Constructions des bases d'ondelettes de $L^2([0, 1])$ Estimation du paramètre de longue mémoire par la méthode des ondelettes

Hatem Bibi

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- Definitions of Orthogonal multirésolution & biorthogonal multirésolution analysis on $L^2(\mathbb{R})$

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  - Definitions of Orthogonal multirésolution & biorthogonal multirésolution analysis on $L^2(\mathbb{R})$
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  - Construction of biorthogonal wavelet bases on the interval.
  - The study of regular spaces of functions on the interval.

ii) Adaptive semiparametric wavelet estimator and goodness-of-fit test for long-memory Gaussian processes (Gaussian & linear).
  - Definitions & Assumptions,Wavelet coefficient.
  - CLT for the sample variance of wavelet coefficient.

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  - Adaptive lrd parameter estimation.

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First Part: More general constructions of wavelets on the interval.

Definition 1

A multiresolution analysis of $L^2(\mathbb{R})$ is an increasing sequence $(V_j(\mathbb{R}))$, $j \in \mathbb{Z}$, of closed linear subspaces of $L^2(\mathbb{R})$ with the following properties:

i) $\cap_{j\in\mathbb{Z}} V_j(\mathbb{R}) = \{0\}$ *and* $\cup_{j\in\mathbb{Z}} V_j(\mathbb{R})$ *is dense in* $L^2(\mathbb{R})$

ii) $f(x) \in V_j(\mathbb{R}) \Leftrightarrow f(2x) \in V_{j+1}(\mathbb{R})$ *(dilation invariance)*

iii) $f(x) \in V_0(\mathbb{R}) \Leftrightarrow f(x-1) \in V_0(\mathbb{R})$ *(shift invariance)*

iv) $V_0(\mathbb{R})$ *has a shift-invariant Riesz basis* $(\varphi(x-k))_{k\in\mathbb{Z}}$.

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NOTATIONS. We denote by

- MRA: Multiresolution analysis
- OMRA: Orthogonal multiresolution analysis
- BMRA: Biorthogonal multiresolution analysis.

We consider an OMRA $V_j(\mathbb{R})$ of $L^2(\mathbb{R})$ where the scaling function φ has a compact support $[N_1, N_2]$.
We denote
(1.1) $V_j([0,1])$ =Vect$\{\varphi_{j,k\,/[0,1]},\, \varphi_{j,k} \in V_j(\mathbb{R})\,\}$,
and
(1.2) $v_j([0,1])$ =Vect$\{\varphi_{j,k}$, supp$\varphi_{j,k} \subset [0,1]\,\}$.

Let j_0 be an integer such that $2^{j_0} \geq 2(N_2 - N_1 - 1)$.

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Definition 2

A sequence $\{V_j\}_{j \geq j_0}$ of closed subspaces of $L^2([0,1])$ is called a MRA on $L^2([0,1])$ associated with $V_j(\mathbb{R})$ if we have
i) $\forall j \geq j_0,\ v_j([0,1]) \subset V_j \subset V_j([0,1])$
ii) $\forall j \geq j_0,\ V_j \subset V_{j+1}$.

Definition 3

A sequence (V_j, V_j^*) *of closed subspaces of* $L^2([0,1])$ *associated with a BMRA* $(V_j(\mathbb{R}), V_j^*(\mathbb{R}))$ *of* $L^2(\mathbb{R})$ *is called a BMRA of* $L^2([0,1])$ *if*
i) $v_j([0,1]) \subset V_j \subset V_j([0,1])$ *and* $v_j^*([0,1]) \subset V_j^* \subset V_j^*([0,1])$
ii) $V_j \subset V_{j+1}$ *and* $V_j^* \subset V_{j+1}^*$
iii) $L^2([0,1]) = V_j \oplus (V_j^*)^{\perp}$.

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Orthogonal Multiresolution Analysis on the Interval

We consider an OMRA $V_j(\mathbb{R})$ of $L^2(\mathbb{R})$ such that

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Theorem 4

For $j \geq j_0$, the functions $\varphi_{j,k\,/[0,1]}$, $-N_2 + 1 \leq k \leq 2^j - N_1 - 1$, form a Riesz basis of the space $V_j([0,1])$.

We define

$$V_j^T([0,1]) = \{f \in V_j([0,1]) / f|_T = 0\}$$

where $T \subset \{0,1\}$ and $j \geq j_0$.

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- $D_j^T = \{k | -N_2 + 1 \leq k \leq 2^j - N_1 - 1\}$ if $T = \emptyset$.
- $D_j^T = \{k | -N_1 \leq k \leq 2^j - N_1 - 1\}$ if $T = \{0\}$.
- $D_j^T = \{k | -N_2 + 1 \leq k \leq 2^j - N_2\}$ if $T = \{1\}$.
- $D_j^T = \{k | -N_1 \leq k \leq 2^j - N_2\}$ if $T = \{0,1\}$.

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The space $V_j^T([0,1])$ *has an orthonormal basis* $(\varphi_{j,k}^T)$, $k \in D_j^T$ *where*

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$\varphi_{j,k}^T = \varphi_{j,k-2^j+N_2}^{\beta}$, $(2^j - N_2 + 1 \leq k \leq 2^j - N_1 - 1)$,

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$\varphi_{j,k}^T = \varphi_{j,k-2^j+N_2}^{\beta}$, $(2^j - N_2 + 1 \leq k \leq 2^j - N_1 - 1)$,
- *if* $T = \{1\}$, $\varphi_{j,k}^T = \varphi_{j,k+N_2}^{\alpha}$,
$(-N_2 + 1 \leq k \leq -N_1 - 1)$, $\varphi_{j,k}^T = \varphi_{j,k} = 2^{j/2}\varphi(2^j x - k)$,
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- *if* $T = \emptyset$, $\varphi_{j,k}^T = \varphi_{j,k+N_2}^\alpha$, $(-N_2 + 1 \leq k \leq -N_1 - 1)$, $\varphi_{j,k}^T = \varphi_{j,k} = 2^{j/2}\varphi(2^j x - k)$, $(-N_1 \leq k \leq 2^j - N_2)$, $\varphi_{j,k}^T = \varphi_{j,k-2^j+N_2}^\beta (2^j - N_2 + 1 \leq k \leq 2^j - N_1 - 1)$.

Theorem 6

Let $x_j^T([0,1])$ be a supplement of $V_j^T([0,1])$ into $V_{j+1}^T([0,1])$. Then,
i) dim $x_j^T([0,1]) = 2^j$.
ii) $x_j^T([0,1])$ has the following basis:
$\psi_{j,k}, -N_1 \leq k \leq 2^j - N_2; \varphi_{j+1,-N_1+2k}$ *and* $\varphi_{j+1,2^{j+1}-N_2-2k}$,
$0 \leq k \leq \frac{N_2-N_1-1}{2} - 1.$

Proposition 2.1

i) The norms $||f||_2$ and $||P_{j_0}^T f||_2 + (\sum_{j \geq j_0} ||Q_j^T f||_2^2)^{1/2}$ are equivalent on $L^2([0,1])$.
ii) The norms $||f||_{H^1}$ and $||P_{j_0}^T f||_2 + (\sum_{j \geq j_0} 4^j ||Q_j^T f||_2^2)^{1/2}$ are equivalent on $H^{1,T}([0,1])$.

Biorthogonal Multiresolution Analysis on the Interval

Let $(V_j(\mathbb{R}), V_j^*(\mathbb{R}))$ be a BMRA of $L^2(\mathbb{R})$ with multiscale functions (g, g^*). We assume that $suppg = [N_1, N_2]$, and we denote

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$$P_i^{\alpha}(x) = \sum_{k \leq -N_1 - 1} k^i g(x - k),$$

and

$$P_i^{\beta}(x) = \sum_{k \geq -N_2 + 1} k^i g(x - k).$$

Theorem 7

We consider a BMRA (V_j, V_j^*) *of* $L^2([0,1])$. *We assume that*

i) *g is differentiable and* $g^{'}(x) = \widetilde{g}(x) - \widetilde{g}(x-1)$

ii) V_j *contains the functions*

$P_{0,j}^{\alpha}(x) = P_0^{\alpha}(2^j x)_{/[0,1]}$ *and* $P_{0,j}^{\beta}(x) = P_0^{\beta}(2^j x - 2^j)_{/[0,1]}$.

If we denote $\widetilde{V_j} = \{f \in L^2([0,1]) \;/\; \exists\, g \in V_j,\; f = g^{'}\}$ *and* $\widetilde{V_j}^* = \{f \in L^2([0,1]) \;/\; f^{'} \in V_j^*$ *and* $f(0) = f(1) = 0\}$.
Then $(\widetilde{V_j}, \widetilde{V_j}^*)$ *is a BMRA of* $L^2([0,1])$. *Moreover, if we denote by* P_j *(resp* $\widetilde{P}_j$*) the projector from* $L^2([0,1])$ *onto* V_j *(resp* $\widetilde{V}_j$*) parallel to* $(V_j^*)^{\perp}$ *(resp* $(\widetilde{V_j}^*)^{\perp}$*), then we have the following commutation property*

$$\frac{d}{dx} o P_j = \widetilde{P}_j o \frac{d}{dx}.$$

Corollary 8

Let $V_j(\mathbb{R})$ be the OMRA of $L^2(\mathbb{R})$ with the scaling function φ of class C^m $(m \in \mathbb{N}^*)$. We denote by $V_j^{(m)}(\mathbb{R})$ and $V_j^{*(m)}(\mathbb{R})$ the MRA constructed by m derivations and m integrations. Then $V_j^{(m)}([0,1])$ and $V_j^{*(m)}([0,1]) \cap H_0^m([0,1])$ form a BMRA of $L^2([0,1])$. Moreover, if we denote by $P_j^{(m)}$ the projector on $V_j^{(m)}([0,1])$ parallel to $[V_j^{*(m)}([0,1]) \cap H_0^m([0,1])]^\perp$, we have

$$\frac{d}{dx} o P_j^{(m)} = P_j^{(m+1)} o \frac{d}{dx}.$$

Theorem 9

Assume that φ is a $C^{p+\varepsilon}$ -function, $p \in N^*$, $p \geq m$, $\varepsilon > 0$ and let j_0 be an integer satisfying $2^{j_0} - 1 \geq 2N_2 - 2N_1 - 2 + 2p$. Then we have

i) *for* $f \in L^2([0,1])$, $\|f\|_2 \approx \|P_{j_0}^{(m)} f\|_2 + (\sum_{j \geq j_0} \|Q_j^{(m)} f\|_2^2)^{\frac{1}{2}}$.

ii) *For* $f \in L^2([0,1])$, $\|f\|_2 \approx \|P_{j_0}^{*(m)} f\|_2 + (\sum_{j \geq j_0} \|Q_j^{*(m)} f\|_2^2)^{\frac{1}{2}}$.

iii) *For* $s \in \mathbb{Z}$ *such that* $-m \leq s \leq p - m$, *we have*

- $f \in H^s([0,1]) \Leftrightarrow P_{j_0}^{(m)} f \in L^2([0,1])$ *and* $\sum_{j \geq j_0} 4^{js} \|Q_j^{(m)} f\|_2^2 < +\infty$.
- $f \in H_0^{-s}([0,1]) \Leftrightarrow P_{j_0}^{*(m)} f \in L^2([0,1])$ *and* $\sum_{j \geq j_0} 4^{js} \|Q_j^{*(m)} f\|_2^2 < +\infty$.

(Perspectives)

- Biorthogonal Multiresolution Analysis on a Triangle.
- Wavelet Bases on a Pyramid.
- Numerical Applications.

The Gaussian case

$X = (X_t)_{t\in\mathbb{Z}}$ second-order zero-mean stationary process with :

$$\begin{aligned} r(t) &= \mathbb{E}(X_0 \cdot X_t), \quad \text{for } t \in \mathbb{Z}. \\ f(\lambda) &= \frac{1}{2\pi} \cdot \sum_{k\in\mathbb{Z}} r(k) \cdot e^{-ik}, \quad \lambda \in [-\pi, 0)[\cup]0, \pi] \end{aligned}$$

and

$$f(\lambda) \sim C \cdot \frac{1}{\lambda^D} \quad \text{when } \lambda \to 0, \qquad D < 1, \quad C > 0$$

- **Assumption A1:** X is a zero mean stationary Gaussian process with:

$$f(\lambda) = |\lambda|^{-D} \cdot f^*(\lambda) \text{ for all } \lambda \in [-\pi, 0)[\cup]0, \pi],$$

with $f^*(0) > 0$ and $f^* \in \mathcal{H}(D', C_{D'})$ where $0 < D'$, $0 < C_{D'}$ and

$$\mathcal{H}(D', C_{D'}) = \Big\{ g : [-\pi, \pi] \to \mathbb{R}^+ \ / \ |g(\lambda) - g(0)| \leq C_{D'} \cdot |\lambda|^{D'} \text{ for all } \lambda \in [-\pi, \pi] \Big\}.$$

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$$\mathcal{H}'(D', C_{D'}) = \Big\{ g : [-\pi, \pi] \to \mathbb{R}^+ \ / \ g(\lambda) = g(0) + C_{D'} |\lambda|^{D'} + o\big(|\lambda|^{D'}\big) \text{ when } \lambda \to 0 \Big\}.$$

Define $\tilde{W}(\beta, L)$ for $\beta > 0$ and $L > 0$,

$$\tilde{W}(\beta, L) = \Big\{ g(\lambda) = \sum_{\ell \in \mathbb{Z}} g_\ell e^{2\pi i \ell \lambda} \in \mathbb{L}^2([0,1]) \,/\, \sum_{\ell \in \mathbb{Z}} (1 + |\ell|)^\beta |g_\ell| < \infty \text{ and } \sum_{\ell \in \mathbb{Z}} |g_\ell|^2 \leq L \Big\}.$$

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Assumption $W(\infty)$ **:** $\psi : \mathbb{R} \mapsto \mathbb{R}$ *with* $[0,1]$*-support and such that*

1. ψ *is included in the Sobolev class* $\tilde{W}(\infty, L)$ *with* $L > 0$;
2. $\int_0^1 \psi(t)\, dt = 0$ *and* $\psi(0) = \psi(1) = 0$.

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Consequently :

- for all $p > 0$, $\sup_{\lambda \in \mathbb{R}} |\widehat{\psi}(\lambda)|(1 + |\lambda|)^p < \infty$,
- $\widehat{\psi}(u) \sim C\, u$, for $u \to 0$ with $|C| < \infty$ not depending on u.

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Assumption $W(5/2)$ **:** $\psi : \mathbb{R} \mapsto \mathbb{R}$ *with* $[0,1]$*-support and such that*

1. ψ *is included in the Sobolev class* $\tilde{W}(5/2, L)$ *with* $L > 0$;
2. $\int_0^1 \psi(t)\, dt = 0$ *and* $\psi(0) = \psi(1) = 0$.

The wavelet coefficient

If $Y = (Y_t)_{t\in\mathbb{R}}$ is a continuous-time process for $(a,b) \in \mathbb{R}_+^* \times \mathbb{R}$,

$$d(a,b) = \frac{1}{\sqrt{a}} \int_{\mathbb{R}} \psi(\frac{t}{a} - b) Y_t \, dt .$$

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but $X = (X_t)_{t\in\mathbb{Z}}$ then

$$\begin{aligned} e(a,b) &= \frac{1}{\sqrt{a}} \sum_{k=1}^{a} \psi(\frac{k}{a}) X_{k+ab}, \\ \widehat{T}_N(a) &= \frac{1}{[N/a]} \sum_{k=1}^{[N/a]} e^2(a, k-1). \end{aligned}$$

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then under Assumptions we have:

$$\begin{aligned} \mathbb{E}(e^2(a,0)) &\sim K_{(\psi,D)} \cdot a^D \quad \text{when } a \to \infty, \\ \log(\widehat{T}_N(a_N)) &= D \log(a_N) + \log(f^*(0) K_{(\psi,D)}) + \sqrt{\frac{a_N}{N}} \cdot \varepsilon_N, \end{aligned}$$

Corollary 1

Under:

- *A1' and* $W(\infty)$*;*
- **or** *A1' with* $0 < D < 1$, $0 < D' \leq 2$ *and* $W(5/2)$*;*

then $(e(a,b))_{b\in\mathbb{Z}}$ *is a zero mean Gaussian stationary process and*

$$\mathbb{E}(e^2(a,0)) \underset{a\to\infty}{=} f^*(0)\Big(K_{(\psi,D)}\cdot a^D + C_{D'}K_{(\psi,D-D')}\cdot a^{D-D'}\Big) + o\big(a^{D-D'}\big).$$

Corollary 1

Under:

- *A1' and $W(\infty)$;*
- **or** *A1' with $0 < D < 1$, $0 < D' \leq 2$ and $W(5/2)$;*

then $(e(a,b))_{b\in\mathbb{Z}}$ is a zero mean Gaussian stationary process and

$$\mathbb{E}(e^2(a,0)) \underset{a\to\infty}{=} f^*(0)\Big(K_{(\psi,D)} \cdot a^D + C_{D'} K_{(\psi,D-D')} \cdot a^{D-D'}\Big) + o\big(a^{D-D'}\big).$$

Define

$$\tilde{e}(a,b) = \frac{e(a,b)}{\big(f^*(0)K_{(\psi,D)} \cdot a^D\big)^{1/2}} \quad \text{for } a \in \mathbb{N}^* \text{ and } b \in \mathbb{Z}.$$

$$\tilde{T}_N(a) = \frac{1}{[\frac{N}{a}]} \sum_{k=1}^{[\frac{N}{a}]} \tilde{e}^2(a,k-1).$$

Proposition 1

Define $\ell \in \mathbb{N} \setminus \{0,1\}$ *and* $(r_1,\cdots,r_\ell) \in (\mathbb{N}^*)^\ell$*. Let* $(a_n)_{n\in\mathbb{N}}$ *be such that* $N/a_N \underset{N\to\infty}{\longrightarrow} \infty$ *and* $a_N \cdot N^{-1/(1+2D')} \underset{N\to\infty}{\longrightarrow} \infty$*. Under A1 and* $W(\infty)$*,*

$$\sqrt{\frac{N}{a_N}}\Big(\log \tilde{T}_N(r_i a_N)\Big)_{1\le i\le \ell} \xrightarrow[N\to\infty]{d} \mathcal{N}_\ell\big(0\,;\,\Gamma(r_1,\cdots,r_\ell,\psi,D)\big), \qquad (1)$$

with $\Gamma(r_1,\cdots,r_\ell,\psi,D) = (\gamma_{ij})_{1\le i,j\le \ell}$ *the covariance matrix such that*

$$\gamma_{ij} = \frac{8(r_ir_j)^{2-D}}{K^2_{(\psi,D)}d_{ij}} \sum_{m=-\infty}^{\infty}\Big(\int_0^\infty \frac{\widehat{\psi}(ur_i)\overline{\widehat{\psi}}(ur_j)}{u^D}\cos(u\,d_{ij}m)\,du\Big)^2.$$

Proposition 2

Define $\ell \in \mathbb{N} \setminus \{0,1\}$ *and* $(r_1, \cdots, r_\ell) \in (\mathbb{N}^*)^\ell$. *Let* $(a_n)_{n\in\mathbb{N}}$ *be such that* $N/a_N \underset{N\to\infty}{\longrightarrow} \infty$ *and* $a_N \cdot N^{-1/(1+2D')} \underset{N\to\infty}{\longrightarrow} \infty$. *Under* $W(5/2)$ *and A1 with* $D \in (0,1)$ *and* $D' \in (0,2)$*, the CLT (7) holds.*

Proposition 2

Define $\ell \in \mathbb{N} \setminus \{0,1\}$ and $(r_1,\cdots,r_\ell) \in (\mathbb{N}^*)^\ell$. Let $(a_n)_{n\in\mathbb{N}}$ be such that $N/a_N \underset{N\to\infty}{\longrightarrow} \infty$ and $a_N \cdot N^{-1/(1+2D')} \underset{N\to\infty}{\longrightarrow} \infty$. Under $W(5/2)$ and A1 with $D \in (0,1)$ and $D' \in (0,2)$, the CLT (7) holds.

Corollary 2

Under hypothesis of Proposition (6) or (2) and if ψ is such that $m \in \mathbb{N} \setminus \{0,1\}$ is satisfying,

$$\int t^p \psi(t)\, dt = 0 \text{ for all } p \in \{0,1,\ldots,m-1\}$$

the CLT (7) also holds for any process $X' = (X'_t)_{t\in\mathbb{Z}}$ such that for all $t \in Z$, $\mathbb{E}X'_t = P_m(t)$ with $P_m(t) = a_0 + a_1 t + \cdots + a_{m-1}t^{m-1}$ is a polynomial function and $(a_i)_{0\leq i\leq m-1}$ are real numbers.

From proposition 6

$$\left(\log \widehat{T}_N(r_i a_N)\right)_{1\leq i\leq \ell} = A_N \cdot \begin{pmatrix} D \\ K \end{pmatrix} + \frac{1}{\sqrt{N/a_N}} \left(\varepsilon_i\right)_{1\leq i\leq \ell},$$

with $A_N = \begin{pmatrix} \log(r_1 a_N) & 1 \\ \vdots & \vdots \\ \log(r_\ell a_N) & 1 \end{pmatrix}$, $K = -\log(f^*(0)\cdot K_{(\psi,D)})$ and

$\left(\varepsilon_i\right)_{1\leq i\leq \ell} \xrightarrow[N\to\infty]{d} \mathcal{N}_\ell\left(0\,;\,\Gamma(r_1,\cdots,r_\ell,\psi,D)\right)$.

An (log-log regression) estimator $\begin{pmatrix} \widehat{D}(a_N) \\ \widehat{K}(a_N) \end{pmatrix}$ of $\begin{pmatrix} D \\ K \end{pmatrix}$ such that

$$\begin{pmatrix} \widehat{D}(a_N) \\ \widehat{K}(a_N) \end{pmatrix} = (A_N' \cdot A_N)^{-1} \cdot A_N' \cdot Y_{a_N}^{(r_1,\dots,r_\ell)} \quad \text{with} \quad Y_{a_N}^{(r_1,\dots,r_\ell)} = \left(\log \widehat{T}_N(r_i a_N)\right)_{1\leq i\leq \ell}, \tag{2}$$

Proposition 3

Under the Assumptions of the Proposition 6,

$$\sqrt{\frac{N}{a_N}}\left(\begin{pmatrix} \widehat{D}(a_N) \\ \widehat{K}(a_N) \end{pmatrix} - \begin{pmatrix} D \\ K \end{pmatrix}\right) \xrightarrow[N\to\infty]{d} \mathcal{N}_2(0\,;\,\Lambda), \tag{3}$$

with

$$A = \begin{pmatrix} \log(r_1) & 1 \\ \vdots & \vdots \\ \log(r_\ell) & 1 \end{pmatrix}, \quad \Lambda = (A'\cdot A)^{-1}\cdot A'\cdot\Gamma(r_1,\cdots,r_\ell,\psi,D)\cdot A\cdot(A'\cdot A)^{-1}.$$

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Proposition 4

Let X satisfy Assumption A1' with $D\in(-1,1)$ and ψ the assumption $W(\infty)$. Let (a_N) be a sequence such that $a_N=[N^{1/(1+2D')}]$. Then, the estimator $\widehat{D}(a_N)$ is rate optimal in the minimax sense, i.e.

$$\limsup_{N\to\infty}\ \sup_{D\in(-1,1)}\ \sup_{f^*\in\mathcal{H}(D',C_{D'})} N^{\frac{2D'}{1+2D'}}\cdot\mathbb{E}[\widehat{D}(a_N)-D)^2]<+\infty.$$

take

$$a_N = N^\alpha \text{ with } \alpha^* < \alpha < 1 \text{ and } \alpha^* = \frac{1}{1+2D'}, \text{ but } D' = ?.$$

For $\ell \in \mathbb{N} \setminus \{0,1,2\}$, $(r_1,\dots,r_\ell) = (1,\dots,\ell), \alpha \in (0,1)$, define

- $Y_N(\alpha) = \big(\log \widehat{T}_N(i \cdot N^\alpha)\big)_{1 \le i \le \ell}$;
- $A_N(\alpha) = \begin{pmatrix} \log(N^\alpha) & 1 \\ \vdots & \vdots \\ \log(\ell \cdot N^\alpha) & 1 \end{pmatrix}$;
- $Q_N(\alpha, D, K) =$ $\Big(Y_N(\alpha) - A_N(\alpha) \cdot \begin{pmatrix} D \\ K \end{pmatrix}\Big)' \cdot \Big(Y_N(\alpha) - A_N(\alpha) \cdot \begin{pmatrix} D \\ K \end{pmatrix}\Big).$

$$Q_N(\widehat{\alpha}_N, \widehat{D}(a_N), \widehat{K}(a_N)) = \min_{\alpha \in (0,1), D<1, K \in \mathbb{R}} Q_N(\alpha, D, K).$$

$$\text{for} \quad \widehat{\alpha}_N \in \mathcal{A}_N = \Big\{ \frac{2}{\log N}, \frac{3}{\log N}, \dots, \frac{\log[N/\ell]}{\log N} \Big\}.$$

define $\widehat{Q}_N(\alpha) = Q_N(\alpha, \widehat{D}(a_N), \widehat{K}(a_N))$, then $\widehat{Q}_N(\widehat{\alpha}_N) = \min_{\alpha \in \mathcal{A}_N} \widehat{Q}_N(\alpha)$

Proposition 5

Let X satisfy Assumption A1' and ψ Assumption $W(\infty)$ (or Assumption $W(5/2)$ if $0 < D < 1$ and $0 < D' \leq 2$). Then,

$$\widehat{\alpha}_N = \frac{\log \widehat{a}_N}{\log N} \xrightarrow[N\to\infty]{\mathcal{P}} \alpha^* = \frac{1}{1+2D'}.$$

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Let X satisfy Assumption A1' and ψ Assumption $W(\infty)$ (or Assumption $W(5/2)$ if $0 < D < 1$ and $0 < D' \leq 2$). Then,

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Corollary 3

Under hypothesis of Proposition 5, we have (consistency of $\widehat{D'}_N$)

$$\widehat{D'}_N = \frac{1-\widehat{\alpha}_N}{2\widehat{\alpha}_N} \xrightarrow[N\to\infty]{\mathcal{P}} D'.$$

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$$\widehat{D'}_N = \frac{1-\widehat{\alpha}_N}{2\widehat{\alpha}_N} \xrightarrow[N\to\infty]{\mathcal{P}} D'.$$

define $\widehat{\widehat{D}}_N = \widehat{D}(\widehat{a}_N)$, does not satisfy a CLT : $\Pr(\widehat{\alpha}_N \leq \alpha^*) > 0$

Theorem 1

Let X satisfy A1' and ψ $W(\infty)$ (or $W(5/2)$ if $0 < D < 1$ and $0 < D' \leq 2$). Define,

$$\tilde{\alpha}_N = \widehat{\alpha}_N + \frac{3}{(\ell-2)\widehat{D'}_N} \cdot \frac{\log\log N}{\log N},$$

$$\tilde{a}_N = N^{\tilde{\alpha}_N} = N^{\widehat{\alpha}_N} \cdot \big(\log N\big)^{\frac{3}{(\ell-2)\widehat{D'}_N}} \quad \text{and} \quad \tilde{D}_N = \widehat{D}(\tilde{a}_N).$$

Then, with
$\sigma_D^2 = (1\ 0) \cdot (A' \cdot A)^{-1} \cdot A' \cdot \Gamma(1, \cdots, \ell, \psi, D) \cdot A \cdot (A' \cdot A)^{-1} \cdot (1\ 0)',$

$$\sqrt{\frac{N}{N^{\tilde{\alpha}_N}}}\big(\tilde{D}_N - D\big) \xrightarrow[N\to\infty]{d} \mathcal{N}(0\,;\,\sigma_D^2) \tag{4}$$

$$\text{and} \quad \forall \rho > \frac{2(1+3D')}{(\ell-2)D'}, \quad \frac{N^{\frac{D'}{1+2D'}}}{(\log N)^{\rho}} \cdot \big|\tilde{D}_N - D\big| \xrightarrow[N\to\infty]{\mathcal{P}} 0. \tag{5}$$

Properties of $\widehat{\widehat{D}}_N$ and $\tilde{D}_N$

1. $\widehat{\widehat{D}}_N$ and $\tilde{D}_N$ converge at D with a rate of convergence rate equal to the minimax rate of convergence $N^{\frac{D'}{1+2D'}}$ up to a logarithm factor for all $D < -1$ and $D' > 0$
2. $\tilde{D}_N$ satisfies (9) in the CLT 2 then confidence intervals for D can be computed
3. if ψ has m vanishing moments, then $\widehat{\widehat{D}}_N$ and $\tilde{D}_N$ can also be used for a process X with a polynomial trend of degree $\leq m - 1$.

Simulations

1. (fGn) of parameter $H = (D+1)/2$ (for $-1 < D < 1$) and $\sigma^2 = 1$.
2. FARIMA[p,d,q], $d = D/2 \in (-\frac{1}{2}, \frac{1}{2})\sigma^2 = 1$ and $p, q \in \mathbb{N}$.
3. Gaussian stationary process $X^{(D,D')}$,

$$f_3(\lambda) = \frac{1}{\lambda^D}(1 + \lambda^{D'}) \quad \text{for} \lambda \in [-\pi, \pi],\ D \in (-\infty, 1),\ D' \in (0, \infty). \quad (6)$$

- fGn processes with parameters $H = (D+1)/2$ and $\sigma^2 = 1$;
- FARIMA[0,d,0], $d = D/2$ with standard Gaussian innovations;
- FARIMA[1,d,0], $d = D/2$, standard Gaussian innovations and AR coefficient $\phi = 0.95$;
- FARIMA[1,d,1], $d = D/2$, standard Gaussian innovations and AR coefficient $\phi = -0.3$ and MA coefficient $\phi = 0.7$;
- $X^{(D,D')}$ Gaussian processes with $D' = 1$.

Choice of the mother wavelet ψ:

- For $(D \leq 0)$,$\psi_{SM}(t) = (t^2 - t + a)\exp(-1/t(1-t))$ with $a \simeq 0.23087577$. satisfying $W(\infty)$
- For $(0 < D < 1)$, $\psi_{LM}(t) = 100 \cdot t^2(t-1)^2(t^2 - t + 3/14)\mathbb{I}_{0 \leq t \leq 1}$ satisfying $W(5/2)$.

Choice of the parameter ℓ: $\ell = 15$

$N = 10^3$

	$\sqrt{MSE}$	$\ell = 5$	$\ell = 10$	$\ell = 15$	$\ell = 20$	$\ell = 25$	$\{ \ell$ ℓ
fGn	$\widehat{\widehat{D}}_N, \tilde{D}_N$ $\widehat{\alpha}_N, \tilde{\alpha}_N$	0.16, 0.75 0.12, 0.32	0.14, 0.19 0.07, 0.13	0.13, 0.17 0.05, 0.08	**0.14, 0.15** 0.04, 0.05	**0.14, 0.15** **0.04, 0.04**	0.1 0.0
Farima(0, $\frac{D}{2}$, 0)	$\widehat{\widehat{D}}_N, \tilde{D}_N$ $\widehat{\alpha}_N, \tilde{\alpha}_N$	0.21, 0.81 0.14, 0.34	0.15, 0.20 0.07, 0.13	0.14, 0.17 0.05, 0.09	**0.15, 0.15** 0.05, 0.06	**0.15, 0.15** **0.04, 0.04**	0.1 0.0
Farima(1, $\frac{D}{2}$, 0)	$\widehat{\widehat{D}}_N, \tilde{D}_N$ $\widehat{\alpha}_N, \tilde{\alpha}_N$	0.30, 0.96 0.19, 0.44	0.28, 0.35 0.15, 0.24	**0.27, 0.29** 0.12, 0.17	**0.29, 0.27** 0.11, 0.15	0.30, 0.30 **0.11, 0.12**	0.3 0.1
Farima(1, $\frac{D}{2}$, 1)	$\widehat{\widehat{D}}_N, \tilde{D}_N$ $\widehat{\alpha}_N, \tilde{\alpha}_N$	0.60, 0.92 0.17, 0.38	0.43, 0.41 0.11, 0.18	0.39, 0.35 0.09, 0.12	0.36, 0.35 0.07, 0.09	0.32, 0.33 **0.06, 0.07**	**0.2** 0.0
$X^{(D,D')}, D' = 1$	$\widehat{\widehat{D}}_N, \tilde{D}_N$ $\widehat{\alpha}_N, \tilde{\alpha}_N$	0.33, 0.68 0.10, 0.22	0.29, 0.28 **0.10, 0.07**	0.27, 0.26 0.11, 0.07	0.26, 0.27 0.12, 0.12	**0.25, 0.25** 0.13, 0.13	0.2 0.1

$N = 10^4$

	$\sqrt{MSE}$	$\ell = 5$	$\ell = 10$	$\ell = 15$	$\ell = 20$	$\ell = 25$	$\left\{ \begin{array}{l} \ell_1 \\ \ell_2 \end{array} \right.$
fGn	$\widehat{\widehat{D}}_N, \tilde{D}_N$ $\widehat{\alpha}_N, \tilde{\alpha}_N$	0.08, 0.26 0.08, 0.22	0.05, 0.05 0.05, 0.06	0.05, 0.05 **0.04, 0.05**	**0.04, 0.04** **0.04, 0.05**	**0.04, 0.04** 0.05, 0.05	**0.04** **0.04**
Farima(0, $\frac{D}{2}$, 0)	$\widehat{\widehat{D}}_N, \tilde{D}_N$ $\widehat{\alpha}_N, \tilde{\alpha}_N$	0.08, 0.31 0.09, 0.24	0.06, 0.06 0.05, 0.07	**0.05, 0.05** **0.04, 0.05**	**0.05, 0.05** **0.04, 0.05**	**0.05, 0.05** 0.05, 0.05	**0.05** **0.04**
Farima(1, $\frac{D}{2}$, 0)	$\widehat{\widehat{D}}_N, \tilde{D}_N$ $\widehat{\alpha}_N, \tilde{\alpha}_N$	0.13, 0.57 0.15, 0.36	0.10, 0.10 0.09, 0.16	**0.09, 0.08** 0.08, 0.11	**0.09, 0.08** 0.07, 0.09	0.09, 0.09 **0.06, 0.08**	**0.09** 0.08
Farima(1, $\frac{D}{2}$, 1)	$\widehat{\widehat{D}}_N, \tilde{D}_N$ $\widehat{\alpha}_N, \tilde{\alpha}_N$	0.22, 0.63 0.16, 0.38	0.17, 0.15 0.11, 0.17	0.16, 0.13 0.08, 0.11	0.15, 0.14 0.07, 0.09	0.15, 0.14 **0.06, 0.07**	**0.09** 0.08
$X^{(D,D')}, D' = 1$	$\widehat{\widehat{D}}_N, \tilde{D}_N$ $\widehat{\alpha}_N, \tilde{\alpha}_N$	0.23, 0.36 0.10, 0.18	0.19, 0.15 **0.12, 0.08**	0.18, 0.17 0.13, 0.12	0.17, 0.17 0.14, 0.14	**0.15, 0.14** 0.15, 0.15	**0.15** 0.13

$N = 10^5$

	$\sqrt{MSE}$	$\ell = 5$	$\ell = 10$	$\ell = 15$	$\ell = 20$	$\ell = 25$	$\begin{cases} \ell_1 \\ \ell_2 \end{cases}$
fGn	$\widehat{\widehat{D}}_N, \tilde{D}_N$ $\widehat{\alpha}_N, \tilde{\alpha}_N$	0.04, 0.09 0.07, 0.16	0.03, 0.03 **0.06, 0.04**	0.02, 0.03 0.06, 0.06	**0.02, 0.02** 0.07, 0.07	**0.02, 0.02** 0.07, 0.07	**0.0** 0.0
Farima(0, $\frac{D}{2}$, 0)	$\widehat{\widehat{D}}_N, \tilde{D}_N$ $\widehat{\alpha}_N, \tilde{\alpha}_N$	0.03, 0.13 0.07, 0.18	**0.02, 0.02** 0.04, 0.05	**0.02, 0.02** **0.04, 0.03**	**0.02, 0.02** 0.04, 0.04	**0.02, 0.02** 0.05, 0.05	**0.0** **0.0**
Farima(1, $\frac{D}{2}$, 0)	$\widehat{\widehat{D}}_N, \tilde{D}_N$ $\widehat{\alpha}_N, \tilde{\alpha}_N$	0.05, 0.25 0.12, 0.30	0.05, 0.04 0.07, 0.12	0.04, 0.03 0.05, 0.07	0.04, 0.03 0.04, 0.06	0.04, 0.04 **0.04, 0.05**	**0.0** 0.0
Farima(1, $\frac{D}{2}$, 1)	$\widehat{\widehat{D}}_N, \tilde{D}_N$ $\widehat{\alpha}_N, \tilde{\alpha}_N$	0.08, 0.30 0.13, 0.33	0.06, 0.04 0.09, 0.15	0.05, 0.04 0.08, 0.11	0.05, 0.04 0.07, 0.09	0.05, 0.05 **0.06, 0.08**	**0.0** 0.0
$X^{(D,D')}, D' = 1$	$\widehat{\widehat{D}}_N, \tilde{D}_N$ $\widehat{\alpha}_N, \tilde{\alpha}_N$	0.13, 0.19 0.09, 0.15	0.11, 0.08 **0.10, 0.07**	0.10, 0.08 0.11, 0.09	0.09, 0.09 0.12, 0.11	0.09, 0.09 0.13, 0.13	**0.0** 0.1

Consistency of the estimators $\widehat{\alpha}_N$ and $\tilde{\alpha}_N$:

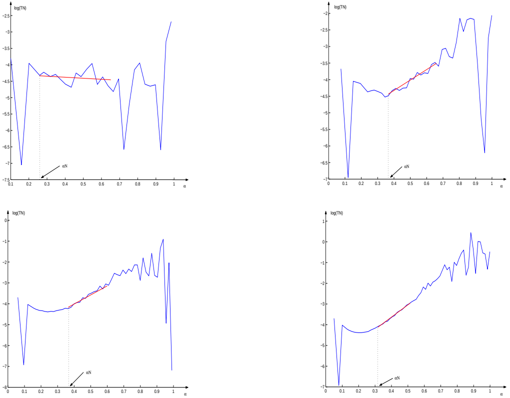

Figure: Log-log graphs for different samples of $X^{(D,D')}$ with $D = 0.5$ and $D' = 1$ when $N = 10^3$ (up and left, $\widehat{\widehat{D}}_N \simeq 1.04$), $N = 10^4$ (up and right, $\widehat{\widehat{D}}_N \simeq 0.66$), $N = 10^5$ (down and left, $\widehat{\widehat{D}}_N \simeq 0.62$) and $N = 10^6$ (down and right, $\widehat{\widehat{D}}_N \simeq 0.54$).

$\log T_N(i \cdot N^\alpha)$ is not a linear function of the logarithm of the scales $\log(i \cdot N^\alpha)$ when N increases and $\alpha < \alpha^*$ (there is a bias). And when $\alpha > \alpha^*$ and α increases, a linear model appears with an increasing error variance

Consistency and distribution of the estimators $\widehat{\widehat{D}}_N$ and $\tilde{D}_N$:

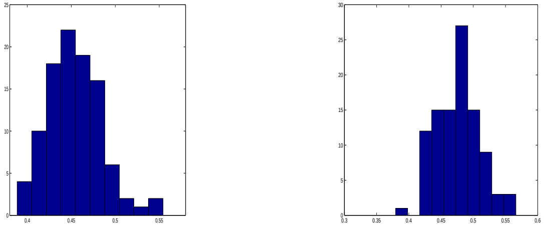

Figure: *Histograms of* $\widehat{\widehat{D}}_N$ *and* $\tilde{D}_N$ *for* 100 *samples of* $FARIMA(1, d, 1)$ *with* $D = 0.5$ *for* $N = 10^5$.

Consistency in case of short memory:

- FARIMA$(0,d,0)$ with $-0.5<d<0$, $(-1<D\leq 0,\ D'=2)$.
- $X^{(D,D')}$ with $D<0$ and $D'>0$.

For different choices of D and D', $N=10^3,\ N=10^4,\ N=10^5$, $\ell_1=15$ and $\ell_2\hat{a}\simeq\frac{N}{10})$.

N		Farima(0,-0.25,0)	$X^{(-1,1)}$	$X^{(-1,3)}$	$X^{(-3,1)}$	$X^{(-3,3)}$
10^3	$\sqrt{MSE}\ \widehat{\widehat{D}}_N,\tilde{D}_N$	0.15, 0.20	0.30, 0.30	0.38, 0.37	0.36, 0.37	0.39, 0.38
10^4	$\sqrt{MSE}\ \widehat{\widehat{D}}_N,\tilde{D}_N$	0.04, 0.04	0.15, 0.14	0.08, 0.08	0.13, 0.14	0.13, 0.13
10^5	$\sqrt{MSE}\ \widehat{\widehat{D}}_N,\tilde{D}_N$	0.03, 0.03	0.06, 0.05	0.04, 0.03	0.04, 0.04	0.03, 0.03

Table: *Estimation of the memory parameter from* 100 *independent samples in case of short memory ($D\leq 0$).*

Robustness of $\widehat{\widehat{D}}_N$, $\tilde{D}_N$:

- $P1$: FARIMA$(0, d, 0)$ with innovations uniform law;
- $P2$: FARIMA$(0, d, 0)$ with innovations distribution with density w.r.t. Lebesgue measure $f(x) = 3/4 * (1 + |x|)^{-5/2}$ for $x \in \mathbb{R}$;
- $P3$! FARIMA$(0, d, 0)$ with innovations Cauchy distribution ;
- $P4$: Gaussian stationary process with a spectral density $f(\lambda) = (|\lambda| - \pi/2)^{-1/2}$ for all $\lambda \in [-\pi, \pi] \setminus \{-\pi/2, \pi/2\}$.

		$P1$	$P2$	$P3$	$P4$
$N = 10^3$	$\sqrt{MSE}$ $\widehat{\widehat{D}}_N$, $\tilde{D}_N$	0.22, 0.23	0.32, 0.41	0.47, 0.76	0.40, 0.41
$N = 10^4$	$\sqrt{MSE}$ $\widehat{\widehat{D}}_N$, $\tilde{D}_N$	0.06, 0.06	0.18, 0.28	0.24, 0.65	0.13, 0.13
$N = 10^5$	$\sqrt{MSE}$ $\widehat{\widehat{D}}_N$, $\tilde{D}_N$	0.02, 0.02	0.02, 0.02	0.14, 0.47	0.03, 0.04

Table: *Estimation of the long-memory parameter from* 100 *independent samples in case of processes* $P1 - 4$ *.*

Comparisons with other estimators

- $\widehat{D}_{BGK}$: "optimal" parametric Whittle estimator (Bhansali it et al., 2006).
- $\widehat{D}_{GRS}$: adaptive local periodogram estimator (Giraitis *et al* (2000)). t
- $\widehat{D}_{MS}$: adaptive global periodogram estimator (Moulines and Soulier (1998, 2003),or FEXP estimator, $\kappa = 2$);
- $\widehat{D}_{R}$: local Whittle estimator (Robinson (1995). $m = N/30$);
- $\widehat{D}_{ATV}$; adaptive wavelet based estimator (Veitch *et al.* (2003) using Db4)
- $\widehat{\widehat{D}}_N$, $\ell_1 = 15$, $\ell_2 = N^{1-\widehat{\alpha}_N}/10$, $\psi(t) = 100 \cdot t^2(t-1)^2(t^2 - t + 3/14)\mathbb{I}_{0 \leq t \leq 1} \qquad \sim W(5/2)$.

$N = 10^3 \longrightarrow$

		$D = 0.1$	$D = 0.3$	$D = 0.5$	$D = 0.7$	$D = 0.9$
fGn ($H = (D+1)/2$)	$\widehat{D}_{BGK}$	**0.089**	0.171	0.259	0.341	0.369
	$\widehat{D}_{GRS}$	0.114	**0.132**	0.147	0.155	0.175
	$\widehat{D}_{MS}$	0.163	0.169	0.181	0.195	0.191
	$\widehat{D}_{R}$	0.211	0.220	0.215	0.218	**0.128**
	$\widehat{D}_{ATV}$	0.176	0.153	0.156	0.164	0.162
	$\widehat{\widehat{D}}_{N}$	0.139	0.147	**0.133**	**0.140**	0.150
FARIMA($0, \frac{D}{2}, 0$)	$\widehat{D}_{BGK}$	**0.094**	0.138	0.239	0.326	0.413
	$\widehat{D}_{GRS}$	0.131	0.139	0.150	0.150	0.162
	$\widehat{D}_{MS}$	0.172	0.167	0.174	0.197	0.188
	$\widehat{D}_{R}$	0.246	0.189	0.223	0.234	0.181
	$\widehat{D}_{ATV}$	0.128	**0.107**	**0.081**	**0.074**	**0.065**
	$\widehat{\widehat{D}}_{N}$	0.161	0.146	0.149	0.149	0.161
FARIMA($1, \frac{D}{2}, 0$)	$\widehat{D}_{BGK}$	**0.146**	**0.203**	0.239	0.236	0.212
	$\widehat{D}_{GRS}$	0.519	0.545	0.588	0.585	0.830
	$\widehat{D}_{MS}$	0.235	0.258	0.256	0.252	0.249
	$\widehat{D}_{R}$	0.242	0.241	**0.234**	**0.202**	0.144
	$\widehat{D}_{ATV}$	0.248	0.267	0.280	0.268	0.375
	$\widehat{\widehat{D}}_{N}$	0.340	0.319	0.314	0.315	0.334
FARIMA($1, \frac{D}{2}, 1$)	$\widehat{D}_{BGK}$	0.204	0.253	0.342	0.363	0.384
	$\widehat{D}_{GRS}$	0.901	0.894	0.866	0.870	0.893
	$\widehat{D}_{MS}$	0.181	**0.175**	**0.180**	**0.185**	0.181
	$\widehat{D}_{R}$	0.204	0.200	0.200	0.191	**0.130**
	$\widehat{D}_{ATV}$	0.392	0.380	0.371	0.343	0.355

$N = 10^4 \longrightarrow$

		$D = 0.1$	$D = 0.3$	$D = 0.5$	$D = 0.7$	$D = 0.9$
fGn $(H = (D+1)/2)$	$\widehat{D}_{BGK}$	0.062	0.143	0.182	0.171	0.182
	$\widehat{D}_{GRS}$	0.040	0.047	0.054	0.068	0.066
	$\widehat{D}_{MS}$	0.069	0.064	0.061	0.071	0.063
	$\widehat{D}_{R}$	0.063	0.055	0.058	0.063	0.052
	$\widehat{D}_{ATV}$	**0.036**	**0.042**	**0.041**	0.047	0.045
	$\widehat{\widehat{D}}_{N}$	0.050	0.040	**0.041**	**0.039**	**0.040**
FARIMA$(0, \frac{D}{2}, 0)$	$\widehat{D}_{BGK}$	0.059	0.141	0.195	0.187	0.178
	$\widehat{D}_{GRS}$	0.042	0.048	0.050	0.046	0.057
	$\widehat{D}_{MS}$	0.072	0.055	0.066	0.059	0.065
	$\widehat{D}_{R}$	0.073	0.053	0.064	0.057	0.059
	$\widehat{D}_{ATV}$	**0.026**	**0.038**	**0.039**	**0.032**	**0.022**
	$\widehat{\widehat{D}}_{N}$	0.053	0.050	0.056	0.055	0.044
FARIMA$(1, \frac{D}{2}, 0)$	$\widehat{D}_{BGK}$	0.085	0.148	0.146	0.164	0.120
	$\widehat{D}_{GRS}$	0.179	0.175	0.182	0.192	0.190
	$\widehat{D}_{MS}$	0.109	0.105	0.099	0.100	0.094
	$\widehat{D}_{R}$	**0.063**	**0.059**	**0.057**	**0.054**	**0.054**
	$\widehat{D}_{ATV}$	0.118	0.101	0.088	0.120	0.081
	$\widehat{\widehat{D}}_{N}$	0.095	0.085	0.093	0.081	0.097
FARIMA$(1, \frac{D}{2}, 1)$	$\widehat{D}_{BGK}$	0.111	0.201	0.189	0.202	0.181
	$\widehat{D}_{GRS}$	0.308	0.321	0.306	0.314	0.311
	$\widehat{D}_{MS}$	0.070	0.064	0.065	**0.064**	0.069
	$\widehat{D}_{R}$	**0.063**	**0.057**	**0.060**	**0.064**	**0.052**
	$\widehat{D}_{ATV}$	0.114	0.118	0.103	0.102	0.093

- $\widehat{D}_{BGK}$ (BIC-criterium). not very satisfactory results except, and time-consuming procedure.
- $\widehat{D}_{GRS}$: good results for fGn and FARIMA$(0, d, 0)$. but not fast enough convergence rate for the other processes.
- Estimators $\widehat{D}_{MS}$ and $\widehat{D}_{R}$: similar properties. They (especially $\widehat{D}_{R}$) are very interesting : good rates of convergence for all processes.
- $\widehat{D}_{ATV}$ and $\widehat{\widehat{D}}_{N}$: similar behavior, fastest convergence rates for fGn and FARIMA$(0, d, 0)$.

The linear case

Assumption A(d, d'):

$$X = (X_t)_{t\in\mathbb{Z}}, \qquad X_t = \sum_{s\in\mathbb{Z}} \alpha(t-s)\xi_s, \quad t\in\mathbb{Z}, \quad \text{where}$$

- $(\xi_s)_{s\in\mathbb{Z}}$ sym iid r.v , $\mathbb{E}\xi_0 = 0$, Var $\xi_0 = 1$, $\mu_4 := \mathbb{E}\xi_0^4 < \infty$;
- $(\alpha(t))_{t\in\mathbb{Z}}$ such $\exists c_d > 0$ and $c_{d'} \in \mathbb{R}$:

$$\begin{aligned} f(\lambda) = \ &= \ 2\pi|\widehat{\alpha}(\lambda)|^2 \\ \widehat{\alpha}(\lambda) &= \ \frac{1}{\lambda^{2d}}\left(c_d + c_{d'}|\lambda|^{d'}(1+\varepsilon(\lambda))\right) \\ & \ \ \ \forall\lambda \in [-\pi, 0)\cup(0,\pi], \quad \varepsilon(\lambda)\to 0(\lambda\to 0) \end{aligned}$$

where $\widehat{\alpha}(\lambda) := \frac{1}{2\pi}\sum_{k\in\mathbb{Z}}\alpha(k)e^{-ik\lambda}$ and $d \in (0, 1/2)$

Assumption $\Psi(k)$:

$\psi : \mathbb{R} \to \mathbb{R}$ *is such that*

1. $Supp(\psi) \subset (0, 1)$;
2. $\int_0^1 \psi(t)\, dt = 0$;
3. $\psi \in \mathcal{C}^k(\mathbb{R})$.

then

- $\psi^{(j)}(0) = \psi^{(j)}(1) = 0$ for any $0 \leq j \leq k$.
- $\widehat{\psi}(u) \sim C\, u^k, \qquad (u \to 0)$
- $\sup_{u \in \mathbb{R}} \left| u^k\, \widehat{\psi}(u) \right| \leq \sup_{x \in [0,1]} |\psi^{(k)}(x)|$.

The wavelet coefficients of X :

$$e(a,b) := \sum_{j=1}^{a} \Big(\frac{1}{\sqrt{a}}\,\psi(\frac{j}{a})\Big)\, X_{b+j} \quad \text{for} \quad (a,b) \in \mathbb{N}^* \times \mathbb{Z}$$

Property 1

Under $A(d,d')$, $d < 1/2$, $d' > 0$, $\psi \sim \Psi(k)$, $k > d' - d + 1/2$, *for* $a \in \mathbb{N}^*$, *then* $(e(a,b))_{b\in\mathbb{Z}}$ *is a zero mean stationary linear process and*

$$\mathbb{E}(e^2(a,0)) \underset{a\to\infty}{=} 2\pi\big(c_d K_{(\psi,2d)}\, a^{2d} + c_{d'} K_{(\psi,2d-d')}\, a^{2d-d'}\big) + o\big(a^{2d-d'}\big),$$

$$\text{with} \quad K_{(\psi,\alpha)} := \int_{-\infty}^{\infty} |\widehat{\psi}(u)|^2\, |u|^{-\alpha} du > 0 \quad \text{for all } \alpha < 1.$$

The sample variances of the wavelet coefficients,

$$T_N(a) := \frac{1}{N-a}\sum_{b=1}^{N-a} e^2(a,b). \quad V_N(a) := \frac{1}{[N/a]}\sum_{b=1}^{[N/a]} e^2(a,ab)$$

The asymptotic variances of $\sqrt{N/a}\,T_N(a)$ and $\sqrt{N/a}\,V_N(a)$ when a, $N \to \infty$ are

$$\begin{cases} \gamma &= 4\pi \frac{1}{K^2_{(\psi,2d)}} \int_{-\infty}^{\infty} \frac{\left|\widehat{\psi}(\lambda)\right|^4}{|\lambda|^{4d}} d\lambda \\ \gamma' &= \frac{2}{K^2_{(\psi,2d)}} \sum_{m=-\infty}^{\infty} \Big(\int_{-\infty}^{\infty} \frac{\left|\widehat{\psi}(u)\right|^2}{|u|^{2d}} \cos(u\,m)\,du \Big)^2, \end{cases}$$

Proposition 6

Define $\ell \in \mathbb{N} \setminus \{0,1\}$ *and* $(r_1, \cdots, r_\ell) \in (\mathbb{N}^*)^\ell$ *with* $0 < r_1 < r_2 < \cdots < r_\ell$. *Under* $A(d,d')$ *with* $d < 1/2$ *and* $d' > 0$, *if* $\psi \sim \Psi(k)$ *with* $k \geq d' - d + 1/2$ *and if* $(a_n)_{n\in\mathbb{N}}$, $N/a_N \underset{N\to\infty}{\longrightarrow} \infty$ *and* $a_N\, N^{-1/(1+2d')} \underset{N\to\infty}{\longrightarrow} \infty$, *then*

$$\sqrt{\frac{N}{a_N}}\Big(\log T_N(r_i a_N) - 2d\log(r_i a_N) - \log\big(2\pi\, c_d\, K_{(\psi,2d)}\big)\Big)_{1\leq i\leq \ell} \underset{N\to\infty}{\overset{d}{\longrightarrow}} \mathcal{N}_\ell\big(0\,;\,\Gamma(r_1,\ldots,r_\ell,\psi,d)\big),$$

with $\Gamma(r_1,\ldots,r_\ell,\psi,d) = (\gamma_{ij})_{1\leq i,j\leq \ell}$ *such as*

$$\gamma_{ij} \;=\; 4\pi\,\frac{(r_i r_j)^{1-2d}}{K^2_{(\psi,2d)}}\int_{-\infty}^{\infty}\frac{\big|\widehat{\psi}(r_i\lambda)\big|^2\big|\widehat{\psi}(r_j\lambda)\big|^2}{\lambda^{4d}}\,d\lambda.$$

$$\widehat{d}_N(a_N) \quad := \quad \left(0\ \frac{1}{2}\right)(Z'_{a_N} Z_{a_N})^{-1} Z'_{a_N}\left(\log T_N(r_i a_N)\right)_{1\leq i\leq \ell} \qquad (7)$$

$$\text{with} \quad Z_{a_N} \quad = \quad \begin{pmatrix} 1 & \log(a_N) \\ 1 & \log(2a_N) \\ \vdots & \vdots \\ 1 & \log(\ell a_N) \end{pmatrix}. \qquad (8)$$

from Proposition 6, $\widehat{d}_N(a_N) \to d$ following a CLT with convergence rate $\sqrt{N/a_N}$ when a_N such $a_N\, N^{-1/(1+2d')} \underset{N\to\infty}{\longrightarrow} \infty$.

But d' is UNKNOWN

Same automatic procedure for choosing the "optimal" scale a_N.
for $\alpha \in (0,1)$, define

$$Q_N(\alpha,c,d) = \Big(Y_N(\alpha) - Z_{N^\alpha}\begin{pmatrix} c \\ 2d \end{pmatrix}\Big)' \cdot \Big(Y_N(\alpha) - Z_{N^\alpha}\begin{pmatrix} c \\ 2d \end{pmatrix}\Big),$$

with $Y_N(\alpha) = \big(\log T_N(iN^\alpha)\big)_{1\leq i\leq \ell}$.

$$\widehat{Q}_N(\alpha) = Q_N\big(\alpha, \widehat{c}(N^\alpha), 2\widehat{d}(N^\alpha)\big)$$

with
$\begin{pmatrix} \widehat{c}(N^\alpha) \\ 2\widehat{d}(N^\alpha) \end{pmatrix} = \big(Z'_{N^\alpha} Z_{N^\alpha}\big)^{-1} Z'_{N^\alpha} Y_N(\alpha);$
Define $\widehat{\alpha}_N$ by:

$$\widehat{Q}_N(\widehat{\alpha}_N) = \min_{\alpha\in\mathcal{A}_N} \widehat{Q}_N(\alpha) \quad \text{where} \quad \mathcal{A}_N = \Big\{\frac{2}{\log N}, \frac{3}{\log N}, \dots, \frac{\log[N/\ell]}{\log N}\Big\}.$$

Under assumptions of Proposition 6:

$$\widehat{\alpha}_N = \frac{\log \widehat{a}_N}{\log N} \xrightarrow[N\to\infty]{\mathcal{P}} \alpha^* = \frac{1}{1+2d'}.$$

then define:

$$\widehat{\widehat{d}_N} := \widehat{d}(N^{\widehat{\alpha}_N}) \quad \text{and} \quad \widehat{\Gamma}_N := \Gamma(1,\ldots,\ell,\widehat{\widehat{d}_N},\psi).$$

we have $\widehat{\widehat{d}_N} \xrightarrow[N\to\infty]{\mathcal{P}} d$ and $\widehat{\Gamma}_N \xrightarrow[N\to\infty]{\mathcal{P}} \Gamma(1,\ldots,\ell,d,\psi)$. We prefer :

$$\widetilde{\alpha}_N := \widehat{\alpha}_N + \frac{6\widehat{\alpha}_N}{(\ell-2)(1-\widehat{\alpha}_N)} \frac{\log\log N}{\log N}.$$

rather than $\widehat{\alpha}_N$ so that :

$$\begin{pmatrix} \widetilde{c}_N \\ 2\widetilde{d}_N \end{pmatrix} := \left(Z'_{N^{\widetilde{\alpha}_N}} \widehat{\Gamma}_N^{-1} Z_{N^{\widetilde{\alpha}_N}}\right)^{-1} Z'_{N^{\widetilde{\alpha}_N}} \widehat{\Gamma}_N^{-1} Y_N(\widetilde{\alpha}_N).$$

Asymptotic behavior of $\widetilde{d}_N$

Theorem 2

Under the assumptions of Proposition 6,

$$\sqrt{\frac{N}{N^{\widetilde{\alpha}_N}}}\big(\widetilde{d}_N - d\big) \xrightarrow[N\to\infty]{d} \mathcal{N}\big(0\,;\,\sigma_d^2(\ell)\big)$$

with $\sigma_d^2(\ell) := \big(0\ \frac{1}{2}\big)\big(Z_1'\big(\Gamma(1,\ldots,\ell,d,\psi)\big)^{-1}Z_1\big)^{-1}\big(0\ \frac{1}{2}\big)'$

and for all $\rho > \dfrac{2(1+3d')}{(\ell-2)d'}$, $\dfrac{N^{\frac{d'}{1+2d'}}}{(\log N)^{\rho}} \times \big|\widetilde{d}_N - d\big| \xrightarrow[N\to\infty]{\mathcal{P}} 0.$

consider the statistic (from PGLS Reg):

$$\widetilde{T}_N := \frac{N}{N^{\widetilde{\alpha}_N}} \left(Y_N(\widetilde{\alpha}_N) - Z_{N^{\widetilde{\alpha}_N}} \begin{pmatrix} \widetilde{c}_N \\ 2\widetilde{d}_N \end{pmatrix} \right)' \widehat{\Gamma}_N^{-1} \left(Y_N(\widetilde{\alpha}_N) - Z_{N^{\widetilde{\alpha}_N}} \begin{pmatrix} \widetilde{c}_N \\ 2\widetilde{d}_N \end{pmatrix} \right).$$

Theorem 3

Under the assumptions of Proposition 6,

$$\widetilde{T}_N \xrightarrow[N\to\infty]{d} \chi^2(\ell - 2).$$

Simulation

Benchmark processes satisfying A(d, d'):

1. fGn, $H = d + 1/2$ for $d \in [0, \frac{1}{2})$ and $\sigma^2 = 1$.
2. FARIMA(p, d, q), $d \in [0, \frac{1}{2})$, $p, q \in \mathbb{N}$.
3. Centered Gaussian stationary process $X^{(d,d')}$, with spectral density

$$f_3(\lambda) = \frac{1}{\lambda^{2d}}(1 + \lambda^{d'}) \quad \text{for } \lambda \in [-\pi, 0) \cup (0, \pi], \tag{9}$$

$d \in [0, \frac{1}{2})$ and $d' \in (0, \infty)$. $X^{(d,d')}$

Particular processes with $d = 0, 0.1, 0.2, 0.3, 0.4$ and ,
$N = 10^3, 10^4$:

- X_1 : fGn processes with parameters $H = d + 1/2$;
- X_2 : FARIMA$(0, d, 0)$, standard Gaussian innovations;
- X_3 : FARIMA$(0, d, 0)$, innovations $\mathcal{U}[-1, 1]$;
- X_4 : FARIMA$(0, d, 0)$, symmetric Burr innovations
- X_5 : FARIMA$(0, d, 0)$, symmetric Burr innovations
- X_6 : FARIMA$(1, d, 1)$, standard Gaussian innovations, MA coefficient $\phi = -0.3$ and AR coefficient $\phi = 0.7$;
- X_7 : FARIMA$(1, d, 1)$, uniform innovations $\mathcal{U}[-1, 1]$, MA coefficient $\phi = -0.3$ and AR coefficient $\phi = 0.7$;
- X_8 : $X^{(d,d')}$ Gaussian processes with $d' = 1$.

Comparison with other estimators

- **Choice of the function ψ:** $\psi(x) = x^4(1-x)^4\left(x^2 - x + \frac{5}{22}\right)\mathbb{I}_{x\in[0,1]}$ satisfying Assumption $\Psi(3)$.
- **Choice of the parameter ℓ:** "beginning" of the linear part of the graph $(\log(ia_N), \log T_N(ia_N))_{1\leq i\leq \ell}$ ($\ell = 13$ for $N = 10^3$ and $\ell = 18$ for $N = 10^4$)

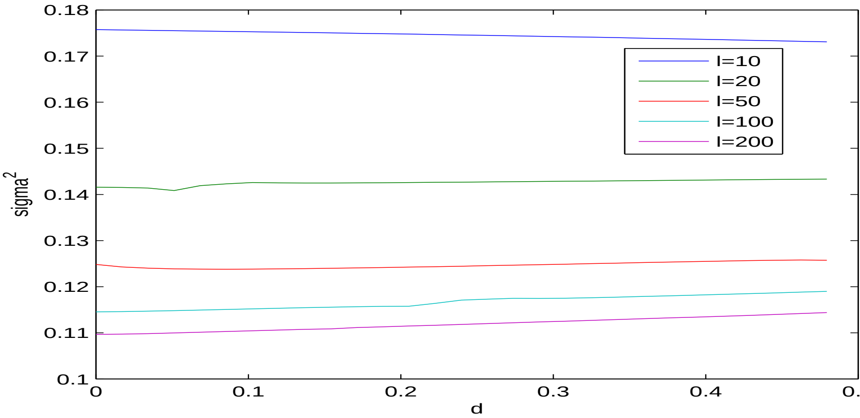

Figure: *Graph of the approximated values of* $\sigma_d^2(\ell)$ *for* $d \in [0, 0.5]$ *and* $\ell = 10, 20, 50, 100$ *and* 200.

- $\widehat{d}_{MS}$, adaptive global log-periodogram estimator (Moulines and Soulier (1998, 2003), FEXP estimator), $\kappa = 2$;
- $\widehat{d}_R$: local Whittle estimator (Robinson (1995)), $m = N/30$.

Conclusions :

- $\widetilde{d}_N$ shows numerically convincing convergence rate.
- $\widehat{d}_R$ and $\widehat{d}_{MS}$, stable results , not sensible to d and to the flatness of the spectral density of the process.
- spectral density of the process notably effects the convergence rate of $\widetilde{d}_N$.
- $\widetilde{d}_N$ is a very accurate and even more efficient for “smooth” spectral densities (fGn and FARIMA$(0, d, 0)$).

$N = 10^3 \longrightarrow$

Model	$\sqrt{MSE}$	$d=0$	$d=0.1$	$d=0.2$	$d=0.3$	$d=0.4$
X_1	$\sqrt{MSE}\ \widehat{d}_{MS}$	0.089	0.091	0.096	0.090	0.100
	$\sqrt{MSE}\ \widehat{d}_R$	0.102	0.114	0.116	0.106	0.102
	$\sqrt{MSE}\ \widetilde{d}_N$	**0.047**	**0.045**	**0.039**	**0.044**	**0.048**
	$\widetilde{p}_n$	0.91	0.82	0.79	0.73	0.68
X_2	$\sqrt{MSE}\ \widehat{d}_{MS}$	0.091	0.094	0.086	0.091	0.099
	$\sqrt{MSE}\ \widehat{d}_R$	0.107	0.105	0.112	0.110	0.097
	$\sqrt{MSE}\ \widetilde{d}_N$	**0.047**	**0.048**	**0.052**	**0.057**	**0.066**
	$\widetilde{p}_n$	0.85	0.86	0.80	0.74	0.68
X_3	$\sqrt{MSE}\ \widehat{d}_{MS}$	0.092	0.094	0.080	0.099	0.096
	$\sqrt{MSE}\ \widehat{d}_R$	0.113	0.113	0.100	0.112	0.095
	$\sqrt{MSE}\ \widetilde{d}_N$	**0.046**	**0.049**	**0.055**	**0.055**	**0.070**
	$\widetilde{p}_n$	0.87	0.85	0.79	0.84	0.71
X_4	$\sqrt{MSE}\ \widehat{d}_{MS}$	0.088	0.079	0.079	0.093	0.104
	$\sqrt{MSE}\ \widehat{d}_R$	0.096	0.100	0.103	0.097	0.095
	$\sqrt{MSE}\ \widetilde{d}_N$	**0.052**	**0.053**	**0.055**	**0.062**	**0.063**
	$\widetilde{p}_n$	0.86	0.82	0.79	0.74	0.71
X_5	$\sqrt{MSE}\ \widehat{d}_{MS}$	0.069	0.067	0.077	0.121	0.143
	$\sqrt{MSE}\ \widehat{d}_R$	0.072	0.078	0.093	0.087	0.074
	$\sqrt{MSE}\ \widetilde{d}_N$	**0.053**	**0.054**	**0.060**	**0.064**	**0.071**
	$\widetilde{p}_n$	0.82	0.80	0.78	0.75	0.74
X_6	$\sqrt{MSE}\ \widehat{d}_{MS}$	**0.096**	**0.091**	**0.090**	**0.086**	**0.093**
	$\sqrt{MSE}\ \widehat{d}_R$	0.111	0.102	0.100	0.101	0.101
	$\sqrt{MSE}\ \widetilde{d}_N$	0.154	0.153	0.143	0.168	0.141
	$\widetilde{p}_n$	0.69	0.68	0.68	0.65	0.57
X_7	$\sqrt{MSE}\ \widehat{d}_{MS}$	**0.085**	**0.096**	**0.086**	**0.093**	0.098

$N = 10^4 \longrightarrow$

Model	$\sqrt{MSE}$	$d = 0$	$d = 0.1$	$d = 0.2$	$d = 0.3$	$d = 0.4$
X_1	$\sqrt{MSE}\ \widehat{d}_{MS}$	0.032	0.029	0.031	0.031	0.036
	$\sqrt{MSE}\ \widehat{d}_R$	0.028	0.028	0.029	0.029	0.032
	$\sqrt{MSE}\ \widetilde{d}_N$	**0.016**	**0.018**	**0.018**	**0.024**	**0.024**
	$\widetilde{p}_n$	0.99	0.97	0.97	0.97	0.95
X_2	$\sqrt{MSE}\ \widehat{d}_{MS}$	0.034	0.030	0.029	0.032	0.028
	$\sqrt{MSE}\ \widehat{d}_R$	0.027	0.027	0.029	0.028	0.023
	$\sqrt{MSE}\ \widetilde{d}_N$	**0.017**	**0.019**	**0.021**	**0.022**	**0.022**
	$\widetilde{p}_n$	0.96	0.97	0.96	0.98	0.94
X_3	$\sqrt{MSE}\ \widehat{d}_{MS}$	0.034	0.034	0.033	0.030	0.031
	$\sqrt{MSE}\ \widehat{d}_R$	0.029	0.028	0.028	0.028	0.029
	$\sqrt{MSE}\ \widetilde{d}_N$	**0.016**	**0.017**	**0.019**	**0.020**	**0.020**
	$\widetilde{p}_n$	0.96	0.97	0.95	0.96	0.96
X_4	$\sqrt{MSE}\ \widehat{d}_{MS}$	0.029	0.060	0.036	0.031	0.031
	$\sqrt{MSE}\ \widehat{d}_R$	0.025	0.027	0.029	0.031	0.029
	$\sqrt{MSE}\ \widetilde{d}_N$	**0.016**	**0.021**	**0.022**	**0.022**	**0.024**
	$\widetilde{p}_n$	0.96	0.94	0.92	0.93	0.95
X_5	$\sqrt{MSE}\ \widehat{d}_{MS}$	0.093	**0.046**	0.039	0.073	0.047
	$\sqrt{MSE}\ \widehat{d}_R$	0.040	0.046	0.035	0.032	**0.024**
	$\sqrt{MSE}\ \widetilde{d}_N$	**0.039**	**0.019**	**0.024**	**0.025**	0.025
	$\widetilde{p}_n$	0.93	0.93	0.90	0.92	0.91
X_6	$\sqrt{MSE}\ \widehat{d}_{MS}$	0.031	0.032	0.033	0.032	0.029
	$\sqrt{MSE}\ \widehat{d}_R$	**0.029**	**0.028**	**0.028**	**0.028**	**0.028**
	$\sqrt{MSE}\ \widetilde{d}_N$	0.044	0.044	0.044	0.042	0.048
	$\widetilde{p}_n$	0.94	0.88	0.90	0.92	0.86
X_7	$\sqrt{MSE}\ \widehat{d}_{MS}$	0.030	0.031	0.037	0.030	0.029

Robustness :

Processes not satisfying $A(d, d')$:

- GARMA$(0, \delta, 0)$ process: Gaussian stationary process, $f(\lambda) = \big||\lambda| - \pi/2\big|^{-2\delta}$ for all $\lambda \in [-\pi, \pi] \setminus \{-\pi/2, \pi/2\}$ a .
- A Gaussian FARIMA$(0, d, 0)$, $(X_t = FARIMA_t + (1 - 2t/N)$, for $t = 1, \ldots, N)$;
- A Gaussian FARIMA$(0, d, 0)$ $(X_t = FARIMA_t + (1 - 2t/N) + \sin(\pi t/6)$, for $t = 1, \ldots, N$).

$N = 10^3 \longrightarrow$

Model	$\sqrt{MSE}$	$d(=\delta) = 0$	$d = 0.1$	$d = 0.2$	$d = 0.3$	$d = 0.4$
GARMA$(0, \delta, 0)$	$\sqrt{MSE}\ \widehat{d}_{MS}$	0.089	0.091	0.123	0.132	0.166
	$\sqrt{MSE}\ \widehat{d}_R$	0.112	0.111	0.119	0.106	**0.106**
	$\sqrt{MSE}\ \widetilde{d}_N$	**0.052**	**0.050**	**0.080**	**0.079**	0.154
	$\widetilde{p}_n$	0.90	0.91	0.85	0.83	0.77
Trend	$\sqrt{MSE}\ \widehat{d}_{MS}$	0.548	0.411	0.292	0.190	0.142
	$\sqrt{MSE}\ \widehat{d}_R$	0.499	0.394	0.279	0.167	0.091
	$\sqrt{MSE}\ \widetilde{d}_N$	**0.044**	**0.045**	**0.040**	**0.044**	**0.041**
	$\widetilde{p}_n$	0.88	0.92	0.90	0.83	0.86
Trend + Seasonality	$\sqrt{MSE}\ \widehat{d}_{MS}$	0.479	0.347	0.233	0.142	0.112
	$\sqrt{MSE}\ \widehat{d}_R$	0.499	0.393	0.279	0.167	**0.091**
	$\sqrt{MSE}\ \widetilde{d}_N$	**0.216**	**0.215**	**0.215**	**0.217**	0.185
	$\widetilde{p}_n$	0.35	0.26	0.18	0.21	0.18

$N = 10^4 \quad \longrightarrow$

Model	$\sqrt{MSE}$	$d(=\delta)=0$	$d=0.1$	$d=0.2$	$d=0.3$	$d=0.4$
GARMA(0, δ, 0)	$\sqrt{MSE}\ \widehat{d}_{MS}$	0.031	0.035	0.039	0.049	0.062
	$\sqrt{MSE}\ \widehat{d}_{R}$	0.028	0.031	**0.030**	**0.030**	**0.034**
	$\sqrt{MSE}\ \widetilde{d}_{N}$	**0.016**	**0.029**	0.032	0.038	0.039
	$\widetilde{p}_n$	0.96	0.96	0.95	0.96	0.90
Trend	$\sqrt{MSE}\ \widehat{d}_{MS}$	0.452	0.286	0.167	0.096	0.056
	$\sqrt{MSE}\ \widehat{d}_{R}$	0.433	0.308	0.191	0.100	0.051
	$\sqrt{MSE}\ \widetilde{d}_{N}$	**0.021**	**0.016**	**0.020**	**0.018**	**0.023**
	$\widetilde{p}_n$	0.96	0.98	0.97	0.97	0.95
Trend + Seasonality	$\sqrt{MSE}\ \widehat{d}_{MS}$	0.471	0.307	0.196	0.123	0.076
	$\sqrt{MSE}\ \widehat{d}_{R}$	0.432	0.305	0.191	0.100	0.052
	$\sqrt{MSE}\ \widetilde{d}_{N}$	**0.042**	**0.046**	**0.043**	**0.048**	**0.052**
	$\widetilde{p}_n$	0.89	0.86	0.87	0.82	0.72

Table: The frequency of acceptation of the adaptive goodness-of-fit test is $\widetilde{p}_n = \frac{1}{n}\#\big(\widetilde{T}_N < q_{\chi^2(\ell-2)}(0.95)\big)$ with $n = 100$.

Robustness of the adaptive goodness-of-fit test

1. MFARIMA, $X_t = FARIMA(0, 0.1, 0)$ for $t = 1, \ldots, N/2$ and $X_t = FARIMA(0, 0.4, 0)$ for $t = N/2 + 1, \ldots, N$.
2. MGN : increments of a multifractional Brownian motion (introduced in Peltier and Lévy-Vehel, 1995).
3. MFGN : increments of a multiscale fractional Brownian motion (introduced in Bardet and Bertrand, 2007).

Model	$N = 10^3$	$N = 10^4$
MFARIMA	$\widetilde{p}_n = 0.42$	$\widetilde{p}_n = 0.90$
MGN	$\widetilde{p}_n = 0.13$	$\widetilde{p}_n = 0.07$
MFGN	$\widetilde{p}_n = 0.03$	$\widetilde{p}_n = 0.06$

Table: The frequency of acceptation of the adaptive goodness-of-fit test is $\widetilde{p}_n = \frac{1}{n} \# \big(\widetilde{T}_N < q_{\chi^2(\ell-2)}(0.95)\big)$ (with $n = 100$ independent replications).