Quelques aspects de la théorie des invariants de type fini en topologie de dimension trois

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Pour Honorine, à la mémoire de ses grand-pères : Dominique et Ernest

Some aspects of the theory of finite-type invariants in three-dimensional topology

Contents

1 Introduction

1.1 A rough overview

The theory of finite-type invariants is part of low-dimensional topology and quantum topology. *Low-dimensional topology* can be characterized by its objects of study, namely manifolds of small dimension (say 2, 3 and 4) and their homeomorphisms; this includes the study of knotted objects in the 3-sphere. There are several approaches to this study, which may involve combinatorial, algebraic, geometric or analytic methods. Among these various approaches, there is *quantum topology* which one can define as the production and study of quantum invariants. Here, by "quantum invariants", we mean all these topological invariants of knots and 3-manifolds, and all these representations of mapping class groups, that have been obtained during the last 25 years after

- V. Jones' discovery in 1984 of his famous knot invariant [Jon87],
- E. Witten's work which interpreted the Jones polynomial as a topological quantum field theory based on the Chern–Simons path integral [Wit89],
- and the work by N. Reshetikhin & V. Turaev who gave a rigourous construction of E. Witten's invariants [RT91].

The mathematical definition of quantum invariants being very algebraic by nature, quantum topology can also be regarded as the place where low-dimensional topology interacts with some fields of algebra (such as the representation theory of quantum groups, or the Lie theory). After its "constructive" period, quantum topology seems now to have taken another direction: the seek for topological interpretations of the quantum invariants that have been obtained so far. Indeed, the mathematical construction of a quantum invariant is usually very indirect: it needs a special way of presenting knots or 3-manifolds, it goes by a finite "state sum" or some kind of integral and, at the end, the topological quantity that is really measured by this invariant is usually unclear. Of course, this understanding of quantum invariants may require to extend their range of definition, to refine them (by adding structures to manifolds) or even to "categorify" them. The challenge is to connect quantum invariants to more classical approaches of low-dimensional topology, which can be provided by algebraic topology or by geometry.

The *theory of finite-type invariants* has its origins in the perturbative approach to E. Witten's quantum invariants. For knots and links in the 3-sphere, the notion of "finite-type invariant" in itself has been introduced independently by V. Vassiliev [Vas90] and M. Goussarov [Gus94] with different motivations. Thanks to the Kontsevich integral and its universal property [Kon93b, BN95a], one knows that the family of finite-type invariants is very rich; besides, this family dominates all Reshetikhin–Turaev quantum invariants (by virtue of the Drinfel'd–Khono theorem [Kas95, LM96]). For $\mathbb{Z}$-homology spheres, the notion of "finite-type invariant" has been introduced by T. Ohtsuki who constructed the first examples by number-theoretical expansions into power series of some Reshetikhin–Turaev invariants [Oht96]. Presenting 3-manifolds by surgery on links in the 3-sphere, T. Le, J. Murakami & T. Ohtsuki built from the Kontsevich integral a universal finite-type invariant of $\mathbb{Z}$-homology spheres [LMO98, Le97]: hence, for 3-manifolds too, the class of finite-type invariants is very rich. Perturbative Chern–Simons theory produces some other universal finite-type invariants: see, in particular, [BN91, BT94, AF97, Poi02] in the case of links and [AS92, BC98, Kon94, KT99] in the case of 3-manifolds. These invariants have intrinsic formulations in terms of configuration space integrals (which are deep generalizations of Gauss' formula for the linking number), and they are conjecturally equal to the Kontsevich integral (in the case of links) and to the LMO invariant (in the case of 3-manifolds). A strong equivalence is established in [Les02a] between the Kontsevich integral and the perturbative Chern–Simons invariant for links in the 3-sphere; these two invariants are also known to coincide in low degrees [Les02b].

T. Ohtsuki's theory of finite-type invariants was subsequently generalized to arbitrary 3-manifolds by T. Cochran & P. Melvin [CM00]. Later, a different theory encompassing arbitrary 3-manifolds and their links was introduced independently by M. Goussarov [Gou99, Gou00] and K. Habiro [Hab00b]. While it contains the Ohtsuki theory for $\mathbb{Z}$-homology spheres and the Vassiliev–Goussarov theory for knots, the Goussarov–Habiro theory is enriched by new surgery techniques and a close relationship with the lower central series of the Torelli group (in the case of 3-manifolds) and that of the pure braid group (in the case of links). In the sequel, we shall concentrate on some aspects of the Goussarov–Habiro theory for compact oriented 3-manifolds. For an introduction to quantum topology, the reader may consult the books [Kas95, Tur94] and, for a more specific treatment of the theory of finite-type invariants, the reader is referred to the books [Oht02b, CDM12].

1.2 Finite-type invariants

In order to define finite-type invariants of 3-manifolds, we need to fix a closed oriented surface R (which may be empty or disconnected). We consider compact connected oriented 3-manifolds M whose boundary is *parameterized* by R, i.e. M comes with an orientation-preserving homeomorphism $R \to \partial M$ which is denoted by the lower-case letter m. Two such manifolds with parameterized boundary M and M' are *homeomorphic* if there is an orientation-preserving homeomorphism $f : M \to M'$ such that $f|_{\partial M} \circ m = m'$. We denote by $\mathcal{V}(R)$ the set of homeomorphism classes of compact connected oriented 3-manifolds with boundary parameterized by R.

Here is one way to modify a manifold $M \in \mathcal{V}(R)$ without modifying its $\mathbb{Z}$-homology type. First, we choose a compact oriented connected surface $S \subset \operatorname{int}(M)$ with one boundary component, and a homeomorphism $s : S \to S$ which is the identity on ∂S and which induces the identity of $H_*(S; \mathbb{Z})$. Then we define

$$M_{(S,s)} := \big(M \setminus \operatorname{int}(S \times [-1,1])\big) \cup_{\tilde{s}} (S \times [-1,1]) \tag{1.1}$$

where $S \times [-1,1]$ is identified with a regular neighborhood of S in M and $\tilde{s}$ is the self-homeomorphism of $\partial(S \times [-1,1])$ which is given by s on $S \times \{1\}$ and is the identity elsewhere. In other words, $M_{(S,s)}$ is obtained by "twisting" M along S with s. The boundary parameterization of $M_{(S,s)}$ is induced by m in the obvious way. The move $M \rightsquigarrow M_{(S,s)}$ in $\mathcal{V}(R)$ is called a *Torelli surgery*.[1]

Let A be an abelian group. A map $f : \mathcal{V}(R) \to A$ is a *finite-type invariant* of *degree at most* d (in the sense of M. Goussarov & K. Habiro) if we have

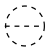
$$\sum_{P \subset \{0,\dots,d\}} (-1)^{|P|} \cdot f(M_P) = 0 \in A \tag{1.2}$$

for any $M \in \mathcal{V}(R)$ and for any pairwise-disjoint surfaces $S_0, \dots, S_d \subset \operatorname{int}(M)$ equipped with self-homeomorphisms $s_0, \dots, s_d$ where each $s_i : S_i \to S_i$ is the identity on ∂S_i and acts trivially in homology. Here $M_P \in \mathcal{V}(R)$ is obtained by simultaneous Torelli surgeries $M \rightsquigarrow M_{(S_p,s_p)}$ for all $p \in P$. In other words, the $(d+1)$-st "formal differential" of f with respect to Torelli surgeries is trivial, so that f should behave like a "polynomial" map of degree at most d with respect to these surgeries.

There are numerous examples of finite-type invariants in the case of closed oriented 3-manifolds (i.e. in the case $R = \varnothing$) and some of them are found among classical invariants. For instance the n-th coefficient of the Conway polynomial of closed oriented 3-manifolds M having $\beta_1(M) = 1$ is a finite-type invariant of degree n [GH00, Lie00]; if one adds spin structures to the theory, the Rochlin invariant is of degree 1 [Mas03b]; if one adds complex spin structures to the theory [DM05], the I-adic reductions of the Reidemeister–Turaev torsion are finite-type invariants too [Mas10]. Yet, the "prototype" of a

[1] This terminology is borrowed from [KT99].

finite-type invariant is certainly the Casson–Walker–Lescop invariant which is of degree 2 by results of Morita [Mor91] and Lescop [Les98]. The LMO invariant, which is constructed in [LMO98] from the Kontsevich integral, can be seen as a far-reaching generalization of the latter. This invariant takes values in the space $\mathcal{A}(\varnothing)$ of *trivalent Jacobi diagrams*: the generators of this $\mathbb{Q}$-vector space are finite trivalent graphs whose vertices are oriented, and the relations are the AS relation (a diagrammatic analogue of the antisymmetry of Lie brackets) and the IHX relation (a diagrammatic analogue of the Jacobi identity). The space $\mathcal{A}(\varnothing)$ is graded by the number of vertices (which is called the *i-degree* of Jacobi diagrams). The LMO invariant $Z(M)$ is mainly interesting for $\mathbb{Q}$-homology spheres M, in which case it coincides with the Aarhus integral introduced by D. Bar-Natan, S. Garoufalidis, L. Rozansky & D. Thurston [BNGRT02a, BNGRT02b][2]. For a $\mathbb{Q}$-homology sphere M, we have

$$Z(M) = \varnothing + \frac{\lambda_{\mathrm{W}}(M)}{4} \cdot$$  $$+ \text{(i-degree} > 2) \ \in \mathcal{A}(\varnothing) \qquad (1.3)$$

where $\lambda_{\mathrm{W}}(M)$ denotes Walker's extension of the Casson invariant as normalized in [Wal92], and the i-degree d part of Z is universal among $\mathbb{Q}$-valued finite-type invariants of degree d [Le97, Hab00b]. The perturbative Chern–Simons invariant constructed by M. Kontsevich [Kon94] and by G. Kuperberg & D. Thurston [KT99] has the same universal property [Les04]. Consequently, these two invariants have the same capacity for distinguishing $\mathbb{Q}$-homology spheres; but it is not known whether they are strictly equal.

1.3 The Torelli group

The theory of finite-type invariants has connections with the study of the Torelli group. Let Σ be a compact connected oriented surface with one boundary component. The *mapping class group* of the surface Σ is the group $\mathcal{M}(\Sigma)$ of isotopy classes of self-homeomorphisms of Σ that are the identity on $\partial\Sigma$. The *Torelli group* $\mathcal{I}(\Sigma)$ is the subgroup of $\mathcal{M}(\Sigma)$ acting trivially on $H_*(\Sigma;\mathbb{Z})$. The study of the Torelli group, from a topological point of view, started with works of J. Birman [Bir71] and was followed by D. Johnson through a series of paper, which notably resulted in a finite generating set for $\mathcal{I}(\Sigma)$ in genus at least 3 [Joh83a]. The reader is referred to D. Johnson's survey [Joh83b] for an account of his work. From the viewpoint of finite-type invariants, one can regard the Torelli group as an analogue of the pure braid group (the latter corresponds to links in the 3-sphere, while the former corresponds to 3-manifolds). However, the Torelli group is not as well understood as the pure braid group.

One way to understand the structure of the Torelli group is to define functions on it using 3-dimensional invariants. For this, one embeds the surface Σ in the interior of a 3-manifold $M \in \mathcal{V}(R)$ and, for any invariant $f : \mathcal{V}(R) \to A$ with values in an abelian group A, one considers the map

$$F : \mathcal{I}(\Sigma) \longrightarrow A, \ s \longmapsto f\left(M_{(\Sigma,s)}\right) - f(M) \qquad (1.4)$$

where $M_{(\Sigma,s)}$ is the result of a Torelli surgery as defined at (1.1). For instance, if M is a $\mathbb{Z}$-homology sphere and if f is the Rochlin invariant (with values in $A := \mathbb{Z}/2\mathbb{Z}$), then F is one of the Birman–Craggs homomorphisms [BC78] which D. Johnson used to compute the abelianization of $\mathcal{I}(\Sigma)$ [Joh85]. If M is still a $\mathbb{Z}$-homology sphere and if f is the Casson invariant (with values in $A := \mathbb{Z}$), the corresponding map F on $\mathcal{I}(\Sigma)$ has been used by S. Morita to compute its second nilpotent quotient [Mor89, Mor91]. The finiteness properties of Rochlin's and Casson's invariants play a crucial role in these computations. Thus D. Johnson's and S. Morita's works somehow prefigure the use of finite-type invariants of $\mathbb{Z}$-homology spheres in the study of the Torelli group. This approach to understand the structure of $\mathcal{I}(\Sigma)$ has been developed in subsequent works of S. Garoufalidis & J. Levine [GL98, GL97].

[2] The LMO invariant $Z(M)$ is denoted by $\hat{\Omega}(M)$ in [LMO98] and by $\mathring{A}(M)$ in [BNGRT02a, BNGRT02b]; the invariant denoted by $\Omega(M)$ in [LMO98] is another normalization which coincides with $\hat{\Omega}(M)$ for a $\mathbb{Z}$-homology sphere M.

Besides, this approach is a way to illustrate the *polynomial* nature of finite-type invariants. We still assume that the surface Σ is embedded in the interior of a 3-manifold $M \in \mathcal{V}(R)$, and we consider any finite-type invariant $f : \mathcal{V}(R) \to \mathbb{Z}$ of degree d. Then the $\mathbb{Z}$-valued function F on $\mathcal{I}(\Sigma)$ defined by (1.4) is bounded by a constant times the d-th power of the word metric on $\mathcal{I}(\Sigma)$ and, in some circumstances, this bound is asymptotically sharp. This has been proved by N. Broaddus, B. Farb & A. Putman when M is a $\mathbb{Z}$-homology sphere and f is the Casson invariant [BFP07]. In general, such a polynomial behaviour of the function $F : \mathcal{I}(\Sigma) \to \mathbb{Z}$ follows from the fact that its linear extension $\mathbb{Z}[F] : \mathbb{Z}[\mathcal{I}(\Sigma)] \to \mathbb{Z}$ to the group ring $\mathbb{Z}[\mathcal{I}(\Sigma)]$ vanishes on the $(d+1)$-st power I^{d+1} of the augmentation ideal $I \subset \mathbb{Z}[\mathcal{I}(\Sigma)]$. Since this does not seem to have been observed before, we have included a proof in the appendix.

1.4 Surgery equivalence relations

Finite-type invariants are strongly connected to some equivalence relations among 3-manifolds, which we now describe. The *lower central series* of a group G is the decreasing sequence of subgroups

$$G = \Gamma_1 G \supset \Gamma_2 G \supset \Gamma_3 G \supset \cdots$$

that are defined inductively by $\Gamma_{i+1} G := [\Gamma_i G, G]$ for all $i \geq 1$. This series is related to the I-adic filtration of the group ring $\mathbb{Z}[G]$ (where I denotes the augmentation ideal of $\mathbb{Z}[G]$) by the implication

$$\forall k \geq 1,\ \forall g \in G, \quad g \in \Gamma_k G \Longrightarrow g - 1 \in I^k. \tag{1.5}$$

The converse is false in general, which constitutes the *dimension subgroup problem* in group theory. As we have observed in the previous paragraph, the notion of finite-type invariant is closely related to the I-adic filtration of the group ring of the Torelli group. This motivates the following definition: let $k \geq 1$ be an integer; two 3-manifolds $M, M' \in \mathcal{V}(R)$ are said to be *Y_k-equivalent* if M' can be obtained from M by a Torelli surgery $M \rightsquigarrow M_{(S,s)} \cong M'$ along a surface $S \subset \operatorname{int}(M)$ with an $s \in \Gamma_k \mathcal{I}(S)$. It follows from (1.5) applied to the Torelli group that, for any integer $d \geq 0$,

$$M \overset{Y_{d+1}}{\sim} M' \Longrightarrow \big(f(M) = f(M') \text{ for any finite-type invariant } f \text{ of degree} \leq d\big). \tag{1.6}$$

The converse is false in general, but it is obviously true for $d = 0$. Note that the Y_k-equivalence relation becomes finer and finer as k increases. In the case of closed oriented 3-manifolds ($R = \varnothing$), for instance, S. Matveev has shown in [Mat87] that the Y_1-equivalence is classified by the isomorphism class of the pair (homology, linking pairing), and it is shown in [Mas03a] that the Y_2-equivalence is classified by the isomorphism class of the quintuplet (homology, space of spin structures, linking pairing, cohomology rings, Rochlin function). But no such characterization of the Y_k-equivalence relation is known for $k \geq 3$ and $R = \varnothing$.

For any closed oriented surface R, the set $\mathcal{V}(R)$ is subdivided into infinitely many Y_1-equivalence classes. Then it is convenient to restrict the theory of finite-type invariants to each of these classes, and to have the following "dual" viewpoint. Let $\mathcal{Y} \subset \mathcal{V}(R)$ be a Y_1-equivalence class. The abelian group $\mathbb{Z}\cdot\mathcal{Y}$ freely generated by the set $\mathcal{Y}$ has the filtration

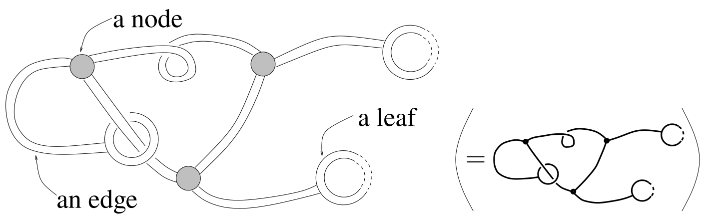
$$\mathbb{Z}\cdot\mathcal{Y} = \mathcal{F}_0(\mathcal{Y}) \supset \mathcal{F}_1(\mathcal{Y}) \supset \mathcal{F}_2(\mathcal{Y}) \supset \cdots \tag{1.7}$$

where, for any $k \geq 1$, $\mathcal{F}_k(\mathcal{Y})$ is the subgroup of $\mathbb{Z}\cdot\mathcal{Y}$ spanned by linear combinations of the form

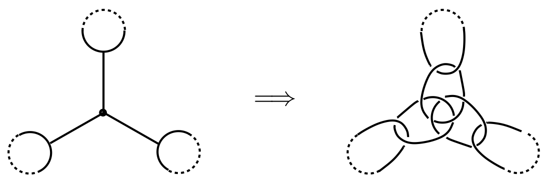
$$\sum_{P \subset \{0,\dots,d\}} (-1)^{|P|} \cdot M_P. \tag{1.8}$$

Here M is any 3-manifold in the class $\mathcal{Y}$, $S_0, \dots, S_d \subset \operatorname{int}(M)$ are pairwise-disjoint surfaces equipped with $s_0 \in \mathcal{I}(S_0), \dots, s_d \in \mathcal{I}(S_d)$ and M_P is the result of doing simultaneously the Torelli surgeries indexed by $p \in P$. Clearly, a map $f : \mathcal{Y} \to A$ is a finite-type invariant of degree at most d in the sense of (1.2) if and only if its linear extension $\mathbb{Z}\cdot f$ to $\mathbb{Z}\cdot\mathcal{Y}$ vanishes on $\mathcal{F}_{d+1}(\mathcal{Y})$.

The Y_k-equivalence relations have been considered in their full generality for the first time by M. Goussarov [Gou99, Gou00] and K. Habiro [Hab00b]. In their works[3], these relations are defined in another way which we briefly recall. In the terminology of [Hab00b], a *graph clasper* in a 3-manifold $M \in \mathcal{V}(R)$ is a compact connected surface $G \subset \text{int}(M)$ which comes decomposed between *leaves*, *nodes* and *edges* according to some rules. Here is an example of a graph clasper with three nodes (which is also drawn using the blackboard framing convention):



The *surgery* in M along the graph clasper G is the usual surgery along a framed link that is derived from G in an appropriate way: the resulting 3-manifold is denoted by M_G. For instance, if G has a single node, then the associated framed link is



and the clasper surgery $M \leadsto M_G$ is equivalent to a "Borromean surgery" in the sense of [Mat87]. A *calculus of claspers* is developed in [Gou00, Hab00b, GGP01], in the sense that some specific "moves" between graph claspers are shown to produce by surgery homeomorphic 3-manifolds. This calculus can be regarded as a topological/embedded version of the commutator calculus in groups [Hab00b]. A clasper surgery $M \leadsto M_G$ (along a graph clasper G with at least one node) is in fact equivalent to a Torelli surgery $M \leadsto M_{(S,s)}$, and the definition of a finite-type invariant can be reformulated in terms of clasper surgeries. Similarly, for any $k \geq 1$, the Y_k-equivalence relation is generated by surgeries along graph claspers with k nodes [Hab00b].[4] It turns out that the calculus of claspers is a very efficient tool to compute finite-type invariants and, at the same time, to study the Y_k-equivalence relations.

Some other surgery equivalence relations can be defined on $\mathcal{V}(R)$ if one replaces the lower central series of the Torelli group by another filtration, which we now define. Let Σ be a compact connected oriented surface with one boundary component. The Torelli group of Σ acts on $\pi := \pi_1(\Sigma, *)$ in the natural way (where $* \in \partial\Sigma$), and the resulting homomorphism

$$\rho : \mathcal{I}(\Sigma) \longrightarrow \text{Aut}(\pi), \ s \longmapsto s_* \tag{1.9}$$

is injective by classical results of M. Dehn and J. Nielsen: we call it the *Dehn–Nielsen representation*. For any integer $k \geq 1$, this representation induces a group homomorphism

$$\rho_k : \mathcal{I}(\Sigma) \longrightarrow \text{Aut}(\pi/\Gamma_{k+1}\pi), \ s \longmapsto (s_* \bmod \Gamma_{k+1}\pi). \tag{1.10}$$

The *Johnson filtration* of the Torelli group is the decreasing sequence of subgroups

$$\mathcal{I}(\Sigma) = \mathcal{I}(\Sigma)[1] \supset \mathcal{I}(\Sigma)[2] \supset \mathcal{I}(\Sigma)[3] \supset \cdots \tag{1.11}$$

[3] The Y_k-equivalence relation is called "$(k-1)$-equivalence" in [Gou99] and "A_k-equivalence" in [Hab00b].

[4] See the appendix of [Mas07] for a proof.

where $\mathcal{I}(\Sigma)[k]$ denotes the kernel of ρ_k for any $k \geq 1$. Then, two 3-manifolds $M, M' \in \mathcal{V}(R)$ are said to be J_k*-equivalent* if M' can be obtained from M by a Torelli surgery $M \rightsquigarrow M_{(S,s)} \cong M'$ along a surface $S \subset \mathrm{int}(M)$ with an $s \in \mathcal{I}(S)[k]$. In the case $R = \varnothing$, for instance, a result of T. Cochran, A. Gerges & K. Orr shows that two closed oriented 3-manifolds are J_2-equivalent if and only if they have isomorphic triplets (homology, linking pairing, cohomology rings) [CGO01]. The lower central series being contained in the Johnson filtration, the equivalence relations on $\mathcal{V}(R)$ are organized as follows:

$$\begin{array}{ccccccccccc} Y_1 & \Longleftarrow & Y_2 & \Longleftarrow & Y_3 & \Longleftarrow & \cdots & Y_k & \Longleftarrow & Y_{k+1} & \Longleftarrow & \cdots \\ \| & & \Downarrow & & \Downarrow & & & \Downarrow & & \Downarrow & & \\ J_1 & \Longleftarrow & J_2 & \Longleftarrow & J_3 & \Longleftarrow & \cdots & J_k & \Longleftarrow & J_{k+1} & \Longleftarrow & \cdots \end{array}$$

1.5 Homology cylinders

A special class of 3-manifolds plays a central role in the Goussarov–Habiro theory of finite-type invariants: this is the monoid of homology cylinders, which can be regarded as a simultaneous generalization of the monoid of $\mathbb{Z}$-homology spheres (with the connected sum operation) and the Torelli group. One can also think of the monoid of homology cylinders as an analogue of the monoid of string-links: the latter contains the pure braid group, while the former contains the Torelli group.

Let Σ be a compact connected oriented surface with one boundary component. A *homology cylinder* over Σ (or a $\mathbb{Z}$*-homology cylinder* over Σ, to be exact) is a compact oriented 3-manifold M with boundary parameterization $m : \partial(\Sigma \times [-1,1]) \to \partial M$ satisfying

$$\begin{tikzcd} H_*(\Sigma \times [-1,1];\mathbb{Z}) \arrow[r, dashed, "\exists", "\simeq"'] & H_*(M;\mathbb{Z}) \\ H_*(\partial(\Sigma \times [-1,1]);\mathbb{Z}) \arrow[u, "\mathrm{incl}_*"] \arrow[r, "\simeq", "m_*"'] & H_*(\partial M;\mathbb{Z}). \arrow[u, "\mathrm{incl}_*"'] \end{tikzcd}$$

In other words, M is a cobordism (with corners) between two copies of the surface Σ, namely $\partial_+ M := m(\Sigma \times \{+1\})$ and $\partial_- M := m(\Sigma \times \{-1\})$, which has the $\mathbb{Z}$-homology type of the usual cylinder $\Sigma \times [-1,1]$. The set of homeomorphism classes of homology cylinders is denoted by $\mathcal{IC}(\Sigma)$ and is a subset of $\mathcal{V}(R)$ with $R := \partial(\Sigma \times [-1,1])$: this is actually the Y_1-equivalence class of $\Sigma \times [-1,1]$ [Hab00b, Hab00a]. The *composition* of two homology cylinders M and M' is defined by "stacking" M' on the top of M, i.e. we define

$$M \circ M' := M \cup_{m_+ \circ (m'_-)^{-1}} M'$$

where we denote $m_\pm := m|_{\Sigma \times \{\pm 1\}} : \Sigma \to \partial_\pm M$, $m'_\pm := m'|_{\Sigma \times \{\pm 1\}} : \Sigma \to \partial_\pm M'$ and $\partial(M \circ M')$ has the obvious parameterization. The set $\mathcal{IC}(\Sigma)$ equipped with the operation $\circ$ is a monoid, whose unit element is $\Sigma \times [-1,1]$ (with the obvious boundary parameterization). For instance, $\mathcal{IC}(\Sigma)$ is isomorphic to the monoid of $\mathbb{Z}$-homology spheres when $\Sigma \cong D^2$.

In positive genus, the *mapping cylinder* construction

$$\mathbf{c} : \mathcal{I}(\Sigma) \longrightarrow \mathcal{IC}(\Sigma), \ s \longmapsto \big(\Sigma \times [-1,1], (\mathrm{Id} \times \{-1\}) \cup (\partial\Sigma \times \mathrm{Id}) \cup (s \times \{+1\})\big) \tag{1.12}$$

defines an embedding of the Torelli group into the monoid of homology cylinders; the image consists of the invertible elements of $\mathcal{IC}(\Sigma)$. This gives 3-dimensional perspectives to the study of the Torelli group. For example, the computation of the abelianization of $\mathcal{I}(\Sigma)$ due to D. Johnson [Joh85] extends to a characterization of the Y_2-equivalence on $\mathcal{IC}(\Sigma)$ [Hab00b, MM03]: in some sense, the Dehn twists computations in D. Johnson's work are replaced by calculus of claspers. The reader may consult the survey [HM12] for an overview of this 3-dimensional approach of the Torelli group.

Another reason to be interested in the monoid $\mathcal{IC}(\Sigma)$ is that it constitutes a class of 3-manifolds with non-trivial $\mathbb{Q}$-homology (provided $\Sigma \not\cong D^2$). Indeed, the theory of finite-type invariants is rather well understood for $\mathbb{Q}$-homology spheres thanks to the existence of the LMO invariant and its universal property. But this is not quite the case for homologically non-trivial 3-manifolds, so that the monoid of homology cylinders should be a "case study" for this theory. Soon after the works of M. Goussarov and K. Habiro, the study of homology cylinders by finite-type invariants has been followed up by S. Garoufalidis & J. Levine [GL05], J. Levine [Lev01], N. Habegger [Hab00a] and others.

1.6 Contents of the dissertation

This dissertation is an exposition of some works of the author in the theory of finite-type invariants for 3-manifolds. Most of these works deal with the monoid of homology cylinders, in relation with the study of the Torelli group of a surface. Some of the results presented below have been obtained in collaboration with **D. Cheptea**, **K. Habiro** and **J.–B. Meilhan**.

In section 2, we present a functorial extension of the LMO invariant to compact oriented 3-manifolds with boundary [CHM08]. The "LMO functor" is defined on the category of "Lagrangian cobordisms" (which contains the monoid of homology cylinders) and it takes values in a certain category of "Jacobi diagrams." This invariant is universal among $\mathbb{Q}$-valued finite-type invariants of Lagrangian cobordisms. We give one application of the LMO functor: the LMO invariant of $\mathbb{Q}$-homology spheres satisfies C. Lescop's "splitting formulas" [Mas12c].

Section 3 specializes to the case of homology cylinders over a surface Σ. First, we consider a graded Lie algebra which is the analogue for the monoid $\mathcal{IC}(\Sigma)$ of the graded Lie algebra associated to the lower central series of a group. Using the LMO functor, we give a diagrammatic description of the graded Lie algebra associated to $\mathcal{IC}(\Sigma)$. We relate through the map $\mathbf{c} : \mathcal{I}(\Sigma) \to \mathcal{IC}(\Sigma)$ this description to R. Hain's presentation of the graded Lie algebra associated to the group $\mathcal{I}(\Sigma)$ [HM09]. Next, we consider the Goussarov–Habiro conjecture which claims that the converse of (1.6) should be true for homology cylinders: we relate this to the dimension subgroup problem in group theory and obtain some weakened versions of the conjecture [Mas07]. Finally, we classify with a few classical invariants the Y_3-equivalence and the J_3-equivalence for homology cylinders and, as an extension of S. Morita's work for $\mathcal{I}(\Sigma)$, we analyse the incidence of the Casson invariant on the structure of $\mathcal{IC}(\Sigma)$ [MM10].

The works of D. Johnson and S. Morita on the Torelli group used in a crucial way the Dehn–Nielsen representation (1.9). In particular, they used this representation to define a sequence of homomorphisms relative to the Johnson filtration (1.11). Section 4 considers certain generalizations of these homomorphisms in relation with the theory of finite-type invariants. We show that, on the monoid $\mathcal{IC}(\Sigma)$, the "tree-reduction" of the LMO functor is equivalent to an "infinitesimal" version of the Dehn–Nielsen representation [Mas12b]. This topological interpretation of a reduction of the LMO functor needs to consider the Malcev Lie algebra of $\pi = \pi_1(\Sigma, *)$ instead of the group itself. Using the same "infinitesimal" approach, we explain how D. Johnson's and S. Morita's homomorphisms can be extended in a canonical way to the Ptolemy groupoid of Σ [Mas12a].

Section 5 concludes this dissertation with a few research directions. The questions, problems and perspectives that we propose here are continuations of the previous works. For a more comprehensive list of open problems in the theory of finite-type invariants, the reader is referred to [Oht02a, §§10–11].

2 A functorial extension of the LMO invariant

2.1 The LMO functor [CHM08]

The LMO invariant has been originally defined for closed oriented 3-manifolds in [LMO98]. Subsequently, it has been extended to 3-dimensional cobordisms in two different ways: by Murakami & Ohtsuki [MO97] and, later, by Cheptea & Le [CL07]. The approach of [CHM08] differs from these works in that it applies to a category of cobordisms between surfaces with one *boundary* component. This avoids the extension of the Kontsevich integral to trivalent graphs in S^3 and allows for monoidal structures. Besides, the combinatorics of the gluing formula given in [CHM08] is easy to describe.

We start by defining the source of the LMO functor and, for this, we need the category $\mathcal{C}ob$ of 3-dimensional cobordisms introduced in [CY99, Ker03]. By definition, an object of $\mathcal{C}ob$ is an integer $g \geq 0$, which one thinks of as the genus of a compact connected oriented surface $\Sigma_{g,1}$ with one boundary component. The surface[5] $\Sigma_{g,1}$ is *fixed* once for all, see Figure 2.1. For any integers $g_+ \geq 0$ and $g_- \geq 0$, a morphism $g_+ \to g_-$ in the category $\mathcal{C}ob$ is a cobordism (M, m) from the surface $\Sigma_{g_+,1}$ to the surface $\Sigma_{g_-,1}$: more precisely, M is a compact connected oriented 3-manifold together with a boundary parameterization $m : \partial C_{g_-}^{g_+} \to \partial M$. Here $C_{g_-}^{g_+} \subset S^3$ is the cube with g_- "tunnels" and g_+ "handles", whose oriented boundary contains one copy of $\Sigma_{g_+,1}$ and one copy of $-\Sigma_{g_-,1}$. The manifold $C_{g_-}^{g_+}$ is also fixed once for all, see Figure 2.2. The boundary parameterization m of M restricts to two embeddings $m_- : \Sigma_{g_-,1} \to M$ and $m_+ : \Sigma_{g_+,1} \to M$, whose images are denoted by $\partial_- M$ and $\partial_+ M$ respectively. The composition $\circ$ in $\mathcal{C}ob$ is given by "vertical" gluing of cobordisms, while the "horizontal" gluing of cobordisms gives a strict monoidal structure $\otimes$.

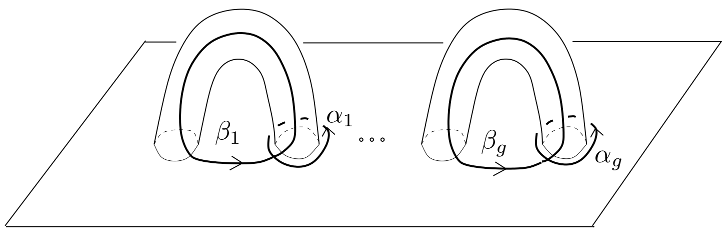


Figure 2.1: The standard surface $\Sigma_{g,1}$ and its system of meridians and parallels (α, β).

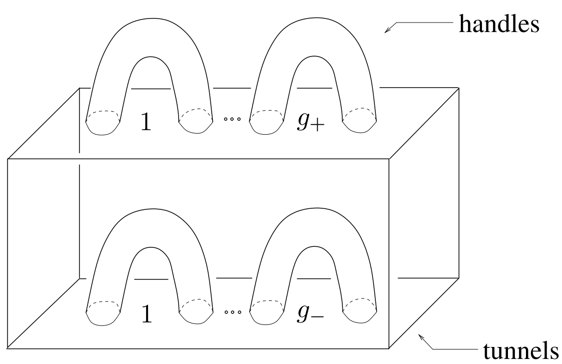


Figure 2.2: The standard cube $C_{g_-}^{g_+}$ with g_- tunnels and g_+ handles.

Unfortunately, the LMO functor of [CHM08] is not defined on the full category $\mathcal{C}ob$, but only on the subcategory $\mathcal{LC}ob$ of *Lagrangian cobordisms* (or $\mathbb{Z}$*-Lagrangian cobordisms*, to be exact). For any

[5] This standard surface is denoted by F_g in [CHM08].

integer $g \geq 0$, let A_g be the subgroup of $H_1(\Sigma_{g,1};\mathbb{Z})$ spanned by the meridians $\alpha_1, \dots, \alpha_g$: this is a Lagrangian subgroup with respect to the homology intersection form. By definition, a cobordism $(M, m) \in \mathcal{C}ob(g_+, g_-)$ belongs to $\mathcal{LC}ob(g_+, g_-)$ if and only if

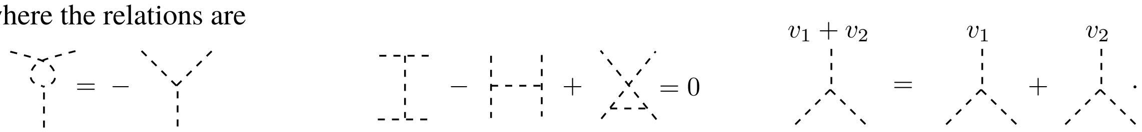
1. $H_1(M;\mathbb{Z}) = m_{-,*}(A_{g_-}) + m_{+,*}\big(H_1(\Sigma_{g_+,1};\mathbb{Z})\big)$,
2. $m_{+,*}(A_{g_+}) \subset m_{-,*}(A_{g_-})$ as subgroups of $H_1(M;\mathbb{Z})$.

For instance, morphisms $0 \to 0$ in the category $\mathcal{LC}ob$ are precisely $\mathbb{Z}$-homology cubes, i.e. $\mathbb{Z}$-homology spheres after gluing of a 3-dimensional ball D^3. A similar category of cobordisms between closed surfaces appears in [CL07]. The source of the LMO functor is not exactly $\mathcal{LC}ob$ but the category $\mathcal{LC}ob_q$ of *Lagrangian q-cobordisms*. An object of $\mathcal{LC}ob_q$ is a (possibly empty) non-associative word in the single letter $\bullet$. For any two such words w_+ and w_-, a morphism $w_+ \to w_-$ in the category $\mathcal{LC}ob_q$ is a Lagrangian cobordism $g_+ \to g_-$ where $g_\pm$ is the length of $w_\pm$. Note that the category $\mathcal{LC}ob_q$ is monoidal in the non-strict sense.

We now describe the target of the LMO functor. For any $\mathbb{Q}$-vector space V, the space of *Jacobi diagrams colored* by V is

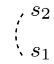
$$\mathcal{A}(V) := \frac{\mathbb{Q} \cdot \left\{ \begin{array}{c} \text{finite uni-trivalent graphs whose trivalent vertices are oriented} \\ \text{and whose univalent vertices are colored by } V \end{array} \right\}}{\text{AS, IHX, multilinearity}} \tag{2.1}$$

where the relations are



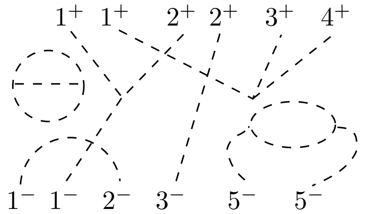
AS IHX multinearity

For any finite set C, we set $\mathcal{A}(C) := \mathcal{A}(\mathbb{Q}\cdot C)$ where $\mathbb{Q}\cdot C$ is the vector space generated by C. In particular, $\mathcal{A}(\varnothing)$ is spanned by trivalent Jacobi diagrams and, as mentionned in §1.2, this is the target of the original LMO invariant. The *degree* of a Jacobi diagram is half the total number of vertices, and its *internal degree* (or, in short, *i-degree*) is the number of trivalent vertices. We shall also need the degree completion of $\mathcal{A}(C)$, which we still denote by $\mathcal{A}(C)$. If S is another finite set, a Jacobi diagram $D \in \mathcal{A}(C \cup S)$ is *S-substantial* if it has no connected component of the form



where $s_1, s_2 \in S$.

The category of *top-substantial Jacobi diagrams* is the linear category ${}^{ts}\!\mathcal{A}$ whose objects are integers $g \geq 0$ and whose space of morphisms ${}^{ts}\!\mathcal{A}(g, f)$ is, for any integers $g \geq 0$ and $f \geq 0$, the subspace of $\mathcal{A}(\lfloor g\rceil^+ \cup \lfloor f\rceil^-)$ spanned by $\lfloor g\rceil^+$-substantial Jacobi diagrams. Here $\lfloor g\rceil^+$ denotes the g-element finite set $\{1^+, \dots, g^+\}$ while $\lfloor f\rceil^-$ denotes the f-element finite set $\{1^-, \dots, f^-\}$. For example, here is a Jacobi diagram defining a morphism $4 \to 5$ in ${}^{ts}\!\mathcal{A}$:



For any integers $f, g, h \geq 0$, the composition law of ${}^{ts}\!\mathcal{A}$

$${}^{ts}\!\mathcal{A}(g, f) \times {}^{ts}\!\mathcal{A}(h, g) \xrightarrow{\quad -\circ- \quad} {}^{ts}\!\mathcal{A}(h, f) \tag{2.2}$$

is defined for any Jacobi diagrams $D \in {}^{ts}\!\mathcal{A}(g,f)$ and $E \in {}^{ts}\!\mathcal{A}(h,g)$ by

$$D \circ E := \left(\begin{array}{c} \text{sum of all ways of gluing } \textit{all} \text{ the } i^+\text{-colored vertices of } D \\ \text{to } \textit{all} \text{ the } i^-\text{-colored vertices of } E, \text{ for every } i \in \{1,\dots,g\} \end{array} \right).$$

With the disjoint union of Jacobi diagrams (and appropriate "shift" of colors), the category ${}^{ts}\!\mathcal{A}$ is monoidal in the strict sense.

Theorem 2.1. [CHM08, Th. 4.13] *There is a tensor-preserving functor*

$$\widetilde{Z} : \mathcal{LC}ob_q \longrightarrow {}^{ts}\!\mathcal{A}$$

whose restriction to $\mathcal{LC}ob_q(\varnothing,\varnothing)$ is the LMO invariant Z of $\mathbb{Z}$-homology spheres.

We call $\widetilde{Z}$ the *LMO functor*. Given some non-associative words v and w of length f and g respectively, the value of $\widetilde{Z}$ on a q-cobordism $M \in \mathcal{LC}ob_q(w,v)$ is constructed by presenting M as tangle γ in a $\mathbb{Z}$-homology cube B; this tangle has g "top" components and f "bottom" components. Next, we consider the Kontsevich–LMO invariant $Z(B,\gamma)$ of the pair (B,γ) and, finally, $\widetilde{Z}(M)$ is defined by normalizing $Z(B,\gamma)$ in an appropriate way. Note that, in this construction of $\widetilde{Z}(M)$, the colors $1^+,\dots,g^+$ refer to the curves $\beta_1,\dots,\beta_g$ (in the "top surface" $\partial_+ M$ of the cobordism M) while the colors $1^-,\dots,g^-$ refer to the curves $\alpha_1,\dots,\alpha_g$ (in the "bottom surface" $\partial_- M$ of the cobordism M). It should be emphasized that the definition of the LMO functor requires two preliminary choices:

1. a rational Drinfel'd associator must be specified (since the Kontsevich integral of tangles is used in the construction);
2. for any $g \geq 0$, a system of meridians and parallels (α,β) is fixed on $\Sigma_{g,1}$ (see Figure 2.1).

A certain reduction of the functor $\widetilde{Z}$ factorizes to the category of Lagrangian cobordisms between *closed* surfaces, and this reduction of $\widetilde{Z}$ recovers the functor defined in [CL07] by Cheptea & Le [CHM08, §6]. The previous constructions also work for the category $\mathbb{Q}\mathcal{LC}ob$ of $\mathbb{Q}$-*Lagrangian cobordisms*, which is defined in the same way as $\mathcal{LC}ob$ by replacing $\mathbb{Z}$-homology with $\mathbb{Q}$-homology. Theorem 2.1 is also valid in this context.

Corollary 2.1. [CHM08, Prop. 8.24] *The i-degree ≤ 2 truncation of $\widetilde{Z} : \mathbb{Q}\mathcal{LC}ob_q \to {}^{ts}\!\mathcal{A}$ is a functorial extension of the Casson–Walker invariant of $\mathbb{Q}$-homology spheres to the category $\mathbb{Q}\mathcal{LC}ob$.*

This is mainly a consequence of (1.3). The i-degree ≤ 2 truncation of $\widetilde{Z}$ is computed in [CHM08, §5.3] on some explicit generators of the monoidal category $\mathcal{LC}ob$ (for the choice of an even Drinfel'd associator).

Just as the LMO invariant of $\mathbb{Z}$-homology spheres [Le97], the LMO functor is universal among $\mathbb{Q}$-valued finite-type invariants of Lagrangian cobordisms. In order to state this universality, we fix a Y_1-equivalence class $\mathcal{Y} \subset \mathcal{LC}ob(g,f)$ of Lagrangian cobordisms, which is obtained by fixing a $\mathbb{Z}$-homology type of Lagrangian cobordisms [CHM08, Cor. 7.7]. We denote by

$$\mathcal{A}^Y(\lfloor g \rceil^+ \cup \lfloor f \rceil^-)$$

the subspace of ${}^{ts}\!\mathcal{A}(g,f) \subset \mathcal{A}(\lfloor g \rceil^+ \cup \lfloor f \rceil^-)$ spanned by Jacobi diagrams with no "strut", i.e. diagrams without component of the form  . We also fix some non-associative words v and w in the single letter $\bullet$ of length f and g respectively. Thus, any cobordism $M \in \mathcal{LC}ob(g,f)$ is promoted to a q-cobordism $M \in \mathcal{LC}ob_q(w,v)$. It turns out that the series of Jacobi diagrams $\widetilde{Z}(M)$ is the disjoint union of two parts: one part only contains struts and encodes the choice of the Y_1-equivalence class $\mathcal{Y}$, while the other part is contained in $\mathcal{A}^Y(\lfloor g \rceil^+ \cup \lfloor f \rceil^-)$ and is denoted by $\widetilde{Z}^Y(M)$.

Theorem 2.2. [CHM08, Th. 7.11] *The linear map*

$$\mathbb{Q}\cdot\mathcal{Y} \longrightarrow \mathcal{A}^Y(\lfloor g \rceil^+ \cup \lfloor f \rceil^-), \quad M \longmapsto \widetilde{Z}^Y(M)$$

sends the $\mathcal{F}$-filtration to the i-degree filtration, and it induces an isomorphism at the graded level:

$$\operatorname{Gr} \widetilde{Z}^Y : \operatorname{Gr}^{\mathcal{F}} \mathbb{Q}\cdot\mathcal{Y} := \bigoplus_{d\geq 0} \mathcal{F}_d(\mathcal{Y})/\mathcal{F}_{d+1}(\mathcal{Y}) \xrightarrow{\simeq} \mathcal{A}^Y(\lfloor g \rceil^+ \cup \lfloor f \rceil^-)$$

Here $\mathcal{F}$ is the filtration (1.7) with coefficients in $\mathbb{Q}$ dual to finite-type invariants. The inverse of $\operatorname{Gr} \widetilde{Z}^Y$ is defined by clasper surgery in a way similar to [Gar02]. In this sense, graph claspers are "topological realizations" of Jacobi diagrams.

The monoid of homology cylinders[6] $\mathcal{IC}(\Sigma_{g,1})$ is a submonoid of $\mathcal{LC}ob(g,g)$, so that the LMO functor can be applied to this class of 3-manifolds. The results of [CHM08] in this direction complete previous works of Garoufalidis & Levine [GL05] and Habegger [Hab00a]. The composition (2.2) of the category ${}^{ts}\!\mathcal{A}$ leads to the following associative multiplication on $\mathcal{A}^Y(\lfloor g \rceil^+ \cup \lfloor g \rceil^-)$:

$$D \star E := \left(\begin{array}{c} \text{sum of all ways of gluing } \textit{some} \text{ of the } i^+\text{-colored vertices of } D \\ \text{to } \textit{some} \text{ of the } i^-\text{-colored vertices of } E, \text{ for every } i \in \{1,\dots,g\} \end{array} \right). \tag{2.3}$$

This diagrammatic multiplication has been discovered in [GL05] from calculus of claspers.

Corollary 2.2. [CHM08, Cor. 8.3 & Cor. 8.6] *The LMO functor induces a monoid homomorphism*

$$\widetilde{Z}^Y : (\mathcal{IC}(\Sigma_{g,1}), \circ) \longrightarrow \big(\mathcal{A}^Y(\lfloor g \rceil^+ \cup \lfloor g \rceil^-), \star\big)$$

which is universal among $\mathbb{Q}$-valued finite-type invariants.

Habegger had already defined in [Hab00a] a universal finite-type invariant of homology cylinders which was deduced in another way from the Kontsevich–LMO invariant [Hab00a], but he did not address the multiplicativity issue which seems to be difficult in his case. Nevertheless, the method used by Habegger to recover Johnson's homomorphisms from the tree-level of his invariant (the so-called "Milnor–Johnson correspondence") works as well for our invariant $\widetilde{Z}^Y$ [CHM08, §8.5]: thus, for any $M \in \mathcal{IC}(\Sigma_{g,1})$, the leading term of the "tree-reduction" of $\widetilde{Z}^Y(M)$ is essentially[7] the first non-vanishing Johnson homomorphism of M. Finally, let us mention that another universal finite-type invariant of homology cylinders has been constructed by Andersen, Bene, Meilhan & Penner [ABMP10] using an extension of the Kontsevich integral [AMR98] to framed links in $\Sigma_{g,1} \times [-1,1]$.

The topic of homology cylinders from the viewpoint of finite-type invariants is the subject of §3.

2.2 Splitting formulas for the LMO invariant [Mas12c]

For $\mathbb{Q}$-homology spheres, there are two universal finite-type invariants: the Le–Murakami–Ohtsuki invariant [LMO98] and the Kontsevich–Kuperberg–Thurston invariant [Kon94, KT99]. These invariants take values in the same space $\mathcal{A}(\varnothing)$ of trivalent Jacobi diagrams, but it is not known whether they are equal. Lescop obtained in [Les04] some relations satisfied by the variations of the KKT invariant when one replaces embedded $\mathbb{Q}$-homology handlebodies by others in a "Lagrangian-preserving" way. It is shown in [Mas12c] that the LMO invariant satisfies exactly the same relations.

[6] This monoid is denoted by $\mathcal{C}yl(F_g)$ in [CHM08].

[7] We do not give a precise statement here, since we shall give in §4.1 a topological interpretation of the *full* tree-reduction of the LMO functor.

We first recall how Lescop's "Lagrangian-preserving" surgeries are defined. A $\mathbb{Q}$-*homology handlebody* of *genus* g is a compact oriented 3-manifold C' that has the same $\mathbb{Q}$-homology as the usual genus g handlebody. The *Lagrangian* of C' is the kernel $L^{\mathbb{Q}}_{C'}$ of the homomorphism $H_1(\partial C';\mathbb{Q}) \to H_1(C';\mathbb{Q})$ induced by the inclusion $\partial C' \subset C'$: this is a Lagrangian subspace of $H_1(\partial C';\mathbb{Q})$ for the intersection form. A $\mathbb{Q}$-*Lagrangian-preserving pair* (or, in short, $\mathbb{Q}$-*LP pair*) is a couple $\mathsf{C} = (C', C'')$ of two $\mathbb{Q}$-homology handlebodies whose boundaries are identified $\partial C' = \partial C''$ in such a way that $L^{\mathbb{Q}}_{C'} = L^{\mathbb{Q}}_{C''}$. The *total manifold* of the $\mathbb{Q}$-LP pair C is the closed oriented 3-manifold

$$C := (-C') \cup_\partial C''.$$

The form $H^1(C;\mathbb{Q})^{\otimes 3} \to \mathbb{Q}$ defined by triple-cup products $(x,y,z) \mapsto \langle x \cup y \cup z, [C]\rangle$ is skew-symmetric: we denote it by

$$\mu(C) \in \mathrm{Hom}_{\mathbb{Q}}\left(\Lambda^3 H^1(C;\mathbb{Q}), \mathbb{Q}\right) \simeq \Lambda^3 H_1(C;\mathbb{Q}).$$

Given a closed oriented 3-manifold M and a $\mathbb{Q}$-LP pair $\mathsf{C} = (C', C'')$ such that $C' \subset M$, one can replace in M the submanifold C' by C'' to obtain a new 3-manifold:

$$M_{\mathsf{C}} := \left(M \setminus \mathrm{int}(C')\right) \cup_\partial C''.$$

The move $M \rightsquigarrow M_{\mathsf{C}}$ between closed oriented 3-manifolds is called a $\mathbb{Q}$-*LP surgery*.

The $\mathbb{Z}$-*LP surgery* is defined in a similar way by replacing $\mathbb{Q}$-homology with $\mathbb{Z}$-homology. A Torelli surgery $M \rightsquigarrow M_{(S,s)}$, as defined in §1.2, is clearly an instance of a $\mathbb{Z}$-LP surgery (since the regular neighborhood of S in M is a handlebody), and the converse is true (since any $\mathbb{Z}$-homology handlebody can be obtained from the usual handlebody of the same genus by doing clasper surgery or, equivalently, by a Torelli surgery [Hab00a]). Therefore, if one replaces Torelli surgeries by $\mathbb{Z}$-LP surgeries in §1.2, one obtains the same notion of finite-type invariant [AL05]. However, the notion differs if one uses $\mathbb{Q}$-LP surgeries instead of $\mathbb{Z}$-LP surgeries: this difference has been recently analyzed by Moussard in the case of $\mathbb{Q}$-homology spheres [Mou12]. Let us observe that, in contrast with $\mathbb{Z}$-LP surgery, $\mathbb{Q}$-LP surgery relates any two $\mathbb{Q}$-homology spheres: therefore $\mathbb{Q}$-LP surgery is more appropriate if one wants to consider $\mathbb{Q}$-homology spheres *all together*.

Suppose that we are now given a $\mathbb{Q}$-homology sphere M and r $\mathbb{Q}$-LP pairs $\mathsf{C} = (\mathsf{C}_1, \dots, \mathsf{C}_r)$ such that $C'_i \subset M$ and $C'_i \cap C'_j = \varnothing$ for all $i \neq j$. We associate to the family C the following tensor:

$$\mu(\mathsf{C}) := \mu(C_1) \otimes \cdots \otimes \mu(C_r) \ \in \bigotimes_{i=1}^{r} \Lambda^3 H_1(C_i;\mathbb{Q}) \subset S^r \Lambda^3 H_1(C;\mathbb{Q}) \tag{2.4}$$

where $C := C_1 \sqcup \cdots \sqcup C_r$ so that $H_1(C;\mathbb{Q}) = H_1(C_1;\mathbb{Q}) \oplus \cdots \oplus H_1(C_r;\mathbb{Q})$. Besides, the linking number in M defines for any $i \neq j$ a linear map

$$c_{i,j} : H_1(C'_i;\mathbb{Q}) \otimes H_1(C'_j;\mathbb{Q}) \longrightarrow \mathbb{Q}$$

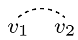
by setting $c_{i,j}([K],[L]) := \mathrm{Lk}_M(K,L)$ for any oriented knots $K \subset C'_i$ and $L \subset C'_j$. Since the space

$$\begin{aligned} \bigoplus_{i,j=1}^{r} \mathrm{Hom}_{\mathbb{Q}}\left(H_1(C'_i;\mathbb{Q}) \otimes H_1(C'_j;\mathbb{Q}), \mathbb{Q}\right) &\simeq \bigoplus_{i,j=1}^{r} \mathrm{Hom}_{\mathbb{Q}}\left(H_1(C_i;\mathbb{Q}) \otimes H_1(C_j;\mathbb{Q}), \mathbb{Q}\right) \\ &\simeq \mathrm{Hom}_{\mathbb{Q}}\left(H_1(C;\mathbb{Q}) \otimes H_1(C;\mathbb{Q}), \mathbb{Q}\right) \\ &\simeq H^1(C;\mathbb{Q}) \otimes H^1(C;\mathbb{Q}) \end{aligned}$$

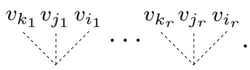
projects onto $S^2H^1(C;\mathbb{Q})$, the sum $\sum_{i\neq j} c_{i,j}$ defines a symmetric tensor which we denote by

$$\mathrm{Lk}_M(\mathsf{C}) \in S^2H^1(C;\mathbb{Q}). \tag{2.5}$$

Next, symmetric 2-tensors such as (2.5), and symmetric products of antisymmetric 3-tensors such as (2.4), can be depicted graphically using the space $\mathcal{A}(V)$ defined at (2.1) for any $\mathbb{Q}$-vector space V. Indeed any $v_1 \cdot v_2 \in S^2V$ can be presented as the Jacobi diagram



and, similarly, any $(v_{i_1} \wedge v_{j_1} \wedge v_{k_1}) \cdots (v_{i_r} \wedge v_{j_r} \wedge v_{k_r}) \in S^r\Lambda^3V$ can be seen as the Jacobi diagram



If the space $\mathcal{A}(V)$ is completed by the degree and has the disjoint union $\sqcup$ as operation, then the exponential $\exp_\sqcup(x) := \sum_{n\geq 0} x^{\sqcup n}/n! \in \mathcal{A}(V)$ is defined for any $x \in \mathcal{A}(V)$ of degree > 0. Let $\mathcal{A}^Y(V)$ be the subspace of $\mathcal{A}(V)$ spanned by diagrams without strut component . The "contraction" pairing

$$\mathcal{A}^Y(V) \otimes \mathcal{A}(V^*) \xrightarrow{\langle -,- \rangle_V} \mathcal{A}(\varnothing) \tag{2.6}$$

assigns to any Jacobi diagrams $D \in \mathcal{A}^Y(V)$ and $E \in \mathcal{A}(V^*)$ the sum of all ways of gluing *all* univalent vertices of D to *all* univalent vertices of E by means of the evaluation pairing $V \otimes V^* \to \mathbb{Q}$.

Theorem 2.3. [Mas12c] *Let M be a $\mathbb{Q}$-homology sphere and let $\mathsf{C} = (\mathsf{C}_1, \dots, \mathsf{C}_r)$ be a family of r $\mathbb{Q}$-LP pairs such that $C_i' \subset M$ and $C_i' \cap C_j' = \varnothing$ for all $i \neq j$. For any $I \subset \{1, \dots, r\}$, let M_I be the 3-manifold obtained from M by simultaneous $\mathbb{Q}$-LP surgeries along those C_i' that are indexed by $i \in I$. Then, the sum*

$$\sum_{I \subset \{1,\dots,r\}} (-1)^{|I|} \cdot Z(M_I) \in \mathcal{A}(\varnothing) \tag{2.7}$$

starts in i-degree r with $\left\langle \mu(\mathsf{C}), \exp_\sqcup\left(\mathrm{Lk}_M(\mathsf{C})/2\right) \right\rangle_{H_1(C;\mathbb{Q})}$.

Theorem 2.3 has been announced in [CHM08, Rem. 7.12] and it is proved in [Mas12c] by means of the LMO functor. The starting point for the proof is that $\mathbb{Q}$-homology handlebodies are $\mathbb{Q}$-Lagrangian cobordisms for appropriate parameterizations of their boundaries: therefore, if a $\mathbb{Q}$-homology sphere M is decomposed as the union of two $\mathbb{Q}$-homology handlebodies, the LMO functor can be applied to such a decomposition in order to compute $Z(M)$. Note that the "contraction" pairing (2.6) is a special case of the composition (2.2) in the category ${}^{ts}\!\mathcal{A}$.

Theorem 2.3 generalizes the fact that the LMO invariant of $\mathbb{Q}$-homology spheres is universal among $\mathbb{Q}$-valued finite-type invariants [Le97, Hab00b]. It is the analogue of Lescop's result for the Kontsevich–Kuperberg–Thurston invariant Z^{KKT}: see [Les04] and [Les09, §3]. Lescop's result for Z^{KKT} generalizes her "sum formula" for the Casson–Walker invariant [Les98]. Indeed, according to [Les04] and [LMO98], we have

$$\text{i-degree 2 part of } Z^{\mathrm{KKT}}(M) = \frac{\lambda_{\mathrm{W}}(M)}{4} \cdot \ominus = \text{i-degree 2 part of } Z(M)$$

where $\lambda_{\mathrm{W}}(M)$ denotes Walker's extension of the Casson invariant as normalized in [Wal92].

3 Finite-type invariants of homology cylinders

3.1 The Lie algebra of homology cylinders [HM09]

Let $\Sigma := \Sigma_{g,1}$ be a compact connected oriented surface of genus g with one boundary component. The mapping cylinder construction $\mathbf{c} : \mathcal{I}(\Sigma) \to \mathcal{IC}(\Sigma)$ defined at (1.12) is an embedding of the Torelli group into the monoid of homology cylinders[8]. The main result of [HM09] is a diagrammatic description of the map $\mathbf{c}$ at the level of graded Lie algebras, which we now define.

On the side of the Torelli group, the lower central series plays a very important role in the works of Johnson and Morita. This series is contained in the Johnson filtration (1.11), which has a trivial intersection since the group $\pi := \pi_1(\Sigma, *)$ is free. Therefore the lower central series of $\mathcal{I}(\Sigma)$ has a trivial intersection too, and it is natural to consider the associated graded Lie ring

$$\mathrm{Gr}^{\Gamma}\, \mathcal{I}(\Sigma) := \bigoplus_{i\geq 1} \frac{\Gamma_i \mathcal{I}(\Sigma)}{\Gamma_{i+1}\mathcal{I}(\Sigma)}.$$

Assuming that $g \geq 3$, Johnson proves in [Joh80] that the homomorphism $\rho_2 : \mathcal{I}(\Sigma) \to \mathrm{Aut}(\pi/\Gamma_3\pi)$ (defined at (1.10)) induces an isomorphism between the abelianization of $\mathcal{I}(\Sigma)$ with rational coefficients and the third exterior power $\Lambda^3 H_{\mathbb{Q}}$ of $H_{\mathbb{Q}} := H_1(\Sigma; \mathbb{Q})$. Therefore there is a graded Lie algebra epimorphism

$$J : \mathfrak{L}\left(\Lambda^3 H_{\mathbb{Q}}\right) \longrightarrow \mathrm{Gr}^{\Gamma}\, \mathcal{I}(\Sigma) \otimes \mathbb{Q}$$

where $\mathfrak{L}\left(\Lambda^3 H_{\mathbb{Q}}\right)$ denotes the Lie algebra freely generated by the vector space $\Lambda^3 H_{\mathbb{Q}}$ in degree 1. According to Hain [Hai97], the ideal of relations $\mathrm{R}(\mathcal{I}(\Sigma)) := \mathrm{Ker}\ J$ is generated by its degree 2 and degree 3 parts and, in genus $g \geq 6$, the degree 2 part is enough. Thus the map J induces a quadratic/cubic presentation of the graded Lie algebra $\mathrm{Gr}^{\Gamma}\, \mathcal{I}(\Sigma) \otimes \mathbb{Q}$.

On the side of homology cylinders, there is a filtration on $\mathcal{IC}(\Sigma)$ which plays a role similar to the lower central series of a group. This is the *Y-filtration*

$$\mathcal{IC}(\Sigma) = Y_1\mathcal{IC}(\Sigma) \supset Y_2\mathcal{IC}(\Sigma) \supset Y_3\mathcal{IC}(\Sigma) \supset \cdots \tag{3.1}$$

where, for any integer $k \geq 1$, $Y_k\mathcal{IC}(\Sigma)$ is the submonoid of homology cylinders that are Y_k-equivalent to $\Sigma \times [-1, 1]$. By results of Goussarov [Gou00] and Habiro [Hab00b],

$$\mathrm{Gr}^{Y}\, \mathcal{IC}(\Sigma) := \bigoplus_{i\geq 1} \frac{Y_i\mathcal{IC}(\Sigma)}{Y_{i+1}}$$

has the structure of a graded Lie ring which is similar to the graded Lie ring associated to an "N-series" of a group [Laz54]. The *Lie algebra of homology cylinders* $\mathrm{Gr}^{Y}\, \mathcal{IC}(\Sigma) \otimes \mathbb{Q}$ has been introduced by Habiro [Hab00b]. Let $\mathcal{A}^{Y,c}(\lfloor g\rceil^+ \cup \lfloor g\rceil^-)$ be the subspace of $\mathcal{A}^{Y}(\lfloor g\rceil^+ \cup \lfloor g\rceil^-)$ spanned by connected Jacobi diagrams, and equip it with the Lie bracket $[-,-]_\star$ induced by the associative multiplication $\star$ defined at (2.3). Corollary 2.2 is equivalent to saying that the map

$$\mathrm{Gr}\ \widetilde{Z}^{Y} : \mathrm{Gr}^{Y}\, \mathcal{IC}(\Sigma) \otimes \mathbb{Q} \longrightarrow \mathcal{A}^{Y,c}(\lfloor g\rceil^+ \cup \lfloor g\rceil^-), \quad \{M\}_{Y_{k+1}} \otimes 1 \longmapsto \widetilde{Z}^{Y}_k(M) \tag{3.2}$$

is a Lie algebra isomorphism, where $\{M\}_{Y_{k+1}}$ denotes the Y_{k+1}-equivalence class of an $M \in Y_k\mathcal{IC}(\Sigma)$ and $\widetilde{Z}^{Y}_k(M)$ denotes the i-degree k part of $\widetilde{Z}^{Y}(M)$. An equivalent diagrammatic description of the Lie algebra $\mathrm{Gr}^{Y}\, \mathcal{IC}(\Sigma) \otimes \mathbb{Q}$ was announced in [Hab00b].

[8]The group $\mathcal{I}(\Sigma)$ is denoted by $\mathcal{I}_{g,1}$ in [HM09] and the monoid $\mathcal{IC}(\Sigma)$ is denoted by $\mathcal{C}_{g,1}$.

Clearly the mapping cylinder construction (1.12) sends the lower central series of $\mathcal{I}(\Sigma)$ to the Y-filtration of $\mathcal{IC}(\Sigma)$, hence a Lie ring homomorphism

$$\mathrm{Gr}\ \mathbf{c} : \mathrm{Gr}^{\Gamma}\, \mathcal{I}(\Sigma) \longrightarrow \mathrm{Gr}^{Y}\, \mathcal{IC}(\Sigma).$$

The interactions between the Johnson–Morita theory and the theory of finite-type invariants are mainly contained in this homomorphism. In order to relate Hain's presentation of $\mathrm{Gr}^{\Gamma}\, \mathcal{I}(\Sigma) \otimes \mathbb{Q}$ to the diagrammatic description (3.2), we need an alternative description of the algebra $\mathcal{A}^{Y}(\lfloor g \rceil^{+} \cup \lfloor g \rceil^{-})$. For this, we consider in the space $\mathcal{A}(H_{\mathbb{Q}})$, which is defined at (2.1), the subspace $\mathcal{A}^{Y}(H_{\mathbb{Q}})$ spanned by Jacobi diagrams without strut component. Given two Jacobi diagrams $D, E \in \mathcal{A}^{Y}(H_{\mathbb{Q}})$, whose sets of univalent vertices are denoted by V and W respectively, we define

$$D \star E := \sum_{\substack{V' \subset V,\ W' \subset W \\ \beta\,:\,V' \xrightarrow{\simeq} W'}} \frac{1}{2^{|V'|}} \cdot \prod_{v \in V'} \omega\big(\mathrm{col}(v), \mathrm{col}(\beta(v))\big) \cdot (D \cup_{\beta} E).$$

Here $\omega : H_{\mathbb{Q}} \times H_{\mathbb{Q}} \to \mathbb{Q}$ is the symplectic form defined by the homological intersection in Σ, the sum is taken over all ways β of identifying a part V' of V with a part W' of W and $D \cup_{\beta} E$ is obtained from the disjoint union $D \sqcup E$ by gluing each vertex $v \in V'$ with $\beta(v) \in W'$. This operation $\star$ on $\mathcal{A}^{Y}(H_{\mathbb{Q}})$ is associative and it can be regarded as a diagrammatic analogue of the Moyal–Weyl product on the symmetric algebra of a symplectic vector space. This analogy is justified by considering the "weight system" associated to a simple Lie algebra [HM09, §3.3]. We call $\big(\mathcal{A}^{Y}(H_{\mathbb{Q}}), \star\big)$ the *algebra of symplectic Jacobi diagrams*.

Lemma 3.1. [HM09, §3.1] *There is an algebra isomorphism*

$$\kappa : \left(\mathcal{A}^{Y}\left(\lfloor g \rceil^{+} \cup \lfloor g \rceil^{-}\right), \star\right) \to \left(\mathcal{A}^{Y}(H_{\mathbb{Q}}), \star\right)$$

defined by

$$\kappa(D) := (-1)^{\chi(D)} \cdot \left(\begin{array}{c} \textit{sum of all ways of } (\times 1/2)\textit{-gluing some of the } i^{-}\textit{-colored vertices of } D \\ \textit{with some of the } i^{+}\textit{-colored vertices of } D\textit{, for every } i \in \{1, \dots, g\} \end{array} \right).$$

Here, $\chi(D)$ is the Euler characteristic of a Jacobi diagram D, a "$(\times 1/2)$-gluing" means the gluing of two vertices and the multiplication of the resulting diagram by $1/2$, and the colors in $\lfloor g \rceil^{+} \cup \lfloor g \rceil^{-}$ are interpreted as colors in $H_{\mathbb{Q}}$ using the rules $j^{+} \mapsto [\beta_j]$ and $j^{-} \mapsto [\alpha_j]$ for any $j \in \{1, \dots, g\}$.

The benefit of switching from the algebra $\mathcal{A}^{Y}(\lfloor g \rceil^{+} \cup \lfloor g \rceil^{-})$ to the algebra $\mathcal{A}^{Y}(H_{\mathbb{Q}})$ is that the former implicitly referred to the system of meridians and parallels (α, β) on the surface Σ, whereas the latter only refers to the $\mathbb{Q}$-homology of Σ and its multiplication is equivariant for the natural action of the symplectic group $\mathrm{Sp}(H_{\mathbb{Q}})$.

We now define the *LMO homomorphism* as the composition

$$Z := \kappa \circ \widetilde{Z}^{Y} : \mathcal{IC}(\Sigma) \longrightarrow \mathcal{A}^{Y}(H_{\mathbb{Q}}). \tag{3.3}$$

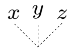
In the sequel, we abuse notation and we simply denote $\mathcal{A}^{Y}(H_{\mathbb{Q}})$ by $\mathcal{A}(H_{\mathbb{Q}})$ as we did in [HM09]. It is shown in [HM09, §4.1] by calculus of claspers that

$$\mathrm{Gr}\, Z = \kappa \circ \mathrm{Gr}\, \widetilde{Z}^{Y} : \mathrm{Gr}^{Y}\, \mathcal{IC}(\Sigma) \otimes \mathbb{Q} \longrightarrow \mathcal{A}^{c}(H_{\mathbb{Q}})$$

does not depend on the preliminary choices that are needed for the construction of the LMO functor (see page 11). Here $\mathcal{A}^{c}(H_{\mathbb{Q}})$ denotes the subspace of $\mathcal{A}(H_{\mathbb{Q}})$ spanned by connected Jacobi diagrams, and it is equipped with the Lie bracket $[-,-]_{\star}$ induced by the associative multiplication $\star$. The following is our algebraic description of $(\mathrm{Gr}\, \mathbf{c}) \otimes \mathbb{Q}$ and answers a question asked by Habiro in [Hab00b].

Theorem 3.1. [HM09, Th. 1.4] *Assume that $g \geq 3$. Then the following diagram is commutative in the category of graded Lie algebras with $\mathrm{Sp}(H_\mathbb{Q})$-actions:*

$$\begin{tikzcd}
\mathrm{Gr}^\Gamma \mathcal{I}(\Sigma) \otimes \mathbb{Q} \arrow[r, "(\mathrm{Gr}\,\mathbf{c})\otimes\mathbb{Q}"] & \mathrm{Gr}^Y \mathcal{IC}(\Sigma) \otimes \mathbb{Q} \arrow[d, "\mathrm{Gr}\, Z", "\simeq"'] \\
\frac{\mathfrak{L}(\Lambda^3 H_\mathbb{Q})}{\mathrm{R}(\mathcal{I}(\Sigma))} \arrow[u, "\overline{J}", "\simeq"'] \arrow[r, "\overline{Y}"] & \mathcal{A}^c(H_\mathbb{Q}).
\end{tikzcd}$$

Here $\overline{Y}$ is induced by the Lie algebra homomorphism $Y : \mathfrak{L}(\Lambda^3 H_\mathbb{Q}) \to \mathcal{A}^c(H_\mathbb{Q})$ which, in degree 1, sends the trivector $x \wedge y \wedge z$ to the diagram 

.

On the one hand, the quadratic relations of the Lie algebra $\mathrm{Gr}^\Gamma \mathcal{I}(\Sigma) \otimes \mathbb{Q}$ can be computed using the representation theory of the symplectic group. This has been done by Hain [Hai97] and Habegger & Sorger [HS00] who showed that (for $g \geq 3$) the $\mathrm{Sp}(H_\mathbb{Q})$-module $\mathrm{R}_2(\mathcal{I}(\Sigma))$ is spanned by the following elements r_1, r_2 of $\mathfrak{L}_2(\Lambda^3 H_\mathbb{Q})$:

$$\begin{aligned}
r_1 &:= \begin{cases} [\alpha_1 \wedge \alpha_2 \wedge \beta_2, \alpha_3 \wedge \alpha_4 \wedge \beta_4] & \text{if } g \geq 4, \\ 0 & \text{if } g = 3, \end{cases} \\
r_2 &:= [\alpha_1 \wedge \alpha_2 \wedge \beta_2, \alpha_g \wedge \omega] \qquad\qquad \text{if } g \geq 3.
\end{aligned}$$

On the other hand, we have the following description of the Lie bracket of $\mathcal{A}^c(H_\mathbb{Q})$ in degree $1+1$, which is also obtained using the representation theory of $\mathrm{Sp}(H_\mathbb{Q})$.

Proposition 3.1. [HM09, §§5.3–5.5] *Assume that $g \geq 3$. The image of $Y_2 : \mathfrak{L}_2(\Lambda^3 H_\mathbb{Q}) \to \mathcal{A}^c_2(H_\mathbb{Q})$ is the subspace spanned by the Theta graph and by the H graphs. Besides its kernel is $\mathrm{Sp}(H_\mathbb{Q})$-spanned by the elements r_1, r_2.*

Then the following is deduced from Theorem 3.1.

Corollary 3.1. [HM09, Cor. 1.6] *If $g \geq 3$, then $(\mathrm{Gr}\,\mathbf{c}) \otimes \mathbb{Q}$ is injective in degree 2.*

We also deduce that $(\mathrm{Gr}\,\mathbf{c}) \otimes \mathbb{Q}$ is not surjective since Phi graphs are not in the image of Y_2. This is a diagrammatic translation of Morita's results [Mor89, Mor91]: the quotient $(\Gamma_2\mathcal{I}(\Sigma)/\Gamma_3\mathcal{I}(\Sigma)) \otimes \mathbb{Q}$ is determined by the restriction of $\rho_3 : \mathcal{I}(\Sigma) \to \mathrm{Aut}(\pi/\Gamma_4\pi)$ to $\Gamma_2\mathcal{I}(\Sigma)$ (i.e. by the second Johnson homomorphism, which corresponds to the H graphs) and by the Casson invariant (which corresponds to the Theta graph).

Finally, the previous results are extended in [HM09, §7] to the case where Σ is replaced by a *closed* connected oriented surface. The statements must be adapted to this case and the proofs become substantially more technical. We shall not describe the closed case here.

3.2 The Goussarov–Habiro conjecture [Mas07]

As shown by Habiro [Hab00b] and by Goussarov [Gou00], the converse of (1.6) is true for $\mathbb{Z}$-homology spheres or, equivalently, for homology cylinders over a disk. Their result provides a surgery description of the "separation power" of finite-type invariants in this particular case. Here, we consider an arbitrary compact connected oriented surface Σ.

Conjecture. (Goussarov–Habiro) *Two homology cylinders over Σ are Y_{d+1}-equivalent if, and only if, they are not distinguished by finite-type invariants of degree at most d.*

The paper [Mas07] identifies the algebra underlying the “Goussarov–Habiro conjecture” (or GHC, for short) by observing that it is a special instance of a general problem in group theory, which is known as the “dimension subgroup problem” (or DSP, for short). Classically, for a given group G, the question is whether the k-th term of the lower central series coincides with the k-th *dimension subgroup*, i.e. the subgroup of G determined by the k-th power of the augmentation ideal I of the group ring $\mathbb{Z}[G]$:

$$\Gamma_k G \stackrel{?}{=} G \cap (1 + I^k). \tag{3.4}$$

The inclusion “$\subset$” is always true, as we have mentioned at (1.5). The DSP can be considered in the more general setting where the lower central series of G is replaced by an *N-series*, i.e. by an arbitrary descending chain of subgroups

$$G = N_1 G \supset N_2 G \supset N_3 G \supset \cdots$$

such that $[N_i G, N_j G] \subset N_{i+j} G$ for any integers $i, j \geq 1$. It is observed in [Mas07] that the GHC can be reduced to the DSP after an algebraic re-formulation of topological results due to Habiro [Hab00b] and Goussarov [Gou00], and which notably led to the above-mentioned result on $\mathbb{Z}$-homology spheres. In particular, they have shown by calculus of claspers that the monoid[9] $\mathcal{IC}(\Sigma)$ quotiented out by the Y_{d+1}-equivalence relation is a group. The N-series to be considered on that group is induced by the Y-filtration (3.1) on the monoid $\mathcal{IC}(\Sigma)$.

Unfortunately, the DSP has been known to be a difficult problem since Rips exhibited a finite 2-group G where (3.4) fails in degree $k = 4$ [Rip72]. Hence the algebraic reduction of the GHC to the DSP does not solve by itself this topological conjecture (other than up to degree $d+1 = 3$). Nevertheless, the DSP has been solved for coefficients in a field. For the lower central series, this is due to Mal’cev, Jennings and Hall in the zero characteristic case [Mal49, Jen55, Hal69] and to Jennings and Lazard in the positive characteristic case [Jen41, Laz54]. These results are generalized in [Mas07, §4.1] to arbitrary N-series by following Passi’s book [Pas79]. Thus, we obtain a version of the GHC for finite-type invariants with values in a given field $\mathbb{F}$, and the statement depends on the characteristic of $\mathbb{F}$.

Theorem 3.2. [Mas07, Th. 1.1] *Let $\mathbb{F}$ be a field of characteristic 0, and let M, M' be two homology cylinders over Σ. Finite-type invariants of degree at most d with values in $\mathbb{F}$ do not distinguish M from M' if, and only if, there exists an integer $n > 0$ such that M^n is Y_{d+1}-equivalent to M'^n.*

Theorem 3.3. [Mas07, Th. 1.2] *Let $\mathbb{F}$ be a field of characteristic $p > 0$, and let M, M' be two homology cylinders over Σ. Finite-type invariants of degree at most d with values in $\mathbb{F}$ do not distinguish M from M' if, and only if, there exist some $C_1, \dots, C_r \in \mathcal{IC}(\Sigma)$ and some integers $e_1, \dots, e_r \geq 0$ such that*

- *C_i is Y_{k_i}-equivalent to $\Sigma \times [-1, 1]$ for some integer $k_i > 0$ such that $k_i p^{e_i} \geq d + 1$,*
- *$M' = M \cdot \prod_{i=1}^{r} C_i^{p^{e_i}}$.*

A weakened version of the GHC is deduced from the above results.

Corollary 3.2. [Mas07, Cor. 1.3] *There exists an integer $\mathbf{d}(\Sigma, d) \geq d$ such that, if two homology cylinders over Σ are not distinguished by finite-type invariants of degree at most $\mathbf{d}(\Sigma, d)$, then they are Y_{d+1}-equivalent.*

Consequently two homology cylinders are Y_k-equivalent *for all* $k \geq 1$ if, and only if, they are not distinguished by finite-type invariants (with values in any abelian group).

[9] This monoid is denoted by $\mathcal{C}yl(\Sigma)$ in [Mas07].

By the same kind of algebraic methods, we prove that the universal enveloping algebra of the Lie algebra of homology cylinders $\mathrm{Gr}^Y \mathcal{IC}(\Sigma) \otimes \mathbb{Q}$ is canonically isomorphic to the algebra dual to $\mathbb{Q}$-valued finite-type invariants, namely

$$\mathrm{Gr}^{\mathcal{F}}\, \mathbb{Q}\cdot\mathcal{IC}(\Sigma) = \bigoplus_{i\geq 0} \frac{\mathcal{F}_i(\mathcal{IC}(\Sigma))}{\mathcal{F}_{i+1}(\mathcal{IC}(\Sigma))}$$

where $\mathcal{F}$ is the filtration (1.7) with coefficients in $\mathbb{Q}$. This remains valid if $\mathbb{Q}$ is replaced by any field $\mathbb{F}$ of characteristic zero. If $\mathbb{F}$ is a field of characteristic $p > 0$, there is a similar result using the notions of *restricted* Lie algebra and *restricted* universal enveloping algebra. All this is proved in [Mas07, §5] by extending a result of Quillen for the lower central series of a group [Qui68] to arbitrary N-series.

3.3 Surgery equivalence relations and the core of the Casson invariant [MM10]

Let $\Sigma := \Sigma_{g,1}$ be a compact connected oriented surface of genus g with one boundary component. As we have mentioned in §1.5, the Y_1-equivalence relation is trivial on $\mathcal{IC}(\Sigma)$ [Hab00b, Hab00a] and the Y_2-equivalence is classified in [Hab00b, MM03]. The paper [MM10] provides the following characterization of the Y_3-equivalence.

Theorem 3.4. [MM10, Th. A] *Let M and M' be two homology cylinders over Σ. The following assertions are equivalent:*

(a) M and M' are Y_3-equivalent;

(b) M and M' are not distinguished by any finite-type invariants of degree at most 2*;*

(c) M and M' share the same invariants ρ_3, λ_j and α;

(d) the LMO homomorphism Z agrees on M and M' up to i-degree 2.

The LMO homomorphism $Z : \mathcal{IC}(\Sigma) \to \mathcal{A}(H_{\mathbb{Q}})$ of condition (d) belongs to quantum topology and has been defined at (3.3). The invariants ρ_3, λ_j and α of condition (c) are more classical invariants which we now describe.

The first invariant is an extension for $k = 3$ of the homomorphism $\rho_k : \mathcal{I}(\Sigma) \to \mathrm{Aut}(\pi/\Gamma_{k+1}\pi)$, defined at (1.10) where $\pi := \pi_1(\Sigma, *)$ and $* \in \partial\Sigma$. Indeed, by virtue of Stallings' theorem [Sta65], ρ_k can be extended to the monoid $\mathcal{IC}(\Sigma)$ for any integer $k \geq 1$:

$$\rho_k : \mathcal{IC}(\Sigma) \longrightarrow \mathrm{Aut}(\pi/\Gamma_{k+1}\pi),\ M \longmapsto (m_{-,*} \mod \Gamma_{k+1}\pi)^{-1} \circ (m_{+,*} \mod \Gamma_{k+1}\pi). \quad (3.5)$$

(Recall that, for any $M \in \mathcal{IC}(\Sigma)$, $m_\pm : \Sigma \to M$ denotes the boundary parameterization of the "top/bottom" surface $\partial_\pm M$.) The definition of the second invariant $\lambda_j : \mathcal{IC}(\Sigma) \to \mathbb{Z}$ needs to choose an embedding $j : \Sigma \hookrightarrow S^3$ such that $j(\Sigma)$ union with a disk splits S^3 into two handlebodies:

$$\forall M \in \mathcal{IC}(\Sigma),\ \lambda_j(M) := \lambda \begin{pmatrix} \mathbb{Z}\text{-homology sphere obtained by "plugging"} \\ M \text{ into } S^3 \text{ in a neighborhood of } j(\Sigma) \end{pmatrix}$$

where λ denotes the Casson invariant.[10] In order to define the third invariant α, we need the Alexander polynomial of homology cylinders relative to their "bottom" boundary. More precisely, we define the *relative Alexander polynomial* of an $M \in \mathcal{IC}(\Sigma)$ as the order of the relative homology group of $(M, \partial_- M)$ with coefficients twisted by $m_{\pm,*}^{-1} : H_1(M;\mathbb{Z}) \to H_1(\Sigma;\mathbb{Z})$:

$$\Delta(M, \partial_- M) := \mathrm{ord}\, H_1(M, \partial_- M; \mathbb{Z}[H]) \in \mathbb{Z}[H]/\pm H \quad \text{where } H := H_1(\Sigma;\mathbb{Z}).$$

[10]Here, the Casson invariant $\lambda(N) \in \mathbb{Z}$ of a $\mathbb{Z}$-homology sphere N is as normalized in [GM92]. We have $\lambda(N) = \lambda_{\mathrm{W}}(N)/2$ where λ_{W} is the normalization of [Wal92].

The multiplicative indeterminacy in $\pm H$ can be fixed by considering a relative version of the Reidemeister–Turaev torsion τ introduced by Benedetti & Petronio [BP01, FJR11]. For this refinement of the relative Alexander polynomial, it is necessary to fix an *Euler structure* on $(M, \partial_- M)$, i.e. a homotopy class of vector fields on M with prescribed behaviour on the boundary. Actually, any $M \in \mathcal{IC}(\Sigma)$ has a preferred Euler structure ξ_0 [MM10, Def. 3.8] so that the class $\Delta(M, \partial_- M)$ has a preferred representative

$$\tau(M, \partial_- M; \xi_0) \in \mathbb{Z}[H].$$

This invariant of homology cylinders features the same finiteness properties as the Reidemeister–Turaev torsion of closed oriented 3-manifolds [Mas10]. More precisely, if we denote by I the augmentation ideal of $\mathbb{Z}[H]$, then the reduction of $\tau(M, \partial_- M; \xi_0)$ modulo I^{k+1} is for every $k \geq 0$ a finite-type invariant of degree at most k. In particular,

$$\alpha(M) \in I^2/I^3 \simeq S^2 H$$

is the "quadratic part" that can be extracted from $\tau(M, \partial_- M; \xi_0)$: we refer to [MM10, §3.2] for details.

In genus $g = 0$, Theorem 3.4 asserts that two $\mathbb{Z}$-homology spheres are Y_3-equivalent if and only if they have the same Casson invariant, which is due to Habiro [Hab00b]. The equivalence between conditions (a) and (b) has already been observed in [Mas07] for any kind of surface Σ. The equivalence between (a), (b) and (d) is proved using the universality of the LMO homomorphism among $\mathbb{Q}$-valued finite-type invariants (see Corollary 2.2), its good behaviour under clasper surgery and the torsion-freeness of a certain space of Jacobi diagrams. Next, the equivalence of condition (c) with the other three follows by determining precisely how the invariants ρ_3, λ_j and α are diagrammatically encoded in the LMO homomorphism. We emphasize that the Birman–Craggs homomorphisms, which are needed to classify the Y_2-equivalence [MM03], do not appear explicitly in the above statement for the Y_3-equivalence: indeed they are determined by the triplet $(\rho_3, \lambda_j, \alpha)$ [MM10, §3.4]. A diagrammatic description of the group $\mathcal{IC}(\Sigma)/Y_3$ is also given in [MM10, §5.3].

Besides, we obtain characterizations of the J_k-equivalence relations for $k = 2$ and $k = 3$.

Theorem 3.5. [MM10, Th. B] *Two homology cylinders M and M' over Σ are J_2-equivalent if and only if we have $\rho_2(M) = \rho_2(M')$.*

In genus $g = 0$, Theorem 3.5 asserts that any $\mathbb{Z}$-homology sphere is J_2-equivalent to S^3. This is already noticed by Morita in [Mor89] and follows from Casson's observation that any $\mathbb{Z}$-homology sphere is obtained from S^3 by a finite sequence of surgeries along (± 1)-framed knots [GM92]. Theorem 3.5 easily follows from the computation of $\mathcal{IC}(\Sigma)/Y_2$ done in [MM03].

Theorem 3.6. [MM10, Th. C] *Two homology cylinders M and M' over Σ are J_3-equivalent if and only if we have $\rho_3(M) = \rho_3(M')$ and $\alpha(M) = \alpha(M')$.*

In genus $g = 0$, we obtain that any $\mathbb{Z}$-homology sphere is J_3-equivalent to S^3, which is due to Pitsch [Pit08]. Theorem 3.6 is deduced from Theorem 3.4.

Although the invariant λ_j is easy to compute by surgery techniques (since it is built from the Casson invariant λ), it is not completely satisfactory in that it depends on the embedding j of the surface Σ in S^3. This phenomenon already appears at the level of the Torelli group, i.e. for the composition $\lambda_j \circ \mathbf{c} : \mathcal{I}(\Sigma) \to \mathbb{Z}$ which has been studied in detail by Morita [Mor89, Mor91]. More precisely, he has shown that its restriction to the *Johnson subgroup* $\mathcal{K}(\Sigma) := \operatorname{Ker} \rho_2$ (i.e. the second term of the Johnson filtration (1.11)) is a group homomorphism which decomposes as

$$-\lambda_j \circ \mathbf{c}|_{\mathcal{K}(\Sigma)} = q_j + \frac{1}{24} d. \tag{3.6}$$

Here the homomorphism $q_j : \mathcal{K}(\Sigma) \to \mathbb{Q}$ is explicitly determined by ρ_3 in a way which involves j, whereas the homomorphism $d : \mathcal{K}(\Sigma) \to \mathbb{Z}$ does not depend on j. The J_3-equivalence relation being trivial for $\mathbb{Z}$-homology spheres [Pit08], formula (3.6) shows that all the information carried by the Casson invariant is contained in this map d: Morita calls it the *core of the Casson invariant*. Let $\mathcal{KC}(\Sigma) := \operatorname{Ker} \rho_2$ be the submonoid of $\mathcal{IC}(\Sigma)$ that acts trivially on $\pi/\Gamma_3\pi$.

Theorem 3.7. [MM10, Th. D] *If $g \geq 3$, then there is a unique extension of the core of the Casson invariant to the monoid $\mathcal{KC}(\Sigma)$*

$$\begin{tikzcd} \mathcal{K}(\Sigma) \arrow[r, "d"] \arrow[d, "\mathbf{c}"'] & 8\mathbb{Z} \\ \mathcal{KC}(\Sigma) \arrow[ur, dashed, "d"'] & \end{tikzcd}$$

that is invariant under Y_3-equivalence and under the action of the mapping class group.

Here the mapping class group $\mathcal{M}(\Sigma)$ acts on $\mathcal{IC}(\Sigma)$ by changing the boundary parameterization of cobordisms (which generalizes the action of $\mathcal{M}(\Sigma)$ on $\mathcal{I}(\Sigma)$ by conjugation). The assumption $g \geq 3$ in Theorem 3.7 can be removed by requiring the invariance under stabilization of the surface Σ [MM10, Th. 7.8]. The unicity of the extension of d is justified by comparing the decomposition of $\frac{\Gamma_2\mathcal{I}(\Sigma)}{\Gamma_3\mathcal{I}(\Sigma)} \otimes \mathbb{Q}$ into irreducible $\operatorname{Sp}(H_\mathbb{Q})$-modules [Hai97, HS00] to that of $\frac{Y_2\mathcal{IC}(\Sigma)}{Y_3} \otimes \mathbb{Q}$ [HM09]. The extension of the homomorphism d to the monoid $\mathcal{KC}(\Sigma)$ can be explicitly defined from the i-degree 2 part of the LMO homomorphism Z, or, in terms of classical invariants as follows:

$$d = -24\,(\lambda_j + q_j) + (\text{something explicitly derived from } \alpha \text{ using } j). \tag{3.7}$$

This generalizes Morita's formula (3.6) since α is trivial on $\mathcal{K}(\Sigma)$.

4 From finite-type invariants to the Dehn–Nielsen representation

4.1 The tree-reduction of the LMO homomorphism [Mas12b]

Let $\Sigma := \Sigma_{g,1}$ be a compact connected oriented surface of genus g with one boundary component. As in the previous sections, we denote $H_\mathbb{Q} := H_1(\Sigma;\mathbb{Q})$ and $\pi := \pi_1(\Sigma, *)$ where $* \in \partial\Sigma$. Recall from §3.1 that the LMO homomorphism $Z : \mathcal{IC}(\Sigma) \to \mathcal{A}(H_\mathbb{Q})$ is a monoid homomorphism which is universal among $\mathbb{Q}$-valued finite-type invariants [CHM08, HM09]. The paper [Mas12b] interprets the "tree reduction" of Z as an "infinitesimal" version of the Dehn–Nielsen representation. In particular, Z contains some "infinitesimal" versions of Morita's homomorphisms.

Let us first recall the definition of Morita's homomorphisms. For every integer $k \geq 1$, we denote by $\mathcal{IC}(\Sigma)[k]$ the kernel of the representation $\rho_k : \mathcal{IC}(\Sigma) \to \operatorname{Aut}(\pi/\Gamma_{k+1}\pi)$ defined at (3.5). Thus we obtain a filtration

$$\mathcal{IC}(\Sigma) = \mathcal{IC}(\Sigma)[1] \supset \mathcal{IC}(\Sigma)[2] \supset \mathcal{IC}(\Sigma)[3] \supset \cdots \tag{4.1}$$

of $\mathcal{IC}(\Sigma)$ by submonoids [Hab00b, GL05]. This is the analogue of the Johnson filtration (1.11) of the Torelli group $\mathcal{I}(\Sigma)$ but, in contrast with the latter case, it is far from being separated: in the case $g = 0$, for instance, we have $\mathcal{IC}(\Sigma)[k] = \mathcal{IC}(\Sigma)$ for every $k \geq 0$. For any integer $k \geq 1$, the *k-th Morita homomorphism* is the map

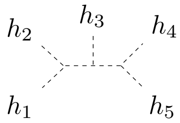
$$M_k : \mathcal{IC}(\Sigma)[k] \longrightarrow H_3\left(\pi/\Gamma_{k+1}\pi;\mathbb{Z}\right), \ M \longmapsto \mu_k(\text{closure of } M).$$

Here the *closure* of M is the closed oriented 3-manifold obtained from M by gluing its top boundary ∂_+M to its bottom boundary ∂_-M, and by gluing a solid torus to the resulting 3-manifold. For any closed oriented 3-manifold C, $\mu_k(C)$ denotes the image[11] of the fundamental class $[C] \in H_3(C;\mathbb{Z})$ in $H_3(\mathrm{K}(\pi_1(C)/\Gamma_{k+1}\pi_1(C),1);\mathbb{Z})$. These homomorphisms have been studied in [Mor93a, Hea06, Sak06]. By considering the simplicial model of the Eilenberg–MacLane space $\mathrm{K}(\pi/\Gamma_{k+1}\pi,1)$, one can give a more algebraic definition of M_k and this was the original approach of Morita. The k-th Morita homomorphism is a refinement of the k-th Johnson homomorphism, which is defined in [Joh83b, Mor93a] for the Torelli group and in [GL05] for homology cylinders.

For every integer $k \geq 1$, we define an *infinitesimal* analogue of the k-th Morita homomorphism

$$m_k : \mathcal{IC}(\Sigma)[k] \longrightarrow H_3\big(\mathfrak{m}(\pi/\Gamma_{k+1}\pi)\big)$$

where $\mathfrak{m}(\pi/\Gamma_{k+1}\pi)$ is the Malcev Lie algebra of the group $\pi/\Gamma_{k+1}\pi$. The definition of m_k given in [Mas12b, §4] imitates the algebraic definition of M_k, the use of the bar resolution for groups being simply replaced by that of the Koszul resolution for Lie algebras. Actually, the homomorphism m_k is equivalent to M_k through a canonical isomorphism $\mathrm{P} : H_3(\pi/\Gamma_{k+1}\pi;\mathbb{Q}) \to H_3\big(\mathfrak{m}(\pi/\Gamma_{k+1}\pi)\big)$ due to Pickel [Pic78]. The advantage of passing from M_k to its infinitesimal version m_k is mainly the possibility of writing its values in a diagrammatic way. Indeed the Lie algebra $\mathfrak{m}(\pi/\Gamma_{k+1}\pi)$ is, in a *non-canonical* way, isomorphic to the quotient $\mathfrak{L}(H_\mathbb{Q})/\mathfrak{L}_{\geq k+1}(H_\mathbb{Q})$ of the Lie algebra $\mathfrak{L}(H_\mathbb{Q})$ freely generated in degree 1 by the vector space $H_\mathbb{Q} \simeq \pi/\Gamma_2\pi \otimes \mathbb{Q}$. (Here $\mathfrak{L}_{\geq k+1}(H_\mathbb{Q})$ denotes the part of $\mathfrak{L}(H_\mathbb{Q})$ of degree at least $k+1$.) Next, any acyclic $H_\mathbb{Q}$-colored connected Jacobi diagram can be transformed to a 3-chain in the Koszul complex of $\mathfrak{L}(H_\mathbb{Q})$ by a simple "fission" process. For instance, the *fission* of the diagram



produces the trivector

$$h_1 \wedge h_2 \wedge [h_3,[h_4,h_5]] + h_3 \wedge [h_4,h_5] \wedge [h_1,h_2] + h_5 \wedge [[h_1,h_2],h_3] \wedge h_4.$$

We observe that this fission process sends the IHX relation of Jacobi diagrams to exact 3-chains in the Koszul complex. Thanks to prior computations of Igusa & Orr [IO01], we show the following.

Theorem 4.1. [Mas12b, Th. 1.5] *Let $\mathcal{A}^{c,t}(H_\mathbb{Q})$ be the subspace of $\mathcal{A}(H_\mathbb{Q})$ spanned by connected tree-shaped Jacobi diagrams. Then, the fission process induces an isomorphism*

$$\Phi : \bigoplus_{d=k}^{2k-1} \mathcal{A}^{c,t}_d(H_\mathbb{Q}) \xrightarrow{\simeq} H_3\Big(\mathfrak{L}(H_\mathbb{Q})/\mathfrak{L}_{\geq k+1}(H_\mathbb{Q})\Big).$$

Thus, one obtains a diagrammatic description of m_k as soon as an identification between the Lie algebras $\mathfrak{m}(\pi/\Gamma_{k+1}\pi)$ and $\mathfrak{L}(H_\mathbb{Q})/\mathfrak{L}_{\geq k+1}(H_\mathbb{Q})$ is fixed. Some of these identifications shall be better for our purposes.

Definition 4.1. [Mas12b, §2.3] *A* symplectic expansion *of π is a multiplicative map $\theta : \pi \to \hat{T}(H_\mathbb{Q})$ with values in the complete tensor algebra generated by $H_\mathbb{Q}$, such that*

(i) for any $x \in \pi$, $\theta(x) = 1 + [x] + (\deg \geq 2)$ where $[x] \in \frac{\pi}{[\pi,\pi]} \otimes \mathbb{Q} \simeq H_\mathbb{Q}$ is seen as a degree 1 element of $\hat{T}(H_\mathbb{Q})$,

(ii) for any $x \in \pi$, $\theta(x)$ is group-like in $\hat{T}(H_\mathbb{Q})$,

(iii) $\theta([\partial\Sigma]) = \exp(-\omega)$ where $\omega \in \Lambda^2 H_\mathbb{Q}$ is the homology intersection form of Σ and is seen as a degree 2 element of $\hat{T}(H_\mathbb{Q})$.

[11]This is sometimes called the *k-th nilpotent homotopy type* of C [Tur84].

Conditions (i) and (ii) are equivalent to asking for a filtration-preserving Lie algebra isomorphism θ between $\mathfrak{m}(\pi)$ and the complete free Lie algebra $\hat{\mathfrak{L}}(H_{\mathbb{Q}})$ spanned by $H_{\mathbb{Q}}$: in particular, this induces for any $k \geq 1$ an isomorphism $\theta : \mathfrak{m}(\pi/\Gamma_{k+1}\pi) \to \mathfrak{L}(H_{\mathbb{Q}})/\mathfrak{L}_{\geq k+1}(H_{\mathbb{Q}})$ as required. The third condition, about the symplectic form ω, is needed for the following construction. We consider, for every $k \geq 1$, the following composition

$$\begin{tikzcd}
\mathcal{IC}(\Sigma) \arrow[r, "\rho_k"] \arrow[rrr, dashed, bend right=12, "\varrho_k^\theta"'] & \mathrm{Aut}(\pi/\Gamma_{k+1}\pi) \arrow[r, tail, "\mathfrak{m}"] & \mathrm{Aut}\big(\mathfrak{m}(\pi/\Gamma_{k+1}\pi)\big) \arrow[r, "\theta\circ - \circ\theta^{-1}", "\simeq"'] & \mathrm{Aut}\big(\mathfrak{L}(H_{\mathbb{Q}})/\mathfrak{L}_{\geq k+1}(H_{\mathbb{Q}})\big),
\end{tikzcd}$$

and we take the limit as $k \to +\infty$ in order to obtain a monoid homomorphism

$$\varrho^\theta : \mathcal{IC}(\Sigma) \longrightarrow \mathrm{IAut}_\omega(\hat{\mathfrak{L}}(H_{\mathbb{Q}})).$$

Here $\mathrm{IAut}_\omega(\hat{\mathfrak{L}}(H_{\mathbb{Q}}))$ denotes the group of filtration-preserving automorphisms of $\hat{\mathfrak{L}}(H_{\mathbb{Q}})$ that fix ω and induce the identity at the graded level. The homomorphism ϱ^θ can be viewed as an *infinitesimal* version of the Dehn–Nielsen representation ρ, which we recalled at (1.9) for the Torelli group. It is convenient to compose ϱ^θ with the following isomorphisms:

$$\mathrm{IAut}_\omega(\hat{\mathfrak{L}}(H_{\mathbb{Q}})) \xrightarrow[\simeq]{\log_\circ} \mathrm{Der}_\omega\left(\hat{\mathfrak{L}}(H_{\mathbb{Q}}), \hat{\mathfrak{L}}_{\geq 2}(H_{\mathbb{Q}})\right) \xleftarrow[\simeq]{\eta} \mathcal{A}^{c,t}\left(H_{\mathbb{Q}}\right).$$

Here $\log_\circ$ is the usual logarithmic series (for the composition $\circ$ of linear maps $\hat{\mathfrak{L}}(H_{\mathbb{Q}}) \to \hat{\mathfrak{L}}(H_{\mathbb{Q}})$); its target is the Lie algebra of filtration-preserving derivations of $\hat{\mathfrak{L}}(H_{\mathbb{Q}})$ that vanish on ω and take values in $\hat{\mathfrak{L}}_{\geq 2}(H_{\mathbb{Q}})$. The map η is the isomorphism appearing in Kontsevich's work [Kon93a] in relation with the cyclic operad of Lie algebras.

Theorem 4.2. [Mas12b, Th. 4.4] *For any symplectic expansion θ of π, the following diagram is commutative:*

$$\begin{tikzcd}
\mathcal{IC}(\Sigma)[k] \arrow[r, "\eta^{-1}\log_\circ \varrho^\theta"] \arrow[d, "-m_k"'] & \mathcal{A}^{c,t}\left(H_{\mathbb{Q}}\right) \arrow[r, two heads] & \bigoplus_{d=k}^{2k-1} \mathcal{A}^{c,t}_d\left(H_{\mathbb{Q}}\right) \arrow[d, "\Phi", "\simeq"'] \\
H_3\big(\mathfrak{m}(\pi/\Gamma_{k+1}\pi)\big) \arrow[rr, "\simeq", "\theta_*"'] & & H_3\big(\mathfrak{L}(H_{\mathbb{Q}})/\mathfrak{L}_{\geq k+1}(H_{\mathbb{Q}})\big).
\end{tikzcd}$$

Since ϱ^θ is explicitly defined from the action of $\mathcal{IC}(\Sigma)$ on the Malcev completion of π, this result allows for explicit computations of the k-th Morita homomorphism. In particular, we obtain that $\mathrm{Ker}\, m_k = \mathrm{Ker}\, \rho_{2k} = \mathcal{IC}(\Sigma)[2k]$, which gives an algebraic proof to Heap's result [Hea06, Sak06] that $\mathrm{Ker}\, M_k = \mathcal{IC}(\Sigma)[2k]$.

Finally, we obtain the following algebraico-topological interpretation of the "tree reduction" of the LMO homomorphism Z.

Theorem 4.3. [Mas12b, Th. 5.13] *The LMO functor induces a symplectic expansion Θ of π which makes the following diagram commutative:*

$$\begin{tikzcd}
 & \mathcal{A}^{c,t}\left(H_{\mathbb{Q}}\right) \\
\mathcal{IC}(\Sigma) \arrow[ur, "\eta^{-1}\log_\circ \varrho^\Theta"] \arrow[dr, "Z"'] & \mathcal{A}^{c}\left(H_{\mathbb{Q}}\right) \arrow[u, two heads] \\
 & \mathrm{GLike}\,\mathcal{A}\left(H_{\mathbb{Q}}\right) \arrow[u, "\log_\star"', "\simeq"]
\end{tikzcd}$$

This symplectic expansion[12] Θ is derived from the LMO functor $\widetilde{Z}$ by encoding π with a very special class of Lagrangians cobordims $g \to (g+1)$ and by reducing in an appropriate way the values of $\widetilde{Z}$ on these cobordisms: the construction of Θ is explicit and done is [Mas12b, §5.2]. In Theorem 4.3, we use the (not yet mentioned) facts that the algebra $(\mathcal{A}(H_{\mathbb{Q}}), \star)$ has a natural Hopf algebra structure whose primitive elements are connected Jacobi diagrams, and Z takes group-like values. Theorem 4.3 also implies, in conjunction with Theorem 4.2, that the k-th Morita homomorphism corresponds to the degree $[k, 2k[$ part of the "tree reduction" of the LMO homomorphism. This theorem is inspired by an analogous result of Habegger & Masbaum for the Kontsevich integral of string-links [HM00]. The correspondence between the two situations is the following:

$\boxed{\Sigma_{0,n+1} : \text{disk with } n \text{ holes}}$	$\boxed{\Sigma_{g,1} : \text{surface of genus } g \text{ with circle boundary}}$
pure braid group	Torelli group
$\hookdownarrow$	$\hookdownarrow$
monoid of string-links	monoid of homology cylinders
Kontsevich integral	LMO homomorphism
$\Downarrow$	$\Downarrow$
Artin representation	Dehn–Nielsen representation
$\Downarrow$	$\Downarrow$
Milnor's invariants	Johnson/Morita's homomorphisms

4.2 Extension of Morita homomorphisms to the Ptolemy groupoid [Mas12a]

Let $\Sigma := \Sigma_{g,1}$ be a compact connected oriented surface of genus g with one boundary component. Morita & Penner considered in [MP08] the problem of extending the first Johnson homomorphism to the Ptolemy groupoid. The same kind of problem was further considered for higher Johnson homomorphisms and other representations of the mapping class group $\mathcal{M}(\Sigma)$ in [BKP09, ABP09]. The paper [Mas12a] extends in a canonical way each of Morita's homomorphisms to the Ptolemy groupoid of Σ. Although it is not directly related to finite-type invariants,[13] this work deals with Morita homomorphisms using the same "infinitesimal" approach as [Mas12b].

Firstly, let us state what may be a *groupoid extension problem* in general. Let Γ be a group, and let K be a CW-complex which is a $\mathrm{K}(\Gamma, 1)$-space and whose fundamental group $\pi_1(K, \centerdot)$ is identified with Γ:

Given $\begin{cases} \text{an abelian group } A \\ \text{a group homomorphism } \varphi : \Gamma \to A \end{cases}$

find $\begin{cases} \text{an abelian group } \widetilde{A} \text{ which contains } A \\ \text{a groupoid homomorphism } \widetilde{\varphi} : \pi_1^{\mathrm{cell}}(K) \to \widetilde{A} \end{cases}$

such that

$$\begin{tikzcd} \pi_1(K, \centerdot) = \Gamma \arrow[r, "\varphi"] \arrow[d, hook] & A \arrow[d, hook] \\ \pi_1^{\mathrm{cell}}(K) \arrow[r, "\widetilde{\varphi}"'] & \widetilde{A}. \end{tikzcd}$$

Of course, solutions to the groupoid extension problem always exist: for instance, by choosing for each vertex v of K a path connecting v to the base vertex $\centerdot$ of K, one easily constructs an extension $\widetilde{\varphi}$ of φ with

[12] The expansion Θ is denoted by $\theta^{\widetilde{Z}}$ in [Mas12b].

[13] Representations of the Ptolemy groupoid are related to the theory of finite-type invariants in [ABMP10, Pen11].

values in $\widetilde{A} := A$. The groupoid extension problem is pertinent in the following situation: the group Γ is not well understood, and the cellular fundamental groupoid $\pi_1^{\rm cell}(K)$ offers a nice combinatorial model in which to embed Γ. Then one seeks a solution $\widetilde{\varphi}$ defined by a *canonical* formula: the simpler the formula, the better the solution $\widetilde{\varphi}$ is. One may need to enlarge A to some $\widetilde{A}$ to achieve this, but $\widetilde{A}/A$ should not be too big. From a cohomological viewpoint, the groupoid extension problem consists in finding a 1-cocycle $\widetilde{\varphi} : \{\text{oriented 1-cells of } K\} \to A$ which represents $[\varphi] \in H^1(\Gamma; A) \simeq H^1(K; A)$: again, the 1-cocycle $\widetilde{\varphi}$ is desired to be canonical, which may require taking coefficients in a larger abelian group $\widetilde{A} \supset A$. (A more general problem can also be stated with twisted coefficients.)

The Ptolemy groupoid is a combinatorial object which has arisen from Teichmüller theory in Penner's work [Pen87, Pen88]. By definition, the *Ptolemy groupoid* $\mathfrak{Pt}(\Sigma)$ of the surface Σ is the cellular fundamental groupoid $\pi_1^{\rm cell}(\widetilde{\mathcal{F}at^b(\Sigma)})$ of a certain cell complex $\widetilde{\mathcal{F}at^b(\Sigma)}$. By results of Harer [Har86] and Penner [Pen87, Pen88], $\widetilde{\mathcal{F}at^b(\Sigma)}$ is the universal cover of a finite cell complex $\mathcal{F}at^b(\Sigma)$ which is a $\mathrm{K}(\mathcal{M}(\Sigma), 1)$-space. The cells of $\mathcal{F}at^b(\Sigma)$ are indexed by graphs of a certain kind (called "bordered fatgraphs"). Then, the groupoid $\mathfrak{Pt}(\Sigma)$ has the following presentation: objects are decorated graphs of a certain kind (called "trivalent bordered fatgraphs") whose edges are marked with elements of $\pi := \pi_1(\Sigma, *)$; morphisms are sequences of elementary moves between such graphs (called "Whitehead moves") modulo some relations (called "involutivity", "commutativity" and "pentagon") [Pen04].

The paper [Mas12a] extends each of Morita's homomorphisms to the Ptolemy groupoid. In a first formulation, we define groupoid extensions which we call "tautological." Indeed, there is a nice 3-dimensional interpretation of the Ptolemy groupoid (due to Penner) which views Whitehead moves as Pachner moves between 2-dimensional triangulations. Since the original definition of Morita's homomorphisms is based on the bar resolution $\mathrm{B}_*(-)$ of groups [Mor93a], it is very natural to extend them to the Ptolemy groupoid using the same simplicial approach. Then, by using the combinatorics of fatgraphs and by giving an explicit value for each kind of Whitehead move, we define in [Mas12a, §2.1] a canonical groupoid homomorphism $\widetilde{M}_k$ which makes the following diagram commute for any choice of a π-marked trivalent bordered fatgraph G:

$$\begin{tikzcd}
\pi_1\left(\widetilde{\mathcal{F}at^b(\Sigma)}/\mathcal{I}(\Sigma)[k], \{G\}\right) \arrow[r, equal, "G"] \arrow[d, hook] & \mathcal{I}(\Sigma)[k] \arrow[r, "M_k"] & H_3\left(\pi/\Gamma_{k+1}\pi; \mathbb{Z}\right) \arrow[d, hook] \\
\pi_1^{\rm cell}\left(\widetilde{\mathcal{F}at^b(\Sigma)}/\mathcal{I}(\Sigma)[k]\right) \arrow[r, equal, ":"] & \mathfrak{Pt}(\Sigma)/\mathcal{I}(\Sigma)[k] \arrow[r, "\widetilde{M}_k"'] & \frac{\mathrm{B}_3(\pi/\Gamma_{k+1}\pi)}{\mathrm{Im}(\partial_4)}.
\end{tikzcd}$$

Thus the groupoid extension problem as it is formulated above, with

$$\Gamma := \mathcal{I}(\Sigma)[k],\ K := \widetilde{\mathcal{F}at^b(\Sigma)}/\mathcal{I}(\Sigma)[k] \quad \text{and} \quad A := H_3(\pi/\Gamma_{k+1}\pi; \mathbb{Z}),\ \varphi := M_k,$$

has the solution

$$\widetilde{A} := \mathrm{B}_3(\pi/\Gamma_{k+1}\pi)/\,\mathrm{Im}(\partial_4), \quad \widetilde{\varphi} := \widetilde{M}_k.$$

But, in contrast with M_k, the groupoid homomorphism $\widetilde{M}_k$ has a target of *infinite* rank. In a second refinement, we improve this groupoid extension by decreasing its target to a finitely generated free abelian group. For this, we replace groups by their Malcev Lie algebras, and we use a homological construction due to Suslin and Wodzicki [SW92]. More precisely, we need their functorial chain map $\mathrm{SW} : \mathrm{B}_*(F) \to \Lambda^*\mathfrak{m}(F)$ between the bar complex of a group F and the Koszul complex of its Malcev Lie algebra $\mathfrak{m}(F)$. If F is finitely generated, torsion-free and nilpotent, the chain map SW induces Pickel's isomorphism $\mathrm{P} : H_*(F; \mathbb{Q}) \to H_*(\mathfrak{m}(F))$ at the level of homology [Pic78].

Theorem 4.4. [Mas12a, Th. 3.1] *The groupoid homomorphism $\widetilde{m}_k$ defined by*

$$\begin{tikzcd} \frac{\mathfrak{Pt}(\Sigma)}{\mathcal{I}(\Sigma)[k]} \arrow[r, "\widetilde{M}_k"] \arrow[rr, dashed, bend right, "\widetilde{m}_k"'] & \frac{\mathrm{B}_3(\pi/\Gamma_{k+1}\pi)}{\mathrm{Im}(\partial_4)} \arrow[r, "\mathrm{SW}_3"] & T_k(\pi) :\!=\!=\!=\; \mathrm{SW}_3\left(\frac{\mathrm{B}_3(\pi/\Gamma_{k+1}\pi)}{\mathrm{Im}(\partial_4)}\right) \subset \frac{\Lambda^3\mathfrak{m}(\pi/\Gamma_{k+1}\pi)}{\mathrm{Im}(\partial_4)} \end{tikzcd}$$

is an extension of M_k to the groupoid $\mathfrak{Pt}(\Sigma)/\mathcal{I}(\Sigma)[k]$:

$$\begin{tikzcd} \pi_1\left(\widetilde{\mathcal{F}at}^b(\Sigma)/\mathcal{I}(\Sigma)[k],\{G\}\right) \arrow[r, equal, "G"] \arrow[d, hook] & \mathcal{I}(\Sigma)[k] \arrow[r, "M_k"] & H_3\left(\pi/\Gamma_{k+1}\pi;\mathbb{Z}\right) \arrow[d, tail, "\mathrm{SW}_3"] \\ \pi_1^{\mathrm{cell}}\left(\widetilde{\mathcal{F}at}^b(\Sigma)/\mathcal{I}(\Sigma)[k]\right) \arrow[r, equal] & \mathfrak{Pt}(\Sigma)/\mathcal{I}(\Sigma)[k] \arrow[r, "\widetilde{m}_k"'] & T_k(\pi) \end{tikzcd}$$

Moreover, the free abelian group $T_k(\pi)$ is finitely generated.

This groupoid extension of M_k is denoted by $\widetilde{m}_k$ in reference to the k-th infinitesimal Morita homomorphism $m_k = \mathrm{P} \circ M_k$ introduced in [Mas12b] and recalled in §4.1. The free abelian group $T_k(\pi)$ is shown to be finitely generated by using Dynkin's formula for the Baker–Campbell–Hausdorff series.

Since Johnson homomorphisms are determined by Morita homomorphisms in an explicit way [Mor93a], the same constructions can be used to extend the former to the Ptolemy groupoid [Mas12a, Th. 4.1]. In the abelian case ($k = 1$), M_1 is equivalent to the first Johnson homomorphism and we exactly recover Morita & Penner's extension of the latter [MP08] with values in

$$\frac{1}{6}\Lambda^3 H = T_1(\pi) \subset \frac{\Lambda^3\mathfrak{m}(\pi/\Gamma_2\pi)}{\mathrm{Im}(\partial_4)} = \Lambda^3(H\otimes\mathbb{Q}) \quad \text{where } H := H_1(\Sigma;\mathbb{Z}).$$

However, for $k > 1$, it does not seem easy to relate in an explicit way our groupoid extension of the k-th Johnson homomorphism to the work of Bene, Kawazumi & Penner [BKP09].

As by-products, we obtain extensions of Morita's homomorphisms to the *full* mapping class group. Note that the choice of a π-marked trivalent bordered fatgraph G gives an injection $\mathcal{M}(\Sigma) \to \mathfrak{Pt}(\Sigma)$, which sends any $f \in \mathcal{M}(\Sigma)$ to a finite sequence of Whitehead moves relating G to $f(G)$.

Corollary 4.1. [Mas12a, Cor. 3.2] *The map $\widetilde{m}_{G,k}$ defined by*

$$\begin{tikzcd} \mathcal{M}(\Sigma) \arrow[r, tail, "G"] \arrow[rrr, dashed, bend right, "\widetilde{m}_{G,k}"'] & \mathfrak{Pt}(\Sigma) \arrow[r, two heads] & \mathfrak{Pt}(\Sigma)/\mathcal{I}(\Sigma)[k] \arrow[r, "\widetilde{m}_k"] & T_k(\pi) \end{tikzcd}$$

is a crossed homomorphism, whose restriction to $\mathcal{I}(\Sigma)[k]$ is m_k and whose cohomology class in $H^1(\mathcal{M}(\Sigma);T_k(\pi))$ does not depend on G.

With the same kind of constructions, we extend Johnson's homomorphisms to crossed homomorphisms on the full mapping class group [Mas12a, Cor. 4.3]. Such extensions of Johnson's homomorphisms exist in prior works by Morita [Mor93b, Mor96], Perron [Per04] and Kawazumi [Kaw05]. Our extensions of Johnson/Morita's homomorphisms to the mapping class group are similar to those obtained by Day in [Day07, Day09]. Day also used Malcev completions of groups, but with the techniques of differential topology. As in the work [Mas12b] relating Morita's homomorphisms to finite-type invariants of 3-manifolds, we use Malcev Lie algebras in the style of Jennings [Jen41] and Quillen [Qui69]: thus, our approach is purely algebraic and it avoids Lie groups. Consequently, our extensions of Johnson/Morita's homomorphisms to the mapping class group are purely combinatorial. The explicit computation of these extensions is subordinate to finding some explicit closed formulas for the Suslin–Wodzicki chain map.

5 Questions, problems and perspectives

This part of the dissertation is not public.

A Finite-type invariants and the word metric on the Torelli group

In this appendix, we prove the following fact which we have mentioned in the introduction: the map on the Torelli group defined by the variation of a finite-type invariant of degree d is bounded by a constant times the d-th power of the word length and, under certain circumstances, this bound is asymptotically sharp. This generalizes a result of Broaddus, Farb & Putman for the Casson invariant of $\mathbb{Z}$-homology spheres [BFP07]. Our motivation is to simply illustrate the polynomial nature of finite-type invariants. Other manifestations of this nature have already been observed in the case of links, see [Dea94, Tra94, BN95b] for instance.

A.1 Polynomial maps on a group and the word metric

Let G be a group and let $F : G \to A$ be a map with values in an abelian group A. For any integer $k \geq 0$, the k-th *formal differential* of F is the map

$$\mathrm{D}^k F : \underbrace{G \times \cdots \times G}_{k} \longrightarrow A$$

defined by

$$\forall g_1, \dots, g_k \in G,\ \mathrm{D}^k F(g_1, \dots, g_k) := \mathbb{Z}[F]\left((1 - g_1) \cdots (1 - g_k)\right)$$

where $\mathbb{Z}[F] : \mathbb{Z}[G] \to A$ is the linear extension of F to the group ring $\mathbb{Z}[G]$. For instance, we have

$$\mathrm{D}^0 F = F(1), \quad \mathrm{D}^1 F(g) = F(1) - F(g), \quad \mathrm{D}^2 F(g_1, g_2) = F(1) - F(g_1) - F(g_2) + F(g_1 g_2), \quad \text{etc.}$$

Note that, for any integer $m \geq 1$ and for any $s_1, \dots, s_m \in G$, we have

$$\begin{aligned} F(s_1 \cdots s_m) &= \mathbb{Z}[F]\big(((s_1 - 1) + 1) \cdots ((s_m - 1) + 1) \big) \\ &= F(1) + \sum_{k=1}^{m} (-1)^k \sum_{1 \leq i_1 < \cdots < i_k \leq m} \mathbb{Z}[F]\big((1 - s_{i_1}) \cdots (1 - s_{i_k})\big) \\ &= \sum_{k=0}^{m} (-1)^k \sum_{1 \leq i_1 < \cdots < i_k \leq m} \mathrm{D}^k F(s_{i_1}, \dots, s_{i_k}). \end{aligned} \tag{A.1}$$

The map $F : G \to A$ is said to be *polynomial* of *degree* d if $\mathrm{D}^{d+1} F = 0$ and $\mathrm{D}^d F \neq 0$. Thus, F is polynomial of degree at most d if and only if $\mathbb{Z}[F]$ vanishes on the $(d+1)$-st power of the augmentation ideal of $\mathbb{Z}[G]$. Polynomial maps have been used by Passi in his study of dimension subgroups [Pas79, Chap. V]. There are plenties of examples. For instance, non-trivial constant maps and non-constant group homomorphisms are polynomial of degree 0 and 1 respectively. If G is the discrete Heisenberg group

$$\left\{ \left(\begin{array}{ccc} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{array} \right) \middle| \, x, y, z \in \mathbb{Z} \right\},$$

then the function $F : G \to \mathbb{Z}$ giving the z-coordinate is a polynomial map of degree 2. Besides it is well-known that, for any integer $d \geq 0$, a map $F : \mathbb{Q}^n \to \mathbb{Q}$ is polynomial of degree d in the previous sense if and only if it is polynomial of degree d in the usual sense.

If the group G is generated by a finite subset S, the *word length* of an element $g \in G$ is the least integer m for which one can find some $s_1, \ldots, s_m \in S \cup S^{-1}$ such that $g = s_1 \cdots s_m$. This length is denoted by $\|g\|_S$. Another choice S' of finitely many generators for G would lead to an equivalent length, in the sense that

$$\exists \lambda > 0,\ \forall g \in G,\ \frac{1}{\lambda} \cdot \|g\|_{S'} \leq \|g\|_S \leq \lambda \cdot \|g\|_{S'}.$$

Given a function $F : G \to \mathbb{Q}$, it can be interesting to bound its absolute values in terms of the word length. The following lemma gives such an estimate when the map F is assumed to be polynomial.

Lemma A.1. *Let G be a finitely generated group, let S be a finite system of generators of G, and let* $F : G \to \mathbb{Q}$ *be a polynomial map of degree* $d > 0$*. Then, there is a constant* $C = C(S, F) > 0$ *such that*

$$\forall g \in G \setminus \{1\}, \quad |F(g)| \leq C \cdot \|g\|_S^d.$$

Moreover, this bound is asymptotically sharp in the sense that one can find a sequence $x = (x_n)_{n \geq N}$ *of elements of G and a constant* $D = D(S, F, x) > 0$ *such that*

$$\lim_{n \to +\infty} \|x_n\|_S = +\infty \quad \textit{and} \quad \forall n \geq N, \quad |F(x_n)| \geq D \cdot \|x_n\|_S^d.$$

Proof. In the sequel, the word length in G with respect to S is simply denoted by $\| \cdot \|$. To prove the first statement of the lemma, we consider an element $g \in G \setminus \{1\}$ with word length $m \geq 1$. Thus, there exist $s_1, \ldots, s_m \in S \cup S^{-1}$ such that $g = s_1 \cdots s_m$. According to (A.1), we have

$$|F(g)| = \left| \sum_{k=0}^{m} (-1)^k \sum_{1 \leq i_1 < \cdots < i_k \leq m} \mathrm{D}^k F\left(s_{i_1}, \ldots, s_{i_k}\right) \right|.$$

For all $k \geq 0$, we introduce the constant

$$C_k := \max\left\{ |\,\mathrm{D}^k F(r_1, \ldots, r_k)| \;\middle|\; r_1, \ldots, r_k \in S \cup S^{-1} \right\}$$

and obtain the following bound:

$$|F(g)| \leq \sum_{k=0}^{m} \binom{m}{k} \cdot C_k.$$

Since F is polynomial of degree at most d, we have $C_{d+1} = C_{d+2} = \cdots = 0$ so that

$$|F(g)| \leq \sum_{k=0}^{\min(m,d)} \binom{m}{k} \cdot C_k.$$

Next, using the fact that

$$\forall k \in \{0, \ldots, \min(m, d)\},\ \binom{m}{k} \leq \frac{m^k}{k!} \leq \frac{m^d}{k!},$$

we obtain

$$|F(g)| \leq \left(\sum_{k=0}^{\min(m,d)} \frac{C_k}{k!} \right) \cdot m^d.$$

Thus, by setting $C := \sum_{k=0}^{d} C_k / k!$, we conclude that $|F(g)| \leq C \cdot \|g\|^d$.
To prove the second statement of the lemma, we now use the assumption that $\mathrm{D}^d F \neq 0$. Thus, we can find an integer $\ell \geq 1$, some elements $h_1, \ldots, h_\ell \in G$ and some integers $i_1, \ldots, i_\ell \geq 1$ such that

$$\mathrm{D}^d F(\underbrace{h_1, \ldots, h_1}_{i_1}, \ldots, \underbrace{h_\ell, \ldots, h_\ell}_{i_\ell}) \neq 0 \tag{A.2}$$

with $i_1 + \cdots + i_\ell = d$ and $h_1 \neq h_2, h_2 \neq h_3, \ldots, h_{\ell-1} \neq h_\ell$. According to (A.1), we have

$$F(h_1^{m_1} \cdots h_\ell^{m_\ell}) = \sum_{k=0}^{d} (-1)^k \sum_{j_1+\cdots+j_\ell=k} \mathrm{D}^k F(\underbrace{h_1, \ldots, h_1}_{j_1}, \ldots, \underbrace{h_\ell, \ldots, h_\ell}_{j_\ell}) \cdot \binom{m_1}{j_1} \cdots \binom{m_\ell}{j_\ell}$$

for any integers $m_1, \ldots, m_\ell \geq d$. In the sequel, we take

$$m_1 := c_1 n, \ldots, m_\ell := c_\ell n$$

where $n \geq d$ is an integer (which will be the index of the sequence x that we are looking for), and where $c_1, \ldots, c_\ell \geq 1$ are integers (which will be fixed soon). For any integer $k \in \{0, \ldots, d\}$ and for any integers $j_1, \ldots, j_\ell \geq 0$ such that $j_1 + \cdots + j_\ell = k$, we have

$$\begin{aligned} \binom{m_1}{j_1} \cdots \binom{m_\ell}{j_\ell} &= \left(\frac{c_1^{j_1} n^{j_1}}{j_1!} + (\deg < j_1 \text{ in } n) \right) \cdots \left(\frac{c_\ell^{j_\ell} n^{j_\ell}}{j_\ell!} + (\deg < j_\ell \text{ in } n) \right) \\ &= \frac{c_1^{j_1} \cdots c_\ell^{j_\ell}}{j_1! \cdots j_\ell!} n^k + (\deg < k \text{ in } n). \end{aligned}$$

We deduce that

$$F\left(h_1^{c_1 n} \cdots h_\ell^{c_\ell n}\right) = (-1)^d \sum_{j_1+\cdots+j_\ell=d} \mathrm{D}^d F(\underbrace{h_1, \ldots, h_1}_{j_1}, \ldots, \underbrace{h_\ell, \ldots, h_\ell}_{j_\ell}) \cdot \frac{c_1^{j_1} \cdots c_\ell^{j_\ell}}{j_1! \cdots j_\ell!} \cdot n^d + P(n)$$

where $P(n)$ is a certain polynomial in n of degree $< d$. By setting

$$\widetilde{D} := \sum_{j_1+\cdots+j_\ell=d} \mathrm{D}^d F(\underbrace{h_1, \ldots, h_1}_{j_1}, \ldots, \underbrace{h_\ell, \ldots, h_\ell}_{j_\ell}) \cdot \frac{c_1^{j_1} \cdots c_\ell^{j_\ell}}{j_1! \cdots j_\ell!},$$

we obtain

$$|F(h_1^{c_1 n} \cdots h_\ell^{c_\ell n})| \geq n^d \cdot |\widetilde{D}| - |P(n)|.$$

Note that $\widetilde{D}$ does not depend on n, but depends polynomially in $c_1, \ldots, c_\ell$. Condition (A.2) insures that, at least, one coefficient of this polynomial is non-zero: so, we can find some integers $c_1, \ldots, c_\ell \geq 1$ for which $\widetilde{D} \neq 0$, and we fix them once for all. The right-hand side of the inequality

$$\frac{|F(h_1^{c_1 n} \cdots h_\ell^{c_\ell n})|}{n^d \cdot |\widetilde{D}|} \geq 1 - \frac{|P(n)|}{n^d \cdot |\widetilde{D}|}$$

goes to 1 as $n \to +\infty$. So, there exists an $N \geq d$ such that

$$\forall n \geq N,\ |F(h_1^{c_1 n} \cdots h_\ell^{c_\ell n})| \geq \frac{|\widetilde{D}|}{2} n^d. \tag{A.3}$$

Then we consider the sequence $x = (x_n)_{n \geq N}$ of elements of G defined by $x_n := h_1^{c_1 n} \cdots h_\ell^{c_\ell n}$. Since we have $\|x_n\| \leq \|h_1\| \cdot c_1 n + \cdots + \|h_\ell\| \cdot c_\ell n$, we deduce from (A.3) that

$$\forall n \geq N,\ |F(x_n)| \geq D \cdot \|x_n\|^d$$

where D is the constant

$$D := \frac{|\widetilde{D}|}{2 \cdot (\|h_1\| c_1 + \cdots + \|h_\ell\| c_\ell)^d}.$$

Furthermore, we deduce from (A.3) and the first assertion of the lemma that

$$\forall n \geq N,\ \frac{|\widetilde{D}|}{2} n^d - |F(1)| \leq |F(x_n)| - |F(1)| \leq C \|x_n\|^d,$$

so that $\|x_n\|$ goes to $+\infty$ when $n \to +\infty$. $\square$

A.2 Restrictions of finite-type invariants to the Torelli group

Let Σ be a compact connected oriented surface with one boundary component, and denote by $\mathcal{I}(\Sigma)$ the Torelli group of Σ. We assume that the genus of Σ is at least 3 so that, by a result of Johnson [Joh83a], the group $\mathcal{I}(\Sigma)$ is finitely generated.

Lemma A.2. *Let $\mathcal{Y}$ be a Y_1-equivalence class of compact oriented 3-manifolds (with parameterized boundary, if any) and let $f : \mathcal{Y} \to A$ be a finite-type invariant of degree d. We choose a 3-manifold $M \in \mathcal{Y}$ and an embedding of Σ in the interior of M. Then the map*

$$F : \mathcal{I}(\Sigma) \longrightarrow A,\ s \longmapsto f\left(M_{(\Sigma,s)}\right) - f(M) \tag{A.4}$$

is polynomial of degree at most d. Moreover the degree of F is exactly d in certain circumstances (which can be specified).

Proof. We consider the group homomorphism $\sigma : \mathbb{Z}[\mathcal{I}(\Sigma)] \to \mathbb{Z}\cdot\mathcal{Y}$ defined by $\sigma(s) := M_{(\Sigma,s)} - M$ for any $s \in \mathcal{I}(\Sigma)$. Let $k \geq 0$ be an integer and let $s_0, \dots, s_k \in \mathcal{I}(\Sigma)$. We identify the regular neighborhood of $\Sigma \subset \operatorname{int}(M)$ with $\Sigma \times [-1,1]$, choose $(k+1)$ points $-1 \leq t_0 < \cdots < t_k \leq 1$ on the interval $[-1,1]$ and consider the $(k+1)$ parallel copies of Σ

$$S_0 := \Sigma \times \{t_0\}, \dots, S_k := \Sigma \times \{t_k\}.$$

We also equip $S_0, \dots, S_k$ with the self-homeomorphisms $s_0, \dots, s_k$ respectively. Then we have

$$\begin{aligned}
\sigma\big((1-s_0)\cdots(1-s_k)\big) &= \sum_{P\subset\{0,\dots,k\}} (-1)^{|P|}\cdot\sigma\Big(\prod_{p\in P} s_p\Big)\\
&= \sum_{P\subset\{0,\dots,k\}} (-1)^{|P|}\cdot(M_P - M)\\
&= \sum_{P\subset\{0,\dots,k\}} (-1)^{|P|}\cdot M_P.
\end{aligned}$$

Here M_P denotes the 3-manifold obtained from M by simultaneous Torelli surgeries along the surfaces S_p indexed by $p \in P$ or, equivalently, it is the 3-manifold obtained by a single surgery along Σ using $\prod_{p\in P} s_p \in \mathcal{I}(\Sigma)$. Therefore, $\sigma : \mathbb{Z}[\mathcal{I}(\Sigma)] \to \mathbb{Z}\cdot\mathcal{Y}$ sends the filtration by powers of the augmentation ideal I to the filtration $\mathcal{F}$ dual to finite-type invariants, as defined at (1.7). It follows, in particular, that $\mathbb{Z}[F] = (\mathbb{Z}\cdot f)\circ\sigma$ vanishes on I^{d+1}, so that F is a polynomial map of degree at most d. At the graded level and in degree d, we have the following commutative triangle:

$$\begin{tikzcd}
I^d/I^{d+1} \arrow[r, "\mathrm{Gr}_d\,\sigma"] \arrow[dr, "\mathrm{Gr}_d\,\mathbb{Z}[F]"'] & \mathcal{F}_d(\mathcal{Y})/\mathcal{F}_{d+1}(\mathcal{Y}) \arrow[d, "\mathrm{Gr}_d\,\mathbb{Z}\cdot f"] \\
& A
\end{tikzcd} \tag{A.5}$$

Consequently, the degree of F is exactly d if and only if $\mathrm{Gr}_d\,\mathbb{Z}\cdot f$ (which we know to be non-zero) is not trivial on the image of $\mathrm{Gr}_d\,\sigma$. $\square$

For example, we can apply Lemma A.2 to a $\mathbb{Z}$-homology sphere M and to the Casson invariant $f := \lambda$, which is of finite type of degree $d := 2$. Morita has given in [Mor91] an explicit formula for $(\mathrm{Gr}_2\,\mathbb{Z}\cdot f)\circ(\mathrm{Gr}_2\,\sigma)$, which results to be non-trivial. Therefore, the map $F : \mathcal{I}(\Sigma) \to \mathbb{Z}$ defined by (A.4) is polynomial of degree 2, exactly, and Lemma A.1 fully applies to this function F. This is the result of [BFP07] which, indeed, is based on Morita's formula. Using Lescop's formula [Les98], the same result applies if M is a $\mathbb{Q}$-homology sphere and $f := \lambda_{\mathrm{W}}$ is Walker's extension of λ [Wal92].

We emphasize that, in the context of Lemma A.2, the polynomial map $F : \mathcal{I}(\Sigma) \to A$ associated to a finite-type invariant $f : \mathcal{Y} \to A$ of degree d can be of degree *strictly less* than d. This is due to the fact that the map $I^d/I^{d+1} \to \mathcal{F}_d(\mathcal{Y})/\mathcal{F}_{d+1}(\mathcal{Y})$ of diagram (A.5) is not surjective in general. If, for instance, $\mathcal{Y}$ is the monoid of homology cylinders $\mathcal{IC}(\Sigma)$, if $\Sigma \subset M$ is the middle surface $\Sigma \times \{0\} \subset \Sigma \times [-1, 1]$ in the usual cylinder, and if we take coefficients in $\mathbb{Q}$ to simplify, then this defect of surjectivity is equivalent to the fact that the Lie algebra of symplectic Jacobi diagrams $\mathcal{A}^c(H_\mathbb{Q})$ is not generated by its degree 1 part.

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N.B. The papers with a "•" are exposed in this dissertation, while those with a "○" are only referred to. The author's Ph.D. thesis was based on the first five listed papers.

En topologie de dimension trois, les invariants de type fini se caractérisent par leur comportement polynomial vis-à-vis de certaines opérations chirurgicales qui préservent l'homologie des variétés. Motivée par l'approche perturbative des « invariants quantiques », la notion d'invariant de type fini a été initialement formulée par T. Ohtsuki qui en contruisit les premiers exemples ; les fondements théoriques des invariants de type fini ont ensuite été posés par plusieurs auteurs dont M. Goussarov et K. Habiro. Grâce à une construction de T. Le, J. Murakami & T. Ohtsuki basée sur l'intégrale de Kontsevich, on dispose pour les sphères d'homologie d'un invariant de type fini universel à valeurs diagrammatiques. Ce mémoire expose d'une manière synthétique certains aspects de la théorie des invariants de type fini, pour les variétés de dimension trois en général, et pour les cylindres d'homologie en particulier. Nous présentons notamment une extension fonctorielle de l'invariant LMO à une certaine catégorie de cobordismes, et nous appliquons ce foncteur à l'étude du monoïde des cylindres d'homologie. Nous expliquons comment nos constructions et résultats se relient aux travaux antérieurs de D. Johnson, S. Morita et R. Hain sur le groupe de Torelli d'une surface. Nous concluons par quelques problèmes et perspectives de recherche. Certains des travaux exposés dans ce mémoire ont été réalisés en collaboration avec D. Cheptea, K. Habiro et J.-B. Meilhan.

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