# A Contribution in Stochastic Control Applied to Finance and Insurance

Moreau Ludovic

**UNIVERSITY OF Paris - Dauphine**
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Ludovic Moreau

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# A Contribution in Stochastic Control Applied to Finance and Insurance

Thesis Advisor: Bruno Bouchard-Denize

***Reviewed and examined by:***

Peter Bank, Professor at Technische Universität Berlin
Nizar Touzi, Professor at École Polytechnique

***and presented to the jury composed of:***

***President:*** Nicole El Karoui, Professor at École Polytechnique

Bruno Bouchard-Denize, Professor at Univerité Paris Dauphine
Luciano Campi, Professor at Université Paris 13
Pierre Cardaliaguet, Professor at Université Paris Dauphine

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# Contents

# Chapter 1

# Introduction

Consider a claim $g$, sold at time $t \geq 0$, of maturity $T \geq t$, with underlying $X_{t,x}$ satisfying $X_{t,x}(t) = x$. In case of a European option, the seller of the claim has to deliver the payoff $g(X_{t,x}(T))$ at terminal date $T$ to the buyer. The natural question arising then is to determine a price $\pi$ to be paid at time $t$ to the seller which will satisfy both the seller and the buyer, so that the risk transfer may occur.

In the so-called *complete market* case of [BS73, AS92, DS94, HP81], the seller may replicate the payoff of the claim by dynamically trading on the market. That is, under good integrability conditions on $g(X_{t,x}(T))$, one can find $y \in \mathbb{R}$ as well as a predictable process $\nu$ such that

$$g\left(X_{t,x}(T)\right) = y + \int_t^T \nu_s \cdot dX(s) \quad \mathbb{P}\text{-a.s.}.$$

The unique fair price is in this case $y$, since it would lead to arbitrage opportunity otherwise.

In the more realistic situation of *incomplete market*, when there are e.g. intrinsic, non-traded sources of risk, both the valuation and the hedging problems may become highly non-trivial issues. Considering the *no-arbitrage* condition leads to an infinity of *viable* prices (see e.g. [DS94]). The risk taker needs thus to define the amount of money he has to invest at time $t$ in some financial portfolio that reduces the risk in an appropriate way. The pricing of contingent claims in incomplete markets thus requires a description of preferences of the agents.

Among the different approaches one could think of, we refer to [BCS98, CPT99, CM96, CK93, EKQ95, KS98] for the super-replication in incomplete markets, [Dav97] for the marginal utility approach, [BL89, DR91, SF85, Sch88, Sch91, Sch99] for the quadratic error minimization approach, [Cvi00, FL99, FL00] for the quantile hedging and shortfall risk minimization point of view.

The aim of this thesis is to contribute to this field.

The first part of this manuscript is dedicated to the stochastic target approach introduced by Soner and Touzi [ST02c, ST00, ST02a, ST03a], and recently developed by Bouchard, Elie and Touzi [BET09] in order to deal with more general frameworks. More specifically, we first provide a generalization of the work of [BET09] in the case of mixed diffusions. This contribution is introduced in Section 1.1.4 below.

Secondly, we establish a game version of the Geometric Dynamic Programming Principle of [ST02a]. This allows us to deal with a more general stochastic target problem in which an adverse player is controlling the diffusion. This is related to hedging problems under Knightian ambiguity. This work is introduced in Section 1.2.1.

We finally focus on the utility indifference pricing framework. Our main aim is to study hybrid claims (see e.g. Section 1.3.1), that is, claims which are in between Finance and Insurance. We provide for the first time in this hybrid framework an asymptotic result for general utility functions defined on the whole real line, when the absolute risk aversion converges uniformly towards 0, and the number of sold claims goes to infinity. This contribution is introduced in Section 1.3.

## 1.1 Stochastic Target in Finance and Insurance

In a geometric form, a stochastic target problem can be formulated as follows. Let $G$ be a Borel subset of a metric space $(\mathcal{Z}, d_{\mathcal{Z}})$, and $Z^{\nu}_{t,z}$ a $\mathcal{Z}$-valued controlled process with initial conditions $Z^{\nu}_{t,z}(t) = z \in \mathcal{Z}$. Consider the so-called *reachability set* $\Lambda(t)$ of initial conditions $z \in \mathcal{Z}$ such that $Z^{\nu}_{t,z}(T) \in G$ $\mathbb{P}$-a.s. for some $\nu \in \mathcal{U}$, with $\mathcal{U}$ the set of *admissible controls*:

$$\Lambda(t) := \left\{ z \in \mathcal{Z} : \text{ there exists } \nu \in \mathcal{U} \text{ s.t. } Z^{\nu}_{t,z}(T) \in G \ \mathbb{P}\text{-a.s.} \right\}. \tag{1.1.1}$$

In [ST02a], Soner and Touzi prove that it satisfies a dynamic programming principle, the so-called Geometric Dynamic Programming Principle (hereafter GDP). This GDP then allows one to perform the derivation of the associated dynamic programming equation, as it is usual in optimal control.

As we shall see below, the GDP opened the door to a wide range of practical applications in finance and insurance. In particular, the results of Chapter 2 heavily rely on this GDP.

### 1.1.1 The Geometric Dynamic Programming and the super-hedging problem

Fix $\mathcal{Z} := \mathbb{R}^d \times \mathbb{R}$. The GDP of Soner and Touzi [ST02a] reads as follows. In Markovian Settings, and under good assumptions on the set of controls $\mathcal{U}$, the reachability set

$$\Lambda(t) = \left\{z \in \mathcal{Z} : Z^{\nu}_{t,z}(T) \in G \ \mathbb{P}\text{-a.s. for some admissible } \nu\right\}$$

coincides with the set $\bar{\Lambda}$

$$\bar{\Lambda}(t) := \left\{z \in \mathcal{Z}, Z^{\nu}_{t,z}(\tau) \in \Lambda(\tau) \ \mathbb{P}\text{-a.s. for some admissible } \nu\right\},$$

for all stopping times $\tau$. Under a "Flow-like" assumption, the first inclusion $\Lambda(t) \subseteq \bar{\Lambda}(t)$ is straightforward, whereas the second is the "*tricky one*". It essentially relies on a measurable selection theorem (see [BS78, Proposition 7.49]), which is made possible by the fact that the map $(t, z, \nu) \in [0, T] \times \mathcal{Z} \times \mathcal{U} \mapsto Z^{\nu}_{t,z}(T)$ is Borel-measurable. We refer the interested reader to [ST02a] for the proof (see [BV10] for an obstacle version).

Fix now $Z := (X, Y)$ and $G := \{z := (x, y) \in \mathbb{R}^d \times \mathbb{R} \text{ s.t. } \Psi(x, y) \geq 0\}$ for some Borel measurable map $\Psi$. Consider furthermore that both $y \mapsto \Psi(\cdot, y)$ and $y \mapsto Y^{\nu}_{t,x,y}(T)$ are non-decreasing, for all $\nu \in \mathcal{U}$. The set $\Lambda(t)$ can then be identified to $\{(x, y) \in \mathbb{R}^d \times \mathbb{R} : y \geq \mathrm{y}(t, x)\}$, with

$$\mathrm{y}(t, x) := \inf\left\{y \in \mathbb{R} : \text{ there exists } \nu \in \mathcal{U} \text{ s.t. } \Psi\left(X^{\nu}_{t,x}(T), Y^{\nu}_{t,x,y}(T)\right) \geq 0 \ \mathbb{P}\text{-a.s.}\right\},$$

whenever the above infimum is achieved.

Formulated as above, this problem may be seen as a generalization of the so-called super-replication problem, see e.g. [EKQ95, CK93, CM96, KS98, BCS98, CPT99].

In the literature, the super-hedging problem is usually solved as follows. The idea is to consider the dual problem, which is a classical optimal control problem, see [JK95, EKQ95, CK93, FK97]. Classical dynamic programming allows to derive the corresponding PDE for the dual value function, which in turns gives a PDE characterization of the value function y.

Soner and Touzi were the first to propose a treatment of this problem in its primal form, that is, to obtain the PDE characterization of y by means of the GDP. The main advantage is that the primal approach of [ST00, ST02c, ST02b, ST03b, ST03b, CST05] applies to general dynamics (such as large investor) or constraints (e.g. gamma constraint), whereas the usual *dual* approach heavily relies on the fact that the coefficients of the wealth dynamics are linear in the control variable, and the stock prices are not influenced by the trading strategy.

This approach was further exploited in Touzi [Tou00], Bouchard and Touzi [BT00], extended to locally bounded jumps in Bouchard [Bou02], and to path dependent constraints in Bouchard and Vu [BV10].

### 1.1.2 The stochastic target with controlled expected loss in Finance

The approach developed in Section 1.1.1 is very powerful to study a large family of non-standard stochastic control problems, in which a target has to be reached with probability one at time $T$. As mentioned above, it provides in particular an extension of the classical super-replication problem. However, in most cases, the super-hedging price leads to an unbearable cost for the buyer, which is not reasonable in practice.

Very recently, Bouchard, Elie and Touzi [BET09] relaxed the $\mathbb{P}$-a.s. criterion $\Psi(Z^{\nu}_{t,z}(T)) \geq 0$ into a moment constraint of the form $\mathbb{E}[\Psi(Z^{\nu}_{t,z}(T))] \geq p$, with $p \in \mathbb{R}$ a given threshold. This new approach has opened the door to a wide range of applications, especially in mathematical finance.

We shall briefly present in this section some possible applications of stochastic target with controlled loss in finance and insurance.

Let $X^{\nu}$ be a process denoting roughly the risks in the portfolio of an agent (one might think of stocks, but also a fixed number of non-tradeable idiosyncratic sources of risks, see Section 1.3.1). Fix $g$, a map defined on $\mathbb{R}^d$ such that $g(X^{\nu}_{t,x}(T))$ has enough regularity. The quantity $g(X^{\nu}_{t,x}(T))$ may be seen as the random payoff of a European claim, given the initial condition $X^{\nu}_{t,x}(t) = x$. The process $Y^{\nu}_{t,x,y}$ represents the wealth of the agent, with initial value $y$ at time $t$, where $\nu$ denotes his strategy in terms of $X^{\nu}$. Consider the value function

$$\mathrm{y}(t,x,p) := \inf\left\{y \in \mathbb{R} : \exists\, \nu \in \mathcal{U} \text{ s.t. } \mathbb{E}\left[\Psi\left(X^{\nu}_{t,x}(T), Y^{\nu}_{t,x,y}(T)\right)\right] \geq p\right\}. \tag{1.1.2}$$

For $p = 1$ and

$$\Psi : (x,y) \longmapsto \mathbb{1}_{\{y \geq g(x)\}},$$

the value function (1.1.2) represents the super-replication price of the claim $g(X^{\nu}_{t,x}(T)$, as discussed above. If $p \in (0,1)$, Equation (1.1.2) may be written as

$$\mathrm{y}(t,x,p) := \inf\left\{y \in \mathbb{R} : \exists\, \nu \in \mathcal{U} \text{ s.t. } \mathbb{P}\left[Y^{\nu}_{t,x,y}(T) \geq g\left(X^{\nu}_{t,x}(T)\right)\right] \geq p\right\}, \tag{1.1.3}$$

and allows one for a treatment of the quantile hedging problem introduced in Föllmer and Leukert [FL99], but in a more general framework, e.g. when the strategy of the agent may influence the value of the risky assets (large investor model). It also permits to deal with more general investment policies. The original treatment of the problem by Fölmer and Leukert relies on the fact that this strategy is linear in the control.

Consider now the case where $p \in \mathbb{R}$ and $\Psi$ belongs to some general class of *utility functions*. More precisely, for an utility function $U : \mathbb{R} \to \mathbb{R}$ and

$$\Psi : (x, y) \in \mathbb{R}^d \times \mathbb{R} \longmapsto U(y - g(x)),$$

the problem (1.1.2) reads

$$\mathrm{y}(t, x, p) := \inf \left\{ y \in \mathbb{R} : \exists\, \nu \in \mathcal{U} \text{ s.t. } \mathbb{E}\left[ U\left( Y^{\nu}_{t,x,y}(T) - g\left( X^{\nu}_{t,x}(T) \right) \right) \right] \geq p \right\}.$$

That is, finding the minimum amount of money the investor has to invest in some strategy $\nu$ in order to have his expected utility above a given threshold $p$. If $p$ happens to be chosen as

$$p := \sup_{\nu' \in \mathcal{U}} \mathbb{E}\left[ U\left( Y^{\nu'}_{t,x,y_o}(T) \right) \right],$$

a straightforward reformulation of this problem defines the value function y as the utility indifference price of the claim $g$:

$$\mathrm{y}(t, x, p) = \inf \left\{ y \in \mathbb{R} :\ \exists\, \nu \in \mathcal{U} \text{ s.t. } \mathbb{E}\left[ \Psi\left( X^{\nu}_{t,x}(T), Y^{\nu}_{t,x,y_o+y}(T) \right) \right] \geq p \right\}.$$

Finally, some minor modifications in the previous reasoning allow us to consider the case where $\Psi$ belongs to some class of *risk functions*,

$$\Psi : (x, y) \in \mathbb{R}^d \times \mathbb{R} \longmapsto -\rho([y - g(x)]^-)$$

for some convex non-decreasing loss function $\rho : \mathbb{R} \to \mathbb{R}$, or the success ratio of Föllmer and Leukert [FL99]

$$\Psi : (x, y) \in \mathbb{R}^d \times \mathbb{R} \longmapsto \mathbb{1}_{\{g(x) \leq y\}}(x, y) + \frac{y}{g(x)} \mathbb{1}_{\{g(x) > y \wedge 0\}}.$$

### 1.1.3 The extension of the Geometric Dynamic Programming Principle to moment constraints

When dealing with stochastic target problems with controlled expected loss, the underlying reachability set (although it is not introduced explicitly in Bouchard, Elie and Touzi [BET09] or [Mor11]) is now

$$\Lambda(t) := \left\{ (z, p) \in \mathbb{R}^d \times \mathbb{R} :\ \text{there exists } \nu \in \mathcal{U} \text{ s.t. } \mathbb{E}\left[ \Psi\left( Z^{\nu}_{t,z}(T) \right) \right] \geq p \right\}.$$

When trying to relate the time-$t$ reachability set to a later time-$\tau$, it is obvious that the value $p \in \mathbb{R}$ has to be incorporated as a part of the state process. The original idea of Bouchard, Elie and Touzi [BET09, Proposition 3.1] (extended to the mixed diffusion case in Proposition 2.3.2) is to apply the martingale representation theorem to the conditional expectation $\mathbb{E}[\Psi(Z^\nu_{t,z}(T))|\mathcal{F}_\cdot]$.

The reachability set may actually be defined as

$$\Lambda(t) := \left\{ \begin{array}{c} (z,p) \in \mathbb{R}^d \times \mathbb{R} : \text{ there exists } \nu \in \mathcal{U} \text{ and } M \in \mathcal{M}_{t,p} \\ \text{s.t. } \tilde{\Psi}\left(Z^\nu_{t,z}(T), M(T)\right) \geq 0 \end{array} \right\}, \tag{1.1.4}$$

where $\tilde{\Psi} : (z,p) \in \mathbb{R}^d \times \mathbb{R} \longmapsto \Psi(z) - p$ and $\mathcal{M}_{t,p}$ denotes a set of martingales $M$ satisfying $M(t) = p$. We thus recover a stochastic target problem in $\mathbb{P}$-a.s. criterion on the state process $(Z, M)$, and the GDP of Soner and Touzi reads in this context

$$\Lambda(t) = \left\{ \begin{array}{c} (z,p) \in \mathbb{R}^d \times \mathbb{R} : \text{ there exists } \nu \in \mathcal{U} \text{ and } M \in \mathcal{M}_{t,p} \\ \text{s.t. } \left(Z^\nu_{t,z}(\tau), M(\tau)\right) \in \Lambda(\tau) \ \mathbb{P}\text{-a.s.} \end{array} \right\}.$$

We are then able to derive the dynamic programming PDE from the GDP of [ST02a], up to non-trivial difficulties, as explained below.

### 1.1.4 The derivation of the PDE in the mixed diffusion case

In Chapter 2, we extend the results of Bouchard, Elie and Touzi [BET09] to the mixed diffusion case. Namely, we consider a filtration $\mathbb{G}$ generated by a Brownian motion $W$ and a $E$-marked right continuous point process $J$. For $0 \leq t \leq T$, we are given two controlled diffusion processes $\left\{X^\nu_{t,x}(s), t \leq s \leq T\right\}$ and $\{Y^\nu_{t,x,y}(s), t \leq s \leq T\}$ taking their values respectively in $\mathbb{R}^d$ and $\mathbb{R}$. These processes satisfy the initial condition $\left(X^\nu_{t,x}(t), Y^\nu_{t,x,y}(t)\right) = (x,y)$, and are $\mathbb{R}^d \times \mathbb{R}$-valued strong solutions of the stochastic differential equations

$$\begin{aligned} dX(s) &= \mu_X\left(X(s), \nu_s\right) ds + \sigma_X\left(X(s), \nu_s\right) dW_s \\ &\quad + \int_E \beta_X\left(X(s-), \nu^1_s, \nu^2_s(e), e\right) J(de, ds) \\ dY(s) &= \mu_Y\left(Z(s), \nu_s\right) ds + \sigma_Y\left(Z(s), \nu_s\right) dW_s \\ &\quad + \int_E \beta_Y\left(Z(s-), \nu^1_s, \nu^2_s(e), e\right) J(de, ds). \end{aligned}$$

In Bouchard, Elie and Touzi [BET09], the filtration $\mathbb{F}$ is generated by the Brownian motion $W$, and $\beta_X \equiv \beta_Y \equiv 0$. We shall see briefly below that this has non-trivial impacts on both the formulation and the derivation of the associated partial differential equations.

For a given measurable map $\Psi$ and threshold $p$, the controller wants to compute:

$$\mathrm{y}(t,x,p) := \inf \left\{ y \in \mathbb{R} : \begin{array}{c} \mathbb{E}\left[\Psi\left(X^\nu_{t,x}(T), Y^\nu_{t,x,y}(T)\right)\right] \geq p \\ \text{for some } \nu \in \mathcal{U} \end{array} \right\}. \tag{1.1.5}$$

As explained in the previous section, increasing the dimension of both the state and the control processes by use of the martingale representation theorem allows to reduce this problem into a standard stochastic target problem.

In the present setting, any martingale $M \in \mathcal{M}_{t,p}$ may be written as

$$M^{\alpha,\chi}_{t,p}(\cdot) = p + \int_t^\cdot \alpha_s \cdot dW_s + \int_t^\cdot \int_E \chi_s(e)\tilde{J}(de,ds), \tag{1.1.6}$$

for some control processes $\alpha$ and $\chi$, with $\tilde{J}(de,ds) := J(de,ds) - \lambda(de)ds$ being the compensated measure associated to $J$. Recalling (1.1.4), we are interested in

$$\mathrm{y}(t,x,p) = \inf \left\{ y \in \mathbb{R} : \begin{array}{c} \text{there exists } (\nu,\alpha,\chi) \in \hat{\mathcal{U}} \\ \text{s.t. } \tilde{\Psi}\left(\hat{X}^{\nu,\alpha,\chi}_{t,x,p}(T), Y^\nu_{t,x,y}(T)\right) \geq 0 \end{array} \right\},$$

where $\hat{X}^{\nu,\alpha,\chi}_{t,x,p}$ stands for the augmented state process $(X^\nu_{t,x}(T), M^{\alpha,\chi}_{t,p})$, and $\hat{\mathcal{U}}$ is the augmented set of controls $(\nu,\alpha,\chi)$.

In order to understand how we can provide a PDE characterization for y, consider the following informal reasoning. In the present settings, $(x,p,y) \in \Lambda(t)$ is equivalent to $y \geq \mathrm{y}(t,x,p)$. Hence, the first part of the GDP (the inclusion $\Lambda(t) \subseteq \bar{\Lambda}(t)$, recall Section 1.1.1) gives that, for $y \geq \mathrm{y}(t,x,p)$, there is $(\nu,\alpha,\chi) \in \hat{\mathcal{U}}$ such that

$$Y^\nu_{t,x,p}(\tau) \geq \mathrm{y}\left(\tau, X^\nu_{t,x}(\tau), M^{\alpha,\chi}_{t,p}(\tau)\right) \ \mathbb{P}\text{-a.s.} \quad \text{for any stopping time } \tau \geq t.$$

Assuming that y is smooth enough and that the above GDP holds even for $y = \mathrm{y}(t,x,p)$, an application of Itô's Lemma around the initial time $t$ shows that the control $(\nu,\alpha,\chi)$ should ensure that

- the volatility of $Y^\nu - \mathrm{y}(\cdot, X^\nu, M^{\alpha,\chi})$ is zero,
- the jumps of $Y^\nu - \mathrm{y}(\cdot, X^\nu, M^{\alpha,\chi})$ are non-negative,
- the drift of $Y^\nu - \mathrm{y}(\cdot, X^\nu, M^{\alpha,\chi})$ is non-negative,

"at the original time $t$". This informal reasoning implies that y is a supersolution of

$$H_{0,0}\mathrm{y}(t,x,p) \geq 0,$$

with

$$H_{\varepsilon,\eta}\mathrm{y}(t,x,p) := \sup_{(u,a,\pi)\in\mathcal{N}_{\varepsilon,\eta}\mathrm{y}(t,x,p)} \left\{\mu_Y(x,\mathrm{y}(t,x,p),u) - \mathcal{L}^{u,a,\pi}_{(X,M)}\mathrm{y}(t,x,p)\right\}, \quad (1.1.7)$$

where $\mathcal{L}^{u,a,\pi}_{(X,M)}$ denotes the Dynkin operator associated to the diffusion $(X, M)$, and, for $\varepsilon > 0, \eta \in [-1, 1]$ and $(t, x, p) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}$,

$$\mathcal{N}_{\varepsilon,\eta}\mathrm{y} := \left\{ \begin{array}{c} (u,a,\pi) \text{ s.t. } |\sigma_Y(\cdot,\mathrm{y},u) - \sigma_X^\top(\cdot,u)\partial_x\mathrm{y} - a\partial_p\mathrm{y}| \le \varepsilon \\ \text{and for } \lambda\text{-a.e. } e \in E \\ \beta_Y(\cdot,\mathrm{y},u,e) - \mathrm{y}(\cdot,\cdot+\beta_X(\cdot,u,e),\cdot+\pi(e)) + \mathrm{y} \ge \eta \end{array} \right\}. \quad (1.1.8)$$

In a Brownian filtration, where the only additional control is $\alpha$, the major difficulty comes from the fact that the process $\alpha$ has a priori no boundedness properties: it comes from the martingale representation theorem. In this context, the operator associated to (1.1.7) typically fails to be semi-continuous.

It is shown in [BET09] that, in the no-jump case, one needs to consider the relaxed semi-limits as $\varepsilon \downarrow 0$ of the operator associated to $H_{\varepsilon,0}$. This relaxation is local, as it only concerns the space point, the gradient and the Hessian matrix of the test function at this point.

In our setting, we need two further relaxations to deal with the non-local term in (1.1.8). Firstly, the semi-limits are taken with respect to the additional parameter $\eta$ as it goes to 0. Secondly, an additional non-local relaxation is performed by considering the semi-continuous envelopes with respect to the test function appearing in the non-local term of (1.1.8), for the topology of the uniform convergence. This adds non-trivial technical difficulties.

The precise statement of the PDE characterization and the associated boundary conditions (in the sense of viscosity solutions) are given in Theorems 2.2.5, 2.2.9 and Corollaries 2.3.7, 2.3.17. In particular, we generalize the convex face-lifting phenomenon in the $p$-variable that was observed in Bouchard, Elie and Touzi [BET09] in the context of quantile hedging problems to much more general situations.

Finally, we provide in Theorem 2.3.14 a boundary condition in the $p$-variable when the function $\Psi$ takes its values in a set of the form $[m, M]$ with $m$ or/and $M$ is/are finite. Theorem 2.3.14 is the counterpart in this framework of [BET09, Theorem 3.1], up to non trivial differences due to the presence of the control $\chi$.

### 1.1.5 Further references and advances in the field of Stochastic Targets

We conclude this section with some references of recent advances in this field. In Bouchard and Dang [BD10], the authors give a PDE characterization of a singular with state constraints version of stochastic target problems. This work allows one to treat the case of market models with proportional transaction costs, and may also be applied to order book liquidation issues.

In [BV11], Bouchard and Vu provide a PDE characterization of the minimal initial endowment required so that the terminal wealth of a financial agent can match a set of constraints in probability. Their original idea was to consider that the agent has a rough idea on the type of P&L he can afford, and that he considers the latter as a target. It was motivated by the fact that, if the attitude of the financial agent toward risk is usually described in academic literature in terms of utility or loss function, this is in practice not so trivial for an agent to characterize precisely his "utility function".

We finally refer to Bouchard, Elie and Reveillac [BER12] for a BSDE formulation of this moment criterion, and to Bouchard, Elie and Imbert [BEI10] and Bouchard and Nutz [BN11] for an optimal stochastic control problem under stochastic target constraint.

## 1.2 A robust version of the stochastic target problems

As exemplified in Section 1.1.2, the stochastic target problems in expectation form allow one to deal with several risk approaches, which is useful in incomplete markets. However, as usual in mathematical finance, the stochastic target problems rely on a choice for the controller of a "mathematical model", that is, a specification of the coefficients $\mu, \sigma$ and $\beta$, as well as their parameters.

In practice, the choice of a model and its calibration give rise to *model risk* (what are the consequences of choosing the wrong model?), or *model uncertainty* (what strategies to employ when no a-priori information on the true coefficient is given?).

One way to tackle the model uncertainty is to consider a situation in which an adverse player, the *nature*, is playing the unknown coefficients against the controller. In the case where the parameters can be observed in a progressive way, this naturally leads to a game version of the stochastic target problems as discussed in the previous sections.

In Chapter 3, we introduce for the first time this new class of differential games, and provide a version of the GDP which allows us to derive the Hamilton-Jacobi-Bellman-Isaacs' (in short HJBI) equation associated to the corresponding reachability set. This requires a game version of the GDP of Soner and Touzi [ST02a].

We refer to [BCR05, Rai07, TH07, MØ08, CR09, BCQ11, Bis10] for advance researches in the field of stochastic differential games.

### 1.2.1 The game version of the Geometric Dynamic Programming Principle

We investigate in Chapter 3 a robust (or game) version of the stochastic target problems. It takes the form of a *one sided* game, defined as follows.

For a given initial position $(t,z) \in [0,T] \times \mathcal{Z}$, the aim of the controller is to find a strategy $\mathfrak{u}[\cdot] \in \mathfrak{U}$, in the sense of differential games (see Section 3.3.1 for a precise definition), such that the controlled state process $Z_{t,z}^{\mathfrak{u}[\nu],\nu}$ reaches a given target at time $T$, whatever the player controlling the adverse controls $\nu \in \mathcal{V}$ could do to prevent it from happening.

We consider a *loss* function $\ell$, and formulate a target in moment in robust form, i.e.

$$\mathbb{E}\left[\ell\left(Z_{t,z}^{\mathfrak{u}[\nu],\nu}(T)\right)\right] \geq p \qquad \text{for all } \nu \in \mathcal{V}.$$

For $t \in [0,T]$, the corresponding reachability set consists in all initial positions $(z,p) \in \mathcal{Z} \times \mathbb{R}$ enabling the controller to find a strategy $\mathfrak{u}$ that allows him to reach the target, for every adverse control $\nu \in \mathcal{V}$:

$$\Lambda(t) := \left\{ \begin{array}{c} (z,p) \in \mathcal{Z} \times \mathbb{R} : \text{there exists } \mathfrak{u} \in \mathfrak{U} \text{ s.t.} \\ J(t,z,\mathfrak{u}) \geq p \end{array} \right\}, \tag{1.2.1}$$

with

$$J(t,z,\mathfrak{u}) := \inf_{\nu \in \mathcal{V}} \mathbb{E}\left[\ell\left(Z_{t,z}^{\mathfrak{u}[\nu],\nu}(T)\right)\right]. \tag{1.2.2}$$

As explained in Section 1.1.3, in the absence of adverse control, one can retrieve the GDP of [ST02a] by considering the martingale $\mathbb{E}[\ell(Z_{t,z}^{\mathfrak{u}}(T))|\mathcal{F}_\cdot]$. Here, the natural counterpart is the family of submartingales $\{S^\nu, \nu \in \mathcal{V}\}$:

$$S^\nu(\cdot) := \operatorname*{ess\,inf}_{\bar{\nu} \in \mathcal{V}} \mathbb{E}\left[\ell\left(Z_{t,z}^{\mathfrak{u}[\nu \oplus_\cdot \bar{\nu}],\nu \oplus_\cdot \bar{\nu}}(T)\right)|\mathcal{F}_\cdot\right], \tag{1.2.3}$$

where $\nu \oplus_s \bar{\nu}$ means that the two adverse controls $\nu$ and $\bar{\nu}$ are pasted at time $s \geq t$. This should be interpreted as the adverse's player value process, if the controller play the strategy $\mathfrak{u}$. Recalling the arguments of Bouchard, Elie and Touzi [BET09] presented in Section 1.1.3, a rough version of the GDP should be that $\Lambda(t)$ coincides with the set of elements $(z,p) \in \mathcal{Z} \times \mathbb{R}$ for which there exist a strategy and an appropriate family of submartingales $\{S^\nu, \nu \in \mathcal{V}\}$, which initial values satisfy $S^\nu(t) = p$ for all $\nu \in \mathcal{V}$, such that

$$\left(Z_{t,z}^{\mathfrak{u}[\nu],\nu}(\tau), S^\nu(\tau)\right) \in \Lambda(\tau) \ \mathbb{P}\text{-a.s.} \quad \text{for all } \nu \in \mathcal{V} \text{ and stopping times } \tau.$$

As we will show in Section 3.2.3, one can actually restrict to the martingale parts of each $S^\nu$:

$$\Lambda(t) = \left\{ \begin{array}{c} (z,p) \in \mathcal{Z} \times \mathbb{R} :\ \exists\, \mathfrak{u} \in \mathfrak{U} \text{ and } \{M^\nu, \nu \in \mathcal{V}\} \subset \mathcal{M}_{t,p} \text{ s.t.} \\ \left( Z_{t,z}^{\mathfrak{u}[\nu],\nu}(\tau), M^\nu(\tau) \right) \in \Lambda(\tau) \ \mathbb{P}\text{-a.s. } \forall\, \nu \in \mathcal{V} \text{ and stopping times } \tau \end{array} \right\},$$

where $\mathcal{M}_{t,p}$ denotes a suitable set of martingale starting from $p$ at time $t$. Neglecting the finite variation parts of the $S^\nu$'s has the advantage of not having to deal with their possible path irregularities. The fact that the martingale part is enough can be understood as follows. The worst situation for the controller playing $\mathfrak{u}$ is when the adverse player plays the optimal adverse control associated to $\mathfrak{u}$. Along an optimal adverse control, $S^\nu$ is a martingale.

Observe that the definitions in (1.2.2) and (1.2.3) do not guarantee that $J(t,z,\mathfrak{u}) = S^\nu(t)$. From the mathematical point of view, one faces the issue of dealing with one nullset for every $\nu \in \mathcal{V}$. One possible answer to handle this problem could be to follow the arguments of Fleming and Souganidis [FS89], and use a discrete time approximation argument. In the context of zero-sum differential games, this provides a Dynamic Programming Principle (DPP) for the approximating problems on the time grids. A limit argument combined with a comparison result for PDEs allows to conclude. Unfortunately, discrete time DPP is not strong enough to derive PDEs in the context of stochastic target problems.

Contrary to Section 1.1.3, we therefore use a formulation of (1.2.2) in terms of essential infimum as in (1.2.3):

$$J(t,z,\mathfrak{u}) := \operatorname*{ess\,inf}_{\nu \in \mathcal{V}} \mathbb{E}\left[ \ell\left( Z_{t,z}^{\mathfrak{u}[\nu],\nu}(T) \right) | \mathcal{F}_t \right]$$

and

$$\Lambda(t) := \left\{ \begin{array}{c} (z,p) \in \mathcal{Z} \times \mathbb{R} : \text{there exists } \mathfrak{u} \in \mathfrak{U} \text{ s.t.} \\ J(t,z,\mathfrak{u}) \geq p \ \mathbb{P}\text{-a.s.} \end{array} \right\}.$$

The consideration of essential infimum is made possible by an argument of Buckdahn and Li [BL08, Proposition 4.1, Lemma 4.1] (see also Buckdahn, Hu and Li [BHL11, Lemmata 3.1 and 3.2] for an extension to jump diffusions), which states that, in a Brownian framework, the random variable

$$K : (t,z) \longmapsto \operatorname*{ess\,sup}_{\mathfrak{u} \in \mathfrak{U}} J(t,z,\mathfrak{u}) \quad \text{is deterministic.} \tag{1.2.4}$$

One major difficulty in establishing a game version of GDP is that we can not apply a measurable selection theorem as in the standard context of [ST02a]. This is due to the presence of strategies for which we do not have a good topological framework. To surround this, we formulate a weak version based on a covering argument in space, in the spirit of Bouchard and Touzi [BT11] or Bouchard and Nutz [BN11]. It only relies on "regularity properties" in the space variable. In particular, we do not impose any time regularity. This issue is solved by a stopping time approximation argument, which leads to a non-trivial additional relaxation. More precisely, our GDP is not stated in terms of

$$\Gamma := \{(t, z, p) \in [0, T] \times \mathcal{Z} \times \mathbb{R} \text{ s.t. } (z, p) \in \Lambda(t)\},$$

but in terms of its interior and its closure.

Note that the absence of measurable selection result adapted to the context of games prevents us to consider more general stochastic target games in which the terminal constraint is stated in the $\mathbb{P}$-a.s. sense. This highly difficult issue is left for further researches.

We finally observe that, if the weak GDP is first stated in Theorem 3.2.1 under strong regularity assumptions on the (deterministic) map $K$ defined in (1.2.4), we show in Corollary 3.2.3 how to relax these assumptions in case of a continuous function $\ell$ with polynomial growth, and when the state process $Z$ satisfies suitable estimates.

### 1.2.2 Derivation of the Hamilton-Jacobi-Bellman-Isaacs' equation

Our weak GDP happens to be sufficient for the derivation of the dynamic programming equation in the viscosity sense. We exemplify this fact with the treatment of a game version of two general problems introduced by Soner and Touzi [ST02c, ST02a] in the context of Brownian controlled SDEs, with controls taking their values in a bounded subset of $\mathbb{R}^d$.

In Theorem 3.3.3, we characterize the reachability set with a Hamilton-Jacobi-Bellman-Isaacs' (HJBI) equation. Namely, we state the PDE satisfied, in the viscosity sense, by the indicator function of the complement of the graph of $\Lambda$:

$$\chi(t, z, p) := 1 - \mathbf{1}_{\Lambda(t)}(z, p).$$

In Theorem 3.3.5, we consider a robust version of the stochastic target problem with controlled expected loss discussed in Section 1.1.2. We hence derive the PDE satisfied by a game version of the problem (1.1.2), i.e.

$$\mathrm{y}(t, x, p) := \inf \left\{ y \in \mathbb{R} : \begin{array}{c} \text{there exists } \mathfrak{u} \in \mathfrak{U} \text{ s.t. for all } \nu \in \mathcal{V} \\ \mathbb{E}\left[\ell\left(Z_{t,z}^{\mathfrak{u}[\nu],\nu}(T)|\mathcal{F}_t\right)\right] \geq p \ \mathbb{P}\text{-a.s.} \end{array} \right\}.$$

This allows us to give a robust characterization of the problems considered in Section 1.1.2.

As discussed in Section 1.1.4, these equations are stated in terms of relaxed HJBI operators, in order to take into account the possible unboundedness of the controls $\alpha$, which come from the martingale representation of the additional state variable.

Finally, we give an example of application to the partial hedging of a European option, in the case where both the drift and the volatility of the underlying are uncertain (controlled by the adverse player, which in that case is the market). We are interested in the problem

$$\mathrm{y}(t,x,p) := \inf\left\{ y \in \mathbb{R} : \begin{array}{c} \exists\, \mathfrak{u} \in \mathfrak{U} \text{ s.t. for all } \nu \in \mathcal{V} \\ \mathbb{E}\left[\Psi\left(Y^{\mathfrak{u},\nu}_{t,x,y}(T) - g\left(X^{\nu}_{t,x}(T)\right)\right)|\mathcal{F}_t\right] \geq p \ \mathbb{P}\text{-a.s.} \end{array} \right\}, \quad (1.2.5)$$

in which $\Psi$ denotes some utility function (concave, non-decreasing), $g(X^{\nu}_{t,x}(T))$ the payoff of the claim, $\nu = (\mu,\sigma)$ stands for the drift and the volatility of the stock price process $X^{\nu}$, whereas $\mathfrak{u}$ is the trading strategy and $Y^{\mathfrak{u}[\nu],\nu}$ is the corresponding wealth process.

In this case, strategies do not take bounded values, and we restrict ourselves to the set of strategies satisfying an integrability condition of the form:

$$\sup_{\nu\in\mathcal{V}} \mathbb{E}\left[\left|\int_0^T |\mathfrak{u}[\nu]_r|^2 dr\right|^{\frac{\bar{q}}{2}}\right] < \infty,$$

for some $\bar{q} > 2$. We extend the PDE characterization obtained for bounded controls to this context. This allows us to give an explicit characterization of the problem (1.2.5). Surprisingly, although the hedging criteria is weak, the result is degenerate. Namely, we prove that

$$\mathrm{y}(t,x,p) = \sup_{\nu\in\mathcal{V}^0} \mathbb{E}\left[g\left(X^{\nu}_{t,x}(T)\right)|\mathcal{F}_t\right] + \Psi^{-1}(p),$$

where $\mathcal{V}^0$ denotes the subset of adverse controls such that the drift $\mu$ is degenerate: $\mu \equiv 0$. This corresponds to the super-hedging price for the shifted option $g(\cdot) + \Psi^{-1}(p)$ in the (driftless) uncertain volatility model.

## 1.3 Utility Asymptotics - Pricing of Hybrid claims

These last years have seen the explosion of the number of liabilities combining pure financial and pure insurancial risks. They typically have the following form: an insurance company sells to each client $i \leq n$ a claim of maturity $T$, whose value depends on the evolution of some tradable financial assets $S = (S_t)_{t\geq 0}$ and some additional idiosyncratic risk $R_i$. The number $n$ introduced above denotes the number of claims sold by the company.

In Chapter 4, we investigate the problem of pricing such claims, in the realistic situation where the $R_i$'s are independent and identically distributed, conditionally to $S$.

Our main result concerns the convergence of the utility indifference price of a claim when the absolute risk aversion of a sequence of general utility functions tends to 0, and the number of sold claims goes to infinity.

### 1.3.1 Examples of hybrid products

The wide range of applications in life or non-life insurance justifies the interest of both insurance and financial mathematics. We list here some examples of such contracts.

The agents are interested in pricing aggregated claims of the following form

$$G_n = \sum_{i=1}^{n} f(S, R_i), \tag{1.3.1}$$

where for each client $i \in \{1, \cdots, n\}$, the $R_i$'s are independent and identically distributed random variables, $n$ denotes the number of unit claims $f(S, R_i)$ sold, and $f$ is some measurable function. In the latter, one could think e.g. of unit-linked contract,

$$f(S, R_i) = \mathbf{1}_{\{R_i > T\}} S_T,$$

or unit-linked with guarantee,

$$f(S, R_i) = \mathbf{1}_{\{R_i > T\}} \max(S_T, K),$$

where $R^i$ denotes in both examples the time of death of the customer $i$, and $S$ a financial index. Contracts with similar features are currently very popular in life insurance.

We might also think of more elaborated claims, with $R_i$ being for example a weather index, or a production yield.

Consider for instance a producer of some good (e.g. wheat), which market price is $S$, expecting to produce the yield $K_G^i$ at time $T$, and to sell each unit of this quantity at least at the price $K_S$. His expected revenue is then $K_S \times K_G^i$, while his realized revenue is $R_i \times S_T$, with $R_i$ his realized production level. In order to cover himself, he can buy a European put on his revenue:

$$f^i(S, R_i) := (K_S \times K_G - S_T R_i)^+ . \tag{1.3.2}$$

These revenue guarantees are already widely sold in the U.S. and are about to be exploited in Europe too.

### 1.3.2 Utility Asymptotics

Consider an insurance company selling to the client $i$ a claim with payoff $g^i$, paid at maturity $T$, whose value depends on the evolution of some tradable financial assets $S = (S_t)_{t\geq 0}$ and some additional idiosyncratic risk. Typically, recall (1.3.1), for every $i \leq n$, each individual contract $g^i$ is of the following form

$$g^i = f(S, R_i).$$

The $g^i$'s are usually not unconditionally independent, but still independent conditionally to $S$. The company is then interested in the unit premium $\pi(G_n)/n$ of the aggregated claim

$$G_n := \sum_{i=1}^{n} g^i,$$

i.e. the premium associated to the global claim $G_n$, $\pi(G_n)$, equally divided by the number of sold contracts $n$.

Such contracts, and especially unit-linked contracts, have been studied by actuaries since the late sixties. While in finance, any pricing rule is fundamentally based on the notion of *no-arbitrage* and the corresponding set of *martingale measures*, the *premium principles* in insurance are mainly motivated by the application of the law of large numbers. (see e.g. [Buh70], [GfIE79] or [BoA86]).

In fact, neither the usual actuarial principles nor the arbitrage arguments seem to be satisfactory to price such claims. Still, it was suggested to combine both. Namely, the valuation principle proposed in Brennan and Schwartz [BS79a, BS79b] consists in combining the law of large numbers with a financial hedging-based valuation. The idea is to replace the insured risks by their expected value, so that the modified claim only contains financial uncertainty. It remains then for the insurer to price and hedge the following modified claim

$$\hat{G}_n = \sum_{i=1}^{N} \mathbb{E}\left[ f\left(S, R_i\right) \middle| S \right].$$

This pricing rule has been widely used in practice, see e.g. [BH03, MP00, MPY06].

The mathematical insight behind this trivial pricing rule is the following. If the $R_i$'s are independent and identically distributed given $S$, then $G_n/n \to \mathbb{E}[f(S, R_1)|S] =: \bar{g}$ a.s. for a large number $n$ of sold contracts. If the financial market formed by the asset $S$ is complete (this is the semi-complete market of Becherer, see [Bec03, Section 4]), then the payoff $\mathbb{E}[f(S, R_1)|S]$ may be replicated from the initial wealth $\mathbb{E}^{\mathbb{Q}}[\mathbb{E}[f(S,R_1)|S]]$ by a suitable trading strategy, where $\mathbb{Q}$ is the unique martingale measure on the (complete) pure financial market.

However, both the theory of pricing in incomplete markets and the usual actuarial principles (recall the notion of *safety loading*) seem to agree on the fact that a linear pricing rule corresponds to a risk neutral agent. Roughly speaking, in our context, selling a large number of claims (necessary for the application of the law of large numbers) entails a bigger exposition on the financial market. If the law of large numbers does not operate well enough, then the losses may be leveraged by an unfavorable evolution of the financial market. A risk-adverse agent shall take this fact into account, so that the (linear) trivial pricing rule should not hold for such agents (see Examples 4.2.3 and 4.2.4 for trivial counterexamples).

This intuitive reasoning leads us to expect this pricing rule to hold only at the limit for a small level of risk aversion and a large number of sold claims. In Chapter 4, we provide conditions under which the limit unit price is given by this linear pricing rule.

Given a locally bounded càdlàg $(\mathbb{F},\mathbb{P})$-semi-martingale $S$, we denote as usual by $\mathcal{M}$ the set of $\mathbb{P}$-equivalent local martingale measures such that $S$ is a $(\mathbb{F},\mathbb{Q})$-local martingale. Let $(U_n)_{n\in\mathbb{N}}$ be a sequence of utility functions defined on the whole real line and satisfying the usual assumptions (Inada, reasonable asymptotic elasticity, see Schachermayer [Sch01]). Assume furthermore that $\mathcal{M}\neq\emptyset$ and that for each $n\in\mathbb{N}$, the corresponding dual problem (see e.g. [Sch01]) is finite, and define the unit utility indifference prices $p_n(G_n,U_n)$:

$$p_n(G_n,U) := \inf\left\{p\in\mathbb{R} : \sup_X \mathbb{E}\left[U(X+np-G_n)\right] \geq \sup_X \mathbb{E}\left[U(X)\right]\right\}, \tag{1.3.3}$$

with $X$ running over the set of achievable terminal wealth. Moreover the optimal *dual probability and multiplier* are given by

$$(y_n^0,\mathbb{Q}_n^0) := \arg\min\left\{\mathbb{E}\left[V_n\left(y\frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right],\ (y,\mathbb{Q})\in(0,\infty)\times\mathcal{M}\right\},$$

in which $V_n$ is the usual convex conjugate of $U_n$. We assume that the sequence of claims $(G_n)_{n\geq1}$ satisfies

$$\sup_{n\geq1}|G_n/n|_{L^\infty}<\infty, \tag{1.3.4}$$

and that

$$n|r_n|_\infty \underset{n\to\infty}{\longrightarrow} 0\,, \qquad \text{with} \quad r_n : x\mapsto -\frac{U_n''(x)}{U_n'(x)}\,, \tag{1.3.5}$$

and $|r_n|_\infty := \sup_{x\in\mathbb{R}}|r_n(x)|$. Observe that Assumption (1.3.4) allows us to consider examples of individual bounded claims, such as the payoff in (1.3.2).

We show in Theorem 4.3.2 that

$$\lim_{n\to\infty} p_n(G_n, U_n) = \lim_{n\to\infty} \mathbb{E}^{\mathbb{Q}_n^0}\left[G_n/n\right]. \tag{1.3.6}$$

One side of the equality (stated in terms of liminf and limsup) is straightforward with the use of the dual problem (see e.g. Owen [Owe02] or Bouchard, Touzi and Zeghal [BTZ04]). Surprisingly, the second inequality is obtained directly from the primal formulation of the problem (contrary to most results on the asymptotic of utility indifference prices, see Section 1.4 below). It relies on a simple second order Taylor expansion of $U_n$, and crucially on Assumptions (1.3.4) and (1.3.5) .

As a byproduct, under the weaker condition $\|r_n\|_\infty \to 0$, and whenever the sequence $(G_n)_{n\geq 1}$ is assumed to be uniformly bounded in $L^\infty$, we also provide a general convergence result for bounded sequences of contingent claims when the absolute risk aversion vanishes in the sup norm, which is of own interest.

Notice that the right hand side term in (1.3.6) is somehow theoretical. In the context of a complete pure financial market (see Definition 4.2.1 for a more precise definition of the so-called Half-Complete Market assumption), a similar reasoning as in [Bec03, Theorem 4.10 and Assertion (4.5)] shows that, if $G_n/n \to \bar{g}$ as $n \to \infty$, with $\bar{g}$ a $\mathcal{F}_T^S$-measurable random variable, then

$$\lim_{n\to\infty} \mathbb{E}^{\mathbb{Q}_n^0}\left[G_n/n\right] = \mathbb{E}^{\mathbb{Q}^*}[\bar{g}],$$

where $\mathbb{Q}^*$ is the pricing measure on the complete pure financial market.

In order to characterize the limit

$$\lim_{n\to\infty} \mathbb{E}^{\mathbb{Q}_n^0}\left[G_n/n\right]$$

in the incomplete market case, we shall restrict the class of utility functions. First note that the fact that $r_n \to 0$ uniformly as $n \to \infty$ entails that there exists a sequence $(\eta_n^1)_{n\geq 1}$ satisfying $\eta_n^1 \to 0$ such that

$$r_n(x) \leq \eta_n^1 \qquad \text{for all } x \in \mathbb{R} \text{ and } n \geq 1.$$

We assume in addition that the convergence $r_n \to 0$ is not too fast: there exists another sequence $(\eta_n^2)_{n\geq 1}$ such that for all $n \geq 1$,

$$0 < \eta_n^2 \leq r_n \leq \eta_n^1 \qquad \text{and} \qquad \eta_n^2/\eta_n^1 \underset{n\to\infty}{\longrightarrow} 1.$$

The sequence $(U_n)_{n\geq 1}$ is "stucked" in between two sequences of exponential utility functions with vanishing asymptotically equivalent risk aversions. We thus are able to show that

$$\mathbb{E}^{\mathbb{Q}_n^0}[G_n/n] \underset{n\to\infty}{\longrightarrow} \mathbb{E}^{\mathbb{Q}^e}[\bar{g}],$$

where $\mathbb{Q}^e$ is the element of $\mathcal{M}$ which minimizes the relative entropy $\mathbb{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}} \log \frac{d\mathbb{Q}}{d\mathbb{P}}\right]$.

## 1.4 Further references on Utility Indifference Price Asymptotics

Asymptotic results for utility indifference prices have been stated for exponential utility function in El Karoui and Rouge [EKR00] for Brownian diffusion models, and in Delbaen et al. [DGR+02] in a general semi martingale setting. In the above quoted papers, it was shown that the utility indifference price converges toward the super-replication price as the absolute risk aversion tends to infinity. A slightly more general class of utility functions is studied in [Bou00]. Carassus and Rásonyi consider general utility functions, in discrete time models, in [CR07, CR06], and deal with the continuous time case in the recent paper [CR11].

Importantly, note that Becherer [Bec03] has studied almost similar problems in the context of exponential utility functions. More precisely, he is interested in the indifference price $p_1(G_n/n; U)$ of the mean claim $G_n/n$, whereas we consider the unit price of the components of $G_n$, $p_n(G_n; U) = p_1(G_n, U)/n$, recall the notation (1.3.3).

However, our result can be recovered in his more restrictive context from the additivity property stated in his Theorem 4.10 and the standard asymptotic result of his Proposition 3.2.

# Notations

In all this manuscript, elements of $\mathbb{R}^n$, $n \geq 1$, are identified to column vectors, the superscript $^\top$ stands for transposition, $\cdot$ denotes the scalar product on $\mathbb{R}^n$, $|\cdot|$ the Euclidean norm, and $\mathbb{M}^n$ denotes the set of $n$-dimensional square matrices. We denote by $\mathbb{S}^n$ the subset of elements of $\mathbb{M}^n$ which are symmetric. For a subset $\mathcal{O}$ of $\mathbb{R}^n$, $n \geq 1$, we denote by $\overline{\mathcal{O}}$ its closure, by $\mathrm{Int}(\mathcal{O})$ its interior and by $\mathrm{dist}(x, \mathcal{O})$ the Euclidean distance from $x$ to $\mathcal{O}$ with the convention $\mathrm{dist}(x, \emptyset) = \infty$. Finally, we denote by $B_r(x)$ the open ball of radius $r > 0$ centered at $x \in \mathbb{R}^n$. Given a square matrix $M \in \mathbb{M}^n$, we denote Tr its trace, that is $\mathrm{Tr}[M] := \sum_{i=1}^n M_{ii}$. For $x, y \in \mathbb{R}$, we will use $x \vee y := \max(x, y)$, $x \wedge y := \min(x, y)$, $x^+ := x \vee 0$ and $x^- := (-x) \vee 0$.

Let $\varphi \in C^2(\mathbb{R}^d; \mathbb{R})$ a smooth function; $D\varphi$ denotes the Jacobian matrix of $\varphi$, i.e. $(D\varphi)_i := \frac{\partial \varphi}{\partial x_i}$, and $D^2\varphi$ its Hessian matrix, i.e. $(D^2\varphi)_{ij} := \frac{\partial^2 \varphi}{\partial x_i x_j}$. In case we wish to denote a partial derivative of $\varphi$ with respect to one or two of its variable(s), we shall use the notation $\partial_{x_i x_j} \varphi := \frac{\partial^2 \varphi}{\partial x_i x_j}$.

Given a locally bounded map $v$ on a subset $B$ of $\mathbb{R}^n$, we define the lower and upper semicontinuous envelopes

$$v_*(b) := \liminf_{B \ni b' \to b} v(b') \qquad v^*(b) := \limsup_{B \ni b' \to b} v(b'), b \in \overline{B}.$$

The convex hull of a function $f$ will be denoted $\circledcirc(f)$, and we recall that it is the greatest convex function lower or equal to $f$. We will use the same notation for the convex hull of a subset, i.e. $\circledcirc(A)$ is the convex hull of the subset $A$, and we recall that it is the smallest convex subset containing $A$, in the sense of inclusion.
In this manuscript, inequalities between random variable have to be understood in the a.s. sense.

# Stochastic Target With Controlled Loss in Jump Diffusion Models - Abstract

**Abstract**

In this chapter, we consider a mixed diffusion version of the stochastic target problem introduced in [BET09]. This consists in finding the minimum initial value of a controlled process which guarantees to reach a controlled stochastic target with a given level of expected loss. It can be converted into a standard stochastic target problem, by increasing both the state space and the dimension of the control. In our mixed-diffusion setting, the main difficulty comes from the presence of jumps, which leads to the introduction of a new kind of controls that take values in an unbounded set of measurable maps. This has non trivial technical impacts on the formulation and derivation of the associated partial differential equations.

**Keywords:** Stochastic target problem, mixed diffusion process, discontinuous viscosity solutions, quantile hedging.

**Note:** The work presented in this chapter is taken from [Mor11], and has been accepted for publication in **SIAM**, *Journal on Control and Optimization*.

# Chapter 2

# Stochastic Target With Controlled Loss in Jump Diffusion Models

**Contents**



## 2.1 Introduction

For $0 \leq t \leq T$, and given two controlled diffusion processes $\left\{X^{\nu}_{t,x}(s), t \leq s \leq T\right\}$ and $\{Y^{\nu}_{t,x,y}(s), t \leq s \leq T\}$ with values respectively in $\mathbb{R}^d$ and $\mathbb{R}$, satisfying the initial condition $\left(X^{\nu}_{t,x}(t), Y^{\nu}_{t,x,y}(t)\right) = (x, y)$. We are interested in finding the minimal initial condition $y$ for which it is possible to find a control $\nu$ satisfying $\mathbb{E}\left[\Psi\left(X^{\nu}_{t,x}(T), Y^{\nu}_{t,x,y}(T)\right)\right] \geq p$ for some given Borel measurable map $\Psi$, non-decreasing in the $y$-variable, and for a threshold $p$. Namely, we want to characterize the value function:

$$\hat{v}(t,x,p) := \inf\left\{y \geq -\kappa : \mathbb{E}\left[\Psi\left(X^{\nu}_{t,x}(T), Y^{\nu}_{t,x,y}(T)\right)\right] \geq p \text{ for some } \nu\right\}, \tag{2.1.1}$$

in the mixed diffusion case. If $\Psi(x,y) := \mathbb{1}_{\{V(x,y)\geq 0\}}$ and $p \in (0,1)$,

$$\hat{v}(t,x,p) = \inf\left\{y \geq -\kappa : \mathbb{P}\left[V\left(X^{\nu}_{t,x}(T), Y^{\nu}_{t,x,y}(T)\right) \geq 0\right] \geq p \text{ for some } \nu\right\}\ , \tag{2.1.2}$$

this problem coincides with the quantile hedging problem discussed in Föllmer and Leukert [FL99], in the context of financial mathematics. In this paper, the process $X$ represents the prices of some given securities. The process $Y$ models the wealth of an investor, based on some portfolio strategy $\nu$. Importantly, the coefficients of the diffusion $Y$ are linear in the control variable and the process $X$ is not affected by the control $\nu$. In this context, Föllmer and Leukert [FL99] used a duality argument to convert this problem into a classical test problem in mathematical statistics.

In order to deal with the problem (2.1.2) in a more general case, Bouchard, Elie and Touzi [BET09] introduced an additional controlled diffusion process $P^{\alpha}_{t,p}$, which appears to (essentially) correspond to the conditional probability of reaching the target $V\left(X^{\nu}_{t,x}(T), Y^{\nu}_{t,x,y}(T)\right) \geq 0$. This allowed them to rewrite the problem 2.1.2 in the form

$$\hat{v}(t,x,p) = \inf\left\{y \geq -\kappa : \mathbf{1}_{\{V\left(X^{\nu}_{t,x}(T), Y^{\nu}_{t,x,y}(T)\right)\geq 0\}} \geq P^{\alpha}_{t,p}(T) \text{ for some } (\nu,\alpha)\right\}\ ,$$

where $\alpha$ is a predictable square integrable process coming from the martingale representation of $\mathbb{P}\left[V\left(X^{\nu}_{t,x}(T), Y^{\nu}_{t,x,y}(T)\right) \geq 0 \mid \mathcal{F}_{\cdot}\right] = P^{\alpha}_{t,p_o} := p_o + \int_t^{\cdot} \alpha_s \cdot dW_s$, for some $p_o \geq p$. The key point is that this reformulation reduces the original problem (2.1.2) into a classical stochastic target problem of the form

$$\hat{v}(t,x,p) := \inf\left\{y \geq -\kappa : \hat{V}\left(X^{\nu}_{t,x}(T), P^{\alpha}_{t,p}(T), Y^{\nu}_{tx,y,}(T)\right) \geq 0 \text{ for some } \nu, \alpha\right\},$$

as studied in Soner and Touzi [ST02a, ST02c], for an augmented system $(X,Y,P)$ and an augmented control $(\nu,\alpha)$. The major difference being that the new control $\alpha$ can no longer be assumed to take values in a compact set, as it is given by the martingale representation theorem.

Up to a non-trivial relaxation, Bouchard, Elie and Touzi [BET09] were able to provide a PDE characterization for the value function $\hat{v}$ in the sense of discontinuous viscosity solutions, for a discontinuous operator which corresponds to the one used in Soner and Touzi [ST02a, ST02c].

The aim of this chapter is to extend the work of Bouchard, Elie and Touzi [BET09] to the setting of jump diffusions, in its more general form (2.1.1).

Diffusing the conditional expectation $\mathbb{E}\left[\Psi\left(X^{\nu}_{t,x}(T), Y^{\nu}_{t,x,y}(T)\right)\middle|\, \mathcal{F}_s\right]$ for $s \in [t,T]$, and considering it as an additional controlled state variable $P^{\alpha,\chi}_{t,p}$ will allow us to convert this problem into a singular stochastic target problem. Here, the additional control $\chi$ comes from the jump part of the martingale representation.

This leads to technical difficulties, mainly because of this new control $\chi$. The first one was already handled in [Bou02], and consists in the consideration of an additional (non-local) term in the PDE characterization. Secondly, part of the control now takes values in an unbounded set of measurable maps, as opposed to a compact subset of $\mathbb{R}^d$. The local relaxation of the associated HJB operator introduced in Bouchard, Elie and Touzi [BET09] will not be sufficient to ensure the semicontinuity needed, and we shall have to introduce a new (non-trivial) relaxation of the non-local part of the associated operator. Furthermore, this non-local operator complicates significantly the discussion of the boundary conditions at $p = m$ and $p = M$ when the map $\Psi$ takes values in $[m, M]$.

Compared to Bouchard, Elie and Touzi [BET09], where they discuss general problem of the form (2.1.1), but state their results for the problem (2.1.2), we aim to state our results for the problem (2.1.1). In particular, we shall see that the convex face-lifting phenomenon in the $p$-variable observed in Bouchard, Elie and Touzi [BET09] for (2.1.2) extends to a much more general context.

This chapter is organized as follows. In Section 2.2, we present the general formulation of stochastic target problem with unbounded measurable map controls, in mixed diffusion case. It contains the statement of the corresponding dynamic programming equation. In Section 2.3, we give the arguments allowing us to translate the problem of expected controlled loss into the case of singular stochastic target problem of the previous section. The boundary conditions for the stochastic target problem with controlled expected loss are discussed in this section.

## 2.2 Singular stochastic target problems

### 2.2.1 Problem formulation

Let $T > 0$ be a fixed time, $E$ a borel subset of $\mathbb{R}_+$, equipped with its Borel $\sigma$-field $\mathcal{E}$, $J(de, dt) = \sum_{i=1}^{d} J^i(de, dt)$ be a $E$-marked right-continuous point process defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $W$ be a $\mathbb{R}^d$-Brownian motion defined on $(\Omega, \mathcal{F}, \mathbb{P})$, such that $W$ and $J$ are independent. We denote by $\mathbb{G} := \{\mathcal{G}_t, 0 \le t \le T\}$ the $\mathbb{P}$-augmented filtration generated by $(W., J(de, \cdot))$. We assume that $\mathcal{G}_0$ is trivial. The random measure $J(de, dt)$ is assumed to have a predictable $(\mathbb{P}, \mathbb{G})-$intensity kernel $\lambda(de)dt$ such that $\lambda(E) < \infty$, and we denote by $\tilde{J}(de, dt) := J(de, dt) - \lambda(de)dt$ the associated compensated random measure. By $\mathbb{H}^2_\lambda$, we denote the set of maps $\chi : \Omega \times [0,T] \times E \to \mathbb{R}$ which are $\mathcal{P} \bigotimes \mathcal{E}$ measurable [1] and such that

$$\|\chi\|_{\mathbb{H}^2_\lambda} := \left( \mathbb{E}\left[ \int_0^T \int_E \chi_t(e)^2 \lambda(de)dt \right] \right)^{\frac{1}{2}} < \infty.$$

We can always assume that $\mathbb{P}\left[J\left(E \setminus \mathrm{supp}(\lambda), [0,T]\right) > 0\right] = 0$, and therefore that $E = \mathrm{supp}(\lambda)$. Let $\mathcal{U}_0 = \mathcal{U}_0^1 \times \mathcal{U}_0^2$ be the collection of predictable processes $\nu = \left(\nu^1, \nu^2\right)$ with $\nu^1 \in L^2\left([0,T] \times \Omega\right)$ and $\nu^2 \in \mathbb{H}^2_\lambda$ $\mathbb{P}$-a.s., and with values in a given closed subset $U = U^1 \times \mathbb{L}^2_\lambda$ of $\mathbb{R}^d \times \mathbb{L}^2_\lambda$. Here $\mathbb{L}^2_\lambda$ denotes the set of measurable functions $\pi : E \to \mathbb{R}$ such that $\|\pi\|^2_\lambda < \infty$, with

$$\|\pi\|^2_\lambda := \int_E |\pi(e)|^2 \lambda(de).$$

For $t \in [0,T]$, $z = (x,y) \in \mathbb{R}^d \times \mathbb{R}$ and $\nu := \left(\nu^1, \nu^2\right) \in \mathcal{U}_0$, we define $Z^\nu_{t,z} := \left(X^\nu_{t,x}, Y^\nu_{t,x,y}\right)$ as the $\mathbb{R}^d \times \mathbb{R}$-valued solution of the stochastic differential equation

$$\begin{aligned} X(s) =& \mu_X\left(X(s), \nu_s\right) ds + \sigma_X\left(X(s), \nu_s\right) dW_s \\ &+ \int_E \beta_X\left(X(s-), \nu^1_s, \nu^2_s(e), e\right) J(de, ds) \\ dY(s) =& \mu_Y\left(Z(s), \nu_s\right) ds + \sigma_Y\left(Z(s), \nu_s\right) dW_s \\ &+ \int_E \beta_Y\left(Z(s-), \nu^1_s, \nu^2_s(e), e\right) J(de, ds) \end{aligned} \tag{2.2.1}$$

satisfying the initial condition $Z(t) = (x,y)$. Here,

$$\begin{aligned} (\mu_X, \sigma_X) &: \mathbb{R}^d \times U \to \mathbb{R}^d \times \mathbb{M}^d \\ (\mu_Y, \sigma_Y) &: \mathbb{R}^d \times \mathbb{R} \times U \to \mathbb{R} \times \mathbb{R}^d \end{aligned}$$

are locally Lipschitz, and are assumed to satisfy, for $u := (u^1, u^2) \in U$,

$$|\mu_Y(x,y,u)| + |\mu_X(x,u)| + |\sigma_Y(x,y,u)| + |\sigma_X(x,u)| \le K(x,y)\left(1 + |u^1| + \left\|u^2\right\|_\lambda\right)$$

where $K$ is a locally bounded map. Moreover

$$\begin{aligned} \beta_X &: \mathbb{R}^d \times U \times E \to \mathbb{R}^d \\ \beta_Y &: \mathbb{R}^d \times \mathbb{R} \times U \times E \to \mathbb{R} \end{aligned}$$

are continuous and are assumed to satisfy, for some $M \geq 0$,

$$\begin{aligned} \int_E \left( |\beta_X(x,u(e),e)|^2 + |\beta_Y(z,u(e),e)|^2 \right) \lambda(de) &\le M\left(1 + |z|^2 + |u|^2\right) \\ \int_E |\beta_X(x,u(e),e) - \beta_X(x',u(e),e)|^2 \lambda(de) &\le M\left|x - x'\right|^2 \\ \int_E |\beta_Y(z,u(e),e) - \beta_Y(z',u(e),e)|^2 \lambda(de) &\le M\left|z - z'\right|^2, \end{aligned} \tag{2.2.2}$$

[1] $\mathcal{P}$ denotes the $\sigma$-algebra of $\mathbb{F}$-predictable subsets of $\Omega \times [0,T]$.

where we have used the notation $u(e) = (u^1, u^2(e))$ and $|u|^2 := |u^1|^2 + \|u^2\|_\lambda^2$. We denote by $\mathcal{U} = \mathcal{U}^1 \times \mathcal{U}^2$ a subset of elements of $\mathcal{U}_0$ for which (2.2.1) admits an unique strong solution for all given initial data. We assume furthermore that any constant controls with values in $U$ belongs to $\mathcal{U}$. We also allow for state constraints and we denote by $\mathbf{X}$ the interior of the support of the controlled process $X$.

Let $V$ be a measurable map from $\mathbb{R}^{d+1}$ to $\mathbb{R}$ such that, for every fixed $x$, the function

$$y \longmapsto V(x,y) \text{ is non-decreasing and right continuous.}$$

We then define the stochastic target problem as follows

$$v(t,x) := \inf\left\{y \geq -\kappa : V\left(X^\nu_{t,x}(T), Y^\nu_{t,x,y}(T)\right) \geq 0 \text{ for some } \nu \in \mathcal{U}\right\}, \tag{2.2.3}$$

with $\kappa \in \mathbb{R}_+ \cup \{+\infty\}$. At this point, the set $U$ may not be bounded, and we will see later that dealing with unbounded controls will be required in the analysis of Section 2.3.

In order to be consistent and avoid the process $Y$ to cross the level $-\kappa$, we shall assume all over this chapter that

$$\begin{gathered}\mu_Y(x,-\kappa,u) \geq 0, \quad \sigma_Y(x,-\kappa,u) = 0 \quad \text{and} \quad \beta_Y(x,y,u,e) \geq -(y+\kappa) \\ \text{for all } (x,y,u,e) \in \mathbf{X} \times \mathbb{R} \times U \times E.\end{gathered} \tag{2.2.4}$$

As usual in this kind of problem, our analysis requires that

$$y' \geq y \text{ and } y \in \Gamma(t,x) \Rightarrow y' \in \Gamma(t,x) \quad \text{for all} \quad (t,x,y,y') \in [0,T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}$$

where

$$\Gamma(t,x) := \left\{y \geq -\kappa : V\left(X^\nu_{t,x}(T), Y^\nu_{t,x,y}(T)\right) \geq 0 \text{ for some } \nu \in \mathcal{U}\right\}.$$

This allows to characterize the closure of $\Gamma(t,x)$ as $[v(t,x), +\infty)$, which will be of important use in the following. Indeed, let us assume that the infimum in the definition of $v$ is attained, and let $y = v(t,x)$. Then we can find some $\nu \in \mathcal{U}$ such that $V\left(X^\nu_{t,x}(T), Y^\nu_{t,x,y}\right) \geq 0$. Hence, if we start with $y' > y$, we should be able to find some $\nu' \in \mathcal{U}$ such that $V(X^{\nu'}_{t,x}(T), Y^{\nu'}_{t,x,y'}) \geq 0$. If this property does not hold, it is not possible to characterize the set $\Gamma(t,x)$ by its lower bound $v(t,x)$.

**Remark 2.2.1.** Let us observe that this problem can be formulated equivalently as

$$v(t,x) := \inf\left\{y \geq -\kappa : Y^\nu_{t,x,y}(T) \geq g\left(X^\nu_{t,x}(T)\right) \text{ for some } \nu \in \mathcal{U}\right\},$$

where $g$ is the generalized inverse of $V$ at 0:

$$g(x) := \inf\{y \geq -\kappa : V(x,y) \geq 0\}, \tag{2.2.5}$$

recall (2.2.4).

**Example 2.2.1.** Consider the case where $\mathbf{X} = (0,\infty)^d$ and $X$ is defined by the stochastic differential equation

$$dX_{t,x}(s) = \mu\left(X_{t,x}(s)\right)ds + \sigma\left(X_{t,x}(s)\right)dW_s + \int_E \beta\left(X_{t,x}(s-), e\right)J(de,ds)$$
$$X_{t,x}(t) = x \in (0,\infty)^d,$$

with $Y^{\nu}_{t,x,y}$ given by

$$Y^{\nu}_{t,x,y}(s) = y + \int_t^s \nu^1_r \cdot dX_{t,x}(r), \quad \text{for} \quad s \geq t \quad \text{and} \quad \nu = \left(\nu^1, \nu^2\right) \in \mathcal{U}.$$

This corresponds to the situation where the process $X$ is not affected by the control:

$$\begin{matrix} \mu_X(x,u) = \mu(x), \quad \sigma_X(x,u) = \sigma(x) \\ \text{and} \quad \beta_X(x,u(e),e) = \beta(x,e) \end{matrix} \qquad \text{are independent of } u$$

and

$$\mu_Y(x,y,u) := u^1 \cdot \mu(x), \quad \sigma_Y(x,y,u) := \sigma^\top(x)u^1, \quad \beta_Y(x,y,u(e),e) := u^1 \cdot \beta(x,e).$$

In financial mathematics, the process $X$ should be interpreted as the price of $d$ risky securities. Because of the jump diffusions, we are in an incomplete market, so that the uniqueness of a $\mathbb{P}$-equivalent martingale measure is not satisfied. The process $Y$ represents the wealth process induced by the trading strategy $\nu$, where $\nu^1_s$ indicates the number of units of the assets in the portfolio at time $s$.

Finally, for some Lipschitz continuous function $g : \mathbb{R} \to \mathbb{R}_+$ and

$$V(x,y) := y - g(x),$$

$v(t,x)$ coincides with the usual superhedging price of the contingent claim $g\left(X_{t,x}(T)\right)$.

### 2.2.2 Main results

The main result of this section is the derivation of the dynamic programming equation corresponding to the stochastic target problem (2.2.3), in the present context of possibly unbounded controls and jumps.

Before stating our main results, we need to introduce additional notations. Given a smooth function $\varphi$, $u \in U$ and $e \in E$, we now define the operators

$$\mathcal{L}^u\varphi(t,x) := \partial_t\varphi(t,x) + \mu_X(x,u)\cdot D\varphi(t,x) + \frac{1}{2}\mathrm{Trace}\left[\sigma_X\sigma_X^\top(x,u)D^2\varphi(t,x)\right]$$
$$\mathcal{G}^{u,e}\varphi(t,x) := \beta_Y\left(x,\varphi(t,x),u(e),e\right) - \varphi\left(t,x+\beta_X(x,u(e),e)\right) + \varphi(t,x),$$

where $\partial_t\varphi$ stands for the partial derivative with respect to $t$, $D\varphi$ and $D^2\varphi$ denote the gradient vector and the Hessian matrix with respect to the $x$ variable. We then define the following relaxed semi-limits

$$\begin{aligned} H^*(\Theta,\varphi) &:= \limsup_{\substack{\varepsilon\searrow 0,\Theta'\to\Theta\\ \eta\to 0,\psi\underset{u.}{\longrightarrow}\varphi}} H_{\varepsilon,\eta}\left(\Theta',\psi\right)\\ H_*(\Theta,\varphi) &:= \liminf_{\substack{\varepsilon\searrow 0,\Theta'\to\Theta\\ \eta\to 0,\psi\underset{u.}{\longrightarrow}\varphi}} H_{\varepsilon,\eta}\left(\Theta',\psi\right), \end{aligned} \tag{2.2.6}$$

where, for $\Theta = (t,x,y,k,q,q',A) \in \mathbb{R}_+\times\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{S}^d, \psi \in C^{1,2}\left([0,T]\times\mathbb{R}^d;\mathbb{R}\right)$, $\varepsilon \geq 0$ and $\eta \in [-1,1]$,

$$H_{\varepsilon,\eta}(\Theta,\psi) := \sup_{u\in\mathcal{N}_{\varepsilon,\eta}(t,x,y,q',\psi)} \mathbf{A}^u(\Theta),$$

with

$$\mathbf{A}^u(\Theta) := \mu_Y(x,y,u) - k - \mu_X(x,u)\cdot q - \frac{1}{2}\mathrm{Trace}\left[\sigma_X\sigma_X^\top(x,u)A\right],$$
$$\mathcal{N}_{\varepsilon,\eta}(t,x,y,q',\psi) := \left\{\begin{array}{c} u\in U \text{ s.t. } |N^u(x,y,q')|\leq\varepsilon \text{ and}\\ \Delta^{u,e}(t,x,y,\psi)\geq\eta \quad \text{for } \lambda\text{-a.e. } e\in E\end{array}\right\},$$
$$N^u(x,y,q') := \sigma_Y(x,y,u) - \sigma_X(x,u)^\top q',$$
$$\Delta^{u,e}(t,x,y,\psi) := \beta_Y(x,y,u(e),e) - \psi\left(t,x+\beta_X(x,u(e),e)\right) + y$$

and the convergence $\psi \underset{u.}{\longrightarrow} \varphi$ in (2.2.6) has to be understood in the sense that $\psi$ converges uniformly towards $\varphi$.

Also notice that, given $\eta\in[-1,1]$, $(\mathcal{N}_{\varepsilon,\eta})_{\varepsilon\geq 0}$ is non-decreasing in $\varepsilon$ so that

$$H_*(\Theta,\varphi) := \liminf_{\substack{\eta\to 0,\Theta'\to\Theta\\ \psi\underset{u.}{\longrightarrow}\varphi}} H_{0,\eta}\left(\Theta',\psi\right).$$

For ease of notations, we shall often simply write $Hv(t,x)$ in place of $H(t,\,x,\,v(t,x)$, $\partial_t v(t,x)$, $Dv(t,x)$, $Dv(t,x)$, $D^2v(t,x)$, $v)$. We shall similarly use the notations $H^*v$ and $H_*v$.

In order to handle the possible unboundedness of the jumps in Section 2.2.3.1, we shall need the following definition of viscosity super solution.

**Definition 2.2.2.** *We say that a l.s.c. (resp. u.s.c.) function $U$ (resp. $V$) is a viscosity supersolution of $H^*U \geq 0$ (resp. subsolution of $H_*V \leq 0$) on $[0,T) \times \mathbb{R}^d$ if for every smooth function $\varphi \in C^{1,2}([0,T] \times \mathbb{R}^d; \mathbb{R})$ of linear growth and $(t_o, x_o) \in [0,T) \times \mathbb{R}^d$ such that $\min_{[0,T]\times\mathbb{R}^d}(U-\varphi) = (U-\varphi)(t_o,x_o) = 0$ (resp. $\max_{[0,T]\times\mathbb{R}^d}(V-\varphi) = (V-\varphi)(t_o,x_o) = 0$), we have*

$$H^*\varphi(t_o, x_o) \geq 0 \qquad (\textit{resp. } H_*\varphi(t_o,x_o) \leq 0).$$

We will need for the proof of the supersolution property on $[0,T] \times \mathbb{R}^d$ (see Sections 2.2.3.1 and 2.2.3.2) the following technical assumption. Define for sake of clarity, for any $\varphi \in C^{1,2}([0,T] \times \mathbb{R}^d; \mathbb{R})$, $u \in U$ and $(t,x,y,z_1,z_2) \in [0,T] \times \mathbb{R}^{2d+2}$

$$\mathcal{L}^u_{X,Z}\bar{\varphi}(t,x,z) := \mathcal{L}^u\varphi(t,x) - \mu_X(x,u)\cdot z_1 - \mu_Y(x,y,u)z_2, \tag{2.2.7}$$

where $z =: (z_1,z_2) \in \mathbb{R}^d \times \mathbb{R}$ and $\bar{\varphi}(t,x,z) := \varphi(t,x) - |z|^2$.

**Assumption 2.2.3.** *For all $\varepsilon > 0, \eta \in [-1,1], (t_o,x_o) \in [0,T]\times\mathbb{R}^d, \varphi \in C^{1,2}([0,T]\times\mathbb{R}^d;\mathbb{R})$ and finite $C_1$ satisfying*

$$\sup_{u\in\mathcal{N}_{\varepsilon,\eta}(t,x,y,D\varphi,\varphi)} \{\mu_Y(x,y,u) - \mathcal{L}^u\varphi(t,x)\} \leq 2C_1$$
$$\textit{for all } (t,x) \in B_\varepsilon(t_o,x_o) \textit{ and } y \in \mathbb{R} \textit{ s.t. } |y-\varphi(t,x)| \leq \varepsilon,$$

*there exists $\varepsilon' > 0, \eta' \in [-1,1]$ and a finite $C_2$ such that*

$$\sup_{u\in\mathcal{N}_{\varepsilon',\eta'}(t,x,y,D\varphi,\varphi)} \left\{\mu_Y(x,y,u) - \mathcal{L}^u_{X,Z}\bar{\varphi}(t,x,z)\right\} \leq 2C_1 + |C_1|$$
$$\textit{for all } (t,x,y,z) \in [0,T]\times\mathbb{R}^{2d+2} \textit{ s.t. } \begin{cases} (t,x,z) \in B_{\varepsilon'}(t_o,x_o,0) \\ y \in \mathbb{R} \textit{ s.t. } |y - \bar{\varphi}(t,x)| \leq \eta', \end{cases} \tag{2.2.8}$$

*and*

$$\frac{\left[\mu_Y(x,y,u) - \mathcal{L}^u_{X,Z}\bar{\varphi}(t,x,z)\right]^+}{1 + |N^u(x,y,D\varphi)|} \leq C_2\left(1 + |\sigma_Y(x,y,u)| + \sum_{i=1}^d \left|\sigma_X^{i,\cdot}(x,u)\right|\right)$$
$$\textit{for all } (t,x,y,z) \in [0,T]\times\mathbb{R}^{2d+2} \textit{ s.t. } \begin{cases} (t,x,z_1,z_2) \in B_{\varepsilon'}(t_o,x_o,0) \\ y \in \mathbb{R} \textit{ s.t. } |y - \bar{\varphi}(t,x)| \leq \eta', \end{cases} \tag{2.2.9}$$
$$\textit{and } u \in U \textit{ such that } \Delta^{u,\cdot}(t,x,y,\varphi) \geq \eta \ \lambda\textit{-a.e.}$$

As in [BET09, ST02a, ST02c], the proof of the subsolution property requires an additional regularity assumption on the set valued map $\mathcal{N}_{0,\eta}(\cdot,f)$.

**Assumption 2.2.4.** ($\textit{Continuity of } \mathcal{N}_{0,\eta}(t,x,y,q,f)$) *For $f \in \mathcal{C}^0\left([0,T]\times\mathbb{R}^d\right)$, $\eta > 0$, let $B$ be a subset of $[0,T)\times\boldsymbol{X}\times\mathbb{R}\times\mathbb{R}^d$ such that $\mathcal{N}_{0,2\eta}(\cdot,f) \neq \emptyset$ on $B$. Then, for every $\varepsilon > 0$, $(t_o,x_o,y_o,q_o) \in Int(B)$, and $u_o \in \mathcal{N}_{0,2\eta}(t_o,x_o,y_o,q_o,f)$, there exists an open neighborhood $B'$ of $(t_o,x_o,y_o,q_o)$ and a locally Lipschitz map $\overline{\nu}$ defined on $B'$ such that $|\overline{\nu}(t_o,x_o,y_o,q_o) - u_o| \leq \varepsilon$ and $\overline{\nu}(t,x,y,q) \in \mathcal{N}_{0,\eta}(t,x,y,q,f)$ on $B'$.*

We also assume that $v$ is locally bounded, so that $v_*$ and $v^*$ are finite. Our first result characterizes $v$ as a discontinuous viscosity solution of the variational inequation (2.2.17) in the following sense.

**Theorem 2.2.5.** *Under Assumption 2.2.3, the function $v_*$ is a viscosity supersolution on $[0,T) \times \boldsymbol{X}$ of*

$$H^* v_* \geq 0. \tag{2.2.10}$$

*Under Assumption 2.2.4 holds, the function $v^*$ is a viscosity subsolution on $[0,T) \times \boldsymbol{X}$ of*

$$\min \{H_* v^*, v^* + \kappa\} \leq 0 \tag{2.2.11}$$

The proof of this result is reported in Section 2.2.3.

**Remark 2.2.6.** 1. Note that the operator $H^*$ would not be upper-semicontinuous in $\varphi$, for the uniform convergence, without the relaxation in the test function on the non-local part. This is the counterpart of the local relaxation introduced in Bouchard, Elie and Touzi [BET09] on the derivatives of the test function.

2. Notice that we impose the Definition 2.2.2 of viscosity solution for integrability issue. This heavily relies on the relaxation of the operator in its test function parameter, in terms of uniform convergence. Indeed, consider the case where the relaxation is stated in terms of uniform convergence on compact sets, and for every $(t_o, x_o) \in [0,T] \times \mathbb{R}^d$ and test function $\varphi$, the family of auxiliary test functions $(\varphi_\iota)_\iota$ defined for each $\iota > 0$ as $\varphi_\iota(t,x) := \varphi(t,x) \pm \iota |x - x_o|^n$, for some $n > 0$. This family converges uniformly on compact subsets towards $\varphi$ as $\iota \to 0$. However, the presence of the jumps may imply that $\varphi_\iota(\cdot, X)$ may fail to be integrable for $n$ large enough.

3. Assumption 2.2.3 is of technical nature, and is needed in the proof of (2.2.10) for integrability issues. It was missing in [BET09, Theorems 2.1 and 2.2, Corollaries 3.1 and 3.2], although it is satisfied in their Section 4. This condition enable us to control the drift $\mu_Y - \mathcal{L}^u \varphi$ in terms of BMO martingales, and thus to define a change of measure with uniformly integrable martingale, see Section 2.2.3.1. Equation (2.2.8) essentially stands in an additional relaxation of the operator. The relaxation in terms of $z_1$ in (2.2.8) is obvious by definition of $H^*$, whereas the relaxation in $z_2$ is new. Equation (2.2.9) is also new, and consists essentially in constraints on the partial derivatives of the test function, as well as a characterization of the controls of the jump part

in the Kernel $\mathcal{N}_{\varepsilon,\eta}$; see the proof in Example 2.3.4 in the particular case of stochastic target under controlled loss.

**Example 2.2.2.** In the context of Example 2.2.1, first notice that the process $X$ is not influenced by the control $\nu$. Hence, Assumption 2.2.3 reduces in this context in a control of $\frac{|\mu_Y(u)|}{|\sigma_Y(u)|}$. It is thus trivially satisfied since these coefficients are linear in $u$. Then, direct computations show that $v_*$ is a viscosity supersolution on $[0,T)\times(0,\infty)^d$ of

$$0 \le \min\left\{-\partial_t\varphi - \frac{1}{2}\sigma^2 D^2\varphi, D\varphi\cdot\beta(\cdot,e) - \varphi\left(\cdot+\beta(\cdot,e)\right)+\varphi\right\}, \quad \text{for } \lambda\text{-a.e. } e\in E$$

and that $v^*$ is a viscosity subsolution of

$$0 \ge \min\left\{-\partial_t\varphi - \frac{1}{2}\sigma^2 D^2\varphi, D\varphi\cdot\beta(\cdot,e) - \varphi\left(\cdot+\beta(\cdot,e)\right)+\varphi\right\} \quad \text{for } e\in E'\in\mathcal{E} \text{ s.t. } \lambda(E')>0.$$

We next discuss the terminal conditions on $\{T\}\times\mathbf{X}$. By the definition of the stochastic target problem, we have

$$v(T,x) = g(x) \text{ for every } x\in\mathbb{R}^d,$$

where $g$ is defined in (2.2.5). However, the possible discontinuities of $v$ might imply that the limits $v_*(T,\cdot)$ and $v^*(T,\cdot)$ do not agree with this boundary condition. We then need to introduce, as in Bouchard, Elie and Touzi [BET09], the set-valued map

$$\mathbf{N}(t,x,y,q,\psi) := \left\{\begin{array}{c}(r,s)\in\mathbb{R}^d\times\mathbb{R} : \exists\, u\in U \text{ s.t. } r = N^u(x,y,q)\\ \text{and } s\le\Delta^{u,e}(t,x,y,\psi) \quad \text{for } \lambda\text{-a.e. } e\in E\end{array}\right\},$$

together with the signed distance function from its complement $\mathbf{N}^c$ to the origin:

$$\delta := \mathrm{dist}(0,\mathbf{N}^c) - \mathrm{dist}(0,\mathbf{N}),$$

where we recall that dist stands for the (unsigned) Euclidean distance. Then,

$$0\in\mathrm{int}\left(\mathbf{N}(t,x,y,q,\psi)\right) \text{ iff } \delta(t,x,y,q,\psi)>0. \tag{2.2.12}$$

The upper and lower-semicontinuous envelopes of $\delta$ are respectively denoted by $\delta^*$ and $\delta_*$, and we will abuse notation by writing $\delta_* v(t,x) = \delta_*\left(t,x,v(t,x),Dv(t,x),v\right)$ and $\delta^* v(t,x) = \delta^*\left(t,x,v(t,x),Dv(t,x),v\right)$. For $\varphi\in\mathcal{C}^2\left(\mathbb{R}^d\right)$, we similarly define $\delta_*\varphi(x) = \delta_*\left(T,x,\varphi(x),D\varphi(x),\varphi\right)$ and the same definition holds for $\delta^*\varphi(x)$.

**Remark 2.2.7.** From the convention $\sup\emptyset = -\infty$ and the supersolution property (2.2.10) in Theorem 2.2.5, it follows that

$$\delta^* v_* \geq 0 \text{ on } [0,T) \times \mathbb{R}^d$$

in the viscosity sense. Then, if $\mathbf{N}^c \neq \emptyset$, this means that $v$ is subject to a gradient constraint.

**Remark 2.2.8.** 1. Assume that for every $(x, y, q)$ and $r \in \mathbb{R}^d$, there is an unique solution $\bar{u}(x, y, q, r)$ to the equation $N^u(x, y, q) = r$, i.e.

$$N^u(x, y, q) = r \quad \text{ iff } \quad u = \bar{u}(x, y, q, r).$$

Assume further that $\bar{u}$ is locally Lipschitz continuous, so that Assumption 2.2.4 trivially holds. For ease of notations, we set $\bar{u}_o(x, y, q) := \bar{u}(x, y, q, 0)$. For a bounded set of controls $U$, it follows that, for any smooth function $\varphi, H^*\varphi(t, x) \geq 0$ implies that

$$\bar{u}_o\left(x, \varphi(t,x), D\varphi(t,x)\right) \in U, \quad \mathbf{A}^{\bar{u}_o}(\cdot, \varphi, \partial_t\varphi, D\varphi, D^2\varphi)(t,x) \geq 0$$
$$\text{and } \Delta^{\bar{u}_o,e}(t, x, \varphi(t,x), \varphi) \geq 0 \quad \text{for } \lambda\text{-a.e. } e \in E.$$

Similarly, $H_*\varphi(t, x) \leq 0$ implies that

$$\text{either } \bar{u}_o\left(x, \varphi(t,x), D\varphi(t,x)\right) \notin \text{int}U,$$
$$\text{or} \quad \mathbf{A}^{\bar{u}_o}(\cdot, \varphi, \partial_t\varphi, D\varphi, D^2\varphi)(t,x) \leq 0$$
$$\text{or } \Delta^{\bar{u}_o,e}(t, x, \varphi(t,x), \varphi) < 0 \quad \text{for} \quad e \in E' \in \mathcal{E} \quad \text{s.t.} \quad \lambda(E') > 0.$$

The following result states that the constraint discussed in Remark 2.2.7 propagates up to the boundary. Here, the main difficulty is due to the unboundedness of the set $U$ and the presence of jumps in the diffusions. As discussed in Section 2.3.4 (see Corollary 2.3.17), the unboundedness of the controls may imply that the condition $\{H^*v_*(T, \cdot) < \infty\}$ is not satisfied.

**Theorem 2.2.9.** *Under Assumption 2.2.3, the function $x \mapsto v_*(T, x)$ is a viscosity supersolution of*

$$\min\left\{(v_*(T,\cdot) - g_*)\,\mathbf{1}_{\{H^*v_*(T,\cdot)<\infty\}}, \delta^* v_*(T,\cdot)\right\} \geq 0 \text{ on } \boldsymbol{X}, \tag{2.2.13}$$

*and, under Assumption 2.2.4, $x \in \boldsymbol{X} \mapsto v^*(T, x)$ is a viscosity subsolution of*

$$\min\left\{v^*(T,\cdot) - g^*, \delta_* v^*(T,\cdot)\right\} \leq 0 \text{ on } \boldsymbol{X}. \tag{2.2.14}$$

We conclude this section by some remarks. Remark 2.2.11 establishes the link between this work and those of Soner and Touzi [ST02c], [Bou02] and Bouchard, Elie and Touzi [BET09]. Remark 2.2.12 was already in Bouchard, Elie and Touzi [BET09], and Remark 2.2.10 will be of important use in the proofs of Section 2.3.5 below.

**Remark 2.2.10.** Assume that

$$\begin{gathered}\operatorname*{ess\,sup}_{u\in\mathcal{N}_\cdot,e\in E}\left\{\left|\beta_X\left(\cdot,u(e),e\right)\right|+\left|\beta_Y\left(\cdot,u(e),e\right)\right|\right\} \text{ is locally bounded,}\\ \text{and } E \text{ is compact.}\end{gathered} \tag{2.2.15}$$

Then, the operator $H$ is continuous for the uniform convergence in its $\psi \in \mathcal{C}^{1,2}$ parameter. In this case, the test function $\psi$ appearing in the form $\psi(t, x + \beta_X(x, u(e), e))$ in the definition of $H^*$ can be replaced by $v_*$ itself. To see this, note that for any $\varepsilon > 0, (t_o, x_o)$ and $\varphi \in \mathcal{C}^{1,2}$ such that $(v_* - \varphi)$ achieves a strict minimum at $(t_o, x_o)$, one can find a sequence of smooth function $\varphi_n^\varepsilon$ such that $\varphi_n^\varepsilon = \varphi$ on $B_\varepsilon(t_o, x_o)$, $\varphi_n^\varepsilon \le v_*$, and $\varphi_n^\varepsilon \uparrow v_*$ uniformly on compact sets of $(B_{2\varepsilon}(t_o, x_o))^c$. This allows to replace the original test function $\varphi$ by $v_*$ on $(B_{2\varepsilon}(t_o, x_o))^c$. It then suffices to send $\varepsilon \to 0$ and use the continuity induced by (2.2.15).

The same remark holds for the subsolution property.

**Remark 2.2.11.** Note that $\delta(x, y, q) \le 0$ whenever $\operatorname{int}(N(x, y, q)) \ne \emptyset$, so that the subsolution property does not carry any information. This would be the case when the control set $U$ has empty interior.

**Remark 2.2.12.** When the set $U$ is bounded, and $\beta_X \equiv \beta_Y \equiv 0$, i.e. there is no jumps, it was proved in Soner and Touzi [ST02c] that the value function $v$ is a discontinuous viscosity solution of

$$\sup_{u\in\mathcal{N}_0(\cdot,v,Dv)(t,x)}\left\{\mu_Y\left(x,v(t,x),u\right)-\mathcal{L}^u v(t,x)\right\}=0, \tag{2.2.16}$$

where

$$\begin{aligned}\mathcal{N}_0\left(x,y,q\right) &:= \left\{u\in U : N^u(x,y,q)=0\right\}\\ \text{and}\quad N^u(x,y,q) &:= \sigma_Y(x,y,u)-\sigma_X(x,u)^\top q,\end{aligned}$$

with the standard convention $\sup\emptyset = -\infty$. In the case of a convex compact set $U$, with jumps and $\mathbb{R}^d$-valued controls, i.e. $\mathcal{U}^2 = \{0\}$, Bouchard [Bou02] showed that $v$ is a viscosity solution of an equation of the form

$$\sup_{u\in\mathcal{N}_0(\cdot,v,Dv)(t,x)}\left\{\min\left\{\mathcal{L}^u\varphi(t,x),\inf_{e\in E}\mathcal{G}^{u,e}\varphi(t,x)\right\}\right\}=0. \tag{2.2.17}$$

Finally the case of unbounded set $U$ with no jumps was considered by Bouchard, Elie and Touzi [BET09]. In this paper, the authors introduced a relaxation on the operator (2.2.16), in order to deal with this unboundedness. This relaxation applies to the space variable $x$, the function $\varphi$, its gradient and its Hessian matrix, at the local point $(t,x)$. Such a relaxation is required in order to ensure that the sub-solution (resp. super-solution) property is stated in terms of a lower semi-continuous (resp. upper semi-continuous) operator. In our jump-diffusion framework, a similar relaxation is required, but it should involve the additional non-local term $\mathcal{G}^{u,e}$ in (2.2.17). One shall note that this relaxation is introduced in the Kernel $\mathcal{N}_\varepsilon$ with $\varepsilon \geq 0$, so that our PDEs do not take the form of (2.2.17). This is however a pure technical consideration, since we recover the same inequalities when considering particular frameworks, see e.g. Example 2.2.2.

### 2.2.3 Derivation of the PDE for singular stochastic target problems

This section is dedicated to the proof of Theorems 2.2.5 and 2.2.9. We first recall the geometric dynamic programming principle of Soner and Touzi [ST02a], see also Bouchard and Vu [BV10]. We next report the proof of the supersolution properties in Sections 2.2.3.1 and 2.2.3.2, and the proof of the subsolution properties in Sections 2.2.3.3 and 2.2.3.4.

**Theorem 2.2.13.** ***(Geometric Dynamic Programming Principle)*** *Fix $(t,x) \in [0,T) \times \boldsymbol{X}$ and let $\{\theta^\nu, \nu \in \mathcal{U}\}$ be a family of $[t,T]-$valued stopping times. Then,*

**(GDPj1)** *If $y > v(t,x)$, then there exists $\nu \in \mathcal{U}$*

$$Y^\nu_{t,x,y}(\theta^\nu) \geq v\left(\theta^\nu, X^\nu_{t,x}(\theta^\nu)\right).$$

**(GDPj2)** *For every $-\kappa \leq y < v(t,x), \nu \in \mathcal{U}$,*

$$\mathbb{P}\left[Y^\nu_{t,x,y}(\theta^\nu) > v\left(\theta^\nu, X^\nu_{t,x}(\theta^\nu)\right)\right] < 1.$$

#### 2.2.3.1 The supersolution property on $[0,T) \times \mathbf{X}$

We follow the arguments of Bouchard, Elie and Touzi [BET09] up to non trivial modifications due to the presence of the jumps, and the consideration of Assumption 2.2.3.

Step 1: Let $(t_o, x_o) \in [0,T) \times \mathbf{X}$ and $\varphi$ be a smooth function of linear growth such that

$$\min_{[0,T)\times\mathbf{X}} (\text{strict})\, (v_* - \varphi) = (v_* - \varphi)(t_o, x_o) = 0.$$

Assume that $H^*\varphi(t_o,x_o) =: -4\eta < 0$ for some $\eta > 0$, and let us work towards a contradiction. We define the family $\{f_\iota, \iota > 0\}$ of real valued functions defined on $\mathbb{R}^d$ for all $\iota > 0$ by

$$f_\iota : x \in \mathbb{R}^d \longmapsto \frac{2\iota}{\pi}\int_0^{\pi|x-x_o|} \sin^2 u du \mathbb{1}_{\{|x-x_o|\leq 1\}} + \iota \mathbb{1}_{\{|x-x_o|>1\}}. \tag{2.2.18}$$

Observe that for each $\iota > 0$,

$$\begin{aligned} &f_\iota \in C^2(\mathbb{R}^d;\mathbb{R}) \text{ is of linear growth,}\\ &0 = f_\iota(x_o) = \min_{x\in\mathbb{R}^d} f_\iota(x), \\ &(f_\iota)_{\iota>0} \text{ converges uniformly towards } 0 \text{ as } \iota \to 0. \end{aligned} \tag{2.2.19}$$

We also notice for later use that for all $\iota > 0$, we have

$$\begin{aligned} &f_\iota(x) \geq \gamma_{\varepsilon,\iota} := \iota\left(\left(\varepsilon - \frac{\sin(2\pi\varepsilon)}{2\pi}\right)\mathbb{1}_{\{|x-x_o|\leq 1\}} + \mathbb{1}_{\{|x-x_o|>1\}}\right) > 0 \\ &\text{for all } \varepsilon > 0 \text{ and } x \in \mathbb{R}^d \text{ such that } |x-x_o| \geq \varepsilon. \end{aligned} \tag{2.2.20}$$

Set $\varphi_\iota(t,x) := \varphi(t,x) - f_\iota(x)$ for $\iota > 0$. By definition of $H^*$ and the fact that $\varphi_\iota \underset{u.}{\longrightarrow} \varphi$ as $\iota \to 0$, we may find $\varepsilon, \iota > 0$ small enough, such that, after possibly changing $\eta > 0$

$$\begin{aligned} &\mu_Y(x,y,u) - \mathcal{L}^u\varphi_\iota(t,x) \leq -2\eta \\ &\text{for all } (t,x,y) \in [0,T)\times\mathbf{X}\times\mathbb{R} \text{ s.t. } \begin{cases} (t,x) \in B_\varepsilon(t_o,x_o) \\ |y - \varphi_\iota(t,x)| \leq \frac{\eta}{2}, \end{cases} \\ &\text{for all } u \in \mathcal{N}_{\varepsilon,-\eta}\left(t,x,y,D\varphi_\iota(t,x),\varphi_\iota\right), \end{aligned}$$

where we recall that $B_\varepsilon(t_o,x_o)$ denotes the ball of center $(t_o,x_o)$ and radius $\varepsilon$. Define now for all $z := (z_1,z_2) \in \mathbf{X}\times\mathbb{R}$ and $(t,x) \in [0,T]\times\mathbf{X}$ the function $\bar\varphi_\iota(t,x,z) := \varphi_\iota(t,x) - |z|^2$, and observe that, since the partial derivatives in $(t,x)$ of $\bar\varphi_\iota$ and $\varphi_\iota$ coincide, we have for every $u \in U, (t,x,y,z) \in [0,T]\times\mathbf{X}\times\mathbb{R}\times\mathbf{X}\times\mathbb{R}$:

$$\mathcal{L}^u\bar\varphi_\iota(t,x,z) = \mathcal{L}^u\varphi_\iota(t,x).$$

We recall from (2.3.7), for every $u \in U$, $(t,x,z) \in [0,T]\times\mathbf{X}^2\times\mathbb{R}$ and $y \in \mathbb{R}$ the definition of the operator

$$\mathcal{L}^u_{X,Z}\bar\varphi_\iota(t,x,z) = \mathcal{L}^u\varphi_\iota(t,x) - \mu_X(x,u)\cdot z_1 - \mu_Y(x,y,u)z_2.$$

By Assumption 2.2.3, there exists then a finite constant $C > 0$ such that, after possibly changing $\varepsilon$ and $\eta > 0$, we have

$$\mu_Y(x,y,u) - \mathcal{L}^u_{X,Z}\bar{\varphi}_\iota(t,x,z) \le -\eta$$

$$\text{for all } (t,x,z,y) \in [0,T) \times \mathbf{X}^2 \times \mathbb{R}^2 \text{ s.t. } \begin{cases} (t,x,z) \in B_\varepsilon(t_o,x_o,0) \\ |y - \bar{\varphi}_\iota(t,x,z)| \le \dfrac{\eta}{4}, \end{cases} \tag{2.2.21}$$

$$\text{for all } u \in \mathcal{N}_{\varepsilon,-\eta}\left(t,x,y,D\varphi_\iota(t,x),\varphi_\iota\right)$$

and

$$\frac{\left[\mu_Y(x,y,u) - \mathcal{L}^u_{X,Z}\bar{\varphi}_\iota(t,x,z)\right]^+}{1 + |N^u(x,y,D\varphi_\iota|} \le C\left(1 + |\sigma_Y(x,y,u)| + \sum_{i=1}^d \left|\sigma_X^{i,\cdot}(x,u)\right|\right)$$
$$\text{for all } (t,x,z) \in B_\varepsilon(t_o,x_o,0) \text{ and } y \in \mathbb{R} \text{ s.t. } |y - \bar{\varphi}_\iota(t,x,z)| \le \frac{\eta}{4}$$
$$\text{and for all } u \in U \text{ s.t. } \Delta^{u,\cdot}(t,x,y,\varphi_\iota) \ge -\eta \;\lambda\text{-a.e.}, \tag{2.2.22}$$

Notice that we still have

$$0 = v_*(t_o,x_o) - \bar{\varphi}_\iota(t_o,x_o,0) = \min_{[0,T)\times\mathbf{X}^2\times\mathbb{R}} (\text{strict})\left(v_* - \bar{\varphi}_\iota\right).$$

Let $\partial_p B_\varepsilon(t_o,x_o,0) := \{t_o+\varepsilon\} \times \overline{B}_\varepsilon(t_o,x_o,0) \cup [t_o,t_o+\varepsilon) \times \partial B_\varepsilon(x_o,0)$ denote the parabolic boundary of $B_\varepsilon(t_o,x_o,0)$. Set

$$\zeta := \min_{\partial_p B_\varepsilon(t_o,x_o,0)} \left(v_* - \bar{\varphi}_\iota\right),$$

and observe that $\zeta > 0$ since the above minimum is strict. We now define $\mathcal{V}_\varepsilon(t_o,x_o,0) := \partial_p B_\varepsilon(t_o,x_o,0) \cup [t_o,t_o+\varepsilon) \times B^c_\varepsilon(x_o) \times B_\varepsilon(0)$, and with $\gamma_{\varepsilon,\iota}$ defined as in (2.2.20), we observe that

$$\left(v_* - \bar{\varphi}_\iota\right)(t,x,z) \ge \zeta \wedge \gamma_{\varepsilon,\iota} =: \xi > 0 \text{ for } (t,x,z) \in \mathcal{V}_\varepsilon(t_o,x_o,0)$$

since $(t_o,x_o,0)$ is a strict minimizer, and $|x - x_o| \ge \varepsilon$ on $B^c_\varepsilon(x_o)$, recall (2.2.20).

step 2: Let $(t_n,x_n)_{n\ge1}$ be a sequence in $[0,T) \times \mathbf{X}$ which converges to $(t_o,x_o)$ and such that $v(t_n,x_n) \to v_*(t_o,x_o)$. Set $y_n := v(t_n,x_n) + n^{-1}$ and observe that

$$\gamma_n := y_n - \bar{\varphi}_\iota(t_n,x_n) \to 0. \tag{2.2.23}$$

For each $n \ge 1$, we have $y_n > v(t_n,x_n)$. Thus, it follows from (GDPj1) that there exists some $\nu^n \in \mathcal{U}$ such that

$$Y^n(t\wedge\theta_n) \ge v\left(t\wedge\theta_n, X^n(t\wedge\theta_n)\right), \quad t \ge t_n, \tag{2.2.24}$$

where

$$\theta_n^o := \{s \geq t_n : (s, X^n(s), Z^n(s)) \notin B_\varepsilon(t_o, x_o, 0)\}$$
$$\theta_n := \left\{s \geq t_n : |Y^n(s) - \bar{\varphi}_\iota\left(s, X^n(s), Z^n(s)\right)| \geq \frac{\eta}{4}\right\} \wedge \theta_n^o, \tag{2.2.25}$$

and

$$(X^n, Y^n, Z^n) := \left(X^{\nu^n}_{t_n,x_n}, Y^{\nu^n}_{t_n,x_n,y_n}, Z^{\nu^n}_{t_n,x_n}\right),$$
$$Z^{\nu^n}_{t_n,x_n}(s) := \frac{1}{2}\int_{t_n}^s \begin{pmatrix} \mu_Y\left(X^n(u), Y^n(u), \nu^n_u\right) \\ \mu_X\left(X^n(u), \nu^n_u\right) \end{pmatrix} du.$$

By the inequalities $v \geq v_* \geq \varphi_\iota \geq \bar{\varphi}_\iota$, this implies that

$$\begin{aligned}
&Y^n(t\wedge\theta_n) - \bar{\varphi}_\iota\left(t\wedge\theta_n, X^n(t\wedge\theta_n), Z^n(t\wedge\theta_n)\right) \\
&\geq \mathbf{1}_{\{t\geq\theta_n\}}\left[Y^n(t\wedge\theta_n) - \bar{\varphi}_\iota\left(t\wedge\theta_n, X^n(t\wedge\theta_n), Z^n(t\wedge\theta_n)\right)\right] \\
&\geq \mathbf{1}_{\{t\geq\theta_n\}}\Big[\left(Y^n(t\wedge\theta_n) - \bar{\varphi}_\iota\left(t\wedge\theta_n, X^n(t\wedge\theta_n), Z^n(t\wedge\theta_n)\right)\right)\mathbf{1}_{\{\theta_n<\theta_n^o\}} \\
&\quad + \left(v_*\left(t\wedge\theta_n, X^n(t\wedge\theta_n)\right) - \bar{\varphi}_\iota\left(t\wedge\theta_n, X^n(t\wedge\theta_n), Z^n(t\wedge\theta_n)\right)\right)\mathbf{1}_{\{\theta_n=\theta_n^o\}}\Big] \\
&\geq \left[\frac{\eta}{4}\mathbf{1}_{\{\theta_n<\theta_n^o\}} + \xi\mathbf{1}_{\{\theta_n=\theta_n^o\}}\right]\mathbf{1}_{\{t\geq\theta_n\}}
\end{aligned}$$

and therefore

$$Y^n(t\wedge\theta_n) - \bar{\varphi}_\iota\left(t\wedge\theta_n, X^n(t\wedge\theta_n), Z^n(t\wedge\theta_n)\right) \geq \left(\frac{\eta}{4}\wedge\xi\right)\mathbf{1}_{\{t\geq\theta_n\}} \geq 0. \tag{2.2.26}$$

step 3: Since $\bar{\varphi}_\iota$ is smooth, recall (2.2.19), it follows from Itô's lemma, (2.2.23), definitions of $Y^n$ and $Z^n$, and (2.2.26), that

$$\begin{aligned}
a_n + \int_{t_n}^{t\wedge\theta_n} b^n_s ds + \int_{t_n}^{t\wedge\theta_n} \psi^n_s dW_s + \int_{t_n}^{t\wedge\theta_n}\int_E c^{n,e}_s J(de, ds) \\
\geq -\left(\frac{\eta}{4}\wedge\xi\right)\mathbf{1}_{\{t<\theta_n\}},
\end{aligned} \tag{2.2.27}$$

with

$$\begin{aligned}
a_n &:= -\left(\frac{\eta}{4}\wedge\xi\right) + \gamma_n, \\
b^n_s &:= \mu_Y(X^n(s), Y^n(s), \nu^n_s) - \mathcal{L}^{\nu^n_s}_{X,Z}\bar{\varphi}_\iota(s, X^n_s, Z^n(s)) \\
c^{n,e}_s &:= \Delta^{\nu^n_s,e}\left(s, X^n(s-), Y^n(s-), \varphi_\iota\right) \\
\psi^n_s &:= N^{\nu^n}\left(Z^n_s, D\varphi_\iota(s, X^n_s)\right).
\end{aligned} \tag{2.2.28}$$

In view of (2.2.23), we have

$$a_n \to -\left(\frac{\eta}{4}\wedge\xi\right) < 0 \text{ for } n\to\infty. \tag{2.2.29}$$

Observe now that, for every $n \geq 1$, the definition of $\theta_n$ implies that for all $s \in [t_n, \theta_n)$, we have

$$|Y^n(s) - \bar{\varphi}_\iota\left(s, X^n_s, Z^n(s)\right)| \leq \frac{\eta}{4}.$$

Hence, we have

$$c_s^{n,e} \geq -\eta \quad \text{for} \quad \lambda\text{-a.e.} e \in E \text{ and } s \in [t_n, \theta_n], \tag{2.2.30}$$

since otherwise we would have

$$Y^n(\theta_n) - \bar{\varphi}_\iota\left(\theta_n, X^n(\theta_n), Z^n(\theta_n)\right) \leq \frac{-3\eta}{4},$$

which is in contradiction with (2.2.24). Hence, by (2.2.21) and the definition of the Kernel $\mathcal{N}_{\varepsilon,-\eta}$, for all $n \geq 1$, $s \in [t_n, \theta_n]$, we have

$$|\psi_s^n| \leq \varepsilon \qquad \Longrightarrow \qquad b_s^n \leq -\eta. \tag{2.2.31}$$

step 4: We now introduce, for each $n \geq 1$, the set

$$A_n := \{s \in [t_n, \theta_n] : b_s^n > -\eta\}.$$

Observe that, for all $n \geq 1$, (2.2.31) implies that the process $\psi^n$ satisfies

$$|\psi_s^n| > \varepsilon \qquad \text{for all} \qquad s \in A_n, \tag{2.2.32}$$

so that we can define the process $\alpha^n$ as

$$\alpha_s^n := \frac{-b_s^n}{|\psi_s^n|^2}\psi_s^n \mathbb{1}_{A_n}(s).$$

**Lemma 2.2.1.** *The stochastic Doleans-Dade exponential*

$$L_\cdot^n := \mathcal{E}\left(\int_{t_n}^{\cdot} \alpha_s^n dW_s\right)_{\cdot\wedge\theta_n}$$

*is well-defined and an uniformly integrable martingale, for all $n \geq 1$.*

This lemma is proved below, and fills a gap in the previous literature, where Assumption 2.2.3 is missing (see Remark 2.2.6). Admitting its result for the moment, by Girsanov's Theorem,

$$\hat{W}_\cdot^n := W_\cdot - \int_{t_n}^{\cdot\wedge\theta_n} \alpha_s^n ds$$

is a $\hat{\mathbb{Q}}^n$-Brownian motion, with $\hat{\mathbb{Q}}^n$ the equivalent probability defined by its density $\left.\frac{d\hat{\mathbb{Q}}^n}{d\mathbb{P}}\right|_{\mathcal{G}_\cdot} := L_\cdot^n$. Recalling (2.2.27), we have

$$\begin{aligned} a_n + \int_{t_n}^{t\wedge\theta_n} b_s^n \mathbb{1}_{A_n^c} ds + \int_{t_n}^{t\wedge\theta_n} \psi_s^n d\hat{W}_s^n + \int_{t_n}^{t\wedge\theta_n} \int_E c_s^{n,e} J(de, ds) \\ \geq -\left(\frac{\eta}{4} \wedge \xi\right) \mathbf{1}_{\{t<\theta_n\}}. \end{aligned} \tag{2.2.33}$$

Define now for each $n \geq 1$ the process

$$M^n_\cdot := \mathcal{E}\left(\int_{t_n}^{\cdot}\int_E \left(\frac{1}{nT\left(|d^n_s|+1\right)}-1\right)\tilde{J}(de,ds)\right)_{\cdot\wedge\theta_n},$$

with

$$d^n_s := \int_E c^{n,e}_s \lambda(de)$$

and where we recall that $\tilde{J}$ is the compensated jump measure. Since

$$\int_{t_n}^{\cdot}\int_E \left(\frac{1}{nT\left(|d^n_s|+1\right)}-1\right)\tilde{J}(de,ds) \geq -1,$$

$M^n$ is a non-negative local-martingale (see e.g. [Bré81, Theorem T10]), and from the fact that

$$\frac{1}{nT\left(|d^n_s|+1\right)} \leq \frac{1}{nT}, \tag{2.2.34}$$

together with $\int_E \lambda(de) < +\infty$, we deduce from [Bré81, Theorems T10 and T11] that $M^n$ is uniformly integrable. We may hence define the equivalent martingale measure $\left.\frac{d\tilde{\mathbb{Q}}^n}{d\hat{\mathbb{Q}}^n}\right|_{\mathcal{G}_\cdot} := M^n_\cdot$, and by Girsanov's Theorem again, we have

$$\int_{t_n}^{\cdot}\int_E \tilde{J}^n(de,ds) := \int_{t_n}^{\cdot}\int_E J(de,ds) - \int_{t_n}^{\cdot}\int_E \frac{1}{nT\left(|d^n_s|+1\right)}\lambda(de)ds$$

is a $\tilde{\mathbb{Q}}^n$-martingale; notice that $\hat{W}^n$ is a $\tilde{\mathbb{Q}}^n$-Brownian motion. Hence, (2.2.33) leads to

$$a_n + \int_{t_n}^{t\wedge\theta_n} b^n_s \mathbb{1}_{A^c_n} + \frac{1}{nT}\frac{d^n_s}{\left(|d^n_s|+1\right)}ds + \int_{t_n}^{t\wedge\theta_n}\psi^n_s d\hat{W}^n_s + \int_{t_n}^{t\wedge\theta_n}\int_E c^{n,e}_s \tilde{J}^n(de,ds)$$
$$\geq -\left(\frac{\eta}{4}\wedge\xi\right)\mathbf{1}_{\{t<\theta_n\}}.$$

Recall from the definition of $\theta_n$ that $\theta_n \leq T$, which combined with (2.2.34) gives

$$\begin{aligned}S^n_t &:= a_n + \frac{1}{n} + \int_{t_n}^{t\wedge\theta_n} b^n_s \mathbb{1}_{A^c_n} + \int_{t_n}^{t\wedge\theta_n}\psi^n_s d\hat{W}^n_s + \int_{t_n}^{t\wedge\theta_n}\int_E c^{n,e}_s \tilde{J}^n(de,ds)\\
&\geq a_n + \int_{t_n}^{t\wedge\theta_n}\left(b^n_s \mathbb{1}_{A^c_n} + \frac{1}{nT}\frac{d^n_s}{\left(|d^n_s|+1\right)}\right)ds + \int_{t_n}^{t\wedge\theta_n}\psi^n_s d\hat{W}^n_s\\
&\quad + \int_{t_n}^{t\wedge\theta_n}\int_E c^{n,e}_s \tilde{J}^n(de,ds)\\
&\geq -\left(\frac{\eta}{4}\wedge\xi\right)\mathbf{1}_{\{t<\theta_n\}},\end{aligned}$$

and from definition of $A_n$, (2.2.2) and the fact that $\bar{\varphi}_\iota$ is a linear growth in its $x$ variable, $S^n$ is local supermartingale, bounded by below, and hence a supermartingale. It follows then that

$$a_n + \frac{1}{n} = S^n_{t_n} \geq \mathbb{E}^{\tilde{\mathbb{Q}}^n}\left[S^n_{\theta_n}|\mathcal{F}_{t_n}\right] \geq -\left(\frac{\eta}{4}\wedge\xi\right)\mathbb{E}^{\tilde{\mathbb{Q}}^n}\left[\mathbf{1}_{\{\theta_n<\theta_n\}}\right] = 0$$

which contradicts (2.2.29) for $n$ large enough.

□

**Proof of Lemma 2.2.1.**

By definition of $\theta_n$, (2.2.32), (2.2.22), (2.2.30), we have,

$$|\alpha_s^n| \le C\left(1 + |\sigma_Y(X^n(s), Y^n(s), \nu_s^n)| + \sum_{i=1}^d \left|\sigma_X^{i,\cdot}(X^n(s), \nu_s^n)\right|\right),$$

for all $s \in [t_n, \theta_n)$. We claim that processes $\int_{t_n}^{\theta_n \wedge \cdot} \sigma_X(X^n(s), \nu_s^n) dW_s$ and $\int_{t_n}^{\theta_n \wedge \cdot} \sigma_Y(X^n(s), Y^n(s), \nu_s^n) dW_s$ are BMO-martingales, so that $\int_{t_n}^{\theta_n \wedge \cdot} \alpha_s^n dW_s$ is itself a BMO-martingale. The required result is then obtained by [Kaz94, Theorem 2.3].

We now prove that $\int_{t_n}^{\cdot} \sigma_X(X^n(s), \nu_s^n) dW_s$ and $\int_{t_n}^{\cdot} \sigma_Y(X^n(s), Y^n(s), \nu_s^n) dW_s$ are BMO-martingales. We shall focus on $\int_{t_n}^{\cdot} \sigma_X(X^n(s), \nu_s^n) dW_s$, the result for $\int_{t_n}^{\cdot} \sigma_Y(X^n(s), Y^n(s), \nu_s^n) dW_s$ following the exact same argument.

Denote for all $n \ge 1$ and $s \in [t_n, \theta_n]$

$$\Delta X^n(s) := X^n(s) - X^n(s-),$$

with $X^n(\cdot-)$ being the left limit of $X^n(\cdot)$. By smoothness of $\bar{\varphi}_\iota$ together with the definition (2.2.25) of $\theta_n$, definition of $Z^n$, (2.2.22) and (2.2.2), for each $n \ge 1$, there exists a constant $K_n$ such that for all $s < \theta_n$

$$\max\left(|X^n(s)|\,; \left|\int_{t_n}^s \mu_X\left(s, X^n(s), \nu_s^n\right) ds\right|; |\Delta X^n(s)|\right) \le K_n. \tag{2.2.35}$$

Being interested in the process $\int_{t_n}^{\theta_n \wedge \cdot} \sigma_X(X^n(s), \nu^n(s)) dW_s$, we may restrict ourselves to stopping times $\tau_n$ taking their values $\mathbb{P}$-a.s. in $[t_n, \theta_n]$. By continuity of the path: $r \in [t_n, \theta_n] \mapsto \int_{t_n}^r \sigma_X(X^n(s), \nu^n(s)) dW_s$, we have, for every $\tau_n \in [t_n, \theta_n)$

$$\begin{aligned}\int_{\tau_n}^{\theta_n} \sigma_X\left(X^n(s), \nu^n(s)\right) dW_s =& X^n(\theta_n -) - X^n(\tau_n) - \int_{\tau_n}^{\theta_n} \mu_X\left(X^n(s), \nu^n(s)\right) ds \\ & - \sum_{\tau_n < s < \theta_n} \Delta X^n(s)\end{aligned}$$

By (2.2.35) together with Jensen's inequality, we thus have

$$\begin{aligned}
&\left\|\mathbb{E}\left[\left\langle\int_{t_n}^{\theta_n}\sigma_X\left(X^n(s),\nu^n(s)\right)dW_s-\int_{t_n}^{\tau_n}\sigma_X\left(X^n(s),\nu^n(s)\right)dW_s\right\rangle|\mathcal{F}_{\tau_n}\right]\right\|_\infty\\
&=\left\|\mathbb{E}\left[\left\langle\int_{\tau_n}^{\theta_n}\sigma_X\left(X^n(s),\nu^n(s)\right)dW_s\right\rangle|\mathcal{F}_{\tau_n}\right]\right\|_\infty\\
&=\left\|\mathbb{E}\left[\left|\int_{\tau_n}^{\theta_n}\sigma_X\left(X^n(s),\nu^n(s)\right)dW_s\right|^2|\mathcal{F}_{\tau_n}\right]\right\|_\infty\\
&\le 4\left\|\mathbb{E}\left[\begin{array}{c}|X^n(\theta_n-)|^2+|X^n(\tau_n)|^2+\left|\int_{\tau_n}^{\theta_n}\mu_X\left(X^n(s),\nu_s^n\right)ds\right|^2\\ +\left|\sum_{\tau_n<s<\theta_n}\Delta X^n(s)\right|^2\end{array}|\mathcal{F}_{\tau_n}\right]\right\|_\infty\\
&\le 12K_n^2\left(1+\left\|\mathbb{E}\left[J(E,[\tau_n,\theta_n))\sum_{\tau_n<s<\theta_n}|\Delta X^n(s)|^2\,|\mathcal{F}_{\tau_n}\right]\right\|_\infty\right)\\
&\le 12K_n^2\left(1+\left\|\mathbb{E}\left[J(E,[\tau_n,\theta_n))^2K_n^2|\mathcal{F}_{\tau_n}\right]\right\|_\infty\right)<\infty,
\end{aligned}$$

since $\lambda(E)<\infty$, and so follows the result.

□

**Remark 2.2.14.** Note that, in the above proof, the relaxation of the non-local part of the operator in term of uniform convergence is required in order to pass from the initial test function $\varphi$ to the penalized one $\varphi_\iota$. It allows to obtain the inequality $v_*\ge\bar{\varphi}_\iota+\xi$ outside of the ball $B_\varepsilon(x_o)$, which is crucial in our proof. This is not required in Bouchard, Elie and Touzi [BET09] where processes are continuous. It is neither required in [Bou02], where the non-local operator is already continuous and the size of the jump is locally bounded.

#### 2.2.3.2 The supersolution property on $\{T\}\times\mathbf{X}$

We split the proof in different lemmas.

**Lemma 2.2.2.** *Let $x_o\in\boldsymbol{X}$ and $\varphi\in\mathcal{C}^2(\boldsymbol{X})$ be such that*

$$0=\left(v_*(T,\cdot)-\varphi\right)(x_o)=\min_{\boldsymbol{X}}(strict)\left(v_*(T,\cdot)-\varphi\right)$$

*then*

$$\delta^*\varphi(x_o)\ge 0.$$

The proof relies on the upper semi-continuity of $\delta^*$, and follows the exact same idea as in [ST02c, Lemma 5.2]. We may however give the main steps of this proof for sake of completeness. As in Soner and Touzi [ST02c], the key idea is to consider an auxiliary test function $\varphi_n$, penalized in both space and time, and to consider local minimizers $(t_n, x_n)$ of $(v_* - \varphi_n)$. After having proved that $(t_n, x_n) \to (T, x_o)$, we prove that $\lim_{n\to\infty} v_*(t_n, x_n) = v_*(T, x_o)$, and then conclude that the viscosity property of $v_*$ holds in $(t_n, x_n)$. We conclude by using the upper semi-continuity of $\delta^*$ and the supersolution property of Theorem (2.2.5) and (2.2.12) on $[0, T) \times \mathbf{X}$.

**Lemma 2.2.3.** *Under Assumption 2.2.3, $v_*$ is a viscosity supersolution of*

$$(v_*(T, \cdot) - g_*)\, \mathbf{1}_{\{H^* v_*(T,\cdot)<\infty\}} \geq 0 \text{ on } \mathbf{X}. \tag{2.2.36}$$

**Proof.** Let $x_o \in \mathbf{X}$ and $\varphi$ be a smooth function of linear growth such that

$$\min_{\mathbf{X}}(\text{strict})\, (v_*(T, \cdot) - \varphi) = (v_*(T, \cdot) - \varphi)\, (x_o).$$

step 1: Assume that $H^* v_*(T, x_o) < \infty$, $\varphi(x_o) = v_*(T, x_o) < g_*(x_o)$, and let us work towards a contradiction. Since $v(T, \cdot) = g$ by the definition of the problem and $g \geq g_*$, there is a constant $\eta > 0$ such that $\varphi - v(T, \cdot) \leq \varphi - g_* \leq -\eta$ on $B_\varepsilon(x_o)$ for some $\varepsilon > 0$. Since $x_o$ is a strict minimizer, we have

$$2\zeta := \min_{x\in\partial B_\varepsilon(x_o)} v_*(T, x) - \varphi(x) > 0,$$

and it follows from the lower semi-continuity of $v_*$ that there exists $r > 0$ such that

$$v(t, x) - \varphi(x) \geq v_*(t, x) - \varphi(x) \geq \zeta > 0$$
$$\text{for all } (t, x) \in [T - r, T] \times \partial B_\varepsilon(x_o),$$

and hence

$$v(t, x) - \varphi(x) \geq \zeta \wedge \eta > 0$$
$$\text{for } (t, x) \in ([T - r, T) \times \partial B_\varepsilon(x_o)) \cup (\{T\} \times B_\varepsilon(x_o)) =: \mathcal{V}_{\varepsilon,r}(T, x_o).$$

Define $\varphi_\iota(x) := \varphi(x) - f_\iota(x)$, for $\iota > 0$ and $f_\iota$ as in (2.2.18). With similar arguments as those of Section 2.2.3.1 and by (2.2.20), we have

$$v(t, x) - \varphi_\iota(x) \geq \zeta \wedge \eta \wedge \gamma_{\varepsilon,\iota} =: 4\xi > 0$$
$$\text{for } (t, x) \in \left([T - r, T] \times \bar{B}^c_\varepsilon(x_o)\right) \cup (\{T\} \times B_\varepsilon(x_o))\,.$$

We now use the fact that $H^*\varphi(x_o) =: \frac{C}{2} < \infty$. Set

$$\tilde{\varphi}_\iota(t, x) := \varphi_\iota(x) + (C + 2\eta)(t - T) \leq \varphi_\iota(x).$$

Then, by (2.2.19), for $r, \iota > 0$ sufficiently small and after possibly changing $\varepsilon, \eta > 0$, we have

$$v(t, x) - \tilde{\varphi}_\iota(t, x) \geq 2\xi > 0 \text{ for } (t, x) \in \mathcal{V}_{\varepsilon,r}(T, x_o) \cup [T - r, T] \times \bar{B}^c_\varepsilon(x_o),$$
$$\mu_Y(x, y, u) - \mathcal{L}^u\tilde{\varphi}_\iota(t, x) \leq -2\eta \text{ for all } u \in \mathcal{N}_{\varepsilon,-\eta}(t, x, y, D\widetilde{\varphi}_\iota(t, x), \widetilde{\varphi}_\iota)$$
$$\text{and } (t, x, y) \in [T - r, T] \times \mathbf{X} \times \mathbb{R} \text{ s.t. } x \in B_\varepsilon(x_o) \text{ and } |y - \widetilde{\varphi}_\iota(t, x)| \leq \frac{\eta}{2}.$$

Indeed, $\mu_Y(x,y,u) - \mathcal{L}^u\tilde{\varphi}_\iota(t,x) = \mu_Y(x,y,u) - \mathcal{L}^u\varphi_\iota(x) - C - 2\eta \le -2\eta$ as soon as $\mu_Y(x,y,u) - \mathcal{L}^u\varphi_\iota(x) \le C$, and we have $\mathcal{N}_{\varepsilon,-\eta}(t,x,y,D\widetilde{\varphi}_\iota(t,x),\varphi_\iota) \subset \mathcal{N}_{\varepsilon,-\eta}(t,x,y,D\bar{\varphi}_\iota(t,x),\widetilde{\varphi}_\iota)$.

We now define for every $(t,x,z) \in [0,T] \times \mathbb{R}^d \times \mathbb{R}^{d+1}$ and $\iota > 0$ the function $\bar{\varphi}_\iota(t,x,z) := \tilde{\varphi}_\iota(t,x) - |z|^2$. By Assumption 2.2.3, and after possibly changing $\varepsilon,\eta > 0$, there is $C' > 0$ such that

$$v(t,x) - \bar{\varphi}_\iota(t,x,z) \ge \xi > 0$$
$$\text{for } (t,x,z) \in \bar{\mathcal{V}}_{\varepsilon,r}(T,x_o,0) \cup [T-r,T] \times \bar{B}^c_\varepsilon(x_o) \times B_\varepsilon(0),$$
$$\mu_Y(x,y,u) - \mathcal{L}^u_{X,Z}\bar{\varphi}_\iota(t,x,z) \le -\eta \text{ for all } u \in \mathcal{N}_{\varepsilon,-\eta}(t,x,y,D\widetilde{\varphi}_\iota(t,x),\widetilde{\varphi}_\iota)$$
$$\text{and } (t,x,y,z) \in [T-r,T] \times \mathbb{R}^{2d+2} \times \mathbb{R} \text{ s.t. } \begin{cases} (x,z) \in B_\varepsilon(x_o,0) \\ |y - \widetilde{\varphi}_\iota(t,x)| \le \dfrac{\eta}{4} \end{cases}$$

and

$$\frac{\left[\mu_Y(x,y,u) - \mathcal{L}^u_{X,Z}\bar{\varphi}_\iota(t,x,z)\right]^+}{|N^u(x,y,D\tilde{\varphi}_\iota|} \le C'\left(1 + |\sigma_Y(x,y,u)| + \sum_{i=1}^d \left|\sigma_X^{i,\cdot}(x,u)\right|\right)$$
$$\text{for all } (t,x,z) \in B_\varepsilon(t_o,x_o,0) \quad \text{and} \quad y \in \mathbb{R} \text{ s.t. } |y - \tilde{\varphi}_\iota(t,x)| \le \frac{\eta}{4}$$
$$\text{and for all } u \in U \text{ s.t. } \Delta_{u,\cdot}(t,x,y,\tilde{\varphi}_\iota) \ge -\eta \ \lambda\text{-a.e.},$$

where $\bar{\mathcal{V}}_{\varepsilon,r}(T,x_o,0)$ is constructed around $(T,x_o,0)$ as $\mathcal{V}_{\varepsilon,r}(T,x_o)$.

step 2: Let $(t_n,x_n)_{n\ge1}$ be a sequence in $[T-r,T] \times \mathbf{X}$ which converges to $(T,x_o,0)$ and such that $v(t_n,x_n) \to v_*(T,x_o)$. Set $y_n := v(t_n,x_n) + n^{-1}$, and observe that

$$\gamma_n := y_n - \bar{\varphi}(t_n,x_n,0) \to 0.$$

For each $n \ge 1$, we have $y_n > v(t_n,x_n)$. Then, by (GDPj1), there exists some $\nu^n \in \mathcal{U}$ such that

$$Y^n(t \wedge \theta_n) \ge v\left(t \wedge \theta_n, X^n(t \wedge \theta_n)\right), \quad t \ge t_n,$$

where

$$\theta^o_n := \{s \ge t_n : (s, X^n(s), Z^n(s)) \notin \mathcal{V}_{\varepsilon,r}(T,x_o,0)\}$$
$$\theta_n := \left\{s \ge t_n : |Y^n(s) - \bar{\varphi}_\iota\left(s, X^n(s), Z^n(s)\right)| \ge \frac{\eta}{4}\right\} \wedge \theta^o_n,$$

and

$$(X^n,Y^n,Z^n) := \left(X^{\nu^n}_{t_n,x_n}, Y^{\nu^n}_{t_n,x_n,y_n}, Z^{\nu^n}_{t_n,x_n}\right),$$
$$Z^{\nu^n}_{t_n,x_n} = \frac{1}{2}\int_{t_n}^s \begin{pmatrix} \mu_Y\left(X^n(u), Y^n(u), \nu^n_u\right) \\ \mu_X\left(X^n(u), \nu^n_u\right) \end{pmatrix} ds.$$

Using the inequalities $v \geq v_* \geq \tilde{\varphi}_\iota \geq \bar{\varphi}_\iota$, this implies that

$$\begin{aligned} & Y^n(t \wedge \theta_n) - \bar{\varphi}_\iota\left(t \wedge \theta_n, X^n(t \wedge \theta_n), Z^n(t \wedge \theta_n)\right) \\ & \geq \left[Y^n(t \wedge \theta_n) - \bar{\varphi}_\iota\left(t \wedge \theta_n, X^n(t \wedge \theta_n), Z^n(t \wedge \theta_n)\right)\right] \mathbf{1}_{\{t \geq \theta_n\}} \\ & \geq \mathbf{1}_{\{t \geq \theta_n\}} \left[\left(Y^n(t \wedge \theta_n) - \bar{\varphi}_\iota\left(t \wedge \theta_n, X^n(t \wedge \theta_n), Z^n(t \wedge \theta_n)\right)\right) \mathbf{1}_{\{\theta_n < \theta_n^o\}} \right. \\ & \qquad \left. + \left(v\left(t \wedge \theta_n, X^n(t \wedge \theta_n)\right) - \bar{\varphi}_\iota\left(t \wedge \theta_n, X^n(t \wedge \theta_n), Z^n(t \wedge \theta_n)\right)\right) \mathbf{1}_{\{\theta_n = \theta_n^o\}}\right] \\ & \geq \left[\varepsilon \mathbf{1}_{\{\theta_n < \theta_n^o\}} + \xi \mathbf{1}_{\{\theta_n = \theta_n^o\}}\right] \mathbf{1}_{\{t \geq \theta_n\}} \end{aligned}$$

and therefore

$$Y^n(t \wedge \theta_n) - \bar{\varphi}_\iota\left(t \wedge \theta_n, X^n(t \wedge \theta_n), Z^n(t \wedge \theta_n)\right) \geq (\varepsilon \wedge \xi)\, \mathbf{1}_{\{t \geq \theta_n\}} \geq 0.$$

By repeating the arguments of steps 3 and 4 of Section 2.2.3.1, we end up to a contradiction.

□

#### 2.2.3.3 The subsolution property on $[0, T) \times \mathbf{X}$

The proof of the subsolution property is a straightforward combination of the arguments of [Bou02] and Bouchard, Elie and Touzi [BET09]. We provide it for completeness.

step 1: Let $(t_o, x_o) \in [0, T) \times \mathbf{X}$ and $\varphi$ be a smooth function of linear growth such that

$$0 = (v^* - \varphi)(t_o, x_o) > (v^* - \varphi)(t, x) \text{ for } (t_o, x_o) \neq (t, x) \in [0, T) \times \mathbf{X}.$$

We assume that $v^*(t_o, x_o) > -\kappa$ and we show that

$$H_* \varphi(t_o, x_o) \leq 0.$$

Assume to the contrary that

$$4\eta := H_* \varphi(t_o, x_o) > 0.$$

By (2.2.6), and after possibly changing $\eta > 0$, we may find $\varepsilon > 0$ and $\iota > 0$ sufficiently small such that

$$\mu_Y(x, y, u) - \mathcal{L}^u \varphi_\iota(t, x) \geq 2\eta$$

for some $u \in \mathcal{N}_{0,\eta}(t, x, y, D\varphi_\iota(t, x), \varphi_\iota)$, for all $(t, x, y) \in [0, T) \times \mathbf{X} \times \mathbb{R}$ such that $(t, x) \in B_\varepsilon(t_o, x_o)$ and $|y - \varphi_\iota(t, x)| \leq \frac{\eta}{4}$, where $\varphi_\iota(t, x) := \varphi(t, x) + f_\iota(x)$, recall (2.2.18) and (2.2.19). Observe that we still have

$$0 = (v^* - \varphi_\iota)(t_o, x_o) = \max_{[0,T) \times \mathbf{X}} (\text{strict})\, (v^* - \varphi_\iota)\,. \tag{2.2.37}$$

For $\varepsilon$ sufficiently small, and after possibly changing $\eta > 0$, Assumption 2.2.4 then implies that

$$\begin{gathered}\min\left\{\begin{array}{c}\mu_Y\left(x,y,\hat{\nu}\left(t,x,y,D\varphi_\iota(t,x)\right)\right)-\mathcal{L}^{\hat{\nu}(t,x,y,D\varphi_\iota(t,x))}\varphi_\iota(t,x),\\ \mathcal{G}^{\hat{\nu}(t,x,y,D\varphi_\iota(t,x)),e}\varphi_\iota(t,x)\end{array}\right\}\geq\eta\\ \text{for } \lambda\text{-a.e. } e\in E \text{ and for all } (t,x,y)\in[0,T)\times\mathbf{X}\times\mathbb{R}\\ \text{s.t. } (t,x)\in B_\varepsilon(t_o,x_o) \text{ and } |y-\varphi_\iota(t,x)|\leq\frac{\eta}{4},\end{gathered} \tag{2.2.38}$$

where $\hat{\nu}$ is a locally Lipschitz map satisfying

$$\hat{\nu}\left(t,x,y,D\varphi_\iota(t,x)\right)\in\mathcal{N}_{0,\eta}\left(t,x,y,D\varphi_\iota(t,x),\varphi_\iota\right) \text{ on } B_\varepsilon(t_o,x_o). \tag{2.2.39}$$

Observe that, since $(t_o,x_o)$ is a strict maximizer in (2.2.37), we have

$$-\zeta:=\max_{\partial_pB_\varepsilon(t_o,x_o)}\left(v^*-\varphi_\iota\right)<0$$

where $\partial_pB_\varepsilon(t_o,x_o)$ denotes the parabolic boundary of $B_\varepsilon(t_o,x_o)$. As in the previous sections, by (2.2.20), we have for all $(t,x)\in[0,T)\times B^c_\varepsilon(x_o)$

$$\left(v^*-\varphi_\iota\right)(t,x)\leq-\gamma_{\varepsilon,\iota}.$$

Thus, for all $(t,x)\in\left([t_o,t_o+\varepsilon)\times B^c_\epsilon(x_o)\right)\cup\left(\{t_o+\varepsilon\}\times\overline{B}_\varepsilon(x_o)\right)$,

$$\left(v^*-\varphi_\iota\right)(t,x)\leq-\left(\gamma_{\varepsilon,\iota}\wedge\zeta\right)=:-\xi<0. \tag{2.2.40}$$

step 2: We now show that (2.2.38), (2.2.39) and (2.2.40) lead to a contradiction of (GDPj2).

Let $(t_n,x_n)_{n\geq1}$ be a sequence in $[0,T)\times\mathbf{X}$ which converges to $(t_o,x_o)$ and such that $v(t_n,x_n)\to v^*(t_o,x_o)$. Set $y_n:=v(t_n,x_n)-n^{-1}$, and observe that

$$\gamma_n:=y_n-\varphi_\iota(t_n,x_n)\to0. \tag{2.2.41}$$

Also notice that $y_n\geq-\kappa$ for $n$ large enough.

Let $Z^n:=(X^n,Y^n)$ denote the solution of (2.2.1) associated to the Markovian control $\hat{\nu}^n:=\hat{\nu}\left(\cdot,X^n,Y^n,D\varphi_\iota(\cdot,X^n)\right)$ and the initial condition $Z^n(t_n)=(x_n,y_n)$. Since $\hat{\nu}$ is locally Lipschitz, this solution is well defined up to the stopping time

$$\theta_n:=\inf\left\{s\geq t_n:\left|Y^n(s)-\varphi_\iota\left(s,X^n(s)\right)\right|\geq\frac{\eta}{4}\right\}\wedge\theta^o_n, \tag{2.2.42}$$

with

$$\theta^o_n:=\inf\left\{s\geq t_n:(s,X^n(s))\notin B_\varepsilon(t_o,x_o)\right\}. \tag{2.2.43}$$

Note that (2.2.38), (2.2.41), and a standard comparison theorem implies that

$$Y^n(\theta_n) - \varphi_\iota(\theta_n, X^n(\theta_n)) \geq \frac{\eta}{4} \quad \text{on} \quad \left\{|Y^n(\theta_n) - \varphi_\iota(\theta_n, X^n(\theta_n))| \geq \frac{\eta}{4}\right\}$$

for $n$ large enough. Indeed, $Y^n(\theta_n) - \varphi_\iota(\theta_n, X^n(\theta_n)) \geq \gamma_n > -\varepsilon$ for $n$ large enough. Since $-v \geq -v^* \geq -\varphi_\iota$, we then deduce from (2.2.40), (2.2.42) and (2.2.43) that

$$\begin{aligned}
&Y^n(\theta_n) - v(\theta_n, X^n(\theta_n)) \\
&\quad \geq \mathbf{1}_{\{\theta_n < \theta_n^o\}} \left(Y^n(\theta_n) - \varphi_\iota(\theta_n, X^n(\theta_n))\right) \\
&\qquad + \mathbf{1}_{\{\theta_n = \theta_n^o\}} \left(Y^n(\theta_n^o) - v^*(\theta_n^o, X^n(\theta_n^o))\right) \\
&\quad \geq \frac{\eta}{4} \mathbf{1}_{\{\theta_n < \theta_n^o\}} + \mathbf{1}_{\{\theta_n = \theta_n^o\}} \left(Y^n(\theta_n^o) - v^*(\theta_n^o, X^n(\theta_n^o))\right) \\
&\quad \geq \frac{\eta}{4} \mathbf{1}_{\{\theta_n < \theta_n^o\}} + \mathbf{1}_{\{\theta_n = \theta_n^o\}} \left(Y^n(\theta_n^o) + \xi - \varphi_\iota(\theta_n^o, X^n(\theta_n^o))\right) \\
&\quad \geq \frac{\eta}{4} \wedge \xi + \mathbf{1}_{\{\theta_n = \theta_n^o\}} \left(Y^n(\theta_n^o) - \varphi_\iota(\theta_n^o, X^n(\theta_n^o))\right).
\end{aligned} \tag{2.2.44}$$

We may continue by using Itô's formula:

$$\begin{aligned}
Y^n(\theta_n) - v(\theta_n, X^n(\theta_n)) \geq \frac{\eta}{4} \wedge \xi + \mathbf{1}_{\{\theta_n = \theta_n^o\}} \Bigg(\gamma_n + \int_{t_n}^{\theta_n} \alpha(s, X_s^n, Y_s^n)\, ds \\
+ \int_{t_n}^{\theta_n} \int_E \delta(s, X_s^n, Y_s^n, e)\, J(de, ds)\Bigg)
\end{aligned}$$

where

$$\begin{aligned}
\alpha(t,x,y) &:= \mu_Y(x, y, \hat{\nu}(t,x,y,D\varphi_\iota(t,x))) - \mathcal{L}^{\hat{\nu}(t,x,y,D\varphi_\iota(t,x))}\varphi_\iota(t,x) \\
\delta(t,x,y,e) &:= \beta_Y(x, y, \hat{\nu}(t,x,y,D\varphi_\iota(t,x))(e), e) \\
&\qquad - \varphi_\iota(t, x + \beta_X(x, \hat{\nu}(t,x,y,D\varphi_\iota(t,x))(e), e)) + \varphi_\iota(t,x)
\end{aligned}$$

and the diffusion coefficient vanishes by (2.2.39). Recalling (2.2.38), the fact that $\gamma_n \to 0$, and that $\eta, \xi > 0$, this implies that

$$Y^n(\theta_n) > v(\theta_n, X^n(\theta_n)) \text{ for sufficiently large } n.$$

Since the initial position of the process $Y^n$ is $y_n = v(t_n, x_n) - n^{-1} < v(t_n, x_n)$, this is clearly in contradiction with (GDPj2).

□

#### 2.2.3.4 The subsolution property on $\{T\} \times \mathbf{X}$

The proof combines arguments used in the two previous sections (2.2.3.2) and (2.2.3.3). The only difference between this proof and the one in Bouchard, Elie and Touzi [BET09] relies on the presence of the jumps. However, it can be handled by following [Bou02]. We then only explain the main steps. Let $x_o \in \mathbf{X}$ and $\varphi$ be a smooth function of linear growth such that

$$\max_{\mathbf{X}} (\text{strict}) \left(v^*(T,\cdot) - \varphi\right) = \left(v^*(T,\cdot) - \varphi\right)(x_o) = 0.$$

Assume that, for some $\eta > 0$,

$$\begin{aligned} &0 < \delta_* \varphi(x_o) \\ &0 < 4\eta < \varphi(x_o) - g^*(x_o) = v^*(T, x_o) - g^*(x_o) \end{aligned}$$

Set $\varphi_\iota(t,x) = \varphi(x) + f_\iota(x) + \iota\sqrt{T-t}$, recall (2.2.18). Since the partial derivatives in $x$ of $\varphi$ and $\varphi_\iota$ are the same for $x = x_o$, by (2.2.12) and Assumption 2.2.4, together with (2.2.19), using the fact that $\varphi_\iota \geq \varphi$, for $\iota > 0$ small enough, after possibly changing $\eta > 0$, we can find $r, \varepsilon > 0$ and a locally Lipschitz map $\hat{\nu}$ satisfying,

$$\hat{\nu}\left(t,x,y,D\varphi_\iota(t,x)\right) \in \mathcal{N}_{0,\eta}\left(t,x,y,D\varphi_\iota(x),\varphi_\iota\right). \tag{2.2.45}$$

such that

$$\begin{aligned} 0 &< \delta_* \varphi_\iota(t,x) \\ 0 < 4\eta &< \varphi_\iota(T,x_o) - g^*(x_o) = v^*(T,x_o) - g^*(x_o) \end{aligned} \tag{2.2.46}$$

for all $(t,x,y) \in [T-r,T) \times \mathbf{X} \times \mathbb{R}$ s.t. $x \in B_r(x_o)$ and $|y - \varphi_\iota(t,x)| \leq \varepsilon$. Since $\partial_t \varphi_\iota \to -\infty$ as $t \to T$, we deduce that, for $r > 0$ small enough,

$$\mu_Y\left(x,y,\hat{\nu}\left(t,x,y,D\varphi_\iota(t,x)\right)\right) - \mathcal{L}^{\hat{\nu}(t,x,y,D\varphi_\iota(t,x))}\varphi_\iota(t,x) \geq \eta \tag{2.2.47}$$

for all $(t,x,y) \in [T-r,T) \times \mathbf{X} \times \mathbb{R}$ s.t. $x \in B_r(x_o)$ and $|y - \varphi_\iota(t,x)| \leq \frac{\eta}{4}$. Also observe that, since $v^* - \varphi_\iota$ is upper-semicontinuous and $(v^* - \varphi_\iota)(T,x_o) = 0$, we can choose $r > 0$ such that

$$v(t,x) \leq \varphi_\iota(t,x) + \frac{\varepsilon}{2} \text{ for all } (t,x) \in [T-r,T) \times B_r(x_o). \tag{2.2.48}$$

Moreover, combining the identity $v(T,x_o) = g(x_o)$, (2.2.20), (2.2.46), (2.2.47), (2.2.48), the fact that $x_o$ achieves a strict maximum, and using similar arguments as those of Section 2.2.3.2 above, recall (2.2.20), we see that

$$v(t,x) - \varphi_\iota \leq -\left(\zeta \wedge \gamma_{\varepsilon,\iota}\right) =: -\xi \tag{2.2.49}$$

for all $(t,x) \in \left([T-r,T] \times \overline{B^c_r}(x_o)\right) \cup \left(\{T\} \times B_r(x_o)\right)$ and for some $r, \varepsilon > 0$ small enough, but so that the above inequalities still hold. By following the arguments in step 2 of Section 2.2.3.3, we see that (2.2.46), (2.2.45), (2.2.48) and (2.2.49) lead to a contradiction of (GDPj2).

□

## 2.3 Target reachability with controlled expected loss

### 2.3.1 Problem reduction

We now turn to the main motivation for the above analysis: the stochastic target problem with controlled expected loss. Let $\Psi$ be a measurable map from $\mathbb{R}^{d+1}$ to $\mathbb{R}$ such that, for every fixed $x$, the function

$$y \longmapsto \Psi(x,y) \text{ is non-decreasing and right continuous.}$$

We define $L$ as the closed convex hull of the image of $\Psi$

$$L := \overline{\circledcirc}\left(\Psi\left(\mathbf{X} \times [-\kappa, \infty)\right)\right) = [m, M],$$

with $m < M$, $m, M \in [-\infty, +\infty]$. For $p \in L$, we define the stochastic target problem with controlled expected loss as follows:

$$\hat{v}(t,x,p) := \inf\left\{y \geq -\kappa : \mathbb{E}\left[\Psi\left(X^\nu_{t,x}(T), Y^\nu_{t,x,y}(T)\right)\right] \geq p \text{ for some } \nu \in \mathcal{U}\right\}, \quad (2.3.1)$$

with $\kappa \in \mathbb{R}_+ \cup \{+\infty\}$.

The aim of this section is to convert the problem (2.3.1) into the class of standard stochastic target problems as defined in Section 2.2. The dynamic programming equation for the target reachability with controlled expected loss will then be deduced directly from Theorem 2.2.5 above.

Following Bouchard, Elie and Touzi [BET09], we introduce an additional controlled state variable

$$P^{\alpha,\chi}_{t,p}(s) := p + \int_t^s \alpha_r \cdot dW_r + \int_t^s \int_E \chi_{s,e} \tilde{J}(de, ds), \quad s \in [t, T],$$

where the additional controls $\alpha, \chi$ are $\mathbb{F}$-predictable measurable processes, with $\chi \in \mathbb{H}^2_\lambda$ and $\alpha$ is $\mathbb{R}^d$-valued and such that $\mathbb{E}\left[\int_0^T |\alpha_s|^2\, ds\right] < \infty$. We denote by $\mathcal{A}$ the collection of such processes $(\alpha, \chi)$. For $\hat{\nu} := (\nu, \alpha, \chi)$, we then set $\hat{X}^{\hat{\nu}} := (X^\nu, P^{\alpha,\chi})$. We also define $\hat{\mathbf{X}} := \mathbf{X} \times L, \hat{U} := U \times \mathbb{R}^d \times \mathbb{L}^2_\lambda$, and denote by $\hat{\mathcal{U}} = \mathcal{U} \times \mathcal{A}$ the corresponding set of admissible controls. Abusing notations, we also set $Y^{\hat{\nu}} = Y^\nu$. Finally, we introduce the function

$$\hat{V}\left(\hat{x}, y\right) := \Psi(x,y) - p, \quad \text{for} \quad y \geq -\kappa \quad \text{and} \quad \hat{x} = (x,p) \in \overline{(\mathbf{X} \times L)}.$$

We make the following assumption, which allows us to use the stochastic integral representation theorem.

**Assumption 2.3.1.** *$\Psi\left(X^{\nu}_{t,x}(T), Y^{\nu}_{t,x,y}(T)\right)$ is square integrable for all initial conditions $(t,x,y) \in [0,T] \times \boldsymbol{X} \times [-\kappa, +\infty)$ and controls $\nu \in \mathcal{U}$.*

Following the arguments of Bouchard, Elie and Touzi [BET09], we can now relate $\hat{v}$ to a stochastic target problem with unbounded controls, and controls taking the form of measurable functions on $E$.

**Proposition 2.3.2.** *For all $(t,\hat{x}) := (t,x,p) \in [0,T] \times \hat{\boldsymbol{X}}$, we have*

$$\hat{v}\,(t,\hat{x}) = u\,(t,\hat{x}) = w\,(t,\hat{x})\,,$$

*where*

$$u(t,x,p) := \inf\left\{y \geq -\kappa : \hat{V}\left(\hat{X}^{\hat{\nu}}_{t,\hat{x}}(T), Y^{\hat{\nu}}_{t,x,y}(T)\right) \geq 0 \text{ for some } \hat{\nu} \in \hat{\mathcal{U}}\right\} \tag{2.3.2}$$

$$w(t,x,p) := \inf\left\{\begin{array}{l} y \geq -\kappa : \hat{V}\left(\hat{X}^{\hat{\nu}}_{t,\hat{x}}(T), Y^{\hat{\nu}}_{t,x,y}(T)\right) \geq 0 \\ \quad \text{and } P^{\alpha,\chi}_{t,p} \in L \text{ for some } \hat{\nu} \in \hat{\mathcal{U}} \end{array}\right\}. \tag{2.3.3}$$

**Proof.**
step 1: We first show that $\hat{v} \geq u$. For $y > \hat{v}(t,x,p)$, we may find $\nu \in \mathcal{U}$ such that $p_o := \mathbb{E}\left[\Psi\left(X^{\nu}_{t,x}(T), Y^{\nu}_{t,x,y}(T)\right)\right] \geq p$. By the stochastic integral representation theorem, recall Assumption (2.3.1), there exists $(\alpha,\chi) \in \mathcal{A}$ such that

$$\Psi\left(X^{\nu}_{t,x}(T), Y^{\nu}_{t,x,y}(T)\right) = p_o + \int_t^T \alpha_s \cdot dW_s + \int_t^T \int_E \chi_{s,e} \tilde{J}(de,ds) = P^{\alpha,\chi}_{t,p_o}(T).$$

Since $p_o \geq p$, it follows that $\Psi\left(X^{\nu}_{t,x}(T), Y^{\nu}_{t,x,y}(T)\right) \geq P^{\alpha,\chi}_{t,p}(T)$, and therefore $y \geq u(t,x,p)$ from the definition of the problem $u$.

step 2: We next show that $u \geq \hat{v}$. For $y > u(t,x,p)$, we have

$$\hat{V}\left(\hat{X}^{\hat{\nu}}_{t,\hat{x}}(T), Y^{\nu}_{t,x,y}(T)\right) \geq 0$$

for some $\hat{\nu} = (\nu,\alpha,\chi) \in \hat{\mathcal{U}}$. It follows that

$$\mathbb{E}\left[\hat{V}\left(\hat{X}^{\hat{\nu}}_{t,\hat{x}}(T), Y^{\hat{\nu}}_{t,x,y}(T)\right)\right] = \mathbb{E}\left[\Psi\left(X^{\nu}_{t,x}(T), Y^{\nu}_{t,x,y}(T)\right) - P^{\alpha,\chi}_{t,p}(T)\right] \geq 0,$$

and since $P^{\alpha,\chi}_{t,p}$ is a martingale

$$\mathbb{E}\left[\Psi\left(X^{\nu}_{t,x}(T), Y^{\nu}_{t,x,y}(T)\right)\right] \geq p = \mathbb{E}\left[P^{\alpha,\chi}_{t,p}(T)\right],$$

so that $y \geq \hat{v}(t,x,p)$ by the definition of $\hat{v}$.
step 3: The inequality $u \leq w$ is obvious. To see that the converse inequality holds, consider some $y > u(t,x,p)$. Then there exists some $\hat{\nu} = (\nu,\alpha,\chi) \in \hat{\mathcal{U}}$ such that

$$\Psi\left(X^{\nu}_{t,x}(T), Y^{\nu}_{t,x,y}(T)\right) \geq P^{\alpha,\chi}_{t,p}(T). \tag{2.3.4}$$

Define

$$\tau := T \wedge \inf\left\{s > t : P^{\alpha,\chi}_{t,p}(s) \le m\right\} \text{ and}$$
$$\tilde{\alpha}_s := \alpha_s \mathbf{1}_{\{s\le\tau\}},$$
$$\tilde{\chi}_{s,e} := \left[-\left(\chi_{s,e} \vee \left(m - P^{\alpha,\chi}_{t,p}(s-)\right)\right)^- + (\chi_{s,e})^+\right] \mathbf{1}_{\{s\le\tau\}} \text{ for } s \in [t,T].$$

Clearly, $P^{\alpha,\chi}_{t,p}(T) = P^{\tilde{\alpha},\tilde{\chi}}_{t,p}(T)$ on the event $\{\tau = T\}$. Since $P^{\tilde{\alpha},\tilde{\chi}}_{t,p}(T) = m$ on the event $\{\tau < T\}$, it follows from (2.3.4) that

$$\Psi\left(X^\nu_{t,x}(T), Y^\nu_{t,x,y}(T)\right) \ge P^{\tilde{\alpha},\tilde{\chi}}_{t,p}(T).$$

We finally observe that $P^{\tilde{\alpha},\tilde{\chi}}_{t,p}(T) \ge m$ by the definition of $\tilde{\alpha}$ and $\tilde{\chi}$, and that the last inequality implies that $P^{\tilde{\alpha},\tilde{\chi}}_{t,p}(T) \le M$. By the martingale property of the process $P^{\tilde{\alpha},\tilde{\chi}}_{t,p}$, we conclude that it is valued in the interval $[m, M] = L$. Hence, $y \ge w(t, x, p)$. □

Let us observe that the problem (2.3.2) can be alternatively formulated as

$$\hat{v}(t,x,p) = \inf\left\{y \ge -\kappa : Y^{\hat{\nu}}_{t,x,y}(T) \ge \hat{g}\left(\hat{X}^{\hat{\nu}}_{t,\hat{x}}(T)\right) \text{ for some } \hat{\nu} = (\nu,\alpha,\chi) \in \hat{\mathcal{U}}\right\}$$

where $\hat{g}$ is the generalized inverse of $\hat{V}$ at 0

$$\hat{g}(\hat{x}) := \inf\left\{y : \widehat{V}(\hat{x}, y) \ge 0\right\}.$$

**Remark 2.3.3.** 1. In the case where the infimum in the definition of $\hat{v}(t,x,p)$ is achieved and there exists a control $\nu \in \mathcal{U}$ satisfying

$$\mathbb{E}\left[\Psi\left(X^\nu_{t,x}(T), Y^\nu_{t,x,y}(T)\right)\right] = p$$

with $y = \hat{v}(t,x,p)$, the above argument shows that the corresponding process $P^{\alpha,\chi}_{t,p}$ coincides with the conditional expectation of $\Psi\left(X^\nu_{t,x}(T), Y^\nu_{t,x,y}(T)\right)$, i.e.

$$P^{\alpha,\chi}_{t,p}(s) = \mathbb{E}\left[\Psi\left(X^\nu_{t,x}(T), Y^\nu_{t,x,y}(T)\right)\middle|\, \mathcal{G}_s\right] \quad \text{for } s \in [t,T].$$

2. Equation (2.3.3) shows that one can restrict to controls $\alpha$ and $\chi$ such that $P^{\alpha,\chi}_{t,p}$ takes values in $L$. This is rather natural since the latter should be interpreted as a conditional expectation of $\Psi$, which convex hull is $L$, and this corresponds to the natural domain $[m, M]$ of the variable $p$. Notice also that the value function $\hat{v}(\cdot, p)$ is constant for $p < m$, and equal $\infty$ for $p > M$. In both cases, the natural domain of $\hat{v}$ is therefore $[0,T] \times \overline{\mathbf{X}} \times [m, M]$.

3. Moreover, in the special case where $m$ and/or $M$ are finite, the fact that $P^{\alpha,\chi}_{t,p}$ takes values in $L$ allows us to consider that the jump coefficient $\chi$ is bounded. This will be useful in the proofs of Section 2.3.5. Indeed we may write in that particular case
$$m - P^{\alpha,\chi}_{t,p}(s-) \le \chi_s \le M - P^{\alpha,\chi}_{t,p}(s-),$$
with $P^{\alpha,\chi}_{t,p}(s-) \in [m, M]$.

**Example 2.3.1.** Given a non-negative function $h$, let us consider the case where $\check{\Psi}(x,y) = \frac{y}{h(x)} \wedge 1$, with the convention $\frac{y}{0} = +\infty$ for $y \in \mathbb{R}_+$. For $\kappa = 0$, we then obtain
$$\check{v}(t,x,p) = \inf\left\{ y \in \mathbb{R}_+ : \mathbb{E}\left[ \frac{Y^\nu_{t,x,y}(T)}{g\left(X^\nu_{t,x}(T)\right)} \wedge 1 \right] \ge p \text{ for some } \nu \in \mathcal{U} \right\},$$
which is the problem of the expected success ratio studied in Föllmer and Leukert [FL99]. Using (2.3.2), we see that the above problem reduces to
$$\check{v}(t,x,p) = \inf\left\{ y \in \mathbb{R}_+ : \check{V}\left( \hat{X}^{\hat{\nu}}_{t,x,p}(T), Y^{\hat{\nu}}_{t,x,y}(T) \right) \ge 0 \text{ for some } \hat{\nu} = (\nu,\alpha,\chi) \in \hat{\mathcal{U}} \right\},$$
where $\check{V}(x,p,y) = \check{\Psi}(x,y) - p$.

**Example 2.3.2.** One can similarly recover the problem of stochastic target under controlled probability of success studied in Bouchard, Elie and Touzi [BET09] and Föllmer and Leukert [FL99]:
$$\tilde{v}(t,x,p) := \inf\left\{ y \in \mathbb{R}_+ : \mathbb{P}\left[ \tilde{\Psi}\left( X^\nu_{t,x}(T), Y^\nu_{t,x,y} \right) \ge 0 \right] \ge p \text{ for some } \nu \in \mathcal{U} \right\},$$
for some measurable map $\tilde{\Psi}$ from $\mathbb{R}^{d+1}$ into $\mathbb{R}$ such that, for every fixed $x \in \mathbb{R}^d$, the function $y \mapsto \tilde{\Psi}(x,y)$ is non-decreasing and right-continuous. The reduction of the problem (2.3.2) leads to
$$\tilde{v}(t,x,p) := \inf\left\{ y \in \mathbb{R}_+ : \tilde{V}\left( \hat{X}^{\hat{\nu}}_{t,x,p}(T), Y^{\hat{\nu}}_{t,x,y} \right) \ge 0 \text{ for some } \hat{\nu} \in \hat{\mathcal{U}} \right\},$$
where $\tilde{V}(x,p,y) = \mathbf{1}_{\{\tilde{\Psi}(x,y)\ge 0\}} - p$.

### 2.3.2 PDE characterization in the domain

In view of Proposition 2.3.2, the PDE characterization of Theorem 2.2.5 can be extended to the problem (2.3.1). Let us first introduce notations associated to the augmented system. For $\hat{u} = (u,\alpha,\pi) \in \hat{U}$ and $\hat{x} = (x,p) \in \hat{\mathbf{X}}$, set
$$\hat{\mu}(\hat{x},\hat{u}) := \begin{pmatrix} \mu_X(x,u) \\ -\displaystyle\int_E \pi(e)\lambda(de) \end{pmatrix}, \quad \hat{\sigma}(\hat{x},\hat{u}) := \begin{pmatrix} \sigma_X(x,u) \\ \alpha^\top \end{pmatrix},$$
$$\hat{\beta}(\hat{x},\hat{u}(e),e) := \begin{pmatrix} \beta_X(x,u(e),e) \\ \pi(e) \end{pmatrix}.$$

We also introduce the following operators

$$\hat{\mathcal{L}}^{\hat{u}}\varphi(t,\hat{x}) := \partial_t\varphi(t,\hat{x}) + \hat{\mu}(\hat{x},\hat{u})\cdot D\varphi(t,\hat{x}) + \frac{1}{2}\mathrm{Tr}\left[\hat{\sigma}\hat{\sigma}^\top(\hat{x},\hat{u})\,D^2\varphi(t,\hat{x})\right]$$
$$\hat{\mathcal{G}}^{\hat{u},e}\varphi(t,\hat{x}) := \beta_Y(x,\varphi(t,\hat{x}),u(e),e) - \varphi\left(t,\hat{x}+\hat{\beta}(\hat{x},\hat{u}(e),e)\right) + \varphi(t,\hat{x}).$$

Recalling point 3 of Remark 2.3.3, we also introduce, for $(x,k,q,A)\in\mathbb{R}\times\mathbb{R}\times\mathbb{R}^{d+1}\times\mathbb{S}^{d+1}$, $\hat{u}=(u,\alpha,\pi)\in\hat{U}$, $\varepsilon>0$ and $\eta\in[-1,1]$,

$$\hat{N}^{\hat{u}}(\hat{x},y,q) := \sigma_Y(x,y,u) - \hat{\sigma}(\hat{x},\hat{u})^\top q = N^u(x,y,q_x) - q_p\alpha,\ \text{for } q=(q_x,q_p)\in\mathbb{R}^d\times\mathbb{R},$$
$$\widehat{\Delta}^{\widehat{u},e}(t,\widehat{x},y,\psi) := \beta_Y(x,y,u,e) - \psi\left(t,\widehat{x}+\widehat{\beta}(\widehat{x},\widehat{u}(e),e)\right) + y$$
$$\hat{\mathcal{N}}_{\varepsilon,\eta}(t,\hat{x},y,q,\psi) := \left\{\begin{array}{c}\hat{u}\in\hat{U}\ \text{s.t.}\ \left|\hat{N}^{\hat{u}}(\hat{x},y,q)\right|\le\varepsilon,\quad p+\pi(e)\in[m,M]\\ \text{and}\quad \widehat{\Delta}^{\widehat{u},e}(t,\widehat{x},y,\psi)\ge\eta\quad\text{for }\lambda\text{-a.e. } e\in E\end{array}\right\} \tag{2.3.5}$$
$$\hat{H}_{\varepsilon,\eta}(\hat{\Theta},\varphi) := \sup_{\hat{u}\in\hat{\mathcal{N}}_{\varepsilon,\eta}(t,\hat{x},y,q,\varphi)} \hat{\mathbf{A}}^{\hat{u}}(\widehat{\Theta}) \tag{2.3.6}$$

where

$$\hat{\Theta} := (t,\hat{x},y,k,q,A)$$
$$\hat{\mathbf{A}}^{\hat{u}}\left(\hat{\Theta}\right) := -k + \mu_Y(x,y,u) - \hat{\mu}(\hat{x},\hat{u})\cdot q - \frac{1}{2}\mathrm{Tr}\left[\hat{\sigma}\hat{\sigma}^\top(\hat{x},\hat{u})\,A\right]$$

and

$$\hat{\mathbf{N}}(t,\widehat{x},y,q,\psi) := \left\{\begin{array}{c}(r,s)\in\mathbb{R}^d\times\mathbb{R} : \exists\ \widehat{u}\in\widehat{U}\ \text{s.t. } r=\widehat{N}^{\widehat{u}}(\widehat{x},y,q)\\ \text{and } s\le\widehat{\Delta}^{\widehat{u},e}(t,\widehat{x},y,\psi)\quad\text{for }\lambda\text{-a.e. } e\in E\end{array}\right\},$$
$$\hat{\delta} := \mathrm{dist}\left(0,\hat{\mathbf{N}}^c\right) - \mathrm{dist}\left(0,\hat{\mathbf{N}}\right).$$

The operators $\hat{H}^*, \hat{H}_*\widehat{\delta}^*$ and $\widehat{\delta}_*$ are constructed from $\hat{H}_{\varepsilon,\eta}$ and $\widehat{\delta}$ exactly as $H^*, H_*, \delta^*$ and $\delta_*$ are defined from $H_{\varepsilon,\eta}$ and $\delta$. Finally, we define the function

$$\hat{g}(\hat{x}) := \inf\left\{y\ge-\kappa : \hat{V}(\hat{x},y)\ge 0\right\},\qquad \hat{x}=(x,p)\in\mathbf{X}\times[m,M].$$

As an almost direct consequence of Theorems 2.2.5 and 2.3.2, we obtain the viscosity property of $\hat{v}$ under the following assumptions, which are the analogs of Assumptions 2.2.3 and 2.2.4 for the augmented control system $\hat{\mathbf{X}}$. Define then as previously for any $\varphi\in C^{1,2,2}([0,T]\times\mathbb{R}^d\times\mathbb{R};\mathbb{R})$, $\hat{u}\in\hat{U}$ and $(t,\hat{x},z_1,z_2)\in[0,T]\times\mathbb{R}^{2d+3}$

$$\hat{\mathcal{L}}^{\hat{u}}_{\hat{X},Z}\bar{\varphi}(t,\hat{x},z) := \hat{\mathcal{L}}^{\hat{u}}\varphi(t,\hat{x}) - \hat{\mu}(\hat{x},u)\cdot z_1 - \mu_Y(\hat{x},y,u)z_2, \tag{2.3.7}$$

where $z=:(z_1,z_2)\in\mathbb{R}^{d+1}\times\mathbb{R}$ and $\bar{\varphi}(t,\hat{x},z) := \varphi(t,\hat{x}) - |z|^2$.

**Assumption 2.3.4.** *For all $\varepsilon > 0, \eta \in [-1,1], (t_o, x_o, p_o) \in [0,T] \times \mathbb{R}^d \times [m,M], \varphi \in C^{1,2}([0,T] \times \mathbb{R}^{d+1}; \mathbb{R})$ and finite $C_1$ satisfying*

$$\begin{gathered}\sup_{\hat{u} \in \mathcal{N}_{\varepsilon,\eta}(t,\hat{x},y,D\varphi,\varphi)} \left\{ \mu_Y(x,y,u) - \hat{\mathcal{L}}^{\hat{u}}\varphi(t,\hat{x}) \right\} \leq 2C_1 \\ \text{for all } (t,\hat{x}) \in B_\varepsilon(t_o,\hat{x}_o) \text{ and } y \in \mathbb{R} \text{ s.t. } |y - \varphi(t,\hat{x})| \leq \varepsilon,\end{gathered} \tag{2.3.8}$$

*there exists $\varepsilon' > 0, \eta' \in [-1,1]$ and a finite $C_2$ such that*

$$\begin{gathered}\sup_{\hat{u} \in \mathcal{N}_{\varepsilon',\eta'}(t,\hat{x},y,D\varphi,\varphi)} \left\{ \mu_Y(x,y,u) - \mathcal{L}^u_{\hat{X},Z}\bar{\varphi}(t,\hat{x},z) \right\} \leq 2C_1 + |C_1| \\ \text{for all } (t,\hat{x},y,z) \in [0,T] \times \mathbb{R}^{2d+4} \text{ s.t. } \begin{cases} (t,\hat{x},z) \in B_{\varepsilon'}(t_o,\hat{x}_o,0) \\ y \in \mathbb{R} \text{ s.t. } |y - \bar{\varphi}(t,\hat{x})| \leq \eta', \end{cases}\end{gathered} \tag{2.3.9}$$

*and*

$$\begin{gathered}\frac{\left[\mu_Y(x,y,u) - \mathcal{L}^{\hat{u}}_{\hat{X},Z}\bar{\varphi}(t,\hat{x},z)\right]^+}{1 + |N^u(x,y,D\varphi)|} \leq C_2 \left(1 + |\sigma_Y(x,y,u)| + \sum_{i=1}^d \left|\hat{\sigma}^{i,\cdot}(\hat{x},u)\right|\right) \\ \text{for all } (t,\hat{x},y,z) \in [0,T] \times \mathbb{R}^{2d+4} \text{ s.t. } \begin{cases} (t,\hat{x},z_1,z_2) \in B_{\varepsilon'}(t_o,\hat{x}_o,0) \\ y \in \mathbb{R} \text{ s.t. } |y - \bar{\varphi}(t,\hat{x},z)| \leq \eta', \end{cases} \\ \text{and } u \in U \text{ such that } \Delta^{u,\cdot}(t,\hat{x},y,\varphi) \geq \eta \ \lambda\text{-a.e.}\end{gathered} \tag{2.3.10}$$

**Assumption 2.3.5.** *(Continuity of $\hat{\mathcal{N}}_{0,\eta}(t,x,p,y,q,f)$) Let $B$ be a subset of $[0,T] \times \boldsymbol{X} \times [m,M] \times \mathbb{R} \times \mathbb{R}^{d+1}$, $f \in \mathcal{C}^0([0,T] \times \boldsymbol{X} \times [m,M])$ and $\eta > 0$ such that $\hat{\mathcal{N}}_{0,2\eta}(\cdot,f) \neq \emptyset$ on $B$. Then, for every $\varepsilon > 0$, $(t_o,x_o,p_o,y_o,q_o) \in Int(B)$ and $\hat{u}_o \in \hat{\mathcal{N}}_{0,2\eta}(t_o,x_o,p_o,y_o,q_o,f)$, there exists an open neighborhood $B'$ of $(t_o,x_o,p_o,y_o,q_o)$ and a locally Lipschitz map $\hat{\nu}$ defined on $B'$ such that $|\hat{\nu}(t_o,x_o,p_o,y_o,q_o) - \hat{u}_o| \leq \varepsilon$, and $\hat{\nu}(t,x,p,y,q) \in \hat{\mathcal{N}}_{0,\eta}(t,x,y,p,q,f)$ on $B'$.*

As in Section 2.2.2, we shall need to define the definition of viscosity solution we shall use in this framework.

**Definition 2.3.6.** *We say that a l.s.c. (resp. u.s.c.) function $U$ (resp. $V$) is a viscosity supersolution of $\hat{H}^*U \geq 0$ (resp. subsolution of $\hat{H}_*V \leq 0$) on $[0,T) \times \mathbb{R}^d \times \mathbb{R}$ if for every smooth function $\varphi \in C^{1,2}([0,T] \times \mathbb{R}^d \times \mathbb{R}, \mathbb{R})$ of linear growth and $(t_o,x_o,p_o) \in [0,T) \times \mathbb{R}^d \times \mathbb{R}$ such that $\min_{[0,T]\times\mathbb{R}^d\times\mathbb{R}}(U - \varphi) = (U - \varphi)(t_o,x_o,p_o) = 0$ (resp. $\max_{[0,T]\times\mathbb{R}^d\times\mathbb{R}}(V - \varphi) = (V - \varphi)(t_o,x_o,p_o) = 0$), we have*

$$H^*\varphi(t_o,x_o,p_o) \geq 0 \qquad (\text{resp. } H_*\varphi(t_o,x_o,p_o) \leq 0).$$

**Corollary 2.3.7.** *Under Assumption 2.3.4, the function $\hat{v}_*$ is a viscosity supersolution of*

$$\hat{H}^*\hat{v}_* \geq \ \text{ on } [0,T) \times \hat{\boldsymbol{X}}. \tag{2.3.11}$$

*Under Assumption 2.3.5, $\hat{v}^*$ is a viscosity subsolution of*

$$\min\left\{\hat{v}^* + \kappa, \hat{H}_* \hat{v}^*\right\} \leq 0 \ \textit{on} \ [0,T) \times \hat{\boldsymbol{X}}. \tag{2.3.12}$$

The supersolution property is a direct consequence of Theorem 2.2.5, the representation (2.3.2) and point 3 of Remark 2.3.3. The subsolution property is obtained similarly.

**Example 2.3.3.** Consider here the context of both Examples 2.3.1 and 2.3.2, with the dynamics of Example 2.2.1. If the conditions of Corollary 2.3.7 are satisfied, direct computations lead that both $\check{v}_*$ and $\tilde{v}_*$ are viscosity supersolution on $[0,T)\times\mathbf{X}$ of

$$\begin{aligned} 0 \leq & -\partial_t \varphi - \frac{1}{2}\sigma^2 D_{xx}\varphi \\ & - \inf_{\substack{\pi\in\Pi(p)\\ \alpha\in\mathbb{R}^d}} \left\{ \begin{array}{c} \frac{1}{2}\left|\alpha\right|^2 D_{pp}\varphi + \mathrm{Tr}\left[\sigma\alpha D_{xp}\varphi\right] \\ -\alpha\left(D_p\varphi\right)^\top \sigma^{-1}\mu - D_p\varphi \displaystyle\int_E \pi(e)\lambda(de) \end{array} \right\}, \end{aligned} \tag{2.3.13}$$

whenever $D_{pp}\varphi > 0$, and with

$$\Pi(p) := \left\{ \begin{array}{c} \pi \in \mathbb{L}^2_\lambda \text{ s.t., for } \lambda\text{-a.e. } e\in E,\ p+\pi\in[0,1] \\ \text{and } \left(D_x\varphi + \sigma^{-1}D_p\varphi\alpha\right)\beta(\cdot,e) - \varphi\left(\cdot,\cdot+\beta(\cdot,e),\cdot+\pi(e)\right) + \varphi \geq 0 \end{array} \right\}.$$

Notice in this particular context that the process $X$ is not influenced by the control $\nu$. Hence, Assumption 2.3.4 allows to control the possible unboundedness of $\mu_Y$ in its $u$-parameter, as well as the possible unboundedness of $\hat{\sigma}\hat{\sigma}^\top$ in its $a$-parameter. Indeed, recall from point 3. of Remark 2.3.3 that we may reduce to $\chi$ bounded. Assumption 2.3.4 essentially states that $\hat{\sigma}\hat{\sigma}^\top(a)$ is controlled in terms of $1+|a|$. We refer to Example 2.3.4 for the argument to handle the possible unboundedness of $\chi$.

**Example 2.3.4.** As we have seen in Example 2.3.3, the fact that we may restrict to processes $\chi$ taking bounded values is crucial for the Assumption 2.3.4 to hold. Consider for sake of simplicity the dynamics of Example 2.2.1 with $d=1$ and the problem

$$\mathrm{y}(t,x,p) := \inf\left\{y \geq -\kappa : \exists\ \nu\in\mathcal{U} \text{ s.t. } \mathbb{E}\left[\rho\left(Y^\nu_{t,x,y}(T) - g\left(X_{t,x}(T)\right)\right)\right] \geq p\right\}$$

with $g:\mathbb{R}\to\mathbb{R}$ a continuous function of polynomial growth, and $\rho:\mathbb{R}\to\mathbb{R}$ (one may think of $\rho$ as an utility function or up to the sign, a loss function) such that $\rho(\mathbb{R}) = \mathbb{R}$. The latter condition entails that the argument allowing to reduce to controls $\chi$ taking bounded values does not hold, and we hence have to characterize more precisely the controls $(u,a,\pi)\in U\times\mathbb{R}\times L^2_\lambda$ such that

$$u\beta(x,e) - \varphi(t,x+\beta(x,e),p+\pi(e)) + \varphi(t,x) \geq \eta \quad \text{for } \lambda\text{-a.e.} e\in E$$

for some $\eta \in [-1,1]$ and $\varphi$ a smooth function. This is done by a suitable use of Assertion (2.3.8). Let the condition of Assumption 2.3.4 hold true for given $\varepsilon > 0, \eta \in [-1,1], (t_o, x_o, p_o) \in [0,T] \times \mathbb{R}^d \times \mathbb{R}, \varphi \in C^{1,2,2}([0,T] \times \mathbb{R}^d \times \mathbb{R})$ and finite $C_1$ (assume without loss of generality that $C_1 > 0$. We recall for $(t, x, p, y, q_x, q_p, \varphi) \in [0,T] \times \mathbb{R}^{2d+3} \times C^{1,2,2}([0,T] \times \mathbb{R}^d \times \mathbb{R})$ the particular form of $\hat{\mathcal{N}}_{\varepsilon,\eta}$ in that case

$$\hat{\mathcal{N}}_{\varepsilon,\eta}(t,x,p,y,q_x,q_p,\varphi) := \left\{ \begin{array}{c} (u,a,\pi) \in \mathbb{R}^2 \times \mathbb{L}^2_\lambda \text{ s.t. } |u\sigma(x) - q_x\sigma(x) - q_p a| \le \varepsilon \\ u\beta(x,e) - \varphi(t, x+\beta(x,e), p+\pi(e)) + \varphi(t,x) \ge \eta \\ \text{for } \lambda\text{-a.e. } e \in E \end{array} \right\}.$$

Hence, by possibly diminishing $\varepsilon > 0$, we have

$$\left\{ \begin{array}{l} u\mu(x) - \partial_t\varphi - \partial_x\varphi\mu(x) + \partial_p\varphi \int_E \pi(e)\lambda(de) \\ -\frac{1}{2}\partial_{xx}\varphi\sigma(x)^2 - \frac{1}{2}\partial_{pp}\varphi a^2_{x,u} - \sigma(x)a_{x,u}\partial_{px}\varphi \end{array} \right\}(t,x,p) \le C_1$$

$$\text{for all } (t,x,p) \in B_\varepsilon(t_o,x_o,p_o),\ \zeta_{t,x,p}, q_p \in \mathbb{R} \text{ s.t. } \left\{ \begin{array}{c} |\zeta_{t,x,p}\sigma(x)| \le \varepsilon \\ q_p \in B_\varepsilon(\partial_p\varphi(t_o,x_o,p_o)) \\ |q_p| \ge \varepsilon/2 \end{array} \right.$$

$$\text{and } (u,\pi) \in \mathbb{R} \times L^2_\lambda \text{ s.t. } \left\{ \begin{array}{c} u\beta(x,e) - \varphi(t, x+\beta(x,e), p+\pi(e)) + \varphi(t,x) \ge \eta, \\ \text{for } \lambda\text{-a.e. } e \in E, \end{array} \right.$$

where

$$a_{x,u} := \frac{\sigma(x)}{q_p}\left(u - \partial_x\varphi(t,x,p) - \zeta_{t,x,p}\right).$$

Hence we have

$$\left(\partial_p\varphi \int_E \pi(e)\lambda(de) + A_{t,x,p} + B_{t,x,p}u - \frac{1}{2}\partial_{pp}\varphi\frac{\sigma(x)^2}{q_p^2}u^2\right)(t,x,p) \le C_1, \quad (2.3.14)$$

with

$$A_{t,x,p} := \left( \begin{array}{c} -\partial_t\varphi - \partial_x\varphi\mu(x) - \frac{1}{2}\partial_{xx}\varphi\sigma(x)^2 \\ +\frac{\sigma(x)^2}{q_p}\partial_{px}\varphi(\varphi_x + \zeta_{t,x,p}) - \frac{\sigma(x)^2}{q_p^2}\partial_{pp}\varphi(\partial_x\varphi + \zeta_{t,x,p}) \end{array} \right)(t,x,p)$$

$$B_{t,x,p} := \left(\mu(x) - \frac{\sigma(x)^2}{q_p}\partial_{px}\varphi + \partial_{pp}\varphi\frac{\sigma(x)^2}{q_p^2}(\partial_x\varphi + \zeta_{t,x,p})\right)(t,x,p),$$

and finally, since $\sigma > 0$, and $\mu, \sigma$ are continuous, there is finite $C' > 0$ such that

$$\left(\partial_p(t,x,p)\varphi \int_E \pi(e)\lambda(de)\right)^+ \leq C'\left(1+u^2\right) \tag{2.3.15}$$

$$\text{for all } (t,x,p) \in B_\varepsilon(t_o,x_o,p_o),\ \zeta_{t,x,p}, q_p \in \mathbb{R} \text{ s.t. } \begin{cases} |\zeta_{t,x,p}\sigma(x)| \leq \varepsilon \\ q_p \in B_\varepsilon(\partial_p\varphi(t_o,x_o,p_o)) \\ |q_p| \geq \varepsilon/2 \end{cases}$$

$$\text{and } (u,\pi) \in \mathbb{R}\times L^2_\lambda \text{ s.t. } u\beta(x) - \varphi(t,x+\beta(x),p+\pi) + \varphi(t,x) \geq \eta.$$

Observe now that (2.3.14) implies that, after possibly changing $\varepsilon > 0$, we have

$$\left(\partial_p\varphi \int_E \pi(e)\lambda(de) + A^\iota_{t,x,p} + B^\iota_{t,x,p}u - \frac{1}{2}\partial_{pp}\varphi\frac{\sigma(x)^2}{q_p^2}u^2\right)(t,x,p) \leq 2C_1,$$

$$\text{for all } (t,x,p) \in B_\varepsilon(t_o,x_o,p_o),\ \zeta_{t,x,p}, q_p, \iota \in \mathbb{R} \text{ s.t. } \begin{cases} |\zeta_{t,x,p}\sigma(x)| \leq \varepsilon \\ q_p \in B_\varepsilon(\partial_p\varphi(t_o,x_o,p_o)) \\ |q_p| \geq \varepsilon/2 \\ |\iota| \leq \varepsilon \end{cases}$$

$$\text{and } (u,\pi) \in \mathbb{R}\times L^2_\lambda \text{ s.t. } \begin{cases} u\beta(x,e) - \varphi(t,x+\beta(x,e),p+\pi(e)) + \varphi(t,x) \geq \eta, \\ \text{for } \lambda\text{-a.e. } e \in E, \end{cases}$$

with

$$A^\iota_{t,x,p} := \begin{pmatrix} -\partial_t\varphi - (\partial_x\varphi+\iota)\mu(x) - \frac{1}{2}\partial_{xx}\varphi\sigma(x)^2 \\ +\frac{\sigma(x)^2}{q_p}\partial_{px}\varphi(\varphi_x+\zeta_{t,x,p}) - \frac{\sigma(x)^2}{q_p^2}\partial_{pp}\varphi(\partial_x\varphi+\zeta_{t,x,p}) \end{pmatrix}(t,x,p)$$

$$B^\iota_{t,x,p} := \left(\mu(x)(1+\iota) - \frac{\sigma(x)^2}{q_p}\partial_{px}\varphi + \partial_{pp}\varphi\frac{\sigma(x)^2}{q_p^2}(\partial_x\varphi+\zeta_{t,x,p})\right)(t,x,p),$$

which entails that (2.3.9) holds true.

We now turn to the verification of Assertion (2.3.9), and will consider two cases.

Case 1: $\partial_p\varphi(t_o,x_o,p_o) \neq 0$. Hence, in view of (2.3.15) and since (2.3.9) holds, after possibly changing $\varepsilon > 0$, we have

$$\frac{\left[\mu_Y(x,y,u) - \mathcal{L}^{\hat{u}}_{\hat{X},Z}\bar{\varphi}(t,\hat{x},z)\right]^+}{1+|N^u(x,y,D\varphi)|} \leq C'\frac{\left(1+|u|^2+|a|^2\right)}{|u\sigma(x) - \partial_x\varphi(t,x,p)\sigma(x) - a\partial_p\varphi(t,x,p)|}$$

$$\text{for all } (t,\hat{x},y,z) \in [0,T]\times\mathbb{R}^{2d+4} \text{ s.t. } \begin{cases} (t,\hat{x},z_1,z_2) \in B_{\varepsilon'}(t_o,\hat{x}_o,0) \\ y \in \mathbb{R} \text{ s.t. } |y-\bar{\varphi}(t,\hat{x},z)| \leq \varepsilon', \end{cases}$$

$$\text{and } u \in U \text{ such that } \Delta^{u,\cdot}(t,\hat{x},y,\varphi) \geq \eta\ \lambda\text{-a.e.,} \tag{2.3.16}$$

where $\bar{\varphi}$ is defined from $\varphi$ as in Assumption 2.3.4, $C'$ is a finite constant, and with $\partial_p\varphi \neq 0$ on $B_\varepsilon(t_o, x_o, p_o)$. If the set $U$ happens to be unbounded, this reasoning is not sufficient to ensure that Assertion (2.3.9) holds, when it is trivial otherwise.

Case 2: Consider now that

$$\partial_p\varphi(t_o, x_o, p_o) = 0. \tag{2.3.17}$$

Recalling the conditions of Assumption 2.3.4, after possibly changing $\varepsilon > 0, \eta \in [-1, 1]$:

$$\left\{-\partial_t\varphi - \mu(x)\partial_x\varphi - \frac{1}{2}\sigma(x)^2\partial_{xx}\varphi - \sigma(x)a\partial_{xp}\varphi - \frac{1}{2}a^2\left(\partial_{pp}\varphi - \varepsilon\right)\right\}(t_o, x_o, p_o) \leq 2C_1$$

for all $(a, \pi) \in \mathbb{R} \times \Delta_\eta(a; t_o, x_o, p_o)$, with

$$\Delta_\eta(a; t, x, p) := \left\{\pi \in L^2_\lambda \text{ s.t. } \left\{\begin{array}{c}\left(\partial_x\varphi(\cdot) + \dfrac{a\partial_p\varphi(\cdot)}{\sigma(x)}\right)\beta(x) \\ -\varphi\left(t, x + \beta(x, e), p + \pi(e)\right) + \varphi(\cdot)\end{array}\right\}(t, x, p) \geq \eta\right\}.$$

Recall from (2.3.17) that $\Delta_\eta(a; t_o, x_o, p_o)$ does not depend on $a \in \mathbb{R}$ so that, for $\pi \in \Delta_\eta(a; t_o, x_o, p_o)$ fixed, we have for all $a \in \mathbb{R}$

$$\left\{-\partial_t\varphi - \mu(x)\partial_x\varphi - \frac{1}{2}\sigma(x)^2\partial_{xx}\varphi - \sigma(x)a\partial_{xp}\varphi - \frac{1}{2}a^2\left(\partial_{pp}\varphi - \varepsilon\right)\right\}(t_o, x_o, p_o) \leq 2C_1,$$

and there is then a finite positive constant $C$ such that

$$-\frac{1}{2}a^2\left(\partial_{pp}\varphi(t_o, x_o, p_o) - \varepsilon\right) \leq C\left(1 + |a|\right).$$

Taking $a$ large enough gives then $\partial_{pp}(t_o, x_o, p_o)\varphi \geq \varepsilon$, and hence, by smoothness of $\varphi$, we have $\partial_{pp}\varphi > 0$ on some neighborhood $B$ of $(t_o, x_o, p_o)$. We finally have

$$\begin{gathered}\frac{\left[\mu_Y(x, y, u) - \mathcal{L}^{\hat{u}}_{\hat{X},Z}\bar{\varphi}(t, \hat{x}, z)\right]^+}{1 + |N^u(x, y, D\varphi)|} \leq C' \frac{\left(1 + |u|^2 - \partial_{pp}\varphi(t, x, p)\,|a|^2\right)}{|u\sigma(x) - \partial_x\varphi(t, x, p)\sigma(x) - a\partial_p\varphi(t, x, p)|} \\ \text{for all } (t, \hat{x}, y, z) \in [0, T] \times \mathbb{R}^{2d+4} \text{ s.t. } \left\{\begin{array}{l}(t, \hat{x}, z_1, z_2) \in B_{\varepsilon'}(t_o, \hat{x}_o, 0) \\ y \in \mathbb{R} \text{ s.t. } |y - \bar{\varphi}(t, \hat{x}, z)| \leq \varepsilon',\end{array}\right. \\ \text{and } u \in U \text{ such that } \Delta^{u,\cdot}(t, \hat{x}, y, \varphi) \geq \eta \ \lambda\text{-a.e.,}\end{gathered} \tag{2.3.18}$$

for some $\varepsilon'$, and again, Assertion (2.3.9) will be obvious in the case where $U$ is bounded.

### 2.3.3 Boundary conditions and state constraint

In our general context, the natural domain of $P$ is $[m, M]$. In the case where $m$ or $M$ are finite, we need to specify the boundary conditions at the end points $m$ and

$M$. By definition of the stochastic target problem with controlled expected loss, we have

$$\hat{v}(\cdot, M) = v \text{ and } \hat{v}(\cdot, m) = -\kappa, \tag{2.3.19}$$

where

$$v(t,x) := \inf\left\{y \geq -\kappa : \Phi\left(X^{\nu}_{t,x}(T), Y^{\nu}_{t,x,y}(T)\right) \geq 0 \text{ for some } \nu \in \mathcal{U}\right\},$$

with

$$\Phi(x,y) := \Psi(x,y) - M. \tag{2.3.20}$$

Also, since $\Psi$ is non-decreasing in $y$, we know that $\hat{v}$ is non-decreasing in $p$. Hence,

$$\begin{aligned} -\kappa \leq \hat{v}_*(\cdot, m) \leq \hat{v}^*(\cdot, p) \leq \hat{v}^*(\cdot, M) \leq v^* \quad &\text{for} \quad p \in [m, M], \\ \hat{v}^*(\cdot, p) = -\kappa \quad \text{for} \quad p < m \quad \text{and} \quad \hat{v}^*(\cdot, p) = \infty \quad &\text{for} \quad p > M, \end{aligned} \tag{2.3.21}$$

and one can naturally expect that $\hat{v}_*(\cdot, m) = -\kappa$ and $\hat{v}^*(\cdot, M) = v^*$. However, the function $\hat{v}$ may have discontinuities at $p = m$ or $p = M$ and, in general, the boundary conditions have to be stated in a weak form, see (2.3.27) and (2.3.61) below. This corresponds to the classical state-space constraint problems, see [Bar94, FS06, Son86a, Son86b] and the references therein.

To obtain a characterization of $\hat{v}$ on these boundaries, we shall appeal to the following additional assumptions. Assumptions 2.3.10 and 2.3.11 already appeared in Bouchard, Elie and Touzi [BET09]. Assumptions 2.3.8, 2.3.9 and 2.3.12 will be used to handled the non-local operator. Also notice that Assumption 2.3.11 linked with Assumption 2.3.4.

**Assumption 2.3.8.** *The following hold.*

**(H1)** *For some integer $\gamma \geq 1, \hat{v}^*(\cdot, m)^+$ satisfies the growth condition*

$$\sup_{[0,T]\times\mathbb{R}^d} \frac{|w(t,x)|}{1 + |x|^{\gamma}} < \infty. \tag{2.3.22}$$

**(H2)** *There is a function $\Lambda$ on $\mathbb{R}^d$ satisfying*

**(H2-i)** *For all $x \in \boldsymbol{X}$ and $y > \Lambda(x)$, there exists $\bar{u} \in U$ such that*

$$\beta_Y(x, y, \bar{u}(e), e) - \Lambda(x + \beta_X(x, \bar{u}(e), e)) + \Lambda(x) > 0 \quad \textit{for } \lambda\textit{-a.e. } e \in E.$$

**(H2-ii)** $\Lambda(x) / |x|^{\gamma} \to +\infty$ *as* $|x| \to \infty$.

**(H2-iii)** $\Lambda \leq -\kappa$ *on* $\boldsymbol{X}$.

**Assumption 2.3.9.** *The set $E$ is finite and $\lambda(e) > 0$ for all $e \in E$.*

**Assumption 2.3.10.** *For all $(x, y, q) \in \boldsymbol{X} \times (-\kappa, \infty) \times \mathbb{R}^d$, we have*

$$\{u \in U : N^u(x, y, q) = 0\} \subsetneq U.$$

We need for the next assumption to introduce the following set, for $(x, y, q) \in \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d$:

$$\tilde{\mathcal{N}}_\varepsilon(x, y, q) := \{u \in U : |N^u(x, y, q)| \le \varepsilon\}. \tag{2.3.23}$$

**Assumption 2.3.11.** *For all compact subset $D$ of $\mathbb{R}^d \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d$, there exists $C > 0$ such that*

$$\sup_{u \in \tilde{\mathcal{N}}_\varepsilon(x,y,q)} \left\{ \mu_Y(x, y, u) - k - \mu_X(x, u) \cdot q - \frac{1}{2} Tr \left[ \sigma_X \sigma_X^\top (x, u) A \right] \right\} \le C \left( 1 + \varepsilon^2 \right)$$

*for all $\varepsilon > 0$ and $(x, y, k, q, A) \in D$.*

**Assumption 2.3.12.** *The maps $\beta_X$, $\beta_Y$ are continuous on $\boldsymbol{X} \times E$ and $\boldsymbol{X} \times \mathbb{R} \times E$ uniformly in $u \in U$. Moreover, $\beta_X$, $\beta_Y$ and $\sigma_X$ satisfy the following condition*

$$\operatorname*{ess\,sup}_{u \in U, e \in E} \{|\beta_X(\cdot, u(e), e)| + |\beta_Y(\cdot, u(e), e)| + |\sigma_X(\cdot, u)|\} \text{ is locally bounded}$$

Since the main concern of this chapter is the analysis of the stochastic target problem under controlled loss with jumps, we do not establish a comparison result of viscosity supersolutions of (2.2.10)-(2.2.13) and subsolutions of (2.2.11)-(2.2.14). Nonetheless, as in Bouchard, Elie and Touzi [BET09], we need such a comparison result in order to establish the boundary conditions of this section.

**Assumption 2.3.13.** *There is a class of functions $\mathcal{C}$ containing all $[-\kappa, +\infty)$ valued functions dominated by $v^*$ such that, for every*

- *$v_1 \in \mathcal{C}$, lower semi-continuous viscosity supersolution of (2.2.10)-(2.2.13) on $[0, T] \times \boldsymbol{X}$*
- *$v_2 \in \mathcal{C}$, upper semi-continuous viscosity subsolution of (2.2.11)-(2.2.14) on $[0, T] \times \boldsymbol{X}$*

*we have $v_1 \ge v_2$.*

The main results of this section shows that the natural boundary conditions (2.3.19) indeed holds true, whenever the comparison principle of Assumption 2.3.13 holds and under the above additional conditions.

**Theorem 2.3.14.** *Assume that Assumptions 2.3.5, 2.3.9 and 2.3.12 hold true.*

(i) *Assume that* $m > -\infty$. *Under Assumptions 2.3.8, and 2.3.10, we have* $\hat{v}^*(\cdot, m) = -\kappa$ *on* $[0, T) \times \boldsymbol{X}$ *and* $\hat{v}_*(\cdot, m) = -\kappa$ *on* $[0, T] \times \boldsymbol{X}$.

(ii) *Assume that* $M < \infty$. *Under Assumptions 2.3.11 and 2.3.4,* $\hat{v}^*(\cdot, M)$ *is a viscosity supersolution of* (2.2.10)-(2.2.13) *on* $[0, T] \times \boldsymbol{X}$. *In particular, if in addition the comparison principle of Assumption 2.3.13 is satisfied, then* $\hat{v}^*(\cdot, M) = \hat{v}_*(\cdot, M) = v_* = v^*$ *on* $[0, T] \times \boldsymbol{X}$.

The proof is reported in Section 2.3.5.

**Remark 2.3.15.** This subsection is similar to the one in Bouchard, Elie and Touzi [BET09], where the authors studied the boundary conditions at $p = 0$ and $p = 1$ in the case of target reachability under controlled probability, i.e. $\Psi$ is of the form $\Psi(x, y) = \mathbf{1}_{\{y \geq g(x)\}}$. In this paper, the natural domain of $P$ is $[0, 1]$, and the authors studied the behavior of the value function $\hat{v}$ when $p \to 0$ and $p \to 1$.

**Remark 2.3.16.** Consider the framework of Example 2.3.4. If the set $U$ happens to be unbounded, then Assumption 2.3.12 is clearly not satisfied. Recalling the discussion in Example 2.3.4, Assumption 2.3.4 is satisfied if $U$ is bounded. However, if we are interested in the case where the set $U$ is unbounded, we might consider the following reasoning.

We introduce the set $\mathcal{U}^n$ consisting in controls of $\mathcal{U}$ taking their values in $U^n$, where $U^n$ is the subset of $u \in U$ satisfying $\|u\| \leq n$, as well as the corresponding value function

$$\mathrm{y}^n(t, x, p) := \inf\left\{y \geq -\kappa : \ \exists\, \nu \in \mathcal{U}^n \text{ s.t. } \mathbb{E}\left[\rho\left(Y^{\nu}_{t,x,y}(T) - g\left(X_{t,x}(T)\right)\right)\right] \geq p\right\}.$$

Considering the problem defined in terms of $\mathrm{y}^n$ instead of $\mathrm{y}$, the previous arguments entails that Assumption 2.3.4 is satisfied, so that, by Corollary 2.3.7, $\mathrm{y}^n$ is a viscosity supersolution of

$$\hat{H}^*_n \mathrm{y}^n_* \geq \text{ on } [0, T) \times \hat{\mathbf{X}}, \tag{2.3.24}$$

where $\hat{H}^*_n$ is the operator defined in terms of controls $u \in U^n$. The sequence of functions $(\mathrm{y}^n)_{n \geq 1}$ is clearly decreasing, and obviously

$$\mathrm{y}^n \geq \mathrm{y}.$$

Defining $\mathrm{y}^\infty := \inf_{n \geq 1} \mathrm{y}^n$, we clearly have $\mathrm{y}^\infty_* \geq \mathrm{y}_*$, where $\mathrm{y}^\infty_*$ and $\mathrm{y}_*$ denote the lower semi-continuous envelopes of $\mathrm{y}^\infty$ and $\mathrm{y}$. On the other hand, standard estimates imply that, for any $\nu \in \mathcal{U}$ and

$$\nu^n := \nu \mathbb{1}_{\{\|\nu\| \leq n\}},$$

we have

$$\mathbb{E}\left[\rho\left(Y^{\nu^n}_{t,x,y}(T)-g\left(X_{t,x}(T)\right)\right)\right] \underset{n\to\infty}{\longrightarrow} \mathbb{E}\left[\rho\left(Y^{\nu}_{t,x,y}(T)-g\left(X_{t,x}(T)\right)\right)\right].$$

For any $\eta>0$ and $y>\mathrm{y}(t,x,p+2\eta)$, we thus have $y\geq \mathrm{y}^n(t,x,p)$ for $n$ large enough, so that $\mathrm{y}_*\geq \mathrm{y}^\infty_*$. If the operator $H^*_n$ corresponding to the controls taking values in $U^n$ happens to have enough regularity, we shall recover from [Bar94, Lemma 4.2] that $\underline{\mathrm{y}}^\infty_*$ is a viscosity supersolution of $H^*\underline{\mathrm{y}}^\infty_*\geq 0$, where

$$\underline{\mathrm{y}}^\infty_*(t,x,p) := \liminf_{\substack{n\to\infty\\(t',x',p')\to(t,x,p)}} \mathrm{y}^n_*(t',x',p').$$

By construction, we have $\underline{\mathrm{y}}^\infty_*\leq \mathrm{y}^\infty_*$, and hence $\underline{\mathrm{y}}^\infty_*\leq \mathrm{y}$. Hence, if we provide an explicit lower bound for $\underline{\mathrm{y}}^\infty_*$ (see e.g. Bouchard, Elie and Touzi [BET09, Section 4]), we shall have the same lower bound for y.

### 2.3.4 On the Terminal Condition

The boundary condition at $T$ for $\hat{v}_*$ and $\hat{v}^*$ can be easily derived from the characterization of Theorem 2.2.9.

**Corollary 2.3.17.** *Under Assumption 2.3.4, the function $\hat{x}\in\hat{\boldsymbol{X}}\mapsto\hat{v}_*(T,\hat{x})$ is a viscosity supersolution of*

$$\min\left\{\left(\hat{v}_*(T,\cdot)-\hat{g}_*\right)\mathbf{1}_{\left\{\hat{H}^*\hat{v}_*(T,\cdot)<\infty\right\}},\hat{\delta}^*\hat{v}_*(T,\cdot)\right\}\geq 0 \text{ on } \hat{\boldsymbol{X}}.$$

*If in addition, Assumption 2.3.5 holds, then $\hat{x}\in\hat{\boldsymbol{X}}\mapsto\hat{v}^*(T,\hat{x})$ is a viscosity subsolution of*

$$\min\left\{\hat{v}^*(T,\cdot)-\hat{g}^*,\hat{\delta}_*\hat{v}^*(T,\cdot)\right\}\leq 0 \text{ on } \hat{\boldsymbol{X}}.$$

The condition $\hat{H}^*\hat{v}_*(T,\cdot)<\infty$ may not be satisfied because the control $(\alpha,\chi)$ appearing in the definition of $\hat{H}$ may not be bounded. It follows that the above boundary condition may be useless in most examples.

The rest of this section is devoted to the discussion of conditions under which a precise boundary condition can be specified.

**Proposition 2.3.18.** (i) *Assume that for all sequence $(t_n,x_n,y_n,p_n,\nu_n)_{n\geq1}$ of $[0,T)\times\boldsymbol{X}\times\mathbb{R}_+\times[m,M]\times\mathcal{U}$ such that $(t_n,x_n,y_n,p_n)\to(T,x,y,p)\in\{T\}\times\boldsymbol{X}\times\mathbb{R}_+\times[m,M]$, there exists a sequence of $\mathbb{P}$-absolutely continuous probability measure $(\mathbb{Q}^n)_{n\geq1}$, defined by $\frac{d\mathbb{Q}^n}{d\mathbb{P}}=:H^n$ for some sequence of non-negative* *random variable* $(H^n)_{n\geq 1}$, *such that*

$$\begin{aligned}\limsup_{n\to\infty} \mathbb{E}^{\mathbb{Q}^n}\left[Y^{\nu_n}_{t_n,x_n,y_n}\right] &\leq y,\\ \limsup_{n\to\infty} \mathbb{E}\left[\left|H^n D^+_p \circledcirc \hat{g}\left(X^{\nu_n}_{t_n,x_n}(T),p_n\right) - D^+_p \circledcirc \hat{g}(x_n,p_n)\right|\right] &= 0 \qquad (2.3.25)\\ \text{and} \quad \liminf_{n\to\infty} \mathbb{E}\left[H^n \circledcirc \hat{g}\left(X^{\nu_n}_{t_n,x_n}(T),p_n\right)\right] &\geq \circledcirc\hat{g}(x,p),\end{aligned}$$

*where* $D^+_p$ *stands for the right derivative in* $p$*. Then,* $\hat{v}_*(T,x,p) \geq \circledcirc\hat{g}(x,p)$ *for all* $(x,p) \in \boldsymbol{X} \times [0,1]$*.*

(ii) *Let the conditions* (ii) *of Theorem 2.3.14 hold true and assume that* $\hat{v}^*$ *is convex in its* $p$*-variable and that* $v^*(T,x) \leq g(x)$*. Then* $\hat{v}^*(T,x,p) \leq \circledcirc\hat{g}(x,p)$ *for all* $(x,p) \in \boldsymbol{X} \times [m,M]$*.*

**Proof.** (i) Given a sequence $(t_n,x_n,p_n)_{n\geq 1}$ in $[0,T) \times \mathbf{X} \times (m,M)$ such that $(t_n,x_n,p_n) \to (T,x,p)$ and $\hat{v}(t_n,x_n,p_n) \to \hat{v}_*(T,x,p)$ as $n \to \infty$, we can find $\hat{\nu}_n = (\nu_n,\alpha_n,\chi_n) \in \hat{\mathcal{U}}$ such that

$$\hat{V}\left(\hat{X}^{\hat{\nu}_n}_{t_n,x_n,p_n}(T), Y^{\hat{\nu}_n}_{t_n,x_n,y_n}(T)\right) \geq 0,$$

where $y_n := \hat{v}(t_n,x_n,p_n) + n^{-1} \to \hat{v}_*(T,x,p)$, recall (2.3.2). This implies that

$$Y^{\hat{\nu}_n}_{t_n,x_n,y_n}(T) \geq \hat{g}\left(\hat{X}^{\hat{\nu}_n}_{t_n,x_n,p_n}(T)\right),$$

and, by the definition of the convex hull of $\hat{g}$,

$$H^n Y^{\hat{\nu}_n}_{t_n,x_n,y_n}(T) \geq H^n \circledcirc \hat{g}\left(\hat{X}^{\hat{\nu}_n}_{t_n,x_n,p_n}(T)\right).$$

Using the convexity of $\circledcirc\hat{g}$ then leads to

$$\begin{aligned}& H^n Y^{\hat{\nu}_n}_{t_n,x_n,y_n}(T)\\ &\geq H^n \circledcirc \hat{g}\left(X^{\nu_n}_{t_n,x_n}(T),p_n\right) + H^n D^+_p \circledcirc \hat{g}\left(X^{\nu_n}_{t_n,x_n}(T),p_n\right)\left(P^{\alpha_n,\chi_n}_{t_n,p_n}(T) - p_n\right)\\ &= H^n \circledcirc \hat{g}\left(X^{\nu_n}_{t_n,x_n}(T),p_n\right) + D^+_p \circledcirc \hat{g}\left(x_n,p_n\right) P^{\alpha_n,\chi_n}_{t_n,p_n}(T)\\ &\quad - H^n p_n D^+_p \circledcirc \hat{g}\left(X^{\nu_n}_{t_n,x_n}(T),p_n\right)\\ &\quad + P^{\alpha_n,\chi_n}_{t_n,p_n}(T)\left[H^n D^+_p \circledcirc \hat{g}\left(X^{\nu_n}_{t_n,x_n}(T),p_n\right) - D^+_p \circledcirc \hat{g}\left(x_n,p_n\right)\right]\\ &\geq H^n \circledcirc \hat{g}\left(X^{\nu_n}_{t_n,x_n}(T),p_n\right) + D^+_p \circledcirc \hat{g}\left(x_n,p_n\right) P^{\alpha_n,\chi_n}_{t_n,p_n}(T)\\ &\quad - H^n p_n D^+_p \circledcirc \hat{g}\left(X^{\nu_n}_{t_n,x_n}(T),p_n\right)\\ &\quad - M\left|H^n D^+_p \circledcirc \hat{g}\left(X^{\nu_n}_{t_n,x_n}(T),p_n\right) - D^+_p \circledcirc \hat{g}\left(x_n,p_n\right)\right|,\end{aligned}$$

where the last inequality follows from the fact that we can always assume that $P^{\alpha_n,\chi_n}_{t_n,p_n}$ takes values in $[m,M]$, see (2.3.3). Taking the expectation under $\mathbb{P}$ and using the fact that $P^{\alpha_n,\chi_n}_{t_n,p_n}$ is a $\mathbb{P}$-martingale, we obtain

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}^n}\left[Y^{\hat{\nu}_n}_{t_n,x_n,y_n}(T)\right] \\
&\geq \mathbb{E}\left[H^n \circledcirc \hat{g}\left(X^{\nu_n}_{t_n,x_n}(T),p_n\right) + p_n\left(D^+_p \circledcirc \hat{g}\left(x_n,p_n\right) - H^n D^+_p \circledcirc \hat{g}\left(X^{\nu_n}_{t_n,x_n}(T),p_n\right)\right)\right. \\
&\qquad \left. -M\left|H^n D^+_p \circledcirc \hat{g}\left(X^{\nu_n}_{t_n,x_n}(T),p_n\right) - D^+_p \circledcirc \hat{g}\left(x_n,p_n\right)\right|\right].
\end{aligned}$$

Passing to the limit, and using (2.3.25) leads to $\hat{v}_*(T,x,p) \geq \circledcirc\hat{g}(x,p)$.

(ii) Using (2.3.21) and the convexity of $\hat{v}^*$ together with the definition of the convex hull of a function lead to the required result. □

**Example 2.3.5.** In the context of Example 2.3.1, we may easily notice that the generalized inverse of $\check{V}$ at 0,

$$\check{g}(x,p) := \inf\left\{y \geq -\kappa : \check{V}(x,p,y) \geq 0\right\},$$

satisfies

$$\check{g}(x,p) = pg(x)$$

and is convex in $p$. Moreover, for the dynamics of Example 2.2.1, the convexity of $\check{v}$ in its $p$-variable is quite obvious, since $Y^{\nu}_{t,x,\mu y}(T) = \mu Y^{\nu}_{t,x,y}(T)$ for any $\mu \in [0,1]$, and the expectation operator is linear.
We have already shown in Section 2.3.2 that $\check{v}_*$ is a supersolution of (2.3.13). we deduce that $\check{v}_*$ satisfies the boundary conditions

$$\begin{aligned}
\check{v}_*(\cdot,1) = v \text{ and } \check{v}_*(\cdot,0) = 0 \text{ on } [0,T) \times \mathbf{X} \\
\text{and } \check{v}_*(T,x,p) \geq pg(x) \text{ on } \mathbf{X} \times [0,1].
\end{aligned} \tag{2.3.26}$$

**Example 2.3.6.** In the context of Example 2.3.2, we define the function

$$\tilde{g}(x,p) := \inf\left\{y \geq -\kappa : \tilde{V}(x,p,y) \geq 0\right\}$$

and let $\tilde{\psi}$ be the generalized inverse of $\tilde{\Psi}$ at 0, i.e.

$$\tilde{\psi}(x) := \inf\left\{y \geq -\kappa : \tilde{\Psi}(x,y) \geq 0\right\}.$$

Then, $\tilde{g}(x,p) = \tilde{\psi}(x)\mathbf{1}_{\{p>0\}}$ for $x \in \mathbf{X}$ and $p \in [0,1]$. The convexity of $\tilde{v}$ is far from being obvious. However, one may notice that the convex hull of $\tilde{g}$ in $p$ is $\circledcirc\left(\hat{g}\right)(x,p) = pg(x)$, with $g = \tilde{\psi}^{-1}$, and that the condition of Corollary 2.3.7 and (i) of Proposition 2.3.18 are satisfied. It follows that, as for the expected success ratio problem of Example 2.3.5 above, $\tilde{v}_*$ is a viscosity supersolution on $[0,T] \times \mathbf{X} \times [0,1]$ of (2.3.13) - (2.3.26).

**Remark 2.3.19.** In Bouchard, Elie and Touzi [BET09], the authors considered the case $\hat{g}(x,p) = g(x)\mathbf{1}_{\{p>0\}}$, so that $\circledcirc\hat{g}(x,p) = pg(x)$, and therefore $D_p^+ \circledcirc \hat{g}(x,p) = g(x)$. Then, Assumption 2.3.25, in the case of Bouchard, Elie and Touzi [BET09], should take the form:

$$\limsup_{n\to\infty} \mathbb{E}\left[\left|H^n g\left(X^{\nu_n}_{t_n,x_n}(T)\right) - g(x)\right|\right] = 0.$$

The Assumption 2.3.25 is then almost the counterpart of the one made in their proposition 3.2. The difference comes from a slight error in their proof [2] where they use the fact that $P^{\alpha_n,\chi_n}_{t_n,p_n}$ is a $\mathbb{Q}$-martingale while it is only a $\mathbb{P}$-martingale, *a priori*.

### 2.3.5 Derivation of the boundary conditions for the stochastic target with controlled expected loss

We now prove Theorem 2.3.14. These boundary conditions need only to be specified in the case where $m$ and/or $M$ are finite.

#### 2.3.5.1 The endpoint $p = M$, finite

In order to show that $\hat{v}_*(\cdot, M)$ is a viscosity supersolution of (2.2.10)-(2.2.13), it suffices to show that $\hat{v}_*(\cdot, M)$ is a viscosity supersolution on $[0,T) \times \mathbf{X}$ of

$$\max\left\{\hat{v}_*(\cdot, M) - v_*, H^*\hat{v}_*(\cdot, M)\right\} \geq 0, \tag{2.3.27}$$

and that $\hat{v}_*(T, \cdot, M)$ is a viscosity supersolution on $\mathbf{X}$ of

$$\max\left\{\begin{array}{c} \hat{v}_*(T,\cdot,M) - v_*, \\ \min\left\{\left(\hat{v}_*(T,\cdot,M) - j_*\right)\mathbf{1}_{\{H^*\hat{v}_*(T,\cdot,M)<\infty\}}, \delta^*\hat{v}_*(T,\cdot,M)\right\} \end{array}\right\} \geq 0, \tag{2.3.28}$$

where $j$ is the generalized inverse of $\Phi$ at 0:

$$j(x) := \inf\left\{y \geq -\kappa : \Phi(x,y) \geq 0\right\},$$

recall (2.3.20).

To convince ourself, let us show for instance that (2.3.27) implies (2.2.10). Let $(t_o, x_o)$ be a local minimizer of $\hat{v}_*(\cdot, M) - \varphi$ for some smooth function $\varphi$ of linear growth. Then

- either $\hat{v}_*(t_o, x_o, M) < v_*(t_o, x_o)$ and then (2.2.10) holds for $\varphi$ at $(t_o, x_o)$

[2] The author would like to thank Bruno Bouchard, Romuald Elie and Nizar Touzi for pointing out this issue and for their explanations on how to fix it in their particular context.

- or $\hat{v}_*(t_o, x_o, M) = v_*(t_o, x_o)$ so that $(t_o, x_o)$ is a local minimizer of $v_* - \varphi$, and (2.2.10) holds for $\varphi$ at $(t_o, x_o)$ by the viscosity property of $v_*$, see Theorem 2.2.5.

step 1: We first show that for any smooth function $\varphi$ of linear growth on $[0,T] \times \mathbf{X} \times [m, M]$ and $(t_o, x_o) \in [0,T) \times \mathbf{X}$ such that

$$\text{(strict)} \min_{[0,T)\times\mathbf{X}\times[m,M]} (\hat{v}_* - \varphi) = (\hat{v}_* - \varphi)(t_o, x_o, M) = 0, \tag{2.3.29}$$

we have

$$\max\left\{\varphi(t_o, x_o, M) - v_*(t_o, x_o), \hat{H}^*\varphi(t_o, x_o, M)\right\} \geq 0.$$

If not, we can find $\eta, \varepsilon, \iota > 0$ such that

$$\begin{aligned} &\max\left\{\varphi_\iota - v_*(t,x), \mu_Y(x,y,u) - \hat{\mathcal{L}}^{\hat{u}}\varphi_\iota(t,x,p),\right\} \leq -2\eta \\ &\text{for all} \quad \hat{u} := (u,\alpha,\pi) \in \hat{\mathcal{N}}_{\varepsilon,-\eta}(t,x,y,D\varphi_\iota(t,x,p),\varphi_\iota) \\ &\text{and} \quad (t,x,p,y) \in [0,T) \times \mathbf{X} \times (m,M] \times \mathbb{R} \\ &\text{s.t.} \quad (t,x,p) \in B_\varepsilon(t_o,x_o) \times [M-\varepsilon, M] \quad \text{and} \quad |y - \varphi_\iota(t,x,p)| \leq \frac{\eta}{2}, \end{aligned} \tag{2.3.30}$$

with $\varphi_\iota(t,x,p) := \varphi(t,x,p) - f_\iota(x) - g_\iota(p)$, $f_\iota$ defined as in (2.2.18), and

$$g_\iota : p \in [m,M] \longmapsto \frac{2\iota}{\pi}\int_0^{\pi|p-M|} \sin^2 u du \mathbb{1}_{\{|p-M|\leq 1\}} + \iota \mathbb{1}_{\{|p-M|>1\}},$$

recall (2.2.19), and observe that the same results hold for $g_\iota$. We now define as previously for all $z \in \mathbf{X} \times [m,M] \times \mathbb{R}$

$$\bar{\varphi}_\iota(t,\hat{x},z) := \varphi_\iota(t,\hat{x}) - |z|^2.$$

By Assumption 2.3.4, there exists a finite constant $C > 0$ such that, after possibly changing $\varepsilon, \eta > 0$, we have

$$\mu_Y(x,y,u) - \hat{\mathcal{L}}^{\hat{u}}_{\hat{X},Z}\bar{\varphi}_\iota(t,\hat{x},z) \leq -\eta$$

$$\text{for all } (t,\hat{x},z,y) \in [0,T) \times (\mathbf{X} \times [m,M])^2 \times \mathbb{R}^2 \text{ s.t. } \begin{cases} (t,\hat{x},z) \in B_\varepsilon(t_o,\hat{x}_o,0) \\ |y - \bar{\varphi}_\iota(t,\hat{x},z)| \leq \dfrac{\eta}{4}, \end{cases}$$

$$\text{for all } u \in \mathcal{N}_{\varepsilon,-\eta}(t,\hat{x},y,D\varphi_\iota(t,x),\varphi_\iota)$$

and

$$\frac{\left[\mu_Y(x,y,u) - \hat{\mathcal{L}}^{\hat{u}}_{\hat{X},Z}\bar{\varphi}_\iota(t,\hat{x},z)\right]^+}{1 + |N^u(x,y,D\varphi_\iota|} \leq C\left(1 + |\sigma_Y(x,y,u)| + \sum_{i=1}^d |\hat{\sigma}^{i,\cdot}(\hat{x},u)|\right)$$

$$\text{for all } (t,\hat{x},z) \in B_\varepsilon(t_o,\hat{x}_o,0) \text{ and } y \in \mathbb{R} \text{ s.t. } |y - \bar{\varphi}_\iota(t,\hat{x},z)| \leq \frac{\eta}{4}$$

$$\text{and for all } u \in U \text{ s.t. } \Delta^{u,\cdot}(t,x,y,\varphi_\iota) \geq -\eta \ \lambda\text{-a.e.}$$

Let $(t_n, x_n, p_n)_n$ be a sequence in $[0,T) \times \mathbf{X} \times (m, M)$ which converges to $(t_o, x_o, M)$ and such that $\hat{v}(t_n, x_n, p_n) \to \hat{v}_*(t_o, x_o, M)$. Set $y_n := \hat{v}(t_n, x_n, p_n) + n^{-1}$ and observe that

$$\gamma_n := y_n - \varphi_\iota(t_n, x_n, p_n) \to 0.$$

For each $n \geq 1$, we have $y_n > \hat{v}(t_n, x_n, p_n)$. Then, by (GDPj1), there exists some $\hat{\nu}^n := (\nu^n, \alpha^n, \chi^n) \in \hat{\mathcal{U}}$ such that

$$Y^n(\theta_n) \geq \hat{v}_* \left(\theta_n, X^n(\theta_n), P^n(\theta_n)\right) \geq \bar{\varphi}_\iota \left(\theta_n, X^n(\theta_n), P^n(\theta_n), Z^n(\theta_n)\right)$$

where

$$\begin{aligned} \theta_n^o &:= \{s \geq t_n : (s, X^n(s), P^n(s), Z^n(s)) \in D\} \\ \theta_n &:= \left\{ s \geq t_n : |Y^n(s) - \varphi_\iota(s, X^n(s), P^n(s))| \geq \frac{\eta}{4} \right\} \wedge \theta_n^o \end{aligned}$$

together with

$$\begin{aligned} (X^n, P^n, Y^n, Z^n) &:= \left( X^{\nu^n}_{t_n, x_n}, P^{\alpha^n, \chi^n}_{t_n, p_n}, Y^{\nu^n}_{t_n, x_n, y_n}, Z^{\hat{\nu}^n}_{t_n, \hat{x}_n} \right), \\ Z^{\hat{\nu}^n}_{t_n, \hat{x}_n}(s) &= \int_{t_n}^s \begin{pmatrix} \hat{\mu}\left(\hat{X}^n(u), \hat{\nu}^n_u\right) \\ \mu_Y\left(\hat{X}^n(u), Y^n(u), \nu^n(u)\right) \end{pmatrix} du \end{aligned}$$

and

$$\begin{aligned} \mathcal{V}_\varepsilon(t_o, x_o, 0) &:= (\{t_o + \varepsilon\} \times B_\varepsilon(x_o, 0)) \cup ([t_o, t_o + \varepsilon) \times \partial B_\varepsilon(x_o, 0)) \\ D &:= (\mathcal{V}_\varepsilon(t_o, x_o, 0) \times [M - \varepsilon, M]) \cup (B_\varepsilon(t_o, x_o) \times [M - \varepsilon, M])^c \times B_\varepsilon(0). \end{aligned}$$

It follows from (2.3.30) and (2.3.29), recall (2.2.20), that

$$\zeta := \inf_D (\hat{v} - \bar{\varphi}_\iota) > 0.$$

Using the definition of $\theta_n$ and $\zeta > 0$, this implies that

$$Y^n(\theta_n) - \bar{\varphi}_\iota \left(\theta_n, X^n(\theta_n), P^n(\theta_n), Z^n(\theta_n)\right) \geq \zeta \wedge \frac{\eta}{4}.$$

By arguing as in Section 2.2.3.1, this leads to a contradiction.

step 2: We now show (2.3.27), i.e. for any smooth function $\varphi$ on $[0,T] \times \mathbf{X}$ and $(t_o, x_o) \in [0,T) \times \mathbf{X}$ such that

$$\text{(strict)} \min_{[0,T) \times \mathbf{X}} (\hat{v}_*(\cdot, M) - \varphi) = (\hat{v}_*(\cdot, M) - \varphi)(t_o, x_o) = 0,$$

we have

$$\max\left\{\varphi(t_o, x_o) - v_*(t_o, x_o), H^*\varphi(t_o, x_o)\right\} \geq 0. \tag{2.3.31}$$

**a.** The first step is similar as in Bouchard, Elie and Touzi [BET09], up to modifications due the need for linear growth test function in $x$. For every $k$, we introduce the smooth function

$$\varphi_k(t,x,p) := \varphi(t,x) - \left(f(x) + (t-t_o)^2 + \psi_k(p)\right),$$

where $f$ is defined as in (2.2.18) with $\iota = 1$, and for some $\rho > 0$,

$$\psi_k(p) := -\rho k \int_p^M \frac{e^{2kM}}{e^{k(r+M)} - e^{2kM+1}} dr, \qquad k > 0. \tag{2.3.32}$$

Observe that

$$\begin{aligned} \psi_k(p) \geq 0 \quad \text{for all} \quad k > 0,\ p \in [m,M], \\ -2\rho k \leq \psi_k'(p) = \rho k \frac{e^{2kM}}{e^{k(p+M)} - e^{2kM+1}} \leq -\frac{\rho k}{2(e-1)} \\ \text{for } k \text{ large enough,} \end{aligned} \tag{2.3.33}$$

$$\psi_k''(p) = -\rho k^2 \frac{e^{k(p+3M)}}{\left(e^{k(p+M)} - e^{2kM+1}\right)^2} < 0 \quad \text{for all } k > 0, \tag{2.3.34}$$

$$\begin{aligned} \lim_{k\to\infty} \frac{(\psi_k'(p_k))^2}{|\psi_k''(p_k)|} = \rho \quad \text{if } (p_k)_k \text{ is a sequence in } [m,M] \\ \text{s.t.} \quad \lim_{k\to\infty} k(M-p_k) = 0. \end{aligned} \tag{2.3.35}$$

Let $(t_k, x_k, p_k)$ be a minimizer of $\hat{v}_* - \varphi_k$ on $[0,T] \times \overline{B_1^{\mathbf{X}}(x_o)} \times [m,M]$, where $B_1^{\mathbf{X}}(x_o) := B_1(x_o) \cap \mathbf{X}$ and $B_1(x_o)$ is the open unit ball centered at $x_o$. Observe that, by definition of $(t_k, x_k, p_k)$ and $(t_o, x_o)$,

$$\begin{aligned} &(\hat{v}_*(\cdot,M) - \varphi)(t_o,x_o) \\ &= (\hat{v}_* - \varphi_k)(t_o,x_o,M) \\ &\geq (\hat{v}_* - \varphi_k)(t_k,x_k,p_k) \\ &= (\hat{v}_*(\cdot,p_k) - \varphi)(t_k,x_k) + \left(f(x_k) + (t_k-t_o)^2 + \psi_k(p_k)\right) \\ &\geq (\hat{v}_*(\cdot,p_k) - \varphi)(t_k,x_k) + \left(f(x_k) + (t_k-t_o)^2 + \frac{\rho k}{2(e-1)}(M-p_k)\right), \end{aligned}$$

where the last inequality follows from (2.3.33), for $k$ large enough, and the fact that $\psi_k(M) = 0$. Since $\hat{v}_* \geq -\kappa$ by construction and $\varphi$ is bounded, this implies that the sequence $(t_k, x_k, p_k)_{k\geq 1}$ is bounded, and therefore converges to some $(t_*, x_*, p_*)$ up to a subsequence. Clearly, $p_* = M$, since otherwise we would have $k(M-p_k) \to \infty$.

By definition of $(t_o, x_o)$, this implies that

$$\begin{aligned}
&\left(\hat{v}_*(\cdot, M) - \varphi\right)(t_o, x_o) \\
&\geq \liminf_{k\to\infty} \left(\hat{v}_* - \varphi_k\right)(t_k, x_k, p_k) \\
&\geq \left(\hat{v}_*(\cdot, M) - \varphi\right)(t_*, x_*) + \left(f(x_*) + (t_* - t_o)^2 + \liminf_{k\to\infty} \frac{\rho k}{2(e-1)}(M - p_k)\right) \\
&\geq \left(\hat{v}_*(\cdot, M) - \varphi\right)(t_o, x_o) + \left(f(x_*) + (t_* - t_o)^2 + \liminf_{k\to\infty} \frac{\rho k}{2(e-1)}(M - p_k)\right).
\end{aligned}$$

This shows that, after possibly passing to a subsequence,

$$\begin{aligned}
(t_k, x_k, p_k) \to (t_o, x_o, M), \quad k(M - p_k) \to 0, \\
\text{and} \quad \hat{v}_*(t_k, x_k, p_k) \to \hat{v}_*(t_o, x_o, M).
\end{aligned} \tag{2.3.36}$$

**b.** We now go on with the arguments of Bouchard, Elie and Touzi [BET09], up to a non trivial adaptation required by the non-local parts of the operator. In order to prove (2.3.27), we assume

$$\hat{v}_*\left(t_o, x_o, M\right) - v_*\left(t_o, x_o\right) < 0 \tag{2.3.37}$$

and we intend to prove that

$$H^*\varphi\left(t_o, x_o\right) \geq 0. \tag{2.3.38}$$

By (2.3.36) and the lower semicontinuity of $\hat{v}_*$, it follows from (2.3.37) that the sequence $(t_k, x_k, p_k)_{k\geq 1}$ of minimizers of the difference $\hat{v}_* - \varphi_k$ satisfies $\varphi_k\left(t_k, x_k, p_k\right) - v_*\left(t_k, x_k\right) < 0$, after possibly passing to a subsequence. By Corollary 2.3.7 together with the result of step 1, Remark 2.2.10, Assumptions 2.3.12 and 2.3.4, and the fact $\varphi_k$ is of linear growth in $x$ and $p$, we deduce that

$$\hat{H}^*\left(t_k, x_k, p_k, \varphi_k, \partial_t\varphi_k, D\varphi_k, D^2\varphi_k, \hat{v}_*\right) \geq 0 \quad \text{for every} \quad k > 1.$$

Now observe that, by (2.3.36), and the definition of $\varphi_k$:

$$\begin{aligned}
&\left(\partial_t\varphi_k, D_x\varphi_k, D^2_{xx}\varphi_k\right)(t_k, x_k, p_k) \underset{k\to\infty}{\longrightarrow} \left(\partial_t\varphi, D_x\varphi, D^2_{xx}\varphi\right)(t_o, x_o) \\
&\left(D_p\varphi_k, D^2_{xp}\varphi_k, D^2_{pp}\varphi_k\right)(t_k, x_k, p_k) = \left(-\psi_k'\left(p_k\right), 0, -\psi_k''\left(p_k\right)\right) \quad \forall\ k > 1.
\end{aligned} \tag{2.3.39}$$

By definition of $\hat{H}^*$, we can find sequences $(\varepsilon_k)_{k\geq 1}$, $\left(\hat{x}^0_k\right)_{k\geq 1}$, $(y_k)_{k\geq 1}$, $(q_k)_{k\geq 1}$, $(A_k)_{k\geq 1}$ such that $\varepsilon_k > 0, \hat{x}^0_k = \left(x^0_k, p^0_k\right) \in \mathbf{X} \times [m, M], y_k \geq -\kappa, q_k = \left(q^x_k, q^p_k\right) \in \mathbb{R}^d \times \mathbb{R}$, $A_k$ is a symmetric matrix of $\mathbb{S}^{d+1}$, with rows $\left(A^{xx}_k, A^{xp}_k\right) \in \mathbb{S}^d \times \mathbb{R}^d$ and $\left(A^{xpT}_k, A^{pp}_k\right) \in \mathbb{R}^d \times \mathbb{R}$,

$$\begin{aligned}
\varepsilon_k \to 0, \quad \hat{x}^0_k \to (x_o, M) \\
\text{and} \left|(y_k, q_k, A_k) - \left(\varphi_k, D\varphi_k, D^2\varphi_k\right)(t_k, x_k, p_k)\right| \leq k^{-1},
\end{aligned} \tag{2.3.40}$$

where $\left(t_k, \hat{x}_k^0\right)$ belongs to a compact neighborhood of $(t_o, x_o, M)$, and

$$\hat{H}_{\varepsilon_k,-k^{-1}}\left(t_k, \hat{x}_k^0, y_k, \partial_t\varphi(t_o, x_o), q_k, A_k, \hat{v}_*\right) \geq -k^{-1}. \tag{2.3.41}$$

By the definition of $\hat{H}_{\varepsilon_k,-k^{-1}}$, we may find a sequence

$$(u_k, \alpha_k, \pi_k) \in \hat{\mathcal{N}}_{\varepsilon_k,-2k^{-1}}\left(t_k, \hat{x}_k^0, y_k, q_k, \widehat{v}_*\right)$$

such that

$$\begin{aligned}-\partial_t\varphi\left(t_o, x_o\right) + \mu_Y\left(x_k^0, y_k, u_k\right) - \mu_X\left(x_k^0, u_k\right)\cdot q_k^x - \frac{1}{2}\mathrm{Tr}\left[\sigma_X\sigma_X^\top\left(x_k^0, u_k\right)A_k^{xx}\right]\\ \geq -2k^{-1} + \frac{1}{2}\left|\alpha_k\right|^2 A_k^{pp} + \sigma_X^\top\left(x_k^0, u_k\right)A_k^{xp}\cdot\alpha_k - \int_E \pi_k(e)\lambda(de)q_k^p\end{aligned} \tag{2.3.42}$$

and

$$\beta_Y\left(x_k^0, y_k, u_k(e), e\right) - \hat{v}_*\left(t_k, x_k^0 + \beta_X\left(x_k^0, u_k(e), e\right), p_k^0 + \pi_k(e)\right) + y_k \geq -2k^{-1}$$
$$\text{for } \lambda\text{-a.e. } e \in E. \tag{2.3.43}$$

Recalling (2.3.23), we observe that $(u_k, \alpha_k, \pi_k) \in \hat{\mathcal{N}}_{\varepsilon_k,-2k^{-1}}(t_k, \hat{x}_k, y_k, q_k, \widehat{v}_*)$ implies that $u_k \in \tilde{\mathcal{N}}_{\varepsilon_k+|q_k^p\alpha_k|}\left(x_k^0, y_k, q_k^x\right)$. We deduce then from Assumption 2.3.11 and (2.3.42) that, for some constant $C > 0$, (which may change from line to line but does not depend on $k$ or $\rho$),

$$\begin{aligned}C\left(1 + \left|q_k^p\alpha_k\right|^2\right) &\geq \frac{1}{2}\left|\alpha_k\right|^2 A_k^{pp} + \sigma_X^\top\left(x_k^0, u_k\right)A_k^{xp}\cdot\alpha_k - \int_E \pi_k(e)\lambda(de)q_k^p\\ &\geq \frac{1}{2}\left|\alpha_k\right|^2 A_k^{pp} - C\left|A_k^{xp}\right|\left|\alpha_k\right| - \int_E \pi_k(e)\lambda(de)q_k^p\end{aligned} \tag{2.3.44}$$

where we have used the condition that $\sup_{u\in U}\left|\sigma_X(\cdot, u)\right|$ is locally bounded. From (2.3.33), (2.3.34), (2.3.35), (2.3.36), (2.3.39) and (2.3.40), it follows that

$$A_k^{pp} \to +\infty, \quad A_k^{xp} \to 0, \quad q_k^p \to +\infty \quad \text{and} \quad \frac{\left(q_k^p\right)^2}{\left|A_k^{pp}\right|} \to \rho \text{ as } k \to \infty. \tag{2.3.45}$$

Recall from (2.3.5) that

$$\pi_k \leq M - p_k \quad \lambda\text{-a.e.}, \tag{2.3.46}$$

where $p_k \in [m, M]$. We may hence consider that $(\pi_k)_{k\geq 1}$ is bounded from above, so that, by (2.3.44) and the fact that $q_k^p, A_k^{pp} > 0$

$$C\left(\frac{1}{A_k^{pp}} + \frac{|q_k^p|^2}{A_k^{pp}}\left|\alpha_k\right|^2\right) \geq \frac{1}{2}\left|\alpha_k\right|^2 - C\frac{\left|A_k^{xp}\right|^2}{A_k^{pp}}\left|\alpha_k\right| - C\frac{q_k^p}{A_k^{pp}}.$$

Hence, (2.3.45) leads to

$$0 \geq \limsup_{k\to\infty} \left( \left( \frac{1}{2} - C\rho \right) |\alpha_k|^2 - C \frac{|A_k^{xp}|^2}{A_k^{pp}} |\alpha_k| \right).$$

Taking $\rho$ small enough implies that

$$|\alpha_k| \underset{k\to\infty}{\to} 0. \tag{2.3.47}$$

Moreover, since $k(M-p_k) \to 0$, see (2.3.36), there exists $\epsilon_k \downarrow 0$ such that $k(M-p_k) \leq \epsilon_k$. Recalling (2.3.46), this implies that $\pi_k \leq \frac{\epsilon_k}{k}$, so that, by (2.3.33),

$$q_k^p \left(\pi_k(e)\right)^+ \to 0 \quad \text{as} \quad k \to \infty \quad \text{for all } e \in E. \tag{2.3.48}$$

Recalling the fact that $\lambda(E) < \infty$ and that $q_k^p > 0$, the above inequalities lead to

$$\left( \int_E \pi_k(e)\lambda(de) q_k^p \right)^+ \to 0. \tag{2.3.49}$$

Also recall that $\frac{|q_k^p|^2}{A_k^{pp}} \to \rho$, see (2.3.45), which combined with (2.3.44), (2.3.45), (2.3.47) and (2.3.49), implies that

$$C\left(1 + \rho A_k^{pp} |\alpha_k|^2\right) \geq \frac{1}{2} |\alpha_k|^2 A_k^{pp} + \left( \int_E \pi_k(e)\lambda(de) q_k^p \right)^-$$

or equivalently

$$C\left(1 + |q_k^p|^2 |\alpha_k|^2\right) \geq \frac{1}{2} |\alpha_k|^2 \frac{|q_k^p|^2}{\rho} + \left( \int_E \pi_k(e)\lambda(de) q_k^p \right)^-$$

for some $\rho > 0$. Taking $\rho$ small enough leads to

$$\begin{aligned} |A_k^{pp}| |\alpha_k|^2 \leq C, \quad |q_k^p|^2 |\alpha_k|^2 \leq C\rho \\ \text{and} \quad C + C\rho \geq \left( \int_E \pi_k(e)\lambda(de) q_k^p \right)^- . \end{aligned} \tag{2.3.50}$$

We then deduce from the right hand side bound of (2.3.33) and (2.3.40) that

$$0 \geq \limsup_{k\to+\infty} \left( \int_E \pi_k(e)\lambda(de) \right)^- .$$

Combined with (2.3.48), this shows that

$$\int_E \pi_k(e)\lambda(de) \to 0 \quad \text{and} \quad \pi_k(e) \to 0 \quad \text{for } \lambda\text{-a.e. } e \in E. \tag{2.3.51}$$

**c.** We now return to (2.3.42) and the middle inequality in (2.3.50) to deduce that

$$-\partial_t\varphi\left(t_o,x_o\right)+\mu_Y\left(x_k^0,y_k,u_k\right)-\mu_X\left(x_k^0,u_k\right)\cdot q_k^x-\frac{1}{2}\mathrm{Tr}\left[\sigma_X\sigma_X^\top\left(x_k^0,u_k\right)A_k^{xx}\right]$$
$$\geq -2k^{-1}+\sigma_X^\top\left(x_k^0,u_k\right)A_k^{xp}\cdot\alpha_k-\left(\int_E\pi_k(e)\lambda(de)q_k^p\right)^+, \tag{2.3.52}$$

and

$$u_k\in\tilde{\mathcal{N}}_{\varepsilon_k+\sqrt{C\rho}}\left(x_k^0,y_k,q_k^x\right). \tag{2.3.53}$$

since $A_k^{pp}>0$.

Consider now (2.3.43), i.e.

$$\begin{aligned}\beta_Y\left(x_k^0,y_k,u_k(e),e\right)-\hat{v}_*\left(t_k,x_k^0+\beta_X\left(x_k^0,u_k(e),e\right),p_k^0+\pi_k(e)\right)+y_k&\\ \geq -2k^{-1}\quad\text{for }\lambda\text{-a.e. }e\in E,&\end{aligned}\tag{2.3.54}$$

Using the upper semi-continuity of $-\hat{v}_*$, the fact that $\beta_Y$ is continuous, (2.3.51), together with $p_k^0\to M$ as $k\to\infty$, we obtain

$$\beta_Y\left(x_k^0,y_k,u_k(e),e\right)-\hat{v}_*\left(t_k,x_k^0+\beta_X\left(x_k^0,u_k(e),e\right),M\right)+y_k\geq -2k^{-1}-\vartheta_k^e$$
$$\text{for }k\text{ large enough and for }\lambda\text{-a.e. }e\in E,$$

with $\vartheta_k^e\geq 0$ such that $\vartheta_k^e\to 0$ as $k\to\infty$ for all $e\in E$. We now use Assumption 2.3.9 to deduce that there exists $\vartheta_k>0$ with $\vartheta_k\to 0$ as $k\to\infty$ such that, for all $e\in E$ and $k$ large enough,

$$\beta_Y\left(x_k^0,y_k,u_k(e),e\right)-\hat{v}_*\left(t_k,x_k^0+\beta_X\left(x_k^0,u_k(e),e\right),M\right)+y_k\geq -2k^{-1}-\vartheta_k. \tag{2.3.55}$$

By combining (2.3.52) (2.3.53) and (2.3.55), we finally obtain

$$H_{\varepsilon_k+\sqrt{C\rho},-2k^{-1}-\vartheta_k}\left(t_k,x_k^0,y_k,\partial_t\varphi(t_o,x_o),q_k^x,A_k^{xx},\hat{v}_*(\cdot,M)\right)$$
$$\geq -2k^{-1}-\left(\sigma_X^\top\left(x_k^0,u_k\right)A_k^{xp}\cdot\alpha_k\right)^--\left(\int_E\pi_k(e)\lambda(de)q_k^p\right)^+,$$

and we deduce the required result (2.3.38) by sending $k\to\infty$ and then $\rho\to 0$, and recalling that $\left(|\alpha_k|,A_k^{xp},\left(\int_E\pi_k(e)\lambda(de)q_k^p\right)^+\right)\to 0$, that $\sigma_X$ is locally bounded uniformly in the control $u$, and that $\hat{v}_*\geq\varphi$.

step 3: It remains to prove (2.3.28). The fact that $\hat{v}_*(T,\cdot,M)$ is a viscosity supersolution

$$\max\left\{\hat{v}_*(T,\cdot,M)-v_*(T,\cdot),\delta^*\hat{v}_*(T,\cdot,M)\right\}\geq 0$$

is deduced from (2.3.31) of the previous step by using the same arguments as in the proof of (2.2.2) in Section 2.2.3.2. It remains to show that $\hat{v}_*(T,\cdot,M)$ is a viscosity supersolution of

$$\max\left\{\hat{v}_*(T,\cdot,M) - v_*(T,\cdot), (\hat{v}_*(T,\cdot,M) - j_*)\,\mathbf{1}_{\{H^*\hat{v}_*(T,\cdot,M)<\infty\}}\right\} \geq 0.$$

By combining the arguments of step 1 with those of Section 2.2.3.2, we first show that for any smooth function $\hat{\varphi}$ on $\mathbf{X}\times[m,M]$ and $x_o \in \mathbf{X}$ such that

$$(\text{strict}) \min_{\mathbf{X}\times[m,M]} (\hat{v}_*(T,\cdot) - \hat{\varphi}) = (\hat{v}_*(T,\cdot) - \hat{\varphi})(x_o, M) = 0,$$

we have

$$\max\left\{\hat{\varphi}(x_o,M) - v_*(T,x_o), (\hat{\varphi}(x_o,M) - \hat{g}_*(x_o))\,\mathbf{1}_{\left\{\hat{H}^*\hat{\varphi}(x_o,M)<\infty\right\}}\right\} \geq 0. \quad (2.3.56)$$

We then consider a smooth function $\varphi$ on $\mathbf{X}$ and $x_o \in \mathbf{X}$ such that

$$(\text{strict}) \min_{\mathbf{X}} (\hat{v}_*(T,\cdot,M) - \varphi) = (\hat{v}_*(T,\cdot,M) - \varphi)(x_o) = 0 \quad (2.3.57)$$

and

$$\varphi(x_o) < \hat{v}(T,x_o), \quad (2.3.58)$$

and we assume that

$$H^*\varphi(T,x_o) < \infty.$$

We next follow the construction of step 2 of the modified test functions

$$\varphi_k := \varphi(x) - (f(x) + \psi_k(p)), \quad (2.3.59)$$

where $\psi_k$ is defined in (2.3.32). As in the above step 2, one can prove that the difference $\hat{v}_*(T,\cdot) - \varphi_k$ has a local minimizer $\hat{x}_k = (x_k, p_k)$ satisfying all estimates derived in the above step 2 (forgetting about the $t$ variable). In particular, since $H^*\varphi_k(x_k) \leq C$ for some constant $C > 0$ independent of $k$, recall (2.3.58), we deduce from the same estimates than in step 2 that $\hat{H}^*\varphi_k(\hat{x}_k) \leq 2C$ for all large $k$. It then follows from Corollary 2.3.17, (2.3.56) and (2.3.58) that $\hat{v}_*(T,\hat{x}_k) \geq \hat{g}_*(\hat{x}_k)$. Sending $k \to \infty$, this provides $\hat{v}_*(T,x_o,M) \geq \hat{g}_*(x_o,M)$, and the proof is completed by observing that $\hat{g}_*(x_o,M) = j_*(x_o)$, by definition of $j$. □

#### 2.3.5.2 The endpoint $p = m$, finite

We organize the proof in four steps. As in the previous section, steps 1, 2 and 3 focus on $t < T$ while step 4 concentrates on $t = T$. Steps 1 and 4 are similar to arguments used in Bouchard, Elie and Touzi [BET09]. The main difference comes from steps 2 and 3.

step 1: We first show that for any smooth function $\hat{\varphi}$ on $[0,T)\times\mathbf{X}\times[m,M]$ and $(t_1,x_1)\in[0,T)\times\mathbf{X}$ such that

$$\text{(strict)} \max_{[0,T)\times\mathbf{X}\times[m,M]} (\hat{v}^*-\hat{\varphi}) = (\hat{v}^*-\hat{\varphi})(t_1,x_1,m) = 0, \tag{2.3.60}$$

we have

$$\min\left\{\hat{v}^*+\kappa,\hat{H}_*\hat{\varphi}\right\}(t_1,x_1,m)\le 0. \tag{2.3.61}$$

The proof is very similar to that of Sections (2.2.3.3) up to the modification explained in the proof of Corollary 2.3.17, and the fact that we have to handle the state constraint $p=m$. For completeness, we report here the entire argument. Assume to the contrary that

$$4\eta := \min\left\{\hat{v}^*+\kappa,\hat{H}_*\hat{\varphi}\right\}(t_1,x_1,m)>0$$

i.e., for some $\varepsilon>0$, and after possibly changing $\eta>0$,

$$\begin{aligned}
&\min\left\{\hat{\varphi}_\iota(t,\hat{x})+\kappa,\mu_Y(x,y,\widehat{u})-\hat{\mathcal{L}}^{\hat{u}}\hat{\varphi}_\iota(t,\hat{x})\right\}\ge 2\eta\\
&\text{for some}\quad \hat{u}=(u,\alpha,\pi)\in\hat{\mathcal{N}}_{0,\eta}(t,\hat{x},y,D\hat{\varphi}_\iota(t,\hat{x}),\widehat{\varphi}_\iota)\\
&\text{for all}\quad (t,\hat{x},y)\in[0,T)\times\hat{\mathbf{X}}\times\mathbb{R}\\
&\text{s.t.}\quad (t,\hat{x})\in B_\varepsilon(t_1,x_1)\times[m,m+\varepsilon], |y-\hat{\varphi}_\iota(t,\hat{x})|\le\varepsilon,
\end{aligned} \tag{2.3.62}$$

where $\hat{\varphi}_\iota(t,\hat{x}) := \hat{\varphi}(t,\hat{x})+f_\iota(x)+g_\iota(p)$ with $\iota$ small enough, for $f_\iota$ and $g_\iota$ defined as in (2.2.18) with $x_1$ and $m$ respectively in place of $x_o$. Then, Assumptions 2.3.5 and 2.3.9 imply that

$$\begin{aligned}
&\min\left\{\begin{array}{c}\hat{\varphi}_\iota(t,\hat{x})+\kappa,\\ \mu_Y(x,y,\hat{\nu}(t,\hat{x},y,D\hat{\varphi}_\iota(t,\hat{x})))-\hat{\mathcal{L}}^{\hat{\nu}(t,\hat{x},y,D\hat{\varphi}_\iota(t,\hat{x}))}\hat{\varphi}_\iota(t,\hat{x}),\\ \min\limits_{e\in E}\hat{\mathcal{G}}^{\hat{\nu}(t,\hat{x},y,D\hat{\varphi}_\iota(t,\hat{x})),e}\hat{\varphi}_\iota(t,\hat{x})\end{array}\right\}\ge\eta\\
&\text{for}\ (t,\hat{x},y)\in[0,T]\times\hat{\mathbf{X}}\times\mathbb{R}\ \text{s.t.}\\
&(t,\hat{x})\in B_\varepsilon(t_1,x_1)\times[m,m+\varepsilon]\quad\text{and}\quad|y-\hat{\varphi}_\iota(t,\hat{x})|\le\frac{\eta}{4},
\end{aligned} \tag{2.3.63}$$

where $\hat{\nu}$ is a locally Lipschitz map satisfying

$$\begin{aligned}
&\hat{\nu}(t,\hat{x},y,D\hat{\varphi}_\iota(t,\hat{x}))\in\hat{\mathcal{N}}_{0,\eta}(t,\hat{x},y,D\hat{\varphi}_\iota(t,\hat{x}),\widehat{\varphi}_\iota)\\
&\text{on}\quad B_\varepsilon(t_1,x_1)\times[m,m+\varepsilon].
\end{aligned} \tag{2.3.64}$$

Observe that, since $(t_1,x_1,m)$ is a strict maximizer in (2.3.60) and by (2.2.20), we have

$$-\xi := -(\zeta\wedge\gamma_{\varepsilon,\iota}) := \max_D(\hat{v}_*-\hat{\varphi}_\iota)<0, \tag{2.3.65}$$

where

$$D := \left(\{t_1+\varepsilon\}\times \overline{B}_\varepsilon(x_1)\times[m,m+\varepsilon]\right)\cup\left([t_1,t_1+\varepsilon)\times\left(B_\varepsilon(x_1)\times[m,m+\varepsilon)\right)^c\right).$$

Also, we deduce from (2.3.62) and the fact that $\hat{v}(\cdot,m)=-\kappa$ by definition, that

$$0>-\eta\geq \max_{B_\varepsilon(t_1,x_1)}\left(\hat{v}-\hat{\varphi}\right)(\cdot,m). \tag{2.3.66}$$

By following the arguments in step 2 of Section 2.2.3.3, we see that (2.3.63), (2.3.64), (2.3.65) and (2.3.66) lead to a contradiction of (GDPj2).

step 2: Let $\varphi$ be a smooth function on $[0,T]\times\mathbf{X}$ and $(t_o,x_o)\in[0,T)\times\mathbf{X}$ such that

$$(\text{strict})\max_{[0,T)\times\mathbf{X}}\left(\hat{v}^*(\cdot,m)-\varphi\right)=\left(\hat{v}^*(\cdot,m)-\varphi\right)(t_o,x_o)=0.$$

By definition, we have $\hat{v}^*(t_o,x_o,m)\geq-\kappa$. Let us assume that

$$\hat{v}^*(t_o,x_o,m)+\kappa=:4\eta>0, \tag{2.3.67}$$

and work towards a contradiction. Define the function $\psi_k$ as in (2.3.32) with $m$ in place $M$:

$$\psi_k(p):=\rho k\int_m^p\frac{e^{2km}}{e^{k(r+m)}-e^{2km+1}}dr,\qquad k>0,$$

and for $f$ defined as in (2.2.18) for $\iota=1$,

$$\varphi_k(t,x,p):=\varphi(t,x)+\Big(f(x)+(t-t_o)^2+\psi_k(p)\Big).$$

Arguing as in step 2 of the preceding section, we see that the difference $\hat{v}^*-\varphi_k$ has a local maximizer $(t_k,x_k,p_k)$ on $\overline{([0,T]\times\mathbf{X}\times[m,M])}$ satisfying

$$(t_k,x_k,p_k)\to(t_o,x_o,m),\quad k(p_k-m)\to 0\quad\text{and}\quad\hat{v}^*(t_k,x_k,p_k)\to\hat{v}^*(t_o,x_o,m),$$

so that

$$\begin{aligned}\left(\partial_t\varphi_k,D_x\varphi_k,D^2_{xx}\varphi_k\right)(t_k,x_k,p_k)&\to\left(\partial_t\varphi,D_x\varphi,D^2_{xx}\varphi\right)(t_o,x_o)\quad\text{as}\quad k\to\infty\\ \left(D_p\varphi_k,D^2_{xp}\varphi_k,D^2_{pp}\varphi_k\right)(t_k,x_k,p_k)&=\left(\psi_k'(p_k),0,\psi_k''(p_k)\right).\end{aligned}$$

Since $\hat{v}^*(t_o,x_o,m)>-\kappa$, we have $\hat{v}^*(t_k,x_k,p_k)>-\kappa$ for all $k$, after possibly passing to a subsequence. Then, it follows from Corollary 2.3.7, step 1 and the arguments of Remark 2.2.10 under Assumption 2.3.12, that

$$\hat{H}_*(\cdot,\varphi_k,\partial_t\varphi_k,D\varphi_k,D^2\varphi_k,\hat{v}^*)(t_k,x_k,p_k)\leq 0\text{ for }k>1.$$

By the definition of $\hat{H}_*$, we deduce that there exist sequences $(\varepsilon_k)_{k\geq1}$, $(\hat{x}_k)_{k\geq1}$, $(y_k)_{k\geq1}$, $(q_k)_{k\geq1}$ and $(A_k)_{k\geq1}$ such that $\varepsilon_k>0$, $\hat{x}^0_k=\left(x^0_k,p^0_k\right)\in\mathbf{X}\times[m,M], y_k\geq$

$-\kappa, q_k = \left(q_k^x, q_k^p\right) \in \mathbb{R}^d \times \mathbb{R}$, and $A_k \in \mathbb{S}^{d+1}$ with rows $\left(A_k^{xx}, A_k^{xp}\right) \in \mathbb{S}^d \times \mathbb{R}^d$ and $\left(A_k^{xpT}, A_k^{pp}\right) \in \mathbb{R}^d \times \mathbb{R}$ satisfying

$$\begin{aligned} &\varepsilon_k \to 0, \quad \hat{x}_k^0 \to (x_o, m), \\ \text{and} \quad &\left|(y_k, q_k, A_k) - \left(\varphi_k, D\varphi_k, D^2\varphi_k\right)(t_k, x_k, p_k)\right| \le k^{-1} \end{aligned} \tag{2.3.68}$$

for which

$$\hat{H}_{\varepsilon_k, k^{-1}}\left(t_k, \hat{x}_k, y_k, \partial_t\varphi(t_o, x_o), q_k, A_k, \hat{v}^*\right) \le k^{-1}. \tag{2.3.69}$$

Fix $u \in U$, $\pi = 0$ and set $\alpha_k := N^u(x_k^0, y_k, q_k^x)/q_k^p$. Since $\pi = 0$, it follows from (2.3.69) together with (2.3.5), (2.3.6) and Assumption 2.3.9 that either $(u, \alpha_k, \pi) \in \hat{\mathcal{N}}_{\varepsilon_k, k^{-1}}\left(t, \widehat{x}_k, y_k, q_k, \widehat{v}^*\right)$ and then

$$\begin{aligned} &\mu_Y\left(x_k^0, y_k, u\right) - \partial_t\varphi(t_o, x_o) - \mu_X\left(x_k^0, u\right) \cdot q_k^x \\ &\quad - \frac{1}{2}\left(\mathrm{Tr}\left[\sigma_X\sigma_X^\top(x_k^0, u)A_k^{xx}\right] + |\alpha|^2 A_k^{pp} + 2\sigma_X^\top(x_k^0, u)A_k^{xp} \cdot \alpha\right) \le k^{-1} \end{aligned} \tag{2.3.70}$$

or

$$\beta_Y(x_k^0, y_k, u(e_k), e_k) - \hat{v}^*\left(t_k, x_k^0 + \beta_X\left(x_k^0, u(e_k), e_k\right), p_k^0\right) + y_k \le k^{-1}, \tag{2.3.71}$$

for some sequence $(e_k)_{k\ge1} \subseteq E$. Using the same kind of arguments as in step 2 of the previous section leads to

$$A_k^{pp} < 0, \quad q_k^p < 0 \text{ for large } k, \quad \lim_{k\to\infty} A_k^{xp} = 0 \quad \text{and} \quad \lim_{k\to\infty} \frac{\left(q_k^p\right)^2}{\left|A_k^{pp}\right|} = \rho. \tag{2.3.72}$$

Consider first the case where (2.3.70) holds along a subsequence. Using (2.3.70) and (2.3.72), we then deduce that

$$\left|A_k^{pp}\right| |\alpha_k|^2 = \frac{\left|A_k^{pp}\right|}{\left(q_k^p\right)^2} \left|N^u\left(x_k^0, y_k, q_k^x\right)\right|^2 \le C,$$

for some $C > 0$ independent of $k$ and $\rho$. Sending $k \to \infty$ in the above inequality, we then deduce from (2.3.68) and (2.3.72) that

$$\rho^{-1}\left|N^u\left(x_o, \varphi(t_o, x_o), D\varphi(t_o, x_o)\right)\right|^2 \le C.$$

Since $\rho > 0$ can be chosen arbitrarily close to 0, this shows that $N^u\left(x_o, \varphi(t_o, x_o), D\varphi(t_o, x_o)\right) = 0$, and the arbitrariness of $u \in U$ is in contradiction with Assumption 2.3.10. This contradicts (2.3.67). Hence, if (2.3.67) holds, then (2.3.71) holds along a subsequence, i.e.

$$\beta_Y(x_k^0, y_k, u(e_k), e_k) - \hat{v}^*\left(t_k, x_k^0 + \beta_X\left(x_k^0, u(e_k), e_k\right), p_k^0\right) + y_k \le k^{-1}.$$

Sending $k \to \infty$, using the arbitrariness of $u \in U$ and Assumption 2.3.9 then leads to

$$\check{G}\hat{v}^*(t_o, x_o, m) \leq 0,$$

where

$$\check{G}\varphi = \sup_{u\in U} \min_{e\in E} \{\beta_Y(\cdot, u(e), e) - \varphi(\cdot + \beta_X(\cdot, u(e), e)) + \varphi\}.$$

Hence

$$\min\left\{\hat{v}^* + \kappa, \check{G}\hat{v}^*\right\}(t_o, x_o, m) \leq 0 \tag{2.3.73}$$

on $[0, T) \times \mathbf{X}$.

step 3: Now observe that, by standard arguments, for every $(t, x) \in [0, T) \times \mathbf{X}$, we may find a sequence of smooth functions $(\varphi^n)_{n\geq 1}$ such that $\varphi^n \downarrow \hat{v}^*$, $(t_n, x_n, p_n)_{n\geq 1}$ converging towards $(t, x, m)$ and such that $(\varphi^n - \hat{v}^*)$ achieves a maximum at $(t_n, x_n, p_n)$. We refer to [Bou02, Lemma 6.1] for the approximation argument by continuous functions. The extension to an approximation by smooth functions is straightforward.

It thus follows from step 2, that $\hat{v}^*(\cdot, m)$ is a classical subsolution of (2.3.73) on $[0, T) \times \mathbf{X}$. In order to conclude the proof, we now appeal to the following easy lemma.

**Lemma 2.3.1.** *Assume that **H2** holds. Let $w$ be a upper semi-continuous subsolution of*

$$\min\left\{w + \kappa, \check{G}w\right\} \leq 0 \text{ on } \boldsymbol{X} \tag{2.3.74}$$

*such that $w^+$ satisfies the growth condition (2.3.22). Then, $w \leq -\kappa$ on $\boldsymbol{X}$.*

Applying Lemma 2.3.1 to $\hat{v}^*(t_o, \cdot, m)$ for an arbitrary $t_o \in [0, T)$ then leads to $\hat{v}^*(\cdot, m) = -\kappa$, since $\hat{v}^*(\cdot, m) \geq -\kappa$ and $\hat{v}^{*-}$ satisfies (2.3.22) by assumption.

step 4: We finally show that $\hat{v}_*(T, \cdot, m) = -\kappa$ on $\mathbf{X}$. Since $\hat{v}^*(t, x, m) = -\kappa$ for $t < T$ and $x \in \mathbf{X}$, we can find a sequence $(t_n, x_n, p_n)_{n\geq 1}$ in $[0, T) \times \mathbf{X} \times (m, M)$ such that $(t_n, x_n, p_n) \to (T, x, m)$ and $-\kappa \leq \hat{v}(t_n, x_n, p_n) \leq -\kappa + \frac{1}{n}$ for all $n \geq 0$. Passing to the limit leads to the required result.

□

**Proof of Lemma 2.3.1.**

We assume that $\sup_{\mathbf{X}}(w + \kappa) > 0$ and work towards a contradiction. It follows from the growth condition (2.3.22) on $w$, **(H2-ii)** and **(H2-iii)** that there is some $x_o \in \mathbf{X}$ such that

$$\max_{\mathbf{X}}(w - \Lambda) = (w - \Lambda)(x_o) =: \xi > 0. \tag{2.3.75}$$

By **(H2-i)**, Assumption 2.3.9 and (2.3.75), there exists some $\bar{u} \in U$ such that

$$\min_{e \in E} \beta_Y\left(x_o, w(x_o), \bar{u}(e), e\right) - \Lambda\left(x_o + \beta_X\left(x_o, \bar{u}(e), e\right)\right) + \Lambda(x_o) > 0. \qquad (2.3.76)$$

Since $w$ is a subsolution on $\mathbf{X}$ of (2.3.74), we have $\check{G}w(x_o) \leq 0$. Recalling Assumption 2.3.9, we may then find $\hat{e} \in E$ such that

$$\beta_Y\left(x_o, w(x_o), \bar{u}(\hat{e}), \hat{e}\right) - w\left(x_o + \beta_X\left(x_o, \bar{u}(\hat{e}), \hat{e}\right)\right) + w(x_o) \leq 0.$$

Combining the last inequality with (2.3.76) leads to

$$w(x_o) - \Lambda(x_o) < w\left(x_o + \beta_X\left(x_o, \bar{u}(\hat{e}), \hat{e}\right)\right) - \Lambda\left(x_o + \beta_X\left(x_o, \bar{u}(\hat{e})\right)\right),$$

which contradicts the definition of $x_o$ in (2.3.75). )

□

# Stochastic Target Games - Abstract

**Abstract**

We study a stochastic game where one player tries to find a strategy such that the state process reaches a target of controlled-loss-type, no matter which action is chosen by the other player. We provide, in a general setup, a relaxed geometric dynamic programming for this problem and derive, for the case of a Brownian controlled SDE, the corresponding dynamic programming equation in the sense of viscosity solutions. As an example, we consider a problem of partial hedging under Knightian uncertainty.

**Keywords:** Stochastic target; Stochastic game; Geometric dynamic programming principle; Viscosity solution

The work presented in this chapter is taken from [BMN12], and has been co-authored with Pr. Bruno Bouchard[3] and Dr. Marcel Nutz[4].

[3]CEREMADE, Université Paris Dauphine and CREST-ENSAE, bouchard@ceremade.dauphine.fr

[4]Dept. of Mathematics, Columbia University, New York, mnutz@math.columbia.edu

# Chapter 3

# Stochastic Target Games

### Contents



## 3.1 Introduction

We study a stochastic (semi) game of the following form. Given an initial condition $(t, z)$ in time and space, we try to find a strategy $\mathfrak{u}[\cdot]$ such that the controlled state process $Z_{t,z}^{\mathfrak{u}[\nu],\nu}(\cdot)$ reaches a certain target at the given time $T$, no matter which control $\nu$ is chosen by the adverse player. The target is specified in terms of expected loss; that is, we are given a real-valued ("loss") function $\ell$ and try to keep the expected loss above a given threshold $p \in \mathbb{R}$:

$$\operatorname*{ess\,inf}_{\nu} \mathbb{E}\left[\ell\left(Z_{t,z}^{\mathfrak{u}[\nu],\nu}(T)\right)|\mathcal{F}_t\right] \geq p \quad \text{a.s.} \tag{3.1.1}$$

Instead of a game, one may also see this as a target problem under Knightian uncertainty; then the adverse player has the role of choosing a worst-case scenario.

Our aim is to describe, for given $t$, the set $\Lambda(t)$ of all pairs $(z, p)$ such that there exists a strategy $\mathfrak{u}$ attaining the target. We provide, in a general abstract framework, a geometric dynamic programming principle (GDP) for this set. To this end, $p$ is seen as an additional state variable and formulated dynamically via a family $\{M^\nu\}$ of auxiliary martingales with expectation $p$, indexed by the adverse controls $\nu$. Heuristically, the GDP then takes the following form: $\Lambda(t)$ consists of all $(z, p)$ such that there exists a strategy $\mathfrak{u}$ and a family $\{M^\nu\}$ satisfying

$$\left(Z_{t,z}^{\mathfrak{u}[\nu],\nu}(\tau), M^\nu(\tau)\right) \in \Lambda(\tau) \quad \text{a.s.}$$

for all adverse controls $\nu$ and all stopping times $\tau \geq t$. The precise version of the GDP, stated in Theorem 3.2.1, incorporates several relaxations that allow us to deal with various technical problems. In particular, the selection of $\varepsilon$-optimal strategies is solved by a covering argument which is possible due a continuity assumption on $\ell$ and a relaxation in the variable $p$. The martingale $M^\nu$ is constructed from the semimartingale decomposition of the adverse player's value process.

Our GDP is tailored such that the dynamic programming equation can be derived in the viscosity sense. We exemplify this in Theorem 3.3.3 for the standard setup where the state process is determined by a stochastic differential equation (SDE) with coefficients controlled by the two players; however, the general GDP applies also in other situations such as singular control. The solution of the equation, a partial differential equation (PDE) in our example, corresponds to the indicator function of (the complement of) the graph of $\Lambda$. In Theorem 3.3.5, we specialize to a case with a monotonicity condition that is particularly suitable for pricing problems in mathematical finance. Finally, in order to illustrate various points made throughout the chapter, we consider a concrete example of pricing an option with partial hedging, according to a loss constraint, in a model where the drift and volatility coefficients of the underlying are uncertain. In a worst-case analysis, the uncertainty corresponds to an adverse player choosing the coefficients; a formula for the corresponding seller's price is given in Theorem 3.4.1.

Stochastic target (control) problems with almost-sure constraints, corresponding to the case where $\ell$ is an indicator function and $\nu$ is absent, were introduced in [ST02a, ST02c] as an extension of the classical superhedging problem [EKQ95] in mathematical finance. Stochastic target problems with controlled loss were first studied in [BET09] and are inspired by the quantile hedging problem [FL99]. The present chapter is the first to consider stochastic target games. The rigorous treatment of zero-sum stochastic differential games was pioneered by [FS89], where the mentioned selection problem for $\varepsilon$-optimal strategies was treated by a discretization and a passage to continuous-time limit in the PDEs. Let us remark, however, that we have not been able to achieve satisfactory results for our problem using such techniques. We have been importantly influenced by [BL08], where the value functions are defined in terms of essential infima and suprema, and then shown to be deterministic. The formulation with an essential infimum (rather than an infimum of suitable expectations) in (3.1.1) is crucial in our case, mainly because $\{M^\nu\}$ is constructed by a method of non-Markovian control, which raises the fairly delicate problem of dealing with one nullset for every adverse control $\nu$.

The remainder of the chapter is organized as follows. Section 3.2 contains the abstract setup and GDP. In Section 3.3 we specialize to the case of a controlled SDE and derive the corresponding PDE, first in the general case and then in the monotone case. The problem of hedging under uncertainty is discussed in Section 3.4.

## 3.2 Geometric dynamic programming principle

In this section, we obtain our geometric dynamic programming principle (GDP) in an abstract framework. Some of our assumptions are simply the conditions we need in the proof of the theorem; we will illustrate later how to actually verify them in a typical setup.

### 3.2.1 Problem statement

We fix a time horizon $T > 0$ and a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ satisfying the usual conditions of right-continuity and completeness. We shall consider two sets $\mathcal{U}$ and $\mathcal{V}$ of controls; for the sake of concreteness, we assume that each of these sets consists of stochastic processes on $(\Omega, \mathcal{F})$, indexed by $[0, T]$, and with values in some sets $U$ and $V$, respectively. Moreover, let $\mathfrak{U}$ be a set of mappings $\mathfrak{u} : \mathcal{V} \to \mathcal{U}$. Each $\mathfrak{u} \in \mathfrak{U}$ is called a strategy and the notation $\mathfrak{u}[\nu]$ will be used for the control it associates with $\nu \in \mathcal{V}$. In applications, $\mathfrak{U}$ will be chosen to consist of mappings that are non-anticipating; see Section 3.3 for an example. Furthermore, we are given a metric space $(\mathcal{Z}, d_{\mathcal{Z}})$ and, for each $(t, z) \in [0, T] \times \mathcal{Z}$ and $(\mathfrak{u}, \nu) \in \mathfrak{U} \times \mathcal{V}$, an adapted càdlàg process $Z^{\mathfrak{u}[\nu],\nu}_{t,z}(\cdot)$ with values in $\mathcal{Z}$ satisfying $Z^{\mathfrak{u}[\nu],\nu}_{t,z}(t) = z$. For brevity, we set

$$Z^{\mathfrak{u},\nu}_{t,z} := Z^{\mathfrak{u}[\nu],\nu}_{t,z}.$$

Let $\ell : \mathcal{Z} \to \mathbb{R}$ be a Borel-measurable function satisfying

$$\mathbb{E}\left[\left|\ell\left(Z^{\mathfrak{u},\nu}_{t,z}(T)\right)\right|\right] < \infty \quad \text{for all } (t, z, \mathfrak{u}, \nu) \in [0, T] \times \mathcal{Z} \times \mathfrak{U} \times \mathcal{V}. \tag{3.2.1}$$

We interpret $\ell$ as a loss (or "utility") function and denote by

$$I(t,z,\mathfrak{u},\nu) := \mathbb{E}\left[\ell\left(Z^{\mathfrak{u},\nu}_{t,z}(T)\right)|\mathcal{F}_t\right], \quad (t,z,\mathfrak{u},\nu) \in [0,T]\times\mathcal{Z}\times\mathfrak{U}\times\mathcal{V}$$

the expected loss given $\nu$ (for the player choosing $\mathfrak{u}$) and by

$$J(t,z,\mathfrak{u}) := \operatorname*{ess\,inf}_{\nu\in\mathcal{V}} I(t,z,\mathfrak{u},\nu), \quad (t,z,\mathfrak{u}) \in [0,T]\times\mathcal{Z}\times\mathfrak{U}$$

the worst-case expected loss. The main object of this chapter is the reachability set

$$\Lambda(t) := \left\{(z,p)\in\mathcal{Z}\times\mathbb{R} : \text{there exists } \mathfrak{u}\in\mathfrak{U} \text{ such that } J(t,z,\mathfrak{u}) \geq p\ \mathbb{P}\text{-a.s.}\right\}. \quad (3.2.2)$$

These are the initial conditions $(z,p)$ such that starting at time $t$, the player choosing $\mathfrak{u}$ can attain an expected loss not worse than $p$, regardless of the adverse player's action $\nu$. The main aim of this chapter is to provide a geometric dynamic programming principle for $\Lambda(t)$. For the case without adverse player, a corresponding result was obtained in [ST02a] for the target problem with almost-sure constraints and in [BET09] for the problem with controlled loss.

As mentioned above, the dynamic programming for the problem (3.2.2) requires the introduction of a suitable set of martingales starting from $p\in\mathbb{R}$. This role will be played by certain families[1] $\{M^\nu, \nu\in\mathcal{V}\}$ of martingales which should be considered as additional controls. More precisely, we denote by $\mathcal{M}_{t,p}$ the set of all real-valued (right-continuous) martingales $M$ satisfying $M(t) = p$ $\mathbb{P}$-a.s., and we fix a set $\mathfrak{M}_{t,p}$ of families $\{M^\nu, \nu\in\mathcal{V}\}\subset\mathcal{M}_{t,p}$; further assumptions on $\mathfrak{M}_{t,p}$ will be introduced below. Since these martingales are not present in the original problem (3.2.2), we can choose $\mathfrak{M}_{t,p}$ to our convenience; see also Remark 3.2.2 below.

As usual in optimal control, we shall need to concatenate controls and strategies in time according to certain events. We use the notation

$$\nu \oplus_\tau \bar{\nu} := \nu\mathbf{1}_{[0,\tau]} + \bar{\nu}\mathbf{1}_{(\tau,T]}$$

for the concatenation of two controls $\nu,\bar{\nu}\in\mathcal{V}$ at a stopping time $\tau$. We also introduce the set

$$\left\{\nu =_{(t,\tau]} \bar{\nu}\right\} := \left\{\omega\in\Omega : \nu_s(\omega) = \bar{\nu}_s(\omega) \text{ for all } s\in(t,\tau(\omega)]\right\}.$$

Analogous notation is used for elements of $\mathcal{U}$.

In contrast to the setting of control, strategies can be concatenated only at particular events and stopping times, as otherwise the resulting strategies would fail to be elements of $\mathfrak{U}$ (in particular, because they may fail to be non-anticipating, see

[1] Of course, there is no mathematical difference between families indexed by $\mathcal{V}$, like $\{M^\nu, \nu\in\mathcal{V}\}$, and mappings on $\mathcal{V}$, like $\mathfrak{u}$. We shall use both notions interchangeably, depending on notational convenience.

also Section 3.3). Therefore, we need to formalize the events and stopping times which are admissible for this purpose: For each $t \le T$, we consider a set $\mathfrak{F}_t$ whose elements are families $\{A^\nu, \nu \in \mathcal{V}\} \subset \mathcal{F}_t$ of events indexed by $\mathcal{V}$, as well as a set $\mathfrak{T}_t$ whose elements are families $\{\tau^\nu, \nu \in \mathcal{V}\} \subset \mathcal{T}_t$, where $\mathcal{T}_t$ denotes the set of all stopping times with values in $[t,T]$. We assume that $\mathfrak{T}_t$ contains any deterministic time $s \in [t,T]$ (seen as a constant family $\tau^\nu \equiv s$, $\nu \in \mathcal{V}$). In practice, the sets $\mathfrak{F}_t$ and $\mathfrak{T}_t$ will not contain all families of events and stopping times, respectively; one will impose additional conditions on $\nu \mapsto A^\nu$ and $\nu \mapsto \tau^\nu$ that are compatible with the conditions defining $\mathfrak{U}$. Both sets should be seen as auxiliary objects which make it easier (if not possible) to verify the dynamic programming conditions below.

### 3.2.2 The geometric dynamic programming principle

We can now state the conditions for our main result. The first one concerns the concatenation of controls and strategies.

**Assumption (C).** *The following hold for all $t \in [0,T]$.*

**(C1)** *Fix $\nu_0, \nu_1, \nu_2 \in \mathcal{V}$ and $A \in \mathcal{F}_t$. Then $\nu := \nu_0 \oplus_t (\nu_1 \mathbf{1}_A + \nu_2 \mathbf{1}_{A^c}) \in \mathcal{V}$.*

**(C2)** *Fix $(\mathfrak{u}_j)_{j\ge 0} \subset \mathfrak{U}$ and let $\{A_j^\nu, \nu \in \mathcal{V}\}_{j\ge1} \subset \mathfrak{F}_t$ be such that $\{A_j^\nu, j \ge 1\}$ forms a partition of $\Omega$ for each $\nu \in \mathcal{V}$. Then $\mathfrak{u} \in \mathfrak{U}$ for*

$$\mathfrak{u}[\nu] := \mathfrak{u}_0[\nu] \oplus_t \sum_{j\ge1} \mathfrak{u}_j[\nu] \mathbf{1}_{A_j^\nu}, \quad \nu \in \mathcal{V}.$$

**(C3)** *Let $\mathfrak{u} \in \mathfrak{U}$ and $\nu \in \mathcal{V}$. Then $\mathfrak{u}[\nu \oplus_t \cdot] \in \mathfrak{U}$.*

**(C4)** *Let $\{A^\nu, \nu \in \mathcal{V}\} \subset \mathcal{F}_t$ be a family of events such that $A^{\nu_1} \cap \{\nu_1 =_{(0,t]} \nu_2\} = A^{\nu_2} \cap \{\nu_1 =_{(0,t]} \nu_2\}$ for all $\nu_1, \nu_2 \in \mathcal{V}$. Then $\{A^\nu, \nu \in \mathcal{V}\} \in \mathfrak{F}_t$.*

**(C5)** *Let $\{\tau^\nu, \nu \in \mathcal{V}\} \in \mathfrak{T}_t$. Then $\{\tau^{\nu_1} \le s\} = \{\tau^{\nu_2} \le s\}$ for $\mathbb{P}$-a.e. $\omega \in \{\nu_1 =_{(0,s]} \nu_2\}$, for all $\nu_1, \nu_2 \in \mathcal{V}$ and $s \in [t,T]$.*

**(C6)** *Let $\{\tau^\nu, \nu \in \mathcal{V}\} \in \mathfrak{T}_t$. Then, for all $t \le s_1 \le s_2 \le T$, $\{\{\tau^\nu \in (s_1, s_2]\}, \nu \in \mathcal{V}\}$ and $\{\{\tau^\nu \notin (s_1, s_2]\}, \nu \in \mathcal{V}\}$ belong to $\mathfrak{F}_{s_2}$.*

The second condition concerns the behavior of the state process.

**Assumption (Z).** *The following hold for all $(t,z,p) \in [0,T] \times \mathcal{Z} \times \mathbb{R}$ and $s \in [t,T]$.*

**(Z1)** *$Z_{t,z}^{\mathfrak{u}_1,\nu}(s)(\omega) = Z_{t,z}^{\mathfrak{u}_2,\nu}(s)(\omega)$ for $\mathbb{P}$-a.e. $\omega \in \{\mathfrak{u}_1[\nu] =_{(t,s]} \mathfrak{u}_2[\nu]\}$, for all $\nu \in \mathcal{V}$ and $\mathfrak{u}_1, \mathfrak{u}_2 \in \mathfrak{U}$.*

**(Z2)** $Z^{\mathfrak{u},\nu_1}_{t,z}(s)(\omega) = Z^{\mathfrak{u},\nu_2}_{t,z}(s)(\omega)$ *for* $\mathbb{P}$*-a.e.* $\omega \in \{\nu_1 =_{(0,s]} \nu_2\}$*, for all* $\mathfrak{u} \in \mathfrak{U}$ *and* $\nu_1, \nu_2 \in \mathcal{V}$.

**(Z3)** $M^{\nu_1}(s)(\omega) = M^{\nu_2}(s)(\omega)$ *for* $\mathbb{P}$*-a.e.* $\omega \in \{\nu_1 =_{(0,s]} \nu_2\}$*, for all* $\{M^\nu, \nu \in \mathcal{V}\} \in \mathfrak{M}_{t,p}$ *and* $\nu_1, \nu_2 \in \mathcal{V}$.

**(Z4)** *There exists a constant* $K(t,z) \in \mathbb{R}$ *such that*

$$\operatorname*{ess\,sup}_{\mathfrak{u}\in\mathfrak{U}} \operatorname*{ess\,inf}_{\nu\in\mathcal{V}} \mathbb{E}\left[\ell(Z^{\mathfrak{u},\nu}_{t,z}(T))|\mathcal{F}_t\right] = K(t,z) \;\; \mathbb{P}\text{-a.s.}$$

The nontrivial assumption here is, of course, (Z4), stating that (a version of) the random variable $\operatorname{ess\,sup}_{\mathfrak{u}\in\mathfrak{U}} \operatorname{ess\,inf}_{\nu\in\mathcal{V}} \mathbb{E}[\ell(Z^{\mathfrak{u},\nu}_{t,z}(T))|\mathcal{F}_t]$ is *deterministic*. For the game determined by a Brownian SDE as considered in Section 3.3, this will be true by a result of [BL08], which, in turn, goes back to an idea of [Pen97] (see also [LP09]). An extension to jump diffusions can be found in [BHL11].

While the above assumptions are fundamental, the following conditions are of technical nature. We shall illustrate later how they can be verified.

**Assumption (I).** *Let* $(t,z) \in [0,T] \times \mathcal{Z}$*,* $\mathfrak{u} \in \mathfrak{U}$ *and* $\nu \in \mathcal{V}$.

**(I1)** *There exists an adapted right-continuous process* $N^{\mathfrak{u},\nu}_{t,z}$ *of class (*D*) such that*

$$\operatorname*{ess\,inf}_{\bar\nu\in\mathcal{V}} \mathbb{E}\left[\ell\left(Z^{\mathfrak{u},\nu\oplus_s\bar\nu}_{t,z}(T)\right)|\mathcal{F}_s\right] \geq N^{\mathfrak{u},\nu}_{t,z}(s) \quad \mathbb{P}\text{-a.s. for all } s \in [t,T].$$

**(I2)** *There exists an adapted right-continuous process* $L^{\mathfrak{u},\nu}_{t,z}$ *such that* $L^{\mathfrak{u},\nu}_{t,z}(s) \in L^1$ *and*

$$\operatorname*{ess\,inf}_{\bar{\mathfrak{u}}\in\mathfrak{U}} \mathbb{E}\left[\ell\left(Z^{\mathfrak{u}\oplus_s\bar{\mathfrak{u}},\nu}_{t,z}(T)\right)|\mathcal{F}_s\right] \geq L^{\mathfrak{u},\nu}_{t,z}(s) \quad \mathbb{P}\text{-a.s. for all } s \in [t,T].$$

*Moreover,* $L^{\mathfrak{u},\nu_1}_{t,z}(s)(\omega) = L^{\mathfrak{u},\nu_2}_{t,z}(s)(\omega)$ *for* $\mathbb{P}$*-a.e.* $\omega \in \{\nu_1 =_{(0,s]} \nu_2\}$*, for all* $\mathfrak{u} \in \mathcal{U}$ *and* $\nu_1, \nu_2 \in \mathcal{V}$.

**Assumption (R).** *Let* $(t,z) \in [0,T] \times \mathcal{Z}$.

**(R1)** *Fix* $s \in [t,T]$ *and* $\varepsilon > 0$*. Then there exist a Borel-measurable partition* $(B_j)_{j\geq 1}$ *of* $\mathcal{Z}$ *and a sequence* $(z_j)_{j\geq1} \subset \mathcal{Z}$ *such that for all* $\mathfrak{u} \in \mathfrak{U}$*,* $\nu \in \mathcal{V}$ *and* $j \geq 1$*,*

$$\left.\begin{aligned} \mathbb{E}\left[\ell(Z^{\mathfrak{u},\nu}_{t,z}(T))|\mathcal{F}_s\right] &\geq I(s,z_j,\mathfrak{u},\nu) - \varepsilon, \\ \operatorname*{ess\,inf}_{\bar\nu\in\mathcal{V}} \mathbb{E}\left[\ell(Z^{\mathfrak{u},\nu\oplus_s\bar\nu}_{t,z}(T))|\mathcal{F}_s\right] &\leq J(s,z_j,\mathfrak{u}[\nu\oplus_s\cdot]) + \varepsilon, \\ K(s,z_j) - \varepsilon \leq K(s,Z^{\mathfrak{u},\nu}_{t,z}(s)) &\leq K(s,z_j) + \varepsilon \end{aligned}\right\} \mathbb{P}\text{-a.s. on } \{Z^{\mathfrak{u},\nu}_{t,z}(s) \in B_j\}.$$

**(R2)** $\lim_{\delta\to 0} \sup_{\nu\in\mathcal{V},\tau\in\mathcal{T}_t} \mathbb{P}\left[\sup_{0\le h\le\delta} d_{\mathcal{Z}}\left(Z^{\mathfrak{u},\nu}_{t,z}(\tau+h), Z^{\mathfrak{u},\nu}_{t,z}(\tau)\right)\ge\varepsilon\right] = 0$ *for all* $\mathfrak{u}\in\mathfrak{U}$ *and* $\varepsilon>0$.

Our GDP will be stated in terms of the closure

$$\bar\Lambda(t) := \left\{ \begin{array}{l} (z,p)\in\mathcal{Z}\times\mathbb{R}: \text{ there exist } (t_n,z_n,p_n)\to(t,z,p) \\ \text{such that } (z_n,p_n)\in\Lambda(t_n) \text{ and } t_n\ge t \text{ for all } n\ge 1 \end{array} \right\}$$

and the uniform interior

$$\mathring{\Lambda}_\iota(t) := \left\{(z,p)\in\mathcal{Z}\times\mathbb{R}:\ (t',z',p')\in B_\iota(t,z,p) \text{ implies } (z',p')\in\Lambda(t')\right\},$$

where $B_\iota(t,z,p)\subset[0,T]\times\mathcal{Z}\times\mathbb{R}$ denotes the open ball with center $(t,z,p)$ and radius $\iota>0$ (with respect to the distance function $d_{\mathcal{Z}}(z,z')+|p-p'|+|t-t'|$). The relaxation from $\Lambda$ to $\bar\Lambda$ and $\mathring{\Lambda}_\iota$ essentially allows us to reduce to stopping times with countably many values in the proof of the GDP and thus to avoid regularity assumptions in the time variable. We shall also relax the variable $p$ in the assertion of (GDP2); this is inspired by [BN11] and important for the covering argument in the proof of (GDP2), which, in turn, is crucial due to the lack of a measurable selection theorem for strategies. Of course, all our relaxations are tailored such that they will not interfere substantially with the derivation of the dynamic programming equation; cf. Section 3.3.

**Theorem 3.2.1.** *Fix* $(t,z,p)\in[0,T]\times\mathcal{Z}\times\mathbb{R}$ *and let Assumptions* (C), (Z), (I) *and* (R) *hold true.*

**(GDP1)** *If* $(z,p)\in\Lambda(t)$*, then there exist* $\mathfrak{u}\in\mathfrak{U}$ *and* $\{M^\nu,\nu\in\mathcal{V}\}\subset\mathcal{M}_{t,p}$ *such that*

$$\left(Z^{\mathfrak{u},\nu}_{t,z}(\tau), M^\nu(\tau)\right)\in\bar\Lambda(\tau) \quad \mathbb{P}\text{-a.s. for all } \nu\in\mathcal{V} \text{ and } \tau\in\mathcal{T}_t.$$

**(GDP2)** *Let* $\iota>0$, $\mathfrak{u}\in\mathfrak{U}$, $\{M^\nu,\nu\in\mathcal{V}\}\in\mathfrak{M}_{t,p}$ *and* $\{\tau^\nu,\nu\in\mathcal{V}\}\in\mathfrak{T}_t$ *be such that*

$$\left(Z^{\mathfrak{u},\nu}_{t,z}(\tau^\nu), M^\nu(\tau^\nu)\right)\in\mathring{\Lambda}_\iota(\tau^\nu) \quad \mathbb{P}\text{-a.s. for all } \nu\in\mathcal{V},$$

*and suppose that* $\left\{M^\nu(\tau^\nu)^+:\ \nu\in\mathcal{V}\right\}$ *and* $\left\{L^{\mathfrak{u},\nu}_{t,z}(\tau')^-:\ \nu\in\mathcal{V},\tau'\in\mathcal{T}_t\right\}$ *are uniformly integrable, where* $L^{\mathfrak{u},\nu}_{t,z}$ *is as in* (I2)*. Then* $(z,p-\varepsilon)\in\Lambda(t)$ *for all* $\varepsilon>0$.

The proof is stated in Sections 3.2.3 and 3.2.4 below.

**Remark 3.2.2.** We shall see in the proof that the family $\{M^\nu,\nu\in\mathcal{V}\}\subset\mathcal{M}_{t,p}$ in (GDP1) can actually be chosen to be non-anticipating in the sense of (Z3). However, this will not be used when (GDP1) is applied to derive the dynamic programming equation. Whether $\{M^\nu,\nu\in\mathcal{V}\}$ is an element of $\mathfrak{M}_{t,p}$ will depend on the definition of the latter set; in fact, we did not make any assumption about its richness. In many application, it is possible to take $\mathfrak{M}_{t,p}$ to be the set of all non-anticipating families in $\mathcal{M}_{t,p}$; however, we prefer to leave some freedom for the definition of $\mathfrak{M}_{t,p}$ since this may be useful in ensuring the uniform integrability required in (GDP2).

We conclude this section with a version of the GDP for the case $\mathcal{Z} = \mathbb{R}^d$, where we show how to reduce from standard regularity conditions on the state process and the loss function to the assumptions (R1) and (I).

**Corollary 3.2.3.** *Let Assumptions* (C), (Z) *and* (R2) *hold true. Assume also that* $\ell$ *is continuous and that there exist constants* $C \geq 0$ *and* $\bar{q} > q \geq 0$ *and a locally bounded function* $\varrho : \mathbb{R}^d \mapsto \mathbb{R}_+$ *such that*

$$|\ell(z)| \leq C(1 + |z|^q), \tag{3.2.3}$$

$$\operatorname*{ess\,sup}_{(\bar{\mathfrak{u}},\bar{\nu})\in\mathfrak{U}\times\mathcal{V}} \mathbb{E}\left[|Z^{\bar{\mathfrak{u}},\bar{\nu}}_{t,z}(T)|^{\bar{q}}|\mathcal{F}_t\right] \leq \varrho(z)^{\bar{q}} \quad \mathbb{P}\text{-a.s.} \quad \text{and} \tag{3.2.4}$$

$$\operatorname*{ess\,sup}_{(\bar{\mathfrak{u}},\bar{\nu})\in\mathfrak{U}\times\mathcal{V}} \mathbb{E}\left[|Z^{\mathfrak{u}\oplus_s\bar{\mathfrak{u}},\nu\oplus_s\bar{\nu}}_{t,z}(T) - Z^{\bar{\mathfrak{u}},\nu\oplus_s\bar{\nu}}_{s,z'}(T)|\;|\mathcal{F}_s\right] \leq C|Z^{\mathfrak{u},\nu}_{t,z}(s) - z'| \quad \mathbb{P}\text{-a.s.} \tag{3.2.5}$$

*for all* $(t, z) \in [0, T] \times \mathbb{R}^d$, $(s, z') \in [t, T] \times \mathbb{R}^d$ *and* $(\mathfrak{u}, \nu) \in \mathfrak{U} \times \mathcal{V}$.

**(GDP1')** *If* $(z, p + \varepsilon) \in \Lambda(t)$ *for some* $\varepsilon > 0$, *then there exist* $\mathfrak{u} \in \mathfrak{U}$ *and* $\{M^\nu, \nu \in \mathcal{V}\} \subset \mathcal{M}_{t,p}$ *such that*

$$\left(Z^{\mathfrak{u},\nu}_{t,z}(\tau), M^\nu(\tau)\right) \in \bar{\Lambda}(\tau) \quad \mathbb{P}\text{-a.s. for all } \nu \in \mathcal{V} \text{ and } \tau \in \mathcal{T}_t.$$

**(GDP2')** *Let* $\iota > 0$, $\mathfrak{u} \in \mathfrak{U}$, $\{M^\nu, \nu \in \mathcal{V}\} \in \mathfrak{M}_{t,p}$ *and* $\{\tau^\nu, \nu \in \mathcal{V}\} \in \mathfrak{T}_t$ *be such that*

$$\left(Z^{\mathfrak{u},\nu}_{t,z}(\tau^\nu), M^\nu(\tau^\nu)\right) \in \mathring{\Lambda}_\iota(\tau^\nu) \quad \mathbb{P}\text{-a.s. for all } \nu \in \mathcal{V}$$

*and assume that* $\{Z^{\mathfrak{u},\nu}_{t,z}(\tau^\nu), \nu \in \mathcal{V}\}$ *is uniformly bounded in* $L^\infty$. *Then* $(z, p - \varepsilon) \in \Lambda(t)$ *for all* $\varepsilon > 0$.

We remark that (GDP2') is usually applied in a setting where $\tau^\nu$ is the exit time of $Z^{\mathfrak{u},\nu}_{t,z}$ from a given ball, so that the boundedness assumption is not restrictive. (Some adjustments are needed when the state process admits unbounded jumps; see also [Mor11].)

### 3.2.3 Proof of (GDP1)

We fix $t \in [0, T]$ and $(z, p) \in \Lambda(t)$ for the remainder of this proof. By the definition (3.2.2) of $\Lambda(t)$, there exists $\mathfrak{u} \in \mathfrak{U}$ such that

$$\mathbb{E}\left[G(\nu)|\mathcal{F}_t\right] \geq p \quad \mathbb{P}\text{-a.s. for all } \nu \in \mathcal{V}, \quad \text{where } G(\nu) := \ell(Z^{\mathfrak{u},\nu}_{t,z}(T)). \tag{3.2.6}$$

In order to construct the family $\{M^\nu, \nu \in \mathcal{V}\} \subset \mathcal{M}_{t,p}$ of martingales, we consider

$$S^\nu(r) := \operatorname*{ess\,inf}_{\bar\nu \in \mathcal{V}} \mathbb{E}\left[G(\nu \oplus_r \bar\nu)|\mathcal{F}_r\right], \quad t \le r \le T. \tag{3.2.7}$$

We shall obtain $M^\nu$ from a Doob-Meyer-type decomposition of $S^\nu$. This can be seen as a generalization with respect to [BET09], where the necessary martingale was trivially constructed by taking the conditional expectation of the terminal reward.

**Step 1:** *We have $S^\nu(r) \in L^1(\mathbb{P})$ and $\mathbb{E}\left[S^\nu(r)|\mathcal{F}_s\right] \ge S^\nu(s)$ for all $t \le s \le r \le T$ and $\nu \in \mathcal{V}$.*

The integrability of $S^\nu(r)$ follows from (3.2.1) and (I1). To see the submartingale property, we first show that the family $\{\mathbb{E}[G(\nu \oplus_r \bar\nu)|\mathcal{F}_r],\ \bar\nu \in \mathcal{V}\}$ is directed downward. Indeed, given $\bar\nu_1, \bar\nu_2 \in \mathcal{V}$, the set

$$A := \{\mathbb{E}\left[G(\nu \oplus_r \bar\nu_1)|\mathcal{F}_r\right] \le \mathbb{E}\left[G(\nu \oplus_r \bar\nu_2)|\mathcal{F}_r\right]\}$$

is in $\mathcal{F}_r$; therefore, $\bar\nu_3 := \nu \oplus_r (\bar\nu_1 \mathbf{1}_A + \bar\nu_2 \mathbf{1}_{A^c})$ is an element of $\mathcal{V}$ by Assumption (C1). Hence, (Z2) yields that

$$\begin{aligned}
\mathbb{E}\left[G(\nu \oplus_r \bar\nu_3)|\mathcal{F}_r\right] &= \mathbb{E}\left[G(\nu \oplus_r \bar\nu_1)\mathbf{1}_A + G(\nu \oplus_r \bar\nu_2)\mathbf{1}_{A^c}|\mathcal{F}_r\right] \\
&= \mathbb{E}\left[G(\nu \oplus_r \bar\nu_1)|\mathcal{F}_r\right]\mathbf{1}_A + \mathbb{E}\left[G(\nu \oplus_r \bar\nu_2)|\mathcal{F}_r\right]\mathbf{1}_{A^c} \\
&= \mathbb{E}\left[G(\nu \oplus_r \bar\nu_1)|\mathcal{F}_r\right] \wedge \mathbb{E}\left[G(\nu \oplus_r \bar\nu_2)|\mathcal{F}_r\right].
\end{aligned}$$

As a result, we can find a sequence $(\bar\nu_n)_{n\ge1}$ in $\mathcal{V}$ such that $\mathbb{E}[G(\nu \oplus_r \bar\nu_n)|\mathcal{F}_r]$ decreases $\mathbb{P}$-a.s. to $S^\nu(r)$; cf. [Nev75, Proposition VI-1-1]. Recalling (3.2.1) and that $S^\nu(r) \in L^1(\mathbb{P})$, monotone convergence yields that

$$\begin{aligned}
\mathbb{E}\left[S^\nu(r)|\mathcal{F}_s\right] &= \mathbb{E}\left[\lim_{n\to\infty} \mathbb{E}\left[G(\nu \oplus_r \bar\nu_n)|\mathcal{F}_r\right]|\mathcal{F}_s\right] \\
&= \lim_{n\to\infty} \mathbb{E}\left[G(\nu \oplus_r \bar\nu_n)|\mathcal{F}_s\right] \\
&\ge \operatorname*{ess\,inf}_{\bar\nu \in \mathcal{V}} \mathbb{E}\left[G(\nu \oplus_r \bar\nu)|\mathcal{F}_s\right] \\
&\ge \operatorname*{ess\,inf}_{\bar\nu \in \mathcal{V}} \mathbb{E}\left[G(\nu \oplus_s \bar\nu)|\mathcal{F}_s\right] \\
&= S^\nu(s),
\end{aligned}$$

where the last inequality follows from the fact that any control $\nu \oplus_r \bar\nu$, where $\bar\nu \in \mathcal{V}$, can be written in the form $\nu \oplus_s (\nu \oplus_r \bar\nu)$; cf. (C1).

**Step 2:** *There exists a family of càdlàg martingales $\{M^\nu, \nu \in \mathcal{V}\} \subset \mathcal{M}_{t,p}$ such that $S^\nu(r) \ge M^\nu(r)$ $\mathbb{P}$-a.s. for all $r \in [t, T]$ and $\nu \in \mathcal{V}$.*

Fix $\nu \in \mathcal{V}$. By Step 1, $S^\nu(\cdot)$ satisfies the submartingale property. Therefore,

$$S_+(r)(\omega) := \lim_{u \in (r,T]\cap\mathbb{Q},\, u \to r} S^\nu(u)(\omega) \quad \text{for } 0 \le r < T \quad \text{and} \quad S_+(T) := S^\nu(T)$$

is well defined $\mathbb{P}$-a.s.; moreover, recalling that the filtration $\mathbb{F}$ satisfies the usual conditions, $S_+$ is a (right-continuous) submartingale satisfying $S_+(r) \geq S^\nu(r)$ $\mathbb{P}$-a.s. for all $r \in [t,T]$ (c.f. [DM82, Theorem VI.2]). Let $H \subset [t,T]$ be the set of points of discontinuity of the function $r \mapsto \mathbb{E}[S^\nu(r)]$. Since this function is increasing, $H$ is at most countable. (If $H$ happens to be the empty set, then $S_+$ defines a modification of $S^\nu$ and the Doob-Meyer decomposition of $S_+$ yields the result.) Consider the process

$$\bar{S}(r) := S_+(r)\mathbf{1}_{H^c}(r) + S^\nu(r)\mathbf{1}_H(r), \quad r \in [t,T].$$

The arguments (due to E. Lenglart) in the proof of [DM82, Theorem 10 of Appendix 1] show that $\bar{S}$ is an *optional modification* of $S^\nu$ and $\mathbb{E}[\bar{S}(\tau)|\mathcal{F}_\sigma] \geq \bar{S}(\sigma)$ for all $\sigma, \tau \in \mathcal{T}_t$ such that $\sigma \leq \tau$; that is, $\bar{S}$ is a strong submartingale. Let $N = N^{\mathfrak{u},\nu}_{t,z}$ be a right-continuous process of class (D) as in (I1); then $S^\nu(r) \geq N(r)$ $\mathbb{P}$-a.s. for all $r$ implies that $S_+(r) \geq N(r)$ $\mathbb{P}$-a.s. for all $r$, and since both $S_+$ and $N$ are right-continuous, this shows that $S_+ \geq N$ up to evanescence. Recalling that $H$ is countable, we deduce that $\bar{S} \geq N$ up to evanescence, and as $\bar{S}$ is bounded from above by the martingale generated by $\bar{S}(T)$, we conclude that $\bar{S}$ is of class (D).

Now the decomposition result of Mertens [Mer72, Theorem 3] yields that there exist a (true) martingale $\bar{M}$ and a nondecreasing (not necessarily càdlàg) predictable process $\bar{C}$ with $\bar{C}(t) = 0$ such that

$$\bar{S} = \bar{M} + \bar{C},$$

and in view of the usual conditions, $\bar{M}$ can be chosen to be càdlàg. We can now define $M^\nu := \bar{M} - \bar{M}(t) + p$ on $[t,T]$ and $M^\nu(r) := p$ for $r \in [0,t)$, then $M^\nu \in \mathcal{M}_{t,p}$. Noting that $\bar{M}(t) = \bar{S}(t) = S^\nu(t) \geq p$ by (3.2.6), we see that $M^\nu$ has the required property:

$$M^\nu(r) \leq \bar{M}(r) \leq \bar{S}(r) = S^\nu(r) \quad \mathbb{P}\text{-a.s. for all } r \in [t,T].$$

**Step 3:** *Let $\tau \in \mathcal{T}_t$ have countably many values. Then*

$$K\left(\tau, Z^{\mathfrak{u},\nu}_{t,z}(\tau)\right) \geq M^\nu(\tau) \quad \mathbb{P}\text{-a.s. } \textit{for all } \nu \in \mathcal{V}.$$

Fix $\nu \in \mathcal{V}$ and $\varepsilon > 0$, let $M^\nu$ be as in Step 2, and let $(t_i)_{i\geq 1}$ be the distinct values of $\tau$. By Step 2, we have

$$M^\nu(t_i) \leq \operatorname*{ess\,inf}_{\bar{\nu}\in\mathcal{V}} \mathbb{E}\left[\ell\left(Z^{\mathfrak{u},\nu\oplus_{t_i}\bar{\nu}}_{t,z}(T)\right)|\mathcal{F}_{t_i}\right] \quad \mathbb{P}\text{-a.s.}, \quad i \geq 1.$$

Moreover, (R1) yields that for each $i \geq 1$, we can find a sequence $(z_{ij})_{j\geq 1} \subset \mathcal{Z}$ and a Borel partition $(B_{ij})_{j\geq 1}$ of $\mathcal{Z}$ such that

$$\operatorname*{ess\,inf}_{\bar{\nu}\in\mathcal{V}} \mathbb{E}\left[\ell\left(Z^{\mathfrak{u},\nu\oplus_{t_i}\bar{\nu}}_{t,z}(T)\right)|\mathcal{F}_{t_i}\right](\omega) \leq J(t_i, z_{ij}, \mathfrak{u}[\nu \oplus_{t_i} \cdot])(\omega) + \varepsilon$$
$$\text{for } \mathbb{P}\text{-a.e. } \omega \in C_{ij} := \{Z^{\mathfrak{u},\nu}_{t,z}(t_i) \in B_{ij}\}.$$

Since (C3) and the definition of $K$ in (Z4) yield that $J(t_i, z_{ij}, \mathfrak{u}[\nu \oplus_{t_i} \cdot]) \leq K(t_i, z_{ij})$, we conclude by (R1) that

$$M^\nu(t_i)(\omega) \leq K(t_i, z_{ij}) + \varepsilon \leq K(t_i, Z^{\mathfrak{u},\nu}_{t,z}(t_i)(\omega)) + 2\varepsilon \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in C_{ij}.$$

Let $A_i := \{\tau = t_i\} \in \mathcal{F}_\tau$. Then $(A_i \cap C_{ij})_{i,j\geq 1}$ forms a partition of $\Omega$ and the above shows that

$$M^\nu(\tau) - 2\varepsilon \leq \sum_{i,j\geq 1} K(t_i, Z^{\mathfrak{u},\nu}_{t,z}(t_i))\mathbf{1}_{A_i\cap C_{ij}} = K(\tau, Z^{\mathfrak{u},\nu}_{t,z}(\tau)) \quad \mathbb{P}\text{-a.s.}$$

As $\varepsilon > 0$ was arbitrary, the claim follows.

<u>**Step 4:**</u> We can now prove (GDP1). Given $\tau \in \mathcal{T}_t$, pick a sequence $(\tau_n)_{n\geq 1} \subset \mathcal{T}_t$ such that each $\tau_n$ has countably many values and $\tau_n \downarrow \tau$ $\mathbb{P}$-a.s. In view of the last statement of Lemma 3.2.1 below, Step 3 implies that

$$\left(Z^{\mathfrak{u},\nu}_{t,z}(\tau_n), M^\nu(\tau_n) - n^{-1}\right) \in \Lambda(\tau_n) \quad \mathbb{P}\text{-a.s. for all } n \geq 1.$$

However, using that $Z^{\mathfrak{u},\nu}_{t,z}$ and $M^\nu$ are càdlàg, we have

$$\left(\tau_n, Z^{\mathfrak{u},\nu}_{t,z}(\tau_n), M^\nu(\tau_n) - n^{-1}\right) \to \left(\tau, Z^{\mathfrak{u},\nu}_{t,z}(\tau), M^\nu(\tau)\right) \quad \mathbb{P}\text{-a.s. as } n \to \infty,$$

so that, by the definition of $\bar{\Lambda}$, we deduce that $(Z^{\mathfrak{u},\nu}_{t,z}(\tau), M^\nu(\tau)) \in \bar{\Lambda}(\tau)$ $\mathbb{P}$-a.s. □

**Lemma 3.2.1.** *Let Assumptions (C2), (C4), (Z1) and (Z4) hold true. For each $\varepsilon > 0$, there exists a mapping $\mu^\varepsilon : [0,T] \times \mathcal{Z} \to \mathfrak{U}$ such that*

$$J\left(t, z, \mu^\varepsilon(t,z)\right) \geq K(t,z) - \varepsilon \quad \mathbb{P}\text{-a.s. for all } (t,z) \in [0,T] \times \mathcal{Z}.$$

*In particular, if $(t,z,p) \in [0,T] \times \mathcal{Z} \times \mathbb{R}$, then $K(t,z) > p$ implies $(z,p) \in \Lambda(t)$.*

**Proof.** Since $K(t,z)$ was defined in (Z4) as the essential supremum of $J(t,z,\mathfrak{u})$ over $\mathfrak{u}$, there exists a sequence $(\mathfrak{u}^k(t,z))_{k\geq 1} \subset \mathfrak{U}$ such that

$$\sup_{k\geq 1} J\left(t, z, \mathfrak{u}^k(t,z)\right) = K(t,z) \quad \mathbb{P}\text{-a.s.} \tag{3.2.8}$$

Set $\Delta^0_{t,z} := \emptyset$ and define inductively the $\mathcal{F}_t$-measurable sets

$$\Delta^k_{t,z} := \left\{J\left(t, z, \mathfrak{u}^k(t,z)\right) \geq K(t,z) - \varepsilon\right\} \setminus \bigcup_{j=0}^{k-1} \Delta^j_{t,z}, \quad k \geq 1.$$

By (3.2.8), the family $\{\Delta^k_{t,z}, k \geq 1\}$ forms a partition of $\Omega$. Clearly, each $\Delta^k_{t,z}$ (seen as a constant family) satisfies the requirement of (C4), since it does not depend on $\nu$, and therefore belongs to $\mathfrak{F}_t$. Hence, after fixing some $\mathfrak{u}_0 \in \mathfrak{U}$, (C2) implies that

$$\mu^\varepsilon(t,z) := \mathfrak{u}_0 \oplus_t \sum_{k\geq 1} \mathfrak{u}^k(t,z)\mathbf{1}_{\Delta^k_{t,z}} \in \mathfrak{U},$$

while (Z1) ensures that

$$\begin{aligned} J(t,z,\mu^\varepsilon(t,z)) &= \operatorname*{ess\,inf}_{\nu\in\mathcal{V}} \mathbb{E}\left[\ell\left(Z_{t,z}^{\mu^\varepsilon(t,z),\nu}(T)\right)|\mathcal{F}_t\right] \\ &= \operatorname*{ess\,inf}_{\nu\in\mathcal{V}} \mathbb{E}\left[\sum_{k\geq 1}\ell\left(Z_{t,z}^{\mathfrak{u}^k(t,z),\nu}(T)\right)\mathbf{1}_{\Delta_{t,z}^k}|\mathcal{F}_t\right] \\ &= \operatorname*{ess\,inf}_{\nu\in\mathcal{V}} \sum_{k\geq 1}\mathbb{E}\left[\ell\left(Z_{t,z}^{\mathfrak{u}^k(t,z),\nu}(T)\right)|\mathcal{F}_t\right]\mathbf{1}_{\Delta_{t,z}^k}, \end{aligned}$$

where the last step used that $\Delta_{t,z}^k$ is $\mathcal{F}_t$-measurable. Since

$$\mathbb{E}\left[\ell\left(Z_{t,z}^{\mathfrak{u}^k(t,z),\nu}(T)\right)|\mathcal{F}_t\right] \geq J(t,z,\mathfrak{u}^k(t,z))$$

by the definition of $J$, it follows by the definition of $\{\Delta_{t,z}^k, k\geq 1\}$ that

$$J(t,z,\mu^\varepsilon(t,z)) \geq \sum_{k\geq 1} J\left(t,z,\mathfrak{u}^k(t,z)\right)\mathbf{1}_{\Delta_{t,z}^k} \geq K(t,z)-\varepsilon \quad \mathbb{P}\text{-a.s.}$$

as required.

**Remark 3.2.4.** Let us mention that the GDP could also be formulated using families of submartingales $\{S^\nu, \nu\in\mathcal{V}\}$ rather than martingales. Namely, in (GDP1), these would be the processes defined by (3.2.7). However, such a formulation would not be advantageous for applications as in Section 3.3, because we would then need an additional control process to describe the (possibly very irregular) finite variation part of $S^\nu$. The fact that the martingales $\{M^\nu, \nu\in\mathcal{V}\}$ are actually sufficient to obtain a useful GDP can be explained heuristically as follows: the relevant situation for the dynamic programming equation corresponds to the adverse player choosing an (almost) optimal control $\nu$, and then the value process $S^\nu$ will be (almost) a martingale.

### 3.2.4 Proof of (GDP2)

In the sequel, we fix $(t,z,p)\in[0,T]\times\mathcal{Z}\times\mathbb{R}$ and let $\iota>0$, $\mathfrak{u}\in\mathfrak{U}$, $\{M^\nu,\nu\in\mathcal{V}\}\in\mathfrak{M}_{t,p}$, $\{\tau^\nu,\nu\in\mathcal{V}\}\in\mathfrak{T}_t$ and $L_{t,z}^{\mathfrak{u},\nu}$ be as in (GDP2). We shall use the dyadic discretization for the stopping times $\tau^\nu$; that is, given $n\geq 1$, we set

$$\tau_n^\nu = \sum_{0\leq i\leq 2^n-1} t_{i+1}^n \mathbf{1}_{(t_i^n,t_{i+1}^n]}(\tau^\nu), \quad \text{where} \quad t_i^n = i2^{-n}T \quad \text{for} \quad 0\leq i\leq 2^n.$$

We shall first state the proof under the additional assumption that

$$M^\nu(\cdot) = M^\nu(\cdot\wedge\tau^\nu) \quad \text{for all } \nu\in\mathcal{V}. \tag{3.2.9}$$

**Step 1:** *Fix $\varepsilon > 0$ and $n \geq 1$. There exists $\mathfrak{u}_n^\varepsilon \in \mathfrak{U}$ such that*

$$\mathbb{E}\left[\ell\left(Z_{t,z}^{\mathfrak{u}_n^\varepsilon,\nu}(T)\right)|\mathcal{F}_{\tau_n^\nu}\right] \geq K\left(\tau_n^\nu, Z_{t,z}^{\mathfrak{u},\nu}(\tau_n^\nu)\right) - \varepsilon \quad \mathbb{P}\text{-a.s. for all } \nu \in \mathcal{V}.$$

We fix $\varepsilon > 0$ and $n \geq 1$. It follows from (R1) and (C2) that, for each $i \leq 2^n$, we can find a Borel partition $(B_{ij})_{j\geq 1}$ of $\mathcal{Z}$ and a sequence $(z_{ij})_{j\geq 1} \subset \mathcal{Z}$ such that, for all $\bar{\mathfrak{u}} \in \mathfrak{U}$ and $\nu \in \mathcal{V}$,

$$\begin{aligned} \mathbb{E}\left[\ell\left(Z_{t,z}^{\mathfrak{u}\oplus_{t_i^n}\bar{\mathfrak{u}},\nu}(T)\right)|\mathcal{F}_{t_i^n}\right](\omega) &\geq I(t_i^n, z_{ij}, \mathfrak{u}\oplus_{t_i^n}\bar{\mathfrak{u}}, \nu)(\omega) - \varepsilon \quad \text{and} && (3.2.10)\\ K\left(t_i^n, z_{ij}\right) &\geq K\left(t_i^n, Z_{t,z}^{\mathfrak{u},\nu}(t_i^n)(\omega)\right) - \varepsilon && (3.2.11)\\ &\text{for } \mathbb{P}\text{-a.e. } \omega \in C_{ij}^\nu := \{Z_{t,z}^{\mathfrak{u},\nu}(t_i^n) \in B_{ij}\}. \end{aligned}$$

Let $\mu^\varepsilon$ be as in Lemma 3.2.1, $\mathfrak{u}_{ij}^\varepsilon := \mu^\varepsilon(t_i^n, z_{ij})$ and $A_{ij}^\nu := C_{ij}^\nu \cap \{\tau_n^\nu = t_i^n\}$, and consider the mapping

$$\nu \mapsto \mathfrak{u}_n^\varepsilon[\nu] := \mathfrak{u}[\nu] \oplus_{\tau_n^\nu} \sum_{j\geq 1, i\leq n} \mathfrak{u}_{ij}^\varepsilon[\nu]\mathbf{1}_{A_{ij}^\nu}.$$

Note that (Z2) and (C4) imply that $\{C_{ij}^\nu, \nu \in \mathcal{V}\}_{j\geq 1} \subset \mathfrak{F}_{t_i^n}$ for each $i \leq 2^n$. Similarly, it follows from (C6) and the definition of $\tau_n^\nu$ that the families $\{\{\tau_n^\nu = t_i^n\}, \nu \in \mathcal{V}\}$ and $\{\{\tau_n^\nu = t_i^n\}^c, \nu \in \mathcal{V}\}$ belong to $\mathfrak{F}_{t_i^n}$. Therefore, an induction argument based on (C2) yields that $\mathfrak{u}_n^\varepsilon \in \mathfrak{U}$. Using successively (3.2.10), (Z1), the definition of $J$, Lemma 3.2.1 and (3.2.11), we deduce that for $\mathbb{P}$-a.e. $\omega \in A_{ij}^\nu$,

$$\begin{aligned} \mathbb{E}\left[\ell\left(Z_{t,z}^{\mathfrak{u}_n^\varepsilon,\nu}(T)\right)|\mathcal{F}_{\tau_n^\nu}\right](\omega) &\geq I\left(t_i^n, z_{ij}, \mathfrak{u}_{ij}^\varepsilon, \nu\right)(\omega) - \varepsilon\\ &\geq J\left(t_i^n, z_{ij}, \mu^\varepsilon(t_i^n, z_{ij})\right)(\omega) - \varepsilon\\ &\geq K\left(t_i^n, z_{ij}\right) - 2\varepsilon\\ &\geq K\left(t_i^n, Z_{t,z}^{\mathfrak{u},\nu}(t_i^n)(\omega)\right) - 3\varepsilon\\ &= K\left(\tau_n^\nu(\omega), Z_{t,z}^{\mathfrak{u},\nu}(\tau_n^\nu)(\omega)\right) - 3\varepsilon. \end{aligned}$$

As $\varepsilon > 0$ was arbitrary and $\cup_{i,j} A_{ij}^\nu = \Omega$ $\mathbb{P}$-a.s., this proves the claim.

**Step 2:** *Fix $\varepsilon > 0$ and $n \geq 1$. For all $\nu \in \mathcal{V}$, we have*

$$\mathbb{E}\left[\ell\left(Z_{t,z}^{\mathfrak{u}_n^\varepsilon,\nu}(T)\right)|\mathcal{F}_{\tau_n^\nu}\right](\omega) \geq M^\nu(\tau_n^\nu)(\omega) - \varepsilon \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in E_n^\nu,$$

*where*

$$E_n^\nu := \left\{\left(\tau_n^\nu, Z_{t,z}^{\mathfrak{u},\nu}(\tau_n^\nu), M^\nu(\tau_n^\nu)\right) \in B_\iota\left(\tau^\nu, Z_{t,z}^{\mathfrak{u},\nu}(\tau^\nu), M^\nu(\tau^\nu)\right)\right\}.$$

Indeed, since $\left(Z_{t,z}^{\mathfrak{u},\nu}(\tau^\nu), M^\nu(\tau^\nu)\right) \in \mathring{\Lambda}_\iota(\tau^\nu)$ $\mathbb{P}$-a.s., the definition of $\mathring{\Lambda}_\iota$ entails that $\left(Z_{t,z}^{\mathfrak{u},\nu}(\tau_n^\nu), M^\nu(\tau_n^\nu)\right) \in \Lambda(\tau_n^\nu)$ for $\mathbb{P}$-a.e. $\omega \in E_n^\nu$. This, in turn, means that

$$K\left(\tau_n^\nu(\omega), Z_{t,z}^{\mathfrak{u},\nu}(\tau_n^\nu)(\omega)\right) \geq M^\nu(\tau_n^\nu)(\omega) \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in E_n^\nu.$$

Now the claim follows from Step 1. (In all this, we actually have $M^\nu(\tau_n^\nu) = M^\nu(\tau^\nu)$ by (3.2.9), a fact we do not use here.)

**Step 3:** *Let* $L^\nu := L_{t,z}^{\mathfrak{u},\nu}$ *be the process from* (I2). *Then*

$$K(t,z) \geq p - \varepsilon - \sup_{\nu\in\mathcal{V}} \mathbb{E}\left[(L^\nu(\tau_n^\nu) - M^\nu(\tau_n^\nu))^- \mathbf{1}_{(E_n^\nu)^c}\right].$$

Indeed, it follows from Step 2 and (I2) that

$$\begin{aligned}
& \mathbb{E}\left[\ell\left(Z_{t,z}^{\mathfrak{u}_n^\varepsilon,\nu}(T)\right)|\mathcal{F}_t\right] \\
\geq\; & \mathbb{E}\left[M^\nu(\tau_n^\nu)\mathbf{1}_{E_n^\nu}|\mathcal{F}_t\right] - \varepsilon + \mathbb{E}\left[\mathbb{E}\left[\ell\left(Z_{t,z}^{\mathfrak{u}_n^\varepsilon,\nu}(T)\right)|\mathcal{F}_{\tau_n^\nu}\right]\mathbf{1}_{(E_n^\nu)^c}|\mathcal{F}_t\right] \\
\geq\; & \mathbb{E}\left[M^\nu(\tau_n^\nu)|\mathcal{F}_t\right] - \mathbb{E}\left[M^\nu(\tau_n^\nu)\mathbf{1}_{(E_n^\nu)^c}|\mathcal{F}_t\right] - \varepsilon + \mathbb{E}\left[L^\nu(\tau_n^\nu)\mathbf{1}_{(E_n^\nu)^c}|\mathcal{F}_t\right] \\
=\; & p - \varepsilon + \mathbb{E}\left[\left(L^\nu(\tau_n^\nu) - M^\nu(\tau_n^\nu)\right)\mathbf{1}_{(E_n^\nu)^c}|\mathcal{F}_t\right].
\end{aligned}$$

By the definitions of $K$ and $J$, we deduce that

$$\begin{aligned}
K(t,z) &\geq J(t,z,\mathfrak{u}_n^\varepsilon) \\
&\geq p - \varepsilon + \operatorname*{ess\,inf}_{\nu\in\mathcal{V}} \mathbb{E}\left[\left(L^\nu(\tau_n^\nu) - M^\nu(\tau_n^\nu)\right)\mathbf{1}_{(E_n^\nu)^c}|\mathcal{F}_t\right].
\end{aligned}$$

Since $K$ is deterministic, we can take expectations on both sides to obtain that

$$K(t,z) \geq p - \varepsilon + \mathbb{E}\left[\operatorname*{ess\,inf}_{\nu\in\mathcal{V}} \mathbb{E}\left[Y^\nu|\mathcal{F}_t\right]\right], \quad \text{where } Y^\nu := \left(L^\nu(\tau_n^\nu) - M^\nu(\tau_n^\nu)\right)\mathbf{1}_{(E_n^\nu)^c}.$$

The family $\{\mathbb{E}[Y^\nu|\mathcal{F}_t], \nu \in \mathcal{V}\}$ is directed downward; to see this, use (C1), (Z2), (Z3), (C5) and the last statement in (I2), and argue as in Step 1 of the proof of (GDP1) in Section 3.2.3. It then follows that we can find a sequence $(\nu_k)_{k\geq 1} \subset \mathcal{V}$ such that $\mathbb{E}[Y^{\nu_k}|\mathcal{F}_t]$ decreases $\mathbb{P}$-a.s. to $\operatorname{ess\,inf}_{\nu\in\mathcal{V}} \mathbb{E}[Y^\nu|\mathcal{F}_t]$, cf. [Nev75, Proposition VI-1-1], so that the claim follows by monotone convergence.

**Step 4:** *We have*

$$\lim_{n\to\infty} \sup_{\nu\in\mathcal{V}} \mathbb{E}\left[(L^\nu(\tau_n^\nu) - M^\nu(\tau_n^\nu))^- 1_{(E_n^\nu)^c}\right] = 0 \quad \mathbb{P}\text{-}a.s.$$

Indeed, since $M^\nu(\tau_n^\nu) = M^\nu(\tau^\nu)$ by (3.2.9), the uniform integrability assumptions in Theorem 3.2.1 yield that $\{(L^\nu(\tau_n^\nu) - M^\nu(\tau_n^\nu))^- : n \geq 1, \nu \in \mathcal{V}\}$ is again uniformly integrable. Therefore, it suffices to prove that $\sup_{\nu\in\mathcal{V}} \mathbb{P}[(E_n^\nu)^c] \to 0$. To see this, note that for $n$ large enough, we have $|\tau_n^\nu - \tau^\nu| \leq 2^{-n}T \leq \iota/2$ and hence

$$\mathbb{P}[(E_n^\nu)^c] \leq \mathbb{P}\left[d_{\mathcal{Z}}\big(Z_{t,z}^{\mathfrak{u},\nu}(\tau_n^\nu), Z_{t,z}^{\mathfrak{u},\nu}(\tau^\nu)\big) \geq \iota/2\right],$$

where we have used that $M^\nu(\tau_n^\nu) = M^\nu(\tau^\nu)$. Using once more that $|\tau_n^\nu - \tau^\nu| \leq 2^{-n}T$, the claim then follows from (R2).

**Step 5:** *The additional assumption* (3.2.9) *entails no loss of generality.*

Indeed, let $\tilde{M}^\nu$ be the stopped martingale $M^\nu(\cdot \wedge \tau^\nu)$. Then $\{\tilde{M}^\nu, \nu \in \mathcal{V}\} \subset \mathcal{M}_{t,p}$. Moreover, since $\{M^\nu, \nu \in \mathcal{V}\} \in \mathfrak{M}_{t,p}$ and $\{\tau^\nu, \nu \in \mathcal{V}\} \in \mathfrak{T}_t$, we see from (Z3) and (C5) that $\{\tilde{M}^\nu, \nu \in \mathcal{V}\}$ again satisfies the property stated in (Z3). Finally, we have that the set $\{\tilde{M}^\nu(\tau^\nu)^+ : \nu \in \mathcal{V}\}$ is uniformly integrable like $\{M^\nu(\tau^\nu)^+ : \nu \in \mathcal{V}\}$, since these sets coincide. Hence, $\{\tilde{M}^\nu, \nu \in \mathcal{V}\}$ satisfies all properties required in (GDP2), and of course also (3.2.9). To be precise, it is not necessarily the case that $\{\tilde{M}^\nu, \nu \in \mathcal{V}\} \in \mathfrak{M}_{t,p}$; in fact, we have made no assumption whatsoever about the richness of $\mathfrak{M}_{t,p}$. However, the previous properties are all we have used in this proof and hence, we may indeed replace $M^\nu$ by $\tilde{M}^\nu$ for the purpose of proving (GDP2).

We can now conclude the proof of (GDP2): in view of Step 4, Step 3 yields that $K(t,z) \geq p - \varepsilon$, which by Lemma 3.2.1 implies the assertion that $(z, p-\varepsilon) \in \Lambda(t)$.
□

### 3.2.5 Proof of Corollary 3.2.3

**Step 1:** *Assume that $\ell$ is bounded and Lipschitz continuous. Then* (I) *and* (R1) *are satisfied.*

Assumption (I) is trivially satisfied; we prove that (3.2.5) implies Assumption (R1). Let $t \leq s \leq T$ and $(\mathfrak{u}, \nu) \in \mathfrak{U} \times \mathcal{V}$. Let $c$ be the Lipschitz constant of $\ell$. By (3.2.5), we have

$$\begin{aligned} \left|\mathbb{E}\left[\ell\left(Z^{\mathfrak{u},\nu}_{t,z}(T)\right) - \ell\left(Z^{\mathfrak{u},\nu}_{s,z'}(T)\right)|\mathcal{F}_s\right]\right| &\leq c\mathbb{E}\left[\left|Z^{\mathfrak{u},\nu}_{t,z}(T) - Z^{\mathfrak{u},\nu}_{s,z'}(T)\right||\mathcal{F}_s\right] \\ &\leq cC\left|Z^{\mathfrak{u},\nu}_{t,z}(s) - z'\right| \end{aligned} \tag{3.2.12}$$

for all $z, z' \in \mathbb{R}^d$. Let $(B_j)_{j\geq 1}$ be any Borel partition of $\mathbb{R}^d$ such that the diameter of $B_j$ is less than $\varepsilon/(cC)$, and let $z_j \in B_j$ for each $j \geq 1$. Then

$$\left|\mathbb{E}\left[\ell\left(Z^{\mathfrak{u},\nu}_{t,z}(T)\right) - \ell\left(Z^{\mathfrak{u},\nu}_{s,z_j}(T)\right)|\mathcal{F}_s\right]\right| \leq \varepsilon \quad \text{on } C^{\mathfrak{u},\nu}_j := \left\{Z^{\mathfrak{u},\nu}_{t,z}(s) \in B_j\right\},$$

which implies the first property in (R1). In particular, let $\bar{\nu} \in \mathcal{V}$, then using (C1), we have

$$\left|\mathbb{E}\left[\ell\left(Z^{\mathfrak{u},\nu\oplus_s\bar{\nu}}_{t,z}(T)\right) - \ell\left(Z^{\mathfrak{u},\nu\oplus_s\bar{\nu}}_{s,z_j}(T)\right)|\mathcal{F}_s\right]\right| \leq \varepsilon \quad \text{on } C^{\mathfrak{u},\nu\oplus_s\bar{\nu}}_j.$$

Since $C^{\mathfrak{u},\nu\oplus_s\bar{\nu}}_j = C^{\mathfrak{u},\nu}_j$ by (Z2), we may take the essential infimum over $\bar{\nu} \in \mathcal{V}$ to conclude that

$$\operatorname*{ess\,inf}_{\bar{\nu}\in\mathcal{V}} \mathbb{E}\left[\ell\left(Z^{\mathfrak{u},\nu\oplus_s\bar{\nu}}_{t,z}(T)\right)|\mathcal{F}_s\right] \leq J(s, z_j, \mathfrak{u}[\nu \oplus_s \cdot]) + \varepsilon \quad \text{on } C^{\mathfrak{u},\nu}_j,$$

which is the second property in (R1). Finally, the last property in (R1) is a direct consequence of (3.2.12) applied with $t = s$.

**Step 2:** We now prove the corollary under the additional assumption that $|\ell(z)| \leq C$; we shall reduce to the Lipschitz case by inf-convolution. Indeed, if we define the functions $\ell_k$ by

$$\ell_k(z) = \inf_{z' \in \mathbb{R}^d} \{\ell(z') + k|z' - z|\}, \quad k \geq 1,$$

then $\ell_k$ is Lipschitz continuous with Lipschitz constant $k$, $|\ell_k| \leq C$, and $(\ell_k)_{k\geq 1}$ converges pointwise to $\ell$. Since $\ell$ is continuous and the sequence $(\ell_k)_{k\geq 1}$ is monotone increasing, the convergence is uniform on compact sets by Dini's lemma. That is, for all $n \geq 1$,

$$\sup_{z\in\mathbb{R}^d,\, |z|\leq n} |\ell_k(z) - \ell(z)| \leq \epsilon_k^n, \tag{3.2.13}$$

where $(\epsilon_k^n)_{k\geq 1}$ is a sequence of numbers such that $\lim_{k\to\infty} \epsilon_k^n = 0$. Moreover, (3.2.4) combined with Chebyshev's inequality implies that

$$\operatorname*{ess\,sup}_{(\mathfrak{u},\nu)\in\mathfrak{U}\times\mathcal{V}} \mathbb{P}\left[|Z_{t,z}^{\mathfrak{u},\nu}(T)| \geq n|\mathcal{F}_t\right] \leq (\varrho(z)/n)^{\bar{q}}. \tag{3.2.14}$$

Combining (3.2.13) and (3.2.14) and using the fact that $\ell_k - \ell$ is bounded by $2C$ then leads to

$$\operatorname*{ess\,sup}_{(\mathfrak{u},\nu)\in\mathfrak{U}\times\mathcal{V}} \mathbb{E}\left[\left|\ell_k\left(Z_{t,z}^{\mathfrak{u},\nu}(T)\right) - \ell\left(Z_{t,z}^{\mathfrak{u},\nu}(T)\right)\right| \; |\mathcal{F}_t\right] \leq \epsilon_k^n + 2C(\varrho(z)/n)^{\bar{q}}. \tag{3.2.15}$$

Let $O$ be a bounded subset of $\mathbb{R}^d$, let $\eta > 0$, and let

$$I_k(t,z,\mathfrak{u},\nu) = \mathbb{E}\left[\ell_k\left(Z_{t,z}^{\mathfrak{u},\nu}(T)\right)|\mathcal{F}_t\right]. \tag{3.2.16}$$

Then we can choose an integer $n_O^\eta$ such that $2C(\varrho(z)/n_O^\eta)^{\bar{q}} \leq \eta/2$ for all $z \in O$ and another integer $k_O^\eta$ such that $\epsilon_{k_O^\eta}^{n_O^\eta} \leq \eta/2$. Under these conditions, (3.2.15) applied to $n = n_O^\eta$ yields that

$$\operatorname*{ess\,sup}_{(\mathfrak{u},\nu)\in\mathfrak{U}\times\mathcal{V}} \left|I_{k_O^\eta}(t,z,\mathfrak{u},\nu) - I(t,z,\mathfrak{u},\nu)\right| \leq \eta \quad \text{for } (t,z) \in [0,T]\times O. \tag{3.2.17}$$

In the sequel, we fix $(t,z,p) \in [0,T]\times\mathbb{R}^d\times\mathbb{R}$ and a bounded set $O \subset \mathbb{R}^d$ containing $z$, and define $J_{k_O^\eta}$, $\Lambda_{k_O^\eta}$, $\mathring{\Lambda}_{k_O^\eta,\iota}$ and $\bar{\Lambda}_{k_O^\eta}$ in terms of $\ell_{k_O^\eta}$ instead of $\ell$.

We now prove (GDP1'). To this end, suppose that $(z, p+2\eta) \in \Lambda(t)$. Then (3.2.17) implies that $(z, p+\eta) \in \Lambda_{k_O^\eta}(t)$. In view of Step 1, we may apply (GDP1) with the loss function $\ell_{k_O^\eta}$ to obtain $\mathfrak{u} \in \mathfrak{U}$ and $\{M^\nu, \nu \in \mathcal{V}\} \subset \mathcal{M}_{t,p}$ such that

$$\left(Z_{t,z}^{\mathfrak{u},\nu}(\tau), M^\nu(\tau) + \eta\right) \in \bar{\Lambda}_{k_O^\eta}(\tau) \quad \mathbb{P}\text{-a.s. for all } \nu \in \mathcal{V} \text{ and } \tau \in \mathcal{T}_t.$$

Using once more (3.2.17), we deduce that

$$\left(Z^{\mathfrak{u},\nu}_{t,z}(\tau), M^{\nu}(\tau)\right) \in \bar{\Lambda}(\tau) \quad \mathbb{P}\text{-a.s. for all } \nu \in \mathcal{V} \text{ and } \tau \in \mathcal{T}_t;$$

i.e., (GDP1') holds for $\ell$. (The last argument was superfluous as $\ell \geq \ell_{k^{\eta}_O}$ already implies $\bar{\Lambda}_{k^{\eta}_O}(\tau) \subset \bar{\Lambda}(\tau)$; however, we would like to refer to this proof in a similar situation below where there is no monotonicity.)

It remains to prove (GDP2'). To this end, let $\iota > 0$, $\mathfrak{u} \in \mathfrak{U}$, $\{M^{\nu}, \nu \in \mathcal{V}\} \in \mathfrak{M}_{t,p}$ and $\{\tau^{\nu}, \nu \in \mathcal{V}\} \in \mathfrak{T}_t$ be such that

$$\left(Z^{\mathfrak{u},\nu}_{t,z}(\tau^{\nu}), M^{\nu}(\tau^{\nu})\right) \in \mathring{\Lambda}_{2\iota}(\tau^{\nu}) \quad \mathbb{P}\text{-a.s. for all } \nu \in \mathcal{V}.$$

For $\eta < \iota/2$, we then have

$$\left(Z^{\mathfrak{u},\nu}_{t,z}(\tau^{\nu}), M^{\nu}(\tau^{\nu}) + 2\eta\right) \in \mathring{\Lambda}_{\iota}(\tau^{\nu}) \quad \mathbb{P}\text{-a.s. for all } \nu \in \mathcal{V}. \tag{3.2.18}$$

Let $\tilde{M}^{\nu} := M^{\nu} + \eta$. Since $\{Z^{\mathfrak{u},\nu}_{t,z}(\tau^{\nu}), \nu \in \mathcal{V}\}$ is uniformly bounded in $L^{\infty}$, we may assume, by enlarging $O$ if necessary, that $B_{\iota}(Z^{\mathfrak{u},\nu}_{t,z}(\tau^{\nu})) \subset O$ $\mathbb{P}$-a.s. for all $\nu \in \mathcal{V}$. Then, (3.2.17) and (3.2.18) imply that

$$\left(Z^{\mathfrak{u},\nu}_{t,z}(\tau^{\nu}), \tilde{M}^{\nu}(\tau^{\nu})\right) \in \mathring{\Lambda}_{k^{\eta}_O,\iota}(\tau^{\nu}) \quad \mathbb{P}\text{-a.s. for all } \nu \in \mathcal{V}.$$

Moreover, as $\ell \leq C$, (3.2.18) implies that $\tilde{M}^{\nu}(\tau^{\nu}) \leq C$; in particular, $\{\tilde{M}^{\nu}(\tau^{\nu})^{+}, \nu \in \mathcal{V}\}$ is uniformly integrable. Furthermore, as $\ell \geq -C$, we can take $L^{\mathfrak{u},\nu}_{t,z} := -C$ for (I2). In view of Step 1, (GDP2) applied with the loss function $\ell_{k^{\eta}_O}$ then yields that

$$(z, p + \eta - \varepsilon) \in \Lambda_{k^{\eta}_O}(t) \quad \text{ for all } \varepsilon > 0. \tag{3.2.19}$$

To be precise, this conclusion would require that $\{\tilde{M}^{\nu}, \nu \in \mathcal{V}\} \in \mathfrak{M}_{t,p+\eta}$, which is not necessarily the case under our assumptions. However, since $\{M^{\nu}, \nu \in \mathcal{V}\} \in \mathfrak{M}_{t,p}$, it is clear that $\{\tilde{M}^{\nu}, \nu \in \mathcal{V}\}$ satisfies the property stated in (Z3), so that, as in Step 5 of the proof of (GDP2), there is no loss of generality in assuming that $\{\tilde{M}^{\nu}, \nu \in \mathcal{V}\} \in \mathfrak{M}_{t,p+\eta}$. We conclude by noting that (3.2.17) and (3.2.19) imply that $(z, p - \varepsilon) \in \Lambda(t)$ for all $\varepsilon > 0$.

**Step 3:** We turn to the general case. For $k \geq 1$, we now define $\ell_k := (\ell \wedge k) \vee (-k)$, while $I_k$ is again defined as in (3.2.16). We also set

$$n_k = \max\left\{m \geq 0 : B_m(0) \subset \{\ell = \ell_k\}\right\} \wedge k$$

and note that the continuity of $\ell$ guarantees that $\lim_{k\to\infty} n_k = \infty$. Given a bounded set $O \subset \mathbb{R}^d$ and $\eta > 0$, we claim that

$$\operatorname*{ess\,sup}_{(\mathfrak{u},\nu)\in\mathfrak{U}\times\mathcal{V}} \left| I_{k^{\eta}_O}(t,z,\mathfrak{u},\nu) - I(t,z,\mathfrak{u},\nu)\right| \leq \eta \quad \text{ for all } (t,z) \in [0,T] \times O \tag{3.2.20}$$

for any large enough integer $k_O^\eta$. Indeed, let $(\mathfrak{u},\nu)\in\mathfrak{U}\times\mathcal{V}$; then

$$\begin{aligned}|I_k(t,z,\mathfrak{u},\nu)-I(t,z,\mathfrak{u},\nu)| &\le \mathbb{E}\left[|\ell-\ell_k|\left(Z^{\mathfrak{u},\nu}_{t,z}(T)\right)|\mathcal{F}_t\right]\\ &= \mathbb{E}\left[|\ell-\ell_k|\left(Z^{\mathfrak{u},\nu}_{t,z}(T)\right)\mathbf{1}Z^{\mathfrak{u},\nu}_{t,z}(T)\notin\{\ell=\ell_k\}|\mathcal{F}_t\right]\\ &\le \mathbb{E}\left[\left|\ell\left(Z^{\mathfrak{u},\nu}_{t,z}(T)\right)\right|\mathbf{1}|Z^{\mathfrak{u},\nu}_{t,z}(T)|>n_k|\mathcal{F}_t\right]\\ &\le C\mathbb{E}\left[\left(1+|Z^{\mathfrak{u},\nu}_{t,z}(T)|^q\right)\mathbf{1}|Z^{\mathfrak{u},\nu}_{t,z}(T)|>n_k|\mathcal{F}_t\right]\end{aligned}$$

by (3.2.3). We may assume that $q>0$, as otherwise we are in the setting of Step 2. Pick $\delta>0$ such that $q(1+\delta)=\bar q$. Then Hölder's inequality and (3.2.4) yield that

$$\begin{aligned}&\mathbb{E}\left[\left|\left(Z^{\mathfrak{u},\nu}_{t,z}(T)\right)\right|^q\mathbf{1}|Z^{\mathfrak{u},\nu}_{t,z}(T)|>n_k|\mathcal{F}_t\right]\\ \le\;& \mathbb{E}\left[\left|\left(Z^{\mathfrak{u},\nu}_{t,z}(T)\right)\right|^{\bar q}|\mathcal{F}_t\right]^{\frac{1}{1+\delta}}\mathbb{P}\left[|Z^{\mathfrak{u},\nu}_{t,z}(T)|>n_k|\mathcal{F}_t\right]^{\frac{\delta}{1+\delta}}\\ \le\;& \rho(z)^{\frac{\bar q}{1+\delta}}\left(\rho(z)/n_k\right)^{\frac{\bar q\delta}{1+\delta}}.\end{aligned}$$

Since $\rho$ is locally bounded and $\lim_{k\to\infty}n_k=\infty$, the claim (3.2.20) follows. We can then obtain (GDP1') and (GDP2') by reducing to the result of Step 2, using the same arguments as in the proof of Step 2. □

## 3.3 The PDE in the case of a controlled SDE

In this section, we illustrate how our GDP can be used to derive a dynamic programming equation and how its assumptions can be verified in a typical setup. To this end, we focus on the case where the state process is determined by a stochastic differential equation with controlled coefficients; however, other examples could be treated similarly.

### 3.3.1 Setup

Let $\Omega=C([0,T];\mathbb{R}^d)$ be the canonical space of continuous paths equipped with the Wiener measure $\mathbb{P}$, let $\mathbb{F}=(\mathcal{F}_t)_{t\le T}$ be the $\mathbb{P}$-augmentation of the filtration generated by the coordinate-mapping process $W$, and let $\mathcal{F}=\mathcal{F}_T$. We define $\mathcal{V}$, the set of adverse controls, to be the set of all progressively measurable processes with values in a compact subset $V$ of $\mathbb{R}^d$. Similarly, $\mathcal{U}$ is the set of all progressively measurable processes with values in a compact $U\subset\mathbb{R}^d$. Finally, the set of strategies $\mathfrak{U}$ consists of all mappings $\mathfrak{u}:\mathcal{V}\to\mathcal{U}$ which are non-anticipating in the sense that

$$\{\nu_1=_{(0,s]}\nu_2\}\subset\{\mathfrak{u}[\nu_1]=_{(0,s]}\mathfrak{u}[\nu_2]\}\quad\text{for all }\nu_1,\nu_2\in\mathcal{V}\text{ and }s\le T.$$

Given $(t,z)\in[0,T]\times\mathbb{R}^d$ and $(\mathfrak{u},\nu)\in\mathfrak{U}\times\mathcal{V}$, we let $Z^{\mathfrak{u},\nu}_{t,z}$ be the unique strong solution of the controlled SDE

$$Z(s)=z+\int_t^s\mu(Z(r),\mathfrak{u}[\nu]_r,\nu_r)\,dr+\int_t^s\sigma(Z(r),\mathfrak{u}[\nu]_r,\nu_r)\,dW_r,\quad s\in[t,T],\tag{3.3.1}$$

where the coefficients

$$\mu : \mathbb{R}^d \times U \times V \to \mathbb{R}^d, \quad \sigma : \mathbb{R}^d \times U \times V \to \mathbb{R}^{d\times d}$$

are assumed to be jointly continuous in all three variables as well as Lipschitz continuous with linear growth in the first variable, uniformly in the two last ones. Throughout this section, we assume that $\ell : \mathbb{R}^d \to \mathbb{R}$ is a continuous function of polynomial growth; i.e., (3.2.3) holds true for some constants $C$ and $q$. Since $Z^{\mathfrak{u},\nu}_{t,z}(T)$ has moments of all orders, this implies that the finiteness condition (3.2.1) is satisfied.

In view of the martingale representation theorem, we can identify the set $\mathcal{M}_{t,p}$ of martingales with the set $\mathcal{A}$ of all progressively measurable $d$-dimensional processes $\alpha$ such that $\int \alpha\, dW$ is a (true) martingale. Indeed, we have $\mathcal{M}_{t,p} = \{P^\alpha_{t,p}, \alpha \in \mathcal{A}\}$, where

$$P^\alpha_{t,p}(\cdot) = p + \int_t^\cdot \alpha_s\, dW_s.$$

We shall denote by $\mathfrak{A}$ the set of all mappings $\mathfrak{a}[\cdot]\colon \mathcal{V} \mapsto \mathcal{A}$ such that

$$\{\nu_1 =_{(0,s]} \nu_2\} \subset \{\mathfrak{a}[\nu_1] =_{(0,s]} \mathfrak{a}[\nu_2]\} \quad \text{for all } \nu_1, \nu_2 \in \mathcal{V} \text{ and } s \leq T.$$

The set of all families $\{P^{\mathfrak{a}[\nu]}_{t,p}, \nu \in \mathcal{V}\}$ with $\mathfrak{a} \in \mathfrak{A}$ then forms the set $\mathfrak{M}_{t,p}$, for any given $(t,p) \in [0,T]\times\mathbb{R}$. Furthermore, $\mathfrak{T}_t$ consists of all families $\{\tau^\nu, \nu \in \mathcal{V}\} \subset \mathcal{T}_t$ such that, for some $(z,p) \in \mathbb{R}^d\times\mathbb{R}$, $(\mathfrak{u},\mathfrak{a}) \in \mathfrak{U}\times\mathfrak{A}$ and some Borel set $O \subset [0,T]\times\mathbb{R}^d\times\mathbb{R}$,

$$\tau^\nu \text{ is the first exit time of } \left(\cdot, Z^{\mathfrak{u},\nu}_{t,z}, P^{\mathfrak{a}[\nu]}_{t,p}\right) \text{ from } O, \quad \text{for all } \nu \in \mathcal{V}.$$

(This includes the deterministic times $s \in [t,T]$ by the choice $O = [0,s] \times \mathbb{R}^d \times \mathbb{R}$.) Finally, $\mathfrak{F}_t$ consists of all families $\{A^\nu, \nu \in \mathcal{V}\} \subset \mathcal{F}_t$ such that

$$A^{\nu_1} \cap \{\nu_1 =_{(0,t]} \nu_2\} \;=\; A^{\nu_2} \cap \{\nu_1 =_{(0,t]} \nu_2\} \quad \text{for all } \nu_1, \nu_2 \in \mathcal{V}.$$

**Proposition 3.3.1.** *The conditions of Corollary 3.2.3 are satisfied in the present setup.*

**Proof.** The above definitions readily yield that Assumptions (C) and (Z1)–(Z3) are satisfied. Moreover, Assumption (Z4) can be verified exactly as in [BL08, Proposition 3.3]. Fix any $\bar{q} > q \vee 2$; then (3.2.4) can be obtained as follows. Let $(\mathfrak{u},\nu) \in \mathfrak{U} \times \mathcal{V}$ and $A \in \mathcal{F}_t$ be arbitrary. Using the Burkholder-Davis-Gundy inequalities, the boundedness of $U$ and $V$, and the assumptions on $\mu$ and $\sigma$, we obtain that

$$\mathbb{E}\left[\sup_{t\leq s\leq\tau} \left|Z^{\mathfrak{u},\nu}_{t,z}(s)\right|^{\bar{q}} \mathbf{1}_A\right] \leq c\mathbb{E}\left[1 + |z|^{\bar{q}}\,\mathbf{1}_A + \int_t^\tau \sup_{t\leq s\leq r} \left|Z^{\mathfrak{u},\nu}_{t,z}(s)\right|^{\bar{q}} \mathbf{1}_A\, dr\right],$$

where $c$ is a universal constant and $\tau$ is any stopping time such that $Z^{\mathfrak{u},\nu}_{t,z}(\cdot \wedge \tau)$ is bounded. Applying Gronwall's inequality and letting $\tau \to T$, we deduce that

$$\mathbb{E}\left[\left|Z^{\mathfrak{u},\nu}_{t,z}(T)\right|^{\bar{q}}\mathbf{1}_A\right] \leq \mathbb{E}\left[\sup_{t\leq u\leq T}\left|Z^{\mathfrak{u},\nu}_{t,z}(u)\right|^{\bar{q}}\mathbf{1}_A\right] \leq c\mathbb{E}\left[1+|z|^{\bar{q}}\mathbf{1}_A\right].$$

Since $A \in \mathcal{F}_t$ was arbitrary, this implies (3.2.4). To verify the condition (3.2.5), we note that the flow property yields

$$\mathbb{E}\left[\left|Z^{\mathfrak{u}\oplus_s\bar{\mathfrak{u}},\nu\oplus_s\bar{\nu}}_{t,z}(T) - Z^{\bar{\mathfrak{u}},\nu\oplus_s\bar{\nu}}_{s,z'}(T)\right|\mathbf{1}_A\right] = \mathbb{E}\left[\left|Z^{\bar{\mathfrak{u}},\nu\oplus_s\bar{\nu}}_{s,Z^{\mathfrak{u},\nu}_{t,z}(s)}(T) - Z^{\bar{\mathfrak{u}},\nu\oplus_s\bar{\nu}}_{s,z'}(T)\right|\mathbf{1}_A\right]$$

and estimate the right-hand side with the above arguments. Finally, the same arguments can be used to verify (R2).

**Remark 3.3.2.** We emphasize that our definition of a strategy $\mathfrak{u} \in \mathfrak{U}$ does not include regularity assumptions on the mapping $\nu \mapsto \mathfrak{u}[\nu]$. This is in contrast to [BY11], where a continuity condition is imposed, enabling the authors to deal with the selection problem for strategies in the context of a stochastic differential game and use the traditional formulation of the value functions in terms of infima (not essential infima) and suprema. Let us mention, however, that such regularity assumptions may preclude existence of optimal strategies in concrete examples (see also Remark 3.4.2).

### 3.3.2 PDE for the reachability set $\Lambda$

In this section, we show how the PDE for the reachability set $\Lambda$ from (3.2.2) can be deduced from the geometric dynamic programming principle of Corollary 3.2.3. This equation is stated in terms of the indicator function of the complement of the graph of $\Lambda$,

$$\chi(t,z,p) := 1 - \mathbf{1}_{\Lambda(t)}(z,p) = \begin{cases} 0 & \text{if } (z,p) \in \Lambda(t) \\ 1 & \text{otherwise,} \end{cases}$$

and its lower semicontinuous envelope

$$\chi_*(t,z,p) := \liminf_{(t',z',p')\to(t,z,p)} \chi(t',z',p').$$

Corresponding results for the case without adverse player have been obtain in [BET09, ST02c]; we extend their arguments to account for the presence of $\nu$ and the fact that we only have a relaxed GDP. We begin by rephrasing Corollary 3.2.3 in terms of $\chi$.

**Lemma 3.3.1.** *Fix* $(t,z,p) \in [0,T] \times \mathbb{R}^d \times \mathbb{R}$.

**(GDP1$_\chi$)** *Assume that $\chi(t,z,p+\varepsilon)=0$ for some $\varepsilon>0$. Then there exist $\mathfrak{u}\in\mathfrak{U}$ and $\{\alpha^\nu,\nu\in\mathcal{V}\}\subset\mathcal{A}$ such that*

$$\chi_*\left(\tau, Z^{\mathfrak{u},\nu}_{t,z}(\tau), P^{\alpha^\nu}_{t,p}(\tau)\right)=0 \quad \mathbb{P}\text{-a.s. for all } \nu\in\mathcal{V} \text{ and } \tau\in\mathcal{T}_t.$$

**(GDP2$_\chi$)** *Let $\varphi$ be a continuous function such that $\varphi\geq\chi$ and let $O\subset[0,T]\times\mathbb{R}^d\times\mathbb{R}$ be a bounded open set containing $(t,z,p)$. Let $(\mathfrak{u},\mathfrak{a})\in\mathfrak{U}\times\mathfrak{A}$ and $\eta>0$ be such that*

$$\varphi\left(\tau^\nu, Z^{\mathfrak{u},\nu}_{t,z}(\tau^\nu), P^{\mathfrak{a}[\nu]}_{t,p}(\tau^\nu)\right)\leq 1-\eta \quad \mathbb{P}\text{-a.s. for all } \nu\in\mathcal{V}, \tag{3.3.2}$$

*where $\tau^\nu$ denotes the first exit time of $\left(\cdot, Z^{\mathfrak{u},\nu}_{t,z}, P^{\mathfrak{a}[\nu]}_{t,p}\right)$ from $O$. Then $\chi(t,z,p-\varepsilon)=0$ for all $\varepsilon>0$.*

**Proof.** After observing that $(z,p+\varepsilon)\in\Lambda(t)$ if and only if $\chi(t,z,p+\varepsilon)=0$ and that $(z,p)\in\bar{\Lambda}(t)$ implies $\chi_*(t,z,p)=0$, (GDP1$_\chi$) follows from Corollary 3.2.3, whose conditions are satisfied by Proposition 3.3.1. We now prove (GDP2$_\chi$). Since $\varphi$ is continuous and $\partial O$ is compact, we can find $\iota>0$ such that

$$\varphi<1 \quad \text{on a } \iota\text{-neighborhood of } \partial O\cap\{\varphi\leq 1-\eta\}.$$

As $\chi\leq\varphi$, it follows that (3.3.2) implies

$$\left(Z^{\mathfrak{u},\nu}_{t,z}(\tau^\nu), M^\nu(\tau^\nu)\right)\in\mathring{\Lambda}_\iota(\tau^\nu) \quad \mathbb{P}\text{-a.s. for all } \nu\in\mathcal{V}.$$

Now Corollary 3.2.3 yields that $(z,p-\varepsilon)\in\Lambda(t)$; i.e., $\chi(t,z,p-\varepsilon)=0$.

Given a suitably differentiable function $\varphi=\varphi(t,z,p)$ on $[0,T]\times\mathbb{R}^{d+1}$, we shall denote by $\partial_t\varphi$ its derivative with respect to $t$ and by $D\varphi$ and $D^2\varphi$ the Jacobian and the Hessian matrix with respect to $(z,p)$, respectively. Given $u\in U$, $a\in\mathbb{R}^d$ and $v\in V$, we can then define the Dynkin operator

$$\mathcal{L}^{u,a,v}_{(Z,P)}\varphi:=\partial_t\varphi+\mu_{(Z,P)}(\cdot,u,v)^\top D\varphi+\frac{1}{2}\mathrm{Tr}\left[\sigma_{(Z,P)}\sigma^\top_{(Z,P)}(\cdot,u,a,v)D^2\varphi\right]$$

with coefficients

$$\mu_{(Z,P)}:=\begin{pmatrix}\mu\\0\end{pmatrix}, \quad \sigma_{(Z,P)}(\cdot,a,\cdot):=\begin{pmatrix}\sigma\\a\end{pmatrix}.$$

To introduce the associated relaxed Hamiltonians, we first define the relaxed kernel

$$\mathcal{N}_\varepsilon(z,q,v)=\left\{(u,a)\in U\times\mathbb{R}^d:\ \left|\sigma^\top_{(Z,P)}(z,u,a,v)q\right|\leq\varepsilon\right\}, \quad \varepsilon\geq 0$$

for $(t,z)\in[0,T]\times\mathbb{R}^d$, $q\in\mathbb{R}^{d+1}$ and $v\in V$, as well as the set $N_{Lip}(z,q)$ of all continuous functions

$$(\hat{u},\hat{a}):\ \mathbb{R}^d\times\mathbb{R}^{d+1}\times V\to U\times\mathbb{R}^d, \quad (z',q',v')\mapsto(\hat{u},\hat{a})(z',q',v')$$

that are locally Lipschitz continuous in $(z', q')$, uniformly in $v'$, and satisfy

$$(\hat{u}, \hat{a}) \in \mathcal{N}_0 \quad \text{on } B \times V, \quad \text{for some neighborhood } B \text{ of } (z, q).$$

The local Lipschitz continuity will be used to ensure the local wellposedness of the SDE for a Markovian strategy defined via $(\hat{u}, \hat{a})$. Setting

$$F(\Theta, u, a, v) := \left\{ -\mu_{(Z,P)}(z, u, v)^\top q - \frac{1}{2}\mathrm{Tr}\left[\sigma_{(Z,P)}\sigma_{(Z,P)}^\top(z, u, a, v)A\right]\right\}$$

for $\Theta = (z, q, A) \in \mathbb{R}^d \times \mathbb{R}^{d+1} \times \mathbb{S}^{d+1}$ and $(u, a, v) \in U \times \mathbb{R}^d \times V$, we can then define the relaxed Hamiltonians

$$H^*(\Theta) \;:=\; \inf_{v\in V} \limsup_{\varepsilon\searrow 0, \Theta'\to\Theta} \sup_{(u,a)\in\mathcal{N}_\varepsilon(\Theta',v)} F(\Theta', u, a, v), \tag{3.3.3}$$

$$H_*(\Theta) \;:=\; \sup_{(\hat{u},\hat{a})\in N_{Lip}(\Theta)} \inf_{v\in V} F(\Theta, \hat{u}(\Theta, v), \hat{a}(\Theta, v), v). \tag{3.3.4}$$

(In (3.3.4), it is not necessary to take the relaxation $\Theta' \to \Theta$ because $\inf_{v\in V} F$ is already lower semicontinuous.) The question whether $H^* = H_*$ is postponed to the monotone setting of the next section; see Remark 3.3.6.

We are now in the position to derive the PDE for $\chi$; in the following, we write $H^*\varphi(t, z, p)$ for $H^*(z, p, D\varphi(t, z, p), D^2\varphi(t, z, p))$, and similarly for $H_*$.

**Theorem 3.3.3.** *The function $\chi_*$ is a viscosity supersolution on $[0, T) \times \mathbb{R}^{d+1}$ of*

$$(-\partial_t + H^*)\varphi \geq 0.$$

*The function $\chi^*$ is a viscosity subsolution on $[0, T) \times \mathbb{R}^{d+1}$ of*

$$(-\partial_t + H_*)\varphi \leq 0.$$

**Proof.**

**Step 1:** *$\chi_*$ is a viscosity supersolution.*

Let $(t_o, z_o, p_o) \in [0, T) \times \mathbb{R}^d \times \mathbb{R}$ and let $\varphi$ be a smooth function such that

$$(\text{strict}) \min_{[0,T)\times\mathbb{R}^d\times\mathbb{R}} (\chi_* - \varphi) = (\chi_* - \varphi)(t_o, z_o, p_o) = 0. \tag{3.3.5}$$

We suppose that

$$(-\partial_t + H^*)\varphi(t_o, z_o, p_o) \leq -2\eta < 0 \tag{3.3.6}$$

for some $\eta > 0$ and work towards a contradiction. Using the continuity of $\mu$ and $\sigma$ and the definition of the upper-semicontinuous operator $H^*$, we can find $v_o \in V$ and $\varepsilon > 0$ such that

$$\begin{gathered} -\mathcal{L}^{u,a,v_o}_{(Z,P)}\varphi(t, z, p) \leq -\eta \\ \text{for all } (u, a) \in \mathcal{N}_\varepsilon(z, p, D\varphi(t, z, p), v_o) \text{ and } (t, z, p) \in B_\varepsilon, \end{gathered} \tag{3.3.7}$$

where $B_\varepsilon := B_\varepsilon(t_o, z_o, p_o)$ denotes the open ball of radius $\varepsilon$ around $(t_o, z_o, p_o)$. Let

$$\partial B_\varepsilon := \{t_o + \varepsilon\} \times B_\varepsilon(z_o, p_o) \cup [t_o, t_o + \varepsilon) \times \partial B_\varepsilon(z_o, p_o)$$

denote the parabolic boundary of $B_\varepsilon$ and set

$$\zeta := \min_{\partial B_\varepsilon}(\chi_* - \varphi).$$

In view of (3.3.5), we have $\zeta > 0$.

Next, we claim that there exists a sequence $(t_n, z_n, p_n, \varepsilon_n)_{n\geq 1} \subset B_\varepsilon \times (0,1)$ such that

$$(t_n, z_n, p_n, \varepsilon_n) \to (t_o, z_o, p_o, 0) \quad \text{and} \quad \chi(t_n, z_n, p_n + \varepsilon_n) = 0 \quad \text{for all } n \geq 1. \tag{3.3.8}$$

In view of $\chi \in \{0, 1\}$, it suffices to show that

$$\chi_*(t_o, z_o, p_o) = 0. \tag{3.3.9}$$

Suppose that $\chi_*(t_o, z_o, p_o) > 0$, then the lower semicontinuity of $\chi_*$ yields that $\chi_* > 0$ and therefore $\chi = 1$ on a neighborhood of $(t_o, z_o, p_o)$, which implies that $\varphi$ has a strict local maximum in $(t_o, z_o, p_o)$ and thus

$$\partial_t\varphi(t_o, z_o, p_o) \leq 0, \quad D\varphi(t_o, z_o, p_o) = 0, \quad D^2\varphi(t_o, z_o, p_o) \leq 0.$$

This clearly contradicts (3.3.7), and so the claim follows.

For any $n \geq 1$, the equality in (3.3.8) and (GDP1$_\chi$) of Lemma 3.3.1 yield $\mathfrak{u}^n \in \mathfrak{U}$ and $\{\alpha^{n,\nu}, \nu \in \mathcal{V}\} \subset \mathcal{A}$ such that

$$\chi_*\left(t \wedge \tau_n, Z^n(t \wedge \tau_n), P^n(t \wedge \tau_n)\right) = 0, \quad t \geq t_n, \tag{3.3.10}$$

where

$$(Z^n(s), P^n(s)) := \left(Z^{\mathfrak{u}^n, v_o}_{t_n, z_n}(s), P^{\alpha^{n,v_o}}_{t_n, p_n}(s)\right)$$

and

$$\tau_n := \inf\left\{s \geq t_n : (s, Z^n(s), P^n(s)) \notin B_\varepsilon\right\}.$$

(In the above, $v_o \in V$ is viewed as a constant element of $\mathcal{V}$.) By (3.3.10), (3.3.5) and the definitions of $\zeta$ and $\tau_n$,

$$-\varphi(\cdot, Z^n, P^n)(t \wedge \tau_n) = (\chi_* - \varphi)(\cdot, Z^n, P^n)(t \wedge \tau_n) \geq \zeta \mathbf{1} t \geq \tau_n \geq 0.$$

Applying Itô's formula to $-\varphi(\cdot, Z^n, P^n)$, we deduce that

$$S_n(t) := S_n(0) + \int_{t_n}^{t\wedge\tau_n} \delta_n(r)\, dr + \int_{t_n}^{t\wedge\tau_n} \Sigma_n(r)\, dW_r \geq -\zeta \mathbf{1} t < \tau_n, \tag{3.3.11}$$

where

$$\begin{aligned} S_n(0) &:= -\zeta - \varphi(t_n, z_n, p_n), \\ \delta_n(r) &:= -\mathcal{L}_{(Z,P)}^{\mathfrak{u}_r^n[v_o],\alpha_r^{n,v_o},v_o} \varphi\left(r, Z^n(r), P^n(r)\right), \\ \Sigma_n(r) &:= -D\varphi\left(r, Z^n(r), P^n(r)\right)^\top \sigma_{(Z,P)}\left(Z^n(r), \mathfrak{u}_r^n[v_o], \alpha_r^{n,v_o}, v_o\right). \end{aligned}$$

Define the set

$$A_n := [\![t_n, \tau_n]\!] \cap \{\delta_n > -\eta\};$$

then (3.3.7) and the definition of $\mathcal{N}_\varepsilon$ imply that

$$|\Sigma_n| > \varepsilon \quad \text{on } A_n. \tag{3.3.12}$$

**Lemma 3.3.2.** *After diminishing $\varepsilon > 0$ if necessary, the stochastic exponential*

$$E_n(\cdot) = \mathcal{E}\left(-\int_{t_n}^{\cdot\wedge\tau_n} \frac{\delta_n(r)}{|\Sigma_n(r)|^2}\Sigma_n(r)\mathbf{1}_{A_n}(r)\,dW_r\right)$$

*is well-defined and a true martingale for all $n \geq 1$.*

This lemma is proved below. Admitting its result for the moment, integration by parts yields

$$\begin{aligned} (E_nS_n)(t\wedge\tau_n) &= S_n(0) + \int_{t_n}^{t\wedge\tau_n} E_n\delta_n\mathbf{1}_{A_n^c}\,dr \\ &\quad + \int_{t_n}^{t\wedge\tau_n} E_n\left(\Sigma_n - S_n\frac{\delta_n}{|\Sigma_n|^2}\Sigma_n\mathbf{1}_{A_n}\right)dW. \end{aligned}$$

As $E_n \geq 0$, it then follows from the definition of $A_n$ that $E_n\delta_n\mathbf{1}_{A_n^c} \leq 0$ and so $E_nS_n$ is a local supermartingale; in fact, it is a true supermartingale since it is bounded from below by the martingale $-\zeta E_n$. In view of (3.3.11), we deduce that

$$-\zeta - \varphi(t_n, z_n, p_n) = (E_nS_n)(t_n) \geq \mathbb{E}\left[(E_nS_n)(\tau_n)\right] \geq -\zeta\mathbb{E}\left[\mathbf{1}\tau_n < \tau_nE_n(\tau_n)\right] = 0,$$

which yields a contradiction due to $\zeta > 0$ and the fact that, by (3.3.9),

$$\varphi(t_n, z_n, p_n) \to \varphi(t_o, z_o, p_o) = \chi_*(t_o, z_o, p_o) = 0.$$

**Step 2:** $\chi^*$ *is a viscosity subsolution.*

Let $(t_o, z_o, p_o) \in [0,T) \times \mathbb{R}^d \times \mathbb{R}$ and let $\varphi$ be a smooth function such that

$$\max_{[0,T)\times\mathbb{R}^d\times\mathbb{R}} (\chi^* - \varphi) = (\chi^* - \varphi)(t_o, z_o, p_o) = 0.$$

In order to prove that $(-\partial_t + H_*)\varphi(t_o, z_o, p_o) \leq 0$, we assume for contradiction that

$$(-\partial_t + H_*)\varphi(t_o, z_o, p_o) > 0. \tag{3.3.13}$$

An argument analogous to the proof of (3.3.9) shows that $\chi^*(t_o, z_o, p_o) = 1$. Consider a sequence $(t_n, z_n, p_n, \varepsilon_n)_{n\geq 1}$ in $[0, T) \times \mathbb{R}^d \times \mathbb{R} \times (0, 1)$ such that

$$(t_n, z_n, p_n - \varepsilon_n, \varepsilon_n) \to (t_o, z_o, p_o, 0) \quad \text{and} \quad \chi(t_n, z_n, p_n - \varepsilon_n) \to \chi^*(t_o, z_o, p_o) = 1.$$

Since $\chi$ takes values in $\{0, 1\}$, we must have

$$\chi(t_n, z_n, p_n - \varepsilon_n) = 1 \tag{3.3.14}$$

for all $n$ large enough. Set

$$\tilde{\varphi}(t, z, p) := \varphi(t, z, p) + |t - t_o|^2 + |z - z_o|^4 + |p - p_o|^4.$$

Then the inequality (3.3.13) and the definition of $H_*$ imply that we can find $(\hat{u}, \hat{a})$ in $N_{Lip}(\cdot, D\tilde{\varphi})(t_o, z_o, p_o)$ such that

$$\inf_{v\in V} \left( -\mathcal{L}_{(Z,P)}^{(\hat{u},\hat{a})(\cdot,D\tilde{\varphi},v),v} \tilde{\varphi} \right) \geq 0 \quad \text{on } B_\varepsilon := B_\varepsilon\left(t_o, z_o, p_o\right), \tag{3.3.15}$$

for some $\varepsilon > 0$. By the definition of $N_{Lip}$, after possibly changing $\varepsilon > 0$, we have

$$(\hat{u}, \hat{a}) \in \mathcal{N}_0(\cdot, D\tilde{\varphi}, \cdot) \quad \text{ on } B_\varepsilon \times V. \tag{3.3.16}$$

Moreover, we have

$$\tilde{\varphi} \geq \varphi + \eta \quad \text{on } \partial B_\varepsilon \tag{3.3.17}$$

for some $\eta > 0$. Since $\tilde{\varphi}(t_n, z_n, p_n) \to \varphi(t_o, z_o, p_o) = \chi^*(t_o, z_o, p_o) = 1$, we can find $n$ such that

$$\tilde{\varphi}(t_n, z_n, p_n) \leq 1 + \eta/2 \tag{3.3.18}$$

and such that (3.3.14) is satisfied. We fix this $n$ for the remainder of the proof.

For brevity, we write $(\hat{u}, \hat{a})(t, z, p, v)$ for $(\hat{u}, \hat{a})(z, p, D\tilde{\varphi}(t, z, p), v)$ in the sequel. Exploiting the definition of $N_{Lip}$, we can then define the mapping $(\hat{\mathfrak{u}}, \hat{\mathfrak{a}})[\cdot] : \mathcal{V} \to \mathcal{U} \times \mathcal{A}$ implicitly via

$$(\hat{\mathfrak{u}}, \hat{\mathfrak{a}})[\nu] = (\hat{u}, \hat{a}) \left(\cdot, Z_{t_n,z_n}^{\hat{\mathfrak{u}}[\nu],\nu}, P_{t_n,p_n}^{\hat{\mathfrak{a}}[\nu]}, \nu\right) \mathbf{1}_{[t_n,\tau^\nu]},$$

where

$$\tau^\nu := \inf\left\{r \geq t_n : \left(r, Z_{t_n,z_n}^{\hat{\mathfrak{u}}[\nu],\nu}(r), P_{t_n,p_n}^{\hat{\mathfrak{a}}[\nu]}(r)\right) \notin B_\varepsilon\right\}.$$

We observe that $\hat{\mathfrak{u}}$ and $\hat{\mathfrak{a}}$ are non-anticipating; that is, $(\hat{\mathfrak{u}}, \hat{\mathfrak{a}}) \in \mathfrak{U} \times \mathfrak{A}$. Let us write $(Z^\nu, P^\nu)$ for $(Z_{t_n,z_n}^{\hat{\mathfrak{u}},\nu}, P_{t_n,p_n}^{\hat{\mathfrak{a}}[\nu]})$ to alleviate the notation. Since $\chi \leq \chi^* \leq \varphi$, the continuity of the paths of $Z^\nu$ and $P^\nu$ and (3.3.17) lead to

$$\varphi\left(\tau^\nu, Z^\nu(\tau^\nu), P^\nu(\tau^\nu)\right) \leq \tilde{\varphi}\left(\tau^\nu, Z^\nu(\tau^\nu), P^\nu(\tau^\nu)\right) - \eta.$$

On the other hand, in view of (3.3.15) and (3.3.16), Itô's formula applied to $\tilde{\varphi}$ on $[t_n, \tau^\nu]$ yields that

$$\tilde{\varphi}\left(\tau^\nu, Z^\nu(\tau^\nu), P^\nu(\tau^\nu)\right) \leq \tilde{\varphi}\left(t_n, z_n, p_n\right).$$

Therefore, the previous inequality and (3.3.18) show that

$$\varphi\left(\tau^\nu, Z^\nu(\tau^\nu), P^\nu(\tau^\nu)\right) \leq \tilde{\varphi}\left(t_n, z_n, p_n\right) - \eta \leq 1 - \eta/2.$$

By (GDP2$_\chi$) of Lemma 3.3.1, we deduce that $\chi(t_n, z_n, p_n - \varepsilon_n) = 0$, which contradicts (3.3.14).

To complete the proof of the theorem, we still need to show Lemma 3.3.2. To this end, we first make the following observation.

**Lemma 3.3.3.** *Let $\alpha \in L^2_{loc}(W)$ be such that $M = \int \alpha \, dW$ is a bounded martingale and let $\beta$ be an $\mathbb{R}^d$-valued, progressively measurable process such that $|\beta| \leq c(1+|\alpha|)$ for some constant $c$. Then the stochastic exponential $\mathcal{E}(\int \beta \, dW)$ is a true martingale.*

**Proof.** The assumption clearly implies that $\int_0^T |\beta_s|^2 \, ds < \infty$ $\mathbb{P}$-a.s. Since $M$ is bounded, we have in particular that $M \in BMO$; i.e.,

$$\sup_{\tau \in \mathcal{T}_0} \left\| \mathbb{E}\left[ \int_\tau^T |\alpha_s|^2 \, ds \, |\mathcal{F}_\tau \right] \right\|_\infty < \infty.$$

In view of the assumption, the same holds with $\alpha$ replaced by $\beta$, so that $\int \beta \, dW$ is in $BMO$. This implies that $\mathcal{E}(\int \beta \, dW)$ is a true martingale; cf. [Kaz94, Theorem 2.3].

**Proof.** [Proof of Lemma 3.3.2] Consider the process

$$\beta_n(r) := \frac{\delta_n(r)}{|\Sigma_n(r)|^2} \Sigma_n(r) \mathbf{1}_{A_n}(r);$$

we show that

$$|\beta_n| \leq c(1 + |\alpha^{n,v_o}|) \quad \text{on } [\![t_n, \tau_n]\!] \tag{3.3.19}$$

for some $c > 0$. Then, the result will follow by applying Lemma 3.3.3 to $\alpha^{n,v_o} \mathbf{1}_{[\![t_n,\tau_n]\!]}$; note that the stochastic integral of this process is bounded by the definition of $\tau_n$. To prove (3.3.19), we distinguish two cases.

*Case 1:* $\partial_p \varphi(t_o, z_o, p_o) \neq 0$. Using that $\mu$ and $\sigma$ are continuous and that $U$ and $B_\varepsilon$ are bounded, tracing the definitions yields that

$$|\delta_n| \leq c\big\{1 + |\alpha^{n,v_o}| + |\alpha^{n,v_o}|^2 |\partial_{pp}\varphi(\cdot, Z^n, P^n)|\big\} \quad \text{on } [\![t_n, \tau_n]\!],$$

while

$$|\Sigma_n| \geq -c + |\alpha^{n,v_o}||\partial_p \varphi(\cdot, Z^n, P^n)| \quad \text{on } [\![t_n, \tau_n]\!],$$

for some $c > 0$. Since $\partial_p\varphi(t_o, z_o, p_o) \neq 0$ by assumption, $\partial_p\varphi$ is uniformly bounded away from zero on $B_\varepsilon$, after diminishing $\varepsilon > 0$ if necessary. Hence, recalling (3.3.12), there is a cancelation between $|\delta_n|$ and $|\Sigma_n|$ which allows us to conclude (3.3.19).

*Case 2:* $\partial_p\varphi(t_o, z_o, p_o) = 0$. We first observe that

$$\delta_n^+ \leq c(1 + |\alpha^{n,v_o}|) - c^{-1}|\alpha^{n,v_o}|^2 \partial_{pp}\varphi(\cdot, Z^n, P^n) \quad \text{on } [\![t_n, \tau_n]\!]$$

for some $c > 0$. Since $\delta_n^-$ and $|\Sigma_n|^{-1}$ are uniformly bounded on $A_n$, it therefore suffices to show that $\partial_{pp}\varphi \geq 0$ on $B_\varepsilon$. To see this, we note that (3.3.6) and the relaxation in the definition (3.3.3) of $H^*$ imply that there exists $\iota > 0$ such that, for every $v \in V$ and all small $\varepsilon > 0$,

$$-\partial_t\varphi(t_o, z_o, p_o) + F(\Theta^\iota, u, a, v) \leq -\eta \quad \text{for all } (u, a) \in \mathcal{N}_\varepsilon(\Theta^\iota), \tag{3.3.20}$$

where $\Theta^\iota = (z_0, p_0, D\varphi, A^\iota)$ and $A^\iota$ is the same matrix as $D^2\varphi(t_o, z_o, p_o)$ except that the entry $\partial_{pp}\varphi(t_o, z_o, p_o)$ is replaced by $\partial_{pp}\varphi(t_o, z_o, p_o) - \iota$. Going back to the definition of $\mathcal{N}_\varepsilon$, we observe that $\mathcal{N}_\varepsilon(\Theta^\iota)$ does no depend on $\iota$ and, which is the crucial part, the assumption that $\partial_p\varphi(t_o, z_o, p_o) = 0$ implies that $\mathcal{N}_\varepsilon(\Theta^\iota)$ is of the form $\mathcal{N}^U \times \mathbb{R}^d$; that is, the variable $a$ is unconstrained. Now (3.3.20) and the last observation show that

$$-(\partial_{pp}\varphi(t_o, z_o, p_o) - \iota)|a|^2 \leq c(1 + |a|)$$

for *all* $a \in \mathbb{R}^d$, so we deduce that $\partial_{pp}\varphi(t_o, z_o, p_o) \geq \iota > 0$. Thus, after diminishing $\varepsilon > 0$ if necessary, we have $\partial_{pp}\varphi \geq 0$ on $B_\varepsilon$ as desired. This completes the proof.

**Remark 3.3.4.** Lemma 3.3.2 consists in an alternative proof to fix the integrability issue in the previous literature (see Assumption 2.2.3 and Remark 2.2.6 of the previous chapter). More specifically, this result should be related to Assumption 2.3.4, where Assumption 2.2.3 and Lemma 2.2.1 allows to deal with the more general framework of Bouchard, Elie and Touzi [BET09, Section 2] or [Mor11, Section 2]).

### 3.3.3 PDE in the monotone case

We now specialize the setup of Section 3.3.1 to the case where the state process $Z$ consists of a pair of processes $(X, Y)$ with values in $\mathbb{R}^{d-1} \times \mathbb{R}$ and the loss function

$$\ell : \mathbb{R}^{d-1} \times \mathbb{R} \to \mathbb{R}, \quad (x, y) \mapsto \ell(x, y)$$

is nondecreasing in the scalar variable $y$. This setting, which was previously studied in [BET09] for the case without adverse control, will allow for a more explicit description of $\Lambda$ which is particularly suitable for applications in mathematical finance.

For $(t,x,y) \in [0,T] \times \mathbb{R}^{d-1} \times \mathbb{R}$ and $(\mathfrak{u},\nu) \in \mathcal{U} \times \mathcal{V}$, let $Z^{\mathfrak{u},\nu}_{t,x,y} = (X^{\mathfrak{u},\nu}_{t,x}, Y^{\mathfrak{u},\nu}_{t,x,y})$ be the strong solution of (3.3.1) with

$$\mu(x,y,u,v) := \begin{pmatrix} \mu_X(x,u,v) \\ \mu_Y(x,y,u,v) \end{pmatrix}, \quad \sigma(x,y,u,v) := \begin{pmatrix} \sigma_X(x,u,v) \\ \sigma_Y(x,y,u,v) \end{pmatrix},$$

where $\mu_Y$ and $\sigma_Y$ take values in $\mathbb{R}$ and $\mathbb{R}^{1\times d}$, respectively. The assumptions from Section 3.3.1 remain in force; in particular, the continuity and growth assumptions on $\mu$ and $\sigma$. In this setup, we can consider the real-valued function

$$\gamma(t,x,p) := \inf\{y \in \mathbb{R} : (x,y,p) \in \Lambda(t)\}.$$

In mathematical finance, this may describe the minimal capital $y$ such that the given target can be reached by trading in the securities market modeled by $X^{\mathfrak{u},\nu}_{t,x}$; an illustration is given in the subsequent section. In the present context, Corollary 3.2.3 reads as follows.

**Lemma 3.3.4.** *Fix $(t,x,y,p) \in [0,T] \times \mathbb{R}^{d-1} \times \mathbb{R} \times \mathbb{R}$ and assume that $\gamma$ is locally bounded.*

**(GDP1$_\gamma$)** *Assume that $y > \gamma(t,x,p+\varepsilon)$ for some $\varepsilon > 0$. Then there exist $\mathfrak{u} \in \mathfrak{U}$ and $\{\alpha^\nu, \nu \in \mathcal{V}\} \subset \mathcal{A}$ such that*

$$Y^{\mathfrak{u},\nu}_{t,x,y}(\tau) \geq \gamma_*\left(\tau, X^{\mathfrak{u},\nu}_{t,x}(\tau), P^{\alpha^\nu}_{t,p}(\tau)\right) \quad \mathbb{P}\text{-a.s. for all } \nu \in \mathcal{V} \text{ and } \tau \in \mathcal{T}_t.$$

**(GDP2$_\gamma$)** *Let $\varphi$ be a continuous function such that $\varphi \geq \gamma$ and let $O \subset [0,T] \times \mathbb{R}^{d-1} \times \mathbb{R}$ be a bounded open set containing $(t,z,p)$. Let $(\mathfrak{u},\mathfrak{a}) \in \mathfrak{U} \times \mathfrak{A}$ and $\eta > 0$ be such that*

$$Y^{\mathfrak{u},\nu}_{t,x,y}(\tau^\nu) \geq \varphi\left(\tau, X^{\mathfrak{u},\nu}_{t,x}(\tau^\nu), P^{\mathfrak{a}[\nu]}_{t,p}(\tau^\nu)\right) + \eta \quad \mathbb{P}\text{-a.s. for all } \nu \in \mathcal{V},$$

*where $\tau^\nu$ is the first exit time of $(\cdot, X^{\mathfrak{u},\nu}_{t,x}, Y^{\mathfrak{u},\nu}_{t,x,y}, P^{\mathfrak{a}[\nu]}_{t,p})$ from $O$. Then $y \geq \gamma(t,x,p-\varepsilon)$ for all $\varepsilon > 0$.*

**Proof.** Noting that $y > \gamma(t,x,p)$ implies $(x,y,p) \in \Lambda(t)$ and that $(x,y,p) \in \Lambda(t)$ implies $y \geq \gamma(t,x,p)$, the result follows from Corollary 3.2.3 by arguments similar to the proof of Lemma 3.3.1.

The Hamiltonians $G^*$ and $G_*$ for the PDE describing $\gamma$ are defined like $H^*$ and $H_*$ in (3.3.3) and (3.3.4), but with

$$F(\Theta,u,a,v) := \mu_Y(x,y,u,v) - \mu_{(X,P)}(x,u,v)^\top q - \frac{1}{2}\mathrm{Tr}\left[\sigma_{(X,P)}\sigma_{(X,P)}^\top(x,u,a,v)A\right]$$

where $\Theta := (x, y, q, A) \in \mathbb{R}^{d-1} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d$ and

$$\mu_{(X,P)}(x, u, a, v) := \begin{pmatrix} \mu_X(x, u, v) \\ 0 \end{pmatrix}, \quad \sigma_{(X,P)}(x, u, a, v) := \begin{pmatrix} \sigma_X(x, u, v) \\ a \end{pmatrix},$$

with the relaxed kernel $\mathcal{N}_\varepsilon$ replaced by

$$\mathcal{K}_\varepsilon(x, y, q, v) := \left\{ (u, a) \in U \times \mathbb{R} : \left| \sigma_Y(x, y, u, v) - q^\top \sigma_{(X,P)}(x, u, a, v) \right| \leq \varepsilon \right\},$$

and $N_{Lip}$ replaced by a set $K_{Lip}$, defined like $N_{Lip}$ but in terms of $\mathcal{K}_0$ instead of $\mathcal{N}_0$. We then have the following result for the semicontinuous envelopes $\gamma^*$ and $\gamma_*$ of $\gamma$.

**Theorem 3.3.5.** *Assume that $\gamma$ is locally bounded. Then $\gamma_*$ is a viscosity supersolution on $[0, T) \times \mathbb{R}^{d-1} \times \mathbb{R}$ of*

$$(-\partial_t + G^*)\varphi \geq 0$$

*and $\gamma^*$ is a viscosity subsolution on $[0, T) \times \mathbb{R}^{d-1} \times \mathbb{R}$ of*

$$(-\partial_t + G_*)\varphi \leq 0.$$

**Proof.** The result follows from Lemma 3.3.4 by adapting the proof of [BET09, Theorem 2.1], using the arguments from the proof of Theorem 3.3.3 to account for the game-theoretic setting and the relaxed formulation of the GDP. We therefore omit the details.

We shall not discuss in this generality the boundary conditions as $t \to T$; they are somewhat complicated to state but can be deduced similarly as in [BET09]. Obtaining a comparison theorem at the present level of generality seems difficult, mainly due to the presence of the sets $\mathcal{K}_\varepsilon$ and $K_{Lip}$ (which depend on the solution itself) and the discontinuity of the nonlinearities at $\partial_p \varphi = 0$. It seems more appropriate the treat this question on a case-by-case basis. In fact, once $G^* = G_*$ (see also Remark 3.3.6), the challenges in proving comparison are similar as in the case without adverse player. For that case, comparison results have been obtained, e.g., in [BV11] for a specific setting (see also the references therein for more examples).

**Remark 3.3.6.** Let us discuss briefly the question whether $G^* = G_*$. We shall focus on the case where $U$ is convex and the (nondecreasing) function $\gamma$ is strictly increasing with respect to $p$; in this case, we are interested only in test functions $\varphi$ with $\partial_p \varphi > 0$. It is not hard to see that for such functions, the relaxation $\varepsilon \searrow 0, \Theta' \to \Theta$ in (3.3.3) is superfluous, so we are left with the question whether

$$\inf_{v \in V} \sup_{(u,a) \in \mathcal{K}_0(\Theta, v)} G(\Theta, u, a, v) = \sup_{(\hat{u}, \hat{a}) \in K_{Lip}(\Theta)} \inf_{v \in V} G(\Theta, \hat{u}(\Theta, v), \hat{a}(\Theta, v), v).$$

The inequality "$\geq$" is clear. The converse inequality will hold if, say, for each $\varepsilon > 0$, there exists a locally Lipschitz mapping $(\hat{u}_\varepsilon, \hat{a}_\varepsilon) \in K_{Lip}$ such that

$$G(\cdot, (\hat{u}_\varepsilon, \hat{a}_\varepsilon)(\cdot, v), v) \geq \sup_{(u,a)\in\mathcal{K}_0(\cdot,v)} G(\cdot, u, a, v) - \varepsilon \text{ for all } v \in V.$$

Conditions for the existence of $\varepsilon$-optimal *continuous* selectors can be found in [KN87, Theorem 3.2]. If $(u_\varepsilon, a_\varepsilon)$ is an $\varepsilon$-optimal continuous selector, the definition of $\mathcal{K}_0$ entails that $a_\varepsilon^\top(\Theta, v)q_p = -\sigma_X^\top(x, u_\varepsilon(\Theta, v), v)q_x + \sigma_Y(x, y, u_\varepsilon(\Theta, v), v)$, where we use the notation $\Theta = (x, y, p, (q_x^\top, q_p)^\top, A)$. Then $u_\varepsilon$ can be further approximated, uniformly on compact sets, by a locally Lipschitz function $\hat{u}_\varepsilon$. We may restrict our attention to $q_p > 0$; so that, if we assume that $\sigma^\top$ is (jointly) locally Lipschitz, the mapping $\hat{a}_\varepsilon^\top(\Theta, v) := (q_p)^{-1}\left(-\sigma_X^\top(x, \hat{u}_\varepsilon(\Theta, v), v)q_x + \sigma_Y(x, y, \hat{u}_\varepsilon(\Theta, v), v)\right)$ is locally Lipschitz and then $(\hat{u}_\varepsilon, \hat{a}_\varepsilon)$ defines a sufficiently good, locally Lipschitz continuous selector: for all $v \in V$,

$$G(\cdot, (\hat{u}_\varepsilon, \hat{a}_\varepsilon)(\cdot, v), v) \geq G(\cdot, (u_\varepsilon, a_\varepsilon)(\cdot, v), v) - O_\varepsilon(1) \geq \sup_{(u,a)\in\mathcal{K}_0} G(\cdot, u, a, v) - \varepsilon - O_\varepsilon(1)$$

on a neighborhood of $\Theta$, where $O_\varepsilon(1) \to 0$ as $\varepsilon \to 0$. One can similarly discuss other cases; e.g, when $\gamma$ is strictly concave (instead of increasing) with respect to $p$ and the mapping $(x, y, q_x, u, v) \mapsto -\sigma_X^\top(x, u, v)q_x + \sigma_Y(x, y, u, v)$ is invertible in $u$, with an inverse that is locally Lipschitz, uniformly in $v$.

## 3.4 Application to hedging under uncertainty

In this section, we illustrate our general results in a concrete example, and use the opportunity to show how to extend them to a case with unbounded strategies. To this end, we shall consider a problem of partial hedging under Knightian uncertainty. More precisely, the uncertainty concerns the drift and volatility coefficients of the risky asset and we aim at controlling a function of the hedging error; the corresponding worst-case analysis is equivalent to a game where the adverse player chooses the coefficients. This problem is related to the $G$-expectation of [Pen07, Pen08], the second order target problem of [STZ10] and the problem of optimal arbitrage studied in [FK11]. We let

$$V = [\underline{\mu}, \overline{\mu}] \times [\underline{\sigma}, \overline{\sigma}]$$

be the possible values of the coefficients, where $\underline{\mu} \leq 0 \leq \overline{\mu}$ and $\overline{\sigma} \geq \underline{\sigma} \geq 0$. Moreover, $U = \mathbb{R}$ will be the possible values for the investment policy, so that, in contrast to the previous sections, $U$ is not bounded.

The notation is the same as in the previous section, except for an integrability condition for the strategies that will be introduced below to account for the unboundedness of $U$, moreover, we shall sometimes write $\nu = (\mu, \sigma)$ for an adverse control $\nu \in \mathcal{V}$. Given $(\mu, \sigma) \in \mathcal{V}$ and $\mathfrak{u} \in \mathfrak{U}$, the state process $Z^{\mathfrak{u},\nu}_{t,x,y} = (X^{\nu}_{t,x}, Y^{\mathfrak{u},\nu}_{t,y})$ is governed by

$$\frac{dX^{\nu}_{t,x}(r)}{X^{\nu}_{t,x}(r)} = \mu_r\,dr + \sigma_r\,dW_r, \quad X^{\nu}_{t,x}(t) = x$$

and

$$dY^{\mathfrak{u},\nu}_{t,y}(r) = \mathfrak{u}[\nu]_r\big(\mu_r\,dr + \sigma_r\,dW_r\big), \quad Y^{\mathfrak{u},\nu}_{t,y}(t) = y.$$

To wit, the process $X^{\nu}_{t,x}$ represents the price of a risky asset with unknown drift and volatility coefficients $(\mu, \sigma)$, while $Y^{\mathfrak{u},\nu}_{t,y}$ stands for the wealth process associated to an investment policy $\mathfrak{u}[\nu]$, denominated in monetary amounts. (The interest rate is zero for simplicity.) We remark that it is clearly necessary to use strategies in this setup: even a simple stop-loss investment policy cannot be implemented as a control.

Our loss function is of the form

$$\ell(x, y) = \Psi(y - g(x)),$$

where $\Psi, g : \mathbb{R} \to \mathbb{R}$ are continuous functions of polynomial growth. The function $\Psi$ is also assumed to be strictly increasing and concave, with an inverse $\Psi^{-1} : \mathbb{R} \to \mathbb{R}$ that is again of polynomial growth. As a consequence, $\ell$ is continuous and (3.2.3) is satisfied for some $q > 0$; that is,

$$|\ell(z)| \le C(1 + |z|^q), \quad z = (x, y) \in \mathbb{R}^2. \tag{3.4.1}$$

We interpret $g(X^{\nu}_{t,x}(T))$ as the random payoff of a European option written on the risky asset, for a given realization of the drift and volatility processes, while $\Psi$ quantifies the disutility of the hedging error $Y^{\mathfrak{u},\nu}_{t,y}(T) - g(X^{\nu}_{t,x}(T)$. In this setup,

$$\gamma(t, x, p) = \inf\big\{y \in \mathbb{R} :\ \exists\, \mathfrak{u} \in \mathfrak{U} \text{ s.t. } \mathbb{E}\big[\Psi(Y^{\mathfrak{u},\nu}_{t,y}(T) - g(X^{\nu}_{t,x}(T))|\mathcal{F}_t\big] \ge p \ \mathbb{P}\text{-a.s. } \forall\, \nu \in \mathcal{V}\big\}$$

is the minimal price for the option allowing to find a hedging policy such that the expected disutility of the hedging error is controlled by $p$.

We fix a finite constant $\bar{q} > q \vee 2$ and define $\mathfrak{U}$ to be the set of mappings $\mathfrak{u} : \mathcal{V} \to \mathcal{U}$ that are non-anticipating (as in Section 3.3) and satisfy the integrability condition

$$\sup_{\nu \in \mathcal{V}} \mathbb{E}\left[\left|\int_0^T |\mathfrak{u}[\nu]_r|^2 dr\right|^{\frac{\bar{q}}{2}}\right] < \infty. \tag{3.4.2}$$

The conclusions below do not depend on the choice of $\bar{q}$. The main result of this section is an explicit expression for the price $\gamma(t, x, p)$.

**Theorem 3.4.1.** *Let* $(t,x,p) \in [0,T] \times (0,\infty) \times \mathbb{R}$. *Then* $\gamma(t,x,p)$ *is finite and given by*

$$\gamma(t,x,p) = \sup_{\nu \in \mathcal{V}^0} \mathbb{E}\left[g\left(X^{\nu}_{t,x}(T)\right)\right] + \Psi^{-1}(p), \qquad \text{where } \mathcal{V}^0 = \{(\mu,\sigma) \in \mathcal{V} : \mu \equiv 0\}. \tag{3.4.3}$$

In particular, $\gamma(t,x,p)$ coincides with the superhedging price for the shifted option $g(\cdot) + \Psi^{-1}(p)$ in the (driftless) uncertain volatility model for $[\underline{\sigma}, \overline{\sigma}]$; see also below. That is, the drift uncertainty has no impact on the price, provided that $\underline{\mu} \leq 0 \leq \overline{\mu}$. Let us remark, in this respect, that the present setup corresponds to an investor who knows the present and historical drift and volatility of the underlying. It may also be interesting to study the case where only the trajectories of the underlying (and therefore the volatility, but not necessarily the drift) are observed. This, however, does not correspond to the type of game studied in this chapter.

### 3.4.1 Proof of Theorem 3.4.1

**Proof.** [Proof of "$\geq$" in (3.4.3).] We may assume that $\gamma(t,x,p) < \infty$. Let $y > \gamma(t,x,p)$; then there exists $\mathfrak{u} \in \mathfrak{U}$ such that

$$\mathbb{E}\left[\Psi\left(Y^{\mathfrak{u},\nu}_{t,y}(T) - g\left(X^{\nu}_{t,x}(T)\right)\right)\right] \geq p \quad \text{for all } \nu \in \mathcal{V}.$$

As $\Psi$ is concave, it follows by Jensen's inequality that

$$\Psi\left(\mathbb{E}\left[Y^{\mathfrak{u},\nu}_{t,y}(T) - g\left(X^{\nu}_{t,x}(T)\right)\right]\right) \geq p \quad \text{for all } \nu \in \mathcal{V}.$$

Since the integrability condition (3.4.2) implies that $Y^{\mathfrak{u},\nu}_{t,y}$ is a martingale for all $\nu \in \mathcal{V}^0$, we conclude that

$$\Psi\left(y - \mathbb{E}\left[g\left(X^{\nu}_{t,x}(T)\right)\right]\right) \geq p \quad \text{for all } \nu \in \mathcal{V}^0$$

and hence $y \geq \sup_{\nu \in \mathcal{V}^0} \mathbb{E}\left[g\left(X^{\nu}_{t,x}(T)\right)\right] + \Psi^{-1}(p)$. As $y > \gamma(t,x,p)$ was arbitrary, the claim follows.

We shall use Theorem 3.3.5 to derive the missing inequality in (3.4.3). Since $U = \mathbb{R}$ is unbounded, we introduce a sequence of approximating problems $\gamma_n$ defined like $\gamma$, but with strategies bounded by $n$:

$$\gamma_n(t,x,p) := \inf\left\{y \in \mathbb{R} : \exists\, \mathfrak{u} \in \mathfrak{U}^n \text{ s.t. } \mathbb{E}\left[\ell\left(Z^{\mathfrak{u},\nu}_{t,x,y}(T)\right)|\mathcal{F}_t\right] \geq p \ \mathbb{P}\text{-a.s. } \forall\, \nu \in \mathcal{V}\right\},$$

where

$$\mathfrak{U}^n = \{\mathfrak{u} \in \mathfrak{U} : |\mathfrak{u}[\nu]| \leq n \text{ for all } \nu \in \mathcal{V}\}.$$

Then clearly $\gamma_n$ is decreasing in $n$ and

$$\gamma_n \geq \gamma, \quad n \geq 1. \tag{3.4.4}$$

**Lemma 3.4.1.** *Let* $(t,z) \in [0,T] \times (0,\infty) \times \mathbb{R}$, $\mathfrak{u} \in \mathfrak{U}$, *and define* $\mathfrak{u}_n \in \mathfrak{U}$ *by*

$$\mathfrak{u}_n[\nu] := \mathfrak{u}[\nu]\mathbf{1}|\mathfrak{u}[\nu]| \leq n, \quad \nu \in \mathcal{V}.$$

*Then*

$$\operatorname*{ess\,sup}_{\nu\in\mathcal{V}} \left|\mathbb{E}\left[\ell\left(Z^{\mathfrak{u}_n,\nu}_{t,z}(T)\right) - \ell\left(Z^{\mathfrak{u},\nu}_{t,z}(T)\right)|\mathcal{F}_t\right]\right| \to 0 \quad \textit{in } L^1 \textit{ as } n \to \infty.$$

**Proof.** Using monotone convergence and an argument as in the proof of Step 1 in Section 3.2.3, we obtain that

$$\begin{aligned}\mathbb{E}&\left\{\operatorname*{ess\,sup}_{\nu\in\mathcal{V}} \left|\mathbb{E}\left[\ell\left(Z^{\mathfrak{u}_n,\nu}_{t,z}(T)\right) - \ell\left(Z^{\mathfrak{u},\nu}_{t,z}(T)\right)|\mathcal{F}_t\right]\right|\right\}\\ &= \sup_{\nu\in\mathcal{V}} \mathbb{E}\left\{\left|\ell\left(Z^{\mathfrak{u}_n,\nu}_{t,z}(T)\right) - \ell\left(Z^{\mathfrak{u},\nu}_{t,z}(T)\right)\right|\right\}.\end{aligned}$$

Since $V$ is bounded, the Burkholder-Davis-Gundy inequalities show that there is a universal constant $c > 0$ such that

$$\begin{aligned}\mathbb{E}\left\{\left|Z^{\mathfrak{u}_n,\nu}_{t,z}(T) - Z^{\mathfrak{u},\nu}_{t,z}(T)\right|\right\} &\leq c\mathbb{E}\left[\int_t^T |\mathfrak{u}[\nu]_r - \mathfrak{u}_n[\nu]_r|^2\, dr\right]^{\frac{1}{2}}\\ &= c\mathbb{E}\left[\int_t^T |\mathfrak{u}[\nu]_r\mathbf{1}|\mathfrak{u}[\nu]_r| > n|^2\, dr\right]^{\frac{1}{2}}\end{aligned}$$

and hence (3.4.2) and Hölder's inequality yield that, for any given $\delta > 0$,

$$\sup_{\nu\in\mathcal{V}} \mathbb{P}\left\{\left|Z^{\mathfrak{u}_n,\nu}_{t,z}(T) - Z^{\mathfrak{u},\nu}_{t,z}(T)\right| > \delta\right\} \leq \delta^{-1} \sup_{\nu\in\mathcal{V}} \mathbb{E}\left\{\left|Z^{\mathfrak{u}_n,\nu}_{t,z}(T) - Z^{\mathfrak{u},\nu}_{t,z}(T)\right|\right\} \to 0 \tag{3.4.5}$$

for $n \to \infty$. Similarly, the Burkholder-Davis-Gundy inequalities and (3.4.2) show that $\{|Z^{\mathfrak{u}_n,\nu}_{t,z}(T)| + |Z^{\mathfrak{u},\nu}_{t,z}(T)|,\ \nu \in \mathcal{V}, n \geq 1\}$ is bounded in $L^{\bar{q}}$. This yields, on the one hand, that

$$\sup_{\nu\in\mathcal{V},\, n\geq 1} \mathbb{P}\left\{\left|Z^{\mathfrak{u}_n,\nu}_{t,z}(T)\right| + \left|Z^{\mathfrak{u},\nu}_{t,z}(T)\right| > k\right\} \to 0 \tag{3.4.6}$$

for $k \to \infty$, and on the other hand, in view of (3.4.1) and $\bar{q} > q$, that

$$\left\{\ell\left(Z^{\mathfrak{u}_n,\nu}_{t,z}(T)\right) - \ell\left(Z^{\mathfrak{u},\nu}_{t,z}(T)\right) : \nu \in \mathcal{V}, n \geq 1\right\} \quad \text{is uniformly integrable.} \tag{3.4.7}$$

Let $\varepsilon > 0$; then (3.4.6) and (3.4.7) show that we can choose $k > 0$ such that

$$\sup_{\nu\in\mathcal{V}} \mathbb{E}\left[\left|\ell\left(Z^{\mathfrak{u}_n,\nu}_{t,z}(T)\right) - \ell\left(Z^{\mathfrak{u},\nu}_{t,z}(T)\right)\right| \mathbf{1}_{\{|Z^{\mathfrak{u}_n,\nu}_{t,z}(T)|+|Z^{\mathfrak{u},\nu}_{t,z}(T)|>k\}}\right] < \varepsilon$$

for all $n$. Using also that $\ell$ is uniformly continuous on $\{|z| \leq k\}$, we thus find $\delta > 0$ such that

$$\begin{aligned}&\sup_{\nu\in\mathcal{V}} \mathbb{E}\left[\left|\ell\left(Z^{\mathfrak{u}_n,\nu}_{t,z}(T)\right) - \ell\left(Z^{\mathfrak{u},\nu}_{t,z}(T)\right)\right|\right]\\ &\quad\leq 2\varepsilon + \sup_{\nu\in\mathcal{V}} \mathbb{E}\left[\left|\ell\left(Z^{\mathfrak{u}_n,\nu}_{t,z}(T)\right) - \ell\left(Z^{\mathfrak{u},\nu}_{t,z}(T)\right)\right| \mathbf{1}_{\{|Z^{\mathfrak{u}_n,\nu}_{t,z}(T)-Z^{\mathfrak{u},\nu}_{t,z}(T)|>\delta\}}\right].\end{aligned}$$

By (3.4.5) and (3.4.7), the supremum on the right-hand side tends to zero as $n \to \infty$. This completes the proof of Lemma 3.4.1.

**Proof.** [Proof of "$\leq$" in (3.4.3).] It follows from the polynomial growth of $g$ and the boundedness of $V$ that the right-hand side of (3.4.3) is finite. Thus, the already established inequality "$\geq$" in (3.4.3) yields that $\gamma(t,x,p) > -\infty$. We now show the theorem under the hypothesis that $\gamma(t,x,p) < \infty$ for all $p$; we shall argue at the end of the proof that this is automatically satisfied.

**Step 1:** *Let $\gamma_\infty := \inf_n \gamma_n$. Then the upper semicontinuous envelopes of $\gamma$ and $\gamma_\infty$ coincide: $\gamma^* = \gamma^*_\infty$.*

It follows from (3.4.4) that $\gamma^*_\infty \geq \gamma^*$. Let $\eta > 0$ and $y > \gamma(t,x,p+2\eta)$. We show that $y \geq \gamma_n(t,x,p)$ for $n$ large; this will imply the remaining inequality $\gamma^*_\infty \leq \gamma^*$. Indeed, the definition of $\gamma$ and Lemma 3.4.1 imply that we can find $\mathfrak{u} \in \mathfrak{U}$ and $\mathfrak{u}_n \in \mathfrak{U}^n$ such that

$$J(t,x,y,\mathfrak{u}_n) \geq J(t,x,y,\mathfrak{u}) - \epsilon_n \geq p + \eta - \epsilon_n \quad \mathbb{P}\text{-a.s.},$$

where $\epsilon_n \to 0$ in $L^1$. If $K_n$ is defined like $K$, but with $\mathfrak{U}^n$ instead of $\mathfrak{U}$, then it follows that $K_n(t,x,y) \geq p + \eta - \epsilon_n$ $\mathbb{P}$-a.s. Recalling that $K_n$ is deterministic (cf. Proposition 3.3.1), we may replace $\epsilon_n$ by $\mathbb{E}[\epsilon_n]$ in this inequality. Sending $n \to \infty$, we then see that $\lim_{n\to\infty} K_n(t,x,y) \geq p+\eta$, and therefore $K_n(t,x,y) \geq p+\eta/2$ for $n$ large enough.

**Step 2:** *The relaxed semi-limit*

$$\bar{\gamma}^*_\infty(t,x,p) := \limsup_{\substack{n\to\infty \\ (t',x',p')\to(t,x,p)}} \gamma^*_n(t',x',p')$$

*is a viscosity subsolution on $[0,T) \times (0,\infty) \times \mathbb{R}$ of*

$$-\partial_t \varphi + \inf_{\sigma \in [\underline{\sigma},\overline{\sigma}]} \left\{ -\frac{1}{2}\sigma^2 x^2 \partial_{xx}\varphi \right\} \leq 0 \tag{3.4.8}$$

*and satisfies the boundary condition $\bar{\gamma}^*_\infty(T,x,p) \leq g(x) + \Psi^{-1}(p)$.*

We first show that the boundary condition is satisfied. Fix $(x,p) \in (0,\infty) \times \mathbb{R}$ and let $y > g(x) + \Psi^{-1}(p)$; then $\ell(x,y) > p$. Let $(t_n,x_n,p_n) \to (T,x,p)$ be such that $\gamma_n(t_n,x_n,p_n) \to \bar{\gamma}^*_\infty(T,x,p)$. We consider the strategy $\mathfrak{u} \equiv 0$ and use the arguments from the proof of Proposition 3.3.1 to find a constant $c$ independent of $n$ such that

$$\operatorname*{ess\,sup}_{\nu\in\mathcal{V}} \mathbb{E}\left[ |Z^{0,\nu}_{t_n,x_n,y}(T) - (x,y)|^{\bar{q}} \,|\mathcal{F}_{t_n} \right] \leq c\left( |T-t_n|^{\frac{\bar{q}}{2}} + |x-x_n|^{\bar{q}} \right).$$

Similarly as in the proof of Lemma 3.4.1, this implies that there exist constants $\varepsilon_n \to 0$ such that

$$J(t_n,x_n,y,0) \geq \ell(x,y) - \varepsilon_n \quad \mathbb{P}\text{-a.s.}$$

In view of $\ell(x,y) > p$, this shows that $y \geq \gamma_n(t_n,x_n,p_n)$ for $n$ large enough, and hence that $y \geq \bar{\gamma}^*_\infty(T,x,p)$. As a result, we have $\gamma^*_\infty(T,x,p) \leq g(x) + \Psi^{-1}(p)$.

It remains to show the subsolution property. Let $\varphi$ be a smooth function and let $(t_o,x_o,p_o) \in [0,T) \times (0,\infty) \times \mathbb{R}$ be such that

$$(\bar{\gamma}^*_\infty - \varphi)(t_o,x_o,p_o) = \max(\bar{\gamma}^*_\infty - \varphi) = 0.$$

After passing to a subsequence, [Bar94, Lemma 4.2] yields $(t_n,x_n,p_n) \to (t_o,x_o,p_o)$ such that

$$\lim_{n\to\infty} (\gamma^*_n - \varphi)(t_n,x_n,p_n) = (\bar{\gamma}^*_\infty - \varphi)(t_o,x_o,p_o),$$

and such that $(t_n,x_n,p_n)$ is a local maximizer of $(\gamma^*_n - \varphi)$. Applying Theorem 3.3.5 to $\gamma^*_n$, we deduce that

$$\sup_{(\hat{u},\hat{a}) \in K^n_{Lip}(\cdot,D\varphi)} \inf_{(\mu,\sigma)\in V} G\varphi(\cdot,(\hat{u},\hat{a})(\mu,\sigma),(\mu,\sigma))(t_n,x_n,p_n) \leq 0, \tag{3.4.9}$$

where

$$G\varphi(\cdot,(u,a),(\mu,\sigma)) := u\mu - \partial_t\varphi - \mu x \partial_x \varphi - \frac{1}{2}\left(\sigma^2 x^2 \partial_{xx}\varphi + a^2 \partial_{pp}\varphi + 2\sigma x a \partial_{xp}\varphi\right)$$

and $K^n_{Lip}(\cdot,D\varphi)(t_n,x_n,p_n)$ is the set of locally Lipschitz mappings $(\hat{u},\hat{a})$ with values in $[-n,n] \times \mathbb{R}$ such that

$$\sigma\hat{u}(x,q_x,q_p,\mu,\sigma) = x\sigma q_x + q_p \hat{a}(x,q_x,q_p,\mu,\sigma) \quad \text{for all } \sigma \in [\underline{\sigma},\overline{\sigma}]$$

for all $(x,(q_x,q_p))$ in a neighborhood of $(x_n, D\varphi(t_n,x_n,p_n))$. Since the mapping

$$(0,\infty) \times \mathbb{R}^2 \times [\underline{\mu},\overline{\mu}] \times [\underline{\sigma},\overline{\sigma}] \to \mathbb{R}^2, \quad (x,q_x,q_p,\mu,\sigma) \mapsto (xq_x, 0)$$

belongs to $K^n_{Lip}(\cdot,D\varphi)(t_n,x_n,p_n)$ for $n$ large enough, (3.4.9) leads to

$$-\partial_t\varphi + \inf_{\sigma\in[\underline{\sigma},\overline{\sigma}]} \left\{-\frac{1}{2}\sigma^2 x^2 \partial_{xx}\varphi\right\}(t_n,x_n,p_n) \leq 0$$

for $n$ large. Here the nonlinearity is continuous; therefore, sending $n \to \infty$ yields (3.4.8).

<u>**Step 3:**</u> *We have $\bar{\gamma}^*_\infty \leq \pi$ on $[0,T] \times (0,\infty) \times \mathbb{R}$, where*

$$\pi(t,x,p) := \sup_{\nu\in\mathcal{V}^0} \mathbb{E}\left[g\left(X^\nu_{t,x}(T)\right)\right] + \Psi^{-1}(p)$$

*is the right hand side of* (3.4.3).

Indeed, our assumptions on $g$ and $\Psi^{-1}$ imply that $\pi$ is continuous with polynomial growth. It then follows by standard arguments that $\pi$ is a viscosity supersolution on $[0,T) \times (0,\infty) \times \mathbb{R}$ of

$$-\partial_t\varphi + \inf_{\sigma\in[\underline{\sigma},\overline{\sigma}]} \left\{-\frac{1}{2}\sigma^2 x^2 \partial_{xx}\varphi\right\} \geq 0,$$

and clearly the boundary condition $\pi(T,x,p) \geq g(x) + \Psi^{-1}(p)$ is satisfied. The claim then follows from Step 2 by comparison.

We can now deduce the theorem: We have $\gamma \leq \gamma^*$ by the definition of $\gamma^*$ and $\gamma^* = \gamma^*_\infty$ by Step 1. As $\gamma^*_\infty \leq \bar{\gamma}^*_\infty$ by construction, Step 3 yields the result.

It remains to show that $\gamma < \infty$. Indeed, this is clearly satisfied when $g$ is bounded from above. For the general case, we consider $g_m = g \wedge m$ and let $\gamma_m$ be the corresponding value function. Given $\eta > 0$, we have $\gamma_m(t,x,p+\eta) < \infty$ for all $m$ and so (3.4.3) holds for $g_m$. We see from (3.4.3) that $y := 1 + \sup_m \gamma_m(t,x,p+\eta)$ is finite. Thus, there exist $\mathfrak{u}_m \in \mathfrak{U}$ such that

$$\mathbb{E}\left[\Psi\left(Y^{\mathfrak{u}_m,\nu}_{t,y}(T) - g_m\left(X^\nu_{t,x}(T)\right)\right)|\mathcal{F}_t\right] \geq p + \eta/2 \quad \text{for all } \nu \in \mathcal{V}.$$

Using once more the boundedness of $V$, we see that for $m$ large enough,

$$\mathbb{E}\left[\Psi\left(Y^{\mathfrak{u}_m,\nu}_{t,y}(T) - g\left(X^\nu_{t,x}(T)\right)\right)|\mathcal{F}_t\right] \geq p \quad \text{for all } \nu \in \mathcal{V},$$

which shows that $\gamma(t,x,p) \leq y < \infty$.

**Remark 3.4.2.** We sketch a probabilistic proof for the inequality "$\leq$" in Theorem 3.4.1, for the special case without drift ($\underline{\mu} = \overline{\mu} = 0$) and $\underline{\sigma} > 0$. We focus on $t = 0$ and recall that $y_0 := \sup_{\nu \in \mathcal{V}^0} \mathbb{E}[g(X^\nu_{0,x}(T))]$ is the superhedging price for $g(\cdot)$ in the uncertain volatility model. More precisely, if $B$ is the coordinate-mapping process on $\Omega = C([0,T];\mathbb{R})$, there exists an $\mathbb{F}^B$-progressively measurable process $\vartheta$ such that

$$y_0 + \int_0^T \vartheta_s \frac{dB_s}{B_s} \geq g(B_T) \quad P^\nu\text{-a.s. for all } \nu \in \mathcal{V}^0,$$

where $P^\nu$ is the law of $X^\nu_{0,x}$ under $P$ (see, e.g., [NS11]). Seeing $\vartheta$ as an adapted functional of $B$, this implies that

$$y_0 + \int_0^T \vartheta_s(X^\nu_{0,x}) \frac{dX^\nu_{0,x}(s)}{X^\nu_{0,x}(s)} \geq g(X^\nu_{0,x}(T)) \quad P\text{-a.s. for all } \nu \in \mathcal{V}^0.$$

Since $X^\nu_{0,x}$ is non-anticipating with respect to $\nu$, we see that $\mathfrak{u}[\nu]_s := \vartheta_s(X^\nu_{0,x})$ defines a non-anticipating strategy such that, with $y := y_0 + \Psi^{-1}(p)$,

$$y + \int_0^T \mathfrak{u}[\nu]_r \frac{dX^\nu_{0,x}(s)}{X^\nu_{0,x}(s)} \geq g(X^\nu_{0,x}(T)) + \Psi^{-1}(p);$$

that is,

$$\Psi\left(Y^{\mathfrak{u},\nu}_{0,y}(T) - g(X^\nu_{0,x}(T)\right) \geq p$$

holds even $P$-almost surely, rather than only in expectation, for all $\nu \in \mathcal{V}^0$, and $\mathcal{V}^0 = \mathcal{V}$ because of our assumption that $\underline{\mu} = \overline{\mu} = 0$. In particular, we have the *existence of an optimal strategy* $\mathfrak{u}$. (We notice that, in this respect, it is important that our definition of strategies does not contain regularity assumptions on $\nu \mapsto \mathfrak{u}[\nu]$.)

Heuristically, the case with drift uncertainty (i.e., $\underline{\mu} \neq \overline{\mu}$) can be reduced to the above by a Girsanov change of measure argument; e.g., if $\mu$ is deterministic, then we can take $\mathfrak{u}[(\mu, \sigma)] := \mathfrak{u}[(0, \sigma^\mu)]$, where $\sigma^\mu(\omega) := \sigma(\omega + \int \mu_t \, dt)$. However, for general $\mu$, there are difficulties related to the fact that a Girsanov Brownian motion need not generate the original filtration (see, e.g., [FS97]), and we shall not enlarge on this.

# Utility based pricing: Asymptotic Risk diversification - Abstract

**Abstract**

In principle, liabilities combining both insurancial risks (e.g. mortality/longevity, crop yield,...) and pure financial risks cannot be priced neither by applying the usual actuarial principles of diversification, nor by arbitrage-free replication arguments. Still, it has been often proposed in the literature to combine these two approaches by suggesting to hedge a pure financial payoff computed by taking the mean under the historical/objective probability measure on the part of the risk that can be diversified. Not surprisingly, simple examples show that this approach is typically inconsistent for risk adverse agents. We show that it can nevertheless be recovered asymptotically if we consider a sequence of agents whose absolute risk aversions go to zero and if the number of sold claims goes to infinity simultaneously. This follows from a general convergence result on utility indifference prices which is valid for both complete and incomplete financial markets. In particular, if the underlying financial market is complete, the limit price corresponds to the hedging cost of the mean payoff. If the financial market is incomplete but the agents behave asymptotically as exponential utility maximizers with vanishing risk aversion, we show that the utility indifference price converges to the expectation of the discounted payoff under the minimal entropy martingale measure.

**Keywords:** Utility indifference pricing, diversification, risk aversion, entropy.

**Note:** The work presented in this chapter is taken from Bouchard, Elie and Moreau [BEM12], and has been accepted for publication in *Mathematics and Financial Economics*. This work has been co-authored with Pr. Bruno Bouchard[2] and Dr. Romuald Elie[3].

[2]CEREMADE, Université Paris Dauphine
and CREST-ENSAE, bouchard@ceremade.dauphine.fr
[3]CEREMADE, Université Paris Dauphine and CREST-ENSAE, elie@ceremade.dauphine.fr

# Chapter 4

# Utility based pricing: Asymptotic Risk diversification

## Contents



## 4.1 Introduction

These last years have seen the explosion of the number of liabilities combining pure financial and pure insurancial risks. They typically have the following form: an insurance company sells to the client $i$ a claim with discounted payoff $g^i$ paid at maturity $T$ whose value depends on the evolution of some tradable financial assets $S = (S_t)_{t\geq 0}$ and some additional idiosyncratic risk. The $g^i$'s are usually not unconditionally independent, but still independent conditionally to $S$.

It is (essentially) the case of many variable annuities schemes in which death times or withdrawals policies can be assumed to be independent conditionally to the financial market's behavior, see e.g. [BKR08]. This is also the case for crop revenue insurance schemes that depend on the production yield of the farmer and the market price of the crop, see e.g. [GK02]. More examples can be found in [EC00] or [Bec03].

In such a situation, and if the financial market formed by the assets $S$ is complete, it is tempting to play on the ability to diversify the conditionally idiosyncratic risks and cover the systemic pure financial risk by dynamically trading on the market. If the $g^i$'s are independent and identically distributed given $S$, then the price of each of these contingent claims could be defined as $\bar{p} := \mathbb{E}^{\mathbb{Q}}\left[\bar{g}(S)\right]$ where $\bar{g}(S) := \mathbb{E}\left[g^i|S\right]$ does not depend on $i$, and $\mathbb{Q}$ denotes the unique martingale measure on the pure financial market (i.e. restricted to $S$). The rationality behind this is the following: by an informal application of the law of large numbers conditionally to $S$, we obtain the convergence $G_n/n := \sum_{i=1}^{n} g^i/n \to \bar{g}(S)$ a.s. for a large number $n$ of sold contracts. In the above, the payoff $\bar{g}(S)$ only depends on $S$ and can thus be hedged dynamically by trading on the (complete) pure financial market. Hence, by replicating the mean payoff $\bar{g}(S)$, we end up with a zero net position in mean (under the initial probability measure $\mathbb{P}$).

This solution has been originally proposed by Brennan and Schwartz [BS79a, BS79b], and then applied several times, in particular in the literature on variables annuities, see e.g. [BKR08], [BH03], [MP00] or [MPY06]. However, it seems to ignore the fact that playing with the law of large numbers on the diversifiable part of the risk requires selling a large number of contracts, and therefore may lead to huge positions on the financial market. If the law of large numbers does not operate well enough, then the losses may be leveraged by an unfavorable evolution of the financial market. More generally, the classical central limit theorem that allows to control the asymptotic distribution of the risk in terms of the Gaussian law will in general not apply in this context.

One classical solution for pricing such claims is to use the indifference pricing rule of Hodge-Neuberger [HN89], see e.g. [Bec03] for the exponential utility case. As expected, it typically does not lead to the price $\bar{p}$ defined as above, see Section 4.2.2 for trivial counter-examples. However, one should intuitively recover it asymptotically when the number of sold contracts is large, so that the conditional law of large numbers can operate, and the risk aversion is small.

In this note, we provide sufficient conditions under which the above holds true. Namely, we consider a family of utility functions $(U_n)_n$, defined on $\mathbb{R}$, with corresponding absolute risk aversion $(r_n)_n$ and indifference prices $(np_n)_n$ for the aggregate claims $(G_n)_n$. Under mild assumptions detailed in Section 4.3.2, we show that $n \to \infty$ and $n|r_n|_\infty \to 0$ implies $p_n \to \bar{p}$, whenever the underlying financial market formed by the liquid financial assets $S$ is complete. This follows from a more general asymptotic result derived in Section 4.3.1, which provides a formulation for the asymptotic unit price $\lim_n p_n$ in terms of the sequence of martingales measures minimizing the corresponding dual problems. The latter applies to incomplete financial markets without providing a clear identification of the asymptotic pricing measure, except when $(U_n)_n$ behaves asymptotically like a sequence of exponential utility functions. In this case, we show that $p_n \to p^e := \mathbb{E}^{\mathbb{Q}^e}[\bar{g}(S)]$ where $\mathbb{Q}^e$ is the martingale measure with minimum relative entropy, see Section 4.3.3. This generalizes to our setting the well-known property that the exponential utility indifference price of a claim converges to the risk neutral price under $\mathbb{Q}^e$ for vanishing risk aversion, see [EKR00] and [Bec03]. Note that a similar result is obtained in [Bec03] for the indifference price of the mean payoff $\bar{G}_n := G_n/n = \sum_{i=1}^n g^i/n$ as $n$ goes to infinity and the risk aversion is fixed, which is a completely different situation.

In the following, any assertion involving random variables has to be understood in the a.s. sense. Given a probability measure $\mathbb{Q}$ and a sigma-algebra $\mathcal{G}$, we denote by $L^1(\mathbb{Q}, \mathcal{G})$ (resp. $L^\infty(\mathbb{Q}, \mathcal{G})$) the space of $\mathbb{Q}$-integrable (resp. $\mathbb{Q}$-essentially bounded) random variables that are $\mathcal{G}$-measurable. We omit the argument $\mathbb{Q}$ or $\mathcal{G}$ if it is clearly given by the context.

## 4.2 Diversification based pricing rules and risk aversion

In this section, we describe the financial market and elaborate on the relation between diversification and utility indifference pricing.

### 4.2.1 The market model

From now on, we fix a time horizon $T > 0$ to avoid unnecessary technical issues, although for some applications (e.g. mortality/longetivity linked contracts) it should in principle be infinite. We consider a model of a security market which consists of $d$ stocks with price process described by a locally bounded càdlàg semi-martingale $(S^i)_{1 \le i \le d}$ defined on some complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} := (\mathcal{F}_t)_{0 \le t \le T}, \mathbb{P})$, with $\mathbb{F}$ satisfying the usual assumptions and $\mathcal{F} = \mathcal{F}_T$. As usual, we normalize the risk free rate to 0 for simplicity, which can always been done by considering discounted values.

A (self financing) strategy is defined as an element $\vartheta = (\vartheta^i)_{1 \le i \le d}$ of the set $\Theta$ of $\mathbb{F}$-predictable $S$-integrable processes. Given an initial endowment $x \in \mathbb{R}$ and a strategy $\vartheta \in \Theta$, the induced wealth process $X^{x,\vartheta} = (X_t^{x,\vartheta})_{0 \le t \le T}$ is given by

$$X_t^{x,\vartheta} = x + \int_0^t \vartheta_u \cdot dS_u \ , \qquad 0 \le t \le T \ .$$

In order to avoid *doubling strategies*, we restrict as usual to strategies leading to bounded from below wealth processes. We denote by $\mathfrak{X}^b(x)$ the family of terminal values of wealth processes starting from $x$ such that the above holds:

$$\mathfrak{X}^b(x) := \left\{ X_T^{x,\vartheta} : \vartheta \in \Theta,\ X^{x,\vartheta} \geq -\kappa \text{ on } [0,T] \text{ for some } \kappa \in \mathbb{R} \right\}. \tag{4.2.1}$$

Note that $\mathfrak{X}^b(x) = x + \mathfrak{X}^b(0)$.

As usual, a probability measure $\mathbb{Q}$ is called an equivalent local martingale measure if it is equivalent to $\mathbb{P}$ and if $S$ is a $(\mathbb{F},\mathbb{Q})$-local martingale. The family of equivalent local martingales will be denoted by $\mathcal{M}$. We assume throughout this chapter that

$$\mathcal{M} \neq \emptyset, \tag{4.2.2}$$

which ensures the absence of arbitrage opportunities (in the no-free lunch with vanishing risk sense), see [DS94] for details. In the following, we will often use the notation $\mathbb{Q}^*$ to denote a fixed element of $\mathcal{M}$.

Note that we do not impose that $\mathcal{F}_T$ equals $\mathcal{F}_T^S$, where $\mathbb{F}^S = (\mathcal{F}_t^S)_{t \leq T}$ is the completion of the right-continuous filtration generated by $S$, in order to allow for additional randomness.

However, we shall often consider that the *pure financial market* is complete in the following sense.

**Definition 4.2.1.** *We say that the pure financial market is complete, in short* **(HCM)** *holds, if*

$$\mathbb{E}^{\mathbb{Q}^*}[\xi] = \mathbb{E}^{\mathbb{Q}}[\xi] \;\; \textit{for all } \mathbb{Q} \in \mathcal{M} \textit{ and } \xi \in L^\infty(\mathcal{F}_T^S),$$

*where $L^\infty(\mathcal{F}_T^S)$ denotes the set of essentially bounded $\mathcal{F}_T^S$-measurable random variables.*

**Remark 4.2.2.** Under **(HCM)**, we must have $\xi \in \mathfrak{X}^b(\mathbb{E}^{\mathbb{Q}}[\xi])$ for all $\xi \in L^\infty(\mathcal{F}_T^S)$. See [DS94].

**Remark 4.2.3.** Note that, if $\mathcal{F}_T^S \subsetneq \mathcal{F}_T$, then **(HCM)** only implies that the pure financial market is complete in the sense of Remark 4.2.2, and not that $\mathcal{M}$ is reduced to a singleton. As an example, assume that we can find $A \in \mathcal{F}_T$ such that $\mathbb{P}[A] > 0$ and $A$ is independent of $\mathcal{F}_T^S$ given $\mathcal{F}_s$ for all $s \in [0,T]$ under $\mathbb{P}$. Set $H^* := d\mathbb{Q}^*/d\mathbb{P}$, $H^*(S) := \mathbb{E}\left[H^* | \mathcal{F}_T^S\right]$ and $H_A^\varepsilon := \varepsilon H^* + (1-\varepsilon) H^*(S) \mathbb{1}_A / \mathbb{P}[A]$ for some $\varepsilon \in (0,1]$. Then, for any increasing sequence of $\mathbb{F}^S$-stopping times $(\tau_k)_{k \geq 1}$ such that $S^{(k)} :=$

$S_{\cdot\wedge\tau_k}$ is bounded on $[0,T]$ for all $k \geq 1$, we have

$$\begin{aligned}
\mathbb{E}\left[H_A^\varepsilon S_t^{(k)}|\mathcal{F}_s\right] &= \varepsilon\mathbb{E}\left[H^*|\mathcal{F}_s\right]S_s^{(k)} + (1-\varepsilon)\mathbb{E}\left[\mathbb{E}\left[H^* S_t^{(k)}|\mathcal{F}_T^S\right]\mathbb{1}_A|\mathcal{F}_s\right]/\mathbb{P}\left[A\right] \\
&= \varepsilon\mathbb{E}\left[H^*|\mathcal{F}_s\right]S_s^{(k)} + (1-\varepsilon)\mathbb{E}\left[\mathbb{E}\left[H^* S_t^{(k)}|\mathcal{F}_T^S\right]|\mathcal{F}_s\right]\mathbb{E}\left[\mathbb{1}_A|\mathcal{F}_s\right]/\mathbb{P}\left[A\right] \\
&= \left(\varepsilon\mathbb{E}\left[H^*|\mathcal{F}_s\right] + (1-\varepsilon)\mathbb{E}\left[H^*(S)|\mathcal{F}_s\right]\mathbb{P}\left[A|\mathcal{F}_s\right]/\mathbb{P}\left[A\right]\right)S_s^{(k)} \\
&= \mathbb{E}\left[H_A^\varepsilon|\mathcal{F}_s\right]S_s^{(k)}
\end{aligned}$$

for $0 \leq s \leq t \leq T$, which shows that the measure $\mathbb{Q}_A^\varepsilon$ defined by $d\mathbb{Q}_A^\varepsilon/d\mathbb{P} = H_A^\varepsilon$ belongs to $\mathcal{M}$. In general $\mathbb{Q}_A^\varepsilon \neq \mathbb{Q}^*$.

**Remark 4.2.4.** The same arguments as in Remark 4.2.3 imply that $\mathbb{Q}^*(S)$ defined by $d\mathbb{Q}^*(S) = \mathbb{E}\left[d\mathbb{Q}^*/d\mathbb{P}|\mathcal{F}_T^S\right]d\mathbb{P}$ belongs to $\mathcal{M}$. Note for later use that $d\mathbb{Q}^*(S) = \mathbb{E}\left[d\mathbb{Q}/d\mathbb{P}|\mathcal{F}_T^S\right]d\mathbb{P}$ for any $\mathbb{Q} \in \mathcal{M}$ when **(HCM)** holds.

**Remark 4.2.5.** Assume that $\mathbb{F}$ can be written as $(\mathcal{F}_t^S \vee \mathcal{F}_t^\perp)_{t\leq T}$ for some filtration $\mathbb{F}^\perp = (\mathcal{F}_t^\perp)_{t\leq T}$ independent of $\mathbb{F}^S$ under $\mathbb{P}$, and satisfying the usual conditions. Then, any $A \in \mathcal{F}_T^\perp$ is independent of $\mathcal{F}_T^S$ given $\mathcal{F}_s$ for all $s \in [0,T]$ under $\mathbb{P}$. Indeed, under the above assumption, $\mathcal{F}_s$ is generated by elements of the form $B_s^S \cap B_s^\perp$ with $B_s^S \in \mathcal{F}_s^S$ and $B_s^\perp \in \mathcal{F}_s^\perp$. Given $\xi \in L^\infty(\mathcal{F}_T^S)$, we then have

$$\mathbb{E}\left[\xi\mathbb{1}_{B_s^S\cap B_s^\perp}\right] = \mathbb{E}\left[\mathbb{E}\left[\xi|\mathcal{F}_s^S\right]\mathbb{1}_{B_s^S}\right]\mathbb{E}\left[\mathbb{1}_{B_s^\perp}\right] = \mathbb{E}\left[\mathbb{E}\left[\xi|\mathcal{F}_s^S\right]\mathbb{1}_{B_s^S\cap B_s^\perp}\right],$$

so that $\mathbb{E}\left[\xi|\mathcal{F}_s^S\right] = \mathbb{E}\left[\xi|\mathcal{F}_s\right]$. Similarly, $\mathbb{E}\left[\mathbb{1}_A|\mathcal{F}_s^\perp\right] = \mathbb{E}\left[\mathbb{1}_A|\mathcal{F}_s\right]$. Moreover,

$$\mathbb{E}\left[\xi\mathbb{1}_A\mathbb{1}_{B_s^S\cap B_s^\perp}\right] = \mathbb{E}\left[\xi\mathbb{1}_{B_s^S}\right]\mathbb{E}\left[\mathbb{1}_A\mathbb{1}_{B_s^\perp}\right] = \mathbb{E}\left[\mathbb{E}\left[\xi|\mathcal{F}_s^S\right]\mathbb{1}_{B_s^S}\right]\mathbb{E}\left[\mathbb{E}\left[\mathbb{1}_A|\mathcal{F}_s^\perp\right]\mathbb{1}_{B_s^\perp}\right]$$

which, combined with the above assertions, leads to

$$\begin{aligned}
\mathbb{E}\left[\xi\mathbb{1}_A\mathbb{1}_{B_s^S\cap B_s^\perp}\right] &= \mathbb{E}\left[\mathbb{E}\left[\xi|\mathcal{F}_s^S\right]\mathbb{1}_{B_s^S}\mathbb{E}\left[\mathbb{1}_A|\mathcal{F}_s^\perp\right]\mathbb{1}_{B_s^\perp}\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\xi|\mathcal{F}_s\right]\mathbb{E}\left[\mathbb{1}_A|\mathcal{F}_s\right]\mathbb{1}_{B_s^S\cap B_s^\perp}\right].
\end{aligned}$$

Hence, $\mathbb{E}\left[\xi\mathbb{1}_A|\mathcal{F}_s\right] = \mathbb{E}\left[\xi|\mathcal{F}_s\right]\mathbb{E}\left[\mathbb{1}_A|\mathcal{F}_s\right]$.

### 4.2.2 Diversification and utility based pricing

We are interested in the pricing by utility indifference, see [HN89], of aggregated claims of the form

$$G_n := \sum_{i=1}^n g^i,\ n \geq 1,$$

where $(g^i)_{i\geq 1}$ is a given sequence of random variables.

Although the specific structure of the $G_n$'s is not so important from the mathematical point of view, we have in mind that each $g^i$ corresponds to a contingent claim sold by an insurance company to a specific agent $i$, and that the $g^i$'s have the same law and are independent conditionally to $\mathcal{F}_T^S$ under $\mathbb{P}$. This means that the $g^i$'s may depend of two sources or risks. One related to the pure financial market behavior, i.e. $S$, the other one coming from an external source of randomness which only depends on the agent $i$.

**Example 4.2.1** (Revenue insurance). Let $S^1$ denote the spot price of one quintal of wheat on the financial market and let $Y^i$ denotes the number of quintals produced by the farmer $i$ at time $T$. The payoff of a revenue insurance takes the form $g^i = [K - Y^i S_T]^+$ for some strike $K > 0$ fixed in advance. It compensates the losses incurred by the farmer $i$ if his revenue $Y^i S_T$, induced by the sale of the production at the spot price $S_T$ at time $T$, is less than a targeted level $K$. If the wheat market contains enough futures and provides enough liquidity, we can consider that it is complete. Moreover, we can also consider that the global level of production (at the level of a sufficiently large area) is already reflected into the prices so that the $Y^i$'s can be assumed to be independent given $\mathcal{F}_T^S$.

**Example 4.2.2** (Mortality derivatives). A simple example takes the form $g^i = f(S, \zeta^i)$ where $f$ is a real valued measurable map on $\mathbb{D} \times ([0,T] \cup \{\infty\})$, with $\mathbb{D}$ denoting the set of càdlàg $\mathbb{R}^d$-valued functions on $[0,T]$ (endowed with the Skorohod topology), and $\zeta^i$ is a $[0,T]$-valued random variable denoting the time of death of $i$ if it is before $T$ and taking the value $\infty$ otherwise. Again, one source of randomness comes from the financial market, while the $\zeta^i$'s can generally be assumed to be independent and with the same law (at least among a given sub-population).

Under the above interpretation, the global liability of the insurance company is $G_n$ if the contracts have been subscribed by the clients 1 to $n$. If the insurance company does not differentiate its clients, then it has to fix the same price $p_n$ to each of them.

If the global market was complete, meaning that $\mathcal{M} = \{\mathbb{Q}^*\}$, and the law of $G_n/n$ under $\mathbb{Q}^*$ was independent on $n$, then $p_n$ should be equal to $\bar{p} := \mathbb{E}^{\mathbb{Q}^*}[G_n/n]$. Obviously the completeness of the global market typically fails for the examples we have in mind. Still, as explained in the introduction, this pricing rule has been proposed in the literature for the case where **(HCM)** holds and $(g^i)_{i\geq 1}$ is a sequence of independent and identically distributed random variables given $\mathcal{F}_T^S$. The latter has essentially to be understood in the sense that one can appeal to the law of large numbers, at least conditionally to $\mathcal{F}_T^S$, so that

$$G_n/n \to \bar{g} \;\; \mathbb{P}\text{-a.s. for some } \bar{g} \in L^\infty(\mathcal{F}_T^S). \tag{4.2.3}$$

Under **(HCM)**, one can indeed find $\vartheta \in \Theta$ such that

$$X_T^{\bar{p},\vartheta} = \bar{g} \quad \text{for} \quad \bar{p} = \mathbb{E}^{\mathbb{Q}^*}[\bar{g}],$$

recall Remark 4.2.2. Then, (4.2.3) implies that

$$\frac{X_T^{n\bar{p},n\vartheta} - G_n}{n} \to 0 \quad \mathbb{P}\text{-a.s.}$$

i.e. by replicating $n\bar{g}$ we end up with a hedging error that converges $\mathbb{P}$-a.s. to 0. This is achieved by considering the strategy $n\vartheta$ and starting from the $n$ initial premiums, each equal to $\bar{p}$.

This is however inconsistent with the typical behavior of a risk adverse agent. In particular, it has no reason to be in accordance with a (unit) utility indifference price defined by

$$p_n(G_n, U) := \inf\{p \in \mathbb{R} : \sup_{X \in \mathfrak{X}^b(np)} \mathbb{E}[U(X - G_n)] \geq \sup_{X \in \mathfrak{X}^b(0)} \mathbb{E}[U(X)]\}, \quad (4.2.4)$$

where $U$ is a concave non-decreasing function viewed as a utility function, and where we restrict to a 0 initial endowment (before selling the claims) without loss of generality since any fixed initial wealth can be incorporated in the utility function by a simple translation argument.

We conclude this section with simple counter-examples, where we observe that the limit of the asymptotic utility indifference price $p_n(G_n, U)$ indeed does not coincide with the price of $\bar{g}$ computed under $\mathbb{Q}^*$. In order to find conditions under which this commonly used pricing rule holds at the limit, we will therefore consider in the next section agents with "almost zero" risk aversion. The first example concerns utility functions with bounded from below domains.

**Example 4.2.3** (Utility with bounded from below domain)**.** Let $U$ be concave non-decreasing with values in $\mathbb{R} \cup \{-\infty\}$ such that $|U(0)| + |U(\infty)| < \infty$ and $U(-c) = -\infty$ for some $c > 0$. Let $p_n := p_n(G_n, U)$ be defined as in (4.2.4). Since $U(0) > -\infty$ and $U$ is bounded from above, for each $\varepsilon > 0$, there must exist $\xi^{n,\varepsilon} \in \mathfrak{X}^b(0)$ such that $np_n + \varepsilon + \xi^{n,\varepsilon} \geq G_n - c$. This implies that $p_n \geq \sup_{\mathbb{Q} \in \mathcal{M}} \mathbb{E}^{\mathbb{Q}}[G_n/n] - c/n$ and therefore

$$\liminf_{n \to \infty} p_n \geq \liminf_{n \to \infty} \sup_{\mathbb{Q} \in \mathcal{M}} \mathbb{E}^{\mathbb{Q}}[G_n/n]. \quad (4.2.5)$$

Let us now concentrate on the case where $(g^i)_{i \geq 1}$ is defined as in Example (4.2.2) with $f$ bounded. Assume that the $\zeta^i$'s form an independent family with a common law under $\mathbb{P}$, and that $\sigma(\zeta^i, i \geq 1) \subset \mathcal{F}_T^{\perp}$ with $\mathbb{F}^{\perp}$ satisfying the conditions of Remark 4.2.5. Note that

$$\mathbb{E}\left[H^*(S)f(S,\zeta^i)|\mathcal{F}_T^{\perp}\right] = \mathbb{E}\left[H^*(S)f(S,\zeta^i)|\zeta^i\right], \quad i \geq 1,$$

where $H^*(S)$ is defined as in Remark 4.2.3. Set $\psi := \operatorname{ess\,sup} \mathbb{E}\left[H^*(S)f(S,\zeta^1)|\zeta^1\right]$. For $k, n \geq 1$ and $\varepsilon \in (0,1)$, we define $A_k^n := \{\mathbb{E}\left[H^*(S)f(S,\zeta^i)|\zeta^i\right] \geq \psi - k^{-1}$, for all $i \leq n\}$ and $H_{n,k}^{\varepsilon} := \varepsilon H^* + (1-\varepsilon)H^*(S)\mathbb{1}_{A_k^n}/\mathbb{P}\left[A_k^n\right]$. Note that $\mathbb{P}\left[A_k^n\right] > 0$ since the $\zeta^i$'s are independent and have the same law under $\mathbb{P}$. Then, according to Remarks 4.2.5 and 4.2.3, $\mathbb{Q}_{n,k}^{\varepsilon} := H_{n,k}^{\varepsilon} \cdot \mathbb{P} \in \mathcal{M}$. Recalling (4.2.5), this implies that

$$\liminf_{n\to\infty} p_n \geq \liminf_{n\to\infty} \lim_{k\to 0} \lim_{\varepsilon\to 0} \mathbb{E}^{\mathbb{Q}_{n,k}^{\varepsilon}}[G_n/n] = \psi.$$

Clearly, the above lower bound is typically strictly larger than $\mathbb{E}^{\mathbb{Q}^*}\left[\mathbb{E}\left[f(S,\zeta^1)|\mathcal{F}_T^S\right]\right]$, while applying the law of large numbers conditionally to $\mathcal{F}_T^S$ implies that $G_n/n \to \bar{g} = \mathbb{E}\left[f(S,\zeta^1)|\mathcal{F}_T^S\right]$ $\mathbb{P}$-a.s. For instance, if $f$ is lower-semicontinuous, non-decreasing with respect to its second parameter, and if each $\zeta^i$ has a support equal to $[y_{min}, y_{max}] \subset \mathbb{R}$ under $\mathbb{P}$, then $\psi = \mathbb{E}\left[H^*(S)f(S,y_{max})\right] = \mathbb{E}^{\mathbb{Q}^*}\left[f(S,y_{max})\right] = \mathbb{E}^{\mathbb{Q}^*}\left[\max\{f(S,y), y_{min} \leq y \leq y_{max}\}\right]$. Under **(HCM)**, the later is the hedging price of $\max\{f(S,y), y_{min} \leq y \leq y_{max}\}$, recall Remark 4.2.2, so that $p_n \leq \psi$ for all $n \geq 1$, and therefore $p_n \to \psi$.

In general, going through utility function with unbounded domain does not help, as show in our second example.

**Example 4.2.4** (Exponential utility function)**.** Let $U$ be an exponential utility function of the form $U^{\eta}(y) = -e^{-\eta y}$, $\eta > 0$. Assume that the $g^i$'s have the form taken in Example (4.2.3), that $\mathcal{F}_t^{\perp} = \sigma(\mathcal{F}_t^{\perp,i}, i \geq 1)$, $t \leq T$, for some filtrations $(\mathbb{F}^{\perp,i})_{i\geq 1}$ such that $\mathcal{F}_T^S$, $\mathcal{F}_T^{\perp,1}$, $\mathcal{F}_T^{\perp,2}, \ldots$ are independent under $\mathbb{P}$, and that $\zeta^i$ is $\mathcal{F}_T^{\perp,i}$-measurable for each $i \geq 1$. Then, it can be shown that $p_n^{\eta} := p_n(G_n, U^{\eta}) = p_1(g^1, U^{\eta}) =: p^{\eta}$ whenever there exists a unique element of $\mathcal{M}$ with finite relative entropy and whose density is $\mathcal{F}_T^S$-measurable, see [Bec03, Theorem 4.10]. It is in particular the case if **(HCM)** holds and $\mathbb{Q}^*(S)$ defined in Remark 4.2.4 has a finite relative entropy. On the other hand, it is well-known that $p^{\eta}$ converges to the super-hedging price of $g^1$ as $\eta \to \infty$, see [DGR+02] and [Bec03]. This implies that, for all $\varepsilon > 0$, we can find $\eta_{\varepsilon}$ large enough so that $\lim_n p_n^{\eta_{\varepsilon}} = p^{\eta_{\varepsilon}} \geq \sup_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}^{\mathbb{Q}}\left[g^1\right] - \varepsilon$. For $\varepsilon > 0$ small enough, this is again typically greater than $\mathbb{E}^{\mathbb{Q}^*}\left[\mathbb{E}\left[f(S,\zeta^1)|\mathcal{F}_T^S\right]\right]$.

## 4.3 Asymptotic diversification rule

As pointed out in the previous examples, the convergence of the mean aggregated claim $G_n/n$ to a replicable claim $\bar{g} \in L^{\infty}(\mathcal{F}_T^S)$ is not enough in order to ensure the convergence of its unit utility indifference price $p_n(G_n, U)$ to $\mathbb{E}^{\mathbb{Q}^*}[\bar{g}]$. Intuitively, this can be recovered only if the risk aversion vanishes and the number of sold claims goes to infinity.

Hence, we consider from now on a sequence of utility functions $(U_n)_{n\geq 1}$, which depends on the number $n$ of sold claims $(g^i)_{0\leq i\leq n}$, so as to model the asymptotic situation in which the risk aversion is almost zero and the number of claims is very large : the purpose of this section is to investigate the asymptotic behavior of the corresponding unit utility indifference price when $n$ times the absolute risk aversion of $U_n$ vanishes to 0 as $n$ goes to $\infty$.

We first provide a general characterization of the asymptotic unit utility indifference price in terms of the sequence of associated dual pricing measures. Whenever the pure financial market is complete, i.e. **(HCM)** is satisfied, this limit identifies to the risk neutral price of $\bar{g}$. When the pure financial market is incomplete, it does not seem possible to obtain a precise characterization of the limit price, except when $(U_n)_{n\geq 1}$ behaves asymptotically like a sequence of exponential utility functions with vanishing absolute risk aversion. In this case, we prove that the asymptotic price coincides with the price of $\bar{g}$ under the minimal entropy martingale measure.

### 4.3.1 General convergence result

In this section, we consider a sequence of twice continuously differentiable, strictly concave and increasing utility functions $(U_n)_{n\geq 1}$ defined on the whole real line and satisfying the Inada conditions:

$$\begin{aligned} U_n'(-\infty) = \lim_{x\to-\infty} U_n'(x) = \infty \\ \text{and} \qquad U_n'(\infty) = \lim_{x\to\infty} U_n'(x) = 0 \,, \end{aligned} \qquad n \geq 1 \,. \tag{4.3.1}$$

Besides, we suppose that all the utility functions have a *reasonable asymptotic elasticity* as defined in [Sch01], i.e.

$$\limsup_{x\to\infty} \frac{xU_n'(x)}{U_n(x)} < 1 \,, \qquad \liminf_{x\to-\infty} \frac{xU_n'(x)}{U_n(x)} > 1 \,, \qquad n \geq 1 \,. \tag{4.3.2}$$

We finally introduce the convex conjugates of the $U_n$'s defined by

$$V_n \;:\; y \in (0,\infty) \mapsto \sup_{x\in\mathbb{R}} \{U_n(x) - xy\},$$

and assume that the dual problems are finite:

$$\{(y,\mathbb{Q}) \in (0,\infty) \times \mathcal{M} \;:\; \mathbb{E}\left[V_n(yd\mathbb{Q}/d\mathbb{P})\right] < \infty\} \neq \emptyset \;\text{ for all }\; n \geq 1. \tag{4.3.3}$$

Under the additional uniform boundedness assumption

$$\sup_{n\geq 1} |G_n/n|_{L^\infty} < \infty, \tag{4.3.4}$$

the unit utility indifference prices $p_n(G_n, U_n)$ given by (4.2.4) are well defined for any $n \geq 1$ and existence for the optimal *dual probability and multiplier*, given by

$$(y_n^0, \mathbb{Q}_n^0) := \arg\min \left\{ \mathbb{E}\left[ V_n\left( y\frac{d\mathbb{Q}}{d\mathbb{P}} \right) \right],\ (y, \mathbb{Q}) \in (0,\infty) \times \mathcal{M} \right\}, \tag{4.3.5}$$

is guaranteed, see e.g. Bouchard, Touzi and Zeghal [BTZ04, Theorem 3.1, Remark 3.3 and Proposition 3.1] (see also [Sch01] for a fomulation in terms of absolutely continuous local martingale measures).

In the rest of this section, we work under the standing assumption:

**Assumption 4.3.1.** *The conditions* (4.3.1)*,* (4.3.2)*,* (4.3.3) *and* (4.3.4) *hold.*

In order to derive the asymptotic behavior of the unit indifference price, we shall work under the following additional condition on the asymptotic absolute risk aversion:

$$n|r_n|_\infty \underset{n\to\infty}{\longrightarrow} 0, \qquad \text{with} \quad r_n : x \mapsto -\frac{U_n''(x)}{U_n'(x)}, \tag{4.3.6}$$

and $|r_n|_\infty := \sup_{x\in\mathbb{R}} |r_n(x)|$.

**Theorem 4.3.2.** *Let* (4.3.1) *and* (4.3.6) *hold. Then, the sequence of utility indifference prices satisfies*

$$\liminf_{n\to\infty} p_n(G_n, U_n) = \liminf_{n\to\infty} \mathbb{E}^{\mathbb{Q}_n^0}\left[G_n/n\right]$$
$$\text{and} \quad \limsup_{n\to\infty} p_n(G_n, U_n) = \limsup_{n\to\infty} \mathbb{E}^{\mathbb{Q}_n^0}\left[G_n/n\right].$$

**Proof.** We set $p_n := p_n(G_n, U_n)$ for ease of notations. We only provide the proof for the lim inf, the other one being similar.

**1.** Given $n \geq 1$, it follows from standard duality arguments, see [Owe02] and Bouchard, Touzi and Zeghal [BTZ04], that

$$\begin{aligned}\inf_{y>0,\mathbb{Q}\in\mathcal{M}} \mathbb{E}\left[ V_n\left( y\frac{d\mathbb{Q}}{d\mathbb{P}} \right) + ynp_n - y\frac{d\mathbb{Q}}{d\mathbb{P}}G_n \right] &= \inf_{y>0,\mathbb{Q}\in\mathcal{M}} \mathbb{E}\left[ V_n\left( y\frac{d\mathbb{Q}}{d\mathbb{P}} \right) \right] \\ &= \mathbb{E}\left[ V_n\left( y_n^0\frac{d\mathbb{Q}_n^0}{d\mathbb{P}} \right) \right],\end{aligned}$$

recall (4.3.5). Taking in particular $(y, \mathbb{Q}) = \left(y_n^0, \mathbb{Q}_n^0\right)$, this implies that

$$\mathbb{E}\left[ V_n\left( y_n^0\frac{d\mathbb{Q}_n^0}{d\mathbb{P}} \right) + y_n^0 np_n - y_n^0\frac{d\mathbb{Q}_n^0}{d\mathbb{P}}G_n \right] \geq \mathbb{E}\left[ V_n\left( y_n^0\frac{d\mathbb{Q}_n^0}{d\mathbb{P}} \right) \right],$$

and therefore $p_n \geq \mathbb{E}^{\mathbb{Q}_n^0}[G_n/n]$.

**2.** On the other hand, it follows from [Sch01] that, for $n \geq 1$, we can find $\hat{X}^n \in L^0(\mathcal{F}_T)$ such that

$$\sup_{X \in \mathfrak{X}^b(0)} \mathbb{E}\left[U_n\left(X_T\right)\right] = \mathbb{E}\left[U_n\left(\hat{X}^n\right)\right]$$

and for which there exists a sequence of optimizers $\left(X^{k,n}\right)_{k\geq 1} \subset \mathfrak{X}^b(0)$ such that

$$U_n\left(X_T^{n,k}\right) \xrightarrow[k\to\infty]{L^1(\mathbb{P})} U_n\left(\hat{X}^n\right). \tag{4.3.7}$$

In order to upper-bound $p_n$, we then introduce the following candidate

$$\pi_n := \inf\left\{p \in \mathbb{R} : \mathbb{E}\left[U_n\left(np + \hat{X}^n - G_n\right)\right] \geq \mathbb{E}\left[U_n\left(\hat{X}^n\right)\right]\right\}.$$

**a.** We first check that

$$\pi_n \geq p_n, \quad \text{for all} \quad n \geq 1. \tag{4.3.8}$$

To this purpose, it suffices to fix $n \in \mathbb{N}$ and show that

$$U_n\left(np + X_T^{n,k} - G_n\right) \xrightarrow[k\to\infty]{L^1(\mathbb{P})} U_n\left(np + \hat{X}^n - G_n\right) \in L^1(\mathbb{P}), \quad \text{for all} \quad p \in \mathbb{R}\,. \tag{4.3.9}$$

To see that the above holds, first note that

$$\begin{aligned} &\left|U_n\left(np + X_T^{n,k} - G_n\right) - U_n\left(np + \hat{X}^n - G_n\right)\right| \\ &\quad = \left|\int_{\hat{X}^n}^{X_T^{n,k}} U_n'(np + t - G_n)dt\right|. \end{aligned} \tag{4.3.10}$$

Consider now the relation

$$\log U_n'(np + t - G_n) - \log U_n'(t) \leq n \left\|\frac{U_n''}{U_n'}\right\|_\infty \left|p - \frac{G_n}{n}\right|, \qquad t, p \in \mathbb{R},$$

which, together with (4.3.4) and (4.3.6) leads to the existence of a constant $C_p$ (which only depends on $p$) such that

$$U_n'(np + t - G_n) \leq C_p U_n'(t), \text{ for all } t \in \mathbb{R}. \tag{4.3.11}$$

Plugging (4.3.11) into (4.3.10) gives

$$\left|U_n\left(np + X_T^{n,k} - G_n\right) - U_n\left(np + \hat{X}^n - G_n\right)\right| \leq C_p \left|U_n\left(X_T^{n,k}\right) - U_n\left(\hat{X}^n\right)\right|.$$

Hence, (4.3.9) follows from (4.3.7), which proves (4.3.8).

**b.** We now conclude the proof by providing an upper bound for $\liminf_{n\to\infty} \pi_n$. By definition of $\pi_n$, the continuity of the non-increasing function $U_n$, (4.3.9) and the monotone convergence theorem, we have

$$\mathbb{E}\left[U_n\left(n\pi_n + \hat{X}^n - G_n\right)\right] = \mathbb{E}\left[U_n\left(\hat{X}^n\right)\right]. \tag{4.3.12}$$

This implies that

$$\mathbb{E}\left[U_n\left(\hat{X}^n\right)\right] = \mathbb{E}\left[U_n\left(\hat{X}^n\right) + U_n'\left(\hat{X}^n\right)(n\pi_n - G_n) + \frac{1}{2}U_n''(\xi_n)(n\pi_n - G_n)^2\right],$$

where $\xi_n$ is a random variable lying in the (random) interval $I_n$ formed by $\hat{X}^n$ and $n\pi_n + \hat{X}^n - G_n$. We now use the fact that $U_n'\left(\hat{X}^n\right) = y_n^0 \frac{d\mathbb{Q}_n^0}{d\mathbb{P}}$, recall (4.3.5) and see [Sch01] and Bouchard, Touzi and Zeghal [BTZ04], to deduce that

$$\mathbb{E}^{\mathbb{Q}_n^0}\left[\pi_n - \frac{G_n}{n} - \frac{nr_n(\xi_n)}{2}\frac{U_n'(\xi_n)}{U_n'\left(\hat{X}^n\right)}\left(\pi_n - \frac{G_n}{n}\right)^2\right] = 0. \tag{4.3.13}$$

We shall prove in c. below that

$$\sup_{n\geq 1}\left|\frac{U_n'(\xi_n)}{U_n'\left(\hat{X}^n\right)}\left(\pi_n - \frac{G_n}{n}\right)^2\right| \leq C, \tag{4.3.14}$$

for some constant $C > 0$. Combining this last estimate with (4.3.6) and (4.3.13) leads to

$$\liminf_{n\to\infty} \pi_n \leq \liminf_{n\to\infty} \mathbb{E}^{\mathbb{Q}_n^0}\left[G_n/n\right],$$

which together with (4.3.8) and step 1. concludes the proof.

**c.** It remains to prove the claim (4.3.14). To see that it holds, we first appeal to (4.3.12) to deduce that $\mathbb{E}\left[U_n'(\tilde{\xi}_n)(n\pi_n - G_n)\right] = 0$ for some random variable $\tilde{\xi}_n$ such that $\tilde{\xi}_n \in I_n$. Since $U_n$ is strictly increasing, we deduce from (4.3.4) that

$$\begin{aligned} |\pi_n - G_n/n| &= \frac{\mathbb{E}\left[U_n'(\tilde{\xi}_n)|G_n|/n\right]}{\mathbb{E}\left[U_n'(\tilde{\xi}_n)\right]} + |G_n/n| \\ &\leq C'\frac{\mathbb{E}\left[U_n'(\tilde{\xi}_n)\right]}{\mathbb{E}\left[U_n'(\tilde{\xi}_n)\right]} + C' = 2C', \qquad n \geq 1, \end{aligned} \tag{4.3.15}$$

for some constant $C' > 0$. Similarly, since $\xi_n \in I_n$, we have

$$\begin{aligned} \log\left(U_n'(\xi_n)\right) - \log\left(U_n'\left(\hat{X}^n\right)\right) &\leq \left\|\frac{U_n''}{U_n'}\right\|_\infty \left|\xi_n - \hat{X}^n\right| \\ &\leq n\left\|\frac{U_n''}{U_n'}\right\|_\infty \left|\pi_n - \frac{G_n}{n}\right| \quad, \quad n \geq 1, \end{aligned}$$

which is bounded uniformly in $n$ thanks to (4.3.6) and (4.3.15). □

**Remark 4.3.3.** Let $(p_{\varphi(n)})_{n\geq 1}$ be a convergent subsequence of $(p_n(G_n, U_n))_{n\geq 1}$. Then, the same arguments as above show that $\lim_{n\to\infty} p_{\varphi(n)} = \lim_{n\to\infty} \mathbb{E}^{\mathbb{Q}^0_{\varphi(n)}}[G_{\varphi(n)}/\varphi(n)]$ whenever $\varphi(n)r_{\varphi(n)} \to 0$.

Observe that a straightforward adaptation of the previous argumentation allows to obtain the convergence of the indifference prices $p_1(G_n, U_n)$ under the weaker condition $\|r_n\|_\infty \to 0$, whenever the sequence $(G_n)_{n\geq 1}$ is assumed to be uniformly bounded in $L^\infty$. This provides in (4.3.4) below a general convergence result for bounded sequences of contingent claims when the absolute risk aversion vanishes in the sup norm, which is of own interest.

**Theorem 4.3.4.** *Let Assumptions 4.3.1, 4.3.2 and 4.3.3 hold. Assume further that* $\|r_n\|_\infty \underset{n\to\infty}{\longrightarrow} 0$ *and that* $\sup_{n\geq 1} \|G_n\|_{L^\infty} < \infty$. *Then, the sequence of utility indifference prices satisfies*

$$\liminf_{n\to\infty} p_1(G_n, U_n) = \liminf_{n\to\infty} \mathbb{E}^{\mathbb{Q}_n^0}[G_n] \quad \textit{and} \quad \limsup_{n\to\infty} p_1(G_n, U_n) = \limsup_{n\to\infty} \mathbb{E}^{\mathbb{Q}_n^0}[G_n].$$

**Remark 4.3.5.** In particular, for a constant sequence $(G_n)_{n\geq 1}$, $G_n = G$ for all $n$, (4.3.4) provides a new insight on the asymptotic behavior of the indifference price of any bounded contingent claim $G$ as the risk aversion of a general utility function goes to zero. The sequence of dual minimizers $(\mathbb{Q}_n^0)_n$ can be analyzed in such a situation along the lines of arguments presented in the next sections.

### 4.3.2 Semi-complete markets

The representation of the asymptotic unit utility indifference price presented in (4.3.2) does not provide a-priori an exact formulation, except in particular cases. When the pure financial market is complete, i.e. **(HCM)** is satisfied, and (4.2.3) holds, we verify hereafter that it coincides with the price under the risk neutral measure $\mathbb{Q}^*$ of the replicable claim $\bar{g}$.

**Corollary 4.3.6.** *Let the conditions of* (4.3.2) *hold. Assume further that* (4.2.3) *and* **(HCM)** *are satisfied. Then the sequence of unit utility indifference prices satisfies*

$$\lim_{n\to\infty} p_n(G_n, U_n) = \mathbb{E}^{\mathbb{Q}^*}[\bar{g}].$$

**Proof.** In view of (4.3.2), it suffices to show that **(HCM)** implies that the minimal dual measure $\mathbb{Q}_n^0$, see (4.3.5), coincides with $\mathbb{Q}^*(S)$ as constructed in Remark 4.2.4, and to apply the dominated convergence theorem, recall (4.3.4) and (4.2.3):

$$\lim_{n\to\infty} p_n(G_n, U_n) = \lim_{n\to\infty} \mathbb{E}^{\mathbb{Q}_n^0}[G_n/n] = \lim_{n\to\infty} \mathbb{E}^{\mathbb{Q}^*(S)}[G_n/n] = \mathbb{E}^{\mathbb{Q}^*(S)}[\bar{g}] = \mathbb{E}^{\mathbb{Q}^*}[\bar{g}],$$

where the last equality follows from the fact that $\bar{g}$ is $\mathcal{F}_T^S$-measurable.

To see this, we use the convexity of $V_n$ to obtain that $\mathbb{E}\left[V_n(y_n^0 \frac{d\mathbb{Q}_n^0}{d\mathbb{P}})\right] \geq \mathbb{E}\left[V_n(y_n^0 H_n^0(S))\right]$ where $H_n^0(S) := \mathbb{E}\left[\frac{d\mathbb{Q}_n^0}{d\mathbb{P}} | \mathcal{F}_T^S\right] = d\mathbb{Q}^*(S)/d\mathbb{P}$ by Remark 4.2.4, under **(HCM)**. Since $\mathbb{Q}^*(S) \in \mathcal{M}$ by Remark 4.2.4 again, this proves the claim.
□

**Remark 4.3.7.** If the $g^i$'s have the same law and form an independent family conditionally to $\mathcal{F}_T^S$ under $\mathbb{P}$, then applying the law of large numbers conditionally to $\mathcal{F}_T^S$ implies $\bar{g} = \mathbb{E}\left[g^1|\mathcal{F}_T^S\right]$, so that $\lim_{n\to\infty} p_n(G_n, U_n) = \mathbb{E}^{\mathbb{Q}^*}\left[\mathbb{E}\left[g^1|\mathcal{F}_T^S\right]\right]$. We thus retrieve asymptotically the hybrid pricing rule which consists in taking the mean on the part of the risk that can be diversified and computing the hedging price of the resulting replicable claim.

**Remark 4.3.8.** In [Bec03], the author refers to *semi-complete product models* to designate situations where the filtration $\mathbb{F}$ and $\mathcal{M}$ have the structure specified in Example 4.2.4, in particular:

**(HCM$^{\mathrm{e}}$)**: there exists only one element $\mathbb{Q}^{\mathrm{e}}(S)$ of $\mathcal{M}$ with finite relative entropy and whose density with respect to $\mathbb{P}$ is $\mathcal{F}_T^S$-measurable.

This later condition is weaker than **(HCM)**, up to the restriction related to the finite relative entropy. However, if we are only interested in utility functions of exponential type, the same argument as in the proof of (4.3.6) above shows that $\mathbb{Q}_n^0 = \mathbb{Q}^{\mathrm{e}}(S)$ whenever **(HCM$^{\mathrm{e}}$)** holds. Assuming further that the conditions of (4.3.2) and (4.2.3) hold, this leads to $\lim_{n\to\infty} p_n(G_n, U_n) = \mathbb{E}^{\mathbb{Q}^{\mathrm{e}}(S)}[\bar{g}]$.

### 4.3.3 Incomplete markets and asymptotically exponential utility behaviors

In a general incomplete framework, it seems to be hopeless to interpret the limit of $\mathbb{E}^{\mathbb{Q}_n^0}[G_n/n]$ as the expectation under some martingale measure of $\bar{g}$, the limit of $G_n/n$, except if the $U_n$'s are all of exponential type, compare with Remark 4.3.8.

In this section, we show that the convergence result of Remark 4.3.8 remains true even if the utility functions do not have a constant absolute risk aversion but only asymptotically behave like a sequence of exponential utility functions in the following sense.

**Assumption 4.3.9.** *There exist two sequences of strictly positive numbers* $\left(\eta_n^1\right)_{n\geq 1}$ *and* $\left(\eta_n^2\right)_{n\geq 1}$ *converging toward* $0$ *such that*

$$0 < \eta_n^2 \leq r_n(x) \leq \eta_n^1 \quad \textit{for all} \;\; x \in \mathbb{R} \;\; \textit{and} \;\; n \geq 1, \tag{4.3.16}$$

$$\lim_{n\to\infty} \eta_n^2/\eta_n^1 = 1. \tag{4.3.17}$$

**Remark 4.3.10.** The existence of the sequence $(\eta_n^1)_{n\geq 1}$ converging to zero is exactly the content of the assumption (4.3.6). (4.3.9) implies that the function $r_n$ is asymptotically bounded in between two sequences converging to zero with the same first order convergence rate. In particular, this assumption includes the case of agents with utility functions of the form $U_n : x \mapsto -(\lambda_n)^{-1}e^{-\lambda_n x} - (\mu_n)^{-1}e^{-\mu_n(x+x_0)}$, for $n \geq 1$ and $x_0 > 0$ as long as the positive sequences $(\lambda_n)_{n\geq 1}$ and $(\mu_n)_{n\geq 1}$ are equivalent as $n$ goes to $\infty$.

It follows from (4.3.1) below that (4.3.3) is equivalent to

$$\left\{\mathbb{Q} \in \mathcal{M} \ : \ \mathbb{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}} \log\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right] < \infty\right\} \neq \emptyset, \tag{4.3.18}$$

whenever (4.3.9) holds. Hence, (4.3.1) can now be formulated as

**Assumption 4.3.11.** *The conditions* (4.3.1), (4.3.2), (4.3.18) *and* (4.3.4) *hold.*

In the following, we denote by $\mathbb{Q}^e$ the element of $\mathcal{M}$ that minimizes the relative entropy,

$$\mathbb{E}\left[\frac{d\mathbb{Q}^e}{d\mathbb{P}} \log\left(\frac{d\mathbb{Q}^e}{d\mathbb{P}}\right)\right] \quad = \quad \inf_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}} \log\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right],$$

and whose existence is guaranteed by Theorem 2.2 in [Csi75].

**Remark 4.3.12.** The map $y > 0 \mapsto y \log y$ being strictly convex, it follows from Remark 4.2.4 that $d\mathbb{Q}^e/d\mathbb{P}$ is $\mathcal{F}_T^S$-measurable, recall the argument used in the proof of (4.3.6). However, we do not impose in (4.3.18) the uniqueness condition of **(HCM$^e$)** in Remark 4.3.8.

**Theorem 4.3.13.** *Let* (4.3.9), (4.3.11) *and* (4.2.3) *be in force. Then the sequence of unit utility indifference prices satisfies*

$$\lim_{n\to\infty} p_n(G_n, U_n) = \mathbb{E}^{\mathbb{Q}^e}[\bar{g}] \ .$$

**Remark 4.3.14.** When the $g^i$'s satisfy the conditions of Remark 4.3.7, the above result shows that the unit indifference prices converges to $\mathbb{E}^{\mathbb{Q}^e}[\mathbb{E}\left[g^1|\mathcal{F}_T^S\right]]$. Again, this consists in taking the mean over the part of the risk that can be diversified and computing the price under the minimal entropy martingale measure of the pure financial remaining claim.

In the rest of this section, we provide the proof of (4.3.13). In view of (4.3.2), it would suffice to show that

$$\lim_{n\to\infty} \mathbb{E}^{\mathbb{Q}_n^0}[G_n/n] = \mathbb{E}^{\mathbb{Q}^e}[\bar{g}] \ , \tag{4.3.19}$$

where we recall that $\mathbb{Q}_n^0$ is defined in (4.3.5).

In the following, we actually prove that

$$\lim_{n\to\infty} \mathbb{E}^{\mathbb{Q}_n^\alpha}[G_n/n] = \mathbb{E}^{\mathbb{Q}^e}[\bar{g}] \ , \tag{4.3.20}$$

where $(y_n^\alpha, \mathbb{Q}_n^\alpha)$ are defined as $(y_n^0, \mathbb{Q}_n^0)$ in (4.3.5) but with

$$U_n^\alpha : x \mapsto \alpha_n \frac{U_n(x) - U_n(0)}{U_n'(0)}, \qquad n \geq 1$$

in place of $U_n$. In the above, $(\alpha_n)_{n\geq 1}$ is a sequence of positive numbers to be chosen later on, see the proof of (4.3.2) below. This trick is inspired from [CR11] and allows to reduce to the case where

$$U_n^\alpha(0) = 0 \quad \text{and} \quad [U_n^\alpha]'(0) = \alpha_n,\ n \geq 1. \tag{4.3.21}$$

Obviously, since $U_n^\alpha$ is an affine transformation of the utility function $U_n$, we have

$$p_n(G_n, U_n) = p_n(G_n, U_n^\alpha). \tag{4.3.22}$$

Recalling (4.3.2), (4.3.20) is thus sufficient to deduce the result of (4.3.13).

We first provide upper and lower bounds for the Fenchel transform $V_n^\alpha$ of $U_n^\alpha$ in terms of Fenchel transforms of exponential utility functions with risk aversion $\eta_n^1$ and $\eta_n^2$.

**Lemma 4.3.1.** *Let (4.3.9) hold. Then, for each $n \geq 1$,*

$$V_n^1(y) \ \leq\ V_n^\alpha(y) \ \leq\ V_n^2(y)\,, \qquad y \in (0,\infty)\,, \tag{4.3.23}$$

*where the functions $V_n^1$ and $V_n^2$ are defined by*

$$V_n^i(y) := \frac{y}{\eta_n^i}\log\left(\frac{y}{\alpha_n}\right) + \frac{\alpha_n - y}{\eta_n^i}, \qquad y \in (0,\infty),\ i = 1,2.$$

**Proof.** It follows from the definition of $(V_n^i)_{i=1,2}$ and (4.3.21) that

$$V_n^\alpha(\alpha_n) = V_n^i(\alpha_n) = 0 \quad \text{and} \quad [V_n^\alpha]'(\alpha_n) = [V_n^i]'(\alpha_n) = 0,\ i = 1,2. \tag{4.3.24}$$

Since $r_n = -[U_n^\alpha]''/[U_n^\alpha]'$ and $[U_n^\alpha]'' \circ ([U_n^\alpha]')^{-1} = 1/[V_n^\alpha]''$, we deduce from (4.3.16) in (4.3.9) that

$$\eta_n^2 \ \leq\ \frac{1}{y[V_n^\alpha]''(y)} \ \leq\ \eta_n^1\,, \qquad y \in (0,\infty).$$

Together with the strict convexity of $V_n^\alpha$, of each $V_n^i$, and the relation $\eta_n^1[V_n^1]''(y) = \eta_n^2[V_n^2]''(y) = 1/y$, this shows that

$$[V_n^1]'' \leq [V_n^\alpha]'' \leq [V_n^2]''. \tag{4.3.25}$$

We now simply deduce from the right-hand side of (4.3.24) and (4.3.25) that

$$[V_n^2]' \leq [V_n^\alpha]' \leq [V_n^1]' \qquad \text{on} \qquad (0,\alpha_n]$$

$$\text{and} \qquad [V_n^2]' \geq [V_n^\alpha]' \geq [V_n^1]' \qquad \text{on} \qquad [\alpha_n,\infty)\,.$$

We conclude by using the left-hand side of (4.3.24). □

We can now use the fact that $V_n^1$ and $V_n^2$ interpret as Fenchel transforms of exponential utility functions, and that the dual probability associated to any exponential utility function is the minimal entropic one $\mathbb{Q}^e$, to deduce from (4.3.23) that the sequence of dual martingale measures $(\mathbb{Q}_n^\alpha)_{n\geq 1}$ associated to $(U_n^\alpha)_{n\geq 1}$ achieves asymptotically the mininal relative entropy, under (4.3.9) and for a suitable choice of the sequence $(\alpha_n)_{n\geq 1}$. We shall see later that this implies convergence to $\mathbb{Q}^e$ is the total variation norm.

**Lemma 4.3.2.** *Let (4.3.9) and (4.3.18) hold. Then, there exists a sequence of positive numbers $(\alpha_n)_{n\geq 1}$ such that*

$$\lim_{n\to\infty} \mathbb{E}\left[\frac{d\mathbb{Q}_n^\alpha}{d\mathbb{P}} \log\left(\frac{d\mathbb{Q}_n^\alpha}{d\mathbb{P}}\right)\right] \quad = \quad \mathbb{E}\left[\frac{d\mathbb{Q}^e}{d\mathbb{P}} \log\left(\frac{d\mathbb{Q}^e}{d\mathbb{P}}\right)\right]. \tag{4.3.26}$$

**Proof.** For $i = 1, 2$, direct computations leads to

$$\begin{aligned} &\inf_{y>0,\mathbb{Q}\in\mathcal{M}} \mathbb{E}\left[V_n^i\left(y\frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right] \\ &= \inf_{y>0,\mathbb{Q}\in\mathcal{M}} \left\{\frac{y}{\eta_n^i}\mathbb{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}\log\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right] + \frac{y}{\eta_n^i}\log\left(\frac{y}{\alpha_n}\right) + \frac{\alpha_n - y}{\eta_n^i}\right\} \\ &= \inf_{y>0} \left\{\frac{y}{\eta_n^i}\mathbb{E}\left[\frac{d\mathbb{Q}^e}{d\mathbb{P}}\log\left(\frac{d\mathbb{Q}^e}{d\mathbb{P}}\right)\right] + \frac{y}{\eta_n^i}\log\left(\frac{y}{\alpha_n}\right) + \frac{\alpha_n - y}{\eta_n^i}\right\} \\ &= \mathbb{E}\left[V_n^i\left(\hat{y}_n\frac{d\mathbb{Q}^e}{d\mathbb{P}}\right)\right], \end{aligned} \tag{4.3.27}$$

where the common minimizer $\hat{y}_n > 0$ is given by

$$\hat{y}_n := \alpha_n e^{-\hat{\rho}} \qquad \text{with} \quad \hat{\rho} := \mathbb{E}\left[\frac{d\mathbb{Q}^e}{d\mathbb{P}}\log\left(\frac{d\mathbb{Q}^e}{d\mathbb{P}}\right)\right]. \tag{4.3.28}$$

Also note that

$$\mathbb{E}\left[V_n^2\left(\hat{y}_n\frac{d\mathbb{Q}^e}{d\mathbb{P}}\right) - V_n^1\left(\hat{y}_n\frac{d\mathbb{Q}^e}{d\mathbb{P}}\right)\right] = \quad (1-e^{-\hat{\rho}})\alpha_n\left(\frac{1}{\eta_n^2} - \frac{1}{\eta_n^1}\right),$$

so that

$$\begin{gathered} \lim_{n\to\infty} \mathbb{E}\left[V_n^2\left(\hat{y}_n\frac{d\mathbb{Q}^e}{d\mathbb{P}}\right) - V_n^1\left(\hat{y}_n\frac{d\mathbb{Q}^e}{d\mathbb{P}}\right)\right] = 0 \\ \text{whenever} \quad \alpha_n\left(\tfrac{1}{\eta_n^2} - \tfrac{1}{\eta_n^1}\right) \underset{n\to\infty}{\longrightarrow} 0. \end{gathered} \tag{4.3.29}$$

On the other hand, Lemma 4.3.1 implies

$$\mathbb{E}\left[V_n^1\left(y\frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right] \leq \mathbb{E}\left[V_n^\alpha\left(y\frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right] \leq \mathbb{E}\left[V_n^2\left(y\frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right], \qquad y > 0, \tag{4.3.30}$$

for any $\mathbb{Q} \in \mathcal{M}$. Picking in particular $\mathbb{Q} = \mathbb{Q}_n^\alpha$, we deduce

$$\inf_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}\left[V_n^1\left(y\frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right] \leq \mathbb{E}\left[V_n^1\left(y\frac{d\mathbb{Q}_n^\alpha}{d\mathbb{P}}\right)\right] \leq \mathbb{E}\left[V_n^\alpha\left(y\frac{d\mathbb{Q}_n^\alpha}{d\mathbb{P}}\right)\right], \qquad y > 0.$$

Taking the infimum over $y > 0$ and recalling (4.3.27), this shows that

$$\mathbb{E}\left[V_n^1\left(\hat{y}_n\frac{d\mathbb{Q}^e}{d\mathbb{P}}\right)\right] \le \mathbb{E}\left[V_n^1\left(y_n\frac{d\mathbb{Q}_n^\alpha}{d\mathbb{P}}\right)\right] \le \inf_{y>0}\mathbb{E}\left[V_n^\alpha\left(y\frac{d\mathbb{Q}_n^\alpha}{d\mathbb{P}}\right)\right], \qquad (4.3.31)$$

where the minimizer $y_n > 0$ associated to the middle term is given by

$$y_n := \alpha_n e^{-\rho_n} \qquad \text{with} \quad \rho_n := \mathbb{E}\left[\frac{d\mathbb{Q}_n^\alpha}{d\mathbb{P}}\log\left(\frac{d\mathbb{Q}_n^\alpha}{d\mathbb{P}}\right)\right]. \qquad (4.3.32)$$

Similarly, (4.3.27) and (4.3.30) imply

$$\inf_{y>0}\mathbb{E}\left[V_n^\alpha\left(y\frac{d\mathbb{Q}_n^\alpha}{d\mathbb{P}}\right)\right] = \inf_{y>0,\mathbb{Q}\in\mathcal{M}}\mathbb{E}\left[V_n^\alpha\left(y\frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right] \le \mathbb{E}\left[V_n^2\left(\hat{y}_n\frac{d\mathbb{Q}^e}{d\mathbb{P}}\right)\right],$$

which, combined with (4.3.31), entails

$$\mathbb{E}\left[V_n^1\left(\hat{y}_n\frac{d\mathbb{Q}^e}{d\mathbb{P}}\right)\right] \le \mathbb{E}\left[V_n^1\left(y_n\frac{d\mathbb{Q}_n^\alpha}{d\mathbb{P}}\right)\right] \le \mathbb{E}\left[V_n^2\left(\hat{y}_n\frac{d\mathbb{Q}^e}{d\mathbb{P}}\right)\right].$$

If $\alpha_n\left(\frac{1}{\eta_n^2}-\frac{1}{\eta_n^1}\right) \underset{n\to\infty}{\longrightarrow} 0$, (4.3.29) thus implies that

$$0 \le \mathbb{E}\left[V_n^1\left(y_n\frac{d\mathbb{Q}_n^\alpha}{d\mathbb{P}}\right) - V_n^1\left(\hat{y}_n\frac{d\mathbb{Q}^e}{d\mathbb{P}}\right)\right] \underset{n\to\infty}{\longrightarrow} 0,$$

which, by the definitions in (4.3.28) and (4.3.32) of $\hat{y}_n$ and $y_n$, is equivalent to

$$\frac{\alpha_n}{\eta_n^1}\left(e^{-\hat{\rho}} - e^{-\rho_n}\right) \underset{n\to\infty}{\longrightarrow} 0 \quad \text{whenever} \quad \alpha_n\left(\frac{1}{\eta_n^2}-\frac{1}{\eta_n^1}\right) \underset{n\to\infty}{\longrightarrow} 0\,. \qquad (4.3.33)$$

We now choose the sequence $(\alpha_n)_{n\ge1}$ and pick $\alpha_n := \eta_n^1$, so that (4.3.9) implies

$$\alpha_n\left(\frac{1}{\eta_n^2}-\frac{1}{\eta_n^1}\right) = \frac{\eta_n^1}{\eta_n^2} - 1 \underset{n\to\infty}{\longrightarrow} 0\,.$$

Hence, we deduce from (4.3.33) that $\rho_n \to \hat{\rho}$ as $n \to \infty$, i.e. (4.3.26) holds. □

We are now in position to complete the proof of (4.3.13).

**Proof of (4.3.13).** In the following, we let $(\alpha_n)_{n\ge1}$ be as in Lemma 4.3.2.

1. We first deduce from Lemma 4.3.2 that $(\mathbb{Q}_n^\alpha)_{n\ge1}$ converges to $\mathbb{Q}^e$ in the norm of total variation. Since $\mathbb{Q}^e$ minimizes the entropy with respect to $\mathbb{P}$ over the convex set $\mathcal{M}$, it follows from Theorem 2.2 in [Csi75] that

$$\mathbb{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}\log\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right] \ge \mathbb{E}^{\mathbb{Q}^e}\left[\frac{d\mathbb{Q}}{d\mathbb{Q}^e}\log\left(\frac{d\mathbb{Q}}{d\mathbb{Q}^e}\right)\right] + \mathbb{E}\left[\frac{d\mathbb{Q}^e}{d\mathbb{P}}\log\left(\frac{d\mathbb{Q}^e}{d\mathbb{P}}\right)\right]$$
for any $\mathbb{Q} \in \mathcal{M}$.

In particular,

$$0 \le \mathbb{E}^{\mathbb{Q}^e}\left[\frac{d\mathbb{Q}_n^\alpha}{d\mathbb{Q}^e}\log\left(\frac{d\mathbb{Q}_n^\alpha}{d\mathbb{Q}^e}\right)\right] \le \mathbb{E}\left[\frac{d\mathbb{Q}_n^\alpha}{d\mathbb{P}}\log\left(\frac{d\mathbb{Q}_n^\alpha}{d\mathbb{P}}\right)\right] - \mathbb{E}\left[\frac{d\mathbb{Q}^e}{d\mathbb{P}}\log\left(\frac{d\mathbb{Q}^e}{d\mathbb{P}}\right)\right] , \quad n \ge 1 .$$

Hence, (4.3.26) implies that

$$\mathbb{E}^{\mathbb{Q}^e}\left[\frac{d\mathbb{Q}_n^\alpha}{d\mathbb{Q}^e}\log\left(\frac{d\mathbb{Q}_n^\alpha}{d\mathbb{Q}^e}\right)\right] \underset{n\to\infty}{\longrightarrow} 0 .$$

The fact that

$$\mathbb{E}\left[\left|\frac{d\mathbb{Q}_n^\alpha}{d\mathbb{P}} - \frac{d\mathbb{Q}^e}{d\mathbb{P}}\right|\right] \underset{n\to\infty}{\longrightarrow} 0 \tag{4.3.34}$$

then follows from Pinsker's inequality, see e.g. [PP09].

2. Combining (4.3.4) and (4.3.34) implies that

$$\left|\mathbb{E}^{\mathbb{Q}_n^\alpha}\left[\frac{G_n}{n}\right] - \mathbb{E}^{\mathbb{Q}^e}\left[\frac{G_n}{n}\right]\right| = \left|\mathbb{E}\left[\frac{G_n}{n}\left(\frac{d\mathbb{Q}_n^\alpha}{d\mathbb{P}} - \frac{d\mathbb{Q}^e}{d\mathbb{P}}\right)\right]\right| \le C\,\mathbb{E}\left[\left|\frac{d\mathbb{Q}_n^\alpha}{d\mathbb{P}} - \frac{d\mathbb{Q}^e}{d\mathbb{P}}\right|\right] \underset{n\to\infty}{\longrightarrow} 0,$$

with $C := \sup_{n\ge1}\|G_n/n\|_{L^\infty}$. Besides, (4.2.3) and (4.3.4) imply that $\mathbb{E}^{\mathbb{Q}^e}[G_n/n] \underset{n\to\infty}{\longrightarrow} \mathbb{E}^{\mathbb{Q}^e}[\bar{g}]$. This shows that (4.3.20) holds. We conclude by using (4.3.22) and (4.3.2). □

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**Abstract:** The aim of this thesis is to investigate some solutions to the pricing of contingent claims in incomplete markets. We first consider the stochastic target introduced by Soner and Touzi (2002) for the general super-replication problem, and extended by Bouchard, Elie and Touzi (2009) in order to deal with more general approaches. We first generalize the work of Bouchard *et al* to a framework where the diffusions are subject to jumps. In our particular settings, we need to consider a control taking the form of unbounded maps, which has non-trivial impacts on the derivation of the associated PDE. Our second contribution consists in establishing a version of stochastic target problems which is robust to model uncertainty. We provide, in a general setup, a relaxed geometric dynamic programming principle for this problem and derive, for the case of a controlled SDE, the corresponding dynamic programming equation in the sense of viscosity solutions. We consider an example of partial hedging under Knightian uncertainty. Finally, we focus on the problem of pricing *hybrid* claims. More specifically, we intend to give a sufficient condition for a (very popular) pricing rule, combining actuarial diversification with arbitrage free replication arguments, to hold.
**Résumé:** Le but de cette thèse est d'apporter une contribution à la problématique de valorisation de produits dérivés en marchés incomplets. Nous considérons tout d'abord les cibles stochastiques introduites par Soner et Touzi (2002) afin de traiter le problème de sur-réplication, et récemment étendues afin de traiter des approches plus générales par Bouchard, Elie et Touzi (2009). Nous généralisons le travail de Bouchard *et al* à un cadre plus général où les diffusions sont sujettes à des sauts. Nous devons considérer dans ce cas des contrôles qui prennent la forme de fonctions non bornées, ce qui impacte de façon non triviale la dérivation des EDP correspondantes. Notre deuxième contribution consiste à établir une version des cibles stochastiques qui soit robuste à l'incertitude de modèle. Dans un cadre abstrait, nous établissons une version faible du principe de programmation dynamique géométrique de Soner et Touzi (2002), et nous dérivons, dans un cas d'EDS controllées, l'équation aux dérivées partielles correspondantes, au sens des viscosités. Nous nous intéressons ensuite à un exemple de couverture partielle sous incertitude de Knightian. Finalement, nous nous concentrons sur le problème de valorisation de produits dérivées *hybrides* (produits dérivés combinant finance de marché et assurance). Nous cherchons plus particulièrement à établir une condition suffisante sous laquelle une règle de valorisation (populaire dans l'industrie), consistant à combiner l'approches actuarielle de mutualisation avec une approche d'arbitrage, soit valable.