# Étude de la conjecture de Seymour sur le second voisinage

Salman Ghazal

N°d'ordre: 356 - 2011 Année 2011

THESE DE L'UNIVERSITE DE LYON

délivrée par

L'UNIVERSITE CLAUDE BERNARD LYON 1

et préparée en cotutelle avec

L'UNIVERSITE LIBANAISE

Ecole Doctorale Informatique et Mathématiques (France)

Ecole Doctorale des Sciences et Technologie (Liban)

THESE DE DOCTORAT

(arrêté du 7 août 2006/arrêté du 6 janvier 2005)

Discipline: Mathématiques

soutenue publiquement le 15 Décembre 2011 par

GHAZAL Salman

TITRE : Etude de la Conjecture de Seymour sur le Second Voisinage

Directeur de thèse: EL SAHILI Amine

Co-encadrement de la thèse: BROCHET Jean-Michel

**JURY**:

M. KHEDDOUCI Hamamache, Professeur, GAMA, Université Lyon 1, Président.

M. BOUDABBOUS Youssef, Professeur, King Saud University, Rapporteur.

M. HAVET Fréderic, Chargé de recherche avec HDR, INRIA, Sophia-Antipolis, Rapporteur.

Mlle. KOUIDER Mekkia, Maître de Conference, LRI, Université Paris 11, Examinateur.

M. KHURI-MAKDISI Kamal, Professeur, American University of Beirut, Examinateur.

M. ABED ALKADER Issam, Professeur, Islamic University of Lebanon, Examinateur.

M. EL SAHILI Amine, Professeur, Université Libanaise, Directeur.

M. MRAD Bassam, Maître de Conference, Université Libanaise, Examinateur.

# Résumé

Soit $D$ un digraphe simple (sans cycle orienté de longueur 2 ). En 1990, P. Seymour a conjecturé que $D$ a un sommet $v$ avec un second voisinage extérieur au moins aussi grand que son (premier) voisinage extérieur [14]. Cette conjecture est connue sous le nom de la conjecture du second voisinage du Seymour (SNC). Cette conjecture, si elle est vraie, impliquerait, un cas spécial plus faible (mais important) de la conjecture de Caccetta et Häggkvist [11] proposé en 1978: tout digraphe $D$ avec un degré extérieur minimum au moins égale à $|V(D)|/k$ a une cycle orienté de longueur au plus $k$. Le cas particulier est $k = 3$, et le cas faible exige les deux: le degré extérieur minimum et le degré intérieur minimum de $D$ sont au moins égaux à $|V(D)|/k$.

La conjecture de Seymour restreinte au tournoi est connue sous le nom de conjecture de Dean [14]. En 1996, Fisher [4] a prouvé la conjecture de Dean en utilisant un argument de probabilité.

En 2003, Chen, Shen et Yuster [8] ont démontré que tout digraphe a un sommet $v$ tel que $d^+(v) \leq \gamma d^{++}(v)$ où $\gamma$=0.657298..... est l'unique racine de l'équation $2x^3 + x^2 - 1 = 0$.

En 2000, Havet et Thomassé [7] ont donné une preuve combinatoire de la conjecture de Dean, en utilisant un outil appelé l'ordre médian. Ils ont démontré que le dernier sommet d'un tel ordre a toujours un second voisinage extérieur au moins aussi grand que son voisinage extérieur.

En 2007, Fidler et Yuster [3] ont utilisé l'ordre médian et une autre outil qui s'appelle le digraphe de dépendance afin de prouver la conjecture de Seymour pour tout digraphe $D$ ayant un degré minimum $|V(D)| - 2$. Ils l'ont montré pour tout tournoi où manque un autre sous-tournoi.

El Sahili a conjecturé que pour tout $D$, il exist un completion $T$ de $D$ et un ordre médian de $T$ tel que le denier sommet a un second voisinage extérieur au moins aussi grand que son voisinage extérieur (EC). Il est clair que, EC implique SNC. Cependant, EC propose une méthode pour afin de résoudre la SNC. En général, on orient les non arcs de $D$ en manière appropriée, afin d'obtenir un tournoi $T$ et on essaie de trouver un sommet particulier (le denier sommet d'un ordre médian) avec la propriété desirée.

Clairement, grace aux resultats de [7] et [3], la EC est valable pour tourni, et tout tournoi où manque un autre sous-tournoi. Nous allons vérifier EC pour tout digraphe $D$ ayant un degré minimum $|V(D)| - 2$. Alors, EC est vraie pour tout digraphe où la SNC est déjà connue d'être vraie non trivialement. Nous sommes aussi intéressé à la version pondérée de SNC et EC. En réalité, Fidler et Yuster [3] utilisé les digraphes de dépendance comme un outil supplémentaire et le fait que la SNC pondérée est vraie pour les tournois afin de prouver la SNC pour tout digraphe $D$ ayant un degré minimum $|V(D)| - 2$.

Nous allons définir le digraphe de dépendance de façon plus générale et qui convient à n'importe quel digraphe. Nous allons utiliser le digraphe de dépendance et l'ordre médian comme des outils dans nos contributions à cette conjecture.

Suivant la méthode proposée par la EC, nous démontrons la version pondérée de EC, et par conséquent la SNC, pour les classes des digraphes suivants : Digraphes où manque une étoile généralisée, soleil, étoile, ou un graphe complète. En outre, nous prouvons la EC, et par conséquent la SNC, pour digraphes où manque un peigne et digraphe où manque un graphe complet moins 2 arêtes indépendantes ou moins les arêtes d'une cycle de longueur 5. Par ailleurs, nous prouvons la EC, et par conséquent la SNC, pour les digraphes où manque $n$ étoiles disjointes, sous certaines conditions sur les deux degrés minimum du digraphe de dépendance. Des conditions plus faible sont exigées dans le cas $n = 1, 2, 3$. Dans certaines cas, on trouve au moins deux sommets avec la propriété désirée.

**Title : A Study of Seymour's Second Neighborhood Conjecture**

## Abstract

Let $D$ be a digraph without digons (directed cycles of length 2). In 1990, Seymour [14] conjectured that $D$ has a vertex whose first out-neighborhood is at most as large as its second out-neighborhood. Such a vertex is said to have the second neighborhood property (SNP). This conjecture is known as the second neighborhood conjecture (SNC). This conjecture, if true, would imply a weakening of a particular case (but important) of a long standing conjecture proposed by Caccetta and Häggkvist in 1978, which states that every digraph $D$ with minimum out-degree at least $|V(D)|/k$ has a directed cycle of length at most $k$. The special case is when $k = 3$ and the weakening requires both minimum out-degree and minimum in-degree at least $|V(D)|/k$ [11].

Seymour's conjecture restricted to tournaments is known as Dean's conjecture [14]. In 1996, Fisher [4] gave a probabilistic proof to Dean's conjecture.

In 2003 Chen, Shen and Yuster [8] proved that every digraph contains a vertex $v$ such that $d^+(v) \leq \gamma d^{++}(v)$, where $\gamma = 0.657298...$ is the unique real root of the equation $2x^3 + x^2 - 1 = 0$.

In 2000, another proof of Dean's conjecture was given by Havet and Thomassé using a tool called median order [7]. They proved that the last vertex of this order, called a feed vertex, has second out-neighborhood at least as large as its first out-neighborhood. Median order is found to be a useful tool not only for the class of tournaments but for other classes of digraphs.

In 2007, Fidler and Yuster [3] used also median orders to prove Seymour's conjecture for the class of digraphs with minimum degree $|V(D)|-2$ (i.e. $D$ is a digraph missing a matching) and tournaments minus another

subtournament.

El Sahili conjectured that for every digraph $D$ there is a completion $T$ of $D$ and a median order of $T$ whose feed vertex has the SNP in $D$. Clearly, El Sahili's conjecture (EC) implies SNC. However, as one can observe, EC suggests a method (an approach) for solving the SNC, which we will call the completion approach. In general, following this approach, we orient the missing edges of $D$ in some 'proper' way, to obtain a tournament $T$. Then we consider a particular feed vertex (clearly, it has the SNP in $T$) and try to prove that it has the SNP in $D$ as well. Clearly, the result of Havet and Thomassé shows that EC is true for tournaments and the result of Fidler and Yuster [3] shows that EC holds for tournaments minus another subtournament. We will verify EC for the class of tournaments missing a matching. So EC is verified for all the classes of digraphs where the SNC is known to hold non trivially. We will be interested also in the weighted version of EC and SNC. In reality, Fidler and Yuster [3] used dependency digraphs as a supplementary tool for proving the SNC for digraphs missing a matching and the fact that the weighted SNC holds for tournaments.

We define dependency digraphs in a more general way, which is suitable to any digraph, and use them in our contribution to Seymour's conjecture. We also use the median order as a tool in our contribution. Using these two tools, and following the completion approach, we prove the weighted version of EC, and consequently the SNC, for several classes of digraphs: Digraphs missing a generalized star, sun, star or a complete graph. In addition, we prove EC, and consequently the SNC for digraphs missing a comb, and digraphs whose missing graph is a complete graph minus two independent edges or the edges of a cycle of length five. Moreover, we prove it for digraphs missing $n$ disjoint stars under some conditions. Weaker conditions are required for $n = 1, 2, 3$. In some cases, we exhibit at least two vertices with the SNP.

## Keywords

**MOTS-CLES:** *graphe, digraphe, tournoi, voisinage, second voisinage, degrés, chemin, cycle, arbre, étoile, ordre médian, digraphe de dépendance.*

**KEYWORDS:** *graph, digraph, tournament, neighborhood, second neighborhood, degrees, path, cycle, tree, star, median order, dependency digraph.*

# Résumé Substantiel

## Définitions Préliminaires

**Graphe.** Un graphe $G$ est un couple de deux ensembles disjoints finis $(V(G), E(G))$ tel que $E(G) \subseteq \{\{x, y\}; x, y \in V(G)\}$. $V(G)$ est l'ensemble des sommets de $G$ et $E(G)$ est l'ensemble des arêtes de $G$. On ecrit $xy$ (ou $yx$) au lieu de $\{x, y\}$ et on peut écrire $v \in G$ et $e \in G$ au lieu de $v \in V(G)$ et $e \in E(G)$.
Quand $e = xy \in E(G)$, on dit que $x$ et $y$ sont voisins ou adjacents, et ils sont les extrémités de $e$. Le voisinage d'un sommet $x$, noté par $N_G(x)$, est l'ensemble des voisins de $x$. On definit le voisinage d'un ensemble $X \subseteq V(G)$ par $N_G(X) := \bigcup_{x \in X} N_G(x) \setminus X$. Le degré d'un sommet $x$ est $d_G(x) = |N_G(x)|$.

**Sous-graphe.** Un sous-graphe de $G$ est un graphe $G' = (V', E')$ tel que $V' \subseteq V(G)$ et $E' \subseteq E(G)$. Lorsque, pour tout $x, y \in V'$, on a $xy \in E'$ si et seulement si $xy \in E(G)$, $G'$ est appelé sous-graphe engendré par $V'$ et on le note par $G[V']$.

**Digraphe.** Un digraphe $D$ est un couple de deux ensembles disjoints finis $(V(D), E(D))$ où $E(D) \subseteq \{(x, y); x, y \in V(G)\}$. $V(D)$ est l'ensemble des sommets de $D$, et $E(D)$ est l'ensemble des arcs de $D$. On peut écrire $v \in G$ et $e \in D$ au lieu de $v \in V(D)$ et $e \in E(D)$.
Lorsque $(x, y) \in E(D)$, on écrit $x \rightarrow y$ et on dit que $y$ (resp. $x$) est un voisin extérieur (resp. intérieur) de $x$ (resp. $y$). Le voisinge extérieur (resp. intérieur) d'un sommet $x$, noté par $N_D^+(x)$ (resp. $N_D^-(x)$), est l'ensemble des voisins extérieur (resp. intérieur) de $x$. Pour $X \subseteq V(D)$, le voisinge extérieur de $X$ est l'ensemble $N_D^+(X) := \bigcup_{x \in X} N_D^+(x) \setminus X$. En particulier, le second voisinage extérieur d'un sommet $x$ est l'ensemble $N_D^{++}(x) = N_D^+(N_D^+(x))$. Alors, les deux ensembles $N_D^+(x)$ et $N_D^{++}(x)$ sont disjoints. Le degré extérieur (resp. intérieur ) de sommet $x$ est $d_D^+(x) = |N_D^+(x)|$ (resp. $d^-(x) = |N_D^-(x)|$) et le degré de sommet $x$ est $d_D(x) = d_D^+(x) + d^-(x)$. Une source est un sommet dont le degré intérieur est nul. Un puits est un sommet dont le degré extérieur est nul. Une feuille est un sommet de degré 1. Une feuille intérieure (resp. extérieure) est une feuille $x$ telle que $d^-(x) = 1$ (resp. $d^+(x) = 1$). Un sommet $x$ est complet si $d_D(x) = |V(D)| - 1$.

**Digraphe pondéré.** Un digraphe pondéré est un couple $(D, \omega)$, où $D$ est un digraphe et $\omega : V(D) \longrightarrow \mathcal{R}_+$ est une fonction réelle positive, dite fonction de poids. Cette fonction induit une autre fonction, notée aussi par $\omega$, sur les arcs de $D$: pour tout $e = (x, y) \in E(D)$ on pose $\omega(x, y) := \omega(y)$. Le poids d'un ensemble $U \subseteq V(D)$ est $\omega(U) := \Sigma_{u \in U} \omega(u)$ et le poid de $F \subseteq E(D)$ est $\omega(F) := \Sigma_{e \in F} \omega(e)$.

**Graphe orienté.** Un boucle est un arc de la forme $(x, x)$ et un digon est un ensemble de 2 arcs de la forme $(x, y)$ and $(y, x)$. Un digraphe est un graphe orienté ssi il ne contient pas de boucle ni digon.

**Graphe sous-jacent.** Le graphe sous-jacent d'un digraphe $D$ est un graphe noté par $G(D)$, avec $V(G(D)) = V(D)$ et $xy \in E(G(D))$ si $(x, y) \in D$ ou $(y, x) \in D$.

**Graphe manquant.** Le graphe manquant de digraph $D$, noté par $G_D$, est un graphe definit comme suit: Les sommets de $G_D$ sont les sommets de $D$ et $xy \in E(G_D)$ si $(x, y)$ et $(y, x) \notin E(D)$. Une arête $xy \in G_D$ est dite arête manquante de $D$.

**Sous-digraphe.** Un sous-digraphe de $D$ est un digraphe $D' = (V', E')$ avec $V' \subseteq V$ et $E' \subseteq E$. Lorsque, pour tout $x, y \in V'$, on a $(x, y) \in E'$ si et seulement si $(x, y) \in E(D)$, $D'$ est appelé sous-digraphe engendré par $V'$ et on le note par $D[V']$. Pour $F \subseteq E(D)$, $D - F$ (resp. $D + F$) est le digraphe $(V(D), E(D) \backslash F)$ (resp. $(V(D), E(D) \cup F)$). Lorsque $F = \{e\}$, on le note $D - e$ (resp. $D + e$). Pour $A \subseteq V(D)$, $D - A$ est le digraphe $D[V(D) \backslash A]$.

Dans la suite, on suppose que les digraphes ne contiennent pas de boucle ni digon.

**Chemin.** Un chemin $P$ est un graphe avec un ensemble des sommets $\{v_1, ..., v_n\}$ et un ensemble d'arêtes $v_i v_{i+1}$ pour $i < n$. Un chemin noté par $v_1 v_2 ... v_n$ est dit un $v_1 v_n$-chemin ou un chemin de $v_1$ à $v_n$.

**Chemin orienté.** Un chemin orienté $P$ est un digraphe avec $V(P) = \{v_1, ..., v_n\}$ et $E(D) = \{(v_i, v_{i+1}), i < n\}$. Un tel chemin est dit $v_1 v_n$-chemin orienté. On écrit $P = v_1 ... v_n$.

**Cycle.** Un cycle $C$ est un graphe de sommets $\{v_1, ..., v_n\}$ et d'arêtes $v_i v_{i+1}$ pour $i < n$ plus l'arête $v_n v_1$. On écrit $C = v_1 ... v_n$.

**Circuit.** Un circuit $C$ est un digraphe avec un ensemble de sommets

$\{v_1, ..., v_n\}$ et d'arcs $(v_i, v_{i+1})$ pour $i < n$ plus l'arc $(v_n, v_1)$. On écrit $C = v_1...v_n$.

**Connexe.** Un graphe $G$ est connexe si tous deux sommets sont liés par un chemin.

**Fortement connexe.** Un digraphe $D$ est fortement connexe si pour tous deux sommets $x$ et $y$, il existe un $xy$-chemin orienté .

**Arbre.** Un arbre est un graphe connexe sans cycle.

**Arboresence.** Une arbresence sortante (resp. rentrante) est un arbre orienté tel que, tous les sommets, sauf exactement un sommet sont de degré intérieur 1 (resp. extérieur 1).

**Etoile.** Une étoile est un arbre formé par un sommets et ses voisins.

**Couplage.** Un couplage est un ensemble d'arcs (ou d'arêtes) deux à deux disjoints.

**Stable.** Un stable est un ensemble de sommets deux à deux non adjacents.

**Graphe complet.** Un graphe est dit complet si $xy \in E(G)$ pour tous 2 sommets distincts $x$ et $y$ de $V(G)$.

**Tournoi.** Un tournoi est une orientation d'un graphe complet.

**Triangle.** Un triangle est une cycle ayant 3 sommets. Un triangle dans un digraphes est dit cyclique s'il est un circuit. Sinon, il est acyclique.

**Carré.** Un carré est une cycle ayant 4 sommets exactement.

**Roi.** Un roi dans un digraphe $D$ est un sommet $x$ tel que $\{x\} \cup N_D^+(x) \cup N_D^{++}(x) = V(D)$.

**Degrés de digraphe.** Le degré minimum de $D$ est $\delta_D = min\{d(x); x \in V(D)\}$.
Le degré extérieur minimum de $D$ est $\delta_D^+ = min\{d^+(x); x \in V(D)\}$.
Le degré extérieur maximum de $D$ est $max\{d^+(x); x \in V(D)\}$.
Le degré intérieur minimum de $D$ est $\delta_D^- = min\{d^-(x); x \in V(D)\}$.

Le degré intérieur maximum de $D$ est $max\{d^-(x); x \in V(D)\}$.

**Longueur.** La longueur d'un chemin (resp. chemin orienté) ou d'un cycle (resp. circuit) est le nombre de ses arêtes (resp. arcs).

**Maille.** Le maille d'un graphe $G$ (resp. digraphe $D$) est la longueur minimale d'un cycle minimale (resp. circuit) dans $G$ (resp. $D$).

**Nombre chromatique.** Le nombre chromatique de digraphe (ou graphe) $D$, noté par $\chi(D)$, est le plus petit $k$, tel que $V(D)$ est l'union de $k$ stables.

## Ordre Médian

## Définition et propriétés

**Acyclic.** Soit $(D, \omega)$ un digraphe pondéré. $D$ est dit acyclique s'il ne contient aucun circuit. Par exemple, les arbres orientés sont des digraphes acyclique. Un sous-digraphe acyclique $D'$ est maximum lorsque $\omega(E(D'))$ est maximum.

**Feedback set.** Un ensemble des arcs de digraphe $D$ est appelé un feedback arc set si $D - F$ est acyclique. $F$ est minimum feedback arc set lorsque $D - F$ est acyclique et $\omega(F)$ est minimum.

Clairement, $D'$ est un sous-digraphe acyclic maximum de $D$ ssi $E(D) - E(D')$ est un feedback arc set minimum.

**Ordre median.** Soient $(D, \omega)$ un digraphe pondéré et $L = x_1 x_2 ... x_n$une énumération des sommets de $D$. On dit que $e = (x_i, x_j)$ est un arc avant si $i < j$, sinon $e$ est un arc retour. Les ensembles des arcs avants et retours sont notés par $A(L)$ et $R(L)$ respectivement. Le poids de $L$ est $\omega(L, D) := \omega(A(L))$. Une énumération $L$, avec $\omega(L, D)$ est maximum, est dite un ordre médian pondéré de $(D, \omega)$. Notons que, $L$ est un ordre médian ssi $R(L)$ est un feedback arc set minimum, $A(L)$ est donc maximal.

Pour $i \leq j$, $[i, j]$ ou $[x_i, x_j]$ est (le digraphe engendré par) l'ensemble $\{x_i, x_{i+1}, ..., x_j\}$.

**Propriété de Feedback.** Soit $L = x_1x_2...x_n$un ordre médian de $D$. Pour $1 \leq i \leq j \leq n$, dans $D[i, j]$ on a:

$$\omega(N^+(x_i)) \geq \omega(N^-(x_i))$$

et

$$\omega(N^-(x_j)) \geq \omega(N^+(x_j)).$$

Lorsque $D$ est un tournoi, la Propriété de feedback est équivalente à

$$d^+(x_i) \geq \frac{j-i}{2}$$

et

$$d^-(x_j) \geq \frac{j-i}{2}$$

dans $D[i, j]$, pour tout $i \leq j$.

**Ordre médian local.** Une énumération $L = x_1x_2...x_n$vérifiant la Propriété de feedback est dit un ordre médian penderé local. Dans ce cas, $x_n$ est appelé un feed sommet de $(D, \omega)$. Soient $j < n$ et $x_j \notin N^+(x_n)$. S'il exist $i < j$ tel que $x_n \to x_i \to x_j$ on dit que $x_j$ est un bon sommet, sinon, $x_j$ est mauvais. L'ensemble des bons sommets sera noté par $G_L^D$ (ou simplement par $G_L$, s'il n'y a pas de confusion).

Il est clair que tout ordre médian ponderé est un ordre médian ponderé local.

Lorsque $\omega = 1$, on obtient la défintion d'ordre médian (local) de digraphe (non ponderé).

**Proposition 0.0.1.** *Soit $L = x_1x_2...x_n$un order médian ponderé (local) d'un digraphe $(D, \omega)$. Soit $D' = D + F - B$ où $F \subseteq \{(x_i, x_j) \notin D; i > j\}$ et $B \subseteq R(L)$. Alors $L$ est un median order ponderé (local) de $(D', \omega)$.*

## La Conjecture de Sumner sur les Tournoi

L'ordre médian est un outil inductif: si $L = x_1x_2...x_n$est un ordre médian (local) de $D$ alors, $I = x_ix_{i+1}...x_j$ est un ordre médian (local) de $D' = D[x_i, x_j]$. Il est utilisé par Havet et Thomassé [7] pour démontrer la CSV pour les tournois, et aussi pour démontrer la conjecture de Sumner pour les arboresences.

**Conjecture 1.** *(La conjecture de Sumner[13]) Tout tournoi sur* $2k-2$ *sommets* $(k>1)$ *contient toute arbre orienté sur* $k$ *sommets.*

Soient $A$ et $D$ deux digraphes et $L = x_1x_2...x_n$un order médian de $D$. Un plongement de $A$ dans $D$ est une fonction injective $f : V(A) \longrightarrow V(D)$ telle que $(f(v_i), f(v_j)) \in E(D)$ si $(v_i, v_j) \in E(A)$. Un *L-plongement* de $A$ dans $D$ est un plongement $f$ de $A$ dans $D$ telle que, pour tous intervalle de $L$ de la forme $[x_{i+1}, x_n]$,

$$|f(A) \cap [x_{i+1}, x_n]| < \frac{1}{2}|[x_{i+1}, x_n]| + 1.$$

Dans ce cas, on dit que, $A$ est $L$-plongeable dans $D$.

**Proposition 0.0.2.** *([1]) Soient* $T$ *un tournoi ayant au moins 3 sommets et* $L = x_1x_2...x_n$*un order médian de* $T$*. Posons* $T' = T - \{v_1, v_2\}$ *et* $L' = x_1x_2...x_{n-2}$*. Soit* $A$ *un digraphe avec un feuille entrante* $y$ *et suppose que* $A' = A - y$ *a un* $L'$*-plongement* $f'$ *dans* $T'$*. Alors,* $A$ *a un* $L$*-plongement* $f$ *dans* $T$ *qui prolonge*$f'$*.*

**Corollaire 0.0.1.** *([7]) Tout tournoi sur* $2k-2$ *sommets contient toute arboresence sur* $k$ *sommets* $(k>1)$*.*

En effet, il est prouvé que, pour tout $k > 0$, pour tout tournoi $T$ sur $2k-2$ sommets, pour tout order médian $L = x_1...x_{2k-2}$ de $T$, pour toute arbre $A$ sur $k$ sommets, on a $A$ est $L$-plongeable [7].

En utilisant un argument similaire on a:

**Théorèm 0.0.1.** *([7]) Tout tournoi sur* $4k-6$ *sommets contient tout arbre orienté sur* $k$ *sommet* $(k>1)$*.*

El Sahili a utilisé aussi l'ordre médian pour prouver:

**Théorèm 0.0.2.** *([1]) Tout tournoi sur* $3k-3$ *sommets contient tout arbre orienté sur* $k$ *sommets* $(k>1)$*.*

Récemment, le conjecture de Sumner est prouvé pour $k$ suffisament grand.

**Théorèm 0.0.3.** *([5]) Il existe* $k_0$ *tel que pour tout* $k \geq k_0$*, tout tournoi sur* $2k-2$ *sommets contient toute arbre orientée sur* $k$ *sommets.*

## Conjecture de Seymour sur le Second Voisinage

On dit qu'un sommet $v$ a la propriété du second voisinage (PSV) dans un digraphe si $d^+(v) \leq d^{++}(v)$. En 1990 P. Seymour a donné la conjecture suivante.

**Conjecture 2.** *[14](**Conjecture de Second Voisinage(CSV)**) Tout digraphe a un sommet avec le PSV.*

On dit qu'un sommet a la propriété du second voisinage pondéré (PSV pondéré) si $\omega(N^+(v)) \leq \omega(N^{++}(v))$. On sait que la CSV est équivalent à sa version ponderé: tout digraphe ponderé a un sommet avec la PSV ponderé.

## Conjecture de Dean - une preuve de probabiliste

La conjecture de Dean [14] est celle de Seymour restrainte aux tournois. En 1996, Fisher [4] a démontré la conjecture de Dean.

**Théorèm 0.0.4.** *Tout tournoi a un sommet avec la PSV.*

Fisher a utilisé un argument de probabiliste, il a prouvé que pour tout digraphe $D$ il exist une fonction de probabilité $p : V(D) \longrightarrow [0,1]$ telle que pour chaque sommet $v$, $p(N^-(v)) \leq p(N^+(v))$. En outre, il a observé que cette probabilité verifie $p(N^-(v)) \leq p(N^{--}(v))$ pour tout sommet $v$ de $D$, lorsque $D$ est un tournoi. Poson $V(D) = \{v_1, v_2, ..., v_n\}$. Alors, on a

$$E_p(d^+) := \Sigma_{v_i \in V(D)} p(v_i) d^+(v_i) = \Sigma_{v_i \in V(D)} p(N^-(v_i))$$

et

$$E_p(d^{++}) := \Sigma_{v_i \in V(D)} p(v_i) d^+(v_i) = \Sigma_{v_i \in V(D)} p(N^-(v_i)).$$

Donc $E_p(d^+) \leq E_p(d^{++})$ et par conséquent il existe un sommet $v_i$ tel que $d^+(v_i) \leq d^{++}(v_i)$.

# Conjecture de Dean - une preuve combinatoire

En 2000, Havet et Thomassé ont donné une preuve combinatoire de la conjecture de Dean, en utilisant l'ordre médian.

**Théorèm 0.0.5.** *[7] Soit $L = x_1x_2...x_n$un ordre médian local de tournoi $T$. Alors, $x_n$ a la PSV.*

En fait, ils ont démontré que $d^+(x_n) \leq |G_L|$. La version pondérée du théorème précédent est facile à obtenir.

**Théorèm 0.0.6.** *[3] Soit $L = x_1x_2...x_n$un ordre médian pondéré local de tournoi $(T, \omega)$. Alors, $x_n$ a la PSV pondéré.*

En outre, ils ont prouvé que tout tournoi sans puits a au moins deux sommets avec la PSV en utilisant le notin de sedimnetation d'un ordre médian.

**Definition 0.0.1.** *Soit $L = x_1x_2...x_n$un ordre médian local de tournoi $T$.*
*Si $|N^+(x_n)| < |G_L|$, Sed(L)=L.*
*Si $|N^+(x_n)| = |G_L|$, on dénote par $b_1, ..., b_k$ les sommets mauvais de (T,L) at par $v_1, ..., v_{n-1-k}$ les sommet qui appartiennent à $N^+(x_n) \cup G_L$, énumérés dans l'ordre croissant par rapport à L. Dans ce cas, $Sed(L)$ est l'ordre $b_1...b_kx_nv_1...v_{n-1-k}$ de $T$ ([7]).*

**Théorèm 0.0.7.** *([7]) $Sed(L)$ est un ordre médian de $T$.*

**Théorèm 0.0.8.** *([7]) Un tournoi qui n'a pas des puits contient 2 sommets ayants le PSV.*

## Une approche approximitive

Une autre approche du CSV est introduite dans [8]. Cette approche cherche à trouver la valeur maximale de $\gamma$ telle que pour tout digraphe on a: il existe un sommet $v$ avec $d^{++}(v) \geq \gamma d^{+}(v)$. Comme $d^{++}(v) = d^{+}(v)$ pour les sommets dans un circuit, on a $\gamma \leq 1$. La conjecture de Seymour dit que $\gamma = 1$.

Chen, Shen and Yuster [8]ont démontré le résultat suivant:

**Théorèm 0.0.9.** *Tout digraphe a un sommet $v$ tel que $d^{++}(v) \geq \gamma d^{+}(v)$, où $\gamma = 0.657298...$ est la racine unique de l'equation $2x^3 + x^2 - 1 = 0$.*

Pour les digraphes qui n'ont pas des sous-digraphes qui sont tournoi sur $k$ sommets, une amélioration est établie. Ces digraphes sont appelés $K_{k+1}$-free digraphes.

**Théorèm 0.0.10.** *([3]) Soit $D$ un $K_{k+1}$-free digraphe. Alors, $D$ a un sommet $v$ avec $d^{++}(v) \geq \gamma d^{+}(v)$, où $\gamma$ est plus grand ou égale à la racine de $f(x) = \frac{2k-2}{k}x^3 + \frac{k-2}{k}x^2 - 1$.*

Par example, pour $k = 3$, $\gamma \geq 0.8324$.

## La conjecture de Caccetta et Häggkvist

En 1978, Caccetta et Häggkvist ont proposé la conjecture suivante:

**Conjecture 3.** *[11] Tout digraphe $D$ ayant un degré extérieur minimum $|V(D)|/k$ est de maille au plus $k$.*

Cette conjecture est toujours non resolue meme pour $k = 3$.

**Conjecture 4.** *Tout digraphe $D$ ayant un degré extérieur minimum et un degré intérieur minimum $|V(D)|/3$, est de maille au plus 3.*

Le CSV implique la conjecture précédente. En effet, dans ce cas, il suffit de considérer un sommet ayant le PSV et les ensembles $N^-(v)$, $N^+(v)$ et $N^{++}(v)$. Chaqu'un de ces ensembles, auraient au moins $|V(D)|/3$ sommets. Si $D$ n'est pas de maille 3, donc ces ensemble sont deux à deux disjoints, et par conséquent $D$ a plus que $|V(D)|$ sommets, contradiction.

En utilisant un argument similaire, on a:

**Proposition 0.0.3.** *[8] Si $\gamma$ est un nombre positif tel que, pour tout digraphe $D$, il existe un sommet $v$ tel que $d^{++}(v) \geq \gamma.d^+(v)$, donc tout digraphe $D$ sur $n$ sommets avec $min\{\delta_D^+, \delta_D^-\} \geq \frac{n}{2+\gamma}$ est de maille 3.*

**Corollaire 0.0.2.** *Tout digraphe avec $min\{\delta_D^+, \delta_D^-\} \geq \frac{n}{2+\gamma}$ est de maille 3, où $\gamma = 0.657298...$ est la racine unique de l'equation $2x^3 + x^2 - 1 = 0$.*

Dans ce cas, $\frac{n}{2+\gamma} \approx 0.3764n$.

**Théorèm 0.0.11.** *[9] Tout digraphe $D$ sur $n$ sommets et avec $\delta_D^+ \geq 0.3465n$ est de maille 3.*

**Théorèm 0.0.12.** *[15] Tout digraphe $D$ sur $n$ sommets et avec $min\{\delta_D^+, \delta_D^-\} \geq 0.343545n$ est de maille 3.*

On termine par les remarques suivantes:

**Proposition 0.0.4.** *Tout digraphe sans triangles acyclic satisfait la CSV.*

**Corollaire 0.0.3.** *Tout digraphe de graphe sous-jacent de maille 4, satisfait le CSV.*

## Contres exemples minimaux

Soit $D$ un digraphe. Un intervalle de $D$ est un ensemble $K \subseteq V(D)$ tel que pour tous $u, v \in K$ on a: $N^+(u) \backslash K = N^+(v) \backslash K$ et $N^-(u) \backslash K = N^-(v) \backslash K$. L'ensemble vide et l'ensemble $V(D)$ sont des interavlles triviaux. $D$ est dit indécompsable si tous ses intervalles sont triviaux. Dans l'autre cas, on dit que $D$ est décomposable. Un digraphe indécomposable est critique s'il est indécompsable, et pour tout $u \in V(D)$ le digraphe $D - u$ est décomposable.

Dans [10] (corollaire 5.8), les digraphes critiques sont caractérisés. Pour $r \geq 2$ les 5 digraphes suivantes sont définit.

Les digraphes $\mathcal{P}_r$ et $\mathcal{P'}_r$ ont mème ensembles de sommets $\{a_1, ..., a_r, b_1 ..., b_r\}$, les arc de $\mathcal{P}_r$ sont les arcs $(a_i, b_j)$ où $i \geq j$, les arcs de $\mathcal{P'}_r$ sont $(a_i, a_j), (b_i, b_j)$ et $(a_i, b_j)$ où $i < j$.
Le tournoi $\mathcal{T}_r^1$ a un ensemble des sommets $\{c_0, c_1, ..., c_{2r}\}$ et un ensemble des arcs $(c_i, c_{i+k})$, $k = 1, ..., r$, où la somme $i + k$ est mod $2r + 1$.
Le tournoi $\mathcal{T}_r^2$ et le digraphe $\mathcal{D}_r$ ont $\{a_0, a_1, ..., a_r, b_1 ..., b_r\}$ comme un ensemble des sommets et les arcs du premier digraphe sont $(a_i, a_j), (b_j, a_i), (a_j, b_i), (b_j, b_i)$ si $i < j$ et les arcs $(b_j, a_j)$. Les arcs du second digraphe sont $(a_i, b_j), (b_i, b_j), (b_i, a_j)$ si $i < j$ et les arcs $(b_j, a_j)$.
Le tournoi $\mathcal{T}_r^3$ a $\{b, a_1, a_2, ..., a_r\}$ comme un ensemble de sommet et ses arcs sont $(a_i, a_j)$ pour $i < j$, $(b, a_i)$ pour $i$ impair et $(a_i, b)$ pour $i$ pair.

**Lemme 0.0.1.** *[10] Tout digraphe critique est isomorphe à $\mathcal{P}_r$, $\mathcal{P'}_r$, $\mathcal{T}_r^1$, $\mathcal{T}_r^2$, $\mathcal{T}_r^3$ ou $\mathcal{D}_r$, pour $r \geq 2$.*

Remarquons que $\mathcal{P}_r$, $\mathcal{P'}_r$ et $\mathcal{D}_r$ ont un puits, $\mathcal{T}_r^1$, $\mathcal{T}_r^2$ et $\mathcal{T}_r^3$ sont tournois. Donc les digraphes critiques satisfont le CSV ponderé.

Un contre exemple du CSV ponderé est un digraphe ponderé qui n'est pas de sommet avec le PSV pondéré. Il est minimal s'il a le minimum nombre de sommets.

**Proposition 0.0.5.** *Un contre exemple du CSV ponderé (s'il existe ) est fortement connexe, indécomposable et n'est pas critique.*

## Digraphe de Dépendance

**Motivation.** Supposons que $D = T - e$ est un digraphe où manque exactement une arête $e = ab$. L'orientation de $e$ nous donne un tournoi qui est une completion de $D$. Cependant, on cherche une orientation particulière. Supposons que $(i)$ il existe $v \in V\backslash\{a,b\}$ avec $v \to a$ et $b \notin N^+(v) \cup N^{++}(v)$ et $(ii)$ il existe $u \in V\backslash\{a,b\}$ avec $u \to b$ et $a \notin N^+(u) \cup N^{++}(u)$. Par définition, $v$ et $u$ sont distincts et $uv \neq e$. Alors, $uv$ n'est pas une arête manquante de $D$. D'où posons $u \to v$, ceci implique $u \to v \to a$ et contredit $(ii)$, $v \to u$ implique $v \to u \to b$ et contredit $(i)$. Donc
$(i)$ $(\forall v \in V\backslash\{a,b\})[(v \to a) \Rightarrow (b \in N^+(v) \cup N^{++}(v))]$ est vrai ou $(ii)$ $(\forall v \in V\backslash\{a,b\})[(v \to b) \Rightarrow (a \in N^+(v) \cup N^{++}(v))]$ est vrai.

Si $(i)$ est vrai, on ajoute l'arc $(a,b)$ à $D$, sinon, on ajoute $(b,a)$ à $D$. On suppose, sans perte de généralité que, $(i)$ est vrai et soit $T = D + (a,b)$ et on considère un ordre médian local (pondéré) de $T$ et soit $f$ son feed sommet. $f$ a le PSV dans $T$. Mais, on va vérifier qu'il a la PSV dans $D$. On suppose que $f \notin \{a,b\}$, i.e. $f$ est un sommet complet. Si $f \to x \to y$ dans $T$. L'arc $f \to x$ est dans $D$. Si $x \to y$ dans $D$ alors $y \in N^{++}(f) \cup N^+(f)$. Si non, $(x,y) = (a,b)$. Par définition de $(i)$, $y = b \in N^{++}(f) \cup N^+(f)$. Ce qui prouve que $f$ a la même second voisinage dans $D$ et $T$. Alors, $f$ a le PSV dans $D$. Supposons que $f$ est non complet, i.e., $f \in \{a,b\}$. Soit $T'$ le tournoi obtenu de $T$ en orientant $ab$ vers $f$. $L$ est aussi un ordre médian de $T'$. D'où $f$ a le PSV dans $T'$. $f$ a le même voisinage extérieur dans $D$ et $T'$ a le même deuxième voisinage extérieur dans $D$ et $T'$. D'où, $f$ a la PSV dans $D$.

Ce qui nous a motivé de donner la définition suivante:

**Definition 0.0.2.** ***(arête manquante et orientation convenable)***
*Une arête manquant ab est bonne si:*
$(i)$ $(\forall v \in V\backslash\{a,b\})[(v \to a) \Rightarrow (b \in N^+(v) \cup N^{++}(v))]$ *ou*
$(ii)$ $(\forall v \in V\backslash\{a,b\})[(v \to b) \Rightarrow (a \in N^+(v) \cup N^{++}(v))]$.
*Si ab satisfait $(i)$ on dit que $(a,b)$ est une orientation convenable de ab.*
*Si ab satisfait $(ii)$ on dit que $(b,a)$ est une orientation convenable de ab.*

**Motivation.** Supposons qu'une arête manquante $ab$ de $D$ n'est pas bonne. Donc $(i)$ et $(ii)$ ne sont pas vrais. Alors, il existe $v \in V\backslash\{a,b\}$ avec $v \to a$ et $b \notin N^+(v) \cup N^{++}(v)$ et il existe $u \in V\backslash\{a,b\}$ tele que $u \to b$ et $a \in N^+(u) \cup N^{++}(u))$. Dans ce cas, $vu$ doit être une arête manquante de

$D$.

**Dépend.** On dit qu'une arête manquante $x_1y_1$ de digraphe $D$ dépend d'une arête $x_2y_2$ si:
$x_1 \to x_2$, $y_2 \notin N^+(x_1) \cup N^{++}(x_1)$, $y_1 \to y_2$ et $x_2 \notin N^+(y_1) \cup N^{++}(y_1)$.

**Digraphe de dépendance.** Le digraphe de dépendance d'un digraphe $D$ noté par $\Delta_D$ (ou simplement par $\Delta$) est définit de la façon suivante: Ses sommets sont les arêtes manquantes de $D$ et $(ab, cd) \in E(\Delta)$ si $ab$ dépend de $cd$. Notons que $\Delta$ peut avoir des digons.

**Proposition 0.0.6.** *Soient $D$ un digraphe et $\Delta$ son digraphe de dépendance. Une arête manquante $ab$ est bonne si et seulement si son degré intérieur dans $\Delta$ est nul.*

## Bon ordre médian et intervalles

Soit $D$ un digraphe (pondéré) et soit $\Delta$ son digraphe de dépendance. Soit $C$ une composante connex de $\Delta$; On pose $K(C) = \{u \in V(D)$; il existe $v \in V(D)$ t.q. $uv$ est une arête manquante et $uv \in C$ $\}$. Le graphe intervalle de $D$, noté par $\mathcal{I}_D$, est définit de la façon suivante: ses sommets sont les composants connexes de $\Delta$ et 2 sommets $C_1$ et $C_2$ sont voisins si $K(C_1) \cap K(C_2) \neq \phi$. Soit $\xi$ une compsante connexe de $\mathcal{I}_D$. On pose $K(\xi) = \cup_{C \in \xi} K(C)$. Clairement, si $uv$ est une arête manquante de $D$, alors il existe une composante connexe unique $\xi$ de $\mathcal{I}_D$ telle que $u$ et $v$ appartiennent à $K(\xi)$. Soit $f \in V(D)$. Si $f$ est complet, on pose $J(f) = \{f\}$, sinon on pose $J(f) = K(\xi)$, où $\xi$ est l'unique composante connexe de $\mathcal{I}_D$ telle que $f \in K(\xi)$. Il est clair que si $x \in J(f)$ alors $J(f) = J(x)$ et si $x \notin J(f)$ alors, pour tout $y \in J(f)$, on a $x$ et $y$ sont voisins dans $D$.

Soit $L = x_1x_2...x_n$un ordre médian (local) pondéré de $D$. L'ensemble $[i, j] := [x_i, x_j] := \{x_i, x_{i+1}, ..., x_j\}$ est dit un intervalle de $L$. Rappellons qu'un intervalle de $D$ est un ensemble $K \subseteq V(D)$ tel que pour tous $u, v \in K$ on a: $N^+(u)\backslash K = N^+(v)\backslash K$ and $N^-(u)\backslash K = N^-(v)\backslash K$. La proposition suivante montre une relation entre ces deux définitions de l'intervalle.

**Proposition 0.0.7.** *Soit $\mathcal{I} = \{I_1, ..., I_r\}$ un ensemble d'intervalles de $D$, deux à deux disjoints. Pour tout ordre médian (pondéré) $L$ de $D$, il existe un ordre médian $L'$ de $D$ tel que: $L$ et $L'$ ont le même feed sommet et tout intervalle dans $\mathcal{I}$ est un intervalle de $L'$.*

On dit qu'un digraphe est bon si les ensembles $K(\xi)$ sont des intervalles de $D$. La propostion ci-dessus montre que, tout bon digraphe a un ordre médian (local) pondéré $L$ où les $K(\xi)$ sont des intervalles de $L$. Cet ordre est appelé un bon ordre médian (local) pondéré de $D$.

**Théorèm 0.0.13.** *Soient $(D, \omega)$ un bon digraphe pondéré et $L$ un bon ordre médian de $(D, \omega)$, avec un feed sommet $f$. Pour tout $x \in J(f)$, on a $\omega(N^+(x)\backslash J(f)) \leq \omega(G_L\backslash J(f))$. En plus, si $x$ a le PSV pondéré dans $(D[J(f)], \omega)$ donc $x$ a la PSV pondéré dans $D$.*

Le théorème ci-dessus implique qu'un tournoi satisfait le CSV pondéré.

Soient $L$ un bon ordre médian d'un bon digraphe $D$ et $f$ son feed vertex. Pour tout $x \in J(f)$ on a $\omega(N^+(x)\backslash J(f)) \leq \omega(G_L\backslash J(f))$. Soient $b_1, \cdots, b_r$ les sommets mauvais de $L$ qui n'appartiennent pas à $J(f)$ et $v_1, \cdots, v_s$ les autres sommets qui n'appartiennent pas à $J(f)$, énumérés dans l'ordre

croissant dans $L$.
Si $\omega(N^+(f)\backslash J(f)) < \omega(G_L\backslash J(f))$, $Sed(L) := L$. Si $\omega(N^+(f)\backslash J(f)) = \omega(G_L\backslash J(f))$, $sed(L) := b_1 \cdots b_r J(f) v_1 \cdots v_s$.

**Lemme 0.0.2.** *Soit $L$ un bon ordre médian ponderé d'un bon digraphe $(D, \omega)$. Donc $Sed(L)$ est un bon ordre médian ponderé de $(D, \omega)$.*

On définit inductivement $Sed^0(L) = L$ et $Sed^{q+1}(L) = Sed(Sed^q(L))$. S'il existe $q$ telle que $Sed^q(L) = y_1 ... y_n$ et $\omega(N^+(y_n)\backslash J(y_n)) < \omega(G_{Sed^q(L)}\backslash J(y_n))$, on dit que $L$ est stable. Sinon, on dit que $L$ est périodique. Ces ordres sont utilisés pour prouver que certains digraphes ont 2 sommets ayant le PSV.

## L'approche par complétion

Dans cette section $D = T - M$, où T est un tournoi et $M$ est un couplage, c'est à dire, $D$ est un digraphe où manque un couplage, et $\Delta$ son digraphe de dépendance.

**Lemme 0.0.3.** *[3] $\Delta$ est formé de chemins orientés et circuits disjoints.*

**Lemme 0.0.4.** *[3] $K(C)$ est un intervalle de $D$, pour tout circuit $C$ de $\Delta$.*

**Lemme 0.0.5.** *[3] Pour tout circuit $C$ de $\Delta$, $D[K(C)]$ a un sommets $v$ avec le PSV.*

Pour tout chemin maximal $P = a_1b_1, ...a_kb_k$ de $\Delta$, précisément $a_i \rightarrow a_{i+1}$, $b_i \rightarrow b_{i+1}$, $b_{i+1} \notin N^+(a_i) \cup N^{++}(a_i)$ et $b_{i+1} \notin N^+(a_i) \cup N^{++}(a_i)$ pour $i = 1, ..., k-1$. Le degré intérieur de $a_1b_1$ est zéro, donc $a_1b_1$ est une bonne arête manquante de $D$. On peut supposer que $(a_1, b_1)$ est une orientation convenable. On ajoute à $D$ les arêtes $(a_i, b_i)$ pour tout $i$. Soient $D'$ le digraphe obtenu et $\Delta'$ son digraphe de dépendance. $\Delta'$ est formé de circuits disjoints. Alors $D'$ est un bon digraphe.

Soit $L$ un bon ordre médian de $D'$ avec un feed vertex $f$. $f$ a le PSV dans $D'$. En plus, si $f$ est complet dans $D'$ on peut vérifier que $f$ a le PSV dans $D$. Sinon, il existe un circuit $C$ de $\Delta'$ (et $\Delta$) tel que $f \in K(C) = J(f)$. D'après le lemme 0.0.5, il existe un sommet $v$ ayant le PSV dans $D[K(C)]$. Mais $|N^+(v) \backslash K(C)| \leq |G_L|$. Alors,

**Théorèm 0.0.14.** *[3] $D$ satisfait le CSV.*

On peut voir qu'on a utilisé un autre digraphe $D'$ qui contient $D$, pour vérifier que le dernier satisfait le CSV. A. El Sahili a proposé la conjecture suivante:

**Conjecture 5.** *(EC) Tout digraphe $D$ a un completion avec un feed vertex ayant le PSV dans $D$.*

Il est clair que EC implique le CSV. On ne sait pas s'ils sont équivalents.

La versionponderé de la conjecture de El Sahili sera:

**Conjecture 6.** *(GC) Tout digraphe ponderé $(D, \omega)$ a une completion avec un feed vertex ayant le PSV ponderé dans $(D, \omega)$.*

# Forcing Graphe

## Définitions

Soit $\mathcal{H}$ un ensemble de digraphes ( les digons sont permis) et soit $G$ un graphe. On dit que $G$ est $\mathcal{H}$-forcing, si $\Delta_D \in \mathcal{H}$ pour tout digraphe $D$ où manque $G$. L'ensemble de tous les $\mathcal{H}$-forcing graphes sera noté par $\mathcal{F}(\mathcal{H})$.

Les digraphes qui n'ont pas des arêtes sont appelés triviaux.

**Proposition 0.0.8.** *Soit $\mathcal{H}$ un ensemble de digraphes. On a $\mathcal{F}(\mathcal{H})$ est non vide si et seulement si $\mathcal{H}$ a un digraphe trivial.*

**Problèm 0.0.1.** *Soient $\mathfrak{S}$ l'ensemble de tous les digraphes triviaux et $\vec{\mathcal{P}}$ l'ensemble de tous les digraphes composés de chemins orientés disjoints seulement. Caractériser $\mathcal{F}(\mathfrak{S})$ et $\mathcal{F}(\vec{\mathcal{P}})$.*

## $\mathfrak{S}$-forcing graphe

On va Caractériser $\mathcal{F}(\mathfrak{S})$ et prouver que les digraphes où manque un élément de $\mathcal{F}(\mathfrak{S})$ satisfont la CSV $\mathfrak{S}$). Clairement, si $G$ est le graphe manquant de $D$ et $G \in \mathcal{F}(\mathfrak{S})$ alors toutes les arêtes manquantes de $D$ sont bonnes. On va prouver que les digraphes où manque un élément de $\mathcal{F}(\mathfrak{S})$ satisfont le CSV.

**Théorèm 0.0.15.** *Soient $(D, \omega)$ un digraphe pondéré. Si toutes les arêtes manquantes de $D$ sont bonnes donc $D$ satisfait la GC.*

**Definition 0.0.3.** *Une n-étoile généralisée $G_n$ est un graphe définit par:*

**1)** $V(G_n) = \bigcup_{i=1}^{n}(X_i \cup A_{i-1})$, *où les $A_i$ et $X_i$ sont deux à deux disjoints*

**2)** $G_n[\bigcup_{i=1}^{n} X_i]$ *est un graphe complet et les $X_i$ sont non vides*

**3)** $\bigcup_{i=1}^{n} A_{i-1}$ *est un stable et $A_i$ est non vide pour tout $i > 0$*

**4)** $N(A_0) = \phi$ *et pour tout $i > 0$, pour tout $a \in A_i$, $N(a) = \bigcup_{1 \leq j \leq i} X_j$.*

**Théorèm 0.0.16.** *Soit $G$ un graphe. Les proposition suivantes sont équivalentes:*

**(A)** *$G$ est une n-étoile généralisée.*

**(B)** *Les extrémités de deux arêtes non adjacentes de $G$, n'engendrent pas un carré.*

**(C)** *Toutes les arêtes de tout digraphe où manque $G$ sont bons.*

Un *soleil* $G$ est un graphe formé d'un graphe complet $T$ et un stable $S$ tel que pour tout $s \in S$ on a $N(s) = V(T)$. Clairement, $G$ est une 2-étoile généralisée où une 1-étoile généralisée. Si $V(T)$ a un seul élément donc $G$ est une étoile et si $S$ est vide donc $G$ est un graphe complet.

**Corollaire 0.0.4.** *Tout digraphe pondéré où manque une n-étoile généralisée satisfait (GC).*

**Corollaire 0.0.5.** *Tout digraphe pondéré où manque un soleil satisfait (GC).*

**Corollaire 0.0.6.** *Tout digrapheponderé où manque un graphe complet satisfait (GC).*

**Corollaire 0.0.7.** *Tout digraphe ponderé où manque une étoile satisfait (GC).*

En particulier, ces digraphes satisfont (EC) et donc (CSV).

## $\vec{\mathcal{P}}$-forcing graphe

Un peigne $G$ est un graphe définit par:

**1)** $V(G)$ est l'union disjoint de trois ensembles $A$, $X$ et $Y$.

**2)** $G[X \cup Y]$ est un graphe complet.

**3)** $A$ est un stable où $N(A) = X$ et $N(a) \cap N(b) = \phi$ pour tout 2 sommets distincts $a, b \in A$.

**4)** Pour tout $a \in A$, $d(a) = 1$.

Remarquons que les arêtes avec un extémité dans $A$ forme un couplage.

**Proposition 0.0.9.** *Les peignes sont $\vec{\mathcal{P}}$-forcing.*

**Théorèm 0.0.17.** *Tout digraphe où manque un peigne satisfait (EC).*

Un $\tilde{K}^4$ est un graphe obtenu d'un graphe complet en supprimant deux arêtes non adjacentes. Si $xy$ et $uv$ les arêtes supprimées donc la restriction de $\tilde{K}^4$ à $\{x, y, u, v\}$ est un carré.

**Proposition 0.0.10.** *Les graphes $\tilde{K}^4$ sont $\vec{\mathcal{P}}$-forcing.*

**Théorèm 0.0.18.** *Tout digraphe où manque un $\tilde{K}^4$ satisfait (EC).*

Un $\tilde{K}^5$ est un graphe obtenu d'un graphe complet en supprimant les arêtes d'un cycle de longueur 5. Notons que la restriction de $\tilde{K}^5$ aux sommets du cycle supprimé est encore un cycle de longueur 5.

**Proposition 0.0.11.** *Les graphes $\tilde{K}^5$ sont $\vec{\mathcal{P}}$-forcing.*

**Théorèm 0.0.19.** *Tout digraphe où manque un $\tilde{K}^5$ satisfait (EC).*

Digraphes où manque un couplage sont digraphes de degré minimum $|V(D)| - 2$. Ces digraphes satisfont (CSV). Une généralization de ces digraphes sont les digraphes de degré minimum $|V(D)| - 3$, c'est à dire, le graphe manquant est formé de cycles et chemins disjoints. $P_3$ est le chemin de longueur 3, $C_3$, $C_4$ et $C_5$ sont les cycles de longueur 3, 4 et 5 respectivement.

**Corollaire 0.0.8.** *Tout digraphe où manque un $P_3$, $C_3$, $C_4$ ou un $C_5$ satisfait (EC).*

## Tournoi où manque $n$ étoiles

Rappellons qu'un roi d'un digraphe $D$ est un sommet tel que $\{x\} \cup N^+(x) \cup N^{++}(x) = V(T)$. On sait que tout tournoi a un roi. En plus, pour chaque entier positive $n \notin \{2, 4\}$, il exist un tournoi $T_n$ avec $n$ sommets, tel que chaque sommet est un roi.

On dit que $n$ étoiles sont disjoints si ses ensembles de sommets sont deux à deux disjoints.

**Théorèm 0.0.20.** *Soit $D$ un digraphe où manque $n$ étoiles disjointes et $\Delta$ son digraphe de dépendance. Supposons que, dans le tournoi engendré par les centres de ces étoiles, chaque sommet est un roi. Si $\delta^-_\Delta > 0$ donc $D$ satisfait le EC.*

On a besoin du lemme suivant.

**Lemme 0.0.6.** *Soit $D$ un digraphe où manque $n$ étoiles disjoints. Si les composantes connexes de son digraphe de dépendance sont fortements connexes et non triviaux alors $D$ est un bon digraphe.*

**Théorèm 0.0.21.** *Soit $D$ un digraphe où manque une étoile et un couplage. Si les composants connexes du digraphe de dépendance qui contiennet un arête de l'étoile supprimée ont un degré extérieur et intérieur minimum non nulles, alors $D$ satisafait EC.*

Soit $D$ un digraphe tel que son graphe manquant est l'union disjoint d'un couplage $M$ et un étiole $S_x$ de centre $x$. $\Delta$ et $\mathcal{I}_D$ dénotent le digraphe de dépendance et le graphe intervalle de $D$. En plus, on suppose que les composantes connexes du digraphe de dépendance qui contiennent une arête de l'étoile supprimé ($x$ est un éxtrémité de cet arête), ont un degré extérieur et intérieur minimum non nulles.

On va prouver que $D$ satisfait EC et par conséquent CSV.

Soient $P$ une composante connexe de $\Delta$ ou $\mathcal{I}_D$ et $v$ un sommet de $D$. On dit que $v$ apparaît dans $P$ si $v \in K(P)$. Si non, on dit que $v$ n'apparaît pas dans $P$.

Il n'est pas dificile de démontrer que le degré extérieur et intérieur dans $\Delta$ de chaque arête $ax$ de l'étoile $S_x$ est exactement 1 et si une arête $uv \in M$ a un degré extérieur (resp. intérieur) plus que 1 donc $N^+_\Delta(uv) \subseteq E(S_x)$ (resp, $N^-_\Delta(uv) \subseteq E(S_x)$). Par suite, toute composante connexe de $\Delta$, où $x$ n'apparaît pas, est un chemin orienté, ou un circuit.

On note par $\xi$ la compsante connexe unique de $\mathcal{I}_D$ où $x$ apparaît. On note que $\mathcal{I}_D$ est formé de $\xi$ et des autres sommets isolés (ayant degré zéro dans $\mathcal{I}_D$).

Soit $P = a_1b_1a_2b_2 \cdots a_kb_k$ une compsante connexe de $\Delta$ qui est un chemin maximal et où $x$ n'apparaît pas, précisément $a_i \to a_{i+1}, b_i \to b_{i+1}$ pour $i = 1, ..., k-1$. Parce que $a_1b_1$ est un bon arête manquant, $(a_1, b_1)$ ou $(b_1, a_1)$ est une orientation convenable. On peut supposer que $(a_1, b_1)$ est une orientation convenable. On ajoute à $D$ les arêtes $(a_i, b_i)$ pour tout $i$. On fait de même pour tous les chemins orientés maximals de $\Delta$. On note l'ensemble de nouveaux arcs par $F$ et pose $D' = D + F$.

**Lemme 0.0.7.** *$D'$ est un bon digraphe.*

**Lemme 0.0.8.** *$D[K((\xi)]$ satisfait EC.*

Dans la suite, $C = a_1b_1...a_kb_k$ dénote un circuit de $\Delta$ où $x$ n'apparaît pas, précisément $a_i \to a_{i+1}$, $b_{i+1} \notin N^{++}(a_i) \cup N^+(a_i)$, $b_i \to b_{i+1}$ et $a_{i+1} \notin N^{++}(b_i) \cup N^+(b_i)$.

**Lemme 0.0.9.** *Dans $D[K(C)]$ on a:*
*Si $k$ est impair:*

$$N^+(a_1) = N^-(b_1) = \{a_2, b_3, \cdots, a_{k-1}, b_k\}$$
$$N^-(a_1) = N^+(b_1) = \{b_2, a_3, \cdots, b_{k-1}, a_k\},$$

*Si $k$ est pair:*

$$N^+(a_1) = N^-(b_1) = \{a_2, b_3, \cdots, b_{k-1}, a_k\}$$
$$N^-(a_1) = N^+(b_1) = \{b_2, a_3, \cdots, a_{k-1}, b_k\}.$$

**Lemme 0.0.10.** *Dans $D[K(C)]$ on a: $N^+(a_i) = N^-(b_i)$, $N^-(a_i) = N^+(b_i)$, $N^{++}(a_i) = N^-(a_i) \cup \{b_i\} \backslash \{b_{i+1}\}$ et $N^{++}(b_i) = N^-(b_i) \cup \{a_i\} \backslash \{a_{i+1}\}$ pour tout $i = 1, ..., k$ où $a_{k+1} := a_1$, $b_{k+1} := b_1$ si $k$ est impair $a_{k+1} := b_1$, $b_{k+1} := a_1$ si $k$ est pair. Donc $d^{++}(v) = d^+(v) = d^-(v) = k - 1$ pour tout $v \in K(C)$.*

**Corollaire 0.0.9.** *Tout digraphe où manque un couplage satisfait EC.*

Notons que notre méthode montre que le sommet $f$ trouvé ayant la PSV est un feed vertex d'un digraphe qui contient $D$, mais c'est pas le cas par la methode présenté dans [3].

Rappellons que $F$ est l'ensemble des arcs ajoutés à $D$ afin d'obtenir le digraphe $D'$. Donc, si $F = \phi$ alors $D$ est un bon digraphe.

**Théorèm 0.0.22.** *Soit $D$ un bon digraphe où manque un couplage et supposons que $F = \phi$. Si $D$ ne contient pas de puits alors $D$ a au moins deux sommets ayants le PSV.*

On va étudier les cas où le graphe manquant de $D$ est soit deux étoiles disjoints, soit trois étoiles disjoints.

**Théorèm 0.0.23.** *Soient $D$ un digraphe où manque 2 étoiles et $\Delta$ son digraphe de dépendance. Si $\delta_\Delta > 0$ alors $D$ satisfait EC.*

**Théorèm 0.0.24.** *Soient $D$ un digraphe où manque 2 étoiles et $\Delta$ son digraphe de dépendance. Si $\delta^+_\Delta > 0$, $\delta^-_\Delta > 0$ et $D$ n'a pas de puits, alors $D$ a au moins 2 sommets ayants le PSV.*

**Théorèm 0.0.25.** *Soit $D$ un digraphe où manque trois étoiles telles que le triangle engendré par ses centres est cyclique. Soit $\Delta$ le digraphe de dépendance de $D$. Si $\delta_\Delta > 0$ alors $D$ satisfait (EC).*

**Théorèm 0.0.26.** *Soit $D$ un digraphe où manque trois étoiles telles que le triangle engendré par ses centres est cyclique. Soit $\Delta$ le digraphe de dépendance de $D$. Si $\delta^+_\Delta > 0$, $\delta^-_\Delta > 0$ et $D$ n'a pas des puits, alors $D$ a au moins 2 sommets ayants la PSV.*

This dissertation is prepared at:

Institute Camille Jordan
Université Claude Bernard Lyon 1
Département de Mathématiques
43 BD du 11 Novembre 1918
69622 Villeurbanne, France

and at:

Département de Mathématiques
Faculté des Sciences I
Université Libanaise
Campus Universitaire de Rafic Hariri
Beyrouth-Hadath, Liban

## Acknowledgements

It would not have been possible to write this thesis without the help of many people. I am grateful for each one of them.

First, I would like to thank my advisor professor A. El Sahili for many useful scientific discussions.

In addition, I would like to express my gratitude to Dr. Hamza Si Kaddour for his advices and help concerning my stay at Lyon and Claude Bernard Lyon 1 University, France.

I would like to thank also my family: My mother and my sisters. Above all, my father. Words can not express my gratitude for his support and help.

# Contents

# Part I

# Introduction

# Chapter 1

# Preliminary Definitions

## 1.1 Graphs

**Graph.** A *graph* $G$ is a pair of two disjoint finite sets $(V(G), E(G))$ where the elements of $E(G)$ belongs to the set $\{\{x, y\}; x, y \in V(G)\}$. $V(G)$ is called the *vertex set* and its elements are the *vertices* of $G$, while $E(G)$ is called the *edge set* and its elements are the *edges* of $G$. We write $xy$ (or $yx$) instead of the edge $\{x, y\}$. We may write $v \in G$ and $e \in G$ instead of $v \in V(G)$ and $e \in E(G)$.

**Empty graphs.** An *empty graph* is a one with no edge.

**Adjacent.** If $e = xy \in E(G)$, we say that $x$ and $y$ are *neighbors* or *adjacent* and they are the *endpoints* of $e$ and $e$ is *incident* to $x$ and $y$. If $xy \notin E(G)$, we say $x$ and $y$ are *non-adjacent*. Two edges $e$ and $f$ are *adjacent* if they have a common endpoint, otherwise *non-adjacent* or *independent*.

**Neighborhood.** The *neighborhood* of a vertex $x$ is the set of neighbors of $x$ and is denoted by $N_G(x)$. For $X \subseteq V(G)$, $N_G(X) := \bigcup_{x \in X} N_G(x) \backslash X$.

**Degree.** The *degree* of a vertex $x$ is the number $|N_G(x)|$ and is denoted by $d_G(x)$. $x$ is a *whole vertex* if it is adjacent to all other vertices of $G$, that is $d_G(x) = |V(G)| - 1$. Otherwise, $x$ is a *non-whole* vertex. A vertex is *isolated* if its degree is zero.

The following statement is sometimes called the "The First Theorem of Graph Theory " or " The Hand Shaking Lemma".

**Proposition 1.1.1.** ***(Degree-Sum Formula)*** *Let $G$ be a graph. We have $\Sigma_{x\in V(D)} d_G(x) = 2|E(G)|$.*

Indeed, every edge $xy$ is counted twice. Once in $d_G(x)$ and once in $d_G(y)$.

**Complement graph.** For $x, y \in V(G)$, $xy$ is called a *missing edge* if $xy \notin G$. The *complement graph* $\overline{G}$ of $G$ is the one whose vertices are non-whole ones of $G$ and whose edges are the missing edges of $G$.

**Subgraphs.** A *subgraph* of $G$ is a graph $G'$ such that $V(G') \subseteq V(G)$ and $E(G') \subseteq E(G)$. It is a *spanning subgraph* if $V(G') = V(G)$. An *induced subgraph* of $G$ is a subgraph of $G'$ such that $xy \in E(G')$ whenever $x, y \in V(G')$ and $xy \in E(G)$. For $U \subseteq V(G)$, the subgraph induced by $U$, is the induced subgraph of $G$ whose vertex set is $U$, and it is denoted by $G[U]$. For $F \subseteq E(G)$, the subgraph of $G$ induced by $F$, denoted by $G[F]$, is the subgraph induced by the endpoints of the edges in $F$.

**Isomorphism.** Two graphs $G$ and $G'$ are said to be *isomorphic* if there is a bijective function $f : V(G) \longrightarrow V(G')$ such that for every $x, y \in V(G)$, $xy \in E(G)$ if and only if $f(x)f(y) \in E(G')$. Such a function is called an *isomorphism*.
We say that $G$ contains a copy of $G'$ if the second is isomorphic to a subgraph of the first.

**$H$-free graphs.** Let $G$ and $H$ be two graphs. We say that $G$ is an $H$-free graph if $G$ does not contain a copy of $H$. In this case $H$ is forbidden in $G$. Forbidden graphs plays important role in characterizing some families of graphs. We will use them in characterizing a family of graph defined in Chapter 5.

A *loop* is an edge of the form $xx$. In this thesis, we suppose that all our graphs do not have any loop since they do not play a role. When the graph is clear from the context we remove the subscript $G$ from the above notations.

## 1.2 Digraphs

**Digraph.** A *digraph* $D$ is a pair of two disjoint finite sets $(V(D), E(D))$ where $E(D) \subseteq \{(x,y); x,y \in V(G)\}$. $V(D)$ is called the *vertex set* and its members are the *vertices* of $D$, while $E(D)$ is the *arc set* and its members are the *arcs* of $D$. If $f$ is a vertex or an arc of $D$, we may write $f \in D$ when no confusion is possible. For short, we write $x \rightarrow y$ if $(x,y) \in E(D)$.

**Empty digraphs.** An *empty digraph* is a one with no arc.

**Adjacent.** For $x, y \in V(D)$, if $e = (x,y) \in E(D)$, we say that $y$ (resp. $x$) is an *out-neighbor* (resp. *in-neighbor*) of $x$ (resp. $y$), $x$ and $y$ are *adjacent* or *neighbors* and they are the *endpoints* of the arc $e$, $x$ is the *tail* of $e$ and $y$ is its *head* and $e$ is incident to $x$ and $y$. If $(x,y)$ and $(y,x) \notin E(D)$, then $x$ and $y$ are non-adjacent. Two arcs are *non-adjacent* or *independent* if they do not have a common endpoint.

**Neighborhood.** The *out-neighborhood* (resp. *in-neighborhood*) of a vertex $x$ is the set of its out-neighbors (in-neighbors) and is denoted by $N_D^+(x)$ (resp. $N_D^-(x)$). For $X \subseteq V(D)$, $N_D^+(X)$ is the set $\bigcup_{x\in X} N_D^+(x)\backslash X$ and $N_D^-(X)$ is the set $\bigcup_{x\in X} N_D^-(x)\backslash X$. In particular, the *second out-neighborhood* of a vertex $x$ is $N_D^{++}(x) = N_D^+(N_D^+(x))$. So the two sets $N_D^+(x)$ and $N_D^{++}(x)$ are disjoint.

**Matching.** A set of independent edges or arcs is called a *matching*.

**Degree.** The *out-degree* (resp. in-degree) of a vertex $x$ is $d_D^+(x) = |N_D^+(x)|$ (resp. $d^-(x) = |N_D^-(x)|$). A vertex $x$ is said to be *whole* if its degree $d_D(x) := d_D^+(x) + d_D^-(x) = |V(D)| - 1$, i.e. $x$ is adjacent to all the other vertices of $D$. Otherwise, $x$ is a non-whole vertex. A vertex with zero degree is an *isolated* vertex.

The digraph analogue of the degree-sum formula for graphs is the following:

**Proposition 1.2.1.** *For a digraph $D$ we have $\Sigma_{x\in V(D)} d_D^+(x) = \Sigma_{x\in V(D)} d_D^-(x)$ $= |E(D)|$.*

Indeed, in both sums, every arc is counted once.

**Sink, source and leaf.** A *sink* is a vertex of zero out-degree. In the opposite side, a *source* is a vertex with zero in-degree. A *leaf* is a vertex of degree one. An *out-leaf* is a sink of degree one, i.e. it is a sink and a leaf. An *in-leaf* is a source of degree one, i.e. it is a source and a leaf.

**Oriented graph.** The arcs $(x, y)$ and $(y, x)$ are called *orientations* of a given edge $xy$. An *orientation* $D$ of a graph $G$, is a digraph with vertex set $V(G)$ and with one orientation of every edge of $G$. Such a digraph is called an *oriented graph.* A *loop* is an arc of the form $(x, x)$ and a *digon* is a set of two arcs of the form $(x, y)$ and $(y, x)$. Clearly, a digraph with no loop and no digon is an oriented graph.

**Underlying graph.** The *underlying graph* of a digraph $D$ is a graph, denoted by $G(D)$, whose vertex set is $V(D)$ and $xy$ is an edge of $G(D)$ if $(x, y)$ or $(y, x)$ is in $D$.

**Missing graph.** The *missing graph* of a digraph $D$, denoted by $G_D$, is the complement graph of its underlying graph, i.e. $G_D = \overline{G(D)}$. In other words, its vertices are the non-whole vertices of $D$ and $xy \in E(G_D)$ if $(x, y)$ and $(y, x) \notin E(D)$.

**Subdigraph.** A *subdigraph* $D'$ of a digraph $D$ is a digraph with $V(D') \subseteq V(D)$ and $E(D') \subseteq E(D)$. It is a *spanning* subdigraph if $V(D') = V(D)$. An *induced* subdigraph $D'$ of $D$ is a subdigraph such that for $x, y \in V(D')$, $(x, y) \in E(D')$ whenever $(x, y) \in E(D)$. For $U \subseteq V(D)$, the subdigraph induced by $U$, denoted by $D[U]$, is the induced subdigraph of $D$ whose vertex set is $U$. For $F \subseteq E(D)$, the subdigraph induced by $F$ is the subdigraph induced by the set of end-points of the arcs in $F$. For a set of arcs $F$, $D\backslash F$ (resp. $D \cup F$) denote the spanning subdigraph of $D$ whose arc set is $E(D)\backslash F$ (resp. $E(D) \cup F$). When $F$ is a singleton $\{e\}$, we write $D\backslash e$ (resp. $D \cup e$) instead. For $A \subseteq V(D)$, $D - A$ is by definition $D[V(D)\backslash A]$.

**Isomorphism.** Two digraphs $D$ and $D'$ are said to be *isomorphic* if there is a bijective function $f : V(D) \longrightarrow V(D')$ such that for every $x, y \in V(D)$, $(f(x), f(y)) \in E(D')$ if and only if $(x, y) \in E(D)$. Such a function is called an *isomorphism*. We say that $D$ contains a copy of $D'$ (or $D'$ is contained in $D$) if the second is isomorphic to a subdigraph of the first.

**Weighted digraph.** A weighted digraph is a couple $(D, \omega)$ where $D$ is a digraph and $\omega : V(D) \longrightarrow \mathcal{R}_+$ is a non negative real valued function on the vertex set of $D$. Such a function $\omega$ is called a weight function. The value $\omega(x)$ is the weight of $x$, where $x \in V(D)$. An edge-weighted digraph is a couple $(D, \omega)$ where $D$ is a digraph and $\omega : E(D) \longrightarrow \mathcal{R}_+$ is a non-negative real valued function on the edge set of $D$. Such a function $\omega$ is called an edge-weight function.

Let $(D, \omega)$ be a weighted digraph. The weight function $\omega$ induces an edge-weight function, denoted also by $\omega$ and defined as follows: for an arc $(x, y) \in E(D)$, its weight is $\omega(x, y) := \omega(y)$.
For $U \subseteq V(D)$ the weight of $U$ is $\omega(U) := \Sigma_{u \in U} \omega(u)$ and the weight of $F \subseteq E(D)$ is $\omega(F) := \Sigma_{e \in F} \omega(e)$. Since vertices of weight zero have no role we suppose that $\omega$ is positively valued.

**Contraction.** Let $D$ be a digraph and let $S \subseteq V(D)$. $D/S$ denotes the digraph obtained from $D$ after contracting $S$, defined as follows. Its vertex set is $V(D) \cup v_S \backslash S$ and its arc set is $\{(x, y) \in E(D); x, y \in V(D) \backslash S\} \bigcup \{(v_S, y)$; there is $s \in S$ with $(s, y) \in E(D)\} \bigcup \{(x, v_S)$; there is $s \in S$ with $(x, s) \in E(D)\}$. In other words, $D/S$ is obtained from $D - S$ by adding an extra vertex $v_S$ adjacent to the neighbors of $S$. We say that the contraction is well defined if $D/S$ does not contain digons. For a collection $\mathcal{S}$ of pairwise disjoint subsets of $V(D)$, $\mathcal{S} = \{S_1, ..., S_r\}$, $D/\mathcal{S}$ is the digraph obtained by contracting the sets $S_i$, $1 \leq i \leq r$, successively.

In the rest of this thesis, digraphs contain neither loops nor digons (unless, otherwise stated). So they are oriented graphs. When there is no confusion, we omit the subscript in the above notations.

## 1.3 Usual structures

**Path.** A *path* $P$ is a graph with vertex set $\{v_1, ..., v_n\}$ and edge set $\{v_i v_{i+1}, i < n\}$. Such a path is denoted by $v_1 v_2 ... v_n$ and is called a $v_1, v_n$-path or a path between $v_1$ and $v_n$. A spanning subgarph which is a path is called a *Hamiltonian path*.

**Oriented path.** An *oriented path* is simply an orientation of a path.

**Directed path.** A *directed path* $P$ is a digraph with vertex set $\{v_1, ..., v_n\}$ and arc set composed of the arcs $\{(v_i, v_{i+1}), i < n\}$. Such a directed path is denoted by $v_1 ... v_n$. If $P = v_1 ... v_n$ is a (directed) path, we say that $P$ is a (directed) $v_1, v_n$-path. A spanning subdigraph of a digraph $D$ which is a directed path is called a *Hamiltonian path* of $D$.

**Cycle.** A *cycle* $C$ is a graph with vertex set $\{v_1, ..., v_n\}$ and edge set $\{v_i v_{i+1}, i < n\} \cup \{v_n v_1\}$. Such a cycle is also denoted by $v_1 ... v_n$. A spanning cycle of a graph is called a *Hamiltonian cycle*.

**Oriented cycle.** An *oriented cycle* is simply an orientation of a cycle.

**Directed cycle.** A *directed cycle* $C$ is a digraph with vertex set $\{v_1, ..., v_n\}$ and arc set $\{(v_i, v_{i+1}) i < n\} \cup \{(v_n, v_1)\}$. Such a directed cycle is also denoted by $v_1 ... v_n$. A spanning directed cycle is called a *Hamiltonian cycle*.

**Connected components.** A graph $G$ is *connected* if for any two vertices $x$ and $y$, there is an $x, y$-path contained in $G$. A *connected component* (or component) $U$ of $G$ is a maximal subset of $V(G)$ such that $G[U]$ is connected. A digraph is connected if its underlying graph is connected and its connected components are those of its underlying graph.

**Strongly connected components.** A digraph $D$ is *strongly connected* (or strong) if for any two vertices $x$ and $y$ there is a directed $x, y$-path in $D$. A *strongly connected component* (or strong component) $U$ of $D$ is a maximal subset of $V(D)$ such that $D[U]$ is strong.

Clearly, the connected components of a (di)graph form a partition of the vertex set. In addition, the strong components of a digraph form a partition of the vertex set. For convenience and with a slight abuse of notation, we do not distinguish between a connected component $K$ and the sub(di)graph $G[K]$ (or $D[K]$). For example, we might say a path in $K$ instead of a path in $G[K]$.

**Tree.** A *tree* is connected graph without any cycle. It is well known that every connected graph has a spanning tree: A connected spanning subgraph $T$ of a graph $G$ with the minimum number of edges is a tree. Indeed, if $T$ has a cycle, then deleting one edge (but not its endpoints) of the cycle from $T$ yields a spanning connected subgraph of $G$ with number of edges less than that of $T$, a contradiction. So, between any two vertices of a tree there is exactly one path.

**Oriented tree.** An *oriented tree* is simply an orientation of a tree.

**In-tree and out-tree.** An *out-tree* is an oriented tree with exactly one source vertex and in which every other vertex has in-degree one. So, an out-tree $T$ with source $s$ satisfies that for every $x \in V(T)$ there is a unique directed $s, x$-path.
An *in-tree* is an oriented tree with exactly one sink vertex and every other vertex has out-degree one. So, an in-tree $T$ with sink $s$ satisfies that for every $x \in V(T)$ there is a unique directed $x, s$-path.

**Forest.** A *forest* is a graph whose connected components are trees.

**Oriented forest.** An *oriented forest* is simply an orientation of a forest.

**In-forest and out-forest.** An *out-forest* is an oriented forest whose connected components are out-trees.
An *in-forest* is an oriented forest whose connected components are in-trees.

**Star.** A *star* is tree with one vertex, called the center, adjacent to all the other vertices. So every edge of the star is incident to its center.

We say that $n$ stars are disjoint if their sets of vertices are pairwise disjoint. So, they form a forest of stars.
An out-star is an out-tree whose underlying graph is a star.

**Complete graph.** A *complete graph* is a graph $G$ such that $xy \in E(G)$ for any two distinct $x, y \in V(G)$. A complete graph with $n$ vertices is denoted by $K_n$.

**Tournament.** A *tournament* is an oriented graph $T$ such that any two distinct vertices are adjacent.

**Transitive tournament.** A *transitive tournament* $T$ is a tournament with vertex set, say $\{v_1, v_1, ..., v_n\}$ with the arcs $(v_i, v_j)$ whenever $i < j$.

**Completion.** A tournament $T$ is said to be a *completion* of a digraph $D$ if $V(T) = V(D)$ and $E(D) \subseteq E(T)$.

**Triangle, cyclic and acyclic triangles.** A *triangle* is a $K_3$. Equivalently, it is a cycle having three vertices only. If an oriented triangle is a directed cycle, then it is called a *cyclic triangle*, otherwise it is called *acyclic triangle*. For example $(\{x, y, z\}, \{(x, y), (y, z), (z, x)\})$ is a cyclic triangle, while $(\{x, y, z\}, \{(x, y), (y, z), (x, z)\})$ is an acyclic triangle.

**Square.** A *square* is a cycle whose vertex set consists of four vertices only. It is denoted by $C_4$.

**King.** A *king* in a digraph $D$ is a vertex $x$ such that $\{x\} \cup N_D^+(x) \cup N_D^{++}(x) = V(D)$.

Clearly, when $T$ is a transitive tournament, namely with vertex set $\{v_1, v_1, ..., v_n\}$ and arcs $(v_i, v_j)$ whenever $i < j$, it has a king, which is the vertex $v_1$ (in fact, it is the unique king).
However, every tournament $T$ has at least one king: Every vertex $x \in V(T)$ with maximum out-degree is a king. Indeed, suppose that there is a vertex $y \in V(T) - \{x\} \cup N^+(x) \cup N^{++}(x)$. Then $\{x\} \cup N^+(x) \subseteq N^+(y)$, whence $d^+(y) \geq d^+(x) + 1$, a contradiction to the fact that $x$ has maximum out-degree.

Moreover, if $T$ has no source, then it has at least three kings. Indeed, the subdigraph $A$ induced by the set $N^{++}(x)$ (this set is not empty because $x$ is a king and not a source) is a tournament, and hence it has a king $y$. Since $y \to x$ then $y$ is also a king in $T$. By the same reasoning, there is a vertex $z \in N^{++}(y)$ which is a king in $T$. Sine $x \notin N^{++}(y)$, the three vertices $x, y$ and $z$ are pairwise distinct.
Furthermore, it is known that for every nonzero natural number $n \notin \{2, 4\}$, there is a tournament $T_n$ on $n$ vertices, such that every vertex is a king for this tournament. Indeed for $n = 1$ it is trivial and for $n = 3$, the cyclic triangle $C_3 = (\{x, y, z\}, \{(x, y), (y, z), (z, x)\})$ satisfies this. Add to $C_3$ two vertices $a$ and $b$ with $b \to a \to t \to b$, for every $t \in V(C_3)$. The obtained tournament on five vertices, denote it by $F$, satisfies the desired statement. By adding to $F$ a vertex $c$, with $a \to c \to b$ and $t \to c$ for every $t \in V(C_3)$ we obtain a tournament on six vertices in which every vertex is a king. Now let $n \geq 7$. There is a tournament $A$ on $n - 2$ vertices in which every vertex is a king. The tournament on $n$ vertices, obtained by adding two vertices $g$ and $h$ to $A$, with $h \to g \to t \to h$ for every $h \in V(A)$, has the desired property. This proves the aforementioned statement.

## 1.4 Classical functions on graphs and digraphs

In this section $G$ and $D$ denote a graph and a digraph respectively.

**Degrees of a graph.** The *minimum degree* of $G$, denoted by $\delta_G$, is the minimum of $\{d(x); x \in V(G)\}$.
The *maximum degree* of $G$, denoted by $\Delta_G$, is the maximum of $\{d(x); x \in V(G)\}$.

**Degrees of a digraph.** The *minimum degree* of $D$, denoted by $\delta_D$, is the minimum of $\{d(x); x \in V(D)\}$.
The *maximum degree* of $D$, denoted by $\Delta_D$, is the maximum of $\{d(x); x \in V(D)\}$.
The *minimum out-degree* of $D$, denoted by $\delta^+_D$, is the minimum of $\{d^+(x); x \in V(D)\}$.
The *maximum out-degree* of $D$, denoted by $\Delta^+_D$, is the maximum of $\{d^+(x); x \in V(D)\}$.
The *minimum in-degree* of $D$, denoted by $\delta^-_D$, is the minimum of

$\{d^-(x); x \in V(D)\}$.
The *maximum in-degree* of $D$, denoted by $\Delta^-_D$, is the maximum of $\{d^-(x); x \in V(D)\}$.

**Length of paths and cycles**. The *length* of a path or a cycle $S$ is the number of its edges. The *length* of a directed path or cycle $S$ is the number of its arcs. For example, if $v_1 ... v_n$ is a (directed) path, then its length is $n-1$, while, if it is a (directed) cycle, then its length is $n$. The length of such an $S$ is denoted by $l(S)$. An odd (resp. even) cycle is a cycle of odd (resp. even) length.

**Distances.** The *distance* between two vertices in a connected $G$ is the length of a shortest path between them. The maximum distance between two vertices is called the *diameter* of $G$. The *distance* from $x$ to $y$ in $D$ is the length of a shortest directed $x, y$-path.

**Circumference and girth.** The *circumference* of $D$ (resp. $G$) is the length of a longest directed cycle (resp. cycle) in $D$ (resp. $G$). On the other hand, the *girth* of $D$ (resp $G$), denoted by $g(D)$ (resp. $g(G)$), is the length of a shortest directed cycle (resp. cycle) in $D$ (resp. in $G$). If there is no (directed) cycle the circumference and the girth are by convention zero and infinity, respectively.

**Clique number.** Let $W$ be a subset of $V(H)$ where $H$ is a digraph (resp. graph). $W$ is a *clique* of $H$ if $H[W]$ is a tournament (resp. complete graph). The *clique number* of $H$ is the maximum $k$ such that $H$ contains a clique of size $k$.

**Stability.** A set of vertices is called *stable* if its elements are pairwise non-adjacent. The *stability* (or the independence number) of a (di)graph is the maximum size of a stable set.

**Chromatic number.** The *chromatic number* of a digraph (or a graph) $H$, denoted by $\chi(H)$, is the smallest integer $k$ such that $V(H)$ is a union of $k$ stable sets. In this case, we say that $H$ is $k$-chromatic. For example, $\chi(K_n) = n$, the chromatic number of trees is 2, the chromatic number of cycles of even length is 2, while that of odd length is 3. 2-chromatic (di)graphs are called *bipartite* graphs.

**Universal digraphs.** A digraph $H$ is called *$n$-universal* if it is contained in every $n$-chromatic digraph.

**Theorem 1.4.1.** *([16]) For any $g$ and $k$, there is a graph with girth at least $g$ and chromatic number at least $k$.*

This theorem shows that universal digraphs are oriented trees. Burr considered [18] the function $f$ such that every oriented tree on $k$ vertices is $f(k)$-universal. He proved that $f(k) \leq (k-1)^2$ and conjectured that $f(k) = 2k-2$ remarking that $f(k) \geq 2k-2$ since a regular tournament $T$ on $2k-3$ vertices (i.e. $T$ is a tournament on $2k-3$ vertices with $d^+(x) = d^-(x) = k-2$ for every vertex $x$) has no vertex with out-degree $k-1$ and thus $T$ is $(2k-3)$-chromatic digraph but it does not contain the out-star on $k$ vertices.

**Conjecture 1.** *(Burr's conjecture [18]) Every oriented tree with $k$ vertices is $(2k-2)$-universal ($k > 1$).*

# Chapter 2

# Median Order

## 2.1 Acyclic Digraphs and Feedback Sets

**Acyclic.** Let $(D, \omega)$ be an edge-weighted digraph. $D$ is said to be *acyclic* if it does not have any directed cycle. For example, oriented trees and forests are acyclic. A maximum acyclic subdigraph $D'$ of $D$, is an acyclic subdigraph of $D$ such that $\omega(E(D'))$ is maximum.

**Feedback sets.** Let $F \subseteq E(D)$ be a set of edges. We say that $F$ is a *feedback arc set* if $D \backslash F$ is acyclic. $F$ is a *minimum feedback arc set* if $\omega(F)$ is minimum among all the feedback arc sets. Similarly, a subset $A \subseteq V(D)$ is called a *feedback vertex set* if $D - V$ is an acyclic digraph. Such a set is a *minimum feedback vertex set* if its weight is minimum among all the feedback vertex sets of $D$.

Clearly, $D'$ is acyclic subdigraph of $D$ if and only if $E(D) \backslash E(D')$ is a feedback arc set. In addition $D'$ is maximum if and only if $E(D) \backslash E(D')$ is minimum. A result of Karp[17] asserts that finding a minimum feedback arc set is NP-Hard. It is even NP-Hard for tournaments [12].

Let $\mathcal{S}(D)$ be the set of all strongly connected components of $D$. Recall that, $D/\mathcal{S}(D)$ is the digraph obtained from $D$ by contracting each strongly connected component into a single vertex, successively. So we may suppose that, its vertex set is $\mathcal{S}(D)$ and for two distinct strong components $S$ and $S'$ , $(S, S')$ is an arc if there is $u \in S$ and $u' \in S'$ with $(u, u') \in E(D)$.

$D/\mathcal{S}(D)$ is a well defined digraph. It has no loops by definition. It has no directed cycles of length two. Indeed, it has no directed cycle. Otherwise, if $S_1, ..., S_r$ is a directed cycle then $\bigcup_{1\leq i\leq r} S_i$ is again a strongly connected component containing $S_i$, a contradiction to the fact that strong components are pairwise disjoint. So, $D/\mathcal{S}(D)$ is an acyclic digraph.
Note that every acyclic digraph has a sink and a source. In fact, the first vertex of every longest directed path is a source, while the last is a sink.

## 2.2 Definition of median order

**Median order.** Let $(D,\omega)$ be a weighted digraph. Let $L = x_1x_2...x_n$ be an enumeration of the vertices of $D$. Suppose that $(x_i, x_j) \in E(D)$. It is *forward* if $i < j$, otherwise it is *backward*. The set of forward arcs and backward arc are denoted by $F(L)$ and $B(L)$ respectively. Clearly, these two sets are feedback arc sets and the two digraphs $(V(D), F(L))$ and $(V(D), B(L))$ are acyclic. The weight of an enumeration $L$ is $\omega(L, D) := \omega(F(L))$ (when there is no confusion we write $\omega(L)$). An enumeration $L$ such that $\omega(L)$ is maximum is called a *median order*. Observe that, $L$ is a median order if and only if $B(L)$ is a minimum feedback arc set and $(V(D), F(L))$ is a maximum acyclic subdigraph of $D$. Thus finding a median order is NP-Hard even for tournaments.

For $i \leq j$, the interval $[x_i, x_j]$ or $[i, j]$ of $L$ is the set $\{x_i, x_{i+1}, ..., x_j\}$.

Median orders have many properties. The following property is called the ***feedback property*** [7, 3].

**Proposition 2.2.1.** *Let $L = x_1x_2...x_n$ be a median order of a weighted digraph $(D,\omega)$. Then for every $1 \leq i \leq j \leq n$, in $D[x_i, x_{i+1}, ..., x_j]$ we have: $\omega(N^+(x_i)) \geq \omega(N^-(x_i))$ and $\omega(N^-(x_j)) \geq \omega(N^+(x_j))$.*

*Proof.* Suppose that the first inequality is false. Then $\omega(F(L')) = \omega(F(L)) + \omega(N^-(x_i)) - \omega(N^+(x_i)) > \omega(F(L))$, where $L'$ is obtained from $L$ by inserting $x_i$ just after $x_j$. A contradiction. Similar argument is used to prove the second inequality . □

As a corollary, if $D$ is a weighted tournament, then by the feedback property $x_1x_2...x_n$ is a directed path of $D$. In the case of a (non weighted) tournament $D$, the feedback property is equivalent to say that $d^+(x_i) \geq \frac{j-i}{2}$ and $d^-(x_j) \geq \frac{j-i}{2}$, in $D[i,j]$, for all $i \leq j$, because every two distinct vertices are adjacent in tournaments.

**Local median order.** An enumeration $L = x_1x_2...x_n$ that satisfies the feedback property is called a *local median order*. So every median order is a local median order.

When all the weights are units, we get the definition of the (local) median order of a (non weighted) digraph $D$.

**Proposition 2.2.2.** *Let $L = x_1x_2...x_n$be a (local) median order of a digraph $D$. Let $D' = D \cup F \backslash B$ where $F \subseteq \{(x_i, x_j) \notin D; i > j\}$ and $B \subseteq B(L)$. Then $L$ is a (local) median order of $D'$.*

*Proof.* If $L$ is a local median order then adding forward arcs and deleting backward arcs strengthens the feedback property. Now suppose that $L$ is a median order and suppose $e = (x, y) \in B$ and let $D' = D \backslash e$. Since $e$ is backward arc, then it does not contribute to $\omega(L, D)$. Let $L'$ be a median order of $D'$. If, in $L'$, the index of $y$ is smaller than the one of $x$, then we have $\omega(L', D') = \omega(L', D) \leq \omega(L, D) = \omega(L, D') \leq \omega(L', D')$. Thus $\omega(L, D') = \omega(L', D')$, whence $L$ is a median order of $D'$.

Otherwise, in $L'$, the index of $x$ is smaller than the one of $y$. In this case $\omega(L', D) = \omega(L', D') + w(x, y) \leq w(L', D')$. So we have, $\omega(L', D') \leq w(L', D) \leq \omega(L, D) = \omega(L, D') \leq \omega(L', D')$. So, $\omega(L, D') = \omega(L', D')$, whence $L$ is a median order of $D'$.

Now, suppose that $e = (x, y) \in \{(x_i, x_j) \notin D; i > j\}$ and let $D' = D \cup e$. Let $L'$ be a median order of $D'$. Clearly, $\omega(L, D') = \omega(L, D) + w(x, y) \geq \omega(L, D)$. If $e \in B(L')$, then by the above argument $L'$ is a median order of $D$. We get $\omega(L', D') = w(L', D) \leq \omega(L, D) \leq \omega(L, D') \leq \omega(L', D')$. Thus $\omega(L, D') = \omega(L', D')$ and $L$ is a median order of $D'$.

Otherwise, $e \in F(L')$. We have $\omega(L, D) + w(x, y) = \omega(L, D') \leq \omega(L', D') = \omega(L', D) + w(x, y) \leq \omega(L, D) + w(x, y)$. Thus $\omega(L, D') =$

$\omega(L', D')$ and $L$ is a median order of $D'$.

To conclude we use induction on the number of the deleted and added arcs.

□

**Back and feed vertices.** Let $L = x_1x_2...x_n$be a (local) median order of a (weighted) digraph $D$. The first vertex $x_1$ is called a *back vertex*, while the last vertex $x_n$ is called a *feed vertex* of $D$.

**Proposition 2.2.3.** *([7]) The back vertex $x_1$ is a king when $D$ is a tournament.*

*Proof.* Let $i > 1$. We have by the feedback property, $|N^+_{[1,i]}(x_1)| \geq \frac{i-1}{2}$ and $|N^-_{[1,i]}(x_i)| \geq \frac{i-1}{2}$. If $x_i \notin N^+_{[1,i]}(x_1) \cup N^{++}_{[1,i]}(x_1)$, then the set $\{x_1, x_i\}$, $N^+_{[1,i]}(x_1)$ and $N^{++}_{[1,i]}(x_1)$ are pairwise disjoint. Thus, $i = |[1,i]| \geq \{x_1, x_i\} \cup N^+_{[1,i]}(x_1) \cup |N^{++}_{[1,i]}(x_1)| \geq 2 + \frac{i-1}{2} + \frac{i-1}{2} = i+1$, a contradiction. Thus, $x_i \notin N^+_{[1,i]}(x_1) \cup N^{++}_{[1,i]}(x_1)$ and $x_1$ is a king.

□

**$G_L$.** Among the vertices not in $N^+(x_n)$ two types are distinguished: A vertex $x_j$ is *good* if there is $i \leq j$ such that $x_n \rightarrow x_i \rightarrow x_j$, otherwise $x_j$ is a *bad* vertex. The set of good vertices of $L$ is denoted by $G^D_L$ ( or $G_L$ if there is no confusion ). Clearly, $G_L \subseteq N^{++}(x_n)$ [7].

## 2.3 Median orders and the chromatic number

Suppose that $B$ is a minimum feedback arc set of a weighted digraph $(D, \omega)$ and consider the acyclic weighted digraph $D' = D - B$. Let $S_0$ be the (possibly empty) set of isolated vertices of $D'$. We define inductively, for $i > 0$, the set $S_i$ as follows. $S_i$ is the set of sources of $D' - \bigcup_{j<i} S_j$. Let $n$ be the greatest integer such that $S_n \neq \phi$. Note that for all $0 < i \leq n$, $S_i \neq \phi$. Enumerate, for every $0 \leq i \leq n$, the vertices of $S_i$ in arbitrary way as
$v_{j_{i-1}+1}\ v_{j_{i-1}+2}\ ......v_{j_i}$ with $j_{-1} + 1 := 1$ and $j_i = \Sigma_{k \leq i}|S_k|$. Denote by $L$ the overall enumeration. By construction, the sets $S_i$ are stable sets

in $D'$, $F(L) = E(D')$ and $B(L) = B$. Thus $L$ is a median order of $D$. However,

**Proposition 2.3.1.** *Every set $S_i$ is stable in $D$. Therefore $\chi(D) \leq n$.*

*Proof.* Since the sets $S_i$ are stable sets in $D'$ and $F(L) = E(D')$ we have the following. If $(x, y) \in E(D')$, then there is $i < j$ such that $x \in S_i$ and $y \in S_j$. Indeed, fix $i \geq 0$ and assume that there is $(x, y) \in E(D)$ with $x, y \in S_i$. Then $(x, y) \in B$. Hence $D' + (x, y)$ has a cycle $y = u_1u_2...u_{k-1}u_k = x$. Every arc of this cycle, except $(x, y)$, is in $D'$ (hence forward), whence there is $j > i$ such that $x \in S_j$. A contradiction. □

The proof of the above proposition shows the following:

**Proposition 2.3.2.** *If $i < j$ and $(x, y) \in E(D)$ with $y \in S_i$ and $x \in S_j$, then there is a directed $y, x$-path formed of forward arcs of $L$. Therefore, $(x, y)$ is an arc of a directed cycle of $D$ of length at most $j - i + 1$.*

**Proposition 2.3.3.** *$D$ has a directed path of length at least $n - 1$.*

*Proof.* We construct a directed path $v_1v_2...v_n$ with $v_i \in S_i$. As a basis step, we choose a vertex $v_n$ from $S_n$. Suppose that such a directed path $v_{i+1}v_{i+2}...v_n$ is constructed. Since $v_{i+1}$ is not a source in $D'[\bigcup_{j\geq i} S_j]$ and since the arcs of $D'$ are in $F(L)$, there is $v_i \in S_i$ such that $v_i \to v_{i+1}$. □

So we obtain Galli-Roy theorem.

**Corollary 2.3.1.** *([19],[2]) $D$ has a directed path of length at least $\chi(D) - 1$.*

## 2.4 Median orders and Sumner's conjecture

Median order is a powerful inductive tool: if $L = x_1x_2...x_n$is a median order of a weighted digraph $D$, then for $i < j$, also $I = x_ix_{i+1}...x_j$ is a median order of the digraph $D' = D[x_i, x_j]$. To see this, observe that if $J$ is a median order of $D'$ then $\omega(L') = \omega(L) + \omega(J) - \omega(I)$, where $L' = x_1...x_{i-1}Jx_{j+1}...x_n$. Now, $I$ is not a median order if and only if $\omega(J) > \omega(I)$ if and only if $\omega(L') > \omega(L)$.

As an example for the importance of (local) median orders as an inductive tool, we present a short argument of [7] to show how they are used in order to reach an upper bound for Burr's conjecture restricted to tournaments. Burr's conjecture restricted to the class of tournaments is the following one, known as Sumner's Universal Conjecture.

**Conjecture 2.** *([13]) Every tournament on $2k-2$ vertices ($k>1$) contains a copy of every oriented tree with $k$ vertices.*

Let $A$ and $D$ be digraphs, and let $L = x_1x_2...x_n$be a local median order of $D$. An embedding of $A$ in $D$ is an injective function $f : V(A) \longrightarrow V(D)$ such that $(f(v_i), f(v_j)) \in E(D)$ whenever $(v_i, v_j) \in E(A)$. An *$L$-embedding* of $A$ in $D$ is an embedding $f$ of $A$ in $D$ such that, for every interval of $L$ of the form $[x_{i+1}, x_n]$,

$$|f(A) \cap [x_{i+1}, x_n]| < \frac{1}{2}|[x_{i+1}, x_n]| + 1.$$

In this case, we say that $A$ is $L$-embeddable.

**Proposition 2.4.1.** *([1]) Let $T$ be a tournament on at least three vertices and let $L = x_1x_2...x_n$be a local median order of $T$. Set $T' = T - \{v_{n-1}, v_n\}$ and $L' = x_1x_2...x_{n-2}$. Let $A$ be a digraph with an in-leaf $y$ and suppose that $A' = A - y$ has an $L'$-embedding $f'$ in $T'$. Then $A$ has an $L$-embedding $f$ in $T$ which extends $f'$.*

*Proof.* Let $x$ denote the in-neighbor of $y$. Let $f'$ be an $L'$- embedding of $A'$ in $T'$. Suppose that $f'(x) = x_i$. We have $|f'(A') \cap [x_{i+1}, x_{n-2}]| < \frac{1}{2}|[x_{i+1}, x_{n-2}]|+1$. Since $L$ is a median order of $T$ and $T$ is a tournament, $|N^+_T(x_i) \cap [x_{i+1}, x_n]| \geq \frac{1}{2}|[x_{i+1}, x_n]| = \frac{1}{2}|[x_{i+1}, x_{n-2}]| + 1$. Therefore $|N^+_T(x_i) \cap [x_{i+1}, x_n]| > |f'(A') \cap [x_{i+1}, x_{n-2}]| = |f'(A') \cap [x_{i+1}, x_n]|$. In other words, $x_i$ has an out-neighbor $x_j \in I \backslash f'(A')$. We define $f : V(A) \longrightarrow V(T)$ by $f(v) := f'(v)$ for $v \in V(A')$ and $f(y) := x_j$. □

An immediate consequence of the above proposition is that out-trees (and in-trees) satisfies Sumner's conjecture.

**Corollary 2.4.1.** *([7]) Every tournament on $2k-2$ vertices contains a copy of every out-tree on $k$ vertices ($k>1$).*

In fact, it is proved that, for every $k > 0$, for every tournament $T$ on $2k-2$ vertices, for every local median order $L = x_1...x_{2k-2}$ of $T$, for every out-tree $A$ on $k$ vertices we have that $A$ is $L$-embeddable [7].

Using a similar argument Havet and Thomassé proved the following:

**Theorem 2.4.1.** *([7]) Every tournament on $4k - 6$ vertices contains a copy of every oriented tree on $k$ vertices ($k > 1$).*

The prove of this result differs from the previous one by taking $L' = x_3x_4...x_{2k-8}$ in the inductive step instead of $L' = x_1x_2...x_{2k-8}$ because the existence of an out-leaf of $A$ is not guaranteed. However, a leaf must exist in $A$ (in or out), which makes the induction successful.

El Sahili also used the notion of median order to prove the best known bound for general $k$.

**Theorem 2.4.2.** *([1]) Every tournament on $3k - 3$ vertices contains a copy of every oriented tree on $k$ vertices ($k > 1$).*

Recently, it is proved that for sufficiently large $k$ we have $f(k) = 2k - 2$.

**Theorem 2.4.3.** *([5]) There exists $k_0$ such that for every $k \geq k_0$, every tournament on $2k - 2$ vertices contains a copy of every oriented tree on $k$ vertices.*

# Chapter 3

# The second neighborhood conjecture

A vertex $v$ is said to have the *second neighborhood property* (SNP) if $d^+(v) \leq d^{++}(v)$.

## 3.1 The conjecture stated

In 1990, P. Seymour conjectured [14] the following statement:

**Conjecture 3.** ***(The Second Neighborhood Conjecture (SNC))*** *Every digraph has a vertex with the SNP.*

The SNC was verified for digraphs with minimum out-degrees at most 6 [20].

A vertex $v$ is said to have the *weighted second neighborhood property* (weighted SNP) if $\omega(N^+(v)) \leq \omega(N^{++}(v))$. It is known that the SNC is equivalent to its weighted version: *Every weighted digraph has a vertex with the weighted SNP*.
One of the two implications is obvious. For the non obvious one, let $(D, \omega)$ be a weighted digraph. First we suppose that the weights are non negative integers. We construct a new (non weighted) digraph $D'$. We replace every vertex $v$ by a stable set $S_v$ with $\omega(v)$ vertices (each new vertex has weight 1). Let $u'$ and $v'$ be two vertices of $D'$. There are two vertices $u$ and $v$ such that $u' \in S_u$ and $v' \in S_v$. Now $(u', v') \in E(D')$ if and only if $(u, v) \in E(D)$. Now $D'$ has a vertex $v'$ with SNP. This is equivalent to say that $v$ has the weighted SNP in $D$,

where $v$ is the vertex of $D$ such that $v' \in S_v$. Multiplying by a common denominator this can be extended to digraphs weighted by rationales. Since rationales are dense in the set of real numbers, we can extend this to digraphs weighted by real numbers.

## 3.2 Dean's Conjecture

Seymour's conjecture restricted to tournaments is known as Dean's conjecture [14]. In 1996, Fisher [4] proved Dean's conjecture, thus asserting the SNC for tournaments.

**Theorem 3.2.1.** *([4]) Every tournament has a vertex with the SNP.*

Fisher's proof uses Farkas' Lemma (see for example, [6] page 279). The symbols **0** and **1** denote the vectors whose all components are 0 and 1 respectively and for vectors $x$ and $y$ we write $x \geq y$ if $x_i \geq y_i$, the $i - th$ entries of $x$ and $y$ respectively, for every $i$. **I** denotes the identity matrix.

**Lemma 3.2.1.** ***(Farkas' Lemma)*** *Given a Matrix* ***M*** *and a vector* ***b****, exactly one of these systems has a solution:*

*(i)* $Mx{=}b$ *with* $x \geq \mathbf{0}$*;*

*(ii)* $M^T y \geq \mathbf{0}$ *with* $b^T.y < \mathbf{0}$*.*

Now, let $D$ be a digraph with vertex set $\{v_1, v_2, ..., v_n\}$. A (probability) *density* $p$ on $D$ is a weight function such that $p(V(D)) = 1$. For such $p$ let $\mathbf{p}$ be the vector $(p(v_i); i = 1, ..., n)^T$. A density $p$ is called *winning* if for every vertex $v_i$ we have $p(N^-(v_i)) \geq p(N^+(v_i))$. On the other hand, it is a losing density if for every vertex $v_i$ we have $p(N^-(v_i)) \leq p(N^+(v_i))$.

The matrix $K(D)$ (or simply $\mathbf{K}$) is the $n \times n$ matrix $(k_{ij})_{ij}$, with $k_{ij} = 1$ if $v_i \rightarrow v_j$, $k_{ij} = -1$ if $v_j \rightarrow v_i$ and $k_{ij} = 0$ otherwise.

Note that $K$ is a skew-symmetric matrix and the $i$-th component $(K.p)_i$ equals $p(N^+(v_i)) - p(N^-(v_i))$. So $p$ is a winning density if $K.p \geq 0$ and a losing density if $K.p \leq 0$.

**Lemma 3.2.2.** *([4]) Any digraph $D$ has a winning (resp. losing) density. Further for a winning (resp. losing) density $p$, if $p(v_i) > 0$, then $p(N^+(v_i)) = p(N^-(v_i))$.*

*Proof.* Suppose $D$ has no winning density. Then this system has no solution:
$\begin{bmatrix} K(D) & I \\ 1^T & 0^T \end{bmatrix} \begin{pmatrix} w \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ with $\begin{pmatrix} w \\ z \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Since $K^T = -K$, Farkas' Lemma shows that this system has a solution:
$\begin{bmatrix} -K & I \\ 1^T & 0^T \end{bmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ with $\begin{pmatrix} 0^T & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} < 0.$

Thus $\mathbf{u} \geq \mathbf{0}$ and $K\mathbf{u} \leq v\mathbf{1}$ with $v < 0$. So $K\mathbf{u} < \mathbf{0}$ and hence $\mathbf{p} = (1^T u)^{-1}\mathbf{u}$ is the associated vector of a winning density, a contradiction. Therefore $D$ has a winning density.
Now let $p$ be a winning density of $D$ with associated vector denoted also by $p$. Then $K.p \leq \mathbf{0}$ and $p \geq \mathbf{0}$ and hence $p_i(K.p)_i \leq 0$ for every $i$. Since $K$ is skew-symmetric, $p^T.K.p = 0$. Thus $p_i(K.p)_i \leq 0$ for every $i$. Therefore, if $p_i := p(v_i) > 0$, then $0 = (K.p)_i = p(N^+(v_i)) - p(N^-(v_i))$, whence $p(N^+(v_i)) = p(N^-(v_i))$.

Since a losing density on $D$ is a winning density on the digraph formed by reversing its arcs, the statement holds.

□

**Lemma 3.2.3.** *([4]) Let $p$ be a losing density on a tournament $T$. Then $p(N^-(v_i)) \leq p(N^{--}(v_i))$ for every vertex $v_i$ of $T$.*

*Proof.* Let $v_i$ be a vertex of $T$. Since $p$ is a losing density, $p(N^-(v_i)) \leq \frac{1}{2}$. If $p(N^{--}(v_i)) \geq \frac{1}{2}$ we are done. Otherwise let $Q$ be the tournament induced by the set $V(T) - N^-(v_i) \cup N^{--}(v_i)$. Note that $p(V(Q)) > 0$. Within $Q$ we have

$$\Sigma_{v_j \in V(Q)} p(v_j) p(N_Q^-(v_j)) = \Sigma_{v_k \in V(Q)} p(v_k) p(N_Q^+(v_k)).$$

Since $p(V(Q)) > 0$, there is $v_h \in V(Q)$ with $p(v_h) > 0$ and $p(N_Q^-(v_h)) \geq p(N_Q^+(v_h))$.

Since $v_h \notin N^-(v_i) \cup N^{--}(v_i)$ and $T$ is a tournament, we have $N^-(v_i) \cup N^-(v_h)$. So

$$p(N^-(v_h)) = p(N_Q^-(v_h)) + p(N^-(v_h) \cap (N^-(v_i) \cup N^{--}(v_i)))$$
$$\geq p(N_Q^-(v_h)) + p(N^-(v_i)).$$

Similarly $N^+(v_h) \cup N^+(v_i)$ and hence $N^+(v_h) \cap (N^-(v_i) \cup N^{--}(v_i)) = N^+(v_h) \cap N^{--}(v_i) \subseteq N^{--}(v_i)$. So

$$p(N^+(v_h)) = p(N_Q^+(v_h)) + p(N^+(v_h) \cap (N^-(v_i) \cup N^{--}(v_i)))$$
$$\leq p(N_Q^+(v_h)) + p(N^{--}(v_i)).$$

Since $p(v_h) > 0$, by Lemma 3.2.2 $p(N^+(v_h)) = p(N^-(v_h))$. Hence

$$p(N_Q^-(v_h)) + p(N^-(v_i)) \leq p(N^-(v_h)) = p(N^+(v_h)) \leq p(N_Q^+(v_h)) + p(N^{--}(v_i)).$$

But $p(N_Q^-(v_h)) \geq p(N_Q^+(v_h))$ then $p(N^-(v_i)) \leq p(N^{--}(v_i))$.

□

The above lemma applies only for losing densities on tournaments. For example, let $D$ be the directed cycle $x_1x_2x_3x_4$ with the losing density $p$ defined by $p(v_1) = p(v_3) = \frac{1}{2}$ and $p(v_2) = p(v_4) = 0$. However, we have $p(N^-(v_4)) = p(v_3) = \frac{1}{2}$ while $p(N^{--}(v_4)) = p(v_2) = 0$.

Let $p$ be a probability density on $D$ and let $f$ be any real valued function defined on $V(D)$. The expected value of $f$ on a random pick from the density $p$ is

$$E_p(f) = \Sigma_{v_i \in V(D)} p(v_i) f(v_i).$$

**Proof of Theorem 3.2.1 ([4]):** When $f = d^+ : V(D) \longrightarrow \mathcal{R}$, is the function that associates to every vertex its first out-degree, we have

$$E_p(d^+) = \Sigma_{v_i \in V(D)} p(v_i) d^+(v_i) = \Sigma_{v_i \in V(D)} p(N^-(v_i)).$$

When $f = d^{++}$ the function that associates to every vertex it second out-degree, we have

$$E_p(d^{++}) = \Sigma_{v_i \in V(D)} p(v_i) d^+(v_i) = \Sigma_{v_i \in V(D)} p(N^-(v_i)).$$

By Lemma 3.2.3, we have $p(N^-(v_i)) \leq p(N^{--}(v_i))$ for every vertex $v_i$ of $T$, then $E_p(d^+) \leq E_p(d^{++})$. Hence there is a vertex $v_i$ with $d^+(v_i) \leq d^{++}(v_i)$.

## 3.3 Dean's Conjecture - A combinatorial proof

Another proof to Dean's conjecture was achieved in 2000 by F. Havet and S. Thomassé. It is a constructive and a combinatorial proof that uses median orders. We will see in the next chapter (Lemma 4.2.1) another proof similar to a proof given by F. Havet and S. Thomassé. They have proved that every feed vertex (the last vertex of every local median order) of a tournament has the SNP. In fact, they have proved that $d^+(x_n) \leq |G_L|$. Moreover, they have exhibited two vertices with the SNP if the tournament does not have a sink [7].

**Theorem 3.3.1.** *([3]) Let $L = x_1x_2...x_n$be a local median order of a weighted tournament $T$. Then $x_n$ has the SNP.*

The following proposition, proved by Fidler and Yuster, is an extension of the above theorem to weighted tournaments.

**Proposition 3.3.1.** *([3]) Let $L = x_1x_2...x_n$be a local median order of a tournament $(T, \omega)$. Then $x_n$ has the weighted SNP.*

*Proof.* We shall prove, by induction on $n$, that $\omega(N^+(x_n)) \leq \omega(G_L)$. If $L$ has no bad vertices then $N^-(x_n) = G_L$, whence $\omega(N^+(x_n)) \leq \omega(N^-(x_n)) = \omega(G_L)$, where the first inequality is by the feedback property. Otherwise, let $x_i$ be the bad vertex of $L$ with minimal $i$. Since $i$ is minimal, for all $s \leq i-1$, $x_s \in N^+_{[1,i]}(x_n)$ or $x_s \in G_L \cap [x_1, x_i]$. Let $x_s \in N^+_{[1,i]}(x_n)$, then $x_s \in N^+_{[1,i]}(x_i)$, hence $N^+_{[1,i]}(x_n) \subseteq N^+_{[1,i]}(x_i)$, equivalently, $N^-_{[1,i]}(x_i) \subseteq G_L \cap [x_1, x_i]$. So $\omega(N^+_{[1,i]}(x_n)) \leq \omega(N^+_{[1,i]}(x_i)) \leq \omega(N^-_{[1,i]}(x_i)) \leq \omega(G_L \cap [x_1, x_i])$, where the second inequality is by the feedback property. By induction, $\omega(N^+_{[i+1,n]}(x_n)) \leq \omega(G_{L'})$ where $L' = x_{i+1} \cdots x_n$ which is a median order of $T' = T[x_{i+1}, x_n]$. However $G_{L'} \subseteq$

$G_L \cap [x_{i+1}, x_n]$, then $\omega(N^+(x_n)) = \omega(N^+_{[1,i]}(x_n)) + \omega(N^+_{[i+1,n]}(x_n)) \leq \omega(G_L \cap [x_1, x_i]) + \omega(G_L \cap [x_{i+1}, x_n]) = \omega(G_L)$ □

A natural question is to seek another vertex with the SNP. Obviously, this is not always possible: consider for instance a transitive tournament. The sole vertex with the SNP is its sink vertex. In [7], it is proved that a tournament always has two vertices with the SNP, provided that every vertex has out-degree at least one. The notion of local median orders turns out to be too weak for that purpose, so median orders are used.
The notion of *sedimentation* of a median order $L = x_1...x_n$ of $T$ is introduced in [7], denoted by $Sed(L)$. We recall that, by the proof of Theorem 3.3.1,
$|N^+(x_n)| \leq |G_L|$.

**Definition 3.3.1.** *If $|N^+(x_n)| < |G_L|$, then Sed(L)=L.*
*If $|N^+(x_n)| = |G_L|$, we denote by $b_1, ..., b_k$ the bad vertices of (T,L) and by $v_1, ..., v_{n-1-k}$ the vertices of $N^+(x_n) \cup G_L$, both enumerated in increasing order with respect to their index in L. In this case $Sed(L)$ is the order $b_1...b_k x_n v_1...v_{n-1-k}$ of $T$ ([7]).*

**Lemma 3.3.1.** *([7]) The order Sed(L) is a median order of T.*

*Proof.* If Sed$(L) = L$, there is nothing to prove. Otherwise, we assume that $|N^+(x_n)| = |G_L|$. The proof is by induction on $k$ the number of bad vertices. If $k = 0$, all the vertices are good or in $N^+(x_n)$, in particular $N^-(x_n) = G_L$. Thus, $|N^+(x_n)| = |N^-(x_n)|$ and the order $Sed(L) = x_n x_1, ..., x_{n-1}$ is a median order of $T$. (Note that this is not true for local median orders.) Now, assume that $k$ is a positive integer. Let $i$ be the index of the vertex $b_1$ in $L$ (that is $x_i = b_1$). By the previous proof $d^+_{[1,i]}(x_n) \leq |G_L \cap [x_1, x_i]|$ and $d^+_{[i,n]}(x_n) \leq |G_L \cap [x_i, x_n]|$. But $|d^+(x_n)| = |G_L|$, then equality holds in both previous inequalities. Let $t \leq i-1$. If $v_t \in N^+(x_n)$, then $v_t \in N^+(x_i)$, since $x_i$ is bad vertex. If $x_t \in N^-(x_i)$, then $x_t \in N^+(x_n) \subseteq G_l \cap [x_1, x_i]$, since $x_i$ is the bad vertex with minimal index $i$. Therefore, $d^+_{[1,i]}(x_n) \leq d^+_{[1,i]}(x_i) \leq d^-_{[1,i]}(x_i) \leq |G_L \cap [x_1, x_i]|$, then $d^+_{[1,i]}(x_i) = d^-_{[1,i]}(x_i)$, whence $L' = x_i x_1 \cdots x_{i-1} x_{i+1} \cdots x_n$ is a median order of $T$. The bad vertices of $L$ and $L'$ are the same. To conclude apply induction to $L" = x_1 \cdots x_{i-1} x_{i+1} \cdots x_n$ which is a median order of $T - v_i$, hence $b_2...b_k x_n v_1...v_{n-1-k}$ is median order of $T - v_i$. Hence, $Sed(L)$ is a median order of $T$.

□

Define now inductively $Sed^0(L) = L$ and $Sed^{q+1}(L) = Sed(Sed^q(L))$. If the process reaches a rank $q$ such that $Sed^q(L) = y_1...y_n$ and $|N^+(y_n)| < |G_{Sed^q(L)}|$, call the order $L$ stable. Otherwise call $L$ periodic [7].

**Theorem 3.3.2.** *[7] A tournament with no sink vertex has at least two vertices with the SNP.*

*Proof.* Let $L = x_1...x_n$ be a median order of $T$. By Theorem 3.3.1, $x_n$ has the SNP, so we need to find another vertex with this property. Consider the restriction of $(T, L)$ to the interval $[x_1, ..., x_{n-1}]$, and denote it by $(T', L')$. Suppose first that $L'$ is stable, and consider an integer $q$ for which $Sed^q(L')= y_1...y_{n-1}$ and $|N^+_{T'}(y_{n-1})| < |G_{Sed^q(L')}|$. Note that $y_1...y_{n-1}x_n$ is a median order of $T$, and consequently $y_{n-1} \to x_n$. Thus,

$$|N^+(y_{n-1})| = |N^+_{T'}(y_{n-1})| + 1 \leq |G_{Sed^q(L')}| \leq N^{++}(y_{n-1})|.$$

So $y_{n-1}$ has the SNP in $T$. Now assume that $L'$ is periodic. Since $T$ has no dominated vertex, $x_n$ has an outneighbor $x_j$. Note that for every integer $q$, the feed vertex of $Sed^q(L')$ is an in-neighbor of $x_n$. So $x_j$ is not the feed vertex of any $Sed^q(L')$. Observe also that, since $L'$ is periodic, $x_j$ must be a bad vertex of some $Sed^q(L')$, otherwise the index of $x_j$ would always increase during the sedimentation process. Now, fix this value of $q$. Let $Sed^q(L')$=$y_1...y_{n-1}$. We claim that $y_{n-1}$ has the SNP in $T$: on the one hand we have

$$|N^+(y_{n-1})| = |N^+_{T'}(y_{n-1})| + 1 = |G_{Sed^q(L')}| + 1$$

and on the other hand we have $y_{n-1} \to x_n \to x_j$, so the second neighborhood of $y_{n-1}$ in $T$ contains $G_{Sed^q(L')} \cup \{x_j\}$, hence, it contains at least $|G_{Sed^q(L')}|$ +1 elements.

□

We will prove in the next chapter (Lemma 4.2.1 ) a result, which yields Theorem 3.3.1 and Proposition 3.3.1.

## 3.4 The approximation approach

Another approach to the SNC is to find the maximum value of $\gamma$ such that every digraph has a vertex $v$ with $d^{++}(v) \geq \gamma d^{+}(v)$. Since every vertex $v$ of a directed cycles satisfies $d^{++}(v) = d^{+}(v)$, we have $\gamma \leq 1$. The conjecture is $\gamma = 1$.

Chen, Shen and Yuster [8] proved the following:

**Theorem 3.4.1.** *Every digraph has a vertex $v$ such that $d^{++}(v) \geq \gamma d^{+}(v)$, where $\gamma = 0.657298...$ is the unique real root of the equation $2x^3 + x^2 - 1 = 0$.*

*Proof.* For 2 disjoint sets $A$ and $B$, $e(A,B)$ is the number of arcs with tail in $A$ and head in $B$. Note that $e(A,B) + e(B,A) \leq ab$.
The proof is by induction on $n = |V(D)|$. The theorem is trivial for digraphs with one or two vertices. Suppose that $D$ is a digraph on $n$ vertices. Assume, to the contrary, that $D$ does not contain any vertex $v$ such that $d^{++}(v) \geq \gamma d^{+}(v)$.
Let $u$ be a vertex with minimum out-degree, i.e. $d^{+}(u) = \delta^{+}(D)$. Let $D'$ be the subdigraph induced by $N^{+}(u)$. Set $A = N^{+}(u)$, $B = N^{++}(u)$, $a = |A|$ and $b = |B|$. It will be shown that $e(A,B) + e(B,A) > ab$, a contradiction to the fact that $D$ does not contain any digon.
By the assumption, the following inequality holds,

$$b = d^{++}(u) < \gamma d^{+}(u) = \gamma a. \tag{3.1}$$

Since $a = d^{+}(u) = \delta^{+}(D)$, then $d^{+}_{A}(x) + d^{+}_{B}(x) = d^{+}(x) \geq d^{+}(u) = a$ for every $x \in A$. Since $D$ does not contain any digon, we have that $\Sigma_{x\in A} d^{+}_{D'}(x) \leq a(a-1)/2$. Thus,

$$e(A,B) = \Sigma_{x\in A}|N^{+}(x) \cap B| \geq_{x\in A} (a - d^{+}_{A}(x)) \geq a^2 - a(a-1)/2 > a^2/2. \tag{3.2}$$

Since $|V(D')| = a < n$, then by induction hypothesis, there is a vertex $x \in N^{+}(u)$ such that $d^{++}_{D'}(x) \geq \gamma d^{+}_{D'}(x)$. Let $X = N^{+}(x) \cap A$, $Y = N^(x) - A = N^{+}(x) \cap B$ and $d = |Y|$. Since $|A - X| \geq N^{++}_{D'}(x)$, then $(1-\gamma)|X| \leq a$. Thus,

$$|X| \leq \frac{1}{1+\gamma} a \leq \frac{2a}{3}$$

where the last inequality follows since $\gamma \geq 1/2$. Since $d^+(x) \geq \delta^+(D) = a$,

$$d = |Y| = |N^+(x)| - |X| \geq a - \frac{2a}{3} = \frac{a}{3}. \tag{3.3}$$

For every $y \in Y$, since $d^{++}(x) < \gamma d^+(x)$ and $d^{++}_{D'}(x) \geq \gamma d^+_{D'}(x)$, we have

$$d^+_{D[V-A-Y]}(y) \leq d^{++}(x) - d^{++}_{D'} < \gamma d^+(x) - \gamma d^+_{D'}(x) = \gamma|Y| = \gamma d$$

Using the inequalities

$$d^+(y) \geq a \tag{3.4}$$

and

$$\Sigma_{y\in Y} d^+_{D[Y](y)} \leq d(d-1)/2, \tag{3.5}$$

We obtain the following inequalities.

$$e(Y, A) = \Sigma_{y\in Y}|N^+(y) \cap A| \tag{3.6}$$

$$\geq (a - d^+_{D[V-A-Y]}(y) - d^+_{D[Y]}(y)) \tag{3.7}$$

$$\geq (a - \gamma d)d - \Sigma_{y\in Y} d^+_{D[Y]}(y) \tag{3.8}$$

$$\geq (a - \gamma d)d - d(d-1)/2 \tag{3.9}$$

$$> (a - \gamma d - d/2)d. \tag{3.10}$$

Combining the inequalities 3.1, 3.2 and 3.10, we obtain that

$$\gamma a^2 > ab \geq e(A, B)+e(B, A) \geq e(A, B)+e(Y, A) \geq a^2/2+(a-\gamma d-d/2)d. \tag{3.11}$$

Let $f(z) = a^2/2 + (a - \gamma z - z/2) = -(\gamma + 1/2)z^2 + az + a^2/2$. Since $f(z)$ is a quadratic fucntion with negative leading coefficient, the following inequality holds.

$$f(z) > min\{f(a/3), f(\gamma a)\}$$

for all $z \in (a/3, \gamma a)$.
Thus, $\gamma a^2 > min\{f(a/3), f(\gamma a)\}$.

A simple calculation gives that

$$f(a/3) = a^2(7 - \gamma)/9.$$

Solving $\gamma a^2 > a^2(7 - \gamma)/9$, we obtain that $\gamma > 0.7$, a contradiction. Also, a simple calculation gives that

$$f(\gamma a) = \frac{1}{2}a^2(-2\gamma^3 - \gamma^2 + 2\gamma + 1).$$

Solving the inequality

$$\gamma a^2 > \frac{1}{2}a^2(-2\gamma^3 - \gamma^2 + 2\gamma + 1)$$

we obtain that $2\gamma^3 + \gamma^2 - 1 > 0$, which contradicts that $\gamma$ is the unique real root of the equation $2x^3 + x^2 - 1 = 0$. □

For $K_{k+1}$-free digraphs, since $D[A]$ and $D[B]$ are $K_k$-free digraphs, greater values (in terms of $k$) of $e(A, B)$ and $e(B, A)$ are found, leading to better results of $\gamma = \gamma(k)$.

**Theorem 3.4.2.** *([3]) Let $D$ be a $K_{k+1}$-free digraph. Then $D$ has a vertex $v$ with $d^+(v) \geq \gamma d^+(v)$, where $\gamma$ is greater than or equal the real root of $f(x) = \frac{2k-2}{k}x^3 + \frac{k-2}{k}x^2 - 1$.*

For example for $k = 3$, $\gamma \geq 0.8324$.

In Chapter 4, we will see another approach, which is based on the completion of a digraph in some way, and proving that a feed vertex of such completion has the SNP in the original digraph.

## 3.5 Intervals and minimal counterexamples

Let $D$ be a digraph. A set of vertices $I \subseteq V(D)$ is said to be an *interval* of $D$ if for every $x, y \in I$ we have $N^+(x)\backslash I = N^+(y)\backslash I$ and $N^-(x)\backslash I = N^+(y)\backslash I$. The empty set, $V(D)$ and singletons are intervals for any digraph $D$. Such interval are called *trivial intervals.* An interval which is not trivial is called a *non-trivial interval.* $D$ is said to be *indecomposable* if all its intervals are trivial. Otherwise, $D$ is said to be a *decomposable* digraph. $D$ is said to be *critically indecomposable* if it is indecomposable but for every $u \in V(D)$ the dighraph $D - u$ is decomposable.

In [10] (Corollary 5.8), critically indecomposable digraphs are characterized. For $r \geq 2$ the following five digraphs are defined. The digraphs $\mathcal{P}_r$ and $\mathcal{P}'_r$ are defined as follows: There vertex set is $\{a_1, ..., a_r, b_1 ..., b_r\}$, the edge set of $\mathcal{P}_r$ is composed of the arcs $(a_i, b_j)$ where $i \geq j$, the edge set of $\mathcal{P}'_r$ consists of the arcs $(a_i, a_j), (b_i, b_j)$ and $(a_i, b_j)$ where $i < j$.
The tournament $\mathcal{T}_r^1$ has vertex set $\{c_0, c_1, ..., c_{2r}\}$ and arc set $(c_i, c_{i+k})$ for $k = 1, ..., r$, where the sum $i + k$ is mod $2r + 1$.
The tournament $\mathcal{T}_r^2$ and the digraph $\mathcal{D}_r$ have $\{a_0, a_1, ..., a_r, b_1 ..., b_r\}$ as a vertex set and the arcs of the first are $(a_i, a_j), (b_j, a_i)$,$(a_j, b_i), (b_j, b_i)$ if $i < j$ and the arcs $(b_j, a_j)$. While the arcs of the second are $(a_i, b_j), (b_i, b_j), (b_i, a_j)$ if $i < j$ and the arcs $(b_j, a_j)$.
The tournament $\mathcal{T}_r^3$ has $\{b, a_1, a_2, ..., a_r\}$ as a vertex set and its arc set consists of the arcs $(a_i, a_j)$ for $i < j$, $(b, a_i)$ for odd $i$ and $(a_i, b)$ for even $i$.

**Lemma 3.5.1.** *[10] Every critically indecomposable digraph is isomorphic to a* $\mathcal{P}_r$, $\mathcal{P}'_r$, $\mathcal{T}_r^1$, $\mathcal{T}_r^2$, $\mathcal{T}_r^3$ *or* $\mathcal{D}_r$, *for* $r \geq 2$.

Remark that $\mathcal{P}_r$, $\mathcal{P}'_r$ and $\mathcal{D}_r$ have a sink, $\mathcal{T}_r^1$, $\mathcal{T}_r^2$ and $\mathcal{T}_r^3$ are tournaments. So critically indecomposable digraphs satisfies the weighted SNC.

A *counterexample* to the weighted SNC is a weighted digraph that does not have a vertex with the weighted SNP. A *minimal counterexample* to the the weighted SNC is a one with the least number of vertices.

**Proposition 3.5.1.** *A minimal counterexample to the weighted SNC (if exists) is strongly connected, indecomposable and not critically indecomposable.*

*Proof.* Suppose that $(D, \omega)$ is a minimal counterexample to the the weighted SNC. By the above remark, $D$ is not a critically indecomposable. Assume that $D$ is decomposable. There is a non-trivial interval $I$ of $D$. Let $D' = D/I$ denote the well defined digraph obtained from $D$ by contracting $I$ into a single vertex $v_I$. Define a weight on $D'$ as follows: $\omega'(v_I) = \omega(I)$ and $\omega'(u) = \omega(u)$ if $u \notin I$. Since $I$ is a non-trivial interval, $(D', \omega')$ has fewer vertices than $D$, so it is not a counterexample of the weighted SNC, i.e. it has a vertex $x$ with the weighted

SNP. By construction of $(D', \omega')$, $x$ has the weighted SNP in $(D, \omega)$, a contradiction.
Now, suppose for contradiction that $D$ is not strongly connected. Since the digraph $D/\mathcal{S}(D)$ is acyclic, $D$ has a strongly connected component $S$ which is a sink in $D/\mathcal{S}(D)$. Now the weighted digraph $(D[S], \omega)$ induced by $S$ has fewer vertices than $D$, whence it is not a counterexample. Thus there is $v \in S$ with $d^+_{D[S]}(v) \leq d^{++}_{D[S]}(v)$. However, $d^+_{D[S]}(v) = d^+_D(v)$ and $d^{++}_{D[S]}(v) = d^{++}_D(v)$, because every vertex in $S$ has no out-neighbor outside $S$. The rest is due to Lemma 3.5.1 and the remark that follows it. □

## 3.6 Caccetta-Häggkvist's Conjecture

For completeness, we introduce the following related conjecture, which was proposed in 1978 by Caccetta and Häggkvist [11]. Recall that the girth of a digraph is the length of a shortest directed cycle contained in $D$

**Conjecture 4.** *If $D$ is a digraph with minimum out-degree at least $|V(D)|/k$, then its girth is at most $k$.*

This conjecture is still open even for the particular case $k = 3$. Moreover, the following weakening is not yet proved.

**Conjecture 5.** *If $D$ is a digraph with both minimum out-degree and minimum in-degree at least $|V(D)|/3$, then its girth is at most 3.*

Seymour's Conjecture, if true, would imply this weakening. In fact, in such a case consider a vertex $v$ with the SNP in the digraph $D$ and the sets $N^-(v)$, $N^+(v)$ and $N^{++}(v)$. Each of these sets have size at least $|V(D)|/3$. If $D$ does not have any directed triangle, then these three sets would be pairwise disjoint, whence $D$ would have more than $|V(D)|$ vertices, a contradiction.

By simple observation we have:

**Proposition 3.6.1.** *[8] If $\gamma$ is a positive real number such that, for every digraph $D$, there exists a vertex $v$ such that $d^{++}(v) \geq \gamma . d^+(v)$, then every digraph $D$ on $n$ vertices with $min\{\delta^+_D, \delta^-_D\} \geq \frac{n}{2+\gamma}$ has a directed triangle.*

*Proof.* Consider the settings of the statement. By given, there is a vertex $v$ with $d^{++}(v) \geq \gamma.d^{+}(v)$. So, we have the following three sets $N^{-}(v)$, $N^{+}(v)$ and $N^{++}(v)$ has sizes at leat $\frac{n}{2+\gamma}$, $\frac{n}{2+\gamma}$ and $\gamma.\frac{n}{2+\gamma}$. If $D$ has no triangle, then $|V(D)| \geq |\{v\} \cup N^{-}(v) \cup N^{+}(v) \cup N^{++}(v)| > n$, a contradiction. □

Replacing $\gamma$ by 1, the above proposition shows that Seymour's conjecture implies Conjecture 5. Moreover, combining Proposition 3.6.1 and Theorem 3.4.1 yields the following corollary.

**Corollary 3.6.1.** *[8] Every digraph with min*$\{\delta^{+}_{D}, \delta^{-}_{D}\} \geq \frac{n}{2+\gamma}$ *contains a directed triangle, where* $\gamma = 0.657298...$ *is the unique real root of the equation* $2x^3 + x^2 - 1 = 0$.

In this case, $\frac{n}{2+\gamma} \approx 0.3764n$. However, much better results are found recently.

**Theorem 3.6.1.** *[9] Every digraph* $D$ *on* $n$ *vertices with* $\delta^{+}_{D} \geq 0.3465n$, *contains a directed triangle.*

**Theorem 3.6.2.** *[15] Every digraph* $D$ *on* $n$ *vertices with min*$\{\delta^{+}_{D}, \delta^{-}_{D}\} \geq 0.343545n$, *contains a directed triangle.*

We end this section by the following easy to proof observation.

**Proposition 3.6.2.** *Every digraph without transitive triangle satisfies SNC.*

*Proof.* Consider a vertex $v$ with minimum out-degree. If $v$ is a sink then it, clearly, has the SNP. Otherwise, let $x \in N^{+}(v)$. Since $D$ has no transitive triangles then $N^{+}(x) \subseteq N^{++}(v)$. So, $|N^{++}(v)| \geq |N^{+}(x)| \geq \delta^{+}_{D} = |N^{+}(v)|$. □

**Corollary 3.6.2.** *Every digraph whose underlying graph has girth at least 4 satisfies SNC.*

*Proof.* Digraphs whose underlying graph has girth at least 4 do not have oriented triangles and in particular do not have transitive triangles. □

# Part II

# Some Approaches to Seymour's Second Neighborhood Conjecture

# Chapter 4

# Dependency digraph

## 4.1 Definition of the dependency digraph

**Motivation.** Suppose that $D$ is a digraph missing exactly one edge $e = ab$. By giving an orientation to $e$ and adding it to $D$, we obtain a tournament completing $D$ ($T$ is a completion of $D$). However, we shall orient $e$ in some "convenient" way. Suppose $(i)$ there is $v \in V\backslash\{a,b\}$ with $v \to a$ and $b \notin N^+(v) \cup N^{++}(v)$ and $(ii)$ there is $u \in V\backslash\{a,b\}$ with $u \to b$ and $a \in N^+(u) \cup N^{++}(u)$. From the definition, $v$ and $u$ are distinct and $uv \neq e$. So $uv$ is not a missing edge. Whence, either $u \to v$ and this implies $u \to v \to a$ which contradicts $(ii)$, or $v \to u$ and this implies $v \to u \to b$ which contradicts $(i)$. Then at least one of the following holds:
$(i)$ $(\forall v \in V\backslash\{a,b\})[(v \to a) \Rightarrow (b \in N^+(v) \cup N^{++}(v))]$ or $(ii)$ $(\forall v \in V\backslash\{a,b\})[(v \to b) \Rightarrow (a \in N^+(v) \cup N^{++}(v))]$.

If $(i)$ is true, orient $e = ab$ as $(a,b)$, otherwise $(ii)$ holds and in this case orient $e$ as $(b,a)$. Suppose without loss of generality that $(i)$ holds and let $T = D + (a,b)$ and consider a local median order of $T$ and let $f$ denote the feed vertex obtained. By theorem 3.3.1, $f$ has the SNP in $T$. Suppose $f \notin \{a,b\}$, i.e. $f$ is a whole vertex. $f$ has the same out-neighbor in $T$ and $D$. Suppose $f \to x \to y$ in $T$. The first arc is in $D$. If the second arc is also in $D$, then $y \in N^{++}(f) \cup N^+(f)$. Otherwise, $(x,y) = (a,b)$. By definition of $(i)$, $b \in N^{++}(f) \cup N^+(f)$. This proves that $f$ has also the same second out-neighbor in $D$ and $T$. Therefore, $f$ has the SNP in $D$. Suppose that $f$ is not whole, i.e.

$f \in \{a, b\}$. Let $T'$ be obtained from $T$ by reorienting the missing edge $ab$ in a forward direction with respect to $L$, i.e. the head of this oriented arc is $f$. By Proposition 2.2.2 the same enumeration $L$ is also a local median order of $T'$, whence $f$ has the SNP in $T'$. However, $f$ has the same out-neighbor in $D$ and $T$ and the same second out-neighbor in $D$ and $T$. Therefore, $f$ has the SNP in $D$.

This motivates us to the following definition:

**Definition 4.1.1.** ***Good missing edges and convenient orientations.*** *A missing edge $ab$ is called* good *if:*
*(i)* $(\forall v \in V\backslash\{a, b\})[(v \to a) \Rightarrow (b \in N^+(v) \cup N^{++}(v))]$ *or*
*(ii)* $(\forall v \in V\backslash\{a, b\})[(v \to b) \Rightarrow (a \in N^+(v) \cup N^{++}(v))]$.
*If $ab$ satisfies $(i)$ we say that $(a, b)$ is a* convenient orientation *of $ab$.*
*If $ab$ satisfies $(ii)$ we say that $(b, a)$ is a* convenient orientation *of $ab$.*

We will see in Chapter 5, that when all the missing edges of $D$ are good, then $D$ has a vertex with the SNP property, using the above argument as a base.

**Motivation.** Suppose that a missing edge $e = ab$ is not a good missing edge of a digraph $D$. Then $(i)$ and $(ii)$ do not hold. Then there is $v \in V\backslash\{a, b\}$ with $(v \to a)$ and $b \notin N^+(v) \cup N^{++}(v)$ and there is $u \in V\backslash\{a, b\}$ with $u \to b$ and $a \in N^+(u) \cup N^{++}(u))$. In this case $vu$ should be also a missing edge of $D$.

**Loses.** We say that a missing edge $x_1y_1$ *loses to* a missing edge $x_2y_2$ if: $x_1 \to x_2$, $y_2 \notin N^+(x_1) \cup N^{++}(x_1)$, $y_1 \to y_2$ and $x_2 \notin N^+(y_1) \cup N^{++}(y_1)$.

**Dependency digraph.** The *dependency digraph* $\Delta_D$ (or $\Delta$ if there is no confusion) of $D$ is defined as follows: Its vertex set consists of all the missing edges and $(ab, cd) \in E(\Delta)$ if $ab$ loses to $cd$. Note that $\Delta$ may contain digons (directed cycles of length 2).

These digraphs were used in [3] to prove SNC for tournaments missing a matching. However, our definition is general and is suitable for any digraph, as we shall see.

The following holds by the definition of good missing edges and losing relation between them.

**Lemma 4.1.1.** *Let $D$ be a digraph and let $\Delta$ denote its dependency digraph. A missing edge $ab$ is good if and only if its in-degree in $\Delta$ is zero.*

## 4.2 Intervals and good median orders

Let $D$ be a (weighted) digraph and let $\Delta$ denote its dependency digraph. Let $C$ be a connected component of $\Delta$. Set $K(C) = \{u \in V(D);$ there is a vertex $v$ of $D$ such that $uv$ is a missing edge and belongs to $C$ $\}$. The *interval graph of* $D$, denoted by $\mathcal{I}_D$ is defined as follows. Its vertex set consists of the connected components of $\Delta$ and two vertices $C_1$ and $C_2$ are adjacent if $K(C_1) \cap K(C_2) \neq \phi$. So $\mathcal{I}_D$ is the intersection graph of the family $\{K(C); C$ is a connected component of $\Delta$ $\}$. Let $\xi$ be a connected component of $\mathcal{I}_D$. We set $K(\xi) = \cup_{C \in \xi} K(C)$. Clearly, if $uv$ is a missing edge in $D$ then there is a unique connected component $\xi$ of $\mathcal{I}_D$ such that $u$ and $v$ belongs to $K(\xi)$. If $f \in V(D)$, we set $J(f) = \{f\}$ if $f$ is a whole vertex, and $J(f) = K(\xi)$ otherwise, where $\xi$ is the unique connected component of $\mathcal{I}_D$ such that $f \in K(\xi)$. Clearly, if $x \in J(f)$, then $J(f) = J(x)$ and if $x \notin J(f)$, then $x$ is adjacent to every vertex in $J(f)$.

Let $L = x_1 \cdots x_n$ be a median order of a weighted digraph $D$. Recall, for $i < j$, the set $[i,j] := [x_i, x_j] := \{x_i, x_{i+1}, ..., x_j\}$ and $]i,j[= [i,j] \backslash \{x_i, x_j\}$ are called *intervals* of $L$. We recall also that $K \subseteq V(D)$ is an interval of $D$ if for every $u, v \in K$ we have: $N^+(u) \backslash K = N^+(v) \backslash K$ and $N^-(u) \backslash K = N^-(v) \backslash K$. The following shows a relation between the intervals of $D$ and the intervals of $L$.

**Proposition 4.2.1.** *Let $\mathcal{I} = \{I_1, ..., I_r\}$ be a set of pairwise disjoint intervals of $D$. Then for every median order $L$ of $(D, \omega)$, there is a median order $L'$ of $(D, \omega)$ such that: $L$ and $L'$ have the same feed vertex and every interval in $\mathcal{I}$ is an interval of $L'$.*

*Proof.* Let $L = x_1 x_2 ... x_n$be a median order of a weighted digraph $(D, \omega)$ and let $\mathcal{I} = \{I_1, ..., I_r\}$ be a set of pairwise disjoint intervals of $D$. We will use the feedback property to prove it. Suppose

$a, b \in I_1$ with $a = x_i$, $b = x_j$, $i < j$ and $[x_i, x_j] \cap I_1 = \{x_i, x_j\}$. Since $I_1$ is an interval of $D$, we have $N^+(x_i) = N^+_{]i,j}[(x_j)$ and $N^-_{]i,j[}(x_i) = N^-_{]i,j[}(x_j)$. So, $\omega(N^-_{]i,j[}(x_i)) \leq \omega(N^+_{]i,j[}(x_i)) = N^+_{]i,j[}(x_j) \leq \omega(N^-_{]i,j[}(x_j)) = \omega(N^-_{]i,j[}(x_i))$, where the two inequalities are by the feedback property. Whence, all the quantities in the previous statement are equal. In particular, $\omega(N^+_{]i,j[}(x_i)) = \omega(N^-_{]i,j[}(x_i))$. Let $L_1$ be the enumeration $x_1...x_{i-1}x_{i+1}...x_{j-1}x_{x_i}x_jx_{j+1}...x_n$. Then $\omega(L_1) = \omega(L) + \omega(N^-_{]i,j[}(x_i)) - \omega(N^+_{]i,j[}(x_i)) = \omega(L)$. Thus, $L_1$ is a median order of $(D, \omega)$. By successively repeating this argument, we obtain a weighted median order in which $I_1$ is an interval of $L$. Again, by successively repeating the argument for each $I \in \mathcal{I}$, we obtain the desired order. □

We say that $D$ is *good* if the sets $K(\xi)$'s are intervals of $D$. By the previous proposition, every good digraph has a median order $L$ such that the $K(\xi)$'s form intervals of $L$. Such an enumeration is called a *good median order* of the good weighted digraph $(D, \omega)$.

**Lemma 4.2.1.** *Let $(D, \omega)$ be a good weighted digraph and let $L$ be a good median order of $(D, \omega)$, with feed vertex say $f$. Then for every $x \in J(f)$, $\omega(N^+(x)\backslash J(f)) \leq \omega(G_L\backslash J(f))$. So if $x$ has the weighted SNP in $(D[J(f)], \omega)$, then it has the weighted SNP in $D$.*

*Proof.* The proof is by induction $n$, the number of vertices of $D$. It is trivial for $n = 1$. Let $L = x_1...x_n$ be a good median order of $(D, \omega)$. Since $J(f)$ is an interval of $D$, we may assume that $J(x_n) = \{x_n\}$. If $L$ does not have any bad vertex then $N^-(x_n) = G_L$. Whence, $\omega(N^+(x_n)) \leq \omega(N^-(x_n)) = \omega(G_L)$ where the inequality is by the feedback property. Now suppose that $L$ has a bad vertex and let $i$ be the smallest such that $x_i$ is bad. Since $J(x_i)$ is an interval of $D$ and $L$, then every vertex in $J(x_i)$ is bad and thus $J(x_i) = [x_i, x_p]$ for some $p < n$. For $j < i$, $x_j$ is either an out-neighbor of $x_n$ or a good vertex, by definition of $i$. Moreover, if $x_j \in N^+(x_n)$, then $x_j \in N^+(x_i)$. So $N^+(x_n) \cap [1, i] \subseteq N^+(x_i) \cap [1, i]$. Equivalently, $N^-(x_i) \cap [1, i] \subseteq G_L \cap [1, i]$. Therefore, $\omega(N^+(x_n) \cap [1, i]) \leq \omega(N^+(x_i) \cap [1, i]) \leq \omega(N^-(x_i) \cap [1, i]) \leq \omega(G_L \cap [1, i])$, where the second inequality is by the feedback property. Now $L' = x_{p+1}...x_n$ is good also. By induction, $\omega(N^+(x_n) \cap [p+1, n]) \leq \omega(G_{L'})$. Note that $G_{L'} \subseteq G_L \cap [p+1, n]$. Whence $\omega(N^+(x_n)) = \omega(N^+(x_n) \cap [1, i]) + \omega(N^+(x_n) \cap [p + 1, n]) \leq$

$\omega(G_L \cap [1,i]) + \omega(G_L \cap [p+1,n]) = \omega(G_L)$. The second part of the statement is obvious. □

So Theorem 3.3.1 and Proposition 3.3.1 are corollaries of Lemma 4.2.1. In fact, we have, in our proof of the above lemma, used nearly the same ideas of [7] used to prove theorem 3.3.1, where the difference in ours is that we treated intervals of the given good weighted digraph as a single vertex, but with weight equals to that of the interval it represents. In addition, similar to an idea of [7], we can define the sedimentation of a good median order of a good weighted digraph.

Let $L$ be a good median order of a good digraph $(D,\omega)$ and let $f$ denote its feed vertex. We have for every $x \in J(f)$, $\omega(N^+(x)\backslash J(f)) \leq \omega(G_L\backslash J(f))$. Let $b_1, \cdots, b_r$ denote the bad vertices of $L$ not in $J(f)$ and $v_1, \cdots, v_s$ denote the non bad vertices of $L$ not in $J(f)$, both enumerated in increasing order with respect to their index in $L$.
If $\omega(N^+(f)\backslash J(f)) < \omega(G_L\backslash J(f))$, then we set $Sed(L) = L$. If $\omega(N^+(f)\backslash J(f)) = \omega(G_L\backslash J(f))$, then we set $sed(L) = b_1 \cdots b_r J(f) v_1 \cdots v_s$.

**Lemma 4.2.2.** *Let L be a good median order of a good weighted digraph $(D,\omega)$. Then $Sed(L)$ is a good median order of $(D,\omega)$.*

*Proof.* Let $L = x_1...x_n$ be a good median order of $(D,\omega)$. If $Sed(L) = L$, then there is nothing to prove. Otherwise, we may assume that $\omega(N^+(x_n)\backslash J(x_n)) = \omega(G_L\backslash J(x_n))$. The proof is by induction on $r$, the number of bad vertices not in $J(x_n)$. Set $J(x_n) = [x_t, x_n]$. If $r = 0$, then for every $x \in J(x_n)$ we have $N^-(x)\backslash J(x_n) = G_L\backslash J(x_n)$. Whence, $\omega(N^+(x)\backslash J(x_n)) = \omega(G_L\backslash J(x_n)) = \omega(N^-(x)\backslash J(x_n))$. Thus, $Sed(L) = J(x_n)x_1...x_{t-1}$ is a good median order. Now suppose $r > 0$ and let $i$ be the smallest such that $x_i \notin J(x_n)$ and is bad. As before, $J(x_i) = [x_i, x_p]$ for some $p < n$, $\omega(N^+(x_n)\cap[1,i]) \leq \omega(N^+(x_i)\cap[1,i]) \leq \omega(N^-(x_i)\cap[1,i]) \leq \omega(G_L\cap[1,i])$ and $\omega(N^+(x_n)\cap[p+1,t-1]) \leq \omega(G_L\cap [p+1,t-1])$. However, $\omega(N^+(x_n)\backslash J(x_n)) = \omega(G_L\backslash J(x_n))$, then the previous inequalities are equalities. In particular, $\omega(N^+(x_i) \cap [1,i]) = \omega(N^-(x_i) \cap [1,i])$. Since $J(x_i)$ is an interval of $L$ and $D$, then for every $x \in J(x_i)$ we have $\omega(N^+(x) \cap [1,i]) = \omega(N^-(x) \cap [1,i])$. Thus $J(x_i)x_1...x_{i-1}x_{p+1}...x_n$ is a good median order. To conclude, apply the induction hypothesis to the good median order $x_1...x_{i-1}x_{p+1}...x_n$.

□

Define now inductively $Sed^0(L) = L$ and $Sed^{q+1}(L) = Sed(Sed^q(L))$. If the process reaches a rank $q$ such that $Sed^q(L) = y_1...y_n$ and $\omega(N^+(y_n)\backslash J(y_n)) < \omega(G_{Sed^q(L)}\backslash J(y_n))$, call the order $L$ stable. Otherwise call $L$ periodic. These new order are used by Havet and Thomassé to exhibit a second vertex with the SNP in tournaments that do not have any sink. We will use them for the same purpose but for other classes of digraphs.

## 4.3 The completion approach

In this section we present the proof of Fidler and Yuster of the SNC restricted to digraphes missing matchings [3] and then we present a new conjecture proposed by El Sahili.

In what follows $D$ is a digraph whose missing edges form a matching $M$ and $\Delta$ denotes it dependency digraph.

An induced directed cycle $(a_1, a_2, b_1, b_2)$ is called a *losing cycle* if $b_2 \notin N^{++}(a_1)$ and $a_2 \notin N^{++}(b_1)$ [3]. Clearly, $a_1b_1 \in M$ *loses* to $a_2b_2 \in M$.

**Lemma 4.3.1.** *[3] The maximum out-degree of $\Delta$ is 1, and the maximum in-degree of $\Delta$ is 1. Thus $\Delta$ is composed of vertex disjoint directed paths and directed cycles.*

*Proof.* Assume that $(a_1, a_2, b_1, b_2)$ and $(a_1, a'_2, b_1, b'_2)$ are two losing cycles. The edge $a'_2b_2$ is not a missing edge of $D$. If $a'_2 \rightarrow b_2$ then $b_1 \rightarrow a'_2 \rightarrow b_2$, a contradiction. If $b_2 \rightarrow a'_2$ then $b_1 \rightarrow b_2 \rightarrow a'_2$, a contradiction. Thus, the maximum out-degree of $\Delta$ is 1. Similarly, the maximum in-degree is 1. □

This lemma shows that $\Delta$ is composed of vertex-disjoint directd paths and directed cycles. Let $C = a_1b_1, ...a_kb_k$ be a maximal path or a cycle in $\Delta$ (possibly $k = 1$). Namely, $(a_i, a_{i+1}, b_i, b_{i+1})$ forms a losing cycle for $i = 1, ..., k-1$. We have:

**Lemma 4.3.2.** *[3] If $C$ is an odd cycle then $(a_k, a_1, b_k, b_1)$ is a losing cycle. If $C$ is an even cycle then $(a_k, b_1, b_k, a_1)$ is a losing cycle.*

*Proof.* Assume first that $C$ is odd. Consider the tournament induced by the vertices $a_1, b_2, ..., b_{k-1}, a_k$. This tournament has a king. Since

$b_2 \notin N^{++}(a_1 \cup N^+(a_1)$, $a_1$ is not a king. Similarly, $b_2, a_3, ..., b_{k-1}$ are not kings. So $a_k$ is the king. If $(a_k, b_1, b_k, a_1)$ were a losing cycle, then $a_1 \notin N^{++}(a_k \cup N^+(a_k)$, a contradiction. So $(a_k, a_1, b_k, b_1)$ is the losing cycle between the edges $a_k b_k$ and $a_1 b_1$. Similar argument is used for the even case. □

**Lemma 4.3.3.** *[3] If $C$ is a cycle, then $K(C)$ is an interval of $D$.*

*Proof.* Let $(a_1, a_2, b_1, b_2)$ be a losing cycle and let $f \notin K := \{a_1, a_2, b_1, b_2\}$. $f$ is adjacent to every vertex in $K$. If $a_1 \to f$ then $b_2 \to f$, since otherwise $b_2 \in N^{++}(a_1 \cup N^+(a_1)$ which is a contradiction. So $N^+(a_1)\backslash K \subseteq N^+(b_2)\backslash K$. Applying this to every losing cycle of $C$ yields $N^+(a_1)\backslash K(C) \subseteq N^+(b_2)\backslash K(C) \subseteq N^+(a_3)\backslash K(C)... \subseteq N^+(b_k)\backslash K(C) \subseteq N^+(b_1)\backslash K(C) \subseteq N^+(a_2)\backslash K(C)... \subseteq N^+(a_k)\backslash K(C) \subseteq N^+(a_1)\backslash K(C)$ if $k$ is even. So these inclusion are equalities. An analogous argument proves the same result for odd cycles. □

If $C$ is a path, then by Lemma 4.1.1 $a_1 b_1$ is a good missing edge. Assume without loss of generality that $(a_1, b_1)$ is a convenient orientation. Then for all $1 \leq i \leq k$, add the arc $(a_i, b_i)$ to $D$ (in the other case add $(b_i, a_i)$). If $C$ is a path add $(a_i, b_i)$ for every $i$. We do this for every such $C$ of $\Delta$. The obtained digraph $T$ is a tournament. Let $T^*$ be the tournament obtained from $T$ by contracting the sets $K(C)$ whenever $C$ is a cycle in $\Delta$. Assign the new vertices weight $2k$ where $k$ is the length of the corresponding cycle of $\Delta$ (this is equal the size of the contracted set), while assign weight 1 to the non-contracted vertices. note that the contraction is well defined since the contracted set are intervals of $T$. We obtain a weighted tournament, denote it by $(T^*, \omega)$. Let $L$ be weighted median order of the new tournament and let $f^*$ denote its feed vertex. $f^*$ has the weighted SNP in $T^*$ [3].

**Lemma 4.3.4.** *[3] If $C$ is a cycle of $\Delta$, then $D[K(C)]$ has a vertex with the SNP.*

Set $f = f^*$ if the feed vertex is not a contracted vertex. Otherwise, let $C$ be the cycle corresponding to the set contracted into $f^*$. Set $f$ to be a vertex of $K(C)$ having the SNP in $D[K(C)]$.

**Corollary 4.3.1.** *[3] $f$ has the SNP in $D$.*

This proves that digraphs missing a matching satisfies the SNC.

El Sahili conjectured the following statement:

**Conjecture 6.** *(EC) Every digraph $D$ has a completion with a feed vertex having the SNP in $D$.*

Clearly, El Sahili's Conjecture implies the Second Neighborhood Conjecture. We do not know if the reverse is holds. As one can observe, that EC suggests a method (an approach) for solving the SNC, which we will call the completion approach. In general, following this approach, we orient the missing edges of $D$ in some 'proper' way, to obtain a tournament $T$. Then we consider a particular feed vertex (clearly, it has the SNP in $T$) and try to prove that it has the SNP in $D$ as well.

Not so far from EC, we conjecture the following:

**Conjecture 7.** *(GC) Every weighted digraph $(D, \omega)$ has a completion with a feed vertex having the weighted SNP in $(D, \omega)$.*

Clearly, GC is just the weighted version of EC and the first implies the later one. We do not know whether these two statements are equivalent or not. Theorem 3.3.1 and Proposition 3.3.1 shows that EC and GC hold for tournaments. For the class of digraphs missing a matching, the above proof does not guarantee that the vertex found with the SNP is a feed vertex of some completion of $D$. In Chapter 7, we refine this proof to guarantee that the vertex found with the SNP is a feed vertex of some completion of $D$, and thus proving EC for these digraphs. However, in Chapter 5, we prove GC, and thus EC and SNC, for digraphs missing a generalized star, sun, star or a complete graph. Moreover, we prove EC, and thus SNC, for digraphs missing a comb and digraphs whose missing graph is a complete graph minus two independent edges or the edges of a cycle of length five. In addition, we prove EC, and thus SNC for digraphs missing $n$ disjoint stars under some conditions. Weaker conditions are required for $n = 1, 2, 3$.

## 4.4 Forcing graphs

Let $\mathcal{H}$ be a family of digraphs (digons are allowed) and let $G$ be a given graph. We say that $G$ is *$\mathcal{H}$-forcing* if the dependency digraph of every digraph missing $G$ is a member of $\mathcal{H}$. The set of all $\mathcal{H}$-forcing graphs is denoted by $\mathcal{F}(\mathcal{H})$.

**Proposition 4.4.1.** *Let $\mathcal{H}$ be a family of digraphs. Then $\mathcal{F}(\mathcal{H})$ is non-empty if and only if $\mathcal{H}$ has a trivial digraph.*

*Proof.* Let $G$ be a graph and let $D$ be any digraph missing it. Suppose $xy \to uv$ in $\Delta$, the dependency digraph of $D$, namely $v \notin N^+(x) \cup N^{++}(x)$. We add to $D$ an extra whole vertex $\alpha$ such that $x \to \alpha \to v$. This breaks the arc $(xy, uv)$. Hence, by adding a sufficient number of such vertices, one obtains a digraph whose missing graph is $G$ and such that its dependency digraph is trivial. This establishes the necessary condition.
The converse holds, by observing that the dependency digraph of any digraph missing a star is trivial. □

**Problem 4.4.1.** *Let $\mathfrak{S}$ denote the class of all empty digraphs and let $\vec{\mathcal{P}}$ be the family of all digraphs composed of vertex disjoint directed paths only. Characterize $\mathcal{F}(\mathfrak{S})$ and $\mathcal{F}(\vec{\mathcal{P}})$.*

We will characterize $\mathcal{F}(\mathfrak{S})$ and prove when such a graph is the missing graph of $D$, then $D$ satisfies GC, and thus EC and SNC. For the second class, we give some examples and prove when such an example is the missing graph of $D$, then $D$ satisfies EC, and thus the SNC.

# Chapter 5

# $\mathfrak{S}$-forcing graphs

As we can see from sections 3.5 and 4.3, the weighted version of Seymour's second neighborhood conjecture is also important. For example, in 4.3 the truthfulness of the weighted SNC in weighted tournaments was used to proved that every digraph missing a matching satisfies the SNC. In this chapter we prove GC, and thus the weighted SNC, for digraphs whose missing graphs are in $\mathcal{F}(\mathfrak{S})$. When the missing graph of $D$ is in $\mathcal{F}(\mathfrak{S})$, then by definition, all the missing edges of $D$ are good.

**Theorem 5.0.1.** *Let $(D, \omega)$ be a weighted digraph. If all the missing edges of $D$ are good then it satisfies GC.*

*Proof.* We give every missing edge a convenient orientation and add it to $D$. The obtained digraph is a tournament $T$. Consider a weighted local median order $L$ of $(T, \omega)$ and let $f$ denote its feed vertex. We modify $T$ by reorienting all the missing edges incident to $f$ towards $f$, if any exists. Let $T'$ denote the new obtained tournament. $L$ is also a weighted local median order of $(T', \omega)$, by Proposition 2.2.2. We have that $f$ has the weighted SNP in $T'$, by Proposition 3.3.1. Note that $N^+(f) = N^+_{T'}(f)$. Suppose $f \to u \to v$ in $T'$. Either $(u, v) \in E(D)$ or a convenient orientation. Thus $v \in N^+(f) \cup N^{++}(f)$. Whence, $N^{++}(f) = N^{++}_{T'}(f)$. Therefore, $f$ has the weighted SNP in $(D, \omega)$ as well. $\square$

This shows when the missing graph of a given graph is an $\mathfrak{S}$-forcing graph, then the digraph satisfies the weighted SNC.

## 5.1 Characterization

In this subsection, we characterize the class $\mathcal{F}(\mathfrak{S})$. We begin by the following definition

**Definition 5.1.1.** *An n-generalized star $G_n$ is a graph defined as follows:*

**1)** $V(G_n) = \bigcup_{i=1}^{n}(X_i \cup A_{i-1})$, *where the $A_i$'s and $X_i$'s are pairwise disjoint sets*

**2)** $G_n[\bigcup_{i=1}^{n} X_i]$ *is a complete graph and $X_i$'s are non-empty*

**3)** $\bigcup_{i=1}^{n} A_{i-1}$ *is a stable set and $A_i$ is non-empty for all $i > 0$*

**4)** $N(A_0) = \phi$ *and for all $i > 0$, for all $a \in A_i$,* $N(a) = \bigcup_{1 \leq j \leq i} X_j$.

Recall that a square is a cycle of length 4.

**Theorem 5.1.1.** *Let $G$ be a simple graph. The following are equivalent:*

**(A)** *$G$ is a generalized star.*

**(B)** *Any two non-adjacent edges of $G$ do not induce a subgraph of square.*

**(C)** *All the missing edges of every digraph whose missing graph is $G$ are good missing edges.*

*Proof.* $A \Rightarrow B$: By the definition of a generalized star.

$B \Rightarrow A$: By setting $A_0$ the set of isolated vertices, we may assume that $G$ has no isolated vertices. Let $S$ be a stable set in $G$ with the maximum size. Set $T = V(G) - S$. We Show that $T$ is a clique. By the maximality of $S$, every element of $T$ has a neighbor in $S$. Suppose $x, y \in T$. If $(N(x) \bigcup N(y)) \cap S = \{a\}$, then $xy \in E(G)$, since otherwise

the stable set $S \bigcup \{x, y\} - \{a\}$ is lager than $S$ which is a contradiction. Otherwise, there's distinct vertices $a$, $b \in S$ such that $ax$ and $by$ are in $E(G)$. By hypothesis, these two edges do not induce a subgraph of a square, then at least one of them has at least an endpoint which is adjacent to the endpoints of the other. Assume, without loss of generality, that this edge is $ax$. Since $S$ is stable, $x$ is the endpoint which is adjacent to $b$ and $y$. In particular, $xy \in E(G)$. Thus $T$ is a clique.
Suppose $a, b \in S$ with $d(a) \leq d(b)$. We prove $N(a) \subseteq N(b)$. Suppose there is $x \in N(a) - N(b)$. Since $d(a) \leq d(b)$ there is $y \in N(b) - N(a)$. Thus the path $axyb$ is the induced graph in $G$ by the two non-adjacent edges $ax$ and $by$, which is a subgraph of a square, a contradiction. Whence, $N(a) \subseteq N(b)$. Finally, let $d_1 < ... < d_s$ be the list of distinct degrees of vertices of $S$. Set $A_i = \{a \in S; d(v) = d_i\}$ and $X_i = \{x \in T;$ there is $a \in A_i$ such that $ax \in E(G)\} \backslash \bigcup_{j<i} X_j$. From these two families of sets, we can show that $G$ is an $s$ or $s+1$-generalized star.

$B \Rightarrow C$: Let $D$ be a digraph whose missing graph is $G$ and let $ab$ be a missing edge. Suppose, to the contrary, that $ab$ is not good. Then there is $u, v \in V(D) - \{a, b\}$ such that $u \rightarrow a$, $b \notin N^+(u) \cup N^{++}(u)$, $v \rightarrow b$ and $a \notin N^+(v) \cup N^{++}(v)$. In this case, also $uv$ is a missing edge and not adjacent to $ab$. Clearly, These 2 missing edges induce a subgraph of a square. A contradiction.

$C \Rightarrow B$: Suppose to the contrary that there is two non-adjacent edges in $G$, say $xy$ and $uv$, that induce in $G$ a subgraph of square. We may assume without lose of generality that $xu$ and $yv$ are not in $E(G)$. We construct a digraph $D$ whose missing graph is $G$ and such that $xy$ is not good as follows: $V(D) = V(G)$. For a vertex $w$ with $wu \notin E(G)$ (resp. $wv \notin E(G)$), $(w, u) \in E(D)$ (resp. $(w, v) \in E(D)$), with exception when $w = x$ (resp. $w = y$), $(u, x) \in E(D)$ (resp. $(v, y) \in E(D)$). For any two non-adjacent vertices $w, t$ in $G$ both not in $\{u, v\}$, we give $wt$ any orientation to be in $E(D)$. By construction of $D$, $u \rightarrow x$, $y \notin N^+(u) \cup N^{++}(u)$, $v \rightarrow y$ and $x \notin N^+(v) \cup N^{++}(v)$. Whence, $xy$ is not a good missing edge of $D$. A contradiction.

□

So the graphs of $\mathcal{F}(\mathfrak{S})$ are the generalized stars.

## 5.2 Corollaries

A *sun* $G$ is a graph formed of a complete graph $T$ and a stable set $S$ such that for every $s \in S$ we have $N(s) = V(T)$. Clearly, $G$ is a 2-generalized star or a 1-generalized star. If $V(T)$ is a singleton, then $G$ is a star and if $S$ is empty, then $G$ is a complete graph.

Now, the previous two theorems imply the following statements.

**Corollary 5.2.1.** *Every weighted digraph whose missing graph is a generalized star satisfies GC.*

**Corollary 5.2.2.** *Every weighted digraph whose missing graph is a sun satisfies GC.*

**Corollary 5.2.3.** *Every weighted digraph whose missing graph is a star satisfies GC.*

**Corollary 5.2.4.** *Every weighted digraph whose missing graph is a complete graph satisfies GC.*

In particular, the aforementioned digraphs satisfies EC and thus SNC.

# Chapter 6

# Some $\vec{\mathcal{P}}$-forcing graphs

## 6.1 Removing a comb

A *comb* $G$ is a graph defined as follows:

**1)** $V(G)$ is disjoint union of three sets $A$, $X$ and $Y$ with —A—=—X—.

**2)** $G[X \cup Y]$ is a complete graph.

**3)** $A$ is stable set.

**4)** The bipartite graph induced by the edges $A$ and $X$ is a perfect matching.

Observe that the edges with an end in $A$ form a matching, say $M$.

**Proposition 6.1.1.** *Combs are $\vec{\mathcal{P}}$-forcing.*

*Proof.* Let $D$ be a digraph missing a comb $G$. We follow the previous notations. The only possible arcs of $\Delta$ occur between the edges in $M$. For $i = 1, 2, 3$ let $a_ix_i \in M$ with $a_i \in A$ and $x_i \in X$. Suppose $a_1x_1$ loses to the two others. Then we have $a_1 \to x_3$, $x_1 \to a_2$, $a_2 \notin N^{++}(a_1) \cup N^+(a_1)$ and $x_3 \notin N^{++}(x_1) \cup N^+(x_1)$. Since $a_2x_3$ is not a missing edge then either $a_2 \to x_3$ or $a_2 \leftarrow x_3$. Whence, either $x_3 \in N^{++}(x_1) \cup N^+(x_1)$ or $a_2 \in N^{++}(a_1) \cup N^+(a_1)$. A contradiction. Therefore, the maximum out-degree in $\Delta$ is 1. Similarly, the maximum in-degree is 1. Thus, $\Delta$ is composed of vertex-disjoint directed paths and directed cycles. Now it is enough to prove that it has no directed cycles. Suppose that $C = a_0x_0a_1x_1...a_nx_n$ is a cycle in $\Delta$. Then we have

$a_{i+1} \notin N^{++}(a_i)$ and $a_i \leftarrow a_{i+1}$ for all $i < n$. We prove, by induction on $i$, that $a_i \rightarrow a_n$ for all $i < n$. In particular, $a_{n-1} \rightarrow a_n$, a contradiction. The case $i = 1$ holds since $a_n x_n$ loses to $a_1 x_1$. Now let $1 < i < n$. By induction hypothesis, $a_{i-1} \rightarrow a_n$. Since $a_i \notin N^{++}(a_{i-1})$ and $a_i a_n$ is not a missing edge we must have $(a_i, a_n) \in D$.

□

**Theorem 6.1.1.** *Every digraph missing a comb satisfies EC.*

*Proof.* Let $D$ be a digraph missing a comb $G$. We follow the previous notations. Let $P = a_0 x_0 a_1 x_1 ...$ be a maximal directed path in $\Delta$ ($a_i \in A$ and $a_i x_i$ is a vertex in $\Delta$). By Lemma 4.1.1, $a_0 x_0$ has a convenient orientation. Suppose $(a_0, x_0)$ is a convenient orientation. In this case add $(a_{2i}, x_{2i})$ and $(x_{2i+1}, a_{2i+1})$ to $D$. Otherwise, we orient in the reverse direction. We do this for all such paths of $\Delta$. The obtained digraph $D'$ is missing the complete graph $G[X \cup Y]$. Clearly, all the missing edges of $D'$ are good (in $D'$), so we give each one a convenient orientation and add it to $D'$. The obtained digraph $T$ is a tournament. Let $L$ be a local median order of $T$ and let $f$ denote its feed vertex. By Theorem 3.3.1, $f$ has the SNP in $T$. We claim that $f$ has the SNP in $D$ as well.

Suppose $f$ is a whole vertex. We show that $f$ gains no vertex in its second out-neighborhood and hence our claim holds. Assume $f \rightarrow u \rightarrow v \rightarrow f$ in $T$. Since $f$ is whole, $f \rightarrow u$ in $D$. If $u \rightarrow v$ in $D' - D$, then it is either a convenient orientation and hence $v \in N^{++}(f)$ or there is a missing edge $rs$ that loses to $uv$, namely $s \rightarrow v$ and $u \notin N^{+}(s) \cup N^{++}(s)$. However, $fs$ is not a missing edge, then we must have $f \rightarrow s$. Whence $v \in N^{++}(f)$. Now, if $u \rightarrow v$ in $T - D'$, then $v \in N^{++}_{D'}(f)$. But this case is already discussed. This argument is used implicitly in the rest of the proof.

Suppose $f \in A$. There is a maximal directed path $P = a_0 x_0 ... a_i x_i ... a_k x_k$ with $f = a_i$. If $(x_i, a_i) \in D'$, then $d^{+}(f) = d^{+}_{T}(f) \leq d^{++}_{T}(f) = d^{++}(f)$. In fact $f$ gains no new first nor second out-neighbor. Otherwise $(a_i, x_i) \in D'$. If $i < k$, $f$ gains only $x_i$ as a first out-neighbor and at least $a_{i+1}$ as a second out-neighbor. If $i = k$, then we reorient $a_k x_k$ as $(x_k, a_k)$. The same order $L$ is also a local median order of $T'$ the modified tournament. Now $f$ gains no vertex in its second out-neighborhood.

Suppose $f \in X$. There is a maximal directed path $P = a_0x_0...a_ix_i...a_kx_k$ with $f = x_i$. If $(a_i, x_i) \in D'$ we reorient all the missing edges incident to $x_i$ towards $x_i$. In this case $f$ gains no new first nor second out-neighbor in the modified tournament. Otherwise $(x_i, a_i) \in D'$. If $i = k$, we reorient all the missing edges incident to $x_i$ towards $x_i$. In this case $f$ gains no new first nor second out-neighbor in the modified tournament. If $i < k$, we reorient all the missing edges incident to $x_i$ towards $x_i$ except $(x_i, a_i)$. In this case $f$ gains only $a_i$ (resp. $x_{i+1}$ ) as a first (resp. second ) out-neighbor in the modified tournament.

Suppose $f \in Y$. Reorient all the missing edges incident to $y$ towards $y$. In the modified tournament $f$ gains no vertex in its second out-neighborhood.

Therefore $D$ satisfies EC.

□

## 6.2 Removing a $\tilde{K}^4$

A $\tilde{K}^4$ is a graph obtained from the complete graph by removing 2 non adjacent edges. If $xy$ and $uv$ are the removed edges then $\tilde{K}^4$ restricted to $\{x, y, u, v\}$ is a cycle of length 4.

**Proposition 6.2.1.** *The graphs $\tilde{K}^4$ are $\vec{\mathcal{P}}$-forcing.*

*Proof.* This is clear because the dependency digraph can have at most one arc. □

**Theorem 6.2.1.** *Every digraph whose missing graph is a $\tilde{K}^4$ satisfies EC.*

*Proof.* Let $D$ be a digraph missing a $\tilde{K}^4$. If $\Delta$ has no arc then $D$ satisfies SNC, by theorem 5.0.1 . Otherwise, it has exactly one arc, say $xy \to uv$ with $x \to u$ and $v \notin N^{++}(v)$. Note that the cycle $C = xyuv$ is an induced cycle in the missing graph. We may suppose that $(x, y)$ is a convenient orientation. Add $(x, y)$ and $(u, v)$ to $D$. The rest of the missing edges are good missing edges. So we give them a convenient orientation and add to $D$. The obtained digraph $T$ is a tournament.

Let $L$ be a local median order of $T$ and let $f$ denote its feed vertex. Now $f$ has the SNP in $T$. We discuss according to $f$.

Suppose $f$ is a whole vertex. Then $f$ gains no vertex in its second out-neighborhood.

Suppose $f = x$. Reorient all the missing edges incident to $x$ towards $x$ except $(x, y)$. The same order $L$ is a local median order of the modified tournament $T'$. The only new first (resp. second) out-neighbor of $f$ is $y$ (resp. $v$).

Suppose $f = y$, $u$, $v$ or a non-whole vertex that does not belong $C$. Reorient all the missing edges incident to $f$ towards $f$. In the modified tournament, $f$ gains no vertex in its second out-neighborhood.

□

## 6.3 Removing a $\tilde{K}^5$

A $\tilde{K}^5$ is a graph obtained from the complete graph by removing a cycle of length 5. Note that $\tilde{K}^5$ restricted to the vertices of the removed cycle is also a cycle of length 5.

In the following $ab \to cd$ means $ab$ loses to $cd$, namely, $a \to c$ and $b \to d$ (the order of the endpoints is considered). Let $D$ be a digraph missing $\tilde{K}^5$ and let $\Delta$ denote its dependency digraph. Let $C = xyzuv$ be the induced cycle of length five in $\tilde{K}^5$. Checking by cases, we find that $\Delta$ has at most three arcs. If $\Delta$ has exactly three arcs, then its arcs are (isomorphic to) $uv \to xy \to zu \to vx$ or $uv \to xy \to zu$ and $xv \to zy$.
If $\Delta$ has exactly two arcs, then they are (isomorphic to) $uv \to xy \to zu$ or $uv \to xy$ and $vx \to yz$.
If $\Delta$ has exactly one arc, then it is (isomorphic to) $uv \to xy$. So we have the following.

**Proposition 6.3.1.** *The graphs $\tilde{K}^5$ are $\vec{\mathcal{P}}$-forcing.*

**Theorem 6.3.1.** *Every digraph whose missing graph is a $\tilde{K}^5$ satisfies EC.*

*Proof.* Let $D$ be a digraph missing a $\tilde{K}^5$. Let $C = xyzuv$ be the induced cyle of length five in $\tilde{K}^5$. If $\Delta$ has no arcs, then $D$ satisfies SNC, by Theorem 5.0.1.

Suppose $\Delta$ has exactly one arc $uv \to xy$. Without loss of generality, we may assume that $(u, v)$ is a convenient orientation. Add $(u, v)$ and $(x, y)$ to $D$. We give the rest of the missing edges (they are good) convenient orientations and add them to $D$. Let $L$ be a local median order of the obtained tournament $T$ and let $f$ denote its feed vertex. $f$ has the SNP in $T$. If $f = u$, then the only new first (resp. second) out-neighbor of $u$ is $v$ (resp. $y$). Whence $f$ has the SNP in $D$. Otherwise, we reorient all the missing edges incident to $f$ towards $f$, if any exist. The same order $L$ is a local median order of the new tournament $T'$ and $f$ has the SNP in $T'$. However, $f$ gains neither a new first out-neighbor nor a new second out-neighbor. So $f$ has the SNP in $D$.

Suppose $\Delta$ has exactly two arcs, say $uv \to xy$ and $vx \to yz$. We may assume that $(u, v)$ is a convenient orientation. Add $(u, v)$ and $(x, y)$ to $D$. If $(v, x)$ is a convenient orientation, then we add $(v, x)$ and $(y, z)$ to $D$, otherwise we add their reverse. We give the rest of the missing edges (they are good) convenient orientations and add them to $D$. Let $L$ be a local median order of the obtained tournament $H$ and let $f$ denote its feed vertex. We reorient every missing edge incident to $f$, whose other endpoint is not in $\{u, v, x\}$, towards $f$ if any exists . The same $L$ is a local median order of the new tournament $T$ and $f$ has the SNP in $T$.

If $f \notin \{u, v, x\}$, then it gains neither a new first out-neighbor nor a new second out-neighbor. So $f$ has the SNP in $D$.

If $f = u$, then the only new first (resp. second ) out-neighbor of $f$ is $v$ (resp. $y$), whence $f$ has the SNP in $D$.

If $f = v$, then either $v \to x$ in $T$ and in this case the only new first (resp. second ) out-neighbor of $v$ is $x$ (resp. $z$) or $x \to v$ and in this case $f$ gains neither a new first out-neighbor nor a new second out-neighbor. Whence $f$ has the SNP in $D$.

If $f = x$, then we reorient $xy$ as $(y, x)$. The same $L$ is a local median order of the new tournament $T'$ and $f$ has the SNP in $T'$. If $v \to x$ in $T'$, then $f$ gains neither a new first out-neighbor nor a new second out-neighbor. Otherwise, $x \to v$ in $T'$ then the only new first (resp.

second ) out-neighbor of $f$ is $v$ (resp. $y$). Whence $f$ has the SNP in $D$.

Suppose $\Delta$ has exactly 2 arcs with $uv \to xy \to zu$. We may assume that $(u, v)$ is a convenient orientation. Add $(u, v)$, $(x, y)$ and $(z, u)$ to $D$. We give the rest of the missing edges (they are good) convenient orientations and add them to $D$. Let $L$ be a local median order of the obtained tournament $H$ and let $f$ denote its feed vertex. We reorient every missing edge incident to $f$, whose other endpoint is not in $\{u, v, x, y, z\}$, towards $f$, if any exists . The same $L$ is a local median order of the new tournament $T$ and $f$ has the SNP in $T$.

If $f \notin \{u, v, x, y, z\}$, then it gains neither a new first out-neighbor nor a new second out-neighbor. So $f$ has the SNP in $D$.

If $f = u$, then the only new first (resp. second ) out-neighbor of $f$ is $v$ (resp. $y$), whence $f$ has the SNP in $D$.

If $f = v$, then we orient $xv$ as $(x, v)$. The same $L$ is a local median order of the new tournament $T'$ and $f$ has the SNP in $T'$.

If $f = x$, then we orient $xv$ as $(v, x)$. The same $L$ is a local median order of the new tournament $T'$ and $f$ has the SNP in $T'$. The only new first (resp. second) out-neighbor of $f$ is $y$ (resp. $u$). Whence $f$ has the SNP in $D$.

If $f = y$, then we orient $yz$ as $(z, y)$. The same $L$ is a local median order of the new tournament $T'$ and $f$ has the SNP in $T'$. $f$ gains neither a new first out-neighbor nor a new second out-neighbor. So $f$ has the SNP in $D$.

If $f = z$, then we orient $yz$ and $zu$ towards $z$. The same $L$ is a local median order of the new tournament $T'$ and $f$ has the SNP in $T'$. $f$ gains neither a new first out-neighbor nor a new second out-neighbor. So $f$ has the SNP in $D$.

Suppose $\Delta$ has exactly three arcs with $uv \to xy \to zu \to vx$. We may assume that $(u, v)$ is a convenient orientation. Add $(u, v)$, $(x, y)$, $(z, u)$ and $(v, x)$ to $D$. We give the rest of the missing edges (they are good) convenient orientations and then add to $D$. Let $L$ be a local median order of the obtained tournament $H$ and let $f$ denote its feed vertex. We reorient every missing edge incident to $f$, whose other endpoint is not in $\{u, v, x, y, z\}$, towards $f$ if any exists . The same $L$ is a local median order of the new tournament $T$ and $f$ has the SNP in $T$.

If $f \notin \{u, v, x, y, z\}$, then it gains neither a new first out-neighbor nor a new second out-neighbor. So $f$ has the SNP in $D$.

If $f = u$, then the only new first (resp. second ) out-neighbor of $f$ is $v$ (resp. $y$), whence $f$ has the SNP in $D$.

If $f = v$, then we orient $xv$ as $(x, v)$. The same $L$ is a local median order of the new tournament $T'$ and $f$ has the SNP in $T'$. In this case $f$ gains neither a new first out-neighbor nor a new second out-neighbor. So $f$ has the SNP in $D$.

If $f = x$, then the only new first (resp. second ) out-neighbor of $f$ is $y$ (resp. $u$). Whence $f$ has the SNP in $D$.

If $f = y$, then we orient $yz$ as $(z, y)$. The same $L$ is a local median order of the new tournament $T'$ and $f$ has the SNP in $T'$. $f$ gains neither a new first out-neighbor nor a new second out-neighbor. So $f$ has the SNP in $D$.

If $f = z$, then we orient $yz$ towards $z$. The same $L$ is a local median order of the new tournament $T'$ and $f$ has the SNP in $T'$. The only new first (resp. second) out-neighbor of $f$ is $u$ (resp. $x$). Whence $f$ has the SNP in $D$.

Finally, suppose $\Delta$ has exactly three arcs with $uv \rightarrow xy \rightarrow zu$ and $xv \rightarrow zy$. We may assume that $(u, v)$ is a convenient orientation. Add $(u, v)$, $(x, y)$ and $(z, u)$ to $D$. Note that $xv$ is a good missing edge. If $(x, v)$ is a convenient orientation, then add it with $(z, y)$, otherwise, we add the reverse of these arcs. We give the rest of the missing edges (they are good) convenient orientations and add them to $D$. Let $L$ be a local median order of the obtained tournament $H$ and let $f$ denote its feed vertex. We reorient every missing edge incident to $f$, whose other endpoint is not in $\{u, v, x, y, z\}$, towards $f$, if any exists. The same $L$ is a local median order of the new tournament $T$ and $f$ has the SNP in $T$.

If $f \notin \{u, v, x, y, z\}$, then it gains neither a new first out-neighbor nor a new second out-neighbor. So $f$ has the SNP in $D$.

If $f = u$, then the only new first (resp. second ) out-neighbor of $f$ is $v$ (resp. $y$), whence $f$ has the SNP in $D$.

If $f = v$, then either $x \rightarrow f = v$ in $T$ and in this case it gains neither a new first out-neighbor nor a new second out-neighbor or $f = v \rightarrow x$ and in this case the only new first (resp. second ) out-neighbor of $f$ is $x$ (resp. $z$). Whence $f$ has the SNP in $D$.

If $f = x$, then either $v \rightarrow f = x$ in $T$ and in this case the only new first (resp. second ) out-neighbor of $f$ is $y$ (resp. $z$) or $f = x \rightarrow v$ and in this case the only new first (resp. second ) out-neighbor of $f$ are $y$ and $v$ (resp. $z$). Whence $f$ has the SNP in $D$.

If $f = y$ or $z$, then we orient the missing edges incident to $f$ towards $f$. The same $L$ is a local median order of the new tournament $T'$ and $f$ has the SNP in $T'$. $f$ gains neither a new first out-neighbor nor a new second out-neighbor. So $f$ has the SNP in $D$.

□

Digraphs missing a matching are the digraphs with minimum degree $|V(D)| - 2$. These digraphs satisfies SNC (also EC 7.1.1). A more general class of digraphs is the class of digraphs with minimum degree at least $|V(D)| - 3$. The missing graph of such a digraph is composed of vertex disjoint directed paths and directed cycles. $P_3$ is the path of length 3 and $C_3$, $C_4$ and $C_5$ are the cycles of length 3, 4 and 5 respectively. Theorems 6.1.1, 6.2.1 and 6.3.1 imply the following statement.

**Corollary 6.3.1.** *Every digraph whose missing graph is $P_3$, $C_3$, $C_4$ or a $C_5$ satisfies EC.*

# Chapter 7

# Digraphs missing disjoint stars

## 7.1 Removing n stars

We recall that a vertex $x$ in a tournament $T$ is a king if $\{x\} \cup N^+(x) \cup N^{++}(x) = V(T)$. It is well known that every tournament has a king. However, for every natural number $n \notin \{2, 4\}$, there is a tournament $T_n$ on $n$ vertices, such that every vertex is a king for this tournament.

**Theorem 7.1.1.** *Let $D$ be a digraph obtained from a tournament by deleting the edges of disjoint stars. Suppose that, in the induced tournament by the centers of the missing stars, every vertex is a king. If $\delta^-_\Delta > 0$ then $D$ satisfies EC.*

*Proof.* Orient all the missing edges towards the centers of the missing stars. Let $L$ be a median order of the obtained tournament $T$ and let $f$ denote its feed vertex. We have $d^+_T(f) \leq d^{++}_T(f)$. It is easy to prove that if $f$ is a whole vertex, then it has the SNP in $D$.

Suppose that $f$ is the center of a missing star. In this case $N^+(f) = N^+_T(f)$. Suppose $f \to u \to v$ in $T$. If $(u, v) \in D$ then $v \in N^+(f) \cup N^{++}(f)$. Otherwise, $uv$ is a missing edge, hence $v$ is the center of a missing star, whence $v \in N^+(f) \cup N^{++}(f)$ because $f$ is a king for the centers of the missing stars. Thus $N^{++}(f) = N^{++}_T(f)$. Therefore $f$ has the SNP in $D$.

Now suppose that $fx$ is a missing edge belonging to some missing star of center $x$. Suppose, first, that $fx$ loses to a missing edge $by$, say $y$ is the center of the missing star containing $by$. Assume $f \to x \to q$ in $T$ with $q \neq y$, then $b \to y$, whence, $f \to b \to q$. Assume that $f \to c \to z$ in $T$, for some missing edge $cz$ with $z \neq y$. Since $\delta^-_\Delta > 0$ there is a missing edge $uv$, with $x \notin \{u, v\}$ that loses to $cz$, namely, $v \to z$ and $c \notin N^+(v) \cup N^{++}(v)$. But $f \to c$ then $f \to v$, hence $f \to v \to z$ and $z \in N^+(f) \cup N^{++}(f)$. Thus $y$ is the only new second out-neighbor of $f$. Note that $f$ have lost $x$ as a second out-neighbor and became a first out-neighbor. Therefore, $d^+(f) + 1 = d^+_T(f) \leq d^{++}_T(f) = d^{++}(f)$.
Suppose that $fx$ does not lose to any edge. Reorient $fx$ from $x$ to $f$. The same order $L$ is a median order for the new tournament $T'$ and $N^+(f) = N^+_{T'}(f)$. Suppose that $f \to c \to z$ with $cz$ is a missing edge and $z \notin N^+(f) \cup N^{++}(f)$. Assume that $ax$ is a missing edge that loses to $cz$. Then $x \to z$ and $c \notin N^+(z) \cup N^{++}(z)$. Whence, $fx$ loses to $cz$, a contradiction. Since $\delta^-_\Delta > 0$ there is a missing edge $by$, with $x \notin \{b, y\}$ that loses to $cz$, namely, $y \to z$ and $c \notin N^+(y) \cup N^{++}(y)$. But $f \to c$ then $f \to y$, hence $f \to y \to z$ and $z \in N^+(f) \cup N^{++}(f)$. Thus, $N^{++}(f) = N^{++}_{T'}(f)$. Therefore, $f$ has the SNP in $D$. □

We will need the following lemma in this chapter.

**Lemma 7.1.1.** *Let $D$ be a digraph missing disjoint stars such that the connected components of its dependency digraph are non-trivial strongly connected. Then $D$ is a good digraph.*

*Proof.* Let $\xi$ be a connected component of $\mathcal{I}_D$. Assume first that $K(\xi) = K(C)$ for some directed cycle $C$ of $\Delta$, say $C = a_1b_1...a_nb_n)$, namely $a_i \to a_{i+1}$ and $b_{i+1} \notin N^+(a_i) \cup N^{++}(a_i)$ ($a_ib_i$ is considered as a vertex). If the set of edges $\{a_ib_i\}_i$ forms a matching then by Lemma 4.3.3, we have the desired result. So, we will suppose that a center $x$ of a missing star appears twice in the list $a_1, b_1, ..., a_n, b_n$ and assume without loss of generality that $x = a_1$. Suppose $n$ is even. Set $K_1 = \{a_1, b_2, ..., a_{n-1}, b_n\}$ and $K_2 = K(C)\backslash K_1$. Suppose that $a_n \to b_1$ and $a_1 \notin N^+(a_n) \cup N^{++}(a_n)$. Then by following the proof of Lemma 4.3.3 in [3] we obtain the desired result. Suppose $a_n \to a_1$ and $b_1 \notin N^+(a_n) \cup N^{++}(a_n)$. By using the same argument of Lemma 4.3.3 in [3], we have that $K_1$ and $K_2$ are intervals of $D$. Assume, for contradiction, that $K_1 \cap K_2 \neq \phi$ and let $i > 1$ be the smallest index for which $x$ is incident to $a_ib_i$. Clearly $i > 2$. However, $b_3 \notin K_1$ and

$x = a_1 \to a_2 \to a_3$ implies that $i > 3$. Suppose that $x = a_i$. Since $b_2 \to a_1 = x = a_i$ and $a_3 \notin N^+(b_2) \cup N^{++}(b_2)$ then $a_3 \to x$. Similarly $b_4, a_5, ..., b_{i-1}$ are in-neighbors of $x$. However, $b_{i-1}$ is an out-neighbor of $a_i = x$, a contradiction. Suppose that $x = b_i$. Similarly, $a_3, b_4, ..., a_{i-1}$ are in-neighbors of $x$. However, $a_{i-1}$ is an out-neighbor of $x$, a contradiction. Thus $K_1 \cap K_2 \neq \phi$. whence, the desired result follows. Similar argument is used to prove it when $C$ is an odd directed cycle.
This result can be easily extended to the case when $K(\xi) = K(C)$ and $C$ is a non-trivial (having more than one vertex) strongly connected component of $\Delta$, because between any two missing edges $uv$ and $zt$ there is directed path from $uv$ to $zt$ and a directed path from $zt$ to $uv$. These two directed paths will form many directed cycles that are used to prove the desired result. This also is extended to the case when $K(\xi) = \cup_{C\in\xi} K(C)$: Let $u, u'$ be 2 vertices in $K(\xi)$. There is non-trivial (having more than one vertex) strongly connected components $C$ and $C'$ containing $u$ and $u'$ respectively. Since $\xi$ is a connected component of there is a directed path $C = C_0, C_1, ..., C_n = C'$. For all $i > 0$, there is $u_i \in K(C_{i-1}) \cap K(C_n)$. Therefore, we have: $N^+(u)\backslash K(\xi) = N^+(u_1)\backslash K(\xi) = ... = N^+(u_i)\backslash K(\xi) = ... = N^+(u_n)\backslash K(\xi) = N^+(u')\backslash K(\xi)$ and $N^-(u)\backslash K(\xi) = N^-(u_1)\backslash K(\xi) = ... = N^-(u_i)\backslash K(\xi) = ... = N^-(u_n)\backslash K(\xi) = N^-(u')\backslash K(\xi)$.

□

**Theorem 7.1.2.** *Let $D$ be a digraph whose missing graph is disjoint union of one star and a matching. If every connected component of the dependency digraph containing an edge of the missing star, has positive minimum out-degree and positive minimum in-degree, then $D$ satisfies EC.*

Let $D$ be a digraph such that its missing graph is disjoint union of a star $S_x$ of center $x$ and a matching $M$. $\Delta$ and $\mathcal{I}_D$ denote the dependency digraph and the interval graph of $D$ respectively. In addition, we suppose that each connected component of $\Delta$ containing a missing edge of $D$ incident to $x$ (edge of the missing star) has positive minimum out-degree and positive minimum in-degree. In what follows, we prove that $D$ satisfies EC.

Let $P$ be a connected component of $\Delta$ or $\mathcal{I}_D$ and let $v$ be a vertex of $D$. We say that $v$ appears in $P$ if $v \in K(P)$. Otherwise, we say $v$ does not appear in $P$.

Note that we can use the same argument of Lemma 4.3.1 to prove that the in-degree and out-degree in $\Delta$ of every edge $ax$ of the missing star $S_x$ is exactly one, and that if an edge $uv$ of $M$ has out-degree (resp. in-degree) more than one then $N^+_\Delta(uv) \subseteq E(S_x)$ (resp, $N^-_\Delta(uv) \subseteq E(S_x)$). So every connected component of $\Delta$, in which $x$ does not appear, is either a directed path or a directed cycle.

We denote by $\xi$ the unique connected component of $\mathcal{I}_D$ in which $x$ appears. So $\mathcal{I}_D$ is composed of the connected component $\xi$ and other isolated vertices.

Let $P = a_1b_1a_2b_2 \cdots a_kb_k$ be a connected component of $\Delta$, which is also a maximal path in $\Delta$ in which $x$ does not appear, namely $a_i \to a_{i+1}, b_i \to b_{i+1}$ for $i = 1, ..., k-1$. Since $a_1b_1$ is a good edge then $(a_1, b_1)$ or $(b_1, a_1)$ is a convenient orientation. If $(a_1, b_1)$ is a convenient orientation , we orient $(a_i, b_i)$ for $i = 1, ..., k$. Otherwise we orient $a_ib_i$ as $(b_i, a_i)$. We do this for every such a path of $\Delta$. Denote the set of these new arcs by $F$. Set $D' = D + F$.

**Lemma 7.1.2.** *$D'$ is a good digraph.*

*Proof.* Lemma 4.3.3 proves that every set $K(C)$ is an interval of $D$ whenever $C$ is a directed cycle of $\Delta$ in which $x$ does not appear.

Now we prove for all $u \in K(\xi)$, we have $N^+(u)\backslash K(\xi) = N^+(x)\backslash K(\xi)$. Let $u \in K(\xi)$ and let $C$ denote the connected component of $\Delta$ in which $u$ appears. Note that also $x$ appears in $C$. If $u$ appears in a non-trivial strongly connected component then by the proof of Lemma 7.1.1 the result follows. Otherwise, due to the condition that $C$ has positive minimum out-degree and positive minimum in-degree, there is a directed path $P = u_1v_1, ..., u_kv_k$ joining two non-trivial strongly connected components $C_1$ and $C_2$ contained in $C$ such that $u$ appears in $P$. The vertex $x$ must appear in $C_1$ and $C_2$. By the proof of Lemma 7.1.1, for all $a \in K(C_1) \cup K(C_2)$, we have $N^+(a)\backslash K(\xi) = N^+(x)\backslash K(\xi)$. Due to the definition of losing relations between missing edges (precisely, the beginning of the proof of Lemma 4.3.3), we can easily show that for all $a \in K(C_1)$, $b \in K(P)$ and $c \in K(C_2)$ we have $N^+(a)\backslash K(\xi) \subseteq N^+(b)\backslash K(\xi) \subseteq N^+(c)\backslash K(\xi)$, in particular, for $a = x = c$ and $b = u$. So $K(\xi)$ is an interval of $D$.

This shows also that the dependency digraph $\Delta'$ of $D'$ is obtained from $\Delta$ by deleting the components that are directed paths not containing $x$. So the above intervals of $D$ are also intervals of $D'$. Whence $D'$ is a good digraph.

□

**Lemma 7.1.3.** *$D[K((\xi)]$ satisfies EC.*

*Proof.* Set $A = V(S_x) - x$. For all $a \in A$, orient $ax$ as $(a, x)$. Let $uv \in M$ such that $u, v \in K(\xi)$. Let $P$ be the shortest path in $\Delta$ starting with an edge of the star $S_x$ and ending in $uv$, namely, $P = ax, u_1v_1, ..., u_nv_n$ with $x \to v_1$, $v_i \to v_{i+1}$ for all $i < n$ and $u_nv_n = uv$. We orient $uv$ from $u_n$ to $v_n$. We do this for all the missing edges of $D[K(\xi)]$. We denote the obtained tournament by $T[K(\xi)]$.

Let $L$ be a median order of $T[K(\xi)]$ which maximizes $\alpha$ the index of $x$ and let $g$ denote its feed vertex. In addition to the fact that $g$ has the SNP in $T[K(\xi)]$, $g$ has the SNP in $D[K(\xi)]$. In fact, if $g = x$ then clearly $g$ gains no out-neighbor. Moreover, $g$ does not gain any new second out-neighbor. Suppose that $g \to u \to v \to g$, with $uv \in M$. Since $x \to u$ and $uv$ is oriented from $u$ to $v$, then for every $a \in A$, $ax \to uv$ in $\Delta$, whence there is a missing edge $u'v'$ that loses to $uv$, say, $v' \to v$ and $u \notin N^{++}(v')$. But $x \to u$, then $x \to v'$, whence $x \to v' \to v$ in $D$. So $x$ gains no new second out-neighbor, so it has the SNP in $D[K(\xi)]$ also. Suppose that $g = a \in A$. Then $a$ gains only $x$ in its first out-neighbor. There is a unique missing $rs$ with $ax \to rs$, say $a \to r$ and $s \notin N^+(a) \cup N^{++}(a)$. Then $(r, s) \in T[K(\xi)]$. Suppose that $a \to u \to v \to a$ in $T[K(\xi)]$ with $uv \in M - rs$. There is a missing edge $u'v'$ that loses to $uv$, say, $v' \to v$ and $u \notin N^{++}(v')$. But $a \to u$, then $a \to v' \to v$. Suppose that $a \to x \to q$ in $T[K(\xi)]$ with $q \neq s$. Since $x \to q$ in $D$ and $r \notin N^{++}(x)$ then $r \to q$, whence $a \to r \to q$ in $D[K(\xi)]$. Note that $a$ loses $x$ as second out-neighbor in $T[K(\xi)]$. We get $d^+_{D[K(\xi)]}(a) + 1 = d^+_{T[K(\xi)]}(a) \leq d^{++}_{T[K(\xi)]}(a) = d^{++}_{D[K(\xi)]}(a)$, whence, $a$ has the SNP in $D[K(\xi)]$. Similar argument can be used in the case when $g$ is incident to a missing edge of $M$, that is oriented out of $g$ , to show $g$ has the SNP in $D[K(\xi)]$. Suppose that $g$ is incident to a missing edge of $M$, that is oriented towards $g$. We can use similar arguments as above, to show that $x$ is the only possible new second out-neighbor of $g$. If $x \in G_L$ and $d^+_{T[K(\xi)]}(g) = |G_L|$ then $sed(L)$ is a median order of $T[K(\xi)]$, in which the index of $x$ is greater than

$\alpha$, a contradiction. Otherwise, $x \notin G_L$ or $d^+_{T[K(\xi)]}(g) < |G_L|$, whence, $d^+_{D[K(\xi)]}(g) = d^+_{T[K(\xi)]}(g) \leq d^{++}_{D[K(\xi)]}(g)$, hence $g$ has the SNP in $D[K(\xi)]$ in this case. So $g$ has the SNP in $D[K(\xi)]$ and $D[K(\xi)]$ satisfies EC. □

In the following, $C = a_1b_1...a_kb_k$ denotes a directed cycle of $\Delta$ in which $x$ does not appear, namely $a_i \to a_{i+1}$, $b_{i+1} \notin N^{++}(a_i) \cup N^+(a_i)$, $b_i \to b_{i+1}$ and $a_{i+1} \notin N^{++}(b_i) \cup N^+(b_i)$.

**Lemma 7.1.4.** *In $D[K(C)]$ we have:*
*$k$ is odd:*

$$N^+(a_1) = N^-(b_1) = \{a_2, b_3, \cdots, a_{k-1}, b_k\}$$
$$N^-(a_1) = N^+(b_1) = \{b_2, a_3, \cdots, b_{k-1}, a_k\},$$

*$k$ is even:*

$$N^+(a_1) = N^-(b_1) = \{a_2, b_3, \cdots, b_{k-1}, a_k\}$$
$$N^-(a_1) = N^+(b_1) = \{b_2, a_3, \cdots, a_{k-1}, b_k\}.$$

*Proof.* Suppose that $k$ is odd. Since $(a_k, a_1, b_k, b_1)$ is a losing cycle, then $b_k \in N^+_{D[K(C)]}(a_1)$. Since $(a_{k-1}, a_k, b_{k-1}, b_k)$ is a losing cycle and $(a_1, b_k) \in E(D)$ then $(a_1, a_{k-1}) \in E(D)$ and so $a_{k-1} \in N^+_{D[K(C)]}(a_1)$, since otherwise $(a_{k-1}, a_1) \in E(D)$ and so $b_k \in N^{++}_{D[K(C)]}(a_{k-1})$, contradiction to the definition of the losing cycle $(a_{k-1}, a_k, b_{k-1}, b_k)$. And so on $b_{k-2}, a_{k-3}, ..., b_3, a_2 \in N^+_{D[K(C)]}(a_1)$. Again, since $(a_1, a_2, b_1, b_2)$ is a losing cycle then $b_2 \in N^-_{D[K(C)]}(a_1)$. Since $(a_2, a_3, b_2, b_3)$ is a losing cycle and $(b_2, a_1) \in E(D)$ then $(a_3, a_1)\ inE(D)$ and so $a_3 \in N^-_{D[K(C)]}(a_1)$. And so on, $b_4, a_5, ..., b_{k-1}, a_k \in N^-_{D[K(C)]}(a_1)$. We use the same argument for finding $N^+_{D[k(C)]}(b_1)$ and $N^-_{D[k(C)]}(b_1)$. Also we use the same argument when $k$ is even. □

**Lemma 7.1.5.** *In $D[K(C)]$ we have: $N^+(a_i) = N^-(b_i)$, $N^-(a_i) = N^+(b_i)$,*
*$N^{++}(a_i) = N^-(a_i) \cup \{b_i\} \backslash \{b_{i+1}\}$ and $N^{++}(b_i) = N^-(b_i) \cup \{a_i\} \backslash \{a_{i+1}\}$ for all $i = 1, ..., k$ where $a_{k+1} := a_1$, $b_{k+1} := b_1$ if $k$ is odd and $a_{k+1} := b_1$, $b_{k+1} := a_1$ if $k$ is even. So $d^{++}(v) = d^+(v) = d^-(v) = k - 1$ for all $v \in K(C)$.*

*Proof.* The first part is due to the previous lemma and the symmetry in these cycles. For the second part it is enough to prove it for $i = 1$ and $a_1$. Suppose first that $k$ is odd. By definition of losing relation between $a_1b_1$ and $a_2b_2$ we have $b_2 \notin N^{++}(a_1) \cup N^+(a_1)$. Moreover $a_1 \to a_2 \to b_1$, whence $b_1 \in N^{++}(a_1)$. Note that for $i = 1, ..., k-1$, $a_i \to a_{i+1}$ and $b_i \to b_{i+1}$. Combining this with the previous lemma we find that $N^{++}(a_1) = N^-(a_1) \cup \{b_1\} \backslash \{b_2\}$. Similar argument is used when $k$ is even.

□

**Proof of Theorem 7.1.2:** Let $L$ be a good median order of the good digraph $D'$ and let $f$ denote its feed vertex.

Suppose $f$ is a whole vertex then $J(f) = \{f\}$ and by Lemma 4.2.1, $|N^+_{D'}(f)| \leq |G_L \backslash J(f)|$. We show that $f$ has the SNP in $D$. Clearly, $f$ gains no new first out-neighbor. Suppose $f \to u \to v$ in $D''$ with $(u, v) \notin D$. If $(u, v)$ is a convenient orientation, then $v \in N^+(f) \cup N^{++}(f)$. Otherwise, there is a missing edge $rs$ that loses to $uv$, namely $s \to v$ and $u \notin N^+(s) \cup N^{++}(s)$. But $f \to u$ then $f \to s$, whence, $f \to s \to v$. So $f$ gains no new second out-neighbor and thus $f$ has the SNP in $D$.

Suppose that $J(f) = K(C)$ for some cycle $C$ of $\Delta$. By Lemma 4.2.1 and Lemma 7.1.3, $f$ has the SNP in $D'$. We use the same argument of the above case to prove that $f$ has the SNP in $D$.

Suppose that $f \in K(\xi)$, i.e. $J(f) = K(\xi)$. Consider the tournament $T[K(\xi)]$ established in the proof of lemma 7.1.5 and let $L'$ be a median order of $T[K(\xi)]$ which maximizes $\alpha$ the index of $x$. Let $g$ denote the feed vertex of $L'$. In $L$, replace $L$ restricted to $K(\xi)$ by $L'$. The new enumeration $L''$ is again a median order of the digraph $D''$ obtained from $D'$ by adding the new arcs of $T[K(\xi)]$, since both $L$ and $L'$ are median orders. By Lemma 4.2.1 and the fact that $L$ and $L''$, $D'$ and $D''$ coincides outside $K(\xi)$, we have $|N^+_{D''}(g) \backslash K(\xi)| \leq |G_{L''} \backslash K(\xi)|$. By the proof of Lemma 7.1.5, $g$ has the SNP in $T[K(\xi)]$. Thus $g$ has the SNP in $D''$. We use the same argument of the first case to show that $g$ also have the SNP in $D$.

Now assume that $(z, f) \in F$ for some $z$. Then $f$ gains neither a new first out-neighbor nor a new second out-neighbor. Now assume that $(f, z) \in F$. If $fz$ is the last vertex of the directed path in $\Delta$, then we reorient it as $(z, f)$. The same $L$ is a median order of the new digraph $D''$, however, $f$ gains neither a new second out-neighbor nor a new first out-neighbor. The last case to consider is when $f = a_i$ and $(a_i, b_i) \in F$ and $a_i b_i \rightarrow a_{i+1} b_{i+1}$ in $\Delta$. In this case, $f$ gains only $b_i$ as new first out-neighbor and only $b_{i+1}$ as a new second out-neighbor. Thus, $f$ has the SNP in $D$.

To conclude, in each case, we add to our final digraph obtained its missing edges after orienting them in forward direction with respect to the final median order considered. So $D$ satisfies EC.

**Corollary 7.1.1.** *Every digraph missing a matching satisfies EC.*

We note that our method guarantees that the vertex $f$ found with the SNP is a feed vertex of some digraph containing $D$ (by orienting the missing edges towards $f$ we obtain that $f$ is a feed vertex of a completion of $D$). This is not guaranteed by the proof presented in [3]. Recall that $F$ is the set of the new arcs added to $D$ to obtain the good digraph $D'$. So, if $F = \phi$, then $D$ is a good digraph.

**Theorem 7.1.3.** *Let $D$ be a digraph missing a matching and suppose $F = \phi$. If $D$ does not have any sink then it has two vertices with the SNP.*

*Proof.* Consider a good median order $L = x_1 ... x_n$ of $D$. If $J(x_n) = K(C)$ for some directed cycle $C$ of $\Delta$ then by Lemma 4.2.1 and Lemma 7.1.5 the result holds. Otherwise, $x_n$ is a whole vertex (i.e. $J(x_n) = \{x_n\}$). By Lemma 4.2.1, $x_n$ has the SNP in $D$. So we need to find another vertex with SNP. Consider the good median order $L' = x_1 ... x_{n-1}$. Suppose first that $L'$ is stable. There is $q$ for which $Sed^q(L') = y_1 ... y_{n-1}$ and $\mid N^+(y_{n-1}) \backslash J(y_{n-1}) \mid < \mid G_{Sed^q(L')} \backslash J(y_{n-1}) \mid$. Note that $y_1 ... y_{n-1} x_n$ is also a good median order of $D$. By Lemma 4.2.1 and Lemma 7.1.5, $y := y_{n-1}$ has the SNP in $D[y_1, y_{n-1}]$. So $\mid N^+(y) \mid = \mid N^+_{D[y_1, y_{n-1}]}(y) \mid +1 \leq \mid G_{Sed^q(L')} \mid \leq \mid N^{++}(y) \mid$. Now suppose that $L'$ is periodic. Since $D$ has no sink then $x_n$ has an out-neighbor $x_j$. Choose $j$ to be the greatest (so that it is the last vertex of its corresponding interval). Note that for every $q$, $x_n$ is an out-neighbor of the feed vertex of $Sed^q(L')$. So $x_j$ is not the feed vertex of any $Sed^q(L')$. Since $L'$ is periodic, $x_j$ must be a bad vertex of $Sed^q(L')$ for some integer $q$, otherwise the index of $x_j$ would always increase during the sedimentation process. Let $q$ be such an integer. Set $Sed^q(L') = y_1 ... y_{n-1}$. Lemma 7.1.5 and Lemma

4.2.1 guarantees that the vertex $y := y_{n-1}$ with the SNP in $D[y_1, y_{n-1}]$. Note that $y \rightarrow x_n \rightarrow x_j$ and $G_{Sed^q(L')} \cup \{x_j\} \subseteq N^{++}(y)$. So $| N^+(y) |=| N^+_{D[y_1,y_{n-1}]}(y) | +1 =| G_{Sed^q(L')} + 1 |=| G_{Sed^q(L')} \cup \{x_j\} |\leq| N^{++}(y) |$.

□

## 7.2 Removing one star and an Erratum

A more general statement to the following theorem is proved in Chapter 5. Here we present, yet, another prove that uses the sedimentation technique of a median order.

**Theorem 7.2.1.** *Let $D$ be a digraph obtained from a tournament by deleting the edges of a star. Then $D$ satisfies EC.*

*Proof.* Orient all the missing edges of $D$ towards the center $x$ of the missing star. The obtained digraph is a tournament $T$ completing $D$. Let $L$ be a median order of $T$ that maximizes $\alpha$ the index of $x$ in $L$ and let $f$ denote its feed vertex. If $f = x$, then clearly, $d^+(f) = d^+_{T'}(f) \leq |G^{T'}_L| \leq d^{++}_{T'}(f) = d^{++}(f)$. Now suppose that $f \neq x$. Reorient the missing edges incident to $f$ towards $f$ (if any). $L$ is also a median order of the new tournament $T'$. Note that $N^+(f) = N^+_{T'}(f)$ and we have $d^+_{T'}(f) \leq |G^{T'}_L|$. If $x \in G^{T'}_L$ and $d^+_{T'}(f) = |G^{T'}_L|$ then $sed(L)$ is a median order of $T'$ in which the index of $x$ is greater than $\alpha$, and also greater than the index of $f$. So we can give the missing edge incident to $f$ (if it exists it is $xf$) its initial orientation (as in $T$) such that $sed(L)$ is a median order of $T$, a contradiction to the fact that $L$ maximizes $\alpha$. So $x \notin G^{T'}_L$ or $d^+T'(f) < |G^{T'}_L|$. We have that $x$ is the only possible gained second out-neighbor vertex for $f$. If $x \notin G^{T'}_L$ then $G^{T'}_L \subseteq N^{++}(f)$, whence the result follows. If $d^+_{T'}(f) < |G^{T'}_L|$, then $d^+(f) = d^+_{T'}(f) \leq |G^{T'}_L| - 1 \leq d^{++}(f)$. So $f$ has the SNP in $D$. □

In the paper entitled: "*Remarks on the second neighborhood problem* "[3], the authors' proof of their true statement ("every digraph missing a star satisfies SNC"), is false.
They gave every missing edge a convenient orientation and then they added them to $D$. They claimed that the obtained tournament $T$ satisfies that for every $v \in V(D) - \{x\}$, where $x$ is the center of the missing star, we have $N^+_T(x) = N^+_D(x)$, and continued their proof based on this claim. However their claim is false. When the edge $xy$ is

a missing edge and $(y, x)$ is the added arc, outgoing neighbors of $x$ may become new second outgoing neighbors of $y$. For example, consider the digraph $D$, with vertex set $V = \{x, y, z, t, q\}$ and edge set $E = \{(y, q), (q, x), (z, q), (x, z), (z, y), (t, q), (t, z), (t, y)\}$. We have that $xy$ and $xt$ are the missing edges, $x$ is the center of the missing star, ($q \in Q$, $R = \phi$, $z \notin N^{++}(y)$ these are the notations of the author's proof), however, adding the convenient arc $(y, x)$ to $D$, makes $z$ a new second outgoing neighbor of $y$.

## 7.3 Removing two stars

In this section, $D$ is a digraph obtained from a tournament by deleting the edges of two disjoint stars. Let $S_x$ and $S_y$ be the two missing disjoint stars with centers $x$ and $y$ respectively, $A = V(S_x)\backslash x$, $B = V(S_y)\backslash y$, $K = V(S_x) \cup V(S_y)$ and assume without loss of generality that $x \rightarrow y$. In the chapter 5 it is proved that if the dependency digraph of any digraph consists of isolated vertices only then it satisfies SNC. Here we consider the case when the dependency digraph of $D$ has no isolated vertex.

**Theorem 7.3.1.** *Let $D$ be a digraph obtained from a tournament by deleting the edges of two disjoint stars. If $\delta_\Delta > 0$, then $D$ satisfies EC.*

*Proof.* Assume without loss of generality that $x \rightarrow y$. We note that the condition $\delta_\Delta > 0$ implies that for every $a \in A$ and $y \in B$ we have $y \rightarrow a$ and $b \rightarrow x$. We shall orient the missing edges to obtain a completion of $D$. First, we give every good edge a convenient orientation. For the other missing edges, let the orientation be towards the center of the two missing stars $S_x$ or $S_y$. The obtained digraph is a tournament $T$ completing $D$. Let $L$ be a median order of $T$ such that the index $k$ of $x$ is maximum and let $f$ denote its feed vertex. We know that $f$ has the SNP in $T$. We have only five cases:
Suppose that $f$ is a whole vertex. In this case $N^+(f) = N^+_T(f)$. Suppose $f \rightarrow u \rightarrow v$ in $T$. Clearly $(f, u) \in D$. If $(u, v) \in D$ or is a convenient orientation, then $v \in N^+(f) \cup N^{++}(f)$. Otherwise there is a missing edge $zt$ that loses to $uv$ with $t \rightarrow v$ and $u \notin N^+(t) \cup N^{++}(t)$. But $f \rightarrow u$, then $f \rightarrow t$, whence $f \rightarrow t \rightarrow v$ in D. Therefore, $N^{++}(f) = N^{++}_T(f)$ and $f$ has the SNP in $D$ as well.

Suppose $f = x$. Orient all the edges of $S_x$ towards the center $x$. $L$ is a median order of the modified completion $T'$ of $D$. We have $N^+(f) = N^+_{T'}(f)$. Suppose $f \to u \to v$ in $T'$. If $(u, v) \in D$ or is a convenient orientation, then $v \in N^+(f) \cup N^{++}(f)$. Otherwise, $(u, v) = (b, y)$ for some $b \in B$, but $f = x \to y$. Thus, $N^{++}(f) = N^{++}_{T'}(f)$ and $f$ has the SNP in $T'$ and $D$.
Suppose $f = b \in B$. Orient the missing edge $by$ towards $b$. Again, $L$ is a median order of the modified tournament $T'$ and $N^+(f) = N^+_{T'}(f)$. Suppose $f \to u \to v$ in $T'$. If $(u, v) \in D$ or is a convenient orientation, then $v \in N^+(f) \cup N^{++}(f)$. Otherwise $(u, v) = (b', y)$ for some $b' \in B$ or $(u, v) = (a, x)$ for some $a \in A$, however $x, y \in N^{++}(f) \cup N^+(f)$ because $f = b \to x \to y$ in $D$. Thus, $N^{++}(f) = N^{++}_{T'}(f)$ and $f$ has the SNP in $T'$ and $D$.
Suppose $f = y$. Orient the missing edges towards $y$ and let $T'$ denote the new tournament. We note that $B \subseteq N^{++}(y) \cap N^{++}_{T'}(y)$ due to the condition $\delta_\Delta > 0$. Also, $x$ is the only possible new second neighbor of $y$ in $T'$. If $B \cup \{x\} \nsubseteq G_L$ or $d^+_{T'}(y) < d^{++}_{T'}(y)$, then $d^+(y) = d^+_{T'}(y) \leq d^{++}_{T'}(y) - 1 \leq d^{++}(y)$. Otherwise, $B \cup \{x\} \subseteq G_L$ and $d^+_{T'}(y) = |G_L|$. In this case we consider the median order $Sed(L)$ of $T'$. Now the feed vertex of $sed(L)$ is different from $y$, the index of $x$ had increased, and the index of $y$ became less than the index of any vertex of $B$ which makes $Sed(L)$ a median order of $T$ also, in which the index of $x$ is greater than $k$, a contradiction.
Suppose $f = a \in A$. Orient the missing edge $ax$ as $(x, a)$ and let $T'$ denote the new tournament. Note that $y$ is the only possible new second out-neighbor of $a$ in $T'$ and not in $D$. Also $x \in N^{++}_T(a) \cap N^{++}(a)$. If $d^+_{T'}(a) < d^{++}_{T'}(a)$, then $d^+(a) = d^+_{T'}(a) \leq d^{++}_{T'}(a) - 1 \leq d^{++}(a)$, hence $a$ has the SNP in $D$. Otherwise, $d^+_{T'}(a) = |G_L| = d^{++}_{T'}(a)$ and in particular $x \in G_L$. In this case we consider $sed(L)$ which is a median order of $T'$. Note that the feed vertex of $Sed(L)$ is different from $a$ and the index of $a$ is less than the index of $x$ in the new order $Sed(L)$. Hence $Sed(L)$ is a median of $T$ as well, in which the index of $x$ is greater than $k$, a contradiction.
So in all cases $f$ has the SNP in $D$. Therefore $D$ satisfies EC. □

**Theorem 7.3.2.** *Let $D$ be a digraph obtained from a tournament by deleting the edges of two disjoint stars. If $\delta^+_\Delta > 0$, $\delta^-_\Delta > 0$ and $D$ does not have any sink, then $D$ has at least two vertices with the SNP.*

*Proof.* First, we show that $D[K]$ has at least two vertices with the

SNP. The condition $\delta_\Delta > 0$ implies that for every $a \in A$ and $b \in B$ we have $y \to a$ and $b \to x$. Clearly, $N^+(x) = \{y\}$, $N^+(y) = A$, $d^+(x) \leq 1 \leq |A| \leq d^{++}(x)$, thus $x$ has the SNP in $D[K]$. Let $H$ be the tournament $D - \{x, y\}$. Then $H$ has a vertex $v$ with the SNP in $H$. If $v \in A$, then $d^+v = d^+_H(v) \leq d^{++}_H(v) = d^{++}(v)$. If $v \in B$, then $d^+(v) = d^+_H(v) + 1 \leq d^{++}_H(v) + 1 = d^{++}(v)$. Whence, $v$ also has the $SNP$ in $D[K]$.

Next, we show that $D$ is a good digraph. Let $\mathcal{I}_\mathcal{D}$ be the interval graph of $D$. Let $C_1$ and $C_2$ be two distinct connected components of $\Delta$. Then the centers $x$ and $y$ appear in each of the these two connected components, whence $K(C_1) \cap K(C_2) \neq \phi$. Therefore, $\mathcal{I}_\mathcal{D}$ is a connected graph ( more precisely, it is a complete graph), having only one connected component $\xi$. Then, $K = K(\xi)$.

So, if $\Delta$ is composed of non-trivial strongly connected components, then the result holds by Lemma 7.1.1.
Due to the condition $\delta^+_\Delta > 0$ and $\delta^-_\Delta > 0$, $\Delta$ has a non-trivial strongly connected component, hence $N^+(x)\backslash K = N^+(y)\backslash K$. Now let $v \in K$ and assume without loss of generality that $xv$ is the missing edge incident to $v$. Due to the condition $\delta^+_\Delta > 0$ and $\delta^-_\Delta > 0$, we have that either $xv$ belongs to a non-trivial strongly connected component of $\Delta$, and in this case $N^+(v)\backslash K = N^+(x)\backslash K = N^+(y)\backslash K$, or $xv$ belongs to a directed path $P = xa_1, yb_1, \cdots, xa_p, yb_p$ joining two non-trivial strongly connected components $C_1$ and $C_2$ with $xa_1 \in C_1$ and $yb_p \in C_2$. There is $i > 1$ such that $v = a_i$. $L = xa_{i-1}, yb_{i-1}, xa_i, yb_i$ is a path in $\Delta$. By the definition of losing cycles, we have $N^+(x)\backslash K \subseteq N^+(b_{i-1})\backslash K \subseteq N^+(a_i)\backslash K \subseteq N^+(y)\backslash K = N^+(x)\backslash K$. Hence $N^+(x)\backslash K = N^+(v)\backslash K$ for all $v \in K$. Since every vertex outside $K$ is adjacent to every vertex in $K$ we also have $N^-(x)\backslash K = N^-(v)\backslash K$ for all $v \in K$.

Now, consider a good median order $L = x_1...x_n$ of $D$. If $J(x_n) = K$, then by Lemma 4.2.1 the result holds. Otherwise, $x_n$ is a whole vertex (i.e. $J(x_n) = \{x_n\}$). By Lemma 4.2.1, $x_n$ has the SNP in $D$. So we need to find another vertex with SNP. Consider the good median order $L' = x_1...x_{n-1}$. Suppose first that $L'$ is stable. There is $q$ for which $Sed^q(L') = y_1...y_{n-1}$ and $\mid N^+(y_{n-1})\backslash J(y_{n-1}) \mid < \mid G_{Sed^q(L')}\backslash J(y_{n-1}) \mid$. Note that $y_1...y_{n-1}x_n$ is also a good median order of $D$. Lemma 4.2.1

guarantees the existence of a vertex $y$ with the SNP in $D[y_1, y_{n-1}]$. Since $y_{n-1} \rightarrow x_n$ and $y \in J(y_{n-1})$ which is an interval of $D$, then $y \rightarrow x_n$. So $\mid N^+(y) \mid = \mid N^+_{D[y_1,y_{n-1}]}(y) \mid +1 \leq \mid G_{Sed^q(L')} \mid \leq \mid N^{++}(y) \mid$. Now suppose that $L'$ is periodic. Since $D$ has no sink, then $x_n$ has an out-neighbor $x_j$. Note that for every $q$, $x_n$ is an out-neighbor of the feed vertex of $Sed^q(L')$. So $x_j$ is not the feed vertex of any $Sed^q(L')$. Since $L'$ is periodic, $x_j$ must be a bad vertex of $Sed^q(L')$ for some integer $q$, otherwise the index of $x_j$ would always increase during the sedimentation process. Let $q$ be such an integer. Set $Sed^q(L') = y_1...y_{n-1}$. Lemma 4.2.1 guarantees the existence of a vertex $y$ with the SNP in $D[y_1, y_{n-1}]$. Since $y_{n-1} \rightarrow x_n$ and $y \in J(y_{n-1})$ which is an interval of $D$, then $y \rightarrow x_n \rightarrow x_j$. Note that $G_{Sed^q(L')} \cup \{x_j\} \subseteq N^{++}(y)$. So $\mid N^+(y) \mid = \mid N^+_{D[y_1,y_{n-1}]}(y) \mid +1 = \mid G_{Sed^q(L')} \mid +1 = \mid G_{Sed^q(L')} \cup \{x_j\} \mid \leq \mid N^{++}(y) \mid$.

□

## 7.4 Removing three stars

In this section, $D$ is obtained from a tournament missing the edges of three disjoint stars $S_x$, $S_y$ and $S_z$ with centers $x$, $y$ and $z$ respectively. Set $A = V(S_x) - x$, $B = V(S_y) - x$, $C = V(S_z) - z$ and $K = A \cup B \cup C \cup \{x, y, z\}$. Let $\Delta$ denote the dependency digraph of $D$. The triangle induced by the vertices $x$, $y$ and $z$ is either a transitive triangle or a directed triangle.
First we will deal with the case when this triangle is directed, and assume without loss of generality that $x \rightarrow y \rightarrow z \rightarrow x$. This is a particular case of the case when the missing graph is a disjoint union of stars such that, in the induced tournament by the centers of the missing stars, every vertex is a king.

**Theorem 7.4.1.** *Let $D$ be a digraph obtained from a tournament by deleting the edges of three disjoint stars whose centers form a directed triangle. If $\delta_\Delta > 0$, then $D$ satisfies EC.*

*Proof.* Note that $xa$ can not lose to $zc$ because $z \rightarrow x$ and $z \in N^{++}(x)$. Similarly $yb$ can not lose to $xa$ and $zc$ can not lose to $yb$. So the only possible arcs in $\Delta$ have the forms $xa \rightarrow yb$ or $yb \rightarrow zc$ or $zc \rightarrow xa$, where $a \in A$, $b \in B$ and $c \in C$.

Orient the good missing edges in a convenient way and orient the other edges toward the centers. The obtained digraph $T$ is a tournament. Let $L$ be a median order of $T$ such that the sum of the indices of $x, y$ and $z$ is maximum. Let $f$ denote the feed vertex of $L$. Due to symmetry, we may assume that $f$ is a whole vertex or $f = x$ or $f = a \in A$. Suppose $f$ is a whole vertex. Clearly, $N^+(f) = N^+_T(f)$. Suppose $f \rightarrow u \rightarrow v$ in $T$. If $(u, v) \in E(D)$ or $uv$ is a good missing edge, then $v \in N^+(f) \cup N^{++}(f)$. Otherwise, there is missing edge $rs$ that loses to $uv$ with $r \rightarrow v$ and $u \notin N^{++}(r) \cup N^+(r)$. But $f \rightarrow u$, then $f \rightarrow r$, whence $f \rightarrow r \rightarrow v$ and $v \in N^+(f) \cup N^{++}(f)$. Thus, $N^{++}_T(f) = N^{++}(f)$ and $f$ has the SNP in $D$.
Suppose $f = x$. Reorient all the missing edges incident to $x$ toward $x$. In the new tournament $T'$ we have $N^+(x) = N^+_{T'}(x)$. Since $y \in N^+(x)$ and $z \in N^{++}(x)$ we have that $N^{++}(x) = N^{++}_{T'}(x)$. Thus $x$ has the SNP in $D$.
Suppose that $f = a \in A$. Reorient $ax$ toward $a$. Suppose $a \rightarrow u \rightarrow v$ in the new tournament $T'$ with $v \neq y$. If $(u, v) \in E(D)$ or $uv$ is a good missing edge then $v \in N^+(a) \cup N^{++}(a)$. Otherwise, there is $b \in B$ and $c \in C$ such that $(u, v) = (c, z)$ and $by$ loses to $cz$, then $f \rightarrow c$ implies that $a \rightarrow y$, but $y \rightarrow z$, whence $z \in N^{++}(a) \cup N^+(a)$. So the only possible new second out-neighbor of $a$ is $y$, hence if $y \notin N^{++}_{T'}(a)$ then $a$ has the SNP in $D$. Suppose $y \in N^{++}_{T'}(a)$. If $d^+_{T'}(a) < d^{++}_{T'}(a)$ then $d^+(a) = d^+_{T'}(a) \leq d^{++}_{T'}(a) = d^{++}_{(}a)$, hence $a$ has the SNP in $D$. Otherwise, $d^+_{T'}(a) = |G_L|$ and $G_L = N^{++}_{T'}(a)$. So $x, y$ and $z$ are not bad vertices, hence the index of each increases in the median order $Sed(L)$ of $T'$. But the index of $a$ is less than the index of $x$, then we can give $ax$ its initial orientation as in $T$ and the same order $Sed(L)$ is a median order of $T$. However, the sum of indices of $x, y$ and $z$ have increased. A contradiction. Thus $f$ has the SNP in $D$ and $D$ satisfies EC. □

**Theorem 7.4.2.** *Let $D$ be a digraph obtained from a tournament by deleting the edges of three disjoint stars whose centers form a directed triangle. If $\delta^+_\Delta > 0$ and $\delta^-_\Delta > 0$ and $D$ does not have any sink, then it has at least two vertices with SNP.*

*Proof.* Due to the condition $\delta^+_\Delta > 0$ and $\delta^-_\Delta > 0$ and the fact that the only possible arcs in $\Delta$ have the forms $xa \rightarrow yb$ or $yb \rightarrow zc$ or $zc \rightarrow xa$, where $a \in A$, $b \in B$ and $c \in C$, the following holds: For every $a \in A$, $b \in B$ and $c \in C$ we have:

$b \to x \to c \to y \to a \to z \to b.$ .

First we show that $D[K]$ has at least three vertices with the SNP. Let $H = D - \{x, y, z\}$. $H$ is a tournament with no sink (dominated vertex). Then $H$ has two vertices $u$ and $v$ with SNP in $H$. Without loss of generality we may assume that $u \in A$. But $y \to u \to z$, then adding the vertices $x, y$ and $z$ makes $u$ gains only one vertex to its first out-neighborhood and $x$ to its second out-neighborhood. Thus, also $u$ has the SNP in $D[K]$. Similarly, $v$ has the SNP in $D[K]$. Suppose, without loss of generality, that $|A| \geq |C|$. We have $C \cup \{y\} = N^+(x)$ and $A \cup \{z\} = N^{++}(x)$. Hence, $d^+(x) = |C| + 1 \leq |A| + 1 \leq d^{++}(x)$, whence, $x$ has the SNP in $D$.

Next, we show that $D$ is a good digraph. Let $\mathcal{I}_\mathcal{D}$ be the interval graph of $D$. Let $C_1$ and $C_2$ be two distinct connected components of $\Delta$. The three centers of the missing disjoint stars appear in each of the these two connected components, whence $K(C_1) \cap K(C_2) \neq \phi$. Therefore, $\mathcal{I}_\mathcal{D}$ is a complete graph, having only one connected component $\xi$. Then, $K = K(\xi)$. So if $\Delta$ is composed of non-trivial strongly connected components, then the result holds by Lemma 7.1.1. Due to the condition $\delta^+_\Delta > 0$ and $\delta^-_\Delta > 0$, $\Delta$ has a non-trivial strongly connected component $C$. Since $x$, $y$ and $z$ appear in $C$ we have $N^+(x)\backslash K = N^+(y)\backslash K = N^+(z)\backslash K$. Now let $v \in K$. If $v$ appears in a non-trivial strongly connected component of $\Delta$, then $N^+(v)\backslash K = N^+(x)\backslash K = N^+(y)\backslash K = N^+(z)\backslash K$. Otherwise, due to the condition $\delta^+_\Delta > 0$ and $\delta^-_\Delta > 0$, $v$ appears in a directed path $P$ of $\Delta$ joining two non-trivial strongly connected components of $\Delta$. By the definition of losing relations, we can prove easily that for all $a \in K(C_1)$, $b \in K(P)$ and $c \in K(C_2)$ we have $N^+(a)\backslash K(\xi) \subseteq N^+(b)\backslash K(\xi) \subseteq N^+(c)\backslash K(\xi)$. In particular, for $a = x = c$ and $b = v$, So the result follows.

To conclude, we apply the same argument of the proof of Theorem 7.3.2. □

Now we will deal with the case when the triangle induced by the vertices $x$, $y$ and $z$ is a transitive triangle and assume without loss of generality that $x \to y \to z \leftarrow x$. An *alternating path* in $\Delta$ is a directed path of length two $u \to v \to w$ where $\{u, v, w\} = \{xa, yb, zc\}$ for some $a \in A$, $b \in B$ and $c \in C$.

**Theorem 7.4.3.** *Let $D$ be a digraph obtained from a tournament by deleting the edges of three disjoint stars whose centers form a transitive triangle. If $\delta^+_\Delta > 0$ and $\delta^-_\Delta > 0$ then $D$ satisfies EC.*

*Proof.* Observe that the only possible alternating paths in $\Delta$ have the forms:
$xa \rightarrow zc \rightarrow yb$, $yb \rightarrow xa \rightarrow zc$ and $zc \rightarrow yb \rightarrow xa$. Indeed, suppose there is an alternating path of the form $xa \rightarrow yb \rightarrow zc$. Then $b \notin N^+(x) \cup N^{++}(x)$. But $x \rightarrow z \rightarrow b$, a contradiction. Suppose there is a path of the form $zc \rightarrow xa \rightarrow yb$. Since $\delta^+_\Delta > 0$ and the previous path is forbidden, there is $a' \in A$ such that $yb$ loses to $xa'$, whence $x \notin N^{++}(y)$. However, $y \rightarrow a$ and $a \notin N^+(c) \cup N^{++}(c)$, then $y \rightarrow c$, but $c \rightarrow x$, hence $x \in N^{++}(y)$, a contradiction. Suppose there is a path of the form $yb \rightarrow zc \rightarrow xa$. Since $\delta^+_\Delta > 0$ and the previous path is forbidden, there is $c' \in C$ such that $xa$ loses to $zc'$, whence $z \notin N^+(a) \cup N^{++}(a)$. Since $a \rightarrow c$ and $c \notin N^+(y) \cup N^{++}(y)$, then $a \rightarrow y$, whence $a \rightarrow y \rightarrow z$, a contradiction.

Remark that $\Delta$ has no directed cycle $P$ such that for all $b \in B$, $yb \notin P$. Indeed suppose that there is such a directed cycle. Then it can be easily showed, as before, that $N^+(x)\backslash(A\cup C) = N^+(y)\backslash(A\cup C)$. But $x \rightarrow y \rightarrow z$, a contradiction.

Orient the missing edges toward the centers. The obtained digraph is a tournament $T$. Let $L$ be a median order of $T$ and let $f$ denote its feed vertex. If $f$ is a whole vertex or $f = x$, then it is easy to show that $d^+(f) = d^+_T(f) \leq d^{++}_T(f) = d^{++}(f)$. Suppose $f = a \in A$. Let $q \in V(D)-\{y,z\}$ with $a \rightarrow x \rightarrow q \rightarrow a$. Since $\delta^+_\Delta > 0$ there is a missing edge $uv$ such that $ax$ loses to it, with $x \rightarrow v$ and $u \notin N^+(x) \cup N^{++}(x)$. Hence, $u \rightarrow q$ and $q \in N^{++}(a)$. So $N^{++}_T(a) - \{y,z\} \subseteq N^{++}(a) - \{x\}$ and $N^+_T(a) = N^+(a) \cup \{x\}$, whence, $d^+(a) + 1 = d^+_T(a) \leq d^{++}_T(a) \leq d^{++}(a) + 1$. So $a$ has SNP in $D$. Suppose $f = b \in B$. We first show that $x \notin N^{++}_T(b) - (N^{++}(b) \cup N^+(b))$. Otherwise, there is $a \in A$ such that $b \rightarrow a \rightarrow x$ in $T$. Since $\delta^-_\Delta > 0$ there is a missing edge $cz$ that loses to $ax$. Since $b \rightarrow a$ and $a \notin N^{++}(c)$ then $b \rightarrow c$, whence $b \rightarrow c \rightarrow x$, a contradiction. Using the argument of the previous case, we can prove that $N^{++}_T(b) - \{z\} \subseteq N^{++}(b) - y$. However, $N^+_T(b) = N^+(b) \cup \{y\}$. Therefore, $d^+(b) + 1 = d^+_T(b) \leq d^{++}_T(b) \leq d^{++}(b)$. Thus $b$ has the SNP in $D$. Suppose $f = c \in C$. Using the same argument of the case $f = b$, we show that $x$ and $y$ are not in $N_T^{++}(c) - N^+(c) \cup N^{++}(c)$. Hence, $d^+(c)+1 = d_T^+(c) \leq d_T^{++}(c) \leq d^{++}(c)$ and $c$ has the SNP in $D$. Finally, suppose that $f = z$. Assume that $x \notin N_T^{++}(z) - N^{++}(z)$. Then $y$ is the only possible new second out-neighbor of $z$. If $d_T^+(z) < |G_L|$, then $d^+(z) = d_T^+(z) \leq d_T^{++}(z) - 1 \leq d^{++}(z)$. Otherwise, $d_T^+(z) = |G_L|$, whence the feed vertex, which is not $z$, of the median order $Sed(L)$ of $T$ has the SNP in $D$. Now assume that $x \in N_T^{++}(z) - N^{++}(z)$. If $\Delta$ has an alternating path, then $x \in N^{++}(z)$, a contradiction. So $\Delta$ has no alternating path. There is $a \in A$ such that $z \rightarrow a \rightarrow x$ in $T$, because $x \notin N^{++}(z)$. Let $Q$ be a connected component of $\Delta$ containing $xa$. If for every $c \in C$, $(cz, ax) \notin E(\Delta)$, then there is $b \in B$ such that $(by, ax) \in E(\Delta)$. But $z \rightarrow a$ and $a \notin N^+(b) \cup N^{++}(b)$, then $z \rightarrow b$, hence $z \rightarrow b \rightarrow x$, a contradiction. So there is $c \in C$ such that $cz \in Q$. There is $b \in B$ such that $yb \in Q$, since otherwise $Q$ must contain a directed cycle $P$ such that for all $b \in B$, $yb \notin P$. Hence, $Q$ must contain an alternating path. Whence $x \in N^{++}(z)$, a contradiction. □

**Corollary 7.4.1.** *Let $D$ be a digraph obtained from a tournament by deleting the edges of three disjoint stars. If $\delta_\Delta^+ > 0$ and $\delta_\Delta^- > 0$, then $D$ satisfies EC.*

# Bibliography


[1] A. El Sahili, *Trees in tournaments*. J. Combin. Theory Ser. B 92 (2004), no. 1, 183–187.

[2] B. Roy, Nombre chromatique et plus longs chemins d'un graphe, Rev Française Automat. Informat. Recherche Opérationelle Sér. Rouge 1 (1967), 127-132.

[3] D. Fidler and R. Yuster, *Remarks on the second neighborhood problem*, J. Graph Theory 55 (2007), 208-220.

[4] D. Fisher, *Squaring a tournament: a proof of Dean's conjecture*, J. Graph Theory 23 (1996), 43-48.

[5] D. Kühn, R. Mycroft, and D. Osthus, *A proof of Sumner's universal tournament conjecture for large tournaments* Proc. London Math. Soc. (2010) pdq035 first published online November 12, 2010 doi:10.1112/plms/pdq035

[6] D. Solow, *Linear programming: An introduction to finite improvement algorithms*. North-Holland, Amsterdam (1984).

[7] F. Havet and S. Thomassé, *Median Orders of Tournaments: A Tool for the Second Neighborhood Problem and Sumner's Conjecture*, J. Graph Theory 35 (2000), 244-256.

[8] G. Chen, J. Shen, R. Yuster, *Second Neighborhood via First Neighborhood*, Annals of Combinatorics 7 (2003), 15-20.

[9] J. Hladký, D. Král, and S. Norin, Counting flags in triangle-free digraphs, Electronic Notes in Discrete Mathematics, 34 (2009), pp. 621-625.

[10] J. Shmerel, W. Trotter, *Critically indecomposable partailly ordered sets, graphs, tournaments and other binary relational structures*, Discrete Mathematics 113 (1993), 191-205.

[11] L. Caccetta, R. Häggkvist: *On minimal digraphs with given girth.* Proc. 9th S-E Conf. Combinatorics, Graph Theory and Computing 181-187 (1978)

[12] N. Alon, Ranking tournaments, SIAM J. Discrete Math. 20 (2006), 137-142.

[13] N.C. Wormald, *Subtrees of large tournaments*, in: Combinatorial Mathematics, X (Adelaide, 1982). Lecture Notes in Mathematics, Vol. 1036, Springer, Berlin, 1983, pp. 417-419.

[14] N. Dean and B. J. Latka, *Squaring the tournament: an open problem*, Congress Numerantium 109 (1995), 73-80.

[15] N. Lichiardopol, *A new bound for a particular case of the Caccetta-Hggkvist conjecture*, Discrete Mathematics, 23 (2010), 3368-3372.

[16] P. Erdös, Graph theory and probability, Canad. J. Math. 11 (1959), 34-38.

[17] R. Karp, Reducibility among combinatorial problems, Proc. Sympos. IBM Thomas J. Watson Res. Center, Yorktown Heights, N.Y., 85-103, 1972.

[18] S.A. Burr, *Subtrees of directed graphs and hypergraphs*, Proceedings of the Eleventh Southeastern Conference on Combinatorics, Graph Theory and Computing, Boca Raton, Congr. Numer., 28 (1980), 227-239.

[19] T. Gallai, On directed paths and circuits, Theory of Graphs, P.Erdös and G.Katona (Editors), Academic press, New York, 1968, pp. 115-118.

[20] Y. Kaneko and S.C. Locke, *The minimum degree approach for Paul Seymour's distance 2 conjecture*, Congressus Numerantium 148 (2001), 201-206.