta(G) + 1$ is likely to exist, unless P = NP.

There is a linear-time algorithm to compute the Grundy number of partial k-trees [116], which are precisely the graphs of tree-width at most k. A natural question that may arise is the complexity of the problem for chordal graphs, that contains the k-trees, for all $k \geq 1$. In Theorem 3.3.1 we prove that GRUNDY NUMBER is NP-complete even for a chordal graph.

We end the chapter by considering some open problems that we would like to study in the future.

3.2 Bipartite graphs

We now prove that the Grundy number of a bipartite graph cannot be computed in polynomial time unless P = NP. An implication of our reductions is that any characterization of the graphs whose Grundy number equals to $\Delta(G)+1$ is unlikely to be checkable in polynomial time, under the classical complexity assumptions. In other words, no Brooks' type theorem shall exists for the Grundy number.

We need the following lemma before proving our result.

Lemma 3.2.1. *Let G be a graph and v a vertex of G. If there is a greedy colouring c such that v is coloured p, then, for any $1 \leq i \leq p$, there is a greedy colouring such that v is coloured i.*

Proof. For $1 \leq i \leq p$, let S_i be the set of vertices coloured i by c. Then for any $1 \leq i \leq p$, $(S_{p-i+1}, \ldots, S_p)$ is a greedy i-colouring of $G[\bigcup_{j=p-i+1}^{p} S_j]$ in which v is coloured i. This partial greedy colouring of G may be extended into a greedy colouring of G in which v is coloured i. □

Theorem 3.2.1. *It is NP-complete to decide if a bipartite graph G satisfies $\Gamma(G) = \Delta(G) + 1$.*

Proof. The problem is in NP because a greedy colouring using $\Delta(G) + 1$ colours is a certificate. To show that it is also NP-complete, we present a reduction from 3-edge-colourability of 3-regular graphs, which is NP-complete [67]. In this problem we are given a 3-regular graph and want to assign colours in $\{1, 2, 3\}$ to the edges of the graph in a way that edges sharing one endpoint are to be assigned distinct colours.

Let G be a 3-regular graph with $n - 3$ vertices. Set $V(G) = \{v_4, v_5, \dots, v_n\}$ and $E(G) = \{e_1, \dots, e_m\}$. Let I be the vertex-edge incidence graph of G, that is the bipartite graph with vertex set $V(I) = V(G) \cup E(G)$ in which an edge of G is adjacent to its two endvertices. See Figure 3.2. Also, let $M_{p,p}$ denote the graph obtained from the complete bipartite graph $K_{p,p}$ by removing a perfect matching. It can be easily checked that $\Gamma(M_{p,p}) = p$. We construct from I a new bipartite graph H as follows. For each vertex $e_i \in E(G)$, we add a copy $M_{3,3}(e_i)$ of $M_{3,3}$ and identify one of its vertices with e_i. We add a new vertex w adjacent to all the vertices of $V(G)$. We add copies $M^w_{1,1}$, $M^w_{2,2}$, $M^w_{3,3}$, $M^w_{n+1,n+1}$ of K_1, K_2, $M_{3,3}$, $M_{n+1,n+1}$ and we choose arbitrary vertices v_1, v_2, v_3, v_{n+1} respectively from each copy and add the edges v_1w, v_2w, v_3w, $v_{n+1}w$. Finally, for every $5 \le i \le n$, we do the following: for every $4 \le j \le i - 1$, we add a copy $M^i_{j,j}$ of $M_{j,j}$, choose an arbitrary vertex v^i_j of it and add the edge $v_iv^i_j$. See Figure 3.2.

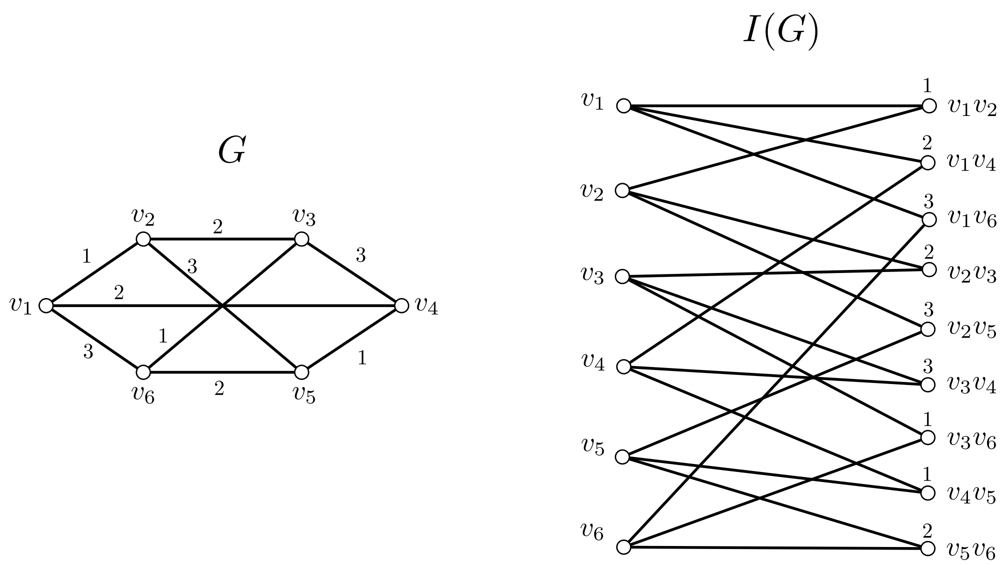


Figure 3.1: A 3-regular graph G with a 3-edge colouring and its incidence graph $I(G)$.

Observe that:

(i) $d_H(w) = n + 1$,

(ii) $d_H(v_i) = 1 + (i - 1) = i$, for $4 \le i \le n + 1$.

(iii) $d_H(e_j) = 4$, for $1 \le j \le m$, since e_i has two neighbours in I and two in $M_{3,3}(e_i)$.

(iv) $d_H(v^i_j) = j$, for $5 \le i \le n$ and $4 \le j < i$, since a vertex in $M^i_{j,j}$ has degree $j - 1$ and v^i_j is adjacent to v_i.

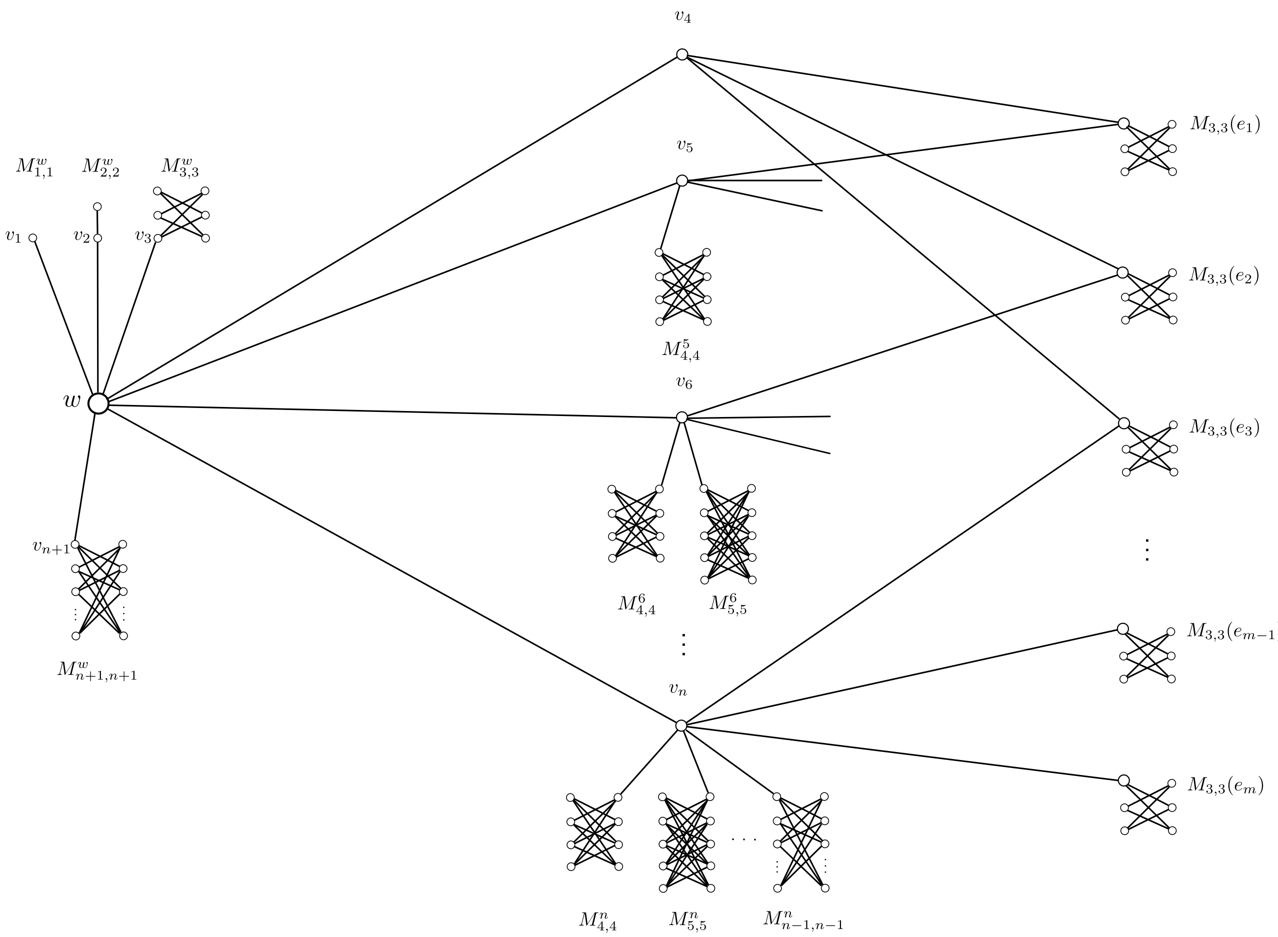


Figure 3.2: The graph H of the reduction in Theorem 3.2.1

(v) $\Delta(H) = n + 1$ and the only vertices with degree $n + 1$ are w and v_{n+1}.

Let us show that $\Gamma(H) = \Delta(H) + 1 = n + 2$ if and only if G is 3-edge-colourable.

Assume first G has a 3-edge-colouring ϕ. By Lemma 3.2.1, for every $1 \leq i \leq m$, there is a greedy colouring of the copy of $M_{3,3}(e_i)$ associated with e_i where e_i is coloured $\phi(e_i)$. Then in I every vertex in $V(G)$ has one neighbour of each colour in $\{1, 2, 3\}$. There is a greedy colouring of $M^i_{j,j}$, $4 \leq j < i \leq n$ such that v^i_j is coloured j.

Then we greedily extend the union of these colourings to v_i, $4 \leq i \leq n$, so that v_i is coloured i. We also greedily colour $M^w_{1,1}$, $M^w_{2,2}$, $M^w_{3,3}$, $M^w_{n+1,n+1}$ in such a way that w has one neighbour coloured i, for all $i \in \{1, 2, 3, n + 1\}$. Finally, w has one neighbour of each colour j, $1 \leq j \leq n + 1$. So we colour it with $n + 2$.

Hence $\Gamma(H) \geq n + 2$ and so $\Gamma(H) = n + 2$ because $\Delta(H) = n + 1$.

Let us now show that if $\Gamma(H) = n + 2$ then G is 3-edge-colourable. Assume that c is a greedy $(n + 2)$-colouring of H.

Claim 3.2.1. $\{c(w), c(v_{n+1})\} = \{n + 1, n + 2\}$

Proof. Let u be a vertex such that $c(u) = n + 2$. Then, u must have one neighbour coloured with each of the other $n + 1$ colours and then $d(u) \geq n + 1$. Hence, by Observation (v), u is either w or v_{n+1}.
Case 1: $u = w$.
Then, $c(v_{n+1}) = n + 1$, since the only neigbours of w with degree at least n are v_n and v_{n+1}, $d(v_n) = n$, and v_n is adjacent to w which is already coloured $n + 2$.
Case 2: $u = v_{n+1}$. The only neighbour of v_{n+1} that could be coloured $n + 1$ is w, since all its neighbours in $M^w_{n+1,n+1}$ have degree n and are adjacent to v_{n+1} which is coloured $n + 2$. □

Claim 3.2.2. *For* $1 \leq i \leq n$*,* $c(v_i) = i$*.*

Proof. By Claim 3.2.1, $\{c(w), c(v_{n+1})\} = \{n+1, n+2\}$. Since $d_H(w) = n+1$, w has one neighbour coloured i, for each $1 \leq i \leq n$. A neighbour of w which is coloured n must have degree at least n. So, by Observation (ii), it must be v_n. And so on, by decreasing induction, we show that $c(v_i) = i$, for $1 \leq i \leq n$. □

We now prove that c induces a proper 3-edge-colouring of G.

Consider vertex v_i, $4 \leq i \leq n$. By Claim 3.2.2, it is coloured i, and by Observation (ii) it has degree i. Since it is adjacent to w, which by Claim 3.2.1 has a colour greater than i, there are only $i - 1$ vertices remaining for the other $i - 1$ colours. So, v_i has exactly one neighbour coloured j, for each $1 \leq j \leq i - 1$ and therefore the three edges incident to v_i in G, which are adjacent to v_i in I, have different colours. Furthermore, for $1 \leq i \leq m$, the edge vertex e_i has at most two neighbours in H with a colour at most 3. Thus, $c(e_i) \in \{1, 2, 3\}$. □

As a corollary to Theorem 3.2.1, it is NP-hard to compute the Grundy number of a bipartite graph.

Corollary 3.2.1. *Given a bipartite graph G and an integer k, it is NP-complete to decide if* $\Gamma(G) \geq k$*.*

3.3 Chordal graphs

We now prove the NP-completeness of deciding the Grundy number of a chordal graph.

Theorem 3.3.1. *Given a chordal graph G and an integer k, deciding if* $\Gamma(G) \geq k$ *is NP-complete.*

Proof. Clearly, one can verify in polynomial time if a colouring is a greedy colouring and so the problem is in NP. To show that it is also NP-complete, we present a reduction from 3-edge-colourability.

Let G be a 3-regular graph with n vertices, where $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $E(G) = \{e_1, \ldots, e_m\}$. Let I be the vertex-edge incidence graph of G, that is the bipartite graph with vertex set $V(I) = V(G) \cup E(G)$ in which $e_i \in E(G)$ is adjacent to $v_j \in V(G)$ if and only if vertex v_j is an endpoint of e_i. In order to avoid confusion, we will use V to denote the vertices in $V(I)$ corresponding to vertices in $V(G)$ and E to denote the vertices in $V(I)$ corresponding to the edges in $E(G)$.

We construct from I a new graph H as follows. Start with a copy of I. Add an edge between every pair of vertices of V in H, thus making V a clique. Finally, for every vertex $e \in E$, add a copy of the binomial trees $\mathcal{B}_1$ and $\mathcal{B}_2$ and make their roots adjacent to e, so that the graph induced by e and the vertices of the binomial trees that were added is isomorphic to $\mathcal{B}_3$. We use the notation $\mathcal{B}_3(e)$ to refer to the copy of $\mathcal{B}_3$ to which e belongs.

Since $|V| = n$ and because G is a 3-regular graph, we have that $d_H(v) = n - 1 + 3 = n + 2$, for $v \in V$. A vertex $e \in E$ corresponds to an edge of G, and therefore has degree 2 in I. Since in H e

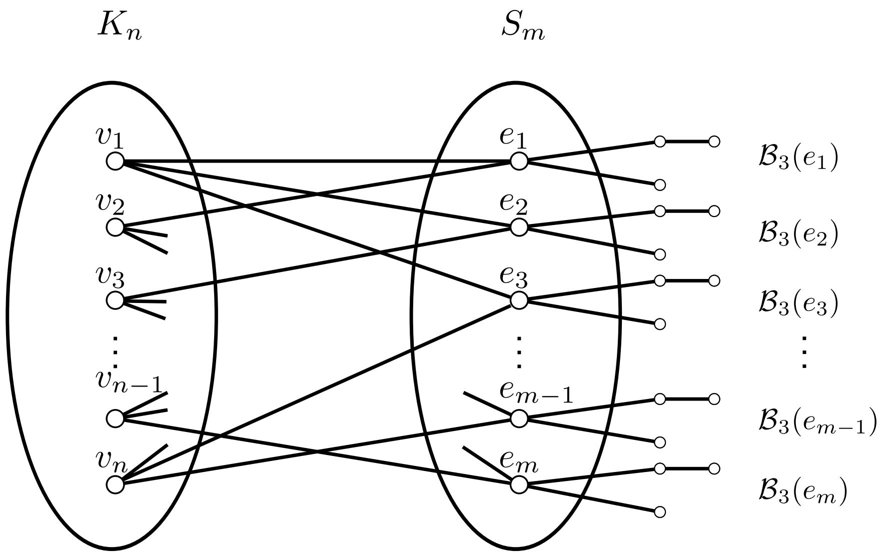


Figure 3.3: The graph H of the reduction in Theorem 3.3.1

is also adjacent to two vertices in $\mathcal{B}_3(e)$, we have that $d_H(e) = 4$. All the other vertices are in $\mathcal{B}_3(e')$ for some $e' \in E$ and therefore have degree at most 2.

Consequently, $\Delta(H) = n + 2$, implying that $\Gamma(H) \leq n + 3$. In H, V is a clique and E is an independent set, so $H[V \cup E]$ is a split graph, and so chordal. The vertices that are not in $V \cup E$ are from one of the binomial trees that were linked with a vertex in E, so no induced cycle of size greater than 3 can pass through one of these vertices and therefore H is chordal.

We now prove that G admits a 3-edge-colouring if and only if $\Gamma(H) = n + 3$.

Let c be a 3-edge-colouring of E that uses colours $\{1, 2, 3\}$. We shall construct a greedy colouring c' of H with $n + 3$ colours. Let $c'(u) = c(u)$, for $u \in E$, and $c'(v_i) = i + 3$, for $1 \leq i \leq n$, $v_i \in V(G)$. Note that in this partial colouring, the vertices in V have neighbours with all the colours that are smaller than their colour, and as a consequence they are Grundy vertices. The vertices in E can all be made Grundy vertices by giving an appropriate colouring of the binomial trees that are linked to them, using Lemma 3.2.1. Then, c' is a greedy colouring of H with $n + 3$ colours.

Now, let c' be a greedy colouring of H that uses $n+3$ colours. A vertex with a colour in $\{6, \dots, n+3\}$ has to be in V, as it must have degree at least 5. Assume, without loss of generality, that $c'(v_i) = i + 3$, for $3 \leq i \leq n$.

Claim 3.3.1. $\{c(v_1), c(v_2)\} = \{4, 5\}$.

Proof. Suppose $5 \notin \{c(v_1), c(v_2)\}$. Observe that vertex v_i, $3 \leq i \leq n$, should have neighbours with colours 4 and 5, as it is coloured $i + 3$. Therefore vertex v_n should be adjacent to a vertex $e \in E$, such that $c(e) = 5$. But $c(v_n) = n + 3 > 4$ and since e should have neighbours with colours 3 and 4, one of its neighbours in $\mathcal{B}_3(e)$ has colour at least 3. But if a neighbour of e in $\mathcal{B}_3(e)$ has colour at least 3 it implies that the restriction of c to a copy of either $\mathcal{B}_1$ or $\mathcal{B}_2$ is a greedy colouring that uses at least 3 colours, what contradicts the fact they have Grundy number at most 2. Therefore, $5 \in \{c(v_1), c(v_2)\}$.

Now assume without loss of generality that $c(v_2) = 5$ and suppose $c(v_1) \neq 4$. Every vertex in $\{v_2, \dots, v_n\}$ should be adjacent to a vertex in E that is coloured 4, and so this vertex has to be adjacent to a vertex with colour 3. This vertex can only be v_1, since if a neighbour of $e \in E$ in $\mathcal{B}_3(e)$

is coloured 3 we get a contradiction as before. But a vertex in E is adjacent to only 2 vertices in V, therefore assuming $n \geq 4$ we already get a contradiction. □

Given a vertex $v \in V(G)$, $d_H(v) = n + 2$ and we showed that all its neighbours in V have colours greater than 3. Since $c(v) > 3$, its 3 neighbours in E have to be coloured with distinct colours in $\{1, 2, 3\}$. This implies that c is a proper 3-edge-colouring of G, thus completing the proof. □

3.4 Open problems

In view of the existing results showing that GRUNDY NUMBER is NP-complete even for bipartite graphs, complement of bipartite graphs, P_5-free graphs and chordal graphs, a natural direction for further investigation is the complexity of this problem when restricted to subclasses of these graph classes or to classes that are in the intersection of these. In particular, we are interested in the following graph classes.

P_5-free graphs

Given a graph $G = (V, E)$, we say that $S \subseteq V$ is a *dominating set* if every vertex $v \in V \backslash S$ has a neighbour in S. The following characterization of P_5-free graphs was given by Bacsó and Tuza [6]:

Theorem 3.4.1. *[6] A connected graph G is P_5-free if, and only if, for every induced subgraph $H \subseteq G$, H has either a dominating clique or a dominating cycle on five vertices.*

A direct consequence of this characterization is the following:

Corollary 3.4.1. *If G is a connected P_5-free bipartite graph, then there exists a dominating edge in G.*

Lemma 3.4.1. *Let G be a P_5-free bipartite graph $G = (A \cup B, E)$ and uv a dominating edge. Then in any greedy colouring with k colours, $\{c(u), c(v)\} = \{k, k-1\}$.*

Proof. Let c be a greedy k-colouring of G and suppose by contradiction that one of u, v has a colour smaller than $k - 1$. We may assume without loss of generality that $c(u) = i \leq k - 2$ and that $u \in A$. In this case, since uv is a dominating edge, u dominates all the vertices in B, and therefore no vertex of B can have colour i. Now, let w be a vertex with colour k. it must have a neighbour with colour $k - 1$ and therefore we may assume that there is a vertex $w' \in A$ (which may be the same as w) that has colour at least $k - 1$. But then w' should have a neighbour coloured i and we have a contradiction, since all its neighbours are in B. □

Finally, we are able to determine the Grundy number of a P_5-free bipartite graph.

Lemma 3.4.2. *Let G be a P_5-free bipartite graph $G = (A \cup B, E)$. Then, $\Gamma(G) \leq 3$.*

Proof. Let G be a minimal counter-example, with respect to the number of vertices. Then G is connected and $\Gamma(G) = 4$. Let c be a greedy 4-colouring of G. Because of Corollary 3.4.1, G has a dominating edge, say uv. Assume without loss of generality that $u \in A$ and $v \in B$. Because of Lemma 3.4.1 we may also assume without loss of generality that $c(u) = 4$ and $c(v) = 3$. Since the colouring is greedy and $c(u), c(v) > 2$, u and v should have neighbours with colour 2. Let $b_2 \in N(u)$ and $a_2 \in N(v)$ be such that $c(b_2) = c(a_2) = 2$. Then, since the colouring is greedy, b_2 and a_2 should have at least one neighbour with colours 1, say $b_1 \in N(a_2)$ and $a_1 \in N(b_2)$. Since the colouring is proper, $b_2a_2, b_1a_1 \notin E(G)$. This implies that G is not P_5-free, since $a_1b_2ub_1a_2$ induces a P_5. □

In Corollary 2.3.1 it was proved that a connected graph has Grundy number 2 if and only if it is a complete bipartite graph. To determine the Grundy number of a connected P_5-free bipartite graph, we just have to check if it is a complete bipartite graph or not. In the former case the Grundy number is 2 and in the later one it is 3, by Lemma 3.4.2.

That case being solved, a natural question is the following:

Problem 3.1. Determine the Grundy number of a triangle-free P_5-free graph.

If a class of graphs is such that the Grundy number is bounded by a constant, say p, then the Grundy number of this class of graphs can be computed in polynomial time. To see this, observe that given a graph G from the class, in order to determine $\Gamma(G)$ it suffices to check for every $k \leq p$ if G contains a k-atom. Therefore the following is a related question:

Problem 3.2. Does there exists triangle-free P_5-free graphs with arbitrarily large Grundy number?

Chordal bipartite graphs

A graph is called *chordal bipartite* if it is bipartite and it contains no induced cycle of length greater than four. Although this name has been used by many authors, these graphs are precisely the *weakly chordal bipartite* graphs, where a graph is *weakly chordal* if it does not contain C_k or $\overline{C_k}$ as an induced subgraph, for $k \geq 5$. Observe that a chordal bipartite graph is not necessarily chordal. On the other hand, these graphs admits a characterization which is similar to the one of chordal graphs in terms of the existence of a perfect elimination ordering of the vertices. In the following paragraph we describe this characterization.

Given a bipartite graph $G = (V, E)$, an edge uv is called a *bisimplicial edge* if $N(u) \cup N(v)$ induces a complete bipartite subgraph. An ordering $e_1, \ldots, e_m$ of E is called a *perfect edge elimination ordering* if the edge e_i is bisimplicial in G_i, where $G_0 = G$ and $G_i = G_{i-1} - e_i$ for $1 \leq i \leq m$. A graph is chordal bipartite if and only if it has a perfect edge elimination ordering. This characterization appears for example in [14].

Observe that the graph in the reduction from Theorem 3.2.1 is not a chordal bipartite graph, and it is not evident if this reduction can be changed in order to prove that GRUNDY NUMBER is NP-complete when restricted to chordal bipartite graphs.

Problem 3.3. Determine the computational complexity of GRUNDY NUMBER for chordal bipartite graphs.

Approximation algorithms

In Section 2.3.1 we presented some results concerning approximation algorithms for the Grundy number of graphs. No constant-factor approximation algorithm exists for the Grundy number of a general graph, and we leave that as an open problem:

Problem 3.4. Find a constant-factor approximation algorithm for the Grundy number of a graph, or prove that no such algorithm exists, unless P = NP.

For some classes like complement of bipartite graphs, interval graphs and complement of chordal graphs there are constant-factor approximation algorithms. But for some important graph classes no result is known.

Problem 3.5. Find constant-factor approximation algorithms for the Grundy number of particular classes of graphs, like bipartite graphs and chordal graphs, for which determining the Grundy number is an NP-hard problem.

Chapter 4

The b-chromatic number of graphs

4.1 Introduction

We remind the reader that the *m-degree* $m(G)$ of a graph is the largest value m such that there are m vertices with degree at least $m-1$. We recall that a vertex is *dense* if its degree is at least $m(G)-1$. Consider the following class of graphs.

Definition (tight graph)**.** A graph G is *tight* if it has exactly $m(G)$ vertices of degree $m(G)-1$.

Observe that $m(G)$ can be computed in polynomial time following the definition. So it can be checked in polynomial time if a graph is tight. In this chapter we will be interested in the complexity of the following decision problem:

TIGHT b-CHROMATIC PROBLEM
INPUT : A tight graph G.
OUTPUT : Is $\chi_b(G)$ equal to $m(G)$?

A direct consequence of the NP-completeness result shown by Kratochvíl, Tuza and Voigt [90] is the following:

Corollary 4.1.1. *The* TIGHT b-CHROMATIC PROBLEM *is NP-complete for connected bipartite graphs.*

We recall that a graph G is *P_4-sparse* if every set of five vertices of G induces at most one P_4. Bonomo et al. [12] proved that the b-chromatic number of P_4-sparse graphs can be determined in polynomial time, and asked if this result could be extended to *distance-hereditary graphs*, that are graphs in which every induced path is a shortest path. We answer in the negative to this question by showing that TIGHT b-CHROMATIC PROBLEM is NP-complete for chordal distance-hereditary graphs, in Theorem 4.2.1. We then consider split graphs, that are a subclass of chordal graphs, and show in Theorem 4.2.2 that determining the b-chromatic number of a split graph can be done in polynomial time.

In Section 4.3, we introduce the *b-closure* G^* of a tight graph G. We show that for a tight graph G, $\chi_b(G) = m(G)$ if and only if $\chi(G^*) = m(G)$. We introduce the definition of a *pivoted tight graph*, generalizing the pivoted trees defined by Irving and Manlove [70]. We show in Theorem 4.3.1 that a tight graph G is pivoted if and only if $\omega(G^*) > m(G)$. This gives a necessary condition for G

to have $\chi_b(G) < m(G)$. Since, for a tight graph G, $\chi_b(G) = m(G)$ if and only if $\chi(G^*) = m(G)$, if one can determine the chromatic number of G^* in polynomial time, then one can also solve TIGHT b-CHROMATIC PROBLEM in polynomial time. We show that it is the case for tight complement of bipartite graphs, by proving that the closures of such graphs are also complements of bipartite graphs and making use of the fact that the chromatic number of the latter graphs can be determined in polynomial time, as they are perfect graphs.

The method of computing the b-closure of a graph and then its chromatic number does not yield polynomial-time algorithms to solve TIGHT b-CHROMATIC PROBLEM for all classes of tight graphs, since it is not always easy to compute the chromatic number of the closure. However, for some of them, we show in Section 4.4, that the TIGHT b-CHROMATIC PROBLEM may be solved in polynomial time using a slight modification of the closure, the *partial closure*. It is the case for block graphs and P_4-sparse graphs.

4.2 Chordal graphs

We start this section by showing that TIGHT b-CHROMATIC PROBLEM is NP-complete for chordal distance hereditary graphs.

Theorem 4.2.1. *The* TIGHT b-CHROMATIC PROBLEM *is NP-complete for connected chordal distance-hereditary graphs.*

Proof. Clearly, one can verify in polynomial time if a colouring is a b-colouring and so the problem is in NP. To show that it is also NP-complete, we present a reduction from 3-EDGE-COLOURABILITY of 3-regular graphs. Let G be a 3-regular graph with n vertices. Set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $E(G) = \{e_1, \ldots, e_m\}$. Let I be the vertex-edge incidence graph of G, that is, I is the bipartite graph with vertex set $V(I) = V(G) \cup E(G)$ in which $e_i \in E(G)$ is adjacent to $v_j \in V(G)$ if and only if vertex v_j is an endpoint of e_i.

We construct from I a new graph H as follows. First, add to H a copy of I. Now, add an edge between every pair of vertices of $V(G)$ in H and then, finally, add to H three disjoint copies of $K_{1,n+2}$. See Figure 4.2.

One can easily see that $d_H(v) = n - 1 + 3 = n + 2$, for $v \in V(G)$, and that $d_H(u) = 2$, for $u \in E(G)$. Moreover, each copy of $K_{1,n+2}$ has exactly one vertex with degree equal to $n + 2$. Consequently, $m(H) = n + 3$ and H is tight. In H, $V(G)$ is a clique and $E(G)$ is an independent set, so $H[A \cup B]$ is a split graph, and so it is chordal. As the disjoint copies of $K_{1,n+2}$ are themselves chordal graphs, we get that the entire graph H is chordal. One can easily check that H is also distance-hereditary. We now prove that G admits a 3-edge-colouring if and only if $\chi_b(H) = m(H) = n + 3$.

Let c be a 3-edge-colouring of $E(G)$ that uses colours $\{1, 2, 3\}$. We shall construct a b-colouring c' of H with $n + 3$ colours. Let $c'(u) = c(u)$, for $u \in E(G)$, and $c'(v_i) = i + 3$, for $1 \leq i \leq n$, $v_i \in V(G)$. Note that in this partial colouring, the vertices in $V(G)$ are b-vertices of their respective colours. To obtain the remaining b-vertices, one just have to appropriately colour the copies of $K_{1,n+2}$, which can be easily done. Then, c' is a b-colouring of H with $m(H) = n + 3$ colours.

Now, let c' be a b-colouring of H that uses $n + 3$ colours. Since $V(G)$ is a clique, we may assume that $c'(v_i) = i + 3$, for $1 \leq i \leq n$. Since there are only $n + 3$ vertices of degree $n + 2$ in H, each vertex in $V(G)$ is an b-vertex. But then, since every vertex in $V(G)$ has degree exactly $n + 2$ in H, all its neighbours must have distinct colours. As a consequence, since no vertex in $V(G)$ is coloured with one of the colours in $\{1, 2, 3\}$, for every vertex in $V(G)$, its 3 neighbours in $E(G)$ are coloured with

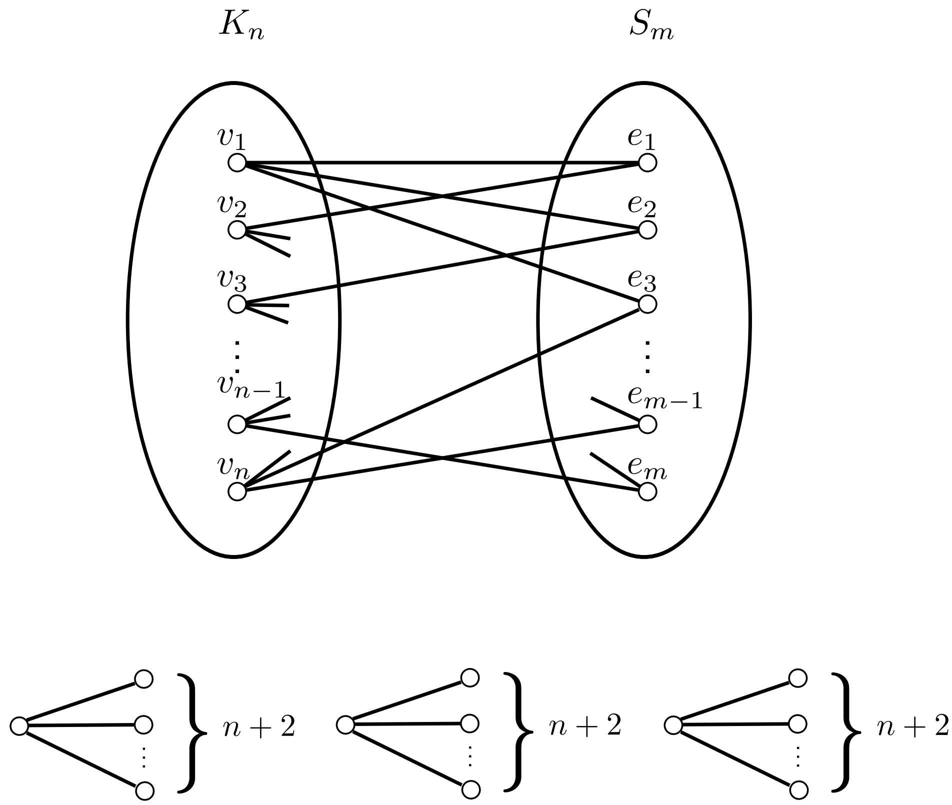


Figure 4.1: The graph H of the reduction in Theorem 4.2.1

distinct colours in $\{1, 2, 3\}$. This implies that G admits a 3-edge-colouring of G, thus completing the proof.

It is easy to see that one can change the reduction in order to obtain a connected graph by joining one of the leafs of each component that is a star to a vertex in $E(G)$. Therefore, the result holds for connected graphs. □

The complexity of determining the b-chromatic number of a distance-hereditary graph was asked by Bonomo et al. [12], who proved that this can be done in polynomial time for a P_4-sparse graph. There are many other graph classes defined in terms of induced P_4's of the graph. A graph $G = (V, E)$ is P_4-*laden* [47] if for every set $S \subseteq V$ of six vertices, the subgraph induced by S contains at most one induced P_4 or is a split graph. G is said to be *extended* P_4-*laden* if for every set $S \subseteq V$ of six vertices, the subgraph induced by S contains at most one induced P_4 or is a *pseudo-split graph*, that is a $\{C_4, 2K_2\}$-free graph. From these definitions we get that every P_4-laden graph is extended P_4-laden. It is easy to see that that the graph in the reduction of Theorem 4.2.1 is P_4-laden, so TIGHT b-CHROMATIC PROBLEM is NP-complete for P_4-laden graphs.

The class of the extended P_4-laden graphs contains many graph classes with few induced P_4's. In particular, it contains the class of P_4-*tidy graphs* [48] which in turn contains the ones of P_4-lite [73], P_4-extendible [75] and P_4-reducible graphs [74]. A graph is P_4-*tidy* if for every set A inducing a P_4 there is at most one vertex x such that the subgraph induced by $A \cup \{x\}$ has more than one induced P_4. Bonomo, Koch, and Velasquez [118] proved that the b-chromatic number of a P_4-tidy graph can be determined in polynomial time, thus extending the result in [12].

Finally, one can remark that the graph in the reduction of Theorem 4.2.1 is such that each connected component is a split graph. It is natural to ask the complexity of determining the b-chromatic number of a split graph, what is done in the following.

Theorem 4.2.2. *If G is a split graph, then $\chi_b(G) = m(G)$. Hence, the b-chromatic number of a split graph can be determined in polynomial time.*

Proof. Let G be a split graph and (K, S) a partition of $V(G)$ with K a clique and S an independent set such that $|K|$ is maximum. Every vertex in K has degree at least $|K| - 1$ and every vertex s in S has degree at most $|K| - 1$ otherwise $(K \cup \{s\}, S \setminus \{s\})$ would contradict the maximality of $|K|$. Hence $m(G) = |K|$.

Colouring the vertices in K with $|K|$ distinct colours and then extending it greedily to the vertices of S (This is possible since every vertex in S has degree smaller than $|K|$.) gives a b-colouring of G with $m(G) = |K|$ colours. □

4.3 b-closure

Definition (b-closure)**.** Let G be a tight graph. The *b-closure* of G, denoted by G^*, is the graph with vertex set $V(G^*) = V(G)$ and edge set $E(G^*) = E(G) \cup \{uv \mid u$ and v are non-adjacent dense vertices$\} \cup \{uv \mid u$ and v are vertices with a common dense neighbour$\}$.

See Figure 4.3 for examples of application of the closure. The next theorem proves the relation, for a tight graph G, between the parameters $\chi_b(G)$ and $\chi(G^*)$:

Lemma 4.3.1. *Let G be a tight graph. Then $\chi_b(G) = m(G)$ if and only if $\chi(G^*) = m(G)$.*

Proof. Set $m = m(G)$. Suppose that $\chi_b(G) = m$, and let c be a b-colouring of G with m colours. It is easy to see that the m dense vertices form a clique in G^* and so $\chi(G^*) \geq m$. Let us show that c is a proper colouring for G^*. Let $uv \notin G$ be such that $uv \in E(G^*)$. If both u and v are dense, as there are exactly m dense vertices in G, they must have distinct colours in c. Now, suppose that u or v is not a dense vertex. By the definition of G^*, u and v have a common dense neighbour, say d, in G. Since all dense vertices of G have degree $m - 1$ and c is a b-colouring, u and v must have been assigned distinct colours in c. Hence, $\chi(G^*) = m$.

Conversely, let c' be a proper colouring of G^* with m colours. In this case, since $E(G) \subseteq E(G^*)$, c' is also a proper colouring of G. It only remains to show that every colour of c' has a b-vertex. As the dense vertices of G form a clique in G^*, they have distinct colours in c'. Moreover, for a dense vertex d of G, we have that $N_{G^*}(d)$ is a clique. As a consequence, d is a b-vertex. Therefore, $\chi_b(G) = m$. □

If G is a graph such that $\omega(G^*) > m$, then $\chi(G^*) > m$. As a consequence of this and Lemma 4.3.1 we have:

Corollary 4.3.1. *Let G be a tight graph. If $\chi_b(G) = m(G)$, then $\omega(G^*) = \chi(G^*) = m(G)$.*

4.3.1 Complement of bipartite graphs

By Lemma 4.3.1, it is interesting to consider the b-closure of a tight graph G if the chromatic number of its closure can be determined in polynomial time. Indeed if so, one can decide in polynomial time if $\chi_b(G) = m(G)$. We now show that it is the case if G is the complement of a bipartite graph.

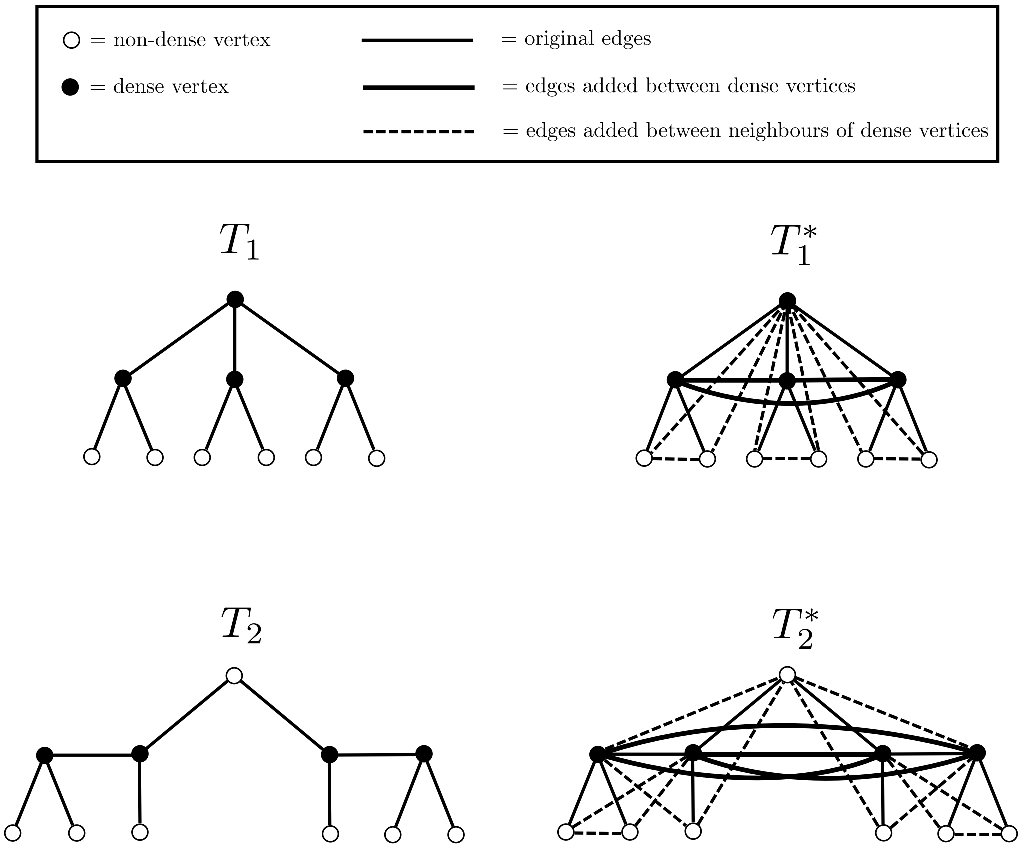


Figure 4.2: The closure of tight trees T_1 and T_2. Observe that T_2 is a pivoted tree and there is a clique of $m(T_2)$ vertices in T_2^*

.

Lemma 4.3.2. *The b-closure of a tight complement of a bipartite graph is a complement of a bipartite graph.*

Proof. Let G be a tight complement of a bipartite graph. Let $V(G) = X \cup Y$ where X and Y are two disjoint cliques in G. As $V(G^*) = V(G)$, and since $E(G) \subseteq E(G^*)$, the sets X and Y are cliques in G^*. So they also form a partition of $V(G^*)$ into two cliques. □

Computing the chromatic number of the complement G of a bipartite graph $\overline{G}$ is equivalent to compute the maximum size of a matching in this bipartite graph. Hence it can be done in $O(\sqrt{|V(G)|}\cdot |E(\overline{G}|)$ by the algorithm of Hopcroft and Karp [68] and in $O(|V(G)|^{2.376})$ using an approach based on the fast matrix multiplication algorithm [103].

Corollary 4.3.2. *Let G be a tight complement of a bipartite graph. It can be decided in $O(\max\{\sqrt{|V(G)|}\cdot |E(\overline{G})|, |V(G)|^{2.376}\})$ if $\chi_b(G) = m(G)$.*

4.3.2 Pivoted graphs

In the study of the b-chromatic number of trees, Irving and Manlove [70] introduced the notion of a *pivoted tree*, and showed that a tree T satisfies $\chi_b(T) < m(T)$ if and only if it is pivoted. We generalize this notion and show how our generalization is related to the b-chromatic number of tight graphs.

Definition (Pivoted Graph)**.** Let G be a tight graph. We say that G is *pivoted* if there is a set N of non-dense vertices, with $|N| = k$, and a set of dense vertices D, with $|D| = m(G) - k + 1$, satisfying:

1. For every pair $u, v \in N$, u is adjacent to v, or there is a dense vertex w that is adjacent to both u and v.

2. For every pair $u \in N$, $d \in D$, either u is adjacent to d or u and d are both adjacent to a dense vertex w (not necessarily in D).

Theorem 4.3.1. *Let G be a tight graph. Then G is a pivoted graph if and only if $\omega(G^*) > m(G)$.*

Proof. First, assume that G is a pivoted graph. Then Definitions 4.3 and 4.3.2 immediately imply that $N \cup D$ is a clique of size $m + 1$ in G^*.

Reciprocally, assume that $\omega(G^*) > m$. Let $S \subseteq V(G^*)$ be a clique of size $m + 1$ in G^*. Let $N = \{v \in S \mid v \text{ is not dense in } G\}$ and $D = \{v \in S \mid v \text{ is dense in } G\}$. Let $u, v \in S$. If $u, v \in D$, there is nothing to show, since Definition 4.3.2 imposes no restrictions between dense vertices in G. If $u \in N, v \in D \cup N$, we have that either $uv \in E(G)$, or $ud, vd \in E(G)$, for a dense vertex $d \in V(G)$. So, it is easy to see that the sets N and D satisfy the requirements of Definition 4.3.2. □

The tree T_2 in Figure 4.3 ilustrates the statement of Theorem 4.3.1. Lemma 4.3.1 and Theorem 4.3.1 have the following corollary.

Corollary 4.3.3. *Let G be a tight graph. If G is a pivoted graph, then $\chi_b(G) < m(G)$.*

Proof. As G is pivoted, Theorem 4.3.1 implies that $\omega(G^*) > m(G)$, and therefore $\chi(G^*) > m(G)$. Then, by Lemma 4.3.1, $\chi_b(G) < m(G)$. □

There are graphs satisfying $\chi(G^*) > m(G)$ but not $\omega(G^*) > m(G)$. Figure 4.3 shows a chordal non-pivoted graph G with exactly $m(G) = 7$ dense vertices, each of degree 6, such that $\chi_b(G) < m(G)$. In contrast to what happens with pivoted graphs, where a clique of size greater than m is formed in their b-closures, the graph of Figure 4.3 has clique number 7, but its b-closure produces an odd hole (by the five non-dense vertices in the bigger component) which causes $\chi(G^*) > 7$.

4.4 Partial b-closure

Definition (partial b-closure)**.** Let G be a tight graph. The *partial b-closure* of G, denoted G_p^*, is the graph with vertex set $V(G^*) = V(G)$ and edge set $E(G^*) = E(G) \cup \{uv \mid u \text{ and } v \text{ are vertices with a common dense neighbour}\}$.

Lemma 4.4.1. *Let G_p^* be the partial b-closure of a graph G, and let D be the set of $m(G)$ dense vertices of G. Then $\chi_b(G) = m(G)$ if and only if G_p^* admits a $m(G)$-colouring where all the vertices in D have distinct colours.*

Proof. The proof is similar to the one of Lemma 4.3.1. In this case, since we do not add edges between all the pairs of dense vertices in G_p^*, we need the requirement that the $m(G)$-colouring of G_p^* is such that all dense vertices have distinct colours. □

By Lemma 4.4.1, whenever the constrained colouring of the partial closure G_p^* can be obtained in polynomial time, deciding if $\chi_b(G) = m$ is polynomial-time solvable. In particular, it is the case if PRECOLOURING EXTENSION can be solved in polynomial time for G. We show that this is the case for block graphs and P_4-sparse graphs.

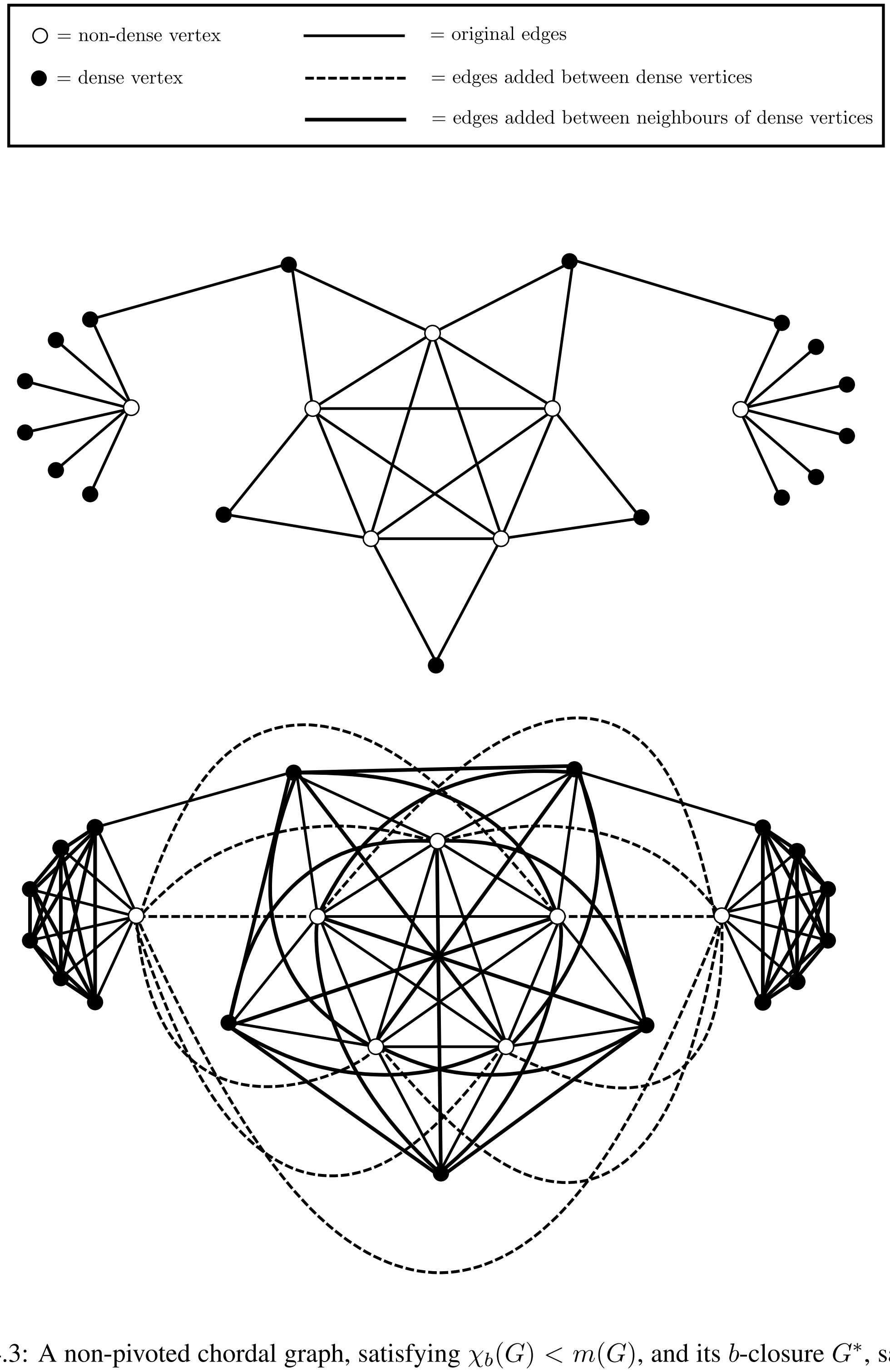


Figure 4.3: A non-pivoted chordal graph, satisfying $\chi_b(G) < m(G)$, and its b-closure G^*, satisfying $\chi(G^*) > \omega(G^*) = m(G)$
.

4.4.1 Block graphs

We remind the reader that a graph $G = (V, E)$ is *a block graph* if each of its blocks (maximal 2-connected subgraphs) is a complete graph. An example of block graph is in Figure 4.4.

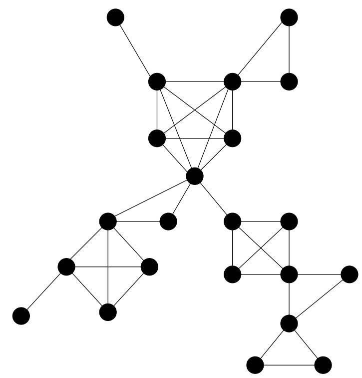


Figure 4.4: A block graph.

Lemma 4.4.2. *The partial b-closure of a block graph is chordal.*

Proof. By contradiction, assume that the partial b-closure G_p^* of a block graph G is not chordal. Then it has an induced cycle $C = (v_1, v_2, \ldots, v_k)$ of length $k \geq 4$. For every edge v_iv_{i+1} of C (indices must be taken modulo k) either $v_iv_{i+1} \in E(G)$ or there is a dense vertex $w_i \in V(G)$ such that $v_iw_i, w_iv_{i+1} \in E(G)$. In the latter case, the vertex w_i is not adjacent to any v_j in G, $j \notin \{i, i+1\}$, otherwise both v_jv_i and v_jv_{i+1} would be edges of G_p^* and C would not be induced. Furthermore, this implies that $w_i \neq w_j$, $i \neq j$. Let C' be the cycle obtained from C by replacing each edge v_iv_{i+1} by $v_iw_iv_{i+1}$ whenever $v_iv_{i+1} \notin E(G)$. Observe that C' is a cycle of G.

But, since G is a block graph, the vertices of any cycle (in particular, C') form a clique in G and thus also in G_p^*. Hence the vertices of C form a clique in G_p^*, a contradiction. □

Marx [100] showed that the 1-PRECOLOURING EXTENSION, the version of PRECOLOURING EXTENSION in which the k colours are used at most once is solvable in time $O(k \cdot |V(G)|^3)$ for a chordal graph G. Hence,

Corollary 4.4.1. *the* TIGHT b-CHROMATIC PROBLEM *can be decided in time $O(m(G)|V(G)|^3)$ for tight block graphs.*

Remark. A tree is a block graph, so by using the partial closure method we obtain that the TIGHT b-CHROMATIC PROBLEM for tight trees can be solved in $O(m(G)|V(G)|^3)$-time . However, Irving and Manlove [70] gave a linear-time algorithm to compute the b-chromatic number of any tree. Hence the TIGHT b-CHROMATIC PROBLEM can be solved in linear time for trees.

4.4.2 P_4-sparse graphs

Lemma 4.4.3. *The partial b-closure of a P_4-sparse graph is P_4-sparse.*

Proof. Let G be a P_4-sparse graph. Suppose, by way of contradiction, that G_p^* is not P_4-sparse. Then there is at least one induced P_4 in G_p^* that is not in G. Let $P = (v_1, v_2, v_3, v_4)$ be such a P_4 in G_p^*. We will show that there are 5 vertices that induces two P_4's in G, thus getting a contradiction. By symmetry, it is enough to consider the following five cases.

Case 1 : $v_1v_2 \in E(G)$, $v_2v_3 \in E(G)$ and $v_3v_4 \notin E(G)$.
Then, v_3 and v_4 are both adjacent to a dense vertex $w \in V(G)$ (by the definition of the partial b-closure). Note that $v_1w \notin E(G)$ (resp. $v_2w \notin E(G)$) otherwise $v_1v_4 \in E(G_p^*)$ (resp. $v_2v_4 \in E(G_p^*)$). Hence, $\{v_1, v_2, v_3, w, v_4\}$ induces a P_5 which contains two induced P_4.

Case 2 : $v_1v_2 \in E(G)$, $v_2v_3 \notin E(G)$ and $v_3v_4 \in E(G)$.
In this case, v_2 and v_3 are both adjacent to a dense vertex $w \in V(G)$. Again, since $v_1w, v_4w \notin E(G)$, $\{v_1, v_2, w, v_3, v_4\}$ is an induced P_5 in G.

Case 3 : $v_1v_2 \notin E(G)$, $v_2v_3 \in E(G)$ and $v_3v_4 \notin E(G)$.
As $v_1v_2 \notin E(G)$, there exists vertices $w_1, w_2 \in V(G)$ such that $w_1v_1, w_1v_2 \in E(G)$, $w_1v_3, w_1v_4 \notin E(G)$, $w_2v_3, w_2v_4 \in E(G)$ and $w_2v_1, w_2v_2 \notin E(G)$. Note that $w_1 \neq w_2$, since $w_1v_4 \notin E(G)$. If $w_1w_2 \notin E(G)$, then $\{v_1, w_1, v_2, v_3, w_2\}$ is an induced P_5 in G. If $w_1w_2 \in E(G)$, then $\{v_1, w_1, v_2, w_2, v_4\}$ induces two P_4's in G.

Case 4 : $v_1v_2 \notin E(G)$, $v_2v_3 \notin E(G)$ and $v_3v_4 \in E(G)$.
Using arguments similar to the ones in the previous cases, we obtain that there are distinct dense vertices $w_1, w_2 \in V(G)$ satisfying $v_1w_1, v_2w_1, v_2w_2, v_3w_2 \in E(G)$, and v_1w_2, v_4w_2, $v_3w_1, v_4w_1 \notin E(G)$. If $w_1w_2 \in E(G)$ then $\{v_1, w_1, w_2, v_3, v_4\}$ induces a P_5 in G. If $w_1w_2 \notin E(G)$, then the set $\{v_1, w_1, v_2, w_2, v_3\}$ induces a P_5 in G.

Case 5 : $v_1v_2 \notin E(G)$, $v_2v_3 \notin E(G)$ and $v_3v_4 \notin E(G)$.
Again, by similar arguments to the ones used in the previous cases, there are distinct dense vertices $w_1, w_2, w_3 \in V(G)$ such that $v_1w_1, v_2w_1, v_2w_2, v_3w_2, v_3w_3, v_4w_3 \in E(G)$, and v_3w_1, v_4w_1, v_1w_2, v_4w_2, $v_1w_3, v_2w_3 \notin E(G)$. If $w_1w_3 \in E(G)$, the set $\{v_1, w_1, w_3, v_3, v_4\}$ induces two P_4's in G. Henceforth we may assume that $w_1w_3 \notin E(G)$. If $w_1w_2, w_2w_3 \in E(G)$, then the set $\{v_1, w_1, w_2, w_3, v_4\}$ induces a P_5 in G. Hence by symmetry, we may assume that $w_2w_3 \in E(G)$. If $w_1w_2 \in E(G)$, then the set $\{v_1, w_1, v_2, w_2, v_3\}$ induces two P_4's in G. If $w_1w_2 \notin E(G)$ the set $\{v_1, w_1, v_2, w_2, w_3\}$ induces two P_4's in G.

□

Babel et al. [5] showed that PRECOLOURING EXTENSION is linear-time solvable for $(q, q-4)$-graphs, which are graphs where no set of at most q vertices induces more than $q-4$ different P_4's. Hence,

Corollary 4.4.2. *The* TIGHT b-CHROMATIC PROBLEM *can be decided in linear time for tight P_4-sparse graphs.*

Consequently, for tight P_4-sparse graphs, this algorithm is faster than the $O(|V|^3)$-time algorithm given in [12], that solves the more general case where the input graph is not necessarily tight.

4.5 Open problems

Complement of bipartite graphs

In Corollary 4.3.2 we proved that the b-chromatic number of a tight complement of a bipartite graph can be computed in polynomial time. Kouider and Zaker [87] gave a characterization of the complement of bipartite graphs that admit a b-colouring with k colours. Let $G = (X \cup Y, E)$ be the complement of a bipartite graph and suppose X and Y can be partitioned into three subsets $X = A_1 \cup B_1 \cup C_1$ and $Y = A_2 \cup B_2 \cup C_2$ such that the following properties hold:

1. The subgraph induced by $A_1 \cup A_2 \cup B_2$ is a clique, and so is the one induced by $A_1 \cup A_2 \cup C_1$.
2. $|B_1| = |B_2|$ and there is a perfect anti-matching between B_1 and B_2.
3. $|C_1| = |C_2|$ and there is a perfect anti-matching between C_1 and C_2.

Finally, let the weight of the partition be the value $k = |A_1 \cup A_2| + |B_1| + |C_1| = |X| + |A_2|$. By colouring the vertices in $X \cup A_2$ with different colours and then using the anti-matchings to give the same colours to B_2 as in B_1 and to C_2 as in C_2, we obtain a b-colouring of G with t colours. Kouider and Zaker proved that if G is the complement of a bipartite graph, then G admits a b-colouring with k colours if and only if either it admits a partition as before with weight k or $\omega(G) = k$.

It is not evident how this characterization can lead to a polynomial-time algorithm to compute the b-chromatic number of the complement of a bipartite graph. Therefore we believe that the following problem is an interesting one.

Problem 4.1. Can it be decided in polynomial time if $\chi_b(G) \geq k$, where G is the complement of a bipartite graph?

b-colouring of tight graphs and the Erdös-Faber-Lovász conjecture

In a research report [60] co-autored with Havet and Linhares-Sales we tried to connect the following well-known conjecture of Erdös-Faber-Lovász with the b-colouring of a particular class of tight bipartite graphs.

Conjecture 4.5.1 (Erdös-Faber-Lovász Conjecture [35])**.** *Let $\mathcal{K}_m$ be the class of graphs $H = \bigcup_{i=1} K_m^i$, where K_m^i is a complete graph of m vertices for $1 \leq i \leq m$ and $|K_m^i \cap K_m^j| \leq 1$ for $i \neq j$. If $G \in \mathcal{K}_m$, then $\chi(G) = m$.*

In [60] we presented the following conjecture and gave a wrong proof that it was equivalent to the one of Erdös-Faber-Lovász.

Conjecture 4.5.2. *Let G be a tight graph such that:*

1. *For every edge $uv \in E(G)$ one of its endpoints is dense, and the other non-dense, and*
2. *If d' and d'' are dense vertices in G, then $|N(d') \cap N(d'')| \leq 1$.*

Then, $\chi_b(G) = m(G)$.

Lin and Chang [93] proposed a new conjecture and related it with the Erdös-Faber-Lovász Conjecture. Let $\mathcal{B}_m$ denote the class of tight bipartite graphs G with $m(G) = m$, in which D and $D' = \bigcup_{x \in D} N_G(x)$ are stable sets and $|N_G(x) \cap N_G(x')| \leq 1$ for any two distinct dense vertices x and x'. They proved that:

Theorem 4.5.1 ([93]). *If Erdös-Faber-Lovász Conjecture is true, then $\chi_b(G) = m$ or $m - 1$ for any $G \in \mathcal{B}_m$.*

Theorem 4.5.2 ([93]). *If for every $G \in \mathcal{B}_m$, $\chi_b(G) = m$, then the Erdös-Faber-Lovász Conjecture is true.*

They also proved that there are graphs $G \in \mathcal{B}_m$ such that $\chi_b(G) = m - 1$, therefore proving that Conjecture 4.5.2 is not true. Finally, they proposed the following conjecture which is weaker than the Erdös-Faber-Lovász Conjecture, and which we leave as an open problem.

Problem 4.2. Is it true that if $G \in \mathcal{B}_m$ then $\chi_b(G) \geq m - 1$?

Approximation algorithms

There is no known approximation algorithm for the b-chromatic number of graphs. We already mentioned in Section 2.3.2 that Corteel, Valencia-Pabon and Vera [27] proved that there is no $\epsilon > 0$ for which the b-chromatic number can be approximated by a factor of $\frac{120}{133} - \epsilon$ in polynomial time, unless P = NP. On the other hand, appart from this result no other results exists about the approximability of the b-chromatic number, so the following questions are interesting:

Problem 4.3. Find a constant-factor approximation algorithm for the b-chromatic number of a graph, or prove that no such algorithm exists, unless P = NP.

Problem 4.4. Find constant-factor approximation algorithms for the b-chromatic number of particular classes of graphs, like chordal and bipartite graphs, for which determining the b-chromatic number is an NP-hard problem.

Chapter 5

Fixed-parameter-complexity of the colouring parameters

5.1 Introduction

In this chapter we investigate some problems associated with the Grundy and b-chromatic numbers from the point of view of the parameterized complexity theory. We refer the reader to [30] or [106] for an introduction to this theory. The parameterized complexity of the following parameterized problems is investigated in this chapter:

DUAL OF GREEDY COLOURING
INPUT : A graph G and an integer k.
PARAMETER : k
OUTPUT : $\Gamma(G) \geq |V(G)| - k$?

DUAL OF b-COLOURING
INPUT : A graph G and an integer k.
PARAMETER : k.
OUTPUT : $\chi_b(G) \geq |V(G)| - k$?

We show in Section 5.2.1 that DUAL OF GREEDY COLOURING is Fixed Parameter Tractable(FPT). A similar approach is used to show that DUAL OF b-COLOURING is also FPT (Section 5.2.2). In Section 5.3 we consider the parameterization by the maximum degree of the graph and give FPT algorithms for deciding if $\Gamma(G) = \Delta(G) + 1$ or if $\chi_b(G) = \Delta(G) + 1$. Finally, open problems and further research are discussed in Section 5.4.

5.2 Parameterizations from below the number of vertices

Consider the following parameterized version of the graph colouring problem.

PROPER COLOURING
INPUT : A graph G and an integer k.
PARAMETER : k
OUTPUT : $\chi(G) \leq k$?

We already mentioned that the problem of deciding if a given graph admits a colouring with 3 colours is NP-complete. As a consequence, PROPER COLOURING is not in XP, and so it is not in FPT. On the other hand, Telle (See [30], Exercise 3.2.7) proved that the following parameterized version of the graph colouring problem is Fixed Parameter Tractable (FPT).

DUAL OF PROPER COLOURING
INPUT : A graph G and an integer k.
PARAMETER : k
OUTPUT : $\chi(G) \leq |V(G)| - k$?

Observe that from the classical complexity point of view, DUAL OF COLOURING is as hard as PROPER COLOURING, since one can esily reduce one problem to the other.

In this Section we consider DUAL OF GREEDY COLOURING and DUAL OF b-COLOURING , which may be seen as analog versions of DUAL OF PROPER COLOURING for the Grundy number and the b-chromatic number, respectively, and show that both problems are FPT.

5.2.1 Dual of Greedy Colouring

The aim of this Section is to prove the following Theorem.

Theorem 5.2.1. DUAL OF GREEDY COLOURING *can be solved in time* $O\left((2k)^{2k} \cdot |E| + 2^{2k}k^{3k+5/2}\right)$.

We remind the reader that a *vertex cover* of a graph G is a set $C \subseteq V(G)$ such that for every $e \in E(G)$, at least one of the endvertices of e is in C. A vertex cover is said to be *minimal* if there is no vertex cover $C' \subset C$. We also remind that the *complement* of a graph G, denoted $\overline{G}$, is the graph with the same vertex set and such that $uv \in E(\overline{G})$ if and only if $uv \notin E(G)$.

The proof of Theorem 5.2.1 may be outlined as follows. We first show that a graph $G = (V, E)$ has Grundy number at least $|V| - k$ if and only if its complement has a vertex cover with certain properties and in particular size at most $2k$. If G has a Grundy number at least $|V| - k$ and c is a corresponding greedy colouring, the vertices which are not alone in their colour classes in c will be shown to form such a vertex cover in $\overline{G}$. We then give an algorithm that runs in in time $O\left(k^{2k} \cdot |E| + k^{3k+5/2}\right)$ that decides if a given minimal vertex cover of $\overline{G}$ is contained in a vertex cover having the desired properties.

There are at most 2^{2k} minimal vertex covers of size at most $2k$ and we can enumerate them in time $O\left(2^{2k} \cdot |V|\right)$ using a search tree (see for example Section 8.2 of [106]). Hence applying the above-mentioned algorithm for each minimal vertex cover yields an algorithm in time $O\left((2k)^{2k} \cdot |E| + 2^{2k}k^{3k+5/2}\right)$ for DUAL OF GREEDY COLOURING.

Lemma 5.2.1. *Let* $G = (V, E)$ *be a graph and* $k \geq 0$ *an integer. Then,* $\Gamma(G) \geq |V| - k$ *if and only if there is a vertex cover* C *of* $\overline{G}$ *such that* $G[C]$ *admits a greedy colouring* $(C_1, C_2, \ldots, C_{k'})$ *with the following properties:*

P1*:* $|C| - k \leq k' \leq k$*;*

P2*:* $|C_i| \geq 2$*, for every* $1 \leq i \leq k'$*;*

P3*: For each* $v \in V \backslash C$ *and for every* $1 \leq i \leq k'$*, there is* $u \in C_i$ *such that* $uv \in E$*.*

Proof. ($\Rightarrow$) Assume that $\Gamma(G) \geq |V| - k$ and consider a greedy $\Gamma(G)$-colouring c. Let C be the set of vertices that are in a colour class with more than one vertex. Then $V \backslash C$ is the set of vertices that are alone in their colour classes.

We claim that $V \backslash C$ is a clique in G. If this is not the case, let u and v be two non-adjacent vertices in $V \setminus C$. Without loss of generality we may assume that $c(u) > c(v)$. Then, as c is a greedy colouring, u has a neighbour coloured $c(v)$, which must be v. So $uv \in E$ and we get a contradiction. Now, since $V \backslash C$ is a clique in G, it is a stable set in $\overline{G}$. Consequently, C is a vertex cover in $\overline{G}$.

Let c' be the greedy colouring $(C_1, C_2, \ldots, C_{k'})$ of $G[C]$ induced by c. Clearly, $|C_i| \geq 2$, for every $1 \leq i \leq k'$, and Property P2 is satisfied. By definition of C, we have $\Gamma(G) = |V| - |C| + k'$. Since $\Gamma(G) \geq |V| - k$, we obtain that $k' \geq |C| - k$. Property P2 implies that $|C| \geq 2k'$, therefore $\Gamma(G) = |V| - |C| + k' \geq |V| - k'$, and again because $\Gamma(G) \geq |V| - k$, we get $k' \leq k$. As a consequence, Property P1 is satisfied.

Finally, let $v \in V \backslash C$ and $1 \leq i \leq k'$. If the colour of the vertices of C_i in c is smaller than $c(v)$, then v is adjacent to at least one vertex of C_i because c is greedy. If not then every vertex of C_i is adjacent to v because it is the sole vertex coloured $c(v)$. In both cases, v is adjacent to at least one vertex in C_i, so c' also has Property P3.

($\Leftarrow$) Let C be a vertex cover of $\overline{G}$ such that there is a greedy colouring $c' = \Big(C_1, C_2, \ldots, C_{k'}\Big)$ of $G[C]$ having Properties P1, P2 and P3. One can extend c' to the entire graph G by assigning $|V| - |C|$ distinct colours to the vertices of $V \backslash C$. As a consequence of P3 and the fact that $V \backslash C$ is a stable set in $\overline{G}$ and therefore a clique in G, the obtained colouring is greedy. Because of P1, it uses $k' + |V| - |C| \geq (|C| - k) + |V| - |C| = |V| - k$ colours.

□

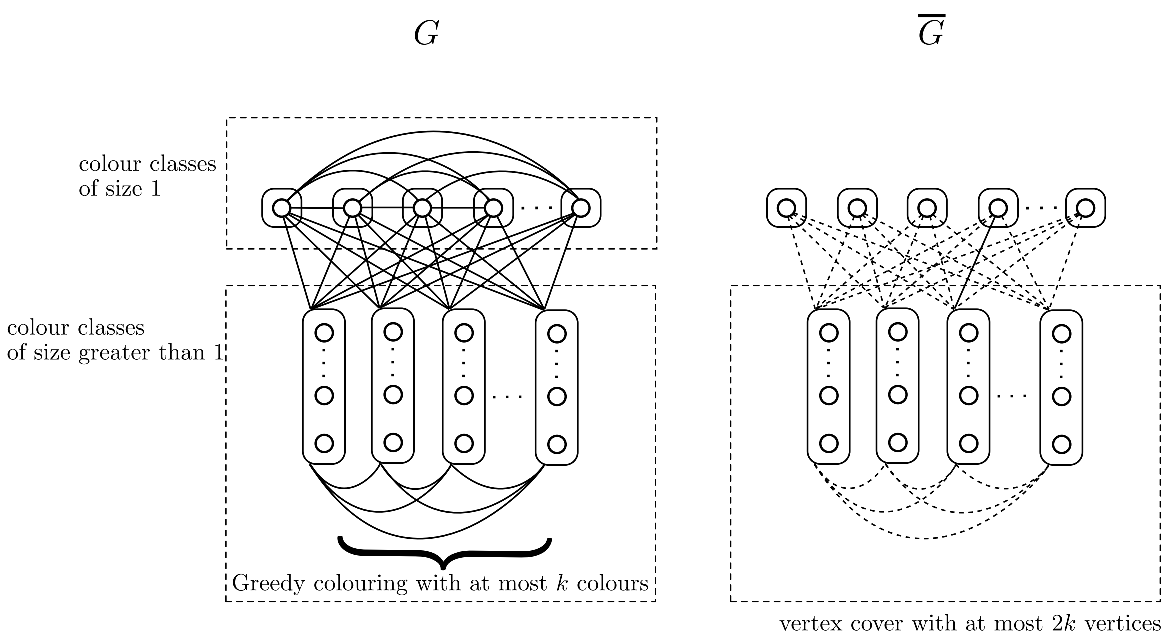


Figure 5.1: An ilustration of the characterization from Lemma 5.2.1

Let C be a vertex cover of $\overline{G}$. A greedy colouring $(C_1, C_2, \ldots, C_{k'})$ of $G[C]$ having the Properties

P1, P2 and P3 of Lemma 5.2.1 is said to be *good*. C is *suitable* if $G[C]$ has a good greedy colouring. Observe that Property P1 implies that a suitable vertex cover has cardinality at most $2k$.

Proposition 5.2.1. *Let C be a suitable vertex cover of $\overline{G}$, $(C_1, C_2, \ldots, C_{k'})$ a good greedy colouring of $G[C]$ and $C_{min} \subseteq C$ a minimal vertex cover. Then for all $1 \leq i \leq k'$, $|C_i \setminus C_{min}| \leq 1$.*

Proof. Each colour class C_i, $1 \leq i \leq k'$, is a stable set of size at least 2 in G. So it is a clique of size at least 2 in $\overline{G}$. Since C_{min} is a vertex cover in $\overline{G}$, $|C_i \cap C_{min}| \geq |C_i| - 1$, so $|C_i \setminus C_{min}| \leq 1$. □

Lemma 5.2.2. *Let k be an integer, $G = (V, E)$ a graph and C_{min} a minimal vertex cover of $\overline{G}$ of size at most $2k$. It can be determined in time $O\left(k^{2k}|E| + k^{3k+5/2}\right)$ if C_{min} is contained in a suitable vertex cover C.*

Proof. In order to determine if C_{min} is contained in a suitable vertex cover, we enumerate all possible proper colourings of $G[C_{min}]$ with k' colours, $|C_{min}| - k \leq k' \leq k$. For each of them, we then check in time $O(|E| + k^{k+5/2})$ if it can be extended into a good greedy colouring of a suitable vertex cover. There are at most $k^{|C_{min}|} \leq k^{2k}$ proper colourings of C_{min} with at most k colours and they can be enumerated in time $O(k^{2k})$. Hence the running time of our algorithm is $O\left(k^{2k}|E| + k^{3k+5/2}\right)$.

Let us now detail an algorithm that, for a proper colouring $c = (C_1, C_2, \ldots, C_{k'})$ of $G[C_{min}]$, decides if it can be extended into a good greedy colouring of a suitable vertex cover in time $O(|E| + k^{k+5/2})$. By Proposition 5.2.1, for such an extension at most one vertex of $V \setminus C_{min}$ is added in each colour class.

If c is a good colouring of C_{min} then we are done. So we may assume that it is not. We say that a colour class S_i is *defective* with respect to a colouring $s = (S_1, \ldots, S_l)$ of $S \subseteq V$ if at least one of the following holds:

(i) $|S_i| < 2$;

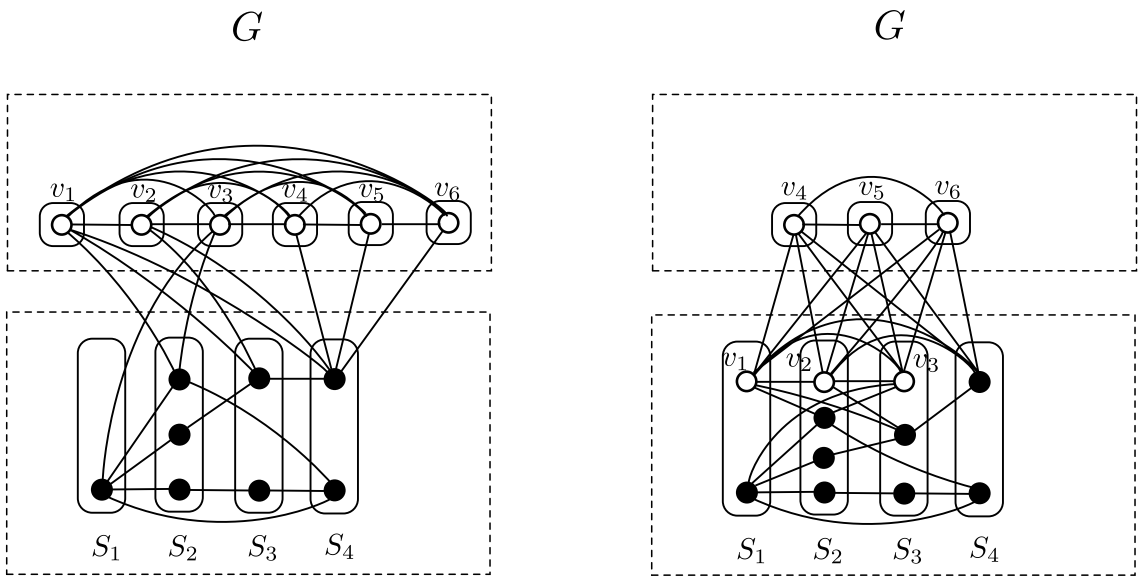
(ii) For some $j > i$, there is $v \in S_j$ with no neighbour in S_i;

(iii) There is $v \in V \setminus S$ such that v has no neighbours in S_i.

Let S_i be a defective colour class with respect to s. An *i-candidate* with respect to s is a vertex $v \in V \setminus \bigcup_{j=1}^{l} S_j$ such that $S_i \cup \{v\}$ is a stable set which is not defective with respect to the colouring $(S_1, \ldots, S_{i-1}, S_i \cup \{v\}, S_{i+1}, \ldots, S_l)$. We denote by $X_s(i)$ the set of i-candidates with respect to s and D_s the set of defective colour classes with respect to s. If S_i is not defective, then $X_s(i)$ is defined to be the empty set. If $|X_s(i)| \geq k$, we say that i is a colour class of *type 1*. Otherwise, we say that it is of *type 2*. It is easy to see that the set of defective colour classes of c and their candidates can be computed in time $O(|E|)$.

Clearly, if c can be extended into a good colouring, it means that we can place candidates into some of its colour classes and obtain a colouring without defective colour classes. See Figure 5.2.1. Because of Proposition 5.2.1, we are only allowed to place at most one vertex in each colour class. As we will show later, the only defective colour classes that may not receive a candidate in the extension of c to a good colouring are those of type 2.

Claim 5.2.1. *Let $C \supseteq C_{min}$, s be a k'-colouring of C, i one of its defective colour classes and v an i-candidate. Let s' be the extension of s where we place v in colour class i. Then, for every colour class $j \neq i$, $X_{s'}(j) = X_s(j) \setminus \{v\}$.*



Colouring s of a vertex cover C_{min} of $\overline{G}$ | Extension of s into a good colouring

$|V(G)| = 14,\qquad k = 7,\ \ k' = 4.$

$s = (S_1, \ldots, S_4).$

$X_s(1) = \{v_1, v_2\}$, S_1 is defective because of (i), (ii) and (iii).

$X_s(2) = \{v_2, v_4, v_5, v_6\}$, S_2 is defective because of (ii) and (iii).

$X_s(3) = \{v_3, v_4, v_5, v_6\}$, S_3 is defective because of (iii).

$X_s(4) = \{v_3\}$, S_4 is defective because of (iii).

Figure 5.2: Example of a minimal vertex cover together with a colouring with defective colour classes and one extension of it into a good colouring.

Proof. First, assume that j is not defective in s. If it does become defective in s' it is due to condition (ii), since it cannot satisfy (i) or (iii) after the insertion of v into the colour class i. But then, since j does not satisfy (ii) in s, v is the only vertex in i that may have no neighbours coloured j, which implies that j satisfies condition (iii) in s, a contradiction.

Now assume that j is defective in s. We first prove that $X_s(j)\backslash\{v\} \subseteq X_{s'}(j)$. In this case, again we have that (i) and (iii) remain unchanged after the insertion of v, in the sense that if a vertex different from v is a candidate for j in s because of one of these conditions, the same will happen in s'. Regarding condition (iii), the only thing that may change in s' is that v is now one of the vertices with no neighbours in j. Since v is not in C_{min}, it is adjacent to every j-candidate. Then, every vertex distinct from v that was a candidate for j in s remains a candidate for j in s'.

The converse, that is that every vertex in $X_{s'}(j)$ is also in $X_s(j)\backslash\{v\}$, is trivial.

□

In particular, Claim 5.2.1 shows that the insertion of a candidate in a defective colour classes does not create a new defective colour class.

Claim 5.2.2. *Let $s = (S_1, \ldots, S_{k'})$ be a k'-colouring of C_{min} and assume it can be extended into a good colouring $s' = (S'_1, \ldots, S'_{k'})$ of a suitable vertex cover. If j is a defective colour class in s and $|S'_j \backslash S_j| = 0$, then j is of type 2 in s. Moreover, j is defective only because of (iii).*

Proof. Let j be such that $S'_j = S_j$. If j satisfies (i) then there is only one vertex coloured j in s', and thus s' cannot be a good colouring, a contradiction. If j satisfies (ii) then there is a vertex v coloured $j' > j$ with no neighbours in S_j. Since no vertices were added to S_j in s', vertex v also has no neighbours coloured j in s', a contradiction. Hence j can be defective only because of condition (iii). Observe that every vertex in $X_s(j)$ is moved to some colour class, since otherwise colour class j would still be defective in s' because of (iii). But then, if $|X_s(j)| \geq k$, as we add at most one vertex to each colour class when extending s to s', at least one vertex in $X_s(j)$ is not in any colour class of s'. Hence, there is a vertex that is not coloured in s' and has no neighbours with colour j, which implies that j is a defective colour class in s', a contradiction.

□

In order to determine if c can be properly extended, in a first step we consider all possible extensions of the type 2 colour classes. For such a colour class of type 2, we can choose to add to it either one of its candidates or none. By Claim 5.2.2 this later case is possible if the colour class satisfies only (iii). There are at most k defective colour classes of type 2. Moreover each of these colour classes has less than k candidates, and so the number of possible ways to extend each colour class is bounded by k. Hence, we can enumerate all the possible extensions of the type 2 colour classes in time $O(k^k)$. In a second step, for each possible extension, we check if the type 1 colour classes could be extended in order to obtain a good greedy colouring.

Let c' be one possible extension of c as considered in the last paragraph. If a colour class S_i of type 2 has not been extended, it may still be defective. If it is defective because it satisfies (i) or (ii), then it will remain defective after the second step in which we add some candidate to colour classes of type 1. Hence, we can stop, it will never lead to a good colouring. If it is defective because it satisfies (iii) (and only (iii)) then all the vertices $v \in V \setminus \bigcup_{j=1}^{l} S_j$ such that v has no neighbours in S_i must be placed into some type 1 colour class in any extension to a good colouring. In particular, they need to be candidates of at least one colour class of type 1. We call such vertices v *necessary candidates*.

Let $D_1 \subseteq D_{c'}$ be the set of defective colour classes in c' that are of type 1 in c. Also let N be the set of necessary candidates in c' and C the vertex cover given by the vertices coloured in c'. Remember that a suitable vertex cover C must satisfy $|C| - k \leq k'$, and so if $|C| + |D_1| - k > k'$ there is no way of properly extending c', since by Claim 5.2.2 we need to place one candidate in each of the $|D_1|$ defective colour classes of type 1. The number of colour classes in c is at most k, and after a candidate is placed in a defective colour class, the colour class is no longer defective. So, since the type 1 colour classes have at least k candidates in c and because of Claim 5.2.1, there are at least $|D_{c'}|$ candidates for each of the $|D_1|$ defective colour classes of type 1 in c. As a consequence, there are enough candidates to place in each colour class in D_1. But we also have to ensure that every necessary candidate is placed in a defective colour class. This is equivalent to finding a matching in the bipartite graph H with vertex set $V(H) = D_1 \cup N$ and with edge set $E(H) = \{(i \in D_1, v \in N) \mid \text{vertex } v \text{ is an } i\text{-candidate}\}$ such that every vertex in N is saturated. This can be done in time $O\left((|V(H)| + |E(H)|)\sqrt{|V(H)|}\right) = O(k^{5/2})$ by the algorithm of Hopcroft and Karp [68].

If such a matching does not exist, then we cannot properly extend c' by adding candidates to the vertices in D_1, and so we may reject c'. If such a matching exists, since each type 1 colour class has more than $|D_1|$ candidates, we can greedily extend c' to a good colouring.

Hence one can check in time $O(|E| + k^{k+5/2})$ if a proper colouring of $G[C_{min}]$ can be extended into a good greedy colouring of G. □

We are now able to prove Theorem 5.2.1.

Proof of Theorem 5.2.1. Let G be an instance of the problem. To answer the question, we enumerate all minimal vertex covers of $\overline{G}$, and check, for each one, if it is contained in a suitable vertex cover. To enumerate all minimal vertex covers takes time $O(2^{2k})$. For each of these at most 2^{2k} minimal vertex covers, we check if it is contained in a suitable vertex cover. By Lemma 5.2.2, it can be done in $O\left(k^{2k} \cdot |E| + k^{3k+5/2}\right)$. The total running time is $O\left((2k)^{2k}|E| + 2^{2k}k^{3k+5/2}\right)$. □

5.2.2 Dual of b-Colouring

The main result of this section is the following Theorem, which implies that DUAL OF b-COLOURING is FPT.

Theorem 5.2.2. DUAL OF b-COLOURING *can be solved in time* $O\left(2^{2k}3^k k^{2k}(k+1)!|V(G)|\right)$.

The proof is very similar to the one of the previous section. We first state a lemma analogous to Lemma 5.2.1.

Lemma 5.2.3. *Let $G = (V, E)$ be a graph and $k \geq 0$ an integer. Then, $\chi_b(G) \geq |V| - k$ if and only if there is a vertex cover C of $\overline{G}$ such that $G[C]$ admits a b-colouring $(C_1, C_2, \ldots, C_{k'})$ with the following properties:*

P1*:* $|C| - k \leq k' \leq k$*;*

P2*:* $|C_i| \geq 2$*, for every* $1 \leq i \leq k'$*;*

P3*: For every* $1 \leq i \leq k'$ *there is a b-vertex* $v_i \in C_i$ *such that* $uv_i \in E$ *for all* $u \in V \backslash C$.

Proof. (⇒) Assume that $\chi_b(G) \geq |V| - k$ and consider a b-colouring c with $\chi_b(G)$ colours. Let C be the set of vertices that are in a colour class with more than one vertex and $V \backslash C$ is the set of vertices that are alone in their colour classes.

We claim that $V \backslash C$ is a clique in G. If this is not the case, let u and v be two non-adjacent vertices in $V \setminus C$. Since c is a b-colouring and u and v are alone in their colour classes, they must be b-vertices. As a consequence, u should have a neighbour coloured $c(v)$ and therefore this neighbour must be v, a contradiction. Since $V \backslash C$ is a clique in G, it is a stable set in $\overline{G}$. Consequently, C is a vertex cover in $\overline{G}$.

Let c' be the b-colouring $(C_1, C_2, \ldots, C_{k'})$ of $G[C]$ induced by c. By definition of C, we have $|C_i| \geq 2$, for every $1 \leq i \leq k'$, and therefore Property P2 is satisfied. Moreover, $\chi_b(G) = |V| - |C| + k'$, and since $\chi_b(G) \geq |V| - k$, we obtain that $k' \geq |C| - k$. Because of property P2, $|C| \geq 2k'$, and again since $\chi_b(G) \geq |V| - k$, then $k' \leq k$ and so Property P1 is satisfied.

Finally, let v be a b-vertex from Ci, $1 \leq i \leq k'$. Then, for every $u \in V \backslash C$, u is the only vertex in its colour class, so v must be adjacent to u. Therefore, c' has Property P3.

($\Leftarrow$) Let C be a vertex cover of $\overline{G}$ such that there is a b-colouring $c' = \left(C_1, C_2, \ldots, C_{k'}\right)$ of $G[C]$ having Properties P1, P2 and P3. One can extend c' to the entire graph G by assigning $|V| - |C|$ new distinct colours to the vertices of $V \backslash C$. As a consequence of P3 and the fact that $V \backslash C$ is a stable set in $\overline{G}$ and therefore a clique in G, the obtained colouring is a b-colouring. Because of P1, it uses $k' + |V| - |C| \geq (|C| - k) + |V| - |C| = |V| - k$ colours. □

Let C be a vertex cover of $\overline{G}$. We say that a b-colouring $(C_1, C_2, \ldots, C_{k'})$ of $G[C]$ having the Properties P1, P2 and P3 of Lemma 5.2.3 is *good*. C is *suitable* if $G[C]$ has a good b-colouring. Property P1 implies that a suitable vertex cover has cardinality at most $2k$.

We now prove the analogue of Proposition 5.2.1.

Proposition 5.2.2. *Let C be a suitable vertex cover of $\overline{G}$, $(C_1, C_2, \ldots, C_{k'})$ a good b-colouring of $G[C]$ and $C_{min} \subseteq C$ a minimal vertex cover. Then for all $1 \leq i \leq k'$, $|C_i \setminus C_{min}| \leq 1$.*

Proof. Each colour class C_i, $1 \leq i \leq k'$, is a stable set of size at least 2 in G. So it is a clique of size at least 2 in $\overline{G}$. Since C_{min} is a vertex cover in $\overline{G}$, $|C_i \cap C_{min}| \geq |C_i| - 1$, so $|C_i \setminus C_{min}| \leq 1$. □

Then in order to determine if $\chi_b(G) \geq |V| - k$ one may enumerate all minimal vertex covers of $\overline{G}$ with size at most $2k$ and verify, for each such vertex cover, if it is suitable or if it may be extended to a suitable vertex cover.

Lemma 5.2.4. *Let k be an integer, $G = (V, E)$ a graph and C_{min} a minimal vertex cover of $\overline{G}$ of size at most $2k$. It can be determined in time $O\left(3^k k^{2k} (k+1)! |V(G)|\right)$ if C_{min} is contained in a suitable vertex cover C.*

Proof. In order to determine if C_{min} is contained in a suitable vertex cover, we first enumerate all possible proper colourings of $G[C_{min}]$ with k' colours, $|C_{min}| - k \leq k' \leq k$. For each of them, we check in time $O\left(3^k (k+1)! |V(G)|\right)$ if it can be extended into a good b-colouring of a suitable vertex cover. There are at most $k^{|C_{min}|} \leq k^{2k}$ proper colourings of C_{min} with at most k colours and they can be enumerated in time $O\left(k^{2k}\right)$. Hence our algorithm runs in time $O\left(3^k k^{2k} (k+1)! |V(G)|\right)$.

Let us now detail an algorithm that, for a proper colouring $c = (C_1, C_2, \ldots, C_{k'})$ of $G[C_{min}]$, decides if it can be extended into a good b-colouring of a suitable vertex cover in FPT time. If c is a good b-colouring of C_{min} then we are done. Otherwise, we need to check if we can extend c to vertices in $V \backslash C_{min}$ and obtain a good b-colouring. By Proposition 5.2.2, for such an extension at most one vertex of $V \setminus C_{min}$ is added in each colour class.

In order to check if c can be extended to a good b-colouring, we must guess for each colour which vertex will be a b-vertex in such an extension. For each colour i such a b-vertex may either already be in C_i or be in $V \setminus C_{min}$. Hence we try all possible sets $B \subset C_{min}$ such that $|B \cap C_i| \leq 1$ for all $1 \leq i \leq k'$ and check if one can extend c into a good b-colouring *highlighting* B, such that all vertices of B are b-vertices. The number of possible choices of B is given by $p = \prod_{i=1}^{k'} (|C_i| + 1)$. Since $\sum_{i=1}^{k'} |C_i| = n$ and p is maximized when the $|C_i|$ are all equal, we have $p \leq \left(\frac{n + k'}{k'}\right)^{k'} \leq \left(1 + \frac{n}{k'}\right)^{k'}$. As $f(x) = \left(1 + \frac{n}{x}\right)^x$ is an increasing function, we get that $\left(1 + \frac{n}{k'}\right)^{k'} \leq \left(1 + \frac{n}{k}\right)^k \leq \left(1 + \frac{2k}{k}\right)^k = 3^k$.

For all i such that $C_i \cap B \neq \emptyset$, let F_i be the set of vertices of $V \backslash C_{min}$ which are not adjacent to the vertex b_i of $C_i \cap B$. In any extension of c to a good b-colouring highlighting B, all the vertices in

F_i are to be moved to one of the colour classes because of (P3). The vertices of $F = \bigcup_{\{i|b_i \neq \emptyset\}} F_i$ are called *forced* vertices. If $|F| > k'$, c cannot be extended to a good b-colouring because at most one vertex of $V \setminus C_{min}$ may be added to each colour by Proposition 5.2.2. Hence we consider the case when $|F| \leq k'$. We enumerate all possible ways of moving the forced vertices into different colour classes, every forced vertex v being moved to a colour class C such that $\{v\} \cup C$ is stable. There are at most $k'! \leq k!$ such possibilities.

Let c^* be one of the so obtained colourings of $C^* = C \cup F$. If a vertex v is moved to a colour class C_i such that $C_i \cap B = \emptyset$, then it must be the b-vertex of this class because at most one vertex can be added to each colour class. Hence we have to look for a good b-colouring extending c^* and highlighting B^* the set containing all the vertices of B and the forced vertices moved to colour classes not intersecting B.

A colour class S_i is *defective* with respect to a colouring $s = (S_1, \ldots, S_l)$ of $S \subseteq V$ and a set $B \subset S$ if at least one of the following holds:

(i) $B \cap S_i = \emptyset$;

(ii) $|S_i| < 2$;

(iii) There is a vertex $b \in B \setminus S_i$ with no neighbour in S_i;

Let S_i be a defective colour class with respect to (s, B). An *i-candidate* with respect to (s, B) is a vertex $v \in V \setminus S$ such that $S_i \cup \{v\}$ is a stable set which is not defective with respect to $(S_1, \ldots, S_{i-1}, S_i \cup \{v\}, S_{i+1}, \ldots, S_l), B_i)$ where $B_i = B$ if $B \cap S_i \neq \emptyset$ and $B_i = B \cup \{v\}$ otherwise. We denote by $X_{s,B}(i)$ the set of i-candidates with respect to (s, B) and $D_{s,B}$ the set of defective colour classes with respect to (s, B).

Claim 5.2.3. *Let $C \supset C^*$, s be a k'-colouring of C, B a subset of S with at most one vertex in each colour class, i a defective colour class with respect to (s, B) and v an i-candidate. Let s' be the extension of s where we place v in colour class i. Then, for every colour class $j \neq i$, $X_{s',B_i}(j) = X_{s,B}(j) \backslash \{v\}$.*

Proof. If j is not defective with respect to (s, B), then it cannot become defective with repect to (s', B_i), since the insertion of v in colour i cannot cause j to satisfy one of the conditions (i), (ii) or (iii).

Then, we assume that j is defective in s. To see that $X_{s,B}(j) \backslash \{v\} \subseteq X_{s',B_i}(j)$, we have again that (i) (ii) and (iii) remain unchanged after the insertion of v, in the sense that if a vertex different from v is a j-candidate with respect to (s, B) because of one of these conditions, the same will happen with respect to (s', B_i).

Finally, it follows directly from the definition that every vertex in $X_{s',B_i}(j)$ is also in $X_{s,B}(j) \backslash \{v\}$. $\square$

In particular, Claim 5.2.1 shows that the insertion of a candidate in a defective colour class does not create a new defective colour class.

Claim 5.2.4. *Let $s = (S_1, \ldots, S_{k'})$, be a k'-colouring of C^* and assume it can be extended into a good b-colouring $s' = (S'_1, \ldots, S'_{k'})$ highlighting B^*. If j is a defective colour class in s then $|S'_j \backslash S_j| = 1$.*

Proof. If j satisfies (i) for (s, B^*), then S'_j contains the b-vertex of colour j which is not in S_j. If j satisfies (ii), then since $|S'_j| \geq 2$ by Property P2, $|S'_j| > |S_j|$. If j satisfies (iii) then a vertex $b \in B \setminus S_j$ has no neighbours in S_j. But S'_j contains a neighbour of b since it highlights B. Again $|S'_j| > |S_j|$.

In all cases S'_j has more vertices than S_j. So by Proposition 5.2.2, $|S'_j \setminus S_j| = 1$.

□

Recall that if C' is suitable it must satisfy $|C'| - k \leq k'$, and so if $|C^*| + |D_{c^*,B^*}| - k > k'$ there is no way extend c^* into a good b-colouring. The number of colour classes in c^* is at most k', and after a candidate is placed in a defective colour class, the colour class is no longer defective. As a consequence, to decide if c^* can be extended to a good b-colouring highlighting B^* is equivalent to find a matching in the bipartite graph H with vertex set $V(H) = D_{c^*,B^*} \cup \{v \in X_{c^*,B^*}(i),\ 1 \leq i \leq k'\}$ and with edge set $E(H) = \{(i \in D_{c^*,B^*}, v \in V \setminus C^*) \mid \text{vertex } v \text{ is an } i\text{-candidate}\}$ such that every vertex in D_{c^*,B^*} is saturated. This can be done in time $O\left((|V(H)| + |E(H)|)\sqrt{|V(H)|}\right) = O\left(k|V(G)|^{3/2}\right)$, since $|V(H)| = O(|V(G)|)$ and $|E(H)| = O(k|V(H)|) = O(k|V(G)|)$ and because of the algorithm of Hopcroft and Karp [68].

If such a matching does not exist, then we cannot properly extend c^* by adding candidates to the colour classes, and we must reject c. If such a matching exists, we can move each saturated candidate to the matched defective colour class to obtain a good b-colouring highlighting B^*.

Hence one can check in time $O\left(3^k (k+1)! |V(G)|\right)$ if a proper colouring of $G[C_{min}]$ can be extended into a good b-colouring of G.

□

Proof of Theorem 5.2.2. Let G be an instance of the problem. To answer the question, we enumerate all minimal vertex covers of $\overline{G}$, and check, for each one, if it is contained in a suitable vertex cover. To enumerate all minimal vertex covers takes time $O(2^{2k})$. For each of these at most 2^{2k} minimal vertex covers, we check if it is contained in a suitable vertex cover. By Lemma 5.2.4, it can be done in time $O\left(3^k k^{2k} (k+1)! |V(G)|\right)$. Hence the total running time is $O\left(2^{2k} 3^k k^{2k} (k+1)! |V(G)|\right)$.

□

5.3 Parameterizations based on the maximum degree

5.3.1 Greedy colourings with $\Delta(G) + 1$ colours

Consider the problem of deciding, given a graph G, if $\Gamma(G) = \Delta(G) + 1$. This problem is NP-complete even for a bipartite or a chordal graph, as it was shown in Theorems 3.2.1 and 3.3.1, respectively. On the other hand, a consequence of Corollary 2.3.2 is that if $\Delta(G)$ is bounded by a constant then the problem is solvable in polynomial time. A natural question that arises is the fixed parameter complexity of the following problem.

> $(\Delta + 1)$-GREEDY-COL
> INPUT : A graph G.
> PARAMETER : $\Delta(G)$
> OUTPUT : $\Gamma(G) = \Delta(G) + 1$?

Theorem 5.3.1. $(\Delta + 1)$-GREEDY-COL *can be solved in FPT time.*

Proof. Set $\Delta = \Delta(G)$. By Theorem 2.3.4, $\Gamma(G) = \Delta + 1$ if and only if G contains a $(\Delta + 1)$-atom. From the definition of an atom, we know that there is a finite number, say $f(\Delta)$, of $(\Delta + 1)$-atoms. Moreover, a $(\Delta + 1)$-atom has at most 2^{Δ} vertices.

Now consider a greedy colouring c of G with $\Delta + 1$ colours and let v be a vertex coloured $\Delta + 1$. Clearly, $d(v) = \Delta$. Now, observe that since $c(v) = \Delta + 1$, there is a $(\Delta + 1)$-atom such that all its vertices are contained in $N^{\Delta}[v]$, where $N^1[v] = \{v\} \cup N(v)$ and $N^i[v] = N^{i-1}[v] \cup N(N^{i-1}[v])$. Moreover, since Δ is the maximum degree of the graph, $|N^{\Delta}[v]| \leq 1+\Delta+\Delta^2+\ldots+\Delta^{\Delta} = O(\Delta^{\Delta})$.

Hence checking the existence of a $(\Delta + 1)$-atom in $N^{\Delta}[v]$ can be done by brute force in time $g(\Delta) = O(f(\Delta)(\Delta^{2^{\Delta}\Delta}))$.

To decide if $\Gamma(G) = \Delta + 1$, it suffices to consider each vertex v of degree Δ and check for the existence of a $(\Delta + 1)$-atom in $N^{\Delta}[v]$, what by the previous remarks can be done in time $O(g(\Delta) \cdot n)$. □

5.3.2 b-colourings with $\Delta(G) + 1$ colours

The problem of deciding if $\chi_b(G) = \Delta(G) + 1$, given a graph G, is NP-complete even for a chordal graph, as shown in Theorem 4.2.1. On the other hand, it is not evident if the problem is solvable in polynomial time or not if we assume that $\Delta(G)$ is bounded. We show that this problem is polynomial and FPT, by giving a polynomial kernel to it.

$(\Delta + 1)$-b-COL
INPUT : A graph G.
PARAMETER : $\Delta(G)$
OUTPUT : $\chi_b(G) = \Delta(G) + 1$?

Theorem 5.3.2. $(\Delta + 1)$-b-COL *admits a kernel of size* $\Delta^5(G)$.

Proof. Let D be the set of vertices with degree equal to $\Delta(G)$. Clearly, in any b-colouring of G with $\Delta(G) + 1$ colours, all the b-vertices are from D, and therefore if $|D| < \Delta(G) + 1$, the answer is no.

If there is a set B of vertices of degree $\Delta(G)$ such that $B = \{v_1, v_2, \ldots, v_{d+1}\}$ and for every $u, v \in B$, $dist(u, v) \geq 4$, we can make $c(v_i) = i$ and colour the neighbours of v_i with distinct colours. This partial colouring is proper, because of the distance constraint, and v_i is the b-vertex of colour i. Now, since there are $\Delta(G) + 1$ available colours and every vertex has degree at most $\Delta(G)$, we may easily extend this partial colouring by giving to every vertex one colour that does not appear in its neighbourhood.

We claim that if $|D| \geq \Delta^4(G)$, there is a set B as in the last paragraph. B may be obtained as follows. First choose an arbitrary vertex v_1 from D and then remove the vertices that are at distance at most 3 from it. The number of removed vertices from D that are removed is at most $1 + d + d(d-1) + d(d-1)(d-1) < d^3$. We then proceed in the same way on the resulting graph in order to choose v_2, and so on to choose $v_3, \ldots, v_d$. Since the number of removed vertices that belong to D altogether is smaller than d^4, there remains at least one vertex that may be chosen as v_{d+1} and the claim follows.

As a consequence, if $|D| \geq \Delta^4(G)$, then $\chi_b(G) = \Delta(G) + 1$, and the answer to the problem is yes.

Now it remains to consider the case $\Delta(G)+1 \leq |D| < \Delta^4(G)$. If there is a b-colouring of G with $\Delta(G)+1$ colours, then clearly there is a set $B \subseteq D$, $|B| = \Delta(G)+1$, such that the graph induced by $B \cup N(B)$ admits a b-colouring with $\Delta(G)+1$ colours. The converse is also true, if $B \cup N(B)$ admits a b-colouring with $\Delta(G) + 1$ colours, this colouring can be easily extended to the rest of the graph since all vertices have degree at most $\Delta(G)$ and there are $\Delta(G) + 1$ available colours. Therefore, in order to solve $(\Delta+1)-b-$Col on G it suffices to solve it on the graph $D \cup N(D)$, what gives the kernel of size $\Delta^5(G)$.

□

5.4 Open problems

We now discuss some open problems concerning the fixed-parameter-complexity of problems associated with the Grundy and b-chromatic numbers.

Problem 5.1. Is the following problem FPT?

> Greedy Colouring
> INPUT : A graph G and an integer k.
> PARAMETER : k.
> OUTPUT : $\Gamma(G) \geq k$?

The last problem is in XP, that is for fixed k it can be solved in polynomial time, as it was shown in Corollary 2.3.2. However, this algorithm runs in time $O(n^{2^{k-1}})$ and therefore it is not fixed-parameter-tractable. We believe that finding a faster algorithm, for example in time $O(|V(G)|^{k^c})$ with c a constant, would already be interesting.

The analogue of Problem 5.1 for b-colourings is also open.

Problem 5.2. Is the following problem FPT?

> b-Chromatic
> INPUT : A graph G and an integer k.
> PARAMETER : k.
> OUTPUT : $\chi_b(G) \geq k$?

We now consider the following problem and prove it is not in XP.

> b-Colouring
> INPUT : A graph G and an integer k.
> PARAMETER : k.
> OUTPUT : Does G admits a b-colouring with k colours ?

Proposition 5.4.1. *Given a graph G and a fixed integer $k \geq 3$, deciding if G admits a b-colouring with k colours is an NP-complete problem.*

Proof. The problem is clearly in NP. We first show the NP-hardness of the case $k = 3$. In order to do so, we give a reduction from the problem of deciding if a 4-regular graph G admits a 3-colouring, that is known to be NP-complete [28]. Let G be an instance of this problem. Let G' be the graph obtained from G by adding 3 copies of $K_{1,2}$. If G admits a proper 3-colouring c, one can obtain a b-colouring of G' with 3 colours by colouring the copies of $K_{1,2}$ in a way that each of the 3 colours has a b-vertex and then colouring the component isomorphic to G using c. The converse is trivial, since a b-colouring with 3 colourings is by definition a proper colouring. Therefore G admits a 3-colouring if and only if G' admits a b-colouring with 3 colours, and the reduction is completed.

The case when $k > 3$ can be reduced to the case $k = 3$. To see this, observe that a graph G admits a b-colouring with 3 colours if and only if the graph $H_{k'} = G \oplus K_{k'-3}$, $k' \geq 1$, admits a b-colouring with k colours, where $k = 3 + k'$. □

Although b-COLOURING is not in XP, it is not clear if there is an equivalence between this and the b-CHROMATIC problem. It is not obvious how being able to decide if a graph G satisfies $\chi_b(G) \geq k$ can help in deciding the existence of a b-colouring with k colours. Therefore it is not even known if b-CHROMATIC belongs to XP.

Problem 5.3. Is the following problem polynomial?

> k-b-CHROMATIC
> INPUT : A graph G.
> OUTPUT : $\chi_b(G) \geq k$?

On the other hand, b-COLOURING can be show to be in XP for any class of graphs in which PRECOLOURING EXTENSION is in XP. Given a graph G from such a class of graphs, in order to solve b-COLOURING one need first to consider all possible sets of k vertices with degree at least $k-1$, since these are the only vertices that could be b-vertices in a k-colouring of G. Clearly, there are at most $O(n^k)$ possibilities for this first choice. Let $v_1, v_2, \ldots, v_k$ be one such set of vertices. Then, given a vertex v_i, we need to consider the $O\left(\binom{n}{k-1}\right)$ subsets of size $k-1$ from its neighbours and enumerate all the possible $O((k-1)!)$ colourings with colours $\{1, 2, \ldots, i-1, i+1, \ldots k\}$ for each subset. As a consequence, there are $O\left((k-1)!\binom{n}{k-1}\right)^k$ colourings to be considered in this step. Finally, for each of the $O\left(n^k(k-1)!(\binom{n}{k-1}))^k\right) = O(n^{k+k^k})$ possible partial b-colourings, we can use the algorithm for PRECOLOURING EXTENSION and decide if the colouring can be extended to the rest of the graph, what under our assumptions can be done in polynomial time.

In particular the remark of the last paragraph implies that b-COLOURING is in XP for chordal graphs, since Marx [101] proved that PRECOLOURING EXTENSION can be solved in time $O(kn^{k+2})$ for a chordal graph. In the same paper he proves that PRECOLOURING EXTENSION parameterized by k is a W[1]-hard. Therefore a natural question is the following.

Problem 5.4. Is b-COLOURING FPT for chordal graphs?

Telle showed that DUAL OF COLOURING is FPT by showing that it has a quadratic kernel. It is well-known that every FPT problem has a kernel, but it is not necessarily a polynomial one. Then, other natural questions in view of Theorems 5.2.1 and 5.2.2 are the following.

Problem 5.5. Does DUAL OF GREEDY COLOURING have a polynomial kernel?

Problem 5.6. Does DUAL OF b-COLOURING have a polynomial kernel?

Chapter 6

Relation between the colouring parameters

In this Chapter we discuss the complexity of comparing the different colouring parameters that were presented in Chapter 2 for a given graph. For a fixed value $c \geq 1$, we want to determine the complexity of the problem of deciding if $\phi(G) \leq c\psi(G)$, where $\phi(G), \psi(G) \in \{\omega(G)$, $\chi(G)$, $\chi_b(G)$, $\Gamma(G)$, $\partial\Gamma(G)$, $\zeta(G)$, $\Delta(G) + 1\}$. In Table 6 our results are summarized. This table should be read as follows. At the cell with row $\phi(G)$ and column $\psi(G)$, the complexity of the problem of deciding if $\phi(G) \leq c\psi(G)$ is given. The abbreviations have the following meaning:

(=): $\phi(G) = \psi(G)$.

(≤): the problem is trivial because $\phi(G) \leq \psi(G)$;

(NPC): for every $c \geq 1$, the problem is NP-complete;

(=-NPC): for $c = 1$ the problem is NP-complete;

(>-NPC): for $c > 1$ the problem is NP-complete;

(coNPC): for every $c \geq 1$, the problem is co-NP-complete;

(=-coNPC): for $c = 1$ the problem is co-NP-complete;

(>-coNPC): for $c > 1$ the problem is co-NP-complete;

(NPH): for every $c \geq 1$, the problem is NP-hard;

(=-NPH): for $c = 1$ the problem is NP-hard;

(>-NPH): for $c > 1$ the problem is NP-hard;

(Poly): the problem is polynomial-time solvable.

In our proofs we use some of the reductions that were presented in the previous chapters. When doing so we use the following notation:

	$\omega(G)$	$\chi(G)$	$\chi_b(G)$	$\Gamma(G)$	$\partial\Gamma(G)$	$\zeta(G)$	$\Delta(G)+1$
$\omega(G)$	=	$\leq$	$\leq$	$\leq$	$\leq$	$\leq$	$\leq$
$\chi(G)$	NPC	=	$\leq$	$\leq$	$\leq$	$\leq$	$\leq$
$\chi_b(G)$	NPH	coNPC	=	NPH	$\leq$	$\leq$	$\leq$
$\Gamma(G)$	NPH	coNPC	NPH	=	$\leq$	$\leq$	$\leq$
$\partial\Gamma(G)$	Unknown	$>$-NPH	=-NPH	NPH	=	$\leq$	$\leq$
$\zeta(G)$	Unknown	$>$-NPH	=-NPC	NPC	=-NPC	=	$\leq$
$\Delta(G)+1$	Unknown	$>$-coNPC [1]	NPC	NPC	NPC	Poly	=

Table 6.1: Complexity of deciding equality between the parameters.

Reduction 1 **($\mathcal{B}(G)$):** Let G be a 3-regular graph with n vertices. Set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $E(G) = \{e_1, \ldots, e_m\}$. Let $\mathcal{I}(G)$ be the vertex-edge incidence graph of G, that is the bipartite graph with vertex set $V(\mathcal{I}(G)) = V(G) \cup E(G)$ in which an edge of G is adjacent to its two end-vertices.

The graph $\mathcal{B}(G)$ is constructed from $\mathcal{I}(G)$ as follows. First, we add an edge between every pair of vertices in $V(G)$ and then, we add three disjoint copies of $K_{1,n+2}$. We denote the vertices of degree $n+2$ in each copy of $K_{1,n+2}$ by v_{n+1}, v_{n+2} and v_{n+3}. One can easily see that $d_{\mathcal{B}(G)}(v) = n - 1 + 3 = n + 2$, for $v \in V(G)$, and that $d_{\mathcal{B}(G)}(u) = 2$, for $u \in E(G)$. Moreover, each copy of $K_{1,n+2}$ has exactly one vertex with degree equal to $n+2$. Consequently, $m(\mathcal{B}(G)) = n + 3$.

As a consequence of the proof of Theorem 4.2.1, a 3-regular graph G is 3-edge-colourable if and only if $\chi_b(\mathcal{B}(G)) = m(\mathcal{B}(G)) = n + 3 = \Delta(\mathcal{B}(G)) + 1$.

Reduction 2 **($\mathcal{G}(G)$):** Let G be a 3-regular graph with $n-4$ vertices. Set $V(G) = \{v_4, \ldots, v_n\}$ and $E(G) = \{e_1, \ldots, e_m\}$. Let $M_{p,p}$ denote the graph obtained from the complete bipartite graph $K_{p,p}$ by removing a perfect matching.

The bipartite graph $\mathcal{G}(G)$ is constructed from $\mathcal{I}(G)$ as follows. For each vertex $e_i \in E(G)$, we add a copy $M_{3,3}(e_i)$ of $M_{3,3}$ and identify one of its vertices with e_i. We add a new vertex w adjacent to all the vertices of $V(G)$. We add copies $M^w_{1,1}$, $M^w_{2,2}$, $M^w_{3,3}$, $M^w_{n+1,n+1}$ of K_1, K_2, $M_{3,3}$, $M_{n+1,n+1}$ and we choose arbitrary vertices v_1, v_2, v_3, v_{n+1} respectively from each copy and add the edges v_1w, v_2w, v_3w, $v_{n+1}w$. Finally, for every $5 \leq i \leq n$, we do the following: for every $4 \leq j \leq i-1$, we add a copy $M^i_{j,j}$ of $M_{j,j}$, choose an arbitrary vertex v^i_j of it and add the edge $v_iv^i_j$.

As a consequence of the proof of Theorem 3.2.1, a 3-regular graph G is 3-edge-colourable if and only if $\Gamma(\mathcal{G}(G)) = \Delta(\mathcal{G}(G)) + 1 = n + 2$.

6.1 The complexity of comparing two parameters

$\chi(G)$ and $\omega(G)$

For any fixed integer $c \geq 1$, we will prove that the problem of deciding if a given graph G satisfies $\chi(G) \leq c\omega(G)$ is NP-complete. To see it is in NP, it suffices to observe that a proper colouring with

[1] For $c = 1$ the problem can be solved in polynomial time.

χ' colours and a clique with ω' vertices such that $\chi' \leq c\omega'$ is a proof that $\chi(G) \leq c\omega(G)$, since $\chi(G) \leq \chi' \leq c\omega' \leq c\omega(G)$.

In what follows we prove the NP-hardness of the problem. A graph is *triangle-free* if it does not contains K_3 as a subgraph. As a consequence of this definition, a triangle-free graph has clique number at most 2. Now consider the following problem.

> k-TRIANGLE-FREE COLOURABLITY
> INPUT : A triangle-free graph G.
> OUTPUT : Is $\chi(G) \leq k$?

The following proof is directly adapted from the one of Kral' et al. [89] for $k = 3$.

Theorem 6.1.1. *For any $k \geq 3$, k-TRIANGLE-FREE COLOURABILITY is NP-complete.*

Proof. The problem is clearly in NP since any k-colouring of G is a certificate that $\chi(G) \leq k$.

To prove that it is NP-complete we give a reduction of k-COLOURABILITY. The case $k = 3$ is well known to be NP-complete (See [46]), while the case $k > 3$ can be easily reduced to it.

Let G be a graph. We remind the reader that a graph is *k-critical* if it is not k-colourable but all its proper subgraphs are k-colourable. The existence of triangle-free graphs with arbitrarily large chromatic number is a well known result in graph theory [34]. Let H be a triangle-free *k-critical* graph, whose existence is assured by the previous observation. Take an edge ab of H, delete it and add a new extra vertex a' and make it adjacent to b. Call this new graph $H(a, a')$. Since H was triangle-free, so does $H(a, a')$ and also every path from a to a' has length at least 3. Since H was k-critical, $H(a, a')$ is k-colourable, but in every k-colouring, a and b have the same colour and so a and a' have different colours. Return to G and construct a graph G' by replacing every edge uv by a copy of $H(u, v)$. Thus G' is triangle-free and it is k-colourable if and only if G is k-colourable.

Hence k-TRIANGLE-FREE COLOURABLITY is NP-complete. □

Now, we prove that in deciding if $\chi(G) = \omega(G)$, given a graph G, is NP-hard. Let G be an instance of k-TRIANGLE-FREE COLOURABLITY, for a fixed $k \geq 3$, and let H be the graph obtained from G by adding a disjoint copy of K_k. Clearly, $\chi(H) \geq \omega(H) = k$, and $\chi(H) = k$ if an only if G is k-colourable. But then, $\chi(H) = \omega(H)$ if and only if G is k-colourable, and we get the reduction.

For a fixed integer $c \geq 2$, it suffices to observe that if G is a triangle-free graph, then $\chi(G) \leq c\omega(G)$ if and only if G is $(2c)$-colourable. Therefore, deciding if $\chi(G) \leq c\omega(G)$ is an NP-hard problem.

$\chi_b(G)$ and $\omega(G)$

We now show, for a fixed $c \geq 1$, that the problem of deciding if a given graph G satisfies $\chi_b(G) \leq c\omega(G)$ is NP-hard. The reduction is from the problem of deciding if a bipartite graph G satisfies $\chi_b(G) = \Delta(G) + 1$, which was shown to be NP-complete in [90]. Let G be an instance of the later problem, and G' be the graph obtained by adding a disjoint copy of $K_{\lfloor \frac{\Delta(G)}{c} \rfloor}$. Clearly, $\Delta(G') = \Delta(G)$, and since G is bipartite, $\omega(G') = \lfloor \frac{\Delta(G)}{c} \rfloor$. Moreover, if $\chi_b(G) = \Delta(G)+1$ then $\chi_b(G') = \Delta(G')+1$, since any b-colouring of G with $\Delta(G) + 1$ colours can be easily extended to the vertices in the component isomorphic to $K_{\lfloor \frac{\Delta(G)}{c} \rfloor}$ in G'. The converse is also true. If there is a b-colouring of G' with $\Delta(G')+1$ colours, then no b-vertex in that colouring belongs to the copy of $K_{\lfloor \frac{\Delta(G)}{c} \rfloor}$, since these vertices have degree at most $\Delta(G)-1$. As a consequence, the restriction of the colouring to G remains a b-colouring with $\Delta(G)+1$ colours. Therefore $\chi_b(G) = \Delta(G)+1$ if and only if $\chi_b(G') = \Delta(G')+1$.

Now it suffices to observe that $\chi_b(G) \leq c\omega(G)$ if and only if $\chi_b(G) \leq c\lfloor\frac{\Delta(G)}{c}\rfloor < \Delta(G) + 1$. Since the b-chromatic number is an integer value, this completes the reduction.

$\chi_b(G)$ and $\chi(G)$

Let $c \geq 1$ be a fixed integer. We show that the problem of deciding if a given a graph G satisfies $\chi_b(G) \leq c\chi(G)$ is co-NP-complete. To see it is in co-NP, observe that a b-colouring with χ'_b colours and a proper colouring with χ' colours satisfying $\chi'_b > c\chi'$ is a certificate that $\chi_b(G) > c\chi(G)$, since $\chi_b(G) \geq \chi'_b > c\chi' \geq c\chi(G)$.

It remains to prove that the problem is NP-hard. In [90] it is proved that given a bipartite graph G, the problem of deciding if $\chi_b(G) = \Delta(G) + 1$ is NP-complete. Let G be an instance of this problem. Moreover, let G' be the graph obtained from G by adding a disjoint copy of K_d, where $d = \lfloor\frac{\Delta(G)}{c}\rfloor$. Since G is bipartite, then clearly $\chi(G') = d$. Moreover, $\chi_b(G') = \Delta(G) + 1$ if and only if $\chi_b(G) = \Delta(G) + 1$, because a vertex in the copy of K_d cannot be a b-vertex in a colouring with $\Delta(G) + 1$ colours, since it has degree at most $d - 1$, and $d \leq \Delta(G)$. If $\chi_b(G') \leq cd$, then $\chi_b(G') \leq \Delta(G)$ and so $\chi_b(G) < \Delta(G) + 1$. On the other hand, if $\chi_b(G) = \Delta(G) + 1$ then $\chi_b(G') = \Delta(G) + 1 > cd$. As a consequence $\chi_b(G') \leq c\chi(G') = cd$ if and only if $\chi_b(G) < \Delta(G) + 1$, and we get the desired result.

$\chi_b(G)$ and $\Gamma(G)$

For any fixed integer $c \geq 1$, deciding if $\chi_b(G) \leq c\Gamma(G)$ for a given a bipartite graph G is an NP-hard problem. In order to prove that, we give a reduction from the problem of deciding if $\Gamma(G) = \Delta(G)+1$, for a given bipartite graph G, which was proved to be NP-complete in Theorem 3.2.1. Let G be a bipartite graph. Set $r = c(\Delta(G) + 1)$. Let H be the disjoint union of G and r copies $S_1, S_2, \ldots S_r$ of the star $K_{1,r-1}$. Clearly H is bipartite. A b-colouring of H with r colours may be obtained by colouring the vertex of degree $r - 1$ from S_i with colour i, $1 \leq i \leq r$, colouring its neighbours with the remaining $r - 1$ colours, and finally colouring the component corresponding to G with 2 arbitrary colours. It is easy to see that $m(H) = r$. Therefore $\chi_b(H) = r$. Finally $\Gamma(H) = \Gamma(G)$, since the Grundy number of a disconnected graph is the maximum Grundy number of its components and $\Gamma(K_{1,r}) = 2$. Hence $\chi_b(H) \leq c\Gamma(H)$ if and only if $\Gamma(G) = \Delta(G) + 1$ and we get the desired result.

$\Gamma(G)$ and $\omega(G)$

For any fixed $c \geq 1$, we show that the problem of deciding if a given a graph G satisfies $\Gamma(G) \leq c\omega(G)$ is NP-hard.

To do so we use again Theorem 3.2.1, which states that given a bipartite graph G, the problem of deciding if $\Gamma(G) = \Delta(G) + 1$ is NP-complete. Let G be an instance of this problem and G' be obtained from G by adding a disjoint copy of K_d, where $d = \lfloor\frac{\Delta(G)}{c}\rfloor$. Because G is bipartite, $\omega(G') = d$ and so $\Gamma(G') \geq d$. Since the Grundy number of a disconnected graph is the maximum Grundy number of its components and $\Gamma(K_d) = d$, we have that $\Gamma(G') = \Delta(G') + 1$ if and only if $\Gamma(G) = \Delta(G) + 1$. If $\Gamma(G') \leq cd$, then $\Gamma(G') \leq \Delta(G)$ and so $\Gamma(G) < \Delta(G) + 1$. On the other hand, if $\Gamma(G) = \Delta(G) + 1$ then $\Gamma(G') = \Delta(G) + 1 > cd$. This shows that $\Gamma(G') \leq c\omega(G') = cd$ if and only if $\Gamma(G) < \Delta(G) + 1$, and therefore we get the desired result.

$\Gamma(G)$ and $\chi(G)$

Let $c \geq 1$ be fixed. We now show that given a graph G, the problem of deciding if $\Gamma(G) \leq c\chi(G)$ is co-NP-Complete.

To see that the problem is in co-NP observe that a certificate that $\Gamma(G) > c\chi(G)$ is a proper colouring with χ' colours and a greedy colouring with Γ' colours such that $\Gamma' > c\chi'$.

The NP-hardness proof is very similar to the one of the last one that was presented. We use again Theorem 3.2.1, which states that given a bipartite graph G, the problem of deciding if $\Gamma(G) = \Delta(G) + 1$ is NP-complete. Let G be an instance of the later problem and G' be obtained from G by adding a disjoint copy of K_d, where $d = \lfloor \frac{\Delta(G)}{c} \rfloor$. Since $\chi(G) = 2$, we have that $\chi(G') = d$. Because the Grundy number of a disconnected graph is the maximum Grundy number of its components and $\Gamma(K_d) = d$, we have that $\Gamma(G') = \Delta(G') + 1$ if and only if $\Gamma(G) = \Delta(G) + 1$. If $\Gamma(G') \leq cd$, then $\Gamma(G') \leq \Delta(G)$ and so $\Gamma(G) < \Delta(G) + 1$. On the other hand, if $\Gamma(G) = \Delta(G) + 1$ then $\Gamma(G') = \Delta(G) + 1 > cd$. This shows that $\Gamma(G') \leq c\chi(G') = cd$ if and only if $\Gamma(G) < \Delta(G) + 1$, and we get the desired result.

$\Gamma(G)$ and $\chi_b(G)$

We now prove that for a fixed $c \geq 1$, the problem of deciding if $\Gamma(G) \leq c\chi_b(G)$ for a given a graph G is NP-hard.

Let M^p be the graph obtained from the complete bipartite graph $K_{p,p}$ by removing a matching of size $p-1$. We have already shown in Proposition 2.3.8 that $\chi_b(M^p) = 2$ and in Proposition 2.4.1 that $\Gamma(M^p) = p + 1$.

Proposition 6.1.1. *Let $p \geq 2$. In any proper colouring of M^p there are at most two b-vertices with distinct colours.*

Proof. Let c be a proper colouring of M^p, A and B its parts, and $u \in A$ and $v \in B$ be its only vertices of degree p. We may assume without loss of generality that $c(u) = 1$ and $c(v) = 2$, since u and v are adjacent. To our purposes it is sufficient to show that there are no b-vertices of colours distinct from 1 and 2. Suppose by contradiction that w is a b-vertex such that $c(w) \notin \{1, 2\}$, and assume without loss of generality that $c(w) = 3$. We may assume that $w \in A$, the proof of the case in which $w \in B$ being analogous to this one. Since w is a b-vertex, it should have a neighbours with colours 1 and 2. But all its neighbours are in B, and since u is coloured 1 and is adjacent to every vertex in B, no vertex in B is coloured 1, therefore u is not a b-vertex. □

We now consider a modified version of the graph $\mathcal{B}(G)$. Replace the three stars in the definition of $\mathcal{B}(G)$ by one copy of $M^{c(n+3)-1}$ and one copy of $K_{1,n+2}$, and denote this graph by $\mathcal{B}'(G)$.

Proposition 6.1.2. *Given a 3-regular graph G, $\chi_b(\mathcal{B}(G)) = n + 3$ if and only if $\chi_b(\mathcal{B}'(G)) = n + 3$.*

Proof. If $\chi_b(\mathcal{B}(G)) = n + 3$, let c be a b-colouring of $\mathcal{B}(G)$ with $\chi_b(\mathcal{B}(G))$ colours. Let c' be the restriction of c to the vertices from $\mathcal{B}'(G)$ that also belong to $\mathcal{B}(G)$. The colouring c' is a partial colouring, as the vertices from the component isomorphic to $M^{c(n+3)-1}$ are still uncoloured. Clearly in c' there are only two colour classes without b-vertices; since the only vertices from $\mathcal{B}(G)$ that are not in $\mathcal{B}'(G)$ are the ones from the two copies of $K_{1,n+2}$. Now it suffices to see that the component $M^{c(n+3)-1}$ can be easily coloured so that we obtain the two missing b-vertices. Therefore $\chi_b(\mathcal{B}'(G)) = n + 3$

Now suppose $\chi_b(\mathcal{B}'(G)) = n + 3$ and let c be a b-colouring of $\mathcal{B}'(G)$ with $\chi_b(\mathcal{B}'(G))$ colours. In this case, Proposition 6.1.1 implies that there are at most two b-vertices in the component isomorphic to $M^{c(n+3)-1}$. Since there are only $n+1$ vertices of degree at least $n+2$ that are not in $M^{c(n+3)-1}$, then in c there are exactly two b-vertices that belong to the component isomorphic to $M^{c(n+3)-1}$. So we can restrict c to the vertices from $\mathcal{B}(G)$ that also belong to $\mathcal{B}'(G)$ and then colour the two copies of $K_{1,n+2}$ in order to make them b-vertices of the two colours that have no b-vertices yet, thus obtaining that $\chi_b(\mathcal{B}(\mathcal{G})) = n+3$. □

We are now able to show a reduction from 3-edge-colourability of 3-regular graphs, in order to show the desired result. Let G be an instance of this problem, that is a 3-regular graph. By Theorem 4.2.1, to decide if G admits a 3-edge-colouring is equivalent to deciding if $\chi_b(\mathcal{B}(G)) = n+3$. By Proposition 6.1.2, $\chi_b(\mathcal{B}(G)) = n+3$ if and only if $\chi_b(\mathcal{B}'(G)) = n+3$. Since the Grundy number of a disconnected graph is the maximum Grundy number of its components, $\Gamma(\mathcal{B}'(G)) = \Gamma(M^{c(n+3)-1}) = c(n+3)$. Now it suffices to observe that $\Gamma(\mathcal{B}'(\mathcal{G})) \leq c\chi_b(\mathcal{B}'(\mathcal{G}))$ if and only if $\chi_b(\mathcal{B}'(\mathcal{G})) \geq \frac{\Gamma(\mathcal{B}'(\mathcal{G}))}{c} = \frac{c(n+3)}{c} = n+3$, and we obtain the desired result.

$\partial\Gamma(G)$ and $\chi(G)$

Let $c \geq 2$ be a fixed integer. We now show that given a graph G, the problem of deciding if $\partial\Gamma(G) \leq c\chi(G)$ is co-NP-complete. To see that the problem belongs to co-NP, observe that a proper colouring with χ' colours and a partial greedy colouring with $\partial\Gamma'$ colours such that $\partial\Gamma' > c\chi'$ is sufficient to show that $\partial\Gamma(G) > c\chi(G)$.

It remains to show the problem is NP-hard. The reduction is from the problem of deciding if $\chi(G) \geq 4$ for a given 4-regular graph G satisfying $\chi(G) \geq 3$.

Proposition 6.1.3. *Given a 4-regular graph G such that $\chi(G) \geq 3$, the problem of deciding if $\chi(G) \geq 4$ is NP-hard.*

Proof. Dailey [28] proved that the problem of deciding if a planar 4-regular graph G admits a 3-colouring is NP-complete. Let G be an instance of the later problem. It can be decided in polynomial time if $\chi(G) = 2$, since this corresponds to verifying if G is bipartite. Therefore if we assume that $\chi(G) \geq 3$ the problem of deciding if G admits a 3-colouring remains NP-complete. In this case, it is easy to see that $\chi(G) \geq 4$ if and only if G does not admits a 3-colouring, and so the problem of deciding if a 4-regular graph G satisfies $\chi(G) \geq 4$ is NP-hard. □

Let G be a 4-regular graph such that $\chi(G) \geq 3$. Consider the graph G' obtained from G by adding disjoint copies of the stars $K_{1,2}, \dots, K_{1,4c-1}$. A partial greedy colouring of G' with $4c$ colours may be obtained by colouring the vertex of degree i in the copy of $K_{1,i}$ with colour i, colouring its neighbours with the colours $1, 2, \dots, i-1$, and then colouring the bipartite component isomorphic to G with colours 1 and 2. Since $c \geq 2$, $\Delta(G') = 4c-1$ and so $\partial\Gamma(G') = 4c$. Moreover, since $\chi(G) \geq 3$ and $\chi(K_{1,p}) = 2$, for every $p \geq 1$, we have that $\chi(G) = \chi(G')$. As a consequence, $\partial\Gamma(G') \leq c\chi(G')$ if and only if $\chi(G') \geq \frac{\partial\Gamma(G')}{c} = \frac{4c}{c} = 4$, and the reduction is completed.

$\partial\Gamma(G)$ and $\chi_b(G)$

We prove that deciding if $\partial\Gamma(G) = \chi_b(G)$ for a given a graph G is an NP-hard problem. In order to do so we consider a modified version of the graph $\mathcal{B}(G)$. Replace the three stars in the definition of $\mathcal{B}(G)$ by one copy of a $(n+3)$-binomial tree and one copy of $K_{1,n+2}$, and denote this graph by $\mathcal{B}''(G)$.

Proposition 6.1.4. *Given a 3-regular graph G, $\chi_b(\mathcal{B}(\mathcal{G})) = n+3$ if and only if $\chi_b(\mathcal{B}''(G)) = n+3$*

Proof. If $\chi_b(\mathcal{B}(G)) = n+3$, let c be a b-colouring of $\mathcal{B}(G)$ with $\chi_b(\mathcal{B}(G))$ colours. Let c' be the restriction of c to the vertices from $\mathcal{B}''(G)$ that also belong to $\mathcal{B}(G)$. The colouring c' is a partial colouring, as the vertices from the component isomorphic to the binomial tree $\mathcal{B}_{n+3}$ are still uncoloured. Denote this component by H. Clearly in c' there are only two colour classes without b-vertices; since the only vertices from $\mathcal{B}(G)$ that are not in $\mathcal{B}'(G)$ are the ones from the two copies of $K_{1,n+2}$. Now it suffices to see that the component H can be easily coloured so that we obtain the two missing b-vertices. Therefore $\chi_b(\mathcal{B}'(G)) = n+3$.

Now suppose $\chi_b(\mathcal{B}''(G)) = n+3$ and let c be a b-colouring of $\mathcal{B}''(G)$ with $\chi_b(\mathcal{B}''(G))$ colours. Since there are only two vertices in H with degree $n+2$, there are at most two b-vertices in c that belong to H. So we can restrict c to the vertices from $\mathcal{B}(G)$ that also belong to $\mathcal{B}''(G)$ and then colour the two copies of $K_{1,n+2}$ in order to make them the eventually missing b-vertices, thus obtaining that $\chi_b(\mathcal{B}(\mathcal{G})) = n+3$. □

Since $\Gamma(\mathcal{B}_{n+3}) = n+3$, we get that $\partial\Gamma(\mathcal{B}''(G) = n+3$. Given a 3-regular graph G, we already shown in Theorem 4.2.1 that the problem of deciding if G admits a 3-edge colouring can be reduced to the one of deciding if $\chi_b(\mathcal{B}(G)) = n+3$. Therefore, because of Proposition 6.1.4, if we could decide in polynomial time if $\chi_b(\mathcal{B}''(G)) = \partial\Gamma(\mathcal{B}''(G))$ then we could solve the 3-edge-colourability problem, which is NP-complete. This gives the desired result.

$\partial\Gamma(G)$ and $\Gamma(G)$

For a fixed integer $c \geq 1$, the problem of deciding if a given a bipartite graph G satisfies $\partial\Gamma(G) \leq c\Gamma(G)$ is NP-hard. In order to prove that, we give a reduction from the problem of deciding if $\Gamma(G) = \Delta(G)+1$, for a given bipartite graph G, that was proved to be NP-complete in Theorem 3.2.1. Let G be a bipartite graph and H be the disjoint union of G and the copies of the stars $K_{1,2}, \ldots, K_{1,c(\Delta(G)+1)-1}$. A partial greedy colouring of H with $c(\Delta(G)+1)$ colours may be obtained by colouring the vertex of degree i in the copy of $K_{1,i}$ with colour i, colouring its neighbours with the colours $1, 2, \ldots, i-1$, and then colouring the bipartite component isomorphic to G with colours 1 and 2. The maximum degree of H is clearly $c(\Delta(G)+1)-1$, and so $\partial\Gamma(H) = c(\Delta(G)+1)$. Moreover $\Gamma(H) = \Gamma(G)$, since the Grundy number of a disconnected graph is the maximum Grundy number of its components and $\Gamma(K_{1,p}) = 2$, for every $p \geq 1$. Hence $\partial\Gamma(G) \leq c\Gamma(H)$ if and only if $\Gamma(H) \geq \frac{\partial\Gamma(G)}{c} = \frac{c(\Delta(G)+1)}{c} = \Delta(G)+1$ and we get the desired result.

$\zeta(G)$ and $\chi(G)$

Let $c \geq 2$ be a fixed integer. We now show that given a graph G, the problem of deciding if $\zeta(G) \leq c\chi(G)$ is co-NP-complete. To see that it is in co-NP, observe that to show that $\zeta(G) > c\chi(G)$ it suffices to provide a feasible Grundy sequence of cardinality ζ' and a proper colouring with χ' colours, where $\zeta' > c\chi'$.

To prove it is NP-hard, we show a reduction from the problem of deciding if $\chi(G) \geq 4$ for a 4-regular graph G such that $\chi(G) \geq 3$ (see Proposition 6.1.3 for the NP-completeness of this problem). Let G be an instance of this problem. Consider the graph G' obtained from G by adding disjoint copies of the stars $K_{1,1}, K_{1,2}, \ldots, K_{1,4c-1}$. Clearly, $\zeta(G') \geq 4c$. Since $c \geq 2$, $\Delta(G') = 4c-1$ and so $\zeta(G') = 4c$. Moreover, since $\chi(G) \geq 3$ and $\chi(K_{1,p}) = 2$, for every $p \geq 1$, we have that $\chi(G) = \chi(G')$. As a consequence, $\zeta(G') \leq c\chi(G')$ if and only if $\chi(G') \geq \frac{\zeta(G')}{c} = \frac{4c}{c} = 4$, and the reduction is completed.

$\zeta(G)$ and $\chi_b(G)$

We now show that given a graph G, deciding if $\zeta(G) = \chi_b(G)$ is an NP-complete problem. The stair factor of a graph can be computed in polynomial time [113] and $\chi_b(G) \leq \zeta(G)$ (see Section 2.4). Therefore it can be checked in polynomial time if a b-colouring uses $\zeta(G)$ colours, and the problem is in NP.

To see it is NP-hard we show a reduction from the 3-edge-colourability problem on 3-regular graphs. Let G be a 3-regular graph with n vertices. Consider the graph $\mathcal{B}(G)$ and observe that the sequence of vertices $(v_{n+3}, v_{n+2}, v_{n+1}, v_n, \ldots, v_1)$ defines a feasible Grundy sequence, so $\zeta(\mathcal{B}(G)) \geq n + 3$. Now, since $\Delta(\mathcal{B}(G)) = n + 2$ and $\zeta(\mathcal{B}(G)) \leq \Delta(\mathcal{B}(G)) + 1$ we have that $\zeta(\mathcal{B}(G)) = n + 3$.

As a consequence, deciding if $\chi_b(\mathcal{B}(G)) = \zeta(\mathcal{B}(G))$ is equivalent to decide if $\chi_b(\mathcal{B}(G)) = m(\mathcal{B}(G)) = n + 3$. Finally, because of Theorem 4.2.1 the problem of deciding if G admits a 3-edge colouring can be reduced to the one of deciding if $\chi_b(\mathcal{B}(G)) = n + 3$, and so given a 3-regular graph G, the problem of deciding if $\chi_b(\mathcal{B}(G)) = \zeta(\mathcal{B}(G))$ is NP-hard.

$\zeta(G)$ and $\Gamma(G)$

For a fixed integer $c \geq 1$, the problem of deciding if a given bipartite graph G satisfies $\zeta(G) \leq c\Gamma(G)$ is an NP-hard problem. To see it is in NP, observe that because the stair factor of a graph can be computed in polynomial time, it can be verified in polynomial time if a greedy colouring with Γ' colours is such that $\Gamma' \geq \frac{\zeta(G)}{c}$.

In order to prove that the problem is NP-hard, we give a reduction from the problem of deciding if $\Gamma(G) = \Delta(G) + 1$, for a given bipartite graph G, which was proved to be NP-complete in Theorem 3.2.1. Let G be a bipartite graph and G' be the disjoint union of G and the copies of the stars $K_{1,0}, K_{1,1}, K_{1,2}, \ldots, K_{1,c(\Delta(G)+1)-1}$. For each $0 \leq i \leq c(\Delta(G) + 1) - 1$, let v_i be the vertex of degree i in the copy of $K_{1,i}$ in G'. It is easy to see that the sequence of vertices $(v_{c(\Delta(G)+1)-1}, v_{c(\Delta(G)+1)-2}, \ldots, v_0)$ defines a feasible Grundy sequence, so $\zeta(G') \geq c(\Delta(G) + 1)$. The maximum degree of G' is $c(\Delta(G) + 1) - 1$, and so $\zeta(G') = c(\Delta(G) + 1)$. Moreover $\Gamma(G') = \Gamma(G)$, since the Grundy number of a disconnected graph is the maximum Grundy number of its components and $\Gamma(K_{1,p}) \leq 2$, for every $p \geq 0$. Hence $\zeta(G) \leq c\Gamma(G')$ if and only if $\Gamma(G') \geq \frac{\zeta(G)}{c} = \frac{c(\Delta(G)+1)}{c} = \Delta(G) + 1$ and the reduction is completed.

$\zeta(G)$ and $\partial\Gamma(G)$

Given a bipartite graph G, deciding if $\zeta(G) = \partial\Gamma(G)$ is NP-complete To see it is in NP, observe that because the stair factor of a graph can be computed in polynomial time, it can be verified in polynomial time if a partial greedy colouring with $\partial\Gamma'$ colours is such that $\partial\Gamma' = \zeta(G)$.

The fact that the problem is NP-hard is a consequence of the reduction presented by Shi et al. [113]. The graph in the reduction, say H, is such that $\zeta(H) = \Delta(H) + 1$, and they prove that deciding if $\partial\Gamma(H) = \Delta(H) + 1$ is an NP-complete problem.

$\Delta(G) + 1$ and $\chi(G)$

Observe first that the case $c = 1$ is equivalent to the problem of deciding if $\chi(G) = \Delta(G) + 1$, that can be solved in polynomial time, as a consequence of Brooks' Theorem.

It remains to consider a fixed $c \geq 2$. We will show that the problem of deciding if $\Delta(G) + 1 \leq c\chi(G)$ for a given a graph G is co-NP-complete. To see that the problem belongs to co-NP, observe that a proper colouring with $\chi' < \frac{\Delta(G)+1}{c}$ colours can be used to show that $\Delta(G) + 1 > c\chi(G)$.

It remains to show that the problem is NP-hard. The reduction is from the problem of deciding if $\chi(G) \geq 4$ for a planar 4-regular graph G such that $\chi(G) \geq 3$ (see Proposition 6.1.3 for the NP-hardness of this problem). Let G be an instance of this problem. Consider the graph G' obtained from G by adding a disjoint copy of $K_{1,4c-1}$. Since $c \geq 2$, then $4 = \Delta(G) < \Delta(G') = 4c - 1$. Moreover, since $\chi(G) \geq 3$ and $\chi(K_{1,p}) = 2$, for every $p \geq 1$, we have that $\chi(G) = \chi(G')$. Finally, $\Delta(G) + 1 \leq c\chi(G')$ if and only if $\chi(G') \geq \frac{\Delta(G')+1}{c} = \frac{4c}{c} = 4$, and we get the desired result.

$\Delta(G) + 1$ and $\chi_b(G)$

For a fixed integer $c \geq 1$, the problem of deciding if a given a graph G satisfies $\Delta(G) + 1 \leq c\chi_b(G)$ is NP-complete. To see that the problem belongs to NP it suffices to observe that one can check in polynomial time if a given colouring of G is a b-colouring with at least $\frac{\Delta(G)+1}{c}$ colours.

It remains to show that the problem is NP-hard. Again, let M^p be the graph obtained from the complete bipartite graph $K_{p,p}$ by removing a matching of size $p - 1$. We have already shown in Proposition 2.3.8 that $\chi_b(M^p) = 2$. Moreover, in Proposition 6.1.1 we proved that in any proper colouring of M^p there are at most 2 b-vertices with distinct colours.

Now consider another modified version of the graph $\mathcal{B}(G)$. Replace the three stars in the definition of $\mathcal{B}(G)$ by one copy of $M^{c(n+2)}$ and one copy of $K_{1,n+2}$, and denote this graph by $\mathcal{B}'''(G)$. Observe that $\Delta(\mathcal{B}'''(G)) = c\Delta(G) = c(n + 2)$.

The proof of the following proposition is very similar to the one of Proposition 6.1.2, so we omit its proof here.

Proposition 6.1.5. *Given a 3-regular graph G, $\chi_b(\mathcal{B}(G)) = n+3$ if and only if $\chi_b(\mathcal{B}'''(G)) = n+3$.*

Now we are able to prove that deciding if $\Delta(G) + 1 \leq c\chi_b(G)$ is NP-hard. The reduction is again from 3-edge-colourability of 3-regular graphs. Let G be an instance of this problem. Clearly, $\Delta(\mathcal{B}'''(G)) + 1 \leq c\chi_b(\mathcal{B}'''(\mathcal{G}))$ if and only if $\chi_b(\mathcal{B}'''(G)) \geq \frac{\Delta(\mathcal{B}'''(G))+1}{c} = \frac{c(\Delta(G))+1}{c}$. Since the b-chromatic number of a graph is always a positive value, $\chi_b(\mathcal{B}'''(G)) \geq \frac{c\Delta(G)+1}{c}$ if and only if $\chi_b(\mathcal{B}'''(G)) \geq \Delta(G) + 1 = n + 3$. Finally, because of Proposition 6.1.5, $\chi_b(\mathcal{B}'''(G)) = n + 3$ if and only if $\chi_b(\mathcal{B}(\mathcal{G})) = n + 3$, and we get the desired result.

$\Delta(G) + 1$ and $\Gamma(G)$

Let $c \geq 1$ be a fixed integer. Given a graph G, the problem of deciding if $\Delta(G) + 1 \leq c\Gamma(G)$ is NP-complete. Clearly the problem is in NP, since given a colouring of G, one can check in polynomial time if it is a greedy colouring with at least $\frac{\Delta(G)+1}{c}$ colours. To prove it is NP-hard we use a reduction to the problem of deciding if $\Gamma(G) = \Delta(G) + 1$, given a graph G, that we proved to be NP-complete in Theorem 3.2.1. Let G be an instance of the later problem. Consider the graph G' obtained from G by adding a disjoint copy of $K_{c\Delta(G),c\Delta(G)}$. Since $\Gamma(K_{c\Delta(G),c\Delta(G)}) = 2$, $\Gamma(G') = \Gamma(G)$. Moreover, $\Delta(G') = c\Delta(G)$ Now, $\Delta(G') + 1 \leq c\Gamma(G')$ if and only if $\Gamma(G') \geq \frac{\Delta(G')+1}{c} = \frac{c\Delta(G)+1}{c}$. Since the Grundy number of a graph is always a positive value, $\Gamma(G') \geq \frac{c\Delta(G)+1}{c}$ if and only if $\Gamma(G') \geq \Delta(G) + 1$.

$\Delta(G) + 1$ and $\partial\Gamma(G)$

For a fixed integer $c \geq 1$, we now show that the problem of deciding if a given a graph G satisfies $\Delta(G) + 1 \leq c\partial\Gamma(G)$ is NP-complete. Clearly the problem is in NP, since given a colouring of G, one can check in polynomial time if it is a partial greedy colouring with at least $\frac{\Delta(G)+1}{c}$ colours.

To prove it is NP-hard we use a reduction from the problem of deciding if a graph G satisfies $\partial\Gamma(G) = \Delta(G) + 1$, which was shown to be NP-complete in [113]. Let G be an instance of the later problem. Consider the graph G' obtained from G by adding a disjoint copy of $K_{1,c(\Delta(G)+1)+1}$, and let w be the vertex of degree $c(\Delta(G) + 1) + 1$ in this copy. It is easy to see that $\zeta(G') = \zeta(G) + 1 = \Delta(G)+2$, and that therefore $\partial\Gamma(G') \leq \Delta(G)+2$. Now, if G' admits a partial greedy colouring c with $\Delta(G)+2$ colours, then the only Grundy vertex of colour $c\Delta(G)+2$ is w, since all other vertices have degree at most $\Delta(G)$. No vertex from the copy of $K_{1,c(\Delta(G)+1)+1}$ different from w is a Grundy vertex, except if it is coloured one, since all vertices different from w have degree one. As a consequence, the restriction of c to the component isomorphic to G contains one Grundy vertex of each colour, and so it is a partial greedy colouring with $c\Delta(G) + 1$ colours. On the other hand, if c is a partial greedy colouring of G with $\Delta(G) + 1$ colours, then a partial greedy colouring of G' may be obtained from c by giving colour $\Delta(G) + 2$ to w and colouring its neighbours in a way that it has one neighbour with each colour in $\{1, \ldots, \Delta(G)+1\}$. Therefore, $\partial\Gamma(G) = \Delta(G)+1$ if and only if $\partial\Gamma(G) = \Delta(G)+2$. Now observe that $\Delta(G') + 1 \leq c\partial\Gamma(G')$ if and only if $\partial\Gamma(G') \geq \frac{\Delta(G')+1}{c} = \frac{c(\Delta(G)+1)+1}{c}$. Since the partial Grundy number of a graph is always a positive value, $\Gamma(G') \geq \frac{\Delta(G')+1}{c}$ if and only if $\Gamma(G') \geq \Delta(G) + 2$, and this concludes the reduction.

$\Delta(G) + 1$ and $\zeta(G)$

In [113] it is shown that the stair factor of a graph can be computed in linear time. Therefore given a graph and a fixed integer $c \geq 1$, it can be decided in polynomial time if $\Delta(G) + 1 \leq c\zeta(G)$.

6.2 Extensions of Reed's conjecture

Reed [110] considered the problem of bounding the chromatic number of a graph by it's natural upper and lower bounds, the maximum degree and the clique number. He conjectured that

$$\chi(G) \leq \left\lceil \frac{\omega(G) + \Delta(G) + 1}{2} \right\rceil.$$

As an evidence of this conjecture, he proved that there is an $0 < \epsilon < 1$ such that $\chi(G) \leq \epsilon\omega(G) + (1 - \epsilon)(\Delta(G) + 1)$.

In this section we consider some extensions of Reed's conjecture for the Grundy number and b-chromatic number. We will make use of the following notation to indicate some graphs that already appeared before in this thesis.

$K^*_{p,p}$: The graph obtained from the complete bipartite graph $K_{l,l}$ by removing a perfect matching.

$K'_{p,p}$: The graph obtained from the complete bipartite graph $K_{l,l}$ by removing a matching of size $p - 1$.

S^p: The graph obtained from the disjoint union of p copies of $K_{1,p}$

The Table 6.2 contains the values of the parameters $\omega(G)$, $\chi(G)$, $\chi_b(G)$, $\Gamma(G)$, $\partial\Gamma(G)$, $\zeta(G)$ and $\Delta(G) + 1$ on these graphs. These values are either trivial to compute or were already computed before, in Chapter 2.

	$\omega(G)$	$\chi(G)$	$\chi_b(G)$	$\Gamma(G)$	$\partial\Gamma(G)$	$\zeta(G)$	$\Delta(G)+1$
$K^*_{p,p}$	2	2	p	p	p	p	p
$K'_{p,p}$	2	2	2	$p+1$	$p+1$	$p+1$	$p+1$
S^p	2	2	p	2	p	p	p

Table 6.2: The values of the different parameters for $K^*_{p,p}, K_{l,l}$ and S^p.

6.2.1 Grundy number

Let G be any graph. In this section we consider $\phi(G) \in\{\omega(G), \chi(G)\}$ and $\psi(G) \in\{\partial\Gamma(G), \zeta(G), \Delta(G)+1\}$ and investigate the problem of weather there is $0<\epsilon<1$ such that:

$$\Gamma(G) \leq \lceil \epsilon\phi(G) + (1-\epsilon)\psi(G) \rceil$$

.

or

$$\Gamma(G) \geq \lfloor \epsilon\phi(G) + (1-\epsilon)\psi(G) \rfloor$$

.

Theorem 6.2.1. *Let ϵ be a fixed value satisfying $0<\epsilon<1$. There exists a graph G such that for $\phi(G) \in\{\omega(G), \chi(G)\}$ and $\psi(G) \in\{\partial\Gamma(G), \zeta(G), \Delta(G)+1\}$,*

$$\Gamma(G) > \lceil \epsilon\phi(G) + (1-\epsilon)\psi(G) \rceil .$$

Moreover, there exists a graph H such that for $\phi(H) \in\{\omega(H), \chi(H)\}$ and $\psi(H) \in\{\partial\Gamma(H), \zeta(H), \Delta(H)+1\}$,

$$\Gamma(H) < \lfloor \epsilon\phi(H) + (1-\epsilon)\psi(H) \rfloor .$$

Proof. Consider an integer $p \geq 3$ and set $G = K^*_{p,p}$. Then, $\omega(G) = \chi(G) = 2$, $\Gamma(G) = p$, and $\partial\Gamma(G) = \zeta(G) = \Delta(G)+1 = p$. Now, let $\phi(G) \in\{\omega(G), \chi(G)\}$ and $\psi(G) \in\{\partial\Gamma(G), \zeta(G), \Delta(G)+1\}$. Then, $\lceil \epsilon\phi(G) + (1-\epsilon)\psi(G) \rceil \leq 2\epsilon + (1-\epsilon)p + 1 < p\epsilon + (1-\epsilon)p = \Gamma(G)$.

Now, let p be an integer such that $p \geq 4$ and set $H = S^p$. In this case, $\omega(H) = \chi(H) = 2$, $\Gamma(H) = 2$, and $\partial\Gamma(H) = \zeta(H) = \Delta(H)+1 = p$. Therefore, for $\phi(H) \in\{\omega(H), \chi(H)\}$ and $\psi(H) \in\{\partial\Gamma(H), \zeta(H), \Delta(H)+1\}$, we have that $\lfloor \epsilon\phi(H) + (1-\epsilon)\psi(H) \rfloor \geq 2\epsilon + (1-\epsilon)p - 1$. Since $p \geq 4$, $2\epsilon + (1-\epsilon)p - 1 > 2 = \Gamma(H)$, and we have the result. □

6.2.2 b-chromatic number

Let G be any graph. In this section we consider $\phi(G) \in\{\omega(G), \chi(G)\}$ and $\psi(G) \in\{\partial\Gamma(G), \zeta(G), \Delta(G)+1\}$ and investigate the problem of weather there is $0<\epsilon<1$ such that:

$$\chi_b(G) \leq \lceil \epsilon\phi(G) + (1-\epsilon)\psi(G) \rceil$$

.

or

$$\chi_b(G) \geq \lfloor \epsilon\phi(G) + (1-\epsilon)\psi(G) \rfloor$$

.

Theorem 6.2.2. *Let ϵ be a fixed value satisfying $0 < \epsilon < 1$. There exists a graph G such that for $\phi(G) \in \{\omega(G), \chi(G)\}$ and $\psi(G) \in \{\partial\Gamma(G), \zeta(G), \Delta(G)+1\}$,*

$$\chi_b(G) > \lceil \epsilon\phi(G) + (1-\epsilon)\psi(G) \rceil .$$

Moreover, there exists a graph H such that for $\phi(H) \in \{\omega(H), \chi(H)\}$ and $\psi(H) \in \{\partial\Gamma(H), \zeta(H), \Delta(H)+1\}$,

$$\chi_b(H) < \lfloor \epsilon\phi(H) + (1-\epsilon)\psi(H) \rfloor .$$

Proof. Consider an integer $p \geq 3$ and set $G = K^*_{p,p}$. Then, $\omega(G) = \chi(G) = 2$, $\chi_b(G) = p$, and $\partial\Gamma(G) = \zeta(G) = \Delta(G) + 1 = p$. Now, let $\phi(G) \in \{\omega(G), \chi(G)\}$ and $\psi(G) \in \{\partial\Gamma(G), \zeta(G), \Delta(G)+1\}$. Then, $\lceil \epsilon\phi(G) + (1-\epsilon)\psi(G) \rceil \leq 2\epsilon + (1-\epsilon)p + 1 < p\epsilon + (1-\epsilon)p = \chi_b(G)$.

Now, let p be an integer such that $p \geq 3$ and set $H = K'_{p,p}$. In this case, $\omega(H) = \chi(H) = 2$, $\chi_b(H) = 2$, and $\partial\Gamma(H) = \zeta(H) = \Delta(H) + 1 = p + 1$. Therefore, for $\phi(H) \in \{\omega(H), \chi(H)\}$ and $\psi(H) \in \{\partial\Gamma(H), \zeta(H), \Delta(H)+1\}$, we have that $\lfloor \epsilon\phi(H) + (1-\epsilon)\psi(H) \rfloor \geq 2\epsilon + (1-\epsilon)(p+1) - 1 > 2 = \chi_b(H)$.

□

6.2.3 Partial Grundy number

Let G be any graph. In this section we consider $\phi(G) \in \{\omega(G), \chi(G), \chi_b(G), \Gamma(G)\}$ and $\psi(G) \in \{\zeta(G), \Delta(G)+1\}$ and investigate the problem of weather there is $0 < \epsilon < 1$ such that:

$$\partial\Gamma(G) \leq \lceil \epsilon\phi(G) + (1-\epsilon)\psi(G) \rceil$$

.

or

$$\partial\Gamma(G) \geq \lfloor \epsilon\phi(G) + (1-\epsilon)\psi(G) \rfloor$$

.

Theorem 6.2.3. *Let ϵ be a fixed value satisfying $0 < \epsilon < 1$. There exists a graph G such that for $\phi(G) \in \{\omega(G), \chi(G), \chi_b(G)\}$ and $\psi(G) \in \{ \zeta(G), \Delta(G)+1\}$,*

$$\partial\Gamma(G) > \lceil \epsilon\phi(G) + (1-\epsilon)\psi(G) \rceil .$$

There exists a graph G' such that for $\psi(G') \in \{\zeta(G'), \Delta(G')+1\}$,

$$\partial\Gamma(G') > \lceil \epsilon\Gamma(G') + (1-\epsilon)\psi(G') \rceil .$$

Moreover, there exists a graph H such that for $\phi(H) \in \{\omega(H), \chi(H)\}$ and $\psi(H) \in \{\partial\Gamma(H), \Delta(H)+1\}$,

$$\partial\Gamma(H) < \lfloor \epsilon\phi(H) + (1-\epsilon)\psi(H) \rfloor .$$

Proof. Consider an integer $p \geq 3$ and set $G = K'_{p,p}$. Then, $\omega(G) = \chi(G) = \chi_b(G) = 2$, $\partial\Gamma(G) = p+1$, and $\zeta(G) = \Delta(G)+1 = p+1$. Now, let $\phi(G) \in \{\omega(G), \chi(G), \chi_b(G)\}$ and $\psi(G) \in \{\zeta(G), \Delta(G)+1\}$. Then, $\lceil \epsilon\phi(G) + (1-\epsilon)\psi(G) \rceil \leq 2\epsilon + (1-\epsilon)(p+1)+1$. Now, since $p \geq 3$, $2\epsilon + (1-\epsilon)(p+1)+1 < (p+1)\epsilon + (1-\epsilon)(p+1) = \partial\Gamma(G)$.

Consider an integer $p \geq 4$ and set $G = S^p$. Then, $\Gamma(G) = 2$, $\partial\Gamma(G) = p$, and $\zeta(G) = \Delta(G)+1 = p$. Let $\psi(G) \in \{\zeta(G), \Delta(G)+1\}$. Then, $\lceil \epsilon\Gamma(G) + (1-\epsilon)\psi(G) \rceil \leq 2\epsilon + (1-\epsilon)p+1$, and since $p \geq 4$, $2\epsilon + (1-\epsilon)p + 1 < p\epsilon + (1-\epsilon)p = \partial\Gamma(G)$.

Now, let p be an integer such that $p \geq 3$ and set $H = K_{1,p-1}$. Then, $\omega(H) = \chi(H) = \chi_b(H) = \Gamma(H) = 2$, $\partial\Gamma(H) = 2$, and $\Delta(G)+1 = p$. Therefore, for $\phi(H) \in \{\omega(H), \chi(H), \chi_b(H) = \Gamma(H)\}$, we have that $\lfloor \epsilon\phi(H) + (1-\epsilon)(\Delta(H)+1) \rfloor \geq 2\epsilon + (1-\epsilon)p - 1 > 2 = \Gamma(H)$. □

6.3 Open problems

6.3.1 Complexity of comparing the parameters

The complexity of some of the problems of the form $\phi(G) \leq c\psi(G)$ is still unknown. We leave these as open problems.

Problem 6.1. Given a graph G and for a fixed $c \geq 1$, what is the complexity of deciding if $\partial\Gamma(G) \leq c\omega(G)$?

Problem 6.2. Given a graph G, what is the complexity of deciding if $\partial\Gamma(G) = \chi(G)$?

Problem 6.3. Given a graph G and for a fixed $c > 1$, what is the complexity of deciding if $\partial\Gamma(G) \leq c\chi_b(G)$?

Problem 6.4. Given a graph G and for a fixed $c \geq 1$, what is the complexity of deciding if $\zeta(G) \leq c\omega(G)$?

Problem 6.5. Given a graph G, what is the complexity of deciding if $\zeta(G) = \chi(G)$?

Problem 6.6. Given a graph G and for a fixed $c > 1$, what is the complexity of deciding if $\zeta(G) \leq c\chi_b(G)$?

Problem 6.7. Given a graph G and for a fixed $c > 1$, what is the complexity of deciding if $\zeta(G) \leq c\partial\Gamma(G)$?

Problem 6.8. Given a graph G and for a fixed $c > 1$, what is the complexity of deciding if $\Delta(G)+1 \leq c\omega(G)$?

6.3.2 Extensions of Reed's conjecture

The following problem remains unsolved:

Problem 6.9. Let G be any graph and $\phi(G) \in \{\omega(G), \chi(G)\}$. Is it true that $\partial\Gamma(G) < \lfloor \epsilon\phi(G) + (1-\epsilon)\zeta(G) \rfloor$?

Other question related with the later one is if the stair factor of a graph can be bounded as a function of its partial Grundy number.

Problem 6.10. Does there exists M such that for any graph G, $\zeta(G) - \partial\Gamma(G) \leq M$?

Problem 6.11. Does there exists C such that for any graph G, $\frac{\zeta(G)}{\partial\Gamma(G)} \leq C$?

Chapter 7

Conclusion

In this thesis we investigated the computational complexity of the problem of computing the Grundy number and the b-chromatic number of a graph. We considered these problems from the point-of-view of the classical complexity theory and from the parameterized complexity theory.

In the case of the Grundy number, we determined the complexity of computing this parameter for bipartite and chordal graphs, showing in each case that the problem is NP-hard. We proved that deciding if $\Gamma(G) \geq |V(G)| - k$ for a given graph G and with k being the parameter is a FPT problem. We also proved that deciding if $\Gamma(G) = \Delta(G) + 1$, given a graph G and $\Delta(G)$ being the parameter, is an FPT problem.

In the case of the b-chromatic number, we determined the complexity of computing this parameter for tight chordal graphs, tight distance-hereditary graphs, and tight P_4-laden graphs, showing in each case that the problem is NP-hard. We defined the closure and the partial closure of a tight graph, and used these operations to obtain polynomial algorithms for deciding if $\chi_b(G) = m(G)$, for a given tight complement of bipartite graph, tight block graph or a tight P_4-sparse graph. The method that was used to obtain these results is general and may be possible to prove similar results for other subclasses of tight graphs. The problem of deciding if $\chi_b(G) \geq |V(G)| - k$ for a given graph G and with k being the parameter was considered and proven to be FPT. We also proved that deciding if $\chi_b(G) = \Delta(G) + 1$, given a graph G and $\Delta(G)$ being the parameter, is an FPT problem.

The complexity of problems related to comparing the colouring parameters for a given graph were considered. For a fixed $c \geq 1$, we investigated the complexity of the problem of deciding if $\phi(G) \leq c\psi(G)$, where $\phi(G), \psi(G) \in \{\omega(G),\ \chi(G),\ \chi_b(G),\ \Gamma(G),\ \partial\Gamma(G),\ \zeta(G),\ \Delta(G) + 1\}$. In most of the cases the complexity of the problem was completely determined.

Finally, we considered analogue versions of the Reed's conjecture involving the parameters $\Gamma(G)$, $\chi_b(G)$ and $\partial\Gamma(G)$, and their upper and lower bounds. Most of the analogue versions of the conjecture that were considered were proven to be false.

Up to a few exceptions, the previously-mentioned results are new and therefore this thesis contributed to the state of the art of the problems that were investigated. The perspectives on further research were presented in the end of each chapter, where open problems related to its contents are given.

Chapter 8

Appendices

INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

On the hull number of some graph classes.

J. Araujo — V. Campos — F. Giroire — N. Nisse — L. Sampaio — R. Soares

N° 7567

Septembre 2011

Domaine 3

Rapport de recherche

ISSN 0249-6399 ISRN INRIA/RR--7567--FR+ENG

INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

INRIA

centre de recherche SOPHIA ANTIPOLIS - MÉDITERRANÉE

On the hull number of some graph classes. *

J. Araujo † ‡ , V. Campos ‡ , F. Giroire † , N. Nisse † , L. Sampaio † § , R. Soares † ‡

Thème : Réseaux, systèmes et services, calcul distribué
Équipe-Projet Mascotte

Rapport de recherche n° 7567 — Septembre 2011 — 19 pages

Abstract:

In this paper, we study the geodetic convexity of graphs focusing on the problem of the complexity to compute inclusion-minimum hull set of a graph in several graph classes.

For any two vertices $u,v \in V$ of a connected graph $G=(V,E)$, the *closed interval* $I[u,v]$ of u and v is the the set of vertices that belong to some shortest (u,v)-path. For any $S \subseteq V$, let $I[S] = \bigcup_{u,v\in S} I[u,v]$. A subset $S \subseteq V$ is *geodesically convex* if $I[S]=S$. In other words, a subset S is convex if, for any $u,v \in S$ and for any shortest (u,v)-path P, $V(P) \subseteq S$. Given a subset $S \subseteq V$, the *convex hull* $I_h[S]$ of S is the smallest convex set that contains S. We say that S is a *hull set* of G if $I_h[S]=V$. The size of a minimum hull set of G is the *hull number* of G, denoted by $hn(G)$. The HULL NUMBER problem is to decide whether $hn(G) \leq k$, for a given graph G and an integer k. Dourado *et al.* showed that this problem is NP-complete in general graphs.

In this paper, we answer an open question of Dourado et al. [12] by showing that the HULL NUMBER problem is NP-hard even when restricted to the class of bipartite graphs. Then, we design polynomial time algorithms to solve the HULL NUMBER problem in several graph classes. First, we deal with the class of complements of bipartite graphs. Then, we generalize some results in [1] to the class of $(q,q-4)$-graphs and to the class of cacti. Finally, we prove tight upper bounds on the hull numbers. In particular, we show that the hull number of an n-node graph G without simplicial vertices is at most $1+\lceil\frac{3(n-1)}{5}\rceil$ in general, at most $1+\lceil\frac{n-1}{2}\rceil$ if G is regular or has no triangle, and at most $1+\lceil\frac{n-1}{3}\rceil$ if G has girth at least 6.

Key-words: graph convexity, hull number, bipartite graph, cobipartite graph, cactus graph, $(q,q-4)$-graph.

* Research supported by the INRIA Equipe Associée EWIN.

† {julio.araujo, frederic.giroire, nicolas.nisse, leonardo.sampaio_rocha, ronan.pardo_soares}@inria.fr - MASCOTTE Project, I3S (CNRS & UNS) and INRIA, INRIA Sophia-Antipolis, 2004 route des Lucioles, BP 93, 06902 Sophia-Antipolis Cedex France.

‡ {campos@lia.ufc.br} ParGO Research Group - Universidade Federal do Ceará - UFC - Campus do Pici, Bloco 910. 60455-760 - Fortaleza, CE - Brazil.

§ Partially supported by CAPES/Brazil and ANR Blanc AGAPE.

Le nombre enveloppe de quelques classes de graphes

Résumé : Dans cet article nous étudions une notion de convexité dans les graphes. Nous nous concentrons sur la question de la compléxité du calcul de l'enveloppe minimum d'un graphe dans le cas de diverses classes de graphes.

Étant donné un graphe $G=(V,E)$, l'intervalle $I[u,v]$ entre deux sommets $u,v \in V$ est l'ensemble des sommets qui appartiennent à un plus court chemin entre u et v. Pour un ensemble $S \subseteq V$, on note $I[S]$ l'ensemble $\bigcup_{u,v \in S} I[u,v]$. Un ensemble $S \subseteq V$ de sommets est dit *convexe* si $I[S] = S$. L'*enveloppe convexe* $I_h[S]$ d'un sous-ensemble $S \subseteq V$ de G est défini comme le plus petit ensemble convexe qui contient S. $S \subseteq V$ est une *enveloppe* de G si $I_h[S] = V$. Le *nombre enveloppe* de G, noté $hn(G)$, est la cardinalité minimum d'une enveloppe de graphe G.

Nous montrons que décider si $hn(G) \leq k$ est un problème NP-complet dans la classe des graphes bipartis et nous prouvons que $hn(G)$ peut être calculé en temps polynomial pour les cobipartis, $(q,q-4)$-graphes et cactus. Nous montrons aussi des bornes supérieures du nombre enveloppe des graphes en général, des graphes sans triangles et des graphes réguliers.

Mots-clés : convexité des graphes, nombre enveloppe, graphes bipartis, graphes cobipartis, graphes cactus, $(q,q-4)$-graphes

1 Introduction

A classical example of convexity is the one defined in Euclidean spaces. In an Euclidean space E, a set $S \subseteq E$ is said *convex* if for any two points x and y of S, $[x,y] \subseteq S$, i.e., the set of points lying in the straight line segment between x and y also belongs to S. Note that if two convex sets $X,Y \subseteq E$ contain a given set $S \subseteq E$ of points, then their intersection $X \cap Y$ is also a convex set of E containing S. Hence, we can define the *convex hull* of S as the inclusion-minimum convex set that contains S. Reciprocally, given a convex set S of E, a *hull set* of S is any subset S' of S such that S is the convex hull of S'. A naive way to compute the convex hull H of a set S consists in starting with $H = S$ and, while it is possible, adding $[x,y]$ to H for any $x,y \in H$. However there exist more efficient algorithms. For instance, for any set S of a d-dimensional euclidean space, the *gift wrapping algorithm* computes the convex hull and a minimum-inclusion hull set of S in polynomial-time in the size of S (d being fixed). For more results concerning the convexity in Euclidean spaces, we refer to [19].

In order to capture the abstract notion of convexity, [16] defines an *alignment* over a set X as a family $\mathcal{C}$ of subsets of X that is closed under intersection and that contains both X and the empty set. The members of $\mathcal{C}$ are called the *convex sets* of X. The pair $(X,\mathcal{C})$ is then called an *aligned space*. An example of aligned space $(E,\mathcal{C})$ is the one where E is an euclidean space and $\mathcal{C} = \{H \subseteq E \ : \ \forall x,y \in H, \ [x,y] \subseteq H\}$. Given an aligned space $(X,\mathcal{C})$, the definitions of convex hull and hull set are generalized as follows. For any $S \subseteq X$, the *convex hull* of S is the smallest member of $\mathcal{C}$ containing S. For any $S \in \mathcal{C}$, a *hull set* of S is a set $S' \subseteq S$ such that S is the convex hull of S'.

Various notions of convexity can be defined in graphs as specific alignments over the set of vertices. This paper is devoted to the study of the *geodetic convexity* of graphs. Let $G = (V,E)$ be a connected undirected graph. For any $u,v \in V$, let the *closed interval* $I[u,v]$ of u and v be the the set of vertices that belong to some shortest (u,v)-path. The closed interval of a set of vertices can be seen as an analog to segments in Euclidian spaces. For any $S \subseteq V$, let $I[S] = \bigcup_{u,v \in S} I[u,v]$. A subset $S \subseteq V$ is *geodesically convex* if $I[S] = S$. In this paper convexity refers to the geodesical variant. In other words, a subset S is convex if, for any $u,v \in S$ and for any shortest (u,v)-path P, $V(P) \subseteq S$. That is, the geodetic convexity can be defined as the alignment $\mathcal{C}$ over V where $\mathcal{C} = \{S \subseteq V \ : \ I[S] = S\}$.

Given a subset $S \subseteq V$, the *convex hull* $I_h[S]$ of S is the smallest convex set that contains S. We say that S is a *hull set* of G if $I_h[S] = V$. That is, S is a hull set of G if, starting from the vertices of S and successively adding in S the vertices in some shortest path between two vertices in S, we eventually obtain V. The size of a minimum hull set of G is the *hull number* of G, denoted by $hn(G)$. The HULL NUMBER problem is to decide whether $hn(G) \leq k$, for a given graph G and an integer k [15]. This problem is known to be NP-complete in general graphs [12]. In this paper, we consider the problem of the complexity to compute inclusion-minimum hull set of a graph in several graph classes.

Our results. We first answer an open question of Dourado et al. [12] by showing that the HULL NUMBER problem is NP-hard even when restricted to the class of bipartite graphs (Section 3). Then, we design polynomial time algorithms to solve the HULL NUMBER problem in several graphs' classes. In Section 4, we deal with the class of complements of bipartite graphs. In Section 5 we generalize some results in [1] to the class of $(q,q-4)$-graphs. Section 6 is devoted to the class of cacti. Finally, we prove tight upper bounds on the hull number of graphs in Section 7. In particular, we show that the hull number of an n-node graph G without simplicial vertices is at most $1 + \lceil \frac{3(n-1)}{5} \rceil$ in general, at most $1 + \lceil \frac{n-1}{2} \rceil$ if G is regular or has no triangle, and at most $1 + \lceil \frac{n-1}{3} \rceil$ if G has girth at least 6.

Related work. In the seminal work [15], the authors present some upper and lower bounds on the hull number of general graphs and characterize the hull number of some particular graphs. The corresponding minimization problem has been shown to be NP-complete [12]. Dourado *et al.* also proved that the hull number of unit interval graphs, cographs and split graphs can be computed in polynomial time [12]. Bounds on the hull number of triangle-free graphs are shown in [13]. The hull number of the cartesian and the strong product of two connected graphs is studied in [5, 11]. In [18], the authors have studied the relationship between the *Steiner number* and the hull number of a given graph. An oriented version of the HULL NUMBER problem is studied in [8, 17].

Other parameters related to the geodetic convexity have been studied in [9, 10]. Variations of graph convexity have been further proposed and studied. For instance, the *monophonic convexity* that deals with induced paths instead of shortest paths is studied in [14, 16]. Another example is the P_3-*convexity* where just paths of order three are considered [6, 16]. Other variants of graph convexity and other parameters are mentioned in [7].

2 Preliminaries

In this paper, we adopt the graph terminology defined in [4]. Otherwise stated, all graphs considered in this work are simple, undirected and connected. Let $G = (V,E)$ be a graph. Given a vertex $v \in V$, $N(v)$ denotes the (open) neighborhood of v, i.e., the set of neighbors of v. Let $N[v] = N(v) \cup \{v\}$ be the closed neighborhood of v. A vertex v is *universal* if $N[v] = V$. A vertex is *simplicial* if $N[v]$ induces a complete subgraph in G. Finally, a subgraph H of G is *isometric* if, for any $u,v \in V(H)$, the distance $dist_H(u,v)$ between u and v in H equals $dist_G(u,v)$.

This section is devoted to basic lemmas on hull sets. These lemmas will serve as cornerstone of most of the results presented in this paper.

Lemma 1 ([15]). *For any hull set S of a graph G, S contains all simplicial vertices of G.*

Lemma 2 ([12]). *Let G be a graph which is not complete. No hull set of G with cardinality $hn(G)$ contains a universal vertex.*

Lemma 3 ([12]). *Let G be a graph, H be an isometric subgraph of G and S be any hull set of H. Then, the convex hull of S in G contains $V(H)$.*

Lemma 4 ([12]). *Let G be a graph and S a proper and non-empty subset of $V(G)$. If $V(G) \setminus S$ is convex, then every hull set of G contains at least one vertex of S.*

3 Bipartite graphs

In this section, we answer an open question of Dourado et al. [12] by showing that the Hull Number Problem is NP-complete in the class of bipartite graphs. Since the Hull Number Problem is in NP, as proved in [12], it only remains to prove the following theorem:

Theorem 1. *The* HULL NUMBER *problem is NP-hard in the class of bipartite graphs.*

Proof. To prove this theorem, we adapt the proof presented in [12]. We reduce the 3-SATisfiability Problem to the HULL NUMBER problem in bipartite graphs. Let us consider the following instance of 3-SAT. Given a formula in the conjunctive normal form, let $\mathcal{F} = \{C_1, C_2, \dots, C_m\}$ be the set of its 3-clauses and $X = \{x_1, x_2, \dots, x_n\}$ the set of its boolean variables. We may assume that $m = 2^p$, for a positive integer $p \geq 1$, since it is possible to add dummy variables and clauses without changing the satisfiability of $\mathcal{F}$ and such that the size of the instance is at most twice the size of the initial instance. Moreover, we also assume, without loss of generality, that each variable x_i and its negation appear at least once in $\mathcal{F}$ (otherwise the clauses where x_i appeared could always be satisfied).

Let us construct the bipartite graph $G(\mathcal{F})$ as follows. First, let T be a full binary tree of height p rooted in r with $m = 2^p$ leaves, and let $L = \{c_1, c_2, \dots, c_m\}$ be the set of leaves of T. We then construct a graph H as follows. First, let us add a vertex u that is adjacent to every vertex in L. Then, any edge $\{u,v\} \in E(T)$ with u the parent of v is replaced by a path with $2^{h(v)}$ edges, where $h(v)$ is the distance between v and any of its descendent leaves. Note that, in H, the distance between r and any leaf is $\sum_{i=0}^{p-1} 2^i = 2^p - 1 = m - 1$. Moreover, it is easy to see that $|V(H)| = O(m \cdot \log m)$.

The following claims are proved in [12].

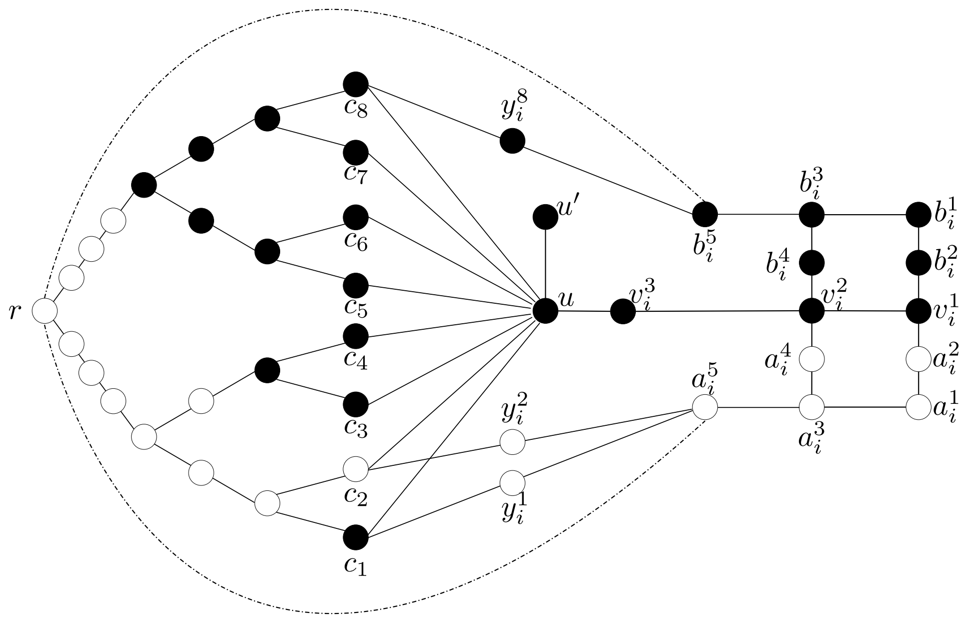


Figure 1: Subgraph of the bipartite instance $G(\mathcal{F})$ containing the gadget of a variable x_i that appears positively in clauses C_1 and C_2, and negatively in C_8. If x_i appears positively in C_j, link a_i^5 to c_j through y_i^j. If it appears negatively, we use b_i^5 instead of a_i^5.

Claim 1. *Let $v,w \in V(T)\setminus\{r\}$. The closed interval of v,w in H contains the parents of v in T if and only if v and w are siblings in T.*

Claim 2. *The set L is a* minimal *hull set of H.*

Then, let H' be obtained by adding a one degree vertex u' adjacent to u in H. Finally, we build a graph $G(\mathcal{F})$ from H' by adding, for any variable x_i, $i \leq n$, the gadget defined as follows.

Let us start with a cycle $\{a_i^1,a_i^2,v_i^1,b_i^2,b_i^1,b_i^3,b_i^4,v_i^2,a_i^4,a_i^3\}$ plus the edge $\{v_i^2,v_i^1\}$. Then, add the vertex v_i^3 as common neighbor of v_i^2 and u. Add a neighbor b_i^5 (resp., a_i^5) adjacent to b_i^3 (resp., a_i^3) and a path of length $2^{h(r)}-3 = m-3$ edges between b_i^5 (resp., a_i^5) and r. Let D be the set of internal vertices of all these $2n$ paths between a_i^5, resp., b_i^5, and r, $i \leq n$. Finally, for any clause C_j in which x_i appears, if x_i appears positively (resp., negatively) in C_j then add a common neighbor y_i^j between c_j and a_i^5 (resp., b_i^5). See an example of such a gadget in Figure 1. Note that $|V(G(\mathcal{F}))| = O(m\cdot(n+\log m))$.

Lemma 5. *$G(\mathcal{F})$ is a bipartite graph.*

Proof. Let us present a proper 2-coloring c of $G(\mathcal{F})$. Let $c(r)=1$, and for each vertex w in $V(H)$, define $c(w)$ as 1 if w is in an even distance from r, and 2 otherwise. Clearly, c is a partial *proper* coloring of $G(\mathcal{F})$ and moreover we have $c(u)=1$ and $c(c_j)=2$, for any $j\in\{1,\dots,m\}$ (Indeed, any c_i is at distance $m-1$ (odd) of r in H). Let $c(u')=2$. For every $i\in\{1,\dots,n\}$ and for any j such that $x_i\in C_j$, let $c(y_i^j)=1$. For any $i\leq n$, for any $x\in\{b_i^5,a_i^5,v_i^3,b_i^4,a_i^4,b_i^1,v_i^1,a_i^1\}$, $c(x)=2$.

$c(b_i^5)=c(a_i^5)=c(v_i^3)=2$. Again, this partial coloring of $G(\mathcal{F})$ is proper. One can easily verify that this coloring can be extended to $\{a_i^1,a_i^2,v_i^1,b_i^2,b_i^1,b_i^3,b_i^4,v_i^2,a_i^4,a_i^3\}$ for any $i\leq n$. Moreover, since $c(r)=1$ and $c(a_i^5)=2$ ($c(b_i^5)=2$), for every $i\in\{1,\dots,n\}$, and since the path that we add in $G(\mathcal{F})$ between r and a_i^5 (b_i^5) is of odd length $m-3$, one can completely extend c in order to get a proper 2-coloring of $G(\mathcal{F})$. ◇

Claim 3. *The set* $V(G(\mathcal{F}))\backslash\{a_i^1, a_i^2, v_i^1, b_i^1, b_i^2\}$ *is convex, for any* $i \in \{1,\dots,n\}$.

Proof. Denote $W_i = \{a_i^1, a_i^2, v_i^1, b_i^1, b_i^2\}$, for some $i \in \{1,\dots,n\}$, and $W_i' = \{a_i^3, b_i^3, v_i^2\}$. By contradiction, suppose that there exists an (x,y)-shortest path containing a vertex of W_i, for some $x,y \in V(G(\mathcal{F}))\backslash W_i$. Observe that it implies that that there are $x', y' \in W_i'$ such that $I[x',y']$ contains a vertex of W_i, since W_i' contains all the neighbors of W_i in $V(G(\mathcal{F}))\backslash W_i$. However, it is easy to verify that for any pair $x,y \in W_i'$, $I[x,y]$ contains no vertex of W_i. This is a contradiction. ◇

Lemma 6. $hn(G(\mathcal{F})) \geq n+1$.

Proof. Let S be any hull set of $G(\mathcal{F})$. Clearly $u' \in S$, because u' is a simplicial vertex of $G(\mathcal{F})$ (Lemma 1). Furthermore, Claim 3 and Lemma 4 imply that S must contain at least one vertex w_i of the set $\{a_i^1, a_i^2, v_i^1, b_i^1, b_i^2\}$, for every $i \in \{1,\dots,n\}$. Hence, $|S| \geq n+1$. ◇

The main part of the proof consists in showing:

Lemma 7. $\mathcal{F}$ *is satisfiable if and only if* $hn(G(\mathcal{F})) = n+1$.

First, consider that $\mathcal{F}$ is satisfiable. Given an assignment A that turns $\mathcal{F}$ true, define a set S as follows. For $1 \leq i \leq n$, if x_i is true in A add a_i^1 to S, otherwise add b_i^1 to S. Finally, add u' to S. Note that $|S| = n+1$. We show that S is a hull set of $G(\mathcal{F})$. First note that $a_i^5, c_j \in I[a_i^1, u']$, for every clause C_j containing the positive literal of x_i. Similarly, observe that $b_i^5, c_j \in I[b_i^1, u']$, for every clause C_j containing the negative literal of x_i. Since A satisfies $\mathcal{F}$, it follows $L \subseteq I_h[S]$. Therefore, H being an isometric subgraph of $G(\mathcal{F})$, Lemma 3 and Claim 3 imply that $V(H) \subseteq I_h[S]$. Furthermore, the shortest paths between r and u have length m, which implies that all vertices a_i^5, b_i^5, y_i^j $(i \leq n)$ and all vertices in D are included in $I_h[S]$. It remains to observe that $I_h[a_i^5, b_i^5, w, u']$, where $w \in \{a_i^1, b_i^1\}$, contains the variable subgraph of x_i. Therefore we have that S is a hull set of $G(\mathcal{F})$.

We prove the sufficiency by contradiction. Suppose that $G(\mathcal{F})$ contains a hull set S with $n+1$ vertices and that $\mathcal{F}$ is not satisfiable.

Recall that, by Lemma 1, $u' \in S$. For any $i \leq n$, let W_i as defined in Claim 3. Recall also that there must be a vertex $w_i \in W_i \cap S$, for any $i \leq n$. Since $v_i^1 \in I[u', a_i^1]$, $v_i^1 \in I[u', b_i^1]$, $a_i^2 \in I[u', a_i^1]$ and $b_i^2 \in I[u', b_i^1]$, we can assume, without loss of generality, that $w_i \in \{a_i^1, b_i^1\}$, for every $i \in \{1,\dots,n\}$ (indeed, if $w_i \in \{v_i^1, a_i^2\}$, it can be replaced by a_i^1, and if $w_i = b_i^2$, it can be replaced by b_i^1). Therefore S defines the following truth assignment $\mathcal{A}$ to $\mathcal{F}$. If $w_i = a_i^1$ set x_i to true, otherwise set x_i to false. As $\mathcal{F}$ is not satisfiable, there exists at least one clause C_j not satisfied by $\mathcal{A}$.

Using the hypothesis that $\mathcal{F}$ is not satisfiable, we complete the proof by showing that there is a non empty set U such that $V(G(\mathcal{F}))\backslash U$ is a convex set and $U \cap S = \emptyset$. That is, we show that $I_h[S] \subseteq V(G(\mathcal{F}))\backslash U$ for some $U \neq \emptyset$, contradicting the fact that S is a hull set.

For any clause C_j, let us define the subset U_j of vertices as follows. Let P_j be the path in T between c_j and r, let X_j be the p vertices in $V(T) \setminus V(P_j)$ that are adjacent to some vertex in P_j. Then, U_j is the union of the vertices that are either in P_j or that are internal vertices of the paths resulting of the subdivision of the edges $\{x,y\}$ where $x,y \in P_j \cup X_j$. Another way to build the set U_j is to start with the set of vertices of the (unique) shortest path between c_j and r in H and then add successively to this set, the vertices of $V(H) \setminus (V(T) \cup \{u\})$ that are adjacent to some vertex of the current set.

Now, let $U' = \cup_{j \in J} U_j$ where J is the (non empty) set of clauses that are not satisfied by $\mathcal{A}$. Note that $r \in U'$.

For any $i \leq n$, let W_i be defined as follows. If $w_i = a_i^1$ (x_i assigned to true by $\mathcal{A}$), then W_i is the union of $\{b_i^\ell : \ell \leq 5\}$ with the set of the y_i^k that are adjacent to b_i^5. Otherwise, $w_i = b_i^1$ (x_i assigned to false by $\mathcal{A}$), then W_i is the union of $\{a_i^\ell : \ell \leq 5\}$ with the set of the y_i^k that are adjacent to a_i^5.

Finally, let $U = U' \cup (\bigcup_{i \leq n} W_i) \cup D$. In Figure 1, U is depicted by the white vertices, assuming that clause C_2 is false and that x_i is set to false by $\mathcal{A}$. Observe that $U \cap S = \emptyset$.

It remains to prove that $V(G(\mathcal{F}))\backslash U$ is a convex set. Consider the partition $\{A_1, A_2, A_3\}$ of $V(G(\mathcal{F}))\backslash U$ where $A_1 = V(H)\backslash(U \cup \{u\})$, $A_2 = \{u, u'\}$ and $A_3 = V(G(\mathcal{F}))\backslash(U \cup A_1 \cup A_2)$. To prove that $V(G(\mathcal{F}))\backslash U$ is convex, let

$w \in A_i$ and $w' \in A_j$ for some $i, j \in \{1,2,3\}$. We show that $I[w,w'] \cap U = \emptyset$ considering different cases according to the values of i and j. Recall that $V(H) \setminus \{u\}$ induces a tree T' rooted in r and that, if a vertex of T' is in A_1, then, by definition of U', all its descendants in T' are also in A_1 (i.e., if $v \in U \cap V(T')$, then all ancestors of v in T' are in U). It is important to note that, for any vertex v in A_1, the shortest path in $G(\mathcal{F})$ from v to any leaf ℓ of T' is the path from v to ℓ in T' (in particular, such a shortest path does not pass through r and any vertices in D).

- The case $i = j = 2$, i.e., $m, m' \in \{u, u'\}$, is trivial;
- First, let us assume that $w \in A_1 = V(H) \setminus (U \cup \{u\})$ and $w' \in A_2 = \{u, u'\}$. If $w' = u$ (resp., if $w' = u'$) then $I_h[w,w']$ consists of the subtree of T' rooted in w union u (resp., union u and u'). Hence, $I_h[w,w'] \cap U = \emptyset$ because no descendants of w in T' are in U.
- Second, let $w, w' \in A_1$. If one of them, say w, is an ancestor of the other in T', then $I_h[w,w']$ consists of the path between them in T' (remember that $r \in U$ so $w \neq r$). Since no descendants of w in T' are in U, $I_h[w,w'] \cap U = \emptyset$. Otherwise, there are three cases: (1) either $I_h[w,w']$ consists of the path P between w and w' in T', or (2) $I_h[w,w']$ consists of the union of the subtree R of T' rooted in w, the subtree R' of T' rooted in w' and u, or (3) $I_h[w,w'] = R \cup R' \cup P \cup \{u\}$. Again, $(R \cup R' \cup \{u\}) \cap U = \emptyset$ because no descendants of w and w' in T' are in U. Hence, it only remains to prove that when $P \subseteq I_h[w,w']$ then $P \cap U = \emptyset$. It is easy to check that $P \subseteq I_h[w,w']$ only in the following case: there exist $x, y, z \in V(T)$ such that x is the parent of y and z in T, and w (resp., w') is a vertex of the path resulting from the subdivision of $\{x,y\}$ (resp., $\{x,z\}$). In this case, it means that all clause-vertices that are descendants of y and z are not in U. Therefore $x \notin U$ and hence no descendants of x are in U. In particular, $P \cap U = \emptyset$.
- Assume now that $w \in A_3$. Let $i \leq n$ such that w belongs to the gadget G_i corresponding to variable x_i. Let us assume that $w_i = b_i^1$. The case $w_i = a_i^1$ can be handled in a similar way by symmetry. Then, by definition, U contains $\{a_i^1, \cdots, a_i^5\}$ and the y_i^j's adjacent to a_i^5. With this setting, x_i is set to false in the assignment $\mathcal{A}$. If there is a vertex y_i^j adjacent to b_i^5, let C_j be the other neighbor of j_i^j. By definition, it means that clause C_j contains the negation of variable x_i. Since x_i is set to false, it means that clause C_j is satisfied and so $C_j \notin U$.

  Let $x \in V(G_i) \setminus U$. Then, any shortest path P from w to x either passes through $V(G_i) \setminus U$ or, there is y_i^j adjacent to b_i^5 such that P passes through y_i^j, C_j, u and v_i^3 (the latter case may occur if $a \in \{y_i^j, b_i^5\}$ and $b = v_i^3$, or $a = y_i^j$ and $b \in \{v_i^3, v_i^2\}$ where $\{a,b\} = \{x,w\}$). Hence, such a path P avoid U, and the result holds if $x = w' \in A_3 \cap G_i$.

  Similarly, if $x \in \{u,u'\}$, then, any shortest path P from w to x either passes through $V(G_i) \setminus U$ or through y_i^j, C_j, u with y_i^j adjacent to b_i^5. In particular, if $x = w' \in \{u,u'\} = A_2$, then the result holds.

  Now, let $x = C_{j'}$ be a leaf of T' that is not in U. Then, any shortest path P from w to x either passes through u or through y_i^j, C_j and, if $j \neq j'$, through u. In any case, P avoids U. If $w' \in A_3 \setminus G_i$, any path between w and w' passes through u or through one or two leaves that are not in U. Finally, if $w' \in A_1$, let R be the subtree of T' rooted in w'. $V(R) \subseteq I_h[w,w']$. Moreover, any shortest path from w to w' path through a leaf of R, i.e., a leaf not in U. By previous remarks, in all these cases, the shortest paths between w and w' avoid u, and $I_h[w,w']$ are disjoint from U.

□

We conclude this section by showing one *approximability result.* Let $IG(G)$ be the *incidence graph* of G, obtained from G by subdividing each edge once. That is, let us add one vertex s_{uv}, for each edge $uv \in E(G)$, and replace the edge uv by the edges $us_{uv}, s_{uv}v$.

Proposition 2. $hn(IG(G)) \leq hn(G) \leq 2hn(IG(G))$.

Proof. Let $IG(G)$ be the incidence graph of G. Observe that any hull set of G is a hull set of $IG(G)$, since for any shortest path, $P = \{v_1, \dots, v_k\}$ in G there is a shortest path $P' = \{v_1, s_{v_1 v_2}, v_2, \dots, s_{v_{k-1} v_k}, v_k\}$ in $IG(G)$ (the edges were subdivided). Consequently, $hn(IG(G)) \leq hn(G)$. However, given a hull set S_h of $IG(G)$, one may find a hull set of G by simply replacing each vertex of S_h that represents an edge of G by its neighbors (vertices of G). Thus, $hn(G) \leq 2hn(IG(G))$.

□

Corollary 1. *If there exists a k-approximation algorithm B to compute the hull number of bipartite graphs, then B is a* $2k$*-approximation algorithm for any graph.*

4 Complement of bipartite graphs

A graph $G = (V, E)$ is a complement of a bipartite graph if there is a partition $V = A \cup B$ such that A and B are cliques. In this section, we give a polynomial-time algorithm to compute a hull set of G with size $hn(G)$. We start with some notations.

Given the partition (A, B) of V, we say that an edge $uv \in E$ is a *crossing-edge* if $u \in A$ and $v \in B$. Denote by S the set of simplicial vertices of G, by $S_A = S \cap A$ and by $S_B = S \cap B$. Let U be the set of universal vertices of G. Note that, if G is not a clique, $U \cap S = \emptyset$. Let H be the graph obtained from G by removing the vertices in S and U, and removing the edges intra-clique, i.e., $V(H) = V \setminus (U \cup S)$ and $E(H) = \{\{u, v\} \in E \ : \ u \in A \cap V(H) \textit{ and } v \in B \cap V(H)\}$. Let $\mathcal{C} = \{C_1, \cdots, C_r\}$ $(r \geq 1)$ denote the set of connected components C_i of H. Observe that, if G is neither one clique nor the disjoint union of A and B, H is not empty and each connected component C_i has at least two vertices, for every $i \in \{1, \dots, r\}$. Indeed, any vertex in $A \setminus S_A$ (resp., in $B \setminus S_B$) has a neighbor in $B \cap V(H)$ (resp. in $A \cap V(H)$).

Theorem 3. *Let* $G = (A \cup B, E)$ *be the complement of a bipartite n-node graph. There is an algorithm that computes* $hn(G)$ *and a hull set of this size in time* $O(n^7)$.

Proof. We use the notations defined above. Recall that, by Lemma 1, S is contained in any hull set of G. In particular, if G is a clique or G is the disjoint union of two cliques A and B, then $hn(G) = n$. From now on, we assume it is not the case. By Lemma 2, no vertices in U belong to any minimal hull set of G. Now, several cases have to be considered.

Claim 4. *If* $U = \emptyset$, $S_A \neq \emptyset$ *and* $S_B \neq \emptyset$, *then S is a minimum hull set of G and thus* $hn(G) = |S|$.

Proof. Since G has no universal vertex, a simplicial vertex in S_A (in S_B) has no neighbor in B (resp., in A). Since G is not the disjoint union of two cliques, every vertex $u \in A \backslash S_A$ has a neighbor $v \in B \backslash S_B$ and vice-versa. Thus, $s_a u v s_b$ is a shortest (s_a, s_b)-path, for any $s_a \in A$ and $s_b \in B$, and then $u, v \in I_h[S]$. □

Hence, from now on, let us assume that $U \neq \emptyset$ or, w.l.o.g., $S_B = \emptyset$.

Again, if there is some simplicial vertex in G, i.e., if $S_A \neq \emptyset$, all the vertices of S belong to any hull set of G and thus $hn(G) \geq |S|$. In fact, for each connected component of H, we prove that it is necessary to choose at least one of its vertices to be part of any hull set of G.

Claim 5. *If* $U \neq \emptyset$ *or* $S_B = \emptyset$ *or* $S_A = \emptyset$, *then* $hn(G) \geq |S| + r$.

Proof. Again, all vertices of S belong to any hull set of G. We show that, for any $1 \leq i \leq r$, $V \backslash C_i$ is a convex set. Thus, by Lemma 4, any hull set of G contains at least one vertex of C_i for any $i \leq r$.

It is sufficient to show that no pair $u, v \in V(G) \backslash C_i$ can generate a vertex v_i of C_i. By contradiction, suppose that there exists a pair of vertices $u, v \in V(G) \backslash C_i$ such that there is a shortest (u, v)-path P containing a vertex v_i of C_i. Consequently, u and v must not be adjacent and we consider that $u \in A$ and $v \in B$. If $U = \emptyset$, then, w.l.o.g.,

$S_B = \emptyset$ and v is not simplicial and has at least one neighbor in A. Hence, since $U \neq \emptyset$ or $S_b = \emptyset$, u and v are at distance two. Consequently, $P = uv_iv$. However, if $v_i \in A$, v belongs to C_i, because of the crossing edge v_iv, otherwise, $u \in C_i$. In both cases we reach a contradiction. □

Now, two cases remain to be considered. We recall that $U \neq \emptyset$ or $S_B = \emptyset$.

1. If $r \geq 2$, then $hn(G) = |S| + r$, and we can build a minimum convex hull by taking the vertices in S, one arbitrary vertex in $A \cap C_i$ for all $i < r$ and one arbitrary vertex in $B \cap C_r$.

   Let $R = \{v_1, \ldots, v_r\}$ such that $v_i \in C_i \cap A$ for any $i < r$ and $v_r \in C_r \cap B$.

   Claim 6. *$S \cup R$ is a hull set of G.*

   Proof. Since all vertices in U are generated by v_1 and v_r (that are not adjacent, since they are in different components), it is sufficient to show that $S \cup R$ generates all the vertices in C_i, for any $i \in \{1, \ldots, r\}$. Actually, we show that R generates all the vertices in C_i.

   By contradiction, suppose that there is a vertex $z \notin I_h[R]$. Let $i \leq r$ such that $z \in C_i$. Because C_i contains one vertex in R and is connected, we can choose z and $w \in C_i \cap I_h[R]$ linked by a crossing edge. We will show that $z \in I_h[R]$ (a contradiction), hence, w.l.o.g., we may assume that $z \in A$. If $i = r$, then v_1zw is a shortest (v_1, w)-path and $z \in I_h[R]$.

   Otherwise, recall that $N(v_r) \cap A \cap C_r \neq \emptyset$ and, for any $i < r$, $N(v_i) \cap B \cap C_i \neq \emptyset$ because v_i is not simplicial for any $i \leq r$. Let $x \in N(v_r) \cap A \cap C_r$ and $y_i \in N(v_i) \cap B \cap C_i$. Note that $x \in I_h[R]$ because v_1xv_r is a shortest (v_r, v_1)-path, and $y_i \in I_h[R]$ because $v_iy_iv_r$ is a shortest (v_r, v_i)-path. Hence, since xzy_i is a shortest (x, y_i)-path, we have $z \in I_h[R]$. □

   As $|R| = r$, we conclude by Claim 5 that $hn(G) = |S| + r$.

2. If $r = 1$, then $hn(G) \leq |S| + 4$, and any minimum convex hull contains at most 4 vertices not in S.

   Again, S is included in any hull set of G by Lemma 1, and no vertices in U belong to some hull set by Lemma 2. In this case, when H has just one connected component $C_1 = C$, one vertex of C may not suffice to generate this component, as in the previous case. However, we prove that at most 4 vertices in C are needed.

   (a) If $S_A \neq \emptyset$ and $S_B \neq \emptyset$ (and thus $U \neq \emptyset$ because Claim 4 applies otherwise), then $hn(G) = |S| + 1$.

   By Claim 5, we know that $hn(G) \geq |S| + 1$. Let v be an arbitrary vertex of C. We claim that $S \cup \{v\}$ is a minimum hull set of G. By contradiction, let $z \notin I_h[S \cup \{v\}]$. Since C is a connected component of H, we may choose z such that there is $w \in N(z) \cap C \cap I_h[S \cup \{v\}]$. Moreover, we may assume w.l.o.g. that $z \in A$, and thus $w \in B$. In that case, since $S_A \neq \emptyset$, there is $v_A \in S_A$ and as $v_Aw \notin E(G)$ (indeed, any vertex in $N(v_A) \cap B$ must be universal because v_A is simplicial, which is not the case since w is not universal because it belongs to C), z is generated by v_A and w.

   (b) If $S_A \neq \emptyset$ and $S_B = \emptyset$, then $hn(G) \leq |S| + 2$.

   Let $v_A \in A \cap C$ be such that $|N(v_A) \cap B \cap C|$ is maximum. Since v_A is not universal in G, there exists $x \in B$ such that $v_Ax \notin E(G)$. Note that $x \in C$ since x is not universal and $S_B = \emptyset$. Let $R = \{v_A, x\}$. Observe that $N(v_A) \cap B \cap C \subseteq I_h[R \cup S]$ since $v_Ax \notin E$.

   By contradiction, assume $V(G) \setminus I_h[R \cup S] \neq \emptyset$. Let $z \in V(G) \setminus I_h[R \cup S]$. First, suppose that $z \in A$. Since C is connected in H, we may assume that z has a neighbor $w \in I_h[R \cup S] \cap B \cap C$. As $S_A \neq \emptyset$, there is $v \in S_A$ and as $vw \notin E(G)$ (because otherwise w would be universal in G and not in C), z is generated by v and w. Now suppose that $z \in B$, and now it has a neighbor $w \in I_h[R \cup S] \cap A \cap C$. Observe that $I_h[R \cup S] \cap B \subseteq N(w)$, otherwise z would be in $I_h[R \cup S]$. However, since $N(v_A) \cap B \cap$

$C \subset (N(v_A) \cap B \cap C) \cup \{x\} \subseteq I_h[R \cup S] \cap B$, we get that $N(v_A) \cap B \cap C \subset N(w) \cap B \cap C$, contradicting the maximality of $|N(v_A) \cap B \cap C|$.

(c) If $S_A = \emptyset$ and $S_B = \emptyset$, then $hn(G) \leq 4$.

Let $v_A \in A \cap C$ be such that $|N(v_A) \cap B \cap C|$ is maximum and $v_B \in B \cap C$ be such that $|N(v_B) \cap A \cap C|$ is maximum. Since v_A is not universal in G and $S_B = \emptyset$, there exists $y \in C \cap B \setminus N(v_a)$, and similarly there exists $x \in C \cap A \setminus N(v_B)$. Let $R = \{v_A, v_B, x, y\}$. Observe that $N(v_A) \cap B \subseteq I_h[R]$ and $N(v_B) \cap A \subseteq I_h[R]$, since $v_A y \notin E$ and $v_B x \notin E$.

By contradiction, assume $V(G) \setminus I_h[R] \neq \emptyset$. Let $z \in V(G) \setminus I_h[R]$. First, suppose that $z \in A$. As in the previous case, since C is connected in H, we may assume that z has a neighbor $w \in I_h[R] \cap B \cap C$. Observe that $I_h[R] \cap A \cap C \subseteq N(w)$, otherwise z would be in $I_h[R]$. However, since $N(v_B) \cap A \cap C \subset (N(v_B) \cap A \cap C) \cup \{x\} \subseteq I_h[R] \cap A \cap C$, we get that $N(v_B) \cap A \cap C \subset N(w) \cap A \cap C$, contradicting the maximality of $|N(v_B) \cap A \cap C|$.

Whenever $z \in B$, one can use the same arguments to reach a contradiction on the maximality of $|N(v_A) \cap B \cap C|$.

Since $|S| + 1 \leq hn(G) \leq |S| + 4$, S is included in any hull set of G and no vertices in U belong to some hull set, there exist a subset R of at most 4 vertices in C such that $S \cup R$ is a minimum hull set of G. There are $O(|V|^4)$ subsets to be tested and, for each one, its convex hull can be computed in $O(|V||E|)$ time [12]. This leads to the announced result.

□

5 Graphs with few P_4's

A graph $G = (V, E)$ is a $(q, q-4)$-graph, for a fixed $q \geq 4$, if for any $S \subseteq V$, $|S| \leq q$, S induces at most $q-4$ paths on 4 vertices [2]. Observe that cographs and P_4-sparse graphs are the $(q, q-4)$-graphs for $q = 4$ and $q = 5$, respectively. The hull number of a cograph can be computed in polynomial time [12]. This result is improved in [1] to the class of P_4-sparse graphs. In this section, we generalize these results by proving that for any fixed $q \geq 4$, computing the hull number of a $(q, q-4)$-graph can be done in polynomial time. Our algorithm runs in time $O(2^q n^2)$ and is therefore a Fixed Parameter Tractable for any graph G, where the number of induced P_4's of G is the parameter.

5.1 Definitions and brief description of the algorithm

The algorithm that we present in this section uses the canonical decomposition of $(q, q-4)$-graphs, called *Primeval Decomposition*. For a survey on Primeval Decomposition, the reader is referred to [3]. In order to present this decomposition of $(q, q-4)$-graphs, we need the following definitions.

Let G_1 and G_2 be two graphs. $G_1 \cup G_2$ denotes the disjoint union of G_1 and G_2. $G_1 \oplus G_2$ denotes the join of G_1 and G_2, i.e., the graph obtained from $G_1 \cup G_2$ by adding an edge between any two vertices $v \in V(G_1)$ and $w \in V(G_2)$. A *spider* $G = (S, K, R, E)$ is a graph with vertex set $V = S \cup K \cup R$ and edge set E such that

1. (S, K, R) is a partition of V and R may be empty;
2. the subgraph $G[K \cup R]$ induced by K and R is the join $K \oplus R$, and K separates S and R, i.e., any path from a vertex in S to a vertex in R contains a vertex in K;
3. S is a stable set, K is a clique, $|S| = |K| \geq 2$, and there exists a bijection $f : S \to K$ such that, either $N(s) \cap K = K - \{f(s)\}$ for all vertices $s \in S$, or $N(s) \cap K = \{f(s)\}$ for all vertices $s \in S$. In the latter case or if $|S| = |K| = 2$, G is called *thin*, otherwise G is *thick*.

A graph $G = (S,K,R,E)$ is a *pseudo-spider* if it satisfies only the first two properties of a spider. A graph $G = (S,K,R,E)$ is a *q-pseudo-spider* if it is a pseudo-spider and, moreover, $|S \cup K| \leq q$. Note that q-pseudo-spiders and spiders are pseudo-spiders.

We now describe the decomposition of $(q,q-4)$-graphs.

Theorem 4 ([2])**.** *Let $q \geq 0$ and let G be a $(q,q-4)$-graph. Then, one of the following holds:*

1. *G is a single vertex, or*
2. *$G = G_1 \cup G_2$ is the disjoint union of two $(q,q-4)$-graphs G_1 and G_2, or*
3. *$G = G_1 \oplus G_2$ is the join of two $(q,q-4)$-graphs G_1 and G_2, or*
4. *G is a spider (S,K,R,E) where $G[R]$ is a $(q,q-4)$-graph if $R \neq \emptyset$, or*
5. *G is a q-pseudo-spider (H_2,H_1,R,E) where $G[R]$ is a $(q,q-4)$-graph if $R \neq \emptyset$.*

Theorem 4 leads to a tree-like structure $T(G)$ (the *primeval tree*) which represents the Primeval Decomposition of a $(q,q-4)$-graph G. $T(G)$ is a rooted binary tree where any vertex v corresponds to an induced $(q,q-4)$-subgraph G_v of G and the root corresponds to G itself. Moreover, the vertices of subgraphs corresponding to the leaves of $T(G)$ form a partition of $V(G)$, i.e., $\{V(G_\ell)\}_{\ell \text{ } leaf \text{ } of \text{ } T(G)}$ is a partition of $V(G)$.

For any leaf ℓ of $T(G)$, G_ℓ is either a spider $(S,K,\emptyset,E)$, or has at most q vertices. Moreover, any internal vertex v has its label following one of the four cases in Theorem 4 corresponds to G_v. More precisely, let v be an internal vertex of $T(G)$ and let u and w be its two children. v is a *parallel node* if $G_v = G_u \cup G_w$. v is a *series node* if $G_v = G_u \oplus G_w$. v is a *spider node* if u is a leaf with G_u is a spider $(S,K,\emptyset,F)$ and G_v is the spider (S,K,R,E) where $G_v[R] = G_w$ and $G_v[S \cup K] = G_u$. Finally, v is a *small node* if u is a leaf with $|V(G_u)| \leq q$ and G_v is the q-pseudo-spider (S,K,R,E) where $G_v[R] = G_w$ and $G_v[S \cup K] = G_u$.

This tree can be obtained in linear-time [3].

We compute $hn(G)$ by a post-order traversal in $T(G)$. More precisely, given $v \in V(T(G))$, let H_v be an optimal hull set of G_v and let H^*_v be an optimal hull set of G^*_v, the graph obtained by adding a universal vertex to G_v. We show in next subsection that we can compute (H_ℓ, H^*_ℓ) for any leaf ℓ of $T(G)$ in time $O(2^q n)$. Moreover, for any internal vertex v of $T(G)$, we show that we can compute (H_v, H^*_v) in time $O(2^q n)$, using the information that was computed for the children and grand children of v in $T(G)$.

Theorem 5. *Let $q \geq 0$ and let G be a n-node $(q,q-4)$-graph. An optimal hull set of G can be computed in time $O(2^q n^2)$.*

Before going into the details of the algorithm in next subsection, we prove some useful lemmas.

Lemma 8 ([1])**.** *Let $G = (S,K,R,E)$ be a pseudo-spider with R neither empty nor a clique. Then any minimum hull set of G contains a minimum hull set of the subgraph $G[K \cup R]$.*

Proof. Let H be a minimum hull set of G. Let $H_S = H \cap S$ and $H_R = H \setminus H_S$. We prove that H_R is a minimum hull set of $G[K \cup R]$.

Let H' be any minimum hull set of $G[K \cup R]$. Note that $H' \subseteq R$ because K is a set of universal vertices in $G[K \cup R]$ and by Lemma 2. Moreover, By Lemma 3, because $G[K \cup R]$ is an isometric subgraph of G, the convex hull of H' in G contains $G[K \cup R]$. Hence, $H_S \cup H'$ is a hull set of G and $hn(G) \leq |H_S| + hn(G[K \cup R])$.

Now it remains to prove that H_R is a hull set of $G[K \cup R]$. Clearly, if H_R generate all vertices of R in $G[K \cup R]$ then H_R is a hull set of $G[K \cup R]$ since there are at least two non adjacent vertices in R and any vertex in K is adjacent to all vertices in R. For purpose of contradiction, assume H_R does not generate R in $G[K \cup R]$. This means that there is a vertex $v \in R$, that is generated in G by a vertex in $S \cup K$, i.e., $v \in R$ is an internal vertex of a shortest path between $s \in S \cup K$ and some other vertex, which is not possible, since we have all the edges between K and R. Hence, $hn(G[K \cup R]) \leq |H_R|$.

Therefore, $|H_S| + |H_R| = hn(G) \leq |H_S| + hn(G[K \cup R]) \leq |H_S| + |H_R|$. So, $hn(G[K \cup R]) = |H_R|$, i.e., H_R is a minimum hull set of $G[K \cup R]$ contained in H. □

The next lemma is straightforward by the use of isometry.

Lemma 9. *Let G be a graph which is not complete and that has a universal vertex. Let H obtained from G by adding some new universal vertices. A set is a minimum hull set of G if, and only if, it is a minimum hull set of H.*

5.2 Dynamic programming and correctness

In this section, we detail the algorithm presented in previous section and we prove its correctness. Let $v \in V(T(G))$, which may therefore be either a leaf, a parallel node, a series node, a spider node or a small node. For each of these five cases, we describe how to compute (H_v, H_v^*), in time $O(2^q n)$.

Let us first consider the case when v is a leaf of $T(G)$.

If G_v is a singleton $\{w\}$, then $H_v = V(G_v) = \{w\}$ and $H_v^* = V(G_v^*)$. If G_v is a spider $(S, K, \emptyset, E)$ then $H_v = S$ since S is a set of simplicial vertices (so it has to be included in any hull set by Lemma 1) and it is sufficient to generate G_v. One may easily check that if G_v is a thick spider, S is also a minimum hull set of G_v^*, i.e., $S = H_v^*$. However, in case G_v is a thin spider, S does not suffice to generate G_v^* and in this case it is easy to see that this is done by taking any extra vertex $k \in K$, in which case we have $H_v^* = S \cup \{k\}$. Finally, if G_v has at most q vertices, H_v and H_v^* can be computed in time $O(2^q)$ by an exhaustive search.

Now, let v be an internal node of $T(G)$ with children u and w.

If v is a parallel node, then $G_v = G_u \cup G_w$. Then, (H_v, H_v^*) can be computed in time $O(1)$ from (H_u, H_u^*) and (H_w, H_w^*) thanks to Lemma 10.

Lemma 10 ([12])**.** *Let $G_v = G_u \cup G_w$. Then $(H_v, H_v^*) = (H_u \cup H_w, H_u^* \cup H_w^*)$.*

Proof. The fact that $H_u \cup H_w$ is an optimal hull set for G_v is trivial. The second part comes from the fact that H_u^* (resp., H_w^*) is an isometric subgraph of H_v^* and from Lemma 3. □

Now, we consider the case when v is a series node.

Lemma 11. *If $G_v = G_u \oplus G_w$, then (H_v, H_v^*) can be computed from the sets (H_x, H_x^*) of the children or grand children x of v in $T(G)$, in time $O(2^q n)$.*

Proof. If G_u and G_w are both complete, then G_v is a clique and $(H_v, H_v^*) = (V(G_v), V(G_v^*))$.

If G_u and G_w are both not complete, let x, y be any two non adjacent vertices in G_u. Then, we claim that $H_v = H_v^* = \{x, y\}$. Indeed, in G_v, x and y generate all vertices in $V(G_w)$ (resp., of G_w^*). In particular, two non adjacent vertices $z, r \in V(G_w)$ are generated. Symmetrically, z, r generate all vertices in $V(G_u)$ (resp., in $V(G_u^*)$).

Without loss of generality, we suppose now that G_u is a complete graph and that G_w is a non-complete $(q, q-4)$-graph. First, observe that no vertex of G_u belongs to any minimum hull set of G_v, since they are universal (Lemma 2). Note also that, by Lemma 9 and since G_v is not a clique and has universal vertices, we can make $H_v = H_v^*$. Hence, in what follows, we consider only the computation of H_v. Let us consider all possible cases for w in $T(G)$.

- w is a series node. G_w is the join of two graphs. We claim that $H_v = H_w$.

  In this case, G_w is an isometric subgraph of G_v. Thus, by Lemma 3, any minimum hull set of G_w generates all vertices of $V(G_w)$ in G_v. Finally, since G_w has two non-adjacent vertices they generate all vertices of G_u in G_v.

- w is a parallel node. G_w is the disjoint union of two graphs. Let x and y the children of w in $T(G)$. Then $G_w = G_x \cup G_y$. Let $X = H^*_x$ if G_x is not a clique and $X = V(G_x)$, otherwise, let $Y = H^*_y$ if G_y is not a clique and $Y = V(G_y)$, otherwise. We claim that $H_v = X \cup Y$.

  Clearly, if G_x (resp., G_y) is a clique, all its vertices are simplicial in G_v and then must be contained in any hull set by Lemma 1. Moreover, recall that, by Lemma 2, no vertex of G_u belongs to any minimum hull set of G.

  Now, let $z \in \{x,y\}$ such that G_z is not complete. It remains to show that it is necessary and sufficient to also include any minimum hull set H^*_z of G^*_z, in any minimum hull set of G.

  The necessity can be easily proved by using Lemma 8 to every G_z that is not a complete graph.

  The sufficiency follows again from the fact that G_u is generated by two non adjacent vertices of G_w and since, in all cases, $X \cup Y$ contains at least one vertex in G_x and one vertex in G_y, all vertices in G_u will be generated.

- w is a spider node and G_w is a thin spider $(S,K,\emptyset,E')$. Then, $H_v = S \cup \{k\} = G^*_w$ where k is any vertex in K.

  All vertices in S are simplicial in G_v, hence any hull set of G_v must contain S by Lemma 1. Now, in G_v, the vertices in S are at distance two and no shortest path between two vertices in S passes through a vertex in K, since there is a join to a complete graph. Therefore, S is not a hull set of G_v. However, since $|S| \geq 2$, it is easy to check that adding any vertex $k \in K$ to S is sufficient to generate all vertices in G_v. So $S \cup \{k\}$ is a minimum hull set of G_v.

  Note that, in that way, $H_v = S \cup \{k\} = G^*_w$

- w is a spider node and G_w is a spider (S,K,R,E') that is either thick or $R \neq \emptyset$ and R induces a $(q,q-4)$-graph. Then, $H_v = H_w$.

  If $R = \emptyset$, then G_w is thick. In this case, it is easy to check that the only minimum hull set of G_w is S (because it consists of simplicial vertices) and it is also a minimum hull set for G_v. Hence, $H_v = H_w = S$.

  If $R \neq \emptyset$, then by Lemma 1 any minimum hull set of G_w contains S. Moreover, by Lemma 8 any minimum hull set of G_w contains a minimum hull set of $K \cup R$ which is composed by vertices of R.

  By the same lemmas, a minimum hull set of G_w is a minimum hull set of G_v since, by Lemma 2, no vertex of G_u belongs to any minimum hull set of G_v and G_u is generated by non-adjacent vertices of G_w.

- w is a small node. G_w is a q-pseudo-spider (H_2,H_1,R,E') and R induces a $(q,q-4)$-graph.

  If $R = \emptyset$, G_v is the join of a clique G_u with a graph G_w that has at most q vertices. No vertex of G_u belongs to any minimum hull set of G_v, since they are universal. Thus, H_v can be computed in time $O(2^q)$ by testing all the possible subsets of vertices of G_w.

  Similarly, if R is a clique, all vertices in R are simplicial in G_v so they must belong to any hull set of G_v. Moreover, no vertices in G_u belong to any minimum hull set of G_v. So H_v can be computed in time $O(2^q)$ by testing all the possible subsets of vertices of $H_1 \cup H_2$ and adding R to them.

  In case $R \neq \emptyset$ nor a clique, two cases must be considered. By definition of the decomposition, there exists a child r of w in $T(G)$ such that $V(G_r) = R$.

  - If $G[H_1]$ is a clique, then, $G_v = (H_2, H_1 \cup V(G_u), R, E)$ is a pseudo-spider that satisfies the conditions in Lemma 8. Hence, any minimum hull set of G_v contains a minimum hull set of $P = G[H_1 \cup V(G_u) \cup R]$. Let Z be a minimum hull set of G_v and let $Z' = Z \cap H_2$. By Lemma 8, we have $|Z'| \leq hn(G_v) - hn(P)$.

    By Lemma 9, H^*_r is a minimum hull set of $G[H_1 \cup V(G_u) \cup R]$. Now, $G[H_1 \cup V(G_u) \cup R]$ is an isometric subgraph of G_v. Hence, by Lemma 3, H^*_r generates all vertices of $G[H_1 \cup V(G_u) \cup R]$ in

G_v. Therefore, $H_r^* \cup Z'$ will generate all vertices of G_v. Since $|H_r^*| = hn(P)$, we get that $|H_r^* \cup Z'| \leq hn(G_v)$ and then $H_r^* \cup Z'$ is a minimum hull set of G_v.

So, we have shown that there exists a minimum hull set for G_v that can be obtained from H_r^* by adding some vertices in $H_1 \cup H_2$. Since $|H_1 \cup H_2| \leq q$, such a subset of $H_1 \cup H_2$ can be found in time $O(2^q)$.

– In case $G[H_1]$ is not a clique, let x and y be two non adjacent vertices of H_1. We claim in this case that there exists a minimum hull set of G_v containing at most one vertex of R. Let S be a minimum hull set of G_v containing at least two vertices in R. Observe that $S' = (S \backslash R) \cup \{x, y\}$ is also a hull set of G_v since x and y are sufficient to generate all vertices in R. Consequently, $|S'| \leq |S|$ and S' is minimum.

  Since no hull set of G_v contains a vertex in $V(G_u)$, there always exists a minimum hull set of G_v that consists of only vertices in $H_1 \cup H_2$ plus at most one vertex in R. Therefore an exhaustive search can be performed in time $O(n2^q)$.

□

Now, we consider the case when v is a spider node or a small node. That is $G_v = (S, K, R, E)$. If $R \neq \emptyset$, let r be the child of v such that $V(G_r) = R$.

Lemma 12. *Let $G_v = (S, K, R, E)$ be a spider such that R induces a $(q, q-4)$-graph.*
Then, $H_v = H_v^* = S \cup H_r^*$ if $R \neq \emptyset$ and R is not a clique, and $H_v = H_v^* = S \cup R$, otherwise.

Proof. Since all the vertices in S are simplicial vertices in G_v and in G_v^*, we apply Lemma 1 to conclude that they are all contained in any hull set of G_v (resp., of G_v^*).

By the structure of a spider, every vertex of K (and the universal vertex in G_v^*) belongs to a shortest path between two vertices in S and are therefore generated by them in any minimum hull set of G_v (resp., of G_v^*). Consequently, if $R = \emptyset$, S is a minimum hull set of G_v (resp., of G_v^*). If R is a clique, $S \cup R$ is the set of simplicial vertices of G_v (resp., of G_v^*) and also a minimum hull set of G_v (resp., of G_v^*).

Finally, if $R \neq \emptyset$ and R is not a clique, then G_v is a pseudo-spider satisfying the conditions of Lemma 8. Similarly, G_v^* is a pseudo-spider (by including the universal vertex in K). Then, by Lemma 8, any hull set of G_v (resp., of G_v^*) contains a minimum hull set of $G[K \cup R]$ (resp., of $G_v^* \setminus S$. Moreover, any hull set contains all vertices in S since they are simplicial. Hence, $hn(G_v) = hn(G_v^*) = |S| + hn(G[K \cup R])$ (recall that, by Lemma 9, $hn(G[K \cup R]) = hn(G_v^* \setminus S)$). Finally, since $G[K \cup R]$) is an isometric subgraph of G_v, then H_r^* (which is a minimum hull set of $G[K \cup R]$ by Lemma 9) generates $G[K \cup R]$ in G_v (resp., in G_v^*).

Hence, $S \cup H_r^*$ is a hull set of G_v and G_v^*. Moreover, it has size $|S| + hn(G[K \cup R])$, so it is optimal.

□

Lemma 13. *Let $G_v = (H_2, H_1, R, E)$ be a q-pseudo-spider such that R is a $(q, q-4)$-graph. Then, H_v and H_v^* can be computed in time $O(2^q n)$.*

Proof. All the arguments to prove this lemma are in the proof of Lemma 11. Moreover, the following arguments hold both for G_v and G_v^*: they allow to compute both H_v and H_v^*.

If $R = \emptyset$, G_v has at most q vertices, for a fixed positive integer q. Thus, its hull number can be computed in $O(2^q)$-time.

Otherwise, if H_1 is a clique, by Lemma 8, any minimum hull set of G_v contains a minimum hull set of $G[H_1 \cup R]$. Moreover, by the same arguments as in Lemma 11, we can show that there is an optimal hull set for G_v that can be obtained from H_r^* (minimum hull set of $G[H_1 \cup R]$) and some vertices in H_2.

If H_1 is not a clique, two non-adjacent vertices of H_1 can generate R. Thus, we conclude that there exists a minimum hull set of G_v containing at most one vertex of R. Then, a minimum hull set of G_v can be found in $O(2^q n)$-time, where $n = |V(G_v)|$. □

6 Hull Number via 2-connected components

In this section, we introduce a generalized variant of the hull number of a graph. Let $G=(V,E)$ be a graph and $S\subseteq V$. Let $hn(G,S)$ denote the minimum size of a set $U\subseteq V\setminus S$ such that $U\cup S$ is a hull set for G. We prove that to compute the hull number of a graph, it is sufficient to compute the generalized hull number of its 2-connected components (or *blocks*). This extends a result in [15].

Theorem 6. *Let G be a graph and $G_1,\dots,G_n$ be its 2-connected components. For any $i\leq n$, let $S_i\subseteq V(G_i)$ be the set of cut-vertices of G in G_i. Then,*

$$hn(G)=\sum_{i\leq n} hn(G_i,S_i).$$

Proof. Clearly, the result holds if $n=1$, so we assume $n>1$.

A block G_i is called a *leaf-block* if $|S_i|=1$. Note that, for any leaf-block G_i, $G[V\setminus(V(G_i)\setminus S_i)]$ is convex, so by Lemma 4, any hull set of G contains at least one vertex in $V(G_i)\setminus S_i$. Moreover,

Claim 7. *For any minimum hull set S of G, $S\cap(\cup_{i\leq n}S_i)=\emptyset$.*

Proof. For purpose of contradiction, let us assume that a minimum hull set S of G contains a vertex $v\in S_i$ for some $i\leq n$. Note that there exist two leaf-blocks G_1 and G_2 such that v is on a shortest path between vertices in $V(G_1)$ and $V(G_2)$ or $\{v\}=V(G_1)\cap V(G_2)$. By the remark above, there exist $x\in(V(G_1)\setminus S_1)\cap S$ and $y\in(V(G_2)\setminus S_2)\cap S$. Hence, v is on a shortest (x,y)-path, i.e., $v\in I[x,y]\subseteq I_h[S\setminus\{v\}]$. Hence, $V\subseteq I_h[S]\subseteq I_h[S\setminus\{v\}]$ and $S\setminus\{v\}$ is a hull set of G, contradicting the minimality of S. ◇

Claim 8. *Let S be a hull set of G. Then $S'=(S\cap V(G_i))\cup S_i$ is a hull set of G_i.*

Proof.

For purpose of contradiction, assume that $I_h[S']=V(G_i)\setminus X$ for some $X\neq\emptyset$. Then, there is $v\in X\cap I[a,b]$ for some $a\in V(G)\setminus V(G_i)$ and $b\in V(G)\setminus X$. Then, there is a shortest (a,b)-path P containing v. Hence, there is $u\in S_i$ such that u is on the subpath of P between a and v. Moreover, let $w=b$ if $b\in G_i$, and else let w be a vertex of S_i on the subpath of P between v and b. Hence, $v\in I[u,w]\subseteq I_h[S']$, a contradiction.

◇

Let X be any minimum hull set of G. By Claim 7, $X\cap(\cup_{i\leq n}S_i)=\emptyset$, hence we can partition $X=\cup_{i\leq n}X_i$ such that $X_i\subseteq V(G_i)\setminus S_i$ and $X_i\cap X_j=\emptyset$ for any $i\neq j$. Moreover, by Claim 8, $X_i\cup S_i$ is a hull set of G_i, i.e., $|X_i|\geq hn(G_i,S_i)$. Hence, $hn(G)=|X|=\sum_{i\leq n}|X_i|\geq\sum_{i\leq n}hn(G_i,S_i)$.

It remains to prove the reverse inequality. For any $i\leq n$, let $X_i\subseteq V(G_i)\setminus S_i$ such that $X_i\cup S_i$ is a hull set of G_i and $|X_i|=hn(G_i,S_i)$. We prove that $S=\cup_{i\leq n}X_i$ is a hull set for G. Indeed, for any $v\in S_i$, there are two leaf-blocks G_1,G_2 such that v is on a shortest path between G_1 and G_2 or $\{v\}=V(G_1)\cap V(G_2)$. So, there exist $x\in X_1$ and $y\in X_2$ such that v is on a shortest (x,y)-path, i.e., $v\in I[x,y]\subseteq I_h[S]$. Hence, $\cup_{i\leq n}S_i\subseteq I_h[S]$ and therefore, $V=\cup_{i\leq n}I_h[X_i\cup S_i]\subseteq I_h[\cup_{i\leq n}(X_i\cup S_i)]\subseteq I_h[\cup_{i\leq n}(X_i)]=I_h[S]$. □

A *cactus* G is a graph in which every pair of cycles have at most one common vertex. This definition implies that each block of G is either a cycle or an edge. By using the previous result, one may easily prove that:

Corollary 2 ([1])**.** *In the class of cactus graphs, the hull number can be computed in linear time.*

7 Bounds

In this section, we use the same techniques as presented in [12, 15] to prove new bounds on the hull number of several graphs classes. These techniques mainly rely on a greedy algorithm for computing a hull set of a graph and that consists of the following: given a connected graph $G = (V,E)$ and its set S of simplicial vertices, we start with $H = S$ or $H = \{v\}$ (v is any vertex of V) if $S = \emptyset$, and $C_0 = I_h[H]$. Then, at each step $i \geq 1$, if $C_{i-1} \subset V$, the algorithm greedily chooses a subset $X_i \subseteq V \setminus C_{i-1}$, add X_i to H and set $C_i = I_h[H]$. Finally, if $C_i = V$, the algorithm returns H which is a hull set of G.

Claim 9. *If for every $i \geq 1$, $|C_i \setminus (C_{i-1} \cup X_i)| \geq c \cdot |X_i|$, for some constant $c > 0$, then $|H| \leq \max\{1,|S|\} + \left\lceil \frac{|V|-\max\{1,|S|\}}{1+c} \right\rceil$.*

In the following, we keep the notation used to describe the algorithm.

Claim 10. *Let G be a connected graph. Then, before each step $i \geq 1$ of the algorithm, for any $v \in V \setminus C_{i-1}$, $N(v) \cap C_{i-1}$ induces a clique. Moreover, any connected component induced by $V \setminus C_{i-1}$ has at least 2 vertices.*

Proof. Let $v \in V \setminus C_{i-1}$ and assume v has two neighbors u and w in C_{i-1} that are not adjacent. Then, $v \in I[u,w] \subseteq C_{i-1}$ because C_{i-1} is convex, a contradiction. Note that, at any step $i \geq 1$ of the algorithm, $V \setminus C_{i-1}$ contains no simplicial vertex. By previous remark, if v has only neighbors in C_{i-1}, then v is simplicial, a contradiction. □

Claim 11. *If G is a connected C_3-free graph, then, at every step $i \geq 1$ of the algorithm, a vertex in $V \setminus C_{i-1}$ has at most one neighbor in C_{i-1}.*

Proof. Assume that $v \in V \setminus C_{i-1}$ has two neighbors $u,w \in C_{i-1}$. $\{u,w\} \notin E$ because G is triangle-free. This contradicts Claim 10. □

Lemma 14. *For any C_3-free connected graph G and at step $i \geq 1$ of the algorithm, either $C_{i-1} = V$ or there exists $X_i \subset V \setminus C_{i-1}$ such that $|C_i \setminus (C_{i-1} \cup X_i)| \geq |X_i|$.*

Proof. If there is $v \in V \setminus C_{i-1}$ at distance at least 2 from C_{i-1}, let $X_i = \{v\}$ and the result clearly holds. Otherwise, let v be any vertex in $V \setminus C_{i-1}$. By Claim 10, v has a neighbor u in $V \setminus C_{i-1}$. Moreover, because no vertices of $V \setminus C_{i-1}$ are at distance at least 2 from C_{i-1}, v and u have some neighbors in C_{i-1}. Finally, u and v have no common neighbors because G is triangle-free. Hence, by taking $X_i = \{v\}$, we have $u \in C_i$ and the result holds. □

Recall that the *girth* of a graph is the length of its smallest cycle.

Lemma 15. *Let G connected with girth at least 6. Before any step $i \geq 1$ of the algorithm when $C_{i-1} \neq V$, there exists $X_i \subset V \setminus C_{i-1}$ such that $|C_i \setminus (C_{i-1} \cup X_i)| \geq 2|X_i|$.*

Proof. If there is $v \in V \setminus C_{i-1}$ at distance at least 3 from C_{i-1}, let $X_i = \{v\}$ and the result clearly holds. Otherwise, let v be a vertex in $V \setminus C_{i-1}$ at distance two from any vertex of C_{i-1}. Let $w \in V \setminus C_{i-1}$ be a neighbor of v that has a neighbor $z \in C_{i-1}$. Since v is not simplicial, v has another neighbor $u \neq w$ in $V \setminus C_{i-1}$. If u is at distance two from C_{i-1}, let $y \in V \setminus C_{i-1}$ be a neighbour of u that has a neighbor $x \in C_{i-1}$. In this case, since the girth of G is at least six, $z \neq x$ and, there is a shortest (v,z)-path containing w and a shortest (v,x)-path containing u and y. Consequently, by setting $X_i = \{v\}$ we obtain the desired result. The same happens in case u has a neighbor $x \in C_{i-1}$. One may use again the hypothesis that the girth of G is at least six to conclude that, by setting $X_i = \{v\}$ we obtain that $w,u \in C_i$.

Finally, we claim that no vertex remains in $V \setminus C_{i-1}$. By contradiction, suppose that it is the case and that there are in $V \setminus C_{i-1}$ and all these vertices have a neighbor in C_{i-1}. Let v be a vertex in $V \setminus C_{i-1}$ that has a neighbor z in C_{i-1}. Again, v has a neighbor $u \in V \setminus C_{i-1}$, since it is not simplicial. The vertex u must have a neighbor x in C_{i-1}.

Observe that x and z are at distance 3, since the girth of G is at least six. Consequently, v and u are in a shortest (x,z)-path should not be in $V \setminus C_{i-1}$, that is a contradiction.

□

Lemma 16. *Let G be a connected graph. Before any step $i \geq 1$ of the algorithm when $C_{i-1} \neq V$, there exist $X_i \subset V \setminus C_{i-1}$ such that $|C_i \setminus (C_{i-1} \cup X_i)| \geq 2|X_i|/3$.*

Moreover, if G is k-regular ($k \geq 1$), there exist $X_i \subset V \setminus C_{i-1}$ such that $|C_i \setminus (C_{i-1} \cup X_i)| \geq |X_i|$.

Proof. By Claim 10, all connected component of $V \setminus C_{i-1}$ contains at least one edge.

- If there is $v \in V \setminus C_{i-1}$ at distance at least 2 from C_{i-1}, let $X_i = \{v\}$ and $|C_i \setminus (C_{i-1} \cup X_i)| \geq |X_i|$.
- Now, assume all vertices in $V \setminus C_{i-1}$ are adjacent to some vertex in C_{i-1}. If there are two adjacent vertices u and v in $V \setminus C_{i-1}$ such that there is $z \in C_{i-1} \cap N(u) \setminus N(v)$, then let $X_i = \{v\}$. Therefore, $u \in C_i$ and $|C_i \setminus (C_{i-1} \cup X_i)| \geq |X_i|$. So, the result holds.
- Finally, assume that for any two adjacent vertices u and v in $V \setminus C_{i-1}$, $N(u) \cap C_{i-1} = N(v) \cap C_{i-1} \neq \emptyset$.

  We first prove that this case actually cannot occur if G is k-regular. Let $v \in V \setminus C_{i-1}$. By Claim 10, $K = N(v) \cap C_{i-1}$ induces a clique. Moreover, for any $u \in N(v) \setminus C_{i-1}$, $N(u) \cap C_{i-1} = K$. Note that $k = |K| + |N(v) \setminus C_{i-1}|$. Let $w \in K$. Then, $A = (K \cup N(v) \cup \{v\}) \setminus \{w\} \subseteq N(w)$ and since $|A| = k$, we get that $A = N(w)$. Moreover, $N[u]$ cannot induce a clique since $V \setminus C_{i-1}$ contains no simplicial vertices, $i \geq 1$. Hence, there are $x, y \in N(v) \setminus C_{i-1}$ such that $\{x,y\} \notin E$. Because G is k-regular, there is $z \in N(x) \setminus (N(v) \cup C_{i-1})$. However, $N(z) \cap C_{i-1} = N(x) \cap C_{i-1} = K$. Hence, $z \in N(w) \setminus A$, a contradiction.

  Now, assume that G is a general graph. Let v be a vertex of minimum degree in $V \setminus C_{i-1}$. Recall that, by Claim 10, $N(v) \cap C_{i-1}$ induces a clique. Because any neighbor $u \in V \setminus C_{i-1}$ of v has the same neighborhood as v in C_{i-1} and because v is not simplicial, then there must be $u, w \in N(v) \setminus C_{i-1}$ such that $\{u, w\} \notin E$. Now, by minimality of the degree of v, there exists $y \in N(u) \setminus (N(v) \cup C_{i-1}) \neq \emptyset$. Similarly, there exists $z \in N(w) \setminus (N(v) \cup C_{i-1}) \neq \emptyset$. Let us set $X_i = \{v, z, y\}$. Hence, $u, w \in C_i \setminus (C_{i-1} \cup X_i)$ and the result holds.

□

Theorem 7. *Let G be a connected n-node graph with s simplicial vertices. All bounds below are tight:*

- $hn(G) \leq \max\{1,s\} + \left\lceil \frac{3(n-\max\{1,s\})}{5} \right\rceil$;
- *If G is C_3-free or k-regular ($k \geq 1$), then* $hn(G) \leq \max\{1,s\} + \left\lceil \frac{n-\max\{1,s\}}{2} \right\rceil$;
- *If G has girth ≥ 6, then* $hn(G) \leq \max\{1,s\} + \left\lceil \frac{1(n-\max\{1,s\})}{3} \right\rceil$.

Proof. First statement follows from Claim 9 and first statement in Lemma 16. The second statement follows from Claim 9 and Lemma 14 (case C_3-free) and second part of Lemma 16 (case regular graphs) . Last statement follows from Claim 9 and Lemma 15.

All bounds are reached in case of complete graphs. In case with no simplicial vertices: the first bound is reached by the graph obtained by taking several disjoint C_5 and adding a universal vertex, the second bound is obtained for a C_5, and the third one is reached by a C_7.

□

The first statement of the previous theorem improves another result in [15]:

Corollary 3. *If G is a graph with no simplicial vertex, then:*

$$\limsup_{|V(G)|\to\infty} \frac{hn(G)}{|V(G)|} = \frac{3}{5}.$$

It it important to remark that the second statement of Theorem 7 is closely related to a bound of Everett and Seidman proved in Theorem 9 of [15]. However, the graphs they consider do not have simplicial vertices and, consequently, they do not have vertices of degree one, which is not a constraint for our result.

8 Conclusions

In this paper, we simplified the reduction of Dourado et al. [12] to answer a question they asked about the complexity of computing the hull number of bipartite graphs. We presented polynomial-time algorithms for computing the hull number of cobipartite graphs, $(q,q-4)$-graphs and cactus graphs. Finally, we presented upper bounds for general graphs and two particular graph classes.

The result in Section 5 provides an FPT algorithm where the parameter is the number of induced P_4's in the input graph. It would be nice to know about the paramerized complexity of HULL NUMBER when the parameter is the size of the hull set.

Another question of Dourado *et al.* [12], concerning to the complexity of this problem for interval graphs and chordal graphs, remains open. Up to the best of our knowledge, determining the complexity of the HULL NUMBER problem on planar graphs is also an open problem.

References

[1] J. Araujo, V. Campos, F. Giroire, L. Sampaio, and R. Soares. On the hull number of some graph classes. In *Proceedings of the European Conference on Combinatorics, Graph Theory and Applications*, EuroComb'11, Budapest, Hungary, 2011.

[2] Luitpold Babel and Stephan Olariu. On the isomorphism of graphs with few p4s. In *Proceedings of the 21st International Workshop on Graph-Theoretic Concepts in Computer Science*, WG '95, pages 24–36, London, UK, 1995. Springer-Verlag.

[3] Luitpold Babel and Stephan Olariu. On the p-connectedness of graphs - a survey. *Discrete Applied Mathematics*, 95(1-3):11 – 33, 1999.

[4] J. A. Bondy and U. S. R. Murty. *Graph Theory*. Graduate Texts in Mathematics. Springer, 2008.

[5] Gilbert B. Cagaanan, Sergio R. Canoy, and Jr. On the hull sets and hull number of the cartesian product of graphs. *Discrete Mathematics*, 287(1-3):141 – 144, 2004.

[6] Manoj Changat and Joseph Mathew. On triangle path convexity in graphs. *Discrete Mathematics*, 206(1-3):91 – 95, 1999.

[7] Manoj Changat, Henry Martyn Mulder, and Gerard Sierksma. Convexities related to path properties on graphs. *Discrete Mathematics*, 290(2-3):117 – 131, 2005.

[8] Gary Chartrand, John Frederick Fink, and Ping Zhang. The hull number of an oriented graph. *International Journal of Mathematics and Mathematical Sciences*, 2003(36):2265–2275, 2003.

[9] Gary Chartrand, Frank Harary, and Ping Zhang. On the geodetic number of a graph. *Networks*, 39(1):1–6, 2002.

[10] Gary Chartrand, Curtiss E. Wall, and Ping Zhang. The convexity number of a graph. *Graphs and Combinatorics*, 18:209–217, 2002. 10.1007/s003730200014.

[11] J. Cáceres, C. Hernando, M. Mora, I.M. Pelayo, and M.L. Puertas. On the geodetic and the hull numbers in strong product graphs. *Computers Mathematics with Applications*, 60(11):3020 – 3031, 2010.

[12] Mitre C. Dourado, John G. Gimbel, Jan Kratochvíl, Fábio Protti, and Jayme L. Szwarcfiter. On the computation of the hull number of a graph. *Discrete Mathematics*, 309(18):5668 – 5674, 2009. Combinatorics 2006, A Meeting in Celebration of Pavol Hell's 60th Birthday (May 1-5, 2006).

[13] Mitre C. Dourado, Fábio Protti, Dieter Rautenbach, and Jayme L. Szwarcfiter. On the hull number of triangle-free graphs. *SIAM J. Discret. Math.*, 23:2163–2172, January 2010.

[14] Mitre C. Dourado, Fábio Protti, and Jayme L. Szwarcfiter. Complexity results related to monophonic convexity. *Discrete Applied Mathematics*, 158(12):1268 – 1274, 2010. Traces from LAGOS'07 IV Latin American Algorithms, Graphs, and Optimization Symposium Puerto Varas - 2007.

[15] Martin G. Everett and Stephen B. Seidman. The hull number of a graph. *Discrete Mathematics*, 57(3):217 – 223, 1985.

[16] Martin Farber and Robert E Jamison. Convexity in graphs and hypergraphs. *SIAM J. Algebraic Discrete Methods*, 7:433–444, July 1986.

[17] Alastair Farrugia. Orientable convexity, geodetic and hull numbers in graphs. *Discrete Appl. Math.*, 148:256–262, June 2005.

[18] Carmen Hernando, Tao Jiang, Mercè Mora, Ignacio M. Pelayo, and Carlos Seara. On the steiner, geodetic and hull numbers of graphs. *Discrete Mathematics*, 293(1-3):139 – 154, 2005. 19th British Combinatorial Conference.

[19] R. T. Rockafellar. *Convex analysis*. Princeton Mathematical Series, No. 28. Princeton University Press, Princeton, N.J., 1970.

Centre de recherche INRIA Sophia Antipolis – Méditerranée
2004, route des Lucioles - BP 93 - 06902 Sophia Antipolis Cedex (France)

Centre de recherche INRIA Bordeaux – Sud Ouest : Domaine Universitaire - 351, cours de la Libération - 33405 Talence Cedex
Centre de recherche INRIA Grenoble – Rhône-Alpes : 655, avenue de l'Europe - 38334 Montbonnot Saint-Ismier
Centre de recherche INRIA Lille – Nord Europe : Parc Scientifique de la Haute Borne - 40, avenue Halley - 59650 Villeneuve d'Ascq
Centre de recherche INRIA Nancy – Grand Est : LORIA, Technopôle de Nancy-Brabois - Campus scientifique
615, rue du Jardin Botanique - BP 101 - 54602 Villers-lès-Nancy Cedex
Centre de recherche INRIA Paris – Rocquencourt : Domaine de Voluceau - Rocquencourt - BP 105 - 78153 Le Chesnay Cedex
Centre de recherche INRIA Rennes – Bretagne Atlantique : IRISA, Campus universitaire de Beaulieu - 35042 Rennes Cedex
Centre de recherche INRIA Saclay – Île-de-France : Parc Orsay Université - ZAC des Vignes : 4, rue Jacques Monod - 91893 Orsay Cedex

Éditeur
INRIA - Domaine de Voluceau - Rocquencourt, BP 105 - 78153 Le Chesnay Cedex (France)
http://www.inria.fr
ISSN 0249-6399

INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

Hull number: P_5-free graphs and reduction rules

J. Araujo — G. Morel — L. Sampaio — R. Soares — V. Weber

N° 8045

Août 2012

Domaine 2

Rapport de recherche

ISSN 0249-6399 ISRN INRIA/RR--8045--FR+ENG

INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

INRIA

centre de recherche
SOPHIA ANTIPOLIS - MÉDITERRANÉE

Hull number: P_5-free graphs and reduction rules*

J. Araujo †‡ , G. Morel † , L. Sampaio †§¶ , R. Soares †‡¶ , V. Weber ‖

Thème : Algorithmique, programmation, logiciels et architectures
Équipe-Projet Mascotte

Rapport de recherche n° 8045 — Août 2012 — 10 pages

Abstract: In this paper, we study the (geodesic) hull number of graphs. For any two vertices $u, v \in V$ of a connected undirected graph $G = (V, E)$, the closed interval $I[u, v]$ of u and v is the set of vertices that belong to some shortest (u, v)-path. For any $S \subseteq V$, let $I[S] = \bigcup_{u,v \in S} I[u, v]$. A subset $S \subseteq V$ is (geodesically) convex if $I[S] = S$. Given a subset $S \subseteq V$, the convex hull $I_h[S]$ of S is the smallest convex set that contains S. We say that S is a hull set of G if $I_h[S] = V$. The size of a minimum hull set of G is the hull number of G, denoted by $hn(G)$.

First, we show a polynomial-time algorithm to compute the hull number of any P_5-free triangle-free graph. Then, we present four reduction rules based on vertices with the same neighborhood. We use these reduction rules to propose a fixed parameter tractable algorithm to compute the hull number of any graph G, where the parameter can be the size of a vertex cover of G or, more generally, its neighborhood diversity, and we also use these reductions to characterize the hull number of the lexicographic product of any two graphs.

Key-words: Graph Convexity, Hull Number, Geodesic Convexity, P_5-free Graphs, Lexicographic Product, Parameterized Complexity, Neighborhood Diversity.

* This work was partly supported by ANR Blanc AGAPE ANR-09-BLAN-0159 and the INRIA/FUNCAP exchange program.

† {julio.araujo, gregory.morel, leonardo.sampaio_rocha, ronan.pardo_soares}@inria.fr - MASCOTTE Project, I3S (CNRS & UNS) and INRIA, INRIA Sophia Antipolis, 2004 route des Lucioles, BP 93, 06902 Sophia Antipolis Cedex France.

‡ ParGO Research Group, Universidade Federal do Ceará, Campus do Pici, Bloco 910. 60455-760, Fortaleza, Ceará, Brazil.

§ Partially supported by CAPES/Brazil

¶ Partially supported by ANR Blanc AGAPE ANR-09-BLAN-0159.

‖ valentin.weber@g-scop.grenoble-inp.fr / Grenoble-INP / UJF-Grenoble 1 / CNRS, G-SCOP UMR5272 Grenoble, F-38031, France.

Centre de recherche INRIA Sophia Antipolis – Méditerranée
2004, route des Lucioles, BP 93, 06902 Sophia Antipolis Cedex
Téléphone : +33 4 92 38 77 77 — Télécopie : +33 4 92 38 77 65

Nombre enveloppe : graphes sans P_5 et règles de réduction

Résumé : Dans cet article, nous étudions le *nombre enveloppe* (géodésique) des graphes. Pour deux sommets u et $v \in V$ d'un graphe connexe non orienté $G = (V, E)$, l'intervalle fermé $I[u, v]$ de u et v est l'ensemble des sommets qui appartiennent à une plus courte chaîne reliant u et v. Pour tout $S \subseteq V$, on note $I[S] = \bigcup_{u,v \in S} I[u, v]$. Un sous-ensemble $S \subseteq V$ est (géodésiquement) convexe si $I[S] = S$. Étant donné un sous-ensemble $S \subseteq V$, l'enveloppe convexe $I_h[S]$ de S est le plus petit ensemble convexe qui contient S. On dit que S est un *ensemble enveloppe* de G si $I_h[S] = V$. La taille d'un ensemble enveloppe minimum de G est le nombre enveloppe de G, noté $hn(G)$.

Tout d'abord, nous donnons un algorithme polynomial pour calculer le nombre enveloppe d'un graphe sans P_5 et sans triangle. Ensuite, nous présentons quatre règles de réductions basées sur des sommets ayant même voisinage. Nous utilisons ces règles de réduction pour proposer un algorithme FPT pour calculer le nombre enveloppe de n'importe quel graphe G, ou le paramètre peut être la taille d'un transversal de G ou, plus généralement sa diversité de voisinage ; nous utilisons également ces règles pour caractériser le nombre enveloppe du produit lexicographique de deux graphes.

Mots-clés : Convexité dans les graphes, Nombre enveloppe, Convexité géodésique, Graphes sans P_5, Produit lexicographique, Complexité paramétrée, Diversité de voisinage.

1 Introduction

All graphs in this work are undirected, simple and loop-less. Given a connected graph $G = (V, E)$, the closed interval $I[u, v]$ of any two vertices $u, v \in V$ is the set of vertices that belong to some u-v geodesic of G, i.e. some shortest (u, v)-path. For any $S \subseteq V$, let $I[S] = \bigcup_{u,v\in S} I[u, v]$. A subset $S \subseteq V$ is *(geodesically) convex* if $I[S] = S$. Given a subset $S \subseteq V$, the *convex hull* $I_h[S]$ of S is the smallest convex set that contains S. We say that a vertex v is *generated* by a set of vertices S if $v \in I_h[S]$. We say that S is a *hull set* of G if $I_h[S] = V$. The size of a minimum hull set of G is the *hull number* of G, denoted by $hn(G)$ [9].

It is known that computing $hn(G)$ is an NP-hard problem for bipartite graphs [3]. Several bounds on the hull number of triangle-free graphs are presented in [8]. In [7], the authors show, among other results, that the hull number of any P_4-free graph, i.e. any graph without induced path with four vertices, can be computed in polynomial time. In Section 3, we show a linear-time algorithm to compute the hull number of any P_5-free triangle-free graph.

In Section 4, we show four reduction rules to obtain, from a graph G, another graph G^* that has one vertex less than G and which satisfies either $hn(G) = hn(G^*)$ or $hn(G) = hn(G^*) + 1$, according to the used rule. We then first use these rules to obtain a fixed parameter tractable (FPT) algorithm, where the parameter is the neighborhood diversity of the input graph. For definitions on Parameterized Complexity we refer to [10]. Given a graph G and vertices $u, v \in V(G)$, we say that u and v are of the *same type* if $N(v)\setminus\{u\} = N(u)\setminus\{v\}$. The *neighborhood diversity* of a graph is k, if its vertex set can be partitioned into k sets $S_1, \ldots, S_k$, such that any pair of vertices $u, v \in S_i$ are of the same type. This parameter was proposed by Lampis [12], motivated by the fact that a graph of bounded vertex cover also has bounded neighborhood diversity, and therefore the later parameter can be used to obtain more general results. To see that a graph of bounded vertex cover has bounded neighborhood diversity, let G be a graph that has a vertex cover $S \subseteq V(G)$ of size k, and let $I = V(G) \setminus S$. Since S is a vertex cover, observe that I is an independent set. Therefore, vertices in I can be partitioned in at most 2^k sets (one for each possible subset of S), where each of these sets contains vertices of the same type, i.e. vertices having the same neighborhood in S. Moreover, the vertices in S may be partitioned in k sets of singletons, what gives a partition of the vertices of the graph into $k + 2^k$ sets of vertices having the same type. Then, the neighborhood diversity of the graph is at most $k+2^k$. Many problems have been show to be FPT when the parameter is the neighborhood diversity [11].

Finally, we use these rules to characterize the hull number of the lexicographic product of any two graphs. Given two graphs G and H, the *lexicographic product* $G \circ H$ is the graph whose vertex set is $V(G \circ H) = V(G) \times V(H)$ and such that two vertices (g_1, h_1) and (g_2, h_2) are adjacent if, and only if, either $g_1g_2 \in E(G)$ or we have that both $g_1 = g_2$ and $h_1h_2 \in E(G)$.

It is known in the literature a characterization of the (geodesic) convex sets in the lexicographic product of two graphs [1] and a study of the pre-hull number for this product [13]. There are also some results concerning the hull number of the Cartesian and strong products of graphs [6, 5].

2 Preliminaries

Let us recall some definitions and lemmas that we use in the sequel.

We denote by $N_G(v)$ (or simply $N(v)$) the neighborhood of a vertex. A vertex v is *simplicial* (resp. *universal*) if $N(v)$ is a clique (resp. is equal to $V(G)\setminus\{v\}$). Let $d_G(u,v)$ denote the *distance* between u and v, i.e. the length of a shortest (u,v)-path. A subgraph $H \subseteq G$ is *isometric* if, for each $u,v \in V(H)$, $d_H(u,v) = d_G(u,v)$. A P_k (resp. C_k) in a graph G denotes an induced path (resp. cycle) on k vertices. Given a graph H, we say that a graph G is H-free if G does not contain H as an induced subgraph. Moreover, we consider that all the graph in this work are connected. Indeed, if a graph G is not connected, its hull number can be computed by the sum of the hull numbers of its connected components, as observed by Dourado et al. [7].

Lemma 1. *[9] For any hull set S of a graph G, S contains all simplicial vertices of G.*

Lemma 2. *[7] Let G be a graph which is not complete. No hull set of G with cardinality $hn(G)$ contains a universal vertex.*

Lemma 3. *[7] Let G be a graph, H be an isometric subgraph of G and S be any hull set of H. Then, the convex hull of S in G contains $V(H)$.*

Lemma 4. *[7] Let G be a graph and S a proper and non-empty subset of $V(G)$. If $V(G)\setminus S$ is convex, then every hull set of G contains at least one vertex of S.*

3 Hull number of P_5-free triangle-free graphs

In this section, we present a linear-time algorithm to compute $hn(G)$, for any P_5-free triangle-free graph G. To prove the correctness of this algorithm, we need to recall some definitions and previous results:

Definition 1. *Given a graph $G=(V,E)$, we say that $S \subseteq V$ is a* dominating set *if every vertex $v \in V \setminus S$ has a neighbor in S.*

It is well known that:

Theorem 1. *[4] G is P_5-free if, and only if, for every induced subgraph $H \subseteq G$ either $V(H)$ contains a dominating C_5 or a dominating clique.*

As a consequence, we have that:

Corollary 1. *If G is a connected P_5-free bipartite graph, then there exists a dominating edge in G.*

Theorem 2. *The hull number of a P_5-free bipartite graph $G=(A\cup B,E)$ can be computed in linear time.*

Proof. By Corollary 1, G has at least one dominating edge. Observe that the dominating edges of a bipartite graph can be found in linear time by computing the degree of each vertex and then considering the sum of the degrees of the endpoints of each edge. For a dominating edge, this sum is equal to the number of vertices.

- Consider first the case in which G has at least two dominating edges. Let $uv, xy \in E(G)$ be such dominating edges. Consider that $u, x \in A$ and $v, y \in B$.

  If $x \neq u$ and $v \neq y$, then we claim that $\{u, x\}$ is a minimum hull set of G. Indeed, since u and x are not adjacent and every vertex in B is a common neighbor of u and x, and then $\{u, x\}$ generate all the vertices in B, particularly v and y. Similarly, all the vertices of A are in a shortest (v, y)-path. Thus, $I_h(\{u, x\}) = V(G)$.

  Assume now, w.l.o.g., that $u \neq x$ and $v = y$. Again, $B \subseteq I_h(\{u, x\})$. Observe that, if there are simplicial vertices in $V(G)$, they must all belong to A, since u and x are not neighbors, but they are adjacent to all vertices in B. In case $|B| = 1$, then all vertices in A are simplicial vertices, and therefore A is the minimum hull set of G. Then, consider now that $|B| \geq 2$.

  In case there is no simplicial vertex in A, $\{u, x\}$ is a minimum hull set, since $B \subseteq I_h(\{u, x\})$ and every vertex in A has at least two neighbors in B. In case there are simplicial vertices in A, we claim that $S \cup \{b\}$ is a minimum hull set of G, where $S \subset A$ is the set of simplicial vertices of G and b is a vertex in B distinct from v. Indeed, by Lemma 1, we know that S must be part of any hull set of G and observe that $I_h(S) = S \cup \{v\}$ (the only neighbor of each simplicial vertex is exactly v). Consequently, since $|B| \geq 2$, at least one more vertex must be chosen to be part of a minimum hull set of G. We claim that if we choose $b \in B$ such that $b \neq v$, then $S \cup \{b\}$ is a minimum hull set of G. Indeed, let $s \in S$. Since $sb \notin E$ and xv, uv are dominating edges, x, u and v are generated by $\{s, b\}$. But then, as $B \subseteq I_h(\{u, x\})$, B is generated. Finally, every vertex in A is either simplicial, in case it belongs to S, or is adjacent to two vertices in B and therefore is generated by its neighbors.

- Consider now that G has only one dominating edge uv and that, w.l.o.g., $u \in A$ and $v \in B$. Let $H = G[V \setminus \{u, v\}]$. We may assume H is not the empty graph, for otherwise G is trivial. Let $C_1, \ldots, C_k$ be the connected components of H. We claim that $V \setminus C_i$ is a convex set of G, for every $i \in \{1, \ldots, k\}$.

  Since C_i is a connected component in H, the only vertices in $V \setminus C_i$ that may be adjacent to a vertex in C_i are u and v. Suppose a shortest (s, t)-path P such that $s, t \in V \setminus C_i$ and containing at least one vertex of C_i. It would pass through u and v. But there is an edge between u and v, so there is a contradiction because P would not be a shortest path.

  Consequently, by Lemma 4, for each connected component C_i of H at least one vertex of C_i must be chosen to be part of a minimum hull set of G (observe that simplicial vertices are the particular case in which $|C_i| = 1$).

  If $k = 1$, observe that G is not a complete bipartite graph, as we are assuming there is exactly one dominating edge. Let $w \in A$ and $z \in B$ be two non-adjacent vertices of C_1. In this case, we claim that $\{w, z\}$ is a minimum hull set of G. By contradiction, suppose that there exists a vertex $p \notin I_h(\{w, z\})$. First observe that u and v belong to $I_h(\{w, z\})$. Then, w.l.o.g., we may assume that p has a neighbor q in $I_h(\{w, z\})$ which is not in $\{u, v\}$, since C_1 is a connected component in H. However, since uv is a dominating edge, either qpu or qpv is a shortest path between two vertices of $I_h\{w, z\}$ and p should belong to $I_h(\{w, z\})$, a contradiction.

Now, suppose that $k > 1$. Let $W = \{w_1, \ldots, w_k\} \subseteq V(G)$ be such that $W \cap A \neq \emptyset$, $W \cap B \neq \emptyset$ and $w_i \in C_i$, for every $i \in \{1, \ldots, k\}$. We claim that W is a minimum hull set of G. By Lemma 4, all these vertices are required, so it suffices to show that $I_h(W) = V(G)$. Observe that u and v belong to $I_h(W)$, since $W \cap A \neq \emptyset$ and $W \cap B \neq \emptyset$. Then, by contradiction, suppose that there exists a vertex $p \notin I_h(W)$ and let C_p be its connected component in H. Again, we may assume that p has a neighbor q in $I_h(\{w, z\})$ which belongs to C_p, since C_p is a connected component and $C_p \cap W \neq \emptyset$. However, since uv is a dominating edge, either qpu or qpv is a shortest path in G and p should belong to $I_h(\{w, z\})$, a contradiction.

Finally, observe that all these cases can be checked in linear time and thus $hn(G)$ can be computed in linear time. □

For the next result, recall that the complexity of finding the convex hull of a set of vertices $S \subseteq V(G)$ of a graph G is $\mathcal{O}(|S||E(G)|)$, as described in [7]. We can relax the constraint of G being bipartite to obtain the following:

Corollary 2. *If G is a P_5-free triangle-free graph, then $hn(G)$ can be computed in polynomial time.*

Proof. By Theorem 1, G either has a dominating C_5 or a dominating clique of size at most two, since it is triangle-free.

In case it has a dominating $C_5 = v_1, \ldots, v_5$, we claim that $\{v_1, v_3, v_5\}$ is a hull set of G. To prove this fact, first observe that $I_h(\{v_1, v_3, v_5\}) \supseteq V(C_5)$. Moreover, since G is connected, and it has no induced P_5 and no triangle, we conclude that any vertex $w \in V(G) \setminus V(C_5)$ has two non-adjacent neighbors in C_5, and so $w \in I_h(\{v_1, v_3, v_5\})$. Thus, if G has a dominating C_5, we can test if there is a minimum hull set of size two in $\mathcal{O}(|V(G)|^2|E(G)|)$. Otherwise, we have that $hn(G) = 3$ and $\{v_1, v_3, v_5\}$ is a minimum hull set of G.

If G has a dominating clique of size one, then G must be a star since it is triangle-free. Thus, $hn(G) = |V(G)| - 1$.

Finally, if G has a dominating edge uv, we claim that G is bipartite. Since G is triangle-free and uv is a dominating edge, we have that $N(u)$ and $N(v)$ are stable sets and that $N(u) \cap N(v) = \emptyset$. Thus, G is bipartite and, by Theorem 2, we can compute its hull number in linear time. □

4 Neighborhood Diversity and Lexicographic Product

In this section, we present four reduction rules to compute the hull number of a graph. We need to introduce some definitions.

Given a set S, let $I^0[S] = S$ and $I^k[S] = I[I^{k-1}[S]]$, for $k > 0$. We say that v is generated by S *at step* $t \geq 1$, if $v \in I^t[S]$ and $v \notin I^{t-1}[S]$. Observe that the convex hull $I_h(S)$ of a given set of vertices S is equal to $I^{|V(G)|}[S]$.

Given a graph G, we say that two vertices v and v' are *twins* if $N(v) \setminus \{v'\} = N(v') \setminus \{v\}$. If v and v' are adjacent, we call them *true* twins, otherwise we say that they are *false* twins.

Let G be a graph and v and v' be two of its vertices. The *identification* of v' into v is the operation that produces a graph G' such that $V(G') = V(G) \setminus \{v'\}$ and $E(G') = (E(G) \setminus \{v'w \mid w \in N_G(v')\}) \cup \{vw \mid v'w \in E(G) \text{ and } w \neq v\}$.

Lemma 5. *Let G be a graph and v and v' be non-simplicial and twin vertices. Let G' be obtained from G by the identification of v' into v. Then, $hn(G) = hn(G')$.*

Proof. Let u and w be two non-adjacent neighbors of v and thus also of v' in G. In order to show that $hn(G) \leq hn(G')$, let S be a minimum hull set of G'. Since G' is an isometric subgraph of G, $V(G) \setminus \{v'\} \subseteq I_h(S)$ by Lemma 3. Moreover, $\{v'\} \subseteq I_G[u, w]$, hence S is a hull set of G.

To prove that $hn(G) \geq hn(G')$, let S be a minimum hull set of G. We may assume that S does not contain both v and v', because if there exists a minimum hull set containing both of them, then we can replace v and v' by u and w obtaining a hull set of same size, since $v, v' \in I_G[u, w]$.

Suppose first that $v, v' \notin S$. Let $\{x, y\} \neq \{v, v'\}$ and let P be a shortest (x, y)-path. Observe that P cannot contain both v and v'. In case v' (resp. v) is contained in P, then one can replace it by v (resp. v') and obtain another shortest path, as v and v' have the same neighborhood. In particular, this implies that the minimum k such that $v' \in I_G^k[S]$ is equal to the minimum k' such that $v \in I_{G'}^{k'}[S]$, and therefore for $i < k$, $I_{G'}^i[S] = I_G^i[S]$. It also implies that $I_G[v', w] \setminus \{v'\} = I_{G'}[v, w] \setminus \{v\}$, $w \notin \{v, v'\}$, and therefore for $i \geq k$ we have that $I_{G'}^i[S] = I_G^i[S] \setminus \{v'\}$. As a consequence, S is a hull set of G'.

Finally, suppose that either v or v' is in S. We may assume w.l.o.g. that $v \in S$. Then we can use the same argument as in the last paragraph to show that for every $1 \leq i \leq n$ its true that $I_{G'}^i[S] = I_G^i[S] \setminus \{v'\}$ and then again we have that S is a hull set of G'. □

Lemma 6. *Let G be a graph and v, v', v'' be simplicial and pairwise false twin vertices. Let G' be obtained from G by the identification of v'' into v. Then, $hn(G) = hn(G') + 1$.*

Proof. In order to show that $hn(G) \leq hn(G')+1$, observe that G' is an isometric subgraph of G and that v'' is simplicial. Consequently, any hull set S of G' is such that $I_h(S) = V(G) \setminus \{v''\}$, hence $S \cup \{v''\}$ is a hull set of G, by Lemmas 1 and 3.

To show that $hn(G) \geq hn(G')+1$. Let S be a hull set for G and $S' = S \setminus \{v''\}$. Since v, v' and v'' are simplicial, we know that $\{v, v', v''\} \subseteq S$. Any shortest (v'', u)-path, with $u \in V \setminus \{v', v''\}$ is still a shortest path if v'' is replaced by v', so $I[v'', u] = I[v', u]$. In the case of the shortest (v'', v')-path, replacing v'' by v is still a shortest path and $I[v'', v'] = I[v, v']$. Therefore $I_h(S') = I_h(S) \setminus \{v''\}$ and then S' is a hull set of G'. □

Observe that we cannot simplify the statement of Lemma 6 to consider any pair of simplicial false twin vertices instead of triples. As an example, consider the graph obtained by removing an edge uv from a complete graph with more than 3 vertices.

Lemma 7. *Let G be a graph and v, v' be simplicial and true twin vertices. Let G' be obtained from G by the identification of v' into v. Then, $hn(G) = hn(G')+1$.*

Proof. In order to show that $hn(G) \leq hn(G')+1$, observe that G' is an isometric subgraph of G and that v' is simplicial. Let S be a hull set of G'. Then $S \cup \{v'\}$ is a hull set of G, by Lemmas 1 and 3.

Now, we show that $hn(G) \geq hn(G') + 1$. Let S be a hull set of G. Since v and v' are simplicial, by Lemma 1 we know that $v, v' \in S$. Observe that, for every $w \in V(G')$, we have $I_G[v', w] \setminus \{v'\} \subseteq I_{G'}[v, w]$. Thus, $S \setminus \{v'\}$ is a hull set of G' and the result follows. □

According to Lampis [12], for a given graph G and vertices $u, v \in V(G)$, u and v are of the *same type* if $N(v) \setminus \{u\} = N(u) \setminus \{v\}$. This is exactly the same definition of *twin* vertices. Recall that the *neighborhood diversity* of a graph is k, if its vertex set can be partitioned into k sets $S_1, \ldots, S_k$, such that any pair of vertices $u, v \in S_i$ are of the same type. Now, we use this partition to obtain the following result:

Theorem 3. *Let G be a graph whose neighborhood diversity is at most k. Then, there exists an FPT algorithm to compute $hn(G)$ in $\mathcal{O}(4^k poly(|V(G)|))$-time.*

Proof. Lampis proved that a neighborhood partition of G can be found in $\mathcal{O}(poly(|V(G)|))$-time [12]. Observe that each part is either an independent set of false twin vertices or a clique of true twin vertices. We now use Lemmas 5, 6 and 7 to reduce each of these parts to at most two vertices.

First, in case there are parts of size greater than one consisting of non-simplicial vertices, we reduce these parts to a single vertex by the identification of its vertices. This procedure generates a graph G' whose hull number is equal to $hn(G)$, by Lemma 5.

Observe that if a vertex is simplicial, then its part is composed of simplicial vertices. In the sequence, we reduce each part of size greater than two containing only independent simplicial false twins to two vertices, by applying Lemma 6. If c identifications are done in this procedure, then the hull number of the graph G'' obtained after this procedure is $hn(G'') = hn(G') - c = hn(G) - c$.

Then, we reduce all the parts composed of pairwise adjacent simplicial true twins to one vertex, by applying Lemma 7. In the end of this procedure, we obtain a graph G''' such that $hn(G''') = hn(G'') - c' = hn(G) - c - c'$, where c' is the number of identifications that were made in this last procedure.

Observe that G''' has at most $2k$ vertices, since the neighborhood partition is of size at most k and each part is reduced to at most two vertices. Finally, we can enumerate all the subsets of $V(G''')$ (there are at most 2^{2k} of them) and test for each of these sets whether it is a hull set. Hence, we obtain $hn(G''')$ and therefore $hn(G)$.

Recall that this proof provides a kernelization algorithm and G''' is a kernel of linear size. □

As pointed before, a graph of bounded vertex cover size has also bounded neighborhood diversity, therefore the previous result also holds for this parameter.

Now, we use Lemma 5 and Lemma 7 to determine the lexicographic product of two graphs. Recall that the lexicographic product of two graphs G and H is the graph whose vertex set is $V(G \circ H) = V(G) \times V(H)$ and such that two vertices (g_1, h_1) and (g_2, h_2) are adjacent if, and only if, either $g_1 g_2 \in E(G)$ or we have both $g_1 = g_2$ and $h_1 h_2 \in E(G)$. For a vertex $g \in V(G)$, let its *H-layer* in $G \circ H$ be the set $H(g) = \{(g, h) \in V(G \circ H) \mid h \in V(H)\}$. Let $S(G)$ denote the set of simplicial vertices of G.

Observe that if G has a single vertex, then $hn(G \circ H) = hn(H)$. Else, we have that:

Theorem 4. *Let G be a connected graph, such that $|V(G)| \geq 2$, and let H be an arbitrary graph. Thus,*

$$hn(G \circ H) = \begin{cases} 2, & \textit{if } H \textit{ is not complete;} \\ (|V(H)| - 1)|S(G)| + hn(G), & \textit{otherwise.} \end{cases}$$

Proof. If H is not complete, since G is connected and it has at least two vertices, any two non-adjacent vertices in the same H-layer suffice to generate all the vertices of $G \circ H$.

We consider now that H is a complete graph on k vertices. First, observe that all the vertices in the same H-layer are all simplicial vertices or they are all non-simplicial vertices. Moreover, a vertex is simplicial in G if, and only if, its corresponding H-layer in $G \circ H$ is composed of simplicial vertices.

First, we obtain from $G \circ H$ a graph F by reducing each H-layer composed of non-simplicial vertices to a single vertex. By Lemma 5, $hn(G \circ H) = hn(F)$. Then, we apply Lemma 7 to reduce each H-layer of simplicial vertices to a single vertex obtaining a graph F'. Observe that we have $|V(H)||S(G)|$ simplicial vertices in $G \circ H$ and, thus, $(|V(H)| - 1)|S(G)|$ identifications are done in this procedure. Finally, since all the H-layers were reduced to a single vertex, observe that $F' \cong G$ and we have that $hn(G \circ H) = hn(F) = hn(F') + (|V(H)| - 1)|S(G)| = hn(G) + (|V(H)| - 1)|S(G)|$.

□

5 Conclusions

In this work, we first presented a linear time algorithm to compute the hull number of any P_5-free triangle-free graph. Although, the computational complexity of determining the hull number of a P_5-free graph and also of a triangle-free graph is still unknown. More generally, we propose the following open question:

Question 1. *For a fixed k, what is the computational complexity of determining $hn(G)$, for a P_k-free graph G?*

In the second part of this paper, we introduced four reduction rules that we use to present an FPT algorithm to compute the hull number of any graph, where the parameter is its neighborhood diversity, and a characterization of the lexicographic product of any two graphs. It is already known in the literature another FPT algorithm to compute the hull number of any graph, where the parameter is the number of its induced P_4's [2]. To the best our knowledge, the following is also open:

Question 2. *Given a graph G, is there an FPT algorithm to determine whether $hn(G) \leq k$, for a fixed k?*

References

[1] Bijo Anand, Manoj Changat, Sandi Klavžar, and Iztok Peterin. Convex sets in lexicographic products of graphs. *Graphs and Combinatorics*, pages 1–8, feb 2011.

[2] J. Araujo, V. Campos, F. Giroire, N. Nisse, L. Sampaio, and R. Soares. On the hull number of some graph classes. Technical Report RR-7567, INRIA, September 2011.

[3] J. Araujo, V. Campos, F. Giroire, L. Sampaio, and R. Soares. On the hull number of some graph classes. *Electronic Notes in Discrete Mathematics*, 38(0):49 – 55, 2011. The Sixth European Conference on Combinatorics, Graph Theory and Applications, EuroComb 2011.

[4] G. Bacsó and Z. Tuza. Dominating cliques in p_5-free graphs. *Periodica Mathematica Hungarica*, 21(4):303–308, 1990.

[5] J. Cáceres, C. Hernando, M. Mora, I.M. Pelayo, and M.L. Puertas. On the geodetic and the hull numbers in strong product graphs. *Computers and Mathematics with Applications*, 60(11):3020 – 3031, 2010.

[6] Gilbert B. Cagaanan, Sergio R. Canoy, and Jr. On the hull sets and hull number of the cartesian product of graphs. *Discrete Mathematics*, 287(1-3):141 – 144, 2004.

[7] Mitre C. Dourado, John G. Gimbel, Jan Kratochvíl, Fbio Protti, and Jayme L. Szwarcfiter. On the computation of the hull number of a graph. *Discrete Mathematics*, 309(18):5668 – 5674, 2009. Combinatorics 2006, A Meeting in Celebration of Pavol Hell's 60th Birthday (May 1-5, 2006).

[8] Mitre C. Dourado, Fábio Protti, Dieter Rautenbach, and Jayme L. Szwarcfiter. On the hull number of triangle-free graphs. *SIAM J. Discret. Math.*, 23:2163–2172, January 2010.

[9] Martin G. Everett and Stephen B. Seidman. The hull number of a graph. *Discrete Mathematics*, 57(3):217 – 223, 1985.

[10] J. Flum and M. Grohe. *Parameterized Complexity Theory*. Texts in Theoretical Computer Science. An EATCS Series. Springer, 1 edition, March 2006.

[11] Robert Ganian. Using neighborhood diversity to solve hard problems, 2012.

[12] Michael Lampis. Algorithmic meta-theorems for restrictions of treewidth. *Algorithmica*, 64:19–37, 2012.

[13] Iztok Peterin. The pre-hull number and lexicographic product. *Discrete Mathematics*, 312(14):2153 – 2157, 2012. Special Issue: The Sixth Cracow Conference on Graph Theory, Zgorzelisko 2010.

Centre de recherche INRIA Sophia Antipolis – Méditerranée
2004, route des Lucioles - BP 93 - 06902 Sophia Antipolis Cedex (France)

Centre de recherche INRIA Bordeaux – Sud Ouest : Domaine Universitaire - 351, cours de la Libération - 33405 Talence Cedex
Centre de recherche INRIA Grenoble – Rhône-Alpes : 655, avenue de l'Europe - 38334 Montbonnot Saint-Ismier
Centre de recherche INRIA Lille – Nord Europe : Parc Scientifique de la Haute Borne - 40, avenue Halley - 59650 Villeneuve d'Ascq
Centre de recherche INRIA Nancy – Grand Est : LORIA, Technopôle de Nancy-Brabois - Campus scientifique
615, rue du Jardin Botanique - BP 101 - 54602 Villers-lès-Nancy Cedex
Centre de recherche INRIA Paris – Rocquencourt : Domaine de Voluceau - Rocquencourt - BP 105 - 78153 Le Chesnay Cedex
Centre de recherche INRIA Rennes – Bretagne Atlantique : IRISA, Campus universitaire de Beaulieu - 35042 Rennes Cedex
Centre de recherche INRIA Saclay – Île-de-France : Parc Orsay Université - ZAC des Vignes : 4, rue Jacques Monod - 91893 Orsay Cedex

Éditeur
INRIA - Domaine de Voluceau - Rocquencourt, BP 105 - 78153 Le Chesnay Cedex (France)
http://www.inria.fr
ISSN 0249-6399

Nomenclature

$\alpha(G)$ Maximum size of a stable set

$\mathcal{A}_k$ The family of the k-atoms (see p.24)

$\mathcal{B}_k$ Binomial tree of order k (see p.20)

$\chi(G)$ Chromatic number (see p.13)

$\chi_b(G)$ b-chromatic number (see p.26)

$\Delta(G)$ Maximum degree of a vertex

$\Gamma(G)$ Grundy number (see p.19)

$\omega(G)$ Maximum size of a clique

$\partial\Gamma(G)$ Partial Grundy number (see p.33)

$\zeta(G)$ Stair factor (see p.34)

$dist(u, v)$ Distance between u and v

K_n The complete graph of order n

$K_{p,q}$ The complete bipartite graph with parts of order p and q

$m(G)$ The m-degree (see p.28)

$N(u)$ Neighbourhood of u

Index

Bibliography

[1] J. Araujo, V. Campos, F. Giroire, L. Sampaio, and R. Soares. On the hull number of some graph classes. In *The Sixth European Conference on Combinatorics, Graph Theory and Applications, EuroComb 2011*, volume 38 of *Electronic Notes in Discrete Mathematics*, pages 49 – 55, 2011.

[2] J. Araujo and C. Linhares Sales. On the grundy number of graphs with few p_4's. *Discrete Applied Mathematics*, 2011. In Press.

[3] S. Arora and B. Barak. *Computational Complexity: A Modern Approach*. Cambridge University Press, New York, NY, USA, 1st edition, 2009.

[4] M. Asté, F. Havet, and C. Linhares-Sales. Grundy number and products of graphs. *Discrete Mathematics*, 310(9):1482–1490, 2010.

[5] L. Babel, T. Kloks, J. Kratochvíl, D. Kratsch, H. Müller, and S. Olariu. Efficient algorithms for graphs with few p_4's. *Discrete Mathematics*, 235(23):29 – 51, 2001.

[6] G. Bacsó and Z. Tuza. Dominating cliques in p_5-free graphs. *Periodica Mathematica Hungarica*, 21(4):303–308, 1990.

[7] R. Balakrishnan and T. Kavaskar. Color chain of a graph. *Graphs and Combinatorics*, 27(4):487–493, 2011.

[8] D. Barth, J. Cohen, and T. Faik. On the b-continuity property of graphs. *Discrete Applied Mathematics*, 155(13):1761–1768, 2007.

[9] A. Beutelspacher and P.-R. Hering. Minimal graphs for which the chromatic number equals the maximal degree. *Ars Combinatoria*, 18:201–216, 1984.

[10] T. Beyer, S. M. Hedetniemi, and S. T. Hedetniemi. A linear algorithm for the grundy number of a tree. In *Proceedings of the Thirteenth Southeastern Conference on Combinatorics, Graph Theory and Computing*, Utilitas Mathematica, pages 351–363, 1982.

[11] M. Blidia, F. Maffray, and Z. Zemir. On b-colorings in regular graphs. *Discrete Applied Mathematics*, 157(8):1787 – 1793, 2009.

[12] F. Bonomo, G. Durán, F. Maffray, J. Marenco, and M. Valencia-Pabon. On the b-coloring of cographs and p_4-sparse graphs. *Graphs and Combinatorics*, 25:153–167, 2009.

[13] O. V. Borodin and A. V. Kostochka. On an upper bound of a graph's chromatic number, depending on the graph's degree and density. *Journal of Combinatorial Theory Series B*, 23(2-3):247–250, 1977.

[14] A. Brandstädt. Special graph classes - a survey. Technical Report SM-DU-199, Universität Duisburg Gesamthochschule, 1991.

[15] R. L. Brooks. On colouring the nodes of a network. *Mathematical Proceedings of the Cambridge Philosophical Society*, 37:194–197, 1941.

[16] S. Cabello and M. Jakovac. On the b-chromatic number of regular graphs. *Discrete Applied Mathematics*, 159(13):1303 – 1310, 2011.

[17] V. Campos, de V. Farias, and A. Silva. b-coloring graphs with large girth. *Journal of the Brazilian Computer Society*, pages 1–4, 2012.

[18] V. Campos, A. Gyárfás, F. Havet, C. Linhares Sales, and F. Maffray. New bounds on the grundy number of products of graphs. *Journal of Graph Theory*, 71(1):78–88, 2012.

[19] V. Campos, C. Linhares Sales, F. Maffray, and A. Silva. b-chromatic number of cacti. *Electronic Notes in Discrete Mathematics*, 35(0):281–286, 2009.

[20] V. Campos, C. Linhares Sales, A. Maia, and R. Sampaio. On b-colorings of graphs with few p4's. In *8th French Combinatorial Conference*, 2010.

[21] V. Campos, C. Linhares Sales, K. Maia, N. Martins, and R. Sampaio. Restricted coloring problems on graphs with few p4's. *Electronic Notes in Discrete Mathematics*, 37:57–62, 2011.

[22] F. Chow and J. Hennessy. Register allocation by priority-based coloring. *ACM SIGPLAN Notices*, 19:222–232, 1984.

[23] F. Chow and J. Hennessy. The priority-based coloring approach to register allocation. *ACM Transactions on Programming Languages and Systems*, 12:501–536, 1990.

[24] C. A. Christen and S. M. Selkow. Some perfect coloring properties of graphs. *Journal of Combinatorial Theory, Series B*, 27(1):49–59, 1979.

[25] M. Chrobak and M. Ślusarek. On some packing problem related to dynamic storage allocation. *Informatique Théorique et Applications/Theoretical Informatics and Applications*, 22(4):487–499, 1988.

[26] D.G. Corneil, H. Lerchs, and L.Stewart Burlingham. Complement reducible graphs. *Discrete Applied Mathematics*, 3(3):163–174, 1981.

[27] S. Corteel, M. Valencia-Pabon, and J. C. Vera. On approximating the b-chromatic number. *Discrete Applied Mathematics*, 146(1):106 – 110, 2005.

[28] David P. Dailey. Uniqueness of colorability and colorability of planar 4-regular graphs are np-complete. *Discrete Mathematics*, 30(3):289–293, 1980.

[29] Reinhard Diestel. *Graph Theory*, volume 173 of *Graduate Texts in Mathematics*. Springer-Verlag, Heidelberg, 2005.

[30] R. G. Downey and M. R. Fellows. *Parameterized Complexity*. Monographs in Computer Science. Springer-Verlag, New York, first edition, 1999.

[31] J. E. Dunbar, S. M. Hedetniemi, S. T. Hedetniemi, D. P. Jacobs, J. Knisely, R.C. Laskar, and D. F. Rall. Fall coloring of graphs. *Journal of Combinatorial Mathematics and Combinatorial Computing*, 33:257–273, 2000.

[32] H. Elghazel, V. Deslandres, M.-S. Hacid, A. Dussauchoy, and H. Kheddouci. A new clustering approach for symbolic data and its validation: Application to the healthcare data. In *Foundations of Intelligent Systems*, volume 4203 of *Lecture Notes in Computer Science*, pages 473–482. Springer Berlin / Heidelberg, 2006.

[33] T. Emden-Weinert, S. Hougardy, and B. Kreuter. Uniquely colourable graphs and the hardness of colouring graphs of large girth. *Combinatorics, Probability and Computing*, 7(4):375–386, 1998.

[34] P. Erdős. Graph theory and probability. *Canadian Journal of Mathematics*, 11:34–38, 1959.

[35] P. Erdös. On the combinatorial problems which i would most like to see solved. *Combinatorica*, 1:25–42, 1979.

[36] P. Erdös, W. R. Hare, S. T. Hedetniemi, and R. C. Laskar. On the equality of the grundy and ochromatic numbers of graphs. *Journal of Graph Theory*, 11:157–159, 1987.

[37] P. Erdös, S. T. Hedetniemi, R. C. Laskar, and G. C. E. Prins. On the equality of the partial grundy and upper ochromatic numbers of graphs. *Discrete Mathematics*, 272:53–64, 2003.

[38] T. Faik. About the b-continuity of graphs. In *Workshop on Graphs and Combinatorial Optimization*, volume 17, pages 151–156, 2004.

[39] T. Faik. *La b-continuité des b-colorations: complexité, propriétés structurelles et algorithmiques*. PhD thesis, Université Paris XI Orsay, 2010.

[40] B. Farzad, M. Molloy, and B. Reed. $(\delta - k)$-critical graphs. *Journal of Combinatorial Theory Series B*, 93(2):173–185, 2005.

[41] Frobenius. *Über zerlegbare determinanten*. Sitzungsber. König. Preuss. Akad. Wiss., 1917.

[42] D. R. Fulkerson and O. A. Gross. Incidence matrices and interval graphs. *Pacific Journal of Mathematics*, 15(3):835–855, 1965.

[43] D. Gaceb, V. Eglin, F. Lebourgeois, and H. Emptoz. Graph b-coloring for automatic recognition of documents. In *ICDAR'09 - 10th International Conference on Document Analysis and Recognition*, pages 261–265. IEEE, 2009.

[44] T. Gallai. über extreme punkt- und kantenmengen. *Annales Universitatis Scientarium Budapestinensis de Rolando Eötvös Nominatae Sectio Mathematica*, 2:133–138, 1959.

[45] A. Gamst. Some lower bounds for the class of frequency assignment problems. *IEEE Transactions on Vehicular Technology*, 35(8–14), 1986.

[46] M. R. Garey, D. S. Johnson, and L. J. Stockmeyer. Some simplified np-complete graph problems. *Theoretical Computer Science*, 1(3):237–267, 1976.

[47] V. Giakoumakis. p_4-laden graphs: a new class of brittle graphs. *Information processing letters*, 60:29–36, 1996.

[48] V. Giakoumakis, F. Russel, and H. Thuillier. On p_4-tidy graphs. *Discrete Mathematics and Theoretical Computer Science*, 1:17–41, 1997.

[49] J.L. Gonzalez-Velarde and M. Laguna. Tabu search with simple ejection chains for coloring graphs. *Annals of Operations Research*, 117:165–174(10), November 2002.

[50] N. Goyal and S. Vishvanathan. Np-completeness of undirected grundy numbering and related problems. Unpublished Manuscript, 1997.

[51] G. R. Grimmett and C. J. H. McDiarmid. On colouring random graphs. *Mathematical Proceedings of the Cambridge Philosophical Society*, 77(02):313–324, 1975.

[52] M . Grotschel, L. Lovasz, and A. Schrijver. *Geometric Algorithms and Combinatorial Optimization*. Algorithms and Combinatorics. Springer, 1994.

[53] P. M. Grundy. Mathematics and games. *Eureka*, 2:6–8, 1939.

[54] A. Gyárfás and J. Lehel. On-line and first fit colorings of graphs. *Journal of Graph Theory*, 12(2):217–227, 1988.

[55] P. Hall. On representatives of subsets. *Journal London Mathematical Society*, 10(1):26–30, 1935.

[56] M. Halldórsson. Parallel and on-line graph coloring algorithms. In Toshihide Ibaraki, Yasuyoshi Inagaki, Kazuo Iwama, Takao Nishizeki, and Masafumi Yamashita, editors, *Algorithms and Computation*, volume 650 of *Lecture Notes in Computer Science*, pages 61–70. Springer Berlin / Heidelberg, 1992.

[57] M. Halldórsson and M. Szegedy. Lower bounds for on-line graph coloring. In *Proceedings of the third annual ACM-SIAM symposium on Discrete algorithms*, SODA '92, pages 211–216, Philadelphia, PA, USA, 1992. Society for Industrial and Applied Mathematics.

[58] M. M. Halldórsson. A still better performance guarantee for approximate graph coloring. *Information Processing Letters*, 45(1):19–23, 1993.

[59] J. Hamiez and J. Hao. Scatter search for graph coloring. In *Selected Papers from the 5th European Conference on Artificial Evolution*, pages 168–179. Springer-Verlag, 2002.

[60] F. Havet, C. Linhares Sales, and L. Sampaio. b-coloring of tight graphs. Technical Report 7241v1, INRIA, 2010.

[61] F. Havet, C. Linhares Sales, and L. Sampaio. b-coloring of tight graphs. *Discrete Applied Mathematics*, 2011.

[62] F. Havet and L. Sampaio. On the grundy and b-chromatic numbers of a graph. *Algorithmica*, pages 1–15, 2011.

[63] C. Hoàng. *Perfect graphs*. PhD thesis, School of Computer Science, McGill University, Montreal, 1995.

[64] C. T. Hoang and M. Kouider. On the b-dominating coloring of graphs. *Discrete Applied Mathematics*, 152(1-3):176–186, 2005.

[65] C. T. Hoàng, C. Linhares Sales, and F. On minimally b-imperfect graphs. *Discrete Applied Mathematics*, 157(17):3519–3530, 2009.

[66] T. C. Hoàng, F. Maffray, and M. Mechebbek. A characterization of b-perfect graphs. 2009.

[67] I. Holyer. The NP-completeness of edge-coloring. *SIAM Journal on Computing*, 10(4):718–720, 1981.

[68] J. E. Hopcroft and R. M. Karp. An $o(n^{5/2})$ algorithm for maximum matchings in bipartite graphs. *SIAM Journal on Computing*, 2(4):225–231, 1973.

[69] M. Hujter and Z. Tuza. Precoloring extension iii: Classes of perfect graphs. *Combinatorics, Probability and Computing*, 5(01):35–56, 1996.

[70] R. W. Irving and D. F. Manlove. The b-chromatic number of a graph. *Discrete Applied Mathematics*, 91(1-3):127–141, 1999.

[71] M. Jakovac and S. Klavžar. The b-chromatic number of cubic graphs. *Graphs and Combinatorics*, 26(1):107–118, 2010.

[72] M. Jakovac and I. Peterin. On the b-chromatic number of strong, lexicographic, and direct product. *Studia Scientiarum Mathematicarum Hungarica*, 2009. To appear.

[73] B. Jamison and S. Olariu. A new class of brittle graphs. *Studies in Applied Mathematics*, 81:89–92, 1989.

[74] B. Jamison and S. Olariu. p_4-reducible graphs - a class of uniquely tree representable graphs. *Studies in Applied Mathematics*, 81:79–87, 1989.

[75] B. Jamison and S. Olariu. On a unique tree representation for p4-extendible graphs. *Discrete Applied Mathematics*, 34:151–164, 1991.

[76] R. Javadi and B. Omoomi. On b-coloring of the kneser graphs. *Discrete Mathematics*, 309(13):4399–4408, 2009.

[77] R. Javadi and B. Omoomi. On b-coloring of cartesian product of graphs. *Ars Combinatoria*, 2010. To appear.

[78] T. R. Jensen and B. Toft. *Graph coloring problems*. Wiley-Interscience Series in Discrete Mathematics and Optimization. John Wiley & Sons, Inc., New-York, 1995.

[79] J. Kara, Z. Tuza, and M. Voigt. b-continuity. Technical report, Technical University Ilmenau, Faculty of Mathematics and Natural Sciences, 2004.

[80] R. M. Karp. Reducibility among combinatorial problems. In R. E. Miller and J. W. Thatcher, editors, *Complexity of Computer Computations*, pages 85–103. Plenum Press, 1972.

[81] H. A. Kierstead. The linearity of first-fit coloring of interval graphs. *SIAM Journal on Discrete Mathematics*, 1(4):526–530, 1988.

[82] H.A. Kierstead and Jun Qin. Coloring interval graphs with first-fit. *Discrete Mathematics*, 144(1-3):47–57, 1995.

[83] D. König. Gráfok és mátrixok. *Matematikai és Fizikai Lapok*, 38:116–119, 1931.

[84] G. Kortsarz. A lower bound for approximating grundy number. *Discrete Mathematics and Theoretical Computer Science*, 9(1), 2007.

[85] M. Kouider. b-chromatic number of a graph, subgraphs and degrees. Technical Report 1392, LRI, Université Paris Sud, 2002.

[86] M. Kouider and A. E. Sahili. About b-colouring of regular graphs. Technical Report 1432, LRI, Université Paris Sud, 2002.

[87] M. Kouider and M. Zaker. Bounds for the b-chromatic number of some families of graphs. *Discrete Mathematics*, 306(7):617 – 623, 2006.

[88] Mekkia Kouider and Maryvonne Mahéo. Some bounds for the b-chromatic number of a graph. *Discrete Mathematics*, 256(1-2):267–277, 2002.

[89] D. Král, J. Kratochvíl, Z. Tuza, and G. J. Woeginger. Complexity of coloring graphs without forbidden induced subgraphs. In *Proceedings of the 27th International Workshop on Graph-Theoretic Concepts in Computer Science*, pages 254–262, London, UK, 2001. Springer-Verlag.

[90] J. Kratochvíl, Z. Tuza, and M. Voigt. On the b-chromatic number of graphs. In *Graph-Theoretic Concepts in Computer Science*, volume 2573 of *Lecture Notes in Computer Science*, pages 310–320. Springer Berlin / Heidelberg, 2002.

[91] Luděk Kučera. The greedy coloring is a bad probabilistic algorithm. *Journal of Algorithms*, 12(4):674–684, 1991.

[92] W. T. Trotter L. Lovász, M. Saks. An online graph coloring algorithm with sublinear performance ratio. *Discrete Mathematics*, 75:319–325, 1989.

[93] W. Lin and G. Chang. b-coloring of tight graphs and the erdös-faber-lovász conjecture. *Discrete Applied Mathematics*. Submitted.

[94] L. Lovász and M.D. Plummer. *Matching Theory*. North Holland, 1986.

[95] L. Lovász. Three short proofs in graph theory. *Journal of Combinatorial Theory Series B*, 19:111–113, 1975.

[96] C. Lucet, F. Mendes, and A. Moukrim. An exact method for graph coloring. *Comput. Oper. Res.*, 33(8):2189–2207, 2006.

[97] C. Lund and M. Yannakakis. On the hardness of approximating minimization problems. *Journal of the ACM*, 41(5):960–981, 1994.

[98] F. Maffray and A. Silva. b-colouring the cartesian product of trees and some other graphs. *Discrete Applied Mathematics*, 2011. In Press.

[99] Frédéric Maffray and Ana Silva. b-colouring outerplanar graphs with large girth. *Discrete Mathematics*, 312(10):1796–1803, 2012.

[100] D. Marx. Precoloring extension on chordal graphs. In *Graph Theory in Paris. Proceedings of a Conference in Memory of Claude Berge, Trends in Mathematics*, pages 255–270, Birkhäuser, 2004.

[101] D. Marx. Parameterized coloring problems on chordal graphs. *Theoretical Computer Science*, 351(3):407–424, 2006.

[102] M. Molloy and B. Reed. Colouring graphs when the number of colours is nearly the maximum degree. In *Proceedings of the Thirty-Third Annual ACM Symposium on Theory of Computing*, pages 462–470, New York, 2001.

[103] M. Muchaand and P. Sankowski. Maximum matchings via gaussian elimination. In *Proc. 45st IEEE Symp. Foundations of Computer Science*, pages 248–255, 2004.

[104] N. Narayanaswamy and R. Babu. A note on first-fit coloring of interval graphs. *Order*, 25(1):49–53, 2008.

[105] G. L. Nemhauser and L. A. Wolsey. *Integer and combinatorial optimization*. Wiley-Interscience, New York, NY, USA, 1988.

[106] R. Niedermeier. *Invitation to Fixed-Parameter Algorithms*. Oxford University Press, 2006.

[107] A. L. Rubin P. Erdös and H. Taylor. Choosability in graphs. In *Proceedings of the West Coast Conference on Combinatorics, Graph Theory and Computing*, volume 26 of *Congressus Numerantium*, pages 125–157, 1979.

[108] S. V. Pemmaraju, R. Raman, and K. Varadarajan. Buffer minimization using max-coloring. In *Proceedings of the fifteenth annual ACM-SIAM symposium on Discrete algorithms*, SODA '04, pages 562–571, Philadelphia, PA, USA, 2004. Society for Industrial and Applied Mathematics.

[109] R. Raman. *Chromatic scheduling*. PhD thesis, University of Iowa, 2007.

[110] B. Reed. ω, δ, and χ. *Journal of Graph Theory*, 27(4):177–212, 1998.

[111] B. Reed. A strengthening of Brooks' theorem. *Journal of Combinatorial Theory Series B*, 76(2):136–149, 1999.

[112] Y. Saad. *Iterative Methods for Sparse Linear Systems*. PWS Publishing Company, Boston, MA, USA, 1996.

[113] Z. Shi, W. Goddard, S.T. Hedetniemi, K. Kennedy, R. Laskar, and A. McRae. An algorithm for partial grundy number on trees. *Discrete Mathematics*, 304:108–116, 2005.

[114] A. Silva. *The b-chromatic number of some tree-like graphs*. PhD thesis, Université Joseph-Fourier, Grenoble I, 2010.

[115] G. J. Simmons. On the ochromatic number of a graph. *Congressus Numerantium*, 40:339–366, 1983.

[116] J. A. Telle and A. Proskurowski. Algorithms for vertex partitioning problems on partial k-trees. *SIAM Journal on Discrete Mathematics*, 10:529–550, 1997.

[117] Z. Tuza. Graph colorings with local constraints - a survey. *Math. Graph Theory*, 17:161–228, 1997.

[118] C. I. B. Velasquez, F. Bonomo, and I. Koch. On the b-coloring of p4-tidy graphs. *Discrete Applied Mathematics*, 159(1):60–68, 2011.

[119] S. Vishwanathan. Randomized online graph coloring. *Journal of Algorithms*, 13(4):657–669, 1992.

[120] D. Werra. An introduction to timetabling. *European Journal of Operations Research*, 19:151–161, 1985.

[121] M. Zaker. The grundy chromatic number of the complement of bipartite graphs. *Australasian Journal of Combinatorics*, 31:325–329, 2005.

[122] M. Zaker. Results on the grundy chromatic number of graphs. *Discrete Mathematics*, 306(23):3166–3173, 2006.

[123] David Zuckerman. Linear degree extractors and the inapproximability of max clique and chromatic number. *Theory of Computing*, 3(6), 2007.