Complex boundaries for the Totally Asymmetric Simple Exclusion process

Nicky Sonigo

- École normale supérieure de LYON -
Laboratoire de l'Unité de Mathématiques Pures et Appliquées

THÈSE

en vue d'obtenir le grade de

Docteur de l'Université de Lyon - École Normale Supérieure de Lyon

Spécialité : Mathématiques

au titre de l' École doctorale de Mathématiques et Informatique fondamentale

soutenue publiquement le 2 novembre 2011 par

Nicky SONIGO

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Directeur de thèse :	Alice	GUIONNET	
Co-encadrant de thèse :	Vincent	BEFFARA	
Proposition de jury de thèse :			
	Christophe	BAHADORAN	Rapporteur
	Vincent	BEFFARA	Membre
	Stefan	GROSSKINSKY	Rapporteur
	Alice	GUIONNET	Membre
	Pierre	MATHIEU	Membre
	Christophe	SABOT	Membre
	Vladas	SIDORAVICIUS	Rapporteur

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Introduction au processus d'exclusion

1.1 Présentation du modèle

Le processus d'exclusion simple est un processus de Markov en temps continu sur un espace d'états de la forme $X := \{0,1\}^S$, où S est dénombrable. Il est formellement défini de la manière suivante. On se donne un noyau de Markov $(p(x,y))_{x,y\in S}$ sur S, *i.e.* $\forall x, y \in S$, $p(x,y) \geqslant 0$ et

$$\forall x \in S, \quad \sum_{y\in S} p(x,y) = 1.$$

Une configuration du système décrit la présence ou l'absence d'une particule en chaque *site* (élément de S). La *règle d'exclusion* n'autorise qu'au plus une particule par site. Chaque particule effectue une marche aléatoire en temps continu sur S selon le noyau $(p(x,y))_{x,y\in S}$ et interagit avec les autres particules en n'effectuant pas les sauts violant la règle d'exclusion. Plus précisément, une particule située au site x attend un temps de loi exponentielle de paramètre 1 puis choisit un site y avec probabilité $p(x,y)$. Si ce site est libre, la particule saute au site y, sinon elle reste en x et attend de nouveau un temps de même loi pour effectuer une nouvelle tentative de saut.

Étant donné qu'il peut y avoir une infinité de particules (dans le cas où S est infini), il n'est pas clair que le processus d'exclusion soit toujours bien défini. En effet, il se pourrait *a priori* qu'il existe une suite de temps strictement décroissante $(t_{x_n})_{n\geqslant 1}$ telle que pour tout $n \geqslant 1$, à l'instant t_{x_n} une particule est au site x_n et tente de sauter au site y. Dans ce cas, il n'y a pas de moyen de déterminer la « première particule » qui saute en y.

Nous allons voir qu'il y a deux manières équivalentes de construire le processus d'exclusion : la première est analytique et la seconde probabiliste. Ceci nous donnera deux points de vue différents pour aborder l'étude de ce processus. Dans cette introduction, nous allons brièvement voir ces deux méthodes car elles seront toutes les deux utilisées dans la suite de cette thèse.

Dans toute cette thèse, X est muni de la topologie produit ce qui en fait un espace compact. On note $C(X)$ l'ensemble des fonctions réelles continues sur X. On dira qu'une fonction (à valeurs réelles) f sur X est *cylindrique* si elle ne dépend que de l'état d'un nombre fini de sites, *i.e.* si il existe $p \geqslant 0$, $x_1, \dots, x_p \in S$ et une fonction $g : \{0,1\}^{\{x_1,\dots,x_p\}} \longrightarrow \mathbb{R}$ tels que pour tout $\eta \in X$, $f(\eta) = g(\eta_{|\{x_1,\dots,x_p\}})$. Soit $\mathcal{P}$ l'ensemble des mesures de probabilité sur X. On munit $\mathcal{P}$ de la topologie de la convergence faible : $\mu_n \longrightarrow \mu$ dans $\mathcal{P}$ si et seulement si pour toute fonction $f \in C(X)$, on a $\int f d\mu_n \longrightarrow \int f d\mu$. On notera alors souvent $\mu_n \Longrightarrow \mu$.

1.1.1 Construction à partir du générateur infinitésimal

Commençons par donner quelques définitions générales sur les processus de Markov. Soit $(\eta_t)_{t\geqslant 0}$ un processus de Markov sur X. On définit son semi-groupe $(S(t))_{t\geqslant 0}$ comme étant la famille d'opérateurs sur $C(X)$ définis par

$$S(t)f(\eta) := E^\eta\left[f(\eta_t)\right], \quad \forall f \in C(X), \eta \in X, t \geqslant 0,$$

où E^η désigne l'espérance sous la loi P^η du processus $(\eta_t)_{t\geqslant 0}$ conditionné à partir de la configuration initiale $\eta_0 = \eta$. On a alors le théorème fondamental suivant :

Théorème 1.1 (Hille-Yosida). *Soit*

$$\mathcal{D}(\Omega) := \left\{ f \in C(X) : \lim_{t\downarrow 0} \frac{S(t)f - f}{t} \ \textit{existe} \right\},$$

et

$$\Omega f := \lim_{t\downarrow 0} \frac{S(t)f - f}{t}, \quad \textit{pour toute fonction } f \in \mathcal{D}(\Omega).$$

Alors pour toute fonction $f \in C(X)$ et tout $t \geqslant 0$,

$$S(t)f = \lim_{n\to\infty} \left(I - \frac{t}{n}\Omega \right)^{-n} f.$$

De plus si $f \in \mathcal{D}(\Omega)$, alors $S(t)f \in \mathcal{D}(\Omega)$ et

$$\frac{d}{dt} S(t)f = \Omega S(t) f = S(t)\Omega f.$$

L'opérateur Ω est appelé *générateur infinitésimal* du processus de Markov $(\eta_t)_{t\geqslant 0}$.

Une première manière de définir un processus d'interaction de particules est de donner son générateur infinitésimal. En utilisant le théorème de Hille-Yosida, il est facile de voir que si le processus d'exclusion est bien défini, alors son générateur est

$$\Omega f(\eta) := \sum_{x,y\in S} p(x,y)\eta(x)\left(1 - \eta(y)\right)\left[f(\eta_{x,y}) - f(\eta)\right],$$

où

$$\eta_{x,y}(z) := \begin{cases} \eta(y) & \text{si } z = x, \\ \eta(x) & \text{si } z = y, \\ \eta(z) & \text{sinon.} \end{cases}$$

D'après un théorème de Liggett (cf. [44]), ce générateur est bien le générateur d'un processus de Markov si et seulement si

$$\sup_{y \in S} \sum_{x \in S} p(x,y) < \infty. \tag{1.1}$$

L'interprétation de (1.1) est que le taux d'apparition d'une particule en un site $y \in S$ est borné ce qui empêche l'existence d'une infinité de particules voulant sauter au même site en temps fini.

Remarque. *Le théorème de Liggett dans [44] est en fait plus général et s'applique pour une classe plus large de processus d'interaction de particules.*

L'avantage de cette première construction est que l'on a un critère (le théorème de Liggett) pour montrer qu'un processus est bien défini. De plus, le générateur est facile à écrire à partir de la description formelle du processus à l'aide des taux de transitions. Elle permet également, grâce au théorème de Hille-Yosida, de faire certains calculs explicites. En particulier, nous verrons plus loin que ce théorème permet de vérifier si une mesure est invariante ou non pour un processus de Markov.

1.1.2 Construction graphique

Nous allons voir maintenant une manière de construire le processus d'exclusion (et de manière analogue d'autres processus d'interaction de particules) entièrement probabiliste. Cette construction est appelée « construction graphique » et elle est due à Harris (cf. [29]).

Soit $\eta \in X$ une configuration (éventuellement aléatoire) et soit $\mathcal{N} := (\mathcal{N}_{x,y} : x, y \in S)$ une famille de processus ponctuels de Poisson, que l'on appellera *horloges*, sur $\mathbb{R}_+^*$ indépendants deux à deux et indépendants de η. Le processus de Poisson $\mathcal{N}_{x,y}$ a pour intensité $p(x,y)$ (si $p(x,y) = 0$, $\mathcal{N}_{x,y} := \varnothing$). Puisque S est dénombrable, la probabilité que deux horloges distinctes aient un point en commun est 0. On fera donc l'hypothèse que toutes les horloges sont disjointes quitte à exclure un ensemble de réalisations de probabilité 0.

Le processus d'exclusion $(\eta_t)_{t \geqslant 0}$ de configuration initiale η est alors construit comme une fonction (déterministe) de η et de $\mathcal{N}$: si $t \in \mathcal{N}_{x,y}$ et si la condition $(\eta_{t-}(x), \eta_{t-}(y)) = (1,0)$ est satisfaite, alors au temps t, la particule du site x saute au site y et la configuration devient $(\eta_t(x), \eta_t(y)) = (0,1)$. Si le saut n'est pas autorisé, c'est à dire si $(\eta_{t-}(x), \eta_{t-}(y)) \neq (1,0)$, alors rien ne se passe au temps t aux sites x et y.

Cette construction ne permet pas *a priori* de bien définir le processus d'exclusion pour les même raisons que précédemment. En effet, pour déterminer si le site x est occupé au temps t, on a besoin de regarder en arrière dans le temps, tous les sites desquels une particule aurait pu sauter en x. Cette construction ne s'arrête pas si il existe une suite infinie $t > t_{x_1} > t_{x_2} > \cdots > 0$ telle que pour tout $k \geqslant 1$, $t_{x_k} \in \mathcal{N}_{x_k,x_{k-1}}$, avec $x_0 := x$. Ce problème est résolu par l'argument de percolation suivant. Pour $t > 0$ on définit le graphe aléatoire $\mathcal{G}_t$ dont l'ensemble des sommets est S et pour lequel l'arête $\langle x, y\rangle$ est présente si et seulement si $(\mathcal{N}_{x,y} \cup \mathcal{N}_{y,x}) \cap [0, t] \neq \varnothing$. Alors sous l'hypothèse (1.1), il existe $t_0 > 0$ (déterministe) tel que presque sûrement, le graphe $\mathcal{G}_{t_0}$ n'a pas de composante connexe infinie. Ainsi, si $t \geqslant t_0$, une telle suite décroissante ne peut pas exister presque sûrement. On peut donc construire sans problème le processus jusqu'au temps t_0. En itérant la construction, on obtient finalement le processus $(\eta_t)_{t\geqslant 0}$ pour tout temps $t \geqslant 0$.

L'origine de la terminologie « construction graphique » vient de la figure suivante (Figure 1.1) : on place en abscisse les sites de S et en ordonnée le temps. À chaque site $x \in S$, on attache un axe temporel vertical orienté vers le haut et pour chaque $t \in \mathcal{N}_{x,y}$, on trace une flèche de (x, t) à (y, t) (il est plus pratique de faire le dessin dans le cas où $S = \mathbb{Z}$ et $(p(x, y))_{x,y\in\mathbb{Z}}$ est une marche aléatoire simple comme dans le cas de la Figure 1.1). On dessine alors la configuration initiale sur le plan $\{t = 0\}$ et pour obtenir la configuration au temps $t > 0$, on fait suivre les flèches aux particules (tant qu'elles ne sont pas bloquées par d'autres particules). On obtient ainsi tout le processus de manière graphique ainsi que toutes les trajectoires spatio-temporelles des particules.

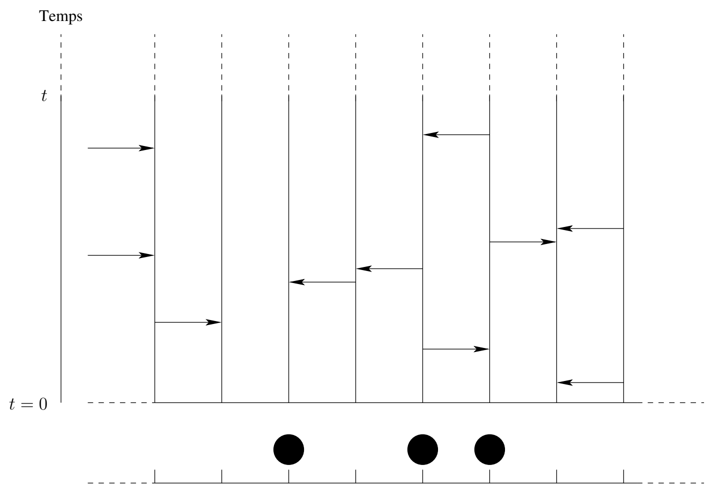

FIG. 1.1 – *Construction graphique du processus d'exclusion.*

1.2 Mesures invariantes

Soit μ une loi sur X. On note $\mu S(t)$ la loi de η_t lorsque η_0 a pour loi μ :

$$\int f d\,[\mu S(t)] := \int S(t) f d\mu,$$

pour toute fonction $f \in C(X)$.

Définition 1.1. *On dit que μ est invariante pour le processus $(\eta_t)_{t\geqslant 0}$ si pour tout $t \geqslant 0$, $\mu S(t) = \mu$.*

On a alors la caractérisation des probabilités invariantes suivante :

Théorème 1.2. *Soit Ω un générateur infinitésimal d'un processus de Markov sur X, et soit μ une loi sur X. Alors μ est invariante si et seulement si pour toute fonction cylindrique f on a*

$$\int \Omega f d\mu = 0.$$

Ce dernier théorème sera très utile pour déterminer si une probabilité est invariante ou non. En revanche, dans la plupart des cas, il ne permet pas, à lui seul, de déterminer toutes les mesures invariantes. Ce problème est en général difficile à résoudre mais nous verrons que pour le processus d'exclusion, sous certaines hypothèses, on peut déterminer entièrement l'ensemble des mesures invariantes du processus. Les preuves de ce genre de résultats suivent en général le plan suivant :

- identifier les possibles mesures invariantes ;
- vérifier, à l'aide du Théorème 1.2, qu'elles sont bien invariantes ;
- montrer, à l'aide de *couplages*, que ce sont les seules.

Nous noterons $\mathcal{I}$ l'ensemble des probabilités invariantes du processus $(\eta_t)_{t\geqslant 0}$.

Propriété. *L'ensemble $\mathcal{I}$ satisfait les propriétés suivantes :*

(i) $\mathcal{I}$ est un convexe compact non vide dans l'ensemble $\mathcal{P}$ des lois sur X. D'après le théorème de Krein-Milman, $\mathcal{I}$ est l'enveloppe convexe de ses points extrémaux.

(ii) si $\mu \in \mathcal{P}$ et si $\mu S(t)$ converge faiblement vers μ_∞, alors $\mu_\infty \in \mathcal{I}$.

1.3 Couplage et particules de deuxième classe

Dans cette partie, nous allons voir l'une des principales techniques utilisées dans cette thèse : le *couplage*. On considère deux configurations $\eta, \xi \in X$ et une famille $\mathcal{N} := (\mathcal{N}_{x,y} : x, y \in S)$ d'horloges de Poisson. On effectue alors la construction graphique avec la même famille d'horloges pour obtenir deux processus d'exclusion $(\eta_t)_{t\geqslant 0}$ et $(\xi_t)_{t\geqslant 0}$ issus respectivement des configurations η et ξ. On appellera ce couplage, le *couplage standard*.

Considérons le cas particulier suivant : on se place dans le cas $S = \mathbb{Z}^d$, $d \geqslant 1$. Soit $\eta \in X$ tel que $\eta(0) = 0$ et

$$\xi(z) := \begin{cases} \eta(z) & \text{si } z \neq 0, \\ 1 & \text{si } z = 0. \end{cases}$$

On considère le couplage standard $(\eta_t, \xi_t)_{t\geqslant 0}$ issu de ces deux configurations. Alors pour tout $t \geqslant 0$, il existe un unique $Q(t) \in \mathbb{Z}^d$ tel que $\eta(Q(t)) = 0$ et $\xi(Q(t)) = 1$. On peut donc coder ce couplage avec un seul processus de Markov $(\zeta_t)_{t\geqslant 0}$ sur $\{0,1,2\}^{\mathbb{Z}^d}$ défini par

$$\zeta_t(x) := \begin{cases} \eta_t(x) & \text{si } x \neq Q(t), \\ 2 & \text{si } x = Q(t). \end{cases}$$

La particule notée 2 est appelée *particule de deuxième classe*. Les particules notées 1 sont appelées *particules de première classe*. Cette terminologie provient de la remarque suivante : le déplacement des particules est régi par les mêmes lois que pour le processus d'exclusion excepté que les particules de première classe ont *priorité* sur la particule de seconde classe au sens où une particule de première classe peut sauter sur un site occupé par la particule de deuxième classe. Dans ce cas de figure, les deux particules échangent leurs positions. Au contraire, la particule de deuxième classe ne peut pas sauter sur un site occupé par une autre particule.

En considérant plus de deux processus, ou bien de manière formelle, on peut définir les particules de classe $k \geqslant 1$ de sorte que si $1 \leqslant k < l$ alors une particule de classe k a priorité sur une particule de classe l.

1.4 Monotonie

De nombreux arguments dans cette thèse, et de manière générale dans le domaine des processus d'interaction de particules, sont basés sur des propriétés de monotonie. L'espace d'état X est naturellement muni d'un ordre partiel. On notera $\eta \leqslant \xi$ si pour tout $x \in S$, $\eta(x) \leqslant \xi(x)$.

Définition 1.2. *Une fonction $f : X \longrightarrow \mathbb{R}$ est dite croissante (resp. décroissante) si*

$$\eta \leqslant \xi \Longrightarrow f(\eta) \leqslant f(\xi),$$

respectivement si

$$\eta \leqslant \xi \Longrightarrow f(\eta) \geqslant f(\xi).$$

Cela permet de définir la notion de *monotonie stochastique* pour les lois sur X.

Définition 1.3. *Soit μ, ν deux lois sur X. On notera $\mu \prec \nu$ si pour toute fonction f croissante*

$$\int f d\mu \leqslant \int f d\nu.$$

Cette dernière notion est fortement liée à celle de couplage. En effet, deux lois, μ et ν sur X sont stochastiquement ordonnées, *i.e.* $\mu \prec \nu$ si et seulement si on peut coupler deux configurations aléatoires η et xi sur un même espace de probabilité, de sorte que η a pour loi μ et ξ a pour loi ν et, presque sûrement, $\eta \leqslant \xi$.

Regardons maintenant le lien entre la monotonie stochastique et les processus de Markov sur X.

Définition 1.4. *Un processus de Markov de semi-groupe $S(t)$ est dit* monotone *ou* attractif *si l'une des deux conditions équivalentes suivantes est vérifiée :*

- *pour toute fonction croissante f et pour tout $t \geqslant 0$, $S(t)f$ est croissante ;*
- *$\mu \prec \nu$ implique que $\mu S(t) \prec \nu S(t)$ pour tout $t \geqslant 0$.*

La propriété d'attractivité joue un rôle très important dans cette thèse puisqu'elle permet dans certains cas d'obtenir des convergences en loi automatiques lorsqu'elle est vérifiée :

Théorème 1.3. *Soit $(\eta_t)_{t\geqslant 0}$ un processus de Markov attractif sur X et soit $S(t)$ son semi-groupe. Soit μ^0, respectivement μ^1, la mesure de Dirac de la configuration ne contenant aucune particule, respectivement de la configuration ne contenant aucun site vide. Alors $\mu^0 S(t)$ et $\mu^1 S(t)$ convergent faiblement quand t tend vers l'infini vers respectivement μ^0_∞ et μ^1_∞ qui sont des mesures invariantes pour $(\eta_t)_{t\geqslant 0}$. De plus si μ est invariante pour $(\eta_t)_{t\geqslant 0}$, alors $\mu^0_\infty \prec \mu \prec \mu^1_\infty$.*

La preuve de ce résultat se trouve dans le Chapitre 3.

1.5 Théorie ergodique

Commençons par rappeler brièvement les définitions liées à l'ergodicité d'un processus.

Définition 1.5. *Un processus $(\eta_t)_{t\geqslant 0}$ sur X est dit stationnaire si pour tout $n \geqslant 0$ et $t_1, \ldots, t_n \geqslant 0$, la loi de $(\eta_{t_1+t}, \ldots, \eta_{t_n+t})$ est indépendante de t.*

Il est dit ergodique si de plus il satisfait la propriété suivante : soit T un événement sur l'espace $D[0,\infty)$ des fonctions càdlàg de $[0,\infty)$ dans X, invariant par décalage temporel. Alors $P((\eta_t)_{t\geqslant 0} \in T) \in \{0,1\}$.

Le seul résultat qui sera utilisé par la suite au sujet de l'ergodicité est le théorème ergodique de Birkhoff :

Théorème 1.4. *Si $(\eta_t)_{t\geqslant 0}$ est stationnaire et ergodique, alors pour toute fonction f mesurable bornée sur X*

$$\frac{1}{t}\int_0^t f(\eta_s)ds \underset{t\to\infty}{\longrightarrow} E\left(f(\eta_0)\right) \quad \textit{presque sûrement.}$$

Afin de montrer qu'un processus de Markov stationnaire est ergodique, on utilisera à de nombreuses reprises le résultat suivant (Théorème B52 de [45]) :

Théorème 1.5. *Soit $(\eta_t)_{t\geqslant 0}$ un processus de Markov stationnaire sur X de loi $\mu \in \mathcal{I}$ en tout temps fixé. Alors $(\eta_t)_{t\geqslant 0}$ est ergodique si et seulement si $\mu \in \mathcal{I}_e$ l'ensemble des points extrémaux de $\mathcal{I}$.*

Chapter 2

Introduction

2.1 Statistical mechanics

Statistical mechanics is a branch of physics that aims to explore and describe physical systems consisting of a large number of components, also called *particles*. Given their number, the particles are therefore very small compared to the whole system so that there is a clear separation between the scale of the observed system, called the *macroscopic scale*, and that of its constituents, called the *microscopic scale*. However, the size of a single particle is not important in itself so that the studied systems can be very diverse. For example, such physical systems can be a gas, a strand of DNA or a forest and the corresponding particles are molecules (or atoms), nucleic acids and trees.

The basic principle of statistical mechanics is the following. Since the studied system is always composed of a very large number of microscopic components, understanding of a macroscopic phenomenon involves the description of the behavior of a significant number of its constituents. But it is not possible to study the microscopic trajectory of each individual particle: firstly because it would require a huge number of computations and secondly because the result would not be satisfactory since it would be impossible to interpret such a large number of trajectories. It is at this stage that probabilities come in: using simple rules, generally derived from quantum mechanics, we describe, from a statistical point of view, the movement and interactions of the components. Then, we deduce the most likely behavior of the whole system or its probability distribution on the space of possible configurations. Hence, using probabilistic tools, we are able to study such physical systems at the microscopic level and deduce information about the macroscopic level.

The foundations of statistical mechanics were laid down in the late 1800s. Its birth is derived from the desire of physicists to explain the nature of gases and interpret quantities such as heat, pressure or work. In 1738, Daniel Bernoulli published “Hydrodynamica” [9] which laid the basis for the kinetic theory of gases. In this work, Bernoulli posited the argument, still used to this day, that gases consist of great numbers of molecules moving in all directions, that their impact on a surface causes the gas pressure that we feel, and that what we experience as heat is simply the kinetic energy of their motion. This kinetic theory of gases will grow over the next century with the works of Herapath [30] and Joule [33]. In 1859, after reading a paper on the diffusion of molecules by Rudolf Clausius [17], physicist James Clerk Maxwell [46] formulated the Maxwell distribution of molecular velocities, which gives the proportion of molecules having a certain velocity in a specific range. This was the first ever statistical law in physics. A few years later, Ludwig Boltzmann [11, 12] will be so inspired by Maxwell's paper that he will spent much of his life developing the subject further. But it took the work of Gibbs [26] for a formulation of the theory as we know it today.

Even today, statistical mechanics is an active branch of science that a whole century was not enough to fully explore. This is probably due to the immense variety of the models and observed phenomena: the relationship with quantum mechanics, phase transitions, shock waves, diffusions, etc.

Models of statistical mechanics can be divided into two categories according to the physical system they model: equilibrium statistical mechanics and out-of-equilibrium statistical mechanics.

Equilibrium statistical mechanics is the study of physical systems in equilibrium thermodynamics. In this context, we seek to explain and interpret macroscopic quantities such as temperature, pressure or magnetization from the behavior of microscopic particles. One of the most important examples of a model at equilibrium is the Ising model which will be developed later in this introduction.

We say that a model of statistical mechanics is out-of-equilibrium when the underlying physical system is not in thermodynamic equilibrium. In particular, it presents macroscopic currents of particles or energy. The techniques used to study such models are different from those models at equilibrium. One example is the exclusion process that is the subject of this thesis; another classical one is that of a system confined between two sources at different temperatures.

2.2 The exclusion process

In this section, I will present the main results established on the exclusion process since its introduction in 1970 by Spitzer [53]. This overview in not intended to be exhaustive but aims to be as self-contained as possible and to direct the interested reader to the existing literature on the subject.

2.2.1 Notation, definitions and first results

Let S be a finite or countable set and let $(p(x,y))_{x,y\in S}$ be the transition function of a Markov chain on S:

$$\forall x,y \in S,\ p(x,y) \geqslant 0,$$
$$\forall x \in S,\ \sum_{y\in S} p(x,y) = 1.$$

Let $X = \{0,1\}^S$ be the set of all configurations of the exclusion process, endowed with the product topology, and $C(X)$ be the set of continuous real-valued functions on X. We denote by $\mathcal{P}$ the set of probability measures on X endowed with the topology of the weak convergence: the sequence of measures $(\mu_n)_{n\geqslant 0}$ on X converges weakly to μ if and only if $\int f d\mu_n$ converges to $\int f d\mu$ for every $f \in C(X)$ (for more background on weak convergence, see for instance Billingsley [10]).

The physical interpretation of $\eta \in X$ is that a site $x \in S$ is *occupied* (by a particle) if $\eta(x) = 1$ and *vacant* or *empty* if $\eta(x) = 0$. The exclusion rule states that there is at most one particle per site at a given time.

The exclusion process on S can be informally defined as follows. Each particle individually tries to perform a random walk on S according to the transition kernel $p(.,.)$, *i.e.*, when it is at site $x \in S$, the particle waits for an exponential time of parameter 1, chooses a site $y \in S$ with probability $p(x,y)$ and tries to jump from x to y. The interaction between particles appears when a jump attempt violates the exclusion rule. In this case, the jump is canceled and the particle simply remains where it was until its next attempt at jumping.

When the set S is finite, one can easily turn this description into a rigorous definition of the exclusion process on S as a Markov process on X. If S is infinite, the construction of such a process requires more work and will be discussed below.

It will be useful to consider the following partial order on the space of configurations. For $\eta, \xi \in X$, we write $\eta \leqslant \xi$ if for all $x \in S$, $\eta(x) \leqslant \xi(x)$. Let f be a real-valued function on X. We say that f is *increasing* if it satisfies:

$$\eta \leqslant \xi \Rightarrow f(\eta) \leqslant f(\xi).$$

Finally, consider two probability measures μ and ν on X. We say that μ is *stochastically dominated* by ν, and we write $\mu \prec \nu$, if for every increasing function f:

$$\int f d\mu \leqslant \int f d\nu.$$

We denote by $\mathcal{I}$ the set of invariant probability measures of the exclusion process on S (see Chapter 1 or [44] for a definition of an invariant measure of a Markov process). $\mathcal{I}$ is a non-empty, convex compact subset of $\mathcal{P}$ (see for example [44]); hence, by the Krein-Milman Theorem, $\mathcal{I}$ is the convex hull of its extremal points. We denote by $\mathcal{I}_e$ the set of extremal points of $\mathcal{I}$.

In the study of particle systems such as the exclusion process, two important questions occur naturally. The first is the determination of all extremal invariant probability measures of the process. The second is to find the attraction domain of each of them, *i.e.*, to find for which initial states the process converges to a given invariant measure.

As mentioned in the beginning of the section, the exclusion process was introduced by Spitzer in the seminal paper [53]. The starting point of this article is a theorem of Doob [19] and Derman [18] on the *independent particles process* defined as follows. Consider a certain number (possibly infinite) of particles distributed on the sites of the set S. Each particle, independently of the others, performs a random walk on S according to the transition kernel $(p(x,y))_{x,y\in S}$. Since the particles are indistinguishable and since they perform random walks with the same transition kernel, the process described above is Markovian. Doob and Derman's theorem states that for each $\lambda > 0$, the Poisson measure μ_λ on $\mathbb{N}^S$ for which the random variables $\{\eta(x), x \in S\}$ are independent and for all $x \in S, k \in \mathbb{N}$,

$$\mu_\lambda\{\eta : \eta(x) = k\} = e^{-\lambda}\frac{\lambda^k}{k!},$$

is invariant for this process. Furthermore, an ergodic theorem holds which states that, under suitable aperiodicity assumptions on p, given any initial distribution of the particles, their distribution at time t converges to some Poisson measure as t goes to infinity.

In [53], Spitzer tried to break away from the sphere of influence of the Poisson point processes. For that, he considered five kinds of generalization of the independent particles process, adding interaction between particles, and among which we find the exclusion process and the zero-range process.

2.2.2 Construction

After Spitzer's paper, it was necessary to answer the question of the construction of the exclusion process in the case where S and the initial number of particles are both infinite. The main difficulty to transform the informal description of the process into a rigorous definition is the following. Let $\eta \in X$ be a configuration with infinitely many particles, $x \in S$ a site and $t > 0$. In order to construct the exclusion process starting from the configuration η, we must be able to describe under which conditions the site x is occupied at time t. But this event depends, *a priori*, on the trajectories, between times 0 and $t-$, of all particles. Hence it is not clear how to proceed since in the time interval $[0,t)$, the number of jump attempts is infinite. Somehow, we have to show that two particles which are far apart have very little influence on each other in a short time, so that the state of a site at time t effectively depends on the behavior of only finitely many particles.

There are two different ways to make this construction. One is probabilistic and the other is analytic. This former consist by defining the process as the Markov process with a given infinitesimal generator. The difficulty is then to prove that this infinitesimal generator exists.

The first work in this direction is the one by Holley [32] in which he develops method for constructing such a stochastic process on the integers. His method uses the Trotter-Kato Theorem [56] to obtain the convergence of a sequence of semi-groups to the semi-group associated with the exclusion process. Then Harris [28] found a direct probabilistic construction of the exclusion process in the case $S = \mathbb{Z}^d$ and with nearest-neighbor interactions. His construction can easily be extended to more general graphs S and to finite-range interactions. The Harris construction, also called the *graphical construction*, is described in Chapters 1 and 3. Finally, Liggett [38, 44] uses the semi-group method to construct a large class of processes including the exclusion process.

The exclusion process is then well defined on any countable set S if the associated transition function $(p(x,y))_{x,y\in S}$ satisfies

$$\sup_{y\in S}\sum_{x\in S} p(x,y) < \infty. \tag{2.1}$$

Condition (2.1) can be understood as follows: for any site $y \in S$, the total rate at which all particles in the system try to jump to y is bounded.

2.2.3 The symmetric exclusion process

The symmetric exclusion process refers to the exclusion process with a symmetric transition function, *i.e.*, $p(x,y) = p(y,x)$ for all $x, y \in S$. In this case, Spitzer proved the following important result known as *duality*.

Theorem 2.1 (Spitzer [53])**.** *Suppose $p(x,y)$ is a symmetric function of x and y. If $\xi_1, \xi_2 \in X$ and $\sum_x \xi_2(x) < \infty$, then*

$$\mathbf{P}^{\xi_1}(\eta_t \geqslant \xi_2) = \mathbf{P}^{\xi_2}(\eta_t \leqslant \xi_1), \tag{2.2}$$

where $\mathbf{P}^{\xi}$ is the distribution of the exclusion process starting from the configuration ξ and $(\eta_t)_{t\geqslant 0}$ is the exclusion process.

This result often permits the reduction of a problem involving an infinite number of particles into a problem involving a finite number of particles for which one can make explicit computations. It is the main reason why the symmetric case is much simpler as the asymmetric case. For example, this theorem has an immediate consequence on the invariant measure problem:

Corollary 2.1. *Assume $p(x,y)$ is symmetric. If $\mu \in \mathcal{P}$, define a function g on X by $g(\eta) := \mu\{\xi | \xi \geqslant \eta\}$. Then μ is invariant for the exclusion process $(\eta_t)_{t\geqslant 0}$ if and only if $\mathbf{E}^{\eta}[g(\eta_t)] = g(\eta)$ for all $\eta \in X$ for which $\sum_x \eta(x) < \infty$.*

Using these results, Liggett [39] solved the two main problems, *i.e.*, describe the set $\mathcal{I}_e$ and prove ergodic theorems, for the symmetric case with the additional hypothesis that $p(x,y)$ is irreducible and transient. More precisely, let H be the set of $[0,1]$-valued harmonic functions, *i.e.*,

$$H := \{\alpha : S \longrightarrow [0,1] : \forall x \in S, \sum_{y\in S} p(x,y)\alpha(y) = \alpha(x)\}.$$

For $\alpha \in H$, we define the probability measure ν^{α} on X as the product measure with marginals

$$\nu^{\alpha}\{\eta : \eta(x) = 1\} = \alpha(x),\ \forall x \in S. \tag{2.3}$$

Theorem 2.2 (Liggett [39]). *Suppose that $p(x,y)$ is symmetric, irreducible and transient and let $S(t)$ be the semi-group of the exclusion process on S. Then*

(i) $\mu_\alpha := \lim_{t\to\infty} \nu^\alpha S(t)$ *exists for every* $\alpha \in H$ *and* $\mu_\alpha \in \mathcal{I}_e$*;*

(ii) $\mu_\alpha\{\eta : \eta(x) = 1\} = \alpha(x)$, $\forall x \in S$*;*

(iii) $\mathcal{I}_e = \{\mu_\alpha : \alpha \in H\}$*;*

(iv) $\mu_\alpha = \nu^\alpha$ *if and only if* α *is constant.*

In the same article, Liggett gives the following ergodic theorem.

Theorem 2.3 (Liggett [39]). *Let*

$$p_t(x,y) := e^{-t} \sum_{k=0}^{\infty} \frac{t^k}{k!} p^{(k)}(x,y), \tag{2.4}$$

where $p^{(k)}(x,y)$ is the k-step transition probability associated to $p(x,y)$. Let $\mu \in \mathcal{P}$ and $\alpha \in H$. Then under the hypothesis of Theorem 2.2,

$$\mu S(t) \underset{t\to\infty}{\longrightarrow} \mu_\alpha$$

if and only if for all $x \in S$,

$$\sum_{y\in S} p_t(x,y)[\eta(y) - \alpha(y)] \underset{t\to\infty}{\longrightarrow} 0 \quad \text{in } \mu\text{-probability.} \tag{2.5}$$

Following this, Spitzer [54] solves the symmetric case under a slightly stronger assumption than the recurrence:

> Let two discrete Markov chains move according to $p(x,y)$ in the following way: at each unit time, one of them is selected at random and makes a transition according to $p(.,.)$. Then they will sooner or later occupy the same point of S with probability one. (2.6)

After a reduction into a problem involving finitely many particles, Spitzer studies the bounded harmonic functions using coupling techniques and proves that the invariant probability measures are exactly the exchangeable probability measures on X. He concludes his proof using De Finetti's Theorem [31]. The results obtained are the following:

Theorem 2.4 (Spitzer [54]). *Assume that $p(.,.)$ is symmetric, irreducible and recurrent, and suppose that (2.6) is satisfied. Then $\mathcal{I}_e = \{\mu_\alpha : \alpha \in [0,1]\}$.*

We can remark that this theorem extends Theorem 2.2 since in the recurrent case, H consists only of constant functions.

For the statement of the ergodic theorem, we need additional notation. If $(\eta_t)_{t\geqslant 0}$ is an exclusion process, we denote by A_t the set of occupied sites at time t. For

$A \subset S$, we denote by P^A, and the corresponding expectation by E^A, the distribution of the exclusion process starting from the configuration η defined by $\eta(x) = 1$ if and only if $x \in A$. Finally for a probability $\mu \in \mathcal{P}$, and a set $A \subset S$,

$$\hat{\mu}(A) := \mu\{\eta : \forall x \in A, \eta(x) = 1\}.$$

Theorem 2.5 (Spitzer [54]). *Let $\mu \in \mathcal{P}$ and $\alpha \in [0, 1]$. Then under the assumption of Theorem 2.4,*

$$\mu S(t) \xrightarrow[t\to\infty]{} \mu_\alpha$$

if and only if for all $x, y \in S$, $x \neq y$,

$$\lim_{t\to\infty} E^{\{x\}}\hat{\mu}(A_t) = \alpha \tag{2.7}$$

and

$$\lim_{t\to\infty} E^{\{x,y\}}\hat{\mu}(A_t) = \alpha^2. \tag{2.8}$$

It is not trivial at all that assumption (2.6) is not equivalent to the recurrence of $p(., .)$. A counter-example was found by Liggett [40]. In the same paper, Liggett proves Theorems 2.2 and 2.3 in the case where $p(., .)$ is recurrent and (2.6) does not hold.

To summarize, the three papers [39, 54, 40] completely solve the main problems for the symmetric exclusion process. Although the results are similar, the methods used to solve the three cases are different and each of them does not apply in any other case.

2.2.4 Asymmetric exclusion processes on $\mathbb{Z}^d$

In this section, we do not assume anymore that the transition function $p(., .)$ is symmetric. Theorem 2.1 does not apply in this general case. Therefore, since we can't simplify problems on the infinite particle system into problems about finite system, we need techniques which deal with the infinite system directly. The main technique used in this thesis is the *coupling technique*. It consists of constructing two copies of the process on the same probability space in such a way as to derive some properties of the process itself.

Invariant probability measures

We want to describe the set of invariant probability measures $\mathcal{I}$ for the exclusion process. Recall that it is sufficient to describe the set $\mathcal{I}_e$ of extremal invariant probability measures. The verification that a given probability measure is invariant is usually a straightforward computation using the characterization of invariant probabilities through the infinitesimal generator. In most cases, one has a collection of invariant probabilities and the difficult question is to show that the process has no other invariant measure.

There are two cases in which one has an explicit collection of invariant measures. First assume that $p(.,.)$ is doubly stochastic, *i.e.*,

$$\forall y \in S, \sum_{x \in S} p(x, y) = 1.$$

Recall that ν^α denotes the product measure on X such that $\nu^\alpha\{\eta : \eta(x) = 1\} = \alpha$ for all $x \in S$. Then

$$\{\nu^\alpha : \alpha \in [0, 1]\} \subset \mathcal{I}. \tag{2.9}$$

Secondly, suppose that $p(.,.)$ is reversible, *i.e.*, there exists some positive $\pi(.)$ on S such which satisfies $\pi(x)p(x, y) = \pi(y)p(x, y)$ for all $x, y \in S$. We denote by $\nu^{(\rho)}$ the product measure on X such that

$$\forall x \in S, \, \nu^{(\rho)}\{\eta : \eta(x) = 1\} = \frac{\rho\pi(x)}{1 + \rho\pi(x)}.$$

Then it was shown in [40] that

$$\{\nu^{(\rho)} : \rho \in [0, \infty]\} \subset \mathcal{I}. \tag{2.10}$$

Note that, when $p(.,.)$ is symmetric, these two classes of invariant measures coincide.

From now on, we look at the exclusion process on $S = \mathbb{Z}^d$, for $d \geqslant 1$, and we assume that $p(.,.)$ is translation invariant, *i.e.*, there exists a function $p(.)$ such that for all $x, y \in S$, $p(x, y) = p(y - x)$. For the asymmetric case, we will need a more flexible definition of *irreducibility* for the random walk kernel $p(.)$:

Definition 2.1. *$p(.)$ is said* irreducible *if for all $x \in S$ there exists some $n \geqslant 1$ such that $p^{(n)}(x) + p^{(n)}(-x) > 0$.*

If it makes sense, let $\mu := \sum_x xp(x)$ be the mean of $p(.)$.

In [42], Liggett uses the coupling technique to partially answer the invariant probability problem. Let $\mathcal{S}$ be the set of translation invariant probability measures on X.

Theorem 2.6 (Liggett [42])**.**

$$(\mathcal{I} \cap \mathcal{S})_e = \{\nu^\alpha, \alpha \in [0, 1]\}. \tag{2.11}$$

The above theorem begs the question whether there exists some extremal invariant measure which is not translation invariant. In the same paper, Liggett gives an answer in two cases. The first one is the one-dimensional mean-zero exclusion process where he proves that, as in the symmetric case, all invariant measures are translation invariant:

Theorem 2.7 (Liggett [42])**.** *Suppose $d = 1$ and $\mu = 0$. Then*

$$\mathcal{I}_e = \{\nu^\alpha, \alpha \in [0, 1]\}. \tag{2.12}$$

The second one is the one-dimensional nearest-neighbor asymmetric exclusion process. In this case, there is an other one-parameter collection of invariant probability which are not translation invariant:

Theorem 2.8 (Liggett [42]). *Suppose $d = 1$, $p(x, x+1) = p$, $p(x, x-1) = q$, $p + q = 1$ and $p > \frac{1}{2}$. Then*

$$\mathcal{I}_e = \{\nu^\alpha, \alpha \in [0,1]\} \cup \{\gamma^{(n)}, n \in \mathbb{N}\} \tag{2.13}$$

In the above theorem, the measures $\gamma^{(n)}$ are *blocking measures*. That means that they put mass on *blocking configurations*, *i.e.*, configurations for which all sites are occupied far enough to the right and all sites are empty far enough to the left. In particular, a blocking measure ν is a *profile measure*, that is, $\nu\{\eta : \eta(x) = 1\}$ goes to 1 as x goes to ∞ and $\nu\{\eta : \eta(x) = 0\}$ goes to 0 as x goes to $-\infty$.

The outline of the proof of Theorem 2.7 is the following. Liggett defines the coupled process on $X \times X$ consisting with two copies $(\eta_t)_{t \geqslant 0}$ and $(\xi_t)_{t \geqslant 0}$ of the exclusion process. This coupling has the informal property that if a particle for the process $(\eta_t)_{t \geqslant 0}$ is at the same site as a particle for the process $(\xi_t)_{t \geqslant 0}$, then both particles will jump together as long as possible. Then he shows, using the infinitesimal generator, that any invariant measure ν for the coupled process satisfies $\nu\{(\eta, \xi) : \eta \geqslant \xi \text{ or } \eta \leqslant \xi\} = 1$. If in addition ν is extremal, then $\nu\{(\eta, \xi) : \eta \geqslant \xi\} = 1$ or $\nu\{(\eta, \xi) : \eta \leqslant \xi\} = 1$. Finally, if $\mu_1, \mu_2 \in \mathcal{I}_e$, then one can find an extremal invariant measure for the coupled process which has marginals μ_1 and μ_2 respectively. This proves that the set $\mathcal{I}_e$ is ordered.

Still in dimension 1, the existence of stationary blocking measures has been proved by Ferrari, Lebowitz and Speer [25] for a restricted class of $p(.)$ with $\mu > 0$. They find some sufficient inequalities on transition functions to deduce the existence of stationary blocking measures for a process from the existence of stationary blocking measures for an other process. Next, Bramson and Mountford [16] have shown that if $p(.)$ has finite range and $\mu > 0$, then the associated exclusion process has a stationary invariant blocking measure. Finally, Bramson, Liggett and Mountford [15] have proved results in the opposite direction:

Theorem 2.9. *Assume $d = 1$.*

(i) If $p(.)$ is irreducible and $\mu \in (0, \infty)$, then the only possible extremal non-translation invariant stationary measures consist of a profile measure ν, together with its translates;

(ii) If $p(.)$ has finite mean and satisfies

$$\sum_{x<0} x^2 p(x) = \infty, \tag{2.14}$$

then no stationary blocking measure exist;

(iii) If $\mu \in (0, \infty)$ and if $p(.)$ satisfies

$$\begin{aligned} &\sum_{x<0} x^2 p(x) < \infty, \\ &p(y) \leqslant p(x) \text{ and } p(-y) \leqslant p(-x) \quad \text{for} \quad 1 \leqslant x \leqslant y, \\ &p(-x) \leqslant p(x) \quad \text{for} \quad x \geqslant 1, \end{aligned} \tag{2.15}$$

then there exists a stationary blocking measure ν satisfying

$$\sum_x \nu\{\eta : \eta(x) = 1, \eta(x+1) = 0\} < \infty. \tag{2.16}$$

For $d \geqslant 2$ very few results have been proved. The only paper which has made significant progress in this direction is the one of Bramson and Liggett [14]. In this paper, the authors start by giving a very simple characterization of invariant product measures for the exclusion process for general S and $p(.,.)$. Then, they apply this characterization to the case $S = \mathbb{Z}^d$ and $p(.,.)$ translation invariant to get the following result:

Theorem 2.10. *Let $\alpha : \mathbb{Z}^d \longrightarrow (0,1)$ and $\pi(x) := \alpha(x)/(1-\alpha(x))$. Suppose that $\pi(x) = \pi(0)e^{\langle x,v\rangle}$ for all $x \in \mathbb{Z}^d$ and $v \in \mathbb{R}^d$. Then ν^α is stationary for the exclusion process if and only if*

$$p(z) = e^{\langle z,v\rangle} p(-z) \text{ for all } z \text{ such that } \langle z,v\rangle \neq 0, \tag{2.17}$$

Then, the authors prove that under a weak assumption, which is trivially necessary, π must have the exponential form assumed in Theorem 2.10:

Theorem 2.11. *Assume that the transition function $p(.)$ has the following property: there is no proper subgroup of $\mathbb{Z}^d$ that contains $P = \{u \in \mathbb{Z}^d : p(u) > 0\}$. If α is such that ν^α is stationary for the exclusion process, then there exists $v \in \mathbb{R}^d$ so that $\pi(x) = \pi(0)e^{\langle x,v\rangle}$ for all $x \in \mathbb{Z}^d$.*

Finally, after a generalization of the concept of profile measure in dimension $d \geqslant 2$, the authors give necessary conditions for the existence of a profile measure.

Some important open problems remain after this paper in high dimension. Here is an non-exhaustive list of such problems:

(i) If $\sum_x xp(x) = 0$ then the extremal stationary measures are exactly the product measures with constant density.

(ii) If $\sum_x xp(x) \neq 0$ there exists an extremal stationary measure that is not a product measure.

(iii) There exists $p(.)$ for $d \geqslant 2$ such that there exists an extremal stationary measure that is not a product measure.

Ergodic theorems

In this section, we are interested in ergodic theorems for the asymmetric exclusion process. Indeed, once we have described the set of extremal stationary measures, one important problem is to determine the initial measures from which the process converges to a given extremal stationary measure.

The first article in this direction is the one by Liggett [41] on the one-dimensional asymmetric simple exclusion process. In this paper, Liggett gives an almost complete description of the limit behavior for initial measure with asymptotic densities on both sides. Let $p(x, x+1) = p$, $p(x, x-1) = q$ with $p + q = 1$ and $p > \frac{1}{2}$. Consider a product measure μ on X such that

$$\lim_{x\to-\infty} \mu\{\eta : \eta(x) = 1\} = \lambda, \text{ and } \lim_{x\to\infty} \mu\{\eta : \eta(x) = 1\} = \rho. \tag{2.18}$$

Liggett first proves that $\mu S(t)$ converges, as t goes to infinity, to a stationary measure μ_∞, except possibly if $0 < \lambda < \frac{1}{2}$ and $\lambda + \rho = 1$. Furthermore, the limit distribution μ_∞ is explicit and described in the diagram of Figure 2.1.

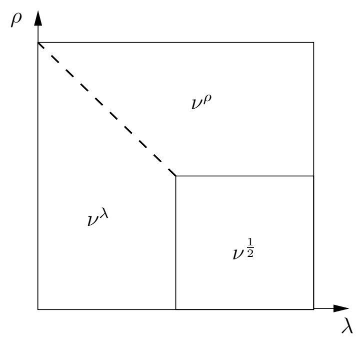

Figure 2.1: *Phase diagram for the ASEP.*

For the proof of this theorem, Liggett starts by studying the following auxiliary process. Let $S = \mathbb{Z}_+$ and consider the exclusion process on S with the same transition function $p(.)$ for which we add a creation/destruction mechanism: at jump times of a Poisson process with intensity $p\lambda$, a particle is created at site 0 if this site is empty; at jump times of a Poisson process with intensity $q(1-\lambda)$, if a particle is at site 0, then it disappears. This process is called the one-dimensional semi-infinite ASEP. If μ is a product probability on X such that

$$\lim_{x\to\infty} \mu\{\eta : \eta(x) = 1\} = \rho, \tag{2.19}$$

then $\mu S(t)$ converges, as t goes to infinity, to a stationary measure given by the diagram of Figure 2.2. The measures appearing in Figure 2.2 are invariant for the semi-infinite ASEP and have the following properties:

Theorem 2.12. *(i) $\mu_{\lambda,\rho}$ behaves like ν^ρ at ∞,* i.e., *for every $x_1, \ldots, x_n \in \mathbb{Z}$ and all $n \geqslant 1$,*

$$\lim_{x\to\infty} \mu\{\eta : \eta(x_1 + x) = 1, \ldots, \eta(x_n + x) = 1\} = \rho^n; \tag{2.20}$$

(ii) $\mu_{\lambda,\lambda} = \nu^\lambda$*;*

(iii) $\mu_{\lambda,\rho}$ *is jointly continuous in* λ *and* ρ*;*

(iv) If $0 < \lambda < \frac{1}{2}$*, then* $\lim_{\rho\downarrow 1-\lambda} \mu_{\lambda,\rho} = \nu^\lambda$.

Liggett's method to prove these theorems is to start from an exclusion process on the finite set $\{1,\dots,N\}$ with a creation/destruction mechanism, to make some computations on the generator for this process and to pass to the limit. Liggett also uses coupling methods and monotonicity. These methods will be used in this thesis in order to compute a certain survival probability of a *second class particle* in the semi-infinite simple exclusion process (cf. Appendix A). In [43], Liggett extends these results without the nearest neighbor hypothesis, *i.e.*, he only assumes that $\sum_x |x|p(x) < \infty$ and that $\mu = \sum_x xp(x) > 0$.

One reason for which the case $0 < \lambda < \frac{1}{2}$ and $\lambda + \rho = 1$ is more complicated is that the hypothesis "μ product measure having right and left asymptotic densities" is not sufficient to prove that $\mu S(t)$ converges as t goes to infinity. Indeed, it was shown in [41] that there exists μ with this property such that $\mu S(t)$ have both ν^λ and $\nu^\rho = \nu^{1-\lambda}$ as weak limit points. However, Liggett conjectures in the same paper that if we add the hypothesis:

$$\sum_{x<0} |\mu\{\eta : \eta(x) = 1\} - \lambda| < \infty \text{ and } \sum_{x>0} |\mu\{\eta : \eta(x) = 1\} - \rho| < \infty, \tag{2.21}$$

then $\mu S(t)$ converges weakly to $\frac{1}{2}\nu^\lambda + \frac{1}{2}\nu^\rho$. In 1986, Andjel [2] proves a weak form of this conjecture using coupling and attractivity methods. The obtained result is that the Cesaro convergence of $\mu S(t)$ to $\frac{1}{2}\nu^\lambda + \frac{1}{2}\nu^\rho$ under the condition (2.21). Andjel also shows that the convergence of $\mu S(t)$ occurs outside a set of zero asymptotic density. Finally, in 1988, Andjel, Bramson and Liggett [3] prove the full conjecture using coupling and symmetry considerations. The first part of the proof consists in showing that any weak limit point of $\mu S(t)$ is translation invariant. For that, one can couple the process starting from μ with the process starting from a translation

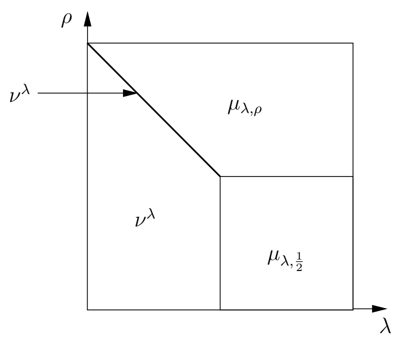

Figure 2.2: *Phase diagram for the semi-infinite ASEP. For* $\lambda = 0$*,* $\rho = 1$ *and* $p = 1$*, the additional assumption that* $\sum_{x\geqslant 1} \mu\{\eta : \eta(x) = 0\} = \infty$ *is required.*

of μ and show that all discrepancies eventually disappear. Once this is done, a result of Andjel [2] allows to prove that any weak limit point ν of $\mu S(t)$ has the following form:

$$\nu = \int_{\lambda}^{1-\lambda} \nu^{\alpha}\gamma(d\alpha), \tag{2.22}$$

where γ is a probability measure on $[\lambda, 1-\lambda]$. Then, they show that the flow is at any time and any point smaller than the one in equilibrium. This part is done using again a coupling argument and the semi-group theory of a Markov process. This implies that γ put mass on λ and $1-\lambda$. Finally, a symmetry argument permits to conclude.

For ergodic theorems with non-product initial measures, a natural first step is to look at the case where the initial measure is invariant under translations. Under this hypothesis, Andjel [1] obtained two results: the first one in $\mathbb{Z}^d$ and the second one for the nearest-neighbor case in $\mathbb{Z}$.

Theorem 2.13. *Assume that $S = \mathbb{Z}^d$, $p(.)$ is irreducible and μ is a translation invariant probability measure on X. Then there exists a probability measure γ on $[0,1]$ such that*

$$\mu S(t) \underset{t\to\infty}{\longrightarrow} \int_0^1 \nu^{\rho}\gamma(d\rho). \tag{2.23}$$

Furthermore, $\int_0^1 \rho\gamma(d\rho) = \mu\{\eta : \eta(0) = 1\}$.

Theorem 2.14. *Assume that $S = \mathbb{Z}$, $p(1) = p$, $p(-1) = q$ with $p + q = 1$ and μ is a translation invariant ergodic probability measure on X such that $\mu\{\eta : \eta(0) = 1\} = \rho_0$. Then*

$$\mu S(t) \underset{t\to\infty}{\longrightarrow} \nu^{\rho_0}. \tag{2.24}$$

Then Seppäläinen [51] proved Theorem 2.14 in the finite-range case. Mountford [47] also improved Theorem 2.14 using a very different method. Its key idea is to use the results of Rezakhanlou [49] on the hydrodynamical limit of the exclusion process with coupling methods. This leads to the following theorem which is valid without the finite-range condition but which needs the existence of a nonzero mean:

Theorem 2.15. *Assume that $S = \mathbb{Z}$, $p(.)$ is irreducible and has finite positive mean,* i.e., *$\sum_x |x|p(x) < \infty$ and $m := \sum_x xp(x) > 0$. If μ is a translation invariant probability measure on X such that for μ-almost every $\eta \in X$,*

$$\frac{1}{2n+1}\sum_{|x|\leqslant n} \eta(x) \underset{n\to\infty}{\longrightarrow} \alpha \in [0,1], \tag{2.25}$$

then $\mu S(t)$ converges weakly to ν^{α} as t goes to infinity.

In [49], Rezakhanlou derives the hydrodynamical limit for the exclusion process and for other particle systems. His result on the exclusion process is as follows.

Theorem 2.16. *Let f be a positive function that is bounded by 1 and continuous except at finitely many points. Let $(\eta_t^N)_{t\geqslant 0}$ be a sequence of exclusion processes on $\mathbb{Z}/N\mathbb{Z}$ with initial distribution such that the random variables $(\eta_0^N(x))$ are independent with Bernoulli distribution: $\mathbf{P}\left[\eta_0^N(x)=1\right]=f(x)$ for all $x\in\mathbb{Z}/N\mathbb{Z}$. Then for every interval I we have $\forall\gamma>0$,*

$$\mathbf{P}\left[\left|\frac{1}{N}\sum_{x\in(\mathbb{Z}/N\mathbb{Z}):\,\frac{x}{N}\in I}\eta_{Nt}^N(x)-\int_I u(t,x)dx\right|>\gamma\right]\underset{N\to\infty}{\longrightarrow}0, \tag{2.26}$$

where u is the unique entropy solution to

$$\frac{\partial u}{\partial t}+m\frac{\partial(u(1-u))}{\partial x}=0, \tag{2.27}$$

with initial condition

$$u(0,x)=f(x). \tag{2.28}$$

In this theorem, the exclusion process $(\eta_t^N)_{t\geqslant 0}$ on $\mathbb{Z}/N\mathbb{Z}$ has the following transition function: a particle at site x jumps to site y at rate $p(N(y-x))$. Roughly speaking, the theorem above says that when we divide the length between two neighbor sites by N and we multiply the time by N, the particle distribution of the exclusion process starting from the product distribution associated to the function f looks like, at time t, the entropy solution of the equations (2.27) and (2.28). (2.27) is called *Burgers' equation*. For more about hydrodynamic limit of the exclusion process, see [36].

Finally, Bahadoran and Mountford [5] also use the hydrodynamic technique and coupling ideas to prove convergence without the translation invariance hypothesis. But their theorem requires both finite-range and non-zero mean assumptions:

Theorem 2.17. *Suppose that $S=\mathbb{Z}$, $p(.)$ has finite support and is irreducible, and $m:=\sum_x xp(x)\neq 0$. Let μ be a probability measure on X such that for some $\alpha\in[0,1]$, the limits*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{x=0}^{N}\eta(x)=\alpha \tag{2.29}$$

and

$$\lim_{N\to\infty}\frac{1}{N}\sum_{x=0}^{N}\eta(x)=\alpha, \tag{2.30}$$

hold in probability with respect to the initial distribution of the process. Then $\mu S(t)$ converges weakly to ν^α as t goes to infinity.

Tagged particle and second-class particle

In this section, we are interested in the behavior of some individual particle instead of the whole process. We will see that this point of view allows for the microscopical localization of some macroscopic effects (in Burgers' equation).

The tagged particle is an individual particle in the exclusion process. Of course, the process consisting only of the position of the tagged particle is not Markovian since the trajectory of this particle depends on the whole system. Hence, in order to keep the Markov property, we have to keep track of the occupancy of all other sites. One way to do that is to consider the process consisting of the configuration at time t "seen from the point of view of the tagged particle", *i.e.*, the exclusion process shifted in order to have the tagged particle at site 0 at any time. Let $S = \mathbb{Z}^d$ and consider the tagged particle space

$$\hat{X} := X \cap \{\eta \in X : \eta(0) = 1\}.$$

Then the tagged particle process is the process with state space $\hat{X}$ and infinitesimal generator

$$\begin{aligned} \hat{\Omega} f(\eta) := & \sum_{y \in S} (1-\eta(y)) p(y) [f(\theta^y \eta_{0,y}) - f(\eta)] \\ & + \sum_{x,y \neq 0} \eta(x)(1-\eta(y)) p(y-x) [f(\eta_{x,y}) - f(\eta)], \end{aligned} \tag{2.31}$$

where

$$\eta_{x,y}(z) := \begin{cases} \eta(y) & \text{if } z = x, \\ \eta(x) & \text{if } z = y, \\ \eta(z) & \text{otherwise.} \end{cases} \tag{2.32}$$

and $\theta^y \eta(z) := \eta(z+y)$. The existence of the tagged particle process can be obtained using Liggett's existence criteria [38].

As we have seen before, the determination of stationary measures which are translation invariant is easier than the determination of all stationary measures. Hence it will be useful to generalize the notion of translation invariance in the state space $\hat{X}$. For a probability measure μ on X, we define the measure $\hat{\mu}$ on $\hat{X}$ by:

$$\hat{\mu} := \mu(.|\eta(0) = 1).$$

Let $\mathcal{S}$ be the set of translation invariant probability measures on X and

$$\hat{\mathcal{S}} := \{\hat{\mu} : \mu \in \mathcal{S}\}.$$

We define as $\hat{\mathcal{I}}$ the set of stationary measures for the tagged particle process. In [20], Ferrari proves the following theorem on this process:

Theorem 2.18.

$$(\hat{\mathcal{I}} \cap \hat{\mathcal{S}})_e = (\widehat{\mathcal{I} \cap \mathcal{S}})_e = \{\widehat{\nu^\rho} : \rho \in [0,1]\}. \tag{2.33}$$

In the same article, he obtains a precise description of $\hat{\mathcal{I}}_e$ in the one-dimensional nearest-neighbor case analogous to the one of the simple exclusion process.

These results describe the configuration around the tagged particle but do not give information about its trajectory. We will see that the behavior of the tagged particle position depends on the transition function in a very complex way in dimension one. Recall that if X_t is a Markov process with transition function $p(.)$ with finite mean, then it satisfies the following law of large numbers:

$$\frac{X_t}{t} \underset{t\to\infty}{\longrightarrow} \sum_{x\in S} xp(x) =: v_0 \text{ almost surely,} \tag{2.34}$$

and the following central limit theorem if $p(.)$ has a finite second moment:

$$\frac{X_t - v_o t}{\sqrt{t}} \underset{t\to\infty}{\longrightarrow} \mathcal{N}(0,\sigma^2) \text{ in distribution,} \tag{2.35}$$

where $\sigma^2 = \sum_{x\in S} x^2 p(x) - v_0^2$.

Consider now an exclusion process on $\mathbb{Z}^d$ with distribution ν^ρ conditioned to have a particle at site 0. We denote by X_t the position of this particle at time t. Then the process X_t is not a Markov process anymore in the general case. We have seen that, seen from the position X_t, the process is stationary with a product distribution. In the sequel, we say that X_t satisfies a law of large numbers if there exists some constant $v(p,\rho) \in \mathbb{R}$ such that

$$\frac{X_t}{t} \underset{t\to\infty}{\longrightarrow} v(p,\rho) \text{ almost surely.} \tag{2.36}$$

We say that X_t satisfies a central limit theorem if, furthermore, there exists $\sigma > 0$ such that

$$\frac{X_t - v(p,\rho)t}{\sqrt{t}} \underset{t\to\infty}{\longrightarrow} \mathcal{N}(0,\sigma^2) \text{ in distribution.} \tag{2.37}$$

For the particular case where $p(1) = 1$, X_t is still a Markov process as was pointed out by Kesten. Spitzer [53] uses this remark to prove that X_t satisfies a law of large numbers with $v(p,\rho) = 1-\rho$ and a central limit theorem with $\sigma = 1-\rho$.

For the general case, a law of large numbers has been proved by Kipnis [35] when $p(.)$ is nearest-neighbor in dimension one and then by Saada [50] in every dimension under the assumption that $\sum_x |x|p(x) < \infty$. The speed obtained is $v(p,\rho) := (1-\rho)\sum_x xp(x)$ which is the one conjectured by Liggett in [44] (see Chapter VIII, Section 7). Kipnis uses the correspondence between the exclusion process and the zero-range process and Saada uses the extremality of Bernoulli measures conditioned to have a particle at site 0 for the tagged particle process.

The central limit theorem problem is more delicate. In dimension $d \geqslant 2$, nothing spectacular is expected. It was proved in [37] that for $p(.)$ symmetric, X_t satisfies a central limit theorem with an unknown $\sigma > 0$. For non-symmetric $p(.)$, the same kind of results is conjectured. If $d = 1$, it was also proved in [37] that X_t satisfies a central limit theorem if $p(.)$ has at least 4 points in its support. On the other hand, if $p(1) = p(-1) = \frac{1}{2}$, then Arratia [4] has proved that X_t does not satisfies a central limit theorem. Indeed, the correct renormalization in this case is $t^{\frac{1}{4}}$ and we have

$$\frac{X_t}{t^{\frac{1}{4}}} \underset{t\to\infty}{\longrightarrow} \mathcal{N}(0,\sigma^2) \text{ in distribution.} \tag{2.38}$$

Still in [35], Kipnis has obtained a central limit theorem for X_t for the non-symmetric nearest neighbor case with an unknown σ satisfying:

$$\sigma \geqslant (1-\rho)(\sqrt{2p}-1), \tag{2.39}$$

where $p(1) = p = 1 - p(-1)$.

Now, consider two initial configurations η^1 and η^2 such that for all $x \in S \setminus \{0\}$, $\eta^1(x) = \eta^2(x)$, and $\eta^2(0) = 1 - \eta^1(0) = 1$. Performing the graphical construction from both configurations using the same Poisson processes $(\mathcal{N}_{x,y})$, one obtains a coupled process $(\eta_t^1, \eta_t^2)_{t\geqslant 0}$ in such a way that almost surely for every time t there exists a unique $Y_t \in S$ such that $\eta^2(Y_t) = 1 - \eta^1(Y_t) = 1$. This coupling can be interpreted in the following way. Sites in which there is an η^1-particle contain a *first-class particle* and the site Y_t contains a *second-class particle*. All other sites are empty. The second-class particle moves as a standard particle in the exclusion process excepted that if a first-class particle tries to jump from a site x to Y_t (the location of the second-class particle), then the jump occurs and the two particles exchange their positions.

The second-class particle has been introduced by Ferrari, Kipnis and Saada [24, 21] with the aim to identify the microscopic location of the shock in Burger's equation (2.27). Let $\lambda, \rho \in [0, 1]$ and $1/2 < p \leqslant 1$. Define the measure $\nu^{\lambda,\rho}$ on $\{0, 1, 2\}^{\mathbb{Z}}$ as follows: the random variables $(\eta(x), x \in \mathbb{Z})$ are independent,

$$\nu^{\lambda,\rho}\{\eta : \eta(x) = 1\} = \begin{cases} \lambda & \text{if } x < 0, \\ \rho & \text{if } x > 0, \\ 0 & \text{if } x = 0, \end{cases} \tag{2.40}$$

and $\eta(0) = 2$ almost surely. The following theorem is a summary of some results proved in [24, 23, 48].

Theorem 2.19. *(i) If $\lambda \leqslant \rho$, then Y_t/t converges almost surely to $2p-1-\lambda-\rho$.*

(ii) If $\lambda > \rho$ and $p = 1$, then Y_t/t converges almost surely to a uniform random variable on the interval $[1 - 2\lambda, 1 - 2\rho]$.

This result will be used a lot in the sequel since the speed of a second-class particle plays an important role in techniques used in this thesis.

Concerning fluctuations of the second-class particle, Balázs and Seppäläinen [7] have proved the following theorem which states that these are of order $t^{2/3}$ for the stationary ASEP. They used the exact connection between currents and second-class particles.

Theorem 2.20. *Assume that $\frac{1}{2} < p \leqslant 1$. Consider the ASEP starting with the product distribution with density $\rho \in (0, 1)$ on $\mathbb{Z}^*$ and with a second class particle at site 0. There exist constants $0 < t_0, C < \infty$ such that for every $1 \leqslant m < 3$ and $t \geqslant t_0$*

$$C^{-1} \leqslant \mathbf{E}\left[\left|\frac{Y_t - (2p-1-2\rho)t}{t^{2/3}}\right|^m\right] \leqslant C. \tag{2.41}$$

In an other direction, some exact results have been proved by Ferrari, Gonçalves and Martin [22]. Consider the followings initial configurations:

$$\eta(x) := \begin{cases} 1 & \text{if } x \leqslant 0, \\ 0 & \text{if } x > 0, \end{cases} \tag{2.42}$$

and

$$\xi(x) := \begin{cases} 1 & \text{if } x < 0 \text{ or } x = 1, \\ 0 & \text{else.} \end{cases} \tag{2.43}$$

With the first/second-class particles interpretation, it is equivalent to consider the configuration for which all negative sites contain a first-class particle, sites 0 and 1 both contain a second-class particle and all other sites are empty. They proved the following theorem.

Theorem 2.21. *The probability that the second-class particle at site* 0 *eventually tries to jump on a site occupied by the other second-class particle is* $\frac{1+p}{3p}$.

We can remark that this event is the coupling event of processes $(\eta_t)_{t\geqslant 0}$ and $(\xi_t)_{t\geqslant 0}$, *i.e.*, the event that configurations η_t and ξ_t become eventually the same. In the same paper, they also prove that the coupling probability becomes $(1+2p^2)/6p^2$ if the initial configurations are η and $\xi_{1,2}$.

2.3 Non-uniqueness for specifications

Let A be a finite alphabet and let $\mathcal{P}(A)$ be the set of probability distributions on A.

Definition 2.2. *A* specification *(also known as g-*function*) is a measurable function g from $A^{\mathbb{N}}$ to $\mathcal{P}(A)$.*

A *Gibbs measure* for a specification g is a probability measure μ on $A^{\mathbb{Z}}$ such that

- μ is shift-invariant;
- if $(x_n)_{n\in\mathbb{Z}}$ is distributed according to μ, then for every $i \in \mathbb{Z}$ and $a \in A$,

$$\mu(x_i = a | x_{i-1}, x_{i-2}, \dots) = g_{x_{i-1},x_{i-2},\dots}(a). \tag{2.44}$$

Assume g has a range 1, *i.e.*, if $g_{x_{-1},x_{-2},\dots}$ depends only on x_{-1}. If μ is a probability measure on $A^{\mathbb{Z}}$ satisfying (5.1) and if $(x_n)_{n\in\mathbb{Z}}$ is distributed according to μ, then $(x_n)_{n\in\mathbb{Z}}$ is a Markov process. Hence, in this case, a Gibbs measure is an invariant measure and reciprocally. It is well known that if the Markov chain is ergodic, then it admits a unique Gibbs measure.

For more general cases, the existence of a Gibbs measure is ensured when g is continuous or if g defines a monotone Markov chain. For the question of uniqueness we have to assume that g is *regular*, *i.e.*, that g is bounded away from 0.

For $k \geqslant 1$, we define the *k-variation* of g by

$$var_k(g) := \sup\{||g_x - g_y|| : x_1 = y_1, \dots, x_k = y_k\}. \tag{2.45}$$

We can remark that the continuity of g is equivalent to the condition that $var_k(g) \to 0$ as k goes to infinity. An old result of Keane and Walter [34, 55] is the following:

Theorem 2.22. *If $(var_k(g))_{k\geqslant 1}$ is l^1, then g admits a unique Gibbs measure.*

However, the continuity is not sufficient to ensure the uniqueness as it was proved by Bramson and Kalikow [13]:

Theorem 2.23. *There exists a continuous regular specification that admits multiple Gibbs measures.*

2.4 Contents of the thesis

In my thesis, I am interested in the exclusion process on $\mathbb{Z}_+$. I only considered the totally asymmetric nearest neighbor exclusion process. However, most of the results generalize to the partially asymmetric nearest neighbor case but the notations which would be needed would be much more tedious. As seen before, Liggett [41] has first studied this process in order to prove ergodic theorems for the exclusion process in $\mathbb{Z}_+$ and in $\mathbb{Z}$.

Let $S := \mathbb{Z}_+$, $p(x, x+1) := 1$ for all $x \in S$. At site 0, we add a creation mechanism in the following way. Let $\mathcal{N}_b$ be a Poisson point process on $\mathbb{R}_+^*$ with intensity 1 independent of the Harris system in the bulk. Let r be a measurable function from X to $[0, 1]$. For each time $t \in \mathcal{N}_b$ such that $\eta_{t-}(0) = 0$, we create a particle at site 0 at time t with probability $r(\eta_{t-})$. We will call r the creation rate function.

In [41], Liggett considers the particular case $r \equiv \lambda$ for some $\lambda \in [0, 1]$. He obtains ergodic theorems for this process given by the phase diagram in Figure 2.2. We see that the product measure with density λ is still invariant for the process. When r is not a constant function this is not true anymore: the process does not have an invariant measure which is product.

In Chapter 3 we consider the particular example where the creation function r is given by

$$r(\eta) := \begin{cases} \alpha_0 & \text{if } \eta(1) = 0, \\ \alpha_1 & \text{if } \eta(1) = 1, \end{cases} \tag{2.46}$$

where $\alpha_0, \alpha_1 \in [0, 1]$ and $\alpha_0 \neq \alpha_1$. This is the simplest example for which r is not constant. In this chapter, we use coupling methods and second-class particles to compare this process with semi-infinite TASEP with constant creation density α_0 and α_1. Consider the process $(\eta_t)_{t\geqslant 0}$ starting from the empty configuration:

$$\eta_0(x) = 0 \quad \text{for all } x \in \mathbb{Z}_+. \tag{2.47}$$

Let N_t be the number of created particles between times 0 and t. The main result of this chapter is the following strong law of large numbers:

$$\frac{N_t}{t} \longrightarrow v(\alpha_0, \alpha_1) \quad \text{as } t \to \infty, \tag{2.48}$$

almost surely, where $v(\alpha_0, \alpha_1)$ is a constant. Furthermore, if $\alpha = \lambda \in [0, 1/2)$ and $\alpha_1 = \lambda + \epsilon \in [0, 1/2)$ with $\epsilon > 0$, then

$$v(\lambda, \lambda + \epsilon) = \lambda(1 - \lambda)\,[1 + p(\lambda)\epsilon] + o(\epsilon), \tag{2.49}$$

as $\epsilon \downarrow 0$, where $p(\lambda)$ is the probability of a certain event which will be discussed later.

The subject of Chapter 4 is the study of the TASEP with a complex creation mechanism for which the creation function r is of finite-range, *i.e.*, we assume there exists some $R \in \mathbb{Z}_+$ such that for any two configurations $\eta, \xi \in X$ such that

$$\eta(x) = \xi(x) \quad \text{for all } x \geqslant R + 1, \tag{2.50}$$

we have $r(\eta) = r(\xi)$. In other words, the creation function depends only on the first R sites. Under this assumption, there is no phase transition at low density. More precisely, assume that the initial configuration is distributed according to a probability measure μ which is dominated by some product probability ν^α with $\alpha < 1/2$. Then the process converges in distribution to a stationary probability μ_∞ which is independent of μ. For constant creation function $r \equiv \lambda \in [0, 1/2)$, the result holds but in this case $\mu_\infty = \nu^\lambda$. The method used to prove this result is based on the following idea. We couple two TASEP with a complex boundary mechanism $(\eta_t)_{t\geqslant 0}$ and $(\xi_t)_{t\geqslant 0}$ using the standard coupling. We assume that η_0 and ξ_0 are distributed according to the probability measures μ and ν respectively, which are both dominated by ν^α with $\alpha < 1/2$. As usual, we interpret the discrepancies as second-class particles. The key argument of the proof is a consequence of a theorem of Ferrari, Kipnis and Saada [24, 23]: a second-class particle in a stationary TASEP with density less than $1/2$ goes to infinity at positive speed. From this fact, a second-class particle will eventually leave the system from site 0 (when a first class particle is created when it is located at this site) or leave the box $\{0, \dots, R\}$ from site R and never return to it. Furthermore, for every time t, there is a positive probability that no new second-class particle is created before time t. Hence, the event that no second-class particle is created at any time has also positive probability. This implies that there exists a finite time after which all particle created are the same for both processes which gives the result.

The result obtained in Chapter 4 can be generalized, without major changes in the proof, to the case where the transition rate depends on the whole process in the following way. Let N be a random variable on $\mathbb{N}$ with finite mean and α be a measurable function from $\mathbb{N} \times X$ to $[0,1]$ such that for every $n \in \mathbb{N}$, α_n is a creation function with range (at most) n. We consider the creation function

$$r(\eta) := \mathbf{E}\left[\alpha_N(\eta)\right], \tag{2.51}$$

where the expectation is taken relatively to the distribution of N. This function is not of finite range when N is unbounded.

There are extreme cases where the main result of Chapter 4 does not hold. For example, consider the process for which the creation rate function is

$$r(\eta) := \liminf_{y\to\infty} \frac{1}{x} \sum_{x=1}^{y} \eta(x). \tag{2.52}$$

In this case, every mixture of Bernoulli product measures is invariant for the process. Of course, this case is too extreme and it is not in the case of (2.51). In Chapter 5, we use the methods of [13] to investigate the construction of a counter example with a creation rate of the form (2.51) and where N has infinite mean. Unfortunately, we were not able to give a fully satisfactory answer to the question, and had to instead introduce two intermediate toy models for which a phase transition can be observed. In the first one, we look at a model with two particles types but with the same priority. We obtain that the limiting behavior of the process depends on the initial configuration if the distribution of N is sufficiently heavy tailed. In other words, the process does not forget the initial configuration, as the time goes to infinity, since an i.i.d. sequence of random variable distributed as N takes sufficiently often large values. The second toy-model is a mean-field version of the initial model, which turns out to be a modified Pòlya urn model and exhibits two kinds of behavior according to the value of a parameter playing the same role as the tail of N.

Finally, in the appendix, we complete the result obtained in Chapter 3, computing the exact value of $p(\lambda)$ the probability that a second-class particle *survives*, which means that it does never be replaced by a first-class particle at site 0. The developed method for this problem allows to compute this probability for a large class of initial distribution.

Chapter 3

Semi-infinite TASEP with a Complex Boundary Mechanism

We consider a totally asymmetric simple exclusion process on the positive half-line. When particles enter the system according to a Poisson source, Liggett has computed all the limit distributions when the initial distribution has an asymptotic density. In this paper, we consider systems for which particles enter according to a complex mechanism depending on the current configuration in a finite neighborhood of the origin. For this kind of models, we prove a strong law of large numbers for the number of particles which have entered the system at a given time. Our main tool is a new representation of the model as a multi-type particle system with infinitely many particle types.

3.1 Introduction

The simple exclusion process $\eta_. = (\eta_t)_{t\geqslant 0}$ on a countable space S, with random walk kernel $p(.)$, is a continuous time Markov process on $X := \{0,1\}^S$. For a configuration $\eta \in X$, we say that the site x is *occupied* (by a particle) if $\eta(x) = 1$, and is *empty* if $\eta(x) = 0$. A particle "tries" to move from an occupied site x to an empty site y at rate $p(x,y)$, or in an equivalent way, waits for an exponential time of parameter 1 and then chooses a site y randomly with probability $p(x,y)$ and "tries" to jump on y. If the site y is already occupied, the jump is cancelled and the particle stays at x, otherwise it jumps to y. In this way, there is always at most one particle at any given site. Formally, the exclusion process $\eta_.$ is defined as the Feller process with generator

$$\Omega f(\eta) := \sum_{x,y\in S} p(x,y)\eta(x)\left(1-\eta(y)\right)\left[f(\eta_{x,y}) - f(\eta)\right], \tag{3.1}$$

for all cylindrical functions f, where

$$\eta_{x,y}(z) := \begin{cases} \eta(y) & \text{if } z = x, \\ \eta(x) & \text{if } z = y, \\ \eta(z) & \text{otherwise.} \end{cases}$$

A natural question is to describe the set of invariant probability measures $\mathcal{I}$, which is the set of probability measures μ on S such that, if $\eta_0 \sim \mu$ then for all $t \geqslant 0$, $\eta_t \sim \mu$. These measures are characterized by the equations:

$$\int \Omega f \mu(d\eta) = 0,$$

for any cylindrical functions f (see e.g. [44] for a review). We denote by $\mathcal{I}_e$ the set of extreme points of $\mathcal{I}$. In the case $S = \mathbb{Z}$, the set of extremal, translation-invariant stationary measures is exactly the set of translation-invariant Bernoulli product measures on $\mathbb{Z}$ (see [42]).

In this paper, we consider the case $S := \mathbb{Z}_+$ and $p(x, x+1) := 1$, *i.e.*, the totally asymmetric nearest neighbor case. In $\mathbb{Z}_+$, one has to add some boundary mechanism to make the model non trivial. The simplest way to do this is to add a particle reservoir at site 0 with a certain density $\lambda > 0$. This means that a new particle is created at site 1 according to a Poisson process with rate λ when this site is empty. We call the model on $\mathbb{Z}_+$ TASEP(λ), and we denote by Ω_λ its generator and by $S_\lambda(t)$ its semi-group:

$$\begin{aligned} \Omega_\lambda f(\eta) &:= \lambda \left(1 - \eta(1)\right) \left[f(\eta_1) - f(\eta)\right] \\ &\quad + \sum_{x=1}^{\infty} \eta(x) \left(1 - \eta(x+1)\right) \left[f(\eta_{x,x+1}) - f(\eta)\right], \end{aligned} \tag{3.2}$$

for all cylindrical functions f, where

$$\eta_1(z) := \begin{cases} 1 - \eta(1) & \text{if } z = 1, \\ \eta(z) & \text{otherwise.} \end{cases}$$

In (3.2) we see two parts for the generator: one is due to the boundary mechanism and we will call it the *boundary part*; the other one, which has the form given by (3.1) for $S = \mathbb{Z}_+$, is due to the exclusion process and we will call it the *bulk part*.

Let us introduce some notation. In the following, we denote by ν^λ the product measure on $\mathbb{Z}_+$ with density λ and by θ the shift. θ acts on configurations $\eta \in X$ by

$$\theta\eta(x) := \eta(x+1), \forall x \in \mathbb{Z}_+,$$

on functions $f : X \longrightarrow \mathbb{R}$ by

$$\theta f(\eta) := f(\theta\eta), \forall \eta \in X,$$

and on measures μ on X by

$$\int f d\theta\mu := \int \theta f d\mu, \forall f \in L^1(\mu).$$

For a measure μ on S and $f \in L^1(\mu)$, we will denote $\langle f \rangle_\mu := \int f d\mu$.

We are interested in the asymptotic behavior of the distribution when t goes to infinity. For this model, we have a good understanding about what happens at equilibrium. Indeed, Liggett has shown in [41] the following ergodic theorem, which gives the limit measure for an initial measure with a product form and an asymptotic density:

Theorem 3.1 (Liggett [41]). *Let π be a product measure on $\mathbb{Z}_+$ for which $\rho := \lim_{x\to\infty} \langle \eta(x) \rangle_\pi$ exists.*

$$\text{If } \lambda \geqslant \frac{1}{2} \text{ then } \lim_{t\to\infty} \pi S_\lambda(t) = \begin{cases} \mu_\rho^\lambda, & \text{if } \rho \geqslant \frac{1}{2} \text{ (bulk dominated)}, \\ \mu_{\frac{1}{2}}^\lambda, & \text{if } \rho \leqslant \frac{1}{2} \text{ (maximum current)}. \end{cases}$$

$$\text{If } \lambda \leqslant \frac{1}{2} \text{ then } \lim_{t\to\infty} \pi S_\lambda(t) = \begin{cases} \mu_\rho^\lambda, & \text{if } \rho > 1-\lambda \text{ (bulk dominated)}, \\ \nu^\lambda, & \text{if } \rho \leqslant 1-\lambda \text{ (boundary dominated)}, \end{cases}$$

where the μ_ρ^λ's, for $\rho \geqslant \frac{1}{2}$, are stationary measures and asymptotically product with density ρ, i.e., $\lim_{x\to\infty} \theta^x \mu_\rho^\lambda = \nu^\rho$ *(in a weak sense with test functions $f \in C(X, \mathbb{R})$). We also have $\mu_\lambda^\lambda = \nu^\lambda$.*

To describe the set of invariant probability measures in the cases $S = \mathbb{Z}$ and $S = \mathbb{Z}_+$, Liggett uses that the Bernoulli product measures are invariant and for these measures one can make explicit computations. In this paper, we study Markov processes with no invariant product measure. We consider a TASEP on $\mathbb{Z}_+$ for which the boundary rate depends on the current configuration. We limit ourselves to finite range boundary mechanisms, *i.e.*, systems for which there exist some $R \in \mathbb{Z}_+$ such that the boundary part of the generator vanishes on every cylindrical function with support in $\{R+1, \ldots\}$. This idea was first introduced by Großkinsky in chapter 3 of his PhD Thesis [27] where he defines the following Feller process:

$$\begin{aligned} \Omega f(\eta) := \sum_{x \in \mathbb{Z}_+^*} \eta(x)\,(1-\eta(x+1))\,[f(\eta_{x,x+1}) - f(\eta)] \\ + \sum_{\xi \in X_R} d_{\eta_{|S_R},\xi} \left[f(\xi \cup \eta_{|^c S_R}) - f(\eta) \right], \end{aligned} \tag{3.3}$$

for all cylindrical functions f where $S_R := \{1, \ldots, R\}$, $X_R := \{0,1\}^{S_R}$, $\eta_{|S_R}$ and $\eta_{|^c S_R}$ are the configuration η restricted to S_R and ${}^c S_R = \mathbb{Z}_+^* \backslash S_R$ respectively, $\xi \cup \eta_{|^c S_R}$ is the natural concatenation of configurations on S_R and on ${}^c S_R$, and $(d_{\xi,\xi'})_{\xi,\xi' \in X_R}$ are non-negative rates.

Assuming the existence of an invariant measure which is product outside of the box $\{1, \ldots, R\}$ with a non-trivial density leads to relations which the boundary rates have to satisfy – we will refer to such models as *almost classic*. These are still within the reach of Theorem 3.1, at least for suitable choices of λ and ρ. From now on, we will assume that at least one of these relations is not satisfied by our boundary mechanism.

Remark. The reason for which we only treat the finite range case is that when we are not in this case, pathological things can occur. For example, consider the following dynamic with a non-local boundary mechanism. Define the asymptotic density of a configuration $\eta \in X$ by

$$\rho(\eta) := \liminf_{x\to\infty} \frac{1}{x} \sum_{i=1}^{x} \eta(i);$$

we consider now a TASEP on $\mathbb{Z}_+$ for which the rate of apparition of a particle in site 1 is $\rho(\eta)$ where η is the current configuration. More formally, the boundary part of the generator is

$$\rho(\eta)(1-\eta(1))\left[f(\eta_1) - f(\eta)\right].$$

In this example, every mixture of Bernoulli product measures is invariant for the process. Admittedly this case is too extreme; probably, suitable decay of dependency would create a behavior similar to the finite dependency case.

For this generalized boundary mechanism, we will not have an exact solution as for the TASEP(λ). Indeed, one can check that this process is not almost classic and then does not have any invariant measure which is of product form. Our approach is to study the number of particles which have entered the system by time t. We will see that it grows linearly with an almost sure speed equal to the stationary current $j_\infty := \mu_\infty\{\eta \in X : \eta(1) = 1, \eta(2) = 0\}$ for an invariant measure μ_∞. Define ρ_∞ as the root of $\rho(1-\rho) = j_\infty$ in $[0, 1/2[$. We believe that the process has a stationary measure which is asymptotically product with density ρ_∞; but we are still unable to prove it.

The rest of the paper is organized as follows: in Section 3.2 we give a construction of the process defined above using a graphical representation similar to that introduced by Harris [29]. We also introduce the basic coupling technique which is the main tool used in the paper; in Section 3.3 we give some general results on the asymptotic behavior of the TASEP with complex boundary mechanism. In particular, we show that, starting from the empty configuration, the process converges in distribution to an invariant ergodic measure μ_∞; finally, in Section 3.4 we study a particular example: take a TASEP(λ) on $\mathbb{Z}_+$ and add a source (independent of everything) with density $\epsilon > 0$ which is activated only when site 2 is occupied. For this model, let N_t be the number of particles which have entered the system between 0 and t. Then the main result of this paper is the following strong law of large numbers:

Theorem 3.2. *Let $0 \leqslant \lambda < \frac{1}{2}$, $\epsilon > 0$. Then starting from μ_∞,*

$$\lim_{t\to\infty} \frac{N_t}{t} = \lambda(1-\lambda) + \lambda(1-\lambda)p(\lambda)\epsilon + o(\epsilon),$$

with probability one, where $p(\lambda)$ is a positive constant (depending only on λ) for which we give a natural probabilistic interpretation.

It should be noted that this particular choice of boundary mechanism is rather arbitrary, and that our method is robust enough to be used in a much larger generality. However, the notations which would be needed would be much more tedious, while providing very little additional insight into the model — so we choose to limit ourselves to one representative case.

Acknowledgments

I wish to thank Vincent Beffara for his help and for suggesting this work, and Stefan Großkinsky for very useful discussions. I thank the reviewers for their numerous comments and suggestions, which greatly improved the final version of the paper.

3.2 The Harris construction

We will use the method developed by Harris [29] to construct our process. Let

$$\mathcal{N} := \left(\mathcal{N}_x, \mathcal{N}_{\eta,\eta'}; x \in \mathbb{Z}_+, \eta, \eta' \in \{0,1\}^{\{1,\dots,R\}}\right),$$

be a family of independent Poisson point processes on $\mathbb{R}_+^*$ constructed on the same probability space $(\Gamma, \mathcal{F}, \mathbf{P})$, such that the rate of the processes indexed by $\mathbb{Z}_+$ is 1 and the rate of the process indexed by (η, η') is $d_{\eta,\eta'} \geqslant 0$. By discarding a $\mathbf{P}$–null set, we may assume that

each poisson point process in $\mathcal{N}$ has only finitely many jump times in every bounded interval $[0, T]$, and no two distinct processes have a jump in common. (3.4)

We denote

$$\mathcal{N}_0 := \bigcup_{\eta,\eta' \in \{0,1\}^{\{1,\dots,R\}}} \mathcal{N}_{\eta,\eta'}.$$

Fix $T > 0$ and $\eta \in X$. The process $(\eta_t)_{0 \leqslant t \leqslant T}$ starting from η is now constructed as follows. Consider the following subgraph of $\mathbb{Z}_+$:

$$\begin{aligned}\mathcal{G}_T := &\Big\{\{x, x+1\} : x \geqslant R, \mathcal{N}_x \cap [0,T] \neq \varnothing\Big\} \\ &\bigcup \Big\{\{x, x+1\} : x \in \{0, \dots, R-1\}\Big\}.\end{aligned}$$

It is easy to see that every connected component of $\mathcal{G}_T$ is almost surely finite. Let Γ_0 be the subset of Γ such that (3.4) and the above condition hold for all $T \geqslant 0$. Then we have $\Gamma_0 \in \mathcal{F}$ and $\mathbf{P}[\Gamma_0] = 1$. We consider now only $\omega \in \Gamma_0$. For every connected component $\mathcal{C}$ of $\mathcal{G}_T$, the set $(\cup_{x \in \mathcal{C}} \mathcal{N}_x) \bigcap [0, T]$ is finite so its elements can be ordered chronologically $\tau_1 < \ldots < \tau_n$ and we need only to describe the action of each of them. We start with the configuration η:

$$\eta_t(x) := \eta(x)$$

for all $x \in \mathcal{C}$ and $0 \leqslant t < \tau_1$.
Suppose that the process is constructed on $\mathcal{C}$ for $0 \leqslant t < \tau_k$ and $k \in \{1, \dots, n\}$. Then:

- if $\tau_k \in \mathcal{N}_{\xi,\xi'}$ and if $\eta_{\tau_k^-|S_R} = \xi$ then $\eta_{\tau_k|S_R} := \xi'$ and $\eta_{\tau_k}(x) := \eta_{\tau_k^-}(x)$ for all $x \in \mathcal{C} \backslash S_R$,
- if $\tau_k \in \mathcal{N}_{\xi,\xi'}$ and if $\eta_{\tau_k^-|S_R} \neq \xi$ then $\eta_{\tau_k}(x) := \eta_{\tau_k^-}(x)$ for all $x \in \mathcal{C}$,
- if $\tau_k \in \mathcal{N}_x$ and $\eta_{\tau_k^-}(x)(1 - \eta_{\tau_k^-}(x+1)) = 1$ then $\eta_{\tau_k} := (\eta_{\tau_k^-})_{x,x+1}$ on $\mathcal{C}$,
- if $\tau_k \in \mathcal{N}_x$ and $\eta_{\tau_k^-}(x)(1 - \eta_{\tau_k^-}(x+1)) \neq 1$ then $\eta_{\tau_k} := \eta_{\tau_k^-}$ on $\mathcal{C}$,

Finally, we put $\eta_t := \eta_{\tau_k}$ on $\mathcal{C}$ for $\tau_k \leqslant t < \tau_{k+1}$ if $k < n$ and for $\tau_n \leqslant t \leqslant T$ if $k = n$. We make the same construction on every connected component of $\mathcal{G}_T$ and then let T go to infinity to get the process $(\eta_t)_{t \geqslant 0}$ for every $\omega \in \Gamma_0$.

The usefulness of such a construction is that, using the same Harris process, we can construct two or more realizations of the process on the same probability space starting from different initial configurations. We will refer to this coupling as the *basic coupling.*

3.3 The attractive case

Recall the usual definition of *attractiveness* (or *monotonicity*). Define a partial order on X as follows:

$$\eta \leqslant \xi \quad \text{iff} \quad \forall x \in \mathbb{Z}_+^*,\ \eta(x) \leqslant \xi(x).$$

A function f on X is called *increasing* if $\eta \leqslant \xi$ implies $f(\eta) \leqslant f(\xi)$. This leads to the usual definition of the stochastic monotonicity: $\mu_1 \prec \mu_2$ iff $\langle f \rangle_{\mu_1} \leqslant \langle f \rangle_{\mu_2}$ for every increasing function f. We say that a process on X is *attractive* (or *monotone*) if one of the following equivalent statements hold:

$$\text{for every increasing function } f,\ S(t)f \text{ is also increasing for all } t \geqslant 0,$$

and

$$\mu_1 \prec \mu_2 \text{ implies } \mu_1 S(t) \prec \mu_2 S(t) \text{ for all } t \geqslant 0.$$

In this section, we consider the process with generator (3.3) and we assume the process attractive.

3.3.1 The stationary measure

Proposition 3.1. *Assume that the process is attractive (or monotone). We start from the empty configuration and we denote by μ_t the distribution of the process at time t. Then, the process $(\mu_t)_{t \geqslant 0}$ is stochastically increasing and converges to a measure $\mu_\infty \in \mathcal{I}$, which is the smallest invariant measure of the dynamic. Furthermore, $\mu_\infty \in \mathcal{I}_e$ and μ_∞ is ergodic.*

Proof. Let $0 \leqslant s < t$. We have $\delta_0 \prec \mu_{t-s}$, where δ_0 is the measure charging the empty configuration. Thus by monotonicity of the process, we have $\delta_0 S(s) \prec \mu_{t-s} S(s)$, *i.e.*, $\mu_s \prec \mu_t$. Hence, by monotonicity, μ_t converges weakly to an invariant measure μ_∞.

For all $\nu \in \mathcal{I}$, we have $\delta_0 \prec \nu$, which implies that $\mu_t \prec \nu$ for all $t \geqslant 0$, and then $\mu_\infty \prec \nu$. Assume now that $\mu_\infty = \lambda \nu_1 + (1-\lambda)\,\nu_2$, with $\nu_1, \nu_2 \in \mathcal{I}$ and $\lambda \in]0,1[$. We have $\mu_\infty = \lambda\nu_1 + (1-\lambda)\,\nu_2 \succ \mu_\infty$, thus $\nu_1 = \nu_2 = \mu_\infty$ and μ_∞ is extremal. Finally, by Theorem B52 of [45], μ_∞ is also ergodic. ∎

Proposition 3.2. *$\theta^R \mu_\infty$ is stochastically dominated by the measure $\mu^1_{1/2}$ of Theorem 3.1.*

Proof. Define $\mathcal{N}' := (\mathcal{N}'_x, x \in \mathbb{Z}_+)$, where $\mathcal{N}'_x := \mathcal{N}_{x+R}$. Then $\mathcal{N}'$ defines a TASEP (ξ_t) on $\mathbb{Z}_+$ with rate 1 of particle apparition in 1. By Theorem 3.1, starting from the empty configuration, the distribution at time t converges to $\mu^1_{1/2}$. In this coupling, we have $\xi_t(x) \geqslant \eta_t(x+R)$ almost surely for all $t \geqslant 0$ and $x \geqslant 1$. Thus the restriction of μ_∞ to $\{R+1, R+2, \ldots\}$ is stochastically dominated by $\mu^1_{1/2}$. ∎

3.3.2 Asymptotic measures

Let us extend the measure μ_∞ to a measure on $\{0,1\}^{\mathbb{Z}}$ by

$$\overline{\mu}_\infty(A) := \mu_\infty \{\eta \in X : \tilde{\eta} \in A\},$$

where

$$\tilde{\eta}(x) := \begin{cases} \eta(x) \text{ if } x \geqslant 1, \\ 0 \text{ otherwise,} \end{cases}$$

for all A in the product σ-field of $\{0,1\}^{\mathbb{Z}}$. By a slight abuse of notation, we still denote this measure by μ_∞. Let $\mu^k := \theta^k \mu_\infty$ and consider any weak limit μ^∞ of this sequence; let $k_i \uparrow \infty$ such that:

$$\lim_{i \to \infty} \mu^{k_i} = \mu^\infty.$$

Proposition 3.3. *The measure μ^∞ is a translation invariant stationary measure for TASEP on $\mathbb{Z}$. Consequently, it is a mixture of Bernoulli product measures,* i.e., *there exists a probability measure σ on $[0,1]$ such that*

$$\mu^\infty = \int_0^1 \nu^\lambda \sigma(d\lambda).$$

Proof. Let Ω^e be the generator of the TASEP on $\mathbb{Z}$. For any cylindrical function $f : \{0,1\}^{\mathbb{Z}} \to \mathbb{R}$, let $x \in \mathbb{Z}_+$ large enough such that *supp* $\theta^x f \subset \{R+1, R+2, \ldots\}$, where *suppf* is the support of f. Thus $\theta^x f$ could be considered has a function on

$\mathbb{Z}_+$ and we can apply the generator Ω to this function. We get $\Omega\theta^y f = \Omega^e\theta^y f$ for all $y \geqslant x$. But it is easy to see that Ω^e and θ commute, thus we have

$$\begin{aligned}\int \Omega\theta^y f \mu_\infty(d\eta) = 0 &= \int \theta^y \Omega^e f \mu_\infty(d\eta), \\ &= \int \Omega^e f \mu^y(d\eta).\end{aligned}$$

Hence for i large enough, $\langle \Omega^e f\rangle_{\mu^i} = 0$, which implies that $\langle \Omega^e f\rangle_{\mu^\infty} = 0$. This is true for arbitrary f thus μ^∞ is invariant for the TASEP on $\mathbb{Z}$. We know that for this model we have $\mathcal{I}_e = \left\{\nu^\lambda, \lambda \in [0,1]\right\} \cup \{\nu_n, n \in \mathbb{Z}\}$, where $\nu_n = \theta^n \nu_0$ and ν_0 is the Dirac measure of the configuration for which all the sites $x \geqslant 0$ are occupied and all the sites $x < 0$ are empty (see [42]). Using Proposition 3.2, since $\mu^1_{1/2}$ is asymptotically product with density $\frac{1}{2}$, μ^∞ is stochastically dominated by $\nu^{1/2}$. Thus μ^∞ is translation invariant and is a mixture of Bernoulli product measures. ■

3.3.3 A strong law of large numbers

Let μ be an invariant and ergodic measure for the process with generator given by (3.3). Fix ξ_0, ξ and ξ' three configurations on S_R and consider

$$N(t) := \sharp\left(\mathcal{N}_{\xi,\xi'} \cap I_t\right),$$

with $I_t := \overline{\left\{s \in [0,t] : \eta_{s|S_R} = \xi_0\right\}}$, where we denote by $\overline{A}$ the closure of a set $A \subset \mathbb{R}_+$. We show a strong law of large numbers for $N(t)$ which will be useful in the sequel.

Proposition 3.4. *If η_0 is distributed according to μ and if $\xi' \neq \xi_0$, then almost surely:*

$$\lim_{t\to\infty} \frac{N(t)}{t} = d_{\xi,\xi'}\mu\left\{\eta \in X : \eta_{|S_R} = \xi_0\right\}.$$

Proof. Let

$$T_t := \int_0^t \mathbf{1}_{\eta_{s|S_R}=\xi_0} ds,$$

and

$$\psi(t) := \inf\{s \geqslant 0 : T_s = t\}.$$

Since μ is ergodic, $T_t/t \underset{t\to\infty}{\longrightarrow} \mu\left\{\eta \in X : \eta_{|S_R} = \xi_0\right\}$ almost surely. Let

$$I := \overline{\{t > 0 : \eta_{t|S_R} = \xi_0\}}.$$

$\psi : \mathbb{R}_+^* \to I$ is a one to one map, since it is increasing, thus we can define $\mathcal{M} := \psi^{-1}(\mathcal{N}_{\xi,\xi'} \cap I)$ and $N'(t) := \sharp\left(\mathcal{M} \cap]0,t]\right)$ the associated counting process. We have $N'(t) = N(\psi(t))$ almost surely.

Claim: $\mathcal{M}$ is a Poisson point process with parameter $d_{\xi,\xi'}$.

Let $\tilde{\tau}_0 := 0$ and for $i \geqslant 1$:

$$\tau_i := \inf\{t > \tilde{\tau}_{i-1} : \eta_{t|S_R} = \xi_0\},\ \tilde{\tau}_i := \inf\{t > \tau_i : \eta_{t|S_R} \neq \xi_0\} \text{ and } J_i = [\tau_i, \tilde{\tau}_i]\,.$$

$(\tau_i)_{i\geqslant 1}$ and $(\tilde{\tau}_i)_{i\geqslant 1}$ are stopping times for the process $(\mathcal{N}\cap[0,t])$. To prove the claim we need to distinguish two cases.

Case $\xi \neq \xi_0$**:** (see Figure 3.1) In this case, the points of $\mathcal{N}_{\xi,\xi'} \cap I$ have no effect on the configuration. Hence for each $i \geqslant 1$, with the strong Markov property, τ_i and the length of J_i are independent of $\mathcal{N}_{\xi,\xi'} \cap [\tau_i, \infty[$. Consequently, conditionally to J_i, $\mathcal{N}_{\xi,\xi'} \cap J_i$ is a Poisson point process with parameter $d_{\xi,\xi'}$. Again with the strong Markov property, $(\mathcal{N}_{\xi,\xi'} \cap J_i)_{i\geqslant 1}$ are independent conditionally to I. Hence, the claim follows.

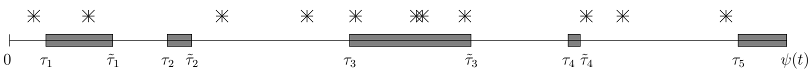

Figure 3.1: *On the time interval* $[0, \psi(t)]$ *we see the set* $I_{\psi(t)}$ *in grey. The total length of the grey part is* t*. The stars are points of the process* $\mathcal{N}_{\xi,\xi'}$*. In this example,* $N'(t) = 5$*.*

Case $\xi = \xi_0$**:** (see Figure 3.2) In this case, each $\mathcal{M}_i := \mathcal{N}_{\xi,\xi'} \cap J_i$ has, almost surely, at most 1 point, thus we have to argue in a different way. For $i \geqslant 1$, let

$$\sigma_i := \inf \mathcal{N}_{\xi,\xi'} \cap [\tau_i, \infty[$$

and

$$\sigma_i' := \inf \bigcup_{\xi'' \in X_R \setminus \{\xi_0, \xi'\}} (\mathcal{N}_{\xi_0,\xi''} \cap [\tau_i, \infty[) \bigcup_{\substack{x\in\{1,\dots,R\}: \\ \eta_{\tau_i}(x)=1, \eta_{\tau_i}(x+1)=0}} (\mathcal{N}_x \cap [\tau_i, \infty[)\,.$$

The interpretation of σ_i and σ_i' is simple: if $\sigma_i < \sigma_i'$, then the time interval J_i ends with a jump in $\mathcal{N}_{\xi,\xi'}$ and $\mathcal{M}_i$ contains one point ($\mathcal{M}_i = \{\tilde{\tau}_i\}$); if $\sigma_i > \sigma_i'$, then the time interval J_i ends with an other jump and $\mathcal{M}_i$ is empty. By the strong Markov property, the sequence $(\sigma_i)_{i\geqslant 1}$ is i.i.d. with distribution exponential with parameter $d_{\xi,\xi'}$. Furthermore, because of the independence of the Poisson point processes in $\mathcal{N}$, $(\sigma_i)_{i\geqslant 1}$ and $(\sigma_i')_{i\geqslant 1}$ are independent. By construction, $\inf \mathcal{M} = \sum_{i\leqslant I} \min(\sigma_i, \sigma_i')$, where $I := \min\{i \geqslant 1 : \sigma_i < \sigma_i'\}$. Hence, using basic properties of Poisson processes, it is easy to see that $\inf \mathcal{M}$ is an exponential random variable with parameter $d_{\xi,\xi'}$. Finally, using again the strong Markov property, the claim follows.

Then, for every $\epsilon > 0$, $\psi(T_t) \leqslant t \leqslant \psi(T_t + \epsilon)$. Since $N(t)$ is non-decreasing, we get $N'(T_t) \leqslant N(t) \leqslant N'(T_t + \epsilon)$. Consequently:

$$\frac{N'(T_t)}{T_t}\frac{T_t}{t} \leqslant \frac{N(t)}{t} \leqslant \frac{N'(T_t+\epsilon)}{T_t+\epsilon}\frac{T_t+\epsilon}{t}.$$

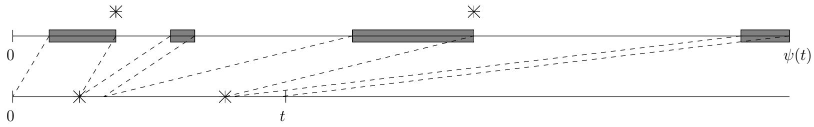

Figure 3.2: *In the first line, we see the time interval* $[0, \psi(t)]$*: the set* $I_{\psi(t)}$ *is in grey; the stars are points of the process* $\mathcal{N}_{\xi,\xi'}$*. They are always at the end of intervals of* $I_{\psi(t)}$ *since they change the current configuration. In the second line, we see the Poisson point process* $\mathcal{N}_{\xi,\xi'}$ *viewed from* $I_{\psi(t)}$*,* i.e.*, the set* $\mathcal{M} \cap]0, t]$*.*

Since both sides converge to $d_{\xi,\xi'}\mu \left\{\eta \in X : \eta_{|S_R} = \xi_0\right\}$ almost surely, it leads to the conclusion. ■

3.4 A particular case and the Multi-Species model

In this section, we are interested in a particular case of TASEP with a complex boundary mechanism: let $\lambda, \epsilon > 0$ such that $\lambda + \epsilon < \frac{1}{2}$. Particles are created at site 1 with rate $\lambda + \epsilon\eta(2)$, where η is the current configuration and the bulk dynamic is the one of the TASEP. This model has a generator given by:

$$\begin{aligned} \Omega f(\eta) := \sum_{x\in\mathbb{Z}_+} &\eta(x)\,(1-\eta(x+1))\,[f(\eta_{x,x+1}) - f(\eta)] \\ &\times (1-\eta(1))\,(\lambda + \epsilon\eta(2))\,[f(\eta_1) - f(\eta)]\,, \end{aligned} \tag{3.5}$$

for all cylindrical functions f on X. As it is explained in the introduction, the choice of the model is rather arbitrary, and the methods that we use are quite robust (at least as long as the system can be dominated by a Bernoulli product measure of intensity lower than 1/2 — which is indeed the case here).

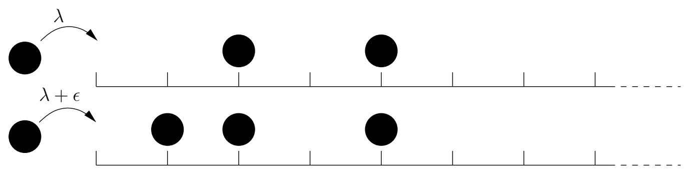

Figure 3.3: *Particles enter with additional rate* ϵ *when the site* 2 *is occupied.*

In this model, the range of the boundary mechanism is $R = 2$. The hypothesis $\epsilon > 0$ implies that the process is monotone, thus we can define the smallest stationary measure $\mu_\infty = \mu_\infty(\lambda, \epsilon)$ of the model. Using the Harris representation, we can couple this process with $\eta^\lambda_\cdot$, a TASEP(λ), and $\eta^{\lambda+\epsilon}_\cdot$, a TASEP($\lambda+\epsilon$), in such a way that if $\eta^\lambda_0 \leqslant \eta_0 \leqslant \eta^{\lambda+\epsilon}_0$ then for all $t \geqslant 0$, $\eta^\lambda_t \leqslant \eta_t \leqslant \eta^{\lambda+\epsilon}_t$. This proves that $\nu^\lambda \prec \mu_\infty \prec \nu^{\lambda+\epsilon}$ and then $\nu^\lambda \prec \tilde{\mu}_\infty \prec \nu^{\lambda+\epsilon}$.

3.4.1 Some estimates about the particle flux

Here we will see another way to see the process with generator given by (3.5). For any $i \geqslant 1$, let

$$\mathcal{X}_i := \{\infty, 1, \dots, i\}^{\mathbb{Z}_+}.$$

We define

$$\begin{aligned}\Omega^{(i)} f(\eta) :=& \lambda \mathbf{1}_{\eta(1)\geqslant 2} \left[f\left(\eta_{1\to 1}\right) - f(\eta)\right] \\ &+ \sum_{j=2}^{i} \epsilon \mathbf{1}_{\eta(1)\geqslant j+1} \mathbf{1}_{\eta(2)=j-1} \left[f\left(\eta_{j\to 1}\right) - f(\eta)\right] \\ &+ \epsilon \mathbf{1}_{\eta(1)=\infty} \mathbf{1}_{\eta(2)=i} \left[f\left(\eta_{i\to 1}\right) - f(\eta)\right] \\ &+ \sum_{x=1}^{\infty} \mathbf{1}_{\eta(x+1)>\eta(x)} \left[f\left(\eta_{x,x+1}\right) - f(\eta)\right],\end{aligned} \tag{3.6}$$

for all cylindrical function $f : \mathcal{X}_i \to \mathbb{R}$, where

$$\eta_{j\to 1}(x) := \begin{cases} j \text{ if } x = 1, \\ \eta(x) \text{ otherwise,}\end{cases}$$

for $j \in \{1, \dots, i\}$.

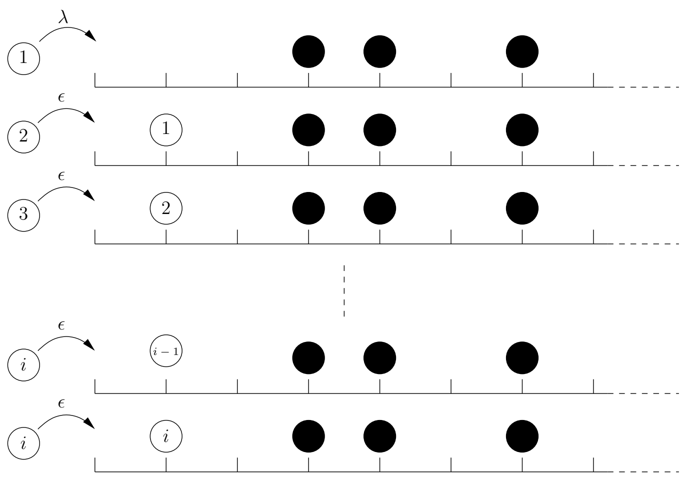

Figure 3.4: *First class particles enter with rate λ whatever is the configuration in $\{2, 3, \dots\}$ and second class particles enter with rate ϵ if the site 2 is occupied by a first class particle. Particles in black are indistinguishable particles (their class has no influence on the rate of the source in the current configuration).*

We fix $i \geqslant 2$ for the sequel. The new description is described in Figure 3.4 and in the following. We put the particles into a certain number of classes. For a configuration $\eta \in \mathcal{X}_i$ and for a site $x \in \mathbb{Z}_+$, the number $\eta(x)$ designates the class of the particle at site x if it exists, *i.e.*, if $\eta(x) \neq \infty$, and is equal to ∞ if the site is empty. We use here another notation for empty sites because it allows us to have a simpler expression for the generator and we can also interpret holes as particles of class infinity. The evolution is the same as before, except that if a particle of the k-th class (or of type k) attempts to jump on a site occupied by a particle of the j-th class (or of type j), then it is not allowed to do so if $k \geqslant j$, and the particles exchange positions if $k < j$. We say that a particle of class $k \in \{1, 2, \dots\}$ has priority over all particles of classes greater than k. In this way, a particle of type k behaves as a hole for particles of type $j < k$.

Now we will explain how we affect classes to the particles. First class particles enter the system (at site 1) at rate λ. As they have priority over other particles, they are not affected by them, so the process of first class particles is simply a TASEP(λ) on $\mathbb{Z}_+$. Next, particles of class $2 \leqslant j \leqslant i-1$ enter the system with rate ϵ, if the site 2 is occupied by a particle of class $j-1$ and with rate 0 otherwise. Finally, particles of class i enter the system with rate ϵ if the site 2 is occupied by a particle of class $i-1$ or i and with rate 0 otherwise. For each configuration of the system, at most 2 types of particles are allowed to enter the system. We can also remark that if we consider the process consisting with particles of class $1, \dots, i$, then it has the generator given by (3.5).

In terms of the Harris system, we define $\mathcal{N}$ the collection of the following independent Poisson point processes on $\mathbb{R}_+^*$: let $(\mathcal{N}_x, x \geqslant 1)$ be Poisson point processes of rate 1; let $(\mathcal{N}_j^b, j \geqslant 1)$ be Poisson point processes of rate λ for $\mathcal{N}_1^b$ and of rate ϵ for the others. In the sequel, we consider holes as particles of class infinity. The mechanism is then the following: if $t \geqslant 0$ is a jump time of $\mathcal{N}_x$ and if at time t^- we have $\eta(x+1) > \eta(x)$ (*i.e.*, the particle at x has higher priority than the one at $x+1$), then the particles at x and $x+1$ swap; if $t \geqslant 0$ is a jump time of $\mathcal{N}_1^b$ and if at time t^- we have $\eta(1) \geqslant 2$, then a first class particle appears at site 1; if $t \geqslant 0$ is a jump time of $\mathcal{N}_j^b$ with $2 \leqslant j \leqslant i-1$ and if at time t^- we have $\eta(1) \geqslant j+1$ and $\eta(2) = j-1$, then a j-particle appears at site 1; finally, if $t \geqslant 0$ is a jump time of $\mathcal{N}_i^b$ and if at time t^- we have $\eta(1) = \infty$ and $\eta(2) \in \{i-1, i\}$, then an i-particle appears at site 1.

We denote by $S^{(i)}(t)$ the semi-group corresponding to the generator $\Omega^{(i)}$ and by $(\eta_t^{(j)})_{t \geqslant 0}$ the process of the j-th class particles for $j = 1, \dots, i$, *i.e.*, $\eta_t^{(j)}(x) := \mathbf{1}_{\eta_t(x)=j}$. The process is attractive, thus we can define $\mu_\infty^{(i)}$ as the weak limit of $\delta_0 S^{(i)}(t)$. As in Proposition 3.1, this measure is extremal, ergodic and the smallest invariant measure of the system. For all $1 \leqslant j \leqslant i$, we denote $\bar{\eta}_t^{(j)} := \sum_{k=1}^j \eta_t^{(k)}$. Remark that the process $(\bar{\eta}_t^{(i)})_{t \geqslant 0}$ is exactly the process that we want to study, *i.e.*, it has the generator given by (3.5). Furthermore, for all $j \geqslant 1$ the distribution of the process $(\bar{\eta}_t^{(j)})_{t \geqslant 0}$ is the same for all $i \geqslant j+1$, *i.e.*, the generator of this process is independent of i since changing the value of i is equivalent to adding or removing some particles with lower priority.

In order to compare the processes $(\bar{\eta}_t^{(i-1)})_{t \geqslant 0}$ and $(\bar{\eta}_t^{(i)})_{t \geqslant 0}$, we need to control the number of particles of a given type in the system at a given time. Let $N_t^{(j)}$ be the number of j-particles which have entered the system between times 0 and t, and define

$$T_t^{(j)} := \int_0^t \eta_s^{(j)}(2)ds$$

and

$$\tilde{T}_t := \int_0^t \eta_s^{(1)}(2)\left(1-\eta_s^{(1)}(1)\right)ds,$$

for all $j \in \{1, \ldots, i\}$.

$T_t^{(j)}$ is the time spent by j-particles in site 2 during $[0,t]$, and $\tilde{T}_t$ is the length of the subset of $[0,t]$ for which 2-particles can enter site 1 with rate ϵ (excepted if the site 1 is already occupied by another 2-particle). The following lemma says that we have a uniform control on the total time spent by a particular particle of type $\geqslant 2$ at site 2. Let $T_\infty^{(j),k}$ be the total time spent in site 2 by the k-th particle of type $j \geqslant 2$ which have entered the system.

Lemma 3.1. *There exists a constant $C_\lambda \in]0,+\infty[$, independent of ϵ, such that for all $k \geqslant 1$ and all $j \geqslant 2$ we have*

$$\mathbf{E}\left[T_\infty^{(j),k}\right] \leqslant C_\lambda.$$

Proof. Let E_t be the event that, between times t and $t+1$, a first class particle enters (or tries to enter) the system, then jumps, if it is possible, to site 2, and finally another first class particle tries to enter the system. We also assume that in E_t there is no other jump time for $\mathcal{N}_1$, $\mathcal{N}_2$ and $\mathcal{N}_1^b$ between 0 and t. In particular, if E_t occurs and if there was a particle of type greater or equal to 2 in site 2 at time t, then it has disappeared at time $t+1$. $q(\lambda) := \mathbf{P}\left[E_t\right]$ does not depend on t neither on ϵ and $q(\lambda) > 0$.

On the event $\left\{T_\infty^{(j),k} > t\right\}$, there exists a time τ such that the k-th particle of type j is at the site 2 and it has spent exactly time t in this site between 0 and τ. We have $E_\tau \subset \left\{T_\infty^{(j),k} \leqslant t+1\right\}$. Hence

$$\mathbf{P}\left[E_\tau | T_\infty^{(j),k} > t\right] \leqslant \mathbf{P}\left[T_\infty^{(j),k} \leqslant t+1 | T_\infty^{(j),k} > t\right]. \tag{3.7}$$

But τ is a stopping time for the Markov process $(\eta_t^{(l)}, l = 1, \ldots, j)_{t \geqslant 0}$ and the event E_τ depends only on the poisson processes of the Harris system for times between τ and $\tau+1$, so, conditionally to $\{\tau < \infty\}$, E_τ has the same law as E_0 by the strong Markov property. Hence the left-hand side of (3.7) is equal to $q(\lambda)$. Finally, we have

$$\mathbf{P}\left[T_\infty^{(j),k} > t+1\right] \leqslant (1-q(\lambda))\,\mathbf{P}\left[T_\infty^{(j),k} > t\right].$$

The last inequality implies that there exist some deterministic positive constants a_1, a_2, depending only on λ, such that almost surely and for all $t \geqslant 0$ we have

$$\mathbf{P}\left[T_\infty^{(j),k} > t\right] \leqslant a_1 e^{-a_2 t}.$$

The result follows with $C_\lambda := \int_0^\infty a_1 e^{-a_2 t} dt$. ∎

Finally, the following theorem gives the estimates that we need:

Theorem 3.3. *For each $1 \leqslant j \leqslant i$ and $k \geqslant i$, $T_t^{(j)}/t$ converges almost surely to a deterministic value if the process starts under $\mu_\infty^{(k)}$. Furthermore, for all $\epsilon < \frac{1}{2C_\lambda}$, where C_λ is as in Lemma 3.1, we have*

$$\limsup_{t\to\infty} \frac{N_t^{(j)}}{t} \leqslant c_{j-1}\epsilon^{j-1}, \quad \lim_{t\to\infty} \frac{T_t^{(j)}}{t} \leqslant c_j\epsilon^{j-1},$$

for $1 \leqslant j \leqslant i-1$, and

$$\limsup_{t\to\infty} \frac{N_t^{(i)}}{t} \leqslant 2c_{i-1}\epsilon^{i-1}, \quad \lim_{t\to\infty} \frac{T_t^{(i)}}{t} \leqslant 2c_i\epsilon^{i-1},$$

where $(c_j)_{j=1,\dots,i}$ are constants (depending only on λ) such that $c_0 := \lambda(1-\lambda)$ and $c_j := C_\lambda^{j-1}c_0$.

Proof. We have seen that every $\mu_\infty^{(k)}$ is stationary and ergodic, so by the ergodic theorem, we have almost surely

$$\frac{T_t^{(j)}}{t} \underset{t\to\infty}{\longrightarrow} \mu_\infty^{(k)}\{\eta \in \mathcal{X}_k : \eta(2) = j\} \tag{3.8}$$

and

$$\frac{\tilde{T}_t}{t} \underset{t\to\infty}{\longrightarrow} \mu_\infty^{(k)}\{\eta \in \mathcal{X}_k : \eta(1) \geqslant 2, \eta(2) = 1\}. \tag{3.9}$$

Since the distribution of the first class particles is ν^λ under every $\mu_\infty^{(k)}$, the right-hand side of (3.8) is λ if $j = 1$ and the right-hand side of (3.9) is $\lambda(1-\lambda)$. Using Proposition 3.4, $N_t^{(1)}/t$ converges to $\lambda(1-\lambda)$ almost surely.

Let

$$M_t^{(2)} := \sharp\left\{s \in \mathcal{N}_2^b \cap [0,t] : \eta_s^{(1)}(2)\left(1-\eta_s^{(1)}(1)\right) = 1\right\}.$$

Then almost surely $N_t^{(2)} \leqslant M_t^{(2)}$ and applying Proposition 3.4:

$$\limsup_{t\to\infty} \frac{N_t^{(2)}}{t} \leqslant \epsilon\lambda(1-\lambda) = \lim_{t\to\infty} \frac{M_t^{(2)}}{t}. \tag{3.10}$$

Now, we need to find an upper bound for $\lim_{t\to\infty} T_t^{(2)}/t$. First, we can remark that $T_t^{(2)}$ can be decomposed into two parts: the time spent by initial second class particles, *i.e.*, particles present at time 0, denoted by $T_{t,1}^{(2)}$, plus the time spent by the new second class particles in site 2, denoted by $T_{t,2}^{(2)}$. But, since $T_{t,1}^{(2)}$ is bounded by a random variable that is almost surely finite, it is sufficient to study $\lim_{t\to\infty} T_{t,2}^{(2)}/t$. Indeed, using $\lambda + \epsilon < 1/2$, it can be shown that every initial second class particle has a probability uniformly bounded from below by a positive constant, to never go behind its starting point (see [50]). Thus the number of initial second class particles visiting the site 2 is finite and each of them spent a finite time in this site as a consequence of Lemma 3.1.

As we have seen previously, $\mu_\infty^{(k)} \prec \nu^{\lambda+\epsilon}$. The idea is that since we know the number of second class particles created up to time t, it is sufficient to bound the time spent in site 2 by one of them in the environment $\nu^{\lambda+\epsilon}$ where it is slower. But there are some difficulties. For example, at the moment where a second class particle is created, the environment in $\{2,3,\dots\}$ is not dominated anymore by a Bernoulli product measure with density $\lambda+\epsilon$ because we know that a first class particle has to be in site 2. To avoid this problem, we will use the following fact: if a particle of a class different than 1 is at site 2 at time t then it has a positive probability (depending only on λ) to be out of the system at time $t+1$. This implies Lemma 3.1 which says:

$$\mathbf{E}\left[T_\infty^{(2),l}\right] \leqslant C_\lambda, \tag{3.11}$$

where C_λ is a constant. Take any $\beta > \epsilon\lambda(1-\lambda)$ and

$$\tau := \inf\left\{t \geqslant 0 : \forall s \geqslant t, N_s^{(2)} \leqslant \beta s\right\}.$$

We have that τ is almost surely finite by (3.10) and

$$\frac{T_{t,2}^{(2)}}{t}\mathbf{1}_{\{\tau\leqslant t\}} \leqslant \frac{1}{t}\sum_{k=1}^{N_t^{(2)}} T_\infty^{(2),k}\mathbf{1}_{\{\tau\leqslant t\}} \leqslant \frac{1}{t}\sum_{k=1}^{\lfloor\beta t\rfloor} T_\infty^{(2),k}\mathbf{1}_{\{\tau\leqslant t\}}. \tag{3.12}$$

Taking expectation in both sides, it leads to

$$\mathbf{E}\left[\frac{T_{t,2}^{(2)}}{t}\mathbf{1}_{\{\tau\leqslant t\}}\right] \leqslant \frac{1}{t}\sum_{k=1}^{\lfloor\beta t\rfloor}\mathbf{E}\left[T_\infty^{(2),k}\mathbf{1}_{\{\tau\leqslant t\}}\right] \underset{(3.11)}{\leqslant} \frac{\lfloor\beta t\rfloor}{t}C_\lambda. \tag{3.13}$$

Hence, by dominated convergence we have almost surely

$$\lim_{t\to\infty}\frac{T_t^{(2)}}{t} = \lim_{t\to\infty}\frac{T_{t,2}^{(2)}}{t} = \lim_{t\to\infty}\mathbf{E}\left[\frac{T_{t,2}^{(2)}}{t}\mathbf{1}_{\{\tau\leqslant t\}}\right] \leqslant \beta C_\lambda. \tag{3.14}$$

The above inequality is true for all $\beta > \epsilon\lambda(1-\lambda)$, thus we also have

$$\lim_{t\to\infty}\frac{T_t^{(2)}}{t} \leqslant \epsilon\lambda(1-\lambda)C_\lambda.$$

Let now $c_2 := C_\lambda c_1$ and by induction, using exactly the same arguments, we have for all $1 \leqslant j \leqslant i-1$:

$$\limsup_{t\to\infty}\frac{N_t^{(j)}}{t} \leqslant c_{j-1}\epsilon^{j-1},$$

and

$$\lim_{t\to\infty}\frac{T_t^{(j)}}{t} \leqslant c_j\epsilon^{j-1},$$

where $c_j := C_\lambda^{j-1}\lambda\,(1-\lambda)$.

Finally, let $\alpha := \limsup_{t\to\infty} N_t^{(i)}/t$. Doing the same computation as in (3.12), (3.13) and (3.14), we get:

$$\lim_{t\to\infty}\frac{T_t^{(i)}}{t} \leqslant \alpha C_\lambda.$$

Consequently,

$$\lim_{t\to\infty}\frac{T_t^{(i-1)}+T_t^{(i)}}{t} \leqslant c_{i-1}\epsilon^{i-2}+\alpha C_\lambda,$$

which implies as in (3.10):

$$\limsup_{t\to\infty}\frac{N_t^{(i)}}{t} = \alpha \leqslant (c_{i-1}\epsilon^{i-2}+\alpha C_\lambda)\epsilon.$$

Since $\epsilon < \frac{1}{2C_\lambda}$, we have $\alpha \leqslant 2c_{i-1}\epsilon^{i-1}$ and

$$\lim_{t\to\infty}\frac{T_t^{(i)}}{t} \leqslant 2c_i\epsilon^{i-1}.$$

■

Now, let $\bar{N}_t^{(i-1)}$ and $\bar{N}_t^{(i)}$ be the number of particles which have entered the system between 0 and t for the processes $(\bar{\eta}_t^{(i-1)})_{t\geqslant 0}$ and $(\bar{\eta}_t^{(i)})_{t\geqslant 0}$. We deduce from the above theorem that

$$\limsup_{t\to\infty}\frac{\bar{N}_t^{(i)}-\bar{N}_t^{(i-1)}}{t} = \limsup_{t\to\infty}\frac{N_t^{(i)}}{t} = O\left(\epsilon^{i-1}\right).$$

3.4.2 The asymptotic flux at the first order

In this section, we consider the particle system with generator given by (3.6) for $i = 3$ (see Figure 3.5). In order to differentiate it from particle systems we will define below, we will now refer to this system as the *true* process. In the previous section we have seen that in order to compute $\lim_{t\to\infty}\bar{N}_t^{(i)}/t$ up to order ϵ, it is sufficient to compute this limit only for first and second class particles. In other words, if $\mathbf{N}_t^{(j)}$ denotes the number of new j-particles, *i.e.*, the number of j-particles at time t which was not in the system at time 0, then:

$$\begin{aligned}\limsup_{t\to\infty}\frac{\mathbf{N}_t^{(1)}+\mathbf{N}_t^{(2)}+\mathbf{N}_t^{(3)}}{t} &= \limsup_{t\to\infty}\frac{\mathbf{N}_t^{(1)}+\mathbf{N}_t^{(2)}}{t}+o(\epsilon),\\ &= \lambda(1-\lambda)+\limsup_{t\to\infty}\frac{\mathbf{N}_t^{(2)}}{t}+o(\epsilon).\end{aligned}$$

In the following we denote by N_t, rather than by $\mathbf{N}_t^{(2)}$, the number of new second class particles because there will be no possible confusion. The aim of this section is to prove a law of large numbers for N_t and to compute the limit up to order ϵ. First we introduce some notation. Let $c > 0$ such that $\lambda + c < \frac{1}{2}$ and $\epsilon \in [0, c]$. Consider the point process $\mathcal{N}_2^b \cap \{t \geqslant 0 : \eta_t(1) \neq 1, \eta_t(2) = 1\}$, and denote its elements ordered chronologically by $\tau_1^e < \ldots < \tau_i^e < \ldots$ By construction, at each time τ_i^e, a second class particle tries to enter the system. We denote by $X_i(t)$ the position at time t of this particle, with the convention $X_i(t) := 0$ if the corresponding particle is not in the system at time t. We define

$$\tau_i^s := \inf\{t \geqslant \tau_i^e : X_i(t) = 0\},\ S_i(t) := \mathbf{1}_{X_i(t) \geqslant 1}, \text{ and } S_i := \mathbf{1}_{\tau_i^s = \infty}.$$

Remark that there is a positive probability that $\tau_i^s = \tau_i^e$. This happens if $\eta_{\tau_i^e}(1) = 2$. In this case, $X_i(t) = 0$ for all $t \geqslant 0$.

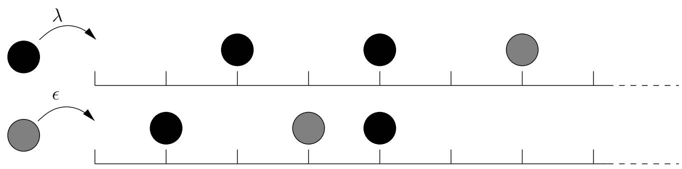

Figure 3.5: *First class particles, in black, enter with rate λ whatever is the configuration in $\{2, 3, \ldots\}$ and second class particles, in grey, enter with rate ϵ if the site 2 is occupied by a first class particle.*

In order to have simpler estimates in the sequel, we consider the process $(\eta_t)_{t \geqslant 0}$ on $\mathcal{X}_3$ starting with the measure $\mu_\infty^{(3)}(.|\eta(1) \neq 1, \eta(2) = 1)$. Of course, the limit that we obtain in this case is the same as the one we would get if we started from $\mu_\infty^{(3)}$. Moreover, the estimates of Theorem 3.3 also hold in this case. Indeed, the distribution of the process converges to $\mu_\infty^{(3)}$. In the sequel, we denote $\bar{\eta}_t(x) := \mathbf{1}_{\eta_t(x) \neq \infty}$ the process with indistinguishable particles associated to $(\eta_t)_{t \geqslant 0}$.

Since $\nu^\lambda \prec \mu_\infty \prec \nu^{\lambda + c}$ and the dynamic is monotone, we can make a basic coupling with a TASEP(λ), denoted $\eta_.^{inf}$, and a TASEP($\lambda + c$), denoted $\eta_.^{sup}$, such that:

- η_0^{inf} has distribution $\nu^\lambda(.|\eta(1) = 0, \eta(2) = 1)$,
- η_0^{sup} has distribution $\nu^{\lambda + c}(.|\eta(1) = 0, \eta(2) = 1)$,
- almost surely $\eta_t^{inf} \leqslant \bar{\eta}_t \leqslant \eta_t^{sup}$, for all $t \geqslant 0$.

The process without interaction

We define a new particle system with state space $\{0, 1, (2, i)_{i \geqslant 1}\}^{\mathbb{Z}_+}$ and the following generator:

$$\begin{aligned} \bar{\Omega}_\nu f(\eta) :=& \nu \mathbf{1}_{\eta(1) \neq 1} \left(f(\eta_{1 \to 1}) - f(\eta)\right) \\ &+ \epsilon \mathbf{1}_{\eta(1) \neq 1, \eta(2) = 1} \left(f(\eta_{2 \to 1}) - f(\eta)\right), \\ &+ \sum_{x=1}^{\infty} \mathbf{1}_{\eta(x) \neq 0, \eta(x+1) \neq 1} \left(f(\eta^{x, x+1}) - f(\eta)\right), \end{aligned} \tag{3.15}$$

for all cylindrical function f, where

$$\eta_{1\to 1}(z) := \begin{cases} 1 & \text{if } z = 1, \\ \eta(z) & \text{otherwise,} \end{cases}$$

$$\eta_{2\to 1}(z) := \begin{cases} (2,1) & \text{if } z = 1 \text{ and } \eta(1) = 0, \\ (2, i+1) & \text{if } z = 1 \text{ and } \eta(1) = (2,i), \\ \eta(z) & \text{otherwise,} \end{cases}$$

and

$$\eta^{x,x+1}(z) := \begin{cases} \eta_{x,x+1}(z) & \text{if } \eta(x) = 1, \\ 0 & \text{if } \eta(x) \neq 1 \text{ and } z = x, \\ (2, i+j) & \text{if } z = x+1, \eta(x) = (2,i) \text{ and } \eta(x+1) = (2,j), \\ \eta(z) & \text{otherwise,} \end{cases}$$

with the convention $(2,0) := 0$ (j can be equal to 0). We will refer to this process as the process *without interaction.*

This particle system has the following description: there are two classes of particles; first class particles perform a TASEP(ν); second class particles enter with rate ϵ if a first class particle is in site 2 and with rate 0 otherwise; they have lower priority than first class particles; and, contrary to the process of Section 3.4.1, second class particles are allowed to jump on a site containing one or more second class particles. Once some particles (necessarily of type 2) are on the same site at a given time, they will always jump together since they use the same Harris system. Another possible choice would be to put a Poisson clock on particles instead of sites. This would lead to the same asymptotic results.

To link our process to the above one, we proceed as follows. We construct a process $\xi^{inf}_{\cdot}$ on $\{0, 1, (2,i)_{i\geqslant 1}\}^{\mathbb{Z}_+}$, with generator $\bar{\Omega}_\lambda$, in such a way that the process $\eta^{inf}_{\cdot}$ defined above is exactly the process of first class particles of $\xi^{inf}_{\cdot}$. Furthermore, at each time τ^e_i, we add a second class particle in $\xi^{inf}_{\cdot}$ at site 1 and we denote by $X^{inf}_i(t)$ its trajectory. This particle will behave as a second class particle in the system with generator (3.15), *i.e.*, it has a lower priority than first class particles but it can jump on a site already occupied by an other second class particle. As a consequence, we can remark that, contrary to $X_i(\tau^e_i)$, we have almost surely $X^{inf}_i(\tau^e_i) \geqslant 1$. By construction we almost surely have $X_i(t) \leqslant X^{inf}_i(t)$ for all $t \geqslant 0$. Indeed, ξ^{inf} and η have the same first class particles and contrary to X^{inf}_i, the particle X_i is blocked by other second class particles thus it stays behind X^{inf}_i. In order to bound from below the trajectory $X_i(t)$, we now construct a process $\xi^{sup}_{\cdot}$ on $\{0,1,2\}^{\mathbb{Z}_+}$ such that for all $t \geqslant 0, x \geqslant 1$, $\mathbf{1}_{\xi^{sup}_t(x)\neq 0} = \eta^{sup}_t(x)$, by affecting the type 2 to particles of $\eta^{sup}_{\cdot}$ entering at times $(\tau^e_i)_{i\geqslant 1}$. In the same way, we denote by $X^{sup}_i(t)$ their trajectory and we have almost surely for all $t \geqslant 0$, $X^{sup}_i(t) \leqslant X_i(t)$. We define analogously the quantities N^{inf}_t, N^{sup}_t, $\tau^{s,inf}_i$, $\tau^{s,sup}_i$, etc.

Consider the following initial configuration: at time 0, first class particles are distributed on $\mathbb{Z}_+ \setminus \{1,2\}$ according to ν^λ (the Bernoulli product measure with density λ), and we put one first class particle in site 2 and one second class particle in site 1. We show in Proposition 3.6 below that this is exactly the distribution of the configuration $\eta^{inf}_{\tau^e_i}$ for all $i \geqslant 1$. Then first class particles enter site 1 with rate λ and they have priority over the second class particle. Two cases can occur: either the second class particle survives, or it dies. Let $p(\lambda)$ be the probability that the second class particle survives. p is a non-increasing function, $p(0) = 1$, $p(\frac{1}{2}) = 0$ and $p(\lambda) > 0$ for all $\lambda < \frac{1}{2}$. Indeed, for the last point, it can be shown that if the second class particle survives, then it has a positive speed $1 - 2\lambda$ (see e.g. [50]). The exact expression of $p(\lambda)$ is unknown. However, simulations indicate that $p(\lambda) = 1 - 2\lambda$ for $\lambda \in [0, \frac{1}{2}]$. We have by construction and with results of Section 3.4.2 below, $\mathbf{P}[S^{inf}_i = 1] = p(\lambda)$ and $\mathbf{P}[S^{sup}_i = 1] = p(\lambda + c)$ for all $i \geqslant 1$. Consequently, $p(\lambda + c) \leqslant \mathbf{P}[S_i = 1] \leqslant p(\lambda)$.

The aim of Section 3.4.2 is to prove the following law of large numbers:

Theorem 3.4. *Almost surely,* $\lim_{\epsilon \downarrow 0} \frac{1}{\epsilon} \lim_{t\to\infty} \frac{N_t}{t} = \lambda(1-\lambda)\, p(\lambda)$.

With the discussion at the beginning of Section 3.4.2, Theorem 3.2 follows.

The idea is the following: when ϵ is very small, second class particles do not interact before they are very far from the left boundary and if a second class particle is far enough from this boundary, then it survives with high probability. In other words, the effect on N_t of interaction goes to 0 with ϵ. The first step in the proof will be to find estimates for the process without interaction and to prove the theorem in this case. Next, we will show, for the true process, that if two second class particles meet, they both survive with a probability going to 1 as ϵ goes to 0; this implies the theorem.

Distribution of the process at time τ^e_i

In this section we prove that at each time τ^e_i, the configuration $\eta^{inf}_{\tau^e_i}$ has distribution $\nu^\lambda(.|\eta_0(2)(1-\eta_0(1)) = 1)$. For that we need some preliminary results about the motion of a tagged particle in a TASEP. It is convenient to regard the exclusion process as a Markov process (X_t, η_t) on the space $V := \{(x,\eta) \in \mathbb{Z}_+ \times X : \eta(x) = 1\}$, so that x is the position of the tagged particle and η is the entire configuration. Consider the generator

$$\begin{aligned}\Omega f(x,\eta) := \sum_{y\in\mathbb{Z}_+, y\neq x} &\eta(y)(1-\eta(y+1))\,[f(x,\eta_{y.y+1}) - f(x,\eta)] \\ &+ (1-\eta(x+1))\,[f(x+1,\eta_{x.x+1}) - f(x,\eta)]\,,\end{aligned} \tag{3.16}$$

for all cylindrical functions. Suppose that initially, the tagged particle is placed at some point $x \in \mathbb{Z}_+$ and other particles are placed according to the Bernoulli product measure with density λ on $\mathbb{Z}_+ \setminus \{x\}$. Then the system is stationary when viewed from the position of the tagged particle. In other words, for all $t \geqslant 0$,

$$\mathbf{E}\left[\prod_{y\in A} \eta_t(\phi_t(y))\right] = \lambda^{|A|},$$

where A is a finite subset of $\mathbb{Z}_+$ and

$$\phi_t(y) := \begin{cases} y & \text{if } y < X_t, \\ y+1 & \text{if } y \geqslant X_t. \end{cases}$$

Moreover, the random variable $\prod_{y\in A} \eta_t(\phi_t(y))$ is independent of X_t for each t. Consequently, it can be shown that $X_t - X_0$ is a Poisson process with parameter $1-\lambda$ (see [44]).

The following proposition will be useful to describe the process at a random time.

Proposition 3.5. *Let X_t be the position of a tagged particle starting at site 0. The other particles are initially distributed according to a Bernoulli product measure with density λ on $\{1, 2, \ldots\}$. Let $H_0 := 0$, and for $i \geqslant 1$, let $H_i := \inf\{t \geqslant 0 : X_t = i\}$. Then for all $i \geqslant 0$, $(\eta_{H_i}(X_{H_i} + x))_{x\geqslant 0}$ has the same distribution as η_0.*

Proof. By the strong Markov property, it is sufficient to prove it for $i = 1$ since it is true for $i = 0$ by hypothesis. Define $X_t^0 := X_t$ and for $i \geqslant 1$, X_t^i is the position of the $i-$th particle to the right of X_t (X_t^i always exists if $\lambda > 0$ and if not the result is obvious). The result will follow if we can prove that $X_{H_1}^1 - X_{H_1}^0, \ldots, X_{H_1}^L - X_{H_1}^{L-1}$ are i.i.d. random variables with geometric distribution with parameter λ for all $L \geqslant 1$. Since $X_t - X_0$ is a Poisson process with parameter $1-\lambda$, the process $\xi_t(i) := X_t^{i+1} - X_t^i - 1$, for $i = 0, \ldots, L-1$, is a totally asymmetric Zero Range process on $\{0, \ldots, L-1\}$ with generator

$$\Omega f(\xi) := \sum_{y=0}^{L-1} \mathbf{1}_{\xi(y)\geqslant 1} \left[f(\xi^y) - f(\xi)\right] + (1-\lambda)\left[f(\xi^L) - f(\xi)\right], \tag{3.17}$$

where

$$\xi^y(z) := \begin{cases} \xi(z) & \text{if } z \notin \{y-1, y\}, \\ \xi(y) - 1 & \text{if } z = y, \\ \xi(y-1) + 1 & \text{if } z = y-1. \end{cases}$$

Let μ be the product measure on $\mathbb{N}^L$ such that $\mu\{\xi : \xi(0) = k\} = \lambda(1-\lambda)^k$. μ is invariant for $\xi_.$ and $\xi_0 \sim \mu$. We also have

$$H_1 = \inf\{t \geqslant 0 : \xi_{t^-}(0) = \xi_t(0) + 1\},$$

i.e., H_1 is the first time at which a particle leaves the system (from 0). We need to prove that ξ_{H_1} has distribution μ. Let $q(\xi, \zeta)$ be the rate for which $(\xi_t)_{t\geqslant 0}$ goes from ξ to ζ, for every $\xi, \zeta \in \mathbb{N}^L$, and let $q(\xi) := \sum_\zeta q(\xi, \zeta)$.

Fix a configuration $\gamma \in \mathbb{N}^L$ and let $\phi(\xi) := \mathbf{P}\left[\xi_{H_1} = \gamma | \xi_0 = \xi\right]$. Conditioning on the first step we get:

$$\phi(\xi) = \sum_\zeta \frac{q(\xi, \zeta)}{q(\xi)} \mathbf{1}_{\zeta(0)\geqslant\xi(0)} \phi(\zeta) + \frac{q(\xi, \gamma)}{q(\xi)} \mathbf{1}_{\gamma=\xi^0}. \tag{3.18}$$

Moreover, since μ is invariant, $\int \Omega\phi(\xi)d\mu = 0$, thus

$$\begin{aligned}\sum_{\xi,\zeta}\mu(\xi)q(\xi,\zeta)\phi(\zeta) &= \sum_{\xi,\zeta}\mu(\xi)q(\xi,\zeta)\phi(\xi),\\ &= \sum_{\xi}\mu(\xi)q(\xi)\phi(\xi),\\ &\overset{(3.18)}{=} \sum_{\xi,\zeta}\mu(\xi)q(\xi,\zeta)\mathbf{1}_{\zeta(0)\geqslant\xi(0)}\phi(\zeta)\\ &\quad+\sum_{\xi}\mu(\xi)q(\xi,\gamma)\mathbf{1}_{\gamma=\xi^0}.\end{aligned} \tag{3.19}$$

But $q(\xi,\zeta)\mathbf{1}_{\zeta(0)<\xi(0)} = 1$ if $\zeta = \xi^0$ and 0 otherwise, hence

$$\begin{aligned}\sum_{\xi}\mu(\xi)\phi(\xi^0)\mathbf{1}_{\xi(0)\geqslant 1} &= \sum_{\xi}\mu(\xi)q(\xi,\gamma)\mathbf{1}_{\gamma=\xi^0},\\ &= \mu(\xi:\xi^0=\gamma) = (1-\lambda)\mu(\gamma).\end{aligned} \tag{3.20}$$

Finally, the left-hand side of (3.20) is equal to

$$(1-\lambda)\sum_{\xi}\mu(\xi^0)\phi(\xi^0)\mathbf{1}_{\xi(0)\geqslant 1} = (1-\lambda)\int\phi(\xi)d\mu,$$

which leads to $\mathbf{P}\left[\xi_{H_1}=\gamma\right] = \mu(\gamma)$. ■

Corollary 3.1. *Consider the TASEP on $\mathbb{Z}_+$ starting from $\nu^\lambda(.|\eta(2)(1-\eta(1))=1)$. Let H_i be the time at which the first particle created is at site i, for $i\geqslant 1$. Then $(\eta_{H_i}(i+x))_{x\geqslant 1}$ has distribution ν^λ.*

Proof. By Proposition 3.5, it is sufficient to treat the case $i=1$. The distance d between the initial particle at site 2 and the new particle evolves as follows: it increases by 1 with rate $1-\lambda$ and decreases by 1 with rate λ until the new particle is at site 1. Hence, at this time, $d+1$ is distributed as a geometric random variable with parameter λ. Using again Proposition 3.5, the configuration in front of the first particle has for distribution a Bernoulli product measure with parameter λ. Therefore, it is the same for the new particle. ■

Now we can give the distribution of $\eta^{inf}_{\tau^e_i}$.

Proposition 3.6. *For each $i\geqslant 1$, $\eta^{inf}_{\tau^e_i}$ has distribution $\nu^\lambda(.|\eta_0(2)(1-\eta_0(1))=1)$. In particular, it does not depend on ϵ.*

Proof. We use the following compact notation for initial measures: the conditioned measure $\nu^\lambda(.|\eta_0(2)(1-\eta_0(1))=1)$ will be denoted by $0\,1\,\nu^\lambda$, and $\nu^\lambda(.|\eta_0(1))=1)$ by $1\,\nu^\lambda$.

Let f be a bounded function on $\{0,1\}^{\mathbb{Z}_+}$. Conditioning on the type of the first new particle and using the above corollary with the Markov property we get:

$$\mathbf{E}^{01\nu^\lambda}\left[f(\theta^2\eta_{\tau_i^e})\right] = \frac{\epsilon}{1-\lambda+\epsilon}\mathbf{E}^{21\nu^\lambda}\left[f(\theta^2\eta_0)\right] + \frac{1-\lambda}{1-\lambda+\epsilon}\mathbf{E}^{1\nu^\lambda}\left[f(\theta^2\eta_{\tau_i^e})\right].$$

The first expectation on the right-hand side is equal to $\langle f\rangle_{\nu^\lambda}$ and, using Proposition 3.5, the second expectation is equal to $\mathbf{E}^{01\nu^\lambda}\left[f(\theta^2\eta_{\tau_i^e})\right]$. Hence $\mathbf{E}^{01\nu^\lambda}\left[f(\theta^2\eta_{\tau_i^e})\right] = \langle f\rangle_{\nu^\lambda}$. ■

Proof in the case "without interaction"

Consider a family $\left(\mathcal{N}_\lambda^b\right)_{0\leqslant\lambda<\frac{1}{2}}$ of Poisson point processes such that the parameter of $\mathcal{N}_\lambda^b$ is λ and for all $0\leqslant\lambda\leqslant\mu<\frac{1}{2}$, $\mathcal{N}_\lambda^b\subset\mathcal{N}_\mu^b$ and $\mathcal{N}_\mu^b\backslash\mathcal{N}_\lambda^b$ is independent of $\mathcal{N}_\lambda^b$. Take also a family $\left(\eta_0^\lambda\right)_{0\leqslant\lambda<\frac{1}{2}}$ of initial configurations such that $\eta_0^\lambda(2)\left(1-\eta_0^\lambda(1)\right)=1$ for all $\lambda\in\left[0,\frac{1}{2}\right[$, the distribution of η_0^λ on $\{3,4,\dots\}$ is ν^λ, and for all $x\geqslant 3$ and all $0\leqslant\lambda\leqslant\mu<\frac{1}{2}$, $\eta_0^\lambda(x)\leqslant\eta_0^\mu(x)$ almost surely. Then using the same Poisson point processes $(\mathcal{N}_x, x\geqslant 1)$ for the bulk dynamic we construct, as in Section 3.2, the family of TASEP $(\eta_\cdot^\lambda)_{0\leqslant\lambda<\frac{1}{2}}$ such that $\eta_\cdot^\lambda$ is a TASEP(λ) and for all $t\geqslant 0$ and all $0\leqslant\lambda\leqslant\mu<\frac{1}{2}$, $\eta_t^\lambda\leqslant\eta_t^\mu$ almost surely. At time 0 we add a second class particle in site 1 to each of these processes and we denote by $X_\lambda(t)$ the position at time t of the particle in the process $\eta_\cdot^\lambda$ (with the convention $X_\lambda(t):=0$ if the particle has left the system). We define

$$S^\lambda := \mathbf{1}_{X_\lambda \text{ survives}},$$

and

$$H_x^\lambda := \inf\{t\geqslant 0 : X_\lambda(t)=x\},$$

for all $x\geqslant 1$.

Since we use the basic coupling, the following inequality holds almost surely:

$$X_\lambda(t)\geqslant X_\mu(t),$$

for all $\lambda\leqslant\mu$ and all $t\geqslant 0$. This easily implies that, for all $\lambda\leqslant\mu$ and all $x\geqslant 1$, $S^\lambda\geqslant S^\mu$ and $H_x^\lambda\leqslant H_x^\mu$. Furthermore, by definition of $p(.)$, S^λ is a Bernoulli random variable with parameter $p(\lambda)$.

We start with an intuitive lemma which will be useful to propagate results from the process without interaction to the true process.

Lemma 3.2. *The function $p:[0,1]\to[0,1]$ is right-continuous.*

Proof. Since $p(\lambda)=0$ for $\lambda\geqslant\frac{1}{2}$, it is sufficient to prove it on $\left[0,\frac{1}{2}\right[$. Let $0\leqslant\lambda<\frac{1}{2}$, $\epsilon'>0$ and $0<c<\frac{1}{2}-\lambda$. There exists some $x\geqslant 1$ such that

$$\mathbf{P}\left[S^{\lambda+c}=0|H_x^{\lambda+c}<\infty\right]<\epsilon'.$$

Indeed, if $M := \max\{X_{\lambda+c}(t), t \geqslant 0\}$ then conditionally to $\{S^{\lambda+c} = 0\}$, M is almost surely finite. Thus there exists $x \geqslant 1$ such that

$$\mathbf{P}\left[M \geqslant x | S^{\lambda+c} = 0\right] < \epsilon' \frac{p(\lambda+c)}{1-p(\lambda+c)}.$$

Then, using

$$\{H_x^{\lambda+c} < \infty\} = \{M \geqslant x\},$$

and

$$\mathbf{P}[M \geqslant x] \geqslant \mathbf{P}\left[S^{\lambda+c} = 1\right] = p(\lambda+c),$$

this implies

$$\mathbf{P}\left[S^{\lambda+c} = 0 | H_x^{\lambda+c} < \infty\right] = \mathbf{P}\left[M \geqslant x | S^{\lambda+c} = 0\right] \frac{\mathbf{P}\left[S^{\lambda+c} = 0\right]}{\mathbf{P}[M \geqslant x]} < \epsilon'.$$

Furthermore for all $\epsilon \in [0, c]$,

$$\begin{aligned}
\mathbf{P}\left[S^{\lambda+\epsilon} = 0 | H_x^{\lambda+\epsilon} < \infty\right] &= \frac{\mathbf{P}\left[H_x^{\lambda+\epsilon} < \infty\right] - \mathbf{P}\left[S^{\lambda+\epsilon} = 1\right]}{\mathbf{P}\left[H_x^{\lambda+\epsilon} < \infty\right]}, \\
&= \frac{\mathbf{P}\left[S^{\lambda+\epsilon} = 0\right] - 1}{\mathbf{P}\left[H_x^{\lambda+\epsilon} < \infty\right]} + 1, \\
&\leqslant \frac{\mathbf{P}\left[S^{\lambda+c} = 0\right] - 1}{\mathbf{P}\left[H_x^{\lambda+c} < \infty\right]} + 1, \\
&= \mathbf{P}\left[S^{\lambda+c} = 0 | H_x^{\lambda+c} < \infty\right] < \epsilon'.
\end{aligned} \tag{3.21}$$

Now let $t_0 \geqslant 0$ such that

$$\mathbf{P}\left[\sup_{t \in [0,t_0]} X_\lambda(t) \geqslant x | H_x^\lambda < \infty\right] > 1 - \epsilon'. \tag{3.22}$$

We can find $0 < c' \leqslant c$ such that

$$\mathbf{P}\left[\sum_{i=1}^{x} \left(\eta_0^{\lambda+c'}(i) - \eta_0^\lambda(i)\right) = 0,\ \left(\mathcal{N}_{\lambda+c'}^b \backslash \mathcal{N}_\lambda^b\right) \cap [0, t_0] = \varnothing\right] > 1 - \epsilon'. \tag{3.23}$$

We define the events

$$B := \left\{\sum_{i=1}^{x} \left(\eta_0^{\lambda+c'}(i) - \eta_0^\lambda(i)\right) = 0,\ \left(\mathcal{N}_{\lambda+c'}^b \backslash \mathcal{N}_\lambda^b\right) \cap [0, t_0] = \varnothing\right\},$$

and

$$A := \left\{\sup_{t \in [0,t_0]} X_\lambda(t) \geqslant x\right\} \bigcap B.$$

By the Harris construction of the process, the event B is independent of $\{H_x^\lambda < \infty\}$ and $\{\sup_{t\in[0,t_0]} X_\lambda(t) \geqslant x\}$. Moreover, $A \subset \{H_x^{\lambda+c'} < \infty\}$, thus using (3.22) and (3.23)

$$\begin{aligned} \mathbf{P}\left[H_x^{\lambda+c'} < \infty | H_x^\lambda < \infty\right] &\geqslant \mathbf{P}\left[A | H_x^\lambda < \infty\right], \\ &= \mathbf{P}\left[\sup_{t\in[0,t_0]} X_\lambda(t) \geqslant x | H_x^\lambda < \infty\right] \mathbf{P}\left[B\right], \\ &\geqslant (1-\epsilon')^2 > 1 - 2\epsilon'. \end{aligned} \tag{3.24}$$

Finally, with (3.21) and (3.24), we get

$$\begin{aligned} p(\lambda) - p(\lambda + c') &= \mathbf{P}\left[S^{\lambda+c'} = 0, S^\lambda = 1\right], \\ &= \mathbf{P}\left[S^{\lambda+c'} = 0, S^\lambda = 1, H_x^{\lambda+c'} < \infty\right] \\ &\quad + \mathbf{P}\left[S^\lambda = 1, H_x^{\lambda+c'} = \infty\right], \\ &\leqslant \mathbf{P}\left[S^{\lambda+c'} = 0, H_x^{\lambda+c'} < \infty\right] + \mathbf{P}\left[H_x^\lambda < \infty, H_x^{\lambda+c'} = \infty\right], \\ &\leqslant \mathbf{P}\left[S^{\lambda+c'} = 0 | H_x^{\lambda+c'} < \infty\right] + \mathbf{P}\left[H_x^{\lambda+c'} = \infty | H_x^\lambda < \infty\right], \\ &< 3\epsilon'. \end{aligned}$$

■

Now we prove Theorem 3.4 in the case without interaction.

Proposition 3.7. *$\frac{N_t}{t}$ and $\frac{N_t^{inf}}{t}$ both have almost sure limits as t goes to infinity and*

$$\lim_{\epsilon\to 0} \frac{1}{\epsilon} \lim_{t\to\infty} \frac{N_t^{inf}}{t} = \lambda\,(1-\lambda)\,p(\lambda).$$

Proof. We use the coupling of $\eta_.$ with the processes $\xi_.^{inf}$ and $\xi_.^{sup}$ defined in Section 3.4.2.

Recall that N_t is the number of 2-particles at time t which are not in the system at time 0 for the *true* process $(\eta_t)_{t\geqslant 0}$, *i.e.*, the number of X_i for which $\tau_i^e \leqslant t < \tau_i^s$. N_t^{inf} and N_t^{sup} have the same definition as N_t but for processes $\xi_.^{inf}$ and $\xi_.^{sup}$ respectively, or for particles X_i^{inf} and X_i^{sup} respectively.

Let $\bar{N}_t := \sharp\left(\mathcal{N}_2^b \cap \{t \geqslant 0 : \eta_t(1) \neq 1, \eta_t(2) = 1\}\right)$. $\bar{N}_t$ is the number of 2-particles (in the process $\eta_.$) which have entered the system between time 0 and time t, *i.e.*, the number of X_i for which $\tau_i^e \leqslant t$.

The convergence to almost sure limits is a consequence of Proposition 3.4. Indeed, for example N_t counts the number of elements of $\mathcal{N}_2^b$ for which $\eta_t(1) \neq 1$ and $\eta_t(2) = 1$ minus the number of elements of $\mathcal{N}_1^b$ for which $\eta_t(1) = 2$. Furthermore, by Proposition 3.4, $\bar{N}_t/t$ converges almost surely to $\lambda(1-\lambda)\epsilon$.

We denote by $(t_n)_{n\geqslant 1}$ the successive times at which the $\bar{N}_{t_n}$-th 2-particle of the *true* process is exactly the n-th particle which will survive. Then:

$$\frac{N_{t_n}^{inf}}{t_n} = \frac{1}{t_n}\sum_{i=1}^{\bar{N}_{t_n}} S_i^{inf} = \frac{n}{t_n}.$$

Thus if $n_t := \sup\{n \geqslant 1 : t_n \leqslant t\}$ then, since t_n/n converges almost surely to

$$(\lim_{t\to\infty} N_t^{inf}/t)^{-1},$$

we almost surely have:

$$\lim_{t\to\infty} \frac{n_t - N_t^{inf}}{t} = 0. \tag{3.25}$$

On the other hand, n_t is exactly the number of 2-particles which are in the system at time t and which will survive, *i.e.*, $n_t = \sum_{i=1}^{\bar{N}_t} S_i^{inf}$ almost surely. Hence:

$$\lim_{t\to\infty} \frac{1}{t}\sum_{i=1}^{\bar{N}_t} S_i^{inf} = \lim_{t\to\infty} \frac{N_t^{inf}}{t}, \text{ a.s.}$$

Next we compute the limit of $1/t\sum_{i=1}^{\bar{N}_t} S_i^{inf}$ in expectation. Let $\epsilon' > 0$ and

$$\tau := \inf\left\{t \geqslant 0 : \forall s \geqslant t, \left|\frac{\bar{N}_t}{t} - \lambda(1-\lambda)\epsilon\right| < \epsilon'\right\}.$$

τ is almost surely finite and, using $\mathbf{E}\left[S_i^{inf}\right] = p(\lambda)$,

$$\begin{aligned}
\mathbf{E}\left[\frac{1}{t}\sum_{i=1}^{\bar{N}_t} S_i^{inf}\right] &\geqslant \frac{1}{t}\sum_{i=1}^{\lfloor((\lambda(1-\lambda)\epsilon-\epsilon')t\rfloor} \mathbf{E}\left[S_i^{inf}\mathbf{1}_{\tau\leqslant t}\right], \\
&\geqslant \frac{\lfloor((\lambda(1-\lambda)\epsilon-\epsilon')\,t\rfloor}{t}p(\lambda) - \frac{1}{t}\sum_{i=1}^{\lfloor((\lambda(1-\lambda)\epsilon-\epsilon')t\rfloor} \mathbf{E}\left[S_i^{inf}\mathbf{1}_{\tau>t}\right].
\end{aligned}$$

Since S_i^{inf} is bounded by 1

$$\mathbf{E}\left[\frac{1}{t}\sum_{i=1}^{\bar{N}_t} S_i^{inf}\right] \geqslant \frac{\lfloor((\lambda(1-\lambda)\epsilon-\epsilon')\,t\rfloor}{t}\left(p(\lambda) - \mathbf{P}\left[\tau > t\right]\right).$$

Let t go to infinity, then ϵ' go to 0:

$$\lim_{t\to\infty} \frac{1}{t}\sum_{i=1}^{\bar{N}_t} S_i^{inf} \geqslant \lambda(1-\lambda)p(\lambda)\epsilon.$$

On the other hand

$$\begin{aligned}\mathbf{E}\left[\frac{1}{t}\sum_{i=1}^{\bar{N}_t} S_i^{inf}\mathbf{1}_{\tau\leqslant t}\right] &\leqslant \frac{1}{t}\sum_{i=1}^{\lfloor((\lambda(1-\lambda)\epsilon+\epsilon')t\rfloor}\mathbf{E}\left[S_i^{inf}\mathbf{1}_{\tau\leqslant t}\right],\\ &\leqslant \frac{\lfloor((\lambda(1-\lambda)\epsilon+\epsilon')\,t\rfloor}{t}p(\lambda),\end{aligned}$$

which gives the reverse inequality letting t go to infinity and ϵ' go to 0. Finally

$$\lim_{t\to\infty}\frac{1}{t}\sum_{i=1}^{\bar{N}_t} S_i^{inf} = \lim_{t\to\infty}\frac{N_t^{inf}}{t} = \lambda(1-\lambda)p(\lambda)\epsilon.$$

■

Interaction implies survival

The following lemma states that if a second class particle goes far enough, then it survives with high probability.

Lemma 3.3. *For all $\epsilon' > 0$, there exists x_0 (depending only on λ and ϵ') such that if c is small enough, then for all $i \geqslant 1$*

$$\mathbf{P}\left[\tau_i^s < \infty, \exists t \geqslant 0, X_i(t) \geqslant x_0\right] < \epsilon'.$$

Proof. We start by proving the same result for X^{inf}. Let

$$M := \sup\left\{X_i^{inf}(t), t \geqslant 0\right\}.$$

Conditionally on $\left\{\tau_i^{s,inf} < \infty\right\}$, M is almost surely finite, thus we can choose x_0 such that

$$\mathbf{P}\left[M \geqslant x_0 | \tau_i^{s,inf} < \infty\right] < \frac{\epsilon'}{2}.$$

Hence

$$\begin{aligned}\mathbf{P}\left[\tau_i^{s,inf} < \infty, \exists t \geqslant 0, X_i^{inf}(t) \geqslant x_0\right] &= \mathbf{P}\left[M \geqslant x_0 | \tau_i^{s,inf} < \infty\right]\\ &\quad\times \mathbf{P}\left[\tau_i^{s,inf} < \infty\right],\\ &< \frac{\epsilon'}{2}.\end{aligned}$$

Furthermore, since the law of X_i^{inf} is the same for all $i \geqslant 1$ (because they enter in the same environment), we can choose the same x_0 for all $i \geqslant 1$. Then we have, using Lemma 3.2:

$$\begin{aligned}\mathbf{P}\left[\tau_i^s < \infty, \exists t \geqslant 0, X_i(t) \geqslant x_0\right] &\leqslant \mathbf{P}\left[\tau_i^{s,inf} < \infty, \exists t \geqslant 0, X_i^{inf}(t) \geqslant x_0\right] \\ &\quad + \mathbf{P}\left[S_i \neq S_i^{inf}\right], \\ &< \frac{\epsilon'}{2} + p(\lambda) - p(\lambda + c), \\ &< \epsilon',\end{aligned}$$

if c is small enough. ■

For $x \geqslant 1$, let $H_x := \inf\{t \geqslant 0 : X_i(t) = x\}$ (we omit the dependence on i in the notation because there will be no possible confusion). We can deduce from this lemma a stronger form of the same estimate:

Corollary 3.2. *Let $x \geqslant 1$. For all $\epsilon' > 0$, there exists x_1 depending only on λ, ϵ' and x such that if c is small enough, then for all $i \geqslant 1$*

$$\mathbf{P}\left[H_{x_1} < \infty, \exists t \geqslant H_{x_1}, X_i(t) \leqslant x\right] < \epsilon'.$$

Proof. We will use the same method as in Lemma 3.1. Let E_t be the following event on the Poisson point processes of the Harris system during the time space $[t, t+1]$:

- one first class particle enters site 1 and moves to site x;
- then one first class particle enters and moves to site $x-1$;
- we continue in the same way until x first class particles have entered the system and they have moved until that the box $\{1, \ldots, x\}$ is full;
- finally we impose that $\mathcal{N}_x \cap [t, t+1] = \varnothing$.

Then $q_x(\lambda) := \mathbf{P}[E_t]$ depends only on λ and x, is positive and, under this event, every second class particle which was in the box $\{1, \ldots, x\}$ at time t has left the system at time $t+1$.

Now let x_1 be given by Lemma 3.3 such that

$$\mathbf{P}\left[\tau_i^s < \infty, \exists t \geqslant 0, X_i(t) \geqslant x_1\right] < \epsilon' q_x(\lambda),$$

and define $H_x^+ := \inf\{t \geqslant H_{x_1} : X_i(t) = x\}$. Then

$$\mathbf{P}\left[\tau_i^s < \infty | H_x^+ < \infty\right] \geqslant q_x(\lambda).$$

This implies

$$\begin{aligned}\mathbf{P}\left[H_{x_1} < \infty, \exists t \geqslant H_{x_1}, X_i(t) \leqslant x\right] = \mathbf{P}\left[H_x^+ < \infty\right] &= \frac{\mathbf{P}\left[\tau_i^s < \infty, H_x^+ < \infty\right]}{\mathbf{P}\left[\tau_i^s < \infty | H_x^+ < \infty\right]}, \\ &< \epsilon'.\end{aligned}$$

■

The next lemma states that if we fix $x \geqslant 1$, then the probability that two second class particles meet in the box $\{1, \ldots, x\}$ goes to 0 with ϵ.

Lemma 3.4. *Let $\tau_{i+1\to i}$ be the first time at which the $(i+1)$-th second class particle tries to jump on the site occupied by the i-th second class particle. Then for all fixed $x \geqslant 1$,*

$$\mathbf{P}\left[\tau_{i+1\to i} < \infty, X_i(\tau_{i+1\to i}) \leqslant x\right] \underset{\epsilon\to 0}{\longrightarrow} 0, \qquad \textit{uniformly in } i.$$

Proof. Fix $\epsilon' > 0$ and let x_1 and $0 < c_0 < \frac{1}{2} - \lambda$ be given by Corollary 3.2 such that

$$\mathbf{P}\left[H_{x_1} < \infty, \exists t \geqslant H_{x_1}, X_i(t) \leqslant x\right] < \epsilon', \qquad \text{for all } \epsilon \leqslant c_0. \tag{3.26}$$

Then x_1 and c_0 depend only on λ and ϵ' (and x). We have:

$$\begin{aligned}\mathbf{P}\left[\exists s \geqslant t, X_i(s) \in \{1, \ldots, x\}\right] \leqslant\ & \mathbf{P}\left[\exists s \geqslant t, X_i(s) \in \{1, \ldots, x\}, H_{x_1} \leqslant t\right] \\ & + \mathbf{P}\left[X_i(t) \geqslant 1, H_{x_1} > t\right], \\ \leqslant\ & \mathbf{P}\left[H_{x_1} < \infty, \exists s \geqslant H_{x_1}, X_i(s) \leqslant x\right] \\ & + \mathbf{P}\left[X_i(s) \in \{1, \ldots, x_1\}, \forall s \in [0,t]\right].\end{aligned} \tag{3.27}$$

As in Lemma 3.1, we have

$$\begin{aligned}\mathbf{P}\left[X_i(s) \in \{1, \ldots, x_1\}, \forall s \in [0,t+1]\right] \leqslant (1 - q_{x_1}(\lambda)) \times \\ \mathbf{P}\left[X_i(s) \in \{1, \ldots, x_1\}, \forall s \in [0,t]\right],\end{aligned}$$

which implies the existence of a constant $C > 0$ depending only on λ and ϵ' such that

$$\mathbf{P}\left[X_i(s) \in \{1, \ldots, x_1\}, \forall s \in [0,t]\right] \leqslant e^{-Ct}.$$

Finally, using (3.26) and (3.27), there exists some deterministic $t_0 \geqslant 0$, depending only on λ and ϵ', such that

$$\mathbf{P}\left[\exists s \geqslant t, X_i(s) \in \{1, \ldots, x\}\right] < 2\epsilon',$$

for all $t \geqslant t_0$ and $\epsilon \leqslant c_0$.

Besides, if we define σ as the time elapsed between τ_i^e and the first jump time of $\mathcal{N}_2^b$ greater than τ_i^e, then σ is an exponential random variable with parameter ϵ independent of the trajectory of X_i. As a consequence, we have

$$\begin{aligned}\mathbf{P}\left[\tau_{i+1\to i} < \infty, X_i(\tau_{i+1\to i}) \leqslant x\right] &\leqslant \mathbf{P}\left[\exists t \geqslant \sigma, X_i(t) \in \{1, \ldots, x\}\right], \\ &\leqslant \mathbf{P}\left[\exists t \geqslant \sigma, X_i(t) \in \{1, \ldots, x\}, \sigma > t_0\right] \\ &\quad + \mathbf{P}\left[\sigma \leqslant t_0\right], \\ &< 2\epsilon' + 1 - e^{-\epsilon t_0}.\end{aligned}$$

Finally we have $\mathbf{P}\left[\tau_{i+1\to i} < \infty, X_i(\tau_{i+1\to i}) \leqslant x\right] \underset{\epsilon\to 0}{\longrightarrow} 0$ uniformly in i. ∎

Now we are able to prove that when a second class particle meets another one, both survive with a probability going to 1 as ϵ goes to 0.

Corollary 3.3.

$$\mathbf{P}\left[\tau_{i+1\to i}<\infty,\tau^s_{i+1}<\infty\right]\underset{\epsilon\to 0}{\longrightarrow}0,\qquad \textit{uniformly in } i. \tag{3.28}$$

Proof. Fix $\epsilon'>0$ and let x_0 be given by Lemma 3.3. We have

$$\begin{aligned}\mathbf{P}\left[\tau_{i+1\to i}<\infty,\tau^s_{i+1}<\infty\right]&=\mathbf{P}\left[\tau_{i+1\to i}<\infty,\tau^s_{i+1}<\infty,X_i(\tau_{i+1\to i})\leqslant x_0\right]\\&\quad+\mathbf{P}\left[\tau_{i+1\to i}<\infty,\tau^s_{i+1}<\infty,X_{i+1}(\tau_{i+1\to i})\geqslant x_0\right],\\&\leqslant\mathbf{P}\left[\tau_{i+1\to i}<\infty,X_i(\tau_{i+1\to i})\leqslant x_0\right]\\&\quad+\mathbf{P}\left[\tau^s_{i+1}<\infty,\exists t\geqslant 0,X_{i+1}(t)\geqslant x_0\right],\\&<2\epsilon',\end{aligned}$$

if ϵ is small enough. ■

The proof of Theorem 3.4

Fix $\epsilon'>0$ and use (3.28) to find $\epsilon>0$ small enough to have

$$\mathbf{P}\left[\tau_{i+1\to i}<\infty,\tau^s_{i+1}<\infty\right]<\epsilon'.$$

We have already seen that both N_t/t and N^{inf}_t/t converge to almost sure limits and that $\frac{1}{\epsilon}\lim_{t\to\infty}N^{inf}_t/t$ converges almost surely to $\lambda\,(1-\lambda)\,p(\lambda)$ as ϵ goes to 0. Recall the definition of $\bar N_t$ at the beginning of the proof of Proposition 3.7. We have $\lim_{t\to\infty}\bar N_t/t=\lambda\,(1-\lambda)\,\epsilon$. Thus if we define

$$\tau:=\inf\left\{t\geqslant 0:\forall s\geqslant t,\frac{\bar N_s}{s}\leqslant(\lambda\,(1-\lambda)+1)\,\epsilon\right\},$$

then τ is almost surely finite and $N^{inf}_t-N_t=\sum_{i=1}^{\bar N_t}\mathbf{1}_{S^{inf}_i(t)=1,S_i(t)=0}$ which implies

$$\begin{aligned}\mathbf{E}\left[\frac{N^{inf}_t-N_t}{t}\mathbf{1}_{\tau\leqslant t}\right]&\leqslant\frac{1}{t}\sum_{i=1}^{\lfloor(\lambda(1-\lambda)+1)\epsilon t\rfloor}\mathbf{P}\left[S^{inf}_i(t)=1,S_i(t)=0,\tau\leqslant t\right],\\&\leqslant\frac{1}{t}\sum_{i=2}^{\lfloor(\lambda(1-\lambda)+1)\epsilon t\rfloor+1}\mathbf{P}\left[\tau_{i\to i-1}<\infty,\tau^s_i<\infty\right],\\&\leqslant\frac{\lfloor(\lambda\,(1-\lambda)+1)\,\epsilon t\rfloor+1}{t}\epsilon',\end{aligned}$$

and, by dominated convergence theorem, the left-hand side of the above inequality converges to $\lim_{t\to\infty}N^{inf}_t/t-\lim_{t\to\infty}N_t/t$ as t goes to infinity. Hence, dividing by ϵ, we get

$$0\leqslant\frac{1}{\epsilon}\lim_{t\to\infty}\frac{N^{inf}_t}{t}-\frac{1}{\epsilon}\lim_{t\to\infty}\frac{N_t}{t}\leqslant(\lambda\,(1-\lambda)+1)\,\epsilon'.$$

Since ϵ' was arbitrary we can conclude.

Chapter 4

Complex Boundary Mechanism: the general case

Let $\lambda \in [0,1]$. Recall the TASEP(λ) on $S := \mathbb{N}$ is the Feller process on $X := \{0,1\}^S$ with generator

$$\Omega_\lambda f(\eta) := \lambda(1-\eta(0))\,[f(\eta_0) - f(\eta)] + \Omega_{bulk} f(\eta),$$

for all cylindrical function $f : X \to \mathbb{R}$, where

$$\Omega_{bulk} f(\eta) := \sum_{x=0}^{\infty} \eta(x)(1-\eta(x+1))\,[f(\eta_{x,x+1}) - f(\eta)]\,,$$

and

$$\eta_0(y) = \begin{cases} \eta(y) & \text{if } y \neq 0, \\ 1-\eta(0) & \text{if } y = 0, \end{cases}$$
$$\eta_{x,x+1}(y) = \begin{cases} \eta(y) & \text{if } y \notin \{x, x+1\}, \\ \eta(x+1) & \text{if } y = x, \\ \eta(x) & \text{if } y = x+1. \end{cases}$$

In terms of particle system, particles are created at site 0 at rate λ and then, for all $i \geqslant 0$, move from site i to site $i+1$ at rate 1 if the target site does not contain any particle. Otherwise, the jump does not occur.

Let $R \in \mathbb{N}$, $S_R := \{0, \dots, R\}$ and $X_R := \{0,1\}^{S_R}$. For a configuration $\eta \in X$, we denote by η^R its restriction to X_R and by $\bar{\eta}^R$ its restriction to $\{0,1\}^{S_R^c}$, where $S_R^c := \mathbb{N} \backslash S_R$. Conversely, for $\xi \in X_R$ and $\eta \in \{0,1\}^{S_R^c}$ we denote by $\xi \cup \eta$, or by $\xi\eta$, the concatenation of the two configurations into a configuration of X.

Consider the TASEP on $\mathbb{N}$ with a complex boundary mechanism, *i.e.*, the Feller process with generator

$$\Omega f(\eta) := \sum_{\xi \in X_R} d_{\eta_{|S_R},\xi}\,\left[f(\xi \cup \eta_{|S_R^c}) - f(\eta)\right] + \Omega_{bulk} f(\eta),$$

where $(d_{\xi,\xi'})_{\xi,\xi'\in X_R}$ are non-negative rates. In this process, we allow every possible transition in the box S_R respecting the exclusion condition of at most one particle per site. We assume that, for all $\eta \in X_R, d_{\eta,\eta} = 0$. We will also assume that the dynamic is dominated by a subcritical TASEP, *i.e.*, there exist some $\lambda \in \left[0, \frac{1}{2}\right[$ and a Feller process $(\eta_t, \xi_t)_{t\geqslant 0}$, with generator $\tilde{\Omega}_\lambda$, such that:

$$\begin{aligned} &\bullet\ (\eta_t)_{t\geqslant 0} \text{ has generator } \Omega,\\ &\bullet\ (\xi_t)_{t\geqslant 0} \text{ is a TASEP}(\lambda) \text{ (and has generator } \Omega_\lambda),\\ &\bullet \text{ if } \eta_0 \leqslant \xi_0, \text{ then, almost surely for all } t \geqslant 0, \eta_t \leqslant \xi_t, \end{aligned} \tag{4.1}$$

where we used the usual partial order on X: $\eta \leqslant \xi \Leftrightarrow \forall x \in S, \eta(x) \leqslant \xi(x)$. A function $f : X \longrightarrow \mathbb{R}$ is said to be *increasing* if $\eta \leqslant \xi$ implies $f(\eta) \leqslant f(\xi)$. For μ,ν two probability measures on X, we say that μ is stochastically dominated by ν, denoted by $\mu \prec \nu$, if for every increasing function f, $\int f d\mu \leqslant \int f d\nu$. Finally, we say that a Feller process is *attractive* (or *monotone*) if one of the following two equivalent statements hold:

$$\text{for every increasing function } f,\ S(t)f \text{ is increasing for all } t \geqslant 0,$$

and

$$\mu \prec \nu \text{ implies } \mu S(t) \prec \nu S(t) \text{ for all } t \geqslant 0.$$

In Chapter 3, we studied the TASEP with a complex boundary mechanism in the attractive case. We proved that there exist a smallest (in the sense of stochastic domination) invariant probability measure which is extremal and ergodic, and that the stationary process starting with this distribution satisfies a strong law of large numbers for the number of particles created. Furthermore, we computed the value of the limit in a particular case in terms of the probability of survival of a single second class particle in a stationary environment.

In this paper, we do not assume any attractivity of the process. Our main assumption is the hypothesis (4.1) which allow us to bound from above the motion of a tagged particle in this process with the one of a tagged particle in the TASEP(λ). This hypothesis is satisfied in most cases of interest, for example, the processes for which the rate at which the particles enter the system in the first site is a function of the configuration in the box $\{0, \dots, R\}$ and is bounded by some $\lambda < \frac{1}{2}$. For this kind of processes we have $d_{\eta,\eta_0} = r(\eta) \leqslant \lambda$ for all $\eta \in X_R$ such that $\eta(0) = 0$ and $d_{\eta,\xi} = 0$ otherwise.

Definition 4.1. *We say that a probability measure μ on X is subcritical if $\mu \prec \nu^\lambda$ for some $\lambda < \frac{1}{2}$.*

We say that a probability measure ν on X^2 is subcritical if both of its marginals are subcritical.

One of the main result in this paper is the following ergodic theorem:

Theorem 4.1. *Under the assumption (4.1), there exist a unique subcritical invariant probability measure μ for the process with generator Ω. Furthermore, μ is extremal and ergodic and if ν is a subcritical probability measure, the process starting from ν converges weakly to μ.*

As in Chapter 3, we use a description of the process in terms of particles with different classes. The particles in the system are divided into two (or possibly more) classes. The evolution is the same as before, except that if a second class particle attempts to jump on a site occupied by a first class particle, it is not allowed to do so, while if a first class particle attempts to jump on a site occupied by a second class particle, the two particles exchange positions. In other words, a first class particle has priority over a second class particle. In particular, if a second class particle is at site 0 while a first class particle is created, then the second class particle disappears and we say it *dies*. If a second class particle never dies, then we say it *survives*.

Consider the following initial configuration. A second class particle is at site 0 and the configuration on $\mathbb{N}^*$ contains only first class particles distributed according to the product measure with uniform density λ, denoted by ν^λ. Then we consider a TASEP(λ) starting from this configuration for which we create only first class particles. We denote by $p(\lambda)$ the probability that the second class particle survives.

Theorem 4.2. *For all $\lambda \in [0,1]$, $p(\lambda) = \frac{1-2\lambda}{1-\lambda}\mathbf{1}_{\lambda<\frac{1}{2}}$.*

We can remark that if we consider a random walk on $\mathbb{Z}$ starting from 0 with jump $+1$ with probability $1-\lambda$ and -1 with probability λ, then the probability that the walker never visits the site -1 is equal to $p(\lambda)$. In Chapter 3, the probability appearing in the law of large numbers was the probability of survival of a second class particle starting in the same configuration conditioned to have a first class particle at site 1. Denote by $q(\lambda)$ this probability.

Corollary 4.1. *For all $\lambda \in [0,1]$, $q(\lambda) = (1-2\lambda)\mathbf{1}_{\lambda<\frac{1}{2}}$.*

This gives the exact value at the first order of the limit in the law of large number of Chapter 3.

The paper is organized as follows: in Section 4.1, we prove the Theorem 4.1. Then, we prove that the stationary process associated with this distribution satisfies a strong law of large numbers for the number of particles created up to time t.

4.1 The coupled process

Define the coupled process by

$$\begin{aligned}\tilde{\Omega} f(\eta,\xi) := &\sum_{\eta',\xi' \in X_R} t\left((\eta^R,\xi^R),(\eta',\xi')\right)\left[f(\eta'\cup\bar{\eta}^R,\xi'\cup\bar{\xi}^R) - f(\eta,\xi)\right] \\ &+ \tilde{\Omega}_{bulk} f(\eta,\xi),\end{aligned} \tag{4.2}$$

where

$$\begin{aligned}\tilde{\Omega}_{bulk} f(\eta,\xi) := & \sum_{\substack{x:\eta(x)=\xi(x)=1,\\ \eta(x+1)=\xi(x+1)=0}} \left[f(\eta_{x,x+1},\xi_{x,x+1}) - f(\eta,\xi)\right] \\ & + \sum_{\substack{x:\eta(x)>\xi(x),\\ \eta(x+1)=0}} \left[f(\eta_{x,x+1},\xi) - f(\eta,\xi)\right] \\ & + \sum_{\substack{x:\eta(x)<\xi(x),\\ \xi(x+1)=0}} \left[f(\eta,\xi_{x,x+1}) - f(\eta,\xi)\right],\end{aligned} \tag{4.3}$$

and $\left(t\left((\eta,\xi),(\eta',\xi')\right)\right)_{(\eta,\xi),(\eta',\xi')\in X_R^2}$ are non-negative rates defined as follows: for all $(\eta,\xi)\in X_R$,

- $t\left((\eta,\xi),(\eta,\eta)\right) := d_{\xi,\eta}$,
- $t\left((\eta,\xi),(\xi,\xi)\right) := d_{\eta,\xi}$,
- $\forall \zeta \in X_R \backslash \{\eta,\xi\}, t\left((\eta,\xi),(\zeta,\zeta)\right) := \min(d_{\eta,\zeta}, d_{\xi,\zeta})$,
- $\forall \eta' \in X_R \backslash \{\eta,\xi\}, t\left((\eta,\xi),(\eta',\xi)\right) := d_{\eta,\eta'} - \min(d_{\eta,\eta'}, d_{\xi,\eta'})$,
- $\forall \xi' \in X_R \backslash \{\eta,\xi\}, t\left((\eta,\xi),(\eta,\xi')\right) := d_{\xi,\xi'} - \min(d_{\eta,\xi'}, d_{\xi,\xi'})$,
- $t\left((\eta,\xi),(\eta',\xi')\right) := 0$ otherwise.

This coupled process is constructed in order to make agree at most as possible the processes $(\eta_t)_{t\geqslant 0}$ and $(\xi_t)_{t\geqslant 0}$. It is easy to check that both marginals of the coupled process have generator Ω. In the sequel, we denote by $\mathcal{I}$ (resp. $\tilde{\mathcal{I}}, \mathcal{I}_\lambda$) the set of invariants probability measures of the process with generator Ω (resp. $\tilde{\Omega}, \Omega_\lambda$). Finally, we denote by $S(t)$ (resp. $\tilde{S}(t), S_\lambda(t)$) the semi-group associated to the generator Ω (resp. $\tilde{\Omega}, \Omega_\lambda$).

Proposition 4.1 (Liggett [42]). *Let $\mu_1,\mu_2 \in \mathcal{I}$. There exists $\nu \in \tilde{\mathcal{I}}$ with marginals μ_1 and μ_2.*

Proof. Let ν_1 be the product measure with marginals μ_1 and μ_2. The marginals of $\frac{1}{t}\int_0^t \nu_1 \tilde{S}(s)ds$ are μ_1 and μ_2 for all $t \geqslant 0$. Take any subsequence $t_n \uparrow \infty$ such that $\frac{1}{t_n}\int_0^{t_n} \nu_1 \tilde{S}(s)ds$ converges weakly and denote ν its limit. Then, $\nu \in \tilde{\mathcal{I}}$ (see [44]) and has μ_1 and μ_2 for marginals. ■

From now on, $(\eta_t,\xi_t)_{t\geqslant 0}$ is a process with generator $\tilde{\Omega}$. We will only precise when we change the initial distribution. Define the following multi-types exclusion process:

$$\zeta_t(x) := \begin{cases} 1 & \text{if } \eta_t(x) = \xi_t(x) = 1, \\ 2^+ & \text{if } \eta_t(x) > \xi_t(x), \\ 2^- & \text{if } \eta_t(x) < \xi_t(x), \\ 0 & \text{otherwise.} \end{cases} \tag{4.4}$$

If $\zeta_t(x) = 1$, we say that there is a first class particle at site x (at time t). If $\zeta_t(x) = 2^+$ or 2^-, we say that there is a second class particle at site x. The ζ-process has the same evolution as the exclusion process, excepted that if a second class particle attempts to jump on a site occupied by a first class particle, it is not allowed to do so, while if a first class particle attempts to jump on a site occupied by a second class particle, the two particles exchange positions. In other words, a first class particle has priority over a second class particle. Finally, if a second class particle attempts to jump on a site occupied by another second class particle, then nothing appends excepted if the two particles are labelled with different signs. In this case, the jump occurs and the particles merge into a first class particle. We resume the possible transitions in the following diagram:

$$\begin{array}{cccccc} 1 & \rightarrow & 2^{\pm} & = & 2^{\pm} & 1 \\ 2^{\pm} & \rightarrow & 1 & = & 2^{\pm} & 1 \\ 2^{+} & \rightarrow & 2^{-} & = & 0 & 1 \\ 2^{-} & \rightarrow & 2^{+} & = & 0 & 1 \end{array} \tag{4.5}$$

In the ζ-process, particles are created in the box $\{0, \dots, R\}$ according to a complex mechanism. However, we can remark that to create a second class particle at time t, it is necessary to have an other second class particle in the box S_R at this time. In other words, if η_t and ξ_t agreed on the box S_R, then the rate of creating a second class particle is 0. Consequently, the following random time will be useful:

$$T := \inf \{t \geqslant 0 : \forall s \geqslant t, \forall x \in S_R, \eta_s(x) = \xi_s(x)\}. \tag{4.6}$$

Hence, if T is finite, then for all $t \geqslant T$, η_t and ξ_t agreed on the box S_R and no more second class particle can be created. In the following proposition, we show that if the process start from an invariant distribution, then T is either null or infinite.

Proposition 4.2. *Let $\nu \in \tilde{\mathcal{I}}$. We have:*

$$\nu\{(\eta, \xi) : \eta = \xi\} = \mathbf{P}^\nu [\eta_0 = \xi_0] = \mathbf{P}^\nu [T < \infty] = \mathbf{P}^\nu [T = 0]. \tag{4.7}$$

Proof. We may assume that $\mathbf{P}^\nu [T < \infty] > 0$ since otherwise $\mathbf{P}^\nu [\eta_0 = \xi_0] = 0$ because $\{\eta_0 = \xi_0\} \subset \{T = 0\}$.

Let $\tilde{\nu}$ be the distribution of (η_0, ξ_0) when the process $(\eta_t, \xi_t)_{t \geqslant 0}$ is under the measure $\mathbf{P}^\nu [.|T < \infty]$. Then $\tilde{\nu}$ is invariant for $\tilde{\Omega}$. Indeed, for $t \geqslant 0$, we define the following random variable:

$$T(t) := \inf\{u \geqslant t : \forall s \geqslant u, \forall x \in S_R, \eta_s(x) = \xi_s(x)\}, \tag{4.8}$$

and let $A \subset X^2$ measurable. Let $\mathcal{F}_t := \sigma((\eta_s, \xi_s)_{0 \leqslant s \leqslant t})$, the canonical filtration associated to the process $(\eta_t, \xi_t)_{t \geqslant 0}$. Since $\{T < \infty\} = \{T(t) < \infty\}$, we have:

$$\begin{aligned} \mathbf{P}^\nu [(\eta_t, \xi_t) \in A \text{ and } T < \infty] &= \mathbf{E}^\nu [\mathbf{P}^\nu [(\eta_t, \xi_t) \in A \text{ and } T(t) < \infty | \mathcal{F}_t]], \\ &= \mathbf{E}^\nu \left[\mathbf{P}^{(\eta_t, \xi_t)} [(\eta_0, \xi_0) \in A \text{ and } T < \infty]\right], \\ &= \mathbf{P}^\nu [(\eta_0, \xi_0) \in A \text{ and } T < \infty], \end{aligned} \tag{4.9}$$

where we have used the Markov property at the second line and the invariance of ν at the third line. Hence, dividing by $\mathbf{P}^\nu[T<\infty]$ and using the definition of $\tilde{\nu}$ we get:

$$\tilde{\nu}\tilde{S}(t)\{(\eta,\xi)\in A\}=\tilde{\nu}\{(\eta,\xi)\in A\}. \tag{4.10}$$

Now take any $x\geqslant 0$ and define:

$$T_x:=\inf\{t\geqslant 0:\forall s\geqslant t,\forall y\in S_x,\eta_s(y)=\xi_s(y)\}.$$

Consider the stationary process starting from $\tilde{\nu}$. After the time T (which is almost surely finite for this process), there is no more second class particle created and all second class particles present at this time will survive. Hence there is a finite time for which the left-most second class particle will be in $\{x+1,x+2,\ldots\}$ for ever after. This time is exactly T_x. In summary, $T_x<\infty$ almost surely under $\mathbf{P}^{\tilde{\nu}}$. Consequently:

$$\begin{aligned}\tilde{\nu}\{(\eta,\xi):\eta_{|S_x}=\xi_{|S_x}\} &=\tilde{\nu}\tilde{S}(t)\{(\eta,\xi):\eta_{|S_x}=\xi_{|S_x}\},\\ &=\mathbf{P}^{\tilde{\nu}}\left[\eta_{t|S_x}=\xi_{t|S_x}\right],\\ &\geqslant\mathbf{P}^{\tilde{\nu}}\left[t\geqslant T_x\right],\\ &\xrightarrow[t\to\infty]{}1.\end{aligned} \tag{4.11}$$

Since this is true for all $x\geqslant 0$, we have $\tilde{\nu}\{(\eta,\xi):\eta=\xi\}=1$ and $T=0$ almost surely under $\mathbf{P}^{\tilde{\nu}}$. Finally:

$$\begin{aligned}\mathbf{P}^\nu[T<\infty] &=\mathbf{P}^\nu[T=0],\\ &=\mathbf{P}^\nu[\eta_0=\xi_0],\\ &=\nu\{(\eta,\xi):\eta=\xi\}.\end{aligned} \tag{4.12}$$

■

The next proposition says that if the initial distribution is subcritical, then T is finite with positive probability.

Proposition 4.3. *Let $\nu\in\tilde{\mathcal{I}}$ be a subcritical measure and T defined as above. Assume that $\nu\{(\eta,\xi):\eta\neq\xi\}>0$. Then $\mathbf{P}^\nu[T<\infty|\eta_0\neq\xi_0]>0$.*

Proof. Let $\lambda\in\left[0,\frac{1}{2}\right[$ be such that both marginals of ν are dominated by ν^λ and such that the process $\tilde{\Omega}_\lambda$ satisfies the hypothesis (4.1). Since ν is subcritical, there exist some measure γ on X^2, invariant for the standard coupled process of the TASEP(λ), such that $\nu\prec\gamma$ and both marginals of γ are ν^λ. Hence, we can construct four configurations $(\eta_0,\xi_0,\alpha_0,\beta_0)$ on the same probability space, such that $\eta_0\leqslant\alpha_0$ and $\xi_0\leqslant\beta_0$ almost surely, (η_0,ξ_0) has distribution ν, and (α_0,β_0) has distribution γ. Using (4.1), we can couple the four processes $(\eta_t,\xi_t,\alpha_t,\beta_t)_{t\geqslant 0}$ such that $(\eta_t,\xi_t)_{t\geqslant 0}$ is the stationary process with generator $\tilde{\Omega}$ and distribution ν, $(\alpha_t)_{t\geqslant 0}$ and $(\beta_t)_{t\geqslant 0}$ are two TASEP(λ), and almost surely for all $t\geqslant 0$, $\eta_t\leqslant\alpha_t$ and $\xi_t\leqslant\beta_t$.

Let $X^+(t)$ be the position of the left-most second class particle of $(\eta_t)_{t\geqslant 0}$ and $X^-(t)$ be the position of the left-most second class particle of $(\xi_t)_{t\geqslant 0}$ (with the convention $X^\pm(t)=\infty$ if the particle does not exists). Let $Y^+(t)$ be the position of a second class particle in the α-process starting at the same site as X^+ at time 0. In the same way, define Y^- for the β-process. Then, while none second class particle is created, we have almost surely $Y^+(t)\leqslant X^+(t)$ and $Y^-(t)\leqslant X^-(t)$. Furthermore, since a second class particle in a stationary TASEP of density λ has a positive speed (cf. [24, 23]), with positive probability:

$$\min(Y^-(0),Y^+(0))\leqslant \min(Y^-(t),Y^+(t))\leqslant \min(X^-(t),X^+(t)), \tag{4.13}$$

for all $t\geqslant 0$. It remains to prove that

$$\min(Y^-(0),Y^+(0))=\min(X^-(0),X^+(0))\geqslant R+1, \tag{4.14}$$

with positive probability. Since $\lambda<\frac{1}{2}$, there exist $x_0\in\mathbb{N}$ such that

$$\mathbf{P}\left[\sharp\{y\leqslant x_0:\eta_0(y)=\xi_0(y)=0\}\geqslant R+1\right]=\mathbf{P}\left[A_{x_0}\right]>0.$$

Consider the following event denoted by B_{x_0}: there exist $0<t_1^1<t_1^2<t_2^2<\ldots<t_1^{x_0}<\ldots<t_{x_0}^{x_0}<1$ such that

- $\forall k=1,\ldots,x_0, t_1^k\in\mathcal{N}_{x_0-k}, t_2^k\in\mathcal{N}_{x_0-k+1},\ldots,t_k^k\in\mathcal{N}_{x_0-1}$,
- $(\mathcal{N}_{boundary}\cup\mathcal{N}_0\cup\ldots\cup\mathcal{N}_{x_0})\cap[0,1]=\left\{t_i^k:k=1,\ldots,x_0,i=1,\ldots,k\right\}$.

We have $\mathbf{P}\left[B_{x_0}\right]>0$ and B_{x_0} is independent of A_{x_0} by construction. Hence

$$\mathbf{P}\left[A_{x_0}\cap B_{x_0}\right]>0.$$

But it is easy to see that under the event $A_{x_0}\cap B_{x_0}$, we have almost surely

$$\min(Y^-(1),Y^+(1))\geqslant R+1.$$

Thus, since γ is invariant, (4.14) is proved. Finally:

$$\begin{aligned}\mathbf{P}^\nu\left[T=0|\eta_0\neq\xi_0\right] &\geqslant \mathbf{P}\left[\min(X^-(t),Y^+(t))\geqslant R+1 \text{ for all } t\geqslant 0|\eta_0\neq\xi_0\right],\\ &>0.\end{aligned} \tag{4.15}$$

■

Now we can make the proof of the first part of Theorem 4.1:

Proof. Suppose μ_1 and μ_2 are subcritical invariant probability measure for the process with generator Ω. We use the Proposition 4.1 to construct $\nu\in\tilde{\mathcal{I}}$ subcritical with marginals μ_1 and μ_2. Using now Proposition 4.2:

$$\mathbf{P}^\nu\left[T<\infty\right]=\mathbf{P}^\nu\left[T<\infty\right]+\mathbf{P}^\nu\left[T<\infty|\eta_0\neq\xi_0\right]\mathbf{P}^\nu\left[\eta_0\neq\xi_0\right]. \tag{4.16}$$

Finally with Proposition 4.3, we get:

$$\mathbf{P}^\nu\left[\eta_0\neq\xi_0\right]=\nu\{(\eta,\xi):\eta\neq\xi\}=0. \tag{4.17}$$

Hence $\mu_1=\mu_2$.

For the existence, we just have to check that any weak limit of $\frac{1}{t}\int_0^t\delta_0 S(s)ds$ is an invariant subcritical probability measure for Ω. ■

4.2 Ergodicity

In order to finish the proof of Theorem 4.1, it remains to prove that μ is an extremal measure in $\mathcal{I}$. This will implies that it is ergodic (see [45] Theorem B52). Assume $\mu = \alpha\mu_1 + (1-\alpha)\mu_2$ with $\mu_1, \mu_2 \in \mathcal{I}$ and $\alpha \in]0,1[$.

There exists $t_n \uparrow \infty$ such that $\frac{1}{t_n}\int_0^{t_n} \mu_1 S_\lambda(s)ds$ and $\frac{1}{t_n}\int_0^{t_n} \mu_2 S_\lambda(s)ds$ converge weakly to γ and γ' respectively, as t goes to ∞. Then γ and γ' are invariant for the TASEP(λ). Furthermore, $\delta_0 \prec \alpha\gamma + (1-\alpha)\gamma' \prec \nu^\lambda$, where δ_0 is the Dirac measure of the empty configuration. Since $\lambda < \frac{1}{2}$, $\delta_0 S_\lambda(t)$ converges weakly to ν^λ. Hence $\alpha\gamma + (1-\alpha)\gamma' = \nu^\lambda$ and finally, by extremality of ν^λ, $\gamma = \gamma' = \nu^\lambda$ which proves that both $\frac{1}{t}\int_0^t \mu_1 S_\lambda(s)ds$ and $\frac{1}{t}\int_0^t \mu_2 S_\lambda(s)ds$ converge to ν_λ as t goes to ∞.

Now, by hypothesis (4.1), we have for $i = 1, 2$, $\mu_i = \mu_i S(t) \prec \frac{1}{t}\int_0^t \mu_i S_\lambda(s)ds$. Letting t go to ∞, we get $\mu_i \prec \nu^\lambda$. Finally, using the unicity of an subcritical invariant probability measure, we have $\mu_1 = \mu_2 = \mu$ which proves that μ is extremal.

4.3 Law of large numbers

In this section, we consider the process $(\eta_t)_{t\geqslant 0}$ with generator Ω starting from the unique subcritical invariant probability measure μ_∞. However, the main result of this section is still true when we start from any subcritical measure or, more generally, any measure in the domain of attraction of μ_∞.

Let N_t be the number of particles entering the system between times 0 and t and N'_t be the number of particles which have jumped throw the edge $\langle R, R+1\rangle$ between times 0 and t. We have for all $t \geqslant 0$,

$$N'_t - R \leqslant N_t \leqslant N'_t + R.$$

Theorem 4.3. *Almost surely*

$$\frac{N_t}{t} \underset{t\to\infty}{\longrightarrow} j_\infty := \mu_\infty\{\eta : \eta(R) = 1, \eta(R+1) = 0\}.$$

Proof. Define

$$I_t := \{s \in [0,t] : \eta_s(R) = 1, \eta_s(R+1) = 0\}.$$

By ergodicity, $\frac{1}{t}\int_0^t \mathbf{1}_{\eta_s(R)=1,\eta_s(R+1)=0}ds$ converges almost surely to j_∞. Furthermore, $N'_t = \sharp(\mathcal{N}_{R,R+1} \cap I_t)$. Now, doing the same proof as the one of Proposition 3.4 in Chapter 3, we get N'_t/t, and also N_t/t, converge almost surely to j_∞. ■

Chapter 5

Metastability and specifications

5.1 Introduction

In this chapter, we focus on the opposite aspect of the preceding chapters. A natural question is to find a complex source for which the system's behavior is qualitatively different from the behavior of the classic TASEP. The goal is then to construct a process with a phase transition when the creation rate is small, which means that the distribution of the system, as the time goes to infinity, still depends on the initial configuration. According to the main result of Chapter 4, it is necessary to have a long range source of particles.

A similar problem is that of the *specifications* about the uniqueness of Gibbs measures. For this model, Bramson and Kalikow have proposed an example for which this uniqueness does not hold [13]. This example gives a natural candidate for our problem.

We start, in Section 2, by giving a description of the specification model. We also describe the example found by Bramson and Kalikow for which there are more than one Gibbs measure. In the third section, we study a TASEP on $\mathbb{Z}_+$ with two types of particle. Contrary to previous chapters, for this process, all particles have the same priority. The mechanism of creation is similar to the one given by the Bramson and Kalikow example. In the last section, we look at a model that is a mean field version of the original question.

5.2 The specifications problem

Let A be a finite alphabet and let $\mathcal{P}(A)$ be the set of probability distributions on A.

Definition 5.1. *A* specification *(also known as g-*function*) is a measurable function g from $A^{\mathbb{N}}$ to $\mathcal{P}(A)$. It is said* regular *if there exists some $\epsilon > 0$ such that for every $x \in A^{\mathbb{N}}$, $g(x) \geqslant \epsilon$.*

A *Gibbs measure* for a specification g is a probability measure μ on $A^{\mathbb{Z}}$ such that

- μ is shift-invariant;
- if $(x_n)_{n\in\mathbb{Z}}$ is distributed according to μ, then for every $i \in \mathbb{Z}$ and $a \in A$,

$$\mu(x_i = a | x_{i-1}, x_{i-2}, \dots) = g_{x_{i-1},x_{i-2},\dots}(a). \tag{5.1}$$

Assume that g has range 1, *i.e.*, that $g_{x_{-1},x_{-2},\dots}$ depends only on x_{-1}. If μ is a probability measure on $A^{\mathbb{Z}}$ satisfying (5.1) and if $(x_n)_{n\in\mathbb{Z}}$ is distributed according to μ, then $(x_n)_{n\in\mathbb{Z}}$ is a Markov process. Hence, in this case, a Gibbs measure is an invariant measure and reciprocally. It is well known that if the Markov chain is ergodic, then it admits a unique Gibbs measure.

For more general cases, the existence of a Gibbs measure is ensured when g is continuous or if g defines a monotone Markov chain. For the question of uniqueness we have to assume that g is *regular*.

For $k \geqslant 1$, define the *variation of g at distance k* of g by

$$var_k(g) := \sup\{||g_x - g_y|| : x_1 = y_1, \dots, x_k = y_k\}. \tag{5.2}$$

We can remark that the continuity of g is equivalent to the condition that $var_k(g) \to 0$ as k goes to infinity. An old result of Keane and Walter [34, 55] is the following:

Theorem 5.1. *If $(var_k(g))_{k\geqslant 1}$ is summable, then g admits a unique Gibbs measure.*

However, the continuity is not sufficient to ensure the uniqueness as it was proved by Bramson and Kalikow :

Theorem 5.2 ([13])**.** *There exists a continuous regular specification that admits multiple Gibbs measures.*

Berger, Hoffman and Sidoravicius [8] have then construct a counter-example for which the variation of g is in ℓ^p for every $p > 2$.

We now explain the idea of this counter-example. Let $A := \{0, 1\}$ and $\epsilon > 0$. Let N be some random variable on $\mathbb{N}$. Given $(x_i)_{i<0}$, define the random variable:

$$W(x) := \begin{cases} 1-\epsilon & \text{if } \sum_{i=1}^N x_{-i} > \frac{N}{2}, \\ \frac{1}{2} & \text{if } \sum_{i=1}^N x_{-i} = \frac{N}{2}, \\ \epsilon & \text{if } \sum_{i=1}^N x_{-i} < \frac{N}{2}. \end{cases} \tag{5.3}$$

Then the probability distribution $g_{x_{-1},x_{-2},\dots}$ is given by

$$g_{x_{-1},x_{-2},\dots}(1) = \mathbf{E}(W(x)). \tag{5.4}$$

In [13], Bramson and Kalikow prove that if the distribution of N is sufficiently heavy tailed, then the process starting from $x_i = 1$ for every $i < 0$ has an asymptotic density greater than $\frac{1}{2}$. This implies, by symmetry, that there are at least two Gibbs measures.

In the next section, we use this idea to construct a TASEP on $\mathbb{N}$ with two types of particles, such that, for every creation rate, there are multiple invariant measures.

5.3 The bicolor-process

Let us consider the state space $X := \{0, B, R\}^{\mathbb{N}}$. A configuration $\xi \in X$ describes the presence or the absence in each site $x \in \mathbb{N}$ of a *blue-particle* (if $\xi(x) = B$) or a *red-particle* (if $\xi(x) = R$). Fix $\epsilon \in (0, \frac{1}{2})$.

We start with some notation. For $\eta \in X$ and $n \in \mathbb{N}^*$, let $N_n(\xi)$, resp. $B_n(\xi)$, $R_n(\xi)$, be the number of particles, resp. blue-particles, red-particles, in the box $\{1, \dots, n\}$ in the configuration ξ:

$$\begin{aligned} N_n(\xi) &:= \textstyle\sum_{i=1}^n \mathbf{1}_{\xi(i)\neq 0}, \\ B_n(\xi) &:= \textstyle\sum_{i=1}^n \mathbf{1}_{\xi(i)=B}, \\ R_n(\xi) &:= \textstyle\sum_{i=1}^n \mathbf{1}_{\xi(i)=R}. \end{aligned}$$

Then, define $b_n(\xi)$, resp. $r_n(\xi)$, as the relative proportion of blue-particles, resp. of red-particles, in this box:

$$\begin{aligned} b_n(\xi) &:= \begin{cases} \frac{B_n(\xi)}{N_n(\xi)} & \text{if } N_n(\xi) \neq 0, \\ 0 & \text{otherwise,} \end{cases} \\ r_n(\xi) &:= \begin{cases} \frac{R_n(\xi)}{N_n(\xi)} & \text{if } N_n(\xi) \neq 0, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Let $(p_j)_{j\geqslant 1}$ be a decreasing sequence of positive numbers such that

$$\sum_{j=1}^{\infty} p_j = 1 \text{ and } \forall k \geqslant 1, \sum_{j>k} p_j \geqslant 2p_k.$$

An example of such a probability is given by $p_k := cr^k$, where $r \in (\frac{2}{3}, 1)$ and $c := \frac{1-r}{r}$. Finally, let $(m_j)_{j\geqslant 1}$ be an increasing sequence of integers and consider the random variable N with distribution given by

$$P\left[N = m_j\right] = p_j, \forall j \geqslant 1.$$

Let $(N_i)_{i\geqslant 1}$ be an i.i.d. sequence of random variables with the same distribution as N and for each $\xi \in X$, $i \geqslant 1$:

$$W_i(\xi) := \begin{cases} 1-\epsilon & \text{if } b_{N_i}(\xi) > r_{N_i}(\xi), \\ \epsilon & \text{if } b_{N_i}(\xi) < r_{N_i}(\xi), \\ \frac{1}{2} & \text{if } b_{N_i}(\xi) = r_{N_i}(\xi). \end{cases}$$

In this section, we consider the *bicolor-process* $(\xi_t)_{t\geqslant 0}$ on $\mathbb{N}$ defined as follows. The process $(\mathbf{1}_{\xi_t(x)\neq 0})_{x\geqslant 0}$ is a TASEP(λ) on $\mathbb{N}$. Let $\mathcal{N} = \{0 < t_1 < t_2 < \dots\}$ be a Poisson point process with parameter λ. At each time t_i, if the site 0 is free, then we create a particle at this site, otherwise we do nothing. If a particle is created, its color is chosen independently of everything and is blue with probability $W_i(\xi_{t_i})$ and red with probability $1 - W_i(\xi_{t_i})$.

For $\xi \in X$ we denote by $\mathbf{P}^\xi$ the distribution of the process starting from the configuration ξ. If ξ is random and has distribution μ, we denote the distribution of the process by $\mathbf{P}^\mu$.

It will be useful to introduce the following partial order on X. For $\xi^1, \xi^2 \in X$, we note $\xi^1 \prec \xi^2$ if for every $x \geqslant 0$

- $\mathbf{1}_{\xi^1(x)\neq 0} = \mathbf{1}_{\xi^2(x)\neq 0}$;
- $\mathbf{1}_{\xi^1(x)=B} \leqslant \mathbf{1}_{\xi^2(x)=B}$.

The bicolor-process is monotone in the following sense. If $\xi^1 \prec \xi^2$, then we can couple (using the standard coupling) two bicolor-processes starting from ξ^1 and ξ^2 in such a way that almost surely for every $t \geqslant 0$, $\xi^1_t \prec \xi^2_t$.

Let $\mu^{(B)}$ be the product product probability measure such that for every $x \geqslant 0$, $\mu^{(B)}\{\eta : \eta(x) = B\} = \lambda$ and $\mu^{(B)}\{\eta : \eta(x) = 0\} = 1 - \lambda$. Define similarly $\mu^{(R)}$ for red-particles. The aim of this section is to prove that the limit distributions of the bicolor-process (which exist by monotonicity) differ if the process starts from $\mu^{(B)}$ or from $\mu^{(R)}$. In order to show this fact, our strategy is to compare the bicolor-process with a TASEP with finite range boundary mechanism with a creation rate of blue-particles greater than the one of red-particles. For that, we will use the methods of [13] on specifications.

5.3.1 Auxiliary processes

In this section, we define an auxiliary process with finite range boundary mechanism and we obtain estimates on the density of blue-particles for the stationary process associated. Fix $k \geqslant 1$. For every $\xi \in X$, let

$$W_i^k(\xi) := \begin{cases} W_i(\xi) & \text{if } N_i \in \{m_1, \dots, m_{k-1}\}, \\ \epsilon & \text{if } N_i = m_k, \\ 1-\epsilon & \text{if } N_i \in \{m_{k+1}, m_{k+2}, \dots\}, \end{cases}$$

and

$$\tilde{W}_i^k(\xi) := \begin{cases} W_i(\xi) & \text{if } N_i \in \{m_1, \dots, m_{k-1}\}, \\ \frac{1}{2} & \text{otherwise.} \end{cases}$$

We define the processes $(\xi^k_t)_{t\geqslant 0}$, resp. $(\tilde{\xi}^k_t)_{t\geqslant 0}$, using the random variables $(W^k_i)_{i\geqslant 1}$, resp. $(\tilde{W}^k_i)_{i\geqslant 1}$, as we did for the process $(\xi_t)_{t\geqslant 0}$ with the random variables $(W_i)_{i\geqslant 1}$. We call the process $(\xi^k_t)_{t\geqslant 0}$ the k-process. Both processes $(\xi^k_t)_{t\geqslant 0}$ and $(\tilde{\xi}^k_t)_{t\geqslant 0}$ are monotone. Hence, starting from the distribution $\mu^{(B)}$, the law of the processes at time t converges weakly to invariant measures $\mu^{k,(B)}_\infty$ and $\tilde{\mu}^{k,(B)}_\infty$. Similarly, if they start from $\mu^{(R)}$, the law of the processes converges weakly to invariant measures $\mu^{k,(R)}_\infty$ and $\tilde{\mu}^{k,(R)}_\infty$. Using Theorem 4.1 of Chapter 4, we have $\mu^{k,(B)}_\infty = \mu^{k,(R)}_\infty := \mu^k_\infty$ and $\tilde{\mu}^{k,(B)}_\infty = \tilde{\mu}^{k,(R)}_\infty := \tilde{\mu}^k_\infty$ and both stationary processes $(\xi^k_t)_{t\geqslant 0}$ and $(\tilde{\xi}^k_t)_{t\geqslant 0}$ with marginals μ_∞^k and $\tilde{\mu}_\infty^k$ respectively are ergodic. The process $(\tilde{\xi}_t^k)_{t\geqslant 0}$ is symmetric in blue/red-particles, hence so is $\tilde{\mu}_\infty^k$. We start with two limit theorems for this symmetric process and we deduce the estimates we need for the k-process. For a probability measure μ on X, we write $\mu \sim \nu^\lambda$ if $(\mathbf{1}_{\eta(x)\neq 0})_{x\geqslant 0}$ has distribution ν^λ when η has distribution μ.

Let B_t^k be the number of blue-particles created between times 0 and t for the process $(\xi_t^k)_{t\geqslant 0}$. Similarly define R_t^k for the number of red-particles and $N_t^k := B_t^k + R_t^k$ for the number of particles. Define also the analogous numbers $\tilde{B}_t^k$, $\tilde{R}_t^k$ and $\tilde{N}_t^k$ for the process $(\tilde{\xi}_t^k)_{t\geqslant 0}$.

Lemma 5.1. *If $\mu \sim \nu^\lambda$, then*

$$\mathbf{P}^\mu \left[\frac{\tilde{B}_t^k}{t} \underset{t\to\infty}{\longrightarrow} \frac{\lambda(1-\lambda)}{2} \right] = 1.$$

Proof. By ergodicity and using the Proposition 3.4 of Chapter 3, under $\mathbf{P}^{\tilde{\mu}_\infty^k}$, $\tilde{B}_t^k/t$ converges almost surely to

$$\lambda \sum_{i=1}^{k-1} p_i E^{\tilde{\mu}_\infty^k} \left[\left((1-\epsilon)\mathbf{1}_{b_{m_i}(\xi)>\frac{1}{2}} + \epsilon \mathbf{1}_{b_{m_i}(\xi)<\frac{1}{2}} + \frac{1}{2}\mathbf{1}_{b_{m_i}(\xi)=\frac{1}{2}} \right) \mathbf{1}_{\xi(0)=0} \right]$$
$$+ \frac{\lambda(1-\lambda)}{2} \sum_{i=k}^{\infty} p_i.$$

But by symmetry of the colors for $\tilde{\mu}_\infty^k$ and since $\tilde{\mu}_\infty^k \sim \nu^\lambda$, this is equal to $\frac{\lambda(1-\lambda)}{2}$.

Now, if $\mu \sim \nu^\lambda$, then the process starting from μ converges weakly to $\tilde{\mu}_\infty^k$. This easily implies

$$\mathbf{P}^\mu \left[\frac{\tilde{B}_t^k}{t} \underset{t\to\infty}{\longrightarrow} \frac{\lambda(1-\lambda)}{2} \right] = 1.$$

■

Lemma 5.2. *Under $\tilde{\mu}_\infty^k$, $B_m(\xi)/m$ converges almost surely to $\frac{\lambda}{2}$ as m goes to infinity.*

Proof. Let x_t be the position of the left-most initial particle, *i.e.*, the left-most particle at time 0. $x_t - x_0$ is a Poisson process with parameter $1-\lambda$, hence x_t/t converges to $1-\lambda$ almost surely as t goes to infinity. By definition

$$B_t = B_{x_t - 1}.$$

Hence B_{x_t}/x_t converges to $\frac{\lambda}{2}$ almost surely which implies

$$\liminf_{m\to\infty} \frac{B_m}{m} \geqslant \frac{\lambda}{2}.$$

But since

$$\limsup_{m\to\infty} \frac{B_m}{m} = \lambda - \liminf_{m\to\infty} \frac{R_m}{m}$$

and since B_m and R_m have the same distribution, we get that almost surely

$$\limsup_{m\to\infty} \frac{B_m}{m} \leqslant \frac{\lambda}{2}.$$

Finally, almost surely

$$\lim_{m\to\infty} \frac{B_m}{m} = \frac{\lambda}{2}.$$

■

Now we compare the k-process with the symmetric process $(\tilde{\xi}_t^k)_{t\geqslant 0}$. Take $\xi \in X$. For every $i \geqslant 1$

$$\begin{aligned} E\left(W_i^k(\xi) - \tilde{W}_i^k(\xi)\right) &= (\frac{1}{2} - \epsilon)(\sum_{i=k+1}^{\infty} p_i - p_k), \\ &\geqslant (\frac{1}{2} - \epsilon)p_k := \gamma_k > 0. \end{aligned} \tag{5.5}$$

Hence, if we modify the process $(\tilde{\xi}_t^k)_{t\geqslant 0}$ in such a way that at each time we create a red-particle, we re-sample its color and the particle becomes blue with probability γ_k, then by (5.5) we can find a monotone coupling between this modified process and $(\xi_t^k)_{t\geqslant 0}$. This gives the following corollary.

Corollary 5.1. *If $\mu \sim \nu^\lambda$, then*

$$\mathbf{P}^\mu\left[\liminf_{t\to\infty} \frac{B_t^k}{t} \geqslant \lambda(1-\lambda)\left(\frac{1}{2} + \gamma_k\right)\right] = 1.$$

As we done for Lemma 5.2 with Lemma 5.1, we deduce, in the following lemma, an estimate on the stationary occupancy of blue-particle from the estimate on the creation rate obtained in the above corollary. We get an estimate on the mixing time for the k-process.

Lemma 5.3. *For every $\epsilon' > 0$, there exists $t_0 > 0$ such that for all $t \geqslant t_0$, if $m = \lfloor (1-\lambda)t \rfloor$, then for all $\mu \sim \nu^\lambda$*

$$\mathbf{P}^\mu\left[b_m(\xi_t^k) < \frac{1}{2}(1+\gamma_k)\right] \leqslant \epsilon'. \tag{5.6}$$

Proof. Fix $\delta, \epsilon' > 0$. By monotonicity, it suffices to show the result for $\mu = \mu^{(R)}$.

Let x_t be the position of the left-most initial particle at time t. Since the process is stationary, $x_t - x_0$ is a Poisson process with parameter $1-\lambda$. Hence, using the Corollary 5.1, there exists t_0 such that

$$\mathbf{P}^{\mu}\left[\exists t \geqslant t_0 : \frac{B_t^k}{t} < \left(\frac{1}{2}+\delta'\right)\lambda(1-\lambda)\left(1+\gamma_k\right) \text{ or } \left|\frac{x_t}{t}-(1-\lambda)\right| > \delta\right] \leqslant \frac{\epsilon'}{2}, \tag{5.7}$$

where $0 < \delta' < [2\gamma_k(1+\gamma_k)]^{-1}$.

Let $m_0 := \lfloor (1-\lambda)t_0 \rfloor$ and denote by A_{t_0} the event appearing in (5.7). Without loss of generality we can assume δ small enough and then m_0 large enough (t_0/m_0 is bounded from above) so that

$$\frac{1}{m_0} + \delta\frac{t_0}{m_0} \leqslant \frac{\delta'}{2}\lambda\left(1+\gamma_k\right), \tag{5.8}$$

and

$$\mathbf{P}^{\mu}\left[N_{m_0}^k(\xi_{t_0}^k) > (1+\delta')\lambda m_0\right] = \mathbf{P}\left[\mathcal{B}(m_0,\lambda) > (1+\delta')\lambda m_0\right] \leqslant \frac{\epsilon'}{2}. \tag{5.9}$$

Under the event ${}^cA_{t_0}$

$$\left|B_{m_0}(\xi_{t_0}^k) - B_{t_0}^k\right| = \left|B_{m_0}(\xi_{t_0}^k) - B_{x_{t_0}-1}^k(\xi_{t_0}^k)\right| \leqslant |m_0+1-x_{t_0}| \leqslant 1+\delta t_0.$$

Hence using (5.8)

$$\left|\frac{B_{m_0}(\xi_{t_0}^k)}{m_0} - \frac{B_{t_0}^k}{m_0}\right| \leqslant \frac{\delta'}{2}\lambda\left(1+\gamma_k\right). \tag{5.10}$$

Also

$$\frac{B_{t_0}^k}{m_0} = \frac{B_{t_0}^k}{t_0}\frac{t_0}{m_0} \geqslant \left(\frac{1}{2}+\delta'\right)\lambda\left(1+\gamma_k\right). \tag{5.11}$$

Using (5.10) and (5.11)

$$\frac{B_{m_0}(\xi_{t_0}^k)}{m_0} \geqslant \frac{\lambda}{2}(1+\delta')\left(1+\gamma_k\right). \tag{5.12}$$

Finally, if the event appearing in (5.9) does not hold, then

$$b_{m_0}(\xi_{t_0}^k) = \frac{B_{m_0}(\xi_{t_0}^k)}{m_0}\frac{m_0}{N_{m_0}(\xi_{t_0}^k)} \geqslant \frac{1}{2}\left(1+\gamma_k\right).$$

We get

$$\mathbf{P}^{\mu}\left[b_{m_0}(\xi_{t_0}^k) < \frac{1}{2}\left(1+\gamma_k\right)\right] \leqslant \epsilon'.$$

The computations above are also valid for any $t \geqslant t_0$ and $m = \lfloor (1-\lambda)t \rfloor$ which finishes the proof. ∎

5.3.2 Comparison with the k-process

We now describe more explicitly how to choose the sequence $(m_k)_{k\geqslant 1}$. We define it recursively as follows: given $m_j = \lfloor (1-\lambda)t_j \rfloor$ for $j = 1, \ldots, k-1$, we choose $m_k = \lfloor (1-\lambda)t_k \rfloor > m_{k-1}$ large enough so that

(i) $\mathbf{P}\left[\mathcal{B}(m_k,\lambda) < \frac{\lambda}{2}m_k\right] < 3^{-k-1}\gamma_k$;

(ii) $\frac{8t_{k-1}}{\lambda\gamma_k m_k} < 3^{-k-1}\gamma_k$;

(iii) $\forall \mu \sim \nu^\lambda, \mathbf{P}^\mu\left[b_{m_k}(\xi^k_{t_k}) < \frac{1}{2}(1+\gamma_k)\right] < \frac{1}{2}3^{-k-1}\gamma_k$,

where we used the Lemma 5.3 and the fact the k-process does not depends on m_j for $j \geqslant k$ for condition (iii).

Lemma 5.4. $\forall t \geqslant 0, \forall k \geqslant 1$,

$$\mathbf{P}^{\mu^{(B)}}\left[b_{m_k}(\xi_t) \leqslant \frac{1+\gamma_k}{2}\right] \leqslant 3^{-k}\gamma_k. \tag{5.13}$$

Proof. It suffices to show that if μ is a probability measure on X satisfying

- $\mu \sim \nu^\lambda$;
- for every $k \geqslant 1$, $\mu\left\{\xi : b_{m_k}(\xi) \leqslant \frac{1+\gamma_k}{2}\right\} \leqslant 3^{-k}\gamma_k$;

then for every $k \geqslant 1$,

$$\mathbf{P}^{\mu}\left[b_{m_k}(\xi_{t_k}) \leqslant \frac{1+\gamma_k}{2}\right] \leqslant 3^{-k}\gamma_k.$$

Indeed, applying this result iteratively and using monotonicity, we get the result for all $t \geqslant 0$.

Fix $k \geqslant 1$ and let

$$G := \{\xi \in X : \exists j > k, b_{m_j}(\xi) \leqslant \frac{1+\gamma_j}{2}\}.$$

By hypothesis

$$\mu\{G\} \leqslant \sum_{j>k} 3^{-j}\gamma_j \leqslant \frac{3}{2}3^{-k-1}\gamma_k, \tag{5.14}$$

since the sequence $(\gamma_k)_k$ is non-increasing.

Using the standard coupling, we consider $(\xi_t, \xi^k_t)_{t\geqslant 0}$ a bicolor-process and a k-process starting both from μ. Let C be the event that for all $t \in [0, t_k]$ and $x \in \{0, \ldots, m_k\}$

$$\mathbf{1}_{\xi_t(x)=B} \geqslant \mathbf{1}_{\xi^k_t(x)=B},$$

i.e., the event that the processes stay ordered up to time t_k. The event C is realized if for all $j > k$, the number of blue-particles that leave the box $\{1, \ldots, m_j\}$ by time

t_k plus the number of red-particles created by time t_k is less than $\frac{\gamma_j}{2} N_{m_j}(\xi_0)$. Hence, using (5.14), (i), (ii) and the Chebychev inequality

$$\begin{aligned}
\mathbf{P}^{\mu}\left[{}^{c}C\right] &\leqslant \mu\left\{G\right\} + \sum_{j>k} \mu\left\{N_{m_j}(\xi) < \frac{\lambda}{2} m_j\right\} + \sum_{j>k} \mathbf{P}\left[N(t_k) \geqslant \frac{\lambda}{4}\gamma_j m_j\right], \\
&\leqslant \frac{1}{2} 3^{-k}\gamma_k + \sum_{j>k} 3^{-j-1}\gamma_j + \sum_{j>k} \frac{8t_k}{\lambda\gamma_j m_j}, \\
&\leqslant 2.3^{-k-1}\gamma_k + \frac{1}{2}3^{-k-1}\gamma_k, \\
&\leqslant \frac{5}{2}\gamma_k 3^{-k-1},
\end{aligned} \tag{5.15}$$

where $(N(t))_{t\geqslant 0}$ is a Poisson process with intensity 2 (greater than the number of creations plus the number of leaves of the box by time t).

Finally, using (5.15) and (iii)

$$\begin{aligned}
\mathbf{P}^{\mu}\left[b_{m_k}(\xi_{t_k}) \leqslant \frac{1+\gamma_k}{2}\right] &\leqslant \mathbf{P}^{\mu}\left[{}^{c}C\right] + \mathbf{P}^{\mu}\left[b_{m_k}(\xi^k_{t_k}) \leqslant \frac{1+\gamma_k}{2}\right], \\
&\leqslant 3^{-k}\gamma_k.
\end{aligned}$$

■

5.3.3 Consequences

Theorem 5.3. *The invariant probability measure $\mu_\infty^{(B)}$ is ergodic and B_t/t converges almost surely to a constant $\tau^{(B)} > \lambda(1-\lambda)/2$ almost surely w.r.t. $\mathbf{P}^{\mu_\infty}$.*

Consequently, $\mu_\infty^{(B)} \neq \mu_\infty^{(R)}$.

Proof. As in Chapter 4, the ergodicity of $\mu_\infty^{(B)}$ is a consequence of its extremality in the convex compact set of the stationary probability measures of the bicolor-process.

Indeed, assume that $\mu_\infty^{(B)} = (1-\alpha)\mu_0 + \alpha\mu_1$ where $\alpha \in (0,1)$ and μ_0, μ_1 are invariant probability measures for the bicolor process. Denote by ν_0, ν_1 the distributions of $(\mathbf{1}_{\eta(x)\neq 0})_{x\geqslant 0}$ when η has distribution μ_0 and μ_1 respectively. Since $\mu_\infty^{(B)} \sim \nu^\lambda$

$$\nu^\lambda = (1-\alpha)\nu_0 + \alpha\nu_1.$$

Furthermore, ν_0 and ν_1 are invariant for the TASEP(λ) and ν^λ is extremal for this process (see for example [44]). Hence $\nu_0 = \nu_1 = \nu^\lambda$ which implies that $\mu_0 \sim \nu^\lambda$ and $\mu_1 \sim \nu^\lambda$.

But $\mu_\infty^{(B)}$ is maximal in the sense that for every $\mu \sim \nu^\lambda$ invariant for the bicolor-process, we can couple two configurations ξ with distribution $\mu_\infty^{(B)}$ and ξ' with distribution μ in such a way that, almost surely, $\xi' \prec \xi$. This property implies that $\mu_\infty^{(B)} = \mu_0 = \mu_1$.

By the ergodic Theorem, B_t/t converges almost surely to a constant $\tau^{(B)}$ under $\mathbf{P}^{\mu_\infty^{(B)}}$. Again by ergodicity, if we let

$$T_t := \int_0^t \mathbf{1}_{\forall k\geqslant 1, b_{m_k}(\xi_s) > \frac{1+\gamma_k}{2}} ds,$$

then T_t/t converges almost surely to

$$\mu_\infty^{(B)} \left\{ \xi : \forall k \geqslant 1, b_{m_k}(\xi) > \frac{1+\gamma_k}{2} \right\} \geqslant 1 - \frac{1}{2}\gamma_1 > \frac{1}{2},$$

by Lemma 5.4. Finally, using the same techniques as in the proof of Proposition 3.4 of Chapter 3, we get

$$\begin{aligned} \tau^{(B)} \geqslant \lambda(1-\lambda) & \left[(1-\epsilon)\mu_\infty^{(B)} \left\{ \xi : \forall k \geqslant 1, b_{m_k}(\xi) > \frac{1+\gamma_k}{2} \right\} \right. \\ & \left. +\epsilon\mu_\infty^{(B)} \left\{ \xi : \exists k \geqslant 1, b_{m_k}(\xi) \leqslant \frac{1+\gamma_k}{2} \right\} \right], \\ > & \frac{\lambda(1-\lambda)}{2}. \end{aligned}$$

Now, by symmetry, B_t/t converges almost surely to a constant $\tau^{(R)} < \lambda(1-\lambda)/2$ under $\mathbf{P}^{\mu_\infty^{(R)}}$. Hence $\mu_\infty^{(B)} \neq \mu_\infty^{(R)}$. ■

As we did for Lemma 5.2, we can deduce from this the following corollary.

Corollary 5.2. *Under $\mu_\infty^{(B)}$, $\liminf_{m\to\infty} B_m/m > \lambda/2$.*

5.4 A toy-model

In this section, we try to break the geometry of the previous model, in order to be able to make explicit computations.

Let $\alpha \in [0,1]$. For $n \in \mathbb{N}$, let $N_n = 2 + \lfloor n^\alpha \rfloor$. We consider a generalized Pòlya urn, containing N_n balls at the end of the step n, and such that the initial distribution of the urn is one ball of each color blue and red. Then to proceed to step $n+1$,

- We draw one ball uniformly from the urn and we denote by C its color. The ball is then placed back in the urn.
- We add an other ball with color C in the urn.
- If $N_{n+1} = N_n$, then we draw again one ball uniformly from the urn but we don't place it back in the urn.

Remark that, by construction, $N_{n+1} - N_n \in \{0, 1\}$. We denote by B_n (resp. R_n) the number of blue (resp. red) balls at step n. Hence, for all $n \in \mathbb{N}$, $B_n + R_n = N_n$.

Comparing with the previous model, the urn plays the role of the particles "seen by the source" when the n^{th} particle is created. Let $\beta > 0$. If N has distribution that satisfies

$$\mathbf{P}[N > n] \sim n^{-\beta}, \quad \text{as } n \text{ goes to infinity,}$$

and if $(N_k)_{k \in \mathbb{N}}$ are i.i.d. with the same distribution as N, then $\max(N_1, \ldots, N_n)$ behaves like $n^{1/\beta}$. Hence, the parameter α in the mean-field version is the inverse of the parameter β of the distribution of N.

If $\alpha = 1$, this model is exactly the Pòlya urn model. In this case, we know that the proportion of blue balls B_n/N_n converges almost surely to a random variable with Beta distribution.

If $\alpha = 0$, then the box will be mono-color eventually. Hence, B_n/N_n converges almost surely to a random variable with Bernoulli distribution.

For the intermediate models, we get the following result:

Theorem 5.4. *B_n/N_n converges almost surely to a random variable which has Bernoulli distribution if $\alpha < \frac{1}{2}$ and has a distribution with support $[0, 1]$ if $\alpha > \frac{1}{2}$.*

Proof. We start by proving the almost sure convergence. Let A be the subset of $\mathbb{N}$ of times such that the size of the box increases, *i.e.*

$$n \in A \Leftrightarrow N_{n+1} = N_n + 1.$$

By definition of the model

$$\begin{aligned} \mathbf{E}[B_{n+1}|B_n] &= \mathbf{E}[B_{n+1} - B_n|B_n] + B_n, \\ &= B_n \left(\frac{1}{N_n} \mathbf{1}_{n \in A} + 1 \right). \end{aligned} \tag{5.16}$$

Let

$$M_n = B_n \prod_{k=0}^{n-1} \left(\frac{1}{N_k} \mathbf{1}_{k \in A} + 1 \right)^{-1}.$$

Using (5.16), (M_n) is a bounded martingale, hence M_n converges almost surely to some random variable. Remarking that

$$\prod_{k=0}^{n-1} \left(\frac{1}{N_k} \mathbf{1}_{k \in A} + 1 \right) = \frac{N_{n-1} + 1}{2},$$

we get the first part of the result. We denote by X the limit of B_n/N_n.

Assume now that $\alpha < \frac{1}{2}$. If $n \notin A$, then almost surely:

$$\begin{aligned} &\mathbf{P}[B_{n+1} = B_n + 1|B_n] = \mathbf{P}[B_{n+1} = B_n - 1|B_n] \\ &= \frac{1}{2}(1 - \mathbf{P}[B_{n+1} = B_n|B_n]) = \frac{B_n R_n}{N_n(N_n + 1)}. \end{aligned} \tag{5.17}$$

Hence the random variable $B_{n+1} - B_n$ has a symmetric distribution. Furthermore, $B_n R_n / N_n(N_n + 1)$ converges almost surely to $X(1 - X)$ as n goes to infinity.

Assume that the event $\{X \notin \{0, 1\}\}$ has positive probability. Then, on this event,

$$\frac{1}{\sqrt{X(1-X)n}} \sum_{k=0}^{n-1} (B_{k+1} - B_k) \mathbf{1}_{k \notin A}$$

converges in distribution to $\mathcal{N}(0, 1)$.

On the other side, the sum of non-symmetric terms

$$\sum_{k=0}^{n-1} (B_{k+1} - B_k) \mathbf{1}_{k \in A}$$

is at most n^α. Hence, on the event $X \notin \{0, 1\}$, since

$$\frac{B_n}{n^\alpha} = \frac{1}{n^\alpha} \left(\sum_{k=0}^{n-1} (B_{k+1} - B_k) \mathbf{1}_{k \notin A} + \sum_{k=0}^{n-1} (B_{k+1} - B_k) \mathbf{1}_{k \in A} + 1 \right), \tag{5.18}$$

B_n / n^α is unbounded with probability 1 which is a contradiction. Finally, by symmetry, X is a Bernoulli random variable with parameter $1/2$.

In the case $\alpha > \frac{1}{2}$, with probability 1, the symmetric parts of the process

$$\sum_{k=0}^{n-1} (B_{k+1} - B_k) \mathbf{1}_{k \notin A}$$

is less than $\sqrt{n}$ for n large enough. Let

$$X_n := \sum_{k=0}^{n-1} (B_{k+1} - B_k) \mathbf{1}_{k \in A}.$$

Using (5.18),

$$\lim_{n \to \infty} \frac{X_n}{n^\alpha} = X.$$

Fix some (large) n_0 and assume that $X_{n_0} / N_{n_0} \in [x, y]$, where $0 \leqslant x < y \leqslant 1$. We consider two urns of size N_{n_0}: in the first one, we start with $\lceil x N_{n_0} \rceil$ blue balls and we denote by Y_k the number of blue balls in this urn at time $k \geqslant n_0$; in the second one, we start with $\lfloor y N_{n_0} \rfloor$ blue balls and we denote by Z_k the number of blue balls in this urn at time $k \geqslant n_0$. For both urns, at each time $k \in A$, we draw a ball at random and add a new ball of the same color as in the Pòlya urn and at each time $k \notin A$, we do nothing. In this way, we can couple the processes $(X_k)_{k \geqslant n_0}$, $(Y_k)_{k \geqslant n_0}$ and $(Z_k)_{k \geqslant n_0}$ in such a way that while $X_k / N_k \in [x, y]$, $Y_k \leqslant X_k \leqslant Z_k$. But it is well known that

$$\mathbf{P} \left[\forall k \geqslant n_0, x \leqslant \frac{Y_k}{N_k} \leqslant \frac{Z_k}{N_k} \leqslant y \right] > 0.$$

Hence $\mathbf{P} \left[\forall k \geqslant n_0, x \leqslant \frac{X_k}{N_k} \leqslant y \right] > 0$ which implies $\mathbf{P} \left[X \in [x, y] \right] > 0$. ∎

Appendix A

Survival Probability

A.1 Introduction

The *simple exclusion process* is an interacting particle system introduced by Spitzer in 1970 as a model of lattice gas. It became a central model of statistical mechanics out of equilibrium since it combines a very simple microscopic description (which allows the derivation of exact results) and very complex macroscopic properties (phase transitions, shocks, long-range correlations, etc.). It is defined as follows. On a given graph, particles are arranged on sites according to the *exclusion rule*: At a given time, there is at most one particle in each site; each of these particles waits for an exponential time with parameter 1 and then chooses one of its neighboring site, according to some probability distribution, and tries to jump to it. The jump occurs if and only if the site is empty, *i.e.* if it respects the exclusion rule.

More specifically, let S be a countable set and consider transition probabilities $(p(x,y))_{x,y\in S}$ of a discrete-time Markov chain on S. The exclusion process on S is the Feller process on $X := \{0,1\}^S$ with generator Ω defined on every cylindrical function $f : X \to \mathbb{R}$ by

$$\Omega f(\eta) := \sum_{x,y\in S} p(x,y)\eta(x)(1-\eta(y))\,[f(\eta_{x,y}) - f(\eta)],$$

where $\eta_{x,y}$ is the result of a swap between sites x and y:

$$\eta_{x,y}(z) := \begin{cases} \eta(z) & \text{if } z \notin \{x,y\}, \\ \eta(y) & \text{if } z = x, \\ \eta(x) & \text{if } z = y. \end{cases}$$

In terms of a particle system, each particle tries to perform a continuous-time random walk on S with transition rates $(p(x,y))_{x,y\in S}$, interacting with the others through the exclusion rule. The resulting process $(\eta_t)_{t\geqslant 0}$ is a Markov process on X. A site $x \in S$ is said to be *occupied* (resp. *empty*) at time t if $\eta_t(x) = 1$ (resp. $\eta_t(x) = 0$). If $p(.,.)$ is doubly stochastic, then the product Bernoulli measure ν^λ

with constant density $\lambda \in [0,1]$ is an invariant probability for the exclusion process (see [44]).

A convenient and graphical way to construct the exclusion process is using the so-called *Harris system*. Consider a family of independent Poisson processes $\{\mathcal{N}_{x,y} : x, y \in S\}$, where $\mathcal{N}_{x,y}$ has intensity $p(x,y)$. If $t \in \mathcal{N}_{x,y}$, and if at time t there is a particle at site x, then it attempts to jump to y. The jump is realized if and only if site y is empty. This construction is easily seen to be equivalent to the one using generator theory (see [44]).

Consider two initial configurations η^1 and η^2 such that $\eta^1 \leqslant \eta^2$, *i.e.*, such that $\eta^1(x) \leqslant \eta^2(x)$ for all $x \in S$. Performing the previous graphical construction from both using the same Poisson processes $(\mathcal{N}_{x,y})$, one obtains a coupled process $(\eta^1_t, \eta^2_t)_{t \geqslant 0}$ in such a way that $\eta^1_t \leqslant \eta^2_t$ for all $t \geqslant 0$ with probability 1. This coupling is called the *standard coupling*. Define the *two-species process* on $\{0,1,2\}^S$ by

$$\eta_t(x) := \begin{cases} 0 & \text{if } \eta^1_t(x) = \eta^2_t(x) = 0, \\ 1 & \text{if } \eta^1_t(x) = \eta^2_t(x) = 1, \\ 2 & \text{if } \eta^1_t(x) = 0 \text{ and } \eta^2_t(x) = 1 \end{cases}$$

for all $x \in S$ and $t \geqslant 0$. Particles labeled 1 are named *first-class particles* and particles labelled 2, *second-class particles*. The reason for this terminology is the following remark: If a first-class particle tries to jump to a site occupied by a second-class particle, then the particles exchange positions, whereas if a second-class particle tries to jump on a site occupied by a first-class particle, then the particles do not move. In other words, first-class particles have *priority* on second-class particles.

The motion of second-class particles for the exclusion process gives a lot of information about the process itself: At the microscopic level, second-class particles describe the discrepancies between two or more coupled exclusion processes; at the macroscopic level, they can be used to localize the propagation of a shock [21]. Moreover, since the presence of second-class particles encodes the way in which two coupled systems differ, information about their survival has to do with the speed of convergence to equilibrium of the initial process.

In this paper, we are interested in the TASEP (Totally Asymmetric Simple Exclusion Process) on $\mathbb{N}$ where

$$p(x,y) = \begin{cases} 1 & \text{if } y = x+1, \\ 0 & \text{otherwise.} \end{cases}$$

In order to make the long-time behavior of the model non-trivial, we add a Poisson *source* at the boundary, as follows. Let $\lambda \in [0,1]$ and consider the Feller process on $X = \{0,1\}^{\mathbb{Z}_+}$, denoted by TASEP($\lambda$), with generator

$$\Omega_\lambda f(\eta) := \lambda(1-\eta(0))\,[f(\eta_0) - f(\eta)] + \Omega_{bulk} f(\eta),$$

for every cylindrical function $f : X \to \mathbb{R}$; here,

$$\Omega_{bulk} f(\eta) := \sum_{x=0}^{\infty} \eta(x)(1-\eta(x+1)) \left[f(\eta_{x,x+1}) - f(\eta) \right],$$

and

$$\eta_0(y) := \begin{cases} \eta(y) & \text{if } y \neq 0, \\ 1 & \text{if } y = 0. \end{cases}$$

(η_0 corresponds to adding a particle at the origin if there is not one already.)

Consider the initial state in which there is a second-class particle at site 0 and every positive site contains a first class particle with probability λ independently of the others. Let $(\eta_t)_{t\geqslant 0}$ the two-species process starting from this configuration and denote by $X(t)$ the position of the second-class particle at time t with the convention $X(t) := -1$ if the particle has *died*, *i.e.*, if it has left the system. This can only happen when a first class particle is created while a second-class particle was at site 0. If the second-class particle never leaves the system, we say that it *survives*.

We are interested in the exact value of the probability $p(\lambda)$ that the second-class particle survives; we denote by S this event. Using the result of [24], it is easy to see that if $\lambda > \frac{1}{2}$, then $p(\lambda) = 0$; as in [22], this probability can be viewed as the coupling probability of processes η^1 and η^2. Our main result is the following:

Theorem A.1. *For all* $\lambda \in [0,1]$, $p(\lambda) = \frac{1-2\lambda}{1-\lambda} \mathbf{1}_{\lambda < \frac{1}{2}}$.

The following remark is rather surprising: If we consider a random walk on $\mathbb{Z}$ starting from 0, and jumping to the right (resp. to the left) with probability $1-\lambda$ (resp. λ), then the probability that the walker never visits site -1 is equal to $p(\lambda)$. However, the trajectory of the second-class particle is not the same as that of the walker: Indeed, the fluctuations of $X(t)$ are of order $t^{2/3}$ (see [6]), which says that the motion of a second-class particle in a stationary TASEP is *super-diffusive*. Furthermore, the process seen from the position of the second-class particle is not stationary. We were not able to get an intuition of this coincidence.

The reason why we are studying this particular survival probability comes from our previous paper [52], where it appears in the statement of a law of large numbers. More precisely, in [52] we consider the two-species process for which the rate of creation of second-class particles (at site 0) is $\epsilon > 0$ if site 0 is empty and site 1 contains a particle, whatever its class is, and 0 otherwise. If the process starts from the empty configuration, and if N_t denotes the number of particles in the system at time t, we proved that N_t/t converges almost surely to a constant and that the limit is equal, at the first order in ϵ, to

$$\lambda(1-\lambda)(1+q(\lambda)\epsilon) + o(\epsilon)$$

where $q(\lambda) := \mathbf{P}\left[S|\eta_0(1) = 1\right]$. A corollary of the proof of the main theorem is the exact value of $q(\lambda)$:

Corollary A.1. *For all* $\lambda \in [0,1]$, $q(\lambda) = (1-2\lambda)\mathbf{1}_{\lambda<\frac{1}{2}}$.

A few others exact computations involving second-class particles are possible; we now describe some of them. Let $p \in (\frac{1}{2}, 1]$, $\rho \in [0,1]$ and $\lambda \in [0,1]$ and consider the one-dimensional ASEP (Asymmetric Simple Exclusion Process) on $S = \mathbb{Z}$ with transition probabilities

$$p(x,y) = \begin{cases} p & \text{if } y = x+1, \\ 1-p & \text{if } y = x-1, \\ 0 & \text{otherwise.} \end{cases}$$

We start from the configuration with the product distribution with density ρ on $\mathbb{Z}_-^*$ and λ on $\mathbb{Z}_+^*$ of first class particles, and with a second-class particle at site 0. Denote by $X(t)$ the location of this particle at time t. If $\rho \leqslant \lambda$, Ferrari, Kipnis and Saada [24] proved that, almost surely, $X(t)/t$ converges to $2p-1-\lambda-\rho$. In the case $\rho > \lambda$ and $p = 1$, Ferrari and Kipnis [23] obtained the convergence in distribution of $X(t)/t$ to the uniform distribution on $[1-2\rho, 1-2\lambda]$. Then, Mountford and Guiol [48] proved this convergence is almost sure.

In a different setup, consider the two-species process starting from the configuration

$$\eta(x) = \begin{cases} 1 & \text{if } x < 0, \\ 2 & \text{if } x \in \{0,1\}, \\ 0 & \text{if } x > 1. \end{cases}$$

Ferrari *et al.* [22] proved that the probability that the two second-class particles collide (*i.e.*, that one of them tries to jump on the other) is equal to $(1+p)/3p$. They also proved that this probability becomes $(1+2p^2)/6p^2$ if the process starts from $\eta_{1,2}$ instead of η. These probabilities can be interpreted as coupling probabilities of two ASEPs starting from

$$\eta^1(x) = \begin{cases} 1 & \text{if } x \leqslant 0, \\ 0 & \text{if } x > 0, \end{cases}$$

and

$$\eta^2(x) = \begin{cases} 1 & \text{if } x < 0 \text{ or } x = 1, \\ 0 & \text{if } x = 0 \text{ or } x > 1, \end{cases}$$

for the first probability and from $(\eta^1, \eta^2_{1,2})$ for the second probability.

The plan of the article is the following. In Section A.2, we study the survival probability of a second-class particle in a TASEP on $\{1,\ldots,N\}$. (In this case, we say that a particle survives if it leaves the system from site N.) Denoting by $P_N(\eta)$ the probability of survival when the process starts from the configuration η, we obtain recursion relations for the $(P_N(\eta))_{N,\eta}$ for all η with exactly one second-class particle. This allow us to compute the probability of survival if the process starts from the product Bernoulli distribution with parameter λ on $\{2,\ldots,N\}$ and a second-class particle at site 1. In Section A.3, we compute the limit, as N goes to infinity, of the probability obtained in Section A.2 and we show that this limit is the survival probability in the infinite volume.

A.2 Finite sized systems

A.2.1 A recursion relation

Let $\lambda \in [0,1]$ and $N \geqslant 1$. We define the two state-spaces X_N and Y_N by:

$$X_N := \{0,1\}^N,$$
$$Y_N := \{\eta \in \{0,1,2\}^N : \sum_{x=1}^{N} \mathbf{1}_{\eta(x)=2} = 1\}.$$

Consider the multi-type TASEP(λ) with generator:

$$\begin{aligned}\Omega_N f(\eta) := {} & \lambda \mathbf{1}_{\eta(1)\neq 1}\left[f(\eta_1) - f(\eta)\right] \\ & + \sum_{x=1}^{N-1} \left[\mathbf{1}_{\eta(x)=1,\eta(x+1)\neq 1} + \mathbf{1}_{\eta(x)=2,\eta(x+1)=0}\right]\left[f(\eta_{x,x+1}) - f(\eta)\right] \\ & + (1-\lambda)\mathbf{1}_{\eta(N)\neq 0}\left[f(\eta_N) - f(\eta)\right],\end{aligned} \tag{A.1}$$

where

$$\eta_1(y) = \begin{cases} \eta(y) & \text{if } y \neq 1, \\ 1 & \text{if } y = 1, \end{cases}$$
$$\eta_N(y) = \begin{cases} \eta(y) & \text{if } y \neq N, \\ 0 & \text{if } y = N, \end{cases}$$
$$\eta_{x,x+1}(y) = \begin{cases} \eta(y) & \text{if } y \notin \{x, x+1\}, \\ \eta(x+1) & \text{if } y = x, \\ \eta(x) & \text{if } y = x+1. \end{cases}$$

For an initial configuration $\eta \in Y_N$, we are interested in the probability that the second-class particle leaves the system from the right (*i.e.*, from the site N) instead from the left (*i.e.*, from the site 1). If it does, we say that the particle survives; otherwise, we say that it dies.

Notation. In the sequel, we will use the shorthand 0^a (resp. 1^a) to denote a sequence of a empty sites (resp. a sites occupied by a first class particle).

Let $\eta \in Y_{N+1}$ be a configuration for which there exists $x_0 \in \{1,\dots,N+1\}$ such that $\eta(x_0) = 0$. Define the configuration $\eta^0 \in Y_N$ in the following way:

- if $\eta = \xi 0 1^k$ with $0 \leqslant k \leqslant N-1$, $\xi \in Y_{N-k}$, then $\eta^0 := \xi 1^k$;
- if $\eta = \xi 0 1^k 2 1^l$ with $k, l \geqslant 0$, $k + l \leqslant N-1$, $\xi \in Y_{N-k-l-1}$, then $\eta^0 := \xi 2 1^{k+l}$.

We denote by η' the configuration of $\{0,1,2\}^N$ obtained by taking out the site $N+1$, *i.e.*, $\eta'(x) = \eta(x)$ for all $x \in \{1,\dots,N\}$. Note that $\eta' \in Y_N$ if $\eta(N+1) \neq 2$ and $\eta' \in X_N$ otherwise.

Finally, we will denote by $P_N(\eta)$ the probability that the second class particle survives if the process starts from the configuration $\eta \in Y_N$. Furthermore, if $\eta \in X_N$ we define by convention $P_N(\eta) := 1$.

Theorem A.2. *The family $(P_N(\eta))_{\eta\in Y_N}$ satisfies the followings recursion relations:*

1) $P_1(2) = 1 - \lambda$,

2) $\forall N \geqslant 1$, $\forall \eta \in Y_{N+1}$ *with at least one empty site:*

$$P_{N+1}(\eta) = \lambda P_N(\eta^0) + (1-\lambda)P_N(\eta'),$$

3) $\forall N \geqslant 1$, $\forall \eta \in Y_{N+1}$ *with no empty site:*

$$P_{N+1}(\eta) = (1-\lambda)P_N(\eta').$$

Proof. First, remark that the equations 1) to 3) define all the $P_N(\eta)$ for $N \geqslant 1$ and $\eta \in Y_N$. We will prove by induction that the numbers defined by these equations are the probability searched.

For $N = 1$, the probability of survival is the probability that the second-class particle jumps before a first class particle enter the system. Hence $P_1(2) = 1 - \lambda$.

Let $N \geqslant 1$. For $\eta, \xi \in Y_N$, define

$$q(\eta,\xi) := \begin{cases} \lambda & \text{if } \eta(1) = 0 \text{ and } \xi = \eta_1, \\ 1 & \text{if } \eta(x) = 1, \eta(x+1) \neq 1 \text{ or } \eta(x) = 2, \eta(x+1) = 0 \\ & \text{and } \xi = \eta_{x,x+1} \text{ for } x \in \{1,\dots,N-1\}, \\ 1-\lambda & \text{if } \eta(N) = 1 \text{ and } \xi = \eta_N, \\ 0 & \text{otherwise,} \end{cases}$$

the rate to go from the configuration η to ξ, and

$$q(\eta) := \sum_{\xi\in Y_N} q(\eta,\xi) + \lambda \mathbf{1}_{\eta(1)=2} + (1-\lambda)\mathbf{1}_{\eta(N)=2},$$

the rate of leaving the configuration η.

By construction of the process, the $(P_N(\eta))_{\eta\in Y_N}$ satisfy the following linear system:

$$\forall \eta \in Y_N,\ q(\eta)P_N(\eta) = \sum_{\xi\in Y_N} q(\eta,\xi)P_N(\xi) + (1-\lambda)\mathbf{1}_{\eta(N)=2}. \tag{A.2}$$

Furthermore, since the matrix associated to this system is irreducibly diagonally dominant, $(P_N(\eta))_{\eta\in Y_N}$ is the unique solution of (A.2).

Assume now that for all $k \in \{1,\dots,N\}$, $(P_k(\eta))_{\eta\in Y_k}$ satisfies the equations 1) to 3) and define $(Q(\eta))_{\eta\in Y_{N+1}}$ as follows:

$$Q(\eta) := \lambda P_N(\eta^0) + (1-\lambda)P_N(\eta'), \tag{A.3}$$

if η has at least one empty site, and

$$Q(\eta) := (1-\lambda)P_N(\eta'), \tag{A.4}$$

otherwise. We will show that this family is solution of (A.2) and hence $Q(\eta) = P_{N+1}(\eta)$ for all $\eta \in Y_{N+1}$. It will be convenient to define the following function: $\forall N \geqslant 1, \forall k \in \{0,\dots,N\}, \forall \xi \in Y_N$,

$$A_k(\xi) := \lambda \mathbf{1}_{\xi(1)=0} P_N(\xi_1) + \sum_{x=1}^{k-1} \mathbf{1}_{\xi(x)\succ\xi(x+1)} P_N(\xi_{x,x+1}),$$

where we used the *priority notation*: For all $i,j \in \{0,1,2\}$, we write $i \succ j$ if and only if ($i=1$ and $j \neq 1$) or ($i=2$ and $j=0$). We proceed by splitting the possible configuration into several cases which we study separately.

Case I: $\eta = \xi 0$

We have $\eta^0 = \eta' = \xi$. Let $a := \xi(N) = \eta(N) \in \{0,1,2\}$. Then $q(\eta) = q(\xi) + \lambda \mathbf{1}_{a\neq 0}$. Hence:

$$\begin{aligned}
q(\eta)Q(\eta) &= [q(\xi) + \lambda\mathbf{1}_{a\neq 0}]\, P_N(\xi), \\
&= \sum_{\zeta\in Y_N} q(\xi,\zeta)P_N(\zeta) + (1-\lambda)\mathbf{1}_{a=2} + \lambda\mathbf{1}_{a\neq 0}P_N(\xi), \\
&= A_N(\xi) + (1-\lambda)\mathbf{1}_{a=1}P_N(\xi'0) + (1-\lambda)\mathbf{1}_{a=2} \\
&\quad + \lambda\mathbf{1}_{a\neq 0}P_N(\xi), \\
&= A_N(\xi) + \mathbf{1}_{a\neq 0}\left[\lambda P_N(\xi) + (1-\lambda)P_N(\xi'0)\right].
\end{aligned} \tag{A.5}$$

On the other hand,

$$\begin{aligned}
\sum_{\zeta\in Y_{N+1}} q(\eta,\zeta)Q(\zeta) &= \lambda\mathbf{1}_{\xi(1)=0}Q(\xi_1 0) + \sum_{x=1}^{N-1}\mathbf{1}_{\xi(x)\succ\xi(x+1)}Q(\xi_{x,x+1}0) \\
&\quad + \mathbf{1}_{a\neq 0}Q(\xi'0a), \\
&= A_N(\xi) + \mathbf{1}_{a\neq 0}\left[\lambda P_N(\xi) + (1-\lambda)P_N(\xi'0)\right], \\
&= q(\eta)Q(\eta).
\end{aligned} \tag{A.6}$$

Case II: $\eta = \xi 0 1^k$ **with** $k \in \{1,\dots,N-1\}, N \geqslant 3$

We have $\eta^0 = \xi 1^k$ and $\eta' = \xi 0 1^{k-1}$. Let $a := \xi(N) = \eta(N) \in \{0,1,2\}$. Then $q(\eta) = q(\eta') + (1-\lambda)\mathbf{1}_{k=1} = q(\eta^0) + \mathbf{1}_{a\neq 0}$. Hence:

$$\begin{aligned}
q(\eta)Q(\eta) &= [q(\xi) + \lambda\mathbf{1}_{a\neq 0}]\, P_N(\xi), \\
&= \sum_{\zeta\in Y_N} q(\xi,\zeta)P_N(\zeta) + (1-\lambda)\mathbf{1}_{a=2} + \lambda\mathbf{1}_{a\neq 0}P_N(\xi), \\
&= A_N(\xi) + (1-\lambda)\mathbf{1}_{a=1}P_N(\xi'0) + (1-\lambda)\mathbf{1}_{a=2} \\
&\quad + \lambda\mathbf{1}_{a\neq 0}P_N(\xi), \\
&= A_N(\xi) + \mathbf{1}_{a\neq 0}\left[\lambda P_N(\xi) + (1-\lambda)P_N(\xi'0)\right].
\end{aligned} \tag{A.7}$$

On the other hand,

$$\begin{aligned}\sum_{\zeta\in Y_{N+1}} q(\eta,\zeta)Q(\zeta) &= \lambda \mathbf{1}_{\xi(1)=0} Q(\xi_1 0) + \sum_{x=1}^{N-1} \mathbf{1}_{\xi(x)\succ\xi(x+1)} Q(\xi_{x,x+1}0)\\ &\quad + \mathbf{1}_{a\neq 0} Q(\xi' 0a),\\ &= A_N(\xi) + \mathbf{1}_{a\neq 0}\left[\lambda P_N(\xi) + (1-\lambda)P_N(\xi' 0)\right],\\ &= q(\eta)Q(\eta).\end{aligned} \tag{A.8}$$

Case III: $\eta = \xi 02$ **with** $N \geqslant 2$

The case $N = 2$ is easy and left to the reader. Assume that $N \geqslant 3$. We have $\eta' = \xi 0$ and $\eta^0 = \xi 2$. Let $a := \xi(N-1) = \eta(N-1) \in \{0,1\}$. Then $q(\eta^0) = q(\eta)$ and:

$$\begin{aligned} q(\eta)Q(\eta) &= \lambda q(\eta^0)P_N(\eta^0) + (1-\lambda)q(\eta),\\ &= \lambda \sum_{\zeta\in Y_N} q(\eta^0,\zeta)P_N(\zeta) + \lambda(1-\lambda) + (1-\lambda)q(\eta),\\ &= \lambda A_{N-1}(\xi 2) + \lambda a P_N(\xi' 21) + \lambda(1-\lambda) + (1-\lambda)q(\eta).\end{aligned} \tag{A.9}$$

On the other hand,

$$\begin{aligned}\sum_{\zeta\in Y_{N+1}} q(\eta,\zeta)Q(\zeta) &= \lambda \mathbf{1}_{\xi(1)=0} Q(\xi_1 02) + \sum_{x=1}^{N-2} \mathbf{1}_{\xi(x)\succ\xi(x+1)} Q(\xi_{x,x+1}02)\\ &\quad + aQ(\xi' 012),\\ &= \lambda A_{N-1}(\xi 2) + \lambda a P_N(\xi' 21)\\ &\quad + (1-\lambda)\Big[\lambda \mathbf{1}_{\xi(1)=0} + \sum_{x=1}^{N-2} \mathbf{1}_{\xi(x)\succ\xi(x+1)} + a\Big],\\ &= \lambda A_{N-1}(\xi 2) + \lambda a P_N(\xi' 21) + (1-\lambda)q(\eta)\\ &\quad - (1-\lambda)^2,\\ &= q(\eta)Q(\eta) - (1-\lambda).\end{aligned} \tag{A.10}$$

Case IV: $\eta = \xi 0 2 1^k$ **with** $k \in \{1, \dots, N-1\}, N \geqslant 3$

We have $\eta' = \xi 0 2 1^{k-1}$ and $\eta^0 = \xi 2 1^k$. Let $a := \xi(N-1-k) = \eta(N-1-k) \in \{0,1\}$, with $a := 0$ if $k = N-1$. Then $q(\eta) = q(\eta^0) = q(\eta')$ and:

$$\begin{aligned}
q(\eta)Q(\eta) &= \lambda q(\eta^0) P_N(\eta^0) + (1-\lambda) q(\eta') P_N(\eta'), \\
&= \lambda \sum_{\zeta \in Y_N} q(\eta^0, \zeta) P_N(\zeta) + \lambda \sum_{\zeta \in Y_N} q(\eta', \zeta) P_N(\zeta) \\
&\quad + (1-\lambda)^2 \mathbf{1}_{k=1}, \\
&= \lambda A_{N-1-k}(\xi 2 1^k) + (1-\lambda) A_{N-1-k}(\xi 0 2 1^{k-1}) \\
&\quad + \lambda a P_N(\xi' 2 1^{k+1}) + \lambda(1-\lambda) P_N(\xi 2 1^{k-1} 0) \\
&\quad + (1-\lambda) a P_N(\xi' 0 1 2 1^{k-1}) + (1-\lambda)^2 \mathbf{1}_{k \geqslant 2} P_N(\xi 0 2 1^{k-2} 0) \\
&\quad + (1-\lambda)^2 \mathbf{1}_{k=1}.
\end{aligned} \tag{A.11}$$

On the other hand,

$$\begin{aligned}
\sum_{\zeta \in Y_{N+1}} q(\eta, \zeta) Q(\zeta) &= \lambda A_{N-1-k}(\xi 2 1^k) + (1-\lambda) A_{N-1-k}(\xi 0 2 1^{k-1}) \\
&\quad + a Q(\xi' 0 1 2 1^k) + (1-\lambda) Q(\xi 0 2 1^{k-1} 0), \\
&= \lambda A_{N-1-k}(\xi 2 1^k) + (1-\lambda) A_{N-1-k}(\xi 0 2 1^{k-1}) \\
&\quad + \lambda a P_N(\xi' 2 1^{k+1}) + (1-\lambda) a P_N(\xi' 0 1 2 1^{k-1}) \\
&\quad + (1-\lambda) P_N(\xi 0 2 1^{k-1}), \\
&= q(\eta) Q(\eta) + (1-\lambda) P_N(\xi 0 2 1^{k-1}) \\
&\quad - \lambda(1-\lambda) P_N(\xi 2 1^{k-1} 0) \\
&\quad - (1-\lambda)^2 \mathbf{1}_{k \geqslant 2} P_N(\xi 0 2 1^{k-2} 0) - (1-\lambda)^2 \mathbf{1}_{k=1}, \\
&= q(\eta) Q(\eta),
\end{aligned} \tag{A.12}$$

using for the last equality

$$\begin{aligned}
P_N(\xi 0 2 1^{k-1}) &= \lambda P_{N-1}(\xi 2 1^{k-1}) + (1-\lambda) \Big[\mathbf{1}_{k \geqslant 2} P_{N-1}(\xi 0 2 1^{k-2}) + \mathbf{1}_{k=1} \Big], \\
&= \lambda P_N(\xi 2 1^{k-1} 0) + (1-\lambda) \Big[\mathbf{1}_{k \geqslant 2} P_N(\xi 0 2 1^{k-2} 0) + \mathbf{1}_{k=1} \Big].
\end{aligned}$$

Case V: $\eta = \xi 0 1^k 2 1^l$ **with** $k \geqslant 1, l \geqslant 0, k + l \leqslant N-1, N \geqslant 3$

The computation, very similar to that of the previous case, gives

$$q(\eta) Q(\eta) = \sum_{\zeta \in Y_{N+1}} q(\eta, \zeta) Q(\zeta) + (1-\lambda) \mathbf{1}_{l=0}. \qquad \blacksquare$$

For $\eta \in X_N$, let $a(\eta)$ be the number of occupied sites, $x(\eta)$ be the rightmost occupied site if $\eta \neq 0^N$ and 0 otherwise, and $d(\eta)$ be the number of empty sites to the left of $x(\eta)$:

$$d(\eta) := \sum_{x=1}^{x(\eta)} (1-\eta(x)) = x(\eta) - a(\eta).$$

Corollary A.2. *For all $N \geqslant 0, \eta \in X_N$, $P_{N+1}(2\eta) = (1-\lambda)^{a(\eta)+1} Q_\eta(\lambda)$ where Q_η is a polynomial of degree $d(\eta)$ satisfying the followings recurrence relations: $\forall \eta \in X_N$,*

$$\begin{aligned} &Q_\varnothing = 1, \\ &Q_\eta = Q_{\eta'}, \text{ if } \eta(N) = 0 \text{ or } \eta = 1^N, \\ &Q_\eta = Q_{\eta'} + \lambda Q_{\eta^0}, \text{ if } \eta(N) = 1. \end{aligned} \tag{A.13}$$

Proof. It is a straightforward induction using Theorem A.2. ■

Corollary A.3. *For all $N \geqslant 0, \eta \in X_{N+1}$ such that $\eta(N+1) = 1$:*

$$Q_\eta = \frac{Q_{\eta'} - Q_{\eta'}(1)\lambda^{d(\eta)+1}}{1-\lambda}.$$

Proof. If $\eta = 1^{N+1}$ then $Q_\eta = 1 = Q_{1^N}$ and the equality is verified. We will show the corollary for other cases by induction. For $N = 0$, there is only one case for which $\eta = 1$. Since $Q_\eta = 1 = Q_{\eta'} = Q_\varnothing$, the equality is again verified.

Assume the corollary true for all $\eta \in X_{N+1}$ and let $\eta \in X_{N+2}$ such that $\eta(N+2) = 1$ and $\eta \neq 1^{N+2}$. We can write $\eta = \xi 1$ with $\xi \neq 1^{N+1}$. Let $\tilde{\xi}$ be the configuration of X_N obtained from ξ by taking out the last empty site: $\tilde{\xi} = \xi'$ if $\xi(N+1) = 0$ and $\tilde{\xi} = \xi^0$ otherwise. Using the previous proposition $Q_\eta = Q_\xi + \lambda Q_{\tilde{\xi}1}$.

Again we distinguish two cases. If $\xi(N+1) = 0$,

$$\begin{aligned} (1-\lambda)Q_\eta &= (1-\lambda)Q_\xi + \lambda(1-\lambda)Q_{\xi'1}, \\ &= (1-\lambda)Q_{\xi'} + \lambda Q_{\xi'} - Q_{\xi'}(1)\lambda^{d(\xi'1)+2}, \\ &= Q_\xi - Q_\xi(1)\lambda^{d(\eta)+1}, \end{aligned} \tag{A.14}$$

since $Q_{\xi'} = Q_\xi$ and $d(\xi'1) = d(\eta) - 1$, using Corollary A.2. If $\xi(N+1) = 1$,

$$\begin{aligned} (1-\lambda)Q_\eta &= (1-\lambda)Q_\xi + \lambda(1-\lambda)Q_{\xi^0 1}, \\ &= Q_{\xi'} - Q_{\xi'}(1)\lambda^{d(\xi)+1} + \lambda Q_{\xi^0} - Q_{\xi^0}(1)\lambda^{d(\xi^0 1)+2}. \end{aligned} \tag{A.15}$$

Since $d(\xi) = d(\eta)$ and $d(\xi^0 1) = d(\eta) - 1$, we get:

$$\begin{aligned} (1-\lambda)Q_\eta &= Q_{\xi'} + \lambda Q_{\xi^0} - (Q_{\xi'}(1) + Q_{\xi^0}(1))\lambda^{d(\eta)+1}, \\ &= Q_\xi - Q_\xi(1)\lambda^{d(\eta)+1}, \end{aligned} \tag{A.16}$$

since $Q_\xi = Q_{\xi'} + \lambda Q_{\xi^0}$, using Corollary A.2.

Finally, in both cases we have $(1-\lambda)Q_\eta = Q_{\eta'} - Q_{\eta'}(1)\lambda^{d(\eta)+1}$. ■

A.2.2 Random configurations

Now we consider the probability $p_N(\lambda)$ of survival of a second class particle starting from site 1 in a system of size $N+1$ for which the initial configuration of first class particles in $\{2,\dots,N+1\}$ is a random configuration with the product measure distribution of density λ. Using Corollary A.2, we obtain the following form for $p_N(\lambda)$:

$$\begin{aligned} p_N(\lambda) &= \sum_{\eta\in X_N} \nu^\lambda(\eta) P_{N+1}(2\eta), \\ &= \sum_{\eta\in X_N} \lambda^{a(\eta)}(1-\lambda)^{N-a(\eta)}(1-\lambda)^{a(\eta)+1} Q_\eta(\lambda), \\ &= (1-\lambda)^{N+1}\sum_{\eta\in X_N} \lambda^{a(\eta)} Q_\eta(\lambda). \end{aligned} \tag{A.17}$$

Proposition A.1. *For all $N \geqslant 0$,*

$$\sum_{\eta\in X_N} \lambda^{a(\eta)} Q_\eta(\lambda) = 1 + \sum_{i=1}^{N} \frac{N+1-i}{i}\binom{N+i}{N+1}\lambda^i.$$

Proof. Let $a_N(\lambda) := \sum_{\eta\in X_N} \lambda^{a(\eta)} Q_\eta(\lambda)$. Using Corollary A.3 and splitting the sum according to the value of $\eta(N+1)$,

$$a_{N+1} = a_N + \sum_{\eta\in X_N} \lambda^{a(\eta 1)} \frac{Q_\eta(\lambda) - Q_\eta(1)\lambda^{d(\eta 1)+1}}{1-\lambda}. \tag{A.18}$$

Then $d(\eta 1) + a(\eta 1) = N+1$, therefore

$$(1-\lambda)a_{N+1} = a_N - a_N(1)\lambda^{N+2}. \tag{A.19}$$

Furthermore, the polynomial family $(a_N(\lambda))_{N\geqslant 0}$ is the unique family satisfying (A.19) with $a_0(\lambda) = 1$.

On the other hand, if $b_N(\lambda)$ is the polynomial on the right-hand side of (A.18), then for all $N \geqslant 1$

$$\begin{aligned} &(1-\lambda)b_{N+1} - b_N \\ &= 1 - \lambda + \sum_{i=1}^{N+1} \frac{N+2-i}{i}\binom{N+1+i}{N+2}\lambda^i \\ &\quad - \sum_{i=2}^{N+2} \frac{N+3-i}{i-1}\binom{N+i}{N+2}\lambda^i \\ &\quad - 1 - \sum_{i=1}^{N} \frac{N+1-i}{i}\binom{N+i}{N+1}\lambda^i, \end{aligned}$$

$$
\begin{aligned}
&= (-1+N+1-N)\lambda \\
&\quad + \sum_{i=2}^{N} \left[\frac{N+2-i}{i}\binom{N+1+i}{i-1} - \frac{N+3-i}{i-1}\binom{N+i}{i-2} \right. \\
&\quad \left. - \frac{N+1-i}{i}\binom{N+i}{i-1} \right] \lambda^i \\
&\quad + \left[\frac{1}{N+1}\binom{2N+2}{N+2} - \frac{2}{N}\binom{2N+1}{N+2} \right] \lambda^{N+1} \\
&\quad + \left[2\binom{2N+2}{N} - \binom{2N+3}{N+1} \right] \lambda^{N+2}, \\
&= -\frac{1}{N+1}\binom{2N+2}{N+2}\lambda^{N+2}.
\end{aligned}
$$

Evaluating the above equation for $\lambda = 1$, we get $b_N(1) = \frac{1}{N+2}\binom{2N+2}{N+1}$ and $(1-\lambda)b_{N+1} - b_N = -b_N(1)\lambda^{N+2}$. Finally, since $b_0(\lambda) = 1$ and $b_1(\lambda) = 1+\lambda$, for all $N \geqslant 0$, $(1-\lambda)b_{N+1} - b_N = -b_N(1)\lambda^{N+2}$ which implies that $b_N(\lambda) = a_N(\lambda)$. ■

A.3 Survival probabilities in infinite volume

Theorem A.3. *For all $\lambda \in [0,1]$, $p_N(\lambda)$ converges to $\frac{1-2\lambda}{1-\lambda}\mathbf{1}_{\lambda<\frac{1}{2}}$ as N goes to infinity. Furthermore:*

- *if $\lambda \neq \frac{1}{2}$, then there exist some constants $c_1, c_2 > 0$ such that for all $N \geqslant 0$, $|p_N(\lambda) - \frac{1-2\lambda}{1-\lambda}\mathbf{1}_{\lambda<\frac{1}{2}}| \leqslant c_1 e^{-c_2 N}$,*

- *if $\lambda = \frac{1}{2}$, then there exist some constant $c_3 > 0$ such that for all $N \geqslant 0$, $|p_N(\frac{1}{2})| \leqslant \frac{c_3}{\sqrt{N}}$.*

Proof. Let

$$F_N(\lambda) := \sum_{i=0}^{N} \binom{N+i}{N}\lambda^i,$$

and

$$u_N(\lambda) := (1-\lambda)^N F_N(\lambda).$$

An easy computation gives

$$a_N(\lambda) = (1-2\lambda)F_{N+1}(\lambda) + R_N(\lambda), \tag{A.20}$$

where

$$R_N(\lambda) := \lambda^{N+1}\left[2\binom{2N+1}{N} - (1-2\lambda)\binom{2N+2}{N+1}\right].$$

From $\binom{2N}{N} \sim_{N\to\infty} 4^N/\sqrt{\pi N}$, we get for all $\lambda \in]0,1]$

$$R_N(\lambda) \sim_{N\to\infty} 2\lambda^{N+2}\frac{4^{N+1}}{\sqrt{\pi N}}.$$

Using $\lambda(1-\lambda) \leqslant \frac{1}{4}$ for $\lambda \in [0,1]$, there exist $c_1', c_2' > 0$ such that for all $N \geqslant 0$

$$(1-\lambda)^{N+1}R_N(\lambda) \leqslant c_1' e^{-c_2' N} \quad \text{if } \lambda \neq \frac{1}{2}, \tag{A.21}$$

and if $\lambda = \frac{1}{2}$, then there exist $c_3' > 0$ such that for all $N \geqslant 0$

$$\frac{1}{2^{N+1}}R_N\left(\frac{1}{2}\right) \leqslant c_3' N^{-\frac{1}{2}}.$$

Therefore

$$p_N\left(\frac{1}{2}\right) = \frac{1}{2^{N+1}}R_N\left(\frac{1}{2}\right) \leqslant c_3' N^{-\frac{1}{2}} \overset{N\to\infty}{\longrightarrow} 0.$$

For all $0 \leqslant p \leqslant N$, $\binom{p}{p} + \cdots + \binom{N}{p} = \binom{N+1}{p+1}$, hence

$$\begin{aligned}
F_{N+1}(\lambda) &= \sum_{i=0}^{N+1}\sum_{k=0}^{i}\binom{N+k}{k}\lambda^i, \\
&= \sum_{k=0}^{N+1}\sum_{i=k}^{N+1}\binom{N+k}{k}\lambda^i, \\
&= \sum_{k=0}^{N+1}\binom{N+k}{k}\lambda^k\frac{1-\lambda^{N+2-k}}{1-\lambda}, \\
&= \frac{1}{1-\lambda}\sum_{k=0}^{N+1}\binom{N+k}{k}(\lambda^k - \lambda^{N+2}).
\end{aligned}$$

Therefore

$$(1-\lambda)F_{N+1}(\lambda) = F_N(\lambda) + \binom{2N+1}{N}\lambda^{N+1} - \binom{2N+2}{N+1}\lambda^{N+2},$$

hence

$$u_{N+1}(\lambda) - u_N(\lambda) = (1-2\lambda)\binom{2N+1}{N}\lambda^{N+1}(1-\lambda)^N.$$

Finally, for every $\lambda \in [0,1]$, $\lambda \neq 1/2$, $u_N(\lambda)$ converges to

$$\begin{aligned}
l_\lambda &= 1 + \lambda(1-2\lambda)\sum_{k=0}^{\infty}\binom{2k+1}{k}[\lambda(1-\lambda)]^k, \\
&= \begin{cases} \frac{1}{1-\lambda} & \text{if } 0 \leqslant \lambda < \frac{1}{2}, \\ 0 & \text{if } \frac{1}{2} < \lambda \leqslant 1, \end{cases}
\end{aligned}$$

as N goes to infinity. From this and using (A.17), (A.20) and (A.21) we get the first part of the result. It remains to show that the convergence of $u_N(\lambda)$ is exponentially fast if $\lambda \neq \frac{1}{2}$. For that, let

$$G_N(\lambda) := \sum_{k=N}^{\infty} \binom{2k+1}{k} [\lambda(1-\lambda)]^k.$$

Since for all k, $\binom{2k+3}{k+1} \leqslant 4\binom{2k+1}{k}$, we have

$$G_{N+1}(\lambda) \leqslant 4\lambda(1-\lambda)G_N(\lambda).$$

Hence there exist some constants $c_4', c_5' > 0$ such that $|u_N(\lambda) - l_\lambda| \leqslant c_4' e^{-c_5' N}$ for every $N \geqslant 0$. ■

In order to finish the proof of Theorem A.1, it remains to show that $p_N(\lambda)$ converges to $p(\lambda)$ as N goes to infinity. In Chapter 3, we proved that if a second-class particle is not died at time t large, then it survives with high probability. Hence, if we consider the standard coupling between the finite (with a large size) and the infinite system, then the two second-class particles will remain coupled for a long time and consequently, the event the second-class particle in the finite system survives and the other dies has very small probability. Hence

$$\lim_{N\to\infty} p_N(\lambda) \leqslant p(\lambda).$$

On the other hand, since, using Theorem A.2, for every $M \geqslant N+1$,

$$p_N(\lambda) = P_M(2\eta 0^{M-N-1}),$$

where η is a random configuration on X_N with product distribution with density λ, it is easy to see that $p_N(\lambda) \geqslant p(\lambda)$ for all $N \geqslant 0$. Hence

$$\lim_{N\to\infty} p_N(\lambda) = p(\lambda) = \frac{1-2\lambda}{1-\lambda}\mathbf{1}_{\lambda \leqslant \frac{1}{2}}.$$

The Corollary A.1 follows using again Theorem A.2 which implies that for all $\eta \in X_N$,

$$P_{N+2}(21\eta) = (1-\lambda)P_{N+1}(2\eta).$$

Therefore, taking η random with product distribution with density λ and letting N go to infinity, we get

$$q(\lambda) = (1-\lambda)p(\lambda).$$

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