École doctorale IAEM

Informatique

Transversal Helly numbers, pinning theorems and projection of simplicial complexes

Habilitation thesis

Xavier Goaoc

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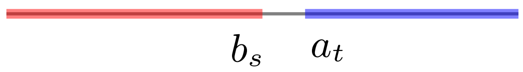

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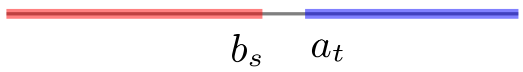

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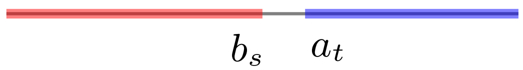

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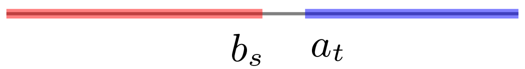

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If $\mathcal{F}$ has empty intersection then two of its members already have empty intersection.

This is what Helly numbers capture:

situations where empty intersection of arbitrary large families
can be traced back to constant-size sub-families.

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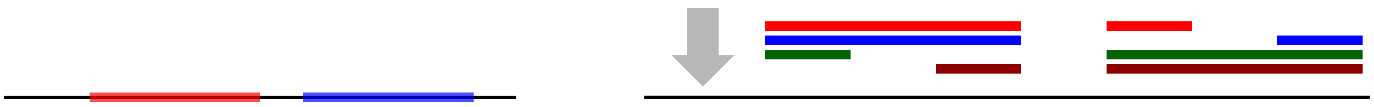

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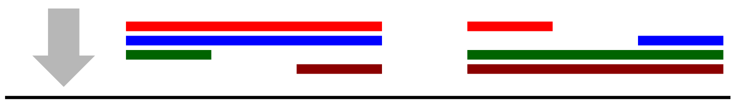

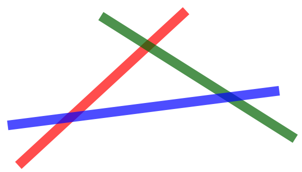

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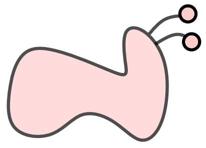

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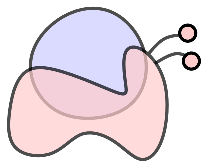

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Helly numbers $\simeq$ notion of combinatorial dimension in generalized linear programming.

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Helly numbers also arise naturally in discrete geometry, topology, algebra...

This presentation discusses Helly numbers of sets of line transversals.

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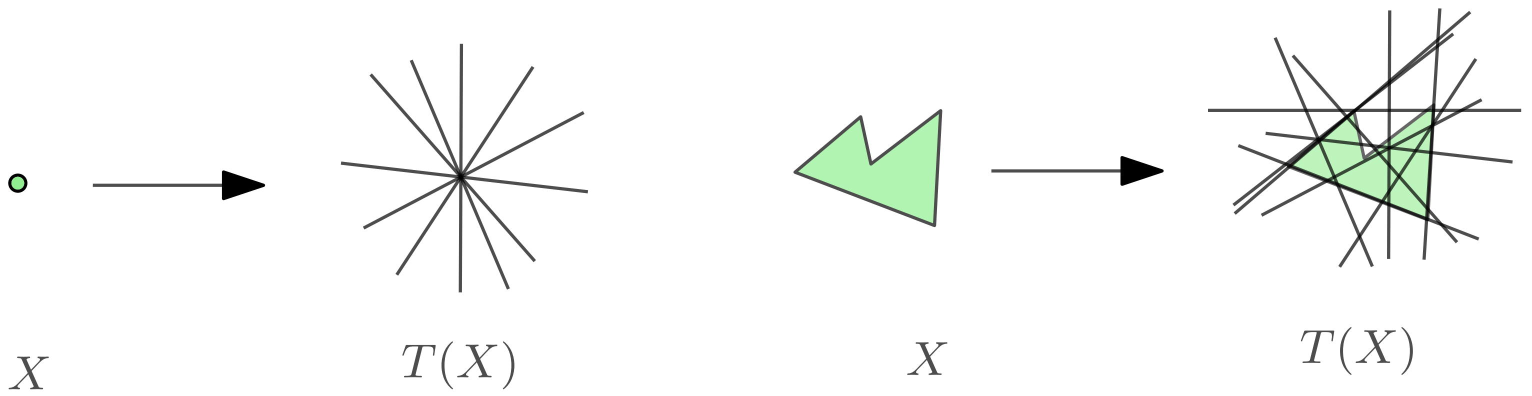

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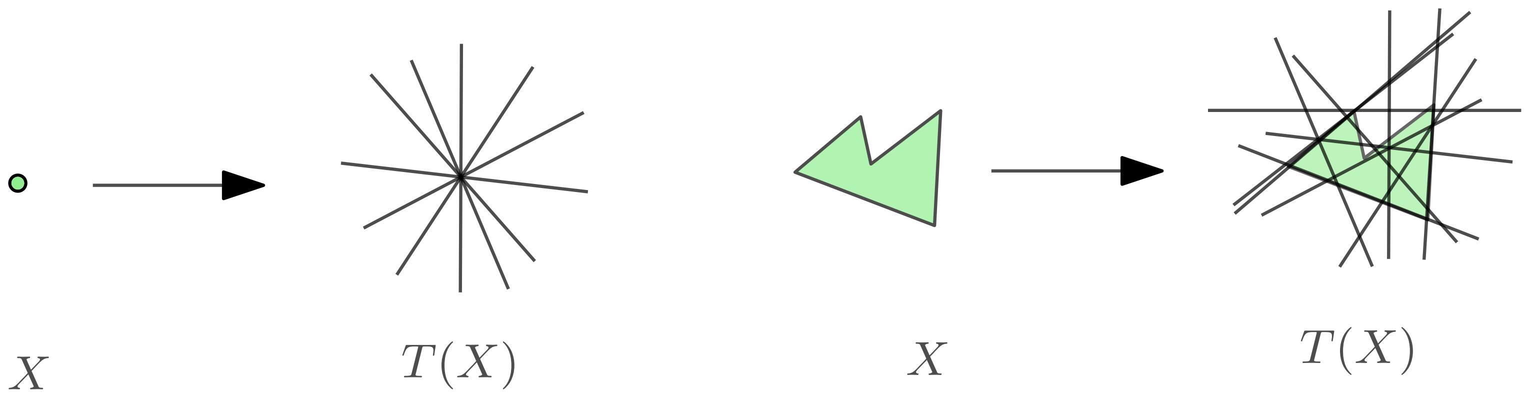

Conjecture (Danzer, 1957). For any $d \geq 2$ there exists $\mathcal{H}_d \in \mathbb{N}$ such that the following holds: for any $n \in \mathbb{N}$, for any family $\{B_1, \ldots, B_n\}$ of pairwise disjoint unit balls in $\mathbb{R}^d$, the Helly number of $\{T(B_1), \ldots, T(B_n)\}$ is at most $\mathcal{H}_d$.

"if any $\mathcal{H}_d$ balls in a family can be stabbed by a line, the whole family can be stabbed by one and the same line."

In this presentation...

An overview of a proof of Danzer's conjecture

Show how "everything fits together" (high-level).

A follow-up: a new homological conditions for bounding Helly numbers

Show a "nice machinery in motion" (more in-depth).

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Quick panorama of my research activity of these last years

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Some research perspectives

Panorama of research activities

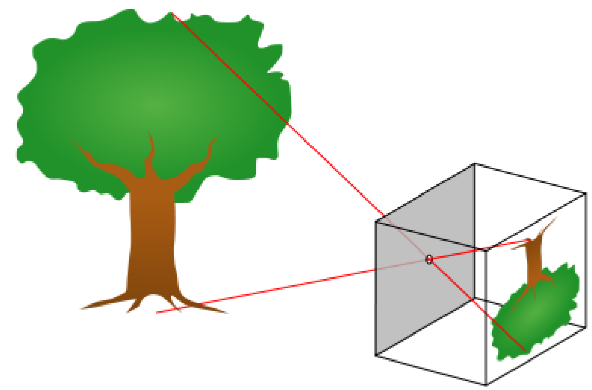
At the interface between computer science and mathematics.

Line geometry for visibility and imaging

How can line geometry help understand light propagation and models of imaging systems?

Lehtinen et al. 2008

Wikipedia



⋆ Shadow boundaries & topological visual event surfaces.

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[PhD Demouth], [Msc Batog], [PhD Batog], [Msc Jang]
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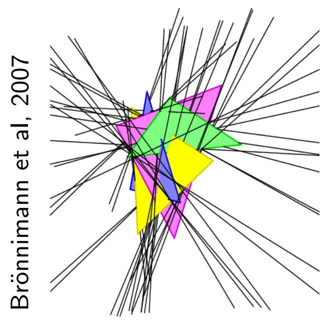
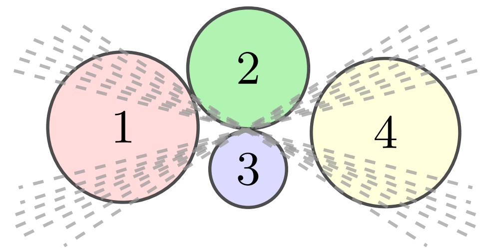
Geometric transversal theory

How does the geometry of an object determine the structure of its geometric transversals?





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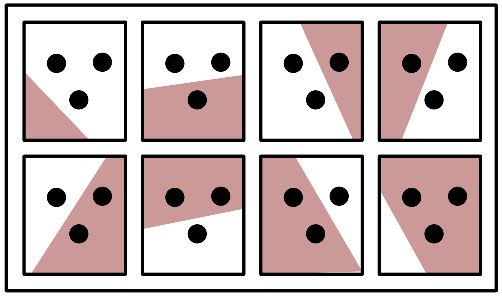
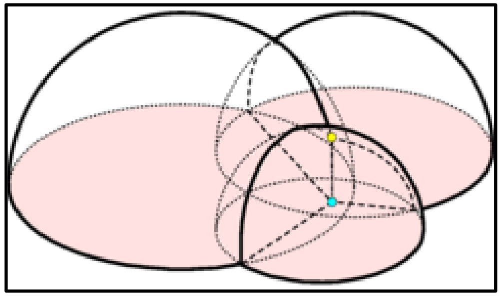
Combinatorics of geometric structures

How does the geometry shape the combinatorial structure underlying geometric objects?





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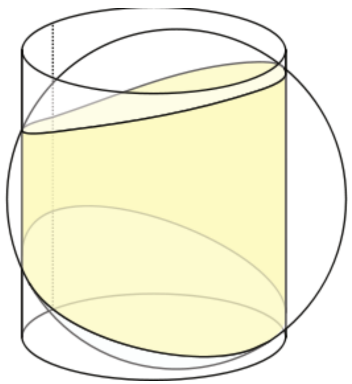
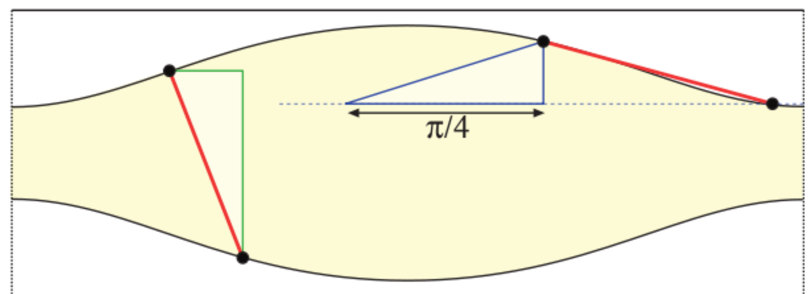
Complexity of random geometric structures

How can probabilistic analysis help refine overly pessimistic worst-case analysis?





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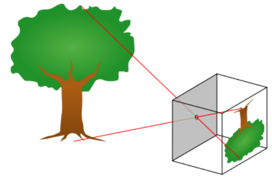
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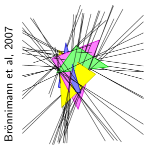
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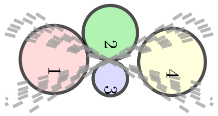

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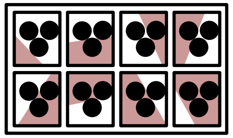

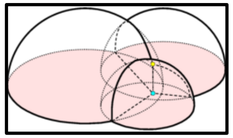

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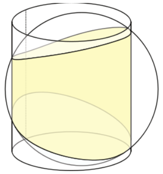

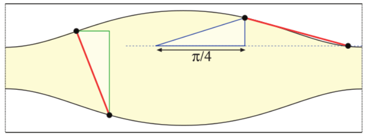

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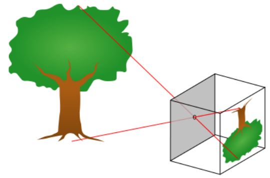

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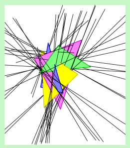

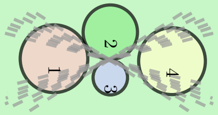

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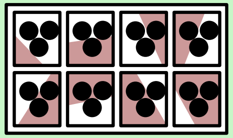
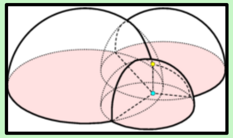
How does the geometry shape the combinatorial structure underlying geometric objects?



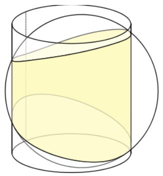


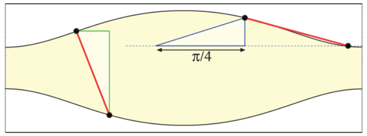
- ⋆ Helly numbers for approximate covering
- ⋆ asymptotic of shatter functions of hypergraphs and families of permutations
- ⋆ Helly numbers from generalized nerve theorems

[PhD Demouth]
[SoCG 2008]
Collaboration with O. Devillers, M. Glisse, O. Cheong, C. Nicaud, E. Colin de Verdière, G. Ginot

Complexity of random geometric structures

How can probabilistic analysis help refine overly pessimistic worst-case analysis?





- ⋆ Delaunay triangulation of random samples of a surface
- ⋆ Smoothed complexity of convex hulls

[ANR "Projet Blanc" 2012-2016 with stochastic geometers]
[SODA 2008]
Collaboration with O. Devillers, J. Erickson, M. Glisse, D. Attali

An overview of the proof of Danzer's conjecture

Where does Danzer's conjecture come from?

Helly's theorem reformulates as

"The Helly number of sets of point transversals to convex sets is at most $d+1$."

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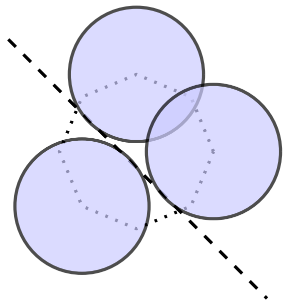
No: 

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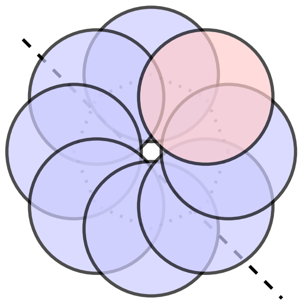
No: 

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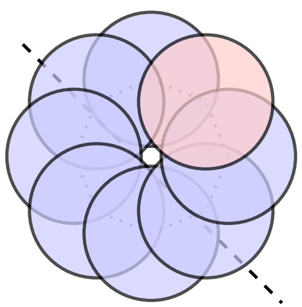
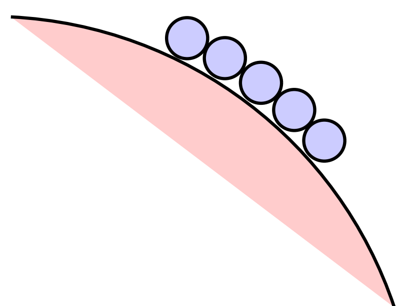
No:  

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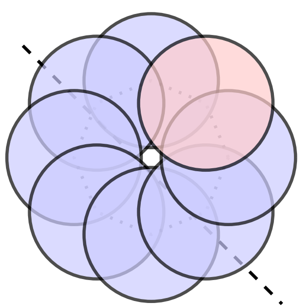
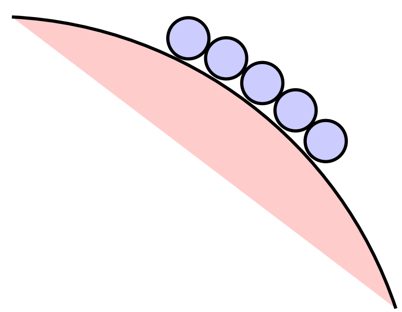
No:  

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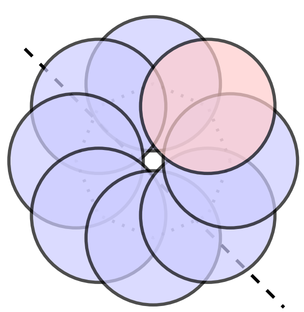

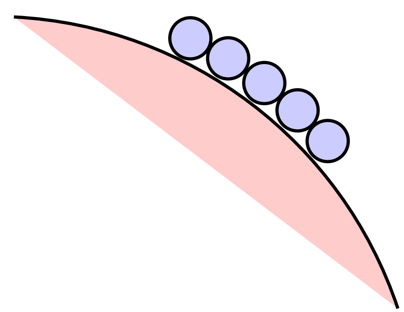

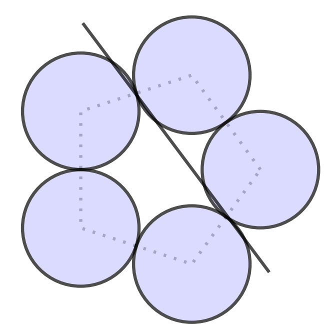

Two "sources of trouble": non-disjointedness and disparity in size.

Theorem (Danzer, 1957). For any $n \in \mathbb{N}$, for any family $\{B_1, \ldots, B_n\}$ of pairwise disjoint unit discs in $\mathbb{R}^2$, the Helly number of $\{T(B_1), \ldots, T(B_n)\}$ is at most 5.

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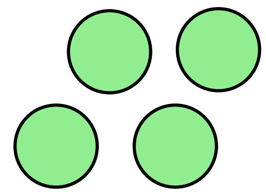
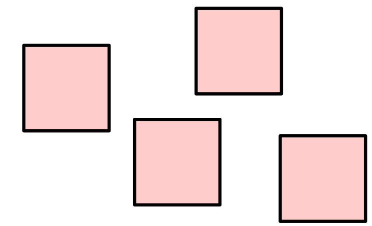
Grünbaum extended Danzer's theorem from  to 

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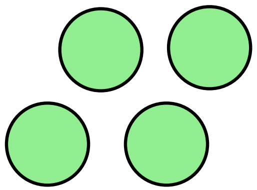
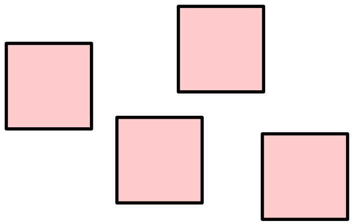
Grünbaum extended Danzer's theorem from  to 

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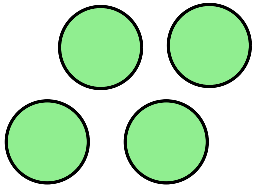
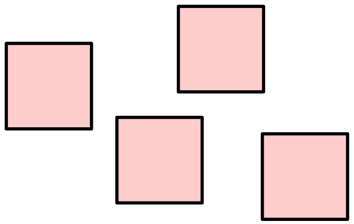
Grünbaum extended Danzer's theorem from  to 

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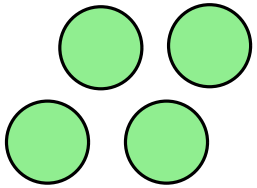
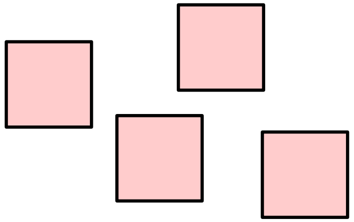
Grünbaum extended Danzer's theorem from  to 

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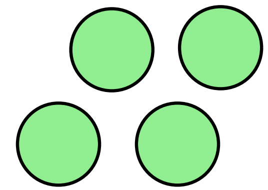
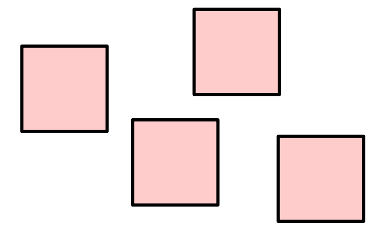
Grünbaum extended Danzer's theorem from  to 

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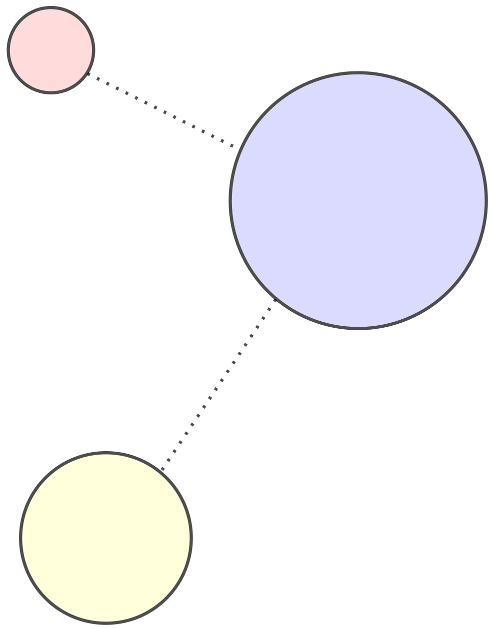
Results on Danzer's conjecture up to 2004. (1/2)

⋆ True for collections of thinly distributed balls in $\mathbb{R}^d$.

[Hadwiger 1959] and [Grünbaum 1960]

Thinly distributed means that the distance between any two balls is at least the sum of their radii.



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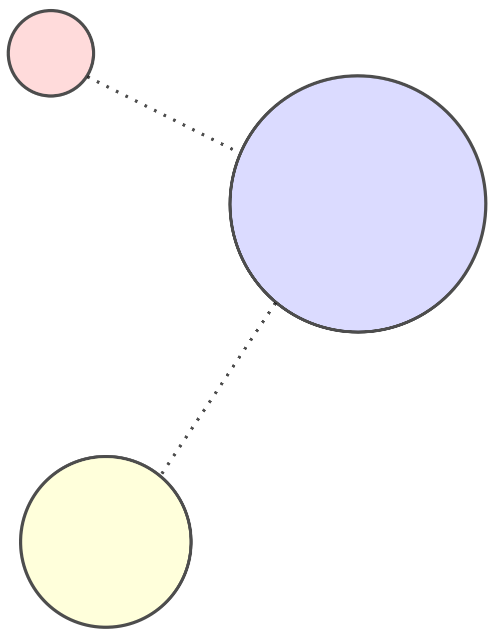

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Given $\mathcal{F} = \{B_1, \ldots, B_n\}$ put $T(\mathcal{F}) = \bigcap_i T(B_i)$

Let π map each line to its orientation in $\mathbb{RP}^{d-1}$

$\mathcal{K}(\mathcal{F}) = \pi(T(\mathcal{F}))$ is the cone of directions.

If $\mathcal{F}$ is thinly distributed then $\mathcal{K}(\mathcal{F})$ is convex (Hadwiger).

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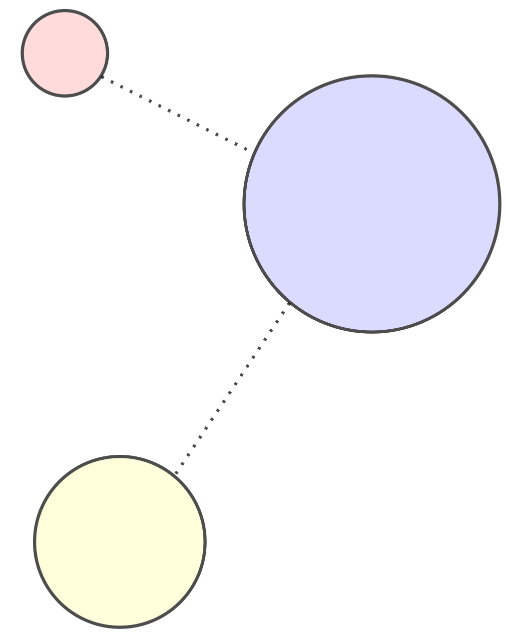

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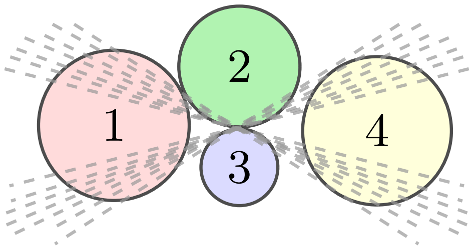
Thus $\{T(B_1), \ldots, T(B_n)\}$ form a good cover (Grünbaum).

Helly's topological theorem $\Rightarrow$ Helly number of $\{T(B_1), \ldots, T(B_n)\} \leq 2d - 1$.

Results on Danzer's conjecture up to 2004. (2/2)

⋆ True for collections of disjoint unit balls in $\mathbb{R}^3$.

[Holmsen-Katchalski-Lewis 2003]



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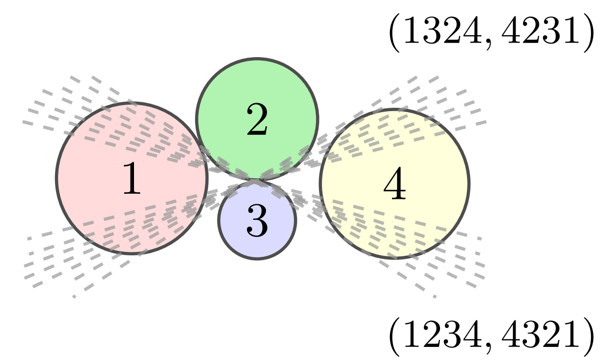

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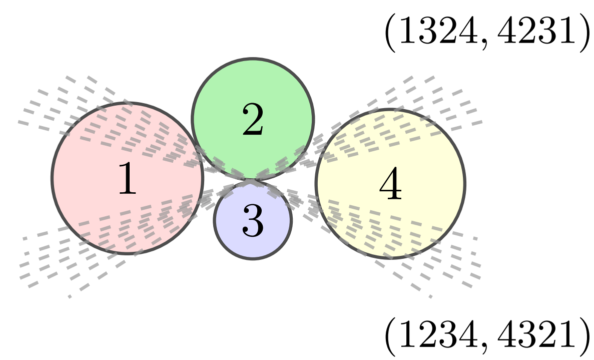

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Combinatorial restrictions on geometric permutations of disjoint unit balls

$\Rightarrow$ Helly number of $\{T(B_1), \ldots, T(B_n)\} \leq 46$.

Ingredients of our proof

⋆ Generalized the convexity structure of cones of directions.

[SoCG 2007] [DCG]
Joint work with C. Borcea and S. Petitjean

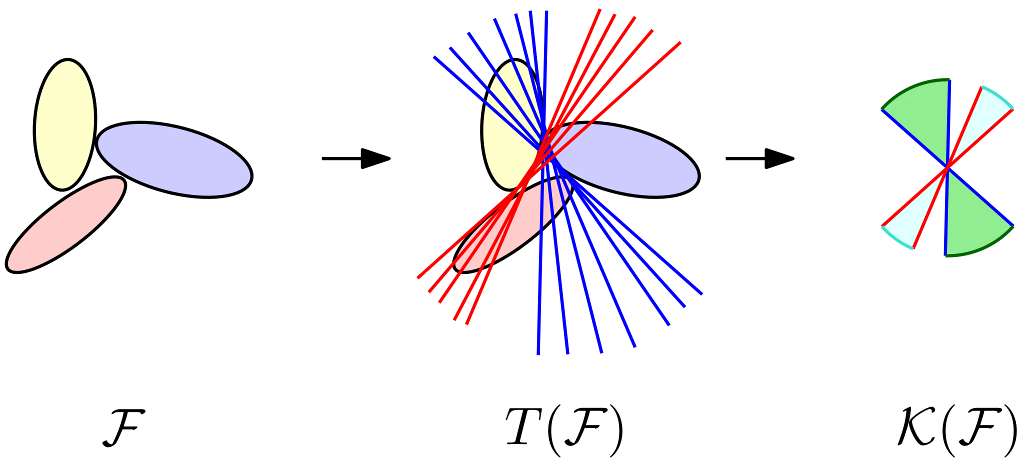
⋆ Clarified the structure of sets of geometric permutations.

[CGTA]
Joint work with O. Cheong and H.S. Na

⋆ Added a new ingredient: pinning theorems.

[SoCG 2005] [DCG]
Joint work with O. Cheong and A. Holmsen

What are cones of directions?



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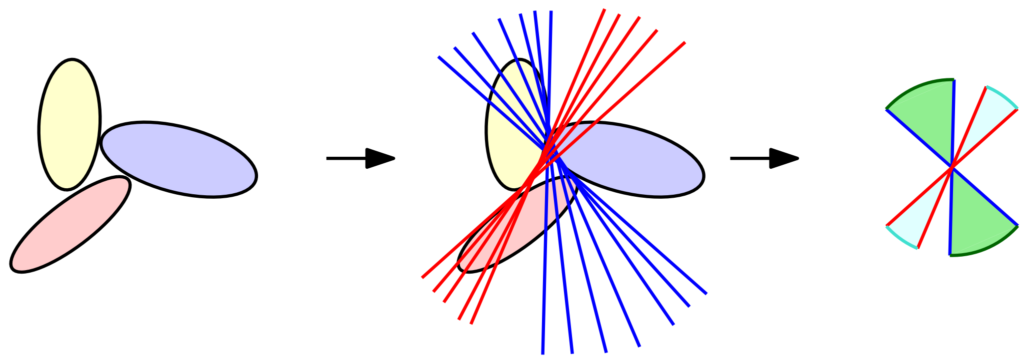
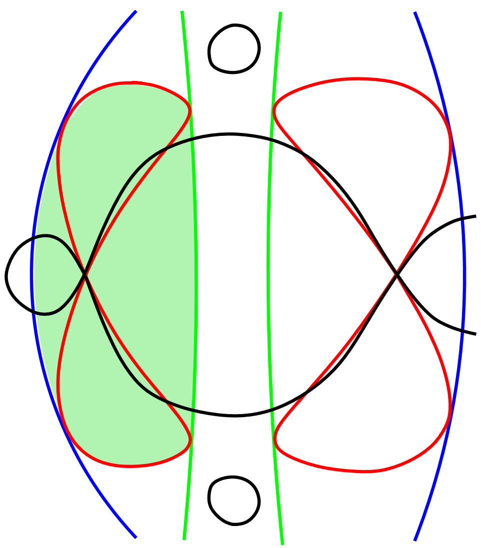


$\mathcal{F}$ $\qquad T(\mathcal{F})$ $\qquad \mathcal{K}(\mathcal{F})$

How to prove that cones of directions are convex?



⋆ we analyzed the geometry of the curves bounding $\mathcal{K}(\mathcal{F})$

Arcs of conics and sextics.

Track the inflexion/singular points.

Characterization of the arcs of sextic on $\partial\mathcal{K}(\mathcal{F})$

No inflexion/singular point on $\partial\mathcal{K}(\mathcal{F})$

[SoCG 2007] [DCG]

Joint work with C. Borcea and S. Petitjean

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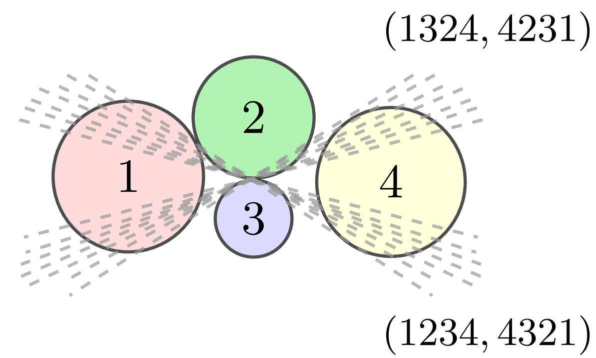

An oriented line transversal to $\mathcal{F}$

$\hookrightarrow$ a permutation of $\{B_1, \ldots, B_n\} \simeq \{1, \ldots, n\}$.

A line transversal to $\mathcal{F}$

$\hookrightarrow$ a pair of permutations of $\{1, \ldots, n\}$, one reverse of the other.

The geometric permutations of $\mathcal{F}$ are the pairs of permutations realizable by a line transversal.

[CGTA]

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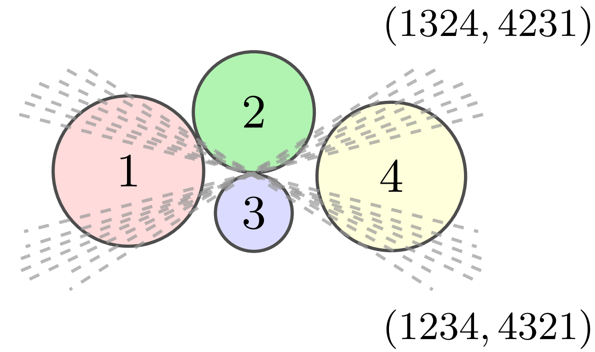

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Theorem (Cheong-G-Na, 2004) Let $\mathcal{F}$ be a family of n disjoint unit balls in $\mathbb{R}^d$. If $n \geq 9$ then $\mathcal{F}$ has at most 2 distinct geometric permutations that differ in the exchange of 2 adjacent elements. If $n \leq 8$ then $\mathcal{F}$ has at most 3 distinct geometric permutations.

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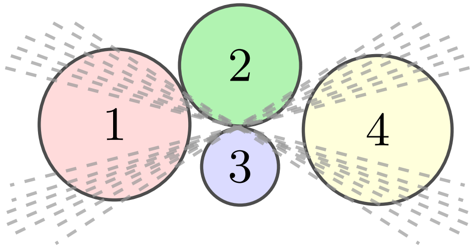

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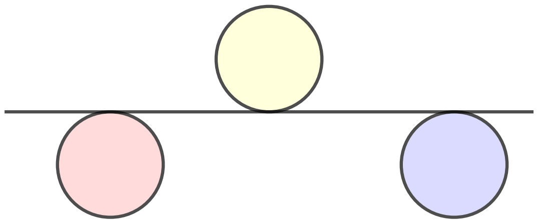
$\star$ geometry $\Rightarrow$ excluded pairs of patterns (in the Stanley-Wilf sense).

$\star$ combinatorial extrapolation.

[CGTA]

Joint work with O. Cheong and H.S. Na

A family $\mathcal{F} = \{B_1, \ldots, B_n\}$ of sets in $\mathbb{R}^d$ pins a line ℓ
$\Leftrightarrow \ell$ is an isolated point in $T(\mathcal{F})$.



[SoCG 2005] [DCG]
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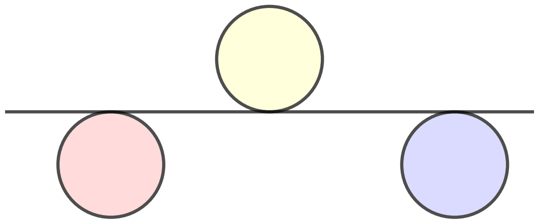

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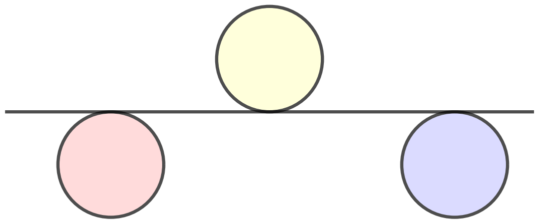

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Pinning theorem, a local analogue of a Helly number.

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Joint work with O. Cheong and A. Holmsen

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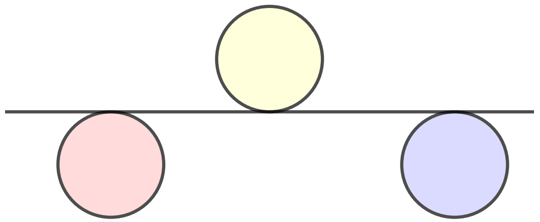

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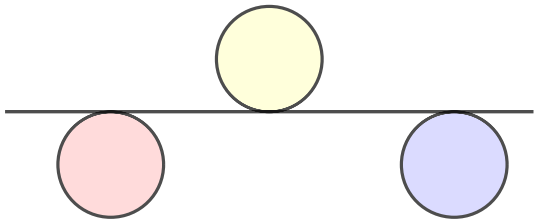

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- ⋆ Argue that locally near ℓ the $T(B_i)$ form a good cover.
- ⋆ Conclude using Helly's topological theorem.

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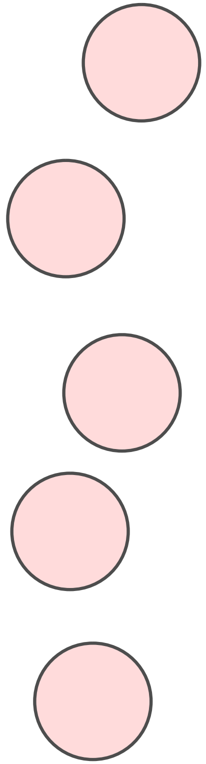

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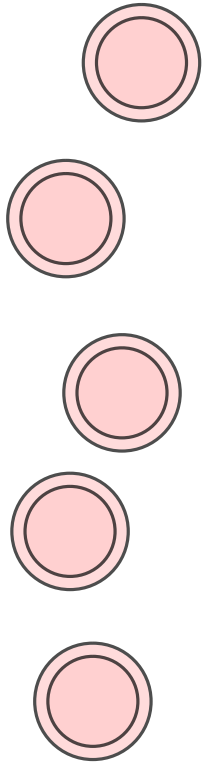

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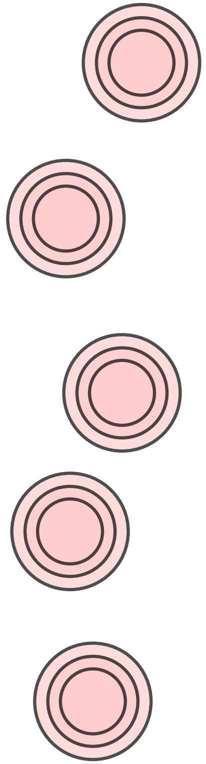

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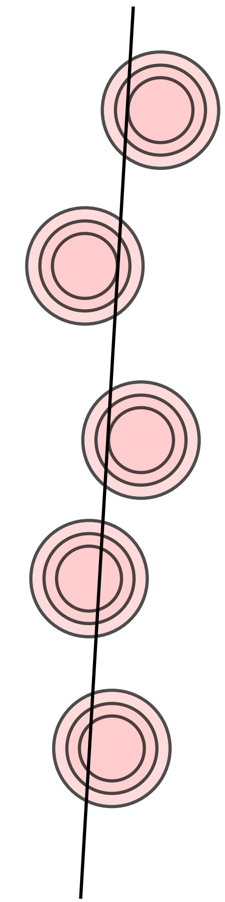

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$\Rightarrow \ell$ is pinned and the only transversal of $G^* \subseteq G$ of size at most $4d-2$.

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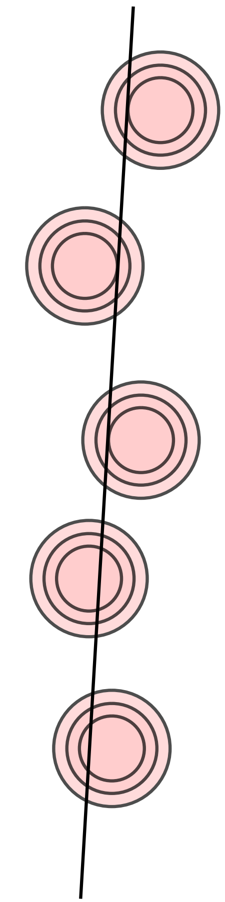

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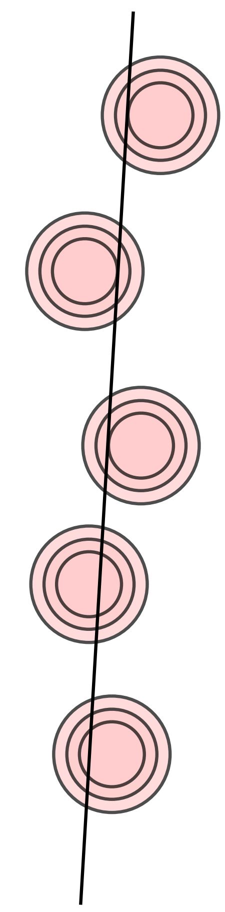
⋆ Or G has two line transversals ℓ_1 and ℓ_2, which it pins.

Similar (but slightly more complicated) arguments.



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Convexity of $\mathcal{K}(\mathcal{F})$ for disjoint balls in $\mathbb{R}^d$

$\Longrightarrow$ Vietoris-Begle mapping theorem

Contractibility of the set of transversals to disjoint balls in $\mathbb{R}^d$ in a given order

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Pinning theorems for other shapes (polytopes and ovaloids).

Stable pinning

- $\star$ tight lower bound for the pinning theorem
- $\star$ lower bound of $2d-1$ for the Helly number
- $\star$ relation to transversality of intersection

Application: computing a smallest enclosing cylinder (1/3)

Smallest enclosing cylinder (SEC) problem: given n points in $\mathbb{R}^d$, compute the cylinder with minimum radius that contains all the points.

Here a cylinder is the set of points within bounded distance from a given line (the axis).

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In the Real RAM model:

For $d = 2$ the worst-case complexity of SEC is $\Theta(n \log n)$.
[Avis-Robert-Wenger, 1989] and [Egyed-Wenger, 1989]

For $d = 3$, for any $\epsilon > 0$ there is an algorithm that solves SEC in $O(n^{3+\epsilon})$.
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In the Turing machine model:

SEC is NP-hard when the dimension d is part of the input.

[Meggido 1990]

Application: computing a smallest enclosing cylinder (2/3)

Let P be a set of n points in $\mathbb{R}^d$.

Let $\phi : \begin{cases} 2^P & \rightarrow & \mathbb{R} \times \mathbb{N} \\ Q & \mapsto & (r_Q, n_Q) \text{ where } \begin{cases} r_Q = \text{radius of the SEC of } Q \\ n_Q = \#\text{enclosing cylinders of Q of radius } r_Q \end{cases} \end{cases}$

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Proposition. (P, ϕ) is a LP-type problem.

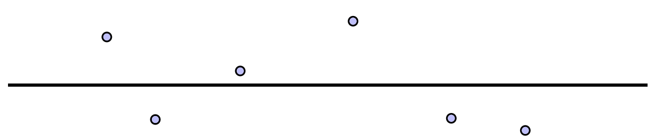
The combinatorial dimension of (P, ϕ) is the maximum size of a subset $Q \subseteq P$ such that $\forall x \in Q, \quad \phi(Q \setminus \{x\}) \neq \phi(Q)$.

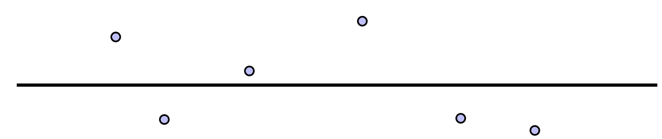
For any LP-type problem (P, ϕ) with constant combinatorial dimension, $\phi(P)$ can be computed in randomized time linear in $|P|$.

[Matoušek-Sharir-Welzl, 1992], [Seidel 1991], [Clarkson 1995]

Application: computing a smallest enclosing cylinder (3/3)

Q is enclosed by the cylinder with axis ℓ and radius r $\iff$ The line ℓ is a transversal to the balls of radius r centered in the points of Q.





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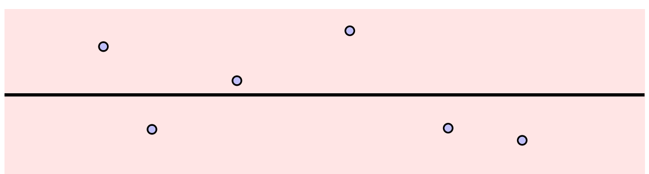

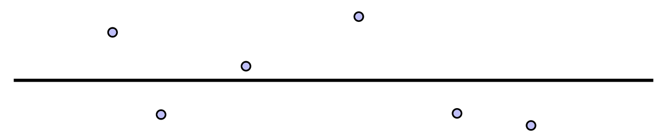

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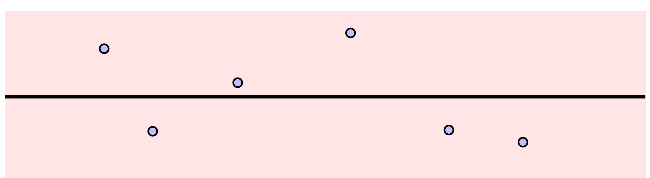

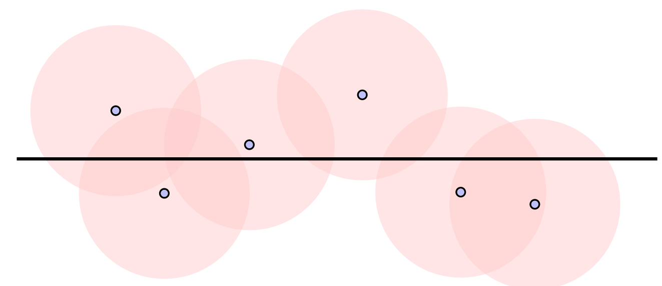

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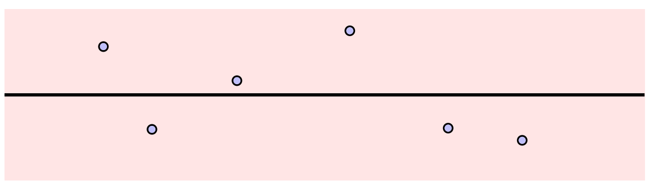

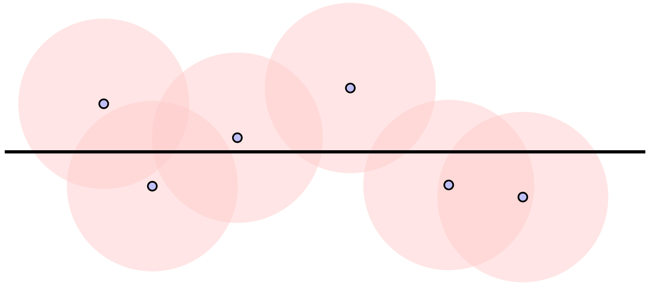

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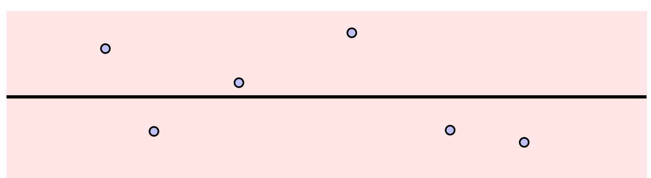

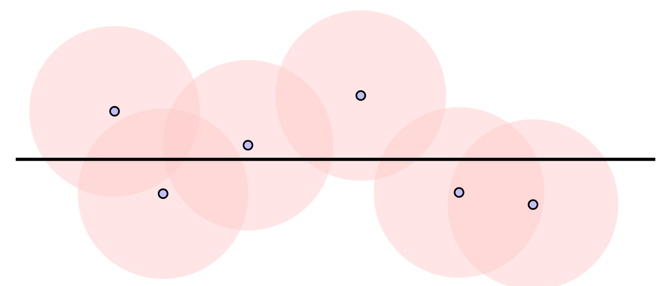

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Corollary. If P is sparse then (P, ϕ) has combinatorial dimension at most $4d-1$ and the SEC of P can be computed in randomized linear time.

(in any fixed dimension d)

Summary

Complete proof of Danzer's conjecture.

Algorithmic consequences.

The proof uses a combination of techniques from...

- $\star$ convex and euclidean geometry
- $\star$ topology
- $\star$ (classical) algebraic geometry
- $\star$ combinatorics

... and opens new perspectives

- $\star$ Topology of $\mathcal{K}(\mathcal{F})$ for disjoint convex sets in $\mathbb{R}^d$.
- $\star$ Pinning theorems for disjoint convex sets in $\mathbb{R}^3$.

Helly numbers from homological conditions

An interesting pattern...

Sets of line transversals with bounded Helly number...

Disjoint translates of a convex figure in $\mathbb{R}^2$ [Tverberg, 1989]

Disjoint unit balls

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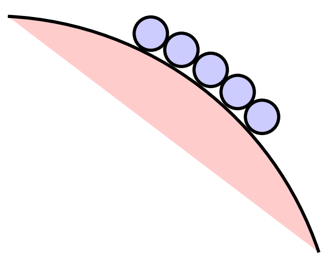

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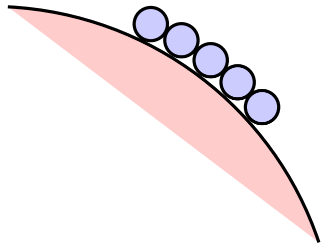

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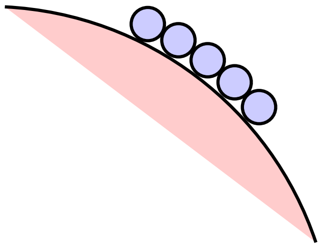

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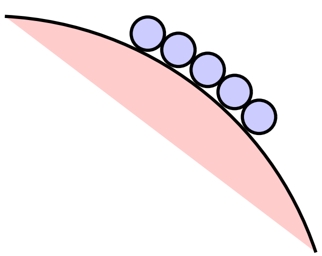

The proofs all rely on ad hoc geometric arguments

Can we bring them under the same (topological) umbrella?

There are two topological Helly-type theorems for non-connected sets.

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Theorem (Matoušek, 1999). For any $d \geq 2$ and $r \geq 1$ there exists a constant $h(d,r)$ such that the following holds: any finite family of subsets of $\mathbb{R}^d$ such that the intersection of every subfamily has at most r connected components, each $\lceil \frac{d}{2} \rceil$-connected, has Helly number at most $h(d,r)$.

The topological condition ressemble what we are looking for but...

- $\star$ Very large bound.
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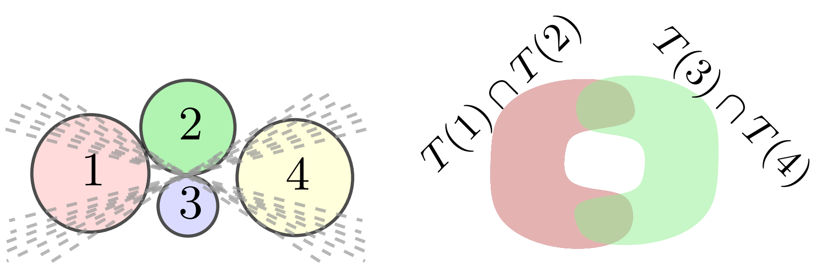
Theorem (Kalai-Meshulam, 2008). Let G be an open good cover in $\mathbb{R}^d$. Any family $\mathcal{F}$ such that the intersection of every subfamily is a disjoint union of at most r elements of G has Helly number at most $r(d+1)$.

The bound look like what we'd like to have but...

- ⋆ Not the kind of topological conditions we have.



Theorem (Colin de Verdière-Ginot-G, 2011). If $\mathcal{F}$ is a finite family of open subsets of $\mathbb{R}^d$ such that the intersection of every subfamily has at most r connected components, each a homology cell, then $\mathcal{F}$ has Helly number at most $r(d+1)$.

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In fact, we prove a more general statement where:

⋆ the ambient space is any (locally connected) topological space Γ,

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⋆ only families of cardinality at least t need to intersect in at most r connected components,

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This hammer implies Tverberg's theorem and Danzer's conjecture.

The nerve $N(F)$ of a family $\mathcal{F}$ of sets is:

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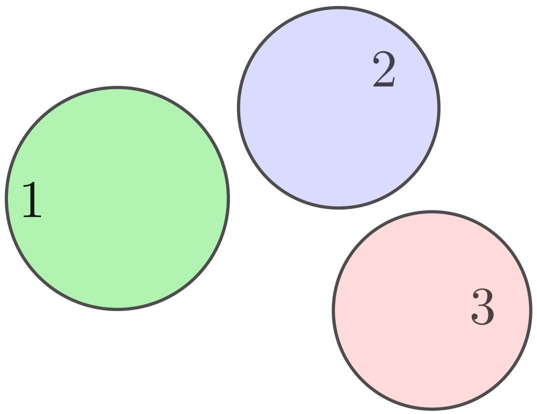

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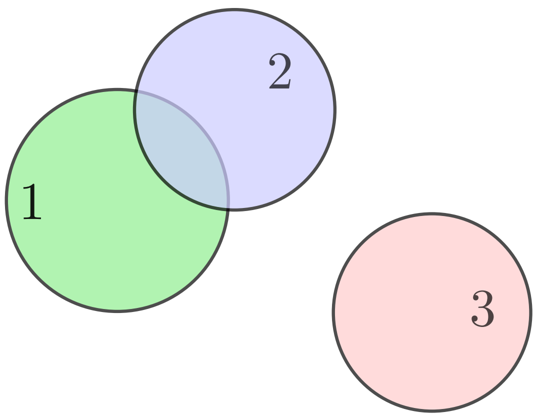

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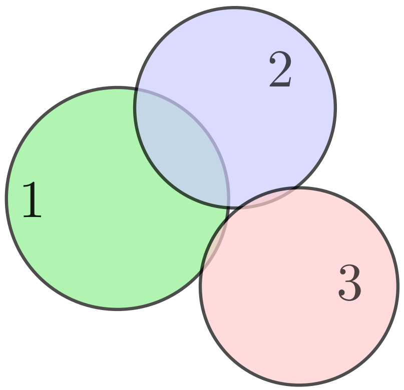

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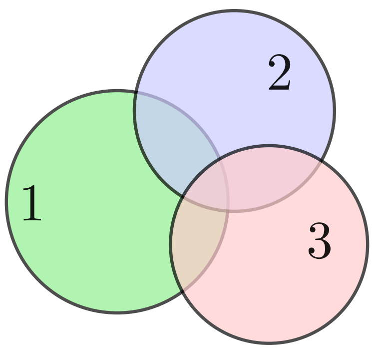

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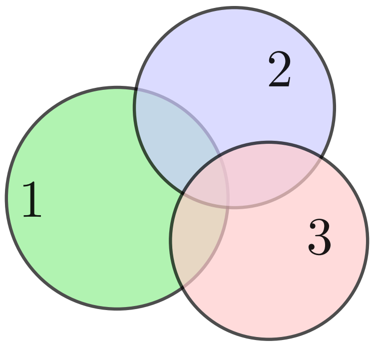

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F

$N(F)$

It is an abstract simplicial complex.

(= a family of finite sets closed under taking subsets).

(= a monotone hypergraph / set system).

Geometric realization of an abstract simplicial complex.

k-tuple $\mapsto$ $(k-1)$-dim. ball (with boundary conditions).

Can be done linearly in sufficiently high dimension.

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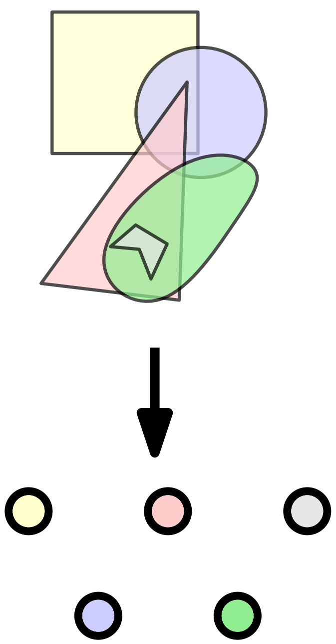

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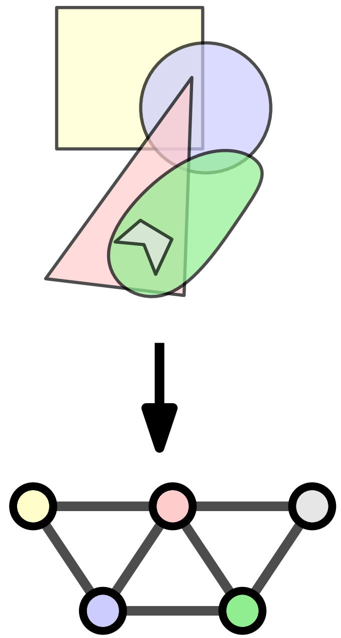

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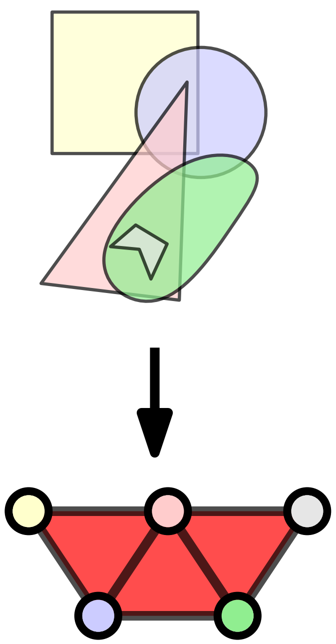

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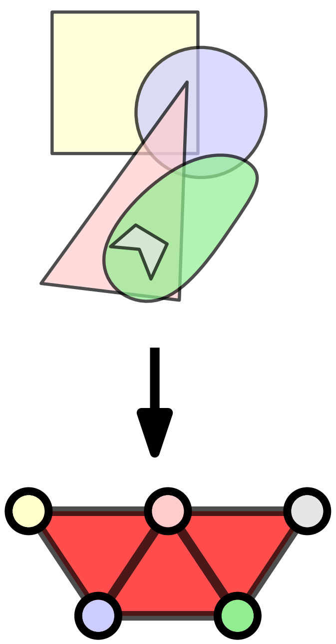

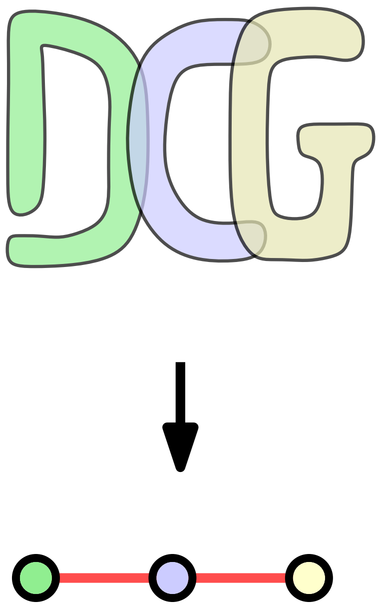

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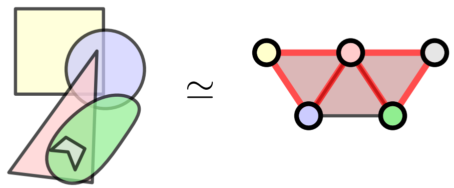

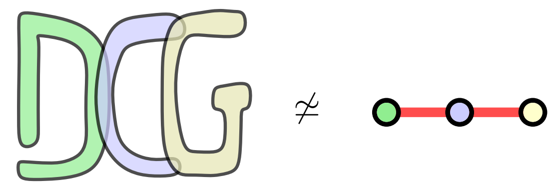

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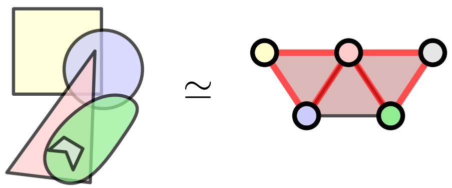

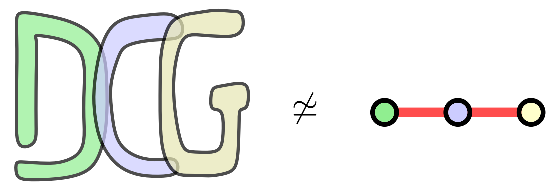

Nerve Theorem $\Rightarrow$ Helly's topological theorem

Let F be an open good cover in $\mathbb{R}^d$,

and $G \subseteq F$ an inclusion-minimal subfamily with empty intersection.

$\bigcup_G$ must have non-vanishing homology in dimension $|G| - 2$.

Minimality $\Rightarrow N(G) = 2^G \setminus \{G\} \simeq \mathbb{S}^{|G|-2}$

Nerve Theorem $\Rightarrow \bigcup_G \simeq N(G)$.

Open subsets of $\mathbb{R}^d$ have vanishing homology in dimension $\geq d$.

$|G| - 2 < d \Rightarrow |G| \leq d + 1$.

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(Here I mean "Čech complex" in the sense of algebraic topology, which is different from what is called "Čech complex" in computational topology.)

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We introduce a combinatorial structure that...

- ⋆ is close enough to a simplicial complex that Helly numbers are "within sight",
- ⋆ retains "enough" of the Čech complex that its homology is controlled by the union.

Let $\mathcal{F}$ be a family of subsets of a topological space.

The multinerve $M(\mathcal{F})$ of $\mathcal{F}$ is the poset

$$M(\mathcal{F}) = \{(G, X) \mid G \subseteq \mathcal{F}, X \text{ is a connected component of } \bigcap_G\}$$

ordered by $(G, X) \prec (G', X')$ iff $G \subset G'$ and $X \supset X'$.

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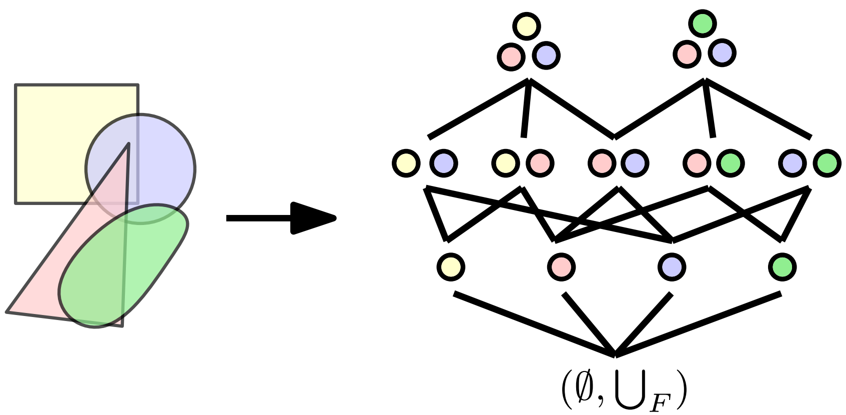

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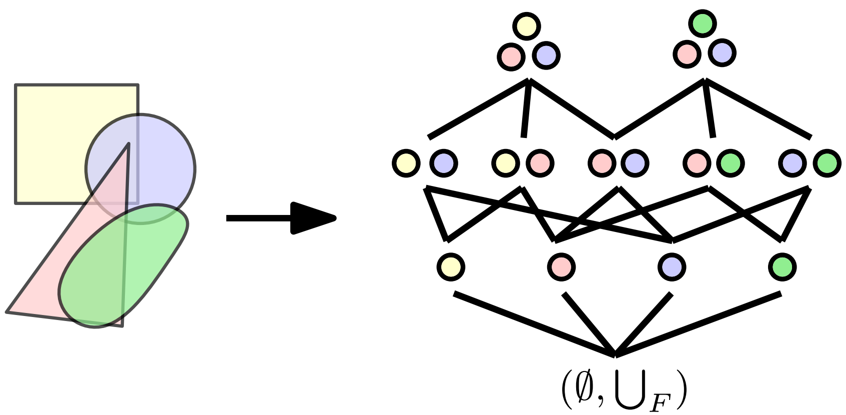

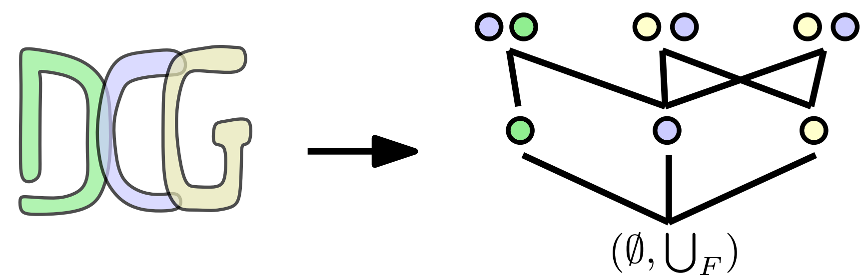

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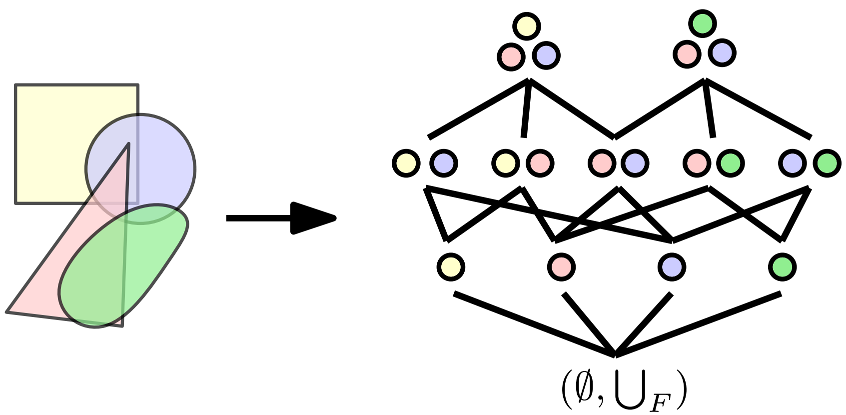

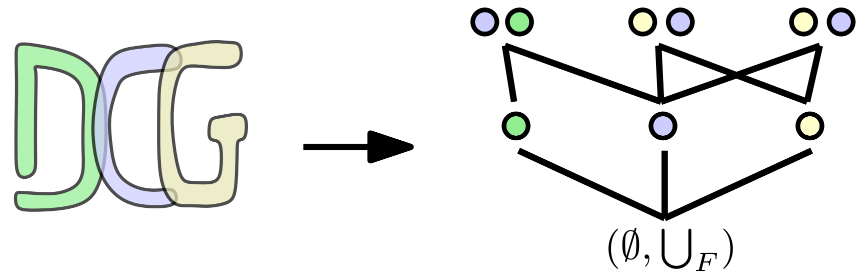

$M(\mathcal{F})$ is a simplicial poset: Unique minimum element. Every lower interval is isomorphic to the face lattice of a simplex.

Geometric realizations, homology... extend to simplicial posets.

Theorem (Colin de Verdière-Ginot-G). If $\mathcal{F}$ is a family of open sets such that the connected components of the intersection of any subfamily is a homology cell then $M(\mathcal{F})$ and $\bigcup_{\mathcal{F}}$ have isomorphic (reduced) homology groups (over $\mathbb{Q}$).

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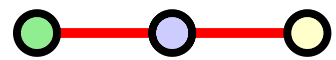

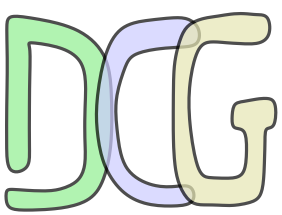

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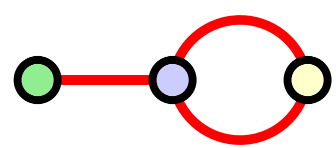

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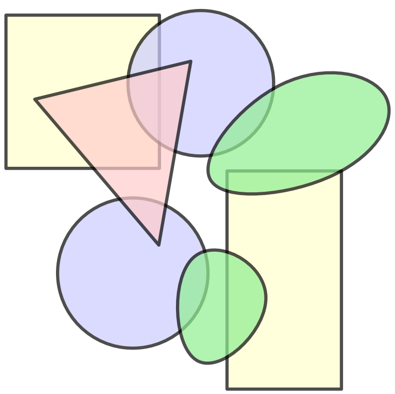

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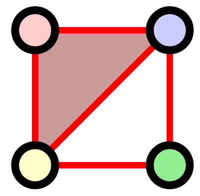

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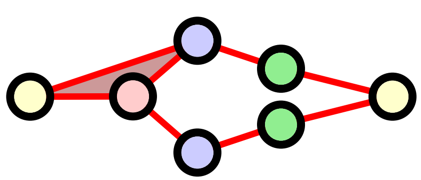

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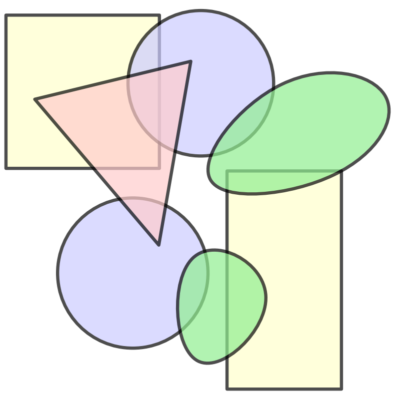

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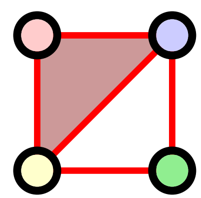

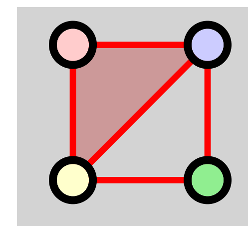

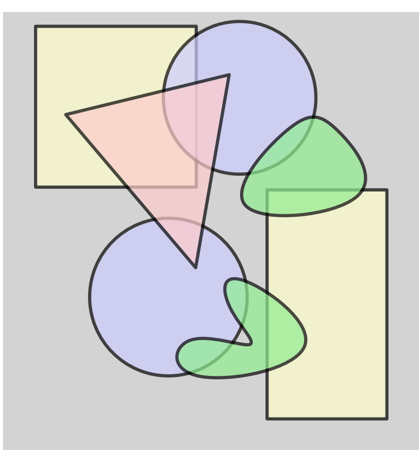

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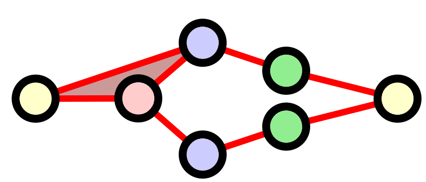

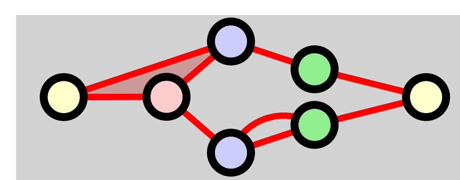

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Goal: "induced subcomplexes of $N(\mathcal{F})$ have trivial homology in dimension $\geq h$" (would imply that the Helly number of $\mathcal{F} \leq h+1$).

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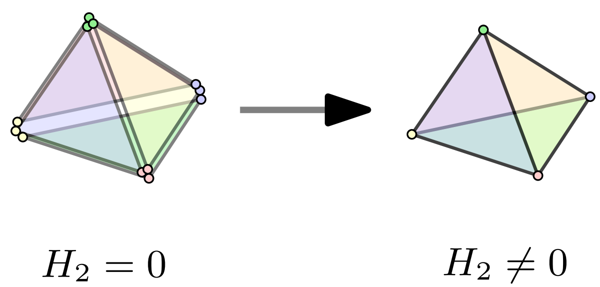

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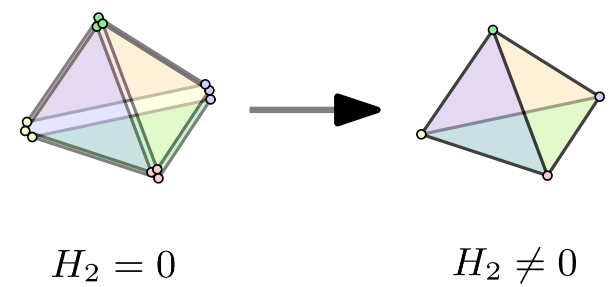

⋆ the projection preserves dimension and is at most r-to-one
$\Rightarrow$ induced subcomplexes of $N(\mathcal{F})$ have trivial homology in dimension $\geq rd + r - 1$.

Coming back to Danzer's conjecture...

If $\mathcal{F} = \{B_1, \ldots, B_n\}$ is a family of disjoint balls in $\mathbb{R}^d$ and $G \subseteq \mathcal{F}$ then...

$T(B_i) \simeq \mathbb{RP}^{d-1}$,

$T(G)$ has contractible connected components if $|G| \geq 2$,

The number of connected components of $T(G)$ is at most 3 in general and at most 2 if $|G| \geq 9$.

The ambient space Γ is a compact subset of $\mathbb{RG}_{2,d+1}$ which is of dimension $2d-2$.

Applying our homological Helly-type theorem we obtain:

For $d \geq 6$ the Helly number of $\{T(B_1), \ldots, T(B_n)\}$ is at most $2(2d-1)$.

(to use $r = 2$ in our bound we need $2d - 2 \geq 9$, hence the condition $d \geq 6$).

Summary

Refinement of the classical nerve that enjoys a similar "Nerve Theorem".

"Combinatorial interface" to Leray's acyclic cover theorem.

Homological Helly-type theorem that

essentially generalizes those of Matoušek and Kalai-Meshulam,

(re)proves in a unified manner Helly numbers in geometric transversal theory.

Raises questions on the combinatorics of simplicial complexes and posets.

dimension-preserving projections

Some perspectives

Short/medium term

Simplify...

⋆ projection of simplicial complexes

Generalize...

⋆ Topology of sets of k-dimensional transversals to convex sets in $\mathbb{R}^d$.

⋆ Tangents to convex sets and transversality

Apply...

⋆ Efficient computation of SEC, from sparse to general point sets.

⋆ Other uses of multinerves for Hadwiger-type theorems, Stanley-Riesner ideals...

In a more distant future

More territory to map...

- ⋆ Does a "recursively" bounded sum of Betti numbers imply a bounded Helly number?
- ⋆ Algorithmic applications of bounded "local combinatorial dimension"?

And some "hard nuts" to break...

- ⋆ Geometric permutations of disjoint convex sets in $\mathbb{R}^d$.
- ⋆ Pinning theorem for disjoint convex sets in $\mathbb{R}^3$? $\mathbb{R}^d$?

Other directions

⋆ Combinatorics of geometric structures

Shatter functions, VC-dimension & excluded patterns for geometric permutations.

Topological combinatorics (inclusion-exclusion formulas...)

Random generation of combinatorial structures underlying geometric objects (order-types...).

⋆ Complexity of random geometric structures

Average-case analysis (random polytopes, Delaunay of points on a surface).

Smoothed complexity (convex hull, Delaunay triangulation...).

Co-workers (2004-2011)

Geometric transversals
Boris Aronov (NYU-Poly)
Ciprian Borcea (Rider University)
Otfried Cheong (KAIST)
Andreas Holmsen (KAIST)
Stefan Koenig (TU Munchen)
Sylvain Petitjean (INRIA)
Günter Rote (FU Berlin)

Line geometry for visibility & imaging
Guillaume Batog (U. Nancy 2)
Julien Demouth (U. Nancy 2)
Jeong-Hwan Jang (KAIST)
Bruno Levy (INRIA)
Jean Ponce (ENS)

Random geometric structures
Dominique Attali (CNRS)
Olivier Devillers(INRIA)
Jeff Erickson (UIUC)
Marc Glisse (INRIA)
⋮

Combinatorics of geometric structures
Otfried Cheong (KAIST)
Éric Colin de Verdière (CNRS)
Grégory Ginot (U. Paris 6)
Cyril Nicaud (U. Marne la Vallée)
⋮

And also... Mark de Berg (TU Eindhoven), Veronique Cortier (CNRS), Hyo-Sil Kim (KAIST), Jan Kratochvil (Charles U.), Sylvain Lazard (INRIA), Mira Lee (KAIST), Hyeon-Suk Na (Soongsil U.), Yoshio Okamoto (JAIST), Chan-Su Shin (HUFS), Frank Van der Stappen (Utrecht U.), Alexander Wolff (U. Würzburg).

(student at the time of the collaboration)