Modeling and Imaging of Attenuation in Biological Media

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Problem Statement

Let F be compactly supported in a bounded smooth domain $\Omega \subset \mathbb{R}^d$ with $d = 2, 3$ and boundary $\partial\Omega$.

Problem

Find $supp\{F(x,t)\}$ *given* $\{g_a(y,t) := p_a(y,t) : (y,t) \in \partial\Omega \times [0,T]\}$ *such that :*

$$\begin{cases} \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta - L_a\right) p_a(x,t) = F(x,t), & (x,t) \in \mathbb{R}^d \times \mathbb{R}, \\ p_a(x,t) = 0 = \frac{\partial p_a(x,t)}{\partial t}, & x \in \mathbb{R}^d, t \ll 0, \end{cases}$$

for T sufficiently large.

- $L_a[p_a(x,\cdot)](t) := \frac{1}{\sqrt{2\pi}} p_a * \int_{\mathbb{R}} \left(\kappa^2(\omega) - \frac{\omega^2}{c_0^2}\right) e^{i\omega t} d\omega$
- $\kappa(\omega) = \frac{\omega}{c(\omega)} + ia|\omega|^{\xi},\ 1 \leq \xi \leq 2.$

Problem Statement

Let $\vec{F}$ be compactly supported in a bounded smooth domain $\Omega \subset \mathbb{R}^d$ with $d = 2, 3$ and boundary $\partial\Omega$.

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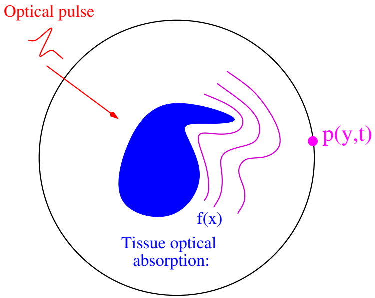
Find $supp\{\vec{F}(x,t)\}$ *given* $\{\vec{g}_a(y,t) := \vec{p}_a(y,t) : (y,t) \in \partial\Omega \times [0,T]\}$ *such that :*

$$\begin{cases} \left(\frac{\partial^2}{\partial t^2} - \frac{\partial}{\partial t}\mathcal{L}_{\eta_\lambda,\eta_\mu} - \mathcal{L}_{\lambda,\mu}\right)\vec{p}_a(x,t) = \vec{F}(x,t), & (x,t) \in \mathbb{R}^d \times \mathbb{R}, \\ \vec{p}_a(x,t) = \vec{0} = \frac{\partial}{\partial t}\vec{p}_a(x,t), & x \in \mathbb{R}^d, t \ll 0, \end{cases}$$

for T *sufficiently large.*

- (λ, μ) : Lamé parameters,
- (η_λ, η_μ) : visco-elastic moduli,
- $\mathcal{L}_{\alpha,\beta}[\vec{p}_a] = (\alpha + \beta)\nabla\nabla \cdot \vec{p}_a - \beta\Delta\vec{p}_a,$

Motivation



- Multi-Physics Imaging
  - *Photo-acoustic imaging, Magneto-acoustic imaging, Acoustic radiation force imaging, Elasticity Imaging,...*
  - *Temporally localized sources i.e.* $F(x,t) = \partial_t \delta(t) f(x)$
- Noise Source Localization
  - *Robotics, Passive Elastography, ...*
  - $F(x,t)$ *is stationary Gaussian process with mean zero.*

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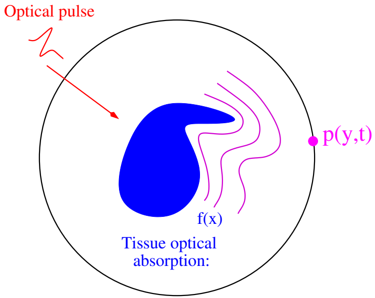

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- Other Applications
  - *Earthquake sources (spatially localized),*
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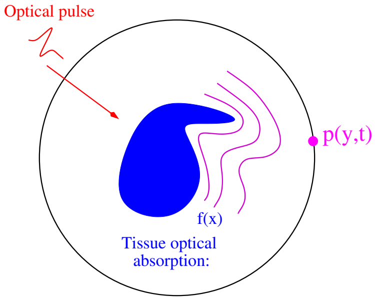

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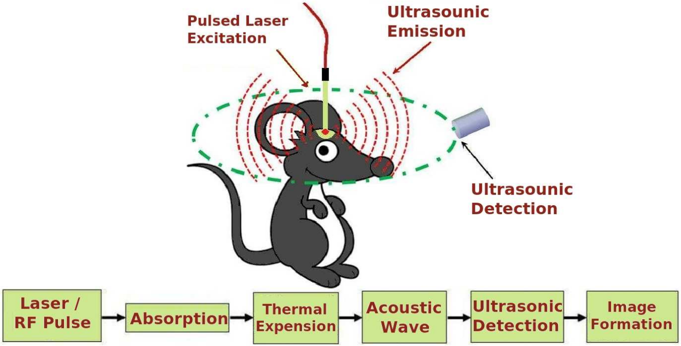
Outlines

- Photo-acoustic Imaging
  - *Radon transform based algorithms*
  - *Attenuation correction*
- Acoustic Time Reversal
  - *Mathematical analysis*
  - *Extension to attenuating media*
  - *Preprocessing techniques*
- Elastic Time Reversal
  - *Weighted time reversal*
  - *Visco-elastic extension*
- Noise Source Localization
  - *Lossless media*
  - *Attenuating media*
  - *Spatial correlation*
- Conclusions and Perspectives

Photo-acoustic Imaging

Photo-acoustic Imaging

Photo-acoustic Imaging
Problem Formulation



Mathematical Formulation

Find the absorbed energy density $f(x)$ *given* $g_0(y,t) := p_0(y,t)$ *for all* $(y,t) \in \partial\Omega \times [0,T]$ *such that*

$$\begin{cases} \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta\right) p_0(x,t) = \frac{\partial \delta(t)}{\partial t} f(x), & (x,t) \in \mathbb{R}^d \times \mathbb{R}, \\ p_0(x,t) = 0 = \frac{\partial p_0(x,t)}{\partial t}, & x \in \mathbb{R}^d, t \ll 0, \end{cases}$$

Photo-acoustic Imaging
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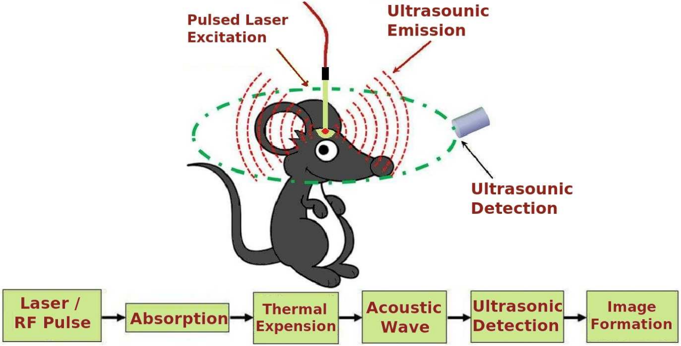

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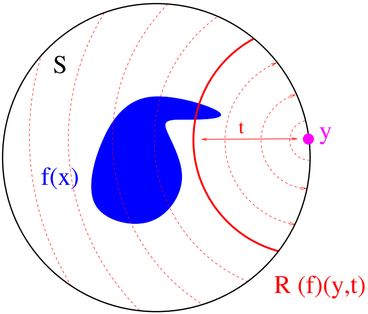
Photo-acoustic Imaging

Spherical Radon transform : $a = 0$, $2D$



- Spherical Radon transform

$$\mathcal{R}_\Omega[f](y,t) = \int_S t f(y + t\omega) d\sigma(\omega)$$

- Kirchhoff formula implies

$$\begin{cases} g_0(y,t) & = \dfrac{1}{2\pi}\dfrac{\partial}{\partial t}\displaystyle\int_0^t \frac{\mathcal{R}_\Omega[f](y,cr)}{\sqrt{t^2 - r^2}} dr, \\ \mathcal{R}_\Omega[f](y,r) & = 4r \displaystyle\int_0^r \frac{g_0(y,t/c)}{\sqrt{r^2 - t^2}} dt. \end{cases}$$

- Therefore,

$$\mathcal{R}_\Omega[f](y,r) = \mathcal{W}[g_0](y,r) := 4r \int_0^r \frac{g_0(y,t/c)}{\sqrt{r^2 - t^2}} dt.$$

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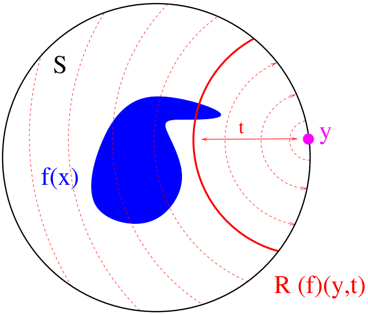

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Photo-acoustic Imaging

Filtered back-projection : $a = 0$, $2D$

- Filtered back-projection formula [04Finch], [07Haltmeier], [07Kunyansky], [09Nguyen]

$$
\begin{aligned}
f(x) &= \frac{1}{(4\pi^2)} \int_{\partial\Omega} \int_0^2 \left[\frac{d^2}{dr^2} \mathcal{R}_\Omega[f](y,r) \right] \ln |r^2 - (y-x)^2| dr\, d\sigma(y) \\
&= \frac{1}{4\pi^2} \mathcal{R}^*_\Omega \left[\mathcal{B}\mathcal{W}[g_0] \right](x),
\end{aligned}
$$

with $\mathcal{R}^*_\Omega[g](x) = \int_{\partial\Omega} g(y, |y-x|) d\sigma(y)$ and $\mathcal{B}[g](y,t) := \int_0^2 \left[\frac{d^2}{dr^2} g(y,r) \right] \ln |r^2 - t^2| dr$

for all $g : \Omega \times \mathbb{R}^+ \to \mathbb{R}$.

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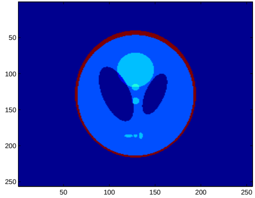


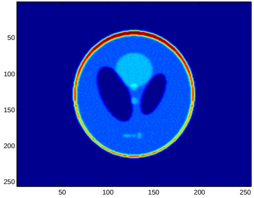
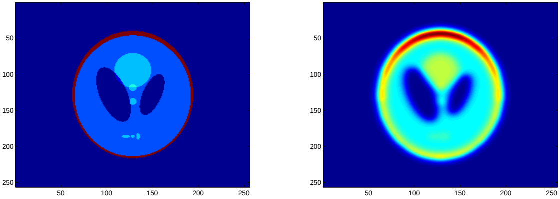


Photo-acoustic Imaging
Acoustic attenuation : $a > 0$



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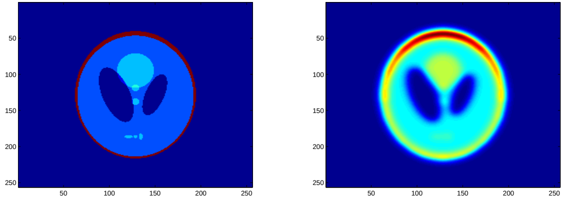

- To take into account acoustic attenuation, let p_a be the solution to

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta - L_a\right) p_a(x,t) = \frac{\partial \delta(t)}{\partial t} f(x)$$

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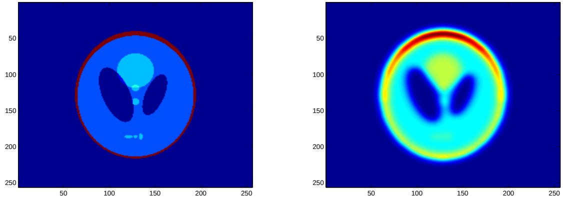

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- The loss term $L_a[p_a]$ is given by [04Sushilov]

$$L_a[p_a(x,\cdot)](t) := \frac{1}{\sqrt{2\pi}} p_a * \int_{\mathbb{R}} \left(\kappa^2(\omega) - \frac{\omega^2}{c^2} \right) e^{i\omega t} d\omega$$

with complex wave number $\kappa(\omega) = \frac{\omega}{c(\omega)} + ia|\omega|^{\xi}, \ 1 \le \xi \le 2.$

Photo-acoustic Imaging

Spherical Radon Transform : $a > 0$

Remark that $\widehat{p}_0(x,\omega)$ and $\widehat{p}_a(x,\omega)$ satisfy

$$\left(\kappa^2(\omega) + \Delta\right)\widehat{p}_a(x,\omega) = \frac{i\omega}{\sqrt{2\pi}c^2}f(x), \quad \text{and} \quad \left(\frac{\omega^2}{c^2} + \Delta\right)\widehat{p}_0(x,\omega) = \frac{i\omega}{\sqrt{2\pi}c^2}f(x).$$

Therefore, $\widehat{p}_a(x,\omega) = \frac{\omega}{c\kappa(\omega)}\widehat{p}_0(x, c\kappa(\omega))$ or $p_a(x,t) = \mathcal{L}[p_0(x,\cdot)](t)$ where

Attenuation Operator

$$\mathcal{L}[\phi](t) = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{\omega}{c\kappa(\omega)} e^{-i\omega t}\left\{\int_{\mathbb{R}+} \phi(s)e^{ic\kappa(\omega)s}ds\right\}d\omega.$$

Photo-acoustic Imaging

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- A natural definition of the spherical Radon transform when $a > 0$ is

$$\mathcal{R}_{\Omega,a}[f] := \mathcal{W}[p_a] = \mathcal{W}\left[\mathcal{L}[p_0]\right]$$

- Then, a *pseudo-inverse* $\mathcal{R}_{\Omega,a}^{-1}$ may be given by

$$\mathcal{R}_{\Omega,a}^{-1} = \mathcal{R}_{\Omega}^{-1}\mathcal{W}\mathcal{L}^{-1}\mathcal{W}^{-1}$$

Photo-acoustic Imaging
SVD-Approach

- Consider a singular value decomposition of $\mathcal{L}$:

$$\mathcal{L}[\phi] = \sum_l \sigma_l \langle \phi, \widetilde{\psi}_l \rangle \psi_l,$$

where $(\widetilde{\psi}_l)$ and (ψ_l) are two orthonormal bases of $L^2(0,T)$ and σ_l are positives eigenvalues such that

$$\begin{cases} \mathcal{L}^*[\phi] & = \sum_l \sigma_l \langle \phi, \psi_l \rangle \widetilde{\psi}_l, \\ \mathcal{L}^*\mathcal{L}[\phi] & = \sum_l \sigma_l^2 \langle \phi, \widetilde{\psi}_l \rangle \widetilde{\psi}_l, \\ \mathcal{L}\mathcal{L}^*[\phi] & = \sum_l \sigma_l^2 \langle \phi, \psi_l \rangle \psi_l. \end{cases}$$

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- An ϵ-approximation inverse of $\mathcal{L}$ is then given by [08Modgil]

$$\mathcal{L}_\epsilon^{-1}[\phi] = \sum_l \frac{\sigma_l}{\sigma_l^2 + \epsilon^2} \langle \phi, \psi_l \rangle \widetilde{\psi}_l,$$

where $\epsilon > 0$.

Photo-acoustic Imaging
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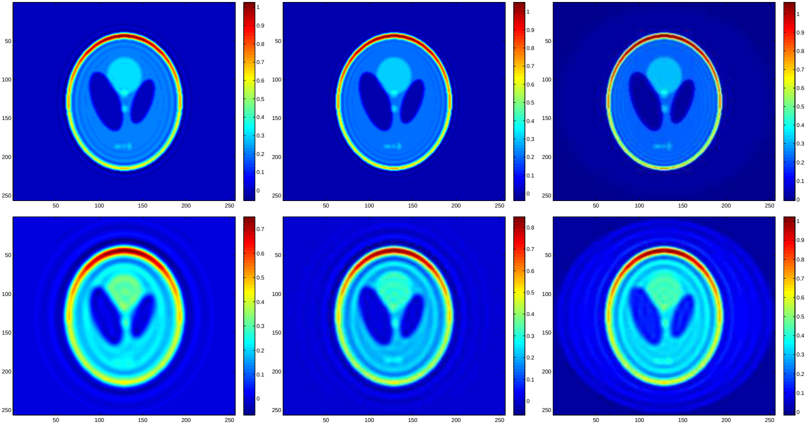
- An ϵ-approximation inverse of spherical Radon transform is given by

$$\mathcal{R}_{\Omega,a,\epsilon}^{-1} = \mathcal{R}_\Omega^{-1} \mathcal{W} \mathcal{L}_\epsilon^{-1} \mathcal{W}^{-1}.$$

Photo-acoustic Imaging

SVD-Approach : Reconstruction using $\mathcal{L}_\epsilon^{-1}$



Top : $a = 0.0005$; Bottom : $a = 0.0025$. Left to right : $\epsilon = 0.01$, $\epsilon = 0.001$ and $\epsilon = 0.0001$.
$N = 256$, $N_R = 200$ and $N_\theta = 200$.

Photo-acoustic Imaging

Asymptotic approach : Quadratic losses ($\xi = 2$)

Approximate thermo-viscous Model : $a \ll \frac{1}{\omega}$, $\xi = 2$, $\kappa(\omega) \simeq \frac{\omega}{c} + ia\frac{\omega^2}{2}$

$$\begin{aligned}
\mathcal{L}[\phi](t) &= \frac{1}{2\pi}\int_{\mathbb{R}+}\phi(s)\int_{\mathbb{R}}\left(1 - i\frac{ac\omega}{2}\right)e^{i\omega(s-t)}e^{-\frac{1}{2}ac\omega^2 s}d\omega ds \\
&= \left(1 + \frac{ac}{2}\frac{\partial}{\partial t}\right)\underbrace{\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}+}\phi(s)\frac{1}{\sqrt{acs}}\exp\left\{-\frac{1}{2}\frac{(t-s)^2}{acs}\right\}ds}_{:=\widetilde{\mathcal{L}}[\phi](t)}.
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Photo-acoustic Imaging

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Theorem (Stationary Phase Theorem [03Hormander])

For $K \subset [0,\infty)$ *compact,* X *its open neighbourhood,* $k \in \mathbb{N}$, $\psi \in \mathcal{C}_0^{2k}(K)$ *and* $h \in \mathcal{C}_0^{3k+1}(X)$ *such that :*

$$\Im m\{h(t)\} \geq 0, \quad \Im m\{h(t_0)\} = 0, \quad h'(t_0) = 0, \quad h''(t_0) \neq 0, \quad \textit{and} \quad h' \neq 0 \textit{ in } K\backslash\{t_0\}$$

we have $$\left|\int_K \psi(t)e^{ih(t)/\epsilon}dt - e^{ih(t_0)/\epsilon}\left(h''(t_0)/2i\pi\epsilon\right)^{-1/2}\sum_{j<k}\epsilon^j D_j[\psi]\right| \leq C\epsilon^k \sum_{\alpha\leq 2k}\sup_x\left|\psi^{(\alpha)}(x)\right|$$

for $\epsilon > 0$, *where*

$$D_j[\psi] = \sum_{\nu-\mu=j}\sum_{2\nu\geq 3\mu} i^{-j}(-1)^\nu \frac{1}{2^\nu\,\nu!\,\mu!}h''(t_0)^{-\nu}\left(\theta_{t_0}^\mu\psi\right)^{(2\nu)}(t_0)$$

with

$$\theta_{t_0}(t) := h(t) - h(t_0) - \frac{1}{2}h''(t_0)(t-t_0)^2.$$

Photo-acoustic Imaging

Asymptotic approach : Stationary phase analysis

Theorem (Asymptotic expansion)

For a sufficiently smooth function ϕ

$$\widetilde{\mathcal{L}}[\phi](t) = \sum_{j=0}^{k} \frac{(ac)^j}{2^j\, j!} \mathcal{D}_j[\phi](t) + o\left(a^k\right) \quad \textit{where} \quad \mathcal{D}_j[\phi] = \left(t^j \phi(t)\right)^{(2i)}.$$

Moreover, an approximate inverse of order k *of* $\widetilde{\mathcal{L}}$ *is given by*

$$\widetilde{\mathcal{L}}_k^{-1}[\psi] = \sum_{j=0}^{k} a^j \psi_{k,j}, \quad \textit{where} \quad \begin{cases} \psi_{k,0} = \psi \\ \psi_{k,j} = -\displaystyle\sum_{m=1}^{j} \frac{c^m}{2^m\, m!} \mathcal{D}_m[\psi_{k,j-m}], \quad \textit{for all} \quad j \le k. \end{cases}$$

Photo-acoustic Imaging

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Moreover, an approximate inverse of order k of $\widetilde{\mathcal{L}}$ is given by

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Consequently,

- An approximate inverse of order k to the spherical Radon transform is given by

$$\mathcal{R}_{\Omega,a,k}^{-1} = \mathcal{R}_{\Omega}^{-1} \mathcal{W} \mathcal{L}_k^{-1} \mathcal{W}^{-1} \quad \text{with} \quad \mathcal{L}_k^{-1} = \widetilde{\mathcal{L}}_k^{-1} \left(1 + \frac{ac}{2} \frac{\partial}{\partial t}\right)^{-1}.$$

Photo-acoustic Imaging

Asymptotic approach : $1 \leq \xi < 2$

- In general

$$\widetilde{\mathcal{L}}[\phi](t) = \int_{\mathbb{R}+} \phi(s) \int_{\mathbb{R}} e^{i\omega(s-t)} e^{-acs|\omega|^{\xi}} d\omega\, ds$$

- Its adjoint $\widetilde{\mathcal{L}}^*$ is given by

$$\widetilde{\mathcal{L}}^*[\phi](t) = \int_{\mathbb{R}+} \phi(s) \int_{\mathbb{R}} e^{i\omega(t-s)} e^{-act|\omega|^{\xi}} d\omega\, ds = \frac{1}{\pi} \int_{\mathbb{R}+} \phi(s) \frac{act}{(act)^2 + (t-s)^2} ds$$

Photo-acoustic Imaging

Asymptotic approach : $1 \leq \xi < 2$

- In general

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Theorem

For a sufficiently smooth function ϕ,

$$\widetilde{\mathcal{L}}[\phi](t) = \phi(t) + C_{\xi}\, ac\mathcal{D}_t^{\xi/2}(t\phi(t)) + o(a),$$

and

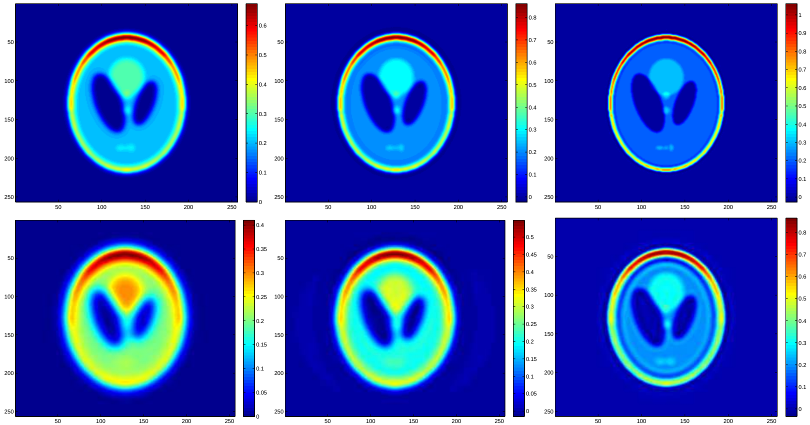
$$\widetilde{\mathcal{L}}^*[\phi](t) = \phi(t) + C_{\xi}\, ac\, t\mathcal{D}_t^{\xi/2}(\phi(t)) + o(a),$$

where C_{ξ} is a constant, depending only on ξ and $\mathcal{D}_t^{\xi/2}$ is defined by

$$\mathcal{D}_t^{\xi/2}[\phi](t) := \frac{1}{\pi} p.v. \int_{\mathbb{R}} \frac{\phi(s) - \phi(t)}{(s-t)^{1+\xi}} ds$$

Photo-acoustic Imaging

Asymptotic approach : Reconstruction using $\widetilde{\mathcal{L}}_k^{-1}$



Top : $a = 0.0005$; Bottom : $a = 0.0025$. Left to Right : $k = 0$; $k = 1$ and $k = 8$.
$N = 256$, $N_R = 200$ and $N_\theta = 200$.

Acoustic Time Reversal

Acoustic Time Reversal

Acoustic Time-reversal

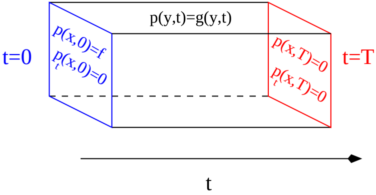
Idea of time-reversal

- The wave equation

$$\partial_{tt} p_0(x,t) - \Delta p_0(x,t) = 0$$

is invariant under time-transformation $t \to T - t$.

- By reciprocity principle, we can re-focus on a (temporally localized) source location. [97Fink], [07Fouque]



Adjoint Wave

Let v be the solution of the wave equation

$$\begin{cases} \partial_{tt} v(x,t) - \Delta v(x,t) = 0, & (x,t) \in \Omega \times (0,T) \\ v(x,0) = 0, \quad \partial_t v(x,0) = 0, & x \in \Omega \\ v(x,t) = g_0(x, T-t), & (x,t) \in \partial\Omega \times [0,T] \end{cases}$$

Then,

$$v(x,t) = p_0(x, T-t), \quad \forall (x,t) \in \Omega \times [0,T], \quad \textit{and} \quad v(x,T) = f(x)$$

Acoustic Time-reversal
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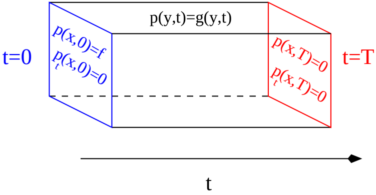

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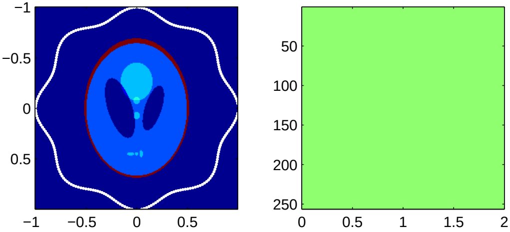
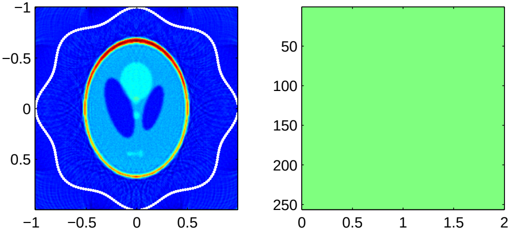
Acoustic Time-reversal
Time-reversal experiment



Simulations carried out by E. Bretin(INSA-Lyon)

Acoustic Time-reversal

Time-reversal experiment



Simulations carried out by E. Bretin(INSA-Lyon)

Acoustic Time-reversal

Integral formulation

Exact Integral Formulation

Green's theorem and integration by parts yield

$$f(x) = v(x, T) = \int_0^T \int_{\partial\Omega} \frac{\partial G_D(x, y, t - T)}{\partial \nu_y} g_0(y, t - T) d\sigma(y) \quad \forall x \in \Omega$$

where G_D is the Dirichlet Green function and v is the adjoint wave.

Modified TR-functional

Let $G_0(x, y, t)$ be the outgoing fundamental solution and $v_s(x, t)$ be such that

$$\begin{cases} \partial_{tt} v_s(x, t) - \Delta v_s(x, t) = \partial_t \delta_s(t) g_0(x, T - s) \delta_{\partial\Omega}(x), & \forall (x, t) \in \mathbb{R}^d \times \mathbb{R}, \\ v_s(x, t) = 0, \quad \partial_t v_s(x, t) = 0 & \forall x \in \mathbb{R}^d, \quad t \ll s. \end{cases}$$

Then, a modified time-reversal functional is given by

$$\mathcal{I}(x) := \int_0^T v_s(x, T) ds = \int_0^T \int_{\partial\Omega} \partial_t G_0(x, y, T - s) g_0(y, T - s) d\sigma(y) ds \quad \forall x \in \Omega.$$

Acoustic Time-reversal

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Acoustic Time-reversal

Integral formulation II

- Remark that $\widehat{g}_0(y) = -i\omega \int_\Omega \widehat{G}_0(z,y) f(z) dz$ for all $y \in \partial\Omega$
- Helmholtz-Kirchhoff Identity : For $x, z \in \Omega$ sufficiently far from $y \in \partial\Omega$

$$\int_{\partial\Omega} \widehat{G}_0(x,y)\overline{\widehat{G}_0}(z,y) d\sigma(y) = \frac{1}{\omega} \Im m\left\{\widehat{G}_0(x,z)\right\}$$

- $\int_{\mathbb{R}} \omega \, \Im m\left\{\widehat{G}_0(x,z)\right\} d\omega = \delta_x(z)$

Acoustic Time-reversal

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- Therefore,

$$\begin{aligned}
\mathcal{I}(x) &= -\frac{1}{2\pi} \int_{\mathbb{R}} \int_{\partial\Omega} i\omega \widehat{G}_0(x,y) \overline{\widehat{g}_0}(y) d\sigma(y) d\omega \\
&= \frac{1}{2\pi} \int_{\mathbb{R}^d} f(z) \int_{\mathbb{R}} \int_{\partial\Omega} \omega^2 \widehat{G}_0(x,y) \overline{\widehat{G}_0}(z,y) d\sigma(y) d\omega dz \\
&\simeq \frac{1}{2\pi} \int_{\mathbb{R}^d} f(z) \int_{\mathbb{R}} \omega \, \Im m\left\{\widehat{G}_0(x,z)\right\} d\omega dz
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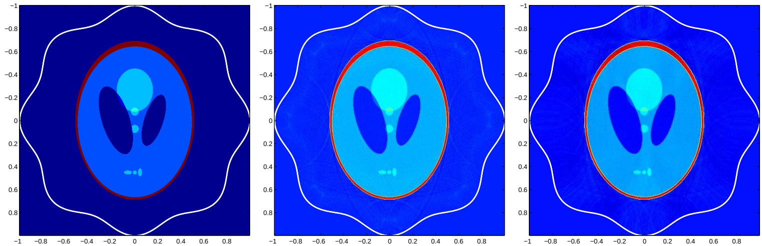
&\simeq \frac{1}{2\pi} \int_{\mathbb{R}^d} f(z) \int_{\mathbb{R}} \omega \, \Im m \{ \widehat{G}_0(x,z) \} d\omega dz
\end{aligned}$$

Theorem

For x far from $\partial\Omega$ (w.r.t. wavelength), we have $\mathcal{I}(x) \simeq f(x)$.

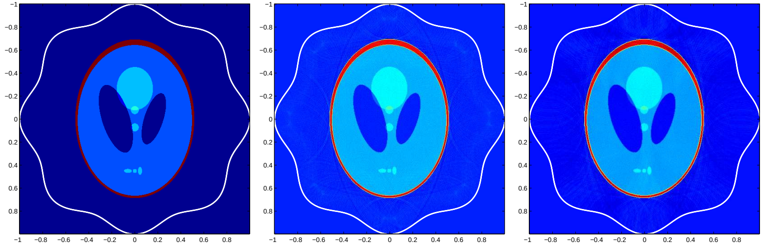
Acoustic Time-reversal

Reconstructions



Left to Right : Initial image, Exact time-reversal, Modified time-reversal

Acoustic Time-reversal
Reconstructions



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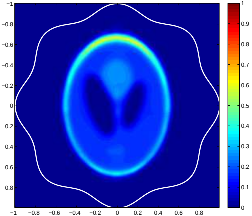

TR in Attenuating Media

Acoustic Time-reversal

TR in attenuating media

Consider the thermo-viscous wave equation

$$\begin{cases} \partial_{tt} p_a(x,t) - \Delta p_a(x,t) - a\partial_t(\Delta p_a(x,t)) = 0 \\ p_a(x,0) = f(x), \quad \text{and} \quad \partial_t p_a(x,0) = 0. \end{cases}$$

Attenuated TR-functional

Define

$$\mathcal{I}_a(x) = \int_0^T v_{s,a}(x,T)ds \quad \forall\, x \in \Omega$$

where $v_{s,a}(x,t)$ is the *solution* of the adjoint attenuated wave equation [07Burgholzer], [10Treeby]

$$\partial_{tt} v_{s,a}(x,t) - \Delta v_{s,a}(x,t) + a\partial_t(\Delta v_{s,a}(x,t)) = \partial_t \delta_s(t) g_a(x, T-s)\delta_{\partial\Omega}(x).$$

Acoustic Time-reversal

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- Highly unstable,
- Order of correction,
- Mathematical justifications.

Acoustic Time-reversal

TR in attenuating media : Truncated functional

- Let $\widehat{\widehat{G}}_a$ be the fundamental solution of the Lossy Helmholtz equation

$$\omega^2 \widehat{\widehat{G}}_a(x, y) + (1 + ia\omega)\,\Delta_y \widehat{\widehat{G}}_a(x, y) = -\delta_x(y) \quad \text{in} \quad \mathbb{R}^d.$$

- Let $\widetilde{G}_{a,\rho}(x, y, t) := \frac{1}{2\pi} \int_{|\omega|<\rho} \widehat{\widehat{G}}_a(x, y) \exp(-i\omega t) d\omega$
- Consider an approximation $v_{s,a,\rho}(x, t)$ of $v_{s,a}(x, t)$ given by :

$$v_{s,a,\rho}(x, t) = \int_{\partial\Omega} \partial_t \widetilde{G}_{a,\rho}(x, y, t - s) g_a(y, T - s) d\sigma(y)$$

Acoustic Time-reversal

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We define :

Truncated TR-Functional

$$\mathcal{I}_{a,\rho}(x) := \int_0^T v_{s,a,\rho}(x, T) ds = \int_0^T \int_{\partial\Omega} \partial_t \widetilde{G}_{a,\rho}(x, y, T - s) g_a(y, T - s) d\sigma(y) ds, \quad x \in \Omega.$$

Acoustic Time-reversal

Attenuation operators

- We have $p_a(x,t) = \mathcal{L}[p(x,\cdot)](t)$, where

$$\mathcal{L}[\phi](t) = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{\kappa(\omega)}{\omega}\left\{\int_{\mathbb{R}} \phi(s) e^{i\kappa(\omega)s} ds\right\} e^{-i\omega t} d\omega.$$

with $\kappa(\omega) = \frac{\omega}{\sqrt{1-ia\omega}}$.

Acoustic Time-reversal
Attenuation operators

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- Moreover we define operator $\widetilde{\mathcal{L}}_\rho$ associated with $\widetilde{\kappa}(\omega) = \frac{\omega}{\sqrt{1+ia\omega}}$ by
$$\widetilde{\mathcal{L}}_\rho[\phi](t) := \frac{1}{2\pi}\int_0^\infty \phi(s)\left\{\int_{|\omega|\le\rho} \frac{\widetilde{\kappa}(\omega)}{\omega} e^{i\widetilde{\kappa}(\omega)s} e^{-i\omega t} d\omega\right\} ds,$$

Acoustic Time-reversal

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- We denote its adjoint operator by $\widetilde{\mathcal{L}}_\rho^*$ given by

$$\widetilde{\mathcal{L}}_\rho^*[\phi](t) = \frac{1}{2\pi}\int_{|\omega|\le\rho} \frac{\widetilde{\kappa}(\omega)}{\omega} e^{i\widetilde{\kappa}(\omega)t}\left\{\int_0^\infty \phi(s)e^{-i\omega s}ds\right\} d\omega.$$

Remark that

$$\partial_t \widetilde{G}_{a,\rho}(x,y,t) = \widetilde{\mathcal{L}}_\rho\left[\partial_t G_0(x,y,.)\right](t)$$

Acoustic Time-reversal

Asymptotic approximation : $a \ll \omega^{-1}$

Proposition

For $\kappa(\omega) \simeq \omega + \frac{ia\omega^2}{2}$ and $a \to 0$ following results hold :

- *Let $\phi(t) \in \mathcal{S}([0,\infty[)$, then*

$$\mathcal{L}[\phi](t) = \phi(t) + \frac{a}{2}\left(t\phi'\right)'(t) + o(a).$$

- *Let $\phi(t) \in \mathcal{D}([0,\infty[)$, then for all $\rho > 0$*

$$\widetilde{\mathcal{L}}^*_\rho[\phi](t) = S_\rho[\phi](t) - \frac{a}{2} S_\rho[(t\phi')'] + o(a).$$

- *Let $\phi(t) \in \mathcal{D}([0,\infty[)$ and $\rho > 0$, then*

$$\widetilde{\mathcal{L}}^*_\rho \mathcal{L}\phi(t) = S_\rho[\phi](t) + o(a).$$

where $\mathcal{S}$ is the Schwartz space, $\mathcal{D}$ is the space of $\mathcal{C}^\infty-$functions of compact support and

$$S_\rho[\phi](t) = \frac{1}{2\pi}\int_{|\omega|\le\rho} e^{-i\omega t}\widehat{\phi}(\omega)d\omega$$

Acoustic Time-reversal

Analysis of truncated functional

Consequently we have

$$\begin{aligned}\mathcal{I}_{a,\rho}(x) &= \int_0^T \int_{\partial\Omega} \partial_t G_0(x,y,t)\widetilde{\mathcal{L}}_\rho^* \left[\mathcal{L}\left[g_0(y,\cdot)\right]\right](t)d\sigma(y)dt \\ &= \int_0^T \int_{\partial\Omega} \partial_t G_0(x,y,t) S_\rho \left[g_0(y,\cdot)\right](t)d\sigma(y)dt + o(a)\end{aligned}$$

Acoustic Time-reversal

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Finally remark that

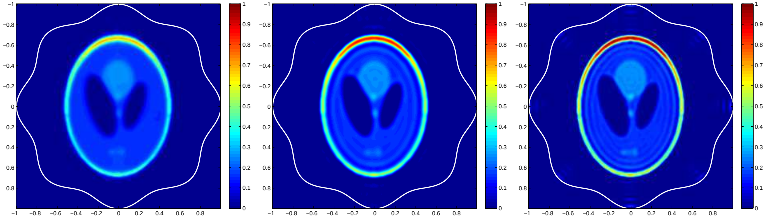
$$\delta_{\rho,x}(z) = \frac{1}{2\pi} \int_{|\omega| \leq \rho} \omega \Im m\left\{\widehat{G}_0(x,z)\right\} d\omega \longrightarrow \delta_x(z) \quad \text{as} \quad \rho \to +\infty.$$

Therefore,

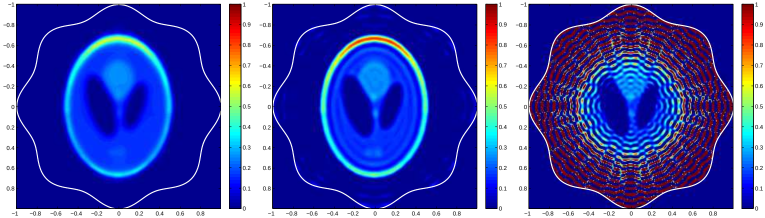
$$\begin{aligned} \mathcal{I}_{a,\rho}(x) &\simeq \delta_{\rho,x}(y) * f(y) + o(a) \\ &\xrightarrow{\rho \to \infty} f(x) + o(a). \end{aligned}$$

Acoustic Time-reversal

Truncated TR-functional : Reconstruction



Test with $a = 0.0005$. Left to Right : Without correction, with correction & $\rho = 15$, with correction & $\rho = 20$.



Test with $a = 0.001$. Left to Right : Without correction, with correction & $\rho = 15$, with correction & $\rho = 20$.

Acoustic Time-reversal
Pre-processing TR-scheme

- As $g_a(y,t) = \mathcal{L}[g_0(y,.](t)$, an alternative strategy is to
  - pre-process the measured data $g_a(y,t)$ using a pseudo-inverse of $\mathcal{L}$ as a filter
  - apply the ideal time-reversal functional $\mathcal{I}(x)$ to identify source location.

Acoustic Time-reversal
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  - apply the ideal time-reversal functional $\mathcal{I}(x)$ to identify source location.
- Using higher order asymptotics :

$$\mathcal{L}[\phi](t) = \sum_{m=0}^{k} \frac{a^m}{m!\, 2^m} \left(t^m \phi'\right)^{(2m-1)}(t) + o(a^k)$$

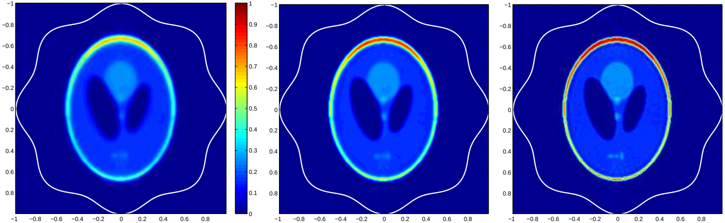
$$\mathcal{L}_k^{-1}[\phi](t) = \sum_{m=0}^{k} a^m \phi_{k,m}(t) \quad \text{such that} \quad \mathcal{L}_k^{-1}\mathcal{L}[\phi](t) = \phi(t) + o(a^k).$$

and $\phi_{k,m}$ verify

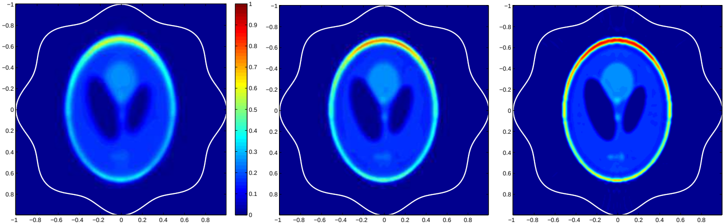
$$\begin{cases} \phi_{k,0} &= \phi \\ \phi_{k,m} &= -\sum_{l=1}^{m} \mathcal{D}_l[\phi_{k,m-l}], \end{cases} \quad \text{and} \quad \mathcal{D}_m\phi(t) = \frac{1}{m!\, 2^m} \left(t^m \phi'\right)^{(2m-1)}(t).$$

Acoustic Time-reversal

Pre-processing TR-scheme : Reconstruction



Test with $a = 0.0005$. Left to Right : Without correction, with correction & $k = 1$, with correction & $k = 4$.



Test with $a = 0.001$. Left to Right : Without correction, with correction & $k = 1$, with correction & $k = 4$.

Elastic Time-reversal

Elastic Time-reversal

Elastic Time-reversal

Ideas of elastic time-reversal

Problem

Find $supp\{\mathbf{f}(x)\}$ *given* $\{\mathbf{g}_0(y,t) := \mathbf{u}_0(y,t) : (y,t) \in \partial\Omega \times [0,T]\}$ *such that :*

$$\begin{cases} (\partial_{tt} - \mathcal{L}_{\lambda,\mu})\,\mathbf{u}_0(x,t) = \partial_t\delta_0(t)\mathbf{f}(x), & (x,t) \in \mathbb{R}^d \times \mathbb{R}, \\ \mathbf{u}_0(x,t) = \mathbf{0},\ \partial_t\mathbf{u}_0(x,t) = \mathbf{0}, & x \in \mathbb{R}^d, t \ll 0, \end{cases}$$

for T *sufficiently large and*

$$\mathcal{L}_{\alpha,\beta}[\mathbf{u}] = (\alpha+\beta)\nabla\nabla\cdot\mathbf{u} - \beta\Delta\mathbf{u}.$$

Elastic TR-functional

An elastic time-reversal functional is given by

$$\mathcal{I}(x) := \int_0^T \mathbf{v}_s(x,T)ds,$$

where $v_s(x,t)$ *is the adjoint elastic wave :*

$$\begin{cases} \partial_{tt}\mathbf{v}_s(x,t) - \mathcal{L}_{\lambda,\mu}\mathbf{v}_s(x,t) = \partial_t\delta_s(t)\mathbf{g}_0(x,T-s)\delta_{\partial\Omega}(x), & \forall(x,t) \in \mathbb{R}^d \times \mathbb{R}, \\ \mathbf{v}_s(x,t) = \mathbf{0}, \quad \partial_t\mathbf{v}_s(x,t) = \mathbf{0} & \forall x \in \mathbb{R}^d, \quad t \ll s. \end{cases}$$

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Elastic Time-reversal

Integral formulation and Green's Tensors

Integral formulation

$$\mathcal{I}(x) := \Re e \left\{ \frac{1}{2\pi} \int_{\mathbb{R}^d} \int_{\mathbb{R}} \omega^2 \left[\int_{\partial\Omega} \widehat{\mathbb{G}}(x,y) \overline{\widehat{\mathbb{G}}}(y,z) d\sigma(y) \right] d\omega \mathbf{f}(z)\, dz \right\}$$

- We have defined $\widehat{\mathbb{G}}(x,y) := \widehat{\mathbb{G}}_0(x-y)$ such that $\widehat{\mathbb{G}}_0(x-y)$ is the fundamental solution of the time harmonic wave equation *i.e.*

$$\left(\mathcal{L}_{\lambda,\mu} + \omega^2\right) \widehat{\mathbb{G}}_0(x) = -\delta_{\mathbf{0}}(x)\mathbb{I}, \quad x \in \mathbb{R}^d.$$

- It can be expressed as

$$\widehat{\mathbb{G}}_0(x) = \frac{1}{\mu\kappa_s^2} \left(\kappa_s^2 \widehat{G}_0^s(x)\mathbb{I} + \mathbb{D}\left(\widehat{G}_0^s - \widehat{G}_0^p\right)(x) \right), \qquad x \in \mathbb{R}^d,$$

where

- $\mathbb{D} = (\partial_i \partial_j)_{i,j=1}^d$,
- $\kappa_s^2 = \omega^2 \mu^{-1}$ and $\kappa_p^2 = \omega^2(\lambda + 2\mu)^{-1}$,
- $[\Delta + \kappa_\alpha^2]\widehat{G}_0^\alpha(x) = -\delta(x)$ in $\mathbb{R}^d$,
- $\alpha = p, s$.

Elastic Time-reversal

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Elastic Time-reversal
Helmholtz-Kirchhoff identities

Let $\widehat{\mathbb{G}}^p$ and $\widehat{\mathbb{G}}^s$ be the divergence and curl free parts of $\widehat{\mathbb{G}}$ such that $\widehat{\mathbb{G}}(x) = \widehat{\mathbb{G}}^p(x) + \widehat{\mathbb{G}}^s(x)$. Then

Proposition (Elastic HK-identities)

For all $x, z \in \Omega$, we have

1. $$\int_{\partial\Omega} \left[\frac{\partial \widehat{\mathbb{G}}(x,y)}{\partial \nu} \overline{\widehat{\mathbb{G}}}(y,z) - \widehat{\mathbb{G}}(x,y) \frac{\partial \overline{\widehat{\mathbb{G}}}(y,z)}{\partial \nu} \right] d\sigma(y) = 2i\Im m \left\{ \widehat{\mathbb{G}}(x,z) \right\}.$$
2. $$\int_{\partial\Omega} \left[\frac{\partial \widehat{\mathbb{G}}^{\alpha}(x,y)}{\partial \nu} \overline{\widehat{\mathbb{G}}^{\alpha}}(y,z) - \widehat{\mathbb{G}}^{\alpha}(x,y) \frac{\partial \overline{\widehat{\mathbb{G}}^{\alpha}}(y,z)}{\partial \nu} \right] d\sigma(y) = 2i\Im m \left\{ \widehat{\mathbb{G}}^{\alpha}(x,z) \right\}, \quad \alpha = p, s.$$
3. $$\int_{\partial\Omega} \left[\frac{\partial \widehat{\mathbb{G}}^{s}(x,y)}{\partial \nu} \overline{\widehat{\mathbb{G}}^{p}}(y,z) - \widehat{\mathbb{G}}^{s}(x,y) \frac{\partial \overline{\widehat{\mathbb{G}}^{p}}(y,z)}{\partial \nu} \right] d\sigma(y) = 0.$$

where the co-normal derivative in the outward unit normal direction $\mathbf{n}$ is defined by

$$\frac{\partial \mathbf{u}}{\partial \nu} := \lambda(\nabla \cdot \mathbf{u})\mathbf{n} + \mu\left(\nabla \mathbf{u}^T + (\nabla \mathbf{u}^T)^T\right)\mathbf{n}.$$

Elastic Time-reversal

Helmholtz-Kirchhoff identities II

Proposition

If $\mathbf{n} = \widehat{y - x}$ *and* $|x-y| \gg 1$ *then*

$$\frac{\partial \widehat{\mathbb{G}}^{\alpha}}{\partial \nu}(x,y) = i\omega c_{\alpha} \widehat{\mathbb{G}}^{\alpha}(x,y) + o\left(|x-y|^{-1}\right), \quad \alpha = p, s.$$

where $c_s^2 = \mu$ *and* $c_p^2 = \lambda + 2\mu$ *are shear and pressure wave speeds.*

Lemma

Let $\Omega \subset \mathbb{R}^d$ *be a ball with large radius (w.r.t. wavelength). Then, for all* $x, z \in \Omega$ *sufficiently far from the boundary* $\partial\Omega$*, we have*

1. $\Re e \left\{ \int_{\partial\Omega} \widehat{\mathbb{G}}^{\alpha}(x,y) \overline{\widehat{\mathbb{G}}^{\alpha}}(y,z) d\sigma(y) \right\} \simeq \frac{1}{\omega c_{\alpha}} \Im m \left\{ \widehat{\mathbb{G}}^{\alpha}(x,z) \right\}, \quad \alpha = p, s.$
2. $\Re e \left\{ \int_{\partial\Omega} \widehat{\mathbb{G}}^{s}(x,y) \overline{\widehat{\mathbb{G}}^{p}}(y,z) d\sigma(y) \right\} \simeq 0$

Elastic Time-reversal

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Elastic Time-reversal
Analysis of TR-functional

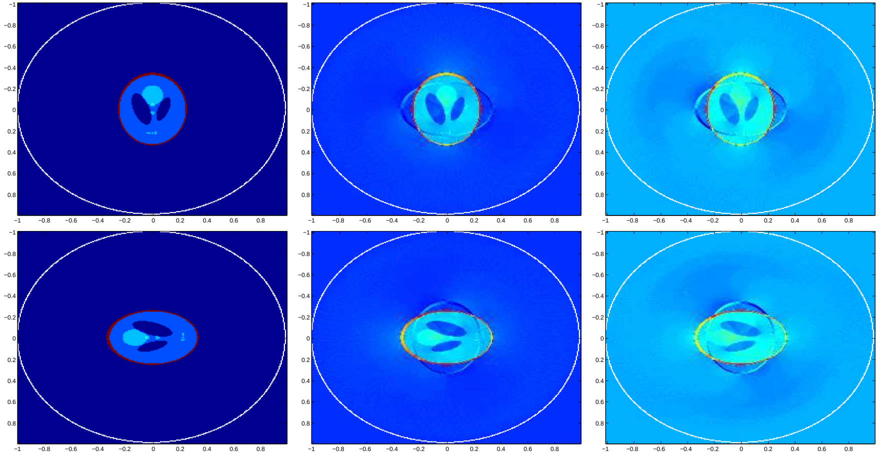
- For x far from $\partial\Omega$,

$$\begin{aligned}
\mathcal{I}(x) &= \Re e\left\{\frac{1}{2\pi}\int_{\mathbb{R}^d}\int_{\mathbb{R}}\omega^2\left[\int_{\partial\Omega}\widehat{\mathbb{G}}(x,y)\overline{\widehat{\mathbb{G}}}(y,z)d\sigma(y)\right]d\omega\mathbf{f}(z)\,dz\right\} \\
&\simeq \frac{c_s+c_p}{c_sc_p}\frac{1}{4\pi}\int_{\mathbb{R}^d}\int_{\mathbb{R}}\omega\,\Im m\left\{\left(\widehat{\mathbb{G}}^p+\widehat{\mathbb{G}}^s\right)(x,z)\right\}d\omega\mathbf{f}(z)\,dz \\
&\quad+\frac{c_s-c_p}{c_sc_p}\frac{1}{4\pi}\int_{\mathbb{R}^d}\int_{\mathbb{R}}\omega\Im m\left\{\left(\widehat{\mathbb{G}}^p-\widehat{\mathbb{G}}^s\right)(x,z)\right\}d\omega\mathbf{f}(z)\,dz \\
&\simeq \frac{c_s+c_p}{2c_sc_p}\mathbf{f}(x)+\frac{c_s-c_p}{2c_sc_p}\int_{\mathbb{R}^d}\mathbb{B}(x,z)\mathbf{f}(z)dz.
\end{aligned}$$

- The operator $\mathbb{B}(x,z) := \frac{1}{2\pi}\int_{\mathbb{R}}\omega\Im m\left\{\left(\widehat{\mathbb{G}}^p-\widehat{\mathbb{G}}^s\right)(x,z)\right\}d\omega$, is not diagonal.
- The reconstruction mixes the components of $\mathbf{f}$ when $c_s \neq c_p$.

Elastic Time-reversal

Elastic TR-functional $\mathcal{I}$: Reconstructions



Left to Right : Initial data, reconstruction with $(\lambda, \mu) = (1, 1)$, with $(\lambda, \mu) = (10, 1)$.

Elastic Time-reversal

Weighted TR-functional

- Let Ψ and Φ be the divergence and the curl free functions respectively such that

$$\mathcal{I} = \nabla \times \Psi + \nabla \Phi.$$

- Define the weighted time-reversal functional by

$$\begin{aligned} \widetilde{\mathcal{I}} &:= c_s \nabla \times \Psi + c_p \nabla \Phi. \\ &= \Re e \left\{ \frac{1}{2\pi} \int_{\mathbb{R}^d} \int_{\mathbb{R}} \omega^2 \left[\int_{\partial\Omega} \left(c_s \widehat{\mathbb{G}}^s(x,y) + c_p \widehat{\mathbb{G}}^p(x,y) \right) \overline{\widehat{\mathbb{G}}}(y,z) d\sigma(y) \right] d\omega \mathbf{f}(z)\, dz \right\} \end{aligned}$$

Theorem

Let $x \in \Omega$ be sufficiently far (w.r.t. wavelength) from the boundary $\partial\Omega$. Then, $\widetilde{\mathcal{I}}(x) \simeq \mathbf{f}(x)$.

Elastic Time-reversal
Weighted TR-functional

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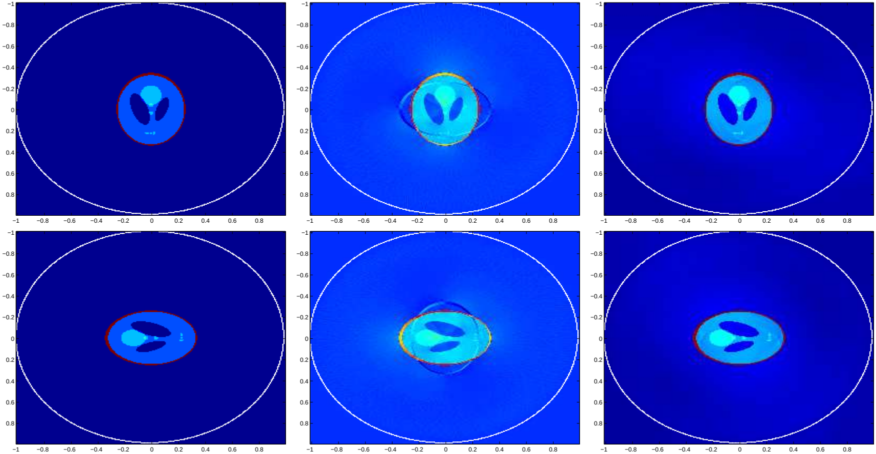
Elastic Time-reversal

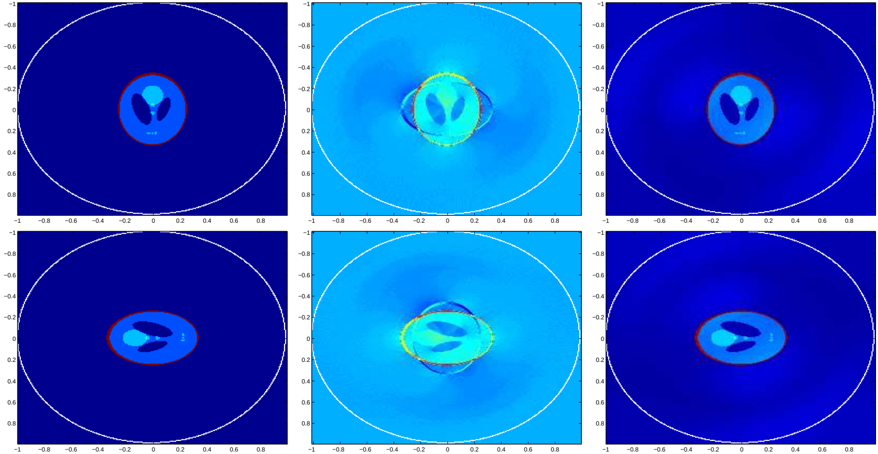
Weighted TR-functional $\widetilde{\mathcal{I}}$: Reconstructions



Reconstruction with $(\lambda, \mu) = (1, 1)$. Left to Right : Initial data, $\mathcal{I}(x)$, $\widetilde{\mathcal{I}}(x)$.

Elastic Time-reversal

Weighted TR-functional $\widetilde{\mathcal{I}}$: Reconstructions



Reconstruction with $(\lambda, \mu) = (10, 1)$. Left to Right : Initial data, $\mathcal{I}(x)$, $\widetilde{\mathcal{I}}(x)$.

Elastic Time-reversal

TR in visco-elastic media

- Consider the Stoke's visco-elastic wave equation with visco-elastic moduli (η_λ, η_μ) *i.e.*

$$\begin{cases} \left(\partial_{tt} - \mathcal{L}_{\lambda,\mu} - \partial_t \mathcal{L}_{\eta_\lambda,\eta_\mu}\right) \mathbf{u}_a(x,t) = \partial_t \delta_0(t) \mathbf{f}(x), & (x,t) \in \mathbb{R}^d \times \mathbb{R}, \\ \mathbf{u}_a(x,0) = \mathbf{0},\ \partial_t \mathbf{u}_a(x,0) = \mathbf{0}, & x \in \mathbb{R}^d, t \ll s. \end{cases}$$

- Define an approximation of the adjoint visco-elastic wave by

$$\mathbf{v}_{s,a,\rho}(x,t) = -\frac{1}{2\pi} \int_{|\omega| \le \rho} \left\{ \int_{\partial\Omega} i\omega \widehat{\mathbb{G}}_{-a}(x,y) \mathbf{g}_a(y, T-s) d\sigma(y) \right\} e^{-i\omega(t-s)} d\omega$$

  where

$$\left(\mathcal{L}_{\lambda,\mu} \pm i\omega \mathcal{L}_{\eta_\lambda,\eta_\mu} + \omega^2\right) \widehat{\mathbb{G}}_{\mp a}(x,y) = -\delta_y(x)\mathbb{I}, \quad x,y \in \mathbb{R}^d.$$

- Define

$$\mathcal{I}_{a,\rho}(x) := \int_0^T \mathbf{v}_{s,a,\rho}(x,T) ds = \int_0^T \int_{\partial\Omega} \partial_t \mathbb{G}_{-a,\rho}(x,y,T-s) \mathbf{g}_a(y,T-s) d\sigma(y) ds,$$

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$$\mathbb{G}_{-a,\rho}(x,y,t) := \frac{1}{2\pi} \int_{|\omega| \le \rho} \widehat{\mathbb{G}}_{-a}(x,y) e^{-i\omega t} d\omega.$$

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$$\widetilde{\mathcal{I}}_{a,\rho}(x) := c_p \nabla \Phi + c_s \nabla \times \Psi$$

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Elastic Time-reversal

Visco-elastic HK-identities

Proposition

Let $\Omega \subset \mathbb{R}^d$ be a ball with large radius. Then,

$$\Re e\left\{\int_{\partial\Omega} \widehat{\mathbb{G}}^s_{-a}(x,y)\overline{\widehat{\mathbb{G}}^p_a}(y,z)d\sigma(y)\right\} \simeq 0$$
$$\Re e\left\{\int_{\partial\Omega} \widehat{\mathbb{G}}^p_{-a}(x,y)\overline{\widehat{\mathbb{G}}^s_a}(y,z)d\sigma(y)\right\} \simeq 0$$

for all $x, z \in \Omega$ sufficiently far from the boundary $\partial\Omega$ w.r.t. wavelength

Theorem

For all $x \in \Omega$ sufficiently far from the boundary $\partial\Omega$, we have

$$\widetilde{\mathcal{I}}_{a,\rho}(x) = \widetilde{\mathcal{I}}_\rho(x) + o(\nu_s^2/c_s^2 + \nu_p^2/c_p^2)$$

where

$$\widetilde{\mathcal{I}}_\rho(x) \overset{\rho\to\infty}{\longrightarrow} \widetilde{\mathcal{I}}(x) \simeq \mathbf{f}(x),$$

ν_s and ν_p are shear and bulk viscosities and

$$\widetilde{\mathcal{I}}_\rho(x) = \int_{\partial\Omega}\int_0^T \partial_t\left[c_s\mathbb{G}^s(x,y,t) + c_p\mathbb{G}^p(x,y,t)\right]S_\rho\left\{\mathbf{g}_0(y,\cdot)\right\}(t)dt\,d\sigma(y)$$

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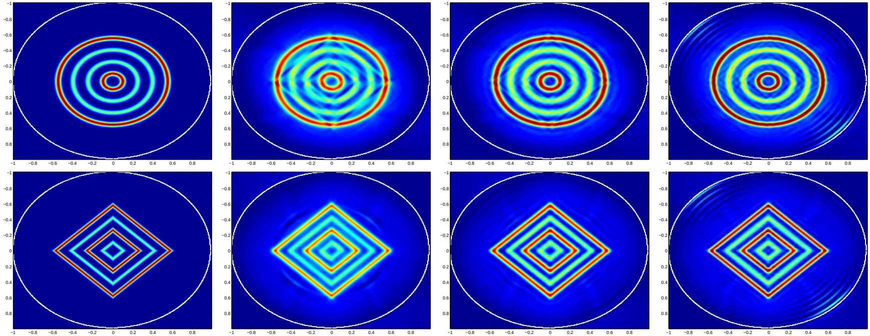
Elastic Time-reversal

Visco-elastic Weighted TR-functional : Reconstructions



Reconstruction with $(\lambda, \mu) = (1, 1)$ and $a = 0.0002$. Left to Right : Initial data, without correction using $\widetilde{\mathcal{I}}(x)$, correction using $\widetilde{\mathcal{I}}_{a,\rho}$ with $\rho = 15$, with $\rho = 20$.

Noise Source Localization

Noise Source Localization

Noise Source Localization

- Let p_0 satisfy the wave equation

$$\begin{cases} \dfrac{1}{c^2(x)} \dfrac{\partial^2}{\partial t^2} p_0(x,t) - \Delta p_0(x,t) = n(x,t), \quad (x,t) \in \mathbb{R}^d \times \mathbb{R} \\ p_0(x,t) = 0, \quad \text{and} \quad \dfrac{\partial}{\partial t} p_0(x,t) = 0, \quad x \in \mathbb{R}^d, t \ll 0, \quad d = 2,3. \end{cases}$$

n is compactly supported in a bounded smooth domain Ω.

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  n is compactly supported in a bounded smooth domain Ω.

- n is a stationary Gaussian process with mean zero and covariance

$$\langle n(x,t)n(y,s)\rangle = F(t-s)K(x)\delta(x-y).$$

Problem

Find $supp\{n\}$ *given* $\{p_0(y,t) : \quad (y,t) \in \partial\Omega \times [0,T]\}$ *for sufficiently large* T.

- $\langle\cdot\rangle$: Statistical average,
- c : Positive, smooth and bounded function,
- F : Time covariance function,
- K : Spatial support of n.

Noise Source Localization

Cross-correlation based functional

Imaging functional[09Garnier]

$$\mathcal{I}(z^S) := \int_{\mathbb{R}} \iint_{\partial\Omega\times\partial\Omega} \widehat{G}_0(x, z^S, \omega)\overline{\widehat{G}_0}(y, z^S, \omega)\widehat{C}_0(x, y, \omega)d\sigma(x)d\sigma(y)d\omega.$$

- $\left(\frac{\omega^2}{c^2(x)} + \Delta\right)\widehat{G}_0(x, y, \omega) = -\delta(x - y), \quad x, y \in \mathbb{R}^d.$
- The statistical cross-correlation C_0 is defined by

$$C_0(x, y, \tau) = \langle p_0(x, t)p_0(y, t + \tau)\rangle = \frac{1}{2\pi}\int_{\mathbb{R}}\left[\int_{\Omega}\overline{\widehat{G}_0}(x, z, \omega)\widehat{G}_0(y, z, \omega)K(z)dz\right]\widehat{F}(\omega)e^{-i\omega\tau}d\omega.$$

Noise Source Localization
Cross-correlation based functional

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- $$\left(\frac{\omega^2}{c^2(x)} + \Delta\right)\widehat{G}_0(x, y, \omega) = -\delta(x-y), \quad x, y \in \mathbb{R}^d.$$
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$$C_0(x, y, \tau) = \langle p_0(x, t)p_0(y, t+\tau)\rangle = \frac{1}{2\pi}\int_{\mathbb{R}}\left[\int_{\Omega}\overline{\widehat{G}_0}(x, z, \omega)\widehat{G}_0(y, z, \omega)K(z)dz\right]\widehat{F}(\omega)e^{-i\omega\tau}d\omega.$$

Theorem

Functional $\mathcal{I}$ gives K up to a smoothing operator, that is

$$\mathcal{I}(z^S) \simeq \int_{\Omega}\mathcal{Q}(z^S, z)K(z)dz,$$

where

$$\mathcal{Q}(z^S, z) = \int_{\mathbb{R}}\frac{\widehat{F}(\omega)}{\omega^2}\Im m\{\widehat{G}_0(z^S, z, \omega)\}^2 d\omega.$$

Noise Source Localization

Weighted imaging functional

- Consider the power spectral density $\mathcal{F}(\omega) = \int_{\partial\Omega} \widehat{C}_0(x, x, \omega) d\sigma(x)$.
- $\widetilde{\mathcal{F}}(\omega) = \frac{1}{\Delta\omega} \int_{\omega-\Delta\omega/2}^{\omega+\Delta\omega/2} \mathcal{F}(\omega')d\omega' \simeq \widehat{F}(\omega) \int_{\Omega} \frac{1}{\omega} \Im m\{\widehat{G}_0(z, z, \omega)\}K(z)dz \simeq \frac{\widehat{F}(\omega)}{4\pi} \int_{\Omega} K(z)dz.$
- Moving frequency window $\Delta\omega$ should be large than $1/T$ and smaller than noise bandwidth.

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- Moving frequency window $\Delta\omega$ should be large than $1/T$ and smaller than noise bandwidth.

Imaging functional

$$\mathcal{I}_W(z^S) := \int_{\mathbb{R}} \frac{W(\omega)}{\widetilde{\mathcal{F}}(\omega)} \iint_{\partial\Omega\times\partial\Omega} \widehat{G}_0(x, z^S, \omega) \overline{\widehat{G}_0}(y, z^S, \omega) \widehat{C}_0(x, y, \omega) d\sigma(x) d\sigma(y) d\omega.$$

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Theorem

$$\mathcal{I}_W(z^S) \simeq \int_{\Omega} \mathcal{Q}_W(z^S, z) \frac{K(z)}{K_0} dz, \quad \textit{with} \quad K_0 = \frac{1}{4\pi} \int_{\Omega} K(z) dz \quad \textit{and}$$

$$\mathcal{Q}_W(z^S, z) = \int_{\mathbb{R}} \frac{W(\omega)}{\omega^2} \Im m\{\widehat{G}_0(z^S, z, \omega)\}^2 d\omega = \begin{cases} \frac{1}{16} \int_{\mathbb{R}} \frac{W(\omega)}{\omega^2} J_0^2(\omega|z|) d\omega, & d = 2 \\ \frac{1}{16\pi^2} \int_{\mathbb{R}} \frac{W(\omega)}{\omega^2} sinc^2(\omega|z|) d\omega, & d = 3. \end{cases}$$

Noise Source Localization
Weighted imaging functional : Remarks

- A potential candidate for W is

$$W(\omega) = \begin{cases} |\omega|^3 \mathbf{1}_{|\omega|<\omega_{\max}}, & d = 2 \\ \omega^2 \mathbf{1}_{|\omega|<\omega_{\max}}, & d = 3. \end{cases}$$

where $\mathbf{1}$ denotes the characteristic function, based on the closure formulae [65Abramowitz]

$$\int_{\mathbb{R}+} \omega J_0^2(\omega|z|) d\omega = \frac{1}{|z|}\delta(z), \quad \text{and} \quad \int_{\mathbb{R}+} \omega^2 sinc^2(\omega|z|) d\omega = \frac{1}{|z|^2}\delta(z).$$

Noise Source Localization
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- $\mathcal{I}_W$ can seen as an application of $\mathcal{I}$ on filtered data $\widetilde{p}_0(x,t)$ where

$$\widehat{\widetilde{p}}_0(x,\omega) := \sqrt{\frac{W(\omega)}{\widetilde{\mathcal{F}}(\omega)}}\widehat{p}_0(x,\omega).$$

where $(-\omega_{\max}, \omega_{\max})$ is the estimated support of $\widetilde{\mathcal{F}}(\omega)$.

Noise Source Localization

Weighted imaging functional : Remarks

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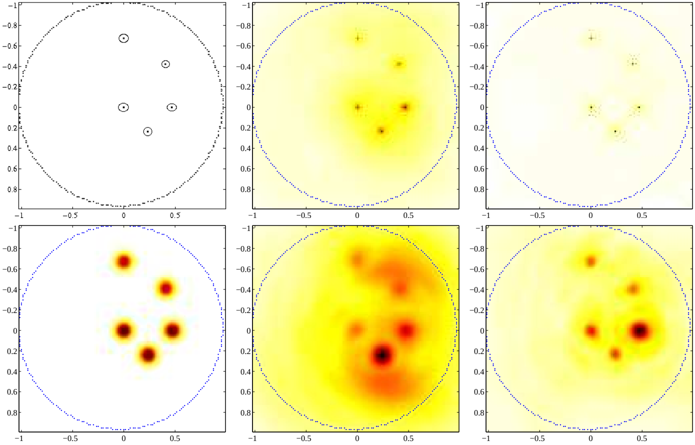
  where $(-\omega_{\max}, \omega_{\max})$ is the estimated support of $\widetilde{\mathcal{F}}(\omega)$.

- Time reversal analogy : Let v be the adjoint wave then,

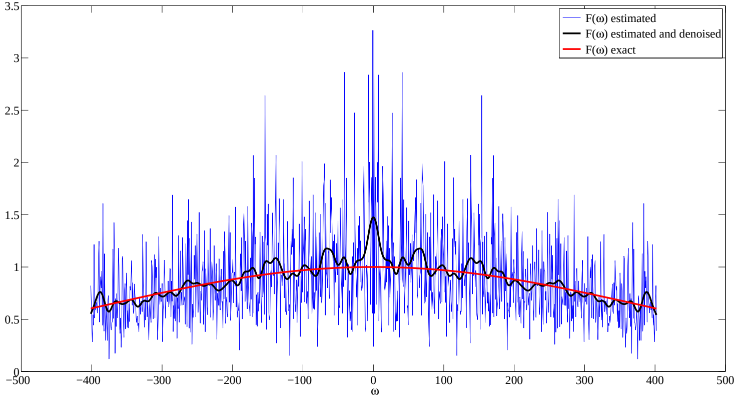
$$\mathcal{I}(z^S) = \int_{\mathbb{R}} \left| \int_{\partial\Omega} \widehat{G}_0(x, z^S, \omega) \overline{\widehat{p}_0}(x,\omega) d\sigma(x) \right|^2 d\omega = 2\pi \int_0^T v(z^S, t)^2 dt,$$

Noise Source Localization
Reconstructions



Top : point sources. Bottom : extended sources. Left to Right : $K(x)$; $\mathcal{I}$; $\mathcal{I}_W$ with $W(\omega) = |\omega|^3 \mathbf{1}_{|\omega|<\omega_{\max}}$.
$T = 8$, $\omega_{\max} = 1000$, $N_x = 2^8$, and $N_t = 2^{11}$.

Noise Source Localization
Estimation of power spectral density



$$\widehat{F}(\omega) = \exp\left(-\pi \frac{\omega^2}{\omega_{\max}^2}\right) \text{ with } \omega_{\max} = 1000.$$

Noise Source Localization
Localization in Attenuating Media

- Let p_a satisfy the thermo-viscous wave equation

$$\begin{cases} \frac{1}{c^2(x)}\frac{\partial^2}{\partial t^2}p_a(x,t) - \Delta p_a(x,t) - \frac{\partial}{\partial t}\Delta p_a(x,t) = n(x,t), & (x,t) \in \mathbb{R}^d \times \mathbb{R} \\ p_a(x,t) = 0, \quad \text{and} \quad \frac{\partial}{\partial t}p_a(x,t) = 0, & x \in \mathbb{R}^d, t \ll 0 \quad d = 2,3. \end{cases}$$

- $C_a(x,y,\tau) = \langle p_a(x,t)p_a(y,t+\tau)\rangle = \frac{1}{2\pi}\int_{\mathbb{R}}\left[\int_{\Omega}\overline{\widehat{G}_a}(x,z,\omega)\widehat{G}_a(y,z,\omega)K(z)dz\right]\widehat{F}(\omega)e^{-i\omega\tau}d\omega.$

- $\left(\frac{\omega^2}{c^2(x)} + (1 \mp ia\omega)\Delta_x\right)\widehat{G}_{\pm a}(x,y,\omega) = -\delta(x-y), \quad x,y \in \mathbb{R}^d.$

Noise Source Localization

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- $\left(\frac{\omega^2}{c^2(x)} + (1 \mp ia\omega)\Delta_x\right)\widehat{G}_{\pm a}(x,y,\omega) = -\delta(x-y), \quad x,y \in \mathbb{R}^d.$
- $\mathcal{I}_a(z^S) := \int_{\mathbb{R}}\iint_{\partial\Omega\times\partial\Omega}\widehat{G}_{-a}(x,z^S,\omega)\overline{\widehat{G}_{-a}}(y,z^S,\omega)\widehat{C}_a(x,y,\omega)d\sigma(x)d\sigma(y)d\omega.$

Truncated imaging functional

$$\mathcal{I}_{a,\rho}(z^S) := \int_{|\omega|\le\rho}\iint_{\partial\Omega\times\partial\Omega}\widehat{G}_{-a}(x,z^S,\omega)\overline{\widehat{G}_{-a}}(y,z^S,\omega)\widehat{C}_a(x,y,\omega)d\sigma(x)d\sigma(y)d\omega.$$

Noise Source Localization

Attenuated Helmholtz-Kirchhoff identities & localization

Lemma

If Ω is a ball with large radius (w.r.t. wavelength) and $c(x) \equiv 1$ outside the ball then

$$\int_{\partial\Omega} \widehat{G}_{-a}(x, z^S, \omega)\overline{\widehat{G}_a}(x, z, \omega) d\sigma(x) \simeq \frac{1}{2i\overline{\kappa_a}(\omega)(1 + ia\omega)} \left(\widehat{G}_{-a}(z, z^S, \omega) - \overline{\widehat{G}_a}(z, z^S, \omega)\right).$$

where

$$\kappa_{\pm a}(\omega) = \frac{\omega}{\sqrt{1 \mp ia\omega}}$$

Noise Source Localization

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where

$$\kappa_{\pm a}(\omega) = \frac{\omega}{\sqrt{1 \mp ia\omega}}$$

Proposition

The truncated imaging functional $\mathcal{I}_{a,\rho}$ satisfies

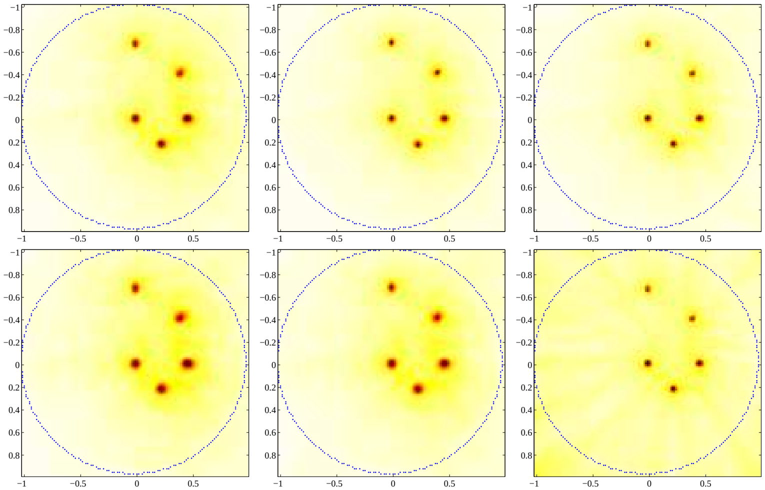
$$\mathcal{I}_{a,\rho}(z^S) := \int_\Omega \mathcal{Q}_\rho(z^S, z)K(z)dz$$

with

$$\mathcal{Q}_\rho(z^S, z) = \int_{|\omega|\le\rho} \frac{\widehat{F}(\omega)}{4\omega^2(1 + a^2\omega^2)^{1/2}}\left|\widehat{G}_{-a}(z, z^S, \omega) - \overline{\widehat{G}_a}(z, z^S, \omega)\right|^2 d\omega.$$

Noise Source Localization

Reconstructions : Point sources in attenuating media



Top : $a = 0.0005$. Bottom :$a = 0.001$. Left to right : $\mathcal{I}_W$, $\mathcal{I}_\rho$ with $\rho = 7.5$, and $\mathcal{I}_\rho$ with $\rho = 15$.
$T = 8$, $\omega_{\max} = 1000$, $N_x = 2^8$, and $N_t = 2^{11}$.

Noise Source Localization
Spatially correlated sources

- Let n be a stationary Gaussian process with mean zero and covariance function
$$\langle n(x,t)n(y,s)\rangle = F(t-s)\Gamma(x,y)$$
where Γ characterizes spatial support and covariance of the sources.

Noise Source Localization

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- $J(z^S, z^{S'}) := \int_{\mathbb{R}} \iint_{\partial\Omega\times\partial\Omega} \widehat{G}_0(x, z^S, \omega)\overline{\widehat{G}_0}(y, z^{S'}, \omega)\widehat{C}_0(x,y,\omega)d\sigma(x)d\sigma(y)d\omega.$

- $C_0(x,y,\tau) = \frac{1}{2\pi}\int_{\mathbb{R}}\left[\iint_{\Omega\times\Omega}\overline{\widehat{G}_0}(x,z,\omega)\widehat{G}_0(y,z',\omega)\Gamma(z,z')dz\,dz'\right]\widehat{F}(\omega)e^{-i\omega\tau}d\omega.$

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- $C_0(x, y, \tau) = \frac{1}{2\pi}\int_{\mathbb{R}} \left[\iint_{\Omega\times\Omega} \overline{\widehat{G}_0}(x, z, \omega)\widehat{G}_0(y, z', \omega)\Gamma(z, z')dz\, dz'\right]\widehat{F}(\omega)e^{-i\omega\tau}d\omega.$

Proposition

$$J(z^S, z^{S'}) := \iint_{\Omega\times\Omega} \underbrace{\int_{\mathbb{R}} \frac{\widehat{F}(\omega)}{\omega^2}\Im m\{\widehat{G}(z, z^S, \omega)\}\Im m\{\widehat{G}(z', z^{S'}, \omega)\}d\omega}_{\Psi(z^S, z^{S'}, z, z')}\,\Gamma(z, z')\,dz\,dz'.$$

In 3D homogeneous media, $\Psi(z^S, z^{S'}, z, z') = \psi(z^S - z, z^{S'} - z')$ *with*

$$\psi(z, z') = \frac{1}{16\pi^2}\int_{\mathbb{R}} \widehat{F}(\omega)sinc(\omega|z|)sinc(\omega|z'|)d\omega.$$

Noise Source Localization

Spatially correlated sources II

- Extended distribution of locally correlated sources : $\Gamma(z,z') = K\left(\frac{z+z'}{2}\right)\gamma(z-z')$. Then

$$\mathcal{I}(z^S) = \int_\Omega \underbrace{\int_{\mathbb{R}} \frac{\widehat{F}(\omega)}{\omega^2} \int_\Omega \Im m\{\widehat{G}(z+\xi/2, z^S, \omega)\} \Im m\{\widehat{G}(z-\xi/2, z^S, \omega)\}\gamma(\xi) d\xi d\omega}_{\Phi(z,z^S)} K(z) dz.$$

Noise Source Localization

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- Correlated point sources : $\Gamma(z,z') = \sum_{i,j=1}^{N_s} \rho_{ij}\delta(z-z_i)\delta(z-z_j)$

  - Find z_i from $\mathcal{I}(z^S) \simeq \sum_{i,j=1}^{N_s} \rho_{ij} \int_{\mathbb{R}} \frac{\widehat{F}(\omega)}{\omega^2} \Im m\{\widehat{G}(z, z^S, \omega)\}^2 d\omega.$
  - Estimate ρ_{ij} from $J(z_i, z_j) = \rho_{ij} \int_{\mathbb{R}} \Im m\{\widehat{G}(z_i, z_i, \omega)\} \Im m\{\widehat{G}(z_j, z_j, \omega)\} d\omega \simeq \rho_{ij} \frac{1}{16\pi} \int_R \widehat{F}(\omega)\omega.$

Noise Source Localization

Spatially correlated sources II

- Extended distribution of locally correlated sources : $\Gamma(z,z') = K\left(\frac{z+z'}{2}\right)\gamma(z-z')$. Then

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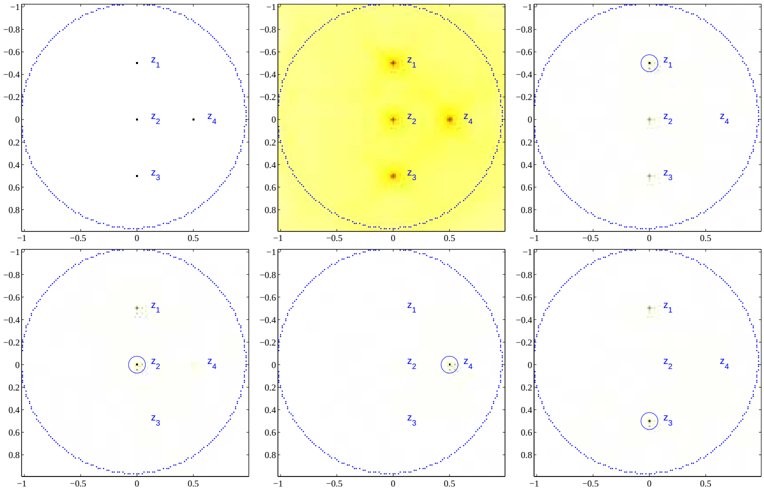
  - Find z_i from $\mathcal{I}(z^S) \simeq \sum_{i,j=1}^{N_s}\rho_{ij}\int_{\mathbb{R}}\frac{\widehat{F}(\omega)}{\omega^2}\Im m\{\widehat{G}(z,z^S,\omega)\}^2 d\omega.$
  - Estimate ρ_{ij} from $J(z_i,z_j) = \rho_{ij}\int_{\mathbb{R}}\Im m\{\widehat{G}(z_i,z_i,\omega)\}\Im m\{\widehat{G}(z_j,z_j,\omega)\}d\omega \simeq \rho_{ij}\frac{1}{16\pi}\int_R \widehat{F}(\omega)\omega.$

$$\rho = \begin{pmatrix} 1 & 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1 & 0 & 0 \\ 1/\sqrt{2} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \widehat{\rho} = \begin{pmatrix} 1.000 & 0.733 & 0.701 & 0.061 \\ 0.733 & 1.000 & 0.049 & 0.061 \\ 0.701 & 0.049 & 1.000 & 0.030 \\ 0.061 & 0.061 & 0.030 & 1.000 \end{pmatrix}$$

Noise Source Localization

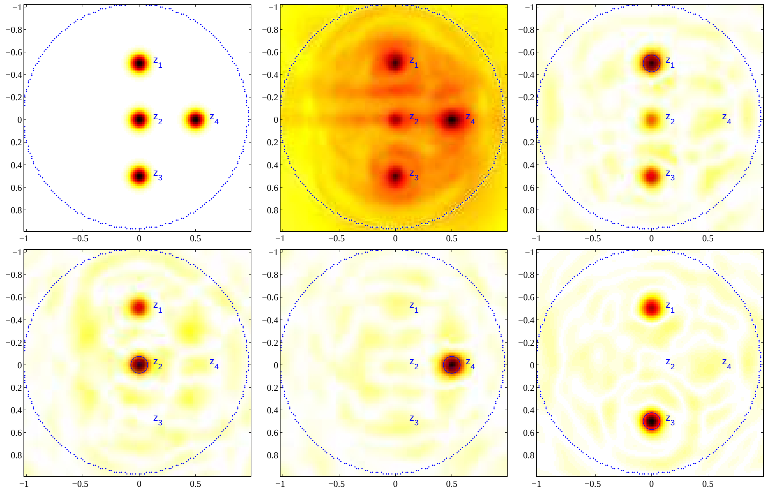
Correlated point sources : Reconstruction



Top : $K(z)$ (left), $\mathcal{I}_W$ with $W(\omega) = |\omega|^3 \mathbf{1}_{|\omega|<\omega_{\max}}$ (middle), and $z \longrightarrow \mathcal{J}_W(z_1, z)$ (right).
Bottom : $z \longrightarrow \mathcal{J}_W(z_2, z)$ (left), $z \longrightarrow \mathcal{J}_W(z_3, z)$ (middle), and $z \longrightarrow \mathcal{J}_W(z_4, z)$ (right).

Noise Source Localization

Extended correlated sources : Reconstruction



Top : $K(z)$ (left), $\mathcal{I}_W$ with $W(\omega) = |\omega|^3 \mathbf{1}_{|\omega|<\omega_{\max}}$ (middle), and $z \longrightarrow \mathcal{J}_W(z_1, z)$ (right).
Bottom : $z \longrightarrow \mathcal{J}_W(z_2, z)$ (left),$z \longrightarrow \mathcal{J}_W(z_3, z)$ (middle), and $z \longrightarrow \mathcal{J}_W(z_4, z)$ (right).

Conclusion and Perspectives

Conclusion and Perspectives

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Conclusion

- Developed Radon transform and time reversal algorithms for Photo-acoustic imaging.
- Justified the use of adjoint of the attenuated wave operator in time reversal.
- Proposed pre-processing technique to compensate for attenuation effects.
- Proposed and justified weighted elastic time-reversal algorithms.
- Proposed weighted algorithms for noise source localization.
- Studied impact of spatial correlation between noise sources.
- Derived Helmholtz-Kirchhoff identities for elastic, viscoelastic and attenuating acoustic media.

Conclusion and Perspectives

Perspectives

- Extension of the algorithms to complex wave propagation models, e.g. by taking into account non-linearity, heterogeneity, anisotropy ...
- Applications : Non-destructive testing, underwater acoustics, telecommunications ...
- Recovery of the attenuation map from attenuated far field measurements ...
- Variable attenuation correction.
- Time reversal with a few transducers : lower bound on the number of transducers for stable reconstructions.
- Passive elastography : Elastic noise source localization in a transversally isotropic medium.

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1. H. Ammari, E. Bretin, J. Garnier, A. Wahab, **Noise source localization in an attenuating medium**, *SIAM Journal on Applied Mathematics*, to appear.

2. H. Ammari, E. Bretin, J. Garnier, A. Wahab, **Time reversal in attenuating acoustic media**, *Mathematical and Statistical Methods for Imaging*, Contemporary Mathematics, vol. 548, pp. 151-163 AMS 2011.

3. H. Ammari, E. Bretin, J. Garnier, A. Wahab, **Time reversal algorithms in viscoelastic media**, *Numerishe Mathematik*, submitted.

4. H. Ammari, E. Bretin, V. Jugnon, A. Wahab, **Photoacoustic imaging for attenuating acoustic media**, *Mathematical Modeling in Biomedical Imaging II*, Lecture Notes in Mathematics, vol. 2035, pp. 57-84 Springer-Verlag, 2011.

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6. E. Bretin, A. Wahab, **Some anisotropic viscoelastic green functions**, *Mathematical and Statistical Methods for Imaging*, Contemporary Mathematics, vol. 548, pp. 129-149, AMS 2011.

It's not the End !

Thank you !