Algebraic and definable closure in free groups

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Introduction

The notion of algebraic closure originated in field theory. It first appeared in the proof by Gauss of the fundamental theorem of algebra in 1799 and laid the basis of the development of Galois (1811-1832) theory. Model theory generalized the notion of algebraic closure to any first-order theory. The **algebraic closure** of a subset A in a structure G, denoted $acl_G(A)$, is the set of elements g such that there exists a formula $\phi(x)$, with parameters from A, such that G satisfies $\phi(g)$ and such the set of elements satisfying ϕ is finite. In model theory, one considers also the notion of **definable closure** which is the set of elements g such that in the previous definition the set of elements g satisfying ϕ is a singleton.

Two related notions are those of existential algebraic closure $acl_G^{\exists}(A)$ and quantifier-free algebraic closure $acl_G^{QF}(A)$, where $\phi(x)$ is restricted to be an existential, respectively quantifier-free formula. As a basic reference for the notion of algebraic closure in model theory, see [Hod97, p.138].

The previous definition generalizes the notion of algebraic closure in two ways. First it considers algebraic closure in an arbitrary model G, not just an algebraically closed field. Second, as not every theory has elimination of quantifiers, it allows $\phi(x)$ to be any first-order formula, not just an equation. In the context of algebraically closed fields, the above model-theoretic notion coincides with the usual one.

Galois theory relies on the fact that algebraic closure coincides with what we call *restricted algebraic closure*. Restricted algebraic closure, denoted $racl(A)$, is the set of elements g such that the orbit

$$\{f(g)|f \in \mathrm{Aut}_A(G)\}$$

is finite, where $\mathrm{Aut}_A(G)$ is the group of automorphisms of G fixing A pointwise. The corresponding notion of *restricted definable closure* is defined in a similar way.

We investigate here algebraic and definable closure in free groups; the main results can be summarized as follows.

1. We prove a constructibility result for torsion-free hyperbolic groups from algebraic closure of some subset. The property of algebraic closure which is relevant for constructibility is the fact that it coincides with its own existential algebraic closure. On the contrary, existential algebraic closure does not have this property - in other words, existential closure operator is not idempotent. Nevertheless, we have proved constructibility of free groups from existential algebraic closure, thanks to Theorem 1.1.24.

2. We looked for some results about the position of $acl_G(A)$ in some decomposition of the group G. At first we focused on $racl_G(A)$, since we have disponibility of results

linking automorphisms and vertex groups of certain decompositions (see Lemmas 3.3.9 and 3.3.10). For torsion-free hyperbolic groups we obtained that $racl_G(A)$ coincides with the vertex group containing A in the *generalized malnormal* cyclic JSJ decomposition of G relative to A (Definitions 3.3.12 and 3.3.13), obtained after ruling out free factors of G not containing A and making edge groups malnormal in adjacent vertex groups. Moreover, for free groups such vertex coincides with strictly-speaking algebraic closure.

3. We studied the possibility of generalizing Bestvina-Paulin method in another direction by considering finitely generated groups acting acylindrically (in the sense of Bowditch) on hyperbolic graphs. The typical formulation of the method supposes a group G acting on a Cayley graph Γ of a group H. Instead of weakening assumptions on G or H, we have dropped the assumption that Γ is a Cayley graph. Informally speaking, a Bowditch-acylindrical action bounds the number of elements of G that move 'little enough' vertices of Γ that are 'far enough from each other'. The results obtained by using this notion seem rather close to those (see [RW10]) obtained making groups act on hyperbolic Cayley graphs.

4. We investigated in relations between the different algebraic closure notions and between algebraic and definable closure. We found that in free groups algebraic closures coincide, the strictly-speaking and the restricted one. Regarding the problem algebraic/definable closure, the knowledge gap between the class of free groups and larger classes is mainly due to Theorem 1.2.39. In 2008, Sela asked whether $acl_F(A) = dcl_F(A)$ for every subset A of a free group F. A positive answer has been given for a free group F of rank smaller than 3: for every subset $A \subseteq F$ we have $acl_F(A) = dcl_F(A)$. Instead, for free groups of rank strictly greater than 3 we found a counterexample. For the free group of rank 3 we found a necessary condition on the form of a possible counterexample.

Rapidly browsing contents, Chapter 1 is a survey of introductive notions. We give basics on combinatorial group theory, starting from free groups and proceeding with the fundamental constructions: free products, amalgamated free products and HNN extensions. We outline a synthesis of Bass-Serre theory, preceded by a survey on Cayley graphs and graphs of groups. After proving the main theorem of Bass-Serre theory, we present its application to the proof of Kurosh subgroup theorem (1.2.36).

Subsequently we recall main definitions and properties of hyperbolic spaces (see [CDP90] and [Gro87]). In Section 1.4 we define algebraic and definable closures and recall a few other notions of model theory related to saturation and homogeneity. The last section of Chapter 1 is devoted to asymptotic cones.

In Chapter 2 we prove a theorem similar to Bestvina-Paulin theorem on the limit of a sequence of actions on hyperbolic graphs. Our setting is more general: we consider Bowditch-acylindrical actions on arbitrary hyperbolic graphs. We prove that edge stabilizers are (finite bounded)-by-abelian, that tripod stabilizers are finite bounded and that unstable edge stabilizers are finite bounded.

In Chapter 3 we introduce the essential notions on limit groups, shortening argument and JSJ decompositions.

In Chapter 4 we present the results on constructibility of a torsion-free hyperbolic group from the algebraic closure of a subgroup. Also we discuss constructibility of a free group from the existential algebraic closure of a subgroup. We obtain a bound to the rank of the algebraic and definable closures of subgroups in torsion-free hyperbolic groups. In Section 4.2 we prove some results about the position of algebraic closures in JSJ decompositions of torsion-free hyperbolic groups and other results for free groups.

Finally, in Chapter 5 we answer the question posed by Sela about equality between algebraic and definable closure in a free group.

Introduction

La notion de la clôture algébrique trouve ses origines en théorie des corps. Elle est apparue pour la première fois dans la preuve de Gauss du théorème fondamental de l'Algèbre en 1799 et elle est à la base du développement de la théorie de Galois (1811-1832). La théorie des modèles a généralisé cette notion aux théories du premier ordre. La **clôture algébrique** d'un sous-ensemble A dans une structure G, notée $acl_G(A)$, est l'ensemble des éléments g tels qu'il existe une formule $\phi(x)$ à paramètres dans A, telle que G satisfait $\phi(g)$ et telle que l'ensemble des éléments qui satisfont ϕ est fini. En théorie des modèles on considère aussi la notion de **clôture définissable** qui est l'ensemble des éléments g tels que dans la définition précédente l'ensemble des éléments qui satisfont ϕ est réduit à g.

Deux autres notions liées sont celles de la clôture algébrique existentielle $acl_G^{\exists}(A)$ et de la clôture algébrique sans-quantificateurs $acl_G^{QF}(A)$, où la formule $\phi(x)$ dans la définition précédente est une formule existentielle, respectivement une formule sans-quantificateurs. Comme référence de base pour la notion de clôture algébrique, on peut consulter [Hod97, p.138].

La définition précédente généralise la notion de clôture algébrique dans deux directions. Premièrement, elle permet de considérer la clôture algébrique dans un modèle arbitraire qui n'est pas nécessairement un corps algébriquement clos. Deuxièmement, comme toute théorie n'a pas nécessairement une élimination de quantificateurs, elle permet de prendre $\phi(x)$ d'être une formule arbitraire pas seulement une équation. Dans le contexte des corps algébriquement clos, la notion de la théorie des modèles coïncide avec la notion usuelle.

La théorie de Galois se base sur le fait que la clôture algébrique coïncide avec ce que nous appelons *la clôture algébrique restreinte*. La clôture algébrique restreinte, notée $racl(A)$, est l'ensemble des éléments g tels que l'orbite

$$\{f(g)|f \in \mathrm{Aut}_A(G)\}$$

est finie, où $\mathrm{Aut}_A(G)$ désigne le groupe des automorphismes de G qui fixent tout élément de A. La notion correspondante de *clôture définissable restreinte* est définie d'une façon similaire.

Nous étudions dans ce mémoire la clôture algébrique et définissable dans les groupes libres. Les résultats principaux peuvent être résumés comme suit.

1. Nous montrons un résultat de constructibilité des groupes hyperboliques sans torsion au-dessus de la clôture algébrique d'un sous-ensemble engendrant un groupe non abélien. Une des propriétés pertinentes de la clôture algébrique utilisée pour obtenir la constructibilité est qu'elle coïncide avec sa propre clôture existentielle. La clôture algébrique existentielle ne possède pas cette propriété, en d'autres termes l'opérateur de la clôture existentielle n'est pas idempotent. Cela dit, nous montrons la constructibilité des groupes libres au-dessus de la clôture algébrique existentielle.

2. Nous avons cherché à comprendre la place qu'occupe la clôture algébrique $acl_G(A)$ dans certaines décompositions de G. Nous nous sommes intéressé en premier lieu à la clôture algébrique restreinte $racl_G(A)$ suite à l'existence des résultats liants les automorphismes du groupe et les groupes sommets de certaines décompositions (voir le Lemme 3.3.9 et d). Pour les groupes hyperboliques sans torsion nous montrons que $racl_G(A)$ coïncide avec le groupe sommet contenant A dans la décomposition JSJ cyclique généralisée de G relativement à A (Définitions 3.3.12 and 3.3.13). Celle-ci est obtenue en enlevant le facteur libre qui ne contient pas A et en rendant les groupes associés aux arêtes malnormaux dans les groupes associés aux sommets adjacents. En outre, pour les groupes libres nous montrons que le groupe sommet contenant A dans cette décomposition JSJ coïncide à proprement parler avec la clôture algébrique.

3. Nous avons étudié la possibilité de la généralisation de la méthode de Bestvina-Paulin dans d'autres directions en considérant les groupes de type fini qui agissent d'une manière acylindrique (au sens de Bowditch) sur les graphes hyperboliques. La formulation typique de la méthode suppose que le groupe G agit sur un graphe de Cayley Γ du groupe H. Au lieu d'affaiblir les hypothèses sur G ou H, nous avons enlevé l'hypothèse que Γ est un graphe de Cayley. D'une façon très informelle, une action acylindrique au sens de Bowditch borne le nombre des éléments de G qui déplacent 'très peu' les sommets de Γ qui sont très éloignés les uns des autres. Les résultats obtenus en utilisant cette notion sont similaires à ceux obtenus dans le cadre des actions sur les graphes de Cayley hyperboliques (voir [RW10]).

4. Nous avons étudié les relations qui existent entre les différentes notions de clôture algébrique et entre la clôture algébrique et la clôture définissable. Nous avons montré que dans un groupe libre la clôture algébrique coïncide avec la clôture algébrique restreinte. En ce qui concerne le problème de la relation entre la clôture algébrique et la clôture définissable, la différence qui existe entre la classe des groupes libres et des classes plus larges de groupes est essentiellement due au Théorème 1.2.39. En 2008, Sela a posé la question de savoir si $acl_F(A) = dcl_F(A)$ pour tout sous-ensemble A du groupe libre F. Une réponse positive est donnée pour les groupes libres de rang plus petit ou égale à 2 : pour tout sous-ensemble $A \subseteq F$ on a $acl_F(A) = dcl_F(A)$. Cependant pour les groupes de rang supérieur ou égale à 4 nous avons construit des contre-exemples. Pour les groupes de rang 3 nous avons trouvé une condition nécessaire sur la forme des possibles contre-exemples.

Le Chapitre 1 est un survole de quelques notions de base. Nous donnons une introduction aux notions de la théorie combinatoire des groupes, en commençant par les groupes libres suivi par les constructions fondamentales : produits libres, produits libres amalgamés et extensions HNN. Nous donnons une synthèse rapide de la théorie de Bass-Serre et les notions liées sur les graphes de Cayley et les graphes de groupes. Après avoir démontré le théorème principal de la théorie de Bass-Serre, nous présentons son application à la preuve du théorème de Kurosh (1.2.36).

Par la suite nous rappelons les définitions et les propriétés essentielles des espaces hyperboliques (voir [CDP90] et [Gro87]). Dans la Section 1.4 nous définissons la clôture algébrique et la clôture définissable et nous reppelons quelques notions de la théorie des modèles liées à la saturation et à l'homogénéité. La dernière section du Chapitre 1 est consacrée aux cônes asymptotiques.

Dans le Chapitre 2 nous montrons un théorème similaire au théorème de Bestvina-Paulin sur les limites des suites d'actions sur les graphes hyperboliques. Notre contexte est plus général : nous considérons les actions acylindriques au sens de Bowditch sur des graphes hyperboliques arbitraires. Nous montrons que les stabilisateurs des segments sont finis-par-abélien, que les stabilisateurs des tripodes sont finis et que les stabilisateurs des segments instables sont finis ; avec une borne uniforme sur le cardinal des groupes finis.

Dans le Chapitre 3 nous rappelons les notions essentielles sur les groupes limites, l'argument de raccourcissement et les décompositions JSJ.

Dans le Chapitre 4 nous présentons les résultats sur la constructibilité des groupes hyperboliques sans torsion au-dessus de la clôture algébrique d'un sous-groupe non abélien. Nous montrons aussi la constructibilité des groupes libres au-dessus de la clôture existentielle. Nous obtenons une borne sur le rang de la clôture algébrique et la clôture existentielle des sous-groupes des groupes libres. Dans la Section 4.2 nous montrons les résultats sur la place qu'occupe la clôture algébrique dans les décompositions JSJ cycliques dans les groupes hyperboliques sans torsion et dans les groupes libres.

Finalement dans le Chapitre 5 nous donnons une réponse à la question de Sela sur l'identité entre la clôture algébrique et la clôture définissable dans les groupes libres.

Introduzione

La nozione di chiusura algebrica ha origine nella teoria dei campi. Apparve per la prima volta nella dimostrazione da parte di Gauss del teorema fondamentale dell'algebra nel 1799, e pose le basi per lo sviluppo della teoria di Galois (1811-1832). La teoria dei modelli ha generalizzato il concetto di chiusura algebrica ad una qualunque teoria del primo ordine. La **chiusura algebrica** di un sottoinsieme A in una struttura G, che denotiamo come $acl_G(A)$, è l'insieme degli elementi g tali per cui esiste una formula $\phi(x)$, con parametri in A, tale per cui G soddisfa $\phi(g)$ e tale per cui l'insieme degli elementi che soddisfano ϕ è finito. Nella teoria dei modelli si considera anche la nozione di **chiusura definibile** che è l'insieme degli elementi g tali per cui nella definizione precedente l'insieme degli elementi g che soddisfano ϕ ha cardinalità 1.

Due concetti correlati sono quello di chiusura algebrica esistenziale $acl_G^{\exists}(A)$ e quello di chiusura algebrica senza quantificatori $acl_G^{QF}(A)$, dove $\phi(x)$ è rispettivamente una formula esistenziale o senza quantificatori. Come referenza di base per la nozione di chiusura algebrica in teoria dei modelli, si può vedere [Hod97, p.138].

La definizione precedente generalizza la nozione di chiusura algebrica in due direzioni. In primo luogo considera la chiusura algebrica in un modello arbitrario G anziché in un campo algebricamente chiuso. In secondo luogo, dato che non tutte le teorie hanno l'eliminazione dei quantificatori, permette che $\phi(x)$ sia una qualsiasi formula del primo ordine, non soltanto un'equazione. Nel contesto dei campi algebricamente chiusi, la nozione di chiusura model-teoretica appena esposta coincide con quella usuale.

La teoria di Galois si fonda sul fatto che la chiusura algebrica coincide con quella che chiamiamo *chiusura algebrica ristretta*. La chiusura algebrica ristretta, indicata con $racl(A)$, è l'insieme degli elementi g tali per cui l'orbita

$$\{f(g) | f \in \mathrm{Aut}_A(G)\}$$

è finita, dove $\mathrm{Aut}_A(G)$ è il gruppo degli automorfismi di G che fissano A punto per punto. La nozione corrispondente di *chiusura definibile ristretta* è definita in maniera simile.

In questa sede studiamo la chiusura algebrica e definibile nei gruppi liberi; qui di seguito riassumiamo i risultati principali.

1. È stato provato un risultato di costruibilità di gruppi iperbolici senza torsione a partire dalla chiusura algebrica di un sottoinsieme dato. La proprietà della chiusura algebrica che è rilevante ai fini della costruibilità è il fatto che essa coincide con la sua chiusura algebrica esistenziale. Invece la chiusura algebrica esistenziale non gode della stessa proprietà - in altre parole, l'operatore di chiusura algebrica esistenziale non è idempotente. Abbiamo tuttavia dimostrato la costruibilità dei gruppi liberi a partire dalla chiusura algebrica esistenziale, grazie al teorema 1.1.24.

2. Abbiamo cercato di capire la posizione occupata da $acl_G(A)$ in una decomposizione sufficientemente significativa del gruppo G. Ci siamo dapprima concentrati su $racl_G(A)$, poiché abbiamo a disposizione risultati che collegano automorfismi e gruppi vertice di certe decomposizioni (si vedano i lemmi 3.3.9 e 3.3.10). Per i gruppi iperbolici senza torsione abbiamo ottenuto che $racl_G(A)$ coincide con il gruppo vertice che contiene A nella decomposizione JSJ ciclica *malnormale generalizzata* di G relativa ad A (definizioni 3.3.12 e 3.3.13), ottenuta da una decomposizione JSJ ciclica non considerando i fattori liberi di G che non contengono A e rendendo i gruppi arco malnormali nei gruppi vertice ad essi adiacenti. Inoltre, per i gruppi liberi tale vertice coincide con la chiusura algebrica propriamente detta.

3. Abbiamo studiato la possibilità di generalizzare il metodo di Bestvina-Paulin in una nuova direzione, considerando azioni acilindriche (nel senso di Bowditch) di gruppi finitamente generati su grafi iperbolici. La formulazione classica di tale metodo suppone che un gruppo G agisca sul grafo di Cayley Γ di un gruppo H. Invece di indebolire le ipotesi su G o H, si è fatta cadere l'assunzione che Γ sia un grafo di Cayley. Informalmente, un'azione acilindrica secondo Bowditch limita il numero di elementi di G che muovono 'abbastanza poco' vertici di Γ che sono 'abbastanza distanti l'uno dall'altro'. I risultati ottenuti utilizzando questa nozione paiono piuttosto vicini a quelli (si veda [RW10]) ottenuti facendo agire gruppi su grafi di Cayley iperbolici.

4. Abbiamo studiato le relazioni tra le diverse nozioni di chiusura algebrica e tra la chiusura algebrica e quella definibile. A questo proposito è stato trovato che nei gruppi liberi la chiusura algebrica propriamente detta coincide con quella ristretta. Per quanto riguarda la relazione tra la chiusura algebrica e quella definibile, l'ostruzione ad una generalizzazione dai gruppi liberi a classi più ampie è principalmente dovuta al teorema 1.2.39. Nel 2008, Sela ha proposto il problema se sia $acl_F(A) = dcl_F(A)$ per ogni sottoinsieme A di un gruppo libero F. È stata fornita una risposta positiva per ogni gruppo libero F di rango minore di 3: per ogni sottoinsieme $A \subseteq F$ si ha $acl_F(A) = dcl_F(A)$. Invece, per gruppi liberi di rango maggiore di 3 è stato trovato un controesempio. Per il gruppo libero di rango 3 è stata trovata una condizione necessaria sulla forma di un possibile controesempio.

Passando ad un breve sguardo sui contenuti dei capitoli, il capitolo 1 è una rassegna di concetti introduttivi. Forniamo un'introduzione alla teoria combinatoria dei gruppi, partendo dai gruppi liberi e procedendo con le costruzioni fondamentali: prodotti liberi, prodotti liberi amalgamati ed estensioni HNN. Viene delineata inoltre una sintesi della teoria di Bass-Serre, preceduta da una breve rassegna sui grafi di Cayley e sui grafi di gruppi. Dopo avere riportato la dimostrazione del teorema principale della teoria di Bass-Serre, presentiamo una sua applicazione alla prova del teorema di Kurosh (1.2.36).

Successivamente ricordiamo le definizioni e le proprietà principali degli spazi iperbolici (si vedano [CDP90] e [Gro87]). Nella sezione 1.4 definiamo la chiusura algebrica e definibile e ricordiamo qualche altra nozione di teoria dei modelli collegata a saturazione ed omogeneità. L'ultima sezione del capitolo 1 è dedicata ai coni asintotici.

Nel capitolo 2 dimostriamo un teorema simile al teorema di Bestvina-Paulin theorem sul limite di una sequenza di azioni su grafi iperbolici. Il nostro contesto è più generale: consideriamo azioni acilindriche secondo Bowditch su grafi iperbolici arbitrari. Abbiamo ottenuto che gli stabilizzatori degli archi sono finiti-per-abeliani, gli stabilizzatori dei tripodi sono finiti e gli stabilizzatori degli archi instabili sono finiti, con cardinalità del gruppo finito uniformemente limitata.

Nel capitolo 3 introduciamo le nozioni essenziali sui gruppi limite, sull'argomento di accorciamento e sulle decomposizioni JSJ.

Nel capitolo 4 presentiamo i risultati di costruibilità di un gruppo iperbolico senza torsione a partire dalla chiusura algebrica di un suo sottogruppo. Discutiamo anche la costruibilità di un gruppo libero a partire dalla chiusura algebrica esistenziale di un suo sottogruppo. Abbiamo ottenuto un limite al rango delle chiusure, algebrica e definibile, nei gruppi iperbolici senza torsione. Nella sezione 4.2 proviamo risultati sulla posizione delle chiusure algebriche nelle decomposizioni JSJ dei gruppi iperbolici senza torsione, con ulteriori risultati per i gruppi liberi.

Infine, nel capitolo 5 rispondiamo alla domanda posta da Sela sull'uguaglianza tra la chiusura algebrica e quella definibile in un gruppo libero.

Chapter 1

Basic notions

In this chapter we introduce the fundamental notions necessary to understand the rest of the work. Since our core problem is of model-theoretic nature, but the methods used to solve it come from geometric group theory, our topic comes to be at the crosspoint between several disciplines. For this reason, this first chapter is articulated in some sections, articulated in further subsections and covering the fundamental aspects of the disciplines involved in this work. In the first section we state the fundamental concepts in combinatorial group theory, namely about presentations of groups, free groups, free products and constructions on groups: amalgamated free product and HNN-extension. Then, we introduce Bass-Serre theory; we give the basics about graphs, Cayley graphs, actions of groups on graphs and graphs of groups. In the following section we give an introduction on hyperbolic groups; to this aim, it is necessary to give some basics about hyperbolic spaces, necessary also to get the notion of asymptotic cone. This gives us the possibility of extending Bass-Serre theory to actions on $\mathbb{R}$-trees in Chapter 2. The last sections are respectively dedicated to a brief introduction to model theory and to asymptotic cones.

The current chapter is not intended to give a complete treatment of the subjects in question; the main references will be given at the beginning of the sections and throughout the chapter, when some result is quoted.

1.1 Basics of combinatorial group theory

The first formal development of group theory, centering around the ideas of Galois, was limited almost entirely to finite groups. The idea of an abstract infinite group is clearly embodied in the work of Cayley on the axioms for a group, but was not immediately pursued to any depth. There developed later a school of group theory, in which Schmidt was prominent, that was concerned in part with developing for infinite groups results parallel to those known for finite groups. Another strong influence on the development of group theory was the recognition, notably by Klein, of the role of groups, many of them infinite, in geometry. In fact, Klein in 1872 with his 'Erlangen program' proposed group theory as a means of formulating and understanding geometrical constructions. The subject of combinatorial group theory, and geometric group theory, which developed from it, might be viewed as a reverse Klein program: geometrical ideas are used to give new insights into group theory. A major stimulus to the study of infinite discrete groups was the development of topology: we mention particularly the work of Poincaré, Dehn and Nielsen. This last influence led naturally to the study of groups presented by generators and relations. In the Fifties of last century, some groups with unsolvable word problem and conjugacy problem were exhibited ([Nov52], [Nov55], [Nov54], [Nov56], [Boo57]); these results in decision theory, strictly linked with logic, have had a great influence on the subject of infinite groups. The book [MKS66] of 1966 sums up the state of the art at that time, witnessing a notable increase of interest in infinite discrete groups, both in the systematic development of the abstract theory and in applications to other areas, especially exploiting the connections with geometry and topology. In more recent years, geometric group theory draws on ideas from across many subjects of mathematics, though two particular sources of inspiration can be identified rather clearly. One is low-dimensional topology, in particular 3-manifold theory; the other is hyperbolic geometry. Thurston [Thu82] in early Eighties shows that these two subjects are intimately linked. The resulting growth of activity might be seen as the birth of geometric group theory as a subject in its own right. In the Eighties, Gromov's works [Gro87] about hyperbolic groups and [Gro93] asymptotic invariants have been particularly influential. In this last paper Gromov explicitly defines asymptotic cones (see Section 1.5), an important tool for studying large-scale structure and quasi-isometry invariants of Cayley graphs. This has led to the works of Paulin [Pau91] and Bestvina [Bes88], giving a method to decompose groups acting on real trees. Makanin in [Mak82] finds a decidability algorithm for existence of solutions of equations in free groups. Razborov in [Raz85] describes the set of solutions for a system of equations in a free group. This construction is now known as Makanin-Razborov diagram. Rips utilized Makanin process to study group actions on real trees, defining what is called Rips machine, which gives a structure theorem for a finitely generated group acting on a real tree, in a similar way as Bass-Serre theory gives a structure theorem for a group acting on a simplicial tree. In this way, Rips succeeds in proving Morgan-Shalen conjecture, formulated in [MS91], according to which any finitely generated group acting freely on a $\mathbb{R}$-tree is a free product of free abelian and surface groups. Kharlampovich and Miasnikov in [KM98] give a combinatorial proof of Makanin-Razborov diagram. Sela in [Sel01, §8] obtains a canonical Makanin-Razborov diagram for systems of equations with parameters utilizing techniques from Diophantine geometry (analysis of projections of sets of solutions to systems of equations) and low-dimensional topology; then, using those results and tools, in [Sel06, Theorem 3] he proves elementary equivalence of non-abelian finitely generated free groups and in [Sel06, Proposition 6] he characterizes the finitely generated groups that are elementary equivalent to a non-abelian finitely generated free group. More recently, Sela in [Sel09, Theorem 1.26] constructs a canonical Makanin-Razborov diagram for torsion-free hyperbolic groups; a result generalized by Weidmann and Reinfeldt ([RW10, Theorem 5.6]) who allow hyperbolic groups to have torsion, under the only assumption of 'weak equational noetherianity'(definition in [RW10, p.74]); at the end (§7.2) of the same paper, they prove that hyperbolic groups are equationally noetherian, so their construction finally applies to the whole class of hyperbolic groups.

Our approach mainly follows [LS77, §I.3,I.4,IV.1,IV.2]; to get a comprehensive vision, the same book of 1977 [LS77] is still now the main reference. Lecture notes [Kap02] are a good reference for geometric group theory, the 'natural evolution' of combinatorial group theory. A more monographical reference, rather focusing on specific problems in geometric group theory, is [DLH00].

1.1.1 Free groups

Notation 1.1.1. Let S be a set of symbols $\{a_i | i \in I\}$. By S^{-1} we denote the set $\{a_i^{-1} | i \in I\}$.

Definition 1.1.2. Let S be a set $\{a_i | i \in I\}$, that may be finite or infinite. A **word** on S is a finite string of elements of S, possibly repeating. By a_i^n we denote the word $\underbrace{a_i \dots a_i}_{n \text{ times}}$.

For any set S, by $W(S)$ we denote the set of words on S.

Definition 1.1.3. Let $w = a_{i_1} \dots a_{i_l}$ be a word on some set $S = \{a_i | i \in I\}$. The **length** of w, that we denote by $|w|$, is the number l.

Consider now $W(S \cup S^{-1})$. If a word has a subword $a_i^{\epsilon} a_i^{-\epsilon}$, where $\epsilon \in \{1, -1\}$, then we may agree to cancel $a_i^{\epsilon} a_i^{-\epsilon}$, obtaining a shorter word.

Definition 1.1.4. Let $S = \{a_i | i \in I\}$. A word on $S \cup S^{-1}$ is **reduced** if it does not contain a subword $a_i^{\epsilon} a_i^{-\epsilon}$, where $\epsilon \in \{1, -1\}$.

Given a word w, a reduced form for w is a word obtained by w by executing all possible cancellations. Clearly, if w is reduced, w is a reduced form for itself.

Definition 1.1.5. Let $S = \{a_i | i \in I\}$. A word on $S \cup S^{-1}$ is **cyclically reduced** if it is reduced and every of its cyclic permutations is reduced.

The following proposition allows to define the equivalence relation $\sim$ in Definition 1.1.7.

Proposition 1.1.6 (Normal form theorem for free groups)**.** *Let $w \in W(S \cup S^{-1})$. Then, there exists a unique reduced form for w.*

Proof. By induction on $|w|$. The base of the induction is the case when w is reduced: in this case the result is clear. Suppose now that w is not reduced. Then, w has the form $\dots bb^{-1} \dots$, where $b \in S \cup S^{-1}$ and the dots to the left or to the right of bb^{-1} may represent the empty word. The inductive step is the following: we claim that we can obtain every reduced form of w by cancelling that pair bb^{-1} first. In this way, the proposition will follow by induction on the shorter word $\dots \not{b}\not{b}^{-1} \dots$.

Let w' be a reduced form of w. So w' is obtained by w by some sequence of cancellations. If the pair bb^{-1} is cancelled at some step in the sequence, then we can reorder the sequence and cancel bb^{-1} first, so this case is settled. If this case does not verify, then the first cancellation involving one element of the pair has the form $\dots \not{b}^{-1}\underline{\not{b}b^{-1}} \dots$ or $\dots \underline{b\not{b}^{-1}}\not{b} \dots$, where our first considered pair is underlined. In this case, note that the word obtained by this cancellation is the same as that obtained by cancelling the pair $\underline{bb^{-1}}$. So we may cancel our pair instead of the other. Since we have exhausted all cases, the proposition is proved.

∎

Definition 1.1.7. Let $w_1, w_2 \in W(S \cup S^{-1})$. Define the equivalence relation $\sim$ as follows: $w_1 \sim w_2$ if and only if w_1 and w_2 have the same reduced form.

Let $F(S)$ denote the quotient of $W(S \cup S^{-1})$ under the equivalence relation $\sim$ of Definition 1.1.7. We are going to show that $F(S)$ can be given a group structure. Group operation is that induced by juxtaposition of words. The following theorem proves that group operation is well-defined, that is, it does not depend on representatives chosen in the equivalence classes.

Proposition 1.1.8. *Let $v_1, v_2, w_1, w_2 \in W(S \cup S^{-1})$ such that $v_1 \sim v_2$ and $w_1 \sim w_2$. Then, $v_1 w_1 \sim v_2 w_2$.*

Proof. To obtain the reduced word equivalent to $v_1 w_1$, we can first cancel as much as possible in v_1 and w_1, to obtain v' and w'. Then we possibly continue cancelling in $v'w'$. Since $v_1 \sim v_2$ and $w_1 \sim w_2$, the same process applied to v_2 and w_2 passes through v' and w' too, so it leads to the same reduced word.

□

Proposition 1.1.8 allows us to define an operation on $F(S)$ as above. To give $F(S)$ a group structure, we must check group properties. Associativity of the operation and identity as the $\sim$-class of the empty word follow from the corresponding facts in $W(S \cup S^{-1})$. As for inverses, for every $w = a_{i_1} \dots a_{i_n} \in W(S \cup S^{-1})$, the class of $a_{i_n}^{-1} \dots a_{i_1}^{-1}$ is the inverse of the class of w.

Definition 1.1.9. The set $F(S)$ with the group structure defined above is the **free group** over the set S. The set S is called a **basis** for $F(S)$.

Definition 1.1.10. The **normal form** of an element $[w]_\sim \in F(S)$ is a sequence $a_1, \dots, a_n$ such that $a_1 \dots a_n$ is the reduced form for w.

Definition 1.1.11. The **length** of an element w of a free group F, that we denote by $|w|_F$, is the length of its reduced form in $W(S \cap S^{-1})$. We will omit the index if it is clear from the context.

Definition 1.1.12. An element of a free group is **cyclically reduced** if its reduced form is cyclically reduced.

Before stating next proposition, we recall that by a^b we denote $b^{-1}ab$, the conjugate of a by b.

Proposition 1.1.13 (Conjugacy theorem for free groups)**.** *Every element of a free group is conjugate to a cyclically reduced element. Moreover, if two elements are conjugate and cyclically reduced, then they are cyclic permutations of each other.*

Proof. Let w be an element of a free group F in reduced form. If w is cyclically reduced, then there is nothing to show. If w is not cyclically reduced, then it is possible to write it as $\alpha w' \alpha^{-1}$, for some $\alpha \in F$. Take α of maximal length. Then, $w' = w^\alpha$ is cyclically reduced, so the first statement is proved. To prove the other statement, let $w, v \in F$ be cyclically reduced and conjugate, say $w = v^\alpha$. Let $w_1 \dots w_n$ and $v_1 \dots v_m$ be cyclically reduced forms for w and v respectively. We argue by induction on $|\alpha|$. If $|\alpha| = 0$, then we have the result by Proposition 1.1.6, so the base of the induction is assured. To prove the inductive step, suppose that $\alpha = \alpha_1 \dots \alpha_k$ is reduced, with $k \geq 1$. Since $w_1 \dots w_n$ is cyclically reduced, equality

$$w_1 \dots w_n = \alpha_k^{-1} \dots \alpha_1^{-1} v_1 \dots v_m \alpha_1 \dots \alpha_k$$

cannot hold if there is no cancellation between α_1^{-1} and v_1 or between v_m and α_1. So, without loss of generality suppose that there is cancellation between v_m and α_1. The equality becomes

$$w_1 \dots w_n = \alpha_k^{-1} \dots \alpha_1^{-1} v_1 \dots v_{m-1} \alpha_2 \dots \alpha_k,$$

but, by the cancellation just applied, we know that $v_m \alpha_1 = 1$, that is $\alpha_1^{-1} = v_m$, so the last equality becomes

$$w_1 \dots w_n = \alpha_k^{-1} \dots v_m v_1 \dots v_{m-1} \alpha_2 \dots \alpha_k,$$

so the inductive step and the whole statement is proved.

□

We recall here the definition of set of generators of a group.

Definition 1.1.14. A set S is a **set of generators** for a group G if every element of G can be written as a word on $S \cup S^{-1}$. We will write $G = \langle S \rangle$.

Definition 1.1.15. A group G is **finitely generated** if there exists a finite set generating G.

The definition of free group we give below has a more categorical flavour. In Proposition 1.1.17 we will prove that it is equivalent to Definition 1.1.9.

Definition 1.1.16. Let S be a subset of a group F. Then F is a free group with basis S if and only if the following holds: if ϕ is any function from the set S into some group G, then there exists a unique extension of ϕ to a homomorphism ϕ^* from F to G.

Proposition 1.1.17. *Definitions 1.1.9 and 1.1.16 are equivalent.*

Proof. Let F be the free group $F(S)$ with basis S in the sense of Definition 1.1.9, let G be a group and let ϕ be a map from S to G. Extend ϕ to a homomorphism from F to G by the following rule. Given an element $w \in F$, write it as $a_1 \dots a_n$, where $a_i \in S \cup S^{-1}$, and set $\phi^*(w) = \phi(a_1) \dots \phi(a_n)$. Let $\phi^*(a^{-1}) = (\phi(a))^{-1}$ for any $a \in S$. This definition of ϕ^* is well-given, since from one representative for w as a word on $S \cup S^{-1}$ we can pass to another representative by a finite number of insertions and deletions of words of the form aa^{-1}, where $a \in S \cup S^{-1}$. This is the unique way to extend ϕ to a homomorphism from F to G.

Now let F be a free group with basis S in the sense of Definition 1.1.16. We claim that $F = F(S)$, where $F(S)$ is the free group over S in the sense of Definition 1.1.9. The identity embedding $S \hookrightarrow F(S)$ can be extended to a homomorphism $F \to F(S)$, hence to a homomorphism $\psi : F \to F$ with image $F(S)$. Both ψ and id_F extend the identity embedding $S \hookrightarrow F$. But by Definition 1.1.16 the extension is unique, so $F(S) = F$.

□

Note that, in Definition 1.1.16, the requirement of existence of the extension corresponds to freeness of F over S, while the requirement of unicity corresponds to the fact that S generates F.

Definition 1.1.16 allows to easily prove the following proposition.

Proposition 1.1.18. *Any group is isomorphic to a quotient of a free group.*

Proof. Let $G = \langle S \rangle$. The identity embedding $S \hookrightarrow G$ extends to a homomorphism $\psi : F(S) \to G$. Since $G = \langle S \rangle = \langle \psi(S) \rangle$, ψ is surjective. So, $G \cong F(S)/\ker\psi$.

□

Proposition 1.1.19. *Let S and S' be two sets. If $F(S) \cong F(S')$, then S and S' have the same cardinality.*

Proof. Denote the underlying set of a group G as $U(G)$. Note that the number of group homomorphism from $F(S)$ to $\mathbb{Z}_2$, the (cyclic) group of order 2, is the same as the number of set functions from S to $U(\mathbb{Z}_2)$, that is $2^{|S|}$. By isomorphism, $2^{|S|} = 2^{|S'|}$. So $|S| = |S'|$.

□

Proposition 1.1.19 allows the following definition.

Definition 1.1.20. Let F be a free group. The **rank** of F, denoted by $rk(F)$, is the cardinality of a basis of F.

Note that a set of generators of a free group F of rank n has cardinality $\geq n$; if it has cardinality n, then it is a basis of F. This observation allows us to define the rank of any finitely generated group.

Definition 1.1.21. Let G be a finitely generated group. Then, the rank of G, denoted by $rk(G)$, is defined as $\min |S|, S$ is a generator set for G.

Let F be a free group of basis $\{a_i | i \in S\}$. The following transformations extend to automorphisms of F, so it is possible to change basis by a composition of them.

Definition 1.1.22. Let $F = F(\{a_i | i \in S\})$. An **elementary Nielsen transformation** is a transformation of one of the following forms:

1. replace a_i by $a_i a_j$, and leave all a_k unchanged for $k \neq i$;
2. replace a_i by a_i^{-1}, and leave all a_k unchanged for $k \neq i$.

Definition 1.1.23. A **Nielsen transformation** is a finite composition of elementary Nielsen transformations.

Since the image by a homomorphism f of a basis of a free group F determines the image by f of the whole F, it is easy to check that an elementary Nielsen transformation extends to an automorphism of F, so a Nielsen transformation extends to an automorphism of F as well. Actually, every automorphism of F extends some Nielsen transformation.

We end this subsection with a theorem that will be used to prove Theorem 4.1.14, a constructibility result.

Theorem 1.1.24. [Tak51, Theorem 2] *Let F be a finitely generated free group and let $\{L_i | i \in \mathbb{N}\}$ be a descending chain of subgroups with bounded rank. Then $\bigcap_i L_i$ is a free factor of L_n for all but finitely many n.*

1.1.2 Free products

Before defining free products, we briefly recall some notions about presentations of groups.

Definition 1.1.25. Let S be a subset of a group G. The **normal closure** of S in G is the smallest normal subgroup $ncl_G(S) \trianglelefteq G$ containing S.

Note that $ncl_G(S)$ is the intersection of all normal subgroups of G containing S.

We can characterize a group using the property given by Proposition 1.1.18. Let G be a group generated by $S = \{a_i | i \in I\}$, and let $F = F(S)$. If $\phi : F \to G$ is the epimorphism extending the identity embedding of S into G, then $G = F/\ker\phi$. If R is a subset of F such that $\ker\phi = ncl_F(R)$, then the expression $\langle S | \{r = 1 | r \in R\}\rangle$ determines the group G up to isomorphism.

Definition 1.1.26. A **presentation by generators and defining relations** of a group G, more briefly a **presentation** of G, is an expression $\langle S|R\rangle$, shorthand for $\langle S|\{r = 1|r \in R\}\rangle$, where $G = \langle S\rangle$ and R is such that $ncl_F(R) = \ker\phi$, where $F = F(S)$ and ϕ is the extension to F of the identity embedding of S into G. R is called the set of defining relations.

Notation 1.1.27. To avoid making too subtle distinctions, that may result to be useless, we will commit a slight abuse of language by identifying a presentation of some group with the group itself. So, we will usually speak about 'the group $\langle S|R\rangle$'. Moreover, sometimes instead of some identity $r = 1$ we write $u = v$, if r has the form uv^{-1} and this makes easier to understand the group.

If $u, v \in S \cup S^{-1}$ represent the same element of a group $G = \langle S\rangle$, then we say that $u = v$ is a relation in G, or uv^{-1} is a relator of G. If G has a presentation $\langle S|R\rangle$, then any relation $u = v$ in G is a consequence of the defining relations in R, that is, any relator uv^{-1} belongs to the normal closure $ncl_F(R)$.

Definition 1.1.28. A group G is **finitely presented** if it has a presentation $\langle S|R\rangle$, where both S and R are finite.

Definition 1.1.29. Let G_1, G_2 be groups with presentations $\langle S_i|R_i\rangle$, where $i \in \{1, 2\}$ respectively, with $S_1 \cap S_2 = \varnothing$. The **free product** $G_1 * G_2$ of G_1 and G_2 is the group with presentation $\langle S_1 \cup S_2|R_1 \cup R_2\rangle$.

More generally, the free product $*_{i\in I}G_i$ of a family $\{G_i|i \in I\}$ of groups is the group whose presentation has the disjoint union of generator sets as generator set and the disjoint union of relation sets as relation set. The groups G_i are called the (free) factors of $*_{i\in I}G_i$.

The following lemma shows that the free product is independent of the choice of the presentations chosen for the factors.

Lemma 1.1.30. *The free product $G = G_1 * G_2$ is independent of the choice of the presentations chosen for G_1 and G_2. Moreover, G is generated by subgroups $\bar{G}_1, \bar{G}_2$ of G which are isomorphic to G_1 and G_2 respectively and have trivial intersection.*

Proof. Let $G_i = \langle S_i|R_i\rangle$ with $i = 1, 2$, be presentations of G_i and let $G'_i = \langle S'_i|R'_i\rangle$, with $i = 1, 2$, also be presentations of G_i. Let the generator sets S_i, S'_j be $\{s_{ik}|k \in K\}, \{s'_{jl}|l \in L\}$ respectively. Let $\psi_i : G_i \to G'_i$ be isomorphisms. The map $\psi_1 * \psi_2 : G_1 * G_2 \to G'_1 * G'_2$ defined by $s_{ik} \mapsto \psi_i s_{ik}$ is a homomorphism since relators are mapped to relators. The map $\psi_1^{-1} * \psi_2^{-1}$ defined by $s'_{ik} \mapsto \psi_i^{-1} s'_{ik}$ is the inverse of $\psi_1 * \psi_2$, giving the isomorphism $G_1 * G_2 \cong G'_1 * G'_2$. Let $\bar{G}_i$ be the subgroup of $G_1 * G_2$ generated by the s_i. We are going to show that $\bar{G}_i \cong G_i$. The group G is generated by $\bar{G}_1 * \bar{G}_2$. Define maps $\eta_i : G_i \to \bar{G}_i$ mapping s_{ik} to s_{ik} and consider the projection maps $\pi_i : G \to G_i$ mapping s_{ik} to s_{ik} and s_{jk} to 1 when $j \neq i$. Then, the map $(\eta_i \circ \pi_i) \restriction G_i$ coincides with id_{G_i} and maps all elements of $\bar{G}_j$ to 1 when $j \neq i$. So, $G_i \cong \bar{G}_i$ and $\bar{G}_1 \cap \bar{G}_2 = \{1\}$.

□

By Lemma 1.1.30, we may identify G_i with $\bar{G}_i$ and consider G_1, G_2 as subgroups of $G_1 * G_2$.

Now we define a normal form for elements of a free product, so to define a length in a free product.

Definition 1.1.31. A **normal form** or reduced sequence in a free product $G * H$ is a sequence $g_1, \ldots, g_n$, $n \geq 0$, of elements of $G * H$ such that each g_i is not 1 and belongs to one of G and H, and g_i, g_{i+1} are not in the same factor for $i \leq n-1$.

Theorem 1.1.32 (Normal form theorem for free products)**.** *Let G, H be groups. Then the following two equivalent statements hold.*

1. *If $w \in G * H$ has a normal form $g_1, \ldots, g_n$, $n > 0$, then $w \neq 1$ in $G * H$.*

2. *Each element $w \in G * H$ can be uniquely expressed as a product $w = g_1 \ldots g_n$, where $g_1, \ldots, g_n$ is a normal form. The product of the elements in the empty sequence is 1.*

Proof. We begin showing the equivalence of the two statements. Since 2. ⇒ 1. is immediate, we show that 1. ⇒ 2.. Assume that 1. holds. Let $g_1 \ldots g_n$ and $h_1 \ldots h_m$ be normal forms for w. Then $1 = g_1 \ldots g_n h_m^{-1} \ldots h_1^{-1}$. In order for the sequence $g_1, \ldots, g_n, h_m^{-1}, \ldots, h_1^{-1}$ not to be reduced, it is necessary that h_m be in the same factor as g_n. For the sequence $g_1, \ldots, g_n h_m^{-1}, \ldots, h_1^{-1}$ not to be reduced it is necessary that $g_n h_m^{-1} = 1$, that is, $g_n = h_m$. By induction, we have $m = n$ and $h_i = g_i$ for $i = 1, \ldots, n$, so we have proved the equivalence of the two statements in the conclusion of the theorem.

The proof follows [Art47], and it uses a homomorphism into a permutation group. Let W be the set of normal forms from $G * H$. For each element $g \in G$, define a permutation $\phi_G g \in Sym(W)$, the group of permutations of W, as follows. If $g = 1$, then let $\phi_G(g) = \mathrm{id}_W$. If $g \neq 1$ and $(g_1, \ldots, g_n)$ is a normal form, then

$$\phi_G(g)((g_1, \ldots, g_n)) \begin{cases} (g, g_1, \ldots, g_n) & \text{if } g_1 \in H; \\ (gg_1, \ldots, g_n) & \text{if } g_1 \in G, g_1 \neq g^{-1}; \\ (g_2, \ldots, g_n) & \text{if } g_1 = g^{-1}. \end{cases}$$

Note that the map ϕ_G is actually a homomorphism of G into $Sym(W)$. In fact, $\phi_G(g^{-1}) = (\phi_G g)^{-1}$ and, if $g, g' \in G$, then $\phi_G(gg') = \phi_G(g) \circ \phi_G(g')$. Define a homomorphism ϕ_H in a similar way. The homomorphisms above defined induce a homomorphism $\phi_G * \phi_H : G * H \to Sym(W)$. Any element $w \in G * H$ can be written as some product $g_1 \ldots g_n$, where $g_1, \ldots, g_n$ is a normal form. Note that $(\phi_G * \phi_H)(w)(1) = (g_1, \ldots, g_n)$.

□

Definition 1.1.33. Let $w \in G * H$ have normal form $(g_1, \ldots, g_n)$. Then, the **length** of w in the free product $G * H$, denoted by $|w|_{G*H}$, is n.

As in the case of a free group, we will omit the index $G * H$ when it is clear from the context.

As for free groups, also for free products we have a conjugacy theorem, whose proof is analogous to that of Proposition 1.1.13.

Theorem 1.1.34 (Conjugacy theorem for free products). *Let G_1, G_2 be finitely generated groups. Then, every element of $G = G_1 * G_2$ is conjugate to a cyclically reduced element. Moreover, if $u = \prod_{1 \leq i \leq n} g_i$ and $v = \prod_{1 \leq j \leq m} h_j$ are cyclically reduced and $u \sim_G v$, then $m = n$ and:*

1. *if $n > 1$, then for every i we have $h_i = g_{\sigma(i)}$ for some cyclic permutation σ of $\{1, \ldots, n\}$;*
2. *if $n \leq 1$, then u and v are in the same factor G_i and $u \sim_{G_i} v$.*

□

The following theorem has been proved independently by Grushko [Gru40] and Neumann [Neu43]. The proof contained in [LS77, Proposition III.3.7] is due to Stallings ([Sta65]).

Theorem 1.1.35 (Grushko-Neumann theorem). *Let F be a free group and let $\phi : F \to *A_i$ be a epimorphism. Then there is a factorization of F as a free product $F = *F_i$, such that $\phi(F_i) = A_i$.*

A consequence of Grushko-Neumann theorem is that, if G_1 and G_2 are finitely generated groups, then $rk(G_1 * G_2) = rk(G_1) + rk(G_2)$, where the rank of a group G is defined as the minimum rank of a free group F such that G is isomorphic to a quotient of F.

Another important consequence is the following

Theorem 1.1.36 (Grushko decomposition theorem). *Let G be a finitely generated group. Then G has a decomposition as*

$$G = *_{1 \leq i \leq r} A_i * F_s,$$

where $s \geq 0, r \geq 0$ and each of the groups A_i is non-trivial, freely indecomposable and not infinite cyclic, and where F_s is a free group of rank s.

Note that there is a canonicity in the decomposition, because the numbers s and r are unique and the groups A_i are unique up to a permutation of their conjugacy classes in G.

Besides its self-contained interest and its many applications, the above theorem has a particular importance as it may be considered the starting point of JSJ theory - see Section 3.3.

We end this subsection with a definition that will be intensively used throughout the whole work.

Definition 1.1.37. Let G be a group and let $A \leq G$. G is **freely decomposable** with respect to A, or freely decomposable relative to A, or freely A-decomposable, if there exist non-trivial subgroups G_1, G_2 such that $G = G_1 * G_2$ and $A \leq G_1$. Otherwise, G is freely A-indecomposable.

When A in Definition 1.1.37 is the trivial group, we simply say 'freely decomposable'.

The following theorem is a freeness criterion for one-relator groups.

Theorem 1.1.38. *[LS77, Proposition II.5.10] Let* $G = \langle x_1, \dots, x_n | r \rangle$. *Then,* G *is free if and only if* $r = 1$ *or* r *is primitive in* $\langle x_1, \dots, x_n | \rangle$.

1.1.3 Amalgamated free products and HNN-extensions

In this subsection we introduce two constructions which are basic to combinatorial group theory. These constructions are the free product with amalgamated subgroup, introduced by Schreier in [Sch26], and Higman-Neumann-Neumann extensions, introduced by G. Higman, B. H. Neumann, and H. Neumann in [HNN49].

Definition 1.1.39. Let G_1, G_2 be two groups of presentations $\langle S_i | R_i \rangle$, where $i = 1, 2$ respectively. Let $A_i \leq G_i$, and let $\phi : A_1 \to A_2$ be an isomorphism. The **free product** of G_1 and G_2, **amalgamating** the subgroups A_1 and A_2 by the isomorphism ϕ is the group

$$\langle S_1, S_2 | R_1, R_2, \{\phi(a) = a | a \in A_1\} \rangle.$$

Note that the amalgamated free product depends on G_i, A_i and ϕ. So, a way to denote it is $G_1 *_{A_1, A_2, \phi} G_2$. When the dependencies are clear from the context, and it is often the case, we use the easier notation $G_1 *_{A_1} G_2$.

Definition 1.1.40. Let G be a group of presentation $\langle S | R \rangle$. Let $A_i \leq G$, where $i = 1, 2$, and let $\phi : A_1 \to A_2$ be an isomorphism. The **HNN-extension**, of G **relative to** the subgroups A_1 and A_2 and the isomorphism ϕ is the group

$$\langle S, t | R, \{a^t = \phi(a) | a \in A_1\} \rangle.$$

The group G is called the base, while the letter t is called the stable letter of the extension.

Like for the amalgamated free product, note that the HNN-extension depends on G, A_i, ϕ and t. So, a way to denote it is $G*_{A_1, A_2, \phi, t}$. When the dependencies are clear from the context, and it is often the case, we use the easier notation $G*_{A_1}$.

We can generalize definitions 1.1.39 and 1.1.40, to an arbitrary number of base groups or subgroups. Let $\{G_i | i \in I\}$ and A be groups, and let $\{\phi_i : A \to G_i | i \in I\}$ be monomorphisms. Then, the amalgamated free product of the G_i over the subgroups $\phi_i(A)$ is the group

$$\langle \coprod S_i | \coprod R_i, \{\phi_i(a) = \phi_j(a) | a \in A, i, j \in I\} \rangle,$$

where G_i have presentations $\langle S_i | R_i \rangle$, respectively.

In an analogous way, let G be a group, let $\{A_i | i \in I\}, \{B_i | i \in I\}$ be families of subgroups of G and let $\{\phi_i : A_i \to B_i | i \in I\}$ be isomorphisms. Then, the HNN-extension of G over the subgroups A_i and B_i and stable letters t_i respectively, is the group

$$\langle S, \{t_i | i \in I\} | R, \{a_i^{t_i} = \phi_i(a_i) | a_i \in A_i\} \rangle,$$

where G has presentation $\langle S | R \rangle$.

In the rest of the subsection we will state some results about canonicity of expressions of elements in HNN-extensions and amalgamated free products. First we will deal with HNN-extensions, then we will show simialr facts for amalgamated free products.

Definition 1.1.41. A sequence $g_0, t^{\epsilon_1}, g_1, \ldots, t^{\epsilon_n}, g_n$, with $n \geq 0$, is said to be **reduced** if there is no consecutive subsequence t^{-1}, g_i, t with $g_i \in A$ or t, g_i, t^{-1} with $g_i \in \phi(A)$.

The following theorem is the analogue of Theorem 1.1.32. Like for free products, also for HNN-extensions there is a notion of normal form, but unlike in free products, unicity of normal forms involves a rather innatural choice of sets of cosets representatives. Since for our purposes what is important is that G is embedded in G^* and that we have a criterion for telling when a word of G^* is not identity in G^*, in our work we will avoid doing such a choice. Due to this fact, the following theorem states only one of the two results of [LS77, Theorem IV.2.1], though actually it is not a great loss, since the two results are equivalent.

In the following theorem, the first statement was proved by Higman, Neumann and Neumann in [HNN49], while the second statement of the same conclusion was proved by Britton in [Bri63]; in fact, it is also known as Britton's lemma.

Before stating the theorem, we give a precisation about some notation that we will use in the theorem and after.

Notation 1.1.42. From now on, to avoid making notations too heavy, we will eventually commit the following abuse of language. If G is a group generated by some set S_G, we will use G instead of S_G in the presentation of some group obtained by G. For instance, we will eventually denote $\langle G, t | A^t = \phi(A) \rangle$ the HNN-extension of G over a subgroup $A \leq G$, instead of the more correct notation $\langle S_G, t | \{a^t = \phi(a) | a \in A\} \rangle$.

Theorem 1.1.43. *Let G^* be the HNN-extension $\langle G, t | A^t = \phi(A) \rangle$. Then the group G is embedded in G^* by identity embedding. Moreover, if $g_0 t^{\epsilon_1} g_1 \ldots t^{\epsilon_n} g_n = 1$ in G^*, then the sequence $g_0, t^{\epsilon_1}, g_1, \ldots, t^{\epsilon_n}, g_n$ is not reduced.*

The proof is, as said above, in [LS77, Theorem IV.2.1].

□

Notation 1.1.44. To avoid making notation too heavy without an enough important reason, we will not formally distinguish between a sequence

$$g_0, t^{\epsilon_1}, g_1, \ldots, t^{\epsilon_n}, g_n$$

and the product

$$g_0 t^{\epsilon_1} g_1 \ldots t^{\epsilon_n} g_n,$$

if it is clear from the context which of the two is actually meant.

By notation 1.1.44, if a word w has the form $g_0 t^{\epsilon_1} g_1 \ldots t^{\epsilon_n} g_n$, we say that w is reduced if the sequence $g_0, t^{\epsilon_1}, g_1, \ldots, t^{\epsilon_n}, g_n$ is reduced.

The following lemma allows to define a length in an HNN-extension, even without unicity of a normal form.

Lemma 1.1.45. *Let G^* be as in Theorem 1.1.43. Let $w = g_0 t^{\epsilon_1} g_1 \ldots t^{\epsilon_n} g_n$ and $v = h_0 t^{\eta_1} h_1 \ldots t^{\eta_m} h_m$ be reduced words, and suppose that $u = v$ in G^*. Then $m = n$ and $\epsilon_i = \eta_i$ for any i.*

Proof. The proof is similar to the proof of 1. ⇒ 2. in Theorem 1.1.32. The base of the induction is clear. To prove the inductive step, since $w = v$, we have

$$g_0 t^{\epsilon_1} g_1 \dots t^{\epsilon_n} g_n h_m^{-1} t^{-\eta_m} \dots h_1^{-1} t^{-\eta_1} h_0^{-1} = 1.$$

Since both w and v are reduced, the only way the sequence can fail to be reduced is that $\epsilon_n = \eta_m$ and $g_n h_m^{-1}$ is in A or $\phi(A)$, depending on the sign ϵ_n. The inductive step can be proved considering one of the following operations, that can reduce a sequence:

1. replace a subword $t^{-1}gt$, where $g \in A$, by $\phi(g)$;
2. replace a subword tgt^{-1}, where $g \in \phi(A)$, by $\phi^{-1}(g)$.

So the inductive step, therefore the lemma also is proved.

□

Definition 1.1.46. Let G^* be as in Theorem 1.1.43, and let $w \in G^*$ have a reduced form $w' = g_0 t^{\epsilon_1} g_1 \dots t^{\epsilon_n} g_n$. Then, the **length** of w in the HNN-extension G^*, denoted by $|w|_{HNN}$ or $|w|_t$, is the number n of occurrences of t^ϵ in w'.

By Lemma 1.1.45, $|w|_t$ is well-defined. By the above definition, all elements of the base G have length 0. As in the previous cases, we will omit the index HNN or t when it is clear from the context.

Also in HNN extensions there is a natural notion of 'cyclically reduced '. An element $w = g_0 t^{\epsilon_1} g_1 \dots t^{\epsilon_n}$ is cyclically reduced if all cyclic permutations of the sequence $g_0, t^{\epsilon_1}, g_1, \dots, t^{\epsilon_n}$ are reduced.

Like for free products and free groups, we have a conjugacy theorem for HNN-extensions. This is due to Collins, who proved it in [Col69].

Theorem 1.1.47 (Conjugacy theorem for HNN-extensions)**.** *Let G^* be as in Theorem 1.1.43. Then, every element of G^* is conjugate to a cyclically reduced element. Moreover, if $u = g_0 t^{\epsilon_1} g_1 \dots t^{\epsilon_n}$ and v are conjugate cyclically reduced elements of G^*, then $|u| = |v|$ and u can be obtained from v by taking a suitable cyclic permutation of v ending in t^{ϵ_n} and then conjugating by an element z belonging to A or $\phi(A)$, if $\epsilon_n = -1$ or $+1$ respectively.*

The proof is in [LS77, Theorem IV.2.5]; its structure is similar to that of Theorem 1.1.34.

□

In the following part of the current subsection we state some results about amalgamated free products.

Also here, a precisation about notation: for the rest of this subsection the group $G = G_1 *_{A_1, A_2, \phi} G_2$ will be an amalgamated free product.

Definition 1.1.48. A sequence $g_1, \dots, g_n$, with $n \geq 0$, of elements of the amalgamated free product G, is said to be **reduced in** G if the following conditions hold:

1. the sequence $g_1, \dots, g_n$ is reduced in the sense of free products (see Definition 1.1.31;
2. if $n > 1$, then no g_i belongs to any A_i.

Note that every element of G can be expressed by a reduced sequence.

The following theorem is the analogue of Theorem 1.1.43. In the following theorem, the first statement was proved by Higman, Neumann and Neumann in [HNN49], while the second statement of the same conclusion was proved by Britton in [Bri63]; in fact, it is also known as Britton's lemma.

In the following theorem, conclusions 1. and 2. are equivalent. The first statement of conclusion 1. was proved by Higman, Neumann and Neumann in [HNN49], while the second statement of the same conclusion was proved by Britton in [Bri63]; in fact, it is also known as Britton's lemma.

We remind that the uses reported in notations 1.1.42 and 1.1.44 keep their validity even in the current context.

Theorem 1.1.49. *Let G be the amalgamated free product $G = G_1 *_{A_1,A_2,\phi} G_2$. Then, G_1 and G_2 are embedded in G by the identity embeddings. Moreover, if $g_1 \dots g_n$, with $n \geq 1$, is reduced, then $g_1 \dots g_n \neq 1$ in G.*

The proof of the above theorem is in [LS77, Theorem IV.2.6]; here we give the parts that differ more from the analogous theorems in the previous cases.

Let G^* be the group $\langle G_1 * G_2, t | A_1^t = A_2 \rangle$. Define $\psi : G \to G^*$ by

$$\psi(g) \begin{cases} g^t & \text{if } g \in G_1, \\ \psi(g) = g & \text{if } g \in G_2. \end{cases}$$

The map ψ is actually a homomorphism since it maps the defining relations of G to 1. If $n = 1$ and $g \in A_1 \setminus \{1\}$, then $\psi(g) = g^t = \phi(g) \neq 1$. In all other cases, ψ maps a reduced sequence $g_1, \dots, g_n$ of elements of G to a reduced sequence of elements of G^*. The result thus follows from Theorem 1.1.43, whose proof, we recall, is in [LS77, Theorem IV.2.1].

□

The above proof shows that ψ is an embedding. Thus G is isomorphic to the subgroup of G^* generated by G_1^t and G_2.

As with HNN-extensions, there is an equivalent statement of the normal form theorem which involves choosing coset representatives for A_i, so to obtain a unique representation; as for HNN-extensions, in amalgamated free products we can define a length: given an element $w \in G$, its length $|w|_{G_1 *_{A_1} G_2}$ is the number n in a reduced form $w' = g_1 \dots g_n$ of w. As in the previous cases, we will omit the index when it is clear from the context.

The notion of 'cyclically reduced' in amalgamated free products is, as naturally as in the previous cases, the following: an element $g_1 \dots g_n$ is cyclically reduced if all of its cyclic permutations are reduced.

Like for free products and free groups, we have a conjugacy theorem for HNN-extensions.

Theorem 1.1.50 (Conjugacy theorem for amalgamated free products)**.** *Let G be as in Theorem 1.1.49. Then, every element of G is conjugate to a cyclically reduced element. Moreover, if $u = g_1 \dots g_n$ is cyclically reduced and $|u| \geq 2$, then every cyclically reduced conjugate of u can be obtained by cyclically permuting $g_1 \dots g_n$ and then conjugating by an element of the amalgamated subgroup A_1.*

The proof uses the embedding ψ defined in the proof of Theorem 1.1.49, then follows the proof of Theorem 1.1.47, that, we recall, is in [LS77, Theorem IV.2.5].

□

1.2 Bass-Serre theory

Bass-Serre theory studies how a group acts on a tree in order to give information about structure of the group. Its key-point may be located in the structure theorem 1.2.35, that gives a decomposition of a group acting on a tree as a graph of groups. The starting point that led to Bass-Serre theory may be found in the classical result stating that a group G acts freely and without inversion on a simplicial tree if and only if G is free. This result has led to the proof of Nielsen-Schreier theorem, affirming that a subgroup of a free group is free, and giving a method to compute its rank whenever it is finitely generated. Bass-Serre theory developes from an original idea by Bass [Bas76] of studying group decompositions by the way of making a group act on a simplicial tree and then studying its quotient under the group action. The theory builds on exploiting and generalizing properties of the two previously treated group-theoretic constructions, amalgamated free product and HNN-extension. A comprehensive reference can be found in [Ser80]; a more recent book, linking this topic with other developments like train tracks theory by Bestvina and Handel about automorphisms of free groups, is [Bog08]. Before dealing with the core topic, we need some preliminaries. We will introduce some notions on graphs, to define Cayley graphs. A Cayley graph is a graph encoding a group structure. The study of Cayley graphs has been one of the first steps in geometric group theory; indeed, it may be considered one of the starting points to the study of 'coarse geometry'. In fact, two Cayley graphs of the same group may be far from being isometric, but they are always quasi-isometric (see next section). A Cayley graph gives the possibility of making a group act on itself. Moreover, it is possible to make a group into a metric space, with the so-called word metric as its distance.

1.2.1 Graphs; Cayley graphs

Definition 1.2.1. A **graph** is a quadruple $\Gamma = (\Gamma^0, \Gamma^1, b, ^-)$, where Γ^0 and Γ^1 are sets, b and $^-$ are applications

$$b\colon \Gamma^1 \to \Gamma^0 \times \Gamma^0$$

$$e \mapsto (\alpha(e), \omega(e))$$

$$^-\colon \Gamma^1 \to \Gamma^1$$

$$e \mapsto \bar{e}$$

such that, for every $e \in \Gamma^1$, we have $\bar{\bar{e}} = e$, $\bar{e} \neq e$ and $\omega(e) = \alpha(\bar{e})$.

Elements of Γ^0 are called **vertices** of Γ and elements of Γ^1 are called **edges** of Γ. $\alpha(e)$ and $\omega(e)$ are the beginning and the end of the edge e, respectively. $\bar{e}$ is the inverse edge of e.

Definition 1.2.2. An orientation of a graph Γ is a subset $\Gamma^{1+} \subset \Gamma^1$ such that $\Gamma^1 = \Gamma^{1+} \amalg \overline{\Gamma^{1+}}$, where $\overline{\Gamma^{1+}} = \{\bar{e} | e \in \Gamma^{1+}\}$. An oriented graph is a graph equipped with an orientation.

Definition 1.2.3. Let Γ be an oriented graph and let $v \in \Gamma^0$. The **degree** of v, denoted by $deg(v)$, is the sum of the number of edges $e \in \Gamma^{1+}$ such that $\alpha(e) = v$, plus the number of edges $e \in \Gamma^{1+}$ such that $\omega(e) = v$.

Let v be a vertex of some graph Γ. If $deg(v) = 0, 1, 2, \geq 3$, then v is called an isolated vertex, an endpoint, an ordinary vertex, a branching point, respectively.

Definition 1.2.4. Let Γ be a graph. A **path** in Γ is a sequence $(e_1, \ldots, e_n)$ of edges such that $\omega(e_i) = \alpha(e_{i+1})$ for every $i = 1, \ldots, n-1$. The **length** of the path is n. A path $(e_1, \ldots, e_n)$ is closed if $\omega(e_n) = \alpha(e_1)$.

Definition 1.2.5. A path $(e_1, \ldots, e_n)$ is **reduced** if $e_{i+1} \neq \bar{e}_i$ for every $i = 1, \ldots, n-1$. A path $(e_1, \ldots, e_n)$ is **cyclically reduced** if it is reduced and $e_n \neq \bar{e}_1$.

Definition 1.2.6. A **circuit** is a closed cyclically reduced path $(e_1, \ldots, e_n)$ of length $n > 0$ such that $\alpha(e_i) \neq \alpha(e_j)$ for every $i \neq j$.

Definition 1.2.7. A graph Γ is **connected** if for every $v_1, v_2 \in \Gamma^0$, there exists a path $(e_1, \ldots, e_n)$ such that $\alpha(e_1) = v_1$, $\omega(e_n) = v_2$.

Definition 1.2.8. A (simplicial) **tree** is a connected graph without circuits.

The following lemma is necessary to introduce the notion 1.2.21 below.

Lemma 1.2.9. *Let T be a maximal subtree (with respect to inclusion) of a connected graph Γ. Then T contains all vertices of Γ.*

Proof. Suppose for a contradiction that this is not true. Then, since Γ is connected, there exists an edge e beginning in T and ending outside T. Adding the edges e and $\bar{e}$ and the vertex $\omega(e)$ to T, we obtain a tree strictly including T, which contradicts to maximality of T.

□

We can give a group a distance:

Definition 1.2.10. Let Γ be a connected graph. A **distance** d on Γ^0 is defined as

$$d(v_1, v_2) = \min\{n | (e_1, \ldots, e_n) \text{ is a path from } v_1 \text{ to } v_2\}.$$

This gives Γ^0 a metric space structure.

Definition 1.2.11. Let G be a group, $S \subseteq G$. An oriented graph $\Gamma(G, S)$ is defined as below:

1. $\Gamma(G, S)^0 = G$;

2. $\Gamma(G, S)^{1+} = G \times S$;

3. for every $g \in G, s \in S$, define $\alpha(g, s) = g, \omega(g, s) = g \cdot_G s$.

In other words, there exists a positively oriented edge from g to h if and only if there exists $s \in S$ such that $h = gs$.

Note that $\Gamma(G, S)$ is connected if and only if S is a generating set for G.

Definition 1.2.12. With the same notation as in Definition 1.2.11, if $\Gamma(G, S)$ is connected then it is called the **Cayley graph** of G with respect to S, and it is denoted by $\mathrm{Cay}(G, S)$.

Given a connected graph, we can make it into a metric space in a quite natural way. Before doing it, we need some definitions. Since we will speak about isometrical embeddings, we recall some fundamental notions about isometries.

Definition 1.2.13. Let $(X, d_X), (Y, d_Y)$ be metric spaces. An **isometry** between X and Y is a map $f : X \to Y$ such that for every $x_1, x_2 \in X$ we have $d_Y(fx_1, fx_2) = d_X(x_1, x_2)$. When $X = Y$, we will say 'an isometry of X'.

Definition 1.2.14. Let (X, d) be a metric space and let f be an isometry of X. The **length** of f is defined as

$$|f| = \inf_{x \in X} d(x, fx),$$

the infimum of the displacements of the points of X by f.

Definition 1.2.15. Let (X, d) be a metric space. An isometry of X is said to be

1. **elliptic** if $|f| = 0$ and it is actually a minimum, that is, if f fixes a point in X;
2. **parabolic** if $|f| = 0$ but this infimum is not reached in X;
3. **hyperbolic**, if $|f| > 0$.

Definition 1.2.16. A metric space X is said to be **geodesic** if for every pair of points $x, y \in X$ there exists a geodesic segment joining x and y.

Definition 1.2.17. Let Γ be a connected graph. The **realization** of Γ, denoted by $real(\Gamma)$, is a geodesic metric space, constructed as follows.

Let $I = [0, 1]$. Give $\Gamma^0 \cup \Gamma^1$ discrete topology. Define

$$U(\Gamma) = (\Gamma^1 \times I) \cup \Gamma^0,$$

where $\Gamma^1 \times I$ is given product topology and $U(\Gamma)$ is given disjoint union topology.

Define an equivalence relation $\sim$ on $U(\Gamma)$ as

$$\{(e, t) \sim (\bar{e}, 1 - t), \alpha(e) \sim (e, 0)\}.$$

Define

$$real(\Gamma) = U(\Gamma) / \sim,$$

with quotient topology.

We can give $real(\Gamma)$ a distance, as follows.

For every $e \in \Gamma^1$, let

$$real(e) = \{[e, t] | t \in I\},$$

where $[e, t]$ is the $\sim$-equivalence class of (e, t).

Let $x, y \in real(\Gamma)$ and let $P(x, y)$ be the set of sequences

$$V = (v_0, e_1, v_1, \dots, e_n, v_n),$$

where $v_0 = x, v_n = y, e_i \in \Gamma^1$ and $v_{i-1}, v_i \in real(e_i)$.

Define

$$L_V = \sum_{i=1,\dots,n} d_{e_i}(v_{i-1}, v_i),$$

$$d_{e_i}([e,t],[e,s]) = |t-s|.$$

Now, define

$$d_\Gamma(x,y) = \inf\{L_V | V \in P(x,y)\};$$

this is our distance in $real(\Gamma)$; in fact, d_Γ actually is a metric for $real(\Gamma)$; $(real(\Gamma), d_\Gamma)$ becomes a geodesic metric space. Γ embeds into $real(\Gamma)$ isometrically, and the function

$$d_\Gamma(x, \Gamma) : real(\Gamma) \to \mathbb{R}$$

is bounded.

Definition 1.2.18. A **simplicial metric space** is a metric space which is the realization of some (connected) graph.

Chiswell in [Chi04, Definition, p.52] defines a polyhedral $\mathbb{R}$-tree as a real tree with finitely many branching points and finitely many endpoints. So, a simplicial $\mathbb{R}$-tree is a polyhedral $\mathbb{R}$-tree where length of every edge is 1.

1.2.2 Graphs of groups, actions on trees

Definition 1.2.19. A **graph of groups** (Γ, G) consists of a connected graph Γ, a vertex group G_v for each vertex $v \in \Gamma^0$, an edge group G_e for each edge $e \in \Gamma^1$ such that $G_e = G_{\bar{e}}$ for every $e \in \Gamma^1$, and monomorphisms $\{\alpha_e : G_e \to G_{\alpha_e} | e \in \Gamma^1\}$.

Sometimes we will consider the underlying graph as oriented; in this case we will also speak of monomorphisms $\{\omega_e : G_e \to G_{\omega_e} | e \in \Gamma^1\}$.

Let $F(\Gamma, G)$ be the group

$$F(\Gamma, G) = \langle \{G_v\}_{v\in\Gamma^0}, \{t_e\}_{e\in\Gamma^1} | \{t_{\bar{e}} = t_e^{-1}\}_{e\in\Gamma^1}, \{\alpha_{\bar{e}}(g) = t_e^{-1}\alpha_e(g)t_e\}_{e\in\Gamma^1, g\in G_e}\rangle.$$

Definition 1.2.20. For each edge e, the element t_e is called the **generating element** of e.

Definition 1.2.21. Let (Γ, G) be a graph of groups and let $T \subseteq \Gamma$ be a maximal subtree. The **fundamental group** of (Γ, G) with respect to T, denoted by $\pi_1(\Gamma, G, T)$, is the group

$$\langle F(\Gamma, G) | \{t_e | e \in T^1\}\rangle.$$

Note that the above definition makes sense, since the existence of a maximal subtree is assured by Lemma 1.2.9.

The fundamental group of a graph of groups can be also defined with respect to a vertex of the underlying graph. Before giving such a definition, we need to define a path in a graph of groups.

Definition 1.2.22. Let (Γ, G) be a graph of groups and let $v, v' \in \Gamma^0$. A **path** of (Γ, G) from v to v' is a sequence $(g_0, t_{e_1}, g_1, \dots, t_{e_n}, g_n)$, where $g_i \in G_{v_i}$, $v_0 = v$, $v_n = v'$, $\alpha(e_i) = v_{i-1}$ and $\omega(e_i) = v_i$.

We say that a path is closed if $v = v'$.

Define an equivalence relation $\approx$ on the set of paths of (Γ, G) as follows: $p \approx p'$ if p' can be obtained from p by a finite chain of the following transformations.

1. Replace a subpath $g, t_e, \omega_e(c), t_{\bar{e}}, g'$ by $g\alpha_e(c)g'$, where $g, g' \in G_{\alpha(e)}$ and $c \in G_e$;
2. the inverse of transformation 1;
3. Replace a subpath g, t_e, g' by $g\alpha_e(c), t_e, (\omega_e(c))^{-1}g'$, where $g \in G_{\alpha(e)}$, $g' \in G_{\omega(e)}$ and $c \in G_e$;
4. the inverse of transformation 3.

Note that, if $p \approx p'$, then p and p' have the same initial vertex and the same terminal vertex in Γ^0.

Definition 1.2.23. Let (Γ, G) be a graph of groups and let $v \in \Gamma^0$. The **fundamental group** of (Γ, G) with respect to v, denoted by $\pi_1(\Gamma, G, v)$, is the group of $\approx$-equivalence classes of closed paths of (Γ, G) based in v, with the $\approx$-quotient of path concatenation as group operation.

Proposition 1.2.24. *[Bog08, Corollary 16.7] The fundamental groups $\pi_1(\Gamma, G, v)$ and $\pi_1(\Gamma, G, T)$ are isomorphic for any choice of $v \in \Gamma^0$ and T a maximal subtree in Γ.*

By Proposition 1.2.24, we denote the fundamental group of a graph of groups simply as $\pi_1(\Gamma, G)$ when there is no need to specify any vertex or subtree.

Definition 1.2.25. Let A be a group. A graph of groups (Γ, G) such that $\pi_1(\Gamma, G) = A$ is said also to be a **splitting** of A.

Definition 1.2.26. Let G a group and let $H \leq G$. A splitting of G **relative to** H is a splitting Γ of G such that H^g is contained in some vertex group of Γ for some $g \in G$.

Definition 1.2.27. Let G be a group, let $H \leq G$ and let χ be a class of groups. By a (χ, H)**-splitting** of G we intend a splitting of G relative to H and having edge groups belonging to class χ.

Remark 1.2.28. When H is the trivial group, we will simply say 'χ-splitting'. For instance, we will say 'cyclic splitting', 'abelian splitting' when χ is the class of cyclic or abelian groups, respectively.

The following proposition allows, under certain conditions, to extend an automorphism of a vertex group of a graph of groups to an automorphism of the fundamental group of the whole graph of groups, by a construction called the **standard extension**.

Proposition 1.2.29. *Let Γ be a graph of groups with fundamental group G, let G_v a vertex group of Γ and let $\phi_v \in \mathrm{Aut}(G_v)$. If for every edge e such that $\alpha(e) = v$ there exists $g_e \in G_v$ such that $\phi_v \restriction \alpha_e(G_e)$ is conjugation by g_e, then we can extend ϕ_v to some $\phi \in \mathrm{Aut}(G)$.*

Proof. Take a maximal subtree T of Γ. For any vertex w of Γ, if the path between w and v in T ends by some edge e, define $\phi \restriction G_w$ to be conjugation by g_e. Let t_f be the generating element of an edge f lying outside T and such that $\alpha(f) = w$ and $\omega(f) = w'$. Let e' be the edge ending the path in T between w' and v. Set $\phi(t_f) = g_{e'}^{-1} t g_e$. It is left to the reader to check that ϕ is well-defined and is an automorphism. Extend progressively ϕ to G; such a ϕ is called the standard extension of ϕ_v to G.

□

Let (Γ, G) be a graph of groups and let $v_0 \in \Gamma^0$. Define an equivalence relation $\sim$ on the set of paths of (Γ, G) from v_0, as follows: $p \sim p'$ if and only if

1. both p and p' end in the same $v_1 \in \Gamma^0$;

2. there exists $a \in G_{v_1}$ such that $p \approx p'a$.

We denote the $\sim$-equivalence class of such a path p as pG_{v_1}. Note that each equivalence class contains a unique representative having form

$$a_0, e_1, a_1, \ldots, a_{n-1}, e_n, 1,$$

with $a_i \in R_{e_{i+1}}$, where R_e is a set of left coset representatives of $\alpha_e(G_e)$ in $G_{\alpha(e)}$.

Define the graph $\widetilde{(\Gamma, v_0)}$ as follows:

1. The vertices of $\widetilde{(\Gamma, v_0)}$ are $\sim$-equivalence classes of paths in Γ from v_0. So, each vertex of $\widetilde{(\Gamma, v_0)}$ has form pG_v, where p is a path in Γ from v_0 to some $v \in \Gamma^0$. Let $x_0 := 1G_{v_0}$.

2. Two vertices $x_1, x_2 \in \widetilde{(\Gamma, v_0)}^0$ are connected by an edge $f = (x_1, e, x_2)$, with $\alpha(f) = x_1, \omega(f) = x_2$ and $e \in \Gamma^1$, and there are expressions for x_1 and x_2 having forms $x_1 = pG_{v_1}$, $x_2 = pgeG_{v_2}$, where p is a path in Γ from v_0 to v_1, $g \in G_{v_1}$, $\alpha(e) = v_1$ and $\omega(e) = v_2$.

3. Edge inversion map is defined as $\overline{(v_1, e, v_2)} = (v_2, \bar{e}, v_1)$.

Remark 1.2.30. If $g_0, e_1, g_1, \ldots, e_k, g_k$ and $g'_0, e'_1, g'_1, \ldots, e'_{k'}, g'_{k'}$ are reduced representatives for vertices v and v' and $k \geq k'$, then v and v' are joined by an edge if and only if $k = k' + 1$ and $g_0, e_1, g_1, \ldots, e_{k'}, g_{k'} \sim g'_0, e'_1, g'_1, \ldots, e'_{k'}, g'_{k'}$. The edge joining v and v' is then (v, e_k, v').

Theorem 1.2.31. *The graph $\widetilde{(\Gamma, v_0)}$ is a tree.*

Proof. Let $p = e_1 \dots e_k$ be a non-trivial closed path in $\widetilde{(\Gamma, v_0)}$. We will show that p is not cyclically reduced.

For each $i = 1, \dots, k$, let $v_{i-1} = \alpha(e_i)$, and for each $i = 0, \dots, k-1$ let p_i be a maximal length reduced path from v_0 to v_{i-1}. After a cyclic permutation of the path p, we may assume that $i \neq 0$.

Since v_i is represented by the reduced path $p_i = g_0, f_1, g_1, \dots, f_k, g_k$ and the length of v_i is maximal, by Remark 1.2.30 both v_{i-1} and v_{i+1} are represented by the path $g_0, f_1, g_1, \dots, f_{k-1}, g_{k-1}$; that is, $v_{i-1} = v_{i+1}$. Moreover, by the same remark v_{i-1} is joined to v_i by the edge $e_i = (v_{i-1}, f_k, v_i)$, and v_{i+1} is joined to v_i by the edge $e_{i+1}^{-} = (v_{i+1}, f_k, v_i) = (v_{i-1}, f_k, v_i)$. Thus, p is not reduced.

□

Definition 1.2.32. The tree defined in Theorem 1.2.31 is called the **Bass-Serre tree** of the graph (Γ, G, v_0).

The important property of Bass-Serre tree is that Bass-Serre tree is a universal covering tree of (Γ, G, v_0). That is, the covering $\widetilde{(\Gamma, v_0)} \to (\Gamma, G, v_0)$ factorizes through any other covering of (Γ, G, v_0).

The fundamental group $G = \pi_1(\Gamma, G, v_0)$ acts in a natural way on its Bass-Serre tree, according to [Hat02, Proposition 1.40], as we show below:

let $g = [q] \in G$, where q is a closed Γ-path from v_0, and let $u = pG_v$, where p is a Γ-path from v_0 to $v \in \Gamma^0$. Define the action as follows:

$$g \cdot u = [q] \cdot pG_v := qpG_v.$$

This action is well-defined on $\widetilde{(\Gamma, v_0)}^0$ and preserves adjacency relation. Therefore, G acts simplicially and without inversions on $\widetilde{(\Gamma, v_0)}$.

Now we recall the definition of (pointwise) stabilizer:

Definition 1.2.33. Let S be a set and let G be a group acting on S. Let A be a subset of S. The **pointwise stabilizer** of A in G, denoted by $\mathrm{Stab}_G(A)$, is the set $\{g \in G | ga = a, \forall a \in A\}$. If $s \in S$, we denote by $\mathrm{Stab}_G(s)$ the pointwise stabilizer of the singleton $\{s\}$ in G.

We can characterize stabilizers of vertices and edges of Bass-Serre tree:

Proposition 1.2.34. *Consider the action of* $\pi_1(\Gamma, G, v_0)$ *on* $\widetilde{(\Gamma, v_0)}^0$. *Then:*

1. *for every vertex* $x = pG_v \in \widetilde{(\Gamma, v_0)}^0$, *the stabilizer* $\mathrm{Stab}_G(x)$ *is* $(G_v)^{p^{-1}}$;

2. *for every edge* $\tilde{e} = (pG_{v_1}, e, pgeG_{v_2}) \in \widetilde{(\Gamma, v_0)}^1$, *the stabilizer* $\mathrm{Stab}_G(\tilde{e})$ *is* $(\alpha_e(G_e))^{(pg)^{-1}}$.

Theorem 1.2.31 allows us to construct a tree on which the fundamental group of a given graph of groups acts. In Theorem 1.2.35 below, we show that for any group G and any G-tree T we can associate a graph of groups, with a construction that is inverse to that of Theorem 1.2.31. This is the key point of Bass-Serre theory.

Let T be a H-tree. We may always assume that H acts on T by automorphisms without inversions. For every $h \in H, e \in T^1$ we have $h\bar{e} = \overline{he}, h\alpha(e) = \alpha(he), h\omega(e) = \omega(he)$ and $he \neq \bar{e}$.

We show how to construct a graph of groups Γ with underlying graph $G = H \setminus T$. By $H \setminus T$ we denote the quotient under the following natural equivalence relation $\sim_H$: $v \sim_H v'$ if there exists $h \in H$ such that $hv = v'$. Let Y be a maximal subtree of G. Then, there exists an injective graph homomorphism $\iota : Y \to T$ such that $\pi \circ \iota = \mathrm{id}_Y$, with $\pi : T \to G$ the canonical projection. We can extend ι to Γ^1 such that for every edge $e \in \Gamma^1 \setminus Y^1$ either $\iota(\alpha(e)) = \alpha(\iota(e))$ or $\iota(\omega(e)) = \omega(\iota(e))$.

Unless G is a tree, the resulting map will not respect graph structure of G. So, if G is not a tree, we may assume that there exists $e \in \Gamma^1 \setminus Y^1$ such that $\iota(\alpha(e)) \neq \alpha(\iota(e))$.

Define $G_v = \mathrm{Stab}_H\, \iota(v)$ for every $v \in Y^0 = \Gamma^0$, and $G_e = \mathrm{Stab}_H\, \iota(e)$ for every $e \in \Gamma^1$. Now define the monomorphisms α_e as follows:

1. for every e such that $\iota(\alpha(e)) = \alpha(\iota(e))$, define α_e as the inclusion map;
2. for every e such that $\iota(\alpha(e)) \neq \alpha(\iota(e))$, choose $h_e \in H$ such that $h_e\iota(\alpha(e)) = \alpha(\iota(e))$ and define α_e mapping g to g^{h_e}.

For every $e \in \Gamma^1 \setminus Y^1$ such that $\iota(\alpha(e)) = \alpha(\iota(e))$, put $h_{e^{-1}} = h_e^{-1}$. As some arbitrary choices were made during the construction of Γ, the result is not unique. However, any two graphs of groups obtained as defined above are equivalent under the following relation $\sim$:

Let Γ_1, Γ_2 be two graphs of groups with underlying graph G. Define the equivalence relation $\sim$ as follows: $\Gamma_1 \sim \Gamma_2$ if and only if there exist a graph isomorphism $j : G_1 \to G_2$, isomorphisms $\eta_e : G_e \to G_{j(e)}$ for every $e \in \Gamma^1$, isomorphisms $\eta_v : G_v \to G_{j(v)}$ for every $v \in \Gamma^0$ an elements $x_e \in G_{\alpha(e)}$ such that $\eta_{\alpha(e)}(\alpha_e(g)) = (\alpha_{j(e)}(\eta_e(g)))^{x_e}$ for every $e \in \Gamma^1$ and $g \in G_e$.

Let Γ, T be as above and $v_0 \in \Gamma^0$. Now we construct an isomorphism

$$\phi : \pi_1(\Gamma, G, v_0) \to H$$

and a ϕ-equivariant graph isomorphism

$$f : \widetilde{(\Gamma, v_0)} \to T.$$

For any $v \in G$, let $p_v = e_{v1}, \dots, e_{vm_v}$ be the unique reduced path in $Y \subset G$ joining v_0 and v. Let $p_v^* = 1, e_{v1}, 1, \dots, 1, e_{vm_v}, 1$ the associated Γ-path. Recall that $\widetilde{(\Gamma, v_0)}$ is generated by the equivalence classes $[(G_v)^{p_v^*}]$ for every $v \in \Gamma^0$ and the elements $h_e := [p_{\alpha(e)}^*, e, \overline{p_{\omega(e)}^*}]$ for every $e \in \Gamma^1 \setminus Y^1$.

Define a homomorphism $\phi : \pi_1(\Gamma, v_0) \to H$ extending the maps $[(g_v)^{p_v^*}] \mapsto g_v$ for every $g_v \in G_v$ and $h_e \mapsto g_e$ for every $e \in \Gamma^1 \setminus Y^1$.

Define $f : \widetilde{(\Gamma, v_0)} \to T$, by mapping $p_v G_v$ to $\iota(v)$ and extending equivariantly to the whole $\widetilde{(\Gamma, v_0)}^0$ by putting $f(g \cdot p_v G_v) = \phi(g) f(p_v G_v)$ for every $g \in \pi_1(\Gamma, G, v_0)$ and $v \in \Gamma^0$. Note that this map is well-defined and extends to $\widetilde{(\Gamma, v_0)}^1$, since adjacency is preserved. Note that ϕ is bijective on edge and vertex stabilizers.

Theorem 1.2.35. *Let ϕ, f be those defined above. Then, ϕ is an isomorphism, f is a tree isomorphism and $f(gx) = \phi(g)f(x)$ for every $g \in \pi_1(\Gamma, G, v_0)$ and $x \in \widetilde{(\Gamma, v_0)}$.*

Proof. Since equivariance follows from construction of ϕ and f, we are left to show that both maps are bijective.

Claim 1. *ϕ is surjective.*

Proof. We have to verify that $H = K$, where

$$K = \langle \bigcup_{v \in Y^0} G_v \cup \{g_e | e \in \Gamma^1 \smallsetminus Y^1\}\rangle.$$

Note that $\mathrm{Stab}_H(v) \subseteq K$ for every $v \in Y^0$. Therefore, to see that $H = K$ it suffices to show that $K(Y^0) = T^0$. Suppose for a contradiction that $K(Y^0) \neq T^0$. Since T is connected and bot $K(Y^0)$ and T^0 are K-invariant, there exist adjacent vertices $v \in Y^0$ and $w \in T^0 \smallsetminus K(Y^0)$. Therefore, there exists an edge e_1 such that $\alpha(e_1) = v$ and $\omega(e_1) = w$. By definition of Y and K, there exists an edge e_2 from v to some $w' \in K(Y^0)$ such that there exists $g \in \mathrm{Stab}_H(v)$ with $ge_1 = e_2$. But since $\mathrm{Stab}_H(v) \leq K$, also $w = gw'$ belongs to $K(Y^0)$, a contradiction.

□ (Claim 1.)

By definition of f, surjectivity of ϕ implies surjectivity of f.

Claim 2. *f is injective.*

Proof. We prove that f is locally injective; this implies injectivity of f. Suppose for a contradiction that f is not locally injective. Then there exist two edges e_1, e_2 from some v such that $f(e_1) = f(e_2)$. In particular $ge_1 = e_2$ for some $g \in \mathrm{Stab}_H(v) \smallsetminus \mathrm{Stab}_H(e_1)$. So, $\phi(g) \in \mathrm{Stab}_H(f(e_1)) = \mathrm{Stab}_H(f(e_2))$, a contradiction with bijectivity of restrictions of ϕ to edge stabilizers.

□ (Claim 2.)

Claim 3. *ϕ is injective.*

Proof. Let $g \in \pi_1(\Gamma, G, v_0) \smallsetminus \{1\}$. If g is elliptic, then $\phi(g) \neq 1$, since ϕ is bijective on stabilizers. If g is hyperbolic, then $gv \neq v$ for some vertex v; therefore $\phi(g)f(v) = f(gv) \neq f(v)$ by injectivity of f. Thus, $\phi(g)$ acts non-trivially on T, so $\phi(g) \neq 1$.

□ (Claim 3 and Theorem 1.2.35.)

An important application of Bass-Serre theory is the following theorem, due to Kurosh. The proof below, utilizing Bass-Serre theory, is easier than the original proof by Kurosh.

Theorem 1.2.36 (Kurosh subgroup theorem)**.** *[Kur37] Let $G = A * B$ and let $H \leq G$. Then there exist subgroups $(A_i \leq A)_{1 \leq i \leq n}$, $(B_j \leq B)_{1 \leq j \leq m}$, $(g_i, g'_j \in G)_{1 \leq i \leq n, 1 \leq j \leq m}$ and a set $X \subseteq G$ such that*

$$H = (*_{1 \leq i \leq n} A_i^{g_i}) * (*_{1 \leq j \leq m} B_j^{g'_j}) * \langle X|\rangle.$$

Proof. Let (Γ, G) be a graph of groups with two vertices a, b such that $G_a = A, G_b = B$, and one edge e such that G_e is trivial. Then, $\pi_1(\Gamma, G) \cong G$. Let $T = \widetilde{(\Gamma, a)}$. Since H acts on T, consider the quotient graph of groups $Z = H \setminus T$. The vertex groups of Z are subgroups of G-stabilizers of vertices of T, so every vertex group of Z is conjugate in G to a subgroup of A or B. The edge groups of Z are trivial since G-stabilizers of edges of T are trivial. By Theorem 1.2.35, we have $H \cong \pi_1(Z)$. Since edge groups of Z are trivial, $H = *_{w \in Z^0} G_w * \pi_1(\underline{Z})$, where $\underline{Z}$ is the underlying graph of Z.

□

Definition 1.2.37. Let Γ be a graph of groups with fundamental group G. Let T be the corresponding Bass-Serre tree. Since vertex groups of Γ are stabilizers of vertices of T, after Definition 1.2.15 and point 1 of Proposition 1.2.34, a subgroup $A \leq G$ is said to be **elliptic** in the splitting Γ if A is a subgroup of a conjugate of some vertex group of Γ. Otherwise, it is said to be **hyperbolic**.

We end this section with the following theorem by Dyer and Scott. It will play a fundamental role for understanding the relation between algebraic and definable closure. The following definition is in some sense dual to that of stabilizer (Definition 1.2.33).

Definition 1.2.38. Let G be a group and let $f \in \mathrm{End}(G)$. The **fixed set** of f in G, denoted by $\mathrm{Fix}_G(f)$, is the set $\{g \in G | f(g) = g\}$.

Theorem 1.2.39. *[DS75, Theorem 2] Let F be a free-by-finite group and let $f \in \mathrm{Aut}(F)$ be of finite order. Then, $\mathrm{Fix}_F(f)$ is a free factor of F.*

Corollary 1.2.40. *Let F be a non-abelian free group and let $A \leq F$. If $\mathrm{Aut}_A(F)$ is finite, then $\mathrm{Aut}_A(F) = 1$, where we denote by $\mathrm{Aut}_A(F)$ the group of A-automorphisms, that is automorphisms of F that fix A pointwise; see Definition 1.4.15 below.*

Proof. Let $f \in \mathrm{Aut}_A(F)$ with $f \neq \mathrm{id}_F$. Then f has a finite order. Hence, by Theorem 1.2.39, $\mathrm{Fix}_F(f)$ is a free factor of F. Since $f \neq \mathrm{id}_F$, $\mathrm{Fix}_F(f)$ is a proper free factor of F. Since $f \in \mathrm{Aut}_A(F)$, we have $A \leq \mathrm{Fix}_F(f)$. Since $\mathrm{Fix}_F(f)$ is a proper free factor of F, we can construct an automorphism of infinite order fixing $\mathrm{Fix}_F(f)$ pointwise, so we have a contradiction and the statement is proved.

□

1.3 Hyperbolic groups

The concept of a group that could be seen, through its Cayley graph, as a hyperbolic metric space appeared first in Gromov's papers [Gro81b] and [Gro84]; a comprehensive reference can be found in his subsequent work [Gro87], where he explicitly gives some different equivalent definitions of hyperbolic groups. The importance of this class of groups lies in the fact that they can in some sense be 'approximated' by free groups, as trees can be seen as approximations of Cayley graphs of hyperbolic groups: thin triangles tend to tripods when seen from far enough. The study of some groups as approximating free groups has a combinatorial counter-part in the previous small cancellation theory (dating back to Dehn, 1911: for a comprehensive reference, see [LS77, Chapter 5]. Indeed, free groups are 'no cancellation groups'.); nevertheless, the power of Gromov's methods lies in the possibility of easing arguments and at the same time to easier generalize arguments from the class of small cancellation groups to the greater class of hyperbolic groups. Sela in [Sel95] finds solvability for isomorphism problem for torsion-free hyperbolic groups. To any hyperbolic space we can associate a so-called boundary. Informally speaking is a kind of 'space at infinity'. Boundary of a hyperbolic space has found applications in many fields; a comprehensive reference can be found in [KB02]. Just to give an example, Gromov in [Gro87] states that every compact metric space is isometric to the boundary of some hyperbolic graph. Connected to the notion of boundary is that of end, a notion by Freudenthal dating back to 1931 ([Fre31]). Informally speaking, an end of a graph is an equivalence class of rays under the relation that two rays are equivalent if no finite vertex set separates them, or equivalently if there are infinitely many disjoint paths joining them (a more formal definition is given throughout the section). In 1944, Hopf ([Hop44]) realizes that the number of ends of the Cayley graph of a finitely generated group is independent of the choice of generating set; therefore, the number of ends of a finitely generated group G is naturally defined as the number of ends of some (every) Cayley graph of G. Moreover, he proves that a finitely generated group G may have 0,1,2 or uncountably many ends, with the case 0 holding if and only if G is finite. Thirty years after, Houghton ([Hou74]) extends the definition of ends of a group to infinitely generated groups. Stallings in [Sta68] and [Sta70] completely classifies finitely generated groups by their number of ends. His main conclusion is that, intuitively, a more-than-one-ended group quite resembles either an infinite cyclic group or a free product. Stallings' result, stated in Theorem 1.3.22, has connected group ends theory - so theory of graph representations of groups - with decomposition of groups, some years before Bass and Serre settle their theory of group actions on trees. In the first subsection we give some preliminaries on hyperbolic spaces, to achieve the notion of quasi-isometry, basic in geometric group theory, since the properties we study - from hyperbolicity itself, to number of ends - are quasi-isometry invariants.

1.3.1 Hyperbolic spaces

Definition 1.3.1. Let (X, d) be a metric space, and let $a, b, c \in X$. The **Gromov product** $(a \cdot b)_c$ of a and b with respect to c is defined as

$$\frac{1}{2}(d(a, c) + d(b, c) - d(a, b)).$$

Definition 1.3.2. Let (X, d) be a metric space, $x_0 \in X$, $\delta \geq 0$. We say that (X, d) is **δ-hyperbolic** with respect to x_0 if

$$(x \cdot y)_{x_0} \geq \min\{(y \cdot z)_{x_0}, (z \cdot x)_{x_0}\} - \delta$$

for every $x, y, z \in X$.

We say that (X, d) is **δ-hyperbolic** if it is δ-hyperbolic with respect to $x_0 \in X$ for every $x_0 \in X$.

We say that (X, d) is **hyperbolic** if it is δ-hyperbolic for some $\delta \geq 0$.

Lemma 1.3.3. *If X is δ-hyperbolic with respect to x_0, then we have*

$$(x \cdot y)_{x_0} + (z \cdot w)_{x_0} \geq \min\{(x \cdot z)_{x_0} + (y \cdot w)_{x_0}, (x \cdot w)_{x_0} + (z \cdot y)_{x_0}\} - 2\delta$$

for every $x, y, z, w \in X$.

Proof. For easiness of notation, we will understand the index x_0. Suppose without loss of generality that $(x \cdot z)$ is maximum of $(x \cdot z), (x \cdot w), (z \cdot y)$. By δ-hyperbolicity we have

$$(x \cdot y) \geq \min\{(y \cdot z), (z \cdot x)\} - \delta \text{ and}$$

$$(z \cdot w) \geq \min\{(w \cdot x), (x \cdot z)\} - \delta.$$

Adding side by side, we obtain the result.

□

The following proposition shows that, while hyperbolicity is an important property since it is independent of the choice of a basepoint, the value of δ practically is of no matter. However, the case $\delta = 0$ carries a notable peculiarity, see Proposition 1.3.11 below.

Proposition 1.3.4. *Let $x_0 \in X$. If X is δ-hyperbolic with respect to x_0, then it is 2δ-hyperbolic with respect to x for every $x \in X$.*

Proof. By Lemma 1.3.3 we have

$$(x \cdot y)_{x_0} + (z \cdot w)_{x_0} \geq \min\{(x \cdot z)_{x_0} + (y \cdot w)_{x_0}, (x \cdot w)_{x_0} + (z \cdot y)_{x_0}\} - 2\delta.$$

Add the following quantity to both sides:

$$\frac{1}{2}(|x - w|_{x_0} + |y - w|_{x_0} + |z - w|_{x_0} - |x|_{x_0} - |y|_{x_0} - |z|_{x_0} - |t|_{x_0}).$$

So we obtain

$$(x \cdot y)_w \geq \min\{(x \cdot z)_w, (z \cdot y)_w\} - 2\delta,$$

that is 2δ-hyperbolicity with respect to w.

□

Recall Definition 1.2.16 of a geodesic metric space; now we give a definition of hyperbolicity for geodesic metric spaces. We will show that it is equivalent to Definition 1.3.2. We need to introduce some definitions about geodesic triangles.

Definition 1.3.5. A **geodesic triangle** is a triangle whose sides are geodesic segments.

By definition of geodesic metric space, for every triple of points x, y, z of a geodesic metric space X there exists a geodesic triangle T with vertices x, y, z. Moreover, if x, y, z are not on the same geodesic, then T is non-degenerate.

Definition 1.3.6. Let X be a geodesic metric space and let $T \subseteq X$ a geodesic triangle with vertices x_1, x_2, x_3. We define a map $\pi : T \to \pi_T$ to a tripod π_T, such that every side of T is mapped by π isometrically.

Definition 1.3.7. A geodesic triangle is **δ-thin** if for every $a \in \pi_T$, for every $x, y \in \pi^{-1}(\pi_T(a))$ we have $d(x, y) \leq \delta$.

Proposition 1.3.8. *Let X be a geodesic metric space. Then,*

1. *if X is δ-hyperbolic, then all geodesic triangles of X are 4δ-thin.*

2. *if all geodesic triangles of X are δ-thin, then X is δ-hyperbolic.*

Proof. We prove now point 1. Let T be a geodesic triangle of vertices x, y, z and let $\alpha \in [x, y], \beta \in [x, z]$ be such that $\pi(\alpha) = \pi(\beta)$. Take x as basepoint: in the follow-on of this proof, the index x will be understood. By δ-hyperbolicity we have

$$(\alpha \cdot \beta) \geq \min\{(\alpha \cdot y), (y \cdot z), (z \cdot \beta)\} - 2\delta.$$

Now we have

$$(\alpha \cdot y) = (z \cdot \beta) = |\alpha| = |\beta| \leq (y \cdot z),$$

hence

$$(\alpha \cdot \beta) = |\alpha| - \frac{1}{2} |\alpha - \beta| \geq |\alpha| - 2\delta,$$

therefore

$$|\alpha - \beta| \leq 4\delta,$$

thus T is 4δ-thin, so point 1. is proved.

To prove point 2., take x as basepoint again. Let $a, b, c \in X$. Consider the three geodesic triangles T_1, T_2, T_3 of vertices $\{x, a, b\}, \{x, b, c\}, \{x, c, a\}$ respectively. Let α, β, γ be points on $[x, a], [x, b], [x, c]$ respectively, such that

$$|\alpha| = |\beta| = |\gamma| = \min\{(a \cdot c), (b \cdot c)\}.$$

(recall that $\min\{(a \cdot c), (b \cdot c)\} \leq \min\{|a|, |b|, |c|\}$). Since the triangles T_2 and T_3 are δ-thin, we have $|\alpha - \gamma| \leq \delta$ and $|\beta - \gamma| \leq \delta$. Hence, by triangle inequality, we have

$$|\alpha - \beta| \leq 2\delta. \tag{1.1}$$

Also by triangle inequality, we have

$$|a - b| \leq |a - \alpha| + |\alpha - \beta| + |\beta - b|,$$

therefore

$$|a - b| \leq |a| + |b| - 2\min\{(a \cdot c), (b \cdot c)\} + |\alpha - \beta|.$$

Using the inequality (1.1) we obtain

$$(a \cdot b) \geq \min\{(a \cdot c), (b \cdot c)\} - \delta,$$

that gives the result.

□

Definition 1.3.9. A **real tree** or $\mathbb{R}$-tree is a geodesic metric space T such that:

1. for every $x, y \in T$ there exists a unique segment e joining x and y. Note that, since T is geodesic, e is a geodesic segment;

2. if e, f are geodesic segments of T having a common endpoint, then $e \cup f$ is a geodesic segment.

Any realization of a simplicial tree is a real tree. On the contrary, there exist real trees that are not realizations of any simplicial tree, like the two following examples.

1. The union in $\mathbb{R}^2$ of the x axis with all lines parallel to the y axis; the distance is
$$d((x_0, y_0), (x_0, y_1)) = |y_1 - y_0|,$$
and for $x_1 \neq x_0$,
$$d((x_0, y_0), (x_1, y_1)) = |y_0| + |x_1 - x_0| + |y_1|.$$
This tree is not the realization of any simplicial tree, since its set of branching points is not discrete in metric topology.

2. The union of the x axis with a line parallel to the y axis for every point $\{x = 1/n | n \in \mathbb{N}\}$; the distance is the same as in the previous example. Neither this tree is the realization of any simplicial tree; in this case, the set of branching points is discrete, but fails to be closed. If we add the y axis, we gain closure, but we lose discreteness.

Even for real trees we can adapt Definition 1.2.3: given a point x of a real tree T, the degree of x in T is the number of connected components of $T \setminus \{x\}$. As an example, a branching point of T is a point $x \in T$ such that $T \setminus \{x\}$ has at least three connected components.

A $\mathbb{R}$-tree is 0-hyperbolic, since its geodesic triangles are tripods, therefore they are 0-thin. To prove the inverse, we need the following lemma.

Lemma 1.3.10. *Let (X, d) be a geodesic $\delta/4$-hyperbolic metric space, and let $Y = \bigcup_{1 \leq i \leq n-1} [x_i, x_{i+1}]$ be a chain of n geodesic segments, with $n \leq 2^k$ and $k \geq 1$. Then, for every point x on a geodesic segment $[x_1, x_n]$, we have $d(x, Y) \leq k\delta$, where $d(x, Y)$ is defined, as usual, as $\inf\{d(x, y) | y \in Y\}$.*

Proof. By subdividing some of the segments $[x_i, x_{i+1}]$, we can reconduce to the case $n = 2^k$. We argue by induction on k.

The base of the induction is assured by the fact that for $k = 1$ we have the result, since every geodesic triangle in X is δ-thin.

We are going to prove the inductive step. Suppose that the result is true for $n = 2^k$; we will show that it is true for $n = 2^{k+1}$.

Consider the two geodesic segments $[x_1, x_{n/2}]$ and $[x_{n/2}, x_n]$. Let $x \in [x_1, x_n]$. In the geodesic triangle $[x_1, x_n] \cup [x_1, x_{n/2}] \cup [x_{n/2}, x_n]$, the point x is at distance at most δ from a point y on $[x_1, x_{n/2}] \cup [x_{n/2}, x_n]$ by δ-thinness. By inductive hypothesis, y is at distance at most $k\delta$ from some point on $\bigcup_{1 \leq i \leq n/2-1} [x_i, x_{i+1}]$ or from some point on $\bigcup_{n/2 \leq i \leq n-1} [x_i, x_{i+1}]$. Hence,
$$d(x, Y) \leq (k+1)\delta,$$
thus the result is proved.

□

The following proposition shows the reason why the value 0 of hyperbolicity constant δ has a special importance.

Proposition 1.3.11. *[CDP90, Theorem 3.4.1] A 0-hyperbolic geodesic metric space is a $\mathbb{R}$-tree.*

Proof. Let X be a 0-hyperbolic geodesic metric space. For any $x, y \in X$, let $[x, y]$ a geodesic segment joining x and y. We are going to show that, if σ is any segment joining x and y, then $\sigma = [x, y]$.

Let $\epsilon > 0$. By compactness of σ, we can find, by uniform continuity, a sequence $x_1, \dots, x_n$ of consecutive points on σ, with $x_1 = x, x_n = y$ and such that $x_i - x_{i+1} \leq \epsilon$. Consider the piecewise geodesic segment $Y = \bigcup_{1 \leq i \leq n-1} [x_i, x_[i + 1]]$. By Lemma 1.3.10, every point on $[x, y]$ is at distance at most $2k\delta$ from Y, where k is an integer verifying $n \leq 2^k$. Since $\delta = 0$, the segment $[x, y]$ is contained into Y. Since every point of Y is at distance at most ϵ from a point of σ, we obtain that $[x, y]$ is contained into the ϵ-neighbourhood of σ. Since this is true for any ϵ, making ϵ go to 0, we obtain that $[x, y]$ is contained into σ, therefore they must coincide.

□

An important property of real trees is the following. Recall Definitions 1.2.15 and 1.2.14 of a hyperbolic isometry and its length, respectively.

Proposition 1.3.12. *([CM87, Theorem 1.3]; in [Chi04, Theorem 1.4] there is a generalization to Λ-trees, where Λ is an ordered abelian group). Let T be a $\mathbb{R}$-tree and let f be a hyperbolic isometry of T. Then there exists a unique line $Ax(f) \subseteq T$, called the* ***axis*** *of f, on which f acts as translation by a length $|f|$.*

Proof. Let $y \in T$. Note that, if $d(y, f^2y) = 2d(y, fy)$, then y, fy, f^2y are on some line L fixed setwise by f. Moreover, note that if such a line L exists, then any f-invariant subtree T', that is fixed setwise by f, must contain L; in fact, also the path connecting T' with L must be fixed setwise by f. Thus, L is an axis and is unique as an f-invariant line.

The proof is constructive, since it allows to effectively find the axis of a hyperbolic isometry. Let $x \in T$. Observe that, if m is the midpoint of the segment $[x, fx]$, then $d(x, fx) \geq d(m, fm)$. So this midpoint looks like a good place to look for the axis.

Consider the tripod of vertices x, fx, f^2x. Let c be its center, that is the intersection of its sides, and let m be the midpoint of $[x, fx]$. If $d(m, x) \geq d(c, x)$, then f fixes m, in contradiction with hyperbolicity of f. Therefore, $d(m, x) < d(c, x)$. Now, it is enough to show that $d(m, f^2m) = 2d(m, fm)$. But $c \in [m, fm]$ and $fc \in [fm, f^2m]$, so we only need to show that $d(c, fc) = 2d(c, fm)$. We have the following equalities:

$$\begin{aligned} d(c, fc) &= d(fx, f^2x) - 2d(c, fx) \\ &= d(x, fx) - 2[\frac{1}{2}d(x, fx) - d(c, fm)] \\ &= 2d(c, fm), \end{aligned}$$

that gives the result.

□

Definition 1.3.13. Let X, Y be metric spaces. A **quasi-isometry** is a map $f : X \to Y$ such that there exist $\lambda \geq 1$ and $K \geq 0$ such that for every $x_1, x_2 \in X$ we have

$$\frac{1}{\lambda} d_X(x_1, x_2) - K \leq d_Y(fx_1, fx_2) \leq \lambda d_X(x_1, x_2) + K.$$

The following theorem is proved in [CDP90, Theorem 3.2.2].

Theorem 1.3.14. *Let X_1, X_2 be two geodesic metric spaces, with X_2 hyperbolic. Let $f : X_1 \to X_2$ be a quasi-isometry. Then X_1 is also hyperbolic.*

In this section we will define the boundary of a hyperbolic space, provided it is proper, that is all closed balls are compact. So, during this section we assume that (X, e) is a δ-hyperbolic proper pointed space, and Gromov products are with respect to e. Observe that an isometry is the particular case of a quasi-isometry, putting $\lambda = 1$ and $K = 0$ in Definition 1.3.13.

Definition 1.3.15. A **ray** in X is an isometry $f : \mathbb{R}^+ \to X$.

Define the equivalence relation $\sim$ on the set of rays of X: $f \sim g$ if the following equivalent conditions hold:

1. $\sup_t |g(t) - f(t)|$ is finite;
2. there exists t_0 such that for every $t \geq t_0$ there exists $t'(t)$ such that $|g(t) - f(t')| \leq 8\delta$.

The equivalence of the above conditions is proved in [GDLH90, Proposition 7.2].

Definition 1.3.16. ([GDLH90, p.119]) Let (X, e) be a pointed hyperbolic space. The **boundary** of X, denoted as ∂X, is the set of equivalence $\sim$-classes of rays of X from e, where a ray from e is a ray f as in Definition 1.3.15 such that $f(0) = e$.

Definition 1.3.17. ([Gro87, Definition 1.8]) Let $(x_i)_{i\in\mathbb{N}}$ be a sequence of points of X. We say that $(x_i)_{i\in\mathbb{N}}$ **diverges** with respect to e if $\lim_{i,j\to\infty}(x_i \cdot x_j)_e = \infty$.

Define the following equivalence relation $\sim'_e$ between divergent sequences of points of X: let $(x_i)_{i\in\mathbb{N}}, (y_i)_{i\in\mathbb{N}}$ be two divergent sequences. Then,

$$(x_i)_{i\in\mathbb{N}} \sim'_e (y_i)_{i\in\mathbb{N}} \text{ if and only if } \lim_{i,j\to\infty}(x_i \cdot y_j)_e = \infty.$$

Since the inequality

$$|(x \cdot y)_e - (x \cdot y)_{e'}| \leq d(e, e')$$

holds for every $x, y, e, e' \in X$, divergence of a sequence and relation $\sim'_e$ are independent of e. Thus we will denote the relation $\sim'_e$ simply as $\sim'$.

Proposition 1.3.18. *([Gro87, Definition,p.98]) The boundary ∂X of X is the set of $\sim'$-equivalence classes of divergent sequences of points of X.*

Define Gromov product on ∂X in the following way: let $a, b \in \partial X$.

$$(a \cdot b) := \sup\{\lim \inf_{i,j\to\infty}(a_i \cdot b_j) | (a_i)_{i\in\mathbb{N}} \to a, (b_i)_{i\in\mathbb{N}} \to b\}.$$

Proposition 1.3.19. *The boundary ∂X is empty if and only if X has finite diameter.*

We may give ∂X a compact and metrizable topological space structure (see [GDLH90, Chapter 7, Section 2]), by defining $\{V_r | r \in \mathbb{Q}^+ \setminus \{0\}\}$ as the fundamental system of neighbourhoods, where $V_r = \{(a, b) \in \partial X \times \partial X | (a \cdot b) \geq r\}$.

Definition 1.3.20. The **ends** of X are the connected components of ∂X.

By ends of a group G we denote the ends of some (any) Cayley graph of G.
We end the section with some important properties of ends.

Theorem 1.3.21. *[Hop44] A finitely generated group has 0,1,2 or uncountably many ends.*

The following theorem gives a complete classification of finitely generated groups by number of their ends.

Theorem 1.3.22. *[Sta70] Let G be a finitely generated group. Then:*

1. *G has no ends if and only if G is finite;*
2. *G has at least two ends if and only if one of the following holds:*
   (a) *there exist $G_1, G_2 \leq G$ such that $G = G_1 *_H G_2$, for some H finite (and clearly different from both G_i); or*
   (b) *there exists $G_1 \leq G$ such that $G = \langle G_1, t | H^t = H' \rangle$ for some H finite, $H' \cong H$.*
3. *G has exactly two ends if and only if G has a normal subgroup N such that the quotient G/N is isomorphic to $\mathbb{Z}$ or to $\mathbb{Z}_2 * \mathbb{Z}_2$.*

1.3.2 Hyperbolic groups

Definition 1.3.23. Let $G = \langle \Sigma \rangle$. Let $f : \langle \Sigma | \rangle \to G$ be the universal homomorphism witnessing property in Definition 1.1.16. The **length** of $g \in G$ with respect to Σ is defined as

$$|g|_\Sigma = \min\{|w|\,|f(w) = g\}.$$

Let $g, h \in G$. Define $d_\Sigma(g, h) = |g^{-1}h|_\Sigma$. (G, d_Σ) becomes a metric space that embeds isometrically into $\mathrm{Cay}(G, \Sigma)$, where we recall that $\mathrm{Cay}(G, \Sigma)$ is the Cayley graph of G with respect to the set of generators Σ, according to Definition 1.2.12. The distance d_Σ is called **word metric**.

Definition 1.3.24. A group G is **hyperbolic** if $G = \langle \Sigma \rangle$ for some finite Σ and $\mathrm{Cay}(G, \Sigma)$ is hyperbolic.

Some examples of hyperbolic groups are finite groups, finitely generated free groups, fundamental groups of surfaces with negative Euler characteristic.

The following proposition shows that a finitely generated group has a unique Cayley graph, modulo quasi-isometries. In particular, hyperbolicity is independent of the choice of generating set.

To prove the proposition, we recall the following definition.

Definition 1.3.25. Let X be a metric space and let Y be a subspace of X. The inclusion of Y into X is said to be C-**quasi-dense** if there exists a constant $C > 0$ such that every point of X lies in the C-neighbourhood of Y. The inclusion of Y into X is said to be **quasi-dense** if it is C-quasi-dense for some C.

Note that a quasi-dense inclusion is a quasi-isometry. This is the property that we will use in the proof of the following proposition.

Proposition 1.3.26. *Let G be a finitely generated group and let Σ_1, Σ_2 be finite generating sets for G. Then:*

1. *(G, d_{Σ_i}) is quasi-isometric to* $\mathrm{Cay}(G, \Sigma_i)$*;*

2. $\mathrm{Cay}(G, \Sigma_1)$ *is quasi-isometric to* $\mathrm{Cay}(G, \Sigma_2)$.

Proof of point 1. The embedding $\iota : (G, d_{\Sigma_i}) \to \mathrm{Cay}(G, \Sigma_i)$ is a quasi-isometry. In fact, let $y \in \mathrm{Cay}(G, \Sigma_i)$. Then, there exists $e \in (\mathrm{Cay}(G, \Sigma_i))^1$ such that $y \in real(e)$. So $d(\omega(e), y) \leq 1$. Therefore, for every $y \in \mathrm{Cay}(G, \Sigma_i)$ there exists $x \in G$ such that $d(\iota(x), y) \leq 1$. By Definition 1.3.25, ι is quasi-dense, so the point is proved.

□ (Point 1.)

Proof of point 2. It is enough to prove that (G, d_{Σ_1}) is quasi-isometric to (G, d_{Σ_2}).

Let $\lambda_i = \max\{|s|_{\Sigma_i} | s \in \Sigma_i\}$ and let $\lambda = \max\{\lambda_1, \lambda_2\}$, and consider the map id_G.

Then we have:

$$d_{\Sigma_2}(\mathrm{id}_G(x), \mathrm{id}_G(y)) = \left|x^{-1}y\right|_{\Sigma_2} \leq \lambda_1 \left|x^{-1}y\right|_{\Sigma_1} \leq \lambda d_{\Sigma_1}(x, y),$$

and the same holds inverting indices 1 and 2.

Therefore we have the inequalities

$$\frac{1}{\lambda} d_{\Sigma_1}(x, y) \leq d_{\Sigma_2}(x, y) \leq \lambda d_{\Sigma_1}(x, y),$$

that prove quasi-isometry of id_G.

□ (Point 2 and Proposition 1.3.26).

Corollary 1.3.27. *Let $G = \langle \Sigma_1 \rangle = \langle \Sigma_2 \rangle$ be a hyperbolic group, and let $\phi_i : G \to$* $\mathrm{Isom}(\mathrm{Cay}(G, \Sigma_i))$ *be the actions of G on its Cayley graphs with respect to the sets of generators $\{\Sigma_i\}$. Then, for every $g \in G$, $\phi_1(g)$ is hyperbolic (respectively parabolic, elliptic, see Definition 1.2.15) if and only if $\phi_2(g)$ is hyperbolic (respectively parabolic, elliptic).*

By Corollary 1.3.27, we can classify elements of a finitely generated group G as hyperbolic, parabolic, elliptic, according to their actions on some (therefore every) Cayley graph of G.

The following result enables us to rule out parabolic elements, when dealing with hyperbolic groups. Recall Definition 1.2.15.

Proposition 1.3.28. *[GDLH90, Theorem 8.29] A hyperbolic group has no parabolic elements.*

An important result, proved by Gromov [Gro87, §2.2] is the following. We will follow a proof due to Howie ([How99, Theorem 2,p.11]).

Theorem 1.3.29. *[Gro87, Corollary 2.2.A] A hyperbolic group G is finitely presented.*

Proof. Fix a finite set of generators Σ. Let d be the metric on $\mathrm{Cay}(G, \Sigma)$. Then, $\mathrm{Cay}(G, \Sigma)$ is δ-hyperbolic for some δ. Let $B_n = B(1, n)$. For each $n \in \mathbb{N}$ define $R_n \subset \langle B_n \rangle$ as

$$R_n = \{xyz | xyz \in B_n, xyz = 1 \text{ in G}\} \cup \{xx^{-1} | x \in B_n\}.$$

Let $G_n = \langle B_n | R_n \rangle$. Then we obtain a sequence of group homomorphisms $(h_i : G_i \to G_{i+1})_{i \in \mathbb{N}}$, whose limit is G.

We will show that there exists N such that h_i are isomorphisms for $i < N$, therefore $G \cong G_i$ for $i < N$, and this proves our theorem.

Claim 1. *The homomorphisms h_i are surjective.*

Proof. Let $g \in B_{n+1} \setminus B_n$. Then there exist elements $u, v \in B_k$ with $uvg = 1$ in G. Since $u, v, g \in B_{k+1}$, we have $uvg \in B_{k+1}$, so $uvg = 1$ in G_{k+1}. Therefore, $\operatorname{im} h_k$ contains the generating set B_{k+1}, so h_k is surjective.

□ (Claim 1.)

Claim 2. *There exists N such that h_i are injective for $i < N$.*

Proof. Let $N \gg 2\delta$. Suppose that $xyz \in R_{N+1}$. That is, let $x, y, z \in B_{N+1}$, with $xyz = 1$ in G. We want to show that the relation can be deduced from those in R_N. The problem is that x, y, z do not in general belong to B_N. To make sense of this, choose, for each $x \in B_{N+1} \setminus B_N$, a splitting $x = x_1 x_2$ with $d(x_i, 1) > \delta$ and $d(x, 1) = d(x_1) + d(x_2, 1)$. Add the generator x and the relation $x_1 x_2 x^{-1}$ to the presentation of G_N, to get an equivalent presentation. Having done this, we show how to deduce $xyz = 1$ from the relations in R_N together with the relations from the splitting $x_1 x_2$.

Two cases may happen:

1. $x, y \in B_N, z \notin B_N$.

   Let P be the point of the geodesic segment corresponding to the splitting $z_1 z_2$ of z. By δ-hyperbolicity, there exists a point Q on one of the other edges of the geodesic triangle T of vertices $1, x, xy$. The geodesic PQ, together with the geodesic from Q to the vertex of T opposite to the edge containing Q, divides the geodesic triangle into three smaller triangles, whose edges are long at most N. Therefore, the relation $xyz_1 z_2 = 1$ can be deduced from three relations in R_N.

2. $y, z \notin B_N$.

   By case 1, we may assume all relations of the form $xyz = 1$ with $x, y \in B_N$ and $z \in B_{N+1}$. We proceed as in case 1, by splitting z. The point Q may belong to an edge of length $N + 1$, in which case it corresponds to a splitting of x or y, say of x. Maybe the splitting is not the same as $x_1 x_2$: say it is $x'_1 x'_2$. However, both x'_1 and x'_2 are shorter than $N + 1$. Therefore, we are allowed to assume the relation $x = x'_1 x'_2$. Let T be a triangle defined as in case 1. Divide T as before. In this situation, it is possible that one of the three triangles has a side of length $N + 1$, but all other sides of the three triangles have length at most N. By case 1, we are done.

We can apply a similar argument to the relations of the form xx^{-1}, with $x \in B_{N+1} \setminus B_N$.

□ (Claim 2 and Theorem 1.3.29.)

At the end of this section, we recall the notion of equational noetherianity, that will be used throughout this work.

Definition 1.3.30. Let G be a group and let $\bar{x}$ be the tuple $(x_1, \ldots, x_n)$. We denote by $G[\bar{x}]$ the free product $G * F(\bar{x})$, where $F(\bar{x})$ is the free group with basis $\{x_1, \ldots, x_n\}$.

Let $s(\bar{x}) \in G[\bar{x}]$ and let $\bar{g}$ be the tuple $(g_1, \ldots, g_n) \in G^n$. We denote by $s(\bar{g})$ the element of G obtained by replacing x_i by g_i, for each i. Let S be a subset of $G[\bar{x}]$. The *algebraic set* over G defined by S, denoted by $V(S)$, is the set $\{\bar{g} \in G^n | s(\bar{g}) = 1 \text{ for all } s \in S\}$. A group G is called **equationally noetherian** if for every $n \geq 1$ and every subset $S \subseteq G[\bar{x}]$ there exists a finite subset $S_0 \subseteq S$ such that $V(S) = V(S_0)$.

Since by the above construction subsets of $G[\bar{x}]$ can be put in correspondence with systems of equations with parameters from G, we may say that a finitely generated group G is equationally noetherian if any system of equations in G is equivalent to a finite subsystem.

In [RW10, §7.2], Weidmann and Reinfeldt have proved equational noetherianity of hyperbolic groups. The proof follows that of [RW10, Corollary 6.13], about weak equational noetherianity of hyperbolic groups. This generalizes the result by Sela ([Sel09, Theorem 1.22]), which stated any system of equations in a torsion-free hyperbolic group is equivalent to a finite subsystem.

1.4 Basics of model theory

Although this section may seem quite detailed, it is not intended to be a reference for all model theory needed to understand the main results. However, we feel the need to give the essentials for non-logicians to understand the motivation of this work, the baseground on which our fundamental questions have been developed. At the end of the section we will give the definitions of algebraic and definable closure(s). A book covering the basics of model theory is [Hod97]; more comprehensive works are [Hod93] and [Mar02]. For this brief introduction, we also refer to [Zam04].

Definition 1.4.1. A first-order **language** or **signature** is a set consisting of:

1. a set L which is the disjoint union of two sets L_F and L_R whose elements we call (symbols for) functions and relations and constants, respectively;
2. a map $a : L \to \mathbb{N}$ from L into the set of the non-negative integers, that we call the arity map. A function of arity 0 is called a constant.

We usually refer to the language only by naming the set L, but we always assume that we are also fixing its partition into functions and relations as well as the arity map.

Definition 1.4.2. A first-order **structure** $\mathcal{M}$ of language L is a pair that consists of

1. a set M, called the domain or support;
2. a mapping, called the interpretation of L, that assigns:

(a) to each symbol $r \in L_R$ a relation $r^{\mathcal{M}} \subseteq M^n$ of arity $n = a(r)$, which is called the interpretation of r in $\mathcal{M}$;

(b) to each symbol $f \in L_F$ a total function $f^{\mathcal{M}} : M^n \to M$ of arity $n = a(f)$, which is called the interpretation of f in $\mathcal{M}$.

By convention, M^0 is the set $\{\varnothing\}$, so the interpretation of a constant c is completely determined by $c^{\mathcal{M}}(\varnothing)$, the element of M that is image of $\varnothing$ under the map $c^{\mathcal{M}}$. This element is usually denoted simply by $c^{\mathcal{M}}$ or, even more simply c, when $\mathcal{M}$ is clear from the context. Since the interpretation of function symbols is required to be a total function, the domain of a structure needs at least to contain the interpretation of the constants.

Remark 1.4.3. Since we shall never meet higher-order structures, it is understood that we always speak about first-order languages and structures.

Definition 1.4.4. Let $\mathcal{M}$ and $\mathcal{N}$ be two structures with the same signature L. We say that $\mathcal{M}$ is a **substructure** of $\mathcal{N}$, and we write $\mathcal{M} \subseteq \mathcal{N}$, if

1. $M \subseteq N$, that is, the domain of $\mathcal{M}$ is contained in the domain of $\mathcal{N}$;

2. the interpretation of relations and functions in $\mathcal{M}$ is the restriction to $\mathcal{M}$ of their interpretation in $\mathcal{N}$, that is,

   (a) for every relation r of L we have $r^{\mathcal{M}} = r^{\mathcal{N}} \cap M^n$, where n is the arity of r; and

   (b) for every function f of L we have $f^{\mathcal{M}} = f^{\mathcal{M}} \restriction M^n$, where n is the arity of f.

Since now the difference between a structure and its support has been clarified, for the sake of a lighter notation we will utilize the notation $M, N, \ldots$ for structures, too.

Some concepts in model theory are easier to formalize using partial functions between structures. For instance, the notions of partial isomorphism and of elementary map. Here partial functions are called maps for short. Precisely, as in the following definition,

Definition 1.4.5. A **map** is a triplet that we denote by $F : M \to N$, where:

1. M is a structure that we call the domain of the map;

2. N is a structure that we call the codomain of the map;

3. F is a function from a subset of M, that we call the domain of definition of F and denote by $dom(F)$, onto a subset of N, that we call the range of F and denote by $rng(F)$.

Definition 1.4.6. A map $F : M \to N$ is **total** if the domain of definition coincides with the domain and it is surjective if the range coincides with the codomain. If $a \in dom(F)$ we write $F(a)$, or Fa when there is no possibility of confusion, for the image of a under F and, if $A \subseteq M$, we write $F[A]$, or FA when there is no possibility of confusion, for the image of A under F, that is the set $\{Fa | a \in A \cap dom(F)\}$. When FA is the domain of a substructure of N, then FA may denote this substructure itself.

Definition 1.4.7. The **composition** of two maps $F : M \to N$ and $H : N \to K$ is the map $HF : M \to K$, where HF is the composition of the functions H and F. The composition of two maps is only defined when the codomain of the first map is the domain of the second. When F is injective, we define the **inverse** of the map $F : M \to N$ to be the map $F^{-1} : N \to M$.

Definition 1.4.8. We say that the map $F' : M' \to N'$ **extends** $F : M \to N$ if M' and N' are superstructures of M and N, respectively, and the function F' is an extension of F.

Definition 1.4.9. Let M and N be two structures with the same signature L. Let $F : M \to N$ be a total, injective map from the domain of M to the domain of N. We say that F is an **embedding** of M in N if

1. $f^N(F\bar{a}) = F(f^M\bar{a})$ for every function $f \in L$ and every tuple $\bar{a}$ of elements of M;

2. $F\bar{a} \in r^N$ if and only if $\bar{a} \in r^M$ for every relation $r \in L$ and every tuple $\bar{a}$ of elements of M.

Note that the symbol F is used to denote also the function that maps the tuple $a_1 \dots a_n$, denoted by $\bar{a}$, to the tuple $Fa_1 \dots Fa_n$. We are also assuming that the arity of the tuple a matches the arity of f and r respectively. Note that, when $\bar{a}$ is empty, the first clause of Definition 1.4.9 says that each constant of M is mapped in the corresponding constant of N.

Definition 1.4.10. We say that M and N are **isomorphic** if there is an embedding $F : M \to N$ which is surjective.

Note that the inverse of an isomorphism is also an isomorphism.

Definition 1.4.11. An isomorphism $F : M \to M$ is called an **automorphism** of M.

The definitions above localize to a set of **parameters**. Parameters are simply elements of some structure, the use of the word is emphatic.

Definition 1.4.12. Let F be a map and let $a \in dom(F)$. We say that F **fixes** a if $Fa = a$.

Definition 1.4.13. Let $A \subseteq dom(F)$. We say that F fixes A **pointwise** if F fixes every element of A.

Definition 1.4.14. Let $A \subseteq dom(F)$. We say that F fixes A **setwise** if $Fa \in A$ for every element a of A.

Definition 1.4.15. Let M and N be structures containing the set A. We say that $F : M \to N$ is an **isomorphism over** A or A**-isomorphism** or even **isomorphism relative to** A if it is an isomorphism that fixes A pointwise. Embeddings, automorphisms, and partial isomorphisms over A are defined similarly.

The language we will use for groups is $(G, \cdot, ()^{-1}, 1_G)$, where G, or whatever other letter, is interpreted as the underlying set, $\cdot$ is interpreted as group operation, $()^{-1}$ is interpreted as inverse and 1_G, or simply 1 when there is no ambiguity, is interpreted as identity. By analogy with Definition 1.4.15, by A-homomorphism we intend a group homomorphism that fixes some subset A pointwise.

Notation 1.4.16. Fix a signature L and a structure M of signature L. We use the symbols $A, B, \dots$ to denote sets of parameters. Fix an infinite set V that we call the set of (free) variables; think of these as placeholders for inputs of functions and relations. We use the letters $x, y, z, \dots$ to denote elements of V. Fix an arbitrary set A that we call the set of parameters. Parameters are elements of some structure(s) that will occur in terms and formulas.

Formally terms are words on the alphabet containing $L \cup V \cup A$ plus two auxiliary elements, that we denote by the symbols (and). These last two objects have the role of delimiters.

Definition 1.4.17. A **term** is defined by induction as follows:

1. each free variable and each parameter is a term;

2. if $\bar{t}$ is a tuple of terms and f is a function symbol with the same arity as t, then $f(\bar{t})$ is a term. By $f(\bar{t})$ we mean the word obtained by concatenating $t_1 \ldots t_n$, prefixing it with the two symbols $f($, and finally postfixing it with the symbol).

Terms obtained from 1. above, that is terms that are either free variables or parameters, are called atomic terms. Sometimes in the literature also constants are called atomic terms. Terms where no free variable occurs are called closed terms. When we want to stress that all parameters occurring in a term t are in A then we may say that t is a **term over** A or A-term; $\varnothing$-terms are also called parameter-free terms.

When $\bar{s}$ is a tuple of terms with the same arity as $\bar{x}$, we write $t(\bar{x}/\bar{s})$ for the term obtained substituting $\bar{s}$ for $\bar{x}$ in t coordinatewise. That $t(\bar{x}/\bar{s})$ is a term is a claim which, strictly speaking, needs to be proved: this we do in Proposition 1.4.18 below. It is convenient in the notation to display the free variables occurring in a term. So we write $t(\bar{x})$ to mean that all free variables of t are among $\bar{x}$; note that $\bar{x}$ may contain more variables. To denote the substitution of $\bar{s}$ for $\bar{x}$ in $t(\bar{x})$, that is when all variables of t have been declared, we often use the abbreviated notation $t(\bar{s})$.

Proposition 1.4.18. *Let $t(\bar{x})$ be a term. Let $\bar{s}$ be a tuple of terms. Then $t(\bar{s})$ is a term.*

Proof. The claim is proved by induction on the syntax of t. That is, checking that the claim is true for t atomic, which is immediate, and proving that if the claim holds for all the terms in the array $\bar{t}$ then it holds for the term $f(\bar{t})$, which is also immediate.

□

As terms name functions, formulas name sets of tuples. We introduce simultaneously also a restricted class of formulas, the quantifier-free formulas. The quantifier-free formulas are words on the alphabet containing $L \cup V \cup A$ together with the symbols $(,), =, \perp, \neg, \wedge$. The last three are called logical connectives; they are called false, negation and conjunction respectively. The idea is to formalize the concept of empty set, complementation and intersection. If we read $\neg$ and $\wedge$ as 'not' and 'and', the intended meaning of the expressions defined below should be clear. To introduce quantifiers we need to use a countable set U of auxiliary variables disjoint of V. The reason is technical and is discussed in remark about notation 1.4.21 below. Variables in U are called bound variables. A formula (non-necessarily quantifier-free) is a word on the alphabet containing $L \cup V \cup A \cup U$, the symbols above, and the symbol *exists* which is called existential quantifier. Here is the inductive definition of formula.

Definition 1.4.19. A **formula** is defined by induction as follows:

1. If $\bar{t}$ is a tuple of terms and r is a relation symbol with the same arity as $\bar{t}$, then $r(\bar{t})$ is a formula;

2. if t and s are terms then $(t = s)$ is a formula;
3. $\perp$ is a formula;
4. if ϕ and ψ are formulas, then $\neg\phi, (\phi \wedge \psi)$ are formulas;
5. if ϕ is a formula, x is a free variable, and u is a bound variable not occurring in ϕ, then $\exists u\ \phi(x/u)$ is a formula, where $\phi(x/u)$ denotes the literal substitution of u for x in ϕ.

Formulas of the form 1. or 2. are called atomic. Formulas obtained without the use of 5. are called quantifier-free formulas. When we want to specify that parameters in the formula come from some set A, then we say 'formula over A' or 'A-formula'. In particular, $\varnothing$-formulas are the parameter-free ones.

Definition 1.4.20. A formula without free variables, that is all of whose variables occur under the scope of a quantifier, is called a closed formula or a **sentence**.

Notation 1.4.21. From now on, we will use a more tolerant notation: we shall make no distinction between bound and free variables and simply write $\exists x\ \phi$ for $\exists u\ \phi(x/u)$. In fact, working all the time with two distinct sorts of variables would overload the notation. We introduced the set of variables U for a technical reason: if we had not kept bound and free variables distinct we would have been forced to give a less straightforward definition of substitution. Consider, for instance, the substitution of the parameters a, b for x, y respectively, in the formula $s(y) \wedge \exists y\ r(x, y)$. But this does not yield any sensible formula. instead, the result we would like to obtain is: $s(b) \wedge \exists y\ r(a, y)$, that is, we want to substitute b for y in the first conjunct, while in the second conjunct we want to leave y unchanged, since it occurs under the scope of the quantifier $\exists y$. Keeping bound variables distinct from free variables we get rid of the problem: literal substitution of any term for *free* variables always yields a well-formed formula.

When $\bar{x}$ is an array of free variables, $\bar{t}$ an array of terms, and ϕ a formula, we write $\phi(\bar{x}/\bar{t})$ for the literal and coordinatewise substitution of $\bar{t}$ for $\bar{x}$ in ϕ. As for terms, we usually introduce a formula together with a tuple of free variables and write: $\phi(\bar{x})$. We agree that when we use this expression we are displaying *all* free variables of ϕ. To denote the substitution of $\bar{t}$ for $\bar{x}$ in $\phi(\bar{x})$ we use the abbreviated notation $\phi(\bar{t})$. The following proposition, analogous to Proposition 1.4.18, proves that substitutions of terms for variables in formulas works well. We use it implicitly in the definition of interpretation of a formula.

Proposition 1.4.22. *Let $\phi(\bar{x})$ be a formula. Let $\bar{t}$ be a tuple of terms. Then $\phi(\bar{t})$ is a formula.*

Proof. The claim is proved by induction on the syntax of ϕ. The claim for atomic formulas follows directly from the analogous claim for terms in Proposition 1.4.18. Induction for the connectives $\neg$ and $\wedge$ is straightforward. So it is enough to prove that if the claim holds for ϕ it holds also for $\exists u\ \phi(y/u)$, where y is a free variable and u is a bound variable not occurring in ϕ. Ideally, to prove that $\exists u\ \phi(y/u)(\bar{x}/\bar{s})$ is a formula we would like to show that it equals $\exists u\ \phi(\bar{x}/\bar{s})(y/u)$ and apply the induction hypothesis. Unfortunately, when y occurs in $\bar{s}$ the two substitutions may not commute. Still, it can easily be checked that $\phi(\bar{x}/\bar{t})(\bar{y}/\bar{s})$ equals $\phi(\bar{y}/\bar{s})(\bar{x}/\bar{t})$ whenever the tuples $\bar{x}$ and $\bar{y}$ have no variable in common, no variable of $\bar{x}$ occurs in $\bar{s}$ and no variable of $\bar{y}$ occurs in $\bar{t}$. So we only need an intermediate substitution: let w be a free variable occurring neither in ϕ nor in $\bar{t}$ nor in $\bar{x}$. By induction hypothesis, $\phi(y/w)$ is a formula: let denote it by ϕ'. Obviously, $\exists u\ \phi(y/u)$ is literally the same formula as $\exists u\ \phi'(w/u)$. Since w does not occur in $\bar{x}$ nor in $\bar{t}$ we obtain that $\phi'(w/u)(\bar{x}/\bar{t})$ equals $\phi'(\bar{x}/\bar{t})(w/u)$. By induction hypothesis, $\phi'(\bar{x}/\bar{t})$ is a formula, so, by Definition 1.4.19 above, $\exists u\ \phi'(\bar{x}/\bar{t})(w/u)$ is a formula. This proves the claim.

□

Now we can establish a link between syntax and semantics, coming to the interpretation of terms and formulas. we will start from terms, then we will give an interpretation to quantifier-free and non-quantifier-free closed formulas, finally we will give an interpretation for any formula.

Definition 1.4.23. Let t be a closed term with parameters in M. We define by induction the interpretation of t in M which we denote by t^M:

1. if t is an atomic term (since t is closed it must be a parameter), then t^M is t itself;

2. if t has been obtained as in 2. of Definition 1.4.17 from the tuple of terms $\bar{s}$ and the function symbol f, then t^M is the element $f^M(\bar{s}^M)$.

The definition above applies only to closed terms. In general, the interpretation of the term $t(\bar{x})$ is the function $t^M(\bar{x})$ that maps the tuple $\bar{a}$ to $t^M(\bar{a})$.

Strictly speaking, we should now prove that every term has a unique interpretation. The interpretation of a term could be non-well-defined if the array $\bar{s}$ in point 2. of Definition 1.4.23 above were not uniquely determined by the term t. So we need to prove that if $\bar{t}$ and $\bar{s}$ are two sequences obtained concatenating the terms $t_1 \dots t_n$ and $s_1 \dots s_n$ respectively, and if $\bar{t} = \bar{s}$ as sequences, then $t_i = s_i$ for every $i = 1, \dots, n$. The delimiters (and) now prove to be useful. In each term, any right parenthesis have to match a left parenthesis: this gives a unique way to split $\bar{t}$ and $\bar{s}$ into their components. We can safely skip the proof of this fact.

Next step is to assign a truth value to closed quantifier-free formulas. As for terms the interpretation of a non-closed formula is derived from that of closed formulas. Non-closed formulas are interpreted as sets, as explained in the paragraph below. So, let ϕ be a quantifier-free formula without free variables and with parameters in M. We define when ϕ is true (in M). If ϕ is not true we say that it is false. We also say that M models ϕ or M does not model ϕ, respectively. We write $M \models \phi$ when M models ϕ and $M \not\models \phi$ when it does not.

Definition 1.4.24. We stipulate the **interpretation of a quantifier-free formula** by induction as follows:

1. If ϕ is the atomic formula $r(\bar{t})$, then we stipulate that ϕ is true if and only if the tuple $\bar{t}$ belongs to the relation r^M.

2. If ϕ is the atomic formula $\bar{t} = \bar{s}$, then we stipulate that ϕ is true if and only if $\bar{t}^M = \bar{s}^M$.

3. The formula $\bot$ is false.

4. If ϕ is of the form $\psi \wedge \xi$, then we stipulate that ϕ is true if and only if both ψ and ξ are true. If ϕ is of the form $\neg\psi$, then we stipulate that ϕ is true if and only if ψ is false.

As above, we define only the interpretation of closed formulas; so let ϕ be a formula without free variables and parameters in M. We extend the definitions above with the following clause:

Definition 1.4.25 (Interpretation of the quantifiers)**.** If ϕ has the form $\exists u\ \psi(y/u)$, then M models ϕ if and only if M models $\psi(y/b)$ for some b in M.

Note that, in contrast to the quantifier-free case, to define the truth value of non-quantifier-free formulas we refer to the whole structure M. In fact, $\exists u$ will be interpreted as 'there is some u in M'. Consequently the truth value of a non-quantifier-free formula is in general dependent on M.

The truth of formulas that are not closed is undefined: we can evaluate the truth of a formula $\phi(\bar{x})$ only after we replace $\bar{x}$ with a tuple $\bar{a}$ from M with the same arity of $\bar{x}$.

Definition 1.4.26. A formula $\phi(\bar{x})$ **holds** in M, or **is valid** in M, if $M \models \phi(\bar{a})$ for every tuple $\bar{a}$ in M^n, where n is the arity of $\bar{x}$.

Definition 1.4.27. A formula $\phi(\bar{x})$ **is consistent** in M if $M \models \phi(\bar{a})$ for some tuple $\bar{a}$ in M^n.

In the following paragraph we introduce some notation that will make easier reading formulas and linking syntactic and semantic aspects together.

Now that we have a clearer distinction between the syntax of terms and formulas and their interpretation, we may adopt a more informal notation. For instance we may omit the outermost parentheses. There are a number of other logical connectives that are introduced as abbreviations to make formulas become more readable. Now we will list them. We write $\phi \vee \psi$ (read: ϕ or ψ) for $\neg(\neg\phi \wedge \neg\psi)$. The connective $\vee$ is called disjunction. We do not use parenthesis around long conjunctions like $\phi_1 \wedge \ldots \wedge \phi_n$, since this connective has an associative meaning; similarly for disjunctions. We write $\exists\bar{x}\ \phi$ for $\exists x_1 \ldots \exists x_n\ \phi$, and $\bar{x} = \bar{y}$ for $x_1 = y_1 \wedge \ldots \wedge x_k = y_k$. We abbreviate $\neg\exists\bar{x}\ \neg\phi$ with $\forall\bar{x}\ \phi$, which reads 'ϕ holds for every $\bar{x}$'. The logical connective $\forall$ is called **universal quantifier**. We abbreviate $\neg\phi \vee \psi$ with $\phi \to \psi$ which reads 'ϕ implies ψ'. The connective $\to$ is called implication. Semantically, implication corresponds to inclusion: the formula $\forall\bar{x}\ [\phi(\bar{x}) \to \psi(\bar{x})]$ holds in M if and only if $\phi(M) \subseteq \psi(M)$, where the notation $\phi(M)$ is defined in Definition 1.4.29 below. We abbreviate $(\phi \to \psi) \wedge (\psi \to \phi)$ with $\phi \leftrightarrow \psi$, which reads: ϕ is logically equivalent to ψ. Semantically, this corresponds to equality: the formula $\forall\bar{x}[\phi(\bar{x}) \leftrightarrow \psi(\bar{x})]$ is true in M if and only if $\phi(M) = \psi(M)$. Finally, for every positive integer n we write $\exists^{\geq n}\bar{x}\ \phi$ for

$$\exists\bar{u}_1 \ldots \bar{u}_n \bigwedge_{1\leq i\leq j\leq n} \bar{u}_i \neq \bar{u}_j \wedge \bigwedge_{1\leq i\leq n} \phi(\bar{x}/\bar{u}_i).$$

This says that there are more than n distinct tuples satisfying ϕ. We write $\exists^{=n}\bar{x}\ \phi$ to abbreviate the formula $\exists^{\geq n}\bar{x}\ \phi \wedge \neg\exists^{\geq n+1}\bar{x}\ \phi$. Both the formal and the intuitive interpretation of these connectives should be clear.

Finally we come to the semantic counter-part of the syntactic notions developed above: definable sets.

Definition 1.4.28. Let $\phi(\bar{x})$ be a formula. A tuple $\bar{a} \subseteq M^n$ such that $M \models \phi(\bar{a})$ is called a **solution** of $\phi(\bar{x})$ or a **witness** of $\exists \bar{x}\ \phi(\bar{x})$.

When it is necessary to make clear where truth is evaluated, we possibly add 'in M'.

Definition 1.4.29. Let $\phi(\bar{x})$ be a formula, possibly with parameters; we define

$$\phi(M) = \{\bar{a} \subseteq M^n | M \models \phi(\bar{a})\},$$

where n is the arity of $\bar{x}$.

Sets of the form $\phi(M)$ where $\phi(\bar{x})$ is an A-formula are called **A-definable** subsets of M.

When $A = M$, we will simply say 'definable'.

The following definition extends definitions 1.4.28 and 1.4.29 to the case of infinite sets of formulas.

Definition 1.4.30. Let $p(\bar{x})$ be an infinite set of formulas with free variables among $\bar{x}$, where this is a possibly infinite tuple of variables. When speaking about infinite sets of formulas, we often use the word **realization** instead of solution: we say that $\bar{a}$ is a realization of $p(\bar{x})$ when $M \models \phi(\bar{a})$ for every formula $\phi(\bar{x})$ in $p(\bar{x})$. We may also say that $\bar{a}$ realizes $p(\bar{x})$. We write $M \models p(\bar{a})$ for short.

When M is not clear from the context, we possibly add 'in M'. We say that $p(\bar{x})$ is consistent in M if it has a realization in M.

Definition 1.4.31. We write

$$p(M) = \{\bar{a} \subseteq M | M \models p(\bar{a})\}$$

for the set of realizations of $p(\bar{x})$. In other words,

$$p(M) = \bigcap_{\phi \in p} \phi(M).$$

Sets of the form $p(M)$ are said to be **type-definable**.

Definition 1.4.32. A set of formulas $p(\bar{x})$ as in Definition 1.4.30 is usually called a **type**.

Definition 1.4.33. Let M be a structure, let $A \subseteq M$ and let $\bar{a}$ be a tuple in M. The **type of** $\bar{a}$ in M over the set of parameters A, or A-type of $\bar{a}$ in M, denoted by $tp_M(\bar{a}/A)$, is the set of formulas $\phi(\bar{x})$ with parameters from A such that $M \models \phi(\bar{a})$.

When M is clear from the context, we avoid putting it in the notation. We also avoid writing the set of parameters, when it is empty.

Similarly, we define the existential type $tp^{\exists}_M(\bar{a}/A)$, the universal type $tp^{\forall}_M(\bar{a}/A)$ and the quantifier-free type $tp^{QF}_M(\bar{a}/A)$, when we consider only existential, universal or quantifier-free formulas, respectively.

Now that we have given the fundamental definitions, we are going to introduce a notion of similarity between structures that involves first-order properties.

Two structures with the same signature may have very little in common. Two isomorphic structures are practically the same. Between this very loose and very strong degree of similarity there is another very important sort of equivalence: being indistinguishable by a first-order sentence. This is the relation of elementary equivalence. To get this notion more precisely, we need some preliminary definitions.

Definition 1.4.34. A set of sentences T is **consistent** if there is a structure M such that $M \models T$, that is, $M \models \phi$ for every $\phi \in T$.

Definition 1.4.35. A consistent set of A-sentences T is called a **theory over** A or simply a theory when A is empty.

Definition 1.4.36. We say that a theory T is **complete** for A-sentences if for every A-formula ϕ, either $\phi \in T$ or $\neg\phi \in T$.

When the set A is not mentioned we understand that it is the set of parameters that occur in T.

Definition 1.4.37. Let M be a structure and let $A \subseteq M$. The set of A-sentences that hold in M is called the **theory of** M **over** A and is denoted by $Th_A(M)$.

Similarly to Definition 1.4.37, the sets of quantifier-free, universal, existential A-sentences that hold in M are called the **quantifier-free theory of** M **over** A, **universal theory of** M **over** A, **existential theory of** M **over** A respectively,and are denoted by $Th_A^{qf}(M)$, $Th_A^{\forall}(M)$, $Th_A^{\exists}(M)$ respectively.

When A is empty we omit it from the notation and the terminology. Note that $Th_A(M)$ is complete for A-sentences, that is, for every A-sentence ϕ, either $\phi \in Th_A(M)$ or $\neg\phi \in Th_A(M)$. Observe that, since the theory of a structure cannot contain both a formula and its negation, the two inclusions $Th_A(M) \subseteq Th_A(N)$ and $Th_A(N) \subseteq Th_A(M)$ are equivalent.

Now we can introduce the notion of elementary equivalence.

Definition 1.4.38. Let M and N be structures and let $A \subseteq M \cap N$. We say that M and N are **elementary equivalent** over A if M and N have the same A-theory, that is, M and N model the same A-sentences. We write $M \equiv_A N$.

We write $M \equiv_A^{qf} N$, $M \equiv_A^{\forall} N$, $M \equiv_A^{\exists} N$ if M and N have the same quantifier-free, universal, existential theory over A respectively. Again when A is the empty set we omit it from the notation and the terminology. Observe that two structures isomorphic over A are elementary equivalent over A. This follows from the fact that isomorphisms preserve definable sets. This fact can be proved by induction on the syntax of formulas, with a method similar to that used in the proof of Proposition 1.4.22, and considering the particular case of a formula $\phi(\bar{x})$ with $\bar{x}$ being the empty tuple.

Definition 1.4.39. We write $M \preceq N$ when $M \subseteq N$ and $M \equiv_M N$. This is the same as requiring that $N \models Th_M(M)$ or that $M \models Th_M(N)$. In words, we say that M is an **elementary substructure** of N or that N is an elementary superstructure of M.

It is clear that the relation of being elementary substructure is transitive. The quantifier-free version of the notion of elementary substructure is simply the notion of substructure. In fact, $M \preceq_{qf} N$ just tells that N and M model the same atomic M-formulas. So it easily follows that they model the same quantifier-free M-formulas, that is, $M \equiv_M^{qf} N$.

Now we are going to define the notions of elementary map and elementary embedding. Let M and N be structures containing A and fix some $\bar{a} \subseteq M$ and $\bar{c} \subseteq N$, possibly infinite. We write $M, \bar{a} \equiv_A N, \bar{c}$ if for all A-formulas we have

$$M \models \phi(\bar{a}) \Leftrightarrow N \models \phi(\bar{c}).$$

When M and N are the same model and this is clear from the context then we write $\bar{a} \equiv_A \bar{c}$. As usual, we omit mentioning A if this is empty.

Definition 1.4.40. We say that $F : M \to N$ is an A**-elementary map** if for every tuple $\bar{a}$ in the domain of definition of F we have $M, \bar{a} \equiv_A N, F\bar{a}$.

A technical remark: though here we do not require that A belongs to the domain of definition of the map, there is always a unique A-elementary extension of $F : M \to N$ defined on $dom(F) \cup A$, namely $\mathrm{id}_A \cup F : M \to N$.

Definition 1.4.41. A total elementary map $F : M \to N$ is called an A**-elementary embedding** of M into N.

It is not difficult to check that elementary maps are injective and that the class of A-elementary maps is closed under inverse and composition, whenever this is defined.

Now we introduce the notion of elementary diagram: we associate to the structure M some set of formulas. Roughly, these formulas are obtained by identifying the elements of M with suitable variables.

Definition 1.4.42. Let $\bar{a}$ be a tuple with range M and let $\bar{x}$ have the same length of $\bar{a}$. Let $A \subseteq M$. Then, the set $p(\bar{x})$ of A-formulas $\phi(\bar{x})$ that is realized by $\bar{a}$ is called the **elementary diagram** of M over A.

Again, when A is empty we omit it from the terminology, as usual.

Like for theories, also for diagrams we have the analogous definitions making restrictions on the number of quantifiers on the considered formulas.

The following proposition is the motivation behind the definition of the diagram.

Proposition 1.4.43. *If N realizes the elementary diagram of M, then there is an elementary embedding of M into N. If N realizes the quantifier-free diagram of M, then there is an embedding of M into N.*

Proof. The proof of the two claims is identical, so we prove only the first one. Let $p(\bar{x})$ be the elementary diagram of M and let $\bar{a}$ be the enumeration of M that realizes $p(\bar{x})$. Let $\bar{c}$ realize $p(\bar{x})$ in N and let $F : M \to N$ map $\bar{a}$ to $\bar{c}$. We check that F is an elementary embedding. Observe first that $a \mapsto c$ is indeed a function. Moreover, an arbitrary M-sentence can be written as $\phi(\bar{a})$, where $\phi(\bar{x})$ is parameter-free. So, if $\phi(\bar{a})$ holds in M, then $\phi(\bar{x})$ is in the elementary diagram, therefore $\phi(\bar{c})$ holds in N.

□

Now we introduce two notions that will be directly used in our work: saturation and homogeneity.

Definition 1.4.44. Let λ be an infinite cardinal. A structure M is λ**-saturated** if for every set $C \subseteq M$ of cardinality less than λ every type with parameters from C is realized in M. A structure M is **saturated** if it is $|M|$-saturated.

Definition 1.4.45. Let λ be an infinite cardinal. A structure M is λ-homogeneous if, for every subset $A \subseteq M$ of cardinality less than λ and for every $a \in M$, any partial elementary map $f : A \to M$ can be extended to a partial elementary map $f^* : A \cup \{a\} \to M$.

When $\lambda = |M|$, we simply say that M is homogeneous.

Proposition 1.4.46. *[Mar02, Proposition 4.2.13] Let M be a homogeneous countable structure and let $\bar{a}, \bar{b} \in M^n$ be finite tuples such that $tp_M(\bar{a}) = tp_M(\bar{b})$. Then there exists an automorphism mapping $\bar{a}$ to $\bar{b}$.*

The converse of the above proposition clearly holds. In fact, $tp_M(\bar{a}) = tp_M(\bar{b})$ implies the existence of a partial elementary map $f : \bar{a} \to M$, mapping $\bar{a}$ to $\bar{b}$. Under the assumption that M is countable, the subsets A in Definition 1.4.45 are the finite ones. Therefore, we can use the result of Proposition 1.4.46 as an equivalent definition of homogeneity for countable structures. In this way we define $\exists$-homogeneity or existential homogeneity: a countable structure M is $\exists$-homogeneous if, for every subset $A \subseteq M$ of cardinality less than $|M|$, for every pair of finite tuples $\bar{a}, \bar{b} \in M^n$, if $tp^{\exists}_M(\bar{a}) = tp^{\exists}_M(\bar{b})$, then there exists an automorphism mapping $\bar{a}$ to $\bar{b}$. Note that $\exists$-homogeneity implies homogeneity, since the existential type of a tuple is a subset of its type.

Saturation and homogeneity are in some way linked. At an intuitive level, saturation implies the existence of many symmetries: every tuple $\bar{a}$ can be mapped, via an automorphism, to any other tuple that is not distinguishable from $\bar{a}$ by a first-order formula; but this is homogeneity. The following proposition gives a more precise proof of this fact. We follow the proof by Marker.

Proposition 1.4.47. *[Mar02, Proposition 4.3.3] Let M be a structure. If M is λ-saturated, then M is λ-homogeneous.*

Proof. Let $A \subseteq M$, $|A| < \lambda$, and let $f : A \to M$ be a partial elementary map. Let $b \in M \setminus A$. Let

$$\Gamma = \{\phi(x, f(\bar{a})) | \bar{a} \in A \text{ and } M \models \phi(b, \bar{a})\}.$$

If $\phi(x, f(\bar{a})) \in \Gamma$, then $M \models \exists x\ \phi(x, \bar{a})$, so, by elementarity of f, $M \models \exists x\ \phi(x, f(\bar{a}))$. Thus, because Γ is closed under conjunction, Γ is consistent. Because M is saturated, there is $c \in M$ realizing Γ. Thus, $f \cup \{(b, c)\}$ is elementary and M is λ-homogeneous.

□

Now we can define the core notions of this work: algebraic and definable closure, together with some similar notions.

Definition 1.4.48. Let M be a structure and let $A \subseteq M$. An element $a \in M$ is said to be **algebraic** over A if there exists a formula ϕ with parameters from A such that $M \models \phi(a)$ and ϕ has finitely many realizations in M.

Definition 1.4.49. Let M be a structure and let $A \subseteq M$. The **algebraic closure** of A in M, denoted with $acl_M(A)$, is the set of algebraic elements over A.

The notion of definable element is the case of Definition 1.4.29, when the set $\phi(M)$ is a singleton. We will nevertheless give a specific definition here, because of the importance of the notion.

Definition 1.4.50. Let M be a structure and let $A \subseteq M$. An element $a \in M$ is said to be **definable** over A if there exists a formula ϕ with parameters from A such that $M \models \phi(a)$ and $\phi(M)$ is a singleton.

Similarly to algebraic closure, we can define definable closure.

Definition 1.4.51. Let M be a structure and let $A \subseteq M$. The **definable closure** of A in M, denoted with $dcl_M(A)$, is the set of definable elements over A.

If we consider only existential formulas, we get the following definitions.

Definition 1.4.52. Let M be a structure and let $A \subseteq M$. An element $a \in M$ is said to be **existential algebraic** or $\exists$-algebraic over A if there exists an existential formula ϕ with parameters from A such that $M \models \phi(a)$ and ϕ has finitely many realizations in M.

Definition 1.4.53. Let M be a structure and let $A \subseteq M$. The **existential algebraic closure** of A in M, denoted with $acl^{\exists}_M(A)$, is the set of existential algebraic elements over A.

Definition 1.4.54. Let M be a structure and let $A \subseteq M$. An element $a \in M$ is said to be **existential definable** over A if there exists an existential formula ϕ with parameters from A such that $M \models \phi(a)$ and $\phi(M)$ is a singleton.

Definition 1.4.55. Let M be a structure and let $A \subseteq M$. The **existential definable closure** of A in M, denoted with $dcl^{\exists}_M(A)$, is the set of existential definable elements over A.

We can consider analogous closure notions based on orbits under automorphisms; we get the notions of restricted closures.

Definition 1.4.56. Let M be a structure and let $A \subseteq M$. The **restricted algebraic closure** of A in M, denoted with $racl_M(A)$, is the set

$$\{b|\{fb|f \in \mathrm{Aut}_A(M)\} \text{ is finite}\}.$$

Definition 1.4.57. Let M be a structure and let $A \subseteq M$. The **restricted definable closure** of A in M, denoted with $rdcl_M(A)$, is the set

$$\{b|\{fb|f \in \mathrm{Aut}_A(M)\} \text{ is a singleton}\},$$

that is

$$\{b|\{fb|f \in \mathrm{Aut}_A(M)\} = \{b\}\}.$$

The following lemma states some properties that correlate the closures defined above.

Lemma 1.4.58. *Let M be a structure of language L and let A, B be subsets of M. Then the following properties hold:*

1. *$acl(A)$, $dcl(A)$, $acl^{\exists}(A)$, $dcl^{\exists}(A)$, $racl(A)$, $rdcl(A)$ are L-substructures of M.*

2. *$dcl(A) \leq acl(A) \leq racl(A)$, $dcl(A) \leq rdcl(A)$.*

3. *$acl(A) = acl^{\exists}(acl(A)) = acl(acl(A)) = acl(dcl(A)) = dcl(acl(A)) = dcl^{\exists}(acl(A))$.*

4. *$A \subseteq B \implies acl(A) \subseteq acl(B)$; similarly for the other closures.*

5. *If $x \in acl(A)$, then there exists a finite subset A_0 of A such that $x \in acl(A_0)$.*

6. *If M is saturated and $|A| < |M|$ then $acl(A) = racl(A)$; similarly for definable closure.*

The following proposition is a bridge between the formula-based and the restricted (automorphism-based) notions of closures.

Proposition 1.4.59. *Let M be an L-structure, $\bar{a}, \bar{b} \in M^n$ and let A be a subset of M.*

1. *$tp(\bar{a}/A) = tp(\bar{b}/A)$ if and only if there exist an elementary extension N of M and an automorphism $f \in Aut(N/A)$ mapping $\bar{a}$ to $\bar{b}$.*

2. *$tp^{\exists}(\bar{a}/A) \subseteq tp^{\exists}(\bar{b}/A)$ if and only if there exist an elementary extension N of M and a monomorphism $f : N \to N$, fixing A pointwise and mapping $\bar{a}$ to $\bar{b}$.*

Proof. While the proof of point 1 can be found in [Mar02, Theorem 4.1.5], we prove point 2 since we did not find a reference. As regards the 'if' implication, if there is some elementary extension N of M and a monomorphism $f : N \to N$ fixing A pointwise and mapping $\bar{a}$ to $\bar{b}$, then $tp^{\exists}(\bar{a}/A) \subseteq tp^{\exists}(\bar{b}/A)$. It remains to show the converse. Set $N_0 = M$ and let N_1 be a $|M|$-saturated elementary extension of M. Using the saturation of N_1, we get a monomorphism $f_0 : N_0 \to N_1$ satisfying $f_0(\bar{a}) = \bar{b}$ and fixing A pointwise. Using a similar argument, we build an elementary chain $(N_i)_{i\in\mathbb{N}}$, $N_i \preceq N_{i+1}$, with a sequence of monomorphisms $(f_i : N_i \to N_{i+1})_{i\in\mathbb{N}}$ such that $f_i \upharpoonright N_i = f_{i+1} \upharpoonright N_i$ for every $i \in \mathbb{N}$. By setting $N = \bigcup_{i\in\mathbb{N}} N_i$ and $f = \bigcup_{i\in\mathbb{N}} f_i$, we get the required elementary extension and the required monomorphism.

□

Ould Houcine in [OH11] has proved that free groups of finite rank are homogeneous and 2-generated torsion-free hyperbolic groups are existential homogeneous. Homogeneity of finite rank free groups has also been proved by Perin and Sklinos in [PS10]. We will use the following theorem in Chapter 4.

Theorem 1.4.60. *[OH11, Proposition 5.9] Let F be a nonabelian free group of finite rank and let $\bar{a}$ be a tuple of F such that F is freely indecomposable relative to the subgroup generated by $\bar{a}$. Let $\bar{s}$ be a basis of F. Then there exists a universal formula $\varphi(\bar{x})$ such that $F \models \varphi(\bar{s})$ and such that for any endomorphism f of F, if $F \models \varphi(f(\bar{s}))$ and f fixes $\bar{a}$ then f is an automorphism. In particular $(F, \bar{a})$ is a prime model of the theory $Th(F, \bar{a})$.*

1.5 Asymptotic cones

An intuitive idea for an asymptotic cone of a metric space is what one sees when one looks at that space from infinitely far away. When the metric space is the Cayley graph of a group with word metric, we refer to an asymptotic cone of a group. This concept was introduced by Gromov as in [Gro81a, §7] in 1981, to prove that a finitely generated group with polynomial growth is virtually nilpotent. Three years after, Van den Dries and Wilkie in [VdDW84, §4] define asymptotic cones via 'nonstandard extensions ', that are ultrapowers, giving the definition used now usually. Asymptotic cones have proven powerful to characterize relevant classes of groups. To give further examples, Gromov in his already cited first paper on that topic states the equivalences, for a finitely generated group, between being virtually nilpotent and having locally compact asymptotic cones, and between being virtually Abelian and having asymptotic cones isometric to the Euclidean space $\mathbb{R}^n$ ([Gro93, 2.B]).

In this section we briefly give the fundamentals about asymptotic cones, to understand their use in next chapter; for more details, a recommended reference is [DS05].

As a convention, by I we will denote an infinite countable set.

Definition 1.5.1. An **ultrafilter** $\mathcal{F}$ over I is a set of subsets of I satisfying the following conditions:

1. if $A, B \in \mathcal{F}$ then $A \cap B \in \mathcal{F}$;
2. if $A \in \mathcal{F}$ and $B \supseteq A$, then $B \in \mathcal{F}$;
3. for every $A \subseteq I$, either $A \in \mathcal{F}$ or $I \setminus A \in \mathcal{F}$.

Definition 1.5.2. Let $\mathcal{F}$ be as in the above definition. Then, $\mathcal{F}$ is said to be **non-principal** if no finite subset of I is in $\mathcal{F}$.

From now on we will use the following shorthands:

1. we fix the index set I, therefore the index set is understood;
2. it is understood that any ultrafilter is non-principal.

Definition 1.5.3. Let $\mathcal{F}$ be an ultrafilter and let $n \in I$. If some statement $P(n)$ holds for all n from a set A belonging to $\mathcal{F}$, we say that $P(n)$ holds **$\mathcal{F}$-almost surely**.

Remark 1.5.4. Let $P_1(n), P_2(n), \ldots, P_m(n)$ be properties such that for any $n \in I$ no two of them can be true simultaneously. If the disjunction of these properties $\bigvee_{1\leq i\leq m} P_i(n)$ holds $\mathcal{F}$-almost surely, then there exists $i \in \{1, 2, \ldots, m\}$ such that $\mathcal{F}$-almost surely $P_i(n)$ holds and all $P_j(n)$ with $j \neq i$ do not hold.

Definition 1.5.5. Let $\mathcal{F}$ be an ultrafilter. For every sequence of points $(x_n)_{n\in I}$ in a topological space X, its **$\mathcal{F}$-limit** $\lim_{\mathcal{F}}(x_n)$ is a point $x \in X$ such that for every neighbourhood U of x the relation $x_n \in U$ holds $\mathcal{F}$-almost surely.

Remark 1.5.6. [Bou65, §8.1, p.I-53; §9.1, a consequence of Definition (C')] Let $X, (x_n)_{n\in I}$ be as in the above definition. If X is Hausdorff and $\lim_{\mathcal{F}} x_n$ exists, then $\lim_{\mathcal{F}}(x_n)$ is unique. Moreover, every sequence of elements in a compact space has a $\mathcal{F}$-limit.

Definition 1.5.7. For every sequence of sets $(X_n)_{n\in I}$, the **ultraproduct** $\prod X_n/\mathcal{F}$ corresponding to an ultrafilter $\mathcal{F}$ consists of $\sim$-equivalence classes of sequences $(x_n)_{n\in I}$ where for every n the point x_n belongs to X_n, where $(x_n) \sim (y_n)$ if $x_n = y_n$ $\mathcal{F}$-almost surely. The $\sim$-equivalence class of a sequence (x_n) in $\prod X_n/\mathcal{F}$ is denoted by $(x_n)^{\mathcal{F}}$. In particular, if all X_n are equal to the same X, the ultraproduct is called the **ultrapower** of X and is denoted by $X^{\mathcal{F}}$.

Recall that if $G_n, n \geq 1$, are groups then $\prod G_n/\mathcal{F}$ is again a group with the operation $(x_n)^{\mathcal{F}}(y_n)^{\mathcal{F}} = (x_n y_n)^{\mathcal{F}}$.

Definition 1.5.8. Let $(X_n, e_n, d_n), n \in I$ be a sequence of metric spaces with basepoints e_n and distances d_n and let $\mathcal{F}$ be an ultrafilter. Consider the ultraproduct $\prod X_n/\mathcal{F}$ and the point $e = (e_n)^{\mathcal{F}}$. For every two points $x = (x_n)^{\mathcal{F}}, y = (y_n)^{\mathcal{F}}$ in $\prod X_n/\mathcal{F}$, let $D(x, y) = \lim_{\mathcal{F}}(d_n(x_n, y_n))$. The function D is a pseudo-metric on $\prod X_n/\mathcal{F}$, that is, it satisfies the triangle inequality and the property $D(x, x) = 0$. It is not a metric yet, because for some $x \neq y$ the number $D(x, y)$ can be 0 or ∞. Let $\prod_e X_n/\mathcal{F}$ be the subset of $\prod X_n/\mathcal{F}$ consisting of elements which are at finite distance from e with respect to D. The $\mathcal{F}$-limit $\lim_{\mathcal{F}}(X_n)_e$ of the metric spaces (X_n, d_n) relative to e is the metric space obtained from $\prod_e X_n/\mathcal{F}$ over the equivalence relation $\sim$ defined as: $x \sim y$ if $D(x, y) = 0$. The equivalence class of a sequence (x_n) in $\lim_{\mathcal{F}}(X_n)_e$ is denoted by $\lim_{\mathcal{F}}(x_n)$.

Note that the limit of metric spaces is independent of the choice of basepoints. That is, if $e, e' \in \prod X_n/\mathcal{F}$ and $D(e, e') < \infty$, then $\lim_{\mathcal{F}}(X_n)_e = \lim_{\mathcal{F}}(X_n)_{e'}$. To prove it, it is sufficient to see that for every $x \in \prod X_n/\mathcal{F}$, we have $D(x, e') \leq D(x, e) + D(e, e')$ and the same with e and e' exchanged. So, $D(x, e)$ is finite if and only if $D(x, e')$ is finite. Thus, we have proved that $\lim_{\mathcal{F}}(X_n)_e$ and $\lim_{\mathcal{F}}(X_n)_{e'}$ are made of the same points. Since the distances in both spaces are induced from the same sequence of distances d_n, we have proved the independence of $\lim_{\mathcal{F}}(X_n)$ of the choice of basepoints.

Definition 1.5.9. Let (X, d) be a metric space, $\mathcal{F}$ be an ultrafilter, $(e_n)_{n \in I}$ be a sequence of points in X, e be the limit $(e_n)^{\mathcal{F}}$. Consider a sequence of numbers $l = (l_n)_{n \in I}$ such that $\lim_{\mathcal{F}} l_n = \infty$. Let $d_n = d/l_n$ The $\mathcal{F}$-limit $\lim_{\mathcal{F}}(X, d_n)_e$ is called the **asymptotic cone** of X relative to $\mathcal{F}$, the sequence of basepoints or observation points (e_n) and the sequence of scaling factors (l_n), and it is a metric space with distance $d_{\mathcal{F}}$. We denote this asymptotic cone as $Con_{\mathcal{F}}(X, e, l)$.

Let G be a group with a fixed set of generators Σ_G. Let X be the Cayley graph $Cay(G, \Sigma_G)$ of G with respect to Σ_G. Sometimes, when we want to put G in evidence rather than its Cayley graph, with some abuse of language we will write $Con_{\mathcal{F}}(G, e, l)$ for $Con_{\mathcal{F}}(X, e, l)$.

We end this section with an easy, already mentioned property of asymptotic cones, that will turn to be useful in Chapter 2.

Proposition 1.5.10. *An asymptotic cone of a hyperbolic space is a $\mathbb{R}$-tree.*

Proof. Let (X, d) be a δ-hyperbolic space and let $(e_n), (l_n)$ be as in Definition 1.5.9. Then, for every ultrafilter $\mathcal{F}$, the distance $d_{\mathcal{F}}$ on $Con_{\mathcal{F}}(X, e, l)$ is $\lim_{\mathcal{F}} d/l_n$. Since d is constant and l_n tends to infinity by Definition 1.5.9, $d_{\mathcal{F}} = 0$. The hyperbolicity constant $\delta_{\mathcal{F}}$ of $Con_{\mathcal{F}}(X, e, l)$ is $\lim_{\mathcal{F}} \delta/l_n$. As all the spaces in the sequence $((X, d/l_n))_{n \in \mathbb{N}}$ have the same δ-hyperbolic support, rescaling for each n by a factor $1/l_n$ makes the hyperbolicity constant of the n^{th} element in the sequence become δ/l_n. Therefore the hyperbolicity constant tends to 0, thus $Con_{\mathcal{F}}(X, e, l)$ is a 0-hyperbolic space, that is a real tree by Proposition 1.3.11.

□

Chapter 2

Bestvina-Paulin method

In this chapter, we study limits of acylindrical (in the sense of Bowditch) actions of finitely generated groups on hyperbolic spaces. Along the line of the method introduced by Paulin in [Pau91] and Bestvina in [Bes88], we will study arc and tripod stabilizers and find stability properties for the limit action.

Here we give a sketch about the historical frame in which this theory has risen and has been developed. The theory of group actions on real trees begins with the classical result of Bass-Serre theory about group actions on simplicial trees, where it is proved that, given a graph of groups Γ, there exists a simplicial tree T on which the fundamental group $G = \pi_1(\Gamma)$ acts in a way such that the quotient graph $G \setminus T$ is isomorphic to Γ and its vertex groups (edge groups, respectively) are conjugate to vertex stabilizers of T (to images of edge stabilizers of T into initial vertices, respectively) that project onto them. Bass-Serre theory allows us to study a group's structure through its action on a tree. For instance, it has been possible to prove that a group acting non-trivially and without inversions on a simplicial tree decomposes as an amalgamated free product or it has an infinite cyclic quotient. In the previous chapter we have also seen a proof of Kurosh subgroup theorem, easier than the traditional one.

Bestvina and Feighn in [BF95] study stable actions of groups on real trees. Up to now, many stability notions have been used, though it is important to keep in mind that the spirit in which they have been elaborated is the same: bounding the length of chains of nested arcs with strictlyincreasing stabilizers. This to approximate the behaviour of simplicial trees. Guirardel in [Gui04] defines another notion of superstability - despite its name, it is weaker than Bestvina and Feighn's. Guirardel's stability has turned to be quite significant after the proof by Paulin ([Pau91]) that a sequence of pairwise non-conjugate actions of a torsion-free hyperbolic group G on its Cayley graph converges to a superstable action of G on a real tree - actually, on its asymptotic cone.

2.1 Acylindrical actions

In [Sel97], Z. Sela defined acylindrical actions on simplicial trees. Let G be a group acting on a simplicial tree X and let $k \in \mathbb{R}$. The action of G on X is said to be **k-acylindrical**, if the diameter of the fixed subgraph of any element $g \in G$ is at most k; the action of G on X is said to be **acylindrical**, if it is k-acylindrical for some k.

In [Bow08, p.284], Bowditch defined acylindrical actions on hyperbolic graphs which can be in fact formulated for any metric space. Throughout this work, by 'acylindrical actions' we will mean acylindricity in the sense of Bowditch.

Definition 2.1.1. Let G be a group acting on a metric space X. The action is said to be **acylindrical** if for any $d \in \mathbb{R}$ there exists N_d and R_d such that for any elements $x, y \in X$, if $d(x,y) \geq R_d$, then the set

$$\{g \in G | d(x, gx) \leq d, d(y, gy) \leq d\}$$

contains at most N_d elements.

An example of acylindrical action is given in the following lemma.

Lemma 2.1.2. *Let G be a finitely generated group and let X be the Cayley graph of G with respect to some finite set of generators Σ_G. Then G acts acylindrically on X.*

Proof. For every $a \in X$ and $d \geq 0$ the cardinality of the ball $B(a,d)$ in X is bounded by $(2|\Sigma_G|)^d$. Since for every $g, g' \in G$ and $a \in X$ we have $ga = g'a \Rightarrow g = g'$ by group law, the cardinality of the set $\{g \in G | d(a, ga) \leq d\}$ is bounded by the cardinality of $B(a,d)$. Therefore it suffices to take, for every d, $R_d = 2d + 1$ and $N_d = (2|\Sigma_G|)^d$.

□

Before stating our main theorem, we need the following definition, generalizing [Per08, Definition 4.29]:

Definition 2.1.3. Let G_1, G_2 be groups and let H be a subgroup of G_1. Let $\mathcal{F}$ be a non-principal ultrafilter on $\mathbb{N}$. A sequence of homomorphisms $(f_n : G_1 \to G_2)_{n\in\mathbb{N}}$ **bounds** H **in the limit with respect to** $\mathcal{F}$ if for any $h \in H$ there exists a finite subset $B(h)$ of G_2 such that the set $\{n \in \mathbb{N} | f_n(h) \in B(h)\}$ belongs to $\mathcal{F}$.

Definition 2.1.4. Let G_1, G_2 be groups and let H be a subgroup of G_1. A sequence of homomorphisms $(f_n : G_1 \to G_2)_{n\in\mathbb{N}}$ **bounds** H **in the limit** if there exists a non-principal ultrafilter $\mathcal{F}$ such that $(f_n)_{n\in\mathbb{N}}$ bounds H in the limit with respect to $\mathcal{F}$.

The two following definitions are classical, but for the reader's convenience we recall them, since they will be used in our main theorem.

Definition 2.1.5. Let $G = \langle\Sigma\rangle$ be a finitely generated group and (X, d) a metric space. Let $(h_i : G \to \mathrm{Isom}(X) | i \in \mathbb{N})$ be a sequence of homomorphisms. We define the **stretching sequence** $(l_i)_{i\in\mathbb{N}}$ by

$$l_i = \inf_{a\in X} \max_{s\in\Sigma}\{d(a, h_i(s)a)\}. \tag{2.1}$$

Definition 2.1.6. Let I be an arc. We say that I is **stable** if $\mathrm{Stab}_{h_{\mathcal{F}}(G)}(I) = \mathrm{Stab}_{h_{\mathcal{F}}(G)}(J)$ for every $J \subseteq I$; otherwise, we say that I is **unstable**.

2.2 Main theorem

The aim of this chapter is the proof of the following theorem. Recall from Definition 1.2.18, by a **simplicial hyperbolic space** we mean the metric realization of a graph which is hyperbolic. Note that when X is a simplicial hyperbolic space then the infimum given in (2.1) is achieved. We let $e_i \in X$ such that $l_i = \max_{s\in\Sigma}\{d(e_i, h_i(s)e_i)\}$.

Theorem 2.2.1. *Let G be a finitely generated group and let $H \leq G$. Let X be a simplicial hyperbolic space and let $(h_i : G \to \mathrm{Isom}(X) | i \in \mathbb{N})$ be a sequence of homomorphisms such that $h_i(G)$ acts acylindrically on X for every i. Suppose that the stretching sequence tends to infinity. Let $\mathcal{F}$ be a non-principal ultrafilter on $\mathbb{N}$. Then, $\lim_{\mathcal{F}} h_i(G)$ acts on $Con_{\mathcal{F}}(X, e, l)$, the asymptotic cone - see Definition 1.5.9 - of X with respect to $\mathcal{F}$, the sequence of observation points $e = (e_i)_{i\in\mathbb{N}}$ and the sequence of scaling factors $l = (l_i)_{i\in\mathbb{N}}$ and $\mathcal{F}$, such that:*

1. *the action is non-trivial;*
2. *there exists $m \in \mathbb{N}$ such that any arc stabilizer is A-by-abelian, where A is a finite group of cardinality at most m.*
3. *there exists $m \in \mathbb{N}$ such that any tripod stabilizer is of cardinality at most m;*
4. *there exists $m \in \mathbb{N}$ such that the stabilizer of any descending chain of arcs stabilizes after at most m steps;*
5. *if $(h_i)_{i\in\mathbb{N}}$ bounds H in the limit with respect to $\mathcal{F}$, then $\lim_{\mathcal{F}} h_i(H)$ is elliptic in $\lim_{\mathcal{F}} h_i(G)$.*

We will prove point 1 in Proposition 2.3.10, point 2 in Proposition 2.3.7, point 3 in Proposition 2.3.8, point 4 in Proposition 2.3.9, and point 5 in Proposition 2.3.11.

In the following theorem we have a sufficient condition to get an unbounded stretching sequence.

Theorem 2.2.2. *Let G be a finitely generated group and let $H \leq G$. Let X be a hyperbolic space such that, for every point $a \in X$, the cardinality of the set of branching points (see Definition 1.2.3) of $B(a, n)$ is bounded by some m independent on a. Let $(h_i : G \to \mathrm{Isom}(X) | i \in \mathbb{N})$ be a sequence of homomorphisms such that $h_i(G)$ acts acylindrically on X for every i and $h_i(g)$ are pairwise non-conjugate in $\mathrm{Isom}(X)$ for every $g \in G$. Suppose that $\mathrm{Isom}(X)$ acts transitively on X. Let $\mathcal{F}$ be a non-principal ultrafilter on $\mathbb{N}$. Then, $\lim_{\mathcal{F}} h_i(G)$ acts on a $\mathbb{R}$-tree in such a way that the conclusions of Theorem 2.2.1 hold.*

The only point which needs a proof is the fact that the stretching sequence tends to infinity. This is provided by the following lemma.

Lemma 2.2.3. *Under the assumptions of Theorem 2.2.2, for every infinite $J \subseteq \mathbb{N}$, we have $\sup_{i\in J} l_i = \infty$.*

Proof. Suppose for a contradiction that there exists an infinite subset $J \subseteq \mathbb{N}$ such that $\sup_{i\in J} l_i = r$. Fix $d \in \mathbb{R}$ and $a, a' \in X$ such that $d(a, a') > R_d$.

As $\mathrm{Isom}(X)$ acts transitively on X, there exist $f_i \in \mathrm{Isom}(X)$ such that $f_i a = e_i$. So, for every $s \in \Sigma, i \in J$ we have $d(a, h_i(s)^{f_i} a) \leq l_i \leq r$. By assumption, $|B(a, r)| \leq m$ for some m. As Σ is finite, there is an infinite set $J' \subseteq J$ such that $h_i(s)^{f_i} a = h_j(s)^{f_j} a$ for every $s \in \Sigma, i, j \in J'$. For every $i \in J'$, we have

$$d(a', h_i(s)^{f_i} a') \leq d(a', a) + d(a, h_i(s)^{f_i} a')$$

$$\leq n + d(a, h_i(s)^{f_i} a')$$

$$\leq n + d(a, h_i(s)^{f_i}a) + d(h_i(s)^{f_i}a, h_i(s)^{f_i}a')$$
$$\leq n + r + n = 2n + r.$$

Since the cardinality of $B(a', 2n + r)$ is finite, there is an infinite set $J'' \subseteq J'$ such that $h_i(s)^{f_i}a' = h_j(s)^{f_j}a'$ for every $s \in \Sigma, i, j \in J''$. Since $d(a, a') > R_d$, by acylindricity we have $h_i(s)^{f_i} = h_j(s)^{f_j}$ for every $s \in \Sigma$, so $h_i(g)^{f_i} = h_j(g)^{f_j}$ for every $g \in G$. Therefore, there are infinitely many $i \in I$ such that h_i are conjugate, in contradiction with assumptions of Theorem 2.2.2.

□

2.3 Proof of main theorem

Assumption 2.3.1. From now on, we assume that X is a δ-hyperbolic simplicial metric space with distance d.

Lemma 2.3.2. *Let $a_1, a_2 \in X$, $D = d(a_1, a_2)$, $f \in \mathrm{Isom}(X)$. Suppose that $d(a_i, fa_i) \leq D/5$. Then, for every $p \in [a_1, a_2]$ such that $d(p, a_i) \geq 2D/5$, we have $d(fp, [a_1, a_2]) \leq 2\delta$.*

Proof. Consider the geodesic triangles of vertices a_1, a_2, fa_1 and a_2, fa_1, fa_2. Let $x_1 \in [fa_1, a_2]$ such that $d(fa_1, x_1) = (a_1 \cdot a_2)_{fa_1}$, $x_2 \in [a_1, a_2]$ such that $d(a_1, x_2) = (a_1 \cdot a_2)_{fa_1}$, $x_3 \in [a_1, fa_1]$ such that $d(a_1, x_3) = (fa_1 \cdot a_2)_{a_1}$, $y_1 \in [fa_1, a_2]$ such that $d(a_2, y_1) = (fa_1 \cdot fa_2)_{a_2}$, $y_2 \in [fa_1, fa_2]$ such that $d(fa_2, y_2) = (fa_1 \cdot a_2)_{fa_2}$, $y_3 \in [a_2, fa_2]$ such that $d(a_2, y_3) = (fa_1 \cdot fa_2)_{a_2}$, $x_1' \in [fa_1, fa_2]$ such that $d(fa_1, x_1') = d(fa_1, x_1)$, $y_1' \in [a_1, a_2]$ such that $d(a_2, y_1') = d(a_2, y_1)$. We have

$$d(fp, fa_i) = d(p, a_i) \geq \frac{2D}{5},$$

$$d(x_1', fa_1) = d(x_1, fa_1) = d(x_3, fa_1) \leq \frac{D}{5} \text{ and}$$

$$d(y_2, fa_2) = d(y_3, fa_2) \leq \frac{D}{5}.$$

Thus $fp \in [x_1', y_2]$, so there exists $p' \in [x_1, y_1] \subseteq [fa_1, a_2]$ such that $d(fp, p') \leq \delta$, therefore there exists $p'' \in [x_2, y_1'] \subseteq [a_1, a_2]$ such that $d(p', p'') \leq \delta$. So we have $d(fp, p'') \leq d(fp, p') + d(p', p'') \leq 2\delta$.

□

Lemma 2.3.3. *Let $a_1, a_2 \in X$, $f_1, f_2 \in \mathrm{Isom}(X)$. Let $D = d(a_1, a_2)$ and let $f_i a_j \in B(a_j, D/5)$ for every i, j. Then, for every $p \in [a_1, a_2]$ such that $d(p, a_i) \geq 2D/5$, we have $d(p, [f_1, f_2]p) \leq 8\delta$.*

Proof. Let $p : [0, D] \to [a_1, a_2]$, $p_i : [0, D] \to [f_i a_1, f_i a_2]$ and $p_{\bar{i}} : [0, D] \to [f_i^{-1}a_1, f_i^{-1}a_2]$ be parametrizations of $[a_1, a_2]$, $[f_i a_1, f_i a_2]$ and $[f_i^{-1}a_1, f_i^{-1}a_2]$, respectively. We have $p = p(s)$ for some $s \in [2D/5, 3D/5]$. Let $x_1 \in [f_i a_1, a_2]$ such that $d(f_i a_1, x_1) = (a_1 \cdot a_2)_{f_i a_1}$, $x_2 \in [a_1, a_2]$ such that $d(a_1, x_2) = (a_1 \cdot a_2)_{f_i a_1}$, $x_3 \in [a_1, f_i a_1]$ such that $d(a_1, x_3) = (f_i a_1 \cdot a_2)_{a_1}$, $x_1' \in [f_i a_1, f_i a_2]$ such that $d(f_i a_1, x_1') = d(f_i a_1, x_1)$.

We have $f_i(p) = p_i(s + k_i)$ and $f_i^{-1}(p) = p_{\bar{i}}(s - k_i)$, where k_i is such that $|k_i| = |d(a_1, x_2) - d(f_i a_1, x_1')|$. We have $d(f_i a_1, x_1') = d(f_i a_1, x_3)$ and $d(a_1, x_2) = d(a_1, x_3)$. Since $x_3 \in [a_1, f_i a_1]$, both $d(f_i a_1, x_1')$ and $d(a_1, x_2)$ are at most $D/5$, so the absolute value of their difference is at most $D/5$. Therefore, all $s + \epsilon_i k_i + \epsilon_j k_j$, where $\epsilon_i, \epsilon_j \in \{+1, -1\}$, belong to $[0, D]$. Thus, all of the following inequalities make sense. By applying Lemma 2.3.2 four times, we obtain:

$$
\begin{aligned}
d(p, [f_1, f_2]p) &\leq d(p(s), f_1 f_2 f_1^{-1} p(s - k_2)) + 2\delta \\
&\leq d(p(s), f_1 f_2 p(s - k_2 - k_1)) + 4\delta \\
&\leq d(p(s), f_1 p(s - k_1)) + 6\delta \\
&\leq d(p(s), p(s)) + 8\delta = 8\delta.
\end{aligned}
$$

□

Definition 2.3.4. Let $T \subseteq X$ be a geodesic triangle of vertices a, b, c. Let $s(T)$ be the size of T, which is defined as

$$
s(T) = \min_{\sigma \in Cycl(a,b,c)} (\sigma(b) \cdot \sigma(c))_{\sigma(a)},
$$

where $Cycl(a_1, \ldots, a_n)$ is the set of cyclic permutations of the ordered tuple $(a_1, \ldots, a_n)$.

Lemma 2.3.5. *Let $T \subseteq X$ be a geodesic triangle of vertices a_1, a_2, a_3. Let $f \in \mathrm{Isom}(X)$. Suppose that $d(a_i, f a_i) \leq s(T)/5$. Let $x_{ij} \in [a_i, a_j]$ such that $d(a_i, x_{ij}) = (a_j \cdot a_k)_{a_i}$. Then, $d(x_{ij}, f x_{ij}) \leq 9\delta$.*

Proof. Without loss of generality, we will show that $d(x_{12}, f x_{12}) \leq 9\delta$. Since $d(a_i, f a_i) \leq s(T)/5$ by our assumption and both $d(x_{12}, a_1)$ and $d(x_{12}, a_2)$ are at least $2s(T)/5$ by Definition 2.3.4, by Lemma 2.3.2 there exists a point $x_{12}' \in [a_1, a_2]$ whose distance from $f x_{12}$ is at most 2δ. Two cases may occur: $x_{12}' \in [x_{12}, a_2]$ or $x_{12}' \in [a_1, x_{12}]$.

1. Case $x_{12}' \in [x_{12}, a_2]$.

   Consider x_{31}. By Definition 2.3.4 and Lemma 2.3.2, there exists $x_{31}' \in [a_3, a_1]$ whose distance from $f x_{31}$ is at most 2δ. Two cases may occur: $x_{31}' \in [x_{31}, a_1]$ or $x_{31}' \in [a_3, x_{31}]$.

   (a) Case $x_{31}' \in [x_{31}, a_1]$.
   Define $x_{31}'' \in [a_1, a_2]$ such that $d(a_1, x_{31}'') = d(a_1, x_{31}')$. We have

$$
\begin{aligned}
d(f x_{12}, x_{12}) &\leq d(f x_{12}, x_{12}') + d(x_{12}', x_{12}) \\
&\leq d(f x_{12}, x_{12}') + d(x_{12}', x_{31}'') \\
&\leq d(f x_{12}, x_{12}') + d(x_{12}', f x_{12}) \\
&\quad + d(f x_{12}, f x_{31}) + d(f x_{31}, x_{31}') + d(x_{31}', x_{31}'') \\
&\leq 2\delta + 2\delta + \delta + 2\delta + \delta = 8\delta.
\end{aligned}
$$

(b) Case $x'_{31} \in [a_3, x_{31}]$.
Define $x''_{31} \in [a_2, a_3]$ such that $d(a_3, x''_{31}) = d(a_3, x'_{31})$ and $x''_{12} \in [a_2, a_3]$ such that $d(a_2, x''_{12}) = d(a_2, x'_{12})$. We have

$$\begin{aligned} d(fx_{12}, x_{12}) &\leq d(fx_{12}, x'_{12}) + d(x'_{12}, x_{12}) \\ &= d(fx_{12}, x'_{12}) + d(x'_{12}, x_{23}) \\ &\leq d(fx_{12}, x'_{12}) + d(x''_{12}, x''_{31}) \\ &\leq d(fx_{12}, x'_{12}) + d(x''_{12}, x'_{12}) + d(x'_{12}, fx_{12}) \\ &+ d(fx_{12}, fx_{31}) + d(fx_{31}, x'_{31}) + d(x'_{31}, x''_{31}) \\ &\leq 2\delta + \delta + 2\delta + \delta + 2\delta + \delta = 9\delta. \end{aligned}$$

2. Case $x'_{12} \in [a_1, x_{12}]$. Analogous to Case 1, exchanging the indices 23 and 31.

Since we have exhausted all cases, we have proved the statement.

□

Lemma 2.3.6. *Let X be a δ-hyperbolic metric space. Let $x_1, x_2 \in X$ and let g be an isometry of X such that $d(x_1, gx_1) \leq 9\delta$. Then, for any $p \in [x_1, x_2]$ satisfying $d(gp, [x_1, x_2]) \leq 2\delta$, we have $d(p, gp) \leq 27\delta$.*

Proof. We claim that we may assume $d(x_1, p) > 9\delta$. Indeed, if $d(x_1, p) \leq 9\delta$ then

$$d(p, gp) \leq d(p, x_1) + d(x_1, gx_1) + d(gx_1, fp) \leq 27\delta,$$

and we get the required result. Hence, we assume that $d(x_1, p) > 9\delta$ as claimed.

Let $c \in [x_1, x_2]$ such that $d(gp, [x_1, x_2]) = d(gp, c)$. Let $\Delta_1(x_1, gx_1, c)$ (resp. $\Delta_2(c, gx_1, gp)$) be a geodesic triangle with vertices x_1, c, gx_1 (resp. c, gx_1, gp).

Let $y_1 \in [x_1, gx_1], y_2 \in [ga, c], x_3 \in [c, x_1]$ be the internal points of Δ_1. Similarly, let $z_1 \in [c, gx_1], z_2 \in [ga, gp], z_3 \in [gp, c]$ be the internal points of Δ_2.

We treat the two cases, whether $c \in [p, x_2]$ or $c \in [x_1, p]$.

1. Case $c \in [p, x_1]$.

   Since $d(x_1, y_3) \leq d(x_1, gx_1) \leq 9\delta$ and $d(x_1, p) > 9\delta$, there exists $p' \in [gx_1, c]$ such that $d(p, c) = d(p', c)$ and $d(p, p') \leq \delta$.

   If $d(c, p) \leq 2\delta$, then $d(p, gp) \leq d(p, c) + d(c, gp) \leq 4\delta$ and we ge the required conclusion. Hence, we assume that $d(c, p) > 2\delta$.

   Therefore $d(p', c) = d(p, c) > d(z_1, c)$ and thus there exists $p'' \in [gx_1, gp]$ such that $d(gx_1, p'') = d(gx_1, p')$ and $d(p', p'') \leq \delta$.

   New we claim that $d(p'', gp) \leq 9\delta$. We have

$$\begin{aligned} d(gx_1, gp) &= d(x_1, p) = d(x_1, y_3) + d(y_3, p) \\ &\leq 9\delta + d(y_3, p) \\ &= 9\delta + d(y_2, p') \leq 9\delta + d(gx_1, p''). \end{aligned}$$

   Therefore, $d(gp, p'') = d(gx_1, gp) - d(gx_1, p'') \leq 9\delta$, as claimed.

   We conclude that $d(p, gp) \leq d(p, p'') + d(p'', gp) \leq 11\delta \leq 27\delta$ and we get the required conclusion in this case.

2. Case $c \in [x_1, p]$.

   We are going to show that $d(gc, gp) \leq 13\delta$.

   We assume that $d(gc, gp) > 2\delta$, as otherwise the result is clear.

   We are going to show that $d(gc, z_2) \leq 9\delta$. If $z_1 \in [gx_1, y_2]$ then

$$d(gc, z_2) \leq d(gx_1, z_2) = d(gx_1, z_1) \leq d(gx_1, y_2) \leq 9\delta$$

   and the result is clear. Hence, we assume that z_1 lies outside $[gx_1, y_2]$.

   We have

$$\begin{aligned} d(gx_1, z_2) &= d(gx_1, z_1) \\ &= d(gx_1, y_2) + d(y_2, z_1) \leq 9\delta + d(y_2, z_1) \leq 9\delta + d(x_1, c), \end{aligned}$$

   and thus

$$d(gx_1, z_2) - d(x_1, c) = d(gc, z_2) \leq 9\delta,$$

   as required.

   Therefore,

$$d(c, p) = d(gc, gp) = d(gc, z_2) + d(z_2, gp) \leq 11\delta.$$

   We conclude that $d(p, gp) \leq d(p, c) + d(c, gp) \leq 13\delta \leq 27\delta$ as required. This ends the proof in this case and at the same time the proof of the lemma.

□

Since X is a simplicial hyperbolic space, the infimum given in l_i by Definition 2.1.5 is achieved in some point e_i. Let $\mathcal{F}$ be a non-principal ultrafilter on $\mathbb{N}$ and let $Con_{\mathcal{F}}(X, e, l)$ be the asymptotic cone of X with respect to $\mathcal{F}$, the sequence of observation points $(e_i | i \in \mathbb{N})$ and the sequence of scaling factors $(l_i | i \in \mathbb{N})$.

Proposition 2.3.7. *Let $a_{\mathcal{F}}, b_{\mathcal{F}} \in Con_{\mathcal{F}}(X, e, l)$ with $a_{\mathcal{F}} \neq b_{\mathcal{F}}$. Then there exists $m \in \mathbb{N}$ such that the set of commutators of* $\mathrm{Stab}_{h_{\mathcal{F}}(G)}([a_{\mathcal{F}}, b_{\mathcal{F}}])$ *has cardinality bounded by m. Moreover, there exists $m' \in \mathbb{N}$ such that any arc stabilizer is A-by-abelian, where A is a finite group of cardinality at most m'.*

Proof. Let C be a finite subset of G such that $h_{\mathcal{F}}(C) \subseteq \mathrm{Stab}_{h_{\mathcal{F}}(G)}([a_{\mathcal{F}}, b_{\mathcal{F}}])$ and set

$$[C] = \{[c_1, c_2] | c_1, c_2 \in C\}.$$

Let $R_{8\delta}$ and $N_{8\delta}$ be the reals given by acylindricity of the action for $d = 8\delta$. We are going to show that the cardinal of the set $h_{\mathcal{F}}([C])$ is bounded by $N_{8\delta}$.

Since C is finite, by definition of $Con_{\mathcal{F}}(X, e, l)$, for every ϵ, there exists $A_C(\epsilon) \in \mathcal{F}$ such that

$$d_i(a_i, h_i(c)a_i) < \epsilon,\ d_i(b_i, h_i(c)b_i) < \epsilon,$$

for every $i \in A_C(\epsilon)$ and $c \in C$.

We have also that for every ϵ there exists $A_{a_{\mathcal{F}}, b_{\mathcal{F}}}(\epsilon) \in \mathcal{F}$ such that

$$|d_{\mathcal{F}}(a_{\mathcal{F}}, b_{\mathcal{F}}) - d_i(a_i, b_i)| < \epsilon,$$

for every $i \in A_{a_{\mathcal{F}}, b_{\mathcal{F}}}(\epsilon)$.

Let $\epsilon < d_{\mathcal{F}}(a_{\mathcal{F}}, b_{\mathcal{F}})/10$ and $A = A_C(\epsilon) \cap A_{a_{\mathcal{F}},b_{\mathcal{F}}}(\epsilon)$.

Put $D_i = d_i(a_i, b_i)$. Then, for every $i \in A$ and $c \in C$, we have

$$d_i(a_i, h_i(c)a_i) < \epsilon = (10\epsilon - \epsilon)/9 \leq (d_{\mathcal{F}}(a_{\mathcal{F}}, b_{\mathcal{F}}) - \epsilon)/9 < d_i(a_i, b_i)/9 < D_i/5,$$

and similarly fo b_i, we have

$$d_i(b_i, h_i(c)b_i) < D_i/5.$$

Let $p_i, q_i \in [a_i, b_i]$ such that $d_i(a_i, p_i) = d_i(q_i, b_i) = 2D_i/5$ and $d_i(p_i, q_i) = D_i/5$.

Since $\lim_F D_i/5 = D/5 > 0$ and $\lim_F(R_{8\delta}/l_i) = 0$, there exists $B \in \mathcal{F}$ such that

$$d_i(p_i, q_i) \geq R_{8\delta}/l_i,$$

for any $i \in B$.

Then, for every $i \in A \cap B$ and $c \in C$, the assumptions of Lemma 2.3.3 hold

$$d_i(a_i, h_i(c)a_i) < D_i/5,$$
$$d_i(b_i, h_i(c)b_i) < D_i/5,$$
$$d_i(a_i, p_i) = d_i(q_i, b_i) \geq 2D_i/5,$$

and moreover, we have

$$d_i(p_i, q_i) \geq R_{8\delta}/l_i.$$

Therefore by Lemma 2.3.3,

$$d_i(p_i, [h_i(c_1), h_i(c_2)]p_i) \leq 8\delta,\ d_i(q_i, [h_i(c_1), h_i(c_2)]q_i) \leq 8\delta_i$$

for every $i \in A \cap B$ and $c_1, c_2 \in C$.

Dividing by l_i we find

$$d(p_i, [h_i(c_1), h_i(c_2)]p_i) \leq 8\delta,$$
$$d(q_i, [h_i(c_1), h_i(c_2)]q_i) \leq 8\delta,$$
$$d(p_i, q_i) \geq R_{8\delta},$$

for every $i \in A \cap B$ and $c_1, c_2 \in C$.

By acylindricity, for every $i \in A \cap B$, the set

$$\{\gamma \in \Gamma | d(p_i, \gamma p_i) \leq 8\delta,\ d(q_i, \gamma q_i) \leq 8\delta\}$$

contains at most $N_{8\delta}$ elements. Therefore, for any $i \in A \cap B$, the set $h_i([C])$ contains at most $N_{8\delta}$ elements.

Suppose for a contradiction that the set of commutators of $\mathrm{Stab}_{h(G)}([a_U, b_U])$ contains more than $N_{8\delta}$ elements. Then, we find a finite subset $C \subseteq G$ for which there exists $D \in \mathcal{F}$ such that the set

$$\{h_i(g) | g \in [C]\}$$

has more than $N_{8\delta}$ elements, a contradiction sincee we get $A \cap B \cap D = \varnothing$. Therefore the set of commutators of $\mathrm{Stab}_{h(G)}([a_{\mathcal{F}}, b_{\mathcal{F}}])$ is finite and bounded by $N_{8\delta}$. By [Neu54, Theorem 3.1], a group with bounded set of commutators has finite derived group, and by [Wie56], the cardinality of derived group is bounded by a function of the bound of the set of commutators. This proves the statement.

□

Proposition 2.3.8. *There exists $m \in \mathbb{N}$ such that for any non-trivial tripod*

$$T \subseteq Con_{\mathcal{F}}(X, e, l),$$

the cardinal of $\mathrm{Stab}_{h_{\mathcal{F}}(G)}(T)$ *is bounded by m.*

Proof. Let $R_{27\delta}$ and $N_{27\delta}$ be the reals given by acylindricity for 27δ. Let $T = T(a_{\mathcal{F}}, b_{\mathcal{F}}, c_{\mathcal{F}})$ be a non-trivial tripod and suppose for a contradiction that $\mathrm{Stab}_{h(G)}(T)$ contains more than $N_{27\delta}$. Hence, we get a finite subset $K \subseteq G$ such that $h(K) \subseteq \mathrm{Stab}_{h(G)}(T)$ and $h(K)$ contains more than $N_{27\delta}$.

Now we are going to show that the cardinal of the set $h(K)$ is bounded by $N_{27\delta}$.

Let $\Delta(a_i, b_i, c_i)$ be the geodesic triangle with vertices a_i, b_i, c_i. Let $x_i \in [a_i, b_i]$, $y_i \in [b_i, c_i]$ and $z_i \in [c_i, a_i]$ be the internal points of $\Delta(a_i, b_i, c_i)$. We see that each of the sequence $(x_i | i \in \mathbb{N})$, $(y_i | i \in \mathbb{N})$, $(z_i | i \in \mathbb{N})$ converges to the center $o_{\mathcal{F}}$ of the tripod T.

Since K is finite, by definition of $Con_{\mathcal{F}}(X, e, l)$, for every ϵ, there exists $A_K(\epsilon) \in \mathcal{F}$ such that

$$d_i(a_i, h_i(k)a_i) < \epsilon,\ d_i(b_i, h_i(k)b_i) < \epsilon, d_i(c_i, h_i(k)c_i) < \epsilon,$$

$$d_i(x_i, h_i(k)x_i) < \epsilon,\ d_i(y_i, h_i(k)y_i) < \epsilon, d_i(z_i, h_i(k)z_i) < \epsilon,$$

for every $i \in A_K(\epsilon)$ and $k \in K$.

We have also that for every ϵ, there exists $A_T(\epsilon) \in \mathcal{F}$ such that

$$|d_{\mathcal{F}}(a_{\mathcal{F}}, b_{\mathcal{F}}) - d_i(a_i, b_i)| < \epsilon,$$

$$|d_{\mathcal{F}}(b_{\mathcal{F}}, c_{\mathcal{F}}) - d_i(b_i, c_i)| < \epsilon,$$

$$|d_{\mathcal{F}}(c_{\mathcal{F}}, a_{\mathcal{F}}) - d_i(c_i, a_i)| < \epsilon,$$

for every $i \in A_T(\epsilon)$.

Let $D = \min\{d(a_{\mathcal{F}}, o_{\mathcal{F}}), d(b_{\mathcal{F}}, o_{\mathcal{F}}), d(c_{\mathcal{F}}, o_{\mathcal{F}})\}$ and $\epsilon = D/6$. Let $A = A_K(\epsilon) \cap A_T(\epsilon)$. Then, for every $i \in A$ and $k \in K$, we have

$$\begin{aligned} d_i(a_i, h_i(k)a_i) &< \epsilon = D/6 = (D - \epsilon)/5 \\ &\leq (d_{\mathcal{F}}(a_{\mathcal{F}}, b_{\mathcal{F}}) - \epsilon)/5 \leq d_i(a_i, b_i)/5, d_i(a_i, c_i)/5, \end{aligned}$$

and similarly fo b_i and c_i. Thus, we conclude that for every $i \in A$ and $k \in K$, the assumptions of Lemma 2.3.2 hold.

Since

$$d_i(a_i, h_i(k)a_i) < \min(d_i(a_i, b_i)/5, d_i(a_i, c_i)/5),$$

we conclude by Lemma 2.3.5

$$d_i(h_i(k)x_i, x_i) \leq 9\delta_i,$$

and similarly for y_i and z_i.

Let $p_i \in [x_i, b_i]$ such that $d_i(x_i, p_i) = d_i(x_i, b_i)/2$.

We have $\lim_{\mathcal{F}} D_i/5 = D/5 > 0$ and $\lim_{\mathcal{F}}(R_{9\delta}/l_i) = 0$. Now there exists $B \in \mathcal{F}$ such that for any $i \in B$ for any $k \in K$, we have

$$9\delta_i < d_i(x_i, b_i)/5,$$

$$d_i(x_i, p_i) \geq R_{9\delta}/l_i.$$

Hence

$$d_i(h_i(k)x_i, x_i) \leq d_i(x_i, b_i)/5, d_i(h_i(k)a_i, a_i) \leq d_i(x_i, b_i)/5,$$

$$d_i(p_i, x_i) = d_i(x_i, a_i)/2 \geq d_i(x_i, a_i)/5$$

and thus for every $i \in A \cap B$ and $k \in K$, the assumptions of Lemma 2.3.2 hold. Therefore

$$d_i(h_i(k)p_i, [x_i, b_i]) \leq 2\delta_i,$$

and by Lemma 2.3.6, we conclude that

$$d_i(h_i(k)p_i, p_i) \leq 27\delta.$$

Finally putting all the pieces together, we find

$$d(x_i, p_i) \geq R_{27\delta}$$

$$d(x_i, h_i(k)x_i) \leq 9\delta \leq 27\delta,$$

$$d(p_i, h_i(k)p_i) \leq 27\delta.$$

By acylindricity, for every $i \in A \cap B$, the set

$$\{\gamma \in \Gamma | d(p_i, \gamma p_i) \leq 27\delta,\ d(x_i, \gamma x_i) \leq 27\delta\}$$

contains at most $N_{27\delta}$ elements. Therefore, for any $i \in A \cap B$, the set $h_i(K)$ contains at most $N_{27\delta}$ elements. Thus $h(K)$ is finite with cardinality bounded by $N_{27\delta}$.

□

Proposition 2.3.9. *There exists $n \geq 1$, such that for any descending chain of non-degenerate segments of $Con_{\mathcal{F}}(X, e, l)$, the corresponding chain of stabilizers in $h_{\mathcal{F}}(G)$ has length at most n.*

Proof. Let $e_1 \supset e_2 \supset \ldots$ be a descending chain of non-degenerate segments of $Con_{\mathcal{F}}(X, e, l)$ and let $S_i = \mathrm{Stab}_{h_{\mathcal{F}}(G)}(e_i)$. Then we have $S_1 \leq S_2 \leq \ldots$. Let m the number given by Proposition 2.3.7. Then the sequence of derived subgroups $([S_i, S_i] | i \in \mathbb{N})$ stabilizes after some $n = n(m)$, that is $[S_i, S_i] = [S_{i+1}, S_{i+1}]$ for every $i > n$. So we have $S_i \trianglelefteq S_{i+1}$. Suppose for a contradiction that $S_i < S_{i+1}$. Let $e_i = [a_{\mathcal{F}}, b_{\mathcal{F}}]$ and let $h_{\mathcal{F}}(g) \in S_{i+1} \setminus S_i$. Then one of the two inequalities $h_{\mathcal{F}}(g)a_{\mathcal{F}} \neq a_{\mathcal{F}}$ and $h_{\mathcal{F}}(g)b_{\mathcal{F}} \neq b_{\mathcal{F}}$ holds. Without loss of generality, let $h_{\mathcal{F}}(g)a_{\mathcal{F}} \neq a_{\mathcal{F}}$.

Claim. The points $a_{\mathcal{F}}, b_{\mathcal{F}}, h_{\mathcal{F}}(g)a_{\mathcal{F}}$ are the vertices of a non-degenerate tripod.

Proof. The point $h_{\mathcal{F}}(g)a_{\mathcal{F}}$ does not belong to any geodesic γ containing $[a_{\mathcal{F}}, b_{\mathcal{F}}]$. Suppose for a contradiction that $h_{\mathcal{F}}(g)a_{\mathcal{F}} \in \gamma$ for some geodesic γ containing $[a_{\mathcal{F}}, b_{\mathcal{F}}]$. By isometry, $d_{\mathcal{F}}(c_{\mathcal{F}}, a_{\mathcal{F}}) = d_{\mathcal{F}}(h_{\mathcal{F}}(g)c_{\mathcal{F}}, h_{\mathcal{F}}(g)a_{\mathcal{F}})$. Since $h_F(g) \in S_2$, we have $d_{\mathcal{F}}(c_{\mathcal{F}}, h_{\mathcal{F}}(g)a_{\mathcal{F}}) = d_{\mathcal{F}}(h_{\mathcal{F}}(g)c_{\mathcal{F}}, h_{\mathcal{F}}(g)a_{\mathcal{F}})$ and thus $d_F(a_{\mathcal{F}}, c_{\mathcal{F}}) = 1/2 d_{\mathcal{F}}(a_f, h_{\mathcal{F}}(g)a_{\mathcal{F}})$; i.e. $c_{\mathcal{F}}$ is the midpoint of $[a_{\mathcal{F}}, h_{\mathcal{F}}(g)a_{\mathcal{F}}]$.

Similarly, we have $d_{\mathcal{F}}(a_{\mathcal{F}}, d_{\mathcal{F}}) = d_{\mathcal{F}}(h_{\mathcal{F}}(g)a_{\mathcal{F}}, d_{\mathcal{F}})$ and thus $d_{\mathcal{F}}$ is the midpoint of $[a_{\mathcal{F}}, h_{\mathcal{F}}(g)a_{\mathcal{F}}]$. Therefore $c_{\mathcal{F}} = d_{\mathcal{F}}$; which is a contradiction. Hence $a_{\mathcal{F}}, b_{\mathcal{F}}, h_{\mathcal{F}}(g)a_{\mathcal{F}}$ are the vertices of a non-degenerate tripod T.

□ (Claim.)

Let $h_{\mathcal{F}}(g') \in S_i$. Then $h_{\mathcal{F}}(g')a_{\mathcal{F}} = a_{\mathcal{F}}$, $h_{\mathcal{F}}(g')b_{\mathcal{F}} = b_{\mathcal{F}}$ and

$$h_{\mathcal{F}}(g')h_{\mathcal{F}}(g)a_{\mathcal{F}} = h_{\mathcal{F}}(g)h_{\mathcal{F}}(g')a_{\mathcal{F}} = h_{\mathcal{F}}(g)a_{\mathcal{F}}.$$

So $h_{\mathcal{F}}(g') \in \mathrm{Stab}_{h_{\mathcal{F}}(G)}(T)$. By Proposition 2.3.8 we obtain that S_i is finite and bounded, and since this holds for every $i > n$, our statement is proved.

□ (Proposition 2.3.9.)

Proposition 2.3.10. *The action on $Con_{\mathcal{F}}(X, e, l)$ induced by $h_{\mathcal{F}}$ is non-trivial.*

Proof. Suppose for a contradiction that $a_{\mathcal{F}}$ is fixed by every $s \in \Sigma$. Then for every $s \in \Sigma$ there exists $A(s) \in \mathcal{F}$ such that for every $i \in A(s)$ we have $d_i(a_i, h_i(s)a_i) < 1/2$. So for $i \in \bigcap_{s\in\Sigma} A(s)$ we have $\max\{d_i(a_i, h_i(s)a_i)|s \in \Sigma\} < 1/2$. But $\max\{d_i(a_i, h_i(s)a_i)|s \in \Sigma\} \geq \max\{d_i(e_i, h_i(s)e_i)|s \in \Sigma\} = 1$, so we have a contradiction.

□

Proposition 2.3.11. *Let $H \leq G$ be such that for every $g \in H$ the set $\{h_i(g)|i \in I\}$ is finite. Then $h_{\mathcal{F}}(H)$ fixes $e_{\mathcal{F}}$.*

Proof. As $\{h_i(g)e_i|i \in I\}$ is finite for every $g \in H$, we have

$$\begin{aligned} d_{\mathcal{F}}(e_{\mathcal{F}}, h_{\mathcal{F}}(g)e_{\mathcal{F}}) &= \lim_{\mathcal{F}} d_i(e_i, h_i(g)e_i) \\ &= \lim_{\mathcal{F}} 1/l_i \; d(e_i, h_i(g)e_i) = 0. \end{aligned}$$

But $d(e_i, h_i(g)e_i)$ is bounded and l_i tends to infinity, so $\lim_{\mathcal{F}} d_i(e_i, h_i(g)e_i) = 0$.

□

Chapter 3

Limit groups, shortening argument, JSJ decompositions

In this chapter we will give a brief survey about some notions and tools that will be used to prove our main results. We will recall the notions of limit group and JSJ decomposition, and we will give an outline of the proof of shortening argument. Some references will be given at the beginning of each section, together with references to results throughout the chapter.

3.1 Limit groups

Limit groups of free groups have been introduced by Sela [Sel01] in order to study equations over free groups. The class of limit groups coincides with the class of *fully residually free groups*, see Definition 3.1.6. A comprehensive reference can be found in [CG05], where Champetier and Guirardel compare the various existing equivalent definitions of limit groups and characterize them more explicitly as *limits* of sequences of marked free groups - where a marked group is a group with a fixed set of generators - in a suitable topology.

We begin the section with some group-theoretic properties that are necessary to understand relevant properties of limit groups in our context.

Definition 3.1.1. Let G be a group. A subgroup $H \leq G$ is said to be **malnormal** in G if, for every $g \in G \setminus \{1\}$, we have $H \cap H^g = \{1\}$.

A class of groups defined by a malnormality property is that of CSA groups. The importance of its definition, given below, is that many results about torsion-free hyperbolic groups depend on the very fact that torsion-free hyperbolic groups are CSA.

Definition 3.1.2. Let G be a group. G is said to be **CSA** - that stands for Conjugately Separated Abelian - if every maximal abelian subgroup of G is malnormal in G.

Note that CSA is a stronger property than the following:

Definition 3.1.3. Let G be a group. G is said to be **commutative transitive** if for every $a, b, c \in G$, if $[a, b] = [b, c] = 1$ then $[a, c] = 1$.

In fact, CSA groups are exactly those in which centralizers of non-trivial elements are abelian and self-normalizing - recall that, given a group G and a subgroup $H \leq G$, the normalizer $N_G(H)$ is the subgroup $\{g \in G | H^g = H\}$ -, while commutative transitive groups are those in which centralizers of non-trivial elements are abelian.

Infinite dihedral group $\langle \rho, \sigma | \sigma^2 = 1, \rho^\sigma = \rho^{-1} \rangle$ is an example of a commutative transitive non-CSA group. In fact, the subgroup $\langle \rho \rangle$ is maximal abelian, but $\langle \rho \rangle \cap \langle \rho \rangle^\sigma = \langle \rho \rangle \neq \{1\}$.

Lemma 3.1.4. *[OH11, Lemma 3.1] Let $G = \langle H, t | U^t = V \rangle$ where U and V are cyclic subgroups of H generated by u and v respectively. Suppose that*

1. *U and V are malnormal in H.*
2. *$U^h \cap V = 1$ for any $h \in H$.*

Let $\alpha, \beta \in H, s \in G$ such that $\alpha^s = \beta, |s|_t \geq 1$. Then one of the following cases holds:

1. *$\alpha = u^{p\gamma}, \beta = v^{p\delta}, s = \gamma^{-1} t \delta$, for some $p \in \mathbb{Z}, \gamma, \delta \in H$.*
2. *$\alpha = v^{p\gamma}, \beta = u^{p\delta}, s = \gamma^{-1} t^{-1} \delta$, for some $p \in \mathbb{Z}, \gamma, \delta \in H$.*

Proof. Let $h_0 t^{\epsilon_0} \dots t^{\epsilon_n} h_{n+1}$ be a normal form for s. Then we have

$$h_{n+1}^{-1} t^{-\epsilon_n} \dots t^{-\epsilon_0} h_0^{-1} \alpha h_0 t^{\epsilon_0} \dots t^{\epsilon_n} h_{n+1} = \beta,$$

thus either $h_0^{-1} \alpha h_0 \in U$ and $\epsilon_0 = 1$ or $h_0^{-1} \alpha h_0 \in V$ and $\epsilon_0 = -1$.

Suppose that the first case occurs. Therefore $\alpha = h_0 u^p h_0^{-1}$ for some $p \in \mathbb{Z}$.

We claim that $n = 0$. Suppose for a contradiction that $n \geq 1$. Then $h_1^{-1} v^p h_1 \in U$ and $\epsilon_1 = 1$ or $h_1^{-1} v^p h_1 \in V$ and $\epsilon_1 = -1$. Since $U^h \cap V = 1$, the first case is impossible. Therefore the second case holds, thus $h_1 \in V$ by the malnormality of V. Hence the sequence $(t^{\epsilon_0}, h_1, t^{\epsilon_1})$ is not reduced, that is a contradiction. Thus $n = 0$ as claimed, hence $\alpha = h_0 u^p h_0^{-1}$, $s = h_0 t h_1$, $\beta = h_1^{-1} v^p h_1$. If $h_0^{-1} \alpha h_0 \in V$ and $\epsilon_0 = -1$, then the proof is similar, with U, u for V, v respectively.

□

The following notions, introduced by Baumslag in [Bau67], generalize the concept of 'group belonging to a class χ'. The sense of these notions is a kind of 'approximating by finite patches'.

Definition 3.1.5. Let G be a group and let χ be a class of groups. G is said to be **residually** χ if for every $g \in G \setminus \{1\}$ there exist $K \in \chi$ and a morphism $f : G \to K$ such that $f(g) \neq 1$.

For instance, we will speak about residually free groups. If χ consists of the singleton K, then we say simply that G is residually K.

Definition 3.1.6. Let G be a group and let χ be a class of groups. G is said to be **fully residually** χ if for every finite subset $X \subseteq G \setminus \{1\}$ there exist $K \in \chi$ and a morphism $f : G \to K$ such that $1 \notin f(X)$.

If χ consists of the singleton K, then we simply say that G is fully residually K.

Let G, K be groups.

Definition 3.1.7. A sequence of homomorphisms $(f_i : G \to K | i \in \mathbb{N})$ is **stable** if, for every $g \in G$, either $f_i(g) = 1$ for all but finitely many i, or $f_i(g) \neq 1$ for all but finitely many i.

Definition 3.1.8. Let $(f_i : G \to K | i \in \mathbb{N})$ be a stable sequence. The **stable kernel** of $(f_i | i \in \mathbb{N})$, denoted $\ker_\infty(f_i)$, is the set

$$\{g \in G | f_i(g) = 1 \text{ for all but finitely many } i\}.$$

Definition 3.1.9. A group L is a K**-limit group** if there exists a group G and a stable sequence $(f_i : G \to K | i \in \mathbb{N})$ such that $L \cong G / \ker_\infty(f_i)$.

The requirement of stability of the kernel may be dropped using the following definition, appeared in [OHV11]. In this way Definition 3.1.9 comes to be the particular case of Definition 3.1.10, with 'for all non-principal ultrafilters' instead of 'there exists a non-principal ultrafilter'.

Definition 3.1.10. Let K be a group and let G be a finitely generated group. Let $\mathcal{F}$ be a non-principal ultrafilter over $\mathbb{N}$ and let $f = (f_i : G \to K)_{i\in\mathbb{N}}$ be a sequence of homomorphisms. Let the $\mathcal{F}$-kernel $\ker_{\mathcal{F}}(f)$ be the set of elements $g \in G$ such that $\{i \in \mathbb{N} | f_i(g) = 1\} \in \mathcal{F}$. A K**-limit group** is a group L such that there exists a finitely generated group G, a non-principal ultrafilter $\mathcal{F}$ and a sequence of homomorphisms $(f_i : G \to K | i \in \mathbb{N})$ such that $L \cong G/ker_{\mathcal{F}}(f)$. □

In Definitions 3.1.9 and 3.1.10, we will simply say that 'L is a limit group' when K is a free group.

Some important facts, connecting group- and model-theoretic properties of limit groups, are collected in the following theorem. Recall definitions 1.5.7, 1.4.37 and 1.3.30 for notions of ultrapower, universal theory of a structure and equationally noetherian group, respectively.

Theorem 3.1.11. *[OH07, Theorem 2.1, points (1) to (4)] Let K be a group. Then*

1. *A countable fully residually K group is a K-limit group.*
2. *A K-limit group is embeddable in all non-principal ultrapowers of K; in particular, it is a model of $Th_A^\forall(K)$.*
3. *Let G be a finitely generated group. Then, the following properties are equivalent:*
   (a) *G is a model of $Th_A^\forall(K)$;*
   (b) *G is embeddable in some non-principal ultrapower of K;*
   (c) *G is embeddable in all non-principal ultrapowers of K.*
4. *If K is equationally noetherian, then for every finitely generated group G the following properties are equivalent:*
   (a) *G is a model of $Th_A^\forall(K)$;*
   (b) *G is fully residually K;*
   (c) *G is a K-limit group.*

Since CSA property is universally axiomatizable, if an equationally noetherian group K is CSA then every K-limit group is CSA by implication (4a) $\Rightarrow$ (4c) of point 4 of Theorem 3.1.11 above. By [MR96, Proposition 12] torsion-free hyperbolic groups are CSA, therefore limit groups of torsion-free hyperbolic groups are CSA too.

Let $A \leq G$ and let there exist a fixed embedding $A \to K$. If in Definition 3.1.9 we take a stable sequence of A-homomorphisms, we say that L is a K-limit group **relative to** A. Recall that a A-homomorphism is a group homomorphism fixing a subset A pointwise, see Definition 1.4.15.

3.2 Shortening argument

The shortening argument has been introduced by Rips and Sela in [RS94]. It basically consists of the following method:

1. we have a stable sequence of actions, with trivial stable kernel, of a group G on some Cayley graph X of a group K, giving a limit action on some asymptotic cone of X;
2. we decompose the limit action into basic blocks, using the Rips decomposition;
3. for each basic type of block we find a suitable modular automorphism that shortens the action on that type of block and leaves the other blocks unaltered;
4. the composition of such modular automorphisms will shorten all but finitely many morphisms of the sequence.

The argument has been formulated by Rips and Sela for G finitely generated and K torsion-free hyperbolic; from then on, some generalization has been carried out; for instance, Perin in [Per08] proves a relative version, and Reinfeldt and Weidmann in [RW10] drop the torsion-freeness assumption for K.

The shortening argument has found important implications. For instance, the facts that modular group has finite index in the group of automorphisms, and shortening quotients are proper quotients. In next section we use this last result to prove constructibility of torsion-free hyperbolic groups over cyclic subgroups from algebraic closure.

Before dealing with the shortening argument, we give the basic notions about modular automorphisms and decomposition of actions on real trees. A reference for Rips decomposition is [Bes02].

Definition 3.2.1. Let $G = \langle \Sigma_G \rangle, K = \langle \Sigma_K \rangle$ be finitely generated groups. Let $X = \mathrm{Cay}(K, \Sigma_K)$ and let e be a basepoint for X. Let $f : G \to K$ be a morphism. The **length** $|f|$ of f is defined as

$$\max_{g \in \Sigma_G} d_X(e, f(g) \cdot e).$$

In particular, if we take $e = 1$, we have $|f| = \max_{g \in \Sigma_G} |f(g)|_X$, where $|\cdot|_X$ is the word metric of X.

Definition 3.2.2. Let G, K, X, e be as in Definition 3.2.1 and let K be torsion-free hyperbolic. Let $H \leq G$. A morphism $f : G \to K$ is **short** with respect to H if for any $\sigma \in \mathrm{Mod}_H(G)$ we have

$$\max_{g \in \Sigma_G} d_X(e, f(g) \cdot e) \leq \max_{g \in \Sigma_G} d_X(e, (f \circ \sigma)(g) \cdot e).$$

Notation 3.2.3. From now on, we will denote as Γ an abelian splitting of a group G relative to a subgroup $H \leq G$ - recall Definition 1.2.26 - such that V, E are the sets of vertices and edges of the underlying graph, respectively.

Definition 3.2.4. Let G be a finitely generated group, and let Γ be a one-edge splitting of G relative to H, with edge group C. Let $c \in Z(C)$. A **Dehn twist** about c is an automorphism $\delta_c \in \mathrm{Aut}(G)$, defined as follows:

1. if $G = A *_C B$, then $\delta_c(a) = a, \delta_c(b) = b^c$ for every $a \in A, b \in B$.
2. if $G = A*_C$, with stable letter t, then $\delta_c \restriction A = \mathrm{id}_A$ and $\delta_c(t) = tc$.

Definition 3.2.5. With Notation 3.2.3 above, let G_v be an abelian vertex group of Γ. Let P be the subgroup of G_v generated by the incident edge groups. By Proposition 1.2.29, any automorphism ϕ_v of G_v fixing H and P pointwise has a standard extension to G. Such an automorphism is called a **modular automorphism of abelian type**.

Definition 3.2.6. With Notation 3.2.3 above, a vertex $v \in V$ is called of **surface type** if G_v is isomorphic to the fundamental group of a compact connected surface S with boundary, which is neither a disk nor a Möbius band nor a cylinder, and such that each edge group G_e incident on v is conjugate to the fundamental group of a boundary component of S.

Definition 3.2.7. With Notation 3.2.3 above, let $v \in V$ be a surface type vertex. By Proposition 1.2.29, any automorphism ϕ_v of G_v that

1. fixes H pointwise and
2. restricts to a conjugation by g_e for every edge e incident in v

has a standard extension to G. Such an automorphism is called a **modular automorphism of surface type**.

Definition 3.2.8. With Notation 3.2.3 above, the **abelian modular group** of Γ relative to H, denoted $\mathrm{Mod}_H(\Gamma)$, is the subgroup of $\mathrm{Aut}_H(G)$ generated by Dehn twists, modular automorphisms of abelian type and modular automorphisms of surface type of Γ relative to H.

Definition 3.2.9. With Notation 3.2.3 above, the **abelian modular group** of G relative to H, denoted $\mathrm{Mod}_H(G)$, is the subgroup of $\mathrm{Aut}_H(G)$ generated by Dehn twists, modular automorphisms of abelian type and modular automorphisms of surface type of Γ relative to H for every abelian splitting Γ of G relative to H.

Now we can state the shortening argument.

Theorem 3.2.10. *Let K be a torsion-free hyperbolic group with a finite generating set Σ_K. Let G be a finitely generated group, with a finite generating set Σ_G and let H be a non-abelian subgroup of G such that G is freely H-indecomposable. Let $(f_i : G \to K)_{i\in\mathbb{N}}$ be a stable sequence of pairwise distinct homomorphisms with trivial stable kernel and which bounds H in the limit. Then, $\{i \in \mathbb{N} | f_i \text{ is not short}\} \in \mathcal{F}$ for any non-principal ultrafilter $\mathcal{F}$.*

Definition 3.2.11. Let σ be an action of a group G on a real tree T. We say that T is **minimal** with respect to σ if it has no proper subtrees whose setwise stabilizer is the whole G.

Definition 3.2.12. With the same notations as Definition 3.2.11, T is said to be **stable** with respect to an action of some group G - or the action of G on T is stable - if, for every descending sequence S of non-degenerate subtrees $T \supseteq T_1 \supseteq T_2 \supseteq \ldots$ such that the pointwise stabilizer $\mathrm{Stab}_G(T)$ is non-trivial, there exists n_S finite such that, in the corresponding sequence of pointwise stabilizers $\mathrm{Stab}_G(T) \subseteq \mathrm{Stab}_G(T_1) \subseteq \mathrm{Stab}_G(T_2) \subseteq \ldots$, $\mathrm{Stab}_G(T_i) = \mathrm{Stab}_G(T_j)$ for every $i, j \geq n_S$.

Definition 3.2.13. We say that the action of G on T is **superstable** if for every subtree T with non-trivial pointwise stabilizer $\mathrm{Stab}_G(T)$, for every subtree $T_1 \subseteq T$, we have $\mathrm{Stab}_G(T_1) = \mathrm{Stab}_G(T)$.

Note that superstability of the action implies stability; moreover, it corresponds to saying that every non-degenerate arc is stable, in the sense of Definition 2.1.6.

Definition 3.2.14. An action σ of a group G on a real tree T is said to be **very small** if σ is non-trivial, T is minimal and stable with respect to σ, pointwise stabilizers of arcs are abelian and pointwise stabilizers of tripods are trivial.

Theorem 3.2.15. *Let G, H, K, f_i be as in Theorem 3.2.10. Then G admits a non-trivial abelian splitting relative to H.*

We only outline the proof of the above theorem, because the structure and details of the proof are quite similar to those of Theorem 2.2.1.

Outline of the proof. Let $X = \mathrm{Cay}(K, \Sigma_K)$. Let $\mathcal{F}$ be a non-principal ultrafilter on $\mathbb{N}$. For every i, let the sequence (X_i, e_i) be the constant sequence whose elements are all equal to $(X, 1)$. Since the given homomorphisms are pairwise distinct, $\lim_{n\to\infty} \lambda_n = \infty$. Then G acts on the asymptotic cone $(Con_{\mathcal{F}}(X, 1, l), d_{\mathcal{F}})$, relative to the constant sequence of observation points 1, the sequence of scaling factors $l = |l_n|_{n\in\mathbb{N}}$ (as in Definition 3.2.1) and the ultrafilter $\mathcal{F}$. By [RW10, Theorem 1.11] in the particular case of a torsion-free K, the action is superstable, with abelian arc stabilizers and trivial tripod stabilizers. Non-triviality of the action follows from Proposition 2.3.10 and ellipticity of H follows from Proposition 2.3.11; note that these points are independent of acylindricity of the action.

By [Gui08, Theorem 5.1 and Corollary 5.2], we get the abelian splitting.

□

Observe that Theorem 2.2.1, the main subject of the previous chapter, gives the same results about action stability and arc and tripod stabilizers, under the assumption that the action is acylindrical; on the other side, it allows X to be an arbitrary hyperbolic graph, instead of a Cayley graph of some torsion-free hyperbolic group. This comparison gives rise to the following

Question 1. Which finitely generated groups act Bowditch-acylindrically on a hyperbolic graph?

To understand the shortening argument, we need some preliminary definitions. We refer to [RS94, Section 10] and [Gui08].

Let G be a finitely generated group and let H be a subgroup of G. By an action of the pair (G, H) on a tree T, we mean an action of G on T such that H fixes a point in T - in other words, H is elliptic.

Now we define some 'basic' types of actions of a group G on a real tree T. We suppose that the action is very small; this assumption allows an easier definition of surface action. The importance of these kinds of actions lies in the fact that, under the actions that we consider, T decomposes into blocks on which G acts in one of the following ways.

Definition 3.2.16. An action of G on T is **simplicial** if T is a polyhedral real tree, in the sense of Definition on page 17.

Definition 3.2.17. An action of G on T is **abelian** if T is isometric to a real line and the action has dense orbits.

Definition 3.2.18. An action of G on T is **surface** if G is an extension of a group fixing T pointwise by a fundamental group of a 2-orbifold.

These types of actions have also other names in literature: we have followed the terminology by Bestvina [BF]; for instance, Sela in [RS94] and throughout his works uses 'discrete, axial, IET (interval exchange transformations)' for 'simplicial, abelian, surface' actions respectively. Guirardel ([Gui08]) uses 'Seifert type' for 'surface'.

Actually it could happen that a fourth type of action arises, the *Levitt* or *thin* action, but this is not our case, since by our assumption G is freely indecomposable with respect to H, while this kind of action corresponds to a free decomposition - possibly relative - of G. See [Bes02, §5.3] for details.

A construction that turns to be useful in splitting of an action is the graph of actions, defined as follows. We follow [Gui08], that is the particular case of the construction defined in [Gui04, Definition 4.3] for a graph of actions over a Λ-tree, where Λ is an ordered abelian group.

Definition 3.2.19. A **graph of actions** of G on $\mathbb{R}$-trees $\mathcal{G}$ is a triple

$$(S, (Y_v)_{v\in S^0}, (p_e)_{e\in S^1}),$$

where:

1. S is a simplicial tree on which G acts without inversions;

2. Y_v are real trees, called *vertex actions*;

3. $p_e \in Y_v$, where $v = \omega(e)$. The points p_e are called *attaching points.*

The graph is subject to the following conditions:

1. G acts on $\coprod_v Y_v$ such that the projection $Y_v \mapsto v$ is equivariant, that is $gY_v = Y_{gv}$ for every $g \in G$ and for every $v \in S^0$;

2. G acts equivariantly on p_e for every e, that is $p_{ge} = gp_e$ for every $g \in G$.

Observe that the equivariance conditions of Definition 3.2.19 imply that the pointwise stabilizer of v stabilizes Y_v setwise, and the pointwise stabilizer of e stabilizes p_e.

We may see a graph of actions as a $\mathbb{R}$-tree $T_{\mathcal{G}}$, obtained as the union of the Y_v quotiented by identifying p_e and $p_{e'}$ for every $e \in S^1$, where $p_{e'}$ belongs to some $Y_{v'}$ and $v' = \alpha(e)$. The formal definition of this construction, called the dual tree to a graph of actions, is in [Gui04, p.1444].

In [Lev94], Levitt defines a graph of actions starting with a graph of groups $\mathbb{G}$ such that $\pi_1(\mathbb{G}) = G$, such that

1. any vertex group G_v acts on a $\mathbb{R}$-tree Y_v;
2. for every edge e of the underlying graph, such that $\omega(e) = v$, there is a point of Y_v fixed by $\omega_e(G_e)$ (recall Definition 1.2.19).

Now we have the elements to outline the proof of the shortening argument.

Outline of the proof of Theorem 3.2.10. Consider $(Cay(K, \Sigma_K), d)$, with the identity vertex e as basepoint. For each $n \in \mathbb{N}$, let $l_n = \max_{g \in \Sigma_G} |f_n(g)|_{\Sigma_K}$ be the length of f_n, as in Definition 3.2.1. Let $\mathcal{F}$ be a non-principal ultrafilter over $\mathbb{N}$. The given homomorphisms are pairwise non-conjugate. In fact, since we are in the relative case with H non-abelian - therefore non-trivial -, the morphisms must be pairwise non-conjugate, otherwise we would have two morphisms f_i and f_i^g, with $g \in G \setminus \{1\}$, both fixing H pointwise, a contradiction. Thus, $\lim_{\mathcal{F}} l_n = \infty$. Then G acts on the asymptotic cone $T = (Con_{\mathcal{F}}(K, e, l), d_{\mathcal{F}})$, which is a real tree, relative to the constant sequence $(e_n = e)_{n \in \mathbb{N}}$ of observation points, the sequence of scaling factors $l = (l_n)_{n \in \mathbb{N}}$ and the ultrafilter $\mathcal{F}$. By Theorem 3.2.15, the action is non-trivial, superstable, with abelian arc stabilizers and trivial tripod stabilizers, and H fixes e.

By [Gui08, Theorem 5.1]), T has a decomposition as a graph of actions

$$\mathcal{A} = (\mathcal{G}(V, E), (T_v)_{v \in V}, (p_e)_{e \in E}),$$

where each vertex action of G_v is either symplicial or surface or abelian.

Let $\Sigma_G = \{g_1, \ldots, g_q\}$. Let I_s be the set of indices i such that the segment $[e, g_i e]$ intersects a surface type component.

By [RS94, Proposition 5.2], it is possible to construct a composition σ_1 of surface type modular automorphisms such that $d_{\mathcal{F}}(e, \sigma_1(g_i)e) < d_{\mathcal{F}}(e, g_i e)$ for all $i \in I_s$ and $\sigma_1(g_i) = g_i$ for all $i \notin I_s$. Let I_a be the set of indices i (not necessarily disjoint from I_s) such that $[e, \sigma_1(g_i)e]$ intersects an abelian component. In that case, it is possible (see [Wil06, Theorem 2.44]) to find a composition σ_2 of abelian type modular automorphisms such that $d_{\mathcal{F}}(e, \sigma_2 \circ \sigma_1(g_i)e) < d_{\mathcal{F}}(e, \sigma_1(g_i)e)$ for all $i \in I_a$ and $\sigma_2 \circ \sigma_1(g_i) = \sigma_1(g_i)$ for all $i \notin I_a$. Finally let I_d be the set of indices i such that $[e, \sigma_2 \circ \sigma_1(g_i)e]$ intersects a simplicial component. In this case it is possible to show that there exists $U_d \in \mathcal{F}$ such that for any $n \in U_d$, there exists a Dehn twist τ_n such that $d_n(e_n, \tau_n \circ \sigma_2 \circ \sigma_1(f_n(g_i))e_n) < d_n(e_n, \sigma_2 \circ \sigma_1(f_n(g_i))e_n)$ for all $i \in I_d$ and $d_n(e_n, \tau_n \circ \sigma_2 \circ \sigma_1(f_n(g_i))e_n) = d_n(e_n, \sigma_2 \circ \sigma_1(f_n(g_i))e_n)$ for all $i \notin I_d$.

By the above characterization of σ_1 there exists $U_s \in \mathcal{F}$ such that for any $n \in U_s$ we have $d_n(e_n, \sigma_1(f_n(g_i))e_n) < d_n(e_n, f_n(g_i)e_n)$ for any $i \in I_s$ and $\sigma_1(f_n(g_i)) = f_n(g_i)$ for all $i \notin I_s$. Similarly, by the above characterization of σ_2 there exists $U_a \in \mathcal{F}$ such that for any $n \in U_a$ we have $d_n(e_n, \sigma_2 \circ \sigma_1(f_n(g_i))e_n) < d_n(e_n, \sigma_1(f_n(g_i))e_n)$ for any $i \in I_a$ and $\sigma_2 \circ \sigma_1(f_n(g_i)) = \sigma_1(f_n(g_i))$ for all $i \notin I_a$. Define $\alpha_n := \tau_n \circ \sigma_2 \circ \sigma_1$ and $U := U_d \cap U_s \cap U_a$. Then we have $U \in \mathcal{F}$, $\alpha_n \in Mod_H(G)$ and $d_n(e_n, \alpha_n(f_n(g_i))e_n) < d_n(e_n, f_n(g_i)e_n)$ for any $g_i \in \Sigma_G$ and for any $n \in U$, which proves the result. For more details, the reader can see [Wil06, §2.4.4 to 2.4.6],[Per08, §4.3 and Chapter 5],[RW10, §4.2].

□

One of the applications of shortening argument is the proof by Rips and Sela ([RS94, Corollary 4.4]) that modular group has a finite index in automorphisms group. This can be slightly generalized as follows (see also [Per08]).

Theorem 3.2.20. *[Per08, Theorem 4.34]Let K be a torsion-free hyperbolic group, G a finitely generated group, H a non-abelian subgroup of G such that G is freely H-indecomposable. Let $e : H \to K$ be an embedding. Suppose that there exists at least an embedding of G in K whose restriction to H is e. Then there exists a finite set $\{f_1, \ldots, f_p\}$ of embeddings of G in K, whose restriction to H coincides with e and such that for any embedding $f : G \to K$, whose restriction to H coincides with e, there exists a modular automorphism $\sigma \in \mathrm{Mod}_H(G)$ such that $f \in \{f_1 \circ \sigma, \ldots, f_p \circ \sigma\}$.*

Proof. Let $(f_n : G \to K)_{n\in\mathbb{N}}$ be the sequence of all embeddings of G in K whose restriction to H is e. For each $n \in \mathbb{N}$, choose a modular automorphism $\sigma_n \in \mathrm{Mod}_H(G)$ such that $f_n \circ \sigma_n$ is short. Suppose for a contradiction that the set $I = \{f_n \circ \sigma_n | n \in \mathbb{N}\}$ is infinite. It is possible to extract a subsequence of pairwise distinct elements from I. Clearly such a subsequence is stable, has trivial stable kernel and bounds H in the limit. Hence, by Theorem 3.2.10 there exists an infinite set $U \subseteq \mathbb{N}$ such that for every $n \in U$, $f_n \circ \sigma_n$ is not short; which is a contradiction.

□

A result derived from the above theorem is a kind of relative co-Hopfianity of torsion-free hyperbolic groups. Recall that a group G is said to be co-Hopfian, or to have co-Hopf property, if any monomorphism of G is an automorphism.

Corollary 3.2.21. *Let K be a torsion-free hyperbolic group and let H be a non-abelian subgroup such that K is freely H-indecomposable. Then any monomorphism $f : K \to K$ fixing H pointwise is an automorphism.*

Proof. By Theorem 3.2.20, there exist $n, m \in \mathbb{N}$ such that $n > m$ and $f^n = f^m \circ \tau$ for some $\tau \in \mathrm{Mod}_H(K)$. Therefore $f^{n-m} = \tau$, thus f is surjective.

□

One of the important concepts in Sela's study of limit groups is the *shortening quotient*. Its importance lies in the fact that homomorphisms in the Makanin-Razborov diagrams - diagrams that encode all homomorphisms from a given group to a free group - factor precisely through shortening quotients. We give the classical definition; recently, Jaligot and Sela in [JS11, Proposition 16] give a version of Theorem 3.2.23 where K is a free product.

Definition 3.2.22. Let K be a torsion-free hyperbolic group, let G be a finitely generated group and let H be a non-abelian subgroup of G such that G is freely H-indecomposable. Let $f = (f_i : G \to K)_{i\in\mathbb{N}}$ be a stable sequence of pairwise distinct homomorphisms bounding H in the limit and such that each f_n is short. The group $SG = G/\ker_\infty(f)$ is called a *shortening quotient* of G.

Theorem 3.2.23. *Every shortening quotient is a proper quotient.*

Proof. If it is not the case, then the stable kernel is trivial and thus by Theorem 3.2.10, f_i is not short for infinitely many i; a contradiction.

□

Another important application in this context of a more general version of the shortening argument is the proof by Sela [Sel09] of the *descending chain condition* of K-limit groups.

Theorem 3.2.24. *[Sel09, Theorem 1.12] Let K be a torsion-free hyperbolic group and $(G_i)_{i\in\mathbb{N}}$ a sequence of K-limit groups. If $(f_i : G_i \to G_{i+1})_{i\in\mathbb{N}}$ is a sequence of epimorphisms, then all but finitely many of them are isomorphisms.*

3.3 JSJ decompositions

A JSJ decomposition of a group G over a class of subgroups χ relative to a subgroup H is a (χ, H)-splitting (see Definition 1.2.27), which describes in a certain sense all other possible (χ, H)-splittings of G.

The notion of JSJ decomposition historically rises as a generalization of previous concepts in two directions, one originating from topology and the other from group theory.

On the topological side, it generalizes JSJ decomposition as classically defined by Jaco and Shalen ([JS79]) and independently by Johannson ([Joh79]): a closed orientable irreducible 3-manifold M is decomposable into a unique - modulo isotopy - minimal collection of incompressible tori, disjointly embedded; this collection is such that any connected component of M obtained by cutting along the tori is either atoroidal or Seifert, that is a bundle with a 2-orbifold as base and a circle as fiber. A good reference for this topic is [Hat07, §1.2]. In a JSJ decomposition of class χ of a group G, G is seen as the fundamental group of a graph of groups, whose edge groups belong to class χ. G generalizes $\pi_1(M)$, vertex groups generalize fundamental groups of the connected components of M after the cuts are done, while edge groups generalize fundamental groups of the surfaces we cut along.

On the group-theoretic side, JSJ decomposition generalizes Grushko decomposition, according to which a finitely generated group G may be decomposed as a free product of a free group and non-infinite cyclic freely indecomposable groups; such a decomposition is canonical modulo conjugation and permutation of the factors. Grushko decomposition may be seen as the particular case of JSJ decomposition over the trivial group.

It is not easy to suggest omnicomprensive references about JSJ decomposition. Rips and Sela in [RS97, Theorem 7.1] find a canonicity notion for cyclic decomposition of finitely presented one-ended groups starting in a quite systematic way from first definitions and analogies with topology and exhausting all cases. Fujiwara and Papasoglu in [FP06, Theorems 5.13 and 5.15] extend JSJ decomposition to finitely presented groups over the class of slender groups, that is, those groups whose subgroups are finitely generated. The existence of cyclic JSJ decompositions of torsion-free hyperbolic finitely generated groups is assured by Theorem 5.13. In the same paper, the Final remarks on p.122 express universality of JSJ decomposition in an explicit way. A relatively easy-to-consult reference, evidencing the important properties of JSJ decomposition, is [Per09, §5.2].

We expose the construction of JSJ decomposition as done by Guirardel and Levitt in [GL09]. Informally speaking, the two required properties of canonicity and universality are satisfied, respectively by defining the decomposition as an equivalence class rather than a single element, and by imposing a requirement of maximality on the number of ‘meaningful vertex groups’.

Given a group G, a subgroup H of G and a class of groups χ, define an order relation on the set of (χ, H)-splittings of G as follows.

Definition 3.3.1. Given a group G and two (χ, H)-splittings Λ_1 and Λ_2 of G, we say that Λ_1 **dominates** Λ_2, denoted with $\Lambda_2 \leq_d \Lambda_1$, if every subgroup of G which is elliptic in Λ_1 is also elliptic in Λ_2.

The relation $\leq_d$ naturally induces an equivalence relation $\sim_d$: we say that $\Lambda_1 \sim_d \Lambda_2$ if $\Lambda_1 \leq_d \Lambda_2$ and $\Lambda_2 \leq_d \Lambda_1$.

Definition 3.3.2. A (χ, H)-splitting Λ of G is said to be **universally elliptic** if for every (χ, H)-splitting Γ of G, for every edge group S of Λ, S is elliptic in Γ.

Let $\leq_d^u$ be the restriction of $\leq_d$ to the set of universally elliptic (χ, H)-splittings of G. Also $\leq_d^u$ induces an equivalence relation $\sim_d^u$.

Definition 3.3.3. A **deformation space** over χ relative to H of a group G is a $\sim_d^u$-class of the set of universally elliptic (χ, H)-splittings of G.

Definition 3.3.4. A **JSJ decomposition** of G over χ relative to H is a maximal (with respect to $\leq_d^u$) universally elliptic (χ, H)-splitting. If χ is the class of abelian or cyclic subgroups, then we simply say *abelian* or *cyclic JSJ decomposition*, respectively.

It is shown in [GL09, Theorem 4.3] ([GL09, Theorem 5.1] for the relative case) that JSJ decompositions exist for finitely presented groups; moreover, they are unique modulo admissible collapses and expansions, that is modulo $\sim_d^u$-equivalence - see [GL09, §5.1, p.12]. We will use here the existence and properties of JSJ decompositions in the framework of finitely generated torsion-free CSA groups proved in [GL10].

Definition 3.3.5. [GL09, §7.1, p.16 and Definition 7.3] Given a surface Σ, a **boundary subgroup** of the fundamental group $\pi_1(\Sigma)$ is a subgroup conjugate to the fundamental group of a boundary component. An **extended boundary subgroup** of $\pi_1(\Sigma)$ is a subgroup of a boundary subgroup.

Definition 3.3.6. [GL09, Definition 7.3] Let G be a group and Λ a (χ, H)-splitting of G. A vertex stabilizer G_v in Λ is called of **QH type** if it is isomorphic to the fundamental group $\pi_1(\Sigma)$ of a surface Σ such that images of incident edge groups are extended boundary subgroups and every conjugate of H intersects G_v in an extended boundary subgroup.

A boundary component C of Σ is **used** if there exists an incident edge group, or a subgroup of G_v conjugate to H whose image in $\pi_1(\Sigma)$ is contained with finite index in $\pi_1(C)$.

Definition 3.3.7. [GL09, Definition 4.2] A vertex stabilizer G_v in Λ is said to be **rigid** if it is elliptic in every (χ, H)-splitting of G. Otherwise it is called **flexible**.

The following theorem is an application of results of [GL10] in our particular context.

Theorem 3.3.8. *[GL10, Theorem 11.1] Let G be a torsion-free finitely generated CSA group and let H be a subgroup of G such that G is H-freely indecomposable. Then abelian JSJ decompositions of G relative to H exist and their non-abelian flexible vertices are of QH type with every boundary component used.*

□

Since boundary subgroups are cyclic, it follows that if H is non-abelian then H is contained in a conjugate of a rigid group in any abelian JSJ decomposition of G relative to H. Hence, without loss of generality, we may assume in the rest of this paper that JSJ decompositions that we are using have the property that H is contained in a rigid vertex group. Since we will use only properties that are satisfied by all of the JSJ decompositions, by misuse of language we will use the term *the JSJ decomposition* rather than a JSJ decomposition. In this work, we will use the next two properties of JSJ decompositions.

Lemma 3.3.9. *Let G be a finitely generated torsion-free CSA group and let H be a non-abelian subgroup of G such that G is H-freely indecomposable. Let Λ be the abelian JSJ decomposition of G relative to H. Let $G(H)$ be the vertex group containing H in Λ. Then any automorphism from $\mathrm{Mod}_H(G)$ fixes $G(H)$ pointwise.*

Proof. Since $G(H)$ is rigid it is elliptic in any abelian splitting of G relative to H. Let $\sigma \in \mathrm{Mod}_H(G)$. Suppose that σ is a Dehn twist and let $G = G_1 *_C G_2$ or $G = L*_C$ be the corresponding one-edge abelian splitting. Since $H \leq G_1$ or $H \leq L$ and $H \leq G(H)$ which is elliptic, it follows that $G(H) \leq G_1$ or $G(H) \leq L$ which is the desired conclusion. Using a similar argument if σ is an automorphism of surface type or abelian type then it fixes $G(H)$ pointwise.

□

Lemma 3.3.10. *Let G be a finitely generated torsion-free CSA group and let H be a non-abelian subgroup of G such that G is H-freely indecomposable. Let $f = (f_i : G \to K)_{i\in\mathbb{N}}$ be a stable sequence of pairwise distinct homomorphisms with trivial stable kernel and which bounds H in the limit. For each $i \in \mathbb{N}$ choose $\sigma_i \in \mathrm{Mod}_H(G)$ such that $f_i \circ \sigma_i$ is short. Let SG be the corresponding shortening quotient and let $\pi : G \to SG$ be the natural projection map. Then the restriction of π to the vertex group $G(H)$ containing H in the abelian JSJ decomposition of G relative to H is injective.*

Proof. By Lemma 3.3.9, for every $g \in G(H)$, $f_n \circ \sigma_n(g) = f_n(g)$ and the required conclusion follows.

□

All the previous properties of JSJ decompositions are widely sufficient in our context of K-limit groups. However, for torsion-free hyperbolic groups we need additional properties. Let G be a group and let Λ be a (χ, H)-splitting of G. We say that a boundary subgroup B of a surface type vertex group G_v is **fully used** if there exists an incident edge group, or a subgroup of G_v conjugate to H, which coincides with B. The following theorem which is sufficient for our purpose summarizes several properties which can be deduced from [GL10].

To give a decomposition of a group K relative to some subgroup H useful to find the placement of algebraic closure, we need to ensure the following points:

1. rule out free factors of K that do not contain H.

2. make edge groups malnormal in their neighbourhood vertex groups.

To get point 1, we can do as follows. First we split K as a free product $K = K_1 * K_2$, where $H \leq K_1$ and K_1 is freely H-indecomposable (relative Grushko decomposition). Then we define a decomposition of K relative to H as the cyclic splitting obtained by adding K_2 as a new vertex group to the cyclic JSJ decomposition of K_1 (relative to H).

To get point 2, we need a further definition.

Definition 3.3.11. Let Γ be a graph of groups of a CSA group K with abelian edge groups. Let $H \neq \{1\}$ be a subgroup of K in Γ. The **elliptic abelian neighbourhood** of H in K is the subgroup of K generated by all the elements of K that are elliptic in Γ and commute with a non-trivial element of H.

By [CG05, Proposition 4.26], given an abelian splitting Γ' of a CSA group K, it is possible to construct an abelian splitting Γ of K whose edge groups are maximal abelian in their neighbourhood vertex groups, and whose edge and vertex groups coincide with their own elliptic abelian neighbourhood. By CSA property, edge groups are malnormal in their neighbourhood vertex groups. In particular, this is possible also when Γ' is a JSJ decomposition. Unfortunately, we are not guaranteed that Γ is JSJ, because universal ellipticity may fail. Nevertheless, if Γ' is JSJ then vertex groups of Γ satisfy lemmas 3.3.9 and 3.3.10, those relevant for our purposes.

Definition 3.3.12. We call **generalized JSJ decomposition** of G relative to H a decomposition obtained applying point 1 above to a JSJ decomposition of G relative to H.

Definition 3.3.13. We call **malnormal JSJ decomposition** of G relative to H a decomposition obtained applying point 2 above to an abelian JSJ decomposition of G relative to H.

Note that we keep giving this last decomposition the name JSJ, even if it is not actually JSJ, as we said above. Our choice is due to the fact that the important properties are still satisfied.

From the previous discussion we deduce the following theorem.

Theorem 3.3.14. *Let G be a torsion-free finitely generated CSA group and let H be a non-abelian subgroup of G such that G is H-freely indecomposable. Suppose that every abelian subgroup of G is cyclic and that G admits at least a non-trivial cyclic splitting relative to H. Then there exist non-trivial generalized malnormal cyclic JSJ decompositions of G relative to H satisfying the following properties:*

1. *Flexible vertices are of QH type with every boundary component fully used.*
2. *Every edge group is maximal abelian in the neighbourhood vertex groups.*
3. *H is contained in a rigid vertex group.*

Chapter 4

The algebraic closure

This chapter is organized into two sections. In the first section we state some results about constructibility, that is the possibility of building a group from some subgroup through a finite sequence of one-edge extensions over some fixed class of edge groups. The possibility to construct a torsion-free hyperbolic group G from the algebraic closure of some subgroup A relies on the fact that algebraic closure coincides with its own existential algebraic closure. To give an easy example of a subgroup having this property, we precede the main results with a more trivial result, about the closures of an abelian subgroup.

The outline of the proof is the following; for the reader willing to see more details, we refer to the proof inside this chapter. Given a group G and a subgroup A, we construct a finite chain of epimorphisms $G \to K_1 \to \ldots \to A$, such that each epimorphism $K_i \to K_{i+1}$

1. either is a retraction on the free factor containing A (in case of free decomposability),

2. or is a A-homomorphism whose restriction to the vertex group containing A in the generalized malenormal cyclic JSJ decomposition of K_i relative to A is injective.

This gives us the possibility of travelling backwards from A to G, ensuring that at every step we can construct the upper group by cyclic extensions.

Unfortunately, since the existential algebraic closure does not coincide with its existential algebraic closure, we are not able to set up this construction starting from it. In this case, we construct a descending sequence of subgroups of bounded rank from G to $acl^{\exists}(A)$ with the following induction method. With K_i as above,

1. either K_i is freely decomposable relative to $acl^{\exists}(A)$. In this case, K_{i+1} is defined as the free factor containing $acl^{\exists}(A)$;

2. otherwise, we construct a suitable sequence of homomorphisms $K_i \to G$, to which we can apply a Paulin argument. In this way we obtain a cyclic decomposition Γ of K_i relative to $acl^{\exists}(A)$. In this case, K_{i+1} is defined as the vertex group of Γ containing $acl^{\exists}(A)$.

To make it all work, what we need is a finiteness condition on the sequence. Up to now, we are bound to a result of Takahasi, dating back to 1951, that ensures us only in the case that G is free.

The second section is devoted to show results about the placement of algebraic closure(s) in the generalized malnormal JSJ decomposition of a torsion-free hyperbolic group. On this subject, we prove that the restricted algebraic closure of a subgroup A of a group

G coincides with the vertex group $G(A)$ containing A in the generalized malnormal cyclic JSJ decomposition of G relative to A. The inclusion $racl(A) \leq G(A)$ is proved by fixing an arbitrary g outside $G(A)$ constructing an infinite sequence of Dehn twists which give g infinitely many distinct images. The reverse inclusion relies on finite index of modular automorphisms in the group of automorphisms, combined with the fact that modular automorphisms fix vertex groups of generalized malnormal JSJ decomposition pointwise. For free groups, we prove that $G(A)$ coincides with the strictly-speaking algebraic closure. The only interesting inclusion $G(A) \leq acl(A)$ is proved utilizing the formula given by Theorem 1.4.60 defined by Ould Houcine in [OH11, Proposition 5.9].

For each theorem we will state the minimum assumptions for it to work, in order to make the reader aware of the key points, even if not reading proofs in detail. As a side-effect, this approach might have the risk of making the work appear rather fragmentary. To minimize this risk, the reader willing to have a more global view is warned that torsion-free hyperbolic groups are our leading framework. All of the results presented here apply to such class, except some stronger facts for free groups.

Recall that, by point 1 of Lemma 1.4.58, given a group G and a subset $A \subseteq G$, we have the equality $cl_G(A) = cl_G(\langle A \rangle)$, where cl stands for any of the closures defined on page 43; thus, from now on, we assume that A is a subgroup of G.

To avoid excessive heaviness in notation, we will simply write $acl(A)$ without any index, and similarly for the other closures, when the group G is clear from the context.

4.1 Constructibility from algebraic closure

Proposition 4.1.1. *Let G be a torsion-free CSA group whose abelian subgroups are cyclic. Let A be a nontrivial abelian subgroup of G. Then $racl(A) = acl(A) = acl^{\exists}(A) = dcl^{\exists}(A) = dcl(A) = rdcl(A) = Z_G(A)$.*

Proof. We first show that $racl(A) \leq Z_G(A)$. Let $g \in racl(A)$, $a \in A$, $g \neq 1$, $a \neq 1$. Let π_n be the conjugation by a^n, $n \in \mathbb{N}$. Hence the set $\{\pi_n(g) | n \in \mathbb{N}\}$ is finite. Thus $[a^{n-m}, g] = 1$ for some $n, m \in \mathbb{N}$, $n \neq m$. Since G is torsion-free and CSA, it is commutative transitive, thus $[g, a] = 1$. Therefore $g \in Z_G(A)$ as required.

Now we show that $Z_G(A) \leq dcl^{\exists}(A)$. Since $Z_G(A)$ is cyclic, there exists $b \in G$ such that $Z_G(A) = \langle b \rangle$. Let $a \in A$, $a \neq 1$ and $m \in \mathbb{Z}$ such that $b^m = a$. Therefore b satisfies the equation $x^m = a$. Since G is torsion-free and commutative transitive, b is the unique element satisfying $x^m = a$. Hence $b \in dcl^{\exists}(A)$; thus $Z_G(A) \leq dcl^{\exists}(A)$ as required. We conclude by the inclusions given by point 2 of Lemma 1.4.58.

□

Observe that, in the case that G is non-abelian and $A = \varnothing$ or $A = \{1\}$, all closures are equal to $\{1\}$. In fact, taking $a, b \in G$ with $[a, b] \neq 1$ we have $acl(1) \leq acl(\langle a \rangle) \cap acl(\langle b \rangle) = \{1\}$.

Definition 4.1.2. Let G be a group, let A be a subgroup and let χ be a class of subgroups. By induction on n, define

$$\mathcal{D}_0 = \{A\}, \mathcal{D}_{n+1} = \mathcal{D}_n \cup \{B_1 *_C B_2, B *_C | B_1, B_2, B \in \mathcal{D}_n, C \in \chi\}.$$

We say that G is **constructible from A over χ**, if there exists $n \in \mathbb{N}$ such that $G \in \mathcal{D}_n$.

Definition 4.1.3. [Gro87, §5.3] Let X be a geodesic metric space. A subspace Y of X is **quasiconvex** if there exists some $k > 0$ such that every geodesic in X connecting a pair of points in Y lies inside the k-neighbourhood of Y.

A subgroup H of a group G is quasiconvex if $Cay(H, \Sigma)$ is quasiconvex in $Cay(G, \Sigma)$ for some set of generators Σ.

By [Gro87, p.139], a quasiconvex subgroup of a hyperbolic group is hyperbolic.

Theorem 4.1.4. *Let G be a torsion-free hyperbolic group and let A be a non-abelian subgroup of G such that $acl^{\exists}(A) = A$. Then G is constructible from A over cyclic subgroups. In particular A is finitely generated and quasiconvex (so it is hyperbolic by the above observation).*

Since for any subset A, $acl^{\exists}(acl(A)) = acl(A)$ by point 3 of Lemma 1.4.58, Theorem 4.1.4 implies the following

Theorem 4.1.5. *Let G be a torsion-free hyperbolic group and let A be a non-abelian subgroup of G. Then G can be constructed from $acl(A)$ by a finite sequence of amalgamated free products and HNN-extensions along cyclic subgroups. In particular, $acl(A)$ is finitely generated and quasiconvex, so it is hyperbolic.*

Recall the following classical model-theoretic definition in model theory:

Definition 4.1.6. [Mar02, Exercise 2.5.17] Let T be a theory. With the convention we introduced on page 34, a model M of T is **existentially closed** if for every model N of T such that $N \supseteq M$ and for every existential formula ϕ with parameters from M, if $N \models \phi$ then $M \models \phi$.

Since for any existentially closed subgroup A we have $acl^{\exists}(A) = A$, Theorem 4.1.4 also implies constructibility of a torsion-free hyperbolic group from any existentially closed subgroup by a finite sequence of amalgamated free products and HNN-extensions along cyclic subgroups.

To prove Theorem 4.1.4 we need some preliminary lemmas.

Lemma 4.1.7. *Let G be an equationally noetherian group and let G^* be an elementary extension of G. Let $P \subseteq G$. Let $K \leq G^*$ be finitely generated and such that $P \subseteq K$. Then there exists $P_0 \subseteq P$ finite such that, for any homomorphism $f : K \to G^*$, if f fixes P_0 pointwise then f fixes P pointwise.*

Proof. Let $K = \langle \bar{g} \rangle$. Enumerate the elements of P as $P = \{p_i | i \in \mathbb{N}\}$. Then for every $i \in \mathbb{N}$ there exists a word $w_i(\bar{x})$ such that $p_i = w_i(\bar{g})$. Since G is equationally noetherian and $P \subseteq G$, there exists $n \in \mathbb{N}$ such that

$$G^* \models \forall \bar{x}((\bigwedge_{0 \leq j \leq n} p_j = w_j(\bar{x})) \to p_i = w_i(\bar{x}))$$

for every $i \in \mathbb{N}$.

Let $P_0 = \{p_0, \ldots, p_n\}$ and let $f : K \to G^*$ be a homomorphism fixing P_0 pointwise. Therefore, $p_i = f(p_i) = w_i(f(\bar{g}))$ for every $i \in \{0, \ldots, n\}$. Hence $p_i = w_i(f(\bar{g}))$ for every $i \in \mathbb{N}$, so $f(p_i) = p_i$ for every $i \in \mathbb{N}$.

□

Proposition 4.1.8. *Let G be a torsion-free hyperbolic group and let A be a non-abelian subgroup of G such that $acl^{\exists}(A) = A$. Let G^* be a non-principal ultrapower of G. Let $K \leq G^*$ be a finitely generated subgroup such that $A \leq K$ and such that K is A-freely indecomposable. Let $K(A)$ be the vertex group containing A in the generalized malnormal abelian JSJ decomposition of K relative to A. Then one of the following cases holds.*

1. *$K(A) = A$, or*
2. *there exist a finitely generated subgroup $L \leq G^*$ such that $A \leq L$ and a non-injective epimorphism $f : K \to L$ satisfying:*
   (a) *f fixes A pointwise;*
   (b) *$f \restriction K(A)$ is injective.*

Proof. Let $\bar{d} = (d_1, \dots, d_p)$ be a finite generating tuple of K. Let $\bar{x} = (x_1, \dots, x_p)$ be a new tuple of variables and set

$$S(\bar{x}) = \{w(\bar{x}) | K \models w(\bar{d}) = 1\},$$

where $w(\bar{x})$ denotes a word on $\bar{x}$ and their inverses.

Since G is equationally noetherian and G^* is an elementary extension of G, there exist words $w_1(\bar{x}), \dots, w_m(\bar{x})$ from $S(\bar{x})$, such that

$$G^* \models \forall \bar{x}(w_1(\bar{x}) = 1 \wedge \dots \wedge w_m(\bar{x}) = 1 \implies w(\bar{x}) = 1),$$

for any $w \in S(\bar{x})$.

By Lemma 4.1.7, there exists a finite subset $P_0 = \{p_1, \dots, p_q\} \subseteq A$, such that for any homomorphism $f : K \to G$, if f fixes P_0 pointwise then f fixes A pointwise. Let $p_1(\bar{x}), \dots, p_q(\bar{x})$ be words such that $p_i(\bar{d}) = p_i$ for every $1 \leq i \leq q$. Set

$$\phi(\bar{x}) := w_1(\bar{x}) = 1 \wedge \dots \wedge w_m(\bar{x}) = 1 \wedge p_1(\bar{x}) = p_1 \wedge \dots \wedge p_q(\bar{x}) = p_q.$$

We conclude that any map $f : K \to G$ satisfying $G \models \phi(f(\bar{d}))$ extends to a A-homomorphism, that we still denote f.

Let $(v_i(\bar{x}))_{i \in \mathbb{N}}$ be the list of reduced words such that $K \models v_i(\bar{d}) \neq 1$. For $m \in \mathbb{N}$, set

$$(*) \qquad \varphi_m(\bar{x}) := \phi(\bar{x}) \wedge \bigwedge_{0 \leq i \leq m} v_i(\bar{x}) \neq 1.$$

Proof of case 1. Suppose first that there exists $m \in \mathbb{N}$, such that for any map $f : K \to G$, if $G \models \varphi_m(f(\bar{d}))$ then f is an embedding. We claim that, in that case, $K(A)$ is exactly A.

Let $\bar{b}$ be a finite generating tuple of $K(A)$. Then there exists a tuple of words $\bar{w}(\bar{x})$ such that $\bar{b} = \bar{w}(\bar{d})$. We claim that the formula

$$\psi(\bar{y}) := \exists \bar{x}(\varphi_m(\bar{x}) \wedge \bar{y} = \bar{w}(\bar{x}))$$

has only finitely many realizations in G.

Let $\bar{c}$ in G such that $G \models \varphi(\bar{c})$. Hence there exists an embedding $f : K \to G$, fixing A pointwise, such that $\bar{c} = \bar{w}(f(\bar{d}))$. Thus the subgroup generated by $\bar{c}$ is the image of $K(A)$ by f.

By Theorem 3.2.20, there exist finitely many embeddings $h_1, \dots, h_k$, fixing A pointwise, such that for any embedding $h : K \to G$, there exists a modular automorphism $\tau \in \mathrm{Mod}_A(G)$ such that $h \circ \tau = h_i$. Since any modular automorphism fixes $K(A)$ pointwise (Lemma 3.3.9), we find $\bar{c} = f(\bar{b}) \in \{h_1(\bar{b}), \dots, h_k(\bar{b})\}$, thus we get the required conclusion. Since $G^* \models \varphi(\bar{b})$, we conclude that $K(A) \leq acl^{\exists}(A) = A$, as claimed.

□ (Case 1.)

Proof of case 2. Now suppose that for every $m \in \mathbb{N}$ there exists a non-injective homomorphism $f : K \to G$ such that $G \models \varphi_m(f(\bar{d}))$. Therefore, we get a stable sequence $(f_m : K \to G)_{m \in \mathbb{N}}$ of pairwise distinct homomorphisms with trivial stable kernel.

For each $n \in \mathbb{N}$, choose a modular automorphism $\tau_n \in \mathrm{Mod}_A(K)$ such that $h_n = f_n \circ \tau_n$ is short relative to A. Hence, we extract a stable subsequence $(h_m : K \to G)_{m \in \mathbb{N}}$ of pairwise distinct homomorphisms. Let L be the corresponding shortening quotient, which is embeddable in G^* and contains A and let $f : K \to L$ be the quotient map. By Theorem 3.2.23 L is a proper quotient. We see also that f fixes A pointwise. Since the stable kernel of $(f_n : K \to G)$ is trivial and since every modular automorphism fixes $K(A)$ pointwise, the restriction of f to $K(A)$ is injective (Lemma 3.3.10).

□ (Case 2 and Proposition 4.1.8.)

Theorem 4.1.4 can be deduced from the following corollary of Proposition 4.1.8.

Corollary 4.1.9. *Let G be a torsion-free hyperbolic group and let A be a non-abelian subgroup of G such that $acl^{\exists}(A) = A$. Let G^* be a non-principal ultrapower of G. Let $K \leq G^*$ be a finitely generated subgroup containing A. Then K is constructible from A over abelian subgroups.*

Proof. We construct a sequence $K = K_0, K_1, \dots, K_n$ of finitely generated subgroups of G^* and a sequence of epimorphisms $f_i : K_i \to K_{i+1}$ satisfying:

1. f_i fixes A pointwise;
2. either K_{i+1} is a free factor of K_i, say $K_i = K_{i+1} * H$ for some H, and f_i is the map killing H, or $f_i \restriction K_i(A)$ is injective, where $K_i(A)$ is the vertex group containing A in the generalized malnormal cyclic JSJ decomposition of K_i relative to A;
3. $K_n(A) = A$.

Suppose that K_i is constructed. If K_i is freely decomposable relative to A, then let $K_i = K_{i+1} * H$ with $A \leq K_{i+1}$ and K_{i+1} freely A-indecomposable. Define $f_i : K_i \to K_{i+1}$ as the retraction killing H.

If K_i is freely A-indecomposable, then one of the cases of Proposition 4.1.8 is satisfied. If case 1 holds, then the construction of the sequence is terminated; otherwise, let $K_{i+1} = L$, where L is the group witnessing case 2 of Theorem 4.1.8. Since descending chain condition (Theorem 3.2.24) holds for G-limit groups, the sequence terminates.

Using the descending chain condition on G-limit groups (Theorem 3.2.24), the sequence terminates. Let K_n be the last element in the sequence. Hence, property 3 is satisfied.

We show the conclusion of the corollary by descending induction on i. The base of the induction holds. In fact, since $A = K_n(A)$, the group K_n can be constructed from A by a sequence of amalgamated free products and HNN-extensions along abelian subgroups.

To prove the inductive step, suppose that K_{i+1} is constructible from A over abelian subgroups. By construction, either K_{i+1} is a free factor of K_i, in which case K_i satisfies the conclusion of corollary, or the restriction $f_i \restriction K_i(A)$ is injective. Since $f_i(K_i(A))$ contains A and f_i fixes A pointwise, $f_i(K_i(A))$ is constructible from A by a sequence of amalgamated free products and HNN-extensions along abelian subgroups. Since the restriction of f_i to $K_i(A)$ is injective, it follows that $K_i(A)$ itself is constructible from A by a sequence of free products and HNN-extensions along abelian subgroups. Recall that $K_i(A)$ is the vertex group containing A in the generalized malnormal cyclic JSJ decomposition of K_i relative to A, so K_i is constructible from $K_i(A)$ by a sequence of free products and HNN-extensions along abelian subgroups. We have proved the inductive step, so $K_0 = K$ satisfies the conclusion of corollary and the result is proved.

□

To complete the proof of Theorem 4.1.4, we are left to show quasiconvexity of A. To this purpose we need the following result.

Theorem 4.1.10. *[KW99, Proposition 4.5] Let G be a hyperbolic group. Suppose that Λ is a cyclic splitting of G with a finite underlying graph. Then all vertex groups of Λ are quasiconvex in G and word-hyperbolic themselves.*

□

Proof of Theorem 4.1.4. The fact that G is constructible from A over cyclic subgroups follows from Corollary 4.1.9 for $K = G$. Since G is finitely generated, any vertex group in any cyclic splitting of G is finitely generated. By a rank argument applied iteratively to the construction of G from A over cyclic subgroups, the subgroup A is finitely generated. By Theorem 4.1.10, A is quasiconvex and in particular hyperbolic.

□

Unlike algebraic closure, in general existential algebraic closure does not coincide with its existential algebraic closure; therefore, in this case Theorem 4.1.4 cannot be applied to obtain a constructibility result. The best result we have obtained from existential algebraic closure is about free groups of finite rank. Before proving the main theorem 4.1.14, we give Corollary 4.1.13, that bounds the rank of algebraic closure in a free group. Besides its independent interest, it is necessary to prove the main result. First we need a definition and a theorem.

Definition 4.1.11. [MV04] A subgroup A of a free group F is *compressed* if whenever $A \leq K$, with K finitely generated, then $rk(A) \leq rk(K)$; here $rk(H)$ denotes the rank of H.

The following result has been proved by Ould Houcine. We use the notation $c \bot_H b$ as a shorthand for '$c \in \langle k_1, \dots, k_n | \rangle$ for some free basis $(b, k_1, \dots, k_n)$ of H'.

Theorem 4.1.12. *[OH10, Theorem 1.1]*

1. *Let $G = G_1 *_{u_1=u_2} G_2$, where G_i are finitely generated and $u_i \neq 1$ in G_i.*

   Then G is free if and only if u_i is primitive in G_i for some i.

2. *Let $G = \langle H, t | u_1^t = u_2 \rangle$, where H is finitely generated and $u_i \neq 1$ in H.*

   Then G is free if and only if u_i is primitive in H for some i and $u_j \perp_H u_i^\alpha$ for some $\alpha \in H$ and $j \neq i$.

Corollary 4.1.13. *Let F be a free group of finite rank and let A be a non-abelian subgroup of F. Then $acl(A)$ is compressed.*

Proof. By Corollary 4.1.9, if $acl(A) \leq K$, with K finitely generated, then K is constructible from $acl(A)$ over cyclic subgroups. Let $K = B_1 *_C B_2$ with $acl(A) \leq B_1$ and let $C = \langle c \rangle$. By Theorem 4.1.12, c is primitive either in B_1 or in B_2. Therefore $rk(B_i) \leq rk(K)$ for $i = 1, 2$. Similarly, if $K = B *_C$ then $rk(B) \leq rk(K)$ by 4.1.12, too. Hence, by induction we obtain $rk(acl(A)) \leq rk(K)$.

□

Theorem 4.1.14. *Let F be a free group of finite rank and let A be a non-abelian subgroup of G. Let K be a finitely generated subgroup of F containing $acl^\exists(A)$. Then K is constructible from $acl^\exists(A)$ over cyclic subgroups.*

First we prove the following proposition, about the construction of a sequence yielding Theorem 3.2.15.

Proposition 4.1.15. *Let G be a finitely generated equationally noetherian group and let A be a subgroup of G. Let $K \leq G$ be finitely generated and suppose that $acl^\exists(A)$ is a proper subgroup of K. Then there exists a stable sequence of pairwise distinct homomorphisms $(h_n : K \to G)_{n \in \mathbb{N}}$ with trivial stable kernel and which bounds $acl^\exists(A)$ in the limit.*

In what follows we fix a finitely generated equationally noetherian group G with a subgroup A. We fix also a finite generating set of G and we denote by B_r the ball of radius r with respect to the word distance induced by the fixed generating set. We denote by $Mon(G/A)$ the monoid of monomorphisms of G fixing A pointwise. We introduce the following definition.

Definition 4.1.16. A stable sequence $(f_n : C \to G | n \in \mathbb{N})$ with trivial stable kernel **strongly converges** to C if it satisfies the following properties:

1. for any $g \in C \cap G$, $f_n(g) = g$ for all but finitely many n;
2. for any $g \in C$, for any $b \in G$, if $f_{n_k}(g) = b$ for some subsequence $(n_k)_{k \in \mathbb{N}}$, then $g = b$.

Lemma 4.1.17. *Let G^* be an elementary extension of G and let $C \leq G^*$ be finitely generated. Then there exists a stable sequence of homomorphisms $(f_n : C \to G | n \in \mathbb{N})$ strongly converging to C.*

Proof. Let
$$C = \langle c_1, \ldots, c_t | w_i(\bar{c}) = 1, i \in \mathbb{N} \rangle$$
be a presentation of C. By equational noetherianity, there exist finitely many words $w_0, \ldots, w_p$ such that
$$G \models \forall \bar{x}((w_0(\bar{x}) = 1 \wedge \ldots \wedge w_p(\bar{x}) = 1) \Rightarrow w_i(\bar{x}) = 1)$$
for any $i \in \mathbb{N}$.

Enumerate the following sets:

$$G \smallsetminus \{1\} = \{a_i | i \in \mathbb{N}\},$$

$$(G \cap C) \smallsetminus \{1\} = \{b_i | i \in \mathbb{N}\} = \{b_i(\bar{c}) | i \in \mathbb{N}\}$$

and

$$C \smallsetminus \{1\} = \{v_i(\bar{c}) | i \in \mathbb{N}\},$$

and

$$C \smallsetminus G = \{d_i(\bar{c}) | i \in \mathbb{N}\}.$$

By elementarity, for any $n \geq 0$ there exists $\bar{c}_n$ in G such that

$$G \models \bigwedge_{0 \leq i \leq p} w_i(\bar{c}_n) = 1 \wedge \bigwedge_{0 \leq i \leq n} v_i(\bar{c}_n) \neq 1 \tag{4.1}$$

and

$$G \models \bigwedge_{0 \leq i \leq n} b_i = b_i(\bar{c}_n) \wedge \bigwedge_{0 \leq i \leq n, 0 \leq j \leq n} d_i(\bar{c}_n) \neq a_j. \tag{4.2}$$

We define $f_n(\bar{c}) = \bar{c}_n$ and we show that the sequence $(f_n)_{n \in \mathbb{N}}$ satisfies properties 1 and 2 of Definition 4.1.16.

The sequence $(f_n | n \in \mathbb{N})$ is stable and has a trivial stable kernel by equation (4.1). Let $g \in C \cap G$. Then there exists m such that $g = b_m = b_m(\bar{c})$. By equation (4.2), we have $f_n(b_m(\bar{c})) = b_m(\bar{c}_n) = b_m$ for any $n \geq m$; thus $f_n(g) = g$ for all but finitely many n, so we have property 1.

Now, let $g \in C$ and $b \in G$ such that there exists a subsequence $(n_k)_{k \in \mathbb{N}}$ with $f_{n_k}(g) = b$ for any $k \geq 0$. Let s be such that $b = a_s$. Suppose first that $g \notin G$. Then there exists r such that $g = d_r(\bar{c})$. Let $n \geq \max\{r, s\}$. By equation (4.2), we have $f_n(g) = f_n(d_r(\bar{c})) = d_r(\bar{c}_n) \neq a_s$. Therefore for n_k large enough we have $f_{n_k}(g) \neq b$; a contradiction.

Hence $g \in G$ and in particular $g \in C \cap G$. By property 1 we get $f_n(g) = g$ for all but finitely many n and in particular $g = b$ as required, so property 2 is proved.

□ (Lemma 4.1.17.)

Lemma 4.1.18. *The following properties are equivalent for any finite subset $C \subseteq G$:*

1. *$C \subseteq acl^{\exists}_G(A)$;*
2. *there exists a finite subset $B(C) \subseteq G$ such that for any elementary extension G^* of G and for any $f \in \mathrm{Mon}_A(G^*)$, $f(C) \subseteq B(C)$;*
3. *there exists $r > 0$ such that for any elementary extension G^* of G, for any $f \in \mathrm{Mon}_A(G^*)$, for any sequence $(g_n : f(G) \to G | n \in \mathbb{N})$ which strongly converges to $f(G)$, $(g_n \circ f)(C) \subseteq B_r$ for all but finitely many n.*

Proof.
Proof of (1) ⇒ (2). This follows immediately from the definition of $acl^{\exists}_F(A)$.
Proof of (2) ⇒ (3). Let $B(C)$ be the given subset. Let

$$r = \max\{|g| \, | g \in B(C)\}.$$

Let $G^* \succeq G$, let $f \in \mathrm{Mon}_A(G^*)$ and let $(g_n : f(G) \to G | n \in \mathbb{N})$ be a sequence strongly converging to $f(G)$. Let $c \in C$. Hence $f(c) = b \in B(C) \subseteq G$ and $b \in G \cap f(G)$. Since $(g_n)_{n\in\mathbb{N}}$ strongly converges to $f(G)$, we have $g_n(b) = b$ for all but finitely many n. Therefore $g_n(f(c)) = b$ for all but finitely many n. Since C is finite, we get $(g_n \circ f)(C) \subseteq B_r$ for all but finitely many n.

Proof of (3) $\Rightarrow$ (2).

Let $c \in C$. Let $G^* \succeq G$ and $f \in \mathrm{Mon}_A(G^*)$. We claim that $f(c) \in B_r$, so to take $B(C) = B_r$.

Let $(g_n : f(G) \to G | n \in \mathbb{N})$ be a sequence strongly converging to $f(G)$; its existence is assured by Lemma 4.1.17. So, there exists $b \in B_r$ such that $g_{n_k}(f(c)) = b$ for some subsequence $(n_k)_{k\in\mathbb{N}}$. Therefore, by property 2 of Definition 4.1.16, we have $f(c) = b$. Hence $f(C) \subseteq B_r$ as claimed.

Proof of (2) $\Rightarrow$ (1). Let $c \in C \setminus acl^{\exists}_G(A)$. Then, any existential formula $\phi(\bar{a}, x) \in tp^{\exists}(c/A)$ has infinitely many realizations. Define the theory $T(d) = Diag_{el}(G) \cup \{\phi(d), d \neq f_i | \phi \in tp^{\exists}(c/A), i \in \mathbb{N}\}$, where $\{f_i | i \in \mathbb{N}\}$ is an enumeration of the elements of G. As $T(d)$ is finitely consistent, there exists an elementary extension $G' \succeq G$ such that $G^* \models T(d)$, $d \in G'$ and $tp^{\exists}(c/A) \subseteq tp^{\exists}(d/A)$. By point 2 of Proposition 1.4.59 there exist an elementary extension $G' \preceq G^*$ and a monomorphism $f \in Mon(G^*/A)$ such that $f(c) = d$. Hence (2) is not true; this ends the proof.

$\square$ (Lemma 4.1.18.)

Proof of Proposition 4.1.15. Let D be a finite generating set of K. Since $acl_G(A) < K$ we have $D \nsubseteq acl_G(A)$. Hence, using the equivalence of points 1 and 3 of Lemma 4.1.18, we have:

($*$) For any $r \geq 0$ there exist an elementary extension G^* of G, a monomorphism $f \in \mathrm{Mon}_A(G^*)$ and a sequence $(g_n : f(G) \to G | n \in \mathbb{N})$ strongly converging to $f(G)$, such that $\max_{d\in D} |(g_n \circ f)(d)| \geq r$ for some subsequence $(n_k)_{k\in\mathbb{N}}$.

Write $K \setminus \{1\}$ as an increasing sequence of finite subsets $(C_i)_{i\in\mathbb{N}}$. Enumerate the elements of $acl_G(A)$: $acl_G(A) = \{b_i | i \in \mathbb{N}\}$. Let $B_{r(i)}$ be the ball witnessing point 3 of Lemma 4.1.18 for b_i.

Claim 1. *For any $m \geq 1$ there exists a homomorphism $h_m : K \to G$ satisfying the following properties:*

1. $1 \notin h_m(C_m)$;
2. $\max_{d\in D} |h_m(d)| \geq m$;
3. $h_m(b_i) \subseteq B_{r(i)}$ *for* $1 \leq i \leq m$.

Proof. Let $m \geq 1$. Let $f \in \mathrm{Aut}_A(G^*)$ and $(g_n : f(G) \to G)_{n\in\mathbb{N}}$ the sequence witnessing ($*$) for m. Since $(g_n : f(G) \to G)_{n\in\mathbb{N}}$ strongly converges to $f(G)$ we have

$$1 \notin (g_n \circ f)(C_m)$$

for all but finitely many n.

Since $b_i \in acl_G(A)$, by the equivalence of points 1 and 3 of Lemma 4.1.18 we have for any $1 \leq i \leq m$,

$$(g_n \circ f)(b_i) \subseteq B_{r(i)}$$

for all but finitely many n.

So, by taking n_k large enough, we obtain:

1. $1 \notin (g_{n_k} \circ f)(C_m)$;
2. $\max_{d \in D} |(g_{n_k} \circ f)(d)| \geq m$;
3. $(g_{n_k} \circ f)(b_i) \subseteq B_{r(i)}$ for $1 \leq i \leq m$.

Let $h_m = g_{n_k} \circ f \upharpoonright K$, with h_0 being the trivial homomorphism. Then, h_m is the desired homomorphism.

□ (Claim 1.)

By point 2 of the above claim and finiteness of balls of finite radius, we can extract a subsequence $(h_{m_n})_{n \in \mathbb{N}}$ of pairwise distinct homomorphisms. Thus, we may assume that the initial sequence consists of pairwise distinct homomorphisms.

Now we are left to show that the sequence $(h_m : K \to G)_{m \in \mathbb{N}}$ satisfies the required properties. By point 1 of Claim 1, the sequence is stable and has a trivial stable kernel.

Now let $b \in acl_G(A)$. Then there exists p such that $b = b_p$. Hence for any $m \geq p$ we have $h_m(b) \in B_{r(p)}$, thus the sequence bounds $acl^{\exists}(A)$ in the limit. Therefore, the sequence satisfies all the required properties.

□ (Proposition 4.1.15.)

Now we can prove the main theorem of this section.

Proof of Theorem 4.1.14. Define a descending sequence $(L_i | i \in \mathbb{N})$ of subgroups of F with bounded rank and containing $acl^{\exists}(A)$, as follows. Let $L_0 = K$. Suppose that L_i is defined. If $L_i = acl^{\exists}(A)$ then this terminates the sequence; put $L_j = L_i$ for any $j \geq i$. If L_i is freely $acl^{\exists}(A)$-decomposable, then set L_{i+1} to be the free factor of L_i containing $acl^{\exists}(A)$ and which is freely $acl^{\exists}(A)$-indecomposable. So, suppose that $acl^{\exists}(A) < L_i$ and L_i is freely $acl^{\exists}(A)$-indecomposable. By Proposition 4.1.15 there exists a stable sequence of pairwise distinct homomorphisms $(h_n : L_i \to F)_{n \in \mathbb{N}}$ with trivial stable kernel and which bounds $acl^{\exists}(A)$ in the limit. Hence by Theorem 3.2.15 L_i admits a non-trivial cyclic splitting relative to $acl^{\exists}(A)$. Then, set L_{i+1} to be the vertex group containing $acl^{\exists}(A)$.

We claim that the sequence stabilizes. Suppose for a contradiction that it does not. Then we get an infinite sequence $(L_i)_{i \in \mathbb{N}}$ such that:

1. $acl^{\exists}(A) \leq L_i$;
2. $rk(L_i) \leq rk(K)$. This bound to the rank of L_i is proved using 4.1.12 as in Corollary 4.1.13;
3. $L_{i+1} < L_i$.

By Theorem 1.1.24, $\bigcap_i L_i$ is a free factor of L_i for all but finitely many n. Hence, for all but finitely many n, L_n is freely decomposable with respect to $acl^{\exists}(A)$, that is a contradiction with the construction of the sequence. Therefore the sequence terminates, as claimed. Let L_p be the last term in the sequence. Then by construction $acl^{\exists}(A) = L_p$. We conclude that K is constructible from $acl^{\exists}(A)$.

□ (Theorem 4.1.14.)

As in the case of the algebraic closure, as a consequence we have a bound to the rank of existential algebraic closure:

Corollary 4.1.19. *Let F be a free group of finite rank and let A be a non-abelian subgroup of F. Then $acl^{\exists}(A)$ is compressed.*

Proof. Identical to that of Corollary 4.1.13, using Theorem 4.1.14 for Theorem 4.1.4.

□

4.2 Algebraic closure in JSJ decomposition

Lemma 4.2.1. *Let G be a torsion-free CSA group whose abelian subgroups are cyclic. Suppose that $G = G_1 * G_2$ with $A \leq G_1$. Then $racl_G(A) \leq racl_{G_1}(A)$.*

Proof. First we show that $racl_G(A) \leq G_1$. We suppose that $g \notin G_1$ and we find a sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathrm{Aut}_A(G)$ such that the orbit $\{f_n(g)|n \in \mathbb{N}\}$ is infinite; this will prove that $g \notin racl(A)$. Depending whether G_2 is abelian or not, we will treat the two cases separately. First suppose that G_2 is abelian. Then G_2 is cyclic; let t be a generating element. Let $\alpha \in G_1$ be non-trivial. Then, let $(f_n)_{n\in\mathbb{N}}$ be the sequence of automorphisms of G defined by being the identity on G_1 and sending t to $\alpha^n t$. Since $g \notin G_1$, g has a normal form $g_0 t^{\epsilon_0} g_1 \cdots g_r t^{\epsilon_r} g_{r+1}$ where $g_i \in G_1$, $\epsilon_i = \pm 1$ and if $g_i = 1$ then $\epsilon_i + \epsilon_{i+1} \neq 0$. If $f_n(g) = f_m(g)$ with $n \neq m$ then a calculation with normal forms shows that $\alpha^{n-m} = 1$ which is a contradiction with torsion-freeness of G. Hence the orbit $\{f_n(g)|n \in \ mathbbN\}$ is infinite, as required.

Suppose now that G_2 is non-abelian. Since $g \notin G_1$, g has a normal form $g = g_1 \cdots g_r$, $r \geq 2$. Let $g_l \in G_2$ appear in the normal form of g. Since G_2 is non-abelian and CSA, there exists an element $\alpha \in G_2$ such that $[g_l, \alpha] \neq 1$. Then, let $(f_n)_{n\in\mathbb{N}}$ be the sequence of automorphisms of G defined by being identity on G_1 and conjugation by α^n on G_2. If $f_n(g) = f_m(g)$ with $n \neq m$, then a calculation with normal forms shows that $[\alpha^{n-m}, g_l] = 1$ which is a contradiction, as G is commutative transitive and $[g_l, \alpha] \neq 1$. Hence the orbit $\{f_n(g)|n \in \mathbb{N}\}$ is infinite, as required.

Now we show that $racl_G(A) \leq racl_{G_1}(A)$. Let $b \in racl_G(A)$ and suppose that $b \notin racl_{G_1}(A)$. Then the orbit $\{f(b)|f \in \mathrm{Aut}_A(G_1)\}$ is infinite; since each element of $\mathrm{Aut}_A(G_1)$ has a natural extension to G, the orbit $\{f(b)|f \in Aut_A(G)\}$ is also infinite, which is a contradiction.

□

Lemma 4.2.2. *Let G be a torsion-free hyperbolic group and let A be a non-abelian subgroup of G. Suppose that $G = G_1 * G_2$ with $A \leq G_1$ and G_1 is freely A-indecomposable. Then $racl_G(A) = racl_{G_1}(A)$.*

Proof. The inequality $racl_G(A) \leq racl_{G_1}(A)$ follows from Lemma 4.2.1, so we are left to show that $racl_{G_1}(A) \leq racl_G(A)$.

Let $f \in \mathrm{Aut}_A(G)$. We claim that $f \restriction G_1 \in \mathrm{Aut}_A(G_1)$. By Grushko decomposition theorem (Theorem 1.1.36) $f(G)$ has a decomposition

$$f(G_1) = G_1^{g_1} \cap f(G_1) * \ldots * G_1^{g_p} \cap f(G_1) * G_2^{h_1} \cap f(G_1) * \ldots * G_2^{h_q} \cap f(G_1) * F,$$

where F is a free group. Since $A \leq f(G_1)$ we have $g_i = 1$ for some i and $A \leq G_1 \cap f(G_1)$ and this last group is a free factor of $f(G_1)$. Since G_1 is freely A-indecomposable, we conclude that $G_1 \cap f(G_1) = f(G_1)$, thus $f(G_1) \leq G_1$. If $f \restriction G_1$ is not an automorphism, then G_1 is freely A-decomposable by Corollary 3.2.21, which is a contradiction. Hence $f \restriction G_1 \in \mathrm{Aut}_A(G_1)$, as claimed.

Therefore, if the orbit $\{f(b)|f \in \mathrm{Aut}_A(G_1)\}$ is finite then the orbit $\{f(b)|f \in \mathrm{Aut}_A(G)\}$ is finite as well, which proves $racl_{G_1}(A) \leq racl_G(A)$.

□

Proposition 4.2.3. *Let G be a torsion-free CSA group and let A be a subgroup of G. Let Λ be an abelian splitting of G relative to A and suppose that each edge group is maximal abelian in the vertex groups at its endpoints. If $G(A)$ is the vertex group containing A then $racl(A) \leq G(A)$ and in particular $acl(A) \leq G(A)$.*

Proof. As in the proof of Lemma 4.2.1, we are going to show that if $g \notin G(A)$ then there exists a sequence $(f_n)_{n\in\mathbb{N}}$ in $Aut_A(G)$ such that the orbit $\{f_n(g)|n \in \mathbb{N}\}$ is infinite; this will prove that $g \notin racl(A)$.

Let $g \notin G(A)$.

Let Λ be the abelian splitting $(\mathcal{G}(V,E), T, \phi)$. To simplify, identify G with $\pi_1(\mathcal{G}(V,E), T)$. Enumerate the edges which lie outside T as $e_1, \dots, e_p$. Let $\mathcal{G}_i(V, E_i)$ be the graph of groups obtained by deleting e_i. Hence G is an HNN-extension of the fundamental group $G_i = \pi(\mathcal{G}_i(V, E_i), T)$.

Suppose that $g \notin G_i$. Write $G = \langle G_i, t | C^t = \varphi(C)\rangle$. Let $c \in C$ be non-trivial. In this case let $(f_n)_{n\in\mathbb{N}}$ be the sequence of Dehn twists around c^n, that is f_n is defined by being identity on G_i and sending t to $c^n t$. As in the previous lemma, g has a normal form $g_0 t^{\epsilon_0} g_1 \dots g_r t^{\epsilon_r} g_{r+1}$; if $f_n(g) = f_m(g)$, with $n \neq m$, we find $\alpha^{n-m} = 1$, a contradiction with torsion-freeness of G. This shows that the orbit $\{f_n(g)|n \in \mathbb{N}\}$ is infinite, as required.

Suppose that $g \in \cap_{1\leq i\leq p} G_i$. Note that $\cap_{1\leq i\leq p} G_i$ is the fundamental group L of the graph of groups $\mathcal{G}(V, E')$ obtained by deleting all the edges $e_1, \dots, e_p$, relative to the maximal subtree T. Let $f_1, \dots, f_q$ be the edges incident to $G(A)$. Hence, for each $1 \leq i \leq q$, L can be written as an amalgamated free product $L = L_{i1} *_{C_i} L_{i2}$ where L_{i1} and L_{i2} are the fundamental groups of the connected components of the graph obtained by deleting e_i and $G(A) \leq L_{i1}$.

Since $g \notin G(A)$, there exists $1 \leq i \leq q$ such that $g \notin L_{i1}$. We claim that there exists a sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathrm{Aut}_A(L)$ such that the orbit $\{f_n(g)|n \in \mathbb{N}\}$ is infinite and such that the restriction of each f_n on any edge group of our initial graph of groups $\mathcal{G}(V,E)$ is a conjugation by an element of L.

Define the sequence $(f_n)_{n\in\mathbb{N}}$ similarly as in the previous case of HNN-extensions and in Lemma 4.2.1 above. Since $g \notin L_1$, g has a normal form $g = g_1 \cdots g_r$, $r \geq 2$. Let $g_l \in L_{i2}$ appear in the normal form of g. Let $c \in C$ be non-trivial. In this case let $(f_n)_{n\in\mathbb{N}}$ be the sequence of Dehn twists around c^n; that is f_n is defined by being identity on L_{i1} and conjugation by c^n on L_{i2}. If $f_n(g) = f_m(g)$ with $n \neq m$, then a calculation with normal forms shows that $[c^{n-m}, g_l] = 1$, thus $[g_l, c] = 1$. Since C_i is maximal abelian, we get $g_l \in C_i$ by CSA property; a contradiction. Hence the orbit $\{f_n(g)|n \in \mathbb{N}\}$ is infinite and the restriction of each f_n on each edge group of $\mathcal{G}(V,E)$ is a conjugation by an element of L, as required.

Each f_n has a standard extension $\hat{f}_n$ to G; thus the sequence $(\hat{f}_n)_{n\in\mathbb{N}}$ is a sequence from $\mathrm{Aut}_A(G)$ with the orbit $\{\hat{f}_n(g)|n \in \mathbb{N}\}$ infinite, as required.

□

Proposition 4.2.4. *Let G be a torsion-free hyperbolic group and let A be a non-abelian subgroup of G. Then $racl_G(A)$ coincides with the vertex group $G(A)$ containing A in the generalized malnormal cyclic JSJ decomposition of G relative to A.*

Proof. Write $G = G_1 * G_2$ with $A \leq G_1$ and G_1 freely A-indecomposable. By Lemma 4.2.1, $racl_G(A) = racl_{G_1}(A)$; thus we must show that $racl_{G_1}(A)$ is $G(A)$.

By Theorem 3.2.20, there exists a finite number of automorphisms $f_1, \dots, f_l$ of G_1 such that for any $f \in \mathrm{Aut}_A(G_1)$, there exists a modular automorphism $\sigma \in \mathrm{Mod}_A(G_1)$ such that $f = f_i \circ \sigma$ for some i.

Let $b \in G(A)$. Since any $\sigma \in \mathrm{Mod}_A(G_1)$ fixes $G(A)$ pointwise (Lemma 3.3.9), for any automorphism $f \in \mathrm{Aut}_A(G_1)$ we have $f(b) \in \{f_1(b), \dots, f_l(b)\}$. Thus $b \in acl_{G_1}(A)$ and $G(A) \leq acl_{G_1}(A)$. The inverse inclusion follows from Proposition 4.2.3 and properties of generalized malnormal JSJ decompositions stated in Theorem 3.3.14.

□

For free groups, we can state the following stronger result.

Theorem 4.2.5. *Let F be a free group of finite rank and let A be a non-abelian subgroup of F. Then $acl(A)$ coincides with the vertex group $F(A)$ containing A in the generalized malnormal cyclic JSJ decomposition of F relative to A.*

Proof. Write $F = F_1 * F_2$ with $A \leq F_1$ and F_1 freely A-indecomposable. Since $F_1 \preceq F$, we have $acl_{F_1}(A) = acl_F(A)$. By Proposition 4.2.3 and properties of generalized malnormal JSJ decompositions stated in Theorem 3.3.14, we have $acl(A) \leq F(A)$. We are left to show that $F(A) \leq acl(A)$. Let $c \in F(A)$ and let $(\bar{d}_1, \bar{d}_2)$ be a tuple generating F_1 with $\bar{d}_1$ generating $F(A)$. Then $c = w(\bar{d}_1)$ for some word w.

By Theorem 4.1.4 $acl(A)$ is finitely generated; let $\bar{b}$ be a finite generating set of $acl(A)$. Let $\varphi(\bar{x}, \bar{y})$ be the formula given by Theorem 1.4.60 with respect to the generating tuple $(\bar{d}_1, \bar{d}_2)$ and to the tuple $\bar{b}$; that is for any endomorphism f of F_1, if $F_1 \models \varphi(f(\bar{d}_1), f(\bar{d}_2))$ and f fixes $\bar{b}$ then f is an automorphism.

By equational noetherianity, there exists a finite system $S(\bar{x}, \bar{y})$ of equations such that for any $(\bar{\alpha}, \bar{\beta})$ if $F_1 \models S(\bar{\alpha}, \bar{\beta})$ then the map sending $(\bar{d}_1, \bar{d}_2)$ to $(\bar{\alpha}, \bar{\beta})$ extends to a homomorphism.

Let $\bar{v}(\bar{x})$ be a tuple of words such that $\bar{b} = \bar{v}(\bar{d}_1)$.

Let

$$\psi(z, \bar{b}) := \exists \bar{x} \exists \bar{y} (\varphi(\bar{x}, \bar{y}) \wedge z = w(\bar{x}) \wedge S(\bar{x}, \bar{y}) \wedge \bar{b} = \bar{v}(\bar{x})).$$

We claim that $\psi(z, \bar{b})$ has only finitely many realizations in F_1. Indeed, if

$$F_1 \models \psi(c', \bar{b}) := \exists \bar{x} \exists \bar{y} (\varphi(\bar{x}, \bar{y}) \wedge c' = w(\bar{x}) \wedge S(\bar{x}, \bar{y}) \wedge \bar{b} = \bar{v}(\bar{x})),$$

then there exists an automorphism f fixing $acl(A)$ pointwise and mapping c to c'. By Proposition 4.2.4 $F(A) = racl(A)$, thus the set $\{f(c) | f \in \mathrm{Aut}_A(F_1)\}$ is finite. Hence $\psi(z, \bar{b})$ has only finitely many realizations as claimed. Thus $c \in acl(acl(A)) = acl(A)$ as required.

□

As a consequence of the above theorem, if A is finitely generated, then the equality $acl^{\exists}(A) = F(A)$ holds, too.

Chapter 5

Algebraic and definable closure

5.1 Main statement and preliminaries

This chapter is dedicated to give an answer to the question posed by Z.Sela in 2008. We will prove the following results:

1. In the free group $\mathbb{F}_2$ of rank 2, for every $A \subseteq \mathbb{F}_2$, every algebraic element over A is definable over A.

2. In the free group $F = \mathbb{F}_n$, with $n > 3$, there exists a subgroup $A \leq F$, such that $dcl_F(A) < acl_F(A)$.

Moreover, we give a partial result about the free group of rank 3: if a counterexample A to the equality $acl_F(A) = dcl_F(A)$ exists for F of rank 3, then F is a cyclic HNN extension with $acl_F(A)$ as its vertex and only one loop.

Theorem 5.1.1. *Let F be a free group of finite rank and A a non-abelian subgroup of F. If $dcl(A) < acl(A)$, then $dcl(A)$ is a free factor of $acl(A)$. Similarly, if $dcl^{\exists}(A) < acl^{\exists}(A)$, then $dcl^{\exists}(A)$ is a free factor of $acl^{\exists}(A)$.*

Proof. By Theorem 4.1.4, $acl(A)$ is finitely generated. Hence, by Grushko decomposition (Theorem 1.1.36), $acl(A)$ freely decomposes as $K * L$, such that K contains $acl(A)$ and it is freely $acl(A)$-indecomposable. We claim that $K = dcl(A)$. Suppose for a contradiction that $dcl(A) < K$ and let $a \in K \setminus dcl(A)$.

Claim 1. *There exists an automorphism h of $acl(A)$, of finite order and fixing $dcl(A)$ pointwise, such that $h(a) \neq a$.*

Proof. Since $a \in acl(A) \setminus dcl(A)$, there exists a formula $\psi(x)$, with parameters from A, such that $\psi(F)$ is finite, contains a and is not a singleton. We claim that there exists $b \in acl(A)$ such that $tp(a/A) = tp(b/A)$ and $a \neq b$. Let $\psi(F) = \{a\} \cup \{b_i | 1 \leq i \leq m\}$. Suppose towards a contradiction that $tp(a/A) \neq tp(b_i/A)$ for every i. Then for every i there exists a formula $\psi_i(x)$, with parameters from A, such that $\psi_i \in tp(b_i/A)$ and $\neg\psi_i \in tp(a/A)$. Thus the formula $\psi(x) \wedge \neg\psi_1(x) \wedge \ldots \wedge \neg\psi_m(x)$ defines a; a contradiction.

Hence, let $b \in \psi(F)$ such that $a \neq b$ and $tp(a/A) = tp(b/A)$. By Proposition 1.4.59, there exist an elementary extension F^* of F and an A-automorphism $f \in \mathrm{Aut}_A(F^*)$ such that $f(a) = b$. Let h be the restriction of f to $acl(A)$. We claim that h has the required properties.

Since h is a restriction of f, we get $h(acl(A)) \leq acl(A)$. Let $b \in acl(A)$ and let $\psi_b(x)$ be a formula with parameters from A such that $\psi_b(F)$ is finite and contains b. Then $h(\psi_b(F)) \leq \psi_b(F)$; since $\psi_b(F)$ is finite and h is injective we get $h(\psi_b(F)) = \psi_b(F)$. Thus h is surjective and in particular h is an automorphism of $acl(A)$. Moreover, since h^n is an automorphism of $acl(A)$ for any n and $h^n(\psi_b(F)) = \psi_b(F)$, there exists $n \in \mathbb{N}$ such that h^n fixes $\psi_b(F)$ pointwise.

Let $\{b_i | 1 \leq i \leq m\}$ be a finite generating set of $acl(A)$. Hence, we get a set $\{n_i | 1 \leq i \leq m\}$ such that $h^{n_i}(b_i) = b_i$. Therefore $h^{\prod_i n_i}(x) = x$ for any $x \in acl(A)$, thus h has finite order.

□ (Claim 1.)

Let h be the automorphism given by the above claim. We claim that $h(K) = K$. We have $h(K) \leq acl(A)$ and by Grushko decomposition

$$h(K) = *_{1 \leq i \leq n} h(K) \cap K^{g_i} * \ldots * *_{1 \leq j \leq m} h(K) \cap L^{h_j} * D,$$

where D is a free group. Since $dcl(A) \leq K \cap h(K)$, it follows that $g_i = 1$ for some i. Since K is $dcl(A)$-freely indecomposable, we find that $h(K) = h(K) \cap K$, thus $h(K) \leq K$. In particular $h(a) \in K$.

If $h(K) < K$, then K is freely $dcl(A)$-decomposable by Corollary 3.2.21; a contradiction. Hence $h(K) = K$.

Since h is a non-trivial automorphism of K of finite order, K is freely $dcl(A)$-decomposable by Theorem 1.2.39; a contradiction. Since in either case we get a contradiction, the equality $dcl(A) = K$ holds, as required.

□ (Theorem 5.1.1 for strictly-speaking closures.)

Concerning the existential counterparts of acl and dcl, the proof follows the same method. We only give a sketch of it, detailing the points where the proof is different. As above, by Theorem 4.1.14 instead of Theorem 4.1.4, $acl^{\exists}(A)$ is finitely generated; hence we get a free decomposition $acl^{\exists}(A) = K * L$, with $dcl^{\exists}(A) \leq K$ and K is freely $acl^{\exists}(A)$-indecomposable. Let $a \in K \setminus dcl^{\exists}(A)$. As before, we also have the following.

Claim 2. *There exists an automorphism h of $acl^{\exists}(A)$, of finite order and fixing $dcl^{\exists}(A)$ pointwise, such that $h(a) \neq a$.*

Proof. The unique different point from Claim 1 is the use of monomorphisms of an elementary extension instead of automorphisms. Since $a \in acl^{\exists}(A) \setminus dcl^{\exists}(A)$, there exists an existential formula $\psi(x)$, with parameters from A, such that $\psi(F)$ is finite, contains a and is not a singleton. The claim here is that there exists $b \in acl^{\exists}(A)$ such that $tp^{\exists}(a/A) \subseteq tp^{\exists}(b/A)$ and $a \neq b$. The details are similar. Then, by Proposition 1.4.59, there exists a monomorphism of an elementary extension F^* of F fixing $dcl^{\exists}(A)$ pointwise such that $f(a) = b$. Let h be the restriction of f to $acl^{\exists}(A)$. The rest of the proof works exactly as in the previous claim.

□ (Claim 2.)

The proof of the equality $dcl^{\exists}(A) = K$ given the finite order automorphism is the same as in the case of *acl*-*dcl*, since it does not involve any kind of argument about types - the only element of difference in Proposition 1.4.59. So the theorem is proved in this case, too.

□ (Theorem 5.1.1 for existential closures.)

5.2 Equality in low ranks

Let F be a free group. Proposition 4.1.1 and its subsequent remark show that for any abelian subgroup A, possibly trivial, we have equality between every algebraic closure and its respective definable closure.

In this section we prove the following theorem, showing point 1 of the main statement announced at the beginning of this chapter.

Theorem 5.2.1. *Let F be the free group of rank 2. Then:*

1. *for any subset A in F, $acl(A) = dcl(A)$ and the same for the other closures;*
2. *$acl^{\exists}_F(A) = acl_F(A) = racl_F(A)$;*
3. *if $\langle A \rangle$ is non-abelian, then $acl(A)$ is the vertex group containing $\langle A \rangle$ in the generalized cyclic JSJ decomposition of F relative to A.*

Proof. Recall that we may assume that A is a subgroup of F, by 1 of Lemma 1.4.58. If $A = \varnothing$ or if A is abelian the result follows from 4.1.1 and the subsequent observation, so we may assume that A is non-abelian. Since the algebraic closures are finitely generated, we may also assume that A is finitely generated.

By Corollary 4.1.13, $rk(acl_F(A)) = 2$. By Theorem 5.1.1, if $dcl_F(A) < acl_F(A)$, then $rk(dcl_F(A)) < rk(acl_F(A))$, a contradiction with non-abelianity of A. So we have proved point 1.

To prove point 2, we claim that the following properties are equivalent:

1. $b \in acl^{\exists}_F(A)$.
2. for any sequence $(f_n | n \in \mathbb{N})$ in $\mathrm{Aut}_A(F)$, there exists a finite subset B such that $f_n(b) \in B$ for any n.

Since the implication (1) ⇒ (2) is clear, we prove the reverse implication.

Let $b \notin acl^{\exists}_F(A)$. We will show that there exists a sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathrm{Aut}_A(F)$ such that for any finite $B \subseteq F$ there exists n such that $f_n(b) \notin B$.

Let $A = \langle a_1, \ldots, a_n \rangle$. Put $F = \langle c, d \rangle$ and $a_i = u_i(c, d)$, $b = v(c, d)$.

Let

$$\psi(\alpha) := \exists x \exists y (\wedge_i a_i = u_i(x, y) \wedge \alpha = v(x, y) \wedge [x, y] \neq 1).$$

Then $F \models \psi(b)$. Since $b \notin acl^{\exists}_F(A)$, ψ has infinitely many realizations.

Let $(B_n)_{n\in\mathbb{N}}$ be an exhaustive enumeration of finite sets of F. For each $n \in \mathbb{N}$, let b_n, c_n, d_n in F such that $b_n \notin B_n$ and

$$F \models \wedge_i a_i = u_i(c_n, d_n) \wedge b_n = v(c_n, d_n) \wedge [c_n, d_n] \neq 1.$$

Define f_n such that $f_n(c) = c_n$ and $f_n(d) = d_n$. Then $f_n \in \mathrm{Aut}_A(F)$ and $f_n(b) \notin B_n$. Thus we have our desired sequence, so $b \notin racl_F(A)$.

Since $acl^{\exists}_F(A) \leq acl_F(A) \leq racl_F(A)$, we have proved point 2.

Point 3 follows by Proposition 4.2.4.

□

5.3 A counterexample in higher ranks

In this section we give a negative answer to the question posed by Sela for free groups with rank at least 4, by constructing a counterexample.

Theorem 5.3.1. *Let F be a free group of finite rank greater than 3. Then there exists $A \leq F$ such that $acl_F(A) = acl_F^{\exists}(A) = racl_F(A)$, $dcl_F(A) = dcl_F^{\exists}(A) = rdcl_F(A) = A$ and $dcl_F(A) < acl_F(A)$.*

Proof. Let $F = \langle H, t | u^t = v \rangle$, where
$v = aybyay^{-1}by^{-1}, H = A * \langle y | \rangle, A = \langle A_0, a, b, u | \rangle$ and A_0 is a finite set, possibly empty.
In the following theorem we will deal with the case $A_0 = \varnothing$; Theorem 5.3.1 will follow quite easily.

Theorem 5.3.2. *Let F be a free group of rank 4. Then there exists $A \leq F$ such that $acl_F(A) = acl_F^{\exists}(A) = racl_F(A)$, $dcl_F(A) = dcl_F^{\exists}(A) = rdcl_F(A) = A$ and $dcl_F(A) < acl_F(A)$.*

Proof. Let $F = \langle H, t | u^t = v \rangle$, where
$v = aybyay^{-1}by^{-1}, H = A * \langle y | \rangle, A = \langle a, b, u | \rangle$.
The following lemma gives a form for the (auto)morphism witnessing non-definability of y.

Lemma 5.3.3. *Let $f \in \mathrm{Hom}_A(F)$.*
Then f maps y to y^{ϵ}, where $\epsilon = \pm 1$.

In the proof, by 'cyclic cancellations' we mean cancellations of the form $w = \not{b} w' \not{b}^{-1} \to w'$, where w is a word, in addition to the usual ones.
Proof. F is free by Theorem 4.1.12.
To prove the statement, we first show that $f(y) \in H$ and $f(t) \notin H$.

Claim 1. *$f(y) \in H$.*

Proof. For better readability, we write w for $f(y)$. Then

$$f(v) = awbwaw^{-1}bw^{-1}.$$

Since $f(u) = u$, $f(v)$ cyclically reduces to u by Lemma 3.1.4. Then, if any t occurs in $f(v)$, it must cyclically cancel. Note that cancellation cannot cross the two first occurrences of a. In fact, no conjugate of a belongs to $\langle u, v \rangle$, because $\langle u, v \rangle_{ab} = \{u^p a^{2q} b^{2q} | p, q \in \mathbb{Z}\}$. So, $wbw \in H$ or $w^{-1}bw^{-1} \in H$. Suppose that the first occurs; when $w^{-1}bw^{-1} \in H$ the argument is similar. We want to prove that $w \in H$, so assume the negation for a contradiction. Write w as $\alpha z \beta$ for some $\alpha, \beta \in H$ and $z \notin H$ a word that begins and ends with t^{ϵ}. Then $f(v)$ becomes

$$a\alpha \underbrace{z\beta b\alpha z}_{z_1} \beta a \beta^{-1} \underbrace{z^{-1}\alpha^{-1} b \beta^{-1} z^{-1}}_{z_2} \alpha^{-1}.$$

Since $wbw \in H$ then $z_1 \in \langle u, v \rangle$. It follows that $z_2 \notin \langle u, v \rangle$, otherwise $z_1 z_2$ would also be in $\langle u, v \rangle$, but Abelianization shows that it is not possible. Then, not all occurrences of t in z_2 cancel and $f(v)$ does not reduce to u, a contradiction.

□ *(Claim 1.)*

Claim 2. *$f(t) \notin H$.*

Proof. Suppose for a contradiction that $f(t) \in H$. Then we have $f(t)^{-1}uf(t) = f(v)$, where $f(v) \in H$. By Abelianizing, we get $u = a^2b^2$, that is impossible, so we have a contradiction.

□ *(Claim 2.)*

Claim 3. *$f(y) = ay^{\epsilon}b$, for some $k, l \in A$.*

By Lemma 3.1.4 with u for α, $f(v)$ for β and $f(t)$ for s, f conjugates v in H and $|f(t)|_t = 1$. To summarize, we have $f(v) = \delta^{-1}v\delta$, $f(t) = t\delta$, $f(y) = w$ for some $\delta, w \in H$, under the condition

$$\delta^{-1}v\delta = awbwaw^{-1}bw^{-1}. \tag{5.1}$$

Write w as kzl for some $k, l \in A$ and z a word that begins and ends with y^{ϵ}, possibly trivial. Then $f(v)$ becomes

$$ak\underbrace{zlbkz}_{z_1}lal^{-1}\underbrace{z^{-1}k^{-1}bl^{-1}z^{-1}}_{z_2}k^{-1}.$$

Also here, no cyclic cancellation may cross the two first occurrences of a. Some cancellation may instead occur in subwords z_1 and z_2. Note that, as z_1z_2 is not trivial, we have $z_1 \neq 1$ or $z_2 \neq 1$. Therefore, we can cancel at most the instances of y in the first two or in the last two occurrences of w. Compare the lengths. Note that $|v|_y = 4$. For the right hand side of (5.1) we have two possibilities of having the same length:

1. one of z_1 and z_2 is trivial and $|w|_y = 2$, that is $f(y) = ky^{\lambda}my^{\mu}l$, for some $m \in A$ and $\lambda, \mu = \pm 1$. Suppose that z_2 is trivial. Then, some cyclic permutation of $aybyay^{-1}by^{-1}$ is equal to $lal^{-1}k^{-1}aky^{\lambda}my^{\mu}lbky^{\lambda}my^{\mu}$, that is impossible, by comparison of the signs of y. Symmetrically, if z_1 is trivial, we get a contradiction for the same reason.

2. The only possibility that remains is that none of z_1 and z_2 is trivial and $|w|_y = 1$. That is, $f(y) = ky^{\epsilon}l$.

□ *(Claim 3.)*

Now we prove the lemma, by showing that $k, l = 1$.

By Lemma 3.1.4, f conjugates v in H. Therefore, by comparison of cyclically reduced words,

$$aybyay^{-1}by^{-1}$$

is a cyclic permutation of the word

$$k^{-1}aky^{\epsilon}lbky^{\epsilon}lal^{-1}y^{-\epsilon}k^{-1}bl^{-1}y^{-\epsilon}.$$

We obtain the equations

- $k^{-1}ak = a$
- $lbk = b$
- $lal^{-1} = a$

- $k^{-1}bl^{-1} = b$.

From the first and the third equation we have $[k,a] = 1$ and $[l,a] = 1$; so $k = a^p$ and $l = a^q$. From the second equation we have $p = q = 0$. Thus, $k = l = 1$. Therefore, $f(y) = y^\epsilon$.

□ *(Lemma 5.3.3.)*

As a corollary, we have the following result, of independent interest.

Corollary 5.3.4. *Let F, H, A as in Lemma 5.3.3. Then* $\mathrm{Hom}_A(F) = \mathrm{Aut}_A(F)$.

The above corollary gives rise to the following

Question 2. Which groups G are such that $\mathrm{Hom}(G) = \mathrm{Aut}(G)$? How big is the gap between this class and $Hopf \cap co\text{-}Hopf$?

Now we can resume the main proof.

Claim 4. $acl_F(A) = acl_F^{\exists}(A) = racl_F(A) = H$.

Proof. Since
$$A \leq acl_F^{\exists}(A) \leq acl_F(A) \leq racl_F(A) \leq H,$$
it is sufficient to show that $y \in acl_F^{\exists}(A)$.

Let
$$\phi(x) := \exists \alpha (u^\alpha = axbxax^{-1}bx^{-1}).$$
Then $F \models \phi(y)$. Let $\gamma \in F$ such that $F \models \phi(\gamma)$. Then the map defined by $f \restriction A = \mathrm{id}_A$, $f(y) = \gamma$ and $f(t) = \alpha$ extends to a homomorphism of F. By Lemma 5.3.3 we have $\gamma = y^\epsilon$. Hence $\phi(x)$ has only finitely many realizations; thus $y \in acl_F^{\exists}(A)$ as desired.

□ *(Claim 4.)*

Claim 5. $dcl_F(A) = dcl_F^{\exists}(A) = rdcl_F(A) = A$.

Proof. Since
$$A \leq dcl_F^{\exists}(A) \leq dcl_F(A) \leq rdcl_F(A) \leq H,$$
it is sufficient to show that $rdcl_F(A) < H$, that is, there exists $f \in \mathrm{Aut}_A(F)$ such that for any $h \in H \smallsetminus A$ we have $f(h) \neq h$.

Define $\hat{f} \in \mathrm{Aut}_A(H)$ as $\hat{f} \restriction A = \mathrm{id}_A$, $\hat{f}(y) = y^{-1}$. Then
$$g(v) = ay^{-1}by^{-1}ayby = (ayby)^{-1}aybyay^{-1}by^{-1}(ayby) = \delta^{-1}v\delta,$$
where $\delta = ayby$. We can extend $\hat{f}$ to an A-automorphism f of F, imposing that $\hat{f}$ maps t to $t\delta$. Let $h \in H \smallsetminus A$. Then $|h|_y \geq 1$, so $f(h) \neq h$, as required.

□ *(Claim 5 and Theorem 5.3.2.)*

As announced before, the simple step to prove Theorem 5.3.1 is the following. Let F be as in Theorem 5.3.2 and let $F' = F * \langle A_0 | \rangle$. Then F' is the desired counterexample in rank $|A_0| + 4$.

□ *(Theorem 5.3.1.)*

5.4 Results in rank 3

For the free group of rank 3, a proof of equality of algebraic and definable closure has not been found yet, neither has been a counterexample; however, we give some restrictions to the form that a possible counterexample must have.

Proposition 5.4.1. *Let F be a free group of rank 3 and let $A \leq F$. If $rdcl_F(A) < racl_F(A)(= acl_F(A))$ then $rdcl_F(A) = dcl_F(A)$ and there exist $u, v, a, y \in F$ such that:*

1. $acl_F(A) = dcl_F(A) * \langle y|\rangle, dcl_F(A) = \langle u, a|\rangle$;
2. *F can be written as $F = \langle acl_F(A), t|u^t = v\rangle$, where* $v = \prod_{i=1,\dots,p}(\alpha_i y)\prod_{i=1,\dots,p}(\alpha_i y^{-1})$ *for some $\alpha_i \in dcl_F(A)$;*
3. *Any automorphism $f \in \mathrm{Aut}_A(F)$ with non-trivial restriction to $acl_F(A)$ maps y to y^{-1}.*

Proposition 5.4.2. *Let F be a free group of rank 3 and let $A \leq F$. If $rdcl_F(A) < acl_F(A)$, then $rk(rdcl_F(A)) = 2$ and $acl_F(A) = rdcl_F(A) * \langle y|\rangle$ for some $y \in F$.*

Proof. By Proposition 5.1.1, $acl_F(A) = rdcl_F(A) * L$ for some $L \leq acl_F(A)$. Since A is non-abelian, $rdcl_F(A)$ is also non-abelian, so $rk(rdcl_F(A)) \geq 2$. Since $rk(acl_F(A)) = 3$ and $rk(rdcl_F(A)) < rk(acl_F(A)$, we obtain that $rdcl_F(A)$ has rank 2 and L is cyclic. Let y be a generator of L.

□

Note that, since $dcl_F(A) \leq rdcl_F(A)$ and $dcl_F(A)$ cannot have finite index in $rdcl_F(A)$, we obtain the equality $rdcl_F(A) = dcl_F(A)$.

The lemmas below are needed to prove Proposition 5.4.1.

Before proving the result, we need the two following lemmas, that hold under the only assumption of torsion-freeness for A. Recall by Definition 1.4.15 that an A-automorphism of a group F is an automorphism of F fixing A pointwise, and that we denote the group of A-automorphisms of F by $\mathrm{Aut}_A(F)$.

Lemma 5.4.3. *Let A be a group and let $f \in \mathrm{Aut}_A(A * \langle y|\rangle)$. Then $f(y) = ay^\epsilon b$ for some $a, b \in A$ and $\epsilon = \pm 1$.*

Proof. Let

$$a_0 \prod_{1\leq i\leq n}(y^{\epsilon_i} a_i)$$

be a normal form for $f(y)$ in $A * \langle y|\rangle$. Since f is an automorphism, y has an inverse image. Let

$$b_0 \prod_{1\leq j\leq m}(y^{\eta_j} b_j)$$

be a normal form for $f^{-1}(y)$ in $A * \langle y|\rangle$. Then we have

$$(f^{-1}\circ f)(y) = a_0 \prod_{1\leq i\leq n}[(b_0 \prod_{1\leq j\leq m}(y^{\eta_j} b_j))^{\epsilon_i} a_i] \quad (5.2)$$
$$= y.$$

By comparing with the Abelianization $A/[A,A]\times\langle y|\rangle$, we obtain that $\sum_{1\leq i\leq n}\epsilon_i\cdot\sum_{1\leq j\leq m}\eta_j = 1$. So, in particular, both n and m are odd. We will show that $n = m = 1$. Suppose for a contradiction that both n and m are greater than 1; then, since $n > 1$, one of the following cases occurs.

1. ϵ_1 and ϵ_2 have the same sign; or

2. ϵ_1 and ϵ_2 have opposite signs.

We will show that both cases lead to contradiction.

Proof of Case 1. Without loss of generality we may assume that both ϵ_1 and ϵ_2 are $+1$, the case $\epsilon_1 = \epsilon_2 = -1$ being similar. Since $b_0\prod_{1\leq j\leq m}(y^{\eta_j}b_j)$ is a normal form, the only cancellations in the expression (5.2) may occur across one occurrence of $b_0\prod_{1\leq j\leq m}(y^{\eta_j}b_j)$ and the next, so we have the equation $b_m a_1 b_0 = 1$. Since we want that only one occurrence of y remains after doing all possible cancellations, and since $m > 1$, we also have the equations $\eta_j = -\eta_{(m+1)-j}$, with j running from 1 to at least $(m+1)/2$, otherwise too many occurrences of $y^{\pm 1}$ would remain in the first occurrence of $b_0\prod_{1\leq j\leq m}(y^{\eta_j}b_j)$ in the expression (5.2). So we have

$$\eta_{\frac{m+1}{2}} = -\eta_{\frac{m+1}{2}},$$

a contradiction with $\eta_j \neq 0$.

□ (Case 1.)

Proof of Case 2. Without loss of generality we may assume that $\epsilon_1 = +1$ and $\epsilon_2 = -1$, the case with opposite signs being similar. For the same reason as in the previous case, we have the equation $b_m a_1 b_m^{-1} = 1$, that has the only solution $a_1 = a$. Therefore in the expression of $f(y)$ we have the subsequence $(\epsilon_1, a_1, \epsilon_2) = (+1, 1, -1)$, in contradiction with being a normal form.

□ (Case 2.)

Since the case $n > 1, m > 1$ leads to contradiction, then either n or m must be 1. Suppose that one of n and m is 1: by constructing the inverse, we will show that the other also must be 1.

Suppose that $n = 1$. Then, f maps y to $ay^\epsilon b$, for some $a, b \in A$ and $\epsilon = \pm 1$. Take the map fixing A pointwise and mapping y to $(a^{-1}yb^{-1})^\epsilon$ as f^{-1}. If $m = 1$ the case is similar, so we have the result.

□ (Lemma 5.4.3.)

Lemma 5.4.4. *Let A be a torsion-free group and let $f \in \mathrm{Aut}_A(A * \langle y|\rangle)$. If f has finite order, then either $f(y) = y$ or $f(y) = ay^{-1}a$ for some $a \in A$.*

Proof. By Lemma 5.4.3, a non-trivial A-automorphism f of $A * \langle y|\rangle$ maps y to $ay^\epsilon b$, for some $a, b \in A$ and $\epsilon = \pm 1$, where $\epsilon = -1$ or $a \neq 1$ or $b \neq 1$.

If $\epsilon = +1$, then

$$f^n(y) = a^n y b^n.$$

By torsion-freeness, $a^n y b^n \neq y$ for every n. So, $\epsilon = -1$, that is, f maps y to $ay^{-1}b$. For every n we have

$$f^{2n}(y) = (ab^{-1})^n y (a^{-1}b)^n$$

and

$$f^{2n+1}(y) = (ab^{-1})^n a y^{-1} b (a^{-1}b)^n.$$

So f has even order and $(ab^{-1})^n = (a^{-1}b)^n = 1$. By torsion-freeness we have $a = b$, so f has order 2.

□

Let F be a free group and let A, H be subgroups of F such that $H = acl_F(A) = dcl_F(A) * \langle y | \rangle$ and $F = \langle H, t | u^t = v \rangle$.

The two following lemmas allow us to assume that u and v are cyclically reduced and that an automorphism $f \in \mathrm{Aut}_A(F)$ with non-trivial restriction to H maps y to y^{-1}.

Lemma 5.4.5. *Let F be a free group and let H be a subgroup of F such that $F = \langle H, t | u^t = v \rangle$, with u cyclically reduced. Then there exist $v' \in H$ cyclically reduced and t' such that $F \cong \langle H, t' | u^{t'} = v' \rangle$.*

Proof. Let $h \in H$ be such that v^h is cyclically reduced. Take the Dehn twist $\delta_h : F' \to F$ that fixes H pointwise and maps t to th. Taking $v' = v^h$ and $t' = th$, δ_h is the desired isomorphism.

□

Lemma 5.4.6. *Let F be a free group and let A, H be subgroups of F such that $H = acl_F(A) = dcl_F(A) * \langle y | \rangle$ and $F = \langle H, t | u^t = v \rangle$. Let $f \in \mathrm{Aut}_A(F)$ be such that $f \restriction H \neq \mathrm{id}_H$. Then there exists $f' \in \mathrm{Aut}_A(F)$ such that $f'(y) = y^{-1}$.*

Proof. By Lemma 5.4.4, f maps y to $ay^{-1}a$, for some $a \in dcl_F(A)$. Consider $\iota \in \mathrm{Aut}_A(F)$ mapping y to $a^{-1}y$ and t to t. Let f' be obtained precomposing f with ι. Thus f' is the desired automorphism.

□

Lemma 5.4.7. *Let F be a free group and let A, H be subgroups of F such that $H = acl_F(A) = dcl_F(A) * \langle y | \rangle$ and $F = \langle H, t | u^t = v \rangle$. Let $f \in \mathrm{Aut}_A(F)$ be such that $f \restriction H \neq \mathrm{id}_H$. If u is primitive and cyclically reduced, then $u \in dcl_F(A)$.*

Proof. Suppose that u is cyclically reduced and $u \notin dcl_F(A)$; we will show that u is not primitive. By Lemmas 5.4.5 and 5.4.6, we may assume that v also is cyclically reduced and f maps y to y^{-1}.Write u in normal form as

$$u = \prod_{i=1,\ldots,n} \alpha_i y^{\eta_i},$$

where $\alpha_i \in dcl_F(A)$ and $\eta_i = \pm 1$ for every i.

Let $f \in \mathrm{Aut}_A(F)$. Then, by Lemma 3.1.4 we have $f(u) \sim_H f(v)$ or $f(u) \sim_H u^\epsilon$ or $f(u) \sim_H v^\epsilon$, where $\epsilon = \pm 1$; recall that $\sim_H$ denotes conjugation in H. If $f(u) \sim_H f(v)$, then $u \sim_H v$, so we have contradiction with freeness of F. For the remaining cases, suppose that $f(u) \sim_H u$; the case $f(u) \sim_H v$ is similar. Note that

1. $f(u)$ is a cyclic permutation of u;

2. if $u = u(y)$, then $f(u) = u(y^{-1})$.

Thus, u has the form

$$\prod_{i=1,\dots,m}(\alpha_i y)\prod_{i=1,\dots,m}(\alpha_i y^{-1}).$$

But $u = (\prod_{i=1,\dots,m}\alpha_i)^2$ in the Abelianization of F, therefore u is not primitive.

□

Proposition 5.4.8. *Let F be a free group of finite rank n, $A \leq F$, $acl_F(A) < F$ and $rk(acl_F(A)) = rk(F)$. Then there exist sequences $(u_1,\dots,u_m),(v_1,\dots,v_m)$ of elements of $acl_F(A)$ such that*

$$F = \langle acl_F(A), t_1,\dots,t_m | \{u_i^{t_i} = v_i | i = 1,\dots,m\}\rangle,$$

with $m < n$.

Proof. By Theorem 4.1.4, we can construct F from $acl_F(A)$ by a finite sequence of cyclic free amalgamated products and cyclic HNN extensions. As $rk(acl_F(A)) = rk(F)$, we obtain F from $acl_F(A)$ from HNN extensions. The general form of the presentation of F is

$$F = \langle \dots\langle\langle acl_F(A), t_1 | u_1^{t_1} = v_1\rangle, t_2 | u_2^{t_2} = v_2\rangle \dots, t_n | u_n^{t_n} = v_n\rangle.$$

As we have only HNN-extensions, we have a one-vertex decomposition. By Proposition 4.2.4, $acl_F(A)$ is the vertex group containing A in the generalized cyclic JSJ decomposition of F relative to A. Therefore, the only vertex must coincide with $acl_F(A)$. The Abelianization F_{ab} of F has the form

$$F_{\mathrm{ab}} = \langle acl_F(A) | u_i = v_i\rangle \times \langle t_i | [t_j, t_k] = 1\rangle.$$

Since t_i is independent from t_j for $i \neq j$, if $m \geq n$, then the Abelian rank of F_{ab} is greater than n, so we have a contradiction.

□

Proposition 5.4.9. *Under the same assumptions as in Lemma 5.4.8, let*

$$H_j = \langle acl_F(A), t_j | u_j^{t_j} = v_j\rangle$$

for $j \in \{1,\dots,n\}$. Let $f \in \mathrm{Aut}_A(F)$. Then $f \restriction H_j \in \mathrm{Aut}_A(H_j)$.

Proof. After a suitable permutation, we may assume $j = 1$. For $i = 1,\dots,n$, define

$$L_1 = H_1 \text{ and } L_i = \langle L_{i-1}, t_i | u_i^{t_i} = v_i\rangle.$$

So we have $F = L_n$. We show that $f(u_1) \sim_{L_1} f(v_1)$ by induction on i.

The base of the induction comes from the fact that f is an automorphism, so we have $f(u_1) \sim_{H_1} f(v_1)$, therefore $f \restriction H_1 \in \mathrm{Aut}_A(H_1)$.

We are left to prove the inductive step. If $f \restriction L_i \in \mathrm{Aut}_A(L_i)$, then $f(u_1) \sim_{L_{i-1}} f(v_1)$. To this purpose, suppose for a contradiction that $f(u_1) \not\sim_{L_{i-1}} f(v_1)$. Then, by Lemma 3.1.4 with $(L_{i-1}, u_i, v_i, u_1, v_1)$ for (H, u, v, α, β), we have $f(u_1) \sim_{L_{i-1}} u_i^{\epsilon}$ and $f(v_1) \sim_{L_{i-1}} v_i^{\epsilon}$ or $f(u_1) \sim_{L_{i-1}} v_n^{\epsilon}$ and $f(v_1) \sim_{L_{i-1}} u_i^{\epsilon}$, where $\epsilon = \pm 1$. Suppose that $f(u_1) \sim_{L_{i-1}} u_i$, the other three cases being analogous. We have $f(v_1) \sim_{L_{i-1}} v_i$ and

$$L_i = \langle L_{i-1}, f(t_i) | f(u_1)^{f(t_i)} = f(v_1)\rangle,$$

but in L_{i-1} we have the relation $u_1^{t_1} = v_1$, so in L_i we have the two relations $u_1^{t_1} = v_1$ and $f(u_1)^{f(t_i)} = f(v_1)$, in contradiction with the fact that f is an automorphism.

□

Proposition 5.4.10. *Under the same assumptions as in Lemma 5.4.8, for every i, either $v_i \in dcl_F(A)$ or v_i has the form $\prod_{j=1,\dots,p}(\alpha_j y)\prod_{j=1,\dots,p}(\alpha_j y^{-1})$, where $\alpha_j \in dcl_F(A)$.*

Proof. By Lemmas 5.4.5 and 5.4.6 we may assume that v_i is cyclically reduced and $f(y) = y^{-1}$. A normal form for v_i in $dcl_F(A) * \langle y|\rangle$ is

$$v_i = \prod_{j=1,\dots,q}(\alpha_j y^{\epsilon_j}).$$

So we have

$$f(v_i) = \prod_{j=1,\dots,q}(\alpha_j y^{-\epsilon_j}).$$

By Lemma 3.1.4, $f(v_i)$ is a cyclic permutation of v_i, so q is even. If $q = 0$ then $v_i \in dcl_F(A)$. If $q > 0$, then v_i has the form

$$v_i = \prod_{j=1,\dots,p}(\alpha_j y^{\epsilon_j}) \prod_{j=1,\dots,p}(\alpha_j y^{-\epsilon_j}).$$

By possibly taking a conjugate, that is a cyclic permutation, of v_i^{ϵ}, we get the result.

□

We can now prove Proposition 5.4.1.

Proof of Proposition 5.4.1. We prove point 1. By Proposition 5.4.2, $dcl_F(A)$ has rank 2 and there exists $y \in F$ such that $acl_F(A) = dcl_F(A) * \langle y|\rangle$. Let u, a be generators for $dcl_F(A)$.

□ (Point 1.)

We prove point 2. By Proposition 5.4.8 the presentation of F is

$$F = \langle acl_F(A), t_1, t_2 | u_1^{t_1} = v_1, u_2^{t_2} = v_2\rangle.$$

Suppose for a contradiction that $n = 2$. Let $H = \langle acl_F(A), t_1 | u_1^{t_1} = v_1\rangle$.

By Theorem 4.1.12, we may assume that u_2 is primitive in H. By Theorem 5.4.7, $u_2 \in dcl_F(A)$. Let $dcl_F(A) = \langle u_2, a\rangle$.

Let $L = \langle H | u_2 = 1\rangle$.

Since u_2 is primitive in H, L is free by Theorem 1.1.38. By Lemmas 5.4.5 and 5.4.6, we may assume that

$$L = \langle a, y, t_1 | t_1^{-1} a^p t_1 = v'\rangle,$$

where $v' = \prod_{i=1,\dots,q}(a^{m_i} y)\prod_{i=1,\dots,q}(a^{m_i} y^{-1})$.

By Theorem 4.1.12, one of a^p and v' is primitive in $\langle a, y|\rangle$. Abelianization shows that v' cannot be, so a^p is primitive. Therefore, $p = \pm 1$. We may assume that $p = 1$. By Theorem 4.1.12 there exists a basis $\{a, b\}$ of $\langle a, y|\rangle$ such that v' is conjugate of b^m for some m. Since v' is root-free, we have $m = \pm 1$, therefore v' is primitive, so we have a contradiction. Thus the statement is proved.

□ (Point 2 and Proposition 5.4.1.)

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