Graph Coloring and Graph Convexity

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UNIVERSIDADE FEDERAL DO CEARÁ
PROGRAMA DE MESTRADO E DOUTORADO EM CIÊNCIA DA COMPUTAÇÃO

P H D T H E S I S

to obtain the title of

Docteur en Sciences

de l'Université de Nice - Sophia Antipolis

Mention : Informatique

Doutor em Ciências

pela Universidade Federal do Ceará

Menção: Computação

Defended by

Julio ARAUJO

Graph Coloring and Graph Convexity

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Defended on September 13, 2012

Jury:

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Acknowledgements

First of all, I would like to thank Ricardo Correa, Frédéric Maffray and Christophe Paul for revising this text. This work would not have been done without the help of several coauthors. I also thank them very much. I am very grateful as well to several professors I had in my undergraduate and graduate courses, specially the ones of the ParGO research group.

Obviously, I first dedicate this thesis to my parents. They provided me everything I needed to succeed and they always supported me during this period. I also dedicate this thesis to my wife, Karol. She has been on my side all these years and she knows how important she was to me in this period. This work is also particularly dedicated to my grandmother Teresinha, the kindest person this world has ever had.

The person who drove me through this PhD path was Claudia Linhares Sales, who I consider my moth... elder sister in research. She advised me to continue my PhD course in France, where I earned a father, Jean-Claude Bermond, and an elder brother, Frédéric Giroire. I am extremely indebted to all my advisors.

When I arrived in France, I also earned at the same time, a mother, a sister and a friend, named Patricia Lachaume. She took care of all my problems these last three years, from the administrative ones to the personal ones. I hope I can help her at least as much as she helped to me.

I cannot mention all of them, but I also thank all the brothers, sisters and friends I lived with in France, specially in MASCOTTE. They made my life so easy that it is still hard to realize that the end of this period has already come.

Pacoti
September 07, 2012

Graph Coloring and Graph Convexity

Abstract: In this thesis, we study several problems of Graph Theory concerning Graph Coloring and Graph Convexity. Most of the results contained here are related to the computational complexity of these problems for particular graph classes.

In the first and main part of this thesis, we deal with Graph Coloring which is one of the most studied areas of Graph Theory. We first consider three graph coloring problems called Greedy Coloring, Weighted Coloring and Weighted Improper Coloring. Then, we deal with a decision problem, called Good Edge-Labeling, whose definition was motivated by the Wavelength Assignment problem in optical networks.

The second part of this thesis is devoted to a graph optimization parameter called (geodetic) hull number. The definition of this parameter is motivated by an extension to graphs of the notions of convex sets and convex hulls in the Euclidean space.

Finally, we present in the appendix other works developed during this thesis, one about Eulerian and Hamiltonian directed hypergraphs and the other concerning distributed storage systems.

Keywords: Graph Theory, Computational Complexity, Coloring, Convexity.

Coloration et convexité dans les graphes

Résumé : Dans cette thèse, nous étudions plusieurs problèmes de théorie des graphes concernant la coloration et la convexité des graphes. La plupart des résultats figurant ici sont liés à la complexité de calcul de ces problèmes pour certaines classes de graphes.

Dans la première, et principale, partie de cette thèse, nous traitons de coloration des graphes qui est l'un des domaines les plus étudiés de théorie des graphes. Nous considérons d'abord trois problèmes de coloration appelés coloration gloutone, coloration pondérée et coloration pondérée impropre. Ensuite, nous traitons un problème de décision, appelé bon étiquetage de arêtes, dont la définition a été motivée par le problème d'affectation de longueurs d'onde dans les réseaux optiques.

La deuxième partie de cette thèse est consacrée à un paramètre d'optimisation des graphes appelé le nombre enveloppe (géodésique). La définition de ce paramètre est motivée par une extension aux graphes des notions d'ensembles et d'enveloppes convexes dans l'espace Euclidien.

Enfin, nous présentons dans l'annexe d'autres travaux développées au cours de cette thèse, l'un sur les hypergraphes orientés Eulériens et Hamiltoniens et l'autre concernant les systèmes de stockage distribués.

Mots clés : Théorie des Graphes, Complexité, Coloration, Convexité.

Coloração e convexidade em grafos

Resumo: Nesta tese, estudamos vários problemas de teoria dos grafos relativos à coloração e convexidade em grafos. A maioria dos resultados contidos aqui são ligados à complexidade computacional destes problemas para classes de grafos particulares.

Na primeira, e principal, parte desta tese, discutimos coloração de grafos que é uma das áreas mais importantes de teoria dos grafos. Primeiro, consideramos três problemas de coloração chamados coloração gulosa, coloração ponderada e coloração ponderada imprópria. Em seguida, discutimos um problema de decisão, chamado boa rotulagem de arestas, cuja definição foi motivada pelo problema de atribuição de frequências em redes óticas.

A segunda parte desta tese é dedicada a um parâmetro de otimização em grafos chamado de número de fecho (geodético). A definição deste parâmetro é motivada pela extensão das noções de conjuntos e fecho convexos no espaço Euclidiano.

Por fim, apresentamos em anexo outros trabalhos desenvolvidos durante esta tese, um em hipergrafos dirigidos Eulerianos e Hamiltonianos e outro sobre sistemas de armazenamento distribuído.

Palavras-chave: Teoria de grafos, complexidade computacional, coloração, convexidade.

Contents

Chapter 1

Introduction

Contents

In this thesis, we present new contributions in two areas of Graph Theory: Graph Coloring and Graph Convexity. Due to the four color problem [AH77] and the modeling of several applications, Graph Coloring is one of the most studied areas of Graph Theory [JT95, MR01, CZ08]. It consists in assigning colors to the vertices or edges of an input graph under various constraints. Graph Convexity studies parameters motivated by the notion of convex sets in the Euclidian space. We refer to Rockafellar [Roc70] as a classical book on convexity in the Euclidian space.

The results presented in this thesis concern the study of various parameters related to the topics above. We study their structural properties with emphasis on the algorithmic aspects. We also consider the parameters for several classes of graphs like graphs without induced P_4 (path on 4 vertices), bipartite graphs, grids, etc. According to the class of graphs, we present either polynomial-time algorithms or show NP-completeness results; in this last case we present approximation algorithms, fixed-parameter tractable algorithms, exponential exact algorithms or heuristics.

For basic definitions on Graph Theory, Algorithms and Computational Complexity we refer to classical books like [BM08a, Ber76, CLRS01, GJ90].

In what follows, we summarize each subject, its motivation and the obtained results.

1.1 Graph Coloring

Vertex Coloring is an important problem in Graph Theory. Many variants of this problem have been considered in the literature like List Coloring [Viz76, ERT80], $L(p_1, \ldots, p_q)$-labeling [Cal06], b-Coloring [IM99] and Star (also known as Circular) Coloring [Vin88]. A book on variations of Vertex Coloring is the one by Jensen and Toft [JT95]. In the first part of this thesis, we study other variants of Vertex Coloring. In order to define them and describe our results, we need to first introduce some definitions and notations.

Let $G = (V, E)$ be a graph. A (vertex) *k-coloring* of G is a function $c : V \to \{1, \dots, k\}$. The coloring c is *proper* if $uv \in E$ implies $c(u) \neq c(v)$. Observe that a proper k-coloring can also be seen as a partition $c = (S_1, \dots, S_k)$ of the vertex set $V(G)$ into *color classes* that are the *independent sets* (also called *stable sets*) S_i, $i \in \{1, \dots, k\}$, i.e. sets of pairwise non-adjacent vertices. If G admits a k-coloring, it is called *k-colorable*. The *chromatic number* of G, denoted by $\chi(G)$, is the minimum integer k such that G admits a proper k-coloring. The goal of the Vertex Coloring problem is to determine $\chi(G)$ for a given graph G. It is a well-known NP-hard problem [Kar72].

We now describe in more details our contribution.

Greedy Coloring Let $G = (V, E)$ be a graph and $\theta = v_1, \dots, v_n$ be an order over $V(G)$. The *greedy* (also known as *first-fit*) *algorithm* for Vertex Coloring problem follows the order θ and assigns to v_i ($1 \leq i \leq n$) the smallest positive integer that was not assigned to one of its already colored neighbors in $N(v_i) \cap \{v_1, \dots, v_{i-1}\}$. The *Grundy number* of a given graph $G = (V, E)$, denoted by $\Gamma(G)$, is the maximum number of colors that the greedy algorithm may use to color G over all possible orderings θ of $V(G)$ [Gru39, CS79].

Observe that the greedy algorithm is an *on-line* heuristic since it colors v_i by only considering the information about $G[\{v_1, \dots, v_{i-1}\}]$, i.e. the subgraph of G induced by the vertices $v_1, \dots, v_{i-1}$. Thus, the study on the Grundy number is motivated by On-line Coloring problem as it gives an upper bound for it [Kie98, KPT94].

It is known in the literature that the Greedy Coloring problem can be solved in polynomial-time for P_4-free graphs [GL88] and it is NP-hard for P_5-free graphs [Zak05]. We present, in Chapter 2, a polynomial-time algorithm to compute the Grundy number of *fat-extended P_4-laden graphs* by using their modular decomposition. This graph class is a generalization of the extended P_4-laden graphs [Gia96]. A graph G is extended P_4-laden if, for all $H \subseteq G$ such that $|V(H)| \leq 6$, the following statement is true: if H contains more than two induced P_4's, then H is a pseudo-split graph, where a *pseudo-split* graph is a $\{C_4, \bar{C}_4\}$-free graph.

The fat-extended P_4-laden graphs properly contain the class of P_4-free graphs, also known as *cographs*, and consequently intersect the class of P_5-free graphs. This result was obtained in a joint work with Cláudia Linhares Sales [AL09, AL12].

Weighted Coloring Given a graph $G = (V, E)$, a weight function $w : V(G) \to \mathbb{R}^*_+$, and a k-coloring $c = (S_1, \dots, S_k)$ of G, let us define the weight of a color S_i as $w(S_i) = \max_{v \in S_i} w(v)$, for every $i \in \{1, \dots, k\}$, and the weight of coloring c as $w(c) = \sum_{i=1}^{k} w(S_i)$. The goal of the Weighted Coloring problem consists, for a given graph G and weight function w, in determining the *weighted chromatic number of* (G, w), denoted as $\chi_w(G)$, which is the minimum weight of a proper coloring of (G, w) [GZ97]. This problem generalizes Vertex Coloring because, in the particular case of unitary weights, we have $\chi(G) = \chi_w(G)$.

Weighted Coloring was defined by Guan and Zhu [GZ97] in order to model an improvement over a distributed dual bus network media access control protocol.

These authors also cite as motivation for Weighted Coloring the dynamic storage allocation problem. Weighted Coloring is also closely related to Greedy Coloring as the maximum number of colors of an optimal weighted coloring of (G, w), over all weight functions w, is $\Gamma(G)$ [GZ97].

In Chapter 3, we first show an extension of the Hajós' Theorem [Haj61] for Weighted Coloring. Hajós' Theorem shows a necessary and sufficient condition for the chromatic number of a graph G be greater than k: it must contain a k-constructible subgraph. The class of k-constructible graphs is obtained from complete graphs on k vertices by recursively applying two well-defined operations. We thus define the class of weighted k-constructible graphs and show that $\chi_w(G) \geq k$ if, and only if, G contains a weighted k-constructible subgraph. This result was also developed together with Claudia Linhares Sales [AL07].

The Weighted Coloring problem, as the previous one, can be solved in polynomial-time for the class of cographs [DdWMP02] and it is NP-hard for the class of P_5-free graphs [MPdW+04]. In the second part of Chapter 3, we extend the result for cographs by showing that a subclass of P_4-sparse graphs that properly contains the cographs and that is properly contained in the P_5-free graphs admits a polynomial-time algorithm to compute the weighted chromatic number. A graph G is P_4-sparse if, for every $V' \subseteq V(G)$ such that $|V'| \leq 5$, then $G[V']$ has at most one induced P_4 [Hoà85]. We also present a 2-approximation algorithm for the class of P_4-sparse graphs.

This result is a joint work with Cláudia Linhares Sales and Ignasi Sau [ALS10].

Weighted Improper Coloring We also studied an extension of the Vertex Coloring problem for edge-weighted graphs motivated by the design of satellite antennas for multi-spot MFTDMA satellites [AAG+05]. Given a graph $G = (V, E)$ and a function $w : E(G) \to \mathbb{R}_+$, a weighted t-improper k-coloring of (G, w) is a k-coloring c of G such that, for every vertex $v \in V(G)$, the following inequality holds:

$$\sum_{\{u \mid u \in N(v) \text{ and } c(u)=c(v)\}} w(uv) \leq t.$$

We define and study two new, up to our best knowledge, parameters that we call *weighted t-improper chromatic number* and *minimum k-threshold* of (G, w). Given (G, w) and a positive real value t, the weighted t-improper chromatic number of (G, w), denoted by $\chi_t(G, w)$, is the minimum value k such that (G, w) has a weighted t-improper k-coloring. On the other hand, the minimum k-threshold corresponds to the minimum value t such that (G, w) admits a weighted t-improper k-coloring, for a given k.

In a joint work with J-.C. Bermond, F. Giroire, F. Havet, D. Mazauric and R. Modrzejewski [ABG+11a, ABG+11b, ABG+12], we presented general upper bounds for both parameters; in particular we show a generalization of Lovász's Theorem [Lov66] for the weighted t-improper chromatic number. We then show how to transform an instance for determining the minimum k-threshold into another equivalent one where the weights are either 1 or M, for a sufficiently large M. Motivated by the original application, we study a special interference model on various grids (square, triangular, hexagonal) where a vertex produces a noise of intensity 1 for its neighbors and a noise of intensity 1/2 for the vertices at distance 2. Consequently, the problem consists in determining the weighted t-improper chromatic number when G is the square of a grid and the weights of the edges are 1 if their end-vertices are adjacent in the grid, and 1/2 if their end-vertices are linked in the square of the grid, but not in the grid. Finally, we model the problem using integer linear programming. We also propose and compare heuristic and exact Branch-and-Bound algorithms on random cell-like graphs, namely the Poisson-Voronoi tessellations.

These results are presented in Chapter 4.

Good Edge-labeling In the Wavelength Division Multiplexing (WDM) problem, the input is a set of paths $\mathcal{P}$ in a network G and the goal is to assign wavelengths to these paths in such a way that if two paths share an edge $e \in E(G)$, then they must receive disjoint wavelengths [Muk97, RS95, BS91]. Given a set of paths $\mathcal{P}$, the load of an edge $e \in E(G)$ is the number of paths that contain e. Observe that the WDM problem corresponds to Vertex Coloring in the conflict graph $G(\mathcal{P})$, where the conflict graph $G(\mathcal{P})$ has one vertex for each path in $\mathcal{P}$ and two vertices are linked if the corresponding paths share an edge. Bermond, Cosnard and Pérennes [BCP09], when studying the WDM problem over particular directed networks in which for any pair of vertices u, v there is most one directed uv-path, defined a problem called Good Edge-labeling. They used good edge-labelings to show that, even in these particular networks, under the assumption that the maximum load of an edge is two, it might be necessary to use an arbitrarily large number of wavelengths in the WDM problem.

Given a graph G, a *good edge-labeling* of G is a function $l : E(G) \to \mathbb{R}$ that associates labels to the edges of G satisfying the following property: for any two vertices $u, v \in V(G)$, there do not exist two $\{u, v\}$-paths with non-decreasing labels. Given a graph G, the Good Edge-labeling problem consists in determining whether G admits or not a good edge-labeling. We say that G is good (resp. bad) if G admits (resp. does not admit) a (resp. any) good edge-labeling.

In Chapter 5, we present the results obtained in collaboration with N. Cohen, F. Giroire and F. Havet about Good Edge-labeling [ACGH09a, ACGH09b, ACGH12]. Bermond et al. asked whether C_3 and $K_{2,3}$ are the unique bad graphs. We answer this question in the negative by showing an infinite class of graphs that do not admit a good edge-labeling. Then we prove that Good Edge-labeling is NP-complete even for bipartite graphs and introduce some classes of good graphs like C_3-free outerplanar graphs, planar graphs of girth at least 6, subcubic graphs. Recall that a graph is *planar* if it admits an embedding in the plane without edge crossings. An *outerplanar* graph is a planar graph which all vertices can be drawn in the unbounded face. A *subcubic* graph G is any graph with maximum degree at most 3. The proof that these are classes of good graphs relies in the observation that critical bad graphs, i.e. bad graphs such that any of its proper subgraphs is good, cannot contain matching-cuts, i.e. a set of pairwise non-incident edges that disconnects the graph. A result of Farley and Proskurowski [FP84] implies that a critical graph G has at least $\frac{3}{2}|V(G)| - \frac{3}{2}$ edges (otherwise, G has a matching-cut). Bonsma [Bon05] proved that the graphs G with no matching-cut and satisfying $|E(G)| = \frac{3}{2}|V(G)| - \frac{3}{2}$ are the ABC- graphs. We also show that $\{C_3, K_{2,3}\}$-free ABC-graphs are good.

1.2 Graph Convexity

One of the basic notions of Geometry in the d-dimensional Euclidian space E^d is the one of convex set. A set of points $S \subseteq E^d$ is *convex* if, for any pair of points $p_1, p_2 \in S$, the points in straight line segment from p_1 to p_2 are included in S. For a given set $S \subseteq E^d$, the *convex hull* of S is the smallest convex set that contains all points in S. These notions are extremely well-known and studied in Geometry and have several applications [Roc70].

The concepts of convex sets were translated to graphs [FJ86]. The principle is the same: a set of vertices $S \subseteq V(G)$ in a graph G is convex if the internal vertices of any *uv-path*, where $u, v \in S$, are also in S. Depending on the kind of *paths* we consider, we study different graph convexities. For example, in case S is convex if the internal vertices of any *shortest* uv-path are also in S, then we are studying the *geodetic convexity*, which is the graph convexity we consider in this thesis. If we consider induced paths instead of shortest paths, we talk about *monophonic convexity* [DPS10]. Another example of graph convexity is the P_3-convexity, in which one just considers paths on three vertices [CM99].

For each graph convexity, several parameters are defined in the literature. For example, the size of a maximum convex set that is properly contained in $V(G)$ is known as the *convexity number* of G [CWZ02]. Another parameter is the *geodetic (resp. monophonic, P_3, etc. depending on the convexity) number* of G, that is the size of a minimum set S of G such that, for every $w \in V(G)$, either $w \in S$ or w is an internal node of a shortest (resp. monophonic, P_3, etc.) uv-path where $u, v \in S$ [CHZ02].

The parameter we study in Chapter 6 is called *hull number* and is the minimum size of a set S whose convex hull equals $V(G)$. More formally, for a given graph $G = (V, E)$, the *closed interval* $I[u, v]$ of two vertices $u, v \in V(G)$ is the set of vertices that belong to some shortest (u, v)-path. For any $S \subseteq V$, let $I[S] = \bigcup_{u,v \in S} I[u, v]$. A subset $S \subseteq V$ is *convex* if $I[S] = S$. Given a subset $S \subseteq V$, the *convex hull* $I_h[S]$ of S is the smallest convex set that contains S. We say that S is a *hull set* of G if $I_h[S] = V$. The size of a minimum hull set of G is the *hull number* of G, denoted by $hn(G)$. The HULL NUMBER problem is to decide whether $hn(G) \leq k$, for a given graph G and an integer k [ES85].

In a joint work with V. Campos, F. Giroire, N. Nisse, L. Sampaio and R. Soares [ACG+11b, ACG+11a, ACG+12], we answer an open question of Dourado et al. [DGK+09] by showing that computing this parameter is an NP-hard problem for bipartite graphs. We then present polynomial-time algorithms for several graph classes: cacti, complements of bipartite and $(q, q-4)$-graphs. We also present new upper bounds for this parameter in the general case and also for particular graph classes like triangle-free graphs, graphs of girth at least 6 and regular graphs.

1.3 Other works

In the appendix, we present other works and techniques that we have used during the period of this thesis.

Eulerian and Hamiltonian de Bruijn Dihypergraphs Together with J-C. Bermond, we supervised the internship of G. Ducoffe in MASCOTTE project about Eulerian and Hamiltonian Dihypergraphs. A *directed hypergraph, or simply dihypergraph* is a pair $H = (\mathcal{V}(H), \mathcal{E}(H))$ where $\mathcal{V}(H)$ is a non-empty set of elements, called *vertices*, and $\mathcal{E}(H)$ is a collection of ordered pairs of subsets of $\mathcal{V}(H)$, called *hyperarcs*. Each hyperarc is represented as $E = (E^-, E^+)$ and E^- and E^+ are, respectively, the in-set and out-set of E.

The notions of Eulerian and Hamiltonian dihypergraphs are simple extensions of these well-known concepts for (directed) graphs. H is Eulerian (resp. Hamiltonian) if there is a directed cycle in H where each hyperarc (resp. vertex) of H appears exactly once. A directed cycle in H is simply a sequence $v_0 E_0 \dots v_k E_k v_0$ where $v_i \in \mathcal{V}(\mathcal{H})$ and $E_i \in \mathcal{E}(\mathcal{H})$, $v_i \in E_i^-$ and $v_{i+1 \pmod k} \in E_i^+$, for every $i \in \{1, \dots, k\}$.

We first show that, in general, it is NP-complete to determine whether H is Eulerian. Then, we present several results concerning Eulerian and Hamiltonian properties for dihypergraphs, specially the case of regular uniform dihypergraphs. Finally, we focus on generalized de Bruijn and Kautz dihypergraphs [BDE97] and we show several results about when these dihypergraphs have a complete Berge cycle, i.e. a cycle that is Eulerian and Hamiltonian. These results can be found in Appendix A.

Distributed Storage Systems We also worked on a completely different topic in collaboration with F. Giroire and J. Monteiro [AGM11]. We studied different erasure coding schemes for distributed storage systems. In such systems, we want to backup data into different servers in a network, but, in order to keep the data safe from disk failures, we introduce redundant data in the network. There are different erasure coding schemes already proposed in the literature to introduce this redundancy, each of them having its own advantages and disadvantages. We propose one new scheme and compare its performance with some other schemes that are well-known in the literature. In order to evaluate these schemes, we model them by Markov Chain Models and, from the stationary state of the chains, we are able to present closed form formulas to estimate the system behavior under certain conditions. This study is presented in Appendix B.

1.4 Publications

We now list the publications that are included in this thesis.

Journals

1. [ACGH12] J. Araujo, N. Cohen, F. Giroire, and F. Havet, *Good edge-labelling of graphs*, Discrete Applied Mathematics (2012), in press.

2. [AL12] J. Araujo and C. Linhares Sales, *On the grundy number of graphs with few p_4's*, Discrete Applied Mathematics (2012), in press.

3. [ABG+12] J. Araujo, J-C. Bermond, F. Giroire, F. Havet, D. Mazauric, and R. Modrzejewski, *Weighted improper colouring*, Journal of Discrete Algorithms, vol. 16, October 2012, pp. 53–66.

Submitted to journals

1. J. Araujo, V. Campos, F. Giroire, N. Nisse, L. Sampaio, and R. Soares, *On the hull number of some graph classes*, Submitted to Theoretical Computer Science, 2012.

2. J. Araujo, J-C. Bermond, and G. Ducoffe, *Eulerian and Hamiltonian Dicycles in Directed Hypergraphs*, Submitted to Journal of Graph Theory, 2012.

International conferences

1. [ABG+11b] J. Araujo, J-C. Bermond, F. Giroire, F. Havet, D. Mazauric, and R. Modrzejewski, *Weighted improper colouring*, Proceedings of International Workshop on Combinatorial Algorithms (IWOCA'11) (Victoria, Canada), Lecture Notes in Computer Science, vol. 7056, Springer-Verlag, June 2011, pp. 1–18.

2. [ACG+12] J. Araujo, V. Campos, F. Giroire, L. Sampaio, and R. Soares, *On the hull number of some graph classes*, Proceedings of the European Conference on Combinatorics, Graph Theory and Applications (Budapest, Hungary), EuroComb'11, 2011.

3. [ACGH09a] J. Araujo, N. Cohen, F. Giroire, and F. Havet, *Good edge-labelling of graphs*, Proceedings of the Latin-American Algorithms, Graphs and Optimization Symposium (LAGOS'09) (Gramado, Brazil), Electronic Notes in Discrete Mathematics, vol. 35, Springer, December 2009, pp. 275–280.

4. [AGM11] J. Araujo, F. Giroire, and J. Monteiro, *Hybrid approaches for distributed storage systems*, Proceedings of Fourth International Conference on Data Management in Grid and P2P Systems (Globe'11) (Toulouse, France), September 2011.

5. [AL09] J. Araujo and C. Linhares Sales, *Grundy Number on P_4-classes*, Proceedings of the 5th Latin-american Algorithms, Graphs and Optimization Symposium (LAGOS), Electronic Notes in Discrete Mathematics, Springer, 2009.

National conferences

1. [ALS10] J. Araujo, C. Linhares Sales, and I. Sau, *Weighted coloring on p_4-sparse graphs*, 11es Journées Doctorales en Informatique et Réseaux (Sophia Antipolis, France), March 2010.

2. [AL07] J. Araujo and C. Linhares Sales, *Teorema de Hajós para Coloração Ponderada*, XXXIX SBPO - Anais do Simpósio, XXXIX Simpósio Brasileiro de Pesquisa Operacional, 2007.

Research reports

1. [ABG+11a] J. Araujo, J-C. Bermond, F. Giroire, F. Havet, D. Mazauric, and R. Modrzejewski, *Weighted Improper Colouring*, Research Report RR-7590, INRIA, Apr 2011.

2. [ACG+11a] J. Araujo, V. Campos, F. Giroire, N. Nisse, L. Sampaio, and R. Soares, *On the hull number of some graph classes*, Tech. Report RR-7567, INRIA, September 2011.

3. [ACGH09b] J. Araujo, N. Cohen, F. Giroire, and F. Havet, *Good edge-labelling of graphs*, Research Report 6934, INRIA, 2009.

Chapter 2

Greedy Coloring

Contents

The Vertex Coloring problem has its *on-line* version, in which the vertices of the input graph are presented to a coloring algorithm one at a time in some arbitrary order. The algorithm must choose a color for each vertex, based only on the colors assigned to the already-processed vertices. The *on-line chromatic number* of a graph G is the minimum number of colors needed to color on-line the vertices of G when they are given in the worst possible order [GL88, GL90]. There are several works in the literature concerning the on-line chromatic number [Kie98, KPT94, GL88, GL90, HS94].

The most popular on-line coloring algorithm is the *greedy algorithm*. Given a graph $G = (V, E)$ and an order $\theta = v_1, \dots, v_n$ over V, the greedy algorithm assigns to v_i the minimum positive integer that was not already assigned to its neighborhood in the set $\{v_1, \dots, v_{i-1}\}$. A greedy coloring is a coloring obtained by this algorithm. The maximum number of colors required by the greedy algorithm to color a graph G, over all the orders θ of $V(G)$, is the *Grundy number* of G and it is denoted by $\Gamma(G)$. Observe that the Grundy number of a graph is an upper bound for its chromatic number as well as its on-line chromatic number.

Determining the Grundy number is NP-hard even for bipartite graphs [HS10] and for complements of bipartite graphs [GV97b, Zak05]. Since every complement of a bipartite graph is P_5-free, the hardness of this problem also holds for P_5-free graphs. Given a graph G and an integer r, it is a $coNP$-complete problem to decide if $\Gamma(G) \leq \chi(G) + r$, or if $\Gamma(G) \leq r \times \chi(G)$, or if $\Gamma(G) \leq c \times \omega(G)$ [AHL08, Zak05], where $\omega(G)$ stands for the size of a maximum clique of G, i.e. a set of pairwise adjacent vertices.

There are polynomial-time algorithms to determine the Grundy number of, for example, the following classes of graphs: P_4-free graphs [GL88], trees [HHB82], k-partial trees [TP97], hypercubes [JT99] and $(q, q-4)$-graphs [CLM+11]. This last result implies a Fixed Parameter Tractable (FPT) algorithm for this problem when the parameter is the number of induced P_4's of the input graph. Another result concerning the Parameterized Complexity of Greedy Coloring shows that the parameterized dual of this problem, i.e., to determine whether $\Gamma(G) \geq |V(G)| - k$ when k is the parameter, is an FPT problem.

By using the notion of k-atoms, Zaker showed that, given a graph $G = (V, E)$ and an integer k, there is an algorithm to determine if $\Gamma(G) \geq k$ with complexity $\mathcal{O}(n^{2^{k-1}})$ [Zak06].

Here, we introduce a new class of graphs, the *fat-extended P_4-laden graphs*, and we present a polynomial-time algorithm to calculate the Grundy number of any graph of this class by using its modular decomposition. Our class intersects the class of the P_5-free graphs class and strictly contains the class of P_4-free graphs. More precisely, our result implies that the Grundy number can be determined in polynomial time for any graph of the following classes: P_4-reducible, extended P_4-reducible, P_4-sparse, extended P_4-sparse, P_4-extendible, P_4-lite, P_4-tidy, P_4-laden and extended P_4-laden, which are all strictly contained in the fat-extended P_4-laden class.

The remaining of this chapter is organized as follows. In Section 2.1, we introduce some basic concepts related to modular decomposition, besides other simple definitions. In Section 2.2, we recall the definition of extended P_4-laden graphs and we define our new class of graphs. We present the algorithm and we prove its correctness and complexity in Section 2.3. Finally, we comment the results in Section 2.4.

2.1 Preliminaries

Let $G = (V, E)$ be a graph and S a subset of $V(G)$. We denote by $G[S]$ the subgraph of G induced by S and denote by $N_G(v)$ the set of neighbors of a vertex v in G (or just $N(v)$ when G is clear in the context).

We say that $M \subseteq V(G)$ is a *module* of a graph G if, for every vertex w of $V \backslash M$, either w is adjacent to all the vertices of M or w is adjacent to none of them. The sets V and $\{x\}$, for every $x \in V$, are *trivial modules*, the latest being called a *singleton* module.

A graph is *prime* if all its modules are trivial. We say that M is a *strong module* of G if, for every module M' of G, either $M' \cap M = \emptyset$ or $M \subset M'$ or $M' \subset M$. The *modular decomposition* of a graph G is a decomposition of G that associates with G a unique *modular decomposition tree* $T(G)$. The modular decomposition tree of G, $T(G)$, is a rooted tree where the leaves are the vertices of G, and such that any maximal set of its leaves having the same least common ancestor v is a strong module of G, which is denoted by $M(v)$.

Let r be an internal node of $T(G)$ and $V(r) = \{r_1, \ldots, r_k\}$ be the set of children of r in $T(G)$. If $G[M(r)]$ is disconnected, then r is called a *parallel* node and $G[M(r_1)], \ldots, G[M(r_k)]$ are its components. If $\bar{G}[M(r)]$ is disconnected then r is called a *series* node and $\bar{G}[M(r_1)], \ldots, \bar{G}[M(r_k)]$ are the components of $\bar{G}[M(r)]$. Finally, if both graphs $G[M(r)]$ and $\bar{G}[M(r)]$ are connected, then r is called a *neighborhood* node and $\{M(r_1), \ldots, M(r_k)\}$ is the unique set of maximal strong submodules of $M(r)$.

The *quotient* graph of $G[M(r)]$, denoted by $G(r)$, is $G[\{v_1, \ldots, v_k\}]$, where $v_i \in M(r_i)$, for $1 \leq i \leq k$. We say that r is a *fat* node, if $M(r)$ is not a singleton module.

More informations about the modular decomposition of graphs can be found in [MS99].

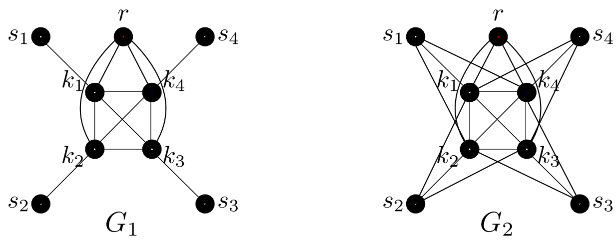


Figure 2.1: examples of thin (G_1) and thick (G_2) spiders with partition $S = \{s_1, s_2, s_3, s_4\}$, $K = \{k_1, k_2, k_3, k_4\}$ and $R = \{r\}$.

A graph is a *spider* (see Figure 2.1) if its vertex set can be partitioned into three sets S, K and R in such a way that S is a stable set, K is a clique, all the vertices of R are adjacent to all the vertices of K and to none of the vertices of S and there exists a bijection $f : S \to K$ such that, for all $s \in S$, either $N(s) = f(s)$ (and it is a *thin spider*) or $N(s) = K - f(s)$ (and it is a *thick spider*). Observe that, by definition, the unique non-trivial maximal strong sub-module of a spider is exactly the set R.

A graph $G = (V = S \cup K, E)$ is *split* if its vertex set can be partitioned into a stable set S and a clique K. Observe that the spiders of Figure 2.1 are also split graphs, since R is a clique and by consequence $V = (S, K \cup R)$ is a partitioning of the vertices of both spiders into a stable set and a clique. Alternately, the vertices of a split graph $G = (V = S \cup K, E)$ can also be partitioned into three disjoint sets $S'(G)$, $K'(G)$ and $R'(G)$, such that every vertex of S which looses at least one vertex in K belongs to $S'(G)$, $K'(G) \subseteq K$ is the neighborhood of the vertices in $S'(G)$ and $R'(G) = V \backslash S'(G) \cup K'(G)$ (see Figure 2.2). It is well-known that a graph is split if, and only if, it is $\{C_5, C_4, \bar{C_4}\}$-free [FH77]. A *pseudo-split* graph is a $\{C_4, \bar{C_4}\}$-free graph.

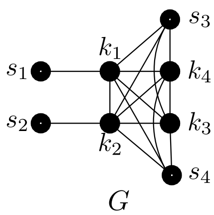


Figure 2.2: example of split graph G with partitioning $S'(G) = \{s_1, s_2\}$, $K'(G) = \{k_1, k_2\}$ and $R'(G) = \{s_3, s_4, k_3, k_4\}$.

2.2 Fat extended P_4-laden graphs

Giakoumakis [Gia96] defined a graph G as *extended P_4-laden graph* if, for all $H \subseteq G$ such that $|V(H)| \leq 6$, the following statement is true: if H contains more than two induced P_4's, then H is a pseudo-split graph. It follows that an extended P_4-laden graph can be completely characterized by its modular decomposition tree, as follows:

Theorem 1. *[Gia96] Let $G = (V, E)$ be a graph, $T(G)$ be its modular decomposition tree and r be any neighborhood node of $T(G)$, with children $r_1, \ldots, r_k$. Then G is extended P_4-laden if and only if $G(r)$ is isomorphic to:*

1. *a P_5 or a $\bar{P}_5$ or a C_5, and each $M(r_i)$, $1 \leq i \leq k$, is a singleton module; or*

2. *a spider $H = (S \cup K \cup R, E)$ and each $M(r_i)$, $1 \leq i \leq k$, is a singleton module, except the one corresponding to R and occasionally another one which may have exactly two vertices; or*

3. *a split graph $H = (S \cup K, E)$, whose modules corresponding to the vertices of S are independent sets and the ones corresponding to the vertices of K are cliques.*

We say that a graph is **fat-extended P_4-laden** if its modular decomposition satisfies Theorem 1, except in the first case, where $G(r)$ is isomorphic to a P_5 or a $\bar{P}_5$ or a C_5, but the maximal strong modules $M(r_i)$, $1 \leq i \leq 5$, of $M(r)$ are not necessarily singleton modules.

Observe that the class of fat-extended P_4-laden graphs contains the class of extended P_4-laden graphs. Figure 2.3 shows us an example of a fat-extended P_4-laden graph that is not an extended P_4-laden graph.

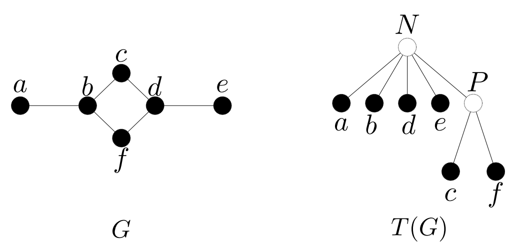


Figure 2.3: Example of a fat-extended P_4-laden graph which is not an extended P_4-laden graph.

Consequently, the class of fat-extended P_4-laden graphs strictly contains all the following classes of graphs: P_4-reducible, extended P_4-reducible, P_4-sparse, extended P_4-sparse, P_4-extendible, P_4-lite, P_4-tidy, P_4-laden and extended P_4-laden. Notice that these classes are all contained in the class of extended P_4-laden graphs [Ped07].

2.3 Grundy number on fat-extended P_4-laden graphs

Let $G = (V, E)$ be a fat-extended P_4-laden graph and $T(G)$ be its modular decomposition tree. Since $T(G)$ can be found in linear time [TCHP08], we propose an algorithm to determine $\Gamma(G)$ that uses a bottom-up strategy. We know that the Grundy number of the leaves of $T(G)$ is equal to one and we show in this section how to determine the Grundy number of $G[M(v)]$, for each inner node v of $T(G)$, based on the Grundy number of its children.

First, observe that for every series node r of $T(G)$, with children $r_1, \ldots, r_k$, the Grundy number of $G[M(r)]$ is equal to the sum of the Grundy numbers of its children, i.e., $\Gamma(G[M(r)]) = \Gamma(G[M(r_1)]) + \ldots + \Gamma(G[M(r_k)])$. However, if r is a parallel node, the Grundy number of $G[M(r)]$ is the maximum Grundy number among its children, i.e., $\Gamma(G[M(r)]) = \max(\Gamma(G[M(r_1)]), \ldots, \Gamma(G[M(r_k)]))$ [GL88].

Thus, it remains to prove that the Grundy number of $G[M(r)]$ can be found in polynomial-time when r is a neighborhood node of $T(G)$. The following definition will be useful:

Definition 1. *Given two graphs G and H, we say that G' is obtained from G by* replacing *a vertex $v \in V(G)$ by H if $V(G') = \{V(G)\backslash\{v\}\} \cup V(H)$ and $E(G') = \{E(G)\backslash\{uv \mid u \in N_G(v)\}\} \cup E(H) \cup \{uh \mid u \in N_G(v)$ and $h \in H\}$.*

The following result and its proof are a simple generalization of a result due to Asté et al. [AHL08] for the Grundy number of the lexicographic product of graphs.

Proposition 1. *Let G, $H_1, \ldots, H_n$ be disjoint graphs. Let $V(G) = \{v_1, \ldots, v_n\}$ and G' be the graph obtained by replacing $v_i \in V(G)$ by H_i, $1 \leq i \leq n$. Then, for every greedy coloring of G', at most $\Gamma(H_i)$ colors appear in $G'[V(H_i)]$.*

Proof. Consider a greedy coloring c of G' and let $c_1, \ldots, c_p$ be the colors occurring in $G'[V(H_k)]$, for some $k \in \{1, \ldots, n\}$. Denote by S_i, $1 \leq i \leq p$, the stable set formed by the vertices of $G'[V(H_k)]$ colored c_i. Let u_i be a vertex of S_i. Since c is a greedy coloring, u_i has at least one neighbor w colored c_j, for all $1 \leq j < i \leq p$.

Now, we claim that $w \in G'[V(H_k)]$. By contradiction, suppose that $w \notin G'[V(H_k)]$. So, $w \in V(G')\backslash V(H_k)$. Let $u_j \in G'[V(H_k)]$ be a vertex colored c_j. By Definition 1, once u_iw is an edge, so is u_jw, contradicting the assumption that c is a proper coloring. Therefore, $w \in G'[V(H_k)]$. It means that c restricted to $G'[V(H_k)]$, with p colors, is a greedy coloring of $G'[V(H_k)]$ and hence $p \leq \Gamma(G'[H_k]) \leq \Gamma(H_k)$. □

Let $G = (H_1 \cup \ldots \cup H_5, E)$ be a graph isomorphic to one of the neighborhood nodes depicted in Figure 2.4. In order to simplify the notation, denote $G[V(H_i)]$ by H_i, $\Gamma(G[H_i])$ by Γ_i and, by θ_i, an order that leads the greedy algorithm to the generation of a greedy coloring of $G[H_i]$ with $\Gamma(G[H_i])$ colors, $i \in \{1, \ldots, 5\}$.

Without loss of generality, we consider, in what follows, that the adjacency between the fat nodes are as depicted in Figure 2.4.

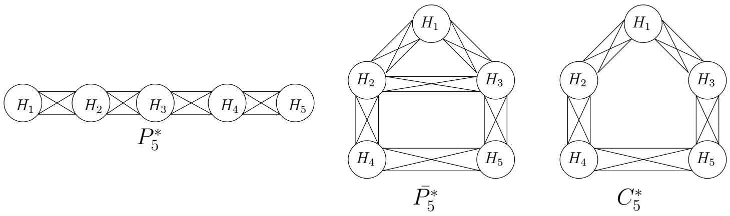


Figure 2.4: Fat neighborhood nodes.

Lemma 1. *Given the Grundy numbers of the graphs $H_1, \ldots, H_5$, the Grundy number of a $P_5^* = (H_1 \cup \ldots \cup H_5, E)$ can be found in constant time.*

Proof. Suppose that $\mathcal{S} = (S_1, \ldots, S_k)$ is a greedy coloring of a P_5^* with $\Gamma(P_5^*)$ colors. So, by definition, each vertex $v \in S_i$ has a neighbor $u \in S_j$, for all $j < i$, $i, j \in \{1, \ldots, k\}$. Let us check all the possible locations of a vertex v colored $\Gamma(G) = k$ in a greedy coloring of G with the maximum number of colors.

1. If there is a vertex $v \in H_1$ colored k, then $\Gamma(P_5^*) = \Gamma_1 + \Gamma_2$.

   In this case, since $N(v) \subseteq V(H_1) \cup V(H_2)$ and $N(v)$ intersects all the stable sets $S_1, \ldots, S_{k-1}$, we have that $\Gamma(P_5^*)$ colors occur in $G[V(H_1) \cup V(H_2)]$. Therefore, by Proposition 1, $k = \Gamma(P_5^*) \leq \Gamma_1 + \Gamma_2$. On the other hand, any ordering over $V(P_5^*)$ that starts by θ_1, followed immediately by θ_2, makes the greedy algorithm generate a greedy coloring of P_5^* with at least $\Gamma_1 + \Gamma_2$ colors.

2. If there is a vertex $v \in V(H_5)$ colored k, then $\Gamma(P_5^*) = \Gamma_4 + \Gamma_5$.

   This case is analogous to the previous one.

3. If there is a vertex $v \in V(H_2)$ colored k, then

$$\Gamma(P_5^*) = \begin{cases} \Gamma_1 + \Gamma_2 + \Gamma_3 & , \textit{if } \Gamma_1 \leq \Gamma_4 \\ \Gamma_1 + \Gamma_2 & , \textit{if } \Gamma_1 > \Gamma_4 \textit{ and } \Gamma_3 \leq s_1 \\ \Gamma_2 + \Gamma_3 + \Gamma_4 & , \textit{if } \Gamma_1 > \Gamma_4 \textit{ and } \Gamma_3 > s_1 \end{cases}$$

   where $s_1 = \Gamma_1 - \Gamma_4$.

   As before, since $N(v) \subseteq V(H_1) \cup V(H_2) \cup V(H_3)$ and $N(v)$ intersects all the stables sets $S_1, \ldots, S_{k-1}$, we have that $\Gamma(P_5^*)$ colors occur in $H_1 \cup H_2 \cup H_3$. Therefore, by Proposition 1, $k = \Gamma(P_5^*) \leq \Gamma_1 + \Gamma_2 + \Gamma_3$.

   If $\Gamma_4 \geq \Gamma_1$, then we claim that $\Gamma(P_5^*) = \Gamma_1 + \Gamma_2 + \Gamma_3$. Observe that there are no edges between $V(H_1)$ and $V(H_4)$ and all the edges between $V(H_3)$ and $V(H_4)$. Therefore, an ordering over the vertices of P_5^* that starts by θ_4, θ_1, θ_3 and θ_2, consecutively in this order, produces a greedy coloring of P_5^* with at least $\Gamma_1 + \Gamma_2 + \Gamma_3$ colors, since the colors used by the greedy algorithm to color H_4 are reused to color H_1, and all the colors occurring in H_3 have to be different from the colors occurring in H_4, and hence, in H_1. The result follows.

Otherwise, if $\Gamma_4 < \Gamma_1$, let $s_1 = \Gamma_1 - \Gamma_4$. We study two subcases. At first, if $\Gamma_3 \leq s_1$, then we prove that $\Gamma(P_5^*) = \Gamma_1 + \Gamma_2$. In order to prove this, consider an ordering over $V(P_5^*)$ that starts by θ_1, θ_4, θ_3 and θ_2, consecutively in this order. We claim the greedy algorithm over this ordering uses at least $\Gamma_1 + \Gamma_2$ colors. Indeed, since there are no edges between H_1 and H_4, clearly Γ_4 colors occurring in H_1 will be reused to color H_4. The other s_1 colors in H_1, more precisely Γ_3 out of them, will be sufficient to color H_3, and a total of Γ_1 colors will have been used thus far. Since all the edges between H_1 and H_2 belong to our P_5^*, another Γ_2 previously unused colors will be necessary to color H_2. We now claim that there is no greedy coloring with more than $\Gamma_1 + \Gamma_2$ colors under these hypothesis. Suppose, by contradiction, that there exists an ordering that makes the greedy algorithm generate a greedy coloring $\mathcal{S}' = \{S'_1, \ldots, S'_p\}$ of P_5^* with $p > \Gamma_1 + \Gamma_2$ colors. By Proposition 1 and by the remark that all the colors occur in $H_1 \cup H_2 \cup H_3$, there exists at least one color i that occurs in H_3 and does not occur in H_1 and in H_2.

Recall that, by hypothesis, $\Gamma_3 + \Gamma_4 \leq \Gamma_1$, i.e., $\mathcal{S}'$ has at least $\Gamma_2 + \Gamma_3 + \Gamma_4 + 1$ colors. Since all the colors of $\mathcal{S}'$ occur in $H_1 \cup H_2 \cup H_3$ and $\Gamma_4 < \Gamma_1$, there exists at least one color j that occurs in H_1 and does not occur in $H_2 \cup H_3 \cup H_4$. This is a contradiction, because the vertices of S'_i in H_3 have no neighbor colored j and the vertices of S'_j in H_1 have no neighbor colored i.

Now suppose that $\Gamma_3 > s_1$. We claim that $\Gamma(P_5^*) = \Gamma_1 + \Gamma_2 + \Gamma_3 - s_1$. Intuitively, if the colors of H_1 not used in H_4 are not enough to color H_3, then all the s_1 colors of H_1 are used in H_3. Consider an ordering over $V(P_5^*)$ that starts by θ_1, θ_4, θ_3 and θ_2, consecutively in this order. Since there is no edge between $V(H_1)$ and $V(H_4)$, then all, but s_1, colors occurring in H_1 will be reused to color H_4. All these s_1 colors will be necessarily used to partially color H_3. To complete the coloring of H_3, at least $\Gamma_3 - s_1$ new colors will be used. Since there all the edges between $V(H_1)$ and $V(H_2)$, this order leads the greedy algorithm to the generation of a greedy coloring with at least $\Gamma_1 + \Gamma_2 + \Gamma_3 - s_1$ colors.

To prove that $\Gamma(P_5^*) \leq \Gamma_1 + \Gamma_2 + \Gamma_3 - s_1$, we use the same idea as in the previous case. Suppose, by contradiction, that there exists a greedy coloring $\mathcal{S}'$ of P_5^* with more than $\Gamma_1 + \Gamma_2 + \Gamma_3 - s_1$ colors. Observe that there exist at least $\Gamma_3 - s_1 + 1$ colors that occur in H_3 and do not occur in $H_1 \cup H_2$. Let i be one of these colors. By hypothesis, $\mathcal{S}'$ has at least $\Gamma_1 + \Gamma_2 + \Gamma_3 - s_1 + 1 = \Gamma_2 + \Gamma_3 + \Gamma_4 + 1$ colors. Then, there is a color j that occurs in H_1 and does not occur in $H_2 \cup H_3 \cup H_4$. The existence of colors i and j leads to a contradiction by the same argument used in the preceding case.

4. If there is a vertex $v \in V(H_4)$ colored k, then

$$\Gamma(P_5^*) = \begin{cases} \Gamma_5 + \Gamma_4 + \Gamma_3 & , if\ \Gamma_5 \leq \Gamma_2 \\ \Gamma_5 + \Gamma_4 & , if\ \Gamma_5 > \Gamma_2\ and\ \Gamma_3 \leq s_5 \\ \Gamma_2 + \Gamma_3 + \Gamma_4 & , if\ \Gamma_5 > \Gamma_2\ and\ \Gamma_3 > s_5 \end{cases}$$

where $s_5 = \Gamma_5 - \Gamma_2$.

The proof of this case is analogous to the previous one.

5. If there is a vertex $v \in V(H_3)$ colored k, then

$$\Gamma(P_5^*) = \begin{cases} \Gamma_2 + \Gamma_3 + \Gamma_4 & , \, if \; \Gamma_1 \geq \Gamma_4 \; or \; \Gamma_5 \geq \Gamma_2 \\ \Gamma_1 + \Gamma_2 + \Gamma_3 & , \, if \; \Gamma_1 < \Gamma_4, \; \Gamma_5 < \Gamma_2 \; and \; \Gamma_2 - s_4 \geq \Gamma_5 \\ \Gamma_3 + \Gamma_4 + \Gamma_5 & , \, if \; \Gamma_1 < \Gamma_4, \; \Gamma_5 < \Gamma_2 \; and \; \Gamma_2 - s_4 < \Gamma_5 \end{cases}$$

where $s_4 = \Gamma_4 - \Gamma_1$.

Again, by Proposition 1 and the fact that there is a vertex colored $k \in V(H_3)$, we have that $\Gamma(P_5^*) \leq \Gamma_2 + \Gamma_3 + \Gamma_4$.

Suppose first that $\Gamma_1 \geq \Gamma_4$ or $\Gamma_5 \geq \Gamma_2$. We will prove that $\Gamma(P_5^*) = \Gamma_2 + \Gamma_3 + \Gamma_4$. In the case $\Gamma_1 \geq \Gamma_4$, consider any ordering that starts by θ_1, θ_4, θ_2 and θ_3, in this sequence. Alternatively, if $\Gamma_5 \geq \Gamma_2$, consider any ordering that starts by θ_5, θ_2, θ_4 and θ_3, in this sequence. In both cases, these orderings produce a greedy coloring of P_5^* with at least $\Gamma_2 + \Gamma_3 + \Gamma_4$ colors and the proposition follows.

Now, we define $s_2 = \Gamma_2 - \Gamma_5$. Assume first that $\Gamma_1 < \Gamma_4$ and $\Gamma_5 < \Gamma_2$. Since $\Gamma_1 < \Gamma_4$, an ordering that starts by θ_1, θ_4, θ_2 and θ_3, makes the greedy algorithm generate a coloring with at least $\Gamma_4 + \Gamma_3 + \Gamma_2 - s_4$ colors. Using the hypothesis that $\Gamma_5 < \Gamma_2$, an ordering that starts by θ_5, θ_2, θ_4 and θ_3, consecutively in this order, leads the greedy algorithm to the generation of a greedy coloring with at least $\Gamma_2 + \Gamma_3 + \Gamma_4 - s_2$ colors.

Now, we need to prove, case by case, that these bounds are also upper bounds. Consider first that $\Gamma_2 - s_4 \geq \Gamma_5$. We claim that $\Gamma(P_5^*) = \Gamma_2 + \Gamma_3 + \Gamma_4 - s_4$. To prove this equality we need only to verify that $\Gamma(P_5^*) \leq \Gamma_4 + \Gamma_3 + \Gamma_2 - s_4$. Suppose, by contradiction, that there is a greedy coloring $\mathcal{S}'$ of P_5^* with more than $\Gamma_4 + \Gamma_3 + \Gamma_2 - s_4$ colors. By Proposition 1 and by hypothesis that $v \in V(H_3)$, there are at least $\Gamma_2 - s_4 + 1$ colors that occur in H_2 and do not occur in $H_3 \cup H_4$. Since, by hypothesis, $\Gamma_5 < \Gamma_2 - s_4 + 1$, there is at least one color i in H_2 that does not occur in $H_3 \cup H_4 \cup H_5$. On the other hand, $\Gamma_2 + \Gamma_3 + \Gamma_4 - s_4 + 1 = \Gamma_1 + \Gamma_2 + \Gamma_3 + 1$, i.e., there is a color j in H_4 that does not occur in $H_1 \cup H_2 \cup H_3$. This is a contradiction, because neither the vertices of S_i in H_2 have a neighbor colored j nor the vertices of S_j in H_4 have a neighbor colored i.

Finally, suppose that $\Gamma_2 - s_4 < \Gamma_5$. We will prove that $\Gamma(P_5^*) = \Gamma_2 + \Gamma_3 + \Gamma_4 - s_2$. To do this, we use again the symmetry of P_5^*. In the analysis of the previous case, we considered the hypothesis of using the colors of H_4 that do not appear in H_1 to color H_2 and we concluded that if the number of colors of H_2 that do not occur in H_4 is at least Γ_5, we know how to determine the Grundy number of P_5^*.

Using the same idea, we can analogously conclude the following fact: if $\Gamma_4 - s_2 \geq \Gamma_1$, then $\Gamma(P_5^*) = \Gamma_4 + \Gamma_3 + \Gamma_2 - s_2$. Under this hypothesis, using the symmetry, we find the result we needed. However, we can easily verify that $\Gamma_4 - s_2 \geq \Gamma_1$ if, and only if, $\Gamma_2 - s_4 < \Gamma_5$, the proof of this complementary case is analogous to the previous case.

By hypothesis, we know the values of $\Gamma_1, \ldots, \Gamma_5$. Then, the value of $\Gamma(P_5^*)$ can be determined by outputting the maximum value found between among all the cases above. Since we have a constant number of cases, the value of $\Gamma(P_5^*)$ can be found in constant time. Observe that since all the possibilities to place a vertex with the greatest color were checked, $\Gamma(P_5^*)$ is correctly computed. □

Lemma 2. *Given the Grundy numbers of $H_1, \ldots, H_5$, the Grundy number of $\bar{P}_5^* = (H_1 \cup \ldots \cup H_5, E)$ can be determined in constant time.*

Proof. Suppose that $\mathcal{S} = (S_1, \ldots, S_k)$ is a greedy coloring of $\bar{P}_5^*$ with $\Gamma(\bar{P}_5^*)$ colors. Analogously to Lemma 1, let us check all the possible cases:

1. There is a vertex $v \in H_1$ colored k, then $\Gamma(\bar{P}_5^*) = \Gamma_1 + \Gamma_2 + \Gamma_3$.

   This case can be easily solved because any ordering over $V(\bar{P}_5^*)$ that contains suborderings θ_1, θ_2 and θ_3 produces a greedy coloring with at least $\Gamma_1+\Gamma_2+\Gamma_3$ colors, since all the colors used in $H_1 \cup H_2 \cup H_3$ must be distinct. Moreover, $\Gamma_1+\Gamma_2+\Gamma_3$ is also an upper bound because of Proposition 1 and the hypothesis that $v \in V(H_1)$.

2. If there is a vertex $v \in H_2$ colored k, then:

$$\Gamma(\bar{P}_5^*) = \begin{cases} \Gamma_1 + \Gamma_2 + \Gamma_3 & , if\ \Gamma_4 \leq \Gamma_3 \\ \Gamma_1 + \Gamma_2 + \Gamma_4 & , if\ \Gamma_4 > \Gamma_3\ and\ \Gamma_1 \leq \Gamma_5 \\ \Gamma_2 + \Gamma_4 + \Gamma_5 & , if\ \Gamma_4 > \Gamma_3,\ \Gamma_1 > \Gamma_5\ and\ \Gamma_4 - s_1 \geq \Gamma_3 \\ \Gamma_1 + \Gamma_2 + \Gamma_3 & , if\ \Gamma_4 > \Gamma_3,\ \Gamma_1 > \Gamma_5\ and\ \Gamma_4 - s_1 < \Gamma_3 \end{cases}$$

   where $s_1 = \Gamma_1 - \Gamma_5$.

   Consider first that $\Gamma_4 \leq \Gamma_3$. We will prove that $\Gamma(\bar{P}_5^*) = \Gamma_1+\Gamma_2+\Gamma_3$. Observe that $\Gamma(\bar{P}_5^*) \geq \Gamma_1 + \Gamma_2 + \Gamma_3$, because of an ordering over $V(\bar{P}_5^*)$ that starts by θ_1, θ_2 and θ_3 leads the greedy algorithm to the generation of a greedy coloring with at least $\Gamma_1 + \Gamma_2 + \Gamma_3$ colors.

   On the other hand, suppose, by contradiction, that there exists a greedy coloring $\mathcal{S}' = \{S'_1, \ldots, S'_p\}$ of $\bar{P}_5^*$ with $p \geq \Gamma_1+\Gamma_2+\Gamma_3+1$ colors. As a consequence of Proposition 1, there is a color i such that $S'_i \subseteq V(H_4)$. Since $\Gamma_4 \leq \Gamma_3$, we conclude that $\mathcal{S}'$ has at least $\Gamma_1 + \Gamma_2 + \Gamma_4 + 1$ colors. Thus, there is a color j such that $S'_j \subseteq V(H_3)$. Consequently, there is no vertex of H_4 colored i adjacent to some vertex of H_3 colored j, i.e., there is no vertex of S'_i adjacent to some vertex of S'_j. This is a contradiction because $\mathcal{S}'$ is a greedy coloring.

Therefore, we can assume that $\Gamma_4 > \Gamma_3$ and set $s_4 = \Gamma_4 - \Gamma_3$. We study two subcases. At first, if $\Gamma_5 \geq \Gamma_1$, then we claim that $\Gamma(\bar{P}_5^*) = \Gamma_1 + \Gamma_2 + \Gamma_4$. Using the hypothesis that $\Gamma_5 \geq \Gamma_1$, we can easily conclude that $\Gamma(\bar{P}_5^*) \geq \Gamma_1 + \Gamma_2 + \Gamma_4$, because an ordering over the vertices of $\bar{P}_5^*$ starting by θ_5, θ_1, θ_4 and θ_2, consecutively in this order, makes the greedy algorithm generate a greedy coloring with at least $\Gamma_1 + \Gamma_2 + \Gamma_4$ colors.

To show that this value is also an upper bound, suppose, by contradiction, that $\bar{P}_5^*$ admits a greedy coloring $\mathcal{S}' = \{S'_1, \ldots, S'_p\}$ with $p \geq \Gamma_1 + \Gamma_2 + \Gamma_4 + 1$ colors. By Proposition 1 and by the hypothesis that $v \in V(H_2)$, there is a color i such that $S'_i \subseteq V(H_3)$ (observe that $S'_i \cap V(H_3) \neq \emptyset$ implies that $S'_i \cap V(H_5) = \emptyset$). Since $\Gamma_4 > \Gamma_3$, $\mathcal{S}'$ has at least $\Gamma_1 + \Gamma_2 + \Gamma_3 + 2$ colors. Thus, there are at least two colors S'_j and S'_l such that $S'_j \cup S'_l \subseteq V(H_4)$. This contradicts the hypothesis that $\mathcal{S}'$ is a greedy coloring, because neither S'_j nor S'_l has a vertex with some neighbor colored i.

As a consequence, we can suppose that $\Gamma_5 < \Gamma_1$, and if $\Gamma_4 - s_1 \geq \Gamma_3$, then we will prove that $\Gamma(\bar{P}_5^*) = \Gamma_1 + \Gamma_2 + \Gamma_4 - s_1$. Using the hypothesis that $\Gamma_4 > \Gamma_3$ and $\Gamma_5 < \Gamma_1$, we can easily check that an ordering over $V(\bar{P}_5^*)$ starting by θ_1, θ_5, θ_4 and θ_2, consecutively in this order, produces a greedy coloring with at least $\Gamma_1 + \Gamma_2 + \Gamma_4 - s_1$ colors.

Suppose, by contradiction, there exists a greedy coloring $\mathcal{S}' = \{S'_1, \ldots, S'_p\}$ of $\bar{P}_5^*$ with $p \geq \Gamma_1 + \Gamma_2 + \Gamma_4 - s_1 + 1$ colors. Since $v \in V(H_2)$, we use Proposition 1 to verify that there are at least $\Gamma_4 - s_1 + 1$ colors that occur only in $H_3 \cup H_4$. Since, by hypothesis, $\Gamma_4 - s_1 \geq \Gamma_3$, there is at least one color i from these $\Gamma_4 - s_1 + 1$ colors that occurs only in H_4. Moreover, since $s_1 = \Gamma_1 - \Gamma_5$, $\mathcal{S}'$ has at least $\Gamma_1 + \Gamma_2 + \Gamma_4 - s_1 + 1 = \Gamma_2 + \Gamma_4 + \Gamma_5 + 1$ colors. Again, the hypothesis that $v \in V(H_2)$ and Proposition 1 imply that there is at least one color j that only occur in $H_1 \cup H_3$. This contradicts the hypothesis that $\mathcal{S}'$ is a greedy coloring because of there are no edges from S'_i to S'_j.

The last case is when $\Gamma_5 < \Gamma_1$ and $\Gamma_4 - s_1 < \Gamma_3$. In this case, $\Gamma(\bar{P}_5^*) = \Gamma_1 + \Gamma_2 + \Gamma_4 - s_4$. In order to prove this, observe that $\Gamma_4 - s_1 < \Gamma_3$ if, and only if, $\Gamma_1 - s_4 > \Gamma_5$. Therefore, in order to simplify the proof of this case, we will prove that if $\Gamma_1 - s_4 > \Gamma_5$, then $\Gamma(\bar{P}_5^*) = \Gamma_1 + \Gamma_2 + \Gamma_4 - s_4$. To see that $\Gamma(\bar{P}_5^*) \geq \Gamma_1 + \Gamma_2 + \Gamma_4 - s_4$, observe that an ordering over $V(\bar{P}_5^*)$ started by θ_4, θ_3, θ_1 and θ_2, consecutively in this order, makes the greedy algorithm generate a greedy coloring with at least $\Gamma_1 + \Gamma_2 + \Gamma_4 - s_4$ colors.

Suppose, by contradiction, that there is a greedy coloring $\mathcal{S}' = \{S'_1, \ldots, S'_p\}$ to $\bar{P}_5^*$ with $p \geq \Gamma_1 + \Gamma_2 + \Gamma_4 - s_4 + 1$ colors. Once $v \in V(H_2)$ and the Proposition 1 holds, there are at least $\Gamma_4 - s_4 + 1$ colors occurring only in $H_1 \cup H_3 \cup H_5$. Since $\Gamma_1 - s_4 > \Gamma_5$, there is at least one color i exclusive to $H_1 \cup H_3$. Recall that $\mathcal{S}'$ has at least $\Gamma_1 + \Gamma_2 + \Gamma_4 - s_4 + 1 = \Gamma_1 + \Gamma_2 + \Gamma_3 + 1$ colors. Then, since $v \in V(H_2)$ and by Proposition 1, there exists a color j such that $S'_j \subseteq V(H_4)$. This is a contradiction because of the same previous arguments.

3. If there is a vertex $v \in V(H_3)$ colored k, then

$$\Gamma(\bar{P}_5^*) = \begin{cases} \Gamma_1 + \Gamma_3 + \Gamma_2 & , if\ \Gamma_5 \leq \Gamma_2 \\ \Gamma_1 + \Gamma_3 + \Gamma_5 & , if\ \Gamma_5 > \Gamma_2\ and\ \Gamma_1 \leq \Gamma_4 \\ \Gamma_3 + \Gamma_4 + \Gamma_5 & , if\ \Gamma_5 > \Gamma_2,\ \Gamma_1 > \Gamma_4\ and\ \Gamma_5 - s_1 \geq \Gamma_2 \\ \Gamma_1 + \Gamma_2 + \Gamma_3 & , if\ \Gamma_5 > \Gamma_2,\ \Gamma_1 > \Gamma_4\ and\ \Gamma_5 - s_1 < \Gamma_2 \end{cases}$$

   where $s_1 = \Gamma_1 - \Gamma_4$.

   The proof of this case is analogous to the previous one, taking $s_5 = \Gamma_5 - \Gamma_2$ to play the role of s_4.

4. If there is a vertex $v \in V(H_4)$ colored k, then

$$\Gamma(\bar{P}_5^*) = \begin{cases} \Gamma_2 + \Gamma_4 + \Gamma_5 & , if\ \Gamma_1 \geq \Gamma_5 \\ \Gamma_4 + \Gamma_5 & , if\ \Gamma_1 < \Gamma_5\ and\ s_5 \geq \Gamma_2 \\ \Gamma_1 + \Gamma_2 + \Gamma_4 & , if\ \Gamma_1 < \Gamma_5\ and\ s_5 < \Gamma_2 \end{cases}$$

   where $s_5 = \Gamma_5 - \Gamma_1$.

   Again, observe that, by Proposition 1, the Grundy number in this case is bounded by $\Gamma_2 + \Gamma_4 + \Gamma_5$.

   First, suppose that $\Gamma_1 \geq \Gamma_5$. Let us prove that $\Gamma(\bar{P}_5^*) = \Gamma_2 + \Gamma_4 + \Gamma_5$.

   In this case, notice that an ordering over $V(\bar{P}_5^*)$ started by θ_1, θ_5, θ_2 and θ_4 leads the greedy algorithm to the generation of a greedy coloring of $\bar{P}_5^*$ with $\Gamma_2 + \Gamma_4 + \Gamma_5$ colors.

   Now, assume that $\Gamma_1 < \Gamma_5$. We have to study two cases. In the first case, consider that $s_5 \geq \Gamma_2$. Then, we claim that $\Gamma(\bar{P}_5^*) = \Gamma_4 + \Gamma_5$. To prove this fact, observe that the same ordering over $V(\bar{P}_5^*)$ of the previous case produces a greedy coloring with at least $\Gamma_4 + \Gamma_5$ colors.

   In order to show that this is also an upper bound, suppose, by contradiction, that there exists a greedy coloring $\mathcal{S}' = \{S'_1, \ldots, S'_p\}$ of $\bar{P}_5^*$ with $p \geq \Gamma_4 + \Gamma_5 + 1$ colors. Since $v \in V(H_4)$ and Proposition 1 holds, there is a color i that occurs in H_2 and does not occur in $H_4 \cup H_5$. Now, the hypothesis that $s_5 \geq \Gamma_2$ implies that $\mathcal{S}'$ has at least $\Gamma_1 + \Gamma_2 + \Gamma_4 + 1$ colors. As a consequence, there are at least $\Gamma_1 + 1$ colors that occur in H_5 and that do not occur in $H_2 \cup H_4$. By Proposition 1, there is at least one color j from these $\Gamma_1 + 1$ colors such that $S'_j \subseteq V(H_5)$. The fact that there are no edges between S'_i and S'_j contradicts the assumption that $\mathcal{S}'$ is a greedy coloring.

   In the complementary case, we have that $\Gamma_1 < \Gamma_5$ and $s_5 < \Gamma_2$. We have to prove now that $\Gamma(\bar{P}_5^*) = \Gamma_4 + \Gamma_5 + \Gamma_2 - s_5$. Observe that the same ordering of the previous case, together with these hypothesis, leads the greedy algorithm to the generation of a greedy coloring of $\bar{P}_5^*$ with at least $\Gamma_4 + \Gamma_5 + \Gamma_2 - s_5$ colors. To verify that this is an upper bound, suppose, by contradiction, that there is a greedy coloring $\mathcal{S}' = \{S'_1, \ldots, S'_p\}$ with $p \geq \Gamma_4 + \Gamma_5 + \Gamma_2 - s_5 + 1$ colors. The hypothesis that $v \in V(H_4)$ and the Proposition 1 imply that there are at least $\Gamma_2 - s_5 + 1$ colors exclusive to H_2. Assume that i is one of these colors. Since $\Gamma_4 + \Gamma_5 + \Gamma_2 - s_5 + 1 = \Gamma_4 + \Gamma_2 + \Gamma_1 + 1$, there is also a color j exclusive to H_5. Again, the fact that there are no edges between S'_i and S'_j contradicts the assumption that $\mathcal{S}'$ is a greedy coloring.

5. If there is a vertex $v \in H_5$ colored k, then

$$\Gamma(\bar{P}_5^*) = \begin{cases} \Gamma_3 + \Gamma_5 + \Gamma_4 & , if\ \Gamma_1 \geq \Gamma_4 \\ \Gamma_5 + \Gamma_4 & , if\ \Gamma_1 < \Gamma_4\ and\ s_4 \geq \Gamma_3 \\ \Gamma_1 + \Gamma_3 + \Gamma_5 & , if\ \Gamma_1 < \Gamma_4\ and\ s_4 < \Gamma_3 \end{cases}$$

where $s_4 = \Gamma_4 - \Gamma_1$.

The proof of this case is analogous to the previous one.

Since there is a fixed number of cases to be checked and the calculus to be made in each of them can be also done in constant time, the Grundy number of $\bar{P}_5^*$, given $\Gamma_1, \ldots, \Gamma_5$, can be determined in constant time. □

Lemma 3. *Given the Grundy numbers of $H_1, \ldots, H_5$, the Grundy number of $C_5^* = (H_1 \cup \ldots \cup H_5, E)$ can be determined in constant time.*

Proof. Suppose that $\mathcal{S} = (S_1, \ldots, S_k)$ is a greedy coloring of C_5^* with $\Gamma(C_5^*)$ colors. It is enough to prove the Lemma for the case where there is a vertex $v \in V(H_1)$ colored k, since all the other cases follow by symmetry. Therefore, suppose that this is the case. Then:

$$\Gamma(C_5^*) = \begin{cases} \Gamma_1 + \Gamma_2 + \Gamma_3 & , if\ \Gamma_5 \geq \Gamma_2\ or\ \Gamma_4 \geq \Gamma_3 \\ \Gamma_1 + \Gamma_2 + \Gamma_4 & , if\ \Gamma_5 < \Gamma_2,\ \Gamma_4 < \Gamma_3\ and\ \Gamma_2 - s_3 \geq \Gamma_5 \\ \Gamma_1 + \Gamma_3 + \Gamma_5 & , if\ \Gamma_5 < \Gamma_2,\ \Gamma_4 < \Gamma_3\ and\ \Gamma_2 - s_3 < \Gamma_5 \end{cases}$$

where $s_3 = \Gamma_3 - \Gamma_4$.

By Proposition 1 and the hypothesis that $v \in V(H_1)$, $\Gamma(C_5^*) \leq \Gamma_1 + \Gamma_2 + \Gamma_3$.

Assume first that $\Gamma_5 \geq \Gamma_2$ *or* $\Gamma_4 \geq \Gamma_3$. We claim that $\Gamma(C_5^*) = \Gamma_1 + \Gamma_2 + \Gamma_3$. To prove this, observe that if $\Gamma_5 \geq \Gamma_2$, an ordering over $V(C_5^*)$ that starts by θ_5, θ_2, θ_3 and θ_1, consecutively in this order, makes the greedy algorithm generate a greedy coloring with exactly $\Gamma_1 + \Gamma_2 + \Gamma_3$ colors and the upper bound is achieved. On the other hand, if $\Gamma_4 \geq \Gamma_3$, an ordering over $V(C_5^*)$ that starts by θ_4, θ_3, θ_2 and θ_1, consecutively in this order, produces a greedy algorithm coloring of C_5^* with $\Gamma_1 + \Gamma_2 + \Gamma_3$ colors and, again, the upper bound is achieved.

As a consequence, we can assume that $\Gamma_5 < \Gamma_2$ *and* $\Gamma_4 < \Gamma_3$. Let us set $s_2 = \Gamma_2 - \Gamma_5$ and consider the following subcases. At first, if $\Gamma_2 - s_3 \geq \Gamma_5$, then we prove that$\Gamma(C_5^*) = \Gamma_1 + \Gamma_2 + \Gamma_3 - s_3$. Observe that an ordering over $V(C_5^*)$ started by θ_3, θ_4, θ_2 and θ_1, consecutively in this order, makes the greedy algorithm generate a greedy coloring of C_5^* having at least $\Gamma_1 + \Gamma_2 + \Gamma_3 - s_3$ colors.

Suppose by contradiction that there is a greedy coloring $\mathcal{S}' = \{S'_1, \ldots, S'_p\}$ of C_5^* with $p \geq \Gamma_1 + \Gamma_2 + \Gamma_3 - s_3 + 1$ colors. By the hypothesis that $v \in V(H_1)$ and Proposition 1, there are at least $\Gamma_2 - s_3 + 1$ colors that occur in H_2 and do not occur in $H_1 \cup H_3$. One of them, let us say i, does not occur in H_5, since $\Gamma_2 - s_3 \geq \Gamma_5$. Moreover, as $\Gamma_1 + \Gamma_2 + \Gamma_3 - s_3 + 1 = \Gamma_1 + \Gamma_2 + \Gamma_4 + 1$, we observe that at least $\Gamma_4 + 1$ colors occur in H_3 and that do not occur in $H_1 \cup H_2$. Among them, at least one, j also does not occur in H_4. These facts contradict the assumption that $\mathcal{S}'$ is a greedy coloring, since there are no edges between S'_i and S'_j.

As the last subcase, suppose that $\Gamma_2 - s_3 < \Gamma_5$. We claim that $\Gamma(C_5^*) = \Gamma_1 + \Gamma_2 + \Gamma_3 - s_2$. Notice that $\Gamma_2 - s_3 < \Gamma_5$ if, and only if, $\Gamma_3 - s_2 > \Gamma_4$ and, since if $\Gamma_3 - s_2 > \Gamma_4$, then $\Gamma_3 - s_2 \geq \Gamma_4$. Therefore, the proof of this case is similar to the proof of the previous one up to symmetry, because we can analogously prove that if $\Gamma_3 - s_2 \geq \Gamma_4$, then $\Gamma(C_5^*) = \Gamma_1 + \Gamma_2 + \Gamma_3 - s_2$.

Again, since there is a fixed number of cases to be checked and the calculus to be made in each of them can be also done in constant time, the Grundy number of C_5^* given $\Gamma_1, \ldots, \Gamma_5$ can be determined in constant time. □

In what follows, the two remaining possible types of neighborhood nodes are treated. Recall that G is a fat-extended P_4-laden graph and that $T(G)$ corresponds to its modular decomposition tree.

Lemma 4. *Let v be a neighborhood node of $T(G)$ such that $G(v)$ is isomorphic to a split graph $H = (S'(H) \cup K'(H) \cup R'(H), E)$. Given $\Gamma(G'[R])$, then the Grundy number of $G[M(v)]$ can be determined in linear time.*

Proof. At first, recall that the partition of the vertices of H into sets $S'(H)$, $K'(H)$ and $R'(H)$ can be found in $\mathcal{O}(V(H))$ [HS81]. Suppose that $\mathcal{S} = (S_1, \ldots, S_k)$ is a greedy coloring of $G[M(v)]$ with the maximum number of colors.

Since the strong modules represented by the vertices of $S'(H)$ and $K'(H)$ are stable sets and cliques, respectively, we denote by $S^*(H)$ ($K^*(H)$) the subgraph of $G[M(v)]$ induced by the union of all the modules represented by the vertices of $S'(H)$ (resp., $K'(H)$). Observe that the subgraph of $G[M(v)]$ induced by $V(S^*(H)) \cup V(K^*(H))$ is a split graph and the vertices of $R'(H)$ are adjacent to all the vertices of $K^*(H)$ and to none of $S^*(H)$.

Notice that for any ordering θ over $M(v)$, the greedy algorithm would never assign distinct colors i and j to the vertices of $S^*(H)$, such that $S_i \cup S_j \subseteq S^*(H)$, since $S^*(H)$ is a stable set and so no vertex of S_i would be adjacent to some vertex of S_j. As there is at most one color exclusive to $S^*(H)$, if $R'(H)$ is empty, then $\Gamma(M(v)) \leq |K^*(H)| + 1$. Moreover, an ordering over $V(M(v))$ such that all the vertices of $S^*(H)$ appear before the ones of $K^*(H)$ produces a greedy coloring with $|K^*(H)| + 1$ colors, because of $K'(H)$ is exactly the neighborhood of $S'(H)$.

On the other hand, if $R'(H)$ is not empty, then any greedy coloring of $G[M(v)]$, in particular, $\mathcal{S}$, should assign distinct colors to the vertices of $R'(H)$ and $K^*(H)$, because there are all the edges between the vertices of both sets. Let j be any color occurring in $R'(H)$. If there is a color i such that $S_i \subseteq S^*(H)$, no vertex of S_i would have a neighbor in S_j, contradicting the assumption that $\mathcal{S}$ is a greedy coloring. Consequently, $\Gamma(M(v)) = |K^*(H)| + \Gamma(R'(H))$. As a consequence, $\Gamma(M(v))$ can be computed in linear time following the equation:

$$\Gamma(M(v)) = \begin{cases} |K^*(H)| + \Gamma(R'(H)) & \text{, if } R'(H) \neq \emptyset \\ |K^*(H)| + 1 & \text{, otherwise.} \end{cases}$$

□

Lemma 5. *Let v be a neighborhood node of $T(G)$ such that $G(v)$ isomorphic to a spider $H = (S \cup K \cup R, E)$, f_r be its child corresponding to R, f_2 be its child corresponding to the module which has eventually two vertices and $\Gamma(R)$ be the Grundy number of $G[M(f_r)]$. Then $\Gamma(G[M(v)])$ can be determined in linear time.*

Proof. Suppose that $\mathcal{S} = (S_1, \ldots, S_k)$ is a greedy coloring of $G[M(v)]$ with $\Gamma(G[M(v)])$ colors. If f_2 is trivial, or if f_2 belongs to S and its vertices are not adjacent, or if f_2 belongs to K and its vertices are adjacent, then the Grundy number of $G[M(v)]$ can be found by using the same arguments of Lemma 4, by replacing S, K and R by $S'(G)$, $K'(G)$ and $R'(G)$, respectively.

For otherwise, let x and w be the vertices of f_2. Again, we denote by S^* (K^*) the subgraph of $G[M(v)]$ induced by the union of all the modules represented by the vertices of S (resp., K). We have to check the following cases:

- f_2 belongs to S and x and w are adjacent.

  We claim that for any greedy coloring of $G[M(v)]$, in particular for $\mathcal{S}$, there are no two distinct colors i and j such that $S_i \cup S_j \subseteq S^*$. To show this fact, suppose the contrary. By similar arguments to those used in the proof of Lemma 4, colors i and j must be assigned to x and w. Without loss of generality, suppose that $x \in S_i$ and $w \in S_j$. Since x and w belong to a same module and because of the definition of a spider, there is at least a vertex $y \in K^*$ which is adjacent to none of x and w. Let us suppose that $y \in S_l$. Observe that $(K^* \cup R) \cap S_l = \{y\}$. Now, let u be any other vertex of S^*. So, u has to be assigned to either a color of a non-neighbor in $K \cup R$ or to the smallest between i and j, say i. These facts imply that there is only one vertex of S^*, which is w, colored j and so $(S^* \cup K^*) \cap S_j = \{w\}$. As a consequence, none of w and y has a neighbor colored l and j, respectively. This contradicts the fact that $\mathcal{S}$ is a greedy coloring.

  Therefore, any greedy coloring of $G[M(v)]$ has at most one color containing only vertices of S^*, and then its Grundy number can be determined in linear time by using similar arguments to those used in Lemma 4.

- f_2 belongs to K and x and w are not adjacent.

  We claim that there are no distinct colors i and j, such that $x \in S_i$ and $w \in S_j$. For otherwise, since x and w are not adjacent and the belong to the same module, either w would not have a neighbor colored i or x would not have a neighbor colored j. Therefore, by similar arguments to those used in the proof of Lemma 4, we can conclude that the Grundy number of $G[M(v)]$ can be found in linear time.

□

Theorem 2. *If $G = (V, E)$ is a fat-extended P_4-laden graph and $|V| = n$, then $\Gamma(G)$ can be found in $\mathcal{O}(n^3)$.*

Proof. The algorithm computes $\Gamma(G)$ by traversing the modular decomposition tree of G in a post-order way and determining the Grundy of each inner node of $T(G)$ based on the Grundy number of its children. The modular decomposition tree can be found in linear time [TCHP08], the post-order traversal can be done in $\mathcal{O}(n^2)$ and the Grundy number of each inner node can be found in linear time, because of Lemmas 1, 2, 3, 4 and 5, and because of the results of Gyárfás and Lehel for cographs [GL88]. □

Corollary 1. *Let G be a graph that belongs to one of the following classes: P_4-reducible, extended P_4-reducible, P_4-sparse, extended P_4-sparse, P_4-extendible, P_4-lite, P_4-tidy, P_4-laden and extended P_4-laden. Then, $\Gamma(G)$ can be determined in polynomial time.*

Proof. According to the definition of these classes [Ped07], they are all strictly contained in the fat-extended P_4-laden graphs and so the corollary follows. □

2.4 Conclusions

We extended the previously known result that states that the Grundy number can be determined in polynomial time for cographs [GL88], which are exactly the P_4-free graphs, to a greater class of graphs that we called fat-extended P_4-laden graphs. In fact, by observing that every complement of a bipartite graph is P_5-free, the result of Zaker [Zak05] implies that determining the Grundy number for a P_5-free graph is also NP-hard.

The problems of finding a minimum vertex coloring, a minimum clique cover, a maximum clique and a maximum independent set can be solved in polynomial time for extended P_4-laden graphs [Gia96, CHMDW87]. We remark that these results can be easily extended to fat-extended P_4-laden graphs. Even though the vertex coloring problem can be solved in polynomial time for fat-extended P_4-laden graphs, the study of the Grundy number also provides bounds to other problems, like Weighted Coloring, whose complexity is not determined even for a subclass of extended P_4-laden graphs called P_4-sparse graphs [GZ97, ALS10] as we present in Chapter 3.

Finally, we observe that, since Lemmas 1, 2 and 3 are proved without the assumption that we are dealing with fat-extended P_4-laden graphs, those results can be useful for any class of graphs whose modular decomposition contains fat neighborhood nodes.

Chapter 3

Weighted Coloring

Contents

As mentioned in Chapter 1, there are many variations of the Vertex Coloring problem that are studied in the literature. In this chapter, we study one that is defined on vertex-weighted graphs.

Let $G = (V, E)$ be a graph and $w : V(G) \to \mathbb{R}^*_+$ be a weight function over the vertices of G. Given a k-coloring $c = (S_1, \ldots, S_k)$ of G, we define the weight of a color S_i as

$$w(S_i) = \max_{v \in S_i} w(v), \qquad \text{for every } i \in \{1, \ldots, k\}.$$

The weight of coloring c is:

$$w(c) = \sum_{i=1}^{k} w(S_i).$$

The goal of Weighted Coloring problem is, for a given graph G and weight function w, determine the *weighted chromatic number of* (G, w), denoted as $\chi_w(G)$, which is the minimum weight of a proper coloring of (G, w) [GZ97].

It is important to remark that an optimal weighted coloring of (G, w) might not use $\chi(G)$ colors (see Figure 3.1). However, the maximum number of colors of an optimal weighted coloring of (G, w) is $\Gamma(G)$ [GZ97]. This is the main relationship between Greedy Coloring and Weighted Coloring.

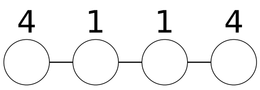


Figure 3.1: Optimal coloring of this weighted P_4 has weight 6 and uses 3 colors: the endpoints must be in the same color class.

The Weighted Coloring problem generalizes Vertex Coloring because, in the particular case of $w : V(G) \to \{1\}$, we have that $\chi(G) = \chi_w(G)$. Weighted Coloring was defined to improve the Distributed Dual Bus Network Media Access Control Protocol, which is a standard IEEE802.6 for metropolitan networks [GZ97].

The complexity of this problem was further studied in some previous works. Weighted Coloring is NP-hard for bipartite graphs [DdWMP02], split graphs [DdWMP02], planar graphs [dWDE+05] and interval graphs [EMP06].

It was shown that $\chi_w(G)$ can be computed in polynomial-time whenever G is a bipartite graph and the image of w has only two different weights [DdWMP02], or G is a P_5-free bipartite graph [dWDE+05], or G is a cograph [DdWMP02].

Approximation algorithms for the Weighted Coloring problem were also proposed when the input graph belongs to different graph classes [DdWMP02, dWDE+05, EMP06].

In Section 3.1, we present an extension of the Hajós' Theorem [Haj61] for Weighted Coloring, i.e. we give a necessary and sufficient condition for $\chi_w(G) \geq k$. Then, in Section 3.2, we address to complexity results in the class of P_4-sparse graphs.

3.1 Hajós-like Theorem for Weighted Coloring

The characterization of the *k-chromatic graphs*, i.e., graphs G such that $\chi(G) = k$, has been a challenging problem for many years. In 1961, Hajós [Haj61] gave a characterization of graphs with chromatic number at least k by proving that they must contain a *k-constructible* subgraph. In order to present the definition of the class of k-constructible graphs and this characterization, we need to recall some definitions. The *identification* of two vertices a and b of a graph G means the removal of a and b followed by the inclusion of a new vertex $a \circ b$ adjacent to $N_G(a) \cup N_G(b)$. A graph $G = (V, E)$ is *complete* if it is simple and if, for every pair of vertices u, v of G, $uv \in E(G)$. We denote by K_n the complete graph with n vertices.

Definition 2. *The set of* k-constructible graphs *is defined recursively as follows:*

1. *The complete graph with k vertices is k-constructible.*
2. ***Hajós' Sum:*** *If G_1 and G_2 are disjoint k-constructible graphs, $a_1b_1 \in E(G_1)$ and $a_2b_2 \in E(G_2)$, then the graph G obtained from $G_1 \cup G_2$ by removing a_1b_1 and a_2b_2, identifying a_1 with a_2, and adding the edge b_1b_2, is a k-constructible graph (see Figure 3.2).*
3. ***Identification:*** *If G is k-constructible and a and b are two non-adjacent vertices of G, then the graph obtained by the identification of a and b is a k-constructible graph.*

Theorem 1 (Hajós [Haj61])**.** *$\chi(G) \geq k$ if, and only if, G is a supergraph of a k-constructible graph.*

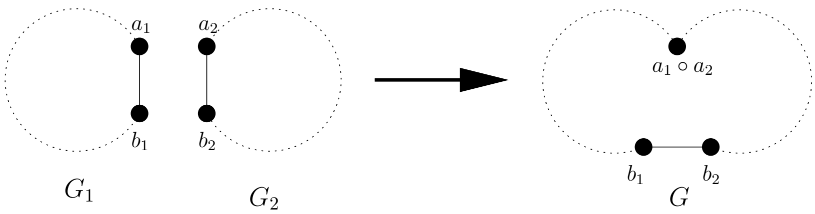


Figure 3.2: Hajós' Sum.

The Hajós' Theorem determines a set of operations that we can use to obtain, from complete graphs with k vertices, all k-chromatic graphs, including the k-critical ones. A graph H is *k-critical* if $\chi(H) \geq k$ and for every proper subgraph I of H, $\chi(I) < k$. Clearly, for a given graph G, the difficulty of determining whether $\chi(G) \geq k$ is equivalent to the difficulty of determining if G contains a k-critical subgraph H. Thanks to the Hajós' Theorem, we know that all the k-critical graphs can be built from complete graphs on k vertices by successive applications of Hajós' Sum and identification of vertices. Because of this, it has been subject of interest to obtain Hajós-like Theorem for several variations of the classical coloring problem. Gravier [Gra96] proved an extension of Hajós' Theorem for List Coloring. Král [Kra04] gave a simplified proof of Gravier's result. Zhu [Zhu03] found an extension of this theorem for circular chromatic number. Mohar [Moh05] demonstrated two new versions of the referred theorem for circular chromatic number and an extension of Hajós' Theorem for the channel assignment problem, i.e., a coloring of edge-weighted graphs.

We now show an extension of Hajós' Theorem for Weighted Coloring.

We deal with simple (vertex-)weighted graphs. We denote by $G = (V, E, w)$ a vertex-weighted graph $G = (V, E)$ together with its weight function $w : V \to \mathbb{R}^*_+$.

Notation 1. *The infinite family of complete weighted graphs $G = (V, E, w)$ with order $n = |V(G)|$ and such that $\sum_{v \in V} w(v) = k$ is denoted by $\mathcal{K}^k_n$.*

Notation 2. *Given a weighted graph $G = (V, E, w)$ and a proper coloring c of G, we choose as the* ***representative*** *of the color i in c, $rep_c(i)$, a unique arbitrary vertex $v \in V$ satisfying the inequality $w(v) \geq w(x)$, for all $x \in V$ such that $c(x) = c(v) = i$.*

Definition 3. *Given two weighted graphs $G = (V, E, w)$, $H = (V', E', w')$, we say that $H \subseteq G$ (H is a* subgraph *of G) if $V' \subseteq V$, $E' \subseteq E$, and, for all $v \in V'$, we have $w'(v) \leq w(v)$.*

Now, we redefine the Hajós' construction for the weighted case:

Definition 4. *The set of* weighted *k-constructible graphs is defined recursively as follows:*

1. *The graphs in $\bigcup_{i \in \mathbb{N}} \mathcal{K}^k_i$ are weighted k-constructible.*

2. ***Weighted Hajós' Sum:*** *If G_1 and G_2 are disjoint weighted k-constructible graphs, $a_1b_1 \in E(G_1)$ and $a_2b_2 \in E(G_2)$, then the graph G obtained from $G_1 \cup G_2$ by removing a_1b_1 and a_2b_2, identifying a_1 with a_2 into a vertex $a_1 \circ a_2$, such that $w(a_1 \circ a_2) = \max\{w(a_1), w(a_2)\}$, and adding the edge b_1b_2 is a weighted k-constructible graph.*

3. ***Weighted Identification:*** *If G is weighted k-constructible and a and b are two non-adjacent vertices of G, then the graph obtained by the identification of a and b into a vertex $a \circ b$, such that $w(a \circ b) = max\{w(a), w(b)\}$, is a weighted k-constructible graph.*

The main result of this section is:

Theorem 2. *Let $G = (V, E, w)$ be a weighted graph and k be a positive real. Then, $\chi_w(G) \geq k$ if, and only if, G has a weighted k-constructible subgraph H.*

Proof. We prove first that if $\chi_w(G) \geq k$, then G has a weighted k-constructible subgraph H. Suppose, by contradiction, that there exists a counter-example $G = (V, E, w)$ with a maximal number of edges. It means that $\chi_w(G) \geq k$, G does not contain any weighted k-constructible subgraph and, for any pair of non-adjacent vertices of G, let us say u, v, $G' = G + uv$ contains a weighted k-constructible subgraph.

We claim that G is not isomorphic to a complete multipartite graph. Suppose the contrary and let p be the number of stable sets in the partition $\mathcal{P}$ of $V(G)$. For each color class C_i of an optimal weighted coloring c of G, there is no pair of vertices of C_i in distinct sets of $\mathcal{P}$, since there is an edge between any two vertices of distinct parts. Moreover, we cannot have more than one color class in one stable set. As a matter of fact, suppose by contradiction that there are two color classes, say C_i and C_j, whose vertices belong to the same stable set in the partition $\mathcal{P}$ of $V(G)$. Without loss of generality, suppose that $w(rep_c(i)) \geq w(rep_c(j))$. Then a coloring c', obtained from c by the union of the color classes C_i and C_j, has cost exactly $\chi_w(G) - w(rep_c(j))$, and this contradicts the optimality of c.

Consequently, vertices with the greatest weight in every part are exactly the representatives of each color class. Observe that the subgraph induced by the representatives is an element of the set $\mathcal{K}_p^k$, because $\chi_w(G) \geq k$ and $p \in \mathbb{N}$. This contradicts the hypothesis that G has no weighted k-constructible subgraph.

Therefore, the counter-example G is not a complete multipartite graph. Thus, there are at least three vertices in G, let us say a, b and c, such that $ab, bc \notin E(G)$ and $ac \in E(G)$. Consider now the graphs $G_1 = G + ab$ and $G_2 = G + bc$. Because of the maximality of G, G_1 and G_2 have each a weighted k-constructible subgraph H_1 and H_2, respectively. Obviously, the edges ab and bc belong, respectively, to H_1 and H_2. Consider then the application of Hajós' Sum to two disjoint graphs isomorphic to H_1 and H_2, respectively. Let us choose edge ab of H_1 and bc of H_2 to remove and let us identify the vertices labeled b. Finally, identify all the vertices in H_1 with their corresponding vertices in H_2, if they exist. Observe that a graph isomorphic to a subgraph of G is obtained at the end of this sequence of operations. Then, G has a weighted k-constructible subgraph, a contradiction.

We prove now that, if G has a weighted k-constructible subgraph H, then $\chi_w(G) \geq k$. First, observe that $\chi_w(G) \geq \chi_w(H)$. Then, we just have to show that $\chi_w(H) \geq k$. The proof is by induction on the number of Hajós' operations applied to obtain H.

If H is isomorphic to a graph $K_i^k \in \mathcal{K}_i^k$, for some $i \in \mathbb{N}$, then its weighted chromatic number is trivially k, since it is a complete graph whose sum of weights is equal to k.

Suppose then that H was obtained by the Weighted Identification of two non-adjacent vertices a and b of a weighted k-constructible graph H' into a vertex $a \circ b$. By induction hypothesis, H' has a weighted chromatic number at least k. Suppose, by contradiction, that $\chi_w(H) < k$ and let c be an optimal weighted coloring of H. Then, a coloring c' of H' can be obtained from c, by assigning to a and b the color assigned to $a \circ b$ in c, and letting all the other vertices of H' be assigned to the same color they have been assigned in c. Observe that, except the color i of $a \circ b$, for all the other colors j, $rep_{c'}(j) = rep_c(j)$. For the color i, the vertex $rep_c(i)$ has weight greater than or equal to the weight of $a \circ b$, that is greater than or equal to the weight of a and b. Therefore, the coloring c' has weight equal to the coloring c, that is less than k. This contradicts the hypothesis of $\chi_w(H') \geq k$.

Finally, suppose that H was obtained from weighted k-constructible graphs H_1 and H_2 using the Weighted Hajós' Sum on the edges (a_1, b_1) and (a_2, b_2) from H_1 and H_2, respectively. Let $a_1 \circ a_2$ be the vertex of H obtained by the identification of a_1 e a_2. Suppose, by contradiction, that $\chi(H) < k$, while $\chi(H_1) \geq k$ and $\chi(H_2) \geq k$. Consider an optimal weighted coloring c of H. Observe that either $c(a_1 \circ a_2) \neq c(b_1)$ or $c(a_1 \circ a_2) \neq c(b_2)$ (because b_1 and b_2 are adjacent). Without loss of generality, suppose that $c(a_1 \circ a_2) \neq c(b_1)$. Then, consider now the restriction c' of c to H_1, assigning to a_1 the color of $a_1 \circ a_2$. We have that, for all color class C_j of c', the weight of $rep_{c'}(j) \leq rep_c(j)$ (including the color class of a_1, because the weight of a_1 is less than or equal to the weight of $a_1 \circ a_2$). Consequently, $w(c') \leq w(c) < k$, contradicting the hypothesis that $\chi_w(H_1) \geq k$. □

Ore [Ore67] has proved that the Hajós construction can be simplified. He has shown that by using only a single operation that collapses the two Hajós operations, one may construct any k-colorable graph from complete graphs of order k.

The same simplification can be done for the weighted case. It is necessary to use the same adaptation we did whenever two vertices u and v are identified, i.e., the weight of the new vertex must be the maximum value between the weight of u and the weight of v.

Moreover, following Urquhart [Urq97] for the non-weighted case, it is also not hard to establish the equivalence between the class of the weighted k-constructible graphs and the class obtained by using the adapted Ore's operation described above.

Finally, there is a well-known problem that is to study the complexity of the construction of k-chromatic graphs of a given size by Hajós operations [MW82, HRT86, JT95]. Since the Vertex Coloring Problem is a particular case of Weighted Coloring, it is obvious that the complexity of this problem in the weighted case is as hard as in the non-weighted one and can also be left as open question.

3.2 P_4-sparse graphs

In the beginning of this chapter, we commented that the Weighted Coloring problem is NP-hard for split graphs, which is a subclass of P_5-free graphs, but can be solved in polynomial-time for cographs, that are exactly the P_4-free graphs. In this section, we study the Weighted Coloring problem in the class of P_4-sparse graphs that strictly contains the cographs and that are strictly contained in the P_5-free graphs.

A graph G is *P_4-sparse* if every subset of five vertices of $V(G)$ induces at most one P_4 [Hoà85]. The structural properties of this class of graphs were well studied by Jamison and Olariu [JO92b, JO92a, JO95, LO98].

Several optimization problems can be solved in polynomial-time on P_4-sparse graphs [JO95]. The algorithms that solve these problems usually compute the desired parameter in a simple post-order traversal in the modular decomposition tree of the graph, as the algorithm we presented for computing the Grundy number of fat-extended P_4-laden graphs in Chapter 2. We use here the same approach to determine the weighted chromatic number of graphs in a subclass of P_4-sparse graphs that properly contains the cographs. Thus, we need to know the structure of the neighborhood nodes of the modular decomposition tree of P_4-sparse graphs.

Theorem 3 ([GV97a]). *G is a P_4-sparse graph if, and only if, the quotient graph of each neighborhood node of its modular decomposition tree $T(G)$ is isomorphic to a spider $H = (S \cup K \cup R, E)$.*

Recall that we defined modular decomposition and spider graphs in Section 2.1.

In the sequel, we present a polynomial-time algorithm to compute the weighted chromatic number of graphs that are contained in the class of P_4-sparse graphs. After that we show that there exists a 2-approximation algorithm for Weighted Coloring on P_4-sparse graphs by using another characterization of this graph class.

3.2.1 Polynomial-Time Algorithm

Let $G = (V, E)$ be a P_4-sparse graph, $w : V(G) \to \mathbb{R}^*_+$ be a function and $T(G)$ be its modular decomposition tree. As announced, we propose an algorithm to determine $\chi_w(G)$ that uses a bottom-up traversal in $T(G)$. We know that the weighted chromatic number of the leaves of $T(G)$ is equal to one. We show in this section how to determine the weighted chromatic number of $G[M(v)]$, for each series or neighborhood inner node v of $T(G)$. This computation is based on the weighted chromatic number of v's children, i.e., of the subgraphs induced by its maximal strong submodules. The algorithm also keeps an optimal weighted coloring of $G[M(v)]$, for every node $v \in T(G)$.

By combining these results with the polynomial-time algorithm for cographs [DdWMP02], we are able to present a polynomial-time algorithm to compute the weighted chromatic number of a subclass of P_4-sparse graphs that strictly contains the cographs. In the end of the section, we discuss the difficulty of computing the weighted chromatic number in the parallel node's case. First, observe that:

Remark 1. *For every series node $v \in T(G)$, with children $v_1, \ldots, v_p$, $\chi_{w_v}(G[M(v)]) = \sum_{i=1}^{p} \chi_{w_{v_i}}(G[M(v_i)])$, where w_v is the function w restricted to the descendant leafs of v. Moreover, an optimal weighted coloring of $G[M(v)]$ is obtained by optimal weighted colorings of $(G[M(v_i)], w_{v_i})$, for every $i \in \{1, \ldots, p\}$.*

Remark 1 implies that it is easy to deal with the *series* nodes in $T(G)$. The rest of this section is dedicated to the *neighborhood* nodes (spiders) and *parallel* nodes (disjoint union) of P_4-sparse graphs.

3.2.1.1 Spiders

Consider that $G = (V = S \cup K \cup R, E)$ is always a spider graph in this subsection. Thus, we many times refer to the sets S, K and R without explicitly repeating they correspond to the sets that define a partition of the vertex set of G, whenever the spider G is clear in the context. Moreover, we always refer to a coloring c as a set of stable sets (color classes), instead of considering c as a function.

As we observed in Section 2.1, the unique non-trivial maximal strong sub-module of a spider is exactly the set R. Thus, the goal of this section is to extend an optimal weighted coloring of $(G[R], w_R)$ to an optimal weighted coloring of (G, w) in polynomial-time, for any weight function $w : V(G) \to \mathbb{R}^*_+$ and w_R being the restriction of w to the elements of R.

Definition 5. *Let $H \subseteq G$ and $c_G = \{S_1, \ldots, S_k\}$ (resp. $c_H = \{S'_1, \ldots, S'_l\}$) be coloring of G (resp. H). We say that c_G is an* extension *of c_H if there is an injective function $f : \{1, \ldots, l\} \to \{1, \ldots, k\}$ such that $S'_i \subseteq S_{f(i)}$, for every $i \in \{1, \ldots, l\}$.*

This notion is commonly used in the PRECOLORING EXTENSION problem [Mar05, Mar06].

Since K is a clique, in any proper k-coloring $c = \{S_1, \ldots, S_k\}$ of G, we have k disjoint colors that appear in the vertices of K. By the definition of a spider, if $u \in K$ and $v \in R$, then $uv \in E(G)$. Consequently, no color class $S_i \in c$ contains vertices from both K and R. Thus, we define two sets of colors $CK(c)$ and $CR(c)$ such that if $S_j \cap K \neq \emptyset$ (resp. $S_j \cap R \neq \emptyset$), then $S_j \in CK(c)$ (resp. $S_j \in CR(c)$). We refer to these sets $CK(c)$ and $CR(c)$ as *colors of* K and *colors of* R in the coloring c. Now, we study in which color classes the vertices of S may appear.

Intuitively, we first prove that it is not necessary to care about vertices in S that have a heavier non-neighbor. Formally, denote by $S_L \subseteq S$ the set of vertices

$s \in S$ such that s has a non-neighbor u in $K \cup R$ satisfying $w(u) \geq w(s)$. Observe that any optimal weighted coloring c of $G[V(G) \backslash S_L]$ can be extended to a coloring c' of G by assigning to each vertex in $s \in S_L$ the color $c(u)$ of its non-neighbor $u \in K \cup R$ satisfying $w(u) \geq w(s)$. By the definition of S_L, we conclude that $w(c) = w(c')$. Since $G[V(G) \backslash S_L]$ is a subgraph of G and c is an optimal weighted coloring of $G[V(G) \backslash S_L]$, we conclude that c' is an optimal weighted coloring of G. Consequently, we consider, with a slight abuse of notation, that:

Remark 2. *If $s \in S$, $u \in K \cup R$ and $su \notin E(G)$, then $w(s) > w(u)$.*

Lemma 6. *Let (G, w) be a weighted spider and $c = \{S_1, \ldots, S_k\}$ be an optimal weighted coloring of (G, w), then:*

1. *if $R = \emptyset$, then there exists at most one color class $S_i \in c$, such that $S_i \notin CK(c)$;*
2. *otherwise,*
   - (a) *$c = CK(c) \cup CR(c)$;*
   - (b) *there exists at most one color class $S_j \in c$ that intersects both S and R;*
   - (c) *if such color S_j exists, then it contains a vertex $r^* \in R \cap S_j$ satisfying $w(r^*) = \max_{r \in R} w(r)$.*

Proof. By contradiction, if $R = \emptyset$ and there are two disjoint colors S_i and S_j in c that just contain vertices of S, the coloring c' obtained from c by just merging S_i and S_j would satisfy $w(c') < w(c)$ and this would be a contradiction. This proves 1.

Now consider that $R \neq \emptyset$. Suppose, by contradiction, that there is a color $S_i \subseteq S$. Since $R \neq \emptyset$, there exists a color $S_j \neq S_i$ such that $S_j \in CR(c)$. Thus, one could merge S_i and S_j and obtain a proper coloring c' satisfying $w(c') < w(c)$ and it contradicts the optimality of c. We derive Statement 2a.

To prove Statement 2b, suppose once more by contradiction that there are two disjoint colors from $CR(c)$. Let these colors be S_j and S'_j, such that S_j and S'_j contain vertices of S. Moreover, without loss of generality, suppose that $w(S_j) \geq w(S'_j)$. By Remark 2, the vertices with the greatest weight in each color class S_j and S'_j belong to S. Thus, the coloring c' obtained from c by recoloring all the vertices in $S \cap S'_j$ with the color S_j would have weight strictly smaller than $w(c)$ and it would again be a contradiction to the optimality of c. Thus, there is at most one color $S_j \in c$ containing vertices from both S and R, in case $R \neq \emptyset$.

If such a color exists and c is an optimal coloring of G, observe that it must contain a vertex r^* of maximum weight in R. Otherwise, if we recolor all the vertices in $S \cap S_j$ with the color of a maximum weight vertex r^*, we obtain a coloring c' that would satisfy $w(c') < w(c)$, by Remark 2, a contradiction. □

By Lemma 6, we denote by the *color of* S, or simply $cS(c)$, the unique possible color class which does not belong to $CK(c)$ in an optimal weighted coloring c of a spider, whenever $R = \emptyset$.

Lemma 7. *If $R \neq \emptyset$, then, for any optimal weighted coloring c_R of $(G[R], w_R)$, there exists an optimal weighted coloring c of (G, w) that is an extension of c_R, where w_R is the function w restricted to R.*

Proof. Without loss of generality, consider that $c_R = \{S_1^r, \ldots, S_t^r\}$ has a vertex r^* of maximum weight $w(r^*) = \max_{r \in R} w(r)$ in S_1^r.

Let $c' = \{S'_1, \ldots, S'_y\}$ be an optimal weighted coloring of G. By Lemma 6, $c' = CK(c') \cup CR(c')$. Without loss of generality, assume that $CK(c') = \{S'_1, \ldots, S'_x\}$ and that $CR(c') = \{S'_{x+1}, \ldots, S'_y\}$, for some $x \in \{1, \ldots, y\}$. Moreover, let S'_{x+1} be the color class of $CR(c')$ that possibly contains vertices of S, according to Lemma 6.

To prove the lemma, we now create another optimal weighted coloring c from c' that is an extension of c_R. Let $c = \{S_1, \ldots, S_{x+t}\}$ be a coloring of G such that $S_i = S'_i$, for every $i \in \{1, \ldots, x\}$, $S_{x+1} = S_1^r \cup \{S'_{x+1} \backslash R\}$ and $S_{x+j} = S_j^r$, for every $j \in \{2, \ldots, t\}$.

Since c_R is an optimal weighted coloring of $G[R]$, observe that c is an optimal weighted coloring to G, because in both colorings c' and c, the color classes of $CK(c)$ are the same as $CK(c')$, $w(S_{x+1}) = w(S'_{x+1})$ and the sum of the remaining colors $S_{x+2}, \ldots, S_{x+t}$ is minimized as c_R is an optimal coloring of $(G[R], w_R)$. Consequently, $w(c) \leq w(c')$, and thus c is optimal and extends c_R. □

Recall that we have all the edges from a vertex of K to a vertex in R and thus K and R receive disjoint colors. Lemma 7 shows us that we can extend any optimal weighted coloring of $(G[R], w_R)$ in order to find an optimal weighted coloring of (G, w). Now, we study which colors the vertices in the stable set S can receive.

Suppose now, without loss of generality, that the vertices of S are labeled $S = \{s_1, \ldots, s_m\}$ satisfying $w(s_1) \leq \ldots \leq w(s_m)$. Let $S(j)^-$ (resp. $S(j)^+$) be the set $\{s_1, \ldots, s_{j-1}\}$ (resp. $\{s_j, \ldots, s_m\}$). Denote by k^* (resp. r^*) a heaviest vertex of K (resp. R) and by k^{**} a second heaviest vertex of K. If G is thin, then denote by s^* the only neighbor of k^* and by k_i the neighbor s_i, for every $k_i \in K$, $k_i \neq k^*$.

Lemma 8. *From any optimal weighted coloring of a spider G, we can obtain another optimal weighted coloring c of G by just recoloring vertices in S such that, for some $j \in \{1, \ldots, m+1\}$, c satisfies:*

1. *each vertex $S(j)^-$ is in the color $cS(c)$ (if $R = \emptyset$) or in the color of r^*; and*

2. *each vertex $S(j)^+$ is in the color of one of its non-neighbors in K. Moreover, if G is thin, we have that either:*

   (a) *each vertex in $S(j)^+ \backslash \{s^*\}$ has the color of k^* and if $s^* \in S(j)$, then it has the color of k^{**}; or*

   (b) *for some vertex $k_i \in K$, each vertex in $S(j)^+ \backslash \{s_i\}$ has the color of $k_i \neq k^*$ and if $s_i \in S(j)^+$, then it has the color of k^*.*

Proof. Consider an optimal weighted coloring c'' of G. By Lemma 6, let $j - 1$ be the highest index of a heaviest vertex of S that is colored either with a color of r^* or with the color $cS(c')$ (consider that if $j-1=0$, then there is no vertex with these colors). Observe that we can obtain a coloring c' from c'' such that $w(c') \leq w(c'')$ by assigning to all $s_1, \ldots, s_{j-1}$ the same color as s_j. This proves the statement 1.

Consequently, we know that c' is an optimal weighted coloring of G and that each vertex $s_j, \ldots, s_m$ is in the color of one of its non-neighbors in K. If G is a thick spider, the lemma is proved as we can choose $c' = c$.

Consider then that G is a thin spider. We now construct another optimal weighted coloring c from c' satisfying the remaining statements of the lemma depending on the following cases.

First, consider that $c'(s_m) = c'(k^*)$. Let c be the coloring obtained from c' by recoloring all the vertices in $\{s_j, \ldots, s_m\} \backslash \{s^*\}$ with the color $c'(k^*) = c'(s_m)$. Observe that all the vertices $\{s_j, \ldots, s_m\} \backslash \{s^*\}$ are not adjacent to a vertex with color $c(k^*)$, thus c is proper. Moreover, $w(c'(k^*)) = w(c(k^*))$, because s_m is a heaviest vertex of S and by hypothesis $c'(s_m) = c'(k^*)$. We also know that all the other color classes do not have a bigger weight in c, because they have just lost some vertices compared to c'. Then, $w(c) \leq w(c')$. If $s^* \notin \{s_j, \ldots, s_m\}$ or $c(s^*) = c(k^{**})$, then c satisfies the lemma. Otherwise, one may observe that we can recolor s_j with the color of k^{**} without increasing the weight of c, thanks to Remark 2.

From now on, consider then that $c'(s_m) \neq c'(k^*)$. If $s_m \neq s^*$, we change the coloring c' by recoloring s_m with the color of k^* and we do not increase the weight of the coloring c', since s_m is the heaviest vertex of S and Remark 2 holds. Then, we are again in the previous case in which $c'(s_m) = c'(k^*)$ and we can find a coloring satisfying the lemma. So, Statement 2a of this lemma holds.

Finally, suppose that $s_m = s^*$ and $c'(s^*) = c'(s_m) = c'(k_i)$, for some $k_i \neq k^*$. In this case, the coloring c obtained from c' by assigning to all the vertices in the set $\{s_j, \ldots, s_m\} \backslash \{s_i\}$ the color of $c'(s_m)$ and to s_i, in case it belongs to $\{s_j, \ldots, s_m\}$, the color of k^* satisfies $w(c) \leq w(c')$ and also the lemma's conditions. □

We can finally state the main result of this section:

Proposition 2. *Let $G = (S \cup K \cup R, E)$ be a spider, $w : V(G) \rightarrow \mathbb{R}^*_+$ be a function and c_R be an optimal weighted coloring of $(G[R], w_R)$, where w_R is the restriction of w to R. Then an optimal weighted coloring of (G, w) can be found in $\mathcal{O}(n^3)$-time.*

Proof. We know, by Lemma 7, that any optimal weighted coloring c_R of R can be extended to an optimal weighted coloring c of G. If we apply to c Lemma 8, we know that there is an optimal weighted coloring c' that extends c_R and for which the vertices in S satisfy the statements of Lemma 8.

Thus, the algorithm we propose (see Algorithm 1) to find an optimal weighted coloring of a spider G, provided an optimal weighted coloring c_R of $G[R]$, returns, among all the possible colorings that extend c_R and satisfy the statements of Lemma 8, the one of minimum weight.

The correctness of the algorithm follows from Lemmas 7 and 8. The vertices of S can be ordered by their weights in $\mathcal{O}(n \log n)$ and the vertices of K and R can be colored in linear-time, provided we are given an optimal weighted coloring c_R of

Algorithm 1: WEIGHTED COLORING of spiders

Input: Spider $G = (S \cup K \cup R, E)$ and an optimal weighted coloring c_R of $G[R]$
Output: Optimal weighted coloring of G

$m \leftarrow |S|$;

Create artificial vertices s_0 and s_{m+1} in S and order them such that $w(s_0) \leq \ldots \leq w(s_{m+1})$;

Choose k^*, k^{**} and r^* and define $c, c' \leftarrow \emptyset$;

```
foreach r ∈ R do
    c'(r) := c_R(r);
foreach k ∈ K do
    c'(k) := a color among the |K| colors of K;
for j = 1, ..., m + 1 do
    for i = 0, ..., j − 1 do
        if R ≠ ∅ then
            c'(s_i) ← c'(r*);
        else
            c'(s_i) ← cS;
    if G is thick then
        for i = j, ..., m do
            c'(s_i) ← the color of its non-neighbor in K (c'(f(s_i)));
        if w(c') < w(c) then
            c ← c';
    else
        for i = j, ..., m do
            if (s_i, k*) ∉ E(G) then
                c'(s_i) ← c'(k*);
            else
                c'(s_i) ← c'(k**);
        if w(c') < w(c) then
            c ← c';
        foreach k_i ∈ K\{k*} do
            for i = j, ..., m do
                if (s_i, k_i) ∉ E(G) then
                    c'(s_i) ← c'(k_i);
                else
                    c'(s_i) ← c'(k*);
            if w(c') < w(c) then
                c ← c';
```

Result: c

$G[R]$. However, to color the vertices of S, we have to try all the colorings satisfying Lemma 8 and this can take $\mathcal{O}(n^3)$-time in the case we have a thin spider. □

Corollary 2. *Let (G, w), $w : V(G) \to \mathbb{R}^*_+$, be a weighted P_4-sparse graph whose modular decomposition tree $T(G)$ satisfies the following statement: if $T(G)$ contains a parallel node v, then $M(v)$ is a cograph. Then, an optimal weighted coloring of (G, w) can be found in $\mathcal{O}(n^3)$ time.*

Proof. At first, the modular decomposition tree of G, $T(G)$, can be found in linear time [TCHP08]. Then, we do a pre-order traversal in $T(G)$ computing $\chi_w(G[M(v)])$ at each node parallel node v. Since $G[M(v)]$ is a cograph, this can be done by using the already known algorithm for cographs [DdWMP02] whose complexity is $\mathcal{O}(n^2)$. Finally, we visit $T(G)$ in a post-order way and use Remark 1 and Proposition 2 to determine $\chi_w(G[M(v)])$ at each series or neighborhood node v of $T(G)$. □

Observe that in Corollary 2, we present an algorithm to solve the WEIGHTED COLORING problem for a subclass of P_4-sparse graphs which strictly contains cographs, since its modular decomposition tree may have modules whose quotient graphs are isomorphic to spiders.

3.2.1.2 Disjoint Union

We know study the parallel nodes v of the tree decomposition $T(G)$ of a P_4-sparse graph G. By definition, $M(v)$ is a disconnected P_4-sparse graph and its connected components correspond to its maximal strong submodules. If we could obtain, from optimal weighted colorings of the components of $M(v)$ an optimal weighted coloring of $M(v)$, then the algorithm we presented in the last section could be extended to the class of P_4-sparse graphs. However, this is not a trivial task.

To illustrate the problem tackled in this section, consider the P_4-sparse graph $G = A \cup B$ of Figure 3.3. An optimal coloring $c_A = \{S_1, \ldots, S_4\}$ of A with weight 5 is given by $S_1 = \{k_1\}$, $S_2 = \{k_2\}$, $S_3 = \{k_3\}$, and $S_4 = \{s_1, s_2, s_3\}$. An obvious optimal coloring $c_B = \{S'_1, S'_2, S'_3\}$ of B with weight 6 is given by $S'_1 = \{u_1\}$, $S'_2 = \{u_2\}$, and $S'_3 = \{u_3\}$.

One simple algorithm to find an optimal weighted coloring of G would be to combine both colorings c_A and c_B by merging the color classes according to their weights.

If we apply this algorithm for our example in Figure 3.3, we obtain a coloring of G with weight 7. However, there exists a better coloring c_G of G with weight 6 given by $S''_1 = \{s_1, k_1, u_1\}$, $S''_2 = \{s_2, k_2, u_2\}$, and $S''_3 = \{s_3, k_3, u_3\}$. Observe that this optimal coloring c_G of G, when restricted to A, it is not an optimal weighted coloring of A because it has weight 6, which is strictly greater than the weight of c_A.

The previous example shows that an optimal weighted coloring of a disconnected graph is not given by optimal weighted colorings of its components.

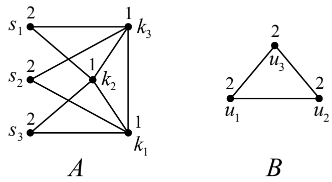


Figure 3.3: An optimal weighted coloring of a disjoint union is not given by merging an optimal weighted colorings of each component.

For the next result, we do another slight abuse of notation and consider that the weight of an empty stable set is equal to zero. We now prove that there exists an optimal weighted coloring c of any disconnected weighted graph (G, w) where the i-th heaviest stable set in the coloring c of G is also the i-th heaviest color when we restrict c to each connected component of G. Recall that the weight of a color (stable set) is the maximum weight of a vertex in this color and that a color may appear in different connected components. Formally, we have:

Proposition 3. *Let (G, w) be a disconnected weighted graph such that $w : V(G) \to \mathbb{R}_+^*$ and $G = G_1 \cup \ldots \cup G_m$, where each G_i is a connected component of G. There exists an optimal weighted coloring $c = \{S_1, \ldots, S_k\}$ of (G, w) such that the coloring $c_{G_i} = \{S_1^i, \ldots, S_k^i\}$ of G_i, where $S_j^i = V(G_i) \cap S_j$ and $j \in \{1, \ldots, k\}$, satisfies $w(S_1^i) \geq \ldots \geq w(S_k^i)$, for every $i \in \{1, \ldots, m\}$.*

Proof. Let $c' = \{S'_1, \ldots, S'_k\}$ be an arbitrary proper k-coloring of G. We prove now that we can obtain a coloring $c = \{S_1, \ldots, S_k\}$ from c' such that $w(c) \leq w(c')$ and c satisfies the proposition. Thus, the proof is completed by considering the particular case in which c' is an optimal weighted coloring.

Without loss of generality, assume that $w(S'_1) \geq \ldots \geq w(S'_k)$. Let $S_j^i = V(G_i) \cap S'_j$, for every $j \in \{1, \ldots, k\}$ and $i \in \{1, \ldots, m\}$. Observe that S'_j is the j-th heaviest stable set of (G, w) thanks to the maximum weight a vertex in this set. Consequently, S_j^i is not necessarily the j-th heaviest stable set in G_i, i.e. in $c'_{G_i} = \{S_j^i \mid j \in \{1, \ldots, k\}\}$.

We now order the set c'_{G_i} by the weights of its elements, thus we define R_j^i as the j-th heaviest element of the set $\{S_j^i \mid j \in \{1, \ldots, k\}\}$, for every $i \in \{1, \ldots, m\}$.

Claim 1. *For every $j \in \{1, \ldots, k\}$, we have:*

$$w(S'_j) \geq \max_{i \in \{1, \ldots, m\}} w(R_j^i)$$

For $j = 1$ the claim is true, since the weight of S'_1 is given by the weight of a heaviest vertex in G, which equals $\max\{w(R_1^1), \ldots, w(R_1^m)\}$. Suppose that the claim is not true for some $j > 2$, i.e., $w(S'_j) < \max\{w(R_j^1), \ldots, w(R_j^m)\}$. Suppose, without loss of generality, that $\max\{w(R_j^1), \ldots, w(R_j^m)\} = w(R_j^1)$. Then, by hypothesis:

$$w(S_j) < w(R_j^1) \leq w(R_{j-1}^1) \leq \ldots \leq w(R_p^1) \leq \ldots \leq w(R_1^1). \tag{3.1}$$

For $p = 1, \ldots, j$, let $S_{q_p} \in \{S'_1, \ldots, S'_k\}$ be the stable set of c containing R_p^1. Observe that, by definition, all these sets S_{q_p} are distinct. Then,

$$w(R_p^1) \leq w(S_{q_p}),\ p = 1, \ldots, j. \tag{3.2}$$

Combining Equations (3.1) and (3.2) we deduce that $w(S_j) < w(S_{q_p})$, for each $p = 1, \ldots, j$. In other words, there exist j color classes with weight strictly greater than $w(S'_j)$ in c', a contradiction to the hypothesis that $w(S'_1) \geq \ldots \geq w(S'_k)$. Thus, claim follows.

Define then a coloring c with color classes $S_1, \ldots, S_k$ as follows:

$$S_j := R_j^1 \cup \ldots \cup R_j^m,\ j = 1, \ldots, k.$$

By the claim, c satisfies the proposition. □

We finish this section by proposing the following conjecture:

Conjecture 1. *There is a polynomial-time algorithm to solve the* WEIGHTED COLORING *problem on P_4-sparse graphs.*

3.2.2 Approximation Algorithm

In the last section, we were not able to present an exact polynomial-time algorithm to compute the weighted chromatic number of any P_4-sparse graph. However, we know a 2-approximation algorithm for this class. The idea is simple, but to present it, let us first consider the special partition of P_4-sparse graphs given by Jamison and Olariu [JO95, LO98]:

Definition 6. *A graph G has a* special partition *if there exists a family $\Sigma = \{S_1, \ldots, S_q\}$ of disjoint stable sets of G with $q \geq 1$ and $|S_i| \geq 2$, for all $i \in \{1, \ldots, q\}$, and there exists an injection $f : \bigcup_{i=1}^q S_i \longrightarrow V - \bigcup_{i=1}^q S_i$ such that the following occurs:*

1. *$K_i = \{z \mid z = f(s)$ for some $s \in S_i\}$ is a clique, for all $i \in \{1, \ldots, q\}$;*
2. *A set of vertices A induces a P_4 in G if, and only if, there exists a subscript $i \in \{1, \ldots, q\}$ and distinct vertices $x, y \in S_i$ such that $A = \{x, y, f(x), f(y)\}$.*

Let us define $\mathcal{S} = \bigcup_{i=1}^q S_i$ and $\mathcal{K} = V - \bigcup_{i=1}^q S_i$. Observe that the graphs induced by S and K are cographs and their weighted chromatic number can be determined in polynomial time [DdWMP02].

Theorem 3 ([JO92b])**.** *A graph is a P_4-sparse graph if, and only if, it is a cograph or it has a special partition.*

Then, we can state the following:

Proposition 4. *There exists a polynomial-time approximation algorithm for* WEIGHTED COLORING *on P_4-sparse graphs with approximation ratio bounded above by 2.*

Proof. Observe that if $H \subseteq G$, then $\chi_w(H) \leq \chi_w(G)$. Let G be a P_4-sparse graph that is not a cograph. By Theorem 3, G contains a special partition (Σ, f). By definition, $G[\mathcal{S}]$ and $G[\mathcal{K}]$ are cographs. Consequently, our algorithm color the cographs $G[\mathcal{S}]$ and $G[\mathcal{K}]$ with disjoint sets of colors by using the polynomial-time algorithm for cographs [DdWMP02]. As $\chi_w(G[\mathcal{S}]) \leq \chi_w(G)$ and $\chi_w(G[\mathcal{K}]) \leq \chi_w(G)$, the proof is completed. □

3.3 Conclusions

In this chapter, we presented new results on the WEIGHTED COLORING problem. First, we showed a necessary and sufficient condition for a weighted graph (G, w) to have weighted chromatic number at least k. Then we give complexity results for P_4-sparse graphs.

The computational complexity of computing the weighted chromatic number of a weighted P_4-sparse graph remains open.

The authors in [EMP06] propose a polynomial-time approximation scheme for partial k-trees. One interesting question related to this result that, up to our best knowledge, remains unsolved is: what is the computational complexity of determining $\chi_w(T)$, when T is a tree?

Chapter 4

Weighted Improper Coloring

Contents

A k-coloring c is *l-improper* if $|\{v \in N(u) \mid c(v) = c(u)\}| \leq l$, for all $u \in V$. Given a non-negative integer l, the *l-improper chromatic number* of a graph G, denoted by $\chi_l(G)$, is the minimum integer k such that G admits an l-improper k-coloring. Given a graph G and an integer l, the Improper Coloring problem consists in determining $\chi_l(G)$ and is also NP-hard [Woo90, CHS09]. Indeed, if $l = 0$, observe that $\chi_0(G) = \chi(G)$. Consequently, Vertex Coloring is a particular case of Improper Coloring.

In this chapter, we define and study a new variation of the Improper Coloring problem for edge-weighted graphs. An edge-weighted graph is a pair (G, w) where $G = (V, E)$ is a graph and $w : E \to \mathbb{R}_+^*$. Given an edge-weighted graph (G, w) and a coloring c of G, the *interference* of a vertex u in this coloring is defined by

$$I_u(G, w, c) = \sum_{\{v \in N(u) | c(v) = c(u)\}} w(u, v).$$

For any non-negative real number t, called *threshold*, we say that c is a *weighted t-improper k-coloring* of (G, w) if c is a k-coloring of G such that $I_u(G, w, c) \leq t$, for all $u \in V$.

Given a threshold $t \in \mathbb{R}_+^*$, the minimum integer k such that the graph G admits a weighted t-improper k-coloring is the *weighted t-improper chromatic number* of (G, w), denoted by $\chi_t(G, w)$. Given an edge-weighted graph (G, w) and a threshold

$t \in \mathbb{R}_+^*$, determining $\chi_t(G, w)$ is the goal of the WEIGHTED IMPROPER COLORING problem. Note that if $t = 0$ then $\chi_0(G, w) = \chi(G)$, and if $w(e) = 1$ for all $e \in E$, then $\chi_l(G, w) = \chi_l(G)$ for any positive integer l. Therefore, the WEIGHTED IMPROPER COLORING problem is clearly NP-hard since it generalizes VERTEX COLORING and IMPROPER COLORING.

On the other hand, given a positive integer k, we define the *minimum k-threshold* of (G, w), denoted by $T_k(G, w)$ as the minimum real t such that (G, w) admits a weighted t-improper k-coloring. Then, for a given edge-weighted graph (G, w) and a positive integer k, the THRESHOLD IMPROPER COLORING problem consists in determining $T_k(G, w)$. The THRESHOLD IMPROPER COLOURING problem is also NP-hard. This fact follows from the observation that determining whether $\chi_l(G) \leq k$ is NP-complete, for every $l \geq 2$ and $k \geq 2$ [CCW86, CGJ95, CHS09]. Consequently, in particular, it is a NP-complete problem to decide whether a graph G admits a weighted t-improper 2-coloring when all the weights of the edges of G are equal to one, for every $t \geq 2$.

Our initial motivation to these problems was the design of satellite antennas for multi-spot MFTDMA satellites [AAG+05]. In this technology, satellites transmit signals to areas on the ground called *spots*. These spots form a grid-like structure which is modeled by an hexagonal cell graph. To each spot is assigned a radio channel or color. Spots are interfering with other spots having the same channel and a spot can use a color only if the interference level does not exceed a given threshold t. The level of interference between two spots depends on their distance. The authors of [AAG+05] introduced a factor of mitigation γ and the interference of remote spots are reduced by a factor $1 - \gamma$. When the interference level is too low, the nodes are considered to not interfere anymore. Considering such types of interference, where nodes at distance at most i interfere, leads to the study of the i-th power of the graph modeling the network and a case of special interest is the power of grid graphs (see Section 4.2).

Our problems are particular cases of the FREQUENCY ASSIGNMENT problem (FAP). FAP has several variations that were already studied in the literature (see [AvHK+07] for a survey). In most of these variations, the main constraint to be satisfied is that if two vertices (mobile phones, antennas, spots, etc.) are close, then the difference between the frequencies that are assigned to them must be greater than some function which usually depends on their distance.

There is a strong relationship between most of these variations and the $L(p_1, \ldots, p_d)$-LABELING problem [Yeh06]. In this problem, the goal is to find a coloring of the vertices of a given graph G, in such a way that the difference between the colors assigned to vertices at distance i is at least p_i, for every $i = 1, \ldots, d$.

In some other variants, for each non-satisfied interference constraint a penalty must be paid. In particular, the goal of the MINIMUM INTERFERENCE FREQUENCY ASSIGNMENT problem (MI-FAP) is to minimize the total penalties that must be paid, when the number of frequencies to be assigned is given. This problem can also be studied for only *co-channel interference*, in which the penalties are applied only if the two vertices have the same frequency. However, MI-FAP under these constraints does not correspond to Weighted Improper Coloring, because we consider the co-channel interference, i.e. penalties, just between each vertex and its neighborhood.

The two closest related works we found in the literature are [MS03] and [FLM+00]. However, they both apply penalties over co-channel interference, but also to the *adjacent channel interference*, i.e. when the colors of adjacent vertices differ by one unit. Moreover, their results are not similar to ours. In [MS03], they propose an enumerative algorithm for the problem, while in [FLM+00] a Branch-and-Cut method is proposed and applied over some instances.

In this work, we study both parameters $\chi_t(G,w)$ and $T_k(G,w)$. We first present general bounds; in particular we show a generalization of Lovász's Theorem for $\chi_t(G,w)$. We after show how to transform an instance of Threshold Improper colouring into an equivalent one where the weights are either one or M, for a sufficiently large M.

Motivated by the original application, we then study a special interference model on various grids (square, triangular, hexagonal) where a node produces a noise of intensity 1 for its neighbors and a noise of intensity 1/2 for the nodes that are at distance two. We derive the weighted t-improper chromatic number for all possible values of t.

Finally, we propose a heuristic and a Branch-and-Bound algorithm to solve Threshold Improper colouring for general graphs. We compare them to an integer linear programming formulation on random cell-like graphs, namely Voronoi diagrams of random points of the plan. These graphs are classically used in the literature to model telecommunication networks [BKLZ97, GK00, HAB+09].

4.1 General Results

In this section, we present some results for Weighted Improper colouring and Threshold Improper colouring for general graphs and general interference models.

4.1.1 Upper bounds

Let (G,w) be an edge-weighted graph with positive real weights given by $w: E(G) \to \mathbb{Q}^*_+$. For any vertex $v \in V(G)$, its *weighted degree* is $d_w(v) = \sum_{u \in N(v)} w(u,v)$. The *maximum weighted degree* of G is $\Delta(G,w) = \max_{v \in V} d_w(v)$.

Given a k-coloring $c: V \to \{1,\ldots,k\}$ of G, we define, for every vertex $v \in V(G)$ and color $i = 1,\ldots,k$, $d^i_{w,c}(v) = \sum_{\{u \in N(v) | c(u)=i\}}(u,v)$. Note that $d^{c(v)}_{w,c}(v) = I_v(G,w,c)$. We say that a k-coloring c of G is *w-balanced* if c satisfies the following property:

$$\text{For any vertex } v \in V(G),\ \ I_v(G,w,c) \le d^j_{w,c}(v),\ \text{ for every } j = 1,\ldots,k.$$

We denote by $\gcd(w)$ the greatest common divisor of the weights of w (observe that $\gcd(w) > 0$ because we just consider positive weights). We use here the generalization of the gcd to non-integer numbers (e.g. in $\mathbb{Q}$) where a number x is said to divide a number y if the fraction y/x is an integer. The important property of $\gcd(w)$ is that the difference between two interferences is a multiple of $\gcd(w)$; in particular, if for two vertices v and u, $d^i_{w,c}(v) > d^j_{w,c}(u)$, then $d^i_{w,c}(v) \geq d^j_{w,c}(u) + \gcd(w)$.

If t is not a multiple of the $\gcd(w)$, that is, there exists an integer $a \in \mathbb{Z}$ such that $a \gcd(w) < t < (a+1)\gcd(w)$, then $\chi^w_t(G) = \chi^w_{a \gcd(w)}(G)$.

Proposition 5. *Let (G, w) be an edge-weighted graph. For any $k \geq 2$, there exists a w-balanced k-coloring of G.*

Proof. Let us color $G = (V, E)$ arbitrarily with k colors and then repeat the following procedure: if there exists a vertex v colored i and a color j such that $d^i_{w,c}(v) > d^j_{w,c}(v)$, then recolor v with color j. Observe that this procedure neither increases (we just move a vertex from one color to another) nor decreases (a vertex without neighbor on its color is never moved) the number of colors within this process. Let W be the sum of the weights of the edges having the same color in their end-vertices. In this transformation, W has increased by $d^j_{w,c}(v)$ (edges incident to v that previously had color j in its endpoint opposite to v), but decreased by $d^i_{w,c}(v)$ (edges that previously had color i in both of their end-vertices). So, W has decreased by $d^i_{w,c}(v) - d^j_{w,c}(v) \geq \gcd(w)$. As $W \leq |E| \max_{e \in E} w(e)$ is finite, this procedure finishes and produces a w-balanced k-coloring of G. □

The existence of a w-balanced coloring gives easily some upper bounds on the weighted t-improper chromatic number and the minimum k-threshold of an edge-weighted graph (G, w). It is a folklore result that $\chi(G) \leq \Delta(G) + 1$, for any graph G. Lovász [Lov66] extended this result for Improper Coloring problem using w-balanced coloring. He proved that $\chi_l(G) \leq \lceil \frac{\Delta(G)+1}{l+1} \rceil$. In what follows, we extend this result to weighted improper coloring.

Theorem 4. *Let (G, w) be an edge-weighted graph with $w : E(G) \to \mathbb{Q}^*_+$, and t a multiple of* $\gcd(w)$*. Then*

$$\chi_t(G, w) \leq \left\lceil \frac{\Delta(G, w) + \gcd(w)}{t + \gcd(w)} \right\rceil.$$

Proof. If t, ω, and G are such that $\chi_t(G, \omega) = 1$, then the inequality is trivially satisfied. Thus, consider that $\chi_t(G, \omega) > 1$.

Observe that, in any w-balanced k-coloring c of a graph G, the following holds:

$$d_w(v) = \sum_{u \in N(v)} w(u, v) \geq k d^{c(v)}_{w,c}(v). \tag{4.1}$$

Let $k^* = \left\lceil \frac{\Delta(G,w)+\gcd(w)}{t+\gcd(w)} \right\rceil \geq 2$ and c^* be a w-balanced k^*-coloring of G. We claim that c^* is a weighted t-improper k^*-coloring of (G, w).

By contradiction, suppose that there is a vertex v in G such that $c^*(v) = i$ and that $d^i_{w,c}(v) > t$. Since c^* is w-balanced, $d^j_{w,c}(v) > t$, for all $j = 1, \ldots, k^*$. By the definition of $\gcd(w)$ and as t is a multiple of $\gcd(w)$, it leads to $d^j_{w,c}(v) \geq t + \gcd(w)$ for all $j = 1, \ldots, k^*$. Combining this inequality with Inequality (4.1), we obtain:

$$\Delta(G, w) \geq d_w(v) \geq k^*(t + \gcd(w)),$$

giving

$$\Delta(G, w) \geq \Delta(G, w) + \gcd(w),$$

a contradiction. The result follows.

□

Note that when all weights are unit, we obtain the bound for the improper coloring derived in [Lov66]. Brooks [Bro41] proved that for a connected graph G, $\chi(G) = \Delta(G) + 1$ if, and only if, G is complete or an odd cycle. One could wonder for which edge-weighted graphs the bound we provided in Theorem 4 is tight. However, Correa *et al.* [CHS09] already showed that it is NP-complete to determine if the improper chromatic number of a graph G attains the upper bound of Lovász, which is a particular case of Weighted Improper colouring, i.e. of the bound of Theorem 4.

We now show that w-balanced colorings also yield upper bounds for the minimum k-threshold of an edge-weighted graph (G, w). When $k = 1$, then all the vertices must have the same color, and $T_1(G, w) = \Delta(G, w)$. This may be generalized as follows, using w-balanced colorings.

Theorem 5. *Let (G, w) be an edge-weighted graph with $w : E(G) \to \mathbb{R}^*_+$, and let k be a positive integer. Then*

$$T_k(G, w) \leq \frac{\Delta(G, w)}{k}.$$

Proof. Let c be a w-balanced k-coloring of G. Then, for every vertex $v \in V(G)$:

$$kT_k(G, w) \leq kd^{c(v)}_{w,c}(v) \leq d_w(v) = \sum_{u \in N(v)} w(u, v) \leq \Delta(G, w)$$

□

Because $T_1(G, w) = \Delta(G, w)$, Theorem 5 may be restated as $kT_k(G, w) \leq \ldots \leq T_1(G, w)$. This inequality may be generalized as follows.

Theorem 6. *Let (G, w) be an edge-weighted graph with $w : E(G) \to \mathbb{R}_+$, and let k and p be two positive integers. Then*

$$T_{kp}(G, w) \leq \frac{T_p(G, w)}{k}.$$

Proof. Set $t = T_p(G, w)$. Let c be a t-improper p-coloring of (G, w). For $i = 1, \ldots, p$, let G_i be the subgraph of G induced by the vertices colored i by c. By definition of improper coloring $\Delta(G_i, w) \leq t$ for all $1 \leq i \leq p$. By Theorem 5, each (G_i, w) admits a t/k-improper k-coloring c_i with colors $\{(i-1)k + 1, \ldots, ik\}$. The union of the c_i's is then a t/k-improper kp-coloring of (G, w). □

Theorem 6 and its proof suggest that to find a kp-coloring with small impropriety, it may be convenient to first find a p-coloring with small impropriety and then to refine it. In addition, such a strategy allows to adapt dynamically the refinement. In the above proof, the vertex set of each part G_i is again partitioned into k parts. However, sometimes, we shall get a better kp-coloring by partitioning each G_i into a number of k_i parts, with $\sum_{i=1}^{p} k_i = kp$. Doing so, we obtain a T-improper kp-coloring of (G, w), where $T = \max\{\frac{\Delta(G_i,w)}{k_i}, 1 \leq i \leq p\}$.

One can also find an upper bound on the minimum k-threshold by considering first the $k-1$ edges of largest weight around each vertex. Let (G, w) be an edge-weighted graph, and let $v_1, \dots, v_n$ be an ordering of the vertices of G. The edges of G may be ordered in increasing order of their weight. Furthermore, to make sure that the edges incident to any particular vertex are totally ordered, we break ties according to the label of the second vertex. Formally, we say that $v_iv_j \leq_w v_iv_{j'}$ if either $w(v_iv_j) < w(v_iv_{j'})$ or $w(v_iv_j) = w(v_iv_{j'})$ and $j < j'$. With such a partial order on the edge set, the set $E_w^k(v)$ of $\min\{|N(v)|, k-1\}$ greatest edges (according to this ordering) around a vertex is uniquely defined. Observe that every edge incident to v and not in $E_w^k(v)$ is smaller than an edge of $E_k(v)$ for $\leq_w$.

Let G_w^k be the graph with vertex set $V(G)$ and edge set $\bigcup_{v\in V(G)} E_w^k(v)$. Observe that every vertex of $E_w^k(v)$ has degree at least $\min\{|N(v)|, k-1\}$, but a vertex may have an arbitrarily large degree. For if any edge incident to v has a greater weight than any edge not incident to v, the degree of v in G_w^k is equal to its degree in G. However we now prove that at least one vertex has degree $k-1$.

Proposition 6. *If (G, w) is an edge-weighted graph, then G_w^k has a vertex of degree at most $k-1$.*

Proof. Suppose for a contradiction, that every vertex has degree at least k, then for every vertex x there is an edge xy in $E(G_w^k) \setminus E_w^k(x)$, and so in $E_w^k(y) \setminus E_w^k(x)$. Therefore, there must be a cycle $(x_1, \dots, x_r)$ such that, for all $1 \leq i \leq r$, $x_ix_{i+1} \in E_w^k(x_{i+1}) \setminus E_w^k(x_i)$ (with $x_{r+1} = x_1$). It follows that $x_1x_2 \leq_w x_2x_3 \leq_w \cdots \leq_w x_rx_1 \leq_w x_1x_2$. Hence, by definition, $w(x_1x_2) = w(x_2x_3) = \cdots = w(x_rx_1) = w(x_1x_2)$. Let m be the integer such that x_m has maximum index in the ordering $v_1, \dots, v_n$. Then there exists j and j' such that $x_m = v_j$ and $x_{m+2} = v_{j'}$. By definition of m, we have $j > j'$. But this contradicts the fact that $x_mx_{m+1} \leq_w x_{m+1}x_{m+2}$. □

Corollary 3. *If (G, w) is an edge-weighted graph, then G_w^k has a proper k-coloring.*

Proof. By induction on the number of vertices. By Proposition 6, G_w^k has a vertex x of degree at most $k-1$. Trivially, $G_w^k - x$ is a subgraph of $(G-x)_w^k$. By the induction hypothesis, $(G-x)_w^k$ has a proper k-coloring, which is also a proper k-coloring of $G_w^k - x$. This coloring can be extended in a proper k-coloring of G_w^k, by assigning to x a color not assigned to any of its $k-1$ neighbors. □

Corollary 4. *If (G, w) is an edge-weighted graph, then $T_k(G, w) \leq \Delta(G \setminus E(G_w^k), w)$.*

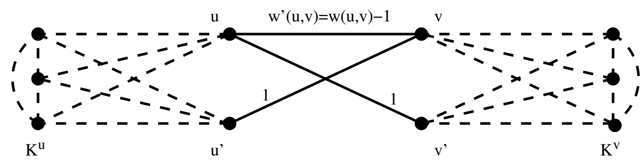


Figure 4.1: Construction of G' from G using edge $uv \in E(G)$ and $k = 4$ colors. Dashed edges represent edges of weight M.

4.1.2 Transformation

In this section, we prove that the Threshold Improper Coloring problem can be transformed into a problem mixing proper and improper coloring. More precisely, we prove the following:

Theorem 7. *Let (G, w) be an edge-weighted graph where w is an integer-valued function, and let k be a positive integer. We can construct an edge-weighted graph (G^*, w^*) such that $w^*(e) \in \{1, M\}$ for any $e \in E(G^*)$, satisfying $T_k(G, w) = T_k(G^*, w^*)$, where $M = 1 + \sum_{e\in E(G)} w(e)$.*

Proof. Consider the function $f(G, w) = \sum_{\{e\in E(G)|w(e)\neq M\}}(w(e) - 1)$.

If $f(G, w) = 0$, all edges have weight either one or M and G has the desired property. In this case, $G^* = G$. Otherwise, we construct a graph G' and a function w' such that $T_k(G', w') = T_k(G, w)$, but $f(G', w') = f(G, w) - 1$. By repeating this operation $f(G, w)$ times we get the required edge-weighted graph (G^*, w^*).

In case $f(G, w) > 0$, there exists an edge $e = uv \in E(G)$ such that $2 \leq w(e) < M$. G' is obtained from G by adding two complete graphs on $k - 1$ vertices K^u and K^v and two new vertices u' and v'. We join u and u' to all the vertices of K^u and v and v' to all the vertices of K^v. We assign weight M to all these edges. Note that, u and u' (v and v') always have the same color, namely the remaining color not used in K^u (resp. K^v).

We also add two edges uv' and $u'v$ both of weight 1. The edges of G keep their weight in G', except the edge $e = uv$ whose weight is decreased by one unit, i.e. $w'(e) = w(e) - 1$. Thus, $f(G', \omega') = f(G, \omega) - 1$ as we added only edges of weights 1 and M and we decreased the weight of e by one unit.

Now consider a weighted t-improper k-coloring c of (G, w). We produce a weighted t-improper k-coloring c' of (G', w') as follows: we keep the colors of all the vertices in G, we assign to u' (v') the same color as u (resp. v), and we assign to K^u (resp. K^v) the $k - 1$ colors different from the one used in u (resp. v).

Conversely, from any weighted improper k-coloring c' of (G', w'), we get a weighted improper k-coloring c of (G, w) by just keeping the colors of the vertices that belong to G.

For such colorings c and c' we have that $I_x(G, w, c) = I_x(G', w', c')$, for any vertex x of G different from u and v. For $x \in K^u \cup K^v$, $I_x(G', w', c') = 0$. The neighbors of u with the same color as u in G' are the same as in G, except possibly v' which has the same color of u if, and only if, v has the same color of u.

Let $\varepsilon = 1$ if v has the same color as u, otherwise $\varepsilon = 0$. As the weight of uv decreases by one and we add the edge uv' of weight 1 in G', we get $I_u(G', w', c') = I_u(G, w, c) - \varepsilon + w'(u, v')\varepsilon = I_u(G, w, c)$. Similarly, $I_v(G', w', c') = I_v(G, w, c)$. Finally, $I_{u'}(G', w', c') = I_{v'}(G', w', c') = \varepsilon$. But $I_u(G', w', c') \geq (w(u, v) - 1)\varepsilon$ and so $I_{u'}(G', w', c') \leq I_u(G', w', c')$ and $I_{v'}(G', w', c') \leq I_v(G', w', c')$. In summary, we have

$$\max_x I_x(G', w', c') = \max_x I_x(G, w, c)$$

and therefore $T_k(G, w) = T_k(G', w')$. □

In the worst case, the number of vertices of G^* is $n + m(w_{max} - 1)2k$ and the number of edges of G^* is $m + m(w_{max} - 1)[(k + 4)(k - 1) + 2]$ with $n = |V(G)|$, $m = |E(G)|$ and $w_{max} = \max_{e \in E(G)} w(e)$.

In conclusion, this construction allows to transform the Threshold Improper Coloring problem into a problem mixing proper and improper coloring. Therefore the problem consists in finding the minimum l such that a (non-weighted) l-improper k-coloring of G^* exists with the constraint that some subgraphs of G^* must admit a proper coloring. The equivalence of the two problems is proved here only for integers weights, but it is possible to adapt the transformation to prove it for rational weights.

4.2 Squares of Particular Graphs

As mentioned in the introduction, Weighted Improper colouring is motivated by networks of antennas similar to grids [AAG+05]. In these networks, the noise generated by an antenna undergoes an attenuation with the distance it travels. It is often modeled by a decreasing function of d, typically $1/d^\alpha$ or $1/(2^{d-1})$.

Here we consider a simplified model where the noise between two neighboring antennas is normalized to 1, between antennas at distance two is $1/2$ and 0 when the distance is strictly greater than two. Studying this model of interference corresponds to study the Weighted Improper colouring of the square of the graph G, that is the graph G^2 obtained from G by joining every pair of vertices at distance two, and to assign weights $w_2(e) = 1$, if $e \in E(G)$, and $w_2(e) = 1/2$, if $e \in E(G^2) \setminus E(G)$. Observe that in this case the interesting threshold values are the non-negative multiples of $1/2$.

Figure 4.2 shows some examples of coloring for the square grid. In Figure 4.2(b), each vertex x has neither a neighbor nor a vertex at distance two colored with its own color, so $I_x(G^2, w_2, c) = 0$ and G^2 admits a weighted 0-improper 5-coloring. In Figure 4.2(c), each vertex x has no neighbor with its color and at most one vertex of the same color at distance 2. So $I_x(G^2, w_2, c) = 1/2$ and G^2 admits a weighted 0.5-improper 4-coloring.

For any $t \in \mathbb{R}_+$, we determine the weighted t-improper chromatic number for the square of infinite paths, square grids, hexagonal grids and triangular grids under the interference model w_2. We also present lower and upper bounds for $\chi_t(T^2, w_2)$, for any tree T and any threshold t.

4.2.1 Infinite paths and trees

In this section, we characterize the weighted t-improper chromatic number of the square of an infinite path, for all positive real t. Moreover, we present lower and upper bounds for $\chi_t(T^2, w_2)$, for a given tree T.

Theorem 8. *Let $P = (V, E)$ be an infinite path. Then,*

$$\chi_t(P^2, w_2) = \begin{cases} 3, & \text{if } 0 \le t < 1; \\ 2, & \text{if } 1 \le t < 3; \\ 1, & \text{if } 3 \le t. \end{cases}$$

Proof. Let $V = \{v_i \mid i \in \mathbb{Z}\}$ and $E = \{(v_{i-1}, v_i) \mid i \in \mathbb{Z}\}$. Each vertex of P has two neighbors and two vertices at distance two. Consequently, the equivalence $\chi_t(P^2, \omega_2) = 1$ if, and only if, $t \ge 3$ holds trivially.

There is a 2-coloring c of (P^2, w_2) with maximum interference 1 by just coloring v_i with color $(i \bmod 2) + 1$. So $\chi_t(P^2, w_2) \le 2$ if $t \ge 1$. We claim that there is no weighted 0.5-improper 2-coloring of (P^2, w_2). By contradiction, suppose that c is such a coloring. If $c(v_i) = 1$, for some $i \in \mathbb{Z}$, then $c(v_{i-1}) = c(v_{i+1}) = 2$ and $c(v_{i-2}) = c(v_{i+2}) = 1$. This is a contradiction because v_i would have interference 1.

Finally, the coloring $c(v_i) = (i \bmod 3) + 1$, for every $i \in \mathbb{Z}$, is a feasible weighted 0-improper 3-coloring. □

Theorem 9. *Let $T = (V, E)$ be a (non-empty) tree. Then,* $\left\lceil \frac{\Delta(T) - \lfloor t \rfloor}{2t+1} \right\rceil + 1 \le \chi_t(T^2, w_2) \le \left\lceil \frac{\Delta(T) - 1}{2t+1} \right\rceil + 2.$

Proof. The lower bound is obtained by two simple observations. First, $\chi_t(H, w) \le \chi_t(G, w)$, for any $H \subseteq G$. Let T be a tree and v be a node of maximum degree in T. Then, observe that the weighted t-improper chromatic number of the subgraph of T^2 induced by v and its neighborhood is at least $\lceil \frac{\Delta(T) - \lfloor t \rfloor}{2t+1} \rceil + 1$. Indeed, the color of v can be assigned to at most $\lfloor t \rfloor$ vertices on its neighborhood. Any other color used in the neighborhood of v cannot appear in more than $2t + 1$ vertices because each pair of vertices in the neighborhood of v is at distance two.

Let us look now at the upper bound. Choose any node $r \in V$ to be the root of T. Color r with color 1. Then, by a breadth-first traversal in the tree, for each visited node v color all the children of v with the $\lceil \frac{\Delta(T)-1}{2t+1} \rceil$ colors different from the ones assigned to v and to its parent in such a way that at most $2t + 1$ nodes have the same color. This is a feasible weighted t-improper k-coloring of T^2, with $k \le \lceil \frac{\Delta(T)-1}{2t+1} \rceil + 2$, since each vertex interferes with at most $2t$ vertices at distance two which are children of its parent. □

For a tree T and the weighted function w^2, Theorem 9 provides upper and lower bounds on $\chi_t(T^2, w_2)$, but we do not know the computational complexity of determining $\chi_t(T^2, w_2)$.

4.2.2 Grids

In this section, we show the optimal values of $\chi_t(G^2, w_2)$, whenever G is an infinite square, hexagonal or triangular grid, for all the possible values of t.

4.2.2.1 Square Grid

The square grid is the graph $\mathfrak{S}$ in which the vertices are all integer linear combinations $ae_1 + be_2$ of the two vectors $e_1 = (1, 0)$ and $e_2 = (0, 1)$, for any $a, b \in \mathbb{Z}$. Each vertex (a, b) has four neighbors: its *down neighbor* $(a, b-1)$, its *up neighbor* $(a, b+1)$, its *right neighbor* $(a+1, b)$ and its *left neighbor* $(a-1, b)$ (see Figure 4.2(a)).

Theorem 10.

$$\chi_t(\mathfrak{S}^2, w_2) = \begin{cases} 5, & \textit{if } t = 0; \\ 4, & \textit{if } t = 0.5; \\ 3, & \textit{if } 1 \leq t < 3; \\ 2, & \textit{if } 3 \leq t < 8; \\ 1, & \textit{if } 8 \leq t. \end{cases}$$

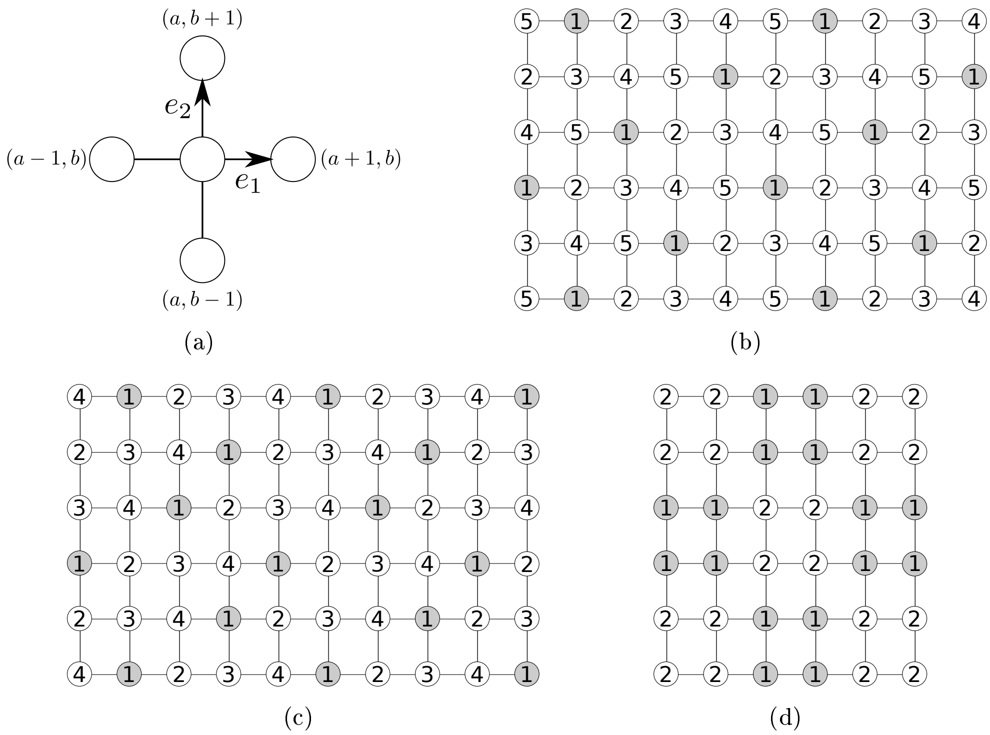


Figure 4.2: Optimal colorings of $(\mathfrak{S}^2, w_2)$: (b) weighted 0-improper 5-coloring of $(\mathfrak{S}^2, w_2)$, (c) weighted 0.5-improper 4-coloring of $(\mathfrak{S}^2, w_2)$, and (d) weighted 3-improper 2-coloring of $(\mathfrak{S}^2, w_2)$.

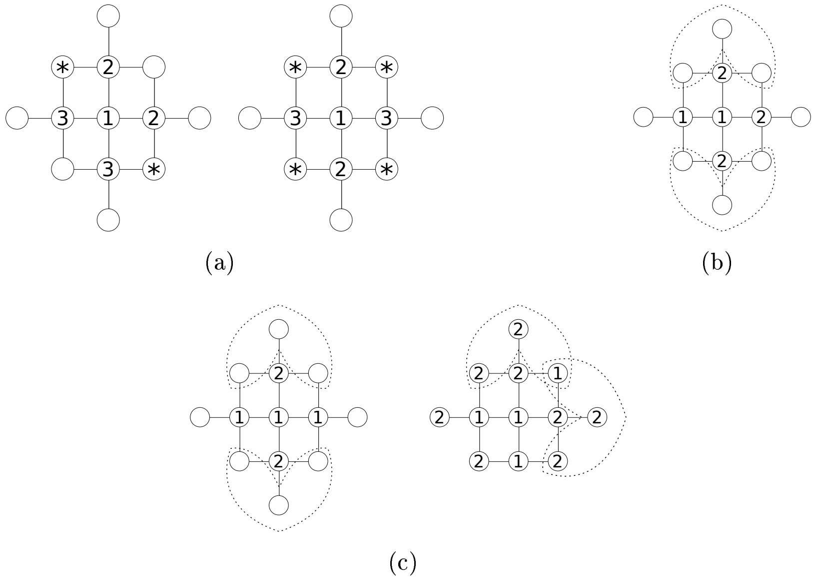


Figure 4.3: Lower bounds for the square grid: (a) if $t \leq 0.5$ and $k \leq 3$, there is no weighted t-improper k-coloring of $(\mathfrak{G}^2, w_2)$; (b) the first case when $t \leq 2.5$ and $k \leq 2$, and (c) the second case.

Proof. If $t = 0$, then the color of vertex (a, b) must be different from the ones used on its four neighbors. Moreover, all the neighbors have different colors, as each pair of neighbors is at distance two. Consequently, at least five colors are needed. The following construction provides a weighted 0-improper 5-coloring of $(\mathfrak{G}^2, w_2)$: for $0 \leq j \leq 4$, let $A_j = \{(j, 0) + a(5e_1) + b(2e_1 + 1e_2) \mid \forall a, b \in \mathbb{Z}\}$. For $0 \leq j \leq 4$, assign the color $j + 1$ to all the vertices in A_j (see Figure 4.2(b)).

When $t = 0.5$, we claim that at least four colors are needed to color $(\mathfrak{G}^2, w_2)$. The proof is by contradiction. Suppose that there exists a weighted 0.5-improper 3-coloring of it. Let (a, b) be a vertex colored 1. None of its neighbors is colored 1, otherwise (a, b) has interference 1. If three neighbors have the same color, then each of them will have interference 1. So two of its neighbors have to be colored 2 and the two other ones 3 (see Figure 4.3(a)). Now consider the four nodes $(a - 1, b - 1)$, $(a - 1, b + 1)$, $(a + 1, b - 1)$ and $(a + 1, b + 1)$. For all configurations, at least two of these four vertices have to be colored 1 (the ones indicated by a * in Figure 4.3(a)). But then (a, b) will have interference at least 1, a contradiction. A weighted 0.5-improper 4-coloring of $(\mathfrak{G}^2, w_2)$ can be obtained as follows (see Figure 4.2(c)): for $0 \leq j \leq 3$, let $B_j = \{(j, 0) + a(4e_1) + b(3e_1 + 2e_2) \mid \forall a, b \in \mathbb{Z}\}$ and $B'_j = \{(j + 1, 2) + a(4e_1) + b(3e_1 + 2e_2) \mid \forall a, b \in \mathbb{Z}\}$. For $0 \leq j \leq 3$, assign the color $j + 1$ to all the vertices in B_j and in B'_j.

If $t = 1$, there exists a weighted 1-improper 3-coloring of $(\mathfrak{G}^2, w_2)$ given by the following construction: for $0 \leq j \leq 2$, let $C_j = \{(j, 0) + a(3e_1) + b(e_1 + e_2) \mid \forall a, b \in \mathbb{Z}\}$. For $0 \leq j \leq 2$, assign the color $j + 1$ to all the vertices in C_j.

Now we prove by contradiction that for $t = 2.5$ we still need at least three colors in a weighted 2.5-improper coloring of $(\mathfrak{G}^2, w_2)$. Consider a weighted 2.5-improper 2-coloring of $(\mathfrak{G}^2, w_2)$ and let (a,b) be a vertex colored 1. Vertex (a,b) has at most two neighbors of color 1, otherwise it will have interference 3. We distinguish three cases:

1. Exactly one of its neighbors is colored 1; let $(a-1,b)$ be this vertex. Then, the three other neighbors are colored 2 (see Figure 4.3(b)). Consider the two sets of vertices $\{(a-1,b-1),(a+1,b-1),(a,b-2)\}$ and $\{(a-1,b+1),(a+1,b+1),(a,b+2)\}$ (these sets are surrounded by dotted lines in Figure 4.3(b)); each of them has at least two vertices colored 1, otherwise the vertex $(a,b-1)$ or $(a,b+1)$ will have interference 3. But then (a,b) having four vertices at distance two colored 1 has interference 3, a contradiction.

2. Two neighbors of (a,b) are colored 1.

(a) These two neighbors are opposite, say $(a-1,b)$ and $(a+1,b)$ (see Figure 4.3(c) left). Consider again the two sets $\{(a-1,b-1),(a+1,b-1),(a,b-2)\}$ and $\{(a-1,b+1),(a+1,b+1),(a,b+2)\}$ (these sets are surrounded by dotted lines in Figure 4.3(c) left); they both contain at least one vertex of color 1 and therefore (a,b) will have interference 3, a contradiction.

(b) The two neighbors of color 1 are of the form $(a,b-1)$ and $(a-1,b)$ (see Figure 4.3(c) right). Consider the two sets of vertices $\{(a+1,b-1),(a+1,b+1),(a+2,b)\}$ and $\{(a+1,b+1),(a-1,b+1),(a,b+2)\}$ (these sets are surrounded by dotted lines in Figure 4.3(c) right); these two sets contain at most one vertex of color 1, otherwise (a,b) will have interference 3. Moreover, each of these sets cannot be completely colored 2, otherwise $(a+1,b)$ or $(a,b+1)$ will have interference at least 3. So vertices $(a+1,b-1)$, $(a+2,b)$, $(a,b+2)$ and $(a-1,b+1)$ are of color 2 and the vertex $(a+1,b+1)$ is of color 1. But then $(a-2,b)$ and $(a-1,b-1)$ are of color 2, otherwise (a,b) will have interference 3. Thus, vertex $(a-1,b)$ has exactly one neighbor colored 1 and we are again in Case 1.

3. All neighbors of (a,b) are colored 2. If one of these neighbors has itself a neighbor (distinct from (a,b)) of color 2, we are in Case 1 or 2 for this neighbor. Therefore, all vertices at distance two from (a,b) have color 1 and the interference in (a,b) is 4, a contradiction.

A weighted 3-improper 2-coloring of $(\mathfrak{G}^2, w_2)$ can be obtained as follows: a vertex of the grid (a,b) is colored with color $(\lfloor \frac{a}{2} \rfloor + \lfloor \frac{b}{2} \rfloor \pmod 2) + 1$, see Figure 4.2(d).

Finally, since each vertex has four neighbors and eight vertices at distance two, there is no weighted 7.5-improper 1-coloring of $(\mathfrak{G}^2, w_2)$ and, whenever $t \geq 8$, one color suffices. □

4.2.2.2 Hexagonal Grid

There are many ways to define the system of coordinates of the hexagonal grid. Here, we use grid coordinates as shown in Figure 4.4. The hexagonal grid graph is then the graph $\mathfrak{H}$ whose vertex set consists of pairs of integers $(a, b) \in \mathbb{Z}^2$ and where each vertex (a, b) has three neighbors: $(a-1, b)$, $(a+1, b)$, and $(a, b+1)$ if $a+b$ is odd, or $(a, b-1)$ otherwise.

Theorem 11.

$$\chi_t(\mathfrak{H}^2, w_2) = \begin{cases} 4, & \textit{if } 0 \leq t < 1; \\ 3, & \textit{if } 1 \leq t < 2; \\ 2, & \textit{if } 2 \leq t < 6; \\ 1, & \textit{if } 6 \leq t. \end{cases}$$

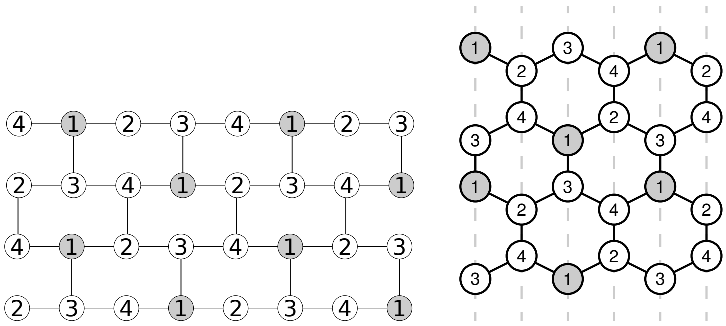
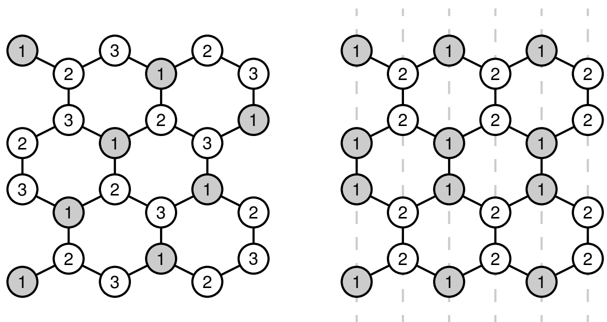


Figure 4.4: Weighted 0-improper 4-coloring of $(\mathfrak{H}^2, w_2)$. Left: Graph with coordinates. Right: Corresponding hexagonal grid in the euclidean space.



(a) $t = 1$, $k = 3$ (b) $t = 2$, $k = 2$

Figure 4.5: (a) weighted 1-improper 3-coloring of $(\mathfrak{H}^2, w_2)$ and (b) weighted 2-improper 2-coloring of $(\mathfrak{H}^2, w_2)$.

Proof. Note first, that when $t = 0$, at least four colors are needed to color the grid, because a vertex and its neighborhood in $\mathfrak{H}$ form a clique of size four in $\mathfrak{H}^2$. The same number of colors are needed if we allow a threshold $t = 0.5$. To prove this fact, let A be a vertex (a, b) of $\mathfrak{H}$ and $B = (a-1, b)$, $C = (a, b-1)$ and $D = (a+1, b)$ be its neighbors in $\mathfrak{H}$. Denote by $G = (a-2, b)$, $E = (a-1, b-1)$, $F = (a-2, b-1)$, $H = (a+1, b-1)$, $I = (a+2, b-1)$ and $J = (a+1, b-2)$ (see Figure 4.6(a)). By contradiction, suppose there exists a weighted 0.5-improper 3-coloring of $\mathfrak{H}^2$. Consider a node A colored 1. Its neighbors B, C, D cannot be colored 1 and they cannot all have the same color. W.l.o.g., suppose that two of them B and C have color 2 and D has color 3. Then E, F and G cannot be colored 2 because of the interference constraint in B and C. If F is colored 3, then G and E are colored 1, creating interference 1 in A. So F must be colored 1 and G and E must be colored 3. Then, H can be neither colored 2 (interference in C) nor 3 (interference in E). So H is colored 1. The vertex I is colored 3, otherwise the interference constraint in H or in C is not satisfied. Then, J can receive neither color 1, because of the interference in H, nor color 2, because of the interference in C, nor color 3, because of the interference in I.

There exists a construction attaining this bound and the number of colors, i.e. a 0-improper 4-coloring of $(\mathfrak{H}^2, w_2)$ as depicted in Figure 4.4. We define for $0 \leq j \leq 3$ the sets of vertices $A_j = \{(j,0) + a(4e_1) + b(2e_1 + e_2) | \forall a, b \in \mathbb{Z}\}$. We then assign the color $j+1$ to the vertices in A_j. This way no vertex experiences any interference as vertices of the same colors are at distance at least three.

For $t = 1.5$ it is not possible to color the grid with less than three colors. By contradiction, suppose that there exists a weighted 1.5-improper 2-coloring. Consider a vertex A colored 1. If all of its neighbors are colored 2, they have already interference 1, so all the vertices at distance two from A need to be colored 1; this gives interference 3 in A. Therefore one of A's neighbors, say D, has to be colored 1 and consider that the other two neighbors B and C are colored 2. B and C have at most one neighbor of color 2. It implies that A has at least two vertices at distance two colored 1. This is a contradiction, because the interference in A would be at least 2 (see Figure 4.6(b)).

Figure 4.5(a) presents a weighted 1-improper 3-coloring of $(\mathfrak{H}^2, w_2)$. To obtain this coloring, let $B_j = \{(j,0) + a(3e_1) + b(e_1 + e_2) \mid \forall a, b \in \mathbb{Z}\}$, for $0 \leq j \leq 2$. Then, we color all the vertices in the set B_j with color $j+1$, for every $0 \leq j \leq 2$.

For $t < 6$, it is not possible to color the grid with one color. As a matter of fact, each vertex has three neighbors and six vertices at distance two in $\mathfrak{H}$. Using one color leads to an interference equal to 6. There exists a 2-improper 2-coloring of the hexagonal grid as depicted in Figure 4.5(b). We define for $0 \leq j \leq 1$ the sets of vertices $C_j = \{(j,0) + a(2e_1) + be_2 | \forall a, b \in \mathbb{Z}\}$. We then assign the color $j+1$ to the vertices in C_j.

□

4.2.2.3 Triangular Grid

The triangular grid is the graph $\mathfrak{T}$ whose vertices are all the integer linear combinations $af_1 + bf_2$ of the two vectors $f_1 = (1,0)$ and $f_2 = (\frac{1}{2}, \frac{\sqrt{3}}{2})$. Thus we may identify the vertices with the ordered pairs (a,b) of integers. Each vertex $v = (a,b)$ has six neighbors: its *right neighbor* $(a+1,b)$, its *right-up neighbor* $(a,b+1)$, its *left-up neighbor* $(a-1,b+1)$, its *left neighbor* $(a-1,b)$, its *left-down neighbor* $(a,b-1)$ and its *right-down neighbor* $(a+1,b-1)$ (see Figure 4.8(a)).

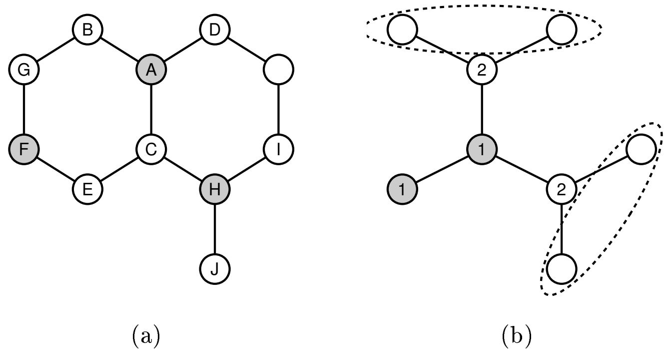


Figure 4.6: Lower bounds for the hexagonal grid. (a) when $t \leq 0.5$ and $k \leq 3$, there is no weighted t-improper k-coloring of $(\mathfrak{H}^2, w_2)$; (b) vertices colored 2 force a vertex colored 1 in each ellipse, leading to interference 2 in central node.

Theorem 12.

$$\chi_t(\mathfrak{T}^2, w_2) = \begin{cases} 7, & \text{if } t = 0; \\ 6, & \text{if } t = 0.5; \\ 5, & \text{if } t = 1; \\ 4, & \text{if } 1.5 \leq t < 3; \\ 3, & \text{if } 3 \leq t < 5; \\ 2, & \text{if } 5 \leq t < 12; \\ 1, & \text{if } 12 \leq t. \end{cases}$$

Proof. If $t = 0$, there is no weighted 0-improper 6-coloring of $(\mathfrak{T}^2, w_2)$, since in $\mathfrak{T}^2$ there is a clique of size seven induced by each vertex and its neighborhood. There is a weighted 0-improper 7-coloring of $(\mathfrak{T}^2, w_2)$ as depicted in Figure 4.7(a). This coloring can be obtained by the following construction: for $0 \leq j \leq 6$, let $A_j = \{(j, 0) + a(7f_1) + b(2f_1 + f_2) \mid \forall a, b \in \mathbb{Z}\}$. For $0 \leq j \leq 6$, assign the color $j + 1$ to all the vertices in A_j.

In what follows, we denote by V_0 a vertex colored 1; by N_0, N_1, N_2, N_3, N_4, N_5 the six neighbors of V_0 in $\mathfrak{T}$ be in a cyclic order. Let Γ^2 be the set of twelve vertices at distance two of V_0 in $\mathfrak{T}$; more precisely $N_{i(i+1)}$ denotes the vertex of Γ^2 adjacent to both N_i and N_{i+1} and by N_{ii} the vertex of Γ^2 joined only to N_i, for every $0 \leq i \leq 5$, $i + 1$ is taken modulo 6 (see Figure 4.8(b)) and we denote by N_{ijk} the vertex at distance three from V_0 adjacent to both N_{ij} and N_{jk} .

We claim that there is no weighted 0.5-improper 5-coloring of $(\mathfrak{T}^2, w_2)$. We prove it by contradiction, thus let c be such a coloring. No neighbor of V_0 can be colored 1, otherwise $I_{V_0}(\mathfrak{T}^2, w_2, c) \geq 1$. As two consecutive neighbors are adjacent, they cannot have the same color. Furthermore, there cannot be three neighbors with the same color (each of them will have an interference at least 1). As there are four colors different from 1, exactly two of them, say 2 and 3, are repeated twice among the six neighbors. So, there exists a sequence of three consecutive neighbors the first

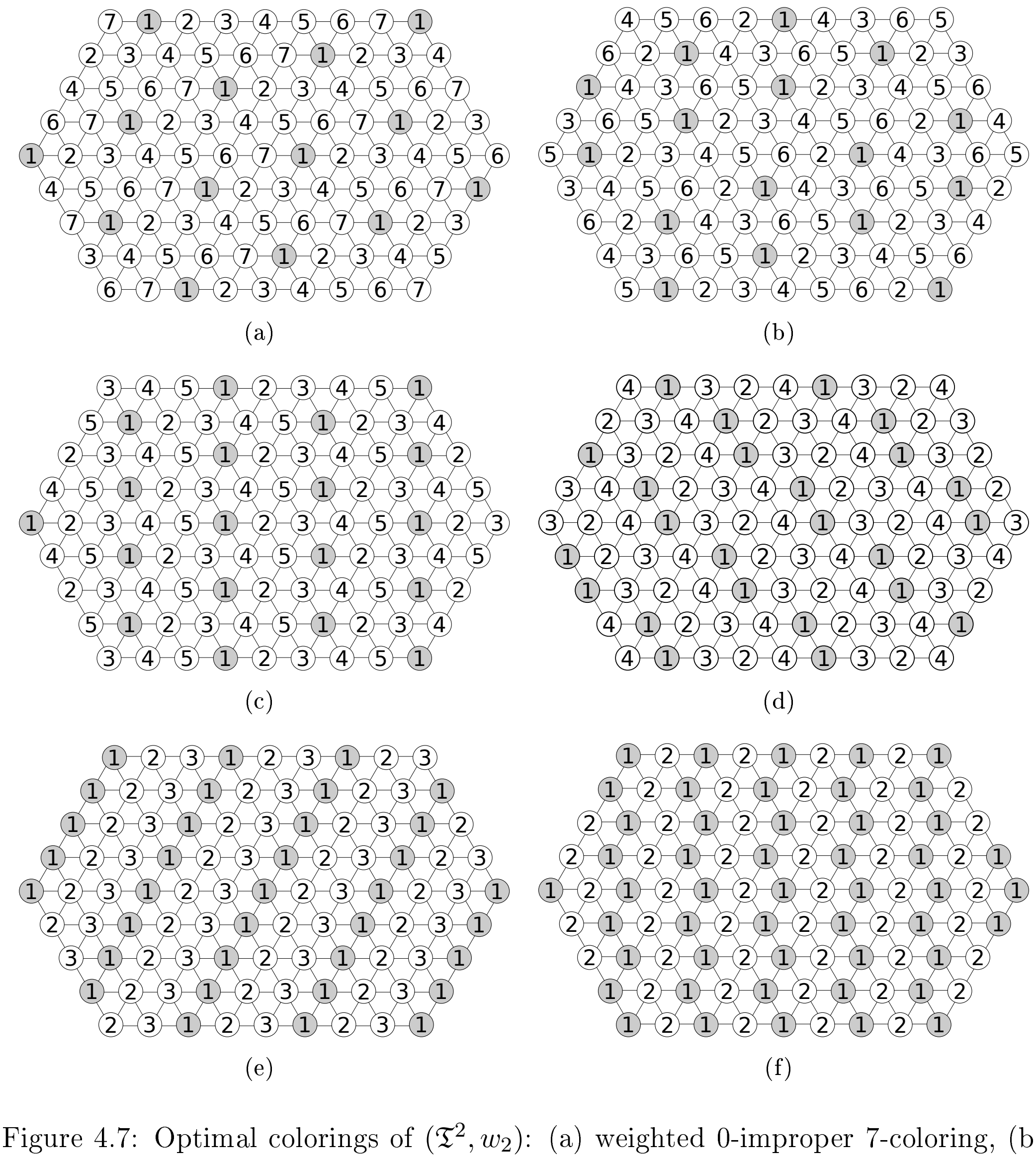
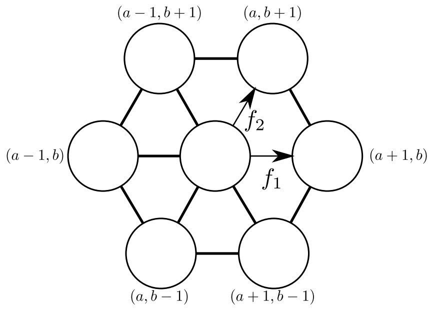
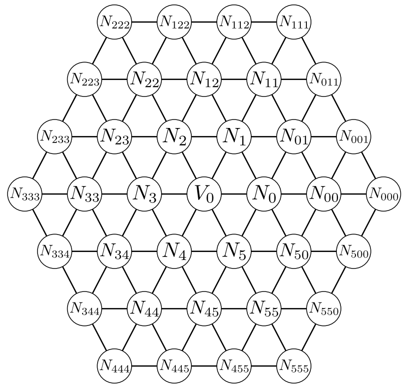


Figure 4.7: Optimal colorings of $(\mathfrak{T}^2, w_2)$: (a) weighted 0-improper 7-coloring, (b) weighted 0.5-improper 6-coloring, (c) weighted 1-improper 5-coloring, (d) weighted 1.5-improper 4-coloring, (e) weighted 3-improper 3-coloring, and (f) weighted 5-improper 2-coloring.



(a)



(b)

Figure 4.8: Notations used in proofs: (a) of existence, and (b) of non-existence; of weighted improper colorings of $(\mathfrak{T}^2, w_2)$.

one with a color different from 2 and 3 and the two others colored 2 and 3. W.l.o.g., let $c(N_5) = 4$, $c(N_0) = 2$, $c(N_1) = 3$.

Note that the vertices colored 2 and 3 have already an interference of 0.5, and so none of their vertices at distance two can be colored 2 or 3. In particular, let $A = \{N_{50}, N_{00}, N_{01}, N_{11}, N_{12}\}$; the vertices of A cannot be colored 2 or 3. At most one vertex in Γ^2 can be colored 1, otherwise $I_{V_0}(\mathfrak{T}^2, w_2, c) \geq 1$. If there is no vertex colored 1 in A, we have a contradiction as we cannot have a sequence of five vertices uniquely colored 4 and 5 (indeed colors should alternate and the vertex in the middle N_{01} will have interference at least 1). Suppose N_4 is colored 3, then N_{45} and N_{55} can only be colored 1 and 5; but, as they have different colors, one is colored 1 and so there is no vertex colored 1 in A. So the second vertex colored 3 in the neighborhood of V_0 is necessarily N_3 (it cannot be N_2 neighbor of N_1 colored 3). Then, N_4 cannot be also colored 5, otherwise N_{45} is colored 1 and again there is no vertex colored 1 in A. In summary $c(N_4) = 2$, $c(N_3) = 3$ and the vertex of Γ^2 colored 1 is in A. But then the five consecutive vertices $A' = \{N_{23}, N_{33}, N_{34}, N_{44}, N_{45}\}$ can only be colored 4 and 5. A contradiction as $I_{N_{34}}(\mathfrak{T}^2, w_2, c) \geq 1$.

A weighted 0.5-improper 6-coloring of $(\mathfrak{T}^2, w_2)$ can be obtained by the following construction (see Figure 4.7(b)): for $0 \leq j \leq 11$, let $B_j = \{(j, 0) + a(12f_1) + b(2f_1 + f_2) \mid \forall a, b \in \mathbb{Z}\}$. For $0 \leq j \leq 5$, assign the color $j + 1$ to all the vertices in B_j, B_6 with color 2, B_7 with color 1, B_8 with color 4, B_9 with color 3, B_{10} with color 6 and B_{11} with color 5.

Now we prove that $(\mathfrak{T}^2, w_2)$ does not admit a weighted 1-improper 4-coloring. Again, by contradiction, suppose that there exists a weighted 1-improper 4-coloring c of $(\mathfrak{T}^2, w_2)$. We analyze some cases:

1. There exist two adjacent vertices in $\mathfrak{T}$ with the same color.

   Let V_0 and one of its neighbors be both colored 1. Note that no other neighbor of V_0, nor the vertices at distance two from V_0 are colored 1 (otherwise, $I_{V_0}(\mathfrak{T}^2, w_2, c) > 1$).

We use intensively the following facts:

Fact 1. *There do not exist three consecutive vertices with the same color (otherwise the vertex in the middle would have interference at least 2).*

Fact 2. *In a path of five vertices there cannot be four of the same color (otherwise the second or the fourth vertex in this path would have interference at least 1.5).*

One color other than 1 should appear at least twice in the neighborhood of V_0. Let this color be denoted 2 (the other colors being denoted 3 and 4).

(a) Two neighbors of V_0 colored 2 are consecutive, say N_0 and N_1. By Fact 1, N_2 is colored 3 w.l.o.g. None of N_{05}, N_{00}, N_{01}, N_{11}, N_{12}, N_{22} and N_{23} can be colored 2, otherwise $I_{N_1}(\mathfrak{T}^2, w_2, c) > 1$. One of N_{12}, N_{22} and N_{23} is colored 3, otherwise we contradict Fact 1 with color 4 and at most one of N_{01}, N_{11}, N_{12}, N_{22} and N_{23} is colored 3, otherwise $I_{N_2}(\mathfrak{T}^2, w_2, c) > 1$; but we have a contradiction with Fact 2.

(b) Two neighbors of V_0 colored 2 are at distance two, say N_0 and N_2. Then N_{50}, N_{00} and N_{01} (respectively N_{12}, N_{22} and N_{23}) are not colored 2, otherwise $I_{N_0}(\mathfrak{T}^2, w_2, c) > 1$ (respectively $I_{N_2}(\mathfrak{T}^2, w_2, c) > 1$). One of N_3 and N_5 is not colored 1, say N_3. It is not colored 2, otherwise $I_{N_3}(\mathfrak{T}^2, w_2, c) > 1$. Let $c(N_3) = 3$. If N_4 or N_{11} is colored 2, then N_{33} and N_{34} are not colored 2, otherwise $I_{N_2}(\mathfrak{T}^2, w_2, c) > 1$ and we have a sequence of five vertices N_{12}, N_{22}, N_{23}, N_{33} and N_{34} contradicting Fact 2 as four are of color 4 (indeed, at most one is colored 3 due to interference in color 3 with N_3 or N_{22}). So N_{11} is colored 3 or 4. If N_1 also is colored 3 or 4, we have a contradiction with Fact 2 applied to the five vertices N_{00}, N_{01}, N_{11}, N_{12} and N_{22}, by the same previous argument. So $c(N_1) = 1$; furthermore N_4 is not colored 1 (at most one neighbor colored 1), nor 2 as we have seen above, nor 3, otherwise we are in the case (a). Therefore $c(N_4) = 4$ and $c(N_5) = 3$, by the same reason. But then $c(N_{23}) = 4$, otherwise the interference in V_0 or N_2 or N_3 is greater than 1. N_{33} and N_{34} can be only colored 2, otherwise V_0, N_3, N_4 or N_{23} will have interference strictly greater than 1, but N_{33} has interference greater than 1, a contradiction.

(c) Two neighbors of V_0 colored 2 are at distance three say N_0 and N_3. Then N_{50}, N_{00} and N_{01} (respectively N_{23}, N_{33} and N_{34}) are not colored 2, otherwise $I_{N_0}(\mathfrak{T}^2, w_2, c) > 1$ (respectively $I_{N_3}(\mathfrak{T}^2, w_2, c) > 1$). W.l.o.g., let N_1 be the vertex colored 1. Among the four vertices N_{12}, N_{22}, N_{44} and N_{45} at most one is colored 2, otherwise $I_{N_3}(\mathfrak{T}^2, w_2, c) > 1$. So, w.l.o.g, we can suppose N_{44} and N_{45} are colored 3 or 4; but we have a set of five consecutive vertices N_{23}, N_{33}, N_{34}, N_{44}, N_{45}, contradicting Fact 2 (indeed at most one can be of the color of N_4).

2. No color appears in two adjacent vertices of $\mathfrak{T}$.

   Let V_0 be colored 1. No color can appear four or more times among the neighbors of V_0, otherwise there are two adjacent neighbors with the same color.

(a) One color appears three times among the neighbors of V_0, say $c(N_0) = c(N_2) = c(N_4) = 2$. W.l.o.g., let $c(N_1) = 3$. No vertex at distance two can be colored 2. N_{01}, N_{11} and N_{12} being neighbors of N_1 cannot be colored 3 and they cannot be all colored 4. So one of N_{01}, N_{11}, N_{12} is colored 1. Similarly one of N_{23}, N_{33}, N_{34} is colored 1 (same reasoning with N_3 instead of N_1) and one of N_{45}, N_{55}, N_{50} is colored 1, so $I_{V_0}(\mathfrak{T}^2, w_2, c) > 1$.

(b) The three colors appear each exactly twice in the neighborhood of V_0.

i. The same color appears in some N_i and N_{i+2}, $0 \leq i \leq 3$. W.l.o.g., let $c(N_0) = c(N_2) = 2$ and $c(N_1) = 3$. Then, $c(N_3) = c(N_5) = 4$ and $c(N_4) = 3$. Then, $c(N_{50}) = 1$ or 3, $c(N_{01}) = 1$ or 4. If $c(N_{50}) = 3$ and $c(N_{01}) = 4$, then $c(N_{00}) = 1$. Among N_{50}, N_{00}, N_{01}, at least one has color 1. Similarly one of N_{12}, N_{22}, N_{23} has color 1. So $I_{V_0}(\mathfrak{T}^2, w_2, c) \geq 1$ and $c(N_{34}) = c(N_{45}) = 2$. Consequently, no matter the color of N_{44} some vertex will have interference greater than 1.

ii. We have $c(N_0) = c(N_3) = 2$, $c(N_1) = c(N_4) = 3$ and $c(N_2) = c(N_5) = 4$. Here we find in each of the sets $\{N_{50}, N_{00}, N_{01}\}$,$\{N_{12}, N_{22}, N_{23}\}$ and $\{N_{34}, N_{44}, N_{45}\}$ a vertex colored 1. Therefore $I_{V_0}(\mathfrak{T}^2, w_2, c) > 1$, a contradiction.

To obtain a weighted 1-improper 5-coloring of $(\mathfrak{T}^2, w_2)$, for $0 \leq j \leq 4$, let $C_j = \{(j, 0) + a(5f_1) + b(2f_1 + f_2) \mid \forall a, b \in \mathbb{Z}\}$. For $0 \leq j \leq 4$, assign the color $j+1$ to all the vertices in C_j. See Figure 4.7(c).

$(\mathfrak{T}^2, w_2)$ has a weighted 1.5-improper 4-coloring as depicted in Figure 4.7(d). Formally, this coloring can be obtained by the following construction: for $0 \leq j \leq 3$, let $D_j = \{(j, 0) + a(4f_1) + b(f_1 + 2f_2) \mid \forall a, b \in \mathbb{Z}\}$; then assign color 4 to all the vertices in D_0, 1 to all the vertices in D_1, 3 to all the vertices in D_2 and 2 to all the vertices in D_3. Now, for $0 \leq j \leq 3$, let $D'_j = \{(j, 1) + a(4f_1) + b(f_1 + 2f_2) \mid \forall a, b \in \mathbb{Z}\}$. Then, for $0 \leq j \leq 3$, assign color $j+1$ to all the vertices in D'_j.

Let us prove that $(\mathfrak{T}^2, w_2)$ does not admit a weighted 2.5-improper 3-coloring. Suppose, by contradiction, that there exists a weighted 2.5-improper 3-coloring c of $(\mathfrak{T}^2, w_2)$. A vertex can have at most two neighbors of the same color as it. Suppose again, w.l.o.g., that $c(V_0) = 1$. We use the following facts:

Fact 3. *No vertex has three neighbors of the same color.*

Fact 4. *If a vertex has two neighbors of the same color, then it has at most one vertex at distance two with its color.*

Fact 5. *There is no path of five vertices of the same color.*

We say that a vertex v is *saturated*, if we know that $I_v(\mathfrak{T}^2, w_2, c) \geq 2.5$.

Let us analyze now each of these cases.

Case: V_0 **has exactly two neighbors colored 1**.

We assume, w.l.o.g., that N_0 is colored 1. We subdivide this case into three subcases according to the position of the second neighbor of V_0 colored 1. Due to the symmetry, we analyze the three possible cases where respectively N_1, N_2 or N_3 is colored 1.

1. **Subcase** $c(N_1) = 1$.

We now show that no coloring is feasible, for all possible different colorings of the vertices N_2, N_3, N_4 and N_5 (up to symmetries). We can have all these vertices of the same color (Case 1a) or three of the same color, say 2, and the other of color 3 (Cases 1b and 1c) and two of color 2 and two of color 3 (Cases 1d, 1e and 1f).

(a) Suppose that $c(N_2) = c(N_3) = c(N_4) = c(N_5) = 2$. Observe that $c(N_{12}) = c(N_{50}) = 3$, thanks to Facts 3 and 5. Since N_3 and N_4 are saturated, we get that all the vertices N_{22}, N_{23}, N_{33}, N_{34}, N_{44}, N_{45} and N_{55} cannot be colored 2. At most one of these vertices is colored 1, due to the interference in V_0. W.l.o.g, we can then consider that $c(N_{22}) = c(N_{23}) = c(N_{33}) = 3$. But then, since N_{23} and N_3 are saturated, we conclude that N_{223}, N_{233}, N_{333}, N_{334} and N_{34} must be all colored 1. This is a contradiction to Fact 5.

(b) Now consider the case in which $c(N_2) = c(N_3) = c(N_4) = 2$ and $c(N_5) = 3$. Observe that N_{12} cannot be colored 1. Let us study the two other cases:

i. Now consider the case in which N_{12} is colored 2. We observe that N_2 and N_3 are saturated.
In case N_{44} is colored 1, we also have that V_0 is saturated and thus all the vertices N_{22}, N_{23}, N_{33} and N_{34} must be colored 3. Consequently, as N_{23} and N_{33} are saturated, we reach a contradiction to Fact 5 as all the vertices N_{222}, N_{223}, N_{233}, N_{333} and N_{334} must be colored 1. Thus, N_{44} is colored 3 (it cannot be colored 2 due to Fact 5).
In case N_{33} is colored 1, we have that V_0 is saturated and all the vertices N_{23}, N_{34} and N_{45} are colored 3. As N_{34} is saturated, the vertices N_{233}, N_{333} and N_{334} must be colored 1. This contradicts Fact 3. Consequently, N_{33} is colored 3. N_{34} cannot be colored 3, because it would imply that $c(N_{45}) = 1$ and, consequently, V_0 is saturated and the vertices N_{22} and N_{23} should be colored 3 and we would have a contradiction to Fact 5. Thus, N_{34} is colored 1. Consequently, N_{22}, N_{23} and N_{45} are colored 3. The vertices N_{334} and N_{344} must then be colored 1 due to the interference constraints on the vertices N_3, N_{33} and N_{44}. However, we reach a contradiction as no color is feasible to vertex N_{233} (and N_{333}).

ii. So we conclude that $c(N_{12}) = 3$.

- Consider first the case $c(N_{22}) = 1$ (and thus V_0 is saturated). We have that N_{23}, N_{33} and N_{34} must be colored 3, thanks to the Facts 3 and 4 and V_0 being saturated. N_{44} cannot be colored 3 as we would have $I_{N_{34}}(\mathfrak{T}^2, w_2, c) \geq 3$. Since V_0 is also saturated, it implies that $c(N_{44}) = 2$. Therefore, N_4 is saturated and so $c(N_{45}) = c(N_{55}) = c(N_{50}) = 3$, but then $I_{N_5}(\mathfrak{T}^2, w_2, c) \geq 3$.
- Thus, consider the case $c(N_{22}) = 2$. Then, N_2 and N_3 are saturated. One of the vertices N_{33}, N_{34}, N_{44} and N_{45} is colored 1, thanks to Fact 5. So V_0 is saturated and $c(N_{01}) = c(N_{11}) = c(N_{23}) = 3$. Then, N_{112} and N_{122} cannot be colored 3, otherwise $I_{N_{12}}(\mathfrak{T}^2, w_2, c) \geq 3$; they cannot be colored 2 as N_2 is saturated; so $c(N_{112}) = c(N_{122}) = 1$, but then we reach a contradiction as $I_{N_1}(\mathfrak{T}^2, w_2, c) \geq 3$.

- We then conclude that $c(N_{22}) = 3$. Due to Facts 3 and 5, at least one of the vertices N_{23}, N_{33} and N_{34} is colored 1 and the two others are colored 3. Consequently, V_0 is saturated. In case N_{44} is colored 2, then N_4 is saturated and the vertices N_{45}, N_{55} and N_{50} are colored 3, contradicting Fact 3. Thus, $c(N_{44}) = 3$.

– If N_{45} is colored 2, N_3 and N_4 are saturated and then, N_{55} and N_{50} are colored 3 and it implies that N_5 is saturated. Consequently, N_{34} is colored 1 and N_{23} and N_{33} are colored 3.
Thus, N_{23} is saturated and the vertices N_{223}, N_{233}, N_{333} and N_{334} are colored 1, contradicting Fact 5.

– Thus, N_{45} is also colored 3 and we obtain $c(N_{55}) = 2$. N_{23} cannot be colored 1, otherwise N_{33} and N_{34} being colored 3, we would contradict Fact 5. If N_{34} is colored 3, N_{44} is saturated and then N_{50} must be colored 2 and N_4 is saturated. In this case, we get a contradiction to Fact 5 because all the vertices N_{334}, N_{344}, N_{444} and N_{445} must be colored 1.
So $c(N_{23}) = c(N_{33}) = 3$, $c(N_{34}) = 1$ and $c(N_{11}) = 2$.
If N_{01} is colored 2, we have that N_2 is saturated and, since N_{22} is saturated, we have that the vertices N_{112}, N_{122}, N_{222}, N_{223} and N_{233} must be all colored 1, contradicting Fact 5. Thus, N_{01} is colored 3 and then N_{50} must be colored 2, due to the interference constraint in N_5.
Consequently, N_4 is saturated and all the vertices N_{344}, N_{444}, N_{445} and N_{455} must be colored 1, due to the interference constraints in N_4, N_{44} and N_{45}. This contradicts Fact 5.

(c) Let us consider now the case $c(N_2) = c(N_3) = c(N_5) = 2$ and $c(N_4) = 3$. Recall that N_{12}, N_{11}, N_{01}, N_{00} and N_{50} cannot be colored 1.

i. Let us study the case $c(N_{12}) = 2$. In this case, N_2 is saturated and thus N_{01} and N_{11} must be colored 3.

- In case N_{34} is colored 1, the vertices N_{22}, N_{23} and N_{33} must be colored 3 as V_0 and N_2 are saturated. Consequently, N_{23} is also saturated. It implies that the vertices N_{122}, N_{222}, N_{223} and N_{233} must be all colored 1. By Fact 5, we conclude that N_{333} must be colored 2 and then N_3 is also saturated. Consequently, $c(N_{44}) = c(N_{45}) = 3$, but then N_4 has interference at least 3, a contradiction.
- Thus we conclude that N_{34} is colored 3, as it cannot be colored 2 thanks to the interference constraint on N_2. Observe that none of the vertices N_{44} and N_{45} can be colored 1, as it would imply that V_0 is saturated and that the vertices N_{22}, N_{23} and N_{33} should be all colored 3, leading to a contradiction to Fact 5. N_{44} and N_{45} can neither be both colored 2 nor 3, due to interference constraints in N_3 and N_4, respectively.
In case $c(N_{44}) = 2$ and $c(N_{45}) = 3$, observe that among N_{23} and N_{33} we have one vertex colored 1 and the other is colored 3. Consequently, V_0 and N_4 are both saturated and N_{55} and N_{50} must be colored 2. But then $I_{N_5}(\mathfrak{T}^2, w_2, c) \geq 3$, a contradiction.
In case $c(N_{44}) = 3$ and $c(N_{45}) = 2$, we conclude that N_{33} is colored 1, thanks to Fact 3, and thus V_0 is saturated; consequently, $c(N_{23}) = 3$ and N_4 is saturated,

but then $c(N_{55}) = c(N_{50}) = 2$ and $I_{N_5}(\mathfrak{T}^2, w_2, c) \geq 3$.

ii. Then consider that N_{12} is colored 3. We claim that neither N_{22} nor N_{23} can receive color 2. For otherwise, suppose the case where at least one of these vertices would be colored 2. As N_2 would be saturated, the vertices N_{01} and N_{11} should be both colored 3. This would imply that N_{112} and N_{122} should be colored 1 and 3, respectively, due to Fact 3 and the interference constraint in N_1 and N_2. Consequently, as N_1 and N_{12} would be both saturated, N_{22} and N_{23} should be both colored 2, a contradiction to Fact 3. Observe that N_{22} and N_{23} cannot be both colored 1 due to the interference in V_0. Let us study the three remaining cases:

- $c(N_{22}) = 1$ and $c(N_{23}) = 3$. At most one of the vertices N_{33} and N_{34} is colored 2, due to Fact 3. If exactly one of them is colored 2 (and thus the other is colored 3 thanks to the interference in V_0), as N_3 is saturated, N_{44} and N_{45} must be colored 3. This is a contradiction as $I_{N_4}(\mathfrak{T}^2, w_2, c) \geq 3$. Thus, N_{33} and N_{34} are both colored 3 and it implies that N_{44} and N_{45} are both colored 2, because of Facts 3 and 5. As N_{45} is saturated, N_{55} and N_{50} are both colored 3 and we reach a contradiction as $I_{N_4}(\mathfrak{T}^2, w_2, c) \geq 3$.
- $c(N_{22}) = 3$ and $c(N_{23}) = 1$. If N_{33} is colored 2, we observe that N_3 is saturated and N_{34}, N_{44} and N_{45} must be all colored 3. This contradicts Fact 3.
  We conclude that $c(N_{33}) = 3$. If N_{34} is colored 2, N_3 is saturated and N_{44} and N_{45} are both colored 3. Then, N_4 is saturated and $c(N_{55}) = c(N_{50}) = 2$. This is a contradiction as $I_{N_5}(\mathfrak{T}^2, w_2, c) \geq 3$. Then, $c(N_{34}) = 3$ and then N_{44} is colored 2. If N_{45} is colored 3, N_4 is saturated and then N_{55} and N_{50} must be both colored 2. This is a contradiction as $I_{N_5}(\mathfrak{T}^2, w_2, c) \geq 3$. So $c(N_{45}) = 2$ and N_5 is saturated. As a consequence, we get $c(N_{55}) = c(N_{50}) = c(N_{00}) = c(N_{01}) = 3$. This is another contradiction as $I_{N_{50}}(\mathfrak{T}^2, w_2, c) \geq 3$.
- $c(N_{22}) = 3$ and $c(N_{23}) = 3$. N_{33} cannot be colored 3 thanks to the interference constraint in N_{23}.
  - If $c(N_{33}) = 2$, then N_3 is saturated. In this case, N_{34}, N_{44} and N_{45} cannot be all colored 3 (Fact 3). So one of them is colored 1 and the two others are colored 3 implying that V_0 and N_4 are saturated and N_{55} and N_{50} are both colored 2. This is a contradiction as $I_{N_5}(\mathfrak{T}^2, w_2, c) \geq 3$.
  - If $c(N_{33}) = 1$, then V_0 is saturated. In case N_{34} is colored 2, N_3 is also saturated and N_{44} and N_{45} must be both colored 3. Then N_4 is saturated and N_{55} and N_{50} are both colored 2. This is a contradiction as $I_{N_5}(\mathfrak{T}^2, w_2, c) \geq 3$.
    Thus we know that $c(N_{34}) = 3$. In case N_{44} is colored 3, N_4 is saturated and N_{45}, N_{55} and N_{50} should be all colored 2. This contradicts Fact 3. Then $c(N_{44}) = 2$. So N_{44} is colored 2 and we know that N_{23} is saturated. Then, among N_{233}, N_{333} and N_{334} there is exactly one vertex colored 2, due to Fact 3 and to the interference in N_3. As N_3 is saturated, we conclude that $c(N_{45}) = 3$. But N_4 is saturated, N_{55} and N_{50} must be colored 2 and we find a contradiction as $I_{N_5}(\mathfrak{T}^2, w_2, c) \geq 3$.

(d) Now, we study the case $c(N_2) = c(N_3) = 2$ and $c(N_4) = c(N_5) = 3$. Observe that colors 2 and 3 are symmetric under these hypothesis. In order to use this symmetry, let us consider the possible colorings of N_{23} and N_{45} (up to the symmetries):

i. In case $c(N_{23}) = 2$ and $c(N_{45}) = 3$, observe that N_{34} is necessarily colored 1, thanks to Fact 3. Consequently, V_0 is saturated, N_{33} is colored 3 and N_{44} is colored 2. It implies that N_3 and N_4 are also saturated and that N_{334} and N_{344} are both colored 1. As N_{34} is also saturated, N_{233} and N_{333} are colored 3. Moreover, N_{22} is also colored 3 as V_0 and N_3 are saturated. This is a contradiction as $I_{N_{33}}(\mathfrak{T}^2, w_2, c) \geq 3$.

ii. Now consider that $c(N_{23}) = 2$ and $c(N_{45}) = 2$. Since N_3 is saturated and Fact 3 holds, among N_{34} and N_{44} we have one vertex colored 1 and the other is colored 3. So V_0 is saturated, N_{33} is colored 3 and N_4 is then saturated. Consequently, N_{334} and N_{344} are colored 1 and N_{55} and N_{50} are colored 2. N_{444} can neither be colored 3 as N_4 is saturated, nor 1 as $I_{N_{344}}(\mathfrak{T}^2, w_2, c) \geq 3$. So $c(N_{45})$ is saturated and N_{445} and N_{455} are both colored 1. This is a contradiction as either $I_{N_{34}}(\mathfrak{T}^2, w_2, c) \geq 3$ or $I_{N_{44}}(\mathfrak{T}^2, w_2, c) \geq 3$.

iii. Let us study the case $c(N_{23}) = 2$ and $c(N_{45}) = 1$. So, $c(N_{33}) = c(N_{34}) = 3$ and $c(N_{44}) = 2$. As N_3, N_4 and N_{34} are saturated, N_{233}, N_{333}, N_{334} and N_{344} are colored 1. As N_3 is saturated, $c(N_{12}) = c(N_{22}) = 3$. N_4 and N_{34} saturated imply that N_{233}, N_{333}, N_{334} and N_{344} are colored 1. So, by Fact 5, $c(N_{233}) = 3$ and N_{22} is saturated. Consequently, $c(N_{11}) = 2$ and N_2 is saturated. Therefore, $c(N_{112}) = c(N_{122}) = 1$, but we have a contradiction as $I_{N_1}(\mathfrak{T}^2, w_2, c) \geq 3$.

iv. We now deal with the case $c(N_{23}) = 1$ and $c(N_{45}) = 2$. Observe that N_{33} cannot be colored 2, because in this case V_0 and N_3 are saturated and we would have a contradiction to Fact 3 as N_{34} and N_{44} should be both colored 3. Consequently, N_{33} is colored 3. In case N_{34} is colored 3, N_4 is saturated and then N_{45}, N_{55} and N_{50} are colored 2. This is a contradiction as $I_{N_{45}}(\mathfrak{T}^2, w_2, c) \geq 3$. So $c(N_{34}) = 2$ and N_3 is saturated. As a consequence, N_{44} is colored 3 and N_4 is also saturated and the vertices N_{55} and N_{50} must be colored 2. It implies that N_{45} is saturated and $c(N_{344}) = c(N_{444}) = c(N_{445}) = c(N_{455}) = 1$. As N_3 and N_4 are saturated, N_{334} should be also colored 1, but this contradicts Fact 5.

v. We now deal with the last subcase in which $c(N_{23}) = 3$ and $c(N_{45}) = 2$ (Recall that colors 2 and 3 are once more symmetric).

- If $c(N_{33}) = 2$, N_3 is saturated. Then N_{34} and N_{44} cannot receive color 2, cannot be both colored 1 (Fact 4 with V_0) and cannot be both colored 3 (Fact 4 with N_4).
  - In case $c(N_{34}) = 1$ and $c(N_{44}) = 3$, N_4 and V_0 are saturated. Consequently, $c(N_{334}) = c(N_{344}) = 1$ and N_{34} is also saturated. Thus, $c(N_{12}) = c(N_{22}) = c(N_{233}) = c(N_{333}) = 3$. This is a contradiction to Fact 5.
  - So $c(N_{34}) = 3$ and $c(N_{44}) = 1$. One more V_0, N_3 and N_4 are saturated. It implies that $c(N_{12}) = c(N_{22}) = 3$ and then N_{23} is also saturated. Consequently, the vertices N_{233}, N_{333}, N_{334} and N_{344} must be all colored 1. This contradicts Fact 5.

As $c(N_{33}) \neq 2$, by symmetry, we conclude that $c(N_{44}) \neq 3$. We use this information in the following subcase.

- If $c(N_{33}) = 3$, observe that N_{34} cannot be colored 3, thanks to Fact 5. Recall that N_{44} is either colored 1 or 2, by symmetry. Moreover, N_{34} and N_{44} cannot be both colored 2 due to Fact 5.

– In case $c(N_{34}) = 2$ and $c(N_{44}) = 1$, V_0 and N_3 are saturated. This implies that $c(N_{12}) = c(N_{22}) = 3$. This is a contradiction as $I_{N_{23}}(\mathfrak{T}^2, w_2, c) \geq 3$.

– So $c(N_{34}) = 1$ and $c(N_{44}) = 2$. N_{55} and N_{50} cannot be both colored 2, otherwise $I_{N_{45}}(\mathfrak{T}^2, w_2, c) \geq 3$. So one is colored 3 and N_4 is saturated. Similarly, N_{12} and N_{22} cannot be both colored 3, otherwise $I_{N_{23}}(\mathfrak{T}^2, w_2, c) \geq 3$. Thus, one of them is colored 2 and N_3 is saturated. Then, $c(N_{334}) = c(N_{344}) = 1$ and N_{34} is saturated. Since N_3 is also saturated, we have that $c(N_{233}) = c(N_{333}) = 3$, but then $I_{N_{33}}(\mathfrak{T}^2, w_2, c) \geq 3$, a contradiction.

As N_{33} cannot be colored 3, again by symmetry we conclude that N_{44} cannot be colored 2. Thus, we have a contradiction to Fact 4 in V_0 as $c(N_{33}) = c(N_{44}) = 1$.

(e) Let us consider now that $c(N_2) = c(N_4) = 2$ and $c(N_3) = c(N_5) = 3$. By Facts 3 and 4, there is at most one vertex in Γ^2 colored 1. By symmetry, we consider w.l.o.g. that this vertex is in $\{N_{22}, N_{23}, N_{33}, N_{34}\}$. So we know that N_{44}, N_{45} and N_{55} are not colored 1.

i. $c(N_{34}) = 1$ (and then V_0 is saturated).

- $c(N_{44}) = c(N_{45}) = 2$. In this case, N_4 is saturated. So, $c(N_{23}) = c(N_{33}) = c(N_{55}) = c(N_{50}) = 3$ and N_3 and N_5 are saturated. We then reach a contradiction because $c(N_{334}) = c(N_{344}) = c(N_{445}) = 1$ and then $I_{N_{34}}(\mathfrak{T}^2, w_2, c) \geq 3$.
- $c(N_{44}) = c(N_{45}) = 3$. So N_{45} is saturated and $c(N_{55}) = c(N_{50}) = 2$. Observe that N_{23} and N_{33} cannot be both colored 3, otherwise $I_{N_3}(\mathfrak{T}^2, w_2, c) \geq 3$. If both N_{23} and N_{33} are colored 2, then N_4 is also saturated and then N_{334}, N_{444}, N_{445} and N_{455} are all colored 1, contradicting Fact 5. So among N_{23} and N_{33} we have one vertex colored 2 and the other is colored 3 and, consequently, N_3 is saturated. So N_{12} and N_{22} are colored 1 and we have a contradiction as $I_{N_2}(\mathfrak{T}^2, w_2, c) \geq 3$.
- Either $c(N_{44}) = 2$ and $c(N_{45}) = 3$, or $c(N_{44}) = 3$ and $c(N_{45}) = 2$. In this case, N_{23} and N_{33} cannot be both colored 3, otherwise $I_{N_3}(\mathfrak{T}^2, w_2, c) \geq 3$. Similarly, N_{55} and N_{50} cannot be both colored 3, otherwise $I_{N_5}(\mathfrak{T}^2, w_2, c) \geq 3$. At most two among N_{23}, N_{33}, N_{55} and N_{50} are colored 2, otherwise $I_{N_4}(\mathfrak{T}^2, w_2, c) \geq 3$. Consequently, one vertex among N_{23} and N_{33} is colored 2 and the other is colored 3, the same happens for vertices N_{55} and N_{50} and, then, N_4 is saturated. N_{12} and N_{22} cannot be both colored 2, otherwise $I_{N_2}(\mathfrak{T}^2, w_2, c) \geq 3$. So one of them is colored 1 and N_3 is saturated, implying that $c(N_{334}) = c(N_{344}) = 1$ and N_{34} is saturated.

  If $c(N_{45}) = 3$, then N_5 is saturated and $c(N_{445}) = 1$, but then $I_{N_{34}}(\mathfrak{T}^2, w_2, c) \geq 3$. If $c(N_{45}) = 2$, we have that $c(N_{44}) = 3$. N_{444} and N_{445} cannot be both colored 3, otherwise $I_{N_{44}}(\mathfrak{T}^2, w_2, c) \geq 3$. So one of them is colored 3 and again $I_{N_{34}}(\mathfrak{T}^2, w_2, c) \geq 3$.

ii. $c(N_{34}) = 2$. Recall that N_{44}, N_{45} and N_{55} are not colored 1. Observe that, by Fact 3, at most one of N_{44} and N_{45} is colored 2. If one of these vertices is colored 2, N_4 is saturated and N_{55} and N_{50} must be both colored 1. It implies a contradiction as $I_{N_5}(\mathfrak{T}^2, w_2, c) \geq 3$. Consequently, N_{44} and N_{45} are both colored 3 and N_{45} is saturated. So N_{55} and N_{50} are colored 2 and N_4 is also saturated implying that $c(N_{344}) = c(N_{444}) = c(N_{445}) = c(N_{455}) = 1$. Since N_{444} is saturated, N_{334} must be colored 3 and then N_{23} and N_{33} cannot receive color 3, otherwise $I_{N_3}(\mathfrak{T}^2, w_2, c) \geq 3$. We obtain a contradiction because N_{23} and N_{33} are both colored 1 and $I_{V_0}(\mathfrak{T}^2, w_2, c) \geq 3$.

iii. $c(N_{34}) = 3$. Observe that N_{44} and N_{45} cannot be both colored 3, due to Fact 5.

- $c(N_{44}) = c(N_{45}) = 2$. In this case, N_4 is saturated and then N_{55} and N_{50} must be colored 3. This is a contradiction because $I_{N_5}(\mathfrak{T}^2, w_2, c) \geq 3$.
- $c(N_{44}) = 2$ and $c(N_{45}) = 3$. Due to the interference in N_5, we have that $c(N_{55}) = c(N_{50}) = 2$ and then N_4 is saturated. However, the vertices N_{23} and N_{33} cannot receive color 3, due to the interference in N_3, and so they are both colored 1 and we have a contradiction as $I_{V_0}(\mathfrak{T}^2, w_2, c) \geq 3$.
- $c(N_{44}) = 3$ and $c(N_{45}) = 2$. In this case, N_{34} is saturated. If N_{23} and N_{33} are both colored 2, N_4 is saturated and N_{55} and N_{50} must be colored 3 and we get $I_{N_5}(\mathfrak{T}^2, w_2, c) \geq 3$. So among N_{23} and N_{33} we have one vertex colored 1 and the other is colored 2.

  N_{55} and N_{50} can neither be both colored 3, otherwise $I_{N_5}(\mathfrak{T}^2, w_2, c) \geq 3$, nor both colored 2, otherwise $I_{N_4}(\mathfrak{T}^2, w_2, c) \geq 3$. So one is colored 2, the other 3 and N_4 and N_5 are saturated. We then get a contradiction to Fact 5 because $c(N_{334}) = c(N_{344}) = c(N_{444}) = c(N_{445}) = c(N_{455}) = 1$.

(f) Now consider that $c(N_2) = c(N_5) = 2$ and $c(N_3) = c(N_4) = 3$. As in Case 1e, we consider w.l.o.g. that N_{44}, N_{45} and N_{55} are not colored 1. Observe that N_{44} and N_{45} cannot be both colored 3, otherwise $I_{N_4}(\mathfrak{T}^2, w_2, c) \geq 3$.

i. Consider first that $c(N_{44}) = c(N_{45}) = 2$. Consequently, $c(N_{55}) = c(N_{50}) = 3$ due to the interference constraints in N_{45} and N_5. If N_{00} is colored 3, N_{50} is saturated and then N_{01} must be colored 2. As a consequence, N_5 is also saturated and N_{550} and N_{500} must be both colored 1. This is a contradiction as $I_{N_0}(\mathfrak{T}^2, w_2, c) \geq 3$. So N_{00} is colored 2 and N_5 is saturated. Thus, $c(N_{01}) = 3$ and N_{550} and N_{500} cannot receive color 2 (interference in N_5) or 3 (interference in N_{50}). So, $c(N_{550}) = c(N_{500}) = 1$, but them $I_{N_0}(\mathfrak{T}^2, w_2, c) \geq 3$.

ii. Either $c(N_{44}) = 2$ and $c(N_{45}) = 3$, or $c(N_{44}) = 3$ and $c(N_{45}) = 2$. In this case, observe that N_{55} and N_{50} can neither be both colored 2 (interference in N_5) nor 3 (interference in N_4). So one is colored 2, the other is colored 3 and N_4 is saturated.

- If $c(N_{44}) = 3$ and $c(N_{45}) = 2$, then N_5 is also saturated and N_{34} must be colored 1. Consequently, V_0 is saturated and $c(N_{23}) = c(N_{33}) = 2$ and $c(N_{00}) = c(N_{01}) = 3$. Due to the interference in N_2, N_{12} and N_{22} must be colored 3 and then, by Fact 5, N_{11} must be colored 2. So N_2 is also saturated and then, due to the interference in N_{12}, N_{112} and N_{122} must be both colored 1. This is a contradiction because $I_{N_1}(\mathfrak{T}^2, w_2, c) \geq 3$.

- So $c(N_{44}) = 2$ and $c(N_{45}) = 3$. Observe that N_{33} and N_{34} cannot be both colored 2, otherwise $I_{N_{34}}(\mathfrak{T}^2, w_2, c) \geq 3$. So one of them is colored 1 and the other is colored 2. Thus, V_0 is saturated and N_{23} must be colored 2. If $c(N_{33}) = 1$ and $c(N_{34}) = 2$, N_{34} is saturated and then $c(N_{334}) = c(N_{344}) = c(N_{444}) = c(N_{445}) = 1$, contradicting Fact 5. So $c(N_{33}) = 2$ and $c(N_{34}) = 1$. Due to the interference in N_2, we have that N_{12} and N_{22} are colored 3 and then N_3 is also saturated. Then, N_{334} must be colored 1 due to the interference in N_3 and N_{33}. If N_{344} is colored 2, N_{33} is saturated and we have a contradiction to Fact 5 because $c(N_{223}) = c(N_{233}) = c(N_{333}) = 1$. So we get $c(N_{344}) = 1$ and N_{34} saturated. This is a contradiction because N_{333} must be colored 2 and then $I_{N_{33}}(\mathfrak{T}^2, w_2, c) \geq 3$.

2. **Subcase** $c(N_2) = 1$.

W.l.o.g., let $c(N_1) = 2$. We deal with the subcases according to the coloring of N_3, N_4 and N_5: they are all colored 2 (Case 2a), two of them are colored 2 (Cases 2b and 2c), only one of them is colored 2 (Cases 2d and 2e) or they are all colored 3 (Case 2f).

(a) Consider first the subcase $c(N_3) = c(N_4) = c(N_5) = 2$. In this case, N_4 is saturated and all the vertices N_{23}, N_{33}, N_{34}, N_{44}, N_{45}, N_{55} and N_{50} cannot be colored 2. Since at most one vertex in Γ^2 is colored 1, this vertex cannot belong to the set $\{N_{23}, N_{33}, N_{55}, N_{50}\}$ as it would imply a contradiction to Facts 5 in color 3. So all the vertices in this set are colored 3, exactly one vertex among N_{34}, N_{44} and N_{45} is colored 1 and V_0 is saturated. By symmetry, we can consider that N_{45} is colored 3.

If N_{01} is colored 2, N_1 is also saturated and all the vertices N_{11}, N_{12} and N_{22} must be colored 3. This is a contradiction to Fact 5. So $c(N_{01}) = 3$.

In order to avoid a P_5 of vertices colored 3, N_{00} must be colored 2. Then, N_{11} and N_{12} must be colored 3, due to the interference constraint in N_1. Thanks to Fact 5, N_{22} must be colored 2 and so N_1 and N_3 are saturated. The vertices N_{112} and N_{233} cannot be colored 3 as we would be in Case 1, then they are both colored 1 and N_2 is also saturated. Consequently, N_{122} must be colored 3 and we reach a contradiction as $I_{N_{12}}(\mathfrak{T}^2, w_2, c) \geq 3$.

(b) Let us now suppose that $c(N_3) = c(N_4) = 2$ and $c(N_5) = 3$. We show that there is no feasible color to N_{44} by examining the three possible cases:

i. Suppose first that N_{44} is colored 2. So N_4 is saturated and then, if $c(N_{55}) = 3$, as either N_{45} or N_{50} must be colored 3, we are in Case 1. Thus, N_{55} is colored 1, V_0 is saturated and N_{23}, N_{33}, N_{34}, N_{45} and N_{50} are colored 3. Consequently, N_5 is saturated and so N_{00} and N_{01} are colored 2. Thus, N_1 is saturated and N_{12} and N_{22} must be colored 3, contradicting Fact 5.

ii. Now consider that $c(N_{44}) = 1$. Thus, V_0 is saturated and N_{34} is colored 3, otherwise we would be in Case 1.

- Suppose that at one of the vertices N_{23} or N_{33} is colored 2. Then, N_3 is saturated and the vertices N_{12}, N_{22} and N_{45} must be all colored 3. So N_{55} is colored 2, as we are no longer in Case 1, and it implies that N_4 is saturated. As a consequence, N_{50} is colored 3 and N_5 is also saturated. Thus, N_{00} and N_{01} must be colored 2, N_1 is saturated and N_{11} is colored 3. Observe that N_{112} and N_{122} are both colored 1, otherwise we are in Case 1. So N_2 is also saturated and no feasible color remains to color N_{223}.

- So N_{23} and N_{33} are both colored 3.
  - If N_{22} is colored 3, N_{12} is colored 2 (Fact 5), N_{11} is colored 3 (as we are not in Case 1) and N_{01} is also colored 3 (interference in V_0 and N_1).
    If $c(N_{00}) = 3$, N_{01} is saturated and then N_{50} is colored 2. It implies that N_1 is saturated and N_{001} and N_{011} must be both colored 1. Consequently, N_0 is saturated and N_{000} and N_{500} are both colored 2. Thus, N_{50} is also saturated and the vertices N_{45} and N_{55} should be both colored 3. But then we are in Case 1.
    So N_{00} is colored 2 and N_1 is saturated. Consequently, N_{50} is colored 3 and N_{55} must be colored 2 as we are no longer in Case 1. But then no feasible color remains to color N_{45}.
  - Thus, we have $c(N_{22}) = 2$. If N_{12} is colored 2, N_1 is saturated and we have a contradiction to Fact 5, because all the vertices N_{50}, N_{00}, N_{01} and N_{11} must be colored 3. So, we conclude that $c(N_{12}) = 3$.
    If N_{01} or N_{11} are colored 2, N_1 is saturated and N_{50} and N_{00} must be colored 3. In this case, N_{45} and N_{55} cannot receive color 3, due to the interference in N_5. So they are both colored 2 and we reach a contradiction as $I_{N_4}(\mathfrak{T}^2, w_2, c) \geq 3$.
    Consequently, N_{01} and N_{11} are both colored 3. Observe that N_{45} is also colored 3, otherwise N_4 is saturated, N_{50} and N_{00} are colored 3 and we are in Case 1. Consequently, N_{55} and N_{50} are colored 2, as we are no longer in Case 1 and we cannot violate the interference constraint in N_5. Moreover, N_{00} is also colored 2, otherwise $I_{N_{01}}(\mathfrak{T}^2, w_2, c) \geq 3$. But then we have a contradiction as $I_{N_{50}}(\mathfrak{T}^2, w_2, c) \geq 3$.

iii. We conclude that N_{44} must be colored 3. Recall that N_{34} cannot be colored 2 as we would be in Case 1.

- Consider first the case in which $c(N_{34}) = 1$ and thus V_0 is saturated. If N_{45} is colored 2, N_4 is saturated and N_{50} and N_{00} should be both colored 3. But then we are in Case 1. So N_{45} is colored 3 and N_{55} must be colored 2.
  Observe that N_{23} and N_{33} cannot be both colored 2, due to Fact 3. In case one of these vertices is colored 2 and the other is colored 3, observe that N_3 and N_4 are saturated. Consequently, N_{50} is colored 3 and N_{45} and N_5 are also saturated. We then reach a contradiction to Fact 5 as all the vertices N_{344}, N_{444}, N_{445} and N_{455} must be colored 1. So we conclude that N_{23} and N_{33} must be both colored 3.
  If N_{50} is colored 3, N_5 is saturated. Then, N_{00} and N_{01} must be colored 2, then N_1 is saturated and we reach a contradiction to Fact 5 as N_{11}, N_{12} and N_{22} must be all colored 3. So N_{50} is colored 2 and N_4 is saturated. Consequently, N_{344} and N_{444} are both colored 1, due to the interference constraints in N_4 and N_{44}. Thus, N_{34} is also saturated and N_{445} must be colored 3. But then we are in Case 1.

- We deduce that $c(N_{34}) = 3$. We now study the possible colorings of N_{45}.

  – If $c(N_{45}) = 2$, N_4 is saturated. The interference constraints in V_0 and N_5 lead us to the conclusion that among N_{55} and N_{50} we have one vertex colored 1 and the other is colored 3. Consequently, V_0 is saturated and N_{23} and N_{33} are both colored 3. This is a contradiction as $I_{N_{34}}(\mathfrak{T}^2, w_2, c) \geq 3$.

  – Now consider that $c(N_{45}) = 1$ (V_0 is saturated). The vertices N_{23} and N_{33} cannot be both colored 2, due to Fact 3. They cannot also be both colored 3, because of the interference constraint in N_{34}. So among N_{23} and N_{33} we have one vertex colored 2 and the other is colored 3 and N_3 is saturated.
    The vertices N_{55} and N_{50} can neither be both colored 2, because of the interference in N_4, nor 3, as we are not in Case 1. So one of them is colored 2 and the other is colored 3. Thus, N_4 is also saturated.
    Similarly, we can conclude that among N_{444} and N_{445} we have one vertex colored 1 and the other is colored 3 (recall that these vertices cannot receive color 2 as N_4 is saturated). Consequently, N_{44} is saturated and the vertices N_{344} and N_{455} must be colored 1. This is a contradiction as $I_{N_{45}}(\mathfrak{T}^2, w_2, c) \geq 3$.

  – So we have $c(N_{45}) = 3$. Consequently, N_{33}, N_{55} and N_{50} cannot receive color 3. We thus conclude that two of these vertices are colored 2 and the other is colored 1, by considering the interference in V_0 and N_4. We then obtain that N_{334}, N_{344}, N_{444}, N_{445} and N_{455} are all colored 1. This contradicts Fact 5.

(c) We now treat the case $c(N_3) = c(N_5) = 2$ and $c(N_4) = 3$. Let us consider the possible colors of N_{23}.

i. Suppose first that N_{23} is colored 1. In this case, V_0 and N_2 are saturated.

- In case N_{33} is colored 2, N_{34} must be colored 3 and N_{44} must be colored 2, otherwise we would be in Case 1. So N_3 is also saturated and N_{45} must be colored 3. Since N_2 and N_3 are both saturated, N_{12}, N_{22}, N_{223} and N_{233} must be all colored 3 and then N_{22} is saturated. It implies that N_{11}, N_{112}, N_{122} are colored 2 and then we reach a contradiction as $I_{N_1}(\mathfrak{T}^2, w_2, c) \geq 3$.

- We conclude that N_{33} must be colored 3. Observe that N_{44} and N_{45} cannot be both colored 3, as we are no longer in Case 1. Thus, at least one of these vertices is colored 2. If N_{34} is colored 2, N_3 is saturated. Then, the vertices N_{12}, N_{22}, N_{223} and N_{233} must be all colored 3. This contradicts Fact 5. Consequently, N_{34} is colored 3 and N_{44} must be colored 2, otherwise we would be in Case 1. Observe that N_{45} cannot be colored 2, because, otherwise N_5 will be saturated, $c(N_{55}) = c(N_{50}) = c(N_{00}) = c(N_{01}) = 3$ and $I_{N_{50}}(\mathfrak{T}^2, w_2, c) \geq 3$. So N_{45} is colored 3, N_4 is saturated and N_{55} and N_{50} are both colored 2. However, we are in Case 1 with N_5.

ii. Now consider that $c(N_{23}) = 2$. Observe that neither N_{33} nor N_{34} are colored 2 due to the interference in N_3.

- Suppose first that $c(N_{33}) = 1$. It implies that V_0 is saturated and that N_{34} is colored 3. Consequently, N_{44} must be colored 2, otherwise we are in Case 1, and then N_3 is saturated. So, N_{12}, N_{22} and N_{45} are colored 3. Observe that among

N_{55} and N_{50}, we must have one vertex colored 2 and the other must be colored 3 (due to Fact 5 and to the hypothesis that we are not in Case 1). So N_4 is also saturated and it implies that N_{334} and N_{344} are colored 1. We conclude that N_{33} is saturated and that the vertices N_{223}, N_{233} and N_{333} should be all colored 3. This contradicts Fact 5.

- Now consider the case in which $c(N_{33}) = 3$ and $c(N_{34}) = 1$. So V_0 is saturated and we can see that N_{44} and N_{45} can neither be both colored 2 (interference in N_3) nor 3 (Case 1 with N_4). Thus, one is colored 2 and the other is colored 3. Consequently, N_3 is saturated and N_{12} and N_{22} are both colored 3. Furthermore, both N_{334} and N_{344} cannot be colored 1 (Case 1 with N_{34}). One of them at least is colored 3. Then N_{55} and N_{50} can neither be both colored 2 (Case 1 with N_5) nor 3 (otherwise $I_{N_4}(\mathfrak{T}^2, w_2, c) \geq 3$). So among N_{55} and N_{50} we have one vertex colored 2 and the other is colored 3. We conclude that N_5 is saturated, N_{00} and N_{01} are colored 3 and, due to Fact 5, N_{11} is colored 2. It implies that N_1 is saturated and N_{122} must be colored 1 (it cannot be colored 3, otherwise we would be in Case 1 with N_{12}). So N_2 is also saturated and N_{223} and N_{233} must be both colored 3. This contradicts Fact 5.
- We obtain that N_{33} and N_{34} are both colored 3. Consequently, N_{44} cannot be colored 3 (Fact 3 with N_{34}).

– Suppose first that N_{44} is colored 1. If $c(N_{45}) = 3$, N_4 is saturated and we are in Case 1 with N_5 instead of V_0, because N_{55} and N_{50} must be both colored 2. So N_{45} is colored 2 and it implies that N_{55} and N_{50} must be both colored 3, due to the interference constraint in V_0 and N_5. Thus, N_4 is saturated. Since N_3 is also saturated, we get that N_{334} and N_{344} are both colored 1. The vertices N_{444} and N_{445} can neither receive color 1, due to the interference in N_{44}, nor color 3, since N_4 is saturated. Thus, they are both colored 2. But then we have a contradiction as $I_{N_{45}}(\mathfrak{T}^2, w_2, c) \geq 3$.

– So we get that $c(N_{44}) = 2$ and then N_3 is saturated. Neither N_{12}, nor N_{22} can be colored 1, otherwise N_2 would also be saturated and it would imply that N_{223} and N_{233} should be colored 3, leading to a contradiction to Fact 5. So we get that $c(N_{12}) = c(N_{22}) = 3$. Consequently, $c(N_{233}) = c(N_{333}) = c(N_{334}) = c(N_{344}) = 1$, due to interference constraints in N_3, N_{33} and N_{34}. So $c(N_{223}) = 3$ and N_{33} is also saturated. It implies that N_{22} is saturated and then N_{11} can either be colored 1 or 2. In case it is colored 1, N_2 is saturated, N_{112}, N_{122} and N_{222} must be colored 2 and we have a contradiction as $I_{N_{122}}(\mathfrak{T}^2, w_2, c) \geq 3$. If N_{11} is colored 2, N_1 is saturated and then N_{112} and N_{122} must be colored 1. However, we get that $I_{N_2}(\mathfrak{T}^2, w_2, c) \geq 3$.

iii. We conclude that N_{23} is colored 3.

- Suppose first that $c(N_{33}) = 1$. Consequently, V_0 is saturated.

– Let us first consider the subcase in which N_{34} is colored 2. Then, N_{44} and N_{45} can neither be both colored 2, due to the interference in N_3, nor 3, since we are no longer in Case 1. So among N_{44} and N_{45} we have one vertex colored 2 and the other is colored 3. It implies that N_3 is saturated. Due to the interference in

V_0 and N_5, we conclude that $c(N_{55}) = c(N_{50}) = 3$. So N_4 is saturated implying N_{334} and N_{344} must be both colored 1. But then N_{33} is also saturated, N_{22} and N_{223} are be both colored 3 and we are in Case 1 with vertex N_{23}.

– We conclude that $c(N_{34}) = 3$. Since we are no longer in Case 1, we get that $c(N_{44}) = 2$. N_{45} and N_{55} can neither be both colored 2 (Fact 3 with N_{45}), nor 3 (interference in N_4). So one of these vertices is colored 2 and the other is colored 3, implying that N_5 is saturated and then that $c(N_{50}) = c(N_{00}) = c(N_{01}) = 3$. However, we get a contradiction as neither N_{45} is colored 3, otherwise $I_{N_4}(\mathfrak{T}^2, w_2, c) \geq 3$, nor N_{55} is colored 3, otherwise $I_{N_{50}}(\mathfrak{T}^2, w_2, c) \geq 3$.

- Let us consider now the case $c(N_{33}) = 2$. Observe that N_{34} cannot be also colored 2, due to the interference constraint in N_3.

– In case $c(N_{34}) = 1$, we have that V_0 is saturated and then N_{44} and N_{45} can neither be both colored 2 (interference in N_3) nor 3 (Case 1 with N_4). So among N_{44} and N_{45} we find one vertex colored 2 and the other is necessarily colored 3. Consequently, N_3 is saturated, N_{12} and N_{22} are colored 3 and then N_{223} must be colored 1. So N_2 is also saturated and N_{233} must be colored 3. This is a contradiction as $I_{N_{23}}(\mathfrak{T}^2, w_2, c) \geq 3$.

– We conclude that N_{34} must be colored 3. Consequently, N_{44} cannot be colored 3, as we are not in Case 1. Let us check the possible colorings of N_{44}.
If N_{44} is colored 1, V_0 is saturated. Then, if N_{45} is colored 3, N_4 is saturated and N_{55} and N_{50} are forced to be colored 2. But then we are in Case 1 with N_5. So N_{45} is colored 2, N_3 is saturated and the vertices N_{55} and N_{50} must be colored 3, due to the interference in N_5. As a consequence, N_4 is also saturated and the vertices N_{334} and N_{344} must be colored 1. As we are no longer in Case 1, N_{444} must be colored 2. Due to the interference constraints in N_4 and N_{45}, we get that N_{445} and N_{455} must be both colored 1. This is a contradiction to Fact 5.
So N_{44} must be colored 2 and then N_3 is saturated. Observe that exactly one of the vertices N_{45}, N_{55} and N_{50} must be colored 1, otherwise N_{45} must be colored 3 (interference in N_3) and N_{55} and N_{50} must be colored 2 (interference in N_4) and we are in Case 1. Then, as N_3 and V_0 are saturated we have $c(N_{12}) = c(N_{22}) = 3$, implying that N_{23} is saturated and so $c(N_{223}) = c(N_{233}) = c(N_{333}) = c(N_{334}) = 1$. However, we have that $I_{N_{233}}(\mathfrak{T}^2, w_2, c) \geq 3$.

- So we have that $c(N_{33}) = 3$. Let us now check the possible colorings of $c(N_{34})$.

– First consider that $c(N_{34}) = 1$. Observe that V_0 is saturated.

∗ If $c(N_{44}) = 3$, we get that $c(N_{45}) = 2$ and, consequently, $c(N_{55}) = c(N_{50}) = 3$ (otherwise, $I_{N_5}(\mathfrak{T}^2, w_2, c) \geq 3$). However, observe that $I_{N_4}(\mathfrak{T}^2, w_2, c) \geq 3$.

∗ So $c(N_{44}) = 2$. If $c(N_{45}) = 2$, N_{45} and N_5 are both saturated implying that N_{55}, N_{50}, N_{00} and N_{01} must be all colored 3. But then we have a contradiction as $I_{N_{50}}(\mathfrak{T}^2, w_2, c) \geq 3$.
So N_{45} is colored 3. N_{55} and N_{50} can neither be both colored 2 (otherwise, Case 1 with N_5), nor 3 (otherwise, $I_{N_4}(\mathfrak{T}^2, w_2, c) \geq 3$). So one of these vertices is colored 2 and the other is colored 3. Thus, N_4 and N_5 are saturated and then N_{00} and N_{01} must be colored 3 and the vertices N_{445} and N_{455} must be

colored 1. In case $c(N_{50}) = 3$, N_{50} is saturated and the vertices N_{555}, N_{550} and N_{500} must be colored 1, contradicting Fact 5. So, we get that $c(N_{55}) = 3$ and $c(N_{50}) = 2$. Observe that N_{555} and N_{550} can neither receive color 2 (since N_5 is saturated) nor 3 (otherwise, $I_{N_{55}}(\mathfrak{T}^2, w_2, c) \geq 3$). Thus, they are both colored 1 and, consequently, N_{500} is colored 3. It implies that N_{00} is saturated and then we get that N_{11} must be colored 2. As a consequence, N_1 is saturated and N_{12} and N_{22} are both colored 3. However, we get that $I_{N_{23}}(\mathfrak{T}^2, w_2, c) \geq 3$.

– Now consider that $c(N_{34}) = 2$. Let us check the possible colorings of N_{44}.

∗ First suppose that $c(N_{44}) = 1$. If N_{45} is colored 2, then N_3 is saturated and we have that N_{12} and N_{22} are colored 3. This is a contradiction as $I_{N_{23}}(\mathfrak{T}^2, w_2, c) \geq 3$. So N_{45} is colored 3. The vertices N_{55} and N_{50} can neither be both colored 2 (otherwise, Case 1 with N_5) nor 3 (otherwise, $I_{N_4}(\mathfrak{T}^2, w_2, c) \geq 3$). So one is colored 2 and the other is colored 3. As a consequence, N_4 and N_5 are both saturated implying that N_{445} and N_{455} are colored 1 and then that N_{444} is colored 2. But then N_{334} and N_{344} must be both colored 1 (interference in N_{34}) and we have a contradiction to Fact 5.

∗ Now let $c(N_{44}) = 2$. Observe that N_3 and N_{34} are saturated and that N_{12} and N_{22} cannot be both colored 3, otherwise $I_{N_{23}}(\mathfrak{T}^2, w_2, c) \geq 3$. So among N_{12} and N_{22} we have one vertex colored 1 and the other is colored 3. It implies that V_0 is saturated. Observe also that the vertices N_{45}, N_{55} and N_{50} cannot receive color 2 due to the interference constraint in N_5. Then, we have a contradiction as N_{45}, N_{55} and N_{50} are all colored 3 and we get $I_{N_4}(\mathfrak{T}^2, w_2, c) \geq 3$.

∗ We conclude that $c(N_{44}) = 3$. If $c(N_{45}) = 1$, V_0 is saturated. In this case, N_{55} and N_{50} can neither be both colored 2 (otherwise, Case 1 with N_5) nor 3 (otherwise, $I_{N_4}(\mathfrak{T}^2, w_2, c) \geq 3$). So one of these vertices is colored 2, the other is colored 3 and we get that N_4 and N_5 are saturated. Thus, N_{445} and N_{455} must be colored 1 and we are in Case 1 for N_{45}.

N_{45} cannot be colored 3 as we are no longer in Case 1, so its color is 2 and N_3 and N_5 are both saturated. N_{55} and N_{50} cannot be both colored 3, otherwise $I_{N_4}(\mathfrak{T}^2, w_2, c) \geq 3$. So one of these vertices is colored 1 implying that V_0 is saturated. Consequently, N_{12} and N_{22} are both colored 3 and we get a contradiction as $I_{N_{23}}(\mathfrak{T}^2, w_2, c) \geq 3$.

– So we conclude that N_{33} and N_{34} are both colored 3 and both saturated. If the vertices N_{12}, N_{44} and N_{45} are not colored 1, they must be all colored 2 and we have that N_3 is saturated and so $c(N_{223}) = c(N_{233}) = c(N_{333}) = c(N_{334}) = c(N_{344}) = 1$, contradicting Fact 5. So one of these vertices is colored 1 and V_0 is saturated. In case N_{12} is colored 1, N_{44} and N_{45} must be colored 2 and N_{45} is saturated. Consequently, N_{55} and N_{50} are colored 3 and we have a contradiction as $I_{N_4}(\mathfrak{T}^2, w_2, c) \geq 3$.

Then, either N_{44} or N_{45} is colored 1 (the other being colored 2) and N_{12} is colored 2. If N_{44} is colored 1, then N_{55} and N_{50} are not colored 2, otherwise $I_{N_5}(\mathfrak{T}^2, w_2, c) \geq 3$. So they are both colored 3, but then $I_{N_4}(\mathfrak{T}^2, w_2, c) \geq 3$.

So we have that N_{44} is colored 2 and N_{45} is colored 1. Consequently, N_{55} and N_{50}

can neither be both colored 2 (interference in N_5) nor 3 (interference in N_4). So one is colored 2 and the other is colored 3 implying that N_4 and N_5 are saturated. Therefore, $c(N_{445}) = c(N_{455}) = 1$ and we are in Case 1.

(d) We now study the case $c(N_3) = 2$ and $c(N_4) = c(N_5) = 3$. Observe that N_{45} cannot be colored 3, otherwise we are in Case 1.

i. First consider that $c(N_{45}) = 1$ (V_0 is saturated). If N_{55} is colored 3, N_{50} must be colored 2 and we are in Case 2b with central vertex N_5. So N_{55} is colored 2.

- In case N_{44} is colored 3, N_{34} must be colored 2 and then N_{33} must be colored 3, because we are not in Case 1. Thus, N_4 is saturated, N_{23} and N_{50} must be colored 2 and N_3 is also saturated. Consequently, N_{334} and N_{344} are both colored 1.

  If N_{00} is colored 2, N_{50} is saturated and then N_{455} must be colored 1, N_{45} is also saturated and N_{01} is colored 3. Since N_{555} and N_{550} must be both colored 3, we reach a contradiction as $I_{N_5}(\mathfrak{T}^2, w_2, c) \geq 3$.

  So we conclude that $c(N_{00}) = 3$. Recall that N_3 is saturated and thus, N_{12} and N_{22} must be both colored 3. N_{01} and N_{11} can neither be both colored 2 (otherwise, Case 1) nor 3 (Fact 5). So one is colored 2 and the other is colored 3. Thus, N_1 is saturated and N_{122} must be colored 1, since we are not in Case 1. The vertices N_{223} and N_{233} cannot receive color 2 as N_3 is saturated, cannot be both colored 3, thanks to Fact 5, and cannot be both colored 1, due to the interference in N_2. So one of these vertices is colored 1 and the other is colored 3; N_2 is saturated and N_{112} must be colored 3. Consequently, N_{11} is colored 2 and N_{01} is colored 3. We then observe that N_{01} and N_5 are saturated and that N_{001} and N_{011} must be both colored 1. It leads to a contradiction as $I_{N_0}(\mathfrak{T}^2, w_2, c) \geq 3$.
- We conclude that N_{44} is colored 2.

– Suppose first that $c(N_{34}) = 2$. In this case, N_{23} and N_{33} are both colored 3 due to the interference in N_3. Observe that N_{12} and N_{22} can neither be both colored 2 (interference in N_3) nor 3 (interference in N_{23}). So one is colored 2 and the other is colored 3. It implies that N_3 is saturated. Thus, N_{223} and N_{233} are both colored 1, due to the interference in N_{23}. So N_2 is also saturated. The vertices N_{333} and N_{334} can neither be both colored 1 (otherwise, $I_{N_{233}}(\mathfrak{T}^2, w_2, c) \geq 3$) nor 3 (Fact 3). So one of them is colored 1 and the other is colored 3. As a consequence, N_{23} is saturated, N_{22} is colored 2 and N_{12} is colored 3. But then N_{122} and N_{222} must be colored 2 and we have a contradiction as $I_{N_{22}}(\mathfrak{T}^2, w_2, c) \geq 3$.

– We obtain that $c(N_{34}) = 3$. N_{23} and N_{33} can neither be both colored 2 (otherwise, Case 1 with N_3) nor 3 (Fact 5). So one of them is colored 2 and the other is colored 3. It implies that N_4 is saturated and N_{55} and N_{50} must be colored 2.

  If $c(N_{00}) = 2$, N_{50} is saturated and thus N_{01} must be colored 3. Observe that N_{550} and N_{500} can neither be both colored 1 (interference in N_0) nor 3 (interference in N_5). So one of these vertices is colored 1 and the other is colored 3. It implies that N_0 and N_3 are both saturated and thus that N_{000} and N_{001} must be both colored 3. Then, N_{11} and N_{011} cannot receive color 1 (N_0 is saturated) neither 3 (otherwise, $I_{N_{01}}(\mathfrak{T}^2, w_2, c) \geq 3$). So they are both colored 2 and we reach a

contradiction as $I_{N_1}(\mathfrak{T}^2, w_2, c) \geq 3$.
We conclude that N_{00} must be colored 3. If N_{01} is colored 3, N_5 is saturated. In this case, N_{550} and N_{500} can neither be both colored 1 (interference in N_0) nor 2 (Fact 3). So one of them is colored 1 and the other is colored 2 and, as a consequence, N_0 and N_{50} are saturated. Thus, N_{000} and N_{001} must be both colored 3 and we reach a contradiction to Fact 3.
So we have that N_{01} must be colored 2 and $c(N_{11}) = c(N_{12}) = 3$, otherwise $I_{N_1}(\mathfrak{T}^2, w_2, c) \geq 3$. In this case, N_{550} and N_{500} cannot receive color 2 (interference in N_{50}). They can neither be both colored 1 (interference in N_0) nor 3 (interference in N_5). Thus, one of these vertices is colored 1, the other is colored 3 and N_0 and N_5 are saturated. It implies that one of the vertices N_{000} or N_{001} must be colored 2 and the other is colored 3, because they can neither be both colored 2 (interference in N_{01}) nor 3 (interference in N_{00}). But then, N_{00} is saturated and it implies that $c(N_{011}) = 2$. This leads to a contradiction as $I_{N_{01}}(\mathfrak{T}^2, w_2, c) \geq 3$.

ii. We then conclude that $c(N_{45}) = 2$. Let us study the possible colorings of N_{44}.

- Suppose now that $c(N_{44}) = 3$. Observe that N_{34} cannot be colored 3, by Fact 3. If $c(N_{34}) = 2$, then we are in Case 2b with N_4.
  So N_{34} is colored 1 and V_0 is saturated. Observe that N_{23} and N_{33} can neither be both colored 2 (otherwise, Case 1 with N_3) nor 3 (interference in N_4). So one of them is colored 2, the other is colored 3 and N_4 is saturated. It implies that N_{55} and N_{50} must be colored 2 and, due to the interference in N_{45}, that N_{445} must be colored 1. Moreover, N_{334} and N_{344} can neither be both colored 1 (interference in N_{34}) nor 2 (interference in N_3). Thus, one of them is colored 1 and the other is colored 2. As a consequence, N_3 and N_{45} are saturated. We obtain that N_{233} and N_{333} are both colored 3. So N_{33} cannot be colored 3, as we are not in Case 1 and then $c(N_{23}) = 3$ and $c(N_{33}) = 2$. Recall that N_3 is saturated and thus N_{12} and N_{22} must be both colored 3. This is a contradiction to Fact 5.
- Suppose now that $c(N_{44}) = 1$ (and thus that V_0 is saturated).
  - If $c(N_{34}) = 3$, then N_{23} and N_{33} can neither be both colored 2 (interference in N_3) nor 3 (Fact 5). So one of them is colored 2 and the other is colored 3, implying that N_4 is saturated. Consequently, N_{55} and N_{50} must be colored 2 and then that N_{445} and N_{455} must be both colored 1 (otherwise, $I_{N_{45}}(\mathfrak{T}^2, w_2, c) \geq 3$). Thus, N_{344} and N_{444} are both colored 2, due to the interference in N_{44}. However, we get that $I_{N_{45}}(\mathfrak{T}^2, w_2, c) \geq 3$.
  - We conclude that N_{34} is colored 2 and thus that N_{23} and N_{33} must be both colored 3, due to the interference in N_3.
    * If $c(N_{22}) = 3$, then N_{23} is saturated and $c(N_{12}) = 2$, implying that N_3 is also saturated. So N_{223} and N_{233} are both colored 1 and N_2 is saturated. Consequently, N_{122} and N_{222} are both colored 2 and we have a contradiction as $I_{N_{12}}(\mathfrak{T}^2, w_2, c) \geq 3$.
    * We obtain that $c(N_{22}) = 2$, and then N_3 is saturated and N_{12} must be colored 3. Consequently, N_{223} and N_{233} must be both colored 1 (interference in N_{45})

and N_2 is also saturated. Since N_{122} and N_{222} cannot be both colored 2 as we are not in Case 1, we conclude that at least one of these vertices is colored 3 and that N_{23} is saturated. But then we get that $c(N_{333}) = c(N_{334}) = 1$ and we have a contradiction as $I_{N_{233}}(\mathfrak{T}^2, w_2, c) \geq 3$.

- So we have that N_{44} must be colored 2. Let us now check the possible colorings of N_{34}.

– In case $c(N_{34}) = 2$, N_3, N_{34} and N_{44} are all saturated. One of N_{12}, N_{22}, N_{23} and N_{33} must be colored 1, otherwise they are all colored 3 and we have $I_{N_{23}}(\mathfrak{T}^2, w_2, c) \geq 3$. So V_0 is also saturated and then N_{55} must be colored 3. Thus, N_{50} is colored 2, by Fact 3, and N_{45} is also saturated.
If both N_{23} and N_{33} are colored 3, N_4 is saturated and then we have a contradiction to Fact 5, because N_{334}, N_{344}, N_{444}, N_{445} and N_{455} should be all colored 1.
So among N_{23} and N_{33} we have one vertex colored 1, the other is colored 3 and then N_{12} and N_{22} must be colored 3.
If N_{23} is colored 1 (and then N_{33} is colored 3), we have that N_2 is saturated and then N_{223} and N_{233} must be colored 3. But then we have a contradiction to Fact 5.
So N_{33} must be colored 1 (and then N_{23} is colored 3). By Fact 3, we have that N_{223} is colored 1 and then N_2 is also saturated. Consequently, N_{233} must be colored 3 and we have a contradiction as $I_{N_{23}}(\mathfrak{T}^2, w_2, c) \geq 3$.

– Suppose now that $c(N_{34}) = 1$ (so V_0 is saturated).

* If $c(N_{33}) = 2$, N_3 is also saturated and then N_{12}, N_{22} and N_{23} must be all colored 3. However, N_{223} and N_{233} must be colored 1, due to interference constraints in N_{22}, N_{23} and N_3, which is a contradiction as $I_{N_2}(\mathfrak{T}^2, w_2, c) \geq 3$.

* So $c(N_{33}) = 3$. Let us check the possible colorings of N_{23}.
If N_{23} is colored 2, N_3 is saturated and then N_{12} and N_{22} are both colored 3. N_{223} and N_{233} can neither be both colored 1 (interference in N_2) nor 3 (Fact 5). So one of them is colored 1, the other is colored 3 and N_2 is saturated. If N_{223} is colored 3 (and then $c(N_{233}) = 1$), N_{22} is also saturated. In this case, the vertices N_{11}, N_{112}, N_{122} and N_{222} must be all colored 2, contradicting Fact 5.
So we conclude that N_{223} is colored 1 and N_{233} is colored 3. Consequently, N_{333} and N_{334} must be colored 1 (interference in N_3 and N_{33}) and then N_{34} is saturated. Thus, N_{344} and N_{445} must be colored 3. It implies that N_4 is saturated and then N_{55} and N_{50} are both colored 2. This is a contradiction as $I_{N_{45}}(\mathfrak{T}^2, w_2, c) \geq 3$.
We conclude that N_{23} is colored 3. If N_{22} is also colored 3, N_{23} is saturated and then N_{12} is colored 2. N_{223} and N_{233} can neither be both colored 1 (interference in N_2) nor 2 (interference in N_3). So one of them is colored 1, the other is colored 2 and N_2 and N_3 are both saturated. It implies that N_{334} and N_{344} are both colored 1, N_{34} is saturated and then N_{344} must be colored 3. But then N_4 is saturated, N_{445} must be colored 2 and we are in Case 1.
So N_{22} must be colored 2. If $c(N_{12}) = 2$, then N_3 is saturated. N_{223} and N_{233} can

neither be both colored 1 (interference in N_2), nor 3 (Fact 3). Thus, one is colored 1, the other is colored 3 and N_2 and N_{23} are both saturated. Consequently, N_{12}, N_{122} and N_{222} must be all colored 2, contradicting Fact 3. Therefore, $c(N_{12}) = 3$, but then N_{223} and N_{233} cannot be colored 3 (otherwise, $I_{N_{23}}(\mathfrak{T}^2, w_2, c) \geq 3$). So they are colored 1 and $I_{N_2}(\mathfrak{T}^2, w_2, c) \geq 3$, a contradiction.

– We conclude that $c(N_{34}) = 3$. Let us study the possible colorings of N_{55}.

∗ Suppose first that $c(N_{55}) = 3$. So N_4 and N_5 are saturated. N_{23} and N_{33} can neither be both colored 1 (interference in V_0) nor 2 (we are not in Case 1). So one of them is colored 1, the other is colored 2 and V_0 and N_3 are also saturated. It implies that N_{334} and N_{344} must be both colored 1 and that N_{50} and N_{00} must be colored 2. Observe then that N_{445} and N_{455} cannot receive color 2 (interference in N_{45}) and 3 (N_4 is saturated). So they are both colored 1 and, by Fact 5, N_{444} must be colored 2. Since N_{45} is also saturated, N_{555} and N_{550} must be colored 1. But then we have a contradiction because N_{500} cannot receive color 1 (Fact 5), 2 (we are not in Case 1) or 3 (N_5 is saturated).

∗ Now consider that $c(N_{55}) = 2$. Observe that N_{45} is saturated. If N_{50} is colored 3, N_4 and N_5 are also saturated and we have a contradiction to Fact 5, because N_{344}, N_{444}, N_{445}, N_{455}, N_{555} and N_{550} are all colored 1.
So N_{50} is colored 1 and V_0 and N_0 are saturated. N_{23} and N_{33} can neither be both colored 2 (we are not in Case 1) nor 3 (Fact 5). So one is colored 2, the other is colored 3 and we have that N_3 and N_4 are both saturated. This leads to a contradiction to Fact 5, because N_{334}, N_{344}, N_{444}, N_{445} and N_{455} are all colored 1.

∗ We then conclude that N_{55} must be colored 1 (and V_0 is saturated). Again N_{23} and N_{33} can neither be both colored 2 nor 3. One of them is colored 2 and the other is colored 3 implying that N_3 and N_4 are both saturated. As a consequence, N_{334} and N_{344} are colored 1 and N_{50} is colored 2. Thus, N_{445} and N_{455} are both colored 1, due to the interference in N_4 and N_{45}. So N_{444} must be colored 2 and N_{45} is saturated. Consequently, N_{555} and N_{550} must be both colored 3 due to the interference in N_{45} and N_{55}. We obtain that N_5 is also saturated and then that N_{00} and N_{01} are both colored 2 and both saturated and N_1 is also saturated. So N_{23} is colored 3 and N_{33} is colored 2. Furthermore, N_{500} is colored 1 and N_0 is saturated. But then N_{011}, N_{11}, N_{12} and N_{22} are colored 3, contradicting Fact 5.

(e) Let us now consider the case $c(N_4) = 2$ and $c(N_3) = c(N_5) = 3$. We study now the subcases concerning the color of N_{45}.

i. First consider that $c(N_{45}) = 1$. Recall that V_0 is saturated.

- In case N_{44} is colored 2, N_{34} is colored 3 and N_{33} is colored 2, as we are no longer in Case 1. N_{55} and N_{50} can neither be both colored 2, otherwise $I_{N_4}(\mathfrak{T}^2, w_2, c) \geq 3$, nor 3, otherwise we would be in Case 1 with N_5. So one of these vertices is colored 2, the other is colored 3 and N_4 is saturated. But then N_{23} is colored 3 and we are in Case 2d with central vertex N_3.
- We conclude that N_{44} is colored 3.

- If N_{34} is colored 3, N_{34} is saturated and $c(N_{23}) = c(N_{33}) = 2$. Then, N_{22} is colored 3, otherwise $I_{N_{23}}(\mathfrak{T}^2, w_2, c) \geq 3$, implying that N_3 is saturated. So $c(N_{12}) = 2$ and N_{23} is also saturated. Thus, N_{223} and N_{233} are both colored 1, N_2 is saturated, which implies that N_{122} and N_{222} must be colored 3. Then, we are in Case 1.
- So N_{34} must be colored 2. In case N_{33} is also colored 2, then we are in one of the cases from 2a to 2d with central vertex N_{34}. Thus, N_{33} must be colored 3 implying that N_{23} is colored 2. N_{12} and N_{22} can neither be both colored 2 (otherwise, $I_{N_{23}}(\mathfrak{T}^2, w_2, c) \geq 3$) nor 3 (otherwise, $I_{N_3}(\mathfrak{T}^2, w_2, c) \geq 3$). So one is colored 2, the other is colored 3, N_3 is saturated and $c(N_{223}) = c(N_{233}) = 1$ (otherwise, $I_{N_{23}}(\mathfrak{T}^2, w_2, c) \geq 3$).
  Thus, N_2 is also saturated. N_{55} and N_{50} cannot be both colored 3, as we are not in Case 1. Consequently, (exactly) one of these vertices is colored 2 and N_4 is saturated. It implies that N_{334} and N_{344} are colored 1 and then N_{333} must be colored 2. Thus, N_{23} is saturated and N_{22} must be colored 3 (otherwise, $I_{N_{23}}(\mathfrak{T}^2, w_2, c) \geq 3$). However, N_{122} and N_{222} must be colored 3 and we are in Case 1 with vertex N_{22}.

By symmetry, we conclude that $c(N_{34}) \neq 1$.

ii. Suppose now that N_{45} is colored 2. Observe that N_{44} cannot be colored 2 as we are no longer in Case 1.

- Consider first the case $c(N_{44}) = 1$ (V_0 is saturated). If N_{34} is colored 2, N_4 is saturated, N_{23} and N_{33} are both colored 3 and we are in Case 1 with N_3. So N_{34} is colored 3 and N_{33} must be colored 2 (otherwise, Case 1 with N_3). N_{55} and N_{50} cannot be both colored 3 (otherwise, Case 1 with N_5). So (exactly) one is colored 2, N_4 is saturated and N_{23} must be colored 3. However, we are in Case 2d with central vertex N_3.
- We conclude that N_{44} must be colored 3. Recall that $c(N_{34}) \neq 1$. In case N_{34} is colored 2, we are in Case 2c with N_4 instead of V_0. So N_{34} is colored 3 and it is saturated. So $c(N_{23}) \neq 3$, $c(N_{33}) \neq 3$, among N_{23}, N_{33}, N_{55} and N_{50} at most one vertex is colored 1 (interference in V_0) and at most two are colored 2 (interference in N_4). Moreover, at most one of the vertices N_{55} and N_{50} is colored 3, otherwise we are in Case 1 with N_5. So, exactly one of the vertices N_{55} and N_{50} is colored 3 and N_5 is saturated; exactly two of the vertices N_{23}, N_{33}, N_{55} and N_{50} are colored 2 and N_4 is saturated. But then we find a contradiction to Fact 5 as N_{334}, N_{344}, N_{444}, N_{445} and N_{455} are all colored 1.

iii. We then conclude that $c(N_{45}) = 3$ and by symmetry that $c(N_{34}) = 3$. N_{44} cannot be colored 3 by Fact 5.

- Suppose first that N_{44} is colored 1 (V_0 is saturated). Consequently, N_{23}, N_{33}, N_{55} and N_{50} must be colored 2 due to the interference constraints in N_3 and N_5. So N_4 is saturated and N_{334}, N_{344}, N_{445} and N_{455} must be colored 1 due to the interference in N_{34} and N_{45}. This is a contradiction to Fact 5.
- We obtain that N_{44} must be colored 2. Among N_{23}, N_{33}, N_{55} and N_{50} at most one vertex is colored 1 (interference in V_0) and none of them is colored 3 (interference

in N_3 and N_5). So at least 3 of them are colored 2 and we get a contradiction as $I_{N_4}(\mathfrak{T}^2, w_2, c) \geq 3$.

(f) Now consider that $c(N_3) = c(N_4) = c(N_5) = 3$. By Fact 3, we know that N_{34}, N_{44} and N_{45} are not colored 3. These vertices are also not colored 1, otherwise we would be in one of the cases from 2a to 2e with vertex N_4 replacing of V_0. So N_{34}, N_{44} and N_{45} are all colored 2. Let us check the possible colorings of N_{55}.

i. First consider that N_{55} is colored 3. If N_{50} is colored 2, then we are in Case 2d with N_5 instead of V_0. So N_{50} is colored 1 and V_0 are saturated. However, we obtain that N_{23} and N_{33} are both colored 2 and we have a contradiction to Fact 5.

ii. Suppose now that $c(N_{55}) = 2$. Observe that N_{50} cannot be colored 2, by Fact 5. If $c(N_{50}) = 1$, V_0 is saturated and as N_{44} is saturated we conclude that N_{33} is colored 3. But then N_3 and N_4 are saturated and we have a contradiction to Fact 5, because all the vertices N_{334}, N_{344}, N_{444}, N_{445} and N_{455} must be colored 1.
So N_{50} is colored 3 and N_4, N_5, N_{44} and N_{45} are saturated. Consequently, we find a contradiction to Fact 5 as N_{344}, N_{444}, N_{445}, N_{455} and N_{555} are all colored 1.

iii. We conclude that $c(N_{55}) = 1$ and V_0 is saturated. N_{23} and N_{33} can neither be both colored 2, nor 3, due to Facts 5 and 3, respectively. If N_{23} is colored 3 and N_{33} is colored 2, we have that N_4, N_{34} and N_{44} are saturated. Thus, N_{334}, N_{344}, N_{444}, N_{445} and N_{455} must be all colored 1, contradicting Fact 5. Consequently, $c(N_{23}) = 2$ and $c(N_{33}) = 3$. But then we are in Case 2d with vertex N_3 replacing V_0.

3. **Subcase** $c(N_3) = 1$.

Observe that the vertices N_{01}, N_{23}, N_{34} and N_{50} cannot be colored 1, otherwise we would be in Case 2. Up to symmetries, we study the possible colorings of N_1, N_2, N_4 and N_5: four of the same color (Case 3a), three of the same color (Case 3b) or two of the same color (Cases 3c and 3d).

(a) Let us consider first the case $c(N_1) = c(N_2) = c(N_4) = c(N_5) = 2$. In this case, N_{01}, N_{23}, N_{34} and N_{50} must be colored 3, due to interference constraints in N_1, N_2, N_4 and N_5, respectively. By symmetry, we consider that if there exists a vertex colored 1 in Γ^2, then it is in the set $\{N_{33}, N_{44}, N_{45}, N_{55}\}$. Thus, the vertices N_{11} and N_{12} must be colored 3 and we are in Case 2 with respect to N_{11}.

(b) Now let $c(N_1) = c(N_2) = c(N_4) = 2$ and $c(N_5) = 3$. Observe that the vertices N_{11}, N_{12} and N_{22} cannot be colored 2, otherwise we would be in the previous Cases 1 or 2. If these vertices are all colored 3, N_{01} and N_{23} cannot receive color 3 as we would be in Case 2. So N_{01} and N_{23} must be both colored 2 and we reach a contradiction as $I_{N_2}(\mathfrak{T}^2, w_2, c) \geq 3$. So one of the vertices N_{11}, N_{12} and N_{22} is colored 1 and V_0 is saturated.

i. If $c(N_{11}) = 1$, N_{12} and N_{22} must be colored 3. So N_{23} is colored 2 (it cannot be colored 3 as we would be in Case 2) and then N_2 is saturated. Consequently, N_{33}

and N_{34} are colored 3. Observe that N_{44} and N_{45} can neither be both colored 2 (Fact 4 with N_4) nor 3 (Fact 5). So N_4 is saturated and N_{55} and N_{50} are both colored 3. Then we find a contradiction as we are in Case 1 with vertex N_5.

ii. In case N_{22} is colored 1 and $c(N_{11}) = c(N_{12}) = 3$, we have that N_{01} is colored 2. So N_1 is saturated, N_{00} and N_{50} must be colored 3 and we are in Case 2 with N_{50}.

iii. So we have that $c(N_{12}) = 1$ and $c(N_{11}) = c(N_{22}) = 3$. If $c(N_{23}) = 2$, we have that N_2 is saturated, N_{33} and N_{34} must be colored 3 and among the vertices N_{44} and N_{45} we have one vertex colored 2 and the other is colored 3. Consequently, N_{55} and N_{50} must be colored 3 and we are in Case 1. So $c(N_{23}) = 3$.
Observe that among N_{34}, N_{44} and N_{45} we have at most one vertex colored 2, otherwise we would be in one of the Cases 1 or 2. Similarly, at most one of the vertices N_{45}, N_{55} and N_{50} is colored 3. In case there is a vertex colored 2 among N_{34} and N_{44}, due to two vertices colored 2 in the set $\{N_{45}, N_{55}, N_{50}\}$, we have a contradiction as $I_{N_4}(\mathfrak{T}^2, w_2, c) \geq 3$. Observe that we cannot have all the vertices N_{34}, N_{44} and N_{45} colored 3 as we would be in Case 2. So, N_{34} and N_{44} are colored 3 and N_{45} is colored 2. Since that there is a vertex in N_{55} and N_{50} colored 2, we conclude that N_4 is saturated and then N_{33} is colored 3. This is a contradiction to Fact 5.

(c) We now study the case $c(N_1) = c(N_2) = 2$ and $c(N_4) = c(N_5) = 3$. By symmetry, we consider that the vertices N_{00}, N_{01}, N_{11}, N_{12}, N_{22} and N_{23} are not colored 1. Then, the vertices N_{11}, N_{12} and N_{22} must colored 3, otherwise we would be in Cases 1 or 2. By the same reason, N_{01} and N_{12} must be colored 2. As N_1 is saturated, N_{00} is colored 3. Consequently, we can neither color N_{50} with colors 1 or 3, because we would be in Case 2, nor color it with color 2, due to the interference in N_1.

(d) Let us consider now that $c(N_1) = 2$, $c(N_2) = 3$ and that among N_4 and N_5 we have one vertex colored 2 and the other is colored 3. By symmetry, we can once more consider that the vertices N_{00}, N_{01}, N_{11}, N_{12}, N_{22} and N_{23} are not colored 1.

i. In case N_{12} is colored 3, all the vertices N_{11}, N_{22} and N_{23} must be colored 2, otherwise we would be in Cases 1 or 2. So N_1 is saturated and N_{00} and N_{01} must be colored 3. Then, as in Case 3c no feasible color remains to color N_{50}.

ii. Thus N_{12} is colored 2. It implies that $c(N_{01}) = c(N_{11}) = c(N_{22}) = 3$, otherwise we would be in Cases 1 or 2. Consequently, N_2 is saturated, N_{23} and N_{34} are colored 2, and thus N_{33} must be colored 1. So N_3 is also saturated and N_{223} and N_{233} must be colored 2. Then we are in Case 1 with N_{23}.

CASE: V_0 **has exactly one neighbor colored 1**.

We also consider that no vertex v has two neighbors with its own color, otherwise we can consider that v is V_0 and we are in the previous case. This fact is extensively used in this proof and many times it is omitted. W.l.o.g, let N_0 be the only neighbor of V_0 colored 1 and let $c(N_1) = 2$.

1. Suppose first that $c(N_2) = 2$. Consequently, $c(N_3) = 3$, otherwise N_2 would have two neighbors colored 2. We have three cases to analyze:

(a) In case $c(N_4) = c(N_5) = 2$, we claim that $c(N_{01}) = c(N_{50}) = 3$. In fact, if not, one of the vertices N_0, N_1 or N_5 would have two neighbors with their colors. By the same reason, we conclude $N_{00} = 2$. At this point, observe that N_1 and N_5 are saturated, thanks to the set $\{N_1, N_2, N_4, N_5, N_{00}\}$. Consequently, the vertices N_{11} and N_{12} cannot receive color 2 and they cannot be both colored 3 as N_{11} would have two neighbors with its color. Similarly, we can conclude that at least one vertex of N_{22} and N_{33} is colored 1 and also one of N_{34} and N_{44} and one of N_{45} and N_{55}. This is a contradiction because $I_{V_0}(\mathfrak{T}^2, w_2, c) \geq 3$.

(b) Suppose now that $c(N_4) = 2$ and $c(N_5) = 3$. Observe that $c(N_{01}) = 3$. By the hypothesis that no vertex has two neighbors with the same color, we conclude that among the vertices N_{11} and N_{12} at least one of them is colored 1, none of them can receive color 2 and they cannot be both colored 3. The same is valid for the vertices N_{22} and N_{23}. Observe also that these four vertices cannot be all colored 1, otherwise $I_{V_0}(\mathfrak{T}^2, w_2, c) \geq 3$. Then consider that three of these vertices are colored 1. Thus, since V_0 is saturated, we must be able color the remaining vertices of Γ^2 with colors 2 and 3. If we consider that $c(N_{33}) = 2$, then all the other colors of vertices in Γ^2 are fixed by the hypothesis that each vertex has no two neighbors with its color. One may check that, in this case, $c(N_{44}) = c(N_{55}) = c(N_{50}) = 2$. Thus, $I_{N_4}(\mathfrak{T}^2, w_2, c) \geq 3$, a contradiction. In case we color N_{33} with color 3, one can check that there is no feasible color for N_{45}. Consequently, we conclude that among N_{11} and N_{12} there is one vertex colored 1 and the other is colored 3; and the same holds for vertices N_{22} and N_{23}.

We now show by contradiction that no color is feasible to N_{55}.

i. First suppose that $N_{55} = 1$. Thus, we already know that V_0 is saturated and we can no longer use color 1 to color vertices in Γ^2. If we suppose that $c(N_{45}) = 2$, we observe that we cannot color the vertices N_{34} and N_{44} with colors 2 and 3. Thus, let $c(N_{45}) = 3$. In this case, $c(N_{50}) = c(N_{44}) = 2$, $c(N_{34}) = 3$ and $c(N_{33}) = 2$. We observe that $I_{N_4}(\mathfrak{T}^2, w_2, c) \geq 3$, a contradiction.

ii. Suppose now that $c(N_{55}) = 2$. Observe that N_{45} cannot be colored 2. Suppose then that $c(N_{45}) = 1$. Again V_0 is saturated and we cannot have color 1 in the remaining vertices of Γ^2. If $c(N_{44}) = 2$, then $c(N_{33}) = 2$ and $I_{N_4}(\mathfrak{T}^2, w_2, c) \geq 3$, a contradiction. Thus, let $c(N_{44}) = 3$. In this case $c(N_{34}) = 2$ and $c(N_{33}) = 3$. Consequently, N_3 and N_4 are saturated. It implies that $c(N_{334}) = c(N_{344}) = 1$. As a consequence, $c(N_{444}) = 3$, $c(N_{445}) = 1$ and, since N_4 is saturated, no color is feasible to color N_{455}.

We must consider then the case in which $c(N_{45}) = 3$. As a consequence we have $c(N_{50}) = 2$. Since $I_{N_4}(\mathfrak{T}^2, w_2, c) \geq 2$, we conclude that $c(N_{44}) = 1$, $c(N_{34}) = 3$ and $c(N_{33}) = 2$. We obtain that N_3 and N_4 are saturated. Consequently, $c(N_{334}) = c(N_{344}) = 1$, but then N_{444} has two neighbors colored 1, a contradiction.

iii. The last subcase to consider is the one in which $c(N_{55}) = 3$. Observe that it implies $c(N_{50}) = 2$ and that N_{45} cannot be colored 3. In case $c(N_{45}) = 1$, V_0

is saturated and then N_{44} cannot be colored 1. Suppose first that $c(N_{44}) = 2$. Observe that N_4 is saturated and that $c(N_{34}) = 3$. Consequently, no feasible color remains to color N_{33}. Then consider that $c(N_{44}) = 3$. Consequently, $c(N_{34}) = 2$ and N_4 and N_5 are saturated. This is a contradiction as the vertices N_{445} and N_{455} should be both colored 1, as they are at distance two from N_4 and N_5, but then N_{45} would have two neighbors with the same color. Thus, $c(N_{45}) = 2$ and N_4 is saturated. If N_{44} is colored 1, then N_{33} and N_{34} should be both colored 3, a contradiction. Consequently, $c(N_{44}) = 3$. In this case, we get $c(N_{33}) = 3$, $c(N_{34}) = 1$ and N_3 is saturated. However, N_{334} and N_{344} should be both colored 1, a contradiction since $c(N_{34}) = 1$.

(c) Now suppose that $c(N_4) = 3$ and $c(N_5) = 2$. First observe that $c(N_{01}) = 3$ and $c(N_{23}) = 1$, thanks to the hypothesis that no vertex has two neighbors with the same color. By the same hypothesis, we can conclude that N_{11} and N_{12} cannot receive color 2 and at most one of them is colored 3. By the same reasoning, we can conclude that at least one of the vertices N_{44} and N_{45} is colored 1. Thus, V_0 is saturated and no other vertex at distance two from V_0 can receive color 1. Consequently, by using this information combined with the hypothesis that no vertex has two neighbors with its color we conclude that $c(N_{33}) = c(N_{34}) = 2$. Thus, we conclude that $c(N_{44}) = 1$ and $c(N_{45}) = 2$. Since $c(N_{45}) = c(N_5) = 2$, we obtain that $c(N_{55}) = c(N_{50}) = 3$. This implies that $c(N_{00}) = 2$. However, $I_{N_5}(\mathfrak{T}^2, w_2, c) \geq 3$, thanks to the vertices N_1, N_2, N_{00}, N_{34} and N_{45}.

(d) Finally, if $c(N_4) = c(N_5) = 3$, then N_4 has two neighbors with its own color and we are in the previous case.

2. Suppose then that $c(N_2) = 3$. We consider the possible colorings of N_3, N_4 and N_5:

(a) First, it is not possible to have $c(N_3) = c(N_4) = c(N_5) = 2$ as N_4 would have two neighbors with its color.

(b) Then, consider the case in which $c(N_3) = c(N_4) = 2$ and $c(N_5) = 3$. Once more we know that N_{50}, N_{00} and N_{01} cannot be colored 1, otherwise N_0 would have two neighbors with its own color. Similarly, none of the vertices N_{23}, N_{33}, N_{34}, N_{44} and N_{45} can receive color 2, otherwise N_3 or N_4 would have two neighbors colored 2. We prove now that no color is feasible for N_{55}.

i. First, consider that $c(N_{55}) = 1$.

- Suppose also that $c(N_{45}) = 1$. Consequently, we get $c(N_{44}) = 3$, otherwise N_{45} has two neighbors with color 1. In case N_{34} is colored 1, V_0 is saturated and we reach a contradiction, because $c(N_{23}) = c(N_{33}) = 3$ and N_{23} would have two neighbors colored 3. Thus, suppose that $c(N_{34}) = 3$. It implies that $c(N_{33}) = 1$ and $c(N_{23}) = 3$. As a consequence, $c(N_{12}) = c(N_{22}) = 2$, because V_0 is saturated and $c(N_{23}) = 3$. We then get a contradiction since N_{12} has two neighbors colored 2.
- We conclude then that N_{45} is colored 3. Since $c(N_5) = 3$, we obtain that $c(N_{44}) = 1$. In case $c(N_{34}) = 1$, we have that V_0 is saturated and both N_{23} and N_{33} should

be colored 3. This would be a contradiction as N_{23} would have two neighbors colored 3. Consequently, $c(N_{34}) = 3$. If N_{33} is colored 1, we have $c(N_{23}) = 3$. Once more $c(N_{12}) = c(N_{22}) = 2$ and we have a contradiction as N_{12} has two neighbors colored 2. So $c(N_{33}) = 3$ and, consequently, $c(N_{23}) = 1$. Since V_0 is saturated and no vertex has two neighbors with its own color, either we have $c(N_{11}) = c(N_{22}) = 2$ and $c(N_{12}) = 3$ or we have $c(N_{11}) = c(N_{22}) = 3$ and $c(N_{12}) = 2$. In the first case, we have a contradiction as $I_{N_1}(\mathfrak{T}^2, w_2, c) \geq 3$ and in the latter case we also have a contradiction as $I_{N_2}(\mathfrak{T}^2, w_2, c) \geq 3$ (recall that $c(N_{50}) = 2$ and in the set $\{N_{00}, N_{01}\}$ we have one vertex colored 2 and the other colored 3).

ii. Suppose then that $c(N_{55}) = 2$. We distinguish three cases.

- $c(N_{44}) = c(N_{45}) = 1$, we have that $c(N_{34}) = 3$. In case $c(N_{33}) = 1$, we have that $c(N_{23}) = 3$ and V_0 is saturated. This is a contradiction as N_{12} and N_{22} have no feasible coloring. Then consider the case $c(N_{33}) = 3$. Observe that $c(N_{344}) = 2$, otherwise N_{34} or N_{44} have two neighbors with their color. Consequently, N_4 is saturated and all the vertices N_{444}, N_{445} and N_{455} should be colored 3, as they all have two adjacent neighbors colored 1 and they are all at distance two from N_4. This is a contradiction as N_{455} would have two neighbors with its own color.
- $c(N_{45}) = 1$ and $c(N_{44}) = 3$. Suppose that $c(N_{33}) = c(N_{34}) = 1$. Thus, V_0 is saturated and $c(N_{23}) = 3$. Once more we get a contradiction as N_{12} and N_{22} should be both colored 2. Thus, consider now that $c(N_{33}) = 3$ and $c(N_{34}) = 1$. Observe that $c(N_{23}) = 1$ and V_0 is saturated. If $c(N_{22}) = 2$, we get that $c(N_{12}) = 3$ and $c(N_{11}) = 2$. Since at least one of the vertices N_{50} and N_{00} must be colored 2, we reach a contradiction as $I_{N_1}(\mathfrak{T}^2, w_2, c) \geq 3$. In case $c(N_{22}) = 3$, we get that $c(N_{12}) = 2$ and $c(N_{11}) = 3$. Since N_2 is saturated, we conclude that $c(N_{01}) = 2$. Once more we obtain a contradiction as $I_{N_1}(\mathfrak{T}^2, w_2, c) \geq 3$. Let us now study the case $c(N_{33}) = 1$ and $c(N_{34}) = 3$. In case $c(N_{50}) = 2$, N_4 is saturated and we obtain a contradiction as all the vertices N_{334}, N_{344} and N_{444} should be colored 1. Thus, consider that $c(N_{50}) = 3$. In this case, N_5 is saturated and we get a contradiction as N_{00} and N_{01} should be both colored 2. Since we do not have the case $c(N_{33}) = 3$ and $c(N_{34}) = 3$ as N_{34} would have two neighbors with its color, we conclude that $c(N_{45}) = 3$.
- So $c(N_{45}) = 3$, then we get that $c(N_{44}) = 1$ (otherwise N_{45} has two neighbors of the same color), $c(N_{50}) = 2$ and $c(N_{00}) = 3$. In this case, we easily obtain a contradiction as N_4 is saturated and the vertices N_{445} and N_{455} have no feasible coloring.

iii. We conclude that $c(N_{55}) = 3$. As a consequence, we get $c(N_{45}) = 1$ and $c(N_{50}) = 2$. If $c(N_{44}) = 1$, then $c(N_{455}) = 2$ and N_4 is saturated. But then all the vertices N_{34}, N_{344}, N_{444} and N_{445} should be colored 3. This would be a contradiction. Consequently, $c(N_{44}) = 3$. In this case, in the set $\{N_{00}, N_{01}\}$ there is exactly one vertex colored 2 and the other is colored 3, thanks to the interference constraint in vertex N_5 and to the hypothesis that no vertex has two neighbors with its own color. Similarly, we can conclude that in the set $\{N_{445}, N_{455}\}$ there is exactly one vertex colored 1 and the other is colored 2. Since N_5 is saturated, we get $c(N_{34}) = 1$. So, N_{45} is saturated and both vertices N_{555} and N_{550} should be colored 2. This would be a contradiction as N_{550} would have two neighbors colored 2.

(c) Now let $c(N_3) = c(N_5) = 2$ and $c(N_4) = 3$. We show now that no color is feasible to N_{55}.

i. Suppose first that $c(N_{55}) = 1$.

- First consider that $c(N_{45}) = 1$. Then N_{44} cannot be colored 1 because we would have $I_{N_{45}}(\mathfrak{T}^2, w_2, c) \geq 3$.

  - Then, suppose that N_{44} is colored 2 and N_{34} is colored 1. Since V_0 is saturated, all the remaining vertices in Γ^2 are not colored 1. In case N_{33} is colored 2, we have that N_3 is saturated and thus $c(N_{22}) = c(N_{23}) = 3$. This is a contradiction to the hypothesis that no vertex has two neighbors with its color as $c(N_2) = 3$. In case N_{33} is colored 3, we have that $c(N_{23}) = 2$, then $c(N_{22}) = 3$ and $c(N_{12}) = 2$. But then, $I_{N_3}(\mathfrak{T}^2, w_2, c) \geq 3$, a contradiction.
  - Consequently, if N_{44} is colored 2, N_{34} must be colored 3 (observe it cannot be colored 2 as it would have two neighbors N_3 and N_{44} colored 2). If N_{50} is colored 2, N_5 is saturated and the vertices N_{445}, N_{455} and N_{555} should be all colored 3 (as N_{45} and N_{55} are both colored 1). This is a contradiction as N_{455} has two neighbors with its own color. Consequently, we have $c(N_{50}) = 3$. Observe that among the vertices N_{445} and N_{455} at least one of them is colored 3. Thus, N_4 is saturated and in the set $\{N_{23}, N_{33}\}$ we have exactly one vertex colored 1 (due to the interference constraint in V_0) and the other is colored 2. Since V_0 and N_3 are saturated, the vertices N_{12} and N_{22} should be both colored 3. This is a contradiction as $c(N_2) = 3$.
  - Thus, $c(N_{44}) = 3$ and N_{34} can be either colored 1 or 2. If $c(N_{34}) = 1$, we get that V_0 is saturated. If N_{33} is colored 2, N_{23} is necessarily colored 3 and N_{12} and N_{22} should be both colored 2. This is a contradiction as N_{12} would have two neighbors colored 2. Thus N_{33} is colored 3. It implies that $c(N_{23}) = 2$, then $c(N_{22}) = 3$, $c(N_{12}) = 2$ and $c(N_{01}) = c(N_{11}) = 3$. This is a contradiction as $I_{N_2}(\mathfrak{T}^2, w_2, c) \geq 3$.
  - We conclude that $c(N_{44}) = 3$ and $c(N_{34}) = 2$. Observe that $c(N_{344}) = 1$ and $c(N_{445}) = 2$, thanks to the hypothesis that no vertex has two neighbors with its color. Since we get that N_{45} is saturated, we have $c(N_{444}) = 2$ and, consequently, $c(N_{455}) = 3$. Observe now that N_{34} and N_4 are saturated (because among N_{33} and N_{334} we have exactly one vertex colored 1 and the other is colored 3). As a consequence, $c(N_{23}) = 1$ and $c(N_{50}) = 2$. At this point the colors of the remaining vertices in Γ^2 are fixed as V_0 is saturated. We have $c(N_{00}) = c(N_{01}) = c(N_{12}) = 3$ and $c(N_{11}) = c(N_{22}) = 2$. Thus we observe that $I_{N_1}(\mathfrak{T}^2, w_2, c) \geq 3$, a contradiction.

- Then, consider now that N_{45} is colored 2. It implies that $c(N_{50}) = 3$ and that among N_{00} and N_{01} we have exactly one vertex colored 2 and the other is colored

3. Consequently, N_5 is saturated and among N_{555} and N_{550} we have exactly one vertex colored 1 and one vertex colored 3. In case N_{44} is colored 1, N_{55} is saturated. Thus, N_{445}, N_{455} and N_{500} are all colored 3. This is a contradiction as $I_{N_{50}}(\mathfrak{T}^2, w_2, c) \geq 3$. If $c(N_{44}) = 3$, we obtain that $c(N_{445}) = 1$ and that $c(N_{455}) = 3$. Consequently, N_{55} is saturated and $c(N_{500}) = 3$. Once more we have a contradiction as $I_{N_{50}}(\mathfrak{T}^2, w_2, c) \geq 3$.

- Suppose then that $c(N_{45}) = 3$.

– If $c(N_{50}) = 2$, we have that $c(N_{00}) = 3$. Since $I_{N_5}(\mathfrak{T}^2, w_2, c) \geq 2$ and $c(N_4) = c(N_{45}) = 3$, we conclude that among N_{34}, N_{44} and N_{445} we have exactly two vertices colored 1 and the other is colored 2. Consequently, we get N_5 is saturated and thus $c(N_{01}) = 3$. This implies that $c(N_{500}) = 1$ and then N_{55} is saturated. Thus, we get a contradiction as we have no feasible coloring for the vertices N_{455} and N_{555}.

– So $c(N_{50}) = 3$. If $c(N_{34}) = c(N_{44}) = 1$, we observe that V_0 is saturated and that among N_{23} and N_{33} we have exactly one vertex colored 2 and one colored 3. Consequently, N_4 is saturated and we reach a contradiction as no coloring is feasible to the vertices N_{334}, N_{344} and N_{444}.
In case N_{44} is colored 1, then N_{34} is colored 2, we observe that among N_{23} and N_{33} we have one vertex colored 1 and the other is colored 3. As a consequence, we get that V_0 and N_4 are saturated. Since $c(N_{334}) = 1$, no coloring is feasible for the vertex N_{344}. If N_{44} is colored 2 (and so N_{34} is colored 1), observe that N_5 is saturated, since there is a vertex colored 2 and another colored 3 in the set $\{N_{00}, N_{01}\}$ and we also find a vertex colored 1 and another colored 2 among vertices N_{445} and N_{455}. Consequently, the vertices N_{555} and N_{550} receive colors 1 and 3 (in some order). Thus, N_{55} is saturated and then $c(N_{500}) = 3$. This is a contradiction as $I_{N_{50}}(\mathfrak{T}^2, w_2, c) \geq 3$. Since no other coloring is feasible for N_{34} and N_{44} as we cannot assign them the color 3, we conclude that the color of N_{55} cannot be 1.

ii. Let us consider now the case $c(N_{55}) = 2$. It implies that $c(N_{50}) = 3$ and, consequently, the vertices N_{00} and N_{01} receive colors 2 and 3 in some order. Thus, N_5 is saturated. In case N_{44} and N_{45} are both colored 1, the vertices N_{34}, N_{445} and N_{455} must be all colored 3. This is a contradiction as $I_{N_4}(\mathfrak{T}^2, w_2, c) \geq 3$. In case $c(N_{44}) = 1$ and $c(N_{45}) = 3$, no coloring is feasible to the vertices N_{445} and N_{455}. If $c(N_{44}) = 3$ and $c(N_{45}) = 1$, observe that $c(N_{34}) = c(N_{445}) = 1$ and that one vertex among N_{555} and N_{550} is colored 1. Thus, $I_{N_{45}}(\mathfrak{T}^2, w_2, c) \geq 3$, a contradiction.

iii. We then conclude that the only possible color for N_{55} is the color 3. Recall N_{50} cannot be colored 1 as N_0 would have two neighbors with its own color.

- Let us first consider the case in which $c(N_{50}) = 2$. As a consequence, we obtain $c(N_{00}) = 3$ and $c(N_{45}) = 1$.

– If $c(N_{01}) = 2$, we can easily check that N_1 and N_5 are saturated. Observe also that N_0 is saturated as N_0 has a neighbor, the vertex V_0, colored 1 and 3 other vertices at distance two also colored 1 which are N_{45}, one vertex in the set $\{N_{11}, N_{12}\}$ and another in the set $\{N_{550}, N_{500}\}$. Consequently, we reach a contradiction as N_{001} and N_{011} should be both colored 3, but then N_{001} would have two neighbors with color 3.

– Thus, $c(N_{01}) = 3$ in this case. It implies that $c(N_{500}) = 1$ and that the color 3 does not appear in the vertices N_{000}, N_{001} and N_{011}. These three vertices can also not be all colored 1 or 2, as N_{001} would have two neighbors of the same color. We cannot have two of these vertices colored 1 as we would have $I_{N_0}(\mathfrak{T}^2, w_2, c) \geq 3$. Consequently, in the set $\{N_{000}, N_{001}, N_{011}\}$ we have one vertex colored 1 and two vertices colored 2. This implies that N_0 and N_1 are saturated. We then reach a contradiction as no feasible color remains to assign to N_{11}.

- Then, we conclude that N_{50} must be colored 3 and then we get $c(N_{00}) = 2$ and $c(N_{01}) = 3$. Observe that if $c(N_{45}) = 2$, we have a contradiction as N_5 is saturated and all the vertices N_{555}, N_{550} and N_{500} should be colored 1. Thus we have that $c(N_{45}) = 1$. Observe that the vertices N_{11} and N_{12} cannot be both colored the same, as we would either violate the interference constraint in N_0 (recall that there is one vertex colored 1 in the set $\{N_{550}, N_{500}\}$) or we would have a vertex with two neighbors of the same color. In case N_{11} and N_{12} are colored 1 and 2, in any order, observe that since N_0 and N_1 are saturated, no coloring is feasible for the vertices N_{001} and N_{011}. We also have no feasible coloring for these vertices in case N_{12} is colored 1 (and then N_0 is saturated) or 2 (in this case N_1 is saturated) and the vertex N_{11} is colored 3.
Thus, $c(N_{12}) = 3$ and suppose first that $c(N_{11}) = 1$. Since N_0 is saturated, the vertices N_{000}, N_{001} and N_{011} can be just colored 2 or 3. In case $c(N_{000}) = 2$, we obtain that $c(N_{001}) = 3$ and $c(N_{011}) = 2$. We reach a contradiction as $I_{N_{00}}(\mathfrak{T}^2, w_2, c) \geq 3$ (observe that one vertex among N_{550} and N_{500} is colored 2). If $c(N_{000}) = 3$, we have that $c(N_{001}) = 2$ and $c(N_{011}) = 3$. Then, we also find a contradiction as $I_{N_{01}}(\mathfrak{T}^2, w_2, c) \geq 3$.
Consequently, $c(N_{11}) = 2$ and N_1 is saturated. In this case, no coloring is feasible for the vertices N_{122}, N_{22} and N_{23} and we complete the proof of this case as no color is feasible for the vertex N_{55}.

(d) In case we have $c(N_3) = 2$ and $c(N_4) = c(N_5) = 3$, we are in a symmetric case to 1b.

(e) If $c(N_3) = 3$ and $c(N_4) = c(N_5) = 2$, we obtain a symmetric case to 1c.

(f) The case $c(N_3) = c(N_5) = 3$ and $c(N_4) = 2$ is symmetric to 2a.

(g) Finally, it is not possible to have $c(N_3) = c(N_4) = 3$ as N_3 would have two neighbors, N_2 and N_4, with its own color.

Case: V_0 **has no neighbor colored 1**.

Now we consider that no vertex has a neighbor with its own color, otherwise we are in the previous case. W.l.o.g, we may conclude that $c(N_0) = c(N_2) = c(N_4) = 2$ and $c(N_1) = c(N_3) = c(N_5) = 3$. Thus, we obtain $c(N_{01}) = c(N_{12}) = c(N_{23}) = c(N_{34}) = c(N_{45}) = c(N_{50}) = 1$. This is a contradiction as $I_{V_0}(\mathfrak{T}^2, w_2, c) \geq 3$.

Now we present the coloring providing the corresponding upper bound.

For a weighted 3-improper 3-coloring of $(\mathfrak{T}^2, w_2)$ set, for $0 \leq j \leq 2$, $E_j = \{(j,0) + a(3f_1) + b(f_2) \mid \forall a, b \in \mathbb{Z}\}$. Then, for $0 \leq j \leq 2$, assign the color $j+1$ to all the vertices in E_j. See Figure 4.7(e).

Now we prove that $(\mathfrak{T}^2, w_2)$ does not admit a weighted 4.5-improper 2-coloring. Again, by contradiction, suppose that there exists a weighted 4.5-improper 2-coloring c of $(\mathfrak{T}^2, w_2)$ with the interference function w_2. A vertex can have at most four neighbors of the same color as it. We analyze some cases:

1. There exists a vertex V_0 with four of its neighbors colored with its own color, say 1. Therefore among the vertices of Γ^2 at most one is colored 1. Consider the two neighbors of V_0 colored 2. First, consider the case in which they are adjacent and let them be N_0 and N_1. In Γ^2, N_0 has three neighbors and four vertices at distance two; since at most one being of color 1, these vertices produce in N_0 an interference equal to 4 and as N_1 is also of color 2, then $I_{N_0}(\mathfrak{T}^2, w_2, c) \geq 5$, a contradiction. In case the two neighbors of V_0 colored 2 are non adjacent, let them be N_i and N_j. At least one of them, say N_i has its three neighbors in Γ^2 colored 2 and it has also at least three vertices at distance two in Γ^2 colored 2; taking into account that N_j is colored 2 and at distance two from N_i, we get $I_{N_i}(\mathfrak{T}^2, w_2, c) \geq 5$, a contradiction.

2. No vertex has four neighbors with its color and there exists at least one vertex V_0 colored 1 that has three neighbors of the same color 1.

(a) The three other neighbors of V_0 colored 2 are consecutive and let them be N_0, N_1 and N_2. N_{34}, N_{44} and N_{45} are all colored 2, otherwise N_4 would have four neighbors colored 1 and we would be in Case 1. At most one of N_{01}, N_{11} and N_{12} has color 2, otherwise N_1 would have four neighbors colored 2 and we would be again in Case 1.

i. N_{11} is colored 2. Then $c(N_{01}) = c(N_{12}) = 1$. As already $I_{V_0}(\mathfrak{T}^2, w_2, c) \geq 4$, there is at most another vertex in Γ^2 colored 1. So either the three vertices N_{22}, N_{23} and N_{33} or the three vertices N_{00}, N_{50} and N_{55} are all colored 2 and then $I_{N_2}(\mathfrak{T}^2, w_2, c) \geq 5$ or $I_{N_5}(\mathfrak{T}^2, w_2, c) \geq 5$, a contradiction.

ii. N_{01} is colored 2 (the case N_{12} is symmetric). Then, $c(N_{11}) = c(N_{12}) = 1$. One of N_{00} and N_{50} is of color 1 otherwise, N_0 has four neighbors of color 2. But then $I_{V_0}(\mathfrak{T}^2, w_2, c) \geq 4.5$ so all the other vertices of Γ^2 are colored 2. Therefore, $I_{N_2}(\mathfrak{T}^2, w_2, c) \geq 5$, a contradiction.

iii. N_{01}, N_{11} and N_{12} all have color 1. In that case $I_{V_0}(\mathfrak{T}^2, w_2, c) \geq 4.5$. Therefore all the other vertices of Γ^2 are colored 2 and $I_{N_0}(\mathfrak{T}^2, w_2, c) \geq 4.5$. So the other vertices at distance two of N_0 are colored 1 and then $I_{N_{01}}(\mathfrak{T}^2, w_2, c) \geq 5$, a contradiction.

(b) Among the three vertices of color 2, only two are consecutive. W.l.o.g., let the three vertices of color 2 be N_0, N_1 and N_3. At least one vertex of N_{50}, N_{00}, N_{01} is colored 1, otherwise N_0 has four neighbors of the same color as it and we would be in the previous case. Similarly at least one vertex of N_{01}, N_{11}, N_{12} is colored 1, otherwise N_1 has four neighbors with its color and we would be in the previous case. At least one vertex of N_{23}, N_{33}, N_{34} is colored 1, otherwise N_3 has three consecutive neighbors of the same color as it and we are in the previous case. Suppose N_{01} is colored 2, then $I_{V_0}(\mathfrak{T}^2, w_2, c) \geq 4.5$ and exactly one of N_{50}, N_{00} and one of N_{11}, N_{12} is colored 1 and N_{45}, N_{55} are colored 2, otherwise $I_{V_0}(\mathfrak{T}^2, w_2, c) \geq 5$. Then $I_{N_0}(\mathfrak{T}^2, w_2, c) \geq 5$, a contradiction. So, $c(N_{01}) = 1$. If both N_{50}, N_{00} are colored 2, then $I_{N_0}(\mathfrak{T}^2, w_2, c) \geq 5$ with three neighbors colored 2 and at least four vertices at distance two colored 2, namely N_3 and three vertices among N_{45}, N_{55}, N_{11}, N_{12} (at most one vertex of these could be of color 1, otherwise $I_{V_0}(\mathfrak{T}^2, w_2, c) \geq 5$). So, one of N_{50}, N_{00} is colored 1 and all the other vertices in $\{N_{11}, N_{12}, N_{22}, N_{44}, N_{45}, N_{55}\}$ are colored 2 implying that $I_{N_3}(\mathfrak{T}^2, w_2, c) \geq 5$, a contradiction.

(c) No two vertices of color 2 are consecutive. W.l.o.g, let these vertices be N_0, N_2, N_4. The three neighbors of N_0 (resp. N_1, N_2) in Γ^2 that are not neighbors of V_0 cannot be all colored 2, otherwise we are in Case (a). So exactly one neighbor of N_0, N_1, N_2 in Γ^2 is colored 1, otherwise $I_{V_0}(\mathfrak{T}^2, w_2, c) \geq 5$. Furthermore all the other vertices of Γ^2 are colored 2. Then, if $c(N_{12}) = c(N_{45}) = 2$, we conclude that $I_{N_0}(\mathfrak{T}^2, w_2, c) \geq 5$, a contradiction. Consequently, w.l.o.g., suppose that $c(N_{12}) = 1$. In this case, N_{23} has at least three neighbors colored 2 and we are in some previous case.

3. No vertex has three neighbors colored with its own color, but there exists at least one vertex, say V_0, of color 1 that has two neighbors colored 1.

(a) These two neighbors are consecutive say N_0 and N_1. The neighbors of N_3 and N_4 in Γ^2 are all colored 1, otherwise they would have at least three neighbors with the same color. Similarly, at least one of N_{12} and N_{22} is colored 1, otherwise N_2 would have at least three neighbors also colored 2. Then, $I_{V_0}(\mathfrak{T}^2, w_2, c) \geq 5$, a contradiction.

(b) These two neighbors are of the form N_i and N_{i+2}, for some $0 \leq i \leq 3$. W.l.o.g., let these neighbors be N_0 and N_2. Thus, the three neighbors of N_4 in Γ^2, N_{34}, N_{44} and N_{45} are colored 1 and at least one vertex of N_{23} and N_{33} (resp. N_{55} and N_{50}) is colored 1. Moreover, at least one vertex of N_{01}, N_{11} and N_{12} must be colored 1, otherwise N_1 would have three neighbors with its color. Consequently, $I_{V_0}(\mathfrak{T}^2, w_2, c) \geq 5$, a contradiction.

(c) These two neighbors are of the form N_i and N_{i+3}, for some $0 \leq i \leq 2$. W.l.o.g., let these neighbors be N_0 and N_3. Again, at least three vertices among N_{01}, N_{11}, N_{12}, N_{22} and N_{23} and at least three other vertices among N_{34}, N_{44}, N_{45}, N_{55} and N_{50} are colored 1. Consequently, $I_{V_0}(\mathfrak{T}^2, w_2, c) \geq 5$, a contradiction.

4. No vertex has two neighbors of the same color. Suppose V_0 is colored 1 and has only one neighbor N_0 colored 1. Then, its other five neighbors are colored 2 and N_2 has two neighbors of the color 2, a contradiction.

A weighted 5-improper 2-coloring of $(\mathfrak{T}^2, w_2)$ is obtained as follows: for $0 \leq j \leq 1$, let $F_j = \{(j, 0) + a(2f_1) + b(f_1 + 2f_2) \mid \forall a, b \in \mathbb{Z}\}$ and $F'_j = \{(j-1, 1) + a(2f_1) +$

$b(f_1 + 2f_2) \mid \forall a, b \in \mathbb{Z}\}$. Then, for $0 \leq j \leq 1$, assign the color $j+1$ to all the vertices in F_j and in F'_j. See Figure 4.7(f).

Since each vertex has six neighbors and twelve vertices at distance two in $\mathfrak{T}$, there is no weighted t-improper 1-coloring of $(\mathfrak{T}^2, w_2)$, for any $t < 12$. Obviously, there is a weighted 12-improper 1-coloring of $\mathfrak{T}^2$. □

4.3 Integer Linear Programming Formulations, Algorithms and Results

In this section, we look at how to solve the WEIGHTED IMPROPER COLOURING and THRESHOLD IMPROPER COLOURING for general instances inspired by the practical motivation. We present integer linear programming models for both problems. These models can be solved exactly for small sized instances using solvers like CPLEX[1]. For larger instances, the solvers can take a prohibitive time to provide exact solutions. It is usually possible to obtain a sub-optimal solution stopping the solver after a limited time. If the time is too short, the quality of the solution may be unsatisfactory. Thus, we introduce two algorithmic approaches to find good solutions for THRESHOLD IMPROPER COLOURING in a short time: a simple polynomial-time greedy heuristic and an exact Branch-and-Bound algorithm. We compare the three methods on different sets of instances, among them Poisson-Voronoi tessellations as they are good models of antenna networks [BKLZ97, GK00, HAB+09].

4.3.1 Integer Linear Programming Models

Given an edge-weighted graph $G = (V, E, w)$, $w : E \to \mathbb{R}^*_+$, and a positive real threshold t, we model WEIGHTED IMPROPER COLORING by using two kinds of binary variables. Variable x_{ip} indicates if vertex i is colored p and variable c_p indicates if color p is used, for every $1 \leq i \leq n$ and $1 \leq p \leq l$, where l is an upper bound for the number of colors needed in an optimal weighted t-improper coloring of G. l can be trivially chosen of value n, but a better value may be given by the results of Section 4.1. The model follows:

$$\begin{array}{lll}
\min & \sum_{p=1}^{l} c_p & \\
\text{subject to} & & \\
& \sum_{ij \in E \text{ and } j \neq i} w(i,j) x_{jp} \leq t + M(1 - x_{ip}) & \forall i \in V, 1 \leq p \leq l \\
& c_p \geq x_{ip} & \forall i \in V, 1 \leq p \leq l \\
& \sum_{p=1}^{l} x_{ip} = 1 & \forall i \in V \\
& x_{ip} \in \{0,1\} & \forall i \in V, 1 \leq p \leq l \\
& c_p \in \{0,1\} & 1 \leq p \leq l
\end{array}$$

where M is a large integer. For instance, it is sufficient to choose $M > \sum_{uv \in E} w(u,v)$.

[1] http://www-01.ibm.com/software/integration/optimization/cplex-optimizer/

For THRESHOLD IMPROPER COLORING, given an edge-weighted graph $G = (V, E, w)$, $w : E \to \mathbb{R}_+^*$, and a number of possible colors $k \in \mathbb{N}^*$, the model we consider is:

$$\begin{array}{lcl} \min & t & \\ \text{subject to} & & \\ & \sum_{ij \in E \text{ and } j \neq i} w(i,j) x_{jp} \leq t + M(1 - x_{ip}) & \forall i \in V, 1 \leq p \leq l \\ & \sum_{p=1}^{k} x_{ip} = 1 & \forall i \in V \\ & x_{ip} \in \{0, 1\} & \forall i \in V, 1 \leq p \leq l \end{array}$$

We give directly these models to the ILP solver CPLEX without using any preprocessing or any other technique to speed the search for an optimal solution.

4.3.2 Algorithmic approach

In this section, we show a Branch-and-Bound algorithm and a randomized greedy heuristic to tackle THRESHOLD IMPROPER COLOURING. Both are based on common procedures to determine the order in which vertices are colored and colors are tried for a single vertex. Although, the Branch-and-Bound needs an ordering of the vertices to be colored as input while the heuristic colors the vertices at the same time the order is being processed.

4.3.2.1 Order of vertices and colors

The order in which the vertices are chosen to be colored follows essentially the same idea as the DSATUR algorithm, created by Daniel Brélaz [Bré79].

Consider a graph $G = (V, E, w)$, $w : E \to \mathbb{R}_+^*$ and a partial coloring $c : U \to \{1, \ldots, k\}$, where $U \subseteq V$. We say that vertex v is *colored* if $v \in U$, otherwise it is *uncolored*. We define the *total potential interference* in vertex v to be:

$$I_{c,v}^{tot} = \sum_{\{u \in V | uv \in E \text{ and } v \notin U\}} w(u, v),$$

which is the sum of interferences for all colors induced in v by all its already colored neighbors.

The idea for both algorithms is to first color vertices with highest total potential interference. Whenever more than one vertex has the highest total potential interference, one of them is chosen at random. At the beginning, when all vertices have $I_{c,v}^{tot} = 0$, one of the highest weighted degree is chosen instead.

Consider the following steps:

1. Color a random vertex with maximal sum of incoming weights.

2. Color a random vertex with maximal total potential interference.

3. If all vertices all colored, stop. Otherwise, repeat step 2.

Note that the total potential interference does not depend on the actual colors assigned to the vertices. Thus, in order to decide which is the next vertex to be colored, both algorithms, Branch-and-Bound and heuristic, use these three steps. However, the Branch-and-Bound algorithm needs an order to color the vertices as input. So, we decide which order to give to the Branch-and-Bound algorithm as input by running these three steps and using a single color.

The procedure above specifies the order of vertices. For the order of colors to try, we define the *potential interference* in vertex v for color x as:

$$I_{c,v,x} = \sum_{\{u \in V | uv \in E \text{ and } c(v)=x\}} w(u,v)$$

Anytime one of our algorithms colors a vertex, it tries the colors in order of increasing potential interference.

4.3.2.2 Branch-and-Bound Algorithm

Having an ordering procedure for both vertices and colors, we construct a simple Branch-and-Bound algorithm using them. The order of vertices to color is fixed before running the algorithm, following the procedure in Section 4.3.2.1. Then, the ordered vertices are colored by a recursive function that tries all the possible colors for each vertex as far as no interference constraint is violated. The order in which the colors are tried is also presented in the previous section. Our algorithm outputs all the feasible colorings it finds and, as all the possible colors are tried, the one using the minimum number of colors is an optimal one.

Here you have a pseudo code for the algorithm:

Algorithm 2: Branch&Bound

input : edge-weighted graph (G, w), number of colors k, partial coloring c, upper bound t and corresponding coloring $\tilde{c}$, order in which vertices should be colored O

output: new upper bound t' and corresponding coloring $\tilde{c}$'

if $\max_{v \in V} I_v(G, w, c) \geq t$ **then**
  return t and $\tilde{c}$
if *all vertices are colored in* c **then**
  return ($\max_{v \in V} I_v(G, w, c)$ and c)
v = next vertex uncolored in c according to O
for $x \in$ *possible colors in order of increasing* $I_{c,v,x}$ **do**
  (t and $\tilde{c}$) = Branch&Bound(G, k, $c \cap (v \leftarrow x)$, t, $\tilde{c}$, O)
return t and $\tilde{c}$

Where by $c \cap (v \leftarrow x)$ we mean a partial coloring where color of vertex v (which was uncolored in c) is set to x, and colors of all other vertices are as in c. The algorithm is first called with all vertices uncolored and $t = \infty$.

This algorithm displays a problematic behavior. Imagine the partial coloring of the first few vertices yields good results locally, but implies a suboptimal interference at a more distant part of the graph. As the solution search takes exponential time in number of vertices, it is easy to envision that the time required to change the coloring of first vertices can be prohibitively long.

4.3.2.3 Greedy Heuristic

Here we propose a randomized greedy heuristic that, repeated multiple, but not exponentially many times, finds similar solutions to the above Branch-and-Bound without the mentioned problem. On the other hand, there are some solutions that are impossible to find with it, no matter the number of tries. An example of such an unobtainable solution is the optimal coloring of infinite square grid with 2 colors.

Algorithm 3: Leveling Heuristic

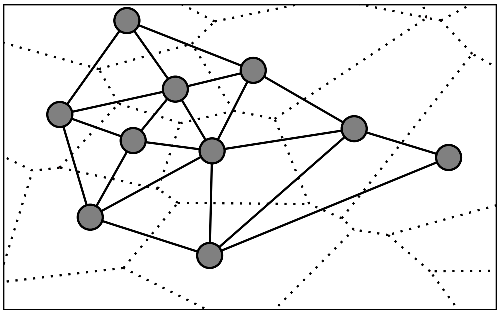
```
input  : edge-weighted graph (G, w), number of colors k, upper bound t
output: failed or a coloring c

c(v) = ∅   ∀v ∈ V
for i ∈ {1, ..., |V|} do
    v = next, in order of increasing I^tot_{c,v}, vertex uncolored in c
    for x ∈ possible colors in order of increasing I_{c,v,x} do
        if coloring v with x does not cause max_{v∈V} I_v(G, w, c) ≥ t then
            c(v) = x
            break the inner loop
    if c(v) = ∅ then
        return failed
return c
```

Note that there is substantial randomness in this algorithm. The first vertex is the one of the ones with highest weighted degree. In the extreme case of regular graphs, this already means any vertex at random. If we use the simple interference function defined in Section 4.2, then the second vertex is a random neighbor of the first vertex. Any time there are multiple vertices with maximum total potential interference, we choose one at random. Similarly, the choice of colors is also random in case of equal potential interference.

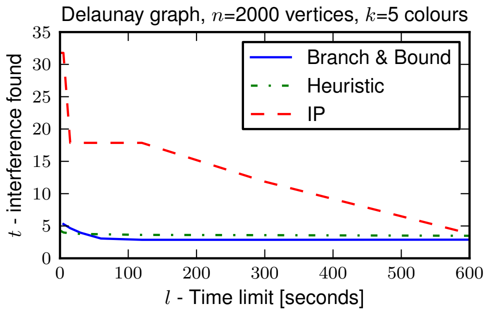
Above algorithm is first called with $t = \infty$. Whenever it returns a coloring, we set $t = \max_{v \in V} I_v(G, w, c)$ for further iterations. It is repeated for a given number of times, or until a time limit is reached. In all instances in the following sections the program is constrained by a time limit. This means that the algorithm is called for an unknown, but probably big number of times (e.g. for a 6-regular grid of 1024 vertices the program performs on average over 500 runs of the algorithm per second).

As a *randomized greedy coloring* heuristic, it has to be ran multiple times to achieve satisfactory results. This is not a practical issue due to low computational cost of each run. The local immutable coloring decision is taken in time $O(k\Delta)$. Then, after each such decision, the interference has to be propagated, which takes linear time in the vertex degree. This gives a computational complexity bound $O(kn\Delta)$-time.

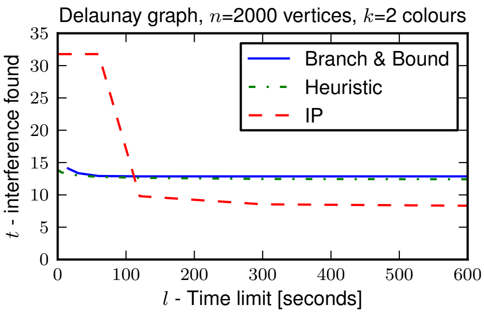
4.3.3 Validation



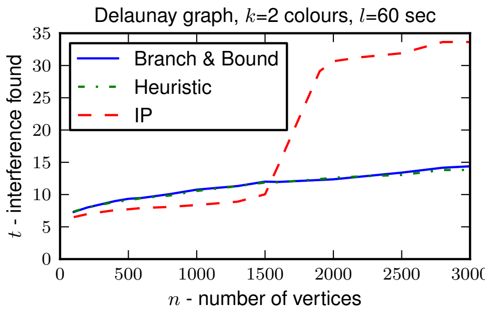
(a) Example Delaunay graph, dotted lines delimit corresponding Voronoi diagram cells



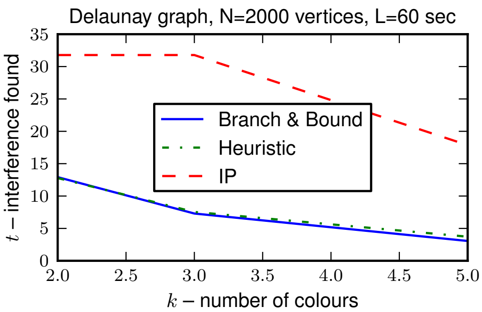
(b) Over time



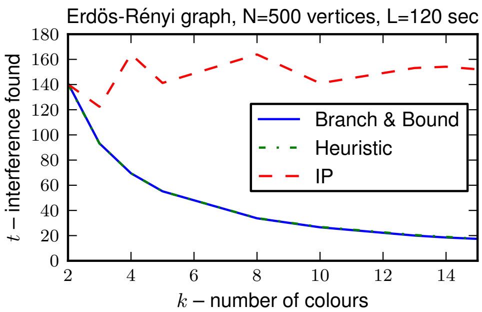
(c) Over time



(d) Over size



(e) Over colors



(f) Over colors

Figure 4.9: Results comparison for Leveling heuristic, Branch-and-Bound algorithm and Integer Linear Programming Formulation.

In this section we validate our algorithmic approaches at Threshold Improper colouring, by examining performance of their implementations. Tests cover a wide range of parameters, mostly on Delaunay graphs (see section 4.3.3.2).

4.3.3.1 Implementation

The ILP model is constructed out of the input graph and given directly to the CPLEX ILP solver. Branch-and-Bound algorithm is implemented in a straightforward way in the Python programming language. The greedy heuristic has a highly optimized implementation in the Cython programming language[2].

In results displayed below, all programs are run simultaneously on the same quad-core enterprise-grade CPU. Both the Branch-and-Bound and greedy heuristic are limited to a single core. CPLEX is allowed to both the remaining cores.

4.3.3.2 Graphs

We consider random Delaunay graphs (dual of Voronoi diagram). This kind of graphs is an intuitive approximation of a network of irregular cells. To obtain a graph in this class, take a set of random points uniformly distributed over a square. These represent the vertices of the graph. To obtain the edges, compute a Delaunay triangulation. This can be done e.g. with Fortune's algorithm described in [For87] in $O(n \log n)$ time.

See Figure 4.9(a) for a depiction of a fragment of such graph. Vertices are arranged according to the positions of original random points. Dotted lines delimit corresponding Voronoi diagram cells. Only edges between vertices visible on the illustration are displayed.

Note that, to follow the model of the physical motivation, we are dealing with very sparse graphs. The average degree in Delaunay graph G converges to six (this results follows from the observation that G is planar and triangulated, thus $|E(G)| = 3|V(G)| - 6$ by Euler's formula). To get an idea about the proposed algorithms' performance in denser graphs, we also run some tests on Erdös-Rényi graphs with expected degree equal to 50.

The interference model we consider in all experiments is the one described in Section 4.2: adjacent nodes interfere by 1 and nodes at distance two interfere by 1/2.

4.3.3.3 Results

Figure 4.9 shows a performance comparison of the above-mentioned algorithms. For all the plots, each data point represents an average over a number (between 24 and 100) of different graphs. The experiment procedure is as follows. For each graph size considered in an experiment, a number of graphs is generated. Each of those graphs is transformed into a set of instances, one for each desired number of allowed colors. All the programs are run on each instance, once for each desired value of time limit. Finally, a data point is created with results and all the parameters, averaged over the number of graphs.

Figures 4.9(b) and 4.9(c) plot how results for a problem instance get enhanced with increasing time limits. Plot 4.9(d) shows how well all the programs scale with

[2] This is the faster implementation envisioned in [ABG+11c].

increasing graph sizes. Plots 4.9(e) and 4.9(f) show decreasing interference along increasing the number of colors allowed.

One immediate observation about both the heuristic and Branch-and-Bound algorithm is that they provide good solutions in relatively short time. On the other hand, with limited time, they fail to improve up to optimal results, especially with a low number of allowed colors. An example near-optimal solution found in around three minutes was not improved by Branch-and-Bound in over six days.

The heuristic, is able to provide good results in sub-second times and scales better with increasing graph sizes than the Branch-and-Bound. It is also not prone to spending a lot time exploring a sub-optimal branch of a decision tree. Still, in many cases it is unable to obtain optimal results and displays a worse end result than an integer linear program, given enough time.

Solving the ILP does not scale with increasing graph sizes as well as our simple algorithms. Furthermore, Figure 4.9(e) reveals one problem specific to ILP. When increasing the number of allowed colors, obtaining small interferences gets easier. But this introduces additional constraints in the formulation, thus increasing the complexity for a solver.

Proposed algorithms also work well for denser graphs. Figure 4.9(f) plots interferences for different numbers of colors allowed found by the programs for an Erdös-Rényi graph with $n = 500$ and $p = 0.1$. This gives us an average degree equal to 50. Both Branch-and-Bound and heuristic programs achieve acceptable, and nearly identical, results. But the large number of constraints makes the integer linear programming formulation very inefficient.

4.4 Conclusion, Open Problems and Future Directions

In this work, we introduced and studied a new coloring problem, Weighted Improper Coloring. This problem is motivated by the design of telecommunication antenna networks in which the interference between two vertices depends on different factors and can take various values. For each vertex, the sum of the interferences it receives should be less than a given threshold value.

We first give general bounds on the weighted-improper chromatic number. We then study the particular case of infinite paths, trees and grids: square, hexagonal and triangular. For these graphs, we provide their weighted-improper chromatic number for all possible values of t.

Finally, we propose a heuristic and a Branch-and-Bound algorithm to find good solutions of the problem. We compare their results with the one of an integer linear programming formulation on cell-like networks, Poisson-Voronoi tessellations.

Many problems remain to be solved:

- The study of the grid graphs, we considered a specific function where vertices at distance one interfere by 1 and vertices at distance two by $1/2$. Other weight functions should be considered. e.g. $1/d^2$ or $1/(2^{d-1})$, where d is the distance between vertices.

- Other families of graphs could be considered, for example hypercubes.
- We showed that the Threshold Improper Coloring problem can be transformed into a problem with only two possible weights on the edges 1 and ∞, that is a mix of proper and improper coloring. This simplify the nature of the graph interferences but at the cost of an important increase of instance sizes. We want to further study this. In particular, let $G = (V, E, w)$ be an edge-weighted graph where the weights are all equal to 1 or M. Let G_M be the subgraph of G induced by the edges of weight M; is it true that if $\Delta(G_M) << \Delta(G)$, then $\chi_t(G, w) \leq \chi_t(G) \leq \left\lceil \frac{\Delta(G,w)+1}{t+1} \right\rceil$? A similar result for $L(p, 1)$-labeling [HRS08] suggests it could be true.

Note that the problem can also be solved *algorithmically* for other classes of graphs and for other functions of interference. We started looking in this direction in [ABG+11a]. The problem can be expressed as a linear program and then be solved exactly using solvers such as CPLEX or Glpk[3] for small instances of graphs. For larger instances, we propose a heuristic algorithm inspired by DSATUR [Bré79] but adapted to the specifics of our coloring problem. We used it to derive coloring with few colors for Poisson-Voronoi tessellations as they are good models of antenna networks [BKLZ97, GK00, HAB+09]. We plan to further investigate the algorithmic side of our coloring problem.

[3]http://www.gnu.org/software/glpk/

Chapter 5

Good Edge-Labeling

Contents

A classical and widely studied problem in WDM (Wavelength Division Multiplexing) networks is the Routing and Wavelength Assignment (RWA) problem [Muk97, RS95, BS91]. It consists in finding routes, and their associated wavelength as well, to satisfy a set of traffic requests while minimizing the number of used wavelengths. This is a difficult problem which is, in general, NP-hard. Thus, it is often split into two distinct problems: First, routes are found for the requests. Then, in a second step, these routes are taken as an input. Wavelengths must be associated to them in such a way that two routes using the same fiber do not have the same wavelength. The last problem can be reformulated as follows: Given a digraph and a set of dipaths, corresponding to the routes for the requests, find the minimal number of wavelengths w needed to assign different wavelengths to dipaths sharing an edge. This problem can be seen as a coloring problem of the *conflict graph* which is defined as follows: It has one vertex per dipath and two vertices are linked by an edge if their corresponding dipaths share an edge. In [BCP09], Bermond et al. studied the RWA problem for UPP-DAG which are acyclic digraphs (or DAG) in which there is at most one dipath from one vertex to another. In such digraph the routing is forced and thus the unique problem is the wavelength assignment one.

In their paper, they introduce the notion of good edge-labeling. An *edge-labeling* of a graph G is a function $\phi : E(G) \to \mathbb{R}$. A path is *increasing* if the sequence of its edge labels is non-decreasing. An edge-labeling of G is *good* if, for any two distinct vertices u, v, there is at most one increasing (u, v)-path. Bermond et al. [BCP09] showed that the conflict graph of a set of dipaths in a UPP-DAG has a *good edge-labeling*. Conversely, for any graph admitting a good edge-labeling one can exhibit a family of dipaths on a UPP-DAG whose conflict graph is precisely this graph. Bermond et al. [BCP09] then use the existence of graphs with a good edge-labeling and large chromatic number to prove that there exist sets of requests on UPP-DAGs with load 2 (an edge is shared by at most two paths) requiring an arbitrarily large number of wavelengths.

To obtain other results on this problem, it may be useful to identify the *good* graphs which admit a good edge-labeling and the *bad* ones which do not. Bermond et al. [BCP09] noticed that C_3 and $K_{2,3}$ are bad. J.-S. Sereni [Ser06] asked whether every $\{C_3, K_{2,3}\}$-*free graph* (i.e., with no C_3 nor $K_{2,3}$ as a subgraph) is good. In Section 5.2, we answer this question in the negative. We give an infinite family of bad graphs none of which is the subgraph of another.

Furthermore, in Section 5.3, we prove that determining if a graph has a good edge-labeling is NP-complete using a reduction from Not-All-Equal 3-SAT.

In Section 5.4, we show large classes of good graphs: forests, C_3-free outerplanar graphs, planar graphs of girth at least 6. To do so, we use the notion of *critical* graph which is a bad graph such that every proper subgraph of which is good. Clearly, a good edge-labeling of a graph induces a good edge-labeling of all its subgraphs. So every bad graph must contain a critical subgraph. We establish several properties of critical graphs. In particular, we show that they have no *matching-cut*. Hence, a result of Farley and Proskurowski [FP84, BFP11] (Theorem 15) implies that a critical graph G has at least $\frac{3}{2}|V(G)| - \frac{3}{2}$ edges.

In Section 5.5, we use the characterization of graphs with no matching-cut and $\lceil \frac{3}{2}|V(G)| - \frac{3}{2} \rceil$ edges given by Bonsma [Bon06, BFP11] to slightly improve this result. We show that a critical graph G has at least $\frac{3}{2}|V(G)| - \frac{1}{2}$ edges unless G is C_3 or $K_{2,3}$.

Finally, we present avenues for future research.

5.1 Preliminaries

In this section, we give some technically useful propositions. Their proofs are straightforward and left to the reader.

A path is *decreasing* if the sequence of its edge labels is non-increasing. Then, a path $u_1u_2\dots u_k$ is decreasing if and only if its reversal $u_ku_{k-1}\dots u_1$ is increasing. Hence an edge-labeling is good if and only if for any two distinct vertices u, v, there is at most one decreasing (u, v)-path. Equivalently, an edge-labeling is good if and only if for any two distinct vertices u, v, there is at most one increasing (u, v)-path and at most one decreasing (u, v)-path.

Let x and y be two vertices of G. Two distinct (x, y)-paths P and Q are *independent* if $V(P) \cap V(Q) = \{x, y\}$. Observe that in an edge-labeled graph G, there are two vertices u, v with two increasing (u, v)-paths if and only if there are two vertices u', v' with two increasing independent (u', v')-paths. Hence the definition of good edge-labeling may be expressed in terms of independent paths.

Proposition 7. *An edge-labeling is good if and only if for any two distinct vertices u and v, there are no two increasing independent (u, v)-paths.*

As above the definition may also be in terms of decreasing independent paths. In this work, we sometimes use Proposition 7 without referring explicitly to it.

Let ϕ be a good edge-labeling of a graph G. If $\phi(E(G)) \subset A$ then for every strictly increasing function $f : A \to B$, $f \circ \phi$ is a good edge-labeling into B. Moreover if ϕ is not injective, one can transform it into an injective one by recursively adding a tiny ε to one of the edges having the same label. Hence we have the following.

Proposition 8. *Let G be a graph and A an infinite set in $\mathbb{R} \cup \{-\infty, +\infty\}$. Then G admits a good edge-labeling if and only if it admits an injective good edge-labeling into A.*

Let ϕ be an injective good edge-labeling into an infinite set in $\mathbb{R} \cup \{-\infty, +\infty\}$ of a graph G. Observe that an injective good edge-labeling ϕ' of G into $\mathbb{R}$ can be easily found by just replacing the label $-\infty$ ($+\infty$) by the smaller (resp., greater) label assigned by ϕ minus (resp., plus) some $\varepsilon > 0$.

5.2 Bad graphs

A path of length one is both increasing and decreasing, and a path of length two is either increasing or decreasing. So C_3 has clearly no good edge-labeling. Also $K_{2,3}$ does not admit a good edge-labeling since there are three paths of length two between the two vertices of degree 3. Hence, in any edge-labeling, two of them are increasing or two of them are decreasing.

Extending this idea, we now construct an infinite family of bad graphs, none of which is the subgraph of another. The construction of this family is based on the graphs H_k defined below. These graphs play the same role as a path of length two because they have two vertices u and v such that any good edge-labeling of H_k has either a (u, v)-increasing path or a (v, u)-increasing path.

For any integer $k \geq 3$, let H_k be the graph defined by

$$\begin{aligned} V(H_k) &= \{u, v\} \cup \{u_i \mid 1 \leq i \leq k\} \cup \{v_i \mid 1 \leq i \leq k\}, \\ E(H_k) &= \{uu_i \mid 1 \leq i \leq k\} \cup \{u_i v_i \mid 1 \leq i \leq k\} \cup \{v_i v \mid 1 \leq i \leq k\}, \\ &\quad \cup \{v_i u_{i+1} \mid 1 \leq i \leq k\} \end{aligned}$$

with $u_{k+1} = u_1$. See Figure 5.1.

Observe that the graph H_k has no $K_{2,3}$ as a subgraph, and for $i \neq k$, H_i is not a subgraph of H_k.

Proposition 9. *Let $k \geq 3$. For every good edge-labeling, the graph H_k has either an increasing (u, v)-path or an increasing (v, u)-path.*

Proof. Suppose, by way of contradiction, that H_k has a good edge-labeling ϕ having no increasing (u, v)-path and no increasing (v, u)-path. By Proposition 8, we may assume that ϕ is injective.

A key component in this proof is the following observation which follows easily from the fact that ϕ is good.

Observation 1. *Suppose $x_1x_2x_3x_4x_1$ is a 4-cycle. Then, either*

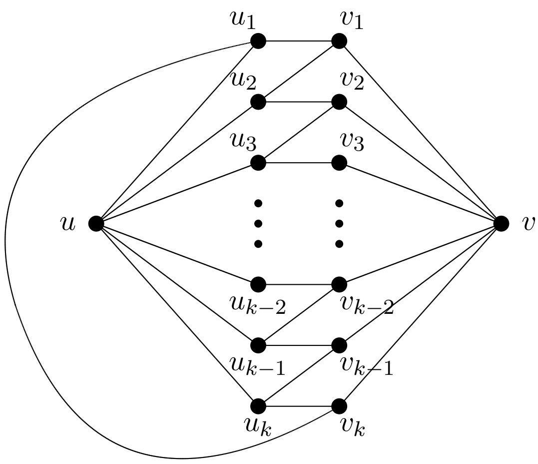


Figure 5.1: Graph H_k

- *$\phi(x_4x_1) < \phi(x_1x_2)$, $\phi(x_2x_3) < \phi(x_1x_2)$, $\phi(x_2x_3) < \phi(x_3x_4)$ and $\phi(x_1x_4) < \phi(x_3x_4)$; or*
- *all those inequalities are reversed.*

By symmetry, we may assume that $\phi(uu_1) < \phi(u_1v_1)$. By Observation 1, $\phi(v_1u_2) < \phi(u_1v_1)$, $\phi(v_1u_2) < \phi(uu_2)$ and $\phi(uu_1) < \phi(uu_2)$. Then, since vv_1u_2u is not increasing, $\phi(u_2v_1) < \phi(v_1v)$. Again by Observation 1, $\phi(v_2v) < \phi(u_2v_2)$. Thus since uu_2v_2v is not increasing $\phi(uu_2) < \phi(u_2v_2)$.

Applying the same reasoning, we obtain that $\phi(uu_2) < \phi(uu_3)$ and $\phi(uu_3) < \phi(u_3v_3)$ and so on, iteratively, $\phi(uu_1) < \phi(uu_2) < \cdots < \phi(uu_k) < \phi(uu_1)$, a contradiction. □

For convenience we denote by H_2 the path of length 2 with end vertices u and v. Let i, j, k be three integers greater than 1. The graph $J_{i,j,k}$ is the graph obtained from disjoint copies of H_i, H_j and H_k by identifying the vertices u of the three copies and the vertices v of the three copies.

Proposition 10. *Let i, j, k be three integers greater than 1. Then $J_{i,j,k}$ is bad.*

Proof. Suppose, by way of contradiction, that $J_{i,j,k}$ admits a good edge-labeling. By Proposition 9, in each of the subgraphs H_i, H_j and H_k, there is either an increasing (u, v)-path or an increasing (v, u)-path. Hence in $J_{i,j,k}$, there are either two increasing (u, v)-paths or two increasing (v, u)-paths, a contradiction. □

5.3 NP-completeness

In this section, we prove that it is an NP-complete problem to decide if a bipartite graph admits a good edge-labeling. We give a reduction from the NOT-ALL-EQUAL (NAE) 3-SAT Problem [Sch78] which is defined as follows:

Instance: A set V of variables and a collection $\mathcal{C}$ of clauses over V such that each clause has exactly 3 literals.

Question: Is there a truth assignment such that each clause has at least one true and at least one false literal?

For sake of clarity, we first present the NP-completeness proof for general graphs.

Theorem 13. *The following problem is NP-complete.*
Instance: *A graph G.*
Question: *Does G have a good edge-labeling?*

Proof. Given a graph G and an injective edge-labeling ϕ into $\mathbb{R}$, one can check in polynomial time if ϕ is good or not using the following algorithm where $(u_1v_1, \dots, u_mv_m)$ is an ordering of the edges of G in increasing order according to their labels.

```
foreach u ∈ V(G) do
    Set V(T) := {u}, E(T) := ∅;
    foreach i=1 to m do
        if {u_i, v_i} ⊂ V(T) then
            return "bad edge-labeling";
        if u_i ∈ V(T) (and v_i ∉ V(T)) then
            V(T) := V(T) ∪ {v_i} and E(T) := E(T) ∪ {u_i v_i};
return "good edge-labeling";
```

Indeed, for each vertex u, the above algorithm grows the tree T of increasing paths from u: at each step i, T is the tree of increasing paths from u with arcs with labels less than $\phi(u_iv_i)$. In particular, there is an increasing (u, v)-path P_v for every $v \in V(T)$. Hence if $u_i \in V(T)$ and $v_i \in V(T)$ then P_{v_i} and $P_{u_i} + u_iv_i$ are two increasing (u, v_i)-paths, so the edge-labeling is not good. If $u_i \in V(T)$ and $v_i \notin V(T)$, then $P_{u_i} + u_iv_i$ is a new increasing path that must be included into T. Finally, if $u_i \notin V(T)$ and $v_i \notin V(T)$, then u_iv_i will not be in any increasing path from u as the edges to be considered after it have larger labels.

Hence the considered problem is in NP.

To prove that the problem is NP-complete, we will reduce the NAE 3-SAT Problem without repetition (i.e. a variable appears at most once in each clause) which is equivalent to NAE 3-SAT Problem (with repetition) to it. (For each repeated variable x, we introduce two other variables y and z. Then the second (third) occurrence of x in a clause is replaced by y (z). Then, x, y, z are forced to have the same truth assignment by adding $\bar{x} \vee y \vee z$, $x \vee \bar{y} \vee z$, $x \vee y \vee \bar{z}$, $\bar{x} \vee \bar{y} \vee z$, $\bar{x} \vee y \vee \bar{z}$, and $x \vee \bar{y} \vee \bar{z}$ to the instance.)

Let $V = \{x_1, \dots, x_n\}$ and $\mathcal{C} = \{C_1, \dots, C_m\}$ be an instance I of the NAE 3-SAT Problem without repetition. We shall construct a graph G_I in such a way that I has an answer yes for the NAE 3-SAT Problem if and only if G_I has a good edge-labeling.

For each variable x_i, $1 \leq i \leq n$, we create a variable graph VG_i defined as follows (See Figure 5.2.):

$$\begin{aligned} V(VG_i) &= \{v_k^{i,j} \mid 1 \leq j \leq m, 1 \leq k \leq 4\} \cup \{r_k^{i,j} \mid 1 \leq j \leq m, 1 \leq k \leq 4\} \\ &\quad \cup \{s_k^{i,j} \mid 1 \leq j \leq m, 1 \leq k \leq 4\}. \\ E(VG_i) &= \{v_k^{i,j} v_{k+1}^{i,j} \mid 1 \leq j \leq m, 1 \leq k \leq 3\} \cup \{v_4^{i,j} v_1^{i,j+1} \mid 1 \leq j \leq m-1\} \\ &\quad \cup \{v_k^{i,j} r_k^{i,j} \mid 1 \leq j \leq m, 1 \leq k \leq 4\} \cup \{v_k^{i,j} s_k^{i,j} \mid 1 \leq j \leq m, 1 \leq k \leq 4\} \\ &\quad \cup \{v_4^{i,j} r_1^{i,j} \mid 1 \leq j \leq m\} \cup \{v_k^{i,j+1} r_{k+1}^{i,j} \mid 1 \leq j \leq m-1, 1 \leq k \leq 3\} \\ &\quad \cup \{v_4^{i,j} s_1^{i,j} \mid 1 \leq j \leq m\} \cup \{v_k^{i,j+1} s_{k+1}^{i,j} \mid 1 \leq j \leq m-1, 1 \leq k \leq 3\}. \end{aligned}$$

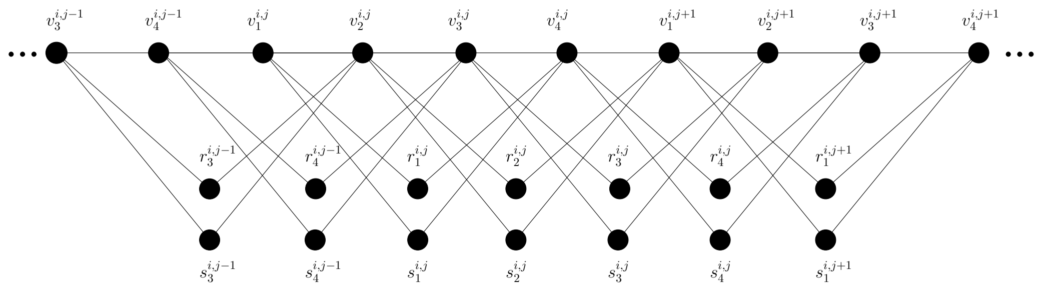


Figure 5.2: The variable graph VG_i

For each clause $C_j = l_1 \vee l_2 \vee l_3$, $1 \leq j \leq m$, we create a clause graph CG_j defined as follows (See Figure 5.3.):

$$\begin{aligned} V(CG_j) &= \{c^j, b_1^j, b_2^j, b_3^j\}; \\ E(CG_j) &= \{c^j b_1^j, c^j b_2^j, c^j b_3^j\}. \end{aligned}$$

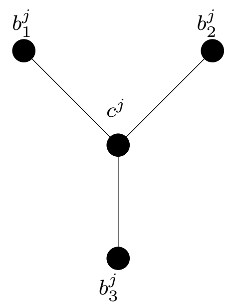


Figure 5.3: The clause graph CG_j.

Now, for each literal l_k, $1 \leq k \leq 3$, if l_k is the non-negated variable x_i, we identify b_k^j, c^j and b_{k+1}^j (index k is taken modulo 3) with $v_1^{i,j}$, $v_2^{i,j}$ and $v_3^{i,j}$, respectively. Otherwise, if l_k is the negated variable $\bar{x}_i$, we identify b_k^j, c^j and b_{k+1}^j with $v_3^{i,j}$, $v_2^{i,j}$ and $v_1^{i,j}$, respectively.

Let us first show that, if G_I has a good edge-labeling ϕ, then there is a truth assignment such that each clause of I has at least one true literal and at least one false literal.

By Proposition 8, we may assume that ϕ is injective.

Claim 2. *Let* $1 \leq i \leq n$. *If* $\phi(v_1^{i,1}v_2^{i,1}) < \phi(v_2^{i,1}v_3^{i,1})$ *then* $\phi(v_1^{i,j}v_2^{i,j}) < \phi(v_2^{i,j}v_3^{i,j})$ *for all* $1 \leq j \leq m$.

Proof. By induction on j. A path of length two is necessarily increasing or decreasing. Now $v_1^{i,j}$ is joined to $v_4^{i,j}$ by two paths of length two via $r_1^{i,j}$ and $s_1^{i,j}$. Since ϕ is good, one of these two paths is increasing and the other one is decreasing. In addition, the path $v_1^{i,j}v_2^{i,j}v_3^{i,j}v_4^{i,j}$ is neither increasing nor decreasing so $\phi(v_2^{i,j}v_3^{i,j}) > \phi(v_3^{i,j}v_4^{i,j})$.

Applying three times this reasoning, we derive $\phi(v_3^{i,j}v_4^{i,j}) < \phi(v_4^{i,j}v_1^{i,j+1})$, $\phi(v_4^{i,j}v_1^{i,j+1}) > \phi(v_1^{i,j+1}v_2^{i,j+1})$ and finally $\phi(v_1^{i,j+1}v_2^{i,j+1}) < \phi(v_2^{i,j+1}v_3^{i,j+1})$. □

Hence we define the truth assignment Λ by $\Lambda(x_i) = \mathit{true}$ if $\phi(v_1^{i,1}v_2^{i,1}) < \phi(v_2^{i,1}v_3^{i,1})$ and $\Lambda(x_i) = \mathit{false}$ otherwise.

Let us show that each clause C_j has at least one true literal or one false literal. Set $C_j = l_1 \vee l_2 \vee l_3$. First observe that, by construction, for all $1 \leq k \leq 3$, l_k is true if $\phi(b_k^jc^j) < \phi(b_{k+1}^jc^j)$ and l_k is false otherwise. Now the three literals are not all true otherwise, $\phi(b_1^jc^j) < \phi(b_2^jc^j) < \phi(b_3^jc^j) < \phi(b_1^jc^j)$, a contradiction. And they are not all false, otherwise $\phi(b_1^jc^j) > \phi(b_2^jc^j) > \phi(b_3^jc^j) > \phi(b_1^jc^j)$, a contradiction. Hence C_j has at least one true literal and one false literal.

Conversely, let us now show that if there is a truth assignment Λ such that each clause of I has at least one true literal and at least one false literal, then G_I has a good edge-labeling.

The idea is to find a good edge-labeling ϕ satisfying the following property $(\star)$: If $\Lambda(x_i) = \mathit{true}$, $\phi(v_1^{i,j}v_2^{i,j}) < \phi(v_2^{i,j}v_3^{i,j})$ for all $1 \leq j \leq m$ and if $\Lambda(x_i) = \mathit{false}$, $\phi(v_1^{i,j}v_2^{i,j}) > \phi(v_2^{i,j}v_3^{i,j})$ for all $1 \leq j \leq m$.

Let $C_j = l_1 \vee l_2 \vee l_3$ be clause. To satisfy $(\star)$, we must label the edges of VG_j such that $\phi(b_k^jc^j) < \phi(b_{k+1}^jc^j)$ if l_k is true and $\phi(b_k^jc^j) > \phi(b_{k+1}^jc^j)$ if l_k is false. As C_j has at least one true and one false literal, there is a unique way to label the three edges $c^jb_k^j$, $1 \leq k \leq 3$, with $\{-1, 0, +1\}$ such that the condition $(\star)$ is fulfilled.

Let us now extend this edge-labeling to the remaining edges of each of the variable graphs VG_i. First, for all $1 \leq j \leq m$ and $1 \leq k \leq 4$, assign -3 and $+3$ alternatingly on the edges of the cycle of length four containing both $r_k^{i,j}$ and $s_k^{i,j}$ such that $\phi(v_k^{i,j}r_k^{i,j}) = -3$. Then, if $\Lambda(x_i) = \mathit{true}$, set $\phi(v_3^{i,j}, v_4^{i,j}) = -2$ and $\phi(v_4^{i,j}, v_1^{i,j+1}) = 2$ for all $1 \leq j \leq m$, and, if $\Lambda(x_i) = \mathit{false}$, set $\phi(v_3^{i,j}, v_4^{i,j}) = 2$ and $\phi(v_4^{i,j}, v_1^{i,j+1}) = -2$ for all $1 \leq j \leq m$.

We claim that ϕ is a good edge-labeling of G_I. Suppose, by way of contradiction, that there is a pair of vertices (x, y) such that two independent increasing (x, y)-paths P_1 and P_2 exist.

A set of two independent paths starting at a vertex of $R = \{r_k^{i,j} \mid 1 \leq j \leq m, 1 \leq k \leq 4\} \cup \{s_k^{i,j} \mid 1 \leq j \leq m, 1 \leq k \leq 4\}$ contains one increasing path (the one starting with the edge labeled -3) and one decreasing path (the one starting with the edge labeled 3). Hence x and y are not in R.

In addition, the union of P_1 and P_2 cannot be one of the four-cycles formed by the edges incident to $r_k^{i,j}$ and $s_k^{i,j}$ for some i,j and k.

Without loss of generality, we may assume that P_1 is at least as long as P_2. As cycles formed by two graphs GV_i and GV_j are of length at least 6, P_1 has length at least 3. Now one can see that P_1 may not contain any vertex of R because every path of length at least 3 with internal vertices in R is not increasing (nor decreasing).

Hence P_1 must contain at least three consecutive edges on one of the paths $Q_i = VG_i - R$. So P_1 is not increasing, a contradiction. □

Observe that the graph G_I constructed in the above proof is not bipartite. However, with a slight modification, we can transform it into a bipartite graph and obtain the following theorem.

Theorem 14. *The following problem is NP-complete.*
Instance: *A bipartite graph G.*
Question: *Does G have a good edge-labeling?*

Proof. Let G'_I be the graph obtained from G_I (described in the proof of Theorem 13) by replacing each path $v_k^{i,j}, r_k^{i,j}, v_{k+3}^{i,j}$ and each path $v_k^{i,j}, s_k^{i,j}, v_{k+3}^{i,j}$, by copies of a graph $H_{k'}$ defined in Section 5.3, for some $k' \geq 3$ and for all $i = 1, \ldots, n$, $j = 1, \ldots, m$ and $k = 1, \ldots, 4$ ($k+3$ is taken modulo 4).

By Proposition 9, it is not difficult to verify that G'_I admits a good edge-labeling if, and only if, G'_I also does. Moreover, each $H_{k'}$ admits a proper 2-coloring such that the vertices u and v have disjoint colors. Thus, G'_I is bipartite, since it admits a proper 2-coloring where all the vertices $v_1^{i,j}$ and $v_3^{i,j}$ belong to the same color class, for all $i = 1, \ldots, n$ and $j = 1, \ldots, m$. □

5.4 Classes of good graphs

Recall that a graph G is *critical* if it is bad but each of its proper subgraphs is good. To prove that every graph in a class $\mathcal{C}$ closed under taking subgraphs has a good edge-labeling, it suffices to prove that $\mathcal{C}$ contains no critical graph.

Lemma 9. *Let G be a graph with a cutvertex x, $C_1, \ldots, C_k$ be the components of $G - x$ and $G_i = G\langle C_i \cup \{x\}\rangle$, $1 \leq i \leq k$. Then G is good if and only if all the G_i are good.*

Proof. Necessity is obvious since a good edge-labeling of G induces a good edge-labeling on each subgraph G_i.

Sufficiency follows from the fact that there are two independent (u, v)-paths in G only if there exists i, $1 \leq i \leq k$, such that u and v are in $V(G_i)$. Hence the union of good edge-labelings of all the G_i is a good edge-labeling of G. □

Corollary 5. *Every critical graph is 2-connected.*

Corollary 6. *Every forest F admits a good edge-labeling.*

Proof. No forest contains a non-trivial 2-connected subgraph, and so contains no critical subgraph. □

Let $G = (V, E)$ be a graph. A K_2-*cut* of G is a set of two adjacent vertices u and v such that the graph $G - \{u, v\}$ (obtained from G by removing u and v and their incident edges) has more connected components than G.

Lemma 10. *Let G be a connected graph and $\{u, v\}$ a K_2-cut in G such that $G - \{u, v\}$ has two connected components C_1 and C_2. If $G_1 = G\langle C_1 \cup \{u, v\}\rangle$ and $G_2 = G\langle C_2 \cup \{u, v\}\rangle$ have a good edge-labeling then G has a good edge-labeling.*

Proof. Let ϕ_1 and ϕ_2 be good edge-labelings of $G\langle C_1 \cup \{u, v\}\rangle$ and $G\langle C_2 \cup \{u, v\}\rangle$ respectively such that $\phi_1(uv) = \phi_2(uv)$.

Then the union of ϕ_1 and ϕ_2 is a good edge-labeling of G. Indeed, suppose by way of contradiction, that there exists x and y and two independent increasing (x, y)-paths P_1 and P_2 in G. W. l. o. g., we may assume that $x \in V(G_1)$. At least one of the paths, say P_1, contains at least one edge e_1 in $E(G_2) \setminus \{uv\}$ since ϕ_1 is good.

Assume first that $y \in V(G_1)$. Then P_1 must go through u and v. Let Q_2 be the shortest (u, v)-subpath of P_1 containing e_1. Then Q_2 is either increasing or decreasing. Hence since uv is both increasing and decreasing, there are either two increasing paths or two decreasing paths in G_2. This contradicts the fact that ϕ_2 is good.

Assume now that $y \in C_2$. Then since P_1 and P_2 are independent without loss of generality, P_1 goes through u and P_2 goes through v. Let Q_1 be the (x, u)-subpath of P_1, R_1 be the (u, y)-subpath of P_1, let Q_2 be the (x, v)-subpath of P_2 and R_2 be the (v, y)-subpath of P_2.

If $\phi(uv)$ is larger than the label of the last edge of Q_1, then Q_1uv and Q_2 are two increasing (x, v)-paths in G_1, a contradiction. If not $\phi(uv)$ is smaller than the label of the first edge of R_1 and vuR_1 and R_2 are two increasing (v, y)-paths in G_2, a contradiction.

□

Let $G = (V, E)$ be a graph. An *edge-cut* is a non-empty set of edges between a set of vertices S and its complement $\overline{S}$. Formally, for any $S \subset V$, the edge-cut $[S, \overline{S}]$ is the set $\{uv \in E \mid u \in S \text{ and } v \in \overline{S})$. An edge cut which is also a matching is called a *matching-cut*.

Lemma 11. *Let G be a graph and $[S, \overline{S}]$ a matching-cut in G. If $G\langle S\rangle$ and $G\langle \overline{S}\rangle$ have a good edge-labeling then G has a good edge-labeling.*

Proof. Let ϕ_1 be a good edge-labeling of $G\langle S\rangle$ and ϕ_2 be a good edge-labeling of $G\langle \overline{S}\rangle$ (in $\mathbb{R}$). Then the edge-labeling ϕ of G defined by $\phi(e) = \phi_1(e)$ if $e \in E(G\langle S\rangle)$, $\phi(e) = \phi_2(e)$ if $e \in E(G\langle \overline{S}\rangle)$ and $\phi(e) = +\infty$ if $e \in [S, \overline{S}]$ is good.

Indeed, suppose by way of contradiction, that it is not good. Then there exist two vertices u and v and two independent increasing (u, v)-paths P_1 and P_2. Since ϕ_1 and ϕ_2 are good, then without loss of generality, we may assume that $u \in S$ and $v \in \overline{S}$. For $i = 1, 2$, the path P_i contains an edge of u_iv_i in $[S, \overline{S}]$. Now, since u_1v_1 and u_2v_2, are labeled $+\infty$ and incident to no edges labeled $+\infty$, u_1v_1 must be the last edge of P_1 and u_2v_2 the last edge of P_2. So $v_1 = v = v_2$, which is impossible as $[S, \overline{S}]$ is a matching. □

Corollary 7. *A critical graph has no matching-cut.*

Corollary 8. *Every C_3-free outerplanar graph admits a good edge-labeling.*

Proof. An easy result of Eaton and Hull [EH99] states that a C_3-free outerplanar graph has either a vertex of degree 1 or two adjacent vertices of degree 2. This implies that it has a matching-cut. Hence by Corollary 7 no C_3-free outerplanar graph is critical, which yields the result. □

A graph is *subcubic* if every vertex has degree at most three.

Lemma 12. *Every subcubic $\{C_3, K_{2,3}\}$-free graph has a matching-cut.*

Proof. Let G be a subcubic graph $\{C_3, K_{2,3}\}$-free. If G has no cycle, then every edge forms a matching-cut. Suppose now that G has a cycle. Let C be a cycle of smallest length in G. If C is a connected component of G (in particular if $C = G$) then any pair of non-adjacent edges of C forms a matching-cut.

If not, let us show that $[V(C), \overline{V(C)}]$ is a matching-cut. Let $e_1 = x_1y_1$ and $e_2 = x_2y_2$ be two distinct edges in $[V(C), \overline{V(C)}]$ with $x_1, x_2 \in V(C)$. Then $x_1 \neq x_2$ because these two vertices have degree (at most) 3 and they have two neighbors in $V(C)$. Suppose by way of contradiction that $y_1 = y_2$. Then x_1 and x_2 are not adjacent since G is C_3-free. Furthermore, there are the two (x_1, x_2)-paths along C are of length at most 2 otherwise C would not be a smallest cycle. Hence C is a cycle of length 4 and the graph induced by $V(C) \cup \{y_1\}$ is a $K_{2,3}$, a contradiction. □

Corollary 7 and Lemma 12 immediately imply that the sole subcubic critical graphs are C_3 and $K_{2,3}$.

Corollary 9. *Every subcubic $\{C_3, K_{2,3}\}$-free graph has a good edge-labeling.*

Farley and Proskurowski [FP84, BFP11] proved that every (multi)graph G on n vertices with less than $\frac{3}{2}(n-1)$ edges has a matching-cut.

Theorem 15 (Farley and Proskurowski [FP84, BFP11])**.** *Let G be a multigraph. If $|E(G)| < \frac{3}{2}|V(G)| - \frac{3}{2}$ then G has a matching-cut.*

Corollary 7 and Theorem 15 yield immediately the following.

Corollary 10. *Every critical graph has at least $\lceil \frac{3}{2}|V(G)| - \frac{3}{2} \rceil$ edges.*

An easy and well-known consequence of Euler's Formula states that every planar graph with girth at least 6 has at most $\frac{3}{2}|V(G)| - 3$ edges and so is not critical.

Corollary 11. *Every planar graph of girth at least 6 has a good edge-labeling.*

5.5 Good edge-labeling of ABC-graphs

Corollary 10 states that every critical graph has at least $\lceil \frac{3}{2}|V(G)| - \frac{3}{2} \rceil$ edges. This is tight since if G is C_3 or $K_{2,3}$ then $|E(G)| = \lceil \frac{3}{2}|V(G)| - \frac{3}{2} \rceil$. We will now show that those two graphs are the unique critical ones satisfying this equality.

Farley and Proskurowski [FP84, BFP11] constructed a class of multigraphs G (called *ABC-graphs*) having $\lceil \frac{3}{2}|V(G)| - \frac{3}{2} \rceil$ edges with no matching-cut. The definition of ABC-graphs is based on the following three operations:

- An *A-operation* on vertex u introduces vertices v and w and edges uv, uw and vw.
- A *B-operation* on edge uv introduces vertices w_1 and w_2 and edges uw_1, vw_1, uw_2 and vw_2, and removes edge uv.
- A *C-operation* on vertices u and v ($u = v$ is allowed) introduces vertex w and edges uw and vw.

Note that the C-operation is the only operation that can introduce parallel edges.

An *ABC-graph* is a graph that can be obtained from K_1 with a sequence of A- and B-operations and at most one C-operation.

It is easy to check that ABC-graphs have no matching-cut. In addition, solving a conjecture of Farley and Proskurowski, Bonsma [Bon06, BFP11] showed that they are the unique extremal examples, i.e., satisfying $|E(G)| = \lceil \frac{3}{2}|V(G)| - \frac{3}{2} \rceil$.

Theorem 16 (Bonsma [Bon06, BFP11])**.** *Let G be a graph such that $|E(G)| = \lceil \frac{3}{2}|V(G)| - \frac{3}{2} \rceil$. Then G has no matching-cut if and only if G is an ABC-graph.*

Our aim is to prove that every $\{C_3, K_{2,3}\}$-free ABC-graph is good. It is easy to check that every 2-connected component of an ABC-graph is an ABC-graph, so by Lemma 9, it suffices to prove it for 2-connected ABC-graphs.

Observe that the C-operation is the only one that changes the parity of the order. Hence an ABC-graph with an odd number of vertices is obtained from K_1 with a sequence of A- and B-operations and no C-operation.

Let G be a graph obtained from a graph H by a B-operation on some edge uv. Let ϕ be an edge-labeling of H. Let ϕ_0 and ϕ_∞ be the edge-labelings of G defined by:

$$\begin{aligned}
&\phi_0(e) = \phi_\infty(e) = \phi(e) \text{ for all } e \in E(H) \setminus \{uv\},\\
&\phi_0(uw_1) = \phi_0(w_2v) = 1/2,\\
&\phi_0(uw_2) = \phi_0(w_1v) = -1/2,\\
&\phi_\infty(uw_1) = \phi_\infty(w_2v) = +\infty,\\
&\phi_\infty(uw_2) = \phi_\infty(w_1v) = -\infty
\end{aligned}$$

Proposition 11. *Let G be a graph obtained from a graph H by a B-operation on some edge uv and ϕ be a good edge-labeling of H.*

(i) If ϕ is injective integer-valued and $\phi(uv) = 0$, then ϕ_0 is a good edge-labeling of G.

(ii) If ϕ is real-valued, then ϕ_∞ is a good edge-labeling of G.

Proof. (i) By contradiction, suppose that ϕ_0 is not a good edge-labeling of G. Then there exist two increasing independent (x, y)-paths P_1 and P_2 on G, for some $x, y \in V(G)$.

Since ϕ is a good edge-labeling of H, by the definition of ϕ_0 at least one edge of the set $E' = \{uw_1, uw_2, vw_1, vw_2\}$ belongs to some of the paths P_1 or P_2. Observe also that an increasing path in H cannot contain more than two edges of E'.

Suppose then that exactly one of the paths, say P_1, contains a non-empty intersection with the set E'. In this case, there would be two increasing paths in the edge-labeling ϕ of H. To prove this fact, let P_1' be the path obtained from P_1 by replacing the edges of the set $E' \cap E(P_1)$ by the edge uv. Observe that P_1' and P_2 would be two increasing paths of H under the edge-labeling ϕ, since $\phi(uv) = 0$.

Hence the paths P_1 and P_2 both contain some edge of the set E'. Suppose first that P_1 and P_2 contain exactly one edge of E' each. As P_1 and P_2 are independent, we assume that $uw_1 \in E(P_1)$ and $vw_1 \in E(P_2)$, without loss of generality. If $w_1 = y$, then the last edge of the (x, u)-subpath of P_1 has a label smaller than 0 (since ϕ is injective) and the same happens for the last edge of the (x, v)-subpath of P_2 (observe that at least one of these subpaths must be non-empty). Consequently, there would be two increasing paths (x, u)-paths or (x, v)-paths in H under the edge-labeling ϕ. Similarly, one may conclude that if $w_1 = x$, then there would also be two increasing paths on ϕ. It is just necessary to verify that the first edges of the (u, y)-subpath of P_1 and of the (v, y)-subpath of P_2 are greater than 0 (at least one of these edges exist) and that there would be two increasing (u, y)-paths or (v, y)-paths in H.

Finally, P_1 and P_2 cannot have both two edges from E' because they are independent.

(ii) The proof that ϕ_∞ is a good edge-labeling of G is similar to the proof of (i). In this case, P_1 and P_2 cannot contain just one edge of E'. Consequently, either $E(P_1) \subset E'$ or $E(P_2) \subset E'$. In any case, there would be and increasing (u, v)-path or an increasing (v, u)-path, which is a contradiction because there would be two increasing paths in H. □

Corollary 12. *If G is a graph obtained from a good graph by a B-operation, then G is good.*

Proof. It follows directly from Proposition 11. □

Lemma 13. *Let G be a 2-connected ABC-graph with an odd number of vertices. If $G \notin \{C_3, K_{2,3}\}$ then G is good.*

Proof. By contradiction, suppose that G is a counter-example to the statement. As every A-operation (with the exception of the transition $K_1 \to C_3$) creates a cut-vertex, by Lemma 9, we may assume that G is obtained from C_3 with a sequence of B-operations. However a B-operation on C_3 at any edge creates a $K_{2,3}$ and a B-operation on $K_{2,3}$ at any edge creates the graph G_1 depicted in Figure 5.4. If $G \notin \{C_3, K_{2,3}\}$ then it is obtained from G_1 with a sequence of B-operations. Now this graph G_1 admits a good edge-labeling (See Figure 5.4). Hence an easy induction and Corollary 12 imply that G has a good edge-labeling, a contradiction.

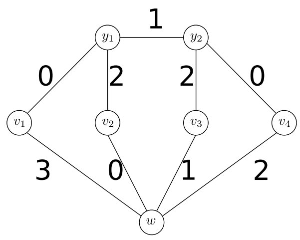


Figure 5.4: The graph G_1 and a good edge-labeling.

□

Since 2-connected components of an ABC-graph with an odd number of vertices are ABC-graphs with an odd number of vertices, we have the following:

Corollary 13. *Every $\{C_3, K_{2,3}\}$-free ABC-graph with an odd number of vertices is good.*

We now would like to prove an analogous statement to the one of Corollary 13 but for ABC-graphs with an even number of vertices.

Let G be a graph and x, y be two distinct vertices of G. An (x, y)-*better edge-labeling* of G is a good edge-labeling of G such that there is no increasing (x, y)-path. Clearly, if x and y are adjacent or if x and y have two neighbors in common then G has no (x, y)-better edge-labeling. A graph is *friendly* if it has a good edge-labeling and for any pair (x, y) of non-adjacent vertices with at most one neighbor in common there exists an (x, y)-better edge-labeling.

Let G_1 be a graph whose vertex set is $\{v_1, v_2, v_3, v_4, w, y_1, y_2\}$ and whose edge set is $\bigcup_{i=1}^{4}\{(w, v_i)\} \cup \{(v_1, y_1), (v_2, y_1), (v_3, y_2), (v_4, y_2)\} \cup \{y_1, y_2\}$ (See Figure 5.4.).

Lemma 14. *G_1 is friendly.*

Proof. Let ϕ be the edge-labeling of G_1 in Figure 5.4. Then ϕ is good.

Let us now prove that for every pair $p = (a, b)$ of two distinct non-adjacent vertices a and b in G_1 such that a and b have at most one common neighbor, there is a better (a, b)-edge-labeling of G_1.

First, observe that the vertex w of G_1 cannot be in such a pair because, for any other vertex of G_1, either w is adjacent to it or they have two common neighbors.

Suppose now that the vertex $y_1 \in p$. Then the other vertex of p must be v_3 or v_4. But ϕ is (v_3, y_1)-better and (y_1, v_4)-better, and so $-\phi$ is (y_1, v_3)-better and (v_4, y_1)-better. Hence in any case, there is a better p-edge-labeling of G_1.

By symmetry, if y_2 is a vertex of p, there exists a p-better edge-labeling.

Suppose that $v_1 \in p$. Then the other vertex of p is v_3 or v_4. ϕ is (v_1, v_4)-better and exchanging the labels of y_2v_3 and y_2v_4 and also the labels of v_3w and v_4w we obtain a (v_1, v_3)-better edge-labeling ϕ'. Thus $-\phi'$ and $-\phi$ are respectively (v_3, v_1)-better and (v_4, v_1)-better. Hence in any case, there is a better p-edge-labeling of G_1.

By symmetry, if v_2, v_3 or v_4 is a vertex of p, there exists a p-better edge-labeling. □

Proposition 12. *Let G be a graph obtained from a graph H by a B-operation on some edge uv. If H is friendly then G is friendly.*

Proof. Let w_1, w_2 be the vertices created by the B-operation. Let x and y be two non-adjacent vertices of G having at most one neighbor in common. Then $|\{x, y\} \cap \{w_1, w_2\}| \leq 1$.

- Suppose first that $\{x, y\} \cap \{w_1, w_2\} = \emptyset$. Then x and y are not adjacent in H.

  Assume first that x and y have at most one common neighbor in H. Let ϕ be an injective integer-valued (x, y)-better edge-labeling of H such that $\phi(uv) = 0$. Then ϕ_0 is a good edge-labeling of G by Proposition 11-(i). Moreover it is easy to check that there is no increasing (x, y)-path in G. Hence ϕ_0 is an (x, y)-better edge-labeling of G.

  Assume now that x and y have two common neighbors in H. As they do not have two common neighbors in G, we can suppose w.l.o.g. that $x = u$ and $N(x) \cap N(y) = \{v, w\}$, for some vertex w. Let ϕ be a real-valued good edge-labeling of H. Free to consider $-\phi$, we may assume that uvy is an increasing path. Hence in $H \backslash uv$ there is no increasing (u, y)-path. By Proposition 11-(ii), ϕ_∞ is a good edge-labeling of G. Moreover it is an (x, y)-better edge-labeling, because there is no increasing (u, y)-path in $H \setminus uv$ and the unique increasing paths containing w_1 and w_2 are uw_2 and uw_2v.

- Suppose now that $|\{x, y\} \cap \{w_1, w_2\}| = 1$. Without loss of generality, we may assume that $x = w_1$ and y is not adjacent to v.

  Assume first that v and y have at most one common neighbor in H. Let ϕ be a (v, y)-better edge-labeling of H. By Proposition 8, we may assume that ϕ is real-valued. By Proposition 11-(ii), ϕ_∞ is a good edge-labeling of G. Moreover, there is no increasing (w_1, y)-path, through u since $\phi(uw_1) = +\infty$, nor through v since there is no increasing (v, y)-path in H. Hence ϕ_∞ is a (w_1, y)-better edge-labeling of G.

  Assume now that v and y have two common neighbors in H.

  - Suppose that y is adjacent to u. Let ϕ be an injective integer-valued good edge-labeling of H such that $\phi(uv) = 0$. Free to consider $-\phi$, we may assume that $\phi(uy) < 0$ and so $\phi(uy) \leq -1$. By Proposition 11-(i), ϕ_0 is a good edge-labeling of G. Moreover it has no increasing (w_1, y)-path and so is (w_1, y)-better. Indeed suppose for a contradiction that there is an increasing (w_1, y)-path P :

  * If u is the second vertex of P then $P - w_1$ is an increasing (u, y)-path. Since $\phi(uy) \leq -1$, $P - w_1$ is not (u, y). So $P - w_1$ and (u, y) are two increasing (u, y)-paths in H a contradiction.
  * If v is the second vertex of P then the path Q in H obtained from P by replacing w_1 with u is an increasing (u, y)-path because the labels of the edges of $P - w_1$ are positive. Thus Q and (u, y) are distinct increasing (u, y)-paths, a contradiction.

– Suppose that y is not adjacent to u. Let t_1 and t_2 be the two common neighbors of v and y. Let ϕ be an injective integer-valued good edge-labeling of H such that $\phi(uv) = 0$. Without loss of generality, we may assume that (v, t_1, y) is increasing and (v, t_2, y) is decreasing. By Observation 1, $\phi(vt_1) < \phi(vt_2)$. Thus, if $\phi(vt_1) > 0$ then $\phi(vt_2) > 0$. So with respect to $-\phi$, (v, t_2, y) is increasing and $-\phi(vt_2) < 0$. Hence, free to consider $-\phi$ (and swap the names of t_1 and t_2), we may assume that $\phi(vt_1) < 0$ and so $\phi(vt_1) \leq -1$. By Proposition 11-(i), ϕ_0 is a good edge-labeling of G. Moreover it has no increasing (w_1, y)-path and so is (w_1, y)-better. Indeed suppose for a contradiction that there is a increasing (w_1, y)-path P :

  * If v is the second vertex of P then $P - w_1$ is an increasing (v, y). Since $\phi(vt_1) \leq -1$, $P - w_1$ is not (v, t_1, y). So there are two increasing (v, y)-paths in H, a contradiction.
  * If u is the second vertex of P then the path P' in H obtained from P by replacing w_1 with v is an increasing (v, y)-path because the labels of the edges of $P - w_1$ are positive. P' is distinct from (v, t_1, y), a contradiction.

□

One can now generalize Lemma 13.

Lemma 15. *Let G be a 2-connected ABC-graph with an odd number of vertices. If $G \notin \{C_3, K_{2,3}\}$ then G is friendly.*

Proof. Similarly as in the proof of Lemma 13, combining Lemma 14 and Proposition 12 yield the result by induction. □

Corollary 14. *Every $\{C_3, K_{2,3}\}$-free ABC-graph with an odd number of vertices is friendly.*

Proof. Let x and y be two non-adjacent vertices of G having at most one common neighbor.

Assume first that x and y are in a same connected 2-component C. By Lemma 15, C has an (x, y)-better edge-labeling and, by Corollary 13, $G \setminus E(C)$

has a good edge-labeling. The union of these two edge-labelings is clearly an (x, y)-better labeling of G.

Suppose now that the 2-connected components containing x do not contain y. Let G_1 be the graph induced by the union of the 2-connected components containing x and $G_2 = G \setminus E(G_1)$. By Corollary 13, the two graphs G_1 and G_2 admit good edge-labelings ϕ_1 and ϕ_2, respectively. Free to add a huge number to all the labels of ϕ_1, we may assume that $\min\{\phi_1(e) \mid e \in E(G_1)\} > \max\{\phi_2(e) \mid e \in E(G_2)\}$. Then the union of ϕ_1 and ϕ_2 is an (x, y)-better labeling of G. □

Lemma 16. *Let G be a 2-connected ABC-graph with an even number of vertices. If G is $\{C_3, K_{2,3}\}$-free, then G is good.*

Proof. We prove this lemma by induction on the number of vertices (or equivalently the number of A-, B- or C-operations). An even ABC-graph is obtained from K_1 with a sequence of A- and B-operations and exactly one C-operation. Since G is 2-connected, no A-operation can be made after a C-operation. Consider a sequence of operations such that the C-operation is done as late as possible. Let u and v be the vertices on which the C-operation is done and w the introduced vertex.

- Suppose that the C-operation is the ultimate one. Note that $u \neq v$ since G has no multiple edges. Since G is $\{C_3, K_{2,3}\}$-free then u and v are not adjacent and u and v have at most one neighbor in common. Hence by Corollary 14, $G - w$ admits a (u, v)-better edge-labeling ϕ (in $\mathbb{R}$). Setting $\phi(uw) = -\infty$ and $\phi(wv) = +\infty$ we obtain a good edge-labeling of G.

- If the C-operation is the penultimate one, then it is followed by a B-operation on one of the introduced edges, because the C-operation is applied as late as possible and G is C_3-free. These two operations together may be seen as a single one on u and v that introduces the vertices t_1, t_2 and w and the edges ut_1, ut_2, t_1w, t_2w and wv.

  Note that u and v are not adjacent since G is $K_{2,3}$-free. Assume first that u and v have at most one neighbor in common. By Corollary 14, $G - \{t_1, t_2, w\}$ admits a (u, v)-better edge-labeling ϕ. Let M be the maximum value of ϕ. Then setting $\phi(ut_1) = \phi(t_2w) = -\infty$, $\phi(ut_2) = \phi(t_1w) = M + 1$ and $\phi(vw) = M + 2$, we obtain a good edge-labeling of G.

  Assume now that u and v have at least two common neighbors. Since G is $K_{2,3}$-free, then u and v have exactly two common neighbors x_1 and x_2. By Corollary 13, $G - \{t_1, t_2, w\}$ admits a good edge-labeling ϕ. By Proposition 8, we may assume that ϕ is injective and real-valued. Without loss of generality, we may suppose that $\phi(vx_1) > \phi(vx_2)$. Let us set $\phi(ut_1) = \phi(t_2w) = +\infty$, $\phi(ut_2) = \phi(t_1w) = -\infty$ and $\phi(vw) = \frac{1}{2}(\phi(vx_1) + \phi(vx_2))$. We claim that ϕ is a good edge-labeling of G. Indeed suppose, by way of contradiction, that it is not the case. Then there exist two vertices a and b and two independent increasing (a, b)-paths P_1 and P_2. Since ϕ is a good edge-labeling of $G - \{t_1, t_2, w\}$ one of these two paths, say P_1 must go through w. Moreover since $\phi(t_1w) = -\infty$

and $\phi(t_2w) = +\infty$ and $d(w) = 3$, then either wt_1 (or t_1w) is the first edge of P_1 or t_2w (or wt_2) is the last edge of P_1. Free to consider $-\phi$ instead of ϕ, we may assume that we are in the first case.

Two cases may occur. Either (a) P_1 starts in t_1 or (b) P_1 starts in w.

(a) In this case, $P_2 = (t_1, u)$ and the third vertex of P_1 is v. Then $Q_1 = P_1 - \{t_1, w\}$ is an increasing (v, u)-path. So by Observation 1 and the assumption that $\phi(vx_1) > \phi(vx_2)$, $Q_1 = vx_2u$ (We recall the reader that another increasing (v, u)-path not going through x_2 cannot exist as ϕ is a good edge-labeling of $G - \{t_1, t_2, w\}$). This is a contradiction because $\phi(wv) > \phi(vx_2)$.

(b) In this case, $P_1 = (w, t_1, u)$, because $\phi(ut_1) = +\infty$. Now the first edge of P_2 is wv. Hence $Q_2 = P_2 - w$ is an increasing (v, u)-path and vx_2 is not the first edge of Q_2 since $\phi(wv) > \phi(vx_2)$. Note that by Observation 1, vx_2u is increasing because $\phi(vx_1) > \phi(vx_2)$. So, in $G - \{t_1, t_2, w\}$, there are two distinct increasing (v, u)-paths. This contradicts the fact that ϕ is a good edge-labeling of $G - \{t_1, t_2, w\}$.

- If there are exactly two B-operations after the C-operation, and if u and v are not adjacent then by the induction hypothesis and Corollary 12, G has a good edge-labeling. If u and v are adjacent, then uv is a K_2-cut. Let C_1 be the component of $G - \{u, v\}$ containing w (i.e., the set of vertices added with the C-operation and the following B-operations). Let $G_1 = G\langle C_1 \cup \{u, v\}\rangle$ and $G_2 = G\langle V(G) \setminus C_1\rangle$. Note that G_1 is obtained from a triangle by performing two B-operations and thus is the graph G_1 depicted Figure 5.4 which has a good edge-labeling. Similarly, G_2 is the graph G taken before performing the C-operation has a good edge-labeling. Hence by Lemma 10, G has a good edge-labeling.

- If there are at least three B-operations after the C-operation, then by the induction hypothesis and Corollary 12, G has a good edge-labeling.

□

Lemma 13 and Lemma 16 imply that every 2-connected $\{C_3, K_{2,3}\}$-free ABC-graph is good. Since 2-connected components of an ABC-graph are ABC-graphs, we have the following.

Corollary 15. *Every $\{C_3, K_{2,3}\}$-free ABC-graph is good.*

In turn, this corollary, together with Corollary 7, Theorems 15 and 16, yield the following.

Theorem 17. *Let G be a critical graph. If $G \notin \{C_3, K_{2,3}\}$ then $|E(G)| \geq \frac{3}{2}|V(G)| - \frac{1}{2}$.*

5.6 Conclusions and further research

We have shown that it is NP-complete to decide if a graph has a good edge-labeling, even for the class of bipartite graphs. It would be nice to find large classes of graphs for which it is polynomial-time decidable. For graphs with treewidth 1, which are the forests, it is the case. But is it also the case for graphs with treewidth at most k?

Problem 1. *Let $k \geq 2$ be a fixed integer. Does there exist a polynomial-time algorithm that decides if a given graph of treewidth at most k has a good edge-labeling?*

We also do not know what is the complexity of the problem when restricted to planar graphs.

Problem 2. *Does there exist a polynomial-time algorithm that decides if a given planar graph has a good edge-labeling?*

We do not even know if there are planar critical graphs distinct from C_3 and $K_{2,3}$.

Problem 3. *Does there exist a $\{C_3, K_{2,3}\}$-free planar graph which is bad?*

If there is no such graphs or only a finite number of them then the answer to Problem 2 will be yes.

Corollary 11 implies that, with the additional condition of girth at least 6, the answer to Problem 3 is no. It would be nice to solve the above problems for planar graphs of smaller girth. In particular, we do not know if there is a planar graph with girth 5 which is bad.

Problem 4. Does every planar graph of girth at least 5 have a good edge-labeling?

Bonsma [Bon09] showed that it is NP-complete to decide if a planar graph of girth at least 5 has a matching-cut. In particular, there are infinitely many planar graphs of girth at least 5 without matching-cut. However, for all such graphs we looked at, we were able to find a good edge-labeling.

The *average degree* of a graph G is $Ad(G) = \frac{\sum_{v \in V(G)} d(v)}{|V(G)|} = \frac{2|E(G)|}{|V(G)|}$.

Theorem 17 implies that for any $c < 3$ there is a finite number of critical graphs with average degree at most c. Actually, we conjectured that the only ones are C_3 and $K_{2,3}$.

Conjecture 2. *Let G be a critical graph. Then $Ad(G) \geq 3$ unless $G \in \{C_3, K_{2,3}\}$.*

However, the authors of [BFT11] recently communicated us that they found a counter-example for Conjecture 2 that is depicted in Figure 5.5.

More generally for any $c < 4$, we conjecture the following.

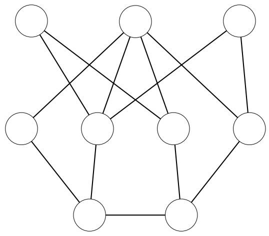


Figure 5.5: Counter-example for Conjecture 2.

Conjecture 3. *For any $c < 4$, there exists a finite list of graphs $\mathcal{L}$ such that if G is a critical graph with $Ad(G) \leq c$ then $G \in \mathcal{L}$.*

The constant 4 in the above conjecture would be tight. Indeed, for all k, the graph $J_{2,2,k}$ defined in Section 5.2 is critical: it is bad according to Proposition 10. Moreover one can easily show that for any edge e, $H_k \setminus e$ has a good edge-labeling with no (u,v)-increasing path and no (v,u)-increasing (just follow the constraint as in the proof of Proposition 9). Extending this labeling by labeling the two H_2 with $-\infty$ and $+\infty$ such that one of them is an increasing (u,v)-path and the other one an increasing (v,u)-path we obtain a good edge-labeling of $J_{2,2,k} \setminus e$. Furthermore $Ad(J_{2,2,k}) = \frac{8k+8}{2k+4} = 4 - \frac{4}{k+2}$. Last, one can easily see that if $k \neq k'$ then $J_{2,2,k}$ is not a subgraph of $J_{2,2,k'}$.

Theorem 17 says that if a graph has no dense subgraphs then it has a good edge-labeling. On the opposite direction one may wonder what is the minimum density ensuring a graph to be bad. Or equivalently,

Problem 5. What is the maximum number $g(n)$ of edges of a good graph on n vertices?

Clearly we have $g(n) = ex(n, \mathcal{C})$ where $\mathcal{C}$ is the set of critical graphs. As $K_{2,3}$ is critical then $g(n) \leq ex(n, K_{2,3}) = \frac{1}{\sqrt{2}} n^{3/2} + O(n^{4/3})$ by a result of Füredi [Für96].

The hypercubes show that g is super-linear. Indeed the hypercube H_k is obtained from two disjoints copies of H_{k-1} by adding a perfect matching between them. Hence an easy induction and Lemma 11 shows that H_k has a good edge-labeling. Since H_k has 2^k vertices and $2^{k-1}k$ edges, $g(2^k) \geq 2^{k-1}k$, so $g(n) \geq \frac{1}{2} n \log n$.

Some works were recently published with further results on good edge-labeling [BFT11, Meh11]. They show some advances related to Conjecture 2 and Problem 5.

CHAPTER 6
Hull Number

Contents

A classical example of convexity is the one defined in Euclidean spaces. In an Euclidean space E, a set $S \subseteq E$ is said *convex* if for any two points x and y of S, $[x, y] \subseteq S$, i.e., the set of points lying in the straight line segment between x and y also belongs to S. Note that if two convex sets $X, Y \subseteq E$ contain a given set $S \subseteq E$ of points, then their intersection $X \cap Y$ is also a convex set of E containing S. Hence, we can define the *convex hull* of S as the minimum convex set that contains S. Reciprocally, given a convex set S of E, a *hull set* of S is any subset S' of S such that S is the convex hull of S'. A naive way to compute the convex hull H of a set S consists in starting with $H = S$ and, while it is possible, adding $[x, y]$ to H for any $x, y \in H$. However there exist more efficient algorithms. For instance, for any set S of a d-dimensional euclidean space, the *gift wrapping algorithm* computes the convex hull and a minimum hull set of S in polynomial-time in the size of S (d being fixed). For more results concerning the convexity in Euclidean spaces, we refer to [Roc70].

In order to capture the abstract notion of convexity, [FJ86] defines an *alignment* over a set X as a family $\mathcal{C}$ of subsets of X that is closed under intersection and that contains both X and the empty set. The members of $\mathcal{C}$ are called the *convex sets* of X. The pair $(X, \mathcal{C})$ is then called an *aligned space*. An example of aligned space $(E, \mathcal{C})$ is the one where E is an euclidean space and $\mathcal{C} = \{H \subseteq E : \forall x, y \in H,\ [x, y] \subseteq H\}$. Given an aligned space $(X, \mathcal{C})$, the definitions of convex hull and hull set are generalized as follows. For any $S \subseteq X$, the *convex hull* of S is the smallest member of $\mathcal{C}$ containing S. For any $S \in \mathcal{C}$, a *hull set* of S is a set $S' \subseteq S$ such that S is the convex hull of S'.

Various notions of convexity can be defined in graphs as specific alignments over the set of vertices. This chapter is devoted to the study of the *geodetic convexity* of graphs. Let $G = (V, E)$ be a connected undirected graph. For any $u, v \in V$, let the *closed interval* $I[u, v]$ of u and v be the the set of vertices that belong to some shortest (u, v)-path. The closed interval of a set of vertices can be seen as an analog to segments in Euclidian spaces. For any $S \subseteq V$, let $I[S] = \bigcup_{u,v \in S} I[u, v]$. A subset $S \subseteq V$ is *geodesically convex* if $I[S] = S$. In this chapter convexity refers to the geodesical variant. In other words, a subset S is convex if, for any $u, v \in S$ and for any shortest (u, v)-path P, $V(P) \subseteq S$. That is, the geodetic convexity can be defined as the alignment $\mathcal{C}$ over V where $\mathcal{C} = \{S \subseteq V \ : \ I[S] = S\}$.

Given a subset $S \subseteq V$, the *convex hull* $I_h[S]$ of S is the smallest convex set that contains S. We say that S is a *hull set* of G if $I_h[S] = V$. That is, S is a hull set of G if, starting from the vertices of S and successively adding in S the vertices in some shortest path between two vertices in S, we eventually obtain V. The size of a minimum hull set of G is the *hull number* of G, denoted by $hn(G)$. The Hull Number problem is to decide whether $hn(G) \leq k$, for a given graph G and an integer k [ES85]. This problem is known to be NP-complete in general graphs [DGK+09]. In this chapter, we consider the problem of the complexity to compute minimum hull set of a graph in several graph classes.

Our results. We first answer an open question of Dourado *et al.* [DGK+09] by showing that the Hull Number problem is NP-hard even when restricted to the class of bipartite graphs (Section 6.2). Then, we design polynomial time algorithms to solve the Hull Number problem in several graph classes. In Section 6.3, we deal with the class of complements of bipartite graphs. In Section 6.4 we generalize some results in [ACG+11b] to the class of $(q, q-4)$-graphs. Section 6.5 is devoted to the class of cacti. Finally, we prove tight upper bounds on the hull number of graphs in Section 6.6. In particular, we show that the hull number of an n-node graph G without simplicial vertices is at most $1 + \lceil \frac{3(n-1)}{5} \rceil$ in general, at most $1 + \lceil \frac{n-1}{2} \rceil$ if G is regular or has no triangle, and at most $1 + \lceil \frac{n-1}{3} \rceil$ if G has girth at least 6.

Related work. In the seminal work [ES85], the authors present some upper and lower bounds on the hull number of general graphs and characterize the hull number of some particular graphs. The corresponding minimization problem has been shown to be NP-complete [DGK+09]. Dourado *et al.* also proved that the hull number of unit interval graphs, cographs and split graphs can be computed in polynomial time [DGK+09]. Bounds on the hull number of triangle-free graphs are shown in [DPRS10]. The hull number of the cartesian and the strong product of two connected graphs is studied in [CCJ04, CHM+10]. In [HJM+05], the authors have studied the relationship between the *Steiner number* and the hull number of a given graph. An oriented version of the Hull Number problem is studied in [CFZ03, Far05].

Other parameters related to the geodetic convexity have been studied in [CHZ02, CWZ02]. Variations of graph convexity have been further proposed and studied. For instance, the *monophonic convexity* that deals with induced paths instead of shortest paths is studied in [FJ86, DPS10]. Another example is the P_3-*convexity* where just paths of order three are considered [FJ86, CM99]. Other variants of graph convexity and other parameters are mentioned in [CMS05].

6.1 Preliminaries

Otherwise stated, all graphs considered in this work are simple, undirected and connected. Let $G = (V, E)$ be a graph. Given a vertex $v \in V$, $N(v)$ denotes the (open) neighborhood of v, i.e., the set of neighbors of v. Let $N[v] = N(v) \cup \{v\}$ be the closed neighborhood of v. A vertex v is *universal* if $N[v] = V$. A vertex is *simplicial* if $N[v]$ induces a complete subgraph in G. Finally, a subgraph H of G is *isometric* if, for any $u, v \in V(H)$, the distance $dist_H(u, v)$ between u and v in H equals $dist_G(u, v)$.

This section is devoted to basic lemmas on hull sets. These lemmas will serve as cornerstone of most of the results presented in this chapter.

Lemma 17 ([ES85])**.** *For any hull set S of a graph G, S contains all simplicial vertices of G.*

Lemma 18 ([DGK+09])**.** *Let G be a graph which is not complete. No hull set of G with cardinality $hn(G)$ contains a universal vertex.*

Lemma 19 ([DGK+09])**.** *Let G be a graph, H be an isometric subgraph of G and S be any hull set of H. Then, the convex hull of S in G contains $V(H)$.*

Lemma 20 ([DGK+09])**.** *Let G be a graph and S a proper and non-empty subset of $V(G)$. If $V(G) \backslash S$ is convex, then every hull set of G contains at least one vertex of S.*

6.2 Bipartite graphs

In this section, we answer an open question of Dourado et al. [DGK+09] by showing that the Hull Number Problem is NP-complete in the class of bipartite graphs. Since the Hull Number Problem is in NP, as proved in [DGK+09], it only remains to prove the following theorem:

Theorem 18. *The* Hull Number *problem is NP-hard in the class of bipartite graphs.*

Proof. To prove this theorem, we adapt the proof presented in [DGK+09]. We reduce the 3-SATisfiability Problem to the Hull Number problem in bipartite graphs. Let us consider the following instance of 3-SAT. Given a formula in the conjunctive normal form, let $\mathcal{F} = \{C_1, C_2, \ldots, C_m\}$ be the set of its 3-clauses and $X = \{x_1, x_2, \ldots, x_n\}$ the set of its boolean variables. We may assume that $m = 2^p$, for a positive integer $p \geq 1$, since it is possible to add dummy variables and clauses without changing the satisfiability of $\mathcal{F}$ and such that the size of the instance is

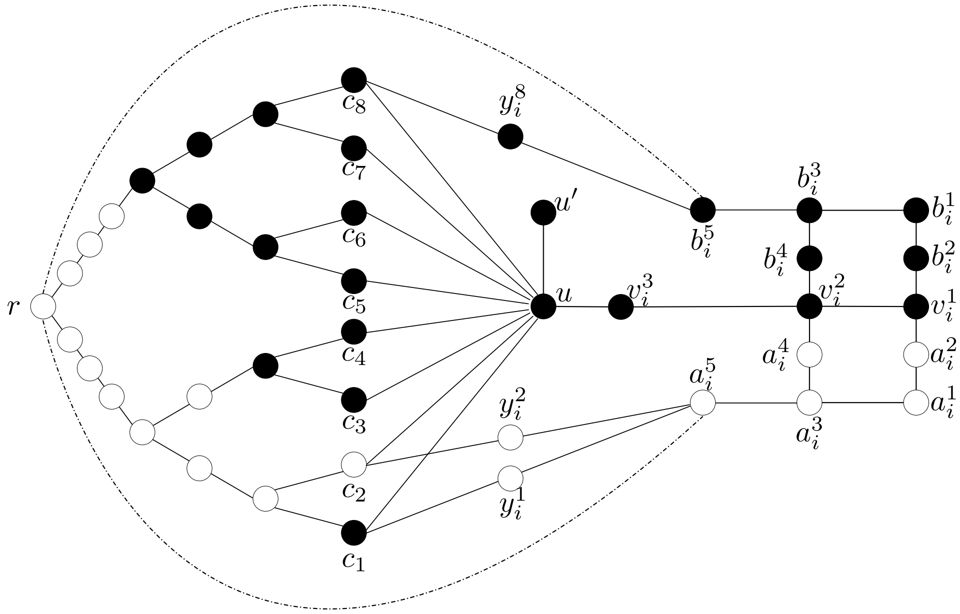


Figure 6.1: Subgraph of the bipartite instance $G(\mathcal{F})$ containing the gadget of a variable x_i that appears positively in clauses C_1 and C_2, and negatively in C_8. If x_i appears positively in C_j, link a_i^5 to c_j through y_i^j. If it appears negatively, we use b_i^5 instead of a_i^5.

at most twice the size of the initial instance. Moreover, we also assume, without loss of generality, that each variable x_i and its negation appear at least once in $\mathcal{F}$ (otherwise the clauses where x_i appeared could always be satisfied).

Let us construct the bipartite graph $G(\mathcal{F})$ as follows. First, let T be a full binary tree of height p rooted in r with $m = 2^p$ leaves, and let $L = \{c_1, c_2, \ldots, c_m\}$ be the set of leaves of T. We then construct a graph H as follows. First, let us add a vertex u that is adjacent to every vertex in L. Then, any edge $\{u, v\} \in E(T)$ with u the parent of v is replaced by a path with $2^{h(v)}$ edges, where $h(v)$ is the distance between v and any of its descendent leaves. Note that, in H, the distance between r and any leaf is $\sum_{i=0}^{p-1} 2^i = 2^p - 1 = m - 1$. Moreover, it is easy to see that $|V(H)| = O(m \cdot \log m)$.

The following claims are proved in [DGK+09].

Claim 3. *Let $v, w \in V(T) \backslash \{r\}$. The closed interval of v, w in H contains the parents of v in T if and only if v and w are siblings in T.*

Claim 4. *The set L is a* minimal *hull set of H.*

Then, let H' be obtained by adding a one degree vertex u' adjacent to u in H. Finally, we build a graph $G(\mathcal{F})$ from H' by adding, for any variable x_i, $i \leq n$, the gadget defined as follows.

Let us start with a cycle $\{a_i^1, a_i^2, v_i^1, b_i^2, b_i^1, b_i^3, b_i^4, v_i^2, a_i^4, a_i^3\}$ plus the edge $\{v_i^2, v_i^1\}$. Then, add the vertex v_i^3 as common neighbor of v_i^2 and u. Add a neighbor b_i^5 (resp.,

a_i^5) adjacent to b_i^3 (resp., a_i^3) and a path of length $2^{h(r)} - 3 = m - 3$ edges between b_i^5 (resp., a_i^5) and r. Let D be the set of internal vertices of all these $2n$ paths between a_i^5, resp., b_i^5, and r, $i \leq n$. Finally, for any clause C_j in which x_i appears, if x_i appears positively (resp., negatively) in C_j then add a common neighbor y_i^j between c_j and a_i^5 (resp., b_i^5). See an example of such a gadget in Figure 6.1. Note that $|V(G(\mathcal{F}))| = O(m \cdot (n + \log m))$.

Lemma 21. *$G(\mathcal{F})$ is a bipartite graph.*

Proof. Let us present a proper 2-coloring c of $G(\mathcal{F})$. Let $c(r) = 1$, and for each vertex w in $V(H)$, define $c(w)$ as 1 if w is in an even distance from r, and 2 otherwise. Clearly, c is a partial *proper* coloring of $G(\mathcal{F})$ and moreover we have $c(u) = 1$ and $c(c_j) = 2$, for any $j \in\{1, \dots, m\}$ (Indeed, any c_i is at distance $m - 1$ (odd) of r in H). Let $c(u') = 2$. For every $i \in\{1, \dots, n\}$ and for any j such that $x_i \in C_j$, let $c(y_i^j) = 1$. For any $i \leq n$, for any $x \in \{b_i^5, a_i^5, v_i^3, b_i^4, a_i^4, b_i^1, v_i^1, a_i^1\}$, $c(x) = 2$.

$c(b_i^5) = c(a_i^5) = c(v_i^3) = 2$. Again, this partial coloring of $G(\mathcal{F})$ is proper. One can easily verify that this coloring can be extended to $\{a_i^1, a_i^2, v_i^1, b_i^2, b_i^1, b_i^3, b_i^4, v_i^2, a_i^4, a_i^3\}$ for any $i \leq n$. Moreover, since $c(r) = 1$ and $c(a_i^5) = 2$ ($c(b_i^5) = 2$), for every $i \in\{1, \dots, n\}$, and since the path that we add in $G(\mathcal{F})$ between r and a_i^5 (b_i^5) is of odd length $m - 3$, one can completely extend c in order to get a proper 2-coloring of $G(\mathcal{F})$. ⋄

Claim 5. *The set $V(G(\mathcal{F}))\setminus\{a_i^1, a_i^2, v_i^1, b_i^1, b_i^2\}$ is convex, for any $i \in\{1, \dots, n\}$.*

Proof. Denote $W_i = \{a_i^1,\ a_i^2,\ v_i^1,\ b_i^1,\ b_i^2\}$, for some $i \in \{1, \dots, n\}$, and $W_i' = \{a_i^3, b_i^3, v_i^2\}$. By contradiction, suppose that there exists an (x, y)-shortest path containing a vertex of W_i, for some $x, y \in V(G(\mathcal{F}))\setminus W_i$. Observe that it implies that that there are $x', y' \in W_i'$ such that $I[x', y']$ contains a vertex of W_i, since W_i' contains all the neighbors of W_i in $V(G(\mathcal{F}))\setminus W_i$. However, it is easy to verify that for any pair $x, y \in W_i'$, $I[x, y]$ contains no vertex of W_i. This is a contradiction. ⋄

Lemma 22. *$hn(G(\mathcal{F})) \geq n + 1$.*

Proof. Let S be any hull set of $G(\mathcal{F})$. Clearly $u' \in S$, because u' is a simplicial vertex of $G(\mathcal{F})$ (Lemma 17). Furthermore, Claim 5 and Lemma 20 imply that S must contain at least one vertex w_i of the set $\{a_i^1, a_i^2, v_i^1, b_i^1, b_i^2\}$, for every $i \in\{1, \dots, n\}$. Hence, $|S| \geq n + 1$. ⋄

The main part of the proof consists in showing:

Lemma 23. *$\mathcal{F}$ is satisfiable if and only if $hn(G(\mathcal{F})) = n + 1$.*

First, consider that $\mathcal{F}$ is satisfiable. Given an assignment A that turns $\mathcal{F}$ true, define a set S as follows. For $1 \leq i \leq n$, if x_i is true in A add a_i^1 to S, otherwise add b_i^1 to S. Finally, add u' to S. Note that $|S| = n + 1$. We show that S is a hull set of $G(\mathcal{F})$. First note that $a_i^5, c_j \in I[a_i^1, u']$, for every clause C_j containing the positive literal of x_i. Similarly, observe that $b_i^5, c_j \in I[b_i^1, u']$, for every clause C_j containing the negative literal of x_i. Since A satisfies $\mathcal{F}$, it follows $L \subseteq I_h[S]$.

Therefore, H being an isometric subgraph of $G(\mathcal{F})$, Lemma 19 and Claim 5 imply that $V(H) \subseteq I_h[S]$. Furthermore, the shortest paths between r and u have length m, which implies that all vertices a_i^5, b_i^5, y_i^j ($i \leq n$) and all vertices in D are included in $I_h[S]$. It remains to observe that $I_h[a_i^5, b_i^5, w, u']$, where $w \in \{a_i^1, b_i^1\}$, contains the variable subgraph of x_i. Therefore we have that S is a hull set of $G(\mathcal{F})$.

We prove the sufficiency by contradiction. Suppose that $G(\mathcal{F})$ contains a hull set S with $n+1$ vertices and that $\mathcal{F}$ is not satisfiable.

Recall that, by Lemma 17, $u' \in S$. For any $i \leq n$, let W_i as defined in Claim 5. Recall also that there must be a vertex $w_i \in W_i \cap S$, for any $i \leq n$. Since $v_i^1 \in I[u', a_i^1]$, $v_i^1 \in I[u', b_i^1]$, $a_i^2 \in I[u', a_i^1]$ and $b_i^2 \in I[u', b_i^1]$, we can assume, without loss of generality, that $w_i \in \{a_i^1, b_i^1\}$, for every $i \in \{1, \ldots, n\}$ (indeed, if $w_i \in \{v_i^1, a_i^2\}$, it can be replaced by a_i^1, and if $w_i = b_i^2$, it can be replaced by b_i^1). Therefore S defines the following truth assignment $\mathcal{A}$ to $\mathcal{F}$. If $w_i = a_i^1$ set x_i to true, otherwise set x_i to false. As $\mathcal{F}$ is not satisfiable, there exists at least one clause C_j not satisfied by $\mathcal{A}$.

Using the hypothesis that $\mathcal{F}$ is not satisfiable, we complete the proof by showing that there is a non empty set U such that $V(G(\mathcal{F}))\backslash U$ is a convex set and $U \cap S = \emptyset$. That is, we show that $I_h[S] \subseteq V(G(\mathcal{F}))\backslash U$ for some $U \neq \emptyset$, contradicting the fact that S is a hull set.

For any clause C_j, let us define the subset U_j of vertices as follows. Let P_j be the path in T between c_j and r, let X_j be the p vertices in $V(T) \setminus V(P_j)$ that are adjacent to some vertex in P_j. Then, U_j is the union of the vertices that are either in P_j or that are internal vertices of the paths resulting of the subdivision of the edges $\{x, y\}$ where $x, y \in P_j \cup X_j$. Another way to build the set U_j is to start with the set of vertices of the (unique) shortest path between c_j and r in H and then add successively to this set, the vertices of $V(H) \setminus (V(T) \cup \{u\})$ that are adjacent to some vertex of the current set.

Now, let $U' = \cup_{j \in J} U_j$ where J is the (non empty) set of clauses that are not satisfied by $\mathcal{A}$. Note that $r \in U'$.

For any $i \leq n$, let Z_i be defined as follows. If $w_i = a_i^1$ (x_i assigned to true by $\mathcal{A}$), then Z_i is the union of $\{b_i^\ell \ : \ \ell \leq 5\}$ with the set of the y_i^k that are adjacent to b_i^5. Otherwise, $w_i = b_i^1$ (x_i assigned to false by $\mathcal{A}$), then Z_i is the union of $\{a_i^\ell \ : \ \ell \leq 5\}$ with the set of the y_i^k that are adjacent to a_i^5.

Finally, let $U = U' \cup (\bigcup_{i \leq n} Z_i) \cup D$. In Figure 6.1, U is depicted by the white vertices, assuming that clause C_2 is false and that x_i is set to false by $\mathcal{A}$. Observe that $U \cap S = \emptyset$.

It remains to prove that $V(G(\mathcal{F}))\backslash U$ is a convex set. Consider the partition $\{A_1, A_2, A_3\}$ of $V(G(\mathcal{F}))\backslash U$ where $A_1 = V(H)\backslash(U \cup \{u\})$, $A_2 = \{u, u'\}$ and $A_3 = V(G(\mathcal{F}))\backslash(U \cup A_1 \cup A_2)$. To prove that $V(G(\mathcal{F}))\backslash U$ is convex, let $w \in A_i$ and $w' \in A_j$ for some $i, j \in \{1, 2, 3\}$. We show that $I[w, w'] \cap U = \emptyset$ considering different cases according to the values of i and j. Recall that $V(H) \setminus \{u\}$ induces a tree T' rooted in r and that, if a vertex of T' is in A_1, then, by definition of U', all its descendants in T' are also in A_1 (i.e., if $v \in U \cap V(T')$, then all ancestors of v in T' are in U). It is important to note that, for any vertex v in A_1, the shortest path in $G(\mathcal{F})$ from v to any leaf ℓ of T' is the path from v to ℓ in T' (in particular, such a shortest path does not pass through r and any vertices in D).

- The case $i = j = 2$, i.e., $m, m' \in \{u, u'\}$, is trivial;

- First, let us assume that $w \in A_1 = V(H)\setminus(U \cup \{u\})$ and $w' \in A_2 = \{u, u'\}$. If $w' = u$ (resp., if $w' = u'$) then $I_h[w, w']$ consists of the subtree of T' rooted in w union u (resp., union u and u'). Hence, $I_h[w, w'] \cap U = \emptyset$ because no descendants of w in T' are in U.

- Second, let $w, w' \in A_1$. If one of them, say w, is an ancestor of the other in T', then $I_h[w, w']$ consists of the path between them in T' (remember that $r \in U$ so $w \neq r$). Since no descendants of w in T' are in U, $I_h[w, w'] \cap U = \emptyset$. Otherwise, there are three cases: (1) either $I_h[w, w']$ consists of the path P between w and w' in T', or (2) $I_h[w, w']$ consists of the union of the subtree R of T' rooted in w, the subtree R' of T' rooted in w' and u, or (3) $I_h[w, w'] = R \cup R' \cup P \cup \{u\}$. Again, $(R \cup R' \cup \{u\}) \cap U = \emptyset$ because no descendants of w and w' in T' are in U. Hence, it only remains to prove that when $P \subseteq I_h[w, w']$ then $P \cap U = \emptyset$. It is easy to check that $P \subseteq I_h[w, w']$ only in the following case: there exist $x, y, z \in V(T)$ such that x is the parent of y and z in T, and w (resp., w') is a vertex of the path resulting from the subdivision of $\{x, y\}$ (resp., $\{x, z\}$). In this case, it means that all clause-vertices that are descendants of y and z are not in U. Therefore $x \notin U$ and hence no descendants of x are in U. In particular, $P \cap U = \emptyset$.

- Assume now that $w \in A_3$. Let $i \leq n$ such that w belongs to the gadget G_i corresponding to variable x_i. Let us assume that $w_i = b_i^1$. The case $w_i = a_i^1$ can be handled in a similar way by symmetry. Then, by definition, U contains $\{a_i^1, \cdots, a_i^5\}$ and the y_i^j's adjacent to a_i^5. With this setting, x_i is set to false in the assignment $\mathcal{A}$. If there is a vertex y_i^j adjacent to b_i^5, let C_j be the other neighbor of j_i^j. By definition, it means that clause C_j contains the negation of variable x_i. Since x_i is set to false, it means that clause C_j is satisfied and so $C_j \notin U$.

  Let $x \in V(G_i) \setminus U$. Then, any shortest path P from w to x either passes through $V(G_i) \setminus U$ or, there is y_i^j adjacent to b_i^5 such that P passes through y_i^j, C_j, u and v_i^3 (the latter case may occur if $a \in \{y_i^j, b_i^5\}$ and $b = v_i^3$, or $a = y_i^j$ and $b \in \{v_i^3, v_i^2\}$ where $\{a, b\} = \{x, w\}$). Hence, such a path P avoid U, and the result holds if $x = w' \in A_3 \cap G_i$.

  Similarly, if $x \in \{u, u'\}$, then, any shortest path P from w to x either passes through $V(G_i) \setminus U$ or through y_i^j, C_j, u with y_i^j adjacent to b_i^5. In particular, if $x = w' \in \{u, u'\} = A_2$, then the result holds.

  Now, let $x = C_{j'}$ be a leaf of T' that is not in U. Then, any shortest path P from w to x either passes through u or through y_i^j, C_j and, if $j \neq j'$, through u. In any case, P avoids U. If $w' \in A_3 \setminus G_i$, any path between w and w' passes through u or through one or two leaves that are not in U. Finally, if

$w' \in A_1$, let R be the subtree of T' rooted in w'. $V(R) \subseteq I_h[w, w']$. Moreover, any shortest path from w to w' path through a leaf of R, i.e., a leaf not in U. By previous remarks, in all these cases, the shortest paths between w and w' avoid u, and $I_h[w, w']$ are disjoint from U.

□

We conclude this section by showing one *approximability result.* Let $IG(G)$ be the *incidence graph* of G, obtained from G by subdividing each edge once. That is, let us add one vertex s_{uv}, for each edge $uv \in E(G)$, and replace the edge uv by the edges $us_{uv}, s_{uv}v$.

Proposition 13. $hn(IG(G)) \leq hn(G) \leq 2hn(IG(G))$.

Proof. Let $IG(G)$ be the incidence graph of G. Observe that any hull set of G is a hull set of $IG(G)$, since for any shortest path, $P = \{v_1, \dots, v_k\}$ in G there is a shortest path $P' = \{v_1, s_{v_1v_2}, v_2, \dots, s_{v_{k-1}v_k}, v_k\}$ in $IG(G)$ (the edges were subdivided). Consequently, $hn(IG(G)) \leq hn(G)$. However, given a hull set S_h of $IG(G)$, one may find a hull set of G by simply replacing each vertex of S_h that represents an edge of G by its neighbors (vertices of G). Thus, $hn(G) \leq 2hn(IG(G))$. □

Corollary 16. *If there exists a k-approximation algorithm B to compute the hull number of bipartite graphs, then B is a $2k$-approximation algorithm for any graph.*

6.3 Complement of bipartite graphs

A graph $G = (V, E)$ is a complement of a bipartite graph if there is a partition $V = A \cup B$ such that A and B are cliques. In this section, we give a polynomial-time algorithm to compute a hull set of G with size $hn(G)$. We start with some notations.

Given the partition (A, B) of V, we say that an edge $uv \in E$ is a *crossing-edge* if $u \in A$ and $v \in B$. Denote by S the set of simplicial vertices of G, by $S_A = S \cap A$ and by $S_B = S \cap B$. Let U be the set of universal vertices of G. Note that, if G is not a clique, $U \cap S = \emptyset$. Let H be the graph obtained from G by removing the vertices in S and U, and removing the edges intra-clique, i.e., $V(H) = V \setminus (U \cup S)$ and $E(H) = \{\{u, v\} \in E \ : \ u \in A \cap V(H) \ and \ v \in B \cap V(H)\}$. Let $\mathcal{C} = \{C_1, \cdots, C_r\}$ ($r \geq 1$) denote the set of connected components C_i of H. Observe that, if G is neither one clique nor the disjoint union of A and B, H is not empty and each connected component C_i has at least two vertices, for every $i \in \{1, \dots, r\}$. Indeed, any vertex in $A \setminus S_A$ (resp., in $B \setminus S_B$) has a neighbor in $B \cap V(H)$ (resp. in $A \cap V(H)$).

Theorem 19. *Let $G = (A \cup B, E)$ be the complement of a bipartite n-node graph. There is an algorithm that computes $hn(G)$ and a hull set of this size in time $O(n^7)$.*

Proof. We use the notations defined above. Recall that, by Lemma 17, S is contained in any hull set of G. In particular, if G is a clique or G is the disjoint union of two cliques A and B, then $hn(G) = n$. From now on, we assume it is not the case. By Lemma 18, no vertices in U belong to any minimal hull set of G. Now, several cases have to be considered.

Claim 6. *If $U = \emptyset$, $S_A \neq \emptyset$ and $S_B \neq \emptyset$, then S is a minimum hull set of G and thus $hn(G) = |S|$.*

Proof. Since G has no universal vertex, a simplicial vertex in S_A (in S_B) has no neighbor in B (resp., in A). Since G is not the disjoint union of two cliques, every vertex $u \in A \backslash S_A$ has a neighbor $v \in B \backslash S_B$ and vice-versa. Thus, $s_a uv s_b$ is a shortest (s_a, s_b)-path, for any $s_a \in A$ and $s_b \in B$, and then $u, v \in I_h[S]$. □

Hence, from now on, let us assume that $U \neq \emptyset$ or, w.l.o.g., $S_B = \emptyset$.

Again, if there is some simplicial vertex in G, i.e., if $S_A \neq \emptyset$, all the vertices of S belong to any hull set of G and thus $hn(G) \geq |S|$. In fact, for each connected component of H, we prove that it is necessary to choose at least one of its vertices to be part of any hull set of G.

Claim 7. *If $U \neq \emptyset$ or $S_B = \emptyset$ or $S_A = \emptyset$, then $hn(G) \geq |S| + r$.*

Proof. Again, all vertices of S belong to any hull set of G. We show that, for any $1 \leq i \leq r$, $V \backslash C_i$ is a convex set. Thus, by Lemma 20, any hull set of G contains at least one vertex of C_i for any $i \leq r$.

It is sufficient to show that no pair $u, v \in V(G) \backslash C_i$ can generate a vertex v_i of C_i. By contradiction, suppose that there exists a pair of vertices $u, v \in V(G) \backslash C_i$ such that there is a shortest (u, v)-path P containing a vertex v_i of C_i. Consequently, u and v must not be adjacent and we consider that $u \in A$ and $v \in B$. If $U = \emptyset$, then, w.l.o.g., $S_B = \emptyset$ and v is not simplicial and has at least one neighbor in A. Hence, since $U \neq \emptyset$ or $S_b = \emptyset$, u and v are at distance two. Consequently, $P = uv_iv$. However, if $v_i \in A$, v belongs to C_i, because of the crossing edge v_iv, otherwise, $u \in C_i$. In both cases we reach a contradiction. □

Now, two cases remain to be considered. We recall that $U \neq \emptyset$ or $S_B = \emptyset$.

1. If $r \geq 2$, then $hn(G) = |S| + r$, and we can build a minimum convex hull by taking the vertices in S, one arbitrary vertex in $A \cap C_i$ for all $i < r$ and one arbitrary vertex in $B \cap C_r$.

   Let $R = \{v_1, \ldots, v_r\}$ such that $v_i \in C_i \cap A$ for any $i < r$ and $v_r \in C_r \cap B$.

   Claim 8. *$S \cup R$ is a hull set of G.*

   Proof. Since all vertices in U are generated by v_1 and v_r (that are not adjacent, since they are in different components), it is sufficient to show that $S \cup R$ generates all the vertices in C_i, for any $i \in \{1, \ldots, r\}$. Actually, we show that R generates all the vertices in C_i.

By contradiction, suppose that there is a vertex $z \notin I_h[R]$. Let $i \leq r$ such that $z \in C_i$. Because C_i contains one vertex in R and is connected, we can choose z and $w \in C_i \cap I_h[R]$ linked by a crossing edge. We will show that $z \in I_h[R]$ (a contradiction), hence, w.l.o.g., we may assume that $z \in A$. If $i = r$, then v_1zw is a shortest (v_1, w)-path and $z \in I_h[R]$.

Otherwise, recall that $N(v_r) \cap A \cap C_r \neq \emptyset$ and, for any $i < r$, $N(v_i) \cap B \cap C_i \neq \emptyset$ because v_i is not simplicial for any $i \leq r$. Let $x \in N(v_r) \cap A \cap C_r$ and $y_i \in N(v_i) \cap B \cap C_i$. Note that $x \in I_h[R]$ because v_1xv_r is a shortest (v_r, v_1)-path, and $y_i \in I_h[R]$ because $v_iy_iv_r$ is a shortest (v_r, v_i)-path. Hence, since xzy_i is a shortest (x, y_i)-path, we have $z \in I_h[R]$. □

As $|R| = r$, we conclude by Claim 7 that $hn(G) = |S| + r$.

2. If $r = 1$, then $hn(G) \leq |S| + 4$, and any minimum convex hull contains at most 4 vertices not in S.

   Again, S is included in any hull set of G by Lemma 17, and no vertices in U belong to some hull set by Lemma 18. In this case, when H has just one connected component $C_1 = C$, one vertex of C may not suffice to generate this component, as in the previous case. However, we prove that at most 4 vertices in C are needed.

   (a) If $S_A \neq \emptyset$ and $S_B \neq \emptyset$ (and thus $U \neq \emptyset$ because Claim 6 applies otherwise), then $hn(G) = |S| + 1$.

   By Claim 7, we know that $hn(G) \geq |S|+1$. Let v be an arbitrary vertex of C. We claim that $S \cup \{v\}$ is a minimum hull set of G. By contradiction, let $z \notin I_h[S \cup \{v\}]$. Since C is a connected component of H, we may choose z such that there is $w \in N(z) \cap C \cap I_h[S \cup \{v\}]$. Moreover, we may assume w.l.o.g. that $z \in A$, and thus $w \in B$. In that case, since $S_A \neq \emptyset$, there is $v_A \in S_A$ and as $v_Aw \notin E(G)$ (indeed, any vertex in $N(v_A) \cap B$ must be universal because v_A is simplicial, which is not the case since w is not universal because it belongs to C), z is generated by v_A and w.

   (b) If $S_A \neq \emptyset$ and $S_B = \emptyset$, then $hn(G) \leq |S| + 2$.

   Let $v_A \in A \cap C$ be such that $|N(v_A) \cap B \cap C|$ is maximum. Since v_A is not universal in G, there exists $x \in B$ such that $v_Ax \notin E(G)$. Note that $x \in C$ since x is not universal and $S_B = \emptyset$. Let $R = \{v_A, x\}$. Observe that $N(v_A) \cap B \cap C \subseteq I_h[R \cup S]$ since $v_Ax \notin E$.

   By contradiction, assume $V(G) \backslash I_h[R \cup S] \neq \emptyset$. Let $z \in V(G) \backslash I_h[R \cup S]$. First, suppose that $z \in A$. Since C is connected in H, we may assume that z has a neighbor $w \in I_h[R \cup S] \cap B \cap C$. As $S_A \neq \emptyset$, there is $v \in S_A$ and as $vw \notin E(G)$ (because otherwise w would be universal in G and not in C), z is generated by v and w. Now suppose that $z \in B$, and now it has a neighbor $w \in I_h[R \cup S] \cap A \cap C$. Observe that

$I_h[R \cup S] \cap B \subseteq N(w)$, otherwise z would be in $I_h[R \cup S]$. However, since $N(v_A) \cap B \cap C \subset (N(v_A) \cap B \cap C) \cup \{x\} \subseteq I_h[R \cup S] \cap B$, we get that $N(v_A) \cap B \cap C \subset N(w) \cap B \cap C$, contradicting the maximality of $|N(v_A) \cap B \cap C|$.

(c) If $S_A = \emptyset$ and $S_B = \emptyset$, then $hn(G) \leq 4$.

Let $v_A \in A \cap C$ be such that $|N(v_A) \cap B \cap C|$ is maximum and $v_B \in B \cap C$ be such that $|N(v_B) \cap A \cap C|$ is maximum. Since v_A is not universal in G and $S_B = \emptyset$, there exists $y \in C \cap B \setminus N(v_a)$, and similarly there exists $x \in C \cap A \setminus N(v_B)$. Let $R = \{v_A, v_B, x, y\}$. Observe that $N(v_A) \cap B \subseteq I_h[R]$ and $N(v_B) \cap A \subseteq I_h[R]$, since $v_A y \notin E$ and $v_B x \notin E$.

By contradiction, assume $V(G) \backslash I_h[R] \neq \emptyset$. Let $z \in V(G) \backslash I_h[R]$. First, suppose that $z \in A$. As in the previous case, since C is connected in H, we may assume that z has a neighbor $w \in I_h[R] \cap B \cap C$. Observe that $I_h[R] \cap A \cap C \subseteq N(w)$, otherwise z would be in $I_h[R]$. However, since $N(v_B) \cap A \cap C \subset (N(v_B) \cap A \cap C) \cup \{x\} \subseteq I_h[R] \cap A \cap C$, we get that $N(v_B) \cap A \cap C \subset N(w) \cap A \cap C$, contradicting the maximality of $|N(v_B) \cap A \cap C|$.

Whenever $z \in B$, one can use the same arguments to reach a contradiction on the maximality of $|N(v_A) \cap B \cap C|$.

Since $|S| + 1 \leq hn(G) \leq |S| + 4$, S is included in any hull set of G and no vertices in U belong to some hull set, there exist a subset R of at most 4 vertices in C such that $S \cup R$ is a minimum hull set of G. There are $\mathcal{O}(|V|^4)$ subsets to be tested and, for each one, its convex hull can be computed in $\mathcal{O}(|V||E|)$ time [DGK+09]. This leads to the announced result.

□

6.4 Graphs with few P_4's

A graph $G = (V, E)$ is a $(q, q-4)$-graph, for a fixed $q \geq 4$, if for any $S \subseteq V$, $|S| \leq q$, S induces at most $q - 4$ paths on 4 vertices [BO95]. Observe that cographs and P_4-sparse graphs are the $(q, q-4)$-graphs for $q = 4$ and $q = 5$, respectively. The hull number of a cograph can be computed in polynomial time [DGK+09]. This result is improved in [ACG+11b] to the class of P_4-sparse graphs. In this section, we generalize these results by proving that for any fixed $q \geq 4$, computing the hull number of a $(q, q-4)$-graph can be done in polynomial time. Our algorithm runs in time $O(2^q n^2)$ and is therefore a Fixed Parameter Tractable for any graph G, where the number of induced P_4's of G is the parameter.

6.4.1 Definitions and brief description of the algorithm

The algorithm that we present in this section uses the canonical decomposition of $(q, q-4)$-graphs, called *Primeval Decomposition*. For a survey on Primeval Decomposition, the reader is referred to [BO99]. In order to present this decomposition of $(q, q-4)$-graphs, we need the following definitions.

Let G_1 and G_2 be two graphs. $G_1 \cup G_2$ denotes the disjoint union of G_1 and G_2. $G_1 \oplus G_2$ denotes the join of G_1 and G_2, i.e., the graph obtained from $G_1 \cup G_2$ by adding an edge between any two vertices $v \in V(G_1)$ and $w \in V(G_2)$. Recall that a *spider* $G = (S, K, R, E)$ is a graph with vertex set $V = S \cup K \cup R$ and edge set E such that

1. (S, K, R) is a partition of V and R may be empty;
2. the subgraph $G[K \cup R]$ induced by K and R is the join $K \oplus R$, and K separates S and R, i.e., any path from a vertex in S to a vertex in R contains a vertex in K;
3. S is a stable set, K is a clique, $|S| = |K| \geq 2$, and there exists a bijection $f : S \to K$ such that, either $N(s) \cap K = K - \{f(s)\}$ for all vertices $s \in S$, or $N(s) \cap K = \{f(s)\}$ for all vertices $s \in S$. In the latter case or if $|S| = |K| = 2$, G is called *thin*, otherwise G is *thick*.

A graph $G = (S, K, R, E)$ is a *pseudo-spider* if it satisfies only the first two properties of a spider. A graph $G = (S, K, R, E)$ is a *q-pseudo-spider* if it is a pseudo-spider and, moreover, $|S \cup K| \leq q$. Note that q-pseudo-spiders and spiders are pseudo-spiders.

We now describe the decomposition of $(q, q-4)$-graphs.

Theorem 20 ([BO95]). *Let $q \geq 0$ and let G be a $(q, q-4)$-graph. Then, one of the following holds:*

1. *G is a single vertex, or*
2. *$G = G_1 \cup G_2$ is the disjoint union of two $(q, q-4)$-graphs G_1 and G_2, or*
3. *$G = G_1 \oplus G_2$ is the join of two $(q, q-4)$-graphs G_1 and G_2, or*
4. *G is a spider (S, K, R, E) where $G[R]$ is a $(q, q-4)$-graph if $R \neq \emptyset$, or*
5. *G is a q-pseudo-spider (H_2, H_1, R, E) where $G[R]$ is a $(q, q-4)$-graph if $R \neq \emptyset$.*

Theorem 20 leads to a tree-like structure $T(G)$ (the *primeval tree*) which represents the Primeval Decomposition of a $(q, q-4)$-graph G. $T(G)$ is a rooted binary tree where any vertex v corresponds to an induced $(q, q-4)$-subgraph G_v of G and the root corresponds to G itself. Moreover, the vertices of subgraphs corresponding to the leaves of $T(G)$ form a partition of $V(G)$, i.e., $\{V(G_\ell)\}_{\ell \text{ leaf of } T(G)}$ is a partition of $V(G)$.

For any leaf ℓ of $T(G)$, G_ℓ is either a spider $(S, K, \emptyset, E)$, or has at most q vertices. Moreover, any internal vertex v has its label following one of the four cases in Theorem 20 corresponds to G_v. More precisely, let v be an internal vertex of $T(G)$ and let u and w be its two children. v is a *parallel node* if $G_v = G_u \cup G_w$.

v is a *series node* if $G_v = G_u \oplus G_w$. v is a *spider node* if u is a leaf with G_u is a spider $(S, K, \emptyset, F)$ and G_v is the spider (S, K, R, E) where $G_v[R] = G_w$ and $G_v[S \cup K] = G_u$. Finally, v is a *small node* if u is a leaf with $|V(G_u)| \leq q$ and G_v is the q-pseudo-spider (S, K, R, E) where $G_v[R] = G_w$ and $G_v[S \cup K] = G_u$.

This tree can be obtained in linear-time [BO99].

We compute $hn(G)$ by a post-order traversal in $T(G)$. More precisely, given $v \in V(T(G))$, let H_v be an optimal hull set of G_v and let H_v^* be an optimal hull set of G_v^*, the graph obtained by adding a universal vertex to G_v. We show in next subsection that we can compute (H_ℓ, H_ℓ^*) for any leaf ℓ of $T(G)$ in time $O(2^q n)$. Moreover, for any internal vertex v of $T(G)$, we show that we can compute (H_v, H_v^*) in time $O(2^q n)$, using the information that was computed for the children and grand children of v in $T(G)$.

Theorem 21. *Let $q \geq 0$ and let G be a n-node $(q, q-4)$-graph. An optimal hull set of G can be computed in time $O(2^q n^2)$.*

Before going into the details of the algorithm in next subsection, we prove some useful lemmas.

Lemma 24 ([ACG+11b]). *Let $G = (S, K, R, E)$ be a pseudo-spider with R neither empty nor a clique. Then any minimum hull set of G contains a minimum hull set of the subgraph $G[K \cup R]$.*

Proof. Let H be a minimum hull set of G. Let $H_S = H \cap S$ and $H_R = H \setminus H_S$. We prove that H_R is a minimum hull set of $G[K \cup R]$.

Let H' be any minimum hull set of $G[K \cup R]$. Note that $H' \subseteq R$ because K is a set of universal vertices in $G[K \cup R]$ and by Lemma 18. Moreover, By Lemma 19, because $G[K \cup R]$ is an isometric subgraph of G, the convex hull of H' in G contains $G[K \cup R]$. Hence, $H_S \cup H'$ is a hull set of G and $hn(G) \leq |H_S| + hn(G[K \cup R])$.

Now it remains to prove that H_R is a hull set of $G[K \cup R]$. Clearly, if H_R generate all vertices of R in $G[K \cup R]$ then H_R is a hull set of $G[K \cup R]$ since there are at least two non adjacent vertices in R and any vertex in K is adjacent to all vertices in R. For purpose of contradiction, assume H_R does not generate R in $G[K \cup R]$. This means that there is a vertex $v \in R$, that is generated in G by a vertex in $S \cup K$, i.e., $v \in R$ is an internal vertex of a shortest path between $s \in S \cup K$ and some other vertex, which is not possible, since we have all the edges between K and R. Hence, $hn(G[K \cup R]) \leq |H_R|$.

Therefore, $|H_S| + |H_R| = hn(G) \leq |H_S| + hn(G[K \cup R]) \leq |H_S| + |H_R|$. So, $hn(G[K \cup R]) = |H_R|$, i.e., H_R is a minimum hull set of $G[K \cup R]$ contained in H. ☐

The next lemma is straightforward by the use of isometry.

Lemma 25. *Let G be a graph which is not complete and that has a universal vertex. Let H obtained from G by adding some new universal vertices. A set is a minimum hull set of G if, and only if, it is a minimum hull set of H.*

6.4.2 Dynamic programming and correctness

In this section, we detail the algorithm presented in previous section and we prove its correctness. Let $v \in V(T(G))$, which may therefore be either a leaf, a parallel node, a series node, a spider node or a small node. For each of these five cases, we describe how to compute (H_v, H_v^*), in time $O(2^q n)$.

Let us first consider the case when v is a leaf of $T(G)$.

If G_v is a singleton $\{w\}$, then $H_v = V(G_v) = \{w\}$ and $H_v^* = V(G_v^*)$. If G_v is a spider $(S, K, \emptyset, E)$ then $H_v = S$ since S is a set of simplicial vertices (so it has to be included in any hull set by Lemma 17) and it is sufficient to generate G_v. One may easily check that if G_v is a thick spider, S is also a minimum hull set of G_v^*, i.e., $S = H_v^*$. However, in case G_v is a thin spider, S does not suffice to generate G_v^* and in this case it is easy to see that this is done by taking any extra vertex $k \in K$, in which case we have $H_v^* = S \cup \{k\}$. Finally, if G_v has at most q vertices, H_v and H_v^* can be computed in time $\mathcal{O}(2^q)$ by an exhaustive search.

Now, let v be an internal node of $T(G)$ with children u and w.

If v is a parallel node, then $G_v = G_u \cup G_w$. Then, (H_v, H_v^*) can be computed in time $O(1)$ from (H_u, H_u^*) and (H_w, H_w^*) thanks to Lemma 26.

Lemma 26 ([DGK+09]). *Let $G_v = G_u \cup G_w$. Then $(H_v, H_v^*) = (H_u \cup H_w, H_u^* \cup H_w^*)$.*

Proof. The fact that $H_u \cup H_w$ is an optimal hull set for G_v is trivial. The second part comes from the fact that H_u^* (resp., H_w^*) is an isometric subgraph of H_v^* and from Lemma 19. □

Now, we consider the case when v is a series node.

Lemma 27. *If $G_v = G_u \oplus G_w$, then (H_v, H_v^*) can be computed from the sets (H_x, H_x^*) of the children or grand children x of v in $T(G)$, in time $O(2^q n)$.*

Proof. If G_u and G_w are both complete, then G_v is a clique and $(H_v, H_v^*) = (V(G_v), V(G_v^*))$.

If G_u and G_w are both not complete, let x, y be any two non adjacent vertices in G_u. Then, we claim that $H_v = H_v^* = \{x, y\}$. Indeed, in G_v, x and y generate all vertices in $V(G_w)$ (resp., of G_w^*). In particular, two non adjacent vertices $z, r \in V(G_w)$ are generated. Symmetrically, z, r generate all vertices in $V(G_u)$ (resp., in $V(G_u^*)$).

Without loss of generality, we suppose now that G_u is a complete graph and that G_w is a non-complete $(q, q-4)$-graph. First, observe that no vertex of G_u belongs to any minimum hull set of G_v, since they are universal (Lemma 18). Note also that, by Lemma 25 and since G_v is not a clique and has universal vertices, we can make $H_v = H_v^*$. Hence, in what follows, we consider only the computation of H_v. Let us consider all possible cases for w in $T(G)$.

- w is a series node. G_w is the join of two graphs. We claim that $H_v = H_w$.

In this case, G_w is an isometric subgraph of G_v. Thus, by Lemma 19, any minimum hull set of G_w generates all vertices of $V(G_w)$ in G_v. Finally, since G_w has two non-adjacent vertices they generate all vertices of G_u in G_v.

- w is a parallel node. G_w is the disjoint union of two graphs. Let x and y the children of w in $T(G)$. Then $G_w = G_x \cup G_y$. Let $X = H_x^*$ if G_x is not a clique and $X = V(G_x)$, otherwise, let $Y = H_y^*$ if G_y is not a clique and $Y = V(G_y)$, otherwise. We claim that $H_v = X \cup Y$.

  Clearly, if G_x (resp., G_y) is a clique, all its vertices are simplicial in G_v and then must be contained in any hull set by Lemma 17. Moreover, recall that, by Lemma 18, no vertex of G_u belongs to any minimum hull set of G.

  Now, let $z \in \{x, y\}$ such that G_z is not complete. It remains to show that it is necessary and sufficient to also include any minimum hull set H_z^* of G_z^*, in any minimum hull set of G.

  The necessity can be easily proved by using Lemma 24 to every G_z that is not a complete graph.

  The sufficiency follows again from the fact that G_u is generated by two non adjacent vertices of G_w and since, in all cases, $X \cup Y$ contains at least one vertex in G_x and one vertex in G_y, all vertices in G_u will be generated.

- w is a spider node and G_w is a thin spider $(S, K, \emptyset, E')$. Then, $H_v = S \cup \{k\} = G_w^*$ where k is any vertex in K.

  All vertices in S are simplicial in G_v, hence any hull set of G_v must contain S by Lemma 17. Now, in G_v, the vertices in S are at distance two and no shortest path between two vertices in S passes through a vertex in K, since there is a join to a complete graph. Therefore, S is not a hull set of G_v. However, since $|S| \geq 2$, it is easy to check that adding any vertex $k \in K$ to S is sufficient to generate all vertices in G_v. So $S \cup \{k\}$ is a minimum hull set of G_v.

  Note that, in that way, $H_v = S \cup \{k\} = G_w^*$

- w is a spider node and G_w is a spider (S, K, R, E') that is either thick or $R \neq \emptyset$ and R induces a $(q, q-4)$-graph. Then, $H_v = H_w$.

  If $R = \emptyset$, then G_w is thick. In this case, it is easy to check that the only minimum hull set of G_w is S (because it consists of simplicial vertices) and it is also a minimum hull set for G_v. Hence, $H_v = H_w = S$.

  If $R \neq \emptyset$, then by Lemma 17 any minimum hull set of G_w contains S. Moreover, by Lemma 24 any minimum hull set of G_w contains a minimum hull set of $K \cup R$ which is composed by vertices of R.

  By the same lemmas, a minimum hull set of G_w is a minimum hull set of G_v since, by Lemma 18, no vertex of G_u belongs to any minimum hull set of G_v and G_u is generated by non-adjacent vertices of G_w.

- w is a small node. G_w is a q-pseudo-spider (H_2, H_1, R, E') and R induces a $(q, q-4)$-graph.

  If $R = \emptyset$, G_v is the join of a clique G_u with a graph G_w that has at most q vertices. No vertex of G_u belongs to any minimum hull set of G_v, since they are universal. Thus, H_v can be computed in time $O(2^q)$ by testing all the possible subsets of vertices of G_w.

  Similarly, if R is a clique, all vertices in R are simplicial in G_v so they must belong to any hull set of G_v. Moreover, no vertices in G_u belong to any minimum hull set of G_v. So H_v can be computed in time $O(2^q)$ by testing all the possible subsets of vertices of $H_1 \cup H_2$ and adding R to them.

  In case $R \neq \emptyset$ nor a clique, two cases must be considered. By definition of the decomposition, there exists a child r of w in $T(G)$ such that $V(G_r) = R$.

  - If $G[H_1]$ is a clique, then, $G_v = (H_2, H_1 \cup V(G_u), R, E)$ is a pseudo-spider that satisfies the conditions in Lemma 24. Hence, any minimum hull set of G_v contains a minimum hull set of $P = G[H_1 \cup V(G_u) \cup R]$. Let Z be a minimum hull set of G_v and let $Z' = Z \cap H_2$. By Lemma 24, we have $|Z'| \leq hn(G_v) - hn(P)$.

    By Lemma 25, H_r^* is a minimum hull set of $G[H_1 \cup V(G_u) \cup R]$. Now, $G[H_1 \cup V(G_u) \cup R]$ is an isometric subgraph of G_v. Hence, by Lemma 19, H_r^* generates all vertices of $G[H_1 \cup V(G_u) \cup R]$ in G_v. Therefore, $H_r^* \cup Z'$ will generate all vertices of G_v. Since $|H_r^*| = hn(P)$, we get that $|H_r^* \cup Z'| \leq hn(G_v)$ and then $H_r^* \cup Z'$ is a minimum hull set of G_v.

    So, we have shown that there exists a minimum hull set for G_v that can be obtained from H_r^* by adding some vertices in $H_1 \cup H_2$. Since $|H_1 \cup H_2| \leq q$, such a subset of $H_1 \cup H_2$ can be found in time $O(2^q)$.

  - In case $G[H_1]$ is not a clique, let x and y be two non adjacent vertices of H_1. We claim in this case that there exists a minimum hull set of G_v containing at most one vertex of R. Let S be a minimum hull set of G_v containing at least two vertices in R. Observe that $S' = (S \backslash R) \cup \{x, y\}$ is also a hull set of G_v since x and y are sufficient to generate all vertices in R. Consequently, $|S'| \leq |S|$ and S' is minimum.

    Since no hull set of G_v contains a vertex in $V(G_u)$, there always exists a minimum hull set of G_v that consists of only vertices in $H_1 \cup H_2$ plus at most one vertex in R. Therefore an exhaustive search can be performed in time $O(n2^q)$.

□

Now, we consider the case when v is a spider node or a small node. That is $G_v = (S, K, R, E)$. If $R \neq \emptyset$, let r be the child of v such that $V(G_r) = R$.

Lemma 28. *Let $G_v = (S, K, R, E)$ be a spider such that R induces a $(q, q-4)$-graph.*

Then, $H_v = H_v^* = S \cup H_r^*$ if $R \neq \emptyset$ and R is not a clique, and $H_v = H_v^* = S \cup R$, otherwise.

Proof. Since all the vertices in S are simplicial vertices in G_v and in G_v^*, we apply Lemma 17 to conclude that they are all contained in any hull set of G_v (resp., of G_v^*).

By the structure of a spider, every vertex of K (and the universal vertex in G_v^*) belongs to a shortest path between two vertices in S and are therefore generated by them in any minimum hull set of G_v (resp., of G_v^*). Consequently, if $R = \emptyset$, S is a minimum hull set of G_v (resp., of G_v^*). If R is a clique, $S \cup R$ is the set of simplicial vertices of G_v (resp., of G_v^*) and also a minimum hull set of G_v (resp., of G_v^*).

Finally, if $R \neq \emptyset$ and R is not a clique, then G_v is a pseudo-spider satisfying the conditions of Lemma 24. Similarly, G_v^* is a pseudo-spider (by including the universal vertex in K). Then, by Lemma 24, any hull set of G_v (resp., of G_v^*) contains a minimum hull set of $G[K \cup R]$ (resp., of $G_v^* \setminus S$. Moreover, any hull set contains all vertices in S since they are simplicial. Hence, $hn(G_v) = hn(G_v^*) = |S| + hn(G[K \cup R])$ (recall that, by Lemma 25, $hn(G[K \cup R]) = hn(G_v^* \setminus S)$). Finally, since $G[K \cup R]$) is an isometric subgraph of G_v, then H_r^* (which is a minimum hull set of $G[K \cup R]$ by Lemma 25) generates $G[K \cup R]$ in G_v (resp., in G_v^*).

Hence, $S \cup H_r^*$ is a hull set of G_v and G_v^*. Moreover, it has size $|S| + hn(G[K \cup R])$, so it is optimal.

□

Lemma 29. *Let $G_v = (H_2, H_1, R, E)$ be a q-pseudo-spider such that R is a $(q, q-4)$-graph. Then, H_v and H_v^* can be computed in time $\mathcal{O}(2^q n)$.*

Proof. All the arguments to prove this lemma are in the proof of Lemma 27. Moreover, the following arguments hold both for G_v and G_v^*: they allow to compute both H_v and H_v^*.

If $R = \emptyset$, G_v has at most q vertices, for a fixed positive integer q. Thus, its hull number can be computed in $\mathcal{O}(2^q)$-time.

Otherwise, if H_1 is a clique, by Lemma 24, any minimum hull set of G_v contains a minimum hull set of $G[H_1 \cup R]$. Moreover, by the same arguments as in Lemma 27, we can show that there is an optimal hull set for G_v that can be obtained from H_r^* (minimum hull set of $G[H_1 \cup R]$) and some vertices in H_2.

If H_1 is not a clique, two non-adjacent vertices of H_1 can generate R. Thus, we conclude that there exists a minimum hull set of G_v containing at most one vertex of R. Then, a minimum hull set of G_v can be found in $\mathcal{O}(2^q n)$-time, where $n = |V(G_v)|$. □

6.5 Hull Number via 2-connected components

In this section, we introduce a generalized variant of the hull number of a graph. Let $G = (V, E)$ be a graph and $S \subseteq V$. Let $hn(G, S)$ denote the minimum size of a set $U \subseteq V \setminus S$ such that $U \cup S$ is a hull set for G. We prove that to compute the hull number of a graph, it is sufficient to compute the generalized hull number of its 2-connected components (or *blocks*). This extends a result in [ES85].

Theorem 22. *Let G be a graph and $G_1, \dots, G_n$ be its 2-connected components. For any $i \leq n$, let $S_i \subseteq V(G_i)$ be the set of cut-vertices of G in G_i. Then,*

$$hn(G) = \sum_{i \leq n} hn(G_i, S_i).$$

Proof. Clearly, the result holds if $n = 1$, so we assume $n > 1$.

A block G_i is called a *leaf-block* if $|S_i| = 1$. Note that, for any leaf-block G_i, $G[V \setminus (V(G_i) \setminus S_i)]$ is convex, so by Lemma 20, any hull set of G contains at least one vertex in $V(G_i) \setminus S_i$. Moreover,

Claim 9. *For any minimum hull set S of G, $S \cap (\cup_{i \leq n} S_i) = \emptyset$.*

Proof. For purpose of contradiction, let us assume that a minimum hull set S of G contains a vertex $v \in S_i$ for some $i \leq n$. Note that there exist two leaf-blocks G_1 and G_2 such that v is on a shortest path between vertices in $V(G_1)$ and $V(G_2)$ or $\{v\} = V(G_1) \cap V(G_2)$. By the remark above, there exist $x \in (V(G_1) \setminus S_1) \cap S$ and $y \in (V(G_2) \setminus S_2) \cap S$. Hence, v is on a shortest (x, y)-path, i.e., $v \in I[x, y] \subseteq I_h[S \setminus \{v\}]$. Hence, $V \subseteq I_h[S] \subseteq I_h[S \setminus \{v\}]$ and $S \setminus \{v\}$ is a hull set of G, contradicting the minimality of S. ◇

Claim 10. *Let S be a hull set of G. Then $S' = (S \cap V(G_i)) \cup S_i$ is a hull set of G_i.*

Proof.

For purpose of contradiction, assume that $I_h[S'] = V(G_i) \setminus X$ for some $X \neq \emptyset$. Then, there is $v \in X \cap I[a, b]$ for some $a \in V(G) \setminus V(G_i)$ and $b \in V(G) \setminus X$. Then, there is a shortest (a, b)-path P containing v. Hence, there is $u \in S_i$ such that u is on the subpath of P between a and v. Moreover, let $w = b$ if $b \in G_i$, and else let w be a vertex of S_i on the subpath of P between v and b. Hence, $v \in I[u, w] \subseteq I_h[S']$, a contradiction.

◇

Let X be any minimum hull set of G. By Claim 9, $X \cap (\cup_{i \leq n} S_i) = \emptyset$, hence we can partition $X = \cup_{i \leq n} X_i$ such that $X_i \subseteq V(G_i) \setminus S_i$ and $X_i \cap X_j = \emptyset$ for any $i \neq j$. Moreover, by Claim 10, $X_i \cup S_i$ is a hull set of G_i, i.e., $|X_i| \geq hn(G_i, S_i)$. Hence, $hn(G) = |X| = \sum_{i \leq n} |X_i| \geq \sum_{i \leq n} hn(G_i, S_i)$.

It remains to prove the reverse inequality. For any $i \leq n$, let $X_i \subseteq V(G_i) \setminus S_i$ such that $X_i \cup S_i$ is a hull set of G_i and $|X_i| = hn(G_i, S_i)$. We prove that $S = \cup_{i \leq n} X_i$ is a hull set for G. Indeed, for any $v \in S_i$, there are two leaf-blocks G_1, G_2 such that v is on a shortest path between G_1 and G_2 or $\{v\} = V(G_1) \cap V(G_2)$. So, there exist $x \in X_1$ and $y \in X_2$ such that v is on a shortest (x, y)-path, i.e., $v \in I[x, y] \subseteq I_h[S]$. Hence, $\cup_{i \leq n} S_i \subseteq I_h[S]$ and therefore, $V = \cup_{i \leq n} I_h[X_i \cup S_i] \subseteq I_h[\cup_{i \leq n}(X_i \cup S_i)] \subseteq I_h[\cup_{i \leq n}(X_i)] = I_h[S]$. □

A *cactus* G is a graph in which every pair of cycles have at most one common vertex. This definition implies that each block of G is either a cycle or an edge. By using the previous result, one may easily prove that:

Corollary 17 ([ACG+11b])**.** *In the class of cactus graphs, the hull number can be computed in linear time.*

6.6 Bounds

In this section, we use the same techniques as presented in [ES85, DGK+09] to prove new bounds on the hull number of several graphs classes. These techniques mainly rely on a greedy algorithm for computing a hull set of a graph and that consists of the following: given a connected graph $G = (V, E)$ and its set S of simplicial vertices, we start with $H = S$ or $H = \{v\}$ (v is any vertex of V) if $S = \emptyset$, and $C_0 = I_h[H]$. Then, at each step $i \geq 1$, if $C_{i-1} \subset V$, the algorithm greedily chooses a subset $X_i \subseteq V \setminus C_{i-1}$, add X_i to H and set $C_i = I_h[H]$. Finally, if $C_i = V$, the algorithm returns H which is a hull set of G.

Claim 11. *If for every $i \geq 1$, $|C_i \setminus (C_{i-1} \cup X_i)| \geq c \cdot |X_i|$, for some constant $c > 0$, then $|H| \leq \max\{1, |S|\} + \left\lceil \frac{|V| - \max\{1,|S|\}}{1+c} \right\rceil$.*

In the following, we keep the notation used to describe the algorithm.

Claim 12. *Let G be a connected graph. Then, before each step $i \geq 1$ of the algorithm, for any $v \in V \setminus C_{i-1}$, $N(v) \cap C_{i-1}$ induces a clique. Moreover, any connected component induced by $V \setminus C_{i-1}$ has at least 2 vertices.*

Proof. Let $v \in V \setminus C_{i-1}$ and assume v has two neighbors u and w in C_{i-1} that are not adjacent. Then, $v \in I[u, w] \subseteq C_{i-1}$ because C_{i-1} is convex, a contradiction. Note that, at any step $i \geq 1$ of the algorithm, $V \setminus C_{i-1}$ contains no simplicial vertex. By previous remark, if v has only neighbors in C_{i-1}, then v is simplicial, a contradiction. □

Claim 13. *If G is a connected C_3-free graph, then, at every step $i \geq 1$ of the algorithm, a vertex in $V \setminus C_{i-1}$ has at most one neighbor in C_{i-1}.*

Proof. Assume that $v \in V \setminus C_{i-1}$ has two neighbors $u, w \in C_{i-1}$. $\{u, w\} \notin E$ because G is triangle-free. This contradicts Claim 12. □

Lemma 30. *For any C_3-free connected graph G and at step $i \geq 1$ of the algorithm, either $C_{i-1} = V$ or there exists $X_i \subset V \setminus C_{i-1}$ such that $|C_i \setminus (C_{i-1} \cup X_i)| \geq |X_i|$.*

Proof. If there is $v \in V \setminus C_{i-1}$ at distance at least 2 from C_{i-1}, let $X_i = \{v\}$ and the result clearly holds. Otherwise, let v be any vertex in $V \setminus C_{i-1}$. By Claim 12, v has a neighbor u in $V \setminus C_{i-1}$. Moreover, because no vertices of $V \setminus C_{i-1}$ are at distance at least 2 from C_{i-1}, v and u have some neighbors in C_{i-1}. Finally, u and v have no common neighbors because G is triangle-free. Hence, by taking $X_i = \{v\}$, we have $u \in C_i$ and the result holds. □

Recall that the *girth* of a graph is the length of its smallest cycle.

Lemma 31. *Let G connected with girth at least 6. Before any step $i \geq 1$ of the algorithm when $C_{i-1} \neq V$, there exists $X_i \subset V \setminus C_{i-1}$ such that $|C_i \setminus (C_{i-1} \cup X_i)| \geq 2|X_i|$.*

Proof. If there is $v \in V \setminus C_{i-1}$ at distance at least 3 from C_{i-1}, let $X_i = \{v\}$ and the result clearly holds. Otherwise, let v be a vertex in $V \setminus C_{i-1}$ at distance two from any vertex of C_{i-1}. Let $w \in V \setminus C_{i-1}$ be a neighbor of v that has a neighbor $z \in C_{i-1}$. Since v is not simplicial, v has another neighbor $u \neq w$ in $V \setminus C_{i-1}$. If u is at distance two from C_{i-1}, let $y \in V \setminus C_{i-1}$ be a neighbour of u that has a neighbor $x \in C_{i-1}$. In this case, since the girth of G is at least six, $z \neq x$ and, there is a shortest (v,z)-path containing w and a shortest (v,x)-path containing u and y. Consequently, by setting $X_i = \{v\}$ we obtain the desired result. The same happens in case u has a neighbor $x \in C_{i-1}$. One may use again the hypothesis that the girth of G is at least six to conclude that, by setting $X_i = \{v\}$ we obtain that $w, u \in C_i$.

Finally, we claim that no vertex remains in $V \setminus C_{i-1}$. By contradiction, suppose that it is the case and that there are in $V \setminus C_{i-1}$ and all these vertices have a neighbor in C_{i-1}. Let v be a vertex in $V \setminus C_{i-1}$ that has a neighbor z in C_{i-1}. Again, v has a neighbor $u \in V \setminus C_{i-1}$, since it is not simplicial. The vertex u must have a neighbor x in C_{i-1}. Observe that x and z are at distance 3, since the girth of G is at least six. Consequently, v and u are in a shortest (x,z)-path should not be in $V \setminus C_{i-1}$, that is a contradiction.

□

Lemma 32. *Let G be a connected graph. Before any step $i \geq 1$ of the algorithm when $C_{i-1} \neq V$, there exist $X_i \subset V \setminus C_{i-1}$ such that $|C_i \setminus (C_{i-1} \cup X_i)| \geq 2|X_i|/3$.*

Moreover, if G is k-regular ($k \geq 1$), there exist $X_i \subset V \setminus C_{i-1}$ such that $|C_i \setminus (C_{i-1} \cup X_i)| \geq |X_i|$.

Proof. By Claim 12, all connected component of $V \setminus C_{i-1}$ contains at least one edge.

- If there is $v \in V \setminus C_{i-1}$ at distance at least 2 from C_{i-1}, let $X_i = \{v\}$ and $|C_i \setminus (C_{i-1} \cup X_i)| \geq |X_i|$.

- Now, assume all vertices in $V \setminus C_{i-1}$ are adjacent to some vertex in C_{i-1}. If there are two adjacent vertices u and v in $V \setminus C_{i-1}$ such that there is $z \in C_{i-1} \cap N(u) \setminus N(v)$, then let $X_i = \{v\}$. Therefore, $u \in C_i$ and $|C_i \setminus (C_{i-1} \cup X_i)| \geq |X_i|$. So, the result holds.

- Finally, assume that for any two adjacent vertices u and v in $V \setminus C_{i-1}$, $N(u) \cap C_{i-1} = N(v) \cap C_{i-1} \neq \emptyset$.

  We first prove that this case actually cannot occur if G is k-regular. Let $v \in V \setminus C_{i-1}$. By Claim 12, $K = N(v) \cap C_{i-1}$ induces a clique. Moreover, for any $u \in N(v) \setminus C_{i-1}$, $N(u) \cap C_{i-1} = K$. Note that $k = |K| + |N(v) \setminus C_{i-1}|$. Let $w \in K$. Then, $A = (K \cup N(v) \cup \{v\}) \setminus \{w\} \subseteq N(w)$ and since $|A| = k$, we get that $A = N(w)$. Moreover, $N[u]$ cannot induce a clique since $V \setminus C_{i-1}$ contains no simplicial vertices, $i \geq 1$. Hence, there are $x, y \in N(v) \setminus C_{i-1}$ such that $\{x, y\} \notin E$. Because G is k-regular, there is $z \in N(x) \setminus (N(v) \cup C_{i-1})$. However, $N(z) \cap C_{i-1} = N(x) \cap C_{i-1} = K$. Hence, $z \in N(w) \setminus A$, a contradiction.

Now, assume that G is a general graph. Let v be a vertex of minimum degree in $V \setminus C_{i-1}$. Recall that, by Claim 12, $N(v) \cap C_{i-1}$ induces a clique. Because any neighbor $u \in V \setminus C_{i-1}$ of v has the same neighborhood as v in C_{i-1} and because v is not simplicial, then there must be $u, w \in N(v) \setminus C_{i-1}$ such that $\{u, w\} \notin E$. Now, by minimality of the degree of v, there exists $y \in N(u) \setminus (N(v) \cup C_{i-1}) \neq \emptyset$. Similarly, there exists $z \in N(w) \setminus (N(v) \cup C_{i-1}) \neq \emptyset$. Let us set $X_i = \{v, z, y\}$. Hence, $u, w \in C_i \setminus (C_{i-1} \cup X_i)$ and the result holds.

□

Theorem 23. *Let G be a connected n-node graph with s simplicial vertices. All bounds below are tight:*

- $hn(G) \leq \max\{1, s\} + \left\lceil \frac{3(n-\max\{1,s\})}{5} \right\rceil$;
- *If G is C_3-free or k-regular ($k \geq 1$), then* $hn(G) \leq \max\{1, s\} + \left\lceil \frac{n-\max\{1,s\}}{2} \right\rceil$;
- *If G has girth ≥ 6, then* $hn(G) \leq \max\{1, s\} + \left\lceil \frac{1(n-\max\{1,s\})}{3} \right\rceil$.

Proof. First statement follows from Claim 11 and first statement in Lemma 32. The second statement follows from Claim 11 and Lemma 30 (case C_3-free) and second part of Lemma 32 (case regular graphs) . Last statement follows from Claim 11 and Lemma 31.

All bounds are reached in case of complete graphs. In case with no simplicial vertices: the first bound is reached by the graph obtained by taking several disjoint C_5 and adding a universal vertex, the second bound is obtained for a C_5, and the third one is reached by a C_7.

□

The first statement of the previous theorem improves another result in [ES85]:

Corollary 18. *If G is a graph with no simplicial vertex, then:*

$$\limsup_{|V(G)| \to \infty} \frac{hn(G)}{|V(G)|} = \frac{3}{5}.$$

It it important to remark that the second statement of Theorem 23 is closely related to a bound of Everett and Seidman proved in Theorem 9 of [ES85]. However, the graphs they consider do not have simplicial vertices and, consequently, they do not have vertices of degree one, which is not a constraint for our result.

6.7 Conclusions

We simplified the reduction of Dourado et al. [DGK+09] to answer a question they asked about the complexity of computing the hull number of bipartite graphs. We presented polynomial-time algorithms for computing the hull number of cobipartite graphs, $(q, q-4)$-graphs and cactus graphs. Finally, we presented upper bounds for general graphs and two particular graph classes.

The result in Section 6.4 provides an FPT algorithm where the parameter is the number of induced P_4's in the input graph. It would be nice to know about the paramerized complexity of Hull Number when the parameter is the size of the hull set.

Another question of Dourado *et al.* [DGK+09], concerning the complexity of this problem for interval graphs and chordal graphs, remains open. Up to the best of our knowledge, determining the complexity of the Hull Number problem on planar graphs is also an open problem.

Chapter 7

Conclusions and further research

In this thesis, we (mainly) studied the computational complexity of different problems related to Graph Coloring and Graph Convexity for particular graph classes. We here briefly describe the obtained results and several open questions for further research.

In Chapter 2, we showed a polynomial-time algorithm to compute the Grundy number of fat extended P_4-laden graphs. This result extends a previous one concerning the Grundy number of cographs. There are several bounds on the Grundy number for particular graph classes, but only few complexity results. For instance, the computational complexity of computing the Grundy number for interval graphs and planar graphs is still not determined. Moreover, the parameterized complexity of determining whether the Grundy number of a graph is at least k, where k is the parameter, is also unknown.

We studied, in Chapter 3, the weighted chromatic number of a given vertex-weighted graph. For this problem, we first presented a generalization of the Hajós' Theorem by showing a necessary and sufficient condition for a weighted graph (G, w) to have weighted chromatic number at least k. Then, we studied the computational complexity of determining this parameter for P_4-sparse graphs. We first presented a polynomial-time algorithm to compute the weighted chromatic number for a subclass of P_4-sparse graphs that properly contains the cographs. Then, we gave a simple 2-approximation algorithm for the class of P_4-sparse graphs. For this problem, we mention two open problems that are the computational complexity of determining the weighted chromatic number of a P_4-sparse graph and of a tree.

After studying vertex-weighted graphs, in Chapter 4 we dealt with a new, up to our best knowledge, variation of the Vertex Coloring problem for edge-weighted graphs. We showed general bounds for both parameters we have introduced and then we studied a particular weight function based on a practical motivation. For that particular weight function, we first computed the weighted improper chromatic number of infinite paths and grids (square, hexagonal and triangular) and we gave upper and lower bounds on this parameter for trees. Finally, we implemented an integer programming formulation on a solver called CPLEX, a heuristic and a branch-and-bound algorithm. Then, we executed these three implementations over instances of practical interest and compared the obtained results. For this problem, it would be interesting to study the computational complexity of computing the weighted improper chromatic number of trees, for the particular weight function we defined. Moreover, it would be useful to deal with more general weight functions for particular graph classes.

In Chapter 5, we studied the good edge-labeling problem. We showed that this problem is NP-complete even for bipartite graphs; then we presented an infinite family of bad graphs (graphs with no good edge-labeling) and showed some families of good graphs. We let open questions in the conclusion of this chapter and we want to emphasize that there is a lot of research that can be done in this topic. For instance, the computational complexity of determining whether a graph is good is an open problem for many particular graph classes like planar graphs, chordal graphs, graphs of bounded tree-width, etc.

Finally, in Chapter 6, we studied a Graph Convexity parameter called (geodetic) hull number. We answered some open questions in the literature by showing that determining the hull number of a graph is NP-hard even for bipartite graphs. Then, we presented several polynomial-time algorithms for different classes of graphs: cacti, complements of bipartite graphs, $(q, q-4)$-graphs. Finally, we gave upper bounds for general graphs and for particular graph classes. An open question, pointed in [DGK+09] that we still do not know how to answer, concerns the computational complexity of determining the hull number for interval graphs or, more generally, for chordal graphs. Up to our best knowledge, no similar result is known for planar graphs. It would also be nice to study the parameterized complexity of determining whether the hull number of a graph is at least k, when k is the parameter. It is also not known in the literature any exponential exact algorithm of complexity better than $O(2^n)$ to compute the hull number of a graph.

state of these chains, we evaluate their average behavior.

We used in this thesis many different techniques: reduction for NP-hardness proofs, integer linear programming, Markov chains, etc. We intend to apply these tools to the open problems pointed in this thesis and, more generally, to other problems on graph theory.

Appendix A

Eulerian and Hamiltonian Dicycles in Directed Hypergraphs

A.1 Introduction

Eulerian and Hamiltonian dicycles are well-known concepts in Graph Theory. An *Eulerian* dicycle in a digraph D is a dicycle C such that each *arc* of D appears exactly once in C. Similarly, a *Hamiltonian* dicycle is a dicycle C such that each *vertex* of D appears exactly once in C (see [BJG10, BM08b]).

We generalize these concepts to *directed* hypergraphs, called shortly *dihypergraphs*. Informally, the difference between an usual digraph D and a dihypergraph H is that (hyper)arcs in H may have multiple heads and multiple tails. Formally, a *dihypergraph* H is a pair $(\mathcal{V}(H), \mathcal{E}(H))$, where $\mathcal{V}(H)$ is a non-empty set of elements, called *vertices*, and $\mathcal{E}(H)$ is a collection of ordered pairs of subsets of $\mathcal{V}(H)$, called *hyperarcs*. It is Eulerian (resp. Hamiltonian) if there is a dicycle containing each hyperarc (resp. each vertex) exactly once.

Eulerian and Hamiltonian (undirected) hypergraphs have already been defined and studied in a similar way [Ber73, LN10]. In fact, if Hamiltonian hypergraphs have received some attention (see [Ber78, KKMO11, KMO10]), Eulerian hypergraphs seem to have been considered in their full generality only recently in [LN10]. A particular case of Eulerian cycles in 3-uniform hypergraphs (called triangulated irregular networks) has been considered in [AHMS94, BG04a, BG04b] motivated by applications in geographic systems or in computer graphics. However, to our best knowledge, Hamiltonian and Eulerian dihypergraphs have not been considered.

Note that there are other definitions of Hamiltonian hypergraphs in the literature. For example, an undirected hypergraph H is called Hamiltonian if there exists a Hamiltonian-l cycle C in H, that is a cycle C where any two consecutive (hyper)edges intersect themselves in exactly l vertices and every vertex of H belongs to exactly one of those intersections [DAR12, KKMO11, KMO10]. Such a notion can also be generalized to dihypergraphs. However, we choose the general definition as otherwise there would be no more a clear connexion between the Eulerian and the Hamiltonian dihypergraphs (with our definition the dual of an Eulerian dihypergraph is Hamiltonian). Furthermore, we are mainly interested in Hamiltonian line dihypergraphs, whose definition is given later, and, in this case, both of these definitions of a Hamiltonian dihypergraph are equivalent.

It is well-known that a strongly connected digraph is Eulerian if, and only if, every vertex has equal in-degree and out-degree. Therefore, deciding whether a digraph is Eulerian can be done in polynomial time; but deciding whether it is Hamiltonian is an NP-complete problem.

In the first part of the chapter, we show that for dihypergraphs the situation is different from that of digraphs. For example, deciding whether a dihypergraph is Eulerian is an NP-complete problem. We show nonetheless that some results about the Eulerian digraphs can be generalized, in the case where the studied dihypergraphs are *uniform* and *regular*. As example, we prove that if H is a weakly-connected, d-regular, s-uniform dihypergraph, then, for every $k \geq 1$, $L^k(H)$ is Eulerian and Hamiltonian. In the second part, we study the Eulerian and Hamiltonian properties of special families of regular uniform dihypergraphs, the generalized de Bruijn and Kautz dihypergraphs [BDE97].

The so called de Bruijn digraphs were introduced to show the existence of de Bruijn sequences, that is circular sequences of d^D elements, such that any subsequence of length D appears exactly once. To prove the existence of such sequences, it was proved that de Bruijn digraphs are both Eulerian and Hamiltonian. These digraphs have been rediscovered many times and their properties have been well studied (see, for example, the survey [BP89]) in particular for the design of interconnection networks. Various generalizations of de Bruijn digraphs have been introduced, like the generalized de Bruijn digraphs (also named Reddy-Pradhan-Kuhl digraphs) presented in [II81, RPK80]. These digraphs are based on arithmetical properties and they exist for any number of vertices. Other generalizations like Kautz digraphs, generalized Kautz digraphs (also called Imase and Itoh digraphs [II81]) and consecutive digraphs [DHH93] have been proposed in the literature.

One generalization concerns hypergraphs and dihypergraphs which are used in the design of optical bus networks [SB99]. In particular, de Bruijn and Kautz dihypergraphs and their generalizations, that were introduced in [BDE97], have several properties that are beneficial in the design of large, dense, robust networks. They have been proposed as the underlying physical topologies for optical networks, as well as dense logical topologies for Logically Routed Networks (LRN) because of ease of routing, load balancing and congestion reduction, that are properties inherent in de Bruijn and Kautz networks. In 2009, J-J. Quisquater brought to our attention the web site (*http://punetech.com/building-eka-the-worlds-fastest-privately-funded-supercomputer/*) where it is explained how these hypergraphs and the results of [BE96] were used for the design of the supercomputer EKA in 2007 (*http://en.wikipedia.or wiki/EKA_(supercomputer)*).

Connectivity properties of generalized de Bruijn dihypergraphs have been studied in [BES11, FP02a, FP02b], but, to our best knowledge, their Hamiltonian and Eulerian properties have not been studied.

More precisely, we first determine when generalized de Bruijn and Kautz dihypergraphs are Hamiltonian and Eulerian. Then, we study the case where their number of hyperarcs is equal to their number of vertices. In that case, we almost characterize when these dihypergraphs have a complete Berge dicycle, i.e. a dicycle both Hamiltonian and Eulerian; in particular, we have a complete characterization when the out-degree of each vertex is equal to the out-size of each hyperarc.

A.2 Definitions and Notations

A.2.1 Dihypergraphs

A *directed hypergraph*, or simply *dihypergraph* is a pair $(\mathcal{V}(H), \mathcal{E}(H))$ where $\mathcal{V}(H)$ is a non-empty set of elements, called *vertices*, and $\mathcal{E}(H)$ is a collection of ordered pairs of subsets of $\mathcal{V}(H)$, called *hyperarcs*. We denote by $n(H)$ (resp. $m(H)$) the number of vertices (resp. hyperarcs) of H. Whenever H is clear in the context, we use shortly n and m. We suppose, to avoid trivial cases, that $n > 1$ and $m > 1$.

Let H be a dihypergraph and $E = (E^-, E^+)$ be a hyperarc in $\mathcal{E}(H)$. Then, the vertex sets E^- and E^+ are called the *in-set* and the *out-set* of the hyperarc E, respectively. The sets E^- and E^+ do not need to be disjoint and they may be empty. The vertices of E^- are said to be *incident to* the hyperarc E and the vertices of E^+ are said to be *incident from* E.

If E is a hyperarc in a dihypergraph H, then $|E^-|$ is the *in-size* and $|E^+|$ is the *out-size* of E. The *maximum in-size* and the *maximum out-size* of H are respectively:

$$s^-(H) = \max_{E \in \mathcal{E}(H)} |E^-| \quad \text{and} \quad s^+(H) = \max_{E \in \mathcal{E}(H)} |E^+|.$$

Note that a *digraph* is a dihypergraph $D = (\mathcal{V}(D), \mathcal{E}(D))$ with $s^-(D) = s^+(D) = 1$.

Let v be a vertex in H. The *in-degree* of v is the number of hyperarcs that contain v in their out-set and it is denoted by $d_H^-(v)$. Similarly, the *out-degree* of vertex v is the number of hyperarcs that contain v in their in-set and it is denoted by $d_H^+(v)$.

The *bipartite representation* $R(H)$ of a dihypergraph H is the bipartite digraph $R(H) = (\mathcal{V}_1(R) \cup \mathcal{V}_2(R), \mathcal{E}(R))$ where $\mathcal{V}_1 = \mathcal{V}(H)$, $\mathcal{V}_2 = \mathcal{E}(H)$ and $\mathcal{E}(R) = \{v_i E_j \mid v_i \in E_j^-\} \cup \{E_j v_i \mid v_i \in E_j^+\}$. This representation digraph is useful for drawing dihypergraphs. To make each figure more readable, we duplicate the vertices and we put in the left part the arcs from $\mathcal{V}_1$ to $\mathcal{V}_2$ and in the right part those from $\mathcal{V}_2$ to $\mathcal{V}_1$. Figure A.1 gives the representation digraph of the de Bruijn dihypergraph $GBH(2, 9, 2, 9)$, where vertex i belongs to the in-set of the hyperarcs E_{2i} and E_{2i+1} and the hyperarc E_j has as out-set the vertices $2j$ and $2j+1$ (all the numbers being taken modulo 9).

Remark that when you inverse the respective roles of $\mathcal{V}_1(R)$ and $\mathcal{V}_2(R)$ in $R(H)$, you intuitively exchange the role of the vertices with the role of the hyperarcs in H. This is an informal notion of the *dual dihypergraph* H^*. Formally, the vertices of the dual dihypergraph H^* are in bijection ϕ_v with the hyperarcs of H and the hyperarcs of H^* are in bijection ϕ_E with the vertices of H. Moreover, for every vertex $v \in \mathcal{V}(H)$ and every hyperarc $E \in \mathcal{E}(H)$, vertex $e = \phi_v(E) \in \mathcal{V}(H^*)$ is in V^-, where $V = \phi_E(v) \in \mathcal{E}(H^*)$, if, and only if, $v \in E^+$ and, similarly, e is in V^+ if, and only if, $v \in E^-$. It is important to notice that a hyperarc $V \in \mathcal{E}(H^*)$ may have an empty in-set (if $d_H^-(v) = 0$) or an empty out-set (if $d_H^+(v) = 0$).

The *underlying multidigraph* $U(H)$ of a dihypergraph H has as vertex set $\mathcal{V}(U(H)) = \mathcal{V}(H)$ and as arc set $\mathcal{E}(U(H))$ that is the multiset of all ordered pairs

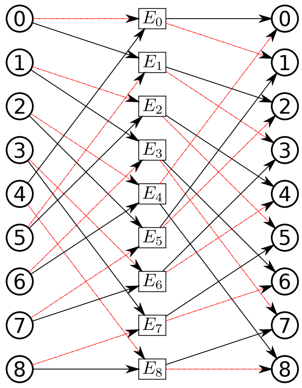


Figure A.1: Bipartite representation of the De Bruijn dihypergraph $GBH(2, 9, 2, 9)$ and a complete Berge dicycle represented by dotted arcs (vertices are drawn twice to better represent all the arcs).

(u, v) such that $u \in E^-$ and $v \in E^+$, for every hyperarc $E \in \mathcal{E}(H)$. We emphasize that $U(H)$ does not need to be simple: the number of arcs from u to v in $U(H)$ is the number of hyperarcs $E = (E^-, E^+)$ in H such that $u \in E^-$ and $v \in E^+$. Observe that the underlying multidigraph of a given dihypergraph is unique. However, a given digraph D can be the underlying digraph of many dihypergraphs H.

A.2.2 Eulerian and Hamiltonian Dicycles in Dihypergraphs

By a *dipath* in a dihypergraph H, we mean a sequence $P = v_0, E_0, \ldots, v_{p-1}, E_{p-1}, v_p$, such that, for all i, j, we have $v_i \in \mathcal{V}(H)$, $E_j \in \mathcal{E}(H)$, $v_i \in E_i^-$ for every $0 \leq i \leq p-1$, and $v_i \in E_{i-1}^+$ for every $1 \leq i \leq p$. We also say that P is a dipath of *length* p. Moreover, the dipath P is called a *dicycle*, or *circuit*, in H if, and only if, we have $v_0 = v_p$. Observe that each dicycle in a dihypergraph H corresponds to a dicycle in its bipartite representation $R(H)$. Note that we allow repetitions of vertices or hyperarcs and some authors prefer to use the word *tour* in this case.

In the same way, we can extend the digraph-theoretic notions of Eulerian dicycles and Hamiltonian dicycles to dihypergraphs:

Definition 7. *Let H be a dihypergraph. We say that H is* Eulerian *(resp. H is* Hamiltonian*) if, and only if, there is a dicycle C in H such that every hyperarc of H (resp. every vertex of H) appears in C exactly once. We call C an* Eulerian dicycle *(resp. a* Hamiltonian dicycle*).*

Our generalization of an Eulerian dicycle to dihypergraphs is close to the extension of an Euler tour to the *undirected* hypergraphs introduced in [LN10].

Definition 8 ([LN10])**.** *Let H_u be an undirected hypergraph. A tour is a sequence $T = v_0, E_0, v_1, \ldots,$
v_{m-1}, E_{m-1}, v_0 where, for all i, $v_i \neq v_{i+1}$ and v_i and v_{i+1} are in the hyperedge E_i*

(indices are taken modulo m). T is called an Euler tour *when every hyperedge of H_u appears exactly once in T. H_u is an Eulerian hypergraph if there exists an Euler tour T in H_u.*

Remark 1. *An Eulerian dicycle in H (resp. a Hamiltonian dicycle in H) is a dicycle in $R(H)$, such that each vertex of $\mathcal{V}_2(R)$ (resp. of $\mathcal{V}_1(R)$) appears exactly once.*

As a consequence, a necessary and sufficient condition for $R(H)$ to be Hamiltonian is that there is a dicycle C in H, such that C is simultaneously an Eulerian dicycle and a Hamiltonian dicycle in H. In reference to the undirected case [Ber73], we call C a *complete Berge dicycle*:

Definition 9. *Let H be a dihypergraph. A* complete Berge dicycle *in H is a dicycle C in H, such that C is both an Eulerian dicycle and a Hamiltonian dicycle in H.*

In the following sections, we focus on Eulerian dihypergraphs. We assume that the studied dihypergraphs have no isolated vertex, without any loss of generality.

A.3 General Results

A.3.1 Some conditions

First, we recall a well-known characterization of Eulerian digraphs:

Theorem 24 ([BP79]). *Let D be a digraph. The following statements are equivalent:*

1. *D is Eulerian;*
2. *D is (strongly) connected and, for all vertex $v \in \mathcal{V}(D)$, $d^-(v) = d^+(v)$;*
3. *D is (strongly) connected and it has a* dicycle decomposition *(i.e. its arcs can be partitioned into arc-disjoint dicycles).*

The digraph-theoretic notions of connectivity can be extended to dihypergraphs [BES11]. We say that H is strongly (resp. weakly) connected if its underlying multidigraph $U(H)$ is strongly (resp. weakly) connected. $U(H)$ is weakly connected if its *associated* multigraph $G_{U(H)}$ (obtained by forgetting the orientation) is a connected multigraph (in Graph Theory this undirected graph is often called the *underlying* graph; we use here a different terminology as we already use the word underlying for the digraph associated to a dihypergraph). The digraph-theoretic notions of vertex-connectivity and arc-connectivity are also generalized by the dihypergraph-theoretic notions of vertex-connectivity and hyperarc-connectivity (see [BES11]). Unlike 1-arc connected digraphs, 1-hyperarc connected dihypergraphs are not always 1-vertex connected.

Remark that unlike an Eulerian digraph, an Eulerian dihypergraph does not need to be strongly connected. Indeed, let H be an Eulerian dihypergraph. If we add a new vertex x in H, such that x is incident to only one hyperarc E of H and $d^-(x) = 0$, then the dihypergraph obtained is still Eulerian, but it is not strongly connected.

On the other hand, we have the following necessary condition:

Proposition 14. *Let H be a dihypergraph. If H is Eulerian, then H is weakly connected.*

Proof. Let $G_{U(H)}$ be the undirected associated multigraph to $U(H)$. We want to prove that $G_{U(H)}$ is connected. Note first that for all hyperarc $E \in \mathcal{E}(H)$, vertices in the subset $E^- \cup E^+$ are in the same connected component in $G_{U(H)}$, by the definition of $U(H)$. Moreover, let E, F be any pair of distinct hyperarcs of H. Since there is an Eulerian dicycle in H, therefore, there exist $u \in E^+$ and $v \in F^-$, such that there is a dipath in H from u to v. Since there is a dipath from u to v in H, therefore there is a dipath P from u to v in $U(H)$ and so a path between u and v in $G_{U(H)}$. Therefore, the subsets $E^- \cup E^+$ and $F^- \cup F^+$ are in the same connected component in $G_{U(H)}$ too. Therefore, $G_{U(H)}$ is connected. □

Recall that a hypergraph is k-uniform if all its hyperedges have the same cardinality k. It was proved in [LN10] that, if H is an Eulerian k-uniform hypergraph, then $|\mathcal{V}_{odd}(H)| \leq (k-2)m(H)$, where $\mathcal{V}_{odd}(H)$ is the set of all the vertices in H with an odd degree and $m(H)$ is the number of hyperedges in H. Using the same idea, we also prove a necessary condition for a dihypergraph H to be Eulerian.

Theorem 25. *Let H be a dihypergraph. If H is Eulerian then:*

$$\sum_{v \in \mathcal{V}(H)} |d^+(v) - d^-(v)| \leq \sum_{E \in \mathcal{E}(H)} (|E^+| + |E^-| - 2).$$

Proof. Let $C = v_0, E_0, v_1, \ldots, v_{m-1}, E_{m-1}, v_0$ be an Eulerian dicycle in H. By definition, a given vertex may appear many times in C, but every hyperarc appears exactly once in the dicycle C. Let us find the maximum number of occurences of a given vertex v in C. For all $i \neq j$ we may have $v_i = v_j$, but we are sure that $E_i \neq E_j$. So a vertex v can appear at most $\min(d^+(v), d^-(v))$ times in C and, as a consequence, we have the following inequality:

$$\sum_{v \in \mathcal{V}(H)} \min(d^+(v), d^-(v)) \geq m$$

Moreover, we know that:

$$\min(d^+(v), d^-(v)) = \frac{1}{2}(d^+(v) + d^-(v) - |d^+(v) - d^-(v)|),$$

$$\sum_{v \in \mathcal{V}(H)} d^+(v) = \sum_{E \in \mathcal{E}(H)} |E^-| \text{ and } \sum_{v \in \mathcal{V}(H)} d^-(v) = \sum_{E \in \mathcal{E}(H)} |E^+|.$$

Therefore, the following inequalities hold:

$$\sum_{v \in \mathcal{V}(H)} (d^+(v) + d^-(v) - |d^+(v) - d^-(v)|) \geq 2m$$

$$\sum_{E \in \mathcal{E}(H)} |E^-| + \sum_{E \in \mathcal{E}(H)} |E^+| - \sum_{v \in \mathcal{V}(H)} |d^+(v) - d^-(v)| \geq \sum_{E \in \mathcal{E}(H)} 2$$

$$\sum_{v \in \mathcal{V}(H)} |d^+(v) - d^-(v)| \leq \sum_{E \in \mathcal{E}(H)} [(|E^+| - 1) + (|E^-| - 1)]$$

□

For a digraph D, Theorem 25 is equivalent to the Euler's condition presented in Theorem 24: for all $v \in \mathcal{V}(D)$, $d^+(v) = d^-(v)$.

Theorem 25 is not a sufficient condition for a strongly connected dihypergraph H to be Eulerian: counter-examples are presented in Figure A.2 and in Figure A.3(b).

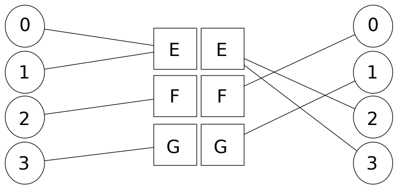


Figure A.2: A regular dihypergraph that is not Eulerian.

Another necessary condition was proposed by N. Cohen (private communication), who transposed the search of an Eulerian dicycle into a PERFECT MATCHING problem (see [Ber73, BM08b]).

Let H be a dihypergraph. If there is a hyperarc $E \in \mathcal{E}(H)$ whose in-set (resp. whose out-set) is empty, then H cannot be Eulerian. Else, let $\varphi : \mathcal{E}(H) \to \mathcal{V}(H) \times \mathcal{V}(H)$ be any function such that, for all E, we have $\varphi(E) \in E^- \times E^+$. By replacing each hyperarc E by the arc $\varphi(E)$ we get a digraph, denoted by $D_\varphi[H] = (\mathcal{V}(H), \varphi(\mathcal{E}(H)))$. Observe that $D_\varphi[H]$ is a subdigraph of $U(H)$ and it can have loops or multiple arcs.

Remark 2. *A dihypergraph H is Eulerian if, and only if, there exists a function φ such that $D_\varphi[H]$ is an Eulerian digraph.*

By Theorem 24, a necessary and sufficient condition for a digraph D to be Eulerian is that D is connected and, for every vertex v, $d^-(v) = d^+(v)$. If D satisfies this degree constraint for every vertex, but is not necessarily connected, we call D a *balanced* digraph.

We will use the well-known Hall's Theorem to prove a necessary and sufficient condition for the digraph $D_\varphi[H]$ to be balanced, for some φ.

Theorem 26 (see [Ber73, BM08b]). *Let $G = (\mathcal{V}_1 \cup \mathcal{V}_2, \mathcal{E})$ be a bipartite graph such that $|\mathcal{V}_1| = |\mathcal{V}_2|$. There is a perfect matching in B if, and only if, for every subset $S \subset \mathcal{V}_1$, $|\Gamma(S)| \geq |S|$, where $\Gamma(S)$ denotes the set of vertices adjacent to some vertex of S.*

Definition 10. *Let X be a subset of $\mathcal{V}(H)$. We denote by $d^+_H(X)$ (shortly $d^+(X)$) the number of hyperarcs $E \in \mathcal{E}(H)$ such that $E^- \cap X \neq \emptyset$ and by $d^-_{s,H}(X)$ (shortly denoted $d^-_s(X)$) the number of hyperarcs E such that $E^+ \subseteq X$.*

We are now able to prove the following result:

Theorem 27. *Let H be a dihypergraph. There exists a function φ such that $D_\varphi[H]$ is a balanced digraph if, and only if, for every subset $X \subseteq \mathcal{V}(H)$, we have $d^-_s(X) \leq d^+(X)$.*

Proof. Let us assume there exists φ such that $D_\varphi[H]$ is a balanced digraph. For every subset $X \subseteq \mathcal{V}(H)$, for every hyperarc E such that $E^+ \subseteq X$, we necessarily have $(u, v) = \varphi(E) \in \mathcal{V}(H) \times X$. Since $D_\varphi[H]$ is balanced, hence there must be some hyperarc F such that $\varphi(F)$ has vertex v as origin, that is $v \in F^-$. So $F^- \cap X$ is not empty. Furthermore, we can associate to two distinct hyperarcs E two distinct hyperarcs F; therefore $d^-_s(X) \leq d^+(X)$.

Conversely, let us assume that for every subset $X \subseteq \mathcal{V}(H)$, $d^-_s(X) \leq d^+(X)$. Consider the bipartite graph $BP(H) = (\mathcal{V}_1(BP) \cup \mathcal{V}_2(BP), \mathcal{E}(BP))$ with $\mathcal{V}_1(BP) = \{E^+_j : E_j \in \mathcal{E}(\mathcal{H})\}$, $\mathcal{V}_2(BP) = \{E^-_j : E_j \in \mathcal{E}(\mathcal{H})\}$ and $\mathcal{E}(BP) = \{E^+_j E^-_{j'} : E^+_j \cap E^-_{j'} \neq \emptyset\}$.

Let $S = \{E^+_{j_1}, E^+_{j_2}, \ldots, E^+_{j_{|S|}}\}$ be a subset of $\mathcal{V}_1(BP)$ and $X = \bigcup_{k=1}^{|S|} E^+_{j_k}$. Observe that $|\Gamma(S)| = d^+(X)$. Since $d^-_s(X) \geq |S|$, we conclude that $|\Gamma(S)| \geq |S|$. Moreover, $\mathcal{V}_1(BP)$ and $\mathcal{V}_2(BP)$ have the same cardinality. By Theorem 26, there is a perfect matching M in $BP(H)$. We now define a function φ as follows: for all edge $E^+_i E^-_j$ of M, one can choose any vertex $v \in E^+_i \cap E^-_j$ as the tail of $\varphi(E_i)$ and the head of $\varphi(E_j)$. Thus, we get a subdigraph $D_\varphi[H]$ that is a balanced digraph. □

Observe that we may define $d^-(X)$ and $d^+_s(X)$ in the same way as $d^+(X)$ and $d^-_s(X)$. Thus, another formulation of Theorem 27 is: there exists φ such that $D_\varphi[H]$ is a balanced digraph if, and only if, for every subset $X \subseteq \mathcal{V}(H)$ $d^+_s(X) \leq d^-(X)$.

By Theorem 27, deciding whether there exists φ such that $D_\varphi[H]$ is a balanced digraph can be done in polynomial time. However, deciding whether there exists φ such that $D_\varphi[H]$ is strongly connected is an NP-complete problem [BJT01].

A.3.2 Duality and Complexity

First, we show that the search of an Eulerian dicycle in H is equivalent to the search of a Hamiltonian dicycle in its dual:

Proposition 15. *A dihypergraph H is Eulerian if, and only if, H^* is Hamiltonian.*

Proof. For each dicycle $C = v_0, E_0, v_1, E_1, \dots, v_p, E_p, v_0$ of H one can find a corresponding dicycle in H^* namely $C^* = e_0, V_1, e_1, \dots, e_p, V_0, e_0$ and vice-versa. Thus, C is an Eulerian dicycle in H (i.e. C contains each hyperarc of H exactly once) if, and only if, C^* contains each vertex of H^* exactly once, i.e. C^* is a Hamiltonian dicycle of H^*. □

As a direct consequence we can observe that, since $(H^*)^* = H$, H is Hamiltonian if, and only if, H^* is Eulerian. Moreover, since deciding whether a di(hyper)graph H is Hamiltonian is an NP-complete problem [BJG10], the following result is not surprising:

Theorem 28. *Deciding whether a dihypergraph H is Eulerian is NP-complete.*

Proof. Let C be a dipath. One can verify, in $O(|\mathcal{E}(H)|)$ operations, whether C is an Eulerian dicycle in H. Consequently, the problem is in NP. Since the dual H^* can be built in $O(|\mathcal{E}(H)| + |\mathcal{V}(H)|)$-time, we conclude the proof directly from Proposition 15 and the NP-completeness of deciding whether a digraph is Hamiltonian. □

In order to check if a given dihypergraph is Hamiltonian, we will often use the following proposition:

Proposition 16. *Let H be a directed hypergraph. H is Hamiltonian if, and only if, its underlying multidigraph $U(H)$ is Hamiltonian.*

Proof. By definition of $U(H)$, any dicycle in H is a dicycle in $U(H)$ with the same vertices, and reciprocally. □

A.3.3 Line Dihypergraphs Properties

The *line dihypergraph* $L(H)$ of a dihypergraph H has as vertices the dipaths of length 1 in H and as hyperarcs the dipaths of length 1 in H^*:

$$\begin{aligned}\mathcal{V}(L(H)) &= \bigcup_{E \in \mathcal{E}(H)} \{(uEv) \mid u \in E^-,\ v \in E^+\},\\ \mathcal{E}(L(H)) &= \bigcup_{v \in \mathcal{V}(H)} \{(EvF) \mid v \in E^+ \cap F^-\};\end{aligned}$$

where the in-set and the out-set of hyperarc (EvF) are $(EvF)^- = \{(uEv) \mid u \in E^-\}$ and $(EvF)^+ = \{(vFw) \mid w \in F^+\}$.

Particularly, when D is a digraph, $L(D)$ is the *line digraph* of D (see [BJG10]).

The following results are used in the sequel:

Theorem 29 ([BES11])**.** *Let H be a dihypergraph. Then,*

1. *the digraphs $R(L(H))$ and $L^2(R(H))$ are isomorphic;*
2. *the digraphs $U(L(H))$ and $L(U(H))$ are isomorphic;*

3. the digraphs $(L(H))^*$ and $L(H^*)$ are isomorphic.

Recall that:

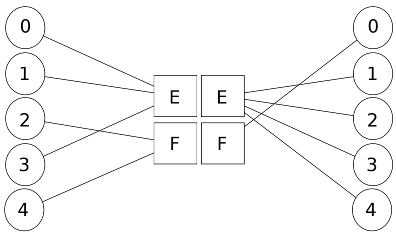
Theorem 30 ([DR94]). *For a given digraph D, the line digraph $L(D)$ is Hamiltonian if, and only if, D is Eulerian.*

This property is useful for some special families of digraphs, e.g. Kautz and de Bruijn digraphs, that are stable by line digraph operation [DR94]. By using induction, one can prove that every digraph of the family is Hamiltonian. It was shown in [BES11] that de Bruijn and Kautz dihypergraphs are also stable by line dihypergraph operation. So, it is natural to wonder whether this property can be generalized to dihypergraphs. However, we only get a weak generalization.

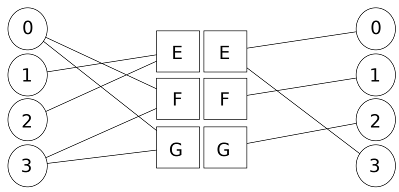
Proposition 17. *Let H be a dihypergraph. Then, $L(H)$ is Hamiltonian if, and only if, $U(H)$ is Eulerian.*

Proof. By Proposition 16, the dihypergraph $L(H)$ is Hamiltonian if, and only if, $U(L(H))$ is Hamiltonian. Moreover, $U(L(H))$ and $L(U(H))$ are isomorphic by Theorem 29. Finally, by Theorem 30 $L(U(H))$ is Hamiltonian if, and only if, $U(H)$ is Eulerian. □

We now show with two counter-examples that both implications of the corresponding version of Theorem 30 to dihypergraphs do not hold. There exist dihypergraphs which are Eulerian such that their line dihypergraph is not Hamiltonian and there also exist dihypergraphs that are not Eulerian such that their line dihypergraph is Hamiltonian.



(a) H_1



(b) H_2

Figure A.3: Counter-examples for extension of Theorem 30 to dihypergraphs.

Consider the dihypergraph $H_1 = (\mathcal{V}(H_1), \mathcal{E}(H_1))$ whose bipartite representation digraph is depicted in Figure A.3(a). Observe that $0, E, 2, F, 0$ is an Eulerian dicycle in H_1. But $d^+_{U(H_1)}(1) = 4$, that is different than $d^-_{U(H_1)}(1) = 3$. As a consequence, $U(H_1)$ cannot be Eulerian, by Theorem 24.

On the other hand, the directed hypergraph $H_2 = (\mathcal{V}(H_2), \mathcal{E}(H_2))$, depicted in Figure A.3(b), is not Eulerian, but $U(H)$ is Eulerian and so $L(H)$ is Hamiltonian. Remark that H_2 verifies the necessary condition of Theorem 25. Furthermore, H_2 is strongly connected. One may observe that its underlying multidigraph $U(H_2)$ is Eulerian (it is even a 2-regular digraph). However, H_2 is not Eulerian because it does not verify the condition of Theorem 27. Indeed, $d_s^-(\{1,2\}) = 2$, which is strictly greater than $d^+(\{1,2\}) = 1$.

We will show, in the next sections, that there are Eulerian dihypergraphs H, which are not digraphs, such that their $U(H)$ is Eulerian.

A.4 Case of d-regular, s-uniform Dihypergraphs

Let (s^-, s^+) be a couple of positive integers. An (s^-, s^+)-uniform dihypergraph H is a dihypergraph such that the in-size (resp. the out-size) of every hyperarc in H equals s^- (resp. equals s^+). When $s^- = s^+ = s$ we also say that H is a s-uniform dihypergraph. Recall that digraphs are 1-uniform dihypergraphs.

Let (d^-, d^+) be a couple of positive integers. A (d^-, d^+)-regular dihypergraph H is a dihypergraph such that the in-degree (resp. the out-degree) of every vertex in H equals d^- (resp. d^+). When $d^- = d^+ = d$, we also say that H is a d-regular dihypergraph. Regular 1-uniform dihypergraphs are exactly regular digraphs. Remark that a dihypergraph H is (p, q)-uniform if, and only if, its dual dihypergraph H^* is (p, q)-regular, for any positive integers p, q.

When the studied dihypergraphs are uniform, Theorem 25 can be reformulated in a very similar way to [LN10]:

Corollary 19. *Let H be an Eulerian dihypergraph. If H is (s^-, s^+)-uniform, then:*

$$\sum_{v \in \mathcal{V}(H)} |d^+(v) - d^-(v)| \leq (s^+ + s^- - 2)\ m$$

Observe that even though d-regular dihypergraphs always verify the necessary condition of Theorem 25, they are not always Eulerian (see Figure A.2).

We recall the following result about regular digraphs:

Theorem 31 ([BJG10])**.** *Deciding whether a 2-regular digraph D is Hamiltonian is an NP-complete problem.*

In [LN10], the authors use a similar result about 3-regular graphs, to prove that deciding whether a k-uniform hypergraph, $k \geq 3$, is Eulerian is an NP-complete problem. We do the same for uniform dihypergraphs. First, observe that if the dihypergraphs are 1-uniform, that is they are digraphs, we know that deciding whether a digraph is Eulerian can be done in polynomial time [DR94].

Theorem 32. *Let (s^-, s^+) be a couple of positive integers. If $s^- \geq 2$ or $s^+ \geq 2$, then deciding whether a (s^-, s^+)-uniform dihypergraph is Eulerian is an NP-complete problem.*

Proof. By symmetry, we only need to prove the case when $s^+ \geq 2$. Furthermore, by Theorem 28, we already know that the problem is in the NP-class. We now reduce the Hamiltonian problem in 2-regular digraphs to the Eulerian problem in (s^-, s^+)-uniform dihypergraphs.

The idea consists in associating in polynomial time to a 2-regular digraph D a dihypergraph H_D, such that H_D is Eulerian if, and only if, D is Hamiltonian and then the result will follow by Theorem 31.

Let $D = (\mathcal{V}(D), \mathcal{E}(D))$ be a 2-regular digraph. We define the dihypergraph H_D with the following rules:

1. $\mathcal{V}(H_D) = \mathcal{V}(D) \cup \{A \times \mathcal{V}(D)\} \cup \{B \times \mathcal{V}(D)\}$, where A and B are two sets satisfying $|A| = s^- - 1$ and $|B| = s^+ - 2$;
2. to each vertex $v \in \mathcal{V}(D)$, we associate a hyperarc $E_v \in \mathcal{E}(H_D)$ such that $E_v^- = \{v\} \cup \{A \times \{v\}\}$ and $E_v^+ = \{w_v, w_v'\} \cup \{B \times \{v\}\}$, where w_v and w_v' are the out-neighbors of v in D.

By construction, H_D is a (s^-, s^+)-uniform dihypergraph. Let us prove now that D is Hamiltonian if, and only if, H_D is Eulerian.

Suppose first that D is Hamiltonian and let $C = v_0, v_1, \ldots, v_{n-1}, v_0$ be a Hamiltonian dicycle in D. From C, we build a dicycle $\mathcal{C}_D$ in H_D, $\mathcal{C}_D = v_0, E_{v_0}, v_1, E_{v_1}, \ldots, v_{n-1}, E_{v_{n-1}}, v_0$, where E_{v_i} is the hyperarc that is induced by v_i. By definition of a Hamiltonian dicycle, for all $v \in \mathcal{V}(D)$, v appears only once in C. Therefore, by construction of H_D, for every $E_v \in \mathcal{E}(H_D)$, E_v appears exactly once in $\mathcal{C}_D$. So, $\mathcal{C}_D$ is an Eulerian dicycle in H_D.

Now, suppose that H_D is Eulerian. Remark that for every $E, F \in \mathcal{E}(H_D)$, by construction of H_D, we have $E^+ \cap F^- \subset \mathcal{V}(D)$. Thus, let $\mathcal{C}_D = v_0, E_0, v_1, E_1, \ldots, v_{m-1}, E_{m-1}, v_0$ be an Eulerian dicycle in H_D. Because of the previous remark, we know that for every i, $v_i \in \mathcal{V}(D)$. However, a vertex $v \in \mathcal{V}(D)$ is incident to only one hyperarc in H_D. As a consequence, for all i, E_i is the hyperarc that is associated to v_i and so, each v_i appears exactly once, and therefore, $C = v_0, v_1, \ldots, v_{m-1}, v_0$ is a Hamiltonian dicycle in D. □

When H is a digraph, we know that:

Theorem 33 ([DR94]). *Let D be a weakly-connected digraph. If D is regular, then all its iterated line digraphs $L^k(D)$, for every $k \geq 1$, are Hamiltonian and Eulerian.*

We now prove that, more generally:

Theorem 34. *Let H be a weakly-connected, d-regular, s-uniform dihypergraph. Then for every $k \geq 1$, $L^k(H)$ is Eulerian and Hamiltonian.*

Proof. Since H is d-regular and s-uniform, then $U(H)$ is a ds-regular multidigraph. As a consequence, for all $k \geq 0$, $L^k(U(H))$ is also ds-regular. By Theorem 29, we have by induction on k that, for all $k \geq 0$, $U(L^k(H))$ is isomorphic to $L^k(U(H))$. So, for all $k \geq 0$, $U(L^k(H))$ is Eulerian (because it is a *regular* multidigraph), that is equivalent, by Proposition 17, to $L(L^k(H)) = L^{k+1}(H)$ be Hamiltonian.

Moreover, H^* is s-regular, d-uniform, and we claim that it is also a weakly connected dihypergraph. Indeed, $s \geq 1$ implies that there is no empty in-set and no empty out-set in H^*. So, the connectivity of H implies the connectivity of H^*.

Therefore, for every $k \geq 1$ $L^k(H^*)$ is also Hamiltonian. Again by Theorem 29, we prove by induction on k that, for all $k \geq 1$, $(L^k(H))^*$ is isomorphic to $L^k(H^*)$. Therefore, by Proposition 15, for every $k \geq 1$ $L^k(H)$ is Eulerian. □

Remark 3. *Theorem 34 holds when H is (d^-, d^+)-regular, H is (s^-, s^+)-uniform, if we add the extra-condition: $d^-s^- = d^+s^+$.*

Recall that a complete Berge dicycle is an Eulerian and Hamiltonian dicycle and that if a dihypergraph has such a dicycle, then its bipartite representation digraph $R(H)$ is Hamiltonian. In the case $s = d$, we are able to prove a slightly more general result:

Proposition 18. *Let H be a d-regular, d-uniform dihypergraph that is weakly connected. There is a complete Berge dicycle in $L(H)$.*

Proof. Because of the d-regularity, d-uniformity of H, its own bipartite representation digraph $R(H)$ is d-regular. Therefore, for every $i \geq 1$, $L^i(R(H))$ is Hamiltonian. By Theorem 29, we know that $L^2(R(H))$ and $R(L(H))$ are isomorphic. Therefore, $R(L(H))$ is Hamiltonian. □

Other results about Eulerian and Hamiltonian dihypergraphs can be found in [Duc12].

A.5 de Bruijn and Kautz Dihypergraphs

In this section, we study the Eulerian and Hamiltonian properties of the generalization of de Bruijn and Kautz digraphs to dihypergraphs.

A.5.1 de Bruijn, Kautz and Consecutive-d digraphs

First, we recall some definitions and previous results on digraphs that we will use in the sequel.

Definition 11 ([II81, RPK80])**.** *The* generalized de Bruijn digraph $GB(d, n)$ *(also called Reddy-Pradhan-Khul digraph), is the digraph whose vertices are labeled with the integers modulo n; there is an arc from vertex i to vertex j if, and only if, $j \equiv di + \alpha \pmod n$, for every α with $0 \leq \alpha \leq d - 1$.*

If $n = d^D$, $GB(d, n)$ is nothing else than the *de Bruijn digraph* $B(d, D)$ (see [BP89, DR94]).

Definition 12 ([II81])**.** *The* generalized Kautz digraph $GK(d, n)$ *(also called Imase-Itoh digraph), is the digraph whose vertices are labeled with the integers modulo n; there is an arc from vertex i to vertex j if, and only if, $j \equiv -di - d + \alpha \pmod n$, for every α with $0 \leq \alpha \leq d - 1$.*

If $n = d^D + d^{D-1}$, $GK(d, n)$ is nothing else than the *Kautz digraph* $K(d, D)$ (see [BP89, DR94]).

Both of those families of digraphs can be generalized in the following way:

Definition 13 ([DH89]). *Let $1 \leq d, q \leq n-1$, and $0 \leq r \leq n-1$, then the* Consecutive-d digraph $G(d,n,q,r)$ *is the digraph whose vertices are labeled with the integers modulo n, such that there is an arc from vertex i to vertex j if, and only if, $j \equiv qi + r + \alpha \pmod n$, for every α with $0 \leq \alpha \leq d-1$.*

Observe that if $q = d$ and $r = 0$, then $G(d,n,d,0) = GB(d,n)$ and that if $q = r = n-d$, then $G(d,n,n-d,n-d) = GK(d,n)$.

Definition 14 ([DH88]). *Let λ be a positive integer, with $1 \leq \lambda \leq d$. Then $GB_\lambda(d,n)$ is the subdigraph of $GB(d,n)$ such that there is a link from i to j if, and only if, $j \equiv di + \alpha \pmod n$, for every $0 \leq \alpha \leq \lambda - 1$.*

Actually, the digraph $GB_\lambda(d,n)$ is nothing else than the Consecutive-d digraph $G(\lambda,n,d,0)$. But the notation of $GB_\lambda(d,n)$ helps to understand that it is a subdigraph of $GB(d,n)$. If $\lambda = d$, $GB_d(d,n) = GB(d,n)$. We can define in a similar way $GK_\lambda(d,n)$.

Consecutive-d digraphs have been intensively studied (see [DH89, DHHZ91, Hwa87, CHT97, CHT99, DHK94, DHP92, DHNP02, CDH[+]98]). Particularly, the characterization of the Hamiltonian Consecutive-d digraphs is nearly complete:

Theorem 35 ([CHT99, DH89, DHHZ91, Hwa87]). *Let $G = G(d,n,q,r)$ be a Consecutive-d digraph.*

- *If $d = 1$, then G is Hamiltonian if, and only if, all of the four following conditions hold:*
  1. $\gcd(n,q) = 1$;
  2. *for every prime number p such that p divides n, then we have p divides $(q-1)$;*
  3. *if 4 divides n, then 4 also divides $(q-1)$;*
  4. $\gcd(n, q-1, r) = 1$.
- *If $d = 2$, then G is Hamiltonian if, and only if, one of the following conditions is verified:*
  1. $\gcd(n,q) = 2$;
  2. $\gcd(n,q) = 1$ *and either $G(1,n,q,r)$ or $G(1,n,q,r+1)$ is Hamiltonian.*
- *If $d = 3$, then:*
  1. *if $\gcd(n,q) \geq 2$, G is Hamiltonian if, and only if, $\gcd(n,q) \leq 3$;*
  2. *if $\gcd(n,q) = 1$ and $1 \leq |q| \leq 3$, G is Hamiltonian.*
- *If $d \geq 4$, then G is Hamiltonian if, and only if, $\gcd(n,q) \leq d$.*

Corollary 20 ([DHHZ91]). *Let $G = G(d,n,q,r)$ be a Consecutive-d digraph. If $\gcd(n,q) \geq 2$, then G is Hamiltonian if, and only if, $\gcd(n,q) \leq d$.*

The only remaining case is when $d = 3$, for which there is only a partial characterization.

In particular, the characterization of the Hamiltonian generalized de Bruijn (resp. Kautz) digraphs is complete:

Theorem 36 ([DH88]). *If $\lambda = \gcd(n, d) \geq 2$, then $GB_\lambda(d, n)$ and $GK_\lambda(d, n)$ are Hamiltonian.*

Theorem 37 ([DHHZ91, DH88]). *$GB(d, n)$ is Hamiltonian if, and only if, one of the following conditions holds:*

1. *$d \geq 3$;*
2. *$d = 2$ and n is even.*

Theorem 38 ([DHHZ91, DH88]). *$GK(d, n)$ is Hamiltonian if, and only if, one of the following conditions holds:*

1. *$d \geq 3$;*
2. *$d = 2$ and n is even;*
3. *$d = 2$ and n is a power of 3.*

$GB(d, n)$ and $GK(d, n)$ are also Eulerian [DH88].

Finally, $GB(d, n)$ and $GK(d, n)$ have interesting line digraph properties. We use the following relations:

Proposition 19 ([DH88]). *If $\gcd(n, d) = \lambda \geq 2$, then*

$$L(GB_\lambda(d, \frac{n}{\lambda})) = GB_\lambda(d, n) \text{ and } L(GK_\lambda(d, \frac{n}{\lambda})) = GK_\lambda(d, n)$$

Particularly:

$$L(GB(d, n)) = GB(d, dn) \text{ and } L(GK(d, n)) = GK(d, dn).$$

A.5.2 Definitions of de Bruijn and Kautz dihypergraphs

We now give the arithmetical definition for the de Bruijn and Kautz dihypergraphs. For other definitions, see [BDE97]. In what follows, the vertices (resp. the hyperarcs) are labeled with the integers modulo n (resp. modulo m); the vertices are denoted $i, 0 \leq i \leq n-1$ and the hyperarcs $E_j, 0 \leq j \leq m-1$. To ease the reading we do not write, when it is clear in the context, the expressions $\pmod{n}$ and $\pmod{m}$.

Definition 15 ([BDE97]). *Let d, n, s and m be four positive integers, such that $dn \equiv 0 \pmod{m}$ and $sm \equiv 0 \pmod{n}$. The* generalized de Bruijn dihypergraph *$GBH(d, n, s, m)$ has as vertex set (resp. hyperarc set) the integers modulo n (resp. modulo m). Any vertex i belongs to the in-set of hyperarcs $E_{di+\alpha \pmod{m}}$, for every $0 \leq \alpha \leq d-1$. Any hyperarc E_j has as out-set the vertices $sj + \beta \pmod{n}$, for every $0 \leq \beta \leq s-1$.*

Note that the condition $dn \equiv 0 \pmod m$ follows from the fact that the vertices i and $i+n$ should be incident to the same hyperarcs $d(i+n)+\alpha \equiv di+\alpha \pmod m$. Similarly $E_j^+ = E_{j+m}^+$ implies $sm \equiv 0 \pmod n$.

Particularly, when $n = m$, it can be useful to remark that in the bipartite digraph $R(GBH(d,n,s,n))$, the incidence relations from $\mathcal{V}_1$ to $\mathcal{V}_2$ are the same as in $GB(d,n)$ and the incidence relations from $\mathcal{V}_2$ to $\mathcal{V}_1$ are the same as in $GB(s,n)$.

Definition 16 ([BDE97]). *Let (d,n,s,m) be four positive integers, such that $dn \equiv 0 \pmod m$ and $sm \equiv 0 \pmod n$. The generalized* Kautz dihypergraph, denoted by *$GKH(d,n,s,m)$, is the dihypergraph whose vertices (resp. hyperarcs) are labeled by the integers modulo n (resp. modulo m), such that a vertex i is incident to hyperarcs $E_{di+\alpha \pmod m}$, for every $0 \le \alpha \le d-1$ and hyperarc E_j has for out-set the vertices $-sj-s+\beta \pmod n$, for every $0 \le \beta \le s-1$.*

By inversing the labeling of the hyperarcs, it has been proposed in [BDE97] an equivalent definition for Kautz dihypergraphs:

Definition 17 ([BDE97]). *Let (d,n,s,m) be four positive integers, such that $dn \equiv 0 \pmod m$ and $sm \equiv 0 \pmod n$. The generalized Kautz dihypergraph, denoted by $GKH(d,n,s,m)$, is the dihypergraph whose vertices (resp. hyperarcs) are labeled by the integers modulo n (resp. modulo m), such that a vertex i is incident to hyperarcs $E_{-di-d+\alpha \pmod m}$, for every $0 \le \alpha \le d-1$ and hyperarc E_j has for out-set the vertices $sj+\beta \pmod n$, for every $0 \le \beta \le s-1$.*

We recall some properties that will be used in Section A.6.

Theorem 39 ([BDE97]). *The underlying multidigraph of $GBH(d,n,s,m)$ (resp. $GKH(d,n,s,m)$) is $GB(ds,n)$ (resp. $GK(ds,n)$).*

Theorem 40 ([BDE97]). *If $H = GBH(d,n,s,m)$ (resp. $GKH(d,n,s,m)$), then $H^* = GBH(s,m,d,n)$ (resp. $GKH(s,m,d,n)$).*

Theorem 41 ([BES11]). *The line dihypergraph of $GBH(d,n,s,m)$ (resp. of $GKH(d,n,s,m)$) is $GBH(d,dsn,s,dsm)$ (resp. is $GKH(d,dsn,s,dsm)$).*

A.5.3 Eulerian and Hamiltonian properties

We now characterize when the generalized de Bruijn and Kautz dihypergraphs are Hamiltonian and Eulerian. Recall that we suppose $n > 1$ and $m > 1$.

Theorem 42. *Let $H = GBH(d,n,s,m)$ be a generalized de Bruijn dihypergraph. H is Hamiltonian if, and only if, one of the following conditions is verified:*

1. *$ds \ge 3$;*

2. *$ds = 2$ and n is even.*

Proof. First, recall that $U(H) = GB(ds,n)$ by Theorem 39. By Theorem 37, we know that the de Bruijn digraph $GB(ds,n)$ is Hamiltonian if, and only if, $ds \ge 3$; or $ds = 2$ and n is even. Therefore, by Proposition 16, Theorem 42 follows. □

Theorem 43. *Let $H = GBH(d, n, s, m)$ be a generalized de Bruijn dihypergraph. H is Eulerian if, and only if, one of the following conditions is verified:*

1. $ds \geq 3$;

2. $ds = 2$ *and* m *is even.*

Proof. By Theorem 40, $H^* = GBH(s, m, d, n)$. Theorem 42 gives a necessary and sufficient condition for H^* to be Hamiltonian. By Proposition 15, this is also a necessary and sufficient condition for H to be Eulerian. □

The method that is used for deciding whether $GBH(d, n, s, m)$ is Eulerian or Hamiltonian can be applied to Kautz dihypergraphs in the same way. By Theorem 38, we have necessary and sufficient conditions for a generalized Kautz digraph to be Hamiltonian. Consequently:

Theorem 44. *Let $H = GKH(d, n, s, m)$ be a generalized Kautz dihypergraph.*

1. *If* $ds \geq 3$, *then* H *is Eulerian and Hamiltonian;*

2. *If* $ds = 2$, *then* H *is Eulerian (resp. Hamiltonian) if, and only if,* m *(resp.* n*) is even or a power of* 3.

A.6 Existence of Complete Berge Dicycles

In this section, we want to determine when there exists a complete Berge dicycle in $GBH(d, n, s, m)$, (i.e a Hamiltonian dicycle in its bipartite representation digraph).

A necessary condition for a dihypergraph H to have a complete Berge dicycle is that $n = m$. Otherwise, $R(H)$ cannot be Hamiltonian. We prove that:

Theorem 45. *There is a complete Berge dicycle in $GBH(d, n, s, n)$ if one of the following conditions is verified:*

1. $d \geq 3$ *and* $s \geq 3$;

2. $d = 2$ *and* $s \geq 4$, *or* $s = 2$ *and* $d \geq 4$;

3. $\{d, s\} = \{2, 3\}$ *and either* n *is even or* n *is a multiple of* 3;

4. $d = s = 2$ *and* n *is even or* n *is a power of* 3 *(otherwise it does not exist);*

5. $d = 1$ *or* $s = 1$ *and* $GB(ds, n)$ *is Hamiltonian (otherwise it does not exist).*

The only remaining case is when $\{d, s\} = \{2, 3\}$ and n and 6 are relatively prime, for which we conjecture $GBH(d, n, s, n)$ has a complete Berge dicycle:

Conjecture 4. *If $\{d, s\} = \{2, 3\}$, then there is a complete Berge dicycle in $GBH(d, n, s, n)$.*

We highlight the particular case when $s = d$, for which we have a complete characterization:

Theorem 46. *There is a complete Berge dicycle in* $GBH(d, n, d, n)$ *if, and only if, one of the following conditions is verified:*

1. $d \geq 3$;
2. $d = 2$ *and* n *is an even number;*
3. $d = 2$ *and* n *is a power of* 3.

Remark that, for $d \geq 2$, these conditions are exactly the same as those implying that $GK(d, n)$ is Hamiltonian (see Theorem 38). It would be interesting to see if there is a relationship between Theorems 38 and 46. We were able to find it only when n is odd (see Lemma 37 in Section A.6.4).

The rest of this section is devoted to the proof of Theorems 45 and 46. In Section A.6.1, we deal with the easy case $d = 1$. In Section A.6.2, we show that Theorem 45 is true when $\gcd(n, d) \geq 2$ and $\gcd(n, s) \geq 2$ using a special product of digraphs and the notion of line digraphs. Then, in Section A.6.3, we consider the opposite case, where $\gcd(n, d) = 1$ or $\gcd(n, s) = 1$, and solve all the cases except $\{d, s\} = \{2, 3\}$, $d = s = 2$ and $d = s = 3$. Section A.6.4 contains the lemma which shows the relation with the generalized Kautz digraphs, and that the conditions of Theorem 46 are sufficient for $d = s = 2$ and n is a power of 3. In Section A.6.5, by using the Euler's function, we show that these conditions are also necessary for $d = s = 2$. Finally, in Section A.6.6, we deal with the remaining case: $d = s = 3$ and $\gcd(n, 3) = 1$ and we solve it using a link-interchange method.

A.6.1 Case $d = 1$

Lemma 33. *If* $d = 1$*, then there is a complete Berge dicycle in* $GBH(1, n, s, n)$ *if, and only if, the de Bruijn digraph* $GB(s, n)$ *is Hamiltonian.*

Proof. If $d = 1$, then every vertex i is only incident to hyperarc E_i. So we may not distinguish the vertices from the hyperarcs and we get a digraph, the relations of incidence of which are the relations of incidence between hyperarcs and vertices in the original dihypergraph. Therefore, Lemma 33 follows. □

By symmetry, observe that the case when $s = 1$ is also solved by Lemma 33.

A.6.2 Case $\gcd(n, d) \geq 2$ and $\gcd(n, s) \geq 2$

In this section, we completely solve the case when $\gcd(n, d) \geq 2$ and $\gcd(n, s) \geq 2$. The proof is involved; in the particular case $d = s$, it can be simplified by using other methods such as the concatenation of digraph dicycles [Duc12].

We will use a subcase of a digraph product introduced in [BH95, BH96]:

Definition 18 ([BH95, BH96]). *Let L_1, L_2 be two digraphs with the same order n and with $\mathcal{V}(L_1) \cap \mathcal{V}(L_2) = \emptyset$ and let ϕ: $\mathcal{V}(L_1) \to \mathcal{V}(L_2)$ be a one-to-one mapping. Then, $L_1 \otimes_\phi L_2$ is the digraph L such that $\mathcal{V}(L) = \mathcal{V}(L_1) \cup \mathcal{V}(L_2)$ and the set of arcs $\mathcal{E}(L)$ is defined by exchanging the out-neighbors of $u \in \mathcal{V}(L_1)$ with the out-neighbors of $\phi(u) \in \mathcal{V}(L_2)$ and vice-versa. More precisely, if $u_2 = \phi(u_1)$, and (u_1, v_1) is an arc of L_1 and (u_2, v_2) is an arc of L_2, then the arcs (u_1, v_2) and (u_2, v_1) belong to $\mathcal{E}(L)$.*

Observe that if L_1 is the generalized de Bruijn digraph $GB(s, n)$, L_2 is the generalized de Bruijn digraph $GB(d, n)$ and ϕ is the identity function, then $L_1 \otimes_\phi L_2$ is the bipartite representation digraph $R(GBH(d, n, s, n))$.

It happens that even if L_1 and L_2 are both strongly connected, $L_1 \otimes_\phi L_2$ may be disconnected. However, it was proven by Barth and Heydemann the following sufficient condition:

Lemma 34 ([BH95]). *If L_1 and L_2 are strongly connected, and if there exist u_1 and u_2 such that $\phi(u_1) = u_2$ and there is a loop $(u_1, u_1) \in \mathcal{E}(L_1)$ and a loop $(u_2, u_2) \in \mathcal{E}(L_2)$, then $L_1 \otimes_\phi L_2$ is strongly connected.*

We now prove a useful lemma:

Lemma 35. *For every $i \in \{1, 2\}$, let D_i be an arbitrary digraph and $L_i = L(D_i)$ be its line digraph. If L_1 and L_2 have the same number of vertices and $\phi : \mathcal{V}(L_1) \to \mathcal{V}(L_2)$ is a one-to-one mapping, then $L_1 \otimes_\phi L_2$ is also a line multidigraph $L(D)$, such that $\mathcal{V}(D) = \mathcal{V}(D_1) \cup \mathcal{V}(D_2)$ and the degree of a vertex in D is the same as the degree of its corresponding vertex in D_1 or D_2.*

Proof. The vertices of L_i $(i = 1, 2)$ are the arcs of D_i and so they are of the form (u_i, v_i), with $u_i, v_i \in \mathcal{V}(D_i)$. Let $\mathcal{V}(D) = \mathcal{V}(D_1) \cup \mathcal{V}(D_2)$. For each (u_1, v_1) of L_1, if $(u_2, v_2) = \phi((u_1, v_1))$ is its image by ϕ, we put in D the arcs (u_1, v_2) and (u_2, v_1).

Now, consider the mapping $\psi : \mathcal{V}(L_1 \otimes_\phi L_2) \to \mathcal{V}(L(D)) = \mathcal{E}(D)$, defined as follows: if (u_1, v_1) is a vertex of L_1 and $(u_2, v_2) = \phi((u_1, v_1))$ is the associated vertex in L_2, then $\psi((u_1, v_1)) = (u_1, v_2)$ and $\psi((u_2, v_2)) = (u_2, v_1)$. Observe that ψ is a one-to-one mapping. To prove the lemma, it suffices to prove that ψ keeps the adjacency relation.

On one side, by definition of the product, the vertex (u_1, v_1) is joined in $L_1 \otimes_\phi L_2$ to the out-neighbors of (u_2, v_2) in L_2 that is to the vertices of the form (v_2, w_2), with (v_2, w_2) an arc of D_2. On the other side, in $L(D)$, the vertex $(u_1, v_2) = \psi((u_1, v_1))$ is joined to the vertices (v_2, y_1), where y_1 is such that there exists x_1 in D_1 and w_2 in D_2, such that (x_1, y_1) is an arc of D_1, $\phi((x_1, y_1)) = (v_2, w_2)$ and (v_2, w_2) is an arc of D_2. But, by definition, $(v_2, y_1) = \psi((v_2, w_2))$. So, $\psi((u_1, v_1))$ is joined in $L(D)$ to all the images by ψ of the out-neighbors of (u_1, v_1) in $L_1 \otimes_\phi L_2$ and then the adjacency relation is kept for the vertices of L_1. The proof is identical for the vertices of L_2. □

When $s = d$, we can prove a stronger result namely that $GB_\lambda(d, n) \otimes_\phi GB_\lambda(d, n)$ is the line digraph of $GB_\lambda(d, \frac{n}{\lambda}) \otimes_\phi GB_\lambda(d, \frac{n}{\lambda})$ [Duc12].

Remark 4. *Note that, if* $L_1 = L(D_1)$ *and* $L_2 = L(D_2)$ *are Hamiltonian digraphs, then* D_1 *and* D_2 *are balanced and so, by Lemma 35,* D *is a balanced digraph, i.e. every vertex of* D *has equal in-degree and out-degree.*

Lemmas 34 and 35 enable us to prove the following theorem:

Theorem 47. *Let* $H = GBH(d, n, s, n)$ *be a generalized de Bruijn dihypergraph. If* $\gcd(d, n) \geq 2$ *and* $\gcd(s, n) \geq 2$*, then there is a complete Berge dicycle in* H*.*

Proof. Let us show that $R = R(GBH(d, n, s, n))$ is a Hamiltonian digraph. We recall that R is isomorphic to $GB(s, n) \otimes_\phi GB(d, n)$, ϕ being the identity function. So, for $\lambda = \gcd(d, n)$ and $\mu = \gcd(s, n)$, the digraph $GB_\mu(s, n) \otimes_\phi GB_\lambda(d, n)$ is isomorphic to a subdigraph of R. As, by Proposition 19, $GB_\mu(s, n)$ and $GB_\lambda(d, n)$ are two line digraphs, then, by Lemma 35, $GB_\mu(s, n) \otimes_\phi GB_\lambda(d, n)$ is also a line digraph $L(D)$. Moreover, since $GB_\mu(s, n)$ and $GB_\lambda(d, n)$ are also Hamiltonian digraphs, by Theorem 36, then D is a balanced digraph by Remark 4. Furthermore, both $GB_\mu(s, n)$ and $GB_\lambda(d, n)$ are strongly connected and those two digraphs have a common loop $(0, 0)$. Consequently, by Lemma 34, $GB_\mu(s, n) \otimes_\phi GB_\lambda(d, n)$ is strongly connected, hence D is strongly connected too.

D is a balanced digraph that is strongly connected. In other words, D is an Eulerian digraph and so $L(D) = GB_\mu(s, n) \otimes_\phi GB_\lambda(d, n)$ is a Hamiltonian digraph. □

A.6.3 Case n and d relatively prime, or n and s relatively prime

In the next proofs, we consider a Hamiltonian dicycle in a Consecutive-d digraph as a circular permutation σ in $\mathbb{Z}_n$. If j is the vertex that follows i in the Hamiltonian dicycle, then $\sigma(i) = j$; if k is the vertex that follows j in the same dicycle, then $\sigma^2(i) = k$ and so on.

Now we deal with the other case $\gcd(n, d) = 1$ or $\gcd(n, s) = 1$ and will prove that the Theorem 45 holds in most of the cases. The proof will rely on the following lemma:

Lemma 36. *Let* n *and* d *be relatively prime. If the Consecutive-*s *digraph* $G(s, n, ds, 0)$ *is Hamiltonian, then there is a complete Berge dicycle in* $GBH(d, n, s, n)$*.*

Proof. Recall that in $G(s, n, ds, 0)$ a vertex i is joined to the vertices $j \equiv dsi + \beta \pmod{n}$, for every β with $0 \leq \beta \leq s - 1$. Let $0, \sigma(0), \sigma^2(0), \ldots, \sigma^{n-1}(0), 0$ be a Hamiltonian dicycle of $G(s, n, ds, 0)$. We construct the following dicycle in $GBH(d, n, s, n)$. Vertex i precedes the hyperarc E_{di}. Since $\gcd(n, d) = 1$, therefore d is invertible in Z_n and $i \to di$ is a bijection between vertices and hyperarcs. The dicycle $0, E_0, \sigma(0), E_{d\sigma(0)}, \ldots, \sigma^{n-1}(0), E_{d\sigma^{n-1}(0)}, 0$ is then a complete Berge dicycle in $GBH(d, n, s, n)$; indeed the vertex $\sigma^{k+1}(0) \equiv ds\sigma^k(0) + \beta \equiv s(d\sigma^k(0)) + \beta$ is in the out-set of $E_{d\sigma^k(0)}$. □

Theorem 48. *Let $H = GBH(d, n, s, n)$ be a generalized de Bruijn dihypergraph such that $d \neq 1$ and $s \neq 1$. If n and d are relatively prime or n and s are relatively prime, then there is a complete Berge dicycle in H if one of the following conditions hold:*

1. *$d \geq 4$ or $s \geq 4$;*

2. *$\{d, s\} = \{2, 3\}$ and n is even or n is a multiple of 3.*

Proof. By Theorem 35, we know that $G(s, n, ds, 0)$ is Hamiltonian if one of the following conditions hold:

- $s \geq 4$ and $gcd(n, ds) \leq s$;
- or $\{s = 3$ and $2 \leq gcd(n, 3d) \leq 3\}$;
- or $\{s = 2$ and $gcd(n, 2d) = 2\}$.

Furthermore, if n and d are relatively prime, we have:

$$\gcd(n, ds) = \gcd(n, s) \leq s \tag{A.1}$$

and so, $2 \leq gcd(n, 3d) \leq 3$ is equivalent to n multiple of 3 and $gcd(n, 2d) = 2$ to n even.

By using these facts and Lemma 36 we get:

- Fact 1: if n and d are relatively prime, then there is a complete Berge dicycle in $GBH(d, n, s, n)$ when $s \geq 4$ or $\{s = 3$ and n is a multiple of $3\}$ or $\{s = 2$ and n is even$\}$.

- Fact 2: (obtained by exchanging d and s) if n and s are relatively prime, then there is a complete Berge dicycle in $GBH(s, n, d, n)$, hence, there is also a complete Berge dicycle in the dual $GBH(d, n, s, n)$, when $d \geq 4$ or $\{d = 3$ and n is a multiple of $3\}$ or $\{d = 2$ and n is even$\}$.

Now, we can conclude as follows:

Let $d \geq 4$. If n and s are relatively prime we conclude by using Fact 2. Otherwise $gcd(n, s) \geq 2$ and n and d are relatively prime. The theorem is proved by using Fact 1 as either $s \geq 4$; or $s = 3$, but then n is a multiple of 3, because $gcd(n, s) \geq 2$; or $s = 2$ and $gcd(n, s) \geq 2$ implies that n is a multiple of 2.

The case $s \geq 4$ can be done similarly (by exchanging d and s, which corresponds to work in the dual).

Now let $d = 3$ and $s = 2$. If n is a multiple of $d = 3$, then by hypothesis n and s are relatively prime and we conclude by Fact 2. If n is a multiple of $s = 2$, then by hypothesis n and d are relatively prime and we conclude by Fact 1. The case $d = 2$ and $s = 3$ is done similarly by exchanging d and s. □

A.6.4 Concatenation of dicycles and relation to generalized Kautz digraphs

If n is an odd number, then Theorem 46 can be partly proven with a concatenation of dicycles.

Lemma 37. *If $GK(d,n)$ is Hamiltonian and n is odd, then there is a complete Berge dicycle in $GBH(d,n,d,n)$.*

Proof. We use a variant for the definition of $GBH(d,n,d,n)$. Indeed as noted in [Cou01], if we label the hyperarc E_j with label E_{n-1-j}, we get the incidence relations of $GK(d,n)$. In other words, $GBH(d,n,d,n)$ can be defined as follows: vertex i is incident to hyperarcs $E_{-di-d+\alpha \pmod{n}}$, for every $0 \leq \alpha \leq d-1$, and hyperarc E_j has as out-set the vertices $-dj-d+\beta \pmod{n}$, for every $0 \leq \beta \leq d-1$.

Now, by Theorem 38, there exists a Hamiltonian dicycle in the Kautz digraph $GK(d,n)$ for n odd, and either $d \geq 3$ or $\{\ d = 2$ and n is a power of $3\}$; let it be $0, \sigma(0), \sigma^2(0), \ldots, \sigma^{n-1}(0), 0$. Let C be the dicycle of $GBH(d,n,d,n)$, where vertex i precedes hyperarc $E_{\sigma(i)}$ and hyperarc E_j precedes vertex $\sigma(j)$. So, $C = 0, E_{\sigma(0)}, \sigma^2(0), \ldots, \sigma^{2h}(0), E_{\sigma^{2h+1}(0)}, \ldots, \sigma^{n-2}(0), E_{\sigma^{n-1}(0)}, 0$, where $0 \leq h \leq n-1$. As n is odd, the n vertices and also the n hyperarcs of the dicycle are all different. Therefore C is a complete Berge dicycle in $GBH(d,n,d,n)$. □

Corollary 21. *If n is odd and $d \geq 3$, then there is a complete Berge dicycle C in $GBH(d,n,d,n)$.*

Corollary 22. *If n is a power of 3 and $d = 2$, then there is a complete Berge dicycle C in $GBH(2,n,2,n)$.*

In the same way, we can prove that, if $GB(d,n)$ is Hamiltonian and n is odd, then there is a complete Berge dicycle in $GBH(d,n,d,n)$. However, even if it would have given the result for $GBH(d,n,d,n)$ with n odd and $d \geq 3$, it would have not been enough to conclude for the case $d = 2$ and n a power of 3. In that case, the proof of Lemma 37 plus the fact that, by [DHHZ91], $\sigma : i \rightarrow -2i-1$ is a Hamiltonian dicycle in $GK(2,n)$, gives the following complete Berge dicycle C in $GBH(2,n,2,n)$ (by renaming the edges with the standard definition). In C, vertex i precedes the hyperarc E_{2i}, and hyperarc E_j precedes the vertex $2j+1$. Thus, C contains as consecutive vertices i and $4i+1$. Figure A.1 shows the dicycle $0, E_0, 1, E_2, 5, E_1, 3, E_6, 4, E_8, 8, E_7, 6, E_3, 7, E_5, 2, E_4, 0$, that is obtained in this way for $n = 9$ (with dotted red arcs).

A.6.5 Case $d = s = 2$

Theorem 47 and Corollary 22 show that there exists a complete Berge dicycle in $GBH(2,n,2,n)$ when n is even, or $\{n$ is odd and n is a power of $3\}$. We still have to prove there is no complete Berge dicycle in the remaining cases. For that, we need to use the Euler function, in the spirit of the proof of [DHHZ91].

Definition 19. *The Euler function, denoted by φ, associates to a positive integer n, the number $\varphi(n)$ of positive integers that are lower than n and relatively prime to n.*

The Euler function satisfies the following properties (the three first ones are immediate consequences of the definition and the fourth one is known as Euler's theorem):

1. $\varphi(1) = 1$;
2. If p is a prime number and $m \geq 1$, then $\varphi(p^m) = (p-1)p^{m-1}$;
3. If a and b are relatively prime, then $\varphi(ab) = \varphi(a)\varphi(b)$;
4. If a and b are relatively prime, then $a^{\varphi(n)} \equiv 1 \pmod{b}$.

Lemma 38. *If n is odd, then there is a complete Berge dicycle in $GBH(2, n, 2, n)$ if, and only if, n is a power of 3.*

Proof. Let us suppose that there is a complete Berge dicycle C in $GBH(2, n, 2, n)$. We distinguish two cases: either there exists a vertex i, such that i precedes in C the hyperarc E_{2i} and we will show this holds for all the vertices; or any vertex i precedes in C the hyperarc E_{2i+1}. To prove this claim, consider that some vertex i precedes E_{2i} in C. Since $\gcd(2, n) = 1$, we have that 2 is invertible in $\mathbb{Z}_n$. Thus, $i' = i - 2^{-1}$ cannot precede $E_{2i'+1}$, as $2i' + 1 = 2i$. So, i' precedes $E_{2i'}$ too. Consequently, since 2^{-1} is a generator element of $\mathbb{Z}_n$, any vertex i precedes in C the hyperarc E_{2i}.

Similarly, we can prove that either every hyperarc E_j in C precedes the vertex $2j$, or every hyperarc E_j in C precedes the vertex $2j + 1$. Therefore, if we consider only the vertices of the dicycle C and we denote by $\sigma(i)$ the successor of i in C, we have exactly four possibilities for σ, namely: $\sigma_k(i) = 4i + k$ with $0 \leq k \leq 3$.

Since $4 \cdot 0 = 0$, the solution $k = 0$ does not generate a complete Berge dicycle. Furthermore, if $\gcd(n, 3) = 1$ the equation $\sigma_k(i) \equiv i \iff 4i + k \equiv i \iff 3i \equiv -k$, has always a solution for $1 \leq k \leq 3$. Therefore, none of the other values of k works, when $\gcd(n, 3) = 1$.

It remains to consider the case $n = c3^p$, $p \geq 1$ and $\gcd(3, c) = 1$. By induction, we have that $\sigma_k^h(0) = \frac{k(4^h-1)}{3} \pmod{n}$.

Let φ be the Euler function; by Properties 2 and 3, $\varphi(n) = \varphi(c)\varphi(3^p) = 2\varphi(c)3^{p-1}$. Since $\gcd(n, 2) = 1$, then, by Property 4, $2^{\varphi(n)} \equiv 1 \pmod{n}$. Therefore, $4^{\varphi(c)3^{p-1}} \equiv 1 \pmod{n}$ and, since $\varphi(c)3^{p-1} < n$, σ_3 never generates a complete Berge dicycle either.

Moreover, since 2 is invertible in $\mathbb{Z}_n$, then we also know that σ_2 generates a complete Berge dicycle if, and only if, this is also the case for solution σ_1. Actually, when we choose σ_1, we choose σ_2 in the dual, and reciprocally. So, let us concentrate now on σ_1. The equation $\sigma_1^h(0) = \sigma_1^{h'}(0)$ is equivalent to $4^h \equiv 4^{h'} \pmod{3n}$. Again, Property 4 of Euler's function implies that $4^{\varphi(c)3^p} \equiv 1 \pmod{3n}$. But the only value of c such that $\varphi(c) = c$ is $c = 1$. Therefore, if n is not a power of 3, σ_1 and σ_2 do not generate a complete Berge dicycle. □

This proof for $d = 2$ and n odd could be shortened using the characterization of the Hamiltonian Consecutive-1 digraphs. Indeed, we prove there are only four possibilities for having a complete Berge dicycle in $GBH(2, n, 2, n)$. In the original proof, we deal with them as applications σ_k of $\mathbb{Z}_n$, for $0 \leq k \leq 3$. But these four solutions are also equivalent to some Consecutive-1 digraphs. They correspond, respectively, to the relations of incidence in $G(1, n, 4, 0)$, $G(1, n, 4, 1)$, $G(1, n, 4, 2)$ and $G(1, n, 4, 3)$. Then, deciding whether one of these four solutions generate a complete Berge dicycle is the same thing as deciding whether one of these four Consecutive-1 digraphs is a Hamiltonian digraph. Furthermore, by Theorem 35, we know whether one of those digraphs is Hamiltonian, depending on the value of n. For all n, $G(1, n, 4, 0)$ and $G(1, n, 4, 3)$ are never Hamiltonian. Moreover, $G(1, n, 4, 1)$ and $G(1, n, 4, 2)$ are Hamiltonian if, and only if, n is a power of 3. Since the Hamiltonicity of at least one of these digraphs is a necessary and sufficient condition for $H = GBH(2, n, 2, n)$ to have a complete Berge dicycle, then there is a complete Berge dicycle in H if, and only if, n is a power of 3.

A.6.6 Case $d = s = 3$

To finish the proof, it remains to deal with the case n even, $d = s = 3$ and n and d relatively prime. Note that, for $d = 3$, we do not know when the Consecutive-3 digraph $G(3, n, 9, 0)$ is Hamiltonian, and so, we cannot use the same proof as in Lemma 36. We will use a method, introduced in [DHHZ91], that is different from the previous ones. This method enables us to merge two disjoint dicycles of $R(H)$ into one dicycle.

Definition 20. *Let C_1, C_2 be two dicycles, that are subdigraphs of the same digraph D. A pair $\{x_1, x_2\}$ with $x_1 \in C_1$ and $x_2 \in C_2$ is called an* interchange pair *if the predecessor y_1 of x_1 in C_1 is incident to x_2 in D, and the predecessor y_2 of x_2 in C_2 is incident to x_1 in D too.*

If $\{x_1, x_2\}$ is an interchange pair, then we can build a dicycle containing all the vertices of $C_1 \cup C_2$ by deleting (y_1, x_1) and (y_2, x_2) and adding the arcs (y_1, x_2) and (y_2, x_1).

Lemma 39. *If n is even and n and 3 are relatively prime, then there is a complete Berge dicycle in $GBH(3, n, 3, n)$.*

Proof. Let R be the bipartite representation digraph of $GBH(3, n, 3, n)$. To every vertex i we associate the hyperarc E_{3i+1} and, similarly, to every hyperarc E_j we associate the vertex $3j + 1$. Since $\gcd(n, 3) = 1$, the digraph R is partitioned into pairwise vertex-disjoint dicycles. If there is only one dicycle in this partition, we are done as it is Hamiltonian. Otherwise, we use interchange pairs to merge successively the dicycles till we have only one. But we have to be careful to do independent interchanges.

Figure A.4 shows an example for the case $n = 8$, where we obtain the 4 dicycles: $C_0 = (0, E_1, 4, E_5)$, $C_1 = (1, E_4, 5, E_0)$, $C_2 = (2, E_7, 6, E_3)$, $C_3 = (3, E_2, 7, E_6)$.

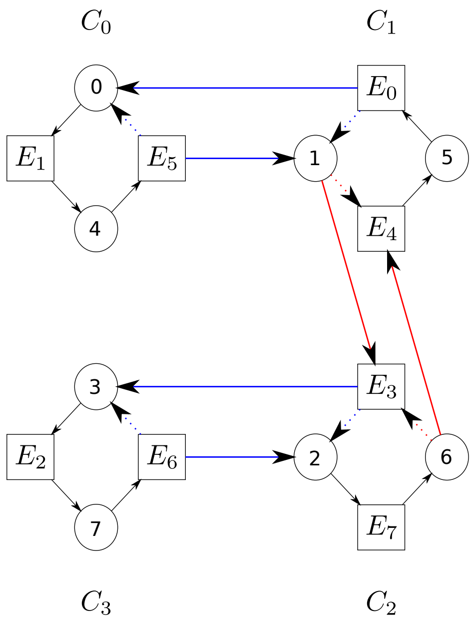


Figure A.4: An application of the link-interchange method to the de Bruijn dihypergraph $GBH(3, 8, 3, 8)$.

We first claim that, if i and $i+1$ belong to two disjoint dicycles C_1 and C_2, then $\{i, i+1\}$ is an interchange pair. Indeed, let E_j be the predecessor of i in C_1 and $E_{j'}$ the predecessor of $i+1$ in C_2. By construction, $3j+1 = i$ and $3j'+1 = i+1$. Consequently, $3j+2 = i+1$ and so there is an arc from E_j to $i+1$. We also have $3j' = i$ and so there is an arc from E'_j to i. Therefore, the claim is proved. For $n = 8$, $\{0, 1\}$ is an interchange pair and we can merge C_0 and C_1 by deleting the arcs $(E_5, 0)$ and $(E_0, 1)$ (in dashed blue in Figure A.4) and adding the arcs $(E_5, 1)$ and $(E_0, 0)$ (in blue).

Similarly, we have that if E_j and E_{j+1} belong to two disjoint dicycles then $\{E_j, E_{j+1}\}$ is an interchange pair. We have to be careful not to use twice the same vertex in an interchange pair, as the predecessor has changed when doing the first merging. Here we will use only some interchange pairs of the form $\{2i, 2i+1\}$ and $\{E_{2j+1}, E_{2j+2}\}$, which are pairwise independent because n is even.

We proceed as follows: if there exists an i such that $2i$ and $2i+1$ belong to different dicycles we merge these dicycles using the interchange pair $\{2i, 2i+1\}$. After at most $n/2$ merge operations, we get a set of disjoint dicycles such that, for all i, $2i$ and $2i+1$ belong to the same dicycle. In the example for $n = 8$, we merge C_0 and C_1 using the interchange pair $\{0, 1\}$ and C_2 and C_3 using the interchange pair $\{2, 3\}$ (see Figure A.4). We now have 2 dicycles.

Then, consider two vertices of the form $2i$ and $2(i+3^{-1})$. Suppose that they belong to two different dicycles C_1 and C_2. The vertex $2i+1$, which is also in C_1 precedes the hyperarc E_{6i+4} in C_1 and the vertex $2(i+3^{-1})$ precedes E_{6i+3} in C_2. Moreover, we claim that $\{E_{6i+3}, E_{6i+4}\}$ is an admissible interchange pair that we can use to merge the two dicycles, because $6i+3$ is odd whereas n is even, and so, $6i+3 \pmod{n}$ is odd. Finally, since 3 and n are relatively prime, 3^{-1} is a generator element in $\mathbb{Z}_n$ and so we can consider successively the possible i such that $2i$ and $2(i+3^{-1})$ belong to two different dicycles and merge all the dicycles.

Observe that for the example in Figure A.4, when $n=8$, we have that $3^{-1}=3$. We now use the construction for $i=0$. Vertices 0 and 6 are in two different dicycles, and $\{E_3, E_4\}$ is an admissible interchange pair. So we can merge the two dicycles by deleting the arcs $(6, E_3)$ and $(1, E_0)$ (in dashed red in Figure A.4) and adding the arcs $(1, E_3)$ and $(6, E_4)$ (in red) to get the final complete Berge dicycle $C = 0, E_1, 4, E_5, 1, E_3, 3, E_2, 7, E_6, 2, E_7, 6, E_4, 5, E_0, 0$. □

A.6.7 Complete Berge dicycles in Kautz Dihypergraphs

The Kautz dihypergraph $GKH(d,n,d,n)$ is close to the dihypergraph $GBH(d,n,d,n)$, but the existence of complete Berge dicycles in it is much harder to prove due to its asymmetry. Indeed, the relations of incidence from its vertices to the hyperarcs are not the same as the relations of incidence from its hyperarcs to the vertices.

Nonetheless, we have been able to show the existence of complete Berge dicycles in $GKH(d,n,d,n)$ for some particular values of (d,n). Remark that $R(GKH(d,n,d,n))$ is isomorphic to the bipartite digraph $BD(d,n)$ (see [GPP98]). The proof of the following theorem uses the same tools as for $GBH(d,n,d,n)$ and can be found in [Duc12]

Theorem 49. *Let $H = GKH(d,n,d,n)$ be a Kautz dihypergraph. There is a complete Berge dicycle in H if one of the following conditions is verified:*

1. *$d \geq 4$;*
2. *$d = 3$ and n is even;*
3. *$d = 2$ and n is even or n is a power of 5 (otherwise it does not exist);*
4. *$d = 1$ and $n \in \{1, 2\}$ (otherwise it does not exist).*

We also have the following conjecture concerning complete Berge dicycles in $GKH(d,n,d,n)$:

Conjecture 5. *Let $H = GKH(d,n,d,n)$ be a Kautz dihypergraph. If $d \geq 3$, then there is a complete Berge dicycle in H.*

A.7 Conclusions

In this chapter, we showed that it is an NP-complete problem to decide whether a dihypergraph is Eulerian (or Hamiltonian). We presented a generalization of some results concerning Eulerian digraphs, in the case where the studied dihypergraphs are *uniform* and *regular*. Then, we studied the Eulerian and Hamiltonian properties of generalized de Bruijn and Kautz dihypergraphs.

We let as open questions Conjectures 4 and 5. It would also be nice to find a relationship between Theorems 38 and 46, since both have similar conditions and different implications.

Appendix B

Distributed Storage Systems

Contents

Distributed or Peer-to-Peer (P2P) storage systems are foreseen as a highly reliable and scalable solution to store digital information [KBC+00, BDET00, BTC+04, CDH+06]. The principle of P2P storage systems is to add redundancy to the data and to spread it onto peers in a network.

There are two classic ways to introduce redundancy: basic replication and erasure codes [Rab89], like the traditional Reed-Solomon (RS) [RS60]. Many studies compare the reliability of replication against erasure codes [WK02, RL05, LCL04]. Erasure codes use less additional storage space to obtain the same reliability as replication. On the other hand, replication has the advantage of having no coding/decoding time, of having an easier and faster access to data, and of being adequate in the presence of high churn.

Furthermore, the reconstruction process of RS systems is costly. In the RS system, the data is divided into small fragments that are sent to different peers. When a fragment of redundancy is lost, the whole original data has to be retrieved to regenerate it. At the opposite, in a system using replication, a repair is done by simply sending again the lost data.

In order to spend less bandwidth in the reconstruction process, the Regenerating Codes were proposed in a recent work [DGWR07] as an improvement of the traditional erasure codes. In this coding scheme, the peers that participate of the reconstruction process send, instead of fragments of the data they have, linear combinations of subfragments of the fragments that they keep, in such a way that total transferred data to the newcomer peer is smaller than the original data. This is possible thanks to previous results on Network Coding [ACyRL+00].

In this work, we investigate in detail the use of two hybrid strategies. The first one is usually called Hybrid Coding and was introduced and studied in [RL05] and [DGWR07], respectively. This strategy combines the use of both replication and coding. It tries to get the best of both worlds: the storage efficiency of RS and the repair efficiency of replication. The idea is to keep one full-replica of the data in one peer along with erasure coded fragments spread in the network.

We also propose a new strategy that we name Double Coding in which we improve the idea of Hybrid Coding. Instead of keeping the full-replica of the data in only one peer of the network, we place a copy of each fragment (including the redundant ones) in different peers in the network.

Here we compare Hybrid Coding and Double Coding with RS systems and Regenerating Codes. We study the bandwidth usage of these systems by considering the availability of the peers under the presence of churn, the data durability and the storage space usage. We show that both hybrid strategies perform better than traditional RS systems and that Double Coding is a good option for system developers since it is simple to implement in practice and can perform close to Regenerating Codes in terms of bandwidth usage.

Related Work.

P2P and large scale distributed storage systems have been analyzed by using Markov chains: for erasure codes in [ADN07, DA06, DGMP09] and for replication in [RP06, CDH+06]. In this work, we model Hybrid Coding, Double Coding and Regenerating Codes with Markovian models. We also introduce a new chain for RS systems that models the failure of the reconstructor during a repair.

Rodrigues and Liskov in [RL05] compare the Hybrid system versus replication in P2P Distributed Hash Tables (DHTs). However, there are no comparisons of the Hybrid system against the traditional erasure codes. Dimakis et al. [DGWR07] study the efficiency of bandwidth consumption for different redundancy schemes, among them the Hybrid Coding. They state that the Hybrid Coding has a better availability/bandwidth trade-off than the traditional erasure codes. Both of these works focus on availability and *they do not consider the durability of the data*. They also do not take into account the time to process the reconstructions. By using Markov chains, we exhibit the impact of this parameter on the average system metrics. Furthermore, they only consider RS using an *eager repair* policy, which is highly inefficient for the bandwidth. In [BTC+04], the authors propose the *lazy repair* mechanism to decrease the bandwidth usage in the reconstruction process. Here, we thus compare Hybrid Coding and an RS system using lazy repair.

In [DA06], Datta and Aberer study analytical models for different lazy repair strategies in order to improve the bandwidth usage under churn. In our work, we employ the lazy repair to minimize the extra-cost in bandwidth even in a system with high availability of peers.

Regenerating Codes [DGWR07] is a promising strategy to reduce the bandwidth usage of the reconstruction of the lost data. There are some studies about these codes like in [RSKR09], [DRW09], [DB09] and [DRWS10]. However, as far as we know, there is no study of the impact of the reconstruction time in these codes. Most of the results in the literature consider only simultaneous failures. In this work, we introduce a Markovian Model to study the impact of the reconstruction time in Regenerating Codes.

Our Contributions.

- We study the *availability and durability of Hybrid systems*. We compare Hybrid solution with RS system and RC systems.
- We propose a new kind of Hybrid codes, that we refer to as *Double coding*. This new code is more efficient than the Hybrid one. Its performance is close to the one of Regenerating Codes in some cases. Furthermore, explicit deterministic constructions of RC are not known for all sets of parameters. *Double Codes* is then an interesting alternative in this case.
- We model these systems by using Markov chains (Section B.2). We derive from these models the *system loss rates* and the *estimated bandwidth usage*. These chains take into account the *reconstruction time* and the more efficient *lazy repair*.
- We analyze different scenarios (Section B.3):
  - When storage is the scarce resource, RS system has a higher durability.
  - When bandwidth is the scarce resource, the Hybrid solution is a better option.
- We compare systems for three metrics durability, availability and bandwidth usage for a given storage space, when other studies focus on only two parameters.

In Section B.1 we present in detail the studied systems. In the following section we describe the Markov Chain Models used to model these systems. Finally, in Section B.3, these systems are compared by an analysis of some estimations on the Markovian models.

B.1 Description

In distributed storage systems using Reed-Solomon (RS) *erasure codes*, each block of data b is divided into s fragments. Then, r fragments of redundancy are added to b in such a way that any subset of s fragments from the $s + r$ fragments suffice to reconstruct the whole information of b. These $s + r$ fragments are then stored in different peers of a network. Observe that, the case $s = 1$ corresponds to the simple *replication*. The codes studied in this work are depicted in Figure B.1.

For comparison, we also study *ideal erasure codes* in which there would also be s original fragments and r redundancy fragments spread in the network, but it would be possible to reconstruct a lost fragment by just sending another fragment of information.

The *Hybrid system* is simply a Reed-Solomon erasure code in which one of the $s+r$ peers stores, besides one of the original s fragments of a block b, also a copy of all the other original fragments. This special peer which contains a full copy of b, namely *full-replica*, is denoted by $p_c(b)$.

Following the idea of the Hybrid system, we propose the *Double Coding* strategy. In Double Coding, each of the $s+r$ fragments has a copy in the network. However, differently from the Hybrid approach, we propose to put the copies of the fragments in different peers of the network, instead of concentrating them in a single peer. Consequently, we need twice the storage space of a Reed-Solomon erasure code and also $2(s+r)$ peers in the network. We show, in Section B.3, that Double Coding performs much better than RS systems in terms of bandwidth usage and probability to lose data and this disadvantage on storage space is worthy.

Finally, in the Regenerating Codes the original data is also divided into $s+r$ fragments and the fragments are also spread into different peers of a network. However, the size of a fragment in these codes depend on two parameters: the *piece expansion index* i and the repair degree d, as explained in [DB09]. These parameters are integer values such that $0 \leq i \leq s-1$ and $s \leq d \leq s+r-1$. Given these parameters, the size of a fragment in a Regenerating Code with parameters (s, r, i, d) is equal to $p(d,i)s$ where

$$p(d,i) = \frac{2(d-s+i+1)}{2s(d-s+1)+i(2s-i-1)}.$$

The repair degree d is the number of peers that are *required* to reconstruct a lost fragment. This parameter also impacts the required bandwidth usage to repair a fragment that was lost as we discuss in the next section.

B.1.1 Reconstruction Process

To ensure fault tolerance, storage systems must have a maintenance layer that keeps enough available redundancy fragments for each block b. In this section, we describe how the lost fragments must be repaired by this maintenance layer in each system.

Reed-Solomon. As stated before, in a Reed-Solomon system the reconstructor $p(b)$ of a block b must download s fragments in the system, in order to rebuild b, before sending the missing fragments to new peers. Most of the works in the literature consider only the case of the *eager reconstruction*, i.e., as soon as a fragment of data is lost the reconstruction process must start. This is highly inefficient in terms of bandwidth usage, because, in most of the cases, s fragments are sent in the network in order to rebuild only one lost fragment.

Here, we assume that the reconstruction process in a RS system uses the *lazy repair* strategy [DA06], which can be much more efficient in terms of bandwidth usage. Given a threshold $0 \leq r_0 < r$, the reconstruction process starts only when the number of fragments of b is less than or equal to $s+r_0$. Observe that the case $r_0 = r-1$ corresponds to the eager reconstruction. Recall that decreasing the value

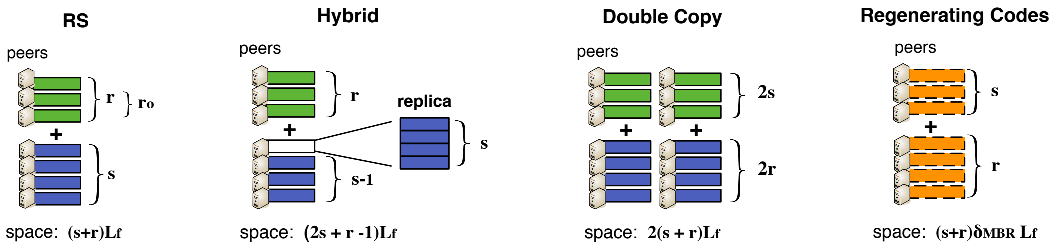


Figure B.1: Description of the redundancy schemes.

of r_0 correspond to increase the probability to lose the block, i.e., to lose at least $r+1$ fragments.

When the reconstruction starts, a peer $p(b)$ is chosen to be the *reconstructor*. Note that, when reconstructing the missing redundancy of b, the peer $p(b)$ possesses a full-replica of the block which is discarded afterwards.

Hybrid Coding. In the Hybrid system, recall that $p_c(b)$ is the peer that contains a full-replica of the block b, hence for each block there are $2s+r-1$ fragments present in the system. When there is a failure, if the peer $p_c(b)$ is still alive, it generates the lost fragments from its full-replica. It then sends the missing fragments to different peers in the network. To be able to do that, the peer only needs to store the initial block or, equivalently, s fragments. As a matter of fact, it can quickly create the other fragments at will.

When the peer $p_c(b)$ fails, a new peer is chosen to maintain the full-replica. In this case, the whole block needs to be reconstructed. This is accomplished by using the traditional Reed-Solomon process, with the addition that the reconstructor keeps a full-replica of the block at the end of the process. From that we see that a Hybrid system can be easily built in practice from an RS system.

Double Coding. Recall that in the Double Coding, for each block there are $2(s+r)$ fragments present in the system. An interesting property of Double Coding is that it keeps the idea of Hybrid Coding, because when a fragment f is lost it is just necessary to ask the peer that contains the other copy of f to send a copy of it to another peer in the network.

Moreover, we can just say that a fragment f is lost in the system if its two copies are lost. In this case, it is necessary to use the Reed-Solomon reconstruction to rebuild at least one of the copies of f. Since this is an expensive process in terms of bandwidth usage, we also adopt a threshold value $0 \leq r_0 < r$ to let this process more efficient. When $r-r_0$ pairs of the same fragments are lost, a peer $p(b)$ is chosen to be the responsible for downloading s disjoint fragments of the system, rebuilding the block b and the r redundant fragments and resending *only* the *first* copies of the fragments that have lost both of their copies. Then, the *second* copies are sent by the peers that contain the first one.

Regenerating Codes. In these codes there is not the figure of the reconstructor. When a fragment f is lost, a peer that is usually called *newcomer* is in charge of downloading linear combinations of subfragments of the block from exactly d peers in the network in order to replace f.

The amount of information that the newcomer needs to download is equal to $d.\delta(d, i).s$ where

$$\delta(d, i) = \frac{2}{2s(d - s + 1) + i(2s - i - 1)}.$$

Recall that d peers are required in the reconstruction process. If there are no d peers available in the beginning of the reconstruction process, but there are still s peers on-line, the reconstruction can be still processed by downloading s complete fragments and reconstructing the original information of b as it happens in a RS system.

There are two special cases of Regenerating Codes: the Minimum Bandwidth Regenerating (MBR) codes and the Minimum Storage Regenerating (MSR) codes. The MBR codes correspond to the case in which $i = s - 1$ and in the MSR ones $i = 0$.

Since the most expensive resource in a network is arguably the bandwidth we use the MBR Regenerating Codes. Observe that these systems have a storage overhead factor δ of $\frac{2d}{2d-s+1}$. That is, each block has $s + r$ fragments, as the RS system, but these fragments are bigger by a overhead factor δ.

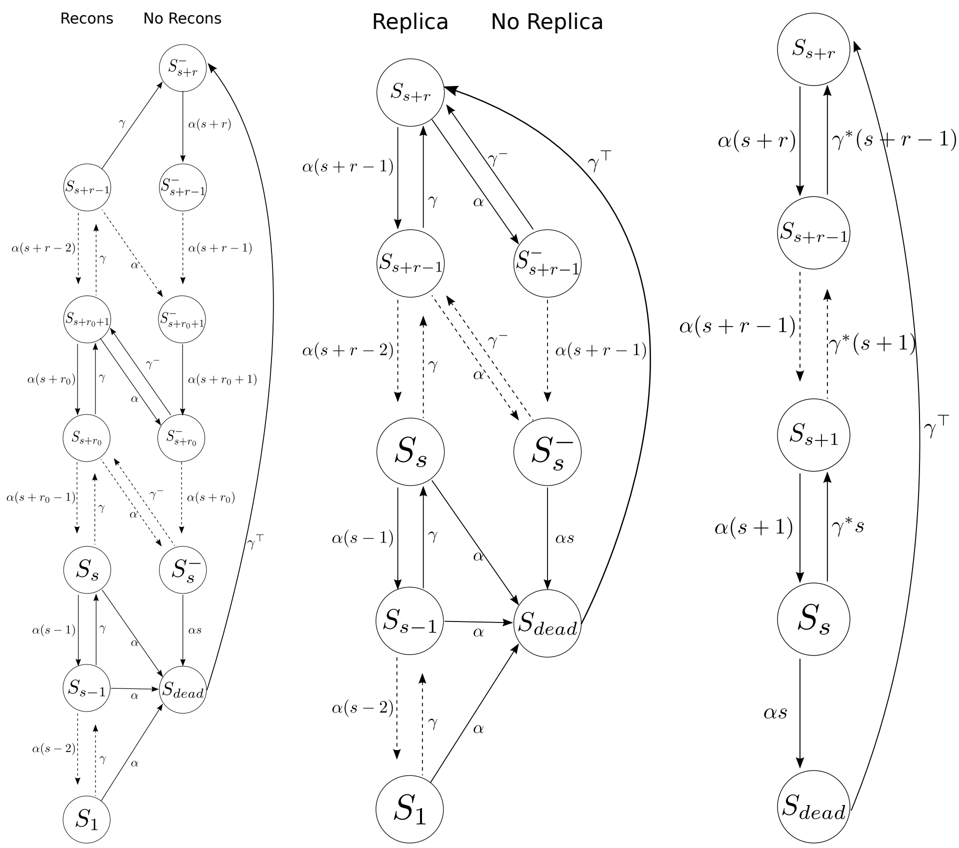
In the following section, we present the Markov Chain Models that we use to study the bandwidth usage and the durability of each system.

B.2 Markov Chain Models

We model the behavior of a block of data in all the cited systems by Continuous Time Markov Chains (CTMCs). From the stability equations of these chains, we derive the bandwidth usage and the system durability.

Model of the Reed-Solomon System. We model the behavior of a block b in a lazy RS system by a CTMC, depicted in Figure B.2(a). We did not use the chains classically used in the literature [ADN07, DA06]. Our chain models the possible loss of the reconstructor $p(b)$ during a reconstruction. In brief, the states of the chain are grouped into two columns. The level in a column represents the number of Reed-Solomon fragments present in the system. The column codes the presence of the reconstructor $p(b)$: present for the left states and absent for the right ones.

Model of the Hybrid System. In Figure B.2(b), it is presented the Markov chain that models the behavior of a block b in the Hybrid system. Recall that, in a Hybrid system, $s + r$ Reed-Solomon fragments and one replica are present inside the system. We draw our inspiration from the chain representing the RS system. We code here the presence of the peer $p_c(b)$ in the system, in a similar way to how we code the presence of the reconstructor $p(b)$ in the RS system.



(a) Reed-Solomon. (b) Hybrid system. (c) Regenerating Codes.

Figure B.2: Markov Chain models for different codes.

Figure B.3: Summary of the notations

s	Number of initial fragments
r	Number of redundancy fragments
r_0	Reconstruction threshold
α	Peer failure rate
MTTF	Mean Time To Failure: $1/\alpha$
a	Peer availability rate
d	Number of available peers to reconstruct (RC)
θ	Average time to send one fragment
γ	Fragment reconstruction rate in Hybrid approaches: $\gamma = 1/\theta$
θ^-	Average time to retrieve the whole block
γ^-	Block reconstruction rate: $\gamma^- = 1/\theta^-$
θ^*	Average time to retrieve a d subfragments in RC
γ^*	Fragment reconstruction rate in RC: $\gamma^* = 1/\theta^*$
$\theta^\top$	Average time to reinsert a dead block in the system
$\gamma^\top$	Dead block reinsertion rate

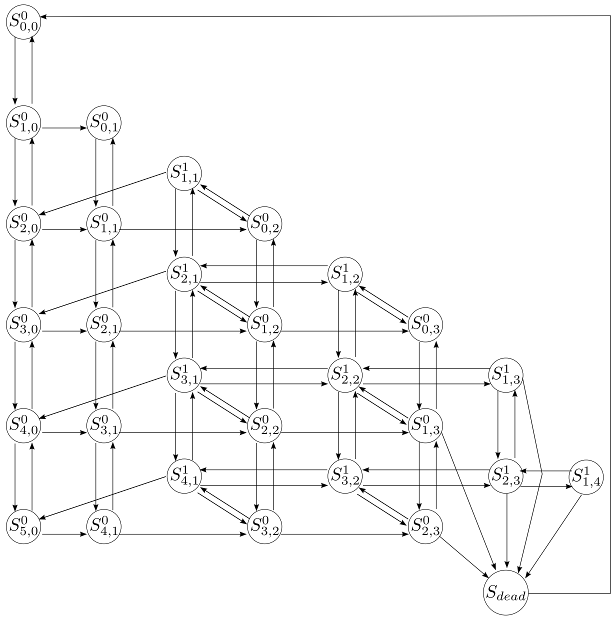


Figure B.4: Markov Chain for Double Coding system for $s = 2$, $r = 3$ and $r_0 = 0$.

Model of the Double Coded System. We also model the behavior of a block in this system by a continuous-time Markov chain (see Figure B.4) to estimate the loss rate of a block and the expected bandwidth usage in the steady state of the system.

Model of Regenerating Codes. Basically, the only difference between the Markov chain that we used to model the RS system and the one that we introduce in this section for Regenerating Codes (see Figure B.2(c)) is that in RC-based systems, we do not have the reconstructor. When a fragment is lost, the newcomer will just download linear combinations of subfragments of the other peers that are present in the system.

Model of Ideal Codes. For the ideal system, the chain is similar to the one that we present for Regenerating Codes, only the estimation of bandwidth usage is different.

B.3 Results

We now use the Markov chains presented in Section B.2 to compare the systems we described from the point of view of *data availability, durability and loss rate*.

The bandwidth usage and loss rate plots are estimations from the chains. To estimate the bandwidth usage, we just observe, in the steady state of the chain, the rate that some data in the reconstruction process is transferred times the amount of transferred data. The loss rate is simply the probability to be in the *dead* state in the stationary distribution.

In Subsections B.3.1, B.3.2 and B.3.3, the plots concerning Regenerating Codes (RC) are estimations taken from the chain where the bandwidth usage is calculated in an *optimal* way, i.e., the estimation considers that the system is a MBR code and, moreover, *all the available peers participate of the reconstruction process.*

Value of the parameters. In the following experiments, we use a set of default parameters for the sake of consistency (except when explicitly stated). We study a system with $N = 10000$ peers. Each of them contributes with $d = 64$ GB of data (total of 640 TB). We choose a system block size of $L_b = 4$ MB, $s = 16$, giving $L_f = L_b/s = 256$ KB. The system wide number of blocks is then $B = 1.6{\cdot}10^8$. The $MTTF$ of peers is set to one year. The disk failure rate follows as $\alpha = 1/MTTF$. The block average reconstruction time is $\theta = \theta^- = \theta^* = \theta^\top = 12$ hours.

Except in the first studied scenario, the availability rate a is chosen to be 0.91 which is exactly the one of PlanetLab [DGWR07].

B.3.1 Systems with same Availability

The first scenario we study is the one we compare the bandwidth usage and the loss rate of the described systems when they have approximately the same availability. Since Ideal, RS and RC systems have the same formula to estimate the availability of each system, they are taken as basis to the hybrid approaches.

In this experiment, we keep the value s constant for all the systems and we increase the availability rate a. For each value of a, we compute the availability for Ideal, RS and RC and, then, we find the value of r for Hybrid coding and also for Double coding that provides the closest value of availability to the one found to Ideal, RS and RC. This experiment provides the results in Figure B.5.

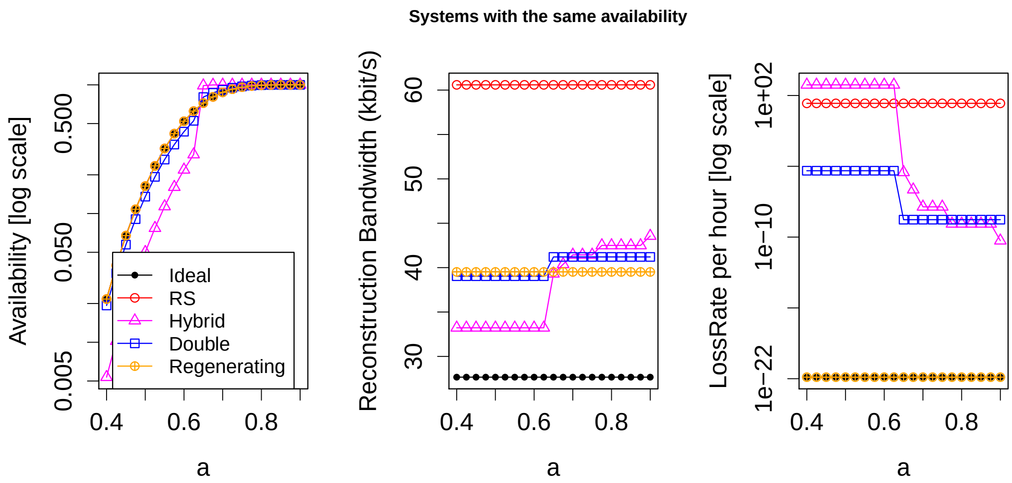


Figure B.5: Systems with same availability.

Since the RS system uses much more bandwidth than the others studied systems, we choose $r_0 = 1$ to provide a lower bandwidth usage. However, one may observe the impact of this choice in the loss rate of this RS system. The Double coding plot has the eager reconstruction strategy, i.e., $r_0 = r - 1$.

Recall that these systems do not use the same storage space, as explained in Section B.1. Observe that the hybrid approaches perform as good as regenerating codes in this case. However, the system loss rate is smaller in the regenerating codes.

B.3.2 Systems with same Durability

In the following experiment, we increase the value of r of an RC system with $s = 16$ and, for each value, the estimation of the system loss rate is taken as a parameter for the others systems.

Given the system loss rate of the RS system, for each other system, the best value of r is considered in order to plot the values of availability and bandwidth usage, i.e., the value of r whose loss rate estimation is the closest to the one of the regenerating code.

In Figure B.6, RS and Double coding are both considered to be in the eager case, i.e., $r_0 = r - 1$.

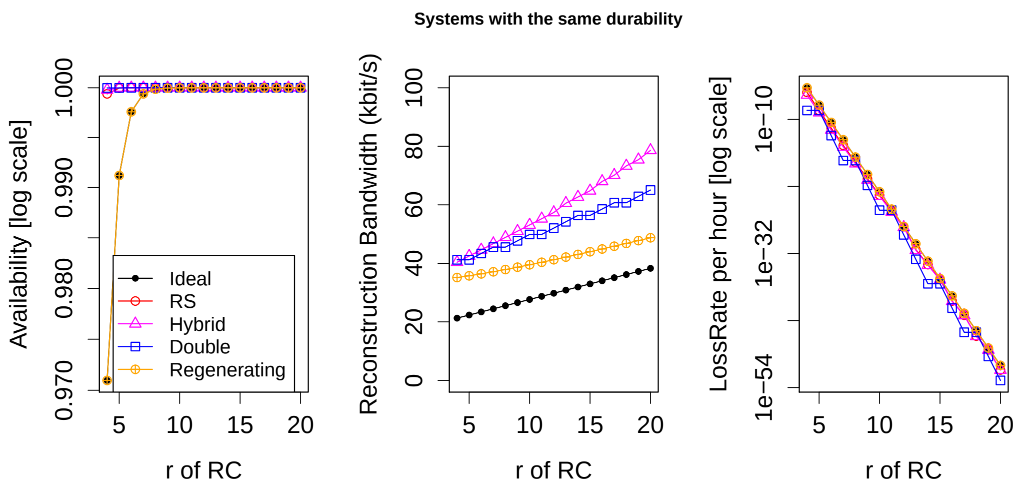


Figure B.6: Systems with same durability.

First, remark that the bandwidth curve of the RS system is not present in the plot since, as commented before, the bandwidth usage in the eager case is much bigger than the bandwidth used by the other systems.

Again, we observe that the hybrid strategies perform well in terms of bandwidth usage when the compared systems have approximately the same loss rate. Recall that these systems do not use the same storage space.

B.3.3 Systems with same Storage Space

Finally, we compare all the systems when they use the same storage space. The RS system is taken as reference and, then, the redundancy of the others systems is set to use only the space of r fragments of the RS system. Recall that the encoded fragments of regenerating codes are bigger than the RS according to the function presented in Section B.1. Consequently, even the regenerating codes have less redundancy fragments in this experiment, when compared with the redundancy of the RS system.

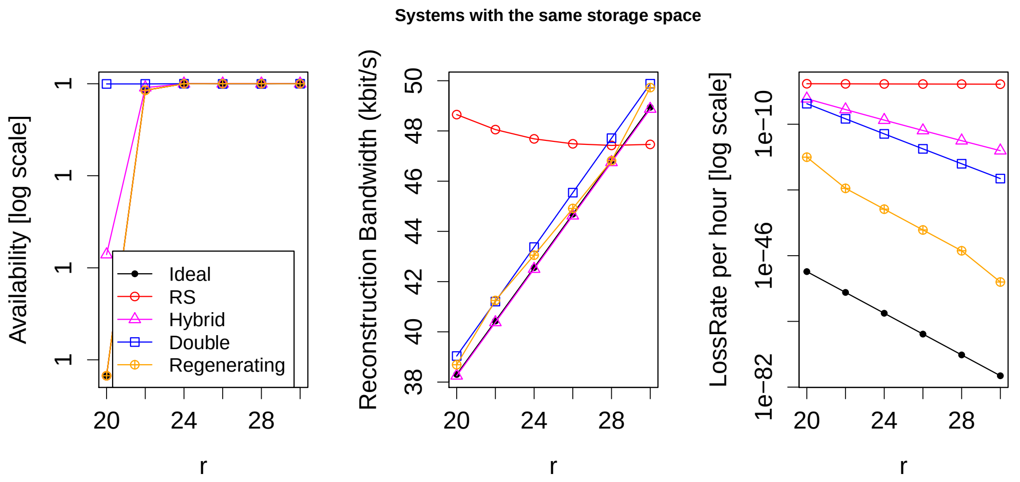


Figure B.7: Systems with same storage space.

The considered RS system has $r_0 = 1$ in order to let the plot of bandwidth usage in the same scale, since the eager policy performs much worse. Again, observe that the system loss rate is affected by this choice.

Another important remark is that even for systems with the same storage space, the hybrid approaches perform as well as the regenerating codes.

Remember that the last three experiments are based in *Optimal* RC systems, where all the available peers participate of the reconstruction process.

B.4 Conclusions

We studied the *availability and durability of Hybrid systems.* We proposed a new kind of Hybrid codes, namely *Double coding.* Then, we compared Hybrid solutions with Reed-Solomon and Regenerating Codes systems.

We modeled these systems by using Markov chains and derived from these models the *system loss rates* and the *estimated bandwidth usage.* Differently from other studies, these chains take into account the *reconstruction time* of a data-block and the use of the more efficient *lazy repair* procedure. We compared these systems for three metrics: durability, availability and bandwidth usage for a given storage space, when other studies focus on only two parameters. We analyzed different scenarios: when the scarce resource is the storage space or when it is bandwidth.

Double Coding is most of the time more efficient than the Hybrid one. Its performance is close to the one of the best theoretical Regenerating Codes in some scenarios. If Reed-Solomon systems have a higher durability when bandwidth is not limited, Double Coding is a better option when it is a scarse resource.

Index of Definitions

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