Viana maps and limit distributions of sums of point measures

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Abstract

This thesis consists of five articles mainly devoted to problems in dynamical systems and ergodic theory. We consider non-uniformly hyperbolic two dimensional systems and limit distributions of point measures, which are absolutely continuous with respect to Lebesgue measure.

Let $f_{a_0}(x) = a_0 - x^2$ be a quadratic map, where the parameter $a_0 \in (1, 2)$ is chosen such that the critical point 0 is pre-periodic (but not periodic). In Papers A and B, we study skew-products $(\theta, x) \mapsto F(\theta, x) = (g(\theta), f_{a_0}(x) + \alpha s(\theta))$, $(\theta, x) \in S^1 \times \mathbb{R}$. The functions $g : S^1 \to S^1$ and $s : S^1 \to [-1, 1]$ are the base dynamics and the coupling functions, respectively, and α is a small, positive constant. Such quadratic skew-products are also called Viana maps. In Papers A and B, we show for several choices of the base dynamics and the coupling function that the map F has two positive Lyapunov exponents and for some cases we further show that F admits also an absolutely continuous invariant probability measure.

In Paper C we consider certain Bernoulli convolutions. By showing that a specific transversality property is satisfied, we deduce absolute continuity of the distributions associated to these Bernoulli convolutions.

In Papers D and E, we consider sequences of real numbers on the unit interval and study how they are distributed. The sequences in Paper D are given by the forward iterations of a point $x \in [0, 1]$ under a piecewise expanding map $T_a : [0, 1] \to [0, 1]$ depending on a parameter a contained in an interval I. Under the assumption that each T_a admits a unique absolutely continuous invariant probability measure μ_a and that some technical conditions are satisfied, we show that the distribution of the forward orbit $T_a^j(x)$, $j \geq 1$, is described by the distribution μ_a for Lebesgue almost every parameter $a \in I$. In Paper E we apply the ideas in Paper D to certain sequences, which are equidistributed in the unit interval and give a geometrical proof of a well-known result by Koksma from 1935.

Sammanfattning

Denna avhandling består av fem artiklar i vilka huvudsakligen problem inom dynamiska system och ergodteori studeras. Vi behandlar ickelikformigt hyperboliska, två dimensionella system och gränsfördelningar av punktmassor som är absolutkontinuerliga med avseende på Lebesguemått.

Låt $f_{a_0}(x) = a_0 - x^2$ vara en kvadratisk avbildning där parametern $a_0 \in (1,2)$ är vald sådan att den kritiska punkten 0 är preperiodisk (men inte periodisk). I artiklarna A and B behandlar vi skevprodukter $(\theta, x) \mapsto F(\theta, x) = (g(\theta), f_{a_0}(x) + \alpha s(\theta))$, $(\theta, x) \in S^1 \times \mathbb{R}$. Funktionerna $g : S^1 \to S^1$ och $s : S^1 \to [-1, 1]$ benämns basavbildningen, respektive kopplingsfunktionen och α är en liten, positiv konstant. Sådana kvadratiska skevprodukter kallas också för Vianaavbildningar. I artiklarna A and B visar vi för olika val av basfunktioner och kopplingsfunktioner att avbildningen F har två positiva Lyapunovexponenter och i några fall visar vi dessutom att F har ett absolutkontinuerligt invariant sannolikhetsmått.

I artikeln C studeras vissa Bernoullifaltningar. Genom att verifiera att en speciell transversalitetsegenskap är uppfylld, visar vi att de till de Bernoullifaltningarna relaterade fördelningarna är absolutkontinuerliga.

I artiklarna D och E behandlar vi följder av reella tal i enhetsintervallet och studerar hur de är fördelade. Följderna i artikeln D ges av framåtiterationer av en punkt $x \in [0, 1]$ under en styckvist expanderande avbildning $T_a : [0, 1] \to [0, 1]$ som är beroende av en parameter a i ett intervall I. Under antagandet att varje avbildning T_a har ett unikt absolutkontinuerligt invariant sannolikhetsmått μ_a och att några tekniska villkor är uppfyllda, visar vi att fördelningen av framåtbanan $T_a^j(x)$, $j \geq 1$, är given av fördelningen μ_a för Lebesgue nästan alla parametrar $a \in I$. I artikeln E tillämpar vi ideerna i artikeln D på några följder, som är likformigt fördelade i enhetsintervallet och vi ger ett geometriskt bevis av ett välkänt resultat av Koksma från 1935.

Contents

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Part I

Introduction and summary

Chapter 1

Introduction

1.1 Viana maps

Consider the quadratic map

$$x \mapsto f_{a_0}(x) = a_0 - x^2, \qquad x \in \mathbb{R},$$

where the parameter $1 < a_0 < 2$ is chosen such that the critical point 0 is pre-periodic (but not periodic). It is well-known that such a quadratic map — also called *Misiurewicz-Thurston* map — has a positive Lyapunov exponent, i.e. there exists a constant $\lambda > 0$ such that

$$\liminf_{n\to\infty} \frac{1}{n} \log |Df^n_{a_0}(x)| \geq \lambda,$$

for Lebesgue almost every $x \in \mathbb{R}$. Consider now the situation when we add after every iteration step a small perturbation to the obtained value and then continue the iteration with this perturbated value. The main question in the first two papers of this thesis is under what kind of perturbations we still have a positive Lyapunov exponent. The perturbations we are considering are correlated. The i.i.d. case is done in [BBM] and [BY]. The maximal size of the perturbation in each step will not exceed the value α, where α is a small, positive real number. More precisely, the maps we are considering are skew-products of the form

$$F : S^1 \times \mathbb{R} \to S^1 \times \mathbb{R},$$
$$(\theta, x) \mapsto (g(\theta), f_{a_0}(x) + \alpha s(\theta)), \tag{1.1}$$

where $s : S^1 \to [-1, 1]$ is some coupling function and $g : S^1 \to S^1$ is the base dynamics in which choice we are mainly interested in. As the title of this section suggests, the study of the ergodic properties of such quadratic skew-product (also called *Viana maps*) traces back to a paper by Viana [Vi]. In this paper the base dynamics g is given by the strongly expanding continuous β-transformation $\theta \mapsto d\theta \bmod 1$, where $d \geq 16$ is an integer, and the coupling function s is a C^2 Morse function, e.g., $s(\theta) = \sin(2\pi\theta)$. Provided that α is sufficiently small, it is shown that the associated map F has almost everywhere w.r.t. Lebesgue measure two positive Lyapunov exponents, i.e. there exists a constant $\lambda > 0$ such that

$$\liminf_{n\to\infty} \frac{1}{n} \log\|DF^n(\theta, x)v\| \geq \lambda,$$

for Lebesgue a.e. $(\theta, x) \in S^1 \times \mathbb{R}$ and every non-zero vector $v \in \mathbb{R}^2$. Based on this result by Viana, Alves [Al] proved that the map F admits an absolutely continuous invariant probability measure (a.c.i.p.) μ (this measure is unique and its basin has full Lebesgue measure in the invariant cylinder $\widehat{J}$ defined below; see [AV]).

Note that the Jacobian matrix of F^n is a lower triangular matrix with the diagonal entries

$$DF^n(\theta, x) = \begin{pmatrix} d^n & 0 \\ * & \prod_{i=0}^{n-1}(-2x_i) \end{pmatrix},$$

where x_i, $i \geq 0$, is defined by $(\theta_i, x_i) = F^i(\theta, x)$. Clearly, if v is a vector in $\mathbb{R}^2$ whose first component is not equal to zero, then

$$\liminf_{n\to\infty} \frac{1}{n} \log\|DF^n(\theta, x)v\| \geq \log d > 0,$$

for all $(\theta, x) \in S^1 \times \mathbb{R}$. Hence, in order to show that F has two positive Lyapunov exponents, it is enough to focus on the *vertical* Lyapunov exponent, i.e. it is enough to show that there exists a constant $\lambda > 0$ such that

$$\liminf_{n\to\infty} \frac{1}{n} \log \prod_{i=0}^{n-1} 2|x_i| \geq \lambda, \tag{1.2}$$

for Lebesgue a.e. $(\theta, x) \in S^1 \times \mathbb{R}$. Of course, if we start with an x far from the origin then x_i tends to infinity as $i \to \infty$ and (1.2) is trivially satisfied. But, since a_0 is strictly smaller than 2, it is easy to check that there is an open interval $(-1, 1) \subset I \subset (-2, 2)$ such that $F(S^1 \times I) \subset S^1 \times I$, provided α is sufficiently small. Thus, it is sufficient to consider the restriction of F to the invariant region

$$\widehat{J} = S^1 \times I.$$

The basic property of F acting on $\widehat{J}$ is that the expansion in the horizontal direction, i.e. in the θ-direction, is dominating. In other words the endomorphism $F : \widehat{J} \to \widehat{J}$ is a partially hyperbolic system by which we mean that there are constants $\lambda > 0$, $C \geq 1$ and a continuous decomposition $T\widehat{J} = E^c \oplus E^u$ with $\dim E^c = \dim E^u = 1$ such that

$$\|DF^n|_{E^u(z)}\| > C^{-1}e^{\lambda n},$$

and

$$\|DF^n|_{E^c(z)}\| < Ce^{-\lambda n}\|DF^n|_{E^u(z)}\|, \tag{1.3}$$

for all $z \in \widehat{J}$ and $n \geq 0$. The subbundles E^c and E^u are called the central and unstable subbundle, respectively. Since $||\partial_x F^n|| \leq 4^n$ on $\widehat{J}$ and $d \geq 16 > 4$, we can choose $E^c(z) \equiv \{0\} \times \mathbb{R}$, $E^u(z) \equiv \mathbb{R} \times \{0\}$, $\lambda = \log d$ and $C = 1$. Notice that by condition (1.3) the central subbundle E^c is forward invariant and uniquely defined but E^u is not. One can show that for the Viana map F, there exist constants $0 < \lambda < \log 2$ and $C < \infty$ (independent on d) such that $||\partial_x F^n|| \leq Ce^{n\lambda}$ on $\widehat{J}$ (see Lemma 3.1 in [BST]). Hence, the system remains partially hyperbolic if we choose d to be an integer greater or equal to 2. In [BST], Buzzi *et al* showed that the corresponding map F still admits two positive Lyapunov exponents.

In Paper A we treat non-continuous Viana maps. More precisely, instead of a continuous base dynamics, we let g be a β-transformation where the expansion d is any real number $d > 1$ chosen so large that d dominates the vertical expansion. Having such a non-continuous map as the base dynamics, we show that we still have a positive vertical Lyapunov exponent and existence of an a.c.i.p. The main technical novelty in Paper A is that we introduce the concept of remainder intervals and show that, roughly speaking, these intervals can be neglected. A remainder interval is a monotonicity interval ω for $g^n : [0, 1) \to [0, 1)$ such that $\omega/d^n < 1$. Observe that if d is an integer then each monotonicity interval ω for g^n satisfies $|\omega| = d^n$, and if d is a non-integer value then $|\omega|/d^n$ may get arbitrarily small (when n increases). The fact that we can neglect remainder intervals enables us then to prove a positive vertical Lyapunov exponent for each considered real d. To ensure the existence of an a.c.i.p., we have to exclude d-values for which the remainder intervals get too small too fast. By using a result due to Schmeling [Sch], we show that the set of d-values we have to exclude is a Lebesgue measure zero set. In fact, in Section 5 of Paper D, we generalize this result of Schmeling used in Paper A to more general β-transformations. Thus, regarding the setting in Paper A, we could even get positive Lyapunov exponents and existence of an a.c.i.p. for certain Viana maps having a C^2-version of the β-transformation as the base dynamics (cf. summary of Paper A, Remark 2.1.4).

In Paper B we prove positive Lyapunov exponents in the case when the base dynamics is given by a sufficiently high iteration of a Misiurewicz-Thurston map (i.e. a quadratic map of the same type as f_{a_0}). More precisely, let $1 < a_1 \leq 2$ be a parameter such that the associated quadratic map f_{a_1} is Misiurewicz-Thurston and let p_1 be the unique negative fixed point for f_{a_1}. In Paper B we prove positive Lyapunov exponents for the map

$$\begin{aligned} F : [p_1, -p_1] \times \mathbb{R} &\to [p_1, -p_1] \times \mathbb{R} \\ (\theta, x) &\mapsto (f_{a_1}^k(\theta), f_{a_0}(x) + \alpha s(\theta)), \end{aligned}$$

where $k \geq 1$ is a sufficiently large integer and the coupling function $s(\theta)$ is chosen in such a way that we can conjugate F to a map which has still a dominating horizontal expansion (note that this makes s dependent on the base dynamics f_{a_1}). Already Viana [Vi] pointed out that it is of interest to study quadratic skew-products where the base dynamics is given by a non-uniformly hyperbolic map. Paper B provides us with a first example of such a system having positive Lyapunov exponents.

We will give a brief sketch of the ideas in the proofs of Papers A and B which are in common with the ideas in [Vi] and [BST]. For simplicity, we consider the setting in [Vi], i.e $g(\theta) = d\theta \bmod 1$, where d is a large integer. The main idea in [Vi] is to make use of certain transversality properties (caused by the Morse function $\sin(2\pi\theta)$ and the partial hyperbolicity) of the so-called admissible curves combined with the mixing property of the underlying base dynamics. In short terms, an *admissible curve* is a non-flat but nearly horizontal curve defined on $S^1 (= [0,1))$ and with image in I. More precisely, if $d \geq 16$ is an integer and $\widehat{X} = \text{graph}(X) = S^1 \times \{x\}$, $x \in I$, is a constant horizontal curve in $\widehat{J}$ and $\widehat{Y} = \text{graph}(Y)$, $Y : [0,1) \to I$ is one of the d^j, $j \geq 1$, curves contained in the image $F^j(\widehat{X})$, then, by the dominating horizontal expansion, Y is almost horizontal (its slope is smaller than α) and, in particular, the curve Y inherits the property of the Morse function $\sin(2\pi\theta)$, i.e. in each point $\theta \in S^1 \setminus \{0\}$ either its first derivative or its second derivative is bounded away from zero. A curve with these properties is called an admissible curve.

Since every point in $F(\widehat{J})$ lies on an admissible curve, in order to prove a positive vertical Lyapunov exponent it is sufficient to show that there exists a constant $\lambda > 0$ such that for an arbitrary admissible curve $\widehat{X} = \text{graph}(X)$, $X : S^1 \to I$,

$$\liminf_{n\to\infty} \frac{1}{n} \log \left| DF^n(\theta, X(\theta)) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right| \geq \lambda, \tag{1.4}$$

for Lebesgue a.e. $\theta \in S^1$. The main dynamical issue in proving (1.4) is recurrence of criticalities, i.e. returns of the forward orbit of a point $(\theta, X(\theta))$ close to the critical line $S^1 \times \{0\}$. Considering the unperturbed quadratic map $x \mapsto f_{a_0}(x)$, the closer some iteration of a point $x \in I$ comes to the critical point 0 the longer the subsequent iterations of x stay close to the forward orbit of the critical point. By assumption, the critical point of the map f_{a_0} is eventually mapped to an expanding periodic point. It is easy to check that this expansion gained by following the critical orbit in fact compensates for the loss of expansion coming too close to the critical point (this is the main ingredient in the proof that the map f_{a_0} has in Lebesgue a.e. point $x \in I$ a positive Lyapunov exponent).

As for the map F, the situation is different. After every iteration we add a perturbation of size smaller or equal than α which causes that even if the iteration of a point $(\theta, X(\theta))$ comes very close to the critical line $S^1 \times \{x\}$ it is possible that the projection to the x-axes of the subsequent iterations of $(\theta, X(\theta))$ follow the forward orbit of the critical point of the unperturbed map f_{a_0} only for a number of iterations proportional to $\log(1/\alpha)$. Hence, the loss of expansion due to close returns to the critical line might not be compensated by the expansion gained by the immediate subsequent iterations. The way to deal with this obstacle is to do a large deviation argument (see [Vi], Section 2.4; or also [BC2]), where one makes use of the mixing property of the base dynamics and the above described non-flatness property of the admissible curves to give a good upper estimate of the measure of θ-values such that the iteration of $(\theta, X(\theta))$ comes too often too close to the critical line.)

In this paragraph we will explain roughly how the non-flatness (or transversality) property of admissible curves and the mixing property of the base dynamics are used in this large deviation argument. Note that the distance between one iteration of the critical point 0 and one iteration of the points $\sqrt{\alpha}$ or $-\sqrt{\alpha}$ by the map f_{a_0} is α. Hence, one can expect that if an iteration of the point $(\theta, X(\theta))$ comes not closer than $\sqrt{\alpha}$ to the critical line then the loss of expansion is compensated during the immediate subsequent iterations as it is the case for the unperturbed map. In fact, as it is shown in the large deviation argument by Viana there exists a constant $0 < \eta < 1/2$ such that the loss of expansion is compensated by the immediate subsequent iterations if an iteration comes not closer than $\alpha^{1-\eta}$ to the critical line. On the other hand it turns out that very close returns to the critical line can be neglected since, by the transversality property of an admissible curve in the image $F^j(\widehat{X})$, only a very small fraction of this admissible curve can lie very close to the critical line. A technical difficulty in the large deviation argument is caused by the shallow returns, i.e. returns of the forward orbit of a point $(\theta, X(\theta))$ roughly speaking in distance less or equal than $\alpha^{1-\eta}$ to the critical line. Recall that the absolute value of the slope of an admissible curve is bounded from above by α and hence it is possible that the whole of an admissible curve is in distance less or equal than $\alpha^{1-\eta}$ from the critical line. To tackle this technical difficulty, one makes use of the mixing property of the base dynamics combined with the property of the underlying Morse function $\sin(2\pi\theta)$. If $\widehat{X} = \text{graph}(X)$, $X : S^1 \to I$, is an admissible curve then, by the non-flatness of $\sin(2\pi\theta)$, one can show that at least two admissible curves Y_1 and Y_2 of the d admissible curves in the image $F(\widehat{X})$ lie in vertical distance a constant times α from each other. This property implies the following. Assuming that the map F is expanding in the vertical direction during the next m iterations, where m is a sufficiently large integer (which on the other hand cannot be larger than a constant times $\log(1/\alpha)$), then it turns out that, roughly speaking, either all admissible curves in $F^m(\widehat{Y}_1)$ or all admissible curves in $F^m(\widehat{Y}_2)$ cannot be closer than $\alpha^{1-\eta}$ to the critical line. By a repeated use of this fact — essentially, the mixing property of the base dynamics is used here — one can show that the fraction of an admissible curve which is mapped by $F^{M(\alpha)}$ (where $M(\alpha)$ is an integer proportional to $\log(1/\alpha)$) closer than $\alpha^{1-\eta}$ to the critical line is sufficiently small to deduce a good large deviation estimate.

While one-dimensional (non-uniformly hyperbolic) dynamical systems, in particular the quadratic family, are quite well-studied by now, there are still many important open questions in higher dimensional dynamical systems. The probably most prominent 2-dimensional example, where very little is known, is the family of standard maps on the two-dimensional torus $\mathbb{T}^2$:

$$(x, y) \mapsto (2x - y + \kappa \sin(2\pi x), x),$$

where κ is a real parameter. Note that this map is area preserving, and it is not hyperbolic nor partially hyperbolic. One open problem for this family is if there is a positive Lebesgue measure set of parameters κ such that the corresponding maps have a positive Lyapunov exponent on a positive Lebesgue measure set on $\mathbb{T}^2$. More is known for higher dimensional dynamical systems having a certain hyperbolical structure as, e.g., Viana maps, Hénon maps or more generally partially hyperbolic maps (for a recent important work on partially hyperbolic endomorphism on the torus see Tsujii [Ts]). In general it seems to be very hard to treat higher dimensional systems without any hyperbolic structure. Regarding the Viana maps, maybe the most interesting but probably very hard case would be the proof of a positive vertical Lyapunov exponent for the map F where the base dynamics is replaced by a rotation, i.e.

$$F(\theta, x) = (\theta + \beta \bmod 1, f_{a_0}(x) + \alpha s(\theta)),$$

for β some generic irrational number (for some results on this map but where one chooses the quadratic map f_{a_0} to have a negative Lyapunov exponent see [Bj]). In general, the philosophy for all skew-products of quadratic maps considered in this section seems to be that the more randomness in the perturbations the easier it is to prove positive Lyapunov exponents. A probably more realistic aim than taking a rotation as the base dynamics would be to keep the β-transformation in the base dynamics and allow the expansion d to be greater but arbitrarily close to the Lyapunov exponent of the unperturbed map f_{a_0}. Even if for such d-values the partial hyperbolicity of the system is not any longer guaranteed, one still can hope to make use of an asymptotic dominating horizontal expansion. Furthermore, the base dynamics in this case is still mixing which is, as we have mentioned above, an essential ingredient of the technical part in [Vi]. Paper A might provide an important step towards the proof of positive Lyapunov exponents and the existence of an a.c.i.p. in this setting.

Regarding non-uniformly hyperbolic base dynamics, an interesting case which is directly related to Paper B would be to drop the dependence of the coupling function on the base dynamics and therefore to break down any possible link to partial hyperbolicity. Furthermore, one preferably would need to have as the base dynamics a quadratic map satisfying the Misiurewicz condition or better (even if probably much harder) a quadratic map with Collet-Eckmann or Benedicks-Carleson parameter.

1.2 Transversality implies absolute continuity

In the late 1930's Erdös considered the random series

$$Y_\lambda = \sum_{n \geq 1} \pm \lambda^n, \qquad 0 < \lambda < 1,$$

where the signs are chosen independently with probability 1/2. A long standing conjecture by Erdös was that, for Lebesgue almost every $\lambda \in [1/2, 1)$, the distribution ν_λ of Y_λ is absolutely continuous with respect to Lebesgue measure m on $\mathbb{R}$. This conjecture was finally proved to be true by Solomyak [So] in 1995 using Fourier transform methods. One year later in 1996, Peres and Solomyak [PS] gave a simpler proof of this result by using differentiation of measures and by taking into account a geometric transversality property of Y_λ. We will roughly explain this transversality property of Y_λ and how it can be used to prove absolute continuity.

Let $\Omega = \{-1, 1\}^{\mathbb{N}}$ be the sequence space equipped with the product topology and μ the Bernoulli measure on Ω with the weights $(1/2, 1/2)$. For $\omega \in \Omega$, we set

$$Y_\lambda(\omega) = \sum_{n \geq 1} \omega_n \lambda^n, \tag{1.5}$$

where ω_n denotes the n-th coordinate of the element ω. Clearly, ν_λ is the distribution of $Y_\lambda : \Omega \to \mathbb{R}$. In [PS] it is shown that there exists a constant $C > 0$ such that for any two different elements ω and ω' in Ω the following holds. If the curves $\lambda \mapsto Y_\lambda(\omega)$ and $\lambda \mapsto Y_\lambda(\omega')$, $\lambda \in [2^{-1}, 2^{-2/3}]$, intersect each other, then the absolute value of the slope of the curve $\lambda \mapsto Y_\lambda(\omega) - Y_\lambda(\omega')$ close to the line $[2^{-1}, 2^{-2/3}] \times \{0\}$ is greater than $C\lambda^k$, where $k = \max\{k \geq 1 \; ; \; \omega_l = \omega'_l, \; 1 \leq l < k\}$ (see Figure 1.1). This transversality property causes that the curves $Y_\lambda(\omega)$, $\omega \in \Omega$, cannot cluster together too much in the strip $[2^{-1}, 2^{-2/3}] \times [-(1 - 2^{-2/3})^{-1}, (1 - 2^{-2/3})^{-1}]$. In other words, if ν is the distribution:

$$\nu(E) = (m \times \mu)\big(\{(\lambda, \omega) \in [2^{-1}, 2^{-2/3}] \times \Omega \; ; \; Y_\lambda(\omega) \in E\}\big),$$

then, by the transversality property of the curves on which ν is supported, ν should have some smoothness or uniformity in the vertical direction and, thus, the measure ν should be absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^2$, which then implies that ν_λ is absolutely continuous for a.e. λ. In fact, having verified the transversality property, this absolute continuity can be proved by a simple argument using differentiation of measures.

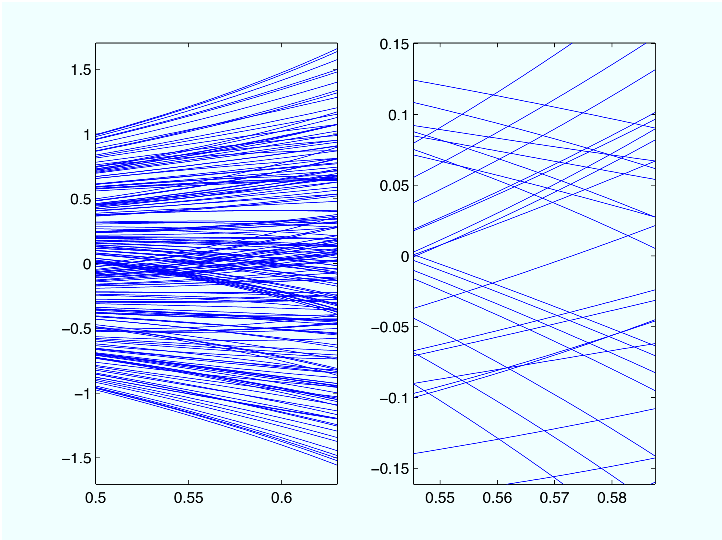

Figure 1.1: Sample of 200 randomly chosen curves $\lambda \mapsto Y_\lambda$, $\lambda \in [2^{-1}, 2^{-2/3}]$, and a zoom on it.

Peres and Solomyak claimed that their simplified proof in [PS] would be better suited to analyze more general random power series. In Paper C we have shown that, indeed, the approach of Peres and Solomyak does apply to certain variants of the original problem — namely, when the λ^n are replaced by $\lambda^{\varphi(n)}$ for certain well-behaved functions $\varphi : \mathbb{N} \to \mathbb{R}$. In the light of the tremendous amount of attention the $\sum \pm\lambda^n$ problems have received and continuous to receive, it is very natural to explore variants such as the ones proposed by Paper C.

To conclude this section, we would like to mention a recent result by Tsujii [Ts] already referred to in Section 1.1. In [Ts], Tsujii applies in an ingenious way the idea in Peres and Solomyak's paper, that a geometric transversality condition implies absolute continuity, to partially hyperbolic surface endomorphisms $F : M \to M$, where M is, say, the two-dimensional torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. We call μ a *physical* measure for F if the set of points $x \in M$ such that

$$\frac{1}{n}\sum_{i=0}^{n-1} \delta_{F^i(x)} \stackrel{\text{weak-}*}{\longrightarrow} \mu, \qquad \text{as } n \to \infty,$$

has positive Lebesgue measure. (In the terms of Paper D, one could say that μ is a physical measure if the set of points $x \in M$ which are *typical* for μ has positive Lebesgue measure.) In [Ts] it is shown that, generically, such a partially hyperbolic surface endomorphism has a finite number of ergodic physical measures, whose basins cover Lebesgue a.e. point of M. Furthermore — here appears essentially the idea in [PS] — these physical measures are absolutely continuous w.r.t. Lebesgue measure on M if the sum of their Lyapunov exponents is positive. This is a true novelty since, usually, absolute continuity is a result of expansion in all directions. To obtain absolute continuity in the case when the central Lyapunov exponent is zero or even negative, Tsujii makes use of a similar geometric transversality property as in [PS], which is generically satisfied in the space of surface endomorphisms. More precisely, the intuitive picture is the following. Let $F : M \to M$ be a partially hyperbolic surface endomorphism and ν an ergodic physical measure with central Lyapunov exponent equal to zero or sufficiently close to zero (it might be negative). It is shown, that, due to the dominating expansion in the unstable directions E^u, ν is attained as a weak-$*$ limit point of the sequence

$$\frac{1}{n}\sum_{i=0}^{n-1} \nu_\gamma \circ F^{-i}, \qquad \text{as } n \to \infty,$$

where ν_γ is a smooth measure on a curve segment γ of an unstable manifold. Zooming in on a small neighborhood of a point in the support of ν, the image $F^n(\gamma)$ should roughly be comparable to the right figure in Figure 1.1. The curves in this neighborhood would not concentrate in the central direction strongly, as the central Lyapunov exponent is nearly neutral (almost everywhere w.r.t. ν). By a certain control of the angles between intersecting curves in $F^n(\gamma)$ — this is generically provided by the transversality property — Tsujii deduces that in fact the measure ν is absolutely continuous.

Heuristically, regarding the Bernoulli convolutions considered in [PS], the numbers $\log 2^{-1}$ and $\log \lambda^{-1}$ correspond to the Lyapunov exponents in [Ts], where $\log 2^{-1}$ corresponds to the central Lyapunov exponent (the 2 comes from the distribution of μ). Their sum is positive for $\lambda > 2^{-1}$, in which case one has indeed almost sure absolute continuity. On the other hand, for $\lambda < 2^{-1}$, ν_λ is singular as it is mentioned in Paper C.

1.3 Absolutely continuous limit distributions of sums of point measures

The results in Papers D and E are essentially inspired by the last chapter, Chapter III, of Benedicks and Carleson's paper [BC1] on the quadratic map $f_a(x) = 1 - ax^2$, $x \in (-1, 1)$, where they prove that for Lebesgue almost every parameter value a in a positive Lebesgue measure set $\Delta_\infty \subset (1, 2)$, constructed in the previous two chapters in [BC1], the map f_a admits an a.c.i.p. μ_a. More precisely, in Chapters I and II in [BC1] they construct in an inductive way a positive Lebesgue measure Cantor set Δ_∞ of a-values such that the associated maps f_a have certain expansion properties along the forward orbit of the critical point 0. An important ingredient of this construction is the fact that, for $j \geq 1$, the a-derivative $\partial_a f_a^j(1)$ and the x-derivative $\partial_x f_a^j(1)$ are comparable if $a \in \Delta_\infty$. In Chapter III these expansion properties in the a-direction are then used to show that for Lebesgue a.e. parameters $a \in \Delta_\infty$ a limit distribution μ_a of the forward orbit of the critical point exists and is absolutely continuous w.r.t. Lebesgue measure. Even if the techniques in Chapter III of [BC1] turn out to be very powerful, they have, to the best of the authors knowledge, not been used in other contexts than the quadratic maps. Papers D and E provide us with several non-trivial, elementary, and important examples where these techniques can be applied.

In the remaining part of this section we will present the main technical ingredient of the result in Chapter III in [BC1] — and as well of the results in Papers D and E — by showing for the doubling map $T(x) = 2x \bmod 1$, $x \in [0,1]$, the well-known fact that for Lebesgue a.e. point $x \in [0,1]$ the weak-$*$ limit of the sequence

$$\frac{1}{n}\sum_{j=1}^{n} \delta_{T^j(x)} \tag{1.6}$$

exists and coincides with the Lebesgue measure m on $[0,1]$. This example provided by the doubling map can serve as a toy model for Paper D as well as for Paper E (cf. summaries of Papers D and E below). The Lebesgue measure m is the unique (and hence ergodic) a.c.i.p. for the map T. Thus, the fact we are going to show with the techniques used in [BC1] follows also straightforward from Birkhoff's ergodic theorem. Now, let

$$\mathcal{B} := \left\{(x-r, x+r) \cap [0,1]\ ;\ x \in \mathbb{Q}, r \in \mathbb{Q}^+\right\},$$

and for each $B \in \mathcal{B}$ consider the function

$$F_n(x) = \frac{1}{n}\sum_{j=1}^{n} \chi_B(T^j(x)), \qquad n \geq 1, \quad x \in [0,1],$$

which counts the average number of visits to the interval B during the first n iterations of x. The main observation is summarized in the following lemma. Its proof is elementary (see, e.g., Lemma A.1 in Paper E).

Lemma 1.3.1. *Let $B \in \mathcal{B}$ and assume that there are positive constants K and C such that for all $h \geq 1$ there is an integer $n_{h,B}$ such that*

$$\int_{[0,1]} F_n(x)^h dx \leq K(C|B|)^h, \tag{1.7}$$

whenever $n \geq n_{h,B}$*. If the sequence* $n_{h,B}$ *can be chosen to grow at most exponentially in* h*, then it follows that*

$$\overline{\lim_{n\to\infty}} F_n(x) \leq C|B|, \tag{1.8}$$

for Lebesgue a.e. $x \in [0,1]$.

If (1.8) holds for all $B \in \mathcal{B}$, then it follows, by standard measure theory, that for a.e. $x \in [0,1]$ every measure μ_x obtained as a weak-$*$ limit point of

$$\frac{1}{n}\sum_{j=1}^{n} \delta_{T^j(x)}$$

has a density, which is bounded above by C and, in particular, μ_x is absolutely continuous. Note that, by construction, μ_x is an invariant probability measure for T. Since the Lebesgue measure m is the unique a.c.i.p. for T, it follows that μ_x in fact coincides with m and that the weak-$*$ limit of (1.6) exists. Thus, we only have to show that for each $B \in \mathcal{B}$ inequality (1.7) is satisfied where the sequence $n_{h,B}$ grows at most exponentially in h. We write

$$\int_{[0,1]} F_n(x)^h dx = \sum_{1\leq j_1,\dots,j_h\leq n} \frac{1}{n^h}\int_{[0,1]} \chi_B(T^{j_1}(x))\cdots\chi_B(T^{j_h}(x))dx. \tag{1.9}$$

Assume that we have proven the following proposition.

Proposition 1.3.2. *For all* $B \in \mathcal{B}$ *and* $h \geq 1$ *there is an integer* $n_{h,B}$*, growing for fixed* B *at most exponentially in* h*, such that, for all* $n \geq n_{h,B}$ *and for all integer* h*-tuples* $(j_1, ..., j_h)$ *with* $1 \leq j_1 < j_2 < ... < j_h \leq n$ *and* $j_l - j_{l-1} \geq \sqrt{n}$*,* $l = 2, ..., h$*, we have*

$$\int_{[0,1]} \chi_B(T^{j_1}(x))\cdots\chi_B(T^{j_h}(x))dx \leq (2|B|)^h. \tag{1.10}$$

Remark 1.3.3. As we are going to see in the proof of this proposition, we could state it in a stronger version. More precisely, we could drop the dependence of $n_{h,B}$ on h and, hence, it would be enough to require that the gaps between two consecutive j_l's are larger than some constant only dependent on B. This is due to the very special properties of the doubling map. However, we are not able to prove such a stronger version in the cases considered in Papers D and E. Since we want to refer to such a toy model as provided by this example with the doubling map, we stated Proposition 1.3.2 in this weaker form.

From a probabilistic point of view, Proposition 1.3.2 says that if the distances between the j_l's are sufficiently large, then the functions $\chi_B(T_B^{j_l}(x))$'s can be seen as independent random variables. In fact, since the doubling map with the invariant measure m is exact and, hence, mixing of all degrees, the integral in (1.10) converges to $|B|^h$ as n tends to infinity (instead of 2 in the right hand side of (1.10), we could take any real number strictly greater than 1). Note that, for $h \geq 2$, the number of h-tuples $(j_1, ..., j_h)$ in (1.9) for which $\min_{k \neq l} |j_k - j_l| < \sqrt{n}$ is bounded by $2h^2 n^{h-1/2}$. Hence, by Proposition 1.3.2, we obtain, for $h \geq 1$,

$$\int_{[0,1]} F_n(x)^h dx \leq (2|B|)^h + \frac{2h^2}{\sqrt{n}} \leq 2(2|B|)^h,$$

if

$$n \geq \max \left\{ n_{h,B}, \left(\frac{2h^2}{(2|B|)^h} \right)^2 \right\}.$$

Since both terms in this lower bound for n grow at most exponentially in h, this concludes the proof of (1.7).

To conclude this section we prove Proposition 1.3.2.

Proof. Set $\tau_B = \log(2/|B|)/\log 2$, and let $n_{h,B}(= n_B)$ be an integer such that $\sqrt{n_{h,B}} \geq \tau_B$. By $\mathcal{P}_j$, $j \geq 1$, we denote the open intervals of monotonicity for $T^j : [0,1] \to [0,1]$, i.e. $\mathcal{P}_j = \{(k/2^j, (k+1)/2^j) \ ; \ 0 \leq k < 2^j\}$. We set $\mathcal{P}_0 = (0,1)$ and if Ω is a subset of monotonicity intervals in $\mathcal{P}_j$, $j \geq 0$, then we write $\mathcal{P}_{j+l}|\Omega$, $l \geq 0$, for the intervals in $\mathcal{P}_{j+l}$, which are also contained in an interval of Ω. Set $\Omega_0 = (0,1)$ and, for $1 \leq l \leq h$, we define

$$\Omega_l = \{\omega \in \mathcal{P}_{j_l + \tau_B}|\Omega_{l-1} \ ; \ T^{j_l}(\omega) \cap B \neq \emptyset\}.$$

Observe that the set we are interested in, i.e. the set

$$\{x \in [0,1] \ ; \ T^{j_l}(x) \in B, \ 1 \leq l \leq h\}$$

is contained in Ω_h (disregarding a finite number of points). If $n \geq n_{h,B}$, then we have $j_l - j_{l-1} \geq \tau_B$, $2 \leq l \leq h$, and we obtain, by the definitions of τ_B and Ω_l, $1 \leq l \leq h$,

$$\Omega_l \subset \{x \in \Omega_{l-1} \ ; \ T^{j_l}(x) \in 2B\},$$

where $2B$ denotes the interval twice as long as B and having the same midpoint as B. Thus, by the piecewise linearity of $T^{j_l - j_{l-1}}$, $1 \leq l \leq h$ (where we set $j_0 = 0$), we get

$$|\Omega_l| \leq 2|B||\Omega_{l-1}|, \tag{1.11}$$

which implies

$$|\Omega_h| \leq (2|B|)^h |\Omega_0| = (2|B|)^h.$$

This concludes the proof of Proposition 1.3.2. $\square$

Chapter 2

Summary

2.1 Overview of Paper A – Non-continuous weakly expanding skew-products of quadratic maps with two positive Lyapunov exponents

We will use the notions from Section 1.1 above. Regarding (1.1), we let F be the Viana map with base dynamics $g(\theta) = d\theta \bmod 1$ and coupling function $s(\theta) = \sin(2\pi\theta)$. In Paper A, instead of integer d's, we allow d to be any real number provided that the expansion in the base dynamics g dominates the vertical expansion. We show that, as in the cases considered in [Vi] and [BST], we still have positive Lyapunov exponents.

Theorem 2.1.1. *There exists $R_0 = R_0(a_0) < 2$ such that for any real number $d > R_0$, for every sufficiently small $\alpha > 0$, F has a positive vertical Lyapunov exponent at Lebesgue almost every point in $\widehat{J}$.*

Furthermore, for a full Lebesgue measure set of d's considered in Theorem 2.1.1, the existence of an a.c.i.p. is shown.

Theorem 2.1.2. *For Lebesgue a.e. $d > R_0$, for every sufficiently small $\alpha > 0$, F admits a unique a.c.i.p. μ, where the basin of μ has full Lebesgue measure in $\widehat{J}$.*

The main technical novelty in Paper A is the introduction of the concept of remainder intervals. Let $g : [0,1) \to [0,1)$ be the map in the base dynamics as defined above, i.e. $g(\theta) = d\theta \bmod 1$, and denote by $\mathcal{P}_n$ the monotonicity intervals of $g^n : [0,1) \to [0,1)$. A *remainder* interval in $\mathcal{P}_n$ is a monotonicity interval of g^n having not full length, i.e. its size is smaller than d^{-n}. The first observation, stated in Lemma 3.2 in Paper A, is that the number of remainder intervals in $\mathcal{P}_n$ can grow in n at most proportionally to the number of *entire* intervals in $\mathcal{P}_n$, which are monotonicity intervals of g^n of full length, i.e. their size is equal to d^{-n}. This implies then that there is a constant $C \geq 1$ such that, for all $d > R_0$,

$$\#\{\text{monotonicity intervals in } \mathcal{P}_n\} \leq Cd^n, \tag{2.1}$$

for all $n \geq 1$. Furthermore, this fact persists when one zooms in on a subinterval I of $[0,1]$, i.e.

$$\#\{\text{monotonicity intervals } \omega \in \mathcal{P}_n \ ; \ \omega \cap I \neq \emptyset\} \leq C'd^n|I|, \tag{2.2}$$

for some constant $C' \geq 1$ and provided that n is sufficiently large (see Lemma 3.3 in Paper A). The facts (2.1) and (2.2) suggest that we can, virtually, assume that the monotonicity intervals of g^n have full length, and, thus, we are in a very similar situation as in the case when d is an integer.

Remark 2.1.3. The property that one can neglect too short intervals is also used in Paper D. It is reflected in condition (III) stated in Paper D, which the one-parameter families therein have to fulfill.

Theorem 2.1.2 follows from a result due to Alves [Al]. In [Al] it is shown that maps contained in a certain family of piecewise expanding maps $\phi : \widehat{J} \to \widehat{J}$, which have not a finite but a countable number of pieces of continuity, admit an a.c.i.p. An essential property of maps ϕ contained in this family is that the image by ϕ of a domain of continuity for ϕ should be large. Regarding the Viana map F, in [Al] a piecewise expanding map ϕ is constructed in a way such that its continuity domains R are of the form $R = \omega \times I$, $\omega \in \mathcal{P}_n$ and I an interval, and such that ϕ restricted to R is identical to the n-th iteration of F, i.e. $\phi|_R \equiv F^n$. Furthermore, the construction is made such that the vertical length of $\phi(R)$ is sufficiently large. Now, the additional difficulty for non-integer d's is that the horizontal length of $\phi(R) = F^n(\omega \times I)$ is small whenever $|\omega| \ll d^{-n}$. To avoid that elements ω associated to the continuity domains of ϕ are too small one has to prevent that points $\theta \in [0, 1]$ are too often contained in a too short monotonicity interval of g^n when n increases. For this purpose it turns out that it is enough to have a good control of the forward orbit by g of the point 1 (see Lemma 8.3 in Paper A and its proof). Due to a result of Schmeling [Sch], the distribution of the forward orbit of 1 coincides with the a.c.i.p. of g for Lebesgue a.e. $d > 1$, which provides us then with a sufficiently good information of the forward orbit of 1 — at least for Lebesgue a.e. $d > 1$.

Remark 2.1.4. The key lemma in proving Theorem 2.1.2 is Lemma 8.3 in Paper A. A similar lemma could be proved for the C^2-versions of the β-transformation considered in Section 5 of Paper D. To do so, one has to replace Lemma 8.4 in Paper A by condition (IIa) in Paper D and Lemma 8.5 in Paper A, which is the above mentioned result due to Schmeling [Sch], by Corollary 5.4 in Paper D, where the map X in this corollary is taken as in its following Remark 5.5. Thus, if we took, instead of the base dynamics g, a one-parameter family T_a as described in Section 5 of Paper D, one could expect to get a result analog to Theorem 2.1.2, provided of course that a corresponding version of Theorem 2.1.1 holds. One obstacle of proving Theorem 2.1.1 in this new setting is that one has to consider high derivatives of the admissible curves if the expansion of T_a is too weak. However, assuming that the family T_a is strongly expanding (as it is done in [Vi]) and taking as the coupling function the linear map $\theta \mapsto 2\theta - 1$, instead of $\sin(2\pi\theta)$, it would be sufficient, in the proof of an analog of Theorem 2.1.1, to look only at the first derivative of an admissible curve. Combined with standard distortion estimates for piecewise expanding interval maps, one should be able to derive a positive vertical Lyapunov exponent also in the case when one has a C^2-version of the β-transformation in the base dynamics.

2.2 Overview of Paper B – Positive Lyapunov exponents for quadratic skew-products over a Misiurewicz-Thurston map

We will use the same notations as in Section 1.1 above. Let $1 < a_1 \leq 2$ be a parameter such that the quadratic map $f_{a_1}(x) = a_1 - x^2$ is Misiurewicz-Thurston. Regarding (1.1), we take the Misiurewicz-Thurston map f_{a_1} as the base dynamics. But in order to have a strong enough horizontal expansion, we choose instead of f_{a_1} a sufficiently high iteration of f_{a_1}, i.e. we set $g(x) = f_{a_1}^k(x)$ for some $k \geq 1$. Let p_1 be the unique negative fixed point for f_{a_1}. In Paper B we consider skew-products

$$\begin{aligned} F : [p_1, -p_1] \times \mathbb{R} &\to [p_1, -p_1] \times \mathbb{R} \\ (\theta, x) &\mapsto (f_{a_1}^k(\theta), f_{a_0}(x) + \alpha s(\theta)), \end{aligned}$$

where $\alpha > 0$ is chosen sufficiently small and the coupling function $s : [p_1, -p_1] \to [-1, 1]$ is *a priori* not fixed. Like for the map considered in Paper A, there is an open interval $(-1, 1) \subset I \subset (-2, 2)$ such that $F([p_1, -p_1] \times I) \subset [p_1, -p_1] \times I$, provided α is sufficiently small. We denote this F-invariant region $[p_1, -p_1] \times I$ by $\widehat{J}$. The main result in Paper B is the following.

Theorem 2.2.1. *There exist a piecewise C^1 coupling function $s : [p_1, -p_1] \to [-1, 1]$ and an integer $k_0 \geq 1$ such that, for all sufficiently small $\alpha > 0$ and all $k \geq k_0$, the map $F : \widehat{J} \to \widehat{J}$:*

$$F(\theta, x) = (f_{a_1}^k(\theta), f_{a_0}(x) + \alpha s(\theta))$$

admits two positive Lyapunov exponents.

For a short illustration of the proof of Theorem 2.2.1 we consider the situation when $a_1 = 2$, in which case $p_1 = -1$ and the map $f_{a_1} : [-1,1] \to [-1,1]$ is conjugated by the map $\varphi(\theta) = 2\pi^{-1} \arcsin(\theta)$, $\theta \in [-1,1]$, to the symmetric tent map with slope 2, i.e. $\varphi \circ f_{a_1} \circ \varphi^{-1}(\theta) = 1-2|\theta|$. Set $T(\theta) = 1-2|\theta|$ and let $h : [-1,1] \to [-1,1]$ be an arbitrary C^1 map whose first derivative is uniformly bounded away from 0, i.e. there exists a constant $K_1 \geq 1$ such that $K_1^{-1} \leq |h'(\theta)| \leq K_1$, $\theta \in [-1,1]$. Now, setting $s(\theta) = h(\varphi(\theta))$ and conjugating the associated function F with the conjugation function $\Phi(\theta, x) = (\varphi(\theta), x)$, we obtain

$$\tilde{F}(\theta, x) = \Phi \circ F \circ \Phi^{-1}(\theta, x) = (T^k(\theta), f_{a_0}(x) + \alpha h(\theta)).$$

Note that if we show two positive Lyapunov exponents for the map $\tilde{F}$, then it immediately follows that F admits two positive Lyapunov exponents. In contrast to F, the base dynamics of $\tilde{F}$ is now a piecewise linear and uniformly expanding map and the coupling function h for $\tilde{F}$ has bounded derivatives. This makes $\tilde{F}$ very similar to the systems studied by Viana and we are able to make use of the methods in [Vi] to prove positive Lyapunov exponents for $\tilde{F}$. The fact that the derivative of h is bounded away from 0 makes it sufficient to look at the first derivative of the admissible curves, provided that T^k is strongly expanding (which is the case if k is chosen so large that $2k \geq 5K_1 + 4$).

For general Misiurewicz-Thurston parameters a_1 we can apply a similar conjugation for f_{a_1} as above to obtain in the base dynamics an expanding map, which expansion is uniformly bounded away from 1. The existence of such a conjugation function was firstly noted by Ognev [Og]. In [Og] it is shown that for each Misiurewicz-Thurston parameter a_1 there exists a piecewise analytic function $\varphi : [p_1, -p_1] \to [-1,1]$ such that for every $D > 1$ there is an integer $k_0 \geq 1$ such that

$$|(\varphi \circ f_{a_1}^k \circ \varphi^{-1})'(\theta)| \geq D,$$

for all $k \geq 1$ and all θ for which the derivative is defined (see Proposition 2.2 in Paper B). By piecewise analytic we mean here that, disregarding a finite number of points, the interval $[p_1, -p_1]$ can be partitioned into a finite number of open intervals on each of which the function φ is analytic. If $a_1 < 2$ then the conjugated function $T(\theta) =$

$\varphi \circ f_{a_1}^k \circ \varphi^{-1}$ is not any longer piecewise linear. Further, in contrast to the case when $a_1 = 2$, there are more than two points in $[p_1, -p_1]$ such that $|\varphi'(\theta)|$ tends to ∞ when approaching them (at least from one side). This causes then that the derivative of T is not any longer bounded and, thus, we have to establish appropriate distortion estimates for the map T. Misiurewicz-Thurston or more generally Misiurewicz maps are well-studied, going back to a fundamental paper by Misiurewicz [Mi]. In this paper we made use of some distortion estimates for Misiurewicz maps due to van Strien [Str] (see Lemma 3.1 in Paper B).

2.3 Overview of Paper C – Almost sure absolute continuity of Bernoulli convolutions

This is joint work with M. Björklund. For a fixed $\alpha > 0$ consider the random series

$$Y_\lambda = \sum_{n \geq 1} \pm \lambda^{n^\alpha}, \qquad 0 < \lambda < 1,$$

where the signs are chosen independently with probability 1/2. In Paper C we are interested in the distribution ν_λ of Y_λ. For the case when $0 < \alpha < 1/2$, Wintner [Wi] considered the Fourier transform of the measure ν_λ, which can be represented as a convergent infinite product: $\hat{\nu}_\lambda(t) = \prod_{n=1}^{\infty} \cos(\lambda^{n^\alpha} t)$. Since $\cos(\lambda^{n^\alpha} t) \leq 2/3$, if $1 \leq \lambda^{n^\alpha} t \leq 2$, it follows that

$$|\hat{\nu}_\lambda(t)| \leq (2/3)^{K(t)},$$

where $K(t) = \#\{n \; ; \; 1 \leq \lambda^{n^\alpha} t \leq 2\}$. A minor calculation yields that, for $0 < \alpha < 1/2$, the term $(2/3)^{K(t)}$ decreases faster than polynomially in t and thus, for each $0 < \lambda < 1$ the distribution of ν_λ is absolutely continuous and the density is smooth. This method seems to break down at $\alpha = 1/2$. Other easy cases are when $\alpha > 1$ and $\lambda \in (0, 1)$ or when $\alpha = 1$ and $\lambda \in (0, 1/2)$. In these cases, the measure ν_λ is singular (see [KW], criteria (10)). In contrast, the situation when $\alpha = 1$ and $\lambda \in (1/2, 1)$ turns out to be much harder. It took over half a century until Solomyak [So] finally settled (with Fourier transform methods) a conjecture by Erdös which claimed that in this case ν_λ is absolutely continuous for Lebesgue a.e. $\lambda \in (1/2, 1)$ (in [So] it is also shown that the density of ν_λ is in L^2). Shortly after, Peres and Solomyak [PS] gave a simpler proof of this result. The techniques of this simpler proof are presented in Section 1.2 above. In Paper C we make use of these methods, developed by Peres and Solomyak, and we are able to extend their result to more general Bernoulli convolutions Y_λ. For instance, we cover the intermediate case when $1/2 \leq \alpha < 1$, in which case we show that ν_λ is absolutely continuous and has a density in L^2, for a.e. $\lambda \in (0, 1)$. More generally, instead of the sequence n^α in the power of λ, we consider sequences $\varphi(n)$ of real numbers and prove the following.

Theorem 2.3.1. *Let ν_λ be the distribution of the random series*

$$Y_\lambda = \sum_{n \geq 1} \pm \lambda^{\varphi(n)},$$

where the signs are chosen independently with probability $1/2$. If

$$\lim_{n \to \infty} \varphi(n+1) - \varphi(n) = 0, \tag{2.3}$$

then ν_λ is absolutely continuous and has an L^2 density, for a.e. $\lambda \in (0, 1)$.

If there exists a constant $0 < \beta < \infty$ such that

$$\lim_{n \to \infty} \frac{\varphi(n)}{n} = \beta,$$

and if the sequence $\varphi(n) - \beta n$ satisfies (2.3) then there exists a (non-empty) interval $I \subset (0, 1)$ such that ν_λ is absolutely continuous and has an L^2 density, for a.e. $\lambda \in I$.

2.4 Overview of Paper D – Typical points for one-parameter families of piecewise expanding maps of the interval

Let $I \subset \mathbb{R}$ be an interval and $T_a : [0,1] \to [0,1]$, $a \in I$, a one-parameter family of maps of the unit interval, which are uniformly expanding and piecewise C^2 or piecewise C^1 with a Lipschitz derivative. We assume that the dependence of the family T_a on the parameter a is 'nice'. For example, for each $x \in [0,1]$ the map $a \mapsto T_a(x)$ is piecewise C^1 on the interval I. Furthermore, we assume that for each parameter $a \in I$ the map T_a has a unique absolutely continuous invariant probability measure μ_a. By Birkhoff's ergodic theorem, the distribution of the forward orbit of μ_a-almost every $x \in [0,1]$ is described by the measure μ_a, i.e.

$$\frac{1}{n}\sum_{i=0}^{n-1} \delta_{T_a^i(x)} \overset{\text{weak-}*}{\longrightarrow} \mu_a, \qquad \text{as } n \to \infty. \tag{2.4}$$

If (2.4) holds for a point $x \in [0,1]$, then we say that x is *typical* for the measure μ_a. In Paper D we address the question whether a similar fact, as this one derived from Birkhoff's ergodic theorem, holds when we fix a point $x \in [0,1]$ and vary the parameter a, i.e. whether a given point $x \in [0,1]$ is typical for μ_a for Lebesgue a.e. parameter values $a \in I$. Or more generally, if $X : I \to [0,1]$ is a C^1 map, what kind of conditions are sufficient to put on X and T_a such that we are able to deduce that $X(a)$ is typical for μ_a, for Lebesgue a.e. $a \in I$?

We will consider some examples. Given a map $X : I \to [0,1]$ we denote by $x_j : I \to [0,1]$, $j \geq 0$, the map $x_j(a) = T_a^j(X(a))$. The most simple example is the case when the family T_a is constant. Regarding the example of the doubling map treated in Section 1.3 above, we set for instance $T_a(x) = 2x \bmod 1$, where a is contained in some interval I. The a.c.i.p. μ_a is in this case the Lebesgue measure m on the unit interval. If $x \in [0,1]$ is a point which is not typical for m and if we choose $X(a) \equiv x$, then $X(a)$ is not typical for m for any parameter

$a \in I$. Note that for this choice of $X(a)$ the derivative of x_j is zero on I for all $j \geq 0$. Let now $X : I \to [0,1]$ be a map whose derivative does not vanish on I. On the one hand, it follows directly from the fact that a.e. $x \in [0,1]$ are typical for m that also $X(a)$ is typical for m for a.e. parameter $a \in I$. On the other hand, considering x_j instead of T^j in Section 1.3, it is possible to show that $X(a)$ is typical for m for a.e. parameter $a \in I$ by almost the very same proof as it is given in Section 1.3 (one only has to replace $T^j : [0,1] \to [0,1]$ by $x_j : I \to [0,1]$ and adjust slightly the proof of Proposition 1.3.2). The reason for this is that the map $x_j : I \to [0,1]$ inherits the properties of $T_a^j : [0,1] \to [0,1]$, by which we mean, in particular, the expanding and the mixing properties. More precisely, one can show that if the derivative of X is uniformly bounded away from 0 then, for j sufficiently large, for any interval ω in I, which is mapped one-to-one onto $[0,1]$ by x_j, the map $x_{j+1} \circ x_j|_\omega^{-1} : [0,1] \to [0,1]$ is almost the doubling map T_a itself. To make a similar kind of comparison of x_j and T_a^j also work for other families, it is sufficient to require that the a-derivative and the x-derivative of $T_a^j(X(a))$ are comparable, i.e. we require that there exists a constant $C \geq 1$ such that

$$\frac{1}{C} \leq \frac{|D_a x_j(a)|}{|\partial_x T_a^j(X(a))|} \leq C, \tag{2.5}$$

for all $a \in I$ for which the derivatives are defined. This is the very basic condition a map $X : I \to [0,1]$ has to satisfy in Paper D in order to obtain almost sure typicality (cf. condition (I) in Paper D).

Let us consider a first non-trivial example obtained by changing the slopes of the doubling map. We define a family $T_a : [0,1] \to [0,1]$, $a \in (0,1)$, as

$$T_a(x) = \begin{cases} \frac{x}{a} & \text{if } x < a, \\ \frac{x-a}{1-a} & \text{otherwise.} \end{cases}$$

(This example is treated in a more general form in Example 7.2 in Paper D.) Observe that, for all $a \in (0,1)$, the a.c.i.p. μ_a for T_a coincides with the Lebesgue measure on $[0,1]$. Let $I \subset (0,1)$ be an interval not adjacent to 0 nor to 1, e.g., we set $I = (\varepsilon, 1-\varepsilon)$ for some small $\varepsilon > 0$. If $X : I \to [0,1]$ has a non-positive derivative then it is shown in

Example 7.2 in Paper D that (2.5) holds. Note that, considering the map T_a, for a fixed $a \in I$, we can follow verbatim the proof given in Section 1.3 to deduce that Lebesgue a.e. $x \in [0,1]$ are typical for μ_a (we only have to replace T by T_a and set τ_B equal to $-\log(2/|B|)/\log(1-\varepsilon)$ instead of $\log(2/|B|)/\log 2$). Thus, since (2.5) holds, we can expect that the same proof also applies for x_j. Indeed, replacing $T^j : [0,1] \to [0,1]$ by $x_j : I \to [0,1]$ in Section 1.3, we only have to adjust slightly the proof of Proposition 1.3.2 (by using (2.5) and by adding some simple distortion estimates), in order to derive that $X(a)$ is typical for μ_a for Lebesgue a.e. parameter values $a \in I$. Since ε in the choice of I was arbitrary we obtain the following result.

Proposition 2.4.1. *If a C^1 map $X : (0,1) \to (0,1)$ satisfies $X'(a) \leq 0$ then $X(a)$ is typical for μ_a, for Lebesgue a.e. parameter $a \in (0,1)$.*

A similar proof applies if we compose the family T_a with a C^2 homeomorphism $g : [0,1] \to [0,1]$ such that $g'(x) \approx 1$, i.e. we obtain the same typicality result for the family $T_a \circ g$, $a \in I$. Note that for this family the a.c.i.p. μ_a is not anymore the Lebesgue measure on the unit interval.

All the examples considered by now consisted of Markov maps (cf. Section 7 in Paper D) and as it is often the case for Markov maps related models they are easier to treat than models or maps where the Markov property is absent. A simple example of a family $T_a : [0,1] \to [0,1]$ which generically consists of non-Markov maps is given by the family of β-transformations, i.e. $T_a(x) = ax \bmod 1$, $a > 1$. In Section 5 in Paper D, we treat C^2-versions of β-transformations. For instance, let $\tilde{T}_a(x) = T_a \circ g(x)$, where $g : [0,1] \to [0,1]$ is a C^2 homeomorphism such that the derivative of g is, say, positive and uniformly bounded away from zero. Let $a_0 \geq 1$ be so large that $\tilde{T}'_{a_0}(x) \geq 1$, for all $x \in [0,1]$, and let $\tilde{\mu}_a$, $a > a_0$, be the a.c.i.p. for $\tilde{T}_a$ (it is straightforward to show that $\tilde{\mu}_a$ is unique). In Section 5 in Paper D, we show the following:

Theorem 2.4.2. *If a C^1 map $X : (a_0, \infty) \to (0,1]$ satisfies $X'(a) \geq 0$, then $X(a)$ is typical for $\tilde{\mu}_a$, for Lebesgue a.e. parameter $a > a_0$.*

Note that we came across the family of β-transformations already in Section 2.1 (cf. also Remark 2.1.4).

In the remaining part of this section, we will consider another example consisting generically of non-Markov maps. It is provided by a family of skew tent maps. For a fixed $0 < c < 1$, let $T : [0,1] \to [0,1]$ be defined as

$$T(x) = \begin{cases} \frac{x}{c} & \text{if } x < c, \\ \frac{1-x}{1-c} & \text{otherwise.} \end{cases}$$

For sufficiently small, positive parameters a, we obtain from this map a one-parameter family $T_a : [0,1] \to [0,1]$ by setting $T_a(x) = T \circ h_a(x)$, where $h_a(x) = (1-a)x + a$ maps the interval $[0,1]$ affinely onto $[a,1]$. The slopes of T_a are $(1-a)/c$ and $-(1-a)/(1-c)$, respectively. Hence, in order that T_a is uniformly expanding, we require that $a \in I = [0, \min\{c, 1-c\})$. It is well-known that for each skew tent map T_a, $a \in I$, there exists a unique a.c.i.p. μ_a (see [LaY] and [LiY]). One main assertion of Paper D is that the turning point $c_a = h_a^{-1}(c)$ of T_a is typical for the a.c.i.p. μ_a, for a.e. $a \in I$.

Theorem 2.4.3. *For Lebesgue a.e. $a \in I$, the turning point of the skew tent map T_a is typical for μ_a.*

We will give a brief sketch of the proof of Theorem 2.4.3. For convenience, we consider only the subinterval of I consisting of parameters a such that T_a is non-renormalizable (what is meant by renormalizable and why we can neglect renormalizable skew tent maps is explained in detail in Section 6 in Paper D). We denote this subinterval again by I. It is shown in Paper D that, for all $a \in I$, the support of μ_a is the whole unit interval. Hence, it follows directly from Birkhoff's ergodic theorem that Lebesgue a.e. $x \in [0,1]$ is typical for μ_a. Let us consider a sketch of the proof of this typicality result for the map T_a by the method in Section 1.3. Until the proof of Proposition 1.3.2, the proof is verbatim the same (just replace T by T_a). Then, after adjusting the constant τ_B, we can follow the proof of Proposition 1.3.2 verbatim until inequality (1.11). To derive an analog to inequality (1.11) in the case of the skew tent map T_a (where $a \neq 0$), we have to be more careful. We proceed in two steps. Let $\omega \in \mathcal{P}_{j_{l-1}+\tau_B}|\Omega_{l-1}$, i.e. ω is a monotonicity interval for $T_a^{j_{l-1}+\tau_B} : \Omega_{l-1} \to [0,1]$. We have the following estimate:

$$\begin{aligned}|T_a^{j_{l-1}+\tau_B}(\{x \in \omega\ ;\ T_a^{j_l}(x) \in B\})|& \\ \leq |\{x \in [0,1]\ ;\ T_a^{j_l - j_{l-1} - \tau_B}(x) \in B\}|.& \quad (2.6)\end{aligned}$$

It is shown in Subsection 6.2 in Paper D that there is a constant $C \geq 1$ such that the density of μ_a is bounded from below by C^{-1} and from above by C. Hence, we derive from the Perron Frobenius equality (see also the paragraph of inequality (3) in Paper D) that

$$\sum_{\substack{x \in [0,1] \\ T_a^j(x) = y}} \frac{1}{|T_a^j{}'(x)|} \leq C^2, \tag{2.7}$$

for a.e. $y \in [0,1]$. Applied to (2.6) we get

$$|T_a^{j_{l-1}+\tau_B}(\{x \in \omega\ ;\ T_a^{j_l}(x) \in B\})| \leq C^2 |B|. \tag{2.8}$$

This estimate is the first step in deriving an analog to inequality (1.11). In the second step we want to pull back this estimate further $j_{l-1} + \tau_B$ iterations. Assuming that $|T_a^{j_{l-1}+\tau_B}(\omega)| \geq \delta$ for some constant $\delta > 0$, then, by (2.8) and the linearity of $T_a^{j_{l-1}+\tau_B}|_\omega$, we obtain

$$|\{x \in \omega\ ;\ T_a^{j_l}(x) \in B\}| \leq \frac{C^2}{\delta} |B||\omega|.$$

If the images by $T_a^{j_{l-1}+\tau_B}$ of all elements in $\mathcal{P}_{j_{l-1}+\tau_B}|\Omega_{l-1}$ were greater than δ, then we would immediately get the following analog to inequality (1.11):

$$|\Omega_l| \leq \frac{C^2}{\delta} |B||\Omega_{l-1}|, \tag{2.9}$$

which would imply then Proposition 1.3.2 (with the constant C^2/δ instead of 2). But for general skew tent maps T_a the image by T_a^j of elements in $\mathcal{P}_j$ can get arbitrarily small as j increases. Hence, in order to obtain an inequality similar to (2.9), we have to show that elements having a too small image are negligible. A sufficient assumption is given in condition (III) in Paper D (condition (III) is formulated for the map x_j instead of T_a^j). That condition (III) is satisfied and, hence, that we are indeed able to neglect too small elements is verified in Subsection 6.4 in Paper D (it is verified for the map x_j). A key idea in this verification is to make use of appropriate approximations by skew tent maps which are Markov. This concludes a sketch of the proof that a.e. $x \in [0,1]$ are typical for μ_a, by the method in Section 1.3.

Finally consider the turning point c_a of T_a and set $X(a) = c_a$. It is shown in Subsection 6.1 of Paper D that (2.5) is satisfied for the associated map $x_j : I \to [0,1]$. Regarding the method of proof in Section 1.3, as in the case for a fixed skew tent map T_a, in order to show that $X(a)$ is typical for μ_a, for a.e. $a \in I$, it is essentially left to prove an inequality analog to (1.11). Since x_j is not the j-th iteration of a fixed map, it is not any longer possible to derive straightforward an inequality similar to (2.7). An analog of inequality (2.7) for the map x_j is formulated in condition (II) in Paper D. The verification that condition (II) is satisfied for the map x_j is a bit cumbersome. It is verified in Subsections 6.2 and 6.3 in Paper D (instead of verifying condition (II) directly, the for condition (II) sufficient conditions (IIa) and (IIb) are verified). Combined with the above mentioned good control of too small partition intervals, this implies then almost sure typicality for the turning point as stated in Theorem 2.4.3.

2.5 Overview of Paper E – Almost sure equidistribution in expansive families

This is joint work with M. Björklund. Using the method presented in Section 1.3, we give a new proof of a well-known result by Koksma [K]. In [K], Koksma studied the distributions of certain sequences of real numbers in the unit interval. The main result in [K] is that, for Lebesgue a.e. $\theta > 1$, the sequence $\theta^j \bmod 1$, $j \geq 1$, is equidistributed in $[0,1)$, i.e.

$$\frac{1}{n}\sum_{j=1}^{n} \delta_{\theta^j \bmod 1} \overset{\text{weak-}*}{\longrightarrow} m, \qquad \text{as } n \to \infty,$$

where m denotes the Lebesgue measure on $[0,1)$.

In Paper E, we consider one-to-one C^1 functions $\tilde{f}_j : I \to \mathbb{R}$, $j \geq 1$ and $I \subset \mathbb{R}$ an interval, which are *expanding*, i.e. there is an at least polynomially growing function $g : \mathbb{N} \to \mathbb{R}^+$, such that, for $j \geq 1$ and $k \geq 1$, we have

$$\frac{|\tilde{f}'_{j+k}(\theta)|}{|\tilde{f}'_j(\theta)|} \geq g(k), \tag{2.10}$$

for all $\theta \in I$; and for which we have good *distortion* estimates, i.e. for all $\varepsilon > 0$ there is an integer j_ε, such that for all $j \geq j_\varepsilon$ we have

$$\frac{|\tilde{f}'_j(\theta)|}{|\tilde{f}'_j(\theta')|} \leq 1 + \varepsilon, \tag{2.11}$$

for all $\theta, \theta' \in \tilde{f}_j^{-1}(T \cap \tilde{f}_j(I))$, where $T = [l, l+1)$ for some $l \in \mathbb{Z}$. In Paper E we prove the following. Let $f_j(\theta) = \tilde{f}_j(\theta) \bmod 1$.

Theorem 2.5.1. *If the functions $\tilde{f}_j : I \to \mathbb{R}$, $j \geq 1$, satisfy (2.10) and (2.11), then the sequence $f_j(\theta)$, $j \geq 1$, is equidistributed in $[0,1)$, for Lebesgue a.e. $\theta \in I$.*

It is straightforward to check that the functions $\tilde{f}_j(\theta) = \theta^j$, restricted to an interval I in $(1, \infty)$ being not adjacent to 1, satisfy (2.10) and (2.11). To include functions like $\tilde{f}_j(\theta) = \theta^{\sqrt{j}}$, it is possible to sharpen the criteria (2.10) (see condition (I) in Paper E). In Section 1.3 above, the proof of Theorem 2.5.1 is illustrated in the case when $\tilde{f}_j(\theta) = 2^j\theta$, $\theta \in [0,1]$ (just replace x by θ). Compared to Paper D the situation in Paper E is much easier. The main reason for this is that the maps $f_j : I \to S^1$ are continuous while in Paper D, after identifying the unit interval with S^1, the maps $x_j : I \to S^1$ can have many discontinuities. As it is the case with Koksma's (Fourier analytic) method, our geometric approach generalizes to higher dimensional sequences (see Theorem 2.1 in Paper E).

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Part II

Scientific Papers

Paper A

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Non-continuous weakly expanding skew-products of quadratic maps with two positive Lyapunov exponents

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Abstract. We study an extension of the Viana map where the base dynamics is a discontinuous expanding map, and prove the existence of two positive Lyapunov exponents.

1. *Introduction*

This work is essentially inspired by the results of Viana [**Vi**] and Buzzi *et al* [**BST**]. Viana [**Vi**] and Buzzi *et al* [**BST**] deal with ergodic properties of the Viana map $F : S^1 \times \mathbb{R} \to S^1 \times \mathbb{R}$:

$$F(\theta, x) = (g(\theta), a - x^2 + \alpha \sin(2\pi\theta)),$$

where $g(\theta) = d\theta \bmod 1$ and $d \geq 2$ is an integer. α is a small positive real number, and the parameter $1 < a < 2$ is such that the map $f_a = a - x^2$ has a pre-periodic (but not periodic) critical point. Viana [**Vi**] showed that, for an integer $d \geq 16$, the map F has almost everywhere with respect to Lebesgue measure two positive Lyapunov exponents. Alves [**Al**] proved that F admits an absolutely continuous invariant probability measure (a.c.i.p.). Buzzi *et al* [**BST**] demonstrate the existence of two positive Lyapunov exponents in the more natural case where d is an integer satisfying $d \geq 2$. In this case there also exists an a.c.i.p. [**Al**]. The purpose of this paper is the extension of these results to the case where d assumes non-integer values, which implies the driving map g is no longer continuous.

2. *Main statements*

The main result in this paper is the following theorem.

THEOREM 2.1. *There exists $R_0 = R_0(a) < 2$ such that for any real number $d > R_0$, for every sufficiently small $\alpha > 0$, F has two positive Lyapunov exponents at Lebesgue almost every point.*

For simplicity, we write

$$\phi(\theta) = \sin(2\pi\theta),$$
$$f(\theta, x) = a - x^2 + \alpha\phi(\theta).$$

For $n \geq 1$, let us define $f_n(\theta, x)$ by

$$F^n(\theta, x) = (g^n(\theta), f_n(\theta, x)).$$

Since for $d > 1$, the horizontal Lyapunov exponent is obviously positive, we only have to focus on the vertical Lyapunov exponent, i.e. we look at

$$\frac{1}{n} \log |\partial_x f_n(\theta, x)| \quad \text{for } (\theta, x) \in S^1 \times \mathbb{R},$$

as n tends to infinity. Furthermore, if p_1 denotes the unique negative fixed point of f_a ($p_1 = (-1 - \sqrt{1 + 4a})/2$), and if we take $\beta \in]a, |p_1|[$, then the interval $B = [-\beta, \beta]$ satisfies: $f_a(B) \subset \text{int}(B)$ and $|f_a'| > 1$ on $\mathbb{R} \setminus \text{int}(B)$. Then, writing $\widehat{J} = S^1 \times B$, for sufficiently small α we have:

- $F(\widehat{J}) \subset \widehat{J}$; and
- $|\partial_x f(\theta, x)| > 1$ outside of $\widehat{J}$.

These facts imply that, for any point (θ, x) on $S^1 \times \mathbb{R}$, either its orbit eventually comes into the invariant strip $\widehat{J}$ or the vertical Lyapunov exponent is positive. Thus, it is enough to consider the restriction of the map F to the cylinder $\widehat{J}$; in other words we show that

$$\liminf_{n \to \infty} \frac{1}{n} \log |\partial_x f_n(\theta, x)| > 0$$

for Lebesgue almost every $(\theta, x) \in \widehat{J}$.

Remark 1. R_0 in Theorem 2.1 is chosen to ensure that, in the invariant strip $\widehat{J}$, the horizontal expansion dominates (after a finite number of iterations) the vertical expansion. This enables us, as in Viana [**Vi**] and Buzzi *et al* [**BST**], to concentrate on nearly horizontal curves: the so-called admissible curves. Since our driving map is not continuous those curves are not necessarily defined on the whole of S^1.

THEOREM 2.2. *For Lebesgue almost every $d > R_0$, for every sufficiently small $\alpha > 0$, F admits an a.c.i.p. μ. In fact, μ is ergodic and the basin of μ has full Lebesgue measure in $\widehat{J}$, i.e. for Lebesgue almost every $(\theta, x) \in \widehat{J}$*

$$\frac{1}{n} \sum_{i=0}^{n-1} \delta_{F^i(\theta,x)} \to \mu \quad \textit{weakly as } n \to \infty.$$

The main part of this paper is dedicated to the proof of Theorem 2.1. In the last section we make some comments on how we can apply the methods of Alves [**Al**] to show the existence of an a.c.i.p. The ergodicity and the fact that the basin has full Lebesgue measure in $\widehat{J}$ can then be derived by applying the methods of Alves and Viana [**AV**, §§6 and 7] verbatim.

3. *The base dynamics*

The map we consider has three parameters $1 < a < 2$, $d > R_0$ and $\alpha > 0$. We assume that d is not an integer since the integer case has been investigated by Viana [**Vi**], respectively by Buzzi *et al* [**BST**]. Henceforth, we fix the parameters a and d and, in almost any cases, we do not specify the dependence on them. In contrast, the parameter α is not fixed but always assumed to be sufficiently small and the dependence on it is always specified.

3.1. *The constant R_0.* As mentioned in Remark 1 the lower bound for d, i.e. R_0, should be so large that the map F is partially hyperbolic in the cylinder $\widehat{J}$. From the proof of Lemma 3.1 of Buzzi *et al* [**BST**] we get a bound which is sufficient for this to happen.

LEMMA 3.1. *Let*

$$R_0 = R_0(a) = \sqrt{2|p_1|}\frac{a/|p_1|}{\sqrt{1-(|p_1|/2)(1-a^2/p_1^2)}}$$

($< \sqrt{2|p_1|}$). Then, for every $R_1 > R_0$ there exists a constant $K_1 < \infty$ such that, when α is sufficiently small, we have

$$|\partial_x f_n(\theta, x)| \le K_1 R_1^n \quad \textit{for all } n \ge 1, \ \textit{and } (\theta, x) \in \widehat{J}.$$

Of course, if f_a has a pre-periodic (but not periodic) critical point, a need to be at least bigger than the value $a_0 = 1.401\ldots$ for which f_{a_0} has Feigenbaum combinatorics. This yields a general lower bound for R_0, and consequently also for d, which can be calculated as

$$d > R_0 > 1.82. \tag{1}$$

3.2. *Partitions of S^1.* We consider the sequences of partitions of $S^1 = [0, 1[$ induced by g:

$$\mathcal{P}_1 = \{[\theta_j, \theta_{j+1}[\mid 0 \le j \le [d]\} \quad \text{where } \theta_j = \begin{cases} j/d & \text{if } j \le [d], \\ 1 & \text{if } j = [d]+1, \end{cases}$$

$$\mathcal{P}_n = \{\text{connected components of } \ g^{-1}(\omega) \mid \omega \in \mathcal{P}_{n-1}\} \quad \text{for } n \ge 2,$$

respectively, for $n = 0$, set $\mathcal{P}_0 = S^1$. If $I \subset S^1$ can be written as a union of elements in $\mathcal{P}_n$ then we use the following notation:

$$\mathcal{P}_n|I = \{\omega \in \mathcal{P}_n \mid \omega \subset I\} \quad \text{and} \quad \mathcal{P}_n|I^c = \{\omega \in \mathcal{P}_n \mid \omega \subset S^1 \setminus I\}.$$

Since d is not an integer there are $\omega \in \mathcal{P}_n$, $n \ge 1$, from which g^n is not a bijection onto S^1. In other words, the length of the ω's in $\mathcal{P}_n$ can differ. We call an element $\omega \in \mathcal{P}_n$ without full length, i.e. $|\omega| < d^{-n}$, a *remainder* interval (rem. int.). An element $\omega \in \mathcal{P}_n$, $n \ge 0$, with full length, i.e. $|\omega| = d^{-n}$, is referred to as *entire* interval (ent. int.). Let

$$\text{rem}_1 = \{d\}, \quad \text{rem}_2 = \{d\{d\}\}, \quad \text{rem}_3 = \{d\{d\{d\}\}\}, \ldots,$$

where $\{d\}$ denotes the fractional part of the real number d. Observe, if ω is a remainder interval in $\mathcal{P}_n$, $n \ge 1$, then there exists $1 \le i \le n$ such that $|\omega| = \text{rem}_i\, d^{-n}$, and, *vice versa*, if $1 \le i \le n$ such that $\text{rem}_i \ne 0$ then there exist $\omega \in \mathcal{P}_n$ such that $|\omega| = \text{rem}_i\, d^{-n}$. We say that $\omega \in \mathcal{P}_n$ is a remainder interval of *type i*, $1 \le i \le n$, if $|\omega| = \text{rem}_i\, d^{-n}$.

Remark 2. We observe the following.

(1) An entire interval in $\mathcal{P}_n$, $n \geq 0$, contains exactly $[d]$ entire interval(s) and one remainder interval of type 1 of the partition $\mathcal{P}_{n+1}$.
(2) An arbitrary remainder interval in $\mathcal{P}_n$, $n \geq 1$, contains maximally one remainder interval of the partition $\mathcal{P}_{n+1}$.
(3) If $1.82 < d < 2$ then a remainder interval of type 1 in $\mathcal{P}_n$, $n \geq 1$, contains exactly one entire interval and one remainder interval of type 2 of the partition $\mathcal{P}_{n+1}$.
(4) The set of $d > 1$ with $\mathrm{rem}_i \neq 0$ for all $i \geq 1$ has full Lebesgue measure in $]1, \infty[$ (cf. e.g. Parry [**Pa**]).

Actually, if in the second point of Remark 2, a remainder interval ω in $\mathcal{P}_n$ contains no remainder interval of $\mathcal{P}_{n+1}$, this means that in $\mathcal{P}_{n+1}$, ω is an entire interval or a union of entire intervals. d's with such a property are called simple β-numbers (cf. e.g. Parry [**Pa**]). They build a countable and dense subset in $]1, \infty[$. In fact, Viana maps where d is a simple β-number would be easier to treat, since for some integer n_0 $\mathrm{rem}_n = 0$ for all $n \geq n_0$, and thus the relative sizes of the elements in $\mathcal{P}_n$, $n \geq 1$, are bounded. This implies that the length of an admissible curve defined in the next section is bounded from below and this case is very similar to that where d is an integer.

LEMMA 3.2. *For $n \geq 0$,*

$$\frac{\#\{\textit{remainder intervals in } \mathcal{P}_n\}}{\#\{\textit{entire intervals in } \mathcal{P}_n\}} \leq 2. \tag{2}$$

In particular, it follows that $\#\{\omega \in \mathcal{P}_n\} \leq 3\#\{\textit{ent. int. in } \mathcal{P}_n\} \leq 3d^n$.

Proof. We prove Lemma 3.2 by induction. We consider first the case if $d > 2$. Equation (2) is obviously true for $\mathcal{P}_0$. Fix $j \geq 0$. By Remark 2, we have

$$\#\{\text{ent. int. in } \mathcal{P}_{j+1}\} \geq [d]\#\{\text{ent. int. in } \mathcal{P}_j\},$$

and

$$\begin{aligned}\#\{\text{rem. int. in } \mathcal{P}_{j+1}\} &\leq \#\{\text{rem. int. in } \mathcal{P}_j\} + \#\{\text{ent. int. in } \mathcal{P}_j\}\\ &\leq 3\#\{\text{ent. int. in } \mathcal{P}_j\},\end{aligned}$$

where in the second inequality we used the induction assumption. Since $[d] \geq 2$ this shows equation (2).

Now assume $1.82 < d < 2$. Fix $q \in \{0, 1\}$. We do the induction considering the partitions $\mathcal{P}_{2j+q}$, $j \geq 0$. Obviously equation (2) is true for $\mathcal{P}_0$ and $\mathcal{P}_1$. Fix $j \geq 0$ and let e_i, $i = 1, 2$, be the number of entire intervals in $\mathcal{P}_{2j+q+i}$ which are contained in remainder intervals in the partition $\mathcal{P}_{2j+q+i-1}$. Hence, we can write

$$\#\{\text{ent. int. in } \mathcal{P}_{2(j+1)+q}\} = \#\{\text{ent. int. in } \mathcal{P}_{2j+q}\} + e_1 + e_2.$$

By Remark 2, we see that, on the one hand, $e_2 \geq \#\{\text{ent. int. in } \mathcal{P}_{2j+q}\}$ and thus

$$\#\{\text{ent. int. in } \mathcal{P}_{2(j+1)+q}\} \geq 2\#\{\text{ent. int. in } \mathcal{P}_{2j+q}\} + e_1,$$

and, on the other hand,

$$\begin{aligned}\#\{\text{rem. int. in } \mathcal{P}_{2(j+1)+q}\} &\le \#\{\text{rem. int. in } \mathcal{P}_{2j+q}\} + 2\#\{\text{ent. int. in } \mathcal{P}_{2j+q}\} + e_1 \\ &\le 4\#\{\text{ent. int. in } \mathcal{P}_{2j+q}\} + e_1,\end{aligned}$$

where in the second inequality we used the induction assumption. Altogether

$$\frac{\#\{\text{rem. int. in } \mathcal{P}_{2(j+1)+q}\}}{\#\{\text{ent. int. in } \mathcal{P}_{2(j+1)+q}\}} \le \frac{4\#\{\text{ent. int. in } \mathcal{P}_{2j+q}\} + e_1}{2\#\{\text{ent. int. in } \mathcal{P}_{2j+q}\} + e_1} \le 2,$$

which proves equation (2). □

Let $I \subset S^1$ be an interval. If the size of I is large compared to an entire interval in $\mathcal{P}_n$, i.e. $|I| \gg d^{-n}$, then the following lemma says that the number of elements in $\mathcal{P}_n$ intersecting I is approximately equal to

$$\#\{\omega \in \mathcal{P}_n\} \cdot |I|.$$

If d were an integer this would be obvious.

LEMMA 3.3. *Let $\gamma > 0$ such that $e^\gamma < d$. Fix $C' > 0$. If $I \subset [0, 1[$ is an interval with $|I| = C'e^{-\gamma j_0}$, for some $j_0 \le l$ and l sufficiently big, then we have*

$$\#\{\omega \in \mathcal{P}_l \mid \omega \cap I \neq \emptyset\} \le 5d^l C' e^{-\gamma j_0}.$$

Proof. We will use the following observation.

CLAIM 1. *Let $n \ge 1$. If $I_0 \subset [0, 1[$ is an arbitrary interval with $[0, d^{-n}[\subset I_0$ then*

$$\frac{\#\{\text{rem. int. } \omega \in \mathcal{P}_n \mid \omega \cap I_0 \neq \emptyset\} + 1}{\#\{\text{ent. int. } \omega \in \mathcal{P}_n \mid \omega \subset I_0\}} \le 3.$$

Proof. Assume $I_0 \neq [0, d^{-n}[$, otherwise the claim is obvious. Let $\eta_1 \ge 0$ be the first time when $1/d \in g^{\eta_1}(I_0)$, j_1 the maximal integer such that $j_1/d \in g^{\eta_1}(I_0)$ and $I_1 = I_0 \setminus g^{-\eta_1}([0, j_1/d[)$. For $k \ge 1$, having defined the intervals $I_1, \ldots, I_k$ and the integers $\eta_1, \ldots, \eta_k$, let η_{k+1} be the first time when $1/d \in g^{\eta_{k+1}}(I_k)$. If $\eta_{k+1} = n$ or $[0, d^{-(n-\eta_k-1)}[\not\subset g^{\eta_k+1}(I_k)$ we stop. Otherwise, let j_{k+1} be the maximal integer such that $j_{k+1}/d \in g^{\eta_{k+1}}(I_k)$ and $I_{k+1} = I_k \setminus g^{-\eta_{k+1}}([0, j_k/d[)$. Note that the left endpoint of $g^{\eta_k+1}(I_k)$ is 0 and $\eta_{k+1} \ge \eta_k + 1$ so the stopping conditions make sense. We end up with a finite sequence of intervals $I_0 \supset I_1 \supset \cdots \supset I_s$ where $\eta_s < n$ and the last interval I_s is contained in one entire or one remainder interval in $\mathcal{P}_n$. Observe that, for $1 \le k \le s$, g^{η_k+1} is a j_k-to-one map from $I_{k-1} \setminus I_k$ to S^1 and the number of entire and remainder intervals of $\mathcal{P}_n|(I_{k-1} \setminus I_k)$ is in j_k-to-one correspondence to the number of entire and remainder intervals in $\mathcal{P}_{n-\eta_k-1}$. Hence we have, by Lemma 3.2, a control of the proportion of the entire intervals in $\mathcal{P}_n|(I_{k-1} \setminus I_k)$:

$$\#\{\text{rem. int. in } \mathcal{P}_n|(I_{k-1} \setminus I_k)\}/\#\{\text{ent. int. in } \mathcal{P}_n|(I_{k-1} \setminus I_k)\} \le 2.$$

Summing up we get

$$\#\{\text{rem. int. in } \mathcal{P}_n|(I_0 \setminus I_s)\}/\#\{\text{ent. int. in } \mathcal{P}_n|(I_0 \setminus I_s)\} \le 2.$$

Now one easily sees that by attaching I_s and adding 1 in the numerator we obtain the claimed estimate. □

Set l large such that $d^{-l} \ll C'e^{-\gamma l} \leq C'e^{-\gamma j_0} = |I|$. Since the size of an element in $\mathcal{P}_l$ is at most d^{-l} and hence much smaller than the size of the interval I we can add to I at the most two elements in $\mathcal{P}_l$ intersecting the boundary of I, which means that we can consider I as a union of elements in $\mathcal{P}_l$. We assume that the left endpoint of I is not 0 since otherwise Lemma 3.3 follows immediately from Claim 1. Moreover, we will only consider the case where $1.82 < d < 2$. The case $d > 2$ is similar.

Set $I_1 = I$ and $\eta_1 = \min\{\eta \geq 0 \mid 1/d \in g^{\eta}(I_1)\}$. We distinguish between the following two situations:

(1) $\#\{\omega \in (\mathcal{P}_{l-\eta_1}|[1/d, 1[) \mid \omega \cap g^{\eta_1}(I_1) \neq \emptyset\} = 1$; and

(2) $\#\{\omega \in (\mathcal{P}_{l-\eta_1}|[1/d, 1[) \mid \omega \cap g^{\eta_1}(I_1) \neq \emptyset\} > 1$.

Define the index $i_1 \in \{1, 2\}$, where $i_1 = 1$ means that we are in the first situation and $i_1 = 2$ means that we are in the second situation. Let $I_2 \subset I_1$ such that

$$g^{\eta_1}(I_2) = g^{\eta_1}(I_1) \setminus [1/d, 1[.$$

Now we repeat the procedure with the interval I_2 and continue until we obtain an interval $I_s \subset I_{s-1}$ such that $g^{\eta_s}(I_s) \cap [0, 1/d[= \emptyset$. Since the left endpoint of $I_1 = I$ was assumed to be different from 0, we know that s is finite, in particular $s < l$. We obtain a sequence of intervals $I_s \subset I_{s-1} \subset \cdots \subset I_1 = I$, a sequence of integers $0 \leq \eta_1 < \eta_2 < \cdots < \eta_s < l$ and a sequence of indices $i_1, \ldots, i_s$. Divide $I = I_1$ into the following s disjoint intervals:

$$\tilde{I}_j = I_j \setminus I_{j-1}, \quad j = 1, \ldots, s-1, \text{ and } \tilde{I}_s = I_s.$$

Note that, by construction, the $\tilde{I}_j$'s can be considered as unions of elements in $\mathcal{P}_l$. Let $1 \leq j \leq s$. If $i_j = 2$ then $g^{\eta_j+1}(\tilde{I}_j) =: I_0$ is an interval with $[0, d^{-(l-\eta_j-1)}[\subset I_0$. Observe if $\omega \in \mathcal{P}_l|\tilde{I}_j$ and ω is not the right-outermost element in $\tilde{I}_j$ then g^{η_j+1} maps ω one-to-one to an element in $\mathcal{P}_{l-\eta_j-1}$. Thus,

$$\#\{\text{rem. int. in } \mathcal{P}_l|\tilde{I}_j\} \leq \#\{\text{rem. int. } \omega \in \mathcal{P}_{l-\eta_j-1} \mid \omega \cap I_0 \neq \emptyset\} + 1$$

(where the 1 on the right-hand side is added for the possibility that the right-outermost element in $\mathcal{P}_l|\tilde{I}_j$ is a remainder interval but is mapped to an entire interval in $\mathcal{P}_{l-\eta_j-1}$). Applying Claim 1, we obtain

$$\#\{\text{rem. int. in } \mathcal{P}_l|\tilde{I}_j\} \big/ \#\{\text{ent. int. in } \mathcal{P}_l|\tilde{I}_j\} \leq 3.$$

Using this result we have, for l sufficiently large,

$$\begin{aligned} \#\{\omega \in \mathcal{P}_l|I\} &= \sum_{j,\, i_j=1} \#\{\omega \in \mathcal{P}_l|\tilde{I}_j\} + \sum_{j,\, i_j=2} \#\{\omega \in \mathcal{P}_l|\tilde{I}_j\} \\ &\leq s + \sum_{j,\, i_j=2} 4\#\{\text{ent. int. in } \mathcal{P}_l|\tilde{I}_j\} \leq l + 4\#\{\text{ent. int. in } \mathcal{P}_l|I\} \\ &\leq l + 4d^l C'e^{-\gamma j_0} \leq 5d^l C'e^{-\gamma j_0}. \end{aligned}$$

□

4. *Admissible curves*

We will analyse the dynamics by focusing on nearly horizontal curves, the so-called *admissible curves*. Although in our case the admissible curves might be shorter, their definition is much the same as in Buzzi *et al* [**BST**, §4].

Choose R_1 such that $R_0 < R_1 < d$ and fix K_1 as it is done in Lemma 3.1. We take an integer $N_0 \geq 1$ such that

$$K_1 R_1^{N_0} < d^{N_0}.$$

Put

$$A_0 = \left(1 - \frac{K_1 R_1^{N_0}}{d^{N_0}}\right)^{-1} \sum_{i=0}^{\infty} \frac{K_1 R_1^i}{d^i}.$$

Then, fix an integer $\xi \geq 2$ so large that

$$d^{\xi - 1} > 400 A_0. \tag{3}$$

(These constants are mainly used in the proofs of Lemmas 4.1 and 4.3.)

In the following, let $T = [0, t[$, $0 < t \leq 1$, denote an arbitrary half-open interval in S^1 having the left endpoint at 0. Consider a curve $\widehat{X} = \mathrm{graph}(X)$, $X : T \to B$, which is C^ξ on $T \setminus \{0\}$ and continuous to the right at 0. If X satisfies

$$|X^{(r)}(\theta)| \leq 2A_0 (2\pi/d)^r \alpha \quad \text{for } \theta \in T \text{ and } 1 \leq r \leq \xi, \tag{4}$$

then X is called *pre-admissible*.

The following lemma, stated slightly differently by Buzzi *et al* [**BST**, Lemma 4.1], allows us to define admissible curves. The proof in Buzzi *et al* [**BST**] can be adapted to our version without any further efforts, so we will omit it.

LEMMA 4.1. *Suppose $\alpha > 0$ is sufficiently small. Let $n \geq N_0$. If $\widehat{X} = \mathrm{graph}(X)$, $X : T \to B$, is a pre-admissible curve. Then for any interval $\tilde{\omega} \subset T$, where $\tilde{\omega}$ is contained in an element $\omega \in \mathcal{P}_n$, the curve Y determined by the image $\widehat{Y} = \mathrm{graph}(Y) = F^n(\widehat{X}|_{\tilde{\omega}})$ satisfies property (4).*

Definition. We say that a curve $\widehat{Y} = \mathrm{graph}(Y)$, $Y : T \to B$, is an *admissible curve* if, for each $0 < n \leq N_0$, there exists an interval $\tilde{\omega}$ (where $\tilde{\omega}$ is contained in an element $\omega \in \mathcal{P}_n$) and a pre-admissible curve $\widehat{X} = \mathrm{graph}(X)$ (where $\tilde{\omega}$ is contained in the domain of X), such that $\widehat{Y} = F^n(\widehat{X}|_{\tilde{\omega}})$.

The main property of this class of nearly horizontal curves is the following corollary.

COROLLARY 4.2. *The curves determined by the image $F^n(\widehat{X})$, $n \geq 1$, of an arbitrary admissible curve $\widehat{X} = \mathrm{graph}(X)$ are again admissible.*

The second fundamental property of admissible curves is that their images are *non-flat*.

LEMMA 4.3. *If $\widehat{X} = \mathrm{graph}(X)$, $X : T \to B$, is an admissible curve, then it satisfies either*

$$|X^{(\xi-1)}(\theta)| > (\pi/d)^{\xi-1}\alpha \quad \text{or} \quad |X^{(\xi)}(\theta)| > (\pi/d)^{\xi}\alpha$$

for each $\theta \in T$. More precisely, T can be divided into at most four closed intervals on each of which one of these two inequalities holds.

We can almost apply word-by-word the proof in Buzzi *et al* [**BST**, Lemma 4.4] to our case, so we omit this proof.

Remark 3. We observe the following.

- A flat curve $X =$ constant is pre-admissible but *not* admissible.
- The image by F^{2N_0} of a pre-admissible curve is admissible. In particular, if we look at the invariant strip $\widehat{J} = S^1 \times B$, which can be considered as a union of flat curves, every point in $F^{2N_0}(\widehat{J})$, and hence every interesting point, lies on an admissible curve.
- In order to prove the claim of Theorem 2.1 on the vertical Lyapunov exponent, it is enough to show for an arbitrary admissible curve $\widehat{X} = \text{graph}(X)$, $X : T \to B$, that

$$\liminf_{n\to\infty} \frac{1}{n} \log |\partial_x f_n(\theta, X(\theta))| > 0$$

for Lebesgue almost every $\theta \in T$.

5. *Critical returns*

We turn now to the central fact about the returns of admissible curves to the critical line $S^1 \times \{0\}$.

5.1. *Statement of the main technical tool.*

First we recall a fact about the map f_a (which in our case satisfies the Misiurewicz condition and has no periodic attractors) [**MS**, Theorem III.6.3].

LEMMA 5.1. *If the parameter a is such that the critical point of f_a is pre-periodic (but not periodic) then there exist constants $\delta > 0$, $\sigma > 1$ and $\mu > 0$ such that $|(f_a^i)'(x)| \geq \mu\sigma^i$ if $f_a^i(x) \in (-\delta, \delta)$.*

Define the constants

$$M(\alpha) = \left\lceil \frac{|\log \alpha|}{\log 32} \right\rceil, \quad \eta = \frac{\log \sigma}{8 \log 32}. \tag{5}$$

Fix $r \geq 0$, and let $J(r) = \{x \in \mathbb{R} \mid |x| \leq \sqrt{\alpha}e^{-r}\}$. Denote the critical strip $S^1 \times J(r)$ by $\widehat{J}(r)$.

Our main technical tool is the following proposition that corresponds to Viana [**Vi**, Lemma 2.6] and Buzzi *et al* [**BST**, Proposition 5.2].

PROPOSITION 5.2. *Let $r_0(\alpha) = (1/2 - 2\eta) \log(1/\alpha)$. There exist $C < \infty$ and $\beta > 0$ such that, for all sufficiently small $\alpha > 0$, for any admissible curve $\widehat{Y} = \text{graph}(Y)$, $Y : T \to B$,*

$$\#\{\omega \in \mathcal{P}_{M(\alpha)} \mid F^{M(\alpha)}(\widehat{Y}|_\omega) \cap \widehat{J}(r_0(\alpha) - 2) \neq \emptyset\} \leq 3d^{M(\alpha)}Ce^{-\beta r_0(\alpha)}, \tag{6}$$

and for any admissible curve $\widehat{Z} = \text{graph}(Z)$, $Z : T' \to B$, and any $r \geq (1/2 + 2\eta) \log(1/\alpha)$,

$$\{\theta \in T' \mid \widehat{Z}(\theta) \in \widehat{J}(r-2)\} \subset \bigcup_{j=1}^{4\cdot 2^\xi} I_j, \tag{7}$$

where the I_j are intervals in T' with

$$m(I_j) \leq Ce^{-\beta r},$$

where m denotes Lebesgue measure on S^1.

5.2. *Preliminary lemmas for the proof of Proposition 5.2.* The main ingredients in the proof of Proposition 5.2 are the following two lemmas. The first is a general result on the size of the pre-image of a small interval for C^s functions, $s \geq 1$, whose sth derivative is bounded away from zero. The second states that the images under F of two certain components of an admissible curve, which is defined on S^1, are well separated over a certain subset of the interval where the domains of these components are mapped to.

LEMMA 5.3. *Let s be a positive integer and let $I \subset \mathbb{R}$ be an interval. Suppose that $h : I \to \mathbb{R}$ is a C^s function such that $|h^{(s)}(\theta)| \geq \delta$ for all $\theta \in I$. Then,*

$$m(\{\theta \in I \mid |h(\theta)| \leq \varepsilon\}) < 2^{s+1}(\varepsilon/\delta)^{1/s}$$

for all $\varepsilon > 0$. More precisely, we have

$$\{\theta \in I \mid |h(\theta)| \leq \varepsilon\} \subset \bigcup_{j=1}^{2^s-1} I_j,$$

where the I_j are intervals with

$$m(I_j) \leq 2\Big(\frac{\varepsilon}{\delta}\Big)^{1/s}.$$

The proof is in Buzzi *et al* [**BST**, Lemma 5.3]. Together with Lemma 4.3, this immediately implies the following corollary.

COROLLARY 5.4. *Let $\widehat{X} = \mathrm{graph}(X)$, $X : T \to B$, be an admissible curve and assume that α is sufficiently small. Then, for $0 < \epsilon < \alpha$, we have*

$$m(\{\theta \in T \mid \widehat{X}(\theta) \in T \times (-\epsilon, \epsilon)\}) \leq C_1 \cdot (\epsilon/\alpha)^{1/\xi},$$

where $C_1 = 4 \cdot 2^{\xi+1}(d/\pi)$.

The next lemma is the analog to Lemma 5.5 in Buzzi *et al* [**BST**] but, for simplicity, we will only state and prove it in a more specific case. Let $\widehat{X} = \mathrm{graph}(X)$ be an admissible curve with domain S^1. For $\omega \in \mathcal{P}_j$, $1 \leq j$, we set $\widehat{X}_1(\omega) = \mathrm{graph}(X_1(\omega, \cdot)) = F(\widehat{X}|_\omega)$. Let $k_d = 2$ if $1.82 < d < 2$ and $k_d = 1$ if $d > 2$.

LEMMA 5.5. *Let m_0 be an integer so that*

$$d^{m_0} > 200\, d^{k_d} 2^{\xi+1},$$

and let α be sufficiently small. There exists a constant $\varepsilon_0 > 0$ such that there are at least two remainder intervals $\omega_1, \omega_2 \in \mathcal{P}_{m_0+1}$ of type k_d with the property that $g(\omega_1) = g(\omega_2)$ and, for all $\theta \in g(\omega_1) = g(\omega_2)$,

$$|\widehat{X}_1(\omega_1, \theta) - \widehat{X}_1(\omega_2, \theta)| \geq \varepsilon_0 \alpha.$$

Remark 4. The reason why we consider in this lemma remainder intervals of type k_d (and not e.g. entire intervals) is that, in view of the proof of Lemma 3.2, we could take away an arbitrary remainder interval ω of type k_d (or also of a higher type) in a partition $\mathcal{P}_m$, $m \geq k_d$, and be sure that we have, as in Lemma 3.2, maximally two times as many remainder intervals as entire intervals in $\mathcal{P}_n|\omega^c$, for all $n \geq m$. We will use this fact in the proof of Proposition 5.2.

Proof. Define

$$j_0 = \begin{cases} ([d]+1)/2 & \text{if } [d] \text{ is odd,} \\ [d]/2 & \text{otherwise.} \end{cases}$$

Recall the definition of θ_j, and set $\widehat{Y} = \text{graph}(Y) = \widehat{X}_1([0, 1/d[)$ and $\widehat{Z} = \text{graph}(Z) = \widehat{X}_1([\theta_{j_0}, \theta_{j_0+1}[)$. Note that Y is defined on S^1 and, since $d > 1.82$, Z is defined at least on $[0, 0.82]$. We only look at the $[0, 1/4]$ part of the domains of Y and Z (the reason for this is that it is only important that ω_1 and ω_2, as sought for in Lemma 5.5, have a fixed size independent from α but not where they are located). We are going to apply Lemma 5.3 to the function $Y(\theta) - Z(\theta)$ on the interval $[0, 1/4]$.

Let $\theta \in [0, 1/4]$. We can write Y as

$$Y(\theta) = a - (X(\theta/d))^2 + \alpha\phi(\theta/d),$$

respectively Z as

$$Z(\theta) = a - (X(\theta/d + \theta_{j_0}))^2 + \alpha\phi(\theta/d + \theta_{j_0}).$$

So, for $1 \le r \le \xi$, the rth derivative of Y can be expressed as

$$Y^{(r)}(\theta) = d^{-r}\alpha\phi^{(r)}(\theta/d) - 2d^{-r}X(\theta/d)X^{(r)}(\theta/d) + \mathcal{O}(\alpha^2) \tag{8}$$

and a similar expression holds for the rth derivative of Z. Note that $\theta/d \in [0, 0.14]$ and $\theta/d + \theta_{j_0} \in [1/3, 3/4]$ as long as $d > 1.82$ and $\theta \in [0, 1/4]$. Let $r = \xi$ or $\xi - 1$ be the odd number. Then we have

$$|\phi^{(r)}(\theta/d) - \phi^{(r)}(\theta/d + \theta_{j_0})| \ge (2\pi)^r(\cos(2\pi \cdot 0.14) - \cos(2\pi \cdot 3/4)) > (2\pi)^r/2.$$

Thus, using equation (8) and condition (3) in the choice of ξ,

$$|Y^{(r)}(\theta) - Z^{(r)}(\theta)| \ge \left(\frac{2\pi}{d}\right)^r \frac{\alpha}{2} - 8d^{-r}2A_0\left(\frac{2\pi}{d}\right)^r \alpha - \mathcal{O}(\alpha^2) > \left(\frac{2\pi}{d}\right)^r \frac{\alpha}{4}.$$

Applying Lemma 5.3 to the function $Y(\theta) - Z(\theta)$ on the interval $[0, 1/4]$ with $\varepsilon = \varepsilon_0\alpha$ and $\delta = (2\pi/d)^r\alpha/4$, we obtain

$$\{\theta \in [0, 1/4] \mid |Y(\theta) - Z(\theta)| \le \varepsilon_0\alpha\} \subset \bigcup_{j=1}^{2^r-1} I_j,$$

where the I_j are intervals with $m(I_j) \le (d/\pi)(4\varepsilon_0)^{1/r}$. We choose ε_0 so small such that $m(I_j) \le (200 \cdot 2^\xi)^{-1}$.

By Remark 2, an entire interval in $\mathcal{P}_{m_0-k_d}$ contains (exactly) one remainder interval of type k_d from the partition $\mathcal{P}_{m_0}$. Thus, to conclude the proof of Lemma 5.5, it is enough to show that there exists an entire interval $\omega_0 \in \mathcal{P}_{m_0-k_d}$ such that $\omega_0 \subset [0, 1/4]$ and ω_0 does not intersect any interval I_j. Then, if $\omega \in \mathcal{P}_{m_0}$ denotes the remainder interval of type k_d in ω_0, we can set ω_1 in the statement of Lemma 5.5 equal to $g^{-1}(\omega) \cap [0, 1/d[$ and ω_2 equal to $g^{-1}(\omega) \cap [\theta_{j_0}, \theta_{j_0+1}[$. On the one hand, we have

$$\#\left\{\text{ent. int. } \omega \in \mathcal{P}_{m_0-k_d} \,\middle|\, \omega \cap \left(\bigcup_{j=1}^{2^r-1} I_j\right) \neq \emptyset\right\}$$
$$\le (2^r - 1)\left(2 + \max\{|I_j|\}/d^{-(m_0-k_d)}\right) \le 2^\xi\left(2 + \frac{d^{m_0-k_d}}{200 \cdot 2^\xi}\right) \le \frac{d^{m_0-k_d}}{100},$$

where in the last inequality we used the condition in the choice of m_0. On the other hand, from the claim in Lemma 3.3, we derive

$$\begin{aligned}\#\{\text{ent. int. } \omega \in \mathcal{P}_{m_0-k_d} \mid \omega \subset [0, 1/4]\} &\geq \frac{1}{4}\#\{\omega \in \mathcal{P}_{m_0-k_d} \mid \omega \subset [0, 1/4]\} \\ &\geq \frac{d^{m_0-k_d}}{16}.\end{aligned}$$

Comparing these two bounds, the claim follows. □

6. *Proof of Proposition 5.2*

To prove claim (7) notice that, for $r \geq (1/2 + 2\eta)\log(1/\alpha)$,

$$1/\sqrt{\alpha} \leq \exp((1+4\eta)^{-1}r) < \exp((1-\eta)r).$$

From Lemmas 4.3 and 5.3 it follows that for any admissible curve $\widehat{Z} = \text{graph}(Z)$, $Z : T' \to B$,

$$\{\theta \in T' \mid \widehat{Z}(\theta) \in \widehat{J}(r-2)\} \subset \bigcup_{j=1}^{4\cdot 2^{\xi}} I_j,$$

where the I_j are intervals with

$$m(I_j) \leq 2(d/\pi)\left(\frac{\sqrt{\alpha}e^{-r+2}}{\alpha}\right)^{1/\xi} < 2(d/\pi)(e^2 e^{-\eta r})^{1/\xi} \leq Ce^{-(\eta/\xi)r},$$

which implies (7).

The remaining part is dedicated to the proof of claim (6). Since the right-hand side of claim (6) does not depend on the size of T, without loss of generality we assume that T has maximal size, i.e. $T = S^1$. We may and do assume that there exists a point $z_0 = (\theta_0, x_0)$ on the admissible curve $\widehat{Y}$ such that $F^{M(\alpha)}(z_0) \in \widehat{J}(0)$ since, otherwise, claim (6) is trivial.

Let $x_i = f_a^i(x_0)$. Consider the horizontal strips

$$S_i = S^1 \times [x_i - 5^i L\alpha, x_i + 5^i L\alpha], \quad i = 0, 1, 2, \ldots,$$

where $L = \max\{1, 2A_0(2\pi/d)\}$. Note that these strips are defined so that the (vertical) width $|S_i|$ satisfies

$$4|S_i| + \alpha \leq |S_{i+1}| \quad \text{and} \quad |S_i| \leq 2\cdot 5^{M(\alpha)}L\alpha < \sqrt{\alpha} \quad \text{for } 0 \leq i \leq M(\alpha),$$

provided that α is sufficiently small. From the first inequality, it follows that $F(S_i) \subset S_{i+1}$. Since the slope of the admissible curve $\widehat{Y}$ is bounded by $2A_0(2\pi/d)\alpha \leq L\alpha$, we have $\widehat{Y} \subset S_0$ and hence $F^i(\widehat{Y}) \subset S^i$ for $0 \leq i \leq M(\alpha)$.

The strips S_i, $0 \leq i \leq M(\alpha) - 1$, do not meet the critical strip $\widehat{J}(0)$. Indeed, otherwise $S_{M(\alpha)}$ would intersect both $\widehat{J}(0)$ and $F^{M(\alpha)-i}(\widehat{J}(0))$. However, this is impossible. In fact, as before, we can see inductively that, for $j \geq 1$, $F^j(\widehat{J}(0))$ is contained in the strip

$$J_j = S^1 \times [f_a^j(0) - 5^j\alpha, f_a^j(0) + 5^j\alpha].$$

The width of J_j is bounded by $\sqrt{\alpha}$ for $1 \leq j \leq M(\alpha)$. Hence, the distance between J_j and $\widehat{J}(0)$ is larger than

$$|f_a^j(0)| - 2\sqrt{\alpha} \geq \text{constant} - 2\sqrt{\alpha} > \sqrt{\alpha} > |S_{M(\alpha)}|$$

(recall that 0 is non-recurrent for f_a).

Choosing α sufficiently small, from $d(S_i, S^1 \times \{0\}) > \sqrt{\alpha}$ and using $\partial_\theta \partial_x f = 0$ and $|\partial_x^2 f| \leq 2$, we obtain the following distortion estimate, for all $0 \leq i \leq M(\alpha) - 1$:

$$\sum_{j=i}^{M(\alpha)-1} \sup_{z,z' \in S_j} \log \left| \frac{\partial_x f(z)}{\partial_x f(z')} \right| \leq M(\alpha) \frac{2 \cdot 2 \cdot 5^{M(\alpha)} L\alpha}{\sqrt{\alpha}} < \log 2, \tag{9}$$

which allows us to consider the maps $F^{M(\alpha)-i}$ as almost linear on S_i. Since $|f_a^{M(\alpha)}(x_0)| < |f_{M(\alpha)}(z_0)| + |S_{M(\alpha)}| < 2\sqrt{\alpha} < \delta$ if α is sufficiently small, Lemma 5.1 gives

$$|\partial_x f_{M(\alpha)-i}(\theta, x)| > \frac{1}{2} \prod_{j=0}^{M(\alpha)-i-1} |f_a'(x_{i+j})| > \frac{\mu \sigma^{M(\alpha)-i}}{2} \tag{10}$$

for any $(\theta, x) \in S_i$ with $0 \leq i \leq M(\alpha) - 1$.

We introduce some more constants (for the definition of m_0 and ε_0 see Lemma 4.3).

- Let $\bar{\sigma} = \sqrt{\sigma} > 1$.
- Fix a constant $\kappa > 4^{m_0}$, independent of α, so that

$$\kappa \varepsilon_0 / 4 - d A_0 (2\pi/d) - 4(1 - \bar{\sigma}^{-1})^{-1} > 1.$$

Set $\lambda_j = |\partial_x f_{M(\alpha)-j}(F^j(z_0))| / \bar{\sigma}^{(M(\alpha)-j)}$ for $0 \leq j \leq M(\alpha) - 1$. Note that we have, from equation (10),

$$\lambda_j > \mu \sigma^{M(\alpha)-j} / 2\bar{\sigma}^{(M(\alpha)-j)} = (\mu/2) \bar{\sigma}^{M(\alpha)-j}. \tag{11}$$

Let $1 \leq t_1 < t_2 < \cdots < t_q \leq M(\alpha) - 1$ be the (finite) sequence of integers defined inductively by

$$t_1 = \min\{s \mid 1 \leq s,\ \lambda_s \geq \lambda_{s+j} \text{ for all } j \geq 1\},$$
$$t_{i+1} = \min\{s \mid t_i < s \leq M(\alpha) - 1,\ \lambda_{t_i} \geq \kappa \lambda_s \text{ and } \lambda_s \geq \lambda_{s+j} \text{ for all } j \geq 1\}.$$

We have $t_{i+1} > t_i + m_0$, for all $1 \leq i < q$, because, by the definition of the λ_j and κ,

$$4^{m_0} < \kappa \leq \frac{\lambda_{t_i}}{\lambda_{t_{i+1}}} = |\partial_x f_{t_{i+1}-t_i}(F^{t_i}(z_0))| \bar{\sigma}^{t_i - t_{i+1}} < 4^{t_{i+1}-t_i}.$$

Let $k_0(\alpha) := \max\{1 \leq i \leq q \mid \lambda_{t_i} \geq 2e^{-r_0(\alpha)+2} / \sqrt{\alpha}\}$.

CLAIM 2. *We have $k_0(\alpha) \geq \gamma r_0(\alpha)$ for $\gamma = \eta / \log(4\kappa)$ if α is sufficiently small.*

Proof. Let α be so small such that $k_0(\alpha) < q$ (observe that $\lambda_{t_q} < \kappa \lambda_{M(\alpha)-1} \leq 4\kappa$ and $2e^{-r_0(\alpha)+2} / \sqrt{\alpha} \sim \alpha^{-2\eta}$).

On the one hand, since $\lambda_j \leq 4\lambda_{j+1}$, we have for $1 \leq i < q$, $\lambda_{t_i} \leq 4\kappa \lambda_{t_{i+1}}$; hence, $\lambda_{t_1} \leq (4\kappa)^{k_0(\alpha)} \lambda_{t_{k_0(\alpha)+1}}$. From equation (11), it follows that

$$\lambda_{t_{k_0(\alpha)+1}} \geq (4\kappa)^{-k_0(\alpha)} \lambda_{t_1} \geq (4\kappa)^{-k_0(\alpha)} (\mu/2) \bar{\sigma}^{M(\alpha)-t_1}.$$

On the other hand, by the definition of $k_0(\alpha)$, $\lambda_{t_{k_0(\alpha)+1}} \leq 2e^{-r_0(\alpha)+2} / \sqrt{\alpha}$. Combining these two bounds on $\lambda_{t_{k_0(\alpha)+1}}$, we derive

$$-r_0(\alpha) - (\log \alpha)/2 \geq M(\alpha) \log \bar{\sigma} - k_0(\alpha) \log(4\kappa) - C,$$

where $C = \log(4\bar{\sigma}^{t_1}/\mu) + 2$ is a constant independent of α. Recall the definitions (5) and that $r_0(\alpha) = (1/2 - 2\eta)\log(1/\alpha)$. We obtain

$$k_0(\alpha)\log(4\kappa) \geq r_0(\alpha) - \left(\frac{1}{2} - 4\eta\right)\log\frac{1}{\alpha} - C = r_0(\alpha)\left(1 - \frac{\frac{1}{2} - 4\eta}{\frac{1}{2} - 2\eta}\right) - C$$
$$= r_0(\alpha)\left(\frac{2\eta}{\frac{1}{2} - 2\eta}\right) - C \geq \eta r_0(\alpha),$$

for sufficiently small α. This proves the claim. □

Let $1 \leq s \leq M(\alpha)$. Two intervals $\tilde{\omega}_1$ and $\tilde{\omega}_2$ in $\mathcal{P}_s$ are said to be *incompatible* if

$$\inf\{|x_1 - x_2| \mid (\theta_1, x_1) \in F^{M(\alpha)}(\widehat{Y}|_{\tilde{\omega}_1}),\ (\theta_2, x_2) \in F^{M(\alpha)}(\widehat{Y}|_{\tilde{\omega}_2})\} > 2e^{-r_0(\alpha)+2}\sqrt{\alpha}.$$

If $\tilde{\omega}_1, \tilde{\omega}_2 \in \mathcal{P}_s$ are incompatible then obviously either all admissible curves contained in $F^{M(\alpha)}(\widehat{Y}|_{\tilde{\omega}_1})$ or all admissible curves contained in $F^{M(\alpha)}(\widehat{Y}|_{\tilde{\omega}_2})$ do not intersect $\widehat{J}(r_0(\alpha) - 2)$. We are going to establish a sufficient condition for incompatibility. Let $1 \leq t \leq M(\alpha)$. For $\omega \in \mathcal{P}_t$ and $1 \leq j \leq t$, we set

$$\widehat{Y}_j(\omega) = \operatorname{graph}(Y_j(\omega, \cdot)) = F^j(\widehat{Y}|_\omega).$$

Let $\omega_0 \in \mathcal{P}_{t_i - 1}$ be an entire interval. It follows that $\widehat{Y}_{t_i-1}(\omega_0)$ is defined on the whole of S^1. We are therefore in a position to apply Lemma 5.5 to this curve. We determine that there exist two remainder intervals $\omega_1, \omega_2 \in \mathcal{P}_{t_i+m_0}|\omega_0$ of type k_d such that $g^{t_i}(\omega_1) = g^{t_i}(\omega_2)$ and, for all $\theta \in g^{t_i}(\omega_1) = g^{t_i}(\omega_2)$,

$$|Y_{t_i}(\omega_1, \theta) - Y_{t_i}(\omega_2, \theta)| \geq \varepsilon_0\alpha.$$

Denote by $\operatorname{inc}(\omega_0)$ all such pairs (ω_1, ω_2) which satisfy the properties above.

CLAIM 3. (A sufficient condition for incompatibility) *Let $1 \leq i < k_0(\alpha)$, $\omega_0 \in \mathcal{P}_{t_i-1}$ an entire interval and $(\omega_1, \omega_2) \in \operatorname{inc}(\omega_0)$. If $\tilde{\omega}_1 \in \mathcal{P}_{t_{i+1}-1}|\omega_1$ and $\tilde{\omega}_2 \in \mathcal{P}_{t_{i+1}-1}|\omega_2$ such that $g^{t_i}(\tilde{\omega}_1) = g^{t_i}(\tilde{\omega}_2)$, then $\tilde{\omega}_1$ and $\tilde{\omega}_2$ are incompatible.*

Proof. Since $(\omega_1, \omega_2) \in \operatorname{inc}(\omega_0)$ we have

$$|Y_{t_i}(\omega_1, \theta) - Y_{t_i}(\omega_2, \theta)| \geq \varepsilon_0\alpha$$

for all $\theta \in g^{t_i}(\omega_1) = g^{t_i}(\omega_2)$. Also, since $g^{t_i}(\tilde{\omega}_1) = g^{t_i}(\tilde{\omega}_2)$, we deduce from the distortion estimate (9) that

$$|F(Y_{t_{i+1}-1}(\tilde{\omega}_1, \theta)) - F(Y_{t_{i+1}-1}(\tilde{\omega}_2, \theta))| \geq \frac{1}{2}\frac{\lambda_{t_i}}{\lambda_{t_{i+1}}}\varepsilon_0\alpha$$

for all $\theta \in g^{t_i}(\tilde{\omega}_1) = g^{t_i}(\tilde{\omega}_2)$. Thus, the vertical distance between $F(\widehat{Y}_{t_{i+1}-1}(\tilde{\omega}_1)) = F^{t_{i+1}}(\widehat{Y}|_{\tilde{\omega}_1})$ and $F(\widehat{Y}_{t_{i+1}-1}(\tilde{\omega}_2)) = F^{t_{i+1}}(\widehat{Y}|_{\tilde{\omega}_2})$ is bounded from below by

$$\frac{1}{2}\frac{\lambda_{t_i}}{\lambda_{t_{i+1}}}\varepsilon_0\alpha - d2A_0(2\pi/d)\alpha,$$

where the second term is a bound for the oscillation of the maximally $[d] + 1$ admissible curves contained in each of these two images.

For $t_{i+1} \le j \le M(\alpha)$, let

$$\Delta_j = \inf\{|x_1 - x_2| \mid (\theta_1, x_1) \in F^j(\widehat{Y}|_{\tilde{\omega}_1}),\ (\theta_2, x_2) \in F^j(\widehat{Y}|_{\tilde{\omega}_2})\}.$$

In other words, Δ_j is the vertical distance between the two images $F^j(\widehat{Y}|_{\tilde{\omega}_1})$ and $F^j(\widehat{Y}|_{\tilde{\omega}_2})$. We have

$$\Delta_{t_{i+1}} \ge \frac{1}{2}\frac{\lambda_{t_i}}{\lambda_{t_{i+1}}}\varepsilon_0\alpha - d2A_0(2\pi/d)\alpha. \tag{12}$$

If we put

$$D_j = \min_{(\theta,x)\in S_j} |\partial_x f(\theta, x)| = \min_{(\theta,x)\in S_j} |f_a'(x)|,$$

for $1 \le j \le M(\alpha) - 1$, then the distances Δ_j satisfy

$$\Delta_{j+1} \ge D_j\Delta_j - 2\alpha,$$

where the last term 2α is the oscillation of $\alpha\phi$. Hence,

$$\Delta_{M(\alpha)} \ge \left(\prod_{j=t_{i+1}}^{M(\alpha)-1} D_j\right)\Delta_{t_{i+1}} - \sum_{j=t_{i+1}}^{M(\alpha)-1}\left(\prod_{l=j}^{M(\alpha)-1} D_l\right)2\alpha.$$

From the definition of the λ_j and the distortion estimate (9), we have

$$\frac{1}{2}\lambda_j\bar{\sigma}^{M(\alpha)-j} \le \prod_{l=j}^{M(\alpha)-1} D_l \le 2\lambda_j\bar{\sigma}^{M(\alpha)-j}.$$

Since, by the definition of t_{i+1}, $\lambda_j \le \lambda_{t_{i+1}}$ for $j \ge t_{i+1}$, we obtain

$$\begin{aligned}
\Delta_{M(\alpha)} &\ge \lambda_{t_{i+1}}\bar{\sigma}^{M(\alpha)-t_{i+1}}\Delta_{t_{i+1}}/2 - \sum_{j=t_{i+1}}^{M(\alpha)-1}\lambda_j\bar{\sigma}^{M(\alpha)-j}4\alpha \\
&\ge \lambda_{t_{i+1}}\bar{\sigma}^{M(\alpha)-t_{i+1}}\left(\Delta_{t_{i+1}}/2 - \sum_{j=t_{i+1}}^{M(\alpha)-1}4\alpha\bar{\sigma}^{-(j-t_{i+1})}\right) \\
&\ge \lambda_{t_{i+1}}\bar{\sigma}^{M(\alpha)-t_{i+1}}(\Delta_{t_{i+1}}/2 - 4\alpha(1-\bar{\sigma}^{-1})^{-1}).
\end{aligned}$$

Recall that $\lambda_{t_{i+1}} \ge 2e^{-r_0(\alpha)+2}/\sqrt{\alpha}$. Using equation (12), the definition of κ and $\lambda_{t_i}/\lambda_{t_{i+1}} \ge \kappa$, we conclude that

$$\begin{aligned}
\Delta_{M(\alpha)} &\ge (2e^{-r_0(\alpha)+2}/\sqrt{\alpha})(\kappa\varepsilon_0\alpha/4 - dA_0(2\pi/d)\alpha - 4\alpha(1-\bar{\sigma}^{-1})^{-1}) \\
&> 2e^{-r_0(\alpha)+2}\sqrt{\alpha}.
\end{aligned}$$

□

Consider an entire interval $\omega_0 \in \mathcal{P}_{t_1-1}$ and one pair $(\omega_1, \omega_2) \in \mathrm{inc}(\omega_0)$. Applying the criteria for incompatibility we can build a union $\tilde{u} \subset (\omega_1 \cup \omega_2)$ of elements in $\mathcal{P}_{t_2-1}$ such that $\tilde{u}$ contains exactly one interval of each interval pair $\tilde{\omega}_1 \in \mathcal{P}_{t_{i+1}-1}|\omega_1$ and $\tilde{\omega}_2 \in \mathcal{P}_{t_{i+1}-1}|\omega_2$ with $g^{t_i}(\tilde{\omega}_1) = g^{t_i}(\tilde{\omega}_2)$, such that $F^{M(\alpha)}(\widehat{Y}|_{\omega})$ does not intersect $\widehat{J}(r_0(\alpha)-2)$ for all $\omega \in \mathcal{P}_{M(\alpha)}|\tilde{u}$. Observe that

$$m(\tilde{u}) = m(\omega_1) = m(\omega_2) = \frac{\mathrm{rem}_{k_d}}{d^{t_1+m_0}}.$$

By Lemma 3.2, at least one-third of all intervals in $\mathcal{P}_{t_1-1}$ are entire intervals. Set $Q_0 = S^1$. Building similar unions as above for each entire interval in $\mathcal{P}_{t_1-1}$ we obtain a set $\overline{Q}_1 \subset Q_0$, which is a union of elements in $\mathcal{P}_{t_2-1}$, such that, for all $\omega \in \mathcal{P}_{M(\alpha)}|\overline{Q}_1$, $F^{M(\alpha)}(\widehat{Y}|_\omega)$ does not intersect $\widehat{J}(r_0(\alpha)-2)$ and

$$m(\overline{Q}_1) \geq \frac{1}{3}\#\{\omega \in \mathcal{P}_{t_1-1}\}\frac{\mathrm{rem}_{k_d}}{d^{t_1+m_0}} \geq \frac{1}{3}\frac{m(Q_0)}{d^{-(t_1-1)}}\frac{\mathrm{rem}_{k_d}}{d^{t_1+m_0}} = \frac{\mathrm{rem}_{k_d}}{3d^{m_0+1}}.$$

We take this set from Q_0 away and continue with the remaining part $Q_1 = Q_0 \setminus \overline{Q}_1$. Obviously,

$$m(Q_1) \leq m(Q_0)\bigg(1 - \frac{\mathrm{rem}_{k_d}}{3d^{m_0+1}}\bigg).$$

As already mentioned in Remark 4 and in the induction step in the proof of Lemma 3.2, we have not used remainder intervals of type k_d. Thus, with a similar argument as in the proof of Lemma 3.2, we see that we still have a control of the proportion of entire intervals in $\mathcal{P}_{t_2-1}|Q_1$ or, more precisely,

$$\#\{\text{rem. int. in } \mathcal{P}_{t_2-1}|Q_1\}\big/\#\{\text{ent. int. in } \mathcal{P}_{t_2-1}|Q_1\} \leq 2.$$

In other words, at least one-third of all intervals in $\mathcal{P}_{t_2-1}|Q_1$ are entire intervals. We can therefore apply the same reasoning again and exclude a 'good' set $\overline{Q}_2 \subset Q_1$, which is a union of elements in $\mathcal{P}_{t_3-1}$, such that, for all $\omega \in \mathcal{P}_{M(\alpha)}|\overline{Q}_2$, $F^{M(\alpha)}(\widehat{Y}|_\omega)$ does not intersect $\widehat{J}(r_0(\alpha)-2)$ and for $Q_2 = Q_1 \setminus \overline{Q}_2$ we have

$$m(Q_2) \leq m(Q_1)\bigg(1 - \frac{\mathrm{rem}_{k_d}}{3d^{m_0+1}}\bigg).$$

We can continue this procedure a further $k_0(\alpha)-3$ times, and we finally obtain a set $Q_{k_0(\alpha)-1} \subset S^1$ such that

$$m(Q_{k_0(\alpha)-1}) \leq m(Q_0)\bigg(1 - \frac{\mathrm{rem}_{k_d}}{3d^{m_0+1}}\bigg)^{k_0(\alpha)-1} = \bigg(1 - \frac{\mathrm{rem}_{k_d}}{3d^{m_0+1}}\bigg)^{k_0(\alpha)-1}$$

and, for all $\omega \in \mathcal{P}_{M(\alpha)}|Q^c_{k_0(\alpha)-1}$, $F^{M(\alpha)}(\widehat{Y}|_\omega)$ does not intersect $\widehat{J}(r_0(\alpha)-2)$. Let $Q = \{\omega \in \mathcal{P}_{M(\alpha)} \mid F^{M(\alpha)}(\widehat{Y}|_\omega) \cap \widehat{J}(r_0(\alpha)-2) \neq \emptyset\}$. By construction, $Q \subset Q_{k_0(\alpha)-1}$ and

$$\#\{\text{rem. int. in } \mathcal{P}_{M(\alpha)}|Q_{k_0(\alpha)-1}\}\big/\#\{\text{ent. int. in } \mathcal{P}_{M(\alpha)}|Q_{k_0(\alpha)-1}\} \leq 2.$$

It follows that

$$\begin{aligned}\#\{\omega \in \mathcal{P}_{M(\alpha)}|Q\} \leq \#\{\omega \in \mathcal{P}_{M(\alpha)}|Q_{k_0(\alpha)-1}\} &\leq 3\frac{m(Q_{k_0(\alpha)-1})}{d^{-M(\alpha)}} \\ &\leq 3d^{M(\alpha)}C\Big(1 - \frac{\mathrm{rem}_{k_d}}{3d^{m_0+1}}\Big)^{k_0(\alpha)}.\end{aligned}$$

As $k_0(\alpha) \geq \gamma r_0(\alpha)$ by Claim 2, this concludes the proof of Proposition 5.2.

7. *Large deviations*

To conclude the proof that the vertical Lyapunov exponent is positive and thus the proof of Theorem 2.1 we can follow the large deviation argument by Viana [**Vi**, §2.4]. Only in estimating the measure of the set $\Omega_q(\rho_0, \dots, \rho_{m_q})$ defined below do we have to be more cautious. Corollary 5.4 and Proposition 5.2 which we proved here are the counterparts to Corollary 2.3 and Lemma 2.6 in Viana [**Vi**]. The other results, in particular Lemmas 2.4 and 2.5 from Viana [**Vi**, §2.4], depend neither on the condition that d is an integer nor that $d \geq 16$; thus, they remain valid including their proofs.

In all that follows we let $n \geq 1$ be fixed and sufficiently large. We define $m \geq 1$ by $m^2 \leq n < (m+1)^2$ and also take $l = 2m - M(\alpha)$. Note that $l/2 \approx m \approx \sqrt{n}$ as long as $n \gg \log(1/\alpha)$. We are considering an arbitrary admissible curve $\widehat{X}_0 = \mathrm{graph}(X_0)$, $X_0 : T \to B$. We can assume that T has maximal measure, i.e. $T = S^1$. Given $0 \leq \nu \leq n$ and $\omega_{\nu+l} \in \mathcal{P}_{\nu+l}$, we set $\gamma = F^\nu(\widehat{X}_0|_{\omega_{\nu+l}})$. Then we say that ν is:

- a I_n-*situation* for $\theta \in \omega_{\nu+l}$ if $\gamma \cap \widehat{J}(0) \neq \emptyset$ but $\gamma \cap \widehat{J}(m) = \emptyset$; and
- a II_n-*situation* for $\theta \in \omega_{\nu+l}$ if $\gamma \cap \widehat{J}(m) \neq \emptyset$.

Note that, by Corollary 4.2, γ is the graph of a function defined on $g^\nu(\omega_{\nu+l}) \in \mathcal{P}_l$ and whose derivative is bounded above by $2A_0(2\pi/d)\alpha$. Therefore, its diameter in the x-direction is bounded by

$$2A_0(2\pi/d)\alpha d^{-l} \ll \sqrt{\alpha}e^{-m} \leq \sqrt{\alpha}(e^{-(m-1)} - e^{-m}) \tag{13}$$

(recall that, by condition (1), $d > 1.82$). This means that, whenever ν is a II_n-situation for $\omega_{\nu+l}$, γ is contained in $\widehat{J}(m-1)$. Recall that $F^n(\widehat{X}_0)$ contains maximally $3d^n$ curves. Let $B_2(n) = \{\theta \in S^1 \mid \text{some } 0 \leq \nu \leq n \text{ is a } \mathrm{II}_n\text{-situation for } \theta\}$. Corollary 5.4 gives

$$\begin{aligned} m(B_2(n)) \leq (n+1)3C_1\left(\frac{|J(m-1)|}{\alpha}\right)^{1/\rho} &\leq \mathrm{const}(n+1)(e^{-m})^{1/\rho} \\ &\leq \mathrm{const}\, e^{-\sqrt{n}/2\rho}. \end{aligned} \tag{14}$$

From now on, we consider only values of $\theta \in S^1 \setminus B_2(n)$, that is, having no II_n-situations in $[0, n]$. Let $0 \leq \nu_1 < \cdots < \nu_s \leq n$ be the I_n-situations for a $\theta \in S^1 \setminus B_2(n)$. For each $\nu = \nu_i$ we fix $r = r_i \in \{1, \dots, m\}$ minimum such that $\gamma \cap \widehat{J}(r) = \emptyset$, and we set $G = \{i \mid r_i \geq (1/2 - 2\eta)\log(1/\alpha)\}$ (note that this set depends on θ). Viana [**Vi**, §2.4] shows that there exists a constant $c > 0$ such that

$$\log|\partial_x f_n(\widehat{X}_0(\theta))| \geq cn \quad \text{for every } \theta \in S^1 \setminus E_n, \tag{15}$$

where $E_n = B_1(n) \cup B_2(n)$ and

$$B_1(n) = \left\{\theta \in S^1 \setminus B_2(n) \;\middle|\; \sum_{i \in G} r_i \geq cn\right\}.$$

In view of equation (14), we are left to prove that

$$m(B_1(n)) \leq \mathrm{const}\, e^{-\gamma\sqrt{n}} \tag{16}$$

for some $\gamma > 0$. First we let $0 \leq q \leq 2m - 1$ be fixed and denote

$$G_q = \{i \in G \mid \nu_i \equiv q \bmod 2m\}.$$

We also take $m_q = \max\{j \mid 2mj + q \leq n\}$ (note that $2m_q \approx m \approx \sqrt{n}$) and, for each $0 \leq j \leq m_q$, we let $\hat{r}_j = r_i$ if $2mj + q = \nu_i$, for some $i \in G_q$, and $\hat{r}_j = 0$ otherwise. Observe that G_q and the $\hat{r}_j$ are, in fact, functions of θ. Then we introduce

$$\Omega_q(\rho_0, \dots, \rho_{m_q}) = \{\theta \in S^1 \setminus B_2(n) \mid \hat{r}_j = \rho_j \text{ for } 0 \leq j \leq m_q\},$$

where for each j either $\rho_j = 0$ or $\rho_j \geq (1/2 - 2\eta)\log(1/\alpha)$; we also assume that the ρ_j are not simultaneously zero. With the help of Proposition 5.2 and Lemma 3.3 we will prove the following lemma.

LEMMA 7.1. *There exists $\beta > 0$ such that*

$$m(\Omega_q(\rho_0, \dots, \rho_{m_q})) \leq C_2^\tau \exp\Big(-\beta \sum_{j=0}^{m_q} \rho_j\Big),$$

where $\tau = \#\{j \mid \rho_j \neq 0\}$ and $C_2 = 60 \cdot 2^\xi C$.

Having shown Lemma 7.1 and renaming β as 5β and C_2 as C_4, we can follow verbatim the remaining large deviation argument in Viana [**Vi**, §2.4] which proves equation (16) and thus the existence of a positive vertical Lyapunov exponent at Lebesgue almost every point. So, it is only left to prove the lemma above.

Proof. We assume $\rho_j \neq 0$, for all $0 \leq j \leq m_q$. The other cases are similar. We introduce the notation $a_j = 2mj + q + l$, for $0 \leq j \leq m_q$. If $\omega \in \mathcal{P}_i$, $i \geq a_j$, then by $\hat{r}_j(\omega) = \rho_j$ we mean that for $\theta \in \omega$, $\hat{r}_j = \rho_j$. Since $\Omega_q(\rho_0, \dots, \rho_{m_q}) \subset \{\omega_{a_{m_q}} \in \mathcal{P}_{a_{m_q}} \mid \hat{r}_j(\omega_{a_{m_q}}) = \rho_j \ \forall\, 0 \leq j \leq m_q\}$ (set $a_{-1} = 0$) we have

$$\begin{aligned} m(\Omega_q(\rho_0, \dots, \rho_{m_q})) &\leq \sum_{\substack{\omega_{a_0} \in \mathcal{P}_{a_0} \\ \hat{r}_0(\omega_{a_0}) = \rho_0}} \sum_{\substack{\omega_{a_1} \in \mathcal{P}_{a_1} | \omega_{a_0} \\ \hat{r}_1(\omega_{a_1}) = \rho_1}} \cdots \sum_{\substack{\omega_{a_{m_q}} \in \mathcal{P}_{a_{m_q}} | \omega_{a_{m_q-1}} \\ \hat{r}_{m_q}(\omega_{a_{m_q}}) = \rho_{m_q}}} m(\omega_{a_{m_q}}) \\ &\leq \frac{1}{d^{a_{m_q}}} \cdot \prod_{j=0}^{m_q} \max_{\omega_{a_{j-1}} \in \mathcal{P}_{a_{j-1}}} \underbrace{\#\{\omega \in (\mathcal{P}_{a_j} | \omega_{a_{j-1}}) \mid \hat{r}_j(\omega) = \rho_j\}}_{(*)}. \end{aligned}$$

Let $1 \leq j \leq m_q$. We claim that

$$(*) \leq d^{2m} C_2 e^{-\beta \rho_j}. \tag{17}$$

Note that, by inequality (13), $F^{2mj+q}(\widehat{X}_0|_\omega) \subset \widehat{J}(\rho_j - 2)$ if $\omega \in \mathcal{P}_{a_j}$ and $\hat{r}_j(\omega) = \rho_j$. Using $a_j - a_{j-1} = 2m$ and $2mj + q = a_{j-1} + M(\alpha)$, it follows that

$$\begin{aligned} (*) &\leq \#\{\omega \in (\mathcal{P}_{a_j} | \omega_{a_{j-1}}) \mid F^{2mj+q}(\widehat{X}_0|_\omega) \subset \widehat{J}(\rho_j - 2)\} \\ &\leq \#\{\omega \in \mathcal{P}_{2m} \mid F^{M(\alpha)}(\widehat{Y}|_\omega) \subset \widehat{J}(\rho_j - 2)\} = (**), \end{aligned}$$

where $\widehat{Y} = \text{graph}(Y) = F^{a_{j-1}}(\widehat{X}_0|_{\omega_{a_{j-1}}})$. To estimate $(**)$ we first consider the case $(1/2 - 2\eta)\log(1/\alpha) \leq \rho_j \leq (1/2 + 2\eta)\log(1/\alpha)$. Recall that $r_0(\alpha) = (1/2 - 2\eta)\log(1/\alpha)$ and $l = 2m - M(\alpha)$. From Lemma 3.2 we derive for $\omega' \in \mathcal{P}_{M(\alpha)}$

$$\#\{\omega \in \mathcal{P}_{2m} | \omega'\} \leq \#\{\omega \in \mathcal{P}_l\} \leq 3d^l.$$

Thus,

$$\begin{aligned} (**) &\leq 3d^l \#\{\omega' \in \mathcal{P}_{M(\alpha)} \mid F^{M(\alpha)}(\widehat{Y}|_{\omega'}) \cap \widehat{J}(\rho_j - 2) \neq \emptyset\} \\ &\leq 3d^l \#\{\omega' \in \mathcal{P}_{M(\alpha)} \mid F^{M(\alpha)}(\widehat{Y}|_{\omega'}) \cap \widehat{J}(r_0(\alpha) - 2) \neq \emptyset\}. \end{aligned}$$

Applying claim (6) in Proposition 5.2 to the last term yields

$$(**) \leq 3d^l 3d^{M(\alpha)} C e^{-\beta r_0(\alpha)} \leq d^{2m} C_2 e^{-\beta r_0(\alpha)} \leq d^{2m} C_2 e^{-(\beta(1/2-2\eta)/(1/2+2\eta))\rho_j}.$$

Renaming $\beta(1/2-2\eta)/(1/2+2\eta)$ by β, this proves claim (17). Now we consider the case when $(\frac{1}{2}+2\eta)\log(1/\alpha) \leq \rho_j \leq m$. Observe that $F^{M(\alpha)}(\widehat{Y})$ consists of maximally $3d^{M(\alpha)}$ admissible curves. We obtain

$$(**) \leq \max_{\widehat{Z} \text{ adm. curve in } F^{M(\alpha)}(\widehat{Y})} 3d^{M(\alpha)} \#\{\omega \in \mathcal{P}_l \mid \widehat{Z}|_\omega \subset \widehat{J}(\rho_j - 2)\}.$$

By claim (7) in Proposition 5.2 we have

$$\{\theta \in T' \mid \widehat{Z}(\theta) \in \widehat{J}(\rho_j - 2)\} \subset \bigcup_{i=1}^{4\cdot 2^\xi} I_i,$$

where the I_i are intervals with $m(I_i) \leq Ce^{-\beta\rho_j}$. Applying Lemma 3.3, it follows that

$$\begin{aligned}&\#\{\omega \in \mathcal{P}_l \mid \widehat{Z}|_\omega \subset \widehat{J}(\rho_j - 2)\}\\ &\qquad \leq 4\cdot 2^\xi \max_{\substack{I\subset[0,1[\text{ interval}\\ m(I)=Ce^{-\beta\rho_j}}} \#\{\omega \in \mathcal{P}_l \mid \omega \cap I \neq \emptyset\} \leq 20\cdot 2^\xi d^l C e^{-\beta\rho_j}.\end{aligned}$$

Thus,

$$(**) \leq 3d^{M(\alpha)} 20 \cdot 2^\xi d^l C e^{-\beta\rho_j} \leq d^{2m} C_2 e^{-\beta\rho_j}.$$

With a similar argument, we obtain for $j = 0$ that $(*) \leq d^{q+l} C_2 e^{-\beta\rho_0}$. (In fact, if $q < M(\alpha)$, this is only true if $(1/2+2\eta)\log(1/\alpha) \leq \rho_0 \leq m$. However, since n is very large this case is negligible.) Altogether, we have

$$\begin{aligned} m(\Omega_q(\rho_0, \ldots, \rho_{m_q})) &\leq \frac{1}{d^{a_{m_q}}} \cdot d^{q+l} C_2 e^{-\beta\rho_0} \cdot d^{2mm_q} C_2^{m_q} \exp\Bigg(-\beta \sum_{j=1}^{m_q} \rho_j\Bigg)\\ &= C_2^\tau \exp\Bigg(-\beta \sum_{j=0}^{m_q} \rho_j\Bigg). \end{aligned}$$

□

8. *Existence of an a.c.i.p.*

To show the existence of an absolutely continuous invariant probability measure we can apply almost verbatim the methods of Alves [**Al**]. However, since the remainder intervals in $\mathcal{P}_n$ can get arbitrarily small we have to be a bit more careful. To ensure that the remainder intervals do not get too small too fast we will exclude, using a result by Schmeling [**Sch**], a zero Lebesgue measure set of parameter values for d.

Where d is assumed to be an integer [**Al**], Alves constructs a 'good' partition $\mathcal{R} = \bigcup_{n\geq p} \mathcal{R}_n$ of $\widehat{J}$ (mod 0), where $p \geq 1$ is some fixed, large integer and each $\mathcal{R}_n$ consists of finitely many partition elements $R \in \mathcal{R}_n$ which are of the form $R = \omega \times I$, where ω is an element in $\mathcal{P}_n$ and $I \subset B$ is some appropriate interval. It is then shown that the map $\phi : \widehat{J} \to \widehat{J}$ defined as

$$\phi|R = F^n|R \quad \text{if } R \in \mathcal{R}_n$$

admits an a.c.i.p. One important property of ϕ is that it is expanding on each element $R \in \mathcal{R}$ and that the size of the images $\phi(R)$ does not fall below a fixed size. Note that if d is an integer then, by the definition of ϕ, the image $\phi(R)$ is stretched along the horizontal direction over the whole of S^1. Thus, one only has to ensure that the images by ϕ of the vertical lines $\theta \times I$, where $R = \omega \times I$ and $\theta \in \omega$, are large enough. To elaborate this expansion, an important tool is the so-called hyperbolic times for points $(\theta, x) \in \widehat{J}$. Let c be the constant in equation (15) and $0 < \varepsilon < c/2$ fixed. We say that $n \geq 1$ is a hyperbolic time for $(\theta, x) \in \widehat{J}$ if for every $0 \leq k < n$ we have

$$\sum_{k \leq j < n} \hat{r}_j(\theta, x) \leq (c + \varepsilon)(n - k),$$

where $\hat{r}_j(\theta, x) = r$ if $F^j(\theta, x) \in \widehat{J}(r - 1) \setminus \widehat{J}(r)$ for some $r \geq (1/2 - 2\eta)\log(1/\alpha)$ and 0 otherwise. Since almost every point in $\widehat{J}$ has infinitely many hyperbolic times (cf. Alves [**Al**, Proposition 2.6]) $\mathcal{R}$ can be constructed in a way such that every element $R \in \mathcal{R}_n$ contains at least one point (θ, x) having a hyperbolic time at n. This is then enough to ensure a large size of the image $\phi(R)$. The estimates obtained by showing the existence of a positive vertical Lyapunov exponent implicate good tail estimates, i.e.

$$m\bigg(\widehat{J} \setminus \bigcup_{j=p}^{n} \mathcal{R}_j\bigg) \leq Ce^{-\gamma\sqrt{n}},$$

which in turn implies that the map F also admits an a.c.i.p.

If d is not an integer we have the additional problem that even if the vertical size of $\phi(R)$ is large enough, following the construction of Alves, the horizontal size of $\phi(R)$, which corresponds to the size of $g^n(\omega)$ where $R = \omega \times I$, might be arbitrarily small. To avoid this problem we provide another version of Proposition 2.6 in Alves [**Al**] on the existence and frequency of hyperbolic times.

For $\theta \in S^1$, let $\omega_n(\theta)$ denote the element in $\mathcal{P}_n$ which contains θ. For $0 < \delta < 1$ we define $H_{\delta,n}$ to be the set of all points $(\theta, x) \in \widehat{J}$ for which n is the first hyperbolic time greater than or equal to p such that $|\omega_n(\theta)| > \delta d^{-n}$. The set $E_n \subset \widehat{J}$ is similarly defined as in the preceding section but with the difference that it is now defined on the whole of $\widehat{J}$ and not just on a single admissible curve.

PROPOSITION 8.1. *For Lebesgue almost every $d > R_0$ the following holds. There is a $\delta = \delta(\varepsilon, d) > 0$ and an integer $n_0 = n_0(p, \varepsilon, \delta, d) \geq p$ such that for every $n \geq n_0$ we have*

$$\widehat{J} \setminus E_n \subset H_{\delta,p} \cup \cdots \cup H_{\delta,n}.$$

Proof. We will use the following lemma due to Pliss [**Pl**].

LEMMA 8.2. *Given $A \geq c_2 > c_1$, let $\kappa = (c_2 - c_1)/(A - c_1)$. Then, given any real numbers $a_1, \ldots, a_n$ such that*

$$\sum_{j=1}^{n} a_j \geq c_2 n \quad \textit{and} \quad a_j \leq A \quad \textit{for all } 1 \leq j \leq n,$$

there are $l \geq \kappa n$ and $1 \leq n_1 < \cdots < n_l \leq n$ so that

$$\sum_{j=k+1}^{n_i} a_j \geq c_1(n_i - k) \quad \textit{for all } 0 \leq k < n_i \textit{ and } i = 1, \ldots, l.$$

Proof. See, for example, Alves *et al* [**ABV**, Lemma 3.1]. □

Set $a_j = -\hat{r}_{j-1}$ for $j \geq 1$, $A = 0$, $c_2 = -c$, $c_1 = -(c+\varepsilon)$ and $\kappa = \varepsilon/(c+\varepsilon)$. For $(\theta, x) \in \widehat{J} \setminus E_n$ we have, by the definition of the set E_n,

$$\sum_{j=0}^{n-1} -\hat{r}_j \geq -cn = c_2 n.$$

Thus we can apply Lemma 8.2 and determine that there are $l \geq \kappa n$ and $0 \leq n_1 < \cdots < n_l \leq n-1$ so that

$$\sum_{k \leq j < n_i} \hat{r}_j \leq -c_1(n_i - k) = (c+\varepsilon)(n_i - k),$$

i.e. every $(\theta, x) \in \widehat{J} \setminus E_n$ has at least κn hyperbolic times between 1 and n.

The next lemma shows that we have—for a full Lebesgue measure set of parameter values for d—a good control of the number of times when the interval $\omega_n(\theta)$ gets too small. We prove this lemma after completing the proof of Proposition 8.1.

LEMMA 8.3. *There exists a set $D \subset \,]1, \infty[$ having full Lebesgue measure such that the following holds. Let $d \in D$. For all $\kappa' > 0$ there is a $\delta > 0$ such that, for every $\theta \in S^1$ and every $n \geq 1$,*

$$\#\{1 \leq k \leq n \mid |\omega_k(\theta)| < \delta d^{-k}\} \leq \kappa' n.$$

Assuming $d > R_0$ lies in D, fixing $\kappa' < \kappa$ and choosing δ as in the lemma above we can now obviously choose n_0 so large that, for all $n \geq n_0$, the number of hyperbolic times between p and n is for all $(\theta, x) \in \widehat{J} \setminus E_n$ greater than the number of times between 1 and n, where the partition element containing θ falls below a fixed size. Hence

$$\widehat{J} \setminus E_n \subset H_{\delta,p} \cup \cdots \cup H_{\delta,n} \quad \text{for all } n \geq n_0.$$

To prove Lemma 8.3 we use the following two results. Let $g^0(1) = 1$, $g^1(1) = \{d\}$ and, for $n \geq 2$, $g^n(1) = g^{n-1}(\{d\})$.

LEMMA 8.4. *Let $d \in \,]1, \infty[$. There exists a unique (ergodic) invariant probability measure μ for the map $g(\theta) = d\theta \bmod 1$ which is equivalent to the Lebesgue measure. The density is given by $h(\theta) = K\tilde{h}(\theta)$, where K is a normalizing constant and*

$$\tilde{h}(\theta) = \sum_{j \geq 0} \frac{1}{d^j} \chi_{[0, g^j(1)[}(\theta).$$

Proof. See, for example, Rényi [**R**] and Parry [**Pa**]. □

Note that $1 - 1/d \leq h(\theta) \leq d/(d-1)$.

LEMMA 8.5. *There exists a full Lebesgue measure set $D \subset \,]1, \infty[$ such that if $d \in D$ then*

$$\frac{1}{n} \sum_{j=1}^{n} \delta_{g^j(1)} \to \mu \quad \textit{weakly as } n \to \infty.$$

Proof. See Schmeling [**Sch**, Theorem C]. □

Proof of Lemma 8.3. From Lemmas 8.4 and 8.5 it follows that, for $\kappa' \in [0, 1]$,

$$\lim_{n\to\infty} \frac{1}{n}\#\{1 \le k \le n \mid g^k(1) \in [0, \kappa'[\} = \mu([0, \kappa'[) \le \frac{d}{d-1}\kappa'.$$

Recall the definition of rem_k, $k \ge 1$, in §3 and set $\mathrm{rem}_0 = 1$. By definition, $g^k(1) \in [0, \kappa'[$ if and only if $\mathrm{rem}_k < \kappa'$. Set N_0 so large that

$$\#\{0 \le k \le l \mid \mathrm{rem}_k < \kappa'\} \le \frac{2d}{d-1}\kappa' l, \tag{18}$$

for all $l \ge N_0$. Set $\delta = \min\{\kappa', \min\{\mathrm{rem}_k \mid k < N_0\}\}$. By the properties of the set D, $\mathrm{rem}_k \ne 0$ for all $k \ge 0$ and therefore $\delta > 0$.

Fix an arbitrary $\theta \in S^1$. By definition, if $\omega_k(\theta)$ is a remainder interval of type $t \ge 0$ then $|\omega_k(\theta)|d^k = \mathrm{rem}_t$ (we consider here an entire interval as a remainder interval of type 0). Furthermore, if $t \ge 1$, then $\omega_{k-j}(\theta)$, $j = 1, \dots, t$, is a remainder interval of type $k - j$. Thus, if $t_k(\theta)$ denotes the remainder type of $\omega_k(\theta)$ then $\{t_k(\theta)\}_{k=0}^n$ is a sequence starting with zero and $t_{k+1}(\theta)$, $k \ge 0$, is either $t_k(\theta) + 1$ or 0, i.e. the sequence is piecewise increasing where the increasing pieces always start with 0 and increase by 1 at each step. Let $0 \le n_1 < \cdots < n_h = n$ be the times where these increasing pieces obtain their maxima, i.e. for all $i = 1, \dots, h-1$, $t_{n_i+1}(\theta) = 0$, and *vice versa*, if $t_k(\theta) = 0$ for some $k = 1, \dots, n$ then there exists $i \in \{1, \dots, h-1\}$ such that $n_i = k - 1$. Observe, if $t_{n_i}(\theta) < N_0$ then, by the definition of δ, $\#\{0 \le j \le t_{n_i}(\theta) \mid |\omega_{n_i-j}(\theta)| < \delta d^{-(n_i-j)}\} = 0$. Further, if $t_{n_i}(\theta) \ge N_0$ then $\#\{0 \le j \le t_{n_i}(\theta) \mid |\omega_{n_i-j}(\theta)| < \delta d^{-(n_i-j)}\}$ is bounded by the left-hand side of equation (18) (setting $l = t_{n_i}(\theta)$) which is in turn bounded by $(2d/(d-1))\kappa' t_{n_i}(\theta)$. Altogether, we obtain

$$\#\{0 \le k \le n \mid |\omega_k(\theta)| < \delta d^{-k}\} \le \frac{2d}{d-1}\kappa' n \quad \text{for all } n \ge 1.$$

This concludes the proof of Lemma 8.3. □

To conclude the proof of Theorem 2.2 we now only have to replace Proposition 3.6 in Alves [**Al**] by our Proposition 8.1, use the sets $H_{\delta,n}$ instead of the sets H_n for the construction of the partition $\mathcal{R}$ and replace δ_1 in Proposition 3.8 in Alves [**Al**] by δ, in the case of $\delta < \delta_1$.

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Paper B

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Positive Lyapunov exponents for quadratic skew-products over a Misiurewicz–Thurston map

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Abstract
We study a class of skew-products of quadratic maps—also called Viana maps—where the base dynamics is given by a high enough iteration of a Misiurewicz–Thurston quadratic map. We show that these systems admit two positive Lyapunov exponents.

Mathematics Subject Classification: 37D99, 37E99, 37C99

1. Introduction

Viana [Vi] studied the ergodic properties of the map $F : S^1 \times \mathbb{R} \to S^1 \times \mathbb{R}$:

$$F(\theta, x) = (d\theta \bmod 1, a_0 - x^2 + \alpha \sin(2\pi\theta)),$$

where $d \geqslant 16$ is an integer, α is a small positive real number and the parameter $1 < a_0 < 2$ is such that the map $f_{a_0}(x) = a_0 - x^2$ has a pre-periodic (but not periodic) critical point. Provided α is sufficiently small, there is an interval $I \subset (-1, 1)$ such that the strip $\widehat{J} = S^1 \times I$ is mapped into itself, i.e. $F(\widehat{J}) \subset \widehat{J}$ (see section 2.2). Points outside $\widehat{J}$ are either eventually mapped into $\widehat{J}$ or F is uniformly expanding along their orbits. Hence, from a dynamical point of view the map F can be restricted to the invariant cylinder $\widehat{J}$ where it is a partially hyperbolic system. Viana [Vi] showed that the map $F : \widehat{J} \to \widehat{J}$ has almost everywhere w.r.t. Lebesgue measure two positive Lyapunov exponents, and Alves [Al] proved that F admits an absolutely continuous invariant probability measure (a.c.i.p.). These results were refined by Buzzi *et al* [BST], respectively Schnellmann [Sch] to maps F as above but with $d \geqslant 2$ an integer, respectively $d > 1$ a real number such that the system remains partially hyperbolic in $\widehat{J}$.

Let now $f_{a_1}(\theta) = a_1 - \theta^2$ be a quadratic map of the same kind as the map f_{a_0} above, i.e. $1 < a_1 \leqslant 2$ is chosen, such that the critical point is pre-periodic but not periodic. Maps with this property are also called *Misiurewicz–Thurston*.

Remark 1.1. In contrast to a_0 we allow a_1 to be equal to 2 in which case the critical point is eventually mapped to the fixed point -1. Furthermore, if $a_1 = 2$ then $f_{a_1} : [-1, 1] \to [-1, 1]$ is conjugated by the conjugation map $\varphi(\theta) = 2\pi^{-1} \arcsin(\theta)$, $\theta \in [-1, 1]$, to the tent map with slope 2, i.e. $\varphi \circ f_{a_1} \circ \varphi^{-1}(\theta) = 1 - 2|\theta|$. This special case can serve as a model along the paper.

Set $S = [f_{a_1}^2(0), f_{a_1}(0)]$, and let the base dynamics be a high enough iteration of f_{a_1}, i.e. $\theta \mapsto f_{a_1}^k(\theta)$, $k \geqslant k_0$, where $k_0 = k_0(a_1) \geqslant 1$ is determined later. The purpose of this paper is to study the map $F : S \times \mathbb{R} \to S \times \mathbb{R}$:

$$F(\theta, x) = (f_{a_1}^k(\theta), f_{a_0}(x) + \alpha s(\theta)),$$

where the coupling function $s : S \to [-1, 1]$ is chosen in such a way that we can establish two positive Lyapunov exponents for this system. We are going to prove the following statement.

Theorem 1.2. *There is a piecewise C^1 function $s : S \to [-1, 1]$ and an integer $k_0 \geqslant 1$ such that, for all sufficiently small $\alpha > 0$ and all $k \geqslant k_0$, the map*

$$F(\theta, x) = (f_{a_1}^k(\theta), f_{a_0}(x) + \alpha s(\theta))$$

admits two positive Lyapunov exponents.

Remark 1.3. Notice that the coupling function s is not *a priori* fixed. In fact it is dependent on the base dynamics f_{a_1}. Roughly speaking the coupling function is identical to a function conjugating $f_{a_1}^k$ to a map whose absolute value of its first derivative is strictly greater than 1. For example, in the case when $a_1 = 2$, $s(\theta)$ can be taken as $h(2\pi^{-1} \arcsin(\theta))$, $\theta \in [-1, 1]$, where $h : [-1, 1] \to [-1, 1]$ is an arbitrary C^1 map whose first derivative is uniformly bounded away from 0 (the closer $|h'(\theta)|$ comes to 0 the larger we have to take k_0).

It would be interesting to drop this dependence of the coupling function on the base dynamics. For example, if we let s to be the affine one-to-one map from S to $[-1, 1]$, can we still prove two positive Lyapunov exponents when the base dynamics is given by a sufficiently high iteration of a Misiurewicz–Thurston map?

2. Preliminaries

2.1. The base dynamics

We list some facts about the Misiurewicz–Thurston map f_{a_1}. The proofs of the statements in this subsection can be found, e.g., in [Og]. Let b_j, $0 \leqslant j \leqslant k_1$, be the ordered points in the forward orbit of the critical point 0, i.e. $\{b_0, \ldots, b_{k_1}\} = \{f_{a_1}^i(0) \,;\, i \geqslant 1\}$ and $b_j < b_{j+1}$. There is a unique (and hence ergodic) a.c.i.p. μ for $f_{a_1} : S \to S$. For the density ρ of μ the following holds. There is a constant $K_0 \geqslant 1$ such that, for all $0 \leqslant j < k_1$, ρ is analytic in (b_j, b_{j+1}), where either $\rho \equiv 0$ or

$$\frac{1}{K_0} \leqslant \rho(\theta) \leqslant K_0 \left(\frac{1}{\sqrt{\theta - b_j}} + \frac{1}{\sqrt{b_{j+1} - \theta}} \right). \tag{1}$$

More precisely, there is an $\varepsilon > 0$ such that, for all $0 \leqslant j < k_1$—and by possibly increasing the constant K_0 from above—the following holds. In the intervals (b_j, b_{j+1}), where $\rho > 0$ we have that, for $\theta \in (b_j, b_j + \varepsilon)$, either

$$\frac{1}{K_0\sqrt{\theta - b_j}} \leqslant \rho(\theta) \leqslant \frac{K_0}{\sqrt{\theta - b_j}} \qquad \text{or} \qquad \frac{1}{K_0} \leqslant \rho(\theta) \leqslant K_0; \tag{2}$$

for $\theta \in (b_{j+1} - \varepsilon, b_{j+1})$, either

$$\frac{1}{K_0\sqrt{b_{j+1}-\theta}} \leqslant \rho(\theta) \leqslant \frac{K_0}{\sqrt{b_{j+1}-\theta}} \qquad \text{or} \qquad \frac{1}{K_0} \leqslant \rho(\theta) \leqslant K_0; \tag{3}$$

and, for $\theta \in (b_j + \varepsilon, b_{j+1} - \varepsilon)$,

$$\frac{1}{K_0} \leqslant \rho(\theta) \leqslant K_0. \tag{4}$$

Observe that by the definition of b_j, the critical point 0 lies inside one of the intervals (b_j, b_{j+1}), i.e. it is not a boundary point. On the interval (b_j, b_{j+1}) which contains the critical point the density ρ is positive.

Let $\varphi : S \to [-1, 1]$ be the piecewise analytic map defined by

$$\varphi(\theta) = 2\int_{b_1}^{\theta} \rho(t)\,\mathrm{d}t - 1.$$

Example 2.1. *If $a_1 = 2$ then the density ρ of the a.c.i.p. of f_{a_1} is given by*

$$\rho(\theta) = \frac{1}{\pi\sqrt{1-\theta^2}},$$

$\theta \in (-1, 1)$. *The above definition of φ gives $\varphi(\theta) = 2\pi^{-1}\arcsin(\theta)$ which is the function conjugating f_{a_1} to the tent map $\theta \mapsto 1 - 2|\theta|$.*

The inverse φ^{-1} is defined everywhere except possibly in the points $\varphi(b_j)$, $0 \leqslant j \leqslant k_1$. For notational reasons define φ^{-1} on the whole of $[-1, 1]$, and in such a way that $\varphi^{-1} : [-1, 1] \to S$ is say left continuous, and $\varphi^{-1}(-1) = b_1 (= f_{a_1}^2(0))$. Note that the closure of the image by φ^{-1} of $[-1, 1]$ is the support of μ.

Proposition 2.2. *The conjugated map $\varphi \circ f_{a_1} \circ \varphi^{-1}$ fulfils*

$$|(\varphi \circ f_{a_1} \circ \varphi^{-1})'(\theta)| \geqslant 1, \tag{5}$$

for all $\theta \in [-1, 1] \setminus \{\varphi(b)\ ;\ b \in \cup_{j=0}^{k_1} f_{a_1}^{-1}\{b_j\}\}$. Furthermore, for every $D > 1$ there is an integer $k_0 \geqslant 1$ such that, for all $k \geqslant k_0$, the map $\varphi \circ f_{a_1}^k \circ \varphi^{-1}$ satisfies

$$|(\varphi \circ f_{a_1}^k \circ \varphi^{-1})'(\theta)| \geqslant D, \tag{6}$$

for all $\theta \in [-1, 1] \setminus \{\varphi(b)\ ;\ b \in \cup_{j=0}^{k_1} f_{a_1}^{-k}\{b_j\}\}$.

Proof. See [Og]. □

Observe that g is a piecewise analytic map where the points in which g is not analytic are contained in $\{\varphi(b)\ ;\ b \in \cup_{j=0}^{k_1} f_{a_1}^{-k}\{b_j\}\}$. The discontinuities of g lie in the points $\varphi(b_j)$, $1 \leqslant j < k_1$, with $\rho|(b_j, b_{j+1}) \equiv 0$.

2.2. The map F

Let $h : [-1, 1] \to [-1, 1]$ be some fixed C^1 function for which there is a constant $K_1 \geqslant 1$ such that $K_1^{-1} \leqslant |h'(\theta)| \leqslant K_1$. Let $k_0 \geqslant 1$ be the integer in (6), such that the expansion of the conjugated map $g = \varphi \circ f_{a_1}^k \circ \varphi^{-1}$, for $k \geqslant k_0$, is at least $5K_1^2 + 4$, i.e.

$$|g'(\theta)| \geqslant 5K_1^2 + 4, \tag{7}$$

for all $\theta \in [-1, 1] \setminus \{\varphi(b) ;\ b \in \cup_{j=0}^{k_1} f_{a_1}^{-k}\{b_j\}\}$. For example, if $h(\theta) = \theta$ then the expansion of g must be at least 9. In theorem 1.2, we will choose $h \circ \varphi$ as the coupling function s, i.e. $s(\theta) = h(\varphi(\theta))$. By this choice we can conjugate F to the map $\tilde{F} : [-1, 1]\times\mathbb{R} \to [-1, 1]\times\mathbb{R}$:

$$\tilde{F}(\theta, x) = (g(\theta), f_{a_0}(x) + \alpha h(\theta)),$$

where the conjugation function Φ is given by $\Phi(\theta, x) = (\varphi(\theta), x)$, i.e. $\tilde{F} = \Phi\circ F\circ\Phi^{-1}$. Note that if we have showed two positive Lyapunov exponents for the map $\tilde{F}$ then it immediately follows that F admits two positive Lyapunov exponents. We will restrict our consideration to the conjugated map $\tilde{F}$. This map has four parameters $1 < a_0 < 2$, $1 < a_1 \leqslant 2$, $k \geqslant k_0$ and $\alpha > 0$. Henceforth, we fix the parameters a_0, a_1 and k and we do not specify the dependence on them. In contrast the parameter α is not fixed but always assumed to be sufficiently small and the dependence on it is always specified.

Set $f_1(\theta, x) = f(\theta, x) = f_{a_0}(x) + \alpha h(\theta)$ and, for $n \geqslant 1$, let us define $f_n(\theta, x)$ by

$$\tilde{F}^n(\theta, x) = (g^n(\theta), f_n(\theta, x)).$$

Observe that by (6) the horizontal Lyapunov exponent is obviously positive. Hence, we have only to focus on the vertical Lyapunov exponent, i.e. we look at

$$\frac{1}{n}\log|\partial_x f_n(\theta, x)|, \qquad \text{for } (\theta, x) \in [-1, 1] \times \mathbb{R},$$

as n tends to infinity. Let p_1 denote the unique negative fixed point of f_{a_0} ($p_1 = (-1 - \sqrt{1 + 4a_0})/2$), and take $a_0 < \beta < |p_1|$. The interval $I = [-\beta, \beta]$ satisfies $f_{a_0}(I) \subset \mathrm{int}(I)$ and $|f'_{a_0}| > 1$ on $\mathbb{R} \setminus \mathrm{int}(I)$. Writing $\widehat{J} = [-1, 1] \times I$, for sufficiently small $\alpha > 0$, we have

- $\tilde{F}(\widehat{J}) \subset \widehat{J}$; and
- $|\partial_x f(\theta, x)| > 1$ outside $\widehat{J}$.

These facts imply that, for any point (θ, x) in $[-1, 1] \times \mathbb{R}$, either its orbit eventually comes into the invariant strip $\widehat{J}$ or the vertical Lyapunov exponent is positive. Thus, it is enough to consider the restriction of the map $\tilde{F}$ to the cylinder $\widehat{J}$, in other words we show that

$$\liminf_{n\to\infty} \frac{1}{n}\log|\partial_x f_n(\theta, x)| > 0$$

for Lebesgue almost every $(\theta, x) \in \widehat{J}$.

3. Partition and distortion

Let B_i, $1 \leqslant i \leqslant k_2 \leqslant k_1$, denote the intervals (b_j, b_{j+1}), $0 \leqslant j < k_1$, on which the density ρ does not vanish. Since f_{a_1} is pre-periodic, these intervals build a Markov partition for f_{a_1}. Set $L_i = \varphi(B_i)$, $1 \leqslant i \leqslant k_2$, and define, for $n \geqslant 1$, the Markov partitions

$$\mathcal{P}_n = \{\omega \subset (-1, 1) ;\ \omega \text{ open interval such that there is } 1 \leqslant i \leqslant k_2 \text{ with } g^n : \omega \to L_i \text{ is one-to-one}\}.$$

For $n = 0$, set $\mathcal{P}_0 = \{L_1, \ldots, L_{k_2}\}$. A set $E \subset \mathcal{P}_n$, $n \geqslant 0$, we consider as a set of partition elements in $\mathcal{P}_n$ as well as an open set in $(-1, 1)$. If $E \subset \mathcal{P}_n$, $n \geqslant 0$, we denote by $\mathcal{P}_{n+m}|E$, $m \geqslant 0$, the restriction of $\mathcal{P}_{n+m}$ to the set E, i.e. $\mathcal{P}_{n+m}|E = \{\omega \in \mathcal{P}_{n+m} ;\ \omega \subset E\}$. Note that for $\omega \in \mathcal{P}_j$, $j \geqslant 1$, $g : \omega \to [-1, 1]$ is analytic.

Misiurewicz maps (i.e. maps with non-recurrent critical points) are well studied, going back to the fundamental paper by Misiurewicz [Mi]. We are going to use a result in [Str].

Lemma 3.1. *For the Misiurewicz–Thurston map $f_{a_1}(\theta) = a_1 - \theta^2$, $f_{a_1} : S \to S$, there exists a constant $K \geqslant 1$ such that, for any $n \geqslant 1$, the following holds. Let $S_n = (p_n, q_n) \subset S$ be a maximal interval for which $f_{a_1}^n | S_n$ is a diffeomorphism. There exist $1 \leqslant l, \hat{l} \leqslant 2$ such that*

$$\frac{1}{K}\frac{|f_{a_1}^n(S_n)|}{|S_n|^l}(\theta - p_n)^{l-1} \leqslant |f_{a_1}^{n}{}'(\theta)| \leqslant K\frac{|f_{a_1}^n(S_n)|}{|S_n|^l}(\theta - p_n)^{l-1},$$

for all $\theta \in S_n$ with $(\theta - p_n) \leqslant |S_n|/2$, and

$$\frac{1}{K}\frac{|f_{a_1}^n(S_n)|}{|S_n|^{\hat{l}}}(q_n - \theta)^{\hat{l}-1} \leqslant |f_{a_1}^{n}{}'(\theta)| \leqslant K\frac{|f_{a_1}^n(S_n)|}{|S_n|^{\hat{l}}}(q_n - \theta)^{\hat{l}-1},$$

for all $\theta \in S_n$ with $(q_n - \theta) \leqslant |S_n|/2$.

Proof. By the Fatou theorem the Misiurewicz–Thurston map f_{a_1} has no periodic attractors. Hence, lemma 3.1 follows immediately from proposition 10.1 in [Str]. □

This very precise information about the map f_{a_1} implies the following distortion estimate for the base dynamics g.

Lemma 3.2. *There are constants $\kappa > 0$ and $K_2 \geqslant 1$ such that the following holds. If $\omega \in \mathcal{P}_n$, $n \geqslant 1$ and $J \subset \omega$ is measurable, then*

$$\frac{1}{K_2}|g^n(J)|^{1/\kappa}|\omega| \leqslant |J| \leqslant K_2|g^n(J)|^{\kappa}|\omega|.$$

Proof. Recall that $g = \varphi \circ f_{a_1}^k \circ \varphi^{-1}$. By the construction of the partitions $\mathcal{P}_n$, for an element $\omega \in \mathcal{P}_n$, $n \geqslant 1$, the interval $\varphi^{-1}(\omega)$ must not be a maximal interval for which $f_{a_1}^{nk}|\varphi^{-1}(\omega)$ is a diffeomorphism. To be able to apply lemma 3.1, we start with a maximal interval $S_{nk} = (p_{nk}, q_{nk}) \subset S$ for which $f_{a_1}^{nk}|S_{nk}$ is a diffeomorphism, and get

$$\frac{|f^{nk}(S_{nk})|}{K}\min\left\{\frac{(\theta - p_{nk})}{|S_{nk}|^2}, \frac{(q_{nk} - \theta)}{|S_{nk}|^2}\right\} \leqslant |f_{a_1}^{nk}{}'(\theta)| \leqslant K\frac{|f_{a_1}^{nk}(S_{nk})|}{|S_{nk}|},$$

for all $\theta \in S_{nk}$. Thus, for a measurable set $J \subset S_{nk}$, it follows that

$$\begin{aligned} |f_{a_1}^{nk}(J)| &= \int_J |f_{a_1}^{nk}{}'(\theta)|\,\mathrm{d}\theta \geqslant \frac{|f_{a_1}^{nk}(S_{nk})|}{K}\int_0^{|J|/2}\frac{\theta}{|S_{nk}|^2}\,\mathrm{d}\theta \\ &\geqslant \frac{|f_{a_1}^{nk}(S_{nk})|}{8K}\left(\frac{|J|}{|S_{nk}|}\right)^2, \end{aligned} \tag{8}$$

and

$$|f_{a_1}^{nk}(J)| \leqslant K\frac{|f_{a_1}^{nk}(S_{nk})|}{|S_{nk}|}|J| \leqslant 2K\frac{|J|}{|S_{nk}|}. \tag{9}$$

Let $B = \min_i |B_i| > 0$, and note that $B \leqslant |f_{a_1}^{nk}(\varphi^{-1}(\omega))|$, for all $\omega \in \mathcal{P}_n$. Considering the set $\varphi^{-1}(\omega)$, where $\omega \in \mathcal{P}_n$ with $\varphi^{-1}(\omega) \subset S_{nk}$, we have $|\varphi^{-1}(\omega)| \leqslant |S_{nk}|$, and by taking $\varphi^{-1}(\omega)$ as the set J in (9) we get $|S_{nk}| \leqslant 2K|\varphi^{-1}(\omega)|/B$. This combined with (8) and (9) implies that for $\omega \in \mathcal{P}_n$ and a measurable set $J \subset \omega$,

$$\frac{B^3}{32K^3}\left(\frac{|\varphi^{-1}(J)|}{|\varphi^{-1}(\omega)|}\right)^2 \leqslant |f_{a_1}^{nk}(\varphi^{-1}(J))| \leqslant 2K\frac{|\varphi^{-1}(J)|}{|\varphi^{-1}(\omega)|}. \tag{10}$$

Set $J' = \varphi^{-1}(J)$ and $\omega' = \varphi^{-1}(\omega)$. We have that $\omega' \subset (b_j, b_{j+1})$ for some $0 \leqslant j < k_1$ and the density ρ is positive on (b_j, b_{j+1}). To estimate $|J'|/|\omega'|$ we use properties (2), (3) and (4) of the density $\rho = \varphi'/2$. For example, if $\omega' \subset (b_j, b_j + \varepsilon)$, and if, for $\theta \in (b_j, b_j + \varepsilon)$,

$$\frac{1}{K_0\sqrt{\theta - b_j}} \leqslant \rho(\theta) \leqslant \frac{K_0}{\sqrt{\theta - b_j}},$$

then, we have, set $\Delta = \mathrm{dist}(\omega', b_j)$,

$$\frac{|J|}{|\omega|} = \frac{|\varphi(J')|}{|\varphi(\omega')|} = \frac{\int_{J'} 2\rho(\theta)d\theta}{\int_{\omega'} 2\rho(\theta)d\theta} \leqslant K_0^2 \frac{\sqrt{|J'|+\Delta} - \sqrt{\Delta}}{\sqrt{|\omega'|+\Delta} - \sqrt{\Delta}} \leqslant K_0^2 \left(\frac{|J'|}{|\omega'|}\right)^{1/2},$$

where the last inequality follows by the observation that the preceding term is a decreasing function in Δ. Furthermore,

$$\begin{aligned}\frac{|J|}{|\omega|} &\geqslant \frac{1}{K_0^2} \frac{\sqrt{|\omega'|+\Delta} - \sqrt{|\omega'| - |J'| + \Delta}}{\sqrt{|\omega'|+\Delta} - \sqrt{\Delta}} \geqslant \frac{1}{K_0^2} \frac{\sqrt{|\omega'|} - \sqrt{|\omega'| - |J'|}}{\sqrt{|\omega'|}} \\ &\geqslant \frac{1}{2K_0^2} \frac{|J'|}{|\omega'|},\end{aligned}$$

where the second inequality follows by the observation that the preceding term is an increasing function in Δ. For the other cases the same or better estimates hold. Combined with (10), we obtain

$$\frac{1}{4KK_0^2} |f_{a_1}^{nk}(\varphi^{-1}(J))||\omega| \leqslant |J| \leqslant \left(\frac{32K^3K_0^8}{B^3}\right)^{1/4} |f_{a_1}^{nk}(\varphi^{-1}(J))|^{1/4}|\omega|.$$

By (1), we get

$$\begin{aligned}|g^n(J)| &= |\varphi(f_{a_1}^{nk}(\varphi^{-1}(J)))| = \int_{f_{a_1}^{nk}(\varphi^{-1}(J))} 2\rho(\theta)\,\mathrm{d}\theta \\ &\leqslant 8K_0 \int_0^{|f_{a_1}^{nk}(\varphi^{-1}(J))|/2} \frac{1}{\sqrt{\theta}}\,\mathrm{d}\theta \leqslant \frac{16K_0}{\sqrt{2}} |f_{a_1}^{nk}(\varphi^{-1}(J))|^{1/2},\end{aligned}$$

and $|g^n(J)| \geqslant \frac{2}{K_0}|f_{a_1}^{nk}(\varphi^{-1}(J))|$.

Altogether, we conclude

$$\frac{1}{K_2}|g^n(J)|^{1/\kappa}|\omega| \leqslant |J| \leqslant K_2|g^n(J)|^{\kappa}|\omega|,$$

where $K_2 = \max\{(16K^3K_0^9/B^3)^{1/4}, 2^9KK_0^4\}$ and $\kappa = 1/4$. □

4. Admissible curves

For an interval $\omega \in \mathcal{P}_1$ consider a C^1 curve $\widehat{X} = \mathrm{graph}(X)$, $X : \omega \to I$, which fulfils $|X'(\theta)| \leqslant \alpha/5K_1$, for $\theta \in \omega$. Let $Y : g(\omega) \to I$ be the curve determined by the image $\widehat{Y} = \mathrm{graph}(Y) = \tilde{F}(\widehat{X}|_\omega)$. Recall that, by (7), we have that $|g'(\theta)| \geqslant 5K_1^2 + 4$, $\theta \in \omega$. Hence, for $\theta \in \omega$, on the one hand

$$\begin{aligned}|Y'(g(\theta))| &= \frac{1}{|g'(\theta)|}|-2X(\theta)X'(\theta) + \alpha h'(\theta)| \\ &\leqslant \frac{1}{5K_1^2+4}\left(\frac{4\alpha}{5K_1} + \alpha K_1\right) = \frac{\alpha}{5K_1},\end{aligned} \tag{11}$$

and on the other hand

$$|Y'(g(\theta))| \geqslant \frac{1}{|g'(\theta)|}\left(\frac{\alpha}{K_1} - 4|X'(\theta)|\right) \geqslant \frac{\alpha}{5K_1|g'(\theta)|}. \tag{12}$$

Definition 4.1. *We say that a curve* $\widehat{Y} = \mathrm{graph}(Y)$*,* $Y : L \to I$*,* $L \in \{L_1, \ldots, L_{k_2}\}$*, is an* admissible curve *if there is a* C^1 *curve* $\widehat{X} = \mathrm{graph}(X)$*,* $X : \omega \to I$*, where* $\omega \in \mathcal{P}_1$ *and* $|X'(\theta)| \leqslant \alpha/5K_1$*,* $\theta \in \omega$*, such that* $\widehat{Y} = \tilde{F}(\widehat{X}|_\omega)$*.*

By (11), if $X : L \to I$ is an admissible curve and $\omega \subset L$ for an element ω in $\mathcal{P}_1$, then $\widehat{Y} = \tilde{F}(\widehat{X}|_\omega)$ is also an admissible curve. The following lemma shows that admissible curves are non-flat.

Lemma 4.2. *There is a constant* $\delta > 0$ *such that if* $Y : L \to I$*,* $L \in \{L_1, \ldots, L_{k_2}\}$*, is an admissible curve then, for* $\theta \in L$*,*

$$|Y'(\theta)| \geqslant \delta(\theta - l)(r - \theta)\alpha, \tag{13}$$

where l *and* r *denote the left and right boundary point of* L*.*

Proof. By the definition of an admissible curve, we can write $\widehat{Y} = \tilde{F}(\widehat{X}|_\omega)$ for some curve $X : \omega \to I$, $\omega \in \mathcal{P}_1$. By (12) we have that, for $\theta \in \omega$,

$$|Y'(g(\theta))| \geqslant \frac{\alpha}{5K_1|g'(\theta)|}.$$

Hence, we have to show that $g^{-1} : L \to \omega$ satisfies, for $\theta \in L$,

$$|g^{-1\prime}(\theta)| \geqslant 5K_1\delta(\theta - l)(r - \theta),$$

for some $\delta > 0$. By the definition of g and by (1), we obtain

$$\begin{aligned} |g^{-1\prime}(\theta)| &= \frac{1}{|g'(g^{-1}(\theta))|} = \frac{1}{|\varphi'(\varphi^{-1}(\theta))||f_{a_1}^{k}{}'(\varphi^{-1}(g^{-1}(\theta)))||\varphi^{-1}{}'(g^{-1}(\theta))|} \\ &\geqslant \frac{1}{2\rho(\varphi^{-1}(\theta))4^k(K_0/2)} = \frac{1}{\rho(\varphi^{-1}(\theta))4^k K_0}. \end{aligned}$$

Take $0 \leqslant j < k_1$ such that $b_j = \varphi^{-1}(l)$ and $b_{j+1} = \varphi^{-1}(r)$. Using (1), it follows, for $\theta \in \varphi^{-1}(L)$,

$$\varphi(\theta) \leqslant l + 4K_0\left(\sqrt{\theta - b_j} - \sqrt{b_{j+1} - \theta} + \sqrt{b_{j+1} - b_j}\right) \leqslant l + 8K_0\sqrt{\theta - b_j}.$$

Thus, for $\theta \in L$, we derive

$$\varphi^{-1}(\theta) \geqslant b_j + \frac{1}{64K_0^2}(\theta - l)^2,$$

and a similar calculation yields

$$\varphi^{-1}(\theta) \leqslant b_{j+1} - \frac{1}{64K_0^2}(r - \theta)^2.$$

Combined with (1), this gives us

$$\rho(\varphi^{-1}(\theta)) \leqslant 8K_0^2\left(\frac{1}{\theta - l} + \frac{1}{r - \theta}\right) \leqslant 8K_0^2\frac{|L|}{(\theta - l)(r - \theta)} \leqslant \frac{16K_0^2}{(\theta - l)(r - \theta)},$$

which concludes the proof by setting $\delta = 5K_1/16K_0^3 4^k$. □

Remark 4.3. The image by $\tilde{F}$ of a constant curve, i.e. $\tilde{F}(\widehat{X})$, where $\widehat{X} = \omega \times \{x_0\}$, for some $\omega \in \mathcal{P}_1$ and $x_0 \in I$, is an admissible curve. It follows that almost every point in $\tilde{F}(\widehat{J})$, and hence every interesting point, lies on an admissible curve. In order to prove the claim for theorem 1.2 on the vertical Lyapunov exponent, it is enough to show that for an arbitrary admissible curve $\widehat{X} = \mathrm{graph}(X)$, $X : L \to I$, $L \in \{L_1, \ldots, L_{k_2}\}$,

$$\liminf_{n\to\infty} \frac{1}{n} \log |\partial_x f_n(\theta, X(\theta))| > 0, \tag{14}$$

for Lebesgue almost every $\theta \in L$.

5. Critical returns

We turn now to the central fact about the returns of admissible curves to the critical line $[-1, 1]\times\{0\}$. First we recall a fact about the map f_{a_0} (which in our case satisfies the Misiurewicz condition and has no periodic attractors) from [MS, theorem III.6.3].

Lemma 5.1. *If the parameter a_0 is such that the critical point of $f_{a_0}(x) = a_0 - x^2$ is pre-periodic (but not periodic) then there exist constants $\delta > 0$, $\sigma > 1$ and $\mu > 0$ such that $|(f_{a_0}^i)'(x)| \geqslant \mu\sigma^i$ if $f_{a_0}^i(x) \in (-\delta, \delta)$.*

Define the constants

$$M(\alpha) = \left[\frac{|\log\alpha|}{\log 32}\right], \qquad \eta = \frac{\log\sigma}{8\log 32}. \tag{15}$$

For $r \geqslant 0$, let $J(r) = \{x \in \mathbb{R}\ ;\ |x| \leqslant \sqrt{\alpha}\mathrm{e}^{-r}\}$ and denote the critical strip $[-1, 1] \times J(r)$ by $\widehat{J}(r)$.

The main technical tool in the proof of theorem 1.2 is the following proposition. It corresponds to lemma 2.6 in [Vi].

Proposition 5.2. *Let $r_0(\alpha) = (1/2 - 2\eta)\log(1/\alpha)$. There exist $C < \infty$ and $\beta > 0$ such that, for all sufficiently small $\alpha > 0$, for any admissible curve $\widehat{Y} = \mathrm{graph}(Y)$, $Y : L \to I$, $L \in \{L_1, \ldots, L_{k_2}\}$,*

$$|\{\omega \in \mathcal{P}_{M(\alpha)}|L\ ;\ \tilde{F}^{M(\alpha)}(\widehat{Y}|_\omega) \cap \widehat{J}(r_0(\alpha) - 2) \neq \emptyset\}| \leqslant C\mathrm{e}^{-\beta r_0(\alpha)}, \tag{16}$$

and furthermore, if $r \geqslant (1/2 + 2\eta)\log(1/\alpha)$,

$$|\{\theta \in L\ ;\ \widehat{Y}(\theta) \in \widehat{J}(r - 2)\}| \leqslant C\mathrm{e}^{-\beta r}. \tag{17}$$

The proof of proposition 5.2 is given in the next section. The following lemma will be an important ingredient in the proof of the estimate (16). Let $\widehat{X} = \mathrm{graph}(X)$, $X : L \to I$, $L \in \{L_1, \ldots, L_{k_2}\}$, be an admissible curve. For $\omega \in \mathcal{P}_t|L$, $t \geqslant 1$, we set $\widehat{X}_j(\omega) = \mathrm{graph}(X_j(\omega, \cdot)) = \tilde{F}^j(\widehat{X}|_\omega)$, $1 \leqslant j \leqslant t$.

Lemma 5.3. *Let α be sufficiently small. There exists a constant $\varepsilon_0 > 0$ such that the following holds. Let $\widehat{X} = \mathrm{graph}(X)$, $X : L \to I$, $L \in \{L_1, \ldots, L_{k_2}\}$, be an admissible curve. There is an integer $1 \leqslant j = j(L) \leqslant k_2$ such that there are at least two elements $\omega_1, \omega_2 \in \mathcal{P}_{j+k_2}|L$ with the property that $g^j(\omega_1) = g^j(\omega_2)$ and, for all $\theta \in g^j(\omega_1)$,*

$$|X_j(\omega_1, \theta) - X_j(\omega_2, \theta)| \geqslant \varepsilon_0\alpha.$$

Proof. Recall that the critical point 0 lies inside one of the intervals B_i, $1 \leqslant i \leqslant k_2$. By the ergodicity of μ, the map f_{a_1} acting on the B_i can be considered as an irreducible Markov system. Hence we deduce that $f_{a_1}^{k_2}$ has at least one critical point (which is a turning point) inside every B_i. It follows that g^{k_2} has at least one turning point inside every L_i, $1 \leqslant i \leqslant k_2$ (instead of g^{k_2} we could also take $g^{[k_2/k]+1}$).

Let $1 \leqslant j = j(L) \leqslant k_2$ be minimal such that $g^j : L \to [-1, 1]$ has at least one turning point. Let p be such a turning point and let $\tilde{\omega}_1, \tilde{\omega}_2 \in \mathcal{P}_j|L$ be the two (disjoint) elements adjacent to p. Since j is minimal $\tilde{\omega}_1$ and $\tilde{\omega}_2$ are contained in the same element of the partition $\mathcal{P}_{j-1}|L$. Furthermore, we have that $g^j(\tilde{\omega}_1) = g^j(\tilde{\omega}_2) = L'$ for some $L' \in \{L_1, \ldots, L_{k_2}\}$, $\mathrm{sign}(X_j'(\tilde{\omega}_1, \theta)) = -\mathrm{sign}(X_j'(\tilde{\omega}_2, \theta))$ for $\theta \in L'$, and

$$\lim_{\substack{\theta \to g(p)\\ \theta \in L'}} X_j(\tilde{\omega}_1, \theta) = \lim_{\substack{\theta \to g(p)\\ \theta \in L'}} X_j(\tilde{\omega}_2, \theta),$$

i.e. the graphs $\widehat{X}_j(\tilde{\omega}_1)$ and $\widehat{X}_j(\tilde{\omega}_2)$ have a common endpoint. Let l' and r' be the left and right endpoint of L'. $g(p)$ is equal to l' or r'; without loss of generality assume that $g(p) = l'$. From lemma 4.2 it follows that

$$|X_j(\tilde{\omega}_1, \theta) - X_j(\tilde{\omega}_2, \theta)| \geqslant 2 \int_{l'}^{\theta} \delta(u - l')(r' - u)\alpha \, \mathrm{d}u$$
$$\geqslant \frac{|L'|\delta}{3}(\theta - l')^2\alpha = \frac{|L'|\delta}{3}(\theta - g(p))^2\alpha.$$

Since $\#\{\omega \in \mathcal{P}_{k_2}|L'\} > 1$, we can fix an element $\omega \in \mathcal{P}_{k_2}|L'$ not adjacent to $g(p)$. Let $\omega_1 = g^{-j}\{\omega\} \cap \tilde{\omega}_1$ and $\omega_2 = g^{-j}\{\omega\} \cap \tilde{\omega}_2$. Note that $\omega_1, \omega_2 \in \mathcal{P}_{j+k_2}|L$. We can now choose an $\varepsilon_0 > 0$ (independent of the interval L and the turning point p we started with) such that

$$\frac{\min_{L' \in \{L_1,\ldots,L_{k_2}\}}|L'|\delta}{3}(\theta - g(p))^2 > \varepsilon_0 \qquad \text{for } \theta \in \omega.$$

Thus, $|X_j(\omega_1, \theta) - X_j(\omega_2, \theta)| \geqslant \varepsilon_0\alpha$, for all $\theta \in g^j(\omega_1)$, which concludes the proof. □

6. Proof of proposition 5.2

We prove first claim (17). This is the easier part. It deals with the deep returns near the critical line $[-1, 1] \times \{0\}$. We obtain the estimate (17) almost directly from the non-flatness property of the admissible curves provided by lemma 4.2. Let $\varepsilon > 0$. We want to estimate the measure of the set $\{\theta \in L\ ;\ X(\theta) \in (-\varepsilon, \varepsilon)\}$. Let l and r denote the left and right boundary point of L. By lemma 4.2, we have

$$|Y'(\theta)| \geqslant \delta(\theta - l)(r - \theta)\alpha.$$

Without loss of generality we assume that $\mathrm{sign}(Y'(\theta)) = +1$. We have,

$$Y(l + \theta) \geqslant Y(l) + \int_0^{\theta} \delta u(|L| - u)\alpha \, \mathrm{d}u \geqslant Y(l) + \frac{\delta|L|}{6}\theta^2\alpha \geqslant Y(l) + 2\varepsilon,$$

as soon as $\theta \geqslant \sqrt{12\varepsilon/\delta|L|\alpha}$. Thus,

$$|\{\theta \in L\ ;\ Y(\theta) \in (-\varepsilon, \varepsilon)\}| \leqslant \left(\frac{12}{\delta|L|}\right)^{1/2} \left(\frac{\varepsilon}{\alpha}\right)^{1/2}.$$

Now, we can take $\varepsilon = \sqrt{\alpha}\mathrm{e}^{-r+2}$. Note that, for $r \geqslant (1/2 + 2\eta)\log(1/\alpha)$,

$$1/\sqrt{\alpha} \leqslant \exp((1 + 4\eta)^{-1}r) < \exp((1 - \eta)r).$$

Thus, we obtain

$$|\{\theta \in L\ ;\ \widehat{Y}(\theta) \in \widehat{J}(r - 2)\}| \leqslant \left(\frac{12\mathrm{e}^2}{\delta \min_{L \in \{L_1,\ldots,L_{k_2}\}}|L|}\right)^{1/2} \mathrm{e}^{-(\eta/2)r},$$

which implies (17).

The remaining part is dedicated to the proof of claim (16). We assume that there exists a point $z_0 = (\theta_0, x_0)$ on the admissible curve $\widehat{Y}$ such that $\tilde{F}^{M(\alpha)}(z_0) \in \widehat{J}(0)$ since, otherwise, claim (16) is trivial. For $i \geqslant 0$, let $x_i = f_{a_0}^i(x_0)$, and we define the horizontal strips

$$S_i = [-1, 1] \times [x_i - 5^i\alpha, x_i + 5^i\alpha].$$

We resume some facts from [BST].

Lemma 6.1. *For $0 \leqslant i \leqslant M(\alpha)$, we have $\tilde{F}^i(\widehat{Y}) \subset S_i$ and the strips S_i, $0 \leqslant i \leqslant M(\alpha)-1$, do not meet the critical strip $\widehat{J}(0)$. Furthermore, for $0 \leqslant i \leqslant M(\alpha)-1$, the map $f_{M(\alpha)-i}$ is in the x-direction almost linear, i.e.*

$$\sum_{j=i}^{M(\alpha)-1} \sup_{(\theta,x),(\theta',x')\in s_j} \log\left|\frac{\partial_x f(\theta,x)}{\partial_x f(\theta',x')}\right| \leqslant \log 2, \tag{18}$$

and $f_{M(\alpha)-i}$ is expanding in the x-direction, i.e.

$$|\partial_x f_{M(\alpha)-i}(\theta,x)| \geqslant \frac{\mu\sigma^{M(\alpha)-i}}{2} \tag{19}$$

for any $(\theta,x) \in S_i$.

Proof. Lemma 6.1 is a summary of what is written in [BST] between page 1409 line 24 and page 1410 line 11 (the constant L in [BST] in our case is equal to 1 and S^1 in our case is equal to $[-1,1]$). lemma 5.1 in [BST] is identical to our lemma 5.1. □

We introduce some more constants (see lemma 5.3 for the definition of ε_0).

- Let $\bar{\sigma} = \sqrt{\sigma} > 1$.
- Fix a constant $\tau > 4^{k_2}$, independent of α, so that

$$\tau\varepsilon_0/4 - 2/5K_1 - 4/(\bar{\sigma}-1) > 1.$$

Put $\lambda_j = |\partial_x f_{M(\alpha)-j}(\tilde{F}^j(z_0))|/\bar{\sigma}^{(M(\alpha)-j)}$ for $0 \leqslant j \leqslant M(\alpha)-1$. Note that we have, from (19),

$$\lambda_j > \frac{\mu\sigma^{M(\alpha)-j}}{2\bar{\sigma}^{(M(\alpha)-j)}} = \frac{\mu\bar{\sigma}^{M(\alpha)-j}}{2}. \tag{20}$$

Let $0 = t_1 < t_2 < \cdots < t_q \leqslant M(\alpha)-1$, be the (finite) sequence of integers defined inductively by

$$t_{i+1} = \min\{t\ ;\ t_i + k_2 < t \leqslant M(\alpha)-1,\ \lambda_{t_i+j} \geqslant \tau\lambda_t,\ 1 \leqslant j \leqslant k_2,$$
$$\text{and } \lambda_t \geqslant \lambda_l \text{ for all } l \geqslant t\}.$$

Remark 6.2. The j in the definition of t_{i+1} corresponds to the j in lemma 5.3, and a t_i in our setting corresponds to a t_i+1 in the definition given in [Vi] (in [Vi] j is constantly equal to 1). Otherwise, disregarding the requirement $\lambda_{t_{i+1}} \geqslant \lambda_l$, for $l \geqslant t_{i+1}$, which is a technical and not an essential requirement, our definition of t_{i+1} and the definition given in [Vi] are the same.

We have $t_{i+1} > t_i + 2k_2$, for all $1 \leqslant i < q$, because, by the definition of the λ_j and τ,

$$4^{k_2} < \tau \leqslant \frac{\lambda_{t_i+k_2}}{\lambda_{t_{i+1}}} = \frac{|\partial_x f_{t_{i+1}-t_i-k_2}(\tilde{F}^{t_i+k_2}(z_0))|}{\bar{\sigma}^{t_{i+1}-t_i-k_2}} < 4^{t_{i+1}-t_i-k_2}.$$

Let $k_0(\alpha) = \max\{1 \leqslant i \leqslant q\ ;\ \lambda_{t_i} \geqslant 2\mathrm{e}^{-r_0(\alpha)+2}/\sqrt{\alpha}\}$.

Sublemma. *We have $k_0(\alpha) \geqslant \gamma r_0(\alpha)$ for $\gamma = \eta/\log(4^{k_2+1}\tau)$ if α is sufficiently small.*

Proof. Observe that $\lambda_j \leqslant 4\lambda_{j+1}$. Let α be so small such that $k_0(\alpha) < q$ (note that $2\mathrm{e}^{-r_0(\alpha)+2}/\sqrt{\alpha} \sim \alpha^{-2\eta}$ and $\lambda_{t_q+j} < \tau\lambda_{M(\alpha)-1} \leqslant 4\tau$, for some $1 \leqslant j \leqslant k_2$, hence $\lambda_{t_q} \leqslant 4^{k_2+1}\tau$). On the one hand, we have for $1 \leqslant i < q$, $\lambda_{t_i+j_i} \leqslant 4\tau\lambda_{t_{i+1}}$, for some $1 \leqslant j_i \leqslant k_2$; hence, $\lambda_0 = \lambda_{t_1} \leqslant (4^{k_2+1}\tau)^{k_0(\alpha)}\lambda_{t_{k_0(\alpha)+1}}$. From (20), it follows that

$$\lambda_{t_{k_0(\alpha)+1}} \geqslant (4^{k_2+1}\tau)^{-k_0(\alpha)}\lambda_0 \geqslant (4^{k_2+1}\tau)^{-k_0(\alpha)}(\mu/2)\bar{\sigma}^{M(\alpha)}.$$

On the other hand, by the definition of $k_0(\alpha)$, $\lambda_{t_{k_0(\alpha)+1}} \leqslant 2\mathrm{e}^{-r_0(\alpha)+2}/\sqrt{\alpha}$. Combining these two bounds on $\lambda_{t_{k_0(\alpha)+1}}$, we derive

$$-r_0(\alpha) - (\log \alpha)/2 \geqslant M(\alpha) \log \bar{\sigma} - k_0(\alpha) \log(4^{k_2+1}\tau) - C,$$

where $C = \log(4/\mu) + 2$ is a constant independent of α. Recall the definitions (15) and that $r_0(\alpha) = (1/2 - 2\eta) \log(1/\alpha)$. We obtain

$$k_0(\alpha) \log(4^{k_2+1}\tau) \geqslant r_0(\alpha) - \left(\frac{1}{2} - 4\eta\right) \log \frac{1}{\alpha} - C = r_0(\alpha)\left(1 - \frac{\frac{1}{2} - 4\eta}{\frac{1}{2} - 2\eta}\right) - C$$

$$= r_0(\alpha)\left(\frac{2\eta}{\frac{1}{2} - 2\eta}\right) - C \geqslant \eta r_0(\alpha),$$

for sufficiently small α. This proves the claim. □

Let $1 \leqslant t \leqslant M(\alpha)$. Two intervals $\tilde{\omega}_1$ and $\tilde{\omega}_2$ in $\mathcal{P}_t$ are said to be *incompatible* if

$$\inf\{|x_1 - x_2| \,;\ (\theta_1, x_1) \in \tilde{F}^{M(\alpha)}(\widehat{Y}|_{\tilde{\omega}_1}),\ (\theta_2, x_2) \in \tilde{F}^{M(\alpha)}(\widehat{Y}|_{\tilde{\omega}_2})\} > 2\mathrm{e}^{-r_0(\alpha)+2}\sqrt{\alpha}.$$

If $\tilde{\omega}_1, \tilde{\omega}_2 \in \mathcal{P}_t$ are incompatible then obviously either all admissible curves contained in $\tilde{F}^{M(\alpha)}(\widehat{Y}|_{\tilde{\omega}_1})$ or all admissible curves contained in $\tilde{F}^{M(\alpha)}(\widehat{Y}|_{\tilde{\omega}_2})$ do not intersect the critical strip $\widehat{J}(r_0(\alpha) - 2)$. We are going to establish a sufficient condition for incompatibility. For $\omega \in \mathcal{P}_t|L$, $t \geqslant 0$, we set $\widehat{Y}_j(\omega) = \mathrm{graph}(Y_j(\omega, \cdot)) = \tilde{F}^j(\widehat{Y}|_\omega)$, $0 \leqslant j \leqslant t$. For $\omega_0 \in \mathcal{P}_{t_i}|L$, $\widehat{Y}_{t_i}(\omega_0)$ is an admissible curve. So we are in a position to apply lemma 5.3 to this curve and we deduce that there exists an integer $1 \leqslant j = j(\omega_0) \leqslant k_2$ and two elements $\omega_1, \omega_2 \in \mathcal{P}_{t_i+j+k_2}|\omega_0$ such that $g^{t_i+j}(\omega_1) = g^{t_i+j}(\omega_2)$ and, for all $\theta \in g^{t_i+j}(\omega_1)$,

$$|Y_{t_i+j}(\omega_1, \theta) - Y_{t_i+j}(\omega_2, \theta)| \geqslant \varepsilon_0\alpha.$$

Denote by $\mathrm{inc}(\omega_0)$ all such triples (j, ω_1, ω_2) which satisfy the properties above.

Sublemma (A sufficient condition for incompatibility). Let $1 \leqslant i < k_0(\alpha)$, $\omega_0 \in \mathcal{P}_{t_i}$ and $(j, \omega_1, \omega_2) \in \mathrm{inc}(\omega_0)$. If $\tilde{\omega}_1 \in \mathcal{P}_{t_{i+1}}|\omega_1$ and $\tilde{\omega}_2 \in \mathcal{P}_{t_{i+1}}|\omega_2$ such that $g^{t_i+j}(\tilde{\omega}_1) = g^{t_i+j}(\tilde{\omega}_2)$, then $\tilde{\omega}_1$ and $\tilde{\omega}_2$ are incompatible.

Proof. Since $(j, \omega_1, \omega_2) \in \mathrm{inc}(\omega_0)$ we have

$$|Y_{t_i+j}(\omega_1, \theta) - Y_{t_i+j}(\omega_2, \theta)| \geqslant \varepsilon_0\alpha.$$

for all $\theta \in g^{t_i+j}(\omega_1) = g^{t_i+j}(\omega_2)$. Since $g^{t_i+j}(\tilde{\omega}_1) = g^{t_i+j}(\tilde{\omega}_2)$, we deduce from the distortion estimate (18) and from the definition of the λ_{t_i} that

$$|Y_{t_{i+1}}(\tilde{\omega}_1, \theta) - Y_{t_{i+1}}(\tilde{\omega}_2, \theta)| \geqslant \frac{1}{2}\frac{\lambda_{t_i+j}}{\lambda_{t_{i+1}}}\varepsilon_0\alpha \geqslant \frac{1}{2}\tau\varepsilon_0\alpha$$

for all $\theta \in g^{t_{i+1}}(\tilde{\omega}_1)$. For $t_{i+1} \leqslant l \leqslant M(\alpha)$, let

$$\Delta_l = \inf\{|x_1 - x_2| \,;\ (\theta_1, x_1) \in \tilde{F}^l(\widehat{Y}|_{\tilde{\omega}_1}),\ (\theta_2, x_2) \in \tilde{F}^l(\widehat{Y}|_{\tilde{\omega}_2})\}.$$

In other words, Δ_l is the vertical distance between the two images $\tilde{F}^l(\widehat{Y}|_{\tilde{\omega}_1})$ and $\tilde{F}^l(\widehat{Y}|_{\tilde{\omega}_2})$. We have

$$\Delta_{t_{i+1}} \geqslant \frac{1}{2}\tau\varepsilon_0\alpha - 4\frac{\alpha}{5K_1}, \tag{21}$$

where the last term is a bound for the oscillations of the two admissible curves $\widehat{Y}_{t_{i+1}}(\tilde{\omega}_1)$ and $\widehat{Y}_{t_{i+1}}(\tilde{\omega}_2)$ (where the length of their domain is at most 2). If we put

$$D_l = \min_{(\theta,x)\in S_l} |\partial_x f(\theta, x)| = \min_{(\theta,x)\in S_l} |f'_{a_0}(x)|,$$

for $t_{i+1} \leqslant l \leqslant M(\alpha) - 1$, then the distances Δ_l satisfy

$$\Delta_{l+1} \geqslant D_l \Delta_l - 2\alpha,$$

where the last term 2α is a bound for the oscillation of $\alpha s(\theta)$. Hence,

$$\Delta_{M(\alpha)} \geqslant \left(\prod_{l=t_{i+1}}^{M(\alpha)-1} D_l \right) \Delta_{t_{i+1}} - \sum_{l=t_{i+1}+1}^{M(\alpha)} \left(\prod_{m=l}^{M(\alpha)-1} D_m \right) 2\alpha.$$

From the definition of λ_l and the distortion estimate (18), we have

$$\frac{1}{2} \lambda_l \bar{\sigma}^{M(\alpha)-l} \leqslant \prod_{m=l}^{M(\alpha)-1} D_m \leqslant 2\lambda_l \bar{\sigma}^{M(\alpha)-l}.$$

Since, by the definition of t_{i+1}, $\lambda_l \leqslant \lambda_{t_{i+1}}$ for $l \geqslant t_{i+1}$, we obtain

$$\begin{aligned} \Delta_{M(\alpha)} &\geqslant \frac{1}{2} \lambda_{t_{i+1}} \bar{\sigma}^{M(\alpha)-t_{i+1}} \Delta_{t_{i+1}} - \sum_{l=t_{i+1}+1}^{M(\alpha)} 2\lambda_l \bar{\sigma}^{M(\alpha)-l} 2\alpha \\ &\geqslant \lambda_{t_{i+1}} \bar{\sigma}^{M(\alpha)-t_{i+1}} \left(\Delta_{t_{i+1}}/2 - 4\alpha \sum_{l=1}^{M(\alpha)-t_{i+1}} \bar{\sigma}^{-l} \right) \\ &\geqslant \lambda_{t_{i+1}} \bar{\sigma}^{M(\alpha)-t_{i+1}} (\Delta_{t_{i+1}}/2 - 4\alpha/(\bar{\sigma} - 1)). \end{aligned}$$

Recall that $\lambda_{t_{i+1}} \geqslant 2\mathrm{e}^{-r_0(\alpha)+2}/\sqrt{\alpha}$. Using (21) and the definition of τ, we conclude that $\Delta_{M(\alpha)} \geqslant (2\mathrm{e}^{-r_0(\alpha)+2}/\sqrt{\alpha})(\tau\varepsilon_0\alpha/4 - 2\alpha/5K_1 - 4\alpha/(\bar{\sigma} - 1)) > 2\mathrm{e}^{-r_0(\alpha)+2}\sqrt{\alpha}$.

It follows that $\tilde{\omega}_1$ and $\tilde{\omega}_2$ are incompatible. □

Let $E_1 = L$ and observe that $\mathcal{P}_{t_1}|L = \mathcal{P}_0|L = L$. Assume we have constructed the set $E_i \subset \mathcal{P}_{t_i}|L$, $1 \leqslant i < k_0(\alpha)$. Consider an element $\omega_0 \in \mathcal{P}_{t_i}|E_i$, and a triple $(j, \omega_1, \omega_2) \in \mathrm{inc}(\omega_0)$. Applying the criteria for incompatibility we can build a union $\tilde{u} \subset (\omega_1 \cup \omega_2)$ of elements in $\mathcal{P}_{t_{i+1}}|\omega_0$ such that $\tilde{u}$ contains exactly one interval of each interval pair $\tilde{\omega}_1 \in \mathcal{P}_{t_{i+1}}|\omega_1$ and $\tilde{\omega}_2 \in \mathcal{P}_{t_{i+1}}|\omega_2$ with $g^{t_i+j}(\tilde{\omega}_1) = g^{t_i+j}(\tilde{\omega}_2)$, and such that $\tilde{F}^{M(\alpha)}(\widehat{Y}|_\omega)$ does not intersect $\widehat{J}(r_0(\alpha) - 2)$ for all $\omega \in \mathcal{P}_{M(\alpha)}|\tilde{u}$. Observe that $|g^{t_i+j}(\tilde{u})| = |g^{t_i+j}(\omega_1)|$. Since $\omega_1 \in \mathcal{P}_{t_i+j+k_2}$, where $j \leqslant k_2$ and k_2 is fixed (it only depends on g) there is a positive number $q_0 > 0$ which is only dependent on the base dynamics g, such that

$$|g^{t_i}(\tilde{u})| \geqslant q_0.$$

By lemma 3.2, we obtain the following lower bound for the measure of the set $\tilde{u} \subset \omega_0 \in \mathcal{P}_{t_i}|E_i$:

$$|\tilde{u}| \geqslant \frac{1}{K_2} |g^{t_i}(\tilde{u})|^{1/\kappa} |\omega_0| \geqslant \frac{1}{K_2} q_0^{1/\kappa} |\omega_0|.$$

Building similar unions as above for each element $\omega_0 \in \mathcal{P}_{t_i}|E_i$ we obtain a set $\tilde{U} \subset E_i$, which is a union of elements in $\mathcal{P}_{t_{i+1}}|E_i$, such that, for all $\omega \in \mathcal{P}_{M(\alpha)}|\tilde{U}$, $\tilde{F}^{M(\alpha)}(\widehat{Y}|_\omega)$ does not intersect $\widehat{J}(r_0(\alpha) - 2)$ and

$$|\tilde{U}| \geqslant \sum_{\omega_0 \in \mathcal{P}_{t_i}|E_i} |\tilde{u}| \geqslant \sum_{\omega_0 \in \mathcal{P}_{t_i}|E_i} \frac{1}{K_2} q_0^{1/\kappa} |\omega_0| = \frac{1}{K_2} q_0^{1/\kappa} |E_i|.$$

We exclude this 'good' set $\tilde{U}$ of partition elements from the partition $\mathcal{P}_{t_{i+1}}|E_i$ and denote the remaining partition elements by $E_{i+1} \subset \mathcal{P}_{t_{i+1}}|E_i$. Obviously,

$$|E_{i+1}| \leqslant \left(1 - \frac{q_0^{1/\kappa}}{K_2} \right) |E_i|,$$

and, thus

$$|E_{k_0(\alpha)}| \leqslant \left(1 - \frac{q_0^{1/\kappa}}{K_2}\right)^{k_0(\alpha)-1} |E_1| = \left(1 - \frac{q_0^{1/\kappa}}{K_2}\right)^{k_0(\alpha)-1} |L|.$$

The set $E_{k_0(\alpha)} \subset \mathcal{P}_{t_{k_0(\alpha)}}|L$ is constructed in such a way that for every element $\omega \in \mathcal{P}_{M(\alpha)}|L$ which does not lie in $\mathcal{P}_{M(\alpha)}|E_{k_0(\alpha)}$, $\tilde{F}^{M(\alpha)}(\widehat{Y}|_\omega)$ does not intersect $\widehat{J}(r_0(\alpha) - 2)$. Hence,

$$|\{\omega \in \mathcal{P}_{M(\alpha)}|L\ ;\ \tilde{F}^{M(\alpha)}(\widehat{Y}|_\omega) \cap \widehat{J}(r_0(\alpha) - 2) \neq \emptyset\}| \leqslant \left(1 - \frac{q_0^{1/\kappa}}{K_2}\right)^{k_0(\alpha)-1} |L|.$$

As $k_0(\alpha) \geqslant \gamma r_0(\alpha)$, by the first sublemma in this section this concludes the proof of claim (16) in proposition 5.2.

7. Large deviations

To show (14) and thus to conclude the proof that the vertical Lyapunov exponent is positive we can follow the large deviation argument by Viana (see [Vi, section 2.4]). proposition 5.2 which we proved here is the counterpart to corollary 2.3 and lemma 2.6 in [Vi] which were the only parts where the properties of the base dynamics g and the coupling function s were relevant. The 'Building expansion' lemmas, lemmas 2.4 and 2.5 in [Vi], remain valid including their proofs. In the following, instead of retyping certain arguments which would be exactly the same in our setting, we will refer to [Vi].

In all that follows let $n \geqslant 1$ be fixed and sufficiently large. Define $m \geqslant 1$ by $m^2 \leqslant n < (m+1)^2$ and take $l = m - M(\alpha)$. Note that $l \approx m \approx \sqrt{n}$ as long as $n \gg \log(1/\alpha)$. We are considering an arbitrary admissible curve $\widehat{X} = \mathrm{graph}(X)$, $X : L \to I$, where $L \in \{L_1, \ldots, L_{k_2}\}$. Given $0 \leqslant \nu \leqslant n$ and $\omega_{\nu+l} \in \mathcal{P}_{\nu+l}|L$, we set $\gamma = \tilde{F}^\nu(\widehat{X}|_{\omega_{\nu+l}})$. Then we say that ν is

- a I_n-*situation* for $\theta \in \omega_{\nu+l}$ if $\gamma \cap \widehat{J}(0) \neq \emptyset$ but $\gamma \cap \widehat{J}(m) = \emptyset$;
- a $\mathrm{I\!I}_n$-*situation* for $\theta \in \omega_{\nu+l}$ if $\gamma \cap \widehat{J}(m) \neq \emptyset$.

Since the $\omega_{\nu+l}$ are open intervals in L there are some $\theta \in L$ for which I_n-situations and $\mathrm{I\!I}_n$-situations are not defined. But they build a zero measure set; thus, they can be neglected. Since γ is a piece of the graph of an admissible curve, its diameter in the x-direction is bounded by

$$\frac{\alpha}{5K_1}|g^\nu(\omega_{\nu+l})| \leqslant \frac{\alpha}{5K_1} 2(5K_1^2+4)^{-l} \ll \sqrt{\alpha}\mathrm{e}^{-m} \leqslant \sqrt{\alpha}(\mathrm{e}^{-(m-1)} - \mathrm{e}^{-m}). \quad (22)$$

This means that, whenever ν is a $\mathrm{I\!I}_n$-situation for $\omega_{\nu+l}$, γ is contained in $\widehat{J}(m-1)$. Hence, the set $A_2(n) = \{\theta \in [-1, 1]\ ;\ \text{some } 0 \leqslant \nu \leqslant n \text{ is a } \mathrm{I\!I}_n\text{-situation for } \theta\}$, is contained in the set

$$\bigcup_{\nu=0}^{n} \bigcup_{\omega \in \mathcal{P}_\nu|L} \{\theta \in \omega\ ;\ \tilde{F}^\nu(\widehat{X}(\theta)) \in \widehat{J}(m-1)\}.$$

Applying the estimate (17) in proposition 5.2 to the admissible curves $\widehat{X}_\nu(\omega) = \mathrm{graph}(X_\nu(\omega, \cdot)) = \tilde{F}^\nu(\widehat{X}|_\omega)$, $\omega \in \mathcal{P}_\nu|L$, we obtain

$$|\{\theta \in g^\nu(\omega)\ ;\ \widehat{X}_\nu(\omega, \theta) \in \widehat{J}(m-1)\}| \leqslant C\mathrm{e}^{-\beta m}.$$

Thus, by the distortion estimate in lemma 3.2,

$$|A_2(n)| \leqslant \sum_{\nu=0}^{n} \sum_{\omega\in\mathcal{P}_\nu|L} C^{1/\kappa} e^{-(\beta/\kappa)m}|\omega| = (n+1)C^{1/\kappa} e^{-(\beta/\kappa)m}|L| \leqslant \text{const}\, e^{-(\beta/2\kappa)\sqrt{n}}. \tag{23}$$

Henceforth, we focus on the I_n-situations for the $\theta \in L$. Let $0 \leqslant \nu_1 < \ldots < \nu_t \leqslant n$ be the I_n-situations for a $\theta \in L$, and let γ_i, $1 \leqslant i \leqslant t$, be the corresponding graph pieces. For each ν_i we fix $r_i \geqslant 1$ minimal such that $\gamma_i \cap \widehat{J}(r_i) = \emptyset$. By the definition of I_n-situations, $r_i \leqslant m$, $1 \leqslant i \leqslant t$. Recall that $r_0(\alpha) = (1/2 - 2\eta)\log(1/\alpha)$ and set $G = \{i\ ;\ r_i \geqslant r_0(\alpha)\}$ (note that this set depends on θ). Viana shows in section 2.4 in [Vi] that there exists a constant $c > 0$ such that

$$\log|\partial_x f_n(\widehat{X}(\theta))| \geqslant cn, \qquad \text{for every } \theta \in L \setminus E_n, \tag{24}$$

where $E_n = A_1(n) \cup A_2(n)$ and

$$A_1(n) = \Big\{\theta \in L\ ;\ \sum_{i\in G} r_i \geqslant cn\Big\}.$$

If we show that

$$|A_1(n)| \leqslant \text{const}\, e^{-\delta\sqrt{n}} \tag{25}$$

for some $\delta > 0$, then together with (23), this implies that the set $E = \cap_{k\geqslant 1} \cup_{n\geqslant k} E_n$ has zero Lebesgue measure and hence, we have shown (14). For the proof of (25) let $0 \leqslant q \leqslant m-1$ be fixed and denote $G_q = \{i \in G\ ;\ \nu_i \equiv q \bmod m\}$. We also take $m_q = \max\{j\ ;\ mj+q \leqslant n\}$ (note that $m_q \approx m \approx \sqrt{n}$) and, for each $0 \leqslant j \leqslant m_q$, we let $\hat{r}_j = r_i$ if $mj + q = \nu_i$, for some $i \in G_q$, and $\hat{r}_j = 0$ otherwise. Observe that G_q and $\hat{r}_j$ are, in fact, functions of θ. With the help of proposition 5.2 we will prove the following lemma.

Lemma 7.1. *Let*

$$\Omega_q(\rho_0, \ldots, \rho_{m_q}) = \{\theta \in L\ ;\ \hat{r}_j = \rho_j \text{ for } 0 \leqslant j \leqslant m_q\},$$

where for each j either $\rho_j = 0$ or $r_0(\alpha) \leqslant \rho_j \leqslant m$. We have

$$|\Omega_q(\rho_0, \ldots, \rho_{m_q})| \leqslant 2(K_2C^{1/\kappa})^\tau e^{-(\beta/2\kappa)\sum_{j=0}^{m_q}\rho_j},$$

where $\tau = \#\{j\ ;\ \rho_j \neq 0\}$.

Having shown lemma 7.1, renaming $\beta/2\kappa$ as 5β and setting $C_4 = 2K_2C^{1/\kappa}$, we can follow verbatim the remaining large deviation argument in [Vi, section 2.4] which proves (25) and thus the existence of a positive vertical Lyapunov exponent at Lebesgue almost every point. So, it is only left to prove the lemma above.

Proof. Fix a sequence $(\rho_0, \ldots, \rho_{m_q})$ as in lemma 7.1. We assume that $\tau \geqslant 1$, i.e. the ρ_j are not simultaneously zero, otherwise the statement is trivial. Let $0 \leqslant j_1 < \ldots < j_\tau \leqslant m_q$ such that $\rho_{j_i} \neq 0$, for $1 \leqslant i \leqslant \tau$. Set $\Omega_q(i) = \{\theta \in L\ ;\ \hat{r}_{j_k} = \rho_{j_k} \text{ for } 1 \leqslant k \leqslant i\}$. Observe that, for $1 \leqslant i < \tau$, $\Omega_q(i+1) \subset \Omega_q(i)$ and $\Omega_q(\rho_0, \ldots, \rho_{m_q}) \subset \Omega_q(\tau)$. Let $a_i = mj_i + q$, for $1 \leqslant i \leqslant \tau$. We can consider $\Omega_q(i)$ as a subset of the partition $\mathcal{P}_{a_i+l}|L$. By inequality (22),

$$\Omega_q(i+1) \subset \bigcup_{\omega\in\mathcal{P}_{a_i+l}|\Omega_q(i)} \{\theta \in \omega\ ;\ \widetilde{F}^{a_{i+1}}(\widehat{X}(\theta)) \in \widehat{J}(\rho_{j_{i+1}} - 2)\}.$$

Notice that $a_{i+1} - (a_i + l) \geqslant M(\alpha)$. If $r_0(\alpha) \leqslant \rho_{j_{i+1}} \leqslant (1/2 + 2\eta)\log(1/\alpha)$, we are in a position to apply claim (16) in proposition 5.2 to the admissible curves $\widehat{X}_{a_{i+1}-M(\alpha)}(\omega)$,

$\omega \in \mathcal{P}_{a_{i+1}-M(\alpha)}|\Omega_q(i)$. Together with the distortion estimate in lemma 3.2, we obtain

$$|\Omega_q(i+1)| \leqslant \sum_{\omega\in\mathcal{P}_{a_{i+1}-M(\alpha)}|\Omega_q(i)} K_2|\{\theta \in g^{a_{i+1}-M(\alpha)}(\omega)\ ;$$
$$\tilde{F}^{M(\alpha)}(\widehat{X}_{a_{i+1}-M(\alpha)}(\omega,\theta)) \in \widehat{J}(\rho_{j_{i+1}}-2)\}|^{1/\kappa}|\omega|$$
$$\leqslant K_2C^{1/\kappa}\mathrm{e}^{-(\beta/\kappa)r_0(\alpha)}|\Omega_q(i)| \leqslant K_2C^{1/\kappa}\mathrm{e}^{-(\beta/2\kappa)\rho_{j_{i+1}}}|\Omega_q(i)|,$$

where in the last inequality we used that $\rho_{j_{i+1}} \leqslant ((1/2+\eta)/(1/2-\eta))r_0(\alpha) \leqslant 2r_0(\alpha)$.

In the other case, when $(1/2+2\eta)\log(1/\alpha) \leqslant \rho_{j_{i+1}} \leqslant m$, claim (17) in proposition 5.2 combined with lemma 3.2 implies

$$|\Omega_q(i+1)| \leqslant \sum_{\omega\in\mathcal{P}_{a_{i+1}}|\Omega_q(i)} K_2|\{\theta \in g^{a_{i+1}}(\omega)\ ;\ \widehat{X}_{a_{i+1}}(\omega,\theta) \in \widehat{J}(\rho_{j_{i+1}}-2)\}|^{1/\kappa}|\omega|$$
$$\leqslant K_2C^{1/\kappa}\mathrm{e}^{-(\beta/\kappa)\rho_{j_{i+1}}}|\Omega_q(i)|.$$

Altogether, we have

$$|\Omega_q(\rho_0,\ldots,\rho_{m_q})| \leqslant |\Omega_q(\tau)| \leqslant (K_2C^{1/\kappa})^{\tau-1}\mathrm{e}^{-(\beta/2\kappa)\sum_{i=2}^{\tau}\rho_{j_i}}|\Omega_q(1)|.$$

With a similar argument we obtain $|\Omega_q(1)| \leqslant K_2C^{1/\kappa}\mathrm{e}^{-\beta\rho_{j_1}/2\kappa}|L|$. (In fact, if $j_1 = 0$ and $q < M(\alpha)$, this is only true if $(1/2+2\eta)\log(1/\alpha) \leqslant \rho_{j_1} \leqslant m$. However, since n is very large this is negligible.) Thus,

$$|\Omega_q(\rho_0,\ldots,\rho_{m_q})| \leqslant 2(K_2C^{1/\kappa})^{\tau}\mathrm{e}^{-(\beta/2\kappa)\sum_{j=0}^{m_q}\rho_j}. \qquad \square$$

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Paper C

ALMOST SURE ABSOLUTE CONTINUITY OF BERNOULLI CONVOLUTIONS

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Abstract. We prove an extension of a result by Peres and Solomyak on almost sure absolute continuity in a class of symmetric Bernoulli convolutions.

1. Introduction

For $\lambda \in (0,1)$, define the random series

$$Y_\lambda = \sum_{n \geq 1} \pm \lambda^n,$$

where the signs are chosen independently with probability 1/2. It is easy to see that the distribution ν_λ of Y_λ is singular for $\lambda < 1/2$, see Kershner and Wintner [2]. Wintner [7] noted that $\nu_{1/2}$ is uniform on $[-1,1]$. For Lebesgue almost every $1/2 < \lambda < 1$, Erdös conjectured that ν_λ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$. This conjecture has attracted a lot of attention during the years, and was finally settled by Solomyak [4] in 1995, who also proved that the densities are in $L^2(\mathbb{R})$. A simpler proof was later given by Peres and Solomyak in [3].

In this paper we discuss one of the many possible applications of the techniques developed in the paper of Peres and Solomyak. We will show generic absolute continuity of the distribution of the random series

$$Y_\lambda = \sum_{n \geq 1} \pm \lambda^{\varphi(n)},$$

where, as above, the signs $\pm$ are chosen independently and with probability 1/2, and where the function $\varphi : \mathbb{N} \to \mathbb{R}$ is assumed to satisfy

$$0 \leq \lim_{n \to \infty} \frac{\varphi(n)}{n} < \infty, \tag{1}$$

and some minor technical conditions. In particular, we treat the cases when $\varphi(n) = n + r(n)$, where r is the logarithm of a slowly varying function, and $\varphi(n) = n^\alpha$ for $0 < \alpha < 1$ (see Example 1 and Example 3).

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If the limit in (1) is infinite it follows that the measure ν_λ is singular, see e.g. [2], Criteria (10).

2. Bernoulli Convolutions and Examples

For a function $\varphi : \mathbb{N} \to \mathbb{R}$ and $0 < \lambda < 1$ we consider the infinite convolution product of $(\delta_{-\lambda^{\varphi(n)}} + \delta_{\lambda^{\varphi(n)}})/2$ for $n \geq 1$. This convolution product converges to a measure ν_λ if and only if

$$\sum_{n\geq 1} \lambda^{2\varphi(n)} < \infty, \tag{2}$$

and the finiteness of (2) implies furthermore that this infinite convolution converges absolutely, i.e. the order of the terms in the convolution is interchangeable (see e.g. Jessen and Wintner [1], Theorem 5 and Theorem 6). Let $\Omega = \{-1, 1\}^{\mathbb{N}}$ be the sequence space equipped with the product topology and μ the Bernoulli measure on Ω with the weights $(1/2, 1/2)$. The measure ν_λ can be written as the push-forward of μ by the random series

$$Y_\lambda(\omega) = \sum_{n\geq 1} \omega_n \lambda^{\varphi(n)}, \tag{3}$$

where ω_n denotes the n-th coordinate of an element ω in Ω. We are interested in the set of λ in the interval $(0, 1)$ for which the measure ν_λ is absolutely continuous with respect to the Lebesgue measure m on $\mathbb{R}$. Our first result deals with the class of random series where

$$\lim_{n\to\infty} \varphi(n + 1) - \varphi(n) = 0. \tag{4}$$

Observe that for functions φ with the property (4), it follows that

$$\lim_{n\to\infty} \frac{\varphi(n)}{n} = 0.$$

We begin by stating the following theorem.

Theorem 2.1. *If $\varphi : \mathbb{N} \to \mathbb{R}$ satisfies property (4) and if there is a $\lambda_1 \in (0, 1]$ such that, for all $\lambda \in (0, \lambda_1)$, condition (2) is fulfilled then, for a.e. $\lambda \in (0, \lambda_1)$, the measure ν_λ induced by the random series (3) is absolutely continuous and has an L^2-density.*

Example 1. If $\varphi(n) = n^\alpha$, $0 < \alpha < 1$, it follows immediately that the distribution of

$$Y_\lambda = \sum_{n\geq 1} \pm\lambda^{n^\alpha}$$

is absolutely continuous for a.e. $\lambda \in (0, 1)$, and that the density is in L^2.

Example 2. Observe that the function $\varphi(n) = n/\log n$ fulfills (4) and hence the distribution of

$$Y_\lambda = \sum_{n\geq 1} \pm\lambda^{n/\log n}$$

is absolutely continuous for a.e. $\lambda \in (0,1)$, and the density is in L^2.

The method used by Wintner in [5], [6] and [7] gives a better result in Example 1 in the case when $0 < \alpha < 1/2$. In fact, if $0 < \alpha < 1/2$, then the distribution of Y_λ is absolutely continuous for *all* $\lambda \in (0,1)$ and, furthermore, the density is smooth. Wintner considered the Fourier transform of the measure ν_λ which can be represented as a convergent infinite product: $\hat{\nu}_\lambda(t) = \prod_{n=1}^{\infty} \cos(\lambda^{\varphi(n)}t)$. Since $\cos(\lambda^{\varphi(n)}t) \leq 2/3$ if $1 \leq \lambda^{\varphi(n)}t \leq 2$, it follows that

$$|\hat{\nu}_\lambda(t)| \leq (2/3)^{K(t)},$$

where $K(t) = \#\{n\ ;\ 1 \leq \lambda^{\varphi(n)}t \leq 2\}$. In Example 1 a minor calculation yields that, for $0 < \alpha < 1/2$, $(2/3)^{K(t)}$ decreases faster than polynomially and thus, ν_λ is absolutely continuous and the density is smooth. To guarantee a sufficiently fast growing of $K(t)$, the function $\varphi(n)$ can not grow too fast. The method seems to break down at $\alpha = 1/2$. However, by taking the slowly growing function $\varphi(n) = \log n$, Wintner's method applies and one gets that the distribution of

$$Y_\alpha = \sum_{n\geq 1} \pm\frac{1}{n^\alpha}$$

is absolutely continuous for *all* $\alpha > 1/2$ and the density is smooth.

3. Absolute Continuity of Bernoulli Convolutions

Theorem 2.1 will be derived from the following result.

Theorem 3.1. *Suppose $\tau : \mathbb{N} \to \mathbb{R}$ is of the form $\tau(n) = \beta n + r(n)$, where the function $r(n)$ satisfies (4). Then the measure η_λ induced by the random series $Z_\lambda = \sum_{n\geq 1} \pm\lambda^{\tau(n)}$, is absolutely continuous and has an L^2-density, for a.e. $\lambda \in (2^{-1/\beta}, 2^{-2/3\beta})$.*

Example 3. If $\tau(n) = n + n^\alpha$, $0 < \alpha < 1$, it follows from Theorem 3.1 that, for a.e. $\lambda \in (2^{-1}, 2^{-2/3})$, the distribution of

$$Z_\lambda = \sum_{n\geq 1} \pm\lambda^{n+n^\alpha}$$

is absolutely continuous and the density is in L^2.

Proof of Theorem 2.1. Let $\{n_j\ ;\ j \geq 1\}$, be a subset of $\mathbb{N}$ such that $\varphi(n_{j+1}) < \varphi(n_j)$, $j \geq 1$, and such that for every $n \geq 1$ there is an

$j \geq 1$ with $\varphi(n) = \varphi(n_j)$, i.e. the sequence n_j should be thought of as the times when φ makes a jump. Observe that we still have

$$\lim_{j\to\infty} \varphi(n_{j+1}) - \varphi(n_j) = 0. \tag{5}$$

Let $\tilde{\varphi} : [1, \infty) \to \mathbb{R}$ be the continuous function which satisfies $\tilde{\varphi}(j) = \varphi(n_j)$ and which is linear on $[j, j+1]$, $j \geq 1$. Fix $0 < \beta < \infty$ and set $\psi(x) = \tilde{\varphi}^{-1}(\beta x)$. Since $\tilde{\varphi}(x+1) - \tilde{\varphi}(x) \to 0$ as $x \to \infty$, we can choose N_0 such that $\psi(x+1) - \psi(x) > 1$, for $x \geq N_0$. Let $[x]$ denote the integer part of the real number x. We split the random series Y_λ into two parts:

$$\begin{aligned} Y_\lambda(\omega) &= \sum_{j\geq N_0} \omega_{n_{[\psi(j)]}} \lambda^{\tilde{\varphi}([\psi(j)])} + \sum_{\substack{n\geq 1 \\ n\notin\{n_{[\psi(j)]}\ ;\ j\geq N_0\}}} \omega_n \lambda^{\varphi(n)} \\ &=: Z_\lambda(\omega) + R_\lambda(\omega). \end{aligned}$$

Note that this is possible since the infinite convolution Y_λ is absolutely convergent. We want to apply Theorem 3.1 to the function $\tau(n) = \tilde{\varphi}([\psi(n)])$. Let $r(n) = \tilde{\varphi}([\psi(n)]) - \beta n$. By the definition of ψ, $r(n) = \tilde{\varphi}([\psi(n)]) - \tilde{\varphi}(\psi(n))$ which, by (5) tends to 0 as $n \to \infty$. Hence, $r(n)$ satisfies trivially condition (4). Let η_λ be the measure induced by the random series Z_λ. It follows from Theorem 3.1, that, for a.e. $\lambda \in (2^{-1/\beta}, 2^{-2/3\beta})$, η_λ is absolutely continuous and has an L^2-density. The random variables Z_λ and R_λ are independent. Hence, for a.e. $\lambda \in (2^{-1/\beta}, 2^{-2/3\beta}) \cap (0, \lambda_1)$, we can write the measure ν_λ as a convolution of two measures where one of them is an absolutely continuous measure having an L^2-density. Thus, the measure ν_λ itself is absolutely continuous and the density of ν_λ is in $L^2(\mathbb{R})$. Since $0 < \beta < \infty$ was arbitrary we can fill out the whole interval $(0, \lambda_1)$, which concludes the proof of Theorem 2.1. □

Remark 1. We have already noted that a function $r : \mathbb{N} \to \mathbb{R}$ satisfying (4) also fulfills $\lim_{n\to\infty} r(n)/n = 0$. Observe furthermore that property (4) implies:

$$\lim_{k\to\infty} r(k+j) - r(k) = 0, \qquad \text{for all } j \geq 1,$$

and

$$\lim_{k\to\infty} \frac{\sup_{j\geq 1} |r(k+j) - r(k)|}{k} = 0.$$

4. Proof of Theorem 3.1

In [3], Peres and Solomyak studied power series of the form,

$$g(\lambda) = 1 + \sum_{j\geq 1} b_j \lambda^j, \quad b_j \in \{-1, 0, 1\},$$

for $\lambda \in (0,1)$, and proved the following lemma:

Lemma 4.1. *Suppose g is of the above form. There is a $\delta > 0$, such that, if $g(\lambda) < \delta$, for some λ in the interval $[0, 2^{-2/3}]$, then $g'(\lambda) < -\delta$.*

We will study slight modifications of these series. Let $r_{k,j}$, $k, j \geq 1$ be any sequence of real numbers, such that, for every $j \geq 1$,

$$r_{k,j} \to 0, \qquad \text{as } k \to \infty, \tag{6}$$

and

$$\lim_{k\to\infty} \frac{\log^+(\sup_{j\geq 1} |r_{k,j}|)}{k} = 0. \tag{7}$$

Define

$$g_k(\lambda) = 1 + \sum_{j\geq 1} b_j \lambda^{j+r_{k,j}} = g(\lambda) + \sum_{j\geq 1} b_j \lambda^j (\lambda^{r_{k,j}} - 1), \tag{8}$$

where $g(\lambda) = 1 + \sum_{j\geq 1} b_j \lambda^j$. Using Lemma 4.1, we can prove:

Lemma 4.2. *There is a positive constant δ' and a positive integer K, such that, if $k \geq K$ and $g_k(\lambda) < \delta'$ for some λ in $[0, 2^{-2/3}]$, then $g_k'(\lambda) < -\delta'$.*

Proof. We have

$$\begin{aligned} g_k'(\lambda) &= \sum_{j\geq 1} (j + r_{k,j}) b_j \lambda^{j-1+r_{k,j}} \\ &= g'(\lambda) + \sum_{j\geq 1} (j + r_{k,j}) b_j \lambda^{j-1} (\lambda^{r_{k,j}} - 1)) + \sum_{j\geq 1} r_{k,j} b_j \lambda^{j-1} \end{aligned}$$

Let δ be the constant in Lemma 4.1. Set $\delta' = \delta/2$ and pick $0 < \varepsilon < \delta/8$. Since $\lambda \leq 2^{-2/3} < 1$ and because of (7), we can choose $j_\varepsilon \geq 1$ such that

$$|\sum_{j\geq j_\varepsilon} (j + r_{k,j}) b_j \lambda^{j-1} (\lambda^{r_{k,j}} - 1)| \leq \varepsilon \quad \text{and} \quad |\sum_{j\geq j_\varepsilon} r_{k,j} b_j \lambda^{j-1}| \leq \varepsilon,$$

and

$$|\sum_{j\geq j_\varepsilon} b_j \lambda^j (\lambda^{r_{k,j}} - 1)| \leq \varepsilon,$$

for all $k \geq 1$. Furthermore, by condition (6), we can choose $K_\varepsilon \geq 1$ such that

$$|\sum_{j=1}^{j_\varepsilon} (j + r_{k,j}) b_j \lambda^{j-1} (\lambda^{r_{k,j}} - 1)| \leq \varepsilon \quad \text{and} \quad |\sum_{j=1}^{j_\varepsilon} r_{k,j} b_j \lambda^{j-1}| \leq \varepsilon,$$

and

$$|\sum_{j=1}^{j_\varepsilon} b_j \lambda^j (\lambda^{r_{k,j}} - 1)| \leq \varepsilon,$$

for all $k \geq K_\varepsilon$. Note that $g_k(\lambda) < \delta'$ and $k \geq K_\varepsilon$ implies, by (8),

$$g(\lambda) \leq \delta' + |\sum_{j\geq 1} b_j \lambda^j (\lambda^{r_{k,j}} - 1)| \leq \delta' + 2\varepsilon < \delta.$$

Hence, if $g_k(\lambda) < \delta'$ and $k \geq K_\varepsilon$, by Lemma 4.1,

$$\begin{aligned} g_k'(\lambda) &\leq -\delta + |\sum_{j\geq 1}(j + r_{k,j}) b_j \lambda^{j-1}(\lambda^{r_{k,j}} - 1)| + |\sum_{j\geq 1} r_{k,j} b_j \lambda^{k-1}| \\ &\leq -\delta + 4\varepsilon < -\delta'. \end{aligned}$$

□

We can now finish the proof of Theorem 3.1. Let $\tau : \mathbb{N} \to \mathbb{R}$ be as in Theorem 3.1. We can without loss of generality assume that $\beta = 1$. The case for general β follows immediately from a simple scaling argument. The proof closely follows the ideas outlined in [3]. Suppose η_λ is the push–forward of the Bernoulli measure on Ω under the map

$$Z_\lambda(\omega) = \sum_{n\geq 1} \omega_n \lambda^{\tau(n)}.$$

Setting $r(n) = \tau(n) - n$, we note that, by Remark 1, the sequence $r_{k,j} = r(k + j) - r(k)$ satisfies condition (6) and (7). Let I denote the interval $[\lambda_0, 2^{-2/3}]$, where $2^{-1} < \lambda_0 < 2^{-2/3}$, and let K and δ' be the constants in Lemma 4.2. It is enough to show that the distribution of the random series

$$\tilde{Z}_\lambda(\omega) = \sum_{n\geq K} \omega_n \lambda^{\tau(n)}$$

is absolutely continuous, for a.e. $\lambda \in I$, and has an L^2-density. Let

$$\Omega_K = \{(\omega_K, \omega_{K+1}, ...) \ ; \ \omega \in \Omega\},$$

and denote by μ_K the Bernoulli measure on Ω_K. Following [3], we need to prove that

$$\begin{aligned} S = \liminf_{r\to 0^+} \frac{1}{r} \int_{\Omega_K} \int_{\Omega_K} \\ m(\{\lambda \in I \ ; \ |\sum_{n\geq K} (\omega_n - \omega_n') \lambda^{\tau(n)}| < r\}) \, d\mu_K(\omega) d\mu_K(\omega') < +\infty. \end{aligned}$$

For $k \geq K$, let $\tilde{\Omega}_k$ denote the subset of elements (ω, ω') in $\Omega_K \times \Omega_K$ such that $\omega_j = \omega_j'$ for all $j \leq k - 1$, and $\omega_k \neq \omega_k'$. Note that

$$(\mu_K \times \mu_K)(\tilde{\Omega}_k) = 2^{-(k+1)+K} \quad \text{and} \quad \tau(k + j) - \tau(k) = j + r_{k,j}.$$

We obtain

$$S \leq \liminf_{r\to 0^+} \frac{1}{r} \sum_{k\geq K} 2^{-(k+1)+K}$$

$$\int_{\tilde{\Omega}_k} m(\{\lambda \in I \ ; \ |g_k(\lambda;\omega,\omega')| < r2^{-1}\lambda_0^{-\tau(k)}\})\, d\mu_K(\omega) d\mu_K(\omega'),$$

where

$$g_k(\lambda;\omega,\omega') = 1 + \sum_{j\geq 1} b_j(k;\omega,\omega')\lambda^{j+r_{k,j}}, \quad b_j(k;\omega,\omega') \in \{-1,0,1\},$$

for $(\omega,\omega') \in \tilde{\Omega}_k$. By Lemma 4.2, the functions g_k satisfy a transversality condition on the interval I, and thus,

$$m(\{\lambda \in I \ ; \ |g_k(\lambda;\omega,\omega')| \leq r2^{-1}\lambda_0^{-\tau(k)}\}) \leq \delta'^{-1} r \lambda_0^{-\tau(k)}.$$

It follows that

$$S \leq \delta'^{-1} 2^{K-1} \sum_{k\geq K} 2^{-k} \lambda_0^{-\tau(k)}.$$

Note now that the right-hand side is finite since $\lambda_0 > 1/2$ and $\tau(k)/k \to 1$ as $k \to \infty$. Hence we have proved Theorem 3.1.

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Paper D

TYPICAL POINTS FOR ONE-PARAMETER FAMILIES OF PIECEWISE EXPANDING MAPS OF THE INTERVAL

Abstract. Let $I \subset \mathbb{R}$ be an interval and $T_a : [0,1] \to [0,1]$, $a \in I$, a one-parameter family of piecewise expanding maps such that for each $a \in I$ the map T_a admits a unique absolutely continuous invariant probability measure μ_a. We establish sufficient conditions on such a one-parameter family such that a given point $x \in [0,1]$ is typical for μ_a for a full Lebesgue measure set of parameters a, i.e.

$$\frac{1}{n}\sum_{i=0}^{n-1}\delta_{T_a^i(x)} \overset{\text{weak-}*}{\longrightarrow} \mu_a, \qquad \text{as } n \to \infty,$$

for Lebesgue almost every $a \in I$. In particular, we consider $C^{1,1}(L)$-versions of β-transformations, skew tent maps, and Markov structure preserving one-parameter families. For the skew tent maps we show that the turning point is almost surely typical.

1. Introduction

Let $I \subset \mathbb{R}$ be an interval and $T_a : [0,1] \to [0,1]$, $a \in I$, a one-parameter family of maps of the unit interval such that, for every $a \in I$, T_a is piecewise C^2 and $\inf_{x\in[0,1]} |\partial_x T_a(x)| \geq \lambda > 1$, where λ is independent on a. Assume that, for all $a \in I$, T_a admits a unique (hence ergodic) absolutely continuous invariant probability measure (a.c.i.p.) μ_a. According to [8] and [9], for Lebesgue almost every $x \in [0,1]$, some iteration of x by T_a is contained in the support of μ_a. From Birkhoff's ergodic theorem we derive that Lebesgue almost every point $x \in [0,1]$ is *typical* for μ_a, i.e.

$$\frac{1}{n}\sum_{i=0}^{n-1}\delta_{T_a^i(x)} \overset{\text{weak-}*}{\longrightarrow} \mu_a, \qquad \text{as } n \to \infty.$$

In this paper we are interested in the question if the same kind of statement holds in the parameter space, i.e. if a chosen point $x \in [0,1]$ is typical for μ_a for Lebesgue a.e. $a \in I$, or more general, if, for some given C^1 map $X : I \to [0,1]$, $X(a)$ is typical for μ_a for Lebesgue a.e. a in I. In Section 2 we try to establish some sufficient conditions on a one-parameter family such that the following statement is true.

For Lebesgue a.e. $a \in I$, $X(a)$ is typical for μ_a.

The method we use in this paper is a dynamical one. It is essentially inspired by the result of Benedicks and Carleson [1] where they prove that for the quadratic family $f_a(x) = 1 - ax^2$ on $(-1,1)$ there is a set $\Delta_\infty \subset (1,2)$ of a-values of positive Lebesgue measure for which f_a admits almost surely an a.c.i.p. and for which the critical point is typical with respect to this a.c.i.p. The main tool in their work is to switch from the parameter space to the dynamical interval by showing that the a-derivative $\partial_a f_a^j(1)$ is comparable to the x-derivative $\partial_x f_a^j(1)$. This will also be the essence of the basic condition on our one-parameter family T_a with an associated map X, i.e. we require that the a- and the x-derivatives of $T_a^j(X(a))$ are comparable (see condition (I) below).

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Some typicality results related to this paper can be found in [14], [4], and [6]. The one-parameter families T_a, $a \in I$, considered in these papers have in common that their slopes are constant for a fixed parameter value, i.e. for each $a \in I$ there is a constant $\lambda_a > 1$ such that $|T_a'| \equiv \lambda_a$ on $[0,1]$. The advantage of our method and the main novelty of this paper is that we can drop this restriction and, thus, we are able to consider much more general families. In the case when $T_a : S^1 \to S^1$ is a smooth expanding map of the circle, there are some recent results by Pollicott [12].

This paper consists mainly of two parts. In the first part, which corresponds to Sections 2-5, we establish a general criteria for typicality. In the second part, which corresponds to Sections 6-8, we apply this criteria to several well-studied one-parameter families and derive various typicality results for these families. Some of these results are presented in the following subsections of this introduction.

We will shortly give a motivation and an overview of our criteria for typicality. Let $B \subset [0,1]$ be a (small) interval and set $x_j(a) = T_a^j(X(a))$, $a \in I$, i.e. $x_j(a)$ is the forward iteration by T_a^j of the points we are interested in. For $h \geq 1$ fixed, the main estimate to be established in the method we apply is roughly of the form:

$$\frac{1}{|I|}|\{a \in I \ ; \ x_{j_1}(a) \in B, ..., x_{j_h}(a) \in B\}| \leq (C|B|)^h, \tag{1}$$

where $1 \leq j_1 < ... < j_h \leq n$ (n large) are h integers with large ($\geq \sqrt{n}$) gaps between each other and $C \geq 1$ is some constant. Such an estimate is easier to establish for a fixed map T_a in the family, i.e. it is easier to verify the estimate

$$|\{x \in [0,1] \ ; \ T_a^{j_1}(x) \in B, ..., T_a^{j_h}(x) \in B\}| \leq (C|B|)^h. \tag{2}$$

(See also inequality (12).) Hence, in order to prove (1), the main idea in the first part of this paper is to imitate the proof of (2). This leads to three rather natural conditions, conditions (I)-(III), on the sequence of maps $x_j : I \to [0,1]$. Condition (I) roughly says that T_a^j and x_j are comparable, i.e. there exists a constant $C \geq 1$ such that

$$C^{-1} \leq \frac{|\partial_x T_a^j(X(a))|}{|D_a x_j(a)|} \leq C,$$

for all $j \geq 1$, and $a \in I$ for which the derivatives are defined. Condition (II) ensures that estimates of the following type apply: There exists a constant $C > 0$ such that for all intervals $B \subset [0,1]$ and $j \geq 1$ we have

$$|\{a \in I \ ; \ x_j(a) \in B\})| \leq C|B|.$$

Since maps in one-parameter families of piecewise expanding maps have in general no finite Markov partition, the image $x_j(\omega)$ of a (maximal) interval of smooth monotonicity ω for x_j might be arbitrarily small. Condition (III) provides us we a certain control of such 'too short' intervals of smooth monotonicity.

Apart from condition (I), in order to apply our result to the examples in Sections 6-8, we will not verify conditions (II) and (III) directly. Instead we will show that two other conditions, conditions (IIa) and (IIb), are satisfied. Conditions (IIa) and (IIb) are described in Section 4. Condition (IIa) should hold for general families. It only requires that the densities for the a.c.i.p. μ_a are uniformly (in a) bounded above and below away from zero. In contrast, condition (IIb) is more restrictive. It requires that there is a kind of order relation in the one-parameter family in the sense that for each two parameter values $a, a' \in I$ satisfying $a < a'$ the following holds. The symbolic dynamics of T_a is contained in the symbolic dynamics of $T_{a'}$ and, furthermore, if ω is a (maximal) interval of smooth monotonicity for T_a^j, then the image of the (maximal) interval of smooth monotonicity ω' for $T_{a'}^j$ with the same combinatorics as ω contains the image of

ω, i.e. $T_a^j(\omega) \subset T_{a'}^j(\omega')$. It is interesting to find other ways of verifying conditions (II) and (III). This might allow to treat more general examples of piecewise expanding one-parameter families than the ones studied in Sections 6-8. The main obstacle seems to be the verification of condition (II). As the labeling of conditions (IIa) and (IIb) suggest, they are aimed for verifying condition (II). More or less as a byproduct they also imply condition (III); see Section 5. For a simple example of a family not satisfying condition (IIb), see Remark 4.1.

1.1. **β-transformations.** The example in Section 6 is a $C^{1,1}(L)$-version of the β-transformation. By saying that a function is $C^{1,1}(L)$, we mean that it is C^1 and its derivative is in Lip(L), i.e. its derivative is Lipschitz continuous with Lipschitz constant L. For a sequence $0 = b_0 < b_1 < ...$ of real numbers such that $b_k \to \infty$ as $k \to \infty$ and a constant $L > 0$, let $T : [0, \infty) \to [0, 1]$ be a right continuous function which is $C^{1,1}(L)$ on each interval (b_k, b_{k+1}), $k \geq 0$. Furthermore, we assume that:

- $T(b_k) = 0$ for each $k \geq 0$.
- For each $a > 1$,

$$1 < \inf_{x \in [0,1]} \partial_x T(ax) \quad \text{and} \quad \sup_{x \in [0,1]} \partial_x T(ax) < \infty.$$

See Figure 1.

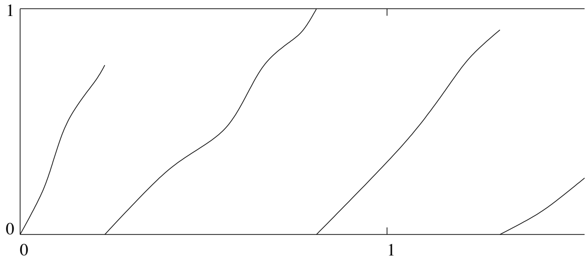

FIGURE 1. A possible beginning of a graph for $T : [0, \infty) \to [0, 1]$.

Given a map T as above, we obtain a $C^{1,1}(L)$-version of the β-transformation $T_a : [0, 1] \to [0, 1]$, $a > 1$, by defining $T_a(x) = T(ax)$, $x \in [0, 1]$. As we will see in Section 6, for each $a > 1$, T_a admits a unique a.c.i.p. μ_a. In Section 6 we will show the following.

Theorem 1.1. *If $X : (1, \infty) \to (0, 1]$ is C^1 and $X'(a) \geq 0$, then $X(a)$ is typical for μ_a for Lebesgue a.e. $a > 1$.*

If we choose $X(a) = b_1/a$ then $X'(a) < 0$ and $T_a^j(X(a)) = 0$ for all $j \geq 1$. Hence, if the condition $X'(a) \geq 0$ in Theorem 1.1 is not satisfied, we cannot any longer guarantee almost sure typicality for the a.c.i.p. For an illustration of the curves on which we have a.s. typicality see Figure 2 (when a is fixed, we can apply Birkhoff's ergodic theorem and get a.s. typicality on the associated vertical line). Observe that if we choose $T : [0, \infty) \to [0, 1]$ by $T(x) = x \bmod 1$, then $T_a(x) = ax \bmod 1$ is the usual β-transformation. Theorem 1.1 generalizes a result due to Schmeling [14] where it is shown that for the usual β-transformation the point 1 is typical for the associated a.c.i.p. for Lebesgue a.e. $a > 1$.

1.2. **Skew tent maps.** In Section 7 we investigate unimodal maps with slopes constant to the left and to the right of the turning point. Let these slopes be α and $-\beta$ where

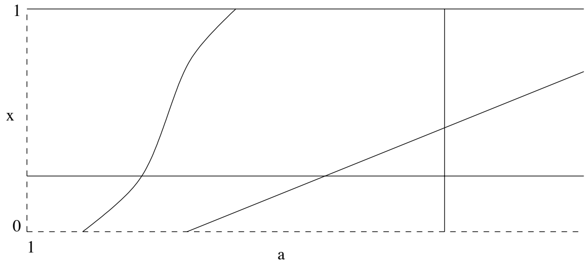

FIGURE 2. Lines and curves on which we have a.s. typicality for the $C^{1,1}(L)$-version of the β-transformation.

$\alpha, \beta > 1$. For instance, let $T_{\alpha,\beta} : [0,1] \to \mathbb{R}_0^+$ be defined by

$$T_{\alpha,\beta}(x) = \begin{cases} \alpha x & \text{if } x \leq \frac{\beta}{\alpha+\beta}, \\ \beta(1-x) & \text{otherwise.} \end{cases}$$

In order that $T_{\alpha,\beta} : [0,1] \to [0,1]$, we have also to assume that $\alpha^{-1} + \beta^{-1} \geq 1$ (see, e.g., Lemma 3.1 in [11]). The map $T_{\alpha,\beta}$ is called a *skew tent map*. Fix two points (α_0, β_0) and (α_1, β_1) in the set $\{(\alpha, \beta)\ ;\ \alpha, \beta > 1 \text{ and } \alpha^{-1} + \beta^{-1} \geq 1\}$ such that $\alpha_1 \geq \alpha_0$, $\beta_1 \geq \beta_0$, and at least one of these two inequalities is sharp. Let

$$\alpha : [0,1] \to [\alpha_0, \alpha_1] \quad \text{and} \quad \beta : [0,1] \to [\beta_0, \beta_1]$$

be functions in $C^1([0,1])$ such that $(\alpha(0), \beta(0)) = (\alpha_0, \beta_0)$, $(\alpha(1), \beta(1)) = (\alpha_1, \beta_1)$, and, for all $a \in [0,1]$, if $\alpha_0 \neq \alpha_1$, then $\alpha'(a) > 0$ and if $\beta_0 \neq \beta_1$, then $\beta'(a) > 0$. Consider the one-parameter family T_a, $a \in [0,1]$, where $T_a : [0,1] \to [0,1]$ is the skew tent map defined by $T_a = T_{\alpha(a),\beta(a)}$. By [9], since T_a has only two intervals of monotonicity, it follows that there exists a unique a.c.i.p. μ_a for T_a. In Section 7 we will show that the turning point is a.s. typical for the a.c.i.p.

Theorem 1.2. *The turning point for the skew tent map T_a is typical for μ_a for Lebesgue a.e. $a \in [0,1]$.*

Theorem 1.2 generalizes a result due to Bruin [4] where almost sure typicality is shown for the turning point of symmetric tent maps (i.e. when $\alpha(a) \equiv \beta(a)$). It is possible to extend the results in Section 7 to certain one-parameter families of $C^{1,1}(L)$ unimodal maps (see Remark 7.5). In Section 7 we will use a slightly different representation of skew tent maps.

1.3. **One-parameter families of Markov maps.** In Section 8 we consider one-parameter families, which preserve a certain Markov structure. A simple example for such a family are the maps $T_a : [0,1] \to [0,1]$ defined by

$$T_a(x) = \begin{cases} \frac{x}{a} & \text{if } x < a, \\ \frac{x-a}{1-a} & \text{otherwise,} \end{cases}$$

where the parameter $a \in (0,1)$. See Figure 3. By [9], since this map has only one point of discontinuity, it admits a unique a.c.i.p. μ_a (which coincides in this case with the Lebesgue measure on $[0,1]$). In Example 8.2 in Section 8 we will show the following.

Proposition 1.3. *If $X : (0,1) \to (0,1)$ is a C^1 map such that $X'(a) \leq 0$, then $X(a)$ is typical for μ_a for a.e. parameter $a \in (0,1)$.*

Observe that if $X(a) = p_a$ where p_a is the unique point of periodicity 2 in the interval $(0,a)$, then $X'(a) > 0$. Hence, if the condition $X'(a) \leq 0$ is violated in Proposition 1.3, we cannot any longer guarantee almost sure typicality for the a.c.i.p. The very simple structure of the example in this subsection makes it to a good candidate for serving the reader as a model along the paper. Example 8.2 in Section 8 is formulated slightly more general by composing T_a with a $C^{1,1}(L)$ homeomorphism $g : [0,1] \to [0,1]$.

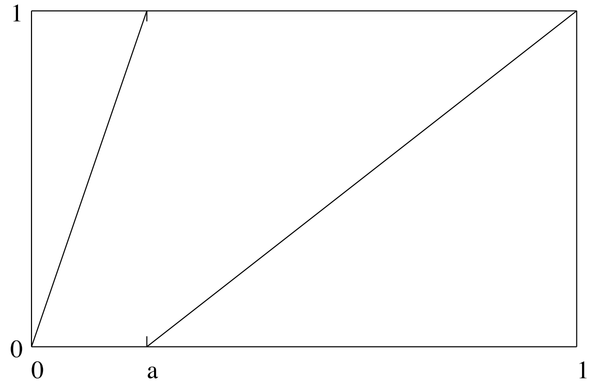

FIGURE 3. A Markov structure preserving one-parameter family T_a where $a \in (0,1)$.

2. Piecewise expanding one-parameter families

2.1. **Preliminaries.** In this subsection we introduce the basic notation and put up a general model for one-parameter families of piecewise expanding maps of the unit interval. A map $T : [0,1] \to \mathbb{R}$ will be called *piecewise* $C^{1,1}(L)$ if there exists a partition $0 = b_0 < b_1 < ... < b_p = 1$ of the unit interval such that for each $1 \leq k \leq p$ the restriction of T to the open interval (b_{k-1}, b_k) is a $C^{1,1}(L)$ function. Observe that, by the Lipschitz property, it follows that T restricted to (b_{k-1}, b_k) can be extended to the closed interval $[b_{k-1}, b_k]$ as a $C^{1,1}(L)$ function. Let $I \subset \mathbb{R}$ be an interval of finite length and $T_a : [0,1] \to [0,1]$, $a \in I$, a one-parameter family of piecewise $C^{1,1}(L)$ maps where the Lipschitz constant $0 < L < \infty$ is independent on the choice of the parameter a. We assume that there are real numbers $1 < \lambda \leq \Lambda < \infty$ such that for every $a \in I$,

$$\lambda \leq \inf_{x\in[0,1]} |\partial_x T_a(x)| \quad \text{and} \quad \sup_{x\in[0,1]} |\partial_x T_a(x)| \leq \Lambda. \tag{3}$$

The parameter dependence is assumed to be piecewise C^1, i.e. for all $x \in [0,1]$, there exists a partition $a_0 < a_1 < ... < a_{p(x)}$ (where a_0 is the left and $a_{p(x)}$ the right boundary point of I) of the parameter interval I such that for each $1 \leq k \leq p(x)$ the restriction of the map $a \mapsto T_a(x)$ to the open interval (a_{k-1}, a_k) is a C^1 function, which can be extended to the closed interval $[a_{k-1}, a_k]$ as a C^1 function. More precise requirements on the parameter dependence will be given shortly (see (i)-(iii) below).

In the sequel, instead of referring to [8] and [9], we will refer to a paper by S. Wong [17] who extended the results in [8] and [9] on piecewise C^2 maps to a broader class of maps containing also piecewise $C^{1,1}(L)$ maps. For a fixed $a \in I$, by [17], there is a finite collection of sets $K_1(a), ..., K_{p_0(a)}(a)$ such that each $K_k(a)$, $1 \leq k \leq p_0(a)$, is a union of finitely many disjoint closed intervals (each of positive length) and, for Lebesgue a.e. $x \in [0,1]$, the accumulation points of the forward orbit of x is identical with one of these $K_k(a)$'s, i.e. to every x in a full Lebesgue measure set of $[0,1]$, there is a $K_k(a)$ such that

$$K_k(a) = \bigcap_{N=1}^{\infty} \overline{\{T_a^n(x)\}_{n=N}^{\infty}}. \tag{4}$$

Furthermore, for each $K_k(a)$ there is a unique (hence ergodic) a.c.i.p. $\mu_{a,k}$ such that $\mathrm{supp}(\mu_{a,k}) = K_k(a)$. Since we are always interested in only one $K_k(a)$, we can without loss of generality assume that $p_0(a) \equiv 1$, $a \in I$. Henceforth, we write $K(a)$ and μ_a instead of $K_1(a)$ and $\mu_{a,1}$, respectively. So, we have that μ_a is the unique a.c.i.p. for T_a and

$$K(a) = \mathrm{supp}(\mu_a).$$

For $a \in I$, let $c_0(a) < c_1(a) < ... < c_{p_1(a)}(a)$ be the associated partition for the piecewise $C^{1,1}(L)$ map $T_a : K(a) \to K(a)$, i.e., if $D_1(a), ..., D_{p_2(a)}(a)$ $(p_2(a) \leq p_1(a))$ denote the (maximal) open intervals in $K(a)$ on which T_a is $C^{1,1}(L)$, then the $c_k(a)$'s are the boundary points of these $C^{1,1}(L)$ domains. For the sake of definition, assume that, for $1 \leq k < p_2$, the domain $D_k(a)$ is to the left of the domain $D_{k+1}(a)$.

We assume that the number of $c_k(a)$'s and $D_k(a)$'s are constant, i.e. $p_1(a) \equiv p_1$ and $p_2(a) \equiv p_2$. Furthermore, we make the following three natural assumptions on our one-parameter family.

(i) For all $0 \leq k \leq p_1$, the map $a \mapsto c_k(a)$ which maps I to $[0,1]$ is $\mathrm{Lip}(L)$, and there is a constant $\delta_0 > 0$ such that
$$|D_k(a)| \geq \delta_0,$$
for all $1 \leq k \leq p_2$ and $a \in I$.

(ii) For each $x \in [0,1]$ and $1 \leq k \leq p_2$, if J denotes the set of parameters $a \in I$ such that $x \in D_k(a)$, then if J is non-empty, it is an interval and the maps $a \mapsto T_a(x)$ and $a \mapsto \partial_x T_a(x)$ from J to $\mathbb{R}$ are $\mathrm{Lip}(L)$.

(iii) For each $x \in [0,1]$ and $1 \leq k \leq p_2$, if J denotes the set of parameters $a \in I$ such that $T_a^{-1}\{x\}$ has a pre-image in $D_k(a)$, then if J is non-empty, it is an interval and the branch of $T_\cdot^{-1}(x)$, which maps J to D_k is $\mathrm{Lip}(L)$.

2.2. **Partitions.** For a fixed parameter value $a \in I$, we denote by $\mathcal{P}_j(a)$, $j \geq 1$, the partition on the dynamical interval consisting of the maximal open intervals of smooth monotonicity for the map $T_a^j : K(a) \to K(a)$. In other words, $\mathcal{P}_j(a)$ denotes the set of open intervals ω in $K(a)$ such that $T_a^j : \omega \to K(a)$ is $C^{1,1}(L)$ and ω is maximal, i.e. for every other open interval $\tilde{\omega} \subset K(a)$ with $\omega \subsetneq \tilde{\omega}$, $T_a^j : \tilde{\omega} \to K(a)$ is no longer $C^{1,1}(L)$. Clearly, $\mathcal{P}_1(a) = \{D_1(a), ..., D_{p_2}(a)\}$. For an open set $H \subset K(a)$, we denote by $\mathcal{P}_j(a)|H$ the restriction of $\mathcal{P}_j(a)$ to the set H. For a set $J \subset K(a)$, which lies completely in one $D_k(a)$, $1 \leq k \leq p_2$, we denote by $\mathrm{symb}_a(J)$ the index (or symbol) k.

We will define similar partitions on the parameter interval I. Let $X : I \to [0,1]$ be a C^1 map from I into the dynamical interval $[0,1]$. The points $X(a)$, $a \in I$, will be our candidates for typical points. The forward orbit of a point $X(a)$ under the map T_a we denote as

$$x_j(a) := T_a^j(X(a)), \qquad j \geq 0.$$

Remark 2.1. Since a lot of informations for the dynamics of T_a is contained in the forward orbits of the partition points $c_k(a)$, $0 \leq k \leq p_1$, it is of interest to know how the forward orbits of these points are distributed. Hence, an evident choice of the map X would be

$$X(a) = \lim_{\substack{x \to c_k(a) \\ x \in \omega}} T_a(x),$$

where $\omega \in \mathcal{P}_1(a)$ is an interval adjacent to $c_k(a)$.

Let J be an open set in the parameter space I. By $\mathcal{P}_j|J$, $j \geq 1$, we denote the partition consisting of all open intervals ω in J such that for each $0 \leq i < j$, there exists $1 \leq k \leq p_2$, such that $x_i(a) \in D_k(a)$, for all $a \in \omega$, and such that ω is maximal, i.e. for every other open interval $\tilde{\omega} \subset J$ with $\omega \subsetneq \tilde{\omega}$, there exist $a \in \tilde{\omega}$, $0 \leq i < j$, and $1 \leq k \leq p_2$

such that $x_i(a) \in \partial D_k(a)$. Observe that this partition might be empty. This is, e.g., the case when $X(a) \notin K(a)$ for all $a \in I$ or when T_a is the usual β-transformation and the map X is chosen to be equivalently equal to 0. However, such trivial situations are excluded by condition (I) formulated in the next subsection. Knowing that condition (I) is satisfied, then the partition $\mathcal{P}_j|J$ can be thought of as the set of the (maximal) open intervals of smooth monotonicity for $x_j : J \to [0,1]$ (in order that this is really true one should also assume that $|x_j'(a)| > L$, for all $j \geq 0$ and parameter values $a \in I$ for which this derivative is defined). We set $\mathcal{P}_0|J = J$, and we will write $\mathcal{P}_j|I$ instead of $\mathcal{P}_j|\operatorname{int}(I)$. If for a set J' in the parameter space and for some integer $j \geq 0$ the symbol $\operatorname{symb}_a(x_j(a))$ exists for all $a \in J'$, then it is constant and we denote this symbol by $\operatorname{symb}(x_j(J'))$. Finally, in view of condition (I) below, observe that if a parameter $a \in I$ is contained in an element of $\mathcal{P}_j|I$, $j \geq 1$, then also the point $X(a)(= x_0(a))$ is contained in an element of $\mathcal{P}_j(a)$.

2.3. **Main statement.** In this subsection we will state our main result, Theorem 2.2. Let n be large. To ensure good distortion estimates we will, in the proof of Theorem 2.2, split up the interval I into smaller intervals $J \subset I$ of size $1/n$. The main idea in this paper is to switch from the map $x_j : J \to [0,1]$, $j \leq n$, to the map $T_a^j : [0,1] \to [0,1]$ where a is, say, the right boundary point of J. By this, since the dynamics of the map T_a is well-understood, we derive similar dynamical properties for the map x_j, which then can be used to prove Theorem 2.2. To be able to switch from x_j to T_a^j, we put further three rather natural conditions, conditions (I)-(III), on our one-parameter family.

In condition (I) we require that the derivatives of x_j and T_a^j are comparable. This is the very basic assumption in this paper. Of course, the choice of the map $X : I \to [0,1]$ plays here an important role. If, e.g., for every parameter $a \in I$, $X(a)$ is a periodic point for the map T_a, then x_j will have bounded derivatives and the dynamics of x_j is completely different from the dynamics of T_a. Henceforth, we will use the notations

$$T_a'(x) = \partial_x T_a(x) \qquad \text{and} \qquad x_j'(a) = D_a x_j(a), \quad j \geq 1.$$

(I) There is a constant $C_0 \geq 1$ such that for $\omega \in \mathcal{P}_j|I$, $j \geq 1$, we have

$$\frac{1}{C_0} \leq \left| \frac{x_j'(a)}{{T_a^j}'(X(a))} \right| \leq C_0,$$

for all $a \in \omega$. Furthermore, the number of $a \in I$, which are not contained in any element $\omega \in \mathcal{P}_j|I$ is finite.

We turn to condition (II). For $a \in I$, let φ_a denote the density for μ_a. Assume for the moment that φ_a is bounded from below and from above by a constant $C \geq 1$, i.e $C^{-1} \leq \varphi_a(x) \leq C$ for a.e. $x \in K(a)$. Note that, since the density φ_a is a fixed point of the Perron-Frobenius operator, we have, for $j \geq 1$,

$$\varphi_a(y) = \sum_{\substack{x \in K(a) \\ T_a^j(x) = y}} \frac{\varphi_a(x)}{|{T_a^j}'(x)|},$$

for a.e. $y \in K(a)$. This implies that

$$\sum_{\substack{x \in K(a) \\ T_a^j(x) = y}} \frac{1}{|{T_a^j}'(x)|} \leq C^2, \tag{5}$$

for a.e. $y \in K(a)$. We require a similar estimate for the map x_j.

(II) There exists a constant $C_1 \geq 1$ such that the following holds. Let $J \subset I$ be an open interval of length $1/n$. For $\tilde{\omega} \in \mathcal{P}_i|J$, $i \geq 1$, and $1 \leq j \leq n$, we have

$$\sum_{\substack{a \in \tilde{\omega} \\ x_{i+j}(a)=y}} \left| \frac{x_i'(a)}{x_{i+j}'(a)} \right| \leq C_1 \tag{6}$$

for a.e. $y \in [0,1]$.

Let $B \subset [0,1]$ be a (small) interval. Recall that in condition (I) it is required that, for $j \geq 1$, there are only finitely many parameter values $a \in I$ not contained in any element of $\mathcal{P}_j|I$. Thus, if in addition to condition (I), condition (II) is satisfied, then it follows that

$$|x_i(\{a \in \tilde{\omega} \ ; \ x_{i+j}(a) \in B\})| = \int_B \sum_{\substack{a \in \tilde{\omega} \\ x_{i+j}(a)=y}} \left| \frac{x_i'(a)}{x_{i+j}'(a)} \right| dy \leq C_1 |B|, \tag{7}$$

which will be the main estimate in the proof of Theorem 2.2. Now, we want to pull back this estimate to the parameter interval I. Assuming that $x_i(\tilde{\omega})$ has a large size, for instance, assuming that $x_i(\tilde{\omega}) = (0,1)$, then, by a minor distortion estimate for x_i, it follows (see (15) below) that

$$|\{a \in \tilde{\omega} \ ; \ x_{i+j}(a) \in B\}| \leq C_1 C_3 |B||\tilde{\omega}|, \tag{8}$$

where C_3 is a bound for the distortion of x_i. If $\tilde{\omega}$ was a very small interval of smooth monotonicity for x_i then it might happen that $\tilde{\omega}$ is mapped by x_{i+j} entirely into the interval B and not just a $|B|$ fraction of it, as it is the case in (8). To avoid such cases we want that the total measure of partition elements with a too small image is negligible.

Condition (III) requires that we are able to exclude elements $\omega \in \mathcal{P}_j|I$, $j \geq 1$, whose length of $x_j(\omega)$ is below a fixed constant, say, $\delta_1 > 0$. However, if $\omega \in \mathcal{P}_j|I$ is an element such that $|x_j(\omega)| < \delta_1$, we will not exclude it immediately but, roughly speaking, we will give ω (or at least a part of it) a chance to grow during the following $\sqrt{n}$ iterations. The formulation of condition (III) is rather technical.

(III) There is a constant $\delta_1 > 0$ such that to every $\varepsilon > 0$ there is an integer n_ε growing at most polynomially in $1/\varepsilon$ such that for $n \geq n_\varepsilon$ the following holds. Let $J \subset I$ be an open interval of length $1/n$ and fix an integer $1 \leq j \leq 2n$. The exceptional set

$$E = \{\omega \in \mathcal{P}_{j+[\sqrt{n}]}|J \ ; \ \nexists\, \tilde{\omega} \in \mathcal{P}_{j+k}|J, \ \ 0 \leq k \leq [\sqrt{n}], \quad \text{such that } \tilde{\omega} \supset \omega \text{ and } |x_{j+k}(\tilde{\omega})| \geq \delta_1\},$$

satisfies

$$|E| \leq \frac{\varepsilon}{n}.$$

Finally, we state the main result of this paper.

Theorem 2.2. *Let $T_a : [0,1] \to [0,1]$, $a \in I$, be a piecewise expanding one-parameter family as described in Subsection 2.1, satisfying properties (i)-(iii), and let $X : I \to [0,1]$ be a C^1 map. If conditions (I)-(III) are fulfilled, then $X(a)$ is typical for μ_a for Lebesgue almost every $a \in I$.*

As already pointed out in the introduction, for all the examples considered in this paper we will not verify conditions (II) and (III) directly. Instead we will verify conditions (IIa) and (IIb) which are described in Section 4. Knowing that these two conditions are satisfied is then sufficient to deduce that also conditions (II) and (III) are satisfied (provided that the basic condition (I) holds); see Propositions 4.5 and 5.1. Furthermore, in the considered examples, we will usually not verify conditions (I), (IIa) and (IIb) for the whole interval I for which the corresponding family is defined. Instead we will cover I by a countable number of smaller intervals and verify these conditions on these smaller intervals.

3. Proof of Theorem 2.2

The idea of the proof of Theorem 2.2 is inspired by Chapter III in [1] where Benedicks and Carleson prove the existence of an a.c.i.p. for a.e. parameter in a certain parameter set (the set Δ_∞). Their argument implies that the critical point is in fact typical for this a.c.i.p.

Let

$$\mathcal{B} := \left\{(q-r, q+r) \cap [0,1] \ ; \ q \in \mathbb{Q}, r \in \mathbb{Q}^+\right\}.$$

We will show that there is a constant $C \geq 1$ such that for each $B \in \mathcal{B}$ the function

$$F_n(a) = \frac{1}{n}\sum_{j=1}^{n} \chi_B(x_j(a)), \quad n \geq 1,$$

fulfills

$$\overline{\lim_{n\to\infty}} F_n(a) \leq C|B|, \qquad \text{for a.e. } a \in I. \tag{9}$$

By standard measure theory (see, e.g., [10]), (9) implies that, for a.e. $a \in I$, every weak-$*$ limit point ν_a of

$$\frac{1}{n}\sum_{j=1}^{n} \delta_{x_j(a)}, \tag{10}$$

has a density which is bounded above by C. In particular, ν_a is absolutely continuous. Observe that, by the definition of $x_j(a)$, the measure ν_a is also invariant for T_a and, hence, ν_a is an a.c.i.p. for T_a. By the uniqueness of the a.c.i.p. for T_a, we finally derive that, for a.e. $a \in I$, the weak-$*$ limit of (10) exists and coincides with the a.c.i.p. μ_a. This concludes the proof of Theorem 2.2.

In order to prove (9), it is sufficient to show that for all (large) integers $h \geq 1$ there is an integer $n_{h,B}$, growing for fixed B at most exponentially in h, such that

$$\int_I F_n(a)^h da \leq \text{const}(C|B|)^h,$$

for all $n \geq n_{h,B}$ (see Lemma A.1 in [2]).

In the remaining part of this section, we assume that $B \in \mathcal{B}$ is fixed. For $h \geq 1$, we have

$$\int_I F_n(a)^h da = \sum_{1\leq j_1,\ldots,j_h\leq n} \frac{1}{n^h}\int_I \chi_B(x_{j_1}(a))\cdots\chi_B(x_{j_h}(a))da. \tag{11}$$

For a fixed parameter a, there exists an integer k such that (T_a^k, μ_a) is exact and, hence, this system is mixing of all degrees (see [13] and [16]). It follows that for sequences of non-negative integers $j_1^r, ..., j_h^r$, $r \geq 1$, with

$$\lim_{r\to\infty} \inf_{i\neq l} |j_i^r - j_l^r| = \infty,$$

one has

$$\int_{[0,1]} \chi_B\Big(T_a^{kj_1^r}(x)\Big)\cdots\chi_B\Big(T_a^{kj_h^r}(x)\Big)\,d\mu_a(x) \tag{12}$$
$$= \mu_a\left(T_a^{-kj_1^r}(B)\cap ... \cap T_a^{-kj_h^r}(B)\right) \stackrel{r\to\infty}{\longrightarrow} \mu_a(B)^h \le (\|\varphi_a\|_\infty |B|)^h .$$

Since the maps T_a^j and x_j are, by conditions (I)-(III), 'comparable', it is natural to expect similar mixing properties for the maps x_j. In fact, in the next subsection, we are going to prove the following statement.

Proposition 3.1. *Under the assumption that conditions (I)-(III) are satisfied, there is a constant $C \ge 1$ such that the following holds. For all $h \ge 1$, there is an integer $n_{h,B}$ growing at most exponentially in h such that, for all $n \ge n_{h,B}$ and for all integer h-tuples $(j_1, ..., j_h)$ with $\sqrt{n} \le j_1 < j_2 < ... < j_h \le n$ and $j_l - j_{l-1} \ge \sqrt{n}$, $l = 2, ..., h$,*

$$\int_I \chi_B(x_{j_1}(a))\cdots\chi_B(x_{j_h}(a))da \le 4|I|(C|B|)^h.$$

Seen from a more probabilistic point of view, Proposition 3.1 says that whenever the distances between consecutive j_i's are sufficiently large, the behavior of the $\chi_B(x_{j_i}(.))$'s is comparable to that of independent random variables. Now, the number of h-tuples $(j_1, ..., j_h)$ in the sum in (11), for which $\min_i j_i < \sqrt{n}$ or $\min_{k\ne l} |j_k - j_l| < \sqrt{n}$, is bounded by $2h^2 n^{h-1/2}$. Hence, by Proposition 3.1,

$$\int_I F_n(a)^h da \le 4|I|(C|B|)^h + \frac{2h^2}{\sqrt{n}}|I| \le 5|I|(C|B|)^h,$$

whenever

$$n \ge \max\left\{ n_{h,B}, \left(\frac{2h^2}{(C|B|)^h}\right)^2 \right\}.$$

Since both terms in this lower bound for n grow at most exponentially in h, this concludes the proof of Theorem 2.2.

3.1. **Proof of Proposition 3.1.** To be able to make use of conditions (II) and (III), we split up the integral in Proposition 3.1 and integrate over smaller intervals of length $1/n$. More precisely, under the assumptions of Proposition 3.1, we are going to show that there exists an integer $n_{h,B}$ growing at most exponentially in h such that, for $n \ge n_{h,B}$, we have

$$\int_J \chi_B(x_{j_1}(a))\cdots\chi_B(x_{j_h}(a))da \le \frac{1}{n}3(C|B|)^h, \tag{13}$$

where $J \subset I$ is an arbitrary interval of length $1/n$. This immediately implies that, for $n \ge n_{h,B}$,

$$\int_I \chi_B(x_{j_1}(a))\cdots\chi_B(x_{j_h}(a))da \le 4|I|(C|B|)^h,$$

(if $n_{h,B} \gg 1/|I|$), which concludes the proof of Proposition 3.1.

Note that by condition (I), for $j \ge 1$, there are only finitely many parameter values not contained in any element of $\mathcal{P}_j|I$. Hence, we can neglect such parameter values and focus on the partitions $\mathcal{P}_j|I$. Let $\tilde\omega \in \mathcal{P}_i|J$, $i \ge 1$, and $1 \le j \le n$. By condition (II) (see (7)), we have

$$|x_i(\{a \in \tilde\omega \ ;\ x_{i+j}(a) \in B\})| \le C_1|B|. \tag{14}$$

We will give a rough idea of how condition (III) can be used to conclude the proof of Proposition 3.1. If our one-parameter family satisfies condition (III), this means that we can neglect too small partition elements, and we can without loss of generality assume that the following preferable picture is true: If $\omega' \in \mathcal{P}_{j_i}|J$, $1 \le i \le h-1$, then we can write the subinterval of ω' which is mapped into B as a disjoint union of intervals $\tilde{\omega}$ such that each $\tilde{\omega}$ is an element of some partition $\mathcal{P}_{j_i+k}|J$, $1 \le k \le j_{i+1} - j_i$, having a large image, i.e. $|x_{j_i+k}(\tilde{\omega})| \ge \delta_1$. By Lemma 4.2 a), which is stated and proved in Section 4 and which follows essentially from condition (I), we have good distortion estimates on $\tilde{\omega}$, i.e.

$$\left| \frac{x'_{j_i+k}(a)}{x'_{j_i+k}(a')} \right| \le C_3,$$

for $a, a' \in \tilde{\omega}$. Hence, combined with (14), we get

$$\begin{aligned} |\{a \in \tilde{\omega} \ ; \ x_{j_{i+1}}(a) \in B\}| &\le C_3 \frac{|x_{j_i+k}(\{a \in \tilde{\omega} \ ; \ x_{j_{i+1}}(a) \in B\})|}{|x_{j_i+k}(\tilde{\omega})|} |\tilde{\omega}| \\ &\le \frac{C_1 C_3}{\delta_1} |B| |\tilde{\omega}|. \end{aligned} \tag{15}$$

So, of each such 'large' $\tilde{\omega}$ only a fraction which is proportional to the length of B can be mapped by $x_{j_{i+1}}$ into B. Observe that the argument for deriving (15) also applies when $\tilde{\omega} \in \mathcal{P}_k|J$, $1 \le k \le j_1$, satisfying $|x_k(\tilde{\omega})| \ge \delta_1$, in which case we obtain

$$|\{a \in \tilde{\omega} \ ; \ x_{j_1}(a) \in B\}| \le \frac{C_1 C_3}{\delta_1} |B| |\tilde{\omega}|. \tag{16}$$

From (16) combined with (15), applied $h-1$ times, we can derive Proposition 3.1. In the remaining part of this subsection, we will work this out in detail.

Fix an integer $\tau \ge 1$ such that $C_0^2 \lambda^{-\tau} \le |B|$, and assume that n is so large that $\sqrt{n} \ge \tau$, which ensures that there are at least τ iterations between two consecutive j_i's. Let $\Omega_0 = J$ and

$$\Omega_i = \{\omega \in \mathcal{P}_{j_i+\tau}|\Omega_{i-1} \ ; \ x_{j_i}(\omega) \cap B \ne \emptyset\},$$

for $1 \le i \le h$. Notice that, by (I) and the assumption on τ, we have $|x_{j_i}(\omega)| \le |B|$ for all $\omega \in \mathcal{P}_{j_i+\tau}|J$ and, thus,

$$\Omega_i \subset \{a \in \Omega_{i-1} \ ; \ x_{j_i}(a) \in 3B\},$$

where $3B$ denotes the interval being three times as long as B and sharing the same midpoint. In each step we will exclude partition intervals with too short images. To this end we define, for $0 \le i \le h-1$, the following exceptional sets (let $j_0 = 0$):

$$\begin{aligned} E_i = \{\omega \in \mathcal{P}_{j_{i+1}}|\Omega_i \ ; \ \nexists\, \tilde{\omega} \in \mathcal{P}_{j_i+k}|\Omega_i, \ \tau \le k \le j_{i+1} - j_i, \\ \text{s.t. } \tilde{\omega} \supset \omega \text{ and } |x_{j_i+k}(\tilde{\omega})| \ge \delta_1\}. \end{aligned}$$

As the $\varepsilon > 0$ in condition (III) we take

$$\varepsilon = \frac{(C|B|)^h}{h},$$

where C is the constant $3C_1C_3/\delta_1$. By (III) we derive that there is an integer $n_{\varepsilon,\tau}$ growing at most polynomially in $1/\varepsilon$ such that for each $0 \le i \le h-1$, $|E_i| \le \varepsilon/(\sqrt{n} - \tau)^2$, for $n \ge n_{\varepsilon,\tau}$. (The need to introduce the integer τ in this subsection is the reason why we require in the formulation of condition (III) that $1 \le j \le 2n$ instead of $1 \le j \le n$.) If $n_{\varepsilon,\tau} \ge (4\tau)^2$, we get that $|E_i| \le 2\varepsilon/n$. τ is only dependent on $|B|$. By the definition of ε, it follows that $n_{h,B} = \max\{n_{\varepsilon,\tau}, (4\tau)^2\}$ grows at most exponentially in h. Disregarding finitely many points, $\Omega_i \setminus E_i$ can be seen as a set of disjoint and open intervals $\tilde{\omega}$ such that each $\tilde{\omega}$ is an element of a partition $\mathcal{P}_{j_i+k}|\Omega_i$, $\tau \le k \le j_{i+1} - j_i$, and $|x_{j_i+k}(\tilde{\omega})| \ge \delta_1$. By (15) and (16), we obtain

$$|\{a \in \tilde{\omega} \ ; \ x_{j_{i+1}}(a) \in 3B\}| \le C|B||\tilde{\omega}|,$$

which in turn implies that, for $n \ge n_{h,B}$,

$$|\Omega_{i+1}| \le C|B||\Omega_i \setminus E_i| + |E_i| \le C|B||\Omega_i| + \frac{2\varepsilon}{n}.$$

Hence, we have

$$|\Omega_h| \le (C|B|)^h |\Omega_0| + h\frac{2\varepsilon}{n} \le \frac{1}{n} 3(C|B|)^h,$$

where in the last inequality we used the definitions of Ω_0 and ε. Since

$$\{a \in J \ ; \ x_{j_1}(a) \in B, ..., x_{j_h}(a) \in B\} \subset \Omega_h,$$

this implies (13).

4. Condition (II)

As already mentioned in Subsection 2.3, in the examples considered in this paper, we will not verify condition (II) directly. Instead we will verify two other conditions, conditions (IIa) and (IIb) described below. We will show in this section that conditions (IIa) and (IIb) imply condition (II). In fact, conditions (IIa) and (IIb) also imply condition (III), see next section.

4.1. **Conditions (IIa) and (IIb).** Recall that for inequality (5) we assumed that the density φ_a is bounded from below and from above. We require that this holds for each density φ_a, $a \in I$, and with a constant independent on a.

(IIa) There is a constant $C_2 \ge 1$ such that for each density φ_a, $a \in I$, we have

$$\frac{1}{C_2} \le \varphi_a(x) \le C_2,$$

for a.e. $x \in K(a)$.

Even if condition (IIa) appears to be a natural requirement on a one-parameter family of piecewise expanding maps, it will take us some effort to verify the lower bound for the examples in Sections 6 and 7. The upper bound follows almost immediately from [17] and [8] (see the proof of Lemma A.1).

We turn to condition (IIb). Let a_1 and a_2 be two arbitrary parameter values in I such that $a_1 < a_2$ and fix an integer $j \ge 1$. We require that, for a.e. $y \in K(a_1)$ the following holds. To each point $x \in K(a_1)$ satisfying $T_{a_1}^j(x) = y$ there is an associated point $x' \in K(a_2)$ satisfying $T_{a_2}^j(x') = y$ and having the same combinatorics as x, i.e. $\text{symb}_{a_2}(T_{a_2}^i(x')) = \text{symb}_{a_1}(T_{a_1}^i(x))$, $0 \le i < j$. In other words, we require that the combinatorics of T_{a_1} should be 'contained' in the combinatorics of T_{a_2} and, furthermore, if $\omega \in \mathcal{P}_j(a_1)$ and $\omega' \in \mathcal{P}_j(a_2)$ have the same combinatorics, then the image by $T_{a_1}^j$ of ω should be contained in the image by $T_{a_2}^j$ of ω'. Condition (IIb) is rather restrictive, see Remark 4.1.

(IIb) For all $a_1, a_2 \in I$, $a_1 \le a_2$, and $j \ge 1$ there is a mapping

$$\mathcal{U}_{a_1,a_2,j} : \mathcal{P}_j(a_1) \to \mathcal{P}_j(a_2),$$

such that, for all $\omega \in \mathcal{P}_j(a_1)$,

$$\text{symb}_{a_1}(T_{a_1}^i(\omega)) = \text{symb}_{a_2}(T_{a_2}^i(\mathcal{U}_{a_1,a_2,j}(\omega))), \quad 0 \le i < j, \tag{17}$$

and, in particular,

$$T_{a_1}^j(\omega) \subset T_{a_2}^j(\mathcal{U}_{a_1,a_2,j}(\omega)). \tag{18}$$

Remark 4.1. A simple example of a one-parameter family not satisfying (IIb) are the maps $T_a(x) = \beta x + \alpha \bmod 1$, $\beta > 1$ and $\alpha \in (0, 1)$, where the parameter a can be chosen to be either β or α. These maps are studied, e.g., in [7] and [6]. However, using the special property that for every fixed map T_a the derivative of T_a is constantly equal to β, it is possible to verify condition (II) directly (at least for large β). The main ingredient in verifying condition (II) for this one-parameter family is that one can show that the number of elements in the partitions $\mathcal{P}_j(a)$, $j \geq 1$, are bounded above by a constant C times β^j where the constant C is independent on the parameter value a. This property, that one, roughly speaking, can switch from considering the maps T_a to counting the number of partition elements, is also used in [15]. To keep this paper in a reasonable size we will not investigate this family T_a. For the parameter choice $a = \beta$, a.s. typicality in the case when $X(a) \equiv x \in [0, 1]$ is shown in [6].

4.2. **Conditions (I), (IIa), and (IIb) imply condition (II).** We prove first a distortion lemma. Let T_a, $a \in I$, be a one-parameter family as described in Subsection 2.1, satisfying properties (i)-(iii), and let $X : I \to [0, 1]$ be a to this family associated C^1 map. Let $J \subset I$ be an interval of length $1/n$. If condition (I) is satisfied, then, for large n, the length of J is huge compared to the length of an element $\omega \in \mathcal{P}_n|J$, which, by (I), can be estimated from above as $|\omega| \leq C_0/\lambda^n$. Nevertheless, as part b) of the following lemma asserts, the interval J is small enough to have good distortion estimates which will enable us to compare the map $x_j : J \to [0, 1]$ with the map $T_{a_J}^j : [0, 1] \to [0, 1]$ where a_J is the right boundary point of J.

Lemma 4.2. *There exists a constant $C_3 \geq 1$ such that the following holds.*

a) *If the one-parameter family T_a, $a \in I$, with the associated map X satisfies condition (I), then for $\omega \in \mathcal{P}_j|I$, $j \geq 1$,*

$$\frac{1}{C_3} \leq \left| \frac{x_j'(a)}{x_j'(a')} \right| \leq C_3,$$

for all $a, a' \in \omega$.

b) *If the one-parameter family T_a, $a \in I$, satisfies condition (IIb), then we have the following distortion estimate. Let $n \geq 1$ and $a_1, a_2 \in I$ such that $a_1 \leq a_2$ and $a_2 - a_1 \leq 1/n$. For $\omega \in \mathcal{P}_j(a_1)$, $1 \leq j \leq 2n$, we have*

$$\frac{1}{C_3} \leq \left| \frac{T_{a_1}^j{}'(x)}{T_{a_2}^j{}'(x')} \right| \leq C_3,$$

for all $x \in \omega$ and $x' \in \mathcal{U}_{a_1,a_2,j}(\omega)$.

Remark 4.3. If $a_1 = a_2$ in Lemma 4.2 b), then we get a well-known distortion estimate for piecewise expanding $C^{1,1}(L)$ maps.

Proof. We proof first part b), which is the more difficult part. Fix $\tau \geq 1$ such that

$$\max\{4L/\tau, L\lambda/(\lambda - 1)\tau\} \ll \delta_0.$$

The constant C_3 in Lemma 4.2 b) will be greater than $(\Lambda/\lambda)^\tau$. Hence, for $2n \leq \tau$, the distortion estimate in b) is trivially satisfied and we can assume that $\tau < j \leq 2n$. Observe that, by (IIb), the set $T_{a_1}^j(\omega)$ is contained in the set $T_{a_2}^j(\mathcal{U}_{a_1,a_2,j}(\omega))$. Fix a point y in $T_{a_1}^j(\omega)$ and let, for $1 \leq i \leq j$,

$$r_i \in T_{a_1}^{j-i}(\omega), \quad s_i \in T_{a_2}^{j-i}(\mathcal{U}_{a_1,a_2,j}(\omega)),$$

be the pre-images of y, i.e. $T_{a_1}^i(r_i) = T_{a_2}^i(s_i) = y$. Note that, by (IIb), we have $\text{symb}_{a_1}(r_i) = \text{symb}_{a_2}(s_i)$. Let $k_i = \text{symb}_{a_1}(r_i)$.

Claim. *The distance between r_i and s_i, $1 \leq i \leq j$, satisfies*

$$|r_i - s_i| \leq \frac{L\lambda}{\lambda - 1}\frac{1}{n}. \tag{19}$$

Proof. In order to show (19), we will show

$$|r_i - s_i| \leq \frac{L}{n}\sum_{l=0}^{i-1}\frac{1}{\lambda^l}, \tag{20}$$

for $1 \leq i \leq j$. Since y has for both parameters a_1 and a_2 a pre-image $r_1 = T_{a_1}^{-1}(y)$ and $s_1 = T_{a_2}^{-1}(y)$, which lies in D_{k_1}, it follows, by (iii) in Subsection 2.1, that y has a pre-image in $D_{k_1}(a)$ for all parameter values a in the interval $[a_1, a_2]$ and, furthermore, the corresponding map $a \mapsto T_a^{-1}$ is Lip(L). Hence, we have

$$|r_1 - s_1| = |T_{a_1}^{-1}(y) - T_{a_2}^{-1}(y)| \leq L(a_2 - a_1) \leq \frac{L}{n}. \tag{21}$$

Assume now that we have shown (20) for some $1 \leq i < j$. Since $a_2 - a_1 \leq 1/n$ it follows, by (i), that the length of the intersection of $D_{k_{i+1}}(a_1)$ and $D_{k_{i+1}}(a_2)$ is at least $\delta_0 - 2L/n$. If z lies in this intersection, then, by (ii), the map $a \mapsto T_a(z)$ is Lip(L) on the interval $[a_1, a_2]$. Thus, the length of the intersection of $T_{a_1}(D_{k_{i+1}}(a_1))$ and $T_{a_2}(D_{k_{i+1}}(a_2))$ is at least $\delta_0 - 4L/n$. Since, by the assumption on τ, $\delta_0 - 4L/n \approx \delta_0$ and $|r_i - s_i| \leq L\lambda/(\lambda - 1)n \ll \delta_0$, we deduce that at least one of the following two situations occurs:

- The branch of $T_{a_1}^{-1}$ which maps r_i to $D_{k_{i+1}}(a_1)$ is defined on the whole interval $[r_i, s_i]$.
- The branch of $T_{a_2}^{-1}$ which maps s_i to $D_{k_{i+1}}(a_2)$ is defined on the whole interval $[r_i, s_i]$.

Assuming the first situation occurs, we obtain, by (3),

$$|T_{a_1}^{-1}(r_i) - T_{a_1}^{-1}(s_i)| \leq \frac{1}{\lambda}|r_i - s_i|,$$

and, as in (21), we derive that

$$|T_{a_1}^{-1}(s_i) - T_{a_2}^{-1}(s_i)| \leq \frac{L}{n}.$$

It follows that

$$|r_{i+1} - s_{i+1}| = |T_{a_1}^{-1}(r_i) - T_{a_2}^{-1}(s_i)| \leq \frac{1}{\lambda}|r_i - s_i| + \frac{L}{n} \leq \frac{L}{n}\sum_{k=0}^{i}\frac{1}{\lambda^k}.$$

We can do a similar calculation when the second situation occurs, which concludes the proof. □

By a similar reasoning as in the proof of (19), we note that at least one of the following two situations occurs:

- $[r_i, s_i] \subset D_{k_i}(a_1)$.
- $[r_i, s_i] \subset D_{k_i}(a_2)$.

If the first situation occurs, we have, by (ii), that the map $a \mapsto T'_a(s_i)$ is Lip(L) on the interval $[a_1, a_2]$. Combined with (19) and since $x \mapsto T_{a_1}(x)$ is $C^{1,1}(L)$ on $[r_i, s_i]$, we obtain

$$\begin{aligned}|T'_{a_1}(r_i)| &\le |T'_{a_1}(s_i)| + L|r_i - s_i| \le |T'_{a_2}(s_i)| + L(a_2 - a_1) + L|r_i - s_i| \\ &\le |T'_{a_2}(s_i)| + \frac{2L\lambda}{\lambda - 1}\frac{1}{n}.\end{aligned}$$

If the second situation occurs, it follows in a similar way that

$$|T'_{a_2}(s_i)| \ge |T'_{a_1}(r_i)| - \frac{2L\lambda}{\lambda - 1}\frac{1}{n}.$$

For $\tau < i \le j$, let $t_i = r_i$ and $\alpha_i = a_1$ if the first situation occurs and $t_i = s_i$ and $\alpha_i = a_2$ otherwise. Altogether, we obtain

$$\begin{aligned}\left|\frac{T^j_{a_1}{}'(x)}{T^j_{a_2}{}'(x')}\right| &\le \left(\frac{\Lambda}{\lambda}\right)^{\tau} \prod_{i=\tau+1}^{j} \frac{|T'_{a_1}(T^{j-i}_{a_1}(x))|}{|T'_{a_2}(T^{j-i}_{a_2}(x'))|} \\ &\le \left(\frac{\Lambda}{\lambda}\right)^{\tau} \prod_{i=\tau+1}^{j} \frac{|T'_{a_1}(r_i)| + L|T^{j-i}_{a_1}(\omega)|}{|T'_{a_2}(s_i)| - L|T^{j-i}_{a_2}(\mathcal{U}_{a_1,a_2,j}(\omega))|} \\ &\le \left(\frac{\Lambda}{\lambda}\right)^{\tau} \prod_{i=\tau+1}^{j} \frac{|T'_{\alpha_i}(t_i)| + 2L\lambda(\lambda-1)^{-1}n^{-1} + L\lambda^{-i}}{|T'_{\alpha_i}(t_i)| - 2L\lambda(\lambda-1)^{-1}n^{-1} - L\lambda^{-i}}. \qquad (22)\end{aligned}$$

(To ensure that the denominators are positive, we should also assume that τ was chosen so large that $2L\lambda(\lambda-1)^{-1}\tau^{-1} + L\lambda^{-\tau} < \lambda$.) Since $j \le 2n$, the product in the last term of inequality (22) is clearly bounded above by a constant independent on n. Hence, this shows the upper bound in the distortion estimate in part b). The lower bound is shown in the same way. This concludes the proof of part b).

The proof of part a) is similar but easier than the proof of part b). We will give only a sketch of the proof. Let $\omega \in \mathcal{P}_j|I$, $j \ge 1$, and $a, a' \in \omega$. By condition (I), we have

$$\left|\frac{x'_j(a)}{x'_j(a')}\right| \le C_0^2 \prod_{i=0}^{j-1} \left|\frac{T'_a(x_i(a))}{T'_{a'}(x_i(a'))}\right|.$$

The distance between $x_i(a)$ and $x_i(a')$, $1 \le i \le j-1$, satisfies, by (I),

$$|x_i(a) - x_i(a')| \le |x_i(\omega)| \le C_0^2 \lambda^{-(j-i)}.$$

This inequality is a counterpart to inequality (19), which is the part in b) where we used condition (IIb). Similarly as in the proof of part b) we derive

$$\left|\frac{x'_j(a)}{x'_j(a')}\right| \le C_0^2 \left(\frac{\Lambda}{\lambda}\right)^{\tau} \prod_{i=1}^{j-\tau} \frac{|T'_{\alpha_i}(t_i)| + 2LC_0^2\lambda^{-(j-i)}}{|T'_{\alpha_i}(t_i)| - 2LC_0^2\lambda^{-(j-i)}},$$

where either $\alpha_i = a$ and $t_i = x_i(a)$ or $\alpha_i = a'$ and $t_i = x_i(a')$, and τ is chosen so large that $2LC_0^2\lambda^{-\tau} < \lambda$. The product in this inequality is clearly bounded above by a constant independent on $j \ge 1$. This concludes the proof of Lemma 4.2. □

To prove Lemma 4.2 b), instead of property (18) in condition (IIb), it would be sufficient to assume that $\operatorname{dist}(T^j_{a_1}(\omega), T^j_{a_2}(\mathcal{U}_{a_1,a_2,j}(\omega))) \le 1/n$. However, to establish inequality (6) in condition (II), property (18) is essential.

Lemma 4.4. *Under the assumption that conditions (I) and (IIb) are satisfied, there is an integer $q \ge 1$ such that the following holds. Let $J \subset I$ be an open interval of length*

$1/n$ such that the right boundary point a_J of J is contained in I, and let $i \geq 0$. For each element $\tilde{\omega} \in \mathcal{P}_i|J$ and integer $1 \leq j \leq 2n$, there is an at most q-to-one map

$$\mathcal{U}_{\tilde{\omega},a_J,j} : \mathcal{P}_{i+j}|\tilde{\omega} \to \mathcal{P}_j(a_J),$$

such that, for $\omega \in \mathcal{P}_{i+j}|\tilde{\omega}$, the image of ω is contained in the image of $\mathcal{U}_{\tilde{\omega},a_J,j}(\omega)$, i.e.

$$x_{i+j}(\omega) \subset T_{a_J}^j(\mathcal{U}_{\tilde{\omega},a_J,j}(\omega)), \tag{23}$$

and we have the following distortion control:

$$\frac{1}{|T_a^j{}'(x_i(a))|} \leq C_3 \frac{1}{|T_{a_J}^j{}'(x)|}, \tag{24}$$

for all $a \in \omega$ and $x \in \mathcal{U}_{\tilde{\omega},a_J,j}(\omega)$.

Proof. We define the map

$$\mathcal{U}_{\tilde{\omega},a_J,j} : \mathcal{P}_{i+j}|\tilde{\omega} \to \mathcal{P}_j(a_J)$$

as follows. Let $\omega \in \mathcal{P}_{i+j}|\tilde{\omega}$ and $a \in \omega$. By the definition of the partitions associated to the parameter interval, $x_{i+l}(a) \notin \{c_0(a), ..., c_{p_1}(a)\}$, for all $0 \leq l < j$. Hence, there exists an element $\omega(x_i(a))$ in the partition $\mathcal{P}_j(a)$ containing the point $x_i(a)$. We set

$$\mathcal{U}_{\tilde{\omega},a_J,j}(\omega) = \mathcal{U}_{a,a_J,j}(\omega(x_i(a))),$$

where $\mathcal{U}_{a,a_J,j} : \mathcal{P}_j(a) \to \mathcal{P}_j(a_J)$ is the map given by (IIb). Note that the element $\omega' = \mathcal{U}_{a,a_J,j}(\omega(x_i(a)))$ has the same combinatorics as ω, i.e.

$$\mathrm{symb}_{a_J}(T_{a_J}^l(\omega')) = \mathrm{symb}(x_{i+l}(\omega)),$$

$0 \leq l < j$. Since there cannot be two elements in $\mathcal{P}_j(a_J)$ with the same combinatorics, the element ω' is independent on the choice of $a \in \omega$. It follows that the map $\mathcal{U}_{\tilde{\omega},a_J,j}$ is well-defined. By property (18) in condition (IIb), we have $T_a^j(\omega(x_i(a))) \subset T_{a_J}^j(\mathcal{U}_{\tilde{\omega},a_J,j}(\omega))$, for all $a \in \omega$. This implies (23). Since $j \leq 2n$ and $a_J - a \leq 1/n$, for $a \in \omega$, inequality (24) follows immediately from the distortion estimate in Lemma 4.2 b). In order to conclude the proof of Lemma 4.4, it is only left to show that the map $\mathcal{U}_{\tilde{\omega},a_J,j}$ is at most q-to-one for some integer $q \geq 1$. Let $i_0 = i_0(C_0, \lambda) \geq 0$ be so large that $|x_i'(a)| \geq L$ for all $i \geq i_0$ and parameter values $a \in I$ for which the derivative is defined (L is the Lipschitz constant introduced in Subsection 2.1). If $i \geq i_0$, using that the partition points $c_0(a), ..., c_{p_1}(a)$ are Lip(L), it is easy to show that the map $\mathcal{U}_{\tilde{\omega},a_J,j} : \mathcal{P}_{i+j}|\tilde{\omega} \to \mathcal{P}_j(a_J)$ is one-to-one. For $0 \leq i \leq i_0$, recall that, by condition (I), the partition $\mathcal{P}_i|I$ consists of only finitely many elements. Hence, setting $q = \#\{\omega \in \mathcal{P}_{i_0}|I\}$ we derive that the map $\mathcal{U}_{\tilde{\omega},a_J,j} : \mathcal{P}_{i+j}|\tilde{\omega} \to \mathcal{P}_j(a_J)$ is at most q-to-one. □

Using Lemma 4.2 and Lemma 4.4, we can easily deduce the main statement of this section.

Proposition 4.5. *If the one-parameter family T_a, $a \in I$, with the associated map X satisfies conditions (I), (IIa), and (IIb), then it satisfies condition (II).*

Proof. Let $J \subset I$ be an open interval of length $1/n$. We assume first that the right endpoint a_J of J lies in I. As in condition (II), let $\tilde{\omega} \in \mathcal{P}_i|J$, $i \geq 1$, and $1 \leq j \leq n$. Observe that, by condition (I), we have $|x_i'(a)|/|x_{i+j}'(a)| \leq C_0^2|T_a^j{}'(x_i(a))|$. Let $\mathcal{U}_{\tilde{\omega},a_J,j} : \mathcal{P}_{i+j}|\tilde{\omega} \to \mathcal{P}_j(a_J)$ be the map provided by Lemma 4.4. By inequality (23), for each $\omega \in \mathcal{P}_{i+j}|\tilde{\omega}$, whenever there is a parameter value $a \in \omega$ such that $x_{i+j}(a) = y$, then there is also a point $x \in \mathcal{U}_{\tilde{\omega},a_J,j}(\omega)$ satisfying $T^j_{a_J}(x) = y$. Combined with the distortion estimate (24), we obtain

$$\sum_{\substack{a\in\tilde{\omega}\\ x_{i+j}(a)=y}} \frac{|x_i'(a)|}{|x_{i+j}'(a)|} \le C_0^2 \sum_{\substack{a\in\tilde{\omega}\\ x_{i+j}(a)=y}} \frac{1}{|T_a^j{}'(x_i(a))|} \le qC_0^2C_3 \sum_{\substack{x\in K(a_J)\\ T^j_{a_J}(x)=y}} \frac{1}{|T^j_{a_J}{}'(x)|},$$

for all but finitely many $y \in [0,1]$ (we exclude points y for which there exists a parameter value $a \in \tilde{\omega}$ which is not contained in any element of $\mathcal{P}_{i+j}|\tilde{\omega}$ and such that $x_{i+j}(a) = y$; by condition (I), the number of such points y is finite). Recall that condition (IIa) implies inequality (5) with the upper bound C_2^2. By applying (5) to the right hand side of the inequality above, we obtain condition (II) with the constant $C_1 = qC_0^2C_2^2C_3$. Since C_1 does not depend on a_J, we can drop the assumption, that the right boundary point of J is contained in I. This concludes the proof. □

5. Condition (III)

As already pointed out in the introduction, even if conditions (IIa) and (IIb) are in particular designed for verifying condition (II), it turns out that if conditions (I), (IIa), and (IIb) are satisfied, then also condition (III) is satisfied — at least on smaller intervals.

Proposition 5.1. *If the one-parameter family T_a, $a \in I$, with the associated map X satisfies conditions (I), (IIa), and (IIb), then, disregarding a finite number of parameter values in I, we can cover I by countably many intervals such that on each such interval condition (III) is satisfied.*

Proof. Fix an integer τ so large that $2^{1/\tau} \le \sqrt{\lambda}$. Observe that, if we set for a fixed $a \in I$,

$$\delta = \min\{|\omega| \ ; \ \omega \in \mathcal{P}_\tau(a)\},$$

then the following is trivially satisfied: For all $\omega \in \mathcal{P}_t(a)$, where $1 \le t \le \tau$, we have

$$|T_a^t(\omega)| \ge \delta.$$

We want that a similar property holds for the partition elements of the parameter space.

Claim. *Disregarding a finite number of parameter values in I, we can cover I by a countable number of intervals $\tilde{I} \subset I$ such that for each interval $\tilde{I}$ there exists a constant $\delta_1 = \delta_1(\tilde{I}) > 0$ such that the following holds. Let $j \ge 1$ and $1 \le t \le \tau$. If $\omega \in \mathcal{P}_{j+t}|\omega'$ for some $\omega' \in \mathcal{P}_j|\tilde{I}$ and if ω is not adjacent to a boundary point of ω', then we have*

$$|x_{j+t}(\omega)| \ge \delta_1. \tag{25}$$

Proof. Even if the proof is rather straightforward, we have to be a bit careful. For $a \in I$, let

$$\kappa(a) = \min\{|\omega| \ ; \ \omega \in \mathcal{P}_\tau(a)\} > 0.$$

Since condition (IIb) holds, we can argue as in the proof of Lemma A.1 (see inequality (48)) and, disregarding a finite number of parameter values in I, we can cover I by a countable number of intervals $\tilde{I} \subset I$ such that for each such interval $\tilde{I}$ there is a constant $\kappa_0 = \kappa_0(\tilde{I}) > 0$ such that

$$\kappa(a) \ge \kappa_0, \tag{26}$$

for all $a \in \tilde{I}$. Fix such a parameter interval $\tilde{I}$ and let $a \in \tilde{I}$. Recall that in Subsection 2.1 we defined $D_k(a)$, $1 \le k \le p_2$, to be the elements of $\mathcal{P}_1(a)$. Let $\mathcal{C}(\tilde{I})$ denote the following set of functions:

$$\mathcal{C}(\tilde{I}) = \{b : \tilde{I} \to [0,1] \ ; \ b(a) = \lim_{\substack{x \to c(a) \\ x \in D_k(a)}} T_a^t(x), \ 1 \le k \le p_2, \ 1 \le t \le \tau,$$

$$\text{and } c(a) \in \partial D_k(a) \text{ (s.t. } c \in C^0(\tilde{I}))\}.$$

By the first claim in the proof of Lemma A.1, the functions in $\mathcal{C}(\tilde{I})$ are continuous. Let $\omega \in \mathcal{P}_t(a)$, $1 \le t \le \tau$. Observe that the image of ω by T_a^t is of the form

$$T_a^t(\omega) = (b_1(a), b_2(a)),$$

for some functions $b_1, b_2 \in \mathcal{C}(\tilde{I})$ (if b_1 and b_2 are not uniquely defined, we choose them such that (27) below holds). By the continuity of the functions in $\mathcal{C}(\tilde{I})$ and by (26), we derive that for each parameter value $a_* \in \tilde{I}$ there exists a to ω associated element $\omega_* \in \mathcal{P}_t(a_*)$ such that

$$T_{a_*}^t(\omega_*) = (b_1(a_*), b_2(a_*)), \quad \text{and} \quad \text{symb}_{a_*}(T_{a_*}^i(\omega_*)) = \text{symb}_a(T_a^i(\omega)), \tag{27}$$

for $0 \le i < t$. By condition (IIb), it follows further that

$$(b_1(a_*), b_2(a_*)) \subset (b_1(a), b_2(a)), \tag{28}$$

for all $a_*, a \in \tilde{I}$ such that $a_* \le a$. Observe that from (26) it follows also that for each function $b \in \mathcal{C}(\tilde{I})$ there exists $1 \le k \le p_2$ such that

$$b(a) \in \text{closure}\{D_k(a)\}, \tag{29}$$

for all $a \in \tilde{I}$.

We turn to the partitions on the parameter space. Let $j_0 = j_0(C_0, \lambda) \ge 1$ be so large that $|x_j'(a)| \ge L$, for all $j \ge j_0$ and parameter values $a \in I$ for which the derivative is defined (L is the Lipschitz constant from Subsection 2.1). We consider first the case when $j \ge j_0$. Let $\omega' \in \mathcal{P}_j|\tilde{I}$, fix an integer $1 \le t \le \tau$, and denote by a_* the left boundary point of $\tilde{I}$. We will construct a map $\mathcal{U} : \mathcal{P}_{j+t}|\omega' \to \cup_{1 \le s \le t} \mathcal{P}_s(a_*)$ such that for each $\omega \in \mathcal{P}_{j+t}|\omega'$ not adjacent to a boundary point of ω' we have

$$T_{a_*}^s(\mathcal{U}(\omega)) \subset x_{j+t}(\omega), \quad \text{for some } 1 \le s \le t. \tag{30}$$

(Observe that for the construction of the map in Lemma 4.4 we consider instead of the left the right boundary point of the parameter interval and the inclusion is in the other direction. The construction regarding (30) is a bit more cumbersome.) Having constructed such a map $\mathcal{U}$, since the sizes of the elements in $\mathcal{P}_s(a_*)$, $1 \le s \le t$, are bounded from below by κ_0, by setting $\delta_1 = \kappa_0$, this immediately implies the assertion of the claim for the case $j \ge j_0$. Fix $\omega \in \mathcal{P}_{j+t}|\omega'$ not adjacent to a boundary point of ω'. Let

$$t_0 = \min\{s \ge 1 \ ; \ \exists\, \tilde{\omega} \in \mathcal{P}_{j+s}|\omega' \text{ s.t. } \omega \subset \tilde{\omega}, \ \partial\tilde{\omega} \cap \partial\omega' = \emptyset \text{ and } \partial\tilde{\omega} \cap \partial\omega \ne \emptyset\},$$

and, for $i \ge 0$, if $\omega \notin \mathcal{P}_{j+t_i}|\omega'$, then let

$$t_{i+1} = \min\{s > t_i \ ; \ \exists\, \tilde{\omega} \in \mathcal{P}_{j+s}|\omega' \text{ s.t. } \omega \subset \tilde{\omega} \text{ and } \tilde{\omega} \notin \mathcal{P}_{j+t_i}|\omega'\};$$

and otherwise, we do not define t_{i+1}. Let $q \le t$ be the maximal integer for which t_q is defined and set $t_{q+1} = t + 1$. For $0 \le i \le q$, let ω_i be the element in $\mathcal{P}_{j+t_i}|\omega'$ containing ω. Observe that ω_i is also an element of $\mathcal{P}_{j+t_{i+1}-1}|\omega'$. In particular, we have $\omega_q = \omega$. We will show that for each ω_i there is an element $\omega_i^* \in \mathcal{P}_{t_i - t_0 + 1}(a_*)$ such that ω_i^* is also an element of $\mathcal{P}_{t_{i+1} - t_0}(a_*)$ and if $b_1, b_2 \in \mathcal{C}(\tilde{I})$ are the functions satisfying

$$T_{a_*}^{t_{i+1} - t_0}(\omega_i^*) = (b_1(a_*), b_2(a_*)), \tag{31}$$

then they satisfy also

(32) $$x_{j+t_{i+1}-1}(\omega_i) = (b_1(a_L), b_2(a_R)) \quad \text{or} \quad x_{j+t_{i+1}-1}(\omega_i) = (b_1(a_R), b_2(a_L)),$$

where a_L and a_R are the boundary points of ω_i. Combined with (28), this immediately implies (30) by setting $\mathcal{U}(\omega) = \omega_q^*$. Recall that by $c_0(a) < c_1(a) < ... < c_{p_1}(a)$ we denote the boundary points of the $D_k(a)$, $1 \leq k \leq p_2$, and these boundary points are Lipschitz in a with Lipschitz constant L. Since $j \geq j_0$ and since ω_0 is not adjacent to a boundary point of ω', we have that the image of ω_0 by x_{j+t_0-1} is of the form

$$x_{j+t_0-1}(\omega_0) = (c_k(a_L), c_{k+1}(a_R)) \quad \text{or} \quad x_{j+t_0-1}(\omega_0) = (c_k(a_R), c_{k+1}(a_L))$$

where a_L and a_R are the boundary points of ω_0 and $0 \leq k < p_1$ (the assumption $j \geq j_0$ we used to avoid that the image is, e.g., of the form $(c_k(a_L), c_k(a_R))$). Now, let $\omega_0^* \in \mathcal{P}_1(a_*)$ be the element of the form $(c_k(a_*), c_{k+1}(a_*))$. By (26) and (29), we derive immediately that ω_0^* is also an element of $\mathcal{P}_{t_1-t_0}(a_*)$ and properties (31) and (32) are satisfied for w_0 and w_0^*. If $q = 0$ then we are done. Otherwise, assume that we have shown (31) and (32) for $0 \leq i < q$. Let a_L and a_R denote the boundary points of ω_{i+1}. We have that one boundary point of $x_{j+t_{i+1}-1}(\omega_{i+1})$ is equal to $c_k(a_L)$ or $c_k(a_R)$ for some $0 < k < p_1$, and the other coincides with a boundary point of $x_{j+t_{i+1}-1}(\omega_i)$ (this latter fact follows since, by the definition of t_0, ω_0 and ω have a common boundary point), i.e. $x_{j+t_{i+1}-1}(\omega_{i+1})$ is of the form $(b(a_L), c_k(a_R))$ or $(b(a_R), c_k(a_L))$ where $b \in \mathcal{C}(\tilde{I})$ (if $c < b$ then we mean by (b, c) the interval (c, b)). Since we assumed that (31) and (32) are satisfied for the element ω_i with an associated element $\omega_i^* \in \mathcal{P}_{t_i-t_0+1}(a_*)$, we can apply (26) and (29) and we deduce that there is an element $\omega_{i+1}^* \in \mathcal{P}_{t_{i+1}-t_0+1}(a_*)$ such that $T_{a_*}^{t_{i+1}-t_0}(\omega_{i+1}^*) = (b(a_*), c_k(a_*))$. Applying (26) and (29) once more, we get that ω_{i+1}^* is also an element of $\mathcal{P}_{t_{i+2}-t_0}(a_*)$ and, furthermore, we deduce that properties (31) and (32) are satisfied for the elements ω_{i+1} and ω_{i+1}^*. This concludes the proof of the claim in the case when $j \geq j_0$. Observe that by condition (I) there are only finitely many elements in a partition $\mathcal{P}_j|\tilde{I}$. Hence, by setting the constant $\delta_1 = \delta_1(\tilde{I})$ equal to

$$\delta_1 = \min\{\kappa_0, \min\{|\omega| \ ; \ \omega \in \mathcal{P}_j|\tilde{I}, \ 1 \leq j \leq j_0\}\},$$

this concludes the proof of the claim. □

In the following, we restrict our considerations to an interval $\tilde{I} \subset I$ with an associated constant $\delta_1 > 0$, as described in the claim above, and verify condition (III) on this interval $\tilde{I}$. As in condition (III), let $J \subset \tilde{I}$ be an open interval of length $1/n$, where we assume $n \gg 1$, and fix an integer $1 \leq j \leq 2n$. For each $\omega' \in \mathcal{P}_j|J$, we define the set

$$E_{\omega'} = \{\omega \in \mathcal{P}_{j+[\sqrt{n}]}|\omega' \ ; \ \nexists \ \tilde{\omega} \in \mathcal{P}_{j+k}|\omega', \quad 0 \leq k \leq [\sqrt{n}], \ \text{s.t. } \tilde{\omega} \supset \omega \text{ and } |x_{j+k}(\tilde{\omega})| \geq \delta_1\}.$$

If $1 \leq t \leq \tau$, we derive from the claim above that

$$\#\{\omega \in \mathcal{P}_{j+t}|\omega' \ ; \ |x_{j+t}(\omega)| < \delta_1\} \leq 2.$$

In other words only the element(s) in $\mathcal{P}_{j+t}|\omega'$ being adjacent to a boundary point of ω' can have a small image. By a repeated use of this fact we derive

$$\#\{\omega \in \mathcal{P}_{j+[\sqrt{n}]}|E_{\omega'}\} \leq 2 \cdot 2^{[\sqrt{n}]/\tau} \leq 2\sqrt{\lambda}^{[\sqrt{n}]},$$

where in the last inequality we used the definition of τ. Applying condition (I), it follows that

$$|x_j(E_{\omega'})| \leq C_0^2 \frac{\#\{\omega \in \mathcal{P}_{j+[\sqrt{n}]}|E_{\omega'}\}}{\lambda^{[\sqrt{n}]}} \leq \frac{2C_0^2}{\sqrt{\lambda}^{[\sqrt{n}]}} =: \gamma_n.$$

The exceptional set E in condition (III) is given by

$$E = \bigcup_{\omega' \in \mathcal{P}_j | J} E_{\omega'}.$$

Let a_J denote the right boundary point of J. Without loss of generality we can assume that $a_J \in I$. Set

$$\mathcal{C}_j := \{ b \ ; \ b \in \partial T^i_{a_J}(\omega), \ 1 \le i \le j, \ \omega \in \mathcal{P}_j(a_J) \}.$$

By (I), we obtain

$$|E| = \sum_{\omega' \in \mathcal{P}_j|J} \int_{x_j(E_{\omega'})} \frac{1}{|x_j'(a_y)|} dy \le \sum_{\omega' \in \mathcal{P}_j|J} C_0 \int_{x_j(E_{\omega'})} \frac{1}{|T^j_{a_y}{}'(X(a_y))|} dy,$$

where $a_y = (x_j|_{\omega'})^{-1}(y)$. Since conditions (I) and (IIb) are satisfied, we can apply Lemma 4.4 in the case where $i = 0$ and get

$$\begin{aligned} |E| &\le \sum_{\omega \in \{\mathcal{U}_{J,a_J,j}(\omega') \ ; \ \omega' \in \mathcal{P}_j|J\}} C_0 C_3 \int_{\Gamma(\omega)} \frac{1}{|T^j_{a_J}{}'(x_y)|} dy \\ &\le q \sum_{\omega \in \mathcal{P}_j(a_J)} C_0 C_3 \int_{\Gamma(\omega)} \frac{1}{|T^j_{a_J}{}'(x_y)|} dy, \end{aligned}$$

where $x_y = (T^j_{a_J}|_\omega)^{-1}(y)$, and

$$\Gamma(\omega) = [b_\omega, b_\omega + |x_j(E_{\omega'})|],$$

where $b_\omega \in \mathcal{C}_j$ denotes the left boundary point of $T^j_{a_J}(\omega)$. Recall that $|x_j(E_{\omega'})| \le \gamma_n$. Finally, we can move the sum over the partition elements inside the integral and we derive that

$$|E| \le q C_0 C_3 \sum_{b \in \mathcal{C}_j} \int_{[b, b+\gamma_n]} \sum_{\substack{x \in K(a_J) \\ T^j_{a_J}(x) = y}} \frac{1}{|T^j_{a_J}{}'(x)|} dy.$$

Since condition (IIa) is satisfied, we can apply inequality (5), and we get that the sum inside the integral above is bounded by the constant C_2^2. Recall that p_2 is the number of (maximal) smooth monotonicity domains for $T_{a_J}|_{K(a_J)}$, and observe that for each $b \in \mathcal{C}_j$ there is such a monotonicity domain $D \in \mathcal{P}_1(a_J)$ and a partition point $c \in \partial D$ such that

$$b = \lim_{\substack{x \to c \\ x \in D}} T^i_{a_J}(x),$$

for some $1 \le i \le j$. Thus, since $j \le 2n$, we have

$$|\mathcal{C}_j| \le |\mathcal{C}_{2n}| \le 2n \cdot 2p_2.$$

Finally, for each $\varepsilon > 0$, we deduce that

$$|E| \le 4 p_2 q C_0 C_2^2 C_3 n \gamma_n \le \frac{\varepsilon}{n},$$

for $n \ge n_\varepsilon$, where n_ε can in fact be taken to grow less than polynomially in $1/\varepsilon$. This concludes the verification of condition (III) on the interval $\tilde{I}$. □

6. β-TRANSFORMATION

We apply Theorem 2.2 to a $C^{1,1}(L)$-version of β-transformations. Let the map $T : [0,\infty) \to [0,1]$ be piecewise $C^{1,1}(L)$ and $0 = b_0 < b_1 < ...$ be the associated partition, where $b_k \to \infty$ as $k \to \infty$. We assume that:

a) T is right continuous and $T(b_k) = 0$, for each $k \geq 0$.
b) For each $a > 1$,

$$1 < \inf_{x\in[0,1]} \partial_x T(ax) \quad \text{and} \quad \sup_{x\in[0,1]} \partial_x T(ax) < \infty.$$

See Figure 1. We define the one-parameter family $T_a : [0,1] \to [0,1]$, $a > 1$, by $T_a(x) = T(ax)$. There exists a unique a.c.i.p. μ_a for each T_a as the following lemma asserts.

Lemma 6.1. *For each $a > 1$ there exists a unique a.c.i.p. μ_a for T_a. The support $K(a)$ is an interval adjacent to 0 and its length $|K(a)|$ is piecewise constant in a where the number of discontinuities is countable. Furthermore, the following holds. Let $I \subsetneq (1,\infty)$ be a parameter interval on which $|K(a)|$ is constant and such that the left endpoint of I does not coincide with 1 or a point of discontinuity for $a \mapsto |K(a)|$. Then, there exists an integer $t \geq 1$ (independent on the parameter value $a \in I$), such that the support $K(a)$, $a \in I$, is obtained by iterating t times the interval of monotonicity adjacent to 0, i.e. $K(a) = \text{closure}\{T_a^t((0, b_1/a))\}$.*

The proof of Lemma 6.1 is not difficult but tedious. For completeness we add the proof in the end of this section. Henceforth in this section, $I \subsetneq (1,\infty)$ will always denote an interval as described in Lemma 6.1 and such that, for $a \in I$, the number of discontinuities of T_a inside $K(a)$ is constant, i.e. the number $\#\{k \geq 1 \; ; \; b_k/a \in \text{int}(K(a))\}$ is constant on I. For a fixed interval I it is now straightforward to check that the one-parameter family T_a, $a \in I$, fits into the model described in Subsection 2.1 fulfilling properties (i)-(iii). Now, we can state the main result of this section.

Theorem 6.2. *If for a C^1 map $X : I \to [0,1]$ condition (I) is satisfied, then $X(a)$ is typical for μ_a for a.e. $a \in I$.*

Remark 6.3. As the family T_a we could also consider other models as, e.g., $x \mapsto ag(x) \bmod 1$ where $g : [0,1] \to [0,1]$ is a $C^{1,1}(L)$ homeomorphism with a strict positive derivative. Even if this model is not included in the families described above, it would be easier to treat since, seen as a map from the circle into itself, it is non-continuous only in the point 0 which, in particular, implies that $K(a) = [0,1]$.

By Theorem 2.2 and Propositions 4.5 and 5.1, in order to proof Theorem 6.2, it is sufficient to check conditions (IIa) and (IIb). As we will show in the following subsection, there is a large class of maps X satisfying condition (I):

Corollary 6.4. *If $X : (1,\infty) \to (0,1]$ is C^1 such that $X'(a) \geq 0$, then $X(a)$ is typical for μ_a for a.e. $a > 1$.*

Remark 6.5. Observe that the map

$$X(a) \equiv \lim_{x\to b_k-} T(x),$$

$a > 1$, satisfies $X(a) > 0$ and $X'(a) \geq 0$, and, hence, Corollary 6.4 can be applied to these from a dynamical point of view important values.

Obviously, we can cover $(1,\infty)$ with a countable number of intervals $I \subset (1,\infty)$ as they are used in Theorem 6.2. Thus, in order to prove Corollary 6.4, it is sufficient to verify condition (I) for the family T_a restricted to such a parameter interval I. Henceforth, we will use the notation of Subsection 2.1 related to the family T_a, $a \in I$. In particular, recall that $\lambda > 1$ is defined to be a lower bound for the expansion in the family, and observe that here we have $c_k(a) = b_k/a$, for $0 < k < p_1$. First we will prove Corollary 6.4.

6.1. **Proof of Corollary 6.4.** Note that if the map X in Corollary 6.4 satisfies $X(a) \notin \mathrm{int}(K(a))$ on I, then, by definition, the partition $\mathcal{P}_j|I$ would be empty for all $j \geq 1$ and, hence, condition (I) is not fulfilled. However, the following calculations in this subsection (see, in particular, (33)) show that, for $j \geq 1$, the derivative of x_j exists and is strictly positive for all but a finite number of points $a \in I$. Combined with property (4), we derive that, disregarding a countable number of points, we can cover I by a countable number of intervals $J \subset I$ such that for each such interval J there is an integer $j \geq 0$ such that $x_j|_J$ is C^1, $x_j'(a) \geq 0$, and $x_j(a) \in \mathrm{int}(K(a))$ for all $a \in J$. Thus, by possibly redefining X as x_j and focusing on smaller parameter intervals, we can without loss of generality assume that $X(a) \in \mathrm{int}(K(a))$ for all $a \in I$.

Let $j \geq 1$ and $\omega \in \mathcal{P}_j|I$. For $a \in \omega$ we have

$$\begin{aligned} x_j'(a) &= D_a T(a x_{j-1}(a)) = T'(a x_{j-1}(a))(x_{j-1}(a) + a x_{j-1}'(a)) \\ &= T_a'(x_{j-1}(a))(x_{j-1}(a)/a + x_{j-1}'(a)), \end{aligned}$$

and, hence, we derive

$$x_j'(a) = \sum_{i=0}^{j-1} T_a^{j-i}{}'(x_i(a)) \frac{x_i(a)}{a} + T_a^j{}'(X(a)) X'(a),$$

(recall that $x_0(a) = X(a)$). Furthermore, we obtain

$$\frac{x_j'(a)}{T_a^j{}'(X(a))} = \sum_{i=0}^{j-1} \frac{1}{T_a^i{}'(X(a))} \frac{x_i(a)}{a} + X'(a).$$

Let $\kappa = \inf_{a \in I} X(a)$ and $M = \sup_{a \in I} X'(a)$. By the assumptions on I and X, we have $\kappa > 0$ and $M < \infty$. Thus, for $a \in \omega$,

$$\frac{\kappa}{a_I} \leq \frac{x_j'(a)}{T_a^j{}'(X(a))} \leq \sum_{i=0}^{j-1} \frac{1}{\lambda^i} + M \leq \frac{\lambda}{\lambda - 1} + M, \tag{33}$$

where a_I denotes the right boundary point of I. This provides us with a lower and an upper bound in (I).

It is only left to show that the number of parameters $a \in I$ not contained in any element of the partition $\mathcal{P}_j|I$ is finite. We show this by induction. Note that the discontinuity points $c_k(a)$, $1 \leq k \leq p_1 - 1$, are equal to b_k/a (the partition points $c_0(a) \equiv 0$ and $c_{p_1}(a)$ are constant) and, thus, strictly decreasing in a. Since $X'(a) \geq 0$ and $X(a) \in \mathrm{int}(K(a))$, for all $a \in I$, it follows that the number of parameters $a \in I$ such that $X(a) = c_k(a)$ for some $0 \leq k \leq p_1$ is finite. Hence, the number of parameters $a \in I$ not contained in any element of $\mathcal{P}_1|I$ is finite. Let $j \geq 1$ and assume that the number of a's not contained in any element of $\mathcal{P}_j|I$ is finite. Let $\omega \in \mathcal{P}_j|I$. By the first inequality in (33), it follows that $x_j'(a) > 0$, $a \in \omega$. Since the partition points $c_k(a)$, $0 \leq k \leq p_1$, of T_a are decreasing or constant in a it follows that the number of $a \in \omega$ such that $x_j(a) = c_k(a)$, $0 \leq k \leq p_1$, is finite. We derive that the number of parameters $a \in I$ not contained in any element of the partition $\mathcal{P}_{j+1}|I$ is finite. This concludes the proof of Corollary 6.4.

6.2. **Condition (IIa).** The verification of condition (IIb) in the next subsection does not make use of condition (IIa). Hence, by Lemma A.1, we can without loss of generality assume that there is a constant $C = C(I) \geq 1$ such that for each $a \in I$ the density φ_a is bounded from above by C and, further, there exists an interval $J(a)$ of length C^{-1} such that φ_a restricted to $J(a)$ is bounded from below by C^{-1} (otherwise, disregarding a finite number of points, by Lemma A.1, we can cover the interval I by a countable number of subintervals on each of which this is true and then proceed with these subintervals instead of I). To conclude the verification of condition (IIa) it is left to show that there exists a lower bound for φ_a on the whole of $K(a)$.

To make the definition of the intervals $J_i(a)$ below work, we assume that the interval $J(a)$ is closed to the left. Recall that, by property (i) in Subsection 2.1, we have $c_k(a) > c_{k-1}(a)+\delta_0$, $1 \leq k \leq p_1$, for some constant $\delta_0 = \delta_0(I) > 0$. Let $\varepsilon = \min\{(\lambda-1)/2C, \lambda\delta_0\}$ and take $l \geq 1$ so large that $\lambda^l/2C > 1$. We claim that $[0, \varepsilon) \subset T_a^l(J(a))$. Let $J_0(a) = J(a)$ and assume that we have defined the interval $J_{i-1}(a) \subset J(a)$, $i \geq 1$, where $J_{i-1}(a)$ is a (not necessarily maximal) interval of monotonicity for T_a^{i-1}. If $[0, \varepsilon) \subset T_a^i(J_{i-1}(a))$, we stop and do not define $J_i(a)$. If $[0, \varepsilon)$ is not contained in $T_a^i(J_{i-1}(a))$ then, since $J_{i-1}(a)$ is a monotonicity interval for T_a^{i-1} and by the definition of ε (combined with property (i) and property a) of T_a), it follows that there can lie at most one partition point $c_k(a)$ in the image $T_a^{i-1}(J_{i-1}(a))$. If there is no partition point in this image then we let $J_i(a) = J_{i-1}(a)$, which is in this case also a monotonicity interval for T_a^i. If there is a partition point $c_k(a) \in T_a^{i-1}(J_{i-1}(a))$, then we define $J_i(a) \subset J_{i-1}(a)$ to be the interval of monotonicity for T_a^i such that $T_a^{i-1}(J_i(a)) = T_a^{i-1}(J_{i-1}(a)) \cap [0, c_k(a))$. Note that $|T_a^{i-1}(J_{i-1}(a)) \cap [c_k(a), 1]| < \varepsilon/\lambda$, since otherwise we would have $[0, \varepsilon) \subset T_a^i(J_{i-1}(a))$. Assuming that $J_l(a)$ is defined, we obtain

$$
\begin{aligned}
|T_a^l(J_l(a))| &\geq \lambda(|T_a^{l-1}(J_{l-1}(a))| - \varepsilon/\lambda) \geq \lambda^l|J_0(a)| - \varepsilon\frac{\lambda^l - 1}{\lambda - 1} \\
&\geq \lambda^l(1/C - 1/2C) \geq \lambda^l/2C > 1,
\end{aligned}
$$

where we used the definitions of ε and l. Since $J_l(a)$ is a monotonicity interval for T_a^l, this is a contradiction and it follows that the maximal integer $i \geq 0$ such that $J_i(a)$ is defined is strictly smaller than l. Hence, $T_a^l(J(a))$ contains $[0, \varepsilon)$ as claimed above. This immediately implies that there is an integer $l' \geq 1$ independent on the parameter $a \in I$ such that $[0, c_1(a)) \subset T_a^{l'}(J(a))$.

Combined with Lemma 6.1 we derive that there is an integer $j \geq 1$ independent on $a \in I$ such that, $K(a) = \text{closure}\{T_a^j(J(a))\}$. Now, by the Perron-Frobenius equality, it follows that, for $a \in I$,

$$
\varphi_a(y) \geq \sum_{\substack{x \in J(a) \\ T_a^j(x) = y}} \frac{\varphi_a(x)}{|T_a^j{}'(x)|} \geq \frac{1}{C\Lambda^j}, \tag{34}
$$

for a.e. $y \in K(a)$. This concludes the proof of a lower bound for φ_a on the whole of $K(a)$.

6.3. **Condition (IIb).** We can verify condition (IIb) by induction over $j \geq 1$. Let $a_1, a_2 \in I$ such that $a_1 \leq a_2$. Note that $\mathcal{P}_1(a) = \{(c_k(a), c_{k+1}(a))\ ;\ 0 \leq k < p_2\}$ where $c_k(a) = b_k/a$ for $0 < k < p_1$. Thus, if $1 \leq k < p_2$ then we clearly have $T_{a_1}((c_{k-1}(a_1), c_k(a_1))) = T_{a_2}((c_{k-1}(a_2), c_k(a_2)))$. The point $c_{p_1}(a) \in (b_{p_1-1}/a, b_{p_1}/a)$ is constant since the length $|K(a)|$ is constant. It follows that $T_a(c_{p_1}(a))$ is increasing in a, which implies that $T_{a_1}((c_{p_1-1}(a_1), c_{p_1}(a_1))) \subset T_{a_2}((c_{p_1-1}(a_2), c_{p_1}(a_2)))$. Hence, (IIb) holds for $j = 1$. Assume that (IIb) holds for $j \geq 1$. Let $\tilde{\omega} \in \mathcal{P}_j(a_1)$ and $\tilde{\omega}' = \mathcal{U}_{a_1,a_2,j}(\tilde{\omega})$ the corresponding element in $\mathcal{P}_j(a_2)$. Note that the image by T_a^i, $i \geq 1$, of an element in $\mathcal{P}_i(a)$ is always adjacent to 0. Since $T_{a_1}^j(\tilde{\omega}) \subset T_{a_2}^j(\tilde{\omega}')$ and the $c_k(a)$'s are decreasing (or constant in the case $k = 0$ and $k = p_1$), it follows immediately that for every element $\omega \in \mathcal{P}_{j+1}(a_1)|\tilde{\omega}$ there is a unique element $\omega' \in \mathcal{P}_{j+1}(a_2)|\tilde{\omega}'$ fulfilling

$\text{symb}_{a_1}(T^i_{a_1}(\omega)) = \text{symb}_{a_2}(T^i_{a_2}(\omega'))$, $0 \le i < j+1$, and $T^{j+1}_{a_1}(\omega) \subset T^{j+1}_{a_2}(\omega')$. Defining $\mathcal{U}_{a_1,a_2,j+1}(\omega) = \omega'$ shows that (IIb) holds also for $j+1$.

6.4. **Proof of Lemma 6.1.** For $a > 1$ let μ_a be an a.c.i.p. for T_a with support $K(a)$ and let $J \subset K(a)$ be an open interval. Since T_a is expanding there exists an integer $j \ge 1$ such that $T^j_a : J \to [0,1]$ is not any longer continuous. It follows that there exists an $\varepsilon > 0$ such that $T^j_a(J)$ contains $[0,\varepsilon)$. If T_a had more than one a.c.i.p. then, by [17], there would exists two a.c.i.p.'s with disjoint supports (disregarding a finite number of points). This shows that the a.c.i.p. μ_a is unique.

For each $a > 1$ we define a number $y(a) \in (0,1]$ and an integer $t(a) \ge 1$. The number $y(a)$ will be the right boundary point of the support $K(a)$ and $t(a)$ will be so large that the image by $T^{t(a)}_a$ of the interval $[0, b_1/a)$ will be equal to $[0, y(a))$. Let

$$y_1(a) = \lim_{x \to b_1-} T(x) \Big(= \lim_{x \to b_1/a-} T_a(x) \Big),$$

and $t_1(a) = 1$. Assume that, for $i \ge 1$, $y_i(a)$ and $t_i(a)$ are defined. Let

$$k_{i,1}(a) = \max \big\{ k \ge 1 \ ; \ \frac{b_k}{a} \le y_i(a) \big\}$$

and set

$$y_{i,1}(a) = \max_{1 \le k \le k_{i,1}(a)} \lim_{x \to b_k-} T(x).$$

Assume that both $k_{i,j}(a)$ and $y_{i,j}(a)$ are defined for $j \ge 1$. Let

$$k_{i,j+1}(a) = \max \big\{ k \ge 1 \ ; \ \frac{b_k}{a} \le y_{i,j}(a) \big\}.$$

If $k_{i,j+1}(a) = k_{i,j}(a)$ we do not define $y_{i,j+1}(a)$. Otherwise, let

$$y_{i,j+1}(a) = \max_{1 \le k \le k_{i,j+1}(a)} \lim_{x \to b_k-} T(x).$$

Since, for fixed $a > 1$, there are only finitely many $k \ge 1$ such that $b_k/a \in [0,1]$, $y_{i,j}(a)$ is only defined for finitely many $j \ge 1$. Let $j \ge 1$ be maximal such that $y_{i,j}(a)$ is defined.

1) If $y_{i,j}(a) = y_i(a)$ we do not define $y_{i+1}(a)$ and $t_{i+1}(a)$.
2) If $y_{i,j}(a) > y_i(a)$ and $\lim_{x \to y_{i,j}(a)-} T_a(x) \le y_{i,j}(a)$ we set $y_{i+1}(a) = y_{i,j}(a)$ and $t_{i+1}(a) = t_i(a) + j$.
3) If we are not in case 1) nor in case 2), it follows that

$$y_{i,j}(a) \in (b_{k_{i,j}(a)}/a, b_{k_{i,j}(a)+1}/a)$$

and $T_a(y_{i,j}(a)) > y_{i,j}(a)$. We set

$$y_{i+1}(a) = \lim_{x \to b_{k_{i,j}(a)+1}-} T(x).$$

Taking l minimal such that $T^l_a(y_{i,j}(a)) \ge b_{k_{i,j}(a)+1}/a$, we set $t_{i+1}(a) = t_i(a) + j + l < \infty$.

Observe that if the cases 1) or 2) occur, $y_{i+2}(a)$ will not be defined. Only when the case 3) occurs, $y_{i+2}(a)$ is possibly defined. Since, for fixed $a > 1$, there are only finitely many $k \ge 1$ such that $b_k/a \in [0,1]$, case 3) can occur only a finite number of times, which implies that $y_i(a)$ and $t_i(a)$ are defined only for a finite number of $i \ge 1$. We set $y(a) = y_i(a)$ and $t(a) = t_i(a)$ where i is the maximal number for which $y_i(a)$ and $t_i(a)$ are defined. (Note that $t(a)$ is finite.) By the constructions of $y(a)$ and $t(a)$ it follows that

$$[0, y(a)) = T^l_a([0, b_1/a)),$$

for all $l \geq t(a)$. Hence, the support $K(a)$ of $\mu(a)$ coincides with the interval $[0, y(a)]$. Furthermore, by the construction of $y(a)$, we deduce that $y(a)$ is not decreasing in a and for each $a > 1$ there exists a $k \geq 1$ such that $y(a) = \lim_{x \to b_k -} T(x)$. It follows that $y(a)$ is piecewise constant. It is straightforward to check that $y(a)$ is right continuous and $\tilde{a} > 1$ is a point of discontinuity for $a \mapsto y(a)$ if and only if $y(\tilde{a})$ is a fixed point for $T_{\tilde{a}}$ and different from 1. Furthermore, $\lim_{a \to \tilde{a}+} t(a) = \infty$ and possibly also $\lim_{a \to 1+} t(a) = \infty$. But on a parameter interval $I \subset (1, \infty)$ where $y(a)$ is constant, $t(a)$ is not increasing in a. Hence, if the left boundary point of I is not adjacent to 1 or a point of discontinuity for $a \mapsto y(a)$ then $t(a)$ is bounded from above on I. This concludes the proof of Lemma 6.1.

7. Tent maps

In this section we apply Theorem 2.2 to skew tent maps. Instead of considering skew tent maps defined on the unit interval as it is done in the introduction, we take the same representation as in [11], i.e. we define the skew tent map with slopes α and $-\beta$ where $\alpha, \beta > 1$, by the formula

$$T_{\alpha,\beta}(x) = \begin{cases} 1 + \alpha x & \text{if } x \leq 0, \\ 1 - \beta x & \text{otherwise,} \end{cases}$$

(see Remark 7.3). The turning point of $T_{\alpha,\beta}$ is 0, $T_{\alpha,\beta}(0) = 1$ and, by Lemma 3.1 in [11], if $\alpha^{-1} + \beta^{-1} \geq 1$ then the interval $[T_{\alpha,\beta}(1), 1] (= [1 - \beta, 1])$ is invariant under $T_{\alpha,\beta}$ (if $\alpha^{-1} + \beta^{-1} < 1$ then there exists no invariant interval of finite positive length). For two parameter couples (α, β) and (α', β') we take the same order relation as the one which appears in [11], i.e. we shall write $(\alpha', \beta') > (\alpha, \beta)$ if $\alpha' \geq \alpha$, $\beta' \geq \beta$, and at least one of these inequalities is sharp. Fix (α_0, β_0) and (α_1, β_1) in the set $\{(\alpha, \beta)\ ;\ \alpha, \beta > 1 \text{ and } \alpha^{-1} + \beta^{-1} \geq 1\}$ such that $(\alpha_1, \beta_1) > (\alpha_0, \beta_0)$. Let

$$\alpha : [0,1] \to [\alpha_0, \alpha_1] \quad \text{and} \quad \beta : [0,1] \to [\beta_0, \beta_1]$$

be functions in $C^1([0,1])$ such that $(\alpha(0), \beta(0)) = (\alpha_0, \beta_0)$, $(\alpha(1), \beta(1)) = (\alpha_1, \beta_1)$, and, for all $a \in [0,1]$, if $\alpha_0 \neq \alpha_1$ then $\alpha'(a) > 0$ and if $\beta_0 \neq \beta_1$ then $\beta'(a) > 0$. Observe that $\alpha(a), \beta(a) > 1$, and $\alpha(a)^{-1} + \beta(a)^{-1} \geq 1$, for all $a \in [0,1]$. We define the one-parameter family T_a as the family of skew tent maps given by

$$T_{\alpha(a),\beta(a)} : [T_{\alpha(a),\beta(a)}(1), 1] \to [T_{\alpha(a),\beta(a)}(1), 1], \qquad a \in [0,1].$$

By [9], since T_a has only two intervals of monotonicity, there exists a unique a.c.i.p. μ_a. Observe that even if T_a is now defined on a larger interval than the one-parameter families described in Subsection 2.1, the definitions of the partitions $\mathcal{P}_j(a)$ and $\mathcal{P}_j|[0,1]$ in Subsection 2.2 still apply. The main statement of this section is the following.

Theorem 7.1. *For a.e. parameter $a \in [0,1]$ the turning point 0 is typical for μ_a.*

In contrast to the β-transformation, it is more difficult to state typicality for other, not so specific points as, e.g., the turning point. However, we will show that conditions (IIa) and (IIb) are satisfied for the one-parameter family of skew tents maps T_a, $a \in [0,1]$. Hence, given a C^1 function $Y : [0,1] \to \mathbb{R}$ (such that $Y(a) \in [T_a(1), 1]$), it is sufficient to check condition (I) in order to obtain a.s. typicality for Y.

Corollary 7.2. *If the one-parameter family T_a, $a \in [0,1]$, with the associated map $a \mapsto Y(a)$ satisfies condition (I), then $Y(a)$ is typical for μ_a, for a.e. $a \in [0,1]$.*

The calculations in Subsection 7.1 below show that if the function $y_j(a) = T_a^j(Y(a))$ has, for some $j \geq 1$, a high enough initial expansion in a, it will imply condition (I) for T_a with the associated map $a \mapsto \tilde{Y}(a) = y_j(a)$. This makes it easy to check condition (I) in Corollary 7.2 numerically.

If $\alpha \le \beta/(\beta^2-1)$ then $T_{\alpha,\beta}$ is renormalizable, see, e.g., [11]. More precisely, $T_{\alpha,\beta}^2(1)$ is greater or equal than the unique fixed point in $(0,1)$ and $T_{\alpha,\beta}^2$ restricted either to the interval $[T_{\alpha,\beta}(1), T_{\alpha,\beta}^3(1)]$ or to the interval $[T_{\alpha,\beta}^2(1), 1]$ is affinely conjugated to $T_{\beta^2,\alpha\beta}$ restricted to the interval $[T_{\beta^2,\alpha\beta}(1), 1]$. Observe that the new slopes $\alpha' = \beta^2$ and $-\beta' = -\alpha\beta$ still satisfy $\alpha', \beta' > 1$ and $(\alpha')^{-1} + (\beta')^{-1} \ge 1$ (the latter inequality follows from the assumption $\alpha \le \beta/(\beta^2-1)$). Since the function $\beta \mapsto \beta/(\beta^2-1)$ is decreasing for $\beta > 1$, we have that if T_0 is not renormalizable then not either T_a, $a \in [0,1]$, is renormalizable. Now, assume for the moment that T_a is renormalizable for each $a \in [0,1]$ and consider the one-parameter family defined by $\tilde{T}_a = T_{\beta(a)^2,\alpha(a)\beta(a)}$. Note that if we show typicality of the turning point for the family $\tilde{T}_a$, for a.e. $a \in [0,1]$, this implies a.s. typicality of the turning point for the original family T_a. Furthermore, if we verify conditions (IIa) and (IIb) for the family $\tilde{T}_a$, this implies that conditions (IIa) and (IIb) also hold for the family T_a. Since the a-derivative of $\alpha(a)\beta(a)$ is positive and the a-derivative of $\beta(a)^2$ is non-negative, the new one-parameter family $\tilde{T}_a$ fits into the family of skew tent maps described in the beginning of this section. Furthermore, it is known that for each $a \in [0,1]$, T_a is at most a finite number of times renormalizable where this number is bounded above by a constant only dependent on (α_0, β_0) and not on the parameter a (this can easily be derived by looking, e.g., at the topological entropy of T_a, see [11] page 137). Altogether, we derive that in order to prove Theorem 7.1 (and therewith also Corollary 7.2) we can without loss of generality restrict ourself to the case when T_a, $a \in [0,1]$, is not renormalizable, i.e. we assume that

$$\alpha_0 > \beta_0/(\beta_0^2 - 1). \tag{35}$$

Observe that it is only possible for the parameter $a = 1$ to satisfy the equality $\alpha(a)^{-1} + \beta(a)^{-1} = 1$. Thus, since we are only interested in Lebesgue almost every parameter we can neglect skew tent maps whose slopes satisfy $\alpha^{-1} + \beta^{-1} = 1$, i.e. we assume that

$$\alpha_1^{-1} + \beta_1^{-1} > 1. \tag{36}$$

For non-renormalizable T_a it will follow from Subsection 7.2 that the support $K(a)$ of the a.c.i.p. μ_a is the whole invariant interval $[T_a(1), 1]$. Hence, if ψ_a is the affine map from $[0,1]$ onto $[T_a(1), 1]$ with, say, positive derivative it is straightforward to check that the one-parameter family $\psi_a \circ T_a \circ \psi_a^{-1} : [0,1] \to [0,1]$, $a \in [0,1]$, fits into the model described in Subsection 2.1, satisfying properties (i)-(iii).

Remark 7.3. Observe that the length of the invariant interval $K(a)$ for T_a is bounded from below by 1 and from above by β_1. Hence, the estimates to be established in conditions (I), (IIa), and (IIb) for the family T_a and the to it affinely conjugated family on the unit interval will differ only by constants, which are uniformly in a bounded above, and below away from zero. Therefore we will continue with the representation T_a and do not switch to skew tent maps defined on the unit interval. The partitions defined in Subsection 2.2 are defined in an analog way for the family T_a.

Let $X(a) = T_a^3(0)$ (if we started with an iteration of 0 lower than the third, then by our definition of the $\mathcal{P}_j|[0,1]$'s, all of these partitions would be empty). To prove Theorem 7.1 it is sufficient to verify conditions (I), (IIa), and (IIb) for the family T_a, $a \in [0,1]$, with the associated map X. Henceforth, we will use the notations of Subsection 2.1 related to the family T_a, $a \in [0,1]$. The constants λ and Λ can be chosen as

$$\lambda = \min\{\alpha_0, \beta_0\} \qquad \text{and} \qquad \Lambda = \max\{\alpha_1, \beta_1\}.$$

The main computation needed for the verifications of (I) and (IIb) is already done in a paper by Misiurewicz and Visinescu [11] (see Lemma 3.3 and 3.4 therein), where they show monotonicity of the kneading sequence for skew tent maps. (A reader not familiar with the basic notions and facts of kneading theory can find them in [5].) For the later use we state here the main result in [11].

Theorem 7.4. *Let* (α,β) *and* (α',β') *be in the set* $\{(\alpha,\beta)\ ;\ \alpha,\beta>1$ *and* $\alpha^{-1}+\beta^{-1}\geq 1\}$. *If* $(\alpha',\beta')>(\alpha,\beta)$ *then the kneading sequence of* $T_{\alpha',\beta'}$ *is strictly greater than the kneading sequence of* $T_{\alpha,\beta}$.

Proof. See Theorem A in [11]. □

Since the derivatives of $\alpha(a)$ and $\beta(a)$ are non-negative and at least one of them is positive, we obtain strict monotonicity of the kneading sequence for our family T_a.

Remark 7.5. We could also formulate and prove certain $C^{1,1}(L)$-versions of Theorem 7.1 and Corollary 7.2. But it is difficult for us to formulate very general statements as it is, e.g., done in Section 6 for the β-transformation. To mention nevertheless an example, one could show, by the methods in this section, almost sure typicality for the turning point of one-parameter families $\tilde{T}_a$ of $C^{1,1}(L)$ unimodal maps, which are of the form $\tilde{T}_a = T_a\circ g$ where T_a is a fixed family of skew tent maps as described above but with a representation such that the T_a's map the unit interval into itself, and $g:[0,1]\to[0,1]$ is a $C^{1,1}(L)$ homeomorphism satisfying $g'(x)\approx 1$. However, to keep this paper in a reasonable size we will here not investigate such possible $C^{1,1}(L)$-versions of skew tent maps.

7.1. **Condition (I).** We note first that, for $j\geq 1$, the number of parameter values $a\in[0,1]$ not contained in any partition element of $\mathcal{P}_j|[0,1]$ is finite. In fact, if $a\in[0,1]$ is not contained in any element of $\mathcal{P}_j|[0,1]$, then $x_i(a)=T_a^{i+3}(0)=0$ for some $0\leq i<j$. Hence, the kneading sequence of T_a ends with C and has length smaller than $j+3$. But, by the strict monotonicity of the kneading sequence, T_a can have such a kneading sequence only for finitely many parameters $a\in[0,1]$.

Let $j\geq 1$, $0\leq\tau<j$, and $\omega\in\mathcal{P}_j|[0,1]$. For $a\in\omega$, we have

$$\begin{aligned} x_j'(a) = D_a(T_a(x_{j-1}(a))) &= \begin{cases} \alpha(a)x_{j-1}'(a)+\alpha'(a)x_{j-1}(a) & \text{if } x_{j-1}(a)\leq 0,\\ -\beta(a)x_{j-1}'(a)-\beta'(a)x_{j-1}(a) & \text{otherwise,}\end{cases}\\ &= T_a'(x_{j-1}(a))\cdot\begin{cases} x_{j-1}'(a)+\frac{1}{T_a'(x_{j-1}(a))}\alpha'(a)x_{j-1}(a) & \text{if } x_{j-1}(a)\leq 0,\\ x_{j-1}'(a)-\frac{1}{T_a'(x_{j-1}(a))}\beta'(a)x_{j-1}(a) & \text{otherwise,}\end{cases}\\ &= T_a^{j-\tau\,\prime}(x_\tau(a))\cdot\\ &\quad\Bigg(x_\tau'(a)+\underbrace{\sum_{i=1}^{j-\tau}\frac{1}{T_a^{i\,\prime}(x_\tau(a))}\cdot\begin{cases}\alpha'(a)x_{\tau+i-1}(a) & \text{if } x_{\tau+i-1}(a)\leq 0,\\ -\beta'(a)x_{\tau+i-1}(a) & \text{otherwise.}\end{cases}}_{(*)}\Bigg).\end{aligned}$$

Let $M=\max_{a\in[0,1]}\{\alpha'(a)|1-\beta_1|,\beta'(a)\}<\infty$ and $M'=\max_{a\in[0,1]}|X'(a)|$. We have

$$|(*)|\leq\frac{M}{\lambda-1},$$

and, by setting $\tau=0$, we obtain in condition (I) the upper bound

$$\left|\frac{x_j'(a)}{T_a^{j\,\prime}(X(a))}\right|\leq M'+\frac{M}{\lambda-1}.$$

To establish a lower bound in condition (I) is more delicate. We will use some derivative estimates given in [11]. To this end we will look at the kneading sequences of the T_a's. Every kneading sequence of a map T_a in our family starts with RL and is smaller or equal than the sequence RL^∞. In fact, by (36) and by the monotonicity of the kneading sequence (Theorem 7.4), the kneading sequence of $T_1(=T_{\alpha_1,\beta_1})$ is strictly smaller than RL^∞. Let $1 \le m_1 < \infty$ be the integer such that the kneading sequence of T_1 starts with $RL^{m_1}R$ or is equal to $RL^{m_1}C$. From [11] we derive the following result.

Proposition 7.6. *There exists a constant $\kappa > 0$ such that for $\omega \in \mathcal{P}_j|[0,1]$, $j \ge 0$, and $a \in \omega$, we have*

$$\left|\partial_\alpha T^{j+3}_{\alpha(a),\beta(a)}(0)\right|, \ \left|\partial_\beta T^{j+3}_{\alpha(a),\beta(a)}(0)\right| \ge \kappa \beta_0^{\left[\frac{j}{m_1}\right]}, \tag{37}$$

and, furthermore,

$$\operatorname{sign}(\partial_\alpha T^{j+3}_{\alpha(a),\beta(a)}(0)) = \operatorname{sign}(\partial_\beta T^{j+3}_{\alpha(a),\beta(a)}(0)) = \operatorname{sign}(T_a^{j+2\,\prime}(1)). \tag{38}$$

Proof. The proof of Proposition 7.6 follows from Lemma 3.3 and 3.4 in [11]. For this note that for each $a \in [0,1]$, the integer $m \ge 1$ such that the kneading sequence of T_a starts with RL^mR or is equal to RL^mC is smaller or equal than m_1. Observe also that x_j in [11] corresponds to x_{j-2} in our setting.

Actually, Lemma 3.4 in [11] is only formulated for the case when $j \ge m$. But considering Lemma 3.4 i) in [11] it is easy to deduce that Proposition 7.6 also holds when $0 \le j < m$. ☐

Property (38) will be essential to verify condition (IIb).

For $j \ge 0$, we have

$$x_j'(a) = \alpha'(a)\partial_\alpha T^{j+3}_{\alpha(a),\beta(a)}(0) + \beta'(a)\partial_\beta T^{j+3}_{\alpha(a),\beta(a)}(0), \tag{39}$$

for all a contained in an element of $\mathcal{P}_j|[0,1]$. Since at least one of the derivatives $\alpha'(a)$ and $\beta'(a)$ is uniformly bounded away from 0, by (37) and (38), $|x_j'|$ is uniformly growing and we can fix an integer $j_0 \ge 1$ such that

$$|x_{j_0}'(a)| \ge \frac{M}{\lambda - 1} + 1.$$

Thus, by setting $\tau = j_0$ in the formula for the derivative of x_j in the beginning of this subsection, we obtain that for all $\omega \in \mathcal{P}_j|[0,1]$, $j \ge j_0$, and $a \in \omega$,

$$|x_j'(a)| \ge |T_a^{j-j_0\,\prime}(x_{j_0}(a))|,$$

which implies

$$\left|\frac{x_j'(a)}{T_a^{j\,\prime}(X(a))}\right| \ge \frac{1}{\Lambda^{j_0}}.$$

Furthermore, by (37), (38), and (39), there is a constant $\kappa' > 0$ such that $|x_j'(a)| \ge \kappa'$, for all $1 \le j < j_0$. This concludes the proof of a lower bound in condition (I).

7.2. **Condition (IIa).** The verification of condition (IIb) in the next subsection does not make use of condition (IIa). Hence, as in the first paragraph in Subsection 6.2, by Lemma A.1, we can without loss of generality assume that there is a constant $C = C([0,1]) \ge 1$ such that for each $a \in [0,1]$ the density φ_a is bounded from above by C and, further, there exists an interval $J(a)$ of length C^{-1} such that φ_a restricted to $J(a)$ is bounded from below by C^{-1}. In the remaining part of this section we will establish a lower bound for φ_a on the whole of $K(a)$.

If $\alpha = \beta/(\beta^2 - 1)$ then the kneading sequence of $T_{\alpha,\beta}$ is RLR^∞ (see Lemma 3.2 and its proof in [11]). By (35) and by the monotonicity of the kneading sequence (Theorem 7.4), there exists a non-negative integer m_0, which is either equal to 0 or even, such that the kneading sequence of $T_0 = T_{\alpha_0,\beta_0}$ starts with $RLR^{m_0}L$ or is equal to $RLR^{m_0}C$. This in turn implies that for each $a \in [0,1]$ there exists an integer $0 \leq m \leq m_0$, which is either equal to 0 or even, such that the kneading sequence of T_a starts with RLR^mL or is equal to RLR^mC. Fix $a \in [0,1]$ and let $0 \leq m \leq m_0$ be the corresponding integer.

Lemma 7.7. *Let $J \subset [T_a(1), 1]$ be an interval adjacent to 0 and let $j \geq 1$ be the first time such that $\#\{\omega \in \mathcal{P}_j(a)|J\} = 2$. If $j \leq m+2$ then $T_a^{m+4}(J) \supset (T_a(1), 1)$.*

Proof. Since $T_a((0,1)) = (T_a(1), 1)$ it is enough to show that $T_a^{m+3}(J) \supset (0,1)$. Observe that, by the definition of j, 0 is contained in $T_a^j(J)$. If $j = 1$ then, since 0 is a boundary point of J, it follows that $T_a(J)$ contains the interval $(0,1)(= (0, T_a(0)))$.

If $j = 2$, then $T_a^2(J)$ contains the interval $(T_a^2(0), 0)$. If $m = 0$ then $T_a^2(0) \leq 0$ and it follows that $T_a^3(J) \supset (0,1)$. If $m \geq 2$ (recall that m is even), then we derive inductively that, for $2 < i \leq m+2$, $T_a^i(J) \supset (T_a^i(0), 1)$ if i is odd and $T_a^i(J) \supset (0, T_a^i(0))$ if i is even. Hence, $T_a^{m+2}(J) \supset (0, T_a^{m+2}(0))$ and, by the fact that $T_a^{m+3}(0) \leq 0$, it follows that $T_a^{m+3}(J) \supset (0,1)$.

If $2 < j \leq m+2$ then $T_a^j(0) > 0$. Hence, $T_a^j(J)$ contains the interval $(0, T_a^j(0))$. Observing that this implies that j is even, we can argue as in the case when $j = 2$ and deduce that $T_a^{m+3}(J) \supset (0,1)$. ☐

Let $J(a)$ be the interval of length C^{-1} such that φ_a restricted to $J(a)$ is bounded from below by C^{-1}. Let $j_0 \geq 1$ be the first time such that $\#\{\omega \in \mathcal{P}_{j_0}(a)|J(a)\} = 2$. We define $J_0(a)$ to be the interval in $\mathcal{P}_{j_0}(a)|J(a)$ satisfying

$$|T_a^{j_0-1}(J_0(a))| \geq |T_a^{j_0-1}(J(a))|/2 \tag{40}$$

(if both intervals in $\mathcal{P}_{j_0}(a)|J(a)$ satisfy this then we choose one arbitrarily). Let $j_1 > j_0$ be the first time such that $\#\{\omega \in \mathcal{P}_{j_1}(a)|J_0(a)\} = 2$. Assume now that $J_{i-1}(a)$ and j_i are defined for some $i \geq 1$. If $j_i - j_{i-1} \leq m+2$ we stop and do not define $J_i(a)$. Otherwise, let $J_i(a)$ be the interval in $\mathcal{P}_{j_i}(a)|J_{i-1}(a)$ satisfying

$$|T_a^{j_i-1}(J_i(a))| \geq |T_a^{j_i-1}(J_{i-1}(a))|/2 \tag{41}$$

(if both intervals in $\mathcal{P}_{j_i}(a)|J_{i-1}(a)$ satisfy this then we choose one arbitrarily). We define $j_{i+1} > j_i$ to be the first time such that $\#\{\omega \in \mathcal{P}_{j_{i+1}}(a)|J_i(a)\} = 2$. The size of the images of the $J_i(a)$'s is growing in i:

Lemma 7.8. *There exists a constant $\tilde{\lambda} > 2$ independent on the parameter a such that if $J_k(a)$ and j_{k+1} are defined for some $k \geq 1$, then, for all $0 \leq i < k$, we have*

$$|T_a^{j_{i+1}-1}(J_i(a))| \geq \tilde{\lambda} |T_a^{j_i-1}(J_i(a))|.$$

Proof. Since there are at least $m+3$ iterations between j_i and j_{i+1} and since the right boundary point of $T_a^{j_i}(J_i(a))$ is 1, we have

$$\begin{aligned} |T_a^{j_{i+1}-1}(J_i(a))| &= |T_a^{j_{i+1}-j_i-1}{}'(1)| \cdot |T_a'|_{T_a^{j_i-1}(J_i(a))}| \cdot |T_a^{j_i-1}(J_i(a))| \\ &= \underbrace{|T_a^{j_{i+1}-j_i-3}{}'(T_a^2(1))| \alpha(a)\beta(a) \min\{\alpha(a), \beta(a)\}}_{(*)} |T_a^{j_i-1}(J_i(a))|. \end{aligned}$$

If $m \geq 2$ (recall that m is even), then $j_{i+1} - j_i - 3 \geq 2$ and the kneading sequence of T_a starts with RLR^2. Therefore $(*) \geq \alpha(a)\beta(a)^3 \min\{\alpha(a), \beta(a)\}$. By (35), $\alpha(a) > \max\{\beta(a)/(\beta(a)^2 - 1), 1\}$. Hence,

$$(*) \geq \inf_{\beta > 1} \left\{ \beta^3 \max\{\beta/(\beta^2-1), 1\} \min\{\max\{\beta/(\beta^2-1), 1\}, \beta\} \right\} \geq 4,$$

where it is straightforward to verify the last inequality.

If $m = 0$ then $(*) = \beta(a)\alpha(a) \min\{\alpha(a), \beta(a)\}$. In the case when $1 < \beta(a) < 2$ we have a better lower bound for $\alpha(a)$ than the one above. Given $1 < \beta < 2$, note that the kneading sequence of $T_{(\beta-1)^{-1},\beta}$ is equal to RLC. Since $m = 0$ it follows that $\alpha(a) \geq \max\{(\beta(a)-1)^{-1}, 1\}$ and we obtain

$$(*) \geq \beta(a)\max\{(\beta(a)-1)^{-1},1\}\min\{\max\{(\beta(a)-1)^{-1},1\},\beta(a)\}.$$

This lower bound is not good enough since the minimum of the function

$$\beta\max\{(\beta-1)^{-1},1\}\min\{\max\{(\beta-1)^{-1},1\},\beta\},$$

which is attained in the point $\beta = 2$, is equal to 2. However, if $\beta(a) \approx 2$ then, since $\alpha(a) \geq \alpha_0 > 1$, we derive that $(*) \geq 2+\varepsilon$ for some $\varepsilon > 0$. Setting $\tilde{\lambda} = \min\{4, 2+\varepsilon\}$, this concludes the proof. □

Assuming that $J_k(a)$ and j_{k+1} are defined for some $k \geq 1$, by (40), (41), and Lemma 7.8, it follows that

$$|T_a^{j_k-1}(J_{k-1}(a))| \geq \Big(\frac{\tilde{\lambda}}{2}\Big)^k \frac{|J(a)|}{2} \geq \Big(\frac{\tilde{\lambda}}{2}\Big)^k \frac{1}{2C},$$

where $(\tilde{\lambda}/2)^k$ is growing in k. The length of the interval $T_a^{j_k-1}(J_{k-1}(a))$ is bounded above by the length of the invariant interval $[T_a(1), 1]$, which in turn is bounded by β_1. This implies that the number $k \geq 0$ for which $J_k(a)$ can be defined, is bounded above by a number independent on $a \in [0,1]$. Let k be maximal such that $J_k(a)$ and j_{k+1} are defined. It follows that $j_{k+1} - j_k \leq m+2$. Since the interval $T_a^{j_k-1}(J_k(a))$ is adjacent to 0, we can apply Lemma 7.7 and we obtain

$$T_a^{j_k+m+3}(J_k(a)) \supset (T_a(1), 1).$$

Clearly, the number of iterations between successive j_i's is bounded above by a number independent on a, and the integer m is bounded above by m_0. Altogether, we derive that there is an iteration $j \geq 1$ (independent on the parameter a), such that for each $a \in [0,1]$, $T_a^j(J(a)) = [T_a(1), 1]$. Finally, we can apply inequality (34) and we obtain a lower bound for φ_a on the whole of $[T_a(1), 1]$. Observe that $T_a^j(J(a)) = [T_a(1), 1]$ implies that $K(a) = [T_a(1), 1]$.

7.3. **Condition (IIb).** The main ingredient in verifying condition (IIb) is property (38) stated in Proposition 7.6. Observe that, by (38), (39), and the definition of x_j, we have

$$\operatorname{sign}(D_a T_a^m(0)) = \operatorname{sign}(T_a^{m-1\,\prime}(1)), \tag{42}$$

for all $m \geq 3$ and parameter values a contained in an element of $\mathcal{P}_{m-3}|[0,1]$.

We verify condition (IIb) by induction over $j \geq 1$. In fact, we will show the following statement. For each $j \geq 1$ there exists a map as described in condition (IIb) and further, if $a_1, a_2 \in [0,1]$ such that $a_1 < a_2$ and $\omega \in \mathcal{P}_{a_1}$, then the boundary points of $T_a^j(\mathcal{U}_{a_1,a,j}(\omega))$ are continuous in $a \in [a_1, a_2]$. For $j = 1$ this statement can easily be verified observing that, by the properties of the maps α and β, $T_a(1)$ is constant or continuously decreasing and $T_a^2(1)$ is continuously increasing in $a \in [0,1]$. Assume now that the statement holds for some $j \geq 1$. Take $\tilde{\omega} \in \mathcal{P}_j(a_1)$ and, for $a \in [a_1, a_2]$, let $\tilde{\omega}(a) = \mathcal{U}_{a_1,a,j}(\tilde{\omega})$ be the to it associated element in $\mathcal{P}_j(a)$. Since $T_{a_1}^j(\tilde{\omega}) \subset T_a^j(\tilde{\omega}(a))$ and the turning point 0 is constant in a, it follows that for each (of the maximal two) element $\omega \in \mathcal{P}_{j+1}(a_1)|\tilde{\omega}$ there is a unique element $\omega(a) \in \mathcal{P}_{j+1}(a)|\tilde{\omega}(a)$ fulfilling

$$\operatorname{symb}_a(T_a^i(\omega(a))) = \operatorname{symb}_{a_1}(T_{a_1}^i(\omega)), \tag{43}$$

for $0 \leq i < j+1$. In particular this is true for $a = a_2$. Setting $\mathcal{U}_{a_1,a_2,j+1}(\omega) = \omega(a_2)$, this verifies property (17) in condition (IIb). Using the induction assumption it is straightforward to derive the continuity of the boundary points of $T_a^{j+1}(\omega(a))$ for $a \in [a_1, a_2]$. So, it is only left to show that property (18) is satisfied, i.e.

$$T_{a_1}^{j+1}(\omega) \subset T_{a_2}^{j+1}(\omega(a_2)). \tag{44}$$

For $a \in [a_1, a_2]$, let x_a be, say, the left boundary point of $\omega(a)$. Observe that, by (43), we have $\operatorname{sign}(T_a^{i}{}'|_{\omega(a)}) \equiv \operatorname{sign}(T_{a_1}^{i}{}'|_{\omega})$ for all $1 \leq i < j+1$. Let $\sigma = \operatorname{sign}(T_{a_1}^{j+1}{}'|_{\omega})$. We have to show that

$$\sigma T_{a_2}^{j+1}(x_{a_2}) \leq \sigma T_{a_1}^{j+1}(x_{a_1}). \tag{45}$$

To this end, we make use of the following general fact for skew tent maps. For $a \in [0,1]$, the image by T_a^i, $i \geq 1$, of a boundary point x of an element in $\mathcal{P}_i(a)$ is of the form

$$T_a^i(x) = T_a^m(0),$$

for some integer $1 \leq m \leq i+2$. Applied to the boundary point x_a, $a \in [a_1, a_2]$, we denote by $1 \leq m(a) \leq j+3$ the minimal integer such that $T_a^{j+1}(x_a) = T_a^{m(a)}(0)$. By the strict monotonicity of the kneading sequence there can only be finitely many $a_0 \in [a_1, a_2]$ such that $T_{a_0}^k(0) = T_{a_0}^l(0)$, for $1 \leq k \neq l \leq j+3$. Hence, by the continuity of $T_a^{j+1}(x_a)$, it follows that, disregarding an at most finite number of points, we can cover $[a_1, a_2]$ by open intervals $J \in [a_1, a_2]$ on which the integer $m(a)$ is constant. Fix such an interval J and let m denote the to it associated integer. In order to show (45), we prove that

$$\sigma T_{a'}^m(0) \leq \sigma T_a^m(0), \tag{46}$$

for all $a, a' \in J$ such that $a < a'$. Since $a \mapsto T_a^{j+1}(x_a)$ is continuous on $[a_1, a_2]$, we can extend (46) to parameters a and a' lying in the closure of J, from which we deduce (45). In order to establish (46), it is sufficient to show that, for all $a \in J$, the derivative $D_a T_a^m(0)$ exists and

$$\operatorname{sign}(D_a T_a^m(0)) = -\sigma \qquad \text{or} \qquad \operatorname{sign}(D_a T_a^m(0)) = 0. \tag{47}$$

The cases when $m = 1, 2$ or 3 are a bit special, so we treat them one by one. If $m = 1$ then $T_a^{j+1}(x_a) = 1$ for all $a \in J$, and (47) is satisfied. If $m = 2$ then, for $a \in J$, we have $T_a^{j+1}(x_a) = T_a(1) = 1 - \alpha(a)$ which is the left boundary point of $K(a)$. It follows that $\sigma = +1$. The derivative of α is non-negative in a and, hence, (47) is satisfied. If $m = 3$ then $T_a^j(x_a) = T_a(1)$ for all $a \in J$. It follows that $\operatorname{sign}(T_a^j{}'|_{\omega(a)})$ and $\operatorname{sign}(T_a'|_{T_a^j(\omega(a))})$ are both equal to $+1$ and, hence, we have $\sigma = +1$. By (42), $\operatorname{sign}(D_a T^3(0)) = -1$ which implies (47).

Finally, we turn to the case when $m > 3$. Observe that since m was chosen minimal, we have that $T_a^i(0) \neq 0$ for all $1 \leq i < m$ and all $a \in J$. It follows that $J \subset \omega'$ for some element $\omega' \in \mathcal{P}_{m-3}|[0,1]$. By (42), we deduce that the derivatives $D_a T^m(0)$ and $D_a T^{m-1}(0)$ exist and are non-zero on J. Note that since $T_a^j(x_a) \neq 0$ this implies that x_a is in fact also the left boundary point of $\tilde{\omega}(a)$. Thus, by the induction assumption, we obtain

$$\operatorname{sign}(T_{a_1}^j{}'|_{\omega}) T_{a'}^j(x_{a'}) \leq \operatorname{sign}(T_{a_1}^j{}'|_{\omega}) T_a^j(x_a),$$

for all $a, a' \in [a_1, a_2]$ such that $a < a'$, which implies that $\operatorname{sign}(D_a T_a^{m-1}(0))$ is equal to $-\operatorname{sign}(T_{a_1}^j{}'|_{\omega})$. On the other hand, by (42), we derive that

$$\operatorname{sign}(D_a T_a^m(0)) \operatorname{sign}(D_a T_a^{m-1}(0)) = \operatorname{sign}(T_a'(T_a^{m-1}(0))).$$

Since $\operatorname{sign}(T_a'(T_a^{m-1}(0))) = \operatorname{sign}(T_{a_1}'|_{T_{a_1}^j(\omega)})$, it follows that $\operatorname{sign}(D_a T_a^m(0)) = -\sigma$, which concludes the proof of (47) in the case when $m > 3$.

If x_a, $a \in [a_1, a_2]$, denotes the right boundary point of $\omega(a)$, we can do an analog argument to show that

$$\sigma T_{a_2}^{j+1}(x_{a_2}) \geq \sigma T_{a_1}^{j+1}(x_{a_1}).$$

Combined with (45) this implies inequality (44) and, thus, this concludes the verification of condition (IIb).

8. Markov partition preserving one-parameter families

Assume that we have a one-parameter family $T_a : [0,1] \to [0,1]$, $a \in I$, as described in Subsection 2.1, satisfying properties (i)-(iii). We require additionally that for each $a \in I$ the intervals $D_1(a), ..., D_{p_2}(a)$ have the following Markov property.

(M) For each $1 \leq k \leq p_2$ there exists $0 \leq i_k^L < i_k^R \leq p_1$ (independent on a), such that, for all $a \in I$,

$$T_a(D_k(a)) = (c_{i_k^L}(a), c_{i_k^R}(a)),$$

and, furthermore, these images are constant, i.e.

$$c_{i_k^L}(a) \equiv c_{i_k^L} \quad \text{and} \quad c_{i_k^R}(a) \equiv c_{i_k^R}.$$

Theorem 8.1. *If the one-parameter family T_a, $a \in I$, satisfies the Markov property (M) and if for a C^1 map $X : I \to [0,1]$ condition (I) is fulfilled, then $X(a)$ is typical for μ_a, for a.e. $a \in I$.*

Example 8.2. Let

$$\tilde{T}_a(x) = \begin{cases} \frac{x}{a} & \text{if } x < a, \\ \frac{x-a}{1-a} & \text{otherwise,} \end{cases}$$

and $g : [0,1] \to [0,1]$ a $C^{1,1}(L)$ homeomorphism such that $\inf_x g'(x) > 0$ and such that the set

$$I = \{a \in (0,1) \ ; \ \inf_x \tilde{T}_a'(g(x))g'(x) > 1\}$$

is non-empty. Clearly, I is an (open) interval. We define the one-parameter family $T_a : [0,1] \to [0,1]$ as

$$T_a(x) = \tilde{T}_a(g(x)), \qquad a \in I.$$

By [17], since T_a has only one point of discontinuity, there exists a unique a.c.i.p. μ_a. From the verification of condition (IIa) in the proof of Theorem 8.1, it will follow that $\text{supp}(\mu_a) = [0,1]$.

Proposition 8.3. *If $X : I \to (0,1)$ is a C^1 map such that $X'(a) \leq 0$, then $X(a)$ is typical for μ_a, for a.e. parameter $a \in I$.*

Proof. To fit the one-parameter family T_a into the model described in Subsection 2.1, we restrict the family to a smaller parameter interval $\tilde{I} \subsetneq I$ such that $\tilde{I}$ does not have a boundary point in common with I. Since I can be covered by a countable number of such intervals $\tilde{I}$, in order to prove Proposition 8.3, it is sufficient to consider the family T_a, $a \in \tilde{I}$. By the choice of $\tilde{I}$, it follows that there exist constants $1 < \lambda \leq \Lambda < \infty$ such that for every $a \in \tilde{I}$,

$$\lambda \leq \inf_{x \in [0,1]} T_a'(x) \qquad \text{and} \qquad \sup_{x \in [0,1]} T_a'(x) \leq \Lambda.$$

Furthermore, for $a \in \tilde{I}$, T_a is piecewise $C^{1,1}(\tilde{L})$ where

$$\tilde{L} = L \cdot \sup_{a \in \tilde{I}} \{a^{-1}, (1-a)^{-1}\}.$$

Now, one checks easily that the one-parameter family T_a, $a \in \tilde{I}$, fits into the model described in Subsection 2.1 satisfying properties (i)-(iii). Hence, we can apply Theorem 8.1 to this family. Clearly, T_a satisfies the Markov property (M). In order to proof a.s. typicality, it is only left to verify condition (I). By a similar calculation as it is done in Subsections 6.1 and 7.1, we derive, for $\omega \in \mathcal{P}_j|\tilde{I}$, $j \geq 1$, the following formula for the derivative $x_j'(a)$, $a \in \omega$:

$$x_j'(a) = T_a^j{}'(X(a))X'(a) - \sum_{i=0}^{j-1} T_a^{j-i}{}'(x_i(a)) \cdot \begin{cases} \frac{g(x_i(a))}{ag'(x_i(a))} & \text{if } g(x_i(a)) < a, \\ \frac{1-g(x_i(a))}{(1-a)g'(x_i(a))} & \text{otherwise.} \end{cases}$$

Note that this derivative is strictly negative. We obtain

$$\frac{x_j'(a)}{T_a^j{}'(X(a))} = X'(a) - \sum_{i=0}^{j-1} \frac{1}{T_a^i{}'(X(a))} \cdot \begin{cases} \frac{g(x_i(a))}{ag'(x_i(a))} & \text{if } g(x_i(a)) < a, \\ \frac{1-g(x_i(a))}{(1-a)g'(x_i(a))} & \text{otherwise.} \end{cases}$$

Set

$$s = \inf_{a \in \tilde{I}} X'(a) \qquad \text{and} \qquad \kappa = \inf_{a \in \tilde{I}} \{g(X(a)), 1 - g(X(a))\}.$$

By the choice of $\tilde{I}$, the constant s is bounded from below and κ is strictly positive. Thus, for $a \in \tilde{I}$ and $j \geq 1$, we deduce that

$$\frac{\kappa}{\sup_x g'(x)} \leq \left| \frac{x_j'(a)}{T_a^j{}'(X(a))} \right| \leq s + \sum_{i=0}^{j-1} \frac{1}{\lambda^i} \cdot \frac{1}{\inf_x g'(x)},$$

where the first term is positive and the last one bounded from above. Hence, to conclude the verification of condition (I), it is only left to show that, for $j \geq 1$, the number of $a \in \tilde{I}$, which are not contained in any element of $\mathcal{P}_j|\tilde{I}$ is finite. By the choice of X and since the point of discontinuity of T_a is strictly increasing in a, there can only be one point in the inner of the interval $\tilde{I}$ not belonging to an element of the partition $\mathcal{P}_1|\tilde{I}$. Assume that for some $j \geq 1$ there are only finitely many points in $\tilde{I}$, which are not contained in any element of the partition $\mathcal{P}_j|\tilde{I}$. For $\omega \in \mathcal{P}_j|\tilde{I}$, we have that $x_j(\omega) \subset (0,1)$ and the derivative of x_j is negative. Hence, there is at most one point $a \in \omega$ satisfying $x_j(a) = a$ and which has to be excluded in the partition $\mathcal{P}_{j+1}|\omega$. It follows that there are only finitely many points not belonging to the partition $\mathcal{P}_{j+1}|\tilde{I}$, which concludes the verification of condition (I) and, hence, the proof of the Proposition 8.3. □

We turn to the proof of Theorem 8.1.

Proof. In order to proof Theorem 8.1, it is sufficient to verify conditions (IIa) and (IIb). We first verify condition (IIa). To verify (IIb) we observe that, since T_a is preserving a Markov structure, there exists even a bijection

$$\mathcal{U}_{a_1,a_2,j} : \mathcal{P}_j(a_1) \to \mathcal{P}_j(a_2),$$

for all $a_1, a_2 \in I$ and $j \geq 1$, satisfying (17). Since, by (M), the images $T_a(D_k(a))$ are constant, we have that, for all $\omega \in \mathcal{P}_j(a_1)$,

$$T_{a_1}^j(\omega) = T_{a_2}^j(\mathcal{U}_{a_1,a_2,j}(\omega)).$$

As in the first paragraph in Subsection 6.2, by Lemma A.1, we can without loss of generality assume that there is a constant $C = C(I) \geq 1$ such that for each $a \in I$ the density φ_a is bounded from above by C and, further, there exists an interval $J(a)$ of length C^{-1} such that φ_a restricted to $J(a)$ is bounded from below by C^{-1}. Since for each $a \in I$ the expansion of T_a is at least λ, we derive that there is an integer $i \geq 1$ independent on a such that the number of elements in $\mathcal{P}_i|J(a)$ is greater or equal than 3. An element $\omega \in \mathcal{P}_i|J(a)$, which is not adjacent to a boundary point of $J(a)$, has, by

(M), image $T_a^i(\omega) = T_a(D_k(a))$ for some $1 \leq k \leq p_2$. By our assumption on the one-parameter family T_a, the measure μ_a is ergodic. It follows that there is an integer $j \geq i$ such that $K(a) = \text{closure}\{T_a^{j-i}(D_k(a))\}$. Furthermore, we can take $j \geq i$ not depending on $1 \leq k \leq p_2$. Thus, for almost every $y \in K(a)$, there exists a point $x \in J(a)$ such that x is mapped to y after j iterations, i.e. $T_a^j(x) = y$. Now, inequality (34) provides us with a lower bound for the density. Note that from this argument follows that $\text{supp}(\mu_a) = [0,1]$ in Example 8.2. □

Appendix A

Lemma A.1. *Let $T_a : [0,1] \to [0,1]$, $a \in I$, be a one-parameter family as described in Subsection 2.1, satisfying properties (i)-(iii) and condition (IIb). Disregarding a finite number of parameters in I, we can cover I by a countable number of intervals $\tilde{I} \subset I$ such that on each interval $\tilde{I}$ the following holds. There exists a constant $C = C(\tilde{I}) \geq 1$ such that for each $a \in \tilde{I}$ the density φ_a of μ_a is bounded above by C and, further, there exists an interval $J(a) \subset [0,1]$ of size C^{-1} such that φ_a restricted to $J(a)$ is bounded from below by C^{-1}.*

Proof. For each $a \in I$ it follows from [17] p.496 line 5 and [8] p.484 line 6, that the variation over the unit interval of the density φ_a is bounded above by a constant

$$C_v(a) = \frac{3}{\kappa(a)(\lambda^\tau - 3)},$$

where the integer $\tau \geq 1$ is chosen so large that $3/\lambda^\tau < 1$ and the number $\kappa(a)$ is given by

$$\kappa(a) = \min\{|\omega| \ ; \ \omega \in \mathcal{P}_\tau(a)\} > 0.$$

Claim. *For $j \geq 1$, let $(s_0, ..., s_{j-1})$ be a sequence of symbols $s_i \in \{1, ..., p_2\}$, $0 \leq i < j$. If $a_0 \in I$ is a parameter value such that there exists an element $\omega(a_0) \in \mathcal{P}_j(a_0)$ satisfying*

$$\text{symb}_{a_0}(T_{a_0}^i(\omega(a_0))) = s_i, \qquad 0 \leq i < j,$$

then there is a neighborhood U of a_0 in I such that for all $a \in U$ there is an element $\omega(a) \in \mathcal{P}_j(a)$ having the same combinatorics as $\omega(a_0)$, i.e. $\text{symb}_a(T_a^i(\omega(a))) = s_i$, $0 \leq i < j$. Furthermore, the boundary points of $\omega(a)$ and $T_a^j(\omega(a))$ depend continuously on $a \in U$.

Proof. We prove the claim by induction over $j \geq 1$. The proof is easy but a bit cumbersome, so we will give only a sketch of it. We do not make use of condition (IIb). For $j = 1$ the elements in $\mathcal{P}_1(a)$ corresponding to the symbols $s_0 \in \{1, ..., p_2\}$ are the intervals $D_k(a)$, $1 \leq k \leq p_2$. The boundary points of these intervals are the partition points $c_k(a)$, $0 \leq k \leq p_1$, which are, by property (i), continuous functions on I. Using property (ii), one can show by an easy calculation that the boundary points of $T_a(D_k(a))$ are continuous on I. Now, assume that the statement holds for some $j \geq 1$. Fix a sequence $(s_0, ..., s_j)$ of symbols in $\{1, ..., p_2\}$. Let $a_0 \in I$ be a parameter such that there exists an element $\omega(a_0) \in \mathcal{P}_{j+1}(a_0)$ satisfying $\text{symb}_{a_0}(T_{a_0}^i(\omega(a_0))) = s_i$, for all $0 \leq i < j+1$ (if there is no such a parameter a_0 for which the element $\omega(a_0)$ exists then there is nothing to show). Let $\tilde{\omega}(a_0) \in \mathcal{P}_j(a_0)$ be the element containing $\omega(a_0)$. By the induction assumption there exists a neighborhood V of a_0 in I such that for all $a \in V$ there is an element $\tilde{\omega}(a) \in \mathcal{P}_j(a)$ having the same combinatorics as $\tilde{\omega}(a_0)$ and the boundary points of $\tilde{\omega}(a)$ and $T_a^j(\tilde{\omega}(a))$ depend continuously on $a \in V$. Note that if $y(a_0)$ is a boundary point of $T_a^j(\omega(a_0))$ then it is equal to a partition point $c_k(a_0)$, $0 \leq k \leq p_1$, or it is a boundary point of $T_{a_0}^j(\tilde{\omega}(a_0))$. By the continuity of the boundary points of $T_a^j(\omega(a))$ on V and the continuity of $a \mapsto c_k(a)$, we deduce that there exists a neighborhood $U \subset V$

of a_0 in I such that for each $a \in U$ there exists an element $\omega(a) \in \mathcal{P}_{j+1}(a)$ having the same combinatorics as $\omega(a_0)$. Since the boundary points of $T_a^j(\omega(a))$ are continuous on U, we can once more apply property (ii) to deduce that also the boundary points of $T_a^{j+1}(\omega(a))$ are continuous on U. The continuity of the boundary points of $\omega(a)$ follows by a repeated use of property (iii). □

Let $(s_0, ..., s_{j-1})$ be a sequence of symbols $s_i \in \{1, ..., p_2\}$, and for each $a \in I$ let $\omega(a) \in \mathcal{P}_j(a)$ be — if it exists — the to it associated element as in the claim above. Writing $|\omega(a)| = 0$ if such an element does not exists it follows immediately from the claim that the map $a \mapsto |\omega(a)|$ is continuous on I. Furthermore, by condition (IIb), if $|\omega(a_0)| > 0$ for some $a_0 \in I$, then $|\omega(a)| > 0$ for all $a \geq a_0$. This implies that the map $a \mapsto \kappa(a)$ is piecewise continuous on I with only a finite number of discontinuities. Hence, disregarding a finite number of parameter values in I, we can cover I by a countable number of intervals $\tilde{I} \subset I$ such that for each such interval $\tilde{I}$ there is a constant $\kappa_0 = \kappa_0(\tilde{I}) > 0$ such that

$$\kappa(a) \geq \kappa_0, \tag{48}$$

for all $a \in \tilde{I}$. It follows that that there is a constant $C_v = C_v(\tilde{I}) \geq 1$ such that the variation of φ_a is bounded from above by C_v for all $a \in \tilde{I}$. Since $\int_0^1 \varphi_a(x)dx = 1$, this immediately implies that φ_a is bounded from above by $C_v + 1$. To establish a lower bound on a subinterval of $K(a)$, we observe the following.

Claim. *If the variation over $[0,1]$ of a function $\varphi : [0,1] \to \mathbb{R}_+$ is bounded from above by a constant $C_v \geq 1$, and if $\int_0^1 \varphi(x)dx = 1$, then there exists an interval J of length $1/2C_v$ such that $\varphi(x) \geq 1/3C_v$ for all $x \in J$.*

Proof. Let $N = [2C_v]$, divide the unit interval into N disjoint intervals $J_1, ..., J_N$ of length $1/N$, and, for $1 \leq l \leq N$, set $m_l = \inf\{\varphi(x) \; ; \; x \in J_l\}$ and $M_l = \sup\{\varphi(x) \; ; \; x \in J_l\}$. Since $1 = \int_0^1 \varphi(x)dx \leq \sum_{l=1}^N M_l/N$, it follows that $N \leq \sum_{l=1}^N M_l$. If $m_l < 1/3C_v$, for all $1 \leq l \leq N$, it would follow that the variation of φ is strictly greater than $\sum_{l=1}^N (M_l - 1/3C_v) \geq N(1 - 1/3C_v) \geq C_v$, where the last inequality follows since $C_v \geq 1$. Hence, at least for one $1 \leq l \leq N$, $m_l \geq 1/3C_v$. □

Setting $C = 3C_v$ this concludes the proof of Lemma A.1. □

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Paper E

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Almost sure equidistribution in expansive families ☆

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ABSTRACT

In this paper we study generic equidistribution in families of sequences of points on tori. We assume that the sequences are parameterized by some subset of a Euclidean space, and we formulate geometric conditions on the dependence so that equidistribution holds almost everywhere with respect to the Lebesgue measure on the parameter space. As a consequence, we can give a new proof of an old result by Koksma.

1. INTRODUCTION

Equidistribution of sequences of real numbers modulo 1 is a very classical and well-studied area of research (see e.g. [2,3,5] and the references therein). The main inspiration for this paper is the following classical and well-known result by Koksma [4] which can also be generalized to a larger class of real sequences: For almost every $\theta > 1$, with respect to the Lebesgue measure on $\mathbb{R}$, the sequence $\theta^j \bmod 1$, $j \geqslant 1$, is equidistributed in $\mathbb{T}$, i.e. for any interval A of the unit circle $\mathbb{T}$,

$$\frac{\#\{j;\, \theta^j \bmod 1 \in A,\, 1 \leqslant j \leqslant N\}}{N} \to |A|, \quad \text{as } N \to \infty,$$

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where $|\cdot|$ denotes the Lebesgue measure on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. Koksma proved this result using Fourier analysis and reduced the statement to the combinatorics of exponential sums.

In this paper we will discuss a geometric method to prove Koksma's result based on the techniques developed by Benedicks and Carleson [1] in one-dimensional dynamics. This method can also be used to establish higher dimensional analogues of Koksma's theorem.

Let I be an open set in $\mathbb{R}^d$, and let $\tilde{f}_j$ be a sequence of maps on I into $\mathbb{R}^d$. Define, for a fixed θ in I, the sequence of points $f_j(\theta) = \tilde{f}_j(\theta) \bmod \mathbb{Z}^d$ in $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$.

What are natural geometric conditions on the sequence $\tilde{f}_j$ so that we can ensure equidistribution of the sequence $f_j(\theta)$ in $\mathbb{T}^d$, for Lebesgue almost every θ in I?

Koksma studied the case $\tilde{f}_j(\theta) = \theta^j$, and $I = (1, \infty) \subset \mathbb{R}$. Two special features of this example are expansion and distortion. Here, expansion refers to the fact that

$$\frac{\theta^{j+k}}{\theta^j} = \theta^k$$

is growing sufficiently fast in k, for all $j \geqslant 1$, and distortion simply means that the quotient of the derivatives of $\tilde{f}_j$, restricted to the pre-image of a unit interval in $(1, \infty)$ is close to 1, provided that j is sufficiently large. We will see that these two properties are sufficient to conclude that, for almost every θ in I, the sequence $f_j(\theta)$ is equidistributed in $\mathbb{T}$. It is straightforward to generalize these two conditions to higher dimensions.

2. MAIN STATEMENT

Let

$$\begin{aligned} \tilde{f} &: \mathbb{N} \times I \to \mathbb{R}^d \\ (j, \theta) &\mapsto \tilde{f}_j(\theta), \end{aligned}$$

where $I \subset \mathbb{R}^d$ open and $d \geqslant 1$. For each $j \geqslant 1$ we assume that $\tilde{f}_j$ is a C^1 function in θ which is one-to-one and whose Jacobian is never 0. We put two conditions on $\tilde{f}$, one concerning the expansion and one concerning the distortion properties of $\tilde{f}$.

(I) There is $0 < \kappa < 1$ and an at least polynomially growing function $g : \mathbb{N} \to \mathbb{R}^+ \cup \{0\}$, i.e.

$$\varliminf_{n \to \infty} \frac{g(n)}{n^\gamma} > 0, \quad \text{for some } \gamma > 0,$$

such that for $1 \leqslant j \leqslant n$ and $k \geqslant n^\kappa$

$$\frac{|D_\theta \tilde{f}_{j+k}(\theta) v|}{|D_\theta \tilde{f}_j(\theta) v|} \geqslant g(n),$$

for all $\theta \in I$ and $v \in \mathbb{R}^d \setminus \{0\}$.

(II) For each $\varepsilon > 0$ and $r > 0$ there is an integer $j_{\varepsilon,r} \geqslant 1$ such that the following holds. Let $B(x,r)$ denote the open ball in $\mathbb{R}^d$ with radius r and center x. For all $x \in \mathbb{R}^d$ and all $\theta, \theta' \in \tilde{f}_j^{-1}(B(x,r) \cap \tilde{f}_j(I))$, $j \geqslant j_{\varepsilon,r}$, we have

$$\frac{|D_\theta \tilde{f}_j(\theta)|}{|D_\theta \tilde{f}_j(\theta')|} \leqslant 1 + \varepsilon,$$

where $|D_\theta \tilde{f}_j(\theta)|$ is the Jacobian of $D_\theta \tilde{f}_j(\theta)$.

Remark. A weaker version of the first condition is:

(I)′ There is an at least polynomially growing function $g : \mathbb{N} \to \mathbb{R}^+ \cup \{0\}$, such that for $j \geqslant 1$ and $k \geqslant 1$,

$$\frac{|D_\theta \tilde{f}_{j+k}(\theta) v|}{|D_\theta \tilde{f}_j(\theta) v|} \geqslant g(k),$$

for all $\theta \in I$ and $v \in \mathbb{R}^d \setminus \{0\}$.

Obviously condition (I)′ implies condition (I) (with an arbitrarily chosen κ). The main reason for stating a refined version in condition (I) is that we want to include examples as $\tilde{f}_j(\theta) = \theta^{\sqrt{j}}$ (see Example 4.1). The introduction of the constant κ in (I) is natural in view of the estimate of the exceptional terms in the sum in (2).

Let Γ be a lattice in $\mathbb{R}^d$. We define $f : \mathbb{N} \times I \to \mathbb{R}^d/\Gamma$ as

$$f_j(\theta) = \tilde{f}_j(\theta) \bmod \Gamma.$$

Theorem 2.1. *If $\tilde{f}$ fulfills conditions* (I) *and* (II)*, then the sequence $f_j(\theta)$, $j \geqslant 1$, is equidistributed in $\mathbb{R}^d/\Gamma$, for Lebesgue almost every $\theta \in I$, i.e.*

$$\frac{1}{n} \sum_{j=1}^{n} \delta_{f_j(\theta)} \overset{weak\text{-}*}{\longrightarrow} m, \quad \text{as } n \to \infty,$$

where m denotes the Haar measure on $\mathbb{R}^d/\Gamma$.

3. PROOF OF THEOREM 2.1

Let $\mathcal{Q}$ be the set of open parallelepipeds in $\mathbb{R}^d$ related to the lattice Γ, i.e. $\mathcal{Q}$ is the set of all open parallelepipeds Q such that the intersection of the closure of Q and Γ is exactly the set of vertices of Q. Since we can cover I by a countable union of open balls $B(x,r)$ we can without loss of generality assume that $I = B(x_0, r_0)$ for some $x_0 \in \mathbb{R}^d$ and $r_0 > 0$.

Fix a parallelepiped $Q_0 \in \mathcal{Q}$. Let

$$\mathcal{B} := \left\{ B(x,r) \bmod \Gamma;\ B(x,r) \subset Q_0, x \in Q_0 \cap \mathbb{Q}^d, r \in \mathbb{Q}^+ \right\}.$$

We will show that, for $B \in \mathcal{B}$, the function

$$F_n(\theta) = \frac{1}{n}\sum_{j=1}^{n} \chi_B\big(f_j(\theta)\big), \quad n \geqslant 1,$$

fulfills

$$\text{(1)} \qquad \varlimsup_{n\to\infty} F_n(\theta) \leqslant m(B), \quad \text{for a.e. } \theta \in I.$$

By a standard argument (see e.g. [6]), (1) implies that, for a.e. $\theta \in I$, every weak-$*$ limit point μ_θ of

$$\frac{1}{n}\sum_{j=1}^{n} \delta_{f_j(\theta)}$$

is absolutely continuous with respect to m in $(\mathbb{R}^d/\Gamma) \setminus \partial Q_0$ where the density satisfies $d\mu_\theta/dm \leqslant 1$. Observe that

$$\Big|\bigcap_{k\geqslant 1}\{\theta \in I;\, f_j(\theta) \in \partial Q_0 \operatorname{mod}\Gamma, \text{ for some } j \geqslant k\}\Big| = 0,$$

where $|\cdot|$ denotes the Lebesgue measure in $\mathbb{R}^d$. Hence, for a.e. $\theta \in I$, the probability measure μ_θ gives zero measure to $\partial Q_0 \operatorname{mod}\Gamma$. It follows that, for a.e. $\theta \in I$, $\mu_\theta = m$ which implies Theorem 2.1.

From now $B \in \mathcal{B}$ is fixed. Let $r_B = \min\{\text{radius}(B), r_0\}$. In order to prove (1) it is sufficient to show that there exists a constant $C > 0$ such that for all $\varepsilon_0 > 0$ and integers $h \geqslant 1$ there is an integer n_{h,ε_0,r_B} growing at most exponentially in h, such that $\int_I F_n(\theta)^h\, d\theta \leqslant C((1+\varepsilon_0)m(B))^h$ for all $n \geqslant n_{h,\varepsilon_0,r_B}$. See 4.2.

Henceforth, fix $h \geqslant 1$.

$$\text{(2)} \qquad \int_I F_n(\theta)^h\, d\theta = \sum_{1\leqslant j_1,j_2,\ldots,j_h\leqslant n} \frac{1}{n^h}\int_I \chi_B\big(f_{j_1}(\theta)\big)\cdots\chi_B\big(f_{j_h}(\theta)\big)\, d\theta.$$

Let $0 < \kappa < 1$ be the constant in condition (I). The number of h-tuples $(j_1, \ldots, j_h)$ in (2), for which $\min_l j_l < n^\kappa$ or $\min_{k\neq l} |j_k - j_l| < n^\kappa$, is bounded by $2h^2 n^{h-(1-\kappa)}$. The sum over these (exceptional) terms in (2) is therefore bounded by $|I| 2h^2 n^{-(1-\kappa)}$. In the following proposition we treat all the other terms in (2), i.e. the terms related to the h-tuples which are most likely to occur.

Proposition 3.1. *For all $\varepsilon_0 > 0$ and $h \geqslant 1$, there is an integer n_{h,ε_0,r_B} growing at most exponentially in h, such that for all $n \geqslant n_{h,\varepsilon_0,r_B}$ and for all h-tuples $(j_1, \ldots, j_h)$ with $1 \leqslant j_1 < j_2 < \cdots < j_h \leqslant n$, $j_1 \geqslant n^\kappa$ and $j_l - j_{l-1} \geqslant n^\kappa$, $l = 2, \ldots, h$,*

$$\int_I \chi_B\big(f_{j_1}(\theta)\big)\cdots\chi_B\big(f_{j_h}(\theta)\big)\, d\theta \leqslant 2|I|\big((1+\varepsilon_0)m(B)\big)^h.$$

From a probabilistic point of view this proposition tells us that whenever the distances between consecutive j_l's are sufficiently large, the behavior of the $\chi_B(f_{j_l}(.))$'s is comparable to that of independent random variables.

Proposition 3.1 implies

$$\int_I F_n(\theta)^h \, d\theta \leqslant 2|I|\big((1+\varepsilon_0)m(B)\big)^h + |I| 2h^2 n^{-(1-\kappa)}$$
$$\leqslant 3|I|\big((1+\varepsilon_0)m(B)\big)^h,$$

for all

$$n \geqslant \max\left\{ n_{h,\varepsilon_0,r_B}, \left(\frac{2h^2}{((1+\varepsilon_0)m(B))^h}\right)^{1/(1-\kappa)} \right\}.$$

Since both terms in this lower bound for n grow at most exponentially in h, this concludes the proof of Theorem 2.1. Now we turn to the proof of Proposition 3.1.

Proof of Proposition 3.1. Let $\tilde{B}$ denote the lift of B to $\mathbb{R}^d$. We have

$$\int_I \chi_B\big(f_{j_1}(\theta)\big) \cdots \chi_B\big(f_{j_h}(\theta)\big) \, d\theta = |\{\theta \in I;\ \tilde{f}_{j_1}(\theta) \in \tilde{B}, \ldots, \tilde{f}_{j_h}(\theta) \in \tilde{B}\}|.$$

For $J \subset I$ open and $j \geqslant 1$ we define the partition $\mathcal{P}_j|J$ on J as

$$\mathcal{P}_j|J := \big\{\tilde{f}_j^{-1}\big(Q \cap \tilde{f}_j(J)\big);\ Q \in \mathcal{Q}\big\}.$$

For $j = 0$ we set $\mathcal{P}_0|J = J$ and $\tilde{f}_0(\theta) = \theta$. We give first a sketch of the proof of Proposition 3.1. Note that by the expansion property of the $\tilde{f}_j$'s we have that for large n a typical partition element $\omega \in \mathcal{P}_{j_1}|I$ is mapped by $\tilde{f}_{j_1}$ onto the whole of a parallelepiped $Q \in \mathcal{Q}$, i.e. we can neglect the elements in $\mathcal{P}_{j_1}|I$ adjacent to the boundary of I (the union of these boundary elements is the exceptional set E_0 defined below). By the distortion property of the $\tilde{f}_j$'s we have that roughly speaking only a $|B|/|Q| (= m(B))$ fraction of the element ω is mapped by $\tilde{f}_{j_1}$ onto $\tilde{B} \cap Q$. Considering only the part J of ω which is mapped onto $\tilde{B} \cap Q$ we can now, by using that $j_2 - j_1$ is large, repeat the argument for the elements in the partition $\mathcal{P}_{j_2}|J$. Going on like this we derive Proposition 3.1. In the remaining part we will work this out in more detail.

We say that an element $\omega \in \mathcal{P}_j|J$, $j \geqslant 1$, is an *entire* element if there is a $Q \in \mathcal{Q}$ such that $\tilde{f}_j(\omega) = Q$. Set

$$I_0 = \{\text{entire } \omega \in \mathcal{P}_{j_1}|I\},$$
$$I_l = \{\text{entire } \omega \in \mathcal{P}_{j_{l+1}}|I_{l-1};\ \tilde{f}_{j_l}(\omega) \subset \tilde{B}\},$$

for $1 \leqslant l < h$, and

$$I_h = \{\theta \in I_{h-1};\ \tilde{f}_{j_h}(\theta) \in \tilde{B}\}.$$

We consider the set I_l, $0 \leqslant l < h$, as a set of partition elements in $\mathcal{P}_{j_{l+1}}|I$ as well as an open set in I. Let

$$E_0 = \{\omega \in \mathcal{P}_{j_1}|I; \omega \notin I_0\},$$

and, for $1 \leqslant l < h$,

$$E_l = \{\omega \in \mathcal{P}_{j_{l+1}}|I_{l-1}; \omega \notin I_l, \tilde{f}_{j_l}(\omega) \cap \tilde{B} \neq \emptyset\}.$$

Observe that the union of these sets contains (modulo a Lebesgue measure zero set) the set we are interested in, i.e.

$$\{\theta \in I; \tilde{f}_{j_1}(\theta) \in \tilde{B}, \ldots, \tilde{f}_{j_h}(\theta) \in \tilde{B}\} \overset{\circ}{\subset} I_h \cup \left(\bigcup_{l=0}^{h-1} E_l\right). \tag{3}$$

We state two lemmas. Provided that n is sufficiently large, the first lemma indicates that we can essentially deal with entire elements only, i.e. the E_l's are exceptional sets which can be neglected, and, thus, if $\omega \in \mathcal{P}_{j_{l+1}}|I$, $1 \leqslant l < h$, and $\tilde{f}_{j_l}(\omega) \cap \tilde{B} \neq \emptyset$ then we can assume – without loss of generality – that ω is an entire element. The main ingredient in the proof of this lemma is condition (I). Using condition (II), the second lemma gives a proof of Proposition 3.1 for the 'nice' set I_h.

Lemma 3.2. *For all $\varepsilon > 0$ and $r > 0$, there is an integer $n_{\varepsilon,r}$ growing at most polynomially in $\frac{1}{\varepsilon}$, such that for $n \geqslant n_{\varepsilon,r}$ the following holds. Assume $j = 0$ or $n^\kappa \leqslant j \leqslant n$, and $k \geqslant n^\kappa$. For $B(x,r) \subset \tilde{f}_j(I)$ set $J = \tilde{f}_j^{-1}(B(x,r))$,*

$$J' = \{\textit{entire } \omega \in \mathcal{P}_{j+k}|J\},$$

and

$$E_J = \left\{\omega \in \mathcal{P}_{j+k}|I; \omega \notin J', \tilde{f}_j(\omega) \cap B(x,r) \neq \emptyset\right\}.$$

We have that

$$|E_J| \leqslant \varepsilon |J|.$$

Proof. By condition (I), for $\omega \in \mathcal{P}_{j+k}|I$, $\operatorname{diam}(\tilde{f}_j(\omega)) \leqslant \operatorname{diam}(Q_0)/g(n)$. Hence,

$$|\tilde{f}_j(E_J)| \leqslant \frac{2\operatorname{diam}(Q_0)\operatorname{Vol}_{d-1}(\partial B(x,r))}{g(n)}.$$

Let $j_{1,r}$ be the integer in condition (II). Take $n_{\varepsilon,r} \geqslant (j_{1,r})^{1/\kappa}$ minimal such that for $n \geqslant n_{\varepsilon,r}$,

$$g(n) \geqslant \frac{4\operatorname{diam}(Q_0)\operatorname{Vol}_{d-1}(\partial B(x,r))}{\varepsilon |B(x,r)|}.$$

Recall that, by (I), $g(n)$ is at least polynomially growing in n, hence $n_{\varepsilon,r}$ is at most polynomially growing in $\frac{1}{\varepsilon}$. For $n \geqslant n_{\varepsilon,r}$, we obtain

$$|\tilde{f}_j(E_J)| \leqslant 2^{-1}\varepsilon|B(x,r)|.$$

If $j = 0$ we are done. Otherwise we have $j \geqslant n^\kappa \geqslant j_{1,r}$, and it follows by the distortion estimate in condition (II) that $|E_J| \leqslant \varepsilon|J|$. □

Lemma 3.3. *Set* $r_\Gamma = \mathrm{diam}(Q_0)/2$ *and let* $j_{\varepsilon_0,r_\Gamma}$ *be the integer in condition* (II). *If* $n^\kappa \geqslant j_{\varepsilon_0,r_\Gamma}$ *then*

$$|I_l| \leqslant (1+\varepsilon_0)m(B)|I_{l-1}|, \quad \textit{for } 1 \leqslant l \leqslant h.$$

Proof. Let $\omega \in I_{l-1}$. By the definition of I_{l-1}, ω is an entire element in $\mathcal{P}_{j_l}|I$. Since $j_l \geqslant n^\kappa \geqslant j_{\varepsilon_0,r_\Gamma}$ we have by condition (II),

$$|\{\theta \in \omega;\, \tilde{f}_{j_l}(\theta) \in \tilde{B}\}| \leqslant (1+\varepsilon_0)\frac{|\tilde{B} \cap Q_0|}{|Q_0|}|\omega| = (1+\varepsilon_0)m(B)|\omega|.$$

Hence,

$$|\{\theta \in I_{l-1};\, \tilde{f}_{j_l} \in \tilde{B}\}| \leqslant (1+\varepsilon_0)m(B)|I_{l-1}|.$$

Since $I_l \subset \{\theta \in I_{l-1};\, \tilde{f}_{j_l}(\theta) \in \tilde{B}\}$ this concludes the proof. □

Recall that $r_B = \min\{\mathrm{radius}(B), r_0\}$. Let $\varepsilon_1 = ((1+\varepsilon_0)m(B))^h/h$ and n_{ε_1,r_B} the integer in Lemma 3.2, and set

$$n_{h,\varepsilon_0,r_B} = \max\{n_{\varepsilon_1,r_B}, (j_{\varepsilon_0,r_\Gamma})^{1/\kappa}\}.$$

Since n_{ε_1,r_B} is at most polynomially growing in $\frac{1}{\varepsilon_1}$, it follows that n_{h,ε_0,r_B} is at most exponentially growing in h. Henceforth, assume $n \geqslant n_{h,\varepsilon_0,r_B}$. Setting $j = 0$, $k = j_1$ and $J = I(= B(x_0,r_0))$ in Lemma 3.2, it follows immediately that $|E_0| \leqslant \varepsilon_1|I|$. Now let $\omega \in I_{l-1}$, $1 \leqslant l < h$. By the definition of I_{l-1}, ω is an entire element in $\mathcal{P}_{j_l}|I$. Set, in Lemma 3.2, $j = j_l$, $k = j_{l+1} - j_l$ and $J = \tilde{f}_{j_l}^{-1}(\tilde{B} \cap \tilde{f}_{j_l}(\omega))$, i.e. J is the part of ω which is mapped by f_{j_l} onto B. Then $J' = I_l \cap \omega$, and we obtain

$$\left|\left\{\omega' \in \mathcal{P}_{j_{l+1}}|\omega;\, \omega' \notin I_l,\, \tilde{f}_{j_l}(\omega') \cap \tilde{B} \neq \emptyset\right\}\right| \leqslant \varepsilon_1|J| \leqslant \varepsilon_1|\omega|.$$

Observe that,

$$E_l = \bigcup_{\omega \in I_{l-1}} \left\{\omega' \in \mathcal{P}_{j_{l+1}}|\omega;\, \omega' \notin I_l,\, \tilde{f}_{j_l}(\omega') \cap \tilde{B} \neq \emptyset\right\}.$$

Thus $|E_l| \leqslant \varepsilon_1|I_{l-1}| \leqslant \varepsilon_1|I|$. By (3), Lemma 3.3 and the choice of ε_1,

$$\begin{aligned}&|\{\theta \in I;\, \tilde{f}_{j_1}(\theta) \in \tilde{B}, \ldots, \tilde{f}_{j_h}(\theta) \in \tilde{B}\}| \\ &\quad \leqslant |I_h| + \sum_{l=0}^{l-1}|E_l| \leqslant \left((1+\varepsilon_0)|B|\right)^h|I_0| + h\varepsilon_1|I| \leqslant 2|I|\left((1+\varepsilon_0)|B|\right)^h,\end{aligned}$$

which concludes the proof of Proposition 3.1. □

4. EXAMPLES

We give two simple examples which can be derived from Theorem 2.1. Note that both examples also can be derived from a combination of Weyl's lemma and Koksma's theorem.

Example 4.1. Let Γ be a lattice in $\mathbb{R}^d$. Fix $\alpha = (\alpha_1, \dots, \alpha_d) \in (0, \infty)^d$. There is a subset X_α of $(1, \infty)^d$ of full Lebesgue measure, such that the sequence $(\theta_1^{j^{\alpha_1}}, \dots, \theta_d^{j^{\alpha_d}}) \bmod \Gamma$, $j \geqslant 1$, is equidistributed in $\mathbb{R}^d/\Gamma$, for all θ in X_α.

Proof. Let $\tilde{f}_j(\theta) = (\theta_1^{j^{\alpha_1}}, \dots, \theta_d^{j^{\alpha_d}})$, defined on $I = (\theta_0, \infty)^d$ for some $\theta_0 > 1$. To apply Theorem 2.1, we have to check that (I) and (II) hold for the functions $\tilde{f}_j : I \to \mathbb{R}^d$. Set $\alpha_0 = \min\{1, \alpha_1, \dots, \alpha_d\}$, and choose $0 < \kappa < 1$ such that $\alpha_0 + \kappa > 1$. Note that for large n,

$$\big(n + n^\kappa\big)^{\alpha_0} - n^{\alpha_0} \geqslant \frac{\alpha_0}{2} \frac{n^{\alpha_0 + \kappa}}{n}.$$

Thus, if $j \leqslant n$ and $k \geqslant n^\kappa$,

$$\frac{|D_\theta \tilde{f}_{j+k}(\theta) v|}{|D_\theta \tilde{f}_j(\theta) v|} \geqslant \theta_0^{(n+n^\kappa)^{\alpha_0} - n^{\alpha_0}},$$

is growing faster than polynomially, for all $v \in \mathbb{R}^d \setminus \{0\}$. This implies (I).

To verify (II), we first observe that any points θ, θ' in $\tilde{f}_j^{-1}(B(x, r) \cap \tilde{f}_j(I))$, are on distance at most $2r / j^{\alpha_0} \theta_0^{j^{\alpha_0} - 1}$. Thus,

$$\frac{|D_\theta \tilde{f}_j(\theta)|}{|D_\theta \tilde{f}_j(\theta')|} \prod_{i=1}^{d} \left(\frac{\theta_i}{\theta_i'}\right)^{j^{\alpha_i} - 1} \leqslant \prod_{i=1}^{d} \left(1 + \frac{2r}{j^{\alpha_0} \theta_0^{j^{\alpha_0} - 1}}\right)^{j^{\alpha_i} - 1},$$

where the right-hand side converges uniformly to 1 when j increases. This implies (II). □

Even if the above result tells us that the sequence $(\theta_1^{j^{\alpha_1}}, \dots, \theta_d^{j^{\alpha_d}}) \bmod \Gamma$, $j \geqslant 1$, is almost surely equidistributed in $\mathbb{R}^d/\Gamma$, it is very hard to determine whether this sequence is equidistributed for some given $\theta \in (1, \infty)^d$. For instance, in one-dimension it is not known whether $(\frac{3}{2})^j$ or e^j modulo 1 are equidistributed in $\mathbb{T}$. However, going in an other direction the next example asserts that j^θ-powers of a fixed $\alpha \in (1, \infty)^d$ are equidistributed, for almost every θ:

Example 4.2. For any $\alpha \in (1, \infty)^d$, there is a full measure subset X_α of $(0, \infty)^d$, such that the sequence $(\alpha_1^{j^{\theta_1}}, \dots, \alpha_d^{j^{\theta_d}}) \bmod \Gamma$, $j \geqslant 1$, is equidistributed in $\mathbb{R}^d/\Gamma$, for all θ in X_α.

Proof. Let $\tilde{f}_j(\theta) = (\alpha_1^{j^{\theta_1}}, \ldots, \alpha_d^{j^{\theta_d}})$, defined on $I = (\theta_0, \infty)^d$ for some $\theta_0 > 0$. Choosing $0 < \kappa < 1$ such that $\theta_0 + \kappa > 1$ condition (I) is verified as in Example 4.1. Let $\alpha_0 = \min\{\alpha_1, \ldots, \alpha_d\}$. To establish condition (II), we note that

$$\frac{|D_\theta \tilde{f}_j(\theta)|}{|D_\theta \tilde{f}_j(\theta')|} = \prod_{i=1}^{d} j^{\theta_i - \theta_i'} \alpha_i^{j^{\theta_i} - j^{\theta_i'}},$$

and for large j, $|D_\theta \tilde{f}_j(\theta) v| \geqslant \alpha_0^{j^{\theta_0}}$ for all $v \in \mathbb{R}^d \setminus \{0\}$. For $\theta, \theta' \in \tilde{f}_j^{-1}(B(x,r) \cap \tilde{f}_j(I))$ the distance between θ and θ' is less than $2r\alpha_0^{-j^{\theta_0}}$. Hence, when $j \to \infty$, $j^{\theta_i - \theta_i'}$ converges to 1 and (assuming $\theta_i' < \theta_i$)

$$j^{\theta_i} - j^{\theta_i'} \leqslant j^{\theta_i}\bigl(\theta_i - \theta_i'\bigr) \log j \to 0, \quad j \to \infty.$$

We conclude that condition (II) holds. □

Remark. We have restrained from making very general statements. The methods in this paper can probably be pushed without too much hard labor to nil-manifolds, with conditions (I) and (II) replaced by natural expansion and distortion properties of the cover maps. With some minor restrictions on the maps involved, the methods should also apply to compact locally symmetric spaces of non-compact type.

APPENDIX

For the sake of completeness we add the following fact from measure theory, which we believe has independent interest. Let $I \in \mathcal{B}(\mathbb{R}^d)$, $e_j : I \to [0, \infty)$, $j \geqslant 1$, measurable functions, and set

$$F_n(\theta) = \frac{1}{n} \sum_{j=1}^{n} e_j(\theta).$$

Lemma A.1. *Assume that for all $h \geqslant 1$ there is an integer n_h such that*

$$\int_I F_n(\theta)^h \, d\theta \leqslant C\varepsilon^h,$$

for all $n \geqslant n_h$, where C is some constant independent of h. If the sequence n_h grows at most exponentially in h then it follows that $\overline{\lim}_{n\to\infty} F_n(\theta) \leqslant \varepsilon$ for Lebesgue a.e. $\theta \in I$.

Proof. By possibly increasing the n_h's we can assume that $n_h = 2^{hk}$, for some fixed integer k. Let $\delta > 0$ and l, H be integers such that $2^{-l}(2^k - 1) \leqslant \delta$, and $Hk \geqslant l$. Consider the sequence m_i, $i \geqslant 0$, defined as

$$m_i = 2^{(H + [i2^{-l}])k} + \bigl(i - [i2^{-l}]2^l\bigr) 2^{(H + [i2^{-l}])k - l} \bigl(2^k - 1\bigr).$$

This sequence of integers is defined such that, for $h \geqslant H$,

(4) $\quad \#\{i; n_h \leqslant m_i < n_{h+1}\} = 2^l,$

and the distance between two successive m_i's lying in the interval $[n_h, n_{h+1}]$ is constant. Furthermore, one easily verifies that

(5) $\quad \dfrac{m_{i+1}}{m_i} \leqslant 1 + 2^{-l}\big(2^k - 1\big) \leqslant 1 + \delta,$

for $i \geqslant 0$. Using (4), we get

$$\begin{aligned}
&\Big|\bigcap_{j\geqslant 0}\bigcup_{i\geqslant j}\{F_{m_i} \geqslant (1+\delta)\varepsilon\}\Big| \\
&\quad\leqslant \lim_{j\to\infty}\sum_{i\geqslant j}|\{F_{m_i} \geqslant (1+\delta)\varepsilon\}| \leqslant \lim_{j\to\infty}\sum_{i\geqslant j}\frac{\int_I F_{m_i}(\theta)^{H+[i2^{-l}]}\,d\theta}{((1+\delta)\varepsilon)^{H+[i2^{-l}]}} \\
&\quad\leqslant \lim_{j\to\infty} C\sum_{i\geqslant j}\left(\frac{\varepsilon}{(1+\delta)\varepsilon}\right)^{H+[i2^{-l}]} \leqslant \lim_{j\to\infty} C2^l\sum_{h\geqslant j}\frac{1}{(1+\delta)^h} = 0.
\end{aligned}$$

It follows that $\overline{\lim}_{i\to\infty} F_{m_i}(\theta) \leqslant (1+\delta)\varepsilon$ for all θ in a set I' which has full Lebesgue measure in I. Fix $\theta \in I'$. For sufficiently large i we have $F_{m_i}(\theta) \leqslant (1+\delta)^2\varepsilon$, and using the definition of F_n and inequality (5), we obtain, for $1 \leqslant j < m_{i+1} - m_i$,

$$F_{m_i+j}(\theta) \leqslant \frac{m_{i+1}}{m_i + j}F_{m_{i+1}}(\theta) \leqslant \frac{m_{i+1}}{m_i}(1+\delta)^2\varepsilon \leqslant (1+\delta)^3\varepsilon.$$

It follows that $\overline{\lim}_{n\to\infty} F_n(\theta) \leqslant (1+\delta)^3\varepsilon$ for all $\theta \in I'$. Thus, since $\delta > 0$ was arbitrary, $\overline{\lim}_{n\to\infty} F_n(\theta) \leqslant \varepsilon$ for a.e. $\theta \in I$. □

To conclude this appendix we give an example showing that n_h cannot grow arbitrarily fast in h. Let $I = [0, 1]$ and consider the super-exponentially growing sequence $n_h = 3^{2^h}$, $h \geqslant 1$. Set $e_j(a) \equiv 0$ for $j < n_1$ and for $n_h \leqslant j < n_{h+1}$, $h \geqslant 1$, let

$$e_j(a) = \chi_{[\frac{k}{2^{h-1}}, \frac{k+1}{2^{h-1}}]}(a),$$

if $3^{2k}n_h \leqslant j < 3^{2k+1}n_h$, $0 \leqslant k < 2^{h-1}$, and $e_j(a) \equiv 0$ otherwise. It can easily be verified that for each $n_h \leqslant n < n_{h+1}$, the average function $F_n(a)$ is smaller or equal than $1/3$ everywhere except on an interval of length $1/2^{h-1}$ and furthermore, for every $a \in I$, there exists an $n_h \leqslant n < n_{h+1}$ such that $F_n(a) \geqslant 2/3$. It follows that $\overline{\lim}_{n\to\infty} F_n(a) \geqslant 2/3$, for all $a \in I$. On the other hand, for $h \geqslant 1$,

$$\int_I F_n(a)^h\,da \leqslant \left(\frac{1}{3}\right)^h + \frac{1}{2^{h-1}} \leqslant 3\left(\frac{1}{2}\right)^h,$$

if $n \geqslant n_h$.

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