Stochastic control methods for optimal transportation and probabilistic numerical schemes for PDEs

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Stochastic control methods for optimal transportation and probabilistic numerical schemes for PDEs

Méthodes de contrôle stochastique pour le problème de transport optimal et schémas numériques de type Monte-Carlo pour les EDP

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Résumé

Cette thèse porte sur les méthodes numériques pour les équations aux dérivées partielles (EDP) non-linéaires dégénérées, ainsi que pour des problèmes de contrôle d'EDP non-linéaires résultants d'un nouveau problème de transport optimal. Toutes ces questions sont motivées par des applications en mathématiques financières. La thèse est divisée en quatre parties.

Dans une première partie, nous nous intéressons à la condition nécessaire et suffisante de la monotonie du θ-schéma de différences finies pour l'équation de diffusion en dimension un. Nous donnons la formule explicite dans le cas de l'équation de la chaleur, qui est plus faible que la condition classique de Courant-Friedrichs-Lewy (CFL).

Dans une seconde partie, nous considérons une EDP parabolique non-linéaire dégénérée et proposons un schéma de type "splitting" pour la résoudre. Ce schéma réunit un schéma probabiliste et un schéma semi-lagrangien. Au final, il peut être considéré comme un schéma Monte-Carlo. Nous donnons un résultat de convergence et également un taux de convergence du schéma.

Dans une troisième partie, nous étudions un problème de transport optimal, où la masse est transportée par un processus d'état type "drift-diffusion" controllé. Le coût associé est dépendant des trajectoires de processus d'état, de son drift et de son coefficient de diffusion. Le problème de transport consiste à minimiser le coût parmi toutes les dynamiques vérifiant les contraintes initiales et terminales sur les distributions marginales. Nous prouvons une formule de dualité pour ce problème de transport, étendant ainsi la dualité de Kantorovich à notre contexte. La formulation duale maximise une fonction valeur sur l'espace des fonctions continues bornées, et la fonction valeur correspondante à chaque fonction continue bornée est la solution d'un problème de contôle stochastique optimal. Dans le cas markovien, nous prouvons un principe de programmation dynamique pour ces problèmes de contrôle optimal, proposons un algorithme de gradient projeté pour la résolution numérique du problème dual, et en démontrons la convergence.

Enfin dans une quatrième partie, nous continuons à développer l'approche duale pour le problème de transport optimal avec une application à la recherche de bornes de prix sans arbitrage des options sur variance étant donnés les prix des options européennes. Après une première approximation analytique, nous proposons un algorithme de gradient projeté pour approcher la borne et la stratégie statique correspondante en options vanilles.

Mots-clés: Contrôle stochastique, transport optimal, borne des prix sans-arbitrage, options sur variance, schéma Monte-Carlo, monotonie.

Abstract

This thesis deals with the numerical methods for a fully nonlinear degenerate parabolic partial differential equations (PDEs), and for a controlled nonlinear PDEs problem which results from a mass transportation problem. The manuscript is divided into four parts.

In a first part of the thesis, we are interested in the necessary and sufficient condition of the monotonicity of finite difference θ-scheme for a one-dimensional diffusion equations. An explicit formula is given in case of the heat equation, which is weaker than the classical Courant-Friedrichs-Lewy (CFL) condition.

In a second part, we consider a fully nonlinear degenerate parabolic PDE and propose a splitting scheme for its numerical resolution. The splitting scheme combines a probabilistic scheme and the semi-Lagrangian scheme, and in total, it can be viewed as a Monte-Carlo scheme for PDEs. We provide a convergence result as well as a rate of convergence.

In the third part of the thesis, we study an optimal mass transportation problem. The mass is transported by the controlled drift-diffusion dynamics, and the associated cost depends on the trajectories, the drift as well as the diffusion coefficient of the dynamics. We prove a strong duality result for the transportation problem, thus extending the Kantorovich duality to our context. The dual formulation maximizes a value function on the space of all bounded continuous functions, and every value function corresponding to a bounded continuous function is the solution to a stochastic control problem. In the Markovian cases, we prove the dynamic programming principle of the optimal control problems, and we propose a gradient-projection algorithm for the numerical resolution of the dual problem, and provide a convergence result.

Finally, in a fourth part, we continue to develop the dual approach of mass transportation problem with its applications in the computation of the model-independent no-arbitrage price bound of the variance option in a vanilla-liquid market. After a first analytic approximation, we propose a gradient-projection algorithm to approximate the bound as well as the corresponding static strategy in vanilla options.

Keywords: Stochastic control, optimal transportation, no-arbitrage bound, variance options, probabilistic numerical scheme, monotonicity.

Notations

- Let $T \in \mathbb{R}^+$ and $d,\ d' \in \mathbb{N}$, we denote $Q_T := [0,T) \times \mathbb{R}^d \times \mathbb{R}^{d'}$, $\overline{Q}_T := [0,T] \times \mathbb{R}^d \times \mathbb{R}^{d'}$ and
$$C^{0,1}(Q_T) \quad := \quad \Big\{ \ \varphi \ : \ Q_T \to \mathbb{R} \text{ such that } |\varphi|_1 < \infty \ \Big\},$$
where $|\varphi|_0 \ := \ \sup_{Q_T} |\varphi(t,x,y)|$ and
$$|\varphi|_1 \ := \ |\varphi|_0 \ + \sup_{Q_T \times Q_T} \frac{|\varphi(t,x,y) - \varphi(t',x',y')|}{|x-x'| + |y-y'| + |t-t'|^{\frac{1}{2}}}.$$

- Given a smooth function φ defined on Q_T and $k \in \mathbb{N}$, by noting $z = (x,y)$, we define
$$|D^k_{z^k}\varphi|_0 \ := \ \sup \ \Big\{ \ \big|D^{\alpha}_{z_1^{\alpha_1}\ldots z_{d+d'}^{\alpha_{d+d'}}}\varphi\big|_0 \ : \ \alpha \in \mathbb{N}^{d+d'}, \ \sum_{i=1}^{d+d'} \alpha_i = k \ \Big\}.$$

- Let μ be a probability measure on $\mathbb{R}^d$, $\phi \in L^1(\mu)$, we then denote
$$\mu(\phi) \quad := \quad \int_{\mathbb{R}^d} \phi(x)\mu(dx).$$

- Let E be a Polish space, we denote by $\mathbf{M}(E)$ the space of all Borel probability measures on E.

- S_d denotes the set of all positive $d \times d$ matrices.

- δ_x denotes the dirac measures on point x.

Sommaire

CHAPITRE 1

Introduction (Français)

La contribution principale de cette thèse porte sur les méthodes numériques pour les équations aux dérivées partielles (EDP) non-linéaires dégénérées ainsi que pour des problèmes de contrôle d'EDP non-linéaires résultants d'un nouveau problème de transport optimal. Nous étudions également des sujets associés, tels que la dualité pour ce problème de transport, le principe de programmation dynamique du problème de contrôle stochastique, et les applications en finance.

La thèse est divisée en quatre parties. La première et la deuxième partie sont consacrées à deux sujets indépendants. Le premier sujet concerne la condition nécessaire et suffisante de monotonie du $\theta-$schéma de différences finies pour l'équation de diffusion en dimension un. Le deuxième porte sur la méthode de type "splitting" pour une EDP parabolique non-linéaire dégénérée.

La troisième partie et la quatrième sont liées. Nous étudions tout d'abord un nouveau problème de transport optimal, où la masse est transportée par un processus d'état de type "drift-diffusion" contrôlé, et nous minimisons un coût de transport parmi toutes les dynamiques vérifiant les contraintes initiales et terminales sur les distributions marginales. Nous prouvons une formule de dualité pour ce problème de transport, étendant ainsi la dualité de Kantorovich à notre contexte. Dans le cas markovien, nous proposons un algorithme de gradient projeté pour la résolution numérique du problème dual et en démontrons la convergence. Ce problème de transport optimal est motivé par le problème de la recherche de bornes de prix sans arbitrage des options exotiques, étant donnés les prix des options de "call" européennes. Nous donnons ainsi un exemple pour les options sur variance dans la quatrième partie, où nous utilisons l'algorithme de gradient projeté pour approximer la borne des prix sans arbitrage des options sur variance. À cause de la structure particulière des options sur variance, nos techniques d'approximation sont différentes de celles utilisées dans le problème de transport optimal.

1.1 Première partie : la monotonie du $\theta-$schéma pour l'équation de diffusion

La monotonie du schéma numérique est un sujet important dans les analyses numériques. Par exemple, dans l'analyse de convergence réalisée dans le chapitre 2 de Allaire [1], l'auteur utilise la monotonie du schéma pour prouver une stabilité pour la norme L^∞.

Dans le cadre de l'étude réalisée par Barles et Souganidis dans [6], la monotonie est un critère clé pour la convergence du schéma numérique.

Dans cette partie, nous nous intéressons à la condition nécessaire et suffissante de la monotonie du $\theta-$schéma pour l'équation de diffusion en dimension un. Considérons l'équation de diffusion

$$\partial_t v(t,x) \;-\; \sigma^2(x) D^2_{xx} v(t,x) \;=\; 0, \tag{1.1.1}$$

avec la condition initiale $v(0,x) = \Phi(x)$. Le $\theta-$schéma de différences finies pour l'équation (1.1.1) est un système linéaire sur la grille $\mathcal{N}$:

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} - \sigma_i^2 \left(\theta \frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{\Delta x^2} + (1-\theta) \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2} \right) = 0, \tag{1.1.2}$$

où nous nous donnons la discrétisation $h = (\Delta t, \Delta x)$, et avec $t_n := n\Delta t$, $x_i := i\Delta x$, u_i^n désignant la solution numérique au point (t_n, x_i), la grille $\mathcal{N}$ est définie par $\mathcal{N} := \{x_i \;:\; i \in \mathbb{N}\}$. Lorsque $\theta = 1$, le schéma ci-dessus est un schéma implicite de différences finies. Lorsque $\theta = 0$, le schéma (1.1.2) devient un schéma explicite.

La condition CFL Il est bien connu (voir par exemple le lemme 2.2.13 de Allaire [1]) que le schéma implicite est monotone inconditionnellement et que le schéma explicite est monotone si et seulement s'il vérifie la condition de Courant-Friedrichs-Lewy (CFL) :

$$\frac{\bar{\sigma}^2 \Delta t}{\Delta x^2} \;\leq\; \frac{1}{2}, \qquad \text{avec} \quad \bar{\sigma} \;:=\; \sup_{i \in \mathbb{Z}} \sigma(x_i).$$

Puisque le $\theta-$schéma est composé d'une partie explicite et d'une partie implicite, une condition suffisante pour la monotonie du $\theta-$schéma est donc

$$\frac{\bar{\sigma}^2 \Delta t}{\Delta x^2} \;\leq\; \frac{1}{2(1-\theta)}, \qquad \text{pour} \quad \bar{\sigma} \;:=\; \sup_{i \in \mathbb{Z}} \sigma(x_i). \tag{1.1.3}$$

Le condition (1.1.3) ci-dessus exige un ratio de discrétisation $\Delta t = O(\Delta x^2)$ pour garantir la monotonie lorsque $\theta < 1$. Une question naturelle est de savoir si cette condition est necessaire.

La condition nécessaire et suffisante de monotonie Nous déduisons la condition nécessaire et suffisante de la monotonie du $\theta-$schéma, confirmant que le ratio $\Delta t = O(\Delta x^2)$ est nécessaire. De plus, dans le cas de l'équation de la chaleur, i.e $\sigma(x) = \sigma_0$ pour une constante $\sigma_0 > 0$, nous obtenons la formule explicite suivante pour la condition nécessaire et suffisante :

$$\frac{\sigma_0^2 \Delta t}{\Delta x^2} \;\leq\; \frac{1}{2(1-\theta)} \;+\; \frac{\theta}{4(1-\theta)^2}, \tag{1.1.4}$$

qui est plus faible que la condition CFL (1.1.3).

1.2 Deuxième partie : une méthode de type “splitting” pour les équations paraboliques non-linéaires dégénérées

1.2.1 Motivation

Les méthodes numériques pour les EDP, telles que les différences finies, les éléments finis, la méthode du semi-lagrangien et la méthode Monte-Carlo, ont été très étudiées dans la litérature. De façon générale, les trois premières methodes sont relativement plus efficaces pour les EDP en petite dimension. Cependant, il est préférable, lorsque cela est possible, d'utiliser la méthode de Monte-Carlo pour les problèmes de grande dimension.

La méthode de “splitting” Outre les méthodes numériques citées ci-dessus, une technique importante en analyse numérique est celle du “splitting”. Dans beaucoup de situations, elle est utilisée pour réduire la dimension et améliorer la précision du schéma. L'idée est de décomposer l'EDP en deux équations, de les traiter ensuite séparément, puis de les réunir. Pour comprendre cette idée, examinons un exemple sur l'équation de la chaleur sur $[0,T]\times\mathbb{R}\times\mathbb{R}$:

$$\partial_t v(t,x,y) \;-\; D^2_{xx}v(t,x,y) \;-\; D^2_{yy}v(t,x,y) \;=\; 0, \tag{1.2.1}$$

avec la condition initiale $v(0,\cdot)=\Phi(\cdot)$.

La méthode de “splitting” décompose tout d'abord l'équation (1.2.1) en deux équations

$$\partial_t v(t,x,y) \;+\; D^2_{xx}v(t,x,y) \;=\; 0 \quad \text{et} \quad \partial_t v(t,x,y) \;+\; D^2_{yy}v(t,x,y) \;=\; 0. \tag{1.2.2}$$

Avec les paramètres de discrétisation $h=(\Delta t,\Delta x,\Delta y)$, nous notons

$$D^{2,h}_{xx}v^h(t_n,x,y) \;:=\; \frac{v^h(t_n,x+\Delta x,y)-2v^h(t_n,x,y)+v^h(t_n,x-\Delta x,y)}{\Delta x^2}$$

et

$$D^{2,h}_{yy}v^h(t_n,x,y) \;:=\; \frac{v^h(t_n,x,y+\Delta y)-2v^h(t_n,x,y)+v^h(t_n,x,y-\Delta y)}{\Delta y^2}.$$

Alors, les schémas explicites de différences finies pour les deux équations (1.2.2) sont donnés par

$$v^h(t_{n+1},\cdot)=v^h(t_n,\cdot)+\Delta t\; D^{2,h}_{xx}v^h(t_n,\cdot) \;\;\text{et}\;\; v^h(t_{n+1},\cdot)=v^h(t_n,\cdot)+\Delta t\; D^{2,h}_{yy}v^h(t_n,\cdot).$$

En introduisant un temps fictif $t_{n+\frac{1}{2}}$, le schéma de “splitting” pour (1.2.1) est donné par

$$v^h(t_{n+\frac{1}{2}},\cdot)=v^h(t_n,\cdot)+\Delta t D^{2,h}_{xx}v^h(t_n,\cdot) \;\;\text{et}\;\; v^h(t_{n+1},\cdot)=v^h(t_{n+\frac{1}{2}},\cdot)+\Delta t D^{2,h}_{yy}v^h(t_{n+\frac{1}{2}},\cdot).$$

Il résulte d'un calcul formel que le schéma de “splitting” s'écrit

$$v^h(t_{n+1},\cdot) \;=\; v^h(t_n,\cdot) \;+\; \Delta t\big(D^{2,h}_{xx}v^h(t_n,\cdot) \;+\; D^{2,h}_{yy}v^h(t_n,\cdot)\big) \;+\; O(\Delta t^2),$$

qui est presque le même que le schéma explicite de l'équation (1.2.1).

La méthode de Monte-Carlo pour les EDP La méthode de Monte-Carlo pour les EDP est liée à la formule de Feynman-Kac. Soient $\mu : [0,T]\times\mathbb{R}^d \to \mathbb{R}^d$ et $\sigma : [0,T]\times\mathbb{R}^d \to S_d$ deux applications lipschitziennes en x et uniformément en t et telles que

$$\int_0^T \left(|\mu(t,0)|^2 \ + \ |\sigma(t,0)\sigma^T(t,0)| \right) dt \quad < \quad \infty.$$

Supposons que la fonction $v(t,x)$ est une solution régulière de l'équation parabolique linéaire

$$-\mathcal{L}^X v(t,x) \ = \ 0 \ \text{ où } \mathcal{L}^X \ := \ \partial_t \ + \ \mu(t,x)\cdot D_x \ + \ \frac{1}{2}\sigma(t,x)\sigma^T(t,x)\cdot D^2_{xx}, \qquad (1.2.3)$$

avec la condition terminale $v(T,x) = \Phi(x)$. Alors par la formule de Feynman-Kac, $v(t,x)$ admet une interprétation probabiliste:

$$v(t,x) \quad = \quad \mathbb{E}\ \Phi(X_T^{t,x}), \qquad (1.2.4)$$

où $X^{t,x}$ est l'unique solution forte de l'équation différentielle stochastique (EDS)

$$X_s^{t,x} = x + \int_t^s \mu(r, X_r^{t,x})dr + \int_t^s \sigma(r, X_r^{t,x})dW_r, \text{ avec le mouvement brownien } W. \qquad (1.2.5)$$

La méthode de Monte-Carlo pour résoudre l'EDP linéaire (1.2.3) consiste donc à simuler la variable aléatoire $X_T^{t,x}$, à approximer $v(t,x)$ avec les simulations $(X_{T,m}^{t,x})_{1\le m\le M}$ par

$$\frac{1}{M}\sum_{m=1}^{M} \Phi(X_{T,m}^{t,x}).$$

Il résulte du théorème de la limite centrale que le taux de convergence est indépendant de la dimension d de l'équation (1.2.3).

En tant qu'extension de la formule de Feynman-Kac, l'équation différentielle stochastique rétrograde (EDSR) ouvre la porte à la résolution numérique de l'EDP semi-linéaire par la méthode de Monte-Carlo. Dans cet esprit, Fahim, Touzi et Warin [32] ont proposé une méthode de Monte-Carlo pour une EDP complètement non-linéaire, qui est liée à l'EDSR du second ordre. Cependant, cette méthode est limitée, car elle ne s'applique que dans le cas non-dégénéré.

Les EDP dégénérées En finance, pour les problèmes d'évaluation d'options financières ou d'optimisation, lorsque la variable sous-jacente considérée n'a pas un générateur de type "diffusion", les équations caractérisées deviennent dégénérées. Par exemple, les problèmes d'évaluation des options asiatiques ou d'évaluation de certains produits d'assurance vie.

Example 1.2.1. *Supposons que le processus de l'actif risqué* S_t *est donné par le modèle de Black-Scholes :* $dS_t = \sigma S_t dW_t$*, où* σ *est le paramètre de volatilité et où* W *est un* *mouvement brownien. Une option asiatique est une option avec "payoff" $g(S_T, A_T)$ à la maturité T, où $A_T = \int_0^T S_t dt$, et son prix est donc caractérisé par l'EDP :*

$$\partial_t v(t,s,a) \;+\; \frac{1}{2}\sigma^2 s^2 D^2_{ss} v(t,s,a) \;+\; s D_a v(t,x,a) \;=\; 0,$$

qui est dégénérée puisque $D^2_{aa} v(t,x,a)$ n'apparaît pas dans l'équation.

Ces applications nous motivent pour développer une méthode de Monte-Carlo pour les EDP non-linéaires dégénérées en grande dimension.

1.2.2 Résultats principaux

1.2.2.1 L'EDP non-linéaire dégénérée et le schéma "splitting"

Nous considérons l'EDP non-linéaire dégénérée suivante :

$$-\;\mathcal{L}^X v(t,x,y) \;-\; F(\cdot, v, D_x v, D^2_{xx} v)(t,x,y) \;-\; H(\cdot, v, D_x v, D_y v)(t,x,y) \;=\; 0, \quad (1.2.6)$$

avec la condition terminale $v(T,\cdot) = \Phi(\cdot)$, où $\mathcal{L}^X$ est définie par (1.2.3), F est une fonction non-linéaire définie sur $[0,T) \times \mathbb{R}^d \times \mathbb{R}^{d'} \times \mathbb{R} \times \mathbb{R}^d \times S_d$, et H est un hamiltonien donné par

$$H(t,x,y,r,p,q) \;:=\; \inf_{\alpha\in\mathcal{A}} \sup_{\beta\in\mathcal{B}} \big(l^{\alpha,\beta}(\cdot) + c^{\alpha,\beta}(\cdot) r + f^{\alpha,\beta}(\cdot)\cdot p + g^{\alpha,\beta}(\cdot)\cdot q\big)(t,x,y).$$

Sur la base du schéma de Monte-Carlo de Fahim, Touzi et Warin [32] et du schéma semi-lagrangien, nous proposons un schéma de "splitting" $\mathbf{S}_h \circ \mathbf{T}_h$ pour l'EDP dégénérée (1.2.6). Soit $(t_n)_{0\le n\le N}$ une grille discrète avec $h := \frac{T}{N}$ et $t_n := nh$, nous définissons le schéma $\mathbf{S}_h \circ \mathbf{T}_h$ par

$$v^h(t_{n+\frac{1}{2}}, x, y) \;:=\; \mathbb{E}\big[v^h(t_{n+1}, \hat{X}^{t_n,x}_h, y)\big] \;+\; hF(t_n, x, y, \mathbb{E}\mathcal{D}_h v^h(t_n,x,y)), \quad (1.2.7)$$

et

$$\begin{aligned} v^h(t_n,x,y) \;&=\; \mathbf{S}_h \circ \mathbf{T}_h[v](t_n,x,y) \\ &:=\; \inf_{\alpha\in\mathcal{A}} \sup_{\beta\in\mathcal{B}} \Big\{ h\, l^{\alpha,\beta}(t_n,x,y) \;+\; h\, c^{\alpha,\beta}(t_n,x,y)\; v^h(t_{n+\frac{1}{2}}, x, y) \\ &\qquad +\; v^h\big(t_{n+\frac{1}{2}},\; x + f^{\alpha,\beta}(t_n,x,y)h,\; y + g^{\alpha,\beta}(t_n,x,y)h\big) \Big\}. \quad (1.2.8) \end{aligned}$$

Dans (1.2.7), $\hat{X}^{t,x}_h$ est défini par le schéma d'Euler de $X^{t,x}$ en (1.2.5) avec

$$\hat{X}^{t,x}_h \;:=\; x \;+\; \mu(t,x)\, h \;+\; \sigma(t,x)\cdot(W_{t+h} - W_t),$$

et

$$\mathbb{E}\mathcal{D}_h v^h(t_n,x,y) \;:=\; \big(\; \mathbb{E}\big[v^h(t_{n+1}, \hat{X}^{t_n,x}_h, y) H^{t_n,x,h}_i(\Delta W_{n+1})\big] : i = 0,1,2 \;\big), \quad (1.2.9)$$

où $\Delta W_{n+1} := W_{t_{n+1}} - W_{t_n}$ et les polynômes d'Hermite sont calculés par l'intégration par parties dans les espérances et donnés par $H^{t,x,h}_0(z) := 1$, $H^{t,x,h}_1(z) := \sigma^T(t,x)^{-1}\frac{z}{h}$ et $H^{t,x,h}_2(z) := \sigma^T(t,x)^{-1}\frac{zz^T - hI_d}{h^2}\sigma(t,x)^{-1}$.

1.2.2.2 Résultats de convergence

Nous donnons deux résultats de convergence du schéma $\mathbf{S}_h \circ \mathbf{T}_h$. Le premier est la convergence locale uniforme dans le contexte de Barles and Souganidis [6].

Hypothèse F : (i) Les fonctions μ et σ sont lipschitziennes en x et continues en t, $\sigma\sigma^T(t,x) > 0$ pour tout $(t,x) \in [0,T] \times \mathbb{R}^d$ et $\int_0^T \big|\sigma\sigma^T(t,0) + \mu(t,0)\big| dt < \infty$.

(ii) L'opérateur non-linéaire $F(t,x,y,r,p,\Gamma)$ est uniformément lipschitzien en (x,y,r,p,Γ), continu en t et $|F(t,x,y,0,0,0)|_\infty < \infty$.

(iii) F est elliptique et vérifie

$$(\sigma\sigma^T)^{-1} \cdot F_\Gamma \;\; \leq \;\; 1 \quad \text{en} \;\; \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^{d'} \times \mathbb{R} \times \mathbb{R}^d \times S_d. \tag{1.2.10}$$

(iv) $F_p \in \text{Image}(F_\Gamma)$ et $\big|F_p^T F_\Gamma^{-1} F_p\big|_\infty < +\infty$.

Hypothèse H : Les coefficients du hamiltonien H sont tous uniformément bornés, i.e.

$$\sup_{(\alpha,\beta)\in\mathcal{A}\times\mathcal{B},\ 1\leq i\leq d,\ 1\leq j\leq d'} \Big\{ \, |l^{\alpha,\beta}|_0 \; + \; |c^{\alpha,\beta}|_0 \; + \; |f_i^{\alpha,\beta}|_0 \; + \; |g_j^{\alpha,\beta}|_0 \, \Big\} \;\; < \;\; \infty.$$

Théorème 1.2.1. *Supposons que les hypothèses* **F** *et* **H** *soient vérifiées et que l'EDP non-linéaire parabolique dégénérée* (1.2.6) *vérifie le principe de comparaison pour des solutions de viscosité bornées. Alors, pour toute fonction terminale* Φ *qui est lipschitzienne, bornée, il existe une fonction bornée* v *telle que*

$$v^h \;\; \longrightarrow \;\; v \quad \textit{localement uniformément lorsque} \;\; h \to 0,$$

où v^h *est la solution numérique du schéma* $\mathbf{S}_h \circ \mathbf{T}_h$ *de* (1.2.8). *De plus,* v *est l'unique solution de viscosité bornée de l'équation* (1.2.6) *avec la condition terminale* $v(T,\cdot) = \Phi(\cdot)$.

L'autre résultat de convergence porte sur le taux de convergence dans le contexte de Barles et Jakobsen [5] où F et H sont tous deux des hamiltoniens concaves.

Hypothèse HJB : L'hypothèse **F** est vérifiée et F est un hamiltonien concave, i.e.

$$\mu \cdot p \; + \; \frac{1}{2} a \cdot \Gamma \; + \; F(t,x,y,r,p,\Gamma) \;\; = \;\; \inf_{\gamma\in\mathcal{C}} \; \mathcal{L}^\gamma(t,x,y,r,p,\Gamma),$$

avec

$$\mathcal{L}^\gamma(t,x,y,r,p,\Gamma) \; := \; l^\gamma(t,x,y) \; + \; c^\gamma(t,x,y) r \; + \; f^\gamma(t,x,y) \cdot p \; + \; \frac{1}{2} a^\gamma(t,x,y) \cdot \Gamma.$$

Et $\mathcal{B} = \{\beta\}$ est un singleton, donc H est aussi un hamiltonien concave, qui peut s'écrire comme

$$H(t,x,y,r,p,q) \;\; = \;\; \inf_{\alpha\in\mathcal{A}} \; \big\{ l^\alpha(t,x,y) \; + \; c^\alpha(t,x,y) r \; + \; f^\alpha(t,x,y) \cdot p \; + \; g^\alpha(t,x,y) \cdot q \big\}.$$

De plus, les fonctions l, c, f, g et σ vérifient

$$\sup_{\alpha\in\mathcal{A},\gamma\in\mathcal{C}} \Big(\, |l^\alpha + l^\gamma|_1 + |c^\alpha + c^\gamma|_1 + |f^\alpha + f^\gamma|_1 + |g^\alpha|_1 + |\sigma^\gamma|_1 \, \Big) \;\; < \;\; \infty.$$

Hypothèse HJB+ : L'hypothèse **HJB** est vérifiée et pour tout $\delta > 0$, il existe un ensemble fini $\{\alpha_i, \gamma_i\}_{i=1}^{I_\delta}$ tel que pour tout $(\alpha, \gamma) \in \mathcal{A} \times \mathcal{C}$:

$$\inf_{1 \leq i \leq I_\delta} |l^\alpha - l^{\alpha_i}|_0 \ + \ |c^\alpha - c^{\alpha_i}|_0 \ + \ |f^\alpha - f^{\alpha_i}|_0 \ + \ |\sigma^\alpha - \sigma^{\alpha_i}|_0 \ \leq \ \delta,$$

et

$$\inf_{1 \leq i \leq I_\delta} |l^\gamma - l^{\gamma_i}|_0 \ + \ |c^\gamma - c^{\gamma_i}|_0 \ + \ |f^\gamma - f^{\gamma_i}|_0 \ + \ |g^\gamma - g^{\gamma_i}|_0 \ \leq \ \delta.$$

Théorème 1.2.2. *Supposons que la condition terminale Φ est bornée et continue lipschitzienne. Alors il existe une constante C telle que*

- *i) sous l'hypothèse* **HJB**, $v - v^h \ \leq \ Ch^{\frac{1}{4}}$,
- *ii) sous l'hypothèse* **HJB+**, $-Ch^{\frac{1}{10}} \ \leq \ v - v^h \ \leq \ Ch^{\frac{1}{4}}$,

où v est l'unique solution de viscosité de (1.2.6).

1.2.2.3 Méthode de simulation-régression

Pour rendre le schéma de "splitting" $\mathbf{S}_h \circ \mathbf{T}_h$ implémentable, nous proposons une méthode de simulation-régression pour estimer les espérances conditionnelles (1.2.9) utilisées dans le schéma $\mathbf{S}_h \circ \mathbf{T}_h$. L'idée est de réécrire (1.2.9) comme

$$\mathbb{E}\left[\varphi(t_{n+1}, \hat{X}_{t_{n+1}}, Y) H_i^{t_n, \hat{X}_{t_n}, h}(\Delta W_{n+1}) \,\Big|\, \hat{X}_{t_n},\ Y\right]_{i=0,1,2}, \tag{1.2.11}$$

où $\hat{X}$ est défini par le schéma d'Euler de X en (1.2.5) :

$$\hat{X}_{t_{n+1}} \ := \ \hat{X}_{t_n} \ + \ \mu(t_n, \hat{X}_{t_n})h \ + \ \sigma(t_n, \hat{X}_{t_n}) \cdot \Delta W_{n+1},$$

et Y est une variable aléatoire avec une distribution continue. Avec M simulations indépendantes $\left((\hat{X}_{t_n}^m)_{0 \leq n \leq N}, (\Delta W_n^m)_{0 < n \leq N}, Y^m\right)_{1 \leq m \leq M}$ de $\hat{X}$, ΔW et Y et une base de fonctions $(e_k(x,y))_{1 \leq k \leq K}$, nous résolvons le problème des moindres carrés :

$$\hat{\lambda}^{i,M} \ = \ \arg\min_{\lambda} \ \sum_{m=1}^{M} \left(\varphi(t_{n+1}, \hat{X}_{t_{n+1}}^m, Y^m) H_i^{t_n, \hat{X}_{t_n}^m, h}(\Delta W_{n+1}^m) \ - \ \sum_{k=1}^{K} \lambda_k e_k(\hat{X}_{t_n}^m, Y^m)\right)^2,$$

qui induit une estimation grossière des espérances conditionnelles (1.2.11) par ces M simulations :

$$\bar{\mathbb{E}}^M\left[\varphi(t_{n+1}, \hat{X}_{t_{n+1}}, Y) H_i^{t_n, \hat{X}_{t_n}, h}(\Delta W_{n+1}) \big| \hat{X}_{t_n}, Y\right] \ := \ \sum_{k=1}^{K} \hat{\lambda}_k^{i,M} e_k(\hat{X}_{t_n}, Y), \ i = 0,1,2.$$

Alors, avec la borne supérieure a priori $\overline{\Gamma}_i(\hat{X}_{t_n}, Y)$ et la borne inférieure a priori $\underline{\Gamma}_i(\hat{X}_{t_n}, Y)$, nous définissons une estimation de régression de (1.2.11) par

$$\begin{aligned} &\hat{\mathbb{E}}^M\left[\varphi(t_{n+1}, \hat{X}_{t_{n+1}}, Y) H_i^{t_n, \hat{X}_{t_n}, h}(\Delta W_{n+1}) \mid \hat{X}_{t_n}, Y\right] \qquad (1.2.12)\\ := \ &\underline{\Gamma}_i(\hat{X}_{t_n}, Y) \ \vee \ \bar{\mathbb{E}}^M\left[\varphi(t_{n+1}, \hat{X}_{t_{n+1}}, Y) H_i^{t_n, \hat{X}_{t_n}, h}(\Delta W_{n+1}) \big| \hat{X}_{t_n}, Y\right] \ \wedge \ \overline{\Gamma}_i(\hat{X}_{t_n}, Y). \end{aligned}$$

Finalement, nous remplaçons les espérances conditionelles (1.2.9) dans le schéma $\mathbf{S}_h \circ \mathbf{T}_h$ par leurs estimations de régression (1.2.12) et obtenons donc le nouveau schéma implementable $\mathbf{S}_h \circ \hat{\mathbf{T}}_h^M$:

$$\hat{v}^h(t_{n+\frac{1}{2}}, x, y) \;:=\; \hat{\mathbb{E}}^M\big[\hat{v}^h(t_{n+1}, \hat{X}_h^{t_n,x}, y)\big] \;+\; h\; F(\cdot, \hat{\mathbb{E}}^M \mathcal{D}\hat{v}^h(\cdot))(t_n, x, y),$$

et

$$\begin{aligned} \hat{v}^h(t_n, x, y) \;&=\; \mathbf{S}_h \circ \hat{\mathbf{T}}_h^M[\hat{v}^h](t_n, x, y) \\ &:= \inf_{\alpha\in\mathcal{A}} \sup_{\beta\in\mathcal{B}} \Big\{ hl^{\alpha,\beta}(t_n, x, y) \;+\; hc^{\alpha,\beta}(t_n, x, y)\hat{v}^h(t_{n+\frac{1}{2}}, x, y) \\ &\qquad + \; \hat{v}^h\big(t_{n+\frac{1}{2}},\; x + f^{\alpha,\beta}(t_n, x, y)h,\; y + g^{\alpha,\beta}(t_n, x, y)h\big)\Big\}, \end{aligned} \tag{1.2.13}$$

où

$$\hat{\mathbb{E}}^M \mathcal{D}_h\varphi(t_n, x, y) \;=\; \big(\; \hat{\mathbb{E}}^M\big[\varphi(t_{n+1}, \hat{X}_h^{t_n,x}, y) H_i^{t_n,x,h}(\Delta W_{n+1})\big] \;:\; i = 0, 1, 2\;\big).$$

Pour obtenir des resultats de convergence des solutions numériques $\hat{v}^h$ du schéma $\mathbf{S}_h \circ \hat{\mathbf{T}}_h^M$, nous imposons deux types de conditions supplémentaires. La première porte sur le choix des fonctions de base $(e_k)_{1\le k\le K}$, et l'autre sur le nombre de simulations M. Nous omettons ici les conditions techniques et donnons juste les résultats de convergence :

Théorème 1.2.3. *Avec des conditions supplémentaires à celles du théorème 1.2.1, on a*

$$\hat{v}^h \quad\to\quad v \qquad \textit{localement uniformément, p.s.}$$

où $\hat{v}^h$ est la solution numérique du schéma $\mathbf{S}_h \circ \hat{\mathbf{T}}_h^M$ définie en (1.2.13), et v est l'unique solution de viscosité bornée de (1.2.6).

Théorème 1.2.4. *Avec des conditions supplémentaires à celles du théorème 1.2.2, il existe une constante C telle que*

$$\|v - \hat{v}^h\|_{L^2(\Omega)} \quad\le\quad Ch^{\frac{1}{10}}.$$

1.2.2.4 Exemples numériques

Nous implémentons notre schéma de "splitting" sur deux exemples. Le premier est le problème d'évaluation des options asiatiques dans un contexte avec volatilité incertaine. Il s'agit d'une équation non-linéaire dégénérée en dimension trois. Le second exemple traite d'un problème de gestion optimale d'une centrale hydro-électrique, qui donne une équation non-linéaire dégénérée en dimension quatre.

1.3 Troisième partie : transport optimal par les dynamiques stochastiques contrôlées

1.3.1 Motivations

Problème de transport de Monge En 1781, Monge [47] proposait un problème de transport optimal. Soient μ_0 et μ_1 deux distributions de masse sur $\mathbb{R}^d$, telles que $\mu_0(\mathbb{R}^d) =$

$\mu_1(\mathbb{R}^d) = 1$, i.e. μ_0 et μ_1 sont des mesures de probabilité sur $\mathbb{R}^d$. On dit qu'une application $T : \mathbb{R}^d \to \mathbb{R}^d$ est un plan de transport admissible si

$$X_1 \;:=\; T(X_0) \;\;\sim\;\; \mu_1 \quad \text{pour toutes les v.a. } X_0 \text{ ayant la distribution } \mu_0.$$

Un plan de transport admissible T peut être interprété comme un plan qui transporte une masse de la distribution μ_0 à la distribution μ_1. Et le problème de Monge cherche la solution de

$$\begin{aligned} & \inf \Big\{ \int_{\mathbb{R}^d} L(x, T(x))\, \mu_0(dx) \;:\; T \text{ plan de transport admissible} \Big\} \\ = \;& \inf \big\{ \,\mathbb{E}\, L(X_0, T(X_0)) \;:\; T \text{ plan de transport admissible et } X_0 \sim \mu_0 \big\}, \end{aligned}$$

où L est une fonction positive sur $\mathbb{R}^d \times \mathbb{R}^d$, et où $L(x,y)$ représente le coût du transport de la position x à la position y. Ce problème est difficile à résoudre à cause de la non-linéarité des contraintes, et est resté ouvert pendant de nombreuses années.

Relaxation de Kantorovich Dans les années 1940, Kantorovich [39] a réalisé une grande avancée sur ce problème. Il a proposé de relaxer les contraintes par la "randomisation" du plan T et a introduit ensuite une formulation duale. Soient μ_0 et μ_1 deux mesures de probabilité sur $\mathbb{R}^d$, un vecteur aléatoire (X_0, X_1) prenant ses valeurs dans $\mathbb{R}^d \times \mathbb{R}^d$ est dit admissible si ses distributions marginales pour X_0 et X_1 sont respectivement μ_0 et μ_1. Il en résulte le problème relaxé de Kantorovich :

$$\inf \big\{ \,\mathbb{E}\, L(X_0, X_1) \;:\; (X_0, X_1) \text{ vecteur admissible} \big\}. \tag{1.3.1}$$

Il est évident qu'avec un plan de transport admissible T et une variable aléatoire $X_0 \sim \mu_0$, on retrouve un vecteur aléatoire admissible (X_0, X_1) par $X_1 := T(X_0)$. Par contre, en général, un vecteur aléatoire admissible n'induit pas un plan de transport admissible.

Kantorovich a donc prouvé une dualité forte entre le problème (1.3.1) et

$$\sup \;\Big\{ \int_{\mathbb{R}^d} \psi(y)\mu_1(dy) \;-\; \int_{\mathbb{R}^d} \varphi(x)\mu_0(dx) \Big\},$$

où le sup est pris sur toutes les paires $(\varphi, \psi) \in L^1(\mu_0) \times L^1(\mu_1)$ vérifiant $\psi(y) - \varphi(x) \le L(x,y)$. L'avantage principal de cette formulation duale est qu'elle s'affranchit de la contrainte non-linéaire, et devient donc plus tractable.

Un mécanisme de tranport stochastique Mikami et Thieullen [46] ont récemment introduit un mécanisme de transport stochastique. Ils ont considéré l'ensemble de toutes les $\mathbb{R}^d$−semi-martingales continues $X = (X_t)_{0 \le t \le 1}$ avec la décomposition canonique :

$$X_t \;=\; X_0 \;+\; \int_0^t \beta_s ds \;+\; W_s, \tag{1.3.2}$$

où W est un mouvement brownien standard de dimension d par rapport à la filtration générée par le processus X. Soient μ_0 et μ_1 deux mesures de probabilité sur $\mathbb{R}^d$, nous définissons $\mathcal{A}(\mu_0, \mu_1)$ comme l'ensemble des semi-martingales X données par (1.3.2) telles que $X_0 \sim \mu_0$ et $X_1 \sim \mu_1$. Leur problème de transport optimal consiste à minimiser un coût de transport associé à la fonction ℓ :

$$V(\mu_0, \mu_1) \;:=\; \inf_{X \in \mathcal{A}(\mu_0,\mu_1)} \mathbb{E} \int_0^1 \ell(s, X_s, \beta_s) ds. \tag{1.3.3}$$

Ils ont prouvé également une dualité forte en fournissant la semi-continuité inférieure et la convexité de l'application $\mu_1 \mapsto V(\mu_0, \mu_1)$.

Généralisation Nous étendons le résultat ci-dessus à une classe plus grande de semi-martingales continues avec caractérisation :

$$X_t \;=\; X_0 + \int_0^t \beta_s ds + \int_0^t \sigma_s dW_s,$$

où le coût de transport dépend de β, σ ainsi que des trajectoires de X.

Premièrement, ce nouveau problème de transport optimal est relié au problème d'immersion de Skorokhod (ou “Skorokhod Embedding Problem” en anglais, SEP). Étant donnés une distribution μ_1 et un mouvement Brownian standard B, le SEP consiste à chercher un temps d'arrêt τ tel que $(B_{t\wedge\tau})_{t\geq 0}$ est uniformément intégrable et $B_\tau \sim \mu_1$. À partir d'une solution du SEP, on peut construire une martingale M par $M_t := B_{\tau \wedge \frac{t}{1-t}}$, et donc $M_1 \sim \mu_1$. Par ailleurs, étant donné une martingale M telle que $M_1 \sim \mu_1$, il résulte d'un argument de changement de temps qu'il induit une solution au SEP. Parmi une infinité de solutions du SEP, quelques-unes sont optimales par rapport à des critères specifiques. Nous nous référons à l'article de Obloj [48].

De plus, comme l'a remarqué Hobson [36], le SEP est connecté au problème de la recherche de bornes de prix sans-arbitrage des options financières exotiques étant donnés les prix des options européennes en maturité T. Dans cet esprit, Galichon, Henry-Labordère et Touzi [33] ont proposé récemment une approche de contrôle stochastique pour la recherche de bornes de prix sans-arbitrage des options exotiques, en considérant toutes les martingales vérifiant les contraintes marginales.

1.3.2 Résultats principaux

1.3.2.1 Le nouveau problème de transport optimal

Soit $\Omega := C([0,1], \mathbb{R}^d)$ l'espace canonique, avec le processus canonique $X_t(\omega) := \omega_t$ et la filtration canonique $\mathbb{F} = (\mathcal{F}_t)_{0\leq t\leq 1}$. Nous considérons toutes les mesures de probabilité $\mathbb{P}$ sur $(\Omega, \mathcal{F}_1)$ sous lesquelles X est une semi-martingale ayant la décomposition continue canonique :

$$X_t \;=\; X_0 \;+\; B_t^{\mathbb{P}} \;+\; M_t^{\mathbb{P}}, \quad t \in [0,1], \quad \mathbb{P}-\text{p.s.} \tag{1.3.4}$$

telle que $B^{\mathbb{P}} = (B_t^{\mathbb{P}})_{0\le t\le 1}$ et $A^{\mathbb{P}} = (A_t^{\mathbb{P}})_{0\le t\le 1} := (\langle M^{\mathbb{P}}\rangle_t)_{0\le t\le 1}$ sont tous p.s. absolument continus en t, et donc

$$A_t^{\mathbb{P}} \;=\; \int_0^t \alpha_s^{\mathbb{P}} ds \quad \text{et} \quad B_t^{\mathbb{P}} \;=\; \int_0^t \beta_s^{\mathbb{P}} ds, \quad t\in[0,1], \quad \mathbb{P}-\text{p.s.} \tag{1.3.5}$$

Soit U un sous-ensemble de $S_d\times\mathbb{R}^d$ fermé et convexe, nous définissons $\mathcal{P}$ comme l'ensemble des mesures de probabilité sous lesquelles X admet la décomposition (1.3.4), et vérifie (1.3.5) avec les caracteristiques $\nu_t^{\mathbb{P}} := (\alpha_t^{\mathbb{P}}, \beta_t^{\mathbb{P}}) \in U,\ d\mathbb{P}\times dt - p.p.$ Étant données deux mesures de probabilité μ_0 et μ_1 sur $\mathbb{R}^d$, on note

$$\mathcal{P}(\mu_0) \;:=\; \big\{\mathbb{P}\in\mathcal{P} \;:\; \mathbb{P}\circ X_0^{-1} = \mu_0\big\} \quad \text{et} \quad \mathcal{P}(\mu_0,\mu_1) \;:=\; \big\{\mathbb{P}\in\mathcal{P}(\mu_0) \;:\; \mathbb{P}\circ X_1^{-1} = \mu_1\big\}.$$

Sous toutes les mesures de probabilité $\mathbb{P}\in\mathcal{P}$, X est une semi-martingale continue et peut être considérée comme un moyen de transporter une masse de la $\mathbb{P}-$distribution de X_0 à la $\mathbb{P}-$distribution de X_1. Soit

$$L : (t,\mathbf{x},u)\in[0,1]\times\Omega\times U \;\mapsto\; L(t,\mathbf{x},u)\in\mathbb{R}^+$$

une fonction positive et convexe en u, nous associons à toutes les mesures $\mathbb{P}\in\mathcal{P}$ un coût de transport et introduisons donc le nouveau problème de transport :

$$V(\mu_0,\mu_1) \;:=\; \inf_{\mathbb{P}\in\mathcal{P}(\mu_0,\mu_1)} J(\mathbb{P}) \quad \text{avec} \quad J(\mathbb{P}) \;:=\; \mathbb{E}^{\mathbb{P}}\int_0^1 L(s,X,\alpha_s^{\mathbb{P}},\beta_s^{\mathbb{P}})ds. \tag{1.3.6}$$

1.3.2.2 Formulation duale

Nous prouvons une dualité forte pour le problème de transport (1.3.6). Pour établir la dualité, nous suivons la méthodologie classique de l'analyse convexe, i.e. montrant la semi-continuité inférieure et la convexité de l'application $\mu_1\mapsto V(\mu_0,\mu_1)$. Ces deux propriétés assurent que V coincïde avec son bi-conjugué, qui est la formulation duale demandée.

En fait, le conjugué de $\mu_1\mapsto V(\mu_0,\mu_1)$ est donné par

$$\begin{aligned} V^*(-\lambda_1) \;&:=\; \sup_{\mu_1}\big(\mu_1(-\lambda_1) - V(\mu_0,\mu_1)\big) \\ &=\; -\inf_{\mathbb{P}\in\mathcal{P}(\mu_0)} \mathbb{E}^{\mathbb{P}}\Big[\int_0^1 L(s,X,\alpha_s^{\mathbb{P}},\beta_s^{\mathbb{P}})ds + \lambda_1(X_1)\Big]. \end{aligned}$$

Il résulte d'un argument de programmation dynamique que

$$V^*(-\lambda_1) \;=-\mu_0(\lambda_0),\ \text{où}\ \lambda_0(x) \;:=\inf_{\mathbb{P}\in\mathcal{P}(\delta_x)} \mathbb{E}^{\mathbb{P}}\Big[\int_0^1 L(s,X,\alpha_s^{\mathbb{P}},\beta_s^{\mathbb{P}})ds + \lambda_1(X_1)\Big]. \tag{1.3.7}$$

Donc, le bi-conjugué de $\mu_1\mapsto V(\mu_0,\mu_1)$ est

$$\mathcal{V}(\mu_0,\mu_1) \;:=\; \sup_{\lambda_1\in C_b(\mathbb{R}^d)} \big(\mu_0(\lambda_0) - \mu_1(\lambda_1)\big). \tag{1.3.8}$$

Avec une condition supplémentaire, nous prouvons que $V(\mu_0, \mu_1)$ de (1.3.6) et $\mathcal{V}(\mu_0, \mu_1)$ de (1.3.8) sont équivalents à une formulation duale faible :

$$\overline{\mathcal{V}}(\mu_0, \mu_1) \;=\; \sup_{\lambda_1 \in C_b^\infty(\mathbb{R}^d)} \big(\mu_0(\lambda_0) - \mu_1(\lambda_1)\big). \tag{1.3.9}$$

1.3.2.3 Programmation dynamique dans le cas markovien

Dans le cas markovien, i.e. $L(t, \mathbf{x}, u) = \ell(t, \mathbf{x}(t), u)$ pour une fonction déterministe ℓ, nous pouvons caractériser la fonction valeur λ_0 donnée en (1.3.7) par une équation de programmation dynamique. En introduisant

$$\lambda(t, x) \;:=\; \inf_{\mathbb{P} \in \mathcal{P}_{t,x}} \mathbb{E}^{\mathbb{P}}\Big[\int_t^1 \ell(s, X_s, \nu_s^{\mathbb{P}})ds + \lambda_1(X_1)\Big],$$

où

$$\mathcal{P}_{t,x} \;:=\; \big\{\mathbb{P} \in \mathcal{P} \;:\; \mathbb{P}(X_s = x,\; 0 \le s \le t) = 1\;\big\} \text{ et } \nu_s^{\mathbb{P}} := (\alpha_s^{\mathbb{P}}, \beta_s^{\mathbb{P}}),$$

nous obtenons $\lambda_0(x) = \lambda(0, x)$, qui est la fonction valeur d'un problème de contrôle standard. Nous prouvons le principe de programmation dynamique :

$$\lambda(t, x) \;=\; \inf_{\mathbb{P} \in \mathcal{P}_{t,x}} \mathbb{E}^{\mathbb{P}}\Big[\int_t^\tau \ell(s, X_s, \nu_s^{\mathbb{P}})ds + \lambda(\tau, X_\tau)\Big], \tag{1.3.10}$$

pour tout $\mathbb{F}-$temps d'arrêt τ prenant sa valeur dans $[t, 1]$, puis nous caractérisons la fonction $\lambda(t, x)$ comme la solution de viscosité de l'équation de programmation dynamique

$$-\partial_t \lambda(t, x) - \inf_{(a,b) \in U} \big(b \cdot D\lambda(t, x) + a \cdot D^2\lambda(t, x) + \ell(t, x, a, b)\big) = 0. \tag{1.3.11}$$

L'idée pour prouver le principe de programmation dynamique (1.3.10) est de décomposer l'égalité (1.3.10) en inégalités "$\ge$" et "$\le$". L'inégalité "$\ge$" est prouvée par un argument de conditionnement, et l'inégalité "$\le$" est essentiellement basée sur la technique de concaténation de mesures de probabilité, où un argument de sélection mesurable est utilisé.

1.3.2.4 Résolution numérique du problème dual

Dans le cas markovien en dimension un, nous donnons un schéma numérique pour résoudre le problème dual (1.3.8). La résolution numérique est basée sur l'observation que l'application $\lambda_1 \mapsto \lambda_0(x)$ est concave car elle peut être représentée comme le minimum d'une classe d'applications linéaires en (1.3.7). Par conséquent, (1.3.8) est un problème de maximisation d'une fonction concave, et donc l'algorithme de gradient projeté est une méthode naturelle de résolution. L'approximation est divisée en quatre étapes.

Première approximation Premièrement, on note Lip_K^0 l'ensemble de fonctions $\phi : \mathbb{R}^d \to \mathbb{R}$ lipschitziennes de module K avec $\phi(0) = 0$, et on note $\mathrm{Lip}^0 := \cup_{K>0} \mathrm{Lip}_K^0$. Il résulte de l'équivalence de (1.3.8) et (1.3.9) que le problème dual (1.3.8) devient

$$V \;=\; \sup_{\lambda_1 \in \mathrm{Lip}^0} v(\lambda_1), \quad \text{avec } \; v(\lambda_1) \;:=\; \mu_0(\lambda_0) - \mu_1(\lambda_1).$$

De ce fait, la première approximation est donnée par

$$V^K \to V, \quad \text{où} \;\; V^K \;\; := \sup_{\lambda_1 \in \mathrm{Lip}_K^0} v(\lambda_1).$$

Deuxième approximation Pour la deuxième approximation, nous introduisons

$$\lambda_0^R(x) \;\; := \;\; \inf_{\mathbb{P}\in\mathcal{P}_{\delta_x}} \mathbb{E}^{\mathbb{P}}\Big[\int_0^{\tau_R\wedge 1} \ell(s, X_s, \alpha_s^{\mathbb{P}}, \beta_s^{\mathbb{P}})ds + \lambda_1(X_{\tau_R\wedge 1})\Big], \tag{1.3.12}$$

où $\tau_R := \inf\{t : \; X_t \notin [-R, R]\}$. Soit

$$V^{K,R} := \sup_{\lambda_1\in \mathrm{Lip}_0^K} v^R(\lambda_1), \;\; \text{où } v^R(\lambda_1) := \mu_0(\lambda_0^R \mathbf{1}_{O_R}) - \mu_1(\lambda_1 \mathbf{1}_{O_R}). \tag{1.3.13}$$

Nous prouvons ensuite un résultat de convergence

$$V^{K,R} \;\to\; V^K \quad \text{lorsque} \quad R \to \infty.$$

Troisième approximation La troisième approximation est une approximation du système discret. Soient $(l, r) \in \mathbb{N}^2$ et $h = (\Delta t, \Delta x) \in (\mathbb{R}^+)^2$ tels que $l\Delta t = 1$ et $r\Delta x = R$. Avec $x_i := i\Delta x$, $t_k := k\Delta t$, nous définissons les grilles :

$$\mathcal{N} := \{x_i \;:\; i \in \mathbb{Z}\}, \qquad \mathcal{N}_R := \mathcal{N} \cap (-R, R),$$

$$\mathcal{M}_{T,R} := \big\{(t_k, x_i) \;:\; (k, i) \in \mathbb{Z}^+ \times \mathbb{Z}\big\} \cap ([0, 1] \times (-R, R)),$$

et également l'ensemble terminal, l'ensemble au bord et l'ensemble intérieur de $\mathcal{M}_{T,R}$

$$\partial_T \mathcal{M}_{T,R} := \big\{(1, x_i) \;:\; x_i \in \mathcal{N}_R\big\}, \;\; \partial_R \mathcal{M}_{T,R} := \big\{(t_k, \pm R) \;:\; k = 0, \cdots, l\big\},$$

$$\mathring{\mathcal{M}}_{T,R} := \mathcal{M}_{T,R} \setminus (\partial_T \mathcal{M}_{T,R} \cup \partial_R \mathcal{M}_{T,R}).$$

Étant donnée une fonction w definie sur $\mathcal{M}_{T,R}$, nous introduisons les dérivées discrètes de w :

$$D^{\pm} w(t_k, x_i) := \frac{w(t_k, x_{i\pm 1}) - w(t_k, x_i)}{\Delta x} \quad \text{et} \quad (bD)w := b^+ D^+ w + b^- D^- w \text{ pour } b \in \mathbb{R},$$

où $b^+ := \max(0, b)$, $b^- := \max(0, -b)$; et

$$D^2 w(t_k, x_i) := \frac{w(t_k, x_{i+1}) - 2w(t_k, x_i) + w(t_k, x_{i-1})}{\Delta x^2}.$$

Alors, l'approximation de différences finies explicite de λ^R en (1.3.12) est donnée par

$$\hat{\lambda}^{h,R}(t_k, x_i) = \Big(\hat{\lambda}^{h,R} + \Delta t \inf_{u=(a,b)\in U} \Big\{\ell(\cdot, u) + (bD)\hat{\lambda}^{h,R} + \frac{1}{2} a D^2 \hat{\lambda}^{h,R}\Big\}\Big)(t_{k+1}, x_i) \;\; \text{sur } \mathring{\mathcal{M}}_{T,R}$$

$$\hat{\lambda}^{h,R}(t_k, x_i) = \hat{\lambda}_1(x_i) \;\; \text{sur } \partial_T \mathcal{M}_{T,R} \cup \partial_R \mathcal{M}_{T,R}, \tag{1.3.14}$$

et donc une approximation naturelle de v^R en (1.3.13) est

$$\hat{v}_h^R(\hat{\lambda}_1) := \mu_0\Big(\mathrm{lin}^R[\hat{\lambda}_0^{h,R}]\Big) - \mu_1\Big(\mathrm{lin}^R[\hat{\lambda}_1]\Big) \quad \text{avec} \quad \hat{\lambda}_0^{h,R} := \hat{\lambda}^{h,R}(0,\cdot), \tag{1.3.15}$$

où étant donnée une fonction ϕ definie sur la grille $\mathcal{N}_R$, on note $\mathrm{lin}^R[\phi]$ l'interpolation linéaire de ϕ étendue par zéro en dehors de $[-R, R]$.

Soit $\mathrm{Lip}_0^{K,R}$ l'ensemble des fonctions sur la grille $\mathcal{N}_R$ définies comme les restrictions des fonctions dans Lip_0^K :

$$\mathrm{Lip}_0^{K,R} \quad := \quad \big\{\hat{\lambda}_1 := \lambda_1|_{\mathcal{N}_R} \quad : \quad \lambda_1 \in \mathrm{Lip}_0^K\big\}.$$

L'approximation ci-dessus pour la fonction valeur λ suggère une approximation naturelle du mininum des coût de transport :

$$V_h^{K,R} \quad := \sup_{\hat{\lambda}_1 \in \mathrm{Lip}_0^{K,R}} \hat{v}_h^R(\hat{\lambda}_1) = \sup_{\hat{\lambda}_1 \in \mathrm{Lip}_0^{K,R}} \mu_0\Big(\mathrm{lin}^R[\hat{\lambda}_0^{h,R}]\Big) - \mu_1\Big(\mathrm{lin}^R[\hat{\lambda}_1]\Big). \tag{1.3.16}$$

Nous montrons alors la convergence :

$$V_h^{K,R} \quad \to \quad V^{K,R} \quad \text{as} \quad h \to 0.$$

Quatrième approximation La quatrième étape est l'algorithme de gradient projeté pour résoudre le système discret (1.3.16). Soient $(\gamma_n)_{n\geq 1}$ une suite de nombres réels, $\nabla\hat{v}_h^R$ le sur-gradient de $\hat{\lambda}_1 \to \hat{v}_h^R(\hat{\lambda}_1)$, et $P_{\mathrm{Lip}_0^{K,R}}(\phi)$ la projection de la fonction ϕ (définie sur $\mathcal{N}_R$) sur l'ensemble $\mathrm{Lip}_0^{K,R}$, l'algorithme de gradient projeté est donné par

$$\hat{\lambda}_1^{n+1} \quad = \quad P_{\mathrm{Lip}_0^{K,R}}\big(\hat{\lambda}_1^n + \gamma_n \nabla\hat{v}_h^R(\hat{\lambda}_1^n)\big). \tag{1.3.17}$$

Nous donnons également un sur-gradient

$$\nabla\hat{v}_h^R(\hat{\lambda}_1) \quad := \quad \big(\mu_0\big(\mathrm{lin}^R[g_0^j]\big) - \mu_1\big(\mathrm{lin}^R[\delta_j]\big)\big)_{-r\leq j\leq r},$$

où g^j est défini comme la solution du système :

$$\begin{cases} g^j(t_k, x_i) = \Big(g^j + \Delta t\Big(\big(\hat{b}_{k,i}(\hat{\lambda}_1)D\big)g^j + \hat{a}_{k,i}(\hat{\lambda}_1)D^2 g^j\Big)\Big)(t_{k+1}, x_i) \text{ sur } \mathring{\mathcal{M}}_{T,R}, \\ g^j(t_k, x_i) = \delta_{i,j}, \quad \text{on } \partial_T\mathcal{M}_{T,R} \cup \partial_R\mathcal{M}_{T,R}, \end{cases}$$

where $\hat{a}_{k,i}$ and $\hat{b}_{k,i}$ are the optimal controls given in (1.3.14).

Enfin, avec une projection simple $P_{\mathrm{Lip}_0^{K,R}}$, on obtient un résultat de convergence de notre algorithme gradient projeté (1.3.17) :

$$\max_{n\leq N} \hat{v}_h^R(\hat{\lambda}_1^n) \quad \to \quad V_h^{K,R}, \quad \text{lorsque } N \to \infty.$$

1.3.2.5 Exemple numérique

Nous implémentons l'algorithme de gradient projeté ci-dessus lorsque $\ell(t,x,a,b) = a$, donc $V = \int_{\mathbb{R}} x^2 \mu_1(dx) - \int_{\mathbb{R}} x^2 \mu_0(dx)$. Avec un ordinateur muni d'un processeur 2.4GHz, la programmation performe 10^5 itérations en 55.2 secondes, et fournit un résultat numérique avec une erreur inférieure à 1%.

1.4 Quatrième partie : une borne des prix sans-arbitrage des options sur variance

1.4.1 Motivations et formulations

Comme mentionné ci-dessus, le nouveau problème de transport optimal de la section 1.3 est motivé par un travail de Galichon, Henry-Labordère et Touzi [33], qui cherche la borne des prix sans-arbitrage des options exotiques dans un marché où les options européennes sont liquides. L'objectif principal de cette partie est de concevoir un schéma numérique pour trouver la borne des prix sans-arbitrage et la stratégie statique correspondante en options vanilla, lorsque cette option exotique est l'option sur variance.

Nous considérons un actif sous-jacent risqué X dont le processus de prix est une martingale de carré intégrable. Soient μ_0 et μ_1 les distributions marginales de X aux temps T_0 et T_1 respectivement, identifiées par les observations des prix des options vanille, nous considérons une option sur variance avec payoff $g(\langle X\rangle_{T_0,T_1}, X_{T_1})$, où g est une fonction continue lipschitzienne. Suivant le cadre de Galichon, Henry-Labordère et Touzi [33], la borne des prix sans-arbitrage peut être formulée comme

$$\inf_{\phi\in\mathrm{Quad}} \sup_{\mathbb{P}\in\mathcal{P}^2(\mu_0)} \left\{\mathbb{E}^{\mathbb{P}}\left[g(\langle X\rangle_{T_0,T_1}, X_{T_1}) - \phi(X_{T_1})\right] + \mu_1(\phi)\right\}, \tag{1.4.1}$$

où X est le processus canonique dans l'espace canonique Ω, $\mathcal{P}^2(\mu_0)$ représente l'ensemble des mesures de probabilité sur Ω sous lesquelles X est une martingale telle que $\mathbb{P}\circ X_0^{-1} = \mu_0$ et $\mathbb{E}^{\mathbb{P}}[X_1^2|X_0] < \infty,\ \mathbb{P}-p.s.$, et

$$\mathrm{Quad} \ := \ \left\{\ \phi:\mathbb{R}\to\mathbb{R} \text{ telle que } \sup_{x\in\mathbb{R}} \frac{|\phi(x)|}{1+|x|^2} < \infty\ \right\}.$$

Le choix de Quad comme l'ensemble des stratégies statiques admissibles est motivé par le fait que l'option "variance swap" (i.e. $g(t,x) = t$) peut être considérée comme une option européenne avec payoff $X_{T_1}^2$.

Par un argument de changement de temps, il est bien connu qu'une martingale locale peut être représentée comme un mouvement brownien changé de temps. Nous reformulons donc le problème (1.4.1) sous forme d'un problème d'arrêt optimal du mouvement brownien. Soient $B = (B_t)_{t\geq 0}$ un mouvement brownien standard de dimension un tel que

$B_0 = 0$, $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ sa filtration naturelle, on note

$$\mathcal{T}^\infty \;:=\; \big\{\ \mathcal{F}-\text{temps d'arrêt } \tau \text{ tel que } \mathbb{E}(\tau) < \infty\ \big\}. \tag{1.4.2}$$

Etant donné $\phi \in \text{Quad}$, nous définissons λ^ϕ et λ_0^ϕ par

$$\lambda^\phi(t,x) \;:=\; \sup_{\tau\in\mathcal{T}^\infty} \mathbb{E}\big[g(t+\tau, x+B_\tau) - \phi(x+B_\tau)\big], \quad \lambda_0^\phi(\cdot) \;:=\; \lambda^\phi(0,\cdot). \tag{1.4.3}$$

Alors, la borne des prix sans-arbitrage est donnée par

$$U \;:=\; \inf_{\phi\in\text{Quad}} u(\phi), \quad \text{avec} \quad u(\phi) \;:=\; \mu_0(\lambda_0^\phi) + \mu_1(\phi). \tag{1.4.4}$$

1.4.2 Résultats principaux

1.4.2.1 Approximation analytique

Dans le but de restreindre le calcul de U en (1.4.4) à un domaine borné, nous donnons tout d'abord une approximation analytique obtenue en quatre étapes.

Première étape Nous introduisons

$$\text{Quad}_0 \;:=\; \big\{\ \phi \in \text{Quad positif, convexe, telle que } \phi(0) = 0\ \big\},$$

et

$$U^K \;:=\; \inf_{\phi\in\text{Quad}_0^K} u(\phi), \quad \text{avec} \quad \text{Quad}_0^K \;:=\; \big\{\ \phi\in\text{Quad}_0 \;:\; \phi(x) \leq K(|x| \vee x^2)\ \big\}.$$

Nous prouvons une équivalence

$$U \;=\; \inf_{\phi\in\text{Quad}_0} u(\phi),$$

et obtenons une convergence naturelle

$$U^K \;\searrow\; U \quad \text{lorsque} \quad K\to\infty, \tag{1.4.5}$$

étant donné que $\text{Quad}_0 = \cup_{K\geq 0}\text{Quad}_0^K$.

Deuxième étape Nous introduisons

$$\text{Quad}_0^{K,M} \;:=\; \big\{\ \phi\in\text{Quad}_0^K \text{ telle que } \phi(x) = Kx^2 \text{ pour } |x|\geq 2M\big\},$$

et prouvons que

$$U^{K,M} \;:=\; \inf_{\phi\in\text{Quad}_0^{K,M}} u(\phi) \;\to\; U^K \quad \text{lorsque } M\to\infty. \tag{1.4.6}$$

Troisième étape Nous définissons

$$\lambda^{\phi,T}(t,x) \;:=\; \inf_{\tau\in\mathcal{T}^\infty,\ \tau\le T-t} \mathbb{E}[g(t+\tau,x+B_\tau)-\phi(x+B_\tau)] \;\text{ et }\; \lambda_0^{\phi,T}(\cdot) := \lambda^{\phi,T}(0,\cdot),$$

et

$$U^{K,M,T} \;:=\; \inf_{\phi\in\mathrm{Quad}_0^{K,M}} u^T(\phi) \;\text{ avec }\; u^T(\phi) \;:=\; \mu_0(\lambda_0^{\phi,T}) + \mu_1(\phi). \quad (1.4.7)$$

Nous prouvons que

$$U^{K,M,T} \;\to\; U^{K,M} \;\text{ lorsque } T\to\infty. \quad (1.4.8)$$

Quatrième étape Enfin, pour la quatrième étape, nous introduisons un temps d'arrêt

$$\tau_x^R \;:=\; \inf\{s \;:\; x+B_s \notin (-R,R)\},$$

et pour tout $R \ge (1+\sqrt{\frac{K}{K-L_0}})M$, nous obtenons une équivalence

$$\lambda^{\phi,T}(t,x) \;=\; \lambda^{\phi,T,R}(t,x) \;:=\; \inf_{\tau\in\mathcal{T}^\infty,\ \tau\le\tau_x^R} \mathbb{E}[g(t+\tau,x+B_\tau)-\phi(x+B_\tau)]. \quad (1.4.9)$$

Il est bien connu que $\lambda^{\phi,T,R}$ peut être caractérisée comme l'unique solution de viscosité de l'inégalité variationnelle

$$\min\Big\{\ \lambda(t,x)-g(t,x)+\phi(x),\quad -\partial_t\lambda - \frac{1}{2}D^2\lambda\ \Big\} \;=\; 0, \quad \text{sur } [0,T)\times(-R,R), \quad (1.4.10)$$

avec condition au bord

$$\lambda(t,x) = g(t,x)-\phi(x), \quad \text{sur } ([0,T]\times\{\pm R\}) \;\cup\; (\{T\}\times[-R,R]).$$

1.4.2.2 Approximation numérique

L'idée principale de l'approximation numérique de $U^{K,M,T}$ de (1.4.7) est d'utiliser le schéma de différences finies pour résoudre l'inégalité variationnelle (1.4.10), et puis utiliser l'algorithme gradient projeté pour résoudre un problème de minimisation d'une fonction convexe.

Soient $h = (\Delta t, \Delta x)$ les paramètres de discrétisation, $x_i = i\Delta x$ et $t_k = k\Delta t$, nous définissons les grilles

$$\mathcal{N}_R \;:=\; \{x_i \;:\, -r\le i\le r\} \quad\text{et}\quad \mathcal{M}_{T,R} \;:=\; \big\{(t_k,x_i) \;:\, 0\le k\le l,\ -r\le i\le r\big\}.$$

Avec une fonction w sur $\mathcal{M}_{T,R}$, nous introduisons une dérivée discrète

$$D_h^2 w(t_k,x_i) \;:=\; \frac{w(t_k,x_{i+1})-w(t_k,x_i)+w(t_k,x_{i-1})}{\Delta x^2}.$$

Puis, pour toute fonction φ sur la grille $\mathcal{N}_R$, nous notons $\lambda_h^{\varphi,T,R}$ la solution numérique du schéma de différences finies de (1.4.10) :

$$\begin{cases} \lambda_h^{T,R}(t_{k+1}, x_i) \; - \; \tilde{\lambda}_h^{T,R}(t_k, x_i) \\ \qquad + \; \frac{1}{2}\Delta t \, \Big(\theta \; D^2 \tilde{\lambda}_h^{T,R}(t_k, x_i) \; + \; (1-\theta) \; D^2 \lambda_h^{T,R}(t_{k+1}, x_i)\Big) \; = \; 0, \\ \lambda_h^{T,R}(t_k, x_i) \; = \; \max\Big(g(t_k, x_i) - \varphi(x_i) \; , \; \tilde{\lambda}_h^{T,R}(t_k, x_i)\Big). \end{cases} \tag{1.4.11}$$

Soit $\mathrm{lin}^R[\varphi]$ (resp. $\mathrm{lin}^R[\lambda_{h,0}^{\varphi,T,R}]$) l'interpolation linéaire de φ (resp. $\lambda_{h,0}^{\varphi,T,R} := \lambda_h^{\varphi,T,R}(0,\cdot)$) étendue par zéro en dehors de l'intervalle $[-R, R]$. Nous obtenons alors une approximation naturelle de $u_T(\phi)$ et $U^{K,M,T}$ en (1.4.7) :

$$u_{h,T}(\varphi) \; := \; \mu_0(\mathrm{lin}^R[\lambda_{h,0}^{\varphi,T,R}]) + \mu_1(\mathrm{lin}^R[\varphi]) \text{ et } U_h^{K,M,T} \; := \; \inf_{\varphi \in \mathrm{Quad}_{0,h}^{K,M}} u_{h,T}(\varphi), \tag{1.4.12}$$

où

$$\mathrm{Quad}_{0,h}^{K,M} \; := \; \big\{ \; \varphi := \phi|_{\mathcal{N}_R} \; : \; \phi \in \mathrm{Quad}_0^{K,M} \; \big\}.$$

Étant donné un résultat de convergence de $\lambda_h^{\varphi,T,R}$ vers $\lambda^{\phi,T,R}$, nous montrons que

$$U_h^{K,M,T} \; \to \; U^{K,M,T} \quad \text{lorsque } h \to 0. \tag{1.4.13}$$

Enfin, nous proposons un algorithme de gradient projeté pour approximer $U_h^{K,M,T}$ dans (1.4.12). Soit $(\gamma_n)_{n\geq 1}$ une suite de réels positifs, alors l'algorithme est donné par l'itération:

$$\varphi_{n+1} := P_{\mathrm{Quad}_{0,h}^{K,M}}\big[\varphi_n - \gamma_n \nabla u_{h,T}(\varphi_n)\big], \tag{1.4.14}$$

où $P_{\mathrm{Quad}_{0,h}^{K,M}}[\varphi]$ est la projection d'une fonction φ en $\mathrm{Quad}_{0,h}^{K,M}$, et $\nabla u_{h,T}(\varphi_n)$ est un sous-gradient de $\varphi \mapsto u_{h,T}(\varphi)$ en φ_n donné par

$$\nabla u_{h,T}(\varphi) \; := \; \big(\; \mu_0(\mathrm{lin}^R[p_0^j]) \; + \; \mu_1(\mathrm{lin}^R[e_j]) \; \big)_{-2m \leq j \leq 2m}$$

avec la solution unique $(p^j, \tilde{p}^j)$ du système linéaire sur $\mathcal{M}_{T,R}$:

$$\begin{cases} p^j(t_k, x_i) \; = \; -\,\delta_{i,j}, & (t_k, x_i) \in \partial_T \mathcal{M}_{T,R} \cup \partial_R \mathcal{M}_{T,R}, \\ p^j(t_{k+1}, x_i) - \tilde{p}^j(t_k, x_i) + \frac{1}{2}\Delta t \, \big(\theta D^2 \tilde{p}^j(t_k, x_i) + (1-\theta) D^2 p^j(t_{k+1}, x_i)\big) = 0, \\ p^j(t_k, x_i) = \begin{cases} \tilde{p}^j(t_k, x_i), & \text{si } \lambda_h^{\varphi,T,R}(t_k, x_i) < g^{\varphi}(t_k, x_i), \\ -\, e_j(x_i), & \text{sinon.} \end{cases} & (t_k, x_i) \in \mathring{\mathcal{M}}_{T,R}. \end{cases}$$

où $e_j \in B(\mathcal{N}_R)$ est donné par $e_j(x_i) := \delta_{i,j} = \begin{cases} 1, & \text{si } \; i = j, \\ 0, & \text{sinon.} \end{cases}$

Enfin, nous analysons la projection $P_{\mathrm{Quad}_{0,h}^{K,M}}$, et donnons un résultat de convergence de l'algorithme de gradient projeté :

$$\min_{n \leq N} \; u_{h,T}(\varphi_n) \; \to \; U_h^{K,M,T}. \tag{1.4.15}$$

1.4.2.3 Exemple numérique

Nous implémentons l'algorithme de gradient projeté dans le cas où l'option sur variance est le "variance swap", i.e. $g(t,x)=t$. Avec un ordinateur muni d'un processeur 2.4GHz, le temps de calcul est 57.24 secondes pour accomplir 4×10^4 itérations, et nous obtenons un résultat avec une erreur inférieure à 1%.

Chapitre 2

Introduction (English)

The main contributions of this thesis are on the numerical methods for a fully nonlinear degenerate parabolic partial differential equations (PDEs), and for a controlled nonlinear PDEs problem which appears naturally in the context of a new mass transportation problem. Some related subjects are also studied, such as the duality of the new mass transportation problem, the dynamic programming principle of stochastic control problems, applications of these numerical methods in finance, etc.

The thesis is divided into four parts. The first and second parts study two independents topics. One is on the monotonicity of finite difference $\theta-$scheme for a one-dimensional diffusion equation, the other is on the splitting numerical scheme for a degenerate nonlinear PDE.

The third and fourth parts are closely related. We first analyze a new mass transportation problem where the mass is transported by the controlled stochastic dynamics, and we minimize the transportation cost among all the dynamics satisfying the initial and terminal constraints. We derive a dual formulation and prove a strong duality of the new transportation problem. This extends the well-known Kantorovich duality to our context. We also propose a gradient-projection algorithm for the numerical resolution of the dual problem, and provide a convergence result. Such a problem is motivated by a problem of finding the optimal no-arbitrage bounds for the prices of exotic options given the observation of the implied volatility curve for some maturity T. We then discuss an example on variance options in the fourth part, where we use the gradient-projection algorithm to approximate the no-arbitrage bound of variance options. Because of the particular structure of the variance options, techniques for the approximations and for the proofs of the convergence may be different from those used in the transportation problem.

2.1 Part one: The monotonicity condition of $\theta-$scheme for diffusion equations

The monotonicity of numerical schemes is an important issue in numerical analysis. For example, in the convergence analysis in Chapiter 2 of Allaire [1], we may use the monotonicity to derive a $L^\infty-$stability of the scheme; in the context of Barles and Souganidis's [6] analysis, the monotonicity is a key criterion for the convergence of numerical schemes.

In this part, we are interested in the necessary and sufficient condition for the monotonicity of $\theta-$scheme for the one-dimensional diffusion equations. Let us consider the diffusion equation

$$\partial_t v(t,x) \;-\; \sigma^2(x) D^2_{xx} v(t,x) \;=\; 0, \tag{2.1.1}$$

with initial condition $v(0,x) = \Phi(x)$. The finite difference $\theta-$scheme for equation (2.1.1) is a linear system on the space grid $\mathcal{N}$:

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} - \sigma_i^2 \left(\theta \frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{\Delta x^2} + (1-\theta) \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2} \right) = 0, \tag{2.1.2}$$

where we are given a time and space discretization $h = (\Delta t, \Delta x)$, and with $t_n := n\Delta t$, $x_i := i\Delta x$, u_i^n denotes the numerical solution at point (t_n, x_i); the grid $\mathcal{N}$ is defined by $\mathcal{N} \;:=\; \{x_i \;:\; i \in \mathbb{N}\}$. When $\theta = 1$, the above scheme is an implicit finite difference scheme; and when $\theta = 0$, the above scheme (2.1.2) becomes to be an explicit scheme.

The CFL condition It is well known (see e.g. Lemma 2.2.13 of Allaire [1]) that the implicit scheme is unconditionally monotone, and the explicit scheme is monotone if and only if it satisfies the Courant-Friedrichs-Lewy (CFL) condition:

$$\frac{\bar\sigma^2 \Delta t}{\Delta x^2} \;\le\; \frac{1}{2}, \qquad \text{for} \quad \bar\sigma \;:=\; \sup_{i\in\mathbb{Z}} \sigma(x_i).$$

Since the $\theta-$scheme can be viewed as a combination of the explicit scheme and implicit scheme, it follows immediately that a sufficient condition of the monotonicity of $\theta-$scheme is

$$\frac{\bar\sigma^2 \Delta t}{\Delta x^2} \;\le\; \frac{1}{2(1-\theta)}, \qquad \text{for} \quad \bar\sigma \;:=\; \sup_{i\in\mathbb{Z}} \sigma(x_i). \tag{2.1.3}$$

The above condition (2.1.3) requires a discretization ratio $\Delta t = O(\Delta x^2)$ for the monotonicity when $\theta < 1$. We ask the question whether it is necessary.

The necessary and sufficient condition of monotonicity We derive the necessary and sufficient condition of the monotonicity of $\theta-$scheme, which confirms that the ratio $\Delta t = O(\Delta x^2)$ is necessary for the monotonicity. Moreover, in the case of heat equation, i.e. $\sigma(x) = \sigma_0$ for some constant σ_0, we get an explicit formula for the necessary and sufficient condition:

$$\frac{\sigma_0^2 \Delta t}{\Delta x^2} \;\le\; \frac{1}{2(1-\theta)} + \frac{\theta}{4(1-\theta)^2}, \tag{2.1.4}$$

which is clearly weaker than the CFL condition (2.1.3).

2.2 Part two: A splitting method for fully nonlinear degenerate parabolic PDEs

2.2.1 Motivations

The numerical methods for PDEs are largely developed in the literature, on finite difference, finite elements, semi-Lagrangian methods and Monte-Carlo methods. In general, the first three methods are relatively more efficient in low dimensional cases, they can be easily implemented and give reliable results. However, in high dimensional cases, the Monte-Carlo method is usually preferred if possible.

Splitting method Besides the numerical schemes cited above, another important technique used in numerical analysis is the splitting method. In many cases, it is used to reduce the dimension of the computation, or to improve the accuracy of the numerical scheme. The idea of splitting method is to split the PDE into two equations, to solve each equation separately and then to combine them together. To illustrate the idea, let us give an example of finite difference splitting method for the heat equation on $[0,T]\times\mathbb{R}\times\mathbb{R}$:

$$\partial_t v(t,x,y) \ - \ D^2_{xx} v(t,x,y) \ - \ D^2_{yy} v(t,x,y) \ = \ 0, \tag{2.2.1}$$

with initial condition $v(0,\cdot)=\Phi(\cdot)$. A splitting method first split the above equation into two equations

$$\partial_t v(t,x,y) \ + \ D^2_{xx} v(t,x,y) \ = \ 0 \quad \text{and} \quad \partial_t v(t,x,y) \ + \ D^2_{yy} v(t,x,y) \ = \ 0. \tag{2.2.2}$$

Given the discretization parameters $h=(\Delta t,\Delta x,\Delta y)$, we denote

$$D^{2,h}_{xx} v^h(t_n,x,y) \ := \ \frac{v^h(t_n,x+\Delta x,y)-2v^h(t_n,x,y)+v^h(t_n,x-\Delta x,y)}{\Delta x^2}$$

and

$$D^{2,h}_{yy} v^h(t_n,x,y) \ := \ \frac{v^h(t_n,x,y+\Delta y)-2v^h(t_n,x,y)+v^h(t_n,x,y-\Delta y)}{\Delta y^2}.$$

Then the explicit finite difference schemes for the above two equations (2.2.2) can be written as

$$v^h(t_{n+1},\cdot)=v^h(t_n,\cdot)+\Delta t\ D^{2,h}_{xx} v^h(t_n,\cdot) \ \text{ and } \ v^h(t_{n+1},\cdot)=v^h(t_n,\cdot)+\Delta t\ D^{2,h}_{yy} v^h(t_n,\cdot).$$

A splitting scheme for (2.2.1) can be written, with a fictitious time $t_{n+\frac{1}{2}}$, as

$$v^h(t_{n+\frac{1}{2}},\cdot)=v^h(t_n,\cdot)+\Delta t D^{2,h}_{xx} v^h(t_n,\cdot) \ \text{ and } v^h(t_{n+1},\cdot)=v^h(t_{n+\frac{1}{2}},\cdot)+\Delta t D^{2,h}_{yy} v^h(t_{n+\frac{1}{2}},\cdot).$$

It follows by a formal calculation that the above splitting scheme turns to be

$$v^h(t_{n+1},\cdot) \ = \ v^h(t_n,\cdot) \ + \ \Delta t\big(D^{2,h}_{xx} v^h(t_n,\cdot) \ + \ D^{2,h}_{yy} v^h(t_n,\cdot)\big) \ + \ O(\Delta t^2),$$

which is almost the same as the non-splitting explicit finite difference scheme for (2.2.1).

Monte-Carlo methods for PDE The Monte-Carlo method for PDE is related by the Feynman-Kac formula. Let $\mu : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : [0,T] \times \mathbb{R}^d \to S_d$ be Lipschitz in x uniformly in t and such that

$$\int_0^T \big(\, |\mu(t,0)|^2 \; + \; |\sigma(t,0)\sigma^T(t,0)| \, \big) \, dt \quad < \quad \infty.$$

Suppose that $v(t,x)$ is a smooth solution of the linear parabolic equation

$$-\mathcal{L}^X v(t,x) \; = \; 0 \text{ where } \mathcal{L}^X \; := \; \partial_t \; + \; \mu(t,x) \cdot D_x \; + \; \frac{1}{2}\sigma(t,x)\sigma^T(t,x) \cdot D^2_{xx}, \quad (2.2.3)$$

with terminal condition $v(T,x) = \Phi(x)$. Then it follows by Feynman-Kac formula that $v(t,x)$ has a probabilistic representation:

$$v(t,x) \quad = \quad \mathbb{E} \; \Phi(X_T^{t,x}), \qquad (2.2.4)$$

where $X^{t,x}$ is the unique strong solution of the stochastic differential equation (SDE)

$$X_s^{t,x} = x + \int_t^s \mu(r, X_r^{t,x}) dr + \int_t^s \sigma(r, X_r^{t,x}) dW_r, \quad \text{with Brownian motion } W. \qquad (2.2.5)$$

The Monte-Carlo method for the resolution of linear PDE (2.2.3) is to simulate random variable $X_T^{t,x}$, and with the simulations $(X_{T,m}^{t,x})_{1 \le m \le M}$, to approximate $v(t,x)$ by

$$\frac{1}{M} \sum_{m=1}^{M} \Phi(X_{T,m}^{t,x}).$$

By central limit theorem (CLT), its convergence rate is independent of the dimension d of the equation (2.2.3).

As an extension of Feynman-Kac formula, the Backward Stochastic Differential Equation (BSDE) opens a door for the resolution of semilinear equations by Monte-Carlo method. In this spirit, Fahim, Touzi and Warin [32] propose a Monte-Carlo method for a fully nonlinear parabolic PDE, which is closely related to second order BSDE. However, one limit of their method is that it only works in the nondegenerate cases.

Degenerate PDE In many financial problems, when the underlying variables involved in the pricing or optimization problems do not have a diffusion generator, their characterization equations may be degenerate. This is the case for Asian option pricing, optimal commodity trading problem, life insurance product pricing etc.

Example 2.2.1. Asian option pricing *Suppose that the price process of a risky asset S_t is defined by the Black-Scholes model: $dS_t = \sigma S_t dW_t$ with volatility σ and a standard Brownian motion W. An Asian option is an option with payoff $g(S_T, A_T)$ at maturity T, where $A_T = \int_0^T S_t dt$, and its price turns out to be characterized by the PDE:*

$$\partial_t v(t,s,a) \; + \; \frac{1}{2}\sigma^2 s^2 D^2_{ss} v(t,s,a) \; + \; s D_a v(t,x,a) \quad = \quad 0,$$

which is degenerate since $D^2_{aa} v(t,x,a)$ does not appear.

These applications motivate us to develop a Monte-Carlo method for high dimensional degenerate nonlinear equations.

2.2.2 Main results

2.2.2.1 The degenerate nonlinear PDE and splitting scheme

We consider the following degenerate nonlinear equation

$$-\ \mathcal{L}^X v(t,x,y)\ -\ F(\cdot,v,D_x v,D^2_{xx}v)(t,x,y)\ -\ H(\cdot,v,D_x v,D_y v)(t,x,y)\ =\ 0, \quad (2.2.6)$$

with terminal condition $v(T,\cdot)=\Phi(\cdot)$, where $\mathcal{L}^X$ is defined in (2.2.3), F is a nonlinear function defined on $[0,T)\times\mathbb{R}^d\times\mathbb{R}^{d'}\times\mathbb{R}\times\mathbb{R}^d\times S_d$, and H is a Hamiltonian defined by

$$H(t,x,y,r,p,q)\ :=\ \inf_{\alpha\in\mathcal{A}}\sup_{\beta\in\mathcal{B}}\ \big(l^{\alpha,\beta}(\cdot)+c^{\alpha,\beta}(\cdot)r+f^{\alpha,\beta}(\cdot)\cdot p+g^{\alpha,\beta}(\cdot)\cdot q\big)(t,x,y).$$

Based on the Monte-Carlo scheme of Fahim, Touzi and Warin [32] and the semi-Lagrangian scheme, we shall propose a splitting scheme $\mathbf{S}_h\circ\mathbf{T}_h$ for the degenerate nonlinear equation (2.2.6). Let $(t_n)_{0\le n\le N}$ be a discrete grid with $h:=\frac{T}{N}$ and $t_n:=nh$, we define our splitting scheme $\mathbf{S}_h\circ\mathbf{T}_h$ by

$$v^h(t_{n+\frac{1}{2}},x,y)\ :=\ \mathbb{E}\big[v^h(t_{n+1},\hat{X}^{t_n,x}_h,y)\big]\ +\ hF(t_n,x,y,\mathbb{E}\mathcal{D}_h v^h(t_n,x,y)), \quad (2.2.7)$$

and

$$\begin{aligned} v^h(t_n,x,y)\ &=\ \mathbf{S}_h\circ\mathbf{T}_h[v](t_n,x,y)\\ &:=\ \inf_{\alpha\in\mathcal{A}}\sup_{\beta\in\mathcal{B}}\Big\{\ h\ l^{\alpha,\beta}(t_n,x,y)\ +\ h\ c^{\alpha,\beta}(t_n,x,y)\ v^h(t_{n+\frac{1}{2}},x,y)\\ &\qquad +\ \ v^h\big(t_{n+\frac{1}{2}},\ x+f^{\alpha,\beta}(t_n,x,y)h,\ y+g^{\alpha,\beta}(t_n,x,y)h\big)\ \Big\}. \quad (2.2.8)\end{aligned}$$

In (2.2.7), $\hat{X}^{t,x}_h$ is defined by the Euler scheme of $X^{t,x}$ in (2.2.5) with

$$\hat{X}^{t,x}_h\ :=\ x\ +\ \mu(t,x)\ h\ +\ \sigma(t,x)\cdot(W_{t+h}-W_t),$$

and

$$\mathbb{E}\mathcal{D}_h v^h(t_n,x,y)\ :=\ \Big(\ \mathbb{E}\big[v^h(t_{n+1},\hat{X}^{t_n,x}_h,y)H^{t_n,x,h}_i(\Delta W_{n+1})\big]:i=0,1,2\ \Big), \quad (2.2.9)$$

where $\Delta W_{n+1}:=W_{t_{n+1}}-W_{t_n}$ and the Hermite polynomials are given by $H^{t,x,h}_0(z):=1$, $H^{t,x,h}_1(z):=\sigma^T(t,x)^{-1}\frac{z}{h}$ and $H^{t,x,h}_2(z):=\sigma^T(t,x)^{-1}\frac{zz^T-hI_d}{h^2}\sigma(t,x)^{-1}$.

2.2.2.2 The convergence results

We shall give two convergence results for the above splitting scheme $\mathbf{S}_h\circ\mathbf{T}_h$. The first one is a local uniform convergence in the context of Barles and Souganidis [6].

Assumption F : (i) The diffusion coefficients μ and σ are Lipschitz in x and continuous in t, $\sigma\sigma^T(t,x) > 0$ for all $(t,x) \in [0,T] \times \mathbb{R}^d$ and $\int_0^T \left|\sigma\sigma^T(t,0) + \mu(t,0)\right| dt < \infty$.

(ii) The nonlinear operator $F(t,x,y,r,p,\Gamma)$ is uniformly Lipschitz in (x,y,r,p,Γ), continuous in t and $|F(t,x,y,0,0,0)|_\infty < \infty$.

(iii) F is elliptic and satisfies

$$(\sigma\sigma^T)^{-1} \cdot F_\Gamma \;\leq\; 1 \quad \text{on } \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^{d'} \times \mathbb{R} \times \mathbb{R}^d \times S_d. \tag{2.2.10}$$

(iv) $F_p \in \text{Image}(F_\Gamma)$ and $\left|F_p^T F_\Gamma^{-1} F_p\right|_\infty < +\infty$.

Assumption H : The coefficients in Hamiltonian H are all uniformly bounded, i.e.

$$\sup_{(\alpha,\beta)\in\mathcal{A}\times\mathcal{B},\ 1\leq i\leq d,\ 1\leq j\leq d'} \left\{ \, |l^{\alpha,\beta}|_0 \;+\; |c^{\alpha,\beta}|_0 \;+\; |f_i^{\alpha,\beta}|_0 \;+\; |g_j^{\alpha,\beta}|_0 \, \right\} \;<\; \infty.$$

Theorem 2.2.1. *Let Assumptions* **F** *and* **H** *hold true, and assume that the degenerate fully nonlinear parabolic PDE* (2.2.6) *satisfies a comparison result for bounded viscosity solutions. Then for every bounded Lipschitz terminal condition function* Φ*, there exists a bounded function* v *such that*

$$v^h \;\longrightarrow\; v \quad \textit{locally uniformly as} \;\; h \to 0,$$

where v^h *is the numerical solution of scheme* $\mathbf{S}_h \circ \mathbf{T}_h$ *in* (2.2.8)*. Moreover,* v *is the unique bounded viscosity solution of the equation* (2.2.6) *with terminal condition* $v(T,\cdot) = \Phi(\cdot)$*.*

The second convergence result is the rate of convergence in the context of Barles and Jakobsen [5] where F and H are both concave Hamiltonians.

Assumption HJB : Assumption **F** holds and F is a concave Hamiltonian, i.e.

$$\mu \cdot p \;+\; \frac{1}{2} a \cdot \Gamma \;+\; F(t,x,y,r,p,\Gamma) \;=\; \inf_{\gamma\in\mathcal{C}} \mathcal{L}^\gamma(t,x,y,r,p,\Gamma),$$

with

$$\mathcal{L}^\gamma(t,x,y,r,p,\Gamma) \;:=\; l^\gamma(t,x,y) \;+\; c^\gamma(t,x,y)r \;+\; f^\gamma(t,x,y)\cdot p \;+\; \frac{1}{2} a^\gamma(t,x,y)\cdot\Gamma.$$

And $\mathcal{B} = \{\beta\}$ is a singleton, hence H is also a concave Hamiltonian, so that it can be written as

$$H(t,x,y,r,p,q) \;=\; \inf_{\alpha\in\mathcal{A}} \left\{ l^\alpha(t,x,y) \;+\; c^\alpha(t,x,y)r \;+\; f^\alpha(t,x,y)\cdot p \;+\; g^\alpha(t,x,y)\cdot q \right\}$$

Moreover, the functions l, c, f, g and σ satisfy that

$$\sup_{\alpha\in\mathcal{A},\gamma\in\mathcal{C}} \left(\, |l^\alpha + l^\gamma|_1 + |c^\alpha + c^\gamma|_1 + |f^\alpha + f^\gamma|_1 + |g^\alpha|_1 + |\sigma^\gamma|_1 \, \right) \;<\; \infty$$

Assumption HJB+ : Assumption **HJB** holds true, and for any $\delta > 0$, there exists a finite set $\{\alpha_i, \gamma_i\}_{i=1}^{I_\delta}$ such that for any $(\alpha, \gamma) \in \mathcal{A} \times \mathcal{C}$:

$$\inf_{1 \le i \le I_\delta} |l^\alpha - l^{\alpha_i}|_0 + |c^\alpha - c^{\alpha_i}|_0 + |f^\alpha - f^{\alpha_i}|_0 + |\sigma^\alpha - \sigma^{\alpha_i}|_0 \le \delta,$$

and

$$\inf_{1 \le i \le I_\delta} |l^\gamma - l^{\gamma_i}|_0 + |c^\gamma - c^{\gamma_i}|_0 + |f^\gamma - f^{\gamma_i}|_0 + |g^\gamma - g^{\gamma_i}|_0 \le \delta.$$

Theorem 2.2.2. *Suppose that the terminal condition function Φ is bounded and Lipschitz-continuous. Then there is a constant C such that*

- *i) under Assumption* **HJB**, *we have* $v - v^h \le C h^{\frac{1}{4}}$,
- *ii) under Assumption* **HJB+**, *we have* $-Ch^{\frac{1}{10}} \le v - v^h \le Ch^{\frac{1}{4}}$,

where v is the unique bouded viscosity solution of (2.2.6).

2.2.2.3 Simulation-regression method and numerical example

To make the splitting scheme $\mathbf{S}_h \circ \mathbf{T}_h$ implementable, we propose a simulation-regression method to estimate the conditional expectations (2.2.9) used in scheme $\mathbf{S}_h \circ \mathbf{T}_h$. The idea is to rewrite (2.2.9) as

$$\mathbb{E}\left[\varphi(t_{n+1}, \hat{X}_{t_{n+1}}, Y) H_i^{t_n, \hat{X}_{t_n}, h}(\Delta W_{n+1}) \;\Big|\; \hat{X}_{t_n},\ Y\right]_{i=0,1,2}, \tag{2.2.11}$$

where $\hat{X}$ is defined by the Euler scheme of X in (2.2.5):

$$\hat{X}_{t_{n+1}} := \hat{X}_{t_n} + \mu(t_n, \hat{X}_{t_n})h + \sigma(t_n, \hat{X}_{t_n})\Delta W_{n+1},$$

and Y is a random variable with continuous probability distribution. With M independent simulations $\big((\hat{X}^m_{t_n})_{0 \le n \le N}, (\Delta W^m_n)_{0 < n \le N}, Y^m\big)_{1 \le m \le M}$ of $\hat{X}$, ΔW and Y, and a function basis $(e_k(x,y))_{1 \le k \le K}$, we solve the least squares problem:

$$\hat{\lambda}^{i,M} = \arg\min_{\lambda} \sum_{m=1}^{M} \Big(\varphi(t_{n+1}, \hat{X}^m_{t_{n+1}}, Y^m) H_i^{t_n, \hat{X}^m_{t_n}, h}(\Delta W^m_{n+1}) - \sum_{k=1}^{K} \lambda_k e_k(\hat{X}^m_{t_n}, Y^m)\Big)^2,$$

which induces a raw regression estimation of conditional expectations (2.2.11) from these M samples:

$$\bar{\mathbb{E}}^M\Big[\varphi(t_{n+1}, \hat{X}_{t_{n+1}}, Y) H_i^{t_n, \hat{X}_{t_n}, h}(\Delta W_{n+1}) \big| \hat{X}_{t_n}, Y\Big] := \sum_{k=1}^{K} \hat{\lambda}_k^{i,M} e_k(\hat{X}_{t_n}, Y),\ i = 0,1,2.$$

Then with a priori upper bounds $\overline{\Gamma}_i(\hat{X}_{t_n}, Y)$ and lower bounds $\underline{\Gamma}_i(\hat{X}_{t_n}, Y)$, we define the regression estimation of (2.2.11) by

$$\begin{aligned} &\hat{\mathbb{E}}^M\Big[\varphi(t_{n+1}, \hat{X}_{t_{n+1}}, Y) H_i^{t_n, \hat{X}_{t_n}, h}(\Delta W_{n+1}) \;\big|\; \hat{X}_{t_n}, Y\Big] \\ := \; & \underline{\Gamma}_i(\hat{X}_{t_n}, Y) \vee \bar{\mathbb{E}}^M\Big[\varphi(t_{n+1}, \hat{X}_{t_{n+1}}, Y) H_i^{t_n, \hat{X}_{t_n}, h}(\Delta W_{n+1}) \big| \hat{X}_{t_n}, Y\Big] \wedge \overline{\Gamma}_i(\hat{X}_{t_n}, Y). \end{aligned} \tag{2.2.12}$$

Finally, we just replace the conditional expectations (2.2.9) in scheme $\mathbf{S}_h \circ \mathbf{T}_h$ by their regression estimations (2.2.12) and get the new numerical splitting scheme $\mathbf{S}_h \circ \hat{\mathbf{T}}_h^M$, which is

$$\hat{v}^h(t_{n+\frac{1}{2}}, x, y) \;:=\; \hat{\mathbb{E}}^M\big[\hat{v}^h(t_{n+1}, \hat{X}_h^{t_n,x}, y)\big] \;+\; h\, F(\cdot, \hat{\mathbb{E}}^M \mathcal{D}\hat{v}^h(\cdot))(t_n, x, y),$$

and

$$\begin{aligned}\hat{v}^h(t_n, x, y) \;&=\; \mathbf{S}_h \circ \hat{\mathbf{T}}_h^M[\hat{v}^h](t_n, x, y) \\ &:=\; \inf_{\alpha\in\mathcal{A}} \sup_{\beta\in\mathcal{B}} \Big\{ h l^{\alpha,\beta}(t_n, x, y) \;+\; h c^{\alpha,\beta}(t_n, x, y)\hat{v}^h(t_{n+\frac{1}{2}}, x, y) \\ &\qquad +\; \hat{v}^h\big(t_{n+\frac{1}{2}},\; x + f^{\alpha,\beta}(t_n, x, y)h,\; y + g^{\alpha,\beta}(t_n, x, y)h\big)\Big\} \end{aligned} \tag{2.2.13}$$

where

$$\hat{\mathbb{E}}^M \mathcal{D}_h\varphi(t_n, x, y) \;=\; \Big(\, \hat{\mathbb{E}}^M\big[\varphi(t_{n+1}, \hat{X}_h^{t_n,x}, y) H_i^{t_n,x,h}(\Delta W_{n+1})\big] \;:\; i = 0, 1, 2 \,\Big).$$

To derive a convergence result for the numerical solution $\hat{v}^h$ of scheme $\mathbf{S}_h \circ \hat{\mathbf{T}}_h^M$, we impose additional assumptions essentially in two aspects. One is the choice of function basis $(e_k)_{1\le k\le K}$, and the other is on the number of simulations M. Here, we shall omit the technical conditions and just cite the convergence results:

Theorem 2.2.3. *With additional assumptions to Theorem 2.2.1, we have*

$$\hat{v}^h \;\to\; v \quad \text{locally uniformly, a.s.}$$

where $\hat{v}^h$ is the numerical solution of scheme $\mathbf{S}_h \circ \hat{\mathbf{T}}_h^M$ defined in (2.2.13), and v is the unique bounded viscosity solution of (2.2.6).

Theorem 2.2.4. *With additional assumptions to Theorem 2.2.2, there a constant C such that*

$$\|v - \hat{v}^h\|_{L^2(\Omega)} \;\le\; C h^{\frac{1}{10}}.$$

2.2.2.4 Numerical examples

We implement our splitting scheme for two examples. One is the Asian option pricing problem in the uncertain volatility model with Hull-White interest rate, which gives a three dimensional (in space) nonlinear degenerate parabolic PDE. The other is a problem of optimal management of a hydropower plant, which involves with a four dimensional nonlinear degenerate parabolic PDE.

2.3 Part three: Optimal transportation under controlled stochastic dynamics

2.3.1 Motivations

Monge's transportation problem In 1781, Monge [47] proposed a mass transportation problem. Let μ_0 and μ_1 be two mass distributions on $\mathbb{R}^d$, such that $\mu_0(\mathbb{R}^d) = \mu_1(\mathbb{R}^d) =$

1, i.e. μ_0 and μ_1 are probability measures on $\mathbb{R}^d$. We say a map $T : \mathbb{R}^d \to \mathbb{R}^d$ is an admissible transport plan if

$$X_1 := T(X_0) \sim \mu_1 \quad \text{whenever the r.v. } X_0 \text{ has the distribution } \mu_0.$$

An admissible transport plan T can be interpreted as a map which transport a mass from distribution μ_0 to distribution μ_1. Then Monge's transportation problem searches for the solution to

$$\begin{aligned} & \inf \Big\{ \int_{\mathbb{R}^d} L(x, T(x))\, \mu_0(dx) \ : \ T \text{ admissible transport plan} \Big\} \\ = \ & \inf \big\{ \mathbb{E}\, L(X_0, T(X_0)) \ : \ T \text{ admissible transport plan and } X_0 \sim \mu_0 \big\}, \end{aligned}$$

where L is a positive function defined on $\mathbb{R}^d \times \mathbb{R}^d$, whose value $L(x, y)$ represents the cost of transportation from x to y. This problem is difficult to solve because of the fully nonlinearity of the constraints, and remained open for many years.

Kantorovich's relaxation A breakthrough was made by Kantorovich [39] in 1940s by relaxing the constraints and then introducing the dual formulation. Let μ_0 and μ_1 be two probability measures on $\mathbb{R}^d$, a random vector (X_0, X_1) taking value in $\mathbb{R}^d \times \mathbb{R}^d$ is said to be admissible if its marginal distribution for X_0 and X_1 are respectively μ_0 and μ_1. Then Kantorovich proposed the relaxed optimal transportation problem:

$$\inf \big\{ \mathbb{E}\, L(X_0, X_1) \ : \ (X_0, X_1) \text{ admissible random vector} \big\}. \tag{2.3.1}$$

Clearly, given an admissible transport plan T as well as a random variable $X_0 \sim \mu_0$, it follows that (X_0, X_1) with $X_1 := T(X_0)$ forms an admissible random vector. However, in general, an admissible random vector may not induce an admissible transport plan.

Kantorovich then proved a strong duality between the problem (2.3.1) and

$$\sup \Big\{ \int_{\mathbb{R}^d} \psi(y)\mu_1(dy) \ - \ \int_{\mathbb{R}^d} \varphi(x)\mu_0(dx) \Big\},$$

where the supremum is taken over all pairs $(\varphi, \psi) \in L^1(\mu_0) \times L^1(\mu_1)$ satisfying $\psi(y) - \varphi(x) \le L(x, y)$. The main advantage of the dual formulation is that it gets rid of the nonlinear constraints, and then becomes solvable.

Stochastic transportation mechanism Recently, Mikami and Thieullen [46] introduced a stochastic transportation mechanism. They considered the collection of all continuous $\mathbb{R}^d$−semimartingales $X = (X_t)_{0 \le t \le 1}$ with canonical decomposition:

$$X_t \ = \ X_0 \ + \ \int_0^t \beta_s ds \ + \ W_s, \tag{2.3.2}$$

where W is a d−dimensional standard Brownian motion with respect to the filtration generated by process X. Let μ_0 and μ_1 be two probability measures on $\mathbb{R}^d$, denote by $\mathcal{A}(\mu_0, \mu_1)$

the collection of all semimartingales X given by (2.3.2) such that $X_0 \sim \mu_0$ and $X_1 \sim \mu_1$, then their optimal transportation problem consists in minimizing the transportation cost defined by a cost function ℓ:

$$V(\mu_0, \mu_1) \quad := \quad \inf_{X \in \mathcal{A}(\mu_0, \mu_1)} \mathbb{E} \int_0^1 \ell(s, X_s, \beta_s) ds. \tag{2.3.3}$$

Finally, they proved a strong duality by providing the lower semi-continuity and convexity of $\mu_1 \mapsto V(\mu_0, \mu_1)$.

The generalization We extend this result to a larger class of continuous semimartingales with characterization:

$$X_t \quad = \quad X_0 + \int_0^t \beta_s ds + \int_0^t \sigma_s dW_s,$$

where the transportation cost depends on the drift and diffusion coefficients as well as the trajectory of X.

First, this new mass transportation problem is intimately connected to the Skorokhod Embedding Problem (SEP). Given a distribution μ_1 and a standard Brownian motion B, the SEP searches for a stopping time τ such that $(B_{t\wedge\tau})_{t\geq 0}$ is uniformly integrable and $B_\tau \sim \mu_1$. From a SEP solution, we can construct a martingale M by $M_t := B_{\tau \wedge \frac{t}{1-t}}$ so that $M_1 \sim \mu_1$. On the other hand, given a martingale M such that $M_1 \sim \mu_1$, it follows by a time-change argument that it gives a solution to the SEP. Among infinite solutions of the SEP, some of them have optimal properties with respect to the specific cost functions. Let us refer to Obloj [48] for a review of the SEP.

Next, as observed by Hobson [36], the SEP is connected to the problem of finding the optimal model-free no-arbitrage bounds of exotic options given the observations of the prices of vanilla options with maturity T and all strikes. Rececently, Galichon, Henry-Labordère and Touzi [33] proposed a framework to compute the model-free price bounds of the exotic options in a vanilla-liquid market. In their model, the marginal distributions of the underlying at some maturities are identified by the observations of vanilla price. They then propose a price bound for exotic options by considering all the martingales satisfying the marginal constraints.

2.3.2 Main results

2.3.2.1 New mass transportation problem

Let $\Omega := C([0,1], \mathbb{R}^d)$ be the canonical space, with canonical process $X_t(\omega) := \omega_t$ and canonical filtration $\mathbb{F} = (\mathcal{F}_t)_{0\leq t\leq 1}$. We shall consider the probability measure $\mathbb{P}$ under which X is a semimartingale having the canonical continuous decomposition:

$$X_t \quad = \quad X_0 \; + \; B_t^{\mathbb{P}} \; + \; M_t^{\mathbb{P}}, \quad t \in [0,1], \quad \mathbb{P} - \text{a.s.} \tag{2.3.4}$$

such that $B^{\mathbb{P}} = (B^{\mathbb{P}}_t)_{0\le t\le 1}$ and $A^{\mathbb{P}} = (A^{\mathbb{P}}_t)_{0\le t\le 1} := (\langle M^{\mathbb{P}}\rangle_t)_{0\le t\le 1}$ are both almost surely absolutely continuous in t, so that

$$A^{\mathbb{P}}_t \;=\; \int_0^t \alpha^{\mathbb{P}}_s ds \quad \text{and} \quad B^{\mathbb{P}}_t \;=\; \int_0^t \beta^{\mathbb{P}}_s ds, \quad t\in[0,1], \quad \mathbb{P}-\text{a.s.} \tag{2.3.5}$$

Let U be a closed and convex subset of $S_d\times\mathbb{R}^d$, we denote by $\mathcal{P}$ the collection of all probability measures on Ω under which X has the decomposition (2.3.4), and satisfies (2.3.5) with characteristics $\nu^{\mathbb{P}}_t := (\alpha^{\mathbb{P}}_t, \beta^{\mathbb{P}}_t) \in U$, $d\mathbb{P}\times dt - a.e.$ Given two probability measures μ_0 and μ_1 on $\mathbb{R}^d$, we also denote

$$\mathcal{P}(\mu_0) \;:=\; \big\{\mathbb{P}\in\mathcal{P} \;:\; \mathbb{P}\circ X_0^{-1}=\mu_0\big\} \quad \text{and} \quad \mathcal{P}(\mu_0,\mu_1) \;:=\; \big\{\mathbb{P}\in\mathcal{P}(\mu_0) \;:\; \mathbb{P}\circ X_1^{-1}=\mu_1\big\}.$$

Under the probability $\mathbb{P}\in\mathcal{P}$, X is a continuous semimartingale, and it represents a vehicle to transport a mass from the $\mathbb{P}-$distribution of X_0 to the $\mathbb{P}-$distribution of X_1. Let

$$L : (t,\mathbf{x},u)\in[0,1]\times\Omega\times U \;\mapsto\; L(t,\mathbf{x},u)\in\mathbb{R}^+$$

be a positive function convex in u. We then associate with every $\mathbb{P}\in\mathcal{P}$ a cost and introduce our mass transportation problem:

$$V(\mu_0,\mu_1) \;:=\; \inf_{\mathbb{P}\in\mathcal{P}(\mu_0,\mu_1)} J(\mathbb{P}) \qquad \text{with} \;\; J(\mathbb{P}) \;:=\; \mathbb{E}^{\mathbb{P}}\int_0^1 L(s,X,\alpha^{\mathbb{P}}_s,\beta^{\mathbb{P}}_s)ds. \tag{2.3.6}$$

2.3.2.2 Dual formulation

We prove a strong duality for the transportation problem (2.3.6). To do this, we follow the general methodology in convex analysis, proving the lower semi-continuity and convexity of $\mu_1\mapsto V(\mu_0,\mu_1)$. These properties ensure that V coincides with its convex bi-conjugate, which is the required dual formulation.

Indeed, we can easily compute the conjugate of $\mu_1\mapsto V(\mu_0,\mu_1)$

$$\begin{aligned} V^*(-\lambda_1) \;&:=\; \sup_{\mu_1}\big(\mu_1(-\lambda_1) - V(\mu_0,\mu_1)\big) \\ &=\; -\inf_{\mathbb{P}\in\mathcal{P}(\mu_0)} \mathbb{E}^{\mathbb{P}}\Big[\int_0^1 L(s,X,\alpha^{\mathbb{P}}_s,\beta^{\mathbb{P}}_s)ds + \lambda_1(X_1)\Big]. \end{aligned}$$

By a dynamic programming argument, we get

$$V^*(-\lambda_1) = -\mu_0(\lambda_0), \;\text{where}\; \lambda_0(x) := \inf_{\mathbb{P}\in\mathcal{P}(\delta_x)} \mathbb{E}^{\mathbb{P}}\Big[\int_0^1 L(s,X,\alpha^{\mathbb{P}}_s,\beta^{\mathbb{P}}_s)ds + \lambda_1(X_1)\Big]. \tag{2.3.7}$$

Then the bi-conjugate of $\mu_1\mapsto V(\mu_0,\mu_1)$ turns out to be

$$\mathcal{V}(\mu_0,\mu_1) \;:=\; \sup_{\lambda_1\in C_b(\mathbb{R}^d)} \big(\mu_0(\lambda_0) - \mu_1(\lambda_1)\big). \tag{2.3.8}$$

With some strengthened conditions, we prove that $V(\mu_0,\mu_1)$ in (2.3.6) and $\mathcal{V}(\mu_0,\mu_1)$ in (2.3.8) are equivalent to a weaker dual formulation

$$\overline{\mathcal{V}}(\mu_0,\mu_1) \;=\; \sup_{\lambda_1\in C_b^\infty(\mathbb{R}^d)} \big(\mu_0(\lambda_0) - \mu_1(\lambda_1)\big). \tag{2.3.9}$$

2.3.2.3 Dynamic programming in the Markovian case

In the Markovian case, i.e. $L(t, \mathbf{x}, u) = \ell(t, \mathbf{x}(t), u)$ for some deterministic function ℓ, we can characterize the value function λ_0 defined in (2.3.7) by a dynamic programming equation. By introducing

$$\lambda(t,x) \quad := \quad \inf_{\mathbb{P}\in\mathcal{P}_{t,x}} \mathbb{E}^{\mathbb{P}}\Big[\int_t^1 \ell(s, X_s, \nu_s^{\mathbb{P}})ds + \lambda_1(X_1)\Big],$$

where

$$\mathcal{P}_{t,x} \quad := \quad \big\{\mathbb{P}\in\mathcal{P} \ : \ \mathbb{P}(X_s = x,\ 0\le s\le t) = 1\ \big\} \text{ and } \nu_s^{\mathbb{P}} := (\alpha_s^{\mathbb{P}}, \beta_s^{\mathbb{P}}),$$

we see that $\lambda_0(x) = \lambda(0,x)$, and it is reduced to be the value function of a standard Markovian control problem. We prove the dynamic programming principle:

$$\lambda(t,x) \quad = \quad \inf_{\mathbb{P}\in\mathcal{P}_{t,x}} \mathbb{E}^{\mathbb{P}}\Big[\int_t^\tau \ell(s, X_s, \nu_s^{\mathbb{P}})ds + \lambda(\tau, X_\tau)\Big], \tag{2.3.10}$$

for every $\mathbb{F}-$stopping time τ taking value in $[t,1]$, and then characterize the function $\lambda(t,x)$ as a viscosity solution of the dynamic programming equation

$$-\partial_t\lambda(t,x) - \inf_{(a,b)\in U}\big(b\cdot D\lambda(t,x) + a\cdot D^2\lambda(t,x) + \ell(t,x,a,b)\big) = 0. \tag{2.3.11}$$

The main idea to prove the dynamic programming principle (2.3.10) is to decompose the equality (2.3.10) into the inequalities "$\ge$" and "$\le$". Then the inequality "$\ge$" can be proved by a conditioning arguments, and the inequality "$\le$" is based on a concatenation technique of probability measures, where a measurable selection argument is used.

2.3.2.4 Numerical resolution of the dual problem

In the one-dimensional Markovian case, we give a numerical scheme to solve the dual problem (2.3.8). The numerical resolution is based on a crucial observation that $\lambda_1 \mapsto \lambda_0(x)$ is a concave mapping since it is represented as the infimum of a class of linear mappings in (2.3.7). Therefore, (2.3.8) turns to be a maximization problem of a concave function and a natural method for its numerical resolution is the gradient projection algorithm. The approximation is divided into four steps.

First approximation First, we define Lip_K^0 as the collection of all bounded $K-$Lipschitz-continuous function $\phi : \mathbb{R}^d \to \mathbb{R}$ with $\phi(0) = 0$, and denote $\mathrm{Lip}^0 := \cup_{K>0}\mathrm{Lip}_K^0$. It follows by the equivalence between (2.3.8) and (2.3.9) that the dual problem (2.3.8) can be reduced to be

$$V \quad = \quad \sup_{\lambda_1\in\mathrm{Lip}^0} v(\lambda_1) \quad \text{with} \quad v(\lambda_1) \ := \ \mu_0(\lambda_0) - \mu_1(\lambda_1).$$

Then the first approximation is given by

$$V^K \to V, \qquad \text{where} \ \ V^K \quad := \quad \sup_{\lambda_1\in\mathrm{Lip}_K^0} v(\lambda_1).$$

Second approximation For the second approximation, we introduce

$$\lambda_0^R(x) \;:=\; \inf_{\mathbb{P}\in\mathcal{P}_{\delta_x}} \mathbb{E}^{\mathbb{P}}\Big[\int_0^{\tau_R\wedge 1} \ell(s,X_s,\alpha_s^{\mathbb{P}},\beta_s^{\mathbb{P}})ds + \lambda_1(X_{\tau_R\wedge 1})\Big], \tag{2.3.12}$$

where $\tau_R := \inf\{t:\ X_t \notin [-R,R]\}$. Let

$$V^{K,R} := \sup_{\lambda_1\in \mathrm{Lip}_0^K} v^R(\lambda_1), \quad \text{where } v^R(\lambda_1) := \mu_0(\lambda_0^R \mathbf{1}_{[-R,R]}) - \mu_1(\lambda_1 \mathbf{1}_{[-R,R]}). \tag{2.3.13}$$

We then derive a convergence result

$$V^{K,R} \;\to\; V^K \quad \text{as} \quad R\to\infty.$$

Third approximation The third approximation is a discrete system approximation. Let $(l,r)\in\mathbb{N}^2$ and $h=(\Delta t,\Delta x)\in(\mathbb{R}^+)^2$ be such that $l\Delta t = 1$ and $r\Delta x = R$. Denote $x_i := i\Delta x$, $t_k := k\Delta t$ and define the discrete grids:

$$\mathcal{N} := \{x_i\ :\ i\in\mathbb{Z}\}, \qquad \mathcal{N}_R := \mathcal{N}\cap(-R,R),$$

$$\mathcal{M}_{T,R} := \big\{(t_k,x_i)\ :\ (k,i)\in\mathbb{Z}^+\times\mathbb{Z}\big\}\cap([0,1]\times(-R,R)).$$

The terminal set, boundary set as well as interior set of $\mathcal{M}_{T,R}$ are denoted by

$$\partial_T\mathcal{M}_{T,R} := \big\{(1,x_i)\ :\ x_i\in\mathcal{N}_R\big\},\ \partial_R\mathcal{M}_{T,R} := \big\{(t_k,\pm R)\ :\ k=0,\cdots,l\big\},$$

$$\mathring{\mathcal{M}}_{T,R} := \mathcal{M}_{T,R}\setminus(\partial_T\mathcal{M}_{T,R}\cup\partial_R\mathcal{M}_{T,R}).$$

For a function w defined on $\mathcal{M}_{T,R}$, we introduce the discrete derivatives of w:

$$D^{\pm}w(t_k,x_i) := \frac{w(t_k,x_{i\pm1})-w(t_k,x_i)}{\Delta x} \quad \text{and} \quad (bD)w := b^+D^+w + b^-D^-w \text{ for } b\in\mathbb{R},$$

where $b^+ := \max(0,b)$, $b^- := \max(0,-b)$; and

$$D^2w(t_k,x_i) := \frac{w(t_k,x_{i+1})-2w(t_k,x_i)+w(t_k,x_{i-1})}{\Delta x^2}.$$

Then an explicit finite difference approximation of λ^R in (2.3.12) is given by the equation

$$\begin{aligned} &\hat\lambda^{h,R}(t_k,x_i) = \Big(\hat\lambda^{h,R} + \Delta t \inf_{u=(a,b)\in U}\Big\{\ell(\cdot,u) + (bD)\hat\lambda^{h,R} + \frac{1}{2}aD^2\hat\lambda^{h,R}\Big\}\Big)(t_{k+1},x_i) \ \text{ on } \mathring{\mathcal{M}}_{T,R} \\ &\hat\lambda^{h,R}(t_k,x_i) = \hat\lambda_1(x_i) \ \text{ on } \partial_T\mathcal{M}_{T,R}\cup\partial_R\mathcal{M}_{T,R}, \end{aligned} \tag{2.3.14}$$

and a natural approximation of v^R in (2.3.13) is given by

$$\hat v_h^R(\hat\lambda_1) := \mu_0\Big(\mathrm{lin}^R[\hat\lambda_0^{h,R}]\Big) - \mu_1\Big(\mathrm{lin}^R[\hat\lambda_1]\Big) \quad \text{with} \quad \hat\lambda_0^{h,R} := \hat\lambda^{h,R}(0,\cdot), \tag{2.3.15}$$

where for all function ϕ defined on the grid $\mathcal{N}_R$, we denote by $\mathrm{lin}^R[\phi]$ the linear interpolation of ϕ extended by zero outside $[-R,R]$.

Let $\mathrm{Lip}_0^{K,R}$ be the collection of all functions on the grid $\mathcal{N}_R$ defined as restrictions of functions in Lip_0^K:

$$\mathrm{Lip}_0^{K,R} \quad := \quad \big\{\hat{\lambda}_1 := \lambda_1|_{\mathcal{N}_R} \text{ for some } \lambda_1 \in \mathrm{Lip}_0^K\big\}.$$

The above approximation of the dynamic value function λ suggests the following natural approximation of the minimal transportation cost value:

$$V_h^{K,R} \quad := \quad \sup_{\hat{\lambda}_1 \in \mathrm{Lip}_0^{K,R}} \hat{v}_h^R(\hat{\lambda}_1) = \sup_{\hat{\lambda}_1 \in \mathrm{Lip}_0^{K,R}} \mu_0\Big(\mathrm{lin}^R[\hat{\lambda}_0^{h,R}]\Big) - \mu_1\Big(\mathrm{lin}^R[\hat{\lambda}_1]\Big). \tag{2.3.16}$$

And we derive a convergence

$$V_h^{K,R} \quad \to \quad V^{K,R} \quad \text{as} \quad h \to 0.$$

Fourth approximation The fourth step is a gradient projection algorithm to solve the discrete system (2.3.16). Let $(\gamma_n)_{n\geq 1}$ be sequence of real numbers, $\nabla \hat{v}_h^R$ denote a super-gradient of $\hat{\lambda}_1 \to \hat{v}_h^R(\hat{\lambda}_1)$, and $P_{\mathrm{Lip}_0^{K,R}}(\phi)$ denote the projection of a function ϕ defined on $\mathcal{N}_R$ to the set $\mathrm{Lip}_0^{K,R}$, the gradient projection algorithm is defined by:

$$\hat{\lambda}_1^{n+1} \quad = \quad P_{\mathrm{Lip}_0^{K,R}}\big(\hat{\lambda}_1^n + \gamma_n \nabla \hat{v}_h^R(\hat{\lambda}_1^n)\big). \tag{2.3.17}$$

We provide also a super-gradient computed by

$$\nabla \hat{v}_h^R(\hat{\lambda}_1) \quad := \quad \big(\mu_0\big(\mathrm{lin}^R[g_0^j]\big) - \mu_1\big(\mathrm{lin}^R[\delta_j]\big)\big)_{-r\leq j\leq r},$$

where g^j is given as the solution to

$$\begin{cases} g^j(t_k, x_i) = \Big(g^j + \Delta t\Big(\big(\hat{b}_{k,i}(\hat{\lambda}_1)D\big)g^j + \hat{a}_{k,i}(\hat{\lambda}_1)D^2 g^j\Big)\Big)(t_{k+1}, x_i) \text{ on } \mathring{\mathcal{M}}_{T,R}, \\ g^j(t_k, x_i) = \delta_{i,j}, \quad \text{on } \partial_T\mathcal{M}_{T,R} \cup \partial_R\mathcal{M}_{T,R}, \end{cases}$$

where $\hat{a}_{k,i}$ and $\hat{b}_{k,i}$ are the optimal controls in (2.3.14).

With a simple projection $P_{\mathrm{Lip}_0^{K,R}}$, we finally get a convergence result of the gradient projection algorithm (2.3.17):

$$\max_{n\leq N} \hat{v}_h^R(\hat{\lambda}_1^n) \quad \to \quad V_h^{K,R}, \quad \text{as } N \to \infty.$$

2.3.2.5 Numerical example

We implement the above gradient projection algorithm, taking $\ell(t,x,a,b) = a$, so that $V = \int_{\mathbb{R}} x^2 \mu_1(dx) - \int_{\mathbb{R}} x^2 \mu_0(dx)$. For a computer with a 2.4GHz CPU, it takes 55.2 seconds to finish 10^5 iterations, which gives a numerical result with an error less than 1%.

2.4 Part four: A model-free no-arbitrage bound for variance options

2.4.1 Motivations and formulation

The new mass transportation problem of Section 2.3 is motivated by the work of Galichon, Henry-Labordère and Touzi [33], which searches for the model-free no-arbitrage bound of prices for exotic options in a vanilla-liquid market. The main objective of this part is to design a numerical scheme to find this no-arbitrage bound as well as the corresponding static strategy in vanilla options when the exotic option is the variance option.

We consider an underlying stock X with price process defined as square integrable martingale. Let μ_0 and μ_1 be the marginal distributions of X at time T_0 and T_1 respectively, which are identified by the observations of the prices of vanilla options, we consider an variance option with payoff $g(\langle X\rangle_{T_0,T_1}, X_{T_1})$, where g is a Lipschitz function. In the framework of Galichon, Henry-Labordère et Touzi [33], its no-arbitrage bound of price can be formulated by

$$\inf_{\phi\in\mathrm{Quad}} \sup_{\mathbb{P}\in\mathcal{P}^2(\mu_0)} \left\{\mathbb{E}^{\mathbb{P}}\big[g(\langle X\rangle_{T_0,T_1}, X_{T_1}) - \phi(X_{T_1})\big] + \mu_1(\phi)\right\}, \tag{2.4.1}$$

where X is the canonical process in the canonical space Ω, $\mathcal{P}(\mu_0)$ represents the collection of all probability measures on Ω under which X is a martingale such that $\mathbb{P}\circ X_0^{-1} = \mu_0$ and $\mathbb{E}^{\mathbb{P}}[X_1^2|X_0] < \infty,\ \mathbb{P}-a.s.$, and

$$\mathrm{Quad} \;:=\; \left\{\ \phi:\mathbb{R}\to\mathbb{R} \text{ such that } \sup_{x\in\mathbb{R}} \frac{|\phi(x)|}{1+|x|^2} < \infty\ \right\}.$$

The choice of Quad as the collection of admissible static strategies is motivated by the fact that variance option (i.e. $g(z,x,) = z$) can be considered as a European payoff $X_{T_1}^2$.

By the time-change martingale theorem, it is well known that a martingale can be represented as a time-changed Brownian motion. Based on this fact, we reformulate the problem (2.4.1) with a standard Brownian motion as well as its stopping times. Let $B = (B_t)_{t\geq 0}$ be a standard one-dimensional Brownian motion such that $B_0 = 0$, $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ be its natural filtration and denote

$$\mathcal{T}^\infty \;:=\; \left\{\ \mathcal{F}-\text{stopping time } \tau \text{ such that } \mathbb{E}(\tau) < \infty\ \right\}. \tag{2.4.2}$$

We define λ^ϕ and λ_0^ϕ with a given $\phi\in\mathrm{Quad}$ by

$$\lambda^\phi(t,x) \;:=\; \sup_{\tau\in\mathcal{T}^\infty} \mathbb{E}\big[g(t+\tau, x+B_\tau) - \phi(x+B_\tau)\big],\quad \lambda_0^\phi(\cdot) \;:=\; \lambda^\phi(0,\cdot). \tag{2.4.3}$$

Then our model-free no-arbitrage bound is given by

$$U \;:=\; \inf_{\phi\in\mathrm{Quad}} u(\phi), \quad \text{with} \quad u(\phi) \;:=\; \mu_0(\lambda_0^\phi) + \mu_1(\phi). \tag{2.4.4}$$

2.4.2 Main results

2.4.2.1 Analytic approximations

In order to restrict the computation of U in (2.4.4) to a bounded domain, we give an analytic approximation consisting of four steps.

First step We introduce

$$\mathrm{Quad}_0 \;:=\; \big\{ \phi \in \mathrm{Quad} \text{ non negative, convex, such that } \phi(0) = 0 \big\},$$

and

$$U^K \;:=\; \inf_{\phi\in\mathrm{Quad}_0^K} u(\phi), \quad \text{with} \quad \mathrm{Quad}_0^K \;:=\; \big\{ \phi \in \mathrm{Quad}_0 \;:\; \phi(x) \le K(|x| \vee x^2) \big\}.$$

Then it follows that

$$U \;=\; \inf_{\phi\in\mathrm{Quad}_0} u(\phi),$$

and we get a natural convergence

$$U^K \;\searrow\; U \quad \text{as} \quad K \to \infty, \tag{2.4.5}$$

by the fact that $\mathrm{Quad}_0 = \cup_{K\ge 0}\mathrm{Quad}_0^K$.

Second step We introduce

$$\mathrm{Quad}_0^{K,M} \;:=\; \big\{ \phi \in \mathrm{Quad}_0^K \text{ such that } \phi(x) = Kx^2 \text{ for } |x| \ge 2M\big\},$$

and prove that

$$U^{K,M} \;:=\; \inf_{\phi\in\mathrm{Quad}_0^{K,M}} u(\phi) \;\to\; U^K \quad \text{as } M \to \infty. \tag{2.4.6}$$

Third step We define

$$\lambda^{\phi,T}(t,x) \;:=\; \inf_{\tau\in\mathcal{T}^\infty,\; \tau\le T-t} \mathbb{E}[g(t+\tau, x+B_\tau) - \phi(x+B_\tau)] \;\text{ and }\; \lambda_0^{\phi,T}(\cdot) := \lambda^{\phi,T}(0,\cdot),$$

as well as

$$U^{K,M,T} \;:=\; \inf_{\phi\in\mathrm{Quad}_0^{K,M}} u^T(\phi) \;\text{ with }\; u^T(\phi) \;:=\; \mu_0(\lambda_0^{\phi,T}) + \mu_1(\phi). \tag{2.4.7}$$

We prove that

$$U^{K,M,T} \;\to\; U^{K,M} \;\text{ as } T \to \infty. \tag{2.4.8}$$

Fourth step Finally, for the fourth step, we introduce a stopping time

$$\tau_x^R \quad := \quad \inf\{s \ : \ x + B_s \notin (-R, R)\},$$

and when $R \geq (1 + \sqrt{\frac{K}{K-L_0}})M$, we get an equivalence

$$\lambda^{\phi,T}(t,x) \quad = \quad \lambda^{\phi,T,R}(t,x) \ := \ \inf_{\tau \in \mathcal{T}^\infty, \ \tau \leq \tau_x^R} \mathbb{E}[g(t+\tau, x+B_\tau) - \phi(x+B_\tau)]. \quad (2.4.9)$$

Clearly, $\lambda^{\phi,T,R}$ can be characterized as the unique viscosity solution of the variational inequality

$$\min\left\{ \ \lambda(t,x) - g(t,x) + \phi(x), \quad -\partial_t \lambda - \frac{1}{2} D^2 \lambda \ \right\} \ = \ 0, \quad \text{on } [0,T) \times (-R,R), (2.4.10)$$

with boundary condition

$$\lambda(t,x) = g(t,x) - \phi(x), \quad \text{on } ([0,T] \times \{\pm R\}) \ \cup \ (\{T\} \times [-R,R]).$$

2.4.2.2 Numerical approximation

The main idea for the numerical approximation of $U^{K,M,T}$ in (2.4.7) is to use a finite difference scheme to solve the variational inequality (2.4.10), and then to use the gradient projection algorithm to solve a minimization problem.

Let $h = (\Delta t, \Delta x)$ be the discretization parameters, denote $x_i = i\Delta x$ and $t_k = k\Delta t$, we define a discrete grid

$$\mathcal{N}_R \ := \ \{x_i \ : -r \leq i \leq r\} \quad \text{and} \quad \mathcal{M}_{T,R} \ := \ \big\{(t_k, x_i) \ : 0 \leq k \leq l, \ -r \leq i \leq r\big\}.$$

For a function w defined on $\mathcal{M}_{T,R}$, we introduce a discrete derivative

$$D_h^2 w(t_k, x_i) \quad := \quad \frac{w(t_k, x_{i+1}) - w(t_k, x_i) + w(t_k, x_{i-1})}{\Delta x^2}.$$

Then given a function φ defined on the grid $\mathcal{N}_R$, we denote by $\lambda_h^{\varphi,T,R}$ the numerical solution of the following finite difference scheme for (2.4.10):

$$\begin{cases} \lambda_h^{T,R}(t_{k+1}, x_i) \ - \ \tilde{\lambda}_h^{T,R}(t_k, x_i) \\ \qquad + \ \frac{1}{2}\Delta t \left(\theta \ D^2 \tilde{\lambda}_h^{T,R}(t_k, x_i) \ + \ (1-\theta) \ D^2 \lambda_h^{T,R}(t_{k+1}, x_i)\right) \ = \ 0, \\ \lambda_h^{T,R}(t_k, x_i) \ = \ \max\left(g(t_k, x_i) - \varphi(x_i) \ , \ \tilde{\lambda}_h^{T,R}(t_k, x_i)\right). \end{cases} \quad (2.4.11)$$

Let $\mathrm{lin}^R[\varphi]$ (resp. $\mathrm{lin}^R[\lambda_{h,0}^{\varphi,T,R}]$) be the linear interpolation of φ (resp. $\lambda_{h,0}^{\varphi,T,R} := \lambda_h^{\varphi,T,R}(0,\cdot)$) extended by zero outside the interval $[-R,R]$. We then get the natural approximations for $u_T(\phi)$ and $U^{K,M,T}$ defined in (2.4.7):

$$u_{h,T}(\varphi) \ := \ \mu_0(\mathrm{lin}^R[\lambda_{h,0}^{\varphi,T,R}]) + \mu_1(\mathrm{lin}^R[\varphi]) \text{ and } U_h^{K,M,T} \ := \ \inf_{\varphi \in \mathrm{Quad}_{0,h}^{K,M}} u_{h,T}(\varphi), (2.4.12)$$

where

$$\mathrm{Quad}_{0,h}^{K,M} \quad := \quad \big\{ \, \varphi := \phi|_{\mathcal{N}_R} \text{ for some } \phi \in \mathrm{Quad}_0^{K,M} \, \big\}.$$

Provided a convergence result of $\lambda_h^{\varphi,T,R}$ to $\lambda^{\phi,T,R}$, we get

$$U_h^{K,M,T} \quad \to \quad U^{K,M,T} \quad \text{as } h \to 0. \tag{2.4.13}$$

Finally, we propose a gradient projection algorithm to solve $U_h^{K,M,T}$ in (2.4.12). Let $(\gamma_n)_{n\geq 1}$ be a sequence of positive real numbers, the algorithm is given by the iteration

$$\varphi_{n+1} := P_{\mathrm{Quad}_{0,h}^{K,M}}\big[\varphi_n - \gamma_n \nabla u_{h,T}(\varphi_n)\big], \tag{2.4.14}$$

where $P_{\mathrm{Quad}_{0,h}^{K,M}}[\varphi]$ denotes the projection of function φ on $\mathrm{Quad}_{0,h}^{K,M}$, and $\nabla u_{h,T}(\varphi_n)$ is a sub-gradient of $\varphi \mapsto u_{h,T}(\varphi)$ at φ_n which is given by

$$\nabla u_{h,T}(\varphi) \quad := \quad \big(\, \mu_0(\mathrm{lin}^R[p_0^j]) \; + \; \mu_1(\mathrm{lin}^R[e_j]) \, \big)_{-2m\leq j\leq 2m}$$

with the unique solution $(p^j, \tilde{p}^j)$ of the following linear system on $\mathcal{M}_{T,R}$:

$$\begin{cases} p^j(t_k,x_i) \; = \; -\,\delta_{i,j}, & (t_k,x_i)\in\partial_T\mathcal{M}_{T,R}\cup\partial_R\mathcal{M}_{T,R}, \\ p^j(t_{k+1},x_i) - \tilde{p}^j(t_k,x_i) + \frac{1}{2}\Delta t\,\big(\theta D^2\tilde{p}^j(t_k,x_i) + (1-\theta)D^2p^j(t_{k+1},x_i)\big) = 0, & \\ p^j(t_k,x_i) = \begin{cases} \tilde{p}^j(t_k,x_i), & \text{if } \lambda_h^{\varphi,T,R}(t_k,x_i) < g^{\varphi}(t_k,x_i), \\ -\,e_j(x_i), & \text{otherwise.}\end{cases} & (t_k,x_i)\in\mathring{\mathcal{M}}_{T,R}.\end{cases}$$

where $e_j \in B(\mathcal{N}_R)$ is defined by $e_j(x_i) := \delta_{i,j} = \begin{cases} 1, & \text{if } \; i=j, \\ 0, & \text{otherwise.}\end{cases}$ Let $p_0^j := p^j(0,\cdot)$.

We finally discuss the projection $P_{\mathrm{Quad}_{0,h}^{K,M}}$, and then provide a convergence result of the gradient projection algorithm:

$$\min_{n\leq N} \; u_{h,T}(\varphi_n) \;\to\; U_h^{K,M,T} \tag{2.4.15}$$

2.4.2.3 Numerical example

We implement the above algorithm in the case where the variance option is the "variance swap", i.e. $g(t,x)=t$. With a 2.4GHz CPU computer, it takes 57.24 seconds to finish 4×10^4 iterations, and we get a result with relative error less than 1%.

Partie I

Some numerical analysis for PDEs

Chapitre 3

The monotonicity condition of $\theta-$scheme for diffusion equations

3.1 Introduction

The monotonicity of a numerical scheme is an important issue in numerical analysis. For example, in the convergence analysis in Chapiter 2 of Allaire [1], the author uses the $L^\infty-$monotonicity to derive the stability of the scheme, which gives a proof of convergence. In the viscosity solution convergence context of Barles and Souganidis [6], the $L^\infty-$monotonicity is a key criterion to guarantee the convergence of the numerical scheme.

We are here interested in the finite difference $\theta-$scheme for the diffusion equation:

$$\partial_t v \;-\; \sigma^2(x)\; D^2_{xx} v \;=\; 0, \qquad (t,x) \in \mathbb{R}^+ \times \mathbb{R}. \tag{3.1.1}$$

with initial condition $v(0,x) = g(x)$.

3.2 The $\theta-$scheme and CFL condition

Let $h = (\Delta t, \Delta x) \in (\mathbb{R}^+)^2$ be the discretization in time and space, denote $t_n := n\Delta t$, $x_i := i\Delta x$, $\sigma_i := \sigma(x_i)$ and by u^n_i the numerical solution of v at point (t_n, x_i), let $\mathcal{N} := \{x_i : i \in \mathbb{Z}\}$ be a discrete grid on $\mathbb{R}$. The finite difference $\theta-$scheme ($0 \leq \theta \leq 1$) for diffusion equation (3.1.1) is a countable infinite dimensional linear system on $\mathcal{N}$:

$$\frac{u^{n+1}_i - u^n_i}{\Delta t} \;-\; \sigma^2_i \Big(\theta \frac{u^{n+1}_{i+1} - 2u^{n+1}_i + u^{n+1}_{i-1}}{\Delta x^2} + (1-\theta) \frac{u^n_{i+1} - 2u^n_i + u^n_{i-1}}{\Delta x^2} \Big) \;=\; 0, \tag{3.2.1}$$

with initial condition $u^0_i = g(x_i)$.

Let $(u^n) := (u^n_i)_{i\in\mathbb{Z}}$ be a $\mathbb{Z}-$dimensional vector, denote $\alpha_i := \frac{\sigma_i^2 \Delta t}{\Delta x^2}$ and $\beta_i := \frac{\theta \alpha_i}{1+2\theta\alpha_i}$, we define $\mathbb{Z} \times \mathbb{Z}$ dimensional matrices I, D, T and E as follows: I is the identity matrix, D is a diagonal matrix with $D_{i,i} = \alpha_i$, T is a tridiagonal matrix with $T_{i,i-1} = T_{i,i+1} = \alpha_i$ and $T_{i,i} = 0$, and $E := \theta[I + 2\theta D]^{-1} T$ which is a tridiagonal matrix with $E_{i,i-1} = E_{i,i+1} = \beta_i$ and $E_{i,i} = 0$. Then the system (3.2.1) can be written as

$$\big[I + 2\theta D - \theta T\big]\; (u^{n+1}) \;=\; \big[I - 2(1-\theta) D + (1-\theta) T\big]\; (u^n),$$

or equivalently

$$\big[I + 2\theta D\big]\; \big[I - E\big]\; (u^{n+1}) \;=\; \big[I - 2(1-\theta) D + (1-\theta) T\big]\; (u^n). \tag{3.2.2}$$

Proposition 3.2.1. *Suppose that the function g is bounded on $\mathcal{N}$ and there is constant $\bar{\sigma} > 0$ such that $|\sigma_i| \le \bar{\sigma}$ for every $i \in \mathbb{Z}$, then the $\mathbb{Z} \times \mathbb{Z}$ matrix $I - E$ is invertible and its inversion B is given by*

$$B \ := \ I \ + \ \sum_{n=1}^{\infty} E^n. \tag{3.2.3}$$

And therefore, there is a unique solution for system (3.2.1) (or (3.2.2)) given by

$$(u^{n+1}) \ = \ B\ \big[I + 2\theta D\big]^{-1}\ \big[I - 2(1-\theta)D + (1-\theta)T\big]\ (u^n). \tag{3.2.4}$$

Proof. First, $(\alpha_i)_{i\in\mathbb{N}}$ defined by $\alpha_i = \frac{\sigma_i^2 \Delta t}{\Delta x^2}$ are uniformly bounded by $\bar{\alpha} := \frac{\bar{\sigma}^2 \Delta t}{\Delta x^2}$ since $(\sigma_i)_{i\in\mathbb{Z}}$ are uniformly bounded by $\bar{\sigma}$. It follows that $\beta_i = \frac{\theta\alpha_i}{1+2\theta\alpha_i} \le \rho := \frac{\theta\bar{\alpha}}{1+2\theta\bar{\alpha}} < \frac{1}{2}$.

Denote by $B(\mathcal{N})$ the space of all bounded functions defined on $\mathcal{N}$, then E can be viewed as an operator on $B(\mathcal{N})$ and its $L^\infty-$norm is defined by

$$\|E\|_\infty \ := \ \sup_{u\in B(\mathcal{N}),u\neq 0} \frac{|Eu|_\infty}{|u|_\infty}.$$

Clearly, $\|E\|_\infty \le 2\rho < 1$, and therefore, B in (3.2.3) is well defined and one can easily verify that B is the inverse of $\big[I - E\big]$. $\square$

Definition 3.2.1. *A numerical scheme for equation (3.1.1) given by $u_i^{n+1} = \mathbf{T}_h[u^n]_i$ is said to be $L^\infty-$monotone if*

$$u_i^{1,n} \ \le \ u_i^{2,n}, \quad \forall i \in \mathbb{Z} \quad \Rightarrow \quad \mathbf{T}_h[u^{1,n}]_i \ \le \ \mathbf{T}_h[u^{2,n}]_i, \quad \forall i \in \mathbb{Z}.$$

Remark 3.2.1. *It is well-known that in the case $\theta = 1$, system (3.2.2) is an implicit scheme, and it is automatically $L^\infty-$monotone for every discretization $(\Delta t, \Delta x)$. When $\theta < 1$, a sufficient condition to gurantee the $L^\infty-$monotonicity of system (3.2.2) is the CFL(Courant-Friedrichs-Lewy) condition*

$$\bar{\alpha} \ := \ \frac{\bar{\sigma}^2 \Delta t}{\Delta x^2} \ \le \ \frac{1}{2(1-\theta)}, \qquad \textit{for } \ \bar{\sigma} := \sup_{i\in\mathbb{Z}} \sigma_i. \tag{3.2.5}$$

The CFL condition is a sufficient condition for the monotonicity of $\theta-$scheme, and it implies a discretization ratio $\Delta t = O(\Delta x^2)$. We shall confirm that this ratio is necessary to guarantee the monotonicity in the following.

3.3 The necessary and sufficient condition

Let $\gamma_i := \frac{(1-\theta)\alpha_i}{1+2\theta\alpha_i} = \frac{(1-\theta)}{\theta}\beta_i$ and $b_{i,j}$ be elements of the matrix B, i.e. $B = (b_{i,j})_{(i,j)\in\mathbb{Z}^2}$. It is clear that $b_{i,j} \ge 0$ for every $(i,j) \in \mathbb{Z}^2$ by the definition of B in (3.2.3). Therefore, it follows from (3.2.4) that the necessary and sufficient condition for monotonicity of system (3.2.1) can be written as :

$$b_{i,j-1}\gamma_{j-1} \ + \ b_{i,j}\big(\frac{1}{1+2\theta\alpha_j} - 2\gamma_j\big) \ + \ b_{i,j+1}\gamma_{j+1} \ \ge \ 0, \qquad \forall (i,j) \in \mathbb{Z}^2. \tag{3.3.1}$$

Theorem 3.3.1. *Suppose that $|\sigma_i| \leq \bar{\sigma} < \infty$ for every $i \in \mathbb{Z}$, and let $\theta \in (0,1)$. Then the necessary and sufficient condition of monotonicity for the $\theta-$scheme in (3.2.1) is*

$$\alpha_i \;=\; \frac{{\sigma_i}^2 \Delta t}{\Delta x^2} \;\leq\; \frac{1}{2(1-\theta)} \;+\; \frac{b_{i,i}-1}{2\theta(1-\theta)}, \qquad \forall i \in \mathbb{Z}. \tag{3.3.2}$$

Proof. First, since B is the inversion of $I-E$, we have $B\,\big[I-E\big] = I$, and it follows that

$$b_{i,j-1}\beta_{j-1} \;+\; b_{i,j+1}\beta_{j+1} \;=\; \begin{cases} b_{ij}-1, & \text{for } i=j, \\ b_{ij}, & \text{for } i \neq j. \end{cases}$$

Therefore, in case that $i \neq j$, (3.3.1) is equivalent to:

$$b_{i,j}\left(\frac{1-\theta}{\theta} + \frac{1}{1+2\theta\alpha_j} - 2\gamma_j\right) \;\geq\; 0. \tag{3.3.3}$$

Since $b_{i,j} \geq 0$, the inequality (3.3.3) holds as soon as

$$(1-\theta)(1+2\theta\alpha_j) \;+\; \theta \;-\; 2\theta(1-\theta)\alpha_j \;=\; 1 \;>\; 0,$$

which is always true.

In case that $i = j$, (3.3.1) is equivalent to:

$$b_{i,i}\left(\frac{1-\theta}{\theta} + \frac{1}{1+2\theta\alpha_i} - 2\gamma_i\right) \;-\; \frac{1-\theta}{\theta} \;\;\geq\;\; 0,$$

i.e.

$$\alpha_i \;\leq\; \frac{1}{2(1-\theta)} \;+\; \frac{b_{i,i}-1}{2\theta(1-\theta)}.$$

which is the required inequality (3.3.2). □

Remark 3.3.1. *Since $b_{i,i} < \infty$ for every $i \in \mathbb{Z}$, it follows from Theorem 3.3.1 that the ratio $\Delta t = O(\Delta x^2)$ is necessary for the monotonicity of $\theta-$scheme ($0 < \theta < 1$) as soon as $\sigma_i \neq 0$ for some $i \in \mathbb{Z}$.*

3.4 The heat equation

In this section, let us suppose that $\sigma(x) \equiv \sigma_0$ with a positive constant σ_0, then the diffusion equation turns to be the heat equation:

$$\partial_t v \;-\; \sigma_0^2\, D^2_{xx} v \;=\; 0, \qquad (t,x) \in \mathbb{R}^+ \times \mathbb{R}. \tag{3.4.1}$$

In this case, we can compute $b_{i,i}$ and then get an explicit formula for the monotonicity condition (3.3.2). Let

A be a $\mathbb{Z} \times \mathbb{Z}$ tridiagonal matrix such that $A_{i,i-1} = A_{i,i+1} = 1$ and $A_{i,i} = 0$, (3.4.2) then clearly, $E = \beta A$ with $\beta = \frac{\theta\alpha}{1+2\theta\alpha} < \frac{1}{2}$, $\alpha = \frac{\sigma_0^2 \Delta t}{\Delta x^2}$ and

$$B \;=\; \Big[I - \beta A\Big]^{-1} \;:=\; \sum_{n=0}^{\infty} \beta^n A^n. \tag{3.4.3}$$

Lemma 3.4.1. *Denote by A^n the $n-th$ exponentiation of matrix A in (3.4.2) for $n \in \mathbb{N}$, we rewritten $A^n = (a_{i,j}^{(n)})_{(i,j)\in\mathbb{Z}\times\mathbb{Z}}$. Then,*

$$a_{i,j}^{(n)} \;=\; \begin{cases} C_n^{(n+i-j)/2}, & \textit{if } \frac{n+i-j}{2} \in \mathbb{Z} \cap [0,n], \\ 0, & \textit{otherwise.} \end{cases} \tag{3.4.4}$$

Proof. We proceed by induction. First, it is clearly that (3.4.4) holds true for $n = 1$. Suppose that the (3.4.4) is true in case that $n = m$. Since $A^{m+1} = A^m\, A$, we then have $a_{i,j}^{m+1} = a_{i,j-1}^m + a_{i,j+1}^m$. It follows from $C_n^k = C_{n-1}^k + C_{n-1}^{k-1}$ that (3.4.4) holds still true for the case $n = m + 1$. We then conclude the proof. □

By Lemma 3.4.1 and equality (3.4.3), we get $b_{i,i} = \sum_{k=0}^{\infty} C_{2k}^k \beta^{2k}$ with the convention that $C_0^0 := 1$. As a result, the monotonicity condition (3.3.2) of $\theta-$scheme reduces to

$$\alpha \;\leq\; \frac{1}{2(1-\theta)} \;+\; \frac{f(\beta)}{2\theta(1-\theta)}, \tag{3.4.5}$$

where

$$f(x) \;:=\; \sum_{k=1}^{\infty} C_{2k}^k\, x^{2k} \qquad \text{for } -\frac{1}{2} < x < \frac{1}{2}.$$

Remark 3.4.1. *We can verify that $C_{2k}^k \approx \frac{1}{\sqrt{\pi k}} 4^k$ as $k \to \infty$ by Stirling's formula, thus the radius of convergence of $f(x)$ is $\frac{1}{2}$.*

Let us now compute the function $f(x)$. Since $C_{2k}^k = 2\,\frac{2k-1}{k}\,C_{2(k-1)}^{k-1}$, it follows that for $|x| < \frac{1}{2}$,

$$\begin{aligned} f'(x) \;&=\; \sum_{k=1}^{\infty} 2k\, C_{2k}^k\, x^{2k-1} \;=\; \sum_{k=1}^{\infty} 4\,(2k-1)\, C_{2(k-1)}^{k-1}\, x^{2k-1} \\ &=\; 4x + \sum_{k=1}^{\infty} (8k+4)\, C_{2k}^k\, x^{2k+1} \;=\; 4x \;+\; 4x^2 f'(x) \;+\; 4x f(x). \end{aligned}$$

We are then reduced to the ordinary differential equation

$$f'(x) \;=\; \frac{4x}{1-4x^2}\,(f(x)+1), \qquad \text{with } f(0) \;=\; 0,$$

whose solution is $f(x) = \frac{1}{\sqrt{1-4x^2}} - 1$. Inserting this solution into (3.4.5), and by a direct manipulation, it follows that (3.4.5) is equivalent to

$$\alpha \;\leq\; \frac{1}{2(1-\theta)} \;+\; \frac{\theta}{4(1-\theta)^2}. \tag{3.4.6}$$

We get the following theorem:

Theorem 3.4.1. *The necessary and sufficient condition for the $L^\infty-$monotonicity of $\theta-$scheme* $(0 < \theta < 1)$ *of the heat equation* (3.4.1) *is*

$$\alpha \ = \ \frac{\sigma_0^2 \Delta t}{\Delta x^2} \ \leq \ \frac{1}{2(1-\theta)} \ + \ \frac{\theta}{4(1-\theta)^2}. \tag{3.4.7}$$

Remark 3.4.2. *In particular, when* $\theta = \frac{1}{2}$*, the CFL condition is* $\alpha = \frac{\sigma_0^2 \Delta t}{\Delta x^2} \leq 1$*, and the necessary and sufficient condition of the monotonicity is* $\alpha = \frac{\sigma_0^2 \Delta t}{\Delta x^2} \leq \frac{3}{2}$.

Chapitre 4

A splitting method for fully nonlinear degenerate parabolic PDEs

4.1 Introduction

Numerical methods for parabolic partial differential equations (PDEs) are largely developed in the litterature, on finite difference scheme, finites elements scheme, semi-Lagrangian scheme, Monte-Carlo method, etc. For nonlinear PDEs, and especially in high dimensional cases, the numerical resolution becomes a big challenge.

A typical kind of nonlinear parabolic PDEs is the Hamilton-Jacobi-Bellman (HJB) equation which characterizes the solution of the optimal control problems. In this context, for finite difference method, one can only use the explicit scheme, since the implicit scheme needs to invert too many matrices. In the one dimensional case, the explicit finite difference scheme can be easily constructed and the monotonicity is guaranteed by the CFL condition. In high dimensional cases, Bonnans and Zidani [11] propose a numerical algorithm to construct a monotone scheme. Another numerical method for HJB equations is the semi-Lagrangian scheme proposed in Debrabant and Jakobsen [24]. It can be easily constructed to be monotone, but they need next to use a finite difference grid as well as an interpolation method to make it implementable. It hence can be viewed as a finite difference scheme.

Generally speaking, finite difference and semi-Lagrangian schemes are easily implemented and perform quite well in low dimensional cases; and in high dimensional cases, the Monte-Carlo method is preferred. Recently, Fahim, Touzi and Warin [32] proposed a probabilistic method for nonlinear parabolic PDEs, which is closely related to the second order backward stochastic differential equation (2BSDE) developed in Cheridito et al. [20] and Soner et al. [53]. With simulations of a diffusion process, they propose the estimations of the value function and its derivatives by conditional expectations, by which they can approximate the nonlinear part of the PDE and then get a convergent scheme. However, their scheme can only be applied in the non-degenerate cases.

We are motivated to generalize the probabilistic scheme of Fahim, Touzi and Warin [32] for degenerate parabolic equations by their applications in finance. For example, in Asian option pricing problems, we must consider the cumulative average stock prices A_t; for lookback options, we consider also the historical maximum and/or minimum stock prices M_t, m_t. They are all degenerate variables without a diffusion generator, and hence the pricing equation turns to be a degenerate parabolic equation. In some optimal commodity trading models(e.g. [3] and [14]), the storage amount of commodities is an important state variable, and the optimization problem induces a PDE which degenerates on storage amount variable. In life insurance, Dai et al. [23] proposed a financial pricing model for a Variable Annuities product Guaranteed Minimum Withdrawal Benefit (GMWB). In their model, the price of GMWB depends on two variables: the reference account and the guaranteed account, where the latter degenerates and the pricing equation is a degenerate parabolic PDE.

For these degenerate PDEs, the degenerate part is separable. Therefore, a natural solution is the splitting scheme. Our idea is to use the probabilistic scheme to treat the non-degenerate part, and use the semi-Lagrangian scheme to solve the degenerate part, and by combining the two methods, we get a splitting scheme. Moreover, in place of the interpolation method, we propose a simulation-regression technique to make the semi-Lagrangian scheme implementable. Then our splitting scheme becomes a Monte-Carlo method for degenerate parabolic nonlinear PDEs, and it is expected to be relatively more efficient in high dimensional cases.

The rest of the chapter is organized as follows. In Section 4.2, we introduce a degenerate PDE and a splitting scheme which combines the probabilistic scheme in [32] and semi-Lagrangian scheme. Then we provide a local uniform convergence result as well as a rate of convergence. In Section 4.3, we propose a simulation-regression technique to approximate the conditional expectations used in the splitting scheme, making the scheme implementable. We shall also discuss the choices of function basis used in the regression and then provide some convergence results for this implementable scheme. Finally, Section 4.4 provides some experimental examples.

Notation: Let $|\eta| := \eta^1 + \cdots + \eta^d$ for $\eta \in \mathbb{N}^d$. Given $T \in \mathbb{R}^+$ and $d,\ d' \in \mathbb{N}$, we denote $Q_T := [0,T) \times \mathbb{R}^d \times \mathbb{R}^{d'}$, $\overline{Q}_T := [0,T] \times \mathbb{R}^d \times \mathbb{R}^{d'}$ and

$$C^{0,1}(Q_T) \;:=\; \left\{ \varphi \;:\; Q_T \to \mathbb{R} \text{ such that } |\varphi|_1 < \infty \right\},$$

where

$$|\varphi|_0 \;:=\; \sup_{Q_T} |\varphi(t,x,y)| \quad \text{and} \quad |\varphi|_1 \;:=\; |\varphi|_0 + \sup_{Q_T \times Q_T} \frac{|\varphi(t,x,y) - \varphi(t',x',y')|}{|x-x'| + |y-y'| + |t-t'|^{\frac{1}{2}}}.$$

In this chapter, the constant C is used in many inequalities, its value may vary from line to line.

4.2 The degenerate PDE and splitting scheme

In this section, we first introduce a nonlinear parabolic PDE which has a separable degenerate part. We next propose a splitting scheme, and for which we provide a local uniform convergence result of the splitting scheme when the PDE satisfies a comparison result for bounded viscosity solutions, as well as a rate of convergence when the nonlinear part of the PDE is a concave Hamiltonian.

4.2.1 A degenerate nonlinear PDE

Let $T \in \mathbb{R}^+$, $\mu : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : [0,T] \times \mathbb{R}^d \to S_d$ be continuous, denote $a(t,x) := \sigma(t,x)\sigma(t,x)^T$, we define a linear operator $\mathcal{L}^X$ on the smooth functions $\varphi : Q_T \to \mathbb{R}$ by

$$\mathcal{L}^X \varphi(t,x,y) \;\; := \;\; \partial_t \varphi(t,x,y) \; + \; \mu(t,x) \cdot D_x \varphi(t,x,y) \; + \; \frac{1}{2} a(t,x) \cdot D^2_{xx} \varphi(t,x,y).$$

We say that $\mathcal{L}^X$ is a linear operator associated to the diffusion process $X = (X_t)_{0 \le t \le T}$ defined by the stochastic differential equation:

$$dX_t \;\; = \;\; \mu(t, X_t) \; dt \; + \; \sigma(t, X_t) \; dW_t, \tag{4.2.1}$$

where $W = (W_t)_{0 \le t \le T}$ is a d-dimensional standard Brownian motion.

Given a nonlinear function

$$F : \; (t,x,y,r,p,\Gamma) \in \mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}^{d'} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}_d \;\; \mapsto \;\; F(t,x,y,r,p,\Gamma) \in \mathbb{R},$$

we then get a nonlinear operator $F(t,x,y,\varphi,D_x\varphi,D^2_{xx}\varphi)$ on φ. We denote by $\mathbb{F}_p$ and F_Γ the derivative of function F w.r.t. p and Γ.

Next, we give the degenerate part which involves with the partial gradient with respect to y. Given functions

$$\big(l^{\alpha,\beta}, \; c^{\alpha,\beta}, \; f_i^{\alpha,\beta}, \; g_j^{\alpha,\beta} \; \big)_{\alpha \in \mathcal{A}, \; \beta \in \mathcal{B}, \; 1 \le i \le d, \; 1 \le j \le d'}$$

defined on Q_T with index space $\mathcal{A}$ and $\mathcal{B}$, we denote $f^{\alpha,\beta} := (f_i^{\alpha,\beta})_{1 \le i \le d}$ and $g^{\alpha,\beta} := (g_j^{\alpha,\beta})_{1 \le j \le d'}$, and define the Lagrangian $\mathcal{L}^{\alpha,\beta}$ by

$$\begin{aligned} \mathcal{L}^{\alpha,\beta}\varphi(t,x,y) \;\; := & \;\; l^{\alpha,\beta}(t,x,y) \; + \; c^{\alpha,\beta}(t,x,y)\varphi(t,x,y) \\ & + \; f^{\alpha,\beta}(t,x,y) \cdot D_x\varphi(t,x,y) \; + \; g^{\alpha,\beta}(t,x,y) \cdot D_y\varphi(t,x,y), \end{aligned}$$

and the Hamiltonian by

$$H(t,x,y,\varphi(t,x,y), D_x\varphi(t,x,y), D_y\varphi(t,x,y)) \;\; := \;\; \inf_{\alpha \in \mathcal{A}} \; \sup_{\beta \in \mathcal{B}} \; \mathcal{L}^{\alpha,\beta}\varphi(t,x,y).$$

Finally, let us introduce the degenerate fully nonlinear parabolic PDE which will be considered throughout the chapter:

$$\big[-\mathcal{L}^X v \; - \; F(\cdot, v, D_x v, D^2_{xx} v) \; - \; H(\cdot, v, D_x v, D_y v) \; \big](t,x,y) \;\; = \;\; 0, \quad \text{ on } \; Q_T, \tag{4.2.2}$$

with terminal condition

$$v(T,x,y) \;\; = \;\; \Phi(x,y). \tag{4.2.3}$$

The PDE (4.2.2) is composed by three separable parts: the linear part $\mathcal{L}^X$, the nonlinear part F, and the first order degenerate part H.

4.2.2 A splitting scheme

As observed above, the three parts in PDE (4.2.2) are separable, we can then propose a splitting numerical scheme to solve it. The idea is to split (4.2.2) into the following two equations:

$$-\ \mathcal{L}^X v(t,x,y)\ -\ F(\cdot,v,D_x v,D^2_{xx}v)(t,x,y)\ =\ 0 \tag{4.2.4}$$

and

$$-\ \partial_t v(t,x,y)\ -\ H(\cdot,v,D_x v,D_y v)(t,x,y)\ =\ 0, \tag{4.2.5}$$

then to solve them separately. Equation (4.2.4) is nonlinear and non-degenerate for every fixed y, then it can be treated by the probabilistic scheme proposed in Fahim et al.[32]. Equation (4.2.5) is a first order HJBI equation, we shall solve it by semi-Lagrangian scheme. Then, combining the two schemes sequentially, we get the splitting scheme.

Let us first give a time discrete grid $(t_n)_{n=0,\cdots,N}$ with $t_n := nh$, where $h := T/N$ for $N \in \mathbb{N}$. As in [32], we define $\hat{X}^{t,x}_h$ by the Euler scheme of the diffusion process X in (4.2.1):

$$\hat{X}^{t,x}_h\ :=\ x\ +\ \mu(t,x)\ h\ +\ \sigma(t,x)\cdot(W_{t+h}-W_t), \quad \forall (t,x)\in[0,T]\times\mathbb{R}^d.$$

Let v^h denote the numerical solution, then the probabilistic scheme of [32] for equation (4.2.4) is given by

$$v^h(t_n,x,y) = \mathbf{T}_h[v^h](t_n,x,y) := \mathbb{E}\big[v^h(t_{n+1},\hat{X}^{t_n,x}_h,y)\big] + hF(t_n,x,y,\mathbb{E}\mathcal{D}_h v^h(t_n,x,y)), \tag{4.2.6}$$

where

$$\mathbb{E}\mathcal{D}_h\varphi(t_n,x,y)\ :=\ \Big(\ \mathbb{E}\big[\varphi(t_{n+1},\hat{X}^{t_n,x}_h,y)H^{t_n,x,h}_i(\Delta W_{n+1})\big] : i=0,1,2\ \Big),$$

with $\Delta W_{n+1} := W_{t_{n+1}} - W_{t_n}$ and the Hermite polynomials are defined by $H^{t,x,h}_0(w) := 1$, $H^{t,x,h}_1(w) := \sigma^T(t,x)^{-1}\frac{w}{h}$ and $H^{t,x,h}_2(w) := \sigma^T(t,x)^{-1}\frac{ww^T-hI_d}{h^2}\sigma(t,x)^{-1}$.

Remark 4.2.1. *The scheme $\mathbf{T}_h$ is well defined as soon as $Det(\sigma(t,x)) \neq 0$ for each $(t,x)\in[0,T)\times\mathbb{R}^d$. When φ is smooth, by integration by parts, one can verify that*

$$\mathbb{E}\Big[\varphi\big(t_{n+1},\hat{X}^{t_n,x}_h,y\big)\ H^{t_n,x,h}_i(\Delta W_{n+1})\Big]\ =\ \mathbb{E}\ D^i_{x^i}\varphi\big(t_{n+1},\hat{X}^{t_n,x}_h,y\big), \quad i=0,1,2.$$

For more details on this fact and of the probabilistic scheme $\mathbf{T}_h$ of (4.2.6), we refer to Fahim et al. [32].

The second PDE (4.2.5) is a first order HJBI equation, its semi-Lagrangian scheme is given by

$$\begin{aligned} v^h(t_n,x,y)\ &=\ \mathbf{S}_h[v^h](t_n,x,y) := \inf_{\alpha\in\mathcal{A}}\ \sup_{\beta\in\mathcal{B}}\Big\{hl^{\alpha,\beta}(t_n,x,y) + hc^{\alpha,\beta}(t_n,x,y)v^h(t_{n+1},x,y) \\ &\quad +\ v^h\big(t_{n+1},\ x+hf^{\alpha,\beta}(t_n,x,y),\ y+hg^{\alpha,\beta}(t_n,x,y)\big)\Big\}. \end{aligned} \tag{4.2.7}$$

Remark 4.2.2. *The semi-Lagrangian scheme* $\mathbf{S}_h$ *is deduced intuitively from the discrete version of equation* (4.2.5)*:*

$$\frac{v^h(t_{n+1},x,y)-v^h(t_n,x,y)}{h} + \inf_{\alpha\in\mathcal{A}} \sup_{\beta\in\mathcal{B}} \Big\{ l^{\alpha,\beta}(t_n,x,y) + c^{\alpha,\beta}(t_n,x,y)v^h(t_{n+1},x,y) + \frac{v^h(t_{n+1},\ x+hf^{\alpha,\beta}(t_n,x,y),\ y+hg^{\alpha,\beta}(t_n,x,y)) - v^h(t_{n+1},x,y)}{h} \Big\} = 0.$$

Finally, we are ready to introduce the splitting scheme $\mathbf{S}_h \circ \mathbf{T}_h$ for the original PDE (4.2.2), (4.2.3). Concretely, with terminal condition

$$v^h(t_N,x,y) \ := \ \Phi(x,y), \tag{4.2.8}$$

we compute $v^h(t_n,\cdot)$ in a backward iteration. Given $v^h(t_{n+1},\cdot)$, we introduce the fictitious time $t_{n+\frac{1}{2}}$ and compute $v^h(t_n,\cdot)$ by

$$v^h(t_{n+\frac{1}{2}},x,y) \ := \ \mathbf{T}_h[v^h](t_n,x,y) \quad \text{with } \mathbf{T}_h \text{ defined in (4.2.6)}, \tag{4.2.9}$$

and

$$\begin{aligned} v^h(t_n,x,y) \ &= \ \mathbf{S}_h \circ \mathbf{T}_h[v](t_n,x,y) \\ &:= \ \inf_{\alpha\in\mathcal{A}} \sup_{\beta\in\mathcal{B}} \Big\{ h\ l^{\alpha,\beta}(t_n,x,y) + h\ c^{\alpha,\beta}(t_n,x,y)\ v^h(t_{n+\frac{1}{2}},x,y) \\ &\qquad + \ v^h\big(t_{n+\frac{1}{2}},\ x+f^{\alpha,\beta}(t_n,x,y)h,\ y+g^{\alpha,\beta}(t_n,x,y)h\big) \Big\}. \end{aligned} \tag{4.2.10}$$

Clearly, when $\mathrm{Det}(\sigma(t,x)) \neq 0$ for every $(t,x) \in [0,T) \times \mathbb{R}^d$, the scheme $\mathbf{S}_h \circ \mathbf{T}_h$ is well defined and it gives a unique numerical solution v^h.

4.2.3 The convergence results

We shall provide two convergence results for the splitting scheme $\mathbf{S}_h \circ \mathbf{T}_h$ in (4.2.10), similar to Fahim et al.[32]. The first one is the local uniform convergence in the context of Barles and Souganidis [6], and the second is a rate of convergence.

We first recall that an upper semicontinuous (resp., lower semicontinuous) function $\underline{v}$ (resp. $\overline{v}$) on Q_T is called a viscosity subsolution (resp., supersolution) of (4.2.2) if, for any $(t,x,y) \in Q_T$ and any smooth function φ satisfying

$$0 = (\underline{v}-\varphi)(t,x,y) = \max_{Q_T}(\underline{v}-\varphi) \qquad \Big(\text{resp., } 0 = (\overline{v}-\varphi)(t,x,y) = \min_{Q_T}(\overline{v}-\varphi)\Big),$$

we have

$$-\mathcal{L}^X\varphi \ - \ F(t,x,y,\varphi,D_x\varphi,D^2_{xx}\varphi) \ - \ H(t,x,y,D_x\varphi,D_y\varphi) \ \leq (\text{resp., } \ \geq) \ 0.$$

Definition 4.2.1. *We say that the PDE (4.2.2) satisfies a comparison result for bounded functions if, for any bounded upper semicontinuous subsolution $\underline{v}$ and any bounded lower semicontinuous supersolution $\overline{v}$ on $\overline{Q}_T$ satisfying*

$$\underline{v}(T,\cdot) \quad \leq \quad \overline{v}(T,\cdot),$$

we have $\underline{v} \leq \overline{v}$.

Let us now give some assumptions on the equation (4.2.2), and then provide a first convergence result.

Assumption F : (i) The diffusion coefficients μ and σ are Lipschitz in x and continuous in t, $\sigma\sigma^T(t,x) > 0$ for all $(t,x) \in [0,T] \times \mathbb{R}^d$ and $\int_0^T \left|\sigma\sigma^T(t,0) + \mu(t,0)\right| dt < \infty$.

(ii) The nonlinear operator F is uniformly Lipschitz in (x,y,r,p,Γ), continuous in t and $\sup_{(t,x,y)\in Q_T} |F(t,x,y,0,0,0)| < \infty$.

(iii) F is elliptic and satisfies

$$a^{-1} \cdot F_\Gamma \quad \leq \quad 1 \qquad \text{on} \;\; \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^{d'} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}_d. \tag{4.2.11}$$

(iv) $F_p \in \text{Image}(F_\Gamma)$ and $\left|F_p^T F_\Gamma^{-1} F_p\right|_\infty < +\infty$.

Remark 4.2.3. *Assumption* **F** *is almost the same as the Assumption* **F** *in [32], here we just add a variable y in the nonlinear operator F.*

Assumption H : The coefficients in Hamiltonian H are all uniformly bounded, i.e.

$$\sup_{(\alpha,\beta)\in\mathcal{A}\times\mathcal{B},\ 1\leq i\leq d,\ 1\leq j\leq d'} \left\{ \, |l^{\alpha,\beta}|_0 \; + \; |c^{\alpha,\beta}|_0 \; + \; |f_i^{\alpha,\beta}|_0 \; + \; |g_j^{\alpha,\beta}|_0 \, \right\} \;\; < \;\; \infty.$$

Assumption M : $F_r - \ \frac{1}{4}\ F_p^T F_\Gamma^{-1} F_p \ \geq \ 0$ and $c^{\alpha,\beta} \geq 0$ for every $\alpha \in \mathcal{A}, \beta \in \mathcal{B}$.

Remark 4.2.4. *Assumption* **M** *is imposed to guarantee the monotonicity of the splitting scheme $\mathbf{S}_h \circ \mathbf{T}_h$. However, it is not crucial as soon as Assumptions* **F** *and* **H** *hold true. In fact, as discussed in Remark 3.13 of [32], since the equation is parabolic, we can introduce a new function $u(t,x,y) := e^{\theta(T-t)} v(t,x,y)$ for some positive constant θ large enough, then the new PDE for $u(t,x,y)$ satisfies Assumption* **M** *under Assumptions* **F** *and* **H**. *Here, we impose this assumption only to simplify the presentation and the arguments.*

Theorem 4.2.1. *Let Assumptions* **F**, **H** *and* **M** *hold true, and assume that the degenerate fully nonlinear parabolic PDE (4.2.2) satisfies a comparison result for bounded viscosity solutions. Then for every bounded Lipschitz terminal condition function Φ, there exists a bounded function v such that*

$$v^h \;\; \longrightarrow \;\; v \quad \textit{locally uniformly as} \;\; h \to 0,$$

where v^h is the numerical solution of scheme $\mathbf{S}_h \circ \mathbf{T}_h$ defined by (4.2.8), (4.2.9) and (4.2.10). Moreover, v is the unique bounded viscosity solution of the equation (4.2.2) with terminal condition (4.2.3).

We next provide a rate of convergence in case that F and H are both concave Hamiltonians. Our analysis stays in the context of the HJB nonlinear PDEs as Barles and Jakobsen [5]. The following strengthen assumptions implies that the nonlinear PDE (4.2.2) satisfies a comparison result for bounded functions, and has a unique bounded viscosity solution given a bounded and Lipschitz continuous function Φ, see e.g. Proposition 2.1 of [5].

Assumption HJB : Assumptions **F** and **M** hold and F is a concave Hamiltonian, i.e.

$$\mu \cdot p \;+\; \frac{1}{2} a \cdot \Gamma \;+\; F(t,x,y,r,p,\Gamma) \;\;=\;\; \inf_{\gamma \in \mathcal{C}} \; \mathcal{L}^{\gamma}(t,x,y,r,p,\Gamma),$$

with

$$\mathcal{L}^{\gamma}(t,x,y,r,p,\Gamma) \;:=\; l^{\gamma}(t,x,y) \;+\; c^{\gamma}(t,x,y)r \;+\; f^{\gamma}(t,x,y)\cdot p \;+\; \frac{1}{2} a^{\gamma}(t,x,y)\cdot \Gamma.$$

And $\mathcal{B} = \{\beta\}$ is a singleton, hence H is also a concave Hamiltonian, so that it can be written as

$$H(t,x,y,r,p,q) \;\;=\;\; \inf_{\alpha \in \mathcal{A}} \; \big\{ l^{\alpha}(t,x,y) \;+\; c^{\alpha}(t,x,y)r \;+\; f^{\alpha}(t,x,y)\cdot p \;+\; g^{\alpha}(t,x,y)\cdot q \big\}$$

Moreover, the functions l, c, f, g and σ satisfy that

$$\sup_{\alpha \in \mathcal{A}, \gamma \in \mathcal{C}} \; \big(\, |l^{\alpha} + l^{\gamma}|_1 + |c^{\alpha} + c^{\gamma}|_1 + |f^{\alpha} + f^{\gamma}|_1 + |g^{\alpha}|_1 + |\sigma^{\gamma}|_1 \, \big) \;\;<\;\; \infty$$

Assumption HJB+ : Assumption **HJB** holds true, and for any $\delta > 0$, there exists a finite set $\{\alpha_i, \gamma_i\}_{i=1}^{I_\delta}$ such that for any $(\alpha, \gamma) \in \mathcal{A} \times \mathcal{C}$:

$$\inf_{1 \le i \le I_\delta} \; \big(\, |l^{\alpha} - l^{\alpha_i}|_0 \;+\; |c^{\alpha} - c^{\alpha_i}|_0 \;+\; |f^{\alpha} - f^{\alpha_i}|_0 \;+\; |\sigma^{\alpha} - \sigma^{\alpha_i}|_0 \, \big) \;\;\le\;\; \delta,$$

and

$$\inf_{1 \le i \le I_\delta} \; \big(\, |l^{\gamma} - l^{\gamma_i}|_0 \;+\; |c^{\gamma} - c^{\gamma_i}|_0 \;+\; |f^{\gamma} - f^{\gamma_i}|_0 \;+\; |g^{\gamma} - g^{\gamma_i}|_0 \, \big) \;\;\le\;\; \delta.$$

Theorem 4.2.2. *Suppose that the terminal condition function Φ is bounded and Lipschitz-continuous. Then there is a constant C such that*

- *i) under Assumption* **HJB**, *we have $v - v^h \;\le\; C h^{\frac{1}{4}}$,*
- *ii) under Assumption* **HJB+**, *we have $-Ch^{\frac{1}{10}} \;\le\; v - v^h \;\le\; C h^{\frac{1}{4}}$,*

where v is the unique bounded viscosity solution of (4.2.2) introduced in Theorem 4.2.1.

Remark 4.2.5. *The above convergence rate is the same as that obtained in Fahim et al.[32]. In fact, it depends essentially on the consistency errors of the scheme.*

4.2.4 Proof of local uniform convergence

To prove the local uniform convergence in Theorem 4.2.1, we shall verify the criteria proposed in Theorem 2.1 of Barles and Souganidis [6]: the monotonicity, the consistency of the scheme and the stability of the numerical solutions. Moreover, as discussed in Remark 3.2 of [32], we need also to show that

$$\liminf_{(t',x',y',h)\to(T,x,y,0)} v^h(t',x',y') \geq \Phi(x,y) \text{ and } \limsup_{(t',x',y',h)\to(T,x,y,0)} v^h(t',x',y') \leq \Phi(x,y). \tag{4.2.12}$$

Remark 4.2.6. *By the definition of the numerical scheme $\mathbf{S}_h \circ \mathbf{T}_h$ in (4.2.10), the numerical solution v^h is only defined on the time grid $(t_n)_{0\leq n\leq n}$ product $\mathbb{R}^d \times \mathbb{R}^{d'}$. However, we can use linear interpolation method to extend v^h on the whole space Q_T.*

Proposition 4.2.1. *Let Assumptions* **F**, **H** *and* **M** *hold true, then for two functions φ and ψ defined on Q_T with exponential growth, we have*

$$\varphi \leq \psi \implies \mathbf{S}_h \circ \mathbf{T}_h[\varphi]\ (t,x,y) \ \leq \ \mathbf{S}_h \circ \mathbf{T}_h[\psi]\ (t,x,y).$$

Proof. By Lemma 3.12 and Remark 3.13 of [32], $\varphi \leq \psi$ implies that $\mathbf{T}_h[\varphi](t,x,y) \leq \mathbf{T}_h[\psi](t,x,y)$. Then since $c^{\alpha,\beta} \geq 0$ according to Assumption **M**, it follows immediately by (4.2.10) that $\mathbf{S}_h \circ \mathbf{T}_h[\varphi](t,x,y) \ \leq \ \mathbf{S}_h \circ \mathbf{T}_h[\psi](t,x,y)$. □

We first define a consistency error function, then prove that our splitting scheme $\mathbf{S}_h \circ \mathbf{T}_h$ is consistent.

Definition 4.2.2. *Given a smooth function φ defined on Q_T, the consistency error function of scheme $\mathbf{S}_h \circ \mathbf{T}_h$ is given by*

$$\Lambda_h^{\varphi}(\cdot) \ := \ \frac{\varphi(\cdot) - \mathbf{S}_h \circ \mathbf{T}_h[\varphi](\cdot)}{h} + \mathcal{L}^X \varphi(\cdot) + F(\cdot, \varphi, D_x\varphi, D^2_{xx}\varphi) + H(\cdot, \varphi, D_x\varphi, D_y\varphi). \tag{4.2.13}$$

And the scheme $\mathbf{S}_h \circ \mathbf{T}_h$ is said consistent if

$$\Lambda_h^{\varphi+c}(t',x',y') \to 0 \quad \text{as } (c,h,t',x',y') \to (0,0,t,x,y), \tag{4.2.14}$$

for every $(t,x,y) \in Q_T$ and every smooth function φ with bounded derivatives.

Proposition 4.2.2. *Let Assumptions* **F**, **H** *and* **M** *hold true, then the scheme $\mathbf{S}_h \circ \mathbf{T}_h$ is consistent. In addition, if μ and σ are uniformly bounded, then the consistency error function Λ_h^{φ} is uniformly bounded by $h\ E(\varphi)$, where*

$$E(\varphi) \ := \ C \left(1 \ + \ |\partial_{tt}\varphi|_0 \ + \ \sum_{i=0}^{2} |\partial_t D^i_{z^i}\varphi|_0 \ + \ \sum_{i=0}^{4} |D^i_{z^i}\varphi|_0 \right) \text{ with } z := (x,y) \in \mathbb{R}^{d+d'},$$

for a constant C independent of φ and h.

Proof. For every $(t,x,y) \in Q_T$, the value $\Lambda_h^\varphi(t,x,y)$ is independent of the value of $(\mu(\bar{t},\bar{x}),\sigma(\bar{t},\bar{x}))$ when $(\bar{t},\bar{x}) \neq (t,x)$. Hence we can always change the value of μ and σ outside the neighborhood of (t,x) without influence on the definition of consistency in (4.2.14). Therefore, without loss of generality, we can just suppose that μ and σ are uniformly bounded and show that for every smooth function φ with bounded derivatives of any order, the consistency error function Λ_h^φ defined in (4.2.13) satisfies

$$\big|\Lambda_h^\varphi(\cdot)\big|_0 \;\leq\; h\ E(\varphi). \tag{4.2.15}$$

First, let us denote

$$\mathcal{L}^{\hat{X}^{t,x}}\varphi(t',x',y) \;:=\; \partial_t\varphi(t',x',y) \;+\; \mu(t,x)\cdot D_x\varphi(t',x',y) \;+\; \frac{1}{2}a(t,x)\cdot D^2_{xx}\varphi(t',x',y),$$

then by Itô's formula,

$$\begin{aligned}
E^h(t,x,y,\varphi) \;&:=\; \mathbf{T}_h[\varphi](t,x,y) \;-\; \varphi(t,x,y) \\
&=\; h\ \big(\ \mathcal{L}^X\varphi(\cdot) \;+\; F(\cdot,\varphi,D_x\varphi,D^2_{xx}\varphi)\ \big)\ (t,x,y) \\
&\quad +\; h^2\ \Big(\ \frac{1}{h^2}\ \mathbb{E}\int_t^{t+h}\int_t^u \mathcal{L}^{\hat{X}^{t,x}}\mathcal{L}^{\hat{X}^{t,x}}\varphi(s,\hat{X}^{t,x}_s,y)\ ds\ du\ \Big) \\
&\quad +\; h^2\ \Big[\ \frac{1}{h}\big(F(\cdot,\mathbb{E}\mathcal{D}_h\varphi)(t,x,y) \;-\; F(\cdot,\varphi,D\varphi,D^2_{xx}\varphi)(t,x,y)\big)\ \Big].
\end{aligned} \tag{4.2.16}$$

Denote $E_1(t,x,y,\varphi) := \mathcal{L}^X\varphi(t,x,y) + F(\cdot,\varphi,D_x\varphi,D^2_{xx}\varphi)(t,x,y)$ and by $E_2(t,x,y,\varphi)$ the last two terms of the above equality (4.2.16) divided by h^2, then $E^h(t,x,y,\varphi)$ can rewritten as

$$E^h(t,x,y,\varphi) \;=\; h\ E_1(t,x,y,\varphi) \;+\; h^2\ E_2(t,x,y,\varphi).$$

Clearly, by the boundedness of μ and σ, together with Assumption **F**, there is a constant C independent of h such that

$$\big|E_2(\cdot,\varphi)\big|_0 \;\leq\; C\ \Big(1 + |\partial_{tt}\varphi|_0 + \sum_{i=0}^{2}|\partial_t D^i_{x^i}\varphi|_0 + \sum_{i=0}^{4}|D^i_{x^i}\varphi|_0\Big),$$

and moreover, E_1 is Lipschitz in $z := (x,y)$ with coefficient

$$L_{E_1} \;\leq\; C\big(\ 1 \;+\; |\partial_t D_z\varphi|_0 \;+\; |D_z\varphi|_0 \;+\; |D^2_{zz}\varphi|_0 \;+\; |D^3_{zzz}\varphi|_0\ \big).$$

By simplifying $\big(c^{\alpha,\beta}(t,x,y), l^{\alpha,\beta}(t,x,y), f^{\alpha,\beta}(t,x,y), g^{\alpha,\beta}(t,x,y)\big)$ into

$(c^{\alpha,\beta}, l^{\alpha,\beta}, f^{\alpha,\beta}, g^{\alpha,\beta})$, we deduce that

$$
\begin{aligned}
& \frac{1}{h}\ \Big(\ \mathbf{S}_h[(\varphi + E^h(\cdot,\varphi))](t,x,y)\ -\ \varphi(t,x,y)\ -\ E^h(t,x,y,\varphi)\ \Big) \\
=\ & \frac{1}{h}\ \inf_{\alpha\in\mathcal{A}}\ \sup_{\beta\in\mathcal{B}}\ \Big[\ hl^{\alpha,\beta}\ +\ hc^{\alpha,\beta}\varphi(t,x,y)\ +\ \varphi(t,x+f^{\alpha,\beta}h,\ y+g^{\alpha,\beta}h)\ -\ \varphi(t,x,y) \\
& \qquad\qquad +\ hc^{\alpha,\beta}E^h(t,x,y,\varphi)\ + E^h(t,x+f^{\alpha,\beta}h, y+g^{\alpha,\beta}h) - E^h(t,x,y,\varphi)\Big] \\
=\ & \inf_{\alpha\in\mathcal{A}}\ \sup_{\beta\in\mathcal{B}}\Big[\ l^{\alpha,\beta}\ +\ c^{\alpha,\beta}\varphi(t,x,y)\ +\ (f^{\alpha,\beta}\cdot D_x\varphi\ +\ g^{\alpha,\beta}\cdot D_y\varphi)(t,x,y) \\
& \qquad +\ \frac{1}{h}\big[\varphi(t,x+f^{\alpha,\beta}h, y+g^{\alpha,\beta}h) - \varphi(t,x,y)\big] - (f^{\alpha,\beta}D_x\varphi + g^{\alpha,\beta}D_y\varphi)(t,x,y) \\
& \qquad\quad +\ c^{\alpha,\beta}\ E^h(t,x,y)\ +\ \frac{1}{h}\big[E^h(t,x+f^{\alpha,\beta}h, y+g^{\alpha,\beta}h,\varphi)\ -\ E^h(t,x,y,\varphi)\big]\Big] \\
=:\ & H(\cdot,\varphi,D_x\varphi,D_y\varphi)(t,x,y) + hE_3(t,x,y,\varphi), \qquad (4.2.17)
\end{aligned}
$$

where $E_3(t,x,y,\varphi)$ is defined by the last equality of (4.2.17), and it satisfies

$$|E_3(t,x,y,\varphi)|\ \leq\ C\ \Big(\ |D^2_{zz}\varphi|_0\ +\ \frac{1}{h}E^h(t,x,y,\varphi)\ +\ 2|E_2(t,x,y,\varphi)|\ \Big)\ +\ L_{E_1}\ \leq\ E(\varphi).$$

Combining the estimations (4.2.16) and (4.2.17), and by (4.2.13) as well as the equality

$$
\begin{aligned}
& \frac{\varphi(t,x,y) - \mathbf{S}_h\circ\mathbf{T}_h[\varphi](t,x,y)}{h} \\
=\ & \frac{\varphi(t,x,y) - \mathbf{T}_h[\varphi](t,x,y)}{h}\ +\ \frac{\varphi(t,x,y) + E^h(t,x,y,\varphi) - \mathbf{S}_h[\varphi + E^h(\cdot,\varphi)](t,x,y)}{h},
\end{aligned}
$$

it follows that (4.2.15) holds true. □

Proposition 4.2.3. *Let Assumptions* **F**, **H** *and* **M** *hold true, and the terminal condition function* Φ *be* L^∞*-bounded, then* $(v^h)_h$ *is* L^∞*-bounded, uniformly in* h *for* h *small enough.*

Proof. Suppose that $|v^h(t_{n+1},\cdot)|_\infty \leq C_{n+1}$, then from Lemma 3.14 of [32], there exists a constant C independent of h such that

$$\big|v^h\big(t_{n+\frac{1}{2}},\cdot\big)\big|_\infty\ \leq\ C_{n+1}(1+hC)\ +\ hC.$$

It follows from (4.2.10) that when $h < C^{-1}$,

$$|v^h(t_n,\cdot)|_\infty\ \leq\ (1+hC)(C_{n+1}(1+hC)+hC)\ +\ hC\ \leq\ (1+3hC)C_{n+1}\ +\ 3hC.$$

Therefore, $|v^h(t_n,\cdot)|_\infty \leq C'e^{C'T}$ for some constant C' (independent of h) from the discrete Gronwall inequality. □

We have shown in the above the monotonicity, consistency and stability of scheme $\mathbf{S}_h\circ\mathbf{T}_h$, the rest is to confirm (4.2.12). In fact, we will provide a little stronger property of $(v^h)_{h>0}$ which implies that

$$\lim_{(t',x',y',h)\to(T,x,y,0)}\ v^h(t',x',y')\ =\ \Phi(x,y).$$

Proposition 4.2.4. *Let Assumptions* **F**, **H** *and* **M** *hold true, and* Φ *be Lipschitz and uniformly bounded. Then* $(v^h)_h$ *is Lipschitz in* (x, y)*, uniformly in* h.

Proof. To prove the that v^h is Lipschitz in (x, y), we shall use the discrete Gronwall inequality as in the proof of Lemma 3.16 of [32].

Suppose that $v^h(t_{n+1}, \cdot)$ is Lipschitz with coefficient L_{n+1}, then by the proof of Lemma 3.16 of [32], the function $v^h(t_{n+\frac{1}{2}}, \cdot) = \mathbf{T}_h[v^h](t_n, \cdot)$ is Lipschitz in x with coefficient $L_{n+1}((1 + Ch)^{1/2} + Ch) + Ch$; moreover, $v^h(t_{n+\frac{1}{2}}, \cdot)$ is Lipschitz in y with coefficient $L_{n+1}(1+Ch)$ by Lemma 3.14 of [32]. It follows that $v^h(t_{n+\frac{1}{2}}, \cdot)$ is Lipschitz in (x, y) with coefficient $L_{n+\frac{1}{2}} \le L_{n+1}((1 + Ch)^{1/2} + Ch) + Ch$.

Next, we can easily verify by (4.2.10) that $v^h(t_n, \cdot)$ is Lipschitz in (x, y) with coefficient $L_n \le L_{n+\frac{1}{2}}(1 + Ch) + Ch$. Therefore, the proof is concluded by the discrete Gronwall inequality. □

We can also prove that v^h is $1/2-$Hölder in t as was done in Lemma 3.17 of [32] for their numerical solution. However, to avoid the heavy calculation in their proof, we shall give a weaker result which is enough to guarantee the condition (4.2.12).

Proposition 4.2.5. *Let Assumptions* **F**, **H** *and* **M** *hold true, and* Φ *be Lipschitz and uniformly bounded. Then* $|v^h(t_n, x, y) - \Phi(x, y)| \le C\sqrt{T - t_n}$.

Proof. We first introduce $\bar{v}^h$ as the numerical solution of (4.2.4) computed by scheme $\mathbf{T}_h$, i.e. $\bar{v}^h(T, \cdot) := \Phi(\cdot)$ and $\bar{v}^h(t_n, \cdot) := \mathbf{T}_h[\bar{v}^h](t_n, \cdot)$. Clearly, by Lemmas 3.14 and 3.17 of [32], $(\bar{v}^h)_{h>0}$ is uniformly bounded and satisfies

$$|\bar{v}^h(t_n, \cdot) - \Phi(\cdot)| \le C(T - t_n)^{1/2}, \quad \text{uniformly in } h. \tag{4.2.18}$$

We claim that

$$|\bar{v}^h(t_n, x, y) - v^h(t_n, x, y)| \le C(T - t_n). \tag{4.2.19}$$

Then by (4.2.18), we conclude the proof. Thus it is enough to prove the claim (4.2.19).

We first recall that by Assumption **F** and (4.2.6), for a constant $c \in \mathbb{R}$, we have $\mathbf{T}_h[v^h + c](t, x, y) \le \mathbf{T}_h[v^h](t, x, y) + c + hF_r|c|$. Suppose that for L large enough,

$$|\bar{v}^h(t_{n+1}, x, y) - v^h(t_{n+1}, x, y)| \le L(T - t_{n+1}).$$

It follows by the monotonicity of $\mathbf{T}_h$ and the uniform boundedness of v^h and $\bar{v}^h$ that

$$|\bar{v}^h(t_n, x, y) - v^h(t_{n+\frac{1}{2}}, x, y)| \le L(T - t_{n+1}) + Ch.$$

And hence by (4.2.10),

$$|\bar{v}^h(t_n, x, y) - v^h(t_n, x, y)| \le L(T - t_{n+1}) + 2Ch \le L(T - t_n),$$

which confirms (4.2.19). □

We remark finally that with Propositions 4.2.1, 4.2.2, 4.2.3, 4.2.4 and 4.2.5 together with Theorem 2.1 of Barles and Souganidis [6], Theorem 4.2.1 holds true.

4.2.5 Proof for rate of convergence

As in [32], our arguments to prove the rate of convergence in Theorem 4.2.2 are based on Krylov's shaking coefficient method, and our analysis stays in the context of Barles and Jakobsen [5]. We first derive some technical Lemmas similar to that in [32].

Lemma 4.2.1. *Let Assumptions* **F**, **H** *and* **M** *hold true and* $h \leq 1$*, define* $\lambda_1 := |F_r|_\infty$*,* $\lambda_2 := \sup_{\alpha,\beta} |c^{\alpha,\beta}|_0$*,* $\lambda := \lambda_1 + \lambda_2 + \lambda_1\lambda_2$*. Then, for every* $(a, b, c) \in \mathbb{R}^3_+$*, and every bounded function* $\varphi \leq \psi$ *defined on* Q_T*, with function* $\delta(t) := e^{\lambda(T-t)}(a + b(T - t)) + c$*, we have*

$$\mathbf{S}_h \circ \mathbf{T}_h[\varphi+\delta](t,x,y) \ \leq \ \mathbf{S}_h \circ \mathbf{T}_h[\psi](t,x,y) \ + \ \delta(t) \ - \ h(b-\lambda c), \ \forall t \leq T-h \text{ and } x \in \mathbb{R}^d.$$

Proof. First, from the proof of Lemma 3.21 in [32], we have

$$\mathbf{T}_h[\varphi + \delta](t,x,y) \quad \leq \quad \mathbf{T}_h[\varphi](t,x,y) \ + \ (1 + h\lambda_1) \ \delta(t+h).$$

It follows by the definition of the splitting scheme $\mathbf{S}_h \circ \mathbf{T}_h$ in (4.2.10) that

$$\mathbf{S}_h \circ \mathbf{T}_h[\varphi + \delta](t,x,y) \quad \leq \quad \mathbf{S}_h \circ \mathbf{T}_h[\varphi](t,x,y) \ + \ (1 + h\lambda_1)(1 + h\lambda_2) \ \delta(t+h).$$

By the monotonicity of the splitting scheme $\mathbf{S}_h \circ \mathbf{T}_h$, we get

$$\mathbf{S}_h \circ \mathbf{T}_h[\varphi+\delta](t,x,y) \leq \mathbf{S}_h \circ \mathbf{T}_h[\psi](t,x,y) + \delta(t) + \zeta(t), \text{ where } \zeta(t) := (1+h\lambda)\delta(t+h) - \delta(t).$$

Finally, using exactly the same arguments as in the proof of Lemma 3.5 of [32], it follows that

$$\zeta(t) \quad \leq \quad - h(b - \lambda c),$$

which concludes the proof. □

Proposition 4.2.6. *Let Assumptions* **F**, **H** *and* **M** *hold true,* $h \leq 1$ *and* φ*,* ψ *be two bounded functions defined on* $\mathbb{Q}_T$ *satisfying*

$$\frac{1}{h}\big(\varphi - \mathbf{S}_h \circ \mathbf{T}_h[\varphi]\big) \ \leq \ g_1 \quad \text{and} \quad \frac{1}{h}(\psi - \mathbf{S}_h \circ \mathbf{T}_h[\psi]) \ \geq \ g_2, \quad \text{on } Q_T$$

for some bounded functions g_1 *and* g_2*. Then for every* $n = 0, \cdots, N$*,*

$$(\varphi - \psi)(t_n, x, y) \ \leq \ e^{\lambda(T-t_n)} |(\varphi - \psi)^+(T, \cdot)|_0 \ + \ (T - h) e^{\lambda(T-t_n)} |(g_1 - g_2)^+|_0,$$

with some constant $\lambda \geq |F_r|_\infty + \sup_{\alpha,\beta} |c^{\alpha,\beta}|_0 + |F_r|_\infty \ \sup_{\alpha,\beta} |c^{\alpha,\beta}|_0$.

Proof. With Lemma 4.2.1, the proof is exactly the same as in Proposition 3.20 of [32]. Note that we replace β by λ in our proposition. □

Now, we are ready to give the

Proof of Theorem 4.2.2 (i). First, under Assumption **HJB**, we can rewrite the original PDE (4.2.2) as a standard HJB

$$- \partial_t v \ - \inf_{\alpha \in \mathcal{A}, \gamma \in \mathcal{C}} \Big\{ \ (l^\alpha + l^\gamma) \ + \ (c^\alpha + c^\gamma) v \ + \ (f^\alpha + f^\gamma) \cdot D_x v \\ + \ g^\alpha \cdot D_y v \ + \ \frac{1}{2} (\sigma^\gamma \sigma^{\gamma T}) \cdot D^2_{xx} v \ \Big\} \ = \ 0.$$

With Assumption **HJB** and the Lipschitz terminal condition, it satisfies a comparison result and admits a unique viscosity solution in $C^{0,1}(Q_T)$ (see e.g. Proposition 2.1 of [5]). Then by the shaking coefficients method, we can construct a bounded subsolution $\underline{v}^\varepsilon \in C^{0,1}(Q_T)$ such that

$$v - \varepsilon \leq \underline{v}^\varepsilon \leq v.$$

Let $\rho \in C_c^\infty(Q_T)$ be a positive function supported in $\{(t,x,y) : t \in [0,1], |x| \leq 1, |y| \leq 1\}$ with unit mass, and define

$$\underline{w}^\varepsilon(t,x,y) := \underline{v}^\varepsilon * \rho^\varepsilon, \quad \text{where } \rho^\varepsilon(t,x,y) := \frac{1}{\varepsilon^{d+d'+2}} \rho\Big(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}, \frac{y}{\varepsilon}\Big).$$

Then $\underline{w}^\varepsilon$ is a smooth subsolution of (4.2.2) and satisfies $|\underline{w}^\varepsilon - v| \leq 2\varepsilon$. Moreover, since $\underline{v}^\varepsilon \in C^{0,1}(Q_T)$ is uniformly Lipschitz in (x,y) and $1/2-$Hölder in t, it follows that

$$\underline{w}^\varepsilon \in C^\infty, \text{ and } \big|\partial_t^{\eta_0} D^{\eta_1+\eta_2}_{x^{\eta_1}y^{\eta_2}} \underline{w}^\varepsilon\big| \leq C\varepsilon^{1-2\eta_0-|\eta_1|-|\eta_2|}, \ \forall (\eta_0,\eta_1,\eta_2) \in N^{1+d+d'} \setminus \{0\}. \tag{4.2.20}$$

Now, let us consider the consistency error function $\Lambda_h^{\underline{w}^\varepsilon}(t,x,y)$ defined in (4.2.13). By Proposition 4.2.2 and (4.2.20), it follows that there exists a constant C independent of ε and h for $0 \leq h \leq 1$ such that

$$|\Lambda_h^{\underline{w}^\varepsilon}|_0 \leq R(h,\varepsilon) := Ch\varepsilon^{-3}. \tag{4.2.21}$$

Moreover, since $\underline{w}^\varepsilon$ is a subsolution of equation (4.2.2), it follows by the definition of $\Lambda_h^{\underline{w}^\varepsilon}$ in (4.2.13) that

$$\underline{w}^\varepsilon \leq \mathbf{S}_h \circ \mathbf{T}_h[\underline{w}^\varepsilon] + Ch^2\varepsilon^{-3}.$$

Finally, by Proposition 4.2.6, we get

$$\underline{w}^\varepsilon - v^h \leq C\,(\varepsilon + h\varepsilon^{-3}), \text{ and } v - v^h = v - \underline{w}^\varepsilon + \underline{w}^\varepsilon - v^h \leq C(\varepsilon + h\varepsilon^{-3})$$

and it follows by a minimization technique on ε that

$$v - v^h \leq C \inf_{\varepsilon>0} \big(\varepsilon + h\varepsilon^{-3} \big) \leq C' h^{\frac{1}{4}}. \tag{4.2.22}$$

□

Proof of Theorem 4.2.2 (ii) : Under Assumption **HJB+**, we can apply the switching system method of Barles and Jakobsen [5] which constructs a smooth supersolution closed to viscosity solution to PDE (4.2.2) and provides the lower bound:

$$v - v^h \geq - \inf_{\varepsilon>0} \big(C\varepsilon^{\frac{1}{3}} + R(h,\varepsilon) \big) = - C' h^{\frac{1}{10}}, \tag{4.2.23}$$

where $R(h,\varepsilon)$ is defined in (4.2.21). □

4.3 Basis projection and simulation-regression method

To get an implementable scheme, we need to specify how to compute the expectations $\mathbb{E}\Big[\varphi(t_{n+1}, \hat{X}_h^{t_n,x}, y) H_i^{t_n,x,h}(\Delta W_{n+1})\Big]_{i=0,1,2}$ in the splitting scheme $\mathbf{S}_h \circ \mathbf{T}_h$. When analytic closed formulas are not available in the concrete examples, we usually use Monte-Carlo simulation-regression method to estimate them. Some estimations were discussed in recent literatures, e.g. Malliavin estimations [15], function basis regression [34] and cubature method [22], etc.

All of these methods need the simulations of X. Given a discrete time grid $(t_n)_{0\leq n\leq N}$, where $t_n := n\ h$ and $h := T/N$, we define a simulative process $\hat{X}$ by the Euler scheme of X

$$\hat{X}_{t_{n+1}} \quad := \quad \hat{X}_{t_n} \ + \ \mu(t_n, \hat{X}_{t_n})h \ + \ \sigma(t_n, \hat{X}_{t_n})\Delta W_{n+1}, \tag{4.3.1}$$

where $\Delta W_{n+1} = W_{t_{n+1}} - W_{t_n}$. Then with simulations of process $\hat{X}$ as well as W, one can estimate the conditional expectations

$$\mathbb{E}\ \Big[\ \varphi(t_{n+1}, \hat{X}_{t_{n+1}}, y)\ H_i^{t_n,\hat{X}_{t_n},h}(\Delta W_{n+1})\ \Big|\ \hat{X}_{t_n}\ \Big]_{i=0,1,2}.$$

However, these methods are usually discussed in a non-degenerate context, in other words, they can be used for a given fixed y, which is not appropriate for the implementation of our splitting scheme $\mathbf{S}_h \circ \mathbf{T}_h$.

One solution is to discretize the space of Y into a discrete grid $(y_i)_{i\in I}$, and then for each fixed y_i, we simulate the diffusion process X and get estimations of the conditional expectations for all x with every fixed y_i, then use the interpolation method to get the estimation of theses expectations for all x and y. This is a finite difference Monte-Carlo method, which may loose the advantages of Monte-Carlo method in high dimensional cases.

Therefore, we propose to simulate the diffusion process X with Euler scheme and to simulate Y with a continuous probability distribution (e.g. normal distribution, uniform distribution, etc.) independent of X. And then we use a regression method like in Longstaff and Schwartz [44] in American option pricing context or Gobet, Lemor and Warin [34] in BSDE context to estimate the conditional expectations

$$\mathbb{E}\ \Big[\ \varphi(t_{n+1}, \hat{X}_{t_{n+1}}, Y) H_i^{t_n,\hat{X}_{t_n},h}(\Delta W_{n+1})\ \Big|\ \hat{X}_{t_n},\ Y\ \Big]_{i=0,1,2}, \tag{4.3.2}$$

with which we shall make the splitting scheme $\mathbf{S}_h \circ \mathbf{T}_h$ implementable.

Remark 4.3.1. *The distribution of Y may be chosen arbitrarily according to the context.*

In the following, we first give a basis projection scheme as well as a similation-regression method to estimate the regression coefficient. Then we discuss the convergence of Monte-Carlo errors in our context.

4.3.1 Basis projection scheme and simulation-regression method

4.3.1.1 The basis projection scheme

To compute the conditional expectations (4.3.2), we first project them on a functional space spanned by the basis functions $(e_k(x,y))_{1\le k\le K}$, where $K \in \mathbb{N}\cup\{+\infty\}$. We recall that $H_2^{t,x,h}$ is a matrix of dimension $d\times d$, $H_1^{t,x,h}$ is a vector of dimension d and $H_0^{t,x,h} = 1$. In order to simplify the presentation, we shall suppose that $d = d' = 1$. All of the results can be easily extended to the case $d > 1$ and/or $d' > 1$. Let

$$\tilde{\lambda}^i \;:=\; \arg\min_{\lambda} \mathbb{E}\,\Big(\varphi(t_{n+1},\hat{X}_{t_{n+1}},Y)H_i^{t_n,\hat{X}_{t_n},h}(\Delta W_{n+1}) \;-\; \sum_{k=1}^{K}\lambda_k e_k(\hat{X}_{t_n},Y)\Big)^2, \quad (4.3.3)$$

then the projected approximation of (4.3.2) is denoted by

$$\tilde{\mathbb{E}}\,\Big[\,\varphi(t_{n+1},\hat{X}_{t_{n+1}},Y)H_i^{t_n,\hat{X}_{t_n},h}(\Delta W_{n+1})\;\Big|\;\hat{X}_{t_n},Y\,\Big] \;:=\; \sum_{k=1}^{K}\tilde{\lambda}_k^i e_k(\hat{X}_{t_n},Y). \quad (4.3.4)$$

Remark 4.3.2. *There are several choices for function basis $(e_k(x,y))_{1\le k\le K}$, for example global polynomials, local hypercubes or local polynomials, we refer to Bouchard and Warin [17] for some interesting discussions.*

We replace the conditional expectations (4.3.2) in scheme $\mathbf{S}_h\circ\mathbf{T}_h$ by their projected approximations (4.3.4), and denote the new splitting scheme by $\mathbf{S}_h\circ\tilde{\mathbf{T}}_h$. Concretely, it is defined as follows:

$$\tilde{\mathbf{T}}_h[\tilde{v}^h](t_n,x,y) \;:=\; \tilde{\mathbb{E}}\,\big[\tilde{v}^h(t_{n+1},\hat{X}_h^{t_n,x},y)\big] \;+\; hF(\cdot,\tilde{\mathbb{E}}\mathcal{D}\tilde{v}^h(\cdot))(t_n,x,y),$$

where

$$\tilde{\mathbb{E}}\mathcal{D}_h\varphi(t_n,x,y) \;=\; \Big(\,\tilde{\mathbb{E}}\,\big[\varphi(t_{n+1},\hat{X}_h^{t_n,x},y)H_i^{t_n,x,h}(\Delta W_{n+1})\big] : i=0,1,2\,\Big),$$

and

$$\begin{aligned}\tilde{v}^h(t_n,x,y) \;&=\; \mathbf{S}_h\circ\tilde{\mathbf{T}}_h[\tilde{v}^h](t_n,x,y)\\ :=&\; \inf_{\alpha}\sup_{\beta}\Big\{\,hl^{\alpha,\beta}(t_n,x,y) \;+\; hc^{\alpha,\beta}(t_n,x,y)\tilde{\mathbf{T}}_h[\tilde{v}^h](t_n,x,y)\\ &\quad+\;\tilde{\mathbf{T}}_h[\tilde{v}^h]\big(t_n,\; x+f^{\alpha,\beta}(t_n,x,y)h,\; y+g^{\alpha,\beta}(t_n,x,y)h\big)\,\Big\}. \quad (4.3.5)\end{aligned}$$

4.3.1.2 Simulation-regression method

Next, we propose to use a simulation-regression method to approximate $\tilde{\lambda}$. We still suppose that $d = d' = 1$ for simplicity.

Let $\big((\hat{X}_{t_n}^m)_{0\le n\le N},(\Delta W_n^m)_{0<n\le N},Y^m\big)_{1\le m\le M}$ be M independent simulations of $\hat{X}$, ΔW and Y, where $\hat{X}$ is defined in (4.3.1), the regression method with function basis

$(e_k(x,y))_{1\le k\le K}$ is to get the solution of the least square problem:

$$\hat{\lambda}^{i,M} = \arg\min_{\lambda} \sum_{m=1}^{M} \Big(\varphi(t_{n+1},\hat{X}^m_{t_{n+1}},Y^m)H_i^{t_n,\hat{X}^m_{t_n},h}(\Delta W^m_{n+1}) - \sum_{k=1}^{K}\lambda_k e_k(\hat{X}^m_{t_n},Y^m)\Big)^2 \quad (4.3.6)$$

A raw regression estimation of the conditional expectations (4.3.2) from these M samples is given by

$$\bar{\mathbb{E}}^M\Big[\varphi(t_{n+1},\hat{X}_{t_{n+1}},Y)H_i^{t_n,\hat{X}_{t_n},h}(\Delta W_{n+1})\big|\hat{X}_{t_n},Y\Big] \;:=\; \sum_{k=1}^{K}\hat{\lambda}^{i,M}_k e_k(\hat{X}_{t_n},Y),\ i=0,1,2. \quad (4.3.7)$$

Then with a priori upper bounds $\overline{\Gamma}_i(\hat{X}_{t_n},Y)$ and lower bounds $\underline{\Gamma}_i(\hat{X}_{t_n},Y)$, we define the regression estimation of (4.3.2):

$$\begin{aligned}
&\hat{\mathbb{E}}^M\Big[\ \varphi(t_{n+1},\hat{X}_{t_{n+1}},Y)H_i^{t_n,\hat{X}_{t_n},h}(\Delta W_{n+1})\ \big|\ \hat{X}_{t_n},Y\ \Big] \quad (4.3.8)\\
&:= \underline{\Gamma}_i(\hat{X}_{t_n},Y)\ \vee\ \bar{\mathbb{E}}^M\Big[\varphi(t_{n+1},\hat{X}_{t_{n+1}},Y)H_i^{t_n,\hat{X}_{t_n},h}(\Delta W_{n+1})\big|\hat{X}_{t_n},Y\Big]\ \wedge\ \overline{\Gamma}_i(\hat{X}_{t_n},Y).
\end{aligned}$$

Remark 4.3.3. *As observed in Bouchard and Touzi [15], the truncation method is an important technique to obtain a L^p-convergence. By Lemma (4.2.5), we can choose $\overline{\Gamma}_0(x,y)=\Gamma_0(x,y)$ and $\underline{\Gamma}_0(x,y)=-\Gamma_0(x,y)$ with a function Γ_0 satisfying*

$$\Gamma_0(x,y)\ \le\ \Phi(x,y)\ +\ C\sqrt{T-t_n}\quad \text{for some constant } C. \quad (4.3.9)$$

Remark 4.3.4. *In Gobet et al. [34], the authors propose the following minimization problem in place of (4.3.6):*

$$\min_{\lambda^0,\lambda^1}\sum_{m=1}^{M}\Big(\varphi(t_{n+1},\hat{X}^m_{t_{n+1}},Y^m)\ -\ \sum_{k=1}^{K}\lambda^0_k e_k(\hat{X}^m_{t_n},Y^m)\ -\ \sum_{k=1}^{K}\lambda^1_k e_k(\hat{X}^m_{t_n},Y^m)\Delta W^m_{n+1}\Big)^2,$$

which gives also a good estimation for $\tilde{\lambda}^i$ by the fact that ΔW_{n+1} is independant of the $\sigma-$field generated by $Y,W_0,\Delta W_1,\cdots,\Delta W_n$.

We replace the conditional expectations (4.3.2) in scheme $\mathbf{S}_h\circ\mathbf{T}_h$ by their regression estimations (4.3.8) and denote the new numerical splitting scheme by $\mathbf{S}_h\circ\hat{\mathbf{T}}^M_h$, which is

$$\hat{\mathbf{T}}^M_h[\hat{v}^h](t_n,x,y)\ :=\ \hat{\mathbb{E}}^M\big[\hat{v}^h(t_{n+1},\hat{X}^{t_n,x}_h,y)\big]\ +\ h\ F(\cdot,\hat{\mathbb{E}}^M\mathcal{D}\hat{v}^h(\cdot))(t_n,x,y),$$

and

$$\hat{\mathbb{E}}^M\mathcal{D}_h\varphi(t_n,x,y)\ =\ \Big(\ \hat{\mathbb{E}}^M\big[\varphi(t_{n+1},\hat{X}^{t_n,x}_h,y)H_i^{t_n,x,h}(\Delta W_{n+1})\big]\ :\ i=0,1,2\ \Big),$$

so that $\mathbf{S}_h\circ\hat{\mathbf{T}}^M_h$ is defined by

$$\begin{aligned}
\hat{v}^h(t_n,x,y)\ &=\ \mathbf{S}_h\circ\hat{\mathbf{T}}^M_h[\hat{v}^h](t_n,x,y)\\
&:=\ \inf_{\alpha\in\mathcal{A}}\sup_{\beta\in\mathcal{B}}\Big\{hl^{\alpha,\beta}(t_n,x,y)\ +\ hc^{\alpha,\beta}(t_n,x,y)\hat{\mathbf{T}}^M_h[\hat{v}^h](t_n,x,y) \quad (4.3.10)\\
&\qquad +\ \hat{\mathbf{T}}^M_h[\hat{v}^h]\big(t_n,\ x+f^{\alpha,\beta}(t_n,x,y)h,\ y+g^{\alpha,\beta}(t_n,x,y)h\big)\Big\}.
\end{aligned}$$

4.3.2 The convergence results of simulation-regression scheme

To get a convergence result of schemes $\mathbf{S}_h \circ \tilde{\mathbf{T}}_h$ and $\mathbf{S}_h \circ \hat{\mathbf{T}}_h^M$, we can no longer use the same arguments as in Fahim et al. [32], since there is no uniform convergence property in L^p for the Monte-Carlo error $(\hat{\mathbb{E}}^M - \mathbb{E})(R)$ as in the Assumption **E** of [32]. To see this, let us consider the extreme case where the equation is totally degenerate (i.e. $d = 0$ and $d' > 0$), and then we need to approximate an arbitrary bounded function in a functional space with finite number of basis functions, which does not give a uniform convergence.

Also, since we are in the viscosity solution analysis context of Barles and Souganidis [6], we can not hope to obtain a probabilistic $L^2(\Omega)-$convergence as in Gobet et al. [34].

However, we can get a convergence result if we choose the local hypercubes as function basis. Let us restrict the numerical resolution on $[0,T]\times D$ instead of Q_T, where $D \subset \mathbb{R}^{d+d'}$ is a bounded domaine. Clearly, we need to assume that the boundary conditions on the domain $D^c := \mathbb{R}^{d+d'} \setminus D$ are available for scheme $\mathbf{S}_h \circ \hat{\mathbf{T}}_h^M$.

Definition 4.3.1. *Given a domaine $D \subseteq \mathbb{R}^{d+d'}$, a class of hypercube sets $(B_k)_{1\le k\le K}$ is called a partition of D whenever $\cup_{k=1}^K B_k = D$ and $B_i \cap B_j = \emptyset$.*

Remark 4.3.5. *The simplest examples of partition of D is the uniform partition. With uniform interval $[x_k, x_k']$ and $[y_k, y_k']$, B_k are of the form $[x_k, x_k'] \times [y_k, y_k']$. Recently, Bouchard and Warin [17] proposed a partition based on the simulations. They first sort all the simulations and then divide the space in a non-uniform way such that they have the same number of simulation particles in every hypercube B_k.*

Remark 4.3.6. *If we use hypercubes $(\mathbf{1}_{B_k})_{\mathbf{1}\le k\le K}$ as basis function in the projections (4.3.3), where $(B_k)_{1\le k\le K}$ is a partition of $D \subseteq \mathbb{R}^{d+d'}$, then the projection approximation is equivalent to taking another conditional expectation on the σ-field generated by $\big\{(X_{t_n}, Y) \in B_k\big\}_{1\le k\le K}$, in other words,*

$$\begin{aligned}
&\tilde{\mathbb{E}}\ \Big[\ \varphi(t_{n+1}, \hat{X}_{t_{n+1}}, Y) H_i^{t_n, \hat{X}_{t_n}, h}(\Delta W_{n+1})\ \Big|\ \hat{X}_{t_n}, Y\ \Big] \qquad (4.3.11)\\
=&\ \sum_{k=1}^{K} \mathbb{E}\Big[\ \varphi(t_{n+1}, \hat{X}_{t_{n+1}}, Y) H_i^{t_n, \hat{X}_{t_n}, h}(\Delta W_{n+1})\ \Big|\ (\hat{X}_{t_n}, Y) \in 1_{B_k}\ \Big]\ \mathbf{1}_{B_k}(\hat{X}_{t_n}, Y).
\end{aligned}$$

Let us use $(e_k)_{1\le k\le K} = (\mathbf{1}_{B_k})_{1\le k\le K}$ as projection basis in (4.3.3) and (4.3.6), where $(B_k)_{1\le k\le K}$ is a partition of D. Given a bounded function φ on D, a process $\hat{X}$ and a random variable Y, we shall consider the random variables of the form

$$R_i(\varphi) \ := \ \varphi(t_{n+1}, \hat{X}_{t_{n+1}}, Y)\ H_i^{t_n, \hat{X}_{t_n}, h}(\Delta W_{n+1}), \quad i = 0, 1, 2, \qquad (4.3.12)$$

and then give an estimation for the regression error $(\hat{\mathbb{E}}^M - \tilde{\mathbb{E}})\ [R_i(\varphi) \mid \hat{X}_{t_n} = x, Y = y]$.

Lemma 4.3.1. *Suppose that the a priori estimations used in (4.3.8) satisfy*

$$\underline{\Gamma}_i(x,y)^2 \ + \ \overline{\Gamma}_i(x,y)^2 \ \le \ C\ \Gamma(x,y)^2\ h^{-i}, \quad \textit{for some function } \Gamma(x,y).$$

Then for every $(x, y) \in B_k$,

$$\mathbb{E}\Big[\ (\hat{\mathbb{E}}^M - \tilde{\mathbb{E}})^2\ [R_i(\varphi) \mid \hat{X}_{t_n} = x, Y = y]\ \Big] \quad \leq \quad C\ \frac{1}{M}\ h^{-i}\frac{|\varphi|_0^2\ +\ \Gamma^2(x, y)}{\mathbb{P}((\hat{X}_{t_n}, Y) \in B_k)}. \quad (4.3.13)$$

The proof is almost the same as that of Theorem 5.1 of Bouchard and Touzi [15], we report it in Appendix for completeness.

Let φ be bounded by constant b, δ denote the longest edge of the hypercubes $(B_k)_{1 \leq k \leq K}$, then the volume of B_k is of order $\delta^{d+d'}$, and $\mathbb{P}((\hat{X}_{t_n}, Y) \in B_k) \approx C\delta^{d+d'}$, where C depends on the density of $(\hat{X}_{t_n}, Y)$. As the total volume of D is fixed and finite, let

$$\hat{C}(\delta) \quad := \quad \sup_{N,n,k,x,y}\ C\ \frac{1}{M}\ h^{-i}\frac{b^2\ +\ \Gamma^2(x, y)}{\mathbb{P}((\hat{X}_{t_n}, Y) \in B_k)}, \qquad (4.3.14)$$

it follows that $\hat{C}(\delta) \approx C\delta^{-(d+d')}$.

Now, let us give a local uniform convergence as well as a rate of convergence for the simulation-regression scheme $\mathbf{S}_h \circ \hat{\mathbf{T}}_h^M$.

Theorem 4.3.1. *Let Assumptions* **F**, **H** *and* **M** *hold true,* F *be uniformly bounded,* Φ *be bounded and Lipschitz continuous, and assume that the PDE* (4.2.2) *satisfies a comparison result for bounded viscosity solutions. In addition, given a time step* h*, there is a* D*-partition hypercubes* $(B_k^h)_{1 \leq k \leq K_h}$ *with edge* δ_h *such that* $\delta_h\ h^{-1} \to 0$ *as* $h \to 0$*. Let the truncation function* Γ_0 *satisfies* (4.3.9)*, and we use hypercubes* $(1_{B_k^h})_{1 \leq k \leq K_h}$ *as projection basis functions and with sample number* $M = M_h$ *such that* $\hat{C}(\delta_h)\ h^{-2} M_h^{-1} \to 0$*, where* $\hat{C}(\delta_h)$ *is defined in* (4.3.14)*. Then there exists a function* v*, such that*

$$\hat{v}^h \quad \to \quad v \qquad \text{locally uniformly, a.s.}$$

where $\hat{v}^h$ *is the numerical solution of scheme* $\mathbf{S}_h \circ \hat{\mathbf{T}}_h^M$ *defined in* (4.3.10) *with terminal condition* Φ*. Moreover,* v *is the unique bounded viscosity solution of* (4.2.2) *and* (4.2.3)*.*

Theorem 4.3.2. *Let Assumption* **HJB+** *hold,* Φ *be bounded and Lipschitz continuous, and assume that we use hypercubes* $(1_{B_k^h})_{1 \leq k \leq K_h}$ *as projection basis functions whose longest edge satisfies* $\delta_h \leq Ch^{\frac{11}{10}}$*, and we choose simulation number* $M = M_h$ *such that*

$$\limsup_{h \to 0}\ h^{-\frac{1}{20} - 2}\ \hat{C}(\delta)\ M^{-1} \quad < \quad \infty.$$

Then there is a constant $C > 0$*, s.t.*

$$\|v - \hat{v}^h\|_{L^2(\Omega)} \quad \leq \quad Ch^{\frac{1}{10}},$$

where $\hat{v}^h$ *is the numerical solution of scheme* $\mathbf{S}_h \circ \hat{\mathbf{T}}_h^M$ *in* (4.3.10) *with terminal condition* Φ *and* v *is the unique bounded viscosity solution of* (4.2.2) *and* (4.2.3)*.*

4.3.3 Some analysis on the basis projection scheme $\mathbf{S}_h \circ \tilde{\mathbf{T}}_h$

In preparation of the proof for Theorems 4.3.1 and 4.3.2, we give some analysis on the scheme $\mathbf{S}_h \circ \tilde{\mathbf{T}}_h$. In general, we shall show that if we use the local hypercubes as projection function basis, then $\mathbf{S}_h \circ \tilde{\mathbf{T}}_h$ still has the monotonicity, consistency and stability.

Proposition 4.3.1. *Let $(B_k)_{1\le k\le K}$ be a partition of domain D, and the three projections ($i = 0, 1, 2$) of (4.3.3) use the same hypercubes $(1_{B_k})_{1\le k\le K}$ as projection function basis. Then under Assumptions* **F**, **H** *and* **M**,

- *i) The basis projection scheme $\mathbf{S}_h \circ \tilde{\mathbf{T}}_h$ is monotone.*
- *ii) If the terminal condition Φ is uniformly bounded, then the numerical solution $\tilde{v}^h$ of scheme $\mathbf{S}_h \circ \tilde{\mathbf{T}}_h$ in (4.3.5) is uniformly bounded for h small enough.*

Proof. In view of Remark 4.3.6, we replace the conditional expectations in $\mathbf{S}_h \circ \mathbf{T}_h$ by the new conditional expectations (4.3.11), and then get the projection scheme $\mathbf{S}_h \circ \tilde{\mathbf{T}}_h$. Therefore, all the arguments still hold in the proof of Lemma 3.2 and 3.3 of [32] for $\tilde{\mathbf{T}}_h$, so do Propositions 4.2.1 and 4.2.3. Therefore, Proposition 4.3.1 holds true. □

Similar to the consistency error function Λ_h^φ for scheme $\mathbf{S}_h \circ \mathbf{T}_h$ defined in (4.2.13), we define the consistency error function $\tilde{\Lambda}_h^\varphi$ for scheme $\mathbf{S}_h \circ \tilde{\mathbf{T}}_h$ by

$$\tilde{\Lambda}_h^\varphi(\cdot) \;:=\; \frac{\varphi(\cdot) - \mathbf{S}_h \circ \tilde{\mathbf{T}}_h[\varphi](\cdot)}{h} + \mathcal{L}^X\varphi(\cdot) + F(\cdot, \varphi, D_x\varphi, D^2_{xx}\varphi) + H(\cdot, \varphi, D_x\varphi, D_y\varphi). \tag{4.3.15}$$

Proposition 4.3.2. *Let δ denote the longest edge of hypercubes $(B_k)_{1\le k\le K}$, then the projection error for a Lipschitz continuous function is proportional to δ. Moreover, if we use hypercubes $(1_{B_k})_{1\le k\le K}$ as projection function basis, then under Assumptions* **F**, **H** *and* **M**, *the consistency error function $\tilde{\Lambda}_h^\varphi$ is uniformly bounded by $\tilde{E}(\varphi)$, where*

$$\tilde{E}(\varphi) \;:=\; E(\varphi) + Ch^{-1}\,\delta\big(\,|D_z\varphi|_0 \;+\; h|D^2_{zz}\varphi|_0 \;+\; h|D^3_{zzz}\varphi|_0\,\big), \quad \text{for } z := (x, y),$$

with $E(\varphi)$ defined in Proposition 4.2.2.

Proof. In view of Remark 4.3.6, the error caused by conditional expectation on hypercube is bounded by $C\delta|D^{i+1}_{z^{i+1}}\varphi|_0$ for $D^i_{z^i}\varphi$. Thus we get immediately the new consistency error $\tilde{E}(\varphi)$ with Proposition 4.2.2. □

Proposition 4.3.3. *Suppose that the three projections in (4.3.3) use the same D-partition hypercubes as projection function basis, then Lemma 4.2.1 and Proposition 4.2.6 hold true if we replace the scheme $\mathbf{S}_h \circ \mathbf{T}_h$ by $\mathbf{S}_h \circ \tilde{\mathbf{T}}_h$.*

Proof. With the Proposition 4.3.1 and under Assumptions **F**, **H** and **M**, we see that all the arguments are still true in the proofs of Lemma 4.2.1 and Proposition 4.2.6 for scheme $\mathbf{S}_h \circ \tilde{\mathbf{T}}_h$, in view of Remark 4.3.6. So we get the same results for the basis projection scheme $\mathbf{S}_h \circ \tilde{\mathbf{T}}_h$. □

4.3.4 The proof for convergence results of scheme $\mathbf{S}_h \circ \hat{\mathbf{T}}_h^M$

To prove Theorem 4.3.1, we shall mimic the proof of Theorem 4.1 in [32], which uses the arguments of [6] in a stochastic context.

Proof of Theorem 4.3.1. Given $\hat{v}^h$ the numerical solution of scheme $\mathbf{S}_h \circ \hat{\mathbf{T}}_h^M$, we denote

$$\hat{v}_*(t,x,y) := \liminf_{(t',x',y',h)\to(t,x,y,0)} \hat{v}^h(t',x',y'), \qquad \hat{v}^*(t,x,y) := \limsup_{(t',x',y',h)\to(t,x,y,0)} \hat{v}^h(t',x',y').$$

First, it is clear by the truncation function (4.3.9) as well as the boundedness of F that $|v(t_n,x,y) - \Phi(x,y)| \le C(T-t_n)$ for some constant C, which implies that $\hat{v}_*(T,x,y) = \hat{v}^*(T,x,y) = \Phi(x,y)$. Then it is enough to prove that $\hat{v}_*$ and $\hat{v}^*$ are respectively viscosity supersolution and subsolution of (4.2.2) to conclude the proof with the comparison assumption. Here we shall only prove the supersolution property, since the subsolution property holds true with the same kind of argument.

Given $(t_0,x_0,y_0) \in Q_T$ and a test function $\varphi \in C_c^\infty(Q_T)$ such that

$$0 = \min(\hat{v}_* - \varphi) = (\hat{v}_* - \varphi)(t_0,x_0,y_0),$$

by uniform boundedness of $\hat{v}^h$ and manipulation on φ, there is a sequence $(t_k,x_k,y_k,h_k) \to (t_0,x_0,y_0,0)$ such that $\hat{v}^{h_k}(t_k,x_k,y_k) \to \hat{v}_*(t_0,x_0,y_0)$ and

$$C_k := \min(\hat{v}^{h_k} - \varphi) = (\hat{v}^{h_k} - \varphi)(t_k,x_k,y_k) \to 0.$$

From the monotonicity of scheme $\mathbf{S}_h \circ \tilde{\mathbf{T}}_h$, it follows that

$$\mathbf{S}_h \circ \tilde{\mathbf{T}}_h[\, \hat{v}^{h_k} \,] \ \ge\ \mathbf{S}_h \circ \tilde{\mathbf{T}}_h[\, \varphi + C_k \,],$$

and hence

$$\begin{aligned} 0 &= \hat{v}^{h_k}(t_k,x_k,y_k) - \mathbf{S}_h \circ \hat{\mathbf{T}}_h^M[\hat{v}^{h_k}](t_k,x_k,y_k) \\ &= \hat{v}^{h_k}(t_k,x_k,y_k) - \mathbf{S}_h \circ \tilde{\mathbf{T}}_h[\hat{v}^{h_k}](t_k,x_k,y_k) + h_k R_k \\ &\le \varphi(t_k,x_k,y_k) + C_k - \mathbf{S}_h \circ \tilde{\mathbf{T}}_h[\varphi + C_k](t_k,x_k,y_k) + h_k R_k, \end{aligned}$$

where $R_k := h_k^{-1}(\mathbf{S}_{h_k} \circ \hat{\mathbf{T}}_{h_k}^M - \mathbf{S}_{h_k} \circ \tilde{\mathbf{T}}_{h_k})[\hat{v}^{h_k}](t_k,x_k,y_k)$. We claim that

$$R_k \to 0 \quad \mathbb{P}\text{-a.s. along some subsequence.} \tag{4.3.16}$$

Then, from the consistence of scheme $\mathbf{S}_h \circ \tilde{\mathbf{T}}_h$ in Proposition 4.3.2,

$$\left[-\mathcal{L}^X\varphi - F(\cdot,\varphi,D_x\varphi,D^2_{xx}\varphi) - H(\cdot,\varphi,D_x\varphi,D_y\varphi)\right](t_0,x_0,y_0) \ \ge\ 0,$$

which is the required supersolution property.

Therefore, it is enough to justify the claim (4.3.16) to conclude the proof. Indeed, by the definition of splitting scheme $\mathbf{S}_h \circ \hat{\mathbf{T}}_h^M$ and $\mathbf{S}_h \circ \tilde{\mathbf{T}}_h$, and the boundedness of $c^{\alpha,\beta}$,

$$\begin{aligned}\mathbb{E}|R_k|^2 &\leq (1+Ch_k)^2 \frac{1}{h_k^2} \mathbb{E} \left[\tilde{\mathbf{T}}_{h_k}[\hat{v}^{h_k}] - \hat{\mathbf{T}}_{h_k}^M[\hat{v}^{h_k}] \right]^2 (t_k, x_k, y_k) \\ &\leq C (1+Ch_k)^2 \frac{1}{h_k^2} (\tilde{\mathbb{E}} - \hat{\mathbb{E}}^M)^2 \Big[R_0(\hat{v}^{h_k}) + h_k R_1(\hat{v}^{h_k}) + h_k R_2(\hat{v}^{h_k}) \Big],\end{aligned}$$

where $R_i(\hat{v}^{h_k})$ is defined in (4.3.12). And therefore by Lemma 4.3.1

$$\begin{aligned}\mathbb{E}|R_k|^2 &\leq C (1+Ch_k)^2 \frac{1}{h_k^2}\Big(\hat{C}(\delta) + h_k^2 \big(\frac{1}{h_k}\hat{C}(\delta) + \frac{1}{h_k^2}\hat{C}(\delta)\big)\Big) \frac{1}{M} \\ &\leq C\, h_k^{-2}\, \hat{C}(\delta)\, M^{-1} \rightarrow 0.\end{aligned}$$

We then proved the claim and the theorem. □

Proof of Theorem 4.3.2. With Proposition 4.3.3, we can proceed as in the proof of Theorem 4.2.2. Then there is a subsolution $\underline{w}^h$ of (4.2.2) such that

$$v \leq \underline{w}^h + C\varepsilon \quad \text{and} \quad \underline{w}^h - \tilde{v}^h \leq C(h\varepsilon^{-3} + h^{-1}\delta + \delta\varepsilon^{-2}).$$

Moreover, since

$$h^{-1}\big(\hat{v}^h - \mathbf{S}_h \circ \tilde{\mathbf{T}}_h[\hat{v}^h]\big) \geq -R_h[\hat{v}^h], \quad \text{where } R_h[\varphi] := \frac{1}{h}\Big|(\mathbf{S}_h \circ \tilde{\mathbf{T}}_h - \mathbf{S}_h \circ \hat{\mathbf{T}}_h^M)[\varphi]\Big|,$$

it follows from Proposition 4.3.3 that $\tilde{v}^h - \hat{v}^h \leq C|R_h[\hat{v}^h]|$, and then

$$v - \hat{v}^h = v - \tilde{v}^h + \tilde{v}^h - \hat{v}^h \leq C\big(\varepsilon + h\varepsilon^{-3} + h^{-1}\delta + \delta\varepsilon^{-2} + \big|R_h[\hat{v}^h]\big|\big).$$

Similarly, we have the other side of the error boundary and get

$$\big|v - \hat{v}^h\big|^2 \leq C\big((\varepsilon^{\frac{1}{3}} + h\varepsilon^{-3} + h^{-1}\delta + \delta\varepsilon^{-2})^2 + \big|R_h[\hat{v}^h]\big|^2 \big). \tag{4.3.17}$$

Finally, it is enough to take expectations on both sides of (4.3.17) and maximize the right side on ε for $\varepsilon_h = h^{\frac{3}{10}}$, which implies that

$$\mathbb{E}\big|v - \hat{v}^h\big|^2 \leq C\Big(h^{\frac{1}{20}} + \frac{1}{M}\frac{1}{h^2}\hat{C}(\delta)\Big) \leq C' h^{\frac{1}{20}}.$$

□

4.4 Numerical examples

We provide here some numerical examples, one is from Asian option pricing problem and the other is from an optimal management problem for hydropower plant. All the examples are implemented by a computer with 2.4GHz CPU and 4G memory, we give also the computation time of each numerical example by this computer.

4.4.1 Asian option pricing

Our first example is to price Asian option in Black-Scholes model, whose pricing equation is a degenerate and linear PDE. The second example is the one in uncertain volatility model (UVM), whose corresponding equation is a degenerate and nonlinear PDE. Then we also consider the problem in UVM with Hull-White interest rate.

4.4.1.1 Asian option pricing in Black-Scholes model: a two-dimensional case

In a Black-Scholes market with interest rate r and volatility σ, the risky asset S_t follows dynamic equation:

$$dS_t \;=\; rS_t dt \;+\; \sigma S_t dW_t.$$

Denote

$$A_t \;:=\; \int_0^t S_u \, du,$$

an Asian option is an option with payoff $g(S_T, A_T)$ at maturity T. Then the value function

$$v(t,s,a) \;:=\; \mathbb{E}\left[\, e^{-r(T-t)} g(S_T, A_T) \;\middle|\; S_t = s,\; A_t = a \,\right]$$

solves the pricing equation

$$\Big(\partial_t v \;+\; rsD_s v \;+\; \frac{1}{2}\sigma^2 s^2 D_s^2 v \;+\; sD_a v \;-\; rv\Big)(t,s,a) \;=\; 0, \tag{4.4.1}$$

with terminal condition $v(T,s,a) = g(s,a)$.

When the payoff is $g(S_T, A_T) = (A_T - K)^+$ for a constant K, Rogers and Shi [50] proposed a dimension reduction method. We use it as well as Monte-Carlo method for comparison.

Let $S_0 = 100$, $K = 100$, $T = 1$, $r = 0.05$ and $\sigma = 0.2$. For Monte-Carlo method, we choose $N = 10^3$, $\Delta t = \frac{1}{N}$ and denote $A_T^N := \sum_{k=1}^N S_{t_k} \Delta t$, where $t_k = k\Delta t$. With 10^6 simulations of A_T^N, the empirical mean value of $e^{-rT}(A_T^N - K)^+$ is 5.7758. For Rogers and Shi's method, we implement an implicit finite difference method with $\Delta\xi = \frac{1}{400}$ and $\Delta t = \frac{1}{8000}$ (for Rogers and Shi's equation on variable ξ, see [50]) and get option price 5.7833.

For our splitting scheme, we launched 50 times the algorithm for every given time step Δt. By computing the empirical mean value and standard deviation, a confidence of interval is also provided. As a comparison to the splitting scheme, we also implemented the Crank-Nicolson finite difference scheme for equation (4.4.1), with $\Delta S = 1$ and $\Delta A = 0.25$. As shown in Figure 4.1, as time step Δt goes to 0, the numerical solution of our splitting scheme as well as the one of the Crank-Nicolson finite difference method converge to reference price.

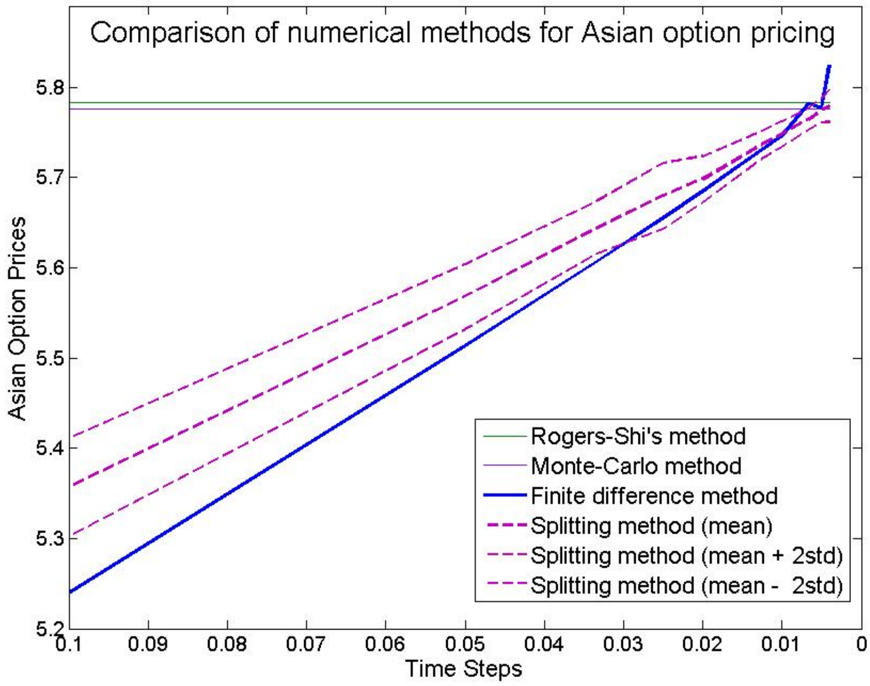

Figure 4.1: The comparaison of some numerical methods for pricing Asian option with payoff $(A_T - K)^+$ in Black-Scholes model, with parameters $S_0 = 100$, $K = 100$, $T = 1$, $r = 0.05$ and $\sigma = 0.2$. When $\Delta t = 0.005$, a single computation takes 2.07 seconds for finite difference method, and 118 seconds for our splitting method using 5×10^5 simulations.

4.4.1.2 Asian option pricing in UVM: a two-dimensional case

In uncertain volatility model, the volatility is uncertain and bounded between the lower volatility $\underline{\sigma}$ and the upper volatility $\overline{\sigma}$. Therefore, the pricing equation for payoff $g(S_T, A_T)$ becomes

$$\Big(\partial_t v \; + \; rsD_s v \; + \; \frac{1}{2} \max_{\underline{\sigma} \leq \sigma \leq \overline{\sigma}} \sigma^2 s^2 D_{ss}^2 v \; + \; sD_a v \; - \; rv \Big)(t, s, a) \quad = \quad 0, \qquad (4.4.2)$$

with terminal condition $v(T, s, a) = g(s, a)$.

To implement the splitting scheme, we rewrite (4.4.2) in form of the equation (4.2.2) with a constant σ_0:

$$\partial_t v \; + \; rsD_s v \; + \; \frac{1}{2}\sigma_0^2 s^s D_{ss}^2 v \; + \; \frac{1}{2} \max_{\underline{\sigma} \leq \sigma \leq \overline{\sigma}} (\sigma^2 - \sigma_0^2) s^2 D_{ss}^2 v \; + \; sD_a v \; - \; rv \; = \; 0. \, (4.4.3)$$

We use a call spread type payoff $g(S, A) = (A - K_1)^+ - (A - K_2)^+$. The numerical results of the splitting scheme are from 50 independent computation. As comparison, we choose $\Delta S = 1$ and $\Delta A = 0.25$ for Crank-Nicolson finite difference scheme of equation (4.4.2). The results of our splitting scheme and Crank-Nicolson scheme for different Δt is given in Figure 4.2. The difference between results of the two schemes are relatively less than 0.3%.

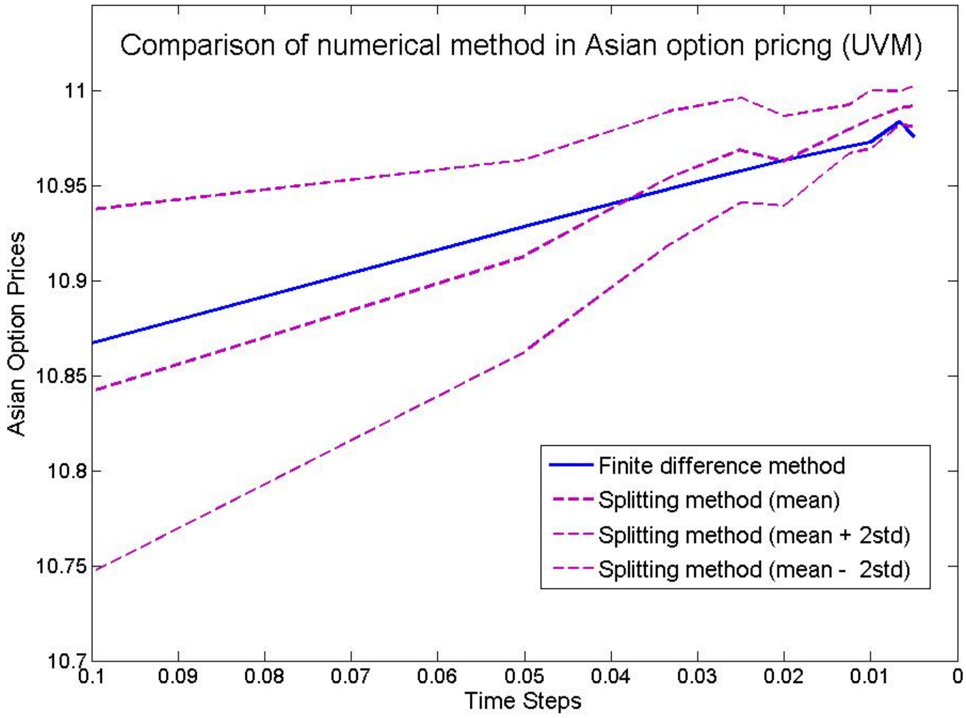

Figure 4.2: The comparison of some numerical methods for pricing Asian option with payoff $(A - K_1)^+ - (A - K_2)^+$ in UVM, with parameters $S_0 = 100$, $K_1 = 90$, $K_2 = 110$, $T = 1$, $r = 0.05$, $\underline{\sigma} = 0.18$, $\overline{\sigma} = 0.22$ and $\sigma_0 = 0.2$. When $\Delta t = 0.005$, a single computation takes 3.74 seconds for finite difference method, and 131.1 seconds for our splitting method using 5×10^5 simulations.

4.4.1.3 Asian option pricing in UVM with Hull-White interest rate: a three-dimensional case

We can also consider the uncertain volatility model with a stochastic interest rate, e.g. Hull-White interest rate (HW-IR). In HW-IR model, the interest rate has dynamic

$$dr_t \;=\; b(\theta_t - r_t)dt \;+\; \sigma^r dB_t,$$

where θ_t is determined by the current interst rate curve, b is the drawback force coefficient and $B = (B_t)_{t \geq 0}$ is another Brownian motion with correlation ρ to Brownian motion W which generates the dynamics of risky asset S. Then the value function

$$v(t, s, a, r) \;:=\; \mathbb{E}\left[e^{-\int_t^T r_s ds} g(S_T, A_T) \;\middle|\; S_t = s,\ A_t = a,\ r_t = r \right]$$

solves the pricing equation

$$\Big(\partial_t v \;+\; rsD_s v \;+\; b(\theta_t - r)D_r v \;+\; \frac{1}{2}(\sigma^r)^2 D^2_{rr} v \;-\; rv \;+\; \max_{\underline{\sigma}\le\sigma\le\overline{\sigma}} \big(\rho\sigma\sigma^r s D^2_{rs} v \;+\; \frac{1}{2}\sigma^2 s^2 D^2_{ss} v\big) \;+\; sD_a v \Big)(t,s,a,r) \;=\; 0,$$

with terminal condition $v(T,s,a,r) = g(s,a)$.

Let $S_0 = 100$, $K_1 = 90$, $K_2 = 110$, $T = 1$, $\underline{\sigma} = 0.15$, $\overline{\sigma} = 0.25$, $r_0 = 0.02$, $b = 0.01$, $\sigma^r = 0.01$, $\rho = 0.2$ and interest rate curve is $f_t = 0.02,\quad \forall t > 0$. As in (4.4.3), we rewrite the pricing equation in form of (4.2.2) with constant σ_0. For $g(S_T, A_T) = (A_T - K_1)^+ - (A_T - K_2)^+$, we implement our splitting method with different constants σ_0, and take the mean value of 50 independent computations. By our numerical resolution, the solution seems to be close to 11.51, see figure 4.3.

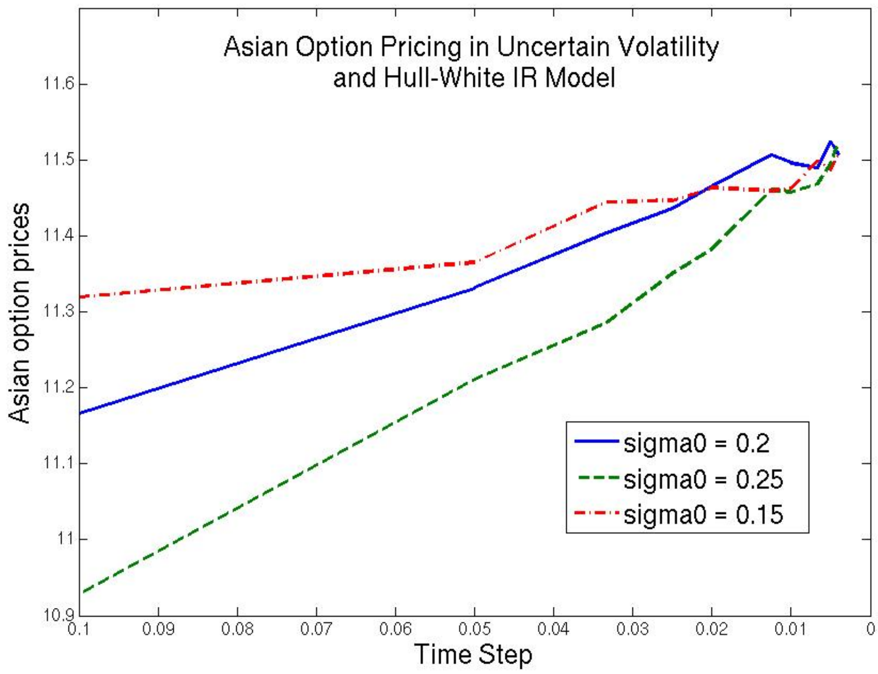

Figure 4.3: The price of Asian option with payoff $(A - K_1)^+ - (A - K_2)^+$ in UVM with HW IR and in BS model with HW IR. In case that $\Delta t = 0.005$, it takes 309.4 seconds for the splitting method using 5×10^5 simulations.

4.4.2 Optimal management of hydropower plant: A four-dimensional case

Let us consider an optimal management problem for a hydropower plant, which generalizes a little the model in Chapter 2 of the thesis of Arnaud Porchet [49].

A hydropower plant manages a dam, which is filled by rain precipitations with non-negative rate A_t, which follows equation

$$dA_t \;=\; \mu_a A_t dt \;+\; \sigma_a A_t dW_t^1.$$

Denote by B_t the volume of water in the dam, then

$$dB_t \;=\; (A_t - q_t)\;dt,$$

where q_t represents the water flow sent at time t to generate electricity. It makes a profit $\int_0^T q_t S_t dt$ in period $[0,T]$, where S_t represents the market electricity price, which follows dynamics

$$dS_t \;=\; \mu_s S_t dt \;+\; \sigma_s S_t dW_t^2.$$

At the same time, the power station invests in electricity market with money θ_t, then the total revenue of the power station X_t follows equation

$$dX_t \;=\; \frac{\theta_t}{S_t} dS_t \;+\; q_t S_t dt \;=\; \theta_t \mu_s dt \;+\; \theta_t \sigma_s dW_t^2 \;+\; q_t S_t dt.$$

The power station optimizes its expected utility $\mathbb{E}U(X_T)$ on the strategy $(q_t)_{0\le t\le T}$ and $(\theta_t)_{0\le t\le T}$. Formally, we get a Bellman equation

$$\begin{aligned} u_t \;&+\; \mu_s s D_s u \;+\; \frac{1}{2}\sigma_s^2 s^2 D_{ss}^2 u \;+\; \mu_a a D_a u \;+\; \frac{1}{2}\sigma_a^2 a^2 D_{aa}^2 u \;+\; \rho\sigma_s\sigma_a s a D_{sa}^2 u \\ &+\; \max_\theta \Big[\; \theta_s \mu D_x u \;+\; \frac{1}{2}\theta^2\sigma_s^2 D_{xx}^2 u \;+\; \theta\rho a\sigma_a\sigma_s D_{ax}^2 u \;+\; \theta\sigma_s^2 s D_{sx}^2 u \;\Big] \\ &+\; \max_q \big[\; (a-q)D_b u \;+\; q s D_x u \;\big] \;=\; 0. \end{aligned}$$

As in the examples in Section 5.2 of [32], we truncate the optimization on θ and rewrite the equation in form of (4.2.2).

$$\begin{aligned} u_t \;&+\; \mu_s s D_s u \;+\; \frac{1}{2}\sigma_s^2 s^2 D_{ss}^2 u \;+\; \mu_a a D_a u \;+\; \frac{1}{2}\sigma_a^2 a^2 D_{aa}^2 u \;+\; \rho\sigma_s\sigma_a s a D_{sa}^2 u \;+\; \frac{1}{2}\sigma_x^2 D_{xx}^2 u \\ &+\; \max_{-n\le\theta\le n} \Big[\; \theta_s \mu D_x u \;+\; \frac{1}{2}\theta^2\sigma_s^2 D_{xx}^2 u \;+\; \theta\rho a\sigma_a\sigma_s D_{ax}^2 u \;+\; \theta\sigma_s^2 s D_{sx}^2 u \;-\; \frac{1}{2}\sigma_x^2 D_{xx}^2 u \;\Big] \\ &+\; \max_q \big[\; (a-q)D_b u \;+\; q s D_x u \;\big] \;=\; 0. \end{aligned}$$

Let $\mu_a = 0$, $\sigma_a = 0.2$, $\mu_s = 0$, $\sigma_s = 0.2$, $\rho = 0$, $n = 5$ and the utility function is given by $U(x) := -e^{-\rho x}$ with $\rho = 0.2$. Using the different choices of σ_x, we report the numerical result in Figure 4.4. It seems that the solution converges to the value -0.66.

4.5 Appendix

We give here the proof for Lemma 4.3.1. Let $(\tilde\lambda_k^i)_{1\le k\le K}$ be the projection coefficients of $R_i(\varphi)$ on basis $(e_k(\hat X_{t_n}, Y))_{1\le k\le K}$ as defined in (4.3.3), and $\hat\lambda_k^{i,M}$ be simulated regression estimations of $\tilde\lambda_k^i$ with M simulations of X, Y as defined in (4.3.6). Then for $(x,y)\in B_k$,

$$\tilde{\mathbb{E}}\;\big[\; R_i(\varphi) \mid \hat X_{t_n} = x, Y = y \;\big] \;=\; \tilde\lambda_k^i$$

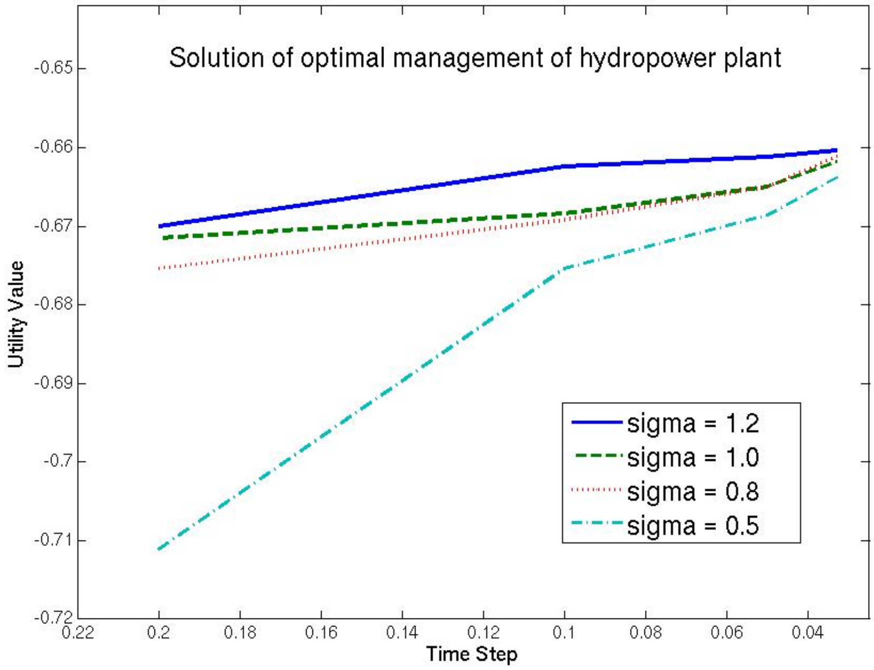

Figure 4.4: Solution of optimal management for a hydropower plant, with $\sigma_x = 1$ and $\sigma_x = 1.5$. Using 2×10^6 simulations, the splitting scheme takes 639.2 seconds for a single calculation when $\Delta t = 0.0333$.

and

$$\hat{\mathbb{E}}^M \left[R_i(\varphi) \mid \hat{X}_{t_n} = x, Y = y \right] \;=\; \underline{\Gamma}_i(x,y) \;\vee\; \hat{\lambda}_k^{i,M} \;\wedge\; \overline{\Gamma}_i(x,y).$$

Moreover,

$$\tilde{\lambda}_k^i \;=\; \frac{\mathbb{E}[R_i(\varphi)e_k(\hat{X}_{t_n}, Y)]}{\mathbb{E}[e_k^2(\hat{X}_{t_n}, Y)]} \quad \text{and} \quad \hat{\lambda}_k^{i,M} \;=\; \frac{\mathbb{E}^M[R_i(\varphi)e_k(\hat{X}_{t_n}, Y)]}{\mathbb{E}^M[e_k^2(\hat{X}_{t_n}, Y)]},$$

where $\mathbb{E}^M$ is the empirical expectation defined as follows: given M simulations $(U^m)_{1 \le m \le M}$ of random variable U, $\mathbb{E}^M[U] := \frac{1}{M}\sum_{m=1}^M U^m$.

Proof of Lemma 4.3.1. We omit the notations $i, k, x, y, \hat{X}_{t_n}, Y$ then simplify the notation as $\tilde{\lambda} = \tilde{\mathbb{E}}[R] = \frac{\mathbb{E}[Re]}{\mathbb{E}[ee]}$ and $\hat{\mathbb{E}}^M[R] = -\Gamma \vee \frac{\mathbb{E}^M[Re]}{\mathbb{E}^M[ee]} \wedge \Gamma$. Denote $\varepsilon^M(Re) := \hat{\mathbb{E}}^M[Re] - \mathbb{E}[Re]$ and $\varepsilon^M(ee) := \hat{\mathbb{E}}^M[ee] - \mathbb{E}[ee]$, then

$$\left| \hat{\mathbb{E}}^M[R] - \tilde{\mathbb{E}}[R] \right|^2 \;\le\; \left| \frac{\hat{\mathbb{E}}^M[Re]}{\hat{\mathbb{E}}^M[ee]} - \frac{\mathbb{E}[Re]}{\mathbb{E}[ee]} \right|^2 \wedge 4\Gamma^2 \;=\; \left| \frac{\varepsilon^M(Re)}{\hat{\mathbb{E}}^M[ee]} + \tilde{\lambda}\frac{\varepsilon^M(ee)}{\hat{\mathbb{E}}^M[ee]} \right|^2 \wedge 4\Gamma^2,$$

and it follows that

$$
\begin{aligned}
\mathbb{E}\big[\hat{\mathbb{E}}^M[R] - \tilde{\mathbb{E}}[R]\big]^2 \quad &\leq \quad \mathbb{E}\left|\frac{\varepsilon^M(Re)}{\hat{\mathbb{E}}^M[ee]} + \tilde{\lambda}\frac{\varepsilon^M(ee)}{\hat{\mathbb{E}}^M[ee]}\right|^2 \;\wedge\; 4\Gamma^2 \\
\leq \;\; & 8\frac{\mathbb{E}[(\varepsilon^M(Re))^2]}{(\mathbb{E}[ee])^2} \;+\; 8\tilde{\lambda}^2\frac{\mathbb{E}[(\varepsilon^M(ee))^2]}{(\mathbb{E}[ee])^2} \;+\; 4\Gamma^2\mathbb{P}\left(\left|\frac{\hat{\mathbb{E}}^M[ee] \;-\; \mathbb{E}[ee]}{\mathbb{E}[ee]}\right| > \frac{1}{2}\right) \\
\leq \;\; & 8\frac{\mathbb{E}[(\varepsilon^M(Re))^2]}{(\mathbb{E}[ee])^2} \;+\; 8\tilde{\lambda}^2\frac{\mathbb{E}[(\varepsilon^M(ee))^2]}{(\mathbb{E}[ee])^2} \;+\; 16\Gamma^2\frac{\mathbb{E}[(\varepsilon^M(ee))^2]}{(\mathbb{E}[ee])^2} \\
= \;\; & \frac{1}{M}\,\frac{8}{(\mathbb{E}[ee])^2}\,\Big[\mathrm{Var}(Re) \;+\; \tilde{\lambda}^2\mathrm{Var}(ee) \;+\; 2\Gamma^2\mathrm{Var}(ee)\Big]. \qquad (4.5.1)
\end{aligned}
$$

When $e = 1_{B_k}$, we have $\mathbb{E}[e^2(\hat{X}_{t_n}, Y)] = \mathbb{E}[e(\hat{X}_{t_n}, Y)] = \mathbb{P}((\hat{X}_{t_n}, Y) \in B_k)$, $\mathbb{E}[eR_i] \leq C\;|\varphi|_0\; h^{i/2}\;\mathbb{E}[e]$ and $\tilde{\lambda} \leq C\;|\varphi|_0\; h^{i/2}$, and then it follows by (4.5.1) that (4.3.13) holds true. □

Partie II

A new mass transportation problem

Chapitre 5

Optimal Transportation under Controlled Stochastic Dynamics

5.1 Introduction

The following stochastic mass transportation mechanism was introduced by Mikami and Thieullen [46] as an extension of the Monge-Kantorovitch optimal transportation problem. Let X be an $\mathbb{R}^d$-continuous semimartingale with decomposition

$$X_t = X_0 + \int_0^t \beta_s ds + W_t, \tag{5.1.1}$$

where W_t is a d-dimensional standard Brownian motion under the filtration $\mathbb{F}^X$ generated by X. The optimal mass transportation problem consists in minimizing the cost of transportation defined by some cost functional ℓ along all transportation plans with initial distribution μ_0 and final distribution μ_1:

$$V(\mu_0, \mu_1) := \inf \mathbb{E} \int_0^1 \ell(s, X_s, \beta_s) ds,$$

where the infimum is taken over all semimartingales given by (5.1.1) satisfying $\mathbb{P} \circ X_0^{-1} = \mu_0$ and $\mathbb{P} \circ X_1^{-1} = \mu_1$. Mikami and Thieullen [46] proved a strong duality result thus extending the classical Kantorovitch duality to this context.

Motivated by a problem in financial mathematics, our main objective is to extend [46] to a larger class of transportation plans defined by continuous semimartingales with absolutely continuous characteristics:

$$X_t = X_0 + \int_0^t \beta_s ds + \int_0^t \sigma_s dW_s,$$

with transportation cost depending on the drift and diffusion coefficients as well as the trajectory of X.

This problem is also intimately connected to the so-called Skorokhod Embedding Problem (SEP), see Obloj [48] for a review. Given a one-dimensional Brownian motion W and a centered $|x|$-integrable probability distribution μ_1 on $\mathbb{R}$, the SEP consists in searching for a stopping time τ such that $W_\tau \sim \mu_1$ and $(W_{t\wedge\tau})_{t\geq 0}$ is uniformly integrable. This problem is well-known to have infinitely many solutions. However, some solutions have been proved to satisfy some optimality with respect to some criterion (Azéma and Yor [2], Root [51] and Rost [52]). This problem can be formulated in our context by restricting the finite variation part to zero, i.e. transportation along a martingale. Indeed, given a solution τ of the SEP, the process $X_t := W_{\tau\wedge\frac{t}{1-t}}$ defines a continuous local martingale satisfying $X_1 \sim \mu_1$. Conversely every continuous local martingale can be represented as time-changed Brownian motion by the Dubins-Schwartz theorem (see e.g. Theorem 4.6, Chapiter 3 of Karatzas and Shreve [40]).

Our extension of Mikami and Thieullen is motivated by Hobson's [36] observation of the connection between the SEP and the problem of finding optimal no-arbitrage bounds for the prices of exotic options (e.g. variance options, lookback option etc.) given the observation of the implied volatility curve for some maturity T, i.e. $T-$maturity European options of all strikes. We refer to Hobson [37] for an overview on some specific applications of the SEP in the context of finance. As observed by Galichon, Henry-Labordère and Touzi [33], our formulation in terms of an optimal transportation problem allows for a systematic treatment of this problem.

Our first main result is to establish the Kantorovitch strong duality for our semimartingale optimal transportation problem. The dual value function consists in the minimization of $\mu_0(\lambda_0) - \mu_1(\lambda_1)$ over all continuous and bounded functions λ_1, where λ_0 is the initial value of a standard stochastic control problem with final cost λ_1. In the Markovian case, the function λ_0 can be characterized as the unique viscosity solution of the corresponding dynamics programming equation with terminal condition λ_1.

Our second main contribution is to exploit the dual formulation for the purpose of numerical approximation of the optimal cost of transportation. In the context of a bounded set of admissible characteristics of the semimartingale, we suggest a numerical scheme which combines finite differences and the gradient projection algorithm. We prove convergence of the scheme, and we derive a rate of convergence.

The chapter is organized as follows. Section 5.2 introduces the optimal mass transportation problem under controlled stochastic dynamics. In Section 5.3, we extend the Kantorovitch duality to our context by using the classical convex duality approach. The convex conjugate of the primal problem turns out to be the value function of a classical stochastic control problem with final condition given by the Lagrange multiplier lying in the space of bounded continuous functions. Then the dual formulation consists in maximizing this value over the class of all Lagrange multipliers. We also show, under some conditions, that the Lagrange multipliers can be restricted to the subclass of $C^\infty-$ functions with bounded derivative of any order. In the Markovian case, we characterize convex dual as the viscosity solution of a dynamic programming equation in the Markovian case in Section 5.4.

Section 5.5 introduces a numerical scheme to approximate the dual formulation in the Markovian case. We first use the probabilistic arguments to restrict the optimal control problem to a bounded domain of $\mathbb{R}^d$, then use the finite difference scheme to solve the control problem. The maximization is approximated by means of the gradient projection algorithm. We provide some general convergence results together with some control of the error. Finally, we provide an implementation of our algorithm in the context of an application in financial mathematics. Namely, we consider the problem of robust hedging *variance swap* derivatives given the prices of options of all strikes. The solution of the last problem is known explicitly and allows to test the accuracy of our algorithm.

Notation: Let E be a Polish space, we denote by $\mathbf{M}(E)$ the space of all Borel probability measures on E, equipped with the weak topology, which is also a Polish space. In particular, $\mathbf{M}(\mathbb{R}^d)$ is the space of all probability measures on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$. S_d denotes the set of $d \times d$ positive symmetric matrices. Given $u = (a, b) \in S_d \times \mathbb{R}^d$, we define $|u|$ by its L^2-norm as an element in $\mathbb{R}^{d^2+d}$.

5.2 The semimartingale transportation problem

Let $\Omega := C([0,1], \mathbb{R}^d)$ be the canonical space, X be the canonical process, i.e.

$$X_t(\omega) := \omega_t \quad \text{for all} \quad t \in [0,1],$$

and $\mathbb{F} = (\mathcal{F}_t)_{1 \leq t \leq 1}$ be the canonical filtration generated by X. We recall that $\mathcal{F}_t$ coincides with the Borel $\sigma-$field on Ω induced by the seminorm $|\omega|_{\infty,t} := \sup_{0 \leq s \leq t} |\omega_s|$, $\omega \in \Omega$. See e.g. the discussions in Section 1.3, Chapter 1 of Stroock and Varadhan [55].

Let $\mathbb{P}$ be a probability measure on $(\Omega, \mathcal{F}_1)$ under which the canonical process X is a $\mathbb{F}-$continuous semimartingale. Then, we have the unique continuous decomposition:

$$X_t = X_0 + B_t^{\mathbb{P}} + M_t^{\mathbb{P}}, \quad t \in [0,1], \quad \mathbb{P} - \text{a.s.} \tag{5.2.1}$$

where $B^{\mathbb{P}} = (B_t^{\mathbb{P}})_{0 \leq t \leq 1}$ is the finite variation part and $M^{\mathbb{P}} = (M_t^{\mathbb{P}})_{0 \leq t \leq 1}$ is the local martingale part satisfying $B_0 = M_0 = 0$. Denote by $A_t^{\mathbb{P}} := \langle M^{\mathbb{P}} \rangle_t$ the quadratic variation of $M^{\mathbb{P}}$ between 0 and t and $A^{\mathbb{P}} = (A_t^{\mathbb{P}})_{0 \leq t \leq 1}$. Then, following Jacod and Shiryaev [38], we say that the $\mathbb{P}-$continuous semimartingale X has characteristics $(A^{\mathbb{P}}, B^{\mathbb{P}})$.

In this chapter, we further restrict to the case where the processes $A^{\mathbb{P}}$ and $B^{\mathbb{P}}$ are absolutely continuous in t w.r.t. Lebesgue measure, $\mathbb{P}-$a.s. Then there are $\mathbb{F}-$progressive processes $\nu^{\mathbb{P}} = (\alpha^{\mathbb{P}}, \beta^{\mathbb{P}})$ (see e.g. Proposition I.3.13 of [38]) such that

$$A_t^{\mathbb{P}} = \int_0^t \alpha_s^{\mathbb{P}} ds, \quad B_t^{\mathbb{P}} = \int_0^t \beta_s^{\mathbb{P}} ds, \quad t \in [0,1] \quad \text{up to a } \mathbb{P} - \text{evanescent set.} \tag{5.2.2}$$

Remark 5.2.1. *By Doob's martingale representation theorem (see e.g. Theorem 4.2 in Chapter 3 of Karatzas and Shreve [40]), we can find a Brownian motion $W^{\mathbb{P}}$ (possibly in an enlarged space) such that X has an Itô representation:*

$$X_t = X_0 + \int_0^t \beta_s^{\mathbb{P}} ds + \int_0^t \sigma_s^{\mathbb{P}} dW_s^{\mathbb{P}},$$

where $\sigma_t^{\mathbb{P}} = (\alpha_t^{\mathbb{P}})^{1/2}$ (i.e. $\alpha_t^{\mathbb{P}} = \sigma_t^{\mathbb{P}} (\sigma_t^{\mathbb{P}})^T$).

Remark 5.2.2. *With the unique processes $(A^{\mathbb{P}}, B^{\mathbb{P}})$, the progressively measurable processes $\nu^{\mathbb{P}} = (\alpha^{\mathbb{P}}, \beta^{\mathbb{P}})$ may not be unique. However, they are unique in sense $d\mathbb{P} \times dt-$a.e.. Since the transportation cost defined below is a $d\mathbb{P} \times dt$ integral, then the choice of $\nu^{\mathbb{P}} = (\alpha^{\mathbb{P}}, \beta^{\mathbb{P}})$ will not change the cost value and then is not essential.*

We next introduce the set U defining some restrictions on the admissible characteristics:

$$U \quad \text{closed and convex subset of} \quad S_d \times \mathbb{R}^d, \tag{5.2.3}$$

and we denote by $\mathcal{P}$ the set of probability measures $\mathbb{P}$ on Ω under which X has the decomposition (5.2.1), and satisfies (5.2.2) with characteristics $\nu_t^{\mathbb{P}} := (\alpha_t^{\mathbb{P}}, \beta_t^{\mathbb{P}}) \in U$, $d\mathbb{P} \times dt-$a.e.

Given two arbitrary probability measures μ_0 and μ_1 in $\mathbf{M}(\mathbb{R}^d)$, we also denote

$$\mathcal{P}(\mu_0) \;:=\; \big\{\mathbb{P} \in \mathcal{P} \;:\; \mathbb{P} \circ X_0^{-1} = \mu_0\big\}, \tag{5.2.4}$$
$$\mathcal{P}(\mu_0, \mu_1) \;:=\; \big\{\mathbb{P} \in \mathcal{P}(\mu_0) \;:\; \mathbb{P} \circ X_1^{-1} = \mu_1\big\}. \tag{5.2.5}$$

Remark 5.2.3. *In general, $\mathcal{P}(\mu_0, \mu_1)$ may be empty. However, in the one-dimensional case $d = 1$ and $U = \mathbb{R}^+ \times \mathbb{R}$, the initial distribution $\mu_0 = \delta_{x_0}$ for some constant $x_0 \in \mathbb{R}$, and the final distribution satisfies $\int_{\mathbb{R}} |x|\mu_1(dx) < \infty$, we now verify that $\mathcal{P}(\mu_0, \mu_1)$ is not empty. First, we can choose any constant in $\mathbb{R}$ for the drift part, hence we can suppose, without loss of generality, that $x_0 = 0$ and μ_1 is centered distributed, i.e. $\int_{\mathbb{R}} x\mu_1(dx) = 0$. Then, given a Brownian motion W, by Skorokhod embedding (see e.g. Section 3 of Obloj [48]), there is a stopping time τ such that $W_\tau \sim \mu_1$ and $(W_{t\wedge\tau})_{t\geq 0}$ is uniformly integrable. Therefore, $M = (M_t)_{0\leq t\leq 1}$ defined by $M_t := W_{\tau\wedge\frac{t}{1-t}}$ is a continuous martingale with marginal distribution $\mathbb{P} \circ M_1^{-1} = \mu_1$. Moreover, its quadratic variation $\langle M\rangle_t = \tau \wedge \frac{t}{1-t}$ is absolutely continuous in t w.r.t Lebesgue for every fixed ω, which can induce a probability on Ω belonging to $\mathcal{P}(\mu_0, \mu_1)$.*

The semimartingale X under $\mathbb{P}$ can be viewed as a vehicle of mass transportation, from the $\mathbb{P}-$distribution of X_0 to the $\mathbb{P}-$distribution of X_1. We then associate $\mathbb{P}$ with a transportation cost

$$J(\mathbb{P}) := \mathbb{E}^{\mathbb{P}} \int_0^1 L(t, X, \nu_t^{\mathbb{P}})dt, \tag{5.2.6}$$

where we denoted by $\mathbb{E}^{\mathbb{P}}$ the expectation under the probability measure $\mathbb{P}$. The above expectation is well defined on $\mathbb{R}^+ \cup \{+\infty\}$ in view of the subsequent Assumption 5.3.1 which states in particular that L is nonnegative.

Our main interest is on the following optimal mass transportation problem, given two probability measures μ_0, $\mu_1 \in \mathbf{M}(\mathbb{R}^d)$:

$$V(\mu_0, \mu_1) := \inf_{\mathbb{P}\in\mathcal{P}(\mu_0,\mu_1)} J(\mathbb{P}), \tag{5.2.7}$$

with the convention $\inf \emptyset = \infty$.

5.3 The duality theorem

The main objective of this section is to prove a duality result for problem (5.2.7) which extends the classical Kantorovitch duality in optimal transportation theory.

This will be achieved by classical convex duality techniques which require to verify that the function V is convex and lower semicontinuous. For general theory on duality analysis in Banach spaces, we refer to Bonnans and Shapiro [12] and Ekeland and Temam [29]. In our context, the value function of the optimal transportation problem is defined on the Polish space of measures on $\mathbb{R}^d$, and our main reference is Deuschel and Stroock [26].

5.3.1 The main duality result

We first formulate the assumptions needed for our duality result.

Assumption 5.3.1. *The function $(t, \mathbf{x}, u) \in [0,1] \times \Omega \times U \mapsto L(t, \mathbf{x}, u) \in \mathbb{R}^+$ is non-negative, continuous in $(t, \mathbf{x}, u)$, and convex in u.*

Notice that we do not impose any progressive measurability for the dependence of L on the trajectory $\mathbf{x}$. However, by immediate conditioning, we may reduce the problem so that such a progressive measurability is satisfied.

The next condition controls the dependence of the cost functional on the time variable.

Assumption 5.3.2. *The function L is uniformly continuous in t in sense that*

$$\Delta_t L(\varepsilon) := \sup \frac{|L(s, \mathbf{x}, u) - L(t, \mathbf{x}, u)|}{1 + L(t, \mathbf{x}, u)} \longrightarrow 0, \quad \text{as} \quad \varepsilon \to 0,$$

where the supremum is taken over all $0 \le s, t \le 1$ such that $|t - s| \le \varepsilon$ and all $\mathbf{x} \in \Omega$, $u \in U$.

We finally need the following coercivity condition on the cost functional.

Assumption 5.3.3. *There are constants $p > 1$ and $C_0 > 0$ such that*

$$|u|^p \le C_0(1 + L(t, \mathbf{x}, u)) < \infty \quad \text{for every} \quad (t, \mathbf{x}, u) \in [0,1] \times \Omega \times U.$$

Remark 5.3.1. *In the particular case $U = \{I_d\} \times \mathbb{R}^d$, the last condition coincides with Assumption A.1 of Mikami and Thieullen [46]. Moreover, whenever U is bounded, Assumption 5.3.3 is a direct consequence of Assumption 5.3.1.*

Let $C_b(\mathbb{R}^d)$ denote the set of all bounded continuous functions on $\mathbb{R}^d$ and

$$\mu(\phi) := \int_{\mathbb{R}^d} \phi(x)\mu(dx) \ \text{ for all } \ \mu \in \mathbf{M}(\mathbb{R}^d) \ \text{ and } \ \phi \in \mathcal{L}^1(\mu).$$

We define the dual formulation of (5.2.7) by

$$\mathcal{V}(\mu_0, \mu_1) := \sup_{\lambda_1 \in C_b(\mathbb{R}^d)} \{\mu_0(\lambda_0) - \mu_1(\lambda_1)\}, \tag{5.3.1}$$

where

$$\lambda_0(x) := \inf_{\mathbb{P}\in\mathcal{P}(\delta_x)} \mathbb{E}^{\mathbb{P}}\Big[\int_0^1 L(s,X,\nu_s^{\mathbb{P}})ds + \lambda_1(X_1)\Big], \tag{5.3.2}$$

with $\mathcal{P}(\delta_x)$ defined in (5.2.4). We notice that $\mu_0(\lambda_0)$ is well defined since λ_0 is bounded from below and measurable by the following Lemma.

Lemma 5.3.1. *Let Assumptions 5.3.1 and 5.3.2 hold true, and assume that λ_0 is locally bounded. Then, λ_0 is measurable w.r.t. the Borel $\sigma-$field on $\mathbb{R}^d$ completed by μ_0, and*

$$\mu_0(\lambda_0) \quad = \inf_{\mathbb{P}\in\mathcal{P}(\mu_0)} \mathbb{E}^{\mathbb{P}}\Big[\int_0^1 L(s,X,\nu_s^{\mathbb{P}})ds + \lambda_1(X_1)\Big].$$

The proof of Lemma 5.3.1 is based on a measurable selection argument, and is reported at the end of Section 5.4.2.

Remark 5.3.2. *The local boundedness of λ_0 can be easily guaranteed under reasonable conditions. For example, suppose that $\int_0^1 L(t,\mathbf{0},a_0,b_0)dt < \infty$ and $|L(t,\mathbf{x},a_0,b_0) - L(t,\mathbf{0},a_0,b_0)| \le e^{C|\mathbf{x}|_\infty}$ with constants $C<\infty$ and $(a_0,b_0)\in U$, for every $0\le t\le 1$ and $\mathbf{x}\in\Omega$. Let the process Z^x be defined by $Z^x_t = x + b_0 t + \sqrt{a_0}W_t$, where W is a $d-$dimensional standard Brownian motion. Then clearly $\mathbb{E}\int_0^1 L(s,Z^x,a_0,b_0)ds \le Ke^{Cx}$ for some constant K independent of x. By the boundedness of λ_1 and positivity of L, considering the probability measure induced by Z^x on Ω, it follows the local boundedness of λ_0.*

We now state the main duality result.

Theorem 5.3.1. *Let Assumptions 5.3.1, 5.3.2 and 5.3.3 hold, and suppose that λ_0 is locally bounded for all $\lambda_1 \in C_b(\mathbb{R}^d)$. Then:*

$$V(\mu_0,\mu_1) = \mathcal{V}(\mu_0,\mu_1) \quad \textit{for all} \quad \mu_0,\ \mu_1 \in \mathbf{M}(\mathbb{R}^d),$$

and the infirmum is achieved by some $\mathbb{P}\in\mathcal{P}(\mu_0,\mu_1)$ for the problem $V(\mu_0,\mu_1)$ of (5.2.7).

The proof of this result is reported in the subsequent subsections.

We finally state a duality result in the space $C_b^\infty(\mathbb{R}^d)$ of all functions with bounded derivatives of any order:

$$\overline{\mathcal{V}}(\mu_0,\mu_1) \quad := \sup_{\lambda_1\in C_b^\infty(\mathbb{R}^d)} \big\{\mu_0(\lambda_0) - \mu_1(\lambda_1)\big\}. \tag{5.3.3}$$

Assumption 5.3.4. *The function L is uniformly continuous in $\mathbf{x}$ in sense that*

$$\Delta_x L(\varepsilon) := \sup \frac{|L(t,\mathbf{x}^1,u) - L(t,\mathbf{x}^2,u)|}{1+L(t,\mathbf{x}^2,u)} \longrightarrow 0, \quad \textit{as} \ \ \varepsilon\to 0,$$

where the supremum is taken over all $0\le t\le 1$, $u\in U$ and all $\mathbf{x}^1,\mathbf{x}^2\in\Omega$ such that $|\mathbf{x}^1-\mathbf{x}^2|_\infty \le \varepsilon$.

Theorem 5.3.2. *Under the conditions of Theorem 5.3.1 together with Assumption 5.3.4, we have* $\mathcal{V} = \overline{\mathcal{V}}$ *on* $\mathbf{M}(\mathbb{R}^d) \times \mathbf{M}(\mathbb{R}^d)$.

The proof of the last result follows exactly the same arguments as those of Mikami and Thieullen [46] in the proof of their Theorem 2.1. We report it in Section 5.3.6 for completeness.

5.3.2 An enlarged space

In preparation of the proof of Theorem 5.3.1, we introduce the enlarged canonical space

$$\overline{\Omega} := C\big([0,1], \mathbb{R}^d \times \mathbb{R}^{d^2} \times \mathbb{R}^d\big) \tag{5.3.4}$$

following the technique used by Haussmann [35] in the proof of his Proposition 3.1.

On $\overline{\Omega}$, we denote the canonical filtration by $\overline{\mathbb{F}} = (\overline{\mathcal{F}}_t)_{0\le t\le 1}$, and the canonical process by (X, A, B), where X, B are $d-$dimensional processes and A is a d^2-dimensional process.

We consider a probability measure $\bar{P}$ on $\overline{\Omega}$ such that X is an $\overline{\mathbb{F}}-$semimartingale characterized by (A, B), and moreover, (A, B) is $\bar{P}-$a.s. absolutely continuous w.r.t. t and $\nu_t \in U$, $d\bar{P} \times dt$ a.e., where $\nu = (\alpha, \beta)$ is defined by:

$$\alpha_t := \limsup_{n\to\infty} n\left(A_t - A_{t-\frac{1}{n}}\right), \quad \text{and} \quad \beta_t := \limsup_{n\to\infty} n\left(B_t - B_{t-\frac{1}{n}}\right). \tag{5.3.5}$$

We also denote by $\overline{\mathcal{P}}$ the set of all the probability measures $\bar{P}$ on $(\overline{\Omega}, \overline{\mathcal{F}}_1)$ satisfying the above conditions, and

$$\overline{\mathcal{P}}(\mu_0) := \big\{\bar{P} \in \overline{\mathcal{P}} : \bar{P} \circ X_0^{-1} = \mu_0\big\}, \quad \overline{\mathcal{P}}(\mu_0, \mu_1) := \big\{\bar{P} \in \overline{\mathcal{P}}(\mu_0) : \bar{P} \circ X_1^{-1} = \mu_1\big\}.$$

Finally, we denote

$$\overline{J}(\bar{P}) \ := \ \mathbb{E}^{\bar{P}} \int_0^1 L(t, X, \nu_t) dt.$$

Lemma 5.3.2. *The function* $\overline{J}$ *is lower semicontinuous on* $\overline{\mathcal{P}}$.

Proof. We follow the lines of arguments for proving the inequality (3.17) of Mikami [45]. Let $(\bar{P}^n)_{n\ge 1}$ be a sequence of probability measures in $\overline{\mathcal{P}}$ which converges weakly to some $\bar{P}^0 \in \overline{\mathcal{P}}$.

First, by Assumption 5.3.2, for every $s \in [0,1)$, $\varepsilon \in (0, 1-s)$, $t \in [s, s+\varepsilon]$, $\mathbf{x} \in \Omega$ and $\mathbb{R}^{d^2+d}-$valued process η,

$$L(s, \mathbf{x}, \eta_t) \le L(t, \mathbf{x}, \eta_t) + \Delta_t L(\varepsilon)(1 + L(t, \mathbf{x}, \eta_t)) = \Delta_t L(\varepsilon) + (1 + \Delta_t L(\varepsilon)) L(t, \mathbf{x}, \eta_t).$$

It follows from the convexity of L in u that

$$L\Big(s, \mathbf{x}, \frac{1}{\varepsilon}\int_s^{s+\varepsilon} \eta_t dt\Big) \le \frac{1}{\varepsilon}\int_s^{s+\varepsilon} L\big(s, \mathbf{x}, \eta_t\big) dt \le \Delta_t L(\varepsilon) + \frac{1 + \Delta_t L(\varepsilon)}{\varepsilon}\int_s^{s+\varepsilon} L(t, \mathbf{x}, \eta_t) dt.$$

Integrating both side on s from 0 to $1-\varepsilon$, we get

$$\begin{aligned}\int_0^{1-\varepsilon} L\Big(s,\mathbf{x},\frac{1}{\varepsilon}\int_s^{s+\varepsilon}\eta_t dt\Big)ds &\le (1-\varepsilon)\Delta_t L(\varepsilon)+\frac{1+\Delta_t L(\varepsilon)}{\varepsilon}\int_0^{1-\varepsilon}\int_s^{s+\varepsilon} L(t,\mathbf{x},\eta_t)dtds\\ &\le (1-\varepsilon)\Delta_t L(\varepsilon)+(1+\Delta_t L(\varepsilon))\int_0^1 L(s,\mathbf{x},\eta_s)ds\end{aligned}$$

by integration by parts formula. Therefore,

$$\int_0^1 L(s,\mathbf{x},\eta_s)ds \ge \frac{1}{1+\Delta_t L(\varepsilon)}\int_0^{1-\varepsilon} L\Big(s,\mathbf{x},\frac{1}{\varepsilon}\int_s^{s+\varepsilon}\eta_t dt\Big)ds-\Delta_t L(\varepsilon).$$

Then replacing $\mathbf{x}$ by X, η by ν defined in (5.3.5), taking expectation under $\bar{P}^n$, by the definition of ν_t as well as the absolute continuity of (A,B) in t, it follows that

$$\begin{aligned}\overline{J}(\bar{P}^n) &= \mathbb{E}^{\bar{P}^n}\int_0^1 L(s,X,\nu_s)ds\\ &\ge \frac{1}{1+\Delta_t L(\varepsilon)}\mathbb{E}^{\bar{P}^n}\Big[\int_0^{1-\varepsilon} L\Big(s,X,\frac{1}{\varepsilon}\int_s^{s+\varepsilon}\nu_t dt\Big)ds\Big]-\Delta_t L(\varepsilon)\\ &= \frac{1}{1+\Delta_t L(\varepsilon)}\mathbb{E}^{\bar{P}^n}\Big[\int_0^{1-\varepsilon} L\Big(s,X,\frac{1}{\varepsilon}(A_{s+\varepsilon}-A_s),\frac{1}{\varepsilon}(B_{s+\varepsilon}-B_s)\Big)ds\Big]-\Delta_t L(\varepsilon).\end{aligned}$$

By Skorokhod's theorem (see e.g. Theorem 3.3 of Billingsley [10]), we may consider a probability space $(\Omega',\mathbb{F}',\mathbb{P}')$ together with a sequence of processes $(X^n,A^n,B^n)_{n\ge 0}$ on it such that (X^n,A^n,B^n) under $\mathbb{P}'$ has the same distribution as (X,A,B) under $\bar{P}^n$ and $(X^n,A^n,B^n)\longrightarrow(X^0,A^0,B^0)$ for a.e. $\omega'\in\Omega'$ as $n\to\infty$ under norm $|\cdot|_\infty$. Then by Fatou's lemma, we get that

$$\liminf_{n\to\infty}\overline{J}(\bar{P}^n)\ge\frac{1}{1+\Delta_t L(\varepsilon)}\mathbb{E}^{\bar{P}^0}\left[\int_0^{1-\varepsilon} L\Big(s,X,\frac{1}{\varepsilon}\int_s^{s+\varepsilon}\nu_t dt\Big)ds\right]-\Delta_t L(\varepsilon).$$

Note that by the absolute continuity assumption of (A,B) in t under $\bar{P}^0$,

$$\frac{1}{\varepsilon}\int_s^{s+\varepsilon}\nu_t(\omega)dt\longrightarrow\nu_s(\omega),\quad\text{as}\quad\varepsilon\to 0,\quad\text{for}\quad d\bar{P}^0\times dt-\text{a.e.}\quad(\omega,s)\in\Omega\times[0,1),$$

and $\Delta_t L(\varepsilon)\to 0$ as $\varepsilon\to 0$ from Assumption 5.3.2, we then finish the proof by sending ε to zero and using Fatou's Lemma. □

Remark 5.3.3. *In the Markovian case $L(t,\mathbf{x},u)=\ell(t,\mathbf{x}(t),u)$, for some deterministic function ℓ, we observe that Assumption 5.3.2 is stronger than Assumption A2 in Mikami [45]. However, we can easily adapt this proof by introducing the trajectory set $\{\mathbf{x}:\sup_{0\le t,s\le 1,|t-s|\le\varepsilon}|\mathbf{x}(t)-\mathbf{x}(s)|\le\delta\}$ and then letting ε, $\delta\to 0$ as in the proof of inequality (3.17) in [45].*

Our next objective is to establish a one-to-one connection between the cost functional J defined on the set $\mathcal{P}(\mu_0,\mu_1)$ of probability measures on Ω and the cost functional $\overline{J}$ defined on the corresponding set $\overline{\mathcal{P}}(\mu_0,\mu_1)$ on the enlarged space $\overline{\Omega}$.

Proposition 5.3.1. (i) *For any probability measure $\mathbb{P} \in \mathcal{P}(\mu_0, \mu_1)$, there exists a probability $\bar{P} \in \overline{\mathcal{P}}(\mu_0, \mu_1)$ such that $J(\mathbb{P}) = \bar{J}(\bar{P})$.*
(ii) *Conversely, let $\bar{P} \in \overline{\mathcal{P}}(\mu_0, \mu_1)$ be such that $\mathbb{E}^{\bar{P}} \int_0^1 |\beta_s| ds < \infty$. Then, under Assumption 5.3.1, there exists a probability measure $\mathbb{P} \in \mathcal{P}(\mu_0, \mu_1)$ such that $J(\mathbb{P}) \leq \bar{J}(\bar{P})$.*

Proof. (i) Given $\mathbb{P} \in \mathcal{P}(\mu_0, \mu_1)$, define the processes $A^{\mathbb{P}}$, $B^{\mathbb{P}}$ from decomposition (5.2.1), and observe that the mapping $\omega \in \Omega \mapsto (X_t(\omega), A_t^{\mathbb{P}}(\omega), B_t^{\mathbb{P}}(\omega)) \in \mathbb{R}^{2d+d^2}$ is measurable for every $t \in [0,1]$. Then the mapping $\omega \in \Omega \mapsto (X(\omega), A^{\mathbb{P}}(\omega), B^{\mathbb{P}}(\omega)) \in \overline{\Omega}$ is also measurable, see e.g. discussions in Chapter 2 of Billingsley [9] at Page 57.

Let $\bar{P}$ be the probability measure on $(\overline{\Omega}, \overline{\mathcal{F}}_1)$ induced by $(\mathbb{P}, (X, A^{\mathbb{P}}(X), B^{\mathbb{P}}(X)))$. In the enlarged space $(\overline{\Omega}, \overline{\mathcal{F}}_1, \bar{P})$, the canonical process X is clearly a continuous semimartingale characterized by $(A^{\mathbb{P}}(X), B^{\mathbb{P}}(X))$. Moreover, $(A^{\mathbb{P}}(X), B^{\mathbb{P}}(X)) = (A, B)$, $\bar{P}-$a.s., where (X, A, B) are canonical processes in $\overline{\Omega}$. It follows that, on the enlarged space $(\overline{\Omega}, \overline{\mathbb{F}}, \bar{P})$, X is a continuous semimartingale characterized by (A, B). Also (A, B) is clearly $\bar{P}-$a.s. absolutely continuous in t, with $\nu^{\mathbb{P}}(X)_t = \nu_t$, $d\bar{P} \times dt-$a.e., where ν is defined in (5.3.5). Then $\bar{P}$ is the required probability in $\overline{\mathcal{P}}(\mu_0, \mu_1)$ and satisfies $\bar{J}(\bar{P}) = J(\mathbb{P})$.
(ii) Let us first consider the enlarged space $\overline{\Omega}$, denote by $\overline{\mathbb{F}}^X = (\overline{\mathcal{F}}_t^X)_{0 \leq t \leq 1}$ the filtration generated by process X. Then for every $\bar{P} \in \overline{\mathcal{P}}(\mu_0, \mu_1)$, $(\overline{\Omega}, \overline{\mathbb{F}}^X, \bar{P}, X)$ is still a continuous semimartingale, by the stability property of semimartingales. It follows from Theorem 5.6.1 in Appendix that the decomposition of X under filtration $\overline{\mathbb{F}}^X = (\overline{\mathcal{F}}_t^X)_{0 \leq t \leq 1}$ can be written as

$$X_t = X_0 + \bar{B}(X)_t + \bar{M}(X)_t = X_0 + \int_0^t \bar{\beta}_s ds + \bar{M}(X)_t,$$

with $\bar{A}(X)_t := \langle \bar{M}(X) \rangle_t = \int_0^t \bar{\alpha}_s ds$, $\bar{\beta}_s = \mathbb{E}^{\bar{P}} \left[\beta_s \middle| \overline{\mathcal{F}}_s^X \right]$ and $\bar{\alpha}_s = \alpha_s$, $d\bar{P} \times dt-$a.e. Moreover, by the convexity property (5.2.3) of set U, it follows that $(\bar{\alpha}, \bar{\beta}) \in U$, $d\bar{P} \times dt-$a.e. Finally, since $\overline{\mathcal{F}}_t^X = \mathcal{F}_t \otimes \{\emptyset, C([0,1], \mathbb{R}^{d^2} \times \mathbb{R}^d)\}$, $\bar{P}$ then induces a probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F}_1)$ by

$$\mathbb{P}[E] := \bar{P}\big[E \times C([0,1], \mathbb{R}^{d^2} \times \mathbb{R}^d)\big], \quad \forall E \in \mathcal{F}_1.$$

Clearly, $\mathbb{P} \in \mathcal{P}(\mu_0, \mu_1)$ and $J(\mathbb{P}) \leq \bar{J}(\bar{P})$ by the convexity of L in b of Assumption 5.3.1 and Jensen's inequality. □

Remark 5.3.4. *Let $\bar{P} \in \overline{\mathcal{P}}$ be such that $\bar{J}(\bar{P}) < \infty$, then from the coercivity property of L in u in Assumption 5.3.3, it follows immediately that*

$$\mathbb{E}^{\bar{P}} \int_0^1 |\beta_s| ds < \infty.$$

5.3.3 Lower semicontinuity and existence

By the correspondence between J and $\overline{J}$ (Proposition 5.3.1) and the lower semicontinuity of $\overline{J}$ (Lemma 5.3.2), we now obtain the corresponding property for V under the crucial Assumption 5.3.3, which guarantees the tightness of any minimizing sequence of our problem $V(\mu_0, \mu_1)$.

Lemma 5.3.3. *Under Assumptions 5.3.1, 5.3.2 and 5.3.3, the map*

$$(\mu_0, \mu_1) \in \mathbf{M}(\mathbb{R}^d) \times \mathbf{M}(\mathbb{R}^d) \quad \longmapsto \quad V(\mu_0, \mu_1) \in \overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$$

is lower semicontinuous.

Proof. Let (μ_0^n) and (μ_1^n) be two sequences in $\mathbf{M}(\mathbb{R}^d)$ converging weakly to $\mu_0, \mu_1 \in \mathbf{M}(\mathbb{R}^d)$, respectively, and let us prove that

$$\liminf_{n\to\infty} V(\mu_0^n, \mu_1^n) \geq V(\mu_0, \mu_1).$$

We focus on the case $\liminf_{n\to\infty} V(\mu_0^n, \mu_1^n) < \infty$ as the result is trivial in the alternative case. Then, after possibly extracting a subsequence, we can assume that $(V(\mu_0^n, \mu_1^n))_{n\geq 1}$ is bounded, and there is a sequence $(\mathbb{P}_n)_{n\geq 1}$ such that $\mathbb{P}_n \in \mathcal{P}(\mu_0^n, \mu_1^n)$ for all $n \geq 1$ and

$$\sup_{n\geq 1} J(\mathbb{P}_n) < \infty, \quad 0 \leq J(\mathbb{P}_n) - V(\mu_0^n, \mu_1^n) \longrightarrow 0 \quad \text{as} \quad n \to \infty. \tag{5.3.6}$$

By Assumption 5.3.3 it follows that $\sup_{n\geq 1} \mathbb{E}^{\mathbb{P}_n} \int_0^1 |\nu_s^{\mathbb{P}_n}|^p ds < \infty$. Then, it follows from Theorem 3 of Zheng [58] that the sequence $(\bar{P}_n)_{n\geq 1}$, of probability measures induced by $(\mathbb{P}_n, X, A^{\mathbb{P}_n}, B^{\mathbb{P}_n})$ on $(\overline{\Omega}, \overline{\mathcal{F}}_1)$, is tight. Moreover, under any one of their limit laws $\bar{P}$, the canonical process X is a semimartingale characterized by (A, B) such that (A, B) are still absolutely continuous in t. Moreover, $\nu \in U$, $d\bar{P} \times dt-$a.e. since $\frac{1}{t-s}(A_t - A_s, B_t - B_s) \in U$, $d\bar{P}-$a.s. for every $t, s \in [0, 1]$, hence $\bar{P} \in \overline{\mathcal{P}}(\mu_0, \mu_1)$. We then deduce from (5.3.6), Proposition 5.3.1, and Lemma 5.3.2 that:

$$\liminf_{n\to\infty} V(\mu_0^n, \mu_1^n) = \liminf_{n\to\infty} J(\mathbb{P}_n) = \liminf_{n\to\infty} \overline{J}(\bar{P}_n) \geq \overline{J}(\bar{P}).$$

By Remark 5.3.4 and Proposition 5.3.1, we may find $\mathbb{P} \in \mathcal{P}(\mu_0, \mu_1)$ such that $\bar{J}(\bar{P}) \geq J(\mathbb{P})$. Hence $\liminf_{n\to\infty} V(\mu_0^n, \mu_1^n) \geq J(\mathbb{P}) \geq V(\mu_0, \mu_1)$, completing the proof. □

Proposition 5.3.2. *Let Assumptions 5.3.1, 5.3.2 and 5.3.3 hold true. Then for every μ_0, $\mu_1 \in \mathbf{M}(\mathbb{R}^d)$ such that $V(\mu_0, \mu_1) < \infty$, existence holds for the minimization problem $V(\mu_0, \mu_1)$. Moreover, the set of minimizers $\{\mathbb{P} \in \mathcal{P}(\mu_0, \mu_1) : J(\mathbb{P}) = V(\mu_0, \mu_1)\}$ is a compact set of probability measures on Ω.*

Proof. We just let $(\mu_0^n, \mu_1^n) = (\mu_0, \mu_1)$ in the proof of Lemma 5.3.3, then the required existence result is proved by following the same arguments. □

5.3.4 Convexity

Lemma 5.3.4. *Let Assumptions 5.3.1 and 5.3.3 hold, then the map $(\mu_0, \mu_1) \mapsto V(\mu_0, \mu_1)$ is convex.*

Proof. Given μ_0^1, μ_0^2, μ_1^1, $\mu_1^2 \in \mathbf{M}(\mathbb{R}^d)$ and $\mu_0 = \theta\mu_0^1 + (1-\theta)\mu_0^2$, $\mu_1 = \theta\mu_1^1 + (1-\theta)\mu_1^2$ with $\theta \in (0,1)$, we shall prove that

$$V(\mu_0,\mu_1) \le \theta V(\mu_0^1,\mu_1^1) + (1-\theta)V(\mu_0^2,\mu_1^2).$$

It is enough to show that for both $\mathbb{P}_i \in \mathcal{P}(\mu_0^i,\mu_1^i)$ such that $J(\mathbb{P}_i) < \infty$, $i = 1,2$, we can find $\mathbb{P} \in \mathcal{P}(\mu_0,\mu_1)$ satisfying

$$J(\mathbb{P}) \le \theta J(\mathbb{P}_1) + (1-\theta)J(\mathbb{P}_2). \tag{5.3.7}$$

As in Lemma 5.3.3, let us consider the enlarged space $\overline{\Omega}$, on which the probability measures $\bar{P}_i$ are induced by $(\mathbb{P}_i, X, A^{\mathbb{P}_i}, B^{\mathbb{P}_i})$, i=1,2. By Proposition 5.3.1, $(\bar{P}_i)_{i=1,2}$ are probability measures under which X is a $\overline{\mathbb{F}}-$semimartingale characterized by the same process (A,B), which is absolutely continuous in t, such that $J(\mathbb{P}_i) = \bar{J}(\bar{P}_i)$, $\quad i = 1,2$.

By Corollary III.2.8 of Jacod and Shiryaev [38], $\bar{P} := \theta\bar{P}_1 + (1-\theta)\bar{P}_2$ is also a probability measure under which X is an $\overline{\mathbb{F}}-$semimartingale characterized by (A,B). Clearly, $\nu \in U$, $d\bar{P}\times dt-$a.e. since it is true $d\bar{P}_i \times dt-$a.e. for $i = 1,2$. Thus $\bar{P} \in \mathcal{P}(\mu_0,\mu_1)$ and it satisfies that

$$\bar{J}(\bar{P}) = \theta\bar{J}(\bar{P}_1) + (1-\theta)\bar{J}(\bar{P}_2) = \theta J(\mathbb{P}_1) + (1-\theta)J(\mathbb{P}_2) < \infty.$$

Finally, by Remark 5.3.4 and Proposition 5.3.1, we can construct $\mathbb{P} \in \mathcal{P}(\mu_0,\mu_1)$ such that $J(\mathbb{P}) \le \bar{J}(\bar{P})$, and it follows that inequality (5.3.7) holds true. □

5.3.5 Proof of the duality result

If $V(\mu_0,\mu_1)$ is infinite for every $\mu_1 \in \mathbf{M}(\mathbb{R}^d)$, then $J(\mathbb{P}) = \infty$ for all $\mathbb{P} \in \mathcal{P}(\mu_0)$. It follows from (5.3.1) and Lemma 5.3.1 that

$$V(\mu_0,\mu_1) = \mathcal{V}(\mu_0,\mu_1) = \infty.$$

Now, suppose that $V(\mu_0,\cdot)$ is not always infinite. Let $\bar{\mathbf{M}}(\mathbb{R}^d)$ be the space of all finite signed measures on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, equipped with weak topology, i.e. the coarsest topology marking $\mu \mapsto \mu(\phi)$ continuous for every $\phi \in C_b(\mathbb{R}^d)$. (See also section 2.2 of Deuschel and Stroock [26].) As indicated in section 3.2 of [26], the topology inherited by $\mathbf{M}(\mathbb{R}^d)$ as a subset of $\bar{\mathbf{M}}(\mathbb{R}^d)$ is its weak topology. We then extend $V(\mu_0,\cdot)$ to $\bar{\mathbf{M}}(\mathbb{R}^d) \supset \mathbf{M}(\mathbb{R}^d)$ by setting $V(\mu_0,\mu_1) = \infty$ when $\mu_1 \in \bar{\mathbf{M}}(\mathbb{R}^d) \setminus \mathbf{M}(\mathbb{R}^d)$, thus $\mu_1 \mapsto V(\mu_0,\mu_1)$ is a convex and lower semicontinuous function defined on $\bar{\mathbf{M}}(\mathbb{R}^d)$. Then, the duality result $V = \mathcal{V}$ follows from Theorem 2.2.15 and Lemma 3.2.3 in [26], together with the fact that for $\lambda_1 \in C_b(\mathbb{R}^d)$:

$$\begin{aligned}
\sup_{\mu_1 \in \mathbf{M}(\mathbb{R}^d)} \big\{\mu_1(-\lambda_1) - V(\mu_0,\mu_1)\big\} &= -\inf_{\substack{\mu_1 \in \mathbf{M}(\mathbb{R}^d) \\ \mathbb{P} \in \mathcal{P}(\mu_0,\mu_1)}} \mathbb{E}^{\mathbb{P}}\Big[\int_0^1 L(s,X,\nu_s^{\mathbb{P}})ds + \lambda_1(X_1)\Big] \\
&= -\inf_{\mathbb{P}\in\mathcal{P}(\mu_0)} \mathbb{E}^{\mathbb{P}}\Big[\int_0^1 L(s,X,\nu_s^{\mathbb{P}})ds + \lambda_1(X_1)\Big] \\
&= -\mu_0(\lambda_0).
\end{aligned}$$

□

5.3.6 Proof of Theorem 5.3.2

Let $\psi \in C_c^\infty([-1,1]^d, \mathbb{R}^+)$ be such that $\int_{\mathbb{R}^d} \psi(x)dx = 1$, and define $\psi_\varepsilon(x) := \varepsilon^{-d}\psi(x/\varepsilon)$. We claim that

$$\overline{\mathcal{V}}(\mu_0,\mu_1) \;\geq\; \frac{\mathcal{V}(\psi_\varepsilon * \mu_0, \psi_\varepsilon * \mu_1)}{1+\Delta_x L(\varepsilon)} - \Delta_x L(\varepsilon). \tag{5.3.8}$$

Since the inequality $\mathcal{V} \geq \overline{\mathcal{V}}$ is obvious, the required result is then obtained by sending $\varepsilon \to 0$, and using Assumption 5.3.4 together with Lemma 5.3.3.

In the rest of this proof, we denote $\delta := \Delta_x L(\varepsilon)$. To prove (5.3.8), we first observe from Assumption 5.3.4 that:

$$L(s,\mathbf{x},u) \geq \frac{L(s,\mathbf{x}+z,u)}{1+\delta} - \delta, \quad \text{for all} \quad z \in \mathbb{R} \;\text{ satisfying }\; |z| \leq \varepsilon.$$

Here, $\mathbf{x}+z := (\mathbf{x}(t)+z)_{0\leq t\leq 1} \in \Omega$. For an arbitrary $\lambda_1 \in C_b(\mathbb{R}^d)$, we denote $\lambda_1^\varepsilon := (1+\delta)^{-1}\lambda_1 * \psi_\varepsilon \in C_b^\infty$, then for every $\mathbb{P} \in \mathcal{P}(\mu_0)$:

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}}\Big[\int_0^1 L(s,X,\nu_s^{\mathbb{P}})ds + \lambda_1^\varepsilon(X_1)\Big] \\
=\;& \int_{\mathbb{R}^d} \mathbb{E}^{\mathbb{P}}\Big[\int_0^1 L(s,X,\nu_s^{\mathbb{P}})ds + \frac{\lambda_1(X_1+z)}{1+\delta}\Big]\psi_\varepsilon(z)dz \\
\geq\;& -\delta + \int_{\mathbb{R}^d} \frac{\psi_\varepsilon(z)}{1+\delta}\mathbb{E}^{\mathbb{P}}\Big[\int_0^1 L(s,X+z,\nu_s^{\mathbb{P}})ds + \lambda_1(X_1+z)\Big]dz.
\end{aligned}$$

Let Z be a r.v. independent of X with distribution defined by the density function ψ_ε under $\mathbb{P}$. Then the probability $\bar{P}_\varepsilon$ on $\overline{\Omega}$ induced by $(\mathbb{P}, X+Z := (X_t+Z)_{0\leq t\leq 1}, A^{\mathbb{P}}, B^{\mathbb{P}})$ is in $\overline{\mathcal{P}}(\psi_\varepsilon * \mu_0)$, and

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}}\Big[\int_0^1 L(s,X,\nu_s^{\mathbb{P}})ds + \lambda_1^\varepsilon(X_1)\Big] \\
\geq\;& -\delta + \frac{1}{1+\delta}\mathbb{E}^{\mathbb{P}}\Big[\int_0^1 L(s,X+Z,\nu_s^{\mathbb{P}})ds + \lambda_1(X_1+Z)\Big] \\
=\;& -\delta + \frac{1}{1+\delta}\mathbb{E}^{\bar{P}_\varepsilon}\Big[\int_0^1 L(s,X,\nu_s)ds + \lambda_1(X_1)\Big] \\
\geq\;& -\delta + \frac{1}{1+\delta}\inf_{\tilde{\mathbb{P}}\in\mathcal{P}(\psi_\varepsilon*\mu_0)} \mathbb{E}^{\tilde{\mathbb{P}}}\Big[\int_0^1 L(s,X,\nu_s^{\tilde{\mathbb{P}}})ds + \lambda_1(X_1)\Big],
\end{aligned}$$

where the last inequality follows from Proposition 5.3.1.

Notice that $\mu_1(\lambda_1^\varepsilon) = (1+\delta)^{-1}(\psi_\varepsilon * \mu_1)(\lambda_1)$ by Fubini's theorem. Then, by the arbitrariness of $\lambda_1 \in C_b(\mathbb{R}^d)$ and $\mathbb{P} \in \mathcal{P}(\mu_0)$, the last inequality implies (5.3.8). □

5.4 PDE characterization of the dual formulation

In the rest of the chapter, we assume that

$$L(t,\mathbf{x},u) \;=\; \ell(t,\mathbf{x}(t),u),$$

where the deterministic function $\ell : (t,x,u) \in [0,1] \times \mathbb{R}^d \times U \mapsto \ell(t,x,u) \in \mathbb{R}^+$ is non-negative and convex in u. Then, the function λ_0 in (5.3.2) is reduced to the value function of a standard Markovian stochastic control problem:

$$\lambda_0(x) \;=\; \inf_{\mathbb{P}\in\mathcal{P}(\delta_x)} \mathbb{E}^{\mathbb{P}}\Big[\int_0^1 \ell(s, X_s, \nu_s^{\mathbb{P}})ds + \lambda_1(X_1)\Big]. \tag{5.4.1}$$

Our main objective is to characterize λ_0 by means of the corresponding dynamic programming equations. We also suppose that $0 \in U$ in purpose of simplification, which is not an essential condition.

We consider the probability measures $\mathbb{P}$ on the canonical space $(\Omega, \mathcal{F}_1)$, under which the canonical process X is a semimartingale on $[t,1]$, characterized by $\int_t^\cdot \nu_s^{\mathbb{P}} ds$ for some progressively measurable process $\nu^{\mathbb{P}}$. As discussed in Remark 5.2.2, $\nu^{\mathbb{P}}$ is unique in sense of $d\mathbb{P}\times dt-$a.e. To simplify the notation, we suppose that U contains the original point 0. Let

$$\mathcal{P}_{t,x} \;:=\; \Big\{\mathbb{P}\in\mathcal{P} : \mathbb{P}[X_s = x,\; 0\le s\le t] = 1\Big\}. \tag{5.4.2}$$

We notice that under probability $\mathbb{P}\in\mathcal{P}_{t,x}$, X is a semimartingale with $\nu^{\mathbb{P}} = 0$, $d\mathbb{P}\times dt-$a.e. on $\Omega\times[0,t]$. The dynamic value function is defined for any $\lambda_1 \in C_b(\mathbb{R}^d)$ by:

$$\lambda(t,x) \;:=\; \inf_{\mathbb{P}\in\mathcal{P}_{t,x}} \mathbb{E}^{\mathbb{P}}\Big[\int_t^1 \ell(s, X_s, \nu_s^{\mathbb{P}})ds + \lambda_1(X_1)\Big]. \tag{5.4.3}$$

As in the previous sections, we also introduce the corresponding probability measures on enlarged space $(\overline{\Omega}, \overline{\mathcal{F}}_1)$. For all $(t,x,a,b) \in [0,1]\times\mathbb{R}^d\times\mathbb{R}^{d^2}\times\mathbb{R}^d$, let

$$\overline{\mathcal{P}}_{t,x,a,b} \;:=\; \Big\{\bar{P}\in\overline{\mathcal{P}} : \bar{P}\big[(X_s, A_s, B_s) = (x,a,b), 0\le s\le t\big] = 1\Big\}. \tag{5.4.4}$$

By similar arguments as in Proposition 5.3.1, we have under Assumption 5.3.1 that

$$\lambda(t,x) = \inf_{\bar{P}\in\overline{\mathcal{P}}_{t,x,a,b}} \mathbb{E}^{\bar{P}}\left[\int_t^1 \ell(s, X_s, \nu_s)ds + \lambda_1(X_1)\right] \quad \text{for all} \quad (a,b)\in\mathbb{R}^{d^2}\times\mathbb{R}^d. \tag{5.4.5}$$

5.4.1 PDE characterization of the dynamic value function

The first step is as usual to establish the dynamic programming principle (DPP). We observe that a weak dynamic programming principle as introduced in Bouchard and Touzi [16] suffices to prove that the dynamic value function λ is a viscosity solution of the corresponding dynamic programming equation. However, our context is slightly different from that of [16], and we will prove the standard dynamic programming principle.

For bounded controls set U and bounded cost functions, the DPP is shown (implicitly) in Haussmann [35]. El Karoui, Nguyen and JeanBlanc [30] considered a relaxed optimal control problem, and provided a scheme of proof without all details. Our approach is to show that the value function (5.4.3) coincides with the corresponding relaxation in the sense of [30], and to provide all details for their scheme of proof.

Proposition 5.4.1. *Let Assumptions 5.3.1, 5.3.2, 5.3.3 hold true, and assume further that λ is locally bounded. Then, for all $\overline{\mathbb{F}}-$stopping time τ with values in $[t,T]$, and all $(a,b)\in\mathbb{R}^{d^2+d}$:*

$$\lambda(t,x) = \inf_{\bar{P}\in\overline{\mathcal{P}}_{t,x,a,b}} \mathbb{E}^{\mathbb{P}}\Big[\int_t^\tau \ell(s,X_s,\nu_s)ds + \lambda(\tau,X_\tau)\Big].$$

The proof is reported in section 5.4.2. The dynamic programming equation is the infinitesimal version of the above dynamic programming principle. Let

$$H(t,x,p,\Gamma) \quad := \quad \inf_{(a,b)\in U}\Big[b\cdot p + \frac{1}{2}a\cdot\Gamma + \ell(t,x,a,b)\Big]. \tag{5.4.6}$$

We observe that H is continuous under Assumptions 5.3.1 and 5.3.3. Indeed, under these two assumptions, for every constant $r>0$, there is a closed bounded domain $D_r\subset S_d\times\mathbb{R}^d$ such that every subgradient $\nabla_u\ell(t,x,u)$ satisfies $|\nabla_u\ell(t,x,u)|\geq r$, for all $(t,x,u)\in[0,T]\times\mathbb{R}^d\times D_r^c$. Therefore, for every (p,Γ) such that $|(p,\Gamma)|\leq r$, the infimum in (5.4.6) can be taken in the compact set $U\cap D_r$. This implies that H is a continuous function.

Theorem 5.4.1. *Let Assumptions 5.3.1, 5.3.2, 5.3.3 hold true, and assume further that λ is locally bounded. Then, λ is a viscosity solution of the dynamic programming equation*

$$-\partial_t\lambda(t,x) - H(t,x,D\lambda,D^2\lambda) \quad = \quad 0,$$

with terminal condition $\lambda(1,x)=\lambda_1(x)$.

The proof is very similar to that of Corollary 5.1 in [16], we report it in Appendix for completeness.

5.4.2 Proof of the dynamic programming principle

We first prove that the dynamic value function λ is measurable and we can choose " in a measurable way" a family of probabilities $(\mathbb{Q}_{t,x,a,b})_{(t,x,a,b)\in[0,1]\times\mathbb{R}^{2d+d^2}}$ which achieves (or achieves with ε error) the infimum in (5.4.5).

There are many versions of the measurable selection theorem in the literature, see e.g. Section 12.1 of Stroock and Varadhan [55], Chapter 7 of Bertsekas and Shreve [8], and Chapiter 3 of Dellacherie and Meyer [25]. In our context, we find it convenient to use a result from El Karoui and TAN [31].

Let λ^* be the upper semicontinuous envelope of the function λ, and

$$\tilde{\mathcal{P}}_{t,x,a,b} := \Big\{\bar{P}\in\overline{\mathcal{P}}_{t,x,a,b} \ : \ \mathbb{E}^{\bar{P}}\left[\int_t^1 \ell(s,X_s,\nu_s)ds + \lambda_1(X_1)\right] \leq \lambda^*(t,x)\Big\},$$

$$\tilde{\mathcal{P}} := \big\{(t,x,a,b,\bar{P}) \ : \ \bar{P}\in\tilde{\mathcal{P}}_{t,x,a,b}\big\}.$$

In the following statement, for the Borel $\sigma-$field $\mathcal{B}([0,1]\times\mathbb{R}^{2d+d^2})$ of $[0,1]\times\mathbb{R}^{2d+d^2}$ with an arbitrary probability measure μ on it, we denote by $\mathcal{B}^\mu([0,1]\times\mathbb{R}^{2d+d^2})$ its $\sigma-$field completed by μ.

Lemma 5.4.1. *Let Assumptions 5.3.1, 5.3.2, 5.3.3 hold true, and assume that* λ *is locally bounded. Then, for any probability measure* μ *on* $\Big([0,1]\times\mathbb{R}^{2d+d^2},\mathcal{B}([0,1]\times\mathbb{R}^{2d+d^2})\Big)$*:*
(i) *the function* $(t,x,a,b)\mapsto\lambda(t,x)$ *is* $\mathcal{B}^{\mu}([0,1]\times\mathbb{R}^{2d+d^2}))-$*measurable,*
(ii) *for any* $\varepsilon>0$*, there is a family of probability* $(\bar{Q}^{\varepsilon}_{t,x,a,b})_{(t,x,a,b)\in[0,1]\times\mathbb{R}^{2d+d^2}}$ *in* $\tilde{\mathcal{P}}$ *such that* $(t,x,a,b)\mapsto\bar{Q}^{\varepsilon}_{t,x,a,b}$ *is a measurable map from* $[0,1]\times\mathbb{R}^{2d+d^2}$ *to* $\mathbf{M}(\overline{\Omega})$ *and*

$$\mathbb{E}^{\bar{Q}^{\varepsilon}_{t,x,a,b}}\Big[\int_t^1\ell(s,X_s,\nu_s)ds+\lambda_1(X_1)\Big]\ \le\ \lambda(t,x)+\varepsilon,\quad \mu-a.s.$$

Proof. By Lemma 5.3.2, the map $\bar{P}\ \mapsto\ \mathbb{E}^{\bar{P}}\big[\int_t^1\ell(s,X_s,\nu_s)ds+\lambda_1(X_1)\big]$ is lower semicontinuous, and therefore measurable. Moreover $\tilde{\mathcal{P}}_{t,x,a,b}$ is non empty for every $(t,x,a,b)\in[0,1]\times\mathbb{R}^{2d+d^2}$. Finally, by using the same arguments as in the proof of Lemma 5.3.3, we see that $\tilde{\mathcal{P}}$ is a closed subset of $[0,1]\times\mathbb{R}^{2d+d^2}\times\mathbf{M}(\overline{\Omega})$. Then, both items of the lemma follow from Corollary 2.19 in El Karoui and TAN [31]. □

We next prove the stability properties of probability measures under conditioning and concatenations at stopping times, which will be the key-ingredients for the proof of the dynamic programming principle.

We first recall some results from Stroock and Varadhan [55] and define some notations.

- For $0\le t\le 1$, let $\overline{\mathcal{F}}_{t,1}:=\sigma((X_s,A_s,B_s):t\le s\le 1)$, and let $\bar{P}$ be a probability measure on $(\overline{\Omega},\overline{\mathcal{F}}_{t,1})$ with $\bar{P}((X_t,A_t,B_t)=\eta_t)=1$ for some $\eta\in C([0,t],\mathbb{R}^{2d+d^2})$. Then, there is a unique probability measure $\delta_\eta\otimes_t\bar{P}$ on $(\overline{\Omega},\overline{\mathcal{F}}_1)$ such that $\delta_\eta\otimes_t\bar{P}[(X_s,A_s,B_s)=\eta_s,0\le s\le t]=1$ and $\delta_\eta\otimes_t\bar{P}[A]=\bar{P}[A]$ for all $A\in\overline{\mathcal{F}}_{t,1}$. In addition, if $\bar{P}$ is also a probability measure on $(\overline{\Omega},\overline{\mathcal{F}}_1)$, under which a process M defined on $\overline{\Omega}$ is a $\overline{\mathbb{F}}-$martingale after time t, then M is still a $\overline{\mathbb{F}}-$martingale after time t in probability space $(\overline{\Omega},\overline{\mathcal{F}}_1,\eta\otimes_t\bar{P})$. In particular, for $t\in[0,1]$, a constant $c_0\in\mathbb{R}^{2d+d^2}$, and $\bar{P}$ satisfying $\bar{P}((X_t,A_t,B_t)=c_0)=1$, we denote $\delta_{c_0}\otimes_t\bar{P}:=\delta_{\eta^{c_0}}\otimes_t\bar{P}$, where $\eta^{c_0}_s=c_0,s\in[0,t]$.

- Let $\bar{Q}$ be a probability measure on $(\overline{\Omega},\overline{\mathcal{F}}_1)$ and τ a $\overline{\mathbb{F}}-$stopping time. Then, there is a family of conditional probability measures $(\bar{Q}_\omega)_{\omega\in\overline{\Omega}}$ w.r.t $\overline{\mathcal{F}}_\tau$ such that $\bar{Q}_\omega((X_t,A_t,B_t)=\omega_t:t\le\tau(\omega))=1$. This is Theorem 1.3.4 of [55], and $(\bar{Q}_\omega)_{\omega\in\overline{\Omega}}$ is called the regular conditional probability distribution (r.c.p.d.)

Lemma 5.4.2. *Let* $\bar{P}\in\overline{\mathcal{P}}_{t,x,a,b}$*,* τ *an* $\overline{\mathbb{F}}-$*stopping time taking value in* $[t,1]$*, and* $(\bar{Q}_\omega)_{\omega\in\overline{\Omega}}$ *be a r.c.p.d. of* $\bar{P}|\overline{\mathcal{F}}_\tau$*. Then there is a* $\bar{P}-$*null set* $N\in\overline{\mathcal{F}}_\tau$ *such that* $\delta_{\omega_{\tau(\omega)}}\otimes_{\tau(\omega)}\bar{Q}_\omega\in\overline{\mathcal{P}}_{\tau(\omega),\omega_{\tau(\omega)}}$ *for all* $\omega\notin N$*.*

Proof. Since $\bar{P}\in\overline{\mathcal{P}}_{t,x,a,b}$, it follows from Theorem II.2.21 of Jacod and Shiryaev [38] that

$$(X_s-B_s)_{t\le s\le 1},\quad\big((X_s-B_s)^2-A_s\big)_{t\le s\le 1}$$

are all local martingales after time t. Then it follows from Theorem 1.2.10 of Stroock and Varadhan [55] together with a localization technique that there is a $\bar{P}-$null set $N_1\in\overline{\mathcal{F}}_\tau$

such that they are still local martingales after time $\tau(\omega)$ both under $\bar{Q}_\omega$ and $\delta_{\omega_{\tau(\omega)}} \otimes_{\tau(\omega)} \bar{Q}_\omega$, for all $\omega \notin N_1$. It is clear, moreover, that $\nu \in U$, $d\bar{Q}_\omega \times dt-$a.e. on $\overline{\Omega} \times [\tau(\omega), 1]$ for $\bar{P}-$a.e. $\omega \in \overline{\Omega}$. Then there is $\bar{P}-$null set $N \in \overline{\mathcal{F}}_\tau$ such that $\delta_{\omega_{\tau(\omega)}} \otimes_{\tau(\omega)} \bar{Q}_\omega \in \overline{\mathcal{P}}_{\tau(\omega),\omega_{\tau(\omega)}}$ for every $\omega \notin N$. □

Lemma 5.4.3. *Let Assumptions 5.3.1, 5.3.2, 5.3.3 hold true, and assume that λ is locally bounded. Let $\bar{P} \in \overline{\mathcal{P}}_{t,x,a,b}$, $\tau \geq t$ a $\overline{\mathbb{F}}-$stopping time, and $(\bar{Q}_\omega)_{\omega \in \overline{\Omega}}$ a family of probability measures such that $\bar{Q}_\omega \in \overline{\mathcal{P}}_{\tau(\omega),\omega_{\tau(\omega)}}$ and $\omega \mapsto \bar{Q}_\omega$ is $\overline{\mathcal{F}}_\tau-$measurable. Then there is a unique probability measure, denoted by $\bar{P} \otimes_{\tau(\cdot)} \bar{Q}_\cdot$, in $\overline{\mathcal{P}}_{t,x,a,b}$, such that*

$$\bar{P} \otimes_{\tau(\cdot)} \bar{Q}_\cdot = \bar{P} \quad on \quad \overline{\mathcal{F}}_\tau, \quad and \ (\delta_\omega \otimes_{\tau(\omega)} \bar{Q}_\omega)_{\omega \in \overline{\Omega}} \ is\ a\ r.c.p.d.\ of \quad \bar{P} \otimes_{\tau(\cdot)} \bar{Q}_\cdot |\overline{\mathcal{F}}_\tau. \tag{5.4.7}$$

Proof. The existence and uniqueness of the probability measure $\bar{P} \otimes_{\tau(\cdot)} \bar{Q}_\cdot$ on $(\overline{\Omega}, \overline{\mathcal{F}}_1)$, satisfying (5.4.7), follows from Theorem 6.1.2 of [55]. It remains to prove that $\bar{P} \otimes_{\tau(\cdot)} \bar{Q}_\cdot \in \overline{\mathcal{P}}_{t,x,a,b}$.

Since $\bar{Q}_\omega \in \overline{\mathcal{P}}_{\tau(\omega),\omega_{\tau(\omega)}}$, X is a $\delta_\omega \otimes_{\tau(\omega)} \bar{Q}_\omega-$semimartingale after time $\tau(\omega)$, characterized by (A, B). Then, the processes $X - B$ and $(X - B)^2 - A$ are local martingales under $\delta_\omega \otimes_{\tau(\omega)} \bar{Q}_\omega$ after time $\tau(\omega)$. By Theorem 1.2.10 of [55] together with a localization argument, they are still local martingales under $\bar{P} \otimes_{\tau(\cdot)} \bar{Q}_\cdot$. Hence, the required result follows from Theorem II.2.21 of [38]. □

We have now collected all the ingredients for the proof of the dynamic programming principle.

Proof of Proposition 5.4.1 Let τ be an $\overline{\mathbb{F}}-$stopping time taking value in $[t, 1]$. We proceed in two steps.

1. For $\bar{P} \in \overline{\mathcal{P}}_{t,x,a,b}$, we denote by $(\bar{Q}_\omega)_{\omega \in \overline{\Omega}}$ a r.c.p.d. of $\bar{P}|\mathcal{F}_\tau$, and $\bar{P}^\omega_\tau := \delta_{\omega_{\tau(\omega)}} \otimes_{\tau(\omega)} \bar{Q}_\omega$. By the representation (5.4.5) of λ, together with the tower property of conditional expectations, we see that

$$\begin{aligned}
\lambda(t,x) &= \inf_{\bar{P} \in \overline{\mathcal{P}}_{t,x,a,b}} \mathbb{E}^{\bar{P}}\Big[\int_t^\tau \ell(s, X_s, \nu_s)ds + \int_\tau^1 \ell(s, X_s, \nu_s)ds + \lambda_1(X_1)\Big] \\
&= \inf_{\bar{P} \in \overline{\mathcal{P}}_{t,x,a,b}} \mathbb{E}^{\bar{P}}\Big[\int_t^\tau \ell(s, X_s, \nu_s)ds + \mathbb{E}^{\bar{P}^\omega_\tau}\Big\{\int_\tau^1 \ell(s, X_s, \nu_s)ds + \lambda_1(X_1)\Big\}\Big] \\
&\geq \inf_{\bar{P} \in \overline{\mathcal{P}}_{t,x,a,b}} \mathbb{E}^{\bar{P}}\Big[\int_t^\tau \ell(s, X_s, \nu_s)ds + \lambda(\tau, X_\tau)\Big],
\end{aligned} \tag{5.4.8}$$

where the last inequality follows from the fact that $\bar{P}^\omega_\tau \in \overline{\mathcal{P}}_{\tau(\omega),\omega_{\tau(\omega)}}$ by Lemma 5.4.2.

2. For $\varepsilon > 0$, let $(\bar{Q}^\varepsilon_{t,x,a,b})_{[0,1]\times\mathbb{R}^{2d+d^2}}$ be the family defined in Lemma 5.4.1, and denote $\bar{Q}^\varepsilon_\omega := \bar{Q}^\varepsilon_{\tau(\omega),\omega_{\tau(\omega)}}$. Then $\omega \mapsto \bar{Q}^\varepsilon_\omega$ is $\overline{\mathcal{F}}_\tau-$measurable. Moreover, for all $\bar{P} \in \overline{\mathcal{P}}_{t,x,a,b}$, we may construct by Lemmas 5.4.1 and 5.4.3 $\bar{P} \otimes_{\tau(\cdot)} \bar{Q}_\cdot \in \overline{\mathcal{P}}_{t,x,a,b}$ such that

$$\mathbb{E}^{\bar{P}\otimes_{\tau(\cdot)}\bar{Q}_\cdot}\left[\int_t^T \ell(s, X_s, \nu_s)ds + \lambda_1(X_1)\right] \leq \mathbb{E}^{\bar{P}}\left[\int_t^\tau \ell(s, X_s, \nu_s)ds + \lambda(\tau, X_\tau)\right] + \varepsilon.$$

By the arbitrariness of $\bar{P} \in \overline{\mathcal{P}}_{t,x,a,b}$ and $\varepsilon > 0$, together with the representation (5.4.5) of λ, this implies that the reverse inequality to (5.4.8) holds true, and the proof is complete. $\square$

We conclude this section by the

Proof of Lemma 5.3.1 By the same arguments as in Lemma 5.4.1, we can easily deduce that λ_0 is $\mathcal{B}^{\mu_0}(\mathbb{R}^d)-$ measurable, and we just need to prove that

$$\mu_0(\lambda_0) = \inf_{\bar{P} \in \overline{\mathcal{P}}(\mu_0)} \mathbb{E}^{\bar{P}}\Big[\int_0^1 \ell(s, X_s, \nu_s)ds + \lambda_1(X_1)\Big].$$

Given a probability measure $\bar{P} \in \overline{\mathcal{P}}(\mu_0)$, we can get a family of conditional probabilities $(\bar{Q}_\omega)_{\omega\in\Omega}$ such that $\bar{Q}_\omega \in \overline{\mathcal{P}}_{0,\omega_0}$, which implies that

$$\mathbb{E}^{\bar{P}}\Big[\int_0^1 \ell(s, X_s, \nu_s)ds + \lambda_1(X_1)\Big] \geq \mu_0(\lambda_0), \quad \forall \bar{P} \in \overline{\mathcal{P}}(\mu_0).$$

On the other hand, for every $\varepsilon > 0$ and $\mu_0 \in \mathbf{M}(\mathbb{R}^d)$, we can select a measurable family of $(\bar{Q}^\varepsilon_x \in \overline{\mathcal{P}}_{0,x,0,0})_{x\in\mathbb{R}^d}$ such that

$$\mathbb{E}^{\bar{Q}^\varepsilon_x}\Big[\int_0^1 \ell(s, X_s, \nu_s)ds + \lambda_1(X_1)\Big] \leq \lambda_0(x) + \varepsilon, \;\; \mu_0 - \text{a.s.}$$

and then construct a probability measure $\mu_0 \otimes_0 \bar{Q}^\varepsilon_\cdot \in \overline{\mathcal{P}}(\mu_0)$ by concatenation such that

$$\mathbb{E}^{\mu_0\otimes_0\bar{Q}^\varepsilon_\cdot}\Big[\int_0^1 \ell(s, X_s, \nu_s)ds + \lambda_1(X_1)\Big] \leq \mu_0(\lambda_0) + \varepsilon, \;\; \forall\varepsilon > 0,$$

which completes the proof. $\square$

5.5 Numerical approximation

In this section, we provide an implementable numerical scheme for the approximation of the value function $V(\mu_0, \mu_1)$ in the Markovian context where $L(t, \mathbf{x}, u) = \ell(t, \mathbf{x}(t), u)$, under Assumptions 5.3.1, 5.3.2, 5.3.3, and 5.3.4. By our duality result of Theorem 5.3.1 together with Theorem 5.3.2, we have that

$$V = \mathcal{V} := \sup_{\lambda_1\in C_b(\mathbb{R}^d)} v(\lambda_1) = \overline{\mathcal{V}} := \sup_{\lambda_1\in C_b^\infty(\mathbb{R}^d)} v(\lambda_1) \quad \text{where} \quad v(\lambda_1) := \mu_0(\lambda_0) - \mu_1(\lambda_1)$$

and the function λ_0 is defined in (5.3.2). We shall require the following additional conditions to hold.

Assumption 5.5.1. $\int_{\mathbb{R}^d} |x|(\mu_0 + \mu_1)(dx) < \infty$.

Assumption 5.5.2. *U is compact, and ℓ is Lipschitz in x uniformly in (t, u).*

Throughout this section, we denote:

$$M \;\; := \sup_{(t,x,u)\in[0,1]\times\mathbb{R}^d\times U} |u| + |\ell(t, 0, u)| + |\nabla_x\ell(t, x, u)|.$$

where $\nabla_x\ell(t, x, u)$ is the gradient of ℓ with respect to x which exists a.e. under Assumption 5.5.2.

5.5.1 First approximations

Let Lip^0_K denote the collection of all bounded $K-$Lipschitz-continuous functions $\phi : \mathbb{R}^d \longrightarrow \mathbb{R}$ with $\phi(0) = 0$, and denote $\mathrm{Lip}^0 := \cup_{K>0}\mathrm{Lip}^0_K$. Since $v(\lambda_1 + c) = v(\lambda_1)$ for any $\lambda_1 \in C_b(\mathbb{R}^d)$ and $c \in \mathbb{R}$, we deduce that:

$$V = \sup_{\lambda_1 \in \mathrm{Lip}^0} v(\lambda_1).$$

As a first approximation, we introduce the function:

$$V^K := \sup_{\lambda_1 \in \mathrm{Lip}^0_K} v(\lambda_1). \tag{5.5.1}$$

Under Assumptions 5.5.1 and 5.5.2, we easily verify that $V < \infty$, see Lemma 5.5.1 below. Then, it is immediate that:

$$(V^K)_{K>0} \text{ is increasing and } V^K \longrightarrow V \quad \text{as} \quad K \to \infty. \tag{5.5.2}$$

Our next approximation restricts the space variable x to the bounded subsets $O_R := (-R, R)^d$ of $\mathbb{R}^d$, $R > 0$. Let τ_R be the first exit time of the canonical process X from O_R:

$$\tau_R := \inf\{t : X_t \notin O_R\},$$

and define for all bounded functions $\lambda_1 \in C_b(\mathbb{R}^d)$:

$$\lambda^R(t,x) := \inf_{\mathbb{P}\in\mathcal{P}_{t,x}} \mathbb{E}^{\mathbb{P}}\Big[\int_t^{\tau_R\wedge 1} \ell(s, X_s, \nu_s^{\mathbb{P}})ds + \lambda_1(X_{\tau_R\wedge 1})\Big].$$

By similar arguments as in Theorem 5.4.1, λ^R is a viscosity solution of equation

$$-\partial_t\lambda^R(t,x) - H(t, x, D\lambda^R, D^2\lambda^R) = 0, \quad (t,x) \in [0,1) \times O_R, \tag{5.5.3}$$

with boundary conditions

$$\lambda^R(t,x) = \lambda_1(x) \quad \text{for all} \quad (t,x) \in ([0,1) \times \partial O_R) \cup (\{1\} \times O_R). \tag{5.5.4}$$

Here ∂O_R denotes the boundary of O_R. Moreover, from discussions in Example 3.6 of Crandall et al. [21], it satisfies a comparison result. Then λ^R is the unique bounded viscosity solution of (5.5.3) with boundary condition (5.5.4).

Lemma 5.5.1. *Under Assumption 5.5.2, let $\lambda_1 \in Lip^0_K$ be arbitrary. Then λ and λ^R are Lipschitz-continuous, and there is a constant C depending on M such that:*

$$|\lambda(t,0)| + |\lambda^R(t,0)| + |\nabla_x\lambda(t,x)| + |\nabla_x\lambda^R(t,x)| \le C(1+K), \quad (t,x) \in [0,1] \times \mathbb{R}^d.$$

Proof. We only provide the estimates for λ, those for λ_R follows from the same arguments. First, by Assumption 5.5.2 together with the fact that λ_1 is $K-$Lipschitz and $\lambda_1(0) = 0$, for every $\mathbb{P} \in \mathcal{P}_{t,0}$,

$$\mathbb{E}^{\mathbb{P}}\Big[\int_t^1 \ell(s, X_s, \nu_s^{\mathbb{P}})ds + \lambda_1(X_1)\Big] \;\leq\; M + (M+K)\sup_{t\leq s\leq 1}\mathbb{E}^{\mathbb{P}}\big|X_s\big|.$$

Recall that X is continuous semimartingale under $\mathbb{P}$ whose finite variation part and quadratic variation of the martingale part are both bounded by constant M. Separating the two parts and using Cauchy-Schwarz's inequality, it follows that $\mathbb{E}^{\mathbb{P}}\big|X_s\big| \leq M + \sqrt{M}$, $\forall t \leq s \leq 1$, and then $|\lambda(t,0)| \leq M + (M+K)(M+\sqrt{M})$.

We next prove that λ is Lipschitz and provide the corresponding estimate. Observe that $\mathcal{P}_{t,y} = \big\{\mathbb{P} := \tilde{\mathbb{P}} \circ (X+y-x)^{-1} \;:\; \tilde{\mathbb{P}} \in \mathcal{P}_{t,x}\big\}$. Then

$$\begin{aligned}
& |\lambda(t,x) - \lambda(t,y)| \\
\leq\;& \sup_{\mathbb{P}\in\mathcal{P}_{t,x}} \mathbb{E}^{\mathbb{P}}\Big|\int_t^1 \ell(s, X_s, \nu_s^{\mathbb{P}}) - \ell(s, X_s + y - x, \nu_s^{\mathbb{P}})ds + \lambda_1(X_1) - \lambda_1(X_1 + y - x)\Big| \\
\leq\;& (M+K)|y-x|
\end{aligned}$$

by the Lipschitz property of ℓ and λ in x. □

Define

$$\lambda_0^R := \lambda^R(0,.),\;\; v^R(\lambda_1) := \mu_0(\lambda_0^R \mathbf{1}_{O_R}) - \mu_1(\lambda_1 \mathbf{1}_{O_R}),\;\text{and}\; V^{K,R} := \sup_{\lambda_1\in \mathrm{Lip}_0^K} v^R(\lambda_1). \quad (5.5.5)$$

In the special case where U is a singleton, equation (5.5.3) degenerates to the heat equation, Barles, Daher and Romano [4] proved that the error $\lambda - \lambda^R$ satisfies a large deviation estimate as $R \to \infty$. The next result extends this estimate to our context.

Proposition 5.5.1. *Let Assumption 5.5.2 and 5.5.1 hold true, we denote* $|x| := \max_{i=1}^d |x_i|$ *for* $x \in \mathbb{R}^d$ *and choose* $R > 2M$*. Then, there is a constant* C *such that:*
(i) *for all* $K > 0$, $\lambda_1 \in Lip_0^K$ *and* $|x| \leq R - M$,

$$|\lambda^R - \lambda|(t,x) \leq C(1+K)e^{-(R-M-|x|)^2/2M},$$

(ii) *for all* $K > 0$*:*

$$\big|V^{K,R} - V^K\big| \;\leq\; C(1+K)\Big(e^{-R^2/8M + R/2} + \int_{O^c_{R/2}} (1+|x|)(\mu_0+\mu_1)(dx)\Big). \quad (5.5.6)$$

Proof. 1. For arbitrary $(t,x) \in [0,1]\times\mathbb{R}^d$ and $\mathbb{P} \in \mathcal{P}_{t,x}$, we denote $Y^i := \sup_{0\leq s\leq 1}|X_s^i|$ where X^i is the $i-$th component of the canonical process X. By the Dubins-Schwartz time-change theorem (see e.g. Theorem 4.6, Chapiter 3 of Karatzas and Shreve [40]), we may represent the continuous local martingale part of X^i as a time-changed Brownian motion W. Since the characteristics of X are bounded by M, we see that:

$$\begin{aligned} S^i(R) := \mathbb{P}[Y^i \ge R] &\le \mathbb{P}\Big[\sup_{0\le t\le M} |W_t| \ge R - |x_i| - M\Big] \\ &\le 2\mathbb{P}\Big[\sup_{0\le t\le M} W_t \ge R - |x_i| - M\Big] \\ &= 4\Big(1 - \mathbf{N}\big(R^M_{|x_i|}\big)\Big), \end{aligned} \tag{5.5.7}$$

where $R^M_{|x_i|} := (R - M - |x_i|)/\sqrt{M}$, $\mathbf{N}$ be the cumulative distribution function of the standard normal distribution $N(0,1)$, and the last equality follows from the reflection principle of the Brownian motion. Then by integration by parts as well as (5.5.7),

$$\begin{aligned} \mathbb{E}^{\mathbb{P}}\Big[Y^i \mathbf{1}_{Y^i\ge R}\Big] &= RS^i(R) + \int_R^\infty S^i(z)dz. \\ &\le 4\int_R^\infty \frac{1}{\sqrt{M}}\frac{1}{\sqrt{2\pi}} \exp\Big(\frac{(z-M-|x_i|)^2}{2M}\Big) z\, dz \\ &= 4(|x_i| + M)\Big(1 - \mathbf{N}\big(R^M_{|x_i|}\big)\Big) + \frac{4\sqrt{M}}{\sqrt{2\pi}} \exp\Big(-\frac{(R^M_{|x_i|})^2}{2}\Big). \end{aligned}$$

We further remark that for any $R > 0$,

$$(1 - \mathbf{N}(R)) = \int_R^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt \le \frac{1}{R}\int_R^\infty \frac{1}{\sqrt{2\pi}} t e^{-\frac{t^2}{2}} dt = \frac{1}{\sqrt{2\pi}}\frac{1}{R} e^{\frac{-R^2}{2}}.$$

2. By definitions of λ, λ^R, it follows that for all (t,x) such that $|x| \le R - M$,

$$\begin{aligned} |\lambda - \lambda^R|(t,x) &\le \sup_{\mathbb{P}\in\mathcal{P}_{t,x}} \mathbb{E}^{\mathbb{P}}\Big[\int_{\tau_R\wedge 1}^1 |\ell(s, X_s, \nu^{\mathbb{P}}_s)|ds + \big|\lambda_1(X_{\tau_R\wedge 1}) - \lambda_1(X_1)\big|\Big] \\ &\le \sup_{\mathbb{P}\in\mathcal{P}_{t,x}} \mathbb{E}^{\mathbb{P}}\Big[\Big(M + \sqrt{d}KR + (M+K)\sup_{t\le s\le 1}|X_s|\Big)\mathbf{1}_{\tau_R<1}\Big] \\ &\le \sup_{\mathbb{P}\in\mathcal{P}_{t,x}} \mathbb{E}^{\mathbb{P}}\Big[\sum_{i=1}^d \Big(M + \sqrt{d}KR + \sqrt{d}(M+K)Y_i\Big)\mathbf{1}_{Y_i\ge R}\Big] \\ &\le C(1+K)e^{-(R^M_{|x|})^2/2}, \end{aligned} \tag{5.5.8}$$

for some constant C depending on M and d. This completes the proof of (i).

3. To prove item (ii) of the proposition, we start with:

$$\begin{aligned} \big|V^{K,R} - V^K\big| &= \Big|\sup_{\lambda_1\in\mathrm{Lip}^0_K} \big\{\mu_0(\lambda^R_0 \mathbf{1}_{O_R}) - \mu_1(\lambda_1\mathbf{1}_{O_R})\big\} - \sup_{\lambda_1\in\mathrm{Lip}^0_K}\big\{\mu_0(\lambda_0) - \mu_1(\lambda_1)\big\}\Big| \\ &\le \sup_{\lambda_1\in\mathrm{Lip}^0_K} \big|\mu_0(\lambda^R_0\mathbf{1}_{O_R}) - \mu_0(\lambda_0)\big| + K\int_{O^c_R} |x|\mu_1(dx). \end{aligned}$$

Now for all $\lambda_1 \in \mathrm{Lip}^0_K$, we estimate that:

$$\begin{aligned} \left|\mu_0(\lambda_0^R \mathbf{1}_{O_R}) - \mu_0(\lambda_0)\right| &\leq \mu_0\Big(\left|\lambda_0^R - \lambda_0\right| \mathbf{1}_{O_{\frac{R}{2}}}\Big) + \mu_0\Big(\left(\left|\lambda_0^R\right| + \left|\lambda_0\right|\right) \mathbf{1}_{(O_{\frac{R}{2}})^c}\Big) \\ &\leq C(1+K)\Big(\int_{O_{\frac{R}{2}}} e^{-(R^M_{|x|})^2/2} \mu_0(dx) + \int_{(O_{\frac{R}{2}})^c} (1+|x|)\mu_0(dx)\Big), \end{aligned}$$

where we used Lemma 5.5.1 together with item (i) of the present proposition. Observing that $(R^M_{|x|})^2 \geq R^2/4M - R + M$ on $O_{\frac{R}{2}}$, this implies that

$$\left|\mu_0(\lambda_0^R \mathbf{1}_{O_R}) - \mu_0(\lambda_0)\right| \leq C(1+K)\Big(e^{-R^2/8M + R/2} + \int_{(O_{\frac{R}{2}})^c} (1+|x|)\mu_0(dx))\Big),$$

and the required estimate follows. □

5.5.2 A finite differences approximation

In the remaining part of this chapter, we restrict the discussion to the one-dimensional case

$$d = 1 \quad \text{so that} \quad O_R = (-R, R).$$

Let $(l, r) \in \mathbb{N}^2$ and $h = (\Delta t, \Delta x) \in (\mathbb{R}^+)^2$ be such that $l\Delta t = 1$ and $r\Delta x = R$. Denote $x_i := i\Delta x$, $t_k := k\Delta t$ and define the discrete grids:

$$\mathcal{N} := \{x_i \ : \ i \in \mathbb{Z}\}, \qquad \mathcal{N}_R := \mathcal{N} \cap (-R, R),$$

$$\mathcal{M}_{T,R} := \big\{(t_k, x_i) \ : \ (k, i) \in \mathbb{Z}^+ \times \mathbb{Z}\big\} \cap ([0,1] \times (-R, R)).$$

The terminal set, boundary set as well as interior set of $\mathcal{M}_{T,R}$ are denoted by

$$\partial_T \mathcal{M}_{T,R} := \big\{(1, x_i) \ : \ x_i \in \mathcal{N}_R\big\}, \ \partial_R \mathcal{M}_{T,R} := \big\{(t_k, \pm R) \ : \ k = 0, \cdots, l\big\},$$

$$\mathring{\mathcal{M}}_{T,R} := \mathcal{M}_{T,R} \setminus (\partial_T \mathcal{M}_{T,R} \cup \partial_R \mathcal{M}_{T,R}).$$

We shall use the finite differences method to solve the dynamic programming equation (5.5.3), (5.5.4) on the grid $\mathcal{M}_{T,R}$. For a function w defined on $\mathcal{M}_{T,R}$, we introduce the discrete derivatives of w:

$$D^{\pm} w(t_k, x_i) := \frac{w(t_k, x_{i\pm 1}) - w(t_k, x_i)}{\Delta x} \quad \text{and} \quad (bD)w := b^+ D^+ w + b^- D^- w \text{ for } b \in \mathbb{R},$$

where $b^+ := \max(0, b)$, $b^- := \max(0, -b)$; and

$$D^2 w(t_k, x_i) := \frac{w(t_k, x_{i+1}) - 2w(t_k, x_i) + w(t_k, x_{i-1})}{\Delta x^2}.$$

We now define the function $\hat{\lambda}^{h,R}$(or $\hat{\lambda}^{h,R,\hat{\lambda}_1}$ to precise its dependence on the boundary condition $\hat{\lambda}_1$) on the grid $\mathcal{M}_{T,R}$ by the following explicit finite differences approximation of the dynamic programming equation (5.5.3):

$$\hat{\lambda}^{h,R}(t_k,x_i) = \Big(\hat{\lambda}^{h,R} + \Delta t \inf_{u=(a,b)\in U}\Big\{\ell(\cdot,u) + (bD)\hat{\lambda}^{h,R} + \frac{1}{2}aD^2\hat{\lambda}^{h,R}\Big\}\Big)(t_{k+1},x_i) \quad \text{on } \mathring{\mathcal{M}}_{T,R}$$
$$\hat{\lambda}^{h,R}(t_k,x_i) = \hat{\lambda}_1(x_i) \quad \text{on } \partial_T\mathcal{M}_{T,R}\cup\partial_R\mathcal{M}_{T,R}, \tag{5.5.9}$$

and we introduce the following natural approximation of v^R:

$$\hat{v}^R_h(\hat{\lambda}_1) := \mu_0\Big(\mathrm{lin}^R[\hat{\lambda}^{h,R}_0]\Big) - \mu_1\Big(\mathrm{lin}^R[\hat{\lambda}_1]\Big) \quad \text{where} \quad \hat{\lambda}^{h,R}_0 := \hat{\lambda}^{h,R}(0,\cdot), \tag{5.5.10}$$

and for all function ϕ defined on the grid $\mathcal{N}_R$ we denote by $\mathrm{lin}^R[\phi]$ the linear interpolation of ϕ extended by zero outside $[-R,R]$.

We shall also assume that the discretization parameters $h=(\Delta t,\Delta x)$ satisfy the CFL condition

$$\Delta t\left(\frac{|b|}{\Delta x}+\frac{|a|}{\Delta x^2}\right) \le 1 \quad \text{for all} \quad (a,b)\in U. \tag{5.5.11}$$

Then the scheme (5.5.9) is $L^\infty-$monotone, so that the convergence of the scheme is guaranteed by the monotonic scheme method of Barles and Souganidis [6]. For our next result, we assume that the following error estimate holds.

Assumption 5.5.3. *There are positive constants $L_{K,R}$, ρ_1, ρ_2 which are independent of $h=(\Delta t,\Delta x)$, such that*

$$\mu_0\Big(\Big|\mathrm{lin}^R[\hat{\lambda}^{h,R}_0] - \lambda_0\mathbf{1}_{[-R,R]}\Big|\Big) \le L_{K,R}(\Delta t^{\rho_1}+\Delta x^{\rho_2}), \forall\lambda_1\in Lip^K_0 \text{ and } \hat{\lambda}_1 = \lambda_1|_{\mathcal{N}_R}.$$

Let $\mathrm{Lip}^{K,R}_0$ be the collection of all functions on the grid $\mathcal{N}_R$ defined as restrictions of functions in Lip^K_0:

$$\mathrm{Lip}^{K,R}_0 \;:=\; \big\{\hat{\lambda}_1 := \lambda_1|_{\mathcal{N}_R} \text{ for some } \lambda_1\in\mathrm{Lip}^K_0\big\}. \tag{5.5.12}$$

The above approximation of the dynamic value function λ suggests the following natural approximation of the minimal transportation cost value:

$$V^{K,R}_h \;:=\; \sup_{\hat{\lambda}_1\in\mathrm{Lip}^{K,R}_0}\hat{v}^R_h(\hat{\lambda}_1) = \sup_{\hat{\lambda}_1\in\mathrm{Lip}^{K,R}_0}\mu_0\Big(\mathrm{lin}^R[\hat{\lambda}^{h,R}_0]\Big) - \mu_1\Big(\mathrm{lin}^R[\hat{\lambda}_1]\Big). \tag{5.5.13}$$

Remark 5.5.1. *Under the additional condition that ℓ is uniformly $\frac{1}{2}-$Hölder in t with constant M, then in spirit of the analysis in Barles and Jakobsen [5], Assumption 5.5.3 holds true with $\rho_1=\frac{1}{4}$, $\rho_2=\frac{1}{10}$ and $L_{K,R}=C(1+K+KR)$ with some constant C depending on M.*

Theorem 5.5.1. *Let Assumption 5.5.3 be true, then with the constants $L_{K,R}$, ρ_1, ρ_2 introduced in Assumption 5.5.3, we have*

$$\left|V_h^{K,R} - V^{K,R}\right| \leq L_{K,R}(\Delta t^{\rho_1} + \Delta x^{\rho_2}) + K\Delta x.$$

Proof. First, given $\lambda_1 \in \mathrm{Lip}_0^K$, we take $\hat{\lambda}_1 := \lambda_1|_{\mathcal{N}_R} \in \mathrm{Lip}_0^{K,R}$, then clearly $|\mathrm{lin}^R[\hat{\lambda}_1] - \lambda_1|_{L^\infty([-R,R])} \leq K\Delta x$, and it follows from Assumption 5.5.3 and (5.5.5) as well as (5.5.10) that $v^R(\lambda_1) \leq \hat{v}_h^R(\hat{\lambda}_1) + L_{K,R}(\Delta t^{\rho_1} + \Delta x^{\rho_2}) + K\Delta x$. Hence,

$$V^{K,R} \leq V_h^{K,R} + L_{K,R}(\Delta t^{\rho_1} + \Delta x^{\rho_2}) + K\Delta x.$$

Next, given $\hat{\lambda} \in \mathrm{Lip}_0^{K,R}$, let $\lambda_1 := \mathrm{lin}[\hat{\lambda}_1] \in \mathrm{Lip}_0^K$ be the linear interpolation of $\hat{\lambda}_1$, it follows from Assumption 5.5.3 that $\hat{v}_h^R(\hat{\lambda}_1) \leq v^R(\lambda_1) + L_{K,R}(\Delta t^{\rho_1} + \Delta x^{\rho_2})$, and therefore,

$$V_h^{K,R} \leq V^{K,R} + L_{K,R}(\Delta t^{\rho_1} + \Delta x^{\rho_2}).$$

□

Remark 5.5.2. *In the $d-$dimensional case, we can use the generalized finite differences method to approximate $V^{K,R}$. To construct the generalized finite difference scheme, we refer to section 5 of Kushner [42] when every $a \in S_d$ for $(a,b) \in U$ are diagonal dominated, and to Bonnans and Zidani [13] as well as Bonnans, Ottenwaelter and Zidani [11] for general cases.*

5.5.3 Gradient projection algorithm

In this section, we suggest a numerical scheme to approximate $V_h^{K,R} = \sup_{\hat{\lambda}_1 \in \mathrm{Lip}_0^{K,R}} \hat{v}_h^R(\hat{\lambda}_1)$ in (5.5.13). The crucial observation for our methodology is the following. By $B(\mathcal{N}_R)$, we denote the set of all bounded function on $\mathcal{N}_R$.

Proposition 5.5.2. *Under the CFL condition (5.5.11), the function $\hat{\lambda}_1 \mapsto \hat{v}_h^R(\hat{\lambda}_1)$ is concave on $B(\mathcal{N}_R)$.*

Proof. Let $\bar{u} = (\bar{u}_{k,i})_{0\leq k<l, -r<i<r}$, with $\bar{u}_{k,i} = (\bar{a}_{k,i}, \bar{b}_{k,i}) \in U$, we introduce $\overline{\lambda}^{h,\bar{u},\hat{\lambda}_1}$ (or just $\overline{\lambda}^{h,\bar{u}}$ if there is no risk of ambiguity) as the unique solution of the discrete linear system on $\mathcal{M}_{T,R}$ with a given $\hat{\lambda}_1$:

$$\begin{cases} \overline{\lambda}^{h,\bar{u}}(t_k, x_i) = \Big(\overline{\lambda}^{h,\bar{u}} + \Delta t\Big(\ell(\cdot, \bar{u}_{k,i}) + (\bar{b}_{k,i} D)\overline{\lambda}^{h,\bar{u}} + \bar{a}_{k,i} D^2 \overline{\lambda}^{h,\bar{u}}\Big)\Big)(t_{k+1}, x_i) \text{ on } \mathring{\mathcal{M}}_{T,R}, \\ \overline{\lambda}^{h,\bar{u}}(t_k, x_i) = \hat{\lambda}_1(x_i), \quad \text{for } (t_k, x_i) \in \partial_T \mathcal{M}_{T,R} \cup \partial_R \mathcal{M}_{T,R}. \end{cases} \tag{5.5.14}$$

Let $\overline{\lambda}_0^{h,\bar{u}} := \overline{\lambda}^{h,\bar{u}}(0, \cdot)$, and define:

$$\overline{v}_h^{R,\bar{u}}(\hat{\lambda}_1) := \mu_0\big(\mathrm{lin}^R[\overline{\lambda}_0^{h,\bar{u}}]\big) - \mu_1\big(\mathrm{lin}^R[\hat{\lambda}_1]\big).$$

We claim that

$$\hat{v}_h^R(\hat{\lambda}_1) \quad = \quad \inf_{\bar{u}\in U^{l(2r-1)}} \overline{v}_h^{R,\bar{u}}(\hat{\lambda}_1). \tag{5.5.15}$$

Indeed, under the CFL condition (5.5.11), the finite difference scheme (5.5.9) as well as (5.5.14) are both $L^\infty-$monotone in sense of Barles and Souganidis [6]. Moreover, the linear interpolation $\hat{\lambda}_0 \mapsto \text{lin}^R[\hat{\lambda}_0]$ is also monotone. Then taking infimum step by step in (5.5.9) and (5.5.13) is equivalent to taking infimum globally in (5.5.15).

Finally, the concavity of $\hat{\lambda}_1 \mapsto \hat{v}_h^R(\hat{\lambda}_1)$ follows from its representation as the infimum of linear maps in (5.5.15). □

By the previous proposition, $V_h^{K,R}$ consists in the maximization of a concave function, and a natural scheme to approximate it is the gradient projection algorithm.

Remark 5.5.3. *Since U is compact by Assumption 5.5.2, then for every function $\hat{\lambda}_1$, we have the optimal control $\hat{u}(\hat{\lambda}_1) = \big(\hat{u}_{k,i}(\hat{\lambda}_1)\big)_{0\le k<l,-r<i<r}$ such that*

$$\hat{\lambda}_0^{h,R} = \overline{\lambda}_0^{h,\hat{u}(\hat{\lambda}_1)} \quad \text{and} \quad \hat{v}_h^R(\hat{\lambda}_1) = \overline{v}_h^{R,\hat{u}(\hat{\lambda}_1)}(\hat{\lambda}_1). \tag{5.5.16}$$

Now we are ready to give the gradient projection algorithm for $V_h^{K,R}$ in (5.5.13). Given a function $\varphi \in B(\mathcal{N}_R)$, we denote by $P_{\text{Lip}_0^{K,R}}(\varphi)$ the projection of φ on $\text{Lip}_0^{K,R}$, where $\text{Lip}_0^{K,R} \subset B(\mathcal{N}_R)$ is defined in (5.5.12). Of course, the projection depends on the choice of the norm equipping $B(\mathcal{N}_R)$ which in turn has serious consequences on the numerics. We shall discuss this important issue later.

Let $\gamma := (\gamma_n)_{n\ge 0}$ be a sequence of positive constants, we propose the following algorithm:

Algorithm 5.5.1. *To solve problem (5.5.13):*

- *1, Let $\hat{\lambda}_1^0 := 0$.*
- *2, Given $\hat{\lambda}_1^n$, compute the super-gradient $\nabla\hat{v}_h^R(\hat{\lambda}_1^n)$ of $\hat{\lambda}_1 \mapsto \hat{v}_h^R(\hat{\lambda}_1)$ at $\hat{\lambda}_1^n$.*
- *3, Let $\hat{\lambda}_1^{n+1} = P_{Lip_0^{K,R}}\big(\hat{\lambda}_1^n + \gamma_n \nabla\hat{v}_h^R(\hat{\lambda}_1^n)\big)$.*
- *4, Go back to step 2.*

In the following, we shall discuss the computation of super-gradient $\nabla\hat{v}_h^R(\hat{\lambda}_1)$, the projection $P_{\text{Lip}_0^{K,R}}$ as well as the convergence of the above gradient projection algorithm.

5.5.3.1 Super-gradient

Let $\hat{\lambda}_1 \in B(\mathcal{N}_R)$ be fixed. Then, by Remark 5.5.3, we may find an optimal control $\hat{u}(\hat{\lambda}_1) = \big(\hat{u}_{k,i}(\hat{\lambda}_1)\big)_{0\le k<l,-r\le i\le r}$, where $\hat{u}_{k,i}(\hat{\lambda}_1) = (\hat{a}_{k,i}(\hat{\lambda}_1), \hat{b}_{k,i}(\hat{\lambda}_1)) \in U$, for system (5.5.15). We

then denote by g^j the unique solution of the following linear system on $\mathcal{M}_{T,R}$, for every $-r \le j \le r$:

$$\begin{cases} g^j(t_k, x_i) = \Big(g^j + \Delta t\Big(\big(\hat{b}_{k,i}(\hat{\lambda}_1)D\big)g^j + \hat{a}_{k,i}(\hat{\lambda}_1)D^2 g^j\Big)\Big)(t_{k+1}, x_i) \text{ on } \mathring{\mathcal{M}}_{T,R}, \\ g^j(t_k, x_i) = \delta_{i,j}, \quad \text{for } (t_k, x_i) \in \partial_T \mathcal{M}_{T,R} \cup \partial_R \mathcal{M}_{T,R}. \end{cases} \tag{5.5.17}$$

Denote $g_0^j := g^j(0, \cdot)$ and δ_j be a function on $\mathcal{N}_R$ defined by $\delta_j(x_i) := \delta_{i,j}$.

Proposition 5.5.3. *Let CFL condition (5.5.11) hold true, then the vector*

$$\nabla \hat{v}_h^R(\hat{\lambda}_1) \quad := \quad \big(\mu_0\big(\mathrm{lin}^R[g_0^j]\big) - \mu_1\big(\mathrm{lin}^R[\delta_j]\big)\big)_{-r \le j \le r} \tag{5.5.18}$$

is a super-gradient of $\varphi \in B(\mathcal{N}_R) \mapsto \hat{v}_h^R(\varphi) \in \mathbb{R}$ at $\hat{\lambda}_1$.

Proof. Consider the system (5.5.14) introduced in the proof of Proposition 5.5.2. Under the CFL condition (5.5.11), by (5.5.15), we have for every perturbation $\Delta\hat{\lambda}_1 \in B(\mathcal{N}_R)$,

$$\hat{v}_h^R(\hat{\lambda}_1 + \Delta\hat{\lambda}_1) = \overline{v}_h^{R,\hat{u}(\hat{\lambda}_1 + \Delta\hat{\lambda}_1)}(\hat{\lambda}_1 + \Delta\hat{\lambda}_1) \le \overline{v}_h^{R,\hat{u}(\hat{\lambda}_1)}(\hat{\lambda}_1 + \Delta\hat{\lambda}_1),$$

which implies that

$$\hat{v}_h^R(\hat{\lambda}_1 + \Delta\hat{\lambda}_1) - \hat{v}_h^R(\hat{\lambda}_1) \quad \le \quad \overline{v}_h^{R,\hat{u}(\hat{\lambda}_1)}(\hat{\lambda}_1 + \Delta\hat{\lambda}_1) - \overline{v}_h^{R,\hat{u}(\hat{\lambda}_1)}(\hat{\lambda}_1).$$

We next observe that for fixed $\hat{\lambda}_1$, the function $\varphi \longmapsto \overline{v}_h^{R,\hat{u}(\hat{\lambda}_1)}(\varphi)$ is linear, it follows that

$$\Big(\overline{v}_h^{R,\hat{u}(\hat{\lambda}_1)}(\hat{\lambda}_1 + \delta_j) - \overline{v}_h^{R,\hat{u}(\hat{\lambda}_1)}(\hat{\lambda}_1)\Big)_{-r \le j \le r} \tag{5.5.19}$$

is a super-gradient of $\varphi \mapsto \hat{v}_h^R(\varphi)$ at $\hat{\lambda}_1$. Finally, by (5.5.14) and (5.5.17), $g^j(t_k, x_i) = \overline{\lambda}^{\hat{u}(\hat{\lambda}_1), \hat{\lambda}_1 + \delta_j}(t_k, x_i) - \overline{\lambda}^{\hat{u}(\hat{\lambda}_1), \hat{\lambda}_1}(t_k, x_i)$, where $\overline{\lambda}^{\hat{u}(\hat{\lambda}_1), \hat{\lambda}_1 + \delta_j}$ is the solution of (5.5.14) with boundary condition $\hat{\lambda}_1 + \delta_j$. By the definition of $\overline{v}_h^{R,\bar{u}}(\hat{\lambda}_1)$ in (5.5.15), it follows that the super-gradient (5.5.19) is equivalent to $\nabla \hat{v}_h^R(\hat{\lambda}_1)$ defined in (5.5.18). □

5.5.3.2 Projection

To compute the projection $P_{\mathrm{Lip}_0^{K,R}}(\varphi)$, $\forall \varphi \in B(\mathcal{N}_R)$, we need to equip $B(\mathcal{N}_R)$ with a specific norm. In order to obtain a simple projection algorithm, we shall introduce an invertible linear map between $B(\mathcal{N}_R)$ and $\mathbb{R}^{2r+1}$, then equip on $B(\mathcal{N}_R)$ the norm induced by the classical L^2-norm on $\mathbb{R}^{2r+1}$.

Let us define the invertible linear map $\mathcal{T}_R$ from $B(\mathcal{N}_R)$ to $\mathbb{R}^{2r+1}$ as

$$\psi_i = \mathcal{T}_R(\varphi)_i := \begin{cases} \varphi(x_{i+1}) - \varphi(x_i), & i = 1, \cdots, r, \\ \varphi(0), & i = 0, \\ \varphi(x_{i-1}) - \varphi(x_i), & i = -1, \cdots, -r, \end{cases}$$

and define the norm $|\cdot|_R$ on $B(\mathcal{N}_R)$ (easily be verified) by

$$|\varphi|_R := |\mathcal{T}_R(\varphi)|_{L^2(\mathbb{R}^{2r+1})}, \quad \forall \varphi \in B(\mathcal{N}_R).$$

Notice that

$$\begin{aligned}\mathcal{T}_R \mathrm{Lip}_0^{K,R} &:= \left\{ \psi = \mathcal{T}_R \varphi : \; \varphi \in \mathrm{Lip}_0^{K,R} \right\} \\ &= \left\{ \psi = (\psi_i)_{-r \le i \le r} \in [-K\Delta x, K\Delta x]^{2r+1} : \; \psi_0 = 0 \right\}.\end{aligned}$$

Then the projection $P_{\mathrm{Lip}_0^{K,R}}$ from $B(\mathcal{N}_R)$ to $\mathrm{Lip}_0^{K,R}$ under norm $|\cdot|_R$ is equivalent to the projection $P_{\mathcal{T}_R \mathrm{Lip}_0^{K,R}}$ from $\mathbb{R}^{2r+1}$ to $\mathcal{T}_R \mathrm{Lip}_0^{K,R}$ under the L^2-norm, which is simply written as

$$\left(P_{\mathcal{T}_R \mathrm{Lip}_0^{K,R}}(\psi) \right)_i = \begin{cases} 0, & \text{if } \; i = 0, \\ (K\Delta x) \wedge \psi_i \vee (-K\Delta x), & \text{otherwise.} \end{cases}$$

5.5.3.3 Convergence rate

Now, let us give a convergence rate for the above gradient projection algorithm. In preparation, we first provide an estimate for the norm of super-gradients.

Proposition 5.5.4. *Suppose that CFL condition (5.5.11) hold, then* $|\hat{v}_h^R(\varphi_1) - \hat{v}_h^R(\varphi_2)| \le 2|\varphi_1 - \varphi_2|_\infty$ *for every* $\varphi_1, \; \varphi_2 \in B(\mathcal{N}_R)$*. And therefore, the super-gradient* $\nabla \hat{v}_h^R$ *satisfies*

$$\left| \nabla \hat{v}_h^R(\hat{\lambda}_1) \right|_R \le 2\sqrt{\frac{R}{\Delta x} + 1}, \; \textit{for all } \hat{\lambda}_1 \in B(\mathcal{N}). \tag{5.5.20}$$

Proof. Under CFL condition, the scheme (5.5.9) is $L^\infty-$monotone, then $\left| \hat{\lambda}_0^{h,R,\varphi_1} - \hat{\lambda}_0^{h,R,\varphi_2} \right|_\infty \le |\varphi_1 - \varphi_2|_\infty$, and it follows from the definition of $\hat{v}_h^R$ in (5.5.10) that

$$|\hat{v}_h^R(\varphi_1) - \hat{v}_h^R(\varphi_2)| \le 2|\varphi_1 - \varphi_2|_\infty. \tag{5.5.21}$$

Next, by Cauchy-Schwarz inequality,

$$|\varphi_1 - \varphi_2|_\infty \le \max \left(\sum_{i=0}^{r} \left| \mathcal{T}_R(\varphi_1 - \varphi_2)_i \right|, \sum_{i=0}^{-r} \left| \mathcal{T}_R(\varphi_1 - \varphi_2)_i \right| \right) \le \sqrt{r+1} |\varphi_1 - \varphi_2|_R.$$

Together with (5.5.21), this implies that (5.5.20) holds for every super-gradient $\nabla \hat{v}_h^R(\hat{\lambda}_1)$. □

Let us finish this section by providing a convergence rate for our gradient projection algorithm. Denote

$$\Pi := \max_{\varphi_1, \varphi_2 \in \mathrm{Lip}_0^{K,R}} |\varphi_1 - \varphi_2|_R^2 \le 2r(K\Delta x)^2 \le 2K^2 R \Delta x,$$

it follows from section 5.3.1 of Ben-Tal and Nemirovski [7] that

$$
\begin{aligned}
0 \le V_h^{K,R} - \max_{n \le N} \hat{v}_h^R(\hat{\lambda}_1^n) &\le \frac{\Pi + \sum_{n=1}^N \gamma_n^2 |\nabla \hat{v}_h^R(\hat{\lambda}_1^n)|_R^2}{\sum_{n=1}^N \gamma_n} \\
&\le \frac{2K^2 R \Delta x + 4\left(\frac{R}{\Delta x} + 1\right) \sum_{n=1}^N \gamma_n^2}{\sum_{n=1}^N \gamma_n}. \qquad (5.5.22)
\end{aligned}
$$

We have several choices for the series $\gamma = (\gamma_n)_{n \ge 1}$:

- Divergent series : $\gamma_n \ge 0$, $\sum_{n=1}^\infty \gamma_n = +\infty$ and $\sum_{n=1}^\infty \gamma_n^2 < +\infty$, then the right hand side of (5.5.22) converges to 0 as $N \to \infty$.
- Optimal stepsizes : $\gamma_n = \frac{\sqrt{2\Pi}}{|\nabla \hat{v}_h^R(\hat{\lambda}_1^n)|_R \sqrt{n}}$, [7] shows that

$$
V_h^{K,R} - \max_{n \le N} \hat{v}_h^R(\hat{\lambda}_1^n) \le C_1 \frac{(\max_{1 \le n \le N} |\nabla \hat{v}_h^R(\hat{\lambda}_1^n)|_R) \cdot \sqrt{2\Pi}}{\sqrt{N}} \le C \frac{K(R + \sqrt{R \Delta x})}{\sqrt{N}},
$$

for some constant C independent of K, R, Δt, Δx and N.

5.5.4 Numerical example

We finally give a numerical example for the above algorithm. Let us consider the one-dimensional case $d = 1$, and $U = U_1 \times \{0\}$ where U_1 is a compact interval in $\mathbb{R}^+$, i.e. X is a one-dimensional martingale under $\mathbb{P}$ for all $\mathbb{P} \in \mathcal{P}(\mu_0, \mu_1)$. Let the transportation cost be given by:

$$
\ell(t, x, a, b) = a \quad \text{so that} \quad J(\mathbb{P}) = \mathbb{E}^{\mathbb{P}} \langle X \rangle_1.
$$

This example is motivated by an application in financial mathematics, where $\langle X \rangle_1$ is the payoff of a financial derivative called *variance swap*. Then, the minimum cost of transportation is the minimum no-arbitrage price of the variance swap given the possibility of dynamic trading the underlying asset, with price X, together with the static trading of the European options maturing at time 1 with all possible strikes.

Suppose that $\mathcal{P}(\mu_0, \mu_1)$ is nonempty, it follows from the duality Theorems 5.3.1 and 5.3.2 that

$$
\begin{aligned}
V(\mu_0, \mu_1) &= \inf_{\mathbb{P} \in \mathcal{P}(\mu_0, \mu_1)} \mathbb{E}^{\mathbb{P}} \int_0^1 \alpha_t^{\mathbb{P}} dt = \inf_{\mathbb{P} \in \mathcal{P}(\mu_0, \mu_1)} \mathbb{E}^{\mathbb{P}} \langle X \rangle_1 \\
&= \inf_{\mathbb{P} \in \mathcal{P}(\mu_0, \mu_1)} \mathbb{E}^{\mathbb{P}} \left[(X_1 - X_0)^2 \right] = \int_{\mathbb{R}} x^2 \mu_1(dx) - \int_{\mathbb{R}} x^2 \mu_0(dx). \quad (5.5.23)
\end{aligned}
$$

We choose μ_i as normal distribution $N(0, \sigma_i^2)$ with $\sigma_0 = 0.1$, $\sigma_1 = 0.2$ and $U = [0, 0.1] \times \{0\}$, we implement the scheme suggested in the previous subsection, and we compare to the explicit solution (5.5.23). The numerical result shows that with 10^5 iterations (which takes no more than 1 minute of calculation on a standard computer), the relative error is less than 1%, see Figure 5.1.

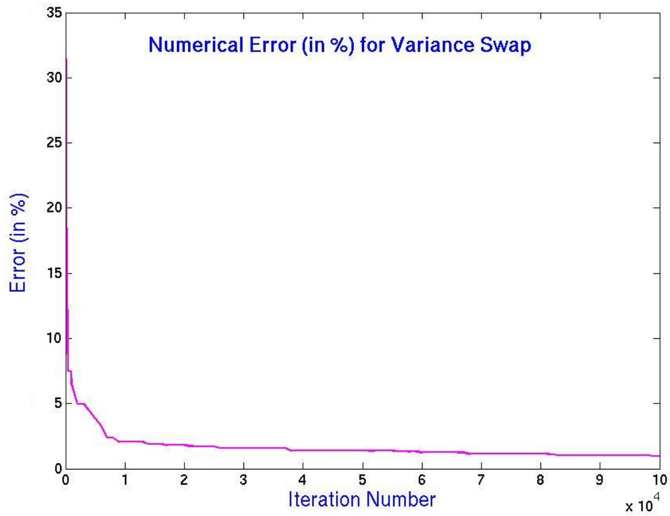

Figure 5.1: Numerical example: $\mu_i = N(0, \sigma_i^2)$ with $\sigma_0 = 0.1$, $\sigma_1 = 0.2$, $U = [0, 0.1] \times \{0\}$, $K = 1.5$, $R = 1$, $\Delta x = 0.1$, $\Delta t = 0.025$.

5.6 Appendix

We first report a theorem which provides the unique canonical decomposition of a continuous semimartingale under different filtrations. In particular, it follows that an Itô process has a diffusion representation, by taking the filtration generated by itself. This is in fact Theorem 7.17 of Liptser and Shiryayev [43] in 1-dimensional case, or Theorem 4.3 of Wong [57] in multi-dimensional case.

Theorem 5.6.1. *In a filtrated space $(\Omega, \mathbb{F} = (\mathcal{F}_t)_{0 \le t \le 1}, \mathbb{P})$ (here Ω is not necessary the canonical space), a process X is a continuous semimartingale with canonical decomposition:*

$$X_t = X_0 + B_t + M_t,$$

where $B_0 = M_0 = 0$, and $B = (B_t)_{0 \le t \le 1}$ is of finite variation and $M = (M_t)_{0 \le t \le 1}$ a local martingale. In addition, suppose that there are measurable and $\mathbb{F}-$adapted processes (α, β) such that

$$B_t = \int_0^t \beta_s ds, \ \int_0^1 \mathbb{E}|\beta_s| ds < \infty, \ \text{and } A_t := \langle M \rangle_t = \int_0^t \alpha_s ds.$$

Let $\mathbb{F}^X = (\mathcal{F}_t^X)_{0 \le t \le 1}$ be the filtration generated by process X and $\bar{\mathbb{F}} = (\bar{\mathcal{F}}_t)_{0 \le t \le 1}$ be a filtration such that $\mathcal{F}_t^X \subseteq \bar{\mathcal{F}}_t \subseteq \mathcal{F}_t$. Then X is still a continuous semimartingale under $\bar{\mathbb{F}}$, whose canonical decomposition is given by

$$X_t = X_0 + \int_0^t \bar{\beta}_s ds + \bar{M}_t \quad \text{with } \bar{A}_t := \langle \bar{M} \rangle_t = \int_0^t \bar{\alpha}_s ds,$$

where

$$\bar{\beta}_t = \mathbb{E}(\beta_t | \bar{\mathcal{F}}_t) \quad \text{and} \quad \bar{\alpha}_t = \alpha_t, \quad d\mathbb{P} \times dt - a.e.$$

Proof of Theorem 5.4.1. The characterization of the value function as viscosity solution is a natural result of the dynamic programming principle. Here, we give a proof, similar to that of Corollary 5.1 in [16], in our context.

1, We first prove the subsolution property. Suppose that $(t_0, x_0) \in [0,1) \times \mathbb{R}^d$ and $\phi \in C_c^\infty([0,1) \times \mathbb{R}^d)$ is a smooth function such that

$$0 = (\lambda - \phi)(t_0, x_0) > (\lambda - \phi)(t, x), \quad \forall (t, x) \neq (t_0, x_0).$$

By adding $\varepsilon\left(|t - t_0|^2 + |x - x_0|^4\right)$ to $\phi(t, x)$, we can suppose that

$$\phi(t, x) \geq \lambda(t, x) + \varepsilon\left(|t - t_0|^2 + |x - x_0|^4\right) \tag{5.6.1}$$

without loosing generality. Assume to the contrary that

$$-\partial_t \phi(t_0, x_0) - H\left(t_0, x_0, D_x \phi(t_0, x_0), D^2_{xx} \phi(t_0, x_0)\right) > 0,$$

we shall derive a contradiction. Indeed, by definition of H, there is $c > 0$ and $(a, b) \in U$ such that

$$-\partial_t \phi(t, x) - b \cdot D_x \phi(t, x) - \frac{1}{2} a \cdot D^2_{xx} \phi(t, x) - \ell(t, x, a, b) > 0, \quad \forall (t, x) \in B_c(t_0, x_0),$$

where $B_c(t_0, x_0) := \{(t, x) \in [0,1) \times \mathbb{R}^d \ : \ |(t, x) - (t_0, x_0)| \leq c\}$. Let $\tau := \inf\{t \geq t_0 \ : (t, X_t) \notin B_c(t_0, x_0)\}$, then

$$\begin{aligned} \lambda(t_0, x_0) = \phi(t_0, x_0) \quad &\geq \inf_{\bar{P} \in \overline{\mathcal{P}}_{t_0, x_0, 0, 0}} \mathbb{E}^{\bar{P}}\left[\int_t^\tau \ell(s, X_s, \nu_s) ds + \phi(\tau, X_\tau)\right] \\ &\geq \inf_{\bar{P} \in \overline{\mathcal{P}}_{t_0, x_0, 0, 0}} \mathbb{E}^{\bar{P}}\left[\int_t^\tau \ell(s, X_s, \nu_s) ds + \lambda(\tau, X_\tau)\right] + \eta, \end{aligned}$$

where η is some positive constant from (5.6.1) and the definition of τ. This is a contradiction to Proposition 5.4.1.

2, For the supersolution property, we assume to the contrary that there is $(t_0, x_0) \in [0,1) \times \mathbb{R}^d$ and smooth function ϕ satisfying

$$0 = (\lambda - \phi)(t_0, x_0) < (\lambda - \phi)(t, x), \quad \forall (t, x) \neq (t_0, x_0).$$

and

$$-\partial_t \phi(t_0, x_0) - H\left(t_0, x_0, D_x \phi(t_0, x_0), D^2_{xx} \phi(t_0, x_0)\right) < 0,$$

We also suppose without loosing generality that

$$\phi(t, x) \leq \lambda(t, x) - \varepsilon\left(|t - t_0|^2 + |x - x_0|^4\right). \tag{5.6.2}$$

By continuity of H, there is $c > 0$ such that for all $(t, x) \in B_c(t_0, x_0)$ and every $(a, b) \in U$,

$$-\partial_t \phi(t, x) - b \cdot D_x \phi(t, x) - \frac{1}{2} a \cdot D^2_{xx} \phi(t, x) - \ell(t, x, a, b) < 0.$$

Let $\tau := \inf\{t \geq t_0 \ : \ (t, X_t) \notin B_c(t_0, x_0)\}$, then

$$\begin{aligned} \lambda(t_0, x_0) = \phi(t_0, x_0) &\leq \inf_{\bar{P} \in \overline{\mathcal{P}}_{t_0, x_0, 0, 0}} \mathbb{E}^{\bar{P}} \left[\phi(\tau, X_\tau) + \int_{t_0}^{\tau} \ell(s, X_s, \nu_s) ds \right] \\ &\leq \inf_{\bar{P} \in \overline{\mathcal{P}}_{t_0, x_0, 0, 0}} \mathbb{E}^{\bar{P}} \left[\lambda(\tau, X_\tau) + \int_{t_0}^{\tau} \ell(s, X_s, \nu_s) ds \right] - \eta \end{aligned}$$

for some $\eta > 0$ by (5.6.2), which is a contradiction to Proposition 5.4.1. □

Chapitre 6

A model-free no-arbitrage price bound for variance options

6.1 Introduction

In a recent work of Galichon, Henry-Labordère and Touzi [33], the authors proposed a framework to compute the optimal model-free no-arbitrage price bound of exotic options in a vanilla-liquid market. Let $\Omega^d := C([0,T],\mathbb{R}^d)$ be the canonical space with canonical process X and canonical filtration $\mathbb{F}^d = (\mathcal{F}_t^d)_{0\le t\le T}$, S_0 be a constant. We denote by $\mathcal{P}(\delta_{S_0})$ the collection of all probability measures $\mathbb{P}$ on $(\Omega^d, \mathcal{F}_T^d)$ under which X is a $\mathbb{F}^d-$martingale and $X_0 = S_0$, $\mathbb{P}-$a.s. As indicated in [33], there is a progressively measurable process $\langle X\rangle_t$ which is pathwise defined and coincides with the $\mathbb{P}$-quadratic variation of X, $\mathbb{P}$-a.s. for every $\mathbb{P}\in\mathcal{P}(\delta_{S_0})$.

The process X is a candidate of underlying stock price, we do not impose any dynamic assumptions on X, but only suppose that it is a martingale. Then for an option with payoff $G\in\mathcal{F}_T^d$, the upper bound of model-free no-arbitrage price is given by

$$\sup_{\mathbb{P}\in\mathcal{P}(\delta_{S_0})} \mathbb{E}^{\mathbb{P}}\big[G\big].$$

Suppose in addition that we are in a market where the vanilla options with maturity T are liquid, so that the investor can identify the marginal distribution μ of X_T. In other words, let $\phi\in\mathbb{L}^1(\mathbb{R}^d,\mu)$, the $T-$maturity European option with payoff $\phi(X_T)$ has a unique no-arbitrage price

$$\mu(\phi) \quad := \quad \int_{\mathbb{R}^d} \phi(x)\mu(dx).$$

Let us use the vanilla option portfolio to hedge G. By buying a portfolio $\phi(X_T)$, we spend $\mu(\phi)$ and so the payoff at maturity T becomes $G-\phi(X_T)$. Therefore, we get a new upper bound of model-free price: $\sup_{\mathbb{P}\in\mathcal{P}(\delta_{S_0})}\mathbb{E}^{\mathbb{P}}\big[G-\phi(X_T)\big]+\mu(\phi)$. By minimizing on the vanilla option portfolio ϕ, the optimal upper bound is then given by

$$\inf_{\phi\in\mathbb{L}^1(\mu)} \sup_{\mathbb{P}\in\mathcal{P}(\delta_{S_0})} \left\{\mathbb{E}^{\mathbb{P}}\big[G-\phi(X_T)\big]+\mu(\phi)\right\}. \tag{6.1.1}$$

As another motivation, we observe that the upper bound (6.1.1) is formally the conjugate dual formulation of problem

$$\sup_{\mathbb{P}\in\mathcal{P}(\delta_{S_0},\mu)} \mathbb{E}^{\mathbb{P}}\big[G\big] \quad = \quad \sup_{\mathbb{P}\in\mathcal{P}(\delta_{S_0})} \inf_{\phi\in\mathbb{L}^1(\mu)} \left\{\mathbb{E}^{\mathbb{P}}\big[G-\phi(X_T)\big]+\mu(\phi)\right\}, \tag{6.1.2}$$

where $\mathcal{P}(\delta_{S_0},\mu)$ denotes the collection of all martingale probability measures $\mathbb{P}\in\mathcal{P}(\delta_{S_0})$ such that $X_T\sim^{\mathbb{P}}\mu$. We remark that the above equality holds since

$$\inf_{\phi\in\mathbb{L}^1(\mu)}\left\{\mathbb{E}^{\mathbb{P}}\big[G-\phi(X_T)\big]+\mu(\phi)\right\} \;=\; \begin{cases}\mathbb{E}^{\mathbb{P}}[G], & \text{if } X_T\sim^{\mathbb{P}}\mu,\\ -\infty, & \text{otherwise.}\end{cases}$$

In this chapiter, we shall consider in particular the no-arbitrage price bound of variance option in a similar framework. Let us restrict to the one-dimensional case $d=1$ and $T_1>T_0\geq 0$ be two constants. We define the corresponding canonical space as $\Omega:=C([0,T_1],\mathbb{R})$ and denote still by X the canonical process, $\mathbb{F}=(\mathcal{F}_t)_{0\leq t\leq T_1}$ the canonical filtration and by $\langle X\rangle$ the progressively measurable process which coincides with the quadratic variation of X under every martingale probability measure $\mathbb{P}$. Suppose that the vanilla options of maturities T_0, T_1 are liquid so that we can identify the marginal distribution μ_0 (resp. μ_1) for X_{T_0} (resp. X_{T_1}). We shall consider the variance option with payoff

$$G \;:=\; g(\langle X\rangle_{T_0,T_1},X_{T_1}) \text{ at maturity } T_1 \text{ for some appropriate function } g,$$

where $\langle X\rangle_{T_0,T_1}:=\langle X\rangle_{T_1}-\langle X\rangle_{T_0}$. Let $\mathcal{P}^2(\mu_0)$ denote the set of all the probability measures $\mathbb{P}$ on $(\Omega,\mathcal{F}_{T_1})$ such that $X_{T_0}\sim^{\mathbb{P}}\mu_0$ and $\mathbb{E}^{\mathbb{P}}\big[\langle X\rangle_{T_0,T_1}\big|\mathcal{F}_{T_0}\big]<\infty$, $\mathbb{P}-$a.s., we define the no-arbitrage price upper bound of variance option $G=g(\langle X\rangle_{T_0,T_1},X_{T_1})$ by

$$\inf_{\phi\in\mathrm{Quad}}\;\sup_{\mathbb{P}\in\mathcal{P}^2(\mu_0)}\;\left\{\mathbb{E}^{\mathbb{P}}\big[g(\langle X\rangle_{T_0,T_1},X_{T_1})-\phi(X_{T_1})\big]+\mu_1(\phi)\right\}, \tag{6.1.3}$$

where Quad denotes the set of functions satisfying a quadratic growth condition, i.e.

$$\mathrm{Quad} \;:=\; \Big\{\;\phi:\mathbb{R}\to\mathbb{R} \text{ such that } \sup_{x\in\mathbb{R}}\frac{|\phi(x)|}{1+|x|^2}<\infty\;\Big\}. \tag{6.1.4}$$

Remark 6.1.1. *The main reason to choose* Quad *is from the observation of Dupire [27] that variance swap (i.e. $g(x,z)=z$) is equivalent to a European option option with payoff X_T^2, see also Remark 6.2.2 and Corollary 6.3.2.*

By the time-change martingale theorem (see e.g. Theorem 3.4.6 of Karatzas and Shreve [40]), we can establish a correspondence between the set of martingale probability measures on $(\Omega,\mathcal{F}_{T_1})$ and the set of stopping times on a Brownian motion, and hence reformulate (6.1.3) as

$$\overline{U} \;:=\; \inf_{\phi\in\mathrm{Quad}}\bar{u}(\phi) \;\text{ with }\; \bar{u}(\varphi) \;:=\; \sup_{\tau\in\mathcal{T}}\mathbb{E}\big[g(\tau,W_\tau)-\phi(W_\tau)\big] \;+\; \mu_1(\phi), \tag{6.1.5}$$

where W is a Brownian motion with its natural filtration $\mathcal{F}^W$ such that $W_0\sim\mu_0$ and

$$\mathcal{T} \;:=\; \big\{\;\mathcal{F}^W-\text{stopping times } \tau \text{ such that } \mathbb{E}\big[\tau\big|W_0\big]<\infty,\ \text{a.s.}\big\}. \tag{6.1.6}$$

In fact, a continuous martingale X can be represented as a time-changed Brownian motion, i.e. $X_t=B_{\langle X\rangle_t}$ with a Brownian motion B and $\langle X\rangle_t$ a stopping time w.r.t. the time-changed filtration. By the strong Markovian property of the Brownian motion, $W_t := B_{\langle X\rangle_{T_0}+t}$ defines a new Brownian motion W such that $W_0 \sim \mu_0$ and $\tau := \langle X\rangle_{T_1} - \langle X\rangle_{T_0}$ is a stopping time in $\mathcal{T}$. On the other hand, given a Brown motion W such that $W_0 \sim \mu_0$ and a stopping time $\tau \in \mathcal{T}$, the process Y, defined by $Y_t := W_{\tau\wedge\frac{t-T_0}{T_1-t}}$ when $t \in [T_0, T_1)$ and $Y_t = Y_{T_0}$ when $t \in [0, T_0)$, turns to be a continuous martingale between T_0 and T_1 which induces a probability measure in $\mathcal{P}^2(\mu_0)$.

We can also derive a dual formulation for (6.1.5) following the same arguments as for deriving (6.1.2). Let $\mathcal{T}(\mu_1)$ denote the set of all stopping times $\tau \in \mathcal{T}$ such that $W_\tau \sim \mu_1$, then the dual formulation of (6.1.5) becomes

$$\sup_{\tau\in\mathcal{T}} \inf_{\phi\in\text{Quad}} \mathbb{E}\Big[g(\tau, W_\tau) - \phi(W_\tau)\Big] + \mu_1(\phi) \quad = \quad \sup_{\tau\in\mathcal{T}(\mu_1)} \mathbb{E}\big[g(\tau, W_\tau)\big]. \tag{6.1.7}$$

Given a Brownian motion W and a distribution μ_1, the problem of finding stopping time τ such that $W_\tau \sim \mu_1$, i.e. $\tau \in \mathcal{T}(\mu_1)$, is called the Skorokhod Embedding Problem (SEP). Then our formulation (6.1.5) is consistent with Hobson's [36] observation of the connection between the SEP and the problem of optimal no-arbitrage bounds of exotic options in a vanilla-liquid market.

The SEP and the optimality property of its solutions as well as their applications in finance are studied in several papers recently, we refer to Obłój [48] and Hobson [37] for a survey. In particular, for the optimization problem (6.1.7), if $g(t, x) = f(t)$ for some function f defined on $\mathbb{R}^+$, it is proved that the maximum is achieved by Root's embedding when f is concave and by Röst's embedding when f is convex (see Root [51] and Rost [52]). However, for general payoff function g, there is no systematic method to find the optimal value of such problems. That is also our main motivation to develop a numerical method to solve these problems.

Our main contribution is then to provide a numerical scheme to approximate the bounds for general variance options.

The rest of the chapter is organized as follows: In Section 6.2, we give an equivalent formulation for the bound $\overline{U}$ in (6.1.5). Then in Section 6.3 we provide an asymptotic analysis of our approximation, which restricts the calculation of $\overline{U}$ to a bounded domain. In Section 6.4, we propose a numerical scheme which combines the gradient projection algorithm and the finite difference method, and we give a general convergence result. Finally, Section 6.5 provides a numerical example on *variance swap*.

Notations: Let μ be a probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, we define

$$\mu(\phi) \;:=\; \int_{\mathbb{R}} \phi(x)\mu(dx), \quad \text{for every } \phi \in \mathbb{L}^1(\mu).$$

6.2 An equivalent formulation of the bound

We will fix the payoff function $g : (t, x) \in \mathbb{R}^+ \times \mathbb{R} \mapsto g(t, x) \in \mathbb{R}$ of the variance option as well as the marginal distributions μ_0, μ_1, and then reformulate the price bound problem

(6.1.5). To make the problem be well posed, let us first make some assumptions on the marginal distributions μ_0, μ_1 and the payoff function g.

Assumption 6.2.1. *The probability measures μ_0, μ_1 on $\mathbb{R}$ have finite second moment, i.e.*

$$\mu_0(\phi_0) \ + \ \mu_1(\phi_0) \ < \ \infty, \quad \textit{with} \ \ \phi_0(x) \ := \ x^2.$$

Moreover, $\mu_0 \leq \mu_1$ in the convex order, i.e.

$$\mu_0(\phi) \ \leq \mu_1(\phi), \quad \textit{for every convex function } \phi \textit{ defined on } \mathbb{R}. \tag{6.2.1}$$

Remark 6.2.1. *It is shown in Strassen [54] that the convex order inequality (6.2.1) is a sufficient and necessary condition for the existence of a martingale with marginal distributions μ_0 and μ_1 at time T_0 and T_1 such that $T_0 < T_1$.*

In particular, since the identity function I (where $I(x) := x$) and its opposite $-I$ are both convex, it follows immediately from (6.2.1) that μ_0 and μ_1 have the same first moment, i.e. $\mu_0(I) = \mu_1(I)$.

Assumption 6.2.2. *The payoff function $g(t,x)$ is L_0-Lipschitz in (t,x) with constant $L_0 \in \mathbb{R}^+$.*

Example 6.2.1. *The most popular variance option is the "variance swap", whose payoff function is $g(t,x) = t$. There exist also "volatility swap" with payoff $g(t,x) = \sqrt{t}$, and calls (puts) on variance, or volatility, where the payoff function are $(t-K)^+$ $((K-t)^+)$, or $(\sqrt{t}-K)^+$ $((K-\sqrt{t})^+)$. Of course, the volatility swap payoff function $\sqrt{t}$ is not Lipschitz, which is however can be always approximated by Lipschitz functions.*

In addition to Assumption 6.2.2, we give another assumption on the payoff function g.

Assumption 6.2.3. *The function $g(t,x)$ increases in t, and is convex in x for every fixed $t \in \mathbb{R}^+$. Moreover, for every fixed $t \in \mathbb{R}^+$, $g(t,0) = \min_{x\in\mathbb{R}} g(t,x)$ and $g(t,x)$ is affine in x on $[M_0, \infty)$ and $(-\infty, -M_0]$ with constant $M_0 \in \mathbb{R}^+$.*

Remark 6.2.2. *Assumption 6.2.3 may not be crucial given Assumptions 6.2.1 and 6.2.2. Let $K \in \mathbb{R}$ and $\psi \in$ Quad, denote $g_{K,\psi}(t,x) := g(t,x) + Kt + \psi(x)$. Then by the equality established in Theorem 6.2.1 and Corollary 6.3.2 below, it follows that*

$$\overline{U}(g_{K,\psi}) \quad = \quad \overline{U}(g) + KC_0 + \mu_1(\psi),$$

where $\overline{U}(g)$ (resp. $\overline{U}(g_{K,\psi})$) denotes the upper bound of (6.1.5) associated with the payoff function g (resp. $g_{K,\psi}$), and

$$C_0 \ := \ \mu_1(\phi_0) - \mu_0(\phi_0), \quad \textit{with} \ \ \phi_0(x) \ := \ x^2. \tag{6.2.2}$$

Therefore, for an arbitrary Lipschtiz function g, we can consider, with some contant $K > 0$, the payoff function $g(t,x) + Kt$, which increases in t. And this does not change the nature of the upper bound problem (6.1.5). Similarly, we can make the payoff function be convex in many general cases.

Now we shall give an equivalent formulation of the problem (6.1.5). Let $B = (B_t)_{t\geq 0}$ be a standard one-dimensional Brownian motion such that $B_0 = 0$, $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ be its natural filtration and $\mathcal{T}^\infty$ be a set of $\mathbb{F}-$stopping times defined by

$$\mathcal{T}^\infty \quad := \quad \big\{\mathbb{F} - \text{stopping time } \tau \text{ such that } \mathbb{E}(\tau) < \infty\big\}. \tag{6.2.3}$$

Given a strategy function $\phi \in \text{Quad}$ which is given by (6.1.4), we denote

$$g^\phi(t,x) \quad := \quad g(t,x) \; - \; \phi(x), \tag{6.2.4}$$

and define functions $\lambda^\phi : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ and $\lambda_0^\phi : \mathbb{R} \to \mathbb{R}$ by

$$\lambda^\phi(t,x) \quad := \quad \sup_{\tau \in \mathcal{T}^\infty} \; \mathbb{E}\left[\, g^\phi(t+\tau, x+B_\tau)\,\right], \quad \text{and} \quad \lambda_0^\phi(\cdot) \;:=\; \lambda^\phi(0,\cdot). \tag{6.2.5}$$

Then the new formulation of the model-free no-arbitrage price upper bound is given by

$$U \quad := \quad \inf_{\phi \in \text{Quad}} u(\phi), \quad \text{with} \quad u(\phi) \quad := \quad \mu_0(\lambda_0^\phi) \;+\; \mu_1(\phi). \tag{6.2.6}$$

We notice that $\mu_0(\lambda_0^\phi)$ is well defined under Assumptions 6.2.1 and 6.2.2, by the fact that $\lambda_0^\phi(x) \geq g^\phi(0,x) = g(0,x) - \phi(x) \geq -C(1+x^2)$ for some positive constant C and that $\lambda^\phi(t,x)$ is measurable from the following theorem.

Theorem 6.2.1. *Let Assumptions 6.2.1 and 6.2.2 hold true. Then for every $\phi \in \text{Quad}$, the function $\lambda^\phi(t,x)$ is lower-semicontinuous and hence measurable. Moreover, the problem (6.1.5) and (6.2.6) are equivalent, i.e. $\overline{U} = U$.*

The semicontinuity of λ^ϕ is from the fact that it can be represented as the supremum of a family of continuous function. And the second assertion is a simple consequence of the dynamic programming, we shall report it in the end of Section 6.3.3.

Remark 6.2.3. *Here we only give the upper bound formulation. By the symmetry of the set* Quad *defined in (6.1.4), if we reverse the payoff function to $-g(t,x)$, then with the upper bound $U(-g)$ associated to payoff $-g$, the value $-U(-g)$ is the lower bound for the payoff g.*

When $g(t,x) = (t-K)^+$, i.e. the option is the variance call, Dupire [27], Carr and Lee [19] proposed a systematic scheme to find a non-optimal bound as well as the associated strategy ϕ in a similar context. In their implemented examples, they showed that their bounds are quite close to the optimal bounds from Root's embedding solution.

For general payoff functions $g(t,x)$, when there is no systematic method to solve the problem (6.2.6), we shall propose a numerical scheme to approximate the optimal ϕ as well as the optimal upper bound U. In fact, we can easily observe that $\phi \mapsto \lambda^\phi$ is convex since it is represented as the supremum of a family of linear mapping in (6.2.5). Thus $\phi \mapsto u(\phi)$ is a convex function and the problem of U in (6.2.6) turns out to be a minimization problem of a convex function, as expected for a dual formulation of (6.1.7). We propose to use the finite difference scheme to solve $u(\phi)$ with every given ϕ, and then approximate the minimization problem on ϕ by an iterative algorithm.

6.3 Analytic approximation

In order to make the numerical resolution of U in (6.2.6) possible, we shall restrict the computation to a bounded domain by some analytic approximations. The approximations is divided into four steps, with convergence results cited without proofs in Section 6.3.1. Then before providing the proofs, we give a first analysis with some technical lemmas in Section 6.3.2. Finally, we complete the proofs of the convergence results in Section 6.3.3.

6.3.1 The analytic approximation in four steps

Let us present the analytic approximation in four steps. The first step is to introduce a subset of Quad defined by

$$\mathrm{Quad}_0 \;:=\; \big\{\, \phi \in \mathrm{Quad} \text{ non negative, convex, such that } \phi(0) = 0 \,\big\},$$

and then to prove that it is equivalent to optimize on Quad_0 for problem (6.2.6).

Proposition 6.3.1. *Let Assumptions 6.2.1, 6.2.2 and 6.2.3 hold true, then $|U| < \infty$, and*

$$U \;=\; \inf_{\phi\in\mathrm{Quad}_0} u(\phi). \tag{6.3.1}$$

Our second approximation is on the growth coefficient of ϕ in Quad_0. Let K be a positive constant, we denote

$$U^K \;:=\; \inf_{\phi\in\mathrm{Quad}_0^K} u(\phi) \;\text{ with } \mathrm{Quad}_0^K \;:=\; \big\{\phi \in \mathrm{Quad}_0 \;:\; \phi(x) \le K(|x| \vee x^2)\big\}. \tag{6.3.2}$$

By the convexity of functions in Quad_0, we see that every $\phi \in \mathrm{Quad}_0$ is in fact locally Lipschitz continuous, and hence $\mathrm{Quad}_0 = \cup_{K>0}\mathrm{Quad}_0^K$. Then it follows immediately that

$$U^K \searrow U \;\text{ as }\; K \longrightarrow \infty. \tag{6.3.3}$$

The third approximation is on the tail of functions in Quad_0^K. Given a constant $M \ge M_0$, where M_0 is given in Assumption 6.2.1, we denote

$$\mathrm{Quad}_0^{K,M} \;:=\; \big\{\, \phi \in \mathrm{Quad}_0^K \text{ such that } \phi(x) = Kx^2 \text{ for } |x| \ge 2M \,\big\}, \tag{6.3.4}$$

and

$$U^{K,M} \;:=\; \inf_{\phi\in\mathrm{Quad}_0^{K,M}} u(\phi). \tag{6.3.5}$$

Proposition 6.3.2. *Let Assumptions 6.2.1, 6.2.2 and 6.2.3 hold, then*

$$0 \;\le\; U^{K,M} - U^K \;\le\; \mu_1(\phi_{K,M}), \tag{6.3.6}$$

where

$$\phi_{K,M}(x) \;:=\; 4KM(|x| - M)\mathbf{1}_{M\le|x|\le 2M} \;+\; Kx^2\mathbf{1}_{|x|>2M}. \tag{6.3.7}$$

Clearly, $\phi_{K,M} \in \mathrm{Quad}_0^{K,M}$ and for every fixed $K > 0$, $\mu_1(\phi_{K,M}) \to 0$ as $M \to \infty$ whenever μ_1 satisfies Assumption 6.2.1.

For the fourth step of the analytic approximation, we first introduce

$$\lambda^{\phi,T}(t,x) \;:=\; \sup_{\tau\in\mathcal{T}^\infty,\ \tau\le T-t} \mathbb{E}[g^\phi(t+\tau, x+B_\tau)], \qquad \lambda_0^{\phi,T}(\cdot) \;:=\; \lambda^{\phi,T}(0,\cdot),$$

$$\lambda^{\phi,\tau_R}(t,x) \;:=\; \sup_{\tau\in\mathcal{T}^\infty,\ \tau\le \tau_x^R} \mathbb{E}[g^\phi(t+\tau, x+B_\tau)], \tag{6.3.8}$$

and

$$\lambda^{\phi,T,R}(t,x) \;:=\; \sup_{\tau\in\mathcal{T}^\infty,\ \tau\le \tau_x^R\wedge(T-t)} \mathbb{E}[g^\phi(t+\tau, x+B_\tau)], \tag{6.3.9}$$

where

$$\tau_x^R \;:=\; \inf\{s \,:\, x+B_s \notin (-R,R)\}.$$

Lemma 6.3.1. *Let Assumptions 6.2.2 and 6.2.3 hold true, with constants L_0, M_0 given in the assumptions. Suppose that $K > L_0$, $M \ge M_0$ and $R \ge \big(1+\sqrt{\frac{K}{K-L_0}}\big)M$. Then for every $\phi \in \mathrm{Quad}_0^{K,M}$,*

$$\lambda^\phi(t,x) = \lambda^{\phi,\tau_R}(t,x), \quad \textit{and} \quad \lambda^{\phi,T}(t,x) = \lambda^{\phi,T,R}(t,x), \quad \forall (t,x)\in[0,T]\times\mathbb{R}.$$

With the equivalence between λ^ϕ ($\lambda^{\phi,T}$) and λ^{ϕ,τ_R} ($\lambda^{\phi,T,R}$), we can now make an approximation on coefficient T. Given $\phi \in \mathrm{Quad}_0^{K,M}$, we define

$$U^{K,M,T} \;:=\; \inf_{\phi\in \mathrm{Quad}_0^{K,M}} u^T(\phi), \quad \text{with} \quad u^T(\phi) \;:=\; \mu_0(\lambda_0^{\phi,T}) \;+\; \mu_1(\phi). \tag{6.3.10}$$

Proposition 6.3.3. *Let Assumptions 6.2.1, 6.2.2 and 6.2.3 hold, M_0 and L_0 be constants given in Assumption 6.2.2, $K > L_0$, $M \ge M_0$, $R = \big(1+\sqrt{\frac{K}{K-L_0}}\big)M$ and $L = 2(K+2L_0)(R^2\vee 1)$, we denote*

$$\delta \;:=\; -\log(q(R)) > 0, \quad \textit{where} \quad q(R) \;:=\; \frac{1}{\sqrt{2\pi}}\int_{-2R}^{2R} e^{-x^2/2}\,dx.$$

Then

$$0 \;\le\; U^{K,M} - U^{K,M,T} \;\le\; Le^{-\delta(T-1)}. \tag{6.3.11}$$

Finally, we finish this section by remarking that $U^{K,M,T}$ in (6.3.10) is defined via $\lambda^{\phi,T}$, which is equivalent to $\lambda^{\phi,T,R}$ by Lemma 6.3.1, and $\lambda^{\phi,T,R}$ can be characterized as the viscosity solution of a variational inequality (see e.g. Theorem 6.7 of Touzi [56]).

Proposition 6.3.4. *The function $\lambda^{\phi,T,R}$ defined in (6.3.9) is the unique viscosity solution of variational inequality*

$$\min\Big(\lambda - g^\phi,\; -\frac{1}{2}\frac{\partial^2\lambda}{\partial x^2} - \frac{\partial\lambda}{\partial t}\Big)(t,x) \;=\; 0, \qquad \textit{on} \;\; [0,T)\times(-R,R), \tag{6.3.12}$$

with boundary condition

$$\lambda(t,x) \;=\; g^\phi(t,x), \qquad \textit{on} \;\; \big([0,T]\times\{\pm R\}\big) \,\cup\, \big(\{T\}\times[-R,R]\big).$$

6.3.2 A first analysis

Before proving the convergence results given in Propositions 6.3.1, 6.3.2 and 6.3.3, we first give two well-known properties of the stopping times on a Brownian motion and report their proofs for completeness. We then provide also a first analysis on $u(\phi)$ and U in (6.2.6).

Lemma 6.3.2. *Let $\psi : (t,x) \in \mathbb{R}^+ \times \mathbb{R} \mapsto \psi(t,x) \in \mathbb{R}$ be a function Lipschitz in t and satisfying $\sup_{(t,x)\in\mathbb{R}^+\times\mathbb{R}} \frac{|\psi(t,x)|}{1+x^2} < \infty$. Then for every $\tau \in \mathcal{T}^\infty$,*

$$\mathbb{E}\Big[\psi(\tau, B_\tau)\Big] \;=\; \lim_{t\to\infty} \mathbb{E}\Big[\psi(\tau\wedge t, B_{\tau\wedge t})\Big]. \tag{6.3.13}$$

In particular,

$$\mathbb{E}[B_\tau^2] \;=\; \lim_{t\to\infty}\mathbb{E}[B^2_{\tau\wedge t}] \;=\; \lim_{t\to\infty}\mathbb{E}[\tau\wedge t] \;=\; \mathbb{E}[\tau] \quad and \quad \mathbb{E}[B_\tau] \;=\; 0. \tag{6.3.14}$$

Proof. Given a stopping time $\tau \in \mathcal{T}^\infty$, let $Y_t := B_{\tau\wedge t}$. Then by assumptions on ψ, there is a constant $C > 0$ such that

$$\psi(B_{\tau\wedge t}, \tau\wedge t) \;\le\; C\big(1 + Y_t^2 + \tau\big) \;\le\; C\Big(1 + \sup_{s\ge 0} Y_s^2 + \tau\Big), \quad \forall t \ge 0.$$

We notice that $(Y_t)_{t\ge 0}$ is a continuous uniformly integrable martingale by its definition, and $\mathbb{E}\big[\sup_{s\ge0} Y_s^2\big] \;\le\; 4\mathbb{E}[\tau] \;<\; \infty$ by Doob's inequality. And hence it follows by the dominated convergence theorem that (6.3.13) holds true. □

Given $T > 0$, we denote by $\mathcal{T}^T$ the collection of all $\mathbb{F}-$stopping times taking value in $[0,T]$, i.e.

$$\mathcal{T}^T \;:=\; \big\{\tau\wedge T : \tau \in \mathcal{T}^\infty\big\}. \tag{6.3.15}$$

Lemma 6.3.3. *Let $\psi \in \mathrm{Quad}$ and denote by ψ^{conv} its convex envelope, then*

$$\inf_{\tau\in\mathcal{T}^T} \mathbb{E}\,\psi(B_\tau) \;\to\; \inf_{\tau\in\mathcal{T}^\infty}\mathbb{E}\,\psi(B_\tau) \;=\; \psi^{conv}(0), \quad as \;\; T\to\infty.$$

Proof. Let $a \le 0 \le b$ be two constants and $\tau_{a,b} := \inf\big\{t : B_t \notin (a,b)\big\}$. We first notice that $\tau_{a,b} \in \mathcal{T}^\infty$ since $\mathbb{E}[\tau_{a,b}] = \lim_{t\to\infty}\mathbb{E}[\tau_{a,b}\wedge t] = \lim_{t\to\infty}\mathbb{E}[B^2_{\tau_{a,b}\wedge t}] \le (a^2+b^2) < \infty$. Hence by (6.3.14), $\mathbb{E}[B_{\tau_{a,b}}] = 0$, which implies that $\mathbb{P}(B_{\tau_{a,b}} = a) = \frac{b}{b-a}$ and $\mathbb{P}(B_{\tau_{a,b}} = b) = \frac{-a}{b-a}$. Therefore,

$$\inf_{\tau\in\mathcal{T}^\infty}\mathbb{E}\psi(B_\tau) \;\le\; \inf_{a<0<b}\mathbb{E}\psi(B_{\tau_{a,b}}) \;=\; \inf_{a<0<b}\Big(\frac{b}{b-a}\psi(a) + \frac{-a}{b-a}\psi(b)\Big) \;=\; \psi^{conv}(0).$$

On the other side, for every $\tau \in \mathcal{T}^\infty$, by Jensen's inequality together with the fact that $\mathbb{E}[B_\tau] = 0$ from (6.3.14), it follows that $\psi^{conv}(x) \le \mathbb{E}[\psi^{conv}(x + B_\tau)] \le \mathbb{E}[\psi(x + B_\tau)]$, and therefore,

$$\inf_{\tau\in\mathcal{T}^\infty}\mathbb{E}\psi(B_\tau) \;=\; \psi^{conv}(0).$$

Finally, the convergence of $\inf_{\tau \in \mathcal{T}^T} \mathbb{E}\psi(B_\tau)$ to $\inf_{\tau \in \mathcal{T}^\infty} \mathbb{E}\psi(B_\tau)$ as $T \to \infty$ is a direct consequence of (6.3.13) in Lemma 6.3.2. □

With the above two lemmas, we can now give a first analysis on $u(\phi)$ as well as U defined in (6.2.6).

Corollary 6.3.1. *Let $\phi \in \mathrm{Quad}$ and $(a,b) \in \mathbb{R}^2$, then $u(\phi) = u(\phi_{a,b})$, where $\phi_{a,b}$ is given by $\phi_{a,b}(x) := \phi(x) + ax + b$.*

Proof. By the definition of λ_0^ϕ in (6.2.5) together with Lemma 6.3.2, it follows that $\lambda_0^{\phi_{a,b}}(x) = \lambda_0^\phi(x) + ax + b$. Moreover, as discussed in Remark 6.2.1, $\mu_0(I) = \mu_1(I)$ for the identity function I. Then we get $u(\phi) = u(\phi_{a,b})$ by their definitions in (6.2.6). □

The next result can be viewed as a consequence of Dupire's [27] observation that *variance swap* is equivalent to a European option with payoff function $g(x) = x^2$. We give it in our context.

Corollary 6.3.2. *Let Assumptions 6.2.1, 6.2.2 hold true, $\psi \in \mathrm{Quad}$, $K \in \mathbb{R}$ and $g(t,x)$ be the payoff function, we define another payoff function $g_{K,\psi}$ by $g_{K,\psi}(t,x) := g(t,x) + Kt + \psi(x)$. Denote by $U(g)$ (resp. $U(g_{K,\psi})$) the no-arbitrage price upper bound defined in (6.2.6) associated with the payoff function g (resp. $g_{K,\psi}$). Then*

$$U(g_{K,\psi}) \quad = \quad U(g) \ + \ KC_0 \ + \ \mu_1(\psi), \tag{6.3.16}$$

where C_0 is given by (6.2.2). In particular, the upper bound of "variance swap" option is C_0, and the bound of a European option with payoff function $\psi(x)$ is given by $\mu_1(\psi)$.

Proof. Given $\phi \in \mathrm{Quad}$, we denote $\phi_{K,\psi}(x) := \phi(x) + \psi(x) + Kx^2$ which also belongs to Quad, then by (6.3.14)

$$\mathbb{E}\big[g_{K,\psi}(t+\tau, x+B_\tau) - \phi_{K,\psi}(x+B_\tau)\big] \quad = \quad \mathbb{E}\big[g^\phi(t+\tau, x+B_\tau)\big] - Kx^2, \quad \forall \tau \in \mathcal{T}^\infty.$$

It follows by the definition of U in (6.2.6) that $U(g_{K,\psi}) \geq U(g) + KC_0 + \mu_1(\psi)$. And moreover, by the arbitrariness of $K \in \mathbb{R}$, $\psi \in \mathrm{Quad}$ and symmetric relationship between g and $g_{K,\psi}$, we proved (6.3.16).

For the last statement, it follows by (6.3.16) that we only need to prove that $U(g^0) = 0$ with $g^0 \equiv 0$. Indeed, with the payoff function $g^0 \equiv 0$, we get immediately from (6.2.5) and (6.2.6) as well as Lemma 6.3.3 that

$$u(\phi) \quad = \quad - \ \mu_0(\phi^{conv}) \ + \ \mu_1(\phi) \quad \geq \quad \mu_1(\phi^{conv}) - \mu_0(\phi^{conv}) \quad \geq \quad 0,$$

where the last inequality comes from Assumption 6.2.1. Finally, we conclude with $U(g^0) = 0$ by the fact that $u(g^0) = 0$. □

6.3.3 Proofs of the convergence

Now we are ready to give the proof of the convergence results in Propositions 6.3.1, 6.3.2 and 6.3.3.

Proof of Proposition 6.3.1. First, with the positive constant L_0 given in Assumption 6.2.1, we have

$$g(0,x) \;\le\; g(t,x) \;\le\; g(0,x) \;+\; L_0 t.$$

Moreover, it is clear that U is monotone w.r.t. the payoff function g by its definition in (6.2.6). Then it follows by Corollary 6.3.2 that

$$\mu_1(g(0,\cdot)) \;\le\; U \;\le\; \mu_1(g(0,\cdot)) \;+\; L_0 C_0, \quad \text{with} \;\; C_0 \;\; \text{defined in} \quad (6.2.2).$$

Next, let us prove the equality (6.3.1) for U. Let $T \in \mathbb{R}^+$, $\tau_0 \in \mathcal{T}^T$ and $\phi \in$ Quad. By the dominated convergence theorem, it is easy to see that $x \mapsto \inf_{\tau \in \mathcal{T}^T} \mathbb{E}\phi(x+B_\tau)$ is continuous. This, together with the weak dynamic programming in Theorem 4.1 of Bouchard and Touzi [16], implies the dynamic programming principle:

$$\inf_{\tau_0 \le \tau \le T} \mathbb{E}\phi(x+B_\tau) \;=\; \mathbb{E}\Big[\operatorname{ess}\inf_{\tau_0 \le \tau \le T} \mathbb{E}\big[\phi(x+B_\tau)\big|\mathcal{F}_{\tau_0}\big]\Big].$$

Then for constants $\hat{T} > T$,

$$\lambda_0^\phi(x) \;=\; \sup_{\tau \in \mathcal{T}^\infty} \mathbb{E}\big[\, g^\phi(\tau, x+B_\tau)\,\big] \;\ge\; \sup_{\tau_0 \le \tau \le \hat{T}} \mathbb{E}\big[\, g(\tau, x+B_\tau) - \phi(x+B_\tau)\,\big].$$

By the increase of g in t and its convexity in x from Assumption 6.2.3, we have

$$\mathbb{E}\big[g(\tau, x+B_\tau)\big|\mathcal{F}_{\tau_0}\big] \;\ge\; \mathbb{E}\big[g(\tau_0, x+B_\tau)\big|\mathcal{F}_{\tau_0}\big] \;\ge\; g(\tau_0, x+B_{\tau_0}),$$

and hence

$$\lambda_0^\phi(x) \;\ge\; \mathbb{E}\big[g(\tau_0, x+B_{\tau_0})\big] \;-\; \mathbb{E}\Big[\inf_{\tau_0 \le \tau \le \hat{T}} \mathbb{E}\big[\phi(x+B_\tau)\big|\mathcal{F}_{\tau_0}\big]\Big].$$

Sending $\hat{T}$ to $+\infty$, by Lemma 6.3.3, it follows that

$$\lambda_0^\phi(x) \;\ge\; \mathbb{E}\;\big[\; g(\tau_0, x+B_{\tau_0}) \;-\; \phi^{conv}(x+B_{\tau_0})\;\big].$$

Thus, by arbitrariness of τ_0 in $\mathcal{T}^T$ as well as that of $T \in \mathbb{R}^+$, we get

$$\begin{aligned} \lambda_0^\phi(x) \;&\ge\; \lim_{T\to\infty} \sup_{\tau_0 \in \mathcal{T}^T} \mathbb{E}\;\big[\; g(\tau_0, x+B_{\tau_0}) \;-\; \phi^{conv}(x+B_{\tau_0})\;\big], \\ &=\; \sup_{\tau_0 \in \mathcal{T}^\infty} \mathbb{E}\;\big[\; g(\tau_0, x+B_{\tau_0}) \;-\; \phi^{conv}(x+B_{\tau_0})\;\big], \end{aligned}$$

where the last equality is a direct consequence of Lemma 6.3.2 since ϕ^{conv} is either of quadratic growth or equals to $-\infty$.

Finally, since $\phi \geq \phi^{conv}$, by the definition of u and U in (6.2.6), it is clear that the infimum in (6.2.6) can be taken on the collection of all convex functions in Quad. Moreover, by the property of $u(\phi)$ in Corollary 6.3.1, the infimum can be then taken on the collection of all positive convex functions ϕ in Quad such that $\phi(0) = 0$, i.e. $U = \inf_{\phi \in \mathrm{Quad}_0} u(\phi)$. We then proved (6.3.1). □

Proof of Proposition 6.3.2. Let us first recall that every function $\phi \in \mathrm{Quad}_0^K$ is nonnegative, convex such that $\phi(0) = 0$ and $\phi(x) \leq K(|x| \vee x^2)$. Given $\phi \in \mathrm{Quad}_0^K$, we denote $\phi_M := \phi \vee \phi_{K,M}$. Clearly, ϕ_M lies in $\mathrm{Quad}_0^{K,M}$ and $\lambda^{\phi_M} \leq \lambda^{\phi}$ since $\phi_M \geq \phi$. It follows from the definition of $u(\phi)$ in (6.2.6) and positivity of ϕ that

$$u(\phi_M) \;-\; u(\phi) \;\;\leq\;\; \mu_1(\phi_M) \;-\; \mu_1(\phi) \;\;\leq\;\; \mu_1(\phi_{K,M}).$$

This, together with the arbitrariness of $\phi \in \mathrm{Quad}_0^K$ and the fact that $\phi_M \in \mathrm{Quad}_0^{K,M}$, concludes the proof for (6.3.6). □

In preparation of the proof for Lemma 6.3.1 and Proposition 6.3.3, we first give a property for functions in $\mathrm{Quad}_0^{K,M}$.

Lemma 6.3.4. *Let Assumptions 6.2.2 and 6.2.3 hold true, L_0, M_0 be the constants given in Assumption 6.2.2, $K > L_0$, $M \geq M_0$ and $R = \big(1 + \sqrt{\frac{K}{K-L_0}}\,\big)M$. Given fixed $t \in \mathbb{R}^+$ and $\phi \in \mathrm{Quad}_0^{K,M}$, we denote*

$$\psi(x) \;\;:=\;\; -\; g^{\phi}(t,x) \;-\; L_0 x^2 \;\;=\;\; \phi(x) \;-\; g(t,x) \;-\; L_0 x^2.$$

Then $\psi^{conv}(x) = \psi(x)$ when $x \notin [-R, R]$.

Proof. By Assumption 6.2.2, we know that there are constants C_1, C_2 such that $x \mapsto g(t,x)$ is affine with derivative C_1 when $x \geq M$, and affine with derivative C_2 when $x \leq -M$. For fixed $t \in \mathbb{R}^+$, let χ be a continuous function defined on $\mathbb{R}$ by the following: χ is affine on intervals $[-2M, -M]$, $[-M, 0]$, $[0, M]$, $[M, 2M]$ and

$$\begin{cases} \chi(0) \;:=\; -\,g(t,0), \\ \chi(\pm M) \;:=\; -\,L_0 M^2 \;-\; g(t, \pm M), \\ \chi(\pm 2M) \;:=\; 4(K - L_0)M^2 \;-\; g(t, \pm 2M), \\ \chi(x) \;:=\; (K - L_0)x^2 \;-\; g(t, 2M) \;-\; C_1(x - 2M), \qquad x \geq 2M, \\ \chi(x) \;:=\; (K - L_0)x^2 \;-\; g(t, -2M) - C_2(x + 2M), \qquad x \leq -2M. \end{cases}$$

By Assumptions 6.2.2 and 6.2.3, we can verify that for every $\phi \in \mathrm{Quad}_0^{K,M}$ and the corresponding ψ defined in the statement of the lemma,

$$\psi(x) \begin{cases} \geq \;\chi(x), & \text{when } x \in [-2M, 2M], \\ = \;\chi(x), & \text{when } x \notin [-2M, 2M]. \end{cases}$$

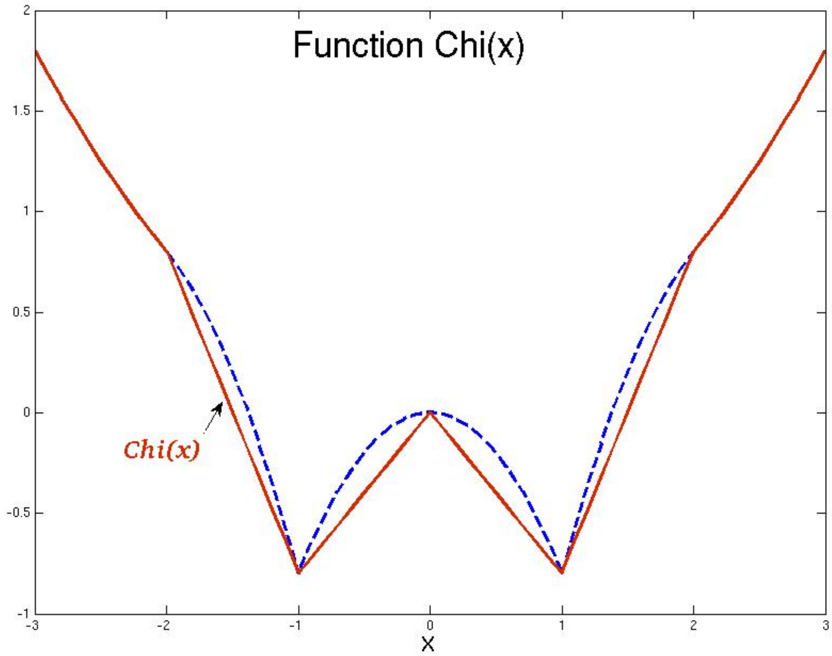

Figure 6.1: An example of function χ when $M = 1$.

Then given $x \notin [-R, R]$, it follows by a simple calculation that $\chi(y) \geq \chi(x) + \chi'(x)(y-x)$ for every $y \in \mathbb{R}$, which implies that $\chi^{conv}(x) = \chi(x)$. And hence $\psi(x) \geq \psi^{conv}(x) \geq \chi^{conv}(x) = \chi(x) = \psi(x)$ for $x \notin [-R, R]$. □

Proof of Lemma 6.3.1. We shall just show that $\lambda^\phi = \lambda^{\phi,\tau_R}$, since $\lambda^{\phi,T} = \lambda^{\phi,T,R}$ holds with the same arguments. Moreover, to prove $\lambda^\phi = \lambda^{\phi,\tau_R}$, it is enough to show that $\lambda^\phi \leq \lambda^{\phi,\tau_R}$ since its inverse inequality is obvious from the definition of λ^{ϕ,τ_R} in (6.3.8).

First, let us fix $t \in \mathbb{R}^+$ and $x \notin (-R, R)$, we denote $\psi_x(y) := -g^\phi(t, y) - L_0 y^2 + L_0 x^2$. Then by Lemma 6.3.4, we have $\psi_x^{conv}(x) = \psi_x(x) = -g^\phi(t, x)$. And it follows that for every $\tau \in \mathcal{T}^\infty$,

$$\begin{aligned} \mathbb{E}\left[\, g^\phi(t+\tau, x+B_\tau) \,\right] \;&\leq\; \mathbb{E}\left[\, g^\phi(t, x+B_\tau) \;+\; L_0\tau \,\right] \\ &=\; \mathbb{E}\left[\, g^\phi(t, x+B_\tau) \;+\; L_0(x+B_\tau)^2 \;-\; L_0 x^2 \,\right] \\ &=\; -\,\mathbb{E}\,\psi_x(x+B_\tau) \;\leq\; -\,\psi_x^{conv}(x) \;=\; g^\phi(t,x), \end{aligned} \tag{6.3.17}$$

which implies that $\lambda^\phi(t, x) \leq \lambda^{\phi,\tau_R}(t, x)$ for every $x \notin (-R, R)$ since in this case $\tau_x^R = 0$.

Next, for every $\tau \in \mathcal{T}^\infty$ and $x \in [-R, R]$, we have according to (6.3.17) that

$$\begin{aligned} &\mathbb{E}\left[\, g^\phi(t+\tau, x+B_\tau) \,\right] \\ =\;& \mathbb{E}\left[g^\phi(t+\tau, x+B_\tau)\mathbf{1}_{\tau\leq\tau_x^R} \right] \;+\; \mathbb{E}\left[\, \mathbb{E}\left[g^\phi(t+\tau, x+B_\tau)\mathbf{1}_{\tau>\tau_x^R} \;\middle|\; \mathcal{F}_{\tau\wedge\tau_x^R} \,\right] \right] \\ \leq\;& \mathbb{E}\left[\, g^\phi(t+\tau\wedge\tau_x^R, x+B_{\tau\wedge\tau_x^R}) \,\right], \end{aligned}$$

which implies that $\lambda^{\phi}(t,x) \le \lambda^{\phi,\tau_R}(t,x)$ for all $x \in [-R,R]$. □

Proof of Proposition 6.3.3. We first derive an estimate on stopping times inferior to τ_x^R, borrowed from Carlier and Galichon's [18] Lemma 5.2. Let $x \in [-R,R]$, then for every stopping time $\tau \le \tau_x^R$, we have

$$\mathbb{P}(\tau \ge T) \;\le\; \mathbb{P}\big(\tau_x^R \ge T\big) \;\le\; \Pi_{1\le n\le T}\mathbb{P}\big(|B_n - B_{n-1}| \le 2R\big) \;\le\; e^{-\delta(T-1)}. \qquad (6.3.18)$$

Recall that $\mathbb{E}\big[(x+B_\tau)^2\big] = x^2 + \mathbb{E}[\tau]$, $\forall \tau \le \tau_x^R$ from (6.3.14). Then by the definitions of λ^{ϕ,τ_R} and $\lambda^{\phi,T,R}$ in (6.3.9), for every $\phi \in \mathrm{Quad}_0^{K,M}$,

$$\begin{aligned}\lambda^{\phi,\tau_R}(0,x) - \lambda^{\phi,T,R}(0,x) \;&\le\; \sup_{\tau\le\tau_x^R} \mathbb{E}\Big[\, g^{\phi}(\tau, x+B_\tau) \;-\; g^{\phi}(\tau\wedge T, x+B_{\tau\wedge T})\,\Big] \\ &=\; \sup_{\tau\le\tau_x^R} \mathbb{E}\Big[\, \psi(\tau\wedge T, x+B_{\tau\wedge T}) - \psi(x+B_\tau,\tau)\,\Big],\end{aligned}$$

where $\psi(t,x) := -g^{\phi}(t,x) - L_0 x^2 + L_0 t$. Clearly, ψ increases in t and $|\psi(t,x_1)-\psi(t,x_2)| \le 2(K+2L_0)(R^2\vee 1)$, $\forall x_1,\ x_2 \in [-R,R]$ by Assumptions 6.2.2 and 6.2.3. Therefore,

$$\begin{aligned}\lambda^{\phi,\tau_R}(0,x) - \lambda^{\phi,T,R}(0,x) \;&\le\; \sup_{\tau\le\tau_x^R} \mathbb{E}\Big[\,\big|\psi(\tau\wedge T, x+B_{\tau\wedge T}) \;-\; \psi(\tau\wedge T, x+B_\tau)\big|\,\Big] \\ &=\; \sup_{\tau\le\tau_x^R} \mathbb{E}\Big[\,\big|\psi(T, x+B_T) \;-\; \psi(T, x+B_\tau)\big|\mathbf{1}_{\tau\ge T}\,\Big] \\ &\le\; \sup_{\tau\le\tau_x^R} 2\,(K+2L_0)\,(R^2\vee 1)\,\mathbb{P}(\tau\ge T) \\ &\le\; Le^{-\delta(T-1)},\end{aligned}$$

where the last inequality is from (6.3.18). Finally, by arbitrariness of $\phi \in \mathrm{Quad}_0^{K,M}$ together with Lemma 6.3.1, we prove (6.3.11). □

Finally, we finish this section by providing the proof of Theorem 6.2.1, where we use the weak dynamic programming technique proposed in Bouchard and Touzi [16].

Proof of Theorem 6.2.1. For the semicontinuity of λ^{ϕ}, we observe that by Assumption 6.2.2, for a fixed $\phi \in \mathrm{Quad}$, there is a constant $C \in \mathbb{R}^+$ such that

$$\big|\, g^{\phi}(t+\tau, x+B_\tau)\,\big| \;\le\; C\big(\,1 + t + \tau + x^2 + B_\tau^2\,\big).$$

Thus for a fixed $\tau \in \mathcal{T}^\infty$, $(t,x) \mapsto \mathbb{E}\big[g^{\phi}(t+\tau, x+B_\tau)\big]$ is continuous by the dominated convergence theorem together with (6.3.14). It follows immediately by its definition in (6.2.5) that λ^{ϕ} is lower-semicontinuous since it is represented as the supremum of a family of continuous function.

Next, to prove $\overline{U} = U$, it is enough to prove that $\sup_{\tau\in\mathcal{T}} \mathbb{E}\big[g(\tau, W_\tau) - \phi(W_\tau)\big] = \mu_0(\lambda_0^{\phi})$, which is in fact the dynamic programming principle for the optimization problem $\bar{u}$ given by (6.1.5). First, given $(W,\tau,\mathbb{P})$ where W is the Brownian motion such that $\mathbb{P}\circ W_0^{-1} = \mu_0$ and $\tau \in \mathcal{T}$ defined by (6.1.6), we consider a family of conditional probability $(\mathbb{P}_x)_{x\in\mathbb{R}}$ of $\mathbb{P}$ w.r.t. W_0. It is clear that for μ−almost every $x \in \mathbb{R}$, $(W - x := (W_t - x)_{t\ge 0}, \tau, \mathbb{P}_x)$ induces a stopping time in $\mathcal{T}^\infty$ on the Brownian motion B. Then it follows that $\mathbb{E}\big[g(\tau, W_\tau) - \phi(W_\tau)\big] \le \mu_0(\lambda_0^\phi)$, and hence that $\sup_{\tau\in\mathcal{T}} \mathbb{E}\big[g(\tau, W_\tau) - \phi(W_\tau)\big] \le \mu_0(\lambda_0^\phi)$.

Now, let us prove the other inequality, using the fact that λ_0^ϕ is lower-semicontinuous. Following the proof of Theorem 4.1 of Bouchard and Touzi [16], for every $\varepsilon > 0$, there is a countable subdivision $\Delta = (\Delta_n)_{n\ge1}$ of $\mathbb{R}$, and a sequence of stopping times $(\tau_n^\varepsilon)_{n\ge1}$ in $\mathcal{T}^\infty$ such that $\mathbb{E}\big[(g^\phi(\tau_n^\varepsilon, x + B_{\tau_n^\varepsilon})\big] \ge \lambda_0^\phi(x) - \varepsilon$, $\forall x \in \Delta_n$. We then construct $\tau^\varepsilon \in \mathcal{T}$ by $\tau^\varepsilon(W) := \sum_{n=1}^\infty \tau_n^\varepsilon(W - W_0)\mathbf{1}_{W_0\in\Delta_n}$, so that $\mathbb{E}\big[g^\phi(\tau^\varepsilon, W_{\tau^\varepsilon})\big] \ge \mu_0(\lambda_0^\phi) - \varepsilon$. By the arbitrariness of $\varepsilon > 0$, we then get that $\sup_{\tau\in\mathcal{T}} \mathbb{E}\big[g(\tau, W_\tau) - \phi(W_\tau)\big] \ge \mu_0(\lambda_0^\phi)$, which concludes the proof. □

6.4 The numerical approximation

We shall propose a numerical method to approximate $U^{K,M,T}$. The idea is to compute $\lambda^{\phi,T,R}$ with a finite difference numerical scheme, and then solve the minimization problem (6.3.10) with an iterative algorithm. Concretely, we shall first propose a discrete system characterized by $h = (\Delta t, \Delta x)$, on which there is a discrete optimization problem with value $U_h^{K,M,T}$ close to $U^{K,M,T}$. Then we use the gradient projection algorithm to solve the discrete optimization problem of $U_h^{K,M,T}$.

6.4.1 A finite difference approximation

Let T, $R > 2M$ be constants in $\mathbb{R}^+$ and $(l, r, m) \in \mathbb{N}^3$, $h = (\Delta x, \Delta t) \in (\mathbb{R}^+)^2$ such that $l\Delta t = T$, $r\Delta x = R$ and $m\Delta x = M$. Denote $x_i := i\Delta x$ and $t_k := k\Delta t$ and define the discrete grid:

$$\mathcal{N} \;:=\; \{x_i \;:\; i \in \mathbb{Z}\}, \qquad \mathcal{N}_R \;:=\; \mathcal{N} \cap [-R, R],$$

$$\mathcal{M}_{T,R} \;:=\; \big\{(t_k, x_i) \;:\; (k,i) \in \mathbb{Z}^+ \times \mathbb{Z}\big\} \cap \big([0,T] \times [-R,R]\big),$$

The terminal set, boundary set as well as interior set of $\mathcal{M}_{T,R}$ are denoted by

$$\partial_T\mathcal{M}_{T,R} \;:=\; \big\{\; (T, x_i) \;: -r \le i \le r \;\big\}, \qquad \partial_R\mathcal{M}_{T,R} \;:=\; \big\{\; (t_k, \pm R) \;:\; 0 \le k \le l \;\big\},$$

$$\mathring{\mathcal{M}}_{T,R} \;\;:=\;\; \mathcal{M}_{T,R} \setminus \big(\partial_R\mathcal{M}_{T,R} \cup \partial_T\mathcal{M}_{T,R}\big).$$

Given a function $w(t,x)$ defined on $\mathcal{M}_{T,R}$, we introduce the discrete derivative of w by

$$D^2 w(t_k, x_i) \;\;:=\;\; \frac{w(t_k, x_{i+1}) - 2w(t_k, x_i) + w(t_k, x_{i-1})}{\Delta x^2}.$$

Then with function φ defined on $\mathcal{N}_R$ and the notation

$$g^\varphi(t_k, x_i) \;\;:=\;\; g(t_k, x_i) \;-\; \varphi(x_i) \tag{6.4.1}$$

as well as $\theta \in [0,1]$, we define $\lambda_h^{\varphi,T,R}$ as the solution of the finite difference scheme of variational inequality (6.3.12) on $\mathcal{M}_{T,R}$:

$$\begin{cases} \lambda_h^{T,R}(t_{k+1},x_i) \;-\; \tilde{\lambda}_h^{T,R}(t_k,x_i) \\ \qquad +\; \frac{1}{2}\Delta t\,\Big(\theta\; D^2\tilde{\lambda}_h^{T,R}(t_k,x_i) \;+\; (1-\theta)\; D^2\lambda_h^{T,R}(t_{k+1},x_i)\Big) \;=\; 0, \\ \lambda_h^{T,R}(t_k,x_i) \;=\; \max\Big(\; g^{\varphi}(t_k,x_i)\;,\; \tilde{\lambda}_h^{T,R}(t_k,x_i)\;\Big), \qquad (t_k,x_i) \in \mathring{\mathcal{M}}_{T,R}, \\ \lambda_h^{T,R}(t_k,x_i) \;=\; g^{\varphi}(t_k,x_i), \qquad\qquad (t_k,x_i) \in \partial_T\mathcal{M}_{T,R} \cup \partial_R\mathcal{M}_{T,R}. \end{cases} \tag{6.4.2}$$

We notice that the above $\theta-$scheme has clearly a unique solution. And it is a consistant scheme for (6.3.12) in sense of Barles and Souganidis [6]. To see this, it is enough to rewrite the second equation of (6.4.2) as

$$\min\Big(\; \lambda_h^{T,R} - g^{\varphi},\;\; \frac{\lambda_h^{T,R} - \tilde{\lambda}_h^{T,R}}{\Delta t}\;\Big)(t_k,x_i) \;\;=\;\; 0.$$

We shall assume in addition that the discretization parameters $h = (\Delta t, \Delta x)$ satisfy the CFL condition

$$(1-\theta)\,\frac{\Delta t}{\Delta x^2} \;\;\le\;\; 1. \tag{6.4.3}$$

Then the finite difference scheme (6.4.2) is monotone in sense of [6], and the numerical solution $\lambda_h^{\varphi,T,R}$ converges to $\lambda^{\phi,T,R}$ given $\varphi := \phi|_{\mathcal{N}}$ by the results of [6].

Remark 6.4.1. *The discrete system (6.4.2) is the θ-scheme for variational inequality (6.3.12) with Dirichlet boundary condition $g(x,t) - \varphi(x)$ on $\partial_T\mathcal{M}_{T,R} \cup \partial_R\mathcal{M}_{T,R}$. It is well-known that when the finite difference scheme is explicit (i.e. $\theta = 0$) and the CFL condition $\frac{\Delta t}{\Delta x^2} \le 1$ holds, it can be interpreted as the dynamic programming principle for a system on a Markov chain Λ (see e.g. Kushner [42]). This interpretation holds also true for general θ-scheme, as we shall see later in the proof of Proposition 6.4.2.*

We next introduce a natural approximation of $u_T(\phi)$ in (6.3.10):

$$u_{h,T}(\varphi) \;\;:=\;\; \mu_0\big(\mathrm{lin}^R[\lambda_{h,0}^{\varphi,T,R}]\big) \;+\; \mu_1\big(\mathrm{lin}^R[\varphi]\big), \tag{6.4.4}$$

where $\lambda_{h,0}^{\varphi,T,R}(\cdot) := \lambda_h^{\varphi,T,R}(0,\cdot)$, and for every function φ defined on $\mathcal{N}_R$, $\mathrm{lin}^R[\varphi]$ denotes the linear interpolation of φ extended by zero outside $[-R,R]$.

Assumption 6.4.1. *There are constants $(\rho_1,\rho_2,L_{K,M,T}) \in (\mathbb{R}^+)^3$ which are independent of $h = (\Delta t, \Delta x)$ such that*

$$\mu_0\Big(\;\Big|\lambda_0^{\phi,T,R}\mathbf{1}_{[-R,R]} - \mathrm{lin}^R[\lambda_{h,0}^{\varphi,T,R}]\Big|\;\Big) \;\;\le\;\; L_{K,M,T}\;\big(\Delta x^{\rho_1} + \Delta t^{\rho_2}\big), \tag{6.4.5}$$

$$\textit{for every } \phi \in \mathrm{Quad}_0^{K,M} \textit{ and } \varphi = \phi|_{\mathcal{N}_R}.$$

Remark 6.4.2. *When $\theta = 1$, (6.4.2) is the implicit scheme for (6.3.12), then Assumption 6.4.1 holds true with $\rho_1 = \frac{1}{2}$ and $\rho_2 = \frac{1}{4}$ in sprirt of the analysis of Krylov [41].*

When $\theta = 0$ and the CFL condition (6.4.3) is true, (6.4.2) is a monotone explicit scheme, then in spirit of Barles and Jakobsen [5], Assumption 6.4.1 holds with $\rho_1 = \frac{1}{10}$ and $\rho_2 = \frac{1}{5}$.

Let $\mathrm{Quad}_{0,h}^{K,M}$ be the collection of all functions on the grid $\mathcal{N}_R$ defined as restrictions of functions in $\mathrm{Quad}_0^{K,M}$:

$$\mathrm{Quad}_{0,h}^{K,M} \quad := \quad \left\{ \varphi := \phi|_{\mathcal{N}_R} \text{ for some } \phi \in \mathrm{Quad}_0^{K,M} \right\}, \tag{6.4.6}$$

we can then provide a discrete approximation for $U^{K,M,T}$ in (6.3.10):

$$U_h^{K,M,T} \quad := \quad \inf_{\varphi \in \mathrm{Quad}_{0,h}^{K,M}} u_{h,T}(\varphi). \tag{6.4.7}$$

Let $B(\mathcal{N}_R)$ denote the set of all bounded functions defined on the grid $\mathcal{N}_R$, then clearly

$$\mathrm{Quad}_{0,h}^{K,M} = \Big\{ \varphi \in B(\mathcal{N}_R) \text{ nonnegative, convex satisfying } \varphi(0) = 0,\ \varphi(x_i) = Kx_i^2, \\ \text{for all } 2m \leq |i| \leq r, \text{ and } |\varphi(x_{i+1}) - \varphi(x_i)| \leq 4KM\Delta x,\ \forall -2m < i \leq 2m \Big\}. \tag{6.4.8}$$

Proposition 6.4.1. *Let Assumptions 6.2.2, 6.4.1 hold true, then with the same constants $L_{K,M,T}$, ρ_1, ρ_2 introduced in Assumption 6.4.1,*

$$\left| U^{K,M,T} - U_h^{K,M,T} \right| \ \leq \ L_{K,M,T}\big(\Delta x^{\rho_1} + \Delta t^{\rho_2}\big) + 4KR\Delta x + (\mu_0 + \mu_1)(\phi_K^R), \tag{6.4.9}$$

where $\phi_K^R(x) := Kx^2 \mathbf{1}_{|x|>R}$.

Proof. First, given $\phi \in \mathrm{Quad}_0^{K,M}$ which is $4KR-$Lipschitz, we introduce $\varphi := \phi|_{\mathcal{N}_R} \in \mathrm{Quad}_{0,h}^{K,M}$ so that $\big|\mathrm{lin}^R[\varphi] - \phi\big|_{L^\infty([-R,R])} \leq 4KR\Delta x$. Then it follows by Assumption 6.4.1 that $|u_T(\phi) - u_{h,T}(\varphi)| \ \leq L_{K,M,T}(\Delta x^{\rho_1} + \Delta t^{\rho_2}) + 4KR\Delta x + \ (\mu_0 + \mu_1)(\phi_K^R)$, and hence

$$U^{K,M,T} - U_h^{K,M,T} \quad \leq \quad L_{K,M,T}\ \big(\Delta x^{\rho_1} + \Delta t^{\rho_2}\big) \ + \ 4KR\Delta x + \ (\mu_0 + \mu_1)(\phi_K^R).$$

Next, given $\varphi \in \mathrm{Quad}_{0,h}^{K,M}$, we take $\phi := \mathrm{lin}^R[\varphi] + \phi_K^R \in \mathrm{Quad}_0^{K,M}$. It follows by Assumption 6.4.1 that $|u_T(\phi) - u_{h,T}(\varphi)| \ \leq L_{K,M,T}(\Delta x^{\rho_1} + \Delta t^{\rho_2}) + \ (\mu_0 + \mu_1)(\phi_K^R)$, and therefore,

$$U_h^{K,M,T} - U^{K,M,T} \quad \leq \quad L_{K,M,T}\ \big(\Delta x^{\rho_1} + \Delta t^{\rho_2}\big) + \ (\mu_0 + \mu_1)(\phi_K^R).$$

□

6.4.2 Gradient projection algorithm

As we can easily observe from its definition in (6.2.6) that $\phi \mapsto u(\phi)$ is convex since it is represented as the supremum of a family of linear map, we shall show that $\varphi \mapsto u_{h,T}(\varphi)$ is also convex, then a natural candidate for the resolution of $U_h^{K,M,T} = \inf_{\varphi \in \mathrm{Quad}_{0,h}^{K,M}} u_{h,T}(\varphi)$ in (6.4.7) is the gradient projection algorithm. Recall that $B(\mathcal{N}_R)$ denotes the collection of all bounded function on $\mathcal{N}_R$.

Proposition 6.4.2. *Under the CFL condition (6.4.3), the function $\varphi \mapsto u_{h,T}(\varphi)$ is convex.*

Proof. Let us first rewrite the finite differences scheme (6.4.2) into a vector system. Denote $\alpha := \frac{\Delta t}{2\Delta x^2}$, $\lambda_k := \big(\lambda_h^{\varphi,T,R}(t_k,x_i)\big)_{-r\le i\le r}$, $\tilde\lambda_k := \big(\tilde\lambda_h^{\varphi,T,R}(t_k,x_i)\big)_{-r\le i\le r}$ and $q_k := \big(g^\varphi(t_k,x_i)\big)_{-r\le i\le r} \in \mathbb{R}^{2r+1}$. Let I_{2r+1} denote the $(2r+1)\times(2r+1)$ identity matrix, Π and $b_k \in \mathbb{R}^{2r+1}$ be defined by

$$\Pi := \begin{pmatrix} 0 & 0 & 0 & 0 & & & 0 \\ 1 & -2 & 1 & 0 & & & \\ 0 & 1 & -2 & 1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & 1 & -2 & 1 & 0 \\ & & & 0 & 1 & -2 & 1 \\ 0 & & & 0 & 0 & 0 & 0 \end{pmatrix}, \ b_k := \begin{pmatrix} q_k(-r) - \lambda_{k+1}(-r) \\ 0 \\ \vdots \\ 0 \\ q_k(r) - \lambda_{k+1}(r) \end{pmatrix},$$

and $\Theta := \big[I_{2r+1} - \theta\alpha\Pi\big]^{-1}\big[I_{2r+1} + (1-\theta)\alpha\Pi\big]$, then scheme (6.4.2) can be rewritten as

$$\tilde\lambda_k \ = \ \Theta\lambda_{k+1} \ + \ b_k, \quad \text{and} \quad \lambda_k \ = \ \tilde\lambda_k \vee q_k. \tag{6.4.10}$$

Under CFL condition (6.4.3), one can verify that the above scheme is monotone, i.e. every element of Θ is positive, and moreover, $\Theta\mathbf{1} = \mathbf{1}$, where $\mathbf{1} := (1,\cdots,1)^T \in \mathbb{R}^{2r+1}$. It follows that Θ can be the probability transition matrix of some Markov chain Λ, whose state space is the grid $\mathcal{N}_R$ with absorbing boundary. Let $\mathcal{T}_h^R$ denote the collection of all stopping times τ_h on Λ, then $\lambda_h^{\varphi,T,R}$ can be represented as solutions of an optimal stopping problem on Λ:

$$\lambda_h^{\varphi,T,R}(t_k,x_i) \ = \ \sup_{\tau_h\in\mathcal{T}_h^R,\ \tau_h\ge t_k} \mathbb{E}\Big[\ g^\varphi(\Lambda_{\tau_h},\tau_h)\ \Big|\ \Lambda_{t_k} = x_i\ \Big].$$

Now given a stopping time $\tau_h \in \mathcal{T}_h^R$, we introduce the function $\lambda_{h,0}^{\varphi,T,R,\tau_h}$ defined on $\mathcal{N}_R$:

$$\lambda_{h,0}^{\varphi,T,R,\tau_h}(x_i) \ := \ \mathbb{E}\Big[\ g^\varphi(\Lambda_{\tau_h},\tau_h)\ \Big|\ \Lambda_0 = x_i\ \Big].$$

Then $u_{h,T}$ has an equivalent representation:

$$u_{h,T}(\varphi) = \sup_{\tau_h\in\mathcal{T}_h^R} \bar u_{h,T}^{\tau_h}(\varphi) \ := \sup_{\tau_h\in\mathcal{T}_h^R} \mu_0\big(\mathrm{lin}^R[\lambda_{h,0}^{\varphi,T,R,\tau_h}]\big) \ + \ \mu_1\big(\mathrm{lin}^R[\varphi]\big). \tag{6.4.11}$$

Clearly, for every τ_h, $\varphi \mapsto \bar u_{h,T}^{\tau_h}(\varphi)$ is linear, and finally it follows by (6.4.11) that $\varphi \mapsto u_{h,T}(\varphi)$ is convex. □

Remark 6.4.3. *In the above Markov chain system (6.4.11), given $\varphi \in B(\mathcal{N}_R)$, one can define an optimal stopping time $\tau_h(\varphi)$ by*

$$\tau_h(\varphi) \ := \ \inf\Big\{\ t_k \ : \ \lambda_h^{\varphi,T,R,\tau_h}(t_k,\Lambda_{t_k}) \ = \ g^\varphi(t_k,\Lambda_{t_k})\ \Big\}, \tag{6.4.12}$$

and clearly,

$$u_{h,T}(\varphi) \ = \sup_{\tau_h\in\mathcal{T}_R^h} \bar u_{h,T}^{\tau_h}(\varphi) \ = \ \bar u_{h,T}^{\tau_h(\varphi)}(\varphi). \tag{6.4.13}$$

Now we are ready to give the gradient projection algorithm for $U_h^{K,M,T}$ in (6.4.7). Given $\varphi \in B(\mathcal{N}_R)$, we denote by $P_{\mathrm{Quad}_{0,h}^{K,M}}[\varphi]$ its projection on $\mathrm{Quad}_{0,h}^{K,M}$. Of course, such a projection depends on the norm equipped on $B(\mathcal{N}_R)$, which is an important issue to be discussed later.

Let $\gamma = (\gamma_n)_{n\geq 0}$ be a sequence of positive real numbers, we propose the following algorithm:

Algorithm 6.4.1. *For optimization problem (6.4.7):*

- *1, Let $\varphi_0 := \phi_{K,M}|_{\mathcal{N}_R}$, where $\phi_{K,M}$ is defined in (6.3.7).*
- *2, Given φ_n, compute $u_{h,T}(\varphi_n)$ and a sub-gradient $\nabla u_{h,T}(\varphi_n)$.*
- *3, Let $\varphi_{n+1} := P_{\mathrm{Quad}_{0,h}^{K,M}}\big[\varphi_n - \gamma_n \nabla u_{h,T}(\varphi_n)\big]$.*
- *4, Go back to step 2.*

In the following, we shall discuss essentially three issues: the computation of sub-gradient $\nabla u_{h,T}(\varphi)$, the projection from $B(\mathcal{N}_R)$ to $\mathrm{Quad}_{0,h}^{K,M}$ and the convergence of the above gradient projection algorithm.

6.4.2.1 Computation of sub-gradient

Let us fix $\varphi \in B(\mathcal{N}_R)$, we then denote by $(p^j, \tilde{p}^j)$ the unique solution of the following linear system on $\mathcal{M}_{T,R}$:

$$\begin{cases} p^j(t_k, x_i) \;=\; -\,\delta_{i,j}, & (t_k, x_i) \in \partial_T \mathcal{M}_{T,R} \cup \partial_R \mathcal{M}_{T,R}, \\ p^j(t_{k+1}, x_i) - \tilde{p}^j(t_k, x_i) + \frac{1}{2}\Delta t\, \big(\theta D^2 \tilde{p}^j(t_k, x_i) + (1-\theta) D^2 p^j(t_{k+1}, x_i)\big) = 0, & \\ p^j(t_k, x_i) = \begin{cases} \tilde{p}^j(t_k, x_i), & \text{if } \lambda_h^{\varphi,T,R}(t_k, x_i) > g^{\varphi}(t_k, x_i), \\ -\,e_j(x_i), & \text{otherwise}. \end{cases} & (t_k, x_i) \in \mathring{\mathcal{M}}_{T,R}. \end{cases} \tag{6.4.14}$$

where $e_j \in B(\mathcal{N}_R)$ is defined by $e_j(x_i) := \delta_{i,j} = \begin{cases} 1, & \text{if } \; i = j, \\ 0, & \text{otherwise}. \end{cases}$ Denote $p_0^j := p^j(0, \cdot)$.

Proposition 6.4.3. *Let CFL condition (6.4.3) hold true, then the vector*

$$\nabla u_{h,T}(\varphi) \quad := \quad \big(\; \mu_0(\mathrm{lin}^R[p_0^j]) \;+\; \mu_1(\mathrm{lin}^R[e_j]) \;\big)_{-2m \leq j \leq 2m} \tag{6.4.15}$$

forms a sub-gradient of map $\varphi \mapsto u_{h,T}(\varphi)$.

Proof. Let us first consider the Markov chain Λ introduced in the proof of Proposition 6.4.2. By (6.4.13), we have for every perturbation $\Delta\varphi \in B(\mathcal{N}_R)$,

$$u_{h,T}(\varphi + \Delta\varphi) \quad = \quad \bar{u}_{h,T}^{\tau_h(\varphi+\Delta\varphi)}(\varphi + \Delta\varphi) \quad \geq \quad \bar{u}_{h,T}^{\tau_h(\varphi)}(\varphi + \Delta\varphi).$$

It follows still by (6.4.13) that

$$u_{h,T}(\varphi+\Delta\varphi) \;-\; u_{h,T}(\varphi) \;\;\geq\;\; \bar{u}_{h,T}^{\tau_h(\varphi)}(\varphi+\Delta\varphi) \;-\; \bar{u}_{h,T}^{\tau_h(\varphi)}(\varphi),$$

which implies that

$$\Big(\; \bar{u}_{h,T}^{\tau_h(\varphi)}(\varphi+e_j) \;-\; \bar{u}_{h,T}^{\tau_h(\varphi)}(\varphi) \;\Big)_{-r\leq j\leq r} \tag{6.4.16}$$

forms a sub-gradient of $u_{h,T}$ at φ since $\psi \mapsto \bar{u}_{h,T}^{\tau(\varphi)}(\psi)$ is linear by its definition in (6.4.11).

Finally, by the definition of $\tau_h(\varphi)$ in (6.4.12) as well as (6.4.2) and (6.4.14), it follows that

$$p^j(t_k,x_i) \;\;=\;\; -\;\mathbb{E}\;\Big[\; e_j\big(\Lambda_{\tau_h(\varphi)}\big) \;\Big|\; \Lambda_{t_k}=x_i \;\Big].$$

And hence the sub-gradient (6.4.16) coincides with $\nabla u_{h,T}(\varphi)$ defined in (6.4.15). □

6.4.2.2 Projection

To compute the projection $P_{\mathrm{Quad}_{0,h}^{K,M}}$ from $B(\mathcal{N}_R)$ to $\mathrm{Quad}_{0,h}^{K,M}$, we still need to specify the norm equipped on $B(\mathcal{N}_R)$. The easiest norm can be the common one defined by $|\varphi|^2 := \sum_{i=-r}^{r}\varphi_i^2$. However, the computation of the projection may be too complicate under this norm. In order to make the projection algorithm simpler, we shall introduce an invertible linear map $\mathcal{L}_R$ from $B(\mathcal{N}_R)$ to $\mathbb{R}^{2r+1}$, then equip on $B(\mathcal{N}_R)$ the norm $|\cdot|_R$ induced by the classical L^2-norm on $\mathbb{R}^{2r+1}$. Let $\mathcal{L}_R : B(\mathcal{N}_R)\to\mathbb{R}^{2r+1}$ be defined by

$$\xi_i \;\;=\;\; \begin{cases} \varphi(x_i) \;-\; \varphi(x_{i-1}), & \text{for } \;0<i\leq r,\\ \varphi(x_0), & \text{for } \;i=0,\\ \varphi(x_i) \;-\; \varphi(x_{i-1}), & \text{for } \;-r\leq i<0. \end{cases} \tag{6.4.17}$$

We define the norm $|\cdot|_R$ on $B(\mathcal{N}_R)$ (easily be verified) by

$$|\varphi|_R \;\;:=\;\; \big|\xi\big|_{L^2(\mathbb{R}^{2r+1})}, \quad \text{with } \xi := \mathcal{L}_R(\varphi), \quad \forall \varphi\in B(\mathcal{N}_R).$$

Denote

$$\begin{aligned} E_0^{K,M} \;&:=\; \Big\{\; \mathcal{L}_R\varphi \;\;:\;\; \varphi\in\mathrm{Quad}_0^{K,M} \;\Big\} \\ &=\; \Big\{\; \xi\in\mathbb{R}^{2r+1} \;:\; 0=\xi_0\leq\xi_{\pm1}\leq\cdots\leq\xi_{\pm 2m}\leq 4KM\Delta x, \\ &\qquad\qquad \xi_{\pm i}=K(x_{i+1}^2-x_i^2),\; \forall 2m<i\leq r \text{ and } \sum_{i=1}^{2m}\xi_i=\sum_{i=-1}^{-2m}\xi_i = 4KM^2 \;\Big\}. \end{aligned}$$

Then the projection $P_{\mathrm{Quad}_{0,h}^{K,M}}$ from $B(\mathcal{N}_R)$ to $\mathrm{Quad}_{0,h}^{K,M}$ under norm $|\cdot|_R$ is equivalent to the projection from $\mathbb{R}^{2r+1}$ to $E_0^{K,M}$ under the L^2-norm, which consists in solving a quadratic minimization problem:

$$\xi^z \;\;:=\;\; \arg\min_{\xi\in E_0^{K,M}} \;\sum_{i=-r}^{r}(z_i-\xi_i)^2, \quad \text{for a given } z\in\mathbb{R}^{2r+1}. \tag{6.4.18}$$

Clearly, for every $z \in \mathbb{R}^{2r+1}$, $\xi_0^z = 0$ and the above optimization problem (6.4.18) can be decomposed into two optimization problems:

$$\min_{\xi \in E_{0,+}^{K,M}} \sum_{i=1}^{2m} (z_i - \xi_i)^2 \quad \text{and} \quad \min_{\xi \in E_{0,-}^{K,M}} \sum_{i=-1}^{-2m} (z_i - \xi_i)^2, \tag{6.4.19}$$

where

$$E_{0,+}^{K,M} := \Big\{ \xi = (\xi_i)_{1 \le i \le 2m} \ : \ 0 \le \xi_1 \le \cdots \le \xi_{2m} \le 4KM\Delta x, \ \sum_{i=1}^{2m} \xi_i = 4KM^2 \Big\},$$

$$E_{0,-}^{K,M} := \Big\{ \xi = (\xi_i)_{-1 \ge i \ge -2m} \ : \ 0 \le \xi_{-1} \le \cdots \le \xi_{-2m} \le 4KM\Delta x, \ \sum_{i=-1}^{-2m} \xi_i = 4KM^2 \Big\}.$$

Here in place of optimization problem (6.4.19), we shall consider a similar but more general optimization problem and give an algorithm for it. Let $a = (a_i)_{1 \le i \le m} \in \mathbb{N}^m$ and $A \in \mathbb{R}^+$ such that $0 < A < \sum_{i=1}^m a_i$, we define

$$\mathcal{K}_m^a := \Big\{ \ \xi = (\xi_i)_{1 \le i \le m} \in \mathbb{R}^m \ : \ \xi_1 \le \cdots \le \xi_m \ \Big\},$$

$$\mathcal{K}_m^A := \Big\{ \ \xi = (\xi_i)_{1 \le i \le m} \in [0,1]^m \ : \ \sum_{i=1}^m a_i \xi_i = A \ \Big\}, \text{ and } \ \mathcal{K}_m^{a,A} := \mathcal{K}_m^a \cap \mathcal{K}_m^A.$$

The projection $P_{\mathcal{K}_m^{a,A}}(z)$ of $z \in \mathbb{R}^m$ to $\mathcal{K}_m^{a,A}$ is to solve the optimization problem

$$\xi_m^{a,A,z} := \arg \min_{\xi \in \mathcal{K}_m^{a,A}} \sum_{i=1}^m a_i (z_i - \xi_i)^2. \tag{6.4.20}$$

Similarly, one can also define the projection $P_{\mathcal{K}_m^a}$(resp. $P_{\mathcal{K}_m^A}$) by the optimization problem (6.4.20), where $\mathcal{K}_m^{a,A}$ in the formula is replaced by $\mathcal{K}_m^a$ (resp. $\mathcal{K}_m^A$), and the projected element $\xi_m^{a,A,z}$ is replaced by $\xi_m^{a,z}$ (resp. $\xi_m^{A,z}$).

In the following, we shall show that

$$P_{\mathcal{K}_m^{a,A}} = P_{\mathcal{K}_m^{a,A}} \circ P_{\mathcal{K}_m^a} = P_{\mathcal{K}_m^A} \circ P_{\mathcal{K}_m^a},$$

and give the algorithms for both $P_{\mathcal{K}_m^a}$ and $P_{\mathcal{K}_m^A}$. With these algorithms, one can deduce easily an algorithm for the projections $P_{E_{K,M}^+}$ and $P_{E_{K,M}^-}$. We just remark that similar algorithms are discussed in Page 143-145 of Edelsbrunner [28] in order to compute the convex envelope of a function defined on a discrete grid.

Given $a \in \mathbb{N}^m$ and $z \in \mathbb{R}^m$, we define $S^{a,z} \in \mathbb{R}^{\sum_{i=1}^m a_i}$ by $S_k^{a,z} := z_j$ for $\sum_{i=1}^{j-1} a_i < k \le \sum_{i=1}^{j} a_i$, and define the function $F^{a,z}$ on the grid $\mathbb{N} \cap [0, 1 + \sum_{i=1}^m a_i]$ by

$$F^{a,z}(0) := 0 \ \text{ and } \ F^{a,z}(k) := \sum_{i=1}^k S_i^{a,z} \ \text{ for } \ k = 1, \cdots, \sum_{i=1}^m a_i. \tag{6.4.21}$$

Lemma 6.4.1. *Suppose that we are given $z \in \mathbb{R}^m$ such that $z_k \geq z_{k+1}$, denote $\xi_m^{a,z} := P_{\mathcal{K}_m^a}(z)$ and $\xi_m^{a,A,z} := P_{\mathcal{K}_m^{a,A}}(z)$, then $(\xi_m^{a,z})_k = (\xi_m^{a,z})_{k+1}$ and $(\xi_m^{a,A,z})_k = (\xi_m^{a,A,z})_{k+1}$. And therefore, in this case, the projections $P_{\mathcal{K}_m^a}(z)$ and $P_{\mathcal{K}_m^{a,A}}(z)$ are equivalent to $P_{\mathcal{K}_{m-1}^{\tilde{a}}}(\tilde{z})$ and $P_{\mathcal{K}_{m-1}^{\tilde{a},A}}(\tilde{z})$ for*

$$\tilde{a}_i = \begin{cases} a_i, & 1 \leq i \leq k-1, \\ a_k + a_{k+1}, & i = k, \\ a_{i+1}, & k+1 \leq i \leq m-1, \end{cases} \quad \text{and} \quad \tilde{z}_i = \begin{cases} z_i, & 1 \leq i \leq k-1, \\ \frac{a_k z_k + a_{k+1} z_{k+1}}{a_k + a_{k+1}}, & i = k, \\ z_{i+1}, & k+1 \leq i \leq m-1, \end{cases} \tag{6.4.22}$$

in sense that $S^{a,\xi_m^{a,z}} = S^{\tilde{a},\xi_{m-1}^{\tilde{a},\tilde{z}}}$ and $S^{a,\xi_m^{a,A,z}} = S^{\tilde{a},\xi_{m-1}^{\tilde{a},A,\tilde{z}}}$, where $\xi_{m-1}^{\tilde{a},\tilde{z}} := P_{\mathcal{K}_{m-1}^{\tilde{a}}}(\tilde{z})$ and $\xi_{m-1}^{\tilde{a},A,\tilde{z}} := P_{\mathcal{K}_{m-1}^{\tilde{a},A}}(\tilde{z})$.

Proof. Given an arbitrary $\xi \in \mathbb{R}^m$ such that $\xi_{k+1} > \xi_k$, then there is $\varepsilon > 0$ satisfying that $\xi_{k+1} = \xi_k + (1 + \frac{a_k}{a_{k+1}})\varepsilon$. Define $\hat{\xi} \in \mathbb{R}^m$ by $\hat{\xi}_i = \begin{cases} \hat{\xi}_k + \varepsilon, & i = k, k+1, \\ \xi_i, & \text{otherwise}, \end{cases}$ one can show that

$$\sum_{i=1}^{m} a_i(\hat{\xi}_i - z_i)^2 < \sum_{i=1}^{m} a_i(\xi_i - z_i)^2. \tag{6.4.23}$$

Thus ξ cannot be the projection of z since $\xi \in \mathcal{K}_m^a$(resp. $\mathcal{K}_m^{a,A}$) implies that $\hat{\xi} \in \mathcal{K}_m^a$ (resp. $\mathcal{K}_m^{a,A}$). It follows by this contradiction that $(\xi_m^{a,z})_k = (\xi_m^{a,z})_{k+1}$ and $(\xi_m^{a,A,z})_k = (\xi_m^{a,A,z})_{k+1}$.

To show the inequality (6.4.23), we can verify that

$$\begin{aligned} &\sum_{i=1}^{m} a_i\ (\xi_i - z_i)^2 - \sum_{i=1}^{m} a_i\ (\hat{\xi}_i - z_i)^2 \\ = &\ a_k\ (\xi_k - z_k)^2 + a_{k+1}\ \left(\xi_k + (1 + \frac{a_k}{a_{k+1}})\varepsilon - z_{k+1}\right)^2 \\ &- a_k\ (\xi_k + \varepsilon - z_k)^2 - a_{k+1}\ (\xi_k + \varepsilon - z_{k+1})^2 \\ = &\ \frac{a_k}{a_{k+1}}\ (a_k + a_{k+1})\ \varepsilon^2 + 2\ a_k\ \varepsilon\ (z_k - z_{k+1}) > 0. \end{aligned}$$

Finally, the equivalence between $P_{\mathcal{K}_m^a}(z)$ (resp. $P_{\mathcal{K}_m^{a,A}}(z)$) and $P_{\mathcal{K}_{m-1}^{\tilde{a}}}(\tilde{z})$ (resp. $P_{\mathcal{K}_{m-1}^{\tilde{a},A}}(\tilde{z})$) is from the fact that for every ξ such that $\xi_k = \xi_{k+1}$, one has the decomposition

$$\sum_{i=1}^{m} a_i(z_i - \xi_i)^2 = \sum_{i=1}^{m-1} \tilde{a}_i(\tilde{z}_i - \tilde{\xi}_i)^2 + a_k z_k^2 + a_{k+1} z_{k+1}^2 - (a_k + a_{k+1})\frac{(z_k + z_{k+1})^2}{4},$$

where $\tilde{\xi}_i = \begin{cases} \xi_i, & i \leq k-1, \\ \xi_k, & i = k, k+1, \\ \xi_{i-1}, & k+2 \leq i \leq m-1. \end{cases}$ □

Lemma 6.4.1 gives an algorithm for projection $P_{\mathcal{K}_m^a}$ which finishes within less than m steps. The algorithm also simplifies the projection $P_{\mathcal{K}_m^{a,A}}$, as we can see in Proposition 6.4.4.

Algorithm 6.4.2. *For projection $P_{\mathcal{K}_m^a}(z)$:*

- *1, Given system parameters (m, a, z), stop if $m = 1$.*
- *2, Find k such that $z_k \geq z_{k+1}$, stop if it does not exist.*
- *3, With the found k in step 2, reduce parameters (m, a, z) to $(m-1, \tilde{a}, \tilde{z})$ as in equation (6.4.22).*
- *4, Go to 1.*

Proposition 6.4.4. *$P_{\mathcal{K}_m^{a,A}} = P_{\mathcal{K}_m^{a,A}} \circ P_{\mathcal{K}_m^a}$, and for every $z \in \mathbb{R}^m$. Moreover, $F^{a,\xi}$ (with $\xi := P_{\mathcal{K}_m^a}(z)$) is the convex envelope of $F^{a,z}$, where the functions $F^{a,\xi}$ and $F^{a,z}$ are define in (6.4.21)*

Proof. Suppose that the entrance data of Algorithm 6.4.2 is (m_1, a_1, z_1) and the exit data is (m_2, a_2, z_2), then clearly $P_{\mathcal{K}_{m_2}^{a_2}}(z_2) = z_2$. And by Lemma 6.4.1, we have $S^{a_1,\xi_1} = S^{a_2,z_2}$ (with $\xi_1 := P_{\mathcal{K}_{m_1}^{a_1}}(z_1)$) and $S^{a_1,\xi_1^A} = S^{a_2,\xi_2^A}$ (with $\xi_1^A := P_{\mathcal{K}_{m_1}^{a_1,A}}(z_1)$ and $\xi_2^A := P_{\mathcal{K}_{m_2}^{a_2,A}}(z_2)$), from which we deduce that $P_{\mathcal{K}_m^{a,A}} = P_{\mathcal{K}_m^{a,A}} \circ P_{\mathcal{K}_m^a}$.
To see that $F^{a,\xi}$ (with $\xi := P_{\mathcal{K}_m^a}(z)$) is the convex envelope of $F^{a,z}$, it is enough to verify that at every step in Algorithm 6.4.2, $F^{\tilde{a},\tilde{z}}$ is greater than the convex envelope of $F^{a,z}$. And at the exit, $F^{a,\xi}$ is a convex function. □

Now, we shall prove that $P_{\mathcal{K}_m^{a,A}} \circ P_{\mathcal{K}_m^a} = P_{\mathcal{K}_m^A} \circ P_{\mathcal{K}_m^a}$, for this prupose, it is enough to show that for every $z \in \mathcal{K}_m^a$, $P_{\mathcal{K}_m^{a,A}}(z) = P_{\mathcal{K}_m^A}(z)$. In fact, we shall give an algorithm of projection $P_{\mathcal{K}_m^A}(z)$ for $z \in \mathcal{K}_m^a$, and then verify that $P_{\mathcal{K}_m^A}(z) \in \mathcal{K}_m^{a,A}$.

Given $\nu \in \mathbb{R}$, denote by $z - \nu$ the sequence $(z_i - \nu)_{1 \leq i \leq m}$, and by z^ν the sequence $(z_i^\nu)_{1 \leq i \leq m} := (0 \vee (z_i - \nu) \wedge 1)_{1 \leq i \leq m}$.

Lemma 6.4.2. *Given $\nu \in \mathbb{R}$, $z \in \mathbb{R}^m$, then $P_{\mathcal{K}_m^{a,A}}(z) = P_{\mathcal{K}_m^{a,A}}(z - \nu)$ and $P_{\mathcal{K}_m^A}(z) = P_{\mathcal{K}_m^A}(z - \nu)$. If in addition $z \in \mathcal{K}_m^a$, then there is $\hat{\nu} \in \mathbb{R}$ such that $\sum_{i=1}^m a_i z_i^{\hat{\nu}} = A$ and $P_{\mathcal{K}_m^A}(z) = P_{\mathcal{K}_m^{a,A}}(z) = z^{\hat{\nu}}$. And it follows that $P_{\mathcal{K}_m^{a,A}} = P_{\mathcal{K}_m^{a,A}} \circ P_{\mathcal{K}_m^a} = P_{\mathcal{K}_m^A} \circ P_{\mathcal{K}_m^a}$.*

Proof. To prove that $P_{\mathcal{K}_m^{a,A}}(z) = P_{\mathcal{K}_m^{a,A}}(z - \nu)$ or $P_{\mathcal{K}_m^A}(z) = P_{\mathcal{K}_m^A}(z - \nu)$, it is enough to see that for every $\xi \in \mathbb{R}^m$ satisfying $\sum_{i=1}^m a_i \xi_i = A$, we have

$$\sum_{i=1}^m a_i (z_i - \nu - \xi_i)^2 \;=\; \sum_{i=1}^m a_i (z_i - \xi_i)^2 \;+\; \nu^2 \sum_{i=1}^m a_i \;-\; 2\nu \Big(\sum_{i=1}^m a_i z_i - A \Big).$$

For the existence of $\hat{\nu}$, it is enough to see that $\nu \mapsto \sum_{i=1}^m a_i z_i^\nu$ is continuous, and that $0 < A < \sum_{i=1}^m a_i$ is supposed at the beginning of the section. Clearly, by its definition, z^ν is the projected element of $z - \nu$ to $[0,1]^m$ in sense that $\xi_0 = z^\nu$ minimizes $\sum_{i=1}^m a_i (z_i - \nu - \xi_i)^2$ among all $\xi \in [0,1]^m$. Then for $z \in \mathcal{K}_m^a$, it is easy to verify that $z^{\hat{\nu}} \in \mathcal{K}_m^{a,A} \subset \mathcal{K}_m^A \subset [0,1]^m$ with the found $\hat{\nu}$. Therefore $P_{\mathcal{K}_m^A}(z) = P_{\mathcal{K}_m^{a,A}}(z) = P_{\mathcal{K}_m^A}(z - \hat{\nu}) = P_{\mathcal{K}_m^{a,A}}(z - \hat{\nu}) = z^{\hat{\nu}}$. □

Algorithm 6.4.3. *To find $\hat{\nu}$ such that $\sum_{i=1}^m a_i z_i^{\hat{\nu}} = A$:*

- *1, Set* $z_0 = -\infty$ *and* $z_{m+1} = \infty$.
- *2, Find* k *such that* $\sum_{i=1}^m a_i z_i^{z_{k-1}} \geq A$ *and* $\sum_{i=1}^m a_i z_i^{z_k} < A$*, then* $z_{k-1} \leq \hat{\nu} < z_k$.
- *3, Find* j *such that* $\sum_{i=1}^m a_i z_i^{z_{j+1}-1} < A$ *and* $\sum_{i=1}^m a_i z_i^{z_j-1} \geq A$*, then* $z_j - 1 \leq \hat{\nu} < z_{j+1} - 1$.
- *4, Set* $\hat{\nu} = \frac{\sum_{i=j+1}^m a_i + \sum_{i=k}^j a_i z_i - A}{\sum_{i=k}^j a_i}$ *when* $k \leq j$*, or* $\hat{\nu} = z_{k-1}$ *when* $k = j+1$.

By the way how to find k and j, it follows that $z_{k-1} \leq \hat{\nu} < z_{j+1} - 1 < z_{j+1}$, hence $k \leq j+1$. Then step 4 of Algorithm 6.4.3 gives the right $\hat{\nu}$ since $z_i^{\hat{\nu}} = \begin{cases} 0, & \text{if } i \leq k-1, \\ 1, & \text{if } i \geq j+1, \\ z_i - \hat{\nu}, & \text{otherwise.} \end{cases}$ for k, j found in step 2 and 3, and hence for $k \leq j$,

$$\sum_{i=k}^{j} a_i(z_i - \hat{\nu}) + \sum_{i=j+1}^{m} a_i = A \implies \hat{\nu} = \frac{\sum_{i=j+1}^m a_i + \sum_{i=k}^j a_i z_i - A}{\sum_{i=k}^j a_i}.$$

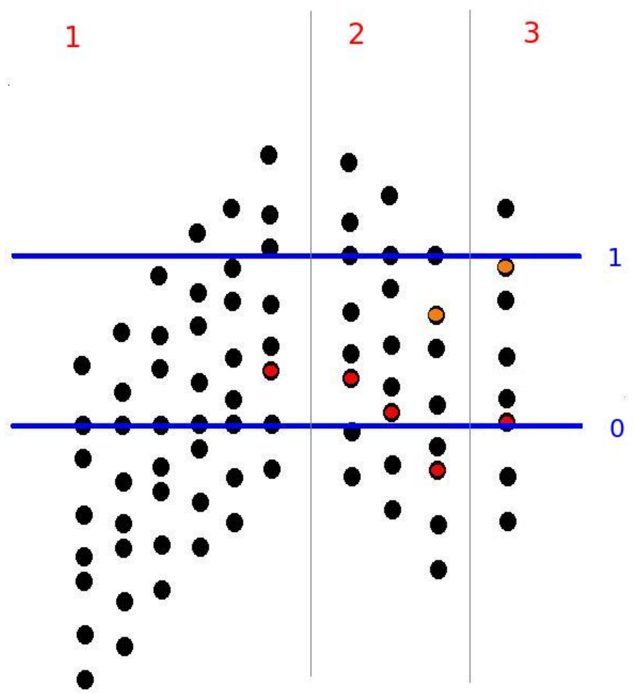

Figure 6.2: An illustration of Algorithm 6.4.3.

Finally, we propose the following algorithm for projection $P_{\mathrm{Quad}_{0,h}^{K,M}}$:

Algorithm 6.4.4. *For projection* $P_{\mathrm{Quad}_{0,h}^{K,M}}$ *in* (6.4.18)*:*

- *1, Compute the convex envelope* $\hat{\varphi}$ *of* φ *on* $[0, 2M]$ *and on* $[-2M, 0]$.

- *2, Set* $z = \mathcal{L}_R(\hat{\varphi}|_{\mathcal{N}_R})$*, use Algorithm 6.4.3 to compute* $P_{E_0^{K,M}}(u)$*.*
- *3, Let* $P_{\mathrm{Quad}_{0,h}^{K,M}}(\varphi) = \mathcal{L}_R^{-1} P_{E_0^{K,M}}(z)$*.*

6.4.2.3 Convergence rate

We shall provide a convergence rate for the gradient projection algorithm. In preparation, let us first give an estimate on the norm of the sub-gradients $\nabla u_{h,T}$.

Proposition 6.4.5. *Let* $\varphi_1, \varphi_2 \in B(\mathcal{N}_R)$*, then under the CFL condition (6.4.3),*

$$\Big|\; u_{h,T}(\varphi_1) \;-\; u_{h,T}(\varphi_2) \;\Big| \;\;\le\;\; 2\;|\varphi_1 - \varphi_2|_\infty, \tag{6.4.24}$$

and it follows that

$$\big|\nabla u_{h,T}(\varphi)\big|_R \;\;\le\;\; 2\sqrt{2m+1} \;\;=\;\; 2\sqrt{\frac{2M}{\Delta x}+1}, \qquad \forall \varphi \in B(\mathcal{N}_R). \tag{6.4.25}$$

Proof. Under the CFL condition (6.4.3), the $\theta-$scheme is monotone, which implies that $|\lambda_h^{\varphi,T,R,\varphi_1} - \lambda_h^{\varphi,T,R,\varphi_2}|_\infty \le |\varphi_1 - \varphi_2|_\infty$, and hence by the definition of $u_{h,T}$ in (6.4.4), the inequality (6.4.24) holds true.

Next, denote $\xi^i := \mathcal{L}_R(\varphi_i)$, $i = 1, 2$, then by Cauchy-Schwarz inequality,

$$|\varphi_1 - \varphi_2|_\infty \le \max\Big(\sum_{i=0}^{2m} \big|\xi_i^1 - \xi_i^2\big|, \sum_{i=0}^{-2m} \big|\xi_i^1 - \xi_i^2\big| \Big) \le \sqrt{2m+1}\cdot \big\|\xi^1 - \xi^2\big\|_{\mathbb{L}^2},$$

which implies immediately (6.4.25). □

Finally, let us finish this section by providing a convergence rate of the proposed gradient projection algorithm (Algorithm 6.4.1). Denote

$$\Phi \;\;:=\;\; \max_{\varphi_1,\varphi_2 \in \mathrm{Quad}_{0,h}^{K,M}} |\varphi_1 - \varphi_2|_R^2 \;\;\le\;\; 4m\;(4KM\Delta x)^2 \;\;\le\;\; 64K^2M^3\Delta x,$$

it follows from Section 5.3.1 of Ben-Tal and Nemirovski[7] that one has the convergence rate:

$$\begin{aligned} \min_{n\le N}\; u_{h,T}(\varphi_n) \;-\; U_h^{K,M,T} \;\;&\le\;\; \frac{\Phi \;+\; \sum_{i=n}^N \gamma_n^2 \big|\nabla u_{h,T}(\varphi_n)\big|_R^2}{2\sum_{n=1}^N \gamma_n} \\ &=\;\; \frac{32K^2M^3\Delta x \;+\; \big(4\frac{M}{\Delta x}+2\big)\; \sum_{i=n}^N \gamma_n^2}{\sum_{n=1}^N \gamma_n}. \end{aligned} \tag{6.4.26}$$

There are several choices for the sequence $\gamma = (\gamma_n)_{n\ge 1}$:

- Divergent Series: $\gamma_n \ge 0$, $\sum_{n=1}^\infty \gamma_n = +\infty$ and $\sum_{n=1}^\infty \gamma_n^2 < +\infty$. Clearly, (6.4.26) converges to 0 as $N \to \infty$.
- Optimal stepsizes: $\gamma_n = \frac{\sqrt{\Phi}}{\big|\nabla u_{h,T}(\varphi_n)\big|_R \sqrt{n}}$, we have by [7] that

$$\min_{n\le N} u_{h,T}(\varphi_n) \;-\; U_h^{K,M,T} \;\;\le\;\; O(1)\frac{16KM\sqrt{2M^2 + M\Delta x}}{\sqrt{N}}.$$

6.5 Numerical example

As shown in Corollary 6.3.2, the model-free price upper bound of variance swap is given by C_0 in (6.2.2). Let $(S_t)_{t\geq 0}$ follow the Black-Scholes dynamics $dS_t = \sigma S_t dW_t$, where $(W_t)_{t\geq 0}$ is a standard Brownian motion, and $\mu_0 \sim S_{\frac{1}{2}}$ and $\mu_1 \sim S_1$. It follows that

$$C_0 \quad = \quad \mathbb{E}\left(S_1^2 - S_{\frac{1}{2}}^2 \right) \quad = \quad \mathbb{E}\int_{\frac{1}{2}}^{1} \sigma^2 S_t^2 dt \quad = \quad \frac{1}{2}\sigma^2 S_0^2.$$

We set $\sigma = 0.2$, $S_0 = 1$, hence $C_0 = 0.02$. In our implemented example, with a 2.40GHz CPU computer, it takes 57.24 seconds to finish 4×10^4 iterations, and we get the numerical upper bound 0.2019, i.e. the relative error is less than 1 %, see also Figure 6.3.

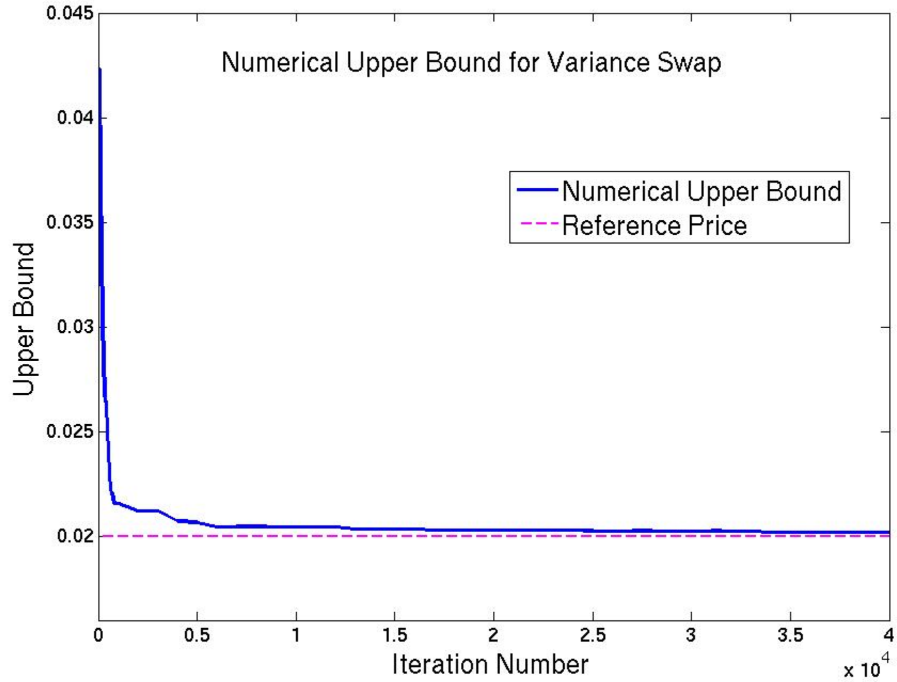

Figure 6.3: Numerical result for variance swap with approximation parameters: $T = 0.1$, $K = 1$, $M = 1$, $R = 2$, $\Delta t = 0.002$, $\Delta x = 0.1$ and $\gamma_n = \sqrt{n}$.

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