Interpolation réelle des espaces de Sobolev sur les espaces métriques mesurés et applications aux inégalités fonctionnelles

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Chapitre 1

Introduction

Comme son titre l'indique, les espaces de Sobolev et l'interpolation réelle représentent les thèmes principaux de cette thèse. Pour celà nous commençons ce chapitre par une petite introduction sur l'origine des espaces de Sobolev et leurs intérêts. Ensuite nous parlons dans la deuxième section de la L_p bornitude de la transformée de Riesz et son inégalité inverse dans le cadre Euclidien, variétés Riemanniennes, groupes de Lie et graphes et leur lien avec l'interpolation des espaces de Sobolev. Nous énonçons dans la troisième section les théorèmes principaux de ce travail et finissons ce chapitre avec un plan de cette thèse. Le chapitre 2 sera un complément à ce chapitre où nous parlerons d'interpolation, des définitions des espaces de Sobolev et des hypothèses que nous allons imposer sur ces espaces pendant tout ce travail avec les propriétés qui en découlent.

1.1 Espaces de Sobolev

L'origine de la théorie des espaces de Sobolev remonte aux travaux effectués par Sobolev ([32]) dans les années 40. On va se restreindre dans cette thèse aux espaces de Sobolev du premier ordre.

Etant donnés Ω un ouvert de $\mathbb{R}^n$, $p \geq 1$ un réel et $u : \Omega \to \mathbb{R}$ une fonction de classe C^∞, on note

$$\|u\|_{1,p}^p = \int_\Omega |u(x)|^p dx + \int_\Omega |\nabla u(x)|^p dx$$

où $\nabla u = (\partial_1 u, ..., \partial_i u, ..., \partial_n u)$.

On définit les espaces de Sobolev non homogènes

$$H_p^1(\Omega) \equiv \text{complété de } \{u \in C^\infty(\Omega);\ \|u\|_{1,p} < \infty\}$$

et

$$W_p^1(\Omega) = \{u \in L_p(\Omega);\ |\nabla u| \in L_p(\Omega)\}$$

où ∇u est le gradient distributionnel de u, muni de la norme $\|u\|_{1,p}$. (Il y a aussi des versions homogènes où on oublie $\|u\|_{L_p}$ dans la définition de la norme.)

Pendant bien longtemps, en fait jusqu'au milieu des années 60, ces deux espaces furent considérés comme distincts. La confusion s'arrête définitivement avec le théorème de Meyers-Serrin [24] dont on trouve une preuve simple dans [1], Chapitre III.

Théorème 1.1.1 (Théorème de Meyers-Serrin). *Pour tout Ω et tout $p \geq 1$, $H^1_p(\Omega) = W^1_p(\Omega)$.*

Le principal intérêt de la théorie des espaces de Sobolev réside dans l'existence des plongements continus de Sobolev, des inégalités de Poincaré, de Gagliardo-Nirenberg, et dans l'existence des plongements compacts de Rellich-Kondrakov. On ajoutera à cette liste, l'existence des théorèmes de régularité très généraux et particulièrement importants dans l'étude des équations aux dérivées partielles linéaires et non-linéaires, des problèmes de bords, calcul des variations. Cependant, l'utilisation et l'importance de ces espaces est plus large, incluant des questions de géométrie différentielle, topologie analytique, analyse complexe et théorie des probabilités.

La théorie des espaces de Sobolev classiques sur l'espace Euclidien s'est généralisée à d'autres cadres géométriques.

La première compréhension des espaces de Sobolev sur les variétés Riemanniennes est due à Thierry Aubin [3] en 1976, voir aussi [4] en 1982. Il a utilisé ses résultats en connection avec les EDP non-linéaires sur les variétés. Les espaces de Sobolev sur les variétés compactes sont utilisés depuis longtemps (voir par exemple les travaux de Ebin). Ils ne diffèrent pas essentiellement des espaces de Sobolev sur une boule de $\mathbb{R}^n$. Le cas des variétés Riemanniennes complètes non-compactes est plus délicat.

Les espaces de Sobolev sur un groupe de Lie G ont été étudiés par Saka [29] dans le cas où G est nilpotent. Peetre [26] a étudié les espaces de Sobolev d'ordre 1 dans le cas général d'un groupe de Lie. Pesenson dans [27] généralise les études de Peetre pour les espaces de Sobolev d'ordre supérieur à 1.

Très récemment, plusieurs notions d'espaces de Sobolev ont été développées sur les espaces métriques mesurés en général par plusieurs auteurs, parmi lesquels on cite: Hajlasz [16], Cheeger [11], Shammugalingan [31], Gol'dschtein et Troyanov [15], Heinonen [17].

Finalement, concernant les espaces de Sobolev sur les graphes, Ostrovskii dans [25] les a défini dans le cas d'un groupe fini. On trouve une définition plus générale dans [15].

Sous certaines hypothèses sur tous ces espaces mentionnés, on retrouve les analogues des théorèmes de plongement de Sobolev, inégalités de Sobolev-Poincaré, plongement de compacité de Rellich-Kondrakov, etc.., déjà connus dans le cadre Euclidien.

1.2 Interpolation et Transformée de Riesz

Une façon de comprendre les espaces de Sobolev homogènes sur les variétés Riemanniennes est via le calcul fonctionnel de l'opérateur de Laplace-Beltrami (positif) Δ. Si l'inégalité a priori sur C_0^∞

$$(E_p) \qquad c_p\|\Delta^{\frac{1}{2}}f\|_p \leq \|\,|\nabla f|\,\|_p \leq C_p\|\Delta^{\frac{1}{2}}f\|_p$$

est vraie pour un $p \in]1,+\infty[$ alors on dispose d'une norme équivalente. Si (E_p) est vraie pour un intervalle de valeurs de p alors on dispose d'une nouvelle norme qui permettrait d'interpoler les espaces de Sobolev entre eux par la méthode de notre choix (réelle ou complexe). Voyons donc ce qui est connu sur (E_p).

Le membre droite de (E_p) qu'on notera par (R_p) n'est autre que l'inégalité traduisant la L_p bornitude de l'opérateur $T = \nabla\Delta^{-\frac{1}{2}}$ appelé transformée de Riesz. L'inégalité de gauche est l'inégalité inverse de (R_p) et sera notée par (RR_p). Notons que $(R_p) \Rightarrow (RR_{p'})$ où p' est le conjugué de p (p' est tel que $\frac{1}{p} + \frac{1}{p'} = 1$) (voir [8], section 4 et [12], Section 2.7). Donc si (R_p) est vérifiée pour $1 < p < p_0$ avec $2 < p_0 \leq \infty$ alors on a (E_p) pour $p_0' < p < p_0$.

Sur $\mathbb{R}^n$, la transformée de Riesz est bornée sur $L_p(\mathbb{R}^n)$ pour $1 < p < \infty$ et par suite (E_p) est vérifiée pour $1 < p < \infty$. C'est l'exemple typique d'un opérateur d'intégrale singulière dit de Calderón-Zygmund.

Dans le cas des variétés compactes, la L_p bornitude de la transformée de Riesz pour $1 < p < \infty$ découle facilement de la théorie des opérateurs pseudo-différentiels et de l'article de Seeley [30].

En 1983, Strichartz pose dans [33] la question suivante: Que peut-on dire sur la L_p bornitude de la transformée de Riesz dans le cadre des variétés Riemanniennes non-compactes? On sait que sur toute variété Riemannienne la transformée de Riesz est bornée sur L_2. En effet on a

$$\|\nabla\Delta^{-\frac{1}{2}} f\|_2 = \|f\|_2.$$

Dans ce même article, il démontre que (E_p) est satisfaite pour $1 < p < \infty$ sur les espaces symétriques de rang 1.

Depuis, l'étude de la L_p bornitude de la transformée de Riesz et son inégalité inverse a attiré beaucoup d'analystes.

En 1985, Lohoué dans [22] considère une variété de Cartan-Hadamard M [1] dont le tenseur de courbure et ses deux premières dérivées covariantes sont bornés. Il suppose de plus que le Laplacien est minoré sur $L_2(M)$: $\|f\|_2 \leq C\|\Delta f\|_2$ pour tout $f \in C_0^\infty(M)$. Il obtient alors sur une telle variété M la validité de (E_p) pour $1 < p < \infty$.

En 1987, Bakry dans [7] prouve que toute variété Riemannienne à courbure de Ricci positive vérifie (E_p) pour $1 < p < \infty$.

Dans sa thèse [19], Li démontre la L_p bornitude de la transformée de Riesz sur toute variété conique à base compacte sans bord sur un intervalle $]1, p_0[$ où p_0 dépend de la première valeur propre non nulle du Laplacien de la base, et publie ce résultat en 1999 ([20]).

En 1999, Coulhon, Duong dans [13] obtiennent le résultat suivant:
Soit M est une variété Riemannienne vérifiant la propriété de doublement (D). Soit p_t le noyau de chaleur (qui n'est autre que le noyau de l'opérateur $e^{-t\Delta}$). Supposons que p_t satisfait

$$(DUE) \qquad |p_t(x,x)| \leq \frac{C}{\mu(B(x,\sqrt{t}))}$$

[1] Une variété Riemannienne est de Cartan-Hadamard si elle est complète, connexe à courbure négative ou nulle.

pour tout $x \in M,\ t > 0$. Alors $T = \nabla \Delta^{-\frac{1}{2}}$ est borné sur L_p pour $1 < p \leq 2$.
Ils ont construit un contre-exemple d'une variété Riemannienne M vérifiant (D) et (DUE) tel que la transformée de Riesz n'est pas bornée sur L_p quand $p > n = dim M$.

En 2004, Auscher, Coulhon, Duong et Hofmann étendent dans [6] l'intervalle de la L_p bornitude de la transformée de Riesz pour $p > 2$ sous des hypothèses plus fortes. Ils obtiennent:
Soit M une variété Riemannienne complète vérifiant (D) et l'inégalité de Poincaré (P_2). On considère l'inégalité

$$(G_p) \qquad \| \, |\nabla e^{-t\Delta}| \, \|_{p \to p} \leq \frac{C}{\sqrt{t}}.$$

Soit $2 < p_0 \leq \infty$. Alors

$$(G_p) \text{ est vérifiée pour tout } p \in]2, p_0[\Leftrightarrow (R_p) \text{ est vérifée pour tout } p \in]2, p_0[.$$

Dans ce même article et comme corollaire à leur résultat et à celui de Coulhon et Duong en 1999, mentionnés ci-dessus, ils obtiennent:

Soit M une variété Riemannienne vérifiant la propriété de doublement (D) et soit p_t le noyau de la chaleur. Supposons que p_t satisfait (DUE) pour tout $x \in M,\ t > 0$ et qu'on a

$$(G) \qquad |\nabla p_t(x,y)| \leq \frac{C}{\mu(B(y, \sqrt{t}))}$$

pour tout $x,\ y \in M,\ t > 0$. Alors $T = \nabla \Delta^{-\frac{1}{2}}$ est borné sur L_p pour $1 < p < \infty$.

Le résultat le plus récent concernant ce sujet sur les variétés Riemanniennes est le suivant:

En 2005, Auscher et Coulhon démontrent dans [5] que si M est une variété Riemannienne complète satisfaisant (D) et (P_2) alors il existe $\varepsilon > 0$ tel qu'on a (R_p) pour $2 < p < 2 + \varepsilon$, par suite pour $1 < p < 2 + \varepsilon$. Dans ce même article, ils prouvent aussi que si M est une variété Riemannienne complète satisfaisant (D) et une inégalité de Poincaré (P_q) pour un $q \in [1, 2[$, alors on a (RR_p) pour $q < p < 2$.

Pour les groupes de Lie à croissance polynomiale (dans le cadre sous-Riemannien), Alexopoulos démontre en 1992 dans [2] la L_p bornitude de la transformée de Riesz du sous-Laplacien pour $1 < p < \infty$.

La transformée de Riesz sur les graphes a été étudiée par Russ dans [28]. Sur un graphe on a les notions de gradient et de Laplacien discrets. Russ démontre dans cet article la L_p bornitude de la transformée de Riesz pour $1 < p \leq 2$ sous les hypothèses (D) et (DUE). Dans le chapitre 6 de cette thèse, en collaboration avec E. Russ, nous parlerons de la L_p bornitude de la transformée de Riesz sur les graphes pour $p > 2$.

Pour les espaces de Sobolev non homogènes, sous des hypothèses locales, l'analogue de (E_p) est

$$(E_p') \qquad c_p \left(\|f\|_p + \|\Delta^{\frac{1}{2}} f\|_p \right) \leq \| \, |\nabla f| \, \|_p + \|f\|_p \leq C_p \left(\|f\|_p + \|\Delta^{\frac{1}{2}} f\|_p \right).$$

On peut commencer par étudier la L_p bornitude de la transformée de Riesz locale $\nabla(\Delta + a)^{-\frac{1}{2}}$ pour un certain $a > 0$ suffisamment grand. Par exemple une variété

Riemannienne complète à courbure de Ricci minorée satisfait (E'_p) pour $1 < p < \infty$ (voir [7]). De même, (E'_p) pour $1 < p < \infty$ sur une variété à géométrie bornée découle du cas Euclidien par recollement. La compréhension de l'opérateur $\nabla(\Delta + a)^{-\frac{1}{2}}$ est encore incomplète en général.

Revenons au problème de l'interpolation. D'une part, les résultats sur (E_p) sont donc loin d'être complets. D'autre part, il est surprenant pour interpoler de devoir faire appel à un objet non local, $\Delta^{\frac{1}{2}}$, alors que le gradient est lui un objet local. Nous voudrons donc développer une théorie d'interpolation des espaces de Sobolev basée sur la définition avec le gradient en oubliant le problème (E_p). En retour, nous verrons que nos résultats ont plus à voir avec l'inégalité (RR_p) en ce sens que l'intervalle des valeurs de p où nous savons démontrer (RR_p) sera celui où nous pourrons interpoler les Sobolev. En fait disposer des résultats d'interpolation permet de simplifier les preuves de (RR_p) et par suite de (R_p) pour les résultats mentionnés ci-dessus. La même chose vaudra pour les espaces de Sobolev non homogènes.

1.3 Résultats et plan de la thèse

Dans cette thèse, nous nous intéresserons surtout à l'étude de l'interpolation réelle des espaces de Sobolev. Pour le cas Euclidien, en 1979, Devore et Scherer démontrent pour la première fois dans [14] que W_p^k, pour $1 < p < \infty$, est un espace d'interpolation entre W_1^k et W_∞^k ($k \in \mathbb{N}^*$). Pour prouver ce théorème d'interpolation ils utilisent une méthode se basant sur les fonctions splines. Une autre preuve plus simple de ce résultat dans le livre de Bennett et Sharpley [9] repose sur le lemme de recouvrement de Whitney. On trouve aussi une preuve due à Calderón et Milman [10] utilisant le lemme d'extension de Whitney.

Durant toute cette thèse, nous allons travailler sur des variétés Riemanniennes, groupes de Lie, graphes et sur des espaces métriques mesurés en général vérifiant la propriété de doublement (D) et des inégalités de Poincaré (voir le chapitre suivant).

Nous définissons les espaces de Sobolev classiques, et aussi des espaces de Sobolev associés à un potentiel positif intervenant dans l'étude des opérateurs de Schrödinger et étudions l'interpolation de ces espaces.

Enonçons nos principaux théorèmes de cette thèse sans rentrer dans les définitions des propriétés et hypothèses utilisées. Pour ces définitions voir le chapitre suivant de préliminaires et les chapitres concernés.

Dans le chapitre 3 nous démontrons le théorème suivant (Theorem 3.1.1 du chapitre 3) qu'on énonce ici dans le cadre des variétés mais qui reste vrai dans le cas des groupes de Lie, des espaces métriques mesurés (voir chapitre 3) et aussi des graphes (voir chapitre 6).

Théorème 1.3.1. *Soit M une variété Riemannienne complète non-compacte satisfaisant la propriété de doublement local (D_{loc}) et une inégalité de Poincaré locale (P_{qloc}), pour un $1 \leq q < \infty$. Alors pour $q < p < \infty$, W_p^1 est un espace d'interpolation entre W_q^1 et W_∞^1.*

Dans ce même chapitre nous démontrons aussi la version homogène de ce théorème (Theorem 3.1.3 du chapitre 3):

Théorème 1.3.2. *Soit M une variété Riemannienne complète non-compacte satisfaisant la propriété de doublement (D) et une inégalité de Poincaré (P_q) pour un $1 \leq q < \infty$. Alors pour $q < p < \infty$, $\dot{W}_p^1$ est un espace d'interpolation entre $\dot{W}_q^1$ et $\dot{W}_\infty^1$.*

Dans le chapitre 4, nous comparons différents espaces de Sobolev définis sur le cône Euclidien et nous parlons du lien de ces espaces avec l'interpolation du chapitre 3. Cela permet de montrer que l'hypothèse sur Poincaré n'est pas nécessaire.

Nous arrivons au chapitre 5 où nous étendons notre résultat du chapitre 3 aux espaces de Sobolev associé à un potentiel positif que nous y définissons par une norme du type $\|u\|_p + \| \, |\nabla u| \, \|_p + \|Vu\|_p$ dans le cas non homogène (resp. $\| \, |\nabla u| \, \|_p + \|Vu\|_p$ dans le cas homogène). Les deux théorèmes que nous obtenons dans le cadre non homogène (respectivement homogène) sont les suivants:

Théorème 1.3.3. *(Theorem 5.1.3 du chapitre 5) Soit M une variété Riemannienne complète vérifiant la propriété de doublement local (D_{loc}). Soit $V \in RH_{qloc}$ pour un $1 < q \leq \infty$. Supposons de plus que M admette une inégalité de Poincaré locale (P_{sloc}) pour un $1 \leq s < q$. Alors pour $s < p < q$, $W_{p,V}^1$ est un espace d'interpolation entre $W_{s,V}^1$ et $W_{q,V}^1$.*

Théorème 1.3.4. *(Theorem 5.1.6 du chapitre 5) Soit M une variété Riemannienne complète vérifiant (D). Soit $V \in RH_q$ pour un $1 < q \leq \infty$ et supposons que M admet une inégalité de Poincaré (P_s) pour un $1 \leq s < q$. Alors, pour $s < p < q$, $\dot{W}_{p,V}^1$ est un espace d'interpolation entre $\dot{W}_{s,V}^1$ et $\dot{W}_{q,V}^1$.*

Nous mentionnons un travail en cours de rédaction, en collaboration avec Besma Ben Ali, où nous appliquons notre résultat d'interpolation des espaces de Sobolev homogènes associé à un potentiel du Chapitre 5. Dans cet article nous étudions la L_p borrnitude de la transformée de Riesz et son inégalité inverse pour les opérateurs de Schrödinger sur les variétés Riemanniennes et groupes de Lie.

Après avoir énoncé les essentiels résultats d'interpolation de cette thèse, nous arrivons aux applications de ces résultats dans la deuxième partie. Dans le chapitre 6, nous obtenons en collaboration avec E. Russ: un résultat d'interpolation des espaces de Sobolev sur les graphes (Theorem 6.1.12 du chapitre 6) et les théorèmes suivants:

Théorème 1.3.5. *(Theorem 6.1.4 du chapitre 6) Soit (Γ, μ) un graphe à poids satisfaisant la propriété de doublement (D), l'inégalité de Poincaré (P_2) et la condition $(\Delta(\alpha))$. Soit $p_0 \in]2, +\infty]$. Alors les propositions suivantes sont équivalentes:*

(i) Pour tout $p \in]2, p_0[$, (G_p) est vraie,

(ii) Pour tout $p \in]2, p_0[$, (R_p) est vraie.

Et pour (RR_p) nous démontrons le théorème suivant:

Théorème 1.3.6. *(Theorem 6.1.6 du chapitre 6) Soit $1 \leq q < 2$. Supposons que (Γ, μ) satisfait (D), $(\Delta(\alpha))$ et une inégalité de Poincaré (P_q). Alors pour $q < p < 2$, (RR_p) est vérifiée.*

Dans ce même article nous démontrons aussi un théorème de L_p bornitude d'une version discrète de la fonction de Littlewood-Paley-Stein g définie pour tout $x \in \Gamma$ par:

$$g(f)(x) = \left(\sum_{l \geq 1} l \left| (I - P) P^l f(x) \right|^2 \right)^{1/2}.$$

Théorème 1.3.7. *(Theorem 6.1.10 du chapitre 6) Supposons que* (Γ, μ) *satisfait* (D), (P_2) *et* $(\Delta(\alpha))$. *Soit* $1 < p < +\infty$. *Il existe alors une constante* $C_p > 0$ *tel que, pour tout* $f \in L_p(\Gamma)$,

$$\|g(f)\|_p \leq C_p \|f\|_p .$$

Dans le dernier chapitre nous utilisons notre résultat d'interpolation des espaces de Sobolev du chapitre 3 et démontrons:

Théorème 1.3.8. *(Theorem 7.1.2 du chapitre 7) Soit* M *une variété Riemannienne complète non-compacte de dimension* n, *satisfaisant* (D), (P_q) *pour un* $1 \leq q < \infty$. *De plus, supposons que* M *satisfait des inégalités de pseudo-Poincaré* (P'_q) *et* (P'_∞). *Alors pour* $q \leq p < l < \infty$ *et pour tout* $f \in W^1_p$

$$\|f\|_l \leq C \| \, |\nabla f| \, \|_p^\theta \|f\|^{1-\theta}_{B^{\frac{\theta}{\theta-1}}_{\infty,\infty}} \tag{1.1}$$

où $\theta = \frac{p}{l}$.

Ce résultat généralise le résultat de Ledoux [18] où il obtient (1.1) sur les variétés Riemanniennes à courbure de Ricci positive.

Sous les mêmes hypothèses du théorème 1.3.8 et en supposant de plus que $1 \leq q < \nu$, avec $\nu > 0$, et que le semi groupe de la chaleur $P_t = e^{t\Delta}$, $t \geq 0$, vérifie $\|P_t\|_{q \to \infty} \leq C t^{-\frac{\nu}{2q}}$, on recouvre l'inégalité de Sobolev

$$\|f\|_{q^*} \leq C \| \, |\nabla f| \, \|_q$$

avec $\frac{1}{q^*} = \frac{1}{q} - \frac{1}{\nu}$.

Plan: Le chapitre d'introduction et le chapitre suivant de préliminaires mis à part, cette thèse comporte deux parties. La première partie est constituée de trois chapitres. Le premier " Real interpolation of Sobolev spaces" (chapitre 3) a été rédigé, sous sa forme originale, en article soumis à "Mathematica Scandinavica". Cependant, dans le présent mémoire, nous avons inclu une version étendue qui clarifie en outre quelques notions et le rend auto-contenu. Nous y démontrons que les espaces de Sobolev W^1_p sur les variétés Riemanniennes vérifiant la propriété de doublement local (D_{loc}) et l'inégalité de Poincaré locale (P_{qloc}) pour un $1 \leq q < \infty$ forment une échelle d'interpolation pour $q \leq p < \infty$. Ensuite, nous démontrons un résultat analogue pour les espaces de Sobolev homogènes $\dot{W}^1_p$ sur les variétés Riemanniennes cette fois avec la propriété de doublement (D) et vérifiant une inégalité de Poincaré (P_q). A la fin de ce chapitre, nous étendons ce résultat aux espaces métriques mesurés, espaces de Carnot-Carathéodory, espaces de Sobolev à poids et finalement aux groupes de Lie. La dernière section est consacrée aux exemples.

Dans le second chapitre (chapitre 4), nous comparons différents espaces de Sobolev définis sur le cône Euclidien et nous parlons de leur lien avec l'interpolation du chapitre précédent (chapitre 3). Dans le troisième chapitre (chapitre 5), nous définissons les espaces de Sobolev $W^1_{p,V}$ associés à un potentiel positif V sur une variété Riemannienne. Nous démontrons que si la variété vérifie la propriété de doublement local (D_{loc}) et une inégalité de Poincaré locale (P_{sloc}) pour un $1 \leq s < \infty$ et si de plus V est localement dans une classe de Hölder inverse RH_{qloc}, pour un $1 \leq s < q$, ces espaces forment aussi une échelle d'interpolation pour $s \leq p < q$. Ensuite nous démontrons un résultat analogue pour les espaces de Sobolev homogènes $\dot{W}^1_{p,V}$ sur les variétés Riemanniennes cette fois avec la propriété de doublement (D) et vérifiant une inégalité de Poincaré (P_s), en imposant que $V \in RH_q$. Nous étendons à la fin ce résultat au cas des groupes de Lie. Il s'agit d'un travail qui a fait l'objet d'un article soumis à "Studia Mathematica".

La deuxième partie est encore composée de deux articles et est considérée comme application aux résultats d'interpolation de la première partie. Le premier article (chapitre 6) en collaboration avec Emmanuel Russ traite des questions sur les graphes. Nous démontrons des théorèmes d'interpolation des espaces de Sobolev homogènes sur les graphes sous des hypothèses de doublement et Poincaré (le cadre non homogène n'est pas traité mais en suivant la même idée on obtient un résultat analogue sous des hypothèses locales). Nous étudions sous les mêmes hypothèses, la L_p bornitude de la transformée de Riesz pour $p > 2$ et son inégalité inverse pour $p < 2$. Nous démontrons aussi sous (D) et (P_2) la L_p bornitude de la version discrète de la fonction g de Littlewood-Paley-Stein pour $1 < p < \infty$.

Dans le deuxième article (chapitre 7), nous démontrons en suivant la méthode de Martin-Milman dans [23] et en utilisant notre résultat d'interpolation du chapitre 3, le théorème 1.3.8 et aussi d'autres inégalités fonctionnelles dans le cadre des variétés Riemanniennes complètes vérifiant le doublement, des inégalités de Poincaré et pseudo-Poincaré. Les résultats de ce chapitre s'appliquent aussi dans le cadre des groupes de Lie et graphes.

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Chapitre 2

Préliminaires

Ce chapitre se compose de trois sections. La première section est consacrée à l'interpolation: définition, méthode K d'interpolation réelle, interpolation des espaces L_p. Nous rappelons dans la deuxième section la propriété de doublement sur un espace métrique mesuré et ce qu'elle implique sur les boules en tant que sous espaces, ainsi que le théorème maximal de Hardy-Littlewood. Nous énonçons un lemme d'extension de Whitney et explicitons sa preuve. Dans cette même section, nous parlons aussi des inégalités de Poincaré et les équivalences entre différentes définitions. Nous finissons ce chapitre avec une troisième section portant sur les définitions des espaces de Sobolev dans différents cadres géométriques.

Notation: Dans tout ce travail, nous noterons par $\mathbb{1}_E$ la fonction caractéristique de E et E^c sera le complément de E. Sur une variété Riemannienne M, C_0^∞ désigne l'ensemble des fonctions définies sur M, C^∞ à support compact. Si X est un espace métrique, Lip est l'ensemble des fonctions Lipschitziennes réelles définies sur X et Lip_0 est l'ensemble des fonctions Lipschitziennes réelles définies sur X à support compact. Pour une boule B dans un espace métrique mesuré, nous noterons λB la boule de même centre que B et de rayon λ fois le rayon de B. Finalement C sera une constante qui peut changer d'une inégalité à une autre et nous noterons $u \sim v$ pour dire qu'il existe deux constantes C_1, $C_2 > 0$ tel que $C_1 u \leq v \leq C_2 u$. Les autres notations seront introduites au fur et à mesure.

2.1 Rappel sur l'interpolation

Pour une étude plus détaillée de la théorie d'interpolation et de toutes les notions qui s'y rapportent se référer à [5], [6], [25].

La principale impulsion pour étudier l'interpolation était les théorèmes classiques d'interpolation: le théorème de Riesz avec la preuve de Thorin, et le théorème de Marcinkiewicz. La preuve de Thorin pour le théorème de Riesz-Thorin contient l'idée derrière la méthode complexe d'interpolation. De même la preuve du théorème de Marcinkiewicz ressemble à la construction de la méthode réelle d'interpolation: nous les citerons pour mémoire.

2.1.1 Espaces L_p

Soit (X,μ) un espace mesuré. On note $L_p(X,\mu)$ ou simplement $L_p(X)$ ou même L_p quand il n'y a pas de confusion, l'espace des fonctions réelles μ-mesurables définies sur X telles que

$$\|f\|_p = \left(\int_X |f(x)|^p d\mu\right)^{\frac{1}{p}}$$

est finie pour $1 \leq p < \infty$.
Pour le cas limite quand $p = \infty$, L_p n'est autre que l'ensemble de toutes les fonctions μ-mesurables bornées muni de la norme

$$\|f\|_\infty = \inf\left\{\lambda; \mu(\{|f| > \lambda\}) = 0\right\}.$$

2.1.2 Fonctions de réarrangements et espaces de Lorentz

Définition 2.1.1. *Soit (X,μ) un espace mesuré et $f : X \to \mathbb{R}$ une fonction μ-mesurable. On note f^* sa fonction de réarrangement définie pour tout $t \geq 0$ par*

$$f^*(t) = \inf\left\{\lambda;\ \mu(\{x :\ |f(x)| > \lambda)\} \leq t\right\}.$$

La fonction f^* est positive, décroissante, continue à droite sur $[0,\infty[$ et elle a la propriété

$$\mu(\{|f^*| > \lambda\}) = \mu(\{|f| > \lambda\}).$$

Définition 2.1.2. *Soit (X,μ) un espace mesuré et $f : X \to \mathbb{R}$ une fonction μ-mesurable. La fonction maximale de f^* notée f^{**} et appelée fonction maximale de réarrangement est définie pour tout $t > 0$ par*

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)ds.$$

La fonction f^{**} est positive, décroissante, continue sur $]0,\infty[$.

Parmi les propriétés de f^* et f^{**} on cite:

1. $f^* \leq f^{**}$.
2. $(f+g)^{**} \leq f^{**} + g^{**}$.
3. $(\mathcal{M}f)^* \sim f^{**}$.
4. $\mu(\{x;\ |f(x)| > f^*(t)\}) \leq t$.
5. Pour tout $1 \leq p \leq \infty$, $\|f^*\|_p = \|f\|_p$.
6. Pour tout $1 < p \leq \infty$

$$\|f^*\|_p \leq \|f^{**}\|_p \leq \frac{p}{p-1}\|f^*\|_p.$$

Des points 5. et 6. on déduit que $\|f^{**}\|_p \sim \|f\|_p$ pour tout $1 < p \leq \infty$. Pour la preuve de ces propriétés et pour d'autres propriétés de f^* et f^{**} voir [5], [6], [7], [25] Chapitre V.

Définition 2.1.3 (Espaces de Lorentz $L(p,q)$)**.** *Soit (X,μ) un espace mesuré. On dit qu'une fonction $f : X \to \mathbb{R}$ μ-mesurable appartient à $L(p,q)$, $1 \leq p \leq \infty$, si et seulement si*

$$\|f\|_{L(p,q)} = \left(\int_0^\infty (t^{\frac{1}{p}} f^*(t))^q \frac{dt}{t}\right)^{\frac{1}{q}} < \infty \quad \textit{quand} \;\; 1 \leq q < \infty,$$

$$\|f\|_{L(p,\infty)} = \sup_t t^{\frac{1}{p}} f^*(t) < \infty \qquad \textit{quand}\, q = \infty.$$

Remarque 2.1.4. *Remarquons que $L(p,p) = L_p$ pour tout $1 \leq p \leq \infty$.*

2.1.3 Théorème de Riesz-Thorin et Théorème de Marcinkiewicz

Théorème 2.1.5 (Théorème de Riesz-Thorin)**.** *([6] p.2) Soit (X,μ) un espace mesuré. Prenons $1 \leq p_0 < p_1 < \infty$, $1 \leq q_0 < q < q_1 \leq \infty$ et soit T un opérateur linéaire tel que*

$$T : L_{p_0} \to L_{q_0}$$

est borné de norme M_0 et

$$T : L_{p_1} \to L_{q_1}$$

est borné de norme M_1. Alors

$$T : L_p \to L_q$$

est borné de norme $M \leq M_0^{1-\theta} M_1^{\theta}$ où $0 < \theta < 1$ et

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1},\ \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

Théorème 2.1.6 (Théorème de Marcinkiewicz)**.** *([6] p.9) Soit (X,μ) un espace mesuré. Prenons $1 \leq p_0 < p_1 < \infty$, $1 \leq q_0 < q_1 \leq \infty$ et soit T un opérateur linéaire tel que*

$$T : L_{p_0} \to L(q_0,\infty)$$

est borné de norme M_0 et

$$T : L_{p_1} \to L(q_1,\infty)$$

est borné de norme M_1. Soit $0 < \theta < 1$ et

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1},\ \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$$

et supposons que $p \leq q$. Alors

$$T : L_p \to L_q$$

est borné de norme $M \leq C_\theta M_0^{1-\theta} M_1^{\theta}$.

2.1.4 Espaces d'interpolation

Soient A_0, A_1 deux espaces vectoriels normés compatibles: deux espaces vectoriels normés A_0 et A_1 sont dits compatibles si A_0 et A_1 s'injectent continûment dans un même espace vectoriel topologique séparé $\mathcal{U}$. On a alors que $A_0 \cap A_1$ est un espace vectoriel normé pour la norme

$$\|a\|_{A_0 \cap A_1} = \max(\|a\|_{A_0},\ \|a\|_{A_1}).$$

De plus $A_0 + A_1$ est un espace vectoriel normé pour la norme

$$\|a\|_{A_0 + A_1} = \inf_{a=a_0+a_1} (\|a_0\|_{A_0} + \|a_1\|_{A_1}).$$

Définition 2.1.7. *Un espace vectoriel normé A est un espace d'interpolation entre deux espaces vectoriels normés A_0 et A_1 si*

1. *$A_0 \cap A_1 \subset A \subset A_0 + A_1$ avec des inclusions continues*
2. *tout opérateur linéaire T borné de A_0 dans A_0, et de A_1 dans A_1, est borné de A dans A.*

2.1.5 La méthode K d'interpolation réelle

On note $\overline{A} = (A_0, A_1)$. Pour tout $a \in A_0 + A_1$ et pour tout réel $t > 0$ on définit

$$K(a, t, \overline{A}) = \inf_{\substack{a=a_0+a_1 \\ a_0 \in A_0,\, a_1 \in A_1}} (\|a_0\|_{A_0} + t\|a_1\|_{A_1}).$$

Pour tout $t > 0$, $K(., t, \overline{A})$ définit une norme équivalente à $\|.\|_{A_0+A_1}$.

Lemme 2.1.8. *Pour tout $a \in A_0 + A_1$, $K(a, ., \overline{A})$ est une fonction positive, croissante et concave. En particulier, pour tout $0 < s,\ t < \infty$,*

$$K(a, t, \overline{A}) \leq \max(1, \frac{t}{s}) K(a, s, \overline{A}).$$

Définition 2.1.9. *Pour tout $0 < \theta < 1$, $1 \leq q \leq \infty$ ou $0 \leq \theta \leq 1$ et $q = \infty$, on définit l'espace d'interpolation $A_{\theta,q} = K_{\theta,q}(\overline{A})$ entre A_0 et A_1, par*

$$A_{\theta,q} = \left\{ a \in A_0 + A_1;\ \varphi_{\theta,q}(K(a, t, \overline{A})) := \left(\int_0^\infty \left(t^{-\theta} K(a, t, \overline{A})\right)^q \frac{dt}{t} \right)^{\frac{1}{q}} < \infty \right\}$$

muni de la norme

$$\|a\|_{\theta,q} = \varphi_{\theta,q}(K(a, t, \overline{A})).$$

Pour $q = \infty$,

$$\|a\|_{\theta,\infty} = \sup_{t>0} t^{-\theta} K(a, t, \overline{A}).$$

Théorème 2.1.10. *$K_{\theta,q}$ est un foncteur exact d'interpolation d'exposant θ, ce qui revient à dire que si $\overline{A}$ et $\overline{B}$ sont deux couples d'espaces vectoriels normés compatibles alors $A_{\theta,q}$ (resp. $B_{\theta,q}$) est un espace d'interpolation entre A_0 et A_1 (resp. entre B_0 et B_1) et de plus, si T est un opérateur linéaire*

$$T : A_0 \to B_0$$

borné avec une norme M_0 et

$$T : A_1 \to B_1$$

borné avec une norme M_1, alors

$$T : A_{\theta,q} \to B_{\theta,q}$$

est borné avec une norme $M \leq M_0^{\theta} M_1^{1-\theta}$.

Théorème 2.1.11 (Théorème de réitération (Theorem 3.5.3 p.50 in [6]))**.** *Soit $\overline{A}$ un couple d'espaces vectoriels normés compatibles. Soit $1 \leq q_i \leq \infty$ et $0 < \theta_i < 1$ pour $i = 0,\, 1$ avec $\theta_0 \neq \theta_1$. Si les A_{θ_i,q_i} sont complets alors*

$$(A_{\theta_0,q_0},\, A_{\theta_1,q_1})_{\eta,q} = A_{\theta,q}$$

où $1 \leq q \leq \infty$, $0 < \eta < 1$ et $\theta = (1-\eta)\theta_0 + \eta\theta_1$.

2.1.6 Interpolation réelle des espaces L_p

Théorème 2.1.12. *([6] p.109) Soit (X, μ) un espace mesuré. Si $f \in L_p + L_\infty$, $0 < p < \infty$, on a alors:*

1. *pour tout $t > 0$, $K(f, t, L_p, L_\infty) \sim \left(\int_0^{t^p} (|f|^*(s))^p ds\right)^{\frac{1}{p}}$ et si $p = 1$ on a égalité;*
2. *pour $0 < p_0 < p_1 \leq \infty$, $(L_{p_0}, L_{p_1})_{\theta,q} = L(p, q)$ avec normes équivalentes, où $\dfrac{1}{p} = \dfrac{1-\theta}{p_0} + \dfrac{\theta}{p_1}$ avec $0 < \theta < 1$ et $p_0 < q \leq \infty$. En particulier $(L_{p_0}, L_{p_1})_{\theta,p} = L(p,p) = L_p$.*

Notons que le point 2. résulte du point 1. et du théorème de réitération (Théorème 2.1.11). Ce résultat nous redonne qualitativement les théorèmes d'interpolation de Riesz-Thorin et le théorème de Marcinkiewicz avec la méthode d'interpolation de Lions-Peetre. Un autre théorème de caractérisation de la fonctionnelle K pour les espaces L_p qui redonne le point 2 du théorème précédent est le suivant:

Théorème 2.1.13. *([18]) Soit (X, μ) un espace métrique mesuré avec μ une mesure non-atomique (ce qui revient à dire que $\mu(\{x\}) = 0$ pour tout $x \in X$). Soit $0 < p_0 < p_1 < \infty$. Alors pour tout $t > 0$,*

$$K(f, t, L_{p_0}, L_{p_1}) \sim \left(\int_0^{t^\alpha} (f^*(u))^{p_0} du\right)^{\frac{1}{p_0}} + t \left(\int_{t^\alpha}^{\infty} (f^*(u))^{p_1} du\right)^{\frac{1}{p_1}},$$

où $\frac{1}{\alpha} = \frac{1}{p_0} - \frac{1}{p_1}$.

2.2 Quelques rappels d'analyse dans les espaces métriques mesurés

2.2.1 Propriété de doublement

Par espace métrique mesuré, on signifie un triplet (X, d, μ) où (X, d) est un espace métrique et μ une mesure borélienne positive. On note par $B(x, r)$ la boule ouverte centrée en x et de rayon $r > 0$ et par $\mu(B(x, r))$ sa mesure.

Définition 2.2.1. *Soit (X, d, μ) un espace métrique mesuré avec μ borélienne. On dit que X satisfait la propriété du doublement local (D_{loc}) s'il existe des constantes $r_0 > 0$, $0 < C = C(r_0) < \infty$, telles que pour tout $x \in X$, $0 < r < r_0$, on a*

$$(D_{loc}) \qquad \mu(B(x, 2r)) \leq C\mu(B(x, r)).$$

Et X vérifie la propriété du doublement (D) si on peut prendre $r_0 = \infty$. On dit aussi que μ est une mesure localement doublante (resp. doublante).

Observons que si X est un espace métrique mesuré vérifiant (D) alors

$$diam(X) < \infty \Leftrightarrow \mu(X) < \infty \ \ ([1]).$$

Définition 2.2.2. *On dit qu'un sous ensemble mesuré E d'un espace métrique mesuré X a la propriété du doublement relatif du volume s'il existe une constante C_E telle que pour tout $x \in E$ et tout $r > 0$ on a*

$$\mu(B(x, 2r) \cap E) \leq C_E \, \mu(B(x, r) \cap E).$$

Ceci revient à dire que $(E, d/E, \mu/E)$ a la propriété du doublement de volume.
La constante C_E est appelée la constante du doublement relatif de E.

Sous certaines conditions, la propriété de doublement local implique le doublement relatif sur les boules. Ce résultat a été démontré dans le cadre des variétés Riemannienes complètes dans [4]. On va l'énoncer ici dans un cadre plus général.

Définition 2.2.3. *Un espace de longueur X est un espace métrique tel que la distance entre n'importe quels points x, $y \in X$ est égale à la borne inférieure des longueurs de toutes les courbes joignant x à y (on suppose implicitement qu'il existe au moins une telle courbe). Une courbe qui joint x à y est une application continue $\gamma : [0, 1] \to X$ avec $\gamma(0) = x$ et $\gamma(1) = y$.*

Lemme 2.2.4. *Soit X un espace de longueur complet vérifiant (D_{loc}). Alors toute boule $B = B(x_1, r_1)$ de rayon $r_1 < \frac{8}{9} r_0$, munie de la distance et de la mesure induites, satisfait la propriété de doublement relative. Ceci revient à dire qu'il existe $C \geq 0$ telle que*

$$(2.1) \qquad \mu(B(x, 2r) \cap B) \leq C \, \mu(B(x, r) \cap B) \quad \forall x \in B, \, r > 0,$$

et

$$(2.2) \qquad \mu(B(x, r)) \leq C\mu(B(x, r) \cap B) \quad \forall x \in B, \, 0 < r \leq 2r_1.$$

Preuve. Soit $x \in B$, $r > 0$. Si $r \geq 2r_1$ il n'y a rien à prouver. Supposons $r < 2r_1$, il existe x_* tel que $B(x_*, \frac{r}{4}) \subset B$ et $d(x, x_*) \leq \frac{r}{4}$. En effet si $d(x, x_1) \leq \frac{r}{4}$ alors $B(x, \frac{r}{4}) \subset B$ et ainsi il suffit de prendre $x_* = x$. Sinon comme X est un espace de longueur, il existe une courbe γ joignant x à x_1 telle que $l(\gamma) \leq d(x, x_1) + \frac{r}{2}$. Sur γ on prend x_* tel que $d(x, x_*) = \frac{r}{4}$ (x_* existe d'après le théorème des valeurs intermédiaires sur un espace métrique connexe) alors $d(x_*, x_1) \leq d(x, x_1) - \frac{r}{4}$ et x_* satisfait $B(x_*, \frac{r}{4}) \subset B \cap B(x, r)$ et $B(x, 2r) \subset B(x_*, \frac{9}{4}r)$. On aura donc

$$\begin{aligned}\mu(B(x,2r)\cap B) &\leq \mu(B(x,2r)) \\ &\leq \mu(B(x_*, \frac{9r}{4})) \\ &\leq C\,\mu(B(x_*, \frac{r}{4})) \\ &\leq C\,\mu(B(x,r)\cap B).\end{aligned}$$

Nous avons utilisé la propriété de doublement du volume pour les boules de rayon inférieur à $\frac{8}{9}r_0$. Ainsi on obtient (2.1) du lemme 2.2.4 et la preuve de (2.2) est contenue dans l'argument. □

Remarque 2.2.5. *Un examen de la preuve montre que la constante C dépend de r_1 mais pas du centre x_1.*

Théorème 2.2.6 (Théorème maximal de Hardy-Littlewood)**.** *([9]) Soit (X, d, μ) un espace métrique mesuré satisfaisant (D). On note par $\mathcal{M}$ la fonction maximale de Hardy-Littlewood non-centrée définie sur les boules de X par*

$$\mathcal{M}f(x) = \sup_{B : x \in B} |f|_B$$

où $f_E := \fint_E f d\mu := \frac{1}{\mu(E)} \int_E f d\mu$. Alors

1. *$\mu(\{x : \mathcal{M}f(x) > \lambda\}) \leq \frac{C}{\lambda} \int_X |f| d\mu$ pour tout $\lambda > 0$;*
2. *$\|\mathcal{M}f\|_{L_p} \leq C_p \|f\|_{L_p}$, pour $1 < p \leq \infty$.*

Théorème 2.2.7. *(Lemme de Whitney) Soit (X, d) un espace métrique et Ω un ouvert de X avec $\Omega \neq X$. Alors, il existe une collection de boules $(\underline{B_i})_{i \in I}$, $\underline{B_i} = B(x_i, r_i)$ et une constante $C_1 > 0$ telles que:*

1. *Les $\underline{B_i}$ sont deux à deux disjointes.*
2. *$\Omega = \cup_{i \in I} B_i$ avec $B_i := C_1 \underline{B_i}$.*
3. *Pour tout i, $\overline{B_i} \cap \Omega \neq \emptyset$ où $\overline{B_i} := 4C_1 \underline{B_i} = 4B_i$.*

Preuve. Soit $\delta(x) = d(x, \Omega^c)$, $x \in X$. Prenons $0 < \varepsilon < \frac{1}{2}$. De la collection de boules $(B(x, \varepsilon\delta(x)))_{x\in\Omega}$ on peut extraire une sous collection maximale de boules deux à deux disjointes (Lemme de Zorn), qu'on note $(B(x_i, r_i))_{i\in I} := (\underline{B_i})_{i\in I}$ avec $r_i = \varepsilon\delta(x_i)$. Posons $C_1 = \frac{1}{2\varepsilon}$, alors $B_i := C_1\underline{B_i} = B(x_i, \frac{1}{2}\delta(x_i)) \subset \Omega$ et $\overline{B_i} := 4C_1\underline{B_i} \cap \Omega^c = B(x_i, 2\delta(x_i)) \cap \Omega^c \neq \emptyset$.

Il reste à démontrer que $\Omega = \cup_{i\in I} B_i$. Sinon, il existe $x \in \Omega \setminus \cup_{i\in I} B_i$. D'où, par la propriété de maximalité des boules $\underline{B_i}$, il existe $k \in I$ tel que $B(x, \varepsilon\delta(x)) \cap B(x_k, \varepsilon\delta(x_k)) \neq \emptyset$. Alors $d(x, x_k) \leq \varepsilon(\delta(x) + \delta(x_k))$. Or $x \in B(x_k, \frac{1}{2}\delta(x_k))^c$, donc $d(x, x_k) \geq \frac{1}{2}\delta(x_k)$. Par conséquent $(\frac{1}{2} - \varepsilon)\delta(x_k) \leq \varepsilon\delta(x)$. Ceci nous donne $B(x_k, 2\delta(x_k)) \subset B(x, 2\delta(x_k) + d(x, x_k)) \subset B(x, C_\varepsilon\delta(x))$ avec $C_\varepsilon = (\frac{2\varepsilon}{\frac{1}{2}-\varepsilon} + \frac{\frac{\varepsilon}{2}}{\frac{1}{2}-\varepsilon})$.

On choisit ε de façon à avoir $C_\varepsilon \leq \frac{1}{2}$. On a ainsi $B(x_k, 2\delta(x_k)) \subset B(x, \frac{1}{2}\delta(x)) \subset \Omega$ ce qui contredit le fait que $\overline{B_k} \cap \Omega^c \neq \emptyset$. □

Corollaire 2.2.8. *Si (X, d, μ) est un espace métrique mesuré vérifiant (D), alors les boules $(B_i)_{i\in I}$ ont la propriété du recouvrement borné.*

Preuve. Fixons $i \in I$. Soit $J_i = \{j \in I : B_i \cap B_j \neq \emptyset\}$. Prenons $j \in J_i$, alors il existe $z \in B_i \cap B_j$. Le fait que $z \in B_j$ nous donne $d(z, x_j) \leq \frac{1}{2}\delta(x_j)$ et $d(z, \Omega^c) \geq \frac{1}{2}\delta(x_j)$. D'autre part , comme $z \in B_i$, $d(z, \Omega^c) \leq d(x_i, \Omega^c) + d(z, x_i) \leq \frac{3}{2}\delta(x_i)$. On a ainsi $\delta(x_j) \leq 3\delta(x_i)$. De la même, on a $\delta(x_i) \leq 3\delta(x_j)$.

On déduit alors que si $j \in J_i$, les boules $B(x_j, \frac{\varepsilon}{3}\delta(x_i))$ sont deux à deux disjointes. De plus, $B(x_j, \frac{\varepsilon}{3}\delta(x_i)) \subset B(x_i, d(x_i, x_j) + \frac{\varepsilon}{3}\delta(x_i)) \subset B(x_i, (2 + \frac{\varepsilon}{3})\delta(x_i))$. Comme X est un espace de type homogène (car il vérifie (D)) (voir [9]) on déduit que le cardinal de J_i est majoré par $C(\frac{2+\frac{\varepsilon}{3}}{\frac{\varepsilon}{3}}, C_0)$ où C_0 est la constante de (D). □

2.2.2 Inégalité de Poincaré

Soit (X, d) un espace métrique. Une courbe γ sur X est une application continue $\gamma : [a, b] \to X$. La longueur de γ est définie par

$$l(\gamma) = \sup \left\{ \sum_{i=0}^{n-1} d(\gamma(t_i), \gamma(t_i + 1)) \right\},$$

où la borne supérieure est prise sur toutes les partitions $a = t_0 < t_1 < ... < t_n = b$. On dit que γ est rectifiable si $l(\gamma) < \infty$.

Proposition 2.2.9. *(Theorem 3.2 de [12]) Si $\gamma : [a, b] \to X$ est une courbe rectifiable, il existe alors une unique courbe (appelée paramètre normal de γ) $\tilde{\gamma} : [0, l(\gamma)] \to X$ telle que*

$$\gamma = \tilde{\gamma} \circ s_\gamma$$

où $s_\gamma : [a, b] \to [0, l(\gamma)]$ est donnée par $s_\gamma(t) = l(\gamma|_{[a,t]})$.

De plus $l(\tilde{\gamma}|_{[0,t]}) = t$ pour tout $t \in [0, l(\gamma)]$. En particulier $\tilde{\gamma} : [0, l(\gamma)] \to X$ est une fonction 1-Lipschitzienne.

Définition 2.2.10. *Soit $\gamma : [a,b] \to X$ une courbe rectifiable et $\rho : X \to [0,\infty]$ une fonction borélienne. On definit*

$$\int_\gamma \rho := \int_0^{l(\gamma)} \rho(\tilde{\gamma}(t))dt.$$

Définition 2.2.11 (Sur-gradient (ou Upper gradient en Anglais))**.** *Soit $u : X \to \mathbb{R}$ une fonction borélienne. On dit qu'une fonction borélienne $g : X \to [0,\infty]$ est un sur-gradient de u si*

$$|u(\gamma(a)) - u(\gamma(b))| \leq \int_\gamma g$$

pour tout $\gamma : [a,b] \mapsto X$ rectifiable.

Remarque 2.2.12. *Si X est une variété Riemannienne et $u \in \mathrm{Lip}(X)$, alors $|\nabla u|$ est un sur-gradient de u et $|\nabla u| \leq g$ pour tout sur-gradient g de u.*

Définition 2.2.13. *Pour $u : X \to \mathbb{R}$ localement Lipschitzienne définie sur un ouvert de X, on définit*

$$\mathrm{Lip}\, u(x) = \begin{cases} \limsup_{\substack{y\to x\\ y\neq x}} \frac{|u(y)-u(x)|}{d(y,x)} & \text{si } x \text{ n'est pas isolé,} \\ 0 \text{ sinon.} \end{cases}$$

Remarque 2.2.14. *$Lip\, u$ est un sur-gradient de u.*

Pour simplifier les notations, nous allons énoncer la définition de l'inégalité de Poincaré dans le cas global. Pour le cas local, nous la définirons au cours des chapitres.

Définition 2.2.15 (Inégalité de Poincaré)**.** *On dit qu'un espace métrique mesuré (X,d,μ) vérifie une inégalité de Poincaré faible $(P_{q,p})$, $1 \leq p, q < \infty$, s'il existe $\lambda \geq 1$, $C_p > 0$ tels que pour toute fonction u continue, tout sur-gradient g de u, et pour toute boule B de rayon $r > 0$ on a:*

$$\Big(\fint_B |u - u_B|^q d\mu\Big)^{\frac{1}{q}} \leq C_p r \Big(\fint_{\lambda B} g^p d\mu\Big)^{\frac{1}{p}} \quad (P_{q,p}).$$

où

$$f_E = \frac{1}{\mu(E)} \int_E f d\mu = \fint_E f d\mu.$$

Si $\lambda = 1$, $(P_{q,p})$ sera appelée Poincaré fort.

Nous noterons $(P_{p,p})$ par (P_p).

Il est clair que si l'espace (X,d,μ) vérifie une inégalité de Poincaré $(P_{q,p})$ [1], alors il vérifie $(P_{q',p})$ pour tout $q' \leq q$ et il vérifie $(P_{q,p'})$ pour tout $p' \geq p$. Donc si l'ensemble des p tels qu'on a $(P_{q',p})$ est non vide, c'est un intervalle non borné à droite. On améliore les exposants dans les inégalités de Poincaré avec les deux théorèmes suivants:

[1] faible ou fort

Théorème 2.2.16. *([13]) Soit (X, d, μ) un espace métrique mesuré avec μ doublante vérifiant une inégalité de Poincaré faible $(P_{1,p})$. Alors il existe $q' > p$ tel que X vérifie $(P_{q,p})$ faible pour tout $1 \leq q < q'$. En particulier X vérifie une inégalité de Poincaré faible (P_p).*

Ainsi $(P_{1,p}) \Leftrightarrow (P_p)$ sous les hypothèses du Théorème 2.2.16. En outre, on a le résultat récent suivant du à Keith et Zhong:

Théorème 2.2.17. *([22]) Soit (X, d, μ) un espace métrique mesuré complet avec μ doublante, satisfaisant (D) et $(P_{1,p})$ faible. Il existe alors $\varepsilon > 0$ tel que (X, d, μ) vérifie $(P_{1,q})$ faible pour tout $q > p - \varepsilon$.*

Définition 2.2.18. *Un espace métrique X est dit quasiconvexe s'il existe $C > 0$ tel que n'importe quels deux points x_1, x_2 de X peuvent être joints par une courbe dont la longueur ne dépasse pas $Cd(x_1, x_2)$.*

Proposition 2.2.19. *([15]) Soit (X, d, μ) un espace métrique mesuré quasiconvexe tel que les boules fermées sont compactes. Si X vérifie $(P_{1,p})$ faible, alors X vérifie $(P_{1,p})$ faible pour toutes les fonctions u mesurables, c'est à dire que u continue est remplacée par u mesurable dans la définition 2.2.15.*

Proposition 2.2.20 ([20])**.** *Soit $1 \leq p < \infty$. On considère (X, d, μ) un espace métrique mesuré complet (ou quasiconvexe propre) avec μ doublante. Alors les propositions suivantes sont équivalentes:*

1. *(X, d, μ) admet une inégalité de Poincaré faible $(P_{1,p})$ pour toutes les fonctions mesurables.*

2. *(X, d, μ) admet une inégalité de Poincaré faible $(P_{1,p})$ pour toutes les fonctions Lipschitziennes à support compact.*

3. *Il existe $C_p > 0$, $\lambda \geq 1$ telles que*

$$\fint_B |u - u_B| \, d\mu \leq C_p \, r \Big(\fint_{\lambda B} (Lip\, u)^p \, d\mu \Big)^{\frac{1}{p}} .$$

   pour tout $u \in \mathrm{Lip}_0$ et pour toute boule B de X.

Définition 2.2.21. *Un espace métrique est dit géodésique si tout x_1, $x_2 \in X$ distincts peuvent être joints par une courbe de longueur égale à $d(x_1, x_2)$.*

Proposition 2.2.22. *(Theorem 9.5 de [16]) Soit $1 \leq p < \infty$. Si (X, d, μ) est géodésique, μ doublante et X vérifie $(P_{1,p})$ faible alors X vérifie $(P_{1,p})$ fort.*

Rappelons qu'une fonction $f : (Y_1, d_1) \mapsto (Y_2, d_2)$ est bi-Lipschitzienne, s'il existe $L > 0$ telle que pour tous x, $y \in Y_1$, on a

$$\frac{1}{L} \, d_1(x, y) \leq d_2(f(x), f(y)) \leq L \, d_1(x, y).$$

Proposition 2.2.23. *(Proposition 6.0.7 de [20]) Soit* (X, d, μ) *un espace métrique mesuré complet, satisfaisant* (D) *et une inégalité de Poincaré faible* $(P_{1,p})$ *pour un* $1 \leq p < \infty$*. Alors* (X, d, μ) *est bi-Lipschitzien à un espace métrique géodésique avec la constante de l'application bi-Lipschitzienne dépendant uniquement de la constante de doublement et celle de Poincaré.*

Corollaire 2.2.24. *Soit* (X, d, μ) *un espace métrique mesuré complet, satisfaisant* (D)*. Si* X *vérifie une inégalité de Poincaré faible* $(P_{1,p})$ *pour un* $1 \leq p < \infty$ *alors* X *vérifie* $(P_{1,p})$ *fort.*

La proposition 2.2.23 découle des lemmes suivants:

Lemme 2.2.25. *(Lemma 6.0.8 de [20]) Soit* (X, d, μ) *un espace métrique mesuré complet, satisfaisant* (D) *et une inégalité de Poincaré faible* $(P_{1,p})$ *pour un* $1 \leq p < \infty$*. Alors* (X, d, μ) *est quasiconvexe avec la constante de quasiconvexité dépendant uniquement de la constante de doublement et celle de Poincaré.*

Lemme 2.2.26. *([12]) Un espace métrique mesuré* (X, d, μ) *complet avec* μ *doublante est propre (c'est à dire les ensembles bornés fermés sont compacts).*

On déduit la preuve de la proposition 2.2.23 à l'aide du lemme suivant:

Lemme 2.2.27. *([16]) Un espace métrique quasiconvexe propre est bi-Lipschitzien à un espace métrique géodésique avec la constante de l'application bi-Lipschitzienne dépendant uniquement de la constante de quasiconvexité.*

Pour terminer, remarquons qu'il faut faire la distinction entre les inégalités de Poincaré relatives et les inégalités de Poincaré locales. Regardons par exemple le cadre des variétés. Soit M une variété Riemannienne complète et E un sous ensemble mesuré de M. On dit que E admet une inégalité de Poincaré relative pour un $1 \leq p < \infty$ s'il existe une constante $C_E > 0$ telle que pour toute boule de rayon $r > 0$, centrée en E, et pour toute fonction f telle que f et $|\nabla f|$ sont p localement intégrables sur E on a

$$\int_{B\cap E} |f - f_{B\cap E}|^p \, d\mu \leq C_E \, r^p \int_{B\cap E} |\nabla f|^p \, d\mu.$$

L'inégalité uniforme de Poincaré relative pour les boules E de rayon $r_1 > 0$ [2] implique une inégalité de Poincaré locale. En effet, soit B une boule de centre x et rayon $s \leq r_1$. Prenons $E = B(x, r_1)$ alors

$$\int_B |f - f_B|^p d\mu = \int_{B\cap E} |f - f_{B\cap E}|^p d\mu \leq C_E \, s^p \int_{B\cap E} |\nabla f|^p d\mu = C \, s^p \int_B |\nabla f|^p d\mu.$$

En revanche, sur un espace métrique mesuré, une inégalité de Poincaré locale (ou globale) n'entraîne pas une inégalité de Poincaré relative pour les boules. Voici un exemple qui nous a été communiqué par Juha Heinonen:
On prend un rectangle fermé dans le plan de sorte qu'il soit long et maigre, disons 100×1. On forme X en enlevant un rectangle ouvert analogue du milieu, disons

[2] c'est à dire la constante C_E ne depend que de r_1 et pas du centre de E.

$98\times 0,5$. On munit X de la distance Euclidienne induite et μ est la mesure de Lebesgue induite. Cet ensemble compact admet une inégalité de Poincaré (P_1), topologiquement c'est un anneau. Prenons une boule centrée sur un des côtés courts et ayant un rayon 90, disons $E = B(y, 90)$. Alors E contient des points x tels que $B(x, 30) \cap E$ n'est pas connexe. Or la connexité est nécessaire à toute inégalité de Poincaré sur $B \cap E$.

2.3 Espaces de Sobolev

Dans cette section, nous allons donner la définition des espaces de Sobolev classiques dans différents cadres géométriques. Nous allons nous restreindre aux espaces de Sobolev non homogènes. Pour la définition des espaces de Sobolev homogènes dans le cadre des variétés Riemanniennes et groupes de Lie voir Chapitre 3. Dans le cas Euclidien la définition des espaces de Sobolev non homogènes a été donnée dans l'introduction.

2.3.1 Espaces de Sobolev non homogènes sur les variétés

On considère une variété Riemannienne M complète, non compacte, μ la mesure Riemannienne, ∇ le gradient Riemannien.

Définition 2.3.1. *([3]) Soit M une variété Riemannienne de dimension n. On appelle E_p^1 l'espace vectoriel des fonctions $\varphi \in C^\infty$ telles que $|\varphi|$ et $|\nabla\varphi| \in L_p$, $1 \leq p < \infty$. On définit l'espace de Sobolev non homogène W_p^1 comme étant le complété de E_p^1 pour la norme*

$$\|\varphi\|_{W_p^1} = \|\varphi\|_p + \|\nabla\varphi\|_p.$$

On note W_∞^1 l'espace de toutes les fonctions Lipschitziennes bornées définies sur M.

Théorème 2.3.2. *Soit M une variété Riemannienne complète alors*

1. *([3]) C_0^∞ est dense dans W_p^1 pour $1 \leq p < \infty$.*
2. *([11]) Lip_0 est dense dans W_p^1 pour $1 \leq p < \infty$.*

2.3.2 Espaces de Sobolev sur les groupes de Lie

Soit G un groupe de Lie connexe. Supposons que G est unimodulaire et soit $d\mu$ une mesure de Haar fixée sur G. Soit $X_1, ..., X_k$ des champs de vecteurs invariants à gauche tel que les X_i satisfont une condition de Hörmander. Dans ce cas la métrique de Carnot-Carathéodory est une distance, G muni de cette distance ρ est complet et cette distance définit la même topologie que celle de G en tant que variété (voir [10] p. 1148).

Définition 2.3.3. *Pour $1 \leq p \leq \infty$, on définit l'espace de Sobolev*

$$W_p^1 = \{f \in L^p;\ |Xf| \in L^p\},$$

Xf étant défini au sens des distributions, muni de la norme:

$$\|f\|_{W_p^1} = \|f\|_{L^p} + \||Xf|\|_{L^p}$$

où $|Xf| = \left(\sum_{i=1}^k |X_i f|^2\right)^{\frac{1}{2}}$.

Remarque 2.3.4. *1. W_p^1 muni de cette norme est un espace de Banach.*

2. W_∞^1 n'est autre que l'ensemble de toutes les fonctions Lipschitziennes bornées définies sur G.

Définition 2.3.5. *Pour $1 \leq p < \infty$, on définit l'espace de Sobolev H_p^1 comme étant le complété des fonctions C^∞ pour la norme :*

$$\|u\|_{H_p^1} := \|u\|_{W_p^1} < \infty.$$

Par convention, on pose $H_\infty^1 = W_\infty^1$.

Proposition 2.3.6. *1. $H_p^1 = W_p^1$ pour tout $1 \leq p < \infty$.*

2. C_0^∞ est dense dans H_p^1 pour $1 \leq p < \infty$.

3. L'ensemble des fonctions Lipschitziennes à support compact est dense dans H_p^1 (voir[11]).

2.3.3 Espaces de Sobolev sur des espaces métriques mesurés

On rappelle que (X, d, μ) désigne un espace métrique muni d'une mesure borélienne positive μ. Dans les dernières années, différents auteurs parmi Hajlasz [14], Cheeger [8], Shammugalingan [24], Gol'dschtein et Troyanov [11], Heinonen [16], [17] se sont intéressés à l'étude des espaces de Sobolev sur des espaces métriques généraux. Ils ont introduit différentes définitions qui coïncident dans certains cas. De ces espaces sur X on cite (les définitions et les propriétés sont prises de [12]):

1. Pour $1 \leq p \leq \infty$, on considère $\tilde{N}_p^1$ la classe de toutes les fonctions boréliennes $u \in L_p$ admettant un sur-gradient $g \in L_p$. On munit $\tilde{N}_p^1$ de la semi-norme

$$\|u\|_{\tilde{N}_p^1} = \|u\|_p + \inf_g \|g\|_p,$$

où la borne inférieure est prise sur tous les sur-gradients g de u.
Pour u, $v \in \tilde{N}_p^1$, on définit la relation d'équivalence $\sim$ par

$$u \sim v \text{ si et seulement si } \|u - v\|_{\tilde{N}_p^1} = 0.$$

L'espace de Sobolev N_p^1 est défini comme l'espace quotient

$$N_p^1(X, d, \mu) := \tilde{N}_p^1(X, d, \mu)/\sim$$

muni de la norme

$$\|u\|_{N_p^1} := \|u\|_{\tilde{N}_p^1}.$$

Remarque 2.3.7. *Dans le cas Euclidien, si Ω est un ouvert de $\mathbb{R}^n$, $N_p^1(\Omega) = W_p^1(\Omega)$ pour tout $1 \leq p < \infty$.*

2. Pour $1 \leq p \leq \infty$, on considère l'ensemble de toutes les fonctions $u \in L_p$ telles que
$$\|u\|_{C_p^1} = \|u\|_p + \inf_{(g_i)} \liminf_{i\to\infty} \|g_i\|_p < \infty$$
où la borne inférieure est prise parmi toutes les suites $(g_i)_i$ de fonctions boréliennes sur-gradient de fonctions $u_i \in L_p$ qui convergent vers u dans L_p.

Lemme 2.3.8. *Les espaces C_p^1 et N_p^1 sont isométriquement isomorphes pour $1 < p < \infty$.*

3. Soit $1 \leq p \leq \infty$. L'espace de Sobolev M_p^1 est l'ensemble de toutes les fonctions $u \in L_p$ telles qu'il existe une fonction mesurable $g \geq 0$, $g \in L_p$, vérifiant
$$|u(x) - u(y)| \leq d(x,y)(g(x) + g(y)) \ \mu - p.p.. \tag{2.3}$$
On munit M_p^1 de la norme
$$\|u\|_{M_p^1} = \|u\|_p + \inf_{g \text{ vérifiant } (2.3)} \|g\|_p.$$

Remarque 2.3.9. *L'ensemble des fonctions Lipschitziennes est dense dans M_p^1.*

4. Soit $1 \leq p < \infty$. On définit P_p^1 comme étant l'ensemble de toutes les fonctions $u \in L_p$ telles qu'il existe $g \in L_p$, $\lambda \geq 1$ vérifiant
$$\forall B \ \text{ boule}, \ \ ⨍_B |u - u_B| d\mu \leq r \left(⨍_{\lambda B} g^p d\mu \right)^{\frac{1}{p}}. \tag{2.4}$$
On munit P_p^1 de la norme
$$\|u\|_{P_p^1} = \|u\|_p + \inf_{g \text{ vérifiant } (2.4)} \|g\|_p.$$
On appelle $P_\infty^1 =$ l'espace des fonctions $u \in L_\infty$ telles qu'il existe une fonction $g \in L_\infty$, $\lambda \geq 1$ avec
$$\forall B \ \text{ boule}, \ \ \|u - u_B\|_{L_\infty(B)} \leq r \|g\|_{L_\infty(\lambda B)}. \tag{2.5}$$
On le munit de la norme
$$\|u\|_{P_\infty^1} = \|u\|_\infty + \inf_{g \text{ vérifiant } (2.5)} \|g\|_\infty.$$

5. Soit $1 \leq p \leq \infty$. On appelle H_p^1 la fermeture de l'espace des fonctions localement Lipschitziennes pour la norme
$$\|u\|_{H_p^1} = \|u\|_p + \|Lip\, u\|_p.$$

Remarque 2.3.10. *Tous les espaces de Sobolev cités sont des espaces de Banach.*

Proposition 2.3.11. *([13]) Si X est un espace métrique mesuré complet satisfaisant (D) et (P_q) pour un $1 < q < \infty$, alors pour $q_0 < p < \infty$, où $q_0 = \inf\{q \in [1, \infty[;\ (P_q)\ \text{vraie}\ \}$ tous les espaces cités sont égaux à H^1_p avec des normes équivalentes.*

Pour $p = \infty$, on a le résultat suivant:

Proposition 2.3.12. *Soit (X, d, μ) un espace métrique mesuré complet vérifiant (D). Soit A l'ensemble de toutes les fonctions Lipschitziennes bornées sur X muni de la norme:*

$$\|u\|_A = \|u\|_\infty + \|u\|_{Lip}$$

où $\|u\|_{\mathrm{Lip}} := \inf\{L;\ u\, L\ \text{Lipschitzienne}\}$.

Si X admet (P_p) pour un $1 \leq p < \infty$, on a $M^1_\infty = P^1_\infty = N^1_\infty = C^1_\infty = H^1_\infty = A$ avec des normes équivalentes.

Preuve. On va montrer que chacun de ces espaces est égal à A.

1. Commençons par $H^1_\infty = A$. Si u est une fonction Lipschitzienne bornée, $\|\operatorname{Lip} u\|_\infty \leq \|u\|_{\mathrm{Lip}}$. D'où $u \in H^1_\infty$ et $\|u\|_{H^1_\infty} \leq \|u\|_A$.
   Maintenant considérons $u \in H^1_\infty$, $u = \lim\limits_n u_n$ dans H^1_∞ avec u_n localement Lipschitzienne pour tout n. Il est clair que u est bornée et $\|u\|_\infty \leq \|u\|_{H^1_\infty}$. Comme $(u_n, \operatorname{Lip} u_n)$ satisfait l'inégalité de Poincaré (P_p) alors pour $\mu \otimes \mu$ presque tout x, $y \in X$ on a

$$\begin{aligned}|u_n(x) - u_n(y)| &\leq C\, d(x,y) \left((\mathcal{M}(\operatorname{Lip} u_n)^p(x))^{\frac{1}{p}} + (\mathcal{M}(\operatorname{Lip} u_n)^p(y))^{\frac{1}{p}} \right) \\ &\leq 2\, C\, d(x,y)\, \|\mathcal{M}(\operatorname{Lip} u_n)^p\|_\infty^{\frac{1}{p}} \\ &\leq 2\, C\, d(x,y)\, \|(\operatorname{Lip} u_n)^p\|_\infty^{\frac{1}{p}} \\ &= 2\, C\, d(x,y)\, \|\operatorname{Lip} u_n\|_\infty\end{aligned}$$

   où on a utilisé dans la première inégalité le Théorème 9.4. de [13]. En passant à la limite quand $n \to \infty$ et en utilisant le fait que $\operatorname{Lip} u \sim |Du|$ pour toute fonction u localement Lipschitzienne avec D opérateur linéaire (voir Theorem 11.6 de [12] ou proposition 3.7.6 de cette thèse) on obtient:

$$|u(x) - u(y)| \leq 2C\, d(x,y)\, \|\operatorname{Lip} u\|_\infty \quad \mu - p.p..$$

   Quitte à modifier u sur un ensemble de mesure nulle, u est une fonction Lipschitzienne bornée avec

$$\|u\|_{\mathrm{Lip}} \leq C \|\operatorname{Lip}\, u\|_\infty.$$

   Ceci implique $H^1_\infty = A$ avec des normes équivalentes.

2. D'après [2], Chapter V, p.90, on a $M^1_\infty = A$. Reprenons rapidement l'argument. On considère u une fonction L Lipschitzienne. En prenant $g = \frac{L}{2}$, on voit que pour tout x, $y \in X$, $|u(x) - u(y)| \leq d(x,y)(g(x) + g(y))$. Ainsi

$$\inf_{g \text{ vérifiant } (2.3)} \|g\|_\infty \leq \frac{1}{2}\|u\|_{\text{Lip}}.$$

On en déduit $\|u\|_{M^1_\infty} \leq \|u\|_A$.
Prenons maintenant $u \in M^1_\infty$ et g vérifiant (2.3), on a

$$|u(x) - u(y)| \leq d(x,y)(g(x) + g(y)) \leq 2d(x,y)\|g\|_\infty$$

pour $\mu \otimes \mu$ presque tout x, $y \in X$. Quitte à modifier u sur un ensemble de mesure nulle, u est une fonction Lipschitzienne avec

$$\|u\|_{\text{Lip}} \leq 2\|g\|_\infty.$$

On en déduit $\|u\|_A \leq 2\|u\|_{M^1_\infty}$.

3. Vérifions maintenant $N^1_\infty = A$. Soit $u \in N^1_\infty$, $g \in L_\infty$ un sur-gradient de u. Soit x, $y \in X$ et γ une courbe rectifiable joignant x à y et $\tilde{\gamma}$ son paramètre normal. On a

$$\begin{aligned} |u(x) - u(y)| &= |u(\gamma(a)) - u(\gamma(b))| \\ &\leq \int_0^{l(\gamma)} g(\tilde{\gamma}(t))dt \\ &\leq \|g\|_\infty \, l(\gamma). \end{aligned}$$

En particulier comme X satisfait (D) et (P_p) pour un $1 \leq p < \infty$, il existe alors γ joignant x à y telle que $l(\gamma) \sim d(x,y)$ (car X est bi-Lipschitzien à un espace géodésique d'après la Proposition 2.2.23). On a donc $|u(x) - u(y)| \leq C\|g\|_\infty \, d(x,y)$. Par suite u est Lipschitzienne bornée et

$$\|u\|_{\text{Lip}} \leq C\|g\|_\infty$$

pour tout sur-gradient g de u. D'où $u \in A$ et $\|u\|_A \leq C\,\|u\|_{N^1_\infty}$.
D'autre part si u est une fonction L Lipschitzienne bornée, alors la fonction constante égale à L est un sur-gradient de u qui appartient à L_∞. Donc $u \in N^1_\infty$ et

$$\inf_{g \text{ sur-gradient de } u} \|g\|_\infty \leq \|u\|_{\text{Lip}}.$$

D'où $\|u\|_{N^1_\infty} \leq \|u\|_A$.

4. La preuve de $C^1_\infty = A$ suit celle de $N^1_\infty = A$. En effet, soit $u \in A$, u L Lipschitzienne bornée. On prend la suite constante des fonctions constantes telles que pour tout i $g_i = L$, et la suite constante définie par $u_i = u$ pour tout i. On a g_i un sur-gradient de u_i et $u_i \to u$ dans L_∞. Par suite $u \in C^1_\infty$ et $\|u\|_{C^1_\infty} \leq \|u\|_A$.

D'autre part, considérons $u \in C^1_\infty$, $(u_i)_i$, $(g_i)_i$ deux suites de fonctions mesurables telles que g_i est un sur-gradient de u_i pour tout i et $u_i \underset{i\to\infty}{\to} u$ dans L_∞. En reprenant l'argument du point 3., on obtient

$$|u_i(x) - u_i(y)| \leq C\|g_i\|_\infty d(x, y)$$

pour tout x, $y \in X$. D'où en passant à la limite inférieure on aura

$$|u(x) - u(y)| \leq C \liminf_{i\to\infty} \|g_i\|_\infty d(x, y). \tag{2.6}$$

En passant à la borne inférieure sur les suites (g_i) dans (2.6), on voit que u est Lipschitzienne bornée avec $\|u\|_A \leq C\|u\|_{C^1_\infty}$.

5. Il reste à montrer que $P^1_\infty = A$. Soit $u \in A$. Il existe $L > 0$ tel que $|u(x)-u(y)| \leq Ld(x, y)$ pour tout x, $y \in X$. Soit B une boule de rayon $r > 0$ et $x \in B$.

$$\begin{aligned}
|u(x) - u_B| &= \left| u(x) - \frac{1}{\mu(B)} \int_B u(y)d\mu(y) \right| \\
&= \left| ⨍_B (u(x) - u(y))d\mu(y) \right| \\
&\leq ⨍_B |u(x) - u(y)|d\mu(y) \\
&\leq L ⨍_B d(x, y)d\mu(y) \\
&\leq 2Lr.
\end{aligned}$$

En passant à la borne supérieure et en prenant $g = 2L \in L_\infty$, on déduit que $u \in P^1_\infty$ et que $\|u\|_{P^1_\infty} \leq 2\|u\|_A$.
Réciproquement, soit $u \in P^1_\infty$, x, $y \in X$. Prenons la plus petite boule B contenant x, y dont le rayon $r \sim d(x, y)$. On a si g vérifie (2.5)

$$\begin{aligned}
|u(x) - u(y)| &\leq |u(x) - u_B| + |u_B - u(y)| \\
&\leq 2\|u - u_B\|_{L_\infty(B)} \\
&\leq 2r\|g\|_\infty \\
&\leq Cd(x, y)\|g\|_\infty.
\end{aligned}$$

D'où u est Lipschitzienne bornée et $\|u\|_A \leq C\|u\|_{P^1_\infty}$.

□

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Partie I

Interpolation réelle des espaces de Sobolev sur les espaces métriques mesurés

Chapitre 3

Real Interpolation of Sobolev Spaces

Ce chapitre représente le travail de ma première année de thèse. Il a été rédigé sous forme d'article " Real Interpolation of Sobolev spaces" mais dans ce chapitre j'ai préféré modifier la version pour le rendre plus complet et en une sorte auto-contenu.

Abstract. In this chapter, we prove that W_p^1 is an interpolation space between $W_{p_1}^1$ and $W_{p_2}^1$ for $q_0 < p_1 < p < p_2 \leq \infty$ on some classes of manifolds and general metric spaces, where q_0 depends on our hypotheses.

Résumé. Dans ce chapitre on démontre que sur certaines variétés Riemanniennes et espaces métriques mesurés, W_p^1 est un espace d'interpolation entre $W_{p_1}^1$ et $W_{p_2}^1$ pour $q_0 < p_1 < p < p_2 \leq \infty$, où q_0 dépend de nos hypothèses.

3.1 Introduction

Do the Sobolev spaces W_p^1 form a real interpolation scale for $1 < p < \infty$? The aim of this chapter is to provide a positive answer for Sobolev spaces on some metric spaces. Let us state here our main theorems, in the case of non homogeneous Sobolev spaces (resp. homogeneous Sobolev spaces) on Riemannian manifolds. For other metric-measure spaces see sections 3.7 to 3.10 below.

Theorem 3.1.1. *Let M be a complete non-compact Riemannian manifold satisfying the local doubling property (D_{loc}) and a local Poincaré inequality (P_{qloc}), for some $1 \leq q < \infty$ (see below for the definitions). Then for $q < p < \infty$, W_p^1 is an interpolation space between W_q^1 and W_∞^1.*

To prove Theorem 3.1.1, we will characterize the K functional of interpolation for non homogeneous Sobolev spaces in the following theorem.

Theorem 3.1.2. *Under the hypotheses of Theorem 3.1.1, there exist C_1, C_2 such that for all $f \in W_q^1 + W_\infty^1$ and all $t > 0$ we have*

$(*_{\text{loc}})$

$$C_1 t^{\frac{1}{q}}\Big(|f|^{q**\frac{1}{q}}(t) + |\nabla f|^{q**\frac{1}{q}}(t)\Big) \leq K(f, t^{\frac{1}{q}}, W_q^1, W_\infty^1) \leq C_2 t^{\frac{1}{q}}\Big(|f|^{q**\frac{1}{q}}(t) + |\nabla f|^{q**\frac{1}{q}}(t)\Big)$$

where $|g|^{q**\frac{1}{q}} = (|g|^{q**})^{\frac{1}{q}}$.

The key point in the proof of this theorem is a Calderón-Zygmund decomposition for Sobolev functions (see Proposition 3.3.5 in section 3.3).

We mention that the fact that M is complete is not necessary to obtain Theorem 3.1.1 and Theorem 3.1.2 (see Remark 3.4.5 in section 3.4).

For $\mathbb{R}^n$, our result was already proved by Devore and Scherer [17] using spline functions, Calderón and Milman [10] using the Whitney extension theorem. Also Bennett and Sharpley [6] presented a simpler proof based on Whitney's covering lemma.

Theorem 3.1.3. *Let M be a complete non-compact Riemannian manifold satisfying the global doubling property (D) and a global Poincaré inequality (P_q) for some $1 \leq q < \infty$. Then for $q < p < \infty$, $\dot{W}_p^1$ is an interpolation space between $\dot{W}_q^1$ and $\dot{W}_\infty^1$.*

For the characterization of the K-functional for homogeneous Sobolev spaces we have a weaker result than Theorem 3.1.2 that will be sufficient for us to prove Theorem 3.1.3:

Theorem 3.1.4. *Under the hypotheses of Theorem 3.1.3 we have that*

1. *there exists C_1 such that for every $F \in \dot{W}_q^1 + \dot{W}_\infty^1$ and all $t > 0$*

$$K(F, t^{\frac{1}{q}}, \dot{W}_q^1, \dot{W}_\infty^1) \geq C_1 t^{\frac{1}{q}} |\nabla f|^{q**\frac{1}{q}}(t) \text{ where } f \in \dot{E}_q^1 + \dot{E}_\infty^1 \text{ and } \overline{f} = F;$$

2. *for $q \leq p < \infty$, there exists C_2 such that for every $F \in \dot{W}_p^1$ and every $t > 0$*

$$K(F, t^{\frac{1}{q}}, \dot{W}_q^1, \dot{W}_\infty^1) \leq C_2 t^{\frac{1}{q}} |\nabla f|^{q**\frac{1}{q}}(t) \text{ where } f \in \dot{E}_p^1 \text{ and } \overline{f} = F.$$

The reiteration theorem, implies another version of Theorem 3.1.1 and Theorem 3.1.3 and interpolation theorems for linear operators between several Sobolev spaces can be proved (see sections 3.4 and 3.5 below).

After finishing this work, Martin and Milman communicated to us their recent paper [33], where they obtain for the Euclidean case ($q = 1$ as (P_1) holds on $\mathbb{R}^n$) that the functional K defined for every $f \in W_1^1 + W_\infty^1$ by

$$K(f,t) = \inf_{\substack{f=f_0+f_1 \\ f_0 \in W_1^1,\, f_1 \in W_\infty^1}} (\| |\nabla f_0| \|_1 + t \| |\nabla f_1| \|_\infty)$$

is equivalent to $t|\nabla f|^{**}(t)$. We point out that this characterization of the functional K played an important role in the symmetrization approach to the sharp Gagliardo-Nirenberg inequalities given in [33]. It seems worth disposing of our more general version to show similar inequalities on more general spaces. We will treat this topic in Chapter 7.

Let us briefly comment on the structure of this Chapter. In section 3.2 we review the notions of doubling property as well as the real K interpolation method. In sections 3.3 to 3.5, we study in detail the case of a complete non-compact Riemannian manifold M satisfying (D) and (P_q) (resp. (D_{loc}) and (P_{qloc})). We briefly mention the case where M is a compact manifold in section 3.6. In sections 3.7 to 3.10 we extend our results respectively to the case of metric-measure spaces, Carnot-Carathéodory spaces, Weighted Sobolev spaces and finally to Lie Groups, with an appropriate definition of W_p^1. Section 3.11 is devoted for some examples to which our results apply.

The initial motivation of this work was to answer the interpolation question for $\dot{W}_p^1$ explicitly posed in [3] (see section 3.5).

Question: (D_{loc}) and (P_{qloc}) are sufficient to interpolate, but are they, and in particular (P_{qloc}), necessary?
In the last example of Section 11, we will see that Poincaré inequality is not a necessary condition to interpolate Sobolev spaces.

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3.2 Preliminaries

Throughout this paper we will denote by $\mathbb{1}_E$ the characteristic function of a set E and E^c the complement of E. If X is a metric space, Lip will be the set of real Lipschitz functions on X and Lip_0 the set of real compactly supported Lipschitz functions on X. For a ball B in a metric space, λB denotes the ball co-centered with B and with radius λ times that of B. Finally, C will be a constant that may change from an inequality to another and we will use $u \sim v$ to say that there exists two constants C_1,$C_2 > 0$ such that $C_1 u \leq v \leq C_2 u$.

3.2.1 The doubling property

By metric-measure space, we mean a triple (X, d, μ) where (X, d) is a metric space and μ a non-negative Borel measure.

Definition 3.2.1. *Let (X, d, μ) be a metric-measure space with μ a Borel measure. Denote by $B(x, r)$ the open ball of center $x \in X$ and radius $r > 0$ and by $\mu(B(x, r))$ its measure. One says that X satisfies the local doubling property (D_{loc}) if there exist constants $r_0 > 0$, $0 < C = C(r_0) < \infty$, such that for all $x \in X$, $0 < r < r_0$ we have*

$$(D_{loc}) \qquad \mu(B(x, 2r)) \leq C\mu(B(x, r)).$$

And X satisfies a global doubling property or simply doubling property (D) if one can take $r_0 = \infty$. We also say that μ is a locally (resp. globally) doubling Borel measure.

Observe that if X is a metric-measure space satisfying (D) then

$$diam(X) < \infty \Leftrightarrow \mu(X) < \infty \;\; ([1]).$$

Theorem 3.2.2 (Maximal theorem). *([13]) Let (X, d, μ) be a metric-measure space satisfying (D). Denote by $\mathcal{M}$ the uncentered Hardy-Littlewood maximal function over open balls of M defined by*

$$\mathcal{M}f(x) = \sup_{B: x \in B} |f|_B$$

where $f_E := \fint_E f d\mu := \frac{1}{\mu(E)} \int_E f d\mu$. *Then*

1. $\mu(\{x : \mathcal{M}f(x) > \lambda\}) \leq \frac{C}{\lambda} \int_X |f| d\mu$ *for every* $\lambda > 0$;
2. $\|\mathcal{M}f\|_{L_p} \leq C_p \|f\|_{L_p}$, *for* $1 < p \leq \infty$.

3.2.2 The K method of real interpolation

The reader is referred to [6], [7] for details on the development of this theory. Here we only recall the essentials to be used in the sequel.

Let A_0, A_1 be two normed vector spaces embedded in a topological Hausdorff vector space V, and define for $a \in A_0 + A_1$ and $t > 0$,

$$K(a, t, A_0, A_1) = \inf_{a = a_0 + a_1} (\|a_0\|_{A_0} + t\|a_1\|_{A_1}).$$

For $0 < \theta < 1$, $1 \leq q \leq \infty$, we denote by $(A_0, A_1)_{\theta,q}$ the interpolation space between A_0 and A_1:

$$(A_0, A_1)_{\theta,q} = \left\{ a \in A_0 + A_1 : \|a\|_{\theta,q} = \left(\int_0^\infty (t^{-\theta} K(a, t, A_0, A_1))^q \frac{dt}{t} \right)^{\frac{1}{q}} < \infty \right\}.$$

It is an exact interpolation space of exponent θ between A_0 and A_1, see [7], Chapter II.

Definition 3.2.3. *Let f be a measurable function on a measure space (X, μ). We denote by f^* its decreasing rearrangement function: for every $t > 0$,*

$$f^*(t) = \inf \{\lambda : \mu(\{x : |f(x)| > \lambda\}) \leq t\}.$$

We denote by f^{**} the maximal decreasing rearrangement of f: for every $t > 0$,

$$f^{**}(t) = \frac{1}{t} \int_0^t f^*(s) ds.$$

It is known that $(\mathcal{M}f)^* \sim f^{**}$ and $\mu(\{x : |f(x)| > f^*(t)\}) \leq t$ for all $t > 0$. We refer to [6], [7], [9] for other properties of f^* and f^{**}.

To end with the preliminaries let us quote the following classical result ([7] p.109):

Theorem 3.2.4. *Let (X, μ) be a measure space where μ is a totally σ-finite positive measure. Let $f \in L_p + L_\infty$, $0 < p < \infty$ where $L_p = L_p(X, d\mu)$, we then have:*

1. $K(f, t, L_p, L_\infty) \sim \left(\int_0^{t^p} (f^*(s))^p ds \right)^{\frac{1}{p}}$ *and equality holds for* $p = 1$;

2. *for* $0 < p_0 < p_1 \leq \infty$, $(L_{p_0}, L_{p_1})_{\theta,q} = L_{pq}$ *with equivalent norms, where* $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ *with* $0 < \theta < 1$, $p_0 < q \leq \infty$ *and* L_{pq} *are the Lorentz spaces. In particular* $(L_{p_0}, L_{p_1})_{\theta,p} = L_{pp} = L_p$.

3.3 Non homogeneous Sobolev spaces on Riemannian manifolds

In this section M denotes a complete non-compact Riemannian manifold. We write μ for the Riemannian measure on M, ∇ for the Riemannian gradient, $|\cdot|$ for the length on the tangent space (forgetting the subscript x for simplicity) and $\|\cdot\|_p$ for the norm on $L_p(M,\mu)$, $1 \leq p \leq +\infty$. Our goal is to prove Theorem 3.1.2.

3.3.1 Non homogeneous Sobolev spaces

Definition 3.3.1 ([2])**.** *Let M be a C^∞ Riemannian manifold of dimension n. Write E_p^1 for the vector space of C^∞ functions φ such that φ and $|\nabla\varphi| \in L_p$, $1 \leq p < \infty$. We define the Sobolev space W_p^1 as the completion of E_p^1 for the norm*

$$\|\varphi\|_{W_p^1} = \|\varphi\|_p + \| \, |\nabla\varphi| \, \|_p.$$

We denote W_∞^1 for the set of all bounded Lipschitz functions on M.

Proposition 3.3.2. *Let M be a complete Riemannian manifold. Then*

1. *([2]) C_0^∞ is dense in W_p^1 for $1 \leq p < \infty$.*
2. *([22]) Lip_0 is dense in W_p^1 for $1 \leq p < \infty$.*

Definition 3.3.3 (Poincaré inequality on M)**.** *We say that a complete Riemannian manifold M satisfying the doubling property admits **a local Poincaré inequality** (P_{qloc}) for some $1 \leq q < \infty$ if there exist constants $r_1 > 0$, $C = C(q, r_1) > 0$ such that, for every function $f \in \mathrm{Lip}_0$ and every ball B of M of radius $0 < r < r_1$, we have*

$$(P_{qloc}) \qquad \fint_B |f - f_B|^q d\mu \leq C r^q \fint_B |\nabla f|^q d\mu.$$

M admits a global Poincaré inequality (P_q) if we can take $r_1 = \infty$ in this definition.

Remark 3.3.4. *By density of C_0^∞ in W_p^1, we can replace Lip_0 by C_0^∞.*

3.3.2 Estimation of the K-functional of interpolation

In a first step, we prove Theorem 3.1.2 in the global case. This will help us to understand the proof of the more general local case.

The global case

Let M be a complete Riemannian manifold satisfying (D) and (P_q), for some $1 \leq q < \infty$. Before we prove Theorem 3.1.2, we make a Calderón-Zygmund decomposition for Sobolev functions inspired by the one done in [3]. For our purposes, we state it for more general spaces (in [3], the authors only needed the decomposition for the f's in C_0^∞). This will be the principal tool in the estimation of the functional K.

Proposition 3.3.5 (Calderón-Zygmund lemma for Sobolev functions)**.** *Let M be a complete non-compact Riemannian manifold satisfying (D). Let $1 \leq q < \infty$ and assume that M satisfies (P_q). Let $q \leq p < \infty$, $f \in W_p^1$ and $\alpha > 0$. Then one can find a collection of balls $(B_i)_i$, functions $b_i \in W_q^1$ and an almost everywhere Lipschitz function g such that the following properties hold:*

$$f = g + \sum_i b_i \tag{3.1}$$

$$|g(x)| \leq C\alpha \ \text{ and } \ |\nabla g(x)| \leq C\alpha \quad \mu - a.e\ x \in M \tag{3.2}$$

$$\operatorname{supp} b_i \subset B_i,\ \int_{B_i} (|b_i|^q + |\nabla b_i|^q) d\mu \leq C\alpha^q \mu(B_i) \tag{3.3}$$

$$\sum_i \mu(B_i) \leq C\alpha^{-p} \int (|f| + |\nabla f|)^p d\mu \tag{3.4}$$

$$\sum_i \chi_{B_i} \leq N \tag{3.5}$$

where C and N only depend on q, p and on the constants in (D) and (P_q).

Proof. Let $f \in W_p^1$, $\alpha > 0$. Consider $\Omega = \{x \in M : \mathcal{M}(|f| + |\nabla f|)^q(x) > \alpha^q\}$. If $\Omega = \emptyset$, then set

$$g = f\ , \quad b_i = 0 \ \text{ for all } i$$

so that (3.2) is satisfied thanks to the Lebesgue differentiation theorem. Otherwise the maximal theorem (Theorem 3.2.2) gives us

$$\begin{aligned}
\mu(\Omega) &\leq C\alpha^{-p} \|(|f| + |\nabla f|)^q\|_{\frac{p}{q}}^{\frac{p}{q}} \\
&\leq C\alpha^{-p}\Big(\int |f|^p d\mu + \int |\nabla f|^p d\mu \Big) \\
&< +\infty.
\end{aligned} \tag{3.6}$$

In particular $\Omega \neq M$ as $\mu(M) = +\infty$. Let F be the complement of Ω. Since Ω is an open set distinct of M, let $(\underline{B_i})$ be a Whitney decomposition of Ω ([14]). That is, the balls $\underline{B_i}$ are pairwise disjoint and there exist two constants $C_2 > C_1 > 1$, depending only on the metric, such that

1. $\Omega = \cup_i B_i$ with $B_i = C_1 \underline{B_i}$ and the balls B_i have the bounded overlap property;

2. $r_i = r(B_i) = \frac{1}{2} d(x_i, F)$ and x_i is the center of B_i;

3. each ball $\overline{B_i} = C_2 B_i$ intersects F ($C_2 = 4C_1$ works).

For $x \in \Omega$, denote $I_x = \{i : x \in B_i\}$. By the bounded overlap property of the balls B_i, we have that $\sharp I_x \leq N$. Fixing $j \in I_x$ and using the properties of the B_i's, we easily see that $\frac{1}{3} r_i \leq r_j \leq 3r_i$ for all $i \in I_x$. In particular, $B_i \subset 7B_j$ for all $i \in I_x$.

Condition (3.5) is nothing but the bounded overlap property of the $\underline{B_i}$'s and (3.4) follows from (3.5) and (3.6). The doubling property and the fact that $\overline{B_i} \cap F \neq \emptyset$ yield:

$$\int_{B_i} (|f|^q + |\nabla f|^q) d\mu \leq \int_{\overline{B_i}} (|f| + |\nabla f|)^q d\mu \leq \alpha^q \mu(\overline{B_i}) \leq C\alpha^q \mu(B_i). \tag{3.7}$$

Let us now define the functions b_i. Let $(\chi_i)_i$ be a partition of unity of Ω subordinated to the covering $(\underline{B_i})$, such that for all i, χ_i is a Lipschitz function supported in B_i with $\| |\nabla \chi_i| \|_\infty \leq \dfrac{C}{r_i}$. To this end it is enough to choose $\chi_i(x) = \psi(\dfrac{C_1 d(x_i, x)}{r_i}) \Big(\sum_k \psi(\dfrac{C_1 d(x_k, x)}{r_k}) \Big)^{-1}$, where ψ is a smooth function, $\psi = 1$ on $[0, 1]$, $\psi = 0$ on $[\frac{1+C_1}{2}, +\infty[$ and $0 \leq \psi \leq 1$. We set $b_i = (f - f_{B_i})\chi_i$. It is clear that $\operatorname{supp} b_i \subset B_i$. Let us estimate $\int_{B_i} |b_i|^q d\mu$ and $\int_{B_i} |\nabla b_i|^q d\mu$:

$$\begin{aligned}
\int_{B_i} |b_i|^q d\mu &= \int_{B_i} |(f - f_{B_i})\chi_i|^q d\mu \\
&\leq C\Big(\int_{B_i} |f|^q d\mu + \int_{B_i} |f_{B_i}|^q d\mu\Big) \\
&\leq C \int_{B_i} |f|^q d\mu \\
&\leq C\alpha^q \mu(B_i).
\end{aligned}$$

where we applied Jensen's inequality in the second estimate, and (3.7) in the last one. Since $\nabla\Big((f - f_{B_i})\chi_i\Big) = \chi_i \nabla f + (f - f_{B_i}) \nabla \chi_i$, we have by (P_q) and (3.7) that

$$\begin{aligned}
\int_{B_i} |\nabla b_i|^q d\mu &\leq C \left(\int_{B_i} |\chi_i \nabla f|^q d\mu + \int_{B_i} |f - f_{B_i}|^q |\nabla \chi_i|^q d\mu \right) \\
&\leq C\alpha^q \mu(B_i) + C \frac{C^q}{r_i^q} r_i^q \int_{B_i} |\nabla f|^q d\mu \\
&\leq C\alpha^q \mu(B_i).
\end{aligned}$$

Thus (3.3) is proved.

Set $g = f - \sum_i b_i$. Since the sum is locally finite on Ω, g is defined almost everywhere on M and $g = f$ on F. Observe that g is a locally integrable function on

M. Indeed, let $\varphi \in L_\infty$ with compact support. Since $d(x,F) \geq r_i$ for $x \in \text{supp } b_i$, we obtain

$$\int \sum_i |b_i|\,|\varphi|\,d\mu \leq \Big(\int \sum_i \frac{|b_i|}{r_i}\,d\mu\Big) \sup_{x\in M}\Big(d(x,F)|\varphi(x)|\Big)$$

and

$$\begin{aligned}\int \frac{|b_i|}{r_i} d\mu &= \int_{B_i} \frac{|f-f_{B_i}|}{r_i}\chi_i d\mu \\ &\leq \Big(\mu(B_i)\Big)^{\frac{1}{q'}}\Big(\int_{B_i} |\nabla f|^q d\mu\Big)^{\frac{1}{q}} \\ &\leq C\alpha\mu(B_i).\end{aligned}$$

We used the Hölder inequality, (P_q) and the estimate (3.7), q' being the conjugate of q. Hence $\displaystyle\int \sum_i |b_i||\varphi| d\mu \leq C\alpha\mu(\Omega) \sup_{x\in M}\Big(d(x,F)|\varphi(x)|\Big)$. Since $f \in L_{1,loc}$, we conclude that $g \in L_{1,loc}$.(Note that since $b \in L_1$ in our case, we can say directly that $g \in L_{1,loc}$. However, for the homogeneous case –section 5– we need this observation to conclude that $g \in L_{1,loc}$.) It remains to prove (3.2). Note that $\displaystyle\sum_i \chi_i(x) = 1$ and $\displaystyle\sum_i \nabla\chi_i(x) = 0$ for all $x \in \Omega$. We have

$$\begin{aligned}\nabla g &= \nabla f - \sum_i \nabla b_i \\ &= \nabla f - (\sum_i \chi_i)\nabla f - \sum_i (f - f_{B_i})\nabla\chi_i \\ &= 1\!\!1_F(\nabla f) + \sum_i f_{B_i}\nabla\chi_i.\end{aligned}$$

By definition of F and the Lebesgue differentiation theorem, we have $1\!\!1_F(|f|+|\nabla f|) \leq \alpha\ \mu-$a.e.. We claim that a similar estimate holds for $h = \sum_i f_{B_i}\nabla\chi_i$. We have $|h(x)| \leq C\alpha$ for all $x \in M$. For this, note first that h vanishes on F and is locally finite on Ω. Then fix $x \in \Omega$ and let B_j be some Whitney ball containing x. For all $i \in I_x$, we have $|f_{B_i} - f_{B_j}| \leq C r_j \alpha$. Indeed, since $B_i \subset 7B_j$, we get

$$\begin{aligned}|f_{B_i} - f_{7B_j}| &\leq \frac{1}{\mu(B_i)}\int_{B_i} |f - f_{7B_j}| d\mu \\ &\leq \frac{C}{\mu(B_j)}\int_{7B_j} |f - f_{7B_j}| d\mu \\ &\leq C r_j \Big(\fint_{7B_j} |\nabla f|^q d\mu\Big)^{\frac{1}{q}} \\ &\leq C r_j \alpha \qquad (3.8)\end{aligned}$$

where we used Hölder inequality, (D), (P_q) and (3.7). Analogously $|f_{7B_j} - f_{B_j}| \leq C r_j\alpha$. Hence

$$|h(x)| = |\sum_{i\in I_x}(f_{B_i} - f_{B_j})\nabla\chi_i(x)|$$

$$
\begin{aligned}
&\le C\sum_{i\in I_x}|f_{B_i}-f_{B_j}|r_i^{-1}\\
&\le CN\alpha.
\end{aligned}
$$

From these estimates we deduce that $|\nabla g(x)| \le C\alpha \ \ \mu - a.e.$. Let us now estimate $\|g\|_\infty$. We have $g = f\mathbb{1}_F + \sum_i f_{B_i}\chi_i$. Since $|f|\mathbb{1}_F \le \alpha$, it remains to estimate $\|\sum_i f_{B_i}\chi_i\|_\infty$. Let $x \in B_i$, we have

$$
\begin{aligned}
(f_{B_i})^q &\le C\Big(\frac{1}{\mu(\overline{B_i})}\int_{\overline{B_i}}|f|d\mu\Big)^q\\
&\le \Big(\mathcal{M}(|f|+|\nabla f|)\Big)^q(y)\\
&\le \mathcal{M}(|f|+|\nabla f|)^q(y)\\
&\le \alpha^q
\end{aligned}
$$

where $y \in \overline{B_i}\cap F$ since $\overline{B_i}\cap F \neq \emptyset$. The second inequality follows from the fact that $(\mathcal{M}f)^q \le \mathcal{M}f^q$ for $q \ge 1$.
Hence $f_{B_i} \le \alpha$. Since $x \in B_i$, then $\sharp I_x \le N$ and

$$
\begin{aligned}
|g(x)| &= |\sum_{i\in I_x} f_{B_i}|\\
&\le \sum_{i\in I_x}|f_{B_i}|\\
&\le N\alpha.
\end{aligned}
$$

Therefore $\|g\|_\infty \le C\,\alpha \quad \mu - a.e.$. Thus the proof of Proposition 3.3.5 is complete. □

Let us prove the following proposition for later use, although it is not necessary for the proof of Theorem 3.1.2.

Proposition 3.3.6. *The function g of Calderón-Zygmund lemma is Lipschitz almost everywhere on M and $|g(x)-g(y)| \le C\alpha d(x,y)$ almost everywhere.*

Proof. To prove this proposition we will distinguish between three cases:

1. Case $(x,y) \in F\times F$. Let B be a ball of minimal radius containing x and y. Then $r(B) \sim d(x,y)$. We set $B_0 = B$ and construct the balls $B_i \subset B$ containing x such that $B_i \subset B_{i-1}$ and $r(B_i) = \frac{1}{2}r(B_{i-1})$. Since $f_{B_i} \underset{i\to\infty}{\longrightarrow} f(x)\ \mu - a.e.$, we have that for $\mu - a.e.$

$$
\begin{aligned}
|f(x)-f_B| &\le \sum_{i=1}^{\infty}|f_{B_i}-f_{B_{i-1}}|\\
&\le C\Big(\mathcal{M}(|\nabla f|^q)(x)\Big)^{\frac{1}{q}}\sum_{i=0}^{\infty} r(B_i)\\
&\le C\alpha r(B).
\end{aligned}
$$

Similarly for $\mu - a.e.\ y \in F$, $|f(y) - f_B| \leq C\alpha r(B)$. Then for $\mu \otimes \mu - a.e.\ (x, y) \in F \times F$,

$$\begin{aligned} |g(y) - g(x)| &= |f(y) - f(x)| \\ &\leq |f(x) - f_B| + |f_B - f(y)| \\ &\leq C\alpha d(x, y). \end{aligned}$$

2. Case $(x, y) \in \Omega \times F$. We have $g(y) = f(y) = \sum_j f(y)\chi_j(x)$ and $g(x) = \sum_j f_{B_j}\chi_j(x)$. Hence $g(y) - g(x) = \sum_{j \in I_x}(f(y) - f_{B_j})\chi_j(x)$. Let us fix $i \in I_x$, where I_x was defined in the proof of Proposition 3.3.5. Recall that for all $j \in I_x$, $B_j \subset 7B_i$. We distinguish two subcases:

   i. If $y \in 7B_i$, we get

$$\begin{aligned} |f(y) - f_{B_j}| &\leq \sum_{k=-\infty}^{-1} |f_{B(y,2^k r_j)} - f_{B(y,2^{k+1} r_j)}| + |f_{B(y,r_j)} - f_{B_j}| \\ &\leq \sum_{k=-\infty}^{-1} \fint_{B(y,2^k r_j)} |f - f_{B(y,2^{k+1} r_j)}| + |f_{B(y,r_j)} - f_{B_j}| \\ &\leq \sum_{k=-\infty}^{-1} \frac{\mu(B(y, 2^{k+1} r_j))}{\mu(B(y, 2^k r_j))} \fint_{B(y,2^{k+1} r_j)} |f - f_{B(y,2^{k+1} r_j)}| + |f_{B(y,r_j)} - f_{B_j}| \\ &\leq C\alpha r_j \sum_{k=-\infty}^{-1} 2^k + |f_{B(y,r_j)} - f_{(7+\frac{r_j}{r_i})B_i}| + |f_{(7+\frac{r_j}{r_i})B_i} - f_{B_j}| \\ &\leq C\alpha r_j + C(r_i + r_j)\alpha \\ &\leq C\alpha r_i \\ &\leq C\alpha d(x, y). \end{aligned}$$

   We used here the same argument as in (3.8) and the fact that for $x \in B_i$, $r_i \leq d(x, F) \leq d(x, y)$. Thus

$$\begin{aligned} |g(y) - g(x)| &\leq \sum_{j \in I_x} |f(y) - f_{B_j}||\chi_j(x)| \\ &\leq NC\alpha d(x, y). \end{aligned}$$

   ii. If $y \notin 7B_i$, we choose $z \in F \cap 4B_i$. Therefore,

$$\begin{aligned} |g(y) - g(x)| &\leq |g(y) - g(z)| + |g(z) - g(x)| \\ &\leq C\alpha d(y, z) + C\alpha d(x, z) \\ &\leq C\alpha d(x, y) + C\alpha d(x, z) \\ &\leq C\alpha d(x, y) + 4C\alpha r_i \\ &\leq C\alpha d(x, y). \end{aligned}$$

   We used in the second inequality the first case and part i. of the second case.

3. Case $(x, y) \in \Omega \times \Omega$. Let L be a constant with $L > 1$.

   i. If $d(x, F) \leq L d(x, y)$, let $\xi \in F$ be such that $d(x, F) < d(x, \xi) \leq 2d(x, F)$. Then

$$\begin{aligned} |g(x) - g(y)| &\leq |g(x) - g(\xi)| + |g(\xi) - g(y)| \\ &\leq C\alpha d(x, \xi) + C\alpha d(\xi, y) \\ &\leq C\alpha d(x, F) + C\alpha(d(\xi, x) + d(x, y)) \\ &\leq C\alpha d(x, y). \end{aligned}$$

   ii. If $d(x, F) > L d(x, y)$, we have then $d(y, F) > (L - 1)d(x, y)$ and for all $i \in I_x$, $d(B_i, F) = r_i$ and $r_i \leq d(x, F) \leq 3r_i$. Similarly for all $j \in I_y$, $r_j \leq d(y, F) \leq 3r_j$. Fix $i \in I_x$. Then $g(z) = \sum_j (f_{B_j} - f_{B_i})\chi_j(z) + f_{B_i}$ for all $z \in \Omega$. Thus

$$|g(x) - g(y)| \leq \sum_{j : x \in B_j \text{ or } y \in B_j} |f_{B_j} - f_{B_i}||\chi_j(x) - \chi_j(y)|.$$

   Since $|\chi_j(x) - \chi_j(y)| \leq \frac{C}{r_j} d(x, y)$, it is enough to prove that $\sum_{j : x \in B_j} |f_{B_j} - f_{B_i}| \leq C r_j \alpha$ and similarly that $\sum_{j : y \in B_j, x \notin B_j} |f_{B_j} - f_{B_i}| \leq C r_j \alpha$. If $j \in I_x$, we have already seen that $B_i \subset 7B_j$ and $|f_{B_j} - f_{B_i}| \leq C r_j \alpha$, hence $\sum_{j ; x \in B_j} |f_{B_j} - f_{B_i}| \leq N C r_j \alpha$. Now for every j such that $y \in B_j$ and $x \notin B_j$, it is enough to prove that $B_i \subset C B_j$ with C a constant independent of x and y. So we obtain like in the case where $j \in I_x$, $|f_{B_j} - f_{B_i}| \leq C r_j \alpha$ and therefore $\sum_{j ; y \in B_j, x \notin B_j} |f_{B_j} - f_{B_i}| \leq N C r_j \alpha$. Indeed, let $z \in B_i$,

$$\begin{aligned} d(x_j, z) &\leq d(x_j, x_i) + d(x_i, z) \\ &\leq d(x_j, y) + d(y, x) + d(x, x_i) + r_i \\ &\leq 2r_i + d(x, y) + r_j \\ &\leq (2 + \frac{3}{L})r_i + r_j \end{aligned}$$

   but

$$\begin{aligned} 3r_j &\geq d(y, F) \\ &\geq (1 - \frac{1}{L})d(x, F) \\ &\geq (1 - \frac{1}{L})r_i. \end{aligned}$$

   Thus $d(x_j, z) \leq C r_j$ and $B_i \subset C B_j$.

Hence g is $C\alpha$ Lipschitz almost everywhere . □

Remark 3.3.7. *The proof of Proposition 3.3.6 has not used any specific property of Riemannian manifolds and still works for general metric-measure spaces.*

Corollary 3.3.8. *Under the same hypotheses as in the Calderón-Zygmund lemma, we have:*

$$W_p^1 \subset W_q^1 + W_\infty^1 \quad \text{for } q \le p < \infty\,.$$

Proof of Theorem 3.1.2. To prove the left inequality we begin applying Theorem 3.2.4, part 1.:

$$K(f, t^{\frac{1}{q}}, L_q, L_\infty) \sim \Big(\int_0^t (f^*(s))^q ds\Big)^{\frac{1}{q}}.$$

On the other hand

$$\begin{aligned}\Big(\int_0^t f^*(s)^q ds\Big)^{\frac{1}{q}} &= \Big(\int_0^t |f(s)|^{q*} ds\Big)^{\frac{1}{q}} \\ &= \Big(t|f|^{q**}(t)\Big)^{\frac{1}{q}}\end{aligned}$$

where in the first equality we used the fact that $f^{*q} = (|f|^q)^*$ and the second follows from the definition of f^{**}. We thus get $K(f, t^{\frac{1}{q}}, L_q, L_\infty) \sim t^{\frac{1}{q}}(|f|^{q**})^{\frac{1}{q}}(t)$. Moreover, we have

$$K(f, t^{\frac{1}{q}}, W_q^1, W_\infty^1) \ge K(f, t^{\frac{1}{q}}, L_q, L_\infty) + K(|\nabla f|, t^{\frac{1}{q}}, L_q, L_\infty)$$

since the operator

$$I + \nabla : (W_q^1, W_\infty^1) \to (L_q, L_\infty)$$

is bounded. These two points yield the left inequality.

We will now prove the reverse inequality. We treat first the case when $f \in W_p^1$, $q \le p < \infty$. Let $t > 0$, we consider the Calderón-Zygmund decomposition of f of Proposition 3.3.5 with $\alpha = \alpha(t) = \Big(\mathcal{M}(|f| + |\nabla f|)^q\Big)^{*\frac{1}{q}}(t)$. Thus we have $f = \sum_i b_i + g = b + g$ where $(b_i)_i$, g satisfy the properties of the proposition. Hence we get the estimate

$$\begin{aligned}\|b\|_q^q &\le \int_M (\sum_i |b_i|)^q d\mu \\ &\le N \sum_i \int_{B_i} |b_i|^q d\mu \\ &\le C\alpha^q(t) \sum_i \mu(B_i) \\ &\le C\alpha^q(t)\mu(\Omega).\end{aligned}$$

This follows from the fact that $\sum_i \chi_{B_i} \le N$ and $\Omega = \bigcup_i B_i$. Therefore $\|b\|_q \le C\alpha(t)\mu(\Omega)^{\frac{1}{q}}$ and similarly we get $\|\,|\nabla b|\,\|_q \le C\alpha(t)\mu(\Omega)^{\frac{1}{q}}$.

Moreover, since $(\mathcal{M}f)^* \sim f^{**}$ and $(f+g)^{**} \le f^{**} + g^{**}$ (see [6]), we obtain

$$\alpha(t) = (\mathcal{M}(|f| + |\nabla f|)^q)^{*\frac{1}{q}}(t) \le C\left(|f|^{q**\frac{1}{q}}(t) + |\nabla f|^{q**\frac{1}{q}}(t)\right).$$

Noting that $\mu(\Omega) \leq t$, we get $K(f, t^{\frac{1}{q}}, W_q^1, W_\infty^1) \leq Ct^{\frac{1}{q}} \left(|f|^{q**\frac{1}{q}}(t) + |\nabla f|^{q**\frac{1}{q}}(t) \right)$ for all $t > 0$ and obtain the desired inequality.

For the general case when $f \in W_q^1 + W_\infty^1$, the same argument used in [17] for the Euclidean case applies. We state it for the sake of completness. Let $f \in W_q^1 + W_\infty^1$, $t > 0$ and $E_0 = \{x \in M; |f|^q(x) > 2|f|^{q*}(t)\}$, $E_1 = \{x \in M; |\nabla f|^q(x) > 2|\nabla f|^{q*}(t)\}$. Take $E = E_0 \cup E_1$, hence $\mu(E) \leq t$. We choose a sequence $\varepsilon_j \to 0$ with the following properties:

1. $\varepsilon_j \leq 2^{-jq}t \quad j = 1, 2, ...$
2. $\varepsilon_j(|f|^q + |\nabla f|^q)^{**}(\varepsilon_j) \leq 2^{-jq}t(|f|^q + |\nabla f|^q)^{**}(t) \quad j = 1, 2, ...$

This is possible since $s(|f|^q + |\nabla f|^q)^{**}(s) \to 0$ when $s \to 0$.
Let us fix $x_0 \in M$. We denote by $B(r)$ the open ball centered at x_0 of radius r and let r_j be a sequence increasing to $+\infty$ such that $\mu(E \cap B(r_j)^c) \leq \varepsilon_j$ and $r_{j+1} - r_j \geq 1$, $j = 1, 2....$ We take a partition of unity ψ_j for $A_j = B(r_j) - B(r_{j-1})$ with $r_0 = 0$. That is

1. $0 \leq \psi_j \leq 1 \quad \forall j$ and $\displaystyle\sum_j \psi_j = 1$ on M.
2. $\operatorname{supp} \psi_j \subset \{x : r_{j-1} \leq d(x_0, x) \leq r_{j+2}\} \quad j = 1, 2... .$
3. $\|\psi_j\|_{W_\infty^1} \leq C \quad j = 1, 2... .$

We have $f = \displaystyle\sum_j f\psi_j = \sum_j f_j$ and

$$|f_j|^q + |\nabla f_j|^q = |f\psi_j|^q + |\nabla f \psi_j + f\nabla\psi_j|^q \leq C(|f|^q + |\nabla f|^q). \tag{3.9}$$

Since M is non-atomic and since $\mu(B(r_{j-1})^c) = +\infty > 2^{-jq}t$ and $\operatorname{supp} f_j^q \subset B(r_{j-1})^c$, there exists F_j a measurable subset of M contained in $B(r_{j-1})^c$ such that $\mu(F_j) = 2^{-jq}t$ and $\int_{F_j} |f_j|^q d\mu = \int_0^{2^{-jq}t} |f_j|^{q*}(u)du$. Similarly there exists F'_j a measurable subset of M contained in $B(r_{j-1})^c$ such that $\mu(F'_j) = 2^{-jq}t$ and $\int_{F'_j} |\nabla f_j|^q d\mu = \int_0^{2^{-jq}t} |\nabla f_j|^{q*}(u)du$ (see [38] Chapter V p.201). So we obtain

$$\begin{aligned}
2^{-jq}t(|f_j|^q)^{**}(2^{-jq}t) &= \int_{F_j} |f_j|^q d\mu \\
&\leq \int_{F_j \cap E} |f_j|^q d\mu + \int_{F_j - E} |f_j|^q d\mu \\
&\leq C \left(\int_{B(r_{j-1})^c \cap E} (|f|^q + |\nabla f|^q) d\mu + 2^{-jq}t(|f|^{q*} + |\nabla f|^{q*})(t) \right) \\
&\leq C \left(\varepsilon_{j-1}(|f|^q + |\nabla f|^q)^{**}(\varepsilon_{j-1}) + 2^{-jq}t(|f|^{q**} + |\nabla f|^{q**})(t) \right) \\
&\leq C2^{-jq}t(|f|^{q**} + |\nabla f|^{q**})(t).
\end{aligned}$$

Indeed, the second inequality follows from (3.9), the third inequality used the fact that $\mu(E \cap B(r_{j-1})^c) \leq \varepsilon_{j-1}$ and the last inequality follows from the property ii) of ε_j and the fact that $(f+g)^{**} \leq f^{**}+g^{**}$ and $f^* \leq f^{**}$. Similarly we have $2^{-jq}t(|\nabla f_j|^q)^{**}(2^{-jq}t) \leq C2^{-jq}t(|f|^{q**}+|\nabla f|^{q**})(t)$. Now each function f_j is in W_q^1. Thus by the first case, there exists g_j such that:

$$\begin{aligned}\|f_j-g_j\|_{W_q^1}+(2^{-jq}t)^{\frac{1}{q}}\|g_j\|_{W_\infty^1} &\leq C2^{-j}t^{\frac{1}{q}}\left(|f_j|^{q**\frac{1}{q}}(2^{-jq}t)+|\nabla f_j|^{q**\frac{1}{q}}(2^{-jq}t)\right)\\ &\leq C2^{-j}t^{\frac{1}{q}}\left(|f|^{q**\frac{1}{q}}(t)+|\nabla f|^{q**\frac{1}{q}}(t)\right).\end{aligned} \tag{3.10}$$

Moreover supp $f_j \subset A_j = B'(r_{j+2})-B(r_{j-1})$ where $B'(r_j)$ is the closed ball centered at x and of radius r_j. Therefore we can require that supp $g_j \subset \tilde{A}_j = B'(r_{j+3})-B(r_{j-2})$. Indeed, set $\tilde{g}_j = \eta_j g_j$ where η_j is a smooth function supported in $\tilde{A}_j$, $\eta_j = 1$ on A_j and $\|\eta_j\|_{W_\infty^1} \leq C$. We have $\|f_j-\tilde{g}_j\|_{W_q^1} = \|f_j-g_j\|_{W_q^1}+\|(1-\eta_j)g_j\|_{W_q^1}$ and since $\|(1-\eta_j)g_j\|_{W_q^1} \leq C\left(\int_{A_j^c}|g_j|^q d\mu+\int_{A_j^c}|\nabla g_j|^q d\mu\right) \leq C\|f_j-g_j\|_{W_q^1}$, we obtain

$$\begin{aligned}&\|f_j-\tilde{g}_j\|_{W_q^1}+(2^{-jq}t)^{\frac{1}{q}}\|\tilde{g}_j\|_{W_\infty^1}\\ &\qquad\leq C\|f_j-g_j\|_{W_q^1}+C2^{-j}t^{\frac{1}{q}}\|g_j\|_{W_\infty^1}\\ &\qquad\leq C2^{-j}t^{\frac{1}{q}}\left(|f|^{q**\frac{1}{q}}(t)+|\nabla f|^{q**\frac{1}{q}}(t)\right)\end{aligned}$$

Thus we assume now that supp $g_j \subset \tilde{A}_j$ and define $g = \sum_{j=1}^{\infty} g_j$. This yields

$$\begin{aligned}\|g\|_{W_\infty^1} &\leq C\sup_j \|g_j\|_{W_\infty^1}\\ &\leq C\left(|f|^{q**\frac{1}{q}}(t)+|\nabla f|^{q**\frac{1}{q}}(t)\right)\end{aligned}$$

because of (3.10). Finally,

$$\begin{aligned}\|f-g\|_{W_q^1}+t^{\frac{1}{q}}\|g\|_{W_\infty^1} &\leq C(\sum_j \|f_j-g_j\|_{W_q^1}+t^{\frac{1}{q}}\|g\|_{W_\infty^1})\\ &\leq Ct^{\frac{1}{q}}\left(|f|^{q**\frac{1}{q}}(t)+|\nabla f|^{q**\frac{1}{q}}(t)\right).\end{aligned}$$

Hence $K(f,t^{\frac{1}{q}},W_q^1,W_\infty^1) \leq C_2 t^{\frac{1}{q}}\left(|f|^{q**\frac{1}{q}}(t)+|\nabla f|^{q**\frac{1}{q}}(t)\right)$ for every $f \in W_q^1+W_\infty^1$, and every $t>0$. Therefore the proof is complete. □

The local case

Let M be a complete non-compact Riemannian manifold satisfying a local doubling property (D_{loc}) and a local Poincaré inequality (P_{qloc}) for some $1 \leq q < \infty$.

Denote by $\mathcal{M}_E$ the Hardy-Littlewood maximal operator relative to a measurable subset E of M, that is, for $x \in E$ and every f locally integrable function on M:

$$\mathcal{M}_E f(x) = \sup_{B:\, x\in B} \frac{1}{\mu(B\cap E)}\int_{B\cap E}|f|d\mu$$

where B ranges over all open balls of M containing x and centered in E. We say that a measurable subset E of M has the relative doubling property if there exists a constant C_E such that for all $x \in E$ and $r > 0$ we have

$$\mu(B(x,2r) \cap E) \leq C_E \mu(B(x,r) \cap E).$$

This is equivalent to saying that the metric-measure space $(E, d/E, \mu/E)$ has the doubling property. On such a set $\mathcal{M}_E$ is of weak type $(1,1)$ and bounded on $L_p(E,\mu)$, $1 < p \leq \infty$.

Proof of Theorem 3.1.2. To fix ideas, we assume without loss of generality $r_0 = 5$, $r_1 = 8$. The lower bound of K in $(*_{\text{loc}})$ is trivial (same proof as for the global case). It remains to prove the upper bound.

For all $t > 0$, take $\alpha = \alpha(t) = \Big(\mathcal{M}(|f| + |\nabla f|)^q\Big)^{*\frac{1}{q}}(t)$. Consider

$$\Omega = \left\{x \in M : \mathcal{M}(|f| + |\nabla f|)^q(x) > \alpha^q(t)\right\}.$$

We have $\mu(\Omega) \leq t$. If $\Omega = M$ then

$$\begin{aligned}\int_M |f|^q d\mu + \int_M |\nabla f|^q d\mu &= \int_\Omega |f|^q d\mu + \int_\Omega |\nabla f|^q d\mu \\ &\leq \int_0^{\mu(\Omega)} |f|^{q*}(s)ds + \int_0^{\mu(\Omega)} |\nabla f|^{q*}(s)ds \\ &\leq \int_0^t |f|^{q*}(s)ds + \int_0^t |\nabla f|^{q*}(s)ds \\ &= t\Big(|f|^{q**}(t) + |\nabla f|^{q**}(t)\Big)\end{aligned}$$

Therefore

$$\begin{aligned}K(f, t^{\frac{1}{q}}, W_q^1, W_\infty^1) &\leq \|f\|_{W_q^1} \\ &\leq Ct^{\frac{1}{q}}\Big(|f|^{q**\frac{1}{q}}(t) + |\nabla f|^{q**\frac{1}{q}}(t)\Big).\end{aligned}$$

We thus obtain the upper bound in this case.

Now assume $\Omega \neq M$. Pick a countable set $\{x_j\}_{j\in J} \subset M$, such that $M = \bigcup_{j\in J} B(x_j, \frac{1}{2})$ and for all $x \in M$, x does not belong to more than N_1 balls $B^j := B(x_j, 1)$. Consider a C^∞ partition of unity $(\varphi_j)_{j\in J}$ subordinated to the balls $\frac{1}{2}B^j$ such that $0 \leq \varphi_j \leq 1$, $\operatorname{supp}\varphi_j \subset B^j$ and $\| \, |\nabla\varphi_j| \, \|_\infty \leq C$ uniformly with respect to j. It is enough to assume $f \in W_p^1$, $q \leq p < \infty$; as the case $f \in W_q^1 + W_\infty^1$ is similar to that of [17]. So consider $f \in W_p^1$. Let $f_j = f\varphi_j$ so that $f = \sum_{j\in J} f_j$. We have for $j \in J$, $f_j \in L_p$ and $\nabla f_j = f\nabla\varphi_j + \nabla f\varphi_j \in L_p$. Hence $f_j \in W_p^1(B^j)$. The balls B^j satisfy the relative doubling property with constant independent of the balls B^j. This follows from the next lemma quoted from [4] p.947 (see Lemma 2.2.4 in Chapter 2 of this thesis for the proof).

Lemma 3.3.9. *Let M be a complete Riemannian manifold satisfying (D_{loc}). Then the balls B^j above, equipped with the induced distance and measure, satisfy the relative doubling property (D), with the doubling constant that may be chosen independently of j. More precisely, there exists $C \geq 0$ such that for all $j \in J$*

$$\mu(B(x,2r)\cap B^j) \leq C\,\mu(B(x,r)\cap B^j) \quad \forall x \in B^j,\ r>0, \tag{3.11}$$

and

$$\mu(B(x,r)) \leq C\mu(B(x,r)\cap B^j) \quad \forall x \in B^j,\ 0<r\leq 2. \tag{3.12}$$

Remark 3.3.10. *Noting that the proof in [4] only used the fact that M is a length space, we see that Lemma 3.3.10 still holds for any length space. Recall that a length space X is a metric space such that the distance between any two points $x, y \in X$ is equal to the infimum of the lengths of all paths joining x to y (we implicitly assume that there exists at least one such path). Here a path from x to y is a continuous map $\gamma : [0,1] \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$.*

Let us return to the proof of the theorem. For any $x \in B^j$ we have

$$\begin{aligned}
\mathcal{M}_{B^j}(|f_j|+|\nabla f_j|)^q(x) &= \sup_{B:\,x\in B,\,r(B)\leq 2} \frac{1}{\mu(B^j\cap B)}\int_{B^j\cap B}(|f_j|+|\nabla f_j|)^q d\mu \\
&\leq \sup_{B:\,x\in B,\,r(B)\leq 2} C\frac{\mu(B)}{\mu(B^j\cap B)}\frac{1}{\mu(B)}\int_B (|f|+|\nabla f|)^q d\mu \\
&\leq C\mathcal{M}(|f|+|\nabla f|)^q(x). \qquad (3.13)
\end{aligned}$$

where we used (3.12) of Lemma 3.3.9. Consider now

$$\Omega_j = \left\{x \in B^j : \mathcal{M}_{B^j}(|f_j|+|\nabla f_j|)^q(x) > C\alpha^q(t)\right\}$$

where C is the constant in (3.13). Ω_j is an open subset of B^j, hence of M, and $\Omega_j \subset \Omega \neq M$ for all $j \in J$. For the f_j's, and for all $t > 0$, we have a Calderón-Zygmund decomposition similar to the one done in Proposition 3.3.5: there exist b_{jk}, g_j supported in B^j, and balls $(B_{jk})_k$ of M, contained in Ω_j, such that

$$f_j = g_j + \sum_k b_{jk} \tag{3.14}$$

$$|g_j(x)| \leq C\alpha(t) \text{ and } |\nabla g_j(x)| \leq C\alpha(t) \quad \text{for } \mu-a.e. \quad x \in M \tag{3.15}$$

$$\operatorname{supp} b_{jk} \subset B_{jk},\ \int_{B_{jk}}(|b_{jk}|^q + |\nabla b_{jk}|^q)d\mu \leq C\alpha^q(t)\mu(B_{jk}) \tag{3.16}$$

$$\sum_k \mu(B_{jk}) \leq C\alpha^{-p}(t)\int_{B^j}(|f_j|+|\nabla f_j|)^p d\mu \tag{3.17}$$

$$\sum_k \chi_{B_{jk}} \leq N \tag{3.18}$$

with C and N depending only on q, p and the constants in (D_{loc}) and (P_{qloc}). The proof of this decomposition will be the same as that of Proposition 3.3.5, taking for all $j \in J$ a Whitney decomposition $(B_{jk})_k$ of $\Omega_j \neq M$ and using the doubling property for balls whose radii do not exceed $3 < r_0$ and the Poincaré inequality for balls whose radii do not exceed $7 < r_1$. For the bounded overlap property (3.18), just note that the radius of every ball B_{jk} is less than 1. Then apply the same argument as for the bounded overlap property of a Whitney decomposition for an homogeneous space, using the doubling property for balls with sufficiently small radii.

By the above decomposition we can write $f = \sum\limits_{j\in J}\sum\limits_k b_{jk} + \sum\limits_{j\in J} g_j = b + g$. Let us now estimate $\|b\|_{W_q^1}$ and $\|g\|_{W_\infty^1}$.

$$\begin{aligned}
\|b\|_q^q &\leq N_1 N \sum_j \sum_k \|b_{jk}\|_q^q \\
&\leq C\alpha^q(t) \sum_j \sum_k (\mu(B_{jk})) \\
&\leq NC\alpha^q(t)\Big(\sum_j \mu(\Omega_j)\Big) \\
&\leq N_1 C\alpha^q(t)\mu(\Omega).
\end{aligned}$$

We used the bounded overlap property of the $(\Omega_j)_{j\in J}$'s and that of the $(B_{jk})_k$'s for all $j \in J$. It follows that $\|b\|_q \leq C\alpha(t)\mu(\Omega)^{\frac{1}{q}}$. Similarly we get $\| \, |\nabla b| \, \|_q \leq C\alpha(t)\mu(\Omega)^{\frac{1}{q}}$.

For g we have

$$\begin{aligned}
\|g\|_\infty &\leq \sup_x \sum_{j\in J} |g_j(x)| \\
&\leq \sup_x N_1 \sup_{j\in J} |g_j(x)| \\
&\leq N_1 \sup_{j\in J} \|g_j\|_\infty \\
&\leq C\alpha(t).
\end{aligned}$$

Analogously $\| \, |\nabla g| \, \|_\infty \leq C\alpha(t)$. Then

$$\begin{aligned}
K(f, t^{\frac{1}{q}}, W_q^1, W_\infty^1) &\leq \|b\|_{W_q^1} + t^{\frac{1}{q}}\|g\|_{W_\infty^1} \\
&\leq C\alpha(t)\mu(\Omega)^{\frac{1}{q}} + Ct^{\frac{1}{q}}\alpha(t) \\
&\leq Ct^{\frac{1}{q}}\alpha(t) \\
&\sim Ct^{\frac{1}{q}}(|f|^{q**\frac{1}{q}}(t) + |\nabla f|^{q**\frac{1}{q}}(t))
\end{aligned}$$

and the proof is therefore complete. □

3.4 Interpolation Theorems

In this section we establish our interpolation Theorem 3.1.1 and some consequences for non homogeneous Sobolev spaces on a complete non-compact Riemannian manifold M satisfying (D_{loc}) and (P_{qloc}) for some $1 \leq q < \infty$. For $q < p < \infty$, we define the interpolation space $W^1_{p,q}$ between W^1_q and W^1_∞ by

$$W^1_{p,q} = (W^1_q, W^1_\infty)_{1-\frac{q}{p},p}.$$

Thanks to the previous results we know that

$$\|f\|_{1-\frac{q}{p},p} \sim \left\{\int_0^\infty (t^{\frac{1}{p}} |f|^{q**\frac{1}{q}}(t))^p \frac{dt}{t}\right\}^{\frac{1}{p}} + \left\{\int_0^\infty (t^{\frac{1}{p}} |\nabla f|^{q**\frac{1}{q}}(t))^p \frac{dt}{t}\right\}^{\frac{1}{p}}.$$

We claim that $W^1_{p,q} = W^1_p$, with equivalent norms. Indeed,

$$\begin{aligned}\|f\|_{1-\frac{q}{p},p} &\sim \left\{\int_0^\infty |f|^{q**\frac{p}{q}}(t)dt\right\}^{\frac{1}{p}} + \left\{\int_0^\infty |\nabla f|^{q**\frac{p}{q}}(t)dt\right\}^{\frac{1}{p}} \\ &= \|f^{q**}\|^{\frac{1}{q}}_{\frac{p}{q}} + \||\nabla f|^{q**}\|^{\frac{1}{q}}_{\frac{p}{q}} \\ &\sim \|f^q\|^{\frac{1}{q}}_{\frac{p}{q}} + \||\nabla f|^q\|^{\frac{1}{q}}_{\frac{p}{q}} \\ &= \|f\|_p + \||\nabla f|\|_p \\ &= \|f\|_{W^1_p},\end{aligned}$$

where we used that for $r > 1$, $\|f^{**}\|_r \sim \|f\|_r$ (see [38] Chapter V: Lemma 3.21 p.191 and Theorem 3.21 p.201). Moreover, from Corollary 3.3.8, we have $W^1_p \subset W^1_q + W^1_\infty$ for $q < p < \infty$. Therefore W^1_p is an interpolation space between W^1_q and W^1_∞ for $q < p < \infty$.

Let us recall some known facts about Poincaré inequality with varying q.
It is known that (P_{qloc}) implies (P_{ploc}) when $p \geq q$ (see [27]). Thus, if the set of q such that (P_{qloc}) holds is not empty, then it is an interval unbounded on the right. A recent result of Keith and Zhong [31] asserts that this interval is open in $[1, +\infty[$.

Theorem 3.4.1. *Let (X, d, μ) be a complete metric-measure space with μ locally doubling and admitting a local Poincaré inequality (P_{qloc}), for some $1 < q < \infty$. Then there exists $\varepsilon > 0$ such that (X, d, μ) admits (P_{ploc}) for every $p > q - \varepsilon$.*

Here, the definition of (P_{qloc}) is that of section 3.7. It reduces to the one of section 3.3 when the metric space is a Riemannian manifold.

Comment on the proof of this theorem. The proof goes as in [31] where this theorem is proved for X satisfying (D) and admitting a global Poincaré inequality (P_q). By using the same argument and choosing sufficiently small radii for the considered balls, (P_{qloc}) will give us $(P_{(q-\varepsilon)loc})$ for every ball of radius less than r_2, for some $r_2 < \min(r_0, r_1)$, r_0, r_1 being the constants given in the definitions of local doubling property and local Poincaré inequality. □

Define $A_M = \{q \in [1,\infty[: (P_{qloc}) \text{ holds }\}$ and $q_{0_M} = \inf A_M$. When no confusion arises, we write q_0 instead of q_{0_M}. We have then

Corollary 3.4.2 (The reiteration theorem)**.** *For all $q_0 < p_1 < p < p_2 \leq \infty$, W_p^1 is an interpolation space between $W_{p_1}^1$ and $W_{p_2}^1$.*

Proof. Since $p_1 > q_0$, there exists $q \in A_M$ such that $q_0 < q < p_1$. Take $0 < \eta < 1$ for which $\frac{1}{p} = (1-\eta)\frac{1}{p_1} + \eta\frac{1}{p_2}$. Then $1 - \frac{q}{p} = (1-\eta)(1-\frac{q}{p_1}) + \eta(1-\frac{q}{p_2})$. The $W_{p_i,q}^1$, $i=1,2$, being complete (because W_q^1 and W_∞^1 are), we have

$$\begin{aligned}(W_{p_1}^1, W_{p_2}^1)_{\eta,p} =& (W_{p_1,q}^1, W_{p_2,q}^1)_{\eta,p} \\ &= (W_q^1, W_\infty^1)_{1-\frac{q}{p},p} \\ &= W_{p,q}^1 \\ &= W_p^1.\end{aligned}$$

□

Remark 3.4.3. *If (P_{1loc}) holds, then $q_0 = 1$ and all the strict inequalities at q_0 become large.*

Theorem 3.4.4. *Let M and N be two complete non-compact Riemannian manifolds satisfying (D_{loc}). Assume that q_{0_M} and q_{0_N} are well defined. Take $q_{0_M} < p_1 \leq p_2 \leq \infty$, $q_{0_N} < r_1, r, r_2 \leq \infty$. Let T be a bounded linear operator from $W_{p_i}^1(M)$ to $W_{r_i}^1(N)$ of norm M_i, $i=1,2$. Then for every couple (p,r) such that $p \leq r$ and $(\frac{1}{p},\frac{1}{r}) = (1-\theta)(\frac{1}{p_1},\frac{1}{r_1}) + \theta(\frac{1}{p_2},\frac{1}{r_2})$, $0<\theta<1$, T is bounded from $W_p^1(M)$ to $W_r^1(N)$ with norm $M \leq CM_0^{1-\theta}M_1^\theta$.*

Proof.

$$\begin{aligned}\|Tf\|_{W_r^1(N)} &\leq C\|Tf\|_{(W_{r_1}^1(N),W_{r_2}^1(N))_{\theta,r}} \\ &\leq CM_0^{1-\theta}M_1^\theta\|f\|_{(W_{p_1}^1(M),W_{p_2}^1(M))_{\theta,r}} \\ &\leq CM_0^{1-\theta}M_1^\theta\|f\|_{(W_{p_1}^1(M),W_{p_2}^1(M))_{\theta,p}} \\ &\leq CM_0^{1-\theta}M_1^\theta\|f\|_{W_p^1(M)}\end{aligned}$$

where we used the fact that $K_{\theta,q}$ is an exact interpolation functor of exponent θ, that $W_p^1(M) = (W_{p_1}^1(M), W_{p_2}^1(M))_{\theta,p}$, $W_r^1(N) = (W_{r_1}^1(N), W_{r_2}^1(N))_{\theta,r}$ with equivalent norms and that $(W_{p_1}^1(M), W_{p_2}^1(M))_{\theta,p} \subset (W_{p_1}^1(M), W_{p_2}^1(M))_{\theta,r}$ if $p \leq r$. □

Remark 3.4.5. *Let M be a Riemannian manifold, not necessarily complete, satisfying (D_{loc}). Assume that for some $1 \leq q < \infty$, a weak local Poincaré inequality holds for all C^∞ functions, that is there exists $r_1 > 0$, $C = C(q,r_1)$, $\lambda \geq 1$ such that for all $f \in C^\infty$ and all ball B of radius $r < r_1$ we have*

$$\Big(\fint_B |f - f_B|^q d\mu\Big)^{\frac{1}{q}} \leq Cr\Big(\fint_{\lambda B} |\nabla f|^q d\mu\Big)^{\frac{1}{q}}.$$

Then, we obtain the characterization of K as in Theorem 3.1.2 and we get by interpolating a result analogous to Theorem 3.1.1.

3.5 Homogeneous Sobolev spaces on Riemannian manifolds

Definition 3.5.1. *Let M be a C^∞ Riemannian manifold of dimension n. For $1 \leq p \leq \infty$, we define $\dot{E}_p^1$ to be the vector space of distributions φ with $|\nabla\varphi| \in L_p$, where $\nabla\varphi$ is the distributional gradient of φ. It is well known that the elements of $\dot{E}_p^1$ are in $L_{p,loc}$. We equip $\dot{E}_p^1$ with the semi norm*

$$\|\varphi\|_{\dot{E}_p^1} = \| \, |\nabla\varphi| \, \|_p.$$

Definition 3.5.2. *We define the homogeneous Sobolev space $\dot{W}_p^1$ as the quotient space $\dot{E}_p^1/\mathbb{R}$ for the norm*

$$\| \, \phi\|_{\dot{W}_p^1} = \inf\left\{ \| \, |\nabla\varphi| \, \|_p, \, \varphi \in \dot{E}_p^1, \, \overline{\varphi} = \phi \right\}.$$

Remark 3.5.3. *For all $\varphi \in \dot{E}_p^1$, $\|\overline{\varphi}\|_{\dot{W}_p^1} = \| \, |\nabla\varphi| \, \|_p$.*

Proposition 3.5.4. *1. ([22]) $\dot{W}_p^1$ is a Banach space.*
2. $C^\infty(M) \cap \dot{W}_p^1$ is dense in $\dot{W}_p^1$ for $1 \leq p < \infty$.
3. Let $1 \leq p < \infty$ and $f \in \dot{E}_p^1$. Then, to every $\varepsilon > 0$ there exists $h \in C^\infty$ such that $\|f - h\|_{W_p^1} < \varepsilon$.

Proof. We prove items 2. and 3. at the same time. Consider locally finite coverings of M, $(U_k)_k$, $(V_k)_k$ with $\overline{U_k} \subset V_k$, V_k being endowed with a real coordinate chart ψ_k. Let $(\varphi_k)_k$ be a partition of unity subordinated to the covering $(U_k)_k$, that is, for all k φ_k is a C^∞ function compactly supported in U_k, $0 \leq \varphi_k \leq 1$ and $\sum_{k=1}^\infty \varphi_k = 1$. Let $F \in \dot{W}_p^1$, $f \in \dot{E}_p^1$, $\overline{f} = F$. We have $f = \sum_k f\varphi_k := \sum_k f_k$. Take $\varepsilon > 0$. The functions $g_k = f_k \circ \psi_k^{-1}$ (which belongs to $W_p^1(\mathbb{R}^n)$ since f and $|\nabla f| \in L_{p,loc}$) can be approximated by smooth functions w_k (standard approximation by convolution) in such a way that $\operatorname{supp} w_k \subset \psi_k(V_k)$ and $\|g_k - w_k\|_{W_p^1} \leq \frac{\varepsilon}{2^k}$. Define

$$h_k(x) = \begin{cases} w_k \circ \psi_k(x) & \text{if } x \in V_k, \\ 0 & \text{otherwise.} \end{cases}$$

Thus $\operatorname{supp} h_k \subset V_k$ and

$$\|f_k - h_k\|_p = \left(\int_{V_k} |f_k - h_k|^p d\mu \right)^{\frac{1}{p}} = \|g_k - w_k\|_p \leq \frac{\varepsilon}{2^k}.$$

$$\| \, |\nabla(f_k - h_k)| \, \|_p = \left(\int_{V_k} |\nabla(f_k - h_k)|^p d\mu \right)^{\frac{1}{p}} = \| \, |\nabla(g_k - w_k)| \, \|_p \leq \frac{\varepsilon}{2^k}.$$

Hence the series $\sum_k (f_k - h_k)$ is convergent in W_p^1. We have also $\sum_k (f_k - h_k) = f - h$ where $h = \sum_k h_k$, so $f - h \in W_p^1$ and $\|f - h\|_{W_p^1} \leq \sum_k \|f_k - h_k\|_{W_p^1} \leq \varepsilon$.
Thus $h = (h - f) + f \in \dot{E}_p^1$ and $\|F - \overline{h}\|_{\dot{W}_p^1} \leq \|f - h\|_{\dot{E}_p^1} \leq \varepsilon$. □

Proposition 3.5.5 (Calderón-Zygmund lemma for Sobolev functions)**.** *Let M be a complete non-compact Riemannian manifold satisfying (D). Let $1 \le q < \infty$. Assume that M admits a Poincaré inequality (P_q) for all $f \in C^\infty$, that is there exists a constant $C > 0$ such that for all $f \in C^\infty$ and for every ball B of M of radius $r > 0$ we have*

$$(P_q) \qquad \left(\fint_B |f - f_B|^q d\mu\right)^{\frac{1}{q}} \le Cr \left(\fint_B |\nabla f|^q d\mu\right)^{\frac{1}{q}}.$$

Let $q \le p < \infty$, $f \in \dot{E}_p^1$ and $\alpha > 0$. Then there exist a collection of balls $(B_i)_i$, functions $b_i \in \dot{E}_q^1$ and a function g Lipschitz almost everywhere such that the following properties hold :

$$f = g + \sum_i b_i \tag{3.19}$$

$$|\nabla g(x)| \le C\,\alpha \quad \mu - a.e. \tag{3.20}$$

$$\operatorname{supp} b_i \subset B_i \text{ and } \int_{B_i} |\nabla b_i|^q d\mu \le C\alpha^q \mu(B_i) \tag{3.21}$$

$$\sum_i \mu(B_i) \le C\alpha^{-p} \int |\nabla f|^p d\mu \tag{3.22}$$

$$\sum_i \chi_{B_i} \le N \tag{3.23}$$

with C and N depending only on q, p and the constant in (D).

Proof. The proof goes as in the case of non homogeneous Sobolev spaces, but taking $\Omega = \{x \in M : \mathcal{M}(|\nabla f|^q)(x) > \alpha^q\}$ as $\|f\|_p$ is not under control. We note that in the non homogeneous case, we used that $f \in L_p$ only to control $g \in L_\infty$ and $b \in L_q$. □

Remark 3.5.6. *It is sufficient for us that the Poincaré inequality holds for all $f \in \dot{E}_p^1$.*

Proposition 3.5.7. *The function g of Calderón-Zygmund lemma is Lipschitz almost everywhere on M and $|g(x) - g(y)| \le C\alpha d(x, y)$ almost everywhere.*

Proof. Same proof as that of Proposition 3.3.6. □

Corollary 3.5.8. *Under the same hypotheses as in the Calderón-Zygmund lemma, we have*

$$\dot{W}_p^1 \subset \dot{W}_q^1 + \dot{W}_\infty^1 \text{ for } q \le p < \infty.$$

Proof of Theorem 3.1.4. The proof of item 1. is the same as in the non homogeneous case. Let us turn to inequality 2.. For $F \in \dot{W}_p^1$ we take $f \in \dot{E}_p^1$ with $\overline{f} = F$. Let $t > 0$ and $\alpha(t) = \left(\mathcal{M}(|\nabla f|^q)\right)^{*\frac{1}{q}}(t)$. By the Calderón-Zygmund decomposition with $\alpha = \alpha(t)$, f can be written $f = b + g$, hence $F = \overline{b} + \overline{g}$, with $\|\overline{b}\|_{\dot{W}_q^1} = \| \, |\nabla b| \, \|_q \leq C\alpha(t)\mu(\Omega)^{\frac{1}{q}}$ and $\|\overline{g}\|_{\dot{W}_\infty^1} = \| \, |\nabla g| \, \|_\infty \leq C\alpha(t)$. Since for $\alpha = \alpha(t)$ we have $\mu(\Omega) \leq t$, we get then $K(F, t^{\frac{1}{q}}, \dot{W}_q^1, \dot{W}_\infty^1) \leq Ct^{\frac{1}{q}} |\nabla f|^{q**\frac{1}{q}}(t)$. □

Proof of Theorem 3.1.3. The proof follows directly from Theorem 3.1.4. Indeed, item 1. of Theorem 3.1.4 yields

$$(\dot{W}_q^1, \dot{W}_\infty^1)_{1-\frac{q}{p},p} \subset \dot{W}_p^1$$

with $\|F\|_{\dot{W}_p^1} \leq C\|F\|_{1-\frac{q}{p},p}$, while item 2. gives us that

$$\dot{W}_p^1 \subset (\dot{W}_q^1, \dot{W}_\infty^1)_{1-\frac{q}{p},p}$$

with $\|F\|_{1-\frac{q}{p},p} \leq C\|F\|_{\dot{W}_p^1}$. Hence

$$\dot{W}_p^1 = (\dot{W}_q^1, \dot{W}_\infty^1)_{1-\frac{q}{p},p}$$

with equivalent norms. □

Corollary 3.5.9 (The reiteration theorem)**.** *Let M be a complete non-compact Riemannian manifold satisfying (D) and (P_q) for some $1 \leq q < \infty$. Define $q_0 = \inf\{q \in [1,\infty[: (P_q)$ holds $\}$. For $q_0 < p_1 < p < p_2 \leq \infty$, $\dot{W}_p^1$ is an interpolation space between $\dot{W}_{p_1}^1$ and $\dot{W}_{p_2}^1$.*

Application

Before we give the application which motivated this work, we need the following proposition:

Proposition 3.5.10. *Let M be a complete Riemannian manifold satisfying (D) and admitting a Poincaré inequality (P_p) for some $1 < p < \infty$ as defined in Proposition 3.5.5. Then,* $\mathrm{Lip} \cap \dot{W}_p^1$ *is dense in* $\dot{W}_p^1$.

Proof. Theorem 3.4.1 proves that M admits a Poincaré inequality (P_q) for some $1 \leq q < p$. Let $F \in \dot{W}_p^1$, $f \in \dot{E}_p^1$ with $\overline{f} = F$. For every $n \in \mathbb{N}^*$, consider the Calderón-Zygmund decomposition of Proposition 3.5.5 with $\alpha = n$. Take a compact K of M. We have

$$\begin{aligned}\int_K |f - g_n|^q d\mu &= \int_{K \cap (\cup_i B_i)} |\sum_i b_i|^q d\mu \\ &= \int_{\cup_i K \cap B_i} |\sum_i b_i|^q d\mu\end{aligned}$$

$$\leq C\sum_i \int_{K\cap B_i} \frac{|f-f_{B_i}|^q}{r_i^q} d(x,F_n)^q d\mu$$
$$\leq C \sup_{x\in K}(d(x,F_n))^q \sum_i \int_{B_i} |\nabla f|^q d\mu$$
$$\leq C \sup_{x\in K}(d(x,F_1))^q \sum_i n^q \mu(B_i)$$
$$\leq C n^{q-p} \| \, |\nabla f| \, \|_p^p.$$

Letting $n\to\infty$, we get that $\int_K |f-g_n|^q d\mu \to 0$. Then $f-g_n \underset{n\to\infty}{\longrightarrow} 0$ in the distributional sense. On the other hand,

$$\int_M |\nabla f - \nabla g_n|^p d\mu = \int_{\Omega_n} |\nabla f - \nabla g_n|^p d\mu \leq C\int_{\Omega_n} |\nabla f|^p d\mu + Cn^p \mu(\Omega_n) \leq C.$$

Thus, $(\nabla f - \nabla g_n)_n$ is bounded in L_p. Since L_p is reflexive for $p>1$, there exists a subsequence, which we denote also by $(\nabla f - \nabla g_n)_n$, converging weakly in L_p to a vector field l. Hence $(\nabla f - \nabla g_n)_n$ converges to l in the distributional sense. The unicity of the limit gives us $l=0$. By Mazur's Lemma, we find a sequence (h_n) of convex combinations of $(\nabla(f-g_n))$ such that $h_n = \sum_{k=1}^n a_{n,k}\nabla(f-g_k)$, $a_{n,k}\geq 0$, $\sum_{k=1}^n a_{n,k}=1$, that converges to 0 in L_p. Since $h_n = \nabla f - \nabla l_n$ with $l_n = \sum_{k=1}^n a_{n,k} g_k$, we obtain $l_n \underset{n\to\infty}{\longrightarrow} f$ in $\dot{E}_p^1$ and then $\overline{l_n} \underset{n\to\infty}{\longrightarrow} F$ in $\dot{W}_p^1$. □

Consider now a complete non-compact Riemannian manifold M satisfying (D) and (P_q) for some $1\leq q<2$. Let Δ be the Laplace-Beltrami operator. Consider the linear operator $\Delta^{\frac{1}{2}}$:

$$\Delta^{\frac{1}{2}} f = c\int_0^\infty \Delta e^{-t\Delta} f \frac{dt}{\sqrt{t}}, \quad f\in C_0^\infty$$

where $c=\pi^{-\frac{1}{2}}$. $\Delta^{\frac{1}{2}}$ can be defined for $f\in \mathrm{Lip}$ as a measurable function (see [3]).

In [3], Auscher and Coulhon proved that on such a manifold, we have

$$\mu\left\{x\in M : |\Delta^{\frac{1}{2}} f(x)| > \alpha\right\} \leq \frac{C}{\alpha^q} \| \, |\nabla f| \, \|_q$$

for $f\in C_0^\infty$, with $q\in[1,2[$. In fact one can check that the argument applies to all $f\in \mathrm{Lip}\cap \dot{E}_q^1$ and since $\Delta^{\frac{1}{2}}1=0$, $\Delta^{\frac{1}{2}}$ can be defined on $\mathrm{Lip}\cap \dot{W}_q^1$ by taking quotient which we keep calling $\Delta^{\frac{1}{2}}$. Moreover, Proposition 3.5.10 gives us that $\Delta^{\frac{1}{2}}$ has a bounded extension from $\dot{W}_q^1$ to $L_{q,\infty}$ for $q>1$. Since we already have

$$\|\Delta^{\frac{1}{2}} f\|_2 \leq \| \, |\nabla f| \, \|_2$$

then by Corollary 3.5.9, we see at once

$$\|\Delta^{\frac{1}{2}} f\|_p \leq C_p \| \, |\nabla f| \, \|_p \tag{3.24}$$

for all $q<p\leq 2$ and $f\in \dot{W}_p^1$, without using the argument in [3]. If $q=1$, we get similarly (3.24) for $1<p\leq 2$, considering $(P_{q+\varepsilon})$ for some $1<q+\varepsilon<p$ instead of (P_q).

3.6 Sobolev spaces on compact manifolds

Let M be a C^∞ compact manifold equipped with a Riemannian metric. M satisfies then the doubling property (D) and the Poincaré inequality (P_1).

Theorem 3.6.1. *Let M be a C^∞ compact Riemannian manifold. There exist C_1, C_2 such that for all $f \in W_1^1 + W_\infty^1$ and all $t > 0$ we have*

$$(*_{\text{comp}}) \quad C_1 t\Big(|f|^{**}(t) + |\nabla f|^{**}(t)\Big) \leq K(f, t, W_1^1, W_\infty^1) \leq C_2 t\Big(|f|^{**}(t) + |\nabla f|^{**}(t)\Big).$$

Proof. It remains to prove the upper bound for K as the lower bound is trivial. Indeed, let us consider for all $t > 0$ and for $\alpha(t) = \left(\mathcal{M}(|f| + |\nabla f|)\right)^* (t)$, $\Omega = \{x \in M; \mathcal{M}(|f| + |\nabla f|)(x) \geq \alpha(t)\}$. If $\Omega \neq M$, we have the Calderón-Zygmund decomposition as in Proposition 3.3.5 with $q = 1$ and the proof will be the same as the proof of Theorem 3.1.2 in the global case. Now if $\Omega = M$, we prove the upper bound by the same argument used in the proof of Theorem 3.1.2 in the local case. Thus we obtain in the two cases the right hand inequality of $(*_{\text{comp}})$ for all $f \in W_1^1 + W_\infty^1$. □

It follows that:

Theorem 3.6.2. *For all $1 \leq p_1 < p < p_2 \leq \infty$, W_p^1 is an interpolation space between $W_{p_1}^1$ and $W_{p_2}^1$.*

3.7 Metric-measure Spaces

In this section we will consider (X, d, μ) a metric-measure space with μ doubling.

3.7.1 Upper gradients and Poincaré inequality

Definition 3.7.1 (Upper gradient)**.** *Let $u : X \to \mathbb{R}$ be a Borel function. We say that a Borel function $g : X \to [0, +\infty]$ is an upper gradient of u if $|u(\gamma(a)) - u(\gamma(a))| \leq \int_a^b g(\gamma(t))dt$ for all 1-Lipschitz curve $\gamma : [a, b] \to X$.* [1]

Remark 3.7.2. *If X is a Riemannian manifold, $|\nabla u|$ is an upper gradient of $u \in$ Lip and $|\nabla u| \leq g$ for all g upper gradient of u.*

Definition 3.7.3. *For every locally Lipschitz continuous function u defined on a open set of X, we define*

$$\operatorname{Lip} u(x) = \begin{cases} \limsup_{\substack{y \to x \\ y \neq x}} \frac{|u(y) - u(x)|}{d(y,x)} & \text{if } x \text{ is not isolated}, \\ 0 & \text{otherwise}. \end{cases}$$

Remark 3.7.4. Lip u *is an upper gradient of u.*

[1] Since every rectifiable curve admits an arc-length parametrization that make curve 1-Lipschiz, the class of 1-Lipschtiz curves coincides with the class of rectifiable curves, modulo a parameter change.

Definition 3.7.5 (Poincaré Inequality). *A metric-measure space (X, d, μ) is said to admit a weak local Poincaré inequality (P_{qloc}) for some $1 \leq q < \infty$, if there exist $r_1 > 0$, $\lambda \geq 1$, $C = C(q, r_1) > 0$, such that for every continuous function u and upper gradient g of u, and for every ball B of radius $0 < r < r_1$ the following inequality holds:*

$$(P_{qloc}) \qquad \Big(\fint_B |u - u_B|^q d\mu \Big)^{\frac{1}{q}} \leq Cr \Big(\fint_{\lambda B} g^q d\mu \Big)^{\frac{1}{q}}.$$

If $\lambda = 1$, we say that we have a strong local Poincaré inequality.
X admits a global Poincaré inequality or simply a Poincaré inequality (P_q) if one can take $r_1 = \infty$.

3.7.2 Interpolation of the Sobolev spaces H_p^1

Before defining the Sobolev spaces H_p^1 it is convenient to recall the following proposition:

Proposition 3.7.6. *(see [26] and [11] Th.4.38) Let (X, d, μ) be a complete metric-measure space, with μ doubling and satisfying a weak Poincaré inequality (P_q) for some $1 < q < \infty$. Then there exist an integer N, $C \geq 1$ and a linear operator D which associates to each locally Lipschitz function u a measurable function $Du : X \to \mathbb{R}^N$ such that :*

1. *if u is L Lipschitz, then $|Du| \leq CL$ $\mu -$ a.e.;*
2. *if u is locally Lipschitz and constant on a measurable set $E \subset X$, then $Du = 0$ $\mu -$ a.e. on E;*
3. *for locally Lipschitz functions u and v, $D(uv) = Du\, v + v\, Du$;*
4. *for each locally Lipschitz function u, $\mathrm{Lip}\, u \leq |Du| \leq C\ \mathrm{Lip}\, u$, and hence $(u, |Du|)$ satisfies the weak Poincaré inequality (P_q) .*

We define now $H_p^1 = H_p^1(X, d, \mu)$ for $1 \leq p < \infty$ as the closure of locally Lipschitz functions for the norm

$$\|u\|_{H_p^1} = \|u\|_p + \| \, |Du| \, \|_p \sim \|u\|_p + \| \mathrm{Lip}\, u\|_p.$$

We denote H_∞^1 for the set of all bounded Lipschitz functions on X.

In the remaining part of this section, we consider (X, d, μ) a complete non-compact metric-measure space with μ doubling. We also assume that X admits a Poincaré inequality (P_q) for some $1 < q < \infty$ as defined in Definition 3.7.5. By [30], Theorem 1.3.4, this is equivalent to say that there exists $C > 0$ such that for all $f \in \mathrm{Lip}$ and for every ball B of X of radius $r > 0$ we have

$$(P_q) \qquad \int_B |f - f_B|^q d\mu \leq Cr^q \int_B |\mathrm{Lip}\, f|^q d\mu.$$

Define $q_0 = \inf \{q \in]1, \infty[: (P_q) \text{ holds } \}$.

Lemma 3.7.7. *Under these hypotheses, and for $q_0 < p < \infty$, $Lip \cap H_p^1$ is dense in H_p^1.*

Proof. In [25] it was proved that for all $1 < p < \infty$, $Lip \cap M_p^1$ is dense in M_p^1 when X is of bounded diameter. The proof follows exactly the same argument if we do not make any assumption on the diameter of X. Noting that under our hypotheses $H_p^1 = M_p^1$ (see [27]), we get the lemma. $\square$

Proposition 3.7.8. ***Calderón-Zygmund lemma for Sobolev functions*** *Let (X, d, μ) be a complete non-compact metric-measure space with μ doubling, admitting a Poincaré inequality (P_q) for some $1 < q < \infty$. Then, Calderón-Zygmund decomposition of Proposition 3.3.5 still holds in the present situation for $f \in \mathrm{Lip} \cap H_p^1$, $q \leq p < \infty$, replacing ∇f by $\mathrm{Lip}\, f$.*

Proof. The proof is similar, replacing ∇f by Df, using that D of Proposition 3.7.6 is linear. Since the χ_i are $\frac{C}{r_i}$ Lipschitz then $\|D\chi_i\|_\infty \leq \frac{C}{r_i}$ by item 1. of Theorem 3.7.6 and the b_i's are Lipschitz. We can see that g is also Lipschitz (see Remark 3.3.7 after Proposition 3.3.6). Using also the finite additivity of D and the property 2. of Proposition 3.7.6, we get the equality $\mu - a.e.$

$$Dg = Df - D(\sum_i b_i) = Df - (\sum_i Db_i).$$

The rest of the proof goes as in Proposition 3.3.5. $\square$

Theorem 3.7.9. *Let (X, d, μ) be a complete non-compact metric-measure space with μ doubling, admitting a Poincaré inequality (P_q) for some $1 < q < \infty$. Then, there exists C_1, C_2 such that for all $f \in H_q^1 + H_\infty^1$ and all $t > 0$ we have*

$(*_{\mathrm{met}})$

$$C_1 t^{\frac{1}{q}} \Big(|f|^{q**\frac{1}{q}}(t) + |\mathrm{Lip}\, f|^{q**\frac{1}{q}}(t) \Big) \leq K(f, t^{\frac{1}{q}}, H_q^1, H_\infty^1) \leq C_2 t^{\frac{1}{q}} \Big(|f|^{q**\frac{1}{q}}(t) + |\mathrm{Lip}\, f|^{q**\frac{1}{q}}(t) \Big).$$

Proof. We have $(*_{\mathrm{met}})$ for all $f \in \mathrm{Lip} \cap H_q^1$ from the Calderón-Zygmund decomposition that we have done. Now for $f \in H_q^1$, by Lemma 3.7.7, $f = \lim_n f_n$ in H_q^1, with f_n Lipschitz and $\|f - f_n\|_{H_q^1} < \frac{1}{n}$ for all n. Since for all n, $f_n \in \mathrm{Lip}$, there exist g_n, h_n such that $f_n = h_n + g_n$ and $\|h_n\|_{H_q^1} + t^{\frac{1}{q}} \|g_n\|_{H_\infty^1} \leq C t^{\frac{1}{q}} \Big(|f_n|^{q**\frac{1}{q}}(t) + |\mathrm{Lip}\, f_n|^{q**\frac{1}{q}}(t) \Big)$. Therefore we find

$$\begin{aligned} \|f - g_n\|_{H_q^1} + t^{\frac{1}{q}} \|g_n\|_{H_\infty^1} &\leq \|f - f_n\|_{H_q^1} + (\|h_n\|_{H_q^1} + t^{\frac{1}{q}} \|g_n\|_{H_\infty^1}) \\ &\leq \frac{1}{n} + C t^{\frac{1}{q}} \Big(|f_n|^{q**\frac{1}{q}}(t) + |\mathrm{Lip}\, f_n|^{q**\frac{1}{q}}(t) \Big). \end{aligned}$$

Letting $n \to \infty$, since $|f_n|^q \underset{n\to\infty}{\longrightarrow} |f|^q$ in L_1 and $|\mathrm{Lip}\, f_n|^q \underset{n\to\infty}{\longrightarrow} |\mathrm{Lip}\, f|^q$ in L_1, it comes $|f_n|^{q**}(t) \underset{n\to\infty}{\longrightarrow} |f|^{q**}(t)$ and $|\mathrm{Lip}\, f_n|^{q**}(t) \underset{n\to\infty}{\longrightarrow} |\mathrm{Lip}\, f|^{q**}(t)$ for all $t > 0$. Hence $(*_{\mathrm{met}})$ holds for $f \in H_q^1$. We prove $(*_{\mathrm{met}})$ for $f \in H_q^1 + H_\infty^1$ by the same argument of [17]. $\square$

Theorem 3.7.10 (Interpolation Theorem)**.** *Let (X, d, μ) be a complete non-compact metric-measure space with μ doubling, admitting a Poincaré inequality (P_q) for some $1 < q < \infty$. Then, for $q_0 < p_1 < p < p_2 \leq \infty$, H_p^1 is an interpolation space between $H_{p_1}^1$ and $H_{p_2}^1$.*

Proof. Theorem 3.7.9 provides us with all the tools needed for interpolating, as we did in the Riemannian case. In particular, we get Theorem 3.7.10. □

We obtain then all the results of section 3.4, but now for metric-measure spaces.

Remark 3.7.11. *Other Sobolev spaces on metric-measure spaces were introduced in the last few years, for instance M_p^1, N_p^1, C_p^1, P_p^1. If X is a complete metric-measure space satisfying (D) and (P_q) for some $1 < q < \infty$, it can be shown that for $q_0 < p \leq \infty$, all the mentioned spaces are equal to H_p^1 with equivalent norms (see [27]). In conclusion our interpolation result carries over to those Sobolev spaces.*

Definition 3.7.12. *A metric space X is called λ-quasi-convex if for all x_1, $x_2 \in X$, there exists a rectifiable curve from x_1 to x_2 of length at most $\lambda d(x_1, x_2)$.*

One can associate, canonically, to a λ-quasi-convex metric space a length metric space, which is λ-bi-Lipschitz to the original one (see [11]).

Remark 3.7.13. *The purpose of this remark is to extend our results to local assumptions. Assume that (X, d, μ) is a complete metric-measure space, with μ locally doubling, and admitting a local Poincaré inequality (P_{qloc}) for some $1 < q < \infty$. Since X is complete and (X, μ) satisfies a local doubling condition and a local Poincaré inequality (P_{qloc}), then according to an observation of David and Semmes (see [11]), every ball $B(z, r)$, with $0 < r < \min(r_0, r_1)$, is $\lambda = \lambda(C(r_0), C(r_1))$ quasi-convex, $C(r_0)$ and $C(r_1)$ being the constants appearing in the local doubling property and in the local Poincaré inequality. Then, for $0 < r < \min(r_0, r_1)$, $B(z, r)$ is λ bi-Lipschitz to a length space. Hence, we get a result analogous to Theorem 3.7.9. Indeed, the proof goes as that of Theorem 3.1.2 in the local case noting that the B^j's considered there are then λ bi-Lipschitz to a length space with λ independent of j. Thus, Lemma 3.3.9 still holds (see Remark 3.3.10). Therefore, we get the characterization $(*_{met})$ of K and by interpolating, we obtain the correspondance analogue of Theorem 3.7.10.*

3.8 Carnot-Carathéodory spaces

We refer to [27] for a survey on the theory of Carnot-Carathéodory spaces.

Let $\Omega \subset \mathbb{R}^n$ be a connected open set, $X = (X_1, ..., X_k)$ a family of vector fields defined on Ω, with real locally Lipschitz continuous coefficient and $|Xu(x)| = \left(\sum\limits_{j=1}^{k} |X_j u(x)|^2\right)^{\frac{1}{2}}$. We equip Ω with the Lebesgue measure $\mathcal{L}^n$ and the Carnot-Carathéodory metric ρ associated to the X_i. Before continuing, let us recall the definition of the Carnot-Carathéodory metric.

Definition 3.8.1. *An absolutely continuous curve $\gamma : [a, b] \mapsto \Omega$ is admissible if there exist measurable functions $c_j(t)$, $a \leq t \leq b$ such that $\sum\limits_{j=1}^{k} c_j^2(t) \leq 1$ and $\gamma'(t) = \sum\limits_{j=1}^{k} c_j(t)\, X_j(\gamma(t))$.*

Definition 3.8.2. ***Carnot-Carathéodory metric:*** *Let x, $y \in \Omega$, we define $\rho(x,y) = \inf\{T > 0;\ \exists \gamma$ admissible $: [0,T] \mapsto \Omega\,,\ \gamma(0) = x\,,\ \gamma(T) = y\}$.
If such a γ does not exist we set $\rho(x,y) = \infty$. If $\rho(x,y) < \infty\ \forall x,\, y \in \Omega$, then ρ defines a distance called Carnot-Carathéodory distance and the map $id : (\Omega,\rho) \mapsto (\Omega, |.|)$ is continuous.*

For the rest of the section, we assume that ρ defines a distance. Then, the metric space (Ω,ρ) is a length space.

Proposition 3.8.3. *$|Xu|$ is an upper gradient of $u \in C^\infty(\Omega)$ on (Ω,ρ).*

Theorem 3.8.4. *Assume that $id : (\Omega,\rho) \to (\Omega,|.|)$ is an homeomorphism. Then the space $(\Omega,\rho,\mathcal{L}^n)$ admits a weak local Poincaré inequality (P_{qloc}), for some $1 \le q < \infty$ as defined in Definition 3.7.5, if and only if there exist $r_1 > 0$, $\lambda \ge 1$ and $C = C(q,r_1)$ such that*

$$\left(\fint_{\widetilde{B}} |u - u_{\widetilde{B}}|^q dx\right)^{\frac{1}{q}} \le Cr \left(\fint_{\lambda\widetilde{B}} |Xu|^q dx\right)^{\frac{1}{q}}$$

for all $\widetilde{B}$ with radius $r < r_1$ and $\lambda\widetilde{B} \subset \Omega$, and $u \in C^\infty(\lambda\widetilde{B})$, where $\widetilde{B}$ is the ball relative to the metric ρ of Carnot-Carathéodory.

Definition 3.8.5. *Let $1 \le p \le \infty$. We define $W^1_{p,X}(\Omega)$ as the set of $f \in L_p(\Omega)$ such that $|Xf| \in L_p(\Omega)$, where Xf is defined in the distributional sense. We equip this space with the norm*

$$\|f\|_{W^1_{p,X}} = \|f\|_{L_p(\Omega)} + \|\,|Xf|\,\|_{L_p(\Omega)}$$

Remark 3.8.6. *([21]) For $1 \le p < \infty$, $W^1_{p,X}(\Omega)$ is nothing but the completion of $C^\infty(\Omega)$ for this norm.*

Theorem 3.8.7. *Consider $(\Omega,\rho,\mathcal{L}^n)$ where Ω is a connected open subset of $\mathbb{R}^n$. Let $1 \le q < \infty$. We assume that $\mathcal{L}^n$ is locally doubling, that id is an homeomorphism and that the space admits a weak local Poincaré inequality (P_{qloc}). Then, there exists C_1, C_2 such that for every $f \in W^1_{q,X} + W^1_{\infty,X}$ and all $t > 0$, we have*

$$C_1 t^{\frac{1}{q}}\Big(|f|^{q**\frac{1}{q}}(t) + |Xf|^{q**\frac{1}{q}}(t)\Big) \le K(f, t^{\frac{1}{q}}, W^1_{q,X}, W^1_{\infty,X}) \le C_2 t^{\frac{1}{q}}\Big(|f|^{q**\frac{1}{q}}(t) + |Xf|^{q**\frac{1}{q}}(t)\Big).$$

Proof. For all $f \in W^1_{q,X}$, the proof goes as in the Riemannian case, using a Calderón-Zygmund decomposition analogous to that done in Proposition 3.3.5, replacing ∇ by X and noting that Lemma 3.3.9 holds in this case because (Ω,ρ) is a length space. For $f \in W^1_{q,X} + W^1_{\infty,X}$, the same argument of [17] works. □

Theorem 3.8.8 (Interpolation of $W^1_{p,X}(\Omega)$)**.** *Let $(\Omega,\rho,\mathcal{L}^n)$ be as in Theorem 3.8.7. Let $q_0 = \inf\{q \in [1,\infty[: (P_q) \text{ holds}\}$. Then, for $q_0 < p_1 < p < p_2 \le \infty$, $W^1_{p,X}$ is an interpolation space between $W^1_{p_1,X}$ and $W^1_{p_2,X}$. We obtain again all the results of section 3.4.*

Proof. Follows from Theorem 3.8.7 and the reiteration theorem. □

3.9 Weighted Sobolev spaces

We refer to [28], [32] for the definitions used in this section.

Let Ω be an open subset of $\mathbb{R}^n$ equipped with the Euclidean distance, $w \in L_{1,loc}(\mathbb{R}^n)$ with $w > 0$, $d\mu = wdx$. In all this section we assume that μ is q-admissible for some $1 < q < \infty$ (see [29] for the definition) which is equivalent to say, (see [27]), that μ is doubling and there exists $C > 0$ such that for every ball $B \subset \mathbb{R}^n$ of radius $r > 0$ and for every function $\varphi \in C^\infty(B)$,

$$(P_q) \qquad \int_B |\varphi - \varphi_B|^q d\mu \leq C r^q \int_B |\nabla \varphi|^q d\mu$$

with $\varphi_B = \frac{1}{\mu(B)} \int_B \varphi d\mu$. The A_q weights, $q > 1$, satisfy these two conditions (see [29], Chapter 15). We recall the defintion of A_q weights:

Definition 3.9.1. *A weight w is a non-negative locally integrable function. We say that $w \in A_p$, $1 < p < \infty$, if there exists a constant C such that for every ball $B \subset \mathbb{R}^n$*

$$(\fint_B w)(\fint_B w^{1-p'})^{p-1} \leq C$$

with $\frac{1}{p} + \frac{1}{p'} = 1$.
For $p = 1$, we say that $w \in A_1$ if there is a constant C such that for every ball $B \subset \mathbb{R}^n$

$$(\fint_B w) \leq C\, w(x), \quad \text{for } \mu - a.e. \quad x \in B\,.$$

Let us now give the defintion of the weighted Sobolev spaces and see how to interpolate them:

Definition 3.9.2. *For $q \leq p < \infty$, we define the Sobolev space $H_p^1(\Omega, \mu)$ to be the closure of $C^\infty(\Omega)$ for the norm:*

$$\|u\|_{H_p^1(\Omega,\mu)} = \|u\|_{L_p(\mu)} + \||\nabla u|\|_{L_p(\mu)}.$$

We denote $H_\infty^1(\Omega, \mu)$ for the set of all bounded Lipschitz functions on Ω.

Theorem 3.9.3. *Let Ω be as in above. Then, there exist C_1, C_2 such that for every $f \in H_q^1(\Omega, \mu) + H_\infty^1(\Omega, \mu)$ and all $t > 0$, we have*

$$C_1 t^{\frac{1}{q}} \left(|f|^{q**\frac{1}{q}} + |\nabla f|^{q**\frac{1}{q}}\right)(t) \leq K(f, t^{\frac{1}{q}}, H_q^1, H_\infty^1) \leq C_2 t^{\frac{1}{q}} \left(|f|^{q**\frac{1}{q}} + |\nabla f|^{q**\frac{1}{q}}\right)(t).$$

Proof. A Calderón-Zygmund decomposition as in Proposition 3.3.5 holds for $u \in H_p^1(\Omega, \mu)$, $q \leq p < \infty$. This decomposition will lead us as in the Riemannian case to the proof of this theorem. □

Theorem 3.9.4 (Interpolation of $H_p^1(\Omega, \mu)$)**.** *Under the same hypotheses, for $q_0 < p_1 < p < p_2 \leq \infty$, $H_p^1(\Omega, \mu)$ is an interpolation space between $H_{p_1}^1(\Omega, \mu)$ and $H_{p_2}^1(\Omega, \mu)$; $q_0 = \inf\{q \in]1, \infty[: (P_q) \text{ holds}\}$.*

Proof. This is a consequence of Theorem 3.9.3 and of the reiteration theorem. □

Definition 3.9.5. *For $q \leq p \leq \infty$, we define*

$$W_p^1(\Omega, \mu) = \{u \in L_{1loc}(\Omega, d\mu) : u \text{ and } \nabla u \in L_p(\Omega, \mu)\}$$

with the norm

$$\|u\|_{W_p^1(\Omega,\mu)} = \|u\|_{H_p^1(\Omega,\mu)}.$$

If $w \in A_p$ for some $1 < p < \infty$, then $H_p^1(\Omega, \mu) = W_p^1(\Omega, \mu)$, but in general $H_p^1(\Omega, \mu) \nsubseteq W_p^1(\Omega, \mu)$ (for counterexamples see [29]).

Theorem 3.9.6. *1.([32]) If $W_p^1(\Omega, \mu)$ is a Banach space, then $H_p^1(\Omega, \mu) \subset W_p^1(\Omega, \mu)$.*
2. The space $H_p^1(\Omega, \mu)$ coincides with $W_p^1(\Omega, \mu)$ if and only if $W_p^1(\Omega, \mu)$ is a Banach space and the Poincaré inequality (P_p) (defined at the beginning of this section) holds for all functions $u \in W_p^1(\mathbb{R}^n, \mu)$.

Corollary 3.9.7. *If there exists $p_1 > q_0$ such that $W_{p_1}^1(\Omega, \mu)$ is a Banach space and the Poincaré inequality (P_{p_1}) holds for all functions $u \in W_{p_1}^1(\mathbb{R}^n, \mu)$, then the $W_p^1(\Omega, \mu)$ interpolate for $p_1 \leq p \leq \infty$.*

Remark 3.9.8. *Equip Ω with the Carnot-Carathéodory distance associated to a family of vector fields with real locally Lipschitz continuous coefficients instead of the Euclidean distance. Under the same hypotheses as at the beginning of this section, just replacing the balls B by the balls $\tilde{B}$ with respect to ρ, and ∇ by X and assuming that $id : (\Omega, \rho) \to (\Omega, |.|)$ is an homeomorphism (see section 3.8), we obtain our interpolation results. As an example we take vectors fields satisfying a Hörmander condition or vectors fields of Grushin type [19].*

3.10 Lie Groups

In all this section, we consider G a connected Lie group. We assume that G is unimodular and we let $d\mu$ be a fixed Haar measure on G. Let $X_1, ..., X_k$ be a family of left invariant vector fields such that the X_i's satisfy a Hörmander condition[2]. In this case the Carnot-Carathéodory metric ρ is a distance, and G equipped with the distance ρ is complete and defines the same topology as that of G as a manifold (see [15] p. 1148). An important result of Guivarc'h [24] says that, either there exists an integer N such that $cr^N \leq V(r) \leq Cr^N$ for all $r > 1$, or $e^{cr} \leq V(r) \leq Ce^{Cr}$ for all $r > 1$ with $V(r) = V(x, r) = \mu(B(x, r))$, for all $x \in G$. In the first case we say that G has polynomial growth, while in the second case G has exponential growth. For small r, a result of [35] implies that there exists an integer n such that $cr^n \leq V(r) \leq Cr^n$ for $0 < r < 1$. Hence G satisfies the local doubling property (D_{loc}) and the global doubling property (D) if it has polynomial growth.

[2]We say that X satisfies a Hörmander condition if there exists an integer p such that familly of commutators of X_i's up to length p span G at every point.

For every smooth function u we have

$$\int_B |u - u_B| d\mu \leq r \frac{V(2r)}{V(r)} \int_{2B} |Xu| d\mu$$

(see [37], [39]). This implies that G admits a local Poincaré inequality (P_{1loc}). If G has polynomial growth, then it admits a global Poincaré inequality (P_1).

3.10.1 Non homogeneous Sobolev spaces on Lie groups

Definition 3.10.1. *For $1 \leq p \leq \infty$, we define the Sobolev space $W_p^1 = \{f \in L_p : |Xf| \in L_p\}$, Xf being defined in the distribution sense. W_p^1 is equipped with the norm:*

$$\|u\|_{W_p^1} = \|u\|_p + \| \, |Xu| \, \|_p$$

where $|Xu| = \left(\sum_{i=1}^k |X_i u|^2\right)^{\frac{1}{2}}$.

Remark 3.10.2. *1. W_p^1 equipped with this norm is a Banach space.*

2. W_∞^1 is the set of all bounded Lipschitz functions on G.

Definition 3.10.3. *For $1 \leq p < \infty$, we define the Sobolev space H_p^1 as the completion of C^∞ functions for the norm:*

$$\|u\|_{H_p^1} := \|u\|_{W_p^1} < \infty.$$

By convention, we note $H_\infty^1 = W_\infty^1$.

Proposition 3.10.4. *1. $H_p^1 = W_p^1$ for all $1 \leq p < \infty$.*

2. C_0^∞ is dense in H_p^1 for $1 \leq p < \infty$.

3. ([22])The set of compactly supported Lipschitz functions is dense in H_p^1 for $1 \leq p < \infty$.

Interpolation of W_p^1:

To interpolate the $W_{p_i}^1$, we distinguish between the polynomial and the exponential growth cases. If G has polynomial growth, then we are in a global case. If G has exponential growth, we are in the local case. In the two cases we obtain the following theorem:

Theorem 3.10.5. *Let G be a connected Lie group as at the beginning of this section. Then, there exist C_1, C_2 such that for every $f \in W_1^1 + W_\infty^1$ and for all $t > 0$ we have*

$$C_1 t\Big(|f|^{**}(t) + |Xf|^{**}(t)\Big) \leq K(f, t, W_1^1, W_\infty^1) \leq C_2 t\Big(|f|^{**}(t) + |Xf|^{**}(t)\Big).$$

Theorem 3.10.6. *Let G be as above. Then, for all $1 \leq p_1 < p < p_2 \leq \infty$, W_p^1 is an interpolation space between $W_{p_1}^1$ and $W_{p_2}^1$, ($q_0 = 1$ here). Therefore, we get all the interpolation theorems of section 3.4.*

Proof. Combine Theorem 3.10.5 and the reiteration theorem. □

3.10.2 Homogeneous Sobolev spaces on Lie groups

Let G be connected Lie group as at the beginning of this section. We define the homogeneous Sobolev space $\dot{W}_p^1$ in the same manner as in section 3.5 for Riemannian manifolds.

Definition 3.10.7. *For $1 \leq p \leq \infty$, we define $\dot{E}_p^1$ to be the vector space of distributions φ with $|X\varphi| \in L_p$, where $X\varphi$ is the distributional gradient of φ. It is well known that the elements of $\dot{E}_p^1$ are in $L_{p,loc}$. We equip $\dot{E}_p^1$ with the semi norm*

$$\|\varphi\|_{\dot{E}_p^1} = \| \, |X\varphi| \, \|_p.$$

Definition 3.10.8. *We define the homogeneous Sobolev space $\dot{W}_p^1$ as the quotient space $\dot{E}_p^1/\mathbb{R}$ for the norm*

$$\| \, \phi\|_{\dot{W}_p^1} = \inf \left\{ \| \, |X\varphi| \, \|_p, \, \varphi \in \dot{E}_p^1, \, \overline{\varphi} = \phi \right\}.$$

Remark 3.10.9. *For all $\varphi \in \dot{E}_p^1$, $\quad \|\overline{\varphi}\|_{\dot{W}_p^1} = \| \, |\nabla\varphi| \, \|_p$.*

Proposition 3.10.10. *1. ([22]) $\dot{W}_p^1$ is a Banach space.*
2. $C^\infty \cap \dot{W}_p^1$ is dense in $\dot{W}_p^1$ for $1 \leq p < \infty$.

Theorem 3.10.11. *Let G be a as above and assume that G has polynomial growth. Then*

1. *there exists C_1 such that for every $f \in \dot{W}_1^1 + \dot{W}_\infty^1$ and all $t > 0$*

$$K(f, t, \dot{W}_1^1, \dot{W}_\infty^1) \geq C_1 t |Xf|^{**}(t);$$

2. *for $1 \leq p < \infty$, there exists C_2 such that for every $f \in \dot{W}_p^1$ and every $t > 0$*

$$K(f, t, \dot{W}_1^1, \dot{W}_\infty^1) \leq C_2 t |Xf|^{**}(t).$$

Theorem 3.10.12. *Under the hypotheses of Theorem 3.10.11, we have that for all $1 \leq p_1 < p < p_2 \leq \infty$, $\dot{W}_p^1$ is an interpolation space between $\dot{W}_{p_1}^1$ and $\dot{W}_{p_2}^1$.*

Proof. Combine Theorem 3.10.11 and the reiteration theorem. □

3.11 Examples

The final section is devoted to present some examples for which our interpolation results apply with the appropriate q_0.

1. $\mathbb{R}^n$ equipped with the Euclidean metric and the Lebesgue measure is a complete Riemannian manifold satisfying (D) and (P_1). Our results are already known for $\mathbb{R}^n$ as we mentioned at the beginning (see [17], [6], [10]).

2. Every complete Riemannian manifold M that is quasi-isometric to a Riemannian manifold with non-negative Ricci curvature (in particular every Riemannian manifold with non-negative Ricci curvature) satisfies (D) and (P_1). If the Ricci curvature is just bounded from below, M satisfies then (D_{loc}) and (P_{1loc}). Indeed if the Ricci curvature is bounded from below that is there exists $a > 0$ such that $R_{ic} \geq -a^2 g$, a Gromov result of [12] show that

$$\mu(B(x, 2r)) \leq 2^n e^{\sqrt{n-1}ar} \mu(B(x, r)) \text{ for all } x \in M \ , \ r > 0 \, .$$

On the other hand, Buser's inequality [8] gives us

$$\int |u - u_B| \, d\mu \leq c(n) e^{ar} r \int_B |\nabla u| \, d\mu \, .$$

Then for all $0 < R < \infty$ we have (D_{loc}) and (P_{1loc}) on all balls of radius less than R. If the Ricci curvature is non-negative $a = 0$ we can take $R = \infty$ (see also [36]), and hence M satisfies (D) and (P_1).

3. A singular conical manifold with closed basis admits a Poincaré inequality (P_2) for C^∞ functions (see [16]). One can also see that it admits a Poincaré inequality (P_1) by using the methods of [23], but such a manifold does not verify (D) necessarily. If the basis is compact then we have (D).

4. The co-compact covering manifolds with polynomial growth deck transformation group satisfy the doubling property and a Poincaré inequality (P_1)(see [37]).

5. For $n \geq 2$, we consider M_n the manifold consisting in two copies of $\mathbb{R}^n \setminus B(0, 1)$ equipped with the Euclidean distance and the Lebesgue measure, glued smoothly along the unit circles. We have then a complete Riemannian manifold. One checks easily that on this manifold the volume has polynomial growth. Hence M_n satisfies (D). It can be proved that M_n admits a Poincaré inequality (P_q) if and only if $q > n$.

6. Consider the Euclidean space $\mathbb{R}^n$ endowed with the measure $\mu = w\mathcal{L}^n$, with w a non-negative A_∞ weight. Recall that $w \in A_\infty$ if for all $\varepsilon > 0$ there exists $\delta > 0$ such that for every ball $B \subset \mathbb{R}^n$ and $E \subset B$ the inequality $\mathcal{L}^n(E) \leq \delta \, \mathcal{L}^n(B)$ implies $\mu(E) \leq \varepsilon \mu(B)$. The measure μ is then doubling. Define the quasi-distance δ as for all x, $y \in \mathbb{R}^n$, $\delta(x, y) = \mu(B_{x,y})^{\frac{1}{n}}$, where $B_{x,y}$ is the Euclidean ball of diameter $|x - y|$, containing x and y. The weight w is strongly A_∞ if the distance δ is equivalent to the geodesic distance d_w associated to the Riemannian metric $w^{\frac{1}{n}} ds$. In this case the Lipschitz functions for the Euclidean distance are also Lipschitz for the distance d_w, and $\operatorname{Lip}_w u(x) = w(x)^{-\frac{1}{n}} \operatorname{Lip} u(x)$. It is also known that the doubling metric-measure space $(\mathbb{R}^n, d_w, \mu)$ admits a Poincaré inequality (P_1)(see [1], [18], [20]).

7. Nilpotent Lie groups have polynomial growth, then they satisfy (D) and (P_1). Among the important nilpotents Lie groups we mention the *Carnot groups*. Recall that a Carnot group is a connected and simply connected Lie group G such that the Lie algebra admits a stratification $\mathcal{G} = V_1 \oplus ... \oplus V_m$, $[V_1, V_i] = V_{i+1}$, $V_i = 0$ for $i > m$. We consider $X_1, ..., X_k$ a basis of V_1. We identify $X_1, ..., X_k$ to a family of left invariant vector fields satisfying a Hörmander condition. An example of a Carnot group is the Heisenberg group $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R}$, whose points are denoted by $[z, t]$, with the group operation: $[z_1, t_1].[z_2, t_2] = [z_1 + z_2, t_1 + t_2 + 2Im(z_1 + \bar{z}_2)]$, z_1, $z_2 \in \mathbb{C}^n$, $t_1, t_2 \in \mathbb{R}$ and $z = (x, y)$ with x, $y \in \mathbb{R}^n$. The distance on $\mathbb{H}^n$ is the Carnot-Carathéodory distance induced by the left invariant vector fields :
$$\begin{cases} X_j(z,t) = X_j(x,y,t) = \partial_{x_j} + 2y_j\partial_t \\ Y_j(z,t) = Y_j(x,y,t) = \partial_{y_j} - 2x_j\partial_t \end{cases}$$
$j = 1, ..., n$ which satisfy Hörmander's condition. The only non trivial commutators are the $[X_j, Y_j] = -4\partial_t$, $j = 1, ..., n$ and the measure is nothing but the Lebesgue measure $\mathcal{L}^n$. $\mathbb{H}^n$ is a Carnot group and then W_p^1 is an interpolation space between $W_{p_1}^1$ and $W_{p_2}^1$ for all $1 \leq p_1 < p < p_2 \leq \infty$.

8. Let $X = \mathbb{R}^2$ be equipped with the Euclidean metric and let μ be the measure generated by the density $d\mu(x) = |x_2|^t dx$, $t > 0$, where x_2 denotes the second coordinate of x. We have the doubling property (D), and a Poincaré inequality (P_q) in the sense of metric-meausre spaces if and only if $q > t + 1$ (see [27] p. 17).

9. Let $X = \left\{(x_1, x_2, ..., x_n) \in \mathbb{R}^n;\ x_1^2 + ... + x_{n-1}^2 \leq x_n^2\right\}$ be equipped with the Euclidean metric of $\mathbb{R}^n$ and with the Lebesgue measure. X consists of two infinite closed cones with a common vertex. X satisfies the doubling property and admits (P_q) in the sense of metric-measure spaces if and only if $q > n$ ([27] p. 17). Denote by Ω the interior of X. Let $H_p^1 := H_p^1(X)$ be the closure of $\mathrm{Lip}_0(X)$ for the norm
$$\|f\|_{H_p^1} = \|f\|_{L_p(\Omega)} + \|\,|\nabla f|\,\|_{L_p(\Omega)}.$$
We define $W_p^1(\Omega)$ as the set of all functions $f \in L_p(\Omega)$ such that $\nabla f \in L_p(\Omega)$ and equip this space with the norm
$$\|f\|_{W_p^1(\Omega} = \|f\|_{H_p^1}.$$
We have seen that the H_p^1 interpolate for $n < p < \infty$. It can be shown that $H_p^1 \subsetneq W_p^1(\Omega)$ for $p > n$ and $H_p^1 = W_p^1(\Omega)$ for $1 \leq p < n$. Hence the H_p^1 interpolate for $1 \leq p < n$ although the Poincaré inequality does not hold on X for this values of p. In this way, we gave a negative answer to the question asked in the end of the introduction. However, we ignore if we can interpolate the H_p^1 for all $1 \leq p < \infty$ (see the next chapter for more details).

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Chapitre 4

Espaces de Sobolev sur le cône Euclidien et lien avec l'interpolation

4.1 Différents espaces de Sobolev

Le but de ce court chapitre est d'étudier un exemple géométrique où il faut faire attention à la façon de définir l'espace de Sobolev. Soit

$$X = \left\{(x_1, x_2, ..., x_n) \in \mathbb{R}^n; x_1^2 + ... + x_{n-1}^2 \leq x_n^2\right\}$$

muni de la distance Euclidienne de $\mathbb{R}^n$ et la mesure de Lebesgue. X consiste en deux cônes fermés infinis avec un point commun. X satisfait la propriété de doublement et admet une inégalité de Poincaré (P_q) au sens des espaces métriques mesurés si et seulement si $q > n$ ([1] p.17). Désignons par Ω l'intérieur de X, Ω_1 le cône supérieur de Ω et Ω_2 le cône inférieur de Ω. Considérons sur X les espaces de Sobolev suivants:

1. $A_p = \left\{F|_\Omega;\ F \in W_p^1(\mathbb{R}^n)\right\} = \left\{f \in W_p^1(\Omega);\ \exists F \in W_p^1(\mathbb{R}^n);\ F|_\Omega = f\right\}$ muni de la norme
$$\|f\|_{A_p} = \inf\left\{\|F\|_{W_p^1(\mathbb{R}^n)};\ F|_\Omega = f\right\}.$$

2. $B_p = \overline{C_0^\infty(\mathbb{R}^n)}$ pour la norme
$$\|f\|_{B_p} = \|f\|_{L_p(\Omega)} + \|\,|\nabla f|\,\|_{L_p(\Omega)}.$$

3. $C_p = \overline{\mathrm{Lip}_0(X)}$ pour la norme $\|.\|_{B_p}$.

4. $D_p = W_p^1(\Omega_1) \oplus W_p^1(\Omega_2)$, muni de la norme $\|f\|_{D_p} = \|f_1\|_{B_p(\Omega_1)} + \|f_2\|_{B_p(\Omega_2)}$.

5. $E_p = W_p^1(\Omega)$ muni de la norme $\|f\|_{W_p^1(\Omega)} = \|f\|_{B_p}$.

Dans les points 4. et 5., le gradient est calculé au sens des distributions sur l'ouvert considéré.
On a des égalités et inclusions évidentes entre ces espaces. On voit immédiatement que pour tout $1 \leq p < \infty$, $A_p \subset B_p \subset C_p \subset D_p = E_p$. Nous allons étudier les inclusions réciproques pour $p < n$ et $p > n$. Ces énoncés sont

Proposition 4.1.1. *Ces espaces sont tous égaux pour* $1 \leq p < n$.

Proposition 4.1.2. *On a* $A_p = B_p = C_p \subsetneq E_p$ *pour* $n < p < \infty$.

4.2 Comparaison des espaces de Sobolev

Pour la facilité des démonstrations, nous prendrons $n = 2$ et par une rotation d'angle $-\frac{\pi}{4}$ on peut considérer que $X = \{(x,y) \in \mathbb{R}^2;\ (x,y) \in (\mathbb{R}_+)^2 \cup (\mathbb{R}_-)^2\}$, $\Omega = (\mathbb{R}_+^*)^2 \cup (\mathbb{R}_-^*)^2$, $\Omega_1 = (\mathbb{R}_+^*)^2$, $\Omega_2 = (\mathbb{R}_-^*)^2$.

Preuve de la Proposition 4.1.1. Nous commençons par démontrer que $E_p \subset B_p$ car nous aurons besoin du fait que $C_p = E_p$ dans la preuve de $E_p \subset A_p$. Nous avons besoin du résultat de densité suivant:

Proposition 4.2.1. *Pour* $1 \leq p < n$, *les fonctions* $f \in E_p$ *qui ont un support ne rencontrant pas une boule centrée en* 0 *sont denses dans* E_p.

Preuve de la Proposition 4.2.1. Soit $f \in E_p$. On écrit $f = f_1 + f_2$ où $f_i = f|_{\Omega_i}$ et on s'intéresse à chaque f_i. On peut donc supposer $f = f_1$. Soit η une fonction C^∞ sur $\mathbb{R}$, $0 \leq \eta \leq 2$, $\eta = 1$ sur $[2, \infty[$ et 0 sur $]-\infty, 1]$. On prend pour $\lambda = (x,y)$ avec $x,\ y \in \mathbb{R}$, $f_k(\lambda) = f(\lambda)\eta(k(x+y))$, $k \in \mathbb{N}^*$. Pour tout k, le support de f_k ne rencontre pas une boule centrée en 0. Ensuite f_k converge vers f pour la norme $L_p(\Omega_1)$. En effet $|f - f_k| \leq g_k|f|$ où g_k est l'indicatrice de $B_k = \{\lambda \in \Omega_1; k(x+y) \leq 2\}$ et on applique le théorème de convergence dominée de Lebesgue. Regardons maintenant les dérivées. On a

$$\partial_x(f - f_k) = \partial_x f \eta(k(x+y)) + k\eta'(k(x+y))f$$

et

$$\partial_y(f - f_k) = \partial_y f \eta(k(x+y)) + k\eta'(k(x+y))f.$$

Le premier terme dans les deux dérivations s'analyse comme avant. Pour le deuxième terme on utilise l'inégalité de Sobolev: on a $\|f\|_{L_{p^*}} < \infty$ où $p^* = \frac{2p}{2-p}$ (en fait, on peut prolonger f dans $W_p^1(\mathbb{R}^2)$ et le prolongement vérifie l'inégalité de Sobolev car $p < 2$ et Ω_1 est un domaine d'extension pour W_p^1). Par l'inégalité de Hölder on a

$$\begin{aligned}\left(\int_{B_k} |k\eta'(k(x+y))f(x)|^p dX\right)^{\frac{1}{p}} &\leq \left(\int_{B_k} |f|^{p*} dX\right)^{\frac{1}{p^*}} \left(\int_{B_k} k^2|\eta'(k(x+y))|^2 dX\right)^{\frac{1}{2}} \\ &\leq C\left(\int_{B_k} |f|^{p*} dX\right)^{\frac{1}{p^*}}.\end{aligned}$$

Comme le terme de droite tend vers 0 par convergence dominée, nous avons bien vérifié que $\nabla(f - f_k)$ tend vers 0 dans L_p. □

Revenons maintenant à l'inclusion $E_p \subset B_p$. Soit $f \in E_p$. Il existe une suite $(f_n) \in E_p$ telle que pour tout n, f_n a un support ne rencontrant pas une boule centrée en 0. Chaque f_n se prolonge donc en une fonction $h_n \in W_p^1(\mathbb{R}^2)$ dont le support possède la même propriété. Ensuite par convolution et troncature on approche h_n dans $W_p^1(\mathbb{R}^2)$ par une fonction $g_n \in C_0^\infty(\mathbb{R}^2)$. Ainsi $\|f - g_n\|_{W_p^1(\Omega)} \to 0$ et on obtient $f \in B_p$. □

Il reste à démontrer que $E_p \subset A_p$ (L'idée de l'argument est due à Michel Pierre que nous remercions). Soit $u \in E_p$. On écrit $r^2 = x^2 + y^2$ et

$$u = \frac{x^2}{r^2}u + \frac{y^2}{r^2}u = u_1 + u_2.$$

Nous allons montrer que $u_1 \in A_p$ et le même raisonnement s'applique à u_2. On a $u_1 \in L_p(\Omega)$ et il vient

$$|\nabla u_1| \leq \frac{x^2}{r^2}|\nabla u| + |\nabla(\frac{x^2}{r^2})|\,|u| \leq |\nabla u| + \frac{C}{r}|u|.$$

Comme $u \in E_p$, $|\nabla u| \in L_p(\Omega)$. Il reste à vérifier que $\frac{C}{r}|u| \in L_p(\Omega)$. Pour cela on démontre l'inégalité suivante:

$$\iint_\Omega |\frac{u}{r}|^p dxdy \leq \left(\frac{p}{2-p}\right)^p \iint_\Omega |\nabla u|^p dxdy \quad \forall u \in E_p. \tag{4.1}$$

Soit $u \in \mathrm{Lip}_0(X)$. On a

$$\begin{aligned}
\iint_{\Omega_1} |\frac{u}{r}|^p d\lambda &= \int_0^{\frac{\pi}{2}} \int_0^\infty r^{-p}|u(r,\theta)|^p r dr d\theta \\
&= \int_0^{\frac{\pi}{2}} \int_0^\infty r^{1-p}|u(r,\theta)|^p dr d\theta \\
&= \int_0^{\frac{\pi}{2}} \left[\frac{1}{2-p} r^{2-p}|u(r,\theta)|^p\right]_0^\infty d\theta - \int_0^{\frac{\pi}{2}} \int_0^\infty \frac{1}{2-p} r^{2-p} p|u|^{p-1}\, sign\, u\, \frac{\partial u}{\partial r} d\theta dr \\
&= -\frac{p}{2-p} \int_0^{\frac{\pi}{2}} \int_0^\infty |\frac{u}{r}|^{p-1}\, sign\, u\, \frac{\partial u}{\partial r} r dr d\theta \\
&\leq \frac{p}{2-p} \left(\int_0^{\frac{\pi}{2}} \int_0^\infty |\frac{u}{r}|^p r dr d\theta\right)^{\frac{p-1}{p}} \left(\int_0^{\frac{\pi}{2}} \int_0^\infty |\frac{\partial u}{\partial r}|^p r dr d\theta\right)^{\frac{1}{p}}.
\end{aligned}$$

Après simplification, on obtient (4.1) sur Ω_1. On fait de même pour l'intégrale sur Ω_2 et donc (4.1) est valable pour tout $u \in \mathrm{Lip}_0(X)$. Par densité on a (4.1) pour tout $u \in C_p$, par suite pour tout $u \in E_p$ car on a déjà démontré $B_p = C_p = E_p$. Donc $u_1 \in E_p$, de même pour u_2.
On considère maintenant la fonction

$$v_1(x,y) = \begin{cases} u_1(x,y) & \text{si } (x,y) \in \Omega, \\ u_1(x,-y) & \text{si } (x,y) \in X^c. \end{cases}$$

On a évidemment que v_1 est définie p.p. sur $\mathbb{R}^2$ et $v_1 \in L_p(\mathbb{R}^2)$. Puisque v_1 est paire en y, on a en scindant l'argument sur Ω_1 et sur Ω_2 que $v_1 \in W_p^1(\mathbb{R}_+^* \times \mathbb{R}) \oplus W_p^1(\mathbb{R}_-^* \times \mathbb{R})$. Ensuite comme u_1 est nulle si $x = 0$ et $y \neq 0$, on montre sans difficultés que la trace de v_1 sur $\{0\} \times \mathbb{R}$ est nulle ce qui entraîne que ∇v_1 est dans $\mathcal{D}'(\mathbb{R}^2)$ et coïncide avec $(\nabla v_1)\mathbb{1}_{x>0} + (\nabla v_1)\mathbb{1}_{x<0}$. Donc $v_1 \in W_p^1(\mathbb{R}^2)$, $v_1|_\Omega = u_1$ et $\|v_1\|_{W_p^1(\mathbb{R}^2)} \leq C\|u\|_{W_p^1(\Omega)}$. De même, il existe $v_2 \in W_p^1(\mathbb{R}^2)$ tel que $v_2|_\Omega = u_2$. D'où en posant $v = v_1 + v_2$ on a $u = v|_\Omega$ avec $v \in W_p^1(\mathbb{R}^2)$. Par suite $E_p \subset A_p$. □

Preuve de la Proposition 4.1.2. En fait on a déjà $A_p \subset B_p \subset C_p$. Et d'après les inclusions de Sobolev, on a $C_p \subset C(X)$, l'espace des fonctions continues et bornées sur X, avec inclusion continue. D'où $C_p \subsetneq E_p$ car une fonction de E_p n'est pas nécessairement continue en 0.

Il reste à démontrer que $C_p \subset A_p$. Ceci revient à démontrer que toute fonction de C_p se prolonge en une fonction de $W_p^1(\mathbb{R}^2)$ avec contrôle de la norme. Soit $u \in C_p$. Soit $\chi \in C_0^\infty(\mathbb{R}^2)$ supportée par la boule unité avec $\chi \equiv 1$ au voisinage de 0. On écrit la décomposition suivante dans C_p:

$$u = (u - u(0))\chi + u(0)\chi + u(1-\chi) = u_0 + u_1 + u_2.$$

On pose $v_1(\lambda) = u(0)\chi(\lambda)$ pour $\lambda \in \mathbb{R}^2$. Alors $v_1 \in W_p^1(\mathbb{R}^2)$, $v_1|_\Omega = u_1$ et $\|v_1\|_{W_p^1(\mathbb{R}^2)} \leq C\|u\|_{B_p}$ car $|u(0)| \leq C\|u\|_{B_p}$ si $p > 2$. Ensuite comme u_2 est nulle au voisinage de 0, il n'est pas difficile de la prolonger dans $W_p^1(\mathbb{R}^2)$. Il reste à prolonger u_0. Pour cela, on reprend la méthode de démonstration de $E_p \subset A_p$ de la Proposition 4.1.1. Il suffit de montrer

$$\iint_\Omega |\frac{u_0}{r}|^p d\lambda \leq C\|u_0\|_{B_p}^p \tag{4.2}$$

car on remarque que $\|u_0\|_{B_p} \leq C\|u\|_{B_p}$. Notons $A = \int\int_{\Omega_1 \cap |\lambda| > \varepsilon} |\frac{u_0}{r}|^p d\lambda$, où $\varepsilon > 0$. D'après le théorème de Morrey, on a $|u_0(\lambda)| \leq C\|u_0\|_{B_p}|\lambda|^\alpha$ avec $\alpha = 1 - \frac{2}{p}$ pour tout $\lambda \in \Omega$. En reprenant le calcul de (4.1) et puisque u_0 a un support compact, on obtient pour tout $\varepsilon > 0$ suffisamment petit

$$\begin{aligned}
A &= \int_0^{\frac{\pi}{2}} \int_\varepsilon^\infty r^{1-p}|u_0(r,\theta)|^p dr d\theta \\
&= \int_0^{\frac{\pi}{2}} \left[\frac{1}{2-p} r^{2-p}|u_0(r,\theta)|^p\right]_\varepsilon^\infty d\theta - \int_0^{\frac{\pi}{2}} \int_\varepsilon^\infty \frac{1}{2-p} r^{2-p} p|u_0|^{p-1}\, sign\, u_0 \,\frac{\partial u_0}{\partial r} d\theta dr \\
&= \frac{\pi}{2}\frac{1}{p-2}\varepsilon^{2-p}|u_0(\varepsilon,\theta)|^p + \frac{p}{p-2}\int_0^{\frac{\pi}{2}} \int_\varepsilon^\infty |\frac{u_0}{r}|^{p-1}\, sign\, u_0 \,\frac{\partial u_0}{\partial r} r dr d\theta \\
&\leq C^p \frac{\pi}{2}\|u_0\|_{B_p}^p + \frac{p}{p-2}\left(\int_0^{\frac{\pi}{2}} \int_\varepsilon^\infty |\frac{u_0}{r}|^p r dr d\theta\right)^{\frac{p-1}{p}} \left(\int_0^{\frac{\pi}{2}} \int_\varepsilon^\infty |\frac{\partial u_0}{\partial r}|^p r dr d\theta\right)^{\frac{1}{p}}
\end{aligned}$$

On a donc

$$A \leq C^p \frac{\pi}{2}\|u_0\|_{B_p}^p + \frac{p}{p-2} A^{\frac{p-1}{p}}\|u_0\|_{B_p}. \tag{4.3}$$

Or $A^{\frac{p-1}{p}}\|u_0\|_{B_p} \leq \delta A^{\frac{p-1}{p}\alpha}+\frac{1}{\delta}\|u_0\|_{B_p}^{\beta}$ pour tout $\delta > 0$ et α, β positifs tels que $\frac{1}{\alpha}+\frac{1}{\beta} = 1$. En prenant $\beta = p$ et $\alpha = \frac{p}{p-1}$, on a donc

$$A^{\frac{p-1}{p}}\|u_0\|_{B_p} \leq \delta A + \frac{1}{\delta}\|u_0\|_{B_p}^p. \tag{4.4}$$

Comme $A < \infty$, en incorporant (4.4) dans (4.3) on obtient

$$A(1 - \frac{p}{p-2}\delta) \leq (C^p\frac{\pi}{2} + \frac{p}{(p-2)\delta})\|u_0\|_{B_p}^p.$$

En choisissant $\delta < \frac{p-2}{p}$, on déduit que

$$A = \int\int_{\Omega_1\cap|X|>\varepsilon} |\frac{u_0}{r}|^p dX \leq C\|u_0\|_{B_p}^p.$$

En faisant tendre ε vers 0 et en faisant de même sur Ω_2, on obtient (4.2). □

4.3 Lien avec l'interpolation

Les E_p s'interpolent pour $1 \leq p \leq \infty$. Soit $f \in E_1 + E_\infty$. On écrit $f = f|_{\Omega_1} + f|_{\Omega_2} = f_1 + f_2$. On a

$$\begin{aligned} K(f,t,E_1,E_\infty) &\leq K(f_1,t,E_1,E_\infty) + K(f_2,t,E_1,E_\infty) \\ &= K(f_1,t,W_1^1(\Omega),W_\infty^1(\Omega)) + K(f_2,t,W_1^1(\Omega),W_\infty^1(\Omega)) \\ &\leq K(f_1,t,W_1^1(\Omega_1),W_\infty^1(\Omega_1)) + K(f_2,t,W_1^1(\Omega_2),W_\infty^1(\Omega_2)) \quad (4.5) \\ &\leq Ct\left(f_1^{**}(t) + |\nabla f_1|^{**}(t) + f_2^{**}(t) + |\nabla f_2|^{**}(t)\right). \quad (4.6) \end{aligned}$$

On a utilisé dans (4.5) les extensions suivantes continues pour $1 \leq p \leq \infty$: $j_1 : W_p^1(\Omega_1) \to W_p^1(\Omega)$ tel que $j_1(f) = g$ avec $g = f$ sur Ω_1 et 0 sur Ω_2 et $j_2 : W_p^1(\Omega_2) \to W_p^1(\Omega)$ tel que $j_2(f) = g$ avec $g = f$ sur Ω_1 et 0 sur Ω_2. Et dans (4.6) on a utilisé la caractérisation de la fonctionnelle K du chapitre 3 pour un espace vérifiant (D) et (P_1) .

Or $\{x \in \Omega_1; |f(x)| > \lambda\} \subset \{x \in \Omega; |f(x)| > \lambda\}$, donc $f_1^*(t) \leq f^*(t)$, de même pour f_2^*. De la même façon on a, $|\nabla f_1|^{**}(t) \leq |\nabla f|^{**}(t)$ et $|\nabla f_2|^{**}(t) \leq |\nabla f|^{**}(t)$. D'où $K(f,t,E_1,E_\infty) \leq 2Ct\left(f^{**}(t) + |\nabla f|^{**}(t)\right)$. L'estimation inférieure étant évidente, on obtient

$$K(f,t,E_1,E_\infty) \sim t\left(f^{**}(t) + |\nabla f|^{**}(t)\right).$$

Comme on a vu au chapitre 3, cette caractérisation de K va nous permettre de conclure que E_p est un espace d'interpolation entre E_1 et E_∞ pour tout $1 < p < \infty$.

Toutefois, les espaces que nous considérons au chapitre 3 sont les C_p (notés H_p^1). On a vu qu'ils s'interpolent pour $n < p \leq \infty$. Puisque $C_p = E_p$ pour $1 \leq p < n$, ils s'interpolent aussi pour $1 \leq p < n$ bien que (P_q) ne soit pas vraie pour $q < n$. On en déduit que l'hypothèse portant sur l'inégalité de Poincaré n'est pas nécessaire pour établir certaines propriétés d'interpolation entre les espaces de Sobolev.

Pour terminer nous ne savons pas déterminer les espaces d'interpolation réels entre C_{p_0} et C_{p_1} avec $1 \leq p_0 < n < p_1 < \infty$. Le fait que $C_{p_i} = A_{p_i}$ ne nous aide pas. Si nous savions trouver un opérateur de prolongement qui est le même dans les deux cas, nous pourrions interpoler car on sait interpoler sur $\mathbb{R}^n$. Le problème est que nous ne savons pas si un tel opérateur existe.

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Chapitre 5

Real Interpolation of Sobolev spaces related to a weight

Abstract. We study the interpolation property of Sobolev spaces of order 1 denoted by $W^1_{p,V}$, arising from Schrödinger operators with positive potential. We show that they form a real interpolation scale for $s_0 < p < q_0$ on some classes of manifolds and Lie groups, where s_0, q_0 depend on our hypotheses.

Résumé. On étudie l'interpolation des espaces de Sobolev d'ordre 1 notés par $W^1_{p,V}$ provenant des opérateurs de Schrödinger avec poids positif. On démontre qu'ils forment, sur certaines variétés Riemanniennes et groupes de Lie, une échelle d'interpolation réelle pour $s_0 < p < q_0$, où s_0, q_0 dépendent de nos hypothèses.

Dans tout ce chapitre, la référence [6] désigne le chapitre 3.

5.1 Introduction

In [2], the Schrödinger operator $-\Delta + V$ on $\mathbb{R}^n$ with $V \in A_\infty$, the Muckenhoupt class (see [14]), is studied and the question whether the spaces defined by the norm $\|f\|_p + \| \, |\nabla f| \, \|_p + \|V^{\frac{1}{2}} f\|_p$ or $(\| \, |\nabla f| \, \|_p + \|V^{\frac{1}{2}} f\|_p)$ interpolate is posed. In fact, it is shown that:

$$\| \, |\nabla f| \, \|_p + \|V^{\frac{1}{2}} f\|_p \sim \|(-\Delta + V)^{\frac{1}{2}} f\|_p$$

whenever $1 < p < \infty$ and $p \leq 2q$, $f \in C_0^\infty(\mathbb{R}^n)$, where $q > 1$ is a Reverse Hölder exponent of V. Hence the question of interpolation can be solved a posteriori using functional calculus and interpolation of L_p spaces. However, it is reasonable to expect a direct proof.

Here we provide such an argument with p lying in an interval depending on the Reverse Hölder exponent of V by estimating the K-functional of interpolation. The particular case $V = 1$ is treated in [6] (also $V = 0$). The method is actually valid on some Lie groups and even some Riemannian manifolds in which we place ourselves.

Let us come to statements:

Definition 5.1.1. *Let M be a Riemannian manifold, $V \in A_\infty$. Consider for $1 \leq p < \infty$, the vector space $E^1_{p,V}$ of C^∞ functions f on M such that f, $|\nabla f|$ and $Vf \in L_p(M)$. We define the Sobolev space $W^1_{p,V}(M) = W^1_{p,V}$ as the completion of $E^1_{p,V}$ for the norm*

$$\|f\|_{W^1_{p,V}} = \|f\|_p + \| \, |\nabla f| \, \|_p + \|Vf\|_p.$$

Definition 5.1.2. *We denote by $W^1_{\infty,V}(M) = W^1_{\infty,V}$ the space of all bounded Lipschitz functions f on M with $\|Vf\|_\infty < \infty$.*

We have the following interpolation theorem for the non homogeneous Sobolev spaces $W^1_{p,V}$:

Theorem 5.1.3. *Let M be a complete Riemannian manifold satisfying a local doubling property (D_{loc}). Let $V \in RH_{qloc}$ for some $1 < q \leq \infty$. Assume that M admits a local Poincaré inequality (P_{sloc}) for some $1 \leq s < q$. Then for $s < p < q$, $W^1_{p,V}$ is an interpolation space between $W^1_{s,V}$ and $W^1_{q,V}$ (see below for definitions).*

Definition 5.1.4. *Let M be a Riemannian manifold, $V \in A_\infty$. Consider for $1 \leq p < \infty$, the vector space $\dot{W}^1_{p,V}$ of distributions f such that $|\nabla f|$ and $Vf \in L_p(M)$. It is well known that the elements of $\dot{W}^1_{p,V}$ are in $L_{p,loc}$. We equip $\dot{W}^1_{p,V}$ with the semi norm*

$$\|f\|_{\dot{W}^1_{p,V}} = \| \, |\nabla f| \, \|_p + \|Vf\|_p.$$

In fact, this expression is a norm since $V \in A_\infty$ yields $V > 0$ $\mu - a.e.$.

Definition 5.1.5. *We denote $\dot{W}^1_{\infty,V}(M) = \dot{W}^1_{\infty,V}$ the space of all Lipschitz functions f on M with $\|Vf\|_\infty < \infty$.*

For the homogeneous Sobolev spaces $\dot{W}^1_{p,V}$, we have

Theorem 5.1.6. *Let M be a complete Riemannian manifold satisfying (D). Let $V \in RH_q$ for some $1 < q \leq \infty$ and assume that M admits a Poincaré inequality (P_s) for some $1 \leq s < q$. Then, for $s < p < q$, $\dot{W}^1_{p,V}$ is an interpolation space between $\dot{W}^1_{s,V}$ and $\dot{W}^1_{q,V}$.*

It is known that if $V \in RH_q$ then $V + 1 \in RH_q$ with comparable constants. Hence part of Theorem 5.1.3 can be seen as a corollary of Theorem 5.1.6. But the fact that $V + 1$ is bounded away from 0 also allows local assumptions in Theorem 5.1.3, which is why we distinguish in this way the non homogeneous and the homogeneous case.

The proof of Theorem 5.1.3 and Theorem 5.1.6 is done by estimating the K-functional of interpolation. We were not able to obtain a characterization of the K-functional. However, this suffices for our needs. When $q = \infty$ (for example V is a positive polynomial on $\mathbb{R}^n$), then there is a characterization. The key tools to estimate the K-functional will be a Calderón-Zygmund decomposition for Sobolev functions and the Fefferman-Phong inequality (see section 5.3).

The paper is organized as follows. In section 5.2, we review the notions of Poincaré inequality, Reverse Hölder classes and summarize some properties for the Sobolev spaces defined above under some additional hypotheses on M and V. After proving the Fefferman-Phong inequality and the Calderón-Zygmund decomposition tools in section 5.3, we estimate in section 5.4 the K-functional of interpolation for non homogeneous Sobolev spaces in two steps: first of all for the global case and secondly for the local case. We interpolate and get Theorem 5.1.3 in section 5.5. Section 5.6 is devoted to prove Theorem 5.1.6. Finally, in section 5.7, we extend our interpolation result to the case of Lie groups with an appropriate definition of $W^1_{p,V}$.

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5.2 Preliminaries

Throughout this paper we write $1\!\!1_E$ for the characteristic function of a set E and E^c for the complement of E. For a ball B in a metric space, λB denotes the ball co-centered with B and with radius λ times that of B. Finally, C will be a constant that may change from an inequality to another and we will use $u \sim v$ to say that there exist two constants C_1,$C_2 > 0$ such that $C_1 u \leq v \leq C_2 u$. Let M denotes a complete non-compact Riemannian manifold. We write μ for the Riemannian measure on M, ∇ for the Riemannian gradient, $|\cdot|$ for the length on the tangent space (forgetting the subscript x for simplicity) and $\|\cdot\|_p$ for the norm on $L_p(M,\mu)$, $1 \leq p \leq +\infty$.

5.2.1 The doubling property and Poincaré inequality

Definition 5.2.1. *Let (M, d, μ) be a Riemannian manifold. Denote by $B(x, r)$ the open ball of center $x \in M$ and radius $r > 0$ and by $\mu(B(x, r))$ its measure. One says that M satisfies the local doubling property (D_{loc}) if there exist constants $r_0 > 0$, $0 < C = C(r_0) < \infty$, such that for all $x \in M$, $0 < r < r_0$ we have*

$$(D_{loc}) \qquad \mu(B(x, 2r)) \leq C\mu(B(x, r)).$$

And M satisfies a global doubling property or simply doubling property (D) if one can take $r_0 = \infty$. We also say that μ is a locally (resp. globally) doubling Borel measure.

Observe that if M satisfies (D) then

$$diam(M) < \infty \Leftrightarrow \mu(M) < \infty \text{ ([1])}.$$

Theorem 5.2.2 (Maximal theorem). *([10]) Let M be a Riemannian manifold satisfying (D). Denote by $\mathcal{M}$ the uncentered Hardy-Littlewood maximal function over open balls of M defined by*

$$\mathcal{M}f(x) = \sup_{B: x\in B} |f|_B$$

where $f_E := \fint_E f d\mu := \dfrac{1}{\mu(E)} \displaystyle\int_E f d\mu$. *Then*

1. $\mu(\{x : \mathcal{M}f(x) > \lambda\}) \leq \frac{C}{\lambda} \int_X |f| d\mu$ *for every* $\lambda > 0$*;*
2. $\|\mathcal{M}f\|_p \leq C_p \|f\|_p$, *for* $1 < p \leq \infty$.

5.2.2 Poincaré inequality

Definition 5.2.3 (Poincaré inequality on M)**.** *Let M be a complete Riemannian manifold, $1 \leq s < \infty$. We say that M admits* ***a local Poincaré inequality*** (P_{sloc}) *if there exist constants $r_1 > 0$, $C = C(r_1) > 0$ such that, for every function $f \in C_0^\infty$, and every ball B of M of radius $0 < r < r_1$, we have*

$$(P_{sloc}) \qquad \fint_B |f - f_B|^s d\mu \leq C r^s \fint_B |\nabla f|^s d\mu.$$

M admits a global Poincaré inequality (P_s) if we can take $r_1 = \infty$ in this definition.

Remark 5.2.4. *By density of C_0^∞ in W_s^1, if (P_{sloc}) holds for every function $f \in C_0^\infty$, then it holds for every $f \in W_s^1$.*

Let us recall some known facts about Poincaré inequality with varying q. It is known that (P_{qloc}) implies (P_{ploc}) when $p \geq q$ (see [17]). Thus, if the set of q such that (P_{qloc}) holds is not empty, then it is an interval unbounded on the right. A recent result for Keith and Zhong [20] asserts that this interval is open in $[1, +\infty[$ in the following sense:

Theorem 5.2.5. *Let (X, d, μ) be a complete metric-measure space with μ locally doubling and admitting a local Poincaré inequality (P_{qloc}), for some $1 < q < \infty$. Then there exists $\varepsilon > 0$ such that (X, d, μ) admits (P_{ploc}) for every $p > q - \varepsilon$ (see [20] and section 3.4 of chapter 3: section 4 in [6]).*

5.2.3 Reverse Hölder classes

Definition 5.2.6. *Let M be a Riemannian manifold. A weight w is a non-negative locally integrable function on M. The reverse Hölder classes are defined in the following way: $w \in RH_q$, $1 < q < \infty$, if there exists a constant C such that for every ball $B \subset M$*

$$\left(\fint_B w^q d\mu \right)^{\frac{1}{q}} \leq C \fint_B w d\mu. \tag{5.1}$$

The endpoint $q = \infty$ is given by the condition: $w \in RH_\infty$ whenever, for any ball B,

$$w(x) \leq C \fint_B w \quad \text{for } \mu - a.e.\ x \in B. \tag{5.2}$$

We say that $w \in RH_{qloc}$ for some $1 < q < \infty$ (resp. $q = \infty$) if there exists $r_2 > 0$ such that (5.1) (resp. (5.2)) holds for all balls B of radius $0 < r < r_2$.
The smallest C is called the RH_q (resp. RH_{qloc}) constant of w.

Proposition 5.2.7.

1. *$RH_\infty \subset RH_q \subset RH_p$ for $1 < p \leq q \leq \infty$.*
2. *If $w \in RH_q$, $1 < q < \infty$, then there exists $q < p < \infty$ such that $w \in RH_p$.*

3. $A_\infty = \bigcup_{1<q\leq\infty} RH_q$.

Proof. These properties are standard, see for instance [14]. □

Proposition 5.2.8. *(see section 11 in [2], [19]) Let V be a non-negative measurable function. Then the following properties are equivalent:*

1. $V \in A_\infty$.

2. For all $r \in]0,1[$, $V^r \in RH_{\frac{1}{r}}$.

3. There exists $r \in]0,1[$, $V^r \in RH_{\frac{1}{r}}$.

Remark 5.2.9. *Propositions 5.2.7 and 5.2.8 still hold in the local case, that is, when the weights are considered in a local reverse Hölder class RH_{qloc} for some $1 < q \leq \infty$.*

5.2.4 The K method of real interpolation

The reader is referred to [7], [8] for details on the development of this theory. Here we only recall the essentials to be used in the sequel.

Let A_0, A_1 be two normed vector spaces embedded in a topological Hausdorff vector space V, and define for $a \in A_0 + A_1$ and $t > 0$,

$$K(a, t, A_0, A_1) = \inf_{a=a_0+a_1} (\|a_0\|_{A_0} + t\|a_1\|_{A_1}).$$

For $0 < \theta < 1$, $1 \leq q \leq \infty$, we denote by $(A_0, A_1)_{\theta,q}$ the interpolation space between A_0 and A_1:

$$(A_0, A_1)_{\theta,q} = \left\{ a \in A_0 + A_1 : \|a\|_{\theta,q} = \left(\int_0^\infty (t^{-\theta} K(a, t, A_0, A_1))^q \frac{dt}{t} \right)^{\frac{1}{q}} < \infty \right\}.$$

It is an exact interpolation space of exponent θ between A_0 and A_1, see [8] Chapter II.

Definition 5.2.10. *Let f be a measurable function on a measure space (X, μ). We denote by f^* its decreasing rearrangement function: for every $t > 0$,*

$$f^*(t) = \inf \{\lambda : \mu(\{x : |f(x)| > \lambda\}) \leq t\}.$$

We denote by f^{**} the maximal decreasing rearrangement of f: for every $t > 0$,

$$f^{**}(t) = \frac{1}{t} \int_0^t f^*(s) ds.$$

It is known that $(\mathcal{M}f)^* \sim f^{**}$ and $\mu(\{x : |f(x)| > f^*(t)\}) \leq t$ for all $t > 0$. We refer to [7], [8], [9] for other properties of f^* and f^{**}.

To end with this subsection let us quote the following theorem ([18]):

Theorem 5.2.11. *Let (X, μ) be a measure space where μ is a non-atomic positive measure. Take $0 < p_0 < p_1 < \infty$. Then*

$$K(f, t, L_{p_0}, L_{p_1}) \sim \left(\int_0^{t^\alpha} (f^*(u))^{p_0} du\right)^{\frac{1}{p_0}} + t\left(\int_{t^\alpha}^\infty (f^*(u))^{p_1} du\right)^{\frac{1}{p_1}},$$

where $\frac{1}{\alpha} = \frac{1}{p_0} - \frac{1}{p_1}$.

5.2.5 Sobolev spaces associated to a weight V

For the definition of the non homogeneous Sobolev spaces $W^1_{p,V}$ and the homogeneous one $\dot{W}^1_{p,V}$ see the introduction. We begin by showing that $W^1_{\infty,V}$ and $\dot{W}^1_{p,V}$ are Banach spaces.

Proposition 5.2.12. *$W^1_{\infty,V}$ equipped with the norm*

$$\|f\|_{W^1_{\infty,V}} = \|f\|_\infty + \|\,|\nabla f|\,\|_\infty + \|Vf\|_\infty$$

is a Banach space.

Proof. Let $(f_n)_n$ be a Cauchy sequence in $W^1_{\infty,V}$. Then it is a Cauchy sequence in W^1_∞ and converges to f in W^1_∞. Hence $Vf_n \to Vf$ $\mu - a.e.$. On the other hand, $Vf_n \to g$ in L_∞, then $\mu - a.e.$ The unicity of the limit gives us $g = Vf$. □

Proposition 5.2.13. *Assume that M satisfies (D) and admits a Poincaré inequality (P_s) for some $1 \leq s < \infty$ and that $V \in A_\infty$. Then, for $s \leq p \leq \infty$, $\dot{W}^1_{p,V}$ equipped with the norm*

$$\|f\|_{\dot{W}^1_{p,V}} = \|\,|\nabla f|\,\|_p + \|Vf\|_p$$

is a Banach space.

Proof. Let $(f_n)_n$ be a Cauchy sequence in $\dot{W}^1_{p,V}$. There exist a sequence of functions $(g_n)_n$ and a sequence of scalar $(c_n)_n$ with $g_n = f_n - c_n$ converging to a function g in $L_{p,loc}$ and ∇g_n converging to ∇g in L_p (see [15]). Moreover, since $(Vf_n)_n$ is a Cauchy sequence in L_p, it converges to a function h $\mu - a.e.$. Lemma 5.3.1 in Section 5.3 below yields

$$\int_B (|\nabla(f_n - f_m)|^s + |V(f_n - f_m)|^s)\, d\mu \geq C(B, V) \int_B |f_n - f_m|^s d\mu$$

for all ball B of M. Thus, $(f_n)_n$ is a Cauchy sequence in $L_{s,loc}$. Since $(f_n - c_n)$ is also Cauchy in $L_{s,loc}$, the sequence of constants $(c_n)_n$ is Cauchy in $L_{s,loc}$ and therefore converges to a constant c. Take $f := g + c$. We have then $g_n + c = f_n - c_n + c \to f$ in $L_{p,loc}$. This yields $f_n \to f$ in $L_{p,loc}$ and so $Vf_n \to Vf$ $\mu - a.e.$. The unicity of the limit gives us $h = Vf$. Hence, we conclude that $f \in \dot{W}^1_{p,V}$ and $f_n \to f$ in $\dot{W}^1_{p,V}$ which finishes the proof. □

In the following proposition we characterize the $W^1_{p,V}$. We have

Proposition 5.2.14. *Let M be a complete Riemannian manifold and let $V \in RH_{qloc}$ for some $1 \leq q < \infty$. Consider, for $1 \leq p < q$,*

$$H^1_{p,V}(M) = H^1_{p,V} = \{f \in L_p : \ |\nabla f| \ \text{and} \ Vf \in L_p\}$$

and equip it with the same norm as $W^1_{p,V}$. Then C_0^∞ is dense in $H^1_{p,V}$ and hence $W^1_{p,V} = H^1_{p,V}$.

Proof. See the Appendix. □

Therefore, under the hypotheses of Proposition 5.2.14, $W^1_{p,V}$ is the set of distributions $f \in L_p$ such that $|\nabla f|$ and Vf belong to L_p.

5.3 Principal tools

We shall use the following form of Fefferman-Phong inequality. The proof is completely analogous to the one in $\mathbb{R}^n$ (see [22], [2]):

Lemma 5.3.1. *(Fefferman-Phong inequality). Let M be a complete Riemannian manifold satisfying (D). Let $w \in A_\infty$ and $1 \leq p < \infty$. We assume that M admits also a Poincaré inequality (P_p). Then there exists a constant $C > 0$ depending only on the A_∞ constant of w, p and the constants in (D), (P_p), such that for all ball B of radius $R > 0$ and $u \in W^1_{p,loc}$*

$$\int_B (|\nabla u|^p + w|u|^p) d\mu \geq C \min(R^{-p}, w_B) \int_B |u|^p d\mu.$$

Proof. Since M admits a (P_p) Poincaré inequality, we have

$$\int_B |\nabla u|^p d\mu \geq \frac{C}{R^p \mu(B)} \int_B \int_B |u(x) - u(y)|^p d\mu(x) d\mu(y).$$

This and

$$\int_B w|u|^p d\mu = \frac{1}{\mu(B)} \int_B \int_B w(x)|u(x)|^p d\mu(x) d\mu(y)$$

lead easily to

$$\int_B (|\nabla u|^p + w|u|^p) d\mu \geq [\min(CR^{-p}, w)]_B \int_B |u|^p d\mu.$$

Now we use that $w \in A_\infty$: there exists $\varepsilon > 0$, independent of B, such that $E = \{x \in B : w(x) > \varepsilon w_B\}$ satisfies $\mu(E) > \frac{1}{2}\mu(B)$. Indeed since $w \in A_\infty$ then there exists $1 \leq p < \infty$ such that $w \in A_p$. Therefore,

$$\frac{\mu(E^c)}{\mu(B)} \leq C \left(\frac{w(E^c)}{w(B)}\right)^{\frac{1}{p}} \leq C\varepsilon^{\frac{1}{p}}.$$

We take $\varepsilon > 0$ such that $C\varepsilon^{\frac{1}{p}} < \frac{1}{2}$. We obtain then

$$[\min(CR^{-p}, w)]_B \geq \frac{1}{2}\min(CR^{-p}, \varepsilon w_B) \geq C' \min(R^{-p}, w_B).$$

This proves the desired inequality and finishes the proof.

□

Now, we will give two versions of a Calderón-Zygmund decomposition:

Proposition 5.3.2. *Let M be a complete non-compact Riemannian manifold satisfying (D). Let $V \in RH_q$, for some $1 < q < \infty$ and assume that M admits a Poincaré inequality (P_s) for some $1 \leq s < q$. Let $f \in W^1_{p,V}$, $s \leq p < q$, and $\alpha > 0$. Then one can find a collection of balls (B_i), functions $g \in W^1_{q,V}$ and $b_i \in W^1_{s,V}$ with the following properties*

$$f = g + \sum_i b_i \tag{5.3}$$

$$\int_{\cup_i B_i} T_q g \, d\mu \leq C\alpha^q \mu(\cup_i B_i) \tag{5.4}$$

$$\operatorname{supp} b_i \subset B_i, \; \int_{B_i} T_s b_i \, d\mu \leq C\alpha^s \mu(B_i) \tag{5.5}$$

$$\sum_i \mu(B_i) \leq \frac{C}{\alpha^p} \int_M T_p f \, d\mu \tag{5.6}$$

$$\sum_i \mathbb{1}_{B_i} \leq N \tag{5.7}$$

where N, C depend only on the constants in (D), (P_s), p and the RH_q constant of V. Denote $T_r f = |f|^r + |\nabla f|^r + |Vf|^r$ for $1 \leq r < \infty$.

Proof. Let $f \in W^1_{p,V}$, $\alpha > 0$. Consider $\Omega = \{x \in M : \mathcal{M}T_s f(x) > \alpha^s\}$. If $\Omega = \emptyset$, then set

$$g = f, \; b_i = 0 \text{ for all } i$$

so that (5.4) is satisfied thanks to the Lebesgue differentiation theorem. Otherwise the maximal theorem (Theorem 5.2.2) and $p \geq s$ give us that

$$\mu(\Omega) \leq \frac{C}{\alpha^p} \int_M T_p f \, d\mu < \infty. \tag{5.8}$$

In particular $\Omega \neq M$ as $\mu(M) = \infty$. Let F be the complement of Ω. Since Ω is an open set distinct of M, let $(\underline{B_i})$ be a Whitney decomposition of Ω ([11]). That is, the balls $\underline{B_i}$ are pairwise disjoint and there exist two constants $C_2 > C_1 > 1$, depending only on the metric, such that

1. $\Omega = \cup_i B_i$ with $B_i = C_1 \underline{B_i}$ and the balls B_i have the bounded overlap property;

2. $r_i = r(B_i) = \frac{1}{2} d(x_i, F)$ and x_i is the center of B_i;

3. each ball $\overline{B_i} = C_2 \underline{B_i}$ intersects F ($C_2 = 4C_1$ works).

For $x \in \Omega$, denote $I_x = \{i : x \in B_i\}$. By the bounded overlap property of the balls B_i, we have that $\sharp I_x \leq N$. Fixing $j \in I_x$ and using the properties of the B_i's, we easily see that $\frac{1}{3} r_i \leq r_j \leq 3r_i$ for all $i \in I_x$. In particular, $B_i \subset 7B_j$ for all $i \in I_x$.

Condition (5.7) is nothing but the bounded overlap property of the B_i's and (5.6) follows from (5.7) and (5.8). Since $V \in RH_q$ implies $V^q \in A_\infty$ (because there exists $\varepsilon > 0$ such that $V \in RH_{q+\varepsilon}$ and hence $V^q \in RH_{1+\frac{\varepsilon}{q}}$) and therefore $V^s \in RH_{\frac{q}{s}}$ by Proposition 5.2.8, Lemma 5.3.1 yields

$$\int_{B_i} (|\nabla f|^s + |Vf|^s) d\mu \geq C \min(V_{B_i}^s, r_i^{-s}) \int_{B_i} |f|^s d\mu. \tag{5.9}$$

We declare B_i of type 1 if $V_{B_i}^s \geq r_i^{-s}$ and of type 2 if $V_{B_i}^s < r_i^{-s}$. One should read $V_{B_i}^s$ as $(V^s)_{B_i}$ but this is also equivalent to $(V_{B_i})^s$ since $V \in RH_q \subset RH_s$.

Let us now define the functions b_i. Let $(\chi_i)_i$ be a partition of unity of Ω subordinated to the covering $(\underline{B_i})$, such that for all i, χ_i is a Lipschitz function supported in B_i with $\| |\nabla \chi_i| \|_\infty \leq \dfrac{C}{r_i}$. To this end it is enough to choose $\chi_i(x) = \psi(\dfrac{C_1 d(x_i, x)}{r_i}) \Big(\sum_k \psi(\dfrac{C_1 d(x_k, x)}{r_k}) \Big)^{-1}$, where ψ is a smooth function, $\psi = 1$ on $[0, 1]$, $\psi = 0$ on $[\frac{1+C_1}{2}, +\infty[$ and $0 \leq \psi \leq 1$. Set

$$b_i = \begin{cases} f\chi_i & \text{if } B_i \text{ of type 1,} \\ (f - f_{B_i})\chi_i & \text{if } B_i \text{ of type 2.} \end{cases}$$

Let us estimate $\int_{B_i} T_s b_i \, d\mu$. We distinguish two cases:

1. If B_i is of type 2, then

$$\begin{aligned} \int_{B_i} |b_i|^s d\mu &= \int_{B_i} |(f - f_{B_i})\chi_i|^s d\mu \\ &\leq C \left(\int_{B_i} |f|^s d\mu + \int_{B_i} |f_{B_i}|^s d\mu \right) \\ &\leq C \int_{B_i} |f|^s d\mu \\ &\leq C \int_{\overline{B_i}} |f|^s d\mu \\ &\leq C \alpha^s \mu(\overline{B_i}) \\ &\leq C \alpha^s \mu(B_i) \end{aligned}$$

where we used that $\overline{B_i} \cap F \neq \emptyset$ and the property (D). The Poincaré inequality (P_s) gives us

$$\begin{aligned}\int_{B_i} |\nabla b_i|^s d\mu &\leq C \int_{B_i} |\nabla f|^s d\mu \\ &\leq C\mathcal{M}T_s f(y)\mu(B_i) \\ &\leq C\alpha^s \mu(B_i)\end{aligned}$$

as y can be chosen in $F \cap \overline{B_i}$. Finally,

$$\begin{aligned}\int_{B_i} |Vb_i|^s d\mu &= \int_{B_i} |V(f - f_{B_i})\chi_i|^s d\mu \\ &\leq \int_{B_i} |Vf|^s d\mu + \int_{B_i} |Vf_{B_i}|^s d\mu \\ &\leq (|Vf|^s)_{B_i}\mu(B_i) + C(V^s)_{B_i}(|f|^s)_{B_i}\mu(B_i) \\ &\leq C\alpha^s\mu(B_i) + \left(|\nabla f|^s + |Vf|^s\right)_{B_i}\mu(B_i) \\ &\leq C\alpha^s\mu(B_i).\end{aligned}$$

We used that $\overline{B_i} \cap F \neq \emptyset$, Jensen's inequality and (5.9), noting that B_i is of type 2.

2. If B_i is of type 1, then

$$\begin{aligned}\int_{B_i} T_s b_i \, d\mu &\leq \int_{B_i} T_s f \, d\mu + r_i^{-s} \int_{B_i} |f|^s d\mu \\ &\leq C \int_{B_i} T_s f \, d\mu \\ &\leq C\alpha^s \mu(B_i)\end{aligned}$$

where we used that $\overline{B_i} \cap F \neq \emptyset$ and that B_i is of type 1.

Set now $g = f - \sum_i b_i$, where the sum is over balls of both types and is locally finite by (5.7). The function g is defined almost everywhere on M, $g = f$ on F and $g = \sum^2 f_{B_i}\chi_i$ on Ω where $\sum^j$ means that we are summing over balls of type j. Observe that g is a locally integrable function on M. Indeed, let $\varphi \in L_\infty$ with compact support. Since $d(x, F) \geq r_i$ for $x \in \text{supp } b_i$, we obtain

$$\int \sum_i |b_i| \, |\varphi| \, d\mu \leq \Big(\int \sum_i \frac{|b_i|}{r_i} \, d\mu\Big) \sup_{x \in M} \Big(d(x, F)|\varphi(x)|\Big)$$

and

$$\begin{aligned}\int \frac{|b_i|}{r_i} d\mu &= \int_{B_i} \frac{|f - f_{B_i}|}{r_i} \chi_i \, d\mu \\ &\leq \Big(\mu(B_i)\Big)^{\frac{1}{s'}} \Big(\int_{B_i} |\nabla f|^s d\mu\Big)^{\frac{1}{s}}\end{aligned}$$

$$\leq C\alpha\mu(B_i).$$

We used the Hölder inequality, (P_s) and that $\overline{B_i} \cap F \neq \emptyset$, s' being the conjugate of s. Hence $\displaystyle\int \sum_i |b_i||\varphi| d\mu \leq C\alpha\mu(\Omega) \sup_{x\in M}\Big(d(x,F)|\varphi(x)|\Big)$. Since $f \in L_{1,loc}$, we conclude that $g \in L_{1,loc}$. (Note that since $b \in L_1$ in our case, we can say directly that $g \in L_{1,loc}$. However, for the homogeneous case –section 5– we need this observation to conclude that $g \in L_{1,loc}$.) It remains to prove (5.4). Note that $\displaystyle\sum_i \chi_i(x) = 1$ and $\displaystyle\sum_i \nabla\chi_i(x) = 0$ for all $x \in \Omega$. A computation of the sum $\sum_i \nabla b_i$ leads us to

$$\nabla g = (\nabla f)1\!\!1_F + \sum{}^2 f_{B_i}\nabla\chi_i.$$

By definition of F and the differentiation theorem, $|\nabla g|$ is bounded by α almost everywhere on F. It remains to control $\|h_2\|_\infty$ where $h_2 = \sum^2 f_{B_i}\nabla\chi_i$. Set $h_1 = \sum^1 f_{B_i}\nabla\chi_i$. By already seen arguments for type 1 balls, $|f_{B_i}| \leq C\alpha r_i$. Hence, $|h_1| \leq C\sum^1 1\!\!1_{B_i}\alpha \leq CN\alpha$ and it suffices to show that $h = h_1 + h_2$ is bounded by $C\alpha$. To see this, fix $x \in \Omega$. Let B_j be a Whitney ball containing x. We may write

$$|h(x)| = \left|\sum_{i\in I_x}(f_{B_i} - f_{B_j})\nabla\chi_i(x)\right| \leq C\sum_{i\in I_x}|f_{B_i} - f_{B_j}| r_i^{-1}.$$

Since $B_i \subset 7B_j$ for all $i \in I_x$, the Poincaré inequality (P_s) and the definition of B_j yield

$$|f_{B_i} - f_{B_j}| \leq Cr_j\left((|\nabla f|^s)_{7B_j}\right)^{\frac{1}{s}} \leq Cr_j\alpha.$$

Thus $\|h\|_\infty \leq C\alpha$.

Let us now estimate $\int_\Omega T_q g\, d\mu$. We have

$$\begin{aligned}\int_\Omega |g|^q d\mu &= \int_M |(\sum{}^2 f_{B_i}\chi_i)|^q d\mu \\ &\leq C\sum{}^2 |f_{B_i}|^q\mu(B_i) \\ &\leq CN\alpha^q\mu(\Omega).\end{aligned}$$

We used the estimate

$$(|f|_{B_i})^s \leq (|f|^s)_{B_i} \leq (\mathcal{M}T_s f)(y) \leq \alpha^s$$

as y can be chosen in $F \cap \overline{B_i}$. For $|\nabla g|$, we have

$$\begin{aligned}\int_\Omega |\nabla g|^q d\mu &= \int_\Omega |h_2|^q d\mu \\ &\leq C\alpha^q\mu(\Omega).\end{aligned}$$

Finally, since by Proposition 5.2.8 $V^s \in RH_{\frac{q}{s}}$, we get

$$\int_\Omega V^q|g|^q d\mu \leq \sum{}^2\int_{B_i} V^q|f_{B_i}|^q d\mu$$

$$\leq C \sum{}^{2} (V_{B_i}^s |f_{B_i}|^s)^{\frac{q}{s}} \mu(B_i).$$

By construction of the type 2 balls and by (5.9) we have $V_{B_i}^s|f_{B_i}|^s \leq V_{B_i}^s(|f|^s)_{B_i} \leq C(|\nabla f|^s + |Vf|^s)_{B_i} \leq C\alpha^s$. Then $\int_\Omega V^q|g|^q d\mu \leq C\sum{}^2 \alpha^q \mu(B_i) \leq NC\alpha^q\mu(\Omega)$.

To finish the proof, we have to verify that $g \in W^1_{q,V}$. For that we just have to control $\int_F T_q g\, d\mu$. As $g = f$ on F, this readily follows from

$$\begin{aligned}
\int_F T_q f d\mu &= \int_F (|f|^q + |\nabla f|^q + |Vf|^q) d\mu \\
&\leq \int_F (|f|^p|f|^{q-p} + |\nabla f|^p|\nabla f|^{q-p} + |Vf|^p|Vf|^{q-p}) d\mu \\
&\leq \alpha^{q-p}\|f\|^p_{W^1_{p,V}}.
\end{aligned}$$

□

Remark 5.3.3. *The estimate $\int_F T_q g\, d\mu$ above is too crude to be used in the interpolation argument. Note that (5.4) only involves control of $T_q g$ on $\Omega = \cup_i B_i$. Compare with (5.11) in the next argument when $q = \infty$.*

Proposition 5.3.4. *Let M be a complete non-compact Riemannian manifold satisfying (D). Let $V \in RH_\infty$ and assume that M admits a Poincaré inequality (P_s) for some $1 \leq s < \infty$. Let $f \in W^1_{p,V}$, $s \leq p < \infty$, and $\alpha > 0$. Then one can find a collection of balls (B_i), functions $b_i \in W^1_{s,V}$ and an almost everywhere Lipschitz function g such that the following properties hold:*

$$f = g + \sum_i b_i \tag{5.10}$$

$$\|g\|_{W^1_{\infty,V}} \leq C\alpha \tag{5.11}$$

$$\text{supp } b_i \subset B_i, \int_{B_i} T_s b_i\, d\mu \leq C\alpha^s\mu(B_i) \tag{5.12}$$

$$\sum_i \mu(B_i) \leq \frac{C}{\alpha^p}\int T_p f\, d\mu \tag{5.13}$$

$$\sum_i \chi_{B_i} \leq N \tag{5.14}$$

where C and N only depend on the constants in (D), (P_s), p and the RH_∞ constant of V.

Proof. The only difference between the proof of this proposition and that of Proposition 5.3.2 is the estimation (5.11). Indeed, as we have seen in the proof of Proposition 5.3.2, we have $|\nabla g| \leq C\alpha$ almost everywhere. By definition of F and the differentiation theorem, $(|g| + |Vg|)$ is bounded by α almost everywhere on F. We have also seen that for all i, $|f|_{B_i} \leq \alpha$. Fix $x \in \Omega$, then

$$\begin{aligned} |g(x)| &= |\sum_{i \in I_x} f_{B_i}| \\ &\leq \sum_{i \in I_x} |f_{B_i}| \\ &\leq N\alpha. \end{aligned}$$

It remains to estimate $|Vg|(x)$. We have

$$\begin{aligned} |Vg|(x) &\leq \sum\nolimits_{i:x\in B_i}^{2} V(x)|f_{B_i}| \\ &\leq C \sum\nolimits_{i:x\in B_i}^{2} (V_{B_i})|f_{B_i}| \\ &\leq C \sum\nolimits_{i:x\in B_i}^{2} ((V^s)_{B_i}(|f|^s)_{B_i})^{\frac{1}{s}} \\ &\leq C \sum\nolimits_{i:x\in B_i}^{2} (|\nabla f|^s + |Vf|^s)_{B_i}^{\frac{1}{s}} \\ &\leq NC\alpha \end{aligned}$$

where we used the definition of RH_∞, and Jensen's inequality as $s \geq 1$. We used also (5.9) and the bounded overlap property of the B_i's. □

5.4 Estimation of the K-functional in the non homogeneous case

Denote for $1 \leq r < \infty$, $T_r f = |f|^r + |\nabla f|^r + |Vf|^r$, $T_{r*}f = |f|^{r*} + |\nabla f|^{r*} + |Vf|^{r*}$, $T_{r**}f = |f|^{r**} + |\nabla f|^{r**} + |Vf|^{r**}$. We have $tT_{r**}f(t) = \int_0^t T_{r*}f(u)du$ for all $t > 0$.

Theorem 5.4.1. *Under the same hypotheses as in Theorem 5.1.3, with $V \in RH_{\infty,loc}$ and $1 \leq s < \infty$, we have for every $f \in W^1_{s,V} + W^1_{\infty,V}$ and every $t > 0$*

$$K(f, t^{\frac{1}{s}}, W^1_{s,V}, W^1_{\infty,V}) \sim \left(\int_0^t T_{s*}f(u)du \right)^{\frac{1}{s}} \sim (tT_{s**}f(t))^{\frac{1}{s}}.$$

Proof. We refer to [6] for an analogous proof. □

Theorem 5.4.2. *We consider the same hypotheses as in Theorem 5.1.3 with $V \in RH_{qloc}$ for some $1 < q < \infty$. Then*

1. *there exists C_1 such that for every $f \in W^1_{s,V} + W^1_{q,V}$ and every $t > 0$*

$$K(f, t, W^1_{s,V}, W^1_{q,V}) \geq C_1 \left\{ \left(\int_0^{t^{\frac{qs}{q-s}}} T_{s*}f(u)du \right)^{\frac{1}{s}} + t \left(\int_{t^{\frac{qs}{q-s}}}^{\infty} T_{q*}f(u)du \right)^{\frac{1}{q}} \right\};$$

2. for $s \leq p < q$, there exists C_2 such that for every $f \in W^1_{p,V}$ and every $t > 0$

$$K(f,t,W^1_{s,V},W^1_{q,V}) \leq C_2 \left\{ \left(\int_0^{t^{\frac{qs}{q-s}}} T_{s*}f(u)du \right)^{\frac{1}{s}} + t \left(\int_{t^{\frac{qs}{q-s}}}^{\infty} (\mathcal{M}T_s f)^{*\frac{q}{s}}(u)du \right)^{\frac{1}{q}} \right\}.$$

Proof. In a first step we prove this theorem in the global case. This will help to understand the proof of the more general local case.

5.4.1 The global case

Let M be a complete Riemannian manifold satisfying (D). Let $V \in RH_q$ for some $1 < q < \infty$ and assume that M admits a Poincaré inequality (P_s) for some $1 \leq s < q$. The principal tool to prove Theorem 5.4.2 in this case will be the Calderón-Zygmund decomposition of Proposition 5.3.2.

We prove the left inequality by applying Theorem 5.2.11 with $p_0 = s$ and $p_1 = q$ which gives for all $f \in L_s + L_q$:

$$K(f,t,L_s,L_q) \sim \left(\int_0^{t^{\frac{qs}{q-s}}} f^{*s}(u)du \right)^{\frac{1}{s}} + t \left(\int_{t^{\frac{qs}{q-s}}}^{\infty} f^{*q}(u)du \right)^{\frac{1}{q}}.$$

Moreover, we have

$$K(f,t,W^1_{s,V},W^1_{q,V}) \geq K(f,t,L_s,L_q) + K(|\nabla f|,t,L_s,L_q) + K(Vf,t,L_s,L_q)$$

since the operator

$$I + \nabla + V : (W^1_{s,V},W^1_{q,V}) \to (L_s,L_q)$$

is bounded.
Hence we conclude with

$$K(f,t,W^1_{s,V},W^1_{q,V}) \geq C \left(\int_0^{t^{\frac{qs}{q-s}}} T_{s*}f(u)du \right)^{\frac{1}{s}} + Ct \left(\int_{t^{\frac{qs}{q-s}}}^{\infty} T_{q*}f(u)du \right)^{\frac{1}{q}}.$$

We prove now item 2. Let $f \in W^1_{p,V}$, $s \leq p < q$ and $t > 0$. We consider the Calderón-Zygmund decomposition of f given by Proposition 5.3.2 with $\alpha = \alpha(t) = (\mathcal{M}T_s f)^{*\frac{1}{s}}(t^{\frac{qs}{q-s}})$. Thus f can be written as $f = b + g$ with $b = \sum_i b_i$ where $(b_i)_i$, g satisfy the properties of the proposition. For the L_s norm of b we have

$$\begin{aligned} \|b\|_s^s &\leq \int_M (\sum_i |b_i|)^s d\mu \\ &\leq N \sum_i \int_{B_i} |b_i|^s d\mu \end{aligned}$$

$$\leq C\alpha^s(t)\sum_i \mu(B_i)$$
$$\leq NC\alpha^s(t)\mu(\Omega_t).$$

This follows from the fact that $\sum_i \chi_{B_i} \leq N$ and $\Omega_t = \Omega = \bigcup_i B_i$. Similarly we get $\| \, |\nabla b| \, \|_s^s \leq C\alpha^s(t)\mu(\Omega_t)$ and $\|Vb\|_s^s \leq C\alpha^s(t)\mu(\Omega_t)$. For g we have $\|g\|_{W^1_{q,V}} \leq C\alpha(t)\mu(\Omega_t)^{\frac{1}{q}} + \left(\int_{F_t} T_q f d\mu\right)^{\frac{1}{q}}$, where $F_t = F$ in the Proposition 5.3.2 with this choice of α.

Moreover, since $(\mathcal{M}f)^* \sim f^{**}$ and $(f+g)^{**} \leq f^{**} + g^{**}$, we obtain

$$\alpha(t) = (\mathcal{M}T_s f)^{*\frac{1}{s}}(t^{\frac{qs}{q-s}}) \leq C(T_{s**}f)^{\frac{1}{s}}(t^{\frac{qs}{q-s}}).$$

Hence, also noting that $\mu(\Omega_t) \leq t^{\frac{qs}{q-s}}$, we get for all $t > 0$

$$K(f,t,W^1_{s,V},W^1_{q,V}) \leq C\left(\int_0^{t^{\frac{qs}{q-s}}} T_{s*}f(u)du\right)^{\frac{1}{s}} + t\left(\int_{F_t} T_q f d\mu\right)^{\frac{1}{q}}. \tag{5.15}$$

Let us estimate $\int_{F_t} T_q f d\mu$. Consider E_t a measurable set such that

$$\Omega_t \subset E_t \subset \left\{x : \mathcal{M}T_s f(x) \geq (\mathcal{M}T_s f)^*(t^{\frac{qs}{q-s}})\right\}$$

and $\mu(E_t) = t^{\frac{qs}{q-s}}$. Remark that $\int_{E_t} (\mathcal{M}T_s f)^r d\mu = \int_0^{t^{\frac{qs}{q-s}}} (\mathcal{M}T_s f)^{*r}(u)du$ for $r \geq 1$ (see [23]: Chapter V, Lemma 3.17). Denote $G_t := E_t - \Omega_t$. We have then

$$\begin{aligned}
\int_{F_t} T_q f d\mu &= \int_{E_t^c} T_q f d\mu + \int_{G_t} T_q f d\mu \\
&\leq C\int_{t^{\frac{qs}{q-s}}}^{\infty} (\mathcal{M}T_s f)^{*\frac{q}{s}}(u)du + C\int_{G_t} (T_{s**}f)^{\frac{q}{s}}(t^{\frac{qs}{q-s}})d\mu \\
&\leq C\int_{t^{\frac{qs}{q-s}}}^{\infty} (\mathcal{M}T_s f)^{*\frac{q}{s}}(u)du + C\mu(E_t)(T_{s**}f)^{\frac{q}{s}}(t^{\frac{qs}{q-s}}) \\
&= C\int_{t^{\frac{qs}{q-s}}}^{\infty} (\mathcal{M}T_s f)^{*\frac{q}{s}}(u)du + Ct^{-q}\left(\int_0^{t^{\frac{qs}{q-s}}} T_{s*}f(u)du\right)^{\frac{q}{s}}.
\end{aligned} \tag{5.16}$$

Hence (5.15) and (5.16) yield

$$K(f,t,W^1_{s,V},W^1_{q,V}) \leq C\left(\int_0^{t^{\frac{qs}{q-s}}} T_{s*}f(u)du\right)^{\frac{1}{s}} + Ct\left(\int_{t^{\frac{qs}{q-s}}}^{\infty} (\mathcal{M}T_s f)^{*\frac{q}{s}}(u)du\right)^{\frac{1}{q}}$$

which finishes the proof in that case.

5.4.2 The local case

Let M be a complete non-compact Riemannian manifold satisfying a local doubling property (D_{loc}). Consider $V \in RH_{qloc}$ for some $1 < q < \infty$ and assume that M admits a local Poincaré inequality (P_{sloc}) for some $1 \leq s < q$.

Denote by $\mathcal{M}_E$ the Hardy-Littlewood maximal operator relative to a measurable subset E of M, that is, for $x \in E$ and every f locally integrable function on M:

$$\mathcal{M}_E f(x) = \sup_{B:\, x\in B} \frac{1}{\mu(B\cap E)} \int_{B\cap E} |f| d\mu$$

where B ranges over all open balls of M containing x and centered in E. We say that a measurable subset E of M has the relative doubling property if there exists a constant C_E such that for all $x \in E$ and $r > 0$ we have

$$\mu(B(x,2r)\cap E) \leq C_E \mu(B(x,r)\cap E).$$

This is equivalent to saying that the metric measure space $(E, d/E, \mu/E)$ has the doubling property. On such a set $\mathcal{M}_E$ is of weak type $(1,1)$ and bounded on $L^p(E,\mu)$, $1 < p \leq \infty$.

We prove now Theorem 5.4.2 in the local case. To fix ideas, we assume $r_0 = 5$, $r_1 = 8$, $r_2 = 2$. The lower bound of K in item 1. is trivial (same proof as for the global case). It remains to prove the upper bound. For all $t > 0$, take $\alpha = \alpha(t) = (\mathcal{M}T_s f)^{*\frac{1}{s}}(t^{\frac{qs}{q-s}})$.

Consider

$$\Omega = \left\{x \in M : \mathcal{M}T_s f(x) > \alpha^s(t)\right\}.$$

We have $\mu(\Omega) \leq t^{\frac{qs}{q-s}}$. If $\Omega = M$ then

$$\begin{aligned}\int_M T_s f\, d\mu &= \int_\Omega T_s f\, d\mu \\ &\leq C \int_0^{\mu(\Omega)} T_{s*} f(r) dr \\ &\leq C \int_0^{t^{\frac{qs}{q-s}}} T_{s*} f(r) dr\end{aligned}$$

Therefore

$$K(f,t,W^1_{s,V},W^1_{q,V}) \leq C t^{\frac{q}{q-s}} (T_{s**}f)^{\frac{1}{q}}(t^{\frac{qs}{q-s}}).$$

We thus obtain item 2. in this case.

Now assume $\Omega \neq M$. Pick a countable set $\{x_j\}_{j\in J} \subset M$, such that $M = \bigcup_{j\in J} B(x_j, \frac{1}{2})$ and for all $x \in M$, x does not belong to more than N_1 balls $B^j := B(x_j, 1)$. Consider a C^∞ partition of unity $(\varphi_j)_{j\in J}$ subordinated to the balls $\frac{1}{2}B^j$ such that $0 \leq \varphi_j \leq 1$, *supp* $\varphi_j \subset B^j$ and $\| |\nabla\varphi_j| \|_\infty \leq C$ uniformly with respect to j. Consider $f \in W^1_{p,V}$, $s \leq p < q$. Let $f_j = f\varphi_j$ so that $f = \sum_{j\in J} f_j$. We have for $j \in J$, f_j, $Vf_j \in L_p$ and $\nabla f_j = f\nabla\varphi_j + \nabla f \varphi_j \in L_p$. Hence $f_j \in W^1_p(B^j)$. The balls B^j satisfy the relative doubling property with the constant independent of the balls B^j. This follows from the next lemma quoted from [3] p. 947 (For the proof, see Lemma 2.2.4 of Chapter 2 of this thesis).

Lemma 5.4.3. *Let M be a complete Riemannian manifold satisfying (D_{loc}). Then the balls B^j above, equipped with the induced distance and measure, satisfy the relative doubling property (D), with the doubling constant that may be chosen independently of j. More precisely, there exists $C \geq 0$ such that for all $j \in J$*

$$\mu(B(x,2r)\cap B^j) \leq C\,\mu(B(x,r)\cap B^j) \quad \forall x \in B^j,\, r>0, \tag{5.17}$$

and

$$\mu(B(x,r)) \leq C\mu(B(x,r)\cap B^j) \quad \forall x \in B^j,\, 0<r\leq 2. \tag{5.18}$$

Let us return to the proof of the theorem. For any $x \in B^j$ we have

$$\begin{aligned}\mathcal{M}_{B^j}T_s f_j(x) &= \sup_{B:\, x\in B,\, r(B)\leq 2} \frac{1}{\mu(B^j\cap B)}\int_{B^j\cap B} T_s f_j d\mu \\ &\leq \sup_{B:\, x\in B,\, r(B)\leq 2} C\frac{\mu(B)}{\mu(B^j\cap B)}\frac{1}{\mu(B)}\int_B T_s f d\mu \\ &\leq C\mathcal{M}T_s f(x). \end{aligned} \tag{5.19}$$

where we used (5.18) of Lemma 5.4.3. Consider now

$$\Omega_j = \left\{x\in B^j : \mathcal{M}_{B^j}T_s f_j(x) > C\alpha^s(t)\right\}$$

where C is the constant in (5.19). The set Ω_j is an open subset of B^j then of M and $\Omega_j \subset \Omega$ for all $j \in J$. For the f_j's, and for all $t>0$, we have a Calderón-Zygmund decomposition similar to the one done in Proposition 5.3.2: there exist b_{jk}, g_j supported in B^j, and balls $(B_{jk})_k$ of M, contained in Ω_j, such that

$$f_j = g_j + \sum_k b_{jk} \tag{5.20}$$

$$\int_{\Omega_j} T_q g_j\, d\mu \leq C\alpha^q(t)\mu(\Omega_j) \tag{5.21}$$

$$supp\ b_{jk} \subset B_{jk},\ \int_{B_{jk}} T_s b_{jk}\, d\mu \leq C\alpha^s(t)\mu(B_{jk}) \tag{5.22}$$

$$\sum_k \mu(B_{jk}) \leq C\alpha^{-p}(t)\int_{B^j} T_p f_j\, d\mu \tag{5.23}$$

$$\sum_k \chi_{B_{jk}} \leq N \tag{5.24}$$

with C and N depending only on q, p and the constant $C(r_0), C(r_1), C(r_2)$ in (D_{loc}) and (P_{sloc}) and the RH_{qloc} condition of V, which is independent of B^j.

The proof of this decomposition will be the same as that of Proposition 5.3.2, taking for all $j \in J$ a Whitney decomposition $(B_{jk})_k$ of $\Omega_j \neq M$ and using the doubling property for balls whose radii do not exceed $3 < r_0$ and the Poincaré inequality for balls whose radii do not exceed $7 < r_1$ and the RH_{qloc} property of V for balls whose radii do not exceed $1 < r_2$. By the above decomposition we can write $f = \sum\limits_{j \in J} \sum\limits_k b_{jk} + \sum\limits_{j \in J} g_j = b + g$. Let us now estimate $\|b\|_{W^1_{s,V}}$ and $\|g\|_{W^1_{q,V}}$.

$$\begin{aligned}
\|b\|_s^s &\leq N_1 N \sum_j \sum_k \|b_{jk}\|_s^s \\
&\leq C\alpha^s(t) \sum_j \sum_k (\mu(B_{jk})) \\
&\leq NC\alpha^s(t) \Big(\sum_j \mu(\Omega_j) \Big) \\
&\leq N_1 C\alpha^s(t)\mu(\Omega).
\end{aligned}$$

We used the bounded overlap property of the $(\Omega_j)_{j\in J}$'s and that of the $(B_{jk})_k$'s for all $j \in J$. It follows that $\|b\|_s \leq C\alpha(t)\mu(\Omega)^{\frac{1}{s}}$. Similarly we get $\| \, |\nabla b| \, \|_s \leq C\alpha(t)\mu(\Omega)^{\frac{1}{s}}$ and $\|Vb\|_s \leq C\alpha(t)\mu(\Omega)^{\frac{1}{s}}$.

For g we have

$$\begin{aligned}
\int_\Omega |g|^q d\mu &\leq N \sum_j \int_{\Omega_j} |g_j|^q d\mu \\
&\leq NC\alpha^q(t) \sum_j \mu(\Omega_j) \\
&\leq N_1 NC\alpha^q(t)\mu(\Omega).
\end{aligned}$$

Analogously $\int_\Omega |\nabla g|^q d\mu \leq C\alpha^q(t)\mu(\Omega)$ and $\int_\Omega |Vg|^q d\mu \leq C\alpha^q(t)\mu(\Omega)$. Then noting that $g \in W^1_{q,V}$ (same argument as in the proof of the global case) we obtain

$$\begin{aligned}
K(f, t, W^1_{s,V}, W^1_{q,V}) &\leq \|b\|_{W^1_s} + t\|g\|_{W^1_q} \\
&\leq C\alpha(t)\mu(\Omega)^{\frac{1}{s}} + Ct\alpha(t)\mu(\Omega)^{\frac{1}{q}} + t\left(\int_{F_t} T_q f d\mu \right)^{\frac{1}{q}} \\
&\leq Ct^{\frac{q}{q-s}} (T_{s**} f)^{\frac{1}{s}} (t^{\frac{qs}{q-s}}) + t\left(\int_{t^{\frac{qs}{q-s}}}^\infty (\mathcal{M}T_s f)^{*\frac{q}{s}}(u) du \right)^{\frac{1}{q}}.
\end{aligned}$$

Thus, we get the desired estimation for $f \in W^1_{p,V}$. □

5.5 Interpolation of non homogeneous Sobolev spaces

Proof of Theorem 5.1.3. The proof of the case when $V \in RH_{\infty loc}$ is the same as the one in section 4 in [6]. Consider now $V \in RH_{qloc}$ for some $1 < q < \infty$. For $s < p < q$, we define the interpolation space $W^1_{p,q,s,V}(M) = W^1_{p,q,s,V}$ between $W^1_{s,V}$ and $W^1_{q,V}$ by

$$W^1_{p,q,s,V} = (W^1_{s,V}, W^1_{q,V})_{\frac{q(p-s)}{p(q-s)},p}.$$

We claim that $W^1_{p,q,s,V} = W^1_{p,V}$ with equivalent norms. Indeed, let $f \in W^1_{p,q,s,V}$. We have

$$\begin{aligned}
\|f\|_{\frac{q(p-s)}{p(q-s)},p} &= \left\{\int_0^\infty \left(t^{\frac{q(s-p)}{p(q-s)}} K(f,t,W^1_{s,V},W^1_{q,V})\right)^p \frac{dt}{t}\right\}^{\frac{1}{p}} \\
&\geq \left\{\int_0^\infty \left(t^{\frac{q(s-p)}{p(q-s)}} t^{\frac{q}{q-s}} (T_{s**}f)^{\frac{1}{s}}(t^{\frac{qs}{q-s}})\right)^p \frac{dt}{t}\right\}^{\frac{1}{p}} \\
&= \left\{\int_0^\infty t^{\frac{qs}{q-s}-1} (T_{s**}f)^{\frac{p}{s}}(t^{\frac{qs}{q-s}}) dt\right\}^{\frac{1}{p}} \\
&= \left\{\int_0^\infty (T_{s**}f)^{\frac{p}{s}}(t) dt\right\}^{\frac{1}{p}} \\
&\geq \|f^{s**}\|^{\frac{1}{s}}_{\frac{p}{s}} + \|\,|\nabla f|^{s**}\|^{\frac{1}{s}}_{\frac{p}{s}} + \|\,|Vf|^{s**}\|^{\frac{1}{s}}_{\frac{p}{s}} \\
&\sim \|f^{s}\|^{\frac{1}{s}}_{\frac{p}{s}} + \|\,|\nabla f|^{s}\|^{\frac{1}{s}}_{\frac{p}{s}} + \|\,|Vf|^{s}\|^{\frac{1}{s}}_{\frac{p}{s}} \\
&= \|f\|_{W^1_{p,V}}
\end{aligned}$$

where we used that for $r > 1$, $\|f^{**}\|_r \sim \|f\|_r$. Therefore $W^1_{p,q,s,V} \subset W^1_{p,V}$, with $\|f\|_{\frac{q(p-s)}{p(q-s)},p} \geq C\|f\|_{W^1_{p,V}}$.

On the other hand, let $f \in W^1_{p,V}$. By the Calderón-Zygmund decomposition of Proposition 5.3.2, $f \in W^1_{s,V} + W^1_{q,V}$. Next,

$$\begin{aligned}
\|f\|_{\frac{q(p-s)}{p(q-s)},p} &\leq C\left\{\int_0^\infty \left(t^{\frac{q(s-p)}{p(q-s)}} t^{\frac{q}{q-s}} (T_{s**}f)^{\frac{1}{s}}(t^{\frac{qs}{q-s}})\right)^p \frac{dt}{t}\right\}^{\frac{1}{p}} \\
&+ C\left\{\int_0^\infty \left(t^{\frac{q(s-p)}{p(q-s)}} t \left(\int_{t^{\frac{qs}{q-s}}}^\infty (\mathcal{M}T_s f)^{*\frac{q}{s}}(u) du\right)^{\frac{1}{q}}\right)^p \frac{dt}{t}\right\}^{\frac{1}{p}} \\
&\leq C\|f\|_{W^1_{p,V}} + C\left\{\int_0^\infty t^{\frac{q(s-p)}{q-s}} t^{p-1} \left(\int_{t^{\frac{qs}{q-s}}}^\infty (\mathcal{M}T_s f)^{*\frac{q}{s}}(u) du\right)^{\frac{p}{q}} dt\right\}^{\frac{1}{p}} \\
&\leq C\|f\|_{W^1_{p,V}} + C\left\{\int_0^\infty t^{-\frac{p}{q}} \left(\int_t^\infty \left(u(\mathcal{M}T_s f)^{*\frac{q}{s}}(u)\right) \frac{du}{u}\right)^{\frac{p}{q}} dt\right\}^{\frac{1}{p}} \\
&\leq C\|f\|_{W^1_{p,V}} + C\left\{\int_0^\infty t^{-\frac{p}{q}} \left(\int_t^\infty \left(u(\mathcal{M}T_s f)^{*\frac{q}{s}}(u)\right)^{\frac{p}{q}} \frac{du}{u}\right) dt\right\}^{\frac{1}{p}} \\
&\leq C\|f\|_{W^1_{p,V}} + \frac{C}{1-\frac{p}{q}}\left\{\int_0^\infty t^{-\frac{p}{q}} (t(t^{\frac{p}{q}-1}(\mathcal{M}T_s f)^{*\frac{p}{s}}(t))) dt\right\}^{\frac{1}{p}} \\
&= C\|f\|_{W^1_{p,V}} + C\|(\mathcal{M}T_s f)^*\|^{\frac{1}{s}}_{\frac{p}{s}} \\
&\leq C\|f\|_{W^1_{p,V}} + C\|\mathcal{M}T_s f\|^{\frac{1}{s}}_{\frac{p}{s}} \\
&\leq C\|f\|_{W^1_{p,V}} + C\|T_s f\|^{\frac{1}{s}}_{\frac{p}{s}}
\end{aligned}$$

$$\leq C\|f\|_{W^1_{p,V}}$$

where we used the monotonicity of $(\mathcal{M}T_s f)^*$ together with $\frac{p}{q} < 1$, the following Hardy inequality

$$\int_0^\infty \left[\int_t^\infty g(u)du\right] t^{r-1}dt \leq \left(\frac{1}{r}\right) \int_0^\infty [ug(u)]\, u^{r-1}du$$

for $r = 1 - \frac{p}{q} > 0$, the fact that $\|g^*\|_l \sim \|g\|_l$ for all $l \geq 1$ and Theorem 5.2.2. Thus, $W^1_{p,V} \subset W^1_{p,q,s,V}$ with $\|f\|_{\frac{q(p-s)}{p(q-s)},p} \leq C\|f\|_{W^1_{p,V}}$. □

Let $A_V = \{q \in]1,\infty] : V \in RH_{qloc}\}$ and $q_0 = \sup A_V$, $B_M = \{s \in [1, q_0[: (P_{sloc}) \text{ holds }\}$ and $s_0 = \inf B_M$.

Corollary 5.5.1. *For all p, p_1, p_2 such that $s_0 < p_1 < p < p_2 < q_0$, $W^1_{p,V}$ is an interpolation space between $W^1_{p_1,V}$ and $W^1_{p_2,V}$.*

Proof. Since $p_1 > s_0$, M admits a Poincaré inequality $(P_{p_1 loc})$ and since $p_2 < q_0$, item 1. of Proposition 5.2.7 gives us that $V \in RH_{p_2 loc}$. Therefore, Theorem 5.1.3 yields the corollary. (We could prove this corollary also using the reiteration theorem.) □

Remark 5.5.2. *If (P_{1loc}) holds, then $s_0 = 1$ and the strict inequality at s_0 in Corollary 5.5.1 becomes large.*

5.6 Interpolation of homogeneous Sobolev spaces

Denote for $1 \leq r < \infty$, $\dot{T}_r f = |\nabla f|^r + |Vf|^r$, $\dot{T}_{r*} f = |\nabla f|^{r*} + |Vf|^{r*}$ and $\dot{T}_{r**} f = |\nabla f|^{r**} + |Vf|^{r**}$. For the estimation of the functional K for homogeneous Sobolev spaces we have the corresponding results:

Theorem 5.6.1. *Under the hypotheses of Theorem 5.1.6 with $q < \infty$ we have that*

1. *there exists C_1 such that for every $f \in \dot{W}^1_{s,V} + \dot{W}^1_{q,V}$ and all $t > 0$*

$$K(f, t, \dot{W}^1_{s,V}, \dot{W}^1_{q,V}) \geq C_1 \left\{ \left(\int_0^{t^{\frac{qs}{q-s}}} \dot{T}_{s*} f(u) du \right)^{\frac{1}{s}} + t \left(\int_{t^{\frac{qs}{q-s}}}^\infty \dot{T}_{q*} f(u) du \right)^{\frac{1}{q}} \right\};$$

2. *for $s \leq p < q$, there exists C_2 such that for every $f \in \dot{W}^1_{p,V}$ and all $t > 0$*

$$K(f, t, \dot{W}^1_{s,V}, \dot{W}^1_{q,V}) \leq C_2 \left\{ \left(\int_0^{t^{\frac{qs}{q-s}}} \dot{T}_{s*} f(u) du \right)^{\frac{1}{s}} + t \left(\int_{t^{\frac{qs}{q-s}}}^\infty \left(\mathcal{M}\dot{T}_s f\right)^{*\frac{q}{s}} (u) du \right)^{\frac{1}{q}} \right\}.$$

Theorem 5.6.2. *Under the hypotheses of Theorem 5.1.6 with $V \in RH_\infty$ we have that*

1. *there exists C_1 such that for every $f \in \dot{W}^1_{s,V} + \dot{W}^1_{\infty,V}$ and every $t > 0$*

$$K(f, t^{\frac{1}{s}}, \dot{W}^1_{s,V}, \dot{W}^1_{\infty,V}) \geq C_1 t^{\frac{1}{s}} (\dot{T}_{s**} f)^{\frac{1}{s}}(t);$$

2. for $s \leq p < \infty$, there exists C_2 such that for every $f \in \dot{W}^1_{p,V}$ and every $t > 0$

$$K(f, t^{\frac{1}{s}}, \dot{W}^1_{s,V}, \dot{W}^1_{\infty,V}) \leq C_2 t^{\frac{1}{s}} (\dot{T}_{s**} f)^{\frac{1}{s}}(t).$$

Before we prove Theorems 5.6.1, 5.6.2 and 5.1.6, we give two versions of a Calderón-Zygmund decomposition.

Proposition 5.6.3. *Let M be a complete non-compact Riemannian manifold satisfying (D). Let $1 \leq q < \infty$ and $V \in RH_q$. Assume that M admits a Poincaré inequality (P_s) for some $1 \leq s < q$. Let $s \leq p < q$ and consider $f \in \dot{W}^1_{p,V}$ and $\alpha > 0$. Then there exist a collection of balls $(B_i)_i$, functions $b_i \in \dot{W}^1_{s,V}$ and a function $g \in \dot{W}^1_{q,V}$ such that the following properties hold:*

$$f = g + \sum_i b_i \tag{5.25}$$

$$\int_{\cup_i B_i} \dot{T}_q g \, d\mu \leq C \, \alpha^q \mu(\cup_i B_i) \tag{5.26}$$

$$\text{supp } b_i \subset B_i \text{ and } \int_{B_i} \dot{T}_s b_i \, d\mu \leq C \alpha^s \mu(B_i) \tag{5.27}$$

$$\sum_i \mu(B_i) \leq C \alpha^{-p} \int \dot{T}_p f \, d\mu \tag{5.28}$$

$$\sum_i \chi_{B_i} \leq N \tag{5.29}$$

with C and N depending only on q, s and the constants in (D), (P_s) and the RH_q condition.

Proposition 5.6.4. *Let M be a complete non-compact Riemannian manifold satisfying (D). Consider $V \in RH_\infty$. Assume that M admits a Poincaré inequality (P_s) for some $1 \leq s < \infty$. Let $s \leq p < \infty$, $f \in \dot{W}^1_{p,V}$ and $\alpha > 0$. Then there exist a collection of balls $(B_i)_i$, functions $b_i \in \dot{W}^1_{s,V}(M)$ and a function g such that the following properties hold :*

$$f = g + \sum_i b_i \tag{5.30}$$

$$\dot{T}_1 g \leq C\alpha \quad \mu - a.e. \tag{5.31}$$

$$\text{supp } b_i \subset B_i \text{ and } \int_{B_i} \dot{T}_s b_i d\mu \leq C \alpha^s \mu(B_i) \tag{5.32}$$

$$\sum_i \mu(B_i) \leq C\alpha^{-p} \int \dot{T}_p f \, d\mu \tag{5.33}$$

$$\sum_i \chi_{B_i} \leq N \tag{5.34}$$

with C and N depending only on q, p and the constant in (D), (P_s) and the RH_∞ condition.

The proof of these two decompositions goes as in the case of non homogeneous Sobolev spaces, but taking $\Omega = \left\{x \in M : \mathcal{M}\dot{T}_s f(x) > \alpha^s\right\}$ as $\|f\|_p$ is not under control. We note that in the non homogeneous case, we used that $f \in L_p$ only to control $b \in L_s$ and $g \in L_\infty$ when $V \in RH_\infty$ and $\int_\Omega |g|^q d\mu$ when $V \in RH_q$ and $q < \infty$.

Proof of Theorem 5.6.1 and 5.6.2. We refer to [6] for the proof of Theorem 5.6.2. The proof of item 1. of Theorem 5.6.1 is the same as in the non homogeneous case. Let us turn to inequality 2. Consider $f \in \dot{W}^1_{p,V}$, $t > 0$ and $\alpha(t) = (\mathcal{M}\dot{T}_s f)^{*\frac{1}{s}}(t^{\frac{qs}{q-s}})$. By the Calderón-Zygmund decomposition with $\alpha = \alpha(t)$, f can be written $f = b + g$ with $\|b\|_{\dot{W}^1_{s,V}} \leq C\alpha(t)\mu(\Omega)^{\frac{1}{s}}$ and $\int_\Omega \dot{T}_q g d\mu \leq C\alpha^q(t)\mu(\Omega)$. Since we have $\mu(\Omega) \leq t^{\frac{qs}{q-s}}$, we get then as in the non homogeneous case

$$K(f, t, \dot{W}^1_{s,V}, \dot{W}^1_{q,V}) \leq Ct^{\frac{q}{q-s}}(\dot{T}_{s**}f)^{\frac{1}{s}}(t^{\frac{qs}{q-s}}) + Ct\left(\int_{t^{\frac{qs}{q-s}}}^\infty (\mathcal{M}\dot{T}_s f)^{*\frac{q}{s}}(u)du\right)^{\frac{1}{q}}.$$

□

Proof of Theorem 5.1.6. We refer to [6] when $q = \infty$. When $q < \infty$, the proof follows directly from Theorem 5.6.1. Indeed, item 1. of Theorem 5.6.1 gives us that

$$(\dot{W}^1_{s,V}, \dot{W}^1_{q,V})_{\frac{q(p-s)}{p(q-s)},p} \subset \dot{W}^1_{p,V}$$

with $\|f\|_{\dot{W}^1_{p,V}} \leq C\|f\|_{\frac{q(p-s)}{p(q-s)},p}$, while item 2. gives us as in section 5.5 for non homogeneous Sobolev spaces, that

$$\dot{W}^1_{p,V} \subset (\dot{W}^1_{s,V}, \dot{W}^1_{q,V})_{\frac{q(p-s)}{p(q-s)},p}$$

with $\|f\|_{\frac{q(p-s)}{p(q-s)},p} \leq C\|f\|_{\dot{W}^1_{p,V}}$. □

Let $A_V = \{q \in]1, \infty] : V \in RH_q\}$ and $q_0 = \sup A_V$, $B_M = \{s \in [1, q_0[: (P_s) \text{ holds }\}$ and $s_0 = \inf B_M$.

Corollary 5.6.5. *For all p, p_1, p_2 such that $s_0 < p_1 < p < p_2 < q_0$, $\dot{W}^1_{p,V}$ is an interpolation space between $\dot{W}^1_{p_1,V}$ and $\dot{W}^1_{p_2,V}$.*

Remark 5.6.6. *If (P_1) holds, then $s_0 = 1$ and the strict inequality at s_0 in Corollary 5.6.5 become large.*

5.7 Interpolation of Sobolev spaces on Lie Groups

Consider G a connected Lie group. Assume that G is unimodular and let $d\mu$ be a fixed Haar measure on G. Let $X_1, ..., X_k$ be a family of left invariant vector fields such that the X_i's satisfy a Hörmander condition. In this case the Carnot-Carathéodory metric ρ is a distance, and G equipped with the distance ρ is complete and defines the same topology as the topology of G as manifold (see [12] p. 1148). It is known that G has an exponential growth or polynomial growth. In the first case, G satisfies the local doubling property (D_{loc}) and admits a local Poincaré inequality (P_{1loc}). In the second case, it admits the global doubling property (D) and a global Poincaré inequality (P_1) (see [12], [16], [21], [24] for more details).

Definition 5.7.1 (Sobolev spaces $W^1_{p,V}$)**.** *For $1 \leq p < \infty$ and for a weight $V \in A_\infty$, we define the Sobolev space $W^1_{p,V}$ as the completion of C^∞ functions for the norm:*

$$\|u\|_{W^1_{p,V}} = \|f\|_p + \| \, |Xf| \, \|_p + \|Vf\|_p$$

where $|Xf| = \left(\sum_{i=1}^k |X_i f|^2\right)^{\frac{1}{2}}$.

Definition 5.7.2. *We denote by $W^1_{\infty,V}$ the space of all bounded Lipschitz functions f on G such that $\|Vf\|_\infty < \infty$ which is a Banach space.*

Proposition 5.7.3. *Let $V \in RH_{qloc}$ for some $1 \leq q < \infty$. Consider, for $1 \leq p < q$,*

$$H^1_{p,V} = \{f \in L_p(G) : \ |\nabla f| \ \text{ and } Vf \in L_p\}$$

and equip it with the same norm as $W^1_{p,V}$. Then as in Proposition 5.2.14 in the case of Riemannian manifolds, C_0^∞ is dense in $H^1_{p,V}$ and hence $W^1_{p,V} = H^1_{p,V}$

Interpolation of $W^1_{p,V}$: Let $V \in RH_{qloc}$ for some $1 < q \leq \infty$. To interpolate the $W^1_{p_i,V}$, we distinguish between the polynomial and the exponential growth cases. If G has polynomial growth and $V \in RH_q$, then we are in the global case. Otherwise we are in the local case. In the two cases we obtain the following theorem:

Theorem 5.7.4. *Let G be a connected Lie group as in the beginning of this section and assume that $V \in RH_{qloc}$ with $1 < q \leq \infty$. Denote $T_1 f = |f| + |Xf| + |Vf|$, $T_{r*} f = |f|^{r*} + |Xf|^{r*} + |Vf|^{r*}$ for $1 \leq r < \infty$.*

a. If $q < \infty$, we have that

1. there exists $C_1 > 0$ such that for every $f \in W^1_{1,V} + W^1_{q,V}$, and every $t > 0$

$$K(f, t, W^1_{1,V}, W^1_{q,V}) \geq C_1 \left\{ \left(\int_0^{t^{\frac{q}{q-1}}} T_{1*} f(u) du \right)^{\frac{1}{s}} + t \left(\int_{t^{\frac{q}{q-1}}}^\infty T_{q*} f(u) du \right)^{\frac{1}{q}} \right\};$$

2. *for $1 \leq p < \infty$, there exists $C_2 > 0$ such that for every $f \in W^1_{p,V}$ and every $t > 0$,*

$$K(f,t,W^1_{1,V},W^1_{q,V}) \leq C_2 \left\{ \int_0^{t^{\frac{q}{q-1}}} T_{1*}f(u)du + t\left(\int_{t^{\frac{q}{q-1}}}^{\infty} (\mathcal{M}T_1 f)^{*q}(u)du\right)^{\frac{1}{q}} \right\}.$$

b. *If $q = \infty$, then for every $f \in W^1_{1,V} + W^1_{\infty,V}$ and for every $t > 0$ we have*

$$K(f,t,W^1_{1,V},W^1_{\infty,V}) \sim \int_0^t T_{1*}f(u)du.$$

Theorem 5.7.5. *Let G be as above, $V \in RH_{qloc}$, for some $1 < q \leq \infty$. Then, for $1 \leq p_1 < p < p_2 < q_0$, $W^1_{p,V}$ is an interpolation space between $W^1_{p_1,V}$ and $W^1_{p_2,V}$ where $q_0 = \sup\{q \in]1,\infty] : V \in RH_{qloc}\}$.*

Proof. Combine Theorem 5.7.4 and the reiteration theorem. □

Remark 5.7.6. *Define the homogeneous Sobolev spaces $\dot{W}^1_{p,V}$ as the vector space of distributions f such that Xf and $Vf \in L_p$ and equip this space with the norm*

$$\|f\|_{\dot{W}^1_{p,V}} = \| \, |Xf| \, \|_p + \|Vf\|_p$$

and $\dot{W}^1_{\infty,V}$ the space of all Lipschitz functions f on G with $\|Vf\|_\infty < \infty$. Theses spaces are Banach spaces. If G has polynomial growth, we obtain the interpolation results analog to those of section 5.6

Examples: For examples of spaces for which our interpolation result applies see section 11 in [6].

Examples of RH_q weights in $\mathbb{R}^n$ for $q < \infty$ are the power weights $|x|^{-\alpha}$ with $-\infty < \alpha < \frac{n}{q}$ and positive polynomials for $q = \infty$. We give an other example for RH_q weights in a Riemannian manifold M: consider f, $g \in L_1(M)$, $1 \leq r < \infty$ and $1 < s \leq \infty$, then $V(x) = (\mathcal{M}f(x))^{-(r-1)} \in RH_\infty$ and $W(x) = (\mathcal{M}g(x))^{\frac{1}{s}} \in RH_q$ for all $q < s$ ($q = s$ if $s = \infty$) and hence $V + W \in RH_q$ for all $q < s$ ($q = s$ if $s = \infty$) (see [5], [4] for details).

Appendix

Proof of Proposition 5.2.14: We follow the method of Davies in [13]. Let $L(f) = L_0(f) + L_1(f) + L_2(f) := \int_M |f|^p d\mu + \int_M |\nabla f|^p d\mu + \int_M |Vf|^p d\mu$. We will prove the proposition in three steps:

1. Let $f \in H^1_{p,V}$. Fix $p_0 \in M$ and let $\varphi \in C_0^\infty(\mathbb{R})$ satisfies $\varphi \geq 0$, $\varphi(\alpha) = 1$ if $\alpha < 1$ and $\varphi(\alpha) = 0$ if $\alpha > 2$. Then put $f_n(x) = f(x)\varphi(\frac{d(x,p_0)}{n})$. Elementary calculations establish that f_n lies in $H^1_{p,V}$. Moreover,

$$L(f - f_n) = \int_M |f(x)\{1 - \varphi(\frac{d(x,p_0)}{n})\}|^p d\mu(x)$$

$$
\begin{aligned}
&+\int_M |\nabla f(x)\{1-\varphi(\frac{d(x,p_0)}{n})\} - n^{-1} f(x)\varphi'(\frac{d(x,p_0)}{n})\nabla(d(x,p_0))|^p d\mu(x) \\
&+\int_M |V^{\frac{1}{2}}(x)f(x)(1-\varphi(\frac{d(x,p_0)}{n}))|^p d\mu(x) \\
&\leq \int_M |f(x)\{1-\varphi(\frac{d(x,p_0)}{n})\}|^p d\mu(x) \\
&+2^{p-1}\int_M |\nabla f(x)\{1-\varphi(\frac{d(x,p_0)}{n})\}|^p d\mu(x) \\
&+2^{p-1}n^{-p}\int_M |f(x)|^p|\varphi'(\frac{d(x,p_0)}{n})|^p d\mu(x) \\
&+\int_M V^p(x)|f(x)|^p|1-\varphi(\frac{d(x,p_0)}{n})|^p d\mu(x).
\end{aligned}
$$

This converges to zero as $n \to \infty$ by the dominated convergence theorem. Thus the the set of functions $f \in H^1_{p,V}$ with compact support is dense in $H^1_{p,V}$.

2. Let $f \in H^1_{p,V}$ with compact support. Let $n > 0$ and $F_n : \mathbb{R} \to \mathbb{R}$ be a smooth increasing function such that

$$
F_n(s) = \begin{cases} s & \text{if } -n \leq s \leq n, \\ n+1 & \text{if } s \geq n+2, \\ -n-1 & \text{if } s \leq -n-2 \end{cases}
$$

and $0 \leq F_n'(s) \leq 1$ for all $s \in \mathbb{R}$. If we put $f_n(x) := F_n(f(x))$ then $|f_n(x)| \leq |f(x)|$ and $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in M$. The dominated convergence theorem yields

$$
\lim_{n\to\infty} L_0(f-f_n) = \lim_{n\to\infty}\int_M |f-f_n|^p d\mu = 0
$$

and

$$
\lim_{n\to\infty} L_2(f-f_n) = \lim_{n\to\infty}\int_M V^p|f-f_n|^p d\mu = 0
$$

Also

$$
\begin{aligned}
\lim_{n\to\infty} L_1(f-f_n) &= \lim_{n\to\infty}\int_M |\nabla f - F_n'(f(x))\nabla f|^p d\mu(x) \\
&= \lim_{n\to\infty}\int_M |1-F_n'(f(x))|^p|\nabla f(x)|^p d\mu(x) \\
&= 0.
\end{aligned}
$$

Therefore the set of bounded functions $f \in H^1_{p,V}$ with compact support is dense in $H^1_{p,V}$.

3. Let now $f \in H^1_{p,V}$ be bounded and with compact support. Consider locally finite coverings of M, $(U_k)_k$, $(V_k)_k$ with $\overline{U_k} \subset V_k$, V_k being endowed with a

real coordinate chart ψ_k. Let $(\varphi_k)_k$ be a partition of unity subordinated to the covering $(U_k)_k$, that is, for all k, φ_k is a C^∞ function compactly supported in U_k, $0 \le \varphi_k \le 1$ and $\sum_{k=1}^\infty \varphi_k = 1$. There exists a finite subset I of $\mathbb{N}$ such that $f = \sum_{k\in I} f\varphi_k := \sum_{k\in I} f_k$. Take $\varepsilon > 0$. The functions $g_k = f_k \circ \psi_k^{-1}$ (which belongs to $W_p^1(\mathbb{R}^n)$ since f and $|\nabla f| \in L_{p,loc}$) can be approximated by smooth functions w_k with compact support (standard approximation by convolution) in such a way that $w_k = g_k * \alpha_k$ where $\alpha_k \in C_0^\infty(\mathbb{R}^n)$ is a standard mollifier, $\operatorname{supp} w_k \subset \psi_k(V_k)$ and $\|g_k - w_k\|_{W_p^1} \le \frac{\varepsilon}{2^k}$. Define

$$h_k(x) = \begin{cases} w_k \circ \psi_k(x) & \text{if } x \in V_k, \\ 0 & \text{otherwise.} \end{cases}$$

Thus $\operatorname{supp} h_k \subset V_k$ and

$$\|f_k - h_k\|_p = \left(\int_{V_k} |f_k - h_k|^p d\mu\right)^{\frac{1}{p}} = \|g_k - w_k\|_p \le \frac{\varepsilon}{2^k}.$$

$$\|\,|\nabla(f_k - h_k)|\|_p = \left(\int_{V_k} |\nabla(f_k - h_k)|^p d\mu\right)^{\frac{1}{p}} = \|\,|\nabla(g_k - w_k)|\,\|_p \le \frac{\varepsilon}{2^k}.$$

Hence the series $\sum_{k\in I}(f_k - h_k)$ is convergent in W_p^1. We have also $\sum_{k\in I}(f_k - h_k) = f - h_\varepsilon$ where $h_\varepsilon = \sum_{k\in I} h_k$, and $\|f - h_\varepsilon\|_{W_p^1} \le \sum_{k\in I} \|f_k - h_k\|_{W_p^1} \le \varepsilon$. If $l_\varepsilon := |f - h_\varepsilon|^p$ then $\lim_{\varepsilon\to 0} \|l_\varepsilon\|_1 = 0$ and there exists a compact set K which contains the support of every l_ε. We have $\|h_\varepsilon\|_\infty \le \sharp I \|f\|_\infty$ for all $\varepsilon > 0$. Indeed

$$\begin{aligned}
\sum_{k\in I} |h_k(x)| &= \sum_{k\in I} \int_{\mathbb{R}^n} |g_k(y)|\, \alpha_k(\psi_k(x) - y) dy \\
&= \int_{\mathbb{R}^n} \sum_{k\in I} |f\varphi_k(\psi_k^{-1}(y))|\, \alpha_k(\psi_k(x) - y) dy \\
&\le \|f\|_\infty \int_{\mathbb{R}^n} \sum_{k\in I} \varphi_k(\psi_k^{-1}(y))\, \alpha_k(\psi_k(x) - y) dy \\
&\le \|f\|_\infty \sum_{k\in I} \int_{\psi_k(U_k)} \varphi_k(\psi_k^{-1}(y))\, \alpha_k(\psi_k(x) - y) dy \\
&\le \|f\|_\infty \sum_{k\in I} \int_{\mathbb{R}^n} \alpha_k(z) dz \\
&\le \sharp I \|f\|_\infty.
\end{aligned}$$

It follows that $\|l_\varepsilon\|_\infty \le 2^{p-1}(1 + \sharp I)\|f\|_\infty^p = C\|f\|_\infty^p$ (C being independent of ε it depends just on f) for all $\varepsilon > 0$. We claim that these facts suffice to deduce that $\lim_{\varepsilon\to 0} \int_M l_\varepsilon V^p d\mu = 0$, that is

$$\lim_{\varepsilon\to 0} L_2(f - l_\varepsilon) = 0.$$

Hence C_0^∞ is dense in $H_{p,V}^1$.

4. It remains to prove the above claim. Since $V \in RH_{ploc}$, there exists $r > p$ such that $V \in RH_{rloc}$ and therefore $V^p \in L_{t,loc}$ where $t = \frac{r}{p} > 1$. Hence, by Hölder inequality we get

$$\begin{aligned} 0 \leq \int_M l_\varepsilon V^p d\mu &= \int_K l_\varepsilon V^p d\mu \\ &\leq \|l_\varepsilon\|_{L_{t'}(K)} \|V^p\|_{L_t(K)} \\ &\leq C\|f\|_\infty^{\frac{p}{r}} \varepsilon^{\frac{1}{t'}} \end{aligned}$$

for all $\varepsilon > 0$, t' being the conjugate exponent of t. The proof of Proposition 5.2.14 is therefore complete.

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Partie II

Applications aux inégalités fonctionnelles

Chapitre 6

Interpolation of Sobolev spaces, Littlewood-Paley inequalities and Riesz transforms on graphs

Ce chapitre est rédigé sous forme d'article en collaboration avec Emmanuel Russ.

Abstract. Let Γ be a graph endowed with a reversible Markov kernel p, and let P the operator defined by $Pf(x) = \sum_y p(x,y)f(y)$. Denote by ∇ the discrete gradient. We give sufficient conditions on Γ in order to compare $\|\nabla f\|_p$ and $\left\|(I-P)^{1/2}f\right\|_p$ uniformly in f for $1 < p < +\infty$. These conditions are different for $p < 2$ and $p > 2$. The proofs rely on recent techniques developed to handle operators beyond the class of Calderón-Zygmund operators. For our purpose, we also prove Littlewood-Paley inequalities and interpolation results for Sobolev spaces in this context, which are of independent interest.

6.1 Introduction and results

It is well-known that, if $n \geq 1$, $\|\nabla f\|_{L^p(\mathbb{R}^n)}$ and $\left\|(-\Delta)^{1/2}f\right\|_{L^p(\mathbb{R}^n)}$ are comparable uniformly in f for all $1 < p < +\infty$. This fact means that the classical Sobolev space $W^{1,p}(\mathbb{R}^n)$ defined by means of the gradient coincides with the Sobolev space defined through the Laplace operator. This is interesting in particular because ∇ is a local operator, while $(-\Delta)^{1/2}$ is not.

Generalizations of this result to geometric contexts can be given. On a Riemannian manifold M, it was asked by Strichartz in [36] whether, if $1 < p < +\infty$, there exists $C_p > 0$ such that, for all function $f \in C_0^\infty(M)$,

$$C_p^{-1}\left\|\Delta^{1/2}f\right\|_p \leq \||\nabla f|\|_p \leq C_p\left\|\Delta^{1/2}f\right\|_p. \tag{6.1}$$

Under suitable assumptions on M, which can be formulated, for instance, in terms of the volume growth of balls in M, uniform L^2 Poincaré inequalities on balls of M, estimates on the heat semigroup (i.e. the semigroup generated by Δ) or the Ricci curvature, each of the two inequalities contained in (6.1) holds for a range of p's (which is, in general, different for the two inequalities). The second inequality in (6.1) means that the Riesz transform $\nabla\Delta^{-1/2}$ is L^p-bounded. We refer to ([4, 6, 9, 16]) and the references therein.

In the present paper, we consider a graph equipped with a discrete gradient and a discrete "Laplacian" and investigate the discrete counterpart of (6.1). To that purpose, we prove, among other things, an interpolation result for Sobolev spaces defined via the gradient, already considered in [31], as well as L^p bounds for Littlewood-Paley functionals.

Let us give precise definitions of our framework. The following presentation is borrowed from [21]. Let Γ be an infinite set and $\mu_{xy} = \mu_{yx} \geq 0$ a symmetric weight on $\Gamma \times \Gamma$. We call (Γ, μ) a weighted graph. In the sequel, we write most of the time Γ instead of (Γ, μ), somewhat abusively. If $x, y \in \Gamma$, say that $x \sim y$ if and only if $\mu_{xy} > 0$. For $x, y \in \Gamma$, a path joining x to y is a finite sequence of edges $x_0 = x, ..., x_N = y$ such that, for all $0 \leq i \leq N-1$, $x_i \sim x_{i+1}$. The length of such a path is N. Assume that Γ is connected, which means that, for all $x, y \in \Gamma$, there exists a path joining x to y. For all $x, y \in \Gamma$, the distance between x and y, denoted by $d(x, y)$, is the shortest length of a path joining x and y. For all $x \in \Gamma$ and all $r \geq 0$, let $B(x, r) = \{y \in \Gamma,\ d(y, x) \leq r\}$. In the sequel, we always assume that Γ is locally uniformly finite, which means that there exists $N \in \mathbb{N}^*$ such that, for all $x \in \Gamma$, $\sharp B(x, 1) \leq N$. If $B = B(x, r)$ is a ball, set $\alpha B = B(x, \alpha r)$ for all $\alpha > 0$, and write $C_1(B) = 4B$ and $C_j(B) = 2^{j+1}B \setminus 2^j B$ for all integer $j \geq 2$.

For any subset $A \in \Gamma$, set

$$\partial A = \{x \in A;\ \exists y \sim x,\ y \notin A\}.$$

For all $x \in \Gamma$, set $m(x) = \sum\limits_{y \sim x} \mu_{xy}$. We always assume in the sequel that $m(x) > 0$ for all $x \in \Gamma$. If $A \subset \Gamma$, define $m(A) = \sum\limits_{x \in A} m(x)$. For all $x \in \Gamma$ and $r > 0$, write $V(x, r)$ instead of $m(B(x, r))$ and, if B is a ball, $m(B)$ will be denoted by $V(B)$.

For all $1 \leq p < +\infty$, say that a function f on Γ belongs to $L^p(\Gamma, m)$ (or $L^p(\Gamma)$) if

$$\|f\|_p := \left(\sum_{x \in \Gamma} |f(x)|^p\, m(x)\right)^{1/p} < +\infty.$$

Say that $f \in L^\infty(\Gamma, m)$ (or $L^\infty(\Gamma)$) if

$$\|f\|_\infty := \sup_{x \in \Gamma} |f(x)| < +\infty.$$

Define $p(x, y) = \mu_{xy}/m(x)$ for all $x, y \in \Gamma$. Observe that $p(x, y) = 0$ if $d(x, y) \geq 2$. Set also

$$p_0(x, y) = \delta(x, y)$$

and, for all $k \in \mathbb{N}$ and all $x, y \in \Gamma$,

$$p_{k+1}(x, y) = \sum_{z \in \Gamma} p(x, z) p_k(z, y).$$

The p_k's are called the iterates of p. Notice that, for all $x \in \Gamma$, there are at most N non-zero terms in this sum. Observe also that, for all $x \in \Gamma$,

$$\sum_{y\in\Gamma} p(x,y) = 1 \tag{6.2}$$

and, for all $x, y \in \Gamma$,

$$p(x,y)m(x) = p(y,x)m(y). \tag{6.3}$$

For all function f on Γ and all $x \in \Gamma$, define

$$Pf(x) = \sum_{y\in\Gamma} p(x,y)f(y)$$

(again, this sum has at most N non-zero terms). The length of the gradient ∇ on Γ is defined, for all function f on Γ and all $x \in \Gamma$, by

$$\nabla f(x) = \left(\frac{1}{2}\sum_{y\in\Gamma} p(x,y)\left|f(y)-f(x)\right|^2\right)^{1/2}$$

(this definition is taken from [17]). Let us now define Sobolev spaces on Γ. Let $1 \leq p \leq +\infty$. Say that a scalar-valued function f on Γ belongs to the (inhomogeneous) Sobolev space $W^{1,p}(\Gamma)$ (see also [31], [25]) if and only if

$$\|f\|_{W^{1,p}(\Gamma)} := \|f\|_{L^p(\Gamma)} + \|\nabla f\|_{L^p(\Gamma)} < +\infty.$$

We will also consider the homogeneous versions of these spaces. For $1 \leq p \leq +\infty$, define $\dot{E}^{1,p}(\Gamma)$ as the space of all scalar-valued functions f on Γ such that $\nabla f \in L^p(\Gamma)$, equipped with the semi-norm

$$\|f\|_{\dot{E}^{1,p}(\Gamma)} := \|\nabla f\|_{L^p(\Gamma)}.$$

Then $\dot{W}^{1,p}(\Gamma)$ is the quotient space $\dot{E}^{1,p}(\Gamma)/(\mathbb{R} \cap L^p(\Gamma))$, equipped with the corresponding norm. Note that, for $1 \leq p < +\infty$, $\mathbb{R} \cap L^p(\Gamma)$ is equal to 0 if $m(\Gamma) = \infty$ and to $\mathbb{R}$ if $m(\Gamma) < +\infty$, while for $p = +\infty$, $\mathbb{R} \cap L^\infty(\Gamma) = \mathbb{R}$. It is then routine to check that both inhomogeneous and homogeneous Sobolev spaces on Γ are Banach spaces.

As in $\mathbb{R}^n$ or in the context of Riemannian manifolds, we wish to compare these Sobolev spaces with the ones defined through a “Laplace” operator. It is easy to see that, if $f \in \dot{E}^{1,2}(\Gamma)$,

$$\begin{aligned} \langle (I-P)f, f\rangle &= \sum_{x,y} p(x,y)(f(x)-f(y))f(x)m(x) \\ &= \frac{1}{2}\sum_{x,y} p(x,y)\left|f(x)-f(y)\right|^2 m(x) \\ &= \|\nabla f\|_2^2, \end{aligned}$$

where we use (6.2) in the second equality and (6.3) in the third one.

Because of (6.3), the operator P is self-adjoint on $L^2(\Gamma)$ and $I-P$, which can be considered as a discrete "Laplace" operator, is non-negative and self-adjoint on $L^2(\Gamma)$. By means of spectral theory, one defines its square root $(I-P)^{1/2}$. The previous computation shows that $\left\|(I-P)^{1/2}f\right\|_2 = \|\nabla f\|_2$. We now focus on L^p versions of this equality.

To that purpose, we consider separately two inequalities, the validity of which will be discussed in the sequel. Let $1<p<+\infty$. The first inequality we look at says that there exists $C_p>0$ such that, for all function f on Γ with, say, bounded support,

$$(R_p) \qquad \|\nabla f\|_p \leq C_p \left\|(I-P)^{1/2}f\right\|_p$$

This inequality means that the sublinear operator $\nabla(I-P)^{-1/2}$, which is nothing but the Riesz transform associated with $(I-P)$, is L^p-bounded[1].
The second inequality under consideration says that there exists $C_p>0$ such that, for all function $f\in \dot{E}^{1,p}(\Gamma)$,

$$(RR_p) \qquad \left\|(I-P)^{1/2}f\right\|_p \leq C_p \|\nabla f\|_p .$$

(The notations (R_p) and (RR_p) are borrowed from [4].) We have just seen that (R_2) and (RR_2) always hold. A well-known fact (see [32] for a proof in this context) is that, if (R_p) holds for some $1<p<+\infty$, then (RR_p) holds with p' such that $1/p+1/p'=1$, while the converse is unclear in this context (it is false in the case of Riemannian manifolds, see [4]). Thus, we have to consider four distinct issues: (R_p) for $p<2$, (R_p) for $p>2$, (RR_p) for $p<2$, (RR_p) for $p>2$.

Let us first consider (R_p) when $p<2$. This problem was dealt with in [32], and we just recall the result proved therein, which involves some further assumptions on Γ. The first one is of geometric nature. Say that (Γ,μ) satisfies the doubling property if there exists $C>0$ such that, for all $x\in\Gamma$ and all $r>0$,

$$(D) \qquad V(x,2r)\leq CV(x,r).$$

Note that this assumption implies that there exist $C,D>0$ such that, for all $x\in\Gamma$, all $r>0$ and all $\theta>1$,

$$(6.4) \qquad V(x,\theta r)\leq C\theta^D V(x,r).$$

Remark 6.1.1. *Observe also that, since (Γ,μ) is infinite, it is also unbounded (since it is locally uniformly finite) so that, if (D) holds, then $m(\Gamma)=+\infty$ (see [30]).*

The second assumption on (Γ,μ) is a uniform lower bound for $p(x,y)$ when $x\sim y$, *i.e.* when $p(x,y)>0$. For $\alpha>0$, say that (Γ,μ) satisfies the condition $\Delta(\alpha)$ if, for all $x,y\in\Gamma$,

$$(\Delta(\alpha)) \qquad (x\sim y \Leftrightarrow \mu_{xy}\geq \alpha m(x)) \text{ and } x\sim x.$$

[1] Say that a sublinear operator T is L^p-bounded, or is of strong type (p,p), if there exists $C>0$ such that $\|Tf\|_p\leq C\|f\|_p$ for all $f\in L^p(\Gamma)$. Say that it is of weak type (p,p) if there exists $C>0$ such that $m(\{x\in\Gamma,\ |Tf(x)|>\lambda\})\leq \frac{C}{\lambda^p}\|f\|_p^p$ for all $f\in L^p(\Gamma)$ and all $\lambda>0$.

The next two assumptions on (Γ,μ) are pointwise upper bounds for the iterates of p. Say that (Γ,μ) satisfies (DUE) (a on-diagonal upper estimate) if there exists $C>0$ such that, for all $x\in\Gamma$ and all $k\in\mathbb{N}^*$,

$$(DUE) \qquad p_k(x,x)\leq \frac{Cm(x)}{V(x,\sqrt{k})}.$$

Say that (Γ,μ) satisfies (UE) (an upper estimate) if there exist $C,c>0$ such that, for all $x,y\in\Gamma$ and all $k\in\mathbb{N}^*$,

$$(UE) \qquad p_k(x,y)\leq \frac{Cm(x)}{V(x,\sqrt{k})}e^{-c\frac{d^2(x,y)}{k}}.$$

Recall that, under assumption (D), estimates (DUE) and (UE) are equivalent (and the conjunction of (D) and (DUE) is also equivalent to a Faber-Krahn inequality, [17], Theorem 1.1). The following result holds:

Theorem 6.1.2. *([32]) Under assumptions (D), $(\Delta(\alpha))$ and (DUE), (R_p) holds for all $1<p\leq 2$. Moreover, the Riesz transform is of weak $(1,1)$ type, which means that there exists $C>0$ such that, for all $\lambda>0$ and all function $f\in L^1(\Gamma)$,*

$$m\left(\left\{x\in\Gamma;\ \nabla(I-P)^{-1/2}f(x)>\lambda\right\}\right)\leq \frac{C}{\lambda}\left\Vert f\right\Vert_1.$$

As a consequence, under the same assumptions, (RR_p) holds for all $2\leq p<+\infty$.

When $p>2$, assumptions (D), (UE) and $(\Delta(\alpha))$ are not sufficient to ensure the validity of (R_p), as the example of two copies of $\mathbb{Z}^2$ linked between with an edge shows (see [32], Section 4). More precisely, in this example, the fact that (R_p) does not hold is related to the absence of an L^2 Poincaré inequality on balls, as explained in Section 4 of [32][2]. Say that (Γ,μ) satisfies a scaled L^2 Poincaré inequality on balls (this inequality will be denoted by (P_2) in the sequel) if there exists $C>0$ such that, for any $x\in\Gamma$, any $r>0$ and any function f locally square integrable on Γ such that ∇f is locally square integrable on Γ,

$$(P_2) \qquad \sum_{y\in B(x,r)}\left|f(y)-f_B\right|^2 m(y)\leq Cr^2\sum_{y\in B(x,r)}\left|\nabla f(y)\right|^2 m(y),$$

where

$$f_B=\frac{1}{V(B)}\sum_{x\in B}f(x)m(x)$$

is the mean value of f on B. Under assumptions (D), (P_2) and $(\Delta(\alpha))$, not only does (UE) hold, but the iterates of p also satisfy a pointwise Gaussian lower bound.

[2]On Riemannian manifolds, the L^2 Poincaré inequality on balls is neither necessary, nor sufficient to ensure that the Riesz transform is L^p-bounded for all $p\in(2,\infty)$, see [4] and the references therein. We do not know if the corresponding assertion holds in the context of graphs.

Namely, there exist $c_1, C_1, c_2, C_2 > 0$ such that, for all $n \geq 1$ and all $x, y \in \Gamma$ with $d(x,y) \leq n$,

$$(LUE) \qquad \frac{c_1 m(x)}{V(x,\sqrt{n})} e^{-C_1 \frac{d^2(x,y)}{n}} \leq p_n(x,y) \leq \frac{C_2 m(x)}{V(x,\sqrt{n})} e^{-c_2 \frac{d^2(x,y)}{n}}.$$

Actually, (LUE) is equivalent to the conjunction of (D), (P_2) and $(\Delta(\alpha))$, and also to a discrete parabolic Harnack inequality, see [21] (see also [5] for another approach of (LUE)).

Let $p > 2$ and assume that (R_p) holds. Then, if $f \in L^p(\Gamma)$ and $n \geq 1$,

$$(G_p) \qquad \|\nabla P^n f\|_p \leq C_p \left\|(I-P)^{1/2} P^n f\right\|_p \leq \frac{C'_p}{\sqrt{n}} \|f\|_p .$$

The first inequality in (G_p) follows from (R_p). The second one is due to the analyticity of P on $L^p(\Gamma)$. More precisely, as was explained in [32], assumption $\Delta(\alpha)$ implies that -1 does not belong to the spectrum of P on $L^2(\Gamma)$. As a consequence, P is analytic on $L^2(\Gamma)$ (see [18], Proposition 3), and since P is submarkovian, P is also analytic on $L^p(\Gamma)$ (see [18], p. 426). Proposition 2 in [18] therefore yields the second inequality in (G_p). Thus, condition (G_p) is necessary for (R_p) to hold. Our first result is that, under assumptions (D), (P_2) and $(\Delta(\alpha))$, for all $q > 2$, condition (G_q) is also sufficient for (R_p) to hold for all $2 < p < q$:

Theorem 6.1.3. *Let $p_0 \in (2, +\infty]$. Assume that (Γ, μ) satisfies (D), (P_2), $(\Delta(\alpha))$ and (G_{p_0}). Then, for all $2 \leq p < p_0$, (R_p) holds. As a consequence, if p'_0 is such that $1/p_0 + 1/p'_0 = 1$, (RR_p) holds for all $p'_0 < p \leq 2$.*

An immediate consequence of Theorem 6.1.3 and the previous discussion is the following result:

Theorem 6.1.4. *Assume that (Γ, μ) satisfies (D), (P_2) and $(\Delta(\alpha))$. Let $p_0 \in (2, +\infty]$. Then, the following two assertions are equivalent:*

(i) for all $p \in (2, p_0)$, (G_p) holds,

(ii) for all $p \in (2, p_0)$, (R_p) holds.

Let us now focus on (RR_p). As already seen, (RR_p) holds for all $p > 2$ under (D), $(\Delta(\alpha))$ and (DUE), and for all $p'_0 < p < 2$ under (D), (P_2), $(\Delta(\alpha))$ and (G_{p_0}) if $p_0 > 2$ and $1/p_0 + 1/p'_0 = 1$. However, we can also give a sufficient condition for (RR_p) to hold for all $p \in (q_0, 2)$ (for some $q_0 < 2$) which does not involve any assumption such that (G_{p_0}). For $1 \leq p < +\infty$, say that (Γ, μ) satisfies a scaled L^p Poincaré inequality on balls (this inequality will be denoted by (P_p) in the sequel) if there exists $C > 0$ such that, for any $x \in \Gamma$, any $r > 0$ and any function f on Γ such that $|f|^p$ and $|\nabla f|^p$ are locally integrable on Γ,

$$(P_p) \qquad \sum_{y \in B(x,r)} |f(y) - f_B|^p \, m(y) \leq C r^p \sum_{y \in B(x,r)} |\nabla f(y)|^p \, m(y).$$

If $1 \leq p < q < +\infty$, then (P_p) implies (P_q) (this is a very general statement on spaces of homogeneous type, *i.e.* on metric measured spaces where (D) holds, see [26]). The converse implication does not hold but an L^p Poincaré inequality still has a self-improvement in the following sense:

Proposition 6.1.5. *Let (Γ, μ) satisfy (D). Then, for all $p \in (1, +\infty)$, if (P_p) holds, there exists $\varepsilon > 0$ such that $(P_{p-\varepsilon})$ holds.*

This deep result actually holds in the general context of spaces of homogeneous type, *i.e.* when (D) holds, see [29].

Assuming that (P_q) holds for some $q < 2$, we establish (RR_p) for $q < p < 2$:

Theorem 6.1.6. *Let $1 \leq q < 2$. Assume that (D), $(\Delta(\alpha))$ and (P_q) hold. Then, for all $q < p < 2$, (RR_p) holds. Moreover, there exists $C > 0$ such that, for all $\lambda > 0$,*

$$m\left(\left\{x \in \Gamma;\ \left|(I-P)^{1/2} f(x)\right| > \lambda\right\}\right) \leq \frac{C}{\lambda^q} \left\|\nabla f\right\|_q^q. \tag{6.5}$$

As a corollary of Theorem 6.1.2, Proposition 6.1.5 and Theorem 6.1.6, we get the following consequence:

Corollary 6.1.7. *Assume that (D), $(\Delta(\alpha))$ and (P_2) hold. Then, there exists $\varepsilon > 0$ such that, for all $2 - \varepsilon < p \leq 2$, $\left\|\nabla f\right\|_p \sim \left\|(I-P)^{1/2} f\right\|_p$.*

In this corollary, the notation $\left\|\nabla f\right\|_p \sim \left\|(I-P)^{1/2} f\right\|_p$ means that there exists $C_p > 0$ such that

$$C_p^{-1} \left\|\nabla f\right\|_p \leq \left\|(I-P)^{1/2} f\right\|_p \leq C_p \left\|\nabla f\right\|_p$$

for all function f.

Let us briefly describe the proofs of our results. Let us first consider Theorem 6.1.3. The operator $T = \nabla(I-P)^{-1/2}$ can be written as

$$T = \nabla\left(\sum_{k=0}^{+\infty} a_k P^k\right),$$

where the a_k's are defined by the expansion

$$(1-x)^{-1/2} = \sum_{k=0}^{+\infty} a_k x^k \tag{6.6}$$

for $-1 < x < 1$. The kernel of T is therefore given by

$$\nabla_x\left(\sum_{k=0}^{+\infty} a_k p_k(x,y)\right).$$

It was proved in [33] that, under (D) and (P_2), this kernel satisfies the Hörmander integral condition, which implies the $H^1(\Gamma) - L^1(\Gamma)$ boundedness of T and therefore its $L^p(\Gamma)$-boundedness for all $1 < p < 2$, where $H^1(\Gamma)$ denotes the Hardy space on Γ

defined in the sense of Coifman and Weiss ([15]). However, the Hörmander integral condition does not yield any information on the L^p-boundedness of T for $p > 2$. The proof of Theorem 6.1.3 actually relies on a theorem due to Auscher, Coulhon, Duong and Hofmann ([6]), which, given some $p_0 \in (2, +\infty]$, provides sufficient conditions for an L^2-bounded sublinear operator to be L^p-bounded for $2 < p < p_0$. Let us recall this theorem here in the form to be used in the sequel for the sake of completeness (see [6], Theorem 2.1, [3], Theorem 2.2):

Theorem 6.1.8. *Let $p_0 \in (2, +\infty]$. Assume that Γ satisfies the doubling property (D) and let T be a sublinear operator acting on $L^2(\Gamma)$. For any ball B, let A_B be a linear operator acting on $L^2(\Gamma)$, and assume that there exists $C > 0$ such that, for all $f \in L^2(\Gamma)$, all $x \in \Gamma$ and all ball $B \ni x$,*

$$\frac{1}{V^{1/2}(B)} \|T(I - A_B)f\|_{L^2(B)} \leq C \left(\mathcal{M}(|f|^2)\right)^{1/2}(x) \tag{6.7}$$

and

$$\frac{1}{V^{1/p_0}(B)} \|TA_B f\|_{L^{p_0}(B)} \leq C \left(\mathcal{M}(|Tf|^2)\right)^{1/2}(x). \tag{6.8}$$

If $2 < p < p_0$ and if, for all $f \in L^p(\Gamma)$, $Tf \in L^p(\Gamma)$, then there exists $C_p > 0$ such that, for all $f \in L^2(\Gamma) \cap L^p(\Gamma)$,

$$\|Tf\|_{L^p(\Gamma)} \leq C_p \|f\|_{L^p(\Gamma)}.$$

Notice that, to simplify the notations in our foregoing proofs, the formulation of Theorem 6.1.8 is slightly different from the one given in [3] and in [6], since the family of operators $(A_r)_{r>0}$ used in these papers is replaced by a family (A_B) indexed by the balls $B \subset \Gamma$, see Remark 5 after Theorem 2.2 in [3]. Observe also that this theorem extends to vector-valued functions (this will be used in Section 6.3). Finally, here and after, $\mathcal{M}$ denotes the Hardy-Littlewood maximal function: for any locally integrable function f on Γ and any $x \in \Gamma$,

$$\mathcal{M}f(x) = \sup_{B \ni x} \frac{1}{V(B)} \sum_{y \in B} |f(y)|\, m(y),$$

where the supremum is taken over all balls B containing x. Recall that, by the Hardy-Littlewood maximal theorem, since (D) holds, $\mathcal{M}$ is of weak type $(1, 1)$ and of strong type (p, p) for all $1 < p \leq +\infty$.
Following the proof of Theorem 2.1 in [6], we will obtain Theorem 6.1.3 by applying Theorem 6.1.8 with $A_B = I - (I - P^{k^2})^n$ where k is the radius of B and n is an integer only depending from the constant D in (6.4).

As far as Theorem 6.1.6 is concerned, note first that (RR_p) cannot be derived from $(R_{p'})$ in this situation (where $1/p + 1/p' = 1$), since we do not know whether $(R_{p'})$ holds or not under these assumptions. Following [4], we first prove (6.5). The proof relies on a Calderón-Zygmund decomposition for Sobolev functions, which is the adaptation to our context of Proposition 1.1 in [4] (see also [2] in the Euclidean case and [7] for the extension to a weighted Lebesgue measure):

Proposition 6.1.9. *Assume that (D) and (P_q) hold for some $q \in [1,\infty)$ and let $p \in [q,+\infty)$. Let $f \in \dot{E}^{1,p}(\Gamma)$ and $\alpha > 0$. Then one can find a collection of balls $(B_i)_{i\in I}$, functions $(b_i)_{i\in I} \in \dot{E}^{1,q}(\Gamma)$ and a function $g \in \dot{E}^{1,\infty}$ such that the following properties hold:*

$$f = g + \sum_{i\in I} b_i, \tag{6.9}$$

$$\|\nabla g\|_\infty \leq C\alpha, \tag{6.10}$$

$$\operatorname{supp} b_i \subset B_i, \ \sum_{x\in B_i} |\nabla b_i|^q(x) m(x) \leq C\alpha^q V(B_i), \tag{6.11}$$

$$\sum_{i\in I} V(B_i) \leq C\alpha^{-q} \sum_{x\in\Gamma} |\nabla f|^q(x) m(x), \tag{6.12}$$

$$\sum_{i\in I} \chi_{B_i} \leq N, \tag{6.13}$$

where C and N only depend on q, p and on the constants in (D) and (P_q).

As in [4], we rely on this Calderón-Zygmund decomposition to establish (6.5). The argument also uses the $L^p(\Gamma)$-boundedness, for all $2 < p < +\infty$, of a discrete version of the Littlewood-Paley-Stein g-function (see [34]), which does not seem to have been stated before in this context and is interesting in itself. For all function f on Γ and all $x \in \Gamma$, define

$$g(f)(x) = \left(\sum_{l\geq 1} l \left|(I-P)P^l f(x)\right|^2 \right)^{1/2}.$$

Observe that this is indeed a discrete analogue of the g-function introduced by Stein in [34], since $(I-P)P^l = P^l - P^{l+1}$ can be seen as a discrete time derivative of P^l and P is a Markovian operator.
It is easy to check that the sublinear operator g is bounded in $L^2(\Gamma)$. Indeed, as already said, the assumption $(\Delta(\alpha))$ implies that the spectrum of P is contained in $[a,1]$ for some $a > -1$. As a consequence, P can be written as

$$P = \int_a^1 \lambda dE(\lambda),$$

so that, for all integer $l \geq 1$,

$$(I-P)P^l = \int_a^1 (1-\lambda)\lambda^l dE(\lambda)$$

and, for all $f \in L^2(\Gamma)$,

$$\left\|(I-P)P^l f\right\|_2^2 = \int_a^1 (1-\lambda)^2 \lambda^{2l} dE_{f,f}(\lambda).$$

It follows that, for all $f \in L^2(\Gamma)$,

$$\begin{aligned}
\|g(f)\|_2^2 &= \sum_{l\geq 1} l \left\|(I-P)P^l f\right\|_2^2 \\
&= \int_a^1 (1-\lambda)^2 \sum_{l\geq 1} l\lambda^{2l} dE_{f,f}(\lambda) \\
&= \int_a^1 \left(\frac{\lambda}{1+\lambda}\right)^2 dE_{f,f}(\lambda) \\
&\leq \|f\|_2^2 .
\end{aligned}$$

It turns out that, as in the Littlewood-Paley-Stein semigroup theory, g is also L^p-bounded for $1 < p < +\infty$:

Theorem 6.1.10. *Assume that (D), (P_2) and ($\Delta(\alpha)$) hold. Let $1 < p < +\infty$. There exists $C_p > 0$ such that, for all $f \in L^p(\Gamma)$,*

$$\|g(f)\|_p \leq C_p \|f\|_p .$$

Actually, this inequality will only be used for $p > 2$ in the sequel, but the result, which is interesting in itself, does hold and will be proved for all $1 < p < +\infty$. The proof for $p > 2$ relies on the vector-valued version of Theorem 6.1.8, while, for $p < 2$, we use the vector-valued version of the following result (see [3], Theorem 2.1 and also [12] for an earlier version:)

Theorem 6.1.11. *Let $p_0 \in [1,2)$. Assume that Γ satisfies the doubling property (D) and let T be a sublinear operator of strong type $(2,2)$. For any ball B, let A_B be a linear operator acting on $L^2(\Gamma)$. Assume that, for all $j \geq 1$, there exists $g(j) > 0$ such that, for all ball $B \subset \Gamma$ and all function f supported in B,*

$$\frac{1}{V^{1/2}(2^{j+1}B)} \|T(I-A_B)f\|_{L^2(C_j(B))} \leq g(j) \frac{1}{V^{1/p_0}(B)} \|f\|_{L^{p_0}} \tag{6.14}$$

for all $j \geq 2$ and

$$\frac{1}{V^{1/2}(2^{j+1}B)} \|A_B f\|_{L^2(C_j(B))} \leq g(j) \frac{1}{V^{1/p_0}(B)} \|f\|_{L^{p_0}} \tag{6.15}$$

for all $j \geq 1$. If $\sum_{j\geq 1} g(j) 2^{Dj} < +\infty$ where D is given by (6.4), then T is of weak type (p_0, p_0), and is therefore of strong type (p,p) for all $p_0 < p < 2$.

Going back to Theorem 6.1.6, once (6.5) is established, we conclude by applying real interpolation theorems for Sobolev spaces, which are also new in this context. More precisely, we prove:

Theorem 6.1.12. *Let $q \in [1, +\infty)$ and assume that (D), (P_q) and ($\Delta(\alpha)$) hold. Then, for all $q < p < +\infty$, $\dot{W}^{1,p}(\Gamma) = \left(\dot{W}^{1,q}(\Gamma), \dot{W}^{1,\infty}(\Gamma)\right)_{1-\frac{q}{p},p}$.*

As an immediate corollary, we obtain:

Corollary 6.1.13 (The reiteration theorem). *Assume that* Γ *satisfies (D),* (P_q) *for some* $1 \leq q < +\infty$ *and* $(\Delta(\alpha))$*. Define* $q_0 = \inf\{q \in [1,\infty) : (P_q) \text{ holds}\}$*. For* $q_0 < p_1 < p < p_2 \leq +\infty$*, if* $\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}$*, then* $\dot{W}^{1,p}(\Gamma) = \left(\dot{W}^{1,p_1}(\Gamma), \dot{W}^{1,p_2}(\Gamma)\right)_{\theta,p}$.

Corollary 6.1.13, in conjunction with (6.5), conclude the proof of Theorem 6.1.6. Notice that, since we know that Sobolev spaces interpolate by the real method, we do not need any argument as the one in Section 1.3 of [4].

Let us finally mention that, in [4], in the context of Riemannian manifolds, the authors also obtain a characterization of the validity of (R_p) for all $p \in (2, p_0)$ (with some $p_0 > 2$) in terms of a reverse Hölder inequality for harmonic functions on balls. We intend to discuss the corresponding issue on graphs in a forthcoming paper.

The plan of the paper is as follows. After recalling some well-known estimates for the iterates of p and deriving some consequences (Section 6.2), we first prove Theorem 6.1.10, which is of independent interest, in Section 6.3. In Section 6.4, we prove Theorem 6.1.3 using Theorem 6.1.8. Section 6.5 is devoted to the proof of Proposition 6.1.9. Theorem 6.1.12 is established in Section 6.6 and, in Section 6.7, we prove Theorem 6.1.6.

6.2 Kernel bounds

In this section, we gather some estimates for the iterates of p and some straightforward consequences of frequent use in the sequel. We always assume that (D), (P_2) and $(\Delta(\alpha))$ hold. First, as already said, (LUE) holds. Moreover, we also have the following pointwise estimate for the discrete "time derivative" of p_l: there exist $C, c > 0$ such that, for all $x, y \in \Gamma$ and all $l \in \mathbb{N}$,

$$|p_l(x,y) - p_{l+1}(x,y)| \leq \frac{Cm(y)}{lV(x,\sqrt{l})} e^{-c\frac{d^2(x,y)}{l}}. \tag{6.16}$$

This "time regularity" estimate, which is a consequence of the L^2 analyticity of P, was first proved by Christ ([13]) by a quite difficult argument. Simpler proofs have been given by Blunck ([11]) and, more recently, by Dungey ([22]).

Thus, if B is a ball in Γ with radius k, f any function supported in B and $i \geq 2$, one has, for all $x \in C_i(B)$ and all $l \geq 1$,

$$\left|P^l f(x)\right| + l\left|(I-P)P^l f(x)\right| \leq \frac{C}{V(B)} e^{-c\frac{4^i k^2}{l}} \|f\|_{L^1}. \tag{6.17}$$

This "off-diagonal" estimate follows from (UE) and (6.16) and the fact that, for all $y \in B$, by (D),

$$V(y,k) \sim V(B) \text{ and } \frac{V(y,k)}{V(y,\sqrt{l})} \leq C \sup\left(1, \left(\frac{k}{\sqrt{l}}\right)^D\right).$$

Similarly, if B is a ball in Γ with radius k, $i \geq 2$ and f any function supported in $C_i(B)$, one has, for all $x \in B$ and all $l \geq 1$,

$$\left|P^l f(x)\right| + l\left|(I-P)P^l f(x)\right| \leq \frac{C}{V(2^i B)} e^{-c\frac{4^i k^2}{l}} \left\|f\right\|_{L^1}. \tag{6.18}$$

Finally, for all ball B with radius k, all $i \geq 2$, all function f supported in $C_i(B)$ and all $l \geq 1$,

$$\left\|\nabla P^l f\right\|_{L^2(B)} \leq \frac{C}{\sqrt{l}} e^{-c\frac{4^i k^2}{l}} \left\|f\right\|_{L^2(C_i(B))}. \tag{6.19}$$

See Lemma 2 in [32]. If one furthermore assumes that (G_{p_0}) holds for some $p_0 > 2$, then, by interpolation between (6.19) and (G_{p_0}), one obtains, for all $p \in (2, p_0)$ and all f supported in $C_i(B)$,

$$\left\|\nabla P^l f\right\|_{L^p(B)} \leq \frac{C_p}{\sqrt{l}} e^{-c\frac{4^i k^2}{l}} \left\|f\right\|_{L^p(C_i(B))}. \tag{6.20}$$

Inequalities (6.19) and (6.20) may be regarded as "Gaffney" type inequalities, in the spirit of [24].

6.3 Littlewood-Paley inequalities

In this section, we establish Theorem 6.1.10.

The case $1 < p < 2$: We apply the vector-valued version of Theorem 6.1.11 with $T = g$ and $p_0 = 1$ and, for all ball B with radius k, A_B defined by

$$A_B = I - (I - P^{k^2})^n,$$

where n is a positive integer, to be chosen in the proof. More precisely, we consider, for $f \in L^2(\Gamma)$ and $x \in \Gamma$,

$$Tf(x) = \left(\sqrt{l}(I-P)P^l f(x)\right)_{l \geq 1},$$

so that T maps $L^2(\Gamma)$ into $L^2(\Gamma, l^2)$.

Let B be a ball and f supported in B. Let us first check (6.14). Using the expansion

$$(I - P^{k^2})^n = \sum_{p=0}^{n} C_n^p (-1)^p P^{pk^2},$$

we obtain

$$T(I - A_B)f = \left(\alpha_l (I-P)P^l f\right)_{l \geq 1}$$

where

$$\alpha_l := \sum_{0 \leq p \leq n;\ l \geq pk^2} C_n^p (-1)^p \sqrt{l - pk^2}.$$

Since it follows from (6.17) that

$$\frac{1}{V(2^{j+1}B)} \left\|(I-P)P^l f\right\|^2_{L^2(C_j(B))} \leq \frac{C}{l^2 V^2(B)} e^{-c\frac{4^j k^2}{l}} \left\|f\right\|^2_{L^1}, \tag{6.21}$$

we will be able to go on thanks to the following estimate:

Lemma 6.3.1. *There exists $C>0$ only depending on n such that, for all $j\geq 2$,*

$$\sum_{l\geq 1}\frac{\left|\alpha_l\right|^2}{l^2}e^{-c\frac{4^jk^2}{l}}\leq C4^{-2nj}.$$

Proof of Lemma 6.3.1: If $mk^2\leq l<(m+1)k^2$ for some integer $0\leq m\leq n$, one obviously has

$$|\alpha_l|\leq Ck\sqrt{m+1}, \tag{6.22}$$

where $C>0$ only depends on n, while, if $l>(n+1)k^2$, one has

$$|\alpha_l|\leq Cl^{-\frac{2n-1}{2}}k^{2n}. \tag{6.23}$$

This estimate follows from the following inequality, valid for any C^n function φ on $[0,+\infty)$:

$$\left|\sum_{p=0}^{n}C_n^p(-1)^p\varphi(t-pk^2)\right|\leq C\sup_{u\geq\frac{t}{n+1}}\left|\varphi^{(n)}(u)\right|k^{2n}, \tag{6.24}$$

where $C>0$ only depends on n (see [23], problem 16, p. 65). It follows from (6.22) that, for all $0\leq m\leq n$,

$$\begin{aligned}\sum_{mk^2<l\leq(m+1)k^2}\frac{\left|\alpha_l\right|^2}{l^2}e^{-c\frac{4^jk^2}{l}} &\leq C\sum_{mk^2<l\leq(m+1)k^2}\frac{(m+1)k^2}{l^2}e^{-c\frac{4^jk^2}{l}}\\ &\leq C\int_{mk^2}^{(m+1)k^2}\frac{(m+1)k^2}{t^2}e^{-c\frac{4^jk^2}{t}}dt\\ &\leq Ce^{-c4^j}\end{aligned}$$

where $C,c>0$ only depend on n. Similarly,

$$\begin{aligned}\sum_{l>(n+1)k^2}\frac{\left|\alpha_l\right|^2}{l^2}e^{-c\frac{4^jk^2}{l}} &\leq C\sum_{l>(n+1)k^2}\frac{l^{-(2n-1)}k^{4n}}{l^2}e^{-c\frac{4^jk^2}{l}}\\ &\leq C\int_{(n+1)k^2}^{+\infty}\frac{k^{4n}}{t^{2n+1}}e^{-c\frac{4^jk^2}{t}}dt\\ &\leq C4^{-2nj}\int_0^{+\infty}\frac{e^{-c/w}}{w^{2n+1}}dw\\ &= C4^{-2nj},\end{aligned}$$

which concludes the proof of Lemma 6.3.1. □

Finally, one obtains

$$\frac{1}{V^{1/2}(2^{j+1}B)}\left\|T(I-A_B)f\right\|_{L^2(2^{j+1}B\setminus 2^jB),l^2}\leq C\frac{4^{-nj}}{V(B)}\left\|f\right\|_{L^1},$$

which means that (6.14) holds with $g(j)=4^{-nj}$, and one just has to choose $n>\dfrac{D}{2}$ in order to have $\sum_j g(j)2^{Dj}<+\infty$.

Let us now check (6.15). Since

$$A_B = \sum_{p=1}^{n} C_n^p (-1)^p P^{pk^2},$$

it is enough to prove that, for all $j \geq 1$ and all $1 \leq p \leq n$,

$$\frac{1}{V^{1/2}(2^{j+1}B)} \left\| P^{pk^2} f \right\|_{L^2(C_j(B))} \leq g(j) \frac{1}{V(B)} \|f\|_{L^1(B)}. \tag{6.25}$$

For all $x \in C_j(B)$, (6.17) yields

$$\left| P^{pk^2} f(x) \right| \leq C \frac{e^{-c' \frac{4^j}{p}}}{V(B)} \|f\|_{L^1(B)}$$

if $j \geq 2$, and

$$\left| P^{pk^2} f(x) \right| \leq \frac{C}{V(B)} \|f\|_{L^1(B)}$$

for $j = 1$. As a consequence,

$$\left\| P^{pk^2} f \right\|_{L^2(C_j(B))} \leq C \frac{e^{-c' \frac{4^j}{p}}}{V(B)} V^{1/2}(2^{j+1}B) \|f\|_{L^1(B)},$$

so that (6.25) holds. This ends the proof of Theorem 6.1.10 when $1 < p < 2$.

The case $2 < p < +\infty$: This time, we apply the vector-valued version of Theorem 6.1.8 with the same choices of T and A_B. Let us first check (6.7), which reads in this situation as

$$\frac{1}{V^{1/2}(B)} \|T(I - A_B) f\|_{L^2(B,l^2)} \leq C \left(\mathcal{M} |f|^2 \right)^{1/2} (y)$$

for all $f \in L^2(\Gamma)$, all ball $B \subset \Gamma$ and all $y \in B$. Fix such an f, such a ball B and $y \in B$. Write

$$f = \sum_{j \geq 1} f \chi_{C_j(B)} := \sum_{j \geq 1} f_j.$$

The L^2-boundedness of g and A_B and the doubling property (D) yield

$$\frac{1}{V^{1/2}(B)} \|T(I - A_B) f_1\|_{L^2(B),l^2} \leq \frac{C}{V^{1/2}(B)} \|f\|_{L^2(4B)} \leq C \left(\mathcal{M} |f|^2 \right)^{1/2} (y).$$

Let $j \geq 2$. Using the same notations as for the case $1 < p < 2$, one has

$$\|T(I - A_B) f_j\|_{L^2(B,l^2)}^2 = \sum_{l \geq 1} |\alpha_l|^2 \sum_{x \in B} \left| (I - P) P^l f_j(x) \right|^2 m(x).$$

For all $x \in B$, it follows from (6.18) and the Cauchy-Schwarz inequality that

$$\begin{aligned} \left| (I - P) P^l f_j(x) \right| &\leq \frac{C}{l} e^{-c' \frac{4^j k^2}{l}} \frac{1}{V(2^j B)} \sum_{z \in 2^{j+1} B} |f_j(z)| \, m(z) \\ &\leq \frac{C}{l} e^{-c' \frac{4^j k^2}{l}} \frac{1}{V^{1/2}(2^{j+1}B)} \left(\sum_{z \in 2^{j+1} B} |f_j(z)|^2 m(z) \right)^{1/2} \\ &\leq \frac{C}{l} e^{-c' \frac{4^j k^2}{l}} \left(\mathcal{M} |f|^2 (y) \right)^{1/2}. \end{aligned}$$

As a consequence, by Lemma 6.3.1,

$$\begin{aligned}\|T(I-A_B)f_j\|^2_{L^2(B,l^2)} &\leq C\left(\sum_{l\geq 1}\frac{|\alpha_l|^2}{l^2}e^{-c\frac{4^jk^2}{l}}\right)\mathcal{M}\left(|f|^2\right)(y)V(B)\\ &\leq CV(B)4^{-2nj}\mathcal{M}\left(|f|^2\right)(y),\end{aligned}$$

which yields (6.7) by summing up on $j\geq 1$.
To prove (6.8), it suffices to establish that, for all $1\leq j\leq n$, all ball $B\subset\Gamma$ and all $y\in B$,

$$\left\|TP^{jk^2}f\right\|_{L^\infty(B,l^2)}\leq C\left(\mathcal{M}\left\|Tf\right\|^2_{l^2}(y)\right)^{1/2}.$$

Let $x\in B$. By Cauchy-Schwarz and the fact that

$$\sum_{y\in\Gamma}p_{jk^2}(x,y)=1$$

for all $x\in\Gamma$, one has, for any function $h\in L^2(\Gamma)$,

$$\left|P^{jk^2}h(x)\right|\leq\left(P^{jk^2}|h|^2(x)\right)^{1/2}.$$

It follows that, for all $l\geq 1$,

$$\left|P^{jk^2}(\sqrt{l}(I-P)P^lf)(x)\right|^2\leq P^{jk^2}\left(l\left|(I-P)P^lf\right|^2\right)(x),$$

so that

$$\begin{aligned}\sum_{l\geq 1}\left|P^{jk^2}(\sqrt{l}(I-P)P^lf)(x)\right|^2 &\leq P^{jk^2}\left(\sum_{l\geq 1}l\left|(I-P)P^lf\right|^2\right)(x)\\ &= P^{jk^2}\left(\|Tf\|^2_{l^2}\right)(x)\\ &\leq C\mathcal{M}\left(\|Tf\|^2_{l^2}\right)(y),\end{aligned}$$

which is the desired estimate. Thus, (6.8) holds and the proof of Theorem 6.1.10 is therefore complete. □

6.4 Riesz transforms for $p>2$

In the present section, we establish Theorem 6.1.3, applying Theorem 6.1.8 with the same choice of A_B as in Section 6.3. One has $\|A_B\|_{2,2}=1$. In view of Theorem 6.1.8, it suffices to show that

$$\frac{1}{V^{1/2}(B)}\left\|T(I-P^{k^2})^nf\right\|_{L^2(B)}\leq C\left(\mathcal{M}(|f|^2)\right)^{1/2}(x) \tag{6.26}$$

and

$$\frac{1}{V^{1/p_0}(B)}\left\|T\left(I-(I-P^{k^2})^n\right)f\right\|_{L^{p_0}(B)}\leq C\left(\mathcal{M}(|Tf|^2)\right)^{1/2}(x) \tag{6.27}$$

for all $f \in L^2(\Gamma)$, all $x \in \Gamma$ and all ball $B \subset \Gamma$ containing x. Fix such data f, x and B.

Proof of (6.34): Set $f_i = f\chi_{C_i(B)}$ for all $i \geq 1$. The L^2-boundedness of $T(I - P^{k^2})^n$ yields

$$\frac{1}{V^{1/2}(B)} \left\| T(I - P^{k^2})^n f_1 \right\|_{L^2(B)} \leq \frac{C}{V^{1/2}(B)} \|f_1\|_{L^2(\Gamma)} \leq C \left(\mathcal{M}(|f|^2) \right)^{1/2} (x). \tag{6.28}$$

Fix now $i \geq 2$. In order to estimate the left-hand side of (6.34) with f replaced by f_i, we use the expansion

$$(I - P)^{-1/2} = \sum_{l=0}^{+\infty} a_l P^l,$$

where the a_l's are defined by (6.6) (observe that, for all $l \geq 0$, $a_l > 0$). Therefore, one has

$$\begin{aligned} (I - P)^{-1/2}(I - P^{k^2})^n f_i &= \sum_{l=0}^{+\infty} a_l P^l (I - P^{k^2})^n f_i \\ &= \sum_{l=0}^{+\infty} a_l \sum_{j=0}^{n} C_n^j (-1)^j P^{l+jk^2} f_i \\ &= \sum_{l=0}^{+\infty} d_l P^l f_i, \end{aligned}$$

where

$$d_l = \sum_{0 \leq j \leq n,\ jk^2 \leq l} (-1)^j C_n^j a_{l-jk^2}.$$

It follows that

$$\left| T(I - P^{k^2})^n f_i(x) \right| \leq \sum_{l=1}^{+\infty} |d_l| \, \nabla P^l f_i(x)$$

for all $x \in B$. Indeed, if $x \in B$ and $l = 0$, $\nabla P^l f_i(x) = \nabla f_i(x) = 0$ because f_i is supported in $C_i(B)$. Thus, one has

$$\left\| T(I - P^{k^2})^n f_i \right\|_{L^2(B)} \leq \sum_{l=1}^{+\infty} |d_l| \left\| \left| \nabla P^l f_i \right| \right\|_{L^2(B)}.$$

According to (6.19), one has

$$\left\| T(I - P^{k^2})^n f_i \right\|_{L^2(B)} \leq C \sum_{l=1}^{+\infty} |d_l| \frac{e^{-c\frac{4^i k^2}{l}}}{\sqrt{l}} \|f\|_{L^2(2^{i+1}B \setminus 2^i B)}. \tag{6.29}$$

We claim that the following estimates hold for the d_l's:

Lemma 6.4.1. *There exists $C > 0$ only depending on n with the following properties: for all integer $l \geq 1$,*

(i) if there exists an integer $0 \leq m \leq n$ such that $mk^2 < l < (m+1)k^2$, $|d_l| \leq \frac{C}{\sqrt{l - mk^2}}$,

(ii) *if there exists an integer* $0 \le m \le n$ *such that* $l = (m+1)k^2$, $|d_l| \le C$,

(iii) *if* $l > (n+1)k^2$, $|d_l| \le Ck^{2n} l^{-n-\frac{1}{2}}$.

We postpone the proof of this lemma to the Appendix and end the proof of (6.34). According to (6.29), one has

$$
\begin{aligned}
\left\| T(I-P^{k^2})^n f_i \right\|_{L^2(B)} &\le C \sum_{m=0}^{n} \sum_{mk^2<l<(m+1)k^2} |d_l| \frac{e^{-c\frac{4^i k^2}{l}}}{\sqrt{l}} \|f\|_{L^2(2^{i+1}B\setminus 2^i B)} \\
&+ C \sum_{m=0}^{n} \left| d_{(m+1)k^2} \right| \frac{e^{-c\frac{4^i}{m+1}}}{k\sqrt{m+1}} \|f\|_{L^2(2^{i+1}B\setminus 2^i B)} \\
&+ C \sum_{l>(n+1)k^2} |d_l| \frac{e^{-c\frac{4^i k^2}{l}}}{\sqrt{l}} \|f\|_{L^2(2^{i+1}B\setminus 2^i B)} \\
&:= S_1 + S_2 + S_3.
\end{aligned} \tag{6.30}
$$

For S_1, Lemma 6.4.1 yields

$$
|S_1| \le C \sum_{m=0}^{n} \sum_{mk^2<l<(m+1)k^2} \frac{e^{-c\frac{4^i k^2}{l}}}{\sqrt{l}\sqrt{l-mk^2}} \|f\|_{L^2(2^{i+1}B\setminus 2^i B)}.
$$

But, for each $1 \le m \le n$,

$$
\begin{aligned}
\sum_{mk^2<l<(m+1)k^2} \frac{e^{-c\frac{4^i k^2}{l}}}{\sqrt{l}\sqrt{l-mk^2}} &\le C \int_{mk^2}^{(m+1)k^2} \frac{e^{-c\frac{4^i k^2}{t}}}{\sqrt{t-mk^2}\sqrt{t}} dt \\
&\le C \int_0^1 \frac{e^{-c\frac{4^i}{n(1+w)}}}{\sqrt{w(w+1)}} dw \\
&\le Ce^{-c4^i},
\end{aligned}
$$

where $C, c > 0$ only depend on n. For $m = 0$,

$$
\sum_{0<l<k^2} \frac{e^{-c\frac{4^i k^2}{l}}}{l} \le \int_0^1 e^{-c\frac{4^i}{u}} \frac{du}{u} \le Ce^{-c4^i}.
$$

Therefore,

$$
|S_1| \le Ce^{-c4^i} \|f\|_{L^2(2^{i+1}B\setminus 2^i B)}. \tag{6.31}
$$

As for S_2, Lemma 6.4.1 gives at once

$$
|S_2| \le Ce^{-c4^i} \|f\|_{L^2(2^{i+1}B\setminus 2^i B)}, \tag{6.32}
$$

where $C, c > 0$ only depend on n once more. Finally, for S_3, Lemma 6.4.1 provides

$$
|S_3| \le Ck^{2n} \sum_{l>(n+1)k^2} l^{-n-\frac{1}{2}} \frac{e^{-c\frac{4^i k^2}{l}}}{\sqrt{l}} \|f\|_{L^2(2^{i+1}B\setminus 2^i B)}.
$$

But one clearly has

$$\begin{aligned}\sum_{l>(n+1)k^2} l^{-n-\frac{1}{2}} \frac{e^{-c\frac{4^i k^2}{l}}}{\sqrt{l}} &\leq \int_{(n+1)k^2}^{+\infty} t^{-n-\frac{1}{2}} \frac{e^{-c\frac{4^i k^2}{t}}}{\sqrt{t}} dt \\ &= (4^i k^2)^{-n} \int_{\frac{n+1}{4^i}}^{+\infty} u^{-n} e^{-\frac{c}{u}} \frac{du}{u} \\ &\leq Ck^{-2n} 4^{-in} \int_0^{+\infty} u^{-n} e^{-\frac{c}{u}} \frac{du}{u} \leq C4^{-in},\end{aligned}$$

so that, since $k \geq 1$,

$$|S_3| \leq C4^{-in} \|f\|_{L^2(2^{i+1}B \setminus 2^i B)}. \tag{6.33}$$

Summing up the upper estimates (6.31), (6.32) and (6.33) and using (6.30), one obtains

$$\left\| T(I - P^{k^2})^n f_i \right\|_{L^2(B)} \leq C4^{-in} \|f\|_{L^2(2^{i+1}B \setminus 2^i B)}. \tag{6.34}$$

The definition of the maximal function and property (6.4) yield

$$\|f\|_{L^2(2^{i+1}B \setminus 2^i B)} \leq V^{1/2}(2^{i+1}B) \left(\mathcal{M}(|f|^2)(x)\right)^{1/2} \leq C2^{(i+1)D/2} V(B)^{1/2} \left(\mathcal{M}(|f|^2)(x)\right)^{1/2}.$$

Choosing now $n > \dfrac{D}{4}$ and summing up over $i \geq 1$, one concludes from (6.28) and (6.34) that

$$\left\| T(I - P^{k^2})^n f \right\|_{L^2(B)} \leq C \left(\sum_{i=0}^{+\infty} 2^{i\left(\frac{D}{2} - 2n\right)} \right) V(B)^{1/2} \left(\mathcal{M}(|f|^2)(x)\right)^{1/2},$$

which ends the proof of (6.34). □

Proof of (6.27): We use the following lemma:

Lemma 6.4.2. *For all $p \in (2, p_0)$, there exists $C, \alpha > 0$ such that, for all ball $B \subset \Gamma$ with radius k, all integer $i \geq 1$ and all function $f \in L^2(\Gamma)$ supported in $C_i(B)$, and for all $j \in \{1, ..., n\}$ (where n is chosen as above), one has*

$$\left(\frac{1}{V(B)^{1/p}} \right) \left\| \nabla P^{jk^2} f \right\|_{L^p(B)} \leq \frac{Ce^{-\alpha 4^i}}{k} \frac{1}{V(2^{i+1}B)^{1/2}} \|f\|_{L^2(\Gamma)}.$$

Proof of Lemma 6.4.2: This proof is very similar to the one of Lemma 3.2 in [6], and we will therefore only indicate the main steps. Consider first the case when $i = 1$. If $j = 2m$ for some integer $m \geq 0$, (6.20) yields

$$\left\| \nabla P^{jk^2} f \right\|_{L^p(B)} \leq \frac{C}{\sqrt{k}} \left\| P^{mk^2} f \right\|_{L^p(\Gamma)}. \tag{6.35}$$

Using (UE), and noticing that, by (D), for $y \in B$, $V(y, k\sqrt{m}) \sim V(B)$, one has, for all $x \in \Gamma$ and all $y \in B$,

$$p_{mk^2}(x, y) \leq \frac{C}{V(B)} \exp\left(-c\frac{d^2(x, y)}{mk^2}\right) m(y).$$

As a consequence, for all $x \in \Gamma$,

$$\left|P^{mk^2} f(x)\right| \leq \frac{C}{V^{1/2}(B)} \|f\|_{L^2(4B)}.$$

The L^2 contractivity of P shows that

$$\left\|P^{mk^2} f\right\|_{L^2(\Gamma)} \leq C \|f\|_{L^2(4B)},$$

so that,

$$\left\|P^{mk^2} f\right\|_{L^p(\Gamma)} \leq CV(B)^{\frac{1}{p}-\frac{1}{2}} \|f\|_{L^2(\Gamma)}. \tag{6.36}$$

Finally, (6.36) and (6.35) yield the conclusion of Lemma 6.4.2 when $i = 1$ and $j = 2m$. If $j = 2m + 1$, argue similarly, writing $j = m + (m + 1)$.

Consider now the case when $i \geq 2$ and assume that $j = 2m$ (one argues similarly if $j = 2m+1 = m+(m+1)$). Let χ_l the characteristic function of $C_l(B)$ for all $l \geq 1$. One has, for all $x \in \Gamma$,

$$\nabla P^{jk^2} f(x) \leq \sum_{l \geq 1} \nabla P^{mk^2} \chi_l P^{mk^2} f(x) =: \sum_{l \geq 1} g_l(x).$$

By (6.35) and (6.4),

$$\begin{aligned} \frac{1}{V^{1/p}(B)} \|g_l\|_{L^p(B)} &\leq C \left(\frac{V(2^{l+1}B)}{V(B)}\right)^{1/p} \frac{e^{-c4^l}}{k} \frac{1}{V^{1/p}(2^{l+1}B)} \left\|P^{mk^2} f\right\|_{L^p(2^{l+1}B \setminus 2^l B)} \\ &\leq C 2^{(l+1)D/p} \frac{e^{-c4^l}}{k} \frac{1}{V^{1/p}(2^{l+1}B)} \left\|P^{mk^2} f\right\|_{L^p(2^{l+1}B \setminus 2^l B)}. \end{aligned}$$

Using (UE) and arguing as in the proof of Lemma 3.2 in [6], one obtains

$$\frac{1}{V(2^{l+1}B)} \left\|P^{mk^2} f\right\|^2_{L^2(C_l)} \leq K_{il} \frac{1}{V(2^{i+1}B)} \|f\|^2_{L^2(C_i)} \tag{6.37}$$

and, for all $x \in 2^{l+1}B \setminus 2^l B$,

$$\left|P^{mk^2} f(x)\right| \leq K_{il} 2^{(i+2)D} \frac{1}{V^{1/2}(2^{i+1}B)} \|f\|_{L^2(2^{i+1}B \setminus 2^i B)}, \tag{6.38}$$

where

$$K_{il} = \begin{cases} Ce^{-c4^i} & \text{if } l \leq i-2, \\ C & \text{if } i-1 \leq l \leq i+1, \\ Ce^{-c4^l} & \text{if } l \geq i+2. \end{cases}$$

Interpolating between (6.37) and (6.38) therefore yields

$$\frac{1}{V^{1/p}(2^{l+1}B)} \left\|P^{mk^2} f\right\|_{L^p(C_l)} \leq K_{il} 2^{(i+2)D\left(1-\frac{2}{p}\right)} \frac{1}{V^{1/2}(2^{i+1}B)} \|f\|_{L^2(C_i)}.$$

Summing up in l, one ends the proof of Lemma 6.4.2 as in [6]. □

To prove (6.27), it is enough to show that, if $p \in (2, p_0)$, there exists $C_p > 0$ such that, for all $j \in \{1, ..., n\}$, all function $f \in L^2_{loc}(\Gamma)$ with $\nabla f \in L^2_{loc}(\Gamma)$, all ball $B \subset \Gamma$ with radius k and any point $x \in B$,

$$\frac{1}{V^{1/p}(B)} \left\| \nabla P^{jk^2} f \right\|_{L^p(B)} \leq C \left(\mathcal{M}(|\nabla f|^2) \right)^{1/2} (x).$$

But, since for all $l \geq 0$, $P^l 1 = 1$, one has

$$\nabla P^l f = \nabla P^l (f - f_{4B}),$$

so that

$$\nabla P^{jk^2} f \leq \sum_{l \geq 1} \nabla P^{jk^2} (\chi_l (f - f_{4B})).$$

One concludes the proof of (6.27) as in [6], using the Poincaré inequality and Lemma 6.4.2. □

6.5 The Calderón-Zygmund decomposition for functions in Sobolev spaces

The present section is devoted to the proof of Proposition 6.1.9, which follows the one of Proposition 1.1 in [4]. Let $f \in \dot{E}^{1,p}(\Gamma)$, $\alpha > 0$. Consider $\Omega = \{x \in \Gamma : \mathcal{M}(|\nabla f|^q)(x) > \alpha^q\}$. If $\Omega = \emptyset$, then set

$$g = f \ , \quad b_i = 0 \ \text{ for all } i \in I$$

so that (6.10) is satisfied thanks to the Lebesgue differentiation theorem and the other properties in Proposition 6.1.9 obviously hold. Otherwise the Hardy-Littlewood maximal theorem gives

$$\begin{aligned} m(\Omega) &\leq C\alpha^{-p} \|(\nabla f)^q\|_{\frac{p}{q}}^{\frac{p}{q}} \\ &= C\alpha^{-p} \Big(\sum_x |\nabla f|^p(x) m(x) \Big) \qquad (6.39) \\ &< +\infty. \end{aligned}$$

In particular, Ω is a proper open subset of Γ, as $m(\Gamma) = +\infty$ (see Remark 6.1.1). Let $(\underline{B_i})_{i \in I}$ be a Whitney decomposition of Ω ([15]). That is, Ω is the union of the $\underline{B_i}$'s, the $\underline{B_i}$'s being pairwise disjoint, and there exist two constants $C_2 > C_1 > 1$, depending only on the metric, such that, if $F = \Gamma \setminus \Omega$,

1. the balls $B_i = C_1 \underline{B_i}$ are contained in Ω and have the bounded overlap property;
2. for each $i \in I$, $r_i = r(B_i) = \frac{1}{2} d(x_i, F)$ where x_i is the center of B_i;
3. for each $i \in I$, if $\overline{B_i} = C_2 \underline{B_i}$, $\overline{B_i} \cap F \neq \emptyset$ ($C_2 = 4C_1$ works).

For $x \in \Omega$, denote $I_x = \{i \in I;\ x \in B_i\}$. By the bounded overlap property of the balls B_i, there exists an integer N such that $\sharp I_x \le N$ for all $x \in \Omega$. Fixing $j \in I_x$ and using the properties of the B_i's, we easily see that $\frac{1}{3} r_i \le r_j \le 3 r_i$ for all $i \in I_x$. In particular, $B_i \subset 7B_j$ for all $i \in I_x$.

Condition (6.13) is nothing but the bounded overlap property of the B_i's and (6.12) follows from (6.13) and (6.39). The doubling property and the fact that $\overline{B_i} \cap F \neq \emptyset$ yield:

$$\sum_{x \in B_i} |\nabla f|^q(x) m(x) \le \sum_{x \in \overline{B_i}} |\nabla f|^q(x) m(x) \le \alpha^q V(\overline{B_i}) \le C \alpha^q V(B_i). \tag{6.40}$$

Let us now define the functions b_i's. Let $(\chi_i)_{i\in I}$ be a partition of unity of Ω subordinated to the covering $(\underline{B}_i)_{i\in I}$, which means that, for all $i \in I$, χ_i is a Lipschitz function supported in B_i with $\|\nabla \chi_i\|_\infty \le \dfrac{C}{r_i}$ and $\displaystyle\sum_{i \in I} \chi_i(x) = 1$ for all $x \in \Gamma$ (it is enough to choose $\chi_i(x) = \psi\left(\dfrac{C_1 d(x_i, x)}{r_i}\right) \left(\displaystyle\sum_k \psi\left(\dfrac{C_1 d(x_k, x)}{r_k}\right)\right)^{-1}$, where $\psi \in \mathcal{D}(\mathbb{R})$, $\psi = 1$ on $[0,1]$, $\psi = 0$ on $[\frac{1+C_1}{2}, +\infty)$ and $0 \le \psi \le 1$). Note that $\nabla \chi_i$ is supported in $2B_i \subset \Omega$. We set $b_i = (f - f_{B_i})\chi_i$. It is clear that $\operatorname{supp} b_i \subset B_i$. Let us estimate $\sum_{x\in B_i} |\nabla b_i|^q(x) m(x)$. Since

$$\nabla b_i(x) = \nabla((f - f_{B_i})\chi_i)(x) \le \max_{y \sim x} \chi_i(y) \nabla f(x) + |f(x) - f_{B_i}| \nabla \chi_i(x)$$

and since $\chi_i(y) \le 1$ for all $y \in \Gamma$, we get by (P_q) and (6.40) that

$$\begin{aligned}
\sum_{x \in B_i} |\nabla b_i|^q m(x) &\le C \left(\sum_{x \in B_i} |\nabla f|^q(x) m(x) + \sum_{x \in B_i} |f - f_{B_i}|^q(x) |\nabla \chi_i|^q(x) m(x) \right) \\
&\le C \alpha^q V(B_i) + C \frac{C^q}{r_i^q} r_i^q \sum_{x \in B_i} |\nabla f|^q(x) m(x) \\
&\le C' \alpha^q V(B_i).
\end{aligned}$$

Thus (6.11) is proved.

Set $g = f - \displaystyle\sum_{i\in I} b_i$. Since the sum is locally finite on Ω, g is defined everywhere on Γ and $g = f$ on F.

It remains to prove (6.10). Since $\displaystyle\sum_{i\in I} \chi_i(x) = 1$ for all $x \in \Omega$, one has

$$g = f \chi_F + \sum_{i \in I} f_{B_i} \chi_i$$

where χ_F denotes the characteristic function of F. We will need the following lemma:

Lemma 6.5.1. *There exists $C > 0$ such that, for all $j \in I$, all $u \in F \cap 4B_j$ and all $v \in B_j$,*

$$|g(u) - g(v)| \le C \alpha d(u, v).$$

Proof: Since $\sum_{i\in I}\chi_i = 1$ on Γ, one has

$$
\begin{aligned}
g(u)-g(v) &= f(u)-\sum_{i\in I} f_{B_i}\chi_i(v)\\
&= \sum_{i\in I}\left(f(u)-f_{B_i}\right)\chi_i(v).
\end{aligned}
\tag{6.41}
$$

For all $i\in I$ such that $v\in B_i$,

$$
\left|f(u)-f_{B_i}\right| \leq \sum_{k=0}^{+\infty}\left|f_{B(u,2^{-k}r_i)}-f_{B(u,2^{-k-1}r_i)}\right| + \left|f_{B(u,r_i)}-f_{B_i}\right|.
$$

For all $k\geq 0$, (P_q) yields
(6.42)

$$
\begin{aligned}
\left|f_{B(u,2^{-k}r_i)}-f_{B(u,2^{-k-1}r_i)}\right| &= \frac{1}{V(u,2^{-k-1}r_i)}\left|\sum_{z\in B(u,2^{-k-1}r_i)}\left(f(z)-f_{B(u,2^{-k}r_i)}\right)m(z)\right|\\
&\leq \frac{C}{V(u,2^{-k}r_i)}\sum_{z\in B(u,2^{-k}r_i)}\left|f(z)-f_{B(u,2^{-k}r_i)}\right|m(z)\\
&\leq \left(\frac{C}{V(u,2^{-k}r_i)}\sum_{z\in B(u,2^{-k}r_i)}\left|f(z)-f_{B(u,2^{-k}r_i)}\right|^q m(z)\right)^{\frac{1}{q}}\\
&\leq C2^{-k}r_i\left(\frac{1}{V(u,2^{-k}r_i)}\sum_{z\in B(u,2^{-k}r_i)}\left|\nabla f(z)\right|^q m(z)\right)^{\frac{1}{q}}\\
&\leq C2^{-k}r_i\left(\mathcal{M}\left(\nabla f\right)^q\right)^{\frac{1}{q}}(u)\\
&\leq C2^{-k}\alpha r_i \leq C2^{-k}\alpha r_j,
\end{aligned}
$$

where the penultimate inequality relies on the fact that $u\in F$ and the last one from the fact that $B_i\cap B_j\neq\emptyset$. Moreover, since $u\in 4B_j$,

$$
\begin{aligned}
B(u,r_i) &\subset B(x_j, r_i+d(u,x_j))\\
&\subset B(x_j, r_i+4r_j)\subset 7B_j.
\end{aligned}
$$

Since one also has $B_i\subset 7B_j$, one obtains, arguing as before,

$$
\begin{aligned}
\left|f_{B(u,r_i)}-f_{B_i}\right| &\leq \left|f_{B(u,r_i)}-f_{7B_j}\right| + \left|f_{7B_j}-f_{B_i}\right|\\
&\leq \frac{C}{V(7B_j)}\sum_{z\in 7B_j}\left|f(z)-f_{7B_j}\right|m(z)\\
&\leq C\alpha r_j.
\end{aligned}
\tag{6.43}
$$

It follows from (6.42) and (6.43) that

$$
\left|f(u)-f_{B_i}\right| \leq C\alpha r_j \leq C\alpha d(u,v),
$$

since

$$\begin{aligned} r_j &= \frac{1}{2}d(x_j,F) \leq \frac{1}{2}d(x_j,u) \leq \frac{1}{2}d(x_j,v) + \frac{1}{2}d(v,u) \\ &\leq \frac{1}{2}r_j + \frac{1}{2}d(v,u). \end{aligned}$$

This ends the proof of Lemma 6.5.1 because of (6.41). □

To prove (6.10), it is clearly enough to check that $|g(x)-g(y)| \leq C\alpha$ for all $x \sim y \in \Gamma$. Let us now prove this fact, distinguishing between three cases:

1. Assume that $x \in \Omega$. Then, $x \in B_j$ for some $j \in I$, and for all $y \sim x$, $y \in 2B_j \subset \Omega$, so that $\chi_F(x) = \chi_F(y) = 0$. It follows that

$$g(y) - g(x) = \sum_{i \in I} \left(f_{B_i} - f_{B_j}\right)(\chi_i(y) - \chi_i(x)),$$

so that $|g(y)-g(x)| \leq \sum_{i\in I} \left|f_{B_i} - f_{B_j}\right| \nabla\chi_i(x) := h(x)$. We claim that $|h(x)| \leq C\alpha$. To see this, note that, for all $i \in I$ such that $\nabla\chi_i(x) \neq 0$, we have $|f_{B_i} - f_{B_j}| \leq Cr_j\alpha$. Indeed, $d(x,B_i) \leq 1$, which easily implies that $r_i \leq 3r_j + 1 \leq 4r_j$, hence $B_i \subset 10B_j$. As a consequence, we have, arguing as before again,

$$\begin{aligned} |f_{B_i} - f_{10B_j}| &\leq \frac{1}{V(B_i)} \sum_{y\in B_i} |f(y) - f_{10B_j}| m(y) \\ &\leq \frac{C}{V(B_j)} \sum_{y \in 10B_j} |f(y) - f_{10B_j}| m(y) \\ &\leq Cr_j \left(\frac{1}{V(10B_j)} \sum_{y\in 10B_j} |\nabla f|^q(y) m(y) \right)^{\frac{1}{q}} \\ &\leq Cr_j\alpha \end{aligned} \tag{6.44}$$

where we used Hölder inequality, (D), (P_q) and the fact that $(|\nabla f|^q)_{10B_j} \leq \mathcal{M}(|\nabla f|)^q(z)$ for some $z \in F \cap \overline{B_j}$. Analogously $|f_{10B_j} - f_{B_j}| \leq Cr_j\alpha$. Hence

$$\begin{aligned} |h(x)| &= \left| \sum_{i\in I;\ x \in 2B_i} (f_{B_i} - f_{B_j}) \nabla\chi_i(x) \right| \\ &\leq C \sum_{i \in I;\ x\in 2B_i} |f_{B_i} - f_{B_j}| r_i^{-1} \\ &\leq CN\alpha. \end{aligned}$$

2. Assume now that $x \in F \setminus \partial F$. In this case $|g(y)-g(x)| = |f(y) - f(x)| \leq C\alpha$ by the definition of F.

3. Assume finally that $x \in \partial F$.

   i. If $y \in F$, we have $|g(y) - g(x)| = |f(x) - f(y)| \leq C\nabla f(x) \leq C\alpha$.

ii. Consider now the case when $y \in \Omega$. There exists $j \in I$ such that $y \in B_j$. Since $x \sim y$, one has $x \in 4B_j$, Lemma 6.5.1 therefore yields

$$|g(x) - g(y)| \leq C\alpha d(x, y) \leq C\alpha.$$

Therefore the proof of Proposition 6.1.9 is complete. □

Remark 6.5.2. *It is easy to get the following estimate for the b_i's: for all $i \in I$,*

$$\frac{1}{V(B_i)} \|b_i\|_1 \leq \frac{1}{V(B_i)^{1/q}} \|b_i\|_q \leq C\alpha r_i.$$

Indeed, the first inequality follows from Hölder and the fact that b_i is supported in B_i. Moreover, by (P_q) and (6.40),

$$\frac{1}{V(B_i)^{1/q}} \|b_i\|_q = \frac{1}{V(B_i)^{1/q}} \|f - f_{B_i}\|_{L^q(B_i)} \leq Cr_i \frac{1}{V(B_i)^{1/q}} \|\nabla f\|_{L^q(B_i)} \leq C\alpha r_i.$$

6.6 An interpolation result for Sobolev spaces

To prove Theorem 6.1.12, we will characterize the K functional of interpolation for homogeneous Sobolev spaces in the following theorem.

Theorem 6.6.1. *Under the same hypotheses as Theorem 6.1.12 we have that*

1. *there exists C_1 such that for every $f \in \dot{W}^{1,q}(\Gamma) + \dot{W}^{1,\infty}(\Gamma)$ and all $t > 0$*

$$K(f, t^{\frac{1}{q}}, \dot{W}^{1,q}, \dot{W}^{1,\infty}) \geq C_1 t^{\frac{1}{q}} \left(|\nabla f|^{q**}\right)^{\frac{1}{q}}(t);$$

2. *for $q \leq p < \infty$, there exists C_2 such that for every $f \in \dot{W}^{1,p}(\Gamma)$ and every $t > 0$*

$$K(f, t^{\frac{1}{q}}, \dot{W}^{1,q}, \dot{W}^{1,\infty}) \leq C_2 t^{\frac{1}{q}} \left(|\nabla f|^{q**}\right)^{\frac{1}{q}}(t).$$

Proof: We first prove item 1. Assume that $f = h + g$ with $h \in \dot{W}^{1,q}$, $g \in \dot{W}^{1,\infty}$, we then have

$$\begin{aligned} \|h\|_{\dot{W}^{1,q}} + t^{\frac{1}{q}} \|g\|_{\dot{W}^{1,\infty}} &\geq \|\nabla h\|_q + t^{\frac{1}{q}} \|\nabla g\|_\infty \\ &\geq K(\nabla f, t^{\frac{1}{q}}, L^q, L^\infty) \\ &\geq C t^{\frac{1}{q}} (|\nabla f|^{q**})^{\frac{1}{q}}(t). \end{aligned}$$

Hence we conclude that $K(f, t^{\frac{1}{q}}, \dot{W}^{1,q}, \dot{W}^{1,\infty}) \geq C_1 t^{\frac{1}{q}} (|\nabla f|^{q**})^{\frac{1}{q}}(t)$.
We prove now item 2. Let $f \in \dot{W}^{1,p}$, $q \leq p < \infty$. Let $t > 0$, we consider the Calderón-Zygmund decomposition of f given by Proposition 6.1.9 with $\alpha = \alpha(t) = \left(\mathcal{M}(|\nabla f|)^q\right)^{*\frac{1}{q}}(t)$. Thus we have $f = \sum_{i \in I} b_i + g = b + g$ where $(b_i)_{i \in I}$, g satisfy the properties of the proposition. We have the estimate

$$\|\nabla b\|_q^q \leq \sum_{x \in \Gamma} \left(\sum_{i \in I} |\nabla b_i| \right)^q (x) m(x)$$

$$\leq CN\sum_{i\in I}\sum_{x\in B_i}|\nabla b_i|^q(x)m(x)$$
$$\leq C\alpha^q(t)\sum_{i\in I}V(B_i)$$
$$\leq C\alpha^q(t)m(\Omega),$$

where the B_i's are given by Proposition 6.1.9 and Ω is defined as in the proof of Proposition 6.1.9. The last inequality follows from the fact that $\sum\limits_{i\in I}\chi_{B_i}\leq N$ and $\Omega=\bigcup\limits_i B_i$. Hence $\|\nabla b\|_q\leq C\alpha(t)m(\Omega)^{\frac{1}{q}}$. Moreover, since $(\mathcal{M}f)^*\sim f^{**}$ (see [10]), we obtain

$$\alpha(t)=(\mathcal{M}(|\nabla f|)^q)^{*\frac{1}{q}}(t)\leq C\left(|\nabla f|^{q**}\right)^{\frac{1}{q}}(t).$$

Hence, also noting that $m(\Omega)\leq t$ (see [10] again), we get $K(f,t^{\frac{1}{q}},\dot{W}^{1,q},\dot{W}^{1,\infty})\leq Ct^{\frac{1}{q}}|\nabla f|^{q**\frac{1}{q}}(t)$ for all $t>0$ and obtain the desired inequality. □

Proof of Theorem 6.1.12: The proof follows directly from Theorem 6.6.1. Indeed, item 1. of Theorem 6.6.1 gives us that

$$(\dot{W}^{1,q},\dot{W}^{1,\infty})_{1-\frac{q}{p},p}\subset\dot{W}^{1,p}$$

with $\|f\|_{\dot{W}^{1,p}}\leq C\|f\|_{1-\frac{q}{p},p}$, while item 2. gives us that

$$\dot{W}^{1,p}\subset(\dot{W}^{1,q},\dot{W}^{1,\infty})_{1-\frac{q}{p},p}$$

with $\|f\|_{1-\frac{q}{p},p}\leq C\|f\|_{\dot{W}^{1,p}}$.
Hence $\dot{W}^{1,p}=(\dot{W}^{1,q},\dot{W}^{1,\infty})_{1-\frac{q}{p},p}$ with equivalent norms. □

6.7 The proof of (RR_p) for $p<2$

In view of Theorem 6.1.12 and since (RR_2) holds, it is enough, for the proof of Theorem 6.1.6, to establish (6.5).

Proof of (6.5): We follow the proof of (1.9) in [4]. Consider such an f and fix $\lambda>0$. Perform the Calderón-Zygmund decomposition of f given by Proposition 6.1.9. We also use the following expansion of $(I-P)^{1/2}$:

$$(I-P)^{1/2}=\sum_{j=0}^{+\infty}a_j(I-P)P^j \tag{6.45}$$

where the (a_j)'s were already considered in Section 6.4. For each $i\in I$, pick the integer k such that $2^k\leq r(B_i)<2^{k+1}$ and define $r_i=2^k$. We split the expansion (6.45) into two parts:

$$(I-P)^{1/2}=\sum_{j=0}^{r_i^2}a_j(I-P)P^j+\sum_{j=r_i^2+1}^{+\infty}a_j(I-P)P^j:=T_i+U_i.$$

We first claim that

$$m\left(\left\{x\in\Gamma;\ \left|(I-P)^{1/2}g(x)\right|>\lambda\right\}\right)\leq\frac{C}{\lambda^q}\left\|\nabla f\right\|_q^q. \tag{6.46}$$

Indeed, one has

$$\begin{aligned} m\left(\left\{x\in\Gamma;\ \left|(I-P)^{1/2}g(x)\right|>\lambda\right\}\right) &\leq \frac{C}{\lambda^2}\left\|(I-P)^{1/2}g\right\|_2^2 \\ &\leq \frac{C}{\lambda^2}\left\|\nabla g\right\|_2^2, \end{aligned}$$

and since $\nabla g\leq C\lambda$ on Γ and $\left\|\nabla g\right\|_q\leq C\left\|\nabla f\right\|_q$, we obtain

$$\left\|\nabla g\right\|_2^2\leq C\lambda^{2-q}\left\|\nabla g\right\|_q^q\leq C\lambda^{2-q}\left\|\nabla f\right\|_q^q,$$

which ends the proof of (6.46).

We now claim that, for some constant $C>0$,

$$m\left(\left\{x\in\Gamma;\ \left|\sum_{i\in I}T_ib_i(x)\right|>\lambda\right\}\right)\leq\frac{C}{\lambda^q}\left\|\nabla f\right\|_q^q. \tag{6.47}$$

To prove (6.47), write

(6.48)

$$m\left(\left\{x\in\Gamma;\ \left|\sum_{i\in I}T_ib_i(x)\right|>\lambda\right\}\right)\leq m\left(\bigcup_i 4B_i\right)+m\left(\left\{x\notin\bigcup_i 4B_i;\ \left|\sum_{i\in I}T_ib_i(x)\right|>\lambda\right\}\right).$$

Observe first that, by (D) and Proposition 6.1.9,

$$m\left(\bigcup_i 4B_i\right)\leq C\sum_{i\in I}V(4B_i)\leq\frac{C}{\lambda^q}\left\|\nabla f\right\|_q^q.$$

As far as the second term in the right-hand side of (6.48) is concerned, it can be estimated by

$$m\left(\left\{x\notin\bigcup_i 4B_i;\ \left|\sum_{i\in I}T_ib_i(x)\right|>\lambda\right\}\right)\leq\frac{1}{\lambda^2}\sum_{x\in\Gamma}\left|\sum_{i\in I}\chi_{\Gamma\setminus 4B_i}(x)T_ib_i(x)\right|^2 m(x).$$

Arguing as in [4, 12, 28], we estimate this last quantity by duality. Fix a function $u\in L^2(\Gamma,m)$ with $\left\|u\right\|_2=1$. One has

$$\left|\sum_{x\in\Gamma}\sum_{i\in I}\chi_{\Gamma\setminus 4B_i}(x)T_ib_i(x)u(x)m(x)\right|\leq\sum_{i\in I}\sum_{j=2}^{+\infty}A_{i,j}$$

where, for all $i\in I$ and all $j\geq 2$,

$$A_{i,j}:=\sum_{x\in 2^{j+1}B_i\setminus 2^jB_i}\left|T_ib_i(x)\right|\left|u(x)\right|m(x).$$

On the one hand, if i, j are fixed,

$$\|T_i b_i\|_{L^2(2^{j+1}B_i \setminus 2^j B_i)} \leq \sum_{k=0}^{r_i^2} |a_k| \left\|(I-P)P^k b_i\right\|_{L^2(2^{j+1}B_i) \setminus 2^j B_i)}.$$

Given $0 \leq k \leq r_i^2$, one has, for all $x \in 2^{j+1}B_i \setminus 2^j B_i$, using (6.16),

$$\left|(I-P)P^k b_i(x)\right| \leq \sum_{y \in B_i} |p_k(x,y) - p_{k+1}(x,y)|\,|b_i(y)| \leq \sum_{y \in B_i} \frac{C}{kV(y,\sqrt{k})} e^{-c\frac{d^2(x,y)}{k}} |b_i(y)|\, m(y).$$

Using (6.4) and arguing exactly as in [4] (relying, in particular, on Remark 6.5.2), we obtain

$$\left\|(I-P)P^k b_i\right\|_{L^2(2^{j+1}B_i \setminus 2^j B_i)} \leq C\frac{r_i}{k}\left(\frac{r_i}{\sqrt{k}}\right)^{2D} e^{-c\frac{4^j r_i^2}{k}} V^{1/2}(2^{j+1}B_i)\lambda.$$

Since

$$a_k \sim \frac{1}{\sqrt{k\pi}}$$

(see Appendix), it follows that

$$\|T_i b_i\|_{L^2(2^{j+1}B_i \setminus 2^j B_i)} \leq C e^{-c4^j} V^{1/2}(2^{j+1}B_i)\lambda.$$

One concludes, as in [4], that (6.47) holds.

What remains to be proved is that

$$m\left(\left\{x \in \Gamma;\ \left|\sum_{i \in I} U_i b_i(x)\right| > \lambda\right\}\right) \leq \frac{C}{\lambda^q} \|\nabla f\|_q^q. \tag{6.49}$$

Define, for all $j \in \mathbb{Z}$,

$$\beta_j = \sum_{i \in I;\ r_i = 2^j} \frac{b_i}{r_i},$$

so that, for all $j \in \mathbb{Z}$,

$$\sum_{i \in I;\ r_i = 2^j} b_i = 2^j \beta_j.$$

One has

$$\begin{aligned}
\sum_{i \in I} U_i b_i &= \sum_{i \in I} \sum_{k > r_i^2} a_k (I-P)P^k b_i \\
&= \sum_{k>0} a_k (I-P)P^k \sum_{i \in I;\ r_i^2 < k} b_i \\
&= \sum_{k>0} a_k (I-P)P^k \sum_{i \in I;\ r_i^2 = 2^{2j} < k} b_i \\
&= \sum_{k>0} a_k (I-P)P^k \sum_{j;\ 4^j < k} 2^j \beta_j.
\end{aligned}$$

For all $k > 0$, define

$$f_k = \sum_{j;\ 4^j < k} \frac{2^j}{\sqrt{k}} \beta_j.$$

It follows from the previous computation and Theorem 6.1.10 that

$$\left\| \sum_{i \in I} U_i b_i \right\|_q \leq C \left\| \left(\sum_{k=1}^{+\infty} \frac{1}{k} |f_k|^2 \right)^{1/2} \right\|_q .$$

To see this, we estimate the left-hand side of this inequality by duality, as in [4] and use the fact that $|a_k| \leq \frac{C}{\sqrt{k}}$ for all $k \geq 1$. Since, by Cauchy-Schwarz,

$$|f_k|^2 \leq 2 \sum_{j;\ 4^j < k} \frac{2^j}{\sqrt{k}} |\beta_j|^2 ,$$

one obtains

$$\left\| \left(\sum_{k=1}^{+\infty} \frac{1}{k} |f_k|^2 \right)^{1/2} \right\|_q \leq \left\| \left(\sum_{k \in \mathbb{Z}} |\beta_k|^2 \right)^{1/2} \right\|_q .$$

By the bounded overlap property,

$$\left\| \left(\sum_{k \in \mathbb{Z}} |\beta_k|^2 \right)^{1/2} \right\|_q^q \leq C \sum_{x \in \Gamma} \sum_{i \in I} \frac{|b_i(x)|^q}{r_i^q} m(x),$$

so that, using Remark 6.5.2, one obtains

$$\sum_{x \in \Gamma} \sum_{i \in I} \frac{|b_i(x)|^q}{r_i^q} m(x) \leq C \lambda^q \sum_{i \in I} V(B_i).$$

As a conclusion,

$$m \left(\left\{ x \in \Gamma;\ \left| \sum_{i \in I} U_i b_i(x) \right| > \lambda \right\} \right) \leq C \sum_{i \in I} V(B_i) \leq \frac{C}{\lambda^q} \left\| \nabla f \right\|_q^q ,$$

which is exactly (6.49). The proof of (6.5) is therefore complete. □

Appendix

We prove Lemma 6.4.1. For all $l \geq 0$, $a_l = \frac{(2l)!}{4^l (l!)^2}$, and, as already used in Section 6.7, the Stirling formula shows $a_l \sim \frac{1}{\sqrt{\pi l}}$. Therefore, there exists $C > 0$ such that, for all $l \geq 1$,

$$0 < a_l \leq \frac{C}{\sqrt{l}}.$$

Assume first that $mk^2 < l < (m+1)k^2$ for some integer $0 \le m \le n$. For each integer $j \ge 0$ such that $jk^2 \le l$, one has $l - jk^2 > 0$ and $j \le m$, so that $|a_{l-jk^2}| \le \frac{C}{\sqrt{l-jk^2}} \le \frac{C}{\sqrt{l-mk^2}}$. It follows at once that

$$|d_l| \le \frac{C}{\sqrt{l - mk^2}}$$

for some $C > 0$ only depending on n.

Assume now that $l = (m+1)k^2$ for some $0 \le m \le n$. For each $j \ge 0$ such that $jk^2 \le l$ and $l - jk^2 > 0$, one has $j \le m$ again, so that $|a_{l-jk^2}| \le \frac{C}{\sqrt{l-mk^2}} = \frac{C}{k} \le C$. Moreover, $a_0 = 1$. One therefore has

$$|d_l| \le C + C_n^{m+1} \le C,$$

where, again, C only depends on n.

Finally, assume that $l > (n+1)k^2$. The classical computation of Wallis integrals shows that

$$a_l = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} (\sin t)^{2l} \, dt = \varphi(l)$$

where, for all $x > 0$, $\varphi(x) = \dfrac{2}{\pi} \displaystyle\int_0^{\frac{\pi}{2}} (\sin t)^{2x} \, dt$. We can then invoke (6.24) and are therefore left with the task of estimating $\varphi^{(n)}$. But, for all $x > 0$,

$$\left|\varphi^{(n)}(x)\right| = \frac{2}{\pi} \left| \int_0^{\frac{\pi}{2}} (2 \log \sin t)^n \, e^{2x \log \sin t} dt \right| \le \frac{2}{\pi} \int_0^{\frac{\pi}{2}} |2 \log \sin t|^n \, e^{2x \log \sin t} dt := \frac{2}{\pi} I_n(x).$$

We now argue as in the "Laplace" method. For all $\delta \in \left(0, \frac{\pi}{2}\right)$, one clearly has, for all $x > 1$,

$$\begin{aligned} 0 \le I_n(x) &\le \int_0^{\frac{\pi}{2}-\delta} |2 \log \sin t|^n \, e^{2x \log \sin t} dt + \int_{\frac{\pi}{2}-\delta}^{\frac{\pi}{2}} |2 \log \sin t|^n \, e^{2x \log \sin t} dt \\ &\le \left(\sin\left(\frac{\pi}{2} - \delta\right)\right)^{2x-2} I_n(1) + J_n(x) = C_{n,\delta} \alpha^{2x-2} + J_n(x) \end{aligned} \tag{6.1}$$

where $C_{n,\delta} > 0$ only depends on n and δ, $0 < \alpha = \sin\left(\frac{\pi}{2} - \delta\right) < 1$ and $J_n(x) := \displaystyle\int_{\frac{\pi}{2}-\delta}^{\frac{\pi}{2}} |2 \log \sin t|^n \, e^{2x \log \sin t} dt$.

Observe now that $J_n(x) = \displaystyle\int_0^{\delta} |2 \log \cos u|^n \, e^{2x \log \cos u} du$. Since $\log(\cos u) \sim -\frac{u^2}{2}$ when $u \to 0$, we fix $\delta > 0$ such that, for all $0 < u < \delta$, $-\frac{3}{4}u^2 \le \log(\cos u) \le -\frac{1}{4}u^2$, which implies

$$\begin{aligned} J_n(x) &\le C \int_0^{\delta} u^{2n} e^{-\frac{1}{2}xu^2} du \\ &\le C \left(\frac{1}{\sqrt{x}}\right)^{2n+1} \int_0^{+\infty} v^{2n} e^{-v^2} dv \le C x^{-n-\frac{1}{2}}. \end{aligned} \tag{6.2}$$

It follows from (6.1) and (6.2) that, for all $x > 1$,

$$\left|\varphi^{(n)}(x)\right| \leq C x^{-n-\frac{1}{2}},$$

which, joined with (6.24), yields assertion (*iii*) in Lemma 6.4.1, the proof of which is now complete. □

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Chapitre 7

Gagliardo-Nirenberg inequalities on manifolds

Abstract. We prove Gagliardo-Nirenberg inequalities on some classes of manifolds, Lie groups and graphs.

Résumé. On démontre des inégalités de Gagliardo-Nirenberg sur certaines classes de variétés Riemanniennes, groupes de Lie et graphes.

7.1 Introduction

The classical Sobolev inequality

$$\|f\|_{p^*} \leq C\| \,|\nabla f|\, \|_p \tag{7.1}$$

with $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$ is well known to hold on $\mathbb{R}^n$ for every $f \in W_p^1(\mathbb{R}^n)$ and for every $1 \leq p < n$. It is also known that (7.1) holds on a compact Riemannian n-manifold M. As an example of complete non-compact Riemannian manifold satisfying (7.1), we can take a complete Riemannian n-manifold M with non-negative Ricci curvature. If there exists $v > 0$ such that for all $x \in M$, $\mu(B(x,1)) \geq v$, then M satisfies (7.1). Here $\mu(B(x,1))$ is the Riemannian volume of the open ball $B(x,1)$. For more general cases where we have (7.1) for some p's depending on the hypotheses, see [22]. Note that if (7.1) holds for some $1 \leq p < n$ then it holds for all $p \leq q < n$ (see Chapter 3 in [22]).

Cohen-Meyer-Oru [7], Cohen-Devore-Petrushev-Xu [6], improved (7.1) in the Euclidean case and obtained the following Gagliardo-Nirenberg type inequality

$$\|f\|_{1^*} \leq C\| \,|\nabla f|\, \|_1^{\frac{n-1}{n}} \|f\|_{B_{\infty,\infty}^{-(n-1)}}^{\frac{1}{n}}. \tag{7.2}$$

for all $f \in W_1^1(\mathbb{R}^n)$, with $1^* = \frac{n}{n-1}$. The proof of (7.2) is rather involved and based on wavelet decompositions, weak l^1 type estimates and interpolation results.

Using a simple method relying on weak type estimates and pseudo-Poincaré inequalities, Ledoux obtained in [19] the following extension of (7.2). He proved that for $1 \leq p < l < \infty$ and for every $f \in W_p^1(\mathbb{R}^n)$

$$\|f\|_l \leq C \| \, |\nabla f| \, \|_p^\theta \|f\|^{1-\theta}_{B^{\frac{\theta}{\theta-1}}_{\infty,\infty}} \tag{7.3}$$

where $\theta = \frac{p}{l}$ and $C > 0$ only depends on l, p and n.

In the same paper, he extended (7.3) to the case of Riemannian manifolds. If $p = 2$ he observed that (7.3) holds without any assumption on M. If $p \neq 2$ he assumed that the Ricci curvature is non-negative and obtained (7.3) with $C > 0$ only depending on l, p when $1 \leq p \leq 2$ and on l, p and n when $2 < p < \infty$.

He also proved that a similar inequality holds on $\mathbb{R}^n$, Riemannian manifolds with non-negative Ricci curvature, Lie groups and Cayley graphs, replacing the $B^{\frac{\theta}{\theta-1}}_{\infty,\infty}$ norm by the $M_\infty^{\frac{\theta}{\theta-1}}$ norm (see definitions below).

Rivière-Strzelecki [21], [24], obtained non linear versions of Gagliardo-Nirenberg inequalities. They got for every $f \in C_0^\infty(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} |\nabla f|^{p+2} \leq C \|f\|^2_{BMO} \int_{\mathbb{R}^n} |\nabla^2 f|^2 |\nabla f|^{p-2}. \tag{7.4}$$

Recently, Martin and Milman [20] developed a new symmetrization approach to obtain the Gagliardo-Nirenberg inequalities (7.3) and, therefore the Sobolev inequalities (7.1) in $\mathbb{R}^n$. They also proved a variant of (7.4). The method of [20] to prove (7.3) is different from that of Ledoux. It relies essentially on an interpolation result for Sobolev spaces and pseudo-Poincaré inequalities in the Euclidean case.

In this paper, we prove analogous results on Riemannian manifolds, Lie groups and graphs making some additional hypotheses on these spaces. This will be done by adapting Martin and Milman's method and making use of our interpolation results in [3]. More precisely we obtain in the case of Riemannian manifolds:

Theorem 7.1.1. *Let M be a complete non-compact Riemannian manifold satisfying (D) and (P_q) for some $1 \leq q < \infty$. Moreover, assume that M satisfies the pseudo-Poincaré inequalities (P'_q) and (P'_∞). Consider $\alpha < 0$. Then, there exists $C > 0$ such that for every $f \in (W_q^1 + W_\infty^1) \cap B^\alpha_{\infty,\infty}$ with $f^*(\infty) = 0$ and $|\nabla f|^*(\infty) = 0$, we have*

$$|f|^{q**\frac{1}{q}}(s) \leq C |\nabla f|^{q**\frac{|\alpha|}{q(1+|\alpha|)}}(s) \|f\|^{\frac{1}{1+|\alpha|}}_{B^\alpha_{\infty,\infty}} \tag{7.5}$$

where $|f|^{q**\frac{1}{q}}$ denotes $(|f|^{q**})^{\frac{1}{q}}$.

Recall that for all $t > 0$

$$f^*(t) = \inf\{\lambda; \mu(\{|f| > \lambda\}) \leq t\};$$

$$f^*(\infty) = \inf\{\lambda; \mu(\{|f| > \lambda\}) < \infty\}$$

and

$$f^{**}(t) = \frac{1}{t} \int_0^t f^*(s) ds.$$

Using this symmetrization result we prove

Theorem 7.1.2. *Let M be a complete Riemannian manifold satisfying the hypotheses of Theorem 7.1.1. Then (7.3) holds for all $q \leq p < l < \infty$.*

Corollary 7.1.3. *Let M be a Riemannian manifold with non-negative Ricci curvature. Then (7.3) holds for all $1 \leq p < l < \infty$.*

This corollary is exactly what Ledoux proved in [19]. We obtain further generalizations.

Corollary 7.1.4. *Consider a complete Riemannian manifold M satisfying (D), (P_1) and assume that there exists $C > 0$ such that for all x, $y \in M$, for all $t > 0$*

$$(G) \qquad |\nabla_x p_t(x, y)| \leq \frac{C}{\sqrt{t}\mu(B(y, \sqrt{t}))}.$$

Then inequality (7.3) holds for all $1 \leq p < l < \infty$.

Note that a Lie group of polynomial growth satisfies the hypotheses of Corollary 7.1.4 (see 4. of section 7.7). Hence on such a group we have (7.3) for all $1 \leq p < l < \infty$. For such group, Ledoux [19] only showed the validity of the variant of (7.3) replacing the Besov norm by the Morrey norm.
Another example that shows the generalization of [19] is given by taking a Galois covering manifold of a compact manifold whose deck transformation group is of polynomial growth (see 3. of section 7.7).

We get also the following Corollary:

Corollary 7.1.5. *Let M be a complete Riemannian manifold satisfying (D) and (P_2). Then (7.3) holds for all $2 \leq p < l < \infty$.*

Note that inequality (7.3) with $p = 2$ needs no assumption on M, so our results are only interesting when $p \neq 2$.

In the following theorem, we show a variant of Theorem 7.1.1 replacing the Besov norm by the Morrey norm.In the Euclidean case, the Morrey space is strictly smaller than the Besov space. So the following Theorem 7.1.6 (resp. Corollary 7.1.7) is weaker than Theorem 7.1.1 (resp. Theorem 7.1.2). For Riemannian manifolds, the Besov and Morrey spaces are not directly comparable in general.

Theorem 7.1.6. *Let M be a complete non-compact Riemannian manifold satisfying (D) and (P_q) for some $1 \leq q < \infty$. Consider $q \leq p < \infty$ and $\alpha < 0$. Then, for every $f \in (W_q^1 + W_\infty^1) \cap M_\infty^\alpha$ we have*

$$|f|^{q**\frac{1}{q}}(s) \leq C|\nabla f|^{q**\frac{|\alpha|}{q(1+|\alpha|)}}(s)\|f\|_{M_\infty^\alpha}^{\frac{1}{1+|\alpha|}}.$$

Corollary 7.1.7. *Under the hypotheses of Theorem 7.1.6, let $q_0 = \inf\{q \in [1, \infty[: (P_q) \text{ holds }\}$ and consider $q_0 < p < l < \infty$[1]. Thus, as above, we recover Ledoux's inequality (7.3) replacing the Besov norm by the Morrey norm.*

[1]if $q_0 = 1$, we allow $1 \leq p < l < \infty$

We finish with the following non linear Gagliardo-Nirenberg theorem:

Theorem 7.1.8. *Let M be a complete non-compact Riemannian manifold satisfying (D) and (P_q) for some $1 \leq q < \infty$. Moreover, assume that M satisfies (P'_q) and (P'_∞). Let $p \geq \max(2, q)$. Then for every $f \in C_0^\infty(M)$*

$$\int_M |\nabla f|^{p+1} d\mu \leq C \|f\|_{B^{-1}_{\infty,\infty}} \int_M |\nabla^2 f|^2 |\nabla f|^{p-2} d\mu.$$

The paper is organized as follows. In section 7.2, we give the definitions on a Riemannian manifold of Besov and Morrey spaces, Sobolev spaces, doubling property, Poincaré and pseudo-Poincaré inequalities. In section 7.3, we see how to obtain under our hypotheses Ledoux's inequality (7.3) and different Sobolev inequalities. Section 7.4 is devoted to prove Theorem 7.1.1 and Theorem 7.1.6. In section 7.5 we give another symmetrization inequality. We prove Theorem 7.1.8 in section 7.6. Finally, in section 7.7 we give some examples and extensions of our result.

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7.2 Preliminaries

Throughout this paper C will be a constant that may change from an inequality to another and we will use $u \sim v$ to say that there exist two constants C_1,$C_2 > 0$ such that $C_1 u \leq v \leq C_2 u$.

Let M be a complete non-compact Riemannian manifold. We write μ for the Riemannian measure on M, ∇ for the Riemannian gradient, $|\cdot|$ for the length on the tangent space (forgetting the subscript x for simplicity) and $\|\cdot\|_p$ for the norm on $L_p(M,\mu)$, $1 \leq p \leq +\infty$. We denote $C_0^\infty := C_0^\infty(M)$ the space of C^∞ functions compactly supported defined on M. Let $P_t = e^{t\Delta}$, $t \geq 0$, be the heat semigroup on M and p_t the heat kernel.

7.2.1 Besov and Morrey spaces

For $\alpha < 0$, we introduce the Besov norm

$$\|f\|_{B^\alpha_{\infty,\infty}} = \sup_{t>0} t^{-\frac{\alpha}{2}} \|P_t f\|_\infty < \infty$$

for measurable functions f such that this makes sense and say $f \in B^\alpha_{\infty,\infty}$ (we shall not try here to give the most general definition of the Besov space).

Lemma 7.2.1. *We have for every $f \in B^\alpha_{\infty,\infty}$*

$$\|f\|_{B^\alpha_{\infty,\infty}} \sim \sup_{t>0} t^{-\frac{\alpha}{2}} \|P_t(f - P_t f)\|_\infty. \tag{7.6}$$

Proof. It is clear that $\sup_{t>0} t^{-\frac{\alpha}{2}} \|P_t(f - P_t f)\|_\infty \leq (1 + 2^{\frac{\alpha}{2}})\|f\|_{B^\alpha_{\infty,\infty}}$. On the other hand

$$t^{-\frac{\alpha}{2}} P_t f = t^{-\frac{\alpha}{2}}(P_t f - P_{2t} f) + 2^{\frac{\alpha}{2}}\,(2t)^{-\frac{\alpha}{2}} P_{2t} f.$$

By taking the supremun over all $t > 0$, we get

$$\|f\|_{B^\alpha_{\infty,\infty}} \leq \sup_{t>0} t^{-\frac{\alpha}{2}} \|P_t(f - P_t f)\|_\infty + 2^{\frac{\alpha}{2}} \|f\|_{B^\alpha_{\infty,\infty}}.$$

Thus, $\|f\|_{B^\alpha_{\infty,\infty}} \leq \frac{1}{1-2^{\frac{\alpha}{2}}} \sup_{t>0} t^{-\frac{\alpha}{2}} \|P_t(f - P_t f)\|_\infty$. □

For $\alpha < 0$, the Morrey space M^α_∞ is the space of locally integrable functions f for which the Morrey norm

$$\|f\|_{M^\alpha_\infty} := \sup_{r>0,\, x\in M} r^{-\alpha} |f_{B(x,r)}| < \infty$$

where $f_B := \fint_B f d\mu = \frac{1}{\mu(B)} \int_B f d\mu$.

7.2.2 Sobolev spaces on Riemannian manifolds

Definition 7.2.2 ([2])**.** *Let M be a C^∞ Riemannian manifold of dimension n. Write E^1_p for the vector space of C^∞ functions φ such that φ and $|\nabla\varphi| \in L_p$, $1 \leq p < \infty$. We define the Sobolev space W^1_p as the completion of E^1_p for the norm*

$$\|\varphi\|_{W^1_p} = \|\varphi\|_p + \|\,|\nabla\varphi|\,\|_p.$$

We denote W^1_∞ for the set of all bounded Lipschitz functions on M.

Proposition 7.2.3. *Let M be a complete Riemannian manifold. Then C^∞_0 is dense in W^1_p for $1 \leq p < \infty$ (see [2]).*

Definition 7.2.4. *Let M be a C^∞ Riemannian manifold of dimension n. For $1 \leq p \leq \infty$, we define $\dot{E}^1_p$ to be the vector space of distributions φ with $|\nabla\varphi| \in L_p$, where $\nabla\varphi$ is the distributional gradient of φ. It is well known that the elements of $\dot{E}^1_p$ are in $L_{p,loc}$. We equip $\dot{E}^1_p$ with the semi norm*

$$\|\varphi\|_{\dot{E}^1_p} = \|\,|\nabla\varphi|\,\|_p.$$

Definition 7.2.5. *We define the homogeneous Sobolev space $\dot{W}^1_p$ as the quotient space $\dot{E}^1_p/\mathbb{R}$ for the norm*

$$\|\,\phi\|_{\dot{W}^1_p} = \inf\left\{\|\,|\nabla\varphi|\,\|_p,\, \varphi \in \dot{E}^1_p,\, \overline{\varphi} = \phi\right\}.$$

Remark 7.2.6. *For all $\varphi \in \dot{E}^1_p$, $\|\overline{\varphi}\|_{\dot{W}^1_p} = \|\,|\nabla\varphi|\,\|_p$.*

7.2.3 Doubling property and Poincaré inequalities

Definition 7.2.7 (Doubling property). *Let (M, d, μ) be a Riemannian manifold. Denote by $B(x, r)$ the open ball of center $x \in M$ and radius $r > 0$ and by $\mu(B(x, r))$ its measure. One says that M satisfies the doubling property (D) if there exists a constant $C_d > 0$ such that for all $x \in M$, $r > 0$ we have*

$$(D) \qquad \mu(B(x, 2r)) \leq C_d \mu(B(x, r)).$$

Observe that if M satisfies (D) then

$$diam(M) < \infty \Leftrightarrow \mu(M) < \infty \text{ (see [1])}.$$

Definition 7.2.8 (Poincaré inequality). *A complete Riemannian manifold M admits a Poincaré inequality (P_q) for some $1 \leq q < \infty$ if there exists a constant C such that for all $f \in C_0^\infty$ and for every ball B of M of radius $r > 0$, we have*

$$(P_q) \qquad \Big(\fint_B |f - f_B|^q d\mu\Big)^{\frac{1}{q}} \leq Cr\Big(\fint_B |\nabla f|^q d\mu\Big)^{\frac{1}{q}}$$

Remark 7.2.9. *Since C_0^∞ is dense in W_q^1, if M admits (P_q) for all $f \in C_0^\infty$ then (P_q) holds for all $f \in W_q^1$. In fact, by Theorem 1.3.4 in [16], M admits (P_q) for all $f \in \dot{E}_q^1$.*

The following recent result of Keith and Zhong [17] improves the exponent of Poincaré inequality:

Theorem 7.2.10. *Let (X, d, μ) be a complete metric-measure space with μ locally doubling and admitting a local Poincaré inequality (P_q), for some $1 < q < \infty$. Then there exists $\varepsilon > 0$ such that (X, d, μ) admits (P_p) for every $p > q - \varepsilon$.*

Definition 7.2.11 (Pseudo-Poincaré inequality for the heat semigroup). *A Riemannian manifold M admits a pseudo-Poincaré inequality for the heat semigroup (P'_q) for some $1 \leq q < \infty$ if there exists a constant C such that for all $f \in C_0^\infty$ and all $t > 0$, we have*

$$(P'_q) \qquad \|f - P_t f\|_q \leq Ct^{\frac{1}{2}} \| \, |\nabla f| \, \|_q.$$

M admits a pseudo-Poincaré inequality (P'_∞) if there exists $C > 0$ such that for every bounded Lipschitz function f we have

$$(P'_\infty) \qquad \|f - P_t f\|_\infty \leq Ct^{\frac{1}{2}} \| \, |\nabla f| \, \|_\infty.$$

Remark 7.2.12. *Again by density of C_0^∞ in W_q^1, if M admits (P'_q) for some $1 \leq q < \infty$ for all $f \in C_0^\infty$ then M admits (P'_q) for all $f \in W_q^1$.*

Definition 7.2.13 (Pseudo-Poincaré inequality for averages). *A complete Riemannian manifold M admits a pseudo-Poincaré inequality for averages (P''_q) for some $1 \leq q < \infty$ if there exists a constant C such that for all $f \in C_0^\infty$ and for every ball B of M of radius $r > 0$, we have*

$$(P''_q) \qquad \|f - f_{B(.,r)}\|_q \leq Cr \| \, |\nabla f| \, \|_q.$$

Remark 7.2.14. *(Lemma 5.3.2 in [22]) If M is a complete Riemannian manifold satisfying (D) and (P_q) for some $1 \leq q < \infty$, then it satisfies (P''_q). Hence (P''_q) holds for all $f \in \dot{E}_q^1$.*

7.3 Ledoux's and Sobolev inequalities

7.3.1 Ledoux's inequality

Proof of Theorem 7.1.2. Let us see how to obtain Ledoux's inequality (7.3) from Theorem 7.1.1. Consider M satisfying the hypotheses of Theorem 7.1.1 and take $q < p < l$. From (7.5), we see that

$$\| \, |f|^{q**\frac{1}{q}} \|_X \leq C \| \, |\nabla f|^{q**\frac{1}{q}} \|_{X_{\frac{|\alpha|}{1+|\alpha|}}}^{\frac{|\alpha|}{1+|\alpha|}} \|f\|_{B^{\alpha}_{\infty,\infty}}^{\frac{1}{1+|\alpha|}}$$

with $X = L_l$ which is a rearrangement invariant space (see [5], section 2 of [20]) and

$$X_a = \left\{ f : |f|^a \in X, \text{with } \|f\|_{X_a} = \| \, |f|^a \, \|_X^{\frac{1}{a}} \right\}.$$

By taking $\alpha = \frac{p}{p-l}$ we get (7.3) for $p > q$. For $q = p$, note that (7.5) implies the weak type inequality (q,l), that is $\mu(\{|f| > \lambda\}) \leq \left(\frac{C}{\lambda} \| \, |\nabla f| \, \|_q^{\frac{q}{l}} \|f\|_{B^{\alpha}_{\infty,\infty}}^{1-\frac{q}{l}} \right)^l$. Consequently the strong type (q,l), that is $\|f\|_l \leq C \| \, |\nabla f| \, \|_q^{\frac{q}{l}} \|f\|_{B^{\alpha}_{\infty,\infty}}^{1-\frac{q}{l}}$, follows by Maz'ya's truncation principle (see [13], [19]). □

Proof of Corollary 7.1.3. Remark that Riemannian manifolds with non-negative Ricci curvature satisfy (D) with $(C_d = 2^n)$, (P_1). They also satisfy (P'_p) for all $1 \leq p \leq \infty$, where the constant C is numerical for $1 \leq p \leq 2$ and only depends on n for $2 < p \leq \infty$ (see [19]). Thus Theorem 7.1.2 applies on such manifolds with $q = 1$. □

Before we prove Corollary 7.1.4, we give the following two lemmas. Let $2 < p \leq \infty$. Consider the following condition: there exists $C > 0$ such that for all $t > 0$

$$(G_p) \qquad \| \, |\nabla e^{t\Delta}| \, \|_{p \to p} \leq \frac{C}{\sqrt{t}}.$$

Lemma 7.3.1. *([11]) Let M be a complete Riemannian manifold M satisfying (D) and the Gaussian heat kernel upper bound, that is, there exist $C, c > 0$ such that for all x, $y \in M$, for all $t > 0$*

$$(7.7) \qquad p_t(x,y) \leq \frac{C}{\mu(B(y,\sqrt{t}))} e^{-c\frac{d^2(x,y)}{t}}.$$

Then (G) holds if and only if (G_∞) holds.

Lemma 7.3.2. *Let M be a complete Riemannian manifold. If the condition (G_p) holds for some $1 < p \leq \infty$ then M admits a pseudo-Poincaré inequality $(P'_{p'})$, p' being the conjugate of p ($\frac{1}{p} + \frac{1}{p'} = 1$).*

Proof. For $f \in C_0^\infty$, we have

$$f - e^{t\Delta} f = -\int_0^t \Delta e^{s\Delta} f \, ds.$$

Remark that (G_p) gives us that $\|\Delta e^{s\Delta} f\|_{p'} \leq \frac{C}{\sqrt{s}} \| \, |\nabla f| \, \|_{p'}$. Indeed

$$\begin{aligned}
\|\Delta e^{s\Delta} f\|_{p'} &= \sup_{\|g\|_p=1} \int_M \Delta e^{s\Delta} f \, g \, d\mu \\
&= \sup_{\|g\|_p=1} \int_M f \, \Delta e^{s\Delta} g \, d\mu \\
&= \sup_{\|g\|_p=1} \int_M \nabla f . \nabla e^{s\Delta} g \, d\mu \\
&\leq \| \, |\nabla f| \, \|_{p'} \sup_{\|g\|_p=1} \| \, |\nabla e^{s\Delta} g| \, \|_p \\
&\leq \frac{C}{\sqrt{s}} \| \, |\nabla f| \, \|_{p'}.
\end{aligned}$$

Therefore

$$\|f - e^{t\Delta} f\|_{p'} \leq C \| \, |\nabla f| \, \|_{p'} \int_0^t \frac{1}{\sqrt{s}} ds = C\sqrt{t} \| \, |\nabla f| \, \|_{p'}$$

which finishes the proof of the lemma. □

Proof of Corollary 7.1.4. The fact that M satisfies (D) and admits (P_1), hence (P_2), gives us the Gaussian heat kernel upper bound (7.7). Since (G) holds, Lemma 7.3.1 gives that (G_∞) holds too. Thus we obtain by Lemma 7.3.2 that M admits a pseudo-Poincaré inequality (P_1'). We claim that (P_∞') holds on M. Indeed, (7.7) yields

$$\begin{aligned}
\|f - e^{t\Delta} f\|_\infty &\leq \sup_{x \in M} \int_M |f(x) - f(y)| p_t(x,y) d\mu(y) \\
&\leq C \| \, |\nabla f| \, \|_\infty \sup_{x \in M} \frac{1}{\mu(B(x,\sqrt{t}))} \int_M d(x,y) e^{-c\frac{d^2(x,y)}{t}} d\mu(y) \\
&\leq C\sqrt{t} \| \, |\nabla f| \, \|_\infty \sup_{x \in M} \frac{1}{\mu(B(x,\sqrt{t}))} \int_M e^{-c'\frac{d^2(x,y)}{t}} d\mu(y) \\
&\leq C\sqrt{t} \| \, |\nabla f| \, \|_\infty \sup_{x \in M} \frac{1}{\mu(B(x,\sqrt{t}))} \mu(B(x,\sqrt{t})) \\
&= C\sqrt{t} \| \, |\nabla f| \, \|_\infty
\end{aligned}$$

where the last estimate is a straightforward consequence of (D). Therefore, we have all we need to apply Theorem 7.1.1 with $q = 1$. The inequality (7.3) for all $1 \leq p < l < \infty$ follows then by Theorem 7.1.2. □

Remark 7.3.3. *Under the hypotheses of Corollary 7.1.4, Theorem 7.1.6 and Theorem 7.1.8 also hold.*

Proof of Corollary 7.1.5. First we know that (G_2) always holds on M then Lemma 7.3.2 gives us that (P_2') holds on M. Secondly (D) and (P_2) yields (P_∞') as we have just seen above. Hence Theorem 7.1.2 applies with $q = 2$. □

7.3.2 The classical Sobolev inequality

Proposition 7.3.4. *Consider a complete non-compact Riemannian manifold satisfying the hypotheses of Theorem 7.1.1 and assume that* $1 \le q < \nu$ *with* $\nu > 0$. *From (7.3) and under the heat kernel bound* $\|P_t\|_{q\to\infty} \le Ct^{-\frac{\nu}{2q}}$, *one recovers the classical Sobolev inequality*

$$\|f\|_{q^*} \le C\| \, |\nabla f| \, \|_q$$

with $\frac{1}{q^*} = \frac{1}{q} - \frac{1}{\nu}$. *Consequently, we get*

$$\|f\|_{p^*} \le C\| \, |\nabla f| \, \|_p$$

with $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{\nu}$ *for* $q \le p < \nu$.

Proof. Recall that $\|f\|_{B^{\alpha}_{\infty,\infty}} \sim \sup_{t>0} t^{-\frac{\alpha}{2}} \|P_t(f - P_t f)\|_\infty$. The pseudo-Poincaré inequality (P'_q), (7.3) and the heat kernel bound $\|P_t\|_{q\to\infty} \le Ct^{-\frac{\nu}{2q}}$ yield

$$\|f\|_{q^*} \le C\| \, |\nabla f| \, \|_q^{\theta} \left(\sup_{t>0} t^{-\frac{1}{2}} \|f - P_t f\|_q \right)^{1-\theta} \le C\| \, |\nabla f| \, \|_q.$$

Thus we get (7.1) with $p = q < \nu$ and $\frac{1}{q^*} = \frac{1}{q} - \frac{1}{\nu}$. □

7.3.3 Sobolev inequalities for Lorentz spaces

For $1 \le p \le \infty$, $0 \le r < \infty$ we note $L(p,r)$ the Lorentz space of functions f such that

$$\|f\|_{L(p,r)} = \left(\int_0^\infty (f^{**}(t) t^{\frac{1}{p}})^r \frac{dt}{t} \right)^{\frac{1}{r}} < \infty$$

and

$$\|f\|_{L(p,\infty)} = \sup_t t^{\frac{1}{p}} f^*(t) < \infty.$$

Consider a complete non-compact Riemannian n-manifold M satisfying (D) and (P_q) for some $1 \le q < \infty$. Moreover, assume that the following global growth condition

$$\mu(B) \ge Cr^{\sigma} \tag{7.8}$$

holds for every ball $B \subset M$ of radius $r > 0$ and for some $\sigma > q$.Using Remark 4 in [15], we get

$$f^{**}(t) - f^*(t) \le Ct^{\frac{1}{\sigma}} |\nabla f|^{q**\frac{1}{q}}(t) \tag{7.9}$$

for every $f \in \dot{E}^1_q$. We can write (7.9) as

$$f^{**}(t) - f^*(t) \le \left[Ct^{\frac{1}{\sigma}} |\nabla f|^{q**\frac{1}{q}}(t) \right]^{1-\theta} (f^{**}(t) - f^*(t))^{\theta}, \; 0 \le \theta \le 1. \tag{7.10}$$

Take $\frac{1}{r} = \frac{1-\theta}{p*} + \frac{\theta}{l}$, $\frac{1}{m} = \frac{1-\theta}{m_0} + \frac{\theta}{m_1}$ with $0 \leq \theta \leq 1$, $p > q$, $m_0 \geq q$ and $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{\sigma}$. Then from (7.10) and Hölder's inequality, we get the following Gagliardo-Nirenberg inequality for Lorentz spaces

$$\|f\|_{L(r,m)} \leq C\| \, |\nabla f| \, \|_{L(p,m_0)}^{1-\theta} \|f\|_{L(l,m_1)}^{\theta}. \tag{7.11}$$

We used also the fact that for $1 < p \leq \infty$ and $1 \leq r \leq \infty$

$$\|f\|_{L(p,r)} \sim \left[\int_0^\infty \left(t^{\frac{1}{p}} f^*(t) \right)^r \frac{dt}{t} \right]^{\frac{1}{r}}$$

to obtain the term $\| \, |\nabla f| \, \|_{L(p,m_0)}$ (see [23] Chapter 5, Theorem 3.21).
If we take $\theta = 0$ and $m_0 = m = p$, $r = p^*$, (7.11) becomes

$$\|f\|_{L(p^*,p)} \leq C\| \, |\nabla f| \, \|_p. \tag{7.12}$$

Noting that $p^* > p$ (hence $\|f\|_{L(p^*,p^*)} \leq C\|f\|_{L(p^*,p)}$), (7.12) yields (7.1) with $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{\sigma}$ and $q < p \leq \sigma$. Using Theorem 7.2.10, we get (7.1) for every $q_0 < p \leq \sigma$ where $q_0 = \inf \{q \in [1,\infty[; (P_q) \text{ holds} \}$. If $q_0 = 1$ the strict inequality at q_0 becomes large.

Remark 7.3.5. *In [22], it was proved that under (D), (P_q'') and (7.8) with $\sigma > q$ we have (7.1) for all $q \leq p < \sigma$. Since (D) and (P_q) yield (P_q'') we cover under our hypotheses this result and we get moreover the limiting case $p = \sigma$.*

7.4 Proof of Theorem 7.1.1 and Theorem 7.1.6

The main tool to prove these two theorems is the following two characterizations of the K functional of interpolation for the homogeneous Sobolev norm.

Theorem 7.4.1. *([3]: Chapter 3 of this thesis) Let M be a complete Riemannian manifold satisfying (D) and (P_q) for some $1 \leq q < \infty$. Consider the K functional of interpolation for the spaces $\dot{W}_q^1$ and $\dot{W}_\infty^1$ defined as*

$$K(F, t, \dot{W}_q^1, \dot{W}_\infty^1) = \inf_{\substack{f=h+g \\ h \in \dot{E}_q^1, \, g \in \dot{E}_\infty^1}} (\| \, |\nabla h| \, \|_q + t \| \, |\nabla g| \, \|_\infty)$$

where $f \in \dot{E}_q^1 + \dot{E}_\infty^1$ such that $F = \overline{f}$.
Then

1. *there exists C_1 such that for every $F \in \dot{W}_q^1 + \dot{W}_\infty^1$ and all $t > 0$*

$$K(F, t^{\frac{1}{q}}, \dot{W}_q^1, \dot{W}_\infty^1) \geq C_1 t^{\frac{1}{q}} |\nabla f|^{q** \frac{1}{q}}(t) \text{ where } f \in \dot{E}_q^1 + \dot{E}_\infty^1 \text{ such that } F = \overline{f};$$

2. *for $q \leq p < \infty$, there exists C_2 such that for every $F \in \dot{W}_p^1$ and every $t > 0$*

$$K(F, t^{\frac{1}{q}}, \dot{W}_q^1, \dot{W}_\infty^1) \leq C_2 t^{\frac{1}{q}} |\nabla f|^{q** \frac{1}{q}}(t) \text{ where } f \in \dot{E}_p^1 \text{ such that } F = \overline{f}.$$

Theorem 7.4.2. *Let M be as in Theorem 7.4.1. For $f \in W_q^1 + W_\infty^1$, consider the functional of interpolation K' defined as follows:*

$$K'(f,t) = K'(f,t,\dot{W}_q^1,\dot{W}_\infty^1) = \inf_{\substack{f=h+g \\ h\in \dot{W}_q^1,\, g\in \dot{W}_\infty^1}} \left(\| \, |\nabla h| \, \|_q + t \| \, |\nabla g| \, \|_\infty \right).$$

Let $f \in W_q^1 + W_\infty^1$ such that $f^*(\infty) = 0$ and $|\nabla f|^*(\infty) = 0$. We have

$$K'(f,t^{\frac{1}{q}}) \sim t^{\frac{1}{q}} (|\nabla f|^{q**})^{\frac{1}{q}}(t) \tag{7.13}$$

where the implicit constants do not depend on f and t. Consequently for such f's,

$$K'(f,t^{\frac{1}{q}}) \sim K(\overline{f}, t^{\frac{1}{q}}, \dot{W}_q^1, \dot{W}_\infty^1).$$

Proof. Obviously

$$t^{\frac{1}{q}} (|\nabla f|^{q**})^{\frac{1}{q}}(t) \leq K(\overline{f}, t^{\frac{1}{q}}, \dot{W}_q^1, \dot{W}_\infty^1) \leq K'(f,t^{\frac{1}{q}})$$

for all $f \in W_q^1 + W_\infty^1$. For the converse estimation, we distinguish three cases:

1. Let $f \in C_0^\infty$. For $t > 0$, we consider the Calderón-Zygmund decomposition given by Proposition 5.5 in [3] (Proposition 3.5.5 in this thesis) with $\alpha(t) = (\mathcal{M}(|\nabla f|^q))^{*\frac{1}{q}}(t) \sim (|\nabla f|^{q**})^{\frac{1}{q}}(t)$. We can write then $f = b + g$ with $\| \, |\nabla b| \, \|_q \leq C\alpha(t) t^{\frac{1}{q}}$ and g Lipschitz with $\| \, |\nabla g| \, \|_\infty \leq C\alpha(t)$ (see also the proof of Theorem 1.4 in [3]: Theorem 3.1.4 in this thesis). One can verify that since $f \in C_0^\infty$ one has in addition $b \in L_q$ hence in W_q^1 and g bounded, hence in W_∞^1. Therefore, we get (7.13).

2. Let $f \in W_q^1$. There exists a sequence $(f_n)_n$ such that for all n, $f_n \in C_0^\infty$ and $\|f - f_n\|_{W_q^1} \to 0$. Since $|\nabla f_n|^q \to |\nabla f|^q$ in L_1, it follows that $|\nabla f_n|^{q**}(t) \to |\nabla f|^{q**}(t)$ for all $t > 0$. We have seen in item 1. that for every n there exists $g_n \in W_\infty^1$ such that $\| \, |\nabla(f_n - g_n)| \, \|_q + t^{\frac{1}{q}} \| \, |\nabla g_n| \, \|_\infty \leq C t^{\frac{1}{q}} (|\nabla f_n|^{q**})^{\frac{1}{q}}(t)$. Then

$$\begin{aligned} \| \, |\nabla(f - g_n)| \, \|_q + t^{\frac{1}{q}} \| \, |\nabla g_n| \, \|_\infty &\leq \| \, |\nabla(f - f_n)| \, \|_q + \Big(\| \, |\nabla(f_n - g_n)| \, \|_q + t^{\frac{1}{q}} \| \, |\nabla g_n| \, \|_\infty \Big) \\ &\leq \varepsilon_n + C t^{\frac{1}{q}} (|\nabla f_n|^{q**})^{\frac{1}{q}}(t) \end{aligned}$$

   where $\varepsilon_n \to 0$ when $n \to \infty$. We let $n \to \infty$ to obtain (7.13).

3. Let $f \in W_q^1 + W_\infty^1$ such that $f^*(\infty) = 0$ and $|\nabla f|^*(\infty) = 0$. Fix $t > 0$ and $p_0 \in M$. Consider $\varphi \in C_0^\infty(\mathbb{R})$ satisfying $\varphi \geq 0$, $\varphi(\alpha) = 1$ if $\alpha < 1$ and $\varphi(\alpha) = 0$ if $\alpha > 2$. Then put $f_n(x) = f(x)\varphi(\frac{d(x,p_0)}{n})$. Elementary calculations establish that f_n lies in W_q^1 hence $K'(f_n, t^{\frac{1}{q}}) \leq C t^{\frac{1}{q}} |\nabla f_n|^{q**\frac{1}{q}}(t)$. It is shown in [3] that

$$K(f, t^{\frac{1}{q}}, W_q^1, W_\infty^1) \sim \left(\int_0^t |f|^{q*}(s) ds \right)^{\frac{1}{q}} + \left(\int_0^t |\nabla f|^{q*}(s) ds \right)^{\frac{1}{q}}.$$

All these ingredients yield

$$
\begin{aligned}
K'(f,t) &\leq K'(f-f_n,t)+K'(f_n,t)\\
&\leq K(f-f_n,t,W_q^1,W_\infty^1)+K'(f_n,t)\\
&\leq C\left(\int_0^t |f-f_n|^{q*}(s)ds\right)^{\frac{1}{q}}+C\left(\int_0^t |\nabla f-\nabla f_n|^{q*}(s)ds\right)^{\frac{1}{q}}\\
&\quad +C\left(\int_0^t |\nabla f_n|^{q*}(s)ds\right)^{\frac{1}{q}}. \qquad (7.14)
\end{aligned}
$$

Now we invoke the following theorem from [18] page 67-68 stated there in the Euclidean case. As the proof is the same, we state it in the more general case:

Theorem 7.4.3. *Let M be a measured space. Consider a sequence of measurable functions $(\psi_n)_n$ and g on M such that $\mu\{|g|>\lambda\}<\infty$ for all $\lambda>0$ with $|\psi_n(x)|\leq|g(x)|$. If $\psi_n(x)\to\psi(x)$ $\mu-a.e.$ then $(\psi-\psi_n)^*(t)\to 0$ $\forall t>0$.*

We apply this theorem three times:

a. with $\psi_n=|f-f_n|^q$, $\psi=0$ and $g=2^qf^q$. Using the Lebesgue dominated convergence theorem we obtain $\int_0^t|f-f_n|^{q*}(s)ds\to 0$.
b. with $\psi_n=|\nabla f-\nabla f_n|^q$, $\psi=0$ and $g=C(|\nabla f|^q+|f|^q)$, where C only depends on q, since

$$
\nabla f_n=\nabla f\mathbb{1}_{B(p_0,n)}+\left(\frac{1}{n}f\varphi'(\frac{d(x,p_0)}{n})\nabla(d(x,p_0))+\nabla f\varphi(\frac{d(x,p_0)}{n})\right)\mathbb{1}_{B(p_0,n)^c}.
$$

So again by the Lebesgue dominated convergence theorem we get $\int_0^t|\nabla f-\nabla f_n|^{q*}(s)ds\to 0$.
c. with $\psi_n=|\nabla f_n|^q$, $\psi=|\nabla f|^q$ and $g=C(|\nabla f|^q+|f|^q)$, C only depending on q, so we get $\int_0^t|\nabla f_n|^{q*}(s)ds\to\int_0^t|\nabla f|^{q*}(s)ds$.

Thus passing to the limit in (7.14) yields $K'(f,t^{\frac{1}{q}})\leq Ct^{\frac{1}{q}}|\nabla f|^{q**\frac{1}{q}}(t)$ and finishes the proof. □

Proof of Theorem 7.1.1. Let $t>0$, $f\in W_q^1+W_\infty^1$ such that $f^*(\infty)=0$ and $|\nabla f|^*(\infty)=0$. Observe that

$$
(f-P_tf)^{q**\frac{1}{q}}(s)\leq Ct^{\frac{1}{2}}|\nabla f|^{q**\frac{1}{q}}(s). \qquad (7.15)
$$

Before proving(7.15), let us see how to conclude from it desired symmetization inequality. Indeed, (7.15) yields

$$
\begin{aligned}
|f|^{q**\frac{1}{q}}(s) &\leq C[|f-P_tf|^{q**\frac{1}{q}}+|P_tf|^{q**\frac{1}{q}}](s)\\
&\leq C[t^{\frac{1}{2}}|\nabla f|^{q**\frac{1}{q}}+t^{\frac{\alpha}{2}}t^{-\frac{\alpha}{2}}|P_tf|^{q**\frac{1}{q}}](s)\\
&\leq Ct^{\frac{1}{2}}|\nabla f|^{q**\frac{1}{q}}(s)+Ct^{\frac{\alpha}{2}}\sup_{t>0}\left(t^{-\frac{\alpha}{2}}|P_tf|^{q**\frac{1}{q}}(s)\right)
\end{aligned}
$$

$$\leq Ct^{\frac{1}{2}}|\nabla f|^{q**\frac{1}{q}}(s) + Ct^{\frac{\alpha}{2}}\sup_{t>0} t^{-\frac{\alpha}{2}}\|P_t f\|_\infty$$
$$= Ct^{\frac{1}{2}}|\nabla f|^{q**\frac{1}{q}}(s) + Ct^{\frac{\alpha}{2}}\|f\|_{B^\alpha_{\infty,\infty}}.$$

Therefore we get

$$|f|^{q**\frac{1}{q}}(s) \leq C\inf_{t>0}\Big(t^{\frac{1}{2}}|\nabla f|^{q**\frac{1}{q}}(s) + t^{\frac{\alpha}{2}}\|f\|_{B^\alpha_{\infty,\infty}}\Big)$$
$$\leq C|\nabla f|^{q**\frac{|\alpha|}{q(1+|\alpha|)}}(s)\|f\|^{\frac{1}{1+|\alpha|}}_{B^\alpha_{\infty,\infty}}.$$

It remains to prove (7.15). The main tool will be the pseudo-Poincaré inequalities (P'_q), (P'_∞) and Theorem 7.4.2.

Let $f \in W^1_q + W^1_\infty$ such that $f^*(\infty) = 0$ and $|\nabla f|^*(\infty) = 0$. Assume that $f = h+g$ with $h \in W^1_q$, $g \in W^1_\infty$, we then have

$$f - P_t f = (h - P_t h) + (g - P_t g).$$

Let $s > 0$. The pseudo-Poincaré inequalities (P'_q) and (P'_∞) yield

$$\|h - P_t h\|_q + s^{\frac{1}{q}}\|g - P_t g\|_\infty \leq Ct^{\frac{1}{2}}(\|\,|\nabla h|\,\|_q + s^{\frac{1}{q}}\|\,|\nabla g|\,\|_\infty).$$

Since

$$K(f, s^{\frac{1}{q}}, L_q, L_\infty) \sim \Big(\int_0^s (f^*(u))^q du\Big)^{\frac{1}{q}} = s^{\frac{1}{q}}|f|^{q**\frac{1}{q}}(s)$$

we obtain

$$s^{\frac{1}{q}}|f - P_t f|^{q**\frac{1}{q}}(s) \sim \inf_{\substack{f - P_t f = h' + g' \\ h' \in L_q,\, g' \in L_\infty}} (\|h'\|_q + s^{\frac{1}{q}}\|g'\|_\infty)$$
$$\leq \inf_{\substack{f = h+g \\ h \in W^1_q,\, g \in W^1_\infty}} (\|h - P_t h\|_q + s^{\frac{1}{q}}\|g - P_t g\|_\infty)$$
$$\leq Ct^{\frac{1}{2}} \inf_{\substack{f = h+g \\ h \in W^1_q,\, g \in W^1_\infty}} (\|\,|\nabla h|\,\|_q + s^{\frac{1}{q}}\|\,|\nabla g|\,\|_\infty)$$
$$= Ct^{\frac{1}{2}}K'(f, s^{\frac{1}{q}}).$$

Applying Theorem 7.4.2, we obtain the desired inequality (7.15). □

Proof of Theorem 7.1.6. The proof of this theorem is similar to that of Theorem 7.1.1. Here the key ingredients will be the pseudo-Poincaré inequality for averages (P''_q) that holds for all $f \in \dot{E}^1_q$ and which follows from (D) and the Poincaré inequality (P_q). We also make use of Theorem 7.4.2. □

7.5 Another symmetrization inequality

In this section we prove another symmetrization inequality which was used in [20] to prove Gagliardo-Nirenberg inequalities with a Triebel-Lizorkin condition.

Theorem 7.5.1. *Let M be a complete non-compact Riemannian manifold satisfying (D) and (P_q) for some $1 \leq q < \infty$. Moreover, assume that M satisfies the pseudo-Poincaré inequalities (P'_q) and (P'_∞). Consider $\alpha < 0$. Then there exists $C > 0$ such that for every $f \in W^1_q + W^1_\infty$ with $f^*(\infty) = 0$ and $|\nabla f|^*(\infty) = 0$ and satisfying $(\sup_{t>0} t^{-\frac{\alpha}{2}} |P_t f(.)|) \in L_q + L_\infty$ we have*

$$|f|^{q**\frac{1}{q}}(s) \leq C |\nabla f|^{q**\frac{|\alpha|}{q(1+|\alpha|)}}(s) \left[\left(\sup_{t>0} t^{-\frac{\alpha}{2}} |P_t f(.)| \right)^{q**\frac{1}{q}}(s) \right]^{\frac{1}{1+|\alpha|}}, \ s > 0. \tag{7.16}$$

Proof. From

$$|f|^q \leq 2^{q-1} \left(|f - P_t f|^q + t^{\frac{\alpha q}{2}} \sup_{t>0} t^{-\frac{\alpha q}{2}} |P_t f|^q \right)$$

we obtain

$$\begin{aligned} |f|^{q**\frac{1}{q}}(s) &\leq C \left(|f - P_t f|^{q**\frac{1}{q}}(s) + t^{\frac{\alpha}{2}} \left(\sup_{t>0} t^{-\frac{\alpha}{2}} |P_t f| \right)^{q**\frac{1}{q}}(s) \right) \\ &\leq C \left(t^{\frac{1}{2}} |\nabla f|^{q**\frac{1}{q}}(s) + t^{\frac{\alpha}{2}} \left(\sup_{t>0} t^{-\frac{\alpha}{2}} |P_t f| \right)^{q**\frac{1}{q}}(s) \right). \end{aligned}$$

It follows that

$$\begin{aligned} |f|^{q**\frac{1}{q}}(s) &\leq C \inf_{t>0} \left(t^{\frac{1}{2}} |\nabla f|^{q**\frac{1}{q}}(s) + t^{\frac{\alpha}{2}} \left(\sup_{t>0} t^{-\frac{\alpha}{2}} |P_t f| \right)^{q**\frac{1}{q}}(s) \right) \\ &\leq C |\nabla f|^{q**\frac{|\alpha|}{q(1+|\alpha|)}}(s) \left(\sup_{t>0} t^{-\frac{\alpha}{2}} |P_t f| \right)^{q**\frac{1}{q(1+|\alpha|)}}(s). \end{aligned}$$

□

7.6 Proof of Theorem 7.1.8

Proof. Let $f \in C_0^\infty$. Since $p + 1 \geq 2$, integrating by parts, we get

$$\| \, |\nabla f| \, \|_{p+1}^{p+1} = - \int_M div(|\nabla f|^{p-1} \nabla f) f d\mu.$$

Moreover we have $div(|\nabla f|^{p-1} \nabla f) \leq C |\nabla f|^{p-1} |\nabla^2 f|$, then

$$\| \, |\nabla f| \, \|_{p+1}^{p+1} \leq C \int_M |\nabla f|^{p-1} |\nabla^2 f| |f| d\mu.$$

Let $I = \int_M |\nabla f|^{p-1} |\nabla^2 f| |f| d\mu$. Then

$$I = \int_0^\infty (|\nabla f|^{p-1} |\nabla^2 f| |f|)^*(s) ds$$

$$
\begin{aligned}
&= \int_0^\infty (|\nabla f|^{\frac{p-2}{2}+\frac{p}{2}}|\nabla^2 f||f|)^*(s)ds \\
&\le \int_0^\infty (|\nabla f|^{\frac{p}{2}})^*(s)|f|^{q*\frac{1}{q}}(s)(|\nabla f|^{\frac{p-2}{2}}|\nabla^2 f|)^*(s)ds \\
&= \int_0^\infty |\nabla f|^{q*\frac{p}{2q}}(s)|f|^{q*\frac{1}{q}}(s)(|\nabla f|^{\frac{p-2}{2}}|\nabla^2 f|)^*(s)ds \\
&\le \int_0^\infty |\nabla f|^{q**\frac{p}{2q}}(s)|f|^{q**\frac{1}{q}}(s)(|\nabla f|^{\frac{p-2}{2}}|\nabla^2 f|)^*(s)ds.
\end{aligned}
$$

Thanks to Theorem 7.1.1, we get

$$
\begin{aligned}
I &\le \|f\|^{\frac{1}{2}}_{B^{-1}_{\infty,\infty}} \int_0^\infty |\nabla f|^{q**\frac{p+1}{2q}}(s)(|\nabla f|^{\frac{p-2}{2}}|\nabla^2 f|^*)(s)ds \\
&\le \|f\|^{\frac{1}{2}}_{B^{-1}_{\infty,\infty}} \left(\int_0^\infty |\nabla f|^{q**\frac{p+1}{q}}(s)ds\right)^{\frac{1}{2}} \left(\int_0^\infty \left((|\nabla f|^{\frac{p-2}{2}}|\nabla^2 f|)^*(s)\right)^2 ds\right)^{\frac{1}{2}} \\
&\le \|f\|^{\frac{1}{2}}_{B^{-1}_{\infty,\infty}} \left(\int_M |\nabla f|^{p+1}d\mu\right)^{\frac{1}{2}} \left(\int_M |\nabla f|^{p-2}|\nabla^2 f|^2 d\mu\right)^{\frac{1}{2}}
\end{aligned}
$$

which finishes the proof. □

Remark 7.6.1. *Let M be a complete Riemannian manifold satisfying (D) and (P_q) for some $1 \le q < \infty$. Then Theorem 7.1.8 holds replacing the Besov norm $B^{-1}_{\infty,\infty}$ by the Morrey norm M^{-1}_∞. This can be proved using Theorem 7.1.6.*

7.7 Examples and Extensions

1. $\mathbb{R}^n$ equipped with the Euclidean metric and the Lebesgue measure is a complete Riemannian manifold satisfying (D) and (P_1), (P'_1), (P'_∞). Thus Theorem 7.1.1, Theorem 7.1.2, Theorem 7.1.6 and Theorem 7.1.8 apply with $q = 1$ and $s = n$. These theorems were already proved in [20] for $\mathbb{R}^n$ as we have mentioned at the beginning.

2. Every complete Riemannian manifold M with non-negative Ricci curvature satisfies (D) with $s = n$ and (P_1), (P'_1), (P'_∞). Then our results apply on such a manifold with $q = 1$ as we have seen in Corollary 7.1.3.

3. Galois manifolds of compact manifolds whose deck transformation group is of polynomial growth satisfy (D), (P_1), (G) (see [12]). Therefore Corollary 7.1.4, Theorem 7.1.6 and Theorem 7.1.8 (with $q = 1$) apply on such manifolds.

4. We consider a connected unimodular Lie group G equipped with a Haar measure and a family of left invariant vertor fields $X_1, ..., X_k$ satisfying a Hörmander condition. If G has polynomial growth, then it satisfies (D), (P_1) and (G) (see [9]). As we have said in the introduction, we have then Corollary 7.1.4 on such groups. Also Theorem 7.1.6 and Theorem 7.1.8 hold with $q = 1$.
   For $n \in [d, D]$, we have $\mu(B) \ge Cr^n$ for any ball B of radius $r > 0$ (d being the

local dimension of G and D the dimension at infinity). Hence subsection 7.3.3 applies in this case.

5. We have the analogous of the theorems mentioned in the introduction if we consider instead of a Riemannian manifold M a connected, infinite locally uniformly finite graph endowed with a positive measure m and satisfying moreover the doubling property (D) and the Poincaré and pseudo-Poincaré inequalities as in the hypotheses of Theorem 7.1.1, Theorem 7.1.6 and Theorem 7.1.8. (For the characterization of the functional K of interpolation of homogeneous Sobolev spaces on such graphs we refer to [4]: Chapter 6 of this thesis. The non homogenous case is not treated in [4] but using the same method we get the characterisation of K as in the case of non homgeneous Sobolev spaces on Riemannian manifolds.) As an example of graphs verifying the hypotheses of Theorem 7.1.6, we consider the Cayley graph of finitely generated group. It was proved in [8], [22], that (P_1) holds on these graphs and as a consequence (P_p) holds for all $1 \leq p \leq \infty$. Hence, we get the analogous of Theorem 7.1.6 with $q = 1$.

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Résumé. Ce mémoire est le rapport de mes deux années de thèse d'octobre 2005 à octobre 2007. Dans cette thèse, nous étudions l'interpolation réelle des espaces de Sobolev et ses applications.

Le manuscrit est constitué de deux parties.

Dans la première partie, nous démontrons au premier chapitre que les espaces de Sobolev non homogènes W_p^1 (resp. homogènes $\dot{W}_p^1$) sur les variétés Riemanniennes complètes vérifiant la propriété de doublement et une inégalité de Poincaré forment une échelle d'interpolation réelle pour un intervalle de valeurs de p. Nous étendons ce résultat à d'autres cadres géométriques.

Dans un deuxième court chapitre, nous comparons différents espaces de Sobolev sur le cône Euclidien et nous regardons le lien de ces espaces avec l'interpolation. Nous montrons sur cet exemple que l'hypothèse de Poincaré n'est pas une condition nécessaire pour pouvoir interpoler les espaces de Sobolev.

Dans le dernier chapitre de cette partie, nous définissons les espaces de Sobolev non homogènes $W_{p,V}^1$ (resp. homogènes $\dot{W}_{p,V}^1$) associé à un potentiel positif V sur une variété Riemannienne. Nous démontrons que si la variété vérifie la propriété de doublement et une inégalité de Poincaré et si de plus V est dans une classe de Hölder inverse, ces espaces forment aussi une échelle d'interpolation réelle pour un intervalle de valeurs de p. Nous étendons ce résultat aux cas des groupes de Lie.

Dans la deuxième partie, dans un premier chapitre en collaboration avec E. Russ, nous étudions sur un graphe vérifiant la propriété de doublement et une inégalité de Poincaré, la L_p bornitude de la transformée de Riesz pour $p > 2$ et son inégalité inverse pour $p < 2$. Pour notre but, nous démontrons aussi des résultats d'interpolation des espaces de Sobolev et des inégalités de Littlewood-Paley.

Dans le deuxième chapitre, nous démontrons en utilisant notre résultat d'interpolation, des inégalités de Gagliardo-Nirenberg sur les variétés Riemanniennes complètes vérifiant le doublement, des inégalités de Poincaré et pseudo-Poincaré. Ce résultat s'applique aussi dans le cadre des groupes de Lie et des graphes.

Mots clés: Interpolation réelle; espaces de Sobolev; inégalité de Poincaré; propriété de doublement; classes de Hölder inverses; variétés Riemanniennes; groupes de Lie; espaces métriques mesurés; graphes; transformée de Riesz, inégalités de Gagliardo-Nirenberg.

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