Tamely ramified geometric Howe correspondence for dual reductive pairs of type II in terms of geometric Langlands program

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La correspondance de Howe géométrique modérément ramifiée pour les paires duales de type II dans le cadre du programme de Langlands géométrique

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À mon ange, mon soleil
ma sangeh sabour.

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Résumé

La correspondance de Howe géométrique modérément ramifiée pour les paires duales de type II dans le cadre du programme de Langlands géométrique

Dans cette thèse on s'intéresse à la correspondance de Howe géométrique pour les paires duales réductives de type II ($G = \mathbf{GL}_n$, $H = \mathbf{GL}_m$) sur un corps local non-Archimédien F de caractéristique différente de 2, ainsi qu'à la fonctorialité de Langlands géométrique au niveau Iwahori. Notons $\mathcal{S}$ la représentation de Weil de $G(F) \times H(F)$ et I_H, I_G des sous groupes d'Iwahori de $H(F)$ et $G(F)$. On considère la version géométrique de la représentation $\mathcal{S}^{I_G \times I_H}$ des algèbres de Hecke-Iwahori $\mathcal{H}_{I_H}$ et $\mathcal{H}_{I_G}$ sur laquelle agissent les foncteurs de Hecke. On obtient des résultats partiels sur la description géométrique de la catégorie correspondante.

Nous proposons une conjecture décrivant le groupe de Grothendieck de cette catégorie comme module sur les algèbres de Hecke affines étendues de G et de H. Notre description est en termes d'un champ attaché aux groupes de Langlands duaux dans le style de l'isomorphisme de Kazhdan-Lusztig. On démontre cette conjecture pour toutes les paires ($\mathbf{GL}_1$, $\mathbf{GL}_m$).

Plus généralement, étant donné deux groupes réductifs connexes G et H et un morphisme $\check{G} \times \mathrm{SL}_2 \to \check{H}$ de groupes de Langlands duaux, on suggère un bimodule sur les algèbres de Hecke affines étendues de G et de H qui pourrait conjecturalement réaliser la fonctorialité de Langlands géométrique locale au niveau Iwahori.

Classification MSC : 22E57, 14D24, 32S60, 16G60, 14D23, 19L47, 14L30.

Mots Clefs : Programme de Langlands géométrique, correspondance de Howe, algèbre de Hecke-Iwahori, variété de Steinberg, faisceaux pervers, variété de drapeaux affine.

Abstract

Tamely ramified Howe correspondence for dual reductive pairs of type II in terms of geometric Langlands program

In this thesis we are interested in the geometric Howe correspondence for the dual reductive pair of type II ($G = \mathbf{GL}_n, H = \mathbf{GL}_m$) over a non-Archimedean field F of characteristic different from 2 as well as in the geometric Langlands functoriality at the Iwahori level. Let $\mathcal{S}$ be the Weil representation of $G(F) \times H(F)$ and I_H, I_G be Iwahori subgroups in $H(F)$ and $G(F)$. We consider the geometric version of the representation $\mathcal{S}^{I_H \times I_G}$ of the Iwahori-Hecke algebras $\mathcal{H}_{I_H}$, $\mathcal{H}_{I_G}$, it is acted on by Hecke functors. We obtain some partial results on the geometric description of the corresponding category.

We propose a conjecture describing the Grothendieck group this category as a module over the affine Hecke algebras for G and H. Our description is in terms of some stack attached to the Langlands dual groups in the style of Kazhdan-Lusztig isomorphism. We prove our conjecture for the pair $(\mathbf{GL}_1, \mathbf{GL}_m)$ for any m.

More generally, given tow reductive connected groups G, H and a morphism $\check{G} \times \mathrm{SL}_2 \to \check{H}$ of Langlands dual groups, we suggest a bimodule over the affine Hecke algebras for H and G that should realize the local geometric Langlands functoriality on the Iwahori level.

MSC classification: 22E57, 14D24, 32S60, 16G60, 14D23, 19L47, 14L30.

Key words: Geometric Langlands program, Howe correspondence, Iwahori-Hecke algebra, Steinberg variety, perverse sheaves, affine flag variety.

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Motivation générale

Le travail dans cette thèse s'inscrit dans le cadre de la théorie géométrique des représentations. Plus précisément, le but principal est d'étudier la correspondance de Howe pour les paires réductives duales de type II et la fonctorialité de Langlands au niveau Iwahori et ses liens avec le programme de Langlands géométrique.

Correspondance de Howe

On fixe un corps fini $\mathbf{k} = \mathbb{F}_q$ de caractéristique différente de 2, on se place sur $F = \mathbf{k}((t))$ et on note $\mathcal{O} = \mathbf{k}[[t]]$ l'anneau des entiers de F. Toutes les représentations considérées sont supposées lisses et sont définies sur $\overline{\mathbb{Q}}_\ell$, où ℓ est un nombre premier différent de la caractéristique de $\mathbf{k}$.

Les notions de base de la correspondance de Howe du point de vue classique ont été présentées dans [MVW87]. Soit $(G(F), H(F))$ une paire réductive duale dans un certain groupe symplectique $Sp(W)(F)$, ψ un caractère additif de F et $\widetilde{Sp}(W)(F)$ le groupe métaplectique qui est un revêtement à deux feuillets du groupe symplectique $Sp(W)(F)$. On dispose d'une représentation lisse irréductible de dimension infinie de $\widetilde{Sp}(W)(F)$ attachée au caractère ψ appelée représentation de Weil. On suppose que le revêtement $\widetilde{Sp}(W)(F) \to Sp(W)(F)$ admet une section au dessus de $G(F)$ et $H(F)$, ce qui est toujours vérifié pour les paires réductives duales de type II auxquelles nous nous intéressons. La correspondance de Howe devient alors une correspondance entre deux familles de représentations de $G(F)$ et $H(F)$.

Il est du plus grand intérêt de connaître la correspondance de Howe de manière explicite et de savoir si cette correspondance entraîne la fonctorialité de Arthur-Langlands. Le cas des représentations cuspidales a été très tôt résolu par Kudla dans [Kud86], mais le cas des représentations générales reste toujours ouvert. Mœglin a donné dans [Mœg89] une description de la dualité de Howe pour certains groupes orthogonaux et symplectiques dans le cas Archimédien en termes de paramètres de Langlands-Vogan. Il semble que si les groupes ont des rangs similaires et si l'on se restreint aux représentations tempérées, la correspondance de Howe respecte la fonctorialité de Langlands. D'une certaine manière, ces hypothèses fournissent les cas les plus simples de la fonctorialité aussi bien sur un corps Archimédien que non-Archimédien. Hormis le cas des groupes **GL** résolu par Mínguez [Mín08], les paramètres de Langlands ne peuvent être conservés dans cette correspondance sauf si les rangs sont quasi égaux. Une explication possible pour ce phénomène est que les paquets de Langlands ne sont pas adaptés ; et il faudrait considérer plutôt les paquets plus larges dont l'existence a été conjecturée par Arthur dans [Art84]. En utilisant le formalisme d'Arthur, Adams [Ada89] a suggéré une formule générale.

D'après les conjectures d'Arthur et Adams, si $\check{G}$ (resp. $\check{H}$) est le groupe dual de Langlands de G (resp. H) sur $\overline{\mathbb{Q}}_\ell$, l'homomorphisme $\kappa : \check{G} \to \check{H}$ devrait être remplacé par un homomorphisme σ de $\check{G} \times \mathrm{SL}_2$ dans $\check{H}$; il faut donc prendre en considération les paquets d'Arthur au lieu des L-paquets usuels. Sous certaines hypothèses, on s'attend à ce que si π est une représentation irréductible de $G(F)$ apparaissant dans la représentation de Weil et si $\pi^{'}$ est la représentation de $H(F)$ associé à la représentation π sous la correspondance de Howe, alors la représentation $\pi^{'}$ appartiendrait à l'image du paquet d'Arthur de π sous le morphisme σ. Pour plus de détails sur ce sujet nous renvoyons le lecteur aux articles suivants [Mœg89],[Mœg09a],[Mœg09b], [Ral82], [Kud86], [Art84].

On s'attend depuis le départ à ce que cette formule d'Adams couvre des cas où les groupes sont de rang très différent. Cependant, comme annoncé par Adams lui même, cette conjecture ne s'applique pas en toutes circonstances. La première difficulté est qu'une représentation quelconque d'un des groupes considérés n'a pas de raison d'être dans un paquet d'Arthur ; et qu'il n'est pas vrai en général que si $(\pi, \pi^{'})$ est dans l'image de la correspondance de Howe et si l'une des représentations est dans un paquet d'Arthur, alors l'autre l'est aussi. L'autre difficulté est qu'une représentation π fixée peut figurer dans plusieurs paquets d'Arthur et la formule d'Adams peut s'appliquer pour certains paquets contenant π mais pas pour tous.

Dans la suite de cette partie on se restreint aux paires réductives duales de type II. Plus précisément, soit L_0 (resp. U_0) un $\mathbf{k}$ espace vectoriel de dimension n (resp. m) avec $n \leq m$, $G = \mathbf{GL}(L_0)$ et $H = \mathbf{GL}(U_0)$. Notons $\Pi(F) = L_0 \otimes U_0(F)$ et $\mathcal{S}(\Pi(F))$ l'espace de Schwartz des fonctions localement constantes à support compact sur $\Pi(F)$ à valeur dans $\overline{\mathbb{Q}}_\ell$. Cet espace réalise la restriction de la représentation de Weil à $G(F) \times H(F)$. D'après les travaux de Mínguez [Mín08] on sait que la correspondance de Howe associe à une représentation lisse irréductible π de $G(F)$ une unique représentation non nulle lisse irréductible de $H(F)$, notée $\theta_{n,m}(\pi)$, telle que $\pi \otimes \theta_{n,m}(\pi)$ soit un quotient de la restriction de la représentation de Weil à $G(F) \times H(F)$. De plus, cette correspondance est décrite explicitement en termes de paramètres de Langlands.

Différentes classes de représentations peuvent être considérées pour l'étude explicite de la correspondance de Howe. On dit qu'une représentation de $G(F)$ est non ramifiée si elle admet un vecteur $G(\mathcal{O})$-invariant non nul. Une première possibilité est de considérer la catégorie des représentations non-ramifiées. Dans ce cas la correspondance de Howe se traduit en termes de modules sous l'action des algèbres de Hecke sphériques. On est alors ramené à étudier la structure de module de $\mathcal{S}^{G(\mathcal{O})\times H(\mathcal{O})}(\Pi(F))$ sous l'action des algèbres de Hecke sphériques de G et H. D'après les travaux non publiés de Howe (cf. [Mín06, Appendice]) on sait que $\mathcal{S}^{G(\mathcal{O})\times H(\mathcal{O})}(\Pi(F))$ est engendré par une certaine fonction s_0 sous l'action de l'algèbre de Hecke sphérique $\mathcal{H}_{G(\mathcal{O})}$ de G et l'algèbre de Hecke sphérique $\mathcal{H}_{H(\mathcal{O})}$ de H. Rappelons que d'après l'isomorphisme de Satake, l'algèbre de Hecke sphérique $\mathcal{H}_{G(\mathcal{O})}$ est isomorphe au groupe de Grothendieck $K(\mathrm{Rep}(\check{G}))$ de la catégorie des représentations de $\check{G}$ sur $\overline{\mathbb{Q}}_\ell$. Dans le cadre de la fonctorialité de Langlands non-ramifiée au niveau classique, on dispose du théorème suivant démontré par Rallis [Ral82] : le morphisme de $\mathcal{H}_{G(\mathcal{O})}$ dans $\mathcal{S}^{G(\mathcal{O})\times H(\mathcal{O})}(\Pi(F))$ envoyant h sur hs_0 est un isomorphisme de $\mathcal{H}_{G(\mathcal{O})}$-module et il existe un homomorphisme $\kappa : \mathcal{H}_{H(\mathcal{O})} \longrightarrow \mathcal{H}_{G(\mathcal{O})}$ tel que l'action de $\mathcal{H}_{H(\mathcal{O})}$ sur $\mathcal{S}^{G(\mathcal{O})\times H(\mathcal{O})}(\Pi(F))$ se factorise par κ. Si $n = m$ l'homomorphisme κ provient du foncteur de restriction $\mathrm{Rep}(\check{H}) \longrightarrow \mathrm{Rep}(\check{G})$ correspondant à l'isomorphisme entre $\check{G}$ et $\check{H}$. Si $n \leq m$, l'homomorphisme κ provient du foncteur de restriction $\mathrm{Rep}(\check{H}) \longrightarrow \mathrm{Rep}(\check{G} \times \mathbb{G}_{\mathfrak{m}})$ correspondant à l'homomorphisme $\check{G} \times \mathbb{G}_{\mathfrak{m}} \longrightarrow \check{H}$. Quand $m > n$, la restriction de cet homomorphisme à $\mathbb{G}_{\mathfrak{m}}$ est non triviale. Remarquons qu'ici $\mathbb{G}_{\mathfrak{m}}$ est le tore maximal du SL_2 d'Arhtur. La version géométrique de ce théorème a été démontrée par Lysenko [Lys11, Proposition 4].

La seconde classe de représentations intéressante à considérer pour l'étude de la correspondance de Howe est la classe des représentations modérément ramifiées. Une représentation de $G(F)$ est appelée modérément ramifiée si l'espace des invariants sous l'action d'un sous groupe d'Iwahori I_G de $G(F)$ est non nul. La catégorie des représentations modérément ramifiées est une sous catégorie pleine de la catégorie des représentations lisses de longueur finie dont tout sous quotient irréductible est modérément ramifié. Considérons le foncteur qui envoie toute représentation modérément ramifiée V de G sur son espace V^{I_G} des invariants sous l'action de I_G. Ce dernier est naturellement un module sous l'algèbre de Hecke-Iwahori $\mathcal{H}_{I_G}$ de G. D'après [Bor76, Théorème 4.10] ce foncteur est une équivalence de catégorie entre la catégorie des représentations modérément ramifiées de G et la catégorie des $\mathcal{H}_{I_G}$-modules de dimension finie. De plus, il est exact sur la catégorie des représentations lisses d'après [IM]. Par conséquent on peut interpréter la correspondance de Howe en termes de modules sous l'action des algèbres de Hecke-Iwahori. L'espace $\mathcal{S}^{I_H \times I_G}(\Pi(F))$ des $I_H \times I_G$-invariants dans l'espace des fonctions de Schwartz $\mathcal{S}(\Pi(F))$ est naturellement un module sous l'action des algèbres de Hecke-Iwahori de G et H. On souhaiterait comprendre cette structure de module à l'aide d'outils géométriques. La géométrisation de la correspondance Howe (et plus généralement de la correspondance theta) a été introduite dans [LL09] et [Lys11] où les auteurs étudient le cas non ramifié pour les paires duales $(\mathrm{Sp}_{2n}, \mathrm{S}\mathbb{O}_{2m})$ et $(\mathbf{GL}_n, \mathbf{GL}_m)$. Dans la continuation de ces deux articles on développe dans cette thèse la correspondance de Howe modérément ramifiée géométrique pour la paire $(\mathbf{GL}_n, \mathbf{GL}_m)$ et on étend les résultats de [Lys11] dans le cas modérément ramifié.

Programme de Langlands géométrique

Une série de conjectures sur le programme de Langlands géométrique (en particulier au niveau Iwahori) a été formulé dans [FG06], plus récemment dans des travaux non publiés de Bezrukavnikov, V. Lafforgue et dans [E.12]. Ces conjectures ont motivé la construction du bimodule réalisant la fonctorialité de Langlands géométrique dans cette thèse. On rappelle brièvement cette motivation. On se place sur un corps algébriquement clos $\mathbf{k}$ de caractéristique p et on note $D^* = \mathrm{Spec}\mathbf{k}((t))$. On considère un groupe réductif connexe G sur $\mathbf{k}$. Frenkel et Gaitstgory conjecturent l'existence d'une catégorie notée $\mathcal{C}_G$ au dessus du champ $LS_{\check{G}}(D^*)$ des $\check{G}$-systèmes locaux sur D^* munie d'une action de $G(F)$. La construction de cette catégorie dans le cas général semble hors de portée mais le cas Iwahori est plus accessible. Notons $\mathcal{N}_{\check{G}}$ le cône nilpotent de $\check{G}$ et $\tilde{\mathcal{N}}_{\check{G}}$ la résolution de Springer de $\mathcal{N}_{\check{G}}$. Un $\check{G}$-système local est dit modérément ramifié s'il admet une singularité régulière à l'origine avec monodromie unipotente. On rappelle que le champ quotient $\mathcal{N}_{\check{G}}/\check{G}$ peut être vu comme le champ des $\check{G}$-systèmes locaux modérément ramifiés. Notons $\mathcal{C}_{\check{G},nilp}$ la catégorie obtenue à partir de $\mathcal{C}_{\check{G}}$ par le changement de la base $\mathcal{N}_{\check{G}}/\check{G} \longrightarrow LS_{\check{G}}(D^*)$. Si on note I_G un sous groupe d'Iwahori de $G(F)$, alors Frenkel et Gaitsgory [FG06, (0.20)] conjecturent l'existence d'un isomorphisme

$$K(\mathcal{C}^{I_G}_{\check{G},nilp}) \xrightarrow{\sim} K(\tilde{\mathcal{N}}_{\check{G}}/\check{G}) \tag{0.0.1}$$

où le membre de gauche est le groupe de Grothendieck des I_G-invariants dans la catégorie $\mathcal{C}_{\check{G},nilp}$ et celui de droite est le groupe de Grothendieck de la catégorie des faisceaux cohérents sur le champ $\tilde{\mathcal{N}}_{\check{G}}/\check{G}$. De plus, cet isomorphisme devrait être compatible avec l'action de l'algèbre de Hecke affine de G.

Dans l'esprit des ces conjectures, considérons les groupes $G = \mathbf{GL}_n$ et $H = \mathbf{GL}_m$. Nous conjecturons l'existence d'un morphisme $\sigma : \check{G} \times \mathbb{G}_{\mathfrak{m}} \longrightarrow \check{H}$ et d'un isomorphisme (après spécialisation et en utilisant l'isomorphisme de Kazhdan-Lusztig) entre $K(DP_{I_H \times I_G}(\Pi(F)))$ et $K(\mathcal{X}) = K((\tilde{\mathcal{N}}_{\check{G}}/(\check{G} \times \mathbb{G}_m)) \times_{\mathcal{N}_{\check{H}}/(\check{H} \times \mathbb{G}_m)} (\tilde{\mathcal{N}}_{\check{H}}/(\check{H} \times \mathbb{G}_m)))$, où $K(DP_{I_H \times I_G}(\Pi(F)))$ est le bimodule sous l'action des deux algèbres de Hecke affines étendues de G et H (muni d'une action de $\mathbb{G}_{\mathfrak{m}}$ par décalage cohomologique). Les objets de la catégorie $DP_{I_H \times I_G}(\Pi(F))$ sont des sommes directes des objets de $P_{I_H \times I_G}(\Pi(F))$ décalés cohomologiquement, ces derniers étant les faisceaux pervers $I_H \times I_G$-équivariants sur $\Pi(F)$. On démontre cette conjecture dans le cas $n = 1$ et $m > 1$. Ceci implique que le bimodule réalisant la correspondance de Howe provient du bimodule réalisant la fonctorialité de Langlands géométrique pour le morphisme σ. Il est important de remarquer que ce théorème dans le cas $n = 1$ et $m > 1$ est nouveau même dans le cadre classique sur un corps fini. La description explicite de Mínguez pourrait être améliorée afin de donner une description du bimodule lui-même en termes du champ $\mathcal{X}$ attaché au morphisme $\sigma : \check{G} \times \mathrm{SL}_2 \to \check{H}$. Ceci fournit de nouvelles perspectives pour les autres paires duales. Particulièrement, il serait très intéressant d'obtenir un résultat similaire pour la paire duale $(\mathrm{Sp}_{2n}, \mathrm{S}\mathbb{O}_{2m})$ ainsi que de donner une approche conceptuelle différente des calculs effectués dans [Aub91]. Un autre développement possible serait de donner une description de la catégorie dérivée $D_{I_G \times I_H}(\Pi(F))$ en termes de catégorie dérivée des faisceaux cohérents $D(\mathcal{X})$ sur le champ $\mathcal{X}$ dans le style de la théorie de Bezrukavnikov [AB09].

Esquisse des résultats

Dans toute la section on fixe $\mathbf{k}$ un corps algébriquement clos de caractéristique différent de 2 et on pose $F = \mathbf{k}((t))$. On note $\mathcal{O}$ l'anneau des entiers $\mathbf{k}[[t]]$ de F. D'après la philosophie de Grothendieck l'équivalent géométrique de la notion de fonction sur les $\mathbf{k}$-points d'une variété V est la notion de complexe de faisceaux ℓ-adiques sur V. On peut associer à tout complexe de faisceaux ℓ-adiques $\mathcal{K}$ une fonction dont la valeur en un point fermé $\overline{x}$ est la somme alternée des traces du Frobenius $Fr_{\overline{x}}$ opérant sur les faisceaux de cohomologie locale $\mathcal{H}^i(\mathcal{K}_{\overline{x}})$. Ce foncteur échange les opérations naturelles sur les complexes de faisceaux avec celles sur les fonctions. Il est préférable de travailler avec la catégorie des faisceaux ℓ-adiques pervers qui est en particulier stable par la dualité de Verdier. Par exemple la version géométrique de l'algèbre de Hecke-Iwahori de G est la catégorie $P_{I_G}(\mathcal{F}l_G)$ des faisceaux pervers I_G-équivariants sur la variété affine de drapeaux $\mathcal{F}l_G$. Suivant cette idée, la version géométrique de l'espace des invariants par $I_H \times I_G$ de l'espace de Schwartz des fonctions sur $\Pi(F)$ est la catégorie des faisceaux pervers $I_H \times I_G$-équivariants sur $\Pi(F)$. Au niveau des fonctions, l'espace $\mathcal{S}^{I_H \times I_G}(\Pi(F))$ est naturellement un module sous l'action de l'algèbre de Hecke-Iwahori de G (resp. H). Cette action par convolution se géométrise et définit une action de la catégorie $P_{I_G}(\mathcal{F}l_G)$ (resp. $P_{I_H}(\mathcal{F}l_H)$) sur $P_{I_H \times I_G}(\Pi(F))$ (de façon un peu plus générale une action sur la catégorie $D_{I_G \times I_H}(\Pi(F))$).

On définit une stratification de l'espace $\Pi(F)$ par les $H(\mathcal{O}) \times I_G$-orbites et on décrit les objets irréductibles de la catégorie $P_{H(\mathcal{O}) \times I_G}(\Pi(F))$ des faisceaux pervers $H(\mathcal{O}) \times I_G$-équivariants sur $\Pi(F)$ qui sont indexés par le réseau des cocaractères de G. Notons Gr_G la Grassmannienne affine $G(F)/G(\mathcal{O})$. Le premier résultat de cette thèse affirme qu'il existe un isomorphisme entre le groupe de Grothendieck de la catégorie $P_{H(\mathcal{O}) \times I_G}(\Pi(F))$ et le groupe de Grothendieck de la catégorie $P_{I_G}(Gr_G)$ munie d'une action à gauche de $K(P_{I_G}(\mathcal{F}l_G))$ et une action de la catégorie $\mathrm{Rep}(\check{H})$ via le foncteur de restriction $\mathrm{Rep}(\check{H}) \longrightarrow \mathrm{Rep}(\check{G} \times \mathbb{G}_\mathfrak{m})$ correspondant à un homomorphisme $\check{G} \times \mathbb{G}_\mathfrak{m} \longrightarrow \check{H}$.

Le second résultat consiste à décrire les éléments irréductibles de la catégorie $P_{I_H \times I_G}(\Pi(F))$ qui sont indexés par $X_G \times S_{n,m}$ où X_G est le réseau des cocaractères de G et $S_{n,m}$ est un ensemble fini. Quand $n = m$, on calcule l'action des objets irréductibles de $P_{I_G}(\mathcal{F}l_G)$ sur un objet spécifique, et on construit une équivalence de catégorie entre $P_{I_G}(\mathcal{F}l_G)$ et $P_{I_H}(\mathcal{F}l_H)$ qui nous permet d'échanger l'action de $P_{I_G}(\mathcal{F}l_G)$ et $P_{I_H}(\mathcal{F}l_H)$. Ce foncteur réalisant l'équivalence de catégories provient d'une anti-involution du groupe $G(F)$ sur lui même. Parmi les résultats partiels obtenus, pour tout $n \leq m$, on définit une graduation sur $P_{I_G}(\mathcal{F}l_G)$ qui est compatible avec le produit de convolution sur $P_{I_G}(\mathcal{F}l_G)$. De plus, on définit une filtration sur $P_{I_H \times I_G}(\Pi(F))$ qui est compatible avec cette graduation et par conséquent avec l'action de $P_{I_G}(\mathcal{F}l_G)$. Ceci nous permet de contrôler l'action de $P_{I_G}(\mathcal{F}l_G)$ sur $P_{I_H \times I_G}(\Pi(F))$.

Quand $n < m$, le calcul de l'action géométrique des éléments irréductibles de $P_{I_G}(\mathcal{F}l_G)$ sur les objets irréductibles de la catégorie $P_{I_H \times I_G}(\Pi(F))$ semble difficile, toutefois on peut calculer cette action sur des objets plus simples (les faisceaux obtenus par extension par zéro) qui se traduit au niveau classique de façon suivante. On construit un sous module Θ dans $K(D_{I_G \times I_H}(\Pi(F)))$ et on montre que c'est un module libre de rang C_m^n sous l'action de l'algèbre de Hecke-Iwahori $\mathcal{H}_{I_G}$ de G en construisant une base explicite de ce module. On définit entièrement l'action de l'algèbre de Hecke-Iwahori $\mathcal{H}_{I_H}$ de H sur cette base et on obtient une description complète de la structure de module de Θ.

Le troisième résultat est de montrer que le bimodule Θ est isomorphe à une induite parabolique d'une sous algèbre parabolique de $\mathcal{H}_{I_H}$. Pour simplifier, la réponse est rédigée plutôt au niveau des groupes de Grothendieck même si la plupart de nos calculs sont faits au niveau des catégories dérivées. Quand $n = 1$ et $m > 1$, les éléments irréductibles sont indexés par $\mathbb{Z}$ et la situation géométrique est plus simple à décrire : on peut calculer entièrement l'action géométrique de $P_{I_H}(\mathcal{F}l_H)$ sur les objets irréductibles de $P_{I_H \times I_G}(\Pi(F))$. On démontre que le groupe de Grothendieck de $DP_{I_H \times I_G}(\Pi(F))$ est un module libre de rang m sous l'action de $\mathrm{R}(\check{G} \times \mathbb{G}_\mathfrak{m})$ et on explicite une base de ce module.

Dans le dernier chapitre, on définit une conjecture générale sur la fonctorialité de Langlands géométrique au niveau Iwahori. Étant donné deux groupes réductifs connexes G et H ainsi qu'un homomorphisme $\sigma : \check{G} \times \mathbb{G}_\mathfrak{m} \longrightarrow \check{H}$, (où $\check{G}$ et $\check{H}$ sont les groupes de Langlands duaux respectifs de G et H sur $\overline{\mathbb{Q}}_\ell$), on construit un champ $\mathcal{X}$ et on conjecture que le groupe de Grothendieck de ce champ est le bimodule, sous l'action des deux algèbres de Hecke affines étendues de G et de H, réalisant la fonctorialité de Langlands géométrique au niveau Iwahori pour l'homomorphisme σ.

On regarde explicitement cette conjecture pour $G = \mathbf{GL}_n$ et $H = \mathbf{GL}_m$ et pour un σ spécifique. En reliant cette conjecture avec la correspondance de Howe géométrique pour la paire $(\mathbf{GL}_n, \mathbf{GL}_m)$, on conjecture que le bimodule $K(\mathcal{X})$ sous l'action des algèbres de Hecke affines étendues de G et de H est isomorphe après spécialisation au groupe de Grothendieck $K(DP_{I_H \times I_G}(\Pi(F))$ de la catégorie $DP_{I_H \times I_G}(\Pi(F))$.

Dans le cas où $n = 1$ et $m > 1$, on obtient une description plus simple du champ $\mathcal{X}$. En effet, $K(\mathcal{X})$ dans ce cas est isomorphe à la K-théorie $\check{G} \times \mathbb{G}_\mathfrak{m}$-équivariante de la fibre de Springer $\mathcal{B}_{\check{H},x}$ au dessus d'un élément nilpotent sous-régulier x du cône nilpotent de $\check{H}$. Finalement, on obtient le résultat le plus important de la thèse qui consiste à démontrer cette conjecture dans le cas $n = 1$ et $m > 1$. Plus précisément, on démontre que la K-théorie $\check{G} \times \mathbb{G}_{\mathfrak{m}}$-équivariante de $\mathcal{B}_{\check{H},x}$ est isomorphe après spécialisation à $K(DP_{I_H \times I_G}(\Pi(F))$ en tant que module à gauche sous l'action de l'algèbre de Hecke affine étendue de H. Les deux membres de cet isomorphisme sont des modules libres de rang m sous l'action de l'anneau des représentations $\mathrm{R}(\check{G} \times \mathbb{G}_{\mathfrak{m}})$ de $\check{G} \times \mathbb{G}_{\mathfrak{m}}$ et l'anneau des représentations de $\check{H}$ agit via le morphisme de restriction $\mathrm{Res}^{\sigma} : \mathrm{R}(\check{H}) \longrightarrow \mathrm{R}(\check{G} \times \mathbb{G}_{\mathfrak{m}})$.

Contenu détaillé de la thèse

Cette thèse est constituée de quatre chapitres. Dans le premier chapitre on présente des rappels ainsi que des résultats déjà connus. Les chapitres deux et trois sont consacrés à la géométrisation et à l'étude de la correspondance de Howe au niveau Iwahori pour la paire réductive duale $(\mathbf{GL}_n, \mathbf{GL}_m)$. L'objet du chapitre quatre est la construction et l'étude de la conjecture sur la fonctorialité de Arthur-Langlands géométrique au niveau Iwahori.

Chapitre 1 : Dans ce chapitre on effectue des rappels (système de racines, algèbre de Hecke, faisceaux pervers, Grassmannienne et variété de drapeaux affines) et on introduit les notations utilisées dans les chapitres suivants.

Chapitre 2 : Ce chapitre est formé des quatre sections suivantes :

§2.1 Cette section est consacrée à rappeler la correspondance de Howe classique (au niveau des fonctions) pour les paires duales de type I et II. Dans le cas des paires duales de type II, on traduit cette correspondance en langage de modules sous l'action des algèbres de Hecke sphériques (dans le cas non ramifié) et des modules sous l'action des algèbres de Hecke-Iwahori (dans le cas modérément ramifié).

§2.2 Cette section est consacrée à la géométrisation des objets classiques. Dans la sous section 2.1.1 on introduit les différentes catégories utilisées ultérieurement. En particulier, on construit la catégorie $P_{I_H \times I_G}(\Pi(F))$ des faisceaux pervers $I_H \times I_G$-équivariants sur $\Pi(F)$ qui est l'analogue géométrique de l'espace des $I_H \times I_G$-invariants dans l'espace des fonctions de Schwartz $\mathcal{S}(\Pi(F))$. Dans la sous section 2.2.2 on introduit les opérateurs de Hecke notés $\overleftarrow{H}_G(,)$ et $\overleftarrow{H}_H(,)$ définissant l'action des catégories $P_{I_G}(\mathcal{F}l_G)$ et $P_{I_H}(\mathcal{F}l_H)$ sur la catégorie dérivée $D_{I_H \times I_G}(\Pi(F))$ des faisceaux ℓ-adiques $I_H \times I_G$-équivariants sur $\Pi(F)$. Dans la sous section 2.2.4 on explique la relation entre notre action géométrique et l'action dans le cas classique des fonctions : pour toute fonction f dans $\mathcal{S}^{I_G}(\Pi(F))$ et pour tout g dans $G(F)$, l'action géométrique est équivalente à l'action au niveau des fonctions donnée pour tout x dans $\Pi(F)$ par : $g.f(x) = |\det(g)|^{-1/2} f(g^{-1}x)$.

§2.3 Dans cette section on étudie la catégorie $P_{H(\mathcal{O}) \times I_G}(\Pi(F))$ des faisceaux pervers $H(\mathcal{O}) \times I_G$-équivariants sur $\Pi(F)$. Dans la sous section 2.3.1 on décrit les objets irréductibles de cette catégorie qui sont indexés par le réseau des cocaractères X_G de G. Notons par Π_λ la strate indexée par λ sous l'action de $H(\mathcal{O}) \times I_G$ dans $\Pi(F)$. La sous section 2.3.2 est consacrée à l'étude de la structure de module de $P_{H(\mathcal{O}) \times I_G}(\Pi(F))$ sous l'action de $P_{I_G}(Gr_G)$ et de la catégorie $\mathrm{Rep}(\check{H})$ des représentations de $\check{H}$. Notons $\mathcal{A}^\lambda$ le faisceau d'intersection de cohomologie du I_G-orbite passant par $t^\lambda G(\mathcal{O})$ dans Gr_G et I_0 le faisceau pervers constant sur Π. On obtient alors le premier résultat de cette thèse : soit $\sigma : \check{G} \times \mathbb{G}_{\mathfrak{m}} \longrightarrow \check{H}$ un homomorphisme fixé (comme dans [Lys11]) et soit $\mathrm{Res}^{\sigma} : \mathrm{Rep}(\check{H}) \longrightarrow \mathrm{Rep}(\check{G} \times \mathbb{G}_{\mathfrak{m}})$ le foncteur de restriction géométrique associé à σ.

Théorème 1 (Théorème 2.3.15 et Corollaire 2.3.17, Chapitre 2)**.** *Pour tout λ dans X_G et $\mathcal{T}$ dans $P_{H(\mathcal{O})}(Gr_H)$ on a l'isomorphisme suivant*

$$\overleftarrow{H}_H(\mathcal{T}, \mathrm{IC}(\Pi_\lambda)) \widetilde{\longrightarrow} \overleftarrow{H}_G(\mathcal{A}^\lambda \star \mathrm{gRes}^\sigma(\mathcal{T}), I_0).$$

Le foncteur $D_{I_G}(Gr_G) \to D_{H(\mathcal{O})\times I_G}(\Pi(F))$ envoyant $\mathcal{A}$ sur $\overleftarrow{H}_G(\mathcal{A}, I_0)$ définit un isomorphisme au niveau des groupes de Grothendieck $K(P_{I_G}(Gr_G)) \widetilde{\longrightarrow} K(P_{H(\mathcal{O})\times I_G}(\Pi(F)))$ commutant avec les actions de $K(P_{H(\mathcal{O})}(Gr_H))$ et $K(P_{I_G}(\mathcal{F}l_G))$.

§2.4 Dans cette section on définit les objets irréductibles de la catégorie $P_{I_H\times I_G}(\Pi(F))$. Ceci est fait en deux étapes. Dans un premier temps, on regarde dans la sous section 2.4.1 le cas $n = m$ et on démontre que les éléments irréductibles de $P_{I_H\times I_G}(\Pi(F))$ sont indexés par le groupe de Weyl affine étendu $\widetilde{W}_G$ de G. Dans la sous section 2.4.2, on considère ensuite le cas général $n \leq m$ et on obtient le second résultat de cette thèse. Notons $S_{n,m}$ l'ensemble des couples (s, I_s) avec I_s un sous-ensemble à n éléments de $\{1, \dots, m\}$ et s une bijection de I_s dans $\{1, \dots, n\}$. Par ailleurs rappelons que $\Pi_{N,r} = t^{-N}\Pi/t^r\Pi$ où N et r sont des entiers tels que $N + r > 0$.

Théorème 2 (Théorème 2.4.8, Chapitre 2)**.**
Les éléments irréductibles de la catégorie $P_{I_H\times I_G}(\Pi(F))$ sont indexés par $X_G\times S_{n,m}$. Pour chaque élément $w = (\lambda, s)$ dans $X_G \times S_{n,m}$, l'objet irréductible associé est le faisceau d'intersection de cohomologie de la $I_H \times I_G$-orbite $\Pi^w_{N,r}$ dans $\Pi_{N,r}$ indexée par (λ, s).

Chapitre 3 : Ce chapitre est formé de six sections suivantes :

§3.1 Dans cette section on regarde le cas $n = m$ et on souhaite définir l'action des éléments irréductibles de $P_{I_G}(\mathcal{F}l_G)$ (resp. $P_{I_H}(\mathcal{F}l_H)$) sur un objet irréductible particulier $\mathcal{I}^{w_0!}$, où w_0 est l'élément le plus long du groupe de Weyl fini W_G de G. On constate que $\overleftarrow{H}_G(L_{w!}, \mathcal{I}^{w_0!})$ (la définition de $L_{w!}$ est donnée dans § 1.5.3) est canoniquement isomorphe à $\mathcal{I}^{ww_0!}$ et $\overleftarrow{H}_H(L_{w!}, \mathcal{I}^{w_0!})$ à $\mathcal{I}^{\overline{ww_0}!}$, où $w \to \overline{w}$ est une anti-involution sur $\widetilde{W}_G$ [Définition 3.1.7].

§3.2 On construit une filtration par $\mathbb{Z}$ sur la catégorie $P_{I_H\times I_G}(\Pi(F))$ valable pour tout $n \leq m$. On définit une graduation sur $P_{I_G}(\mathcal{F}l_G)$ par $\mathbb{Z}$ en considérant les composantes connexes de $\mathcal{F}l_G$ qui sont indexées par $\mathbb{Z}$. Cette graduation est compatible avec le produit de convolution sur $P_{I_G}(\mathcal{F}l_G)$ et la filtration sur $P_{I_H\times I_G}(\Pi(F))$ est compatible avec l'action de $P_{I_G}(\mathcal{F}l_G)$. Ceci nous permettra de contrôler l'action des catégories $P_{I_G}(\mathcal{F}l_G)$ sur $P_{I_H\times I_G}(\Pi(F))$.

§3.3 On se restreint au cas $n = m$ et on complète la description du module engendré par $\mathcal{I}^{w_0!}$ sous l'action des deux algèbres de Hecke-Iwahori associées à G et H. Á l'aide de cette étude on construit une équivalence de catégorie entre $P_{I_G}(\mathcal{F}l_G)$ et $P_{I_H}(\mathcal{F}l_H)$ qui au niveau classique définit un anti-isomorphisme d'algèbre entre $\mathcal{H}_{I_G}$ et $\mathcal{H}_{I_H}$.

Théorème 3 (Théorème 3.3.6, Chapitre 3)**.** *Si $n = m$, il existe une équivalence de catégories σ entre $P_{I_G}(\mathcal{F}l_G)$ et $P_{I_H}(\mathcal{F}l_H)$ envoyant tout objet irréductible L_w [Définition donnée dans § 1.5.3] sur $L_{w_0\overline{w}w_0}$. De plus ce foncteur vérifie les propriétés suivantes : pour tout w et w' dans $\widetilde{W}_G$ on a*

$$\overleftarrow{H}_G(L_w, \mathcal{I}^{w_0!}) \widetilde{\longrightarrow} \overleftarrow{H}_H(\sigma(L_w), \mathcal{I}^{w_0!}) \quad \text{et} \quad \sigma(L_w \star L_{w'}) = \sigma(L_{w'}) \star \sigma(L_w).$$

Le foncteur σ provient d'une anti-involution sur le groupe $G(F)$ envoyant g sur $w_0{}^t g w_0$. On remarque aussi qu'en composant cette anti-involution avec l'anti-involution envoyant g sur g^{-1}, on obtient une involution qui préserve le groupe d'Iwahori I_G et induit un isomorphisme au niveau des algèbres de Hecke-Iwahori. Cette involution sera utilisée plus loin pour la construction du bimodule $\mathcal{S}_0$ quand on considérera l'action à droite de $P_{I_H}(\mathcal{F}l_H)$ au lieu de son action à gauche.

§3.4 On considère le cas général $n \leq m$. Dans un premier temps, on considère dans la sous section 3.4.1, un sous module Θ de $K_{I_H \times I_G}(\Pi(F))$ sous l'action de $K(P_{I_G}(\mathcal{F}l_G))$ engendré par les éléments $\mathcal{I}^{\mu !}$, où I_μ parcourt tout les sous ensembles à n élément dans $\{1, \ldots, m\}$. On démontre que ce sous module est un module libre de rang C_n^m sous l'action de l'algèbre de Hecke-Iwahori $\mathcal{H}_{I_G}$ de G et on le munit d'une base formée des éléments $\mathcal{I}^{\mu !}$, où $(0, \mu)$ est un élément de $X_G \times S_{n,m}$. Dans la sous section 3.4.2, en une série de propositions, on calcule l'action géométrique de $\mathcal{H}_{I_H}$ sur les éléments $\mathcal{I}^\mu$. Finalement on décrit la structure complète de module de Θ sous l'action de $\mathcal{H}_{I_G}$ et $\mathcal{H}_{I_H}$.

§3.5 Le but de cette section est de démontrer que le sous-module Θ est un module sous l'action à droite de $K(P_{I_H}(\mathcal{F}l_H))$ et que Θ est obtenu par une induction parabolique pour un certain sous groupe parabolique de H. Dans cette sous section on travaille plus au niveau des groupes de Grothendieck mais on essaie le plus possible de formuler nos résultats au niveau des catégories dérivées. Afin de pouvoir munir ce sous module d'une structure de $\mathcal{H}_{I_H}$-module on considère l'action à droite de $\mathcal{H}_{I_H}$ qui est liée à l'action à gauche par le morphisme envoyant w sur w^{-1}. Pour w dans $\widetilde{W}_G$ notons par $\mathcal{S}_0$ le sous espace de $K(D_{I_G \times I_H}(\Pi(F))) \otimes \overline{\mathbb{Q}}_\ell$ engendré par les éléments $\mathcal{I}^{(w \centerdot w_0)!}$, où $w \centerdot w_0$ a été défini dans [Définition 2.4.10], sur $\overline{\mathbb{Q}}_\ell$. On rappelle qu'ici un élément $w = t^\lambda \tau$ de $\widetilde{W}_G$ peut être considéré comme un élément (λ, τ) de $X_G \times S_{n,m}$. De cette façon w_0 est un élément de $X_G \times S_{n,m}$ en supposant que le composant sur X_G s'annule. Le sous espace $\mathcal{S}_0$ est un module libre de rang un sous l'action de $\mathcal{H}_{I_G}$. Dans la sous section 3.5.1 on définit l'action des éléments irréductibles correspondants aux éléments du groupe de Weyl fini de G et de H sur $\mathcal{S}_0$. Dans la sous section 3.5.2 on considère le sous groupe de Levi M de H correspondant à la partition $(n, m-n)$ de m et on rappelle brièvement la construction de l'algèbre de Hecke-Iwahori de Levi $\mathcal{H}_{I_M}$ de $\mathcal{H}_{I_H}$ ainsi que quelques unes de ses propriétés figurant dans [Pra05]. Dans la sous section 3.5.3 on muni $\mathcal{S}_0$ d'une action à droite de $\mathcal{H}_{I_M}$ et on obtient les résultat suivants :

Théorème 4 (Théorème 3.5.22, Chapitre 3)**.**
Le sous espace $\mathcal{S}_0$ *est un sous module de* $K(D_{I_G \times I_H}(\Pi(F))) \otimes \overline{\mathbb{Q}}_\ell$ *sous l'action à droite de* $\mathcal{H}_{I_M}$*.*

Théorème 5 (Théorème 3.5.23, Chapitre 3)**.** *Le morphisme* $\alpha : \mathcal{S}_0 \otimes_{\mathcal{H}_{I_M}} \mathcal{H}_{I_H} \to \mathcal{S}^{I_H \times I_G}(\Pi(F))$ *obtenu par adjonction est un morphisme injectif de* $\mathcal{H}_{I_H}$*-modules à droite et de* $\mathcal{H}_{I_G}$*-modules à gauche dont l'image est égale à* $\Theta \otimes \overline{\mathbb{Q}}_\ell$*.*

Le sous groupe de Levi M considéré plus haut est isomorphe à $\mathbf{GL}_n \times \mathbf{GL}_{m-n}$. Dans la suite de cette sous section on démontre que l'action de l'algèbre de Hecke-Iwahori associée au facteur $\mathbf{GL}_n$ du sous groupe de Levi M est liée à l'action de $\mathcal{H}_{I_G}$ via l'anti-involution définie dans Théorème 3. L'action de l'algèbre de Hecke-Iwahori du facteur $\mathbf{GL}_{m-n}$ est par décalage par $[-\ell(w)]$ où ℓ est la fonction longueur définie sur le groupe de Weyl affine étendu.

§3.6 Cette section est consacrée à l'étude du cas $n = 1$ et $m > 1$. Comme dans le cas général on définit les objets irréductibles de la catégorie $P_{I_H \times I_G}(\Pi(F))$ qui sont indexés par $\mathbb{Z}$. Contrairement au cas général, on peut définir l'action complète des éléments irréductibles de la catégorie $P_{I_H}(\mathcal{F}l_H)$ sur les éléments irréductibles de $P_{I_H \times I_G}(\Pi(F))$. On définit d'abord l'action des objets correspondants aux réflexions simples dans le groupe de Weyl fini de H, puis l'action de l'unique réflexion simple affine et enfin on définit l'action des éléments de longueur zéro sur les objets irréductibles de la catégorie $P_{I_H \times I_G}(\Pi(F))$. On affirme que tous les objets irréductibles de la catégorie $P_{I_H \times I_G}(\Pi(F))$ peuvent être obtenus par l'action des éléments de longueur zéro. Dans la suite de cette section on construit un bimodule, encore noté $\mathcal{S}_0$, sous l'action à gauche de $\mathcal{H}_{I_G}$ et de l'action à droite de $\mathcal{H}_{I_H}$.

On définit la catégorie $DP_{I_G \times I_H}(\Pi(F))$ dont les objets sont les sommes directes des objets de $P_{I_G \times I_H}(\Pi(F))[i]$ quand i parcourt $\mathbb{Z}$. On définit l'action du groupe multiplicatif $\mathbb{G}_\mathfrak{m}$ sur $DP_{I_G \times I_H}(\Pi(F))$ par décalage cohomologique -1. Notons g la représentation standard de $\check{G}$. Pour tout entier j la représentation g^j envoie chaque élément irréductible IC^k sur IC^{k-mj}. De cette manière, on munit le groupe de Grothendieck de la catégorie $DP_{I_G \times I_H}(\Pi(F))$ d'une action de l'anneau des représentations $\mathrm{R}(\check{G} \times \mathbb{G}_\mathfrak{m})$ de $\check{G} \times \mathbb{G}_\mathfrak{m}$ et on montre qu'il est un module libre de rang m en explicitant une base de ce module.

On montre aussi que les actions de centres respectifs de $P_{I_G}(\mathcal{F}l_G)$ et $P_{I_H}(\mathcal{F}l_H)$ sont compatibles. Plus précisément, on a :

Théorème 6 (Théorème 3.6.15, Chapitre 3)**.** *L'action du centre de $P_{I_H}(\mathcal{F}l_H)$ et l'action du centre de $P_{I_G}(\mathcal{F}l_G)$ sont compatibles. La catégorie* $\mathrm{Rep}(\check{H})$ *des représentations de $\check{H}$ sur $\overline{\mathbb{Q}}_\ell$ agit via le foncteur de restriction géométrique* $\mathrm{Res}^\sigma : \mathrm{Rep}(\check{H}) \longrightarrow \mathrm{Rep}(\check{G} \times \mathbb{G}_\mathfrak{m})$ *sur les objets irréductibles* IC^k *pour tout k dans $\mathbb{Z}$.*

Chapitre 4 : Ce chapitre est consacré à l'étude de la fonctorialité de Arthur-Langlands géométrique. On y démontre que la correspondance de Howe dans un cas particulier provient de la fonctorialité au niveau Iwahori.

§4.1 La sous section 4.1.1 consiste à construire un bimodule conjectural réalisant la fonctorialité de Langlands géométrique au niveau Iwahori. Plus précisément, on considère deux groupes réductifs G et H connexes et on note leurs groupes Langlands dual sur $\overline{\mathbb{Q}}_\ell$ par $\check{G}$ et $\check{H}$. On suppose de plus que les groupes dérivés respectifs de $\check{G}$ et $\check{H}$ sont simplement connexes. On se donne un homomorphisme $\sigma : \check{G} \times \mathbb{G}_\mathfrak{m} \longrightarrow \check{H}$. Pour construire ce bimodule on utilise le langage K-théorique. On rappelle que d'après l'isomorphisme de Kazhdan-Lusztig, l'algèbre de Hecke affine étendue $\mathbb{H}_H$ associée à H est isomorphe à la K-théorie $\check{H} \times \mathbb{G}_\mathfrak{m}$-équivariante $K^{\check{H} \times \mathbb{G}_m}(Z_{\check{H}})$ de la variété de Steinberg $Z_{\check{H}} = \tilde{\mathcal{N}}_{\check{H}} \times_{\mathcal{N}_{\check{H}}} \tilde{\mathcal{N}}_{\check{H}}$ associée à $\check{H}$. On utilise les résultats et les notations de [CG97]. Le premier énoncé de cette section est la conjecture suivante :

Conjecture 1 (Conjecture 4.1.6, Chapitre 4)**.** *Le bimodule sous l'action de l'algèbre de Hecke affine $K^{\check{G} \times \mathbb{G}_\mathfrak{m}}(\tilde{\mathcal{N}}_{\check{G}} \times_{\mathcal{N}_{\check{G}}} \tilde{\mathcal{N}}_{\check{G}})$ de $\check{G}$ et de l'algèbre de Hecke affine $K^{\check{H} \times \mathbb{G}_\mathfrak{m}}(\tilde{\mathcal{N}}_{\check{H}} \times_{\mathcal{N}_{\check{H}}} \tilde{\mathcal{N}}_{\check{H}})$ de $\check{H}$ réalisant la fonctorialité de Langlands géométrique au niveau Iwahori pour le morphisme $\sigma : \check{G} \times \mathbb{G}_\mathfrak{m} \longrightarrow \check{H}$ s'identifie avec $K(\mathcal{X})$ où $\mathcal{X}$ est le champ quotient*

$$(\tilde{\mathcal{N}}_{\check{G}}/(\check{G} \times \mathbb{G}_\mathfrak{m})) \times_{\mathcal{N}_{\check{H}}/(\check{H} \times \mathbb{G}_\mathfrak{m})} (\tilde{\mathcal{N}}_H/(\check{H} \times \mathbb{G}_\mathfrak{m})).$$

Dans la sous section 4.1.2 on donne différentes descriptions du champ $\mathcal{X}$.

§4.2 Dans cette section on étudie la conjecture énoncée dans la section précédente sous les hypothèses suivantes : $G = \mathbf{GL}_n$ et $H = \mathbf{GL}_m$ et le morphisme σ est obtenu par la composition : $\mathbf{GL}_n \times \mathbb{G}_\mathfrak{m} \to \mathbf{GL}_n \times \mathrm{SL}_2 \to \mathbf{GL}_n \times \mathbf{GL}_{m-n} \to \mathbf{GL}_m$, où $\mathbf{GL}_n$ (resp. $\mathbf{GL}_m$) est considéré comme le groupe de Langlands dual de G (resp. H) sur $\overline{\mathbb{Q}}_\ell$, et le dernier morphisme est l'inclusion de sous groupe de Levi standard associé à la partition $(n, m-n)$ de m et le morphisme $\xi : \mathrm{SL}_2 \to \mathbf{GL}_{m-n}$ correspond à l'orbite principale unipotente. On peut conjecturalement relier le bimodule $K(\mathcal{X})$ à la K-théorie de la catégorie $DP_{I_G \times I_H}(\Pi(F))$, définissant la correspondance de Howe. On obtient alors la conjecture suivante :

Conjecture 2 (Conjecture 4.2.1, Chapitre 4)**.** *Le bimodule $K(\mathcal{X})$ sous chacune des algèbres de Hecke affines $K^{\check{G}\times\mathbb{G}_\mathfrak{m}}(\tilde{\mathcal{N}}_{\check{G}} \times_{\mathcal{N}_{\check{G}}} \tilde{\mathcal{N}}_{\check{G}})$ et $K^{\check{H}\times\mathbb{G}_\mathfrak{m}}(\tilde{\mathcal{N}}_{\check{H}} \times_{\mathcal{N}_{\check{H}}} \tilde{\mathcal{N}}_{\check{H}})$ est isomorphe après spécialisation à $K(DP_{I_H \times I_G}(\Pi(F)))$.*

Nous avons déjà étudié la structure de module d'un des membres de cette conjecture, à savoir la structure de module de $K(DP_{I_H \times I_G}(\Pi(F)))$ sur les algèbres de Hecke affines de G et H. Dans la suite de cette section on définit une stratification sur $\mathcal{X}$ et une filtration sur $K(\mathcal{X})$ qui pourrait être compatible avec celle sur $K(DP_{I_H \times I_G}(\Pi(F)))$ sous l'action des algèbres de Hecke affines de G et H après spécialisation.

§4.3 Dans la sous section 4.3.1 on regarde le cas particulier de Conjecture 4.2.1 quand $n = 1$ et $m > 1$. Dans ce cas le bimodule $K(\mathcal{X})$ est isomorphe à $K^{\check{G}\times\mathbb{G}_m}(\mathcal{B}_{\check{H},x})$, où $\mathcal{B}_{\check{H},x}$ est la fibre de Springer dans $\tilde{\mathcal{N}}_{\check{H}}$ au dessus d'un élément nilpotent sous régulier x de $\check{H}$. On démontre que cette fibre de Springer est une configuration de $m-1$ droites projectives avec m points marqués et en utilisant le Lemme de la fibration cellulaire de Chriss-Ginzburg [CG97, Lemme 5.5] on montre que $K^{\check{G}\times\mathbb{G}_m}(\mathcal{B}_{\check{H},x})$ est un $\mathrm{R}(\check{G}\times\mathbb{G}_\mathfrak{m})$-module libre de rang m et que la structure de $\mathrm{R}(\check{H})$-module sur $K^{\check{G}\times\mathbb{G}_m}(\mathcal{B}_{\check{H},x})$ provient du foncteur de restriction géométrique $\mathrm{Res}^\sigma : \mathrm{R}(\check{H}) \longrightarrow \mathrm{R}(\check{G} \times \mathbb{G}_\mathfrak{m})$ correspondant au morphisme $\sigma : \check{G}\times\mathbb{G}_\mathfrak{m} \longrightarrow \check{H}$. Le résultat majeur de cette thèse est le théorème suivant :

Théorème 7 (Théorème 4.3.3, Chapitre 4)**.** *Supposons $G = \mathbf{GL}_1$ et $H = \mathbf{GL}_m$ avec m>1. Le bimodule $K^{\check{G}\times\mathbb{G}_m}(\mathcal{B}_{\check{H},x})$ est isomorphe après spécialisation à $K(DP_{I_H \times I_G}(\Pi(F)))$ comme module à gauche sous l'action de l'algèbre de Hecke affine $\mathbb{H}_H$. Ces deux bimodules sont des $\mathrm{R}(\check{G}\times\mathbb{G}_\mathfrak{m})$-modules libres de rang m et $\mathrm{R}(\check{H})$ agit via le foncteur de restriction $\mathrm{Res}^\sigma : \mathrm{R}(\check{H}) \longrightarrow \mathrm{R}(\check{G}\times\mathbb{G}_m)$ correspondant au morphisme σ.*

Ce théorème relie les calculs de la section 3.6 du chapitre trois et la conjecture du chapitre quatre dans le cas où $n = 1$ et $m > 1$. Il montre que la correspondance de Howe géométrique provient de la fonctorialité de Arthur-Langlands géométrique au niveau Iwahori pour le morphisme $\sigma : \mathbf{GL}_1 \times \mathbb{G}_m \longrightarrow \mathbf{GL}_m$. La sous section 4.3.2 est consacrée à la démonstration de Théorème 7. La stratégie de la preuve consiste à construire une base de $K^{\check{G}\times\mathbb{G}_\mathfrak{m}}(\mathcal{B}_{\check{H},x})$ de sorte que l'action de l'algèbre de Hecke affine de $\check{H}$ sur cette base soit compatible avec l'action géométrique déjà calculée au chapitre trois.

Appendices : Dans l'appendice A, on fait des rappels sur le produit de convolution en K-théorie. Dans l'appendice B, on présente un analogue géométrique des foncteurs de Jacquet au niveau Iwahori et on démontre que ces foncteurs commutent avec l'action des faisceaux de Wakimoto. On démontre aussi qu'ils préservent les faisceaux pervers de poids zéro.

English introduction

General motivations

This thesis deals with problems in the geometric theory of representations of reductive groups. More precisely, our aim is to study the Howe correspondence for dual reductive pairs of type II and the Langlands functoriality at the Iwahori level in the framework of the geometric Langlands program.

The Howe correspondence

Let $\mathbf{k}$ be a finite field $\mathbb{F}_q$ of characteristic different from 2. We will work over the field $F = \mathbf{k}((t))$, and we will denote by $\mathcal{O} = \mathbf{k}[[t]]$ the ring of integers of F. All representations will be assumed to be smooth and will be defined over $\overline{\mathbb{Q}}_\ell$, where ℓ is a prime number different from the characteristic of F.

The basic notions of the Howe correspondence from the classical point of view have been presented in [MVW87]. Let $(G(F), H(F))$ be a dual reductive pair in some symplectic group $Sp(W)(F)$, ψ be an additive character of F, and $\widetilde{Sp}(W)(F)$ be the metaplectic group which is the twofold topological covering of the symplectic group $Sp(W)(F)$. There exists a smooth irreducible representation of $\widetilde{Sp}(W)(F)$ attached to the character ψ called the Weil representation. We assume that the metaplectic cover $\widetilde{Sp}(W)(F) \to Sp(W)(F)$ admits a section over $G(F) \times H(F)$. This is always verified for dual pairs of type II which are the pairs we are interested in. Hence, the Howe correspondence becomes a correspondence between two classes of representations of $G(F)$ and $H(F)$.

It is of great interest to understand this correspondence explicitly and figure out whether or not it implicates the Arthur-Langlands functoriality. The case of the cuspidal representations has been solved quite early by Kudla in [Kud86], but the case of general representations is still open. Mœglin has given in [Mœg09a] a description of the Howe duality for certain symplectic and orthogonal groups in the Archimedean case in terms of Langlands-Vogan parameters. It seems that if the groups are of similar rank and if we restrict ourselves to the case of the tempered representations, the Howe correspondence respects functoriality. In some way, these hypothesis provide the simplest cases of functoriality in the Archimedean as well as the non-Archimedean case. Except for the linear case solved by Mínguez in [Mín08], the Langlands parameters can not be preserved by the Howe correspondence unless the groups are of almost equal rank. A possible explanation for this phenomenon is that the Langlands packets are not the correct notion here; we should rather consider larger packets whose existence has been conjectured by Arthur in [Art84]. By using Arthur's formalism, Adams in [Ada89] suggested a general formula.

According to Arthur and Adam's conjectures, if $\check{G}$ (resp. $\check{H}$) is the Langlands dual group of G (resp. H) over $\overline{\mathbb{Q}}_\ell$, the homomorphism $\kappa : \check{G} \to \check{H}$ should be replaced by a homomorphism from $\check{G} \times \mathrm{SL}_2$ to $\check{H}$; thus we should take in consideration the Arhtur's packets instead of usual L-packets. Under some assumptions, it is expected that if π is an irreducible representation of $G(F)$ appearing in the Weil representation and π' is the corresponding representation of $H(F)$ under the Howe correspondence then the Arthur packet of π' is the image of the Arthur packet of π by the map σ. For more details we refer the reader to the following papers on this subject: [Mœg89],[Mœg09a],[Mœg09b], [Ral82], [Kud86], [Art84].

We expect from the beginning that Arthur's formula cover the cases where the groups are of very different ranks. However, like Adams explained himself, this conjecture does not apply in all circumstances. The first difficulty is that a given representation of one of the members of the dual pair need not to be in any Arthur packet; and in general it is not true that if (π, π') are in the image of the Howe correspondence and if one of the representations is in some Arthur packet then the other one is too. The other difficulty is that a given representation can appear in several packets and Adam's formula may be applied to some packets but not to all of them.

In the sequel we will restrict ourselves to the dual reductive pairs of type II. More precisely, let L_0 (resp. U_0) be a n-dimensional (resp. m-dimensional) $\mathbf{k}$-vector space with $n \leq m$, and let $G = \mathbf{GL}(L_0)$ and $H = \mathbf{GL}(U_0)$. Denote by $\Pi(F)$ the space $(U_0 \otimes L_0)(F)$ and $\mathcal{S}(\Pi(F))$ the Schwartz space of locally constant functions with compact support on $\Pi(F)$. This space realizes the restriction of the Weil representation to $G(F) \times H(F)$. According to Mínguez [Mín08], we know that the Howe correspondence associates to any smooth irreducible representation of $G(F)$ appearing as a quotient of the restriction of the Weil representation a unique smooth irreducible non-zero representation of $H(F)$, denoted by $\theta_{n,m}(\pi)$, such that $\pi \otimes \theta_{n,m}(\pi)$ is a quotient of the restriction of the Weil representation to $G(F) \times H(F)$. Moreover, he describes this correspondence explicitly in terms of Langlands parameters.

Different classes of representations may be considered for an explicit study of the Howe correspondence. A representation of $G(F)$ is called unramified if it admits a non-zero fixed vector under $G(\mathcal{O})$. The first possibility is to consider the class of unramified representations and translate the Howe correspondence in the language of modules over spherical Hecke algebras. In this case we are reduced to study the module structure of $\mathcal{S}(\Pi(F))^{G(\mathcal{O}) \times H(\mathcal{O})}$ under the action of the spherical Hecke algebras of G and H. According to unpublished work of Howe (cf. [Mín06, Appendix]) we know that $\mathcal{S}(\Pi(F))^{G(\mathcal{O}) \times H(\mathcal{O})}$ is generated by some function s_0 under the action of each spherical Hecke algebra $\mathcal{H}_{G(\mathcal{O})}$ and $\mathcal{H}_{H(\mathcal{O})}$. Let us recall that, according to Satake's isomorphism, the spherical Hecke algebra $\mathcal{H}_{G(\mathcal{O})}$ of G is isomorphic to the Grothendieck group of the category of representations of $\check{G}$ over $\overline{\mathbb{Q}}_\ell$. In the Langlands functoriality framework, we have the following result due to Rallis in [Ral82]: the morphism from $\mathcal{H}_{G(\mathcal{O})}$ to $\mathcal{S}(\Pi(F))^{G(\mathcal{O}) \times H(\mathcal{O})}$ sending h to hs_0 is an isomorphism $\mathcal{H}_{G(\mathcal{O})}$-modules and there exists a homomorphism $\kappa : \mathcal{H}_{H(\mathcal{O})} \to \mathcal{H}_{G(\mathcal{O})}$ such that the action of the spherical algebra $\mathcal{H}_{H(\mathcal{O})}$ factors through κ. If $n = m$, the homomorphism κ comes from the restriction functor $\mathrm{Rep}(\check{H}) \to \mathrm{Rep}(\check{G})$ corresponding to the isomorphism between $\check{G}$ and $\check{H}$. If $n \leq m$, then the homomorphism κ comes from the restriction $\mathrm{Rep}(\check{H}) \to \mathrm{Rep}(\check{G} \times \mathbb{G}_\mathfrak{m})$ corresponding to the homomorphism from $\check{G} \times \mathbb{G}_\mathfrak{m}$ to $\check{H}$. When $m > n$, the restriction of this homomorphism to $\mathbb{G}_\mathfrak{m}$ is non trivial. Let us remark that the factor $\mathbb{G}_\mathfrak{m}$ above is the maximal torus of the Arthur's SL_2 . The geometric version of this result has been proved by Lysenko in [Lys11, Proposition 4]. In the geometric setting, the $\mathbb{G}_\mathfrak{m}$ factor appears in a natural way.

The second interesting class of representations to be considered for the study of the Howe correspondence is the class of tamely ramified representations. A representation of $G(F)$ is said to be tamely ramified if it admits a non zero vector fixed under a Iwahori subgroup I_G of $G(F)$. Let us consider the functor sending any tamely ramified representation V of $G(F)$ to its space of invariants V^{I_G} under I_G. Then, the latter is naturally a module over the Iwahori-Hecke algebra $\mathcal{H}_{I_G}$. According to [Bor76, Theorem 4.10] this functor is an equivalence of categories between the category of tamely ramified admissible representations of G and the category of finite-dimensional modules over $\mathcal{H}_{I_G}$. Moreover, according to [IM], this functor is exact over the category of smooth representations of $G(F)$. Hence, in this case, we can interpret the Howe correspondence in the language of modules over the Iwahori-Hecke algebras. The space $\mathcal{S}^{I_G\times I_H}(\Pi(F))$ of $I_G \times I_H$-invariants in the Schwartz space $\mathcal{S}(\Pi(F))$ is naturally a module over each Iwahori-Hecke algebra $\mathcal{H}_{I_G}$ and $\mathcal{H}_{I_H}$. We would like to understand this module structure by geometric means. The geometrization of the Howe correspondence (more generally of theta correspondence) has been initiated in [LL09] and [Lys11] , where the authors study the unramified case for dual reductive pairs $(\mathrm{Sp}_{2n}, \mathrm{S}\mathbb{O}_{2m})$ and $(\mathbf{GL}_n, \mathbf{GL}_m)$. In the continuation of these two articles, we develop in this thesis the geometric tamely ramified Howe correspondence for dual pairs $(\mathbf{GL}_n, \mathbf{GL}_m)$ and we extend the results in [Lys11] in the tamely ramified case.

The geometric Langlands program

A series of conjectures on geometric Langlands program (in particular in the Iwahori case) has been formulated in [FG06] and more recently in unpublished works of Bezrukavnikov, and V. Lafforgue and in [E.12]. These conjectures have motivated the construction of the bimodule realizing the geometric Langlands functoriality at the Iwahori level in this thesis. Let us briefly recall these motivations. We fix an algebraically closed field $\mathbf{k}$ of characteristic p and we denote by $D^* = \mathrm{Spec}\mathbf{k}((t))$. Let G be a connected reductive group over $\mathbf{k}$. Frenkel and Gaitsgory conjectured the existence of a category $\mathcal{C}_{\check{G}}$ over the stack $LS_{\check{G}}(D^*)$ of $\check{G}$-local systems on D^* endowed with an action of $G(F)$. The construction of this category in the general case seems out of reach but the Iwahori case is more tractable. Denote by $\mathcal{N}_{\check{G}}$ the nilpotent cone of $\check{G}$ and by $\tilde{\mathcal{N}}_{\check{G}}$ its Springer resolution. A $\check{G}$-local system is said to be tamely ramified if it admits a regular singularity at the origin with unipotent monodromy. Let us recall that the stack quotient $\mathcal{N}_{\check{G}}/\check{G}$ can be viewed as the stack of tamely ramified $\check{G}$-local systems. Denote by $\mathcal{C}_{\check{G},nilp}$ the category obtained from $\mathcal{C}_{\check{G}}$ by the base change $\mathcal{N}_{\check{G}}/\check{G} \to LS_{\check{G}}(D^*)$. If I_G denotes a Iwahori subgroup of $G(F)$, then Frenkel and Gaitsgory in [FG06, (2.20)] conjectured the existence of an isomorphism

$$K(\mathcal{C}^{I_G}_{\check{G},nilp}) \widetilde{\longrightarrow} K(\tilde{\mathcal{N}}_{\check{G}}/\check{G}), \tag{0.0.2}$$

where the left hand side is the Grothendieck group of I_G-invariants of the category of $\mathcal{C}_{\check{G},nilp}$ and the right hand side is the Grothendieck group of the category of coherent sheaves on the stack $\tilde{\mathcal{N}}_{\check{G}}/\check{G}$. Moreover, this isomorphism should be compatible with the action of the affine Hecke algebra.

In the spirit of these conjectures consider $G = \mathbf{GL}_n$ and $H = \mathbf{GL}_m$ we conjecture the existence of a morphism $\sigma : \check{G} \times \mathbb{G}_{\mathfrak{m}} \to \check{H}$ and an isomorphism (after specialization by a character of $\mathbb{G}_{\mathfrak{m}}$ in $\overline{\mathbb{Q}}_\ell$) between $K(DP_{I_G\times I_H})(\Pi(F))$ and $K(\mathcal{X}) = K((\tilde{\mathcal{N}}_{\check{G}}/(\check{G}\times\mathbb{G}_{\mathfrak{m}})) \times_{\mathcal{N}_{\check{H}}/(\check{H}\times\mathbb{G}_{\mathfrak{m}})} (\tilde{\mathcal{N}}_{\check{H}}/(\check{H}\times\mathbb{G}_{\mathfrak{m}})))$, where $K(DP_{I_G\times I_H})(\Pi(F))$ is a bimodule under the action of the two affine Hecke algebras of G and H equipped with an action of $\mathbb{G}_{\mathfrak{m}}$ by cohomological shift. The objects of the category $DP_{I_G\times I_H}(\Pi(F))$ are direct sums of the objects of $P_{I_G\times I_H}(\Pi(F))$ shifted cohomologically, the latter being the category of $I_H \times I_G$-equivariant perverse sheaves on $\Pi(F)$. We prove this conjecture in the case $n = 1$ and $m > 1$. This implies that the bimodule realizing geometric Howe correspondence comes from the bimodule realizing geometric Langlands functoriality for the map σ in this special case. We underline the theorem in the case $n = 1$ and $m > 1$ is new even in the classical setting over a finite field. The idea is that the explicit description of the Howe correspondence obtained for the pair $(G = \mathbf{GL}_n, H = \mathbf{GL}_m)$ by Mínguez could be upgraded for a finer description of the bimodule itself in terms of a stack $\mathcal{X}$ attached to the map

$\check{G} \times \mathrm{SL}_2 \to \check{H}$. This opens an important perspective, as the same description should also hold for other dual pairs. Especially, it would be interesting to obtain a similar result for the dual pair $(\mathrm{Sp}_{2n}, \mathrm{S}\mathbb{O}_{2m})$ and thus provide a conceptually new approach to the calculation of Aubert [Aub91]. Another perspective is a hope that the whole derived category $D_{I_G \times I_H}(\Pi(F))$ could possibly be described in terms of the derived category of coherent sheaves $D(\mathcal{X})$ on $\mathcal{X}$ in the style of Bezrukavnikov theory in [AB09].

Summary of results

In the entire section we will assume that $\mathbf{k}$ is an algebraically closed fields of characteristic different from 2 and $F = \mathbf{k}((t))$. According to Grothendieck's philosophy the equivalent geometric notion of a function on the $\mathbf{k}$-points of a variety V is the notion of complex of ℓ-adic sheaves on V. We can associate to any complex of ℓ-adic sheaf $\mathcal{K}$ a function whose value at a closed point $\overline{x}$ is the alternating sum of traces of Frobenius $Fr_{\overline{x}}$ acting on local cohomology sheaves $\mathcal{H}^i(\mathcal{K}_{\overline{x}})$. This functor intertwins the natural operations on complexes of ℓ-adic sheaves and ones on the functions. It is better to work with the category of ℓ-adic perverse sheaves, which is stable under Verdier duality. For instance, the geometric version of the Iwahori-Hecke algebra $\mathcal{H}_{I_G}$ of G is the category $P_{I_G}(\mathcal{F}l_G)$ of I_G-equivariant perverse sheaves on the affine flag variety $\mathcal{F}l_G$. Following this idea, the geometric version of the space of $I_H \times I_G$-invariants of the Schwartz space $\mathcal{S}(\Pi(F))$ is the category of $I_H \times I_G$-equivariant perverse sheaves on $\Pi(F)$. At the level of functions the space $\mathcal{S}^{I_H \times I_G}(\Pi(F))$ is naturally a module over the Iwahori-Hecke algebras of G and H. This action by convolution geometrizes and defines an action of the category $P_{I_G}(\mathcal{F}l_G)$ (resp. $P_{I_H}(\mathcal{F}l_H)$) on $P_{I_H \times I_G}(\Pi(F))$ and more generally on the derived category $D_{I_H \times I_G}(\Pi(F))$.

Denote by Gr_G the affine Grassmannian $G(F)/G(\mathcal{O})$. We define a stratification of the space $\Pi(F)$ by $H(\mathcal{O}) \times I_G$-orbits and describe the irreducible objects of the category $P_{H(\mathcal{O}) \times I_G}(\Pi(F))$ of $H(\mathcal{O}) \times I_G$-equivariant perverse sheaves on $\Pi(F)$, which are indexed by the lattice of cocharacters of G. The first result of this thesis affirms that there exists an isomorphism between the Grothendieck group of the category $P_{H(\mathcal{O}) \times I_G}(\Pi(F))$ and the Grohtendieck group of the category $P_{I_G}(Gr_G)$ equipped with a left action of $K(P_{I_G}(\mathcal{F}l_G))$. The category $\mathrm{Rep}(\check{H})$ acts on both sides via the restriction functor from $\mathrm{Rep}(\check{H})$ to $\mathrm{Rep}(\check{G} \times \mathbb{G}_\mathfrak{m})$ corresponding to a homomorphism $\check{G} \times \mathbb{G}_\mathfrak{m} \to \check{H}$.

The second result consists in describing the irreducible elements of the category $P_{I_G \times I_H}(\Pi(F))$ which are indexed by $X_G \times S_{n,m}$, where X_G is the lattice of cocharacters of G and $S_{n,m}$ is a finite set. When $n = m$, we can calculate the action of the irreducible elements of $P_{I_G}(\mathcal{F}l_G)$ on a specific object, and we construct an equivalence of categories between $P_{I_G}(\mathcal{F}l_G)$ and $P_{I_H}(\mathcal{F}l_H)$. This enables us to exchange the action of $P_{I_G}(\mathcal{F}l_G)$ and $P_{I_H}(\mathcal{F}l_H)$. The functor realizing this equivalence of categories comes from an anti-involution on the group $G(F)$ itself. Among our partial results, for $n \leq m$, we define a grading on $P_{I_G}(\mathcal{F}l_G)$ which is compatible with the convolution product on $P_{I_G}(\mathcal{F}l_G)$. Moreover, we define a filtration on $P_{I_H \times I_G}(\Pi(F))$ which is compatible with the previous grading and hence with the action of $P_{I_G}(\mathcal{F}l_G)$. This enables us to control the action of the category $P_{I_G}(\mathcal{F}l_G)$ on $P_{I_H \times I_G}(\Pi(F))$.

When $n \leq m$, the computation of the geometric action of the irreducible elements of $P_{I_G}(\mathcal{F}l_G)$ on the irreducible elements of the category $P_{I_H \times I_G}(\Pi(F))$ seems complicated, however we can compute this action on some simple objects (the extensions by zero) which can be translated at the level of functions: we construct a submodule Θ in $K(D_{I_H\times I_G}(\Pi(F)))$ and we prove that it is a free module of rank C_m^n over the Iwahori-Hecke algebra $\mathcal{H}_{I_G}$ of G by constructing an explicit basis of this module. We define entirely the action of the Iwahori-Hecke algebra $\mathcal{H}_{I_H}$ of H on this basis and we obtain a complete description of the module structure of this submodule.

The third result consists in showing (at least at the level of Grothendieck groups) that the submodule Θ is isomorphic to a module induced from a parabolic subalgebra of $\mathcal{H}_{I_H}$. The answer is formulated rather at the level of Grothendieck groups for simplicity, however most of our computations are done at the level of derived categories. When $n = 1$ and $m > 1$, the irreducible elements of $P_{I_H\times I_G}(\Pi(F))$ are indexed by $\mathbb{Z}$ and the geometric situation is simpler to describe: we can compute entirely the geometric action of $P_{I_H}(\mathcal{F}l_H)$ on the irreducible elements of $P_{I_H\times I_G}(\Pi(F))$. We show that the Grothendieck group of $DP_{I_H\times I_G}(\Pi(F))$ is a free module of rank m over $\mathrm{R}(\check{G}\times\mathbb{G}_{\mathfrak{m}})$ and we explicit a basis.

In the last chapter, we define a general conjecture on the geometric Langlands functoriality at the Iwahori level. We fix two connected reductive groups G and H and a homomorphism $\sigma : \check{G}\times\mathbb{G}_{\mathfrak{m}} \to \check{H}$, where $\check{G}$ and $\check{H}$ are the respective Langlands dual groups of G and H over $\overline{\mathbb{Q}}_\ell$. We construct a stack $\mathcal{X}$ and conjecture that the Grothendieck group of this stack, which is a bimodule under the natural action of the two affine Hecke algebras of G and H, realizes the geometric Langlands functoriality at the Iwahori level for the homomorphism σ.

We consider this conjecture explicitly when $G = \mathbf{GL}_n$ and $H = \mathbf{GL}_m$ and for a specific σ. By relating this conjecture with the geometric Howe correspondence for the pair $(\mathbf{GL}_n, \mathbf{GL}_m)$, we conjecture that the bimodule $K(\mathcal{X})$ under the action of affine Hecke algebras of G and H is isomorphic after specialization to the Grothendieck group $K(DP_{I_H\times I_G}(\Pi(F)))$ of the category $DP_{I_H\times I_G}(\Pi(F))$.

In the case of $n = 1$ and $m > 1$, we obtain a simpler description of the stack $\mathcal{X}$. In fact, in this case $K(\mathcal{X})$ is isomorphic to the $\check{G}\times\mathbb{G}_{\mathfrak{m}}$-equivariant K-theory of the Springer fibre $\mathcal{B}_{\check{H},x}$ over a nilpotent subregular element x of $\check{H}$. Finally, we get the most important result of this thesis: the $\check{G}\times\mathbb{G}_{\mathfrak{m}}$-equivariant K-theory of the Springer fibre $\mathcal{B}_{\check{H},x}$ is isomorphic after specialization to $K(DP_{I_H\times I_G}(\Pi(F)))$ as a left module under the action of the affine Hecke algebra of H. Both members of this isomorphism are free modules of rank m over the ring of representations $\mathrm{R}(\check{G}\times\mathbb{G}_{\mathfrak{m}})$ of $\check{G}\times\mathbb{G}_{\mathfrak{m}}$, and the ring of representations of $\check{H}$ acts via the restriction morphism $\mathrm{Res}^\sigma : \mathrm{R}(\check{H}) \to \mathrm{R}(\check{G}\times\mathbb{G}_{\mathfrak{m}})$.

Detailed content of the thesis

This thesis consists of four chapters. In the first chapter we present some remindings as well as some already known results. Chapter two and three are devoted to geometrization and to the study of the Howe correspondence at the Iwahori level for the dual reductive pair $(\mathbf{GL}_n, \mathbf{GL}_m)$. In chapter four we establish a conjecture on the geometric Langlands functoriality at the Iwahori level.

Chapter 1 : In this chapter we give some remindings (root datum, Hecke algebras, perverse sheaves, affine Grassmannian and affine flag variety), and we introduce some notation which will be used in the next chapters.

Chapter 2 : This chapter is formed by the four following sections :

§2.1 This section consists of remindings on the classical Howe correspondence (at level of functions), and of an interpretation of this correspondence in the language of modules over spherical Hecke algebras (in the unramified case) and modules over Iwahori-Hecke algebras (in the tamely ramified case).

§2.2 This section is devoted to the geometrization of the classical objects. In subsection 2.1.1 we introduce different categories used afterwards. Especially, we construct the category $P_{I_H \times I_G}(\Pi(F))$ of $I_H \times I_G$-equivariant perverse sheaves on $\Pi(F)$ which is the geometric analog of the space of $I_H \times I_G$-invariants in the space of Schwartz functions $\mathcal{S}(\Pi(F))$. In the subsection 2.2.2 we define the geometric action of the categories $P_{I_G}(\mathcal{F}l_G)$ and $P_{I_H}(\mathcal{F}l_H)$ on the category $P_{I_H \times I_G}(\Pi(F))$, and more generally on the derived category $D_{I_H \times I_G}(\Pi(F))$. These Hecke functors are denoted by $\overleftarrow{H}_G(,)$ and $\overleftarrow{H}_H(,)$. In the subsection 2.2.4 we explain the relation between the geometric action and the classical action at the level of functions. Indeed, for any function f in $\mathcal{S}^{I_G}(\Pi(F))$ and for any g in $G(F)$ then the geometric action is equivalent to the action of $G(F)$ on $\mathcal{S}^{I_G}(\Pi(F))$ given for any x in $\Pi(F)$ by $g.f(x) = |\det(g)|^{-1/2} f(g^{-1}x)$.

§2.3 In this section we study the category $P_{H(\mathcal{O}) \times I_G}(\Pi(F))$ of $H(\mathcal{O}) \times I_G$-equivariant perverse sheaves on $\Pi(F)$. In subsection 2.3.1 we describe the irreducible objects of this category which are parameterized by the lattice of cocharacters X_G of G. We denote by Π_λ the stratum of $\Pi(F)$ under the action of $H(\mathcal{O}) \times I_G$ indexed by λ. The subsection 2.3.2 is devoted to the study of the module structure of $P_{H(\mathcal{O}) \times I_G}(\Pi(F))$ under the action of $P_{I_G}(Gr_G)$ and $\mathrm{Rep}(\check{H})$. Denote by $\mathcal{A}^\lambda$ the constant perverse sheaf of the I_G-orbit passing through $t^\lambda G(\mathcal{O})$ in Gr_G and denote by I_0 the constant perverse sheaf of Π. We obtain the first result of this thesis: let $\sigma : \check{G} \times \mathbb{G}_m \to \check{H}$ be a fixed homomorphism (cf. [Lys11]) and denote by $\mathrm{Res}^\sigma : \mathrm{Rep}(\check{H}) \to \mathrm{Rep}(\check{G} \times \mathbb{G}_m)$ the geometric restriction functor corresponding to σ.

Theorem 1 (Theorem 2.3.15 and Corollary 2.3.17, Chapter 2)**.** *For any λ in X_G and $\mathcal{T}$ in $P_{H(\mathcal{O})}(Gr_H)$ we have the following isomorphism*

$$\overleftarrow{H}_H(\mathcal{T}, \mathrm{IC}(\Pi_\lambda)) \widetilde{\to} \overleftarrow{H}_G(\mathcal{A}^\lambda \star \mathrm{gRes}^\sigma(\mathcal{T}), I_0).$$

Moreover, the functor $D_{I_G}(Gr_G) \to D_{H(\mathcal{O}) \times I_G}(\Pi(F))$ given by $\mathcal{A} \mapsto \overleftarrow{H}_G(\mathcal{A}, I_0)$ yields an isomorphism at the level of Grothendieck groups $K(P_{I_G}(Gr_G)) \widetilde{\to} K(P_{H(\mathcal{O}) \times I_G}(\Pi(F)))$ commuting with the above actions of $K(P_{H(\mathcal{O})}(Gr_H))$ and $K(P_{I_G}(\mathcal{F}l_G))$.

§2.4 We give a description of the irreducible objects of the category $P_{I_H \times I_G}(\Pi(F))$. This is done in two steps. First, in subsection 2.4.1, we consider the case $n = m$ and we show that the irreducible objects are parameterized in this case by the affine extended Weyl group $\widetilde{W}_G$. Let $S_{n,m}$ be the set of pairs (s, I_s), where I_s is a subset of n elements of $\{1, \dots, m\}$ and $s : I_s \longrightarrow \{1, \dots, n\}$ is a bijection. Denote by $\Pi_{N,r} = t^{-N}\Pi / t^r \Pi$, where N and r are two integers such that $N + r > 0$. Then in subsection 2.4.2 we consider the general case $n \leq m$ and we obtain the second principal result of this thesis:

Theorem 2 (Theorem 2.4.8, Chapter 2)**.** *The irreducible objects of the category $P_{I_H \times I_G}(\Pi(F))$ are parameterized by $X_G \times S_{n,m}$. For any element $w = (\lambda, s)$ in $X_G \times S_{n,m}$, the irreducible object of $P_{I_H \times I_G}(\Pi(F))$ indexed by w is the intersection cohomology sheaf of the orbit $\Pi^w_{N,r}$ indexed by w in $\Pi_{N,r}$.*

Chapter 3 : This chapter is formed by the six following sections:

§3.1 In this section we restrict ourselves to the case $n = m$. First, we would like to compute the action of the irreducible objects of $P_{I_G}(\mathcal{F}l_G)$ and $P_{I_H}(\mathcal{F}l_H)$ on some particular object $\mathcal{I}^{w_0!}$, where w_0 is the longest element of the finite Weyl group of G. We observe that $\overleftarrow{H}_G(L_{w!}, \mathcal{I}^{w_0!})$ is isomorphic to $\mathcal{I}^{\overline{w w_0}!}$, where $w \to \overline{w}$ is an anti-involution defined over the affine extended Weyl group $\widetilde{W}_G$ of G.

§3.2 For any $n \leq m$, we define a filtration by $\mathbb{Z}$ on the category $P_{I_H \times I_G}(\Pi(F))$. We also define a grading on $P_{I_G}(\mathcal{F}l_G)$ by $\mathbb{Z}$ by taking in consideration the connected components of the flag variety $\mathcal{F}l_G$. This grading is compatible with the convolution product on $P_{I_G}(\mathcal{F}l_G)$ and the filtration is compatible with the action of $P_{I_G}(\mathcal{F}l_G)$. This enables us to control the action of $P_{I_G}(\mathcal{F}l_G)$ on $P_{I_H \times I_G}(\Pi(F))$.

§3.3 We reconsider the case $n = m$ and complete the description of the module generated by $\mathcal{I}^{w_0!}$ under the action of the Iwahori-Hecke algebras of G and H. By using this study, we construct an equivalence of categories between $P_{I_G}(\mathcal{F}l_G)$ and $P_{I_H}(\mathcal{F}l_H)$ which defines at the level of functions an anti-involution of Iwahori-Hecke algebras $\mathcal{H}_{I_G}$ and $\mathcal{H}_{I_H}$. Moreover, this equivalence of categories enables us to exchange the action of $P_{I_G}(\mathcal{F}l_G)$ and $P_{I_H}(\mathcal{F}l_H)$ while the groups are of the same rank.

Theorem 3 (Theorem 3.3.6, Chapter 3)**.** *Assume $n = m$. There exists an equivalence of categories σ from $P_{I_G}(\mathcal{F}l_G)$ to $P_{I_H}(\mathcal{F}l_H)$ sending L_w (see Definition 1.5.3) to $L_{w_0 \overline{w} w_0}$, where w runs through $\widetilde{W}_G$. Additionally it verifies the following properties: for any w and w' in $\widetilde{W}_G$ we have*

$$\overleftarrow{H}_G(L_w, \mathcal{I}^{w_0!}) \xrightarrow{\sim} \overleftarrow{H}_H(\sigma(L_w), \mathcal{I}^{w_0!}) \text{ and } \sigma(L_w \star L_{w'}) = \sigma(L_{w'}) \star \sigma(L_w).$$

The functor σ comes from an anti-involution of $G(F)$ sending an element g of $G(F)$ to $w_0{}^t g w_0$. Remark that we obtain a true involution by composing this anti-involution with the anti-involution of $G(F)$ sending g to g^{-1}. This involution is used further in the construction of the bimodule $\mathcal{S}_0$ when we consider the right action of $P_{I_H}(\mathcal{F}l_H)$ instead of the usual left action.

§3.4 In this section we consider the general case $n < m$. First, in subsection 3.4.1, we consider a submodule Θ over $K(P_{I_G}(\mathcal{F}l_G))$ of $K(D_{I_H \times I_G}(\Pi(F)))$ generated by the elements $\mathcal{I}^{\mu!}$, where I_μ runs through all possible subsets of n elements in $\{1, \dots, m\}$. We show that the module Θ is free of rank C_m^n over $K(P_{I_G}(\mathcal{F}l_G))$. The elements $\mathcal{I}^{\mu!}$ form a basis of this module over $K(P_{I_G}(\mathcal{F}l_G))$. In subsection 3.4.2, by a series of propositions, we compute the action of $\mathcal{H}_{I_G}$ on $\mathcal{I}^{\mu}$ and we obtain a complete description of the module structure of Θ over $\mathcal{H}_{I_G}$ and $\mathcal{H}_{I_H}$.

§3.5 The purpose of this section is to show that Θ is a submodule with respect to the right $K(P_{I_H}(\mathcal{F}l_H))$-action in $K(D_{I_G \times I_H}(\Pi(F)))$ introduced in Remark 2.2.12 and identify Θ as the induced representation from a parabolic subalgebra of $\mathcal{H}_{I_H}$. The considerations in this subsection are essentially on the level of Grothendieck groups (we formulate them at the level of derived categories however when this is possible, one may assume that we work over a finite field with pure complexes only). In order to be able to endow this module with an action of $P_{I_H}(\mathcal{F}l_H)$, one should consider the right action which is related to the left action via the map sending any element g of $G(F)$ to g^{-1}. For w in $\widetilde{W}_G$ we compute the action of the irreducible objects $L_{w!}$ (see Definition 1.5.3) on $\mathcal{I}^{w_0!}$. Let $\mathcal{S}_0$ be the $\bar{\mathbb{Q}}_\ell$-subspace of $K(D_{I_G \times I_H}(\Pi(F)) \otimes \bar{\mathbb{Q}}_\ell$ generated by the elements $\mathcal{I}^{(w \cdot w_0)!}$ [see Definition 2.4.10], where w runs through $\widetilde{W}_G$. The module $\mathcal{S}_0$ is a free module of rank one over $\mathcal{H}_{I_G}$. In subsection 3.5.1 we define the action of the irreducible objects indexed by the finite Weyl group of G and H on $\mathcal{S}_0$. Let us remind that an element $w = t^\lambda w$ of $\widetilde{W}_G$ is considered as (λ, τ) in $X_G \times S_{n,m}$. In this way the element w_0 is viewed as an element of $X_G \times S_{n,m}$ whose first component vanishes. In subsection 3.5.2 we consider the standard Levi subgroup M of H corresponding to the partition $(n, m-n)$ of m and we recall briefly the construction of the subalgebra $\mathcal{H}_{I_M}$ of the Iwahori-Hecke algebra $\mathcal{H}_{I_H}$ and some properties according to [Pra05]. In subsection 3.5.3, we endow $\mathcal{S}_0$ with a right action of $\mathcal{H}_{I_M}$ and by parabolic induction we construct an induced module. We obtain the two following results:

Theorem 4 (Theorem 3.5.22, Chapter 3).
The space $\mathcal{S}_0$ is a submodule of $K(D_{I_G \times I_H}(\Pi(F))) \otimes \overline{\mathbb{Q}}_\ell$ for the right action of $\mathcal{H}_{I_M}$.

Theorem 5 (Theorem 3.5.23, Chapter 3). *The adjunction map $\alpha : \mathcal{S}_0 \otimes_{\mathcal{H}_{I_M}} \mathcal{H}_{I_H} \to \mathcal{S}^{I_H \times I_G}(\Pi(F))$ is injective and its image equals $\Theta \otimes \overline{\mathbb{Q}}_\ell$.*

The standard Levi subgroup M defined above is isomorphic to $\mathbf{GL}_n \times \mathbf{GL}_{m-n}$. In the rest of this subsection we show that the action of the Iwahori-Hecke algebra of the factor $\mathbf{GL}_n$ of M identifies with the action of $\mathcal{H}_{I_G}$ via the anti-involution defined in Theorem 3. The action of the Iwahori-Hecke algebra of $\mathbf{GL}_{m-n}$ is by shifting by $[-\ell(w)]$, where ℓ denotes the length function on $\widetilde{W}_G$.

§3.6 This section is devoted to an explicit study of the case $n = 1$ and $m > 1$. Similar to the general case we define the irreducible objects of the category $P_{I_H \times I_G}(\Pi(F))$ which are parameterized in this case by $\mathbb{Z}$. Unlike the general case, we are able here to define the action of $P_{I_H}(\mathcal{F}l_H)$ geometrically. First we define the action of objects corresponding to simple reflections in the finite Weyl group of H, then the action of the unique affine simple reflection, and at last the action of elements of length zero. Moreover, we will observe that any irreducible object of the category $P_{I_H \times I_G}(\Pi(F))$ is obtained by the action of some length zero element.

Let $DP_{I_H \times I_G}(\Pi(F))$ be the category whose objects are the direct sums of the objects of the category $P_{I_H \times I_G}(\Pi(F))[i]$, where i runs through $\mathbb{Z}$. We define an action of the multiplicative group $\mathbb{G}_m$ on $DP_{I_H \times I_G}(\Pi(F))$ by cohomological shift -1. Denote by g the standard representation of $\check{G}$. For any integer j, the representation g^j sends each irreducible element IC^k to IC^{k-mj}. In this way, we endow the Grothendieck group of the category $DP_{I_H \times I_G}(\Pi(F))$ with an action of the ring of representations $\mathrm{R}(\check{G} \times \mathbb{G}_m)$ of $\check{G} \times \mathbb{G}_m$ and we show that it is a free module of rank m over this ring. We also show that the action of the respective centers of $P_{I_G}(\mathcal{F}l_G)$ and $P_{I_H}(\mathcal{F}l_H)$ are compatible.

Theorem 6 (Theorem 3.6.15, Chapter 3). *The action of respective centers of $P_{I_H}(\mathcal{F}l_H)$ and $P_{I_G}(\mathcal{F}l_G)$ are compatible. More precisely,* $\mathrm{Rep}(\check{H})$ *acts via the geometric restriction functor* Res^σ *on irreducible objects* IC^k *for all k in $\mathbb{Z}$.*

Chapter 4: This chapter is devoted to the study of the geometric Langlands functoriality at the Iwahori level. We prove that the geometric Howe correspondence in a particular case comes from the functoriality at the Iwahori level.

§4.1 The subsection 4.1.1 consists on the construction of a conjectural bimodule realizing the geometric Langlands functoriality at the Iwahori level for some homomorphism σ. More precisely, we consider two connected reductive groups G and H and we denote by $\check{G}$ and $\check{H}$ their respective Langlands dual groups over $\overline{\mathbb{Q}}_\ell$. We assume that the respective derived groups of $\check{G}$ and $\check{H}$ are simply connected. We fix a homomorphism $\sigma : \check{G} \times \mathbb{G}_\mathfrak{m} \to \check{H}$. For constructing the bimodule, we will use the equivariant K-theory. Let us recall that the affine Hecke algebra associated with H is isomorphic to the $\check{H} \times \mathbb{G}_\mathfrak{m}$-equivariant K-theory $K^{\check{H}\times\mathbb{G}_\mathfrak{m}}(Z_{\check{H}})$ of the Steinberg variety $Z_{\check{H}} = \tilde{\mathcal{N}}_{\check{H}} \times_{\mathcal{N}_{\check{H}}} \tilde{\mathcal{N}}_{\check{H}}$ of $\check{H}$. We use the notation and results of [CG97]. The first conjecture of this section is the following conjecture:

Conjecture 1 (Conjecture 4.1.6, Chapter 4)**.** *The bimodule over the affine Hecke algebras* $K^{\check{G}\times\mathbb{G}_\mathfrak{m}}(\tilde{\mathcal{N}}_{\check{G}} \times_{\mathcal{N}_{\check{G}}} \tilde{\mathcal{N}}_{\check{G}})$ *and* $K^{\check{H}\times\mathbb{G}_\mathfrak{m}}(\tilde{\mathcal{N}}_{\check{H}} \times_{\mathcal{N}_{\check{H}}} \tilde{\mathcal{N}}_{\check{H}})$ *realizing the local geometric Langlands functoriality at the Iwahori level for the map* $\sigma : \check{G} \times \mathbb{G}_\mathfrak{m} \to \check{H}$ *identifies with* $K(\mathcal{X})$*, where*

$$\mathcal{X} = (\tilde{\mathcal{N}}_{\check{G}}/(\check{G} \times \mathbb{G}_m)) \times_{\mathcal{N}_{\check{H}}/(\check{H}\times\mathbb{G}_m)} (\tilde{\mathcal{N}}_H/(\check{H} \times \mathbb{G}_m)).$$

In subsection 4.1.2 we give some different descriptions of the stack $\mathcal{X}$.

§4.2 In this section we study the above conjecture under the following hypothesis: $G = \mathbf{GL}_n$, $H = \mathbf{GL}_m$ and the morphism σ is obtained by the composition: $\mathbf{GL}_n \times \mathbb{G}_\mathfrak{m} \to \mathbf{GL}_n \times \mathrm{SL}_2 \to \mathbf{GL}_n \times \mathbf{GL}_{m-n} \to \mathbf{GL}_m$, where $\mathbf{GL}_n$ (resp $\mathbf{GL}_m$) are considered as dual Langlands groups over $\overline{\mathbb{Q}}_\ell$, the last morphism is the inclusion of the standard Levi subgroup associated to the partition $(n, m-n)$ of m, and the morphism $\xi : \mathrm{SL}_2 \to \mathbf{GL}_{m-n}$ corresponds to the principal unipotent orbit. We can conjecturally relate the bimodule $K(\mathcal{X})$ and the K-theory of the category $DP_{I_H\times I_G}(\Pi(F))$, defining the Howe correspondence. We obtain the following conjecture:

Conjecture 2 (Conjecture 4.2.1, Chapter 4)**.** *The bimodule* $K(\mathcal{X})$ *over each affine Hecke algebra* $K^{\check{G}\times\mathbb{G}_\mathfrak{m}}(\tilde{\mathcal{N}}_{\check{G}} \times_{\mathcal{N}_{\check{G}}} \tilde{\mathcal{N}}_{\check{G}})$ *and* $K^{\check{H}\times\mathbb{G}_\mathfrak{m}}(\tilde{\mathcal{N}}_{\check{H}} \times_{\mathcal{N}_{\check{H}}} \tilde{\mathcal{N}}_{\check{H}})$ *is isomorphic after specialization to* $K(DP_{I_H\times I_G}(\Pi(F)))$.

We have already studied in chapter three the module structure of one of the members of this conjecture, namely the module structure of $K(DP_{I_H\times I_G}(\Pi(F)))$. In the remaining part of this section we define a stratification on $\mathcal{X}$ and a filtration on $K(\mathcal{X})$ which we suspect to be compatible to the one defined on $K(DP_{I_H\times I_G}(\Pi(F)))$ under the action of the affine Hecke algebras of G and H.

§4.3 In subsection 4.3.1, we deal with Conjecture 2 when $n = 1$ and $m > 1$. In this case it is easier to explicit the bimodule $K(\mathcal{X})$. Indeed, this bimodule is isomorphic to the $\check{G} \times \mathbb{G}_\mathfrak{m}$-equivariant K-theory $K^{\check{G}\times\mathbb{G}_m}(\mathcal{B}_{\check{H},x})$ of the Springer fibre of $\check{H}$ over a subregular nilpotent element x of $\mathcal{N}_{\check{H}}$. We show that the Springer fibre $\mathcal{B}_{\check{H},x}$ is a configuration of $m-1$ projective lines with m marked points and (by using the cellular fibration lemma [CG97, Lemma 5.5]) we prove that $K^{\check{G}\times\mathbb{G}_m}(\mathcal{B}_{\check{H},x})$ is a free module of rank m over $\mathrm{R}(\check{G} \times \mathbb{G}_\mathfrak{m})$. We affirm that the $\mathrm{R}(\check{H})$-module structure on $K^{\check{G}\times\mathbb{G}_\mathfrak{m}}(\mathcal{B}_{\check{H},x})$ comes from the functor $\mathrm{Res}^\sigma : \mathrm{R}(\check{H}) \to \mathrm{R}(\check{G} \times \mathbb{G}_m)$ corresponding to the homomorphism $\sigma : \check{G} \times \mathbb{G}_\mathfrak{m} \to \check{H}$. The major result of this thesis is the following theorem:

Theorem 7 (Theorem 4.3.3, Chapter 4)**.** *Assume* $G = \mathbf{GL}_1$ *and* $H = \mathbf{GL}_m$ *with* $m > 1$. *The modules* $K^{\check{G}\times\mathbb{G}_\mathfrak{m}}(\mathcal{B}_{\check{H},x})$ *and* $K(DP_{I_H\times I_G}(\Pi(F)))$ *are free of rank* m *over* $\mathrm{R}(\check{G} \times \mathbb{G}_\mathfrak{m})$, *and* $\mathrm{R}(\check{H})$

acts via $\mathrm{Res}^{\sigma} : \mathrm{R}(\check{H}) \longrightarrow \mathrm{R}(\check{G}\times\mathbb{G}_{\mathfrak{m}})$. *Moreover, the morphism* γ_1 *induces an isomorphism of* $\mathbb{H}_H$*-modules between* $K^{\check{G}\times\mathbb{G}_{\mathfrak{m}}}(\mathcal{B}_{\check{H},x})$ *and* $K(DP_{I_H\times I_G}(\Pi(F)))$, *where the* $\mathbb{Z}[s,s^{-1}]$*-module structure on* $\overline{\mathbb{Q}}_\ell$ *is given by the character sending* s *to* $q^{1/2}$.

This theorem relates the computations of section 3.6 in chapter three and the conjecture of chapter four in the case where $n = 1$ and $m > 1$. It shows that the geometric Howe correspondence comes from the geometric Langlands functoriality at the Iwahori level for the homomorphism σ. The subsection 4.3.2 is devoted to the proof of the Theorem 7. Our strategy consists in constructing an adequate basis of $K^{\check{G}\times\mathbb{G}_{\mathfrak{m}}}(\mathcal{B}_{\check{H},x})$ on which the action of the affine Hecke algebra of H is the same as the one in the geometric. The latter action has already been computed in section 3.6 of chapter three. Surprisingly this basis is different from the basis constructed by Lusztig in [Lus02].

Appendix: In Appendix A, we remind some generalities on the convolution actions. In Appendix B, we present a geometric analog of Jacquet functors in the Iwahori case and we prove that they are compatible with the action of Wakimoto objects. We also show that they preserve pure perverse sheaves of weight zero.

Chapter 1

Remindings

Throughout this chapter we will assume that $\mathbf{k}$ is an algebraically closed field of characteristic p except in § 1.5.3, where $\mathbf{k}$ is assumed to be finite. Let F be $\mathbf{k}((t))$ and $\mathcal{O}$ be $\mathbf{k}[[t]]$, the ring of integers of F. Let ℓ be a prime number different from p.

1.1 Root datum, finite, affine and affine extended Weyl group

The notion of root datum is a slight generalization of the notion of root system, which is quite useful for the theory of reductive groups. For the theory of root system we refer to [Hum78]. For the theory of root datum and reductive groups, the references are [Lus89], [Spr79] and [Bou02]. Let G be a connected reductive group over $\mathbf{k}$ and T be a maximal torus in G. Denote by $(\check{X}, \check{R}, X, R)$ the root datum associated with (G, T). Here and throughout we denote by $\check{X}$ the characters of T and X denotes the cocharacter lattice of T. The set $\check{R}$ is the set of roots and R is the set of coroots.
A choice of a Borel subgroup B containing T defines a unique ordering of $\check{R}$. It follows that one can associate to the triple (G, B, T), a set of positive and negative roots as well as a basis formed by simple roots denoted Δ. The reductive group $\check{G}$ associated with the dual root datum is called the Langlands dual group of G. We will consider it as a reductive group over $\overline{\mathbb{Q}}_\ell$.

Denote by $s_{\check{\alpha}}$ in $\mathrm{Aut}(X)$ the reflection corresponding to the root $\check{\alpha}$. Sometimes we write s_α for $s_{\check{\alpha}}$. The subgroup of $\mathrm{Aut}(X)$ generated by $s_{\check{\alpha}}$ is the finite Weyl group of (G, T). We denote it by W_G. It is a Coxeter group with respect to the set of simple reflections corresponding to a fixed choice of simple roots Δ in $\check{R}$. The finite Weyl group, W_G of the root datum (G, T) naturally identifies with the quotient $N_G(T)/T$, where $N_G(T)$ is the normalizer of T inside G.

Let t be a fixed symbol. Let $\widetilde{W}_G$ be the semi-direct product $W_G \ltimes X$, where W_G acts on X in a natural way. Its elements are wt^x $(w \in W_G, x \in X)$ with multiplication given by

$$wt^x.w^{'}t^{x^{'}} = ww^{'}t^{w^{'-1}(x)+x^{'}}.$$

We will assume additionally that the root datum is irreducible (this is always true in our cases of interest). Then there exists a unique highest root. Let $S_{aff} = \{s_\alpha | \alpha \in \Delta\} \cup \{s_0\}$, where $s_0 = t^{-\alpha_0} s_{\check{\alpha}_0}$, where $\check{\alpha}_0$ the unique highest root. Then the subgroup W_{aff} of $\widetilde{W}_G$ generated by S_{aff} is the affine Weyl group associated with the root datum. Moreover, (W_{aff}, S_{aff}) is a Coxeter system.

There is a length function ℓ defined on the Coxeter group W_{aff}. It extends to a length function on $\widetilde{W}_G$ in the following way.

Definition 1.1.1. *[Lus89, 1.4] We define $\ell : \widetilde{W}_G \longrightarrow \mathbb{N}$ by*

$$\ell(wt^x) = \Sigma_{\check{\alpha}\in\check{R}^+, w(\check{\alpha})\in\check{R}^-}|\langle x, \check{\alpha}\rangle + 1| + \Sigma_{\check{\alpha}\in\check{R}^+, w(\check{\alpha})\in\check{R}^+}|\langle x, \check{\alpha}\rangle|.$$

We have $\ell(w) = \ell(w^{-1})$ for all w in $\widetilde{W}_G$ and the length function verifies the following properties for all x in X and α in Δ :

1. If $\langle x, \check{\alpha}\rangle > 0$ then $\ell(s_\alpha t^x) = \ell(t^x) + 1$ and $\ell(t^x s_\alpha) = \ell(t^x) - 1$.
2. If $\langle x, \check{\alpha}\rangle < 0$ then $\ell(s_\alpha t^x) = \ell(t^x) - 1$ and $\ell(t^x s_\alpha) = \ell(t^x) + 1$.
3. If $\langle x, \check{\alpha}\rangle = 0$ then $\ell(s_\alpha t^x) = \ell(t^x) + 1$ and $\ell(t^x s_\alpha) = \ell(t^x) + 1$.

We define the set of dominant elements in X as follows:

$$\begin{aligned} X^+ &= \{x \in X | \forall\alpha \in \Delta, \langle x, \check{\alpha}\rangle \geq 0\} \\ &= \{x \in X | \forall\alpha \in \Delta, \ell(s_\alpha t^x) = \ell(t^x) + 1\}. \end{aligned}$$

By the definition of dominant elements and the length function, for w in W and x in X^+, one gets $\ell(t^x) = \Sigma_{\check{\alpha}\in\check{R}^+}\langle x, \check{\alpha}\rangle$ and $\ell(wt^x) = \ell(w) + \ell(t^x)$. Additionally, we deduce that for $x, x^{'}$ dominant cocharacters, $\ell(t^x.t^{x^{'}}) = \ell(t^x) + \ell(t^{x^{'}})$.

Let Q denote a subgroup of X generated by coroots. One has $W_{aff} \simeq W_G \rtimes Q$. The subgroup W_{aff} is normal in $\widetilde{W}_G$ and admits a complementary subgroup

$$\Omega = \{w \in \widetilde{W}_G | \ell(w) = 0\},$$

the elements of length zero. It is well known that Ω is an abelian group isomorphic to X/Q. Additionally we have the following exact sequence

$$1 \to W_{aff} \to \widetilde{W}_G \to X/Q \to 1.$$

Thus the affine extended Weyl group $\widetilde{W}_G$ is isomorphic to $W_{aff} \rtimes \Omega$. The algebraic fundamental group $\pi_1(G)$ of G is exactly Ω.

Consider $G = \mathbf{GL}_n$ the linear algebraic group over $\mathbf{k}$. Let T be the maximal torus T of diagonal matrices and B the Borel subgroup of G containing T consisting of upper triangular matrices. We have $B = TU$, where U is the subgroup of upper triangular unipotent matrices in $\mathbf{GL}_n$. The set of characters $X^*(T) = \mathbb{Z}\check{\epsilon}_1 \oplus \cdots \oplus \mathbb{Z}\check{\epsilon}_n$ and cocharacters $X_*(T) = \mathbb{Z}\eta_1 \oplus \ldots \mathbb{Z}\eta_n$, where $\check{\epsilon}_i$ is the i^{th} projection of T onto $\mathbf{k}^*$ and $\eta_i(t)$ is the diagonal matrix, whose i^{th} entry is t and the other entries are 1. So we have $\langle \check{\epsilon}_i, \eta_j\rangle = \delta_i^j$. Hence the lattice of cocharacters is identified with $\mathbb{Z}^n$ and the finite Weyl group with $\mathfrak{S}_n$ the symmetric group of n elements. The set of roots $\check{R}$ is $\{\check{\epsilon}_i - \check{\epsilon}_j | 1 \leq i \neq j \leq n\}$ and the set of coroots R is $\{\eta_i - \eta_j | 1 \leq i \neq j \leq n\}$. This defines a root datum of $\mathbf{GL}_n$. The reflection corresponding to the root $\check{\epsilon}_i - \check{\epsilon}_j$ (resp. to the coroot $\eta_i - \eta_j$) is the permutation who exchanges $\check{\epsilon}_i$ and $\check{\epsilon}_j$ (resp. η_i and η_j). The subgroup Ω is isomorphic to $\mathbb{Z}$.

Denote by $G(F)$ the set of F-points of G. Consider the natural surjection $G(\mathcal{O}) \to G(\mathbf{k})$ (evaluation at $t = 0$). The inverse image of the Borel subgroup B in $G(\mathcal{O})$ is called the Iwahori subgroup of $G(F)$ denoted by I_G. Denote by U the unipotent radical of the Borel subgroup B and let U^- be the subgroup such that its Lie algebra $\mathfrak{u}^-$ is the sum of root subspaces corresponding to the negative roots. Thus one has $B = TU$. Let $I_G^- = I_G \cap U^-(F)$ and $I_G^+ = I_G \cap U(F)$ then one has the Iwahori factorization: $I_G = I_G^- T(\mathcal{O}) I_G^+ = I_G^+ T(\mathcal{O}) I_G^-$.

1.2 Hecke algebras

1.2.1 Finite Hecke algebra

The references for this section are [Lus89], [Lus83a], [Pra05], [Ber84], [IM], [CG97]. Let v be an indeterminate. Let G be a connected reductive group over $\mathbf{k}$ and T a maximal torus in G. Let (W, S) be the Coxeter group associated with the root datum defined on G, where W is the finite Weyl group and S the set of simple reflections (if there is no ambiguity we will omit the subscript G in W_G).

Definition 1.2.1. *[Lus89, 3.2][CG97, 7.1.1] The Hecke algebra of the Coxeter group (W, S) is a $\mathbb{Z}[v^{-1}, v]$-algebra $\mathbb{H}_W$ with generators T_s, $s \in S$, subject to the following relations:*

1. $T_{s_\alpha}.T_{s_\beta}.T_{s_\alpha}\cdots = T_{s_\beta}.T_{s_\alpha}.T_{s_\beta}\ldots,\qquad m(\alpha,\beta)$ *factors.*
2. $(T_{s_\alpha}+1)(T_\alpha - v) = 0.$

In the above definition $m(\alpha, \beta)$ is the order of $s_\alpha s_\beta$ in the Coxeter group. When $v = 1$, these relations specialize to the relations in the Coxeter group (W, S). Thus one can think of $\mathbb{H}_W$ as a v-analogue of $\mathbb{Z}[W]$. This characterization of the Hecke algebra is due to Lusztig but it is hardly used because the verification of the braid relation is rather a hard task. There is another characterization of this algebra in the following way:

Proposition 1.2.2. *[Bou02, Chapter IV, sec.2, Ex 34] [CG97, 7.1.2] The Hecke algebra $\mathbb{H}_W$ has a free $\mathbb{Z}[v^{-1}, v]$-basis $\{T_w, w \in W\}$ such that the following rules hold:*

1. $(T_s + 1)(T_s - v) = 0$ *if $s \in S$ is a simple reflection.*
2. $T_y.T_w = T_{yw}$ *if $\ell(yw) = \ell(y) + \ell(w)$.*

These rules define completely the ring structure of $\mathbb{H}_W$ thus any algebra satisfying the properties of the proposition is isomorphic to the Hecke algebra $\mathbb{H}_W$.

1.2.2 Affine extended Hecke algebra

Denote by $\mathrm{Rep}(\check{G})$ the category of finite-dimensional representations of $\check{G}$ over $\overline{\mathbb{Q}}_\ell$, and denote by $\mathrm{R}(\check{G})$ the ring of representations of $\check{G}$. The group algebra $\mathbb{Z}[X]$ is isomorphic to $\mathrm{R}(\check{T})$, the representation ring of the dual torus to T. We will write e^λ for the element of $\mathrm{R}(\check{T})$ corresponding to the coweight λ in X. The affine extended Hecke algebra associated with G was introduced by Bernstein [Ber84] (it first appeared in [Lus83b]) and is isomorphic to the so-called Iwahori-Hecke algebra of a split p-adic group with connected center. The latter was introduced in [IM] and reflects the structure of $C_c(I_G\backslash G(F)/I_G)$ the space of locally constant compactly supported $\overline{\mathbb{Q}}_\ell$-valued functions on $G(F)$ which are bi-invariant under the action of I_G, where $\mathbf{k}$ is assumed to be a finite field.

Definition 1.2.3. *[CG97, 7.1.9][Affine extended Hecke Algebra] The extended affine Hecke algebra $\mathbb{H}_G$ is a free $\mathbb{Z}[v, v^{-1}]$-module with basis $\{e^\lambda T_w | w \in W, \lambda \in X\}$, such that:*

1. *The $\{T_w\}$ span a sub-algebra of $\mathbb{H}_G$ isomorphic to $\mathbb{H}_W$.*
2. *The $\{e^\lambda\}$ span a $\mathbb{Z}[v, v^{-1}]$-sub-algebra of $\mathbb{H}_G$ isomorphic to $\mathrm{R}(\check{T})[v^{-1}, v]$.*
3. *For any $s_\alpha \in S$ with $\langle \lambda, \check{\alpha}\rangle = 0$ we have $T_{s_\alpha} e^\lambda = e^\lambda T_{s_\alpha}$.*
4. *For any $s_\alpha \in S$ with $\langle \lambda, \check{\alpha}\rangle = 1$ we have $T_{s_\alpha} e^{s_\alpha(\lambda)} T_{s_\alpha} = v e^\lambda$.*

The properties (3),(4) together are equivalent to the following useful formula

$$T_{s_\alpha} e^{s_\alpha(\lambda)} - e^\lambda T_{s_\alpha} = (1-v)\frac{e^\lambda - e^{s_\alpha(\lambda)}}{1-e^{-\alpha}}, \tag{1.2.4}$$

where α is a simple coroot, s_α the corresponding simple reflection and $\lambda \in X$. The properties (1) and (2) give us two canonical embeddings of algebras

$$\mathrm{R}(\check{T})[v^{-1}, v] \hookrightarrow \mathbb{H}_G \qquad \textit{and} \qquad \mathbb{H}_W \hookrightarrow \mathbb{H}_G.$$

The multiplication in $\mathbb{H}_G$ gives rise to a $\mathbb{Z}[v^{-1}, v]$-module isomorphism

$$\mathbb{H}_G \simeq \mathrm{R}(\check{T})[v^{-1}, v] \otimes_{\mathbb{Z}[v^{-1},v]} \mathbb{H}_W.$$

This is a v-analogue of the $\mathbb{Z}$-module isomorphism

$$\mathbb{Z}[\widetilde{W}_G] \simeq \mathrm{R}(\check{T}) \otimes_{\mathbb{Z}} \mathbb{Z}[W_G].$$

The center of the extended affine Hecke algebra $\mathbb{H}_G$ is isomorphic to $\mathrm{R}(\check{T})^W[v^{-1}, v]$ [CG97, 7.1.14]. Hence the center is generated by sums of elements e^λ over the Weyl-orbits of cocharacters λ.

1.2.3 Iwahori-Hecke Algebra

Assume temporary that the ground field $\mathbf{k}$ is the finite field $\mathbb{F}_q$. Consider $C_c(I_G\backslash G(F)/I_G)$, the space of locally constant, I_G-bi-invariant compactly supported $\overline{\mathbb{Q}}_\ell$-valued functions on $G(F)$. We fix a Haar measure dx on $G(F)$ such that I_G be of measure 1. This space endowed with the convolution product is an algebra and is called the Iwahori-Hecke algebra of G denoted by $\mathcal{H}_{I_G}$. For any two functions f_1, f_2 in $\mathcal{H}_{I_G}$ the convolution product is defined by:

$$f_1 \star f_2(g) = \int_{g \in G(F)} f_1(x).f_2(x^{-1}.g)dx$$

There are two well-known presentations of this algebra by generators and relations. The first is due to Iwahori-Matsumoto [IM] and the second is by Bernstein in [Lus89] and [Lus83a].

Denote by T_w the characteristic functions of the double cosets $I_G w I_G$ for $w \in \widetilde{W}_G$ and according to **Iwahori – Matsumoto** presentation the set $\{T_w | w \in \widetilde{W}_G\}$ forms a basis of $\mathcal{H}_{I_G}$. For λ in X we will denote by T_{t^λ} the characteristic function of the double coset $I_G t^\lambda I_G$.

For λ in X, let $\tilde{T}_{t^\lambda}$ equal $q^{-\ell(t^\lambda)/2} T_{t^\lambda}$ the normalized characteristic function of the double coset $I_G t^\lambda I_G$. We define a new function in the Iwahori-Hecke algebra:

$$\theta_\lambda = \tilde{T}_{t^{\lambda_1}} \tilde{T}^{-1}_{t^{\lambda_2}}$$

where $\lambda = \lambda_1 - \lambda_2$ and λ_i is dominant for $i = 1, 2$. The set $\{\theta_\lambda T_w | \lambda \in X, w \in W\}$ forms a basis of $\mathcal{H}_{I_G}$. The elements θ_λ verify the following Bernstein relations: for α a simple coroot, s_α the corresponding simple reflection and λ in X,

$$T_{s_\alpha}\theta_{s_{\alpha(\lambda)}} - \theta_\lambda T_{s_\alpha} = (1-q)\frac{\theta_\lambda - \theta_{s_\alpha(\lambda)}}{1-\theta_{-\alpha}}.$$

One sees that the Bernstein relations are the analogue of the relation 1.2.4 in the affine Hecke algebra when v is specialized. This is the **Bernstein** presentation of the Iwahori Hecke algebra. The elements θ_λ generate a commutative subalgebra of $\mathcal{H}_{I_G}$. The Bernstein presentation reflects the description of $\mathcal{H}_{I_G}$ as an equivariant K-theory of the associated Steinberg variety [CG97, Chapter 5], which plays a role in the classification of the representations of $\mathcal{H}_{I_G}$. We will use the K-theoretic description in Chapter 4. The Bernstein presentation has the advantage that one can construct a basis for the center of $\mathcal{H}_{I_G}$ by summing the elements θ_λ over the Weyl-orbits of cocharacters λ. These functions are called the Bernstein functions. One can write each Bernstein function and as a consequence the center of $\mathcal{H}_{I_G}$ as an explicit linear combination of the Iwahori-Matsumoto basis elements T_w for $w \in \widetilde{W}_G$. In the notation of Definition 1.2.3, the functions θ_λ correspond to e^λ. These are the classical Wakitomo objects, the geometric version of these ones is defined in § 1.5.3. By specializing v, we get the following isomorphism

$$\mathcal{H}_{I_G} \simeq \mathbb{H}_G \otimes_{\mathbb{Z}[v^{-1},v]} \overline{\mathbb{Q}}_\ell.$$

Another way of defining the Iwahori-Hecke algebra is to consider the category $P_{I_G}(\mathcal{F}l_G)$ of I_G-equivariant perverse sheaves on the affine flag variety $\mathcal{F}l_G$ of G. This category endowed with the geometric convolution is a geometric realization of $\mathcal{H}_{I_G}$, i.e., $\mathcal{H}_{I_G}$ is isomorphic to the Grothendieck group of this category tensored with $\overline{\mathbb{Q}}_\ell$.

Denote by H_{sph} the space of compactly supported $G(\mathcal{O})$-bi-invariant $\overline{\mathbb{Q}}_\ell$-valued functions on $G(F)$ with the product defined by convolution. The spherical Hecke algebra H_{sph} is a subalgebra of $\mathcal{H}_{I_G}$. By a theorem of Bernstein [Ber84] the center of the Iwahori-Hecke algebra $Z(\mathcal{H}_{I_G})$ is isomorphic to H_{sph}.
The classical Satake isomorphism states that

$$H_{sph} \simeq K(\mathrm{Rep}(\check{G})) \otimes \overline{\mathbb{Q}}_\ell$$

where K denotes the Grothendieck group. The spherical Hecke algebra H_{sph} has a geometric interpretation as well. This is realized by considering $G(\mathcal{O})$-equivariant perverse sheaves on affine Grassmannian Gr_G of G.

Denote by H_W the subalgebra of $\mathcal{H}_{I_G}$ consisting of functions which are extensions by zero from $G(\mathcal{O})$, that is, $C_c(I_G\backslash G(\mathcal{O})/I_G)$. It is an algebra with respect to the convolution product and generated by $\{T_w | w \in W_G\}$. This subalgebra identifies with $C_c(B(k)\backslash G(k)/B(k))$. By specializing v one gets,

$$H_W = \mathbb{H}_W \otimes_{\mathbb{Z}[v,v^{-1}]} \overline{\mathbb{Q}}_\ell.$$

1.3 Perverse sheaves

Most of our results hold over any algebraically closed fields of characteristic p. For results involving weights or purity, we assume our field to be an algebraic closure of $\mathbb{F}_q$ denoted by $\mathbb{F}$. Let Fr_q be the geometric Frobenius in $\mathrm{Gal}(\mathbb{F}/\mathbb{F}_q)$ which is the inverse of the arithmetic Frobenius sending x to x^q.
Let X_0 be a scheme of finite type defined over a finite field $\mathbb{F}_q$. We refer to [BBD82],[Del80] and also [KW01, Chapter 3], [FK88] for the definitions of the bounded category $D^b_c(X_0, \overline{\mathbb{Q}}_\ell)$ of constructible complexes of $\overline{\mathbb{Q}}_\ell$-sheaves. One may also refer to the complete survey [dCM09].

These categories are stable under the usual operations f^*, f_*, $f^!$, $f_!$, derived Hom and tensor product, duality, vanishing and nearby cycles. The standard and the middle perverse t-structure are also defined, and one obtains the category $P(X_0, \overline{\mathbb{Q}}_\ell)$ of perverse sheaves on X_0. If X is the $\mathbb{F}$-scheme of finite type obtained from X_0 by extending the scalars to $\mathbb{F}$, then we obtain in the same way the categories $D^b_c(X, \overline{\mathbb{Q}}_\ell)$ and $P(X, \overline{\mathbb{Q}}_\ell)$.

Suppression of the index $-_0$ denotes the extension of scalars from $\mathbb{F}_q$ to $\mathbb{F}$. For example, if F_0 is a $\overline{\mathbb{Q}}_\ell$-sheaf on X_0, then we denote its pull-back to X by F. Let $\mathbb{D}$ be the Verdier duality functor. The category of constructible ℓ-adic perverse sheaves $P(X, \overline{\mathbb{Q}}_\ell)$ is a full subcategory of the constructible derived category $D^b_c(X, \overline{\mathbb{Q}}_\ell)$.

Proposition 1.3.1 (Intermediate extension). *[KW01]*

1. *Let U be an open subscheme of X and Y be the complement of U in X. We denote by i_Y (resp. j) the closed (resp. open) embedding of Y (resp. U) in X. Then for every perverse sheaf B on U there exists an extension $\overline{B}$ of B in $P(X, \overline{\mathbb{Q}}_\ell)$ such that $\overline{B}$ has neither quotients nor subobjects of type i_*A for A in $P(Y, \overline{\mathbb{Q}}_\ell)$. This extension is unique up to a unique isomorphism and is denoted by $j_{!*}(B)$. It defines a functor $j_{!*} : P(U, \overline{\mathbb{Q}}_\ell) \to P(X, \overline{\mathbb{Q}}_\ell)$.*
2. *More generally if Z is a locally closed subscheme of X and the immersions*

$$i_Z : Z \xrightarrow{j} \overline{Z} \xrightarrow{i_{\overline{Z}}} X,$$

the functor $i_{Z!*} : P(Z, \overline{\mathbb{Q}}_\ell) \to P(X, \overline{\mathbb{Q}}_\ell)$ is defined by $i_{Z!*} = i_{\overline{Z}*} j_{!*}$.

We recall the theorem describing the structure of simple perverse sheaves.

Theorem 1.3.2. *[KW01] A perverse sheaf B on X is simple if and only if there exists a smooth d-dimensional irreducible locally closed subscheme Z of X as well as an irreducible $\overline{\mathbb{Q}}_\ell$-local system $\mathcal{L}$ on Z such that if i_Z is the immersion of Z in X then B is isomorphic to $i_{Z!*}(\mathcal{L})[d]$, where $i_{Z!*}$ is defined in Proposition* 1.3.1.

Simple perverse sheaves associated with the trivial rank one $\overline{\mathbb{Q}}_\ell$-local systems on locally closed subschemes of X form an important class of perverse sheaves:

Definition 1.3.3. *Let Z be a smooth d-dimensional irreducible locally closed subscheme of X and $i : Z \to X$ be the corresponding immersion, the intersection cohomology sheaf (IC-sheaf for short),* $\mathrm{IC}(Z)$ *is the perverse sheaf $i_{Z!*}(\overline{\mathbb{Q}}_\ell)[d]$.*

The category $P(X, \overline{\mathbb{Q}}_\ell)$ is Abelian, Nœtherian and Artinian. Hence every perverse sheaf is a finite iterated extension of simple perverse sheaves. In the notation of [BBD82], we denote ${}^pD^{\leq 0}(X)$ (resp. ${}^pD^{\geq 0}(X)$) the objects of $D(X)$ whose perverse cohomology sheaves vanishes in degree ≥ 1 (resp. ≤ -1).

1.3.1 Weights and purity

Let F_0 be a $\overline{\mathbb{Q}}_\ell$-sheaf on X_0. Over $\mathbb{F}$, we have the Frobenius morphism $Fr_q : X \to X$ (evaluation to the q^{th} power) and an isomorphism $F_q^* : Fr_q^*(F) \xrightarrow{\sim} F$. For any $n \geq 1$, let $Fr_{q^n} = (Fr_q)^n$ and denote by $F_{q^n}^* : Fr_{q^n}^* F \to F$ the isomorphism induced by iteration of F_q^*. The fixed points of Fr_{q^n} are exactly the points x of X defined over $\mathbb{F}_{q^n}$. For any fixed point x of Fr_{q^n} denote by F_x the stalk of F at x, then $F_{q^n}^*$ is an automorphism of F_x.

Definition 1.3.4 (Pointwise pure and mixed sheaves). *[KW01, Chapter I, Definition 2.1]*

1. *A $\overline{\mathbb{Q}}_\ell$-sheaf F_0 on X_0 is called pointwise pure of weight w ($w \in \mathbb{Z}$) if, for every $n \geq 1$ and every point x of $X_0(\mathbb{F}_{q^n})$, the eigenvalues of the automorphism $F^*_{q^n}$ on F_x are algebraic numbers such that all of their complex algebraic conjugates have absolute value $q^{nw/2}$.*
2. *A $\overline{\mathbb{Q}}_\ell$-sheaf F_0 on X_0 is mixed if it admits a finite filtration with pointwise pure successive quotients. The weights of a mixed F_0 are the weights of the non-zero quotients.*

Definition 1.3.5 (Mixed and pure complexes). *[KW01, Chapter I, § 2]*
The category $D^b_m(X_0, \overline{\mathbb{Q}}_\ell)$ of mixed complexes is the full subcategory of $D^b_c(X_0, \overline{\mathbb{Q}}_\ell)$ given by complexes whose cohomology sheaves are mixed. A complex K_0 in $D^b_m(X_0, \overline{\mathbb{Q}}_\ell)$ is pure of weight w if the cohomology sheaves $H^i(K_0)$ are pointwise pure of weights $\leq w + i$ and the same is true for its Verdier dual $\mathbb{D}K_0$ of K_0.

We will denote by $P_m(X_0, \overline{\mathbb{Q}}_\ell)$ the category of mixed perverse sheaves.

Theorem 1.3.6 (Mixed and simple is pure). *[BBD82, Corollary 5.3.4] Let P be a mixed simple perverse $\overline{\mathbb{Q}}_\ell$-sheaf of weight w in $P_m(X_0, \overline{\mathbb{Q}}_\ell)$, then P is pure of weight w.*

Theorem 1.3.7 (Purity and decompositions). *[BBD82, Theorem 5.4.1, 5.4.5 , 5.4.6, and Corollary 5.3.8]*

1. *Let K_0 in $D^b_m(X_0, \overline{\mathbb{Q}}_\ell)$ then K_0 is pure of weight w if and only if each ${}^pH^i(K_0)$ is a pure perverse sheaf of weight $w+i$ and there is an isomorphism between K and $\oplus_{i \in \mathbb{Z}} {}^pH^i(K)[-i]$ in $D^b_c(X, \overline{\mathbb{Q}}_\ell)$.*
2. *Let $\mathcal{F}_0$ be a pure perverse $\overline{\mathbb{Q}}_\ell$-sheaf on X_0. The pull-back $\mathcal{F}$ of $\mathcal{F}_0$ to X splits in $P(X, \overline{\mathbb{Q}}_\ell)$ as a direct sum of intersection cohomology complexes associated with smooth irreducible $\overline{\mathbb{Q}}_\ell$-local systems on irreducible smooth locally closed subschemes of X.*

Remark that the splittings above do not necessarily hold over X_0.

We denote by ${}^pH^i$ the i^{th}-perverse cohomology sheaf. We will denote the category of pure perverse sheaves of weight 0 on X by $\mathcal{P}(X, \overline{\mathbb{Q}}_\ell)$.

Equivariant perverse sheaves [KW01, III.15] Let X be a scheme of finite type over $\mathbf{k}$. We will work with the bounded derived category $D(X)$ of $\overline{\mathbb{Q}}_\ell$-sheaves on X and the category of perverse sheaves $P(X)$ inside $D(X)$ (we will omit ^{b} and $_c$).
Let G be a connected algebraic group acting on X. The action map is denoted by $a : G \times X \to X$. Let p be the projection to the second factor $G \times X \to X$. An object $\mathcal{F} \in P(X)$ is said to be G-equivariant if the following properties are verified:

1. There exists an isomorphism $\phi : a^*\mathcal{F} \simeq p^*\mathcal{F}$.
2. The pull backs by $id \times a$ and $m \times id$ of the isomorphism ϕ are related by the equation $p^*_{23}\phi \circ (id_G \times a)^*\phi = (m \times id_X)^*\phi$, where m is the multiplication on G and $p_{23} : G \times G \times X \to G \times X$ is the projection along the first factor.
3. $\phi|_{e\times X} = id$, and $\mathcal{F} = a^*\mathcal{F}|_{e\times X} \xrightarrow{\sim} p^*\mathcal{F}|_{e\times X} = \mathcal{F}$, where e denotes the unit of G.

Remark 1.3.8. *Condition* 3) *in the definition above is superfluous and is only given for convenience. Indeed, it can be deduced from* 1) *and* 2) *as follows. Restricting the equality in* 2) *to $e \times e \times X$, one finds $\phi|_{e\times X} \circ \phi|_{e\times X} = \phi|_{e\times X}$. Since $\phi|_{e\times X}$ is an isomorphism, this yields $\phi|_{e\times X} = id$.*

We denote by $P_G(X)$ the full subcategory of $P(X)$ consisting of equivariant perverse sheaves. The derived category of G-equivariant $\overline{\mathbb{Q}}_\ell$-sheaves on X is denoted by $D_G(X)$.
In the following we will also use a subcategory $DP(S)$ of $D(S)$ defined over any scheme or stack S. The objects of $DP(S)$ are the objects of $\oplus_{i\in\mathbb{Z}}P(S)[i]$, and for $K, K^{'} \in P(S)$ and $i, j \in \mathbb{Z}$ the morphisms are:

$$Hom_{DP(S)}(K[i], K^{'}[j]) = \begin{cases} Hom_{P(S)}(K, K^{'}) & \text{if } i = j; \\ 0 & \text{if } i \neq j. \end{cases}$$

1.4 Affine Grassmannian

The references for this section are [BD91], [BL94] and [Gor10].

1.4.1 Affine Grassmannian of $\mathbf{GL}_n$ by lattices and in geometric terms

Consider the linear algebraic group $\mathbf{GL}_n$ over $\mathbf{k}$. Denote by $\mathbf{GL}_n(F)$ its F-points. For any $\mathbf{k}$-algebra R denote by $R[[t]]$ the ring of formal power series over R and by $R((t))$ the ring of Laurent series over R. In this subsection we will introduce the affine Grassmannain $Gr_{\mathbf{GL}_n}$ of $\mathbf{GL}_n$ and we will give two different interpretations. The first interpretation is by means of lattices and will enable us to show that $Gr_{\mathbf{GL}_n}$ is an ind-scheme of ind-finite type, and the second is in a geometrical way by means of vector bundles.

Proposition 1.4.1. *[BL94, Proposition 1.2] The functor sending a $\mathbf{k}$-algebra R to $\mathbf{GL}_n(R[[t]])$ is the group scheme denoted by $\mathbf{GL}_n(\mathcal{O})$. The functor sending a $\mathbf{k}$-algebra R to $\mathbf{GL}_n(R((t)))$ is an ind-scheme that we denote by $\mathbf{GL}_n(F)$.*

Definition 1.4.2. *The affine Grassmannian $Gr_{\mathbf{GL}_n}$ of $\mathbf{GL}_n$ is defined as the quotient $\mathbf{k}$-space $\mathbf{GL}_n(F)/\mathbf{GL}_n(\mathcal{O})$.*

Remark 1.4.3. *The quotient in the category of $\mathbf{k}$-spaces is the sheafification of the presheaf quotient $R \to \mathbf{GL}_n(R((t)))/\mathbf{GL}_n(R[[t]])$ in the fpqc-topology.*

Definition 1.4.4 (Lattices)**.** *[Gor10, Definition 2.8] Let R be a $\mathbf{k}$-algebra. A lattice L in $R((t))^n$ is a $R[[t]]$-submodule such that there exists $N \in \mathbb{Z}_{\geq 0}$ with*

$$t^N R[[t]]^n \subset L \subset t^{-N} R[[t]]^n$$

and the quotient $t^{-N}R[[t]]^n/L$ is locally free of finite rank over R.

Proposition 1.4.5. *[BL94, Proposition 2.3] The R-points of $\mathbf{GL}_n(F)/\mathbf{GL}_n(\mathcal{O})$ are naturally identified with the set of lattices in $R((t))^n$.*

Now we want to describe $\mathbf{GL}_n(F)/\mathbf{GL}_n(\mathcal{O})$ in a geometrical way. Let X be a projective smooth connected curve over the field $\mathbf{k}$. Let x be a closed point in X and let X^* be $X - \{x\}$. Denote by $\mathcal{O}_x$ the completion of the local ring of X at x and by F_x its field of fractions. We choose a local coordinate at the point x, denoted by t and we may identify $\mathcal{O}_x = \mathbf{k}[[t]]$ and $F_x = \mathbf{k}((t))$. For each $\mathbf{k}$-algebra R we set $X_R = X \times_{\mathbf{k}} \mathrm{Spec}(R)$ and $X^*_R = X^* \times_{\mathbf{k}} \mathrm{Spec}(R)$. Let $D_R = \mathrm{Spec}(\mathrm{R}[[\mathrm{t}]])$ and $D^*_R = \mathrm{Spec}(R((t)))$.

Proposition 1.4.6. *[BL94] Consider the functor associating to a* $\mathbf{k}$*-algebra* R *the isomorphism classes of* $\{(E,\rho)\}$*, where* E *is vector bundle of rank* n *on* X_R *and* ρ *is a trivialization of* E *over* X_R^**. Then the affine Grassmannian* $Gr_{\mathbf{GL}_n}$ *identifies canonically with this functor.*

Let G be a linear algebraic group over $\mathbf{k}$. The affine Grassmannian Gr_G associated with G is the quotient $\mathbf{k}$-space $G(F)/G(\mathcal{O})$, see [Remark 1.4.3]. To establish the existence of the affine Grassmannian as an ind-scheme in the general case, we will embed a given linear algebraic group into a general linear group in a suitable way. We use the following lemma due to Beilinson and Drinfeld.

Proposition 1.4.7. *[BD91][Theorem 4.5.1] Let* $H \subset G$ *be linear algebraic groups over* $\mathbf{k}$ *such that the quotient* $U = G/H$ *is quasi-affine. If* Gr_G *is represented by an ind-scheme of ind-finite type then so is* Gr_H*. The morphism* $Gr_H \to Gr_G$ *is a locally closed immersion. If* U *is affine, then this immersion is a closed immersion.*

If the group G is reductive then Gr_G will be of ind-finite type. Indeed, choose an embedding $G \to \mathbf{GL}_n$ for some n then since G is reductive, the quotient $G/\mathbf{GL}_n$ is affine. In the general case one shows that there exists an embedding $G \to \mathbf{GL}_n \times \mathbb{G}_m$ such that the quotient is quasi-affine.

1.4.2 Stratification of Gr_G and convolution product

Let G be a connected reductive group over $\mathbf{k}$ and a T a maximal torus of G. Let B a Borel subgroup of G containing T and let I_G denote the corresponding Iwahori subgroup. The group scheme $G(\mathcal{O})$ acts on the affine Grassmannian with finite-dimensional orbits. The $G(\mathcal{O})$-orbits on Gr_G are parametrized by W-orbits in X. Denote by t^λ the image of t under the map $\lambda : F^* \to G(F)$. Given λ in X, the $G(\mathcal{O})$-orbit associated with $W_G.\lambda$ is $G(\mathcal{O}).t^\lambda$ denoted by Gr_G^λ. Denote by $\overline{Gr_G^\lambda}$ the closure of the $G(\mathcal{O})$-orbit Gr_G^λ. We have the Cartan decomposition of $G(F)$

$$G(F) = \bigcup_{\lambda \in X^+} G(\mathcal{O}) t^\lambda G(\mathcal{O}).$$

For any λ and μ in X^+, we have $Gr_G^\mu \subset \overline{Gr_G^\lambda}$, if and only if $\lambda - \mu$ is a sum of positive coroots and we have

$$\overline{Gr^\lambda} = \bigsqcup_{\mu \leq \lambda} Gr^\mu.$$

For any λ in X^+, the dimension of Gr_G^λ is $\langle 2\check{\rho}, \lambda \rangle$, where $\check{\rho}$ is $\frac{1}{2}\sum_{\check{\alpha} \in \check{R}^+} \check{\alpha}$, the half sum of positive roots.

The Iwahori subgroup I_G acts on the affine Grassmannian Gr_G as well. The I_G-orbits are parametrized by cocharacters λ in X. Each orbit is an affine space. We have the decomposition

$$G(F) = \bigsqcup_{\lambda \in X} I_G t^\lambda G(\mathcal{O}). \tag{1.4.8}$$

Let λ be in X^+, each $G(\mathcal{O})$-orbit Gr_G^λ decomposes into I_G-orbits which are parametrized by $W.\lambda$ and the orbit $I_G t^\lambda G(\mathcal{O})$ is open in Gr_G^λ. For any λ in X denote by O^λ for the I_G-orbit through $t^\lambda G(\mathcal{O})$ in Gr_G. Denote by $\overline{O^\lambda}$ its closure. The scheme $\overline{O^\lambda}$ is stratified by locally closed subschemes O^μ, where μ is in X. Remark that $O^\mu \subset \overline{O^\lambda}$ does not necessarily imply $\mu \leq \lambda$.

Proposition 1.4.9. *[BD91, Proposition 4.5.4] The set of connected components of Gr_G can be identified with $\pi_1(G)$, the algebraic fundamental group of G which is nothing but the elements of length zero in the affine extended Weyl group of G.*

Convolution product on Gr_G One may refer to [MV07, § 4] for details.

Lemma 1.4.10. *[Gai01, Appendix] The space $G(F)\times_{G(\mathcal{O})} Gr_G$ is represented by the functor classifying quadruplets $(E, E^1, \widetilde{\beta}, \beta^1)$, where E and E^1 are G-bundles on D, $\widetilde{\beta}$ is an isomorphism between $E_{|_{D^*}} \simeq E^1_{|_{D^*}}$ on D^* and β^1 is a trivialization of E^1 over D^*.*

The notion of the category of perverse sheaves is well-defined for an ind-scheme of ind-finite type. Indeed let $Y = \varinjlim_i Y_i$, we define the category of perverse sheaves on it as $P(X) = 2\varinjlim_i P(Y_i)$, where the functors $P(Y_i) \to P(Y_{i+1})$ are the direct image functors. This is again an abelian category since the functor of direct image under a closed embedding is exact. Similarly one can define the derived category $D^b(Y) = 2\varinjlim D^b(Y_i)$ which is triangulated category due to the exactness property mentioned above. This construction does not depend on the choice of a presentation of Y as inductive limit of Y_i and $P(Y)$ and $D(Y)$ are intrinsically attached to Y. We emphasize again that a perverse sheaf on an ind-scheme is by definition supported on a closed subscheme of finite type. This means that this notion is essentially finite-dimensional. In the same way one may define the category of equivariant perverse sheaves on an ind-scheme for a nice action of an algebraic group on Y like the case of the affine Grassmannian stratified into finite dimensional orbits under the action of $G(\mathcal{O})$.
Consider the following diagram

$$Gr_G \times Gr_G \xleftarrow{p} G(F)\times Gr_G \xrightarrow{q} G(F)\times_{G(\mathcal{O})} Gr_G \xrightarrow{m} Gr_G. \qquad (1.4.11)$$

The subgroup $G(\mathcal{O})$ acts on $G(F)\times Gr_G$ via the following formula: for (g,x) in $G(F)\times Gr_G$ and h in $G(\mathcal{O})$, we have $h.(g,x) = (gh^{-1}, hx)$. The space $G(F)\times_{G(\mathcal{O})} Gr_G$ is the quotient of $G(F)\times Gr_G$ by $G(\mathcal{O})$ under the above action. The map p sends (g,x) to $(gG(\mathcal{O}), x)$, the map v is the quotient map and the map m is the multiplication.
One may define a geometric convolution product

$$\star : P_{G(\mathcal{O})}(Gr_G)\times P_{G(\mathcal{O})}(Gr_G) \to P_{G(\mathcal{O})}(Gr_G).$$

Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two $G(\mathcal{O})$-equivariant perverse sheaves over Gr_G, the convolution product of these two perverse sheaves is by definition $\mathcal{F}_1\star\mathcal{F}_2 = m_!(\mathcal{F}_1\tilde{\boxtimes}\mathcal{F}_2)$. The sheaf $\mathcal{F}_1\tilde{\boxtimes}\mathcal{F}_2$ is perverse equipped with an isomorphism

$$p^*(\mathcal{F}_1\boxtimes\mathcal{F}_2) \xrightarrow{\sim} q^*(\mathcal{F}_1\tilde{\boxtimes}\mathcal{F}_2). \qquad (1.4.12)$$

Since each perverse sheaf $\mathcal{F}_i$ is supported over a subscheme of finite type, this definition makes sense.
This convolution product at the level of the Grothendieck groups descends, of course, to the usual convolution product on the spherical Hecke algebra H_{sph}. It follows from Lusztig's work in [Lus83a] that this convolution product preserves perversity and makes $P_{G(\mathcal{O})}(Gr_G)$ a monoidal category.
Moreover, the fact that the spherical Hecke algebra H_{sph} is commutative can be lifted to the categorical level. By interpreting the convolution product as a fusion product one can show that

this makes $P_{G(\mathcal{O})}(Gr_G)$ to be a symmetric monoidal category, see [MV07, § 5]. The geometric Satake isomorphism establishes an equivalence of tensor categories $P_{G(\mathcal{O})}(Gr_G) \xrightarrow{\sim} \mathrm{Rep}(\check{G})$, where the latter is the category of representations of the Langlands dual group over $\overline{\mathbb{Q}}_\ell$ and the product is the tensor product of two representations on right hand side and convolution product on left hand side [MV07, Theorem 14.1].
As at the end of Section 1.3, one may also consider the category $DP_{G(\mathcal{O})}(Gr_G) = \oplus_i P_{G(\mathcal{O})}(Gr_G)[i]$. Sometimes, we will use an extension of the Satake equivalence

$$DP_{G(\mathcal{O})}(Gr_G) \xrightarrow{\sim} \mathrm{Rep}(\check{G} \times \mathbb{G}_m). \tag{1.4.13}$$

For $K \in P_{G(\mathcal{O})}(Gr_G)$ and $i \in \mathbb{Z}$ this functor sends $K[i]$ to $Loc(K) \otimes I^{\otimes -i}$, where I is the standard representation of $\mathbb{G}_m$, and $Loc : P_{G(\mathcal{O})}(Gr_G) \xrightarrow{\sim} \mathrm{Rep}(\check{G})$ is the Satake equivalence.

1.5 Affine flag variety

The references for this section are [Gai01], [Gor10].

Definition 1.5.1. *The affine flag variety $\mathcal{F}l_G$ for G is the quotient $\mathbf{k}$-space $G(F)/I_G$.*

1.5.1 Affine flag variety of $\mathbf{GL}_n$ by means of lattices

Definition 1.5.2. *Let R be a $\mathbf{k}$-algebra. A complete periodic flag of lattices inside $R((t))^n$ is a flag*

$$L_{-1} \subset L_0 \subset L_1 \subset \dots$$

such that each L_i is a lattice in $R((t))^n$, each quotient L_{i+1}/L_i is a locally free R-module of rank one and $L_{n+k} = t^{-1}L_k$ for any k in $\mathbb{Z}$.

For $1 \leq i \leq n$, set

$$\Lambda_{i,R} = (\oplus_{j=1}^{i} t^{-1} R[[t]] e_j) \oplus (\oplus_{j=i+1}^{n} R[[t]] e_j).$$

For all i in $\mathbb{Z}$, we set $\Lambda_{i+n,R} = t^{-1}\Lambda_{i,R}$. This defines the standard complete lattice flag

$$\Lambda_{-1,R} \subset \Lambda_{0,R} \subset \Lambda_{1,R} \subset \dots$$

denoted by $\Lambda_{\bullet,R}$ in $R((t))^n$. Each point of $\mathbf{GL}_n(R((t)))$ gives rise to a flag of lattices inside $R((t))^n$ by applying it to the standard lattice flag. The Iwahori subgroup $I_G \subset \mathbf{GL}_n(\mathbf{k}[[t]])$ is precisely the stabilizer of the standard lattice flag $\Lambda_{\bullet,\mathbf{k}}$. This leads to the following proposition:

Proposition 1.5.3. *[Gor10, Proposition 2.13] For any $\mathbf{k}$-algebra R, the set $\mathcal{F}l_{\mathbf{GL}_n}(R)$ is naturally in bijection with the set of complete periodic lattice flags in $R((t))^n$. Besides, $\mathcal{F}l_{\mathbf{GL}_n}$ is an ind-scheme.*

By using this proposition together with Lemma 1.4.7, we obtain that the affine flag variety $\mathcal{F}l_G$ for any reductive group G is an ind-scheme. We can either give an embedding of G into $\mathbf{GL}_n$, or show that it is an ind-scheme by considering the natural projection of $\mathcal{F}l_G$ onto Gr_G which is a fibre bundle whose fibres are all isomorphic to the usual flag variety of G.

1.5.2 Affine flag variety of a reductive group in geometric terms and convolution product

Another interesting and important description of the affine flag variety of a reductive group G can be given in terms of G-torsors. Let X be a smooth connected projective curve and x a closed point in X. Denote by $\mathcal{O}_x$ the completion of the local ring of X at x and by F_x its field of fractions. We choose a local coordinate at the point x, denoted by t and we may identify $\mathcal{O}_x = \mathbf{k}[[t]]$ and $F_x = \mathbf{k}((t))$. Let $D = \mathrm{Spec}(\mathcal{O}_x)$ and $D^* = \mathrm{Spec}(\mathbf{k}((t)))$. Then $\mathcal{F}l_G$ is the ind-scheme classifying the triples $(\mathcal{F}_G, \beta, \epsilon)$, where $\mathcal{F}_G$ is a G-torsor on D, β is a trivialization of $\mathcal{F}_G$ on D^* and ϵ is a reduction of $\mathcal{F}_G|_x$ to a B-torsor. There is a distinguished point $1_{\mathcal{F}l_G}$ that corresponds to the triple $(\mathcal{F}_G^0, \beta^0, \epsilon^0)$, where $(\mathcal{F}_G^0, \beta^0) = 1_{Gr_G}$ and ϵ^0 corresponds to the chosen Borel subgroup B in G.

Theorem 1.5.4. *[Gor10, Theorem 2.18] The affine flag variety decomposes as a disjoint union*

$$\mathcal{F}l_G = \bigcup_{w \in \widetilde{W}_G} I_G w I_G / I_G.$$

The closure of each Schubert cell $I_G w I_G / I_G$ is a union of Schubert cells and the closure relations are given by the Bruhat order:

$$\overline{I_G w I_G / I_G} = \bigcup_{w' \leq w} I_G w' I_G / I_G.$$

For any $w \in \widetilde{W}_G$ we will denote the Schubert cell $I_G w I_G / I_G$ by $\mathcal{F}l_G^w$. It is isomorphic to $\mathbb{A}^{\ell(w)}$.

One can define the category $P_{I_G}(\mathcal{F}l_G)$ of I_G-equivariant perverse sheaves on $\mathcal{F}l_G$ and the bounded derived category $D_{I_G}(\mathcal{F}l_G)$. One has a canonical isomorphism

$$K(P_{I_G}(\mathcal{F}l_G)) \otimes \overline{\mathbb{Q}}_\ell \xrightarrow{\sim} \mathcal{H}_{I_G}, \tag{1.5.5}$$

where $K(P_{I_G}(\mathcal{F}l_G))$ is the Grothendieck group of this category. One may define a convolution product on $\mathcal{F}l_G$ [Gai01, § 1],

$$\star : P_{I_G}(\mathcal{F}l_G) \times P_{I_G}(\mathcal{F}l_G) \longrightarrow D^b_{I_G}(\mathcal{F}l_G)$$

in the following way:

Convolution Product

The Iwahori subgroup I_G acts on $G(F) \times \mathcal{F}l_G$ via the following formula: for (g, x) in $G(F) \times \mathcal{F}l_G$ and h in I_G, we have $h.(g, x) = (gh^{-1}, hx)$. The space $G(F) \times_{I_G} \mathcal{F}l_G$ is the quotient of $G(F) \times \mathcal{F}l_G$ by I_G under the above action.

We have the following diagram

$$\mathcal{F}l_G \times \mathcal{F}l_G \overset{p}{\leftarrow} G(F) \times \mathcal{F}l_G \overset{q}{\rightarrow} G(F) \times_{I_G} \mathcal{F}l_G \overset{m}{\rightarrow} \mathcal{F}l_G,$$

where the map p sends $(g, x) \in G(F) \times \mathcal{F}l_G$ to (gI_G, x), the map q is the quotient map, and the map m is the multiplication map.

One can define the space $G(F) \times_{I_G} \mathcal{F}l_G$ in a geometrical way. It is the ind-scheme classifying the data of 6-tuples $(\mathcal{F}_G, \mathcal{F}_G^1, \tilde{\beta}, \beta^1, \epsilon, \epsilon^1)$, where the objects $(\mathcal{F}_G, \mathcal{F}_G^1, \tilde{\beta}, \beta^1)$ are as in Lemma 1.4.10 and ϵ (resp., ϵ^1) is a reduction of $\mathcal{F}_G|_x$ (resp., $\mathcal{F}_G^1|_x$) to a B-torsor, [Gai01].

For any perverse sheaf $\mathcal{T}$ on $\mathcal{F}l_G$ and any I_G-equivariant perverse sheaf $\mathcal{S}$ on $\mathcal{F}l_G$, let $\mathcal{T}\tilde{\boxtimes}\mathcal{S}$ be the complex of sheaves equipped with an isomorphism

$$q^*(\mathcal{T}\tilde{\boxtimes}\mathcal{S}) \xrightarrow{\sim} p^*(\mathcal{T}\boxtimes\mathcal{S}).$$

This is an object in the derived category $D(G(F)\times_{I_G}\mathcal{F}l_G)$ and we set the convolution of these two perverse sheaves to be $\mathcal{T}\star\mathcal{S} = m_!(\mathcal{T}\tilde{\boxtimes}\mathcal{S})$. However, Lusztig's theorem does not extend to the case of affine flags: the convolution on the affine flag variety does not preserve perversity and for two perverse sheaves $\mathcal{S}_1$ and $\mathcal{S}_2$ there is certainly no isomorphism between $\mathcal{S}_1\star\mathcal{S}_2$ and $\mathcal{S}_2\star\mathcal{S}_1$ since the corresponding equality is not true, even at the Grothendieck group level (the Iwahori-Hecke algebra $\mathcal{H}_{I_G}$ is not commutative, see § 1.2.3).

1.5.3 Some properties of convolution product and Wakimoto sheaves

For $w\in\widetilde{W}_G$, denote by j_w the inclusion of $\mathcal{F}l_G^w$ in $\mathcal{F}l_G$, and let $L_w = j_{w!*}\overline{\mathbb{Q}}_\ell[\ell(w)](\ell(w)/2)$, the IC-sheaf of $\mathcal{F}l_G^w$. We write $L_{w!} = j_{w!}\overline{\mathbb{Q}}_\ell[\ell(w)](\ell(w)/2)$ and $L_{w*} = j_{w*}\overline{\mathbb{Q}}_\ell[\ell(w)](\ell(w)/2)$ for the standard and costandard objects. As j_w is an affine map, both $L_{w!}$ and L_{w*} are perverse sheaves. They satisfy $\mathbb{D}(L_{w*}) = L_{w!}$, where $\mathbb{D}$ denotes the Verdier duality. Remark that in the notation of $L_{w!}$ and L_{w*} we wrote the Tate twists. When working over an algebraically closed field, we will forget the Tate twist.

Assume that $\mathbf{k}$ is finite. To $\mathcal{G}$ in $P_{I_G}(\mathcal{F}l_G)$ we attach a function $[\mathcal{G}] : G(F)/I_G \longrightarrow \overline{\mathbb{Q}}_\ell$ given by $[\mathcal{G}](x) = Tr(Fr_x, \mathcal{G}_x)$, for x a point in $G(F)/I_G$ and where Fr_x is the geometric Frobenius at x. The function $[\mathcal{G}]$ is an element of $\mathcal{H}_{I_G}$. In particular $[L_{w!}] = (-1)^{\ell(w)}q_w^{-1/2}T_w$ and $[L_{w*}] = (-1)^{\ell(w)}q_w^{1/2}T_{w^{-1}}^{-1}$, where $q_w = q^{\ell(w)}$. Here T_w denotes the characteristic function of the double coset I_GwI_G.

In the following we will introduce the Wakimoto sheaves who are the geometrical version of Bernstein functions, § 1.2.1.

Proposition 1.5.6. *[AB09, § 3.2] We have the following two properties:*

1. *If $w_1, w_2\in\widetilde{W}_G$ verify $\ell(w_1w_2) = \ell(w_1)+\ell(w_2)$ then we have a canonical isomorphism*

$$L_{w_1*}\star L_{w_2*} \xrightarrow{\sim} L_{w_1w_2*}. \tag{1.5.7}$$

Under the same assumption and by duality, the same result is true for $L_{w!}$.

2. *Denote by e the identity element of $\widetilde{W}_G$ then for any $w\in\widetilde{W}_G$, we have*

$$L_{w!}\star L_{w^{-1}*} \xrightarrow{\sim} L_{w^{-1}*}\star L_{w!} \xrightarrow{\sim} L_e. \tag{1.5.8}$$

Hence the perverse sheaf $L_{w!}$ is an invertible object of $D_{I_G}(\mathcal{F}l_G)$.

Corollary 1.5.9. *[AB09, § 3.2] The map sending λ to $L_{t^\lambda *}$, for any λ in X^+ extends naturally to a monoidal functor*

$$\mathrm{R}(T) \longrightarrow D_{I_G}(\mathcal{F}l_G).$$

The image of λ under the above functor is usually called a Wakimoto sheaf. There are two conventions for defining the Wakimoto sheaves. The first convention is due to Bezrukavnikov in [AB09]. We will use the convention due to Prasad in [Pra05] by letting $\Theta_\lambda = L_{t^\lambda !}$ for λ dominant and $\Theta_\lambda = L_{t^\lambda *}$ for λ anti-dominant. In any case Wakimoto sheaves verify the following: $\lambda \in X$, if $\lambda = \lambda_1 - \lambda_2$ where λ_i are dominant for $i = 1, 2$, then $\Theta_\lambda \simeq \Theta_{\lambda_1} \star \Theta_{-\lambda_2}$. According to [AB09, Theorem 5], these are actually objects of the category $P_{I_G}(\mathcal{F}l_G)$ (a priori they are defined as objects of the triangulated category $D_{I_G}(\mathcal{F}l_G)$).

Chapter 2

Towards geometrical Howe correspondence at the Iwahori level

2.1 Howe correspondence

2.1.1 Classical Howe correspondence

The standard reference for the Howe correspondence from the classical point of view is [MVW87]. One can also see the survey by Kudla [Kud86]. We are going to recall the construction of the Weil representation of the metaplectic group and the Howe correspondence also known as local theta correspondence. We will overview the Howe conjecture and recall different already known results. Then we will consider the special case of dual reductive pairs of type II and will restrict ourselves to the class of the tamely ramified representations. We will translate the tamely ramified case of the Howe correspondence in the language of representations of Iwahori-Hecke algebras.

Fix a local non-Archimedean field F of characteristic $p > 2$. Let W be a symplectic vector space of dimension $2n$ over F. Let $Sp(W)$ be the symplectic group and $H(W)$ be the Heisenberg group, $H(W) = W \oplus F$ with the multiplication

$$(w_1, t_1)(w_2, t_2) = (w_1 + w_2, t_1 + t_2 + \frac{1}{2}\langle w_1, w_2\rangle),$$

where $\langle , \rangle$ is the symplectic form on W, $w_i \in W$ and $t_i \in F$ for $i = 1, 2$. The symplectic group $Sp(W)$ acts as a group of automorphisms on $H(W)$ by the rule ${}^g(w, t) = (gw, t)$, where g is an element of $Sp(W)$. This action is trivial on the center of $H(W)$. Fix a non-trivial additive character ψ of F with values in $\mathbb{C}^*$. The key fact on the construction of the Weil representation is the following result.

Theorem 2.1.1. *[Stone, Von Neumann][MVW87, § I.2] Up to isomorphism, there is a unique smooth irreducible representation (ρ_ψ, S) of $H(W)$ with central character ψ.*

Since the action of $Sp(W)$ is trivial on the center of $H(W)$, the representation $({}^g\rho_\psi, S)$ given by ${}^g\rho_\psi(h) = \rho_\psi({}^g h)$ for any h in $H(W)$ again has central character ψ. So by Theorem 2.1.1 it is isomorphic to (ρ_ψ, S). In particular, for each g in $Sp(W)$, there is an automorphism $A(g) : S \to S$ such that for any g, h in $Sp(W)$,

$$A(g)\rho_\psi(h)A(g)^{-1} = \rho_\psi({}^g h). \qquad (2.1.2)$$

The automorphism $A(g)$ is unique up to a scalar in $\mathbb{C}^*$. The map $g \to A(g)$ defines a projective representation of $Sp(W)$. For such a projective representation, consider the fibre product

$$\begin{array}{ccc} \widetilde{Sp(W)}_{\psi} & \longrightarrow & \mathbf{GL}(S) \\ \downarrow & & \downarrow \\ Sp(W) & \longrightarrow & \mathbf{GL}(S)/\mathbb{C}^*, \end{array}$$

where

$$\widetilde{Sp(W)}_{\psi} := \{(g, A(g)) \in Sp(W) \times \mathbf{GL}(S) | \quad (2.1.2) \quad holds\}.$$

The group $\widetilde{Sp(W)}_{\psi}$ is then a central extension of $Sp(W)$,

$$1 \to \mathbb{C}^* \to \widetilde{Sp(W)}_{\psi} \to Sp(W) \to 1,$$

such that A may be lifted to a linear representation ω_{ψ} of $\widetilde{Sp(W)}_{\psi}$ given by $\omega_{\psi}(g, A(g)) = A(g)$.

The central extension $\widetilde{Sp(W)}_{\psi}$ does not depend on ψ [Kud86, Theorem 3.1], and is isomorphic to a nontrivial twofold topological extension of $Sp(W)$ called the metaplectic group. The existence of a unique nontrivial twofold topological central extension of $Sp(W)$ follows from [Gel76]. In the following we will omit the subscript ψ in $\widetilde{Sp(W)}_{\psi}$. We call the representation (ω_{ψ}, S) of the metaplectic group $\widetilde{Sp(W)}$ the Weil representation. There are different models of Weil representation. We will be interested in one of these models called the Schrödinger model, [MVW87, Chapter 2, II.6].

Let Y be a maximal isotropic subspace of W. Let X be another maximal isotropic subspace of W such that $W = X \oplus Y$. Such a decomposition is referred to as a complete polarization of W. Denote by $\mathcal{S}(X)$ the space of $\mathbb{C}$-valued locally constant functions on X with compact support. Denote by $(\rho_{\psi}, \mathcal{S}(X))$ the representation of $H(W)$ on $\mathcal{S}(X)$ defined in [MVW87, Chapter 2, I.4] and the corresponding model of the Weil representation of $\widetilde{Sp(W)}$ again will be denoted by (ω_{ψ}, S). This is the Schrödinger model of the Weil representation.

From now on we will replace the field of coefficients $\mathbb{C}$ by $\overline{\mathbb{Q}}_{\ell}$ and consider our representations over $\overline{\mathbb{Q}}_{\ell}$.

Definition 2.1.3. *[MVW87, I.17] A reductive dual pair (G, G') over F in $Sp(W)$ is a pair of closed subgroups G and G' of $Sp(W)$ such that G and G' are reductive groups over F and*

$$\mathrm{Cent}_{Sp(W)}(G) = G', \qquad \mathrm{Cent}_{Sp(W)}(G') = G,$$

where Cent *denotes the centralizer.*

There is a classification of dual reductive pairs in [MVW87, Chapter 1]. There exists two types of dual pairs: dual pairs of type I consisting of symplectic, orthogonal, unitary groups and dual pairs of type II consisting of linear groups.
Fix a reductive dual pair (U_1, U_2) in $Sp(W)$ and for $i = 1, 2$, denote by $\widetilde{U_i}$ the inverse image of U_i in $\widetilde{Sp(W)}$. For $i = 1, 2$. we have the following exact sequences

$$1 \longrightarrow \overline{\mathbb{Q}}_{\ell}^* \longrightarrow \widetilde{U_i} \longrightarrow U_i \longrightarrow 1.$$

The groups $\widetilde{U_1}$ and $\widetilde{U_2}$ are mutual commutators in $\widetilde{Sp}(W)$ [MVW87, Chapter 2, III.1].
Let (π, V) be a smooth irreducible representation of $\widetilde{U_1 \times U_2}$ such that

$$Hom_{\widetilde{U_1 \times U_2}}(S, V) \neq \{0\},$$

then there exists smooth irreducible representations (π_1, V_1) and (π_2, V_2) of $\widetilde{U_1}$ and $\widetilde{U_2}$ respectively unique up to isomorphism such that π is obtained by factorization of the representation $\pi_1 \otimes \pi_2$ of $\widetilde{U_1} \times \widetilde{U_2}$ under the projection $\widetilde{U_1} \times \widetilde{U_2} \to \widetilde{U_1 \times U_2}$. By abuse of notation we will denote $\pi = \pi_1 \otimes \pi_2$.

Denote by $\mathcal{R}(\widetilde{U_i})$ the set of smooth irreducible representations of $\widetilde{U_i}$ appearing as quotients of (ω_ψ, S) restricted to $\widetilde{U_i}$. Let similarly $\mathcal{R}(\widetilde{U_1 \times U_2})$ be the set of smooth irreducible representations of $\widetilde{U_1 \times U_2}$ appearing as quotients of (ω_ψ, S) restricted to $\widetilde{U_1 \times U_2}$. Then $\mathcal{R}(\widetilde{U_1 \times U_2})$ identifies with a subset of $\mathcal{R}(\widetilde{U_1}) \times \mathcal{R}(\widetilde{U_2})$. Here is the Howe duality conjecture.

Conjecture 2.1.4. *[MVW87, Chapter 2, III.2] [How79, Paragraph 6].*
The set $\mathcal{R}(\widetilde{U_1 \times U_2})$ is a graph of a bijection between $\mathcal{R}(\widetilde{U_1})$ and $\mathcal{R}(\widetilde{U_2})$.

We can give a more precise form of this conjecture. For a representation (π_1, V_1) in $\mathcal{R}(\widetilde{U_1})$, let

$$S(\pi_1) = \bigcap \mathrm{Ker}(f), \qquad f \in Hom_{\widetilde{U_1}}(S, V_1).$$

Then $S[\pi_1] = S/S(\pi_1)$ is the maximal quotient of S on which $\widetilde{U_1}$ acts as a multiple of π_1. We refer to the representation $S[\pi_1]$ as a maximal π_1-isotypical quotient of S. The space $S(\pi_1)$ is stable under the action of $\widetilde{U_2}$ and the quotient $S[\pi_1]$ defines a representation of $\widetilde{U_1 \times U_2}$. Now by Lemma [MVW87, Chapter 2, III.4] there is a smooth representation Θ_{π_1} of $\widetilde{U_2}$ such that $S[\pi_1] = \pi_1 \otimes \Theta_{\pi_1}$. The representation Θ_{π_1} is unique up to isomorphism.

Theorem 2.1.5 (Howe duality)**.** *[MVW87, Chapter 3, III] For any representation π_1 in $\mathcal{R}(\widetilde{U_1})$, there exists a unique irreducible quotient of Θ_{π_1} denoted by θ_{π_1}. One similarly defines Θ_{π_2} and θ_{π_2} for any representation π_2 in $\mathcal{R}(\widetilde{U_2})$. Then the map $\pi_1 \to \theta_{\pi_1}$ is a bijection between $\mathcal{R}(\widetilde{U_1})$ and $\mathcal{R}(\widetilde{U_2})$.*

It is of great interest to describe explicitly the bijection between π_1 and θ_{π_1}. Let us mention some known results on the Howe duality.

1. For dual reductive pairs of type I, this conjecture has been proved in [MVW87] and [Wal90] for any p different from 2. The case of unramified representations is originally due to Howe and can be found in [MVW87, Chapter 5].
2. For dual pairs of type I, it has been proved by Kudla in [Kud86] and [MVW87, Chapter 3] that if π_1 is cuspidal then the representation $\Theta(\pi_1)$ is irreducible. Besides, for any irreducible representation π_1 the representation $\Theta(\pi_1)$ is of finite length.
3. For dual reductive pairs of type II, Mínguez in [Mín08] proves this conjecture explicitly in terms of Langlands parameters for any characteristic p.
4. For dual reductive pair $(\mathrm{Sp}_{2n}, \mathrm{S}\mathbb{O}_{2m})$, this conjecture has been proved in tamely ramified case by Aubert in [Aub89].
5. For dual pair of type I, $(\mathrm{O}(V), \mathrm{Sp}_{2n})$, some explicit partial results are known by Muic in [Mui08b], [Mui08a].

6. For dual pairs $(\mathrm{U}_n, \mathrm{U}_m)$, where n and m are small, some explicit partial results can be found in [Pan06].

We will assume that the metaplectic cover $\widetilde{U_i}$ splits over U_i for $i = 1, 2$. This is the case for dual pairs of type II that we are interested in. We fix such a splitting and regard the Weil representation as a representation of $U_1 \times U_2$ which defines the Howe duality between representations of U_1 and U_2.

The theta correspondence in some special cases leads to the Langlands functoriality. We refer to the work of Rallis [Ral82], where he proves the functoriality of the theta correspondence for unramified representations. There are more general conjectures (and partial evidence for them) in the paper [Ada89] by Adams. Moeglin in [Mœg89] gave a description of the Howe duality correspondence for some orthogonal and symplectic groups in the Archimedean case in terms of Langlands-Vogan parameters. It appears that, when the groups have similar rank and we restrict our consideration to only tempered representations, then the Howe duality respects functoriality and it is in certain sense the simplest possible for both Archimedean and non-Archimedean fields.

Later on we will discuss the geometric version of Langlands functoriality and we will give a conjecture on the functoriality at the level of tamely-ramified representations in Chapter 4 which involves Arthur packets.

We will now discuss the case of dual reductive pairs of type II in details. Let $\mathbf{k}$ be a finite field $\mathbb{F}_q$, and set $F = \mathbf{k}((t))$ and $\mathcal{O}$ its ring of integers. When the dual pair is of type II, its Weil representation is described as follows.

Howe duality for dual reductive pairs of type II, [MVW87, Chapter 2, III.1]

Let L_0 be a $\mathbf{k}$-vector space of dimension n and U_0 a $\mathbf{k}$-vector space of dimension m. Let W be equal to $L_0 \otimes U_0 + (L_0 \otimes U_0)^*$. Let $G = \mathbf{GL}(L_0)$ and $H = \mathbf{GL}(U_0)$ and consider the dual reductive pair $(G(F), H(F))$ in $\mathrm{Sp}(W)$. Let $\Pi(F) = L_0 \otimes U_0(F)$ and $\mathcal{S}(\Pi(F))$ be the Schwartz space of locally constant, compactly supported $\overline{\mathbb{Q}}_\ell$-valued functions on $\Pi(F)$. The metaplectic extension splits over $G(F)$ and $H(F)$. The restriction of the Weil representation to $G(F) \times H(F)$ is the representation $\sigma_{n,m} : G(F) \times H(F) \longrightarrow \mathbf{GL}(\mathcal{S}(\Pi(F)))$ defined by

$$\sigma_{n,m}(g, h)(\phi)(x) = |\det g|^{-m/2} |\det h|^{-n/2} \phi(g^{-1} \otimes h^{-1}(x)) \tag{2.1.6}$$

for $g \in G(F)$, $h \in H(F)$, $x \in \Pi(F)$ and $\phi \in \mathcal{S}(\Pi(F))$.

Assume $n \leq m$. Let π be an irreducible smooth representation of $G(F)$. The Howe conjecture, in this case, predicts that there exists a unique non-zero irreducible representation of $H(F)$ denoted by $\theta_{n,m}(\pi)$ such that $\pi \otimes \theta_{n,m}(\pi)$ is a quotient of $\sigma_{n,m}$. Moreover,

$$\dim(Hom_{G(F)\times H(F)}(\sigma_{n,m}, \pi \otimes \theta_{n,m}(\pi))) = 1.$$

A proof of this conjecture and a description of the correspondence in terms of Langlands parameters is given in [Mín08].

2.1.2 Tamely ramified Howe Duality in the language of Hecke algebras

Let $\mathbf{k} = \mathbb{F}_q$, $F = \mathbf{k}((t))$ and $\mathcal{O} = \mathbf{k}[[t]]$. Let (G, H) be a (split) dual reductive pair in some symplectic group over $\mathbf{k}$. Let B_G (resp. B_H) be a Borel subgroup of G (resp. H). Denote by I_G and I_H the corresponding Iwahori subgroup of $G(F)$ and $H(F)$ respectively. Different classes of representations may be considered for a explicit study of the Howe correspondence. We say that a representation of $G(F)$ is unramified if it admits a non-zero $G(\mathcal{O})$-invariant vector. When we restrict ourselves to the class of unramified representations, Howe correspondence may be translated in the language of modules over spherical Hecke algebras. Thus one may study the representations of spherical Hecke algebras associated with a reductive group G instead of studying unramified representations of the reductive group $G(F)$. For both dual pairs of type I and II, this has been done by Howe. In the case of dual pair of type I, one may refer [MVW87, Chapter 5, I.10] and for dual pairs of type II refer to [Mín06, Appendix]. Howe has proved that the $G(\mathcal{O})\times H(\mathcal{O})$-invariant space $S^{G(\mathcal{O})\times H(\mathcal{O})}$, where S is the restriction of the Weil representation to $G(F)\times H(F)$, is a free module of rank one over the spherical Hecke algebra of one of the two groups, say G. Moreover, the action of the spherical Hecke algebra of H on this space factors through a homomoprhism of algebras from the spherical Hecke algebra of H to the spherical Hecke algebra of G.

The other class of representations we are interested in is the class of tamely ramified representations. In this case we may interpret the correspondence in the language of modules over Iwahori-Hecke algebras as follows. An irreducible smooth representation (π, V) of $G(F)$ is called tamely ramified if the space of invariants under Iwahori subgroup I_G is non zero. The category of tamely-ramified representations is the full subcategory of smooth representations of finite length consisting of those representations whose all irreducible subquotients are tamely ramified.
Consider the functor sending a tamely ramified representation V of G to V^{I_G}. The space V^{I_G} is naturally a module over $\mathcal{H}_{I_G}$. According to [Bor76, Theorem 4.10] this functor is an equivalence between the category of tamely ramified representations of $G(F)$ and the category of finite-dimensional $\mathcal{H}_{I_G}$-modules. This functor is exact on the category of smooth representations [IM].

Let $G=\mathbf{GL}(L_0)$ and $H=\mathbf{GL}(U_0)$, we use the notation of § 2.1.1. Denote by $\mathcal{S}^{I_G\times I_H}(\Pi(F))$ the space of $I_H\times I_G$-invariants in the space of Schwartz functions $\mathcal{S}(\Pi(F))$. Remind that we assume $n\leq m$.

Theorem 2.1.7 (Tamely ramified Howe correspondence). *[Mín08] Given an irreducible $\mathcal{H}_{I_G}$-module π, the maximal π-isotypical quotient of $\mathcal{S}^{I_G\times I_H}(\Pi(F))$ is of the form $\pi\otimes\Phi(\pi)$, where $\Phi(\pi)$ is a non zero finite-dimensional representation of $\mathcal{H}_{I_H}$. There is a unique irreducible quotient of $\Phi(\pi)$ denoted $\phi(\pi)$. Define $\Phi(\pi_2)$ and $\phi(\pi_2)$ similarly for an irreducible $\mathcal{H}_{I_H}$-module π_2. The map $\pi\to\phi(\pi)$ is an injection from the set of isomorphism classes of irreducible $\mathcal{H}_{I_G}$-modules to the set of isomorphism classes of irreducible $\mathcal{H}_{I_H}$-modules. If π is an irreducible $\mathcal{H}_{I_G}$-module then*

$$\dim_{\mathcal{H}_{I_H}\otimes\mathcal{H}_{I_G}}(\mathcal{S}^{I_G\times I_H}(\Pi(F)),\pi\otimes\phi(\pi))=1.$$

Our purpose is the study of the space of invariants $\mathcal{S}^{I_G\times I_H}(\Pi(F))$ in the geometrical setting. In the following sections we will define a geometric analogue of $\mathcal{S}^{I_G\times I_H}(\Pi(F))$ which will be the category of $I_G\times I_H$-equivariant ℓ-adic perverse sheaves over $\Pi(F)$ denoted $P_{I_H\times I_G}(\Pi(F))$ and an action of $P_{I_H}(\mathcal{F}l_H)$ and $P_{I_G}(\mathcal{F}l_G)$ on $D_{I_H\times I_G}(\Pi(F))$ by convolution, where $P_{I_G}(\mathcal{F}l_G)$ (resp. $P_{I_H}(\mathcal{F}l_H)$) denotes the category of I_G-equivariant (resp. I_H-equivariant) perverse sheaves on $\mathcal{F}l_G$ (resp. $\mathcal{F}l_H$).

2.2 Geometrization

2.2.1 Geometric analogue of the Schwartz space

Work over an algebraically closed field $\mathbf{k}$ of characteristic $p > 2$. Set $\mathcal{O} = \mathbf{k}[[t]]$ and $F = \mathbf{k}((t))$. Let G be a connected reductive group over $\mathbf{k}$ and assume $M = M_0 \otimes_{\mathbf{k}} \mathcal{O}$, where M_0 is a given finite-dimensional representation of G. Fix a maximal torus T in G and a Borel subgroup B containing T. Denote by I_G the corresponding Iwahori subgroup. The definitions of the derived category $D(M(F))$ of ℓ-adic sheaves on $M(F)$ and the category $P(M(F))$ of ℓ-adic perverse sheaves on $M(F)$ are found in [Lys11]. We remind the definition briefly.

For any two integers $N, r \geq 0$ with $N + r > 0$ set $M_{N,r} = t^{-N}M/t^rM$. Given positive integers $N_1 \geq N_2$, $r_1 \geq r_2$, we have the following Cartesian diagram

$$\begin{array}{ccc} M_{N_2,r_1} & \overset{i}{\hookrightarrow} & M_{N_1,r_1} \\ \downarrow p & & \downarrow p \\ M_{N_2,r_2} & \overset{i}{\hookrightarrow} & M_{N_1,r_2}, \end{array} \tag{2.2.1}$$

where i is the natural closed immersion and p is the projection.

Definition 2.2.2. *[KV04, Definition 4.2.1] Let A be a filtering poset and $(C_\alpha)_{\alpha\in A}$ be a inductive system of categories labelled by A. In other words, for each $\alpha \leq \beta$ we have a functor $i_{\alpha\beta} : C_\alpha \to C_\beta$ and for any $\alpha \leq \beta \leq \gamma$ a natural isomorphism $i_{\beta\gamma} \circ i_{\alpha\beta} \to i_{\alpha\gamma}$ satisfying the natural compatibility conditions. The inductive limit $2\varinjlim C_\alpha$ is the category whose objects are pairs (α, x_α), $\alpha \in A$ and $x_\alpha \in \mathrm{Ob}(C_\alpha)$. The morphisms are*

$$Hom((\alpha, x_\alpha), (\beta, y_\beta)) = \varinjlim_{\gamma\geq\alpha,\beta} Hom_{C_\alpha}(i_{\alpha\gamma}(x_\alpha), i_{\beta\gamma}(y_\beta)).$$

Consider the following functor

$$\begin{aligned} D(M_{N,r_2}) &\longrightarrow D(M_{N,r_1}) \\ K &\longrightarrow p^*K[\dim rel(p)]. \end{aligned} \tag{2.2.3}$$

According to [BBD82, Proposition 4.2.5] the functor (2.2.3) is fully faithful and exact for the perverse t-structure. The functor i_* is fully faithful and exact for the perverse t-structure as well. This yields a commutative diagram of triangulated categories:

$$\begin{array}{ccc} D(M_{N_2,r_1}) & \overset{i_*}{\hookrightarrow} & D(M_{N_1,r_1}) \\ p^*[\dim] \uparrow & & \uparrow p^*[\dim] \\ D(M_{N_2,r_2}) & \overset{i_*}{\hookrightarrow} & D(M_{N_1,r_2}). \end{array} \tag{2.2.4}$$

The derived category $D(M(F))$ is defined as the inductive 2-limit of derived categories $D(M_{N,r})$ as N, r go to infinity. Similarly, $P(M(F))$ is defined as the inductive 2-limit of the categories $P(M_{N,r})$. The category $P(M(F))$ is a geometric analogue of the Schwartz space of locally constant functions with compact support on $M(F)$.

Assume $N+r>0$. The subgroup $G(\mathcal{O})$ acts on $M_{N,r}$ via its finite-dimensional quotient $G(\mathcal{O}/t^{N+r}\mathcal{O})$. Denote by I_s the kernel of the map $G(\mathcal{O})\longrightarrow G(\mathcal{O}/t^s\mathcal{O})$. The Iwahori subgroup I_G acts on $M_{N,r}$ via its finite-dimensional quotient I_G/I_{N+r}. For $s>0$ denote by K_s the quotient I_G/I_s.

Let $r_1\geq N+r>0$, we have the projection $K_{r_1}\twoheadrightarrow K_{N+r}$. This leads to the following projection between stack quotients

$$q: K_{r_1}\backslash M_{N,r}\twoheadrightarrow K_{N+r}\backslash M_{N,r}.$$

This gives rise to an equivalence of equivariant derived categories

$$D_{K_{N+r}}(M_{N,r})\xrightarrow{\sim} D_{K_{r_1}}(M_{N,r}).$$

This is also exact for perverse t-structures. Denote by $D_{I_G}(M_{N,r})$ the derived category of K_{r_1}-equivariant ℓ-adic sheaves $D_{K_{r_1}}(M_{N,r})$ for any $r_1\geq N+r$.

By taking the stack quotient of the diagram (2.2.1) by $K_{N_1+r_1}$, we obtain

$$\begin{array}{ccc} D_{I_G}(M_{N_2,r_1}) & \xhookrightarrow{i_*} & D_{I_G}(M_{N_1,r_1}) \\ {\scriptstyle p^*[dimrel]}\uparrow & & \uparrow{\scriptstyle p^*[dimrel]} \\ D_{I_G}(M_{N_2,r_2}) & \xhookrightarrow{i_*} & D_{I_G}(M_{N_1,r_2}), \end{array} \tag{2.2.5}$$

where each arrow is fully faithful and exact for the perverse t-structure. Define $D_{I_G}(M(F))$ as the inductive 2-limit of $D_{I_G}(M_{N,r})$ as N,r go to infinity. Similarly we define the category $P_{I_G}(M(F))$.

Since the Verdier duality $\mathbb{D}$ is compatible with the transition functors in both diagrams (2.2.1), and (2.2.4) we have the Verdier duality self-functors $\mathbb{D}$ on $D_{I_G}(M(F))$ and $D(M(F))$.

In order to define an action of the Hecke functors on $D_{I_G}(M(F))$, let us first define the equivariant derived category $D_{I_G}(M(F)\times\mathcal{F}l_G)$. Let $s_1,s_2\geq 0$ and set

$${}_{s_1,s_2}G(F)=\{g\in G(F)\mid\ t^{s_1}M\subset gM\subset t^{-s_2}M\}. \tag{2.2.6}$$

Then ${}_{s_1,s_2}G(F)\subset G(F)$ is closed and stable under the left and right multiplication by $G(\mathcal{O})$. Further, ${}_{s_1,s_2}\mathcal{F}l_G={}_{s_1,s_2}G(F)/I_G$ is closed in $\mathcal{F}l_G$. For $s_1'\geq s_1$ and $s_2'\geq s_2$, we have the closed embedding ${}_{s_1,s_2}\mathcal{F}\ell_G\hookrightarrow {}_{s_1',s_2'}\mathcal{F}\ell_G$ and the union of ${}_{s_1,s_2}\mathcal{F}\ell_G$ is the affine flag variety $\mathcal{F}l_G$. The map sending g to g^{-1} yields an isomorphism between ${}_{s_1,s_2}G(F)$ and ${}_{s_2,s_1}G(F)$.
Assume M_0 is a faithful representation of G, then ${}_{s_1,s_2}\mathcal{F}\ell_G\subset\mathcal{F}\ell_G$ is a closed subscheme of finite type.

Lemma 2.2.7. *For any $s_1,s_2\geq 0$, the action of $G(\mathcal{O})$ on ${}_{s_1,s_2}\mathcal{F}l_G$ factors through the quotient $G(\mathcal{O}/t^{s_1+s_2+1}\mathcal{O})$.*

Proof. Choose a Borel B' in $GL(M_0)$ such that $B=G\cap B'$. Denote by

$$M\subset M_1\subset\cdots\subset M_n=t^{-1}M$$

the full flag preserved by $B^{'}$. The Iwahori subgroup I_G consists of the elements g of $G(F)$ preserving M together with the flag M_i above. Hence the map from $\mathcal{F}\ell_G$ to $\mathcal{F}\ell_{GL(M_0)}$ sending a point gI_G to the flag $(gM \subset gM_1 \subset \cdots \subset gM_n)$ is a closed immersion. Thus ${}_{s_1,s_2}\mathcal{F}l_G$ is realized as the closed subscheme in the scheme classifying a lattice $M^{'}$ such that $t^{s_1}M \subset M^{'} \subset t^{-s_2}M$ together with the full flag

$$M^{'} \subset M_1^{'} \subset \cdots \subset M_n^{'} = t^{-1}M^{'}.$$

The action of $G(\mathcal{O})$ on the latter scheme factors through $G(\mathcal{O}/t^{s_1+s_2+1}\mathcal{O})$. □

The action of I_G on ${}_{s_1,s_2}\mathcal{F}l_G$ factors through $K_s = I_G/I_s$ for $s \geq s_1 + s_2 + 1$.
Let $s \geq \max\{N + r, s_1 + s_2 + 1\}$, the group K_s acts on $M_{N,r} \times {}_{s_1,s_2}\mathcal{F}l_G$ diagonally and the category $D_{K_s}(M_{N,r} \times {}_{s_1,s_2}\mathcal{F}l_G)$ is well-defined. For $s^{'} \geq s$ one has a canonical equivalence

$$D_{K_s}(M_{N,r} \times {}_{s_1,s_2}\mathcal{F}l_G) \xrightarrow{\sim} D_{K_{s^{'}}}(M_{N,r} \times {}_{s_1,s_2}\mathcal{F}l_G).$$

Define $D_{I_G}(M_{N,r} \times {}_{s_1,s_2}\mathcal{F}l_G)$ as the category $D_{K_s}(M_{N,r} \times {}_{s_1,s_2}\mathcal{F}l_G)$ for any $s \geq \max\{N+r, s_1 + s_2 + 1\}$.
Define $D_{I_G}(M(F) \times \mathcal{F}l_G)$ as the inductive 2-limit of $D_{I_G}(M_{N,r} \times {}_{s_1,s_2}\mathcal{F}l_G)$ as N, r, s_1, s_2 go to infinity. The subcategory $P_{I_G}(M(F) \times \mathcal{F}l_G) \subset D_{I_G}(M(F) \times \mathcal{F}l_G)$ of perverse sheaves is defined along the same lines.

2.2.2 Hecke functors at the Iwahori level

Denote by $\check{\mu}$ in $\check{X}^+$ the character by which G acts on $\det(M_0)$. The connected components of the affine Grassmannian Gr_G are indexed by the algebraic fundamental group $\pi_1(G)$ of G according to Proposition 1.4.9 in Chapter 1. For θ a cocharacter in $\pi_1(G)$, choose λ in X^+ whose image in $\pi_1(G)$ equals θ. Denote by Gr_G^θ the connected component of Gr_G containing Gr_G^λ.

The affine flag manifold $\mathcal{F}l_G$ is a fibration over Gr_G with the typical fibre G/B. Hence the connected components of the affine flag variety $\mathcal{F}l_G$ are also indexed by $\pi_1(G)$. For θ in $\pi_1(G)$, denote by $\mathcal{F}l_G^\theta$ the preimage of Gr_G^θ in $\mathcal{F}l_G$. Set ${}_{s_1,s_2}\mathcal{F}l_G^\theta = \mathcal{F}l_G^\theta \cap {}_{s_1,s_2}\mathcal{F}l_G$.

Let us now define the convolution action of $P_{I_G}(\mathcal{F}l_G)$ on $D_{I_G}(M(F))$, denoted by

$$\overleftarrow{H}_G : D_{I_G}(\mathcal{F}l_G) \times D_{I_G}(M(F)) \longrightarrow D_{I_G}(M(F)). \tag{2.2.8}$$

Consider the following isomorphism

$$\begin{aligned} \alpha : M(F) \times G(F) &\longrightarrow M(F) \times G(F) \\ (v, g) &\longrightarrow (g^{-1}v, g). \end{aligned}$$

Let $(a, b) \in I_G \times I_G$ act on the source by $(a, b).(v, g) = (av, agb)$ and act on an element of the target $(v^{'}, g^{'})$ by $(a, b).(v^{'}, g^{'}) = (b^{-1}v^{'}, ag^{'}b)$. The map α is $I_G \times I_G$-equivariant with respect to these two actions. Hence this yields a morphism of stacks

$$M(F) \times \mathcal{F}l_G \longrightarrow (M(F)/I_G) \times \mathcal{F}l_G.$$

This enables us to define the following morphism of stack quotients

$$act_q : I_G \backslash (M(F) \times \mathcal{F}l_G) \longrightarrow (M(F)/I_G) \times (I_G \backslash \mathcal{F}l_G),$$

where the action of I_G on $M(F) \times \mathcal{F}l_G$ is the diagonal one.

Lemma 2.2.9. *There exists an inverse image functor*

$$act_q^* : D_{I_G}(M(F)) \times D_{I_G}(\mathcal{F}l_G) \longrightarrow D_{I_G}(M(F) \times \mathcal{F}l_G)$$

which preserves perversity and is compatible with the Verdier duality in the following way: for any $\mathcal{K}$ in $D_{I_G}(M(F))$ and $\mathcal{F}$ in $D_{I_G}(\mathcal{F}l_G)$ we have

$$\mathbb{D}(act_q^*(\mathcal{K}, \mathcal{T})) \xrightarrow{\sim} act_q^*(\mathbb{D}(\mathcal{K}), \mathbb{D}(\mathcal{T})).$$

Proof. Given $N, r, s_1, s_2 \geq 0$ with $r \geq s_1$ and $s \geq \max\{N + r, s_1 + s_2 + 1\}$, one can define the following commutative diagram

$$\begin{array}{ccccc}
 & & M_{N,r} \times {}_{s_1,s_2}G(F) & \xrightarrow{act} & M_{N+s_1,r-s_1} \\
 & & \downarrow{\scriptstyle q_G} & & \downarrow{\scriptstyle q_M} \\
M_{N,r} & \xleftarrow{pr_1} & M_{N,r} \times {}_{s_1,s_2}\mathcal{F}l_G & \xrightarrow{act_q} & K_s\backslash M_{N+s_1,r-s_1} \\
\downarrow & & \downarrow & \nearrow{\scriptstyle act_{q,s}} & \\
K_s\backslash M_{N,r} & \xleftarrow{pr} & K_s\backslash(M_{N,r} \times {}_{s_1,s_2}\mathcal{F}l_G) & \xrightarrow{pr_2} & K_s\backslash\,({}_{s_1,s_2}\mathcal{F}l_G).
\end{array}$$

The action map *act* sends the couple (v, g) to $g^{-1}v$. The maps pr_1, pr_2 and pr are projections. The map q_G sends the couple (v, g) to (v, gI_G). All the vertical arrows are stack quotients for the action of the corresponding group. The group K_s acts diagonally on $M_{N,r} \times {}_{s_1,s_2}\mathcal{F}l_G$ and the map act_q is equivariant with respect to this action. This enables us to define the following functor:

$$D_{I_G}(M_{N+s_1,r-s_1}) \times D_{I_G}({}_{s_1,s_2}\mathcal{F}l_G) \overset{temp}{\longrightarrow} D_{I_G}(M_{N,r} \times {}_{s_1,s_2}\mathcal{F}l_G)$$

sending $(\mathcal{K}, \mathcal{T})$ to

$$(act_{q,s}^*\mathcal{K}) \otimes pr_2^*\mathcal{T}[\dim(K_s) - c + s_1 \dim M_0]$$

where c equals $\langle \theta, \check{\mu} \rangle$ over ${}_{s_1,s_2}\mathcal{F}l_G^\theta$.
Consider $r_1 \geq r_2$ and $s \geq \max\{s_1 + s_2, N + r_1\}$. Then we have the diagram

$$\begin{array}{ccc}
K_s\backslash(M_{N,r_1} \times {}_{s_1,s_2}\mathcal{F}l_G) & \xrightarrow{act_{q,s}} & K_s\backslash(M_{N+s_1,r_1-s_1}) \\
\downarrow & & \downarrow \\
K_s\backslash(M_{N,r_2} \times {}_{s_1,s_2}\mathcal{F}l_G) & \xrightarrow{act_{q,s}} & K_s\backslash(M_{N+s_1,r_2-s_1}).
\end{array} \tag{2.2.10}$$

The functors *temp* and the transition functors in (2.2.10) are compatible. This gives rise to a functor

$$temp_{N,s_1,s_2} : D_{I_G}(M_{N+s_1}) \times D_{I_G}({}_{s_1,s_2}\mathcal{F}l_G) \longrightarrow D_{I_G}(M_N \times {}_{s_1,s_2}\mathcal{F}l_G),$$

where $M_N = t^{-N}M$.
Let $N_1 \geq N + s_2$ then $N \leq N_1 - s_2 \leq N_1 + s_1$ and one has the closed immersion $M_N \hookrightarrow M_{N_1+s_1}$. Thus we have

$$\begin{array}{ccc} D_{I_G}(M_N) \times D_{I_G}({}_{s_1,s_2}\mathcal{F}l_G) & \hookrightarrow & D_{I_G}(M_{N_1+s_1}) \times D_{I_G}({}_{s_1,s_2}\mathcal{F}l_G) \\ & & \downarrow temp_{N_1,s_1,s_2} \\ & & D_{I_G}(M_{N_1} \times {}_{s_1,s_2}\mathcal{F}l_G) \\ & & \downarrow \\ & & D_{I_G}(M(F) \times {}_{s_1,s_2}\mathcal{F}l_G), \end{array} \tag{2.2.11}$$

where the first inclusion is the extension by zero under the closed immersion defined above. For $\mathcal{K}$ in $D_{I_G}(M_N)$ and $\mathcal{T}$ in $D_{I_G}({}_{s_1,s_2}\mathcal{F}l_G)$, the image of $(\mathcal{K}, \mathcal{T})$ under the composition (2.2.11) does not depend on N_1. So we get a functor

$$temp_{s_1,s_2} : D_{I_G}(M_N) \times D_{I_G}({}_{s_1,s_2}\mathcal{F}l_G) \longrightarrow D_{I_G}(M(F) \times {}_{s_1,s_2}\mathcal{F}l_G).$$

For any $s_1' \geq s_1$, and $s_2' \geq s_2$, we have the extension by zero functors

$$D_{I_G}({}_{s_1,s_2}\mathcal{F}l_G) \hookrightarrow D_{I_G}({}_{s_1',s_2'}\mathcal{F}\ell_G),$$

which are compatible with our functor $temp_{s_1s_2}$, so this yields the desired functor

$$act_q^* : D_{I_G}(M(F)) \times D_{I_G}(\mathcal{F}l_G) \overset{temp}{\longrightarrow} D_{I_G}(M(F) \times \mathcal{F}l_G)$$

One checks that $\mathbb{D}(act_q^*(\mathcal{K}, \mathcal{T})) \widetilde{\longrightarrow} act_q^*(\mathbb{D}(\mathcal{K}), \mathbb{D}(\mathcal{T}))$, and act_q^* preserves perversity. □

To define the convolution action, for $N, r, s_1, s_2 \geq 0$ with $s \geq \max\{N + r, s_1 + s_2 + 1\}$, consider the projection

$$pr : K_s\backslash(M_{N,r} \times {}_{s_1,s_2}\mathcal{F}l_G) \longrightarrow K_s\backslash M_{N,r},$$

which gives us

$$pr_! : D_{K_s}(M_{N,r} \times {}_{s_1,s_2}\mathcal{F}l_G) \longrightarrow D_{K_s}(M_{N,r}).$$

These functors are compatible with the transition functors in (2.2.10) and yield a functor

$$pr_! : D_{I_G}(M(F) \times \mathcal{F}l_G) \longrightarrow D_{I_G}(M(F)).$$

For any $\mathcal{K}$ in $D_{I_G}(M(F))$ and $\mathcal{T}$ in $D_{I_G}(\mathcal{F}l_G)$, the Hecke operator $\overleftarrow{H}_G$ (2.2.8), is defined by

$$\overleftarrow{H}_G(\mathcal{T}, \mathcal{K}) = pr_!(act_q^*(\mathcal{K}, \mathcal{T})).$$

Moreover, this functor is compatible with the convolution product on $D_{I_G}(\mathcal{F}l_G)$ defined in § 1.5.2 in Chapter 1. Namely, given $\mathcal{T}_1, \mathcal{T}_2$ in $D_{I_G}(\mathcal{F}l_G)$ and $\mathcal{K}$ in $D_{I_G}(M(F))$, one has naturally

$$\overleftarrow{H}_G(\mathcal{T}_1, \overleftarrow{H}_G(\mathcal{T}_2, \mathcal{K})) \widetilde{\longrightarrow} \overleftarrow{H}_G(\mathcal{T}_1 * \mathcal{T}_2, \mathcal{K}).$$

As at the end of § 1.3, one may also consider the category $DP_{I_G}(\mathcal{F}l_G)$. Sometimes, we will also use the Hecke functors in the form

$$\overleftarrow{H}_G : DP_{I_G}(\mathcal{F}l_G) \times D_{I_G}(M(F)) \longrightarrow D_{I_G}(M(F))$$

defined by $\overleftarrow{H}_G(\mathcal{T}[i], K) = \overleftarrow{H}_G(\mathcal{T}, K)[i]$ for $i \in \mathbb{Z}$ and $\mathcal{T} \in P_{I_G}(\mathcal{F}l_G)$.

Remark 2.2.12. *Let* $* : P_{I_G}(\mathcal{F}l_G) \tilde{\to} P_{I_G}(\mathcal{F}l_G)$ *be the covariant equivalence of categories induced by the map* $G(F) \to G(G)$, $g \mapsto g^{-1}$. *Define* $\overrightarrow{H}_G : D_{I_G}(\mathcal{F}l_G) \times D_{I_G}(\Pi(F)) \to D_{I_G}(\Pi(F))$ *by* $\overrightarrow{H}_G(\mathcal{T}, K) = \overleftarrow{H}_G(*\mathcal{T}, K)$. *We will use this right action in § 3.5.*

2.2.3 Example

Let $R, r \geq 0$ and $t^r M \subset V \subset t^{-R} M$ be an intermediate lattice stable under I_G. Let $K \in P_{I_G}(M_{R,r})$ be a shifted local system on $V/t^r M \subset t^{-R}M/t^r M$. We are going to explain the above construction explicitly in this case. Let $\mathcal{T}$ be in $D_{I_G}({}_{s_1,s_2}\mathcal{F}l_G)$. Choose $r_1 \geq r + s_1$. If g is a point in ${}_{s_1,s_2}\mathcal{F}l_G$ then $t^{r_1}M \subset gV$. So we can define the scheme

$$(V/t^r M)\tilde{\times}_{s_1,s_2}\mathcal{F}l_G$$

as the scheme classifying pairs (gI_G, m) such that gI_G is an element of ${}_{s_1,s_2}\mathcal{F}l_G$ and m is in $(gV)/(t^{r_1}M)$. For a point (m, g) of this scheme we have $g^{-1}m$ is in $V/t^r M$. Assuming $s \geq R + r$ we get the digram

$$M_{R+s_2,r_1} \xleftarrow{p} (V/t^r M)\tilde{\times}_{s_1,s_2}\mathcal{F}l_G \xrightarrow{act_{q,s}} K_s\backslash(V/t^r M),$$

where p is the map sending (gI_G, m) to m. For $gG(\mathcal{O}) \in Gr_G^\theta$, the virtual dimension of V/gV, is $\langle\theta, \check{\mu}\rangle$. The space $(V/t^r M)\tilde{\times}_{s_1,s_2}\mathcal{F}l_G^\theta$ is locally trivial fibration over ${}_{s_1,s_2}\mathcal{F}l_G^\theta$ with fibre isomorphic to an affine space of dimension $\dim(V/t^{r_1}M) - \langle\theta, \check{\mu}\rangle$. Since K is a shifted local system, the tensor product

$$act_{q,s}^* K \otimes pr_2^* \mathcal{T}$$

is a shifted perverse sheaf. Let $K\tilde{\boxtimes}\mathcal{T}$ be the perverse sheaf $act_{q,s}^* K \otimes pr_2^* \mathcal{T}[\dim]$. The shift $[\dim]$ in the definition depends on the dimension of the connected component and hence on $\check{\mu}$ as explained above and is such that the sheaf $act_{q,s}^* K \otimes pr_2^* \mathcal{T}[\dim]$ is perverse. Then

$$\overleftarrow{H}_G(\mathcal{T}, K) = p_!(K\tilde{\boxtimes}\mathcal{T}).$$

2.2.4 The relation between geometrical and classical convolution

Assume temporary that $\mathbf{k} = \mathbb{F}_q$. As in last section, let M_0 be a faithful representation of G. Given K in $D_{I_G}(M(F))$ we can associate with it the following function a_K in the Schwartz space $\mathcal{S}^{I_G}(M(F))$. If K is represented by the ind-pro-system $K_{N,r}$ in $D_{I_G}(M_{N,r})$ then for m in $t^{-N}M_0(\mathcal{O})$ one has

$$a_K(m) = Tr(Fr_{\overline{m}}, K_{N,r,\overline{m}})q^{\frac{rd}{2}},$$

where $d = \dim M_0$, the point $\overline{m}$ is the image of m in $M_{N,r}$, and $Fr_{\overline{m}}$ is the geometric Frobenius at $\overline{m}$. For large enough r, this is independent of r. We will see that the Hecke functors on $D_{I_G}(M(F))$ defined in the previous section geometrize the action of the Hecke operators on $\mathcal{S}^{I_G}(M(F))$ corresponding to the following left action of $G(F)$ on $\mathcal{S}(M(F))$. If g a point in $G(F)$ and f a function in $\mathcal{S}(M(F))$ then

$$g.f(m) = |\det g|^{\frac{-1}{2}} f(g^{-1}m),$$

for any m in $M(F)$.

As in § 1.5.3, to $\mathcal{T} \in P_{I_G}(\mathcal{F}l_G)$ one can associate a function on $G(F)/I_G$ given by $a_{\mathcal{T}}(x) = Tr(Fr_x, \mathcal{T}_x)$ for x in $G(F)/I_G$. For $\mathcal{T}_i \in P_{I_G}(\mathcal{F}l_G)$ denote by f_i the corresponding function then we have

$$Tr(Fr_g, (\mathcal{T}_1 \star \mathcal{T}_2)_g) = \int_{x\in G(F)} f_1(x)f_2(x^{-1}g)dx,$$

where $\star$ is the geometric convolution product on $\mathcal{F}l_G$ defined in [1.5.2, Chapter 1] and dx is the Haar measure on $G(F)$ such that I_G is of volume 1.

Now if $\mathcal{F}$ is in $D_{I_G}(M(F))$, let $K = \overleftarrow{H}_G(\mathcal{T}, \mathcal{F})$ and denote by f the function associated to $\mathcal{F}$, then the function a_K associated to K is

$$a_K(m) = \int_{x\in G(F)} |\det x|^{-\frac{1}{2}} f(x^{-1}m)a_{\mathcal{T}}(x)dx,$$

for any m in $M(F)$.

From now on assume $G = \mathbf{GL}_n$ and $H = \mathbf{GL}_m$. According to the construction in § 2.2, we have the well-defined category of $I_G \times I_H$-equivariant perverse sheaves on $\Pi(F)$ inside the derived category $D_{I_G\times I_H}(\Pi(F))$ and two convolution functors corresponding to the actions of $P_{I_G}(\mathcal{F}l_G)$ and $P_{I_H}(\mathcal{F}l_H)$ on $D_{I_G\times I_H}(\Pi(F))$:

$$\overleftarrow{H}_G : P_{I_G}(\mathcal{F}l_G) \times D_{I_G\times I_H}(\Pi(F)) \longrightarrow D_{I_G\times I_H}(\Pi(F))$$

and

$$\overleftarrow{H}_H : P_{I_H}(\mathcal{F}l_H) \times D_{I_G\times I_H}(\Pi(F)) \longrightarrow D_{I_G\times I_H}(\Pi(F)).$$

2.3 The module structure of $P_{H(\mathcal{O})\times I_G}(\Pi(F))$

The following notation will be used till the end of the chapter: $\mathbf{k}$ is a algebraically closed field of characteristic $p > 2$, F is the field $\mathbf{k}((t))$ and $\mathcal{O}$ is the ring of integers of F. Let $L_0 = \mathbf{k}^n$ and $U_0 = \mathbf{k}^m$. Let $G = \mathbf{GL}(L_0)$ and $H = \mathbf{GL}(U_0)$. We put $\Pi_0 = U_0 \otimes L_0$, $L = L_0(\mathcal{O})$, $U = U_0(\mathcal{O})$, and $\Pi = \Pi_0(\mathcal{O})$. For any $\mathcal{O}$-module of finite rank M and any pair N, r of integers such that $N + r > 0$, we set $M_{N,r} = t^{-N}M/t^rM$. Let T_G (resp. T_H) be the maximal torus of diagonal matrices in G (resp. in H). Let B_G (resp. B_H) be the Borel subgroup of upper-triangular matrices in G (resp. H). Let I_G and I_H be the corresponding Iwahori subgroups. Denote by W_G the finite Weyl group of G, X_G (resp. X_H) the lattice of cocharacters of G (resp. H), and X_G^+ the lattice of dominant cocharacters of G. Let I_0 denote the constant perverse sheaf on Π.

Throughout this section we assume that n is smaller than or equal to m. Let us consider the category $P_{H(\mathcal{O})\times I_G}(\Pi(F))$ of $H(\mathcal{O}) \times I_G$-equivariant perverse sheaves on $\Pi(F)$. The purpose here is to understand the module structure of this category under the action of $P_{I_G}(\mathcal{F}l_G)$ and $P_{H(\mathcal{O})}(Gr_H)$. Remind that according to the Satake isomorphism, $P_{H(\mathcal{O})}(Gr_H)$ is equivalent to the category $\mathrm{Rep}(\check{H})$ of representations of the Langlands dual group $\check{H}$ over $\overline{\mathbb{Q}}_\ell$.

2.3.1 Irreducible objects of $P_{H(\mathcal{O})\times I_G}(\Pi(F))$

Let U^* denote the dual of U. A point v in $\Pi(F)$ may be seen as a $\mathcal{O}$-linear map $v : U^* \to L(F)$. For v in $\Pi_{N,r}$, let $U_{v,r} = v(U^*) + t^r L$. Then $U_{v,r}$ is a $\mathcal{O}$-module in $L(F)$. It is possible to identify Gr_G with the ind-scheme of lattices in $L(F)$, see [§ 1.4.1, Chapter 1]. Then $U_{v,r}$ may be viewed as a point of the affine Grassmannian Gr_G.

Lemma 2.3.1. *The set of $H(\mathcal{O})$-orbits on $\Pi_{N,r}$ identifies with the scheme of lattices R such that $t^r L \subset R \subset t^{-N} L$ via the map sending v to $U_{v,r}$.*

Proof. Let M be a free $\mathcal{O}$-module and M' be any $\mathcal{O}$-module. If f_1 and f_2 are two surjections from M to M', then the kernel $\mathrm{Ker}(f_1)$ of the map f_1 is a free $\mathcal{O}$-submodule of M and $\mathrm{Ext}^1(M, \mathrm{Ker}(f_1)) = 0$. Hence there is an $\mathcal{O}$-linear endomorphism h of M such that $f_1 \circ h = f_2$. Let us now consider two elements v_1 and v_2 of $\Pi_{N,r}$ such that $U_{v_1,r} = U_{v_2,r}$. Adding to v_i a suitable element $t^r\Pi$, we may assume that both $v_i : U^* \to U_{v,r}$ are surjective for $i = 1, 2$. Then the previous argument implies that there exists h in $H(\mathcal{O})$ such that $v_1 \circ h = v_2$. Thus, for v_1 and v_2 in $\Pi_{N,r}$, the $H(\mathcal{O})$-orbits through v_1 and v_2 coincide if and only if $U_{v_1,r} = U_{v_2,r}$. It is straightforward to see that lattices R such that $t^r L \subset R \subset t^{-N} L$ are exactly of the form $U_{v,r}$ for some v in $\Pi_{N,r}$. □

Remark 2.3.2. *The I_G-orbits on Gr_G are parametrized by the cocharacter lattice X_G. For any λ in X_G we denote by O^λ the I_G-orbit through $t^\lambda G(\mathcal{O})$ in Gr_G and its closure by $\overline{O}^\lambda$ see [§ 1.4.2, Chapter 1].*

Let $\check{\omega}_1 = (1, 0 \ldots, 0)$ be the highest weight of the standard representation of G and let w_0 be the longest element of the finite Weyl group W_G.

Lemma 2.3.3. *The $H(\mathcal{O}) \times I_G$-orbits on $\Pi_{N,r}$ are parametrized by elements λ in X_G such that for any ν in $W_G.\lambda$*

$$\langle \nu, \check{\omega}_1 \rangle \leq r \qquad \text{and} \qquad \langle w_0(\nu), \check{\omega}_1 \rangle \leq N. \tag{2.3.4}$$

Denote each orbit indexed by λ as above by $\Pi_{\lambda,r}$. Each $\Pi_{\lambda,r}$ consists of points v such that $U_{v,r}$ lies in $I_G t^\lambda G(\mathcal{O})$.

Proof. Any lattice R satisfying $t^r L \subset R \subset t^{-N} L$ is of the form $U_{v,r}$ for some v in $\Pi_{N,r}$. Consider the lattice $U_{v,r}$ as a point in Gr_G. Then by Lemma 2.3.1 and Remark 2.3.2 the $H(\mathcal{O}) \times I_G$-orbits on $\Pi_{N,r}$ are exactly the locally closed subschemes $(\Pi_{\lambda,r})_{\lambda \in X_G}$ in $\Pi_{N,r}$ such that λ satisfies (2.3.4). □

The closure of the orbit $\Pi_{\lambda,r}$ will always be denoted in the sequel by $\overline{\Pi}_{\lambda,r}$.

Remark 2.3.5. *For a given cocharacter λ, the condition* (2.3.4) *is verified for large enough r. This defines a stratification of $\Pi_{N,r}$ by locally closed subschemes $\Pi_{\lambda,r}$.*

For any λ in X_G, the perverse sheaves $\mathrm{IC}(\Pi_{\lambda,r})$ in $P_{H(\mathcal{O})\times I_G}(\Pi(F))$ are independent of the choice of r provided by the condition $\langle \nu, \check{\omega}_1 \rangle < r$ for any $\nu \in W_G\lambda$. Hence the resulting object of $P_{H(\mathcal{O})\times I_G}(\Pi(F))$ will be denoted by $\mathrm{IC}(\Pi_\lambda)$.

Proposition 2.3.6. *The irreducible objects of $P_{H(\mathcal{O})\times I_G}(\Pi(F))$ are in bijection with X_G. The irreducible object corresponding to a cocharacter λ in X_G is the intersection cohomology sheaf $\mathrm{IC}(\Pi_\lambda)$ of the orbit $\Pi_{\lambda,r}$ for a suitable r.*

2.3.2 The module structure of $P_{H(\mathcal{O})\times I_G}(\Pi(F))$ under $P_{I_G}(\mathcal{F}l_G)$ and $\mathrm{Rep}(\check{H})$

For λ in X_G denote by $\mathcal{A}^\lambda$ the IC-sheaf of O^λ which is an object of $P_{I_G}(Gr_G)$. The constant perverse sheaf I_0 is an object of $D_{H(\mathcal{O})\times I_G}(\Pi(F))$. As in § 2.2.2 one gets the Hecke functor

$$\overleftarrow{H}_G : D_{I_G}(Gr_G) \times D_{H(\mathcal{O})\times I_G}(\Pi(F)) \longrightarrow D_{H(\mathcal{O})\times I_G}(\Pi(F)).$$

It is normalized to commute with the Verdier duality as in Lemma 2.2.9.

Proposition 2.3.7. *For any λ in X_G, the complex $\overleftarrow{H}_G(\mathcal{A}^\lambda, I_0)$ is canonically isomorphic to* $\mathrm{IC}(\Pi_\lambda)$. *Hence any irreducible object of the category $P_{H(\mathcal{O})\times I_G}(\Pi(F))$ is obtained by the action of $\mathcal{A}^\lambda$ on I_0 for some λ in X_G.*

The proof of this proposition will be presented in several steps. First remark that if λ is dominant then $\mathcal{A}^\lambda$ is $G(\mathcal{O})$-equivariant. In this case Proposition 2.3.7 results from [Lys11, Proposition 5]. We use § 2.2.3 to give a concrete description of the complex $\overleftarrow{H}_G(\mathcal{A}^\lambda, I_0)$. Choose two integers N, r, with $N+r > 0$ such that for any $\nu \in W_G.\lambda$, the condition (2.3.4) be satisfied. Let $\Pi_{0,r}\tilde{\times}\overline{O}^\lambda$ be the scheme classifying pairs $(v, gG(\mathcal{O}))$, where $gG(\mathcal{O})$ belongs to $\overline{O}^\lambda$ and v is an $\mathcal{O}$-linear map from U^* to gL/t^rL. Let

$$\pi : \Pi_{0,r}\tilde{\times}\overline{O}^\lambda \longrightarrow \Pi_{N,r} \tag{2.3.8}$$

be the map sending a pair $(v, gG(\mathcal{O}))$ to the composition $U^* \xrightarrow{v} gL/t^rL \hookrightarrow t^{-N}L/t^rL$. This map is proper. The projection $p : \Pi_{0,r}\tilde{\times}\overline{O}^\lambda \to \overline{O}^\lambda$ is a vector bundle of rank $rnm - m\langle\lambda, \check{\omega}_n\rangle$, where $\check{\omega}_n = (1, \dots, 1)$. We obtain in this particular case an isomorphism

$$\overleftarrow{H}_G(\mathcal{A}^\lambda, I_0) \widetilde{\longrightarrow} \pi_!(\overline{\mathbb{Q}}_\ell\tilde{\boxtimes}\mathcal{A}^\lambda), \tag{2.3.9}$$

where the complex $\overline{\mathbb{Q}}_\ell\tilde{\boxtimes}\mathcal{A}^\lambda$ is normalized to be perverse, i.e.

$$\overline{\mathbb{Q}}_\ell\tilde{\boxtimes}\mathcal{A}^\lambda \widetilde{\longrightarrow} p^*\mathcal{A}^\lambda[\dim \mathrm{rel}(p)].$$

According to [Gai01, Theorem 1] the category $P_{G(\mathcal{O})}(Gr_G)$ acts on $P_{I_G}(Gr_G)$ by convolution and this convolution product $\star$ preserves perversity. We want to use this result in order to give a dimension estimate for the objects of $P_{I_G}(Gr_G)$. Let $D = \mathrm{Spec}(\mathbf{k}[[t]])$ and $D^* = \mathrm{Spec}(\mathbf{k}((t)))$. Denote by E^0 the trivial G-torsor on D. Using the notation of § 1.4, the ind-scheme $G(F) \times_{G(\mathcal{O})} Gr_G$ defined in Lemma 1.4.10 classifies quadruplets $(E, E^1, \tilde{\beta}, \beta^1)$. Let m be the multiplication map $G(F) \times_{G(\mathcal{O})} Gr_G \to Gr_G$ defined in [Diagram (1.4.11), Chapter 1]. For $\mathcal{A}$ in $P(Gr_G)$ and $\mathcal{B}$ in $P_{G(\mathcal{O})}(Gr_G)$, the convolution product $m_!(\mathcal{A}\tilde{\boxtimes}\mathcal{B})$ of $\mathcal{A}$ with $\mathcal{B}$ is perverse.

For μ in X_G^+, let $\mathcal{B}^\mu$ be the IC-sheaf associated with the $G(\mathcal{O})$-orbit $t^\mu G(\mathcal{O})$ in Gr_G. According to [Gai01], for any cocharacter λ in X_G the convolution product $\mathcal{A}^\lambda \star \mathcal{B}^\mu$ is perverse. For any ν in X_G, and any point (E^1, β^1) in O^ν, let Y be the fibre of the map m over this point. The fibre Y identifies with the affine Grassmannian Gr_G. For η in X_G and δ in X_G^+, let $Y^{\eta,\delta}$ be the stratum of Y defined by the following two conditions:

1. the G-torsor E is in I_G-position η with respect to the trivial G-torsor E^0.
2. the G-torsor E^1 is in $G(\mathcal{O})$-position δ with respect to E.

Note that the restriction of $\mathcal{A}^\lambda \star \mathcal{B}^\mu$ to O^ν sites in usual degrees smaller than or equal to $-\dim O^\nu$, and the restriction of $\mathcal{A}^\lambda \tilde{\boxtimes} \mathcal{B}^\mu|_{Y^{\eta,\delta}}$ is the constant complex sitting in usual degrees smaller than or equal to $-\dim O^\eta - \dim Gr_G^\delta$.

Lemma 2.3.10. *For any η, ν in X_G and any δ in X_G^+ the following inequality holds:*

$$2\dim Y^{\eta,\delta} - \dim O^\eta - \dim Gr_G^\delta \leq -\dim O^\nu.$$

Proof. Let $\mathcal{B}^{\delta,!}$ (resp. $\mathcal{A}^{\eta,!}$) be the constant perverse sheaf on Gr_G^δ (resp.O^η) extended by zero (in the perverse sense) on Gr_G. The extension by zero functor is right exact for the perverse t-structure. Hence $\mathcal{B}^{\delta,!}$ (resp. $\mathcal{A}^{\eta,!}$) lies in non positive perverse degrees and so does the convolution product $\mathcal{A}^{\eta,!} \star \mathcal{B}^{\delta,!}$. The $*$-restriction of $\mathcal{A}^{\eta,!} \tilde{\boxtimes} \mathcal{B}^{\delta,!}$ to Y is the extension by zero from $Y^{\eta,\delta}$ to Y of the constant complex. Hence this complex lies in degrees $2\dim Y^{\eta,\delta} - \dim O^\eta - \dim Gr_G^\delta + \dim O^\nu$ and so we have the desired inequality. □

Proof of Proposition 2.3.7 : Let λ be in X_G and consider the complex $\pi_!(\overline{\mathbb{Q}}_\ell \tilde{\boxtimes} \mathcal{A}^\lambda)$ appearing in (2.3.9). For ν in X_G, take a $H(\mathcal{O}) \times I_G$-orbit $\Pi_{\nu,r}$ in $\Pi_{N,r}$. If v is in $\Pi_{\nu,r}$, let Y_v be the fibre of the map π over v defined in (2.3.8). The fibre Y_v is the scheme classifying $gG(\mathcal{O})$ in $\overline{O}^\lambda$ such that $U_{v,r}$ is a sublattice of gL. If v is in $\Pi_{\lambda,r}$ then Y_v is just a point and so the map π is an isomorphism over the open subscheme $\Pi_{\lambda,r}$. On one hand this implies directly that $\mathrm{IC}(\Pi_{\lambda,r})$ appears with multiplicity one in the complex of sheaves $\overleftarrow{H}_G(\mathcal{A}^\lambda, I_0)$. On the other hand this gives

$$\dim(\Pi_{\lambda,r}) = rnm - m\langle\lambda, \check{\omega}_n\rangle + \dim O^\lambda.$$

Let U be the open subscheme of $\Pi_{0,r} \tilde{\times} \overline{O}^\lambda$ consisting of pairs $(v, gG(\mathcal{O}))$ such that $gG(\mathcal{O})$ lies in O^λ and $v : U^* \longrightarrow gL/t^r L$ is surjective. The image of U by π is contained in $\Pi_{\lambda,r}$. So, π induces a surjective proper map

$$\pi_\lambda : \Pi_{0,r} \tilde{\times} \overline{O}^\lambda \longrightarrow \overline{\Pi}_{\lambda,r}.$$

For v in $\Pi_{\nu,r}$, we stratify Y_v by locally closed subschemes Y_v^η indexed by cocharacters η in X_G. For any η, the stratum Y_v^η parametrizes elements $gG(\mathcal{O})$ in O^η. The $*$-restriction of $\overline{\mathbb{Q}}_\ell \tilde{\boxtimes} \mathcal{A}^\lambda$ to Y_v^η lives in usual degrees smaller than or equal to $-\dim O^\eta - rnm + m\langle\eta, \check{\omega}_n\rangle$ and the inequality is strict unless $\eta = \lambda$. We will show that

$$2\dim Y_v^\eta - \dim O^\eta - rnm + m\langle\eta, \check{\omega}_n\rangle \leq -\dim \Pi_{\nu,r} \tag{2.3.11}$$

and that the inequality is strict unless $\nu = \lambda$, this would imply our claim. Since we have $\dim(\Pi_{\nu,r}) = rnm - m\langle\nu, \check{\omega}_n\rangle + \dim O^\nu$, the inequality (2.3.11) becomes

$$2\dim Y_v^\eta \leq m\langle\nu - \eta, \check{\omega}_n\rangle + \dim O^\eta - \dim O^\nu. \tag{2.3.12}$$

Considering the map $\pi_\eta : \Pi_{0,r} \tilde{\times} \overline{O}^\eta \longrightarrow \overline{\Pi}_{\eta,r}$, we see that $\Pi_{\nu,r} \subset \overline{\Pi}_{\eta,r}$.
A dominant cocharacter δ in X_G^+ is called *very positive* if

$$\delta = (b_1 \geq \cdots \geq b_n \geq 0).$$

It is natural to stratify Y_v^η by locally closed subschemes $Y_v^{\eta,\delta}$, where δ runs through very positive cocharacters. For any such δ, the stratum $Y_v^{\eta,\delta}$ consists of elements $(v, gG(\mathcal{O}))$ such that the lattice $U_{v,r}$ is in $G(\mathcal{O})$-position δ with respect to the lattice gL. For a point $(v, gG(\mathcal{O}))$ of $Y_v^{\eta,\delta}$ the formula of virtual dimensions $\dim(L/gL) + \dim(gL/U_{v,r}) = \dim(L/U_{v,r})$ gives

$$\langle \delta + \eta - \nu, \check{\omega}_n \rangle = 0.$$

Finally the equation (2.3.12) is equivalent to

$$2 \dim Y_v^{\eta,\delta} \leq n\langle \delta, \check{\omega}_n \rangle + \dim O^\eta - \dim O^\nu.$$

By using Lemma 2.3.10 we are reduced to show that for any very positive δ, $\langle \delta, n\check{\omega}_n - 2\check{\rho}_G \rangle \geq 0$. To prove this inequality notice that

$$n\check{\omega}_n - 2\check{\rho}_G = (1, 3, 5, \ldots, 2n - 1).$$

Thus $n\check{\omega}_n - 2\check{\rho}_G$ is very positive and so for any very positive cocharacter δ we have $\langle \delta, n\check{\omega}_n - 2\check{\rho}_G \rangle \geq 0$. This proves the inequality (2.3.12). Moreover for any very positive δ this inequality is strict unless $\delta = 0$ which is the case if and only if $\nu = \eta$. This finishes the proof. □

The module structure of $P_{H(\mathcal{O})\times G(\mathcal{O})}(\Pi(F))$ under the action of the category $P_{H(\mathcal{O})}(Gr_H)$ has been described in [Lys11, § 5]. Let us recall this result.

Let U_1 (resp. U_2) be the vector subspace of U_0, generated by the first n basis vectors (resp. by the last $m - n$ basis vectors) of U_0. Thus, $U_0 = U_1 \oplus U_2$. Let $P \subset H$ be the parabolic subgroup preserving U_1. Let $M \widetilde{\to} \mathbf{GL}(U_1) \times \mathbf{GL}(U_2)$ the standard Levi factor in P and the map $\kappa : \check{G} \times \mathbb{G}_\mathfrak{m} \to \check{H}$ be the composition

$$\check{G} \times \mathbb{G}_\mathfrak{m} \overset{id\times 2\check{\rho}_{\mathbf{GL}(U_2)}}{\longrightarrow} \check{G} \times \check{\mathbf{GL}}(U_2) = \check{M} \hookrightarrow \check{H}. \tag{2.3.13}$$

Write $\mathrm{gRes}^\kappa : P_{H(\mathcal{O})}(Gr_H) \to DP_{G(\mathcal{O})}(Gr_G)$ for the functor corresponding to the restriction $\mathrm{Rep}(\check{H}) \to \mathrm{Rep}(\check{G} \times \mathbb{G}_\mathfrak{m})$ with respect to κ. We use the extended Satake equivalence (1.4.13).

Proposition 2.3.14. *[Lys11, Proposition 4] The two functors $P_{H(\mathcal{O})}(Gr_H) \to D_{H(\mathcal{O})\times G(\mathcal{O})}(\Pi(F))$ given by*

$$\mathcal{T} \to \overleftarrow{H}_H(\mathcal{T}, I_0) \qquad \textit{and} \qquad \mathcal{T} \to \overleftarrow{H}_G(\mathrm{gRes}^\kappa(\mathcal{T}), I_0)$$

are isomorphic.

Theorem 2.3.15. *For any λ in X_G and $\mathcal{T}$ in $P_{H(\mathcal{O})}(Gr_H)$ we have the following isomorphism*

$$\overleftarrow{H}_H(\mathcal{T}, \mathrm{IC}(\Pi_\lambda)) \widetilde{\to} \overleftarrow{H}_G(\mathcal{A}^\lambda \star \mathrm{gRes}^\kappa(\mathcal{T}), I_0).$$

Proof. Since $P_{I_G}(\mathcal{F}l_G)$ and $P_{H(\mathcal{O})}(Gr_H)$-actions on $P_{H(\mathcal{O})\times I_G}(\Pi(F))$ commute, we get from Proposition 2.3.7 and [Lys11, Proposition 4]

$$\begin{aligned}
\overleftarrow{H}_H(\mathcal{T}, \mathrm{IC}(\Pi_\lambda)) &\widetilde{\longrightarrow} \overleftarrow{H}_H(\mathcal{T}, \overleftarrow{H}_G(\mathcal{A}^\lambda, I_0)) \\
&\widetilde{\longrightarrow} \overleftarrow{H}_G(\mathcal{A}^\lambda, \overleftarrow{H}_H(\mathcal{T}, I_0)) \\
&\widetilde{\longrightarrow} \overleftarrow{H}_G(\mathcal{A}^\lambda, \overleftarrow{H}_G(\mathrm{gRes}^\kappa(\mathcal{T}), I_0)) \\
&\widetilde{\longrightarrow} \overleftarrow{H}_G(\mathcal{A}^\lambda \star \mathrm{gRes}^\kappa(\mathcal{T}), I_0).
\end{aligned}$$

□

From Proposition 2.3.7 it also follows that the functor

$$D_{I_G}(Gr_G) \to D_{H(\mathcal{O})\times I_G}(\Pi(F)) \tag{2.3.16}$$

given by $\mathcal{A} \mapsto \overleftarrow{H}_G(\mathcal{A}, I_0)$ is exact for the perverse t-structures. It is easy to see that neither of the categories $P_{I_G}(Gr_G)$ or $P_{H(\mathcal{O})\times I_G}(\Pi(F))$ is semi-simple. The functor (2.3.16) commutes with the actions of $P_{I_G}(\mathcal{F}l_G)$ by convolutions on the left. Let $P_{H(\mathcal{O})}(Gr_H)$ act on $D_{I_G}(Gr_G)$ via gRes^κ composed with the natural action of $D_{G(\mathcal{O})}(Gr_G)$ by convolutions on the right. According to Proposition 2.3.15, it is natural to expect that (2.3.16) commutes with the actions of $P_{H(\mathcal{O})}(Gr_H)$. From Proposition 2.3.7 and [Lys11, Proposition 4] one easily derives the following.

Corollary 2.3.17. *The functor* (2.3.16) *yields an isomorphism at the level of Grothendieck groups between $K(P_{I_G}(Gr_G))$ and $K(P_{H(\mathcal{O})\times I_G}(\Pi(F)))$ commuting with the above actions of $K(P_{H(\mathcal{O})}(Gr_H))$ and $K(P_{I_G}(\mathcal{F}l_G))$.*

2.4 Description of irreducible objects of $P_{I_H\times I_G}(\Pi(F))$

We use the same notation as in the previous section. Recall that for any cocharacter λ in X_G satisfying (2.3.4), Lemma 2.3.3 defines an $H(\mathcal{O}) \times I_G$-orbit $\Pi_{\lambda,r}$ on $\Pi_{N,r}$. Our goal is to describe the irreducible objects of $P_{I_H\times I_G}(\Pi(F))$. Remind that according to Proposition 2.3.6 the irreducible objects of $P_{H(\mathcal{O})\times I_G}(\Pi(F))$ are in bijection with X_G and the irreducible object corresponding to a cocharacter λ in X_G is the intersection cohomology sheaf $\mathrm{IC}(\Pi_{\lambda,r})$ of the orbit $\Pi_{\lambda,r}$. To do so we study the $I_H \times I_G$-orbits on $\Pi_{\lambda,r}$. It turns out that it is not necessary to do the study for all cocharacters λ. Indeed if $\lambda = (a_1, \dots, a_n)$ we will restrict ourselves to the case where all a_i's are strictly smaller than r. This will be sufficient for our purpose.

Let

$$\mathrm{Stab}_\lambda = \{g \in I_G \mid g(t^\lambda L) = t^\lambda L\}$$

and

$$X^\lambda_{N,r} = \{v \in \Pi_{N,r} \mid U_{v,r} = t^\lambda L + t^r L\}.$$

Describing $I_H \times I_G$-orbits on $\Pi_{\lambda,r}$ is equivalent to describe $I_H \times \mathrm{Stab}_\lambda$-orbits on $X^\lambda_{N,r}$.

2.4.1 Case n=m

Lemma 2.4.1. *The $I_H \times \mathrm{Stab}_\lambda$- orbits on $X^\lambda_{N,r}$ are in bijection with the finite Weyl group W_G.*

Proof. Let $\{e_1, e_2, \dots, e_n\}$ be the standard basis of the vector space L_0 such that the Borel subgroup B_G preserves the standard flag associated with the basis $(e_i)_{1\le i\le n}$. Let $(u_1^*, u_2^* \dots, u_m^*)$ be the standard basis of the dual space U_0^*. We identify L_0 and U_0^* by sending e_{n-i} to u^*_{i+1} for $i = 0, \dots, n-1$. Let v be a point in $X^\lambda_{N,r}$ and consider the induced map

$$\overline{v} : U^*/tU^* \longrightarrow U_{v,r}/(tU_{v,r} + t^r L) = t^\lambda L/t^{\omega_n+\lambda}L, \tag{2.4.2}$$

where $\omega_n = (1, \dots, 1)$. The map $\overline{v}$ is an isomorphism and it may be considered as an element of $\mathrm{Aut}(t^\lambda L/t^{\omega_n+\lambda}L)$. Denote by

$$\dots \subset L_{-1} \subset L_0 \subset L_1 \subset \dots$$

the standard complete flag of lattices inside $L(F)$ preserved by the Iwahori group I_G. Then for any i in $\mathbb{Z}$ the images of $L_i \cap t^\lambda L$ in $t^\lambda L/t^{\omega_n+\lambda}L$ define a complete flag which is preserved by Stab_λ.

Thus the image of Stab_λ in $\mathrm{Aut}(t^\lambda L/t^{\omega_n+\lambda}L)$ is a Borel subgroup of G but not necessary the standard one. Hence the $I_H \times \mathrm{Stab}_\lambda$-orbits on the set of isomorphisms (2.4.2) are parameterized by the finite Weyl group W_G. By Lemma 2.4.3 below each $I_H \times \mathrm{Stab}_\lambda$-orbit on $X^\lambda_{N,r}$ is the preimage of a $I_H \times \mathrm{Stab}_\lambda$-orbit on the scheme of isomorphisms (2.4.2). Finally one gets that $I_H \times \mathrm{Stab}_\lambda$-orbits on $X^\lambda_{N,r}$ are exactly indexed by W_G. □

Hence by this Lemma, any $I_H \times I_G$-orbit on $\Pi_{\lambda,r}$ is parameterized by W_G under the assumption that all a_i's are strictly smaller than r.

Lemma 2.4.3. *Let p, q be two integers such that $p \le q$. Let B be a free $\mathcal{O}$-module of rank p and A be a free $\mathcal{O}$-module of rank q. Let $v_1, v_2 : A \to B$ be surjective $\mathcal{O}$-linear maps such that for $i = 1, 2$ the induced maps $\overline{v}_i : A/tA \to B/tB$ coincide. Then there is $h \in \mathbf{GL}(A)(\mathcal{O})$ with $h = 1$ mod t such that $v_2 \circ h = v_1$.*

Proof. Let A_i be the kernel of v_i for $i = 1, 2$. These are free $\mathcal{O}$-modules of rank $q - p$. Choose a direct sum decomposition $A = A_i \oplus W_i$, where W_i is a free $\mathcal{O}$-module of rank p. Then there is a unique isomorphism $a : W_2 \longrightarrow W_1$ such that $W_2 \xrightarrow{a} W_1 \xrightarrow{v_1} A$ coincides with $W_2 \xrightarrow{v_2} A$. The images of $A_i \otimes_{\mathcal{O}} k$ in $A \otimes_{\mathcal{O}} k$ coincide, therefore there exists an isomorphism of $\mathcal{O}$-modules $b : A_2 \longrightarrow A_1$ such that $\overline{b} : A_2 \otimes_{\mathcal{O}} k \longrightarrow A_1 \otimes_{\mathcal{O}} k$ is identity. Then $a \oplus b$ is the desired map h. □

Lemma 2.4.4. *Any element of w of $\widetilde{W}_G$ defines an $I_H \times I_G$-orbit on $\Pi_{N,r}$ for large enough r denoted by $\Pi^w_{N,r}$. More precisely if $w = t^\lambda \tau$ and $\lambda = (a_1, \dots, a_n)$, all values of r strictly bigger than a_i's are admissible.*

Proof. According to Lemma 2.4.1, we know that for any cocharacter $\lambda = (a_1, \dots, a_n)$ in X_G such that for all i, a_i is strictly smaller than r the set of $I_H \times I_G$-orbits on $\Pi_{\lambda,r}$ is indexed by W_G. For any such λ and any τ in W_G, let $w = t^\lambda \tau$ be the associated element of $\widetilde{W}_G$. Then the $I_H \times I_G$-orbit passing through a point v of $\Pi_{N,r}$ is given by

$$v(u_i^*) = t^{a_{\tau_i}} e_{\tau(i)} \quad for \;\; i = 1, \dots, n.$$

For any $w = t^\lambda \tau$ in $\widetilde{W}_G$ we denote this orbit by $\Pi^w_{N,r}$. Remark that for a given w in $\widetilde{W}_G$ of the form $t^\lambda \tau$, the condition a_i being strictly smaller than r, for $i = 1, \dots, n$ is verified if r is large enough. □

For any w in $\widetilde{W}_G$ denote by $\mathcal{I}^w$ the IC-sheaf of the $I_H \times I_G$-orbit $\Pi^w_{N,r}$ indexed by w, and by $\mathcal{I}^{w!}$ the constant perverse sheaf on $\Pi^w_{N,r}$ extended by zero to $\Pi_{N,r}$. As an object of $P_{I_H \times I_G}(\Pi(F))$ it is independent of r, so that our notation is unambiguous. We underline that this notation is only introduced under the assumption $a_i < r$ for all i.

Proposition 2.4.5. *If n equals m any irreducible object of $P_{I_H \times I_G}(\Pi(F))$ is of the form $\mathcal{I}^w$ for some w in $\widetilde{W}_G$.*

Proof. An irreducible object of $P_{I_H \times I_G}(\Pi(F))$ is the IC-sheaf of an $I_H \times I_G$-orbit $\mathcal{Y}$ on $\Pi_{\lambda,r}$ for some integer r and for some cocharacter λ satisfying (2.3.4). In particular all a_i's are smaller than or equal to r. First we will show that we can restrict ourselves to the case where all a_i's are strictly less than r. Assume that $a_i = r$ for some i. For $s > r$ consider the projection $q : \Pi_{N,s} \to \Pi_{N,r}$. Then the $H(\mathcal{O}) \times I_G$-orbit $\Pi_{\lambda,s}$ is open in $q^{-1}(\Pi_{\lambda,r})$. The map $q : \Pi_{\lambda,s} \to \Pi_{\lambda,r}$ is not surjective but the sheaf $\mathrm{IC}(\mathcal{Y})$ is non-zero over the locus in $q^{-1}(\Pi_{\lambda,r})$ of maps $v : U^* \to t^{-N}L/t^rL$ whose geometric fibre of the image is of maximal dimension n. Hence the IC-sheaf of $\mathcal{Y}$ is a also an IC-sheaf of some $I_H\times I_G$-orbit on $\Pi_{\lambda,s}$. Thus we are reduced to the case where all a_i are strictly less than r.

Next we are going to prove that each $I_H\times I_G$-equivariant local system on $\Pi^w_{N,r}$ is constant. The map $X^\lambda_{N,r}\to \mathrm{Isom}(U^*/tU^*, t^\lambda L/t^{\lambda+\omega_n}L)$ given by $v\to\bar v$ is an affine fibration. The group $\mathrm{Hom}(U^*, t^{\lambda+\omega_n}L/t^rL)$ acts freely and transitively on the fibres of this map. So we are reduced to show that any $B_G\times B_H$-equivariant local system on any $B_H\times B_G$-orbit on $U_0\otimes L_0$ is constant. Indeed this is true because the stabilizer in B_G of a point in the double coset B_GwB_G/B_G for any w in W_G is connected. □

Remark 2.4.6. *If λ is dominant then the image of* Stab_λ *in* $\mathrm{Aut}(t^\lambda L/t^{\lambda+\omega_n}L)$ *is the standard Borel subgroup of G. Thus when $w=t^\lambda$ with λ being dominant we have that $\Pi^w_{N,r}$ is an open subscheme of $\Pi_{\lambda,r}$ and $\mathcal{I}^w=\mathrm{IC}(\Pi_{\lambda,r})$.*

2.4.2 Case $n\le m$

Assume that n is smaller than or equal to m. In this case the map (2.4.2) is not an isomorphism but only a surjection. We may consider the $I_H\times\mathrm{Stab}_\lambda$-orbits on the set of surjections (2.4.2). Let $S_{n,m}$ be the set of pairs (s,I_s), where I_s is a subset of n elements of $\{1,\dots,m\}$ and $s: I_s\longrightarrow\{1,\dots,n\}$ is a bijection. Let $W_1\subset W_2\subset\cdots\subset W_m=U_0^*$ be a complete flag preserved by B_H. We denote by $\overline{W}_i$ for the image of W_i under the map (2.4.2). Then $I_s=\{1\le i\le m|\dim\overline{W}_i>\dim\overline{W}_{i-1}\}$.

Lemma 2.4.7. *Any element of $X_G\times S_{n,m}$ defines an $I_H\times I_G$-orbit on $\Pi_{N,r}$ for large enough r. More precisely if $w=(\lambda,s)$ and $\lambda=(a_1,\dots,a_n)$, all values of r strictly bigger than a_i's are admissible.*

Proof. It is sufficient to describe the $I_H\times\mathrm{Stab}_\lambda$-orbits on the set $X^\lambda_{N,r}$. By Lemma 2.4.3 each $I_H\times\mathrm{Stab}_\lambda$-orbit on $X^\lambda_{N,r}$ is the preimage of a $I_H\times\mathrm{Stab}_\lambda$-orbit on the set of surjections (2.4.2). Let $\lambda=(a_1,\dots,a_n)$. If all a_i's are strictly less than r, the $I_H\times\mathrm{Stab}_\lambda$-orbits on the set of surjections (2.4.2) are indexed by the set $S_{n,m}$. Let $w=(\lambda,s)$ be in $X_G\times S_{n,m}$ then the $I_H\times I_G$-orbit passing through v a point of $\Pi_{N,r}$ is given by

$$\begin{cases} v(u_i^*)=t^{a_{si}}e_{si} & \text{for } i\in I_s;\\ v(u_i^*)=0 & \text{for } i\notin I_s.\end{cases}$$

□

We denote this orbit by $\Pi^w_{N,r}$ and its closure by $\overline{\Pi}^w_{N,r}$. For any $w=(s,\lambda)$ in $X_G\times S_{n,m}$, where λ is in X_G and s is in $S_{n,m}$, denote by $\mathcal{I}^w$ the IC-sheaf of $\Pi^w_{N,r}$.
The corresponding object of $D_{I_H\times I_G}(\Pi(F))$ is well-defined and independent of N,r.

Theorem 2.4.8. *Any irreducible object of $P_{I_H\times I_G}(\Pi(F))$ is of the form $\mathcal{I}^w$ for some w in $X_G\times S_{n,m}$.*

Proof. An irreducible object of $P_{I_H\times I_G}(\Pi(F))$ is the IC-sheaf of an $I_H\times I_G$-orbit $\mathcal{Y}$ on $\Pi_{\lambda,r}$ for some integer r and for some cocharacter $\lambda=(a_1,\dots,a_n)$ satisfying (2.3.4). As in the proof of Proposition 2.4.5 we may assume that all a_i's are strictly less than r. Consider a $I_H\times I_G$-orbit

$\Pi^w_{N,r}$ on $\Pi_{N,r}$ passing through v as defined in Lemma 2.4.7. Let $St(v)$ be the stabilizer of v in $I_H \times I_G$. We are going to show that $St(v)$ is connected. This will imply that any $I_H \times I_G$-equivariant local system on $I_H \times I_G$-orbit $\Pi^w_{N,r}$ is constant.

The stabilizer $St(v)$ of v is a subgroup of $I_H \times \mathrm{Stab}_\lambda$. Let B_λ be the image of Stab_λ in $\mathrm{Aut}(t^\lambda L/t^{\lambda+\omega_n}L)$ then B_λ is a Borel subgroup of $\mathrm{Aut}(t^\lambda L/t^{\lambda+\omega_n}L)$. We define two groups $I_{0,\lambda}$ and $I_{0,H}$ by the exact sequences

$$1 \longrightarrow I_{0,\lambda} \longrightarrow \mathrm{Stab}_\lambda \longrightarrow B_\lambda \longrightarrow 1,$$

and

$$1 \longrightarrow I_{0,H} \longrightarrow I_H \longrightarrow B_H \longrightarrow 1.$$

Note that I_H is semi-direct product of $I_{0,H}$ and B_H. Let $St_0(v)$ be the stabilizer of v in $I_{0,H} \times I_{0,\lambda}$. By Lemma 2.4.3, the $I_{0,H} \times I_{0,\lambda}$-orbit through v on $X^\lambda_{N,r}$ is the affine space of surjections $f : U^* \longrightarrow t^\lambda L/t^r L$ such that $f = v \bmod t$. Thus $St_0(v)$ is connected. Let $\overline{v} : U^*/tU^* \longrightarrow t^\lambda L/t^{\lambda+\omega_n}L$ be the reduction of v mod t. The stabilizer $St(\overline{v})$ of $\overline{v}$ in $B_H \times B_\lambda$ is connected. By Lemma 2.4.3 the reduction map from $St(v)$ to $St(\overline{v})$ is surjective. Using the exact sequence

$$1 \longrightarrow St_0(v) \longrightarrow St(v) \longrightarrow St(\overline{v}) \longrightarrow 1$$

we have $St(v)$ is connected. □

Remark 2.4.9. *In the above description of the irreducible elements of $P_{I_H \times I_G}(\Pi(F))$ above we always assumed that a_i's are strictly less than r for all i. This was enough for defining the irreducible elements but this will not give a description of all orbits. Indeed if we choose a bigger r then the number of $H(\mathcal{O}) \times I_G$-orbits in the closure of $\Pi_{\lambda,r}$ in $\Pi_{N,r}$ will go to infinity as r goes to infinity. We have the same phenomenon for $I_H \times I_G$-orbits.*

We let the affine extended Weyl group $\widetilde{W}_G$ of G act on the set $X_G \times S_{n,m}$ in the following way:

Definition 2.4.10. *Let $w = t^{\lambda_1}\tau_1$ be an element of $\widetilde{W}_G$ and (λ, s) in $X_G \times S_{n,m}$ then we define a left action:*

$$w \centerdot (\lambda, s) = (\lambda_1 + \tau_1(\lambda), \tau_1 s),$$

where $\tau_1 s$ is the composition $I_s \xrightarrow{s} \{1, \dots, n\} \xrightarrow{\tau_1} \{1, \dots, n\}$.

This action will be used in the next chapter. The $\widetilde{W}_G$-orbits on $X_G \times S_{n,m}$ are naturally in bijection with the subsets of n elements in $\{1, \dots, m\}$. Namely to each orbit one associates the subset I_s.

Chapter 3

The module structure of $\mathcal{S}^{I_H \times I_G}(\Pi(F))$

We use the same notation as in [§ 2.3, Chapter 2]. Remind that the space $\mathcal{S}^{I_H\times I_G}(\Pi(F))$ is naturally a module over the Iwahori-Hecke algebras $\mathcal{H}_{I_G}$ and $\mathcal{H}_{I_H}$ of G and H. The action is defined by convolution. The geometric analogue of the $(\mathcal{H}_{I_G}, \mathcal{H}_{I_H})$-bimodule $\mathcal{S}^{I_H\times I_G}(\Pi(F))$ is constructed in [§ 2.2.1, Chapter 2]. Recall that there are two Hecke functors

$$\overleftarrow{H}_G : P_{I_G}(\mathcal{F}l_G) \times D_{I_H\times I_G}(\Pi(F)) \longrightarrow D_{I_H\times I_G}(\Pi(F))$$

and

$$\overleftarrow{H}_H : P_{I_H}(\mathcal{F}l_H) \times D_{I_H\times I_G}(\Pi(F)) \longrightarrow D_{I_H\times I_G}(\Pi(F)).$$

Our main purpose is to understand the corresponding module structure on $D_{I_H\times I_G}(\Pi(F))$.

3.1 Description of the submodule generated by $\mathcal{I}^{w_0}$: (I)

We will assume $n = m$ in the entire section.

Denote by w_0 the longest element of W_G. Let $\{e_1, \dots, e_n\}$ be the standard basis of L_0, $\{u_1, \dots, u_m\}$ be the standard basis of U_0, and $\{u_1^*, \dots, u_m^*\}$ be the dual basis of U_0^* over $\mathbf{k}$. We identify L_0 and U_0^* via the isomorphism sending u_{n-i}^* to e_{i+1}, for all $i = 1, \dots, n-1$. Hence we can identify $L_0 \otimes U_0$ with $\mathrm{End}(L_0)$. Denote by $\mathcal{I}^{w_0}$ the IC-sheaf of the orbit $\Pi_{0,1}^{w_0}$. Let $\Pi_{\mathcal{I}^{w_0}}$ be the affine subspace of $\mathrm{End}(L_0)$ consisting of elements v in $\mathrm{End}(L_0)$ such that v preserves the standard flag $L_i = \mathrm{Vect}(e_1, \dots, e_i)$, for $i = 1, \dots, n$. The space $\Pi_{\mathcal{I}^{w_0}}$ is of dimension $\dim(B_G)$ and it is the closure of the orbit $\Pi_{0,1}^{w_0}$. Thus $\mathcal{I}^{w_0}$ is the constant perverse sheaf on $\Pi_{\mathcal{I}^{w_0}}$.

For any w in the finite Weyl group W_G, let us describe the complex $\overleftarrow{H}_G(L_w, \mathcal{I}^{w_0})$. The projection $p_G : \mathcal{F}l_G \to Gr_G$ is an affine fibration of typical fibre G/B_G. Further more the fibre of p_G over 1 is canonically isomorphic to G/B_G and $\overline{\mathcal{F}l_G^w}$ lies in this fibre. Thus it is the closure of $B_G w B_G/B_G$ in G/B_G.

For any point gI_G in $\overline{\mathcal{F}l_G^w}$ and any integer i between 1 and n, we put $L_i' = gL_i$. Let $\Pi_{\mathcal{I}^{w_0}} \tilde{\times} \overline{\mathcal{F}l_G^w}$ be the scheme classifying pairs (v, gB_G), where gB_G is in $\overline{\mathcal{F}l_G^w}$ and v is an endomorphism of L_0 such that $v(L_i) \subset L_i'$, for $i = 1, \dots, n$. We have the diagram

$$\Pi_{0,1} \xleftarrow{\pi} \Pi_{\mathcal{I}^{w_0}} \tilde{\times} \overline{\mathcal{F}l_G^w} \xrightarrow{pr} \overline{\mathcal{F}l_G^w}, \qquad (3.1.1)$$

where the projection pr is an affine fibration whose fibre is isomorphic to $\Pi_{\mathcal{I}^{w_0}}$ (which is an affine space of dimension $\dim(B_G)$). By definition of the Hecke functor $\overleftarrow{H}_G$ we have

$$\overleftarrow{H}_G(L_w,\mathcal{I}^{w_0}) = \pi_!(pr^*L_w)[\dim(B_G)].$$

The restriction of this complex to the subspace $\mathbf{GL}(L_0)$ of $\mathrm{End}(L_0)$ identifies with $\mathcal{I}^{ww_0}|_{\mathbf{GL}(L_0)}$. For any w in affine extended Weyl group $\widetilde{W}_G$, let us describe $\overleftarrow{H}_G(L_w,\mathcal{I}^{w_0})$. Let $w=t^{\lambda}\tau$, where τ is an element of the finite Weyl group W_G and λ is in X_G. Let N,r be two integers with $N+r\geq 0$ such that the following condition is verified: for any ν in $W_G.\lambda$ we have

$$\langle \nu,\check{\omega}_1\rangle < r \text{ and } \langle -\nu,\check{\omega}_1\rangle \leq N. \tag{3.1.2}$$

For any element gI_G of $\overline{\mathcal{F}l^w_G}$, we put $L'=gL$ and we equip L'/tL' with the flag $L'_i=gL_i$, for $i=1,\dots,n$. Let $\Pi_{\mathcal{I}^{w_0},r}\tilde{\times}\overline{\mathcal{F}l^w_G}$ be the scheme classifying pairs (v,gI_G), where gI_G is in $\overline{\mathcal{F}l^w_G}$ and v is a map from U^* to L'/t^rL such that the induced map

$$\overline{v}: U^*/tU^* \longrightarrow L'/tL' \tag{3.1.3}$$

sends $\mathrm{Vect}(u^*_n,\dots,u^*_{n-i})$ to L'_{i+1}, for $i=0,\dots,n-1$. Let π be the map induced by the first projection

$$\pi: \Pi_{\mathcal{I}^{w_0},r}\tilde{\times}\overline{\mathcal{F}l^w_G} \longrightarrow \Pi_{N,r}.$$

The second projection projection $pr:\Pi_{\mathcal{I}^{w_0},r}\tilde{\times}\overline{\mathcal{F}l^w_G}\longrightarrow \overline{\mathcal{F}l^w_G}$ is a locally trivial fibration with affine fibre. Let $\overline{\mathbb{Q}}_\ell\tilde{\boxtimes}L_w$ be the constant perverse sheaf associated with pr^*L_w, which is the shift of pr^*L_w by the relative dimension of pr. Then by definition

$$\overleftarrow{H}_G(L_w,\mathcal{I}^{w_0})\widetilde{\longrightarrow}\pi_!(\overline{\mathbb{Q}}_\ell\tilde{\boxtimes}L_w).$$

Lemma 3.1.4. *For w in $\widetilde{W}_G$, the restriction of the complex $\overleftarrow{H}_G(L_w,\mathcal{I}^{w_0})$ to the open subscheme $\Pi_{\lambda,r}$ of $\Pi_{N,r}$ is isomorphic to $\mathcal{I}^{ww_0}$.*

Proof. Consider the map

$$\pi: \Pi_{\mathcal{I}^{w_0},r}\tilde{\times}\overline{\mathcal{F}l^w_G} \longrightarrow \Pi_{N,r}$$

and choose a point v in $\Pi_{\lambda,r}$. Then the fibre of π over v is either empty or a single point. More generally for any λ in X_G, write θ for the image of λ in $\pi_1(G)$. If v is a point in $\Pi_{N,r}$ and Gr^θ_G is the connected component containing $U_{v,r}$ then the fibre of π over v is either empty or a single point.

Let V the open subscheme of $\Pi_{\mathcal{I}^{w_0},r}\tilde{\times}\overline{\mathcal{F}l^w_G}$ consisting of pairs (v,gI_G) such that gI_G is in $\mathcal{F}l^w_G$ and that the map $\overline{v}$ defined in (3.1.3) is surjective. Then $\pi(V)$ is contained in $\Pi^{ww_0}_{N,r}$. It follows that the image of π is contained in $\overline{\Pi}^{ww_0}_{N,r}$. Thus the restriction of $\overleftarrow{H}_G(L_w,\mathcal{I}^{w_0})$ to $\overline{\Pi}^{ww_0}_{N,r}\cap\Pi_{\lambda,r}$ identifies with $\mathcal{I}^{ww_0}$. □

The inclusion $\Pi^{w_0}_{0,1}\hookrightarrow\Pi_{\mathcal{I}^{w_0}}$ being an affine map $\mathcal{I}^{w_0!}$ is a perverse sheaf.

Proposition 3.1.5. *For any w in $\widetilde{W}_G$ then the complex $\overleftarrow{H}_G(L_{w!},\mathcal{I}^{w_0!})$ is canonically isomorphic to $\mathcal{I}^{ww_0!}$. Hence the sheaf $\mathcal{I}^{ww_0}$ appears in $\overleftarrow{H}_G(L_w,\mathcal{I}^{w_0})$ with multiplicity one.*

Proof. Let $\Pi^0_{\mathcal{I}^{w_0},r}\tilde{\times}\mathcal{F}l^w_G$ be the open subscheme of pairs (v, gI_G) in $\Pi_{\mathcal{I}^{w_0},r}\tilde{\times}\overline{\mathcal{F}l^w_G}$ such that gI_G is in $\mathcal{F}l^w_G$ and that $\overline{v}$ is an isomorphism. For any points (v, gI_G) in this subscheme the map $\overline{v}$ is an isomorphism between $\mathrm{Vect}(u^*_1, \dots, u^*_{n-i})$ and L'_{i+1}, for $i = 0, \dots, n-1$. Let

$$\pi^0 : \Pi^0_{\mathcal{I}^{w_0},r}\tilde{\times}\mathcal{F}l^w_G \longrightarrow \Pi_{N,r},$$

be the restriction of π to $\Pi^0_{\mathcal{I}^{w_0},r}\tilde{\times}\mathcal{F}l^w_G$. Thus by definition we have

$$\overleftarrow{H}_G(L_{w!}, \mathcal{I}^{w_0!}) \widetilde{\longrightarrow} \pi^0_!(\overline{\mathbb{Q}}_\ell \tilde{\boxtimes} L_w).$$

The image of π^0 is equal to $\Pi^{ww_0}_{N,r}$ and the map π^0 is an isomorphism onto its image. So one gets that $\overleftarrow{H}_G(L_{w!}, \mathcal{I}^{w_0!}) \widetilde{\longrightarrow} \mathcal{I}^{ww_0!}$. $\square$

Remark 3.1.6. *Consider the submodule Θ in $K(D_{I_H\times I_G}(\Pi(F)))$ generated by the objects $\mathcal{I}^{w!}$ for w running through $\widetilde{W}_G$. We see that it is a free module of rank one over the algebra $K(P_{I_G}(\mathcal{F}l_G))$ generated by $\mathcal{I}^{w_0!}$. Actually, $\Theta \otimes \bar{\mathbb{Q}}_\ell$ is strictly smaller than $K(D_{I_H\times I_G}(\Pi(F))) \otimes \bar{\mathbb{Q}}_\ell$. There is a finite filtration of $K(P_{I_H\times I_G}(\Pi(F))) \otimes \bar{\mathbb{Q}}_\ell$ by submodules over both Iwahori-Hecke algebras for G and H studied in [MVW87] such that this is the first term of this filtration.*

Definition 3.1.7. *For λ in X_G and τ in W_G, let $w \longrightarrow \overline{w}$ be the map $\widetilde{W}_G \to \widetilde{W}_G$ defined by*

$$\overline{t^\lambda\tau} \longrightarrow t^{\tau^{-1}(\lambda)}\tau^{-1}$$

This is an anti-automorphism of $\widetilde{W}_G$. Note that $\overline{w_0} = w_0$.

There is another anti-involution ι on $\widetilde{W}_G$ sending an element $t^\lambda\tau$ to $t^{-\lambda}\tau$. The anti-involution ι is linked to the one in Definition 3.1.7 by the formula $\overline{w} = \iota(w^{-1})$.

Proposition 3.1.8. *For w in $\widetilde{W}_H$, the complex $\overleftarrow{H}_H(L_{w!}, \mathcal{I}^{w_0!})$ is canonically isomorphic to $\mathcal{I}^{\overline{ww_0}!}$. Hence the sheaf $\mathcal{I}^{\overline{ww_0}}$ occurs in $\overleftarrow{H}_H(L_w, \mathcal{I}^{w_0})$ with multiplicity one.*

Proof. Let $w = t^\lambda\tau$, where τ is in the finite Weyl group W_H and $\lambda = (a_1, \dots, a_n)$ is a cocharacter in X_H. Then $\tau(\lambda) = (a_{\tau^{-1}(1)}, \dots, a_{\tau^{-1}(n)})$. Since there is a symmetry between L_0 and U_0, by Proposition $refLw!Iw0$ the have an isomorphism between $\overleftarrow{H}_H(L_{w!}, \mathcal{I}^{w_0!})$ and $\mathcal{I}^{w'}!$ for some w' in $\widetilde{W}_H$. Let w be an element in $\widetilde{W}_G$. The $I_H \times I_G$-orbit $\Pi^w_{N,r}$ contains the point $v : L^* \longrightarrow gU/t^rU$ which induces an isomorphism $\overline{v} : L^*/tL^* \longrightarrow gU/t(gU)$ and preserves the corresponding flags on both sides. If $(e^*_i)_{1\le i\le n}$ denotes the dual basis of $(e_i)_{1\le i\le n}$ then $\overline{v}$ sends each vector e^*_i to $t^{a_{\tau w_0(i)}}u_{\tau w_0(i)}$. The transpose of the map v is $u^*_{\tau(i)} \longrightarrow t^{a_{\tau(i)}}e_{w_0(i)}$, that is

$$u^*_j \longrightarrow t^{a_j}e_{w_0\tau^{-1}(j)}.$$

Hence the desired w' is $t^{w_0\tau^{-1}(\lambda)}(w_0\tau^{-1}) = \overline{ww_0}$. $\square$

The following example shows that $\overleftarrow{H}_G(L_w, \mathcal{I}^{w_0})$ is not always irreducible.

Lemma 3.1.9. *Let $1 \le i < n$. Let w be the transposition $(i, i+1)$ in W_G, $\lambda = (0, \dots, 0, 1, 0, \dots, 0)$, where 1 appears on the i^{th} position, and $w' = t^\lambda w_0$. Then*

$$\overleftarrow{H}_G(L_w, \mathcal{I}^{w_0}) \widetilde{\longrightarrow} \mathcal{I}^{ww_0} \oplus \mathcal{I}^{w'}.$$

Proof. The variety $\overline{\mathcal{F}l^w_G}$ classifies lattices L_0' endowed with a complete flag of lattices $L_{-1}' \subset L_0' \subset L_1' \subset \dots$ such that $L'_{i+n} = t^{-1}L_i$ for all i, and $L_j' = L_j$ unless $j = i \bmod n$, where L_j is the standard flag on $L(F)$. So, $\overline{\mathcal{F}l^w_G}$ identifies with the projective space of lines in L_{i+1}/L_{i-1}. Let $\{W_j\}_{j=1,\dots,m}$ be the flag on U_0^* preserved by B_H. Let Y_i be the closed subscheme of $\Pi_{0,1}$ given by $v(W_j) \subset L_j$ for $j \neq i$. Note that Y_i is an affine space. Define a closed subscheme Y_i' of Y_i consisting of elements v of Y_i such that $v(L_i) \subset L_{i-1}$. Then Y_i' is also an affine space.

Let $\Pi_{\mathcal{I}}\tilde{\times}\overline{\mathcal{F}l^w_G}$ be the scheme classifying pairs (v, gI_G), where gI_G is in $\overline{\mathcal{F}l^w_G}$ and $v : U_0^* \to L_0$ such that $v(W_j) \subset gL_j$, for all $1 \leq j \leq n$. We have the diagram

$$Y_i \xleftarrow{\pi} \Pi_{\mathcal{I}}\tilde{\times}\overline{\mathcal{F}l^w_G} \xrightarrow{pr} \overline{\mathcal{F}l^w_G}.$$

By definition of the Hecke operators one has

$$\overleftarrow{H}_G(L_w, \mathcal{I}^{w_0}) = \pi_!(\overline{\mathbb{Q}}_\ell \tilde{\boxtimes} L_w).$$

For a point v in $Y_i \backslash Y_i'$ the fibre of the map π over v is reduced to a point and the map π is an isomorphism over $Y_i \backslash Y_i'$. The restriction of $\overleftarrow{H}_G(L_w, \mathcal{I}^{w_0})$ to $Y_i\backslash Y_i'$ is isomorphic to $\mathrm{IC}(Y_i) = \overline{\mathbb{Q}}_\ell[\dim(Y_i)]$. On the other hand, the space Y_i identifies with $\Pi^{ww_0}_{N,r}$. The fibre of π over a point v of Y_i' is isomorphic to $\mathbb{P}^1$. Since Y_i' is an affine space of codimension 2 in Y_i. This proves the lemma. □

3.2 Filtration on $P_{I_H\times I_G}(\Pi(F))$ and grading on $P_{I_G}(\mathcal{F}l_G)$

Assume that $n \leq m$ in this section. The connected components of Gr_G and $\mathcal{F}l_G$ are indexed by $\pi_1(G)$ see [§ 1.4.9, Chapter 1]. This yields a grading $\oplus_{\theta\in\pi_1(G)} P_{I_G}(\mathcal{F}l^\theta_G) \xrightarrow{\sim} P_{I_G}(\mathcal{F}l_G)$. Denote by $\check{\omega}_n$ the character by which the group G acts on $\det(L_0)$, i.e. $\check{\omega}_n = (1, \dots, 1)$. We may identify $\pi_1(G) \xrightarrow{\sim} \mathbb{Z}$ via the map $\theta \mapsto \langle\theta, \check{\omega}_n\rangle$. This grading is compatible with the convolution product on $P_{I_G}(\mathcal{F}l_G)$. There is also a graduation $\mathcal{H}^k_{I_G}$, $k \in \mathbb{Z}$ of the Iwahori-Hecke algebra $\mathcal{H}_{I_G}$. Besides the isomorphism (1.5.5) becomes a graded isomorphism. More precisely, for any integer θ, the graded piece of index θ on the right hand side is isomorphic to the graded piece of index $\langle\theta, \check{\omega}_n\rangle = k$ on the left hand side. Therefore, the two gradings are linked by the geometric condition $\langle\theta, \check{\omega}_n\rangle = k$.

There exists a filtration on the category $P_{I_H\times I_G}(\Pi(F))$ indexed by $\mathbb{Z}$. Indeed, for an integer a in $\mathbb{Z}$, let Filt^a be the full subcategory in $P_{I_H\times I_G}(\Pi(F))$ defined as the Serre subcategory generated by the objects $\mathcal{I}^w$, where $w = t^\lambda\tau$ are elements of $X_G \times S_{n,m}$ satisfying $\langle\lambda, \check{\omega}_n\rangle \geq a$. We are going to show that this filtration is compatible with the action of $P_{I_G}(\mathcal{F}l_G)$.

Let $w = t^\lambda\tau$ and $u = t^\mu\nu$ be two elements in $X_G \times S_{n,m}$. The condition that $\Pi^u_{N,r}$ lies in the closure of $\Pi^w_{N,r}$ implies that $\langle\mu, \check{\omega}_n\rangle \geq \langle\lambda, \check{\omega}_n\rangle$. Indeed, for any point v of $\Pi_{N,r}$, the dimension of $U_{v,r}/t^rL$ can only decrease under specialization. For the orbit $\Pi^u_{N,r}$ lying in the closure of $\Pi^w_{N,r}$ the number $\langle\mu, \check{\omega}_n\rangle$ can be arbitrary large. This number depends on r is not uniformly bounded.

Lemma 3.2.1. *Let $w_1 = t^{\lambda_1}\tau_1$, and $w_2 = t^{\lambda_2}\tau_2$, be two elements in $X_G \times S_{n,m}$. For $i = 1, 2$ choose two integers N_i and r_i such that the following conditions are satisfied : for any*

$$\nu \in W_G\lambda_i,\ \langle\nu, \check{\omega}_1\rangle \leq r_1,\ \langle\nu, \check{\omega}_1\rangle < r_2, \text{ and } \langle-\nu, \check{\omega}_1\rangle \leq N_i.$$

Let v be an element in $\Pi^{w_2}_{N_2,r_2}\subset\Pi_{\lambda_2,r_2}$ and gI_G be an element in $\mathcal{F}l^{w_1}_G$. For $i=1,2$ let μ_i be a dominant cocharacter lying in $W_G\lambda_i$. Then there exists a cocharacter μ smaller than or equal to $\mu_1+\mu_2$ such that gv belongs to Π_{μ,r_1+r_2}.

Proof. The lattice $U_{v,r_2}=v(U^*)+t^{r_2}L$ lies in $O^{\lambda_2}\subset Gr^{\mu_2}_G$. Thus $g(U_{v,r_2})$ lies in gO^{λ_2}. Since $gG(\mathcal{O})\in O^{\lambda_1}\subset Gr^{\mu_1}_G$, we have

$$I_Gt^{\lambda_1}G(\mathcal{O})t^{\lambda_2}G(\mathcal{O})/G(\mathcal{O})\subset\overline{Gr}^{\mu_1+\mu_2}_G$$

and this implies the assertion. □

Proposition 3.2.2. *Let $w_1=t^{\lambda_1}\tau_1$, and $w_2=t^{\lambda_2}\tau_2$, be two elements in $X_G\times S_{n,m}$. For $i=1,2$ choose two integers N_i and r_i such that the following conditions are satisfied : for any*

$$\nu\in W_G\lambda_i,\ \langle\nu,\check\omega_1\rangle\le r_1,\ \langle\nu,\check\omega_1\rangle<r_2,\ \text{and}\ \langle-\nu,\check\omega_1\rangle\le N_i.$$

Then $\overleftarrow{H}_G(L_{w_1},\mathcal{I}^{w_2})$ lies in Filt^d with $d=\langle\lambda_1+\lambda_2,\check\omega_n\rangle$.

Proof. The sheaf $\mathcal{I}^{w_2}$ is the IC-sheaf of the orbit $\Pi^{w_2}_{N,r}$ which is a subspace of Π_{λ_2,r_2}. In the notation of [2.2.2, Chapter 2] we have $\overline{\mathcal{F}l^{w_1}_G}\subset{}_{r_1,N_1}\mathcal{F}l_G$. Choose $r\ge r_1+r_2$ and $s\ge N_2+r_2$. The space $\Pi_{N_2,r_2}\tilde\times\overline{\mathcal{F}l^{w_1}_G}$ is the scheme classifying pairs (v,gI_G), where gI_G is in $\overline{\mathcal{F}l^{w_1}_G}$ and v is in $t^{-N_2}g\Pi/t^r\Pi$. We have the following diagram

$$\Pi_{N_1+N_2,r}\xleftarrow{\pi}\Pi_{N_2,r_2}\tilde\times\overline{\mathcal{F}l^{w_1}_G}\xrightarrow{act_{q,s}}K_s\backslash(\Pi_{N_2,r_2}),$$

where π is the projection sending (v,gI_G) to v. Let $\mathcal{I}^{w_2}\tilde\boxtimes L_{w_1}$ be the twisted exterior product of $\mathcal{I}^{w_2}$ and L_{w_1} over $\Pi_{N_2,r_2}\tilde\times\overline{\mathcal{F}l^{w_1}_G}$ which is normalized to be perverse. Then by definition

$$\overleftarrow{H}_G(L_{w_1},\mathcal{I}^{w_2})\widetilde{\longrightarrow}\pi_!(\mathcal{I}^{w_2}\tilde\boxtimes L_{w_1}).$$

In our case $\mathcal{I}^{w_2}\tilde\boxtimes L_{w_1}$ is the IC-sheaf of $act^{-1}_{q,s}(\overline{\Pi}^{w_2}_{N_2,r_2})$. For a point v in $\Pi_{N_1+N_2,r}$, let μ be in X_G such that $U_{v,r}$ lies in O^μ. The part of the fibre of the map π over v that contributes to $\pi_!(\mathcal{I}^{w_2}\tilde\boxtimes L_{w_1})$ is

$$\{gI_G\in\overline{\mathcal{F}l^{w_1}_G}|g^{-1}v\in\overline{\Pi}^{w_2}_{N_2,r_2}\}.$$

The latter scheme is empty unless $\langle\mu,\check\omega_n\rangle\ge\langle\lambda_1+\lambda_2,\check\omega_n\rangle$. It follows that $\overleftarrow{H}_G(L_{w_1},\mathcal{I}^{w_2})$ lies in Filt^d with $d=\langle\lambda_1+\lambda_2,\check\omega_n\rangle$. □

Proposition 3.2.3. *Assume $n\le m$. The filtration Filt^d on $P_{I_H\times I_G}(\Pi(F))$ is compatible with the grading on $P_{I_G}(\mathcal{F}l_G)$ defined by the connected components. Namely set $w_1=t^{\lambda_1}\tau$ with τ in $S_{n,m}$, λ_1 in X_G, and let $m_1=\langle\lambda_1,\check\omega_n\rangle$. Then $\overleftarrow{H}_G(L_{w_1},.)$ sends an irreducible object of Filt^d to a direct sum of shifted objects of Filt^{d+m_1}.*

Proof. We use the notation of Lemma 3.2.1. For gI_G in $\overline{\mathcal{F}l^{w_1}_G}$, let $L'=gL$ and equip L' with the flag $L'_i=gL_i$ for $i=1,\dots,n$. Let v be the map from U^* to $t^{-N_2}L'/t^rL$ such that its composition with

$$t^{-N_2}L'/t^rL\longrightarrow t^{-N_2}L'/t^{r_2}L'$$

lies in the closure of the orbit $(U_{N_2r_2}\otimes L^{'})^{w_2}$. The latter scheme is the corresponding orbit on $U_{N_2,r_2}\otimes L^{'}$. The relative dimension formula gives us

$$\dim(L,L^{'})+\dim(L^{'},v(U^*)+t^{r_2}L^{'})=\dim(L,v(U^*)+t^{r_2}L^{'}).$$

Moreover, we have

$$\dim(L^{'},v(U^*+t^{r_2}L^{'}))\geq\langle\lambda_2,\check{\omega}_n\rangle$$

and

$$\dim(L,L^{'})=\langle\lambda_1,\check{\omega}_n\rangle.$$

This leads to $\dim(L,v(U^*+t^{r_2}L^{'}))\geq\langle\lambda_1+\lambda_2,\check{\omega}_n\rangle$. On the other hand we have $t^rL\subset t^{r_2}L^{'}$ so $\dim(L,v(U^*)+t^rL)$ can not be strictly smaller than $\dim(L,v(U^*)+t^{r_2}L^{'})$. □

As a consequence of this proposition $P_{I_H\times I_G}(\Pi(F))$ is a filtered module over $\mathcal{H}_{I_G}$, so that $\oplus_{d\in\mathbb{Z}}\mathrm{Filt}^d/\mathrm{Filt}^{d+1}$ is a left $\mathcal{H}_{I_G}$-module.

3.3 Description of the submodule generated by $\mathcal{I}^{w_0}$: (II)

Now we will go back the case $n=m$ and the description of the submodule $\mathcal{I}^{w_0!}$. The Proposition 3.2.3 in the especial case of $\mathcal{I}^{w_0}$ yields the following:

Corollary 3.3.1. *Let $w=t^\lambda\tau$, where λ in X_G and τ is in W_G. Then if $k=\langle\lambda,\check{\omega}_n\rangle$, there exists K in Filt^{k+1} such that $\overleftarrow{H}_G(L_w,\mathcal{I}^{w_0})$ is isomorphic to $\mathcal{I}^{ww_0}\oplus K$.*

Proof. According to Proposition 3.2.3, we know that $\overleftarrow{H}_G(L_w,.)$ sends an irreducible object of Filt^d to a direct sum of shifted objects of Filt^{d+k}. The constant perverse sheaf $\mathcal{I}^{w_0}$ is in $\mathrm{Filt}^0/\mathrm{Filt}^1$ and additionally by Proposition 3.1.4 we know that $\mathcal{I}^{ww_0}$ occurs with multiplicity at least one in the decomposition of $\overleftarrow{H}_G(L_w,\mathcal{I}^{w_0})$. The assertion follows. □

When $n=m$, the space $\oplus_{d\in\mathbb{Z}}\mathrm{Filt}^d/\mathrm{Filt}^{d+1}$ is a free module of rank one over $\mathcal{H}_{I_G}$ generated by $\mathcal{I}^{w_0}$.

Proposition 3.3.2. *The homomorphism of $\mathcal{H}_{I_G}$-algebras $\mathcal{H}_{I_G}\longrightarrow\mathcal{S}^{I_H\times I_G}(\Pi(F))$ sending $\mathcal{S}$ to $\overleftarrow{H}_G(\mathcal{S},\mathcal{I}^{w_0!})$ is injective. The submodule generated by $\mathcal{I}^{w_0!}$ is a free module of rank one over each one of the Iwahori-Hecke algebra $\mathcal{H}_{I_G}$ and $\mathcal{H}_{I_H}$.*

Remark 3.3.3. *The module generated by $\mathcal{I}^{w_0!}$ is strictly smaller than $\mathcal{S}^{I_H\times I_G}(\Pi(F))$. Also remark that $\mathcal{I}^{w_0!}$ sits Filt^0 and that $\mathcal{I}^{w_0!}$ and $\mathcal{I}^{w_0}$ have the same image in the first graded piece $\mathrm{Filt}^0/\mathrm{Filt}^1$.*

We may stratify $\Pi_{N,r}$ in a slightly different way. Let θ be any element of $\pi_1(G)$ and let λ be a lift of θ in X_G satisfying condition (2.3.4). We define a locally closed subscheme $\Pi^\theta_{N,r}$ of $\Pi_{N,r}$ as follows:

$$v:U^*\rightarrow t^{-N}L/t^rL\ \textit{such that}\ \dim(U_{v,r}/t^rL)=\dim(t^\lambda L/t^rL).$$

This definition is in fact independent of the lift λ.

For a given $w=t^\lambda\tau$ in $\widetilde{W}_G$, let θ be the image of λ in $\pi_1(G)$. Let $\tilde{\mathcal{I}^w}$ be the extension by zero of $\mathcal{I}^w|_{\Pi^\theta_{N,r}}$ on $\Pi_{N,r}$. Then we have the following result:

Proposition 3.3.4. *For any w in $\widetilde{W}_G$, we have two canonical isomorphism*

$$\overleftarrow{H}_G(L_w, \mathcal{I}^{w_0!}) \widetilde{\longrightarrow} \tilde{\mathcal{I}}^{ww_0} \qquad \overleftarrow{H}_H(L_w, \mathcal{I}^{w_0!}) \widetilde{\longrightarrow} \tilde{\mathcal{I}}^{\overline{ww_0}},$$

where the anti-involution $\overline{w}$ is defined in Definition 3.1.7.

Proof. We show the assertion for $\overleftarrow{H}_G$, the case of $\overleftarrow{H}_H$ may be proved similarly. As in Proposition 3.1.5, consider the open subscheme $\Pi^0_{\mathcal{I}^{w_0},r} \tilde{\times} \mathcal{F}l^w_G$ inside $\Pi_{\mathcal{I}^{w_0},r} \tilde{\times} \overline{\mathcal{F}l^w_G}$ given by the additional condition that the map

$$\overline{v} : U^*/tU^* \to L^{'}/tL^{'}$$

is an isomorphism. Thus the restriction

$$\pi^0 : \Pi^0_{\mathcal{I}^{w_0},r} \tilde{\times} \mathcal{F}l^w_G \longrightarrow \Pi_{N,r}$$

of the map π is locally a closed immersion. Therefore by definition

$$\overleftarrow{H}_G(L_w, \mathcal{I}^{w_0!}) \widetilde{\longrightarrow} \pi^0_!(\overline{\mathbb{Q}}_\ell \tilde{\boxtimes} L_w) \widetilde{\longrightarrow} \tilde{\mathcal{I}}^{ww_0}.$$

□

The map from $G(F)$ to $G(F)$ sending any element to its inverse induces an equivalence of categories

$$\star^\sharp : P_{I_G}(\mathcal{F}l_G) \widetilde{\longrightarrow} P_{I_G}(\mathcal{F}l_G).$$

The similar equivalence of categories holds for $P_{I_H}(\mathcal{F}l_H)$. Hence for w in the affine extended Weyl group, we have canonical isomorphisms

$$\star^\sharp(L_w) \widetilde{\longrightarrow} L_{w^{-1}}, \quad \star^\sharp(L_{w!}) \widetilde{\longrightarrow} L_{w^{-1}!}, \quad \star^\sharp(L_{w*}) \widetilde{\longrightarrow} L_{w^{-1}*}.$$

At the level of Iwahori-Hecke algebras $\star^\sharp : \mathcal{H}_{I_G} \to \mathcal{H}_{I_G}$ is an anti-isomorphism of algebras.

Definition 3.3.5. *Assume that $n \leq m$. For any $\mathcal{T}$ in $P_{I_H}(\mathcal{F}l_H)$ and $\mathcal{K}$ in $P_{I_H \times I_G}(\Pi(F))$, we define a right action functor $\overrightarrow{H}_H(\mathcal{T}, \mathcal{K})$ of $P_{I_H}(\mathcal{F}l_H)$ acting on $P_{I_H \times I_G}(\Pi(F))$:*

$$\overrightarrow{H}_H(\mathcal{T}, \mathcal{K}) = \overleftarrow{H}_H(\star^\sharp(\mathcal{T}), \mathcal{K}).$$

Theorem 3.3.6. *Assume $n = m$. There exists an equivalence of categories*

$$\begin{aligned} \sigma : P_{I_G}(\mathcal{F}l_G) &\widetilde{\longrightarrow} P_{I_H}(\mathcal{F}l_H) \\ L_w &\longrightarrow L_{w_0 \overline{w} w_0}, \end{aligned} \tag{3.3.7}$$

Additionally it verifies the following properties: for any w and $w^{'}$ in $\widetilde{W}_G$ we have

$$\overleftarrow{H}_G(L_w, \mathcal{I}^{w_0!}) \widetilde{\longrightarrow} \overleftarrow{H}_H(\sigma(L_w), \mathcal{I}^{w_0!})$$

and

$$\sigma(L_w \star L_{w'}) = \sigma(L_{w'}) \star \sigma(L_w),$$

where $\star$ is the convolution product. Moreover, at the level of functions the functor σ defines an anti-isomorphism of algebras between $\mathcal{H}_{I_G}$ and $\mathcal{H}_{I_H}$.

Proof. The anti-automorphism defined over $\widetilde{W}_G$ in Definition 3.1.7 sending any w to $\overline{w}$ may be extended to an anti-automorphism of the group $G(F)$ itself. It suffices to take the morphism sending any g an element of $G(F)$ to its transpose ${}^t g$. Denote by σ the anti-involution defined over $G(F)$ sending g to $w_0{}^t g w_0$. This anti-involution preserves the Iwahori subgroup I_G and induces an equivalence of categories (still denoted by σ):

$$\sigma : P_{I_G}(\mathcal{F}l_G) \xrightarrow{\sim} P_{I_G}(\mathcal{F}l_G).$$

According to Proposition 3.1.5 and 3.1.8 we have the following canonical isomorphisms

$$\overleftarrow{H}_G(L_{w!}, \mathcal{I}^{w_0!}) \xrightarrow{\sim} \mathcal{I}^{ww_0!} \qquad \overleftarrow{H}_H(L_{w!}, \mathcal{I}^{w_0!}) \xrightarrow{\sim} \mathcal{I}^{\overline{ww_0}!}.$$

By Proposition 3.3.4, we have that

$$\overleftarrow{H}_G(L_w, \mathcal{I}^{w_0!}) \xrightarrow{\sim} \tilde{\mathcal{I}}^{ww_0} \quad \overleftarrow{H}_H(L_w, \mathcal{I}^{w_0!}) \xrightarrow{\sim} \tilde{\mathcal{I}}^{\overline{ww_0}}.$$

Combining these two results, our σ verifies the desired properties.

□

The two anti-isomorphisms σ and $\star^\sharp$ defined above commute and their composition is an algebra isomorphism. We will denote this composition by $\tilde{\sigma}$, i.e. for any g in $G(F)$ $\tilde{\sigma}(g) = w_0{}^t g^{-1} w_0$.

3.4 The module structure of $\mathcal{S}^{I_H \times I_G}(\Pi(F))$ for $n \leq m$

We assume $n \leq m$ for the whole section. Let us recall that the irreducible objects of $P_{I_H \times I_G}(\Pi(F))$ are indexed by the set $X_G \times S_{n,m}$, where elements of $S_{n,m}$ consist of pairs (s, I_s), I_s being a subset of $\{1, \dots, m\}$ with n elements and $s : I_s \to \{1, \dots, n\}$ a bijection [§ 2.4.2, Chapter 2].

We will consider the affine extended Weyl group $\widetilde{W}_G$ as a subset of $X_G \times S_{n,m}$. More precisely to a given $w = t^\lambda \tau$ we associate the element (λ, τ) in $X_G \times S_{n,m}$ with $I_\tau = \{1, \dots, n\}$. Let $I_{w_0} = \{1, \dots, n\}$ be a subset of $\{1, \dots, m\}$ and $w_0 : I_{w_0} \to \{1, \dots, n\}$ be the longest element of the Weyl group W_G, i.e. $w_0(1) = n, \dots, w_0(n) = 1$. By the above convention the element w_0 becomes the element $(0, w_0)$ in $X_G \times S_{n,m}$.

For any strictly decreasing map ν from $\{1, \dots, n\}$ to $\{1, \dots, m\}$, and by I_μ the image of ν and denote by $\mu : I_\mu \to \{1, \dots, n\}$ the inverse of ν. Thus μ can be viewed as an element of $X_G \times S_{n,m}$ by assuming that the corresponding term on X_G vanishes. Let $\overline{\Pi}^\mu_{0,1}$ be the closure of $I_H \times I_G$-orbit $\Pi^\mu_{0,1}$ in $\Pi_{N,r}$, $\overline{\Pi}^\mu_{0,1}$ is an affine space. Denote by $\mathcal{I}^\mu$ the IC-sheaf of $\Pi^\mu_{0,1}$, it is the constant perverse sheaf on its support.

For $i = 1, \dots, n$ let e_i (resp. e_i^*) be a basis of L_0 (resp. of the dual space L_0^*) and for $j = 1, \dots, m$ let u_j (resp. u_j^*) be a basis of U_0 (resp. of the dual space U_0^*). Denote by $U_1 \subset U_2 \subset \cdots \subset U_m = U_0$ the standard flag on U/tU. We consider $\Pi_{0,1}$ the space of maps $v : L^* \to U/tU$ such that the domain and the range are both equipped with a flag preserved by v. Thus $\overline{\Pi}^\mu_{0,1}$ is the space of maps $v : L^* \to U/tU$ such that $v(e_i^*)$ lies in $U_{\nu(i)}$ for all $i = 1, \dots, n$. In other terms the map v sends $\mathrm{Vect}(e_n^*, \dots, e_{n-i}^*)$ to $U_{\nu(n-i)}$ for all $i = 0, \dots, n-1$. An element v lies in $\Pi^\mu_{0,1}$ if additionally the map sending $\mathrm{Vect}(e_n^*, \dots, e_{n-i}^*)$ to $U_{\nu(n-i)}/U_{\nu(n-i)-1}$ is non-zero for all $i = 1, \dots, n$. We may also consider the element v in $\Pi_{0,1}$ as a map from U^* to L/tL, so v lies in $\overline{\Pi}^\mu_{0,1}$ if and only if v sends $\mathrm{Vect}(u_m^*, \dots, u_{1+\nu(j)}^*)$ to L_{j-1} for all $j = 1, \dots, n$. Moreover, the map v lies in $\Pi^\mu_{0,1}$ if in addition $v(u_{\nu(j)}^*) \notin L_{j-1}$ for all $j = 1, \dots, n$.

3.4.1 The action of $\mathcal{H}_{I_G}$ on $\mathcal{S}^{I_H\times I_G}(\Pi(F))$

Let $w=t^\lambda\tau$ be an element of $\widetilde{W}_G$. Choose two integers N and r with $N+r>0$ such that for any ν in $W_G.\lambda$ the following condition is satisfied (condition (3.1.2)):

$$\langle\nu,\check{\omega}_n\rangle<r \quad \text{and} \quad \langle-\nu,\check{\omega}_n\rangle\le N.$$

For a point gI_G in $\overline{\mathcal{F}l_G^w}$, we set $L'=gL$ and equip L'/tL' with the complete flag $L_i'=gL_i$ for $i=1,\dots,n$. Here $(L_1\subset\dots\subset L_n=L/tL)$ is the complete flag on L/tL preserved by B_G.
Let $\overline{\Pi}_r^\mu\tilde{\times}\overline{\mathcal{F}l_G^w}$ be the scheme classifying pairs (v,gI_G), where gI_G is in $\overline{\mathcal{F}l_G^w}$, and $v:U^*\to L'/t^rL$ such that the induced map

$$\overline{v}:U^*/tU^*\longrightarrow L'/tL'$$

sends $\mathrm{Vect}(u_m^*,\dots,u_{\nu(j)+1}^*)$ to L_{j-1}' for all $j=1,\dots,n$. We have a proper map

$$\pi:\overline{\Pi}_r^\mu\tilde{\times}\overline{\mathcal{F}l_G^w}\longrightarrow\Pi_{N,r}$$

sending any element (v,gI_G) to v. By definition of $\overleftarrow{H}_G$,

$$\overleftarrow{H}_G(L_w,\mathcal{I}^\mu)\,\widetilde{\longrightarrow}\,\pi_!(\overline{\mathbb{Q}}_\ell\tilde{\boxtimes}L_w),$$

where $\overline{\mathbb{Q}}_\ell\tilde{\boxtimes}L_w$ is normalized to be perverse.

Proposition 3.4.1. *Let w be an element of $\widetilde{W}_G$. Then $\overleftarrow{H}_G(L_{w!},\mathcal{I}^{\mu!})$ is canonically isomorphic to $\mathcal{I}^{w\centerdot\mu}$, where $w\centerdot\mu$ has been defined in Definition* 2.4.10 *of Chapter* 2.

Proof. Let $w=t^\lambda\tau$ with $\lambda=(a_1,\dots,a_n)$ in X_G and τ in W_G. Let $\Pi_r^\mu\tilde{\times}\mathcal{F}l_G^w$ be the open subscheme of $\overline{\Pi}_r^\mu\tilde{\times}\overline{\mathcal{F}l_G^w}$ given by the additional condition that gI_G lies in $\mathcal{F}l_G^w$ and the map $\overline{v}:\mathrm{Vect}(u_m^*,\dots,u_{\nu(j)}^*)\to L_j'$ is surjective for $j=1,\dots,n$. Denote by π^0 the restriction of π to this open subscheme. The image of π^0 consist of the $I_H\times I_G$-orbit on $\Pi_{N,r}$ through v such that $v(u_{\nu(j}^*))=t^{a_{\tau(j)}}e_{\tau(j)}$ for all $j=1,\dots,n$ and $v(u_k^*)=0$ for $k\in I_\mu$. Therefore the image of the map π^0 is $\Pi_{N,r}^{w\centerdot\mu}$ and π^0 is an isomorphism onto its image. Thus

$$\overleftarrow{H}_G(L_{w!},\mathcal{I}^{\mu!})\,\widetilde{\longrightarrow}\,\mathcal{I}^{(w\centerdot\mu)!}.$$

□

Definition 3.4.2. *Let $\Theta\subset K(D_{I_H\times I_G}(\Pi(F)))$ be the submodule over $K(P_{I_G}(\mathcal{F}l_G))$ generated by the elements $\mathcal{I}^{\mu!}$, where I_μ runs through all possible subsets of n elements in $\{1,\dots,m\}$.*

It is understood that for each such subset I_μ there is a unique strictly decreasing map $\mu:I_\mu\to\{1,\dots,n\}$, so we may view μ as the element $(0,\mu)$ in $X_G\times S_{n,m}$ as above. Our calculation yields the following generalization of Remark 3.1.6.

Proposition 3.4.3. *The module Θ is free of rank C_m^n over $K(P_{I_G}(\mathcal{F}l_G))$. The elements $\mathcal{I}^{\mu!}$, where I_μ runs through all possible subsets of n elements in $\{1,\dots,m\}$, form a basis of this module over $K(P_{I_G}(\mathcal{F}l_G))$.*

The subspace $\Theta\otimes\bar{\mathbb{Q}}_\ell\subset K(D_{I_H\times I_G}(\Pi(F)))\otimes\bar{\mathbb{Q}}_\ell$ is different from $K(D_{I_H\times I_G}(\Pi(F)))\otimes\bar{\mathbb{Q}}_\ell$. For example, each function from $\Theta\otimes\bar{\mathbb{Q}}_\ell$ vanishes at 0 in $\Pi(F)$. Actually, Θ is the first term of the filtration on $K(D_{I_H\times I_G}(\Pi(F)))\otimes\bar{\mathbb{Q}}_\ell$ introduced in [MVW87].

3.4.2 The action of $\mathcal{H}_{I_H}$ on Θ

Let τ be an element of the finite Weyl group W_H of H. Let us describe the complex $\overleftarrow{H}_H(L_\tau,\mathcal{I}^\mu)$. For any point hI_H in $\overline{\mathcal{F}l_H^\tau}$ we write $U_i^{'}=hU_i$ and we fix a complete flag $U_1^{'}\subset\cdots\subset U_m^{'}$ on $U^{'}/tU^{'}$. Let $\overline{\Pi}_1^\mu\tilde{\times}\overline{\mathcal{F}l_H^\tau}$ be the scheme classifying pairs (v,hI_H), where hI_H is in $\overline{\mathcal{F}l_H^\tau}$ and a v is a map from L^* to $U^{'}/tU'$ such that $v(e_i^*)\in U^{'}_{\nu(i)}$ for all $i=1,\dots,n$. Let

$$\pi:\overline{\Pi}_1^\mu\tilde{\times}\overline{\mathcal{F}l_H^\tau}\to\Pi_{0,1} \tag{3.4.4}$$

be the map sending (v,hI_H) to v. By definition we have

$$\overleftarrow{H}_H(L_\tau,\mathcal{I}^\mu)\widetilde{\longrightarrow}\pi_!(\overline{\mathbb{Q}}_\ell\tilde{\boxtimes}L_\tau).$$

Let $\Pi_1^\mu\tilde{\times}\mathcal{F}l_H^\tau$ be the open subscheme in $\overline{\Pi}_1^\mu\tilde{\times}\overline{\mathcal{F}l_H^\tau}$ consisting of pairs (v,hI_H) in $\Pi_1^\mu\tilde{\times}\mathcal{F}l_H^\tau$ such that $v(e_i^*)\notin U^{'}_{\nu(i)-1}$ for all $i=1,\dots,n$. If π^0 is the restriction of π to the open subscheme $\Pi_1^\mu\tilde{\times}\mathcal{F}l_H^\tau$, then we have

$$\overleftarrow{H}_H(L_{\tau!},\mathcal{I}^{\mu!})\widetilde{\longrightarrow}\pi_!^0(\overline{\mathbb{Q}}_\ell\tilde{\boxtimes}L_\tau).$$

For $1\le i<m$ we will denote by τ_i the simple reflection $(i,i+1)$ in W_H.

Proposition 3.4.5. *Let i be an integer such that $1\le i<m$*

1. *If $i\notin I_\mu$ then the complex $\overleftarrow{H}_H(L_{\tau_i},\mathcal{I}^\mu)$ is canonically isomorphic to $\mathcal{I}^\mu\otimes\mathrm{R}\Gamma(\mathbb{P}^1,\overline{\mathbb{Q}}_\ell)[1]$.*
2. *If $i\in I_\mu$ then*

$$\overleftarrow{H}_H(L_{\tau_i},\mathcal{I}^\mu)\widetilde{\longrightarrow}\begin{cases}\mathcal{I}^{\tau_i\circ\nu}\oplus\mathcal{I}^{\tau_{i-1}\circ\nu} & \text{if}\quad i>1 \ \textit{and}\ i-1\notin I_\mu\\ \mathcal{I}^{\tau_i\circ\nu}\oplus\mathrm{IC}(Y^{''}) & \textit{otherwise},\end{cases}$$

where $Y^{''}$ is a specific locally closed subscheme of $\Pi_{0,1}$ (whose construction will be given in the proof below).

Proof. The scheme $\overline{\mathcal{F}l_H^\tau}$ is the projective space of lines in U_{i+1}/U_{i-1}.

1. Consider the projection $p_H:\mathcal{F}l_H\to Gr_H$. Then $p_{H!}(L_{\tau_i})$ is canonically isomorphic to $\mathrm{R}\Gamma(\mathbb{P}^1,\overline{\mathbb{Q}}_\ell)[1]$. This implies that

$$\overleftarrow{H}_H(L_{\tau_i},\mathcal{I}^\mu)\widetilde{\longrightarrow}\mathcal{I}^\mu\otimes\mathrm{R}\Gamma(\mathbb{P}^1,\overline{\mathbb{Q}}_\ell)[1].$$

2. Let us describe the image of the map π in (3.4.4). It is contained in the closed subscheme $Y^{'}$ of $\Pi_{0,1}$ given by the following two conditions:
   a) For $j\neq\nu^{-1}(i)$, $v(e_j^*)\in U_{\nu(j)}$.
   b) For $j=\nu^{-1}(i)$, $v(e_j^*)\in U_{i+1}$.

   Let $Y^{''}$ be the closed subscheme in $Y^{'}$ consisting of elements v such that $v(e_j^*)$ belongs to U_{i-1} if $j=\nu^{-1}(i)$. Over a point of $Y^{''}$ the fibre of the map (3.4.4) is $\mathbb{P}^1$ whence over a point of $Y^{'}-Y^{''}$ the fibre of the map (3.4.4) is a point. Thus

$$\overleftarrow{H}_H(L_\tau,\mathcal{I}^\mu)\widetilde{\longrightarrow}\mathrm{IC}(Y^{'})\oplus\mathrm{IC}(Y^{''}).$$

   Now we need to identify $Y^{'}$ and $Y^{''}$.

If $i+1\notin I_\mu$, let $I_{\mu'}$ be the subset of $\{1,\ldots,m\}$ obtained from I_μ by throwing i away and adding $i+1$. In this case Y' is isomorphic to $\overline{\Pi}_{0,1}^{\mu'}$ so $\mathrm{IC}(Y')$ is canonically isomorphic to $\mathcal{I}^{\mu'}$. Now let $\nu':\{1,\ldots,n\}\longrightarrow\{1,\ldots,m\}$ be strictly decreasing with image $I_{\mu'}$. Considering ν' as an element of $S_{n,m}$, we get that $\Pi_{0,1}^{\nu'}=\Pi_{0,1}^{\mu'}$.

If $i+1\in I_\mu$, let $\nu':\{1,\ldots,n\}\longrightarrow\{1,\ldots,m\}$ be the map ν composed with the permutation τ_i. The image of ν' is the subset I_μ(but ν' is not strictly decreasing). Viewing ν' as an element of $S_{n,m}$ enables us to identify Y' with the closure of $I_H\times I_G$-orbit $\Pi_{0,1}^{\nu'}$.

Thus, in both cases $\nu'=\tau_i\circ\nu$ is the composition $\{1,\ldots,n\}\stackrel{\nu}{\longrightarrow}\{1,\ldots,m\}\stackrel{\tau_i}{\longrightarrow}\{1,\ldots,m\}$ and $\mathrm{IC}(Y')$ is isomorphic to $\mathcal{I}^{\tau_i\circ\nu}$.

If $i>1$ and $i-1\notin I_\mu$ then $\tau_{i-1}\circ\nu:\{1,\ldots,n\}\longrightarrow\{1,\ldots,m\}$ is strictly decreasing and Y'' is isomorphic to the closure of $\Pi_{0,1}^{\tau_{i-1}\circ\nu}$. So we get $\mathrm{IC}(Y'')=\mathcal{I}^{\tau_{i-1}\circ\nu}$. This proves the assertion.

□

Proposition 3.4.6. *Let i be an integer such that $1\le i<m$*

1. *If nor i neither $i+1$ is in I_μ then*
$$\overleftarrow{H}_H(L_{\tau_i},\mathcal{I}^{\mu!})\,\widetilde{\longrightarrow}\,\mathcal{I}^{\mu!}\otimes\mathrm{R}\Gamma(\mathbb{P}^1,\overline{\mathbb{Q}}_\ell)[1]$$
2. *If i is not in I_μ and $i+1$ is in I_μ then the composition $\tau_i\circ\nu$ is again strictly decreasing and there is a distinguished triangle*
$$\mathcal{I}^{\mu!}[-1]\longrightarrow\overleftarrow{H}_H(L_{\tau_i},\mathcal{I}^{\mu!})\longrightarrow\mathcal{I}^{(\tau_i\circ\nu)!}\stackrel{+1}{\longrightarrow}$$
3. *If i is in I_μ and $i+1$ is not an element of I_μ then there is a distinguished triangle*
$$\mathcal{I}^{(\tau_i\circ\nu)!}\longrightarrow\overleftarrow{H}_H(L_{\tau_i},\mathcal{I}^{\mu!})\longrightarrow\mathcal{I}^{\nu!}[1]\stackrel{+1}{\longrightarrow}.$$
4. *If both i and $i+1$ are in I_μ then there is a distinguished triangle*
$$\mathcal{I}^{(\tau_i\circ\nu)!}\longrightarrow\overleftarrow{H}_H(L_{\tau_i},\mathcal{I}^{\mu!})\longrightarrow\mathcal{I}^{\mu!}[1]\stackrel{+1}{\longrightarrow}.$$

Proof. 1. This is straightforward as in Proposition 3.4.5.

2. Let Y' be the locally closed subscheme of $\Pi_{0,1}$ given by the conditions :
   a) For $1\le j\le n$, $v(e_j^*)\in U_{\nu(j)}$.
   b) For $j\ne\nu^{-1}(i+1)$, $v(e_j^*)\notin U_{\nu(j)-1}$.
   c) For $j=\nu^{-1}(i+1)$, $v(e_j^*)\notin U_{\nu(j)-2}$.

   The scheme Y' is the union of two $I_H\times I_G$-orbits corresponding to ν and $\tau_i\circ\nu$. Moreover, we have
$$\dim(\Pi_{0,1}^{\nu})=1+\dim(\Pi_{0,1}^{\tau_i\circ\nu}).$$
   Hence
$$\overleftarrow{H}_H(L_{\tau_i},\mathcal{I}^{\mu!})\,\widetilde{\longrightarrow}\,\mathrm{IC}(Y')[-1]$$
   and the assertion follows.

3. Let Y' be the scheme classifying elements v in $\Pi_{0,1}$ such that :
   a) For $j \neq \nu^{-1}(i)$, $v(e_j^*) \in U_{\nu(j)}$ and $v(e_j^*) \notin U_{\nu(j)-1}$.
   b) For $j = \nu^{-1}(i)$, $v(e_j^*) \in U_{i+1}$ and $v(e_j^*) \notin U_{i-1}$.

   Then Y' is the union of two orbits $\Pi_{0,1}^{\nu}$ and $\Pi_{0,1}^{\tau_i \circ \nu}$. Thus $\overleftarrow{H}_H(L_{\tau_i}, \mathcal{I}^{\mu !})$ is the extension by zero of $\mathrm{IC}(Y')$ from Y' to $\Pi_{0,1}$. Hence we have a distinguished triangle

$$\mathcal{I}^{(\tau_i \circ \nu)!} \longrightarrow \mathrm{IC}(Y') \longrightarrow \mathcal{I}^{\nu !}[1] \xrightarrow{+1} .$$

4. Let Y' be a locally closed subscheme $\Pi_{0,1}$ be the scheme given by the conditions:
   a) For $\nu(j) \neq i, i+1$, $v(e_j^*) \in U_{\nu(j)} - U_{\nu(j)-1}$.
   b) For $j = \nu^{-1}(i)$, $v(e_j^*)$ and $v(e_{j-1}^*)$ belong to U_{i+1}, and their classes modulo U_{i-1} form a basis of U_{i+1}/U_{i-1}.

   Then $\overleftarrow{H}_H(L_{\tau_i}, \mathcal{I}^{\mu})$ is isomorphic to $\mathrm{IC}(Y')$ extended by zero to $\Pi_{0,1}$. The scheme Y' is the union of two orbits $\Pi_{0,1}^{\nu}$ and $\Pi_{0,1}^{\tau_i \circ \nu}$, and we have a distinguished triangle

$$\mathcal{I}^{(\tau_i \circ \nu)!} \longrightarrow \mathrm{IC}(Y') \longrightarrow \mathcal{I}^{\nu !}[1] \xrightarrow{+1} .$$

□

Proposition 3.4.7. *At the level of the functions Propositions* 3.4.3 *and* 3.4.6 *define completely the action of the finite Hecke algebra of G and H on* $\mathcal{S}^{I_H \times I_G}(\Pi(F))$.

Proposition 3.4.8. *Let w be the affine simple reflection, i.e. $w = t^{\lambda}\tau$ where $\lambda = (-1, 0 \dots, 0, 1)$ and $\tau = (1, m)$ is the longest element of W_H.*

1. If nor 1 neither m lies in I_μ then

$$\overleftarrow{H}_H(L_w, \mathcal{I}^{\mu}) \widetilde{\longrightarrow} \mathcal{I}^{\mu !} \otimes R\Gamma(\mathbb{P}^1, \overline{\mathbb{Q}}_\ell)[1].$$

2. If 1 is not in I_μ and m lies in I_μ, let $\lambda' = (-1, 0, \dots, 0)$, and $w' = (\lambda', \tau \circ \nu)$ be an element of $X_G \times S_{n,m}$, then there is a distinguished triangle

$$\mathcal{I}^{w' !} \longrightarrow \overleftarrow{H}_H(L_w, \mathcal{I}^{\mu !}) \longrightarrow \mathcal{I}^{\mu !}[1] \xrightarrow{+1} .$$

3. If 1 is in I_μ, and m is not in I_μ, let $\lambda' = (0, \dots, 0, 1)$, and $w' = (\lambda', \tau \circ \nu)$ be an element of $X_G \times S_{n,m}$, then there is a distinguished triangle

$$\mathcal{I}^{\mu !}[-1] \longrightarrow \overleftarrow{H}_H(L_w, \mathcal{I}^{\mu !)} \longrightarrow \mathcal{I}^{\mu !}[1] \xrightarrow{+1} .$$

4. If 1 and m are both in I_μ, let $\lambda' = (-1, 0, \dots, 0, 1)$ and $w' = (\lambda', \tau \circ \nu)$ then there is a distinguished triangle

$$\mathcal{I}^{w' !} \longrightarrow \overleftarrow{H}_H(L_w, \mathcal{I}^{\mu !}) \longrightarrow \mathcal{I}^{\mu !}[1] \xrightarrow{+1} .$$

Proof. Denote by $U_{-1} \subset U_0 \subset U_1 \subset \cdots \subset U_m = U \subset U_{m+1}$ the standard flag of lattices in $U(F)$, see [1.5.1, Chapter 1]. Assume $m > 1$. A point hI_H in $\overline{\mathcal{F}l_H^w}$ is given by a line U_0'/U_{-1} in U_1/U_{-1}. We set $U_m' = t^{-1}U_0'$. Let $\Pi^{\mu} \tilde{\times} \overline{\mathcal{F}l_H^w}$ be the scheme classifying pairs (v, hI_H), where hI_H is in $\overline{\mathcal{F}l_H^w}$ and v is a map from L^* to U_{m+1}/U_{-1} verifying:

a) For $\nu(j)\neq m$, $v(e_j^*)\in U_{\nu(j)}$.

b) For $\nu(j)\neq 1$, $v(e_j^*)\in U_{\nu(j)}-U_{\nu(j)-1}$.

c) For $m\in I_\mu$, $v(e_1^*)\in U_m^{'}-U_{m-1}^{'}$.
(The condition $m\in I_\mu$ is equivalent to $\nu(1)=m$).

d) If $1\in I_\mu$ then $v(e_n^*)\in U_1-U_0^{'}$.
(The condition $1\in I_\mu$ is equivalent to $\nu(n)=1$).

Now let

$$\pi:\Pi^\mu\tilde{\times}\overline{\mathcal{F}l_H^w}\longrightarrow \mathrm{Hom}_{\mathcal{O}}(L^*,U_{m+1}/U_{-1}) \tag{3.4.9}$$

be the projection sending a couple (v,hI_H) to v. The scheme $\Pi^\mu\tilde{\times}\overline{\mathcal{F}l_H^w}$ is smooth. Write IC for the intersection cohomology sheaf of $\Pi^\mu\tilde{\times}\overline{\mathcal{F}l_H^w}$. The sheaf IC is nothing but the constant sheaf shifted to be perverse. Then

$$\overleftarrow{H}_H(L_w,\mathcal{I}^{\mu !})\widetilde{\longrightarrow}\pi_!(\mathrm{IC}).$$

We can now prove the assertions.

1. This is straightforward as in Proposition 3.4.5.

2. The space $\mathrm{Hom}_{\mathcal{O}}(L^*,U_{m+1}/U_1)$ is an $\mathcal{O}$-module on which t acts trivially, hence it is a vector space. By definition $\overleftarrow{H}_H(L_w,\mathcal{I}^{\mu !})$ may be considered as a complex on $\mathrm{Hom}_{\mathcal{O}}(L^*,U_{m+1}/U_1)$. Let $Y^{'}$ be the subscheme of $\mathrm{Hom}_{\mathcal{O}}(L^*,U_{m+1}/U_1)$ given by the conditions:

   a) For $j>1$, $v(e_j^*)\in U_{\nu(j)}-U_{\nu(j)-1}$.

   b) The vector $v(e_1^*)$ does not vanish in U_{m+1}/U_{m-1}.

   Then $\overleftarrow{H}_H(L_w,\mathcal{I}^{\mu !})$ is isomorphic to $\mathrm{IC}(Y)^{'}$ extended by zero to $\mathrm{Hom}_{\mathcal{O}}(L^*,U_{m+1}/U_1)$. The subscheme $Y^{'}$ is the union of two $I_H\times I_G$-orbits, the closed orbit corresponds to $\Pi^\mu_{N,r}$ and the open orbit passes through maps v given by

$$v(e_n^*)=u_{\nu(n)},\dots,v(e_2^*)=u_{\nu(2)},v(e_1^*)=t^{-1}u_1.$$

   The map v can be written as follows:

   a) For $j\in I_\mu-\{m\}$, $v(u_j^*)=e_{\mu(j)}$.

   b) For all other $k\neq 1$, $v(u_k)=0$; and $v(u_1^*)=t^{-1}e_1$.

   So this open orbit corresponds to the element $w^{'}=(\lambda^{'},\tau\circ\nu)$ in $X_G\times S_{n,m}$, where $\lambda^{'}=(-1,0,\dots,0)$. This leads to the desired distinguished triangle.

3. In this case $\overleftarrow{H}_H(L_w,\mathcal{I}^{\mu !})$ is naturally a complex over $Hom_{\mathcal{O}}(L^*,U_{m-1}/U_{-1})$. The space $Hom_{\mathcal{O}}(L^*,U_{m-1}/U_{-1})$ is an $\mathcal{O}$-module on which t acts trivially, hence is a vector space. Denote by Y' the subscheme of $Hom_{\mathcal{O}}(L^*,U_{m-1}/U_{-1})$ given by the conditions:

   a) $v(e_n^*)\in U_1-U_{-1}$.

   b) $v(e_j^*)\in U_{\nu(j)}-U_{\nu(j)-1}$ for $1\leq j<n$.

   Then the fibres of the map (3.4.9) identify with $\mathbb{A}^1$. So $\overleftarrow{H}_H(L_w,\mathcal{I}^{\mu !})$ is the sheaf $\mathrm{IC}(Y^{'})[-1]$ extended by zero to $Hom_{\mathcal{O}}(L^*,U_{m-1}/U_{-1})$. The scheme $Y^{'}$ is the union of two $I_H\times I_G$-orbits, the open corresponding to ν and the closed one passing though v given by

$$v(e_n^*)=tu_m,v(e_{n-1}^*)=u_{\nu(n-1)},\dots,v(e_1^*)=u_{\nu(1)}.$$

Let $w^{'} = (\lambda^{'}, \tau \circ \nu)$ be in $X_G \times S_{n,m}$ with $\lambda^{'}$ being equal to $(0, \dots, 0, 1)$. Then we have a distinguished triangle

$$\mathcal{I}^{\mu}[-1] \longrightarrow \overleftarrow{H}_H(L_w, \mathcal{I}^{\mu !}) \longrightarrow \mathcal{I}^{w^{'}} \xrightarrow{+1} .$$

4. In this case we have $\nu(1) = m$ and $\nu(n) = 1$. Let $Y^{'}$ be the subscheme of $Hom_{\mathcal{O}}(L^*, U_{m+1}/U_{-1})$ classifying maps v satisfying the conditions:

   a) For all $1 < j < n$, $v(e_j^*) \in U_{\nu(j)} - U_{\nu(j)-1}$.
   b) $v(e_n^*) \in U_1 - U_{-1}$.
   c) $v(e_1^*) \in U_{m+1} - U_{m-1}$.
   d) $\{v(e_n^*), tv(e_1^*)\}$ are linearly independent in U_1/U_{-1}.

   Then the map (3.4.9) is an isomorphism onto Y'. Hence $\overleftarrow{H}_H(L_w, \mathcal{I}^{\mu !})$ identifies with $\mathrm{IC}(Y')$ extended by zero from Y' to $Hom_{\mathcal{O}}(L^*, U_{m+1}/U_{-1})$. The scheme Y' contains the closed subscheme which is the $I_H \times I_G$-orbit corresponding to ν. The complement of the latter scheme in $Y^{'}$ is the $I_H \times I_G$-orbit passing through v given by

$$v(e_1^*) = t^{-1}u_1, v(e_2) = u_{\nu(2)}, \dots, v(e_{n-1}^*) = u_{\nu(n-1)}, v(e_n^*) = tu_m.$$

   Let $w^{'} = (\lambda, \tau_i \circ \nu)$ where $\lambda = (-1, 0, \dots, 0, 1)$. Then there is a distinguished triangle

$$\mathcal{I}^{w^{'}!} \longrightarrow \mathrm{IC}(Y^{'}) \longrightarrow \mathcal{I}^{\mu !}[1] \xrightarrow{+1} .$$

□

3.5 The bimodule Θ

The purpose of this section is to show that Θ is a submodule with respect to the right action of $K(P_{I_H}(\mathcal{F}l_H))$ on $K(D_{I_G \times I_H}(\Pi(F)))$ introduced in Remark 2.2.12 and identify Θ as the induced representation from the parabolic subalgebra. The considerations in this subsection are essentially on the level of Grothendieck groups (we formulate them on the level of derived categories however when this is possible, one may assume that we work over a finite field with pure complexes only).

Let us simply denote $K(D_{I_G \times I_H}(\Pi(F)) \otimes \bar{\mathbb{Q}}_\ell$ by $\mathcal{S}$. Remind that $\mathcal{S}$ is exactly $\mathcal{S}^{I_H \times I_G}(\Pi(F))$. Let $\mathcal{S}_0$ be the $\bar{\mathbb{Q}}_\ell$-subspace of $\mathcal{S}$ generated by the elements $\mathcal{I}^{(w \centerdot w_0)!}$, where w runs through $\widetilde{W}_G$. Remind that $\mathcal{S}_0$ is a free module over $\mathcal{H}_{I_G}$ of rank one.

3.5.1 The action of $\mathcal{H}_{I_G}$ and $\mathcal{H}_{I_H}$ on Θ

We will start by some preliminary lemmas.

Lemma 3.5.1. *For any element w in $\widetilde{W}_G$, we have*

$$\overleftarrow{H}_G(L_{w!}, I^{w_0 !}) \widetilde{\longrightarrow} \mathcal{I}^{(w \centerdot w_0)!},$$

where $w \centerdot w_0$ is defined in [Definition 2.4.10, Chapter 2].

Proof. For a point gI_G in $\overline{\mathcal{F}l^w_G}$, let $L' = gL$ and equip L'/tL' with the flag $L^{'}_i = g(L_i)$, for $1 \leq i \leq n$. Let $\Pi_{\mathcal{I}^{w_0},r}\tilde{\times}\overline{\mathcal{F}l^w_G}$ be the scheme classifying pairs (v, gI_G), where gI_G is in $\overline{\mathcal{F}l^w_G}$, and v is a map from U^* to L'/t^rL such that the induced map

$$\overline{v} : U^*/tU^* \longrightarrow L'/tL'$$

sends $u^*_m, \dots, u^*_{n+1}$ to zero and $Vect(u^*_n, \dots, u^*_{n-i})$ to L'_{i+1} for $i = 0, \dots, n-1$. Let

$$\pi : \Pi_{\mathcal{I}^{w_0},r}\tilde{\times}\overline{\mathcal{F}l^w_G} \longrightarrow \Pi_{N,r} \tag{3.5.2}$$

be the proper map sending a couple (v, gI_G) to v. By definition we have $\overleftarrow{H}_G(L_w, \mathcal{I}^{w_0})\widetilde{\longrightarrow}\pi_!(\overline{\mathbb{Q}}_\ell\tilde{\boxtimes}L_w)$. Let $\Pi^0_{\mathcal{I}^{w_0},r}\tilde{\times}\mathcal{F}l^w_G$ be the open subscheme of $\Pi_{\mathcal{I}^{w_0},r}\tilde{\times}\overline{\mathcal{F}l^w_G}$ consisting of pairs (v, I_G) such that gI_G is in $\mathcal{F}l^w_G$, and the map $\overline{v} : Vect(u^*_n, \dots, u^*_{n-i}) \longrightarrow L'_{i+1}$ is an isomorphism for $i = 0, \dots, n-1$. Thus $\overleftarrow{H}_G(L_{w!}, \mathcal{I}^{w_0!})\widetilde{\longrightarrow}\pi^0_!(\overline{\mathbb{Q}}_\ell\tilde{\boxtimes}L_w)$, where $\pi^0 : \Pi^0_{\mathcal{I},r}\tilde{\times}\mathcal{F}\ell^w_G \longrightarrow \Pi_{N,r}$ is the restriction of π. The image of π^0 equals $\Pi^{w\bullet w_0}_{N,r}$ and π^0 is an isomorphism onto its image. Thus we have

$$\overleftarrow{H}_G(L_{w!}, \mathcal{I}^{w_0!})\widetilde{\longrightarrow}\mathcal{I}^{(w\bullet w_0)!}.$$

□

Now let $w = t^\lambda\tau$ be an element of $\widetilde{W}_H$. The cocharacter λ in X_H is of the form $(a_1, \dots, a_m)$ with a_i in $\mathbb{Z}$. Choose two integers N, r such that $-N \leq a_i < r$ for all i. Denote by $U_1 \subset U_2 \subset \dots \subset U_m$ the standard flag over U/tU. We define the scheme $\Pi_{\mathcal{I}^{w_0},r} \times \overline{\mathcal{F}l^w_H}$ in the same way we did for G. For any point hI_H in $\overline{\mathcal{F}l^w_H}$, we put $U' = hU$ and equip U'/tU' with the complete flag $U^{'}_i = hU_i$. Then $\Pi_{\mathcal{I}^{w_0},r} \times \overline{\mathcal{F}l^w_H}$ is the scheme classifying pairs (v, hI_H), where hI_H is in $\overline{\mathcal{F}l^w_H}$ and v is a map from L^* to U'/t^rU such that the induced map

$$\overline{v} : L^*/tL^* \longrightarrow U'/tU'$$

sends $\mathrm{Vect}(e^*_n, \dots, e^*_{n-i})$ to $U^{'}_{i+1}$ for all $i = 1, \dots, n-1$. Let π be the projection

$$\pi : \Pi_{\mathcal{I}^{w_0},r} \times \overline{\mathcal{F}l^w_H} \longrightarrow \Pi_{N,r}$$

then by definition we obtain $\overleftarrow{H}_H(L_w, \mathcal{I}^{w_0})\widetilde{\longrightarrow}\pi_!(\overline{\mathbb{Q}}_\ell\tilde{\boxtimes}L_w)$. Let $\Pi^0_{\mathcal{I}^{w_0},r} \times \mathcal{F}l^w_H$ be the open subscheme of $\Pi_{\mathcal{I}^{w_0},r}\times\overline{\mathcal{F}l^w_H}$ defined by the additional condition that the above map $\overline{v} : L^*/tL^* \longrightarrow U^{'}_n$ above is an isomorphism. Let

$$\pi^0 : \Pi^0_{\mathcal{I}^{w_0},r} \times \mathcal{F}l^w_H \longrightarrow \Pi_{N,r} \tag{3.5.3}$$

be the restriction of π. Then we have

$$\overleftarrow{H}_H(L_{w!}, \mathcal{I}^{w_0!})\widetilde{\longrightarrow}\pi^0_!(\overline{\mathbb{Q}}_\ell\tilde{\boxtimes}L_w).$$

Lemma 3.5.4. *Assume that $\lambda = (0, \dots, 0, a_{n+1}, \dots, a_m)$ and that the coefficients a_i are non negative. If τ is a permutation acting trivially on $\{1, \dots, n\}$ and permuting $\{n+1, \dots, m\}$, we have*

$$\overleftarrow{H}_H(L_{w!}, \mathcal{I}^{w_0!})\widetilde{\longrightarrow}\mathcal{I}^{w_0!}[n\langle\lambda, \check{\omega}_m\rangle - \ell(w)].$$

Proof. In this case, the image of the map π^0 (3.5.3) is exactly the orbit $\Pi^{w_0}_{0,r}$ and the fibre of π^0 is an affine space. We need to compute the dimension of the fibres. Note that $\Pi^{w_0}_{0,r}$ has dimension $(r-1)mn + \frac{n(n+1)}{2}$. The scheme $\Pi^0_{\mathcal{I}^{w_0},r} \tilde{\times} \mathcal{F}\ell^w_H$ is of dimension $\ell(w) - n\langle\lambda, \check{\omega}_m\rangle + (r-1)nm + \frac{n(n+1)}{2}$. This implies that the dimension of the fibre of π^0 equals $\ell(w) - n\langle\lambda, \check{\omega}_m\rangle$. This yields the result. □

Proposition 3.5.5. *Let τ be in W_H. Then*

$$\overleftarrow{H}_H(L_{\tau !}, \mathcal{I}^{w_0 !}) \widetilde{\longrightarrow} \mathcal{I}^{\nu !}[r],$$

where $\nu = (0, w_0\tau^{-1})$ is an element of $X_H \times S_{n,m}$ and r is the dimension of the fibre of the map π^0 (3.5.3).

Before proving this proposition we will need the following lemma:

Lemma 3.5.6. *Let $U_1 \subset \cdots \subset U_m = U_0$ be a complete flag on U_0. Consider a partial flag*

$$V_1 \subset V_2 \subset \cdots \subset V_n \subset U_0$$

inside U_0. Let τ be a reflection in the finite Weyl group W_H of H. Denote by Y the variety of complete flags

$$V_1^{'} \subset \cdots \subset V_m^{'}$$

which are in relative position τ with respect to the standard complete flag $U_1 \subset \cdots \subset U_m$ such that $V_i^{'} = V_i$ for all $i = 1, \ldots, n$. Then the variety Y is isomorphic to a finite dimensional affine space.

Proof. The stabilizer of the complete flag $U_1 \subset \cdots \subset U_m = U_0$ in H is the Borel subgroup B_H. For $i = 1, \ldots, m$ fix a basis u_i of U_0 such that $U_i = \mathrm{Vect}(u_1, \ldots, u_i)$. The variety H/B_H is identified with the complete flags in U_0. Given a vector subspace V of U_0 of dimension k, we associate to this subspace a subset $I(V)$ of k elements in $\{1, \ldots, m\}$ defined by $I(V) = \{1 \leq i \leq m | \dim(V \cap U_i) > \dim(V \cap U_{i-1})\}$. Thus, for w in W_H, the orbit $B_H w B_H / B_H$ is the variety of complete flags

$$U_1^{'} \subset \cdots \subset U_m^{'}$$

on U_0 such that $1 \leq i \leq m$ we have $I(U_i^{'}) = \{w(1), \ldots w(i)\}$.

Let $\mathcal{V}_n$ be a flag $V_1 \subset \cdots \subset V_n \subset U_0$ such that $\dim(V_i) = i$. The space Y is the variety of complete flags $V_1^{'} \subset \cdots \subset V_m^{'}$ lying in the orbit $B_H w B_H / B_H$ and satisfying $V_i^{'} = V_i$ for all $1 \leq i \leq n$. In order that the space Y be non-empty, we must have $I(V_n) = \{w(1), \ldots w(n)\}$. Assume that this is true. Given a subset of k elements I_k in $\{1, \ldots m\}$, denote by Z_{I_k} the variety of subspaces V of U_0 such that $I(V) = I_k$ (in particular we have $\dim(V) = k$). Given another subset I_{k+1} of $k+1$ elements containing I_k, let $Z_{I_k, I_{k+1}}$ be the variety of pairs $(V \subset V^{'})$, where V lies in Z_{I_k} and $V^{'}$ lies in $Z_{I_{k+1}}$. Denote by π the projection from $Z_{I_k, I_{k+1}}$ onto Z_{I_k} sending $(V \subset V^{'})$ to V. Let us prove that the map π is B_H-equivariant affine fibration.

For V in Z_{I_k} denote by $\overline{U}_i$ the image of U_i under the map $U_0 \to U_0/V$. Then $\overline{U}_i = \overline{U}_{i-1}$ if and only if i lies in I_k. Denote by s the single element of $I_{k+1} - I_k$. The fibre of the map π identifies with the variety of 1-dimensional subspaces $V^{'}/V \subset U_0/V$ such that $V^{'}/V$ is a subset of $\overline{U}_s$ and $V^{'}/V$ is not contained in $\overline{U}_{s-1}$. This fibre is affine and since the space Z_{I_k} is B_H-homogeneous, the map π is a B_H-equivariant affine fibration.

For $r \geq n$ denote by Y_r the variety of flags $V_1 \subset \cdots \subset V_n \subset V'_{n+1} \subset \cdots \subset V'_r$ such that $I(V_i) = \{\tau(1), \ldots, \tau(i)\}$ for $n \leq i \leq r$. We have the forgetful maps

$$Y_m \xrightarrow{f_m} Y_{m-1} \xrightarrow{f_{m-1}} \ldots \xrightarrow{f_{n+1}} Y_n = \mathrm{Spec}(\mathbf{k}).$$

Any of the map f_i above is obtained by a base change from the map π for a suitable pair $(I_k \subset I_{k+1})$. The fibre of a map f_r depends only on V'_{r-1} and not on the smaller V'_j for $j \leq r-2$. As any affine fibration over an affine space is trivial this leads to the result and Y is an affine space of dimension r for some $r \geq n$. □

Proof of Proposition 3.5.5. Let us precise the definition of ν, $\nu = (0, w_0\tau^{-1})$ is an element of $X_H \times S_{n,m}$, where the set $I_{w_0\tau^{-1}}$ is the set $\tau(\{1, \ldots, n\})$ and $w_0\tau^{-1} : I_{w_0\tau^{-1}} \to \{1, \ldots, n\}$ is the corresponding bijection. Consider the map

$$\pi^0 : \Pi^0_{\mathcal{I}^{w_0},1} \times \mathcal{F}l^\tau_H \longrightarrow \Pi_{0,1}.$$

The fibres of the map π^0 are affine spaces according to Lemma 3.5.6. We denote by r their dimension. Since the image of π^0 is the orbit $\Pi^\nu_{0,1}$. We obtain that

$$\overleftarrow{H}_H(L_{\tau!}, \mathcal{I}^{w_0!}) \widetilde{\longrightarrow} \mathcal{I}^{\nu!}[r].$$

□

Remark 3.5.7. *If the permutation τ is a actually a permutation of $\{1, \ldots, n\}$ and acts trivially on $\{n+1, \ldots, m\}$ then the shift in the above formula disappears and the map π^0 will be an isomorphism onto its image $\Pi^\nu_{0,1}$.*

3.5.2 Iwahori-Hecke algebra of a Levi subgroup

Let H be a connected reductive group, B_H be a Borel subgroup in H, and I_H the corresponding Iwahori subgroup. Denote by M a standard Levi subgroup of H, and by W_M the corresponding finite Weyl group. $I_M = M(F) \cap I_H$. The algebra $\mathcal{H}_{I_M}$ is the subalgebra of $\mathcal{H}_{I_H}$ generated by $(T_w)_{w \in W_M}$, and by the Bernstein functions $(\theta_\lambda)_{\lambda \in X_H}$. Our convention for Bernstein functions is defined in [§1.2.3, Chapter 1]. According to [Pra05, § 5.4], each coset $W_M \backslash W_H$ has a unique element of minimal length. Let ${}^M W_H$ be the set of such elements. If Δ_M denotes the simple roots of M then

$${}^M W_H = \{w \in W_H | w(\check{\alpha}) > 0 \text{ for each } \check{\alpha} \textit{ in } \Delta_M\}.$$

Any w in W_H can be written as $w''w'$, where w'' and w' are respective elements of W_M and ${}^M W_H$ satisfying $\ell(w) = \ell(w'') + \ell(w')$. Therefore T_w equals $T_{w''}T_{w'}$. We recall that $\mathcal{H}_{I_H}$ is a free module over $\mathcal{H}_{I_M}$ generated by $\{T_{w'} | w' \textit{ in } {}^M W_H\}$.

Consider the anti-involution on $\widetilde{W}_H$ sending w to w^{-1}. Then the opposite of T_w is $T_{w^{-1}}$ and for any μ in X_H, the opposite of θ_μ is $T^{-1}_{w_0}\theta_{-w_0\mu}T_{w_0}$.

Remark that the realization of $\mathcal{H}_{I_M}$ as a subalgebra of $\mathcal{H}_{I_H}$ does not correspond to the one that comes from viewing M as a closed subgroup of G. In particular, as we noticed above the restriction of the opposition to $\mathcal{H}_{I_M}$ does not correspond to $m \to m^{-1}$ in M. This is clear when M is the torus, the opposite of θ_μ is not $\theta_{-\mu}$. It is hence important to distinguish between the opposition on $\mathcal{H}_{I_H}$ and opposition in $\mathcal{H}_{I_M}$. More details may be found in [Pra05, § 7.3].

3.5.3 The action of Wakimoto sheaves

Consider the subspace $\mathcal{S}_0$ of $\mathcal{S}^{I_H\times I_G}(\Pi(F))$ generated by the elements $\mathcal{I}^{(w\cdot w_0)!}$, where w_0 is the longest element of W_G as usual and w runs through in $\widetilde{W}_G$. According to Lemma 3.5.1, the subspace $\mathcal{S}_0$ is a free module of rank one over $\mathcal{H}_{I_G}$. Let M be the standard Levi subgroup in H corresponding to the partition $(n, m-n)$ of m. Then M is of the form $M_1\times M_2$, where $M_1\widetilde{\longrightarrow}\mathbf{GL}_n$ and $M_2\widetilde{\longrightarrow}\mathbf{GL}_{m-n}$. Write $\mathcal{H}_{I_M}$ for the Iwahori Hecke algebra associated to M viewed as subalgebra of $\mathcal{H}_{I_H}$ defined in § 3.5.2. We have naturally $\mathcal{H}_{I_M}\widetilde{\longrightarrow}\mathcal{H}_{I_{M_1}}\otimes_{\overline{\mathbb{Q}}_\ell}\mathcal{H}_{I_{M_2}}$. We will denote by X_{M_i} the coweight lattice of M_i, for $i=1,2$. The space $\mathcal{S}_0$ is not a $\mathcal{H}_{I_M}$-submodule for the natural left action of $\mathcal{H}_{I_M}$ on $\mathcal{S}^{I_H\times I_G}(\Pi(F))$. For instance, let $\lambda=(1,\dots,1,0\dots,0)$ where 1 appears n times the complex $\overleftarrow{H}_H(L_{t^\lambda !},\mathcal{I}^{w_0!})$ doesn't occur in $\mathcal{S}_0$. Recall that there is a right action of $P_{I_H}(\mathcal{F}l_H)$ on $D_{I_H\times I_G}(\Pi(F))$ which is related to the left action according to Definition 3.3.5. Namely for w in $\widetilde{W}_H$ we have

$$\overrightarrow{H}_H(L_{w!},\mathcal{I}^{w_0!})\widetilde{\longrightarrow}\overleftarrow{H}_H(L_{w^{-1}!},\mathcal{I}^{w_0!}).$$

We will consider this right action and will show that S_0 is a right $\mathcal{H}_{I_M}$-module under this right action. Remind that the right action of $\mathcal{H}_{I_M}$ commutes with the left action of $\mathcal{H}_{I_G}$.

Lemma 3.5.8. *For τ a simple reflection in the finite Weyl group W_M of M, $\overrightarrow{H}_H(L_{\tau!},\mathcal{I}^{w_0!})$ lies in $\mathcal{S}_0$.*

Proof. According to Lemma 3.5.5, we have

$$\overrightarrow{H}_H(L_{\tau!},\mathcal{I}^{w_0!})\widetilde{\longrightarrow}\overleftarrow{H}_H(L_{\tau!},\mathcal{I}^{w_0!})\widetilde{\longrightarrow}\mathcal{I}^{\nu!}[r], \tag{3.5.9}$$

where $\nu=(0,w_0\tau)$ is viewed as an element of $X_H\times S_{n,m}$. Thus $\overrightarrow{H}_H(L_{\tau!},\mathcal{I}^{w_0!})$ occurs in $\mathcal{S}_0$. □

Lemma 3.5.10. *Let $\omega=(1,\dots,1)$ be in X_H, $\mu_1=(a_1,\dots,a_n)$ be in X_{M_1} and $\mu_2=(a_{n+1},\dots,a_m)$ be in X_{M_2}. Then let λ be the coweight $\mu_1+\mu_2$ in X_H and assume that if $m\geq i>n\geq j\geq 1$ then $a_i\geq a_j$. we also fix r,N two integers such that $-N\leq a_i<r$ for all i. Let v be a $\mathcal{O}$-linear map from L^* to $t^\lambda U/t^rU$ such that for $0\leq i<n$, the induced map*

$$\overline{v}: L^*/tL^*\longrightarrow t^\lambda U/t^{\lambda+\omega}U$$

sends $\mathrm{Vect}(e_n^*,\dots,e_{n-i}^*)$ *isomorphically onto* $\mathrm{Vect}(t^{a_1}u_1,\dots t^{a_{i+1}}u_{i+1})$. *Denote by ν the element $(w_0(\mu_1),w_0)$ in $X_G\times S_{n,m}$. Then v is an element of the orbit $\Pi^\nu_{N,r}$.*

Proof. Let $U_1=\mathcal{O}u_1\oplus\dots\oplus\mathcal{O}u_n$ and $U_2=\mathcal{O}u_{n+1}\oplus\dots\oplus\mathcal{O}u_m$. Then the map v can be written as a pair (v_1,v_2), where $v_i: L^*\longrightarrow t^{\mu_i}U_i/t^rU_i$ for $i=1,2$. Let $N_G\subset B_G$ be unipotent radical of the standard Borel subgroup of G. Acting by a suitable element of $N_G\subset I_G$ on v, one may assume that $v_1(e_i^*)=t^{a_{w_0(i)}}u_{w_0(i)}$ modulo $t^{\mu_1+\omega}U_1$.

Furthermore consider the groups

$$I_1=\{g\in\mathbf{GL}(U_1)\mid g=\mathrm{id}\bmod t\}$$

and

$$I_{G,0}=\{g\in\mathbf{GL}(L)\mid g=\mathrm{id}\bmod t\}.$$

Acting by a suitable element of $I_{G,0}\times I_1$ we may assume that $v_1(e_i^*)=t^{a_{w_0(i)}}u_{w_0(i)}$. This implies that $\overline{v}_2$ vanishes. Now viewing v as a map from L^* to $t^{-N}L$, we observe that r can be replaced by $1+\min\{a_{n+1},\dots,a_m\}$. Hence v is an element of $\Pi^\nu_{N,r}$. □

Lemma 3.5.11. *Let $\lambda = (a_1, \dots, a_n, 0, \dots, 0)$ in be a anti-dominant cocharacter in X_H, in particular all a_i's are non positive. Then we have a canonical isomorphism*

$$\overleftarrow{H}_H(L_{t^{\lambda}!}, \mathcal{I}^{w_0!}) \widetilde{\longrightarrow} \mathcal{I}^{\mu!}[\langle \lambda, 2\check{\rho}_G - 2\check{\rho}_H \rangle + (n-m)\langle \lambda, \check{\omega}_m \rangle],$$

where $\mu = (w_0(\lambda), w_0)$ in $X_G \times S_{n,m}$ and w_0 is the longest element of the finite Weyl group of $M_1 \widetilde{\longrightarrow} G$

Proof. Consider the map

$$\pi^0 : \Pi^0_{\mathcal{I}^{w_0},r} \tilde{\times} \mathcal{F}\ell^w_H \longrightarrow \Pi_{N,r}$$

defined in 3.5.2. By applying Lemma 3.5.10 to $w = t^{\lambda}$, we see that the image of π^0 is the $I_H \times I_G$-orbit on $\Pi_{N,r}$ passing through the map v from L^* to $t^{-N}U/t^rU$ given by $v(e^*_i) = t^{a_{w_0(i)}} u_{w_0(i)}$ for $1 \le i \le n$. This orbit corresponds to element $\mu = (w_0(\lambda), w_0)$ in $X_G \times S_{n,m}$. Restricting π^0 to its image we get

$$\pi^0_H : \Pi^0_{\mathcal{I}^{w_0},r} \tilde{\times} \mathcal{F}\ell^{t^{\lambda}}_H \longrightarrow \Pi^{\mu}_{N,r} \tag{3.5.12}$$

whose fibres are affine spaces. One has $\dim(\mathcal{F}\ell^{t^{\lambda}}_H) = \langle \lambda, 2\check{\rho}_H \rangle$. For any point hI_H in $\mathcal{F}\ell^{t^{\lambda}}_H$, let $U' = hU$. Then

$$\dim(\mathrm{Hom}_{\mathcal{O}}(L^*, t^{\lambda+\omega}U/t^rU)) = nm(r-1) - n\langle \lambda, \check{\omega}_m \rangle.$$

Thus the affine space of maps from L^* to hU/t^rU sending $\mathrm{Vect}(e^*_n, \dots, e^*_{n-i})$ to U'_{i+1} for $i = 1, \dots, n-1$ is of dimension $\frac{n^2+n}{2} + nm(r-1) - n\langle \lambda, \check{\omega}_m \rangle$. Finally

$$\dim(\Pi^0_{\mathcal{I}^{w_0},r} \tilde{\times} \mathcal{F}l^{t^{\lambda}}_H) = \langle \lambda, 2\check{\rho}_H \rangle + \frac{n^2+n}{2} + nm(r-1) - n\langle \lambda, \check{\omega}_m \rangle.$$

Moreover, we have the following isomorphism

$$\pi^0_G : \Pi^0_{\mathcal{I}^{w_0},r} \tilde{\times} \mathcal{F}l^{t^{w_0(\lambda)}}_G \widetilde{\longrightarrow} \Pi^{\mu}_{N,r}.$$

By using $\dim(\mathcal{F}\ell^{t^{w_0\lambda}}_G) = \langle \lambda, 2\check{\rho}_G \rangle$, we get that the dimension of $\Pi^{\mu}_{N,r}$ equals

$$\frac{n^2+n}{2} + nm(r-1) - m\langle \lambda, \check{\omega}_m \rangle + \langle \lambda, 2\check{\rho}_G \rangle.$$

and hence the dimension of the fibres of the map (3.5.12) equals

$$\langle \lambda, 2(\check{\rho}_G - \check{\rho}_H) \rangle + (m-n)\langle \lambda, \check{\omega}_m \rangle$$

which allows us to calculate the announced shift in the Lemma. Additionally this proves that $\overleftarrow{H}_H(L_{t^{\lambda}!}, \mathcal{I}^{w_0!})$ lies in $\mathcal{S}_0$. □

Proposition 3.5.13. *Let $\lambda = (a_1, \dots, a_n, 0, \dots, 0)$ an anti dominant cocharacter in X_H, in particular all a_i's are non-positive. Then the complex $\overleftarrow{H}_H(L_{t^{-\lambda^*}}, \mathcal{I}^{w_0!})$ occurs in $\mathcal{S}_0$.*

Proof. According to Lemma 3.5.11, we have

$$\overleftarrow{H}_H(L_{t^{\lambda}!}, \mathcal{I}^{w_0}) \widetilde{\longrightarrow} \mathcal{I}^{\mu!}[d],$$

where the shift d equals $[\langle \lambda, 2\check{\rho}_G - 2\check{\rho}_H \rangle + (n-m)\langle \lambda, \check{\omega}_m \rangle]$. Moreover, according to Proposition [1.5.6 , Chapter 1], we have $L_{t^{-\lambda}*} \star L_{t^{\lambda}!}$ is isomorphic to L_e where e is the identity element in the finite Weyl group of M_1 and $\star$ denotes the convolution product. Combining these two isomorphisms we obtain

$$\begin{aligned}\mathcal{I}^{w_0!} &\xrightarrow{\sim} \overleftarrow{H}_H(L_{t^{-\lambda *}} \star L_{t^{\lambda}!}, \mathcal{I}^{w_0!}) \\ &\xrightarrow{\sim} \overleftarrow{H}_H(L_{t^{-\lambda *}}, \mathcal{I}^{\mu!})[d] \\ &\xrightarrow{\sim} \overleftarrow{H}_H(L_{t^{-\lambda *}}, \overleftarrow{H}_G(L_{t^{w_0(\lambda)}!}, \mathcal{I}^{w_0!}))[d] \\ &\xrightarrow{\sim} \overleftarrow{H}_G(L_{t^{w_0(\lambda)}!}, \overleftarrow{H}_H(L_{t^{-\lambda *}}, \mathcal{I}^{w_0}))[d],\end{aligned} \tag{3.5.14}$$

where the third isomorphism is due to Lemma 3.5.1 and the last one is due the fact that the actions of H and G commute. Applying $\overleftarrow{H}_G(L_{t^{-w_0(\lambda)*}}, .)$ to both sides of (3.5.14), we obtain

$$\overleftarrow{H}_G(L_{t^{-w_0(\lambda)*}}, \mathcal{I}^{w_0!}) \xrightarrow{\sim} \overleftarrow{H}_H(L_{t^{-\lambda *}}, \mathcal{I}^{w_0})[d]. \tag{3.5.15}$$

Since $\mathcal{S}_0$ is a left $\mathcal{H}_{I_G}$-module, the left hand side of (3.5.15) lies in $\mathcal{S}_0$. Thus so does the right hand side. □

Lemma 3.5.16. *Let $\lambda = (0, \dots, 0, a_{n+1}, \dots, a_m)$ be a dominant cocharacter of M. If $a_{n+1} \geq \dots \geq a_m \geq 0$ then*

$$\overleftarrow{H}_H(L_{t^{-\lambda}*}, \mathcal{I}^{w_0!}) \xrightarrow{\sim} \mathcal{I}^{w_0!}[\langle \lambda, 2\check{\rho}_H\rangle - n\langle \lambda, \check{\omega}_m\rangle].$$

Proof. Lemma 3.5.4 applied to $w = t^\lambda$ (τ being the identity) gives us

$$\overleftarrow{H}_H(L_{t^{\lambda}!}, \mathcal{I}^{w_0}) \xrightarrow{\sim} \mathcal{I}^{w_0}[n\langle \lambda, \check{\omega}_m\rangle - \langle \lambda 2\check{\rho}_H\rangle].$$

This implies the assertion for $L_{t^{-\lambda}*}$. □

Proposition 3.5.17. *Let λ be a dominant cocharacter in X_H that can be written as the sum of two cocharacters λ_1 and λ_2 in X_{M_1} and X_{M_2} respectively. If $\nu = (-w_0(\lambda_1), w_0)$ then*

$$\overrightarrow{H}_H(L_{t^{\lambda}!}, \mathcal{I}^{w_0!}) \xrightarrow{\sim} \overleftarrow{H}_H(L_{t^{-\lambda}!}, \mathcal{I}^{w_0!}) \xrightarrow{\sim} \mathcal{I}^{\nu!}[\langle \lambda_1, 2\check{\rho}_G\rangle - \langle \lambda, 2\check{\rho}_H\rangle + \langle m\lambda_1 - n\lambda, \check{\omega}_m\rangle],$$

where we identify M_1 with G and hence $\check{\rho}_{M_1}$ with $\check{\rho}_G$. Thus $\overrightarrow{H}_H(L_{t^{\lambda}!}, \mathcal{I}^{w_0!})$ occurs in $\mathcal{S}_0$.

Proof. Set $-\lambda = (a_1, \dots, a_m)$ and choose two integers r, N such that $-N \leq a_i < r$ for all i. By Lemma 3.5.10 the map

$$\pi^0 : \Pi^0_{\mathcal{I}^{w_0}, r} \tilde{\times} \mathcal{F}\ell_H^{t^{-\lambda}} \longrightarrow \Pi_{N,r}$$

factors through $\Pi^\nu_{N,r}$ by a map π^0_H, where ν is equal $(-w_0(\lambda_1), w_0)$. The dimension of $\Pi^0_{\mathcal{I}^{w_0}, r} \tilde{\times} \mathcal{F}\ell_H^{t^{-\lambda}}$ equals $nm(r-1) + n\langle \lambda, \check{\omega}_m\rangle + \frac{n^2+n}{2} + \langle \lambda, 2\check{\rho}_H\rangle$. We have the isomorphism

$$\Pi^0_{\mathcal{I}^{w_0}, r} \tilde{\times} \mathcal{F}\ell_H^{t^{-w_0(\lambda_1)}} \xrightarrow{\sim} \Pi^\nu_{N,r}$$

and this allows us to calculate the dimension of $I_H \times I_G$-orbit $\Pi^\nu_{N,r}$. Namely

$$\dim(\mathcal{F}\ell_G^{t^{-w_0(\lambda_1)}}) = \langle -w_0(\lambda_1), 2\check{\rho}_G\rangle = \langle \lambda_1, 2\check{\rho}_G\rangle.$$

Remind that w_0 is longest element of the finite Weyl group of $M_1 \xrightarrow{\sim} G$. This yields

$$\dim(\Pi^\nu_{n,r}) = nm(r-1) + m\langle \lambda_1, \check{\omega}_m\rangle + \frac{n^2+n}{2} + \langle \lambda_1, 2\check{\rho}_G\rangle.$$

So the dimension of a fibre of the map π^0_H is $\langle \lambda, 2\check{\rho}_H\rangle - \langle \lambda_1, 2\check{\rho}_G\rangle + \langle n\lambda - m\lambda_1, \check{\omega}_m\rangle$. This justifies the shift in the formula announced above and the assertion follows. □

Remark 3.5.18. *In Lemma* 3.5.17 *if* λ_2 *equals* 0 *then the corresponding map* π_H^0 *is an isomorphism and the shift in the above formula disappears.*

Proposition 3.5.19. *For any* λ *in* X_H^+*, the complex* $\overleftarrow{H}_H(L_{t^\lambda *}, \mathcal{I}^{w_0})[d]$ *occurs in* $\mathcal{S}_0$*. Besides for a dominant cocharacter* μ *in* X_H^+ *then the complex* $\overleftarrow{H}_H(L_{t^{-\mu}!} \star L_{t^\lambda *}, \mathcal{I}^{w_0})$ *occurs in* $\mathcal{S}_0$ *as well.*

Proof. Recall that $L_{t^\lambda *} \star L_{t^{-\lambda}!} \widetilde{\longrightarrow} L_e$, combining this with Proposition 3.5.17, we get :

$$\begin{aligned}\mathcal{I}^{w_0!} &\widetilde{\longrightarrow} \overleftarrow{H}_H(L_{t^\lambda *} \star L_{t^{-\lambda}!}, \mathcal{I}^{w_0!}) \widetilde{\longrightarrow} \overleftarrow{H}_H(L_{t^\lambda *}, \mathcal{I}^\nu)[d] \\ &\widetilde{\longrightarrow} \overleftarrow{H}_H(L_{t^\lambda *}, \overleftarrow{H}_G(L_{t^{-w_0(\lambda)}!}, \mathcal{I}^{w_0!}))[d] \\ &\widetilde{\longrightarrow} \overleftarrow{H}_G(L_{t^{-w_0\lambda}!}, \overleftarrow{H}_H(L_{t^\lambda *}, \mathcal{I}^{w_0!}))[d],. \end{aligned} \tag{3.5.20}$$

The shift d is also the one defined in Proposition 3.5.17. The third isomorphism is due to Lemma 3.5.1 and the fourth holds by using the commutativity of the action of G and H. Applying $\overleftarrow{H}_G(L_{t^{w_0(\lambda)} *}, .)$ to both sides, we get

$$\overleftarrow{H}_G(L_{t^{w_0(\lambda)*}}, \mathcal{I}^{w_0!}) \widetilde{\longrightarrow} \overleftarrow{H}_H(L_{t^\lambda *}, \mathcal{I}^{w_0!})[d].$$

This complex occurs in $\mathcal{S}_0$. Now consider the isomorphism

$$\begin{aligned}\overleftarrow{H}_H(L_{t^{-\mu}!} \star L_{t^\lambda *}, \mathcal{I}^{w_0!}) &\widetilde{\longrightarrow} \overleftarrow{H}_H(L_{t^{-\mu}!}, \overleftarrow{H}_G(L_{t^{w_0(\lambda)} *}, \mathcal{I}^{w_0}))[-d] \\ &\widetilde{\longrightarrow} \overleftarrow{H}_G(L_{t^{w_0(\lambda)} *}, \overleftarrow{H}_H(L_{t^{-\mu}!}, \mathcal{I}^{w_0}))[-d]. \end{aligned} \tag{3.5.21}$$

By Lemma 3.5.17, the complex $\overleftarrow{H}_H(L_{t^{-\mu}!}, \mathcal{I}^{w_0!})$ occurs in $\mathcal{S}_0$. Since $\mathcal{S}_0$ is a $\mathcal{H}_{I_G}$-module we can apply the functor $\overleftarrow{H}_G(L_{t^{w_0(\lambda)} *},)$ to $\overleftarrow{H}_H(L_{t^{-\mu}!}, \mathcal{I}^{w_0!})$. Then the result will still occur in $\mathcal{S}_0$. □

Theorem 3.5.22. *The space* $\mathcal{S}_0$ *is a submodule of* $\mathcal{S}$ *for the right action of* $\mathcal{H}_{I_M}$*.*

Proof. The assertion follows from (3.5.9), Lemmas 3.5.11 and 3.5.16, Propositions 3.5.17, 3.5.13 and 3.5.19. □

The inclusion of $\mathcal{S}_0$ in $\mathcal{S}$ is a homomorphism of right $\mathcal{H}_{I_M}$-modules and left $\mathcal{H}_{I_G}$-modules. By adjoinction, we get a morphism

$$\alpha : \mathcal{S}_0 \otimes_{\mathcal{H}_{I_M}} \mathcal{H}_{I_H} \to \mathcal{S}$$

of right $\mathcal{H}_{I_H}$-modules and left $\mathcal{H}_{I_G}$-modules.

Theorem 3.5.23. *The map* $\alpha : \mathcal{S}_0 \otimes_{\mathcal{H}_{I_M}} \mathcal{H}_{I_H} \to \mathcal{S}$ *is injective, and it's image equals* $\Theta \otimes \overline{\mathbb{Q}}_\ell$*.*

Proof. Lemma 3.5.5 and Proposition 3.4.3 imply that the image of the map α is exactly $\Theta \otimes \overline{\mathbb{Q}}_\ell$. More precisely, if τ runs through ${}^M W_H$ the elements $L_{\tau!}$ form a basis of the left $\mathcal{H}_{I_M}$-module $\mathcal{H}_{I_H}$, § 3.5.2. Hence for w and τ runs through $\widetilde{W}_G$ and ${}^M W_H$ respectively, the objects

$$\overrightarrow{H}_H(L_{\tau!}, \overleftarrow{H}_G(L_{w!}, \mathcal{I}^{w_0!})) \tag{3.5.24}$$

form a basis of $\mathcal{S}_0 \otimes_{\mathcal{H}_{I_M}} \mathcal{H}_{I_H}$ over $\overline{\mathbb{Q}}_\ell$. An element ν in W_H lies in ${}^M W_H$ if and only if ν is strictly increasing on $\{1, \dots, n\}$ and on $\{n+1, \dots, m\}$. For τ in ${}^M W_H$ let $\mu = w_0 \tau$ and $I_\mu = \tau^{-1}(\{1, \dots, n\})$. Consider μ as a map from I_μ to $\{1, \dots, n\}$ and so as an element of

$X_G\times S_{n,m}$. The map $\tau^{-1}w_0:\{1,\dots,n\}\to I_\mu$ is strictly decreasing because $\tau^{-1}:\{1,\dots,n\}\to I_\mu$ is strictly increasing. According to Lemma 3.5.5 we have

$$\overrightarrow{H}_H(L_{\tau!},\mathcal{I}^{w_0!})\tilde{\longrightarrow}\mathcal{I}^{\mu!}[r].$$

By Proposition 3.4.1 we have $\overleftarrow{H}_G(L_{w!},\mathcal{I}^{\mu})\tilde{\longrightarrow}\mathcal{I}^{w\bullet\mu}[d']$ for some d' and hence the image of (3.5.24) under the map α is $\mathcal{I}^{w\bullet\mu!}[d'']$ for some shift d''. □

The Iwahori-Hecke algebra $\mathcal{H}_{I_M}$ identifies canonically with $\mathcal{H}_{I_{M_1}}\otimes_{\overline{\mathbb{Q}}_\ell}\mathcal{H}_{I_{M_2}}$. The right action of $\mathcal{H}_{I_{M_1}}$ and $\mathcal{H}_{I_{M_2}}$ on $\mathcal{S}_0$ commute with each other. We are now going to define the action of the Wakimoto sheaves on $\mathcal{I}^{w_0!}$.

Lemma 3.5.25. *We have the following isomorphisms:*

1. *For any λ in X_{M_2}*

$$\overrightarrow{H}_H(L_{t^\lambda!},\mathcal{I}^{w_0!})\tilde{\longrightarrow}\overleftarrow{H}_H(L_{t^{-\lambda}!},\mathcal{I}^{w_0!})\tilde{\longrightarrow}\mathcal{I}^{w_0!}[-\langle\lambda,2\check{\rho}_{M_2}\rangle].$$

2. *For λ in X_{M_2}*

$$\overrightarrow{H}_H(L_{t^{-\lambda}*},\mathcal{I}^{w_0!})\tilde{\longrightarrow}\mathcal{I}^{w_0!}[\langle\lambda,2\check{\rho}_{M_2}\rangle]. \tag{3.5.26}$$

3. *For any λ in X_{M_2},*

$$\overrightarrow{H}_H(\Theta_\lambda,\mathcal{I}^{w_0!})\tilde{\longrightarrow}\mathcal{I}^{w_0!}[-\langle\lambda,2\check{\rho}_{M_2}\rangle],$$

   where Θ_λ is the Wakimoto sheaf associated to λ defined in 1.5.3 *in Chapter* 1*.*

4. *For w in $\widetilde{W}_G$ and λ in X_{M_2},*

$$\overrightarrow{H}_H(\Theta_\lambda,\mathcal{I}^{(w\bullet w_0)!})\tilde{\longrightarrow}\mathcal{I}^{(w\bullet w_0)!}[-\langle\lambda,\check{\rho}_{M_2}\rangle].$$

Proof. The first formula is obtained by applying Lemma 3.5.17 to the case where $\lambda_1=0$. The second one is obtained from the first and from the isomorphism $L_{t^\lambda*}\star L_{t^{-\lambda}!}\tilde{\longrightarrow}L_e$. The third one holds by definition of Θ_λ using the two first isomorphisms. Finally we prove the fourth one:

$$\begin{aligned}\overrightarrow{H}_H(\Theta_\lambda,\mathcal{I}^{(w\bullet w_0)!})&\tilde{\longrightarrow}\overrightarrow{H}_H(\Theta_\lambda,\overleftarrow{H}_G(L_{w!},\mathcal{I}^{w_0!}))\\ &\tilde{\longrightarrow}\overleftarrow{H}_G(L_{w!},\overrightarrow{H}_H(\Theta_\lambda,\mathcal{I}^{w_0!}))\\ &\tilde{\longrightarrow}\overleftarrow{H}_G(L_{w!},\mathcal{I}^{w_0}[-\langle\lambda,2\check{\rho}_{M_2}\rangle])\tilde{\longrightarrow}\mathcal{I}^{(w\bullet w_0)!}[-\langle\lambda,2\check{\rho}_{M_2}\rangle].\end{aligned} \tag{3.5.27}$$

□

Corollary 3.5.28. *For any object K in $\mathcal{S}_0$ and any λ in X_{M_2} we have*

$$\overrightarrow{H}_H(\Theta_\lambda,K)\tilde{\longrightarrow}K[-\langle\lambda,2\check{\rho}_{M_2}\rangle].$$

Proposition 3.5.29. *For any w in the finite Weyl group of M_2 we have*

$$\overrightarrow{H}_H(L_{w!},\mathcal{I}^{w_0!})\tilde{\longrightarrow}\overleftarrow{H}_H(L_{w^{-1}},\mathcal{I}^{w_0!})\tilde{\longrightarrow}\mathcal{I}^{w_0!}[-\ell(w)].$$

Thus, at the level of the functions, the Iwahori-Hecke algebra $\mathcal{H}_{I_{M_2}}$ acts on $\mathcal{S}_0$ by the character corresponding to the trivial representation of $M_2(F)\tilde{\longrightarrow}\mathbf{GL}_{m-n}(F)$. Moreover,

$$\overrightarrow{H}_H(\Theta_\lambda\star L_\tau,K)\tilde{\longrightarrow}K[-\langle\lambda,2\check{\rho}_{M_2}\rangle-\ell(\tau)].$$

Proof. At the level of functions, for any w in $\widetilde{W}_G$ the character of $\mathcal{H}_{I_G}$ corresponding to the trivial representation sends T_w, the characteristic function of the double coset $I_G w I_G$, to $q^{\ell(w)}$. In our geometric setting, for any w in $\widetilde{W}_G$ this character becomes the functor $\overleftarrow{H}_G(L_{w!},)$ sending K in $\mathcal{S}_0$. to $K[-\ell(w)]$. Remind that the object $L_{w!}$ corresponds to $q^{-\ell(w)/2}T_w$. □

Remind that M_1 is identified with G. Now let us analyse the structure of $\mathcal{S}_0$ as a right $\mathcal{H}_{I_{M_1}}$-module and its relation with the left $\mathcal{H}_{I_G}$-module structure.

Lemma 3.5.30.

1. *For any τ in the finite Weyl group of M_1,*
$$\overrightarrow{H}_H(L_{\tau!}, \mathcal{I}^{w_0}) \widetilde{\longrightarrow} \mathcal{I}^{w_0\tau!}.$$
2. *For any λ in $X^+_{M_1}$,*
$$\overrightarrow{H}_H(L_{t^\lambda!}, \mathcal{I}^{w_0}) \widetilde{\longrightarrow} \mathcal{I}^{w_0 t^{-\lambda}!},$$
and
$$\overleftarrow{H}_G(L_{t^{w_0(\lambda)}*}, \mathcal{I}^{w_0!}) \widetilde{\longrightarrow} \overrightarrow{H}_H(L_{t^{-\lambda}*}, \mathcal{I}^{w_0!}).$$
3. *For any λ in X_{M_1},*
$$\overrightarrow{H}_H(\Theta_\lambda, \mathcal{I}^{w_0!}) \widetilde{\longrightarrow} \overleftarrow{H}_G(\Theta_{-w_0(\lambda)}, \mathcal{I}^{w_0!}).$$
4. *For any τ in the finite Weyl group of M_1 and any λ in X_{M_1},*
$$\overrightarrow{H}_H(\Theta_\lambda \star L_{\tau!}, \mathcal{I}^{w_0}) \widetilde{\longrightarrow} \overleftarrow{H}_G(\Theta_{-w_0\lambda} \star L_{w_0\tau w_0!}, \mathcal{I}^{w_0!}).$$

Proof. The first isomorphism is obtained from Proposition 3.5.5. The second one is a consequence of [1.5.8, Chapter 1.] For the third one, choose λ_1 and λ_2 on $X^+_H \cap X_{M_1}$ such that $\lambda = \lambda_1 - \lambda_2$. By definition of Wakimoto sheaves [§ 1.5.3, Chapter 1], $\Theta_\lambda = L_{t^{-\lambda_2}*} \star L_{t^{\lambda_1}!}$ in $\mathcal{H}_{I_{M_1}}$. The isomorphism in 2) yields that

$$\begin{aligned}
\overrightarrow{H}_H(L_{t^{\lambda_1}!}, \overrightarrow{H}_H(L_{t^{-\lambda_2}*}, \mathcal{I}^{w_0!})) &\widetilde{\longrightarrow} \overrightarrow{H}_H(L_{t^{\lambda_1}!}, \overleftarrow{H}_G(L_{t^{w_0(\lambda_2)}*}, \mathcal{I}^{w_0!}))\\
&\widetilde{\longrightarrow} \overleftarrow{H}_G(L_{t^{w_0(\lambda_2)}*}, \overrightarrow{H}_H(L_{t^{\lambda_1}!}, \mathcal{I}^{w_0!}))\\
&\widetilde{\longrightarrow} \overleftarrow{H}_G(L_{t^{w_0(\lambda_2)}!}, \overleftarrow{H}_G(L_{t^{-w_0(\lambda_2)}!}, \mathcal{I}^{w_0!}).
\end{aligned} \tag{3.5.31}$$

The element $-w_0\lambda_2$ is dominant if λ_2 is dominant. This implies the third assertion. The fourth isomorphism is obtained formally in the following way:

$$\begin{aligned}
\overrightarrow{H}_H(\Theta_\lambda \star L_{\tau!}, \mathcal{I}^{w_0!}) &\widetilde{\longrightarrow} \overrightarrow{H}_H(L_{\tau!}, \overleftarrow{H}_G(\Theta_{-w_0(\lambda)!}, \mathcal{I}^{w_0!}))\\
&\widetilde{\longrightarrow} \overleftarrow{H}_G(\Theta_{-w_0(\lambda)}, \overrightarrow{H}_H(L_{\tau!}, \mathcal{I}^{w_0!}))\\
&\widetilde{\longrightarrow} \overleftarrow{H}_G(\Theta_{-w_0\lambda}, \overleftarrow{H}_G(L_{w_0\tau w_0!}, \mathcal{I}^{w_0!}))\\
&\widetilde{\longrightarrow} \overleftarrow{H}_G(\Theta_{-w_0\lambda} \star L_{w_0\tau w_0!}, \mathcal{I}^{w_0!}).
\end{aligned} \tag{3.5.32}$$

□

Corollary 3.5.33. *The subspace $\mathcal{S}_0$ is a free right $\mathcal{H}_{I_{M_1}}$-module of rank one generated by $\mathcal{I}^{w_0!}$.*

Proof. The assertion follows from Lemma 3.5.30 and the fact that if λ and τ runs through X_{M_1} and W_{M_1} respectively the elements $\Theta_\lambda \star L_{\tau!}$ form a basis of $\mathcal{H}_{I_{M_1}}$. □

Combining Lemma 3.5.30 with Corollary 3.5.33 we obtain the following proposition:

Proposition 3.5.34. *There exists an equivalence of categories*

$$\tilde{\sigma} : P_{I_{M_1}}(\mathcal{F}l_{M_1}) \widetilde{\longrightarrow} P_{I_G}(\mathcal{F}l_G)$$

such that for any w in W_{M_1}, $\tilde{\sigma}$ sends L_w to $L_{w_0\overline{w}^{-1}w_0}$, ($\overline{w}$ is the anti-involution defined in Definition 3.1.7*). Additionally for any $\mathcal{T}$ in $P_{I_{M_1}}(\mathcal{F}l_{M_1})$ we have*

$$\overrightarrow{H}_H(\mathcal{T}, \mathcal{I}^{w_0!}) \widetilde{\longrightarrow} \overleftarrow{H}_G(\tilde{\sigma}(\mathcal{T}), \mathcal{I}^{w_0!}).$$

At last, For any λ a cocharacter of M_1, we have

$$\tilde{\sigma}(\Theta_\lambda \star L_{\tau!}) \widetilde{\longrightarrow} \Theta_{-w_0\lambda} \star L_{w_0\tau w_0!}.$$

In Proposition 3.3.6 when $n = m$, we have constructed an isomorphism $\tilde{\sigma}$ between the Iwahori-Hecke algebra of G and H. This isomorphism is induced by the automorphism β of $G(F)$ sending an element g to $w_0{}^t(g^{-1})w_0$. The isomorphism $\tilde{\sigma}$ in Proposition 3.5.34 and 3.3.6 coincides.

3.6 The case $n = 1$, $m \geq 1$

Assume that $n = 1$ and $m \geq 1$ in the entire section. In this case we will give a complete description of $DP_{I_H \times I_G}(\Pi(F))$ as a module over both $P_{I_H}(Gr_H)$ et $P_{I_G}(Gr_G)$. In this section we work over an algebraically closed field (and ignore the Tate twists).

We use the same notation as in the beginning of this chapter. Additionally for $1 \leq i \leq m$ we denote by ω_i the cocharacter of T_H equal to $(1, \dots, 1, 0, \dots, 0)$, where 1 appears i times. The Iwahori group I_H preserves $t^{-\omega_i}U$ and $t^{\omega_i}U^*$. Let Ω_H be the normal subgroup in the affine extended Weyl group $\widetilde{W}_H$ of elements of length zero, see [§ 1.1, Chapter 1]. Note that $\omega_m = (1, \dots, 1) \in \Omega_H$.

For $1 \leq i \leq m$, let $U^i = t^{-\omega_i}U$. Define U^i for all $i \in \mathbb{Z}$ by the property that $U^{i+m} = t^{-\omega_m}U^i$ for all i. Thus,

$$\dots \subset U^{-1} \subset U^0 \subset U^1 \subset \dots$$

is the standard flag preserved by I_H. For any integer k in $\mathbb{Z}$, we denote by IC^k the IC-sheaf of $U^k \otimes L$.

Proposition 3.6.1. *The irreducible objects of $P_{I_H \times I_G}(\Pi(F))$ are exactly the perverse sheaves* IC^k, $k \in \mathbb{Z}$.

Proof. The assertion follows from [Theorem 2.4.8, Chapter 2]. □

We will denote by $\mathrm{IC}^{k,!}$ the constant perverse sheaf on $U^k \otimes L - U^{k-1} \otimes L$ extended by zero. This is a (non irreducible) perverse sheaf. Denote by $I_0 = \mathrm{IC}^0$ the constant perverse sheaf on Π. Let us describe $\overleftarrow{H}_H(\mathcal{A}^\lambda, I_0)$, for any cocharacter λ of H. Remind that $\mathcal{A}^\lambda$ is the IC-sheaf of the I_H-orbit O^λ through $t^\lambda H(\mathcal{O})$ in Gr_H. Let $\lambda = (a_1 \dots, a_m)$ and choose N, r such that

$-N \leq a_i < r$ for all i. Let $\Pi_{0,r}\tilde{\times}\overline{O}^\lambda$ be the scheme classifying pairs $(v, hH(\mathcal{O}))$, where $hH(\mathcal{O})$ is a point in $\overline{O}^\lambda$ and v is a $\mathcal{O}$-linear map $L^* \to hU/t^rU$. Let

$$\pi : \Pi_{0,r}\tilde{\times}\overline{O}^\lambda \longrightarrow \Pi_{N,r}$$

be the map sending $(v, hH(\mathcal{O}))$ to the composition $L^* \xrightarrow{v} hU/t^rU \longrightarrow t^{-N}U/t^rU$. By definition we have

$$\overleftarrow{H}_H(\mathcal{A}^\lambda, I_0) \widetilde{\longrightarrow} \pi_!(\overline{\mathbb{Q}}_\ell \tilde{\boxtimes} \mathcal{A}^\lambda),$$

where $\overline{\mathbb{Q}}_\ell \tilde{\boxtimes} \mathcal{A}^\lambda$ is normalized to be perverse. Denote by p_H the projection of $\mathcal{F}l_H \to Gr_H$. Note that for any $\mathcal{T}$ in $P_{I_H}(\mathcal{F}l_H)$ we have

$$\overleftarrow{H}_H(\mathcal{T}, I_0) \widetilde{\longrightarrow} \overleftarrow{H}_H(p_{H!}(\mathcal{T}), I_0)$$

For $1 \leq i < m$, let s_i be the simple reflection $(i, i+1)$ in W_H.

Lemma 3.6.2. *For $1 \leq i < m$ we have*

$$\overleftarrow{H}_H(L_{s_i}, I_0) \widetilde{\longrightarrow} I_0 \otimes \mathrm{R}\Gamma(\mathbb{P}^1, \overline{\mathbb{Q}}_\ell) \widetilde{\longrightarrow} I_0 \otimes (\overline{\mathbb{Q}}_\ell[1] \oplus \overline{\mathbb{Q}}_\ell[-1]).$$

Similarly,

$$\overleftarrow{H}_H(L_{s_i!}, I_0) \widetilde{\longrightarrow} I_0[-1].$$

Proof. One has $p_{H!}(L_{s_i}) \widetilde{\longrightarrow} \mathrm{R}\Gamma(\mathbb{P}^1, \overline{\mathbb{Q}}_\ell)[1]$ and the assertion follows. □

Assume that $m > 1$ and let $s_m = t^\lambda \tau$, where $\lambda = (-1, 0, \dots, 0, 1)$ and $\tau = (1, m)$ is the reflection corresponding to the highest root. This is the unique affine simple reflection in $\widetilde{W}_H$.

Lemma 3.6.3. *If $m > 1$, we have the following canonical isomorphisms*

$$\overleftarrow{H}_H(L_{s_m}, I_0) \widetilde{\longrightarrow} \mathrm{IC}^1 \oplus \mathrm{IC}^{-1} \text{ and } \overleftarrow{H}_H(L_{s_m!}, I_0) \widetilde{\longrightarrow} \mathrm{IC}^{1,!} \oplus \mathrm{IC}^{-1}$$

Proof. The composition

$$\overline{\mathcal{F}l_H}^{s_m} \hookrightarrow \mathcal{F}l_H \xrightarrow{p_H} Gr_H$$

is a closed immersion and so $p_{H!}(L_{s_m}) \widetilde{\longrightarrow} \mathcal{A}^\lambda$. Thus we have

$$\overleftarrow{H}_H(L_{s_m}, I_0) \widetilde{\longrightarrow} \overleftarrow{H}_H(\mathcal{A}^\lambda, I_0).$$

In this case the scheme $\overline{O}^\lambda$ classifies lattices U' such that

$$\dots \subset U^{-1} \subset U' \subset U^1 \subset \dots$$

and $\dim(U'/U^{-1}) = 1$. Let $N = r = 1$, then the image of the projection

$$\pi : \Pi_{0,1}\tilde{\times}\overline{O}^\lambda \longrightarrow \Pi_{1,1}$$

is contained in $L \otimes (U^1/tU)$. Let v be a map from L^* to U^1/tU in the image of π. If v factors through U^{-1}/tU then the fibre of π over the point v is $\mathbb{P}^1$, otherwise it is a point. The first claim follows, the second is analogous. □

In a similar way one gets the following. The symmetry in our situation is due to the fact that Ω_H acts freely and transitively on the set of irreducible objects of $P_{I_G\times I_H}(\Pi(F))$.

Lemma 3.6.4. *For for $1\le i\le m$, we have*

$$\overleftarrow{H}_H(L_{s_i},\mathrm{IC}^i)\widetilde{\longrightarrow}\mathrm{IC}^{i+1}\oplus\mathrm{IC}^{i-1}\ \text{ and }\ \overleftarrow{H}_H(L_{s_i!},\mathrm{IC}^i)\widetilde{\longrightarrow}\mathrm{IC}^{i+1,!}\oplus\mathrm{IC}^{i-1}.$$

Proof. The proof follows from Lemma 3.6.2 and 3.6.3. □

For $1\le i\le m$ there is a unique permutation σ_i in W_H such that $t^{-\omega_i}\sigma_i$ is of length zero. Indeed, σ_i is the permutation sending

$$(1,2,\dots,m-i,m-i+1,\dots,m)\longrightarrow(i+1,i+2,\dots,m,1,\dots,i).$$

For $1\le i\le m$ we put $w_i=t^{-\omega_i}\sigma_i$. Extend this definition as follows, for any $i\in\mathbb{Z}$ let $w_i\in\Omega_H$ be the unique element such that $w_iU^r=U^{r+i}$ for any r. For $1\le i\le m-1$ we have $w_1s_iw_1^{-1}=s_{i+1}$ and $w_1s_mw_1^{-1}=s_1$. Thus, the affine Weyl group of H acts on the set $\{s_1,\dots,s_m\}$ by conjugation. As above, one also similarly gets the following:

Lemma 3.6.5. *1) For any i and k in $\mathbb{Z}$ one has canonically*

$$\overleftarrow{H}_H(L_{w_i},\mathrm{IC}^k)\widetilde{\longrightarrow}\mathrm{IC}^{k+i}$$

2) For $1\le i\le m$, $j\in\mathbb{Z}$ with $j\ne i\mod m$ one has

$$\overleftarrow{H}_H(L_{s_i},\mathrm{IC}^j)\,\widetilde{\to}\,\mathrm{IC}^j\otimes(\bar{\mathbb{Q}}_\ell[1]\oplus\bar{\mathbb{Q}}_\ell[-1]).$$

□

Lemma 3.6.4 and 3.6.5 describe completely the action of $P_{I_H}(\mathcal{F}l_H)$ on the irreducible objects IC^k, $k\in\mathbb{Z}$.

For any k in $\mathbb{Z}$ let $I^{k,!}$ be the IC-sheaf of U^k-U^{k-1} extended by zero. Each $I^{k,!}$ is an object of the derived category $D_{I_H\times I_G}(\Pi(F))$. At the level of functions, the objects $I^{k,!}$ generate a subspace of codimension 1 in $\mathcal{S}^{I_H\times I_G}(\Pi(F))$. We are going to describe the submodule generated by $I^{0,!}$ under the action of the Iwahori-Hecke algebra $\mathcal{H}_{I_H}$. For λ in X_H, let us provide an explicit description of the complex $\overleftarrow{H}_H(L_{t^\lambda!},I^{0,!})$. For any hI_H in $\mathcal{F}l_H$, let U' be the lattice hU endowed with a complete flag of lattices

$$\dots U'^{-2}\subset U'^{-1}\subset U'=U'^0\subset U^1\subset\dots,$$

where $U'^k=hU^k$. Let N,r be such that $-N\le a_i<r$ for all i and let $\Pi_{0,r}\tilde{\times}\overline{\mathcal{F}l_H}^{t^\lambda}$ be the scheme classifying pairs (v,HI_H), where hI_H is in $\overline{\mathcal{F}l_H}^{t^\lambda}$ and v is a map from L^* to hU/t^rU preserving the flags in the domain and in the range. If

$$\pi:\Pi_{0,r}\tilde{\times}\overline{\mathcal{F}l_H}^{t^\lambda}\longrightarrow\Pi_{N,r}$$

is the projection, then

$$\overleftarrow{H}_H(L_{t^\lambda!},I^{0,!})\widetilde{\longrightarrow}\pi_!(\overline{\mathbb{Q}}_\ell\tilde{\boxtimes}L_{t^\lambda}).$$

Let $\Pi^0_{0,r}\tilde{\times}\mathcal{F}l_H^{t^\lambda}$ be the open subscheme of $\Pi_{0,r}\tilde{\times}\overline{\mathcal{F}l_H}^{t^\lambda}$ consisting of elements (v, hI_H) in $\Pi^0_{0,r}\tilde{\times}\mathcal{F}l_H^{t^\lambda}$ such that the composition

$$L^* \xrightarrow{v} U'/t^rU \longrightarrow U'/(U'^{-1} + t^rU)$$

does not vanish. Let

$$\pi^0 : \Pi^0_{0,r}\tilde{\times}\mathcal{F}l_H^{t^\lambda} \longrightarrow \Pi_{N,r} \tag{3.6.6}$$

be the restriction of π to the open subscheme $\Pi^0_{0,r}\tilde{\times}\mathcal{F}l_H^{t^\lambda}$. Then we have

$$\overleftarrow{H}_H(L_{t^\lambda!}, I^{0,!})\widetilde{\longrightarrow}\pi^0_!(\overline{\mathbb{Q}}_\ell\tilde{\boxtimes}\mathcal{A}^{\lambda!})\widetilde{\longrightarrow}\pi^0_!(\overline{\mathbb{Q}}_\ell)[\dim(\Pi^0_{0,r}\tilde{\times}\mathcal{F}l_H^{t^\lambda})].$$

Consider the standard Levi subgroup M of H of type $(m-1, 1)$. Then $M\widetilde{\longrightarrow}M_1 \times M_2$, where $M_1 = \mathbf{GL}_{m-1}$ and $M_2 = \mathbf{GL}_1$. Let $\mathcal{H}_{I_M}$ be the Iwahori-Hecke algebra of M viewed as the subalgebra of $\mathcal{H}_{I_H}$ as defined in 3.5.2. We have naturally $\mathcal{H}_{I_M}\widetilde{\longrightarrow}\mathcal{H}_{I_{M_1}} \otimes_{\overline{\mathbb{Q}}_\ell} \mathcal{H}_{I_{M_2}}$. Let X_{M_i} be for the cocharacter lattice of M_i. We are going to define the action of $\mathcal{H}_{I_M}$ on $I^{0,!}$.

Lemma 3.6.7. *Let λ be a dominant cocharacter of H such that $\lambda = (a_1, \ldots, a_{m-1}, 0)$. Then*

$$\overleftarrow{H}_H(L_{t^\lambda!}, I^{0,!})\widetilde{\longrightarrow}I^{0,!}[\langle\lambda, \check{\omega}_m - 2\check{\rho}_H\rangle].$$

Similarly

$$\overleftarrow{H}_H(L_{t^{-\lambda*}}, I^{0,!})\widetilde{\longrightarrow}I^{0,!}[\langle\lambda, 2\check{\rho}_H - \check{\omega}_m\rangle].$$

Proof. First note that the subscheme $\Pi^0_{0,r}\tilde{\times}\mathcal{F}l_H^{t^\lambda}$ is of dimension $\langle\lambda, 2\check{\rho}_H - \check{\omega}_m\rangle + rm$. On the other hand $I^{0,!}$ is the shifted constant sheaf $\overline{\mathbb{Q}}_\ell[rm]$ extended by zero from $\Pi^0_{0,r}$ to $\Pi_{0,r}$. Hence the map π^0 factors as

$$\pi^0_H : \Pi^0_{0,r}\tilde{\times}\mathcal{F}l_H^{t^\lambda} \longrightarrow \Pi^0_{0,r}.$$

Each fibre of π^0_H over a point of $\Pi^0_{0,r}$ is an affine space of dimension $\langle\lambda, \check{\omega}_m - 2\check{\rho}_H\rangle$. Thus the first assertion follows. The second assertion follows from [Proposition 1.5.6, Chapter 1] which states that $L_{t^{-\lambda}*} \star L_{t^\lambda!}\widetilde{\longrightarrow}L_e$. ☐

This defines the action of the Wakimoto sheaves in $\mathcal{H}_{I_{M_1}}$ on $I^{0,!}$. Indeed for any λ in X_{M_1}, we can choose λ_1 and λ_2 dominant such that $\lambda = \lambda_1 - \lambda_2$.

Lemma 3.6.8. *Let w be in the finite Weyl group of M_1. Then*

$$\overleftarrow{H}_H(L_{w!}, I^{0,!})\widetilde{\longrightarrow}I^{0,!}[-\ell(w)].$$

Proof. Indeed (in this case) $\mathcal{F}l_H^w$ is the scheme of complete flags in U/tU that are in relative position w with respect to the standard flag. Hence the map π^0 in (3.6.6) becomes

$$\pi^0 : \Pi^0_{0,1} \times \mathcal{F}l_H^w \longrightarrow \Pi^0_{0,1}.$$

☐

Finally combining Lemmas 3.6.7 and 3.6.8 we obtain:

Proposition 3.6.9. *For any $w = t^\lambda w'$ in the affine extended Weyl group of M_1, we have*

$$\overleftarrow{H}_H(L_{w!}, I^{0,!})\widetilde{\longrightarrow}I^{0,!}[\langle\lambda, \check{\omega}_m\rangle - \ell(w)].$$

Let $\omega=(1,\dots,1)$ be in X_H. As we mentioned before t^ω is of length zero. The element L_{t^ω} is in the center of $\mathcal{H}_{I_H}$ and its action commutes with the action of $\mathcal{H}_{I_{M_1}}$. Moreover, the scheme $\mathcal{F}l_H^{t^\omega}$ is just a point and we obtain the following proposition:

Proposition 3.6.10. *Then for any integer k we have*

$$\overleftarrow{H}_H(L_{t^\omega},I^{k,!})\widetilde{\longrightarrow}I^{k-m,!}.$$

This defines the action of $\mathcal{H}_{I_{M_2}}$ on $I^{k,!}$. Now, denote by $\mathcal{S}_0$ the subspace of $\mathcal{S}^{I_H\times I_G}(\Pi(F))$ over $\overline{\mathbb{Q}}_\ell$ generated by the functions $I^{km,!}$ for k in $\mathbb{Z}$. The subalgebra of $\mathcal{H}_{I_H}$ generated by $\mathcal{H}_{I_{M_1}}$ and L_{t^ω} identifies with $\mathcal{H}_{I_M}$. According to Propositions 3.6.9 and 3.6.10 the subspace $\mathcal{S}_0$ is naturally a $\mathcal{H}_{I_M}$-module. The inclusion of $\mathcal{H}_{I_M}$-modules

$$\mathcal{S}_0\hookrightarrow\mathcal{S}^{I_H\times I_G}(\Pi(F))$$

yields by adjoinction a morphism of $\mathcal{H}_{I_H}$-modules

$$\alpha:\mathcal{H}_{I_H}\otimes_{\mathcal{H}_{I_M}}\mathcal{S}_0\longrightarrow\mathcal{S}^{I_H\times I_G}(\Pi(F)).$$

Theorem 3.6.11. *The map $\alpha:\mathcal{H}_{I_H}\otimes_{\mathcal{H}_{I_M}}\mathcal{S}_0\longrightarrow\mathcal{S}^{I_H\times I_G}(\Pi(F))$ is injective and its image is the $\mathcal{H}_{I_H}$-submodule $\tilde{\mathcal{S}}_0$ generated by $I^{k,!}$ for any k in $\mathbb{Z}$.*

Proof. For any k in $\mathbb{Z}$, denote by s_k the simple reflection $(k,k+1)$ in $\widetilde{W}_H$. The scheme $\overline{\mathcal{F}l_H}^{s_k}$ classifies lines in U^{k+1}/U^{k-1} and we have

$$\overleftarrow{H}_H(L_{s_k!},I^{k,!})\widetilde{\longrightarrow}I^{k+1,!}.$$

For $0\le k<m$ let $\nu_k=s_ks_{k-1}\dots s_0$. This a reduced decomposition of ν_k and so it is of length $k+1$. Hence we have

$$L_{\nu_k!}\widetilde{\longrightarrow}L_{s_k!}\star L_{s_{k-1!}}\star\dots\star L_{s_{0!}},$$

and

$$\overleftarrow{H}_H(L_{\nu_k},I^{0,!})\widetilde{\longrightarrow}I^{k+1,!}.$$

Denote by P the standard parabolic subgroup of H of type $(m-1,1)$, with Levi factor M. Each ν_k is of the form τ_kt^λ, where $\lambda=(1,0\dots,0,-1)$ and $\tau_k=(1,m,k+1,k,\dots,2)$. The set $\{1,\tau_0,\dots\tau_{m-2}\}$ is a system of representatives for the left cosets W_H/W_M. Hence the set $\{1,\nu_0,\dots,\nu_{m-2}\}$ is a system of representatives for the double cosets $I_H\backslash H(F)/P(F)$. Consider the right $\mathcal{H}_{I_M}$-submodule in $\mathcal{H}_{I_H}$ generated by $L_{\nu_{k!}}$. Any of the elements of this module is an extension by zero from an element of $I_H\backslash H(F)/P(F)$. Both $\mathcal{S}_0\otimes_{\mathcal{H}_{I_M}}\mathcal{H}_{I_H}$ and $\tilde{\mathcal{S}}_0$ are free $\mathcal{H}_{I_H}$-modules of rank m and α is surjective on $\tilde{\mathcal{S}}_0$. Thus the map α is injective. □

Remark 3.6.12. *As in § 3.5.2, we have the set ${}^MW_H$ of minimal length representatives in the right cosets W_H/W_M. As w runs through ${}^MW_H$, the elements $L_{w!}$ form a basis of $\mathcal{H}_{I_H}$ as a right $\mathcal{H}_{I_M}$-module. This property is used in Theorem 3.5.23. In the present situation it is not evident to relate ${}^MW_H$ with the system of representatives $\{1,\nu_0,\dots,\nu_{m-2}\}$. This is why we don't the surjectivity of α.*

Let us recall that there is a central functor constructed by Gaitsgory in [Gai01] :

Theorem 3.6.13. *[Gai01, Theorem 1] Let G be a connected reductive group. There exists a functor $\mathcal{Z} : P_{G(\mathcal{O})}(Gr_G) \longrightarrow P_{I_G}(\mathcal{F}l_G)$ verifying the following properties:*

1. *For $\mathcal{S} \in P_{G(\mathcal{O})}(Gr_G)$ and an arbitrary perverse sheaf $\mathcal{T}$ on $\mathcal{F}l_G$, the convolution product $\mathcal{T} \star \mathcal{Z}(\mathcal{S})$ is a perverse sheaf.*
2. *For $\mathcal{S} \in P_{G(\mathcal{O})}(Gr_G)$ and $\mathcal{T} \in P_{I_G}(\mathcal{F}l_G)$ there is a canonical isomorphism between $\mathcal{Z}(\mathcal{S}) \star \mathcal{T}$ and $\mathcal{T} \star \mathcal{Z}(\mathcal{S})$.*
3. *We have $\mathcal{Z}(\delta_{1_{Gr_G}}) = \delta_{\mathcal{F}l_G}$ and for any $\mathcal{S}^1$, $\mathcal{S}^2$ in $P_{G(\mathcal{O})}(Gr_G)$ there is a canonical isomorphism $\mathcal{Z}(\mathcal{S}^1) \star \mathcal{Z}(\mathcal{S}^2) \simeq \mathcal{Z}(\mathcal{S}^1 \star \mathcal{S}^2)$.*

Let $\sigma : \check{G} \times \mathbb{G}_{\mathfrak{m}} \longrightarrow \check{H}$ be given by (2.3.13), this has been recalled in [§ 2.3.2 , Chapter 2]. Denote by $\mathrm{Res}^\sigma : \mathrm{Rep}(\check{H}) \longrightarrow \mathrm{Rep}(\check{G} \times \mathbb{G}_m)$, the corresponding geometric restriction functor. For any $G(\mathcal{O})$-equivariant perverse sheaf $\mathcal{T}$ on Gr_H, $\mathcal{T}$ is naturally isomorphic to $p_{H!}(\mathcal{Z}(\mathcal{T}))$.

The category $\mathrm{Rep}(\check{G} \times \mathbb{G}_{\mathfrak{m}})$ acts on $DP_{I_G \times I_H}(\Pi(F))$ as follows:

$$\begin{cases} \text{If } s \text{ is the standard representation of } \mathbb{G}_m \text{ then } \overleftarrow{H}_G(s^j, \mathrm{IC}^k) \widetilde{\longrightarrow} \mathrm{IC}^k[j]. \\ \text{If } g \text{ is the standard representation of } \check{G} \text{ then } \overleftarrow{H}_G(g^j, \mathrm{IC}^k) \widetilde{\longrightarrow} \mathrm{IC}^{k-mj}. \end{cases} \tag{3.6.14}$$

It follows that the representation ring $\mathrm{R}(\check{G} \times \mathbb{G}_{\mathfrak{m}})$ acts on $K(DP_{I_G \times I_H}(\Pi(F)))$, which becomes in this way a free $\mathrm{R}(\check{G} \times \mathbb{G}_{\mathfrak{m}})$-module of rank m with basis $\{\mathrm{IC}^0, \ldots, \mathrm{IC}^{m-1}\}$.

Theorem 3.6.15. *The respective actions of the centres of $P_{I_H}(\mathcal{F}l_H)$ and of $P_{I_G}(\mathcal{F}l_G)$ on the category $DP_{I_G \times I_H}(\Pi(F))$ are compatible. More precisely, the center of $P_{I_H}(\mathcal{F}l_H)$ acts via the geometric restriction functor $\mathrm{Res}^\sigma : \mathrm{Rep}(\check{H}) \longrightarrow \mathrm{Rep}(\check{G} \times \mathbb{G}_m)$ on the irreducible objects IC^k for any integer k.*

Proof. For any $\mathcal{S}$ in $P_{H(\mathcal{O})}(Gr_H)$, we have

$$\overleftarrow{H}_H(\mathcal{Z}(\mathcal{S}), \mathrm{IC}^0) \widetilde{\longrightarrow} \overleftarrow{H}_H(p_{H!}(\mathcal{Z}(\mathcal{S})), \mathrm{IC}^0) \widetilde{\longrightarrow} \overleftarrow{H}_H(\mathcal{S}, \mathrm{IC}^0) \widetilde{\longrightarrow} \overleftarrow{H}_G(\mathrm{Res}^\sigma(\mathcal{S}), \mathrm{IC}^0), \tag{3.6.16}$$

where the last isomorphism is [Lys11, Proposition 5]. Remind that $\overleftarrow{H}_H(L_{w_k}, \mathrm{IC}^0) \widetilde{\longrightarrow} \mathrm{IC}^k$ for any $k \in \mathbb{Z}$. If $\mathcal{S}$ in $P_{H(\mathcal{O})}(Gr_H)$ then by definition $\mathcal{Z}(\mathcal{S})$ is central so

$$\overleftarrow{H}_H(\mathcal{Z}(\mathcal{S}), \mathrm{IC}^k) \widetilde{\longrightarrow} \overleftarrow{H}_H(L_{w_k}, \overleftarrow{H}_H(\mathcal{Z}(\mathcal{S}), \mathrm{IC}^0)) \widetilde{\longrightarrow} \overleftarrow{H}_G(\mathrm{Res}^\sigma(\mathcal{S}), \mathrm{IC}^k),$$

where the last isomorphism is from (3.6.16). The assertion follows. □

Assume that $\mathbf{k}$ is a finite field $\mathbb{F}_q$. Let us rewrite all useful formulas obtained above with taking in consideration the Tate twists. These formulas will be used in the next chapter.

According to Lemmas 3.6.2 and 3.6.3.

$$\begin{cases} \text{For } 1 \leq i \leq m : \ \overleftarrow{H}_H(L_{s_i}, \mathrm{IC}^i) \widetilde{\longrightarrow} \mathrm{IC}^{i+1} \oplus \mathrm{IC}^{i-1}. \\ \text{For } 1 \leq i \leq m : \ \overleftarrow{H}_H(L_{s_{i!}}, \mathrm{IC}^i) \widetilde{\longrightarrow} \mathrm{IC}^{i+1,!} \oplus \mathrm{IC}^{i-1}. \\ \text{If } j \neq i \bmod m : \overleftarrow{H}_H(L_{s_i}, \mathrm{IC}^j) \widetilde{\longrightarrow} \mathrm{IC}^j(\overline{\mathbb{Q}}_\ell[1](1/2) + \overline{\mathbb{Q}}_\ell[-1](-1/2)). \\ \text{If } j \neq i \bmod m : \overleftarrow{H}_H(L_{s_{i!}}, \mathrm{IC}^j) \widetilde{\longrightarrow} \mathrm{IC}^j[-1](-1/2). \end{cases} \tag{3.6.17}$$

Finally, by Lemma 3.6.5, for any i and k in $\mathbb{Z}$, the action of the element w_i is given by

$$\overleftarrow{H}_H(L_{w_i}, \mathrm{IC}^k) \widetilde{\longrightarrow} \mathrm{IC}^{k+i},$$

in particular we have

$$\overleftarrow{H}_H(L_{w_i}, \mathrm{IC}^0) \widetilde{\longrightarrow} \mathrm{IC}^i. \tag{3.6.18}$$

In the Grothendieck group $K(DP_{I_H\times I_G}(\Pi(F)))$ of the category $P_{I_H\times I_G}(\Pi(F))$ we have

$$I^{k,!} = \mathrm{IC}^k + \mathrm{IC}^{k-1}(1/2).$$

More generally, for $a < b$, denote by $\mathrm{IC}^{a,b,!}$ the sheaf $\overline{\mathbb{Q}}_\ell[b-a]$ on $(U^b/U^a) - \{0\}$ extended by zero to U^b/U^a. This is not perverse in general. In $K(DP_{I_H\times I_G}(\Pi(F)))$ we have

$$\mathrm{IC}^{a,b,!} = \mathrm{IC}^b - \mathrm{IC}^a[b-a].$$

Let $\omega_i = (1, \dots, 1, 0, \dots, 0)$ where 1 appears i times and 0 appears $m - i$ times.

Proposition 3.6.19. *We have a canonical isomorphism in $K(DP_{I_H\times I_G}(\Pi(F)))$:*

$$\overleftarrow{H}_H(L_{t^{\omega_i}!}, \mathrm{IC}^0) \widetilde{\longrightarrow} \mathrm{IC}^{i-m,0,!}[i - \langle\omega_i, 2\check{\rho}_H\rangle] + \mathrm{IC}^{-m}[m - i - \langle\omega_i, 2\check{\rho}_H\rangle].$$

Proof. First remark that

$$\overleftarrow{H}_H(L_{t^{\omega_i}}, \mathrm{IC}^0) \widetilde{\longrightarrow} \overleftarrow{H}_H(\mathcal{A}^{\omega_i!}, \mathrm{IC}^0).$$

Let $N = r = 1$, the scheme O^{ω_i} classifies lattices $tU^0 \subset U' \subset U^0$ such that $\dim(U'/tU^0) = m - i$ and $(U'/tU^0) \cap (U^{m-i}/tU^0) = 0$. Therefore the orbit O^{ω_i} is an affine space of dimension $\ell(t^{\omega_i}) = \langle\omega_i, 2\check{\rho}_H\rangle = (m-i)i$. Let $\Pi_{0,1}\tilde{\times}O^{\omega_i}$ be the scheme classifying pairs (v, U'), where U' is in O^{ω_i} and v is a map from L^* to U'/tU^0. Consider the map

$$\pi : \Pi_{0,1}\tilde{\times}O^{\omega_i} \longrightarrow \Pi_{0,1}$$

sending (v, U') to v. Then we have

$$\overleftarrow{H}_H(\mathcal{A}^{\omega_i!}, \mathrm{IC}^0) \widetilde{\longrightarrow} \pi_! \,\mathrm{IC}(\Pi_{0,1}\tilde{\times}O^{\omega_i})$$

and the assertion follows from the remark above on the elements $\mathrm{IC}^{a,b,!}$.

□

Chapter 4

Geometric Langlands functoriality at the Iwahori level

Remind that our main purpose is to extend, as much as possible, the results of [Lys11] to the Iwahori level. Our main object of study is the bimodule $\mathcal{S}^{I_H \times I_G}(\Pi(F))$ over the Iwahori-Hecke algebras $\mathcal{H}_{I_G}$ and $\mathcal{H}_{I_H}$ introduced in § 2.1.2. In this section we propose a conjectural description of this bimodule in terms of some stack attached to the Langlands dual groups. Our conjecture makes sense for any dual reductive pair.

More generally, assume given two connected reductive groups G, H and a homomorphism $\check{G} \times \mathrm{SL}_2 \to \check{H}$, where $\check{G}$ (resp., $\check{H}$) denotes the Langlands dual group of G (resp., of H). We suggest that there is a bimodule over the affine extended Hecke algebras $\mathbb{H}_G$ and $\mathbb{H}_H$ realizing the Arthur-Langlands functoriality at the Iwahori level for this homomorphism. We propose a definition of this bimodule on this level of generality (given in Conjecture 4.1.6). It is based to a large extent on the Kazhdan-Lusztig isomorphism describing the affine Hecke algebra as the equivariant K-theory of the Steinberg variety.

Consider a dual pair (G, H) in some symplectic group, and assume the metaplectic covering to split over $G(F)$ and $H(F)$, where $F = \mathbf{k}((t))$. In this case our conjecture becomes a precise question. Indeed, one may consider the Weyl category $\mathcal{W}$ introduced in [LL09] as a category with an action of $G(F) \times H(F)$ and ask if the corresponding Grothendieck group of $\mathcal{W}^{I_G \times I_H}$ is indeed isomorphic, as a $\mathcal{H}_{I_G}$ and $\mathcal{H}_{I_H}$-bimodule, to the one given by Conjecture 4.1.6. Our main result is that this is indeed the case for any dual pair $(\mathbf{GL}_1, \mathbf{GL}_m)$.

For basic notions in equivariant K-theory, one can refer to [CG97, Chapter 5]. Let us just recall the Kazhdan-Lusztig isomorphism and fix some notation. Let $\mathbf{k}$ be an algebraically closed field. Let G be a connected reductive algebraic group over $\mathbf{k}$, and $\check{G}$ the Langlands dual over $\bar{\mathbb{Q}}_\ell$. Assume that $[\check{G}, \check{G}]$ is simply connected.

Let $\check{\mathfrak{g}}$ be the Lie algebra of $\check{G}$, $\mathcal{B}_{\check{G}}$ be the variety of Borel subalgebras in $\check{\mathfrak{g}}$, and $\mathcal{N}_{\check{G}}$ be the nilpotent cone in $\check{\mathfrak{g}}$. The Springer resolution $\tilde{\mathcal{N}}_{\check{G}}$ of $\mathcal{N}_{\check{G}}$ is given by

$$\tilde{\mathcal{N}}_{\check{G}} = \{(x, \mathfrak{b}) \in \mathcal{N}_{\check{G}} \times \mathcal{B}_{\check{G}} | x \in \mathfrak{b}\}.$$

Let $\mu : \tilde{\mathcal{N}}_{\check{G}} \to \mathcal{N}_{\check{G}}$ be the Springer map. Let s be the standard coordinate on $\mathbb{G}_{\mathfrak{m}}$. We let $\mathbb{G}_{\mathfrak{m}}$ act on $\check{\mathfrak{g}}$ by requiring that s sends an element x to $s^{-2}x$. We also define an action of $\check{G} \times \mathbb{G}_{\mathfrak{m}}$ on $\tilde{\mathcal{N}}_{\check{G}}$ by the formula

$$(g, s).(x, b) = (s^{-2}gxg^{-1}, g\mathfrak{b}g^{-1}).$$

The map μ is $\check{G} \times \mathbb{G}_\mathfrak{m}$-equivariant. The Steinberg variety is defined by

$$Z_{\check{G}} = \tilde{\mathcal{N}}_{\check{G}} \times_{\mathcal{N}_{\check{G}}} \tilde{\mathcal{N}}_{\check{G}} = \{(x, , \mathfrak{b}, \mathfrak{b}') \in \mathcal{N}_{\check{G}} \times \mathcal{B}_{\check{G}} \times \mathcal{B}_{\check{G}} | x \in \mathfrak{b} \cap \mathfrak{b}'\}.$$

The extended affine Hecke algebra $\mathbb{H}_G$ introduced in Definition 1.2.3 can be considered as $\mathbb{Z}[s, s^{-1}]$-algebra, where $v = s^2$. Viewing $\mathbb{Z}[s, s^{-1}]$ as the representation ring of $\mathbb{G}_m$, one has the following:

Theorem 4.0.1. *[CG97, Theorem 7.2.5] There is a natural $\mathbb{Z}[s, s^{-1}]$-algebras isomorphism*

$$K^{\check{G} \times \mathbb{G}_\mathfrak{m}}(Z_{\check{G}}) \xrightarrow{\sim} \mathbb{H}_G.$$

4.1 Conjectural bimodule for the Langlands functoriality on the Iwahori level

4.1.1 The statement of the conjecture

Let G and H be two connected reductive groups over $\overline{\mathbb{Q}}_\ell$ and denote by $\check{G}$ and $\check{H}$ the respective Langlands dual groups over $\overline{\mathbb{Q}}_\ell$. We assume that the respective derived groups of $\check{G}$ and $\check{H}$ are simply connected. Let $\alpha : \mathbb{G}_m \to \mathrm{SL}_2$ be the standard maximal torus and ξ (resp. η) be a homomorphism from SL_2 (resp. $\check{G}$) to $\check{H}$. We fix a maximal torus T_G (resp. T_H) in G (resp. H) and a Borel subgroup B_G (resp. B_H) in G (resp. H) containing T_G (resp. T_H).
Assume we are given a morphism

$$\sigma : \check{G} \times \mathbb{G}_\mathfrak{m} \longrightarrow \check{H}$$

as the composition

$$\check{G} \times \mathbb{G}_m \xrightarrow{\mathrm{id} \times \alpha} \check{G} \times \mathrm{SL}_2 \xrightarrow{\eta \times \xi} \check{H} \,. \tag{4.1.1}$$

For any element g in $\check{G}$ we will often denote its image $\eta(g)$ in $\check{H}$ by the same letter g. Denote by

$$\overline{\sigma} : \check{G} \times \mathbb{G}_\mathfrak{m} \longrightarrow \check{H} \times \mathbb{G}_\mathfrak{m}$$

the morphism whose first component is σ and whose second component is the second projection $pr_2 : \check{G} \times \mathbb{G}_\mathfrak{m} \longrightarrow \mathbb{G}_\mathfrak{m}$. Let s denote the standard coordinate of $\mathbb{G}_m$. The representation ring of $\check{G} \times \mathbb{G}_\mathfrak{m}$ over $\overline{\mathbb{Q}}_\ell$ is denoted by $\mathrm{R}(\check{G} \times \mathbb{G}_\mathfrak{m})$ and is isomorphic to $\mathrm{R}(\check{G})[s, s^{-1}]$. According to [Lys11], the local Langlands functoriality at the unramified level sends the unramified representation with Langlands parameter γ in $\check{G}$ to the unramified representation with Langlands parameter $\sigma(\gamma, q^{1/2})$ of $\check{H}$. This is realized by the restriction homomorphism

$$\mathrm{Res}^\sigma : \mathrm{Rep}(\check{H}) \longrightarrow \mathrm{Rep}(\check{G} \times \mathbb{G}_\mathfrak{m})$$

induced by σ.
We define an action of the group $\check{G} \times \mathbb{G}_\mathfrak{m}$ on $\tilde{\mathcal{N}}_{\check{G}}$ by the following formula; for any (g, ζ) in $\check{G} \times \mathbb{G}_\mathfrak{m}$ and any $(z, \mathfrak{b})$ in $\tilde{\mathcal{N}}_{\check{G}}$ we have

$$(g, \zeta).(z, \mathfrak{b}) = (\zeta^{-2} g z g^{-1}, g \mathfrak{b} g^{-1}).$$

Remark 4.1.2. *On one hand, it is understood that the standard representation s of $\mathbb{G}_\mathfrak{m}$ corresponds to the cohomological shift -1 in order to have the compatibility with [Lys11]. On the other hand while specializing s, we should think of s as $q^{1/2}$ to makes things compatible with the theory of automorphic forms. The reader may refer to [Lus98] for more details.*

Let e denote the nilpotent element $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ of $\mathrm{Lie}(SL_2)$. If $d\xi : \mathrm{Lie}(SL_2) \to \mathrm{Lie}(\check{H})$ is the linearised morphism associated to ξ, we denote $d\xi(e)$ by x.

Lemma 4.1.3. *The map f from $\mathcal{N}_{\check{G}}$ to $\mathcal{N}_{\check{H}}$ sending any element z in $\mathcal{N}_{\check{G}}$ to $z + x$ is a $\overline{\sigma}$-equivariant map. It defines a morphism of stack quotients*

$$\overline{f} : \mathcal{N}_{\check{G}}/(\check{G} \times \mathbb{G}_{\mathfrak{m}}) \longrightarrow \mathcal{N}_{\check{H}}/(\check{H} \times \mathbb{G}_{\mathfrak{m}}). \tag{4.1.4}$$

Proof. We have the following equality in $\mathrm{Lie}(\mathrm{SL}_2)$

$$ses^{-1} = s^2 e. \tag{4.1.5}$$

This implies that $s^{-2}\xi(s)x\xi(s)^{-1} = x$. For (g,s) in $\check{G} \times \mathbb{G}_{\mathfrak{m}}$, let $(h,s) = \overline{\sigma}(g,s) = (g\xi(s), s)$. Then for any z in $\mathcal{N}_{\check{G}}$

$$s^{-2}gzg^{-1} + x = s^{-2}h(z+x)h^{-1},$$

which implies that f is $\overline{\sigma}$-equivariant and the morphism of stack quotients $\overline{f}$ is well-defined. □

The Springer map $\tilde{\mathcal{N}}_{\check{H}} \longrightarrow \mathcal{N}_{\check{H}}$ is $(\check{H} \times \mathbb{G}_{\mathfrak{m}})$-equivariant. By using this and Lemma 4.1.3 we obtain the following diagram:

$$\begin{array}{ccc}
(\tilde{\mathcal{N}}_{\check{G}}/(\check{G} \times \mathbb{G}_{\mathfrak{m}})) \times_{\mathcal{N}_{\check{H}}/(\check{H} \times \mathbb{G}_{\mathfrak{m}})} (\tilde{\mathcal{N}}_{\check{H}}/(\check{H} \times \mathbb{G}_{\mathfrak{m}})) & \longrightarrow & \tilde{\mathcal{N}}_{\check{G}}/(\check{G} \times \mathbb{G}_{\mathfrak{m}}) \\
\downarrow & & \downarrow \\
\tilde{\mathcal{N}}_{\check{H}}/(\check{H} \times \mathbb{G}_{\mathfrak{m}}) & \longrightarrow & \mathcal{N}_{\check{H}}/(\check{H} \times \mathbb{G}_{\mathfrak{m}}),
\end{array}$$

where the bottom horizontal map is induced from the Springer map for $\check{H}$ and the vertical right arrow is the composition of the $\check{G} \times \mathbb{G}_{\mathfrak{m}}$-equivariant Springer map for $\check{G}$ with the map $\overline{f}$ defined in Lemma 4.1.3. Note that in the left top corner of the diagram we took the fibre product in sense of stacks, see [LMB00, §2.2.2], we denoted it by $\mathcal{X}$. The K-theory $K(\mathcal{X})$ of $\mathcal{X}$ is naturally a module over the associative algebras $K^{\check{G}\times\mathbb{G}_{\mathfrak{m}}}(\tilde{\mathcal{N}}_{\check{G}} \times_{\mathcal{N}_{\check{G}}} \tilde{\mathcal{N}}_{\check{G}})$ and $K^{\check{H}\times\mathbb{G}_{\mathfrak{m}}}(\tilde{\mathcal{N}}_{\check{H}} \times_{\mathcal{N}_{\check{H}}} \tilde{\mathcal{N}}_{\check{H}})$, see Appendix A. The action is by convolution. Thanks to Theorem 4.0.1, these two algebras may be identified with the extended affine Hecke algebras $\mathbb{H}_G$ and $\mathbb{H}_H$ respectively. We may now state the conjecture :

Conjecture 4.1.6. *The bimodule over the algebras $K^{\check{G}\times\mathbb{G}_{\mathfrak{m}}}(Z_{\check{G}})$ and $K^{\check{H}\times\mathbb{G}_{\mathfrak{m}}}(Z_{\check{H}})$ realizing the local geometric Langlands functoriality at the Iwahori level for the map $\sigma : \check{G} \times \mathbb{G}_{\mathfrak{m}} \longrightarrow \check{H}$ identifies with $K(\mathcal{X})$.*

Remark 4.1.7. *If $\check{G} = \check{H}$ then the map ξ is trivial, $\mathcal{X}$ equals $Z_{\check{G}}$ and $K(\mathcal{X})$ identifies with the extended affine Hecke algebra $\mathbb{H}_G$ for G. Thus $K(\mathcal{X})$ is naturally a free module of rank one over both algebras $\mathbb{H}_H$ and $\mathbb{H}_G$.*

4.1.2 Properties of $\mathcal{X}$

Let us explain the following useful construction which we will use in several cases. Let G and H be two algebraic groups, $\phi : G \to H$ be a morphism and X be a G-variety. The induced H-variety with $H \times_G X$ respect to ϕ is the stack quotient $(H \times X)/G$, where G acts on $H \times X$ by

$$g.(h, x) = (h\phi(g)^{-1}, g.x).$$

The group H acts on the stack $H \times_G X$ by

$$h^{'}.(h, x) = (h^{'}h, x),$$

and $(H \times_G X)/H$ is isomorphic to the stack quotient X/G. There exists two functors "res" and "Ind" (restriction and induction) which are mutually inverse and give rise to the following isomorphisms of categories :

$$K^G(X) \leftrightarrows K^H(H \times_G X).$$

Proposition 4.1.8. *The stack $\mathcal{X}$ is naturally isomorphic to $(\tilde{\mathcal{N}}_{\check{G},\check{H}} \times_{\mathcal{N}_{\check{H}}} \tilde{\mathcal{N}}_{\check{H}})/(\check{H} \times \mathbb{G}_{\mathfrak{m}})$, where $\tilde{\mathcal{N}}_{\check{G},\check{H}}$ is defined below.*

Proof. Consider the induced variety with respect to $\overline{\sigma}$ defined by

$$\mathcal{N}_{\check{G},\check{H}} = (\check{H} \times \mathbb{G}_{\mathfrak{m}}) \times_{\check{G}\times\mathbb{G}_{\mathfrak{m}}} \mathcal{N}_{\check{G}}.$$

Since the map f defined in Lemma 4.1.3 is $\overline{\sigma}$-equivariant, it induces a $\check{H} \times \mathbb{G}_{\mathfrak{m}}$-equivariant map

$$f_1 : (\check{H} \times \mathbb{G}_{\mathfrak{m}}) \times_{\check{G}\times\mathbb{G}_{\mathfrak{m}}} \mathcal{N}_{\check{G}} \longrightarrow \mathcal{N}_{\check{H}}. \tag{4.1.9}$$

Similarly we can define the induced space

$$\tilde{\mathcal{N}}_{\check{G},\check{H}} = (\check{H} \times \mathbb{G}_{\mathfrak{m}}) \times_{\check{G}\times\mathbb{G}_{\mathfrak{m}}} \tilde{\mathcal{N}}_{\check{G}}.$$

The map f_1 in (4.1.9) induces a map from $\tilde{\mathcal{N}}_{\check{G},\check{H}}$ to $\mathcal{N}_{\check{H}}$ and we can consider the fibre product $\tilde{\mathcal{N}}_{\check{G},\check{H}} \times_{\mathcal{N}_{\check{H}}} \tilde{\mathcal{N}}_{\check{H}}$. Note that $\tilde{\mathcal{N}}_{\check{G},\check{H}}/(\check{H} \times \mathbb{G}_{\mathfrak{m}})$ is isomorphic to the stack quotient $\tilde{\mathcal{N}}_{\check{G}}/(\check{G} \times \mathbb{G}_{\mathfrak{m}})$.

It follows that the fibre product $\mathcal{X}$ identifies with the stack quotient of $\tilde{\mathcal{N}}_{\check{G},\check{H}} \times_{\mathcal{N}_{\check{H}}} \tilde{\mathcal{N}}_{\check{H}}$ by the action of $\check{H} \times \mathbb{G}_{\mathfrak{m}}$ thanks to the following general fact : If $\phi : X \longrightarrow Z$ and $\psi : Y \longrightarrow Z$ are equivariant morphisms of G-schemes, then the fibre product $X/G \times_{Z/G} Y/G$ in the category of stacks identifies with the quotient stack $(X \times_Z Y)/G$. This yields the desired equivalence of categories. □

If the map σ is the inclusion of $\check{G}$ in $\check{H}$, the natural map

$$\check{H} \times_{\check{G}} \mathcal{N}_{\check{G}} \to (\check{H} \times \mathbb{G}_{\mathfrak{m}}) \times_{\check{G}\times\mathbb{G}_{\mathfrak{m}}} \mathcal{N}_{\check{H},\check{G}} = \mathcal{N}_{\check{G},\check{H}}$$

is an isomorphism. We can identify $\mathcal{N}_{\check{G},\check{H}}$ with the variety of pairs

$$(h\check{G} \in \check{H}/\check{G}, v \in \mathcal{N}_{\check{H}})$$

satisfying $h^{-1}vh \in x + \mathcal{N}_{\check{G}}$ via the map sending any element of (h, z) of $\check{H} \times \mathcal{N}_{\check{G}}$ to $(h\check{G}, v = h(z + x)h^{-1})$. The latter map makes sense because $\check{G}$ centralizes x. Thus the map f_1 (4.1.9) becomes the projection sending any element $(h\check{G}, v)$ of $\mathcal{N}_{\check{G},\check{H}}$ to v. In this case the left $\check{H} \times \mathbb{G}_{\mathfrak{m}}$-action on $\mathcal{N}_{\check{G},\check{H}}$ is such that for any (h_1, s) in $\check{H} \times \mathbb{G}_{\mathfrak{m}}$ and any $(h\check{G}, v)$ in $\mathcal{N}_{\check{G},\check{H}}$ we have

$$(h_1, s).(h\check{G}, v) = (h_1 h\xi(s)^{-1}\check{G}, s^{-2}h_1 v h_1^{-1}).$$

Let us explain the following general result needed in the sequel. Let G be a closed algebraic subgroup of H, Y_1 be a G-scheme, and Y, $\tilde{Y}$ be two H-schemes. Consider the Cartesian diagram

$$\begin{array}{ccc} Y_1 \times_Y \tilde{Y} & \longrightarrow & Y_1 \\ \downarrow & & \downarrow \\ \tilde{Y} & \longrightarrow & Y, \end{array}$$

where the map $\tilde{Y} \to Y$ is H-equivariant and the map $f : Y_1 \to Y$ is G-equivariant, the action of G on Y being induced by the action of defined H.
The group G acts diagonally on the fibre product $Y_1 \times_Y \tilde{Y}$. This allows us to consider the induced space $H \times_G (Y_1 \times_Y \tilde{Y})$. On the other hand, we have a H-equivariant map $f_1 : H \times_G Y_1 \to Y$ given by $f_1(h, y_1) = hf(y_1)$. Consider the cartesian diagram

$$\begin{array}{ccc} (H \times_G Y_1) \times_Y \tilde{Y} & \longrightarrow & H \times_G Y_1 \\ \downarrow & & \downarrow{\scriptstyle f_1} \\ \tilde{Y} & \longrightarrow & Y, \end{array}$$

and let H act diagonally on the fibre product $(H \times_G Y_1) \times_Y \tilde{Y}$.

Lemma 4.1.10. *There is a H-equivariant isomorphism of schemes*

$$H \times_G (Y_1 \times_Y \tilde{Y}) \xrightarrow{\sim} (H \times_G Y_1) \times_Y \tilde{Y}. \tag{4.1.11}$$

Proof. The isomorphism is furnished by the H-equivariant map

$$\begin{aligned} H \times_G (Y_1 \times_Y \tilde{Y}) &\to (H \times_G Y_1) \times_Y \tilde{Y} \\ (h, (y_1, u)) &\to ((h, y_1), hu) \end{aligned}$$

□

Proposition 4.1.12. *There is a natural isomorphism*

$$K(\mathcal{X}) \xrightarrow{\sim} K^{\check{G}\times\mathbb{G}_{\mathfrak{m}}}(\tilde{\mathcal{N}}_{\check{G}} \times_{\mathcal{N}_{\check{H}}} \tilde{\mathcal{N}}_{\check{H}}),$$

and the $\mathrm{R}(\check{H} \times \mathbb{G}_{\mathfrak{m}})$*-module structure on the right hand side is given by the functor* $\mathrm{Res}^{\overline{\sigma}}$: $\mathrm{R}(\check{H} \times \mathbb{G}_{\mathfrak{m}}) \to \mathrm{R}(\check{G} \times \mathbb{G}_{\mathfrak{m}})$.

Proof. The scheme $\tilde{\mathcal{N}}_{\check{G}} \times_{\mathcal{N}_{\check{H}}} \tilde{\mathcal{N}}_{\check{H}}$ classifies couples $((z, \mathfrak{b}_1), \mathfrak{b})$, where $(z, \mathfrak{b}_1)$ lies in $\tilde{\mathcal{N}}_{\check{G}}$ and $\mathfrak{b}$ is Borel subalgebra in Lie(H) containing $z + x$. We define an action of $\check{G} \times \mathbb{G}_{\mathfrak{m}}$ on $\tilde{\mathcal{N}}_{\check{G}} \times_{\mathcal{N}_{\check{H}}} \tilde{\mathcal{N}}_{\check{H}}$ as follows: for any (g, s) in $\check{G} \times \mathbb{G}_{\mathfrak{m}}$ and any $((z, \mathfrak{b}_1, \mathfrak{b})$ in $\tilde{\mathcal{N}}_{\check{G}} \times_{\mathcal{N}_{\check{H}}} \tilde{\mathcal{N}}_{\check{H}}$

$$(g, s).((z, \mathfrak{b}_1), \mathfrak{b}) = (s^{-2}gzg^{-1}, g\mathfrak{b}_1 g^{-1}, g\xi(s)\mathfrak{b}\xi(s)^{-1}g^{-1}).$$

By Lemma 4.1.10 we have an $\check{H} \times \mathbb{G}_{\mathfrak{m}}$-equivariant isomorphism

$$(\check{H} \times \mathbb{G}_{\mathfrak{m}}) \times_{\check{G} \times \mathbb{G}_{\mathfrak{m}}} (\tilde{\mathcal{N}}_{\check{G}} \times_{\mathcal{N}_{\check{H}}} \tilde{\mathcal{N}}_{\check{H}}) \tilde{\longrightarrow} \tilde{\mathcal{N}}_{\check{G},\check{H}} \times_{\mathcal{N}_{\check{H}}} \tilde{\mathcal{N}}_{\check{H}}.$$

Combining this with Proposition 4.1.8 we get the desired isomorphism of categories. □

To understand the correspondence between the irreducible representations induced by this bimodule, one should know in particular whether the following natural map

$$\mathrm{R}(\check{T}_G \times \mathbb{G}_{\mathfrak{m}}) \otimes_{\mathrm{R}(\check{G} \times \mathbb{G}_{\mathfrak{m}})} K^{\check{G} \times \mathbb{G}_{\mathfrak{m}}}(\tilde{\mathcal{N}}_{\check{G}} \times_{\mathcal{N}_{\check{H}}} \tilde{\mathcal{N}}_{\check{H}}) \longrightarrow K^{\check{T}_G \times \mathbb{G}_{\mathfrak{m}}}(\tilde{\mathcal{N}}_{\check{G}} \times_{\mathcal{N}_{\check{H}}} \tilde{\mathcal{N}}_{\check{H}}) \quad (4.1.13)$$

is an isomorphism or not, and if $K(\mathcal{X})$ is a free $\mathrm{R}(\check{G} \times \mathbb{G}_{\mathfrak{m}})$-module.

4.2 Precising the conjecture in the case of $(\mathbf{GL}_n, \mathbf{GL}_m)$

In this section let us assume that $G = \mathbf{GL}_n$ and $H = \mathbf{GL}_m$, where $n \le m$. We provide a geometric candidate for the bimodule $K(\mathcal{X})$ appearing in conjecture 4.1.6.

Remind the category $DP_{I_H \times I_G}(\Pi(F))$ defined in Chapter 3, it is a module over $DP_{I_H}(\mathcal{F}l_H)$ and $DP_{I_G}(\mathcal{F}l_G)$ acting by Hecke functors. According to [IM], the Iwahori-Hecke algebra $\mathcal{H}_{I_H}$ identifies with $\mathbb{H}_H \otimes_{\mathbb{Z}[s,s^{-1}]} \bar{\mathbb{Q}}_\ell$ for the map $\mathbb{Z}[s, s^{-1}] \to \bar{\mathbb{Q}}_\ell$, $s \mapsto q^{\frac{1}{2}}$. This isomorphism is naturally upgraded to an isomorphism

$$K(DP_{I_H}(\mathcal{F}l_H)) \otimes \overline{\mathbb{Q}}_\ell \tilde{\to} \mathbb{H}_H \otimes_{\mathbb{Z}[s,s^{-1}]} \overline{\mathbb{Q}}_\ell$$

such that the multiplication by s in $\mathbb{H}_H$ corresponds to the cohomological shift by -1 in $K(DP_{I_H}(\mathcal{F}l_H))$. It is understood that we view $DP_{I_H}(\mathcal{F}l_H)$ as an abelian category. Our conjecture 4.1.6 now gives rise to the following one.

Conjecture 4.2.1. *The bimodule $K(\mathcal{X})$ over each extended affine Hecke algebra $K^{\check{G} \times \mathbb{G}_{\mathfrak{m}}}(\tilde{\mathcal{N}}_{\check{G}} \times_{\mathcal{N}_{\check{G}}} \tilde{\mathcal{N}}_{\check{G}})$ and $K^{\check{H} \times \mathbb{G}_{\mathfrak{m}}}(\tilde{\mathcal{N}}_{\check{H}} \times_{\mathcal{N}_{\check{H}}} \tilde{\mathcal{N}}_{\check{H}})$ is isomorphic (after specializing s to $q^{1/2}$) to the K-theory of the category $DP_{I_H \times I_G}(\Pi(F))$.*

This conjecture is difficult because $K(\mathcal{X})$ and $K(DP_{I_G \times I_H}(\Pi(F)))$ are essentially of different nature. Recall that we have studied in 3 the module structure of $K(DP_{I_G \times I_H}(\Pi(F)))$ over Iwahori-Hecke algebras $\mathcal{H}_{I_G}$, $\mathcal{H}_{I_H}$. In particular, we provided a filtration on $P_{I_G \times I_H}(\Pi(F))$ compatible with this action. In order to prove Conjecture 4.2.1 it should be necessary we make such a study on $K(\mathcal{X})$. In the remaining of this section we will construct (under mild assumption on σ) a filtration on $K(\mathcal{X})$ compatible with the action of extended affine Hecke algebras $\mathbb{H}_G$ and $\mathbb{H}_H$.

Remind that for any r the Langlands dual group of $\mathbf{GL}_r$ is $\mathbf{GL}_r$ over $\overline{\mathbb{Q}}_\ell$. We will always use the same notation for $\mathbf{GL}_r$ and its Langlands dual over $\overline{\mathbb{Q}}_\ell$. In this setting we choose the morphism η to be the canonical inclusion of $\mathbf{GL}_n$ into $\mathbf{GL}_m$. Since σ is a group morphism, ξ factors through $\mathbf{GL}_{m-n}$. This means that σ is obtained by the composition

$$\mathbf{GL}_n \times \mathbb{G}_{\mathfrak{m}} \to \mathbf{GL}_n \times \mathrm{SL}_2 \xrightarrow{id \times \xi} \mathbf{GL}_n \times \mathbf{GL}_{m-n} \longrightarrow \mathbf{GL}_m,$$

where the last arrow is the inclusion of the standard Levi subgroup associated to the partition $(n, m-n)$ of m and ξ corresponds to the principal unipotent orbit. Then the restriction of the map ξ to $\mathbb{G}_\mathfrak{m}$ is the cocharacter $(0,\dots,0,m-n-1,m-n-3,\dots,1+n-m)$. Let $U_0 = \mathbf{k}^m$ be the standard representation of $\mathbf{GL}_m$, and $\{u_1,\dots,u_m\}$ be the standard basis of U_0. The element $x = d\xi(e)$ is a nilpotent element of $\mathrm{Lie}(\mathbf{GL}_m)$ such that $x(u_i) = 0$ for $1 \le i \le n+1$ and that $x(u_{i+1}) = u_i$ for $n+1 \le i < m$. Such an element is called a subregular nilpotent element in $\mathrm{Lie}(\mathbf{GL}_m)$. Let $G_2 = \mathbf{GL}_{m-n}$ and B_2 be the unique Borel subgroup in G_2 such that x lies in $\mathrm{Lie}(B_2)$.

Let $Z_{G_2}(x)$ be the stabilizer of x in G_2. It acts naturally on $\tilde{\mathcal{N}}_{\check{G}} \times_{\mathcal{N}_{\check{H}}} \tilde{\mathcal{N}}_{\check{H}}$: for any y in $Z_{G_2}(x)$ and any $(z, \mathfrak{b}_1, \mathfrak{b})$ in $\tilde{\mathcal{N}}_{\check{G}} \times_{\mathcal{N}_{\check{H}}} \tilde{\mathcal{N}}_{\check{H}}$: we have

$$y.(z, \mathfrak{b}_1, \mathfrak{b}) = (z, \mathfrak{b}_1, y\mathfrak{b}y^{-1}).$$

For any s is in $\mathbb{G}_\mathfrak{m}$ then the element $\xi(s)$ clearly normalizes $Z_{G_2}(x)$ and the semi-direct product $Z_{G_2}(x) \rtimes \mathbb{G}_\mathfrak{m}$, is a subgroup of G_2. The group $Z_{G_2}(x) \rtimes \mathbb{G}_\mathfrak{m}$ acts on $\tilde{\mathcal{N}}_{\check{G}} \times_{\mathcal{N}_{\check{H}}} \tilde{\mathcal{N}}_{\check{H}}$ and this action commutes with the $\check{G}$-action.

For any $\check{G}$-orbit $\mathbb{O}$ on $\mathcal{N}_{\check{G}}$ we denote by $Y_\mathbb{O}$ the preimage of $\mathbb{O}$ in $\tilde{\mathcal{N}}_{\check{G}} \times_{\mathcal{N}_{\check{H}}} \tilde{\mathcal{N}}_{\check{H}}$ under the projection

$$\tilde{\mathcal{N}}_{\check{G}} \times_{\mathcal{N}_{\check{H}}} \tilde{\mathcal{N}}_{\check{H}} \to \mathcal{N}_{\check{G}}$$

sending $(z, \mathfrak{b}_1, \mathfrak{b})$ to z. We refer the reader to [CG97, § 3.2] for details on nilpotent orbits and stratification of the nilpotent cone $\mathcal{N}_{\check{G}}$ into $\check{G}$-conjugacy classes and the stratification of the Steinberg variety of $\check{G}$. The orbits $Y_\mathbb{O}$ form a $\check{G} \times \mathbb{G}_\mathfrak{m}$-invariant stratification of $\tilde{\mathcal{N}}_{\check{G}} \times_{\mathcal{N}_{\check{H}}} \tilde{\mathcal{N}}_{\check{H}}$, which is also $Z_{G_2}(x)$-invariant. The $\check{G}$-orbit $\mathbb{O}$ is given by a partition $\theta = (n_1 \ge n_2 \ge \cdots \ge n_r \ge 1)$ of n. Let M_θ denote the standard Levi subgroup corresponding to this partition, Namely

$$M_\theta \xrightarrow{\sim} \mathbf{GL}_{n_1} \times \cdots \times \mathbf{GL}_{n_r}.$$

We denote by z_θ the standard upper triangular regular nilpotent element in $\mathrm{Lie}(M_\theta)$; z_θ lies in the orbit $\mathbb{O}$. Let Z_θ be the stabilizer of z_θ in $\check{G} \times \mathbb{G}_\mathfrak{m}$, Z_θ is connected.

Denote by $\mathcal{B}_{\check{G},\theta}$ the preimage of z_θ under the Springer map $\tilde{\mathcal{N}}_{\check{G}} \to \mathcal{N}_{\check{G}}$. Let $\mathcal{B}_{\check{H},\theta}$ be the preimage of $z_\theta + x$ under the Springer map $\tilde{\mathcal{N}}_{\check{H}} \to \mathcal{N}_{\check{H}}$. We have an isomorphism

$$(\check{G} \times \mathbb{G}_\mathfrak{m}) \times_{Z_\theta} (\mathcal{B}_{\check{G},\theta} \times \mathcal{B}_{\check{H},\theta}) \xrightarrow{\sim} Y_\mathbb{O}$$

sending $(g, s, \mathfrak{b}_1, \mathfrak{b})$ to $(s^{-2} g z_\theta g^{-1}, g\mathfrak{b}_1 g^{-1}, g\xi(s)\mathfrak{b}\xi(s)^{-1}g^{-1})$. Hence we have an equivalence of categories

$$K^{\check{G} \times \mathbb{G}_\mathfrak{m}}(Y_\mathbb{O}) \xrightarrow{\sim} K^{Z_\theta}(\mathcal{B}_{\check{G},\theta} \times \mathcal{B}_{\check{H},\theta}). \tag{4.2.2}$$

According to [Spa82] the scheme $\mathcal{B}_{\check{G},\theta}$ and $\mathcal{B}_{\check{H},\theta}$ respectively admit a finite paving by affine spaces stable under the action of Z_θ, see also [Gor10, Theorem 3.4]. Hence (4.2.2) is a free $\mathrm{R}(Z_\theta)$-module of finite type.

We enumerate the nilpotent orbits $\mathbb{O}_1, \mathbb{O}_2, \dots, \mathbb{O}_r$ in $\mathcal{N}_{\check{G}}$ in such an order that

$$\dim(\mathbb{O}_1) \le \dim(\mathbb{O}_2) \le \cdots \le \dim(\mathbb{O}_r).$$

If $\overline{F}^j = \cup_{i \le j} \mathbb{O}_i$, then $\overline{F}^j$ is closed in $\mathcal{N}_{\check{G}}$ and we have a filtration

$$\emptyset = \overline{F}^0 \subset \overline{F}^1 \subset \cdots \subset \overline{F}^r = \mathcal{N}_{\check{G}}.$$

Let F^j be the preimage of $\overline{F}^j$ in $\tilde{\mathcal{N}}_{\check{G}} \times_{\mathcal{N}_{\check{H}}} \tilde{\mathcal{N}}_{\check{H}}$. We get a $\check{G} \times \mathbb{G}_m$-invariant filtration

$$\emptyset = F^0 \subset F^1 \subset \cdots \subset F^r = \tilde{\mathcal{N}}_{\check{G}} \times_{\mathcal{N}_{\check{H}}} \tilde{\mathcal{N}}_{\check{H}}.$$

Let us define a more general version of the cellular fibration [CG97, § 5.5] which we will use in the following.

Lemma 4.2.3 (Cellular fibration). *Let us consider the following general situation: k is an algebraically closed field of arbitrary characteristic and $\mathcal{X}$ is a k-stack of finite type equipped with a filtration*

$$\emptyset = F^0 \subset F^1 \subset \cdots \subset F^r = \mathcal{X},$$

by closed substacks of $\mathcal{X}$. Assume that for $1 \leq i \leq r$ there exists an affine space E^i and a connected linear algebraic group P^i such that

$$F^i - F^{i-1} \xrightarrow{\sim} E^i/P^i,$$

where E^i/P^i is the stack quotient. Let U^i be the unipotent radical of P^i and $G^i = P^i/U^i$ be the reductive quotient. Then the natural sequence

$$0 \longrightarrow K(F^{i-1}) \longrightarrow K(F^i) \longrightarrow K(E^i/P^i) \longrightarrow 0$$

is exact and $K(F^i)$ is a free $\mathbb{Z}$-module.

Proof. Choose a section of the natural projection from P^i to P^i , it yields a map from E^i/G^i to E^i/P^i incducing an isomorphism

$$K(E^i/P^i) \xrightarrow{\sim} K(E^i/G^i) \xrightarrow{\sim} K^{G^i}(\mathrm{Spec}(k)) \xrightarrow{\sim} \mathrm{R}(G^i),$$

where $\mathrm{R}(G^i)$ denotes the ring of representations of G^i (which is a free $\mathbb{Z}$-module.) One has an exact sequence

$$K_1(E^i/P^i) \xrightarrow{\delta} K(F^{i-1}) \longrightarrow K(E^i/P^i) \longrightarrow 0.$$

Let us show that the map δ vanishes. By [CG97, 5.2.18], we have that $K_1^{P^i}(E^i) \xrightarrow{\sim} K_1^{G^i}(E^i)$ and by Thom isomorphism [CG97, 5.4.17] we obtain that $K_1^{G^i}(E^i) \xrightarrow{\sim} K_1^{G^i}(\mathrm{Spec}(k))$. Now, by [Sri08, Corollary 6.12], $K^{G^i}(\mathrm{Spec}(k))$ is isomorphic to $k^* \otimes_{\mathbb{Z}} S$, where S is a free abelian group generated by the irreducible representations of G^i. By induction on i we may assume that $K^i(F^{i-1})$ is a free $\mathbb{Z}$-module. To finish the proof note that for any free $\mathbb{Z}$-module S, one has $\mathrm{Hom}_{\mathbb{Z}}(k^*, S) = 0$. Indeed, Let ϕ be such a morphism and take y in the image of ϕ. Then there exists an element x in k^* such that $\phi(x) = y$. As k is algebraically closed, for any integer n , there exists t in k such that that $t^n = x$. This gives $n\phi(t) = y$ in S. As S is a free abelian group, the element y is dividable by a finite number of integers. Thus, $y = 0$ and ϕ vanishes.

□

Let us go back to the filtration defined on $\tilde{\mathcal{N}}_{\check{G}} \times_{\mathcal{N}_{\check{H}}} \tilde{\mathcal{N}}_{\check{H}}$. We can refine the filtration F^i in such way that the refined filtration be $\check{G} \times \mathbb{G}_m$-stable and the corresponding strata of the stack quotient of $(\tilde{\mathcal{N}}_{\check{G}} \times_{\mathcal{N}_{\check{H}}} \tilde{\mathcal{N}}_{\check{H}})/(\check{G} \times \mathbb{G}_m)$ satisfy the assumptions of Lemma 4.2.3. Then by using this Lemma, we see that for each i the sequence

$$0 \longrightarrow K^{\check{G}\times\mathbb{G}_m}(F^{i-1}) \longrightarrow K^{\check{G}\times\mathbb{G}_m}(F^i) \longrightarrow K^{\check{G}\times\mathbb{G}_m}(Y_{\mathbb{O}_i}) \longrightarrow 0$$

is exact and $K^{\check{G}\times\mathbb{G}_m}(F^i)$, $0 \leq i \leq r$ define a filtration on $K(\mathcal{X})$. Moreover, for each i, $K^{\check{G}\times\mathbb{G}_m}(F^i)$ is a submodule over both extended affine Hecke algebras $K^{\check{G}\times\mathbb{G}_m}(\tilde{\mathcal{N}}_{\check{G}} \times_{\mathcal{N}_{\check{G}}} \tilde{\mathcal{N}}_{\check{G}})$ and $K^{\check{H}\times\mathbb{G}_m}(\tilde{\mathcal{N}}_{\check{H}} \times_{\mathcal{N}_{\check{H}}} \tilde{\mathcal{N}}_{\check{H}})$.

4.3 The conjecture in the special case of $n = 1$, $m \geq 1$

4.3.1 Statement

We will restrict ourselves to the case $n = 1$ and $m \geq 1$. Let $G = \mathbf{GL}_1$ and $H = \mathbf{GL}_m$, where we consider them as Langlands dual groups. The map $\check{G} \times \mathbb{G}_\mathfrak{m} \longrightarrow \check{H}$ is the composition

$$\check{G} \times \mathbb{G}_\mathfrak{m} \longrightarrow \check{G} \times \mathrm{SL}_2 \longrightarrow \check{G} \times \mathbf{GL}_{m-1} \longrightarrow \check{H},$$

where the latter map is the inclusion of the standard Levi subgroup $\mathbf{GL}_1 \times \mathbf{GL}_{m-1}$ in $\check{H}$ and $\xi : \mathrm{SL}_2 \to \mathbf{GL}_{m-1}$ corresponds to the principal unipotent orbit. In particular the inclusion $\check{G}$ in $\check{H}$ is the coweight $(1, 0, \dots, 0)$ of the standard maximal torus of $\check{H}$. The restriction of ξ to the maximal torus $\mathbb{G}_\mathfrak{m}$ of SL_2 is the coweight $(0, m-2, m-4, \dots, 2-m)$ of $\check{H}$. The element $x = d\xi(e)$ in $\mathcal{N}_{\check{H}}$ is the subregular nilpotent element given by $x(u_1) = x(u_2) = 0$ and $x(u_{i+1}) = u_i$ for all $2 \leq i < m$. In this case $\tilde{\mathcal{N}}_{\check{G},\check{H}} = \check{H}/\check{G}$ in such way that the map $f_1 : \tilde{\mathcal{N}}_{\check{G},\check{H}} = \check{H}/\check{G} \longrightarrow \mathcal{N}_{\check{H}}$ defined in (4.1.9) sends $h\check{G}$ to hxh^{-1}. The element s in $\mathbb{G}_\mathfrak{m}$ acts on the left hand side on $\tilde{\mathcal{N}}_{\check{G},\check{H}}$ by sending the right coset $h\check{G}$ to $h\xi(s)^{-1}\check{G}$. The variety $\tilde{\mathcal{N}}_{\check{G},\check{H}} \times_{\mathcal{N}_{\check{H}}} \tilde{\mathcal{N}}_{\check{H}}$ identifies with the variety of pairs $(h\check{G}, \mathfrak{b})$ such that $\mathfrak{b}$ is a Borel subalgebra in $\check{H}$ and hxh^{-1} lies in $\mathfrak{b}$. Any element (h_1, s) in $\check{H} \times \mathbb{G}_\mathfrak{m}$ acts on $\tilde{\mathcal{N}}_{\check{G},\check{H}} \times_{\mathcal{N}_{\check{H}}} \tilde{\mathcal{N}}_{\check{H}}$ by the formula:

$$(h_1, s).(h\check{G}, \mathfrak{b}) = (h_1 h \xi(s)^{-1}\check{G}, h_1 \mathfrak{b} h_1^{-1}).$$

Denote by $\mathcal{B}_{\check{H},x}$ the fibre of the Springer map $\tilde{\mathcal{N}}_{\check{H}} \to \mathcal{N}_{\check{H}}$ over x. The map

$$\overline{\sigma} : \check{G} \times \mathbb{G}_\mathfrak{m} \longrightarrow \check{H} \times \mathbb{G}_\mathfrak{m}$$

sending (g, s) to $(g\xi(s), s)$ identifies $\check{G} \times \mathbb{G}_\mathfrak{m}$ with the stabilizer in $\check{H} \times \mathbb{G}_\mathfrak{m}$ of the right coset of the neutral element in $\check{H}/\check{G}$. Any element (g, s) of $\check{G} \times \mathbb{G}_\mathfrak{m}$ acts on the Springer fibre $\mathcal{B}_{\check{H},x}$ by

$$(g, s).\mathfrak{b}' = (g\xi(s)\mathfrak{b}'\xi(s)^{-1}g^{-1}).$$

This yields an isomorphism

$$K(\mathcal{X}) = K^{\check{H} \times \mathbb{G}_\mathfrak{m}}(\tilde{\mathcal{N}}_{\check{H},\check{G}} \times_{\mathcal{N}_{\check{H}}} \tilde{\mathcal{N}}_{\check{H}}) \widetilde{\longrightarrow} K^{\check{G} \times \mathbb{G}_\mathfrak{m}}(\mathcal{B}_{\check{H},x}).$$

To compute $K(\mathcal{X})$, we provide an explicit description of the Springer fibre $\mathcal{B}_{\check{H},x}$.

Lemma 4.3.1. *The Springer fibre $\mathcal{B}_{\check{H},x}$ is a configuration of projective lines $(V_i)_{1 \leq i \leq m-1}$. For $1 \leq i < j \leq m-1$ the intersection $V_j \cap V_i$ is empty unless $j = i+1$. The fixed locus in $\mathcal{B}_{\check{H},x}$ under the action of $\check{G} \times \mathbb{G}_\mathfrak{m}$ consists of m points $p_1, p_2, \dots, p_{m-1}, p_m$, where p_1 and p_m are distinguished points on V_1 and V_m and for $2 \leq i \leq m-1$, the point p_i is the intersection of V_i with V_{i+1}.*

Proof. Denote by

$$F_1 \subset F_2 \subset \dots \subset F_m = U_0$$

a complete flag on the standard representation U_0 of $\check{H}$ preserved by x. The vector space F_1 is a subspace of $\mathrm{Ker}(x) = \mathrm{Vect}(u_1, u_2)$. We have $\mathrm{Vect}(u_2) = \mathrm{Ker}(x) \cap \mathrm{Im}(x)$. If $F_1 \neq \mathrm{Vect}(u_2)$ then $F_2 = x^{-1}(F_1) = \mathrm{Vect}(u_1, u_2)$, $F_3 = x^{-1}(F_2) = \mathrm{Vect}(u_1, u_2, u_3)$, $\dots$, $F_m =$

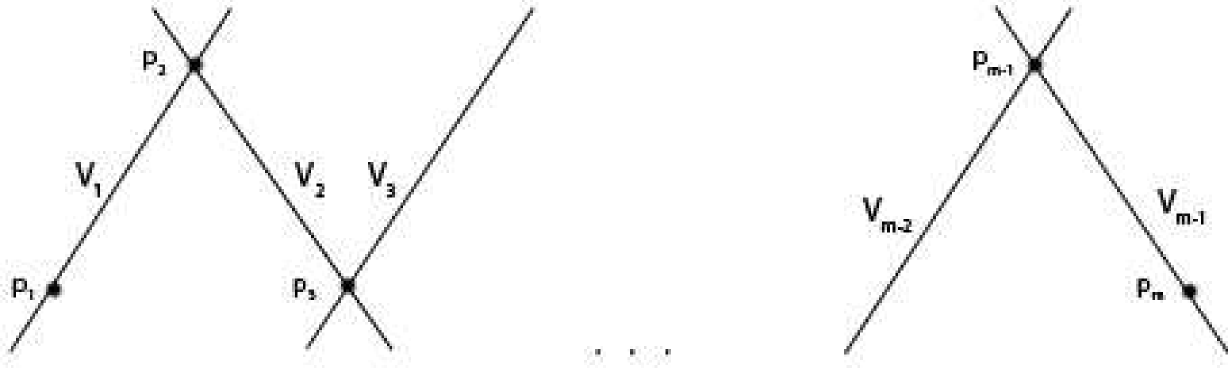

Figure 4.1: Configuration of the Springer fibre $\mathcal{B}_{\check{H},x}$

$x^{-1}(F_{m-1}) = \mathrm{Vect}(u_1, u_2, \dots, u_m) = U_0$. So we may identify V_1 with the projective space of lines in $\mathrm{Vect}(u_1, u_2)$. The point p_2 is $F_1 = \mathrm{Vect}(u_2)$. If $F_1 = \mathrm{Vect}(u_2) \subset \mathrm{Im}(x)$ then $x^{-1}(F_1) = \mathrm{Vect}(u_1, u_2, u_3)$ and V_2 can be identified with the space of lines in $x^{-1}(F_1)/F_1$. Inside $\mathrm{Vect}(u_1, u_2, u_3)$ one has a distinguished subspace $\mathrm{Vect}(u_1, u_2, u_3) \cap \mathrm{Im}(x) = \mathrm{Vect}(u_2, u_3)$. If F_2 is different from this subspace then the whole flag F_i is uniquely defined. So the point p_3 of V_2 corresponds to $F_2 = \mathrm{Vect}(u_2, u_3)$. If now $F_1 = \mathrm{Vect}(u_2)$ and $F_2 = \mathrm{Vect}(u_2, u_3)$ then $x^{-1}(F_2) = \mathrm{Vect}(u_1, u_2, u_3, u_4)$ and D_3 is the space of lines in $x^{-1}(F_2)/F_2$. The point p_4 of V_3 corresponds to $F_3 = \mathrm{Vect}(u_2, u_3, u_4)$, and one can continue the construction till F_m.

The points p_1 is the standard complete flag on U_0 and p_m is the flag $\mathrm{Vect}(u_2) \subset \mathrm{Vect}(u_2, u_3) \subset \dots \subset \mathrm{Vect}(u_2, \dots, u_m) \subset \mathrm{Vect}(u_1, \dots, u_m)$. □

This result combined with the Cellular fibration Lemma in [CG97, § 5.5] implies that $K^{\check{G}\times\mathbb{G}_m}(\mathcal{B}_{\check{H},x})$ is a free $\mathrm{R}(\check{G}\times\mathbb{G}_m)$-module of rank m. Moreover, the $\mathrm{R}(\check{H})$-module structure on $K^{\check{G}\times\mathbb{G}_m}(\mathcal{B}_{\check{H},x})$ comes from $\mathrm{Res}^{\sigma} : \mathrm{R}(\check{H}) \longrightarrow \mathrm{R}(\check{G}\times\mathbb{G}_m)$.

According to [CG97, Lemma 7.6.2] the assignment sending T_w to $s^{\ell(w)}$ for w in W_H, extends by linearity to an algebra homomorphism

$$\epsilon : \mathbb{H}_{W_H} \longrightarrow \mathbb{Z}[s, s^{-1}]$$

and it is known that the induced $\mathbb{H}_H$-module $\mathrm{Ind}_{\mathbb{H}_{W_H}}^{\mathbb{H}_H} \epsilon = \mathbb{H}_H \otimes_{\mathbb{H}_{W_H}} \epsilon$ is isomorphic to the polynomial representation [CG97, 7.6.8]. We have the following crucial set of isomorphisms of $\mathbb{Z}[s, s^{-1}]$-modules [CG97, Formula (7.6.5)]

$$K^{\check{H}\times\mathbb{G}_m}(T^*\mathcal{B}_{\check{H}}) \xrightarrow{\text{Thom}} K^{\check{H}\times\mathbb{G}_m}(\mathcal{B}_{\check{H}}) \xrightarrow{\alpha} \mathrm{R}(\check{T}_H)[s, s^{-1}] \xrightarrow{\beta} \mathrm{Ind}_{\mathbb{H}_{W_H}}^{\mathbb{H}_H} \epsilon,$$

where the first arrow is the Thom isomorphism [CG97, Theorem 5.4.16], the map α is the canonical isomorphism

$$K^{\check{H}\times\mathbb{G}_m}(\mathcal{B}_{\check{H}}) \xrightarrow{\sim} K^{\check{H}\times\mathbb{G}_m}(\check{H}/B_{\check{H}}) \xrightarrow{\sim} K^{B_{\check{H}}\times\mathbb{G}_m}(\mathrm{pt}) \xrightarrow{\sim} \mathrm{R}(\check{T}_H \times \mathbb{G}_m) \xrightarrow{\sim} \mathrm{R}(\check{T}_H)[s, s^{-1}],$$

and the map β is given for any λ by $\beta(e^{\lambda}) = e^{-\lambda}$.

There is a natural action of $\mathbb{H}_H$ on $K^{\check{G}\times\mathbb{G}_{\mathfrak{m}}}(\mathcal{B}_{\check{H},x})$ defined uniquely by the property that the inclusion of $\mathcal{B}_{\check{H},x}$ in $\mathcal{B}_{\check{H}}$ yields a $\mathrm{R}(\check{G}\times\mathbb{G}_{\mathfrak{m}})\otimes_{\mathrm{R}(\check{H}\times\mathbb{G}_{\mathfrak{m}})}\mathbb{H}_H$-equivariant surjection

$$\mathrm{R}(\check{G}\times\mathbb{G}_{\mathfrak{m}})\otimes_{\mathrm{R}(\check{H}\times\mathbb{G}_{\mathfrak{m}})}K^{\check{H}\times\mathbb{G}_{\mathfrak{m}}}(\mathcal{B}_{\check{H}})\widetilde{\longrightarrow}K^{\check{G}\times\mathbb{G}_{\mathfrak{m}}}(\mathcal{B}_{\check{H}})\longrightarrow K^{\check{G}\times\mathbb{G}_{\mathfrak{m}}}(\mathcal{B}_{\check{H},x}).$$

We are now going to construct a morphism

$$\mathfrak{J}: K^{\check{G}\times\mathbb{G}_m}(\mathcal{B}_{\check{H},x})\to K(DP_{I_H\times I_G}(\Pi(F))).$$

Consider the diagram

$$\begin{array}{ccc}\mathbb{H}_H & \xrightarrow{\gamma_1} & K(DP_{I_H\times I_G}(\Pi(F)))\\ \Big\downarrow{\scriptstyle\gamma_2} & & \\ K^{\check{G}\times\mathbb{G}_{\mathfrak{m}}}(\mathcal{B}_{\check{H},x}), & & \end{array}$$

where γ_1 sends $\mathcal{T}$ to $\overleftarrow{H}_H(\mathcal{T},I_0)$, and γ_2 sends $\mathcal{T}$ to the action of $\mathcal{T}$ on the structure sheaf $\mathcal{O}$ of $\mathcal{B}_{\check{H},x}$. Note that γ_1 and γ_2 are surjective. The morphism $\mathfrak{J}$ will be induced by γ_1. From formulas of Section 3.6, one sees that γ_1 factors through the surjective morphism $\bar{\gamma}_1 : \mathbb{H}_H\otimes_{\mathbb{H}_{W_H}}\epsilon\to K(DP_{I_H\times I_G}(\Pi(F)))$ of $\mathbb{H}_H$-modules.

Remark 4.3.2. *If $m=2$ then both $K(DP_{I_H\times I_G}(\Pi(F)))$ and $\mathbb{H}_H\otimes_{\mathbb{H}_{W_H}}\epsilon$ are free $R(\check{G}\times\mathbb{G}_{\mathfrak{m}})$-modules of rank 2, and $\bar{\gamma}_1$ is an isomorphism.*

Theorem 4.3.3. *The morphism γ_1 induces an isomorphism of $\mathbb{H}_H$-modules between $K^{\check{G}\times\mathbb{G}_{\mathfrak{m}}}(\mathcal{B}_{\check{H},x})$ and $K(DP_{I_H\times I_G}(\Pi(F)))$, where the $\mathbb{Z}[s,s^{-1}]$-module structure on $\overline{\mathbb{Q}}_\ell$ is given by the character sending s to $q^{1/2}$. Moreover, the modules $K^{\check{G}\times\mathbb{G}_{\mathfrak{m}}}(\mathcal{B}_{\check{H},x})$ and $K(DP_{I_H\times I_G}(\Pi(F)))$ are free of rank m over $\mathrm{R}(\check{G}\times\mathbb{G}_{\mathfrak{m}})$, and $\mathrm{R}(\check{H})$ acts via $\mathrm{Res}^\sigma:\mathrm{R}(\check{H})\longrightarrow\mathrm{R}(\check{G}\times\mathbb{G}_{\mathfrak{m}})$.*

Note that if $n=m=1$ then one has $I_H=H(\mathcal{O})$ and this theorem can be deduced from [Lys11, Proposition 4].

4.3.2 The proof of the conjecture

The first part of the theorem has essentially been already proved. Indeed:

1. We have seen in §3.6 that the module $K(DP_{I_G\times I_H}(\Pi(F)))$ is free of rank m over $\mathrm{R}(\check{G}\times\mathbb{G}_{\mathfrak{m}})$. In the notation of this section, a basis is given by the elements IC^k for $0\leq k\leq m-1$, and the action of $\mathrm{R}(\check{G}\times\mathbb{G}_{\mathfrak{m}})$ is given on this basis by (3.6.14). Besides, according to Theorem 3.6.15, $\mathrm{R}(\check{H})$ acts via Res^σ.
2. The corresponding properties for $K^{\check{G}\times\mathbb{G}_{\mathfrak{m}}}(\mathcal{B}_{\check{H},x})$ have been proved at the beginning of this section.

In the sequel we will construct a basis of this module (specialized at s by $q^{1/2}$) and we will identify the action of $\mathbb{H}_H$ on both bases. The morphism sending one basis to another will be induced by γ_2.

The representation of $\mathbb{H}_H$ in $\mathrm{R}(\check{T}_H)[s,s^{-1}]$

Consider the polynomial representation of the extended affine Hecke algebra $\mathbb{H}_H$ of H in $\mathrm{R}(\check{T}_H)[s,s^{-1}]$. For v in $\mathbb{H}_H$ and z in $\mathrm{R}(\check{T}_H)[s,s^{-1}]$ write $v * z$ for the action of v on z. Remind that, as in [§1.2.2, Chapter 1] e^λ denotes the element in $\mathrm{R}(\check{\mathrm{T}}_{\mathrm{H}})[s,s^{-1}]$ corresponding to λ; according to [CG97, Formula (7.6.1)], e^λ as an element of $\mathbb{H}_H$ acts on any element u of $\mathrm{R}(\check{T}_H)[s,s^{-1}]$ by

$$e^\lambda * u = e^{-\lambda}u, \tag{4.3.4}$$

and for any simple root α, the action of T_{s_α} on e^λ is given by the formula [CG97, Theorem 7.2.16]:

$$T_{s_\alpha} * e^\lambda = \frac{e^\lambda - e^{s_\alpha(\lambda)}}{e^\alpha - 1} - s^2\frac{e^\lambda - e^{s_\alpha(\lambda)+\alpha}}{e^\alpha - 1}. \tag{4.3.5}$$

This formula was discovered by Lusztig and was the starting point of the K-theoretic approach to Hecke algebras. The formulas (4.3.4) and (4.3.5) together completely determine the polynomial representation of $\mathbb{H}_H$. As mentioned in [§ 1.2.3 in Chapter 1], for λ dominant, the element e^λ corresponds in the Iwahori-Hecke algebra to the function $s^{-\ell(\lambda)}T_{t^\lambda}$, where $\ell(\lambda) = \langle\lambda, 2\check{\rho}_H\rangle$. Denote by ω_i the coweight $(1,\dots,1,0,\dots,0)$, where 1 appears i times. For $1 \le i < m$ there exists a unique permutation σ_i such that $w_i = t^{-\omega_i}\sigma_i$ is of length zero; σ_i is given by

$$\sigma_i(1) = i+1, \sigma_i(2) = i+2, \dots, \sigma_i(m-i) = m, \sigma_i(m-i+1) = 1, \dots, \sigma_i(m) = i.$$

The element w_1 is the generator of the group Ω_H of length zero elements in $\widetilde{W}_H$; for any i in $\mathbb{Z}$, $w_i = w_1^i$. In the extended affine Hecke algebra $\mathbb{H}_H$ we have

$$T_{t^{\omega_i}}T_{w_i} = T_{\sigma_i}.$$

Further we have $\ell(t^{\omega_i}) = \ell(\sigma_i) = \langle\omega_i, 2\check{\rho}_{\check{H}}\rangle = i(m-i)$ and this gives

$$e^{\omega_i} = s^{i(i-m)}T_{t^{\omega_i}}. \tag{4.3.6}$$

In $\mathrm{R}(\check{T}_H)[s,s^{-1}]$, $T_{\sigma_i} * 1 = s^{2i(m-i)}$ and this yields

$$(s^{i(m-i)}e^{\omega_i}T_{w_i}) * 1 = s^{2i(m-i)},$$

and

$$T_{w_i} * 1 = s^{i(m-i)}e^{\omega_i}. \tag{4.3.7}$$

Till now we have described the action of the Wakimoto objects and the elements of length zero. We are going to compute the action of the simple reflections $s_i = (i, i+1)$ and the affine simple reflection $s_m = t^\lambda w_0$, where $\lambda = (-1, 0\dots,0,1)$ and $w_0 = (1,m)$ is the longest element of the finite Weyl group of $\check{H}$. For $1 \le i \le m$ we have $T_{w_1}T_{s_i}T_w^{-1} = T_{s_{i+1}}$ and $T_{w_1}T_{s_m}T_{w_1}^{-1} = T_{s_1}$. For all integer j in $\mathbb{Z}$ set $s_j = s_{j+m}$ and rewrite the above formulas all together as

$$T_{w_1}T_{s_i}T_{w_1}^{-1} = T_{s_{i+1}}$$

Thus, for all i and j in $\mathbb{Z}$

$$T_{w_j}T_{s_i}T_{w_j}^{-1} = T_{s_{i+j}}.$$

For any cocharacter μ we have $w_it^\mu w_i^{-1} = t^{\sigma_i(\mu)}$ and we get

$$T_{w_i}T_{t^\mu}T_{w_i}^{-1} = T_{t^{\sigma_i(\mu)}}.$$

Proposition 4.3.8. *In the polynomial representation the element* T_{s_m} *acts on* 1 *by* $(s^2 - 1) + s^{2(m-1)} e^{\xi + \omega_1}$, *where* $\xi = (0, 0, \dots, 0, -1)$.

Proof. Since $T_{s_m} = T_{w_1}^{-1} T_{s_1} T_{w_1}$, we get using (4.3.7) :

$$T_{s_m} * 1 = (T_{w_1}^{-1} T_{s_1}) * s^{m-1} e^{\omega_1}.$$

Let $\alpha_i = (0, \dots, 0, 1, -1, 0 \dots, 0)$ and $\mu_i = (0, \dots, 0, 1, 0 \dots, 0)$, where 1 appears on i^{th} place then

$$T_{s_1} * e^{\omega_1} = e^{\omega_1 - \alpha_1}.$$

Thus,

$$T_{s_m} * 1 = T_{w_1}^{-1} * s^{m-1} e^{\omega_1 - \alpha_1} \tag{4.3.9}$$

If $\xi = -\sigma_1^{-1} \omega_1 = (0, \dots, 0, -1)$, then ξ is a dominant character, and we have $T_{t^\xi} T_{w_1}^{-1} = T_{\sigma_1}^{-1}$. Thus

$$T_{w_1}^{-1} = s^{1-m} e^{-\xi} T_{\sigma_1^{-1}}.$$

Finally we have to compute

$$T_{s_m} * 1 = s^{1-m} e^{-\xi} T_{\sigma_1^{-1}} * s^{m-1} e^{\omega_1 - \alpha_1}.$$

On one hand the reduced decomposition of σ_1^{-1} is $s_{m-1} \dots s_2 s_1$ and it follows that $T_{\sigma_1^{-1}} = T_{s_{m-1}} \dots T_{s_2} T_{s_1}$. From (4.3.5) we get that $T_{s_1} * e^{\mu_2} = (s^2 - 1) e^{\mu_2} + s^2 e^{\omega_1}$. For $2 \leq i \leq m - 1$ we have $T_{s_i} * e^{\omega_1} = s^2 e^{\omega_1}$. We also have $T_{s_2} * e^{\mu_2} = e^{\mu_2 - \alpha_2} = e^{\mu_3}$ and more generally, for $1 \leq i < m$, $T_{s_i} * e^{\mu_i} = e^{\mu_{i+1}}$. By induction we get

$$T_{\sigma_1^{-1}} * e^{\mu_2} = (s^2 - 1) e^{\mu_m} + s^{2(m-1)} e^{\omega_1}.$$

This implies that

$$T_{s_m} * 1 = (s^2 - 1) + s^{2(m-1)} e^{\xi + \omega_1} \tag{4.3.10}$$

□

In order to prove Theorem 4.3.3 we have to study the $\mathbb{H}_H$-module structure of $K^{\check{G} \times \mathbb{G}_m}(\mathcal{B}_{\check{H},x})$ and compare this action with the results obtained in Chapter 3. Building on a construction due to Lusztig [Lus02], we are going to construct a free family of vectors in $K^{\check{G} \times \mathbb{G}_m}(\mathcal{B}_{\check{H},x})$ which will be a basis after specializing s to $q^{1/2}$ and compute the $\mathbb{H}_H$-action on this basis.

Construction of a basis of $K^{\check{G} \times \mathbb{G}_m}(\mathcal{B}_{\check{H},x})$.

Let us denote by L_λ the line bundle on $\mathcal{B}_{\check{H}}$ corresponding to coweight λ of H as in [CG97, §6.1.11]. The $\check{H}$-module $\mathrm{H}^0(\mathcal{B}_{\check{H}}, L_\lambda)$ vanishes unless $(a_1 \leq \dots \leq a_m)$. Recall that the nilpotent subregular element x in $\mathrm{End}(U_0)$ is such that $x(u_1) = x(u_2) = 0$ and $x(u_i) = u_{i-1}$ for all $3 \leq i \leq m$. The natural morphism from $\mathrm{R}(\check{T}_H)[s, s^{-1}]$ to $K^{\check{G} \times \mathbb{G}_m}(\mathcal{B}_{\check{H},x})$ sends an element e^λ to $L_{-\lambda}$. Besides, any element $\mathcal{L}$ in $K^{\check{H} \times \mathbb{G}_m}(\tilde{\mathcal{N}}_{\check{H}})$ acts on $K^{\check{G} \times \mathbb{G}_m}(\mathcal{B}_{\check{H},x})$ as the tensor product by $\mathcal{L}_{|\mathcal{B}_{\check{H},x}}$.

Let $\{u_1, \dots u_m\}$ be the canonical basis of U_0 and $\{u_1^*, \dots, u_m^*\}$ the corresponding dual basis. For $1 \leq i \leq m$ set

$$U_i = \mathrm{Vect}(u_1, \dots, u_i)$$

and for $1 \leq i \leq m-1$ set

$$U_i' = \mathrm{Vect}(u_2, \dots, u_{i+1}),$$

with U_0' being equal to $\{0\}$. Note that for $0 \leq i \leq m-2$ the element x acts on U_{i+2}/U_i' by zero. For $1 \leq i < m$ let V_i be the projective line classifying flags

$$U_1' \subset \dots U_{i-1}' \subset W_i \subset U_{i+1} \subset \dots \subset U_m,$$

where W_i is i-dimensional. The line V_i is isomorphic to $\mathbb{P}(\mathrm{Vect}(u_1, u_{i+1}))$ via the map sending a line l to the flag given by

$$U_1' \subset \dots U_{i-1}' \subset l \oplus U_{i-1}' \subset U_{i+1} \subset \dots \subset U_m.$$

Then we have $\mathcal{B}_{\check{H},x} = \cup_i V_i$, (see Lemma 4.3.1). Recall that there are m fixed points on $\mathcal{B}_{\check{H},x}$ under the action of $\check{G} \times \mathbb{G}_m$ corresponding to the following flags:

1. $p_1 = U_1 \subset U_2 \subset \dots \subset U_m$.
2. For $2 \leq k \leq m-1$,
$$p_k = U_1' \subset \dots U_{k-1}' \subset U_k \subset \dots \subset U_m.$$
3. $p_m = U_1' \subset U_2' \subset \dots \subset U_{m-1}' \subset U_m$.

Note that for $2 \leq k \leq m-1$, the point p_k equals $V_{k-1} \cap V_k$.

Each line V_i is endowed with a tautological equivariant line bundle $\mathcal{O}_{V_i}(-1)$ which is an equivariant subbundle of $g\mathcal{O}_{V_i} \oplus s^{m-2i}\mathcal{O}_{V_i}$. Note that: for $1 \leq i \leq m-1$

$$\mathcal{O}_{V_i}(-p_i) = s^{2i-m}\mathcal{O}_{V_i}(-1) \quad \text{and} \quad \mathcal{O}_{V_i}(-p_{i+1}) = g^{-1}\mathcal{O}_{V_i}(-1).$$

Thanks to Lusztig [Lus02, §4.7] the elements $\mathcal{O}_{p_1}$, $\mathcal{O}_{V_1}(-1), \dots, \mathcal{O}_{V_{m-1}}(-1)$ define a basis of $K^{\check{G}\times\mathbb{G}_\mathrm{m}}(\mathcal{B}_{\check{H},x})$ over $\mathrm{R}(\check{G})[s, s^{-1}]$.

For $1 \leq i < m$, consider the line bundle L_{ω_i} on $\mathcal{B}_{\check{H},x}$ whose fibre at a point $F_1 \subset \dots \subset F_m$ is $\det(F_i)$. Remind that $\det(U_i') \xrightarrow{\sim} s^{i(m-i-1)}$ as a $\check{G} \times \mathbb{G}_\mathrm{m}$-representation. We also have $L_{\omega_m} = g\mathcal{O}$ in $K^{\check{G}\times\mathbb{G}_\mathrm{m}}(\mathcal{B}_{\check{H},x})$.

Proposition 4.3.11. *The set* $\{\mathcal{O}, L_{-\omega_1}, \dots L_{-\omega_{m-1}}\}$ *forms a basis of* $K^{\check{G}\times\mathbb{G}_\mathrm{m}}(\mathcal{B}_{\check{H},x}) \otimes_{Z[s,s^{-1}]} \overline{\mathbb{Q}}_\ell$.

Proof. For $1 \leq k \leq m-1$ and for any $\check{G} \times \mathbb{G}_\mathrm{m}$-equivariant line bundle L on $\mathcal{B}_{\check{H},x}$, we have the following equality in $K^{\check{G}\times\mathbb{G}_\mathrm{m}}(\mathcal{B}_{\check{H},x})$:

$$L = \sum_{j=1}^{k-1} L_{|V_j}(-p_{j+1}) + L_{|V_k} + \sum_{j=k+1}^{m-1} L_{|V_j}(-p_j)$$

We apply this formula to to L_{ω_k}. Note that :

- If $j < k$, $L_{\omega_k |V_j} = gs^{(k-1)(m-k)}\mathcal{O}_{V_j}$.
- If $j = k$, $L_{\omega_k |V_j} = \mathcal{O}_{V_j}(-1)$.
- If $j > k$, $L_{\omega_k |V_j} = s^{k(m-k-1)}\mathcal{O}_{V_j}$.

Hence we get:

$$L_{\omega_k} = \mathcal{O}_{V_k}(-1) + s^{(k-1)(m-k)}\Big[\sum_{j=1}^{k-1} \mathcal{O}_{V_j}(-1) + \sum_{j=k+1}^{m-1} s^{2(j-k)}\mathcal{O}_{V_j}(-1)\Big].$$

Lastly

$$\mathcal{O} = \mathcal{O}_{p_1} + \sum_{j=1}^{m-1} s^{2j-m}\mathcal{O}_{V_j}(-1).$$

Since $\mathcal{O}_{p_1}$, $\mathcal{O}_{V_1}(-1), \ldots, \mathcal{O}_{V_{m-1}}(-1)$ is a basis of $K^{\check{G}\times\mathbb{G}_m}(\mathcal{B}_{\check{H},x})$, the preceding formulas imply that $\mathcal{O}, L_{\omega_1} \ldots, L_{\omega_{m-1}}$ is a free family which becomes a basis after specializing s to $q^{1/2}$. If we apply the duality functor, we get the same result for the family $\mathcal{O}, L_{-\omega_1} \ldots, L_{-\omega_{m-1}}$.

□

We will now consider the family $\{\mathcal{O}, s^{m-1}L_{-\omega_1}, s^{2(m-2)}L_{-\omega_2}, \ldots, s^{m-1}L_{-\omega_{m-1}}\}$. Thanks to Proposition 4.3.11 this family is also a basis of $K^{\check{G}\times\mathbb{G}_m}(\mathcal{B}_{\check{H},x})$ after specialization. We are going to prove that γ_1 factors through morphism $\mathfrak{J}$ sending this basis to $\{\mathrm{IC}^0, \ldots, \mathrm{IC}^{m-1}\}$.

The action of length zero elements and simple reflections. According to (4.3.7), we have

$$\gamma_2(T_{w_i}) = T_{w_i}(\mathcal{O}) = s^{i(m-i)}e^{\omega_i} = s^{i(m-i)}L_{-\omega_i}.$$

Hence the action of length zero elements is compatible with [(3.6.18), Chapter 3]

The action of the affine simple reflection s_m. If λ be the cocharacter $(-1, 0 \ldots, 0, 1)$, and consider the associated line bundle L_λ (resp. E) on $\mathcal{B}_{\check{H},x}$ whose fibre over a flag $F_1 \subset \cdots \subset F_m = U_m$ is $F_1^* \otimes F_m/F_{m-1}$(resp. F_m/F_{m-1}). The section u_m of the line bundle E yields an exact sequence

$$0 \longrightarrow s^{2-m}\mathcal{O} \longrightarrow E \longrightarrow (L_{m-1,m})_{p_m} \longrightarrow 0.$$

Note that $(E)_{p_m} = g\mathcal{O}_{p_m}$, and $(L_{-\omega_1})_{p_m} = s^{2-m}\mathcal{O}_{p_m}$. Tensoring by $L_{-\omega_1}$, we get the exact sequence on $\mathcal{B}_{\check{H},x}$

$$0 \longrightarrow s^{2-m}L_{-\omega_1} \longrightarrow L_\lambda \longrightarrow gs^{2-m}\mathcal{O}_{p_m} \longrightarrow 0.$$

Consider $u_1^* \wedge \cdots \wedge u_{m-1}^*$ as global section of $L_{-\omega_{m-1}}$ over $\mathcal{B}_{\check{H},x}$. It vanishes only at p_m and gives an exact sequence

$$0 \longrightarrow g^{-1}s^{2-m}\mathcal{O} \longrightarrow L_{-\omega_{m-1}} \longrightarrow \mathcal{O}_{p_m} \longrightarrow 0.$$

Finally we conclude that in $K^{\check{G}\times\mathbb{G}_m}(\mathcal{B}_{\check{H},x})$

$$L_\lambda = s^{2-m}L_{-\omega_1} + gs^{2-m}\mathcal{O}_{p_m}, \quad \text{and} \quad gs^{2-m}\mathcal{O}_{p_m} = gs^{2-m}L_{-\omega_{m-1}} - s^{4-2m}\mathcal{O}.$$

Thus

$$L_\lambda = s^{2-m}L_{-\omega_1} + gs^{2-m}L_{-\omega_{m-1}} - s^{4-2m}\mathcal{O}.$$

From Proposition 4.3.8 we obtain that

$$\gamma_2(T_{s_m}) = T_{s_m}(\mathcal{O}) = (s^2 - 1)\mathcal{O} + s^{2m-2}L_\lambda = -\mathcal{O} + s^mL_{-\omega_1} + gs^mL_{-\omega_{m-1}}. \tag{4.3.12}$$

Finally, $s^{-1}T_{s_m}(\mathcal{O}) + s^{-1}\mathcal{O}$ corresponds to $\overleftarrow{H}_H(L_{s_m}, I_0)$, and the formula (4.3.12) is compatible with 3.6.17 by using the fact that L_{s_m} is isomorphic to $\bar{\mathbb{Q}}_\ell[1](\frac{1}{2})$ over $\overline{\mathcal{F}l}_H^{s_m}$. Moreover, for $1 \leq i < m$ one has $T_{s_i} * 1 = v$ in the polynomial representation, hence $T_{s_i}(\mathcal{O}) = v\mathcal{O}$ in $K^{\check{G}\times\mathbb{G}_m}(\mathcal{B}_{\check{H},x})$. The other relations are readily obtained by symmetry (the action of elements of length zero).

Appendix A

Generalities on convolution actions

A.0.3

For generalities on the K-theory we refer the reader to [CG97]. We work over an algebraically closed field of characteristic zero.

Let G be a linear algebraic group, and X be a scheme of finite type. Denote by $\mathrm{R}(G)$ the ring of representation of G. If Y is a smooth G-variety and $\pi : Y \to X$ is a proper G-equivariant map then according to [CG97, 5.2.20] $K^G(Y \times_X Y)$ is an associative $\mathrm{R}(G)$-algebra.

Let Z be a smooth variety and consider a G-equivariant morphism from Z to X. Then $K^G(Y \times_X Y)$ acts on $K^G(Z \times_X Y)$ by convolutions on the right. This action is $\mathrm{R}(G)$-linear. Namely, for any F in $K^G(Z \times_X Y)$ and any L in $K^G(Y \times_X Y)$, consider the element $p_{12}^* F \boxtimes p_{34}^* L$ in $K^G((Z \times_X Y) \times (Y \times_X Y))$. Let us apply the restriction with supports functor with respect to the smooth closed embedding $\mathrm{id} \times \mathrm{diag} \times \mathrm{id}$ in the following diagram to $p_{12}^* F \boxtimes p_{34}^* L$

$$\begin{array}{ccc} Z \times Y \times Y & \overset{\mathrm{id}\times\mathrm{diag}\times\mathrm{id}}{\to} & Z \times Y \times Y \times Y \\ \cup & & \cup \\ Z \times_X Y \times_X Y & \to & (Z \times_X Y) \times (Y \times_X Y) \end{array},$$

and denote the result by $p_{12}^* F \otimes p_{23}^* L$ in $K^G(Z \times_X Y \times_X Y)$. The projection $p_{13} : Z \times_X Y \times_X Y \to Z \times_X Y$ is proper, and we obtain the convolution product of F and L denoted by

$$F * L = (p_{13})_* (p_{12}^* F \otimes p_{23}^* L) \in K^G(Z \times_X Y).$$

A.0.4

Let H be a closed subgroup of G and assume Y' to be smooth. Let $\pi' : Y' \to X'$ be a proper morphism of H-varieties. Assume that $X \xrightarrow{\sim} G \times_H X'$ and $Y \xrightarrow{\sim} G \times_H Y'$ as G-varieties, and that π is obtained from π' by induction in the sense of [CG97, 5.2.16]. Then $Y \times_X Y \xrightarrow{\sim} G \times_H (Y' \times_{X'} Y')$ as G-varieties. So, we have an isomorphism of stack quotients $(Y \times_X Y)/G \xrightarrow{\sim} (Y' \times_{X'} Y')/H$, and we get an isomorphism of algebras

$$K^G(Y \times_X Y) \xrightarrow{\sim} K^H(Y' \times_{X'} Y')$$

A.0.5

Let Y be a smooth G-variety and $\pi : Y \to X$ a proper G-equivariant morphism. Let $X \to \bar{X}$ and $Z \to \bar{X}$ be G-equivariant morphisms of varieties. Assume Z to be smooth. Then $K^G(Y \times_X Y)$ acts on the left by convolution on $K^G(Y \times_{\bar{X}} Z)$. Indeed, for any F in $K^G(Y \times_{\bar{X}} Z)$ and L in $K^G(Y \times_X Y)$, consider $p_{12}^* L \boxtimes p_{34}^* F$ in $K^G((Y \times_X Y) \times (Y \times_{\bar{X}} Z))$. Apply the restriction with supports with respect to the smooth closed embedding $\mathrm{id} \times \mathrm{diag} \times \mathrm{id}$ in the following diagram to $p_{12}^* L \boxtimes p_{34}^* F$

$$\begin{array}{ccc} Y \times Y \times Z & \overset{\mathrm{id} \times \mathrm{diag} \times \mathrm{id}}{\to} & Y \times Y \times Y \times Z \\ \cup & & \cup \\ Y \times_X Y \times_{\bar{X}} Z & \to & (Y \times_X Y) \times (Y \times_{\bar{X}} Z). \end{array}$$

and denote the result by $p_{12}^* L \otimes p_{34}^* F$ in $K^G(Y \times_X Y \times_{\bar{X}} Z)$. The projection $p_{13} : Y \times_X Y \times_{\bar{X}} Z \to Y \times_{\bar{X}} Z$ is proper, and we obtain the convolution product of L and F denoted by

$$L * F = (p_{13})_* (p_{12}^* L \otimes p_{34}^* F) \in K^G(Y \times_{\bar{X}} Z).$$

Note actually that the essential thing we need is the fact that the structure sheaf $\mathcal{O}_Y$ of the diagonal $Y \subset Y \times Y$ admits a finite G-equivariant resolution by coherent locally free $\mathcal{O}_{Y \times Y}$-modules. Then restrict this resolution with respect to the flat projection $p_{23} : Y \times Y \times Y \times Z \to Y \times Y$. It seems that the smoothness of Z is not necessary. Moreover, assume Z to be smooth and $Z \to \bar{X}$ to be proper, then $K^G(Z \times_{\bar{X}} Z)$ acts on $K^G(Y \times_{\bar{X}} Z)$ by convolutions on the right, and the actions of $K^G(Z \times_{\bar{X}} Z)$ and of $K^G(Y \times_X Y)$ commute.

A.0.6

Let Y be a smooth G-variety, and $\pi : Y \to X$ be a proper G-equivariant morphism. Then $K^G(Y)$ is naturally a left module over $K^G(Y \times_X Y)$. Namely, for any L in $K^G(Y \times_X Y)$ and F in $K^G(Y)$, consider the restriction with supports of $L \boxtimes F$ an element of $K^G((Y \times_X Y) \times Y)$ with respect to the smooth closed embedding

$$\begin{array}{ccc} Y \times Y & \overset{\mathrm{id} \times \mathrm{diag}}{\to} & Y \times Y \times Y \\ \cup & & \cup \\ Y \times_X Y & \to & (Y \times_X Y) \times Y \end{array}$$

and denote the result by $L \otimes p_2^* F \in K^G(Y \times_X Y)$. Then we have $L * F = (p_1)_*(L \otimes p_2^* F) \in K^G(Y)$.

Appendix B

Weak geometric analogue of Jacquet Functors and compatibility with convolution product

Let $\mathbf{k}$ be an algebraically closed field of characteristic $p > 2$, F be the field $\mathbf{k}((t))$, and $\mathcal{O}$ its ring of integers $\mathbf{k}[[t]]$. Let G be a reductive connected group over $\mathbf{k}$, T be a maximal torus of G and fix a Borel subgroup B in G containing T. Denote by I_G the corresponding Iwahori subgroup. Let P be a parabolic subgroup of G containing B and U its unipotent radical. Let L be the Levi subgroup of P isomorphic to P/U. Let M_0 be a faithful representation of G, and let $M = M_0 \otimes_{\mathbf{k}} (\mathcal{O})$.

In the classical setting, an important tool is the Jacquet module $\mathcal{S}(M(F))_{U(F)}$ of coinvariants with respect to $U(F)$. We will define a weak analog of Jacquet functors in the geometric setting. Let V_0 be a P-stable subspace of M_0 endowed with a trivial action of U. Set $V = V_0 \otimes_{\mathbf{k}} (\mathcal{O})$, we have a surjective map of $L(F)$-representations

$$\mathcal{S}(M(F))_{U(F)} \longrightarrow \mathcal{S}(V(F))$$

given by restriction under the inclusion $V(F) \hookrightarrow M(F)$. We will geometrize the composition

$$\mathcal{S}(M(F)) \longrightarrow \mathcal{S}(M(F))_{U(F)} \longrightarrow \mathcal{S}(V(F)),$$

and we will show that in the Iwahori case, geometric Jacquet functors do not commute with the action of the entire Iwahori-Hecke algebra of G but they commute with the action of the Iwahori-Hecke algebra of the Levi subgroup L of P.

Let I_P be the preimage of the Borel subgroup B under the map $P(\mathcal{O}) \to P$. Denote by B_L the image of B in L. It is a Borel subgroup of L. In the same way consider the map $L(\mathcal{O}) \to L$ and denote by I_L the preimage of B_L under this map in $L(\mathcal{O})$. Hence I_L is an Iwahori subgroup of $L(F)$. Finally we have a diagram

$$I_L \longleftarrow I_P \hookrightarrow I_G,$$

where the first map is induced by the natural projection $P(\mathcal{O}) \to L(\mathcal{O})$. According to [§ 2.2.1, Chapter 2], the categories $D_{I_G}(M(F))$ and $D_{I_L}(V(F))$ are well defined. We are going to define the following functors

$$J_P^*, J_P^! : D_{I_G}(M(F)) \longrightarrow D_{I_L}(V(F)).$$

Let N, r be two integers such that $N + r \geq 0$. Set $V_{N,r} = t^{-N}V/t^rV$. Denote by $i_{N,r}$ the natural closed embedding of $V_{N,r}$ in $M_{N,r}$. For any $s \geq 0$, let K_s be the quotient of I_G by the kernel of the map $G\mathcal{O}) \to G(\mathcal{O}/t^s\mathcal{O})$. Let $I_{P,s}$ denote the image of I_P under the inclusion

$$I_P \hookrightarrow P(\mathcal{O}) \longrightarrow P(\mathcal{O}/t^s\mathcal{O}).$$

Similarly let $I_{L,s}$ be the image of I_L under $L(\mathcal{O}) \to L(\mathcal{O}/t^s\mathcal{O})$. We have the following diagram

$$\begin{array}{ccccc} L(\mathcal{O}/t^s\mathcal{O}) & \longleftarrow & P(\mathcal{O}/t^s\mathcal{O}) & \longrightarrow & G(\mathcal{O}/t^s\mathcal{O}) \\ \uparrow & & \uparrow & & \uparrow \\ I_{L,s} & \longleftarrow & I_{P,s} & \longrightarrow & K_s. \end{array}$$

For $s \geq N + r$, we obtain a digram of stack quotients

$$\begin{array}{ccccc} I_{P,s}\backslash V_{N,r} & \xrightarrow{i_{N,r}} & I_{P,s}\backslash M_{N,r} & \xrightarrow{p} & K_s\backslash M_{N,r} \\ \downarrow{\scriptstyle q} & & & & \\ I_{L,s}\backslash V_{N,r}, & & & & \end{array}$$

where p comes from the closed inclusion $I_{P,s} \hookrightarrow I_{G,s}$. Set a equal to $\dim M_0 - \dim V_0$.

Proposition B.0.1. *There exists two well-defined functors*

$$J_P^*, J_P^! : D_{I_G}(M(F)) \longrightarrow D_{I_L}(V(F)). \tag{B.0.2}$$

Proof. For any $s \geq N + r$, we have the following functors:

$$J_{P,N,r}^*, J_{P,N,r}^! : D_{K_s}(M_{N,r}) \longrightarrow D_{I_{L,s}}(V_{N,r})$$

by

$$q^* \circ J_{P,N,r}^*[\dim .\mathrm{rel}(q)] = (i_{N,r})^* p^*[\dim .\mathrm{rel}(p) - ra]$$
$$q^* \circ J_{P,N,r}^![\dim .\mathrm{rel}(q)] = (i_{N,r})^! p^*[\dim .\mathrm{rel}(p) + ra].$$

The sequence

$$1 \longrightarrow U(\mathcal{O}/t^s\mathcal{O}) \longrightarrow I_{P,s} \longrightarrow I_{L,s} \longrightarrow 1$$

is exact. Hence the functor

$$q^*[\dim .\mathrm{rel}(q)] : D_{I_{L,s}}(V_{N,r}) \longrightarrow D_{I_{P,s}}(V_{N,r})$$

is an equivalence of categories and exact for perverse t-structure. The functors $J_{P,N,r}^*$ and $J_{P,N,r}^!$ are well-defined. They are compatible with the transition functors in the ind-system of categories defining $D_{I_G}(M(F))$ and $D_{I_L}(L(F))$ defined in [§ 2.2.1 , Chapter 2]. By the taking the inductive 2-limit, we obtain the two well-defined functors J_P^* and $J_P^!$ (B.0.2) which do not depend on the choice of a section of $P \to P/U$. The Verdier duality functor $\mathbb{D}$ exchanges J_P^* and $J_P^!$, i.e. we have canonically

$$\mathbb{D} \circ J_P^* \xrightarrow{\sim} J_P^! \circ \mathbb{D}.$$

□

As in the case of the affine flag variety, we can define the $\mathbf{k}$-space quotient $P(F)/I_P$ and define $\mathcal{F}\ell_P$ to be the sheaf associated to this presheaf in fpqc-topology. It is an ind-scheme. Let X be a projective smooth connected curve over the field $\mathbf{k}$. Let x be a closed point in X and X^* be equal $X - \{x\}$. Denote by $\mathcal{O}_x$ the completion of the local ring of X at x and by F_x its field of fractions. We choose a local coordinate at the point x, denoted by t and we may identify $\mathcal{O}_x = \mathbf{k}[[t]]$ and $F_x = \mathbf{k}((t))$. Let $D = \mathrm{Spec}(\mathbf{k}[[\mathrm{t}]])$ and $D^* = \mathrm{Spec}(\mathbf{k}((t)))$. Then $\mathcal{F}\ell_P$ classifies $(\mathcal{F}_P, \beta, \epsilon)$, where $\mathcal{F}_P$ is a P-torsor on D, the map β is a trivialization of $\mathcal{F}_P$ over D^*, and ϵ is a reduction of $\mathcal{F}_P|_x$ to a B-torsor. We have the diagram

$$\mathcal{F}\ell_L \xleftarrow{\mathfrak{t}_L} \mathcal{F}\ell_P \xrightarrow{\mathfrak{t}_P} \mathcal{F}\ell_G, \tag{B.0.3}$$

where $\mathfrak{t}_P$ (resp. $\mathfrak{t}_L$) is given by extension of scalars with respect to $P \hookrightarrow G$ (resp. $P \to L$). Let $\mathcal{F}\ell_{P,G}$ be the $P(F)$-orbit through 1 in $\mathcal{F}\ell_G$ viewed as an ind-subscheme with a reduced scheme structure. The reduced ind-scheme $\mathcal{F}\ell_{P,\mathrm{red}}$ gives a stratification of $\mathcal{F}\ell_{P,G}$.

Lemma B.0.4. *There exists a well-defined geometric restriction functor*

$$\mathrm{gRes} : D_{I_G}(\mathcal{F}\ell_G) \longrightarrow D_{I_L}(\mathcal{F}\ell_L).$$

Proof. For $s_1, s_2 \geq 0$, let ${}_{s_1,s_2}P(F) = P(F) \cap {}_{s_1,s_2}G(F)$, and ${}_{s_1,s_2}\mathcal{F}l_P = {}_{s_1,s_2}P(F)/I_P$, , where ${}_{s_1,s_2}G(F)$ was defined in (2.2.6) in Chapter 2. The ind-scheme ${}_{s_1,s_2}\mathcal{F}l_P$ is a closed subscheme of $\mathcal{F}\ell_P$. Similarly we define

$${}_{s_1,s_2}L(F) := \{g \in L(F) | t^{s_1}V \subset gV \subset t^{-s_2}V\},$$

and ${}_{s_1,s_2}\mathcal{F}l_L = {}_{s_1,s_2}L(F)/I_L$. Thus the map $\mathfrak{t}_L$ in (B.0.3) induces a morphism (denoted again by $\mathfrak{t}_L$) from ${}_{s_1,s_2}\mathcal{F}l_P$ to ${}_{s_1,s_2}\mathcal{F}l_L$. For $s \geq s_1 + s_2 + 1$ we have a diagram of stack quotients

$$\begin{array}{ccc}
I_{L,s}\backslash({}_{s_1,s_2}\mathcal{F}l_L) & \xleftarrow{q_L} & I_{P,s}({}_{s_1,s_2}\mathcal{F}l_L) \\
 & & \uparrow \mathfrak{t}_L \\
 & & I_{P,s}\backslash({}_{s_1,s_2}\mathcal{F}l_P) \\
 & & \downarrow \mathfrak{t}_P \\
.K_s\backslash({}_{s_1,s_2}\mathcal{F}l_G) & \xleftarrow{\xi} & I_{P,s}\backslash_{s_1,s_2}\mathcal{F}l_G
\end{array}$$

Moreover, the functor

$$q_L^*[\dim .rel(q_L)] : D_{I_{L,s}}({}_{s_1,s_2}\mathcal{F}l_L) \longrightarrow D_{I_{P,s}}({}_{s_1,s_2}\mathcal{F}l_L)$$

is an equivalence of categories and exact for perverse t-structure. For any perverse sheaf K extension by zero from ${}_{s_1,s_2}\mathcal{F}l_G$ to $\mathcal{F}l_G$, we may define $\mathrm{gRes}(K)$ by the isomorphism

$$q_L^*\mathrm{gRes}(K)[\dim .\mathrm{rel}(q_L)] \xrightarrow{\sim} (\mathfrak{t}_{L!})\mathfrak{t}_P^*\xi^*K[\dim .\mathrm{rel}(\xi)].$$

□

Lemma B.0.5. *For any dominant cocharacter λ of G, we have*

$$\mathrm{gRes}(L_{t^\lambda !}) \xrightarrow{\sim} L_{t^\lambda !}[-\langle \lambda, 2(\check{\rho}_G - \check{\rho}_L)\rangle],$$

where $\check{\rho}_G$ (resp. $\check{\rho}_L$) denote the half sum of positive roots of G (resp. positive roots of L).

Proof. Let U_B be the unipotent radical of B. The space $\mathcal{F}\ell_G^{t^\lambda}$ is the $U_B(\mathcal{O})$-orbit through $t^\lambda I_G$ on $\mathcal{F}\ell_G$. Thus $L_{t^\lambda !}$ is the extension by zero from a connected component of $\mathcal{F}\ell_P$. The map $U_B t^\lambda I_P/I_P \longrightarrow \mathcal{F}l_L^{t^\lambda}$ is a trivial affine fibration with affine fibre of dimension $\langle \lambda, 2(\check{\rho}_G - \check{\rho}_L)\rangle$. Hence we get

$$\mathrm{gRes}(L_{t^\lambda !}) \widetilde{\longrightarrow} L_{t^\lambda !}[-\langle \lambda, 2(\check{\rho}_G - \check{\rho}_L)\rangle].$$

□

Lemma B.0.6. *For any w in the finite Weyl group of L, we have*

$$\mathrm{gRes}(L_w) \widetilde{\longrightarrow} L_w, \quad \mathrm{gRes}(L_{w!}) \widetilde{\longrightarrow} L_{w!}, \quad \mathrm{gRes}(L_{w*}) \widetilde{\longrightarrow} L_{w*}.$$

Proof. The P-orbit through I_P gives a natural closed subscheme $L/B_L \widetilde{\longrightarrow} P/B \hookrightarrow \mathcal{F}l_P$. For any w in the finite Weyl group of L, the double coset BwB is contained in P. Thus $L_{w!}$ initially defined over $\mathcal{F}\ell_G$ is actually an extension by zero from a connected component of $\mathcal{F}\ell_P$. Hence $\mathrm{gRes}(L_{w!}) \widetilde{\longrightarrow} L_{w!}$. Moreover, $\overline{BwB/B}$ in G/B actually lies in P/B, so $\mathrm{gRes}(L_w) \widetilde{\longrightarrow} L_w$. The same result holds for L_{w*}. □

The following proposition will show that the functors $J_P^!, J_P^*$ commute with the action of the subalgebra $\mathcal{H}_{I_L}$ of $\mathcal{H}_{I_G}$. Remind that the subalgebra structure of $\mathcal{H}_{I_L}$ has been defined in [§3.5.2, Chapter 3].

Proposition B.0.7. *Let $\mathcal{T}$ be a perverse sheaf in $P_{I_G}(\mathcal{F}\ell_G)$ which is the extension by zero from a connected component of $\mathcal{F}\ell_P$, and $\mathcal{K}$ be in $D_{I_G}(M(F))$ then we have*

$$J_P^* \overleftarrow{H}_G(\mathcal{T}, \mathcal{K}) \widetilde{\longrightarrow} \overleftarrow{H}_L(\mathrm{gRes}(\mathcal{T}), J_P^* \mathcal{K})[\langle \lambda, \check{\nu} - \check{\mu}\rangle],$$

where λ is the cocharacter whose image in $\pi_1(L)$ is θ, $\check{\nu}$ is the character by which L acts on $\det(V_0)$ *and $\check{\mu}$ is the character by which G acts on* $\det(M_0)$.

Proof. The connected components of $\mathcal{F}\ell_P$ are indexed by $\pi_1(L)$. For θ in $\pi_1(L)$, denote by $\mathcal{F}\ell_P^\theta$ for the corresponding connected component which is the preimage of $\mathcal{F}l_L^\theta$ under the map $\mathfrak{t}_L$ defined in (B.0.3).

Let s_1 and s_2 be two non negative integers and $\mathcal{T}$ be the extension by zero from ${}_{s_1,s_2}\mathcal{F}l_P^\theta = {}_{s_1,s_2}\mathcal{F}l_P \cap \mathcal{F}\ell_P^\theta$. For $N + r \geq 0$ and $s \geq \max\{N + r, s_1 + s_2 + 1\}$ consider the diagram

$$\begin{array}{ccccc}
 & & V_{N,r} \times {}_{s_1,s_2} P(F) & \xrightarrow{act} & V_{N+s_1,r-s_1} \\
 & & \downarrow{\scriptstyle q_P} & & \downarrow{\scriptstyle q_U} \\
V_{N,r} & \xleftarrow{pr} & V_{N,r} \times {}_{s_1,s_2} \mathcal{F}l_P^{\theta} & \xrightarrow{act_{q,P}} & I_{P,s}\backslash V_{N+s_1,r-s_1} \\
\downarrow{\scriptstyle i_{N,r}} & & \downarrow{\scriptstyle i_{N,r}\times id} & & \downarrow{\scriptstyle i_{N+s_1,r-s_1}} \\
M_{N,r} & \xleftarrow{pr} & M_{N,r} \times {}_{s_1,s_2} \mathcal{F}l_P^{\theta} & \xrightarrow{act_{q,P}} & I_{P,s}\backslash M_{N+s_1,r-s_1} \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle p} \\
M_{N,r} & \xleftarrow{pr} & M_{N,r} \times {}_{s_1,s_2} \mathcal{F}l_G^{\theta} & \xrightarrow{act_q} & K_s\backslash M_{N+s_1,r-s_1},
\end{array}$$

where the map act sends (m,p) to $p^{-1}m$, the map q_P sends (m,p) to (m, pI_P), and q_U is the stack quotient under the action of $I_{P,s}$.
Moreover, the second line of this diagram fits in the following diagram

$$\begin{array}{ccc}
V_{N,r} \times {}_{s_1,s_2} \mathcal{F}l_P^{\theta} & \xrightarrow{act_{q,P}} & I_{P,s}\backslash V_{N+s_1,r-s_1} \\
\downarrow{\scriptstyle id\times \mathfrak{t}_L} & & \downarrow{\scriptstyle q} \\
V_{N,r} \times {}_{s_1,s_2} \mathcal{F}l_L^{\theta} & \xrightarrow{act_{q,P}} & I_{L,s}\backslash V_{N+s_1,r-s_1}.
\end{array}$$

At the level of reduced ind-schemes the map ${}_{s_1,s_2}\mathcal{F}l_P^{\theta} \longrightarrow \mathcal{F}l_G$ is a locally closed embedding, thus the perverse sheaf $\mathcal{T}$ may be viewed as a complex over ${}_{s_1,s_2}\mathcal{F}l_P^{\theta}$. For a given $\mathcal{K}$ and large enough N, r, by definition, up to a shift independent of $\mathcal{K}$ and $\mathcal{T}$ we have

$$\overleftarrow{H}_G(\mathcal{T}, K) \widetilde{\longrightarrow} pr_!(act_{q,P}^*(K) \otimes pr_2^*(\mathcal{T})),$$

where $pr : K_s\backslash(M_{N,r}\tilde{\times}_{s_1,s_2}\mathcal{F}l_G) \longrightarrow K_s\backslash_{s_1,s_2}\mathcal{F}l_G$ is defined in [§ 2.2.2, Chapter 2]. Thus by definition of gRes and J_P^* and the commutativity of the diagram above, we get

$$J_P^*\overleftarrow{H}_G(\mathcal{T}, \mathcal{K}) \widetilde{\longrightarrow} \overleftarrow{H}_L(\mathrm{gRes}(\mathcal{T}), J_P^*\mathcal{K})[?],$$

To determine the shift, one may consider the following speacial case, wehre $\mathcal{K}$ is the constant perverse sheaf I_0 on M and $\mathcal{T}$ equals $L_{t^{\lambda}!}$, where λ is a dominant cocharacter of G. For large enough N, r, we have the following diagram

$$\begin{array}{ccc}
M_{N,r} & \xleftarrow{\alpha_M} & M_{0,r}\tilde{\times}\mathcal{F}l_P^{t^{\lambda}} \\
\uparrow{\scriptstyle i_{N,r}} & & \uparrow \\
V_{N,r} & \xleftarrow{\alpha_V} & V_{0,r}\tilde{\times}\mathcal{F}l_P^{t^{\lambda}}.
\end{array} \tag{B.0.8}$$

Remind that $\mathcal{F}l_P^{t^\lambda}$ is the $U_B(\mathcal{O})$-orbit through $t^\lambda I_P$ in $\mathcal{F}l_P$. The scheme $M_{0,r}\tilde{\times}\mathcal{F}l_P^{t^\lambda}$ is the scheme classifying pairs (gI_P, m), where gI_P is in $\mathcal{F}l_P$ and m is in gM/t^rM. Similarly the scheme $V_{0,r}\tilde{\times}\mathcal{F}l_P t^\lambda$ is the scheme classifying pairs (gI_P, v), where gI_P is in $\mathcal{F}l_P^{t^\lambda}$ and v is in gV/t^rV. For large enough r, we have $gV \cap t^rM = t^rV$. So the right vertical arrow in Diagram (B.0.8) is a closed immersion. Denote by IC the IC-sheaf of $M_{0,r}\tilde{\times}\mathcal{F}l_P^{t^\lambda}$. We have canonically

$$\overleftarrow{H}_G(L_{t^\lambda!}, I_0) \widetilde{\longrightarrow} \alpha_{M!}(\mathrm{IC}),$$

and additionally

$$\dim(M_{0,r}\tilde{\times}\mathcal{F}l_P^{t^\lambda}) = \langle\lambda, 2\check{\rho}_G\rangle + r\dim M_0 - \langle\lambda, \check{\mu}\rangle,$$

Hence we have

$$\begin{aligned} J_P^*\overleftarrow{H}_G(L_{t^\lambda!}, I_0) &\widetilde{\longrightarrow} i_{N,r}^*\alpha_{M!}(\mathrm{IC})[-ra] \\ &\widetilde{\longrightarrow} \alpha_{V!}\overline{\mathbb{Q}}_\ell[\langle\lambda, 2\check{\rho}_G\rangle + r\dim V_0 - \langle\lambda, \check{\mu}\rangle]. \end{aligned} \tag{B.0.9}$$

The map α_V factors through

$$V_{0,r}\tilde{\times}\mathcal{F}\ell_P^{t^\lambda} \longrightarrow V_{0,r}\tilde{\times}\mathcal{F}\ell_L^{t^\lambda} \xrightarrow{\alpha_L} V_{N,r}, \tag{B.0.10}$$

where the first map is a trivial affine fibration with an affine fibre of dimension $\langle\lambda, 2(\check{\rho}_G - \check{\rho}_L)\rangle$. This gives us

$$\dim(V_{0,r}\tilde{\times}\mathcal{F}\ell_L^{t^\lambda}) = \langle\lambda, 2\check{\rho}_L\rangle + r\dim V_0 - \langle\lambda, \check{\nu}\rangle.$$

By definition we have

$$\overleftarrow{H}_L(L_{t^\lambda!}, I_0) \widetilde{\longrightarrow} \alpha_{L!}(\mathrm{IC}). \tag{B.0.11}$$

This gives us the desired shift as follows

$$\begin{aligned} J_P^*\overleftarrow{H}_G(L_{t^\lambda!}, I_0) &\widetilde{\longrightarrow} \alpha_{V!}\overline{\mathbb{Q}}_\ell[\langle\lambda, 2\check{\rho}_G\rangle + r\dim V_0 - \langle\lambda, \check{\mu}\rangle] \\ &\widetilde{\longrightarrow} \alpha_{L!}\overline{\mathbb{Q}}_\ell[\langle\lambda, \check{\nu} - \check{\mu} - 2(\check{\rho}_G - \check{\rho}_L)\rangle] \\ &\widetilde{\longrightarrow} \overleftarrow{H}_L(L_{t^\lambda!}, I_0)[\langle\lambda, \check{\nu} - \check{\mu} - 2(\check{\rho}_G - \check{\rho}_L)\rangle] \\ &\widetilde{\longrightarrow} \overleftarrow{H}_L(\mathrm{gRes}(L_{t^\lambda!}), I_0)[\langle\lambda, \check{\nu} - \check{\mu}\rangle], \end{aligned} \tag{B.0.12}$$

where the first isomorphism is due to (B.0.9), the second is due to (B.0.10), the third is due to (B.0.11) and the last one is due to Lemma B.0.5. This shift is compatible with [Lys11, Lemma 5]. □

Let $\delta_U : \mathbb{G}_\mathfrak{m} \times M_0 \to M_0$ be a linear action, whose fixed points set is V_0. Assume δ_U contracts M_0 onto V_0. Let r be a integer, denote by $\nu : \mathbb{G}_m \to L$ the cocharacter of the center of L acting on V by $x \to x^r$. Now consider the case where δ_U is the map sending x in $\mathbb{G}_\mathfrak{m}$ to $\nu(x)x^{-r}$. For any x in $\mathbb{G}_\mathfrak{m}$ and m in $M_{N,r}$ consider the action of $\mathbb{G}_\mathfrak{m}$ on $M_{N,r}$ defined by $(x, m) = xm$. Let K be a $\mathbb{G}_\mathfrak{m}$-equivariant perverse sheaf in $P_{I_G}(M(F))$ which is with respect to this action of $\mathbb{G}_m$ on $M_{N,r}$. Then for any w in $\widetilde{W}_G$, both K and $\overleftarrow{H}_G(L_w, K)$ are $\mathbb{G}_\mathfrak{m}$-equivariant with respect to the δ_U-action on $M(F)$. We get a new version of Proposition B.0.7 as follows:

Corollary B.0.13. *Let K be a $\mathbb{G}_\mathfrak{m}$-equivariant perverse sheaf in $P_{I_G}(M(F))$ for the δ_U-action on $M_{N,r}$, for N, r large enough. Assume that $\mathbf{k}$ admits a $\mathbf{k}'$-structure for some finite subfield $\mathbf{k}'$ of $\mathbf{k}$. Assume that K is pure of weight zero on $\mathbf{k}'$, [see § 1.3.1 , Chapter 1]. Then $J_P^*(K)$ is pure of weight zero.*

This is an analog of [Lys11, Corollary 3] in the Iwahori case which says that the geometric Jacquet functors preserve the pure perverse sheaves of weigth zero.

Bibliography

[AB09] S. Arkhipov et R. Bezrukavnikov : Perverse sheaves on affine flags and Langlands dual group. *Israel J. Math.*, 170:135–183, 2009. With an appendix by Bezrukavrikov and Ivan Mirković.

[Ada89] J. Adams : *L*-functoriality for dual pairs. *Astérisque*, (171-172):85–129, 1989. Orbites unipotentes et représentations, II.

[Art84] J. Arthur : On some problems suggested by the trace formula. *In Lie group representations, II (College Park, Md., 1982/1983)*, volume 1041 de *Lecture Notes in Math.*, pages 1–49. Springer, Berlin, 1984.

[Aub89] A-M. Aubert : Conservation de la ramification modérée par la correspondance de Howe. *Bull. Soc. Math. France*, 117(3):297–303, 1989.

[Aub91] A-M. Aubert : Description de la correspondance de Howe en terms de classification de Kazhdan-Lusztig. *Invent. Math*, 1991.

[BBD82] A. A. Beĭlinson, J. Bernstein et P. Deligne : Faisceaux pervers. *In Analysis and topology on singular spaces, I (Luminy,1981)*, volume 100 de *Astérisque*, pages 5–171. Soc. Math. France, Paris, 1982.

[BD91] A. Beĭlinson et V. Drinfeld : Quantization of Hitchin's integrable systems and Hecke eigensheaves, 1991.

[Ber84] J. N. Bernstein : Le "centre" de Bernstein. *In Representations of reductive groups over a local field*, Travaux en Cours, pages 1–32. Hermann, Paris, 1984. Edited by P. Deligne.

[BL94] A. Beauville et Y. Laszlo : Conformal blocks and generalized theta functions. *Comm. Math. Phys.*, 164(2):385–419, 1994.

[Bor76] A. Borel : Admissible representations of a semi-simple group over a local field with vectors fixed under an Iwahori subgroup. *Invent. Math.*, 35:233–259, 1976.

[Bou02] N. Bourbaki : *Lie groups and Lie algebras. Chapters 4–6.* Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 2002. Translated from the 1968 French original by Andrew Pressley.

[CG97] N. Chriss et V. Ginzburg : *Representation theory and complex geometry.* Birkhäuser Boston Inc., Boston, MA, 1997.

[dCM09] M A. de Cataldo et L. Migliorini : The decomposition theorem, perverse sheaves and the topology of algebraic maps. *Bull. Amer. Math. Soc. (N.S.)*, 46(4):535–633, 2009.

[Del80] P. Deligne : La conjecture de Weil. II. *Inst. Hautes Études Sci. Publ. Math.*, (52):137–252, 1980.

[E.12] Frenkel E. : Langlands program, trace formulas, and their geometrization. *arXiv:1202.2110v1*, 9 Feb 2012.

[FG06] E. Frenkel et D. Gaitsgory : Local geometric Langlands correspondence and affine Kac-Moody algebras. *In Algebraic geometry and number theory*, volume 253 de *Progr. Math.*, pages 69–260. Birkhäuser Boston, Boston, MA, 2006.

[FK88] E. Freitag et R. Kiehl : *Étale cohomology and the Weil conjecture*, volume 13 de *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1988. Translated from the German by Betty S. Waterhouse and William C. Waterhouse, With an historical introduction by J. A. Dieudonné.

[Gai01] D. Gaitsgory : Construction of central elements in the affine Hecke algebra via nearby cycles. *Invent. Math.*, 144(2):253–280, 2001.

[Gel76] S. Gelbart : *Weil's representation and the spectrum of the metaplectic group.* Lecture Notes in Mathematics, Vol. 530. Springer-Verlag, Berlin, 1976.

[Gor10] U. Gortz : Affine springer fibers and affine deligne-lusztig varieties. Schmitt, Alexander (ed.), Affine flag manifolds and principal bundles. Basel: Birkhäuser. Trends in Mathematics, 1-50 (2010)., 2010.

[How79] R. Howe : θ-series and invariant theory. *In Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1*, Proc. Sympos. Pure Math., XXXIII, pages 275–285. Amer. Math. Soc., Providence, R.I., 1979.

[Hum78] J E. Humphreys : *Introduction to Lie algebras and representation theory*, volume 9 de *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1978. Second printing, revised.

[IM] N. Iwahori et H. Matsumoto : On some Bruhat decomposition and the structure of the Hecke rings of p-adic Chevalley groups, journal = Inst. Hautes Études Sci. Publ. Math., fjournal = Institut des Hautes Études Scientifiques. Publications Mathématiques, number = 25, year = 1965, pages = 5–48, issn = 0073-8301, mrclass = 20.70 (14.50), mrnumber = 0185016 (32 #2486), mrreviewer = Rimhak Ree,.

[Kud86] S. Kudla : On the local theta-correspondence. *Invent. Math.*, 83(2):229–255, 1986.

[KV04] M. Kapranov et E. Vasserot : Vertex algebras and the formal loop space. *Publ. Math. Inst. Hautes Études Sci.*, (100):209–269, 2004.

[KW01] R. Kiehl et R. Weissauer : *Weil conjectures, perverse sheaves and ℓ'adic Fourier transform*, volume 42 de *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 2001.

[LL09] V. Lafforgue et S. Lysenko : Geometric Weil representation: local field case. *Compos. Math.*, 145(1):56–88, 2009.

[LMB00] G. Laumon et L. Moret-Bailly : *Champs algébriques*, volume 39 de *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 2000.

[Lus83a] G. LUSZTIG : Singularities, character formulas, and a q-analog of weight multiplicities. *In Analysis and topology on singular spaces, II, III (Luminy, 1981)*, volume 101 de *Astérisque*, pages 208–229. Soc. Math. France, Paris, 1983.

[Lus83b] G. LUSZTIG : Some examples of square integrable representations of semisimple p-adic groups. *Trans. Amer. Math. Soc.*, 277(2):623–653, 1983.

[Lus89] G. LUSZTIG : Affine Hecke algebras and their graded version. *J. Amer. Math. Soc.*, 2(3):599–635, 1989.

[Lus98] G. LUSZTIG : Bases in equivariant K-theory. *Represent. Theory*, 2:298–369 (electronic), 1998.

[Lus02] G. LUSZTIG : Notes on affine Hecke algebras. *In Iwahori-Hecke algebras and their representation theory (Martina-Franca, 1999)*, volume 1804 de *Lecture Notes in Math.*, pages 71–103. Springer, Berlin, 2002.

[Lys11] S. LYSENKO : Geometric theta-lifting for the dual pair $\mathbb{SO}_{2m}, \mathbb{S}\mathrm{p}_{2n}$. *Ann. Sci. Éc. Norm. Supér. (4)*, 44(3):427–493, 2011.

[Mín06] A. MÍNGUEZ : Correspondance de Howe ℓ-modulaire: paires duales de type II. Mémoire de D.E.A., Orsay, 2006.

[Mín08] A. MÍNGUEZ : Correspondance de Howe explicite: paires duales de type II. *Ann. Sci. Éc. Norm. Supér. (4)*, 41(5):717–741, 2008.

[Mœg89] C. MŒGLIN : Correspondance de Howe pour les paires réductives duales: quelques calculs dans le cas archimédien. *J. Funct. Anal.*, 85(1):1–85, 1989.

[Mœg09a] C. MŒGLIN : Comparaison des paramètres de Langlands et des exposants à l'intérieur d'un paquet d'Arthur. *J. Lie Theory*, 19(4):797–840, 2009.

[Mœg09b] C. MŒGLIN : Paquets d'Arthur discrets pour un groupe classique p-adique. *In Automorphic forms and L-functions II. Local aspects*, volume 489 de *Contemp. Math.*, pages 179–257. Amer. Math. Soc., Providence, RI, 2009.

[Mui08a] G. MUIĆ : On the structure of theta lifts of discrete series for dual pairs $(\mathrm{Sp}(n), \mathrm{O}(V))$. *Israel J. Math.*, 164:87–124, 2008.

[Mui08b] G. MUIĆ : Theta lifts of tempered representations for dual pairs $(\mathrm{Sp}_{2n}, \mathrm{O}(V))$. *Canad. J. Math.*, 60(6):1306–1335, 2008.

[MV07] I. MIRKOVIĆ et K. VILONEN : Geometric Langlands duality and representations of algebraic groups over commutative rings. *Ann. of Math. (2)*, 166(1):95–143, 2007.

[MVW87] C. MŒGLIN, M-F. VIGNÉRAS et J-L. WALDSPURGER : *Correspondances de Howe sur un corps p-adique*, volume 1291 de *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1987.

[Pan06] S-Y. PAN : Local theta correspondence for small unitary groups. *Trans. Amer. Math. Soc.*, 358(4):1511–1535 (electronic), 2006.

[Pra05] A. PRASAD : On Bernstein's presentation of Iwahori-Hecke algebras and representations of split reductive groups over non-Archimedean local fields. *Bull. Kerala Math. Assoc.*, (Special Issue):31–51 (2007), 2005.

[Ral82] S. RALLIS : Langlands' functoriality and the Weil representation. *Amer. J. Math.*, 104(3):469–515, 1982.

[Spa82] N. SPALTENSTEIN : *Classes unipotentes et sous-groupes de Borel*, volume 946 de *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1982.

[Spr79] T. A. SPRINGER : Reductive groups. *In Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1*, Proc. Sympos. Pure Math., XXXIII, pages 3–27. Amer. Math. Soc., Providence, R.I., 1979.

[Sri08] V. SRINIVAS : *Algebraic K-theory*. Modern Birkhäuser Classics. Birkhäuser Boston Inc., Boston, MA, second édition, 2008.

[Wal90] J.-L. WALDSPURGER : Démonstration d'une conjecture de dualité de Howe dans le cas p-adique, $p \neq 2$. *In Festschrift in honor of I. I. Piatetski-Shapiro on the occasion of his sixtieth birthday, Part I (Ramat Aviv, 1989)*, volume 2 de *Israel Math. Conf. Proc.*, pages 267–324. Weizmann, Jerusalem, 1990.