Low-Order Controllers for Time-Delay Systems : an Analytical Approach

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Low-Order Controllers for Time-Delay Systems. An Analytical Approach

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Résumé

Les travaux de recherche présentés dans cette thèse concernent des contributions à l’étude de stabilité des systèmes linéaires à retards avec contrôleurs d’ordre réduit. Cette mémoire est partagée en trois parties.

La première partie est axée sur l’étude des systèmes linéaires à retard mono-entré /mono-sortie, bouclées avec un contrôleur de type PID. Inspiré par l’approche géométrique développée par Gu et al. Nous avons proposé une méthode analytique pour trouver la région (ou les régions) de tous les contrôleurs de type PID stabilisant pour le système à retard. Basée sur cette même approche, on a développé un algorithme pour calculer le dégrée de fragilité d’un contrôleur donné de type PID (PI, PD et PID).

La deuxième partie de la thèse est axée sur l’étude de stabilité sous une approche NCS (pour son acronyme en anglais : Networked Control System). Plus précisément, nous avons d’abord étudié le problème de la stabilisation en tenant compte des retards induit par le réseau et les effets induits par la période d’échantillonnages. Pour mener une telle analyse nous avons adopté une approche basée sur la théorie des perturbations.

Finalement, dans la troisième partie de la thèse nous abordons certains problèmes concernant le comportement des zéros d’une certaine classe de systèmes échantillonnés mono-entré /mono-sortie. Plus précisément, étant donné un système à temps continu, on obtient les intervalles d’échantillonnage garantissant l’invariance du nombre de zéros instables dans chaque intervalle. Pour développer cette analyse, nous adoptons une approche basée sur la perturbation aux valeurs propres.

Mots-clefs : stabilité, systèmes linéaires a retards, théorie des perturbations, séries de Puiseux, D-partition, faisceaux matriciels.

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Contents

Introduction

In many physical systems, the rate of variation in the system's state depends not only of its current value, but also of its past history. Systems possessing such a characteristic are called *time-delay* systems (sometimes also called *hereditary* systems, systems with *time-lag* or systems with *aftereffect*). Delay dynamical systems are abundant in nature, and it is worth mentioning that they occur in a wide variety of physical, chemical, engineering, economic and biological systems. In fact, to the best of the author's knowledge, time-delay systems was first introduced in order to describe the behavior of some biological systems and were later found in many engineering systems, such as mechanical transmissions, fluid transmissions, manufacturing processes, transmission lines in pneumatic systems, nuclear reactors, among others (see, for instance, [32], [39], [72], [133], [120], for further details). As a consequence, the problem of *stability analysis* and *control* of time-delay systems has attracted much attention, and considerable effort has been done to different aspects of linear time-delay systems during the last decades [48], [92], [103], [107].

Basic theories describing time-delay systems properties were established in the 1950s and 1960s; they developed topics such as the existence and uniqueness of solutions of the corresponding dynamic equations, stability methods to study the behavior of the trivial solutions, continuity of the characteristic roots with respect to some of the system's parameters, etc. Such works established the foundation for the later analysis and design of controllers for time-delay systems.

The study and design of physical systems can be carried out using empirical methods. In this vein, we can apply several kinds of signals to a physical system and measure its responses. If the performance is not satisfactory, then we can adjust some of the parameters, or connect a compensator to it in order to improve its behavior. However, if the system under study is complex (like nonlinear including or not time-delays, etc.), "expensive", "dangerous" or if it is too important (like systems modeling life science), then experimental methods will became unworkable and the analytical methods will become indispensable.

Even though, there exist several advanced controller design methods as, for example, the finite spectrum assignment, $H_\infty-$synthesis, μ and linear matrix inequalities based synthesis methods for time-delay systems, these methodologies produce generally controllers with orders comparable with order of the system. Therefore, the orders of these controllers tend often to be too high to allow the practical applicability for high-order systems and, as a consequence, simple controllers are often preferred over complex ones. In this context, *low-order controllers* play a relevant role, not only from the practical but also from theoretical point of view.

Among the most popular low-order controllers, we may cite PID-type (P, PI, PD and PID), such controllers have the following mathematical representation in frequency domain:

$$\begin{cases} P-\text{controller:} & C(s) = k_p; \\ PI-\text{controller:} & C(s) = k_p + k_i/s; \\ PD-\text{controller:} & C(s) = k_p + k_d s; \\ PID-\text{controller:} & C(s) = k_p + k_i/s + k_d s, \end{cases}$$

where k_p, k_i and k_d are the proportional, integral and derivative gains, respectively. Generically, PID-type controllers are implemented by feedback, this implies that the controller output signal is calculated by taking into account the available output plant measurements. In this vein, a typical single loop containing a PID-type controller and a linear time-delay SISO (single-input, single-output) system $G(s) = \frac{A(s)}{B(s)}e^{-s\tau}$ is shown in Fig.0.1.

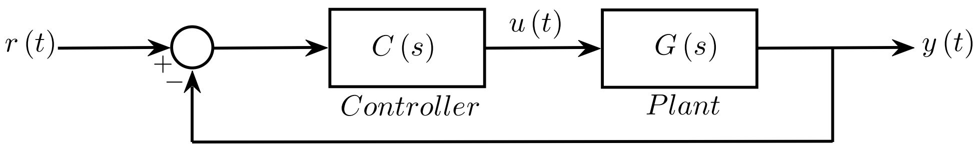


Figure 0.1: Typical feedback control system.

The popularity of the PID-type controllers can be attributed to its relative *simple structure*, which can be easily understood and implemented in practice, and that such controllers have "sufficient" ability of solving many practical control problems (see, for instance, [5], [27], [123], for further details). Surprisingly, despite the "popularity" of PID controllers, the problem of *finding the all set of stabilizing PID controllers* in the (k_p, k_d, k_i) parameters space is still of interest and it becomes quite complicated in the presence of delays in the loop even for the simplest system's structure. In order to illustrate the benefits of such knowledge, consider now the following simple example.

Example 0.1. *Consider the PID stabilization problem of the following plant*

$$G(s) = \frac{s^3 - 4s^2 + s + 2}{s^5 + 8s^4 + 32s^3 + 46s^2 + 46s + 17}e^{-s}.$$

This system has the following stabilizing set of (k_p, k_d, k_i) *parameters (see, Example 7.1, for complete details).*

Taking several controllers within the above stabilizing region, we have the following step response curves.

Then, using the information obtained from the knowledge of the complete stabilizing set of PID controller, could be extremely useful in the design of controllers that must satisfy some requirements for the system performance.

In addition to the previous remarks, the fact that PID controllers have only three tuning parameters, the parameter-space approach have captured the attention of several researchers and, as a consequence, there exists an important amount of results dealing with the analysis of PID-type controllers. In fact, it is worth to mention that there exist several results related with the calculation of the set of stabilizing PID controllers. In delay free case [27, 51], a generalization of the Hermite-Biehler theorem was derived and then used for a given LTI (linear

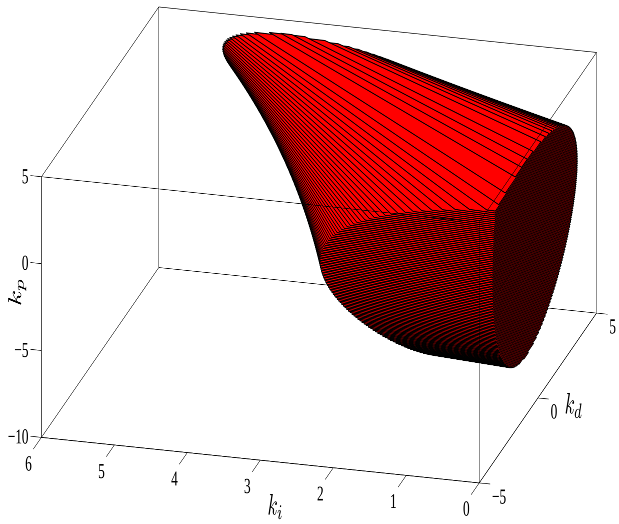


Figure 0.2: Stabilizing set of (k_p, k_d, k_i).

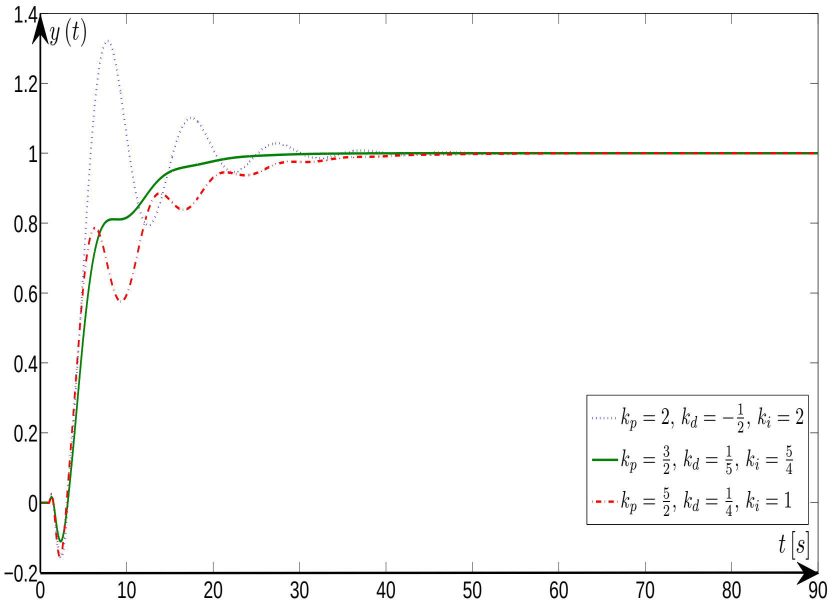


Figure 0.3: Step response curves corresponding to different PID-controllers.

time-invariant) plant. In the delay case, in order to find the set of all stabilizing PID controllers a generalization of the Hermite-Biehler theorem is applied. Under this approach, the works of [112, 110] found necessary and sufficient condition to find the all set of stabilizing PID controllers, however the analysis requires an additional grid on a frequency variable "ω". On the other hand, by using the Neimark [101] decomposition method [79, 80] ($\mathcal{D}-$decomposition) under the assumption that $k_d = k_i/\eta$ (where $\eta \in \mathbb{R}$) derived the set of stabilizing PID controller, under this assumption, clearly the problem was not completely solved. Based on the same approach but without imposing the above restriction, the works of [8, 54] derived the set of stabilizing PID controllers and also they find out that the exact stable region can be described by a finite number of boundaries if the relative degree of the system is larger than 2, yielding to a set of convex polygons as the stable region in the $(k_d, k_i)-$plane.

Among the long list of problems when using PID controllers, we will mention one that, in our opinion, is quite important: the *perfect* knowledge of the gains when implementing the control. This assumption is in some extend valid, since, clearly, the plant uncertainty is the most significant source of uncertainty in the control system, whilst controller are implemented with high-precision hardware. However, there will inevitably be some amount of uncertainty in the controller, a fact that sometimes is ignored in robust and optimal control design [69] (see also, [68], [81], for further comments). If the controller will be implemented by analogue means, then, there will be some tolerance in the components. On the other hand, if the controller is implemented digitally, then, there will be some rounding in the controller parameters. Where, for reasons of security, cost and execution speed, the implementation is with fixed point rather than floating point processors, there will be increased uncertainty in the controller parameters due to the finite length and further uncertainty due to the rounding errors in numerical computations.

Based on these remarks, such controllers have to be designed by considering:

(a) *performance* criteria;

(b) *robustness* issues;

(c) *fragility*.

Roughly speaking, a controller for which the closed-loop system is destabilized by small perturbations in the controller parameters is called "*fragile*". In other words, the fragility describes the deterioration of closed-loop stability due to small variations of the controller parameters.

The problem received a lot of attention in delay free systems, see, e.g., [50] (non-fragile PID control design procedure), [3] (appropriate index to measure the fragility of PID controllers). However, there exists only a few results in the delay case: [124], where only (stable) first-order systems were considered, [9], a non-fragile controller for some classes of non-linear system is proposed, and more recently, [88], where the authors proposed a robust non-fragile control design for a TCP/AQM models.

The above problems constitute the core of the first part of the thesis. More precisely, inspired by the *geometric ideas* introduced by Gu *et al.* [40] we propose a simple method to derive the complete set of stabilizing controllers in the (k_p, k_h) parameter space (where the subindex h stand for the integral i or for the derivative d parameter) or in the (k_p, k_d, k_i) parameter space. Once the complete stabilizing set of controllers is defined, the explicit computation of the distance of some point to the closest stability crossing boundaries is presented. In other words, we introduce a *quantitative fragility measure* for the corresponding controller.

Another distinctive feature that is usually related to delay systems, is that systems possessing delays in the feedback loop are often accompanied with instability or "bad" behaviors (as, for examples, oscillations, bandwidth sensitivity), as pointed out by [39], [92] and the references therein. However, there exist also some situations when the delay has a *positive* effect, that is, the delay may *induce* stability. In order to illustrate such a situation, recall the following:

Example 0.2. *[1] Consider the following simple second-order system:*

$$\ddot{y}(t) + \omega_0^2 + y(t) = u(t)\,. \tag{0.1}$$

It is clear to see, that such class of system is stabilizable by the feedback law

$$u(t) = -k\dot{y}(t), \qquad k \in \mathbb{R}_+,$$

moreover, it is easy to see that there does not exist a proportional controller $u(t) = -ky(t)$, $\forall k \in \mathbb{R}$ *able to stabilize the system* (0.1). *However, using instead a positive delayed output feedback,*

$$u(t) = ky(t-\tau), \qquad k \in \mathbb{R}_+,$$

it is possible to stabilize the system asymptotically. In fact, if we chose (k,τ) *satisfying the following inequalities,*

$$\begin{aligned} 0 \leq \quad k \quad &< \frac{1+4n}{1+4n+8n^2}\omega_0^2 \\ \frac{2n\pi}{\sqrt{\omega_0^2 - k}} < \quad \tau \quad &< \frac{(2n+1)\pi}{\sqrt{\omega_0^2 + k}} \end{aligned}$$

where $n \in \mathbb{N} \cup \{0\}$, *then the closed-loop system will be asymptotically stable (see, [1], for further details).*

For $\omega_0 = 1$, *figure 0.4 illustrate the stability regions in the* (k,τ) *parameter space, as well as, the closed-loop behavior for* $(k,\tau) = (\frac{7}{10}, 1)$.

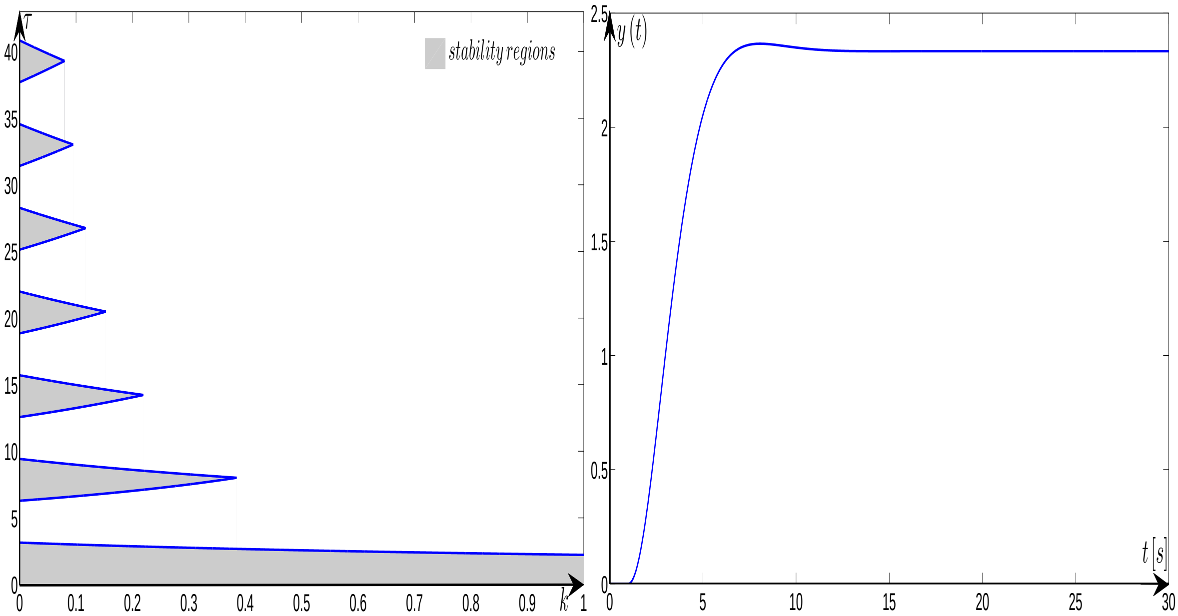


Figure 0.4: (Left) Stability regions in the (k,τ) parameter space. (Right) Step response curve corresponding to $(k,\tau) = (\frac{7}{10}, 1)$.

As pointed out by Michiels & Niculescu [92], the above example opens an interesting perspective in using delays as a *control parameters*. Motivated by this idea, Niculescu & Michiels in their paper [106] solved the problem of stabilizing a chain including n integrators:

$H_{yu}(s) = 1/s^n$ by means of the control law defined by a chain of m distinct delay blocks (k_i, τ_i)

$$u(t) = -\sum_{i=1}^{m} k_i y(t - \tau_i) \qquad k_i \in \mathbb{R},\ \tau_i \in \mathbb{R}_+. \tag{0.2}$$

Furthermore, they have also shown that either $n-$delay blocks or a proportional + $(n-1)-$delay blocks are necessary and sufficient conditions to asymptotically stabilize a chain of n integrators ($n \geq 2$). In addition, they have also shown that these $n-$ delay block are also able to *guarantee* an arbitrary pole placement for the corresponding $n-$rightmost eigenvalues of the closed-loop system.

It is also worth mentioning that there are also other contributions in this direction. For example, Kharitonov *et al.* in [70] gave *necessary conditions* for the existence of a controller with multiple (distinct) delays able to stabilize oscillator systems, Mazenc *et al.* in [87] considered the case of a chain of integrators with bounded input and a single delay.

The aforementioned ideas but in a networked control system framework constitute the core of the second part of the thesis. More precisely, inspired by the results derived by Niculescu & Michiels in [106], we first explore such ideas in studying the problem of stabilization of a chain of integrators by taking into account the network-induced delays and the corresponding sampling period. Next, in the second half of the Part II of the thesis we address the output feedback stabilization problem for a class of linear SISO systems subject to input/output delays. More precisely, we are interested in the characterization of the of delay– gain– and sampling– parameters guaranteeing the stability of the closed-loop system.

Outline

The remaining part of the thesis is organized as follows.

Chapters 1–2 presents the main definitions, preliminary results as well as the main tools that will be consistently used throughout the thesis. More precisely, in Chapter 1 we present the notions of solutions, stability and some existing analytical criteria to verify the stability in the frequency-domain case, such notions are presented first to the continuous-time case and next for the discrete-time case.

Next, Chapter 2 introduce some fundamental results concerning the *perturbation theory of linear operators*. First, eigenvalues are classified according to their characteristics, then several criteria to determine the main coefficients of the series expansion are presented. In order to illustrate how these results can be applied, several illustrative numerical examples have been detailed and complete the presentation.

Chapters 3–8 form the first part of the thesis, where we consider the stabilization problem of a linear time-delay SISO system by means of a PID-type controller as well as the fragility problem of the PID-type controllers. The results presented in this part are collect as a book chapter in a more compact form [97] to be submitted in the forthcoming period.

In Chapter 3, we introduce the basic notations, as well as some general results for a generic controller $h(s, \alpha, \beta)$, which is considered as a transcendental analytical function and where α and β are considered as being the control parameters. The crossing curves in the $(\alpha, \beta)-$parameter space is determined. Next, by means of the *Implicit Function Theorem*

[85], the smoothness property of the crossing curves is proved. Conditions to determine the direction of crossing are also presented.

Chapter 4 recalls some results concerning the Proportional Controller, such results have been presented in [98]. In order to illustrate the applicability of the theoretical results, several numerical example are presented at the end of the chapter. It is worth to mention that such results are presented in the seek of completeness.

In Chapter 5, we summarize the results presented in [99] concerning to the PI stabilization problem of a linear SISO system with I/O delays. Several illustrative examples are presented at the end of the chapter.

Chapter 6 concerns the Geometry of PD controllers for SISO systems with I/O delays. Inspired by the results presented in [99], we extend such an approach to the case of PD controllers. Unlike the PI case, a neutral-type system can be result from the closed-loop system. Then, additional analysis is presented in order to deal with such a situations. We include at the end of the chapter several numerical examples illustrating the proposed method.

In Chapter 7 following the geometric ideas introduced by Gu *et al.* in [40], the geometry of PID controllers for SISO systems with I/O delays is presented. First, we start by developing a simple method to derive the stability regions in the gain parameter space. Then, a classification of the stability crossing boundaries is proposed. Such a classification is established as a function of the kind of the left and right ends of the corresponding frequency crossing interval. In such a case, 8 types of boundaries have been obtained. Following similar ideas as those applied in the PI and PD cases, we propose a criterion to check the crossing directions. The proposed method is illustrated by several numerical examples presented in the last part of the chapter.

Chapter 8 concerns the fragility analysis for the PID-type controllers. The chapter starts by showing through a numerical example the importance of the fragility in the design of a controller. Next, a simple geometrical method for computing the fragility of a PI, PD and PID controller is proposed. Several numerical examples complete the presentation. Such a method is performed in three steps: (i) the construction of the stability crossing boundaries in the $(k_p, k_d, k_i)-$parameter space, (ii) the explicit computation of the crossing directions and (iii) the explicit computation of the distance of some point to the closest stability crossing boundaries.

Chapters 9–10 form the second part of the thesis and concern the stability analysis of a linear SISO system in a networked control system framework.

In Chapter 9 we focus on a chain of integrators system under a NCS framework. Firstly, it is stated the main differences with the continuous case: lack of scaling properties, induced instabilities for small gain values and that $n-$delay are not sufficient to guaranty an arbitrary pole placement. Secondly, a method to construct a controller able to achieve the pole-placement of the closed-loop poles is presented (at least $n+1$ delays are needed). Next, a method to construct a controller (with $n-$delays) able to achieve asymptotic stability is presented. Several numerical examples along the chapter are presented and are helpful in understanding some of the proposed notions and approaches..

Chapter 10 concerns the output feedback stabilization problem for a class of linear SISO systems subject to I/O network delays. We present the characterization of the set of delay-sampling period- and gain- parameters guaranteeing the stability of the closed-loop system.

Such an analysis is performed by adopting an eigenvalue perturbation-based approach. Various numerical examples illustrate the proposed results.

Chapter 11 forms the third part of the thesis. This chapter presents some extension of the eigenvalue perturbation-based approach developed in the chapter 10 to the the analysis of the zero behavior of some class of sampled-data SISO systems . Some illustrative examples are also presented. In the last part of this chapter, some possible future works are mentioned.

Publications

The following publications have been derived from this thesis:

In continuous-time systems:

- C.F. MÉNDEZ-BARRIOS, S.-I. NICULESCU, I.C. MORARESCU AND K. GU, *On the fragility of PI controllers for time-delay SISO systems*, in Proc. 16th IEEE Mediterranean Conference on Control and Automation (MED'08), Ajaccio, France, June 2008, pp.529–534.
- B. LIACU, C.-F. MÉNDEZ-BARRIOS, S.-I. NICULESCU AND S. OLARU, *Some Remarks on the Fragility of PD Controllers for SISO systems with I/O Delays*, in Proc. 14th International Conference on System Theory and Control, Sinaia, Romania, October 2010, pp.287–292.
- I.-C. MORARESCU, C.F. MÉNDEZ-BARRIOS, S.-I. NICULESCU AND K. GU, *Stability crossing boundaries and fragility characterization of PID controllers for SISO systems with I/O delays*, in Proc. of the 2011 American Control Conference (ACC'11), San Francisco, United States, July 2011.
- I.-C. MORĂRESCU, C.-F. MÉNDEZ-BARRIOS, S.-I. NICULESCU AND K. GU: *Geometric Ideas in Controlling Delays Systems*, to be submitted.

In Network Control Systems:

- C.F. MÉNDEZ-BARRIOS, W. MICHIELS AND S.-I. NICULESCU, *Further remarks on stabilizing chains of integrators by using network delays*, in Proc. of the 2009 European Control Conference (ECC'09), Budapest, Hungary, August 2009, p. MoA11.3.
- C.F. MÉNDEZ-BARRIOS, S.-I. NICULESCU AND J. CHEN, *Some remarks on output feedback control for SISO systems with I/O NCS delays*, in Proc. of the 8th IFAC Workshop on Time-Delay Systems, Sinaia, Romania, September 2009, p. WP2-1.

In Sample-Data systems:

- C.F. MÉNDEZ-BARRIOS AND S.-I. NICULESCU, *Some Remarks on the Asymptotic Zeros Behavior of Sampled-Data SISO Systems. An Eigenvalue-Based Approach*, in Proc. of the 4th IFAC Symposium on System, Structure and Control (SSSC'10), Ancona, Italy, September 2010, p. WeAT1.4.

Notations

List of Symbols

$\mathbb{N}$	set of natural numbers, i.e., $\mathbb{N} := \{1, 2, \ldots\}$.
$\mathbb{Z}$	set of integer numbers
$\mathbb{R}_+$	set of strictly positive real numbers
$\mathbb{R}$	field of real numbers
$\mathbb{C}$	field complex numbers
$\mathbb{R}^n$ $(\mathbb{C}^n)$	space of all $n-$dimensional column vectors with components in $\mathbb{R}$ $(\mathbb{C})$
$\mathbb{R}^{n\times m}$ $(\mathbb{C}^{n\times m})$	space of real (complex) matrices of size $n \times m$
$\overline{\lambda}$	complex conjugate of $\lambda \in \mathbb{C}$
$\angle\lambda$	argument of the complex number λ, where $\angle z \in [0, 2\pi)$
$\Re(\cdot)$ $(\Im(\cdot))$	real (imaginary) part of a complex number
e_r	unit vector in the $r-$th direction, with dimension given by the context
$e_r^{(m)}$	$m-$dimensional unit vector in the $r-$th direction
I	identity matrix, with dimension given by the context
A^T	transpose of matrix A
A^{-1}	inverse of matrix A
$\operatorname{diag}(A_1, A_2, \ldots, A_m)$	$\begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_m \end{bmatrix}$
$\operatorname{rank}(\cdot)$	rank of a matrix, or a matrix-valued function
$\det(\cdot)$	determinant of a square matrix
$\operatorname{trace}(\cdot)$	trace of a matrix
$\left[a_{ij}\right]_{i,j=1,\ldots,n}$, $\left[a_{ij}\right]_{i,j=1}^{n}$	square matrix of dimension n
$[a_i]_{i=1}^n$	$n-$dimensional vector
$[a_i]_{i=1}^n$	$n-$dimensional vector
$\sigma(A)$	spectrum of the square matrix $A \in \mathbb{C}^{n\times n}$
$\sigma(A, B)$	set of all generalized eigenvalues, i.e., $\{\lambda \in \mathbb{C} : det(A - \lambda B) = 0\}$
$\langle x, y\rangle$	inner product of the vectors $x, y \in \mathbb{C}^n$
$\\|x\\|$	Euclidean norm of vector $x \in \mathbb{C}^n$
$R(i_1, \ldots, i_\alpha)$	$\left[r_{i_\nu i_\mu}\right]_{\nu,\,\mu=1,\ldots,\alpha}$, for integers $\alpha \in \{1, \ldots, n\}$ and i_ℓ, $1 \le \ell \le \alpha$ with $1 \le i_1 < \cdots < i_\alpha \le n$

$\mathbb{D}$	unit open disk $\{z : z \in \mathbb{C}, \lvert z\rvert < 1\}$
$\overline{\mathbb{D}}$	unit closed disk $\{z : z \in \mathbb{C}, \lvert z\rvert \leq 1\}$
$\partial\mathbb{S}$	boundary of $\mathbb{S}$, where $\mathbb{S}$ is any set. For example, $\partial\mathbb{D}$ is the unit circle $\{z : z \in \mathbb{C}, \lvert z\rvert = 1\}$
$\emptyset$	empty set
$\dot{x}(t)$	derivative of $x(t)$ with respect to time t, $\frac{dx}{dt}$
$\binom{n}{r}$	binomial coefficient defined by $\frac{n!}{r!(n-r)!}$
$csch(z)$	hyperbolic cosecant function $csch : \mathbb{C} \mapsto \mathbb{C}$, defined by $csch(z) := 2/\left(e^z - e^{-z}\right)$
$\boldsymbol{i}$	$\sqrt{-1}$
$\deg(\cdot)$	degree of a polynomial
$:=$, $\triangleq$	equals by definition
$\square$	end of a proof

Abbreviations

AQM	active queue management
CRS	complete regular splitting
DDE	delay differential equations
FE	functional equation
FDE	functional differential equation
LHP	left-half plane
LTI	linear time invariant
NCS	networked control systems
PD	proportional derivative
PI	proportional integral
PID	proportional integral derivative
RFDE	retarded functional differential equation
RHP	right-half plane
SISO	single input, single output
TCP	transmission control protocol

1 Stabilization of Dynamical Systems

1.1 Introduction

The study of time-delay systems (hereditary systems or systems with aftereffect or with time-lag) had its origins in the 18th century, with the works of Euler, Bernoulli, Lagrange, Laplace and others [118], and it received substantial attention in the early 20th century [39]. Such a growth of popularity is related to the fact that delays are natural components of dynamical processes in physics, biology, engineering; for example, in population dynamics systems it has been stated that a delayed logistic model provides a better framework for modeling the dynamical behavior than a model that not taking into account the past history [26, 120, 133]. Among the open problems that required an increasing interest during the last decade, we cite the networked control systems (NCS) [137]. Without discussing the modeling issues, we shall focus on the time-delay systems represented by *functional differential equations*, which are also called *differential equations with deviating arguments*.

According to [72], *functional equations* (FE's) are equations involving an unknown function for different argument values. The equations $x(2t)+2x\left(\dfrac{t}{2}\right)=10$, $x(\sqrt{t})=x(t+1)+5\left[x(t+2)\right]^2$, $x\left(x(t)\right)=x(t-1)^2+2$,etc., are examples of functional equations. The differences between the argument values of an unknown function and "t" in a FE are called *argument deviations*.

Roughly speaking, a simple combination of differential and functional equations leads to functional differential equations or equivalently differential equations with deviating arguments. Thus, this is an equation connecting the unknown function and some of its derivatives for, in general, different argument values.

In this vein, since the first part of the thesis is devoted to the stability analysis of time-delay systems (continuous-time representation), in the first part of this chapter we introduce some basic concepts, as well as some elementary results for time-delay systems. On the other hand, since in the second part of the thesis some notions for discrete-time systems will be needed, in the rest of the chapter we will present some basic concepts and results for such systems too.

1.2 Continuous-Time Systems

Introduce now the basic definitions concerning differential equations with retarded arguments:

Let $\mathcal{C}\left([a,b],\mathbb{R}^n\right)$ be the set of continuous functions mapping the interval $[a,b]$ to $\mathbb{R}^n$. The notation $\mathcal{C}=\mathcal{C}\left([-r,0],\mathbb{R}^n\right)$ will denote the set of continuous functions mapping $[-r,0]$ to $\mathbb{R}^n$. For any $A>0$ and any continuous function of time $\varphi\in\mathcal{C}\left([t_0-r,t_0+A],\mathbb{R}^n\right)$, and $t_0\leq t\leq t_0+A$, let $\varphi_t\left(\theta\right)=\varphi\left(t+\theta\right)$, $-r\leq\theta\leq 0$. The general form of a *retarded functional differential equation* (RFDE) (or functional differential equation of retarded type) is

$$\dot{x}(t)=f(t,x_t), \tag{1.1}$$

where $x(t)\in\mathbb{R}^n$ and $f:\mathbb{R}\times\mathcal{C}\to\mathbb{R}^n$. Since equation (1.1) shows that the derivative of the state variables "x" at the time "t" depends on t and some "past-piece-of-trajectory" $x\left(\xi\right)$ for $t-r\leq\xi\leq t$. Then, in order to determine the future evolution of the state, it is necessary to specify the initial "state" variables $x(t)$ on a time-interval of length r, say, from $t_0 - r$ to t_0, i.e.,

$$x_{t_0} = \varphi, \tag{1.2}$$

where $\varphi \in \mathcal{C}$ is given. In other words, $x(t_0 + \theta) = \varphi(\theta)$, $-r \leq \theta \leq 0$.

It is worth mentioning that since the necessary "amount" to construct the solution is represented by the definition of a function on some appropriated interval, FDEs belong to the class of *infinite-dimensional systems*.

For an $A > 0$, a function x is said to be a solution of (1.1) on the interval $[t_0 - r, t_0 + A)$ if within this interval x is continuous and satisfies the RFDE (1.1). Of course a solution also implies that (t, x_t) is within the domain of the definition of f. If the solution also satisfies the initial condition (1.2), we say that it is a solution of the Cauchy problem (1.1) with initial condition (1.2), or simply a solution through (t_0, φ). A fundamental issue in studying FDE is represented by the existence and uniqueness of a solution;

Theorem 1.1. *[39] Suppose that Ω is an open set in $\mathbb{R} \times \mathcal{C}$, $f : \Omega \to \mathbb{R}^n$ is continuous, and $f(t, \varphi)$ is Lipschitzian in φ in each compact set in Ω, that is for each given compact set $\Omega_0 \subset \Omega$, there exists a constant L, such that:*

$$\| f(t, \varphi_1) - f(t, \varphi_2) \| \leq L \| \varphi_1 - \varphi_2 \|,$$

for any $(t, \varphi_1) \in \Omega_0$ and $(t, \varphi_2) \in \Omega_0$. If $(t_0, \varphi) \in \Omega$, then there exists a unique solution of (1.1) through (t_0, φ).

1.2.1 Stability Notions

Let $y(t)$ be a solution of the RFDE (1.1). Roughly speaking, the stability of the solution concerns the system's behavior when the system trajectory $x(t)$ deviates from $y(t)$. In the following, we will assume without loss of generality that the functional differential equation (1.1) admits the solution $x(t) = 0$, which will be referred to as the *trivial solution*. Indeed, if we are interested to study the stability of a nontrivial solution $y(t)$, then we may sort the standard variable transformation $z(t) \triangleq x(t) - y(t)$, so that the new system

$$\dot{z}(t) = f(t, z_t + y_t) - f(t, y_t) \tag{1.3}$$

has the trivial solution $z(t) = 0$.

For a function $\varphi \in \mathcal{C}([a, b], \mathbb{R}^n)$, lets introduce now the following norm $\|\cdot\|$ by

$$\|\varphi\|_c = \max_{a \leq \theta \leq b} \|\varphi(\theta)\|.$$

In the above definition, the vector norm $\|\cdot\|$ represents the standard 2-norm $\|\cdot\|_2$.

Definition 1.1. *[39] For the system described by (1.1), the trivial solution $x(t) = 0$ is said to be stable if for any $t_0 \in \mathbb{R}$ and any $\varepsilon > 0$, there exist a $\delta = \delta(t_0, \varepsilon) > 0$ such that $\|x_{t_0}\|_c < \delta$ implies $\|x(t)\| < \varepsilon$ for $t \geq t_0$. It is said to be asymptotically stable if it is stable, and for any $t_0 \in \mathbb{R}$ and any $\varepsilon > 0$, there exist a $\delta_a = \delta(t_0, \varepsilon) > 0$ such that $\|x_{t_0}\|_c < \delta_a$ implies $\lim_{t \to \infty} x(t) = 0$. It is said to be uniformly stable if it is stable and $\delta(t_0, \varepsilon)$ can be chosen independently of t_0. It is uniformly asymptotically stable if it is uniformly stable and there exist a $\delta_a > 0$ such that for any $\eta > 0$, there exist a $T = T(\delta_a, \eta)$, such that $\|x_{t_0}\|_c < \delta$ implies $\|x(t)\| < \eta$ for $t \geq t_0 + T$ and $t_0 \in \mathbb{R}$. It is globally (uniformly) asymptotically stable if it is (uniformly) asymptotically stable and δ_a can be an arbitrarily large, finite number.*

Linear Systems

Since in this thesis we mainly focus the linear time-invariant (LTI) case, i.e, when f is linear with respect to x_t, in the following we will give a brief review of general linear time-variant delay systems and next focus on linear time-invariant systems.

A general linear time-delay system can be described by the RFDE:

$$\dot{x}(t) = A(t)x_t + g(t), \tag{1.4}$$

where $A(t)$ is a time-varying linear operator acting on x_t. In this case, it is always possible to find a matrix function $F : \mathbb{R} \times [-\tau, 0] \to \mathbb{R}^{n\times n}$ of bounded variation, such that

$$F(t,0) = 0,$$

and

$$L(t)\varphi = \int_{-\tau}^{0} d_\theta [F(t,\theta)]\varphi(\theta). \tag{1.5}$$

In general, in (1.5) it is required a Stieltjes integral, and in (1.5) the subscript θ means that θ (rather than t) is the integration variable. As such, a general linear RFDE can be represented as

$$\dot{x}(t) = \int_{-\tau}^{0} d_\theta [F(t,\theta)] x(t+\tau) + g(t). \tag{1.6}$$

In particular, many linear RFDEs can be specialized to:

$$\dot{x}(t) = \sum_{k=0}^{N} A_k(t)x(t-\tau_k(t)) + \int_{-\tau}^{0} A(t,\tau)x(t+\theta)\,d\theta, \tag{1.7}$$

where $A_k(t)$ and $A(t,\tau)$ are given $n\times n$ real continuous matrix functions, and $\tau_k(t)$ are given continuous functions representing time-varying delays, which can be ordered with no loss of generality, such that

$$0 = \tau_0(t) < \tau_1(t) < \ldots \tau_N(t) \leq \tau.$$

Such a generic representation covers both classical (time-varying or not) point and distributed delays cases. If the function F in (1.6) is independent of time t, then the system described by (1.6) is linear time-invariant. An LTI RFDE can be written as

$$\dot{x}(t) = \int_{-\tau}^{0} dF(\theta)x(t+\theta). \tag{1.8}$$

Let

$$x_0 = \varphi \tag{1.9}$$

be the initial condition. Taking the Laplace transform of (1.8) with the initial condition (1.9), we get

$$sX(s) - \varphi(0) = \int_{-\tau}^{0} e^{\theta s} dF(\theta)X(s) + \int_{-\tau}^{0} e^{\theta s} dF(\theta) \int_{\theta}^{0} e^{-\alpha s}\varphi(\alpha)d\alpha,$$

where $X(s)$ is the Laplace transform of $x(t)$,

$$X(s) = \mathcal{L}[x(t)] = \int_{0}^{\infty} x(t)e^{-st}dt.$$

Solving for $X(s)$ yields

$$X(s) = \Delta^{-1}(s)\left[\varphi(0) + \int_{-\tau}^{0} e^{\theta s} dF(\theta) \int_{\theta}^{0} e^{-\alpha s}\varphi(\alpha)d\alpha\right], \tag{1.10}$$

where $\Delta : \mathbb{C} \mapsto \mathbb{C}^{n\times n}$

$$\Delta(s) = sI - \int_{-\tau}^{0} e^{\theta s} dF(\theta) \tag{1.11}$$

is the so-called *characteristic matrix*. The equation

$$\det[\Delta(s)] = 0 \tag{1.12}$$

is called the characteristic equation, and the expression $\det[\Delta(s)]$ defines the so-called *characteristic function*, or alternatively, the *characteristic quasipolynomial*. The solutions to (1.12) are called the *characteristic roots* or *poles of the system*.

An useful notion in the study of the stability properties of a given FDE is given by the notion of the *spectral abscissa*

$$\rho(\tau;F) := \sup\left\{\Re(s) \,\middle|\, \det\left[sI - \int_{-\tau}^{0} e^{\theta s} dF(\theta)\right] = 0\right\}. \tag{1.13}$$

In this vein, the key result that enables frequency-domain analysis of stability for the time-delay systems is given by the following result:

Theorem 1.2. *[39] For any real scalar γ, the number of the solutions, counting their multiplicities, to the characteristic equation (1.12) with real parts greater than γ is finite. The following statements are true.*

(i) *The LTI delay system (1.8) is stable if and only if $\rho(\tau;F) < 0$.*

(ii) *For any $\rho_0 > \rho(\tau;F)$, there exist an $L > 1$ such that any solution $x(t)$ of (1.8) and the initial condition (1.9) is bounded by*

$$\|x(t)\| \leq Le^{\rho_0 t}\|\varphi\|_c. \tag{1.14}$$

(iii) *$\rho(\tau;F)$ is continuous with respect to τ_k, for all $\tau_k \geq 0$, $k = 1,2,\cdots,N$.*

Sometimes the spectral abscissa is known as *stability exponent* of the system [39].

Remark 1.1. *Observe that this theorem states that the LTI system (1.8) is stable if and only if its spectral abscissa is strictly negative. This is equivalent to saying that all the poles of the system have negative real parts, which renders the study of the stability of an LTI delay system into the study of the zeros-location of the system's characteristic quasipolynomial.*

A particular class of LTI delay systems under (1.8) is those with pointwise (or concentrated) delays, which can be further simplified to the description

$$\dot{x}(t) = \sum_{k=0}^{N} A_k x(t-\tau_k). \tag{1.15}$$

Here A_k are given $n \times n$ real constant matrices, and τ_k are given real constants, ordered such that

$$0 = \tau_0 < \tau_1 < \cdots < \tau_N = \tau.$$

For such a system, the characteristic quasipolynomial $\Delta(s)$ takes the form

$$\begin{aligned} p\,(s;\tau_1,\ldots,\tau_N) &= \det\left(sI - \sum_{k=0}^{N} e^{-\tau_k s} A_k\right) \\ &= p_0(s) + \sum_{k=1}^{m} p_k(s) e^{-r_k s}, \end{aligned} \tag{1.16}$$

where $p_k(s)$, $k = 1,\ldots,m$ are polynomials, and r_k, $k = 1,\ldots,m$ are sums of some of the delay parameters of the delay parameters τ_k. There exist several situations describing the way the delays are related, rational dependence or not, commensurate or not. For instance, if the ratios between the delays, τ_i/τ_j, may be irrational numbers, in which case the delays are said to be *incommensurate*. When all such ratios are rational numbers, on the other hand, we say that the delays are *commensurate*. In the later case, the delays τ_k (and hence r_i) become integer multiples of a certain positive τ. The characteristic quasipolynomial $\Delta(s)$ of systems with commensurate delays can then be written as

$$p(s;\tau) = \sum_{k=0}^{q} p_k\,(s)\,e^{-k\tau s}. \tag{1.17}$$

On the other hand, based on the relation between delays it is possible to classify delays according to their interdependence. In this vein, the delays τ_1, $\tau_2,\ldots,$ τ_N are called *rationally independent* if and only if

$$\sum_{\ell=1}^{N} z_\ell \tau_\ell = 0, \quad z_\ell \in \mathbb{Z},$$

implies $z_\ell = 0$, $\ell = 1,\ldots,N$.

If the delays τ_1, $\tau_2,\ldots,$ τ_N are *rationally dependent*, then there always exists an integer $p < N$ and a matrix $\Gamma \in \mathbb{Z}^{N\times p}$ of full column rank such that

$$\begin{bmatrix} \tau_1 \\ \vdots \\ \tau_N \end{bmatrix} = \Gamma \begin{bmatrix} s_1 \\ \vdots \\ s_p \end{bmatrix},$$

with the numbers $s_1,\ldots,s_p$ rationally independent. It is worth mentioning that in the case where $p = 1$, the delays τ_1, $\tau_2,\ldots,$ τ_N are commensurate, as they are all multiples of the same number.

For example, the numbers 1, 2 and $\frac{5}{3}$ are commensurate as

$$\begin{bmatrix} 1 \\ 2 \\ \frac{5}{3} \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 5 \end{bmatrix} \left(\tfrac{1}{3}\right).$$

The numbers 1, $\frac{\pi}{2}$ and $\pi + 2$ are rationally dependent, yet not commensurate, as

$$\begin{bmatrix} 1 \\ \frac{\pi}{2} \\ 2+\pi \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ \frac{\pi}{2} \end{bmatrix},$$

with 1 and $\frac{\pi}{2}$ rationally independent.

In order to illustrate the importance of the above concepts, lets relate the commensurate and incommensurate cases with the above notions. In the case of commensurate delays, it is possible to express each delay by a relation of the form

$$(\tau_1,\ldots,\tau_N)=\tau\left(\ell_1,\ldots,\ell_N\right),\quad \ell_i\in\mathbb{N},\, i=1,\ldots,N,$$

where $\tau\in\mathbb{R}_+$ is the free parameter. Geometrically, the parametrization above corresponds to a particular *ray* or direction in the delay-parameter space. Moreover, in order to include incommensurate delays it is possible to relax the above parametrization to the following one

$$(\tau_1,\ldots,\tau_N)=\tau\left(r_1,\ldots,r_N\right),\quad r_i\in\mathbb{R}_+,\, i=1,\ldots,N,$$

with $\tau\in\mathbb{R}_+$ being once again the free parameter.

Based on the above geometric parameterizations, Michiels & Niculescu [92] studied the delay-independent stability when the ray under consideration consists of delay values with *any type of interdependence* (commensurate, rationally (in)dependent delays). Furthermore, they obtained the complete characterization of the so-called *delay-interference phenomenon*, that is, the presence of delay-independent stability along a particular ray, which is not robust against small perturbations of the direction of the ray. In order to introduce such results, lets consider the following notions. Given a direction $\overrightarrow{r}:=(r_1,\ldots,r_N)\in\mathcal{B}_+^N:=\left\{\overrightarrow{r}\in\mathbb{R}_+^N:\|\overrightarrow{r}\|=1\right\}$ in the delay parameter space, the associated ray $\mathcal{T}\left(\overrightarrow{r}\right)$ is defined as follows:

Definition 1.2. *[92] For $\overrightarrow{r}\in\mathcal{B}_+^N$, let $\mathcal{T}\left(\overrightarrow{r}\right):=\{\tau\overrightarrow{r}:\tau\in\mathbb{R}_+\}$.*

Definition 1.3. *[92] The system* (1.15) *is delay-independent stable if and only if its zero solution is asymptotically stable for all $\overrightarrow{\tau}\in\mathbb{R}_+^N$.*

Definition 1.4. *[92] The ray $\mathcal{T}\left(\overrightarrow{r}\right)$ is stable if and only if the zero solution of* (1.15) *is asymptotically stable for all $\overrightarrow{\tau}\in\mathcal{T}\left(\overrightarrow{r}\right)$.*

Definition 1.5. *[92] A stable ray $\mathcal{T}\left(\overrightarrow{r}\right)$ is subject to the delay-interference phenomenon if and only if for all $\varepsilon>0$, there exists a $\overrightarrow{s}\in\mathcal{B}_+^N$ with $\left\|\overrightarrow{r}-\overrightarrow{s}\right\|<\varepsilon$ such that the ray $\mathcal{T}\left(\overrightarrow{s}\right)$ is not stable.*

Consider the following matrix-value functions:

- $L_1:[0,2\pi]^N\mapsto\mathbb{C}^{n\times n}$, given by:

$$L_1\left(\overrightarrow{\theta}\right):=L_1\left(\theta_1,\ldots,\theta_N\right)=A_0+\sum_{\ell=1}^{N}A_\ell e^{-i\theta_\ell},\tag{1.18}$$

- $L_2:\mathbb{R}_+\times\mathbb{R}_+^N\mapsto\mathbb{C}^{n\times n}$, given by:

$$L_2\left(\theta,\overrightarrow{r}\right):=L_2\left(\theta,r_1,\ldots,r_N\right)=A_0+\sum_{\ell=1}^{N}A_\ell e^{-i\theta r_\ell}.\tag{1.19}$$

The following quatities play a major role in the characterization of delay-independent stability and the interference phenomenon:

Definition 1.6. *Let*

$$\begin{aligned}\mathbb{W} &= \bigcup_{\overrightarrow{\theta}\in[0,2\pi]^N} \sigma\left(L_1\left(\overrightarrow{\theta}\right)\right),\\ \alpha_0 &= \sup\{\Re(\lambda):\ \lambda\in\mathbb{W}\}\end{aligned}$$

and for $\overrightarrow{r}\in\mathcal{B}_+^N$, *let*

$$\begin{aligned}\mathbb{V}(\overrightarrow{r}) &= \bigcup_{\theta\geq 0}\sigma\left(L_2\left(\theta,\overrightarrow{r}\right)\right),\\ \alpha(\overrightarrow{r}) &= \sup\{\Re(\lambda):\ \lambda\in\mathbb{V}(\overrightarrow{r})\}.\end{aligned}$$

Proposition 1.1. *[92] If the components of* $\overrightarrow{r}$ *are rationally independent, then*

$$\overline{\mathbb{V}(\overrightarrow{r})}=\mathbb{W},\quad \alpha(\overrightarrow{r})=\alpha_0,$$

where $\overline{(\cdot)}$ *is the closure of the set* $(\cdot)$.

The above concepts and results are illustrated with the scalar equation:

$$\dot{x}(t) = -\frac{12}{5}x(t)-\frac{17}{10}x(t-\tau_1)-\frac{4}{5}x(t-\tau_2), \tag{1.20}$$

example that slightly modifies the one presented in [92]. We have,

$$\begin{aligned}\mathbb{W} &= \bigcup_{(\theta_1,\theta_2)\in[0,2\pi]}\left(-\tfrac{12}{5}-\tfrac{17}{10}e^{-i\theta_1}-\tfrac{4}{5}e^{-i\theta_2}\right) = \left\{\lambda\in\mathbb{C}:\ \frac{9}{10}\leq|\lambda+2.4|\leq\frac{5}{2}\right\},\\ \mathbb{V}(\overrightarrow{r}) &= \bigcup_{\theta\geq 0}\left(-\tfrac{12}{5}-\tfrac{17}{10}e^{-ir_1\theta}-\tfrac{4}{5}e^{-ir_2\theta}\right).\end{aligned}$$

In Figure 1.1 we have shown the sets $\mathbb{V}(\overrightarrow{r})$ (solid curves) and $\mathbb{W}$ (dotted curves) for $\vec{r}=(1,4)$ and $\vec{r}=(10,39)$. As mentioned in [92], when the components of $\vec{r}$ are commensurate, $\mathbb{V}(\overrightarrow{r})$ forms a closed curve. We can consider the ratio $r_2/r_1=3.9$ as a perturbation of $r_2/r_1=4$, and despite the fact that the componetes are still commensurate, they are "more" independent, since the coprime numbers 39 and 10 are larger than 4 and 1. Consequently, the curve

$$\theta\geq 0\mapsto -\frac{12}{5}-\frac{17}{10}e^{-i\theta}-\frac{4}{5}e^{-i\frac{r_2}{r_1}\theta} \tag{1.21}$$

with $r_2/r_1=3.9$ will closes only at $\theta=20\pi$ when θ is increased from zero (instead of $\theta=2\pi$ for $r_2/r_1=4$), and a larger portion of $\mathbb{W}$ is covered by $\mathbb{V}(\vec{r})$. In order to illustrate such situation, the values of (1.21) for $\theta\in[0,2\pi]$ are also plotted in the right side of Fig.1.1 (bold curve). Now, if $r_2/r_1=2$ would perturbed instead to an irrational value, such as $2-\pi/n$, $n\geq 2$, the the corresponding curve $\mathbb{V}(\vec{r})$ would never close and and its points would densely fill $\mathbb{W}$ as indicated in Proposition 1.1 (see, [92], for further details).

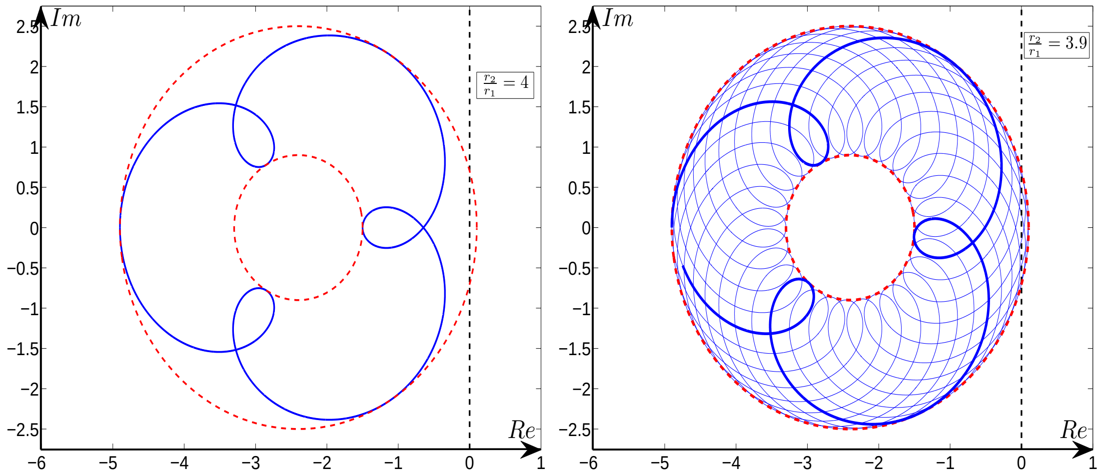


Figure 1.1: The set $\mathbb{V}(\vec{r})$ for the system (1.20). (Left) for $\vec{r} = (1,4)$; (right) for $\vec{r} = (10,39)$. The dotted curves are the boundaries of $\mathbb{W}$.

The delay-interference phenomena is characterized by the following results.

Proposition 1.2. *[92] The ray $\mathcal{T}(\vec{r})$ is stable if and only if*

$$\mathbb{V}(\vec{r}) \subset (\mathbb{C}_{-} \cup \{0\}).$$

Proposition 1.3. *[92] Assume that $\mathbb{W} \not\subset (\mathbb{C}_{-} \cup \{0\})$. If a ray $\mathcal{T}(\vec{r})$ is stable, then it is subjected to the delay-interference phenomenon.*

Theorem 1.3. *[92] Assume that $\mathbb{W} \cap \mathbb{C}_{+} \neq \emptyset$. Then the following holds:*

1. *If the components of $\vec{r}$ are rationally independent, then the ray $\mathcal{T}(\vec{r})$ is unstable.*
2. *If the ray $\mathcal{T}(\vec{r})$ is stable, then it is subjected to the delay-interference phenomenon.*
3. *The set $\left\{\vec{r} \in \mathcal{B}_{+}^{N} : \ \mathcal{T}(\vec{r}) \ \text{stable}\right\}$ is nowhere dense in $\mathcal{B}_{+}^{N}$.*

Neutral time-delay systems

Consider now the linear time-invariant systems with pointwise delays in the form of

$$\sum_{k=0}^{N} \left[A_k \dot{x}(t-\tau_k) + B_k x(t-\tau_k)\right] = 0, \tag{1.22}$$

where A_0 is nonsingular (bearing in mind that $\tau_0 = 0$). With this later assumption and without any loss of generality, we may take that $A_0 = I$. The initial condition condition can again be expressed as

$$x_0 = \varphi. \tag{1.23}$$

In general, the function φ is assumed to be differentiable for the solutions to be well defined, although a relation to be discontinuous solutions is possible (see, [39] for further details). Let the characteristic quasipolynomial $\Delta(s)$ be defined as

$$\Delta(s) = \det\left(\sum_{k=0}^{N} e^{-\tau_k s}\left[sA_k + B_k\right]\right). \tag{1.24}$$

As in the retarded case, the solution of the characteristic equation

$$\Delta(s) = 0 \tag{1.25}$$

are referred to as the *poles* or *characteristic roots* of the system.

For the system (1.22), the spectral abscissa can be defined as

$$\rho\left(\overrightarrow{\tau}; A_0, \ldots, A_N, B_0, \ldots, B_N\right) := \sup\left\{\Re(s) \,\middle|\, \det\left(sI - \sum_{k=0}^{N} e^{-\tau_k s}\left[sA_k + B_k\right]\right) = 0\right\}.$$

Theorem 1.4. *[39] Consider the system described by (1.22). The following statements are true:*

(i) *System (1.22) is stable if* $\rho\left(\overrightarrow{\tau}; A_0, \ldots, A_N, B_0, \ldots, B_N\right) < 0$.

(ii) *For any* $\rho_0 > \rho\left(\overrightarrow{\tau}; A_0, \ldots, A_N, B_0, \ldots, B_N\right)$, *there exist an* $L > 0$ *such that any solution* $x(t)$ *of (1.22) with the initial condition (1.23) is bounded by*

$$\|x(t)\| \leq L m_\varphi e^{\rho_0 t}, \tag{1.26}$$

where

$$m_\varphi = \max_{\tau \leq t \leq 0}\left(\|\varphi(t)\| + \|\dot{\varphi}(t)\|\right).$$

Remark 1.2. *It is important to note that in the neutral case the spectral abscissa is not necessarily continuous with respect to the delay parameters (see, for instance, [92]).*

Analytical Methods

From the above results, we conclude that in order to study the stability of a time-delay we must analyze the zero-location of a given quasipolynomial (1.16), for retarded systems, or given by (1.24) for neutral time-delay systems. Since approximate calculation of all roots of a quasipolynomial is a problem of a great difficulty, different numerically tractable tests of negativity of the real parts of all roots the quasipolynomial have been developed in the literature for avoiding such an issue. Among such tests, most often the following ones are generally employed:

1) the amplitude-phase method and its modifications [72];

2) the method of $\mathcal{D}$-partitions and its modifications [32, 92];

3) the Pontryagin Criterion [115];

4) the method of Meiman and Chebotarev [19].

1. Amplitude-phase method It is well known, from the complex function theory [78], that if $f(s)$ is an analytical function, different from zero on some simple, closed contour Γ (without any self-intersections), and the interior of Γ has only a finite set of polar singularities, then:

$$\frac{1}{2\pi}\int_\Gamma \frac{f'(z)}{f(z)}dz = N_\Gamma - P_\Gamma,$$

where N_Γ is the number of zeros of $f(s)$ in the interior of Γ, counting according to their multiplicities, and P_Γ is the number of poles in the interior of Γ, counted according to their multiplicities. This leads to the so-called *Argument Principle*

$$\frac{1}{2\pi}\Delta_\Gamma \mathrm{Arg} f(s) = N_\Gamma - P_\Gamma. \tag{1.27}$$

Here, $\Delta_\Gamma \mathrm{Arg} f(s)$ is the total increase of the argument of the function $f(s)$ under a single "circuit" of the point s, in a positive direction, around the contour Γ. On the other hand, the difference $N_\Gamma - P_\Gamma$ equals the number of complete revolutions which the vector w perform in the plane, going from the point $w = 0$ to the point $w = f(s)$, when the point s describes the contour Γ in the positive direction.
For obtaining a condition for the absence in the characteristic quasipolynomial $f(s;\tau)$ of roots with positive real parts, it is possible to apply the argument principle to the contour Γ_R, consisting of the segment of the imaginary axis $[-\boldsymbol{i}R, \boldsymbol{i}R]$ and the semi-circle of radius R with center at the origin, lying in the half-plane $\Re(s) > 0$; as preliminary, it is necessary to check that the quasipolynomial does not have any zeros on the imaginary axis.

We note that, in this case, $P_\Gamma = 0$. Using the argument principle, we find from (1.27) N_{Γ_R} and, if $\lim_{R\to\infty} N_{\Gamma_R} = 0$, then all roots s_i of the quasipolynomial satisfy the condition $\Re(s_i) < 0$. To apply this method to the quasipolynomial

$$f(s) = P_n(s) + Q_{n-1}(s)e^{-\tau s},$$

corresponding to the nth order equation with single retarded argument, where $P_n(s)$ and $Q_{n-1}(s)$ are polynomials of degree n and not greater than $n-1$, respectively, it is possible to somewhat simplify the investigation. Instead of the function $f(s)$, it is considered the function

$$\frac{f(s)}{P_n(s)} = 1 + \frac{Q_{n-1}(s)}{P_n(s)}e^{-\tau s},$$

the zeros of which coincide with the zeros of the function $f(s)$ (if $P_n(s)$ and $Q_{n-1}(s)$ do not have common zeros) and which has poles at the zeros of the polynomial $P_n(s)$.

Define now $w_\tau(s) := -\frac{Q_{n-1}(s)}{P_n(s)}e^{-\tau s}$. The limiting position as $R \to \infty$ of the form of the contour Γ_R under the mapping $w_\tau(s)$ is called the *amplitude-phase characteristic*. Since $\frac{f(s)}{P_n(s)} = 1 - w_\tau(s)$, the zeros of the function $\frac{f(s)}{P_n(s)}$ correspond to the points at which $w_\tau(s) = 1$. Therefore, applying the argument principle to the function $w_\tau(s)$, it is necessary to calculate the number of circuits of the amplitude-phase characteristic, not of the point $s = 0$, but of the point $s = 1$. The number of circuits of the amplitude-phase characteristic of the point $s = 1$ equals to the difference $N_\Gamma - P_\Gamma$ and, consequently, in order that $N_\Gamma = 0$, it is necessary that the number of circuits of the amplitude-phase characteristic of the point $s = 1$ equals $-P_\Gamma$. We recall that, for this it is assumed that there are no zeros of the function $f(s)$ on the imaginary axis and that $P_n(s)$ and $Q_{n-1}(s)$ do not have common zeros.

For construction of the amplitude-phase characteristic, it is convenient at first to find the so-called *limiting characteristic*, appearing as the limiting form of the contour Γ_R under the mapping $w_0 : \mathbb{C} \mapsto \mathbb{C}$

$$w_0(s) = -\frac{Q_{n-1}(s)}{P_n(s)}. \tag{1.28}$$

For construction of the form of the imaginary axis under the mapping

$$w_\tau(s) = -\frac{Q_{n-1}(s)}{P_n(s)} e^{-\tau s} = w_0(s) e^{-\tau s} \tag{1.29}$$

or

$$w_\tau(\boldsymbol{i}y) = w_0(\boldsymbol{i}y) e^{-\tau \boldsymbol{i}y},$$

knowing already the limiting characteristic, it suffices to consider the influence of the factor $e^{-\tau \boldsymbol{i}y}$ of rotation without change of modulus, the radius vector of the point of the limiting characteristic corresponding to the value y at the angle $-\tau y$. It is worth mentioning that a particular attention should be given to the points of the limiting characteristic lying on the circle $|s| = 1$ since these points, under rotation by an angle $-\tau y$, may find themselves at the point $s = 1$.

Example 1.1. *A well-studied example in classical stability analysis of time-delay system is given by the first-order delay system*

$$\dot{x}(t) = -ax(t) - bx(t-\tau), \tag{1.30}$$

where $a, b \in \mathbb{R}$ *and* $\tau \in \mathbb{R}_+$. *The characteristic equation for this system has the form*

$$f(s) = s + a + be^{-\tau s}.$$

Then, according to (1.28) *and* (1.29), *we have*

$$w_0(s) = -\frac{b}{s+a}, \tag{1.31}$$

$$w_\tau(s) = -\frac{b}{s+a} e^{-\tau s}. \tag{1.32}$$

Now, in order to analyze the shape of the mapping (1.31), *we observe that*

$$|w_0(\boldsymbol{i}y)| = \frac{|b|}{\sqrt{y^2 + a^2}}. \tag{1.33}$$

It is clear to see from (1.33) *that the modulus is bounded. Moreover, since*

$$\begin{aligned} \frac{d}{dy} |w_0(\boldsymbol{i}y)| &= 0, \\ \Rightarrow -\frac{|b|y}{(y^2+a^2)^{\frac{3}{2}}} &= 0. \end{aligned} \tag{1.34}$$

The later equation implies that $y = 0$ *is the only critical point, and since* $\frac{d^2}{dy^2}|w_0(0)| = -\frac{|b|}{|a|^3} < 0$, *this implies that at* $y = 0$ *the modulus reach its maximum and such value is given by* $\lim_{y \to 0} |w_0(\boldsymbol{i}y)| = \left|\frac{b}{a}\right|$. *Based on these observations, we compute* $\left|w(\boldsymbol{i}y) + \frac{b}{2a}\right|^2 = \frac{b^2}{4a^2}$, *which implies that* (1.31) *maps the imaginary axis into a circle with radius* $\frac{|b|}{2|a|}$ *with center at the point* $s = -\frac{b}{2a}$. *The above discussion is illustrated in Fig1.2.*

Now, if $a > 0$ *the function* $w_\tau(s)$ *has no poles in the RHP, and if in addition* $|b| < a$, *then under each rotation of the points of the circle* $\left|w_0(\boldsymbol{i}y) + \frac{b}{2a}\right| = \frac{|b|}{2|a|}$ *caused by the presence of the term* $e^{-\boldsymbol{i}\tau y}$ *in* $w_\tau(s)$, *the amplitude-phase characteristic will not contain the point* $s = 1$ *as illustrated in Fig.1.3 and consequently, all zeroes of the quasipolynomial* $f(s) = s + a + be^{-\tau s}$ *are located in the LHP. Moreover, under the above assumptions the system will be stable independently of the delay.*

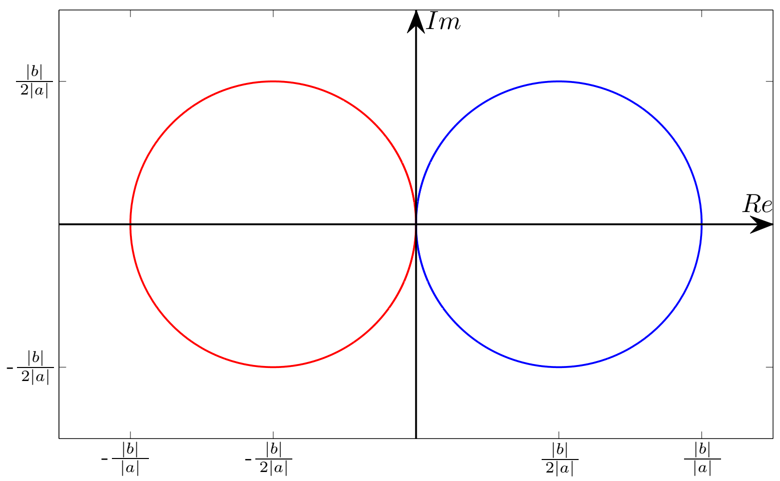


Figure 1.2: The $w_0(\boldsymbol{i}y)$ map. The blue (right) and red (left) circle correspond to the case when *sign* $\left(\frac{b}{a}\right) < 0$, or when *sign* $\left(\frac{b}{a}\right) > 0$, respectively.

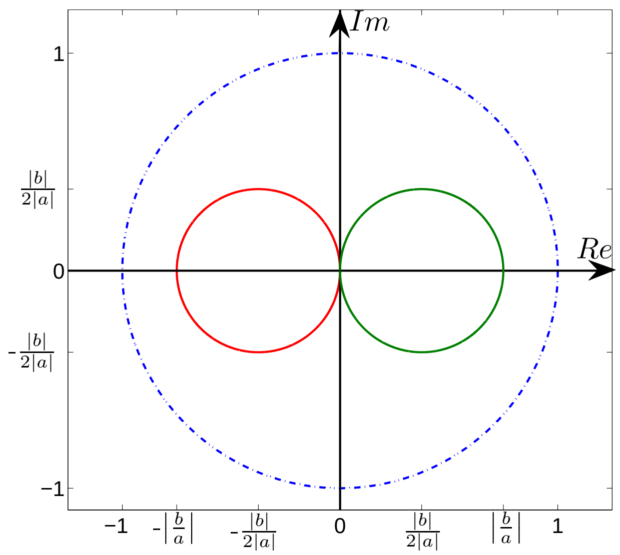


Figure 1.3: The $w_0(\boldsymbol{i}y)$ map for $a > 0$ and $|b| < a$. The green (right) and red (left) circle correspond to the case when *sign* $(b) < 0$, or when *sign* $(b) > 0$, respectively.

2. $\mathcal{D}$-partition Method. The zeros of the characteristic quasipolynomial $f(s)$ for a fixed "deviation" τ are continuous functions of its coefficients (assumed that coefficient of the principal term is not equal to zero, which is always satisfied for equations with retarded argument). This method divides the space of coefficients into regions by hypersurfaces, the points of which correspond to quasipolynomials having at least one zero on the imaginary axis (the cases $s = 0$ is not excluded). The points of each region of such $\mathcal{D}$-partition clearly correspond to a quasipolynomial with the same number of zeros with positive real parts (counting their multiplicities), since under a continuous variation of the coefficients, the number of zeros with positive real parts can change only if a zero passes across the imaginary axis, that is, if the point in the coefficient space passes across the boundary of a region of the $\mathcal{D}$-partition.

Thus, to every region $\mathcal{T}_k$ of the $\mathcal{D}$-partition, it is possible to assign a number k which is the number of zeros with positive real parts of the quasipolynomial defined by the points of this region. Among the regions of this decomposition, we can find some particular regions $\mathcal{T}_0$ (if it exists) corresponding with quasipolynomial which do not have even one root with positive real part. These regions are simply called *stability regions* or *domains*.

Thus, the investigations of stability by the method of $\mathcal{D}-$partition in the space of coefficients (or other parameters on which the coefficients and deviations of the argument depend) reduces to the following scheme: find the $\mathcal{D}-$partition and single out therefrom the region $\mathcal{T}_0$. If the region $\mathcal{T}_0$ is connected, then it may be identified by verifying that at least one of its points corresponds to a quasipolynomial whose roots all have negative real parts.
In order to clarify how the number of roots with positive real parts changes as some boundary of the $\mathcal{D}-$partition is crossed, the differential of the real part of the root is computed, and the decrease or increase of the number of roots wit positive real parts is determined from its algebraic sign.
If $f(s;\alpha_1,\ldots,\alpha_p)=0$ is a characteristic equation containing the parameters $\alpha_1,\ldots,\alpha_p$, then

$$\begin{aligned} \frac{\partial f}{\partial s}ds &= -\sum_{i=0}^{p}\frac{\partial f}{\partial \alpha_i}d\alpha_i, \quad s=x+\boldsymbol{i}y, \\ dx &= -\Re\left(\frac{\sum_{i=0}^{p}\frac{\partial f}{\partial \alpha_i}d\alpha_i}{\frac{\partial f}{\partial s}}\right). \end{aligned} \tag{1.35}$$

Usually dx is computed on some boundary of the $\mathcal{D}-$partition for a change in only one parameter whose changes guarantee passage across the boundary being examined.

Example 1.2. *As an example, consider again the system* (1.30)*, i.e.,*

$$\dot{x}(t) = -ax(t) - bx(t-\tau), \tag{1.36}$$

where $a,b\in\mathbb{R}$ and $\tau\in\mathbb{R}_+$. Adopting the notation presented above, the characteristic quasipolynomial of (1.36) *will be written as*

$$\phi(s) = s + a + be^{-\tau s}. \tag{1.37}$$

The quasipolynomial (1.37) *has a zero root if*

$$a+b=0, \tag{1.38}$$

and such a straight line will belong to the boundary of the $\mathcal{D}-$partition. Now, following the method, find the set of points in the parameters (a,b) such that the quasipolynomial (1.37) *has at least one zero on the imaginary axis. In this vein, consider $s=\boldsymbol{i}\omega$ in* (1.37)*:*

$$(\boldsymbol{i}\omega) + a + be^{-\boldsymbol{i}\omega\tau} = 0,$$

separating the real and imaginary parts we obtain

$$a + b\cos(\omega\tau) = 0 \quad \textit{and} \quad \omega - b\sin(\omega\tau) = 0,$$

respectively. Separating variables, we obtain the parametric equations for the $\mathcal{D}-$partition boundaries:

$$a(\omega) = -\omega\cot(\omega\tau), \tag{1.39a}$$

$$b(\omega) = \frac{\omega}{\sin(\omega\tau)}. \tag{1.39b}$$

From the above equations we have that as $\omega \to 0$, $(a,b) \to \left(-\frac{1}{\tau}, \frac{1}{\tau}\right)$, *which belong to the straight line given in* (1.38)*. Then, the boundaries of the* $\mathcal{D}-$*partition are composed by the straight lines* (1.38) *together with those given by the equations* (1.39)*, these boundaries decompose the plane into regions, as shown in Fig1.4.*

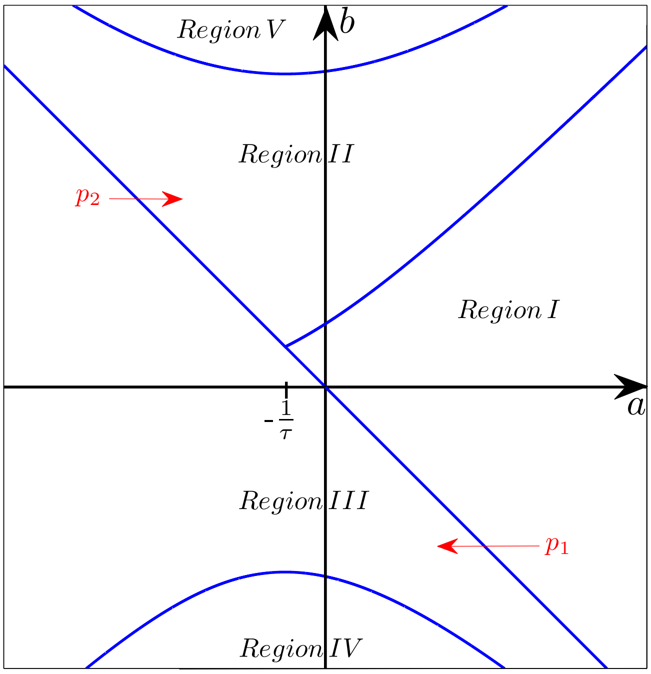


Figure 1.4: The $\mathcal{D}-$partition boundaries for the system (1.36).

Since inside each region we have the same number of unstable roots, then in principle it is sufficient to investigate the stability of a a pair of points of a given region in order to conclude about the stability of the entire region. In this vein, assuming that $a > 0$ *and since* $(a,b) = (a,0)$ *belongs to Region I, we have that for such point the system is asymptotically stable and as a consequence we conclude that Region I is the region of asymptotically stable solutions for the system* (1.36)*. Now, in order to evaluate the crossing direction from region I into region III or from region III into region II, lets evaluate* (1.35)*. To this end, let consider the paths* p_1 *and* p_2 *as show in figure 1.4.*

Now, since in both paths p_1 *and* p_2 *we have a constant* b*, we will have that in such situation* $db = 0$ *and* (1.35) *will be reduced to:*

$$dx = -\Re\left(\frac{da}{1-\tau b e^{-s\tau}}\right). \tag{1.40}$$

Moreover, since both paths are crossing through the straight line (1.38)*, we know that on this line we have one zero at* $s = 0$ *(whereas the real part of the remaining roots of the quasipolynomial approach* $-\infty$*), consequently we can lose (or not) at most one root. In addition, since over this line we have* $s = 0$*, then the condition* (1.40) *can be reduced to*

$$dx = -\frac{da}{1-\tau b}. \tag{1.41}$$

Since, for a crossing from region I to region III through the path p_1 *we have that* $b < \frac{1}{\tau}$ *and* a

decreasing a, therefore $dx > 0$, or in other words the real part of the root $s = 0$ on this straight line receives a positive increment.

Consider now a crossing from the region III into region II through the path p_2. Since in such situation we have that $b > \frac{1}{\tau}$, then from (1.41), we get $dx > 0$ for $da > 0$, therefore in region II will have two roots with positive real part. In conclusion, he have that region I, correspond to the stability region, region II has two roots with positive real part, whereas region III has one root with positive real part. Such conclusion is illustrated in Fig.1.5.

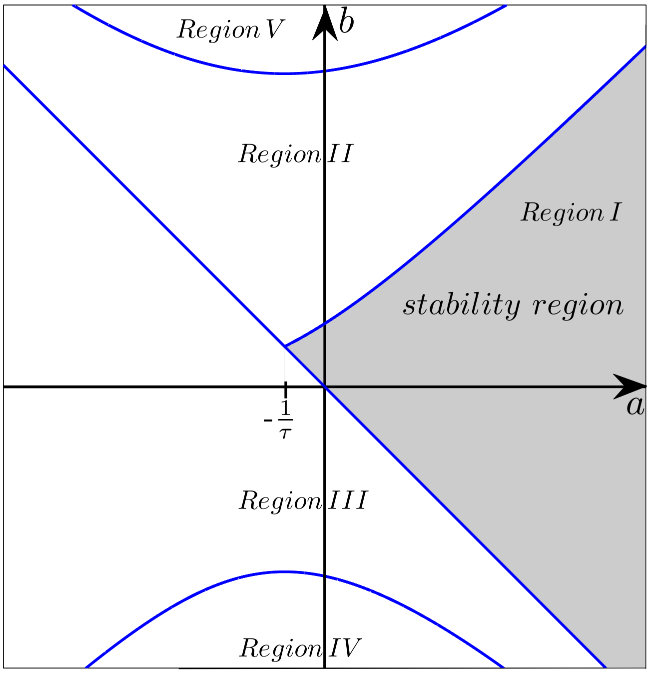


Figure 1.5: The stability regions for system (1.36).

3. Pontryagin Criterion. In the analysis of stability of a time-delay system, Pontryagin [115, 116] obtained some fundamental results concerning the zeros of a quasipolynomial.

Remark 1.3. *Observe that a linear system with single delay or with multiple but commensurable delays, can be associated with a quasipolynomial of the following form*

$$P(s) = \sum_{i=0}^{M} \sum_{j=1}^{N} a_{ij} s^i e^{js}. \tag{1.42}$$

Consider the quasipolynomial (1.42) for $s = \boldsymbol{i}\omega$, where ω is a real number: $P(\boldsymbol{i}\omega) = g(\omega) + \boldsymbol{i}h(\omega)$. The method is summarized in the following

Theorem 1.5 (Pontryagin's Criterion [72])**.** *If quasipolynomial (1.42) has no zeros with positive real part, then all the zeros of the functions $g(\omega)$ and $h(\omega)$ are real, simple and alternating and*

$$\dot{h}(\omega) g(\omega) - \dot{g}(\omega) h(\omega) > 0, \quad -\infty < \omega < \infty. \tag{1.43}$$

For absence of zeros with positive real part of (1.42) one of the following conditions is sufficient:

(1) *All zeros of the functions $g(\omega)$ and $h(\omega)$ are real, simple and alternating, and inequality (1.43) is fulfilled for at least one real ω;*

(2) *all the zeros of the function $g(\omega)$ (or $h(\omega)$) are real and simple and for each zero relation (1.43) is satisfied.*

Example 1.3. *Consider the simplest first-order time-delay system*

$$\dot{x}(t) = -bx(t-\tau), \tag{1.44}$$

where $b \in \mathbb{R}$ and $\tau \in \mathbb{R}_+$. According to (1.42) *the corresponding quasipolynomial for this system is*

$$P(s) = se^{\tau s} + b. \tag{1.45}$$

For $s = \boldsymbol{i}\omega$, we have

$$\begin{aligned} P(\boldsymbol{i}\omega) &= (\boldsymbol{i}\omega)e^{\boldsymbol{i}\omega\tau} + b && (1.46)\\ &= \underbrace{b - \omega\sin(\omega\tau)}_{=:g(\omega)} + \boldsymbol{i}\underbrace{\omega\cos(\omega\tau)}_{=:h(\omega)}. && (1.47) \end{aligned}$$

It is clear to see, that the roots of $h(\omega)$ are given by $\omega_0 = 0$ and $\omega_\ell = \frac{(2\ell-1)}{2\tau}\pi$, with $\ell \in \mathbb{N}$. Now, differentiating $g(\omega)$ and $h(\omega)$ with respect to ω, we obtain

$$\begin{aligned} \dot{g}(\omega) &= -\sin(\omega\tau) - \omega\tau\cos(\omega\tau),\\ \dot{h}(\omega) &= \cos(\omega\tau) - \omega\tau\sin(\omega\tau). \end{aligned}$$

According to the above expressions, we have that

$$\dot{h}(\omega)g(\omega) - \dot{g}(\omega)h(\omega) = b\cos(\omega\tau) + \omega\tau(\omega - b\sin(\omega\tau)).$$

*Then, in order to apply Theorem 1.5***(2)***, we compute first $\dot{h}(0)g(0) - \dot{g}(0)h(0) = b$. Next, for $\omega = \omega_\ell$ we have $\dot{h}(\omega_\ell)g(\omega_\ell) - \dot{g}(\omega_\ell)h(\omega_\ell) = \omega_\ell\tau\left(\omega_\ell - (-1)^{\ell-1}b\right)$. Since, we require the positivity of both quantities, we must have*

$$\begin{aligned} b &> 0,\\ \left(\omega_\ell - (-1)^{\ell-1}b\right) &> 0. \end{aligned}$$

From the last inequality we conclude that $b < \omega_\ell\ \forall \ell \in \mathbb{N}$, i.e.,

$$0 < b < \omega_1 < \omega_2 < \cdots < \omega_\ell < \cdots.$$

Thus, if $0 < b < \frac{\pi}{2\tau}$, the system will be asymptotically stable.

4. Method of Meiman and Chebotarev. This method deals with the stability analysis of quasipolynomials with incommensurable delays by means of a *generalization of the Routh-Hurwitz conditions*. For the quasipolynomial

$$P(s) = \sum_{i=0}^{M}\sum_{j=1}^{N} a_{ij}s^i e^{b_j s} \tag{1.48}$$

expand $P(s)$ in the series $P(s) = a_0 + a_1 s + a_2 s^2 + \cdots$ and define the functions $u(s)$ and $v(s)$:

$$\begin{aligned} P(\boldsymbol{i}s) &= u(s) + \boldsymbol{i}v(s), \qquad u(s) = a_0 - a_2 s^2 + a_4 s^4 - \cdots \\ v(s) &= a_1 s - a_3 s^3 + a_5 s^5 - \cdots \end{aligned}$$

Lets introduce the following determinants Q_m:

$$Q_1 = a_1, \quad Q_2 = \begin{vmatrix} a_1 & a_3 \\ a_0 & a_2 \end{vmatrix} \quad Q_m = \begin{vmatrix} a_1 & a_3 & a_5 & \cdots & a_{2m-1} \\ a_0 & a_2 & a_4 & \cdots & a_{2m-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_m \end{vmatrix}$$

The method is summarized in the following

Theorem 1.6 (Chebotarev-Meiman's Theorem [72]). *Assume that the functions $v(s)$ and $u(s)$ have no common zeros. Then the quasipolynomial (1.48) has no zeros with positive real part if and only if*

$$Q_m > 0, \qquad m = 1, 2, \ldots \tag{1.49}$$

Applications of this theorem are not effective because an infinity number of inequalities (1.49) must be verified.

Example 1.4. *Consider the second-order time-delay system*

$$\ddot{x}(t) - 2\dot{x}\left(t - \frac{\pi}{4}\right) + 8x\left(t - \frac{\sqrt{2}}{2}\right) = 0. \tag{1.50}$$

Now, in order to apply Theorem 1.6, we express the quasipolynomial as

$$P(s) = s^2 e^{\left(\frac{\pi}{4} + \frac{\sqrt{2}}{2}\right)s} - 2se^{\frac{\sqrt{2}}{2}s} + 8e^{\frac{\pi}{4}s}.$$

In such case, we have that $P(\boldsymbol{i}s)$ can be written as

$$P(\boldsymbol{i}s) = u(s) + \boldsymbol{i}v(s),$$

where

$$\begin{aligned} u(s) &= 8 - \left(1 + \tfrac{\pi^2}{4} - \sqrt{2}\right)s^2 + \left(-\tfrac{1}{4} + \tfrac{1}{6\sqrt{2}} - \tfrac{\pi}{4\sqrt{2}} - \tfrac{\pi^2}{32} - \tfrac{\pi^4}{768}\right)s^4 + \cdots \\ v(s) &= 2(-1 + \pi)s - \left(-\tfrac{1}{2} + \tfrac{1}{\sqrt{2}} + \tfrac{\pi}{4} + \tfrac{\pi^3}{48}\right)s^3 + \cdots \end{aligned}$$

Then, in order to evaluate condition (1.49), *lets form Q_1, in this case is given by*

$$Q_1 = 2(-1 + \pi).$$

Since, $Q_1 > 0$, we continue with the procedure. To this end, lets form Q_2,

$$Q_2 = \begin{vmatrix} 2(-1 + \pi) & -\tfrac{1}{2} + \tfrac{1}{\sqrt{2}} + \tfrac{\pi}{4} + \tfrac{\pi^3}{48} \\ 8 & 1 + \tfrac{\pi^2}{4} - \sqrt{2} \end{vmatrix}.$$

Since $Q_2 = 2 - 2\sqrt{2} - 2\sqrt{2}\pi - \frac{\pi^2}{2} + \frac{\pi^3}{3} \approx -4.31357 < 0$, we can see that condition (1.49) *is violated and according to Theorem 1.6 we have that the delay system* (1.50) *is unstable.*

From the above example it is clear to see, that the drawback of the method is that for stable quasipolynomials the method is not conclusive.

Remark 1.4. *There exist other methods to analysis the stability of a time-delay system, like $\tau-$decomposition [39], Yesupovisch-Svirskii [127], Integral Criterion [72], among others. However, in the seek of brevity we will omit such a discussion.*

1.3 Discrete-Time Systems

The general form of a *discrete-time system* is given by

$$x[\ell+1] = f(\ell, x[\ell]), \qquad x(\ell_0) = x_0, \tag{1.51}$$

with $x[\ell] \in \mathbb{R}^n$ is the state vector, $\ell \in \mathbb{N}$ and it is assumed that $f : \mathbb{N} \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function in x. A point x^* is said to be an *equilibrium point* of (1.51) if $f(\ell, x^*) = x^*$ for all $\ell \geq \ell_0$. As in the continuous case x^* is assumed to be the origin 0 and is called the *zero solution*. The justification of this assumption is as follows: Let $y(\ell) = x(\ell) - x^*$. Then, (1.51) becomes

$$y(\ell+1) = f(\ell, y(\ell) + x^*) - x^* = g(\ell, y(\ell)). \tag{1.52}$$

From the above equation it is clear to see that $y = 0$ correspond to $x = x^*$.

1.3.1 Stability Notions

Introduce now various stability notions of the equilibrium point x^* of (1.51).

Theorem 1.7. *[31] The equilibrium point x^* of (1.51) is said to be:*

(i) *Stable (S) if given $\varepsilon > 0$ and $\ell_0 \geq 0$ there exist $\delta = \delta_0(\varepsilon, \ell_0)$ such that $\|x_0 - x^*\| < \delta$ implies $\|x(\ell, \ell_0, x_0) - x^*\| < \varepsilon$ for all $\ell \geq \ell_0$, uniformly stable if δ may be chosen independent of ℓ_0, unstable if it is not stable.*

(ii) *Attracting (A) if there exist $\mu = \mu(\ell_0)$ such that $\|x_0 - x^*\| < \mu$ implies $\lim_{\ell \to \infty} x(\ell, \ell_0, x_0) = x^*$, uniformly attracting (UA) if the choice of μ is independent of ℓ_0. The condition for uniform attracting may be paraphrased by saying that there exists $\mu > 0$ such that for every ε and ℓ_0 such that $\|x(\ell, \ell_0, x_0) - x^*\| < \varepsilon$ for all $\ell > \ell_0 + L$ whenever $\|x_0 - x^*\| < \mu$.*

(iii) *Asymptotically stable (AS) if it is stable and attracting, and uniformly asymptotically stable (UAS) if it is uniformly stable and uniformly attracting.*

(iv) *Exponentially stable (ES) if there exist $\delta > 0$, $M > 0$, and $\eta \in (0,1)$ such that $\|x(\ell, \ell_0, x_0)\| \leq M \|x_0 - x^*\| \eta^{\ell - \ell_0}$, whenever $\|x_0 - x^*\| < \delta$.*

(v) *A solution $x(\ell, \ell_0, x_0)$ is bounded if for some positive constant M, $\|x(\ell, \ell_0, x_0)\| \leq M$ for all $\ell \geq \ell_0$, where M may depend on each solution*

If in parts (ii), (iii) $\mu = \infty$ or in part (iv) $\delta = \infty$, the corresponding stability property is said to be *global*. Observe that in the above definitions, some of the stability properties automatically imply one or more of the others. In general such implications can not be reversed, however, for some special classes of equations, such a implications may be reversed. In particular, the following result holds:

Theorem 1.8. *[31] For the autonomous system*

$$x[\ell+1] = f(x(\ell)), \tag{1.53}$$

the following statements holds for the equilibrium point x^*:

(i) $S \Leftrightarrow US$.

(ii) $AS \Leftrightarrow UAS$.

(iii) $A \Leftrightarrow UA$.

Linear Systems

In the sequel we will discuss briefly the nonautonomous case and then we will focus in the autonomous linear systems.

Nonautonomous Linear Systems

Lets consider the stability properties of the linear system given by

$$x[\ell+1] = A(\ell)\,x(\ell), \qquad \ell \geq \ell_0 \geq 0. \tag{1.54}$$

The following result guarantees the existence and uniqueness of the solutions of (1.54).

Theorem 1.9. *[31] For each $x_0 \in \mathbb{R}^n$ and $\ell_0 \in \mathbb{Z}^+$ there exist a unique solution $x(\ell,\ell_0,x_0)$ of (1.54) with $x(\ell_0,\ell_0,x_0) = x_0$.*

Let $\Psi(\ell)$ be an $n \times n$ matrix whose columns are solutions of (1.54). Then, $\Psi(\ell)$ satisfies the difference equation

$$\Psi(\ell+1) = A(\ell)\,\Psi(\ell). \tag{1.55}$$

Furthermore, the solutions $x_1(\ell), x_2(\ell), \ldots, x_n(\ell)$ are linearly independent if and only if the matrix $\Psi(\ell)$ is nonsingular (i.e., $\det \Psi(\ell) \neq 0$).

Definition 1.7. *[31] If $\Psi(\ell)$ is a matrix that is nonsingular for all $\ell \geq \ell_0$ and satisfies (1.55), then it is said to be a fundamental matrix for systems (1.54)*

One may, in general, write $\Psi(\ell,k) = \Psi(\ell)\Psi^{-1}(k)$ for any two positive integers ℓ, k with $\ell \geq k$. The fundamental matrix $\Psi(\ell,l)$ has the following properties:

(i) $\Psi^{-1}(\ell,k) = \Psi(k,\ell)$;

(ii) $\Psi(\ell,k) = \Psi(\ell,m)\,\Psi(m,k)$;

(iii) $\Psi(\ell,k) = \prod_{i=k}^{\ell-1} A(i)$.

Corollary 1.1. *[31] The unique solution of $x(\ell,\ell_0,x_0)$ of (1.54) with $x(\ell_0,\ell_0,x_0) = x_0$ is given by*

$$x(\ell,\ell_0,x_0) = \Psi(\ell,\ell_0)\,x_0.$$

Theorem 1.10. *[31] Consider system (1.54). Then its zero solution is*

(i) *stable if and only if there exist a positive constant M such that*

$$\|\Psi(\ell)\| \leq M, \qquad \textit{for } \ell \geq \ell_0 \geq 0; \tag{1.56}$$

(ii) *uniformly stable if and only if there exist a positive constant M such that*

$$\|\Psi(\ell,k)\| \leq M, \qquad \textit{for } \ell_0 \leq k \leq \ell < \infty; \tag{1.57}$$

(iii) *asymptotically stable if and only if*

$$\lim_{\ell \to \infty} \|\Psi(\ell)\| = 0; \tag{1.58}$$

(iv) *uniformly asymptotically stable if and only if there exist positive constants M and $\eta \in (0,1)$ such that:*

$$\|\Psi(\ell,k)\| \leq M\eta^{\ell-k}, \qquad \text{for } \ell_0 \leq k \leq \ell < \infty. \tag{1.59}$$

Corollary 1.2. *[31] For the linear system (1.54) the following statements hold:*

(i) *The zero solution is stable if and only if all solutions are bounded.*

(ii) *The zero solution is exponentially stable if and only if it is uniformly asymptotically stable.*

(iii) *Every local stability property of the zero solution implies the corresponding global stability property.*

Autonomous Linear Systems
In the following we will consider the autonomous (time-invariant) system given by

$$x[\ell+1] = Ax[\ell]. \tag{1.60}$$

The next result summarize the main stability results for the linear autonomous systems (1.60).

Theorem 1.11. *[31] The following statements holds:*

(i) *The zero solution of (1.60) is stable if and only if all eigenvalues of A has modulus less or equal than one and the eigenvalues on the unit circle are semisimple.*[1]

(ii) *The zero solution of (1.60) is asymptotically stable if and only if all eigenvalues of A have modulus less than one.*

2 Introductory Remarks to the Perturbation Theory for Linear Operators

Roughly speaking, Perturbation theory studies the behavior of a system subject to small perturbations in its variables. In order to illustrate the purpose of the perturbation theory, consider the particular case of a system represented by a differential equation (or, by a difference equation) $\dot{x}(t) = Ax(t) + f(t)$ (or, $x[\ell+1] = Ax[\ell] + f[\ell]$), for example we can be interested on the solution $x(t)$ (or, $x[\ell]$) if A exhibits a perturbation of the form $A + \varepsilon B$, where ε is a scalar quantity sufficiently small ($\varepsilon << 1$) and will be called the *perturbation factor*. In other words, we may be interested by the way of how a small parameter affects the behavior of some particular systems dynamics.

[1] An eigenvalue is said to be semisimple if the corresponding Jordan block is diagonal. For further details, see the next chapter where we will deeply discuss such ideas.

It should be stressed that *perturbation analysis* starts only after we have already obtained the solution of the original system; which means that the theory is there only to explore the change in the behavior of the system when perturbation take place.

The main goal of this chapter, is to introduce some fundamentals results concerning to the perturbation theory, which will be required in *parts* II-III to obtain some stability properties of the analyzed systems.

The fundamental motivation to use such a perturbation-based approach in *parts* II-III of the thesis, stays in the fact that we will study how a stable equilibrium state (or steady motion) becomes unstable or vice versa with a change of some parameters (and, in particular, the delay parameter). Thus, the *parameter space* is divided into stability and instability domains. On the other hand, we have that perturbation methods produce analytic approximations that often reveal the essential dependence of the exact solution on the parameters.

2.1 Basic Notions

2.1.1 The Eigenvalue Problem

Consider now an eigenvalue problem [135]

$$Au = \lambda u, \tag{2.1}$$

where A is an $n \times n$ real matrix, λ is an *eigenvalue*, and u is the corresponding *eigenvector*. The eigenvalues are determined form the *characteristic equation*

$$\det(A - \lambda I) = 0, \tag{2.2}$$

or, equivalently by

$$\det(\lambda I - A) = 0. \tag{2.3}$$

Since the $\det(A - \lambda I)$ is a polynomial of degree n with respect to λ, there are n eigenvalues, counting multiplicities. Since A is a real matrix, its eigenvalues and corresponding eigenvectors are real or appear in complex conjugate pairs. Multiplicity of an eigenvalue as a root of the characteristic equation is called *algebraic multiplicity*. The eigenvalue λ is called *simple* if its algebraic multiplicity is equal to one. There is a single eigenvector, up to a scaling factor, corresponding to a simple eigenvalue.

2.1.2 Multiple Eigenvalues and the Jordan Canonical Form

A multiple eigenvalue λ of algebraic multiplicity m can have one or several corresponding eigenvectors. The maximal number of linearly independent eigenvectors g is called *geometrical multiplicity* of the eigenvalue, which is less or equal to the algebraic multiplicity [90, 74],

$$g \leq m. \tag{2.4}$$

If the algebraic and geometric multiplicities are equal ($g = m$), then the eigenvalue is called *semi-simple*. If there is a single eigenvector corresponding to λ ($g = 1$), then the eigenvalue is called *non-derogatory* (see [114], for further details). If λ is a non-derogatory eigenvalue, then there exist a set of linearly independent vectors $\{u_0, u_1, \ldots, u_m - 1\}$ satisfying the following equations

$$\begin{aligned} Au_0 &= \lambda u_0, \\ Au_1 &= \lambda u_1 + u_0, \\ Au_2 &= \lambda u_2 + u_1, \\ &\vdots \\ Au_{m-1} &= \lambda u_{m-1} + u_{m-2}. \end{aligned} \tag{2.5}$$

The vectors $u_0, u_1, \ldots, u_{m-1}$ are called *Jordan chain of length m*, where u_0 is the eigenvector and the vectors $u_1, \ldots, u_{m-1}$ are called *generalized eigenvector* (or *associated eigenvectors*). If λ is an eigenvalue having several linearly independent eigenvectors (the *derogatory case*), i.e., $g > 1$, then in this case there are nonnegative integers $1 \leq m_1 \leq \cdots \leq m_g$ such that

$$m_1 + \cdots + m_g = m, \tag{2.6}$$

and linearly independent vectors $u_0^{(i)}, \ldots, u_{m_i-1}^{(i)}$, $i = 1, \ldots, g$, satisfying the Jordan Chain equations

$$\begin{aligned} Au_0^{(i)} &= \lambda u_0^{(i)}, \\ Au_1^{(i)} &= \lambda u_1^{(i)} + u_0^{(i)}, \\ Au_2^{(i)} &= \lambda u_2^{(i)} + u_1^{(i)}, \\ &\vdots \\ Au_{m_i-1}^{(i)} &= \lambda u_{m_i-1}^{(i)} + u_{m_i-2}^{(i)}. \end{aligned} \tag{2.7}$$

The numbers $m_1, \ldots, m_g$ are unique and called *partial multiplicities* of the eigenvalue λ, and the eigenvectors $u_0^{(i)}, \ldots, u_{m_i-1}^{(i)}$ are called the Jordan chain of length m_i. In general, a multiple eigenvalue λ with $1 \leq g < m$ is called a *nonsemisimple* eigenvalue.

Equation (2.7) can be written in the following matrix form

$$AU_\lambda = U_\lambda \begin{pmatrix} J_{m_1}(\lambda) & & \\ & \ddots & \\ & & J_{m_g}(\lambda) \end{pmatrix}, \tag{2.8}$$

where

$$U_\lambda = \left[u_0^{(1)}, \ldots, u_{m_1-1}^{(1)}, \ldots, u_0^{(g)}, \ldots, u_{m_g-1}^{(g)} \right] \tag{2.9}$$

is an $n \times m$ matrix.

2.2 Analytic Perturbations

As mentioned at the beginning of the chapter, the goal of this chapter is to present the tools to investigate how the eigenvalues and eigenvector (or eigenspaces) of a linear operator T change when T is subject to a small perturbation. In dealing with such a problem, it is often convenient to consider a family of operators of the form

$$T(x) = T + x\widetilde{T}. \tag{2.10}$$

where $T(0) = T$ is called the *unperturbed operator* and $\varepsilon\widetilde{T}$ the *perturbation*. In a more general case, we will assume that in the neighborhood of $x = 0$, the perturbed operator $T(x)$ is holomorphic, or equivalently, can be expanded into the power series,

$$T(x) = T(0) + xT'(0) + \frac{1}{2!}x^2 T''(0) + \frac{1}{3!}x^3 T'''(0) + \cdots . \tag{2.11}$$

The eigenvalue of $T(x)$ satisfies the characteristic equation,

$$\det(T(x) - \xi I) = 0. \tag{2.12}$$

This is an algebraic equation in ξ of degree n, with coefficients which are holomorphic in x (see, [67] for further details).

2.3 Perturbations of a Simple and Semisimple Eigenvalues

At a first instance, consider the family of operators given by (2.10), i.e.,

$$T(x) = T + x\widetilde{T}.$$

Then, if T has the eigenvalues $\lambda_1^{(0)}, \lambda_2^{(0)}, \ldots, \lambda_m^{(0)}$ and eigenvectors $u_1, \ldots, u_n$ the eigenvalue problem for the perturbed system can be written as

$$\left(T + x\widetilde{T}\right)\mu(x) = \lambda^{(0)}\mu(x).$$

From the above equation, the vector $u(x)$ has a nontrivial solution only if

$$\det\left(\mu I - T - x\widetilde{T}\right) = 0. \tag{2.13}$$

Obviously, (2.13) can be written in powers of x as follows:

$$\det\left(\mu I - T - x\widetilde{T}\right) = f_0\left(\lambda^{(0)}\right) + x f_1\left(\lambda^{(0)}\right) + \cdots + x^n f_n\left(\lambda^{(0)}\right),$$

where

$$f_0(\lambda) = \det(\mu I - T).$$

The following theorem gives the first order terms in the case of a simple eigenvalue.

Theorem 2.1 ([30])**.** *If $\lambda_i^{(0)}$ is a distinct eigenvalue of a semisimple matrix T with corresponding eigenvector $u_i^{(0)}$, the eigenvalue $\mu_i(x)$ and its corresponding eigenvector $u_i(x)$ of the perturbed matrix $T + x\widetilde{T}$ are given for first order approximation by*

$$\mu_i(x) = \lambda_i^{(0)} + x\left\langle v_i^{(0)}, \widetilde{T}u_i^{(0)}\right\rangle + o(x), \tag{2.14}$$

and

$$u_i(x) \approx u_i^{(0)} + x\sum_{\ell=1,\ell\neq i}^{n} \frac{\left\langle v_\ell^{(0)}, \widetilde{T}u_i^{(0)}\right\rangle}{\lambda_i^{(0)} - \lambda_\ell^{(0)}} u_\ell^{(0)}, \tag{2.15}$$

where $u_1^{(0)}, \ldots, u_n^{(0)}$ are the eigenvectors of T and $v_1^{(0)}, \ldots v_n^{(0)}$ their reciprocal basis.

Example 2.1. *Let T and $\widetilde{T}$ be*

$$T=\begin{bmatrix}4&1&-1\\2&5&-2\\1&1&2\end{bmatrix},\qquad \widetilde{T}=\begin{bmatrix}1&0&-2\\5&1&1\\0&-1&1\end{bmatrix}$$

the eigenvalues of T are $\lambda=5,3,3$. The independent eigenvectors are

$$u_1=\begin{bmatrix}1\\2\\1\end{bmatrix},\qquad u_2=\begin{bmatrix}1\\0\\1\end{bmatrix}\quad\text{and}\quad u_3=\begin{bmatrix}-1\\1\\0\end{bmatrix}.$$

Their reciprocal basis are

$$v_1=\begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\\-\frac{1}{2}\end{bmatrix},\qquad v_2=\begin{bmatrix}-\frac{1}{2}\\-\frac{1}{2}\\\frac{3}{2}\end{bmatrix}\quad\text{and}\quad v_3=\begin{bmatrix}-1\\0\\1\end{bmatrix}.$$

Then, according to (2.14) *the first order approximation for the eigenvalue $\lambda_1^{(0)}=5$ is given by*

$$\begin{aligned}\mu_1(x)&=\lambda_1^{(0)}+x\left\langle v_1,\widetilde{T}u_1\right\rangle+o(x)\\&=5+\left\langle\begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\\-\frac{1}{2}\end{bmatrix},\begin{bmatrix}1&0&-2\\5&1&1\\0&-1&1\end{bmatrix}\begin{bmatrix}1\\2\\1\end{bmatrix}\right\rangle x+o(x)\\&=5+4x+o(x).\end{aligned}$$

The following result cover the case when $\lambda_i^{(0)}$ is a semi-simple eigenvalue.

Theorem 2.2 ([30])**.** *If $\lambda^{(0)}$ is an eigenvalue of multiplicity m of a semi-simple matrix T with corresponding eigenvectors $u_1^{(0)},u_2^{(0)},\ldots,u_m^{(0)}$, the eigenvalue $\mu_i(x)$ and the corresponding eigenvector $u_i(x)$ of the perturbed matrix $T+x\widetilde{T}$ are given for first order approximation by*

$$\mu_i(x)=\lambda_i^{(0)}+x\lambda_i^{(1)}+o(x),\qquad i=1,\ldots,m,\tag{2.16}$$

where $\lambda_i^{(1)}$, are given by the $m-$eigenvalues of

$$S=\begin{bmatrix}v_1^{(0)*}\\\vdots\\v_m^{(0)*}\end{bmatrix}\widetilde{T}\begin{bmatrix}u_1^{(0)}&\cdots&u_m^{(0)}\end{bmatrix},$$

and

$$u_i(x)=v_i+x\sum_{k=m+1}^{n}\frac{\sum_{j=1}^{m}c_{ij}\left\langle v_k^{(0)},\widetilde{T}u_j^{(0)}\right\rangle}{\lambda^{(0)}-\lambda_k^{(0)}},\quad i=1,\ldots,m,$$

where $[\cdot]^*$ is the transpose conjugate of the vector $[\cdot]$, $u_1^{(0)},\ldots,u_n^{(0)}$ are the eigenvectors of T, and $v_1^{(0)},\ldots,v_n^{(0)}$ their reciprocal basis. The values of c_{ij} are obtained by solving the set of linear simultaneous equations

$$\left[\lambda_i^{(1)}I-S\right]\begin{bmatrix}c_{i1}\\\vdots\\c_{im}\end{bmatrix}=0,\quad i=1,2,\ldots,m,$$

and

$$v_i = \sum_{j=1}^{m} c_{ij} u_j^{(0)}.$$

Example 2.2. *Consider the same matrices T and $\widetilde{T}$ treated in example 2.1. Since the eigenvalue $\lambda^{(0)} = 3$, is semi-simple, we can apply the later result. In such a case, we have:*

$$S = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} & \frac{3}{2} \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -2 \\ 5 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{2.17}$$

$$= \begin{bmatrix} -1 & 1 \\ 2 & 0 \end{bmatrix}. \tag{2.18}$$

Simple computations show that matrix S has the eigenvalues $\lambda_i^{(1)} = -2, 1$. Then, according to (2.16) *we conclude that*

$$\mu_2(x) = 3 - 2x + o(x),$$
$$\mu_3(x) = 3 + x + o(x).$$

Consider now a more general operator, i.e., assume that $T(x)$ is holomorphic, or equivalently, admits an expansion of the form (2.11). In this case, we have the following:

Theorem 2.3. *[67] Let $\lambda^{(0)}$ be a semisimple eigenvalue of $T(0)$ with multiplicity m, and P be the eigenprojection for $\lambda^{(0)}$, that is,*

$$P = \frac{1}{2\pi \boldsymbol{i}} \oint_{\Gamma} (\xi I - T(0))^{-1} d\xi, \tag{2.19}$$

where Σ is a positive-oriented closed contour enclosing $\lambda^{(0)}$ but no other eigenvalues of $T(0)$. Then the corresponding eigenvalues of $T(x)$ are analytic in x and have the form:

$$\mu_i(x) = \lambda^{(0)} + \lambda^{(1)} x + o(x^2), \quad i = 1, \ldots, m, \tag{2.20}$$

where $\lambda_i^{(1)}$ are the eigenvalues of $PT'(0)P$.

Without any loss of generality assume that $\lambda^{(0)}$ be ordered as the first eigenvalue of $T(0)$ with multiplicity m. Then, according to the results presented in § 2.1, $T(0)$ can be decomposed as:

$$T(0) = Q\Sigma R = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix} \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix} \begin{bmatrix} R_1 \\ R_2 \end{bmatrix}, \tag{2.21}$$

where Σ_1 is a diagonal matrix with diagonal entries as $\lambda^{(0)}$, $R = Q^{-1} = \begin{bmatrix} r_1^T & \cdots & r_n^T \end{bmatrix}$, and $Q = \begin{bmatrix} q_1 & \cdots & q_n \end{bmatrix}$ consist of the eigenvectors of $T(0)$. Based on this decomposition, the following result shows how $\lambda_i^{(1)}$ may be computed.

Lemma 2.1. *[21] Let $T(0)$ be partitioned as in (2.21). Then the corresponding eigenvalues of $T(x)$ are analytic in x and have the form*

$$\mu_i(x) = \lambda^{(0)} + \lambda_i^{(1)} x + o(x), \quad i = 1, \ldots, m,$$

where $\lambda_i^{(1)}$, $i = 1, \ldots, m$, are the eigenvalues of $R_1 T'(0) Q_1$.

Example 2.3. *Let $T(x) \in \mathbb{C}^{3\times 3}$ be the matrix operator*

$$T(x) = \begin{bmatrix} 4-2\cos(2x) & 16 & 7+e^{\boldsymbol{i}x} \\ 4+x & 14-x^2 & 8 \\ e^{-x}\left(x^2+\sin(x)\right)-8 & -31-\cos(x) & x^2-18 \end{bmatrix}.$$

Simple computations reveal that $T(0)$ has a semisimple eigenvalue at $\lambda_i^{(0)} = -2$, $i \in \{1,2\}$ and a simple eigenvalue at $\lambda_3^{(0)} = 2$. Following Lemma 2.1, we first find

$$T(0) = \begin{bmatrix} 2 & 16 & 8 \\ 4 & 14 & 8 \\ -8 & -32 & -18 \end{bmatrix},$$

which can be decomposed as

$$T(0) = \left[\begin{array}{cc|c} -2 & -4 & -1 \\ 0 & 1 & -1 \\ 1 & 0 & 2 \end{array}\right] \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \left[\begin{array}{ccc} 2 & 8 & 5 \\ -1 & -3 & -2 \\ \hline -1 & -4 & -2 \end{array}\right].$$

Since,

$$T'(0) = \begin{bmatrix} 0 & 0 & -\boldsymbol{i} \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

it follows that

$$\begin{aligned} R_1 T'(0) Q_1 &= \begin{bmatrix} 2 & 8 & 5 \\ -1 & -3 & -2 \end{bmatrix} \begin{bmatrix} 0 & 0 & -\boldsymbol{i} \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} -2 & -4 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} -26-2\boldsymbol{i} & -52 \\ 10+\boldsymbol{i} & 20 \end{bmatrix}. \end{aligned}$$

Simple computations show that $R_1 T'(0) Q_1$ has eigenvalues $\lambda_1^{(1)} = -6$ and $\lambda_2^{(1)} = -2\boldsymbol{i}$. In conclusion, we find

$$\begin{aligned} \mu_1(x) &= -2-6x+o(x), \\ \mu_2(x) &= -2-2\boldsymbol{i}x+o(x). \end{aligned}$$

2.3.1 Second-Order Asymptotic Expansion

The first-order asymptotic series introduced in the previous paragraph can be further developed to include terms of higher orders. In this section, we will derive and recall some formulas for computing the coefficients of the asymptotic series up to the second-order.
Consider first the asymptotic expansion of the operator (2.10). Then, according to [131], for small $\left(\mu - \lambda^{(0)}\right)$, the simple eigenvalues of (2.13) can be expanded as:

$$\mu_i(x) = \lambda_i^{(0)} + x\lambda_i^{(1)} + x^2\lambda_i^{(2)} + \cdots$$

and the problem reduce in the computation of $\lambda^{(1)}, \lambda^{(2)}, \ldots$. In this case, following [131], the eigenvector has the following expansion:

$$u(x) = u_i^{(0)} + xu_i^{(1)} + x^2u_i^{(2)} + \cdots.$$

To compute the shift in the eigenvalues and eigenvectors, we can substitute $\mu_i(x)$ and $u(x)$ in the eigenvalue problem of the perturbed system to obtain:

$$\left(T + x\widetilde{T}\right)\left(u_i^{(0)} + xu_i^{(1)} + x^2u_i^{(2)} + \cdots\right) = \left(\lambda_i^{(0)} + x\lambda_i^{(1)} + x^2\lambda_i^{(2)} + \cdots\right)\left(u_i^{(0)} + xu_i^{(1)} + x^2u_i^{(2)} + \cdots\right).$$

Grouping appropriately the terms in x, we obtain:

$$\begin{cases} Tu_i^{(0)} = \lambda_i^{(0)}u_i^{(0)} \\ Tu_i^{(1)} + \widetilde{T}u_i^{(0)} = \lambda_i^{(0)}u_i^{(1)} + \lambda_i^{(1)}u_i^{(0)} \\ Tu_i^{(2)} + \widetilde{T}u_i^{(1)} = \lambda_i^{(0)}u_i^{(2)} + \lambda_i^{(1)}u_i^{(1)} + \lambda_i^{(2)}u_i^{(0)} \\ \vdots \\ Tu_i^{(k+1)} + \widetilde{T}u_i^{(k)} = \lambda_i^{(0)}u_i^{(k+1)} + \lambda_i^{(1)}u_i^{(k)} + \cdots + \lambda_i^{(k+1)}u_i^{(0)} \\ \vdots \end{cases} \tag{2.22}$$

Since we have already presented the results to compute $\lambda_i^{(1)}$ and $u_i^{(1)}$, we will focus in deriving the computation of $\lambda_i^{(2)}$. Let $v_1^{(0)}, \ldots, v_n^{(0)}$ be the reciprocal basis of $u_1^{(0)}, \ldots, u_n^{(0)}$, then taking the inner product of the third equation in (2.22) with $v_i^{(0)}$ we obtain:

$$\left\langle v_i^{(0)}, Tu_i^{(2)} \right\rangle + \left\langle v_i^{(0)}, \widetilde{T}u_i^{(1)} \right\rangle = \lambda_i^{(0)} \left\langle v_i^{(0)}, u_i^{(2)} \right\rangle + \lambda_i^{(1)} \left\langle v_i^{(0)}, u_i^{(1)} \right\rangle + \lambda_i^{(2)} \left\langle v_i^{(0)}, u_i^{(0)} \right\rangle.$$

However,

$$\left\langle v_i^{(0)}, Tu_i^{(2)} \right\rangle = \lambda_i^{(0)} \left\langle v_i^{(0)}, u_i^{(2)} \right\rangle.$$

Therefore, we have established:

Lemma 2.2. *Let $\lambda_i^{(0)}$ be a distinct simple eigenvalue of T with corresponding eigenvector $u_i^{(0)}$. Then, the corresponding eigenvalues of $T(x) = T + x\widetilde{T}$ have the form*

$$\mu_i(x) = \lambda_i^{(0)} + x\lambda_i^{(1)} + x^2\lambda_i^{(2)} + o\left(x^2\right),$$

where $\lambda_i^{(1)}$ are computing according to (2.14),

$$\lambda_i^{(2)} = \left\langle v_i^{(0)}, \widetilde{T}u_i^{(1)} \right\rangle - \lambda_i^{(1)} \left\langle v_i^{(0)}, u_i^{(1)} \right\rangle \tag{2.23}$$

where $v_i^{(0)}$ is the reciprocal vector of $u_i^{(0)}$ and $u_i^{(1)}$ are computed according to (2.15).

Example 2.4. *Let*

$$T = \begin{bmatrix} 1 & \frac{2}{5} \\ 5 & -1 \end{bmatrix}, \qquad \widetilde{T} = \begin{bmatrix} 1 & \frac{2}{5} \\ 5 & -1 \end{bmatrix}.$$

The eigenvalues of T are $\lambda_1^{(0)} = \sqrt{3}\boldsymbol{i}$ and $\lambda_2^{(0)} = -\sqrt{3}\boldsymbol{i}$. The eigenvectors of T are:

$$u_1^{(0)} = \begin{bmatrix} -\frac{1+\sqrt{3}\boldsymbol{i}}{10} \\ 1 \end{bmatrix} \qquad \textit{and} \qquad u_2^{(0)} = \begin{bmatrix} -\frac{1-\sqrt{3}\boldsymbol{i}}{10} \\ 1 \end{bmatrix}.$$

Their reciprocal basis are:

$$v_1^{(0)} = \begin{bmatrix} -\frac{5}{\sqrt{3}}\boldsymbol{i} \\ \frac{1}{2} - \frac{1}{2\sqrt{3}}\boldsymbol{i} \end{bmatrix} \quad \textit{and} \quad v_2^{(0)} = \begin{bmatrix} \frac{5}{\sqrt{3}}\boldsymbol{i} \\ \frac{1}{2} + \frac{1}{2\sqrt{3}}\boldsymbol{i} \end{bmatrix}.$$

Next, in order to compute the first-order eigenvalue approximation we apply Lemma 2.1, leading to:

$$\begin{aligned} \lambda_1^{(1)} &= \left\langle v_1^{(0)}, \widetilde{T}u_1^{(0)} \right\rangle, \\ &= \left\langle \begin{bmatrix} -\frac{5}{\sqrt{3}}\boldsymbol{i} \\ \frac{1}{2} - \frac{1}{2\sqrt{3}}\boldsymbol{i} \end{bmatrix}, \begin{bmatrix} 1 & \frac{2}{5} \\ 5 & -1 \end{bmatrix} \begin{bmatrix} -\frac{1+\sqrt{3}\boldsymbol{i}}{10} \\ 1 \end{bmatrix} \right\rangle, \\ \lambda_1^{(1)} &= 0. \end{aligned}$$

Similar computations reveal that $\lambda_2^{(1)} = 0$ *concluding that none important information can be obtained from the first order analysis. Then, a second order analysis will be required. In this vein, lets compute first* $u_1^{(1)}$*:*

$$\begin{aligned} u_1^{(1)} &= \sum_{\ell=1, \ell \neq i}^{2} \frac{\left\langle v_\ell^{(0)}, \widetilde{T}u_i^{(0)} \right\rangle}{\lambda_i^{(0)} - \lambda_\ell^{(0)}} u_\ell^{(0)} \\ &= \frac{\left\langle v_2^{(0)}, \widetilde{T}u_1^{(0)} \right\rangle}{\lambda_1^{(0)} - \lambda_2^{(0)}} u_2^{(0)} \\ &= \frac{\left\langle \begin{bmatrix} \frac{5}{\sqrt{3}}\boldsymbol{i} \\ \frac{1}{2} + \frac{1}{2\sqrt{3}}\boldsymbol{i} \end{bmatrix}, \begin{bmatrix} 1 & \frac{2}{5} \\ 5 & -1 \end{bmatrix} \begin{bmatrix} -\frac{1+\sqrt{3}\boldsymbol{i}}{10} \\ 1 \end{bmatrix} \right\rangle}{2\sqrt{3}\boldsymbol{i}} \begin{bmatrix} -\frac{1-\sqrt{3}\boldsymbol{i}}{10} \\ 1 \end{bmatrix} \\ u_1^{(1)} &= \begin{bmatrix} -\frac{1}{20} - \frac{\sqrt{3}}{20}\boldsymbol{i} \\ -\frac{1}{4} + \frac{\sqrt{3}}{4}\boldsymbol{i} \end{bmatrix}. \end{aligned}$$

Next, according to (2.23)*, we have:*

$$\begin{aligned} \lambda_1^{(2)} &= \left\langle v_1^{(0)}, \widetilde{T}u_1^{(1)} \right\rangle - \lambda_1^{(1)} \left\langle v_1^{(0)}, u_1^{(1)} \right\rangle, \\ &= \left\langle v_1^{(0)}, \widetilde{T}u_1^{(1)} \right\rangle, \\ &= \left\langle \begin{bmatrix} -\frac{5}{\sqrt{3}}\boldsymbol{i} \\ \frac{1}{2} - \frac{1}{2\sqrt{3}}\boldsymbol{i} \end{bmatrix}, \begin{bmatrix} 1 & \frac{2}{5} \\ 5 & -1 \end{bmatrix} \begin{bmatrix} -\frac{1}{20} - \frac{\sqrt{3}}{20}\boldsymbol{i} \\ -\frac{1}{4} + \frac{\sqrt{3}}{4}\boldsymbol{i} \end{bmatrix} \right\rangle, \\ \lambda_1^{(2)} &= -\frac{\sqrt{3}}{2}\boldsymbol{i}. \end{aligned}$$

Following the same steps, we find $\lambda_2^{(2)} = \frac{\sqrt{3}}{2}\boldsymbol{i}$*. As a result, the eigenvalues of* $T(x)$ *are given by:*

$$\begin{aligned} \mu_1(x) &= \sqrt{3}\boldsymbol{i} - \frac{\sqrt{3}}{2}\boldsymbol{i}x^2 + o\left(x^2\right), \\ \mu_2(x) &= \sqrt{3}\boldsymbol{i} + \frac{\sqrt{3}}{2}\boldsymbol{i}x^2 + o\left(x^2\right). \end{aligned}$$

In the sequel, assume that $T(x)$ is holomorphic around a neighborhood of $x=0$, that is, it admits an expansion as (2.11).
Define the operator-valued function:

$$\Upsilon(\xi) = (T(0) - \xi I)^{-1}, \tag{2.24}$$

which is known as the resolvent of $T(0)$. It is obvious that the singularities of $\Upsilon(\xi)$ are the eigenvalues of $T(0)$. Let $\lambda^{(0)}$ be a semisimple eigenvalue of $T(0)$. Then $\Upsilon(\xi)$ can be expanded as a Laurent series at $\xi = \lambda^{(0)}$, that is,

$$\Upsilon(\xi) = -(\xi - \lambda^{(0)})^{-1}P - \sum_{n=1}^{\infty}(\xi - \lambda^{(0)})^{-n-1}D_n + \sum_{n=0}^{\infty}(\xi - \lambda^{(0)})^n S_{n+1}, \tag{2.25}$$

where P, D_n and S_{n+1} are the corresponding coefficient matrices. Evidently, the matrix P, known as the eigenprojection for $\lambda^{(0)}$, can be found as

$$P = -\frac{1}{2\pi \boldsymbol{i}}\oint_{\Gamma}\Upsilon(\xi)d\xi = \frac{1}{2\pi \boldsymbol{i}}\oint_{\Gamma}(\xi I - T(0))^{-1}d\xi,$$

where Γ is a positively-oriented closed contour enclosing $\lambda^{(0)}$ but no other eigenvalues of $T(0)$. It was found in [21] that

$$P = Q_1 R_1.$$

The holomorphic part in the Laurent expansion is called the *reduced resolvent* of $T(0)$ with respect to the eigenvalue $\lambda^{(0)}$, denoted as

$$S(\xi) = \sum_{n=0}^{\infty}\left(\xi - \lambda^{(0)}\right)^n S_{n+1}.$$

Let $S = S(\lambda^{(0)})$, namely the value of the reduced resolvent of $T(0)$ at $\xi = \lambda^{(0)}$. Then it is obvious that

$$S = S_1 = \frac{1}{2\pi \boldsymbol{i}}\oint_{\Gamma}\frac{\Upsilon(\xi)}{\xi - \lambda^{(0)}}d\xi.$$

Lemma 2.3. *[36] For any matrix $T(0)$ decomposed in the form of (2.21), where Σ_1 is in Jordan form with diagonal entries as $\lambda^{(0)}$, the reduced resolvent at $\xi = \lambda^{(0)}$ is equal to:*

$$S = Q\begin{bmatrix} 0 & 0 \\ 0 & (\Sigma_2 - \lambda^{(0)}I)^{-1}\end{bmatrix}R = Q_2(\Sigma_2 - \lambda^{(0)}I)^{-1}R_2.$$

The following Lemma given in [67] provides the result on the second-order perturbation of $T(x)$ when all the eigenvalues of $T(0)$ are semisimple.

Lemma 2.4. *[67] Let $\lambda^{(0)}$ be a semi-simple eigenvalue of $T(0)$, $\lambda_i^{(1)}$ be a semi-simple eigenvalue of $PT'(0)P$ with the eigenprojection $P_i^{(1)}$, that is*

$$P_i^{(1)} = \oint_{\Gamma_i}\left(\xi I - PT'(0)P\right)^{-1}d\xi, \tag{2.26}$$

where Γ_i is a positively-oriented closed contour enclosing $\lambda_i^{(1)}$ but no other eigenvalues of $PT'(0)P$. Then $T(x)$ has $d = \dim P_i^{(1)}$ repeated eigenvalues of the form

$$\mu_{ip}(x) = \lambda^{(0)} + x\lambda_i^{(1)} + x^2\mu_{ip}^{(2)} + o(x^2),\ \ p = 1,\cdots,d, \tag{2.27}$$

where $\mu_{ip}^{(2)}$ are the repeated eigenvalues of $P_i^{(1)}T^{(2)}P_i^{(1)}$ with $T^{(2)} = T''(0) - T'(0)ST'(0)$, and

$$P_i^{(1)}T^{(2)}P_i^{(1)} = P_i^{(1)}T''(0)P_i^{(1)} - P_i^{(1)}T'(0)ST'(0)P_i^{(1)}.$$

The eigenvalues of $P_i^{(1)}T^{(2)}P_i^{(1)}$ can be computed in a manner similar to that in the first-order analysis.

Lemma 2.5. *[36] Let $\lambda^{(0)}$ be a semisimple eigenvalue of $T(0)$, $\lambda_i^{(1)}$ be a semisimple eigenvalue of $PT'(0)P$. Let also $T(0)$ be decomposed as in (2.21), and $R_1T'(0)Q_1$ be decomposed as:*

$$\begin{aligned} R_1T'(0)Q_1 &= Q^{(2)}\Sigma^{(2)}R^{(2)} \\ &= \begin{bmatrix} Q_1^{(2)} & Q_2^{(2)} \end{bmatrix} \begin{bmatrix} \Sigma_1^{(2)} & 0 \\ 0 & \Sigma_2^{(2)} \end{bmatrix} \begin{bmatrix} R_1^{(2)} \\ R_2^{(2)} \end{bmatrix}, \end{aligned} \tag{2.28}$$

where $\Sigma_1^{(2)}$ is the Jordan block corresponding to the eigenvalue $\lambda_i^{(1)}$. Then the eigenvalues of $P_i^{(1)}T^{(2)}P_i^{(1)}$ are those of the matrix $R^{(2)}R_1T^{(2)}Q_1Q_1^{(2)}$.

2.4 Perturbations of a Nonsemisimple Eigenvalues

As previously we will consider first in this subsection the family of operators given in (2.10), i.e.,

$$T(x) = T + x\widetilde{T}$$

In this case the corresponding coefficients can be calculated by means of the result:

Theorem 2.4. *[30] If $\lambda^{(0)}$ is an nonsemisimple eigenvalue of multiplicity m of T, with corresponding generalized eigenvectors $u_1, \ldots, u_m$, then the eigenvalues $\mu_\ell(x)$ ($\ell = 1, \ldots, m$) of $T(x) = T + x\widetilde{T}$ will lie, for small enough $|x|$, on the circumference of a circle with center c and radius $r \approx \left|\lambda_\ell^{(1)} \sqrt[m]{x}\right|$, where*

$$c \approx \lambda^{(0)} + \frac{1}{m}\sum_{i=1}^{m} \left\langle v_i, x\widetilde{T}u_i \right\rangle,$$

and

$$\lambda_\ell^{(1)} = \sqrt[m]{\langle v_m, \widetilde{T}u_1\rangle} e^{\frac{2\ell\pi}{m}\mathbf{i}}, \quad \ell = 1, \ldots, m,$$

where $v_1, \ldots, v_m$ are the reciprocal bases for $u_1, \ldots, u_m$. And the eigenvector $u(x)$ of $T(x)$ is given by

$$u(x)_\ell \approx \widehat{u}_\ell + \sqrt[m]{x}\lambda_\ell^{(1)}u_2,$$

where $\widehat{u}_\ell$ is the eigenvector of T, and is assumed that T is non-derogatory without loss of generality.

Example 2.5. *Let T and $\widetilde{T}$ be given by*

$$T = \begin{bmatrix} 3 & -2 & 5 \\ 0 & 1 & 4 \\ 0 & -1 & 5 \end{bmatrix}, \qquad \widetilde{T} = \begin{bmatrix} 1 & -3 & -1 \\ 0 & -3 & 5 \\ 1 & 0 & 5 \end{bmatrix}.$$

The eigenvalues of T are $\lambda^{(0)} = 3,3,3$, with corresponding generalized eigenvectors given by:

$$u_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \quad \textit{and} \quad u_3 = \begin{bmatrix} 1 \\ -3 \\ -1 \end{bmatrix}.$$

The reciprocal basis are

$$v_1 = \begin{bmatrix} 1 \\ 2 \\ -5 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix} \quad \textit{and} \quad v_3 = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}.$$

Next, in order to apply Theorem 2.4, we made the following computations:

$$\begin{aligned} \left\langle v_3, \widetilde{T}u_1 \right\rangle &= \left\langle \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 & -3 & -1 \\ 0 & -3 & 5 \\ 1 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\rangle, \\ &= 2. \end{aligned}$$

Therefore,

$$\begin{aligned} \mu_1(x) &= 3 + \sqrt[3]{2}x^{1/3} + o\left(x^{1/3}\right), \\ \mu_2(x) &= 3 + \sqrt[3]{2}\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}\boldsymbol{i}\right)x^{1/3} + o\left(x^{1/3}\right), \\ \mu_3(x) &= 3 + \sqrt[3]{2}\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}\boldsymbol{i}\right)x^{1/3} + o\left(x^{1/3}\right). \end{aligned}$$

Consider now a more general operator $T(x)$. It is assumed that it is holomorphic around a neighborhood of $x = 0$, or equivalently that admits the expansion (2.11). If $\lambda^{(0)}$ is a non-semisimple eigenvalue of multiplicity m. In this case, according to § 2.1 $T(0)$ admits a Jordan decomposition in which Σ is block diagonal with diagonal Jordan blocks, and Q_1 consists of the generalized eigenvectors associated with $\lambda^{(0)}$. In particular,

$$\Sigma_1 = \begin{bmatrix} \lambda^{(0)} & 1 & \cdots & 0 \\ 0 & \lambda^{(0)} & \ddots & 0 \\ \vdots & \vdots & \ddots & 1 \\ 0 & \cdots & \cdots & \lambda^{(0)} \end{bmatrix}.$$

The eigenvalue of $T(x)$ can no longer be expanded in the form of (2.20), but instead as a Puiseux series [67].

Lemma 2.6. *[21] Let $\lambda^{(0)}$ be a non-semi-simple eigenvalue of $T(0)$ with multiplicity m. Then the corresponding eigenvalues of $T(x)$ have the form*

$$\mu_i(x) = \lambda^{(0)} + \left(\gamma_i^{(1)}\right)^{\frac{1}{m}} x^{\frac{1}{m}} + \cdots, \quad i = 1, \ldots, m, \tag{2.29}$$

where $\gamma_i^{(1)} = r_m T'(0) q_1$.

Illustrate now the above result with the following:

Example 2.6. *Consider a matrix operator $T(x) \in \mathbb{C}^{4\times 4}$ where*

$$T(x) = \begin{bmatrix} 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & 2 \\ \sqrt{2} & 0 & 0 & \boldsymbol{i}2\sqrt{2}e^{-4\boldsymbol{i}x} \\ 0 & 1 & \boldsymbol{i}2\sqrt{2}e^{-4\boldsymbol{i}x}(1-x) & 0 \end{bmatrix} \tag{2.30}$$

A simple computation reveals that $T(0)$ has 2 non-semi-simple eigenvalues at $\pm\boldsymbol{i}\sqrt{2}$, with multiplicity 2. More over, we found:

$$T'(0) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 8\sqrt{2}e^{-4\boldsymbol{i}x} \\ 0 & 0 & -2\sqrt{2}e^{-4\boldsymbol{i}x}((\boldsymbol{i}-4)+4x) & 0 \end{bmatrix},$$

and

$$Q_1 = \begin{bmatrix} -\boldsymbol{i} & \frac{\sqrt{2}}{2} \\ -\boldsymbol{i}\sqrt{2} & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}, \qquad R_1 = \begin{bmatrix} 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{\sqrt{2}}{2} & \frac{1}{2} & \frac{\sqrt{2}}{2}\boldsymbol{i} & \frac{\sqrt{2}}{2}\boldsymbol{i} \end{bmatrix}.$$

According to Lemma 2.6 we have that $r_2 T'(0) q_1 = 2 + 16\boldsymbol{i}$, thus we find

$$\begin{aligned} \mu_1(x) &= \sqrt{2}\boldsymbol{i} + \sqrt{2+16\boldsymbol{i}}x^{1/2} + o\left(x^{1/2}\right), \\ \mu_2(x) &= \sqrt{2}\boldsymbol{i} - \sqrt{2+16\boldsymbol{i}}x^{1/2} + o\left(x^{1/2}\right). \end{aligned}$$

Fig. 2.6 illustrate the eigenvalue behavior.

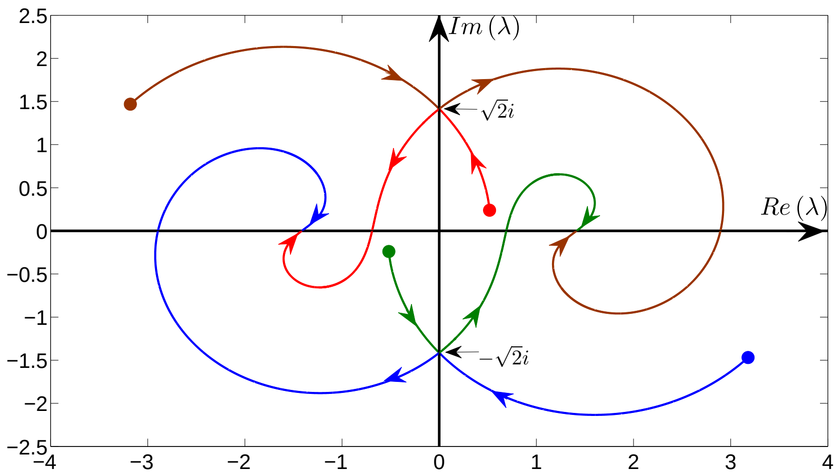


Figure 2.1: Zero loci for $x \in \left(-\frac{1}{4}, 1\right)$.

2.5 The Complete Regular Splitting (CRS) Property

The mean goal of this section is to introduce the notion of the so-called *Complete Regular Splitting* Property and the results that enable us to test when an eigenvalue posses such a behavior.
Consider now the analytic matrix valued function $L(\lambda,\alpha)$. For a fixed value of α, we call λ an *eigenvalue* of $L(\lambda,\alpha)$ whenever $\det L(\lambda,\alpha)=0$. Let $\lambda=\lambda_0$ be an isolated eigenvalue of $L(\lambda,0)$ with partial multiplicities $m_1 \geq \cdots \geq m_N$. Then, *Hryniv and Lancaster* [53] showed that there exists a neighborhood $\mathcal{O}$ of $\lambda=\lambda_0$ such that the spectrum of $L(\lambda,\alpha)$ in $\mathcal{O}$ for all complex α sufficiently close to zero consists of exactly $M \triangleq m_1+\cdots+m_N$ eigenvalues $\lambda_i(\alpha)$, $i=1,\ldots,M$. Furthermore, $\lambda_i(\alpha)$ are algebraic functions of α and can be expressed by all the branches of several Puiseux series [14, 53]:

$$\mu_\nu(\alpha)=c_\nu\alpha^{\frac{1}{q_\nu}}+o(|\alpha|^{\frac{1}{q_\nu}}), \quad \nu=1,\ldots,N',\ q_\nu\in\mathbb{N}, \tag{2.31}$$

where $q_1 \geq \cdots \geq q_{N'}$ and $q_1+\cdots+q_{N'}=M$. A *completely regular splitting* (CRS) [77] property of the eigenvalue $\widetilde{\lambda}=\lambda_0$ at $\alpha=0$ corresponds to: $N=N'$, $q_i=m_i$ and $c_i\neq 0$, $i=1,\ldots,N$.

Definition 2.1. *Let $A(\lambda)$ be an analytic matrix-valued function of a complex variable λ. A vector-valued function $x(\lambda)$ which is analytic in a neighborhood of λ_0 is called a generating function for $A(\lambda)$ of order p at $\lambda=\lambda_0$ if $A(\lambda)x(\lambda)=\mathcal{O}(|\lambda-\lambda_0|^p)$ as $\lambda\to\lambda_0$.*

The following result characterizes the CRS property:

Theorem 2.5 ([53]). *With the notations above, let $\lambda=0$ be an eigenvalue of $L(\lambda,0)$ of geometric multiplicity N and algebraic multiplicity M. Suppose also that for every generating eigenvector x of $L(\lambda,0)$ there exists a generating eigenvector $\hat{x}$ of $(L(\lambda,0))^*$ such that*

$$\left\langle \frac{\partial L}{\partial\alpha}(0,0)x,\hat{x}\right\rangle \neq 0. \tag{2.32}$$

Then the eigenvalue $\lambda=0$ possesses the CRS property.

2.6 The Newton Diagram

The purpose of the section is to present a basic tool employed in the perturbation analysis. Even though, we will use such a technique only in the third part of the thesis and that such a results are very well documented (see for example, [11, 73] and references therein), we decide to present such a notion in order to give a self-contained presentation.
Let $f(x,y)$ be a *pseudo-polynomial* in y, i.e.,

$$f(x,y)=\sum_{k=0}^{n}a_k(x)y^k, \tag{2.33}$$

where the corresponding coefficients are given by,

$$a_k(x)=x^{\rho_k}\sum_{r=0}^{\infty}a_{rk}x^{r/q}, \tag{2.34}$$

a_{rk} are complex numbers, x and y are complex variables, ρ_k are non-negative rational numbers, q is an arbitrary natural number, $a_n(x) \not\equiv 0$, and $a_0(x) \not\equiv 0$.
In one of his treatise, Newton [102] considered the equation (2.33) with complex variables x and y, assuming that $f(x,y)$ can be expanded into a series in positive integral powers of $(x-x_0)$ and $(y-y_0)$. He then sought a solution of (2.33) in the form of a series

$$y = y_0 + \alpha\,(x-x_0)^{\varepsilon} + \alpha'\,(x-x_0)^{\varepsilon'} + \cdots,$$

where ε, ε', ..., is an increasing sequence of rational numbers. To determine the possible values of ε, α, ε', α', ..., Newton made use of a geometrical approach, now known as *Newton's diagram (or Newton's polygon, Newton's parallelogram)*. Since by simple translation any point on a curve can be moved to the origin, we will only consider expansions of the solution of $f(x,y) = 0$ around the origin. In this vein, we will consider solution of (2.33) in the form of the following series:

$$y(x) = y_{\varepsilon}x^{\varepsilon} + y_{\varepsilon'}x^{\varepsilon'} + y_{\varepsilon''}x^{\varepsilon''} + \cdots, \tag{2.35}$$

where $\varepsilon < \varepsilon' < \varepsilon'' < \cdots$, $y_{\varepsilon} \neq 0$, or, in abbreviated form,

$$y(x) = y_{\varepsilon}x^{\varepsilon} + V(x), \tag{2.36}$$

where $V(x) = o(x^{\varepsilon})$ as $x \to 0$.

Definition 2.2 (Newton Diagram). *Given a pseudo-polynomial equation of the form (2.33) with coefficients given by (2.34), plot ρ_k versus k for k for $k = 0,1,\ldots,n$ (if $a_k(\cdot) \equiv 0$, the corresponding point is disregarded). Denote each of these points by $\pi_k = (k,\rho_k)$ and let*

$$\Pi = \{\pi_k \,:\, a_k(\cdot) \neq 0\}$$

be the set of all plotted points. Then, the Newton diagram associated with $f(x,y)$ is the lower boundary of the convex hull of the set Π.

For a given pseudo-polynomial $f(x,y)$, the following figure illustrate *Definition* 2.2.

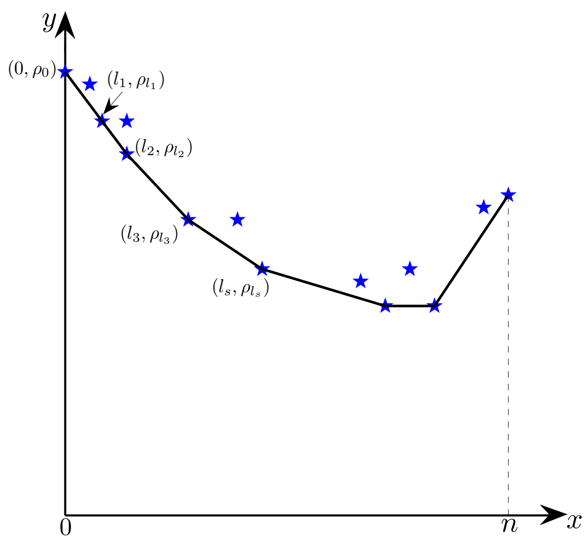


Figure 2.2: Newton Diagram for $f(x,y)$.

Algorithm 2.1 (Newton Diagram Procedure)**.** *Given a pseudo-polynomial (2.33) with coefficients given by (2.34),*

1) *Draw first the associated Newton diagram.*

2) *The leading exponents ε in (2.35) are the different slopes of the segments forming the Newton diagram.*

3) *The number of solutions of order x^{ε} are given by the length of the projection on the horizontal axis of the segment with slope ε.*

4) *The leading coefficients y_{ε} for each solution of order x^{ε} are the non-zero roots of the following polynomial,*

$$\left(\sum_{\ell=i}^{j} a_{0\ell} x_{\varepsilon}^{\ell}\right)^{\natural} = 0, \tag{2.37}$$

where i and j are the end points (i,ρ_i) and (j,ρ_j) of a segment of the Newton's diagram with slope ε, and ($\natural$) signifies that the summation runs over only those (ℓ,ρ_ℓ) lying on this segment, i.e., $\rho_\ell + \ell\varepsilon = \sigma = const$.

5) *To find higher order terms in the expansion (2.35), we substitute (2.36) into $f(x,y)=0$ and repeat steps **(1)**-**(4)**.*

Let us illustrate the above procedure with the following:

Example 2.7. *Let*

$$f(\xi,\lambda) \equiv f_0(\lambda) + f_1(\lambda)\xi + f_3(\lambda)\xi^3 = 0,$$

where

$$\begin{aligned} f_0(\lambda) &:= \lambda^3 & (\rho_0 = 3), \\ f_1(\lambda) &:= -3\lambda & (\rho_1 = 1), \\ f_2(\lambda) &:= 0, & \\ f_3(\lambda) &:= 1 & (\rho_3 = 0). \end{aligned}$$

In order to construct the Newton diagram, we plot the points $(0,3)$, $(1,1)$ *and* $(3,0)$, *leading to the figure Fig.2.3.*

From the diagram (Fig.2.3) we can see that

$$\varepsilon_1 = 2, \qquad \varepsilon_2 = \frac{1}{2}.$$

In order to find ξ_{ε_1} and ξ_{ε_2} we use (2.37) *obtaining the following equations*

$$1 - 3\xi_{\varepsilon_1} = 0 \qquad \textit{and} \qquad -3\xi_{\varepsilon_2} + \xi_{\varepsilon_2}^3 = 0.$$

From these equations, one gets:

$$\xi_{\varepsilon_1} = \frac{1}{3}, \qquad \textit{and} \qquad \xi_{\varepsilon_2} = \pm\sqrt{3}.$$

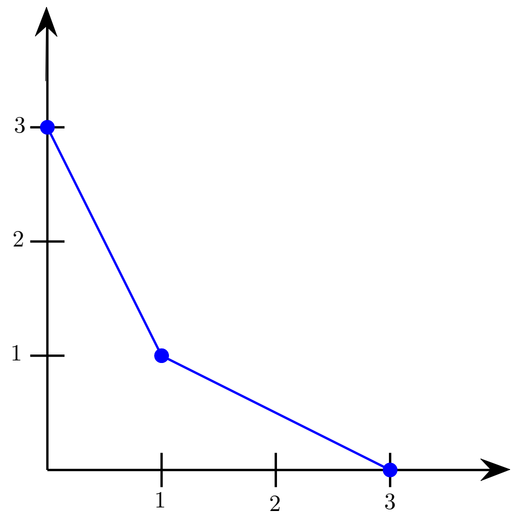


Figure 2.3: Newton Diagram for $f(\xi,\lambda)$.

Thus, in accordance with (2.36)*, we obtain the principal terms of the solutions:*

$$\xi = \frac{1}{3}\lambda^2 + o\left(\lambda^2\right) \qquad \textit{and} \qquad \xi = \pm\sqrt{3}\lambda^{1/2} + o\left(\lambda^{1/2}\right).$$

To find the second-order term of the series in (2.35)*, we substitute each solution into the original equation. We start with the first solution, writing $\xi = \frac{1}{3}\lambda^2 + \zeta$. Substituting this expression into the original equation, we obtain*

$$\frac{1}{27}\lambda^6 + \left(-3\lambda + \frac{1}{3}\lambda^4\right)\zeta + \lambda^2\zeta^2 + \zeta^3 = 0. \tag{2.38}$$

To this equation we again apply the Newton diagram method. We plot the points $(0,6)$, $(1,1)$, $(2,2)$ *and* $(3,0)$ *obtaining figure 2.4.*

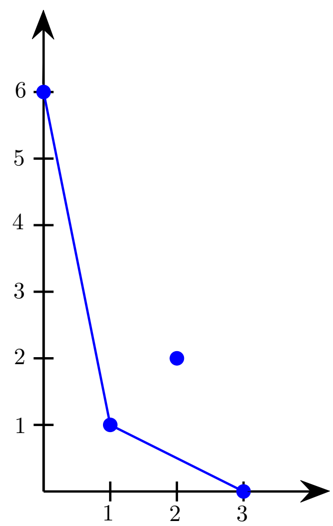


Figure 2.4: Newton Diagram for (2.38).

From the diagram, we obtain:

$$\varepsilon_1' = 5, \qquad \varepsilon_2' = \frac{1}{2}.$$

Since for the series (2.35) *we should have* $\varepsilon' > \varepsilon$, *the expresion obtained for* ε_2' *is inappropriate. as above, we find* $\zeta_{\varepsilon_2'}$ *from the equation*

$$\frac{1}{27} - 3\zeta = 0,$$

it follows that

$$\xi = \frac{1}{3}\lambda^2 + \frac{1}{81}\lambda^5 + o\left(\lambda^5\right).$$

The second terms of the series (2.35) *for the other solutions are found analogously. Substituting*

$$\xi = \pm\sqrt{3}\lambda^{1/2} + \zeta$$

into the original equation, we obtain:

$$\lambda^3 + 6\lambda\zeta \pm \sqrt{3}\lambda^{1/2}\zeta^2 + \zeta^3 = 0. \tag{2.39}$$

We construct the Newton diagram using the points $(0,3)$, $(1,1)$, $\left(2,\frac{1}{2}\right)$, *and* $(3,0)$ *(see Fig.2.5). We obtain* $\varepsilon_1' = 2$ *and* $\varepsilon_2' = \frac{1}{2}$. *As in the previous case, the second value* ε_2' *is inappropriate,*

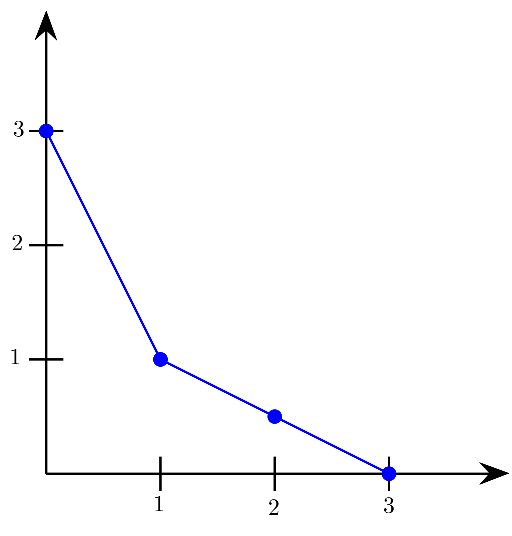


Figure 2.5: Newton Diagram for (2.39).

for it is necessary that $\varepsilon' > \varepsilon$. *We find* $\zeta_{\varepsilon_1'}$ *from the equation*

$$1 + 6\zeta_{\varepsilon_1'} = 0$$

and write

$$\xi = \pm\sqrt{3}\lambda^{1/2} - \frac{1}{6}\lambda^2 + o\left(\lambda^2\right).$$

The higher order terms in the series (2.35), *are found similarly.*

Part I

Continuous-Time Delay Systems

Summary

3 Preliminaries and Problem Formulation

This section presents some general ideas that will be used in the analysis developed in all of the next sections of the first part of the thesis. More precisely, we introduce the basic concepts and describe a part of the analysis using a general equation including all the particular cases that will be treated throughout the chapter. The same ideas and concepts can be used to obtain similar results for different problems as can be seen in [95] or [98]. It is worth mentioning that all the results presented in this part are collected as a book chapter [97] in a more compact form[2].

3.1 Preliminaries

Let us consider the problem of stability analysis of a general class of differential equations (DDE) that can be described in frequency domain by the following characteristic equation $H : \mathbb{C} \times \mathbb{R} \times \mathbb{R} \mapsto \mathbb{C}$

$$H(s;\ \alpha, \beta) = Q(s) + P(s)h(s, \alpha, \beta) = 0, \tag{3.1}$$

where "s" is the variable of the corresponding Laplace transform and α, β are some parameters controlling the behavior of the system. Throughout this chapter P and Q are polynomials and h is a analytic function containing at least an exponential function and making the equation (3.1) transcendental. Furthermore the following statement will be considered as being satisfied:

Assumption 3.1. *The polynomials $P(s)$, $Q(s)$ are such that $P(s)$ and $sQ(s)$ do not have common zeros.*

It is obvious to see that if Assumption 3.1 is violated then the polynomials P and Q have a common factor $c(s) \neq$ constant. Simplifying by $c(s)$ we get a system described by (3.1) which satisfies Assumption 3.1.

The aim of this study is to present the way the closed-loop system described by a characteristic equation of the type (3.1), behaves in the *parameter-space* defined by the pair (α, β). Without any loss of generality, the continuity dependence of the roots of the characteristic function with respect to the parameters α and β (see, for instance, [28] or [32] for the dependence of the roots of DDE's with respect to delay parameters) reduces the (stability) analysis to the following problems:

a) first, to detect crossings with respect to the imaginary axis since such crossing are related to changes in the stability behavior. In other words, we need to compute the frequency crossing set denoted by Ω, which consists of all positive frequencies corresponding to the existence of at least one characteristic root on the imaginary axis. Such a characteristic root will be called *critical*. As we shall see in the cases treated in the next sections, the corresponding frequency crossing set is reduced to a finite collection of intervals. This set will be derived by using geometric arguments.

b) second, to describe the behavior of critical roots under changes of parameters in (α, β) parameter space. More precisely, we will detect switches and reversals corresponding

[2] This book chapter includes I.-C. Morărescu, K. Gu, my Ph.D. advisor and myself as authors

to the situation when the critical characteristic roots cross the imaginary axis towards instability, and stability, respectively. Excepting some explicit computation of the crossing direction, we will also briefly discuss the smoothness properties together with some appropriate classification of the stability crossing boundaries. The geometric arguments will be also useful in defining our classification.

Another useful related concept is represented by the characteristic crossing curves consisting of all pairs (α, β) for which there exists at least one value $\omega \in \Omega$ such that $H(\boldsymbol{i}\omega, \alpha, \beta) = 0$.

Remark 3.1.
- *The first issue pointed out above will be solved separately for each particular class of systems considered in the next sections. Furthermore, in each case we shall give the analytical expression of the crossing curves.*
- *The second issue is presented in this section considering that the frequency crossing set and the stability crossing curves are known.*

3.2 Smoothness of the Crossing Curves and Crossing Direction

In the sequel, let us consider that the frequency crossing set Ω is given and the stability crossing curves are described by

$$\begin{cases} \alpha = \alpha(\omega) \\ \beta = \beta(\omega, \alpha) \end{cases}, \qquad \omega \in \Omega. \tag{3.2}$$

Let us also denote by $\mathcal{T}_h$ an arbitrary crossing curve and consider the following decompositions into real and imaginary parts:

$$\begin{aligned} R_0 + \boldsymbol{i}I_0 &= \boldsymbol{i}\frac{\partial H(s,\alpha,\beta)}{\partial s}\bigg|_{s=\boldsymbol{i}\omega}, \\ R_1 + \boldsymbol{i}I_1 &= -\frac{\partial H(s,\alpha,\beta)}{\partial \beta}\bigg|_{s=\boldsymbol{i}\omega}, \\ R_2 + \boldsymbol{i}I_2 &= -\frac{\partial H(s,\alpha,\beta)}{\partial \alpha}\bigg|_{s=\boldsymbol{i}\omega}. \end{aligned}$$

Then, since $H(s, \alpha, \beta)$ is an analytic function of s, α and β, the Implicit function theorem (see, [85] for further details) indicates that the tangent of $\mathcal{T}_h$ can be expressed as

$$\begin{pmatrix} \dfrac{\mathrm{d}\alpha}{\mathrm{d}\omega} \\ \dfrac{\mathrm{d}\beta}{\mathrm{d}\omega} \end{pmatrix}_{s=\boldsymbol{i}\omega} = \begin{pmatrix} R_1 & R_2 \\ I_1 & I_2 \end{pmatrix}^{-1} \begin{pmatrix} R_0 \\ I_0 \end{pmatrix} = \frac{1}{R_2 I_1 - R_1 I_2} \begin{pmatrix} R_1 I_0 - R_0 I_1 \\ R_0 I_2 - R_2 I_0 \end{pmatrix}, \tag{3.3}$$

provided that

$$R_1 I_2 - R_2 I_1 \neq 0. \tag{3.4}$$

It follows that $\mathcal{T}_h$ is smooth everywhere except possibly at the points where either (3.4) is not satisfied, or when

$$\frac{\mathrm{d}\alpha}{\mathrm{d}\omega} = \frac{\mathrm{d}\beta}{\mathrm{d}\omega} = 0. \tag{3.5}$$

Remark 3.2. *If (3.5) is satisfied then straightforward computations show us that $R_0 = I_0 = 0$. In other words $s = \boldsymbol{i}\omega$ is a multiple solution of (3.1).*

The next paragraph focuses on the characterization of the crossing direction corresponding to the curves defined by (3.2). We will call the direction of the curve that corresponds to increasing ω the *positive direction*. We will also call the region on the left hand side as we head in the positive direction of the curve *the region on the left*.

To establish the direction of crossing we need to consider α and β as functions of $s = \sigma + \boldsymbol{i}\omega$, i.e., functions of two real variables σ and ω, and partial derivative notation needs to be adopted. Since the tangent of $\mathcal{T}_h$ along the positive direction is $\left(\frac{\partial\alpha}{\partial\omega}, \frac{\partial\beta}{\partial\omega}\right)$, the normal to $\mathcal{T}_h$ pointing to the left hand side of positive direction is $\left(-\frac{\partial\beta}{\partial\omega}, \frac{\partial\alpha}{\partial\omega}\right)$. Corresponding to a pair of complex conjugate solutions of (3.2) crossing the imaginary axis along the horizontal direction, (α, β) moves along the direction $\left(\frac{\partial\alpha}{\partial\sigma}, \frac{\partial\beta}{\partial\sigma}\right)$. So, as (α, β) crosses the stability crossing curves from the right hand side to the left hand side, a pair of complex conjugate solutions of (3.2) crosses the imaginary axis to the right half plane, if

$$\left(\frac{\partial\alpha}{\partial\omega}\frac{\partial\beta}{\partial\sigma} - \frac{\partial\beta}{\partial\omega}\frac{\partial\alpha}{\partial\sigma}\right)_{s=\boldsymbol{i}\omega} > 0, \tag{3.6}$$

i.e. the region on the left of $\mathcal{T}_h$ gains two solutions on the right half plane. If the inequality (3.6) is reversed then the region on the left of $\mathcal{T}_h$ loses two right half plane solutions. Similar to (3.3) we can express

$$\begin{aligned} \begin{pmatrix} \frac{\partial\alpha}{\partial\sigma} \\ \frac{\partial\beta}{\partial\sigma} \end{pmatrix}_{s=\boldsymbol{i}\omega} &= \begin{pmatrix} R_2 & R_1 \\ I_2 & I_1 \end{pmatrix}^{-1} \begin{pmatrix} I_0 \\ -R_0 \end{pmatrix} \\ &= \frac{1}{R_2 I_1 - R_1 I_2} \begin{pmatrix} R_0 R_2 + I_0 I_2 \\ -R_0 R_1 - I_0 I_1 \end{pmatrix}. \end{aligned} \tag{3.7}$$

Proposition 3.1. *Assume $\omega \in \Omega$, α, β satisfy (3.2) and $s = \boldsymbol{i}\omega$ is a simple solution of (3.1) and $H(\boldsymbol{i}\omega', \alpha, \beta) \neq 0, \forall\omega' > 0, \omega' \neq \omega$ (i.e. (α, β) is not an intersection point of two curves or different sections of a single curve). Then as (α, β) moves from the region on the right to the region on the left of the corresponding crossing curve, a pair of solutions of (3.1) crosses the imaginary axis to the right (through $s = \pm\boldsymbol{i}\omega$) if $R_2 I_1 - R_1 I_2 > 0$. The crossing is to the left if the inequality is reversed.*

Proof. Straightforward computation shows that

$$\left(\frac{\partial\alpha}{\partial\omega}\frac{\partial\beta}{\partial\sigma} - \frac{\partial\beta}{\partial\omega}\frac{\partial\alpha}{\partial\sigma}\right)_{s=\boldsymbol{i}\omega} = \frac{(R_0^2 + I_0^2)(R_2 I_1 - R_1 I_2)}{(R_2 I_1 - R_1 I_2)^2}.$$

Therefore (3.6) can be written as $R_2 I_1 - R_1 I_2 > 0$. □

Any given direction, (d_1, d_2), is to the left-hand side of the curve if its inner product with the left-hand side normal $\left(-\frac{\partial\beta}{\partial\omega}, \frac{\partial\alpha}{\partial\omega}\right)$ is positive, i.e.,

$$-d_1\frac{\partial\beta}{\partial\omega} + d_2\frac{\partial\alpha}{\partial\omega} > 0, \tag{3.8}$$

from which we have the following result.

Corollary 3.1. *Let ω, α and β satisfy the same condition as Proposition 3.1. Then, as (α,β) crosses the curve along the direction (d_1,d_2), a pair of solutions of (5.4) crosses the imaginary axis to the right if*

$$d_1(R_2I_0 - R_0I_2) + d_2(R_1I_0 - R_0I_1) > 0. \tag{3.9}$$

The crossing is in the opposite direction if the inequality is reversed.

Proof. Writing out the left-hand side, then (3.8) becomes

$$\frac{d_1(R_2I_0 - R_0I_2) + d_2(R_1I_0 - R_0I_1)}{R_2I_1 - R_1I_2} > 0. \tag{3.10}$$

If (d_1,d_2) is in the same side as the left-hand side normal, then, as we move along the (d_1,d_2) direction, the crossing is from the LHP to the RHP if the left-hand sides of (3.10) and $R_1I_2 - R_2I_1$ have the same sign, i.e., their product is positive. □

Remark 3.3. *It is worth to mention, that all above ideas follows the same lines that those developed by Gu et al. in [40].*

4 Output Feedback (Proportional Controller)

In order to give a complete presentation of the low-order controllers for continuous-time systems, we shall recall in the sequel some results concerning to the delayed output feedback case (see, [98], for further details).

Consider the following class of *proper SISO* open-loop transfer function:

$$H_{yu}(s) := \frac{P(s)}{Q(s)} = c^T(sI_n - A)^{-1}b + d \tag{4.1}$$

where (A,b,c^T,d) is a state-space representation of the open-loop system, and consider the *control law*:

$$u(t) = -ky(t-\tau). \tag{4.2}$$

The interest is to find *all* the *pairs* (k,τ) *such that the controller (4.2) stabilizes the corresponding SISO system (4.1).* The corresponding characteristic equation of the closed-loop system simply writes as:

$$Q(s) + kP(s)\mathrm{e}^{-s\tau} = 0. \tag{4.3}$$

Precisely, we deal with a characteristic equation of type (3.1) where the pair of parameters (α,β) is replaced by (k,τ) and $h(s,k,\tau) = k\mathrm{e}^{-s\tau}$. The aim of this section is to illustrate the underlying *stability/instability mechanisms* in presence of delays, that is to see the way the closed-loop system behaves in the *parameter-space* defined by the pair (k,τ).

4.1 Preliminaries and Problem Formulation

As mentioned above, in this section we deal with the following characteristic equation:

$$H(s,k,\tau) = Q(s) + kP(s)\mathrm{e}^{-s\tau} = 0 \tag{4.4}$$

The polynomials P and Q will satisfy the following constraints.

Assumption 4.1. *One assumes* $\deg(Q) \geq \deg(P)$.

Assumption 4.2. $P'(\boldsymbol{i}\omega) \neq 0$ *whenever* $P(\boldsymbol{i}\omega) = 0$.

The Assumption 4.1 is needed in order to ensure that for a fixed value of k, the real part of any characteristic roots is bounded to the right. This assumption implies that k will vary in a limited domain $|k| \leq k_{max}$ since larger gain values will induce instability for infinitesimal delay values. The Assumption 4.2 is imposed to exclude some rare singular cases in order to simplify the presentation. It is noteworthy that Assumption 3.1 is imposed even if it is not explicitly mentioned.

First at all, we briefly present some necessary considerations proposed in [107] using a continuity principle argument for the dependence of the roots of the characteristic equation with respect to some real parameter (the gain k in our study).

Introduce now the following Hurwitz matrix associated to some generic polynomial $A(s) = \sum_{i=0}^{n_a} a_i s^{n_a - i}$:

$$H(A) = \begin{bmatrix} a_1 & a_3 & a_5 & \dots & a_{2n_a-1} \\ a_0 & a_2 & a_4 & \dots & a_{2n_a-2} \\ 0 & a_1 & a_3 & \dots & a_{2n_a-3} \\ 0 & a_0 & a_2 & \dots & a_{2n_a-4} \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_{n_a} \end{bmatrix} \in \mathbb{R}^{n_a \times n_a}, \tag{4.5}$$

where the coefficients a_l are assumed to be zero ($a_l = 0$), for all $l > n_a$.

Consider $H(Q), H(P) \in \mathbb{R}^{n \times n}$ where $\deg(Q) = n > m = \deg(P)$, the coefficients $q_l = 0$ for all $l > n$ and the coefficients $p_l = 0$ for all $l > m$.

Consider also the set $\mathcal{U}$ of the roots of $H(s,k,0)$ located in the closed right half plane, and the quantity k_{max} given by:

$$k_{max} = \begin{cases} \frac{q_0}{p_0}, & \text{if } \deg(Q) = \deg(P) \\ +\infty, & \text{if } \deg(Q) > \deg(P) \end{cases} \tag{4.6}$$

Such a quantity will define the controller's gain domain. It is easy to see that while in the case of a strictly proper transfer ($\deg(Q) > \deg(P)$), we do not have any restriction on the gain, the case of a proper transfer ($\deg(Q) = \deg(P)$) imposes such restrictions. The explanation can be resumed as follows: in this last case, the corresponding closed-loop system is a quasipolynomial of *neutral* type (see, for instance, [48, 103] for further discussions on the topics), and one explicitly needs further constraints on the gain, that is k should satisfy the inequality $|k| < k_{max} = 1/\mid d \mid$ (stability of the corresponding difference operator). Indeed, if this is not the case, larger gain values will induce instability even for infinitesimal small delay values ($\deg(Q) = \deg(P)$ with unstable difference operators) as pointed out by [91].

As a consequence of the remarks above, it is important to point out that for all $k \in \mathbb{R}$, such that $\mid k \mid < k_{max}$, $\text{card}(\mathcal{U})$ is *finite*, where $\text{card}(\cdot)$ denotes the cardinality (number of elements).

Lemma 4.1. *Let* $\lambda_1 < \lambda_2 < ... < \lambda_h$, *with* $h \leq n$ *be the real eigenvalues of the matrix pencil* $\Sigma(\lambda) = \lambda H(P) + H(Q)$ *inside the interval* $(-k_{max}, k_{max})$.

Then, card$(\mathcal{U})$ *remains constant as k varies within each interval* $(\lambda_i, \lambda_{i+1})$. *The same holds for the intervals* $(-k_{max}, \lambda_1)$ *and* (λ_h, k_{max}).

It is worth mentioning that the lemma above gives a simple method to compute card$(\mathcal{U})$ by computing the generalized eigenvalues of the matrix pencil $\Sigma(\lambda)$. Such a quantity card$(\mathcal{U})$ is needed to derive the stability regions in the parameter space defined by the gain and delay parameters (k, τ).

4.2 Stability Crossing Curves Characterization

The following results characterize the stability crossing curves in the (k, τ) parameter space. The presentation is as simple as possible, and intuitive. Some examples illustrating various case study are considered at the end of this section.

4.2.1 Identification of Crossing Points

Let $\mathcal{T}$ denote the set of all $(k, \tau) \in \mathbb{R} \times \mathbb{R}_+$ such that (4.4) has at least one zero on imaginary axis. Any $(k, \tau) \in \mathcal{T}$ is known as a *crossing point*. The set $\mathcal{T}$, which is the collection of all crossing points, is known as the *stability crossing curves*.

We consider also the set Ω of all real number ω such that $\boldsymbol{i}\omega$ satisfy (4.4) for at least one pair $(k, \tau) \in \mathbb{R} \times \mathbb{R}_+$. We will refer to Ω as the *crossing set*.

Remark 4.1. *If ω is a real number and $(k, \tau) \in \mathbb{R} \times \mathbb{R}_+$ then*

$$Q(-\boldsymbol{i}\omega) + kP(-\boldsymbol{i}\omega)e^{\boldsymbol{i}\omega\tau} = \overline{Q(\boldsymbol{i}\omega) + kP(\boldsymbol{i}\omega)e^{-\boldsymbol{i}\omega\tau}}$$

Therefore, in the sequel, we only need to consider positive ω.

Proposition 4.1. *[98] Given any $\omega > 0$, $\omega \in \Omega$ if and only if it satisfies:*

$$|\, P(\boldsymbol{i}\omega)\, | \quad > \quad 0, \tag{4.7}$$

and all the corresponding pairs (k, τ) can be calculated by:

$$k(\omega) \quad = \quad \pm\left|\frac{Q(\boldsymbol{i}\omega)}{P(\boldsymbol{i}\omega)}\right|; \tag{4.8}$$

$$\begin{aligned} \tau_m(\omega) \quad &= \quad \frac{1}{\omega}\left(\angle P(\boldsymbol{i}\omega) - \angle Q(\boldsymbol{i}\omega) + (2m + \varepsilon_k + 1)\pi\right) \\ & \quad\quad m = 0, \pm 1, \pm 2, \ldots \end{aligned} \tag{4.9}$$

where $\varepsilon_k = \begin{cases} 0 \; if \; k \geq 0 \\ -1 \; if \; k < 0 \end{cases}$.

Remark 4.2 (small gain)**.** *[98] If the open-loop SISO system does not include oscillatory modes, that is $Q(s)$ has no roots on the imaginary axis, then some simple algebraic manipulations prove that for all the gains k satisfying the following inequality:*

$$|\, k\, | \quad < \quad \frac{1}{\sup\limits_{\omega > 0}\left\{\frac{|P(\boldsymbol{i}\omega)|}{|Q(\boldsymbol{i}\omega)|}\right\}}, \tag{4.10}$$

the closed-loop system (4.4) is hyperbolic *(see [47, 103] for further details on such a notion), that is there does not exist any crossing roots on the imaginary axis for all positive delays τ. In other words, the closed-loop system is stable (unstable) for all delays value if it is stable (unstable) in the case free of delays. Furthermore, the frequency-sweeping test above (4.10) gives a simple way to exclude some k-interval from the beginning, since in such a case crossing roots can not exist.*

However, it is important to point out that such a frequency-sweeping test (4.10) losses all its interest if if the polynomial $Q(s)$ has roots on the imaginary axis (the corresponding upper bound becomes 0*), that is in the case of linear systems including oscillatory modes (such a case will be considered later in a paragraph dedicated to some illustrative examples).*

4.2.2 Identification of the Crossing Set

In the circumstances presented above, we can assume k within some finite interval $[\alpha,\beta] \subset (-k_{max}, k_{max})$, which contains all generalized eigenvalues λ_i of the matrix pencil $\Sigma(\lambda)$, but excluding the k-interval given by (4.10) if the SISO system does not include oscillatory modes. Lemma 4.1 ensures us that the choice of the interval $[\alpha,\beta]$ includes all the remaining possibilities for the system free of delay. In such a case, define $\ell_l := min\{|\ \alpha\ |, |\ \beta\ |\} \geq 0$, and $\ell_u := max\{|\ \alpha\ |, |\ \beta\ |\} < \infty$. Then, there are only a *finite number of solutions* to each of the following three equations:

$$|\ Q(\boldsymbol{i}\omega)\ | \ = \ \ell_l\ |\ P(\boldsymbol{i}\omega)\ |, \tag{4.11}$$
$$|\ Q(\boldsymbol{i}\omega)\ | \ = \ \ell_u\ |\ P(\boldsymbol{i}\omega)\ |, \tag{4.12}$$

and

$$P(\boldsymbol{i}\omega) \ = \ 0, \tag{4.13}$$

because P, and Q are polynomials satisfying the Assumptions 4.1, 3.1 and 4.2. Therefore, the *crossing set* Ω will be defined by all the frequencies $\omega > 0$ satisfying *simultaneously* the inequalities:

$$\begin{cases} \ell_l\ |\ P(\boldsymbol{i}\omega)\ | \leq |\ Q(\boldsymbol{i}\omega)\ | \leq \ell_u\ |\ P(\boldsymbol{i}\omega)\ |, \\ |\ P(\boldsymbol{i}\omega)\ | > 0. \end{cases} \tag{4.14}$$

In conclusion, due to the form of (4.14), and from the Assumptions 4.1 and 3.1, the corresponding crossing set Ω *consists of a finite number of intervals*. Denote these intervals as: Ω_1, Ω_2, ..., Ω_N. Then:

$$\Omega \ = \ \bigcup_{k=1}^{N} \Omega_k.$$

Remark 4.3 (strictly proper SISO case)**.** *In the case of a strictly proper SISO system $k_{max} = \infty$ (that is no any constraints on the gain k), we note that for $k \in (\beta, \infty)$ (or $k \in (-\infty, \alpha)$) we can still express Ω as a finite number of intervals, but one of them has an infinite end.*

Remark 4.4 (Invariance root at the origin)**.** *If $Q(0)/P(0) \in [\alpha,\beta]$, then 0 will be a characteristic root for all τ if $k = Q(0)/P(0)$, since $e^{-s\tau} = 1$ for $s = 0$, independently of the delay value τ. The last remark allows us to eliminate the value $Q(0)/P(0)$ from Ω if it is the case.*

Remark 4.5 (Crossing characterization)**.** *The frequency-sweeping test (4.14) above gives all the frequency intervals for which crossing roots exist for the corresponding chosen gain interval, but it does not give any information on the* crossing direction*.*

In other words, such a test does not make any distinction between switches *(crossing towards instability) and* reversals *(crossing towards stability) [103]. Such a problem will be considered in the paragraph concerning the direction of crossing.*

4.2.3 Crossing Curves Characterization

In the sequel, we consider $\Omega_i = [\omega_i^l, \omega_i^r]$, for all $i = 1, 2, \ldots, N$. Without any loss of generality, we can order these intervals from left to right, i.e., for any $\omega_1 \in \Omega_{i_1}$, $\omega_2 \in \Omega_{i_2}$, $i_1 < i_2$, we have $\omega_1 < \omega_2$.

We note that ω_1^l can be 0 and in this case Ω_1 is open to the left.

It is clear that $k(\omega_i^l), k(\omega_i^r) \in \{\alpha, \beta\}$ for all $i = 1 \ldots N$ if $\omega_1^l \neq 0$. We will not restrict $\angle Q(\boldsymbol{i}\omega)$ and $\angle P(\boldsymbol{i}\omega)$ to a 2π range. Rather, we allow them to vary continuously within each interval Ω_i. Thus, for each fixed m, (4.8) and (4.9) give us two continuous almost everywhere curves. We can lose the continuity of the curve in the points which correspond to the case $Q(\boldsymbol{i}\omega) = 0$. For example, if $Q(\boldsymbol{i}\omega^*)$ is a real polynomial and its sign is changing at ω^*, then $\angle(Q(\boldsymbol{i}\omega))$ is not continuous in ω^*.

It should be noted that condition (4.8) and k finite, imply $P(\boldsymbol{i}\omega) \neq 0, \quad \forall \omega \in \Omega$. We denote the curves defined by (4.8) and (4.9) with $\mathcal{T}_i^{m\pm}$. Therefore, corresponding to a given interval Ω_i, we have an infinite number of continuous stability crossing curves $\mathcal{T}_i^{m\pm}$, $m = 0, \pm 1, \pm 2, \ldots$.

Finally, is worth mentioning that, for some m, part or the entire curve may be outside of the range $\mathbb{R} \times \mathbb{R}_+$, and therefore, may not be physically meaningful. The collection of all the points in $\mathcal{T}$ corresponding to Ω_i may be expressed as

$$\mathcal{T}_i = \bigcup_{m=-\infty}^{+\infty} \left[\left(\mathcal{T}_i^{m+} \cap (\mathbb{R} \times \mathbb{R}_+) \right) \cup \left(\mathcal{T}_i^{m-} \cap (\mathbb{R} \times \mathbb{R}_+) \right) \right]$$

Obviously,

$$\mathcal{T} = \bigcup_{i=1}^{N} \mathcal{T}_i.$$

Also it is easy to see that, for each Ω_i, we define two curves, one to the right of the $O\tau$ axis and the other to the left. According to the fixed limits α, β of the interval where k varies we can eliminate some of these curves. The end points of these curves are classified as follows:

Type 1. It satisfies the equation $k(\omega) = \alpha$.

Type 2. It satisfies the equation $k(\omega) = \beta$.

Type 3. It equals 0.

Obviously, only ω_1^l can be of type 3. We note that all the crossing curves are situated in the vertical strip $\mathcal{D}$ between the lines $k = \alpha$ and $k = \beta$. Now, let ω_* be an end point of the interval Ω_i. We have already said that each $\mathcal{T}_i^{m+}$ is an continuous almost everywhere curve, so, $(k(\omega_*), \tau_m(\omega_*))$ is an end point of $\mathcal{T}_i^{m\pm}$, and it can be characterized as follows:

- If ω_* is of type 1, then $k(\omega_*) = \alpha$ and $\tau(\omega_*)$ are finite. More precisely, $\mathcal{T}_i^{m+}$ intersects the vertical line $k = \alpha$, which is the left bound of the strip $\mathcal{D}$.
- If ω_* is of type 2 then $k(\omega_*) = \alpha$ and $\tau(\omega_*)$ are finite. Or, we may say that $\mathcal{T}_i^{m+}$ intersects the vertical line $k(\omega) = \beta$, which is the right bound of the strip $\mathcal{D}$.
- If ω_* is of type 2 then τ approaches ∞ and k approaches $Q(0)/P(0)$. In other words, $\mathcal{T}_i^{m+}$ has a vertical asymptote given by $k = Q(0)/P(0)$.

Remark 4.6. *The previous description holds also for $\mathcal{T}_i^{m-}$.*

As defined previously we say that an interval Ω_k is of type lr if its left end is of type l and its right end is of type r. Tehn, it is possible to divide accordingly these intervals into 6 types.

4.2.4 Smoothness of the Crossing Curves and Direction of Crossing

For a given i, we will discuss the smoothness of the curves in $\mathcal{T}_i^{m\pm}$ and thus $\mathcal{T} = \bigcup_{m=-\infty}^{+\infty} \left[\left(\mathcal{T}_i^{m+} \cap (\mathbb{R} \times \mathbb{R}_+) \right) \cup \left(\mathcal{T}_i^{m-} \cap (\mathbb{R} \times \mathbb{R}_+) \right) \right]$. For this purpose, we consider k and τ as implicit functions of $s = \boldsymbol{i}\omega$ defined by (4.4).

Proposition 4.2. *[98] The curve $T_i^{m\pm}$ is smooth everywhere except possibly at the point corresponding to $s = \boldsymbol{i}\omega$ in any one of the following cases:*

1) $s = \boldsymbol{i}\omega$ is a multiple solution of (4.4), and

2) ω is a solution of $Q(\boldsymbol{i}\omega) = 0 \Leftrightarrow k = 0$.

The direction of crossing is characterized as follows:

Proposition 4.3. *[98] Let $\omega \in (\omega_i^l, \omega_i^r)$ and $(k, \tau) \in \mathcal{T}_i$ such that $\boldsymbol{i}\omega$ is a simple solution of (4.4) and $H(\boldsymbol{i}\omega', k, \tau) \neq 0, \forall \omega' > 0, \omega' \neq \omega$ (i.e. (k, τ) is not an intersection point of two curves or different sections of a single curve of $\mathcal{T}$). Then a pair of solutions of (4.4) will cross the imaginary axis to the right, through $s = \pm \boldsymbol{i}\omega$ if $R_2 I_1 - R_1 I_2 > 0$. The crossing is to the left if the inequality is reversed.*

4.3 Illustrative Examples

In the sequel, we consider some classical examples in the literature (first-order, second-order oscillatory systems).

Example 4.1 (Scalar delay case). *Consider the system given by the transfer function*

$$H_{yu}(s) = \frac{1}{s+a} \tag{4.15}$$

subject to the control law $u(t) = -ky(t-\tau)$. The corresponding characteristic equation of the corresponding closed-loop system can be written as:

$$s + a + ke^{-s\tau} = 0. \tag{4.16}$$

For $a > 0$ it is obvious that either for $k = 0$ or $\tau = 0$ and $a + k > 0$, we obtain a stable equation. On the other hand using Proposition 4.3, we derive that all the crossings are towards instability.

It is noteworthy that in a completely different framework, Boese [15] considered $k > 0$ and he proved that for $k \leq a$ one gets a delay independent stable system. He also proved that for $k > a$ one has only one stability interval $[0, \tau_0)$, where τ_0 is a decreasing function of k.

Using the above method for $a = 3$ we can draw the crossing curves and establish the stability region as in figure 4.1. In this case, we have:

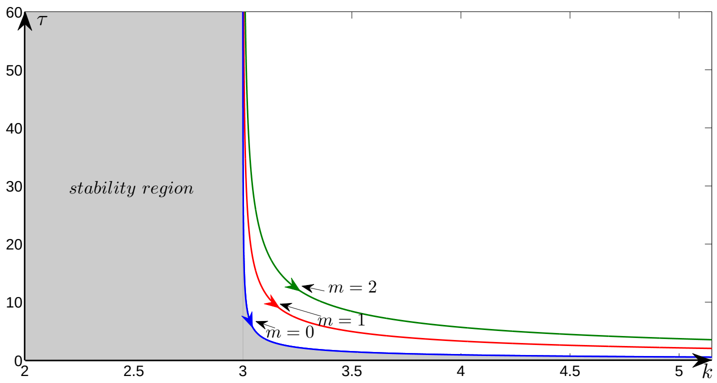


Figure 4.1: $\mathcal{T}_1^{m+}$, $m \in \{0, 1, 2\}$ for the system (4.16)

$$\operatorname{card}(\mathcal{U}) = \begin{cases} 0 \text{ } if \text{ } k > -3, \\ 1 \text{ } if \text{ } k \leq -3, \end{cases} \tag{4.17}$$

and for $k \in [-5, 5]$ the crossing set Ω consists in one interval $(0, 4]$ of type 31. Therefore, we obtain only one *stability interval for $k > 3$, and this interval is $[0, \tau_0)$, where τ_0 is given as a function of the gain "k" by:*

$$\tau_0(k) = \frac{1}{\omega}\left(\pi - \arctan\frac{\omega}{3}\right) := \frac{1}{\sqrt{k^2 - 9}}\left(\pi - \arctan\frac{\sqrt{k^2 - 9}}{3}\right),$$

which is nothing else that the formula given by Boese for the corresponding upper bound of the (closed-loop) stability interval.

Now consider the case $a = -3$ (open-loop system unstable) and $k \in [-5, 5]$, once again we derive $\Omega = (0, 4]$ and

$$\operatorname{card}(\mathcal{U}) = \begin{cases} 0 \text{ } if \text{ } k > 3 \\ 1 \text{ } if \text{ } k \leq 3 \end{cases}.$$

Since all the crossing direction are towards instability, it is sufficient to plot only the first stability crossing curve. As expected (figure 4.2), the system becomes unstable as τ increases.

Example 4.2 (Linear (second-order) oscillators)**.** *Consider the transfer function*

$$H_{yu}(s) = \frac{1}{s^2 + 2} \tag{4.18}$$

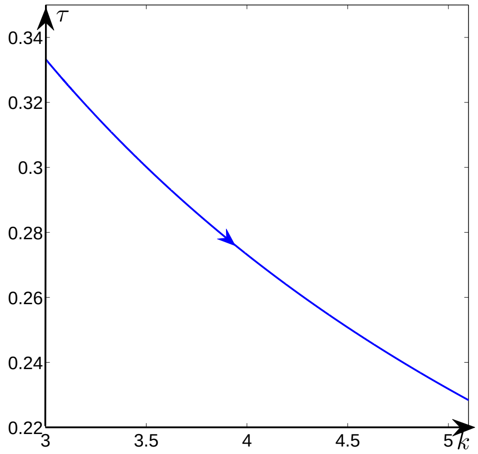


Figure 4.2: $\mathcal{T}_1^{0+}$ for the system (4.16) with $a = -3$

subject to the control law $u(t) = -ky(t-\tau)$. *The corresponding characteristic equation is given by:*

$$s^2 + 2 + ke^{-s\tau} = 0. \tag{4.19}$$

For $k \in (-2, 0)$ *the results regarding stability intervals of the systems can be found in [107] and they simple say that for* $\tau \in \left(0, \frac{\pi}{\sqrt{2+|k|}}\right)$ *the system is stable (see also [1] for a different stability argument). It is easy to see that the number of stabilizing delay interval is a decreasing function of* $|k|$.

Our computation in this case point out that for $k \in (-2, 0)$ *the crossing set* Ω *consists in* one interval $(0, 2]$ *of type 32.*

According to Proposition 4.2, all the crossing curves are discontinuous in the points that correspond to $k = 0$ *and it is easy to see that:*

$$\mathrm{card}\,(\mathcal{U}) = \begin{cases} 1 \text{ if } k < -2, \\ 2 \text{ if } k > -2. \end{cases}$$

Proposition 4.3 simply says that for $k < 0$ *the region on the right hand side of each crossing curve has two more unstable roots.*

Remark 4.7. *If* $\omega \in (0, \sqrt{2})$ *then* $\tau_0(\omega) = 0$ *as we can deduce from the computation below:*

$$\tau_0 = \frac{1}{\omega}(\angle(1) - \angle(2-\omega^2) + (\varepsilon_k + 1)\pi) = 0, \quad \forall \omega \in (0, \sqrt{2}) \tag{4.20}$$

More precisely (see figure 4.3), we recover the result proposed in [105, 107].

Example 4.3 (Third-order unstable system)**.** *This example is only to illustrate that it is possible to have most types of the curves enumerated in the classification section. In the sequel we present a dynamical system with crossing curves of type 11, 22, 31 and 32.*

Consider the transfer function

$$H_{yu}(s) = \frac{1}{s^3 - 2s^2 + 9s - 8}, \tag{4.21}$$

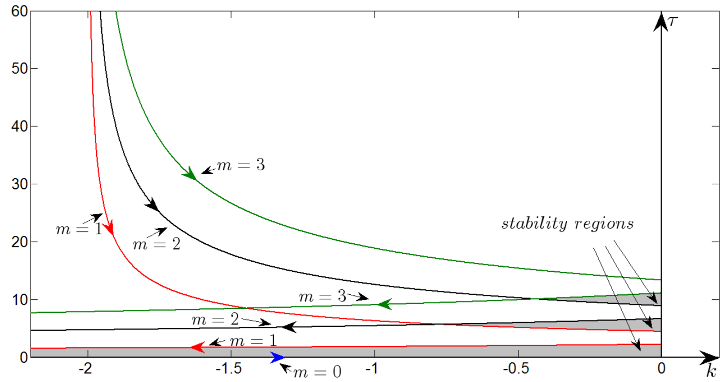


Figure 4.3: τ_i, $m \in \{0,1,2,3\}$ versus k for the system (4.19)

subject to the control law $u(t) = -ky(t-\tau)$. The corresponding characteristic equation of the closed-loop system is given by:

$$s^3 - 2s^2 + 9s - 8 + ke^{-s\tau} = 0. \tag{4.22}$$

We note that this system can not be stabilized by any static output feedback. Indeed, straight-forward computations show us that:

$$\mathrm{card}\,(\mathcal{U}) = \begin{cases} 1 & \textit{if} \quad k < -10, \\ 3 & \textit{if} \quad k \in (-10, 8), \\ 2 & \textit{if} \quad k > 8. \end{cases}$$

Taking $\alpha = -10$ and $\beta = 10$, we get $\Omega = (0,1] \cup [2,3]$ and $\mathcal{T}_1^{m+}$ is of type 32, $\mathcal{T}_1^{m-}$ is of type 31, $\mathcal{T}_2^{m-}$ is of type 11, $\mathcal{T}_2^{m+}$ is of type 22. We present the last three curves in the figures 4.4-4.5.

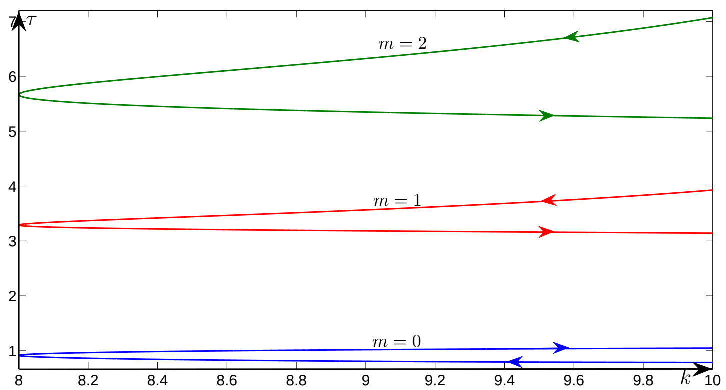


Figure 4.4: $\mathcal{T}_2^{m+}$, $m \in \{0,1,2\}$ for the system (4.22)

Example 4.4 (Sixth-order unstable system)**.** *In this example, we consider a system that can not be stabilized by a static output feedback, but it can be stabilized by a delayed output feedback. This example is borrowed from [107].*

Consider the system:

$$H_{yu}(s) = \frac{1}{s^6 + p_1 s^5 + p_2 s^4 + p_3 s^3 + p_4 s^2 + p_5 s + p_6} \tag{4.23}$$

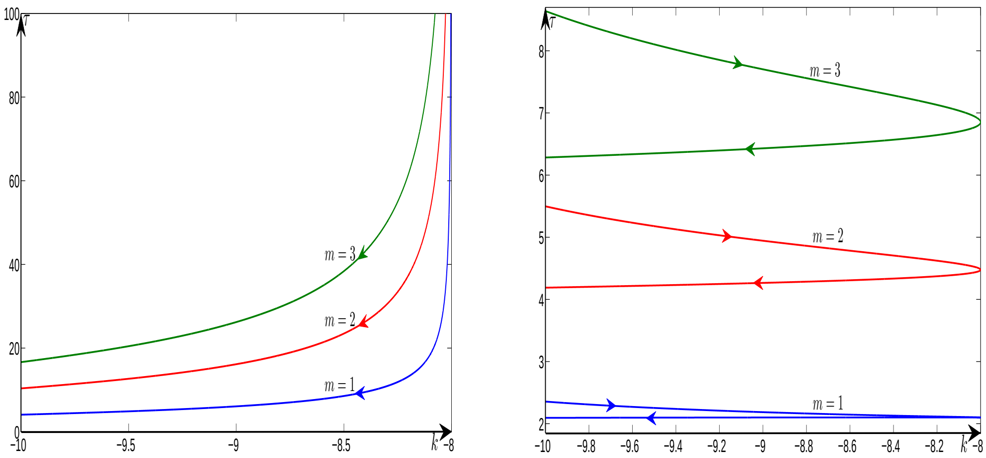


Figure 4.5: $\mathcal{T}_i^{m+}$, $m \in \{1,2,3\}, i \in 1,2$ for the system (4.22)

where

$$p_1 = -6.0000000e-04, \quad p_2 = 1.4081634e+00, \quad p_3 = -5.6326533e-04,$$
$$p_4 = 4.3481891e-01, \quad p_5 = -8,6963771e-05, \quad p_6 = 2.6655565e-02.$$

Using Lemma 1, we obtain:

$$\operatorname{card}(\mathcal{U}) = \begin{cases} 3 & \text{if} \quad k < -0.0707886, \\ 5 & \text{if} \quad k \in (-0.0707886; -0.0266556), \\ 6 & \text{if} \quad (-0.0266556; 0.0120036), \\ 4 & \text{if} \quad k > 0.0120036. \end{cases}.$$

The stability crossing curves and the first two stability region for $k \in (0, 0.16)$ are plotted in figure 4.6, whereas figure 4.7 shows the dependence of the gain k as a function of ω.

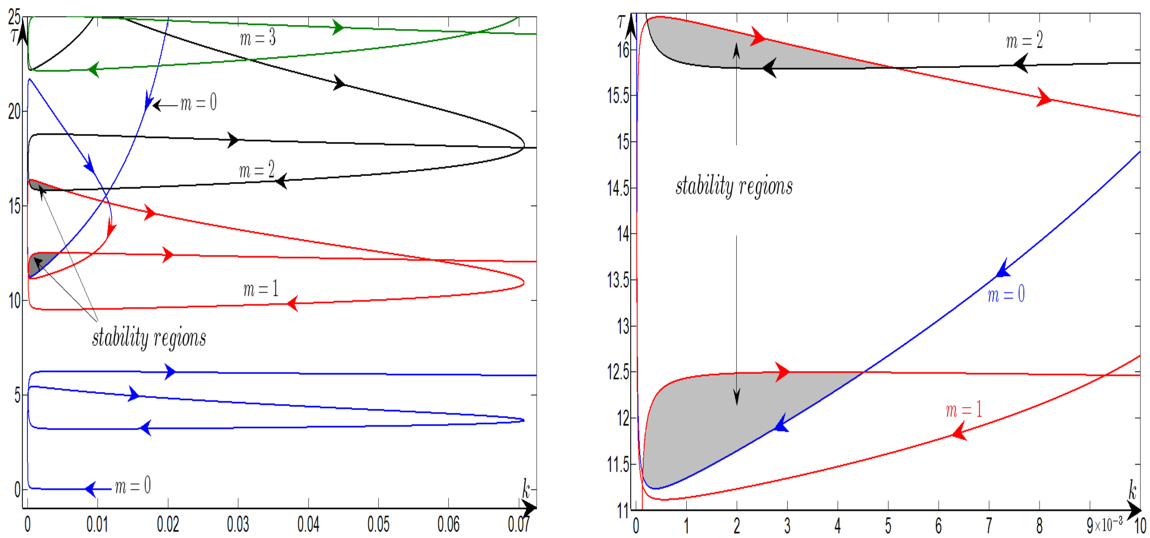


Figure 4.6: Left:Stability crossing curves for the system given by (4.23); Right: Zoom of the stability regions.

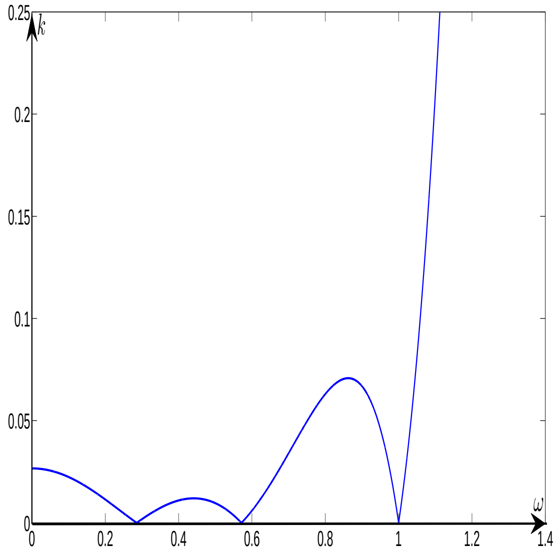


Figure 4.7: The dependence of the gain k as a function of ω for some positive frequencies for the system (4.23).

5 The Geometry of PI Controllers for SISO Systems with Input/Output (I/O) Delays

As discussed in [109], there exists several methods for the controllers construction, and several techniques have been proposed for the analysis of the stability and of the performances of the corresponding closed-loop schemes. Among them, we can mention the computation of stabilizing PI controller's parameters considered by [123, 136] using a Pontryagin approach. More precisely, [123] addresses the control of first-order system with a time-delay in both cases (stable, and unstable delay-free systems), and [136] deals with some robustness issues in terms of delays for the closed-loop system under the assumption that the delay-free system can be stabilized by a proportional controller.

In order to give a complete presentation of the low-order controllers for continuous-time systems, we shall recall in the sequel some results concerning to the PI controller case (see, [99], for further details). In this vein, we are interested in characterizing the stability by the crossing boundaries in the parameter-space defined by the PI controller's parameters. By a crossing boundary, we understand the set of parameters for which the corresponding characteristic equation has at least one root on the imaginary axis. Such a result offers some alternative analysis ways to the approach considered by [123, 136].

5.1 Problem Formulation

Consider the following class of *strictly proper* SISO open-loop systems subject to input delay:

$$\begin{cases} \dot{x}(t) &= Ax(t) + bu(t-\tau), \\ y(t) &= c^T x(t) \end{cases} \qquad x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R} \tag{5.1}$$

with the transfer function:

$$H_{yu}(s) = c^T (sI_n - A)^{-1} b e^{-s\tau} = \frac{P(s)}{Q(s)} e^{-s\tau} \tag{5.2}$$

In this section the loop is closed using a classical PI controller $K(s)$ of the form:

$$K(s) = k\left(1 + \frac{T}{s}\right) = k_p + \frac{k_i}{s}. \tag{5.3}$$

It is noteworthy that in some situation the existence of a time-delay in the actuating input may induce instability or poor performance for the closed-loop scheme (see, for instance, [103] and the references therein). At the same time, there exists situation when the presence of an appropriate time-delay in the actuating input may have the opposite effect (*stabilizing effect*) (see, for instance [1, 107]). However, many problems in process control engineering involve time-delays, thus they can not be neglected. Under the above considerations, the closed-loop system can be expressed as:

$$H(s, k_p, k_i, \tau) = Q(s) + P(s)\left(k_p + \frac{k_i}{s}\right) e^{-s\tau} = 0, \tag{5.4}$$

which is a quasi-polynomial (see, e.g. [39]) with an infinite number of roots [48].

The interest of the this section it to show the explicit conditions on the parameters pair (k_p, k_i), such that the closed-loop system (5.4) is asymptotically stable.

In order to simplify the presentation and without any loss of generality, we consider that the Assumption 3.1 holds, that is the polynomials $P(s)$, $Q(s)$ are such that $P(s)$ and $s\,Q(s)$ do not have common zeros.

5.2 Stability in the Controller Parameter-Space

In the sequel, we are interested in the behavior of the closed-loop system (5.4) for a fixed delay value τ. More precisely, for a given $\tau = \tau^*$ we search the crossing frequencies ω and the corresponding *crossing points* in the parameter space (k_p, k_i) defined by the PI control law such that $H(\boldsymbol{i}\omega, k_p, k_i, \tau^*) = 0$. Since the delay value is fixed, the characteristic equation (5.4) can be seen as a particular form of equation (3.1) where the pair (α, β) is replaced by (k_p, k_i) and $h(s, k_p, k_i) = \left(k_p + \frac{k_i}{s}\right) e^{-s\tau}$.

Remark 5.1. *Using the conjugate of a complex number we get*

$$H(\boldsymbol{i}\omega, k_p, k_i, \tau) = 0 \Leftrightarrow \overline{H(-\boldsymbol{i}\omega, k_p, k_i, \tau)} = 0.$$

Therefore, it is natural to consider only positive frequencies, that is the frequency crossing set $\Omega \subset (0, \infty)$.

5.2.1 Stability Crossing Curves

Considering that the set Ω is known we can easily derive all the crossing points in the parameter space (k_p, k_i).

Proposition 5.1. *[99] For a given $\tau > 0$ and $\omega \in \Omega$, the corresponding crossing point (k_p, k_i) is given by:*

$$k_p = -\Re\left(\frac{Q(\boldsymbol{i}\omega)}{P(\boldsymbol{i}\omega)}e^{i\omega\tau}\right) \tag{5.5}$$

$$k_i = \omega \cdot \Im\left(\frac{Q(\boldsymbol{i}\omega)}{P(\boldsymbol{i}\omega)}e^{i\omega\tau}\right). \tag{5.6}$$

Remark 5.2. *For all $\omega \in \Omega$ we have $P(\boldsymbol{i}\omega) \neq 0$. Indeed, it is easy to see that if $\omega \in \Omega$, then there exists at least one pair (k_p, k_i) such that $H(\boldsymbol{i}\omega, k_p, k_i, \tau) = 0$. Therefore, assuming that $P(\boldsymbol{i}\omega) = 0$ we get also $Q(\boldsymbol{i}\omega) = 0$ which contradicts Assumption 3.1.*

Remark 5.3. *It is important to point out that the controller's gains k_p and k_i include* explicitly *delay information. Furthermore, throughout the chapter, we assume that the corresponding input delay is (perfectly) known, and it is not subject to any uncertainty. The way the delay parameter affects the crossing boundaries can be also analyzed using similar geometric arguments, and, for the sake of brevity, it is not considered here.*

5.2.2 Analytic Characterization of the Crossing Set

In the sequel, we are interested in finding the crossing points (k_p, k_i) such that k_p and k_i are *finite*. This will not restrict the usefulness of the following results since the controller parameters can not be set to some infinite values in practical situation.

Proposition 5.2. *[99] Let $k_p^* > 0$ and $k_i^* > 0$ be given. Let $\Omega_{k_p^*, k_i^*}$ denotes the set of all frequencies $\omega > 0$ such that $s = \boldsymbol{i}\omega$ satisfies equation (5.4) for at least one pair of (k_p, k_i) in the rectangle $|k_p| \leq k_p^*$, $|k_i| \leq k_i^*$. Then, the set $\Omega_{k_p^*, k_i^*}$ consists of a finite number of intervals of finite length.*

5.2.3 Smoothness of the Crossing Curves

When ω varies within some interval Ω_ℓ, the equations (5.5) and (5.6) define a continuous curve. Using the technique developed in Subsection 3.2 (see also [40] and [98]), it can easily be seen that the next result holds.

Proposition 5.3. *[99] The curve $\mathcal{T}_\ell$ is smooth everywhere except possibly at the point corresponding to $s = \boldsymbol{i}\omega$ such that $s = \boldsymbol{i}\omega$ is a multiple solution of (5.4).*

The crossing direction is then obtained directly from Proposition 3.1 by replacing (α, β) with (k_p, k_i).

Remark 5.4. *Straightforward computations show that $R_1 I_2 - R_2 I_1$ is always positive. Thus, a system described by a closed-loop characteristic equation of type (5.4) may have more than one stability region in controller parameter space (k_p, k_i) if one of the following two items is satisfied:*

- *it has one or more crossing curves with some turning points (the direction in controller parameter space changes - see for instance Example 8.7).*
- *it has at least two different crossing curves with opposite direction in (k_p, k_i) - space.*

5.3 Illustrative Examples

In this section, we present several examples in order to point out the usefulness of this methodology in various situations.

Example 5.1 (Scalar system)**.** *First, we validate our results by treating an open-loop stable scalar system already studied in the literature (see for instance [122, 111]). More precisely, we consider*

$$Q(s) = 4s + 1, \quad P(s) = 1, \tag{5.7}$$

and we easily find the corresponding closed-loop characteristic equation

$$H(s,k,T,\tau) = 4s + 1 + (k_p + \frac{k_i}{s})e^{-s\tau}.$$

Taking $\tau = 1$ *(as the authors of [122, 111]) and plotting* k_i *versus* k_p *we obtain the figure 5.1.*

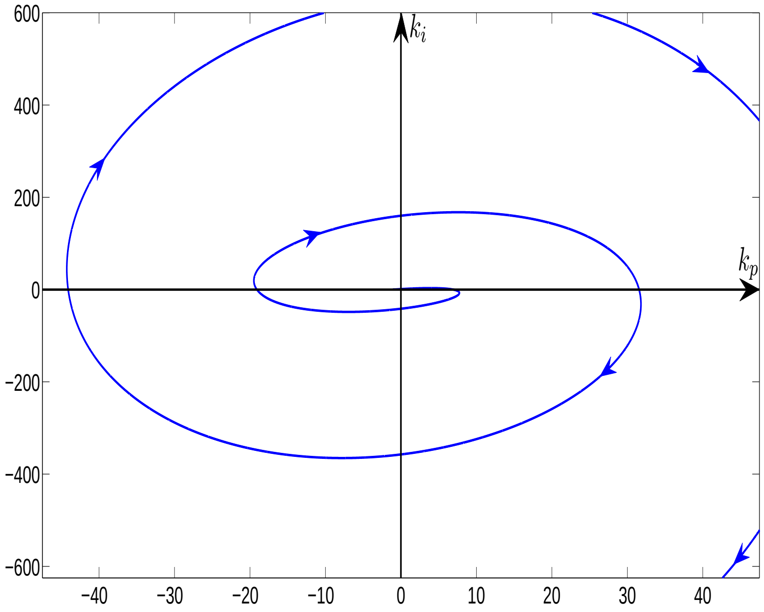


Figure 5.1: Stability crossing curve in the (k_p, k_i) space for the system given by (5.7)

Using Proposition 3.1 we derive that all the crossing direction are towards instability.

On the other hand, the characteristic equation reveals that the system given by (5.7) is stable only if $k_i > 0$*. Therefore, in order to obtain the boundary of the stability region in the* (k_p, k_i) *space, we search the first interval in* Ω *where* $k_i > 0$*. Explicitly, we solve the equation*

$$\omega Im\left((4\boldsymbol{i}\omega + 1)e^{i\omega}\right) > 0,$$

and we get $\omega \in (0, 1.715)$*. Using (5.5) and (5.6) the boundary of the stability region in the* (k_p, k_i) *space is plotted in figure 5.2.*

We note that the same boundary of the stability region has been obtained in [122, 111] by using a different argument.

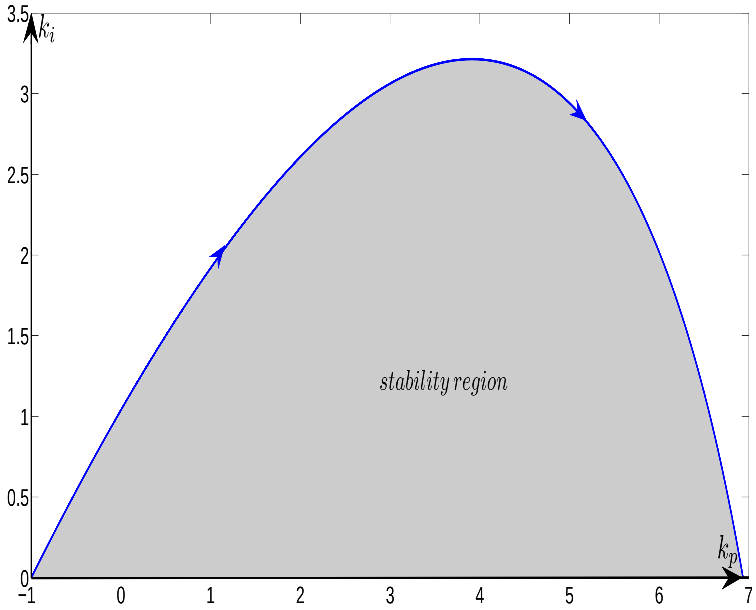


Figure 5.2: The boundary of the stability region in the (k_p, k_i) space for the system given by (5.7)

Example 5.2 (Double integrator subject to input delay)**.** *Consider now the case of a double integrator subject to input delay:*

$$H_{yu}(s) = \frac{e^{-s\tau}}{s^2}.$$

Closing the loop with the PI controller:

$$K(s) = \left(k_p + \frac{k_i}{s}\right),$$

the characteristic equation of the system writes as:

$$s^2 + \left(k_p + \frac{k_i}{s}\right) e^{-s\tau} = 0. \tag{5.8}$$

One obtains:

$$k_p = \omega^2 \cos(\omega\tau), \quad k_i = -\omega^3 \sin(\omega\tau).$$

Thus, k_i and k_p are even functions of ω. In other words it is sufficient to plot k_i versus k_p for positive values of ω. We derive again that the number of unstable roots is getting larger when the distance to the origin increases.

All the crossing directions are towards instability. Taking into account that the system in absence of any control is unstable, we conclude that the system can not be stabilized with a PI controller. The crossing curve for the system is plotted in figure 5.3.

Example 5.3 (An academic example)**.** *In the sequel we consider the unstable system whose dynamics is expressed by the following transfer function ([35]):*

$$H_{yu} = \frac{(s-1)e^{-2s}}{s^2 - 0.5s + 0.5}.$$

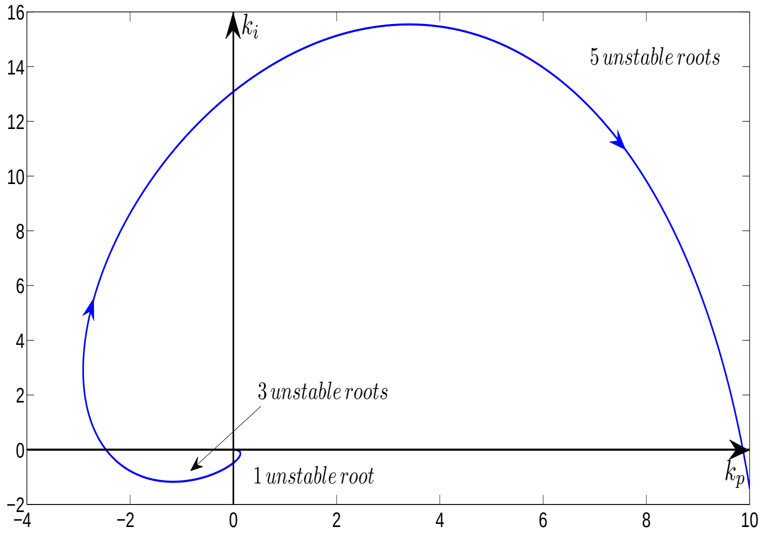


Figure 5.3: Stability crossing curve in the (k_p, k_i) space for the system given by (5.8)

The characteristic equation of the closed-loop system, by using the PI controller is given by

$$s^2 - 0.5s + 0.5 + (s-1)\left(k_p + \frac{k_i}{s}\right)e^{-2s} = 0. \tag{5.9}$$

Straightforward computations show that

$$\begin{aligned} k_p &= \frac{(0.5-0.5\omega^2)\cos 2\omega + \omega^3 \sin 2\omega}{1+\omega^2}, \\ k_i &= \frac{(0.5-0.5\omega^2)\omega \sin 2\omega - \omega^4 \cos 2\omega}{1+\omega^2}. \end{aligned}$$

As we have pointed out before we only need to consider the case $\omega > 0$. Plotting k_i versus k_p, one obtains the border of stability region as illustrated in figure 5.4.

The conclusion in figure 5.4 is obtained taking into account that stability crossing curve, it has a turning point. Analyzing the direction of this curve one can see that all the crossings are towards instability except the one concerning the small stability region pointed out on figure 5.4.

6 The Geometry of PD Controllers for SISO Systems with Input/Output Delays

In this section, we roughly present the characterization of the stability regions in the parameter-space defined by the PD controller's parameters. More precisely we adapt the methodology proposed in the previous section to the case of PD controllers. For the sake of brevity, since the analysis is based on the same arguments (presented in the PI control framework) and does not change in the main aspects, we do not enter too much into details.

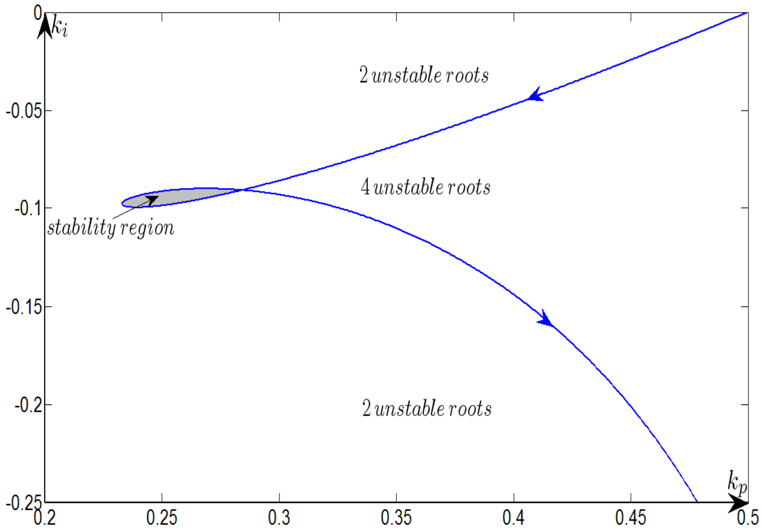


Figure 5.4: The boundary of the stability region in the (k_p, k_i) space for the system given by (5.9)

6.1 Problem Formulation

Consider again the class of *strictly proper* SISO open-loop systems subject to input delay given by (5.1) with the transfer function (5.2). In this section the loop is closed using a classical PD controller $K(s)$ of the form:

$$K(s) = k(1+Ts) = k_p + k_d s. \tag{6.1}$$

Under the above considerations, the closed-loop system can be expressed as:

$$H(s, k_p, k_d, \tau) = Q(s) + P(s)\,(k_p + k_d s)\, e^{-s\tau} = 0. \tag{6.2}$$

The problem considered in this section can be defined as follows:

Problem 6.1. *Find explicit conditions on the parameters pair (k_p, k_d), such that the closed-loop system (6.2) is asymptotically stable.*

The Assumption 3.1: *the polynomials $P(s)$, $Q(s)$ are such that $P(s)$ and $sQ(s)$ do not have common zeros,* makes sense in the PD controller framework and is kept also during this section.

6.2 Stability in the Controller Parameter-Space

In the sequel, we consider that the delay value τ is fixed and we search the crossing frequencies ω and the corresponding *crossing points* in the parameter space (k_p, k_d) defined by the PD control law such that $H(\boldsymbol{i}\omega, k_p, k_d, \tau) = 0$.

As we have pointed out in the previous section, the number of roots in the RHP can change only when some zeros appear and cross the imaginary axis. This time, the *frequency crossing set* Ω consisting of all real positive ω such that there exist at least a pair (k_p, k_d) for which

$$H(\boldsymbol{i}\omega, k_p, k_d, \tau^*) = Q(\boldsymbol{i}\omega) + P(\boldsymbol{i}\omega)\,(k_p + \boldsymbol{i}k_d\,\omega)\,\mathrm{e}^{-\boldsymbol{i}\omega\tau^*} \quad = \quad 0. \tag{6.3}$$

Remark 6.1. *Similarly to the PI controller case, using the conjugate of a complex number we get:*

$$H(\boldsymbol{i}\omega,,k_p,k_d,\tau)=0 \Leftrightarrow \overline{H(-\boldsymbol{i}\omega,k_p,k_d,\tau)}=0.$$

Therefore, it is natural to consider only positive frequencies, that is $\Omega \subset (0,\infty)$.

6.2.1 Stability Crossing Curves

Mimicking the analysis made in the previous section we arrive to the following results:

Proposition 6.1. *For a given* $\tau > 0$ *and* $\omega \in \Omega$ *the corresponding crossing point* (k_p, k_d) *is given by:*

$$k_p = -\Re\left(\frac{Q(\boldsymbol{i}\omega)}{P(\boldsymbol{i}\omega)}e^{\boldsymbol{i}\omega\tau}\right) \tag{6.4}$$

$$k_d = -\frac{1}{\omega}\Im\left(\frac{Q(\boldsymbol{i}\omega)}{P(\boldsymbol{i}\omega)}e^{\boldsymbol{i}\omega\tau}\right). \tag{6.5}$$

Proof. Following similar lines to the ones proposed in *Proposition* 5.1, we have that (6.2) can be rewritten as:

$$\frac{Q(\boldsymbol{i}\omega)}{P(\boldsymbol{i}\omega)}e^{\boldsymbol{i}\omega\tau} = -\left(k_p+\boldsymbol{i}\omega k_d\right).$$

Since, k_p, k_d and ω are real, the previous relation states that the real part of the left hand side is equal to $-k_p$ whereas the imaginary part is $-\omega k_d$. Next, some straightforward computations allow deriving (6.4) and (6.5). ☐

Unlike the PI-controller, in the PD-controller case if $\deg Q(s) = \deg P(s)+1$, some additional attention must be paid. Such a situation is summarized by the following result:

Proposition 6.2. *Let* $\tau \in \mathbb{R}_+$. *Then, the following straight lines belong to the stability crossing curves:*

$$\begin{cases} \left(-\frac{q_0}{p_0},k_d\right) & \text{if} \quad \deg Q(s) > \deg P(s)+1 \\ \left(-\frac{q_0}{p_0},k_d\right),\ \left(k_p,\left|\frac{q_n}{p_{n-1}}\right|\right),\ \left(k_p,-\left|\frac{q_n}{p_{n-1}}\right|\right) & \text{if} \quad \deg Q(s) = \deg P(s)+1 \end{cases} \tag{6.6}$$

where p_i *and* q_i *are the coefficients of the polynomials* $P(s)$ *and* $Q(s)$, *respectively:*

$$P(s)=\sum_{i=1}^{m} p_i s^i, \quad Q(s)=\sum_{i=0}^{n} q_i s^i.$$

Proof. Let us consider first the case $\deg Q(s) > \deg P(s)+1$. In such a situation, we observe from (6.3) that, for $\omega = 0$, the stability crossing curves satisfy:

$$\begin{aligned} Q(0)+P(0)\left(k_p+k_d\cdot 0\right) &= 0, \\ \Rightarrow k_p &= -\frac{q_0}{p_0}. \end{aligned}$$

Since, the above equation holds for all k_d, we conclude that $\left(-\frac{q_0}{p_0},k_d\right)$ belongs to the stability crossing curves.

Next, if $\deg Q(s) = \deg P(s) + 1$, then system become of neutral-type [103] and, in addition to the previous boundary, we must take into account that the closed-loop characteristic equation (6.2) possesses a neutral chain that asymptotically approaches the vertical line [48]:

$$\Re(s) = \frac{1}{\tau} \ln\left(\left|\frac{k_d p_{n-1}}{q_n}\right|\right).$$

Since $\ln(|x|) \geq 0 \Leftrightarrow |x| \geq 1$, this implies that the straight lines $\left(k_p, -\left|\frac{q_n}{p_n}\right|\right)$ and $\left(k_p, \left|\frac{q_n}{p_n}\right|\right)$ belong to the stability crossing curves. □

6.2.2 Analytic Determination of the Crossing Set

In the sequel, we are interested in finding the crossing points (k_p, k_d) such that k_p and k_d are *finite*. This will not restrict the usefulness of the following results since the controller parameters can not be set to some infinite values in practical situation:

Proposition 6.3. *Let $k_p^* > 0$ and $k_d^* > 0$ be given. Let $\Omega_{k_p^*, k_d^*}$ denotes the set of all frequencies $\omega > 0$ satisfying equation (6.3) for at least one pair of (k_p, k_d) in the rectangle $|k_p| \leq k_p^*$, $|k_d| \leq k_d^*$. Then the set $\Omega_{k_p^*, k_d^*}$ consists of a finite number of intervals of finite length.*

Proof. The proof follows the idea presented in Proposition 5.2. The crossing set Ω is derived solving the polynomial inequality:

$$\left|\frac{Q(\boldsymbol{i}\omega)}{P(\boldsymbol{i}\omega)}\right|^2 \leq \left(k_p^*\right)^2 + \left(k_d^*\right)^2 \omega^2. \tag{6.7}$$

For the sake of brevity, we do not present further details here. □

6.2.3 Smoothness of the Crossing Curves

When ω varies within some interval Ω_ℓ satisfying (6.7), the equations (6.4) and (6.5) define a continuous curve. Using the notations introduced in the previous paragraph and the technique developed above, we can easily derive the crossing direction corresponding to this curve.

Precisely, one denotes $\mathcal{T}_\ell$ the crossing curve that corresponds to Ω_ℓ and considers the following decompositions into real and imaginary parts:

$$\begin{aligned}
R_0 + \boldsymbol{i}I_0 &= \boldsymbol{i}\left.\frac{\partial H(s, k_p, k_d, \tau)}{\partial s}\right|_{s=\boldsymbol{i}\omega}, \\
R_1 + \boldsymbol{i}I_1 &= -\left.\frac{\partial H(s, k_p, k_d, \tau)}{\partial k_d}\right|_{s=\boldsymbol{i}\omega}, \\
R_2 + \boldsymbol{i}I_2 &= -\left.\frac{\partial H(s, k_p, k_d, \tau)}{\partial k_p}\right|_{s=\boldsymbol{i}\omega}.
\end{aligned}$$

Then, $\mathcal{T}_\ell$ is smooth everywhere except possibly at the points corresponding to $s = \boldsymbol{i}\omega$ such that $s = \boldsymbol{i}\omega$ is a multiple solution of (6.2).

Finally, the crossing direction is characterized by the following:

Proposition 6.4. *Assume $\omega \in \Omega_\ell$, k_p, k_d satisfy (6.4) and (6.5) respectively, and ω is a simple solution of (6.3) and $H(\mathbf{i}\omega', k_p, k_d, \tau) \neq 0$, $\forall \omega' > 0$, $\omega' \neq \omega$ (i.e. (k_p, k_d) is not an intersection point of two curves or different sections of a single curve). Then as (k_p, k_d) moves from the region on the right to the region on the left of the corresponding crossing curve, a pair of solutions of (6.2) crosses the imaginary axis to the right (through $s = \pm\mathbf{i}\omega$) if $R_2 I_1 - R_1 I_2 > 0$. The crossing is to the left if the inequality is reversed.*

6.3 Illustrative Examples

In the sequel, we present several numerical examples to illustrate the proposed results.

Example 6.1 (Sixth order non-minimal phase system)**.** *Consider a sixth-order, non-minimum phase open-loop system, described by the following transfer function:*

$$H_{yu}(s) = \frac{-s^4 - 7s^3 - 2s + 1}{(s+1)(s+2)(s+3)(s+4)(s^2+s+1)} e^{-\frac{s}{20}}. \tag{6.8}$$

The use of a PD-controller leads to characteristic closed-loop equation:

$$(s+1)(s+2)(s+3)(s+4)\left(s^2+s+1\right) - \left(s^4+7s^3+2s-1\right)e^{-\frac{1}{20}s} = 0. \tag{6.9}$$

In Fig.6.1 is illustrated the stability region in the (k_p, k_d) parameter space for (6.8). *From*

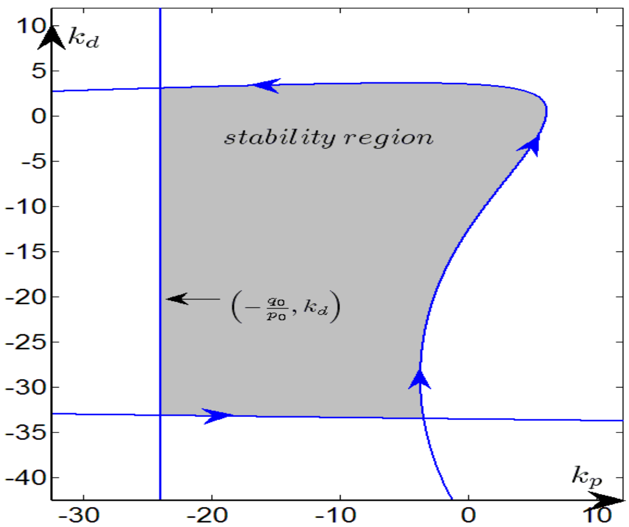


Figure 6.1: Stability region of k_p and k_d for (6.8).

Fig.6.1 we can observe that the stability crossing curves are composed by those of equations (6.4) –(6.5) *together with first equation of* (6.6).

Example 6.2 (Third Order, non-minimal phase, unstable system)**.** *Consider now a system described by the following transfer function:*

$$H_{yu}(s) = \frac{15s^2 + 3s - 20}{125s^3 + 70s^2 + 10s + 8} e^{-2s}. \tag{6.10}$$

The interest of the analysis of this system arises form the fact that the closed-loop characteristic equation behaves as a system of neutral-type. Then, after applying Proposition 6.1 together with Proposition 6.2, we obtain the results depicted in Fig.6.2

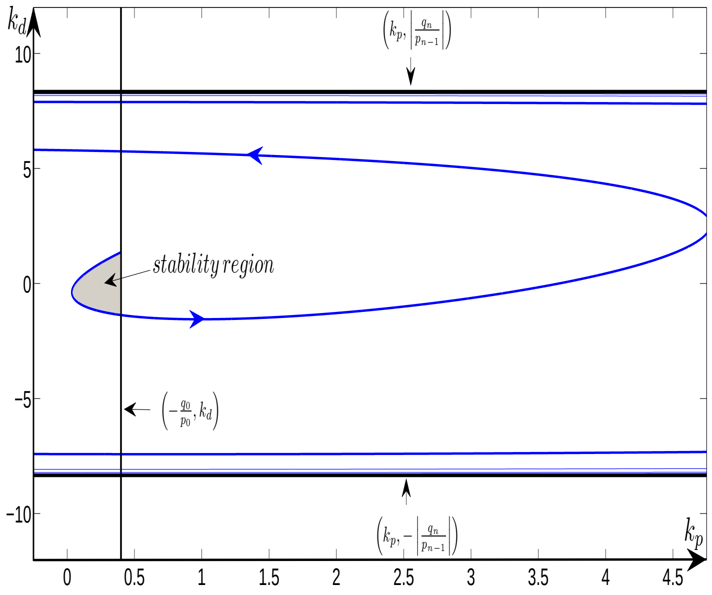


Figure 6.2: Stability region of k_p and k_d for (6.9).

7 The Geometry of PID Controllers for SISO Systems with Input/Output Delays

As it has been emphasized along this first part of the thesis, PID controllers are by far the most applied feedback law for SISO systems in industrial process (see, for instance, [5, 109, 123] and reference therein). The "popularity" of PID controllers can be attributed to their particular distinct features: *simplicity* and *easy implementation*.
Actually, to the best of our knowledge, there exist mainly two approaches enabling to get the set of stabilizing PID controllers for a LTI delay-system. More precisely, we think firstly to the approach based on an extension *Hermite-Biehler Theorem* to the time-delay systems [112, 110]. In this case, using the property of interlacing at high frequencies and solving some linear inequalities, they obtained the set of stabilizing PID controllers. However, the only drawback of the method is the complexity of the algorithm that calculate the appropriate k_p intervals. Second, based on the Neimark $\mathcal{D}-$partition method, the works of [8, 54] derived a method to find the set of stabilizing PID controllers.

As in the previous chapters, the method presented here is inspired by the analysis (based on some geometric arguments) proposed in [40]. Even tough, the proposed method is closely related to the above mentioned works, the method proposed here make use of some different arguments, that in addition will enable us to present a characterization of crossing boundaries. More over, such a technique will be the core, of the next chapter that will deal with fragility analysis of PID-type controllers.

7.1 Problem Formulation

As in the previous paragraphs, consider the following class of strictly proper SISO open-loop systems subject to input/output delays:

$$\left\{\begin{array}{rcl} \dot{x}(t) & = & Ax(t)+bu(t-\tau), \\ y(t) & = & c^T x(t) \end{array}\right. \qquad x(t)\in\mathbb{R}^n,\ u(t)\in\mathbb{R} \tag{7.1}$$

with the transfer function:

$$H_{yu}(s) = c^T (sI_n - A)^{-1} b e^{-s\tau} = \frac{P(s)}{Q(s)} e^{-s\tau} =: G(s). \tag{7.2}$$

The same property holds if the delay acts on the output signals. In this chapter, the loop is closed using a classical PID controller $K(s)$ of the form:

$$K(s) = k\left(1 + T_d s + \frac{1}{T_i s}\right) = k_p + k_d s + \frac{k_i}{s}. \tag{7.3}$$

Under the above considerations, it is clear that the closed-loop dynamics can be characterized by the equation:

$$1 + G(s)K(s) = 0, \tag{7.4}$$

which rewrites as:

$$H(s;k_p,k_d,k_i) = \frac{1}{G(s)} + \left(k_p + k_d s + \frac{k_i}{s}\right) = 0, \tag{7.5}$$

The problem considered in this section can be defined as follows:

Problem 7.1. *Find explicit conditions on the controller parameters (k_p,k_d,k_i), such that the closed-loop system (7.5) is asymptotically stable.*

In order to simplify the presentation and, without any loss of generality, we consider that the Assumption 3.1 holds.

Remark 7.1. *Similarly to the previous chapters, using the conjugate of a complex number we get*

$$H(\boldsymbol{i}\omega;k_p,k_d,k_i) = 0 \Leftrightarrow \overline{H(-\boldsymbol{i}\omega;k_p,k_d,k_i)} = 0. \tag{7.6}$$

Therefore, we only need to consider positive frequencies ω, that is the frequency crossing set $\Omega \subset (0,\infty)$.

7.2 Stability in the Controller Parameter-Space

As in the previous chapters, in the sequel we will focus on the closed-loop behavior of the system (7.5) for a fixed delay value τ. More precisely, we want to derive the *stability crossing boundaries* $\mathcal{T}$ which is the set of parameters $(k_p,k_d,k_i) \in \mathbb{R}^3_+$ such that (7.5) has imaginary solutions. As the parameters (k_p,k_d,k_i) cross the stability crossing boundaries, some characteristic roots cross the imaginary axis. We also consider $\Omega = \left\{\omega \in \mathbb{R} \mid \exists (k_p,k_d.k_i) \in \mathbb{R}^3_+ \text{ such that } H(\boldsymbol{i}\omega;k_p,k_d,k_i) = 0\right\}$ the set of frequencies where the number of unstable roots of (7.5) changes. The set Ω will be called *stability crossing set*.

7.2.1 Stability Crossing Characterizations

Considering that Ω is known, the stability crossing boundaries are simply characterized by:

Proposition 7.1. *The stability crossing boundaries associated to* (7.5) *are described as follows:*

$$\begin{cases} k_p = -\Re\left(\dfrac{Q(\boldsymbol{i}\omega)}{P(\boldsymbol{i}\omega)}e^{\boldsymbol{i}\omega\tau}\right) \\ k_i = k_d\omega^2 + \omega\Im\left(\dfrac{Q(\boldsymbol{i}\omega)}{P(\boldsymbol{i}\omega)}e^{\boldsymbol{i}\omega\tau}\right) \end{cases}, \quad \forall\omega\in\Omega. \tag{7.7}$$

Proof. From its definition, $\mathcal{T}$ is the set of parameters $(k_p,k_d,k_i)\in\mathbb{R}^3$, for which there exists at least one frequency $\omega\in\Omega$ such that $H(\boldsymbol{i}\omega;k_p,k_d,k_i)=0$. Therefore, both the real and the imaginary parts of $H(\boldsymbol{i}\omega;k_p,k_d,k_i)$ have to be zero. Straightforward computation shows that:

$$\Re\left(H(\boldsymbol{i}\omega;\cdot)\right) = k_p + \Re\left(G(\boldsymbol{i}\omega)^{-1}\right),$$

which leads to the first relation stated in (7.7). On the other hand,

$$\Im\left(H(\boldsymbol{i}\omega;\cdot)\right) = \Im\left(G(\boldsymbol{i}\omega)^{-1}\right) + k_d\omega - k_i/\omega,$$

which allows us deriving the second relation in (7.7). □

Remark 7.2. *For any fixed $\omega^*\in\Omega$, one obtains a section of a stability crossing surface which consists in a straight line parallel to the (k_d,k_i) plane and passing through the point $\left(-\Re\left(\dfrac{Q(\boldsymbol{i}\omega)}{P(\boldsymbol{i}\omega)}e^{\boldsymbol{i}\omega\tau}\right),0,\omega\Im\left(\dfrac{Q(\boldsymbol{i}\omega)}{P(\boldsymbol{i}\omega)}e^{\boldsymbol{i}\omega\tau}\right)\right)$. The slope of this line in the (k_d,k_i) plane is always positive and is given by ω^2.*

Remark 7.3. *From the proof of Proposition 7.1, it is clear that $k_i=0$ defines a boundary.*

Remark 7.4. *Let the relative degree of the system (7.1) be $\delta=1$. Then, the closed-loop system (7.1) becomes a system of neutral-type (see, e.g., [48, 92]) and*

$$\left(k_p,\left|\frac{q_n}{p_{n-1}}\right|,k_i\right) \quad \textit{and} \quad \left(k_p,-\left|\frac{q_n}{p_{n-1}}\right|,k_i\right)$$

belong to the stability crossing surfaces. Here, p_{n-1} and q_n represent the leading coefficients of the polynomials $P(s)$ and $Q(s)$, respectively:

$$P(s)=\sum_{i=0}^{n-1}p_is^i, \quad Q(s)=\sum_{i=0}^{n}q_is^i.$$

From *Proposition* 7.1 it is clear to see that for fixed $k_p=k_p^*\in\mathbb{R}$, the second equation in (7.7) defines a set of straight lines in the $(k_d,k_i)-$plane. More over, such a set will define the stability crossing curves in the (k_d,k_i) controller parameter space. Then, it will be interesting (necessary) to find the k_p-intervals, for which the stability characteristics remains unchanged. In this vein, lets introduce the set Λ denoted by,

$$\Lambda := \left\{\omega_\ell\in\mathbb{R}_+\middle|k_p'(\omega_\ell)=0,\ \ell\in\mathbb{N}\right\},$$

where $k_p'(\omega) := \frac{d}{d\omega}k_p(\omega)$. Additionally, define $\omega_\ell^+ \in \Lambda$ if $k_p\left(\omega_\ell^+\right) \geq 0$ and $\omega_\ell^- \in \Lambda$ if $k_p\left(\omega_\ell^-\right) < 0$. According to these definitions lets indexing $\omega_\ell^\pm$ satisfying

$$\begin{aligned} 0 \quad &\leq \quad k_p\left(\omega_1^+\right) < k_p\left(\omega_2^+\right) < k_p\left(\omega_3^+\right) < \ldots && (7.8a) \\ 0 \quad &> \quad k_p\left(\omega_1^-\right) > k_p\left(\omega_2^-\right) > k_p\left(\omega_3^-\right) > \ldots && (7.8b) \end{aligned}$$

Proposition 7.2. *Based on* (7.8)*, lets consider the following intervals:*

$$\begin{cases} \mathcal{K}_1^+ = \left(k_p\left(\omega_1^-\right), k_p\left(\omega_1^+\right)\right), \mathcal{K}_2^+ = \left(k_p\left(\omega_1^+\right), k_p\left(\omega_2^+\right)\right), \ldots \\ \mathcal{K}_1^- = \left(k_p\left(\omega_2^-\right), k_p\left(\omega_1^-\right)\right), \mathcal{K}_2^- = \left(k_p\left(\omega_3^-\right), k_p\left(\omega_2^-\right)\right), \ldots \end{cases} \quad \textit{if} \quad k_p\left(\omega_1^+\right) \neq 0 \ ,$$

or

$$\begin{cases} \mathcal{K}_1^+ = \left(k_p\left(\omega_1^+\right), k_p\left(\omega_2^+\right)\right), \mathcal{K}_2^+ = \left(k_p\left(\omega_2^+\right), k_p\left(\omega_3^+\right)\right), \ldots \\ \mathcal{K}_1^- = \left(k_p\left(\omega_1^-\right), k_p\left(\omega_1^+\right)\right), \mathcal{K}_2^- = \left(k_p\left(\omega_2^-\right), k_p\left(\omega_1^-\right)\right), \ldots \end{cases} \quad \textit{if} \quad k_p\left(\omega_1^+\right) = 0 \ .$$

Then, the number of stability regions in the parameter space defined by the proportional gain, the derivative gain and the integral gain parameters (k_p, k_d, k_i) *remains constant as* k_p *varies within each interval* $\mathcal{K}_\ell^\pm$, $\ell \in \mathbb{N}$.

Proof. According to Proposition 7.1, the stability crossing boundaries are given by:

$$\begin{aligned} k_p \quad &= \quad -\Re\left(\frac{Q(\boldsymbol{i}\omega)}{P(\boldsymbol{i}\omega)}e^{\boldsymbol{i}\omega\tau}\right), && (7.9a) \\ k_i \quad &= \quad k_d\omega^2 + \omega\Im\left(\frac{Q(\boldsymbol{i}\omega)}{P(\boldsymbol{i}\omega)}e^{\boldsymbol{i}\omega\tau}\right). && (7.9b) \end{aligned}$$

Then, the key idea is to express (7.9) as $\mathcal{F}(k_p, k_d, k_i) = 0$ (without depending on ω), in such situation it will be possible to obtain directly the stability region in the (k_p, k_d, k_i) parameter space. In this vein, we can observe from (7.9) that the k_p interval is determined solely by (7.9a), or in other word, k_p depends solely of the frequency ω, that is, it can be seen as a function of one real variable. Bearing the above fact in mind, we have that according to the *Inverse Function theorem* (see, [58], for further details), that there exists a unique continuous function $w : \mathcal{I} \subset \mathbb{R} \mapsto \mathcal{J} \subset \mathbb{R}_+$ (for some appropriate intervals) such that

$$\omega = w(k_p). \tag{7.10}$$

Then, substituting (7.10) into (7.9b) we get

$$\underbrace{w(k_p)k_d - k_i + w(k_p)\Im\left(\frac{Q(\boldsymbol{i}w(k_p))}{P(\boldsymbol{i}w(k_p))}e^{\boldsymbol{i}w(k_p)\tau}\right)}_{=:\mathcal{F}(k_p, k_d, k_i)} = 0.$$

The remaining proof consists in showing that the appropriate sets $\mathcal{I}$, $\mathcal{J}$ are precisely given by the intervals $\mathcal{K}_\ell^\pm$. In this vein, from the construction of $\mathcal{K}_\ell^\pm$ we can observe that if $\omega \in \mathcal{K}_\ell^\pm$ for some $\ell \in \mathbb{N}$, then k_p is continuous and since the sign of k_p' remains constant, this implies the monotonicity of k_p and in consequence k_p will be a one-to-one map for all $\omega \in \mathcal{K}_\ell^\pm$. Then, according to the *Inverse Function theorem*, the above arguments are necessary and sufficient conditions for the existence and uniqueness of the function $w(k_p)$. Finally, at the points where k_p' vanish, two situations may occur: (i) for $k_p = k_p(\omega)$, the map $w(k_p)$ has two (or more) possible solutions, in this situation we can have two (or more) stability regions sharing the same number of unstable roots (see, *Example* 7.2 for an illustrative example); (ii) the second possibility is that $w(k_p)$ is not differentiable, implying that the stability region is reducing to a point on the stability boundary. $\square$

7.2.2 Stability Crossing Sets and Classification of the Stability Crossing Boundaries

In the sequel, we present a practical methodology to derive the stability crossing set. For the sake of brevity, we suppose that the following technical assumption is satisfied:

Assumption 7.1. *There exist some bounds* $\left(\underline{k_p^*}, \overline{k_p^*}\right)$, $\left(\underline{k_d^*}, \overline{k_d^*}\right)$ *and* $\left(\underline{k_i^*}, \overline{k_i^*}\right)$ *of the controller gains.*

These bounds can be arbitrarily fixed and, in principle, they are chosen by the designer according to the physical constraints of the model/controller. In this context, when Assumption 7.1 holds, the section of the stability crossing surface obtained for a fixed $\omega \in \Omega$ reduces to a segment (see Remark 7.2).

Proposition 7.3. *Consider that Assumption 7.1 holds. Then the stability crossing set* Ω *is a union of bounded intervals consisting in all frequencies that simultaneously satisfy the following conditions:*

$$\begin{cases} \underline{k_p^*} \leq -\Re\left(\dfrac{Q(\boldsymbol{i}\omega)}{P(\boldsymbol{i}\omega)} e^{\boldsymbol{i}\omega\tau}\right) \leq \overline{k_p^*} \\ \exists\ \underline{k_d^*} \leq k_d \leq \overline{k_d^*} \text{ s.t. } \underline{k_i^*} \leq k_d\omega^2 + \omega\Im\left(\dfrac{Q(\boldsymbol{i}\omega)}{P(\boldsymbol{i}\omega)} e^{\boldsymbol{i}\omega\tau}\right) \leq \overline{k_i^*}. \end{cases} \tag{7.11}$$

Proof. The characterization of the stability crossing set Ω given by (7.11) follows straightforward from (7.7) and Assumption 7.1. In order to prove the boundedness of the crossing set Ω, we notice that due to the assumption that the transfer $G(s)$ is strictly proper, one has

$$\lim_{\omega \to +\infty} \left|G(\boldsymbol{i}\omega)^{-1}\right| = +\infty.$$

In other words, this means that either

$$\lim_{\omega \to +\infty} \left|\Re\left(G(\boldsymbol{i}\omega)^{-1}\right)\right| = +\infty$$

or

$$\lim_{\omega \to +\infty} \left|\Im\left(G(\boldsymbol{i}\omega)^{-1}\right)\right| = +\infty$$

which contradicts either the first relation in (7.11) or the second one. □

Remark 7.5. *Propositions 7.1 and 7.3 lead to the following algorithm to determine both the stability crossing set* Ω *and the stability crossing boundaries* $\mathcal{T}$*:*

- **Step 1:** *One solves the system* $\underline{k_p^*} \leq -\Re\left(\dfrac{1}{G(\boldsymbol{i}\omega)}\right) \leq \overline{k_p^*}$ *getting a union of intervals.*

- **Step 2:** *For all* ω *derived at the previous step, one computes* k_p *and derive the equation of the line* (k_d, k_i) *given by the second equation in* (7.7).

- **Step 3:** *Finally, one keeps only those frequencies* ω *for which the line* (k_d, k_i) *derived at the previous step intersects the rectangle* $[(\underline{k_d^*}, \underline{k_i^*}); (\overline{k_d^*}, \underline{k_i^*}); (\overline{k_d^*}, \overline{k_i^*}); (\underline{k_d^*}, \overline{k_i^*})]$.

It is worth noting here that k_p, k_d and k_i continuously depend on ω. Therefore, in order to classify the stability crossing boundaries we will first classify the intervals belonging to the stability crossing set. Precisely, a deeper analysis of Proposition 7.3 allows us to say that ω^* is an end of an interval belonging to Ω if and only if one of the following condition is satisfied:

- **Type 1:** $-\Re\left(\frac{1}{G(\boldsymbol{i}\omega^*)}\right) = k_p^*$, where k_p^* is either $k_p^* = \underline{k_p^*}$ or $k_p^* = \overline{k_p^*}$. In this case, $\omega^* \in \Omega$ and the stability crossing surface approaches a segment parallel to the (k_d, k_i) plane given by $k_p = k_p^*$ and
$$k_i = k_d \cdot (\omega^*)^2 + \omega^* \Im\left(\frac{1}{G(\boldsymbol{i}\omega^*)}\right),$$
$$\underline{k_d^*} \leq k_d \leq \overline{k_d^*},\ \underline{k_i^*} \leq k_i \leq \overline{k_i^*}$$
- **Type 2:** $-\frac{1}{\omega^*}\Im\left(\frac{1}{G(\boldsymbol{i}\omega^*)}\right) = k_d^*$. In this case $\omega^* \in \Omega$ and the stability crossing surface ends in the point $\left(-\Re\left(\frac{1}{G(\boldsymbol{i}\omega^*)}\right), -\frac{1}{\omega^*}\Im\left(\frac{1}{G(\boldsymbol{i}\omega^*)}\right), 0\right)$, included in the (k_p, k_d) plane.
- **Type 3:** $\omega^*\Im\left(\frac{1}{G(\boldsymbol{i}\omega^*)}\right) = k_i^*$. In this case $\omega^* \in \Omega$ and the stability crossing surface ends in the point $\left(-\Re\left(\frac{1}{G(\boldsymbol{i}\omega^*)}\right), 0, \omega^*\Im\left(\frac{1}{G(j\omega^*)}\right)\right)$, included in the (k_p, k_i) plane.

Similarly to [40], we classify the stability crossing boundaries in 6 types in function of the kind of the left and right ends of the corresponding frequency crossing interval. Precisely, we say that a crossing surface is of type *ab*, $a, b \in \{1, 2, 3\}$ if it corresponds to a crossing interval (ω_l, ω_r) with ω_l of type a and ω_r of type b. Let us notice that generally the intervals (ω_l, ω_r) are closed.

7.2.3 Crossing Direction

As explained in [25, 132], a pair of imaginary zeros $(\bar{s}, s)$ of the characteristic equation $H(s;\ k_p, k_d, k_i) = 0$ cross the imaginary axis through the "gates" $-\boldsymbol{i}\omega$, $\boldsymbol{i}\omega$ respectively, as (k_p, k_d, k_i) moves from one side of a stability crossing surface to the other side. The direction of crossing may be calculated using implicit function theorem as described in the preceding chapters (see, for further details [40] and reference there in). Precisely, the characteristic equation $H(s;\ k_p, k_d, k_i) = 0$ defines an implicit function s of variables k_p, k_d and k_i. The definition of $H(s;\ k_p, k_d, k_i)$ given by (7.5) allows us to compute the following partial derivatives:

$$\begin{aligned}
\frac{\partial s}{\partial k_p} &= \frac{s^2 G^2(s)}{k_i G^2(s) - k_d s^2 G^2(s) + s^2 G'(s)}, \\
\frac{\partial s}{\partial k_d} &= \frac{s^3 G^2(s)}{k_i G^2(s) - k_d s^2 G^2(s) + s^2 G'(s)}, \\
\frac{\partial s}{\partial k_i} &= \frac{s G^2(s)}{k_i G^2(s) - k_d s^2 G^2(s) + s^2 G'(s)}.
\end{aligned} \tag{7.12}$$

Let $(\widetilde{k}_p, \widetilde{k}_d, \widetilde{k}_i)$ a point belonging to a stability crossing surface and let $s = \boldsymbol{i}\widetilde{\omega}$, $\widetilde{\omega} > 0$ be the corresponding imaginary zero of the characteristic equation. Let $\mathbf{x} = (x_p, x_d, x_i)$ be a unit vector that is not tangent to the surface. Let us also use the following notation $\overrightarrow{k} = (k_p, k_d, k_i)$ and $\overrightarrow{k^*} = (\widetilde{k}_p, \widetilde{k}_d, \widetilde{k}_i)$.

Proposition 7.4. *A pair of zeros of* (7.5) *moves from the left half complex plane (LHP) to the right half complex plane (RHP) as* (k_p, k_d, k_i) *moves from one side of a stability crossing surface to the other side through* $(\widetilde{k}_p, \widetilde{k}_d, \widetilde{k}_i)$ *in the direction of* $\mathbf{x}$ *if*

$$\Re\left(\frac{\partial s}{\partial k_p}x_p + \frac{\partial s}{\partial k_d}x_d + \frac{\partial s}{\partial k_i}x_i\right)\Bigg|_{s=j\widetilde{\omega},\, \overrightarrow{k}=\overrightarrow{k^*}} > 0. \tag{7.13}$$

The crossing is from the RHP to the LHP if the inequality (7.13) *is reversed.*

Proof. The proof follows directly from the fact that the derivative of the implicit function s along the direction given by $\mathbf{x}$ in the point $(\widetilde{k}_p, \widetilde{k}_d, \widetilde{k}_i)$ is

$$\left.\frac{ds}{dx}\right|_{(\widetilde{k}_p, \widetilde{k}_d, \widetilde{k}_i)} = \left(\frac{\partial s}{\partial k_p}x_p + \frac{\partial s}{\partial k_d}x_d + \frac{\partial s}{\partial k_i}x_i\right)_{(\widetilde{k}_p, \widetilde{k}_d, \widetilde{k}_i)}$$

Thus the real part of the previous directional derivative is computed as the right part of (7.13) □

7.3 Illustrative Examples

In order to illustrate the previous results, in the sequel we present several numerical examples borrowed from the literature.

PID Stabilization Problem

Example 7.1. *Consider the PID stabilization problem of the following non-minimal phase system [112]:*

$$G(s) = \frac{s^3 - 4s^2 + s + 2}{s^5 + 8s^4 + 32s^3 + 46s^2 + 46s + 17}e^{-s}. \tag{7.14}$$

Now, in order to apply Proposition 7.2 we plot $k_p'(\omega)$, *obtaining:*

Next, based on Fig.7.1 the ω_i*'s values are summarized in Table 7.1.*

	ω_0	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	$\cdots$
$\omega \approx$	0	0.8542	1.9233	3.5050	5.572	8.019	10.730	$\cdots$
$k_p(\omega_i)$	-8.5	4.6332	-6.610	-29.27	59.433	-108.5	177.45	$\cdots$

Table 7.1: Solution of $k_p'(\omega) = 0$, for the system (7.14).

Now, based on the results showed in Table 7.1, we apply Proposition 7.2, obtaining the k_p- *intervals* $\mathcal{K}_\ell^{\pm}$ *that are summarized in Table 7.2.*

The interval containing the stabilizable parameters is $\mathcal{K}_1^+ = (-6.6109, 4.63329)$, *and the stability region computed is depicted in Fig.7.2.*

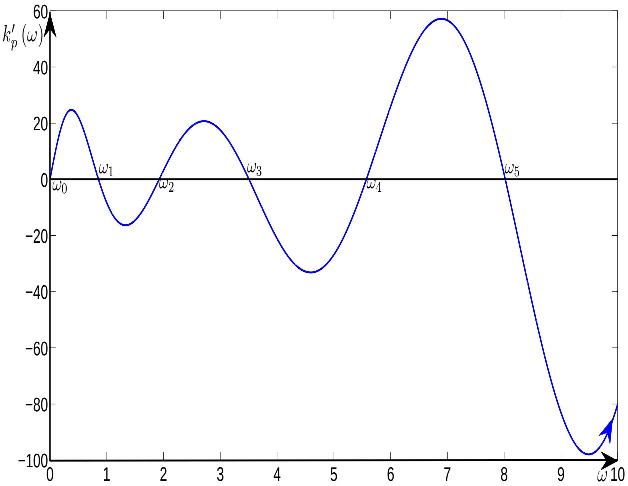


Figure 7.1: Intersections of $k_p'(\omega)$ and 0 for the example (7.14).

$i=$	1	2	3	$\cdots$
$\mathcal{K}_i^+$	$(-6.6109, 4.63329)$	$(4.63329, 59.4333)$	$(59.4333, 177.4514)$	$\cdots$
$\mathcal{K}_i^-$	$(-8.5, -6.6109)$	$(-29.2746, -8.5)$	$(-108.5780, -29.2746)$	$\cdots$

Table 7.2: The k_p-interval for the system (7.14).

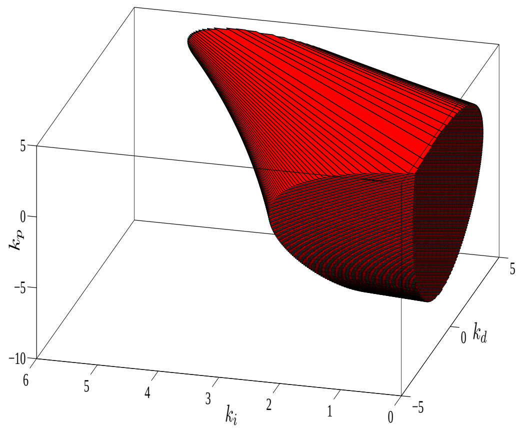


Figure 7.2: Stabilizing set of (k_p, k_d, k_i) for the system (7.14).

Example 7.2 (Non-Minimal Phase System). *Consider now the PID control for a non-minimal phase plant described by the following transfer function [8]:*

$$G(s) = \frac{-s^4 - 7s^3 - 2s + 1}{(s+1)(s+2)(s+3)(s+4)(s^2+s+1)} e^{-\frac{1}{20}s}. \tag{7.15}$$

In order to apply Proposition 7.2 we plot $k_p'(\omega)$, obtaining Fig.7.3. Where the ω_i's values are summarized in the Table 7.3. Based on the results given in Table 7.3, we apply Proposition

	ω_0	ω_1	ω_2	ω_3	ω_4	ω_5	$\cdots$
$\omega \approx$	0	0.511222	0.714606	1.64704	23.236	73.7516	$\cdots$
$k_p(\omega_i)$	-24	4.68073	-3.76712	6.06932	-294.211	4786.59	$\cdots$

Table 7.3: Solution of $k_p'(\omega) = 0$, for the system (7.15).

7.2, obtaining the k_p-intervals summarized in Table 7.4. For this example, we found that the

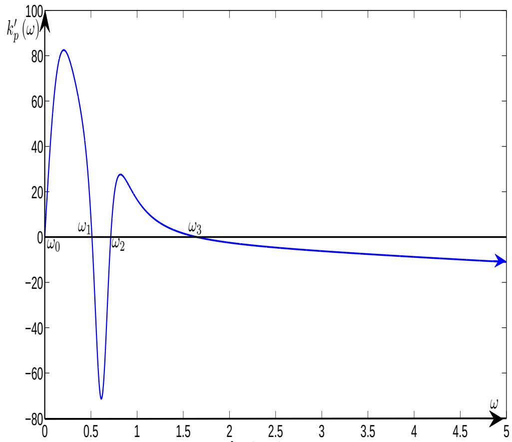


Figure 7.3: Intersections of $k_p'(\omega)$ and 0 for the example (7.15).

$i=$	1	2	3	$\cdots$
$\mathcal{K}_i^+$	$(-3.76712, 4.68073)$	$(4.68073, 6.06932)$	$(6.06932, 4786.59)$	$\cdots$
$\mathcal{K}_i^-$	$(-24, -3.76712)$	$(-294.21, -24)$	$(-16704.7, -294.21)$	$\cdots$

Table 7.4: The k_p-interval for the system (7.15).

stability interval are given by $\mathcal{K}_1^-$, $\mathcal{K}_1^+$ and $\mathcal{K}_2^+$. However, it is interesting to note that these interval possess different characteristics, that are illustrated in the following figures.

The complete stability region $\mathcal{K} = \mathcal{K}_1^- \cup \mathcal{K}_1^+ \cup \mathcal{K}_2^+$ is depicted in Fig.7.5.

Example 7.3 (unstable, non-minimal phase system)**.** *Consider the following plant:*

$$G(s) = \frac{s-2}{s^2 - \frac{1}{2}s + \frac{13}{4}} e^{-\frac{1}{2}s}. \tag{7.16}$$

The interest in the analysis of this system remains in the fact that the closed-loop plant becomes a system of Neutral-Type.

Now, as in the above examples we proceed to plot $k_p'(\omega)$, obtaining Fig.7.16

The main ω_i values are summarized in Table7.5.

	ω_0	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	$\cdots$
$\omega \approx$	0	1.70834	4.73632	10.1297	16.1451	22.3063	28.5204	$\cdots$
$k_p(\omega_i)$	1.625	0.325953	3.26132	-9.42698	15.7038	-21.9869	28.2706	$\cdots$

Table 7.5: Solution of $k_p'(\omega) = 0$, for the system (7.16).

Based in the results of Table7.5, we obtain the mean k_p-intervals summarized in Table7.6.

	1	2	3	$\cdots$
$\mathcal{K}_i^+$	$(-9.42698, 0.325953)$	$(0.325953, 1.625)$	$(1.625, 3.26132)$	$\cdots$
$\mathcal{K}_i^-$	$(-21.9869, -9.42698)$	$(-34.5543, -21.9869)$	$(-47.1214, -34.5543)$	$\cdots$

Table 7.6: The k_p-interval for the system (7.16).

For this example, $\mathcal{K}_2^+$ is the interval containing the stabilizable parameters and the stability region is depicted in Fig.7.7.

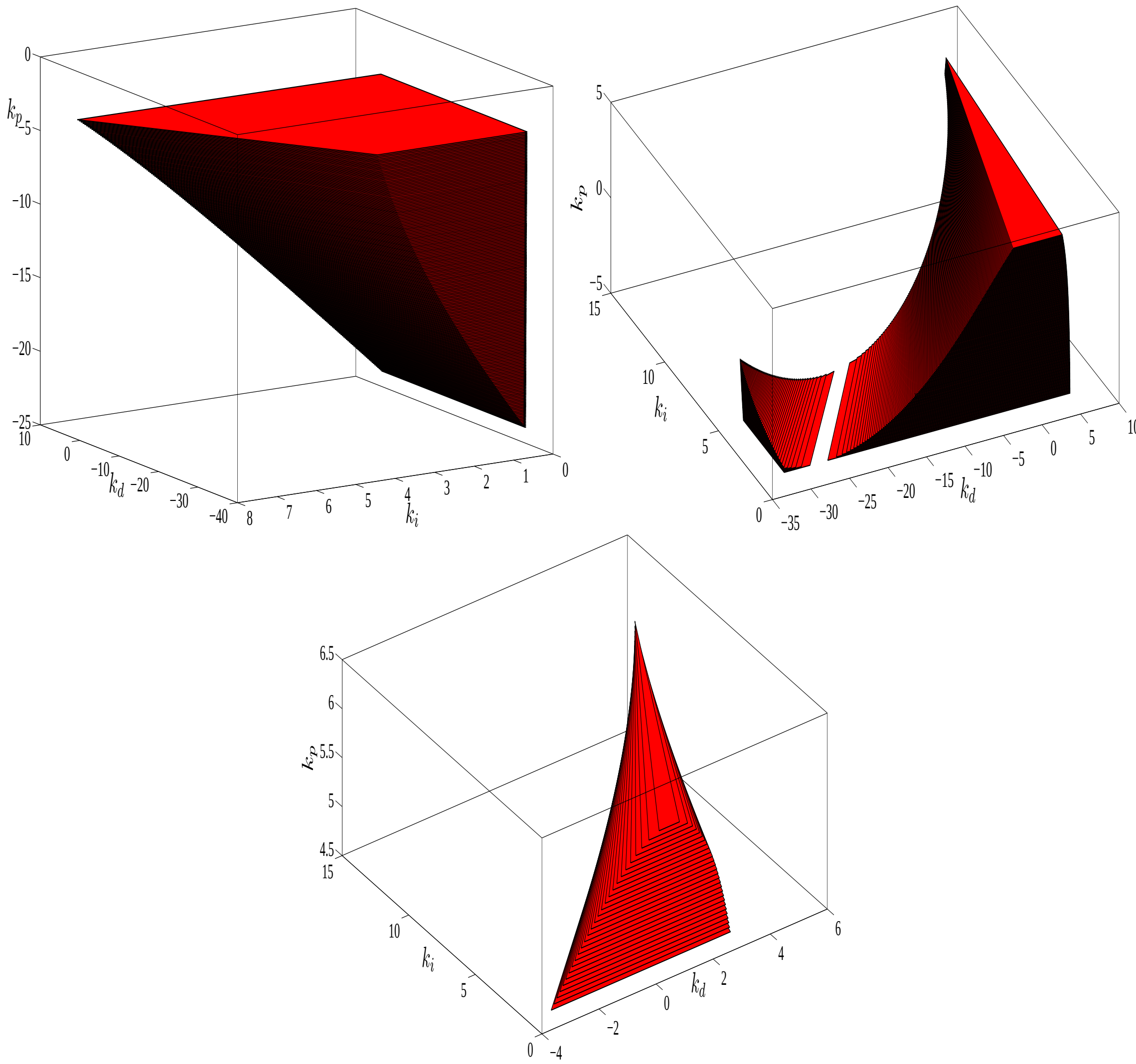


Figure 7.4: The PID stability region for $k_p \in [-5, 5]$. (Upper-Left) One stable region, for $\mathcal{K}_1^-$. (Upper-Right) Two separated stable regions, for $\mathcal{K}_1^+$. (Lower) One stable region, for $\mathcal{K}_2^+$.

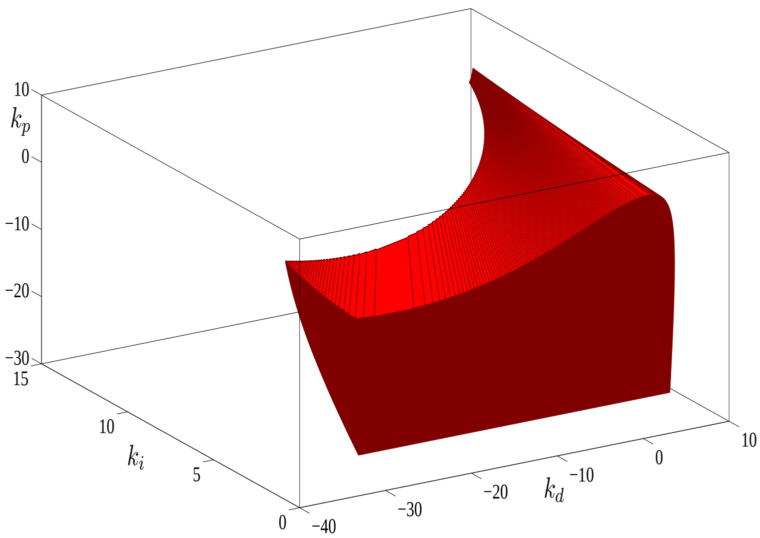


Figure 7.5: The PID stability region for the system (7.15).

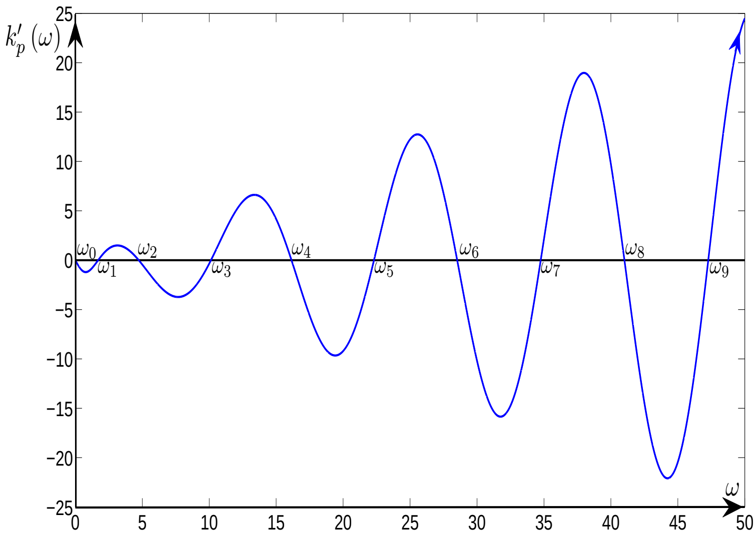


Figure 7.6: Intersections of $k'_p(\omega)$ and 0 for the example (7.16).

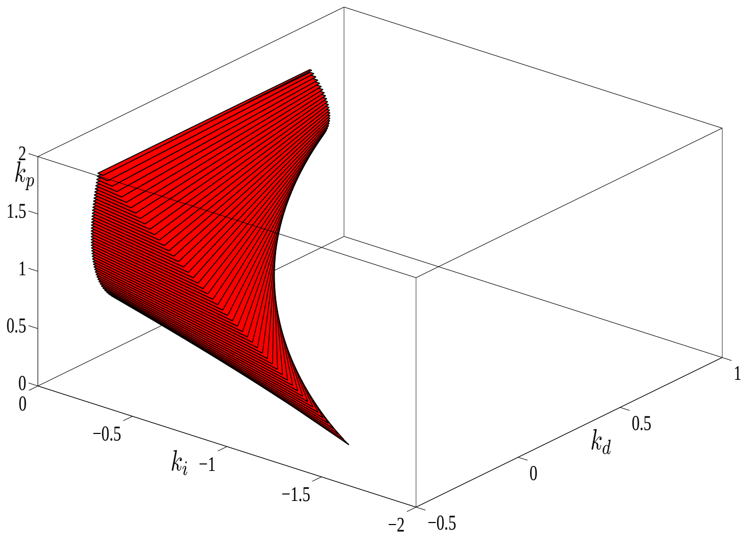


Figure 7.7: The PID stability region of Neutral-Type .

Stability crossing boundaries classification

Now, in order to illustrate the proposed boundaries classification, we consider the following example:

Example 7.4. *Lets consider again the same plant given in Example 7.2, i.e.,*

$$G(s) = \frac{s^3 - 4s^2 + s + 2}{s^5 + 8s^4 + 32s^3 + 46s^2 + 46s + 17} e^{-s}.$$

By choosing the rectangle: $0 \leq k_p \leq 5$, $-12 \leq k_i \leq 5$, $0 \leq k_d \leq 10$, *the table 7.4 summarizes some of the cases cited above.*

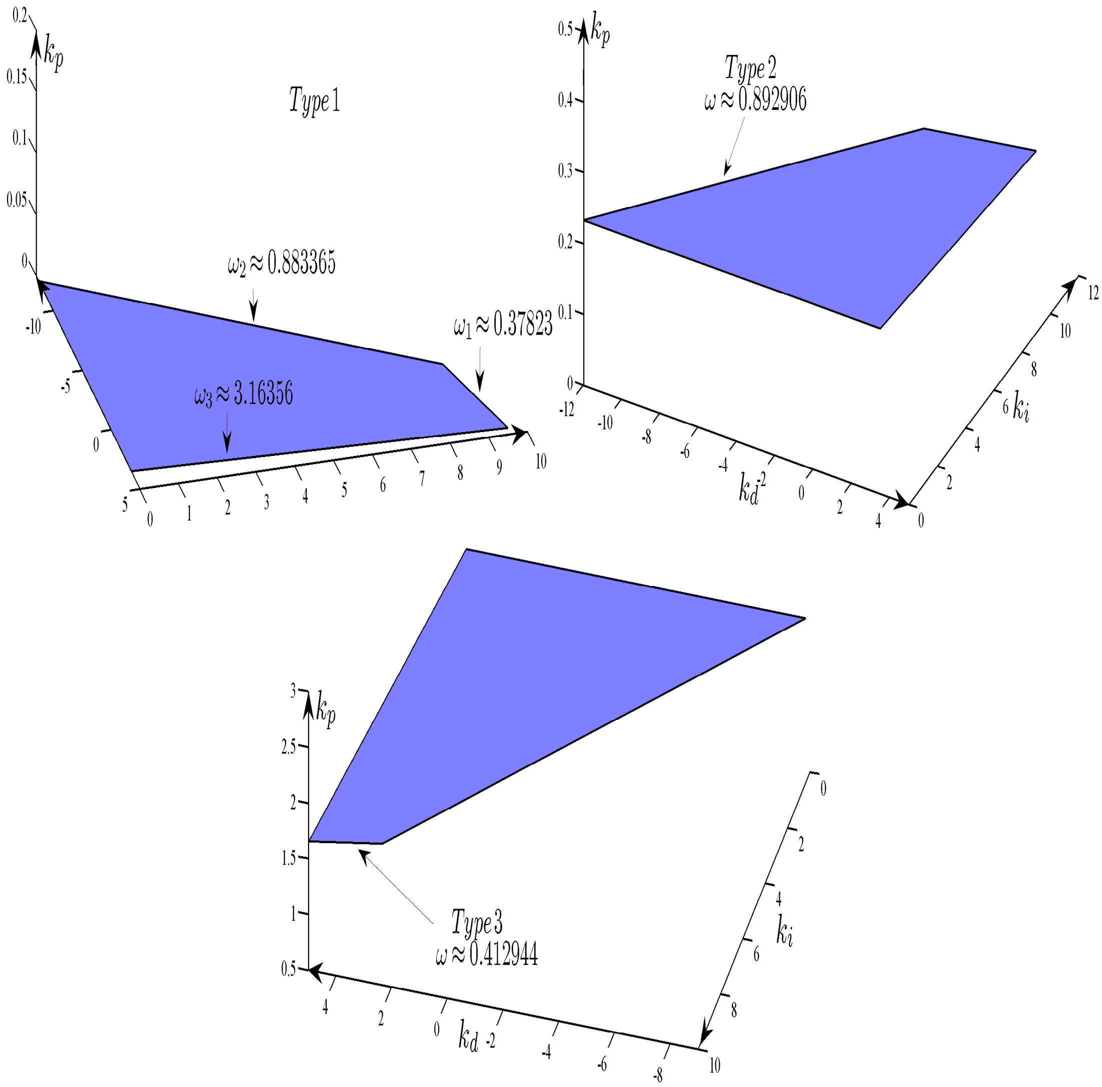


Figure 7.8: Boundary classification for the system (7.15). (Upper-Left) Classification of Type 1. (Upper-Right) Classification of Type 2. (Lower) Classification of Type 3

Interval	**Classification**
$[0.37823, 3.16356]$	Type 11
$[0.37823, 0.89290]$	Type 12
$[0.37823, 0.41294]$	Type 13
$[0.89290, 3.16356]$	Type 21
$[0.41294, 3.16356]$	Type 31
$[0.41294, 0.89290]$	Type 32

Table 7.7: Classification intervals type for the systems (7.15).

8 Fragility Analysis for Low-Order Controllers

Tuning and designing PI/PD/PID controllers is an active research area that has attracted the attention of many researchers during the last decades. A long list of PI, PD and PID tuning methods for controlling processes can be found in [109, 5]. As mentioned by [3], such controllers have to be designed by considering: (a) *performance* criteria; (b) *robustness* issues and, finally, (c) *fragility*. Roughly speaking, a controller for which the closed-loop system is destabilized by small perturbations in the controller parameters is called *"fragile"*. In other words, the fragility describes the deterioration of closed-loop stability due to small variations of the controller parameters.

It is a common assumption in the design of a controller that such a controller can be implemented exactly. This assumption is to some extend valid, since clearly, plant uncertainties are the most important source of uncertainty in the control system, whilst controller are implemented with high-precision hardware. However, there will inevitably be some amount of uncertainty in the controller, a fact that is sometimes ignored in advanced robust control design. If the controller is implemented by analogue means, there are some tolerances in the analogue components. More commonly, the controller will be implemented digitally. Subsequently, there will be some rounding of the controller parameters. Where for reasons of cost and execution speed, the implementation is with fixed point rather than floating point processors, there will be increased uncertainty in the controller parameters due to the finite word length and further uncertainty due to rounding errors in numerical computations [134].

Despite of the preceding arguments, we can still ask: How important could be to consider, for example, $\left(k_p^*, k_d^*, k_i^*\right) + \Delta$, instead of $\left(k_p^*, k_d^*, k_i^*\right)$? (in the case of a PID setting), for some $\|\Delta\| \ll 1$. Obviously, such answer depends on the system to be considered. Then, in order to motivate such analysis, consider the following example:

Example 8.1. *Consider the following plant:*

$$\frac{117187.5}{s^2 + 4.5931s + 2.1486} e^{-0.24s},$$

with a classical PI controller $K(s) = k\left(1 + \frac{1}{T_i s}\right)$. *In this case, for a PI-controller* $(k, T_i) = (0.00012, 2.4)$ *his dynamical behavior is illustrated in Fig.8.1.*

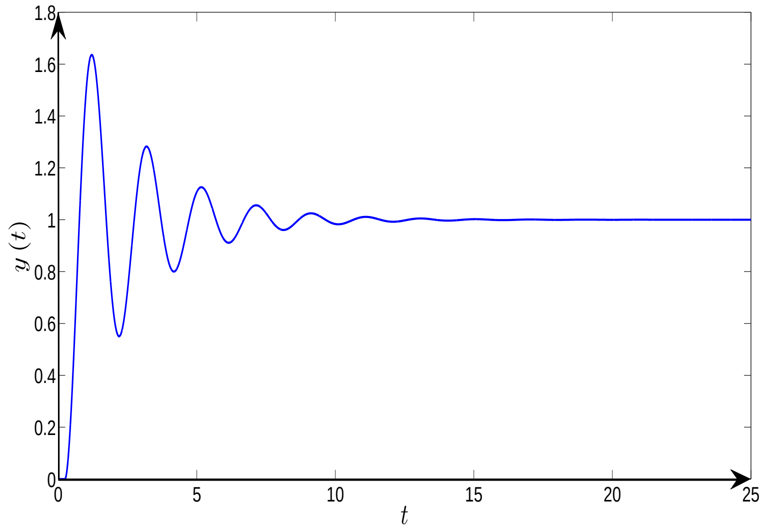


Figure 8.1: Step response curve for $(k, T_i) = (0.00012, 2.4)$.

Now, if instead of the above controller we consider the perturbed controller parameters $\left(\widetilde{k}, T_i\right) = (0.00018, 2.4)$, *we obtain the dynamical behavior depicted in Fig.8.2.*

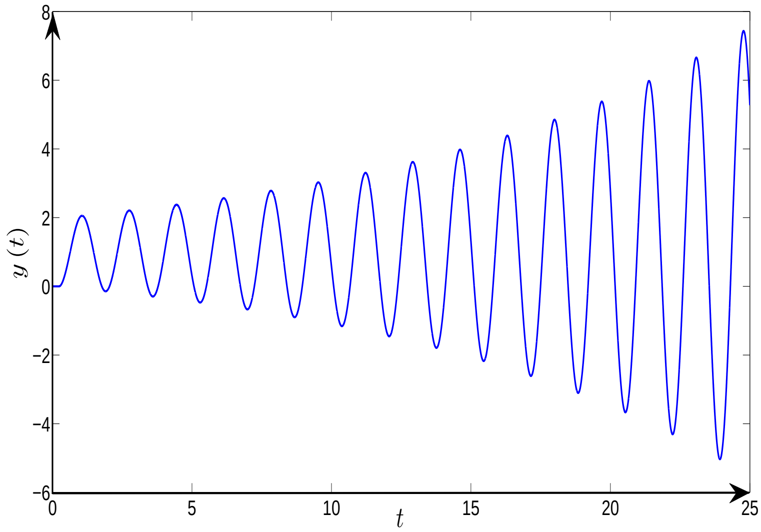


Figure 8.2: Step response curve for $\left(\widetilde{k}, T_i\right) = (0.00018, 2.4)$.

From the above figures, we can appreciate how the dynamics switch from stability to instability for a very small controller parameter perturbation.

Motivated by the above discussion, in this chapter we propose a *simple algorithm* to analyze the fragility of a given low-order controller. More precisely, we will consider the fragility analysis for a Proportional Controller, PI-PD Controller and a PID-Controller. As in the previous chapters, the proposed method is based on the *Implicit Function Theorem* [42] and related properties, and requires three "ingredients":

(i) the construction of the *stability crossing boundaries (surfaces)* in the parameter-space defined by "P" (proportional), "I" (integral) and "D" (derivative) gains,

(ii) the explicit computation of the crossing direction (towards stability or instability) when such a surface is traversed,

(iii) finally, the explicit computation of the distance of some point to the closest stability crossing boundaries.

8.1 Problem Formulation

For the sake of brevity, let us consider now the class of *strictly proper* SISO open-loop systems with I/O delays given by the transfer function:

$$G(s) := \frac{P(s)}{Q(s)} e^{-s\tau} = c^T \left(sI_n - A\right)^{-1} b e^{-s\tau}, \tag{8.1}$$

where $\left(A, b, c^T\right)$ is a state-space representation of the open-loop system. The control law is of PI, PD or PID-type with the following transfer functions:

$$\begin{cases} \text{PI controller:} & K(s) = k_p + \dfrac{k_i}{s}; \\ \text{PD controller:} & K(s) = k_p + k_d s; \\ \text{PID controller:} & K(s) = k_p + k_d s + \dfrac{k_i}{s}. \end{cases} \tag{8.2}$$

Denote its control parameters by $\overrightarrow{k}$, i.e., if $K(s)$ is a PI controller, then $\overrightarrow{k} = (k_p, k_i)$. As mentioned in the Introduction, the aim of this chapter is to compute the maximum controller parameters deviation without loosing the closed-loop stability. In other words, given the parameters $\overrightarrow{k^*}$ such that the roots of the closed-loop characteristic equation:

$$Q(s) + P(s)K(s)\,e^{-s\tau} = 0, \tag{8.3}$$

are located in $\mathbb{C}_-$ (that is the closed-loop system is asymptotically stable), find the m*aximum parameter deviation* $d \in \mathbb{R}_+$ such that the roots of (8.3) stay located in $\mathbb{C}_-$ for all controllers $\overrightarrow{k}$ satisfying:

$$\left\| \overrightarrow{k} - \overrightarrow{k^*} \right\| < d.$$

This problem can be more generally reformulated as: *find the maximum parameter deviation d such that the number of unstable roots of (8.3) remains unchanged*.

8.2 Fragility Analysis of PI-PD Controller

Based on the geometric approach presented in the previous chapter, we present now a simple and user-friendly approach not only to analyze the fragility of PI or PD controllers, but also to provide practical guidelines for the design of *non-fragile* PI or PD controllers. The proposed methodology is illustrated by analyzing several examples encountered in the control literature.

8.2.1 Fragility Analysis of PI-Controllers

Consider the *PI fragility problem*, that is the problem of computing the maximum controller parameters deviation without losing the closed-loop stability, that is given the pair of parameters (k_p^*, k_i^*) such that the roots of the equation:

$$Q(s) + P(s)\left(k_p^* + \frac{k_i^*}{s}\right)e^{-s\tau} = 0, \tag{8.4}$$

are located in $\mathbb{C}_-$ (that is the closed-loop system is asymptotically stable), find the maximum parameter deviation $d_{pi} \in \mathbb{R}_+$ such that the roots of (8.4) stay located in $\mathbb{C}_-$ for all controllers (k_p, k_i) satisfying:

$$\sqrt{(k_p - k_p^*)^2 + (k_i - k_i^*)^2} < d_{pi}.$$

First, let us introduce some notation:

$$\mathcal{T} = \bigcup_{l=1}^{N} \mathcal{T}_l, \quad \mathcal{T}_l = \left\{ (k_p, k_i) \middle| \omega \in \Omega_l \right\}$$

$$\overrightarrow{k(\omega)} = (k_p(\omega), k_i(\omega))^T, \quad \overrightarrow{k^*} = \left(k_p^*, k_i^*\right)^T$$

Let us also denote $d_{\mathcal{T}} = \min_{l \in \{1,\ldots,N\}} d_l$, where

$$d_l = \min\left\{ \sqrt{(k_p - k_p^*)^2 + (k_i - k_i^*)^2} \ \mid (k_p, k_i) \in \mathcal{T}_l \right\}$$

With the notation and the results above, we have:

Proposition 8.1. *The maximum parameter deviation from* (k_p^*, k_i^*)*, without changing the number of unstable roots of the closed-loop equation* (8.4) *can be expressed as:*

$$d_{pi} = \min\left\{ |k_i^*|, \min_{\omega \in \Omega_{f_{pi}}} \left\{ \left\| \overrightarrow{k(\omega)} - \overrightarrow{k^*} \right\| \right\} \right\}, \tag{8.5}$$

where $\Omega_{f_{pi}}$ *is the set of roots of the function* $f_{pi} : \mathbb{R}_+ \mapsto \mathbb{R}$,

$$f_{pi}(\omega) \triangleq \left\langle \left(\overrightarrow{k(\omega)} - \overrightarrow{k^*} \right), \frac{d\overrightarrow{k(\omega)}}{d\omega} \right\rangle. \tag{8.6}$$

Proof. We consider that the pair (k_p^*, k_i^*) belongs to a region generated by the crossing curves. Since the number of unstable roots changes only when (k_p, k_i) get out of this region, our objective is to compute the distance between (k_p^*, k_i^*) and the boundary of the region. Furthermore, the boundary of such a region consists of "pieces"of crossing curves and possibly one segment of the k_p axis. In order to compute the distance between (k_p^*, k_i^*) and a crossing curve we only need to identify the points where the vector $(k_p - k_p^*, k_i - k_i^*)$ and the tangent to the curve are orthogonal. In other words we have to find the solutions of

$$f_{pi}(\omega) = 0,$$

where f_{pi} is defined by (8.6). Taking into account the relation (3.3) (with $\alpha = k_p$ and $\beta = k_i$) we may write (8.6) as

$$f_{pi}(\cdot) = \left(k_p - k_p^*\right)(R_1 I_0 - R_0 I_1) + (k_i - k_i^*)(R_0 I_2 - R_2 I_0).$$

It is noteworthy that $f_{pi}(\omega)$ is a polynomial function and, therefore, it will have a *finite* number of roots. Let us consider $\{\omega_1, \ldots, \omega_M\}$ the set of all the roots of $f_{pi}(\omega)$ when we take into account all the pieces of crossing curves belonging to the region around (k_p^*, k_i^*). Since the distance from (k_p^*, k_i^*) to the $k_p(\omega)$ axis is given by $|k_i^*|$, one obtains:

$$d_{pi} = \min\left\{ |k_i^*|, \min_{\ell = \{1, \ldots, M\}} \left\{ \left\| \overrightarrow{k(\omega_\ell)} - \overrightarrow{k^*} \right\| \right\} \right\},$$

that is just another way to express (8.5). □

The explicit computation of the maximum parameter deviation d can be summarized by the following algorithm:

Algorithm 8.1 (PI-Fragility Algorithm)**.**

Step 1: *First, compute the "degenerate" points of each curve* $\mathcal{T}_l$ *(i.e. the roots of* $R_1 I_2 - R_2 I_1 = 0$ *and the multiple solutions of* (8.4)*).*

Step 2: *Second, compute the set* $\Omega_{f_{pi}}$ *defined by Proposition 8.1 (i.e. the roots of equation* $f_{pi}(\omega) = 0$*, where* f_{pi} *is given by (8.6)).*

Step 3: *Finally, the corresponding maximum parameter deviation* d_l *is defined by (8.5).*

Remark 8.1 (On the gains' optimization)**.** *It is worth mentioning that the geometric argument above can be easily used for solving other* robustness problems. *Thus, for instance, if one of the controller's parameters is fixed (prescribed), we can also explicitly compute the* maximum *interval guaranteeing closed-loop stability with respect to the other parameter. In particular if* T_i *("integral") is fixed, we can derive the corresponding stabilizing* maximum gain interval. *This gives a different insight to the results proposed by [113, 130] by using the small-gain theorem (see, for instance, the illustrative examples below).*

8.2.2 Illustrative Examples

Example 8.2 (Chemical Process)**.** *Consider the problem of controlling a continuous stirred tank reactor (CSTR) as in Fig.8.3 with the numerical values taken from [57] (see, e.g., [65, 125] for more details on CSTR). The goal is to control the reactor composition by manipulating the cool rate through the control signal u. Without getting into details, the transfer function of the system has the form:*

$$H_{yu}(s) = -\frac{1.308}{(13.515s+1)(6.241s+1)}e^{-4.896s}. \tag{8.7}$$

The use of a PI-controller leads to:

$$H(s;k_p,k_i) = (13.515s+1)(6.241s+1) - 1.308\left(k_p + \frac{k_i}{s}\right)e^{-4.896s}. \tag{8.8}$$

The system (8.8) has one stability region plotted in Fig.8.4.

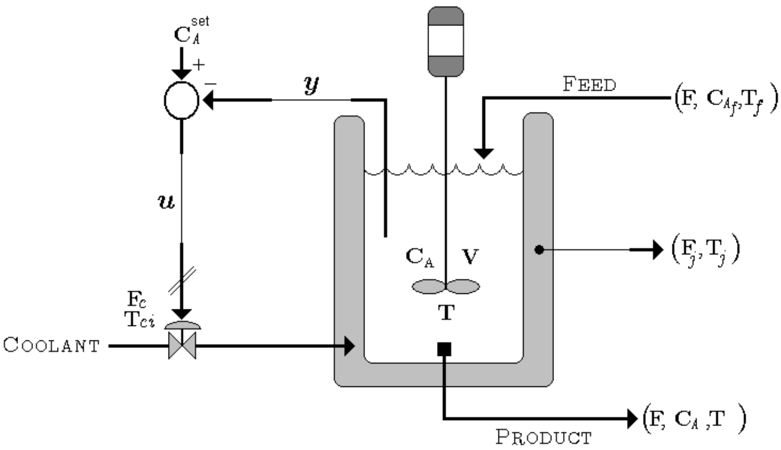


Figure 8.3: A CSTR control system

Next, we will study the fragility of PI-setting for some of the PI controllers proposed in the literature:

- *Huang-Chou-Wuang[57]:* $(k_p^* = -1.6881, k_i^* = -0.0732)$*;*
- *Hwang[60]:* $(k_p^* = -1.2173, k_i^* = -0.0529)$*;*
- *Chao-Lin-Guu-Chang[18]:* $(k_p^* = -1.1294, k_i^* = -0.0387)$*;*
- *Ziegler-Nichols[138]:* $(k_p^* = -1.4702, k_i^* = -0.0601)$.

By applying Proposition 8.1, the derived results are summarized in Table 8.1 and illustrated in the Fig.8.5.

Example 8.3 (A TCP/AQM network model)**.** *Consider the fluid-flow model introduced by [52] for describing the behavior of TCP/AQM networks and subject to PI controllers.*

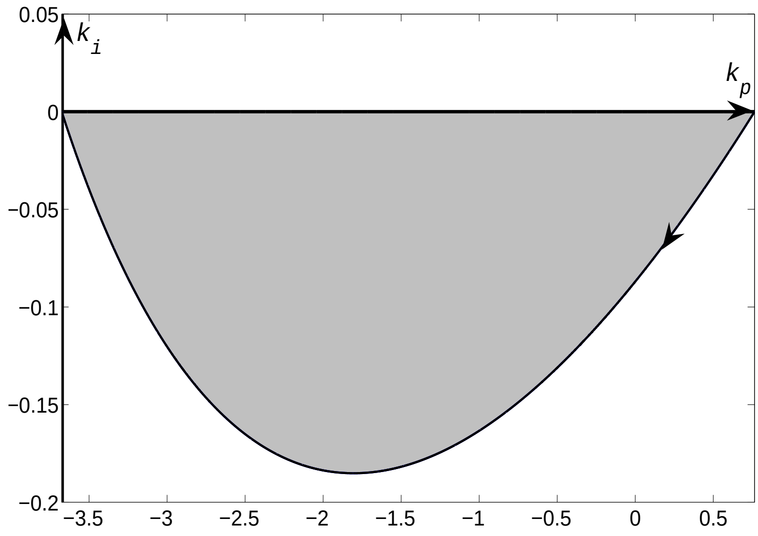


Figure 8.4: The boundary of the stability region in the (k_p, k_i) parameters space for the system (8.7)

	ω	$d_{\mathcal{T}}$	$\min\{d_{\mathcal{T}}, k_i^*\}$
Huang-Chou-Wang	0.1387	0.1114	0.0732
Hwang	0.1225	0.1202	0.0529
Chao-Lin-Guu-Chang	0.1194	0.1308	0.0387
Ziegler-Nichols	0.1323	0.1210	0.0601
Optimal Non-Fragile	0.1405	$0.0925\ldots$	$0.0925\ldots$

Table 8.1: PI fragility comparison for the system (8.7)

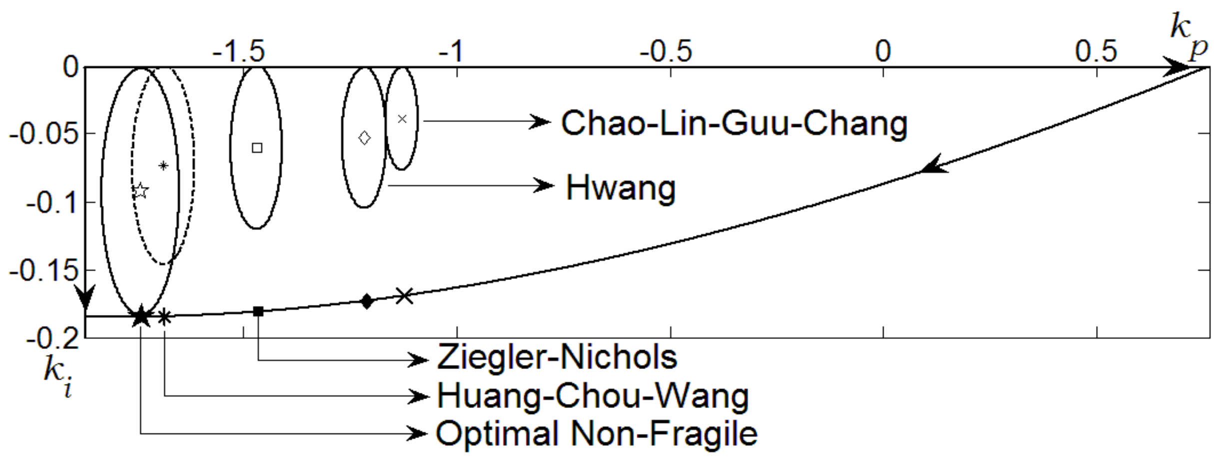


Figure 8.5: The maximum parameter deviation without losing stability for the system (8.7), where the Optimal Non-Fragile controller is given by $k_p^* = -1.7420542840243\ldots$ and $k_i^* = -0.09250851510052\ldots$

As mentioned by [88], the stability of the linearized closed-loop system reduces to the root *location of* $H(s,k_p,k_i,\tau) :=$

$$s^2+\frac{1}{\tau}(1+\frac{n}{\tau c})s+\frac{2n}{\tau^3 c}+\Big[\frac{n}{\tau^2 c}s+\frac{c^2}{2n}(k_p+\frac{k_i}{s})\Big]e^{-\tau s}=0 \quad (8.9)$$

Here, n denotes the load factor (number of TCP sessions), τ *the round-trip time (seconds) and c the link capacity (packets/sec). The crossing curves are given by:*

$$\begin{aligned} k_p &= \frac{2n}{c^2}\left[\left(\omega^2-\frac{2n}{\tau^3 c}\right)\cos(\omega\tau)+\frac{\omega}{\tau}\left(1+\frac{n}{\tau c}\right)\sin(\omega\tau)\right] \\ k_i &= \frac{2n\omega}{c^2}\left[\frac{\omega}{\tau}\left(1+\frac{n}{\tau c}\right)\cos(\omega\tau)+\left(\frac{2n}{\tau^3 c}-\omega^2\right)\sin(\omega\tau)+\frac{n\omega}{\tau^2 c}\right] \end{aligned}$$

Considering the same network parameters as in [52, 88] ($n=60$, $c=3750$, $\tau=0.246$*) and applying Proposition 3.1 we get that all the crossing directions are towards instability. Furthermore, we have* only one stability region. *Consider now some of the controllers proposed in the literature:*

- *Melchor-Niculescu[88]:* $(k_p^*=9.1044\times10^{-5}, k_i^*=6.8\times10^{-5})$*;*
- *Hollot-Misra-Towsley-Gong[52]:* $(k_p^*=1.8485\times10^{-5}, k_i^*=9.7749\times10^{-6})$*;*
- *Üstebay-Özbay[129]:* $(k_p^*=3.5252\times10^{-5}, k_i^*=8.9564\times10^{-6})$*;*
- *Ziegler-Nichols[138]:* $(k_p^*=7.4401\times10^{-5}, k_i^*=5.7057\times10^{-5})$*;*
- *Huang-Chou-Wang[57]:* $(k_p^*=10.0011\times10^{-5}, k_i^*=6.4880\times10^{-5})$.

The results are briefly outlined in the table 8.2 and illustrated in Fig.8.6.

	ω	$d_{\mathcal{T}}$ $[\times10^{-5}]$	$\min\{d_{\mathcal{T}_l}, \lvert k_i^*\rvert\}$ $[\times10^{-5}]$
Melchor and Niculescu	$\omega_1=1.76$ $\omega_2=2.75$ $\omega_3=3.49$	$d_{\mathcal{T}_1}=6.74$ $d_{\mathcal{T}_2}=8.78$ $d_{\mathcal{T}_3}=6.82$	6.7410
Hollot-Misra and Towsley-Gong	$\omega_1=0.72$ $\omega_2=3.00$ $\omega_3=3.69$	$d_{\mathcal{T}_1}=3.00$ $d_{\mathcal{T}_2}=17.0$ $d_{\mathcal{T}_2}=15.6$	0.9774
Üstebay and Özbay	$\omega_1=0.81$ $\omega_2=2.93$ $\omega_3=3.72$	$d_{\mathcal{T}_1}=4.56$ $d_{\mathcal{T}_2}=16.2$ $d_{\mathcal{T}_3}=14.0$	0.8956
Ziegler and Nichols	$\omega_1=1.55$ $\omega_2=2.85$ $\omega_3=3.52$	$d_{\mathcal{T}_1}=5.89$ $d_{\mathcal{T}_2}=10.2$ $d_{\mathcal{T}_3}=8.77$	5.7057
Huang Chou and Wang	$\omega_1=1.79$ $\omega_2=2.68$ $\omega_3=3.53$	$d_{\mathcal{T}_1}=7.65$ $d_{\mathcal{T}_2}=9.07$ $d_{\mathcal{T}_3}=6.10$	6.1094

Table 8.2: PI-fragility comparison for the characteristic equation (8.9)

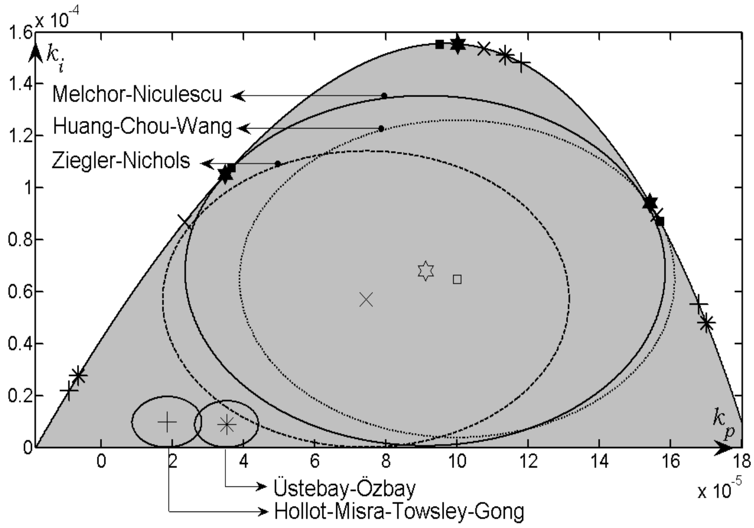


Figure 8.6: Fragility comparison of the PI-controllers for the system

Remark 8.2. *As mentioned in the previous chapter, it is also possible to solve the following problem – given a fixed integral (gain) parameter $T_i = k_p/k_i$, find the* optimal *interval for the gain (integral) parameter $k_p = k$, such that, the resulting closed-loop system is stable for all gain (integral) parameters In this case, it is sufficient to find the "mid-point" of the maximal interval which belong to the stability region. Reconsider the previous controllers:*

- *"optimal" gain (Hollot-Misra-Towsley-Gong): $k = 7.91 \times 10^{-5}$;*
- *"optimal" gain (Üstebay-Özbay): $k = 8.56 \times 10^{-5}$;*
- *"optimal" gain (Melchor-Niculescu): $k = 7.34 \times 10^{-5}$*

It is easy to see that the controller proposed by Üstebay-Özbay is "closer" to the "non-fragile" one than Hollot-Misra-Towsley-Gong. The above results are also depicted in Fig.8.7

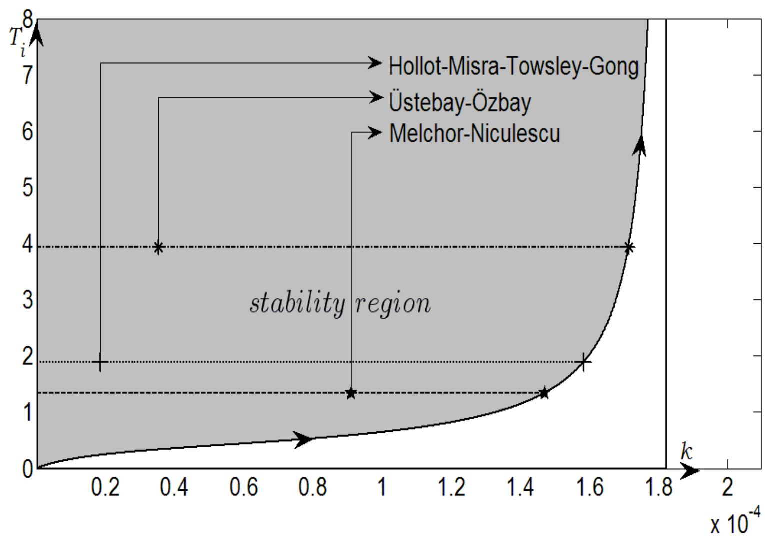


Figure 8.7: Gain fragility comparison of the PI-controllers for the system (8.9)

Example 8.4 (Fourth-order process). *Consider a fourth-order, non-minimum-phase and unstable open-loop system, with the transfer function:*

$$H_{yu}(s) = \frac{(-1.3s+3)e^{-2.8s}}{0.2s^4 - 0.08s^3 + 1.345s^2 - 0.4s + 1.725}. \tag{8.10}$$

Similarly to the previous cases, the problem reduces to the analysis of equation:

$$0.2s^4 - 0.08s^3 + 1.345s^2 - 0.4s + 1.725 + (-1.3s+3)\left(k_p + \frac{k_i}{s}\right)e^{-2.8s} = 0 \tag{8.11}$$

The "optimal"non-fragile PI-controller for the system (8.10) is given by $(k_p^*, k_i^*) =$

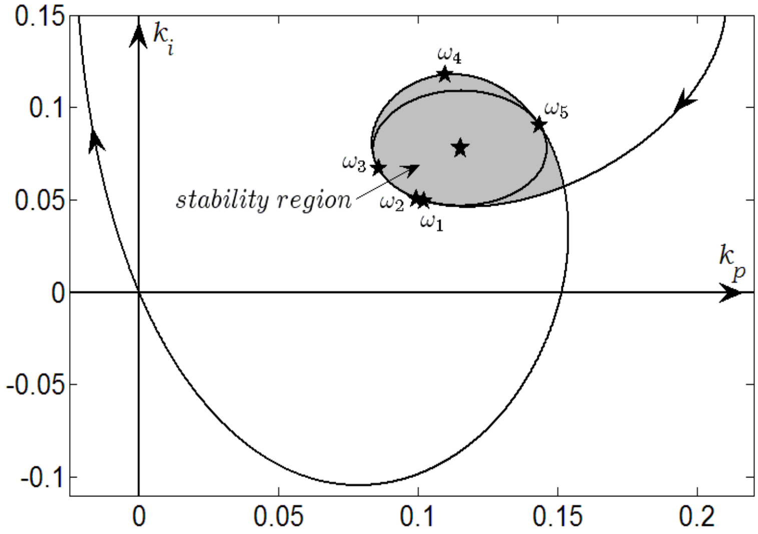


Figure 8.8: The stability crossing curves for the dynamic system (8.10), the boundary of the stability region (shadowed region) in the (k_p, k_i) parameters space and the maximum parameter deviation without loosing stability.

$(0.1149\ldots, 0.0778\ldots)$ *(see also Table 8.3 and Fig.8.8):*

Frequency	$d_{\mathcal{T}_l}$	$\lvert k_i^* \rvert$	$\min\{d_{\mathcal{T}}, \lvert k_i^* \rvert\}$
$\omega_1 = 1.2311$	0.0313616		
$\omega_2 = 1.2422$	0.0313627		
$\omega_3 = 1.3232$	0.0311658	0.077849	0.0311658
$\omega_4 = 1.5556$	0.0400741		
$\omega_5 = 1.7025$	0.0311658		

Table 8.3: Parameters deviation results without loosing the stability.

8.2.3 Fragility Analysis of PD-Controllers

Consider now the PD fragility problem, which is the problem of computing the maximum controller parameters deviation without loosing the closed-loop stability, that is given the pair of parameters (k_p^*, k_d^*) such that the roots of the characteristic equation:

$$Q(s) + P(s)(k_p + k_d s)e^{-s\tau} = 0, \tag{8.12}$$

are located in $\mathbb{C}_-$ (that is the closed-loop system is asymptotically stable), find the maximum parameter deviation $d_{pd} \in \mathbb{R}_+$ such that the roots of (8.12) stay located in $\mathbb{C}_-$ for all controllers (k_p^*, k_d^*) satisfying:

$$\sqrt{\left(k_p - k_p^*\right)^2 + \left(k_d - k_d^*\right)^2} < d_{pd}. \tag{8.13}$$

In order of presenting the main result, lets consider the following notations:

$$\mathcal{T} = \bigcup_{l=1}^{N} \mathcal{T}_l, \qquad \mathcal{T} = \{(k_p, k_d) | \omega \in \Omega_l\}, \tag{8.14}$$

$$\overrightarrow{k(\omega)} = (k_p(\omega), k_d(\omega))^T, \qquad \overrightarrow{k^*} = (k_p^*, k_d^*)^T. \tag{8.15}$$

Let us also denote $d_{\mathcal{T}} = \min\limits_{l \in \{1,\ldots,N\}} d_l$, where:

$$d_l = \min\left\{\sqrt{(k_p - k_p^*)^2 + (k_d - k_d^*)^2} | (k_p, k_d) \in \mathcal{T}_l\right\}. \tag{8.16}$$

With the notation and the results above, we have:

Proposition 8.2. *The maximum parameter deviation from* $\left(k_p^*, k_d^*\right)$, *without changing the number of unstable roots of the closed-loop characteristic equation* (8.12) *can be expressed as:*

$$d_{pd} = \min\left\{k_{d\infty}, |k_p^* - k_p(0)|, \min_{\omega \in \Omega_{f_{pd}}} \left\{\left\|\overrightarrow{k(\omega)} - \overrightarrow{k^*}\right\|\right\}\right\}, \tag{8.17}$$

where

$$k_{d\infty} := \begin{cases} \min\left\{\left|k_d^* - \left|\frac{q_n}{p_m}\right|\right|, \left|k_d^* + \left|\frac{q_n}{p_m}\right|\right|\right\} & \text{if} \quad m = n-1 \\ \emptyset & \text{if} \quad m < n-1 \end{cases}$$

and $\Omega_{f_{pd}}$ is the set of roots of the function $f_{pd} : \mathbb{R}_+ \mapsto \mathbb{R}$,

$$f_{pd}(\omega) \triangleq \left\langle (\overrightarrow{k(\omega)} - \overrightarrow{k^*}), \frac{d\overrightarrow{k(\omega)}}{d\omega} \right\rangle. \tag{8.18}$$

Proof. We consider first that the pair (k_p^*, k_d^*) belongs to some region generated by the crossing curves. Since the number of unstable roots changes only when (k_p, k_d) get out of this region, our objective is to compute the distance between (k_p^*, k_d^*) and the boundary of the region. Now, if $\deg Q(s) = \deg P(s) + 1$ (i.e., we have a neutral-type system) it is well known [48] that the system possesses a neutral chain that asymptotically approach the vertical line

$$\Re(s) = \frac{1}{\tau} \ln\left(\left|\frac{k_d p_{n-1}}{q_n}\right|\right), \tag{8.19}$$

implying that $|k_d| < \left|\frac{q_n}{p_{n-1}}\right|$ must be considered. As a consequence, the boundary of such a region consists of "pieces" of crossing curves and possibly a segment of the shifted axis $k_p + \left|\frac{q_n}{p_{n-1}}\right|$ or $k_p + \left|\frac{q_n}{p_{n-1}}\right|$ for a neutral-type system, and a segment of the shifted axis $k_d + k_p(0)$. In order to compute the distance between (k_p^*, k_d^*) and a crossing curve we only need to identify the points where the vector $(k_p - k_p^*, k_d - k_d^*)$ and the tangent to the boundaries are orthogonal. In other words, we have to find the solutions of:

$$f_{pd}(\omega) = 0,$$

where f_{pd} is defined by (8.18). Taking into account the relation (3.3) (with $\alpha = k_p$ and $\beta = k_d$) we may write (8.18) as:

$$f_{pd}(\cdot) = (k_p - k_p^*)(R_1 I_0 - R_0 I_1) - (k_d - k_d^*)(R_0 I_2 - R_2 I_0). \tag{8.20}$$

It is worth mentioning that the stability region is defined in $(\underline{\omega}, \overline{\omega})$ and, therefore, (8.18) will have a finite number of roots. Let us consider $\{\omega_1, \ldots, \omega_M\}$ the set of all the roots of $f_{pd}(\omega)$ when we take into account all the pieces of crossing curves belonging to the region around (k_p^*, k_d^*). Since the distance from (k_p^*, k_d^*) to the shifted axis $k_d + k_p(0)$ is given by $|k_p^* - k_p(0)|$ and the minimal distance from (k_p^*, k_d^*) to the shifted axis $k_p + \left|\frac{q_n}{p_{n-1}}\right|$ or $k_p + \left|\frac{q_n}{p_{n-1}}\right|$ (for a neutral-type system), is given by $k_{d\infty}$ one obtains that

$$d_{pd} = \min\left\{k_{d\infty}, |k_p^* - k_p(0)|, \min_{\ell=1,\ldots,M}\left\{\left\|\overrightarrow{k(\omega_\ell)} - \overrightarrow{k^*}\right\|\right\}\right\}, \tag{8.21}$$

which are just another way to express (8.17). $\square$

The explicit computation of the maximum parameter deviation d_{pd} can be summarized by the following algorithm:

Algorithm 8.2 (PD-Fragility Algorithm)**.**

Step 1: *First, compute the "degenerate" points of each curve $\mathcal{T}_l$ (i.e. the roots of $R_1 I_2 - R_2 I_1 = 0$ and the multiple solutions of* (8.12)*).*

Step 2: *Second, compute the set $\Omega_{f_{pd}}$ defined by Proposition 8.2 (i.e. the roots of equation $f_{pd}(\omega) = 0$, where f_{pd} is given by (8.18)).*

Step 3: *Finally, the corresponding maximum parameter deviation d_{pd} is defined by* (8.17)*.*

8.2.4 Illustrative Examples

In order to illustrate the previous result, consider now the following example.

Example 8.5 (Gantry crane)**.** *For this example we have chosen a gantry crane model with a time delay of 2 seconds used as slave robot in a teleoperation system [34]:*

$$g(s) = \frac{40s^2 + 2s + 400}{200s^3 + 30s^2 + 2401s + 200} e^{-2s}. \tag{8.22}$$

According to Proposition *8.2, we present the corresponding crossing curves for* (8.22) *in Fig.8.9 .*

After applying the proposed algorithm to analyze the fragility for the controller $(k_p^*, k_d^*) = (3.25, 1.65)$, we summarize the obtained results in table 8.4.

Figure 8.10 illustrate the stability region for the system (8.22) *as well as the maximum parameter deviation for the proposed controller (k_p^*, k_d^*).*

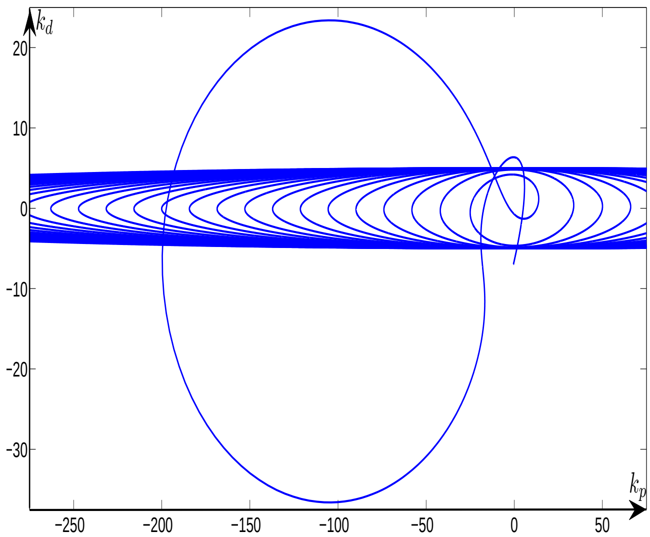


Figure 8.9: Corresponding crossing curves for (8.22).

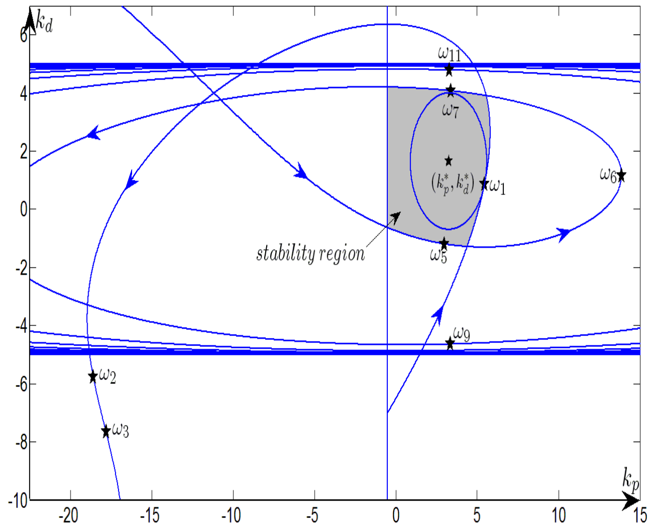


Figure 8.10: Stability region of k_p and k_d for (8.22). $k_p^* = 3.25$, $k_d^* = 1.65$, $d = 2.345980$.

Frequency	d_{T_l}	$k_{d\infty}$	$\lvert k_p^* - k_p(0) \rvert$	$\min\{d_T, k_{d\infty}, k_{p0}\}$
$\omega_1 = 0.9017$	2.3459			
$\omega_2 = 2.7292$	23.0540			
$\omega_3 = 2.8228$	23.0158			
$\omega_4 = 3.1625$	203.161			
$\omega_5 = 3.5744$	2.8603			
$\omega_6 = 4.1134$	10.6317	3.35	3.75	2.34598084836201
$\omega_7 = 4.6386$	2.4229			
$\omega_8 = 5.5736$	28.3525			
$\omega_9 = 6.3485$	6.2916			
$\omega_{10} = 7.1127$	30.5030			
$\omega_{11} = 7.8169$	3.13656			

Table 8.4: Parameters deviation results without losing the stability for system (8.22).

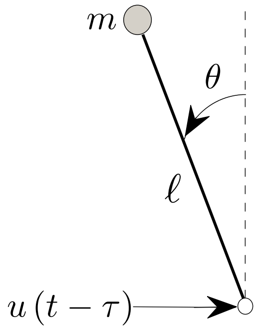


Figure 8.11: Inverted Pendulum

Example 8.6 (Inverted Pendulum)**.** *The linearized dynamics of a normalized inverted pendulum with delayed input (*$\tau = 1/2$*) can be represented by the following transfer function:*

$$g(s) = \frac{1}{s^2 - 1} e^{-\frac{1}{2}s}, \tag{8.23}$$

where the input is acceleration of the pivot and the output is the pendulum angle θ*, as show in Fig.8.11. Now, after applying Proposition 8.2 we obtain the crossing curves depicted in Fig.8.13.*

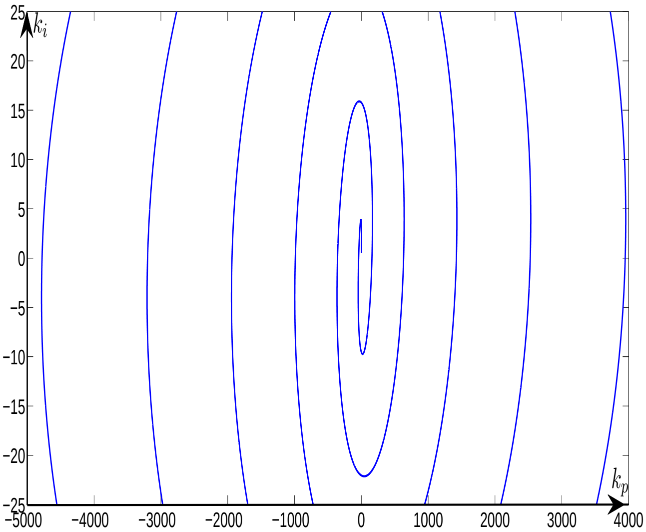


Figure 8.12: Corresponding crossing curves for (8.23)

Next, in order to analyze the fragility for the PD controller $\left(k_p^*, k_d^*\right) = (2.70207, 2.12879)$*, we apply the proposed Algorithm 8.2, we obtain the results summarized in Table 8.5. Figure 8.13 illustrate the stability region for the system* (8.23) *as well as the maximum parameter deviation for the proposed controller* (k_p^*, k_d^*).

In all the above examples that we have seen, the systems possesses at most one stability region. However according to *Remark* 5.4 it is possible to have multi-stability regions if turning points exists. Then, in order to illustrate such statement consider the next example.

Frequency	$d_{\mathcal{T}_l}$	$k_{d\infty}$	$\lvert k_p^* - k_p(0) \rvert$	$\min\{d_T, k_{d\infty}, k_{p0}\}$
$\omega_1 = 1.641505$	0.823298			
$\omega_2 = 2.019206$	0.851033			
$\omega_3 = 2.509846$	0.759900		1.70207	0.759900
$\omega_4 = 7.298481$	49.60874			
$\omega_5 = 9.460565$	11.69581			

Table 8.5: Parameters deviation results without losing the stability for system (8.22).

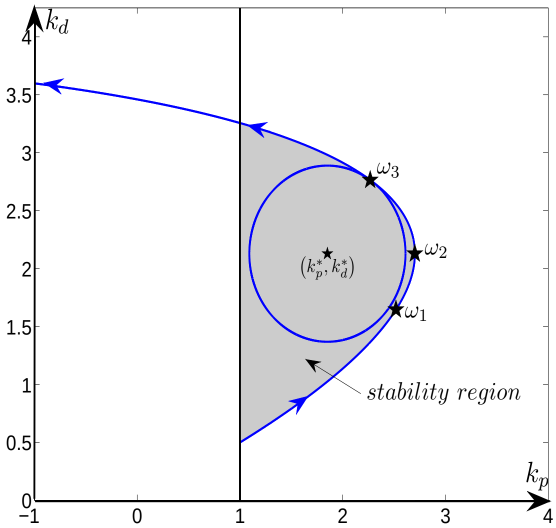


Figure 8.13: Stability region of k_p and k_d for (8.23).

Example 8.7 (multi-stability regions)**.** *Consider the sixth-order unstable open-loop system, with transfer function:*

$$G(s) = \frac{s^5 + 16s^4 + 152s^3 + 824s^2 + 2788s + 4624}{s^6 + 27s^5 + \frac{619}{4}s^4 + \frac{1273}{4}s^3 + \frac{1845}{4}s^2 - \frac{325}{4}s - 125} e^{-\frac{1}{15}s}. \tag{8.24}$$

Applying Proposition 8.2 we obtain the crossing curves depicted in Fig.8.14. Then, in order to analyze the fragility for the PD controller $(k_p^*, k_d^*) = (27.5, 0.25)$*, we apply the proposed Algorithm 8.2, we obtain the results summarized in Table 8.6. Figure 8.15 illustrate the multi-*

Frequency	$d_{\mathcal{T}_l}$	$k_{d\infty}$	$\lvert k_p^* - k_p(0) \rvert$	$\min\{d_T, k_{d\infty}, k_{p0}\}$
$\omega_1 = 7.008604$	3.041536			
$\omega_2 = 11.99477$	9.871934			
$\omega_3 = 20.86991$	0.942710			
$\omega_4 = 31.82475$	9.807249	0.75	27.472967	0.66579911198
$\omega_5 = 41.10371$	0.665799			
$\omega_6 = 75.43110$	104.7725			
$\omega_7 = 99.93986$	1.235575			

Table 8.6: Parameters deviation results without losing the stability for system (8.24).

stability region for the system (8.24) *as well as the maximum parameter deviation for the proposed controller* (k_p^*, k_d^*).

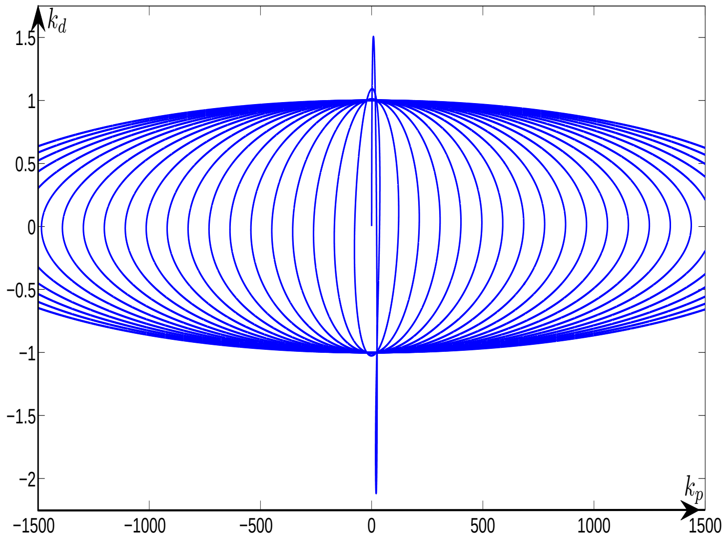


Figure 8.14: Corresponding crossing curves for (8.24)

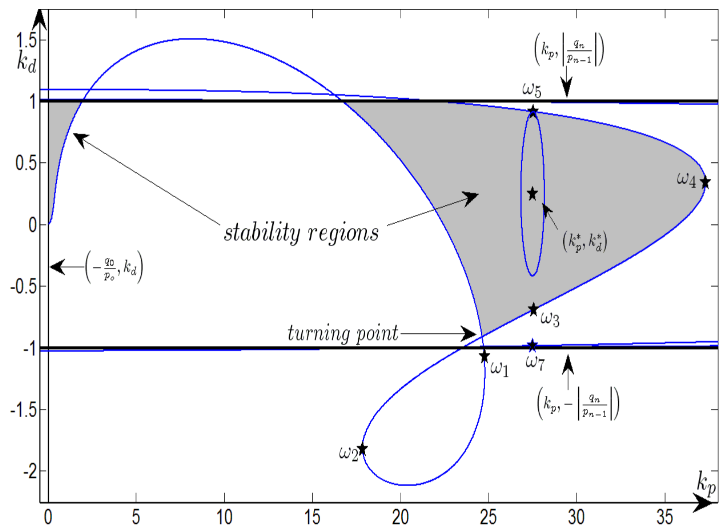


Figure 8.15: Multi-stability region for (8.24).

8.3 PID Controller Fragility Analysis

Finally, in the rest of this chapter we will consider the *PID fragility problem*, that is the problem of computing the maximum controller parameters deviation without loosing the closed-loop stability. In other words, given the parameters (k_p^*, k_d^*, k_i^*) such that the roots of the closed-loop characteristic equation:

$$Q(s) + P(s)\left(k_p^* + k_d^* s + \frac{k_i^*}{s}\right)e^{-s\tau} = 0, \tag{8.25}$$

are located in $\mathbb{C}_-$ (that is the closed-loop system is asymptotically stable), find the maximum parameter deviation $d \in \mathbb{R}_+$ such that the roots of (8.25) stay located in $\mathbb{C}_-$ for all controllers (k_p, k_d, k_i) satisfying:

$$\sqrt{(k_p - k_p^*)^2 + (k_d - k_d^*)^2 + (k_i - k_i^*)^2} < d.$$

First as in the previous section, let us introduce some notations,

$$\begin{aligned}\mathcal{T} &= \bigcup_{l=1}^{N} \mathcal{T}_l, \quad \mathcal{T}_l = \left\{(k_p,k_d,k_i)\middle|\omega\in\Omega_l\right\},\\ \overrightarrow{k(\omega)} &= (k_p(\omega),k_d(\omega),k_i(\omega))^T, \quad \overrightarrow{k^*} = \left(k_p^*,k_d^*,k_i^*\right)^T.\end{aligned}$$

Let us also denote $d_{\mathcal{T}}=\min\limits_{l\in\{1,\dots,N\}} d_l$, where

$$d_l=\min_{(k_p,k_d,k_i)\in\mathcal{T}_l}\left\{\sqrt{(k_p-k_p^*)^2+(k_d-k_d^*)^2+(k_i-k_i^*)^2}\right\}.$$

Finally, observe that for a fixed $\omega^*\in\mathbb{R}_+$ the second equation in (7.7) describes a line in the $(k_d,k_i)-$plane. This observation motivates the following definition:

$$L_\ell(\omega_\ell) := \omega_\ell k_d - k_i + \omega_\ell \Im\left(\frac{Q(\boldsymbol{i}\omega_\ell)}{P(\boldsymbol{i}\omega_\ell)}e^{\boldsymbol{i}\omega_\ell\tau}\right).$$

In order to present the PID fragility algorithm the following result will be needed. Let $k_p = k_p^*\in\mathbb{R}$ be fixed, we have the following:

Proposition 8.3. *The maximum parameter deviation from* (k_d^*,k_i^*), *without changing the number of unstable roots of the closed-loop equation* (8.25) *can be expressed as:*

$$d_{di}^*=\min\left\{|k_i^*|,\min_{\omega_\ell\in\Omega_{k_p^*}}\left\{\left|\frac{\omega_\ell^2 k_d^*-k_i^*+\omega_\ell\Im\left\{\frac{Q(j\omega_\ell)}{P(j\omega_\ell)}e^{j\omega_\ell\tau}\right\}}{\sqrt{(\omega_\ell)^4+1}}\right|\right\}\right\}, \tag{8.26}$$

where $\Omega_{k_p^*}$ *is the set of roots of the function* $f_{k_p^*}:\mathbb{R}\times\mathbb{R}_+\mapsto\mathbb{R}$,

$$f_{k_p^*}\left(k_p^*,\omega\right) \triangleq k_p^* + \Re\left\{\frac{Q(j\omega)}{P(j\omega)}e^{j\omega\tau}\right\}. \tag{8.27}$$

Proof. Let the pair (k_d^*,k_i^*) be inside of a polygon $\mathcal{L}$ formed by the set of lines $L_\ell(\omega_\ell)$ with $\ell\in\{1,2,\dots,N\}$. Then, for each $L_\ell(\omega_\ell)$, the associated perpendicular line passing through the points (k_d^*,k_i^*) will be given by:

$$L_\ell^p(\omega_\ell) = -\frac{1}{\omega_\ell^2}k_d + \left(\frac{\omega_\ell^2 k_i^* + k_d^*}{\omega_\ell^2}\right),$$

more over, the intersection points $(\overline{k_d},\overline{k_i})$ of $L_\ell^p(\omega_\ell)$ with $L_\ell(\omega_\ell)$ are given by:

$$\begin{bmatrix}\overline{k_d}\\ \overline{k_i}\end{bmatrix} = \begin{bmatrix}\frac{\omega_\ell^2k_i^*+k_d^*-\omega_\ell^3\Im\left\{G^{-1}(\omega_\ell)\right\}}{1+\omega_\ell^4}\\ \frac{\omega_\ell^2\left(\omega_\ell^2k_i^*+k_d^*\right)+\omega_\ell\Im\left\{G^{-1}(\omega_\ell)\right\}}{1+\omega_\ell^4}\end{bmatrix}.$$

Then, it is clear to see, that the distance from the point $\left(k_d^*,k_i^*\right)$ to $L_\ell(\omega_\ell)$ is $\sqrt{\left(k_d^*-\overline{k_d}\right)^2+\left(k_i^*-\overline{k_i}\right)^2}$, which is precisely given by the second argument of (8.26). Finally, according to *Remark* 7.3, we know that $k_i=0$ belongs to the boundary set, since the distance from (k_d^*, k_i^*) to the k_d-axis is given by $|k_i^*|$, we have that the minimum distance from (k_d^*, k_i^*) to the boundary of $\mathcal{L}$ is equal to:

$$d_{di}^* = \min\left\{|k_i^*|, \min_{\ell\in\{1,\ldots,N\}}\left\{\left|\frac{\omega_\ell^2 k_d^* - k_i^* + \omega_\ell \Im\left\{\frac{Q(j\omega_\ell)}{P(j\omega_\ell)}e^{j\omega_\ell\tau}\right\}}{\sqrt{(\omega_\ell)^4+1}}\right|\right\}\right\},$$

which is equivalent to (8.26). □

Remark 8.3. *Observe that (8.27) has an uncountable number of solutions, however in Proposition 8.3 we have considered the set including the corresponding (k_d^*, k_i^*) points.*

In order to obtain the obtain the PID fragility we present the following:

Algorithm 8.3 (PID-Fragility Algorithm)**.**

- **Step 1:** *Let $k_{pid}^* \in \mathbb{R}^3$ be fixed. Then, set $d = \min\left\{d_{pi}^*, d_{pd}^*, d_{di}^*\right\}$.*
- **Step 2:** *Sweep over all $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and compute $k_{p\theta}^* = k_p^* + d\sin\theta$.*
- **Step 3:** *Solve $f_{k_p^*}\left(k_{p\theta}^*, \omega\right) = 0$ and denote by Ω_θ the set of solutions.*
- **Step 4:** *Compute,*

$$d_\theta^* = \min_{\omega_\ell\in\Omega_\theta}\left\{\left|\frac{(\omega_\ell)^2 k_d^* - k_i^* + \omega_\ell \Im\left\{\frac{Q(j\omega_\ell)}{P(j\omega_\ell)}e^{j\omega_\ell\tau}\right\}}{\sqrt{(\omega_\ell)^4+1}}\right|\right\}.$$

- **Step 5:** *If $d_\theta^* < d\cos\theta$ then set $d = d_\theta^*/\cos\theta$ and go to step 2. Otherwise continue to step 2.*
- **Step 6:** *If $\theta = \frac{\pi}{2}$, the procedure is finish and d is the PID fragility for the controller $\left(k_p^*, k_d^*, k_i^*\right)$.*

8.3.1 Illustrative Examples

In order to motivate the previous results, we consider in the sequel some numerical examples.

Example 8.8. *Consider now the same plant* (7.14) *as in Example 7.1, i.e.,*

$$G(s) = \frac{s^3 - 4s^2 + s + 2}{s^5 + 8s^4 + 32s^3 + 46s^2 + 46s + 17}e^{-s}.$$

Next, in order to illustrate the proposed PID fragility-algorithm, consider $\left(k_p^*, k_d^*, k_i^*\right) = (2,3,3)$, leading to the values in Table 8.7 and depicted in Fig.8.16.

Example 8.9. *Consider now the plant* (7.15) *considered in Example 7.2, i.e.,*

$$G(s) = \frac{-s^4 - 7s^3 - 2s + 1}{(s+1)(s+2)(s+3)(s+4)(s^2+s+1)}e^{-\frac{1}{20}s}$$

Then, in order to apply the PID fragility-algorithm 8.3, consider $\left(k_p^*, k_d^*, k_i^*\right) = (1,-1,3)$, leading to the following results:

Controller (k_p^*, k_d^*, k_i^*)	**Fragility** (PI, PD, DI)	**Initial** PID-Fragility	**PID-Fragility** $\min\{d^*, d_\theta^*\}$
$(2,3,3)$	$d_{pi}^* = 1.68051$ $d_{pd}^* = 1.33313$ $d_{di}^* = 1.27520$	$d^* = 1.27520$	$d_\theta^* = 1.26295$

Table 8.7: PID fragility for the example (7.14).

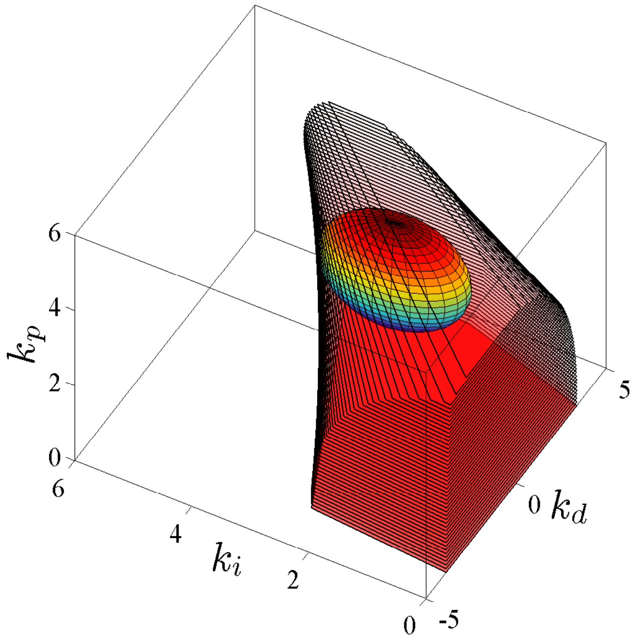


Figure 8.16: PID-fragility for the controller $(k_p^*, k_d^*, k_i^*) = (2,3,3)$.

Controller (k_p^*, k_d^*, k_i^*)	**Fragility** (PI, PD, DI)	**Initial** PID-Fragility	**PID-Fragility** $\min\{d^*, d_\theta^*\}$
$(1,-1,3)$	$d_{pi}^* = 3.00000$ $d_{pd}^* = 3.38832$ $d_{di}^* = 3.00000$	$d^* = 3.00000$	$d_\theta^* = 2.98908$

Table 8.8: PID fragility for the example (7.15).

Example 8.10 (unstable, non-minimal phase system)**.** *Finally, in order to analyze a plant of neutral-type, we consider the plant* (7.16) *given in Example 7.3, i.e.,*

$$G(s) = \frac{s-2}{s^2 - \frac{1}{2}s + \frac{13}{4}} e^{-\frac{1}{2}s}.$$

Now, applying the same procedure as before, and considering $(k_p^*, k_d^*, k_i^*) = \left(\frac{5}{8}, -\frac{1}{10}, -\frac{2}{5}\right)$ *we obtain the results summarized in Table 8.9. Figure 8.18 illustrate such a results.*

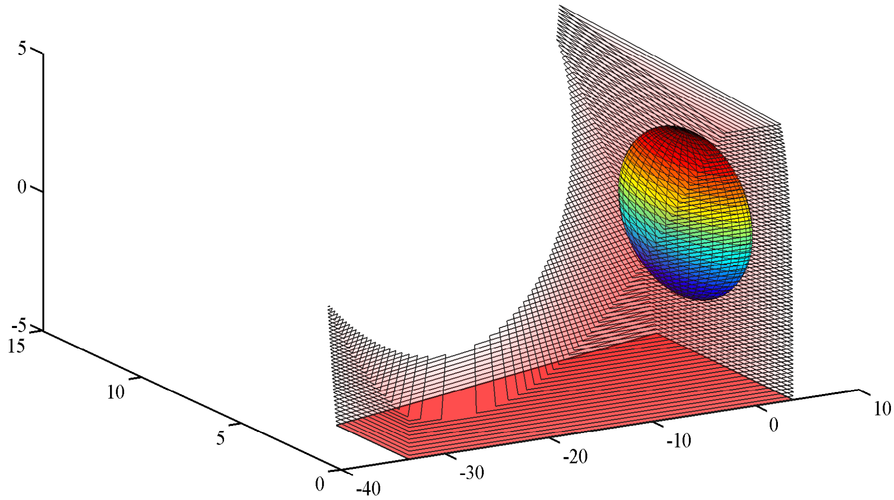


Figure 8.17: PID-fragility for the controller $(k_p^*, k_d^*, k_i^*) = (1, -1, 3)$.

Controller (k_p^*, k_d^*, k_i^*)	**Fragility** (PI, PD, DI)	**Initial** PID-Fragility	**PID-Fragility** $\min\{d^*, d_\theta^*\}$
$\left(\frac{5}{8}, \frac{-1}{10}, \frac{-2}{5}\right)$	$d_{pi}^* = 0.29314$ $d_{pd}^* = 0.16758$ $d_{di}^* = 0.16782$	$d^* = 0.16758$	$d_\theta^* = 0.16453$

Table 8.9: PID fragility for the example (7.16).

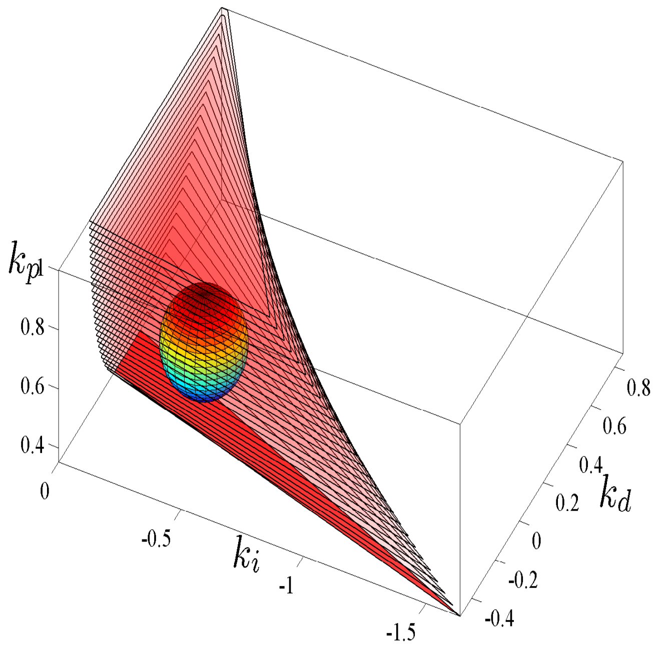


Figure 8.18: PID-fragility for the controller $(k_p^*, k_d^*, k_i^*) = \left(\frac{5}{8}, -\frac{1}{10}, -\frac{2}{5}\right)$.

8.3.2 Conclusions

In this chapter, a simple geometric-based algorithm is derived for computing the fragility of PID-controllers. To prove the efficiency of the proposed methods, several illustrative examples have been considered. It is important to note that such an idea can be easily extended to *proper* SISO systems with I/O delays.

Remark 8.4. *The results presented in this part has been reported in the literature as follows:*

- *The* ***PI Fragility*** *in [89],*
- *The* ***PD Analysis and Fragility*** *in [82],*
- *The* ***PID Analysis and Fragility*** *in [96].*

As mentioned at the beginning of this part, it is worth mentioning that a book chapter [97] covering all these problems is in preparation and will be published after the thesis defense.

Part II

Induced Network Delays in NCS

Summary

9 Chain of Integrators

9.1 Introduction

In this chapter, we introduce first the output feedback stabilization problem of a chain of integrators using multiple delays in the continuous-time case. To this end, we will recall the properties and results developed by *Niculescu and Michiels* (2004) [106]. As mentioned previously, the aim of this chapter is to explore such ideas in the NCS framework, that is to stabilize a chain of integrators by taking into account the *network-induced delays* and the corresponding *sampling period*. Although such an idea sounds simple and easy to analyze, however such a problem was not fully addressed in the control literature. For example, [137] treated the case of a single integrator with one delay block and they derived the stability regions in the parameter-space defined by the delay and the sampling period and, for higher-order systems, they suggested the use of simulations in order to approach a solution for the corresponding stabilization problem. It is well known that, the stability and the performances of NCS are affected by the *network delays* as pointed out by [137], [128]. To overcome such a problem, several approaches have been proposed and, among them we cite: a model-based method [94] for stability analysis or some optimal controllers when the network-induced delay is smaller [108] or longer [56] than the sampling period, or finally, a queuing mechanism [17] used to reshape random NCS delays to deterministic leading to a time-invariant NCS.

In this context, we are interested in deriving closed-loop stability conditions by using the network-induced delays as *control parameters* for the continuous-time process described by the transfer function $H_{yu}(s) = 1/s^n$ $(n \geq 1)$. The corresponding discrete control law is given by:

$$u(t) = -\sum_{\mu=1}^{m} k_\mu y\left(\ell h - \tau_\mu\right) \quad t \in [\ell h + \tau_m, (\ell+1)h + \tau_m),$$

where ℓ is a nonnegative integer and the network-induced delays $\tau_1 < \tau_2 < \ldots < \tau_m$ are positive real numbers. First, considering a small gain value in the control law we will see that, similarly to the continuous-time case [106, 70] one delay block (gain, delay) cannot stabilize a chain of n integrators, with $n \geq 2$. The approach is based on the use of the complete regular splitting (CRS) property of multiple, non-semisimple eigenvalues (see, e.g. [77] and section 2.5 for some prerequisites).

Next, we will explore the cases when multiple delay blocks are able to stabilize the corresponding chain of integrators. More precisely, a perturbation theory approach (see, for instance, [67] or Chapter 2, section 2.5) will allow concluding on the behavior of the characteristic roots of the closed-loop system as a function of some parameters of the system. We will see (sections 9.7–9.8) that the closed-loop stability can be obtained by using n delay blocks, but an arbitrary pole placement requires $(n+1)$ delay blocks. In both cases, the corresponding control law is explicitly derived. In the first case (n delay blocks), the proposed controller leads to some appropriate closed-loop characteristic *lacunary* polynomials (see, e.g. [84]) with nice properties: (a) only one tuning parameter (for improving eventually other performances in closed-loop), (b) particular behaviors of the roots wrt the variations of the corresponding parameter.

9.2 Continuous-Time Systems

As mentioned previously, this chapter is based on the results obtained by *Niculescu & Michiels (2004)* [106]. Then, in order to present the mean difference between the continuous-time cases and the Networked Control System (NCS) case, in the following subsection we will discuss briefly the characteristics and results of the continuous-time case.

9.2.1 Problem Formulation

Given a chain of n integrators:

$$y^{(n)}(t) = u(t). \tag{9.1}$$

Find (necessary and/or sufficient) conditions on the $(2m+1)-$tuple (m, k_i, τ_i), $i = \overline{1,m}$ such that the (output feedback) control law defined by the chain of m distinct delay block (k_i, τ_i)

$$u(t) = -\sum_{i=1}^{m} k_i y(t-\tau_i) \tag{9.2}$$

asymptotically stabilize the system (9.1).

9.2.2 Properties and Motivated Examples

The continuous-time control-law, has the following important property:

Property 9.1 (Scaling)**.** *The control law*

$$u(t) = -\sum_{j=1}^{m} k_j y(t-\tau_j) \tag{9.3}$$

is asymptotically stabilizing if and only if

$$u(t) = -\sum_{j=1}^{m} \frac{k_j}{\delta^n} y(t-\delta\tau_j), \quad \delta > 0 \tag{9.4}$$

is asymptotically stabilizing.

Now, it is clear to see, that system (9.1) can asymptotically be stabilized by a feedback law

$$u(t) = -q_0 y(t) - q_1 y'(t) - \cdots - q_{n-1} y^{(n-1)}(t) \tag{9.5}$$

where $q(s) = s^n + \sum_{i=0}^{n-1} q_i s^i$ is a Hurwitz polynomial. From this observation, the key idea developed in [106] in order to choose appropriated controller gains k_i, consists of approximating the output derivatives in (9.5) with delayed output measurements, for instance,

$$y'(t) \approx \frac{y(t) - y(t-\varepsilon)}{\varepsilon} \tag{9.6}$$

for small ε, corresponding to an approximation

$$s = \frac{1-e^{-\varepsilon s}}{\varepsilon} + \mathcal{O}(\varepsilon s^2)$$

in the frequency domain.
In order to construct a stabilizing controllers, two approaches have been proposed in [106]. Before introducing such a results, lets consider the following Vandermonde matrix

$$T(\tau) := \begin{pmatrix} 1 & \tau_1 & \tau_1^2 & \cdots & \tau_1^{n-1} \\ 1 & \tau_2 & \tau_2^2 & \cdots & \tau_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \tau_n & \tau_n^2 & \cdots & \tau_n^{n-1} \end{pmatrix} \tag{9.7}$$

where $T(\tau)$ is invertible whenever the delays τ_i are all different.

Theorem 9.1 (Interpolation Based-Approach)**.** *[106] Assume that* $0 \leq \tau_1 < \cdots < \tau_n$ *and* $q(s)$ *a Hurwitz polynomials. Then, the control law*

$$u(t) = -\left[\begin{array}{cccccc} \varepsilon^n q_0 & \frac{\varepsilon^{n-1}}{(-1)} q_1 & \frac{2!\varepsilon^{n-2}}{(-1)^2} q_2 & \cdots & \frac{(n-1)!\varepsilon}{(-1)^{n-1}} q_{n-1} \end{array}\right] \cdot T(\tau)^{-1} \begin{bmatrix} y(t-\tau_1) \\ y(t-\tau_2) \\ \vdots \\ y(t-\tau_n) \end{bmatrix} \tag{9.8}$$

achieves asymptotic stability for small values of ε*. As* $\varepsilon \to 0+$*, the n rightmost eigenvalues converge to* $\varepsilon\lambda_i$*,* $i = \overline{1,n}$*, with* λ_i *the zeros of* $q(s)$*.*

The following result presents an alternative approach to design a stabilizing feedback law.

Theorem 9.2 (Exact Pole Placement Based-Approach)**.** *[106] Assume that* $0 \leq \tau_1 < \cdots < \tau_n$ *and let* $T(\tau)$ *be defined by (9.7). Then, the control law*

$$u(t) = (-1)^n \left[\begin{array}{cccc} \varepsilon^n & n\varepsilon^{n-1} & \cdots & n!\varepsilon \end{array}\right] \cdot T(\tau)^{-1} \begin{bmatrix} e^{-\varepsilon\tau_1} & & \\ & \ddots & \\ & & e^{-\varepsilon\tau_n} \end{bmatrix} \begin{bmatrix} y(t-\tau_1) \\ \vdots \\ y(t-\tau_n) \end{bmatrix} \tag{9.9}$$

achieves asymptotic stability for small values of ε*. Moreover, there is a closed-loop eigenvalue at* $\lambda = \varepsilon$ *with multiplicity n.*

Remark 9.1. *Observe that the control law based on interpolation or in pole placement will not necessarily coincide, for example taking* $q(s) = (s+1)^n$ *with the control law (9.8) we get*

$$u(t) = -\left[\begin{array}{ccccc} \varepsilon^n & \frac{n\varepsilon^{n-1}}{(-1)} & \frac{n(n-1)\varepsilon^{n-2}}{(-1)^2} & \cdots & \frac{n!\varepsilon}{(-1)^{n-1}} \end{array}\right] \cdot T(\tau)^{-1} \begin{bmatrix} y(t-\tau_1) \\ \vdots \\ y(t-\tau_n) \end{bmatrix}. \tag{9.10}$$

Clearly, this control law does not coincide with (9.9) because it is based on an asymptotic approximation of $q(s)$*, while (9.9) is based on an exact placement of n eigenvalues.*

9.3 Stabilization of A Chain of Integrators in NCS Framework

In the remaining part of the chapter we will focus on the closed-loop stability of a chain of integrators in a networked-control setting. Similar to the continuous-time case, we will see that a single delay is not sufficient to stabilize a chain having n ($n \geq 2$) integrators, but that n delay-blocks are able to stabilize such a chain without being able to guarantee an arbitrary pole placement for the corresponding system.

9.4 Problem Formulation and Motivating Examples

Consider the following chain of integrators system,

$$y^{(n)}(t) = u(t) \qquad t \in [\ell h + \tau_m, (\ell+1)h + \tau_m) \tag{9.11}$$

with a discrete controller given by

$$u(t) = \sum_{\mu=1}^{m} u_\mu(t), \quad t \in [\ell h + \tau_m, (\ell+1)h + \tau_m), \tag{9.12}$$

where the delay-blocks are defined as $u_\mu(t) := -k_\mu y\left(\ell h - \tau_\mu\right)$, $h \in \mathbb{R}_+$ is the sampling period, τ is the induced network delay and $(d-1)h < \tau =: \tau_1 < \tau_2 < \ldots < \tau_m \leq dh$, for $d \in \mathbb{N}$ (i.e., a positive integer).

9.5 Discretized Delay Case and Some Related Properties

Let h be the sampling period, τ the induced network delay satisfying $\tau = (d-1)h + \widetilde{\tau}$, $d \in \mathbb{N}$ and let τ_μ be chosen satisfying $\tau_\mu = (d-1)h + \widetilde{\tau}_\mu$ where $0 < \widetilde{\tau} =: \widetilde{\tau}_1 < \ldots < \widetilde{\tau}_m < h$. Since the control law (9.12) is piecewise constant, then after some algebraic manipulation the discretized open-loop system can write as (Åström & Wittenmark (1997)) [6]:

$$x[\ell+1] = \Phi(h)x[\ell] + \sum_{\mu=1}^{m} \Gamma(0, h-\tau_\mu)u_\mu[\ell-d+1] + \sum_{\mu=1}^{m} \Gamma(h-\tau_\mu, h)u_\mu[\ell-d], \tag{9.13}$$

where $d \in \mathbb{N}$ and:

$$\Phi(h) \triangleq \left[\phi_{\mu\nu}(h)\right]_{\mu,\nu=1}^{n}, \quad \text{with } \phi_{\mu\nu}(h) \triangleq \begin{cases} \dfrac{h^{\nu-\mu}}{(\nu-\mu)!} & \text{if} \quad \nu \geq \mu \\ 0 & \text{if} \quad \nu < \mu \end{cases} \tag{9.14a}$$

$$\Gamma\left(t_i, t_f\right) \triangleq \left[\sigma_\mu\left(t_i, t_f\right)\right]_{\mu=1}^{n}, \quad \text{with } \sigma_\mu\left(t_i, t_f\right) \triangleq \frac{\left(t_f\right)^{n-\mu+1} - \left(t_i\right)^{n-\mu+1}}{(n-\mu+1)!}. \tag{9.14b}$$

Define now the augmented state vector as $z[\ell] \triangleq \left[x^T[\ell], x_1[\ell-d], x_1[\ell-d+1], \ldots, x_1[\ell-1]\right]^T$ where $x_1[\ell] = e_1^T x[\ell]$, leading to the augmented closed-loop system:

$$z[\ell+1] = \widetilde{\Phi}(h;k,m)z[\ell]. \tag{9.15}$$

Equation (9.13) describes a general situation, that is, when the induced network delay is larger/smaller than the sampling period. Under these observations, we have:

Remark 9.2 (Smaller delay). *Let τ be the induced network delay, such that $\tau =: \tau_1 < \cdots < \tau_m < h$. Then, $d = 1$, $\widetilde{\tau}_\mu = \tau_\mu$ and the transfer matrix of the augmented closed-loop system rewrites as:*

$$\widetilde{\Phi}(h;k,m) \triangleq \begin{bmatrix} \Phi(h) - \Delta_0(k;m)e_1^T & \Delta_1(k;m) \\ -e_1^T & 0 \end{bmatrix}. \tag{9.16}$$

Remark 9.3 (Larger delay). *Let τ be the induced network delay and h the sampling period, such that $d > 1$.Then, $0 < \widetilde{\tau} =: \widetilde{\tau}_1 < \cdots < \widetilde{\tau}_m < h$, and the corresponding transfer matrix becomes:*

$$\widetilde{\Phi}(h;k,m) \triangleq \begin{bmatrix} \Phi(h) & \Delta_1(k;m) & \Delta_0(k;m) & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -e_1^T & 0 & 0 & \dots & 0 \end{bmatrix}, \tag{9.17}$$

where $\Delta_i(k;m)$ $i \in \{0,1\}$ is defined by $\Delta_i(k;m) \triangleq \sum_{\mu=1}^{m} \Gamma\left((h-\widetilde{\tau}_\mu)i, h-(1-i)\widetilde{\tau}_\mu\right) k_\mu$.

The characteristic polynomial of $\widetilde{\Phi}(h;k,m)$ given by:

$$P_{cl}(z) \triangleq z^{n+d} + a_{n+d-1}(h;k,m)\, z^{n+d-1} + \cdots + a_0(h;k,m),$$

describes the general case (larger/smaller delay) for the augmented closed-loop system (9.15).

Assertion 9.1. *Let $d = 1$ (smaller delay case). Then, the coefficients of $P_{cl}(z)$ satisfy the following properties:*

1. *$a_\mu(h;k,m)$ are affine functions in k ($\beta_\mu \in \mathbb{R}$):*

$$a_\mu(h;k,m) = \sum_{\nu=1}^{m} k_\nu \alpha_{\mu,\nu}(h,\tau_\nu) + \beta_\mu, \tag{9.18}$$

2. *$\alpha_{\mu,\nu}(h,\tau)$ is a polynomial function in (h,τ) verifying:*

$$\alpha_{\mu,\nu_i}(h,\tau) = \alpha_{\mu,\nu_j}(h,\tau) \quad \text{for} \quad i \neq j.$$

This assertion follows by a straightforward application of the *Laplace expansion's* rule ([74]) to the last row of:

$$\widetilde{\Phi}_{cl}(z;h,k,m) \triangleq \begin{bmatrix} zI_{n\times n} + \Delta_0(k;m)e_1^T - \Phi(h) & -\Delta_1(k;m) \\ e_1^T & z \end{bmatrix}.$$

Remark 9.4. *By applying the determinant properties [74], it is easy to see that $\alpha_{n,\nu}(h,\tau)$, β_n and $\alpha_{0,\nu}(h,\tau)$, β_0 satisfy:*

$$\alpha_{n,\nu}(h,\tau) = \frac{(h-\tau)^n}{n!},\ \beta_n = -n, \quad \alpha_{0,\nu}(h,\tau) = \frac{\tau^n}{n!},\ \beta_0 = 0.$$

9.6 Motivating examples

In the rest of this section, we present several illustrative examples which show some "singular" behaviors. Specifically, we will see that as in the continuous case, one delay-block cannot stabilize a chain of n integrators, with $n \geq 2$ and that at least $m = n$ delay-blocks should be taken into account. Moreover, it is shown that in contrast with the continuous case, for sufficiently small enough gain parameters the closed-loop system is unstable.

9.6.1 Complete Regular Splitting Property.

In the sequel we present first several motivating examples which illustrate the behavior of the closed-loop system for small gain-parameters. Next, a perturbation-based approach is adopted in order to describe such a behavior in a general context. More precisely, we will see that for $m-$delay blocks the closed-loop system possesses the CRS property. As in the continuous case, we have the following property:

Property 9.2 (Scaling Property)**.** *The control law*

$$u(t^{+}) = -\sum_{\mu=1}^{m} k_{\mu} y(t-\tau_{\mu}), \;\; t \in [\ell h + \tau_m, (\ell+1)h + \tau_m) \tag{9.19}$$

is asymptotically stabilizing if and only if the control law

$$u(t^{+}) = -\sum_{\mu=1}^{m} \frac{k_{\mu}}{\rho^{n}} y(t-\rho\tau_{\mu}) \;\; t \in \rho\,[\ell h + \tau_m, (\ell+1)h + \tau_m)\,, \text{ with } \rho > 0, \tag{9.20}$$

is stabilizing.

Proof. This property can be shown by taking $z_i \triangleq \rho^{n-i} x_i$, $\widetilde{t} \triangleq \rho t$ and $\widetilde{u}^{+} \triangleq \frac{1}{\rho^{n}} \sum_{i=1}^{m} z(\widetilde{t} - \rho\tau_i)$, proving thus the equivalence of the systems. $\square$

In the scalar case, Zhang *et al.* [137] derived the stability region in the $(h(k_1), \tau)$ parameter space for the case of one integrator:

$$\begin{aligned} \dot{y}(t) &= u(t), & t &\in [\ell h + \tau, (\ell+1)h + \tau) \\ u\left(t^{+}\right) &= -k_1 y(t-\tau) & t &\in \{ih + \tau |\, i \in \mathbb{N} \cup \{0\}\}. \end{aligned}$$

The corresponding NCS will be stable *if and only if*:

$$\max\left\{\frac{1}{2}h - \frac{1}{k_1}, 0\right\} < \tau < \min\left\{\frac{1}{k_1}, h\right\}. \tag{9.21}$$

Unfortunately, such a property does not hold for higher-order systems (i.e., $n \geq 2$), as it is shown by the following result:

Proposition 9.1. *The double integrator system*

$$\ddot{y}(t) = u(t), \quad t \in [\ell h + \tau, (\ell+1)h), \quad \tau < h, \tag{9.22}$$

can not be stabilizable with a discrete control law of the form

$$u\left(t^{+}\right) = -k_1 y(t-\tau), \quad t \in \{\ell h + \tau |\; \ell \in \mathbb{N} \cup \{0\}\}, \tag{9.23}$$

for all $k_1 \in \mathbb{R}$.

Proof. Taking the discretization of the system (9.22) with control law (9.23), we derive:

$$z((\ell+1)h) =: z[\ell+1] = \tilde{\Phi}(h, \tau, k_1)\, z[\ell] \tag{9.24}$$

where

$$
\begin{aligned}
z[\ell] &\triangleq \left[\begin{array}{cc} x[\ell]^T & u[\ell-1] \end{array}\right], \\
\tilde{\Phi}(h,\tau,k_1) &\triangleq \left[\begin{array}{ccc} 1-\frac{1}{2}k_1(h-\tau)^2 & h & k_1\left(h-\frac{\tau}{2}\right)\tau \\ -k_1(h-\tau) & 1 & k_1\tau \\ -1 & 0 & 0 \end{array}\right].
\end{aligned}
$$

Applying the Schur-Cohn criteria, the system will be asymptotically stable if and only if the following inequalities hold:

Condition 1:

$$1-\frac{k_1^2\tau^4}{4} > 0 \tag{9.25}$$

Condition 2:

$$\alpha_1^2-\beta_1^2 > 0 \tag{9.26}$$

Condition 3:

$$\alpha_2^2-\beta_2^2 > 0 \tag{9.27}$$

where α_i and β_i, $i=\{1,2\}$ are defined as,

$$
\begin{aligned}
\alpha_1 &\triangleq 2+\frac{h^2k_1}{2}+hk_1\tau-k_1\tau^2 \\
\alpha_2 &\triangleq 1-\frac{k_1^2\tau^4}{4} \\
\beta_1 &\triangleq -2+\frac{h^2k_1}{2}-hk_1\tau+k_1\tau^2 \\
\beta_2 &\triangleq 1+\frac{h^2k_1}{2}+hk_1\tau-\frac{h^2k_1^2\tau^2}{4}+\frac{hk_1^2\tau^3}{2}-\frac{k_1^2\tau^4}{4}
\end{aligned}
$$

It is clear that $k_1 \neq 0$, otherwise (9.26)-(9.27) will not hold. Then, analyzing (9.25)-(9.27) we get the following results:

(i) *Condition 1* will holds if and only if:

$$
\begin{aligned}
4-k_1^2\tau^4 &> 0 && (9.28)\\
\Leftrightarrow -2 < k_1\tau^2 &< 2 && (9.29)
\end{aligned}
$$

(ii-a) *Condition 2* will hold whenever one of the following two cases holds:

- **Case 1** $(\alpha_1+\beta_1)<0 \Rightarrow (\alpha_1-\beta_1)<0$**:** simplifying $(\alpha_1+\beta_1)$ and $(\alpha_1-\beta_1)$ we obtain:

$$
\begin{aligned}
\Rightarrow k_1 &< 0 && (9.30)\\
8+4hk_1\tau &< 4k_1\tau^2 && (9.31)
\end{aligned}
$$

Next, considering from *Condition 3* $(\alpha_2-\beta_2)>0$ and $(\alpha_2+\beta_2)>0$, we get:

$$k_1 < 0 \tag{9.32}$$

$$
\begin{aligned}
&\{8+4hk_1\tau\}+2h^2k_1+2hk_1^2\tau^3- \\
&-h^2k_1^2\tau^2-2k_1^2\tau^4 > 0 && (9.33)
\end{aligned}
$$

from (9.31) we know that $8+4hk_1\tau < 4k_1\tau^2$, then (9.33) can be rewritten as:

$$2hk_1^2\tau^3 > 2k_1^2\tau^4 + h^2k_1^2\tau^2 - 4k_1\tau^2 - 2h^2k_1 > 0 \tag{9.34}$$

From the fact that $h-\tau > 0$, we define the following difference:

$$d \triangleq h - \tau > 0$$

Then, $h = d + \tau$, applying this fact to (9.34), we derive:

$$0 > d^2k_1^2\tau^2 + k_1^2\tau^4 - 4k_1\tau^2 - 2h^2k_1 > 0 \tag{9.35}$$

clearly (9.35) leads to a contradiction, that implies that $(\alpha_1+\beta_1) < 0 \Rightarrow (\alpha_1-\beta_1) < 0$ doesn't hold.

(ii-b) Next we consider in *Condition 2* the case $(\alpha_1-\beta_1) > 0$, that is:

- **Case 2** $(\alpha_1-\beta_1) > 0 \Rightarrow (\alpha_1+\beta_1) > 0$: following the same ideas that in the previous case, we obtain the following inequalities:

$$\begin{aligned} 8+4hk_1\tau &> 4k_1\tau^2 & (9.36)\\ k_1 &> 0 & (9.37)\end{aligned}$$

Taking into account *Condition 3* for $(\alpha_2+\beta_2) < 0$, we get:

$$8+2h^2k_1+4hk_1\tau+2hk_1^2\tau^3 < 2k_1^2\tau^4+h^2k_1^2\tau^2 \tag{9.38}$$

Now, according to (9.28) we have that $4 > k_1^2\tau^4 \Rightarrow 8 > 2k_1^2\tau^4$, with this fact, we have that (9.38) can be stated as:

$$\begin{aligned} 0 < 8+2h^2k_1+4hk_1\tau+2hk_1^2\tau^3 &< 8+h^2k_1^2\tau^2 \\ 0 < 2h^2k_1+4hk_1\tau+2hk_1^2\tau^3 &< 2h^2k_1 \\ 0 < 4hk_1\tau+2hk_1^2\tau^3 &< 0 & (9.39)\end{aligned}$$

In the second line of the previous inequalities, we have used the fact that $h^2k_1^2\tau^2 \equiv h^2k_1(k_1\tau^2) < 2h^2k_1$ (by (9.29)). Once again, we have a contradiction, this means that $\forall k_1 \in \mathbb{R}$ the system (9.22) cannot be stabilized by the control law (9.23).

□

However, the use of a controller involving two delays will be able to stabilize the double integrator system (9.22). Indeed, consider $\tau_1 := \tau < \frac{h}{3}$, $\tau_2 := \tau+\varepsilon$ with $0 < \varepsilon < \frac{2}{3}(h-3\tau)$. Then the control gains:

$$k_1(h,\tau,\varepsilon) \triangleq \frac{6h^2+4h(\tau+\varepsilon)-6(\tau+\varepsilon)^2}{h\varepsilon(4h^2+2\tau(\tau+\varepsilon)-3h(2\tau+\varepsilon)} \tag{9.40}$$

$$k_2(h,\tau,\varepsilon) \triangleq \frac{-6h^2-4h\tau+6\tau^2}{h\varepsilon(4h^2+2\tau(\tau+\varepsilon)-3h(2\tau+\varepsilon)} \tag{9.41}$$

will define a stabilizing control law (9.12), as is illustrated in figure Fig 9.1. The algebraic technique developed in *Proposition* 9.1 is not suitable to deal with the general case ($n > 2$), then in order to prove a general result the following proposition adopt the *perturbation techniques* developed in the *Chapter* 2.

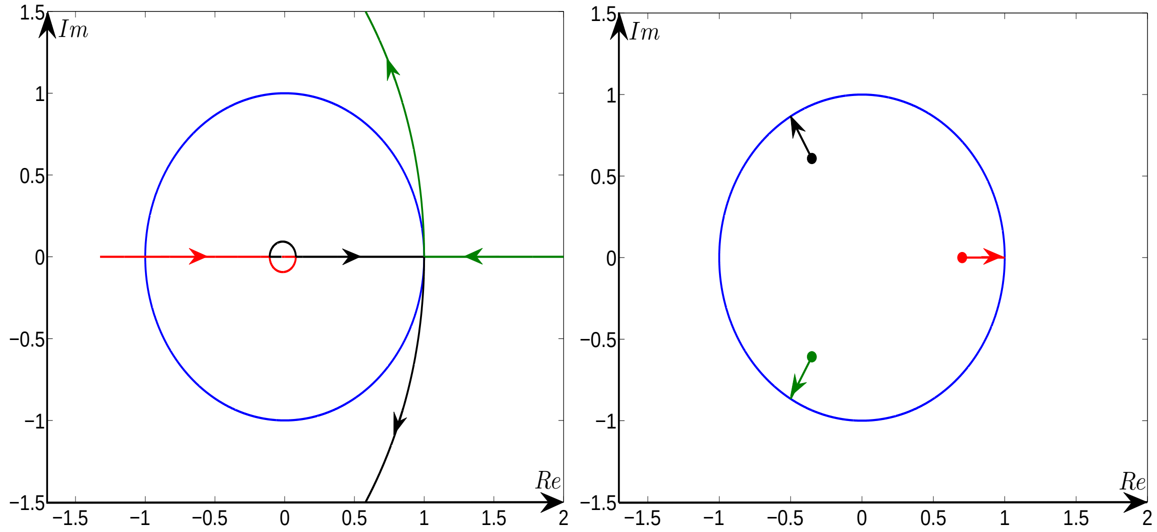


Figure 9.1: Double integrator system (9.22) with $h = \frac{3}{5}$ and $\tau = \frac{1}{10}$. (Left) One delay-block with $k_1 \in \left[-\frac{2!}{\tau_1^2}, \frac{2!}{\tau_1^2}\right]$. (Right) Two delay-blocks, with k_1 and k_2 satisfying (9.40) and (9.41), respectively.

Proposition 9.2. *If $n \geq 2$, then the closed-loop system consisting of a chain of n integrators*

$$y^{(n)}(t) = u(t), \quad t \in [\ell h + \tau, (\ell+1)h + \tau), \quad \tau < h \tag{9.42}$$

and a control law of the form

$$u(t) = -k_1 y(\ell h - \tau), \quad t \in [\ell h + \tau, (\ell+1)h + \tau) \tag{9.43}$$

is unstable for small values of the controller gain k_1.

Proof. The assertion follows from the behavior of the eigenvalue $\lambda_0 = 1$ for $k_1 = 0$ as $|k_1|$ is increased from zero, and is based on Theorem 2.5. To this end, consider

$$L\left(\tilde{\lambda}, k_1\right) = \begin{bmatrix} \Phi(h) - I - \tilde{\lambda} I & 0 \\ -e_1^T & -\tilde{\lambda} - 1 \end{bmatrix} + \begin{bmatrix} -\Delta_0(k_1;1)\, e_1^T & \Delta_1(k_1;1) \\ 0 & 0 \end{bmatrix} \tag{9.44}$$

where $\tilde{\lambda} \triangleq \lambda - 1$. From the definition of $\Phi(h)$ it follows that the algebraic and geometric multiplicity of the eigenvalue $\tilde{\lambda} = 0$ for $k_1 = 0$ is n and 1, respectively. Furthermore the right and left eigenvectors are given by $x = e_1$ and $\hat{x} = e_n$. Next, we have

$$\frac{\partial L}{\partial k_1}(\lambda, k_1) = \begin{bmatrix} -\Gamma(0, h-\tau) & \Gamma(h-\tau, h) \\ 0 & 0 \end{bmatrix}.$$

It is easy to see that (2.32) holds if and only if the equation

$$\begin{aligned} L(0,0)y &= \frac{\partial L}{\partial k_1}(0,0)\, e_1 \\ \Leftrightarrow \begin{bmatrix} \Phi(h) - I & 0 \\ -e_1^T & -1 \end{bmatrix} y &= -\begin{bmatrix} \frac{(h-\tau)^n}{n!} \\ \vdots \\ \frac{h-\tau}{1!} \\ 0 \end{bmatrix} \end{aligned} \tag{9.45}$$

has no solution for y. Since, by hypothesis, $h > \tau$, this is the case and the eigenvalue $\widetilde{\lambda} = 0$ has the CRS property. It follows that for small $|k_1|$ it can be expanded as

$$\widetilde{\lambda}_i(k_1) = ce^{j\frac{2\pi i}{n}} k_1^{\frac{1}{m}} + o(k_1^{\frac{1}{m}}),\ i = 1,\dots,n,$$

for some $c \in \mathbb{C}$. Since $\lambda = 1 + \widetilde{\lambda}$ it follows that the original system always has one eigenvalue outside the unit circle for small values of $|k_1|$. $\square$

Consider now a discrete control law with two parameters gains. As a motivation example, we will consider $n = 3$ and $m = 2$, i.e.,

$$y^{(3)}(t) = u(t), \quad t \in [\ell h + \tau_2, (\ell+1)h + \tau_2), \quad \tau_2 < h, \tag{9.46}$$

where $0 < \tau_1 < \tau_2 < h$ and with the control law,

$$u(t) = -k_1 y(\ell h - \tau_1) - k_2 y(\ell h - \tau_2) \quad t \in [\ell h + \tau_2, (\ell+1)h + \tau_2). \tag{9.47}$$

Without any loss of generality, assume that $k_2 = \alpha k_1$ for some $\alpha \in \mathbb{R} \setminus \{-1\}$. Then, applying the *Newton Diagram's* ([73]) the characteristic roots for "small" control gain $k_1 = \varepsilon$ behave as:

$$\begin{aligned} z_1(\varepsilon) &= w_{0,3}\varepsilon + \frac{w_{0,3}\left(-w_{1,3}^3 + 3h^2 w_{0,1} + 3!hw_{0,2}\right)}{3!}\varepsilon^2 + \mathcal{O}\left(\varepsilon^3\right), \\ z_{\ell+1}(\varepsilon) &= 1 + w_{1,\ell}\varepsilon^{\frac{1}{3}} + \frac{2w_{1,3}^3 + h^2 w_{0,1} - w_{1,\ell}^3}{3w_{1,2}}\varepsilon^{\frac{2}{3}} + \mathcal{O}(\varepsilon), \quad \ell \in \{1,2,3\}, \end{aligned}$$

where $w_{0,\ell} \triangleq \frac{\tau_1^\ell + \alpha\tau_2^\ell}{\ell!}$ and $w_{1,\ell} \triangleq e^{\frac{\ell\pi}{3}j} h(1+\alpha)^{\frac{1}{3}}$. In the particular case $\alpha = -1$, the closed-loop characteristic function can be rewritten as $P_{cl}(z) = (z-1)\widetilde{P}(z)$. Now, applying the same analysis to $\widetilde{P}(z)$, we conclude that the characteristic roots for "small" values of $k_1 = \varepsilon$ behave as:

$$\begin{aligned} \widetilde{z}_1(\varepsilon) &= \frac{\tau_1^3 - \tau_2^3}{3!}\varepsilon + \frac{h(\tau_1-\tau_2)^2(h+\tau_1+\tau_2)\left((\tau_1+\tau_2)^2 - \tau_1\tau_2\right)}{12}\varepsilon^2 + \mathcal{O}\left(\varepsilon^3\right), \\ \widetilde{z}_{2,3}(\varepsilon) &= 1 \pm h\sqrt{\tau_1 - \tau_2}\varepsilon^{\frac{1}{2}} + \frac{h(\tau_1-\tau_2)(h-\tau_1-\tau_2)}{4}\varepsilon + \mathcal{O}\left(\varepsilon^{\frac{3}{2}}\right). \end{aligned}$$

Figure (Fig.9.2) illustrates the closed-loop behavior of the characteristic roots for several values of α. In the above analysis the point $\alpha = -1$ represents a *singular* case which can be generalized as follows:

Property 9.3. *Let $k_i = \alpha_i\varepsilon$, for some $\varepsilon \in \mathbb{R} \setminus \{0\}$ and $\alpha_i \in \mathbb{R}$ for $i = 1,\dots,m$. Assume that $\alpha_1 + \alpha_2 + \cdots + \alpha_m = 0$. Then for the control law*

$$u(t) = -\sum_{\mu=1}^{m} k_\mu y\left(\ell h - \tau_\mu\right), \quad t \in [\ell h + \tau_m, (\ell+1)h + \tau_m), \tag{9.48}$$

the closed-loop system (9.15) has at least one characteristic root on the unit circle at $z = 1$.

Proof. Since $\alpha_1 + \alpha_2 + \cdots + \alpha_m = 0$, this implies that $\sum_i \frac{h^N - (h-\tau_i)^N}{N!}k_i \equiv -\frac{\varepsilon}{N!}\sum_i \frac{(h-\tau_i)^N}{N!}\alpha_i$ for $N = 1,\dots,n$. Then, the first and last columns of the characteristic matrix $\widetilde{\Phi}_{cl}(1;h,\alpha,m)$ are identical. In other words, for all sampling period (h) and for all possible delays (τ_i), $z = 1$ represents a characteristic root of the closed-loop system (9.15). $\square$

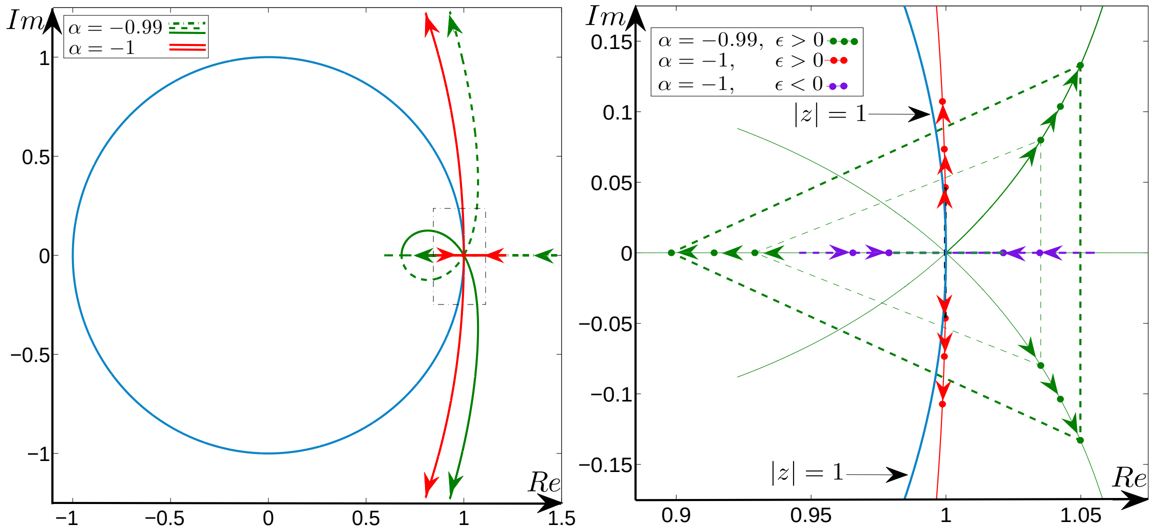


Figure 9.2: *Branches series behavior* for $\alpha = -0.99$ (green) and $\alpha = -1$ (red). (Right) Zoom of the dashed region.

The following result concerns the behavior of the closed-loop eigenvalues for sufficiently "small" controller gains. To this end, we consider m controller parameters satisfying

$$k_i = \alpha_i \varepsilon, \quad i = 1, \ldots, m,$$

where $\varepsilon \in \mathbb{R}$ is a parameter, and the direction $\alpha \triangleq (\alpha_1, \ldots, \alpha_m)$, $\|\alpha\| = 1$ is fixed. For the sake of brevity, we will consider $\varepsilon > 0$.

Proposition 9.3. *Let τ be the induced network delay satisfying $\tau = (d-1)h + \tilde{\tau}$, $d \in \mathbb{N}$, $0 < \tilde{\tau} =: \tilde{\tau}_1 < \cdots < \tilde{\tau}_m < h$, $n > 2$, and assume that $\alpha_1 + \cdots + \alpha_m \neq 0$. Consider the closed-loop system consisting of a chain of n integrators*

$$y^{(n)}(t) = u(t), \quad t \in [\ell h + \tau_m, (\ell+1)h + \tau_m), \tag{9.49}$$

with a control law of the form

$$u(t) = -\sum_{\mu=1}^{m} k_\mu y(\ell h - \tau_\mu), \quad t \in [\ell h + \tau_m, (\ell+1)h + \tau_m). \tag{9.50}$$

Then, for sufficiently small control gains k_μ, the eigenvalues of the closed-loop system possesses the CRS property and behave as:

$$\lambda_\ell(k) = 1 + h e^{\frac{2\ell+1}{n}\pi j} \frac{(\alpha_1 + \alpha_2 + \cdots + \alpha_m)^{\frac{1}{n}}}{\|\alpha\|^{\frac{1}{n}}} \varepsilon^{\frac{1}{n}} + o(|\varepsilon|^{\frac{1}{n}}), \quad \ell = 1, \ldots, n. \tag{9.51}$$

Proof. Consider first the smaller-delay case (i.e. $d = 1$). According to (9.44), we have that the behavior of the zeros of the closed-loop system can be described by,

$$L(\lambda, \varepsilon) \triangleq \begin{bmatrix} \lambda I - \Phi(h) & 0 \\ e_1^T & \lambda \end{bmatrix} + \frac{\varepsilon}{\|\alpha\|} \begin{bmatrix} \Delta_0(\alpha; m) e_1^T & -\Delta_1(\alpha; m) \\ 0 & 0 \end{bmatrix}. \tag{9.52}$$

According to this definition, we have that $\lambda_0 = 1$ is a *non-semi-simple* eigenvalue of $L(\lambda, 0)$ with multiplicity n, then we can apply Theorem 2.4. To this end, observe that the generalized eigenvector of $L(\lambda,0)$ can be written as $x_1 = e_1 - e_{n+1}$, whereas its corresponding reciprocal eigenvector is given by $y_n = \frac{h^{n-1}}{\|\alpha\|} e_n^T$, then the mean coefficient of the Puiseux series for $\lambda(\varepsilon)$ is given by:

$$\begin{aligned}
\langle y_n, A_1 x_1\rangle &= \begin{bmatrix} 0 & 0 & \cdots & \frac{h^{n-1}}{\|\alpha\|} & 0 \end{bmatrix} \begin{bmatrix} -\sum \frac{(h-\tau_\mu)^n \alpha_\mu}{n!} & 0 & \cdots & \sum \frac{h^n - (h-\tau_\mu)^n}{n!} \alpha_\mu \\ -\sum \frac{(h-\tau_\mu)^{n-1} \alpha_\mu}{(n-1)!} & 0 & \cdots & \sum \frac{h^{n-1} - (h-\tau_\mu)^{n-1}}{(n-1)!} \alpha_\mu \\ \vdots & \vdots & \ddots & \vdots \\ -\sum \frac{(h-\tau_\mu)\alpha_\mu}{1!} & 0 & \cdots & \sum \frac{h-(h-\tau_\mu)}{1!} \alpha_\mu \\ 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ -1 \end{bmatrix} \\
&= \begin{bmatrix} -\frac{h^{n-1}}{\|\alpha\|} \sum (h-\tau_\mu)\alpha_\mu & 0 & \cdots & 0 & \frac{h^{n-1}}{\|\alpha\|} \sum \tau_\mu \alpha_\mu \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ -1 \end{bmatrix} \\
\langle y_n, A_1 x_1\rangle &= -\frac{h^n}{\|\alpha\|} \sum_{\mu=1}^{m} \alpha_\mu .
\end{aligned}$$

These arguments complete the proof for the smaller delay case. Consider now the larger delay case (i.e. $d > 1$). According to (9.17), the zero closed-loop behavior is described by the following matrix-valued function,

$$L(\lambda,\varepsilon) = \left[\begin{array}{c|c} \lambda I - \Phi(h) & 0_{n\times d} \\ \hline e_d^{(d)} \times e_1^{(n)T} & \lambda I - J_d(0) \end{array}\right] + \frac{\varepsilon}{\|\alpha\|} \left[\begin{array}{c|cccc} 0_{n\times n} & -\Delta_1(\alpha;m) & -\Delta_0(\alpha;m) & 0_{n\times d-2} \\ \hline 0_{d\times n} & & 0_{d\times d} & \end{array}\right].$$

The remainder of the proof follows the same arguments as those developed in *smaller-delay case* and for the sake of brevity they will be omitted. □

Example 9.1. *Consider first the triple integrator. As mentioned above (Proposition 9.2), it cannot be stabilized by using a single block-delay (see Fig.10.1 (left)). Secondly, consider a fourth-integrator system. As stated in Proposition 9.3, for small controller-gains the system has the CRS property, however by taking $m = n = 4$ block-delays and by increasing the controller-gains it is possible to stabilize the closed-loop system (see Fig.10.1 (right)).*

9.6.2 Admissible Pole Placement.

Next, we will see that unlike the continuous case, taking $m = n$ delay-blocks are not able to place arbitrarily the eigenvalues of the closed-loop system. In order to see such a property, consider the smaller delay case and let $z^{(0)} \triangleq \{z_1^{(0)}, \ldots, z_{n+1}^{(0)}\}$ be the set of roots of $P_{cl}(z)$. Then, the fact that the uncontrolled system has $n+1$ roots on $\mathcal{C}(0,1)$ simply points out that, if $m = n$ in the control law (9.12), then the system does not have sufficient "*degrees-of-freedom*" for an arbitrary pole placement.

Example 9.2. *Consider now the triple integrator:*

$$y^{(3)}(t) = u(t), \quad t \in [\ell h + \tau_m, (\ell+1)h + \tau_m), \quad \tau_m < h.$$

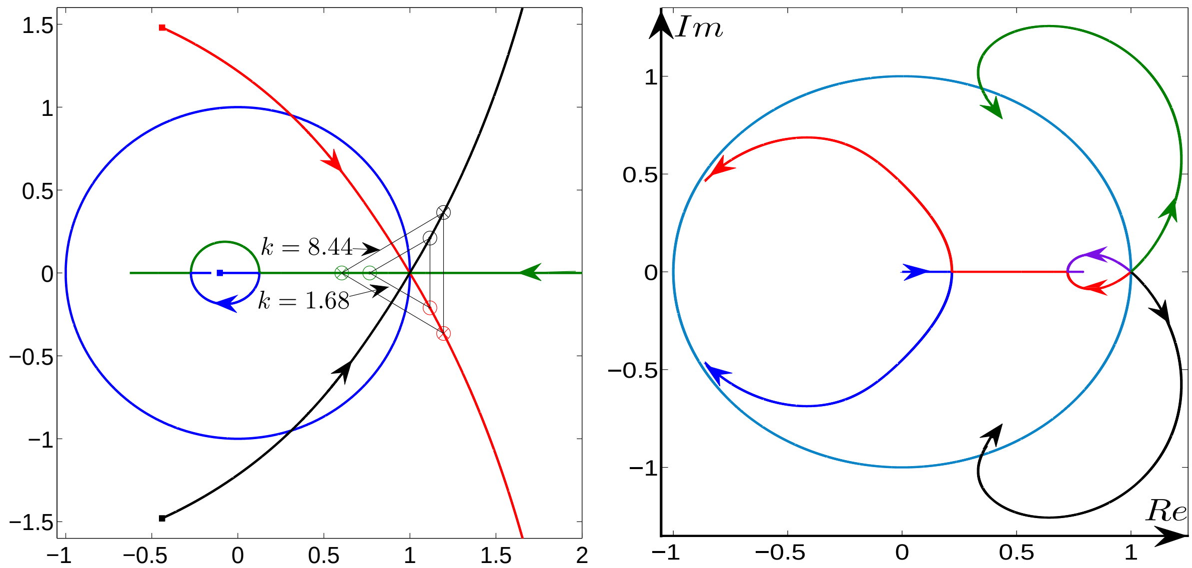


Figure 9.3: *Completely Regular Splitting* property illustrating *Proposition* 9.2 and *Proposition* 9.3 for $\lambda = 1$. (Left) Triple-integrator $(n = 3)$ for $k \in [-\frac{3!}{\tau^3}, \frac{3!}{\tau^3}]$. (Right) Fourth-integrator $(n = 4)$ for $k_\ell = \alpha_\ell \varepsilon$, with $\ell \in \{1, \ldots, 4\}$ and $\varepsilon \in [0, 20]$.

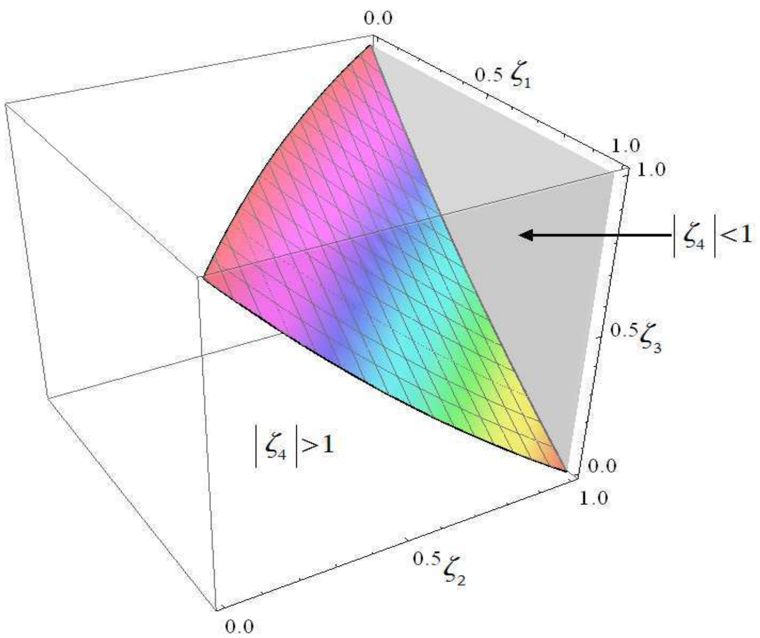


Figure 9.4: Admissible pole-placement for the control law (9.12) with $m = n = 3$

Taking $m = 3$ in (9.12) and denoting the roots of P_{cl} by $\{\zeta_\mu\}$ for $\mu = \{\overline{1,4}\}$ and considering that ζ_4 depend on $(\zeta_1, \zeta_2, \zeta_3)$. Then, the only admissible roots $|\zeta_\mu| < 1$ for $\mu = \{\overline{1,3}\}$ such that $|\zeta_4(\zeta_1, \zeta_2, \zeta_3)| < 1$ are depicted in figure Fig.9.4.

9.7 Control-Law Based on Exact Pole Placement

Denote the set of desired closed-loop roots by $\lambda^{(0)} \triangleq \{\lambda_1, \ldots, \lambda_{n+1}\}$ and the corresponding characteristic polynomial by:

$$P_d(z,c) \triangleq z^{n+1} - c_1 z^n + c_2 z^{n-1} + \cdots + (-1)^{n+1} c_{n+1} \tag{9.53}$$

where c_ℓ is the $\ell - th$ *symmetric function* of $\lambda^{(0)}$ defined as the sum of the product of the eigenvalues taken ℓ at the time:

$$c_\ell \triangleq \sum_{1 \le i_1 \le \cdots \le i_\ell \le n+1} \lambda_{i_1} \cdots \lambda_{i_\ell}$$

Proposition 9.4 (Exact pole-placement)**.** *Assume that* $0 < \tau =: \tau_1 < \tau_2 < \ldots < \tau_{n+1} < h$*. Under the notations above, define the gain:*

$$k = A^{-1} \widetilde{c}^T \tag{9.54}$$

with $A \triangleq \left[\overline{\alpha}_{\mu-1}(h, \tau_\nu)\right]_{\mu,\nu=1}^{n+1}$, $\widetilde{c} \triangleq \left[(-1)^{n+1} c_{n+1} \quad (-1)^n c_n - (-1)^n \binom{n}{0} \quad \cdots \quad -c_1 + \binom{n}{n-1}\right]$
where $\overline{\alpha}_i(h,\tau)$ *is defined recursively by taking* $B_0 \triangleq I$ *and,*

$$\begin{aligned} \overline{\alpha}_\mu(h,\tau) &\triangleq -\tfrac{1}{n-\mu+1} \frac{\partial}{\partial k_\nu} trace\left(\widetilde{\Phi}(h;k,m) B_{n-\mu}\right) \\ B_\mu &\triangleq -\frac{trace\left(\widetilde{\Phi}(h;k,m) B_{\mu-1}\right)}{\mu} I + \widetilde{\Phi}(h;k,m) B_{\mu-1}. \end{aligned}$$

Then the corresponding control law (9.12) guarantees that the closed-loop characteristic roots are located at $\lambda^{(0)}$.

Proof. According to *Assertion 9.1(1)*, the coefficients of $P_{cl}(z)$ satisfy:

$$a_\mu(h;k,m) = \sum_{\nu=1}^{m} k_\nu \alpha_{\mu,\nu}(h,\tau_\nu) + \beta_\mu,$$

that is,

$$\frac{\partial a_\mu(h;k,m)}{\partial k_\nu} \equiv \alpha_{\mu,\nu}(h,\tau_\nu).$$

A straightforward application of *Theorem A.1* leads to:

$$\overline{\alpha}_\mu(h,\tau) = \alpha_{\mu,\nu}(h,\tau_\mu),$$

Assertion 9.1(2) allows concluding that the above equality is true for all ν. On the other hand, from *Assertion 9.1(1)*, $a_\mu(h,\tau,0) = \beta_\mu$. Then, a straightforward application of the *Induction Method* to:

$$\widetilde{\Phi}_{cl}(z;h,0,m) = \begin{bmatrix} zI - \Phi(h) & 0 \\ e_1^T & z \end{bmatrix},$$

shows that $\beta_\mu = \binom{n}{\mu-1}$. With this fact in mind, we have that $a_\mu(h;k,m) = (-1)^{n-\mu+1} s_{n-\mu+1}$, taking $\mu = \overline{0,n}$ and putting this in a matrix form we obtain (9.54). The proof is finished if we show that A is nonsingular. Singularity of A simply means that there exist some dependent row or column vectors. Then, a straightforward application of *Assertion 9.1(2)* and *Remark 9.4* implies that $\tau_1 = \cdots = \tau_{n+1}$. Since, by assumption we have that $\tau_1 < \cdots < \tau_{n+1}$, the proof is completed. □

9.8 Reduced Order Controller

We focus now on finding the control law $k = (k_1, \dots, k_n)$ such that the closed-loop characteristic polynomial becomes:

$$P(z;p,i_p) \triangleq z^{n+1} + p\left(z^{n-i_p+1} + z^{n-i_p} + \cdots + z + 1\right) \tag{9.55}$$

with $1 \le i_p \le n+1$ and $n \ge 1$. It is important to mention that the so-called *lacunary polynomials*, which are of the form (9.55), have received some attention in the literature in the context of delay-difference equations, see, e.g. [66]. Its main interest lies in interesting properties to be exploited in what follows:

Property 9.4. *The following properties hold for $P(z;p,i_p)$:*

(i) *the moduli of the roots increase as $\mid p \mid$ increases.*

(ii) *the roots are inside the unit circle if $|p| < \frac{1}{n-i_p+2}$.*

Proof. First, $P(z;p,i_p) \equiv z^{n+1} + p\frac{z^{n-i_p+2}-1}{z-1}$. Next, for (i), see [66]. (ii) Take now $f(z) \triangleq z^{n+1}$ and $g(z;i_p) \triangleq p\left(\sum_{k=0}^{n-i_p+1} z^k\right)$. For all $z \in \mathbb{C}$, we have that $|g(z;i_p)| = |p||\sum_{k=0}^{n-i_p+1} z^k| \le |p|\sum_{k=0}^{n-i_p+1}|z^k|$. Then, taking $|p| < \frac{1}{n-i_p+2}$ for $|z| = 1$ we have that $|f(z)| > |g(z;i_p)|$. Then, by a straightforward application of Rouché's lemma ([84]), $P(z;p,i_p) \equiv f(z) + g(z;i_p)$ is a Schur-stable polynomial. □

Remark 9.5. *The proof above guarantees not only the existence of some "stabilizing" parameter p, but it also gives a "cheap" way to compute it.*

Define now $\mathcal{X}$ as the set of real zeros of the polynomial:

$$T_{n+1}(x)\sum_{j=0}^{n-i_p} U_j(x) - U_n(x)\sum_{j=0}^{n-i_p+1} T_j(x), \tag{9.56}$$

(see the appendix for the definition of the Chebyshev polynomials T_n and U_n) and introduce the following quantities:

$$p^- = \max_{x^* \in \mathcal{X}}\left\{\frac{-U_n(x^*)}{\sum\limits_{j=0}^{n-i_p} U_j(x^*)} < 0\right\},\ p^+ = \min_{x^* \in \mathcal{X}}\left\{-\frac{U_n(x^*)}{\sum\limits_{j=0}^{n-i_p} U_j(x^*)} > 0\right\}. \tag{9.57}$$

Then we can state the following result:

Proposition 9.5. *The polynomial $P(z;p^*,i_p)$ is Schur stable if and only if*

$$\max\left\{\frac{-1}{n-i_p+2}, p^-\right\} < p^* < p^+. \tag{9.58}$$

Proof. The polynomial $P(z;0,i_p)$ is Schur. Now, since the roots of a polynomial are continuous with respect to their coefficients (see, e.g., [12, 117]), it follows the existence of some real p close to 0 such that $P(z;p,i_p)$ is still Schur stable. Moreover, in the limit case, there exists a $p^* \in [-1,1]$ such that $P(z^{(0)}; p^*, i_p) = 0 \Rightarrow z^{(0)} = e^{i\theta}$, $\theta \in [0,\ 2\pi)$. Then, $\Re\Big(P(z^{(0)}; p^*, i_p)\Big) = 0$ and $\frac{\Im\left(P(z^{(0)}; p^*, i_p)\right)}{\sin\theta} = 0$ lead to:

$$\begin{aligned} T_{n+1}(x) + p^* \sum_{j=0}^{n-i_p+1} T_j(x) &= 0, \qquad (9.59)\\ U_n(x) + p^* \sum_{j=0}^{n-i_p} U_j(x) &= 0, \qquad (9.60)\end{aligned}$$

where $x = \Re(z^{(0)}) = \cos\theta$. Equations (9.59)-(9.60) will give the whole set of solutions, except the singular point $z^{(0)} = 1$. In this last case, p^* can be obtained by solving $P(1; p^*, i_p) = 0$. Some simple algebraic manipulations lead to the conditions (9.58). □

Remark 9.6. *It follows from the first assertion of Property 9.4 that the condition (9.58) defines the whole set of solutions. Notice also that, (9.56) has at most n solutions. Finally, [66] proposed a different argument for proving a similar property.*

We can now state the main result of this subsection:

Proposition 9.6. *Let τ be the induced network delay, $\tau_2, \ldots, \tau_n$ chosen like $\tau_i = \tau + (i-1)\varepsilon$ for $i = \{2, \ldots, n\}$ and p^* be chosen satisfying (9.58) for some $1 \le i_p \le n+1$. Then, the control law (9.12) with,*

$$k(\varepsilon) = \bar{A}^{-1} p^T \tag{9.61}$$

where $\bar{A} \triangleq \left[\overline{\alpha}_\mu(h, \tau_\nu)\right]_{\mu,\nu=1}^{n}$,

$$p \triangleq \left[\begin{array}{ccccccc} p^* - (-1)^n \binom{n}{0} & \cdots & p^* - (-1)^{i_p+1}\binom{n}{n-i_p-1} & (-1)^{i_p+1}\binom{n}{n-i_p} & \cdots & \binom{n}{n-1} \end{array}\right],$$

guarantees the closed-loop stability, whenever ε satisfies,

$$p_0^- < \overline{\alpha}_0(h, \tau_1) k_1(\varepsilon) + \cdots + \overline{\alpha}_0(h, \tau_n) k_n(\varepsilon) < p_0^+, \tag{9.62}$$

for $\varepsilon > 0$, $h > \tau + (n-1)\varepsilon$, and where $p_0^{\pm}$ are given by:

$$p_0^- \triangleq \max\left\{ p_0, \left\{\begin{array}{ll} p^* - 1 & \text{if } n - i_p \in 2\mathbb{N} \\ -1 & \text{otherwise} \end{array}\right. \middle|\, p_0 < p^* \right\} \tag{9.63a}$$

$$p_0^+ \triangleq \min\{p_0 \,|\, p_0 > p^*\} \tag{9.63b}$$

where p_0 is the set given by,

$$p_0 \triangleq -\left\{ T_{n+1}(x^*) + p^* \sum_{l=1}^{n-i_p+1} T_l(x^*) \right\} \tag{9.64}$$

and x^* is a root of the following polynomial

$$U_n(x) + p^* \sum_{l=0}^{n-i_p} U_l(x). \tag{9.65}$$

Proof. According to *Proposition* 9.5, for the invertibility of $\overline{A}$ is sufficient to have $\tau_1 \neq \cdots \neq \tau_n$ and $h > \tau_i$ for all $i = \overline{1,n}$. Since this fact is fulfilled by hypothesis, $k(\varepsilon)$ is well defined. Then, let $k(\varepsilon)$ be given by (9.61). It is clear from *Proposition 9.4* that the closed-loop system will be rewritten as follows:

$$P_{cl}(z) = z^{n+1} + p^* \left(z^{n-i_p+1} + z^{n-i_p} + \cdots + z\right) + \widetilde{p^*}(\varepsilon),$$

where $\widetilde{p^*}(\varepsilon)$ is given by:

$$\widetilde{p^*}(\varepsilon) \triangleq \overline{\alpha}_0(h,\tau_1)\, k_1(\varepsilon) + \cdots + \overline{\alpha}_0(h,\tau_n)\, k_n(\varepsilon).$$

Since by assumption, p^* satisfies (9.58), we have that, for $\widetilde{p^*}(\varepsilon) = p^*$, the closed-loop system is asymptotically stable. Then, similarly to the proof of Proposition 9.5, there exists some interval $\left(p_0^-, p_0^+\right)$ including p^* such that the system remains asymptotically stable. In the limit case, $P_{cl}(e^{i\theta}) = 0$. Taking the corresponding real and imaginary parts ($\Re\left[P_{cl}(e^{i\theta})\right] = 0$, $\Im\left[P_{cl}(e^{i\theta})\right] = 0$) and using the Chebyshev polynomials, we obtain (9.64)-(9.65), respectively. Equation (9.64) gives the set of all possible intervals including p^*, excepting for the singular point $\theta = \pi$. At this point, we must have $p_0^- = p^* - 1$ whenever $n - i_p \in 2\mathbb{N}$. Then, in order to preserve the stability, we must choose the smallest interval, i.e., p_0 must be contained in the interval $\left(p_0^-, p_0^+\right)$ given by equation (9.63). This means that, if $p_0^- < \widetilde{p^*}(\varepsilon) < p_0^+$ the closed-loop system will be asymptotically stable. Since this is equivalent with equation (9.62), the proof is completed. □

9.9 Illustrative Examples

In order to illustrate how the present methodology works, we consider a fourth-order chain of integrators as:

$$y^{(4)}(t) = u(t), \quad t \in [\ell h + \tau_m, (\ell+1)h + \tau_m), \quad \tau_m < h \tag{9.66}$$

where $\tau = 0.1$ is the induced-network delay and $h = 0.6$ is the sampling period. Taking $m = 4$, $\tau_i = \tau + (i-1)\varepsilon$ for $i = \{\overline{2,4}\}$ in the control law (9.12), then applying *Proposition 9.5-9.6* we obtain $p \in (-0.25, 0.4450)$ and $p_0 \in (-0.7181, 0.8158)$ (where the later interval was obtained by choosing $p^* = 0.2$), respectively. Then, according to *Proposition 9.6*, the system (9.66) is asymptotically stable whenever $\varepsilon \in (0, 0.01202)$. In order to illustrate this result graphically, we plot the roots' trajectories for $p_0 \in (-0.7181, 0.8158)$ in Fig.9.5.

9.10 Concluding Remarks

In this chapter, the problem of stabilizing a chain of integrators by using network delays as controller parameters was addressed. Several algorithms and properties have been outlined and various illustrative examples proving the theoretical results have also been proposed. For the sake of brevity, only the case of delays smaller than the sampling period has been considered. However, the approach proposed here works also for the case of larger delays.

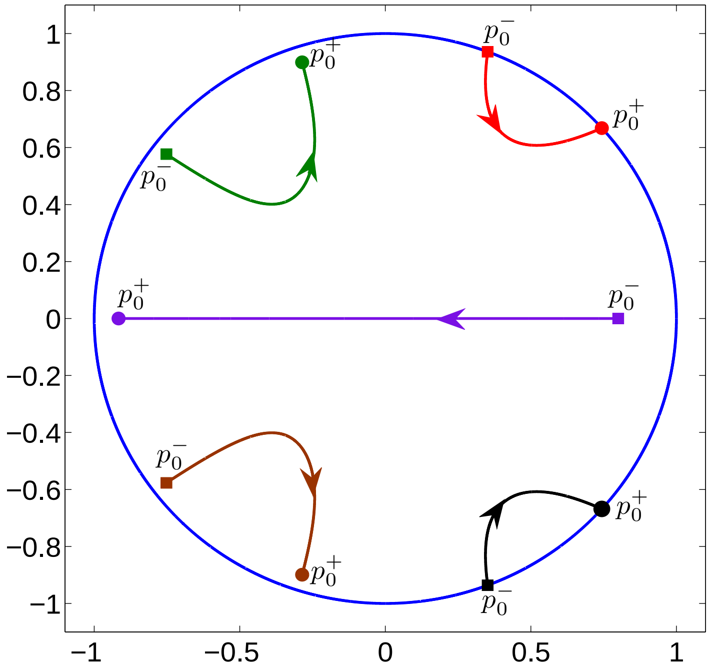


Figure 9.5: Root trajectories for $p_0 \in (-0.7181, 0.8158)$

10 Output Feedback Stabilization. An Eigenvalue Based Approach

As we saw in the previous chapter, in the case of a chain of integrators, the effects induced by the delay presence on the dynamics of NCS appear to be more complicated than expected: *lack of scaling properties* excepting the single integrator case, induced instability for small gain values (see also Fig.10.1 of this chapter, or *Example* 9.1 of the *Chapter* 9), etc. Next, the dependence of the characteristic roots on the delay value may lead to a sequence *stability/instability/stability* if one increases the delay continuously within one sampling period. Such a property is observed for a simple second-order system (see, for instance, system (10.14)). Roughly speaking, the interest of both examples is to point out some *sensitivity* properties of the characteristic roots with respect to the gain- or the delay-parameter, respectively. Such topics will constitute the core of this chapter.

First, inspired by the terminology introduced by [40], we focus on the characterization of the *crossing (frequency) set*, that is, the set of parameters (delay, gain, sampling) (or only (delay, gain) if the sampling is fixed, etc.) for which there exists at least one *critical*[3] *characteristic root*. Next, we explore conditions under which a *gain* may stabilize the corresponding SISO NCS scheme and we will see that the gain stabilization problem is reduced to the *generalized eigenvalues* computation of an appropriate *matrix pencil*. Such a result ca be interpreted as the "discrete-time" version of the analysis proposed by [20] for characterizing the stabilizing gains of "continuous-time" linear LTI SISO systems free of delays. However it is worth to mention that, in the discrete case, not all real generalized eigenvalues defines crossings with respect to the unit circle (see Proposition 10.2).

[3]By a *critical characteristic root*, we mean a root of the corresponding characteristic equation located on the unit circle of the complex plane $\mathbb{C}$.

Next, an *eigenvalue-based perturbation* methodology (see, e.g., [67]) is adopted for characterizing the *crossing direction* towards stability and/or instability. If the analysis with respect to the gain-parameter is relatively simple (see, e.g., Propositions 10.2 and 10.3), the analysis becomes more involved when the delay-parameter is considered. More precisely, if the delay changes around a multiple of the sampling period, additional characteristic roots may appear (or disappear) and a continuity-type argument explains how the characteristic roots behave with respect to the delay parameter (see Proposition 10.5). Finally, the characterization of the crossing direction with respect to the delay-parameter is given in the simple (Corollary 10.1), semi-simple (Proposition 10.8) and non semi-simple (Proposition 10.10) cases. These last results follow closely the arguments proposed by [21] in continuous-time for the stability analysis of delay systems.

10.1 Problem Formulation and Motivating Examples

Consider the following continuous-time linear SISO system:

$$\begin{cases} \dot{x}(t) = Ax(t) + b\widehat{u}(t), & t \in [\ell h + \widetilde{\tau}, (\ell+1)h + \widetilde{\tau}), \\ y(t) = cx(t), \end{cases} \tag{10.1}$$

and the discrete control law,

$$\widehat{u}\left(t^{+}\right) = -ky(t - \widetilde{\tau}), \quad t \in \{\ell h + \widetilde{\tau},\ \ell \in \mathbb{N}\} \tag{10.2}$$

where $\widetilde{\tau}$ is the induced network delay satisfying $\widetilde{\tau} = (r-1)h + \tau$, for $0 < \tau \leq h$, $x \in \mathbb{R}^n$, $k, y \in \mathbb{R}$ and $r \in \mathbb{N}$. As mentioned previously, we are interested in *finding all parameters* (k,h,τ) *such that the controller (10.2) (asymptotically) stabilizes the closed-loop SISO system (10.1).* As in the previous chapter, the system (10.1) includes both continuous- and discrete-time dynamics, the classical analysis consists in discretizing the entire system in order to homogenize the state variables. To this end, we apply similar ideas to the ones proposed by [6, 137], leading to the representation:

$$x[\ell+1] = \Phi(h)x[\ell] + \Phi(h-\tau)\Gamma(\tau)b\widehat{u}[\ell-r] + \Gamma(h-\tau)b\widehat{u}[\ell-r+1] \tag{10.3}$$

where $\Gamma(t) := \int_0^t e^{As}ds$, $\Phi(t) := e^{At}$, $\widetilde{\tau} = (r-1)h + \tau$ with $r \in \mathbb{N}$ and $0 < \tau \leq h$. Define now the augmented state vector as $z[\ell] \triangleq \left[x^T[\ell], u[\ell-r], u[\ell-r+1], \ldots, u[\ell-1]\right]^T$, leading to the augmented closed-loop system:

$$z[\ell+1] = \widetilde{\Phi}(p)z[\ell]. \tag{10.4}$$

The representation (10.3) describes a general situation, that is, when the induced network delay is larger/smaller than the sampling period. Under these observations, we have the following:

Remark 10.1 (Smaller delay). *If the induced network delay* $\widetilde{\tau}$ *satisfies the condition* $0 < \widetilde{\tau} \leq h$*, then,* $\widetilde{\tau} = \tau$*,* $r = 1$ *in (10.3) and the corresponding transfer matrix rewrites as:*

$$\widetilde{\Phi}(p) := \begin{bmatrix} \Phi(h) - k\Gamma(h-\tau)bc & \Phi(h-\tau)\Gamma(\tau)b \\ -k\boldsymbol{c} & 0 \end{bmatrix}. \tag{10.5}$$

Remark 10.2 (Larger delay). *If the induced network delay $\widetilde{\tau}$ satisfies $\widetilde{\tau} = (r-1)h + \tau$ for some positive integer $r > 1$, with $\tau \leq h$, then the corresponding transfer matrix becomes:*

$$\widetilde{\Phi}(p) := \begin{bmatrix} \Phi(h) & \Phi(h-\tau)\Gamma(\tau)b & \Gamma(h-\tau)b & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -kc & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{10.6}$$

10.2 Transfer Function Description

10.2.1 Smaller delay case:

Assume that $0 < \widetilde{\tau} \leq h$, that is $\widetilde{\tau} = \tau$. Then, the transfer function can be written as $H_{yu}(z; p_o) := N(z; p_o)/D(z; p_o) =$

$$= c(zI - \Phi(h))^{-1} \left[\frac{\Phi(h-\tau)\Gamma(\tau)b}{z} + \Gamma(h-\tau)b \right]. \tag{10.7}$$

Now, by taking into account the control law (10.2), the characteristic function of the closed-loop system becomes:

$$F(z; p) = D(z; p_o) + kN(z; p_o). \tag{10.8}$$

10.2.2 Larger delay case:

Now, if $\widetilde{\tau} = (r-1)h + \tau$, with $r > 1$, then the corresponding transfer function is given by:

$$\begin{aligned} \widetilde{H}_{yu} &= c(zI - \Phi(h))^{-1} \left[\Phi(h-\tau)\Gamma(\tau)bz^{-r} + \Gamma(h-\tau)bz^{1-r} \right], \\ \widetilde{H}_{yu} &\equiv z^{1-r} H(z; p_o) \equiv \frac{N(z; p_o)}{z^{r-1} D(z; p_o)}, \end{aligned} \tag{10.9}$$

leading to the following closed-loop characteristic function:

$$F_{r-1}(z; p) = z^{r-1} D(z; p_o) + kN(z; p_o). \tag{10.10}$$

From (10.10) we have that $F_0(z; p) \equiv F(z; p)$. Note also that for $\widetilde{\tau} > h$ the characteristic function of the closed-loop system is affect by z^{r-1}, motivating thus the notation $D_{r-1}(z; p_o) := z^{r-1} D(z; p_o)$.

10.3 Motivating Examples

10.3.1 Chain of Integrators Systems

Even though in the previous chapter we have deeply studied a chain of integrators system, we will recall in the following, some interesting properties. Consider the chain of integrators system,

$$y^{(n)}(t) = \widehat{u}(t^+) \qquad t \in [\ell h + \tau, (\ell+1)h + \tau), \tag{10.11}$$

where $\widehat{u}(t^+)$ is given by,

$$\widehat{u}\left(t^+\right) = -ky(t-\tau), \quad t \in \{\ell h + \tau,\ \ell \in \mathbb{N}\}. \tag{10.12}$$

In the case of NCS stabilization of one integrator ($n=1$) [137] derived the stability region in the $(h(k), \tau)$ parameter space, and the NCS is *stable if and only if*:

$$\max\left\{\frac{1}{2}hk - 1, 0\right\} < k\tau < \min\{1, hk\}. \tag{10.13}$$

Unfortunately, the above result is *not longer valid* for higher-order systems ($n \geq 2$), as it is stated in the previous part of the thesis (*Proposition* 9.2). In other words, a *small gain* is always destabilizing. In order to understand better the property above, consider now the triple-integrator case ($n=3$). In other words, a *small gain* is always destabilizing. In order to understand better the property above, consider now the triple-integrator case ($n=3$).In other words, a *small gain* is always destabilizing. In order to understand better the property above, consider now the triple-integrator case ($n=3$). Then, for small values of the parameter gain,

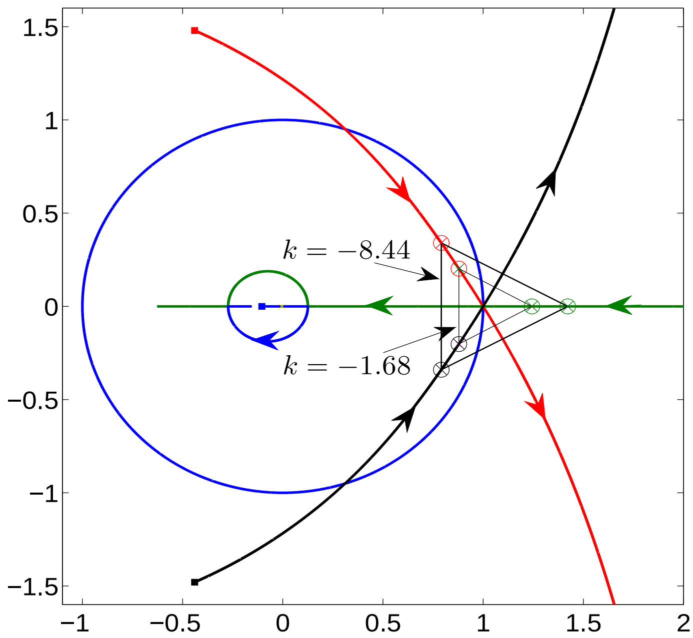


Figure 10.1: Triple-integrator $(n=3)$ subject to $k<0$.

i.e., $k=\varepsilon$, the closed-loop characteristic roots behave as:

$$\begin{aligned} z_1(\varepsilon) &= w_{0,3}\varepsilon + \frac{w_{0,3}\left(-w_{1,3}^3+3h^2w_{0,1}+3!hw_{0,2}\right)}{3!}\varepsilon^2 + \mathcal{O}\left(\varepsilon^3\right) \\ z_{\ell+1}(\varepsilon) &= 1 + w_{1,\ell}\varepsilon^{\frac{1}{3}} + \frac{2w_{1,3}^3+h^2w_{0,1}-w_{1,\ell}^3}{3w_1(\ell)}\varepsilon^{\frac{2}{3}} + \mathcal{O}(\varepsilon) \end{aligned}$$

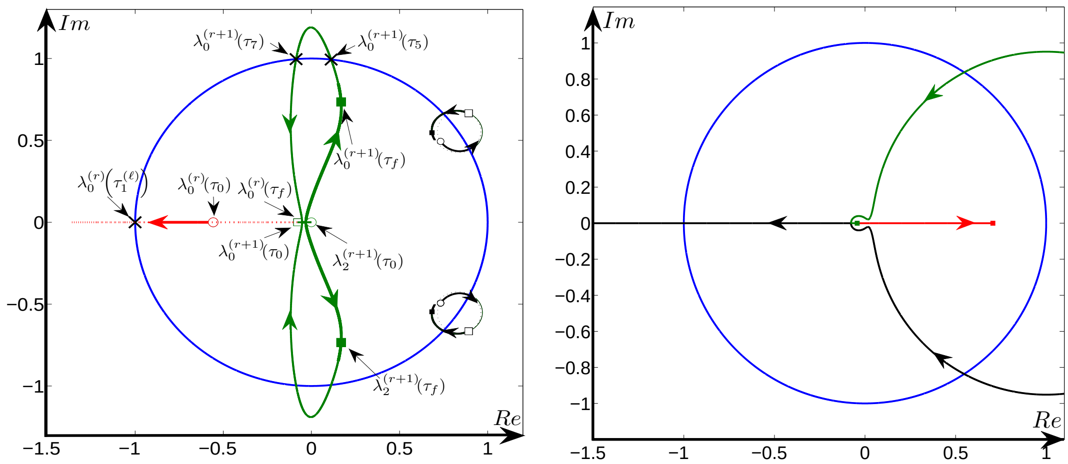
where, $w_{0,\ell} := \frac{\tau^\ell}{\ell!}$, $w_{1,\ell} := e^{\frac{\ell\pi}{3}j}h$ and $\ell = \overline{1,3}$. Fig.10.1 illustrates the above result for $k<0$ (*Chapter* 9 illustrates the closed-loop characteristic root behavior for $k>0$).

10.3.2 Second-Order Systems

In continuous-time, the interest of oscillatory systems is well-known since it is the simplest example pointing out the *stabilizing* effect of the delay in output feedback control schemes (see, for instance, [1, 103] and the comments therein). Consider now the following system:

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} 0 & 1 \\ a_1 & a_2 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \widehat{u}(t),\ t \in [\ell h + \tau, (\ell+1)h + \tau), \\ y(t) = \begin{bmatrix} c_1 & c_2 \end{bmatrix} x(t). \end{cases} \tag{10.14}$$

Now consider that $(a_1, a_2, c_1, c_2) = \left(-\omega_0^2, 0, \omega_0, 0\right)$, with $\omega_0 = 3$, and assume the sampling $h_0 = 6.1$ and the gain $k_0 = 5.2$. Some simple computations prove that if one chooses the delay parameter $\tau_0 = 2.5$, and we take the network delay $\widetilde{\tau} = \tau_0$, then the characteristic function of the closed-loop system $F_0(z;p)$ is stable. Next, if the network delay is assumed to be one sampling larger, i.e. $\widetilde{\tau} = h_0 + \tau_0$, the corresponding characteristic function becomes $F_1(z;p)$ which is still stable. However, if one varies the network delay $\widetilde{\tau}$ between τ_0 and $h_0 + \tau_0$, for some critical delay values, the characteristic roots will cross $\partial\mathbb{D}$ towards instability and for larger network delays they will cross $\partial\mathbb{D}$ back towards stability. Such a situation is depicted in Fig.10.2(a) and reflects the complex behavior of the characteristic roots with respect to the delay parameter. In other words, increasing the network delay parameter leads to a sequence of *stability/instability/stability* within one sampling period. This example will be further discussed in the forthcoming paragraphs. Consider now as a final motivating example, the case:



(a) (b)

Figure 10.2: Characteristic roots behavior as a function of the delay parameter. (a) Second-order oscillator $(a_1, a_2, c_1, c_2) = \left(-\omega_0^2, 0, \omega_0, 0\right)$. (b) Second-order system $(a_1, a_2, c_1, c_2) = (-1, -1, 0, 1)$.

$(a_1, a_2, c_1, c_2) = (-1, -1, 0, 1)$. According to [22], it is known that for the delay $\tau = \pi$, the continuous-time system has a *tangential* root-trajectory, i.e., the root-trajectory (as a function of the delay) touches in a tangential way the imaginary axis. Taking a sampling period $h = 2\pi$, we have that the above property is not longer valid, i.e., we have a simple crossing as it is illustrated in Fig.10.2(b).

10.4 Stability Analysis

10.5 Spectral Radius Properties

In this paragraph we consider first, the continuity property of the spectra of the augmented closed-loop system (10.4) with respect to the delay parameter. To this end, introduce now the following definition:

Definition 10.1. *Let (h^*, k^*) be fixed, the Spectral Radius (function) $\rho : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is defined by:*

$$\rho(\tau) := \sup\left\{|z| : z \in \sigma\left(\widetilde{\Phi}(h^*, \tau, k^*)\right)\right\}.$$

Property 10.1. *The spectral radius function has the following properties:*

- *it always exists;*
- *it is finite;*
- *it is continuous.*

Proof. First observe that for each $\ell \in \mathbb{N}$, the characteristic function of the closed-loop system can be written as,

$$F_{\ell-1}(z;p) = z^{n+\ell} + f_{n+\ell-1}(p) z^{n+\ell-1} + \cdots + f_1(p) z + f_0(p), \tag{10.15}$$

where $f_i(p)$ are analytical functions for all $p \in \mathbb{R}_+^2 \times \mathbb{R}$, then the first two properties follows straightforwardly from the properties of analytic functions (see, for instance [78]). Consider now the continuity property. To this end, we introduce the following intervals $\mathcal{I}_\ell := (\ell h + \tau, (\ell+1)h]$. Observe that when $\widetilde{\tau} \in \mathcal{I}_\ell$ the characteristic function (10.15) has constant degree, since $f_i(p)$ are analytical, then according to [84] we have that the zeros $z_i(p)$ of (10.15) are continuous functions with respect to the parameter p, implying that spectral radius function is continuous for all $\ell \in \mathbb{N}$. Consider now $\widetilde{\tau}_\ell \in \mathcal{I}_\ell$ and $\widetilde{\tau}_{\ell+1} \in \mathcal{I}_{\ell+1}$, then the proof will be complete if $\lim\limits_{\widetilde{\tau}_\ell \to (\ell+1)h^-} \rho(\widetilde{\tau}_\ell) = \lim\limits_{\widetilde{\tau}_{\ell+1} \to (\ell+1)h^+} \rho(\widetilde{\tau}_{\ell+1})$. In order to see this fact, is necessary to consider a *degree-normalization*. To this end and without any loss of generality let $\widetilde{\tau}_\ell \in \mathcal{I}_\ell$, for $\ell > 1$ (i.e., the larger delay case) and denote by $\widetilde{\Phi}(p^*;\ell)$ its respective transfer matrix, we consider the following definition,

$$\overline{\Phi}(h^*, \tau, k^*) := \begin{bmatrix} \Phi(h^*) & 0 & \Phi(h^* - \tau)\Gamma(\tau) b & \Gamma(h^* - \tau) b & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ -kc & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}. \tag{10.16}$$

From (10.16) it is clear to see that $\widetilde{\Phi}(p^*;\ell)$ is *Schur-stable* if and only if $\overline{\Phi}(h^*, \tau, k^*)$ is *Schur-stable*, bearing in mind this fact, we have the following consequences:

$$\begin{aligned} \lim_{\tau \to h} \overline{\Phi}(h^*, \tau, k^*) &= \lim_{\tau \to 0} \widetilde{\Phi}(h^*, \tau, k^*; \ell+1) \\ \Rightarrow \lim_{\widetilde{\tau}_\ell \to (\ell+1)h^-} \rho(\widetilde{\tau}_\ell) &= \lim_{\widetilde{\tau}_{\ell+1} \to (\ell+1)h^+} \rho(\widetilde{\tau}_{\ell+1}). \end{aligned}$$

Then, the proof is complete. □

10.6 Crossing Set Characterization

Denote by $F(z;p)$ the characteristic function of the closed-loop system. As mentioned in the previous section, for smaller/larger networks delays, F can be either $F_{r-1}(z;p) = \sum_{i=0}^{n+r} a_i^{(r-1)}(p) z^i$ (larger delay) or $F_0(z;p)$ (smaller delay). Inspired by [40], we introduce the following notions:

Definition 10.2. *The crossing set Θ, is defined as the collection of all frequencies $\theta \in [0, 2\pi)$ such that there exist a triplet p^* of parameters such that $F_{r-1}\left(e^{i\theta}; p^*\right) = 0$. The collection of all triplets p^* corresponding to crossing frequencies in Θ will define the stability crossing surfaces $\mathcal{S} \subset \mathbb{R}_+^2 \times \mathbb{R}$. Finally, for a fixed sampling h_0, the restriction $\mathcal{S}|_{p=p_{h_0}}$ denotes the corresponding stability crossing curves.*

These critical values can be computed in the following way. Associate now to F_{r-1}, the following parameter-dependent matrices $M_1, M_2 \in \mathbb{R}^{(n+r)\times(n+r)}$:

$$M_1(p) := \begin{bmatrix} a_0(p) & 0 & \cdots & 0 \\ a_1(p) & a_0(p) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+r-1}(p) & a_{n+r-2}(p) & \cdots & a_0(p) \end{bmatrix} \tag{10.17}$$

$$M_2^T(p) := \begin{bmatrix} a_{n+r}(p) & a_{n+r-1}(p) & \cdots & a_1(p) \\ 0 & a_{n+r}(p) & \cdots & a_2(p) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{n+r}(p) \end{bmatrix}, \tag{10.18}$$

and introduce $\mathcal{P} := (h, \tau, k) \subset \mathbb{R}_+^2 \times \mathbb{R}$ as the set of all parameters p satisfying the following equality:

$$\det W_F(p) = 0, \tag{10.19}$$

where

$$W_F(p) := M_2(p) M_2^T(p) - M_1(p) M_1^T(p). \tag{10.20}$$

We have the following:

Proposition 10.1. *Consider the system (10.1) in closed-loop having the characteristic function $F(z;p)$. We have the following properties:*

(a) *p^* is a crossing point ($p^* \in \mathcal{S}$) if and only if the following conditions*

  (i) $0 \in \sigma(W_F(p^*))$;

  (ii) $\sigma\left(\widetilde{\Phi}(p^*)\right) \cap \partial\mathbb{D} \neq \emptyset$.

  hold simultaneously.

(b) *If for some fixed pair (h^*, k^*), $F_\ell(z; h^*, \tau, k^*)|_{\tau=0}$ is Schur-stable, then $(h^*, \tau_{min}, k^*) \in \mathcal{S}$, where τ_{min} is the minimal value of $\Lambda_m := \{\tau \in \mathbb{R}_+ : \sigma(W_{F_\ell}(\tau)) = 0\}$.*

Remark 10.3. *It is important to point out that (i) is necessary but not sufficient for the existence of a crossing since excepting the real crossing points the determinant above vanishes also for symmetric points with respect to the unit circle $\partial\mathbb{D}$[4]. Indeed, the following example illustrate such a situation:*

Consider now the second-order system (10.14), with $(a_1, a_2, c_1, c_2) = (-1, -1, 0, 1)$ and corresponding open-loop transfer function:

$$H_{yu}(z;,p_o) = \frac{\frac{2e^{-\frac{h}{2}+\frac{\tau}{2}}\sin\left[\frac{\sqrt{3}}{2}(h-\tau)\right]}{\sqrt{3}}z^2 + \frac{2e^{-h+\frac{\tau}{2}}\left(-e^{h/2}\sin\left[\frac{\sqrt{3}}{2}(h-\tau)\right]+\sin\left[\frac{\sqrt{3}\tau}{2}\right]\right)}{\sqrt{3}}z - \frac{2e^{-h+\frac{\tau}{2}}\sin\left[\frac{\sqrt{3}\tau}{2}\right]}{\sqrt{3}}}{z^3 - 2e^{-h/2}\cos\left[\frac{\sqrt{3}h}{2}\right]z^2 + e^{-h}z}.$$

Then the characteristic function of the closed-loop system will depend on the parameters $p^* = (h_0, \tau_0, k_0)$ and writes as:

$$F(z;p^*) := D(z,p_o) + k_0 N(z,p_o). \tag{10.21}$$

In this case, by choosing $p^* = (2\pi, \pi, 227.1017)$, $\det W_F$ vanishes without corresponding to the existence of a crossing frequency in Θ. Fig.10.3 illustrates the symmetry mentioned above.

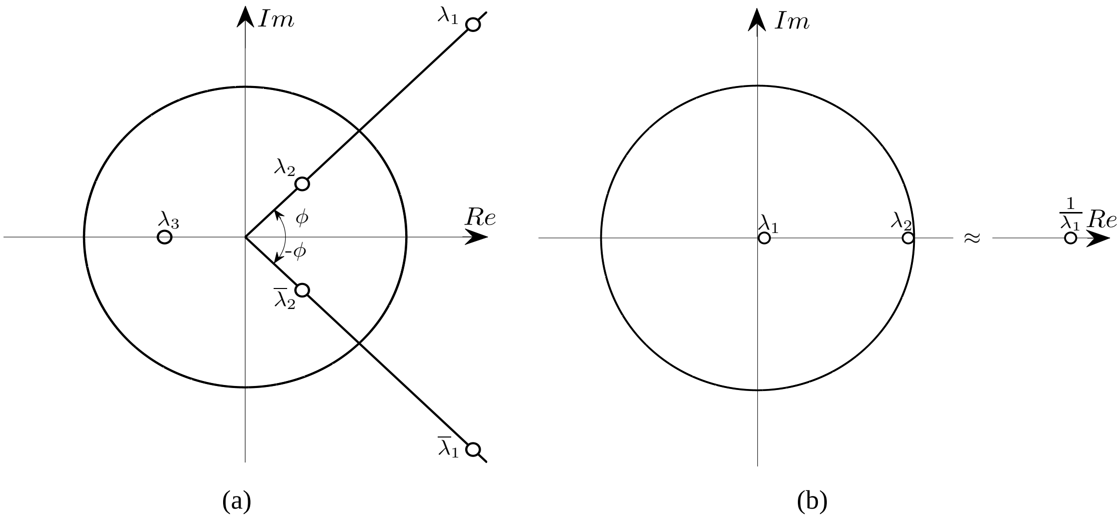


Figure 10.3: Symmetric roots with respect to $\partial\mathbb{D}$. (a) For the open-loop transfer function $H_{yu}(z;\tau_0) = \frac{(25-20e^{\tau_0})z^2+(8-10e^{\tau_0})z+4}{8z^5+(4-20e^{\tau_0})z^4+(50-10e^{\tau_0})z^3}$, with $(k_0, \tau_0) = (1, \log\sqrt{2})$, given the symmetric roots: $|\lambda_1| = 2$, $|\lambda_2| = |\lambda_3| = \frac{1}{2}$ and $\phi = \frac{\pi}{4}$. (b) For the polynomial (10.21), where $|\lambda_1| = 0.0432$, $|\lambda_2| = 0.9624$.

Remark 10.4. *The above result follows from a straightforward application of the Schur-Cohn-Fujiwara result (see, for instance, [10]).*

Remark 10.5. *Note that the above result can be also applied to the computation of the stability crossing curves $\mathcal{S}|_{h=h_0}$ for some sampling h_0.*

[4]Here, we say that two points z_1 and $z_2 \in \mathbb{C}$ are *symmetric* with respect to a circle of radius R and center located at z_0 if: $| z_1 - z_0 | \cdot | z_2 - z_0 | = R^2$, etc.

Proof. (a) According to [84], $F_{r-1}(z;p^*)$ and $F^*_{r-1}(z;p^*) := z^{n+r}\overline{F_{r-1}(\frac{1}{z};p^*)}$ has a common root if and only if the determinant of the following resultant matrix,

$$R_{F_{r-1},F^*_{r-1}} := \begin{bmatrix} M_1(p) & M_2^T(p) \\ M_2(p) & M_1^T(p) \end{bmatrix} \tag{10.22}$$

has the property $\det\left(R_{F_{r-1}}\right) = 0$. Now, a common root of $F_{r-1}(z;p^*)$ and $F^*_{r-1}(z;p^*)$ means that $F_{r-1}(z;p^*)$ can be factored as $F_{r-1}(z;p^*) = \psi(z)f(z)$, with:

$$\psi(z) = \prod_{j}^{\nu}(z-\rho_j e^{i\phi_j})(z-\frac{1}{\rho_j}e^{-i\phi_j}) \tag{10.23}$$

and $f(z)$ collect the rest of the roots belonging to the unit circle. On the other hand, the fact that $M_1(p)$ and $M_2(p)$ commute, implies that $\det R_{F_{r-1},F^*_{r-1}} \equiv -\det W_{F_{r-1}}(p)$. Finally, by observing that condition $\sigma\left(\widetilde{\Phi}(p^*)\right) \cap \partial\mathbb{D} \neq \emptyset$ excludes all solutions of the form $(z-\rho_j e^{i\phi_j})(z-\frac{1}{\rho_j}e^{i\phi_j})$, the proof is complete.

(b) Let us prove this property by contradiction, i.e. assume that $(h^*,\tau_{min},k^*) \notin \mathcal{S}$, then there exist $z^{(0)} \in \mathbb{C}$ such that $F_\ell(z^{(0)};h^*,\tau_{min},k^*) = F_\ell(\frac{1}{\overline{z}^{(0)}};h^*,\tau_{min},k^*) = 0$, where $z^{(0)}$ is symmetric with respect to the unit circle. But this implies that $F_\ell(z^{(0)};h^*,\tau_{min},k^*)$ is unstable, contradicting the fact that the roots of $F_\ell(z;h^*,\tau,k^*)$ are continuous with respect to the delay argument.

□

10.7 Stabilizing Gains and Corresponding Crossing Directions

Consider now that the parameters (h,τ) are fixed. Then we are interested in developing an algorithm to compute the *set of all stabilizing gains* $k \in \mathbb{R}$. By an abuse of notation, we will construct the $(n+r)\times(n+r)$ matrices (10.17)-(10.18) for lower order polynomials by setting the coefficient of higher order terms as zeros. In order to simplify the notations, we will define the resultant of any polynomial $F_{r-1}(z;p)$ with its associated polynomial $F^*_{r-1}(z;p)$ by $R_{F_{r-1}}(p) := R_{F_{r-1},F^*_{r-1}}(p)$. Then, we have the following result:

Proposition 10.2. *Assume (h,τ) a fixed and known pair $p_o := (h_0,\tau_0)$. Introduce the sets $\Lambda = \sigma\left(R_{D_{r-1}}(p_o), -R_N(p_o)\right) \cap \mathbb{R}$,*

$$\Lambda_s := \left\{\xi \in \Lambda : \quad \exists \eta \in \Lambda,\ \xi\eta = 1, \sigma\left(\widetilde{\Phi}(p_o,\xi)\right) \cap \partial\mathbb{D} = \emptyset\right\}.$$

Let $\lambda_1 < \ldots < \lambda_\ell$, with $\ell \leq n+r$ and $\lambda_i \in \Lambda - \Lambda_s$. Then, the system (10.3) cannot be stabilized for any $k = \lambda_i$, $i = 1,2,\ldots,\ell$. Furthermore, the number of unstable roots remains invariant for all $k \in (\lambda_i,\lambda_{i+1})$. The same holds for the intervals $(-\infty,\lambda_1)$ and (λ_ℓ,∞).

Proof. First, by construction, the set Λ_s includes the real spectrum of the matrix pencil $R_F(p_o,\lambda) = R_{D_{r-1}}(p_o) + \lambda R_{N_{r-1}}(p_o)$ having the property of symmetry with respect to $\partial\mathbb{D}$, but without being characteristic roots of the closed-loop system. Thus, the set $\Lambda - \Lambda_s$ collects all the gains for which some crossing with respect to $\partial\mathbb{D}$ exists. Then, the remaining proof is a direct consequence of *Proposition* 10.1. □

Remark 10.6. *The result above can be simply interpreted as the "discrete-time" version of the matrix pencil approach proposed by [20] in the output feedback stabilization of SISO LTI systems.*

Remark 10.7. *Notice that the above characterization for the gain controllers can also be obtained by applying the Neimark $\mathcal{D}$-partition method ([32]). According to this method we have,*

$$\begin{aligned} k(\theta) &= -\frac{e^{i(r-1)\theta} D_{r-1}\left(e^{i\theta}, p_o^*\right)}{N\left(e^{i\theta}, p_o^*\right)}, && (10.24) \\ Im\left(k(\theta)\right) &= 0, && (10.25) \end{aligned}$$

where $\theta \in [0, \pi]$.

In order to illustrate the above results, consider the second-order system (10.14) with $(a_1, a_2, c_1, c_2) = (-\omega_0^2, 0, \omega_0, 0)$ corresponding to the second-order oscillator system (with $\omega_0 = 3$). Taking the induced-network delay as $\widetilde{\tau} = h + \tau$, we obtain the following discretized open-loop transfer function:

$$H_{yu}(z, p_o) = \frac{n_2(p_o)z^2 + n_1(p_o)z + n_0(\tau)}{3z^3 - 6\cos(3h)z^2 + 3z}, \tag{10.26}$$

where $n_2(h,\tau) = 1 - \cos(3h - 3\tau)$, $n_1(h,\tau) = \cos(3h - 3\tau) - 2\cos(3h) + \cos(3\tau)$ and $n_0(\tau) = 1 - \cos(3\tau)$. Then, applying *Proposition* 10.2 with $p_o = \left(\frac{19}{25}, \frac{16}{25}\right)$, we obtain the results summarized in Table 11.3. Let denote by $\mathcal{D}(\ell, n + r - \ell)$ the interval with $\ell-$stable zeros and $(n + r - \ell)-$unstable zeros. Then, according to *Remark* 10.7, we apply Neimark $\mathcal{D}-$*partition* method to compute $k(\theta)$, obtaining Fig.10.4, where we obtain the same results than showed in Table 11.3.

Table 10.1: Generalized eigenvalues for $(h, \tau) = (\frac{19}{25}, \frac{16}{25})$.

	λ_1	λ_2	λ_3	λ_4	λ_5	
	-3	0	2.1061	2.7302	4.2710	
Stability Interval[a]	$(-\infty, -3)$	$(-3, 0)$	$(0, 2.10)$	$(2.11, 2.73)$	$(2.74, 4.27)$	$(4.28, \infty)$

[a] This is not accurate, owing to numerical rounding.

Proposition 10.2 explicitly gives the set of all controller gains with corresponding characteristic roots on the unitary circle $\partial\mathbb{D}$. Then, assuming first that the controller gain parameter is fixed to some critical gain value k^* for which there exists at least one critical characteristic root on $\partial\mathbb{D}$, the characterization of the crossing directions is given as follows:

Proposition 10.3. *Assume that the sampling h and the delay τ are known and fixed at h_0 and τ_0, respectively. Let $k = k^*$ be a critical gain for the crossing frequency $\theta = \theta^*$. Under the assumption that the critical characteristic roots of F are simple, the following statements are equivalent:*

(i) The root $z^* = e^{i\theta^*}$ is crossing $\partial\mathbb{D}$ towards instability (stability).

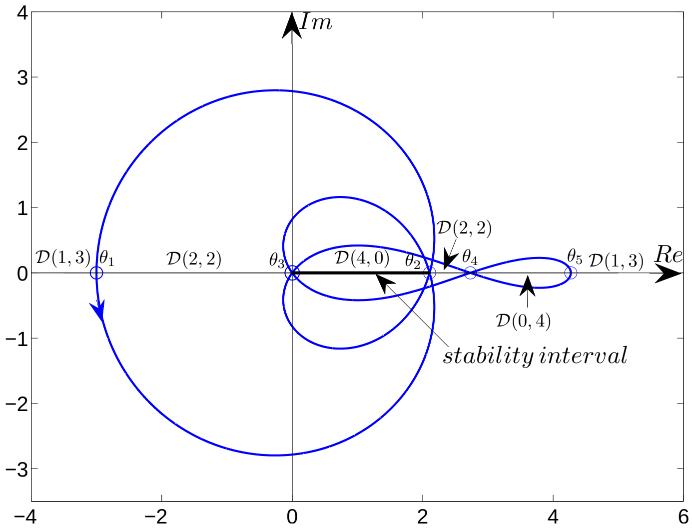


Figure 10.4: Stability intervals $\mathcal{D}(\ell, n+r-\ell)$ for (10.26), where $k(\theta_i) = \lambda_j$ for appropriated i and j.

(ii) The following inequality holds:

$$\left.\frac{d\mid z\mid}{dk}\right|_{k=k^*} > 0 \quad (<0),$$

for any k sufficiently close to k^*, but $k > k^*$.

(iii) The following inequality holds:

$$\Re\left\{\frac{\frac{dF\left(e^{\mathbf{i}\theta^*};k,p_o\right)}{dk}\Big|_{k=k^*}}{z\frac{dF(z,p^*)}{dz}\Big|_{z=e^{\mathbf{i}\theta^*}}}\right\} < 0 \quad (>0), \tag{10.27}$$

then for any k sufficiently close to k^* but $k > k^*$.

Proof. The equivalence between (i) and (ii) is based on the use of the implicit function theorem. The same holds for the equivalence between (ii) and (iii). □

Remark 10.8. *Observe that $F(z;p^*)$ is affine in k. Then the direction of crossing can also be obtained by a straight application of the Neimark $\mathcal{D}$-partition method. To this end, consider $\theta^* \in \Theta$. Then when $k(\theta)$ "crosses" the "point" θ^* a pair of complex roots of the characteristic closed-loop equation will go outside (inside) of $\partial\mathbb{D}$ according to the rule:*

$$\left.\frac{dIm(k(\theta))}{d\theta}\right|_{\theta=\theta^*} > 0 \quad (<0). \tag{10.28}$$

In order to illustrate the previous methodologies, (*Proposition* 10.3 and $\mathcal{D}-$partition method), lets consider example (10.26) with the same parameters $p_o = (\frac{19}{25}, \frac{16}{25})$. After evaluating conditions (10.27) and (10.28), Table 10.2 summarize the results. In some situations the inequality (10.27) will vanish. In such a case, the following corollary provides a second-order analysis.

Table 10.2: Direction of crossing for example, according to the $\mathcal{D}$-partition method and to *Proposition* 10.3.

Method		$\ell = 1$	2	3	4	5
	θ_ℓ	0	1.3190	2.28	2.8422	π
$\mathcal{D}$-Partition	$\frac{d\,Im[k(\theta)]}{d\theta}\Big\vert_{\theta=\theta_\ell}$	-7.1609	4.9809	-2.67247	2.2955	-2.5989
	sign (10.28)	$-$	$+$	$-$	$+$	$-$
	λ_ℓ	-3	0	2.1061	2.7302	4.271
Proposition (10.3)	condition (10.27)	0.1396	0.2451	-0.1858	-0.0439	0.3847
	sign (10.27)	$+$	$+$	$-$	$-$	$+$

Proposition 10.4. *Assume that the sampling h and the delay τ are known and fixed at h_0 and τ_0, respectively. Let $k = k^*$ be a critical gain for the crossing frequency $\theta = \theta^*$. Under the assumption that the critical characteristic roots of F are simple, the following statements are equivalent:*

(i) The root $z^* = e^{i\theta^*}$ stays outside (inside) of the unit circle $\partial\mathbb{D}$.

(ii) The following inequality holds:

$$\left.\frac{d^2 \mid z \mid}{dk^2}\right|_{k=k^*} > 0 \quad (<0),$$

for any k sufficiently close to k^*, but $k > k^*$.

(iii) The following inequality holds:

$$\Re\left\{\left.\frac{2\frac{\partial^2 F}{\partial z \partial k}\frac{\partial F}{\partial k}\frac{\partial F}{\partial z} - \frac{\partial^2 F}{\partial z^2}\left(\frac{\partial F}{\partial k}\right)^2}{z\left(\frac{\partial F}{\partial z}\right)^3}\right|_{\substack{z=z^*\\k=k^*}}\right\} + \Im\left\{\frac{\frac{dF\left(e^{i\theta^*};k,p_o^*\right)}{dk}\big|_{k=k^*}}{z\frac{dF(z;k^*,p_o^*)}{dz}\big|_{z=e^{i\theta^*}}}\right\}^2 > 0 \quad (<0) \quad (10.29)$$

then for any k sufficiently close to k^* but $k > k^*$.

Proof. The proof follows similar lines than *Proposition* 10.3, but with more complicate algebraic manipulations. □

Remark 10.9. *Note that if condition (10.27) doesn't hold, then the curve touches the unitary circle in a tangent way, i.e. it will remain in the same stability domain and the stability property will be given by the second-order analysis (Proposition 10.4).*

Such a situation is illustrated by the following example. Consider the following second-order continuous-time system:

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -\left(\alpha^2+\beta^2\right) & 2\alpha \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \hat{u}(t), \quad t \in [\ell h+\tau, (\ell+1)h+\tau) \\ y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} x(t), \end{cases} \quad (10.30)$$

with corresponding discrete transfer-function:

$$H_{yu}(z;p_0) = \frac{\frac{e^{\alpha(h-\tau)}\sin[\beta(h-\tau)]}{\beta}z^2 + \frac{e^{\alpha(h-\tau)}\left(e^{\alpha h}\sin[\beta\tau]-\sin[\beta(h-\tau)]\right)}{\beta}z - \frac{e^{\alpha(2h-\tau)}\sin[\beta\tau]}{\beta}}{z^3-2e^{\alpha h}\cos[\beta h]z^2+e^{2\alpha h}z},$$

where $\alpha = \frac{\log\alpha_0}{\sqrt{2}\pi}$, $\beta = \frac{2\sqrt{2}\arctan\left(\sqrt{\beta_0}\right)}{\pi}$ with $\alpha_0 \approx 1.17872$ and $\beta_0 \approx 0.168987$.[5] Then, applying *Proposition* 10.2-10.3 and *Proposition* 10.4 for $p_o = (\sqrt{2}\pi, \frac{\pi}{\sqrt{2}})$ we obtain the results summarized in *Table* 10.3. Observe that for λ_2 condition (10.27) vanish, then a second order

Table 10.3: Direction of crossing for the system (10.30), according to *Proposition* 10.3 and *Proposition* 10.4.

Method		$\ell = 1$	2
	λ_ℓ	-5.00872	0.164509
Proposition (10.3)	condition (10.27)	0.0285081	0
	sign (10.27)	$+$	0
Proposition (10.4)	condition (10.29)	$*$	10.2764
	sign (10.29)	$*$	$+$

analysis is required. On the other hand, since the sign of (10.29) is positive, this means that the curve trajectory stays outside of the unit circle. Such a situation is illustrated in Fig10.5.

Remark 10.10. *If instead of inequality (10.29) we have an equality, then higher-order derivatives have to be taken into account.*

Remark 10.11. *Observe, that Proposition 10.3 and Corollary 10.4 are still valid, if instead of the controller gain k, we consider the sampling period h or the induced network-delay $\widetilde{\tau}$.*

10.8 Sensitivity with Respect to the Delay-Parameter

According to *Remark* 10.1, we have that the formulations (10.7)-(10.9) are valid for all $\tau \leq h$. By taking into account this observation, consider now the network delay as being a multiple of the sampling period $\widetilde{\tau} = rh$, with the integer $r > 1$. Then the open-loop system has the transfer function:

$$\begin{aligned} H(z;p_o) &= c\,(zI-\Phi(h))^{-1}\,\Gamma(h)bz^{-r} && (10.31)\\ &= \frac{N(z,h)}{z^r D(z,h)}. && (10.32)\end{aligned}$$

Remark 10.12. *From (10.31), it is easy to see that the polynomials $D(z,h)$ and $N(z,h)$ can be obtained by discretizing directly the continuous-time system (10.1) without taking into account the delay effects.*

[5] Where both, α_0 and β_0 are algebraic numbers which can be computed exactly. α_0 is a root of the polynomial $p_\alpha(x) := x^4+x^3-x^2-x-1$ given by $\alpha_0 = \max\{x \in \mathbb{R} :\ p_\alpha(x) = 0\}$ and β_0 is a root of the polynomial $p_\beta(x) := 13x^8-584x^7+4940x^6-15736x^5+22990x^4-15736x^3+4940x^2-584x+13$ given by $\beta_0 = \min\left\{x \in \mathbb{R} :\ p_\beta(x) = 0\right\}$.

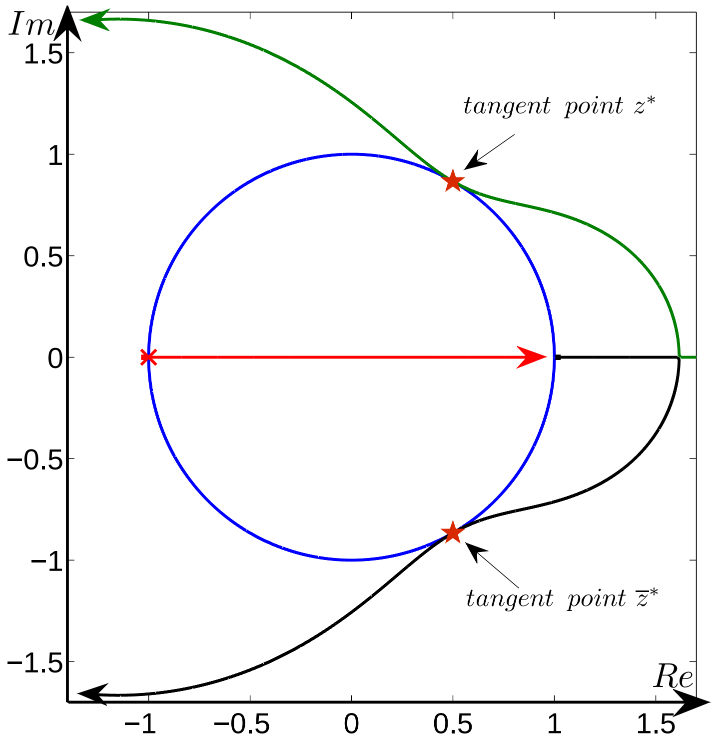


Figure 10.5: Root trajectory for the system (10.30). Tangent point $z^* = \frac{1}{2} \pm j\frac{\sqrt{3}}{2}$.

We have the following result:

Proposition 10.5. *Assume that for some fixed parameter $p^* = (h^*, \tau^*, k^*)$, with $\tau^* = h^*$ the following polynomial,*

$$P_r(z; p^*) = z^r D(z, p_o^*) + k^* N(z, p_o^*). \tag{10.33}$$

is Schur stable. Then, there exists some sufficiently small $\varepsilon > 0$ such that the same property holds for the perturbed polynomials $F_{r-1}(z; p^*_{\tau=h-\varepsilon})$ and $F_r(z; p^*_{\tau=h+\varepsilon})$. Furthermore,

$$\sigma(F_r(z; p_{\tau=h+}) = \sigma(F_{r-1}(z; p_{\tau=h-}) \cup \{0\}, \tag{10.34}$$

where $\tau = h+$ ($\tau = h-$) defines the corresponding right (left) limit.

Proof. Without any lack of generality assume that the closest eigenvalue to $\partial\mathbb{D}$, denoted in the sequel $\lambda^{(0)}$, is simple.

$$\lambda^{(0)} := \left\{ \lambda \in \mathbb{C} : \quad |\lambda| = \max_{\widetilde{\lambda} \in \sigma(\widetilde{\Phi}(p^*))} |\widetilde{\lambda}| \right\}.$$

Without any loss of generality, assume that $r = 1$ (i.e., $\tau = h^*$). Then, since the characteristic polynomial associated to the closed-loop system has different degrees for larger or smaller network delays, we need to consider two independent cases: (i) $\tau^* \to \tau^* - \varepsilon$ and (ii) $\tau^* \to \tau^* + \varepsilon$, respectively. In both situation, we will use a continuity type argument.

Case (i): We have that $P_r(z; h^*, k^*) \equiv F_{r-1}(z; h^*, \tau^* = h^*, k^*)$, then by the continuity of the roots with respect to the coefficients we know that $\exists \varepsilon > 0$ such that $F_{r-1}(z; h^*, h^* - \varepsilon, k^*)$ is Schur stable.

Case (ii): As mentioned above, observe that $P_r(z;h^*,k^*)$ and $F_r(z;h^*,h^*+\varepsilon,k^*)$ has different degrees. Then, in order to normalize the degree is noteworthy to see that the associate transfer matrix of polynomial $zP_r(z;h^*,k^*)$ can be rewritten as:

$$\overline{\Phi}(h^*,k^*) := \begin{bmatrix} \Phi(h^*) & 0 & \Gamma(h^*)b \\ 0 & 0 & 1 \\ -k^*c & 0 & 0 \end{bmatrix}.$$

In the sequel, we interpret ε as a perturbation in the delay parameter and next, we analyze the eigenvalue behavior of $\lambda^{(0)}$ as ε is increased from zero. To this end, consider:

$$T(\varepsilon) := \begin{bmatrix} \Phi(h^*) & \Phi(h^*-\varepsilon)\Gamma(\varepsilon)b & \Gamma(h^*-\varepsilon)b \\ 0 & 0 & 1 \\ -k^*c & 0 & 0 \end{bmatrix}.$$

Since $\tau^* = h^*$, from the above definition we have that $T(\varepsilon) \stackrel{\varepsilon\to 0}{\longrightarrow} \overline{\Phi}(h^*,k^*)$, more over it is clear that $T(\varepsilon)$ is analytic in ε and its first-order derivative is given by:

$$T'(\varepsilon) = \begin{bmatrix} 0 & \Phi(h^*)b - A\Phi(h^*-\varepsilon)\Gamma(\varepsilon)b & -\Phi(h^*-\varepsilon)b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Then, according to *Lemma* 2.3 the eigenvalues of $T(\varepsilon)$ can be expanded in series as (2.20), i.e.:

$$\mu(\varepsilon) = \lambda^{(0)} + \lambda_1^{(1)}\varepsilon + o(\varepsilon^2), \;\; i = 1,\ldots,m,$$

now applying *Lemma* 2.1, we know that $\lambda_1^{(1)} = r_1^T T'(0) q_1$. From the above expansion it is clear that there always exists some sufficiently small $\varepsilon > 0$ such that $F_r(z;h^*,h^*+\varepsilon,k^*)$ is Schur stable. This argument completes the proof for simple characteristic roots. The remaining cases (semi-simple or multiple, but not semi-simple) can be treated by similarity, and thus they are omitted. Finally, the presence of one additional characteristic root at the origin appears naturally from the formulation of T for $\tau = h \pm \varepsilon$. □

The following result, is a direct consequence of *Proposition* 10.5.

Proposition 10.6. *Assume $p^* = (h^*,\tau^*,k^*)$ fixed such that $F_{\ell-1}(z;p^*)$ is Schur-stable. Let $\overline{\tau} := \tau - \tau^*$, $\Lambda_0 := \left\{\overline{\tau}_0 \in \mathbb{R}_+ : \sigma\left(W_{F_{\ell-1}}(\overline{\tau}_0)\right) = 0\right\}$, $\Lambda_1 := \{\tau_1 \in \mathbb{R}_+ : \sigma(W_{F_\ell}(\tau_1)) = 0\}$, and define the minimum elements of Λ_0 and Λ_1 by $\tau^- := \inf \Lambda_0$ and $\tau^+ := \inf \Lambda_1$. Then, the system (10.1) is asymptotically stable for all fixed $\tau \in [(\ell-1)h^* + \tau^*, \ell h^* + \tau^*]$ if and only if the following inequality hold:*

$$(h^* - \tau^*)\tau^* < \tau^- \tau^+. \tag{10.35}$$

Proof. Sufficiency. Since, both τ^- and τ^+ are the minimal elements of Δ_0 and Δ_1, respectively, inequality (10.35) implies that the unstable roots of polynomials $F_{\ell-1}(z;h^*,\tau_1,k^*)$ and $F_\ell(z;h^*,\tau_2,k^*)$ are invariant for all $\tau_1 \in [(\ell-1)h^* + \tau^*, \ell h^*]$ and $\tau_2 \in [\ell h^*, \ell h^* + \tau^*]$. Moreover, the fact that $F_{\ell-1}(z;h^*,\tau^*,k^*)$ is Schur-stable implies that $F_{\ell-1}(z;h^*,\tau = h^*,k^*)$ is Schur-stable, then, straightforwardly by *Proposition* 10.5 we conclude that $F_\ell(z;h^*,\tau,k^*)$ is Schur-stable for all $\tau \in [\ell h^*, \ell h^* + \tau^*]$ implying that system (10.1) is asymptotically stable for all fixed $\tau \in [(\ell-1)h^* + \tau^*, \ell h^* + \tau^*]$.

Necessity. Since the system (10.1) is asymptotically stable for all fixed $\tau \in [(\ell-1)h^* + \tau^*, \ell h^* + \tau^*]$, then the minimal critical delays τ^- and τ^+ satisfy $\tau^- > h^* - \tau^*$ and $\tau^+ > \tau^*$, implying inequality (10.35). □

Remark 10.13. *Observe that the sets Δ_0 and Δ_1 does not exclude the symmetric roots mentioned in Remark 10.3. The main reason is because the polynomial $F_{\ell-1}(z;h^*,\tau^*,k^*)$ is Schur-stable, then according to the Proposition 10.1b, τ^- will correspond to a critical-delay value. Similar arguments can been applied to the set Δ_1.*

Proposition 10.7. *Let the triplet $p^* = (h^*,\tau^*,k^*)$ be fixed and assume that $F_{\ell-1}(z;p^*)$ is a Schur-stable polynomial. Then, $F_m(z;p^*)$ is asymptotically stable for all $m > \ell-1$ if the following inequality holds:*

$$|k^*| < \frac{1}{3}\left|\frac{D(z;p_o^*)}{N(z;p_o^*)}\right|, \quad \forall z \in \partial\mathbb{D}.$$

Proof. The proof is based on the *Rouché's* theorem [23] and in the sake of brevity will be omitted. □

To determine the direction of crossing when the delay is taking into account, we will restrict our analysis to the case when the delay $\widetilde{\tau}$ is smaller that the sampling period h. In this case, $\widetilde{\tau} = \tau$. However, the results are still valid for larger network delay values.

Let p^* be fixed such that τ^* and λ_o^* be a critical pair of critical delay and critical zero of $F(z,p^*)$, i.e., $\lambda_o^* = e^{\boldsymbol{i}\theta^*} \in \sigma(\Phi(p^*))$. Without any loss of generality, let $e^{\boldsymbol{i}\theta^*}$ be ordered as the first eigenvalue of $\Phi(p^*)$, with multiplicity m. Assuming that $e^{\boldsymbol{i}\theta^*}$ is semi-simple we have the following:

Proposition 10.8. *Let $\lambda_o^* = e^{\boldsymbol{i}\theta^*}$ be a semi-simple eigenvalue of $\Phi(p_{\tau=\tau^*})$. Then for any τ sufficiently close to τ^*, the characteristic zeros corresponding to λ_o^* can be expanded by the power series:*

$$\lambda_o^* + \lambda_\ell\left(R_1 T'(0) Q_1\right)(\tau-\tau^*) + o\left((\tau-\tau^*)^2\right), \quad \ell = 1,2,\ldots,m.$$

Thus, for τ sufficiently close to τ^* but $\tau > \tau^*$ there are at least M ($M \le m$) of the characteristic zeros going outside (inside) the unit circle $\partial\mathbb{D}$ if M of the eigenvalues satisfy the condition:

$$\cos(\theta_\ell - \theta^*) > 0 \ (<0), \qquad \ell = 1,\ldots,m,$$

where $\theta_\ell \in [0,2\pi)$ is the phase angle of $\lambda_\ell(R_1T'(0)Q_1) \neq 0$ and $T'(0)$ is given by

$$T'(0) = \begin{bmatrix} k\Phi(h^*)bc & \Phi(h^*)b \\ 0 & 0 \end{bmatrix}. \tag{10.36}$$

Proof. Introduce now the new real variable $\varepsilon := \tau - \tau^*$, and define:

$$T(\varepsilon) := \begin{bmatrix} \Phi(h^*) - k^*\Gamma(h^*-\varepsilon)bc & \Phi(h^*-\varepsilon)\Gamma(\varepsilon)b \\ -kc & 0 \end{bmatrix}. \tag{10.37}$$

Then, the result follows straightforwardly by applying *Lemma* 2.1 and observing that close to $\varepsilon = 0$, the matrix function (10.37) can be expanded as: $T(\varepsilon) = T(0) + \varepsilon T'(0) + o(\varepsilon^2)$. The derivative of (10.37) leads to the expression (10.36). Finally, by observing the increasing (decreasing) properties of the modulus $|\ \lambda_o^*\ |$ in the left- and right-semicircle of $\partial\mathbb{D}$, the proof is complete. □

In the case of a simple root, the result above rewrites as follows:

Corollary 10.1. *Let $\lambda_o^* = e^{i\theta^*}$ be a simple eigenvalue of $\Phi(p_{\tau=\tau^*})$. Then for any τ sufficiently close to τ^*, the characteristic zero corresponding to λ_o^* can be expanded by the power series:*

$$\lambda_o^* + \left(r_1^T T'(0) q_1\right)(\tau - \tau^*) + o\left((\tau - \tau^*)^2\right).$$

Thus, for τ sufficiently close to τ^*, but $\tau > \tau^*$, the corresponding characteristic zero is going outside (inside) the unit circle $\partial\mathbb{D}$ if the following condition is satisfied:

$$\cos(\theta_\ell - \theta^*) > 0 \ (< 0), \qquad \ell = 1, \ldots, m-1$$

where $T'(0)$ is given by

$$T'(0) = \begin{bmatrix} k\Phi(h^*)bc & \Phi(h^*)b \\ 0 & 0 \end{bmatrix}. \tag{10.38}$$

The next results concerns to the case when the first approximation vanish.

Proposition 10.9. *Let $\lambda_o^* = e^{i\theta^*}$ be a semi-simple eigenvalue of $\Phi(p_{\tau=\tau^*})$ with multiplicity m, and let $T(0)$ be partitioned as in (2.21). Let also $\lambda_\ell^{(1)}$ be a semi-simple eigenvalue of $R_1 T'(0) Q_1$ with multiplicity d. Then for any τ sufficiently close to τ^*, the characteristic zeros corresponding to λ_o^* can be expanded into the power series:*

$$\lambda_o^* + \lambda_\ell^{(1)}(\tau - \tau^*) + \mu_{\ell p}^{(2)}(\tau - \tau^*)^2 + o\left((\tau - \tau^*)^3\right), \quad \ell = 1, 2, \ldots, m.$$

with

$$\begin{aligned} \lambda_\ell^{(1)} &= \lambda_\ell\left(R_1 T' Q_1\right), \ \ell = 1, 2, \ldots, m, \\ \mu_{\ell p}^{(2)} &= \lambda_p\left[R_1^{(2)} R_1\left(T''(0) - T'(0) S T'(0)\right) Q_1 Q_1^{(2)}\right], \ p = 1, \ldots, d, \end{aligned}$$

where S is given in Lemma 2.3.

(i) For τ sufficiently close to τ^* but $\tau > \tau^*$ the characteristic zero λ^* cross to the outside (inside) of the unit circle $\partial\mathbb{D}$ if for some $\ell = 1, \ldots, m$,

$$\cos(\theta_\ell - \theta^*) > 0 \ (< 0)$$

(ii) if

$$\cos(\theta_\ell - \theta^*) = 0, \quad \ell = 1, \ldots, m,$$

sufficiently close to τ^* but $\tau > \tau^*$ the characteristic zero λ^* cross to the outside (inside) of the unit circle $\partial\mathbb{D}$ if for some $p = 1, \ldots, d$,

$$\cos(\theta_p - \theta^*) > 0 \ (< 0)$$

where $\theta_p \in [0, 2\pi)$ is the phase angle of $\lambda_p\left[R_1^{(2)} R_1\left(T''(0) - T'(0) S T'(0)\right) Q_1 Q_1^{(2)}\right] \neq 0$ and $T''(0)$ is given by

$$T''(0) = \begin{bmatrix} -k^* A\Phi(h^*)bc & -A\Phi(h^*)b \\ 0 & 0 \end{bmatrix}. \tag{10.39}$$

Corollary 10.2. *Let $\lambda_o^* = e^{i\theta^*}$ be a simple eigenvalue of $\Phi(p_{\tau=\tau^*})$. Then for any τ sufficiently close to τ^*, the characteristic zero λ^* cross to the outside (inside) of the unit circle $\partial\mathbb{D}$ if,*

$$\cos(\theta - \theta^*) > 0 \ (< 0)$$

where $\theta_p \in [0,2\pi)$ is the phase angle of $\lambda_p(r_1 T'(0) q_1)$ where q_1 and r_1 are the right and left eigenvectors associated with λ_o^*. Additionally, if

$$\cos(\theta - \theta^*) = 0,$$

then for any sufficiently close to τ^* but $\tau > \tau^*$ the characteristic zero λ_o^* cross to the outside (inside) of the unit circle $\partial\mathbb{D}$ if

$$\cos(\theta_o - \theta^*) > 0 \ (< 0)$$

where $\theta_o \in [0,2\pi)$ is the phase angle of $\lambda\left[r_1 \left(T''(0) - T'(0) S T'(0)\right) q_1\right]$ and $T''(0)$ is given in Proposition 10.9.

Finally, the next result concerns the case when λ_o^* is not a semi-simple but repeated eigenvalue.

Proposition 10.10. *Let $\lambda_o^* = e^{i\theta^*}$ be a repeated eigenvalue of $\Phi(p_{\tau=\tau^*})$ with multiplicity m. Suppose that λ_o^* is not semi-simple. Then, for any τ sufficiently close to τ^* but $\tau > \tau^*$ the characteristic zeros corresponding to λ_o^* can be expanded by the Puiseux Series*

$$\lambda_o^* + \left|r_m T'(0) q_1\right|^{\frac{1}{m}} e^{i\frac{(2\ell+1)\pi+\theta}{m}} (\tau - \tau^*)^{\frac{1}{m}} + \cdots, \ell = \overline{0, m-1},$$

where $\theta \in [0,2\pi)$ is the phase angle of $r_m T'(0) q_1$. Hence, for τ sufficiently close to τ^* but $\tau > \tau^*$, the number of critical zeros going to outside the unit circle $\partial\mathbb{D}$ (or vice versa) can be determined by the condition

$$cos\left(\frac{(2\ell+1)\pi + \theta - m\theta^*}{m}\right) > 0 \ (< 0), \quad \ell = \overline{0, m-1}. \tag{10.40}$$

The proof is analogous to the previous one, and, for the sake of brevity, it is omitted.

10.9 Illustrative Examples

In order to motivated the previous results, we consider in the rest of this paper the following illustrative numerical examples.

Example 10.1. *Consider the following fifth-order unstable system,*

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} -12.2 & -21.8 & 3.4 & 10.6 & -3.4 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \widehat{u}(t), \\ y(t) = \begin{bmatrix} 0 & 1 & 5 & 4 & 2 \end{bmatrix} x(t). \end{cases} \tag{10.41}$$

Then, applying Proposition 10.1, we get stability region depicted in Fig.10.6(b).

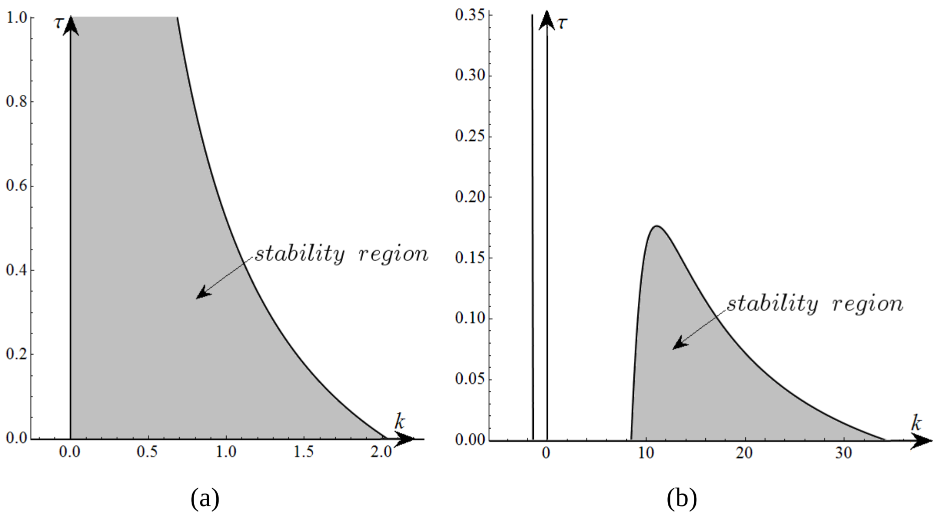


Figure 10.6: Stability crossing curves $\mathcal{S}_{h^*}$. (a) Example 10.2, for the *sampling-period* $h^* = 1$. (b) Example 10.1, for the *sampling-period* $h^* = 0.6$.

Example 10.2. *Consider the following system:*

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} 0 & 1 \\ a_1 & a_2 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ b_1 \end{bmatrix} \widehat{u}(t), t \in [\ell h + \tau, (\ell+1)h + \tau), \\ y(t) = \begin{bmatrix} c_1 & 0 \end{bmatrix} x(t). \end{cases} \tag{10.42}$$

For $(a_1, a_2, b_1, c_1) = (0, -0.1, 0.1, 1)$ *we got the system considered in [137, 55] and for* $h = 1.0$*, we obtain the stability crossing curves illustrated in Fig.10.6(a). Now if* $(a_1, a_2, b_1, c_1) = (-\omega_0^2, 0, 1, \omega_0)$ *we got the oscillator system considered in [103].Considering* $\omega_0 = 1$ *and fixing the control gain at* $k = 0.5$*, the stability regions depicted in Fig.10.7 were obtained. As*

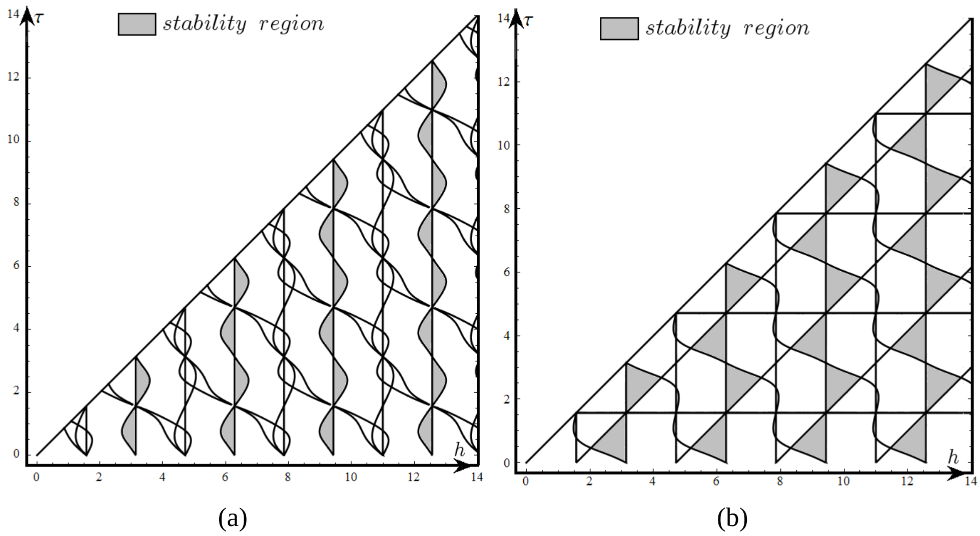


Figure 10.7: Stability crossing curves $\mathcal{S}_{k^*}$ for $k^* = 2$ and $\omega_o = 3$. (a) System $F_{r+1}(z, p)$. (b) System $F_r(z, p)$ with $r = 0$.

shown in Fig.10.7 some pairs (h^*, τ^*) *preserves the stability property for* $\widetilde{\tau} = \tau + h$*. However is noteworthy to see that even in the case when* $F_r(z, p_o^*)$ *and* $F_{r+1}(z, p_o^*)$ *are stable, not* *all* $\tau \in [rh+\tau, (r+1)h+\tau]$ *preserves this property. To illustrate such a situation, consider* $(h,\tau,k) = (6.1, 2.5, 2)$ *and* $\omega_0 = 3$. *Fig.10.8 depicts the root trajectory, when* τ *increases from* $\tau_0 = 2.5$ *to* $\tau_f = 8.60$. *The corresponding delay are enlisted in Table 10.4.*

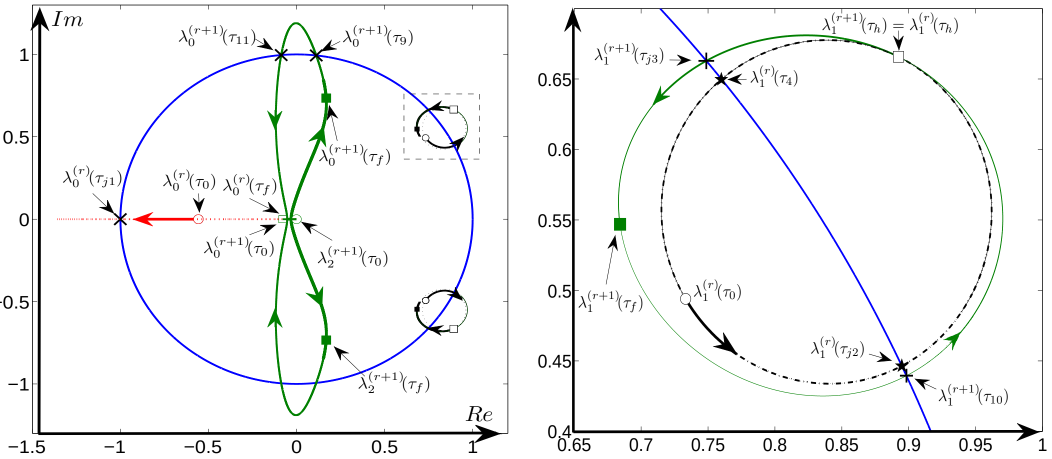


Figure 10.8: (a) Roots trajectories of the example (10.2), for $(a_1,a_2,b_1,c_1) = (-\omega_0^2, 0, 1, \omega_0)$ and $\widetilde{\tau} \in [\tau^*, h^* + \tau^*)$. (b) Zoom of the dashed region.

Table 10.4: Delay stability-crossing values for the oscillator system (10.42).

	Critical Delay Values											
	$\tau_1^{(\ell)}$				$\tau_2^{(\ell)}$		$\tau_3^{(\ell)}$		τ_4	τ_5	τ_6	τ_7
	$\ell=1$	2	3	4	$\ell=1$	2	$\ell=1$	2	$\ell=1$	$\ell=1$	$\ell=1$	$\ell=1$
	2.69	3.4	4.78	5.5	2.98	5.22	6.35	8.44	4.27	6.66	7.21	7.43
Crossing	+	−	+	−	+	+	−	−	−	+	+	−

Example 10.3 (Two-Sampling Period Behavior)**.** *In order to illustrate Proposition 10.6, we will consider as a final example the system (10.42) with* $(a_1,a_2,b_1,c_1) = (-\omega_0^2, 0, 1, \omega_0)$, $\omega_0 =$ 3 *(the oscillator case) and fixed sampling period* $h = \frac{19}{25}$. *Taking* $\ell = 2$, *we will consider the triplets* $p_1^* = (h^*, \tau^*, k_1^*) = \left(\frac{19}{25}, \frac{2}{25}, \frac{1}{5}\right)$ *and* $p_2^* = (h^*, \tau^*, k_2^*) = \left(\frac{19}{25}, \frac{2}{25}, \frac{3}{5}\right)$ *we which we know that* $F_{\ell-1}(z, p_1^*)$ *and* $F_{\ell-1}(z, p_2^*)$ *are both Schur-stable. Then in order to investigate if* $F_{\ell-1}(z, p_i^*)$ $(i \in \{1,2\})$ *stay stable for all* $\tau \in [h^* + \tau^*, 2h^* + \tau^*]$ *we consider Proposition 10.6. The following Table 10.5 summarize the computations. According to Table10.5, we have that*

Table 10.5: Evaluation of *Proposition* 10.6 for $\ell = 2$ and the two triplets $p_1^* = \left(\frac{19}{25}, \frac{2}{25}, \frac{1}{5}\right)$, $p_2^* = \left(\frac{19}{25}, \frac{2}{25}, \frac{3}{5}\right)$.

p_j^*	*Proposition* 10.6				
$\left(h^*, \tau^*, k_j^*\right)$	τ^-	τ^+	$\tau^- \cdot \tau^+$	$(h^* - \tau^*)\,\tau^*$	
$\left(\frac{9}{25}, \frac{2}{25}, \frac{1}{5}\right)$	0.813695	0.136736	0.111262	0.0544	stable
$\left(\frac{9}{25}, \frac{2}{25}, \frac{3}{5}\right)$	0.726888	0.047933	0.034842		unstable

$F_{\ell-1}(z,p_1^*)$ stay stable $\forall\tau \in [h^*+\tau^*, 2h^*+\tau^*]$, whereas $F_{\ell-1}(z,p_2^*)$ does not preserve the same property. Fig.10.9 shows the stability region in the $k-\tau$ parameter space, where the points (k_j,t_j), are given by $k_1 = \frac{1}{5}$, $k_2 = \frac{3}{5}$ and $t_1 = h^*+\tau^*$, $t_2 = 2h^*+\tau^*$.

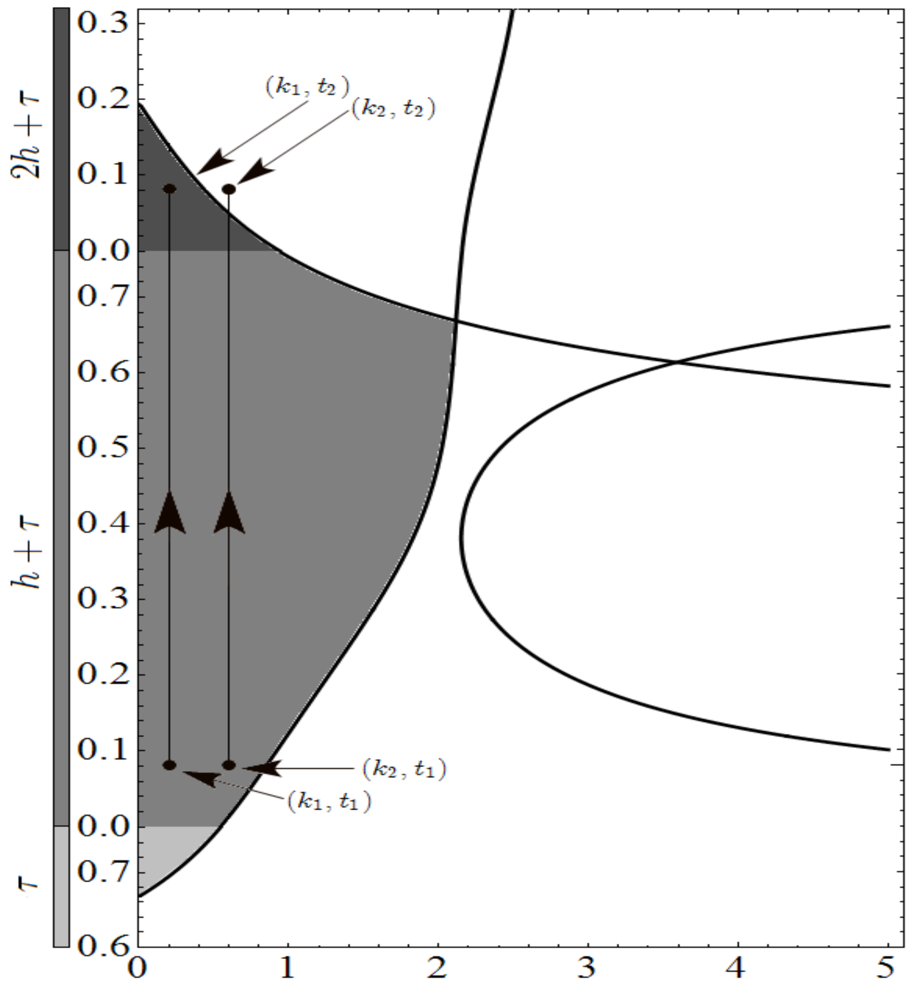


Figure 10.9: Stability region in the $k-\tau$ parameter space, for the oscillator system (10.42).

Part III

Sampled-Data Systems

Summary

11 Perturbation Techniques in Discretization

In the this chapter we will discus some extension of the eigenvalue perturbation based-approach developed in the *Part* II, to the *Sampled-Data Systems*. Then, in order to present the appropriate extension, some preliminaries results will be presented.

11.1 Introduction and Motivating Examples

In the design of linear time-invariant systems, the location of poles and zeros plays a fundamental role in deriving the performance of the closed-loop system (see, e.g., [121], and the references therein). As stated in the literature ([37, 62]), several techniques in adaptive control are based on zero-cancelation and, as a consequence, the corresponding schemes will not work with unstable zeros. As discussed by [4], the poles of a discretized system can be derived from the poles (p_ℓ) of the continuous-time system representation by using the simple transformation $p_\ell \mapsto \exp(p_\ell h)$, where h denotes the sampling period. Unfortunately, there does not exist any explicit map giving the relation between the zeros of a continuous-time system and the zeros of the corresponding sampled-system. Into a different context, in a pioneer work of [4, 6], it was shown that the minimal-phase property for a continuous-time system can be lost, even for a sufficiently small sampling period (h).

The effects of sampling a continuous-time systems with a zero-order hold on the resulting discretized zeros have been largely treated in the literature (see, for instance, [4, 37, 62, 63] and the references therein). However, in all these works, the main results have been established by considering several restrictions. In order to mention some, for example, [4, 37, 62] consider that the system is a strictly proper stable continuous-time system, whereas [63] deals with strictly proper unstable continuous-time system. Finally, although [63] considered a more general case, the corresponding zeros characterization is stated under some additional restrictions. Among them, we cite: *no poles on the imaginary axis* and *all poles are distinct*. Moreover, the author does not pay any attention to the case when *multiple critical samplings* appears on the unit circle. Motivated by these observations and to the best of the authors knowledge, the characterization of all stability intervals for the discretized zeros is still open.

In order to motivate the proposed approach, consider the system:

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \lambda(\sigma^2+\omega^2) & -\sigma(\sigma+2\lambda)-\omega^2 & 2\sigma+\lambda \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} \alpha^2+\beta^2 & -2\alpha & 1 \end{bmatrix} x(t). \end{cases}$$

The corresponding transfer function writes as:

$$G(s;p) = \frac{(s-\alpha)^2+\beta^2}{(s-\lambda)((s-\sigma)^2+\omega^2)}, \tag{11.1}$$

where $p := (\alpha,\beta,\lambda,\sigma,\omega)$ defines the set of parameters. Then, taking $G_i(s) := G(s,p_i)$, $i = \{1,2,3\}$ we have the following cases:

i) Taking $p_1 = (0.5, 1.5, -3, -0.1, 2)$, the sampled-system is minimal-phase for small sampling periods, however increasing h leads to a non-minimal-phase system;

ii) Next, for $p_2 = (-2, 1, 1, 0, 2)$, the sampled-system stays minimal-phase.

iii) Finally, for $p_3 = (0.1, 1, 1, 0.1, 0.25)$ the sampled-system is non-minimal-phase for small sampling periods, however increasing h leads to a minimal-phase system.

Such behaviors are depicted in Fig.11.1. Based on the remarks above, we are interested in

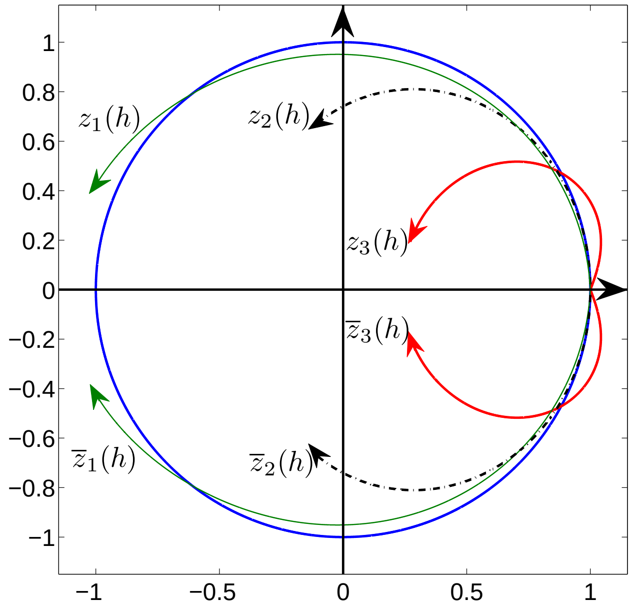


Figure 11.1: Sampled-root trajectories for the system (11.1), where: $z_k(h)$ correspond to the sampled-zero trajectory of $G_k(s)$.

exploring the *minimal-phase property* by using an appropriate formalism to analyze the zero behavior when h is taken as a *free parameter*. Such an approach will give further insights on the zeros behavior of sampled-systems.

11.2 Preliminaries and Problem Formulation

We introduce now some basic prerequisites on operator perturbation theory for matrix eigenvalue problems. More precisely, we discuss the eigenvalues behavior of a matrix function with respect to a small perturbation on some of its parameters. The development is based on [75], [67].

Let $A(\lambda)$ be an $n \times n-$matrix function, defined and analytic in a neighborhood of λ^* and with $\det A(\lambda) \neq 0$. Then, λ^* is an *eigenvalue* of $A(\lambda)$ if $\det A(\lambda^*) = 0$. The *geometric multiplicity* g of λ^* is the dimension of the kernel $\ker A(\lambda^*)$. The function $A(\lambda)$ admits a *local Smith form* at $\lambda = \lambda^*$, thats is there are nonnegative integers $m_1, \ldots, m_n$ satisfying $m_i \leq m_{i+1}$, $i = 1, \ldots, n-1$ with $m_{g+1} = \cdots = m_n = 0$ if $g < n$ and $n \times n-$matrix functions $E(\lambda)$, $F(\lambda)$ which are analytic and invertible in a neighborhood of $\lambda = \lambda^*$ such that:

$$A(\lambda) = E(\lambda)D(\lambda)F(\lambda), \tag{11.2}$$

with $D(\lambda) = diag\left((\lambda-\lambda^*)^{m_1}, \ldots, (\lambda-\lambda^*)^{m_n}\right)$. The numbers m_i, $i = 1, \ldots, g$ are the *partial multiplicities* of the eigenvalue $\lambda = \lambda^*$ of $A(\lambda)$, whereas their sum $m_1 + \cdots + m_g$ is its *algebraic multiplicity*. If there are k groups of mutually equal m_i, the $j-$th group containing n_j elements we have:

$$m_1 = \cdots = m_{n_1} < m_{n_1+1} = \cdots = m_{\widetilde{n}_2} < \cdots < m_{\widetilde{n}_{k-1}+1} = \cdots = m_{\widetilde{n}_k},$$

where $\widetilde{n}_j = n_1 + \cdots + n_j$, hence setting $\widetilde{m}_j := m_{\widetilde{n}_j}$, $j = 1 \ldots, k$, we have $0 < \widetilde{m}_1 < \widetilde{m}_2 < \cdots < \widetilde{m}_k$.

In the sequel, we recall some results concerning the eigenvalues of the perturbed $n \times n-$matrix function,

$$T(\lambda, \varepsilon) := A(\lambda) + B(\lambda, \varepsilon)$$

near $\lambda = \lambda^*$ and for small ε under the assumption that $B(\lambda, \varepsilon)$ is analytic in λ near $(0,0)$ and $B(\lambda, 0) = 0$ for all λ. Since $A(\lambda)$ admits a local Smith form, the study of the eigenvalue perturbation of $T(\lambda, \varepsilon)$ is equivalent to the study of the solutions of the equation $\det \widehat{T}(\lambda, \varepsilon) = 0$ where $\widehat{T}(\lambda, \varepsilon) := D(\lambda) + \widehat{B}(\lambda, \varepsilon)$, with $\widehat{B}(\lambda, \varepsilon) := E^{-1}(\lambda) B(\lambda, \varepsilon) F^{-1}(\lambda)$.

Now, consider a partition of $D(\lambda)$ and $\widehat{B}(\lambda, \varepsilon)$ as

$$D(\lambda) = \begin{bmatrix} D_1(\lambda) & 0 \\ 0 & I \end{bmatrix}, \quad \widehat{B}(\lambda, \varepsilon) = \begin{bmatrix} \widehat{B}_1(\lambda, \varepsilon) & \widehat{B}_2(\lambda, \varepsilon) \\ \widehat{B}_3(\lambda, \varepsilon) & \widehat{B}_4(\lambda, \varepsilon) \end{bmatrix} \tag{11.3}$$

where $D_1(\lambda)$ and $\widehat{B}_1(\lambda, \varepsilon)$ are $g \times g$ matrices. Next, denote

$$H := \frac{\partial \widehat{B}_1}{\partial \varepsilon}(0,0), \tag{11.4}$$

and define for $j = 1, 2 \ldots, k$; $\ell = 1, 2, \ldots, n_j - 1$:

$$\Delta_{j\ell} := \sum \det H\left(\alpha_1, \ldots, \alpha_{n_j-\ell}, \widetilde{n}_j + 1, \widetilde{n}_j + 2, \ldots, g\right), \tag{11.5}$$

where the sum runs over all $\alpha_1, \ldots, \alpha_{n_j-\ell}$ such that $\widetilde{n}_{j-1} < \alpha_1 \leq \ldots \leq \alpha_{n_j-\ell} \leq \widetilde{n}_j$,

$$\Delta_j := \Delta_{j0} := \begin{cases} \det H\left(\widetilde{n}_{j-1} + 1, \widetilde{n}_{j-1} + 2, \ldots, g\right) & \text{if} \quad 1 \leq j \leq k \\ 1 & \text{if} \quad j = k+1 \end{cases} \tag{11.6}$$

Next, for a positive m by $\vartheta_{m,\sigma}$, $\sigma = 1, \ldots, m$, we denote the $m-$th roots of unity.

Theorem 11.1 ([75]). *Let $A(\lambda)$ and $B(\lambda, \varepsilon)$ be as above and assume that the condition $\Delta_1 \cdots \Delta_k \neq 0$ is satisfied.*

i) Then for each $j \in \{1, 2, \ldots, k\}$ there are $n_j \widetilde{m}_j$ eigenvalues of $T(\lambda, \varepsilon) = A(\lambda) + B(\lambda, \varepsilon)$ near $\lambda^* = 0$, satisfying for $\varepsilon \to 0$, the asymptotic relations

$$\lambda_{j\nu\sigma}(\varepsilon) = \gamma_{j\nu} \vartheta_{\widetilde{m}_j, \sigma} \varepsilon^{\frac{1}{\widetilde{m}_j}} + o\left(|\varepsilon|^{\frac{1}{\widetilde{m}_j}}\right),$$

here $\sigma = 1, \ldots, \widetilde{m}_j$; $\nu = 1, \ldots, n_j$ and the numbers $\gamma_{j\nu}$ satisfy $\left(\gamma_{j\nu}\right)^{\widetilde{m}_j} = \xi_{j\nu}$, where $\xi_{j\nu}$ $\nu = 1, \ldots, n_j$, are the solutions of the equation

$$\Delta_j + \sum_{\ell=1}^{n_j-1} \Delta_{j\ell} \xi^{\ell} + \Delta_{j+1} \xi^{n_j} = 0. \tag{11.7}$$

For small ε there are no other eigenvalues of $T(\lambda, \varepsilon)$ near $\lambda^* = 0$.

ii) If for some $j \in \{1, 2 \ldots, k\}$ *the* $j-$*th equation (11.7) has a simple root* $\widehat{\xi}_j$ *then there is a group of* $\widetilde{m}_j$ *eigenvalues* $\widehat{\lambda}_{j\sigma}(\varepsilon)$ *of* $T(\lambda, \varepsilon)$ *near* $\lambda^* = 0$ *having Puiseux expansion in* $\varepsilon^{\frac{1}{\widetilde{m}_j}}$

$$\widehat{\lambda}_{j\sigma}(\varepsilon) = \widehat{\gamma}_j \vartheta_{\widetilde{m}_j,\sigma} \varepsilon^{\frac{1}{\widetilde{m}_j}} + \sum_{\ell=2}^{\infty} \alpha_{j\ell} \vartheta_{\widetilde{m}_j,\sigma}^{\ell} \varepsilon^{\frac{\ell}{\widetilde{m}_j}}, \quad \sigma = 1, 2, \ldots, m_j$$

with $(\widehat{\gamma}_j)^{\widetilde{m}_j} = \widehat{\xi}_j$.

11.2.1 System Description

Consider the following continuous-time linear system:

$$\begin{cases} \dot{x}(t) = Ax(t) + bu(t) \\ y(t) = cx(t) + du(t), \end{cases} \tag{11.8}$$

where $A \in \mathbb{R}^{n \times n}$, and b, $c^T \in \mathbb{R}^n$ are real constant matrices. By discretizing the system (11.8) with a constant sampling period h, we obtain the following discrete-time system:

$$\begin{cases} x[\ell+1] = \Phi(h)x[\ell] + \Gamma(h)u[\ell], \\ \quad y[\ell] = cx[\ell] + du[\ell], \end{cases} \tag{11.9}$$

where $\Phi : \mathbb{R}_+ \mapsto \mathbb{R}^{n \times n}$ and $\Gamma : \mathbb{R}_+ \mapsto \mathbb{R}^n$ are defined by $\Phi(t) := e^{At}$ and $\Gamma(t) := \left(\int_0^t e^{As} ds \right) b$. The corresponding transfer function of (11.9) is given by $H_{yu}(z, h) := N(z, h) / D(z, h)$ where, according to [45, 44],

$$N(z; h) = det \begin{bmatrix} zI - \Phi(h) & -\Gamma(h) \\ c & d \end{bmatrix}, \tag{11.10}$$

$$D(z; h) = det(zI - \Phi(h)). \tag{11.11}$$

11.2.2 Problem Formulation

As mentioned in the Introduction, this chapter will focus on two problems:

(i) first, detecting all critical samplings in $\mathbb{R}^+$, that is the explicit computation of all $h^* \in \mathbb{R}^+$ such that $N\left(e^{i\theta^*}, h^*\right) = 0$ for some $\theta^* \in [0, 2\pi)$, and

(ii) second, computing all intervals $\mathrm{I}_1, \ldots \mathrm{I}_N$ such that for all $h \in \mathrm{I}_\ell$ the number of unstable zeros is *invariant*.

In this way, we will find an appropriate partition

$$\mathrm{I}^S = \bigcup \mathrm{I}_\ell, \qquad \mathrm{I}^U = \bigcup \mathrm{I}_\ell$$

with

$$\mathbb{R}_+ = \mathrm{I}^S \bigcup \mathrm{I}^U,$$

such that the discretized system is *minimal phase* for all $h \in \mathrm{I}^S$ and *non-minimal phase* for all $h \in \mathrm{I}^U$.

11.3 Main Result

We focus in the analysis of the zeros for the polynomial $N(z;h)$ when the sampling period h range over $\mathbb{R}_+$. Despite the fact that, in the *Part* II we have introduced the notion of a critical point and crossing set, we will define such a notions again, but in a different context.

Definition 11.1. *A* stability crossing point *or a* critical sampling h^* *is a sampling period such that there exists at least one critical zero* $z^* \in \partial\mathbb{D}$ *of the corresponding sampled-system (*$N(z^*;h^*) = 0$*). The* stability crossing set $\mathcal{S}$ *is defined as the collection of all critical samplings. Finally, the crossing set* Θ *is defined as the collection of all* $\theta \in [0, 2\pi)$ *such that there exists at least one sampling period* h^* *which a stability crossing point with the critical zero* $z^* = e^{i\theta}$ *(*$N\left(e^{i\theta}; h^*\right) = 0$*).*

Even though the following matrices have been already introduced, we will redefine it, it the seek of completeness. Lets introduce now the following parameter-dependent matrices $M_1, M_2 : \mathbb{R}_+ \mapsto \mathbb{R}^{n\times n}$:

$$M_1(h) \triangleq \begin{bmatrix} a_0(h) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ a_{n-1}(h) & \cdots & a_0(h) \end{bmatrix}; \; M_2(h) \triangleq \begin{bmatrix} a_n(h) & \cdots & a_1(h) \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_n(h) \end{bmatrix},$$

matrices associated to $N(z;h) = \sum_{k=0}^{n} a_k(h) z^k$. Next, introduce $\mathcal{P} := (h) \subset \mathbb{R}_+$ as the set of all points h satisfying the equality $\det(W_N(h)) = 0$, where $W_N(h) := M_2(h) M_2^T(h) - M_1(h) M_1^T(h)$.

Proposition 11.1. *The inclusion* $\mathcal{S} \subset \mathcal{P}$*, holds and is strict.*

Proof. The proof follows from the observation that $\mathcal{P}$ contains not only critical samplings h^*, but also symmetric points, for which the equality $\det(W_N(h)) = 0$ is also valid. □

We have the following:

Proposition 11.2. *Consider the discretized system (11.9) and let the zero transmission behavior be characterized by the polynomial function* $N(z;h)$*. Then the sampling* $h = h^* \in \mathcal{S}$ *is a stability crossing point if and only if the following properties hold simultaneously:*

(i) $h^* \in \mathbb{R}_+$*;*

(ii) $\det(W_N(h^*)) = 0$*;*

(iii) $\sigma(\Phi(h^*)) \cap \partial\mathbb{D} \neq \emptyset$*.*

Remark 11.1. *As mentioned in Chapter 10 we recall here that the condition (ii) is necessary but not sufficient for the existence of a crossing since, excepting the real crossing points, the determinant above vanishes also for symmetric points with respect to the unit circle* $\partial\mathbb{D}$[6]*.*

[6] see, *Proposition* 11.1 and *Remark* 10.3

11.3.1 Crossing Direction Characterization

Proposition 11.2 explicitly gives the set of all sampling periods ($h > 0$) with corresponding characteristic roots on the unit circle $\partial\mathbb{D}$. Then, assuming first that the sampling period parameter is fixed to some critical value h^* for which there exists at least one critical characteristic root on $\partial\mathbb{D}$, the characterization of the crossing directions is given as follows:

Proposition 11.3. *Let $h = h^* \in \mathbb{R}_+$ be a critical sampling for the crossing frequency $z = e^{\boldsymbol{i}\theta^*}$. Under the assumption that the critical characteristic roots of N are simple, the following statements are equivalent:*

(i) The root $z^* = e^{\boldsymbol{i}\theta^*}$ is crossing $\partial\mathbb{D}$ towards instability (stability).

(ii) The following inequality holds:

$$\left.\frac{d \mid z \mid}{dh}\right|_{h=h^*} > 0 \quad (< 0),$$

for any h sufficiently close to h^*, but $h > h^*$.

(iii) The following inequality holds:

$$\Re\left\{\frac{\frac{dN\left(e^{\boldsymbol{i}\theta^*},h\right)}{dh}|_{h=h^*}}{z\frac{dN(z,h^*)}{dz}|_{z=e^{\boldsymbol{i}\theta^*}}}\right\} < 0 \quad (> 0), \tag{11.12}$$

then for any h sufficiently close to h^* but $h > h^*$.

Even in the case of simple eigenvalues the condition (11.12) can vanish, in such a case the following *Proposition* provides a second order analysis:

Proposition 11.4. *Let h^* be a critical sampling-period such that the condition (11.12) vanish. Under the assumption that the critical characteristic roots ($z^* = e^{\boldsymbol{i}\theta^*}$) of N is simple, the following statements are equivalent:*

(i) The root $z^* = e^{\boldsymbol{i}\theta^*}$ stays outside (inside) of the unit circle $\partial\mathbb{D}$.

(ii) The following inequality holds:

$$\left.\frac{d^2 \mid z \mid}{dh^2}\right|_{h=h^*} > 0 \quad (< 0),$$

for any h sufficiently close to h^*, but $h > h^*$.

(iii) The following inequality holds:

$$\Re\left\{\left.\frac{2\frac{\partial^2 N}{\partial z\partial h}\frac{\partial N}{\partial h}\frac{\partial N}{\partial z} - \frac{\partial^2 N}{\partial z^2}\left(\frac{\partial N}{\partial h}\right)^2}{z\left(\frac{\partial N}{\partial z}\right)^3}\right|_{z=z^*,h=h^*}\right\} + \Im\left\{\frac{\frac{\partial N(z,h)}{\partial h}|_{h=h^*}}{z\frac{\partial N(z,h^*)}{\partial z}|_{z=z^*}}\right\}^2 > 0\,(< 0),$$

then for any h sufficiently close to h^* but $h > h^*$.

11.3.2 Multiple critical samplings

Now, in order to analyze a more general situation, i.e. a multiple critical samplings, lets introduce the following notions. Let h^* and $\lambda_0^* = e^{i\theta^*}$, $\theta^* \in [0, 2\pi)$ be a critical pair of critical sampling and critical zero of $N(z;h)$, i.e.,$N(\lambda_0^*;h^*) = 0$. We define, $T : \mathbb{C} \times \mathbb{R}_+ \mapsto \mathbb{C}^{n+1 \times n+1}$ given by:

$$T(\lambda, \varepsilon) := \underbrace{\begin{bmatrix} I(\lambda + \lambda_0^*) - \Phi(\varepsilon^*) & -\Gamma(\varepsilon^*) \\ c & d \end{bmatrix}}_{=:A(\lambda)} + \underbrace{\begin{bmatrix} \Phi(\varepsilon^*) - \Phi(\varepsilon + \varepsilon^*) & -\Phi(\varepsilon^*)\Gamma(\varepsilon) \\ 0 & 0 \end{bmatrix}}_{=:B(\varepsilon)}. \tag{11.13}$$

According to this definition it is clear that $\lambda = 0$ is an eigenvalue of $A(\lambda)$ and that $B(0) \equiv 0$, moreover, it is clear that $T(\lambda, \varepsilon)$ is holomorphic around $\lambda = 0$ and $\varepsilon = 0$. In the rest of this subsection we will adopt the same notations introduced in the *preliminary section* 11.2, i.e., m_i, $i = 1, \ldots, g$ will denote the partial multiplicities of the eigenvalue $\lambda = 0$, $g = \dim(\ker A(0))$ is the geometric multiplicity, k denotes the number of groups mutually equal m_i, where the $j-$th group contain n_j elements (see section 11.2, for more details). In the following we will refer to equation (11.7) as the $j-th$ *polynomial*:

$$P_j(\xi) := \Delta_j + \sum_{\ell=1}^{n_j-1} \Delta_{j\ell}\xi^\ell + \Delta_{j+1}\xi^{n_j}. \tag{11.14}$$

Additionally, we will denote $\widehat{E}(\lambda) := E^{-1}(\lambda)$ and $\widehat{F}(\lambda) := F^{-1}(\lambda)$, where these matrices are partitioned as follow,

$$\widehat{E}(\lambda) = \begin{bmatrix} \widehat{E}_1(\lambda) & \widehat{E}_2(\lambda) \\ \widehat{E}_3(\lambda) & \widehat{E}_4(\lambda) \end{bmatrix}, \ \widehat{F}(\lambda) = \begin{bmatrix} \widehat{F}_1(\lambda) & \widehat{F}_2(\lambda) \\ \widehat{F}_3(\lambda) & \widehat{F}_4(\lambda) \end{bmatrix}. \tag{11.15}$$

Then, in the same spirit of *Part* II (*chapter* 10) we have the following results:

Proposition 11.5. *Assume that $\Delta_1 \cdots \Delta_k \neq 0$. Then for each $j \in \{1, \ldots, k\}$ and any sampling period h sufficiently close to h^*, there are $n_j\widetilde{m}_j$ characteristic zeros of the discretized system (11.9) corresponding to λ_o^* which can expanded by Puiseux series:*

$$\lambda_{j\nu\sigma}(h) = \lambda_o^* + \gamma_{j\nu}\vartheta_{\widetilde{m}_j,\sigma}(h - h^*)^{\frac{1}{\widetilde{m}_j}} + o\left(|h - h^*|^{\frac{1}{\widetilde{m}_j}}\right), \ \sigma = 1, \ldots, \widetilde{m}_j,$$

here $\gamma_{j\nu}$ satisfy $(\gamma_{j\nu})^{\widetilde{m}_j} = \xi_{j\nu}$ where $\xi_{j\nu}$, $\nu = 1, \ldots, n_j$ are the solutions of the $j-th$ polynomial (11.14), with $\Delta_{j\ell}$, Δ_j given by (11.5)-(11.6) and

$$H = -\widehat{E}_1(0)\Phi(h^*)\left[A\widehat{F}_1(0) + b\widehat{F}_3(0)\right].$$

Thus, for h sufficiently close to h^* but $h > h^*$ there are at least M ($M \leq \widetilde{m}_j$) of the characteristic zeros going outside (inside) the unit circle $\partial\mathbb{D}$ if M of the eigenvalues satisfy the condition:

$$\cos(\theta_\ell - \theta^*) > 0 \ (< 0), \qquad \ell = 1, \ldots, \widetilde{m}_j$$

where $\theta_\ell \in [0, 2\pi)$ is the phase angle of $\gamma_{j\nu}\vartheta_{\widetilde{m}_j,\sigma} \neq 0$.

In the case of a simple roots, of the $j-th$ polynomial the result above rewrites as follows:

Proposition 11.6. *Assume that $\Delta_1 \cdots \Delta_k \neq 0$. If for some $j \in \{1, \ldots, k\}$ the $j-th$ polynomial (11.14) has a simple root $\widehat{\xi}_j$, then there is a group of $\widetilde{m}_j$ discretized zeros having a Puiseux expansion:*

$$\widehat{\lambda}_{j\sigma}(h) = \lambda_0^* + \widehat{\gamma}_j \vartheta_{\widetilde{m}_j,\sigma} (h-h^*)^{\frac{1}{\widetilde{m}_j}} + \sum_{\ell=2}^{\infty} \alpha_{j\ell} \vartheta^{\ell}_{\widetilde{m}_j,\sigma} \varepsilon^{\frac{\ell}{\widetilde{m}_j}}, \; \sigma = 1, \ldots, \widetilde{m}_j.$$

with $\left(\widehat{\gamma}_j\right)^{\widetilde{m}_j} = \widehat{\xi}_j$. Thus, for h sufficiently close to h^* but $h > h^*$ there are at least M ($M \leq \widetilde{m}_j$) of the characteristic zeros going outside (inside) the unit circle $\partial\mathbb{D}$ if M of the eigenvalues satisfy the condition:

$$\cos\left(\theta_\ell - \theta^*\right) > 0 \;\; (< 0), \qquad \ell = 1, \ldots, \widetilde{m}_j$$

where $\theta_\ell \in [0, 2\pi)$ is the phase angle of $\widehat{\gamma}_j \vartheta_{\widetilde{m}_j,\sigma} \neq 0$.

Finally, the next result concerns the case when all partial multiplicities are equal and simple.

Corollary 11.1. *Assume that $m_1 = \cdots = m_g =: m$ and let $\mu_1, \ldots, \mu_g$ the eigenvalues of H, such that $\mu_i \neq 0$, for all $i = 1, \ldots, g$. Then, for any h sufficiently close to h^* but $h > h^*$ the characteristic zeros of the discretized system (11.9) can be expanded as:*

$$\lambda_{\ell\sigma}(h) = \lambda_o^* + \sqrt[m]{\mu_\ell} \vartheta_{m,\sigma} (h-h^*)^{\frac{1}{m}} + o\left(|h-h^*|^{\frac{1}{m}}\right), \; \ell = \overline{1,g}, \; \sigma = \overline{1,m}.$$

Hence, for h sufficiently close to h^* but $h > h^*$, the number of critical zeros going to outside the unit circle $\partial\mathbb{D}$ (or vice versa) can be determined by the condition

$$\cos\left(\theta_\ell - \theta^*\right) > 0 \;\; (< 0), \qquad \ell = 1, \ldots, \widetilde{m}_j$$

where $\theta_\ell \in [0, 2\pi)$ is the phase angle of $\sqrt[m]{\mu_\ell} \vartheta_{m,\sigma} \neq 0$.

Remark 11.2. *Observe that in the previous results our main assumption is that all $\Delta_1 \cdots \Delta_k \neq 0$. However, in case that this condition vanish it is still possible to extended the previous results, by applying the similar ideas than [76].*

11.3.3 Numerical Examples

In order to illustrate the previous results, we consider the following examples.

Example 11.1. *Consider a transfer function*

$$G(s) = \frac{b_1 s + b_0}{(s+a)^2} \tag{11.16}$$

where $a \in \mathbb{R} \setminus \{0\}$ and $b_0, b_1 \in \mathbb{R}$ are not both zero. We have the following situations:

i) Let $b_0 = 0$ and $b_1 \neq 0$. In this case we have that $N(z;h) = (b_1 h)e^{-ah}(z-1)$ implying that all sampling periods are critical, i.e. $\mathcal{S} = \mathbb{R}_+$.

ii) *If $b_0 \neq 0$ and $b_1 = 0$, then applying Proposition 11.2 we have that the critical sampling are characterized by the roots of the equation $f(h) = 0$, where:*

$$f(h) = e^{2ah} - 2(ah)e^{ah} - 1. \tag{11.17}$$

Since, $f(h) = 0 \Leftrightarrow h = 0$, we conclude that the only critical sampling is $h^* = 0$.

iii) *Consider now $b_i \neq 0$, $i = 1, 2$. According to Proposition 11.2, the critical samplings are characterized by the solutions of the equation g(x)=0, where:*

$$g(x) = (b_0 - ab_1)\, x\, csch(x) - b_0. \tag{11.18}$$

Then a necessary and sufficient condition *for the existence and uniqueness of a critical sampling $h^* \in \mathbb{R}_+$ is given by:*

$$0 < \frac{b_0}{b_0 - ab_1} < 1. \tag{11.19}$$

If the inequality (11.19) holds, then the stability crossing points (critical sampling) are given by $h^* = \left|\frac{x^*}{a}\right|$, where x^* is a zero of (11.18).

Table 11.1 summarizes the previous discussion.

Table 11.1: Critical sampling and direction of crossing for (11.16). β is given by $\beta = \{x^* \in \mathbb{R} : (b_0 - ab_1)\, x^* csch(x^*) - b_0 = 0\}$.

Case	**Critical Sampling**	$\Re\left\{\frac{\left.\frac{dN}{dh}\right\vert_{h=h^*}}{z\left.\frac{dN}{dz}\right\vert_{z=e^{i\theta^*}}}\right\}$
$\begin{cases} b_0 = 0 \\ b_1 \neq 0 \end{cases}$	$h^* = \mathbb{R}$	$*$
$\begin{cases} b_0 \neq 0 \\ b_1 = 0 \end{cases}$	$h^* = 0$	$\begin{cases} 2a \\ sign(a) \end{cases}$
$\begin{cases} b_0 \neq 0 \\ b_1 \neq 0 \end{cases}$	$h^* = \left\vert\frac{\beta}{a}\right\vert$ if $0 < b_0/(b_0 - ab_1) < 1$	$\begin{cases} -2a \\ -sign(a) \end{cases}$
	$h^* = 0$ otherwise	$\begin{cases} 2b_0/b1 \\ sign(b_0/b1) \end{cases}$

Remark 11.3. *Taking $b_1 = 0$ in (11.16), we recover the transfer function $G_0(s)$ considered in [62, pp.1560]. Although we arrive to a "similar" conclusion here, the analysis proposed by [62] is not complete.*

Example 11.2. *Consider the following non-minimal phase system with the transfer function[43]:*

$$G(s) = \frac{(s - 1/10)^2 + 1}{(s+1)(s + 99/100)(s + 101/100)}. \tag{11.20}$$

Then, evaluating Proposition 11.2, we get that the only critical sampling is $h_1^* = 1.067186$. According to Table 11.3, we conclude that the system is non-minimal phase for all $h \in \mathrm{I}^U = (0, 1.067186]$ and minimal-phase if $h \in \mathrm{I}^S = (1.067186, \infty)$. The interest of this example lies on the fact that the method of Ishitobi [62] does not work in this case, as pointed out by Hagander [43].

Table 11.2: Critical values and direction of crossing for system (11.21).

	Prop. 11.2(ii)	$\{z \in \mathbb{C} \,\vert\, N(z,h^*) = 0\}$	Cond. (11.12)
h_1^*	1.067186	$\{0.369222 \pm \boldsymbol{i}\, 0.929340\}$	0.226438

Example 11.3. *Consider the following unstable and non-minimal phase system with the transfer function:*

$$G(s) = \frac{s-1}{s^3 + \frac{173}{50}s^2 + \frac{143}{50}s - \frac{3}{25}} \tag{11.21}$$

Then, evaluating condition (10.19), we obtain the plot illustrated in Fig.11.2. According to

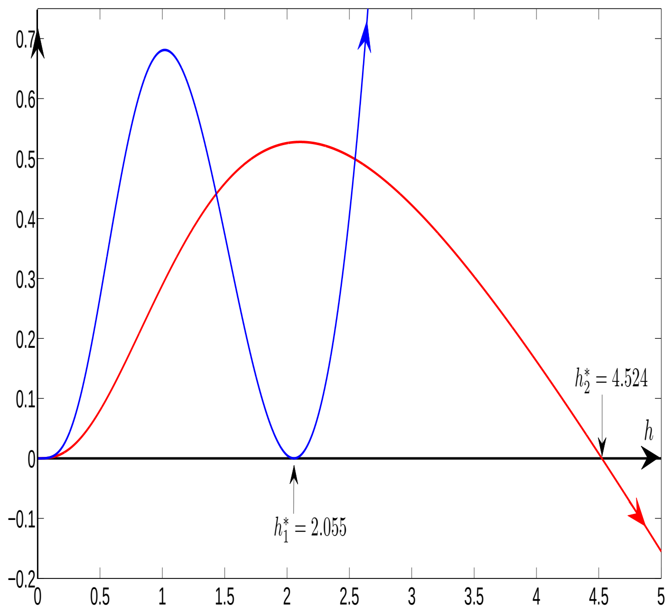


Figure 11.2: Evaluation of (10.19), for the discretized system (11.21).

Fig.11.2, it appears that the possible critical points are $h_1^* = 2.055$ *and* $h_2^* = 4.524$. *However, taking* $h = h_1^*$ *we have that* $N(e^{\boldsymbol{i}\theta}, h_1^*) \neq 0$ *for all* $\theta \in [0,\pi)$, *this implies that* $N(z,h_1^*)$ *has the symmetric property mentioned previously. Indeed, these values are* $z_1 = -0.059835$ *and* $z_2 = 1/z_1 = -16.7126$. *Then, by applying Proposition 11.2, we get that the only critical sampling is* $h = 4.524$. *The next table summarizes the above discussion. According with*

Table 11.3: Critical values and direction of crossing for system (11.21).

	Prop.1(ii)	$\{z \in \mathbb{C} \,\vert\, N(z,h^*) = 0\}$	$\Re\left\{\frac{\frac{dN}{dh}\big\vert_{h=h^*}}{z\frac{dN}{dz}\big\vert_{z=e^{i\theta^*}}}\right\}$
h_1^*	2.0550	$\{-16.71, -0.059\}$	not applicable
h_2^*	4.5242	$\{-1, -0.002\}$	$0.387\,(+)$

Table 11.3 we conclude that the system is non-minimal phase for all $h \in \mathrm{I}^U = (0, 4.5242]$ *and minimal-phase if* $h \in \mathrm{I}^S = (4.5242, \infty)$.

Example 11.4. *Consider now the following non-minimal-phase system:*

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{-1}{16} & \frac{-1}{16} & -1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} 1 & -2 & 1 \end{bmatrix} x(t). \end{cases}$$

The corresponding transfer-function writes as:

$$G(s) = \frac{(s-1)^2}{(s+1)\left(s^2+1/16\right)} \tag{11.22}$$

In this cases we will have an infinity but countable critical samplings, assuming that we are interested only in sampling periods satisfying the inequality $0 < h^* < 100$. Table 11.4 summarizes the results obtained from Proposition 11.2. According to Table 11.4 we observe that

Table 11.4: Crossing direction for the respective critical samplings.

$k \in \mathbb{N}$ $h_\ell^*(k)$	Critical Sampling	$\{z \in \mathbb{C} \mid N(z, h_\ell^*) = 0\}$	$\Re\left\{ \frac{\left.\frac{dN}{dh}\right\|_{h=h^*}}{\left. z\frac{dN}{dz}\right\|_{z=e^{i\theta^*}}} \right\}$
$h_1^*(k)$	2.0617074	$\{-1, 4.151366\}$	-1.029907 $(-)$
$h_2^*(k)$	27.757700	$\{-1, \approx 8.529411\}$	-1.314621 $(-)$
$h_3^*(k)$	52.890441	$\{-1, \approx 8.529411\}$	-1.314621 $(-)$
$h_4^*(k)$	78.023182	$\{-1, \approx 8.529411\}$	-1.314621 $(-)$
$h_5^*(k)$	103.155924	$\{-1, \approx 8.529411\}$	-1.314621 $(-)$
$h_6^*(k)$	$4(2k-1)\pi$	$\left\{-1, \frac{15-2e^{4(2k-1)\pi}}{15e^{4(2k-1)\pi}-2}\right\}$	-1.314621 $(-)$
$h_7^*(k)$	$8k\pi$	$\{1, 1\}$	$*$

the critical sampling $h_7^* = 8k\pi$ has a double zero at $z^* = 1$, then in order to determine the crossing direction we consider the following invertible matrix functions:

$$E(\lambda) = \begin{bmatrix} -\frac{1}{4} & \frac{1}{2} & \frac{3}{4} & \frac{\lambda}{4} \\ \frac{-26-17\lambda\beta}{128} & \frac{17(2+\lambda\beta)}{64} & \frac{94-17\lambda\beta}{128} & \frac{-17\lambda(\lambda+\alpha)}{64\alpha} \\ \frac{17}{4\alpha} & -\frac{17}{2\alpha} & \frac{17}{4\alpha} & -\frac{17\lambda\beta}{8} \\ \frac{17}{8\alpha} & -\frac{17}{4\alpha} & \frac{17}{8\alpha} & \frac{-8-17\lambda\beta}{16} \end{bmatrix}, \tag{11.23}$$

$$F(\lambda) = \begin{bmatrix} -16 & 16 & 1 & 0 \\ -\frac{15}{2} & 8 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \tag{11.24}$$

where $\alpha = 1 - e^{-8\pi}$, $\beta = 1 + \coth(4\pi)$ *and* coth *is the hyperbolic cotangent function. We also have that* $\det E(\lambda) = \frac{17}{128}(1+\coth(4\pi))$ *and* $\det F(\lambda) = 1$. *With these matrices, we have that*

$$D(\lambda) = E(\lambda)^{-1} A(\lambda) F(\lambda)^{-1} = \begin{bmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then, we have $m_1 = m_2 = 1$, $g = 2$, $k = 1$ *and* $n_1 = 2$. *All theses facts together implies that we can apply Corollary 11.1. Then, we obtain that zero as a function of the sampling period around* $h^* = 8\pi$ *behave as:*

$$\lambda_\ell(h) = 1 + \frac{47 \pm j\sqrt{2143}}{128}(h - h^*) + o(|h - h^*|), \quad \ell = 1, 2.$$

Since $\cos(\theta_\ell) = \frac{47}{16\sqrt{17}} > 0$ *for* $\ell = 1, 2$, *implies that the discretized zeros are crossing towards instability. Such a behavior is illustrated in Fig.11.3.*

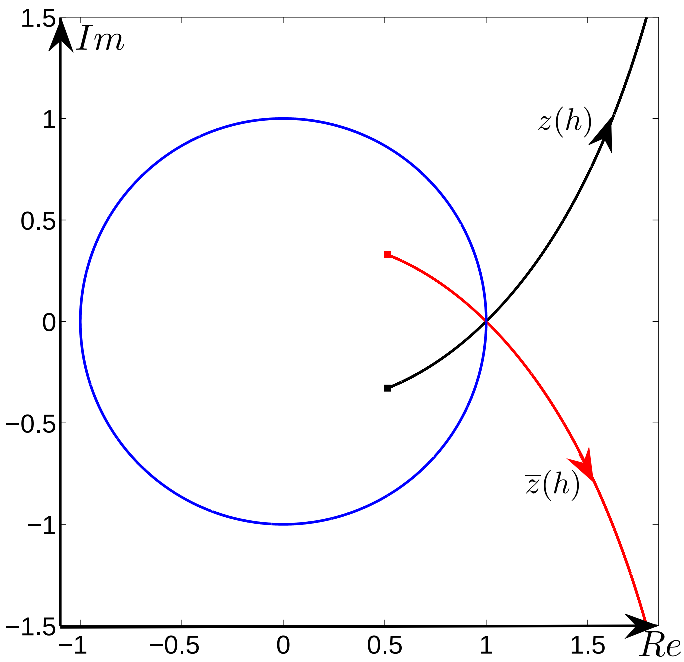


Figure 11.3: Sampled-zero behavior around $h^* = 8\pi$.

11.4 Future Works: Singular Matrix Functions

In the previous sections (*section* 11.2-11.3) we have studied analytically the behavior of the eigenvalue λ^* of the matrix function $T(\lambda, h)$ (see, (11.13)) around some $h^* > 0$, or in other words we have implicitly assumed that the matrix function $T(\lambda, h)$ is *regular*. Here, we say *regular* in spite of the following

Definition 11.2. *A matrix function* $A(\lambda)$ *is singular if for all* $\lambda \in \mathbb{C}$,

$$\det(A(\lambda)) \equiv 0.$$

Otherwise, the matrix-valued function $A(\lambda)$ *is said to be regular.*

Now, if we are interested in the asymptotic eigenvalue behavior around $h^* = 0$ the matrix-valued function $T(\lambda,h)|_{h=0}$ becomes *singular*. The term *limiting zeros* is used to denote the discretized zeros for sufficiently small or large sampling periods. Then, in order to study the asymptotic properties of the limiting zeros when $h \to 0+$, such a zeros have been classified in two categories [4, 44]:

(i) *intrinsic zeros*: these correspond to the zeros of the continuous time system and approach to $z = 1$ as $h \to 0+$. Then, if the continuous-time system is strictly proper, there are at most $n-1$ intrinsic zeros.

(ii) *discretization zeros*: these zeros doesn't have continuous-time counterpart and depend on the relative degree (ρ) of the continuous-time system. Moreover, the behavior as $h \to 0+$ is given by the following Euler polynomial:

$$\begin{aligned} B_\rho(z) &= b_1^{(\rho)} z^{\rho-1} + b_2^{(\rho)} z^{\rho-2} + \cdots + b_\rho^{(\rho)} \\ b_k^{(\rho)} &= \sum_{i=1}^{k} (-1)^{k-i} i^\rho \binom{\rho+1}{k-i}, \qquad k = 1,2,\ldots,\rho. \end{aligned}$$

From the above definition it is not difficult to see that $B_\rho(z)$ is a *symmetric polynomial*, i.e., all their coefficients satisfy $b_k^{(\rho)} = b_{\rho-k+1}^{(\rho)}$ and consequently their roots are symmetric with respect to the unit circle, i.e. since $B_\rho(z) = B_\rho(\frac{1}{z})$, this implies that there the same number of zeros inside and outside of the unit circle (for more details analysis of this polynomial, see[126]).

It is worth to mention that several attempts have been made (see for instance, ([7, 13, 44, 45, 62, 63, 64], and reference therein) in order to characterize asymptotically the zero behavior as $h \to 0+$. However the problem is still open, and the mean reason is that in all the above works they made a description of the limiting zeros, assuming that the continuous zero is simple, and under this assumption they show that the limiting zero $z_d(h)$ admits a Taylor expansion with respect to the sampling period h, i.e., they show that $z_d(h)$ behave as

$$z_d(h) = 1 + \gamma_1 h + \gamma_2 h^2 + \gamma_3 h^3 + \cdots. \tag{11.25}$$

However, if the continuous zeros are multiple, such an expansion is not necessarily valid. In order to illustrate such a situation, lets consider the continuous-time system, given by the following transfer function

$$G(s) = \frac{(s-\gamma)^3}{(s+1)^2\left(s^2 + \frac{1}{4^2}\right)}, \quad \gamma \neq 0. \tag{11.26}$$

In this case, in order to expand in series the limiting zeros $z_d(h)$, we apply the *Newton-Diagram* 2.2, obtaining the diagram depicted in figure 11.4. Now, according with the *Newton-Diagram Procedure* we conclude that the limiting zeros for $0 < h = \varepsilon \ll 1$ behaves as

$$\begin{aligned} z_1(\varepsilon) &= 1 - \gamma\varepsilon + \frac{\left(\gamma(\gamma-1)^2\left(16\gamma^2+1\right)\right)^{1/3} e^{-j\frac{\pi}{3}}}{4\sqrt[3]{3}} \varepsilon^{\frac{5}{3}} + o\left(\varepsilon^{\frac{5}{3}}\right), \\ z_2(\varepsilon) &= 1 - \gamma\varepsilon + \frac{\left(\gamma(\gamma-1)^2\left(16\gamma^2+1\right)\right)^{1/3} e^{j\frac{\pi}{3}}}{4\sqrt[3]{3}} \varepsilon^{\frac{5}{3}} + o\left(\varepsilon^{\frac{5}{3}}\right), \\ z_3(\varepsilon) &= 1 - \gamma\varepsilon + \frac{\left(\gamma(\gamma-1)^2\left(16\gamma^2+1\right)\right)^{1/3} e^{j\pi}}{4\sqrt[3]{3}} \varepsilon^{\frac{5}{3}} + o\left(\varepsilon^{\frac{5}{3}}\right). \end{aligned}$$

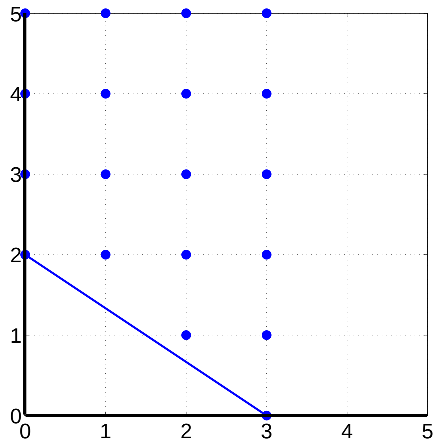


Figure 11.4: Newton Diagram for discretized system (11.26).

Then, by an appropriate extension of the previous results, as a future work can be considered the asymptotic eigenvalue behavior of the matrix function $T(\lambda,h)$ as $h \to 0+$.

A Mathematical Background

A.1 Chebyshev Polynomials Definitions

Definition A.1 ([86]).

(i) The Chebyshev polynomial $T_n(x)$ of the first kind is a polynomial in x of degree n, defined by

$$T_n(x) = \cos n\theta, \quad \textit{when } x = \cos\theta,$$

(ii) The Chebyshev polynomial $U_n(x)$ of the second kind is a polynomial in x of degree n, defined by

$$U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta}, \quad \textit{when } x = \cos\theta.$$

A.2 Leverrier-Sauriau-Frame Algorithm

Theorem A.1 ([90]). *Let the characteristic equation for $A \in \mathbb{R}^{n\times n}$ be given by, $\lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \cdots + c_n = 0$, and define a sequence by taking $B_0 = I$ and $B_i = -\frac{trace(AB_{i-1})}{i}I + AB_{i-1}$ for $i = 1, 2, \ldots, n$. Then, the $i\,th$ coefficient is $c_i = -\frac{trace(AB_{i-1})}{i}$.*

A.3 Rouché's Lemma

In complex analysis, Rouché's Lemma states that if the complex-valued functions f and g are holomorphic inside and on some closed contour $\mathcal{D}$, with $|g(z)| < |f(z)|$ on $\mathcal{D}$, then f and $f + g$ have the same number of zeros inside $\mathcal{D}$, where each zero is counted as many times as its multiplicity. This theorem assumes that the contour $\mathcal{D}$ is simple, that is, without self-intersections.

Theorem A.2. *[78] Let $f(z)$ and $g(z)$ be analytic in a simply connected domain $\mathcal{D}$ containing a Jordan contour $\mathcal{J}$. Let $|f(z)| > |g(z)|$ on $\mathcal{J}$. Then, $f(z)$ and $f(z) + g(z)$ have the same number of zeros inside $\mathcal{J}$.*

A.4 Implicit Function Theorem

Given a set of suitable equations, the *Implicit Function theorem* states that some of the variables can be defined as a functions of the others.

In the general case we shall have a function $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$, and consider the relation,

$$\begin{array}{rcl} F_1(x_1, \ldots, x_n, y_1, \ldots, y_m) & = & 0 \\ . & & . \\ . & & . \\ . & & . \\ F_m(x_1, \ldots, x_n, y_1, \ldots, y_m) & = & 0. \end{array}$$

Then, the Implicit Function theorem guarantees at least locally that we can find a unique differentiable f such that $F(x, f(x)) = 0$. The theorem is as follows.

Theorem A.3 (Implicit Function theorem)**.** *[85, 73] Let $A \subset \mathbb{R}^n \times \mathbb{R}^m$ be an open set and let $F : A \to \mathbb{R}^m$ be a function of class C^p (that is F has p continues derivatives where $p \in \mathbb{N}$). Suppose $(x_0, y_0) \in A$ and $F(x_0, y_0) = 0$. Form*

$$\Delta(x,y) = \begin{vmatrix} \frac{\partial F_1}{\partial y_1} & \cdots & \frac{\partial F_1}{\partial y_m} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \frac{\partial F_m}{\partial y_1} & \cdots & \frac{\partial F_m}{\partial y_m} \end{vmatrix},$$

where $F = (F_1, \ldots, F_m)$. Suppose that $\Delta(x_0, y_0) \neq 0$. Then there is an open neighborhood $U \subset \mathbb{R}^n$ of x_0 and a neighborhood $V \subset \mathbb{R}^m$ of y_0 and a unique function $f : U \to V$ such that

$$F(x, f(x)) = 0$$

for all $x \in U$, Furthermore, f is of class C^p.

Corollary A.1. *[85] In Theorem A.3,$\partial f_j / \partial x_i$ are given by*

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix} = \begin{pmatrix} \frac{\partial F_1}{\partial y_1} & \cdots & \frac{\partial F_1}{\partial y_m} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \frac{\partial F_m}{\partial y_1} & \cdots & \frac{\partial F_m}{\partial y_m} \end{pmatrix}^{-1} \begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \frac{\partial F_m}{\partial x_1} & \cdots & \frac{\partial F_m}{\partial x_n} \end{pmatrix}.$$

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