Anderson Localization in Disordered Systems with Competing Channels

Hong-Yi Xie

Condensed Matter Theory Sector, SISSA, Trieste

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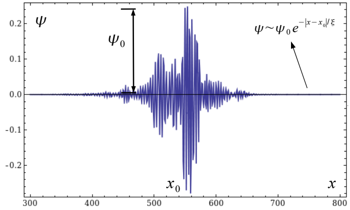
Supervisors:

Michele Fabrizio (*SISSA*)
Vladimir E. Kravtsov (*ICTP*)
Markus Müller (*ICTP*)

Reminder: Anderson localization

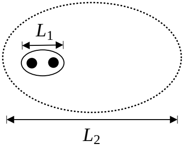
- A single particle can be coherently backscattered in a disordered potential, so that it is trapped in a finite region in real space. — Anderson localization [Anderson 1958]
- In $D \leq 2$ dimensions, in the absence of special symmetries, all one-particle states are localized. In $D > 2$ dimensions, localization-delocalization (Anderson) transition may happen. — Scaling theory of localization [Abrahams *et al.* 1979]
- Profile of a localized state: $\psi(x) \sim \psi_0 e^{-|x-x_0|/\xi}$ for $|x - x_0| \gg \xi$. The localization length ξ is defined by the decay rate of the amplitude, and can be measured by transmission coefficient.



Effect of interactions?

Motivation I: Channel competition in few-particle systems

- Two interacting particles can coherently propagate further than a single particle. [In 1D, Dorokhov 1990, Shepelyansky 1994, Imry 1995, Song & von Oppen 1997; In 2D, Ortuño & Cuevas 1999.]



$$L_2 \gg L_1$$

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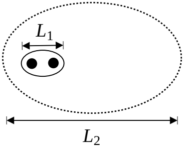

$L_2 \gg L_1$

What happens for more particles?

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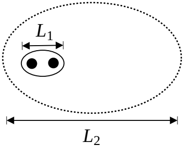
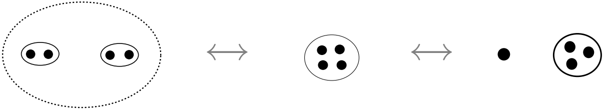


$L_2 \gg L_1$ What happens for more particles?

- But for more particles the competition between "fast" (more delocalized) channel and "slow" (more localized) channel is significant.

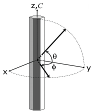


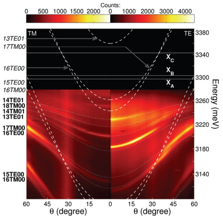
Fast channel Slow channel

Q: Who wins, the fast or the slow?

Motivation II: Anderson localization of a hybrid particle –polariton

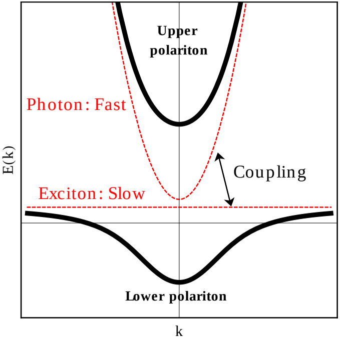
- Cavity polariton in 1D microwires





[Trichet, *et al*, 2011]

$\Longrightarrow$



Q: In the presence of disorder, which component will dominate the localization of polariton, photon (the fast) or exciton (the slow)?

$$m_{\rm ex}/m_{\rm ph} \sim 10^4$$

$$|\mathrm{pl}\rangle = \alpha|\mathrm{ph}\rangle + \beta|\mathrm{ex}\rangle$$

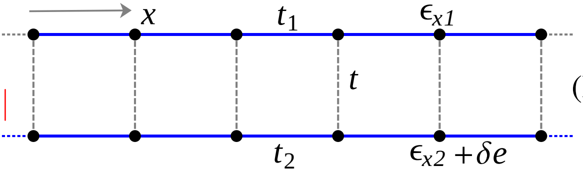
Anderson localization on two coupled lattices

1. Localization on a two-leg ladder ($D = 1$)
   [Phys. Rev. B **86**, 014205 (2012)]

2. Localization on a two-layer Bethe lattice ($D = \infty$)

Two-leg Anderson model and localization lengths

More delocalized (Photon)



$t_1 \gg t_2$
or $|\epsilon_1| \ll |\epsilon_2|$

(Polariton)

More localized (Exciton)

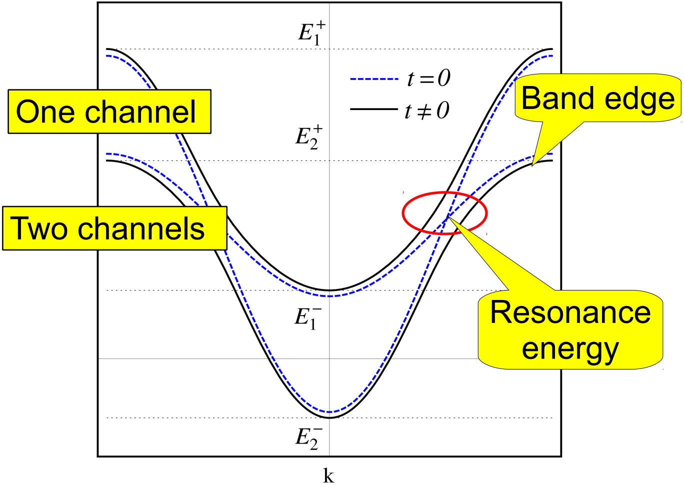
- For uncoupled chains:

$$\xi_1 \gg \xi_2$$

Q: What will be the localization lengths in the presence of coupling?

Two-leg Anderson model and localization lengths

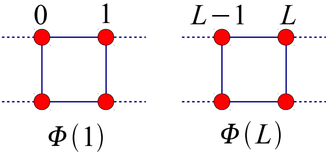
- Energy dispersion in the absence of disorder ($\epsilon_{1,2} = 0$)

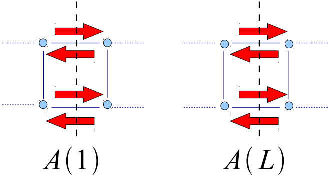


Two-channel regime: Fokker-Planck equation approach

- Transfer matrix in scattering channel basis

Schrödinger equation $\Longrightarrow$ Transfer matrix





$$\Phi(L) = \tilde{M}(L)\Phi(1)$$

$$\tilde{M}(L) = \prod_{x=1}^{L} \tilde{m}_x$$

$$A(L) = M(L)A(1)$$

$$M(L) = \prod_{x=1}^{L} m_x$$

- $m_x \sim I + O(\epsilon) \Rightarrow$ Perturbative analysis in weak disorder.

- T-invarince $\Leftrightarrow M^* = \Sigma_1 M \Sigma_1$ $(\Sigma_1 = \sigma_1 \otimes I)$
- Current conservation $\Leftrightarrow M^\dagger \Sigma_3 M = \Sigma_3$ $(\Sigma_3 = \sigma_3 \otimes I)$

Two-channel regime: Fokker-Planck equation approach

- Parametrization of transfer matrix and localization lengths

  [Mello *et al* 1988]

  - T-invariance ($M^* = \Sigma_1 M \Sigma_1$) and current conservation ($M^\dagger \Sigma_3 M = \Sigma_3$)

$$\Rightarrow \quad M = \begin{pmatrix} \mathbf{u} & 0 \\ 0 & \mathbf{u}^* \end{pmatrix} \begin{pmatrix} \sqrt{\frac{\mathbf{F}+1}{2}} & \sqrt{\frac{\mathbf{F}-1}{2}} \\ \sqrt{\frac{\mathbf{F}-1}{2}} & \sqrt{\frac{\mathbf{F}+1}{2}} \end{pmatrix} \begin{pmatrix} \tilde{\mathbf{u}} & 0 \\ 0 & \tilde{\mathbf{u}}^* \end{pmatrix} \in \mathrm{Sp}(4, \mathbb{R})$$

$\mathbf{F}$ contains two radial variables $\quad \mathbf{u},\ \tilde{\mathbf{u}} \in \mathrm{U}(2)$ contains four angular variables

$$\mathbf{F} = \begin{pmatrix} \frac{2}{T_1} - 1 & 0 \\ 0 & \frac{2}{T_2} - 1 \end{pmatrix} \qquad \mathbf{u}, \tilde{\mathbf{u}} = e^{-i\frac{\phi}{2}} \begin{pmatrix} \cos\frac{\theta}{2} e^{-\frac{i}{2}(\varphi+\psi)} & -\sin\frac{\theta}{2} e^{-\frac{i}{2}(\varphi-\psi)} \\ \sin\frac{\theta}{2} e^{\frac{i}{2}(\varphi-\psi)} & \cos\frac{\theta}{2} e^{\frac{i}{2}(\varphi+\psi)} \end{pmatrix}$$

  - $T_{1,2} \in (0, 1]$ are transmission coefficients.

Two-channel regime: Fokker-Planck equation approach

- Parametrization of transfer matrix and localization lengths
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  - $T_{1,2} \in (0, 1]$ are transmission coefficients.
  - Localization lengths are $\boxed{\xi_{1(2)}^{-1} = -\lim_{L\to\infty} \frac{1}{2} \frac{d}{dL} \langle \ln T_{1(2)} \rangle}$

Two-channel regime: Fokker-Planck equation approach

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  - $R = MM^\dagger$ contains six parameters $\vec{q} = (T_1, T_2, \theta, \psi, \varphi, \phi)$.

Q: What is the probability density $P(\vec{q}; L)$?

Two-channel regime: Fokker-Planck equation approach

- From "Langevin equation" to Fokker-Planck equation
  - "Langevin equation" for the "Brownian motion" of R matrix (6D)

$$R(L+1) = m_{L+1} R(L) m_{L+1}^{\dagger}$$

$$\langle \epsilon_{x\nu} \epsilon_{x'\nu'} \rangle = \mathrm{Var}(\epsilon_{\nu}) \delta_{xx'} \delta_{\nu\nu'}$$

$$\vec{q}_{L+1} = \vec{q}_L + \delta\vec{q}, \quad \delta q \sim O(\epsilon)$$

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$$\vec{q}_{L+1} = \vec{q}_L + \delta\vec{q}, \qquad \delta q \sim O(\epsilon)$$

  - Fokker-Planck (diffusion) equation for $P(\vec{q}; L)$

$$\partial_L P = \sum_i \partial_{q_i} \left(v_i(\vec{q}) P - \sum_j D_{ij}(\vec{q}) \partial_{q_j} P \right)$$

"Diffusion coefficient tensor"

$$D_{ij}(\vec{q}) = \frac{1}{2} \langle \delta q_i \delta q_j \rangle$$

"Stream velocity"

$$v_i(\vec{q}) = \langle \delta q_i \rangle + \frac{1}{2} \partial_{q_j} D_{ij}(\vec{q})$$

Radial variables $(T_1,\ T_2)$ and angular variables $(\theta,\ \psi,\ \varphi,\ \phi)$ are coupled.

Two-channel regime: Fokker-Planck equation approach

- Connection to the DMPK equation

If the angular variables are uniformly distributed (equally mixed channels), $P(\vec{q}; L) \propto P(T_1, T_2; L)$ satisfies the DMPK equation for $N = 2$ [Dorokhov 1982, Mello, Pereyra, & Kumar 1988]:

$$\frac{\partial P}{\partial L} = \mathrm{Var}(\epsilon) \sum_{i=1}^{N(=2)} \frac{\partial}{\partial \lambda_i} \lambda_i (1 + \lambda_i) J(\{\lambda\}) \frac{\partial}{\partial \lambda_i} J^{-1}(\{\lambda\}) P,$$

$$\lambda_i = 1/T_i - 1, \quad J(\{\lambda\}) = \prod_{i<j}^{N(=2)} |\lambda_i - \lambda_j|.$$

In general angular variables are not uniformly distributed!

$\Rightarrow$ "Extended" DMPK equation

Two-channel regime: Fokker-Planck equation approach

- Asymptotic analysis
  - Coarse-graining of space for weak disorder

$$\boxed{1/\Delta k \ll \min\{\xi_1, \xi_2\}} \quad \Delta k = |k_1 - k_2|$$

  - Exponential smallness of transmission coefficients for $L \gg \xi_{1,2}$,

$$T_{1(2)} \sim e^{-L/\xi_{1(2)}} \Rightarrow \quad \boxed{T_{min} \ll T_{max} \ll 1}$$

$\Rightarrow$ Eliminating ϕ, φ, ψ (phase angles) dependence.

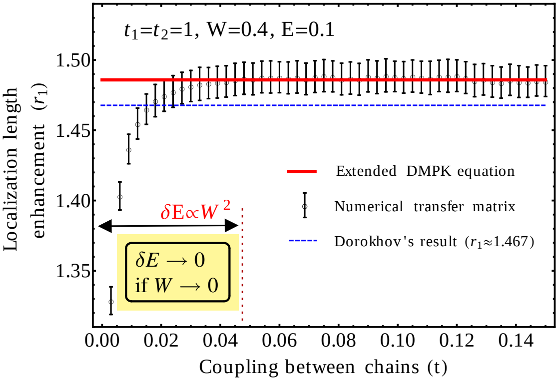
Only the θ dependence is considered.
It controls the distribution of amplitude over the channels.

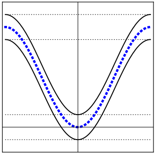
Results

Two-channel regime: resonance and off-resonance

- A benchmark: $t_1 = t_2$, $\mathrm{Var}(\epsilon_1) = \mathrm{Var}(\epsilon_2)$
  [Dorokhov 1982, Kasner & Weller 1988.]

Localization length enhancement: $r_1 = \dfrac{\xi_1}{\xi_1|_{t=0}} \sim O(1)$



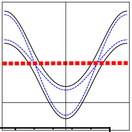


$$\lim_{t\to 0} r_1 = \frac{2\pi}{3(\pi-\sqrt{3})} \approx 1.486 \neq 1 \;\&\; < 2$$

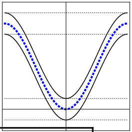
Two-channel regime: resonance and off-resonance

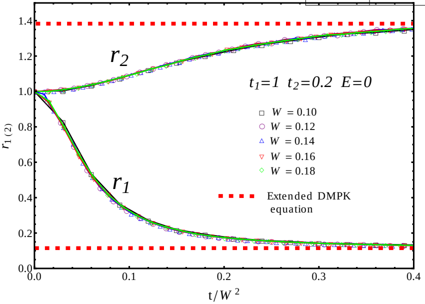
- Resonance: "slow" chain dominates

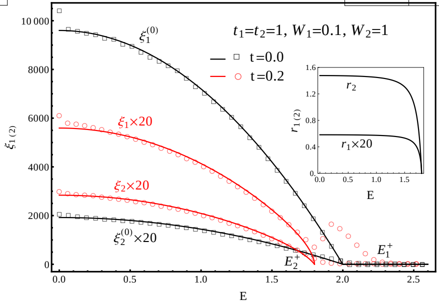
  - $t_1 \gg t_2$, $\mathrm{Var}(\epsilon_1) = \mathrm{Var}(\epsilon_2)$ (polariton) at resonance energy



  - $t_1 = t_2$, $\mathrm{Var}(\epsilon_1) \ll \mathrm{Var}(\epsilon_2)$ (two-channel waveguide) resonance at any energy



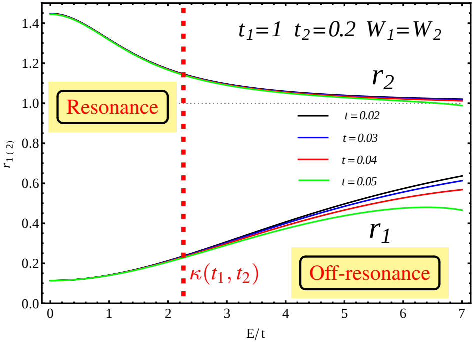


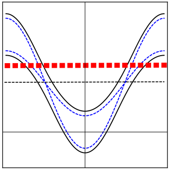


At resonance energy:
ξ_1 is dramatically draged down to the order of $\xi_2|_{t=0}$ ($r_1 \ll 1$ and $\xi_1/\xi_2|_{t=0} \sim 2.972$);
ξ_2 is lifted but still in the same order as $\xi_2|_{t=0}$ ($r_2 \sim 1.507$).

Two-channel regime: resonance and off-resonance

- Off-resonance: "fast" chain dominates
  $t_1 \gg t_2$, $\mathrm{Var}(\epsilon_1) = \mathrm{Var}(\epsilon_2)$ (polariton)





- ▶ Resonance–off-resonance crossover happens at $E \propto t$.
- ▶ Away from resonance $\xi_{1,2}$ approach decoupled values.

Band-edge singularities and one-channel regime

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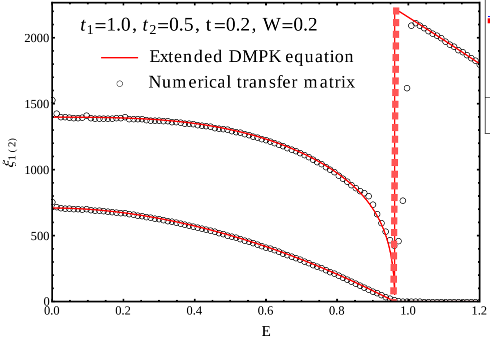
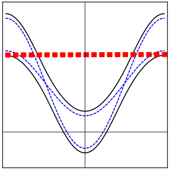




- Below band-edge, "slow" chain (exciton) dominates (zero velocity).
- Above band-edge, ξ_1 recovers to a large value suddenly (one channel).
- In one-channel regime, a propagating channel is coupled to an evanescent channel. For weak disorder, the coupling effect is irrelevant up to $O(\epsilon^2)$.

Conclusion for $D = 1$ dimension

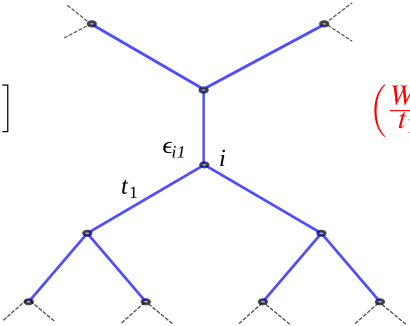
- In weak disorder limit, depending on energy there are four regimes:
  - Near resonance: Slow chain (exciton) dominates.
  - Off-resonance: Fast chain (photon) dominates.
  - Near the band-edge below: Slow chain (exciton) dominates.
  - On-channel regimes: Fast chain (photon) dominates.

Only near resonance or band edges the slow channel dominates. This is a manifestation of the fact that in 1D backscattering rate determines the localization properties of a coupled system!

What will happen in high dimensions?

Anderson model on a two-layer Bethe lattice and Anderson transition

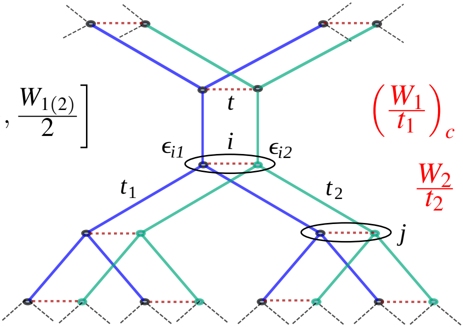
$$\epsilon_1 \in \left[-\frac{W_1}{2}, \frac{W_1}{2}\right]$$



$$\left(\frac{W_1}{t_1}\right)_c \approx 17.3$$

Anderson model on a two-layer Bethe lattice and Anderson transition

$$\epsilon_{1(2)} \in \left[-\frac{W_{1(2)}}{2}, \frac{W_{1(2)}}{2}\right]$$



$$\left(\frac{W_1}{t_1}\right)_c \approx 17.3$$

$$\frac{W_2}{t_2} < 17.3$$

- For $t = 0$, lattice 1 is at the critical point and lattice 2 is localized,

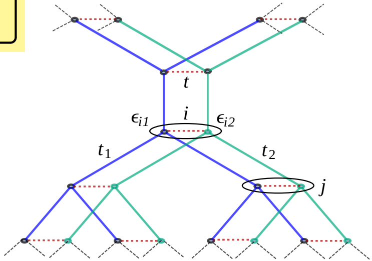
Q: In the presence of coupling, will the system be always delocalized or critical?

Anderson model on a two-layer Bethe lattice and Anderson transition

- Recursion relation for local Green's functions [Abou-Chacra *et al.*, 1973]

$$\hat{G}_j^{(i)} = \left(E + i\eta - \hat{H}_j - \hat{T} \sum_{k\in\partial j|i} \hat{G}_k^{(j)} \hat{T}\right)^{-1}$$

$$\hat{H}_j = \begin{pmatrix} \epsilon_{j1} & -t \\ -t & \epsilon_{j2} \end{pmatrix}, \quad \hat{T} = \begin{pmatrix} -t_1 & 0 \\ 0 & -t_2 \end{pmatrix}, \quad \eta = 0^+$$



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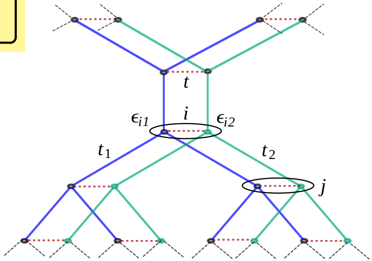


- Self energies, decay rates, and Anderson transition

$$S_{j\nu}(E) = E + i\eta - \epsilon_{j\nu} - 1/G_{j,\nu\nu}^{(i)}$$

$$\Gamma_{j\nu}(E) \equiv \mathrm{Im} S_{j\nu}(E), \quad \nu \in \{1, 2\}$$

- The Anderson transition can be determined by analyzing the stability of the real solution for $S_{j\nu}(E)$'s.
- The recursion relation can be iterated by the population dynamics ("pool" method).

Anderson model on a two-layer Bethe lattice and Anderson transition

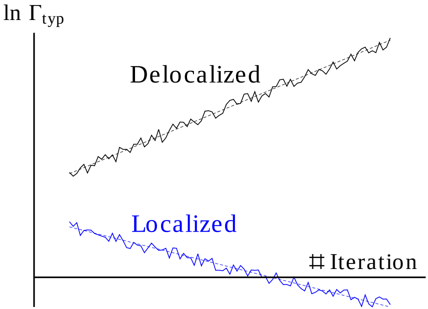
- Stability analysis and Anderson transition
  - The typical value of decay rate

    $$\ln \Gamma^{(n_s)}_{\mathrm{typ},\nu} = \frac{1}{\mathcal{N}} \sum_{j=1}^{\mathcal{N}} \ln \Gamma^{(n_s)}_{j\nu}$$

    $\mathcal{N} = \#$ Sites, $\quad n_s = \#$ Iteration

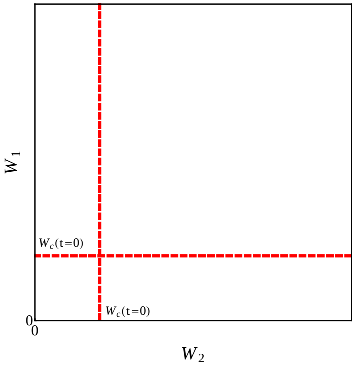
  - The growth rate of $\Gamma^{(n_s)}_{\mathrm{typ},\nu}$

    $$\lambda_{n_s} = \ln \Gamma^{(n_s)}_{\mathrm{typ},\nu} - \ln \Gamma^{(n_s - 1)}_{\mathrm{typ},\nu}$$



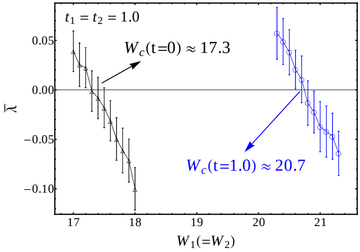
$$\overline{\lambda} = \begin{cases} < 0, & \text{localized}, \\ > 0, & \text{delocalized}, \end{cases} \qquad \delta\lambda \propto 1/\sqrt{\mathcal{N}}.$$

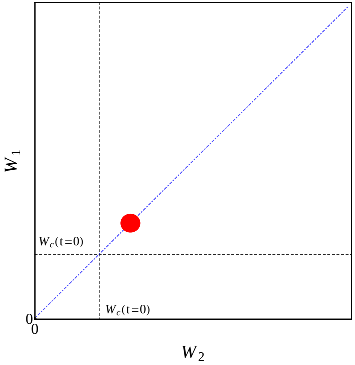
Phase diagram for $t_1 = t_2$ at $E = 0$



Phase diagram for $t_1 = t_2$ at $E = 0$

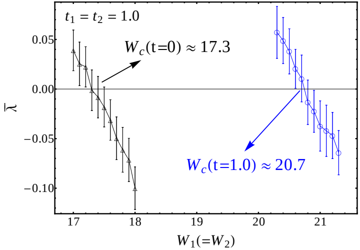
- Statistically identical lattices ($W_1 = W_2$)



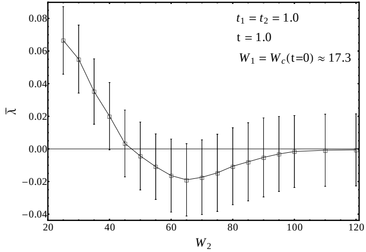


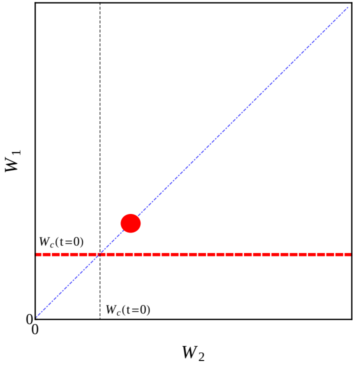
Phase diagram for $t_1 = t_2$ at $E = 0$

- Statistically identical lattices ($W_1 = W_2$)



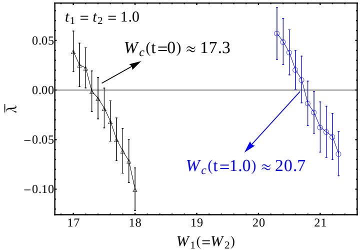
- Parametrically different lattices
  [$W_1 = W_c(t = 0)$]



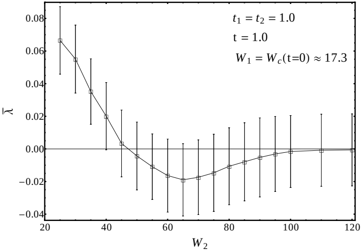


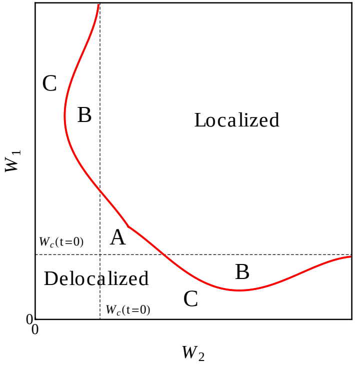
Phase diagram for $t_1 = t_2$ at $E = 0$

- Statistically identical lattices ($W_1 = W_2$)



- Parametrically different lattices
  $[W_1 = W_c(t = 0)]$





The less disordered lattice is not affected much by the more disordered lattice, unless the less disordered lattice is very close to the transition and the more disordered lattice is strongly localized (regime B).

Conclusion for Bethe lattices

- Localization is relevant only if disorder is intermediate or strong. Resonance conditions can not be achieved.

The less disordered lattice (delocalized) is not affected by the more disordered lattice (localized).

Discussion

- For two coupled lattices in D dimensions, We conjecture:

  In $D > 2$ dimensions the physics is similar as that on the Bethe lattices, i.e., the fast channel dominates.
  In $D = 2$ dimensions the physics is similar as that in one dimension, i.e., with a resonant coupling the slow channel dominates.

- For the few-particle problems, we conjecture:

  The fast channel dominates the delocalization of the interacting particles, since the resonance between the fast and slow channels should be an exception rather than a rule.

Thanks to all those who gave me the possibility to complete this dissertation!