Multi-dimensional ”Malliavin-Stein” method on the Poisson space and its applications to limit theorems

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Méthode de "Malliavin-Stein" multi-dimensionelle sur l'espace de Poisson: applications aux théorèmes centraux limites

Sous la direction de Giovanni Peccati et Marc Yor

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To my parents...

Notations and abbreviations

- $(Z, \mathcal{Z}, \mu)$: a separable, σ-finite and non-atomic measure space. — *See the beginning of Section 1.1.*
- $\mathcal{Z}_\mu$: the class of those $A \in \mathcal{Z}$ such that $\mu(A) < \infty$. — *See Notation 1.1.1.*
- $(Z^n, \mathcal{Z}^n)$: abbreviation of $(Z^{\otimes n}, \mathcal{Z}^{\otimes n})$. — *See Notation 1.1.1.*
- $L^2(\mu)$: the abbreviation of $L^2(Z, \mathcal{Z}, \mu)$. — *See Notation 1.1.1.*
- $L^2(\mu^n)$: the abbreviation of $L^2(Z^n, \mathcal{Z}^n, \mu^n) = L^2(Z^{\otimes n}, \mathcal{Z}^{\otimes n}, \mu^{\otimes n})$, the space of real valued functions on $Z^{\otimes n}$ which are square-integrable with respect to $\mu^{\otimes n}$. — *See Notation 1.1.1.*
- $\tilde{f}$: the canonical symmetrization of function $f \in L^2(\mu^n)$. — *See Notation 1.1.1.*
- $L^2_s(\mu^n)$: the closed linear subspace of $L^2(\mu^n)$ composed of *symmetric* functions. — *See Notation 1.1.1.*
- $f(z, \cdot)$: the function defined on Z^{n-1} given by $(z_1, \dots, z_{n-1}) \mapsto f(z, z_1, \dots, z_{n-1})$, for any $f \in L^2_s(\mu^n)$, $(n > 1)$ and $z \in Z$. — *See Notation 1.1.1.*
- $\widetilde{f(z, \cdot)}$: the symmetrization of the $(n-1)$-parameter function $f(z, \cdot)$. — *See Notation 1.1.1.*
- $\mathcal{E}(\mu^n)$: the subset of $L^2(\mu^n)$ composed of elementary functions vanishing on diagonals. — See Notation 1.1.1.
- $\mathcal{E}_s(\mu^n)$: the subset of $L^2_s(\mu^n)$ composed of *symmetric* elementary functions vanishing on diagonals. — *See Notation 1.1.1.*
- $\hat{N}(A)$: a compensated Poisson measure evaluated on set A. *See the beginning of Section 1.1.1.*
- $G(A)$: a Gaussian measure evaluated on set A. — See Definition 1.1.13.
- $L^2(\sigma(G), \mathbb{P})$: the space of square-integrable functionals of some Gaussian measure G. — *See Proposition 1.1.21.*
- $\mathfrak{H}$: a real separable Hilbert space with inner product $\langle \cdot, \cdot \rangle_{\mathfrak{H}}$. — *See Definition 1.1.23.*
- $G(\mathfrak{H})$: abbreviation of $\{G(h) : h \in \mathfrak{H}\}$, an isonormal Gaussian process over $\mathfrak{H}$, with $\mathbb{E}[G(h)G(h')] = \langle h, h' \rangle_{\mathfrak{H}}, \quad \forall h, h' \in \mathfrak{H}$. — *See Definition 1.1.23.*
- $\mathfrak{H}^{\otimes n}$: the n-th tensorial product $\mathfrak{H}^{\otimes n}$. — *See the beginning of Section 1.2.1.*
- $\mathfrak{H}^{\odot n}$: the Hilbert space of n-th symmetric tensors, with the norm $\|f\|^2_{\mathfrak{H}^{\odot n}} = n!\|f\|^2_{\mathfrak{H}^{\otimes n}}$. — *See the beginning of Section 1.2.1.*
- $I^G_q(f)$: the multiple stochastic Wiener-Itô integral (of order q) of $f \in L^2(\mu^q)$ with respect to Gaussian measure G. — *See Section 1.1.2.*

- $I_q(f)$: abbreviation of $I_q^{\hat{N}}(f)$, the multiple stochastic Wiener-Itô integral (of order q) of $f \in L^2(\mu^q)$ with respect to compensated Poisson measure $\hat{N}$. — *See Section 1.1.1.*

- $f \otimes_r g$: contraction of order r between functions $f \in L^2(\mu^p)$ and $g \in L^2(\mu^q)$. — See Definition 1.1.17.

- $f \star_r^l g$: star contraction of order r, l between functions $f \in L_s^2(\mu^p)$ and $g \in L_s^2(\mu^q)$. — See Definition 1.1.6.

- H_q : Hermite polynomial of order q. *See Definition 1.1.19.*

- $\langle \cdot, \cdot \rangle_{H.S.}$: the Hilbert-Schmidt inner product on the class of $d \times d$ real matrices, defined by $\langle A, B \rangle_{H.S.} := Tr(AB^T)$ for every pair of matrices A and B. — *See Definition 1.3.1.*

- $\| \cdot \|_{H.S.}$: the Hilbert - Schmidt norm induced by $\langle \cdot, \cdot \rangle_{H.S.}$. — *See Definition 1.3.1.*

- $\|A\|_{op}$: the operator norm of a $d \times d$ real matrix A given by $\sup_{\|x\|_{\mathbb{R}^d}=1} \|Ax\|_{\mathbb{R}^d}$. — *See Definition 1.3.1.*

- $\| \cdot \|_{\mathbb{R}^d}$: the usual Euclidian norm on $\mathbb{R}^d$. — *See Definition 1.3.1.*

- $\mathrm{Hess}\, g(z)$: the Hessian matrix of g evaluated at a point z. — *See Definition 1.3.1.*

- $\|g\|_{Lip}$: the Lipschitz norm defined by $\sup_{x \neq y} \frac{|g(x) - g(y)|}{\|x - y\|_{\mathbb{R}^d}}$ for every function $g : \mathbb{R}^d \mapsto \mathbb{R}$. — *See Definition 1.3.1.*

- $M_2(g)$: the M_2 norm defined by $\sup_{x \neq y} \frac{\|\nabla g(x) - \nabla g(y)\|_{\mathbb{R}^d}}{\|x - y\|_{\mathbb{R}^d}}$ for $g \in \mathbb{C}^1(\mathbb{R}^d)$. — *See Definition 1.3.1.*

- $M_3(g)$: the M_3 norm defined by $\sup_{x \neq y} \frac{\|\mathrm{Hess}\, g(x) - \mathrm{Hess}\, g(y)\|_{op}}{\|x - y\|_{\mathbb{R}^d}}$ for $g \in \mathbb{C}^2(\mathbb{R}^d)$. — *See Definition 1.3.1.*

- $\|g^{(k)}\|_\infty$: the norm defined by $\max_{1 \leq i_1 \leq \ldots \leq i_k \leq d} \sup_{x \in \mathbb{R}^d} \left| \frac{\partial^k}{\partial x_{i_1} \ldots \partial x_{i_k}} g(x) \right|$ for a positive integer k and a function $g \in \mathbb{C}^k(\mathbb{R}^d)$. — *See Definition 1.3.1.*

- $d_{\mathcal{G}}$: the distance between the laws of two $\mathbb{R}^d$-valued random vectors X and Y such that $\mathbb{E}\|X\|_{\mathbb{R}^d}, \mathbb{E}\|Y\|_{\mathbb{R}^d} < \infty$, given by

$$d_{\mathcal{G}}(X, Y) = \sup_{g \in \mathcal{G}} |\mathbb{E}[g(X)] - \mathbb{E}[g(Y)]|,$$

where $\mathcal{G}$ indicates the collection of some functions. — See *Definition 1.3.3.*

- d_W : the Wasserstein (or Kantorovich- Wasserstein) distance, by taking $\mathcal{G} = \{g : \|g\|_{Lip} \leq 1\}$. — *See Definition 1.3.3.*
- d_{TV} : the total variation distance, by taking $\mathcal{G}$ equal to the collection of all indicators $\mathbf{1}_B$ of Borel sets. — *See Definition 1.3.3.*
- d_{Kol} : the Kolmogorov distance, by taking $\mathcal{G}$ equal to the class of all indicators functions $\mathbf{1}_{(-\infty;z_1]}, \cdots, \mathbf{1}_{(-\infty;z_d]}$, $(z_1, \ldots, z_d) \in \mathbb{R}^d$. — *See Definition 1.3.3.*
- d_2 : the $d_{\mathcal{G}}$ distance, by taking $\mathcal{G} = \{g : g \in \mathbb{C}^2(\mathbb{R}^d), \|g\|_{Lip} \leq 1, M_2(g) \leq 1\}$. — *See Definition 1.3.3.*
- d_3 : the $d_{\mathcal{G}}$ distance, by taking $\mathcal{G} = \{g : g \in \mathbb{C}^3(\mathbb{R}^d), \|g''\|_\infty \leq 1 \ , \ \|g'''\|_\infty \leq 1\}$. — *See Definition 1.3.3.*

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Ma reconnaissance va naturellement aux mes rapporteurs Josep L Solé et David Nualart pour avoir relu attentivement mon manuscrit et pour leurs conseils visant à l'améliorer. Ensuite, je souhaite remercier autres membres de mon jury de thèse Ivan Nourdin et Laurent Decreusefond pour le temps qu'ils m'ont consacré sur ma thèse et soutenance.

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Résumé / Abstract

0.1 Résumé (en Français)

Dans cette thèse nous nous concentrons sur l'établissement de certains théorèmes limite et d'approximations probabilistes. Un théorème limite est un résultat indiquant que la structure à grande échelle de certains systèmes aléatoires peut être véritablement approchée par une distribution de probabilité typique. Les exemples classiques sont le Théorème Central Limite (TCL dans la suite), le principe d'invariance de Donsker, ainsi que les lois circulaire et semi-circulaire de la théorie des matrices aléatoires. D'autre part, nous appelons approximation probabiliste toute formalation mathématique permettant d'évaluer des distances entre les lois de deux éléments aléatoires. Lorsque l'une des distributions est gaussienne, on parle d'approximation normale. Le TCL et l'approximation normale associée sont l'un des thèmes récurrents de toute la théorie des probabilités: voir par exemple [13] pour une introduction à ce sujet.

Au cours des cinq dernières années, I. Nourdin, G. Peccati et d'autres auteurs ont développé une nouvelle théorie d'approximations normales et non normales pour des variables aléatoires sur l'espace de Wiener, qui est basée sur l'utilisation d'un calcul de variations de dimension infinie, connu sous le nom de "calcul de Malliavin", ainsi que la célèbre "méthode de Stein" pour les approximations probabilistes. Leur travail généralise les résultats précédents par D. Nualart et G. Peccati à propos de théorèmes limite portant sur les chaos de Wiener (voir [40]). Après cela, G. Peccati, J. L. Solé, M.S. Taqqu et F. Utzet (voir [46]) ont étendu cette méthode pour obtenir des approximations normales sur l'espace de Poisson.

L'objectif de cette thèse est d'étendre les résultats de [46], afin d'obtenir des théorèmes centraux limites multi-dimensionnels sur l'espace de Poisson, ainsi que plusieurs extensions, comme on le verra dans la suite de ce résumé.

La thèse est organisée comme suit:

Dans le **Chapitre 1** , nous présentons plusieurs résultats classiques préliminaires, y compris la définition d'intégrales stochastiques multiples de Wiener-Itô, les décompositions chaotiques, les opérateurs de contraction, les formules de multiplication, à la fois sur l'espace de Poisson et sur l'espace gaussien. Nous introduisons aussi les deux principaux outils de notre théorie: le calcul de Malliavin et la méthode de Stein. Outre les résultats existants, nous présentons également plusieurs nouveaux lemmes techniques, y compris: le Lemme 1.1.8, qui consiste en une inégalité de type Cauchy-Schwarz; l'inégalité (1.74) dans le lemme 1.3.12, qui donne une estimation de la distance M_3.

Dans le **Chapitre 2**, nous présentons la "méthode de Malliavin-Stein", ainsi qu'une brève description des contributions principales contenues dans cette thèse. La méthode de "Malliavin-Stein" est une méthode d'approximation probabiliste développée par I. Nourdin, G. Peccati et d'autres auteurs, basée sur la combinaison de calcul du Malliavin et de la méthode de Stein. Antérieurement à cette thèse, cette technique a été utilisée pour obtenir le TCL pour des intégrales multiples, dans le cadre unidimensionnel gaussien, ainsi que d'un part dans le cadre multi-dimensionnel gaussien, et aussi d'autre part dans le cadre unidimensionnel de Poisson. Parmi les autres sujets obtenus dans cette thèse, la propriété "d'universalité" dans les chaos de Wiener de type gaussien et les théorèmes centraux limites presques sûrs (TCLPSs dans la suite) ont également été étudiés. A la fin de ce chapitre, nous présentons une introduction aux trois contributions principales originales dans cette thèse: le TCL multi-dimensionnel sur l'espace de Poisson; la propriété "d'universalité" dans les chaos de Wiener-Poisson; le Théorème centrale limite presque sûr sur l'espace de Poisson. Cependant, ces trois contributions sont contenues respectivement dans le chapitre 3 , le chapitre 4, et le chapitre 5.

Dans le **Chapitre 3**, nous étudions certaines approximations normales multi-dimensionnelles sur l'espace de Poisson avec les moyens du calcul de Malliavin, de la méthode de Stein et de l'interpolation "smart path" utilisée beaucoup par M. Talagrand. Nos résultats impliquent de nouveaux théorèmes centraux limites multi-dimensionnels pour des intégrales multiples par rapport aux mesures de Poisson; ainsi nous étendons significativement les précédents résultats dans [46] par G. Peccati, J.L. Solé, M.S. Taqqu et F. Utzet. Plusieurs exemples explicites, en particulier concernant les vecteurs de fonctionnelles linéaires et non linéaires de processus d'Ornstein-Uhlenbeck généralisé, sont discutés en détail. Le contenu principal du chapitre 3 est basé sur l'article publié [50] par G. Peccati et C. Zheng. Les résultats du chapitre 3 ont déjà eu un certain impact dans la littérature, en particulier dans le cadre de la géométrie stochastique (voir [59, 61]) et de la théorie générale des opérateurs de Markov (voir [24]). Voir la section 2.3.1 pour un résumé du chapitre 3.

Dans le **Chapitre 4**, nous étudions la propriété "d'universalité" des sommes homogènes à l'intérieur des chaos de Wiener-Poisson. Ce travail développe l'idée introduite dans l'article [35] écrit par I. Nourdin, G. Peccati et G. Reinert. Par ailleurs, nous montrons aussi que, dans le cas particulier des éléments des chaos de Wiener-Poisson qui sont aussi des sommes homogènes, les conditions suffisantes pour le TCL établies dans le chapitre 3 s'avèrent également être nécessaires. Le contenu principal du chapitre 4 est basé sur l'article en préparation [51] par G. Peccati et C. Zheng. Voir la section 2.3.2 pour un résumé du chapitre 4.

Dans le **Chapitre 5**, nous obtenons certains théorèmes centraux limites presques sûrs pour des fonctionnelles des mesures de Poisson, à la fois dans le cas unidimensionnel et le cas multi-dimensionnel. La source principale d'inspiration pour ce travail vient de l'article [3] par B. Bercu, I. Nourdin et M.S. Taqqu, qui traite TCLPSs sur l'espace de Wiener gaussien. Comme application, nous revisitons les fonctionnelles de processus d'Ornstein-Uhlenbeck généralisé étudiées dans le chapitre 3, et nous construisons des TCLPSs pour elles. Le contenu principal du chapitre 5 est basé sur l'article en préparation [69] par C. Zheng. Voir la section 2.3.3 pour un résumé du chapitre 5.

0.2 Abstract (in English)

In this dissertation we focus on *limit theorems* and *probabilistic approximations*. A "limit theorem" is a result stating that the large-scale structure of some random system can be meaningfully approximated by some typical probability distribution. Distinguished examples are the Central Limit Theorem (CLT in the sequel), the Donsker invariance principle, as well as the circular and semicircular laws in random matrix theory. On the other hand, we call a "probabilistic approximation" any mathematical statement allowing one to assess the distance between the laws of two random elements. When one of the distributions is Gaussian, one refers to a "normal approximation". CLTs and associated normal approximations are one of the recurring themes of the whole theory of probability: see e.g. [13] for an introduction to this topic.

In the last five years, I. Nourdin, G. Peccati and other authors have developed a new theory of normal and non-normal approximations for random variables on the Wiener space, based on the use of an infinite-dimensional calculus of variations, known as the "Malliavin calculus", as well as the well-known "Stein's method" for probabilistic approximations. Their work generalizes previous findings by D. Nualart and G. Peccati about limit theorems on Wiener chaos (see [40]). After that, G. Peccati, J.L. Solé, M.S. Taqqu and F. Utzet (see [46]) extended this method to the framework of normal approximations on the Poisson space.

The aim of this dissertation is to extend the findings of [46] , in order to study multi-dimensional Central Limit Theorems on the Poisson space, as well as several extensions.

The dissertation is organized as follows:

In **Chapter 1**, we present several classic preliminary results, including the definition of the multiple Wiener-Itô stochastic integrals, chaotic decompositions, contraction operators, multiplication formulae, on both the Poisson and Gaussian spaces. We also introduce the two main tools of our theory: Malliavin calculus and Stein's method. Besides the existing results, we also present several new technical lemmas, including: Lemma 1.1.8, which contains a Cauchy-Schwarz type inequality; inequality (1.74) in Lemma 1.3.12, which gives an estimation of the distance M_3.

In **Chapter 2**, we present the "Malliavin-Stein" method, as well as a short description of the main contributions contained in the dissertation. The "Malliavin-Stein" method is a probabilistic approximation method developed by I. Nourdin, G. Peccati and other authors, based on the combination of the Malliavin calculus and Stein's method. Previously to this dissertation, this technique has been used to derive CLTs for multiple integrals, in the one-dimensional Gaussian setting, in the multi-dimensional Gaussian setting, as well as in the one-dimensional Poisson setting. Other topics, including the "Universality" property of Gaussian Wiener chaos and Almost Sure Central Limit Theorems (ASCLTs in the sequel) have also been studied in this framework. At the end of this chapter, we present an introduction to the three main original contributions in this dissertation: multi-dimensional CLTs on the Poisson space; Universality of Poisson Wiener chaos; Almost Sure Central Limit Theorem on the Poisson space. These three contributions are contained respectively, in the subsequent Chapter 3, Chapter 4, and Chapter 5.

In **Chapter 3**, we study multi-dimensional normal approximations on the Poisson space by means of Malliavin calculus, Stein's method and "smart path" interpolations. Our results yield new multi-dimensional central limit theorems for multiple integrals with respect to Poisson measures, thus significantly extending the previous findings in [46] by G. Peccati, J.L. Solé, M.S. Taqqu and F. Utzet. Several explicit examples (including in particular vectors of linear and non-linear functionals of Ornstein-Uhlenbeck Lévy processes) are discussed in detail. The main content of Chapter 3 is based on the published paper [50] by G. Peccati and C. Zheng. Note that the results of Chapter 3 have already had some impact on the literature, in particular in the framework of stochastic geometry (see [59, 61]) and the general theory of Markov operators (see [24]). See Section 2.3.1 for a sketch of Chapter 3.

In **Chapter 4**, we study the "Universality" property of homogeneous sums inside the Poisson Wiener chaos. This work develops the idea introdued in the paper [35] written by I. Nourdin, G. Peccati and G. Reinert. As a by-product, we also show that, in the special case of elements of the Poisson Wiener chaos that are also homogeneous sums, the sufficient conditions for the CLTs established in Chapter 3 turn out to be also necessary. The main content of Chapter 4 is based on the paper in preparation [51] by G. Peccati and C. Zheng. See Section 2.3.2 for a sketch of Chapter 4.

In **Chapter 5**, we obtain Almost Sure Central Limit Theorems for functionals of Poisson measures, in both the one-dimensional and the multi-dimensional cases. The main inspiration for this work comes from the paper [3] by B. Bercu, I. Nourdin and M.S. Taqqu, dealing with ASCLTs on the Wiener space. As an application, we revisit the functionals of Ornstein-Uhlenbeck Lévy processes studied in Chapter 3, and build ASCLTs for them. The main content of Chapter 5 is based on the paper in preparation [69] by C. Zheng. See Section 2.3.3 for a sketch of Chapter 5.

Contents

Chapter 1

Preliminaries

1.1 Preliminaries

Let $(Z, \mathcal{Z}, \mu)$ be a separable, σ-finite and non-atomic measure space. We denote by $\mathcal{Z}_\mu$ the class of those $A \in \mathcal{Z}$ such that $\mu(A) < \infty$. Note that, by σ-additivity, the σ-field generated by $\mathcal{Z}_\mu$ coincides with $\mathcal{Z}$.

Notation 1.1.1

- *For the rest of this dissertation, we shall write* $(Z^n, \mathcal{Z}^n) = (Z^{\otimes n}, \mathcal{Z}^{\otimes n}), n \geq 2$*, and also* $(Z^1, \mathcal{Z}^1) = (Z^{\otimes 1}, \mathcal{Z}^{\otimes 1}) = (Z, \mathcal{Z})$*. Moreover, we set*

$$\mathcal{Z}_\mu^n = \{C \in \mathcal{Z}^n : \mu^n(C) < \infty\}.$$

- *We denote by* $L^2(\mu)$ *the abbreviation of* $L^2(Z, \mathcal{Z}, \mu)$*. And we denote by* $L^2(\mu^n)$ *the abbreviation of* $L^2(Z^n, \mathcal{Z}^n, \mu^n) = L^2(Z^{\otimes n}, \mathcal{Z}^{\otimes n}, \mu^{\otimes n})$*, the space of real valued functions on* $Z^{\otimes n}$ *which are square-integrable with respect to* $\mu^{\otimes n}$*.*

- *For every* $n \geq 1$ *and every* $f, g \in L^2(\mu^n)$*, we note*

$$\langle f, g \rangle_{L^2(\mu^n)} = \int_{Z^n} f(z_1, \ldots, z_n) g(z_1, \ldots, z_n) \mu^n(dz_1, \ldots, dz_n), \quad \|f\|_{L^2(\mu^n)} = \langle f, f \rangle_{L^2(\mu^n)}^{1/2}.$$

- *For every* $f \in L^2(\mu^n)$*,* $n \geq 1$*, and every fixed* $z \in Z$*, we write* $f(z, \cdot)$ *to indicate the function defined on* Z^{n-1} *given by* $(z_1, \ldots, z_{n-1}) \mapsto f(z, z_1, \ldots, z_{n-1})$*. Accordingly,* $\widetilde{f(z, \cdot)}$ *stands for the symmetrization of the function* $f(z, \cdot)$ *(in* $(n-1)$ *variables). Note that, if* $n = 1$*, then* $f(z, \cdot) = f(z)$ *is a constant.*

- *Given a function* $f \in L^2(\mu^n)$*, we denote by* $\tilde{f}$ *its canonical symmetrization, that is*

$$\tilde{f}(z_1, \ldots, z_n) = \frac{1}{n!} \sum_\pi f(z_{\pi(1)}, \ldots, z_{\pi(n)}),$$

where the sum runs over all permutations π *of the set* $\{1, \ldots, n\}$*. Note that, by the triangle inequality,*

$$\|\tilde{f}\|_{L^2(\mu^n)} \leq \|f\|_{L^2(\mu^n)} \tag{1.1}$$

- *We write* $L^2_s(\mu^n)$ *to indicate the closed linear subspace of* $L^2(\mu^n)$ *composed of* symmetric *functions, that is,* $f \in L^2_s(\mu^n)$ *if and only if: (i)* f *is square integrable with respect to* μ^n*, and (ii) for* $d\mu^n$*-almost every* $(z_1, \ldots, z_n) \in Z^n$,

$$f(z_1, \ldots, z_n) = f(z_{\pi(1)}, \ldots, z_{\pi(n)}),$$

for every permutation π *of* $\{1, \ldots, n\}$.

- *We write* $\mathcal{E}(\mu^n)$ *to indicate the subset of* $L^2(\mu^n)$ *composed of elementary functions vanishing on diagonals, that is,* $f \in \mathcal{E}(\mu^n)$ *if and only if* f *is a finite linear combination of functions of the type*

$$(z_1, \ldots, z_n) \mapsto \mathbf{1}_{A_1}(z_1)\mathbf{1}_{A_2}(z_2)\ldots\mathbf{1}_{A_n}(z_n)$$

where the sets A_i *are pairwise disjoint elements of* $\mathcal{Z}_\mu$.

- *We write* $\mathcal{E}_s(\mu^n)$ *to indicate the subset of* $L^2_s(\mu^n)$ *composed of* symmetric *elementary functions vanishing on diagonals, that is,* $g \in \mathcal{E}_s(\mu^n)$ *if and only if* $g = \tilde{f}$ *for some* $f \in \mathcal{E}(\mu^n)$*, where the symmetrization* $\tilde{f}$ *is defined above.*

1.1.1 Poisson space

Poisson measure

We recall the notations at the beginning of this dissertation. Let $(Z, \mathcal{Z}, \mu)$ be a measure space such that Z is a Borel space and μ is a σ-finite non-atomic Borel measure, and $\mathcal{Z}_\mu = \{B \in \mathcal{Z} : \mu(B) < \infty\}$. In what follows, we write $\hat{N} = \{\hat{N}(B) : B \in \mathcal{Z}_\mu\}$ to indicate a *compensated Poisson measure* on $(Z, \mathcal{Z})$ with *control* μ. In other words, $\hat{N}$ is a collection of random variables defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, indexed by the elements of $\mathcal{Z}_\mu$ and such that: (i) for every $B, C \in \mathcal{Z}_\mu$ such that $B \cap C = \varnothing$, the random variables $\hat{N}(B)$ and $\hat{N}(C)$ are independent; (ii) for every $B \in \mathcal{Z}_\mu$, $\hat{N}(B) \overset{(law)}{=} N(B) - \mu(B)$, where $N(B)$ is a Poisson random variable with paremeter $\mu(B)$. A random measure verifying property (i) is customarily called "completely random" or, equivalently, "independently scattered" (see e.g. [67]).

Remark 1.1.2 Due to the assumptions on the space $(Z, \mathcal{Z}, \mu)$, we can always set $(\Omega, \mathcal{F}, \mathbb{P})$ and $\hat{N}$ to be such that

$$\Omega = \left\{ \omega = \sum_{j=0}^{n} \delta_{z_j}, n \in \mathbb{N} \cup \{\infty\}, z_j \in Z \right\}$$

where δ_z denotes the Dirac mass at z, and $\hat{N}$ is the **compensated canonical mapping**

$$\omega \mapsto \hat{N}(B)(\omega) = \omega(B) - \mu(B), \quad B \in \mathcal{Z}_\mu, \quad \omega \in \Omega,$$

(see e.g. [52] for more details). For the rest of the paper, we assume that Ω and $\hat{N}$ have this form. Moreover, the σ-field $\mathcal{F}$ is supposed to be the $\mathbb{P}$-completion of the σ-field generated by $\hat{N}$.

Wiener-Itô integrals on the Poisson space

We introduce here the definition of multiple Wiener-Itô integrals on the Poisson space. The reader is referred e.g. to Peccati and Taqqu [48, Chapter 5] or Privault [56, Chapter 6] for a complete discussion of multiple Wiener-Itô integrals and their properties – see also [41, 67].
Single Wiener-Itô integrals: We first define the single Wiener-Itô integrals. In what follows, we shall use the fact that every elementary function $f \in \mathcal{E}(\mu)$ can be represented as

$$f(z) = \sum_{i=1}^{M} a_i \mathbf{1}_{A_i}(z), \quad z \in Z, \tag{1.2}$$

where $M \geq 1$ is finite, $a_i \in \mathbb{R}$, and the sets A_i are pairwise disjoint elements of $\mathcal{Z}_\mu$. Now consider a compensated Poisson measure $\hat{N}$ on $(Z, \mathcal{Z})$, with control μ. The next result establishes the existence of single Wiener-Itô integrals with respect to $\hat{N}$.

Proposition 1.1.3 *There exists a unique linear isomorphism $f \mapsto \hat{N}(f)$, from $L^2(\mu)$ into $L^2(\mathbb{P})$, such that*

$$\hat{N}(f) = \sum_{i=1}^{M} a_i \hat{N}(A_i)$$

for every elementary function $f \in \mathcal{E}(\mu)$ of the form (1.2).

The proof is standard and thus omitted. Readers may refer to [48, Chapter 5] for the detail.

The random variable $\hat{N}(f)$ is usually written as

$$\int_Z f(z)\hat{N}(dz), \quad \int_Z f d\hat{N}, \quad I_1^{\hat{N}}(f),$$

and it is called the *Wiener-Itô stochastic integral* of f with respect to compensated Poisson measure $\hat{N}$. One can easily verify the following isometric relation

$$\mathbb{E}[\hat{N}(f)\hat{N}(h)] = \int_Z f(z)h(z)\mu(dz) = \langle f, h\rangle_{L^2(\mu)}, \qquad \forall f, h \in L^2(\mu),$$

and property:

$$\hat{N}(f) = \hat{N}(\tilde{f}).$$

Now we consider the space

$$C_1(\hat{N}) = \{\hat{N}(f) : f \in L_s^2(\mu)\}. \tag{1.3}$$

It coincides with the $L^2(\mathbb{P})$-closed linear space generated by $\hat{N}$. One customarily says that (1.3) is the *first Wiener chaos* associated with $\hat{N}$.

Multiple integrals

Based on the discussions above, we may define multiple Wiener-Itô integrals. Fix $n \geq 2$, it is easily seen that every $f \in \mathcal{E}(\mu^n)$ admits a (not necessarily unique) representation of the form

$$f(z_1, \dots, z_n) = \sum_{1 \leq i_1, \dots, i_n \leq M} a_{i_1, \dots, i_n} \mathbf{1}_{A_{i_1}(z_1)} \cdots \mathbf{1}_{A_{i_n}(z_n)}$$

where $M \geq n$, the real coefficients $a_{i_1,\dots,i_n}$ are equal to zero whenever two indices i_k, i_l are equal and $A_1, \dots, A_M$ are pairwise disjoint elements of $\mathcal{Z}_\mu$. Then we define

$$I_n^{\hat{N}}(f) = \sum_{1 \leq i_1,\dots,i_n \leq M} a_{i_1,\dots,i_n} \hat{N}(A_{i_1}) \dots \hat{N}(A_{i_n}), \tag{1.4}$$

and we say that $I_n^{\hat{N}}(f)$ is the *multiple stochastic Wiener-Itô integral* (of order n) of f with respect to compensated Poisson measure $\hat{N}$. Note that $I_n^{\hat{N}}(f)$ has finite moments of all orders, and that the definitions of $I_n^{\hat{N}}(f)$ does not depend on the chosen representation of f. The following result, as well as the fact that $\mathcal{E}(\mu^n)$ is dense in $L^2(\mu^n)$, shows in particular that $I_n^{\hat{N}}$ can be extended to a continuous linear operator from $L^2(\mu^n)$ into $L^2(\mathbb{P})$.

Proposition 1.1.4 *The random variables $I_n^{\hat{N}}(f), n \geq 1, f \in \mathcal{E}(\mu^n)$, enjoy the following properties:*

1. *For every n, the application $f \mapsto I_n^{\hat{N}}(f)$ is linear.*
2. *For every n, one has that $\mathbb{E}[I_n^{\hat{N}}(f)] = 0$ and $I_n^{\hat{N}}(f) = I_n^{\hat{N}}(\tilde{f})$.*
3. *For every $n \geq 2$ and $m \geq 1$, for every $f \in \mathcal{E}(\mu^n)$ and $g \in \mathcal{E}(\mu^m)$,*

$$\mathbb{E}[I_n^{\hat{N}}(f) I_m^{\hat{N}}(g)] = n! \langle \tilde{f}, \tilde{g} \rangle_{L^2(\mu^n)} \mathbf{1}_{(n=m)}, \qquad \textit{(isometric property)} . \tag{1.5}$$

The proof of Proposition 1.1.4 follows almost immediately from the definition in equation (1.4); see e.g. [48, Chapter 5] for a complete discussion. By combining relation (1.5) with inequality (1.1), one infers that $I_n^{\hat{N}}$ can be extended to a linear continuous operator, from $L^2(\mu^n)$ into $L^2(\mathbb{P})$, verifying properties 1, 2 and 3 in the statement of Proposition 1.1.4. Moreover, the second line on the RHS of (1.5) yields that the application

$$I_n^{\hat{N}} : L_s^2(\mu^n) \to L^2(\mathbb{P}) : f \mapsto I_n^{\hat{N}}(f)$$

(that is, the restriction of $I_n^{\hat{N}}$ to $L_s^2(\mu^n)$) is an isomorphism from $L_s^2(\mu^n)$, endowed with the modified scalar product $n!\langle \cdot, \cdot \rangle_{L^2(\mu^n)}$, into $L^2(\mathbb{P})$. For every $n \geq 2$, the $L^2(\mathbb{P})$-closed space

$$C_n(\hat{N}) = \{ I_n^{\hat{N}}(f) : f \in L^2(\mu^n) \} \tag{1.6}$$

is called the *nth Wiener chaos* associated with $\hat{N}$. One conventionally sets

$$C_0(\hat{N}) = \mathbb{R} \tag{1.7}$$

Note that the isometric relation (1.5) implies that $C_n(\hat{N}) \perp C_m(\hat{N})$ for $n \neq m$, where "$\perp$ " indicates orthogonality in $L^2(\mathbb{P})$.

Attention: In the present thesis, without ambiguity, we use I_n as the abbreviation of $I_n^{\hat{N}}$.

The Hilbert space composed of the random variables with the form $I_n(f)$, where $n \geq 1$ and $f \in L_s^2(\mu^n)$, is called the nth *Wiener chaos* associated with the Poisson measure $\hat{N}$. The following well-known *chaotic representation property* is essential in this thesis.

Proposition 1.1.5 (Chaotic decomposition) *Every random variable $F \in L^2(\mathcal{F}, \mathbb{P}) = L^2(\mathbb{P})$ admits a (unique) chaotic decomposition of the type*

$$F = \mathbb{E}[F] + \sum_{n\geq 1}^{\infty} I_n(f_n) \tag{1.8}$$

where the series converges in $L^2(\mathbb{P})$ and, for each $n \geq 1$, the kernel f_n is an element of $L^2_s(\mu^n)$.

The proof of Proposition 1.1.5 will be given in Section "Charlier polynomials".

Star contractions and multiplication formulae

In order to give a simple description of the *multiplication formulae* for multiple Poisson integrals (see formula (1.12)), we here give a formal definition of star contraction operator:

Definition 1.1.6 *We define a contraction kernel $f \star_r^l g$ on $Z^{p+q-r-l}$ for functions $f \in L^2_s(\mu^p)$ and $g \in L^2_s(\mu^q)$, where $p, q \geq 1$, $r = 1, \dots, p \wedge q$ and $l = 1, \dots, r$, as follows:*

$$\begin{aligned} & f \star_r^l g(\gamma_1, \dots, \gamma_{r-l}, t_1, \dots, t_{p-r}, s_1, \dots, s_{q-r}) \\ = & \int_{Z^l} \mu^l(dz_1, \dots, dz_l) f(z_1, \dots, z_l, \gamma_1, \dots, \gamma_{r-l}, t_1, \dots, t_{p-r}) \\ & \qquad \times g(z_1, \dots, z_l, \gamma_1, \dots, \gamma_{r-l}, s_1, \dots, s_{q-r}). \end{aligned}$$

In other words, the star operator "$\star_r^l$" reduces the number of variables in the tensor product of f and g from $p+q$ to $p+q-r-l$: this operation is realized by first identifying r variables in f and g, and then by integrating out l among them. To deal with the case $l = 0$ for $r = 0, \dots, p \wedge q$, we set

$$\begin{aligned} & f \star_r^0 g(\gamma_1, \dots, \gamma_r, t_1, \dots, t_{p-r}, s_1, \dots, s_{q-r}) \\ = & f(\gamma_1, \dots, \gamma_r, t_1, \dots, t_{p-r}) g(\gamma_1, \dots, \gamma_r, s_1, \dots, s_{q-r}), \end{aligned}$$

and

$$f \star_0^0 g(t_1, \dots, t_p, s_1, \dots, s_q) = f \otimes g(t_1, \dots, t_p, s_1, \dots, s_q) = f(t_1, \dots, t_p) g(s_1, \dots, s_q).$$

By using the Cauchy-Schwarz inequality, one sees immediately that $f \star_r^r g$ is square-integrable for any choice of $r = 0, \dots, p \wedge q$, and every $f \in L^2_s(\mu^p)$, $g \in L^2_s(\mu^q)$.

As e.g. in [46, Theorem 4.2], we will sometimes need to work under some specific regularity assumptions for the kernels that are the object of our study.

Definition 1.1.7 *Let $p \geq 1$ and let $f \in L^2_s(\mu^p)$.*

1. *If $p \geq 1$, the kernel f is said to satisfy* **Assumption A**, *if $(f \star_p^{p-r} f) \in L^2(\mu^r)$ for every $r = 1, \dots, p$. Note that $(f \star_p^0 f) \in L^2(\mu^p)$ if and only if $f \in L^4(\mu^p)$.*
2. *The kernel f is said to satisfy* **Assumption B**, *if: either $p = 1$, or $p \geq 2$ and every contraction of the type*

$$(z_1, \dots, z_{2p-r-l}) \mapsto |f| \star_r^l |f|(z_1, \dots, z_{2p-r-l})$$

is well-defined and finite for every $r = 1, \dots, p$, every $l = 1, \dots, r$ and every $(z_1, \dots, z_{2p-r-l}) \in Z^{2p-r-l}$.

The following statement will be used in order to deduce the multivariate CLT stated in Theorem 3.4.9. The proof involves the Cauchy-Schwarz inequality and the Fubini theorem (in particular, Assumption A is needed in order to implicitly apply a Fubini argument – see step (S4) in the proof of Theorem 4.2 in [46] for an analogous use of this assumption).

Lemma 1.1.8 *Fix integers $p, q \geq 1$, as well as kernels $f \in L^2_s(\mu^p)$ and $g \in L^2_s(\mu^q)$ satisfying Assumption A in Definition 1.1.7. Then, for any integers s, t satisfying $1 \leq s \leq t \leq p \wedge q$, one has that $f \star_t^s g \in L^2(\mu^{p+q-t-s})$, and moreover*

1.
$$\|f \star_t^s g\|^2_{L^2(\mu^{p+q-t-s})} = \langle f \star_{p-s}^{p-t} f, g \star_{q-s}^{q-t} g\rangle_{L^2(\mu^{t+s})},$$
(and, in particular,
$$\|f \star_t^s f\|_{L^2(\mu^{2p-s-t})} = \|f \star_{p-s}^{p-t} f\|_{L^2(\mu^{t+s})} \text{);}$$

2.
$$\begin{aligned} \|f \star_t^s g\|^2_{L^2(\mu^{p+q-t-s})} &\leq \|f \star_{p-s}^{p-t} f\|_{L^2(\mu^{t+s})} \times \|g \star_{q-s}^{q-t} g\|_{L^2(\mu^{t+s})} & (1.9)\\ &= \|f \star_t^s f\|_{L^2(\mu^{2p-s-t})} \times \|g \star_t^s g\|_{L^2(\mu^{2q-s-t})}. & (1.10)\end{aligned}$$

Remark 1.1.9

1. Writing $k = p+q-t-s$, the requirement that $1 \leq s \leq t \leq p \wedge q$ implies that $|q-p| \leq k \leq p+q-2$.

2. One should also note that, for every $1 \leq p \leq q$ and every $r = 1, \ldots, p$,
$$\int_{Z^{p+q-r}} (f \star_r^0 g)^2 d\mu^{p+q-r} = \int_{Z^r} (f \star_p^{p-r} f)(g \star_q^{q-r} g) d\mu^r, \quad (1.11)$$
for every $f \in L^2_s(\mu^p)$ and every $g \in L^2_s(\mu^q)$, not necessarily verifying Assumption A. Observe that the integral on the RHS of (1.11) is well-defined, since $f \star_p^{p-r} f \geq 0$ and $g \star_q^{q-r} g \geq 0$.

3. Fix $p, q \geq 1$, and assume again that $f \in L^2_s(\mu^p)$ and $g \in L^2_s(\mu^q)$ satisfy Assumption A in Definition 1.1.7. Then, a consequence of Lemma 1.1.8 is that, for every $r = 0, \ldots, p \wedge q-1$ and every $l = 0, \ldots, r$, the kernel $f(z, \cdot) \star_r^l g(z, \cdot)$ is an element of $L^2(\mu^{p+q-t-s-2})$ for $\mu(dz)$-almost every $z \in Z$.

Proof. Let $\mathbf{x}^n, \mathbf{y}^n, \mathbf{z}^n, \mathbf{w}^n$ and $d\mathbf{x}^n$ be shorthand for $(x_1, \ldots, x_n)$, $(y_1, \ldots, y_n)$, $(z_1, \ldots, z_n)$ and $(w_1, \ldots, w_n)$ and $dx_1 dx_2 \ldots dx_n$ repectively.

We recall that the contraction operator $f \star_t^s g$ transforms two function $f \in L^2_s(\mu^p)$ and $g \in L^2_s(\mu^q)$ into a function of $p+q-t-s$, which is defined by
$$f \star_t^s g(\mathbf{x}^{t-s}, \mathbf{y}^{p-t}, \mathbf{z}^{q-t}) = \int_{Z^s} d\mathbf{w}^s f(\mathbf{w}^s, \mathbf{x}^{t-s}, \mathbf{y}^{p-t}) g(\mathbf{w}^s, \mathbf{x}^{t-s}, \mathbf{z}^{q-t}).$$

Since both f and g are symmetric functions, we have

$$
\begin{aligned}
& \|f \star_t^s g\|^2_{L^2(\mu^k)} \\
= & \int_{Z^k} d\mathbf{x}^{t-s} d\mathbf{x}^{p-t} d\mathbf{z}^{q-t} \int_{Z^s} d\mathbf{w}^s \int_{Z^s} d\mathbf{w}'^s f(\mathbf{w}^s, \mathbf{x}^{t-s}, \mathbf{y}^{p-t}) g(\mathbf{w}^s, \mathbf{x}^{t-s}, \mathbf{z}^{q-t}) \\
& f(\mathbf{w}'^s, \mathbf{x}^{t-s}, \mathbf{y}^{p-t}) g(\mathbf{w}'^s, \mathbf{x}^{t-s}, \mathbf{z}^{q-t}) \\
= & \int_{Z^{t-s}} d\mathbf{x}^{t-s} \int_{Z^s} d\mathbf{w}^s \int_{Z^s} d\mathbf{w}'^s [\int_{Z^{p-t}} d\mathbf{y}^{p-t} f(\mathbf{y}^{p-t}, \mathbf{x}^{t-s}, \mathbf{w}^s) f(\mathbf{y}^{p-t}, \mathbf{x}^{t-s}, \mathbf{w}'^{\mathbf{s}})] \\
& [\int_{Z^{q-t}} d\mathbf{z}^{q-t} g(\mathbf{z}^{q-t}, \mathbf{x}^{t-s}, \mathbf{w}^s) g(\mathbf{z}^{q-t}, \mathbf{x}^{t-s}, \mathbf{w}'^s)] \\
= & \int_{Z^{t-s}} d\mathbf{x}^{t-s} \int_{Z^s} d\mathbf{w}^s \int_{Z^s} d\mathbf{w}'^s [f \star_{p-s}^{p-t} f(\mathbf{x}^{t-s}, \mathbf{w}^s, \mathbf{w}'^s)][g \star_{q-s}^{q-t} g(\mathbf{x}^{t-s}, \mathbf{w}^s, \mathbf{w}'^s)] \\
= & \langle f \star_{p-s}^{p-t} f, g \star_{q-s}^{q-t} g \rangle_{L^2(\mu^{t+s})} \qquad (*) \\
\leq & \|f \star_{p-s}^{p-t} f\|_{L^2(\mu^{t+s})} \times \|g \star_{q-s}^{q-t} g\|_{L^2(\mu^{t+s})} \qquad (**).
\end{aligned}
$$

Relation (*) gives point 1 in the statement, while (**) (which is obtained by Cauchy-Schwartz inequality) yields point 2. ∎

To conclude the section, we present an important *product formula* for Poisson multiple integrals (see [48, Proposition 6.5.1], [22], [66] for the proof).

Proposition 1.1.10 (Product formula) *Let $f \in L^2_s(\mu^p)$ and $g \in L^2_s(\mu^q)$, $p, q \geq 1$, and suppose moreover that $f \star_r^l g \in L^2(\mu^{p+q-r-l})$ for every $r = 1, \ldots, p \wedge q$ and $l = 1, \ldots, r$ such that $l \neq r$. Then,*

$$I_p(f) I_q(g) = \sum_{r=0}^{p\wedge q} r! \begin{pmatrix} p \\ r \end{pmatrix} \begin{pmatrix} q \\ r \end{pmatrix} \sum_{l=0}^{r} \begin{pmatrix} r \\ l \end{pmatrix} I_{p+q-r-l} \left(\widetilde{f \star_r^l g} \right), \tag{1.12}$$

with the tilde $\sim$ indicating a symmetrization, that is,

$$\widetilde{f \star_r^l g}(x_1, \ldots, x_{p+q-r-l}) = \frac{1}{(p+q-r-l)!} \sum_{\sigma} f \star_r^l g(x_{\sigma(1)}, \ldots, x_{\sigma(p+q-r-l)}),$$

where σ runs over all $(p+q-r-l)!$ permutations of the set $\{1, \ldots, p+q-r-l\}$.

Charlier polynomials

In this section, we give a brief introduction of Charlier polynomials based on Chapter 10 of Peccati and Taqqu's book [48].

There are several possible definitions of the Charlier polynomials. We here present the definitions used by Kabanov [22] and Surgailis [66].

Charlier polynomials $\{C_n(x,a), n = 0, 1, 2\ldots\}$ are defined through generating function:

$$\sum_{n=0}^{\infty} \frac{C_n(x,a)}{n!} t^n = e^{-ta}(1+t)^{x+a}. \tag{1.13}$$

By simple calculation, we see that $C_0(x,a) = 1$, $C_1(x,a) = x$, $C_2(x,a) = x^2 - x - a$, etc. It is not difficult to deduce the recursion relation:

$$C_{n+1}(x,a) = (x-n)C_n(x,a) - anC_{n-1}(x,a), \quad n \geq 1. \tag{1.14}$$

Charlier polynomials are good companions with compensated Poisson variables. Indeed, the following propositions illustrate the intimate relation between Charlier polynomials and multiple integrals of compensated Poisson variables.

Proposition 1.1.11 *Let $\hat{N}$ be a centered Poisson measure over $(Z, \mathcal{Z})$ with control measure μ. Then, for every $A \in \mathcal{Z}$ such that $\mu(A) < \infty$, and every $n \geq 1$, one has that*

$$C_n(\hat{N}(A), \mu(A)) = I_n(\mathbf{1}_A^{\otimes n}). \tag{1.15}$$

The proof of the proposition relies on the recursion relation (1.14), and the product formula (1.12). We then complete the proof by induction on n.

Proposition 1.1.12 *Let $A_1, \dots, A_k$ be disjoint sets of finite measure μ. Then,*

$$I_n(\mathbf{1}_{A_1}^{\otimes i_1} \otimes \cdots \otimes \mathbf{1}_{A_k}^{\otimes i_k}) = \prod_{a=1}^{k} C_{i_a}(\hat{N}(A_a), \mu(A_a)) \tag{1.16}$$

Proof. Notice that the functions $\mathbf{1}_{A_a}$, $a = 1, \dots, k$, have mutually disjoint supports, we have, for $a \neq b$,

$$\mathbf{1}_{A_a}^{\otimes i_a} \star_r^l \mathbf{1}_{A_b}^{\otimes i_b} = \begin{cases} \mathbf{1}_{A_a}^{\otimes i_a} \otimes \mathbf{1}_{A_b}^{\otimes i_b}, & \text{if } r = l = 0; \\ 0, & \text{else.} \end{cases} \tag{1.17}$$

We apply the multiplication formula 1.12 repeatedly and then deduce that,

$$\begin{aligned} \prod_{a=1}^{k} C_{i_a}(\hat{N}(A_a), \mu(A_a)) &= \prod_{a=1}^{k} I_{i_a}(\mathbf{1}_{A_a}^{\otimes i_a}) \\ &= I_n(\mathbf{1}_{A_1}^{\otimes i_1} \star_0^0 \cdots \star_0^0 \mathbf{1}_{A_k}^{\otimes i_k}) \\ &= I_n(\mathbf{1}_{A_1}^{\otimes i_1} \otimes \cdots \otimes \mathbf{1}_{A_k}^{\otimes i_k}), \end{aligned}$$

which finishes the proof. ■

To end the section, we present here the proof of "chaotic representation" Proposition 1.1.5.
Proof. By Proposition 1.1.12 and the generating function (1.13), we deduce that every random variable of the type

$$F = \prod_{k=1,\dots,N} (1+t_k)^{\hat{N}(C_k)+\mu(C_k)}, \text{ with } C_k \text{ disjoint,}$$

can be represented as a series of multiple integrals. Notice that the linear span of random variables of type F is dense in $L^2(\sigma(\hat{N}), \mathbb{P})$, we may conclude the proof by using the fact that random variables enjoying the chaotic decomposition (1.8) form a Hilbert space. ■

1.1.2 Some results involving Gaussian space

As pointed out below, many of the results of this dissertation should be compared with those available in a Gaussian framework. This section is devoted to some basic definitions in this context.

Gaussian measure

Definition 1.1.13 *A* ***Gaussian measure*** *on $(Z, \mathcal{Z})$ with* **control** *μ is a centered Gaussian family of the type*

$$G = \{G(A) : A \in \mathcal{Z}_\mu\}$$

verifying the relation

$$\mathbb{E}[G(A)G(B)] = \mu(A \cap B), \quad \forall A, B \in \mathcal{Z}_\mu.$$

The Gaussian measure G is also called a ***white noise based on*** *μ.*

Example 1.1.14 *Fix $d \geq 1$, let $Z = \mathbb{R}^d$, $\mathcal{Z} = \mathcal{B}(\mathbb{R}^d)$, and let λ^d be the Lebesgue measure on $\mathbb{R}^d$. If G is a Gaussian measure with control λ^d, then, for every $A, B \in \mathcal{B}(\mathbb{R}^d)$ with finite Lebesgue measure, one has that*

$$\mathbb{E}[G(A)G(B)] = \int_{A \cap B} \lambda^d(dx_1, \dots, dx_d)$$

It follows that the application

$$(t_1, \dots, t_d) \mapsto \mathbf{W}(t_1, \dots, t_d) \triangleq G([0, t_1] \times \dots \times [0, t_d]), \quad t_i \geq 0,$$

defines a centered Gaussian process such that

$$\mathbb{E}[\mathbf{W}(t_1, \dots, t_d)\mathbf{W}(s_1, \dots, s_d)] = \prod_{i=1}^{d} (s_i \wedge t_i),$$

that is, $\mathbf{W}$ *is a* standard Brownian motion *on $\mathbb{R}$ if $d = 1$, or a* standard Brownian sheet *on $\mathbb{R}^d$ if $d > 1$.*

Wiener-Itô integrals on the Gaussian space

The definition of multiple Wiener-Itô integrals on the Gaussian space is almost the same as its Poisson counterpart. In this section we give a brief introduction on the subject, readers are invited to consult Chapter 1 of Nualart's book [38] for a full description.

For any function $f \in L^2_s(\mu)$, we denote $I^G(f)$ as the standard stochastic integral on f with respect to Gaussian measure G:

$$I^G(f) = \int_Z f(z)G(dz).$$

Now we fix $n \geq 2$, and consider a function $f \in \mathcal{E}(\mu^n)$ with the form

$$f(z_1, \dots, z_n) = \sum_{1 \leq i_1, \dots, i_n \leq M} a_{i_1, \dots, i_n} \mathbf{1}_{A_{i_1}(z_1)} \cdots \mathbf{1}_{A_{i_n}(z_n)}$$

where $M \geq n$, and the real coefficients $a_{i_1,\ldots,i_n}$ are equal to zero whenever two indices i_k, i_l are equal and $A_1,\ldots,A_M$ are pairwise disjoint elements of $\mathcal{Z}_\mu$. Then we define

$$I_n^G(f) = \sum_{1\leq i_1,\ldots,i_n\leq M} a_{i_1,\ldots,i_n} G(A_{i_1})\ldots G(A_{i_n}), \tag{1.18}$$

and we say that $I_n^G(f)$ is the *multiple stochastic Wiener-Itô integral* (of order n) of kernel f with respect to Gaussian measure G. Then we may extend I_n^G to a continuous linear operator from $L^2(\mu^n)$ into $L^2(\mathbb{P})$ with the same procedures as showed in Poisson case. The application thus obtained

$$I_n^G : L_s^2(\mu^n) \to L^2(\mathbb{P}) : f \mapsto I_n^G(f)$$

is an isomorphism from $L_s^2(\mu^n)$, endowed with the modified scalar product $n!\langle\cdot,\cdot\rangle_{L^2(\mu^n)}$, into $L^2(\mathbb{P})$. Indeed, we have

Proposition 1.1.15 *The random variables $I_n^G(f), n \geq 1, f \in L_s^2(\mu^n)$, enjoy the following properties:*

1. *For every n, the application $f \mapsto I_n^G(f)$ is linear.*
2. *For every n, one has that $\mathbb{E}[I_n^G(f)] = 0$ and $I_n^G(f) = I_n^G(\tilde{f})$.*
3. *For every $n \geq 2$ and $m \geq 1$, for every $f \in \mathcal{E}(\mu^n)$ and $g \in \mathcal{E}(\mu^m)$,*

$$\mathbb{E}[I_n^G(f)I_m^G(g)] = n!\langle \tilde{f},\tilde{g}\rangle_{L^2(\mu^n)}\mathbf{1}_{(n=m)}, \qquad \textit{(isometric property)} . \tag{1.19}$$

For every $n \geq 2$, the $L^2(\mathbb{P})$-closed space

$$C_n(G) = \{I_n^G(f) : f \in L^2(\mu^n)\} \tag{1.20}$$

is called the *nth Wiener chaos* associated with Gaussian measure G. And we set

$$C_0(\hat{N}) = \mathbb{R} \tag{1.21}$$

We know that $C_n(\hat{N})\perp C_m(\hat{N})$ for $n \neq m$.

Example 1.1.16 *We consider the case where $(Z,\mathcal{Z}) = (\mathbb{R}_+,\mathcal{B}(\mathbb{R}_+))$ and μ is equal to the Lebesgue measure. As already observed, one has that the process $t \mapsto W_t = G([0,t]), t \geq 0$, is a standard Brownian motion started from zero. Also, for every $f \in L^2(\mu)$,*

$$I_1^G(f) = \int_{\mathbb{R}_+} f(t)G(dt) = \int_0^\infty f(t)dW_t, \tag{1.22}$$

where the RHS of (1.22) indicates a standard Itô integral with respect to W. Moreover, for every $n \geq 2$ and every $n \geq 2$ and every $f \in L^2(\mu^n)$,

$$I_n^G(f) = n!\int_0^1 \left[\int_0^{t_1}\int_0^{t_2}\cdots\int_0^{t_{n-1}} \tilde{f}(t_1,\ldots,t_n)dW_{t_n}\cdots dW_{t_2}\right] dW_{t_1}, \tag{1.23}$$

where the RHS of (1.23) stands for a usual Itô- type stochastic integral, with respect to W, of the stochastic process

$$t \mapsto \phi(t) = n!\int_0^{t_1}\int_0^{t_2}\cdots\int_0^{t_{n-1}} \tilde{f}(t_1,\ldots,t_n)dW_{t_n}\cdots dW_{t_2}, \qquad t_1 \geq 0.$$

Note in particular that $\phi(t)$ is adapted to the filtration $\sigma\{W_u : u \leq t\}, t \geq 0$, and also

$$\mathbb{E}\left[\int_0^\infty \phi^2(t)dt\right] < \infty.$$

Multiplication formulae

Definition 1.1.17 *Let $(Z, \mathcal{Z}, \mu)$ be a separable, σ-finite and non-atomic measure space. For every $p, q \geq 1$, $f \in L^2_s(\mu^p)$, $g \in L^2_s(\mu^q)$ and every $r = 0, \dots, q \wedge p$, the* ***contraction of order*** *r of f and g is the function $f \otimes_r g$ of $p+q-2r$ variables defined as follows: for $r = 1, \dots, q \wedge p$ and $(t_1, \dots, t_{p-r}, s_1, \dots, s_{q-r}) \in Z^{p+q-2r}$,*

$$\begin{aligned} & f \otimes_r g(t_1, \dots, t_{p-r}, s_1, \dots, s_{q-r}) \\ = & \int_{Z^r} f(z_1, \dots, z_r, t_1, \dots, t_{p-r}) g(z_1, \dots, z_r, s_1, \dots, s_{q-r}) \mu^r(dz_1 \dots dz_r), \end{aligned}$$

and , for $r = 0$,

$$\begin{aligned} f \otimes_r g(t_1, \dots, t_p, s_1, \dots, s_q) &= f \otimes g(t_1, \dots, t_p, s_1, \dots, s_q) \\ &= f(t_1, \dots, t_p) g(s_1, \dots, s_q). \end{aligned}$$

Note that, if $p = q$, then $f \otimes_p g = \langle f, g \rangle_{L^2(\mu^p)}$.

By an application of the Cauchy-Schwarz inequality, it is straightforward to prove that, for every $r = 0, \dots, q^p$, the function $f \otimes_r g$ is an element of $L^2(\mu^{p+q-2r})$. Note that $f \otimes_r g$ is in general not symmetric (although f and g are): we shall denote by $f \tilde{\otimes}_r g$ the canonical symmetrization of $f \otimes_r g$.

Theorem 1.1.18 *For every $p, q \geq 1$ and every $f \in L^2(\mu^p)$, $g \in L^2(\mu^q)$,*

$$I_p^G(f) I_q^G(g) = \sum_{r=0}^{p \wedge q} r! \left(\begin{array}{c} p \\ r \end{array} \right) \left(\begin{array}{c} q \\ r \end{array} \right) I_{p+q-2r} \left(\tilde{f} \otimes_r \tilde{g} \right). \tag{1.24}$$

This theorem can be easily established by induction, and we omit its proof.

Hermite polynomials and chaotic decomposition

Definition 1.1.19 *The sequence of* ***Hermite polynomials*** *$\{H_q; q \geq 0\}$ on $\mathbb{R}$, is defined via the following relations: $H_0 \equiv 1$ and, for $q \geq 1$,*

$$H_q(x) = (-1)^q e^{\frac{x^2}{2}} \frac{d^q}{dx^q} e^{-\frac{x^2}{2}}, \quad x \in \mathbb{R}.$$

For instance, $H_1(x) = 1$, $H_2(x) = x^2 - 1$ and $H_3(x) = x^3 - 3x$.

Recall that the sequence $\{(q!)^{-1/2} H_q; q \geq 0\}$ is an orthonormal basis of $L^2(\mathbb{R}, (2\pi)^{-1/2} e^{-x^2/2} dx)$. Several relevant properties of Hermite polynomials can be deduced from the following formula, valid for every $t, x \in \mathbb{R}$,

$$\exp\left(tx - \frac{t^2}{2} \right) = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x). \tag{1.25}$$

For instance, one deduces immediately from the previous expression that

$$\frac{d}{dx} H_n(x) = n H_{n-1}, \quad n \geq 1, \tag{1.26}$$

$$H_{n+1} = x H_n(x) - n H_{n-1}(x), n \geq 1. \tag{1.27}$$

The next result establish an explicit relation between multiple stochastic integrals and Hermite polynomials.

Proposition 1.1.20 *Let $h \in L^2(\mu)$ be such that $\|h\|_{L^2(\mu)} = 1$, and, for $n \geq 2$, define*

$$h^{\otimes n}(z_1, \dots, z_n) = h(z_1) \times \dots h(z_n), \qquad (z_1, \dots, z_n) \in Z^n.$$

Then,

$$I_n^G(h^{\otimes n}) = H_n(G(h)) = H_n(I_1(h)). \tag{1.28}$$

Proof. *Evidently, $H_1(I_1(h)) = I_1(h)$. By the multiplication formula (1.24), one has therefore that, for $n \geq 2$,*

$$I_n^G(h^{\otimes n})I_1(h) = I_{n+1}^G(h^{\otimes n+1}) + nI_{n-1}^G(h^{\otimes n-1})$$

and the conclusion is obtained from (1.27), and by recursion on n. ■

Proposition 1.1.21 *[Chaotic decomposition] Every random variable $F \in L^2(\sigma(G), \mathbb{P})$ (that is, F is a square-integrable functional of G) admits a (unique) chaotic decomposition of the type*

$$F = \mathbb{E}[F] + \sum_{n \geq 1}^{\infty} I_n^G(f_n) \tag{1.29}$$

where the series converges in $L^2(\mathbb{P})$ and, for each $n \geq 1$, the kernels f_n are elements of $L_s^2(\mu^n)$.

Proof. Fix $h \in L^2(\mu)$ such that $\|h\|_{L^2(\mu)} = 1$, as well as $t \in \mathbb{R}$. By using (1.25) and (1.28), one obtains that

$$\exp\left(tG(h) - \frac{t^2}{2}\right) = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(G(h)) = 1 + \sum_{n=0}^{\infty} \frac{t^n}{n!} I_n^G(h^{\otimes n}). \tag{1.30}$$

Since $\mathbb{E}\left[\exp\left(tG(h) - \frac{t^2}{2}\right)\right] = 1$, one deduces that (1.29) holds for every random variable of the form $F = \exp\left(tG(h) - \frac{t^2}{2}\right)$, with $f_n = \frac{t^n}{n!} h^{\otimes n}$. The conclusion is obtained by observing that the linear combinations of random variables of this type are dense in $L^2(\sigma(G), \mathbb{P})$. ■

Remark 1.1.22 Proposition 1.1.15, together with (1.29), implies that

$$\mathbb{E}[F^2] = \mathbb{E}[F]^2 + \sum_{n=1}^{\infty} n! \|f_n\|_{L^2(\mu^n)}^2. \tag{1.31}$$

1.1.3 Isonormal Gaussian process

Most of the results on Gaussian space appeared in this thesis can be generalized to *isonormal Gaussian process*. In this section we give a brief introduction of isonormal Gaussian process, which has been introduced by Dudley in [15]. In particular, the concept of an isonormal Gaussian process can be very useful in the study of fractional fields. See e.g. Pipiras and Taqqu [53, 54, 55], or the second edition of Nualart's book [38, Chapter 1]. For a general approach to Gaussian analysis by means of Hilbert space techniques, and for further details on the subjects discussed in this section, the reader is referred to Janson [21].

Definition 1.1.23 *Let $\mathfrak{H}$ be a real separable Hilbert space with inner product $\langle \cdot, \cdot \rangle_{\mathfrak{H}}$. We denote by*

$$G(\mathfrak{H}) = \{G(h) : h \in \mathfrak{H}\}$$

an ***isonormal Gaussian process*** *over $\mathfrak{H}$. This means that $G(\mathfrak{H})$ is a centered real-valued Gaussian family, indexed by the elements of $\mathfrak{H}$ and such that*

$$\mathbb{E}[G(h)G(h')] = \langle h, h' \rangle_{\mathfrak{H}}, \quad \forall h, h' \in \mathfrak{H}. \tag{1.32}$$

In other words, relation (1.32) means that $G(\mathfrak{H})$ is a centered Gaussian Hilbert space (with respect to the inner product canonically induced by the covariance) isomorphic to $\mathfrak{H}$.

We now present an important example of isonormal Gaussian process.

Example 1.1.24 *Let $(Z, \mathcal{Z}, \mu)$ be a separable, σ-finite and non-atomic measure space. We denote by $\mathcal{Z}_\mu$ the class of those $A \in \mathcal{Z}$ such that $\mu(A) < \infty$. Consider a completely random Gaussian measure $G = \{G(A) : A \in \mathcal{Z}_\mu\}$, as defined in the precedent section. Set $\mathfrak{H} = L^2(Z, \mathcal{Z}, \mu)$ with $\langle h, h' \rangle_{\mathfrak{H}} = \int_Z h(z)h'(z)\mu(dz)$ for every $h, h' \in \mathfrak{H}$, and define $G(h) = I_1(h)$ to be the Wiener-Itô integral of h with respect to G for every $h \in \mathfrak{H}$. Recall that $G(h)$ is a centered Gaussian random variable with variance given by $\|h\|^2_{\mathfrak{H}}$. Then, the collection $G(\mathfrak{H}) = \{G(h) : h \in L^2(Z, \mathcal{Z}, \mu)\}$ is an isonormal Gaussian process over $L^2(Z, \mathcal{Z}, \mu)$.*

In the present thesis, the example above is the only case of isonormal Gaussian process that interests us, though many results also holds in the general case.

1.2 Malliavin operators

Before dealing with Gaussian and Poisson cases, we start with a description of Malliavin operators based on the Fock space. We adopt the definitions in Nualart and Vives's paper [41].

1.2.1 Fock space

Let $\mathfrak{H}$ be a real separable Hilbert space. Consider the n-th tensorial product $\mathfrak{H}^{\otimes n}$. We denote by $\mathfrak{H}^{\odot n}$ the Hilbert space of n-th symmetric tensors, which are invariant under any automorphism. The norm in $\mathfrak{H}^{\odot n}$ is defined as

$$\|f\|^2_{\mathfrak{H}^{\odot n}} = n! \|f\|^2_{\mathfrak{H}^{\otimes n}}.$$

Definition 1.2.1 *The Fock space associated to $\mathfrak{H}$ is the Hilbert space $\Phi(\mathfrak{H}) = \bigoplus_{n=0}^{\infty} \mathfrak{H}^{\odot n}$ with the inner product $\langle h, g \rangle_{\Phi(\mathfrak{H})} = \sum_{n=0}^{\infty} \langle h_n, g_n \rangle_{\mathfrak{H}^{\odot n}}$ for $h = \sum_{n=0}^{\infty} h_n$ and $g = \sum_{n=0}^{\infty} g_n$, where $g_n, h_n \in \mathfrak{H}^{\odot n}$. Here we take $\mathfrak{H}^{\odot 0} = \mathbb{R}$ and $\mathfrak{H}^{\odot 1} = \mathfrak{H}^{\otimes 1} = \mathfrak{H}$.*

In the subsequent sections, we introduce two typical examples of Fock spaces: one associated with isonormal Gaussian process, and the other with Poisson process.

In this thesis, we are in particular interested in the important case $\mathfrak{H} = L^2(Z, \mathcal{Z}, \mu)$. It is not difficult to show that, in this special case, $\mathfrak{H}^{\otimes n}$ is isometric to $L^2(\mu^n) = L^2(Z^n, \mathcal{Z}^{\otimes n}, \mu^{\otimes n})$, while $\mathfrak{H}^{\odot n}$ is the space of square-integrable symmetric functions $L^2_s(\mu^n) = L^2_s(Z^n, \mathcal{Z}^{\otimes n}, \mu^{\otimes n})$ with the norm $\|\cdot\|_{\mathfrak{H}^{\odot n}}$. So, whenever we take $\mathfrak{H} = L^2(\mu)$, then $\mathfrak{H}^{\otimes n}$ and $\mathfrak{H}^{\odot n}$ can be identified as $L^2(\mu^n)$ and $L^2_s(\mu^n)$ respectively.

Now we give the definitions of several Malliavin operators on the Fock space $\Phi(\mathfrak{H})$ for $\mathfrak{H} = L^2(Z, \mathcal{Z}, \mu)$.

I) The derivative operator D.
Taken $\mathfrak{H} = L^2(Z, \mathcal{Z}, \mu)$. For every $F \in \Phi(\mathfrak{H})$, $F = \sum_{n=0}^{\infty} f_n$, where f_n may isometrically be seen as element of $L^2_s(\mu^n)$. We define the derivative of F, DF as the element of $\Phi(\mathfrak{H}) \bigotimes \mathfrak{H} \cong L^2(\mu; \Phi(\mathfrak{H}))$ given by

$$D_z F = \sum_{n=1}^{\infty} n f_n(\cdot, z), \quad \text{for a.e. } z \in Z.$$

DF exists whenever the above sum converges in $L^2(\mu; \Phi(\mathfrak{H}))$, which means

$$\begin{aligned} & \|DF\|^2_{L^2(\mu;\Phi(\mathfrak{H}))} \\ = & \int_Z \|D_z F\|^2_{\Phi(\mathfrak{H})}\, \mu(dz) \\ = & \sum_{n=1}^{\infty} n^2 (n-1)! \int_Z \|f_n(\cdot, z)\|^2_{L^2(\mu^{n-1})}\, \mu(dz) \\ = & \sum_{n=1}^{\infty} n^2 n! \|f_n\|^2_{L^2(\mu^n)} < \infty \end{aligned}$$

We will denote the domain of D by Dom$D$, which is a dense subspace of $\Phi(\mathfrak{H})$.

Moreover, for any $h \in L^2(\mu)$, we can define a closed and unbounded operator D_h from $\Phi(\mathfrak{H})$ to $\Phi(\mathfrak{H})$, by

$$D_h F = \sum_{n=1}^{\infty} n \int_Z f_n(\cdot, z) h(z) \mu(dz)$$

provided that this series converges in $\Phi(\mathfrak{H})$.

II) The Skorohod integral δ.
Consider the Hilbert space $L^2(\mu; \Phi(\mathfrak{H})) \cong \Phi(\mathfrak{H}) \bigotimes L^2(\mu)$. It can be decomposed into the orthogonal sum $\bigoplus_{n=0}^{\infty} \sqrt{n!} \cdot \hat{L}^2(\mu^{n+1})$, where $\hat{L}^2(\mu^{n+1})$ is the subspace of $L^2(\mu^{n+1})$ formed by all square-integrable functions on μ^{n+1} which are symmetric in the first n variables.

Let $u \in L^2(\mu; \Phi(\mathfrak{H}))$ be given by

$$u = \sum_{n \geq 0} u_n, \qquad u_n \in \hat{L}^2(\mu^{n+1}).$$

We define the Skorohod integral of u as the element of $\Phi(\mathfrak{H})$ given by,

$$\delta(u) = \sum_{n \geq 0} \tilde{u}_n,$$

providing that

$$\sum_{n\geq 0}(n+1)!\,\|\tilde{u}_n\|^2_{L^2(\mu^{n+1})} < \infty,$$

where $\tilde{u}_n$ is the symmetrization of u_n with respect to its $n+1$ variables.

We denote by $\mathrm{Dom}\delta$ the set of elements $u \in L^2(\mu;\Phi(\mathfrak{H}))$ verifying the above property.

The following result provide the duality relation between the operator D and δ.

Proposition 1.2.2 (Integration by parts) *Let $u \in \mathrm{Dom}\delta$, and $F \in \mathrm{Dom}D$, then,*

$$\langle u, DF\rangle_{L^2(\mu;\Phi(\mathfrak{H}))} = \langle F, \delta(u)\rangle_{\Phi(\mathfrak{H})}$$

Proof. Suppose that $u = \sum_{n\geq 0} u_n$, and $F = \sum_{n\geq 0} f_n$. Then,

$$\begin{aligned}
&\langle u, DF\rangle_{L^2(\mu;\Phi(\mathfrak{H}))}\\
=\;& \int_Z \langle u(\cdot,z), D_zF\rangle_{\Phi(\mathfrak{H})}\mu(dz)\\
=\;& \sum_{n\geq 0} n!\int_Z \langle u_n(\cdot,z), (n+1)f_{n+1}(\cdot,z)\rangle_{L^2(\mu^n)}\mu(dz)\\
=\;& \sum_{n\geq 0}(n+1)!\int_{Z^{n+1}} u_n(\cdot,z)f_{n+1}(\cdot,z)\mu(dz_1)\dots\mu(dz_n)\mu(dz)\\
=\;& \sum_{n\geq 0}(n+1)!\int_{Z^{n+1}} \tilde{u}_n(\cdot,z)f_{n+1}(\cdot,z)\mu(dz_1)\dots\mu(dz_n)\mu(dz)\\
=\;& \langle F, \delta(u)\rangle_{\Phi(\mathfrak{H})}
\end{aligned}$$

■

1.2.2 Malliavin calculus on the Poisson space

In this section, we shall introduce some Malliavin-type operators associated with the random Poisson measure $\hat{N}$. We follow again the work by Nualart and Vives [41], which is in turn based on the classic definition of Malliavin operators on the Gaussian space. (See e.g. [27],[38, Chapter 1].)

Note that, the square-integrable space $L^2(\mathcal{F},\mathbb{P}) = L^2(\mathbb{P})$ is a realization isometric to the Fock space $\Phi(\mathfrak{H})$ associated with $\mathfrak{H} = L^2(\mu)$. We have $L^2(\mathbb{P}) = \bigoplus_{n=0}^{\infty} C_n$ with $C_n = \{I_n(f), f \in L^2_s(\mu^n)\}$.

I) The derivative operator D.

For every $F \in L^2(\mathbb{P})$, the derivative of F, DF will be defined as an element of $L^2(\mathbb{P};L^2(\mu))$, that is, the space of the measurable random function $u:\Omega\times Z \mapsto \mathbb{R}$ such that $\mathbb{E}[\int_Z u_z^2\mu(dz)] < \infty$.

Definition 1.2.3 *1. The domain of the derivative operator D, denoted $\mathrm{Dom}D$, is the set of all the random variables $F \in L^2(\mathbb{P})$ having the chaotic decomposition (1.8) such that*

$$\sum_{k\geq 1} kk!\|f_k\|^2_{L^2(\mu^k)} < \infty,$$

2. *For any $F \in \mathrm{Dom}D$, the random function $z \mapsto D_zF$ is defined by*

$$D_zF = \sum_{k\geq 1}^{\infty} kI_{k-1}(f_k(z,\cdot)).$$

Since the underlying probability space Ω is assumed to be the collection of discrete measures as described in Remark 1.1.2, we define F_z by $F_z(\omega) = F(\omega + \delta_z), \quad \forall \omega \in \Omega$ for any given random variable F, where δ_z is the Dirac mass. The following lemma, which is essential to the calculations in this paper, provides an elegant representation of D.

Lemma 1.2.4 (Lemma 2.7 in [46],Theorem 6.2 in [41]) *For each $F \in \mathrm{Dom}D$,*

$$D_zF = \Psi_zF, a.e.\text{-}\mu(dz),$$

with the difference transformation $\Psi_zF = F_z - F$.

The proof of the lemma can be found in [41, Theorem 6.2].

Remark 1.2.5 Using twice Lemma 1.2.4, combined with a standard Taylor expansion, we have, for function f with bounded second derivative,

$$\begin{aligned} D_zf(F) &= f(F_z) - f(F) &(1.33)\\ &= f'(F)(F_z - F) + R(F_z - F) &(1.34)\\ &= f'(F)(D_z(F)) + R(D_z(F)) &(1.35)\end{aligned}$$

where the mapping $y \mapsto R(y)$ is such that $R(y) \leq \frac{1}{2}\|f\|_\infty y^2$. The equation (1.35) is the "chain rules" on the Poisson space.

Definition 1.2.6 (see Page 411 in [19]) *Let $\mathbb{D}^{k,2}$, $k = 1, 2, \ldots$, denote the set of $F \in L^2(\mathbb{P})$ such that*

$$\sum_{m=k}^{\infty} m!m(m-1)\cdots(m-k+1)\|f_m\|^2_{L^2(\mu^m)} < \infty. \tag{1.36}$$

D_k is then defined as the closed linear operator from $\mathbb{D}^{k,2}$ to $L^2(\Omega \times Z, \mathcal{Z} \otimes \mathbb{B}(Z^k), \mathbb{P} \otimes \mu^k)$ such that

$$D^k_{t_1,\ldots,t_k}F = \sum_{m=k}^{\infty} m(m-1)\cdots(m-k+1)I_{m-k}(f_m(t_1,\ldots,t_k,\cdot)), \qquad a.s.$$

Furthermore, $\mathbb{E}\|D_kF\|^2_{L^2(\mu^k)}$ equals to LHS of (1.36). In particular, $\mathbb{D}^{1,2} = \mathrm{Dom}D$.

Lemma 1.2.7 *Fix $k \geq 1$. For any $F \in \mathbb{D}^{k,2}$, we have that*

$$D^k_{t_1,\ldots,t_k}F = \Psi^k_{t_1,\ldots,t_k}F,$$

a.s. for all $t_1, \ldots, t_k$ μ-a.e., where $\Psi^k_{t_1,\ldots,t_k}$ is the kth iteration of the difference transformation $\Psi_t = F_z - F$.

II) The Skorohod integral δ.
Thanks to the chaotic representation property of $\hat{N}$, there exists a unique representation for each random function $u \in L^2(\mathbb{P}, L^2(\mu))$:

$$u_z = \sum_{k\geq 0}^{\infty} I_k(f_k(z, \cdot)), \qquad z \in Z \tag{1.37}$$

where the kernel f_k is a function of $k+1$ variables, and $f_k(z, \cdot)$ is an element of $L^2_s(\mu^k)$. The Skorohod integral $\delta(u)$ transforms u into an element of $L^2(\mathbb{P})$.

Definition 1.2.8 *1. The domain of the Skorohod integral operator, denoted by* Domδ, *is the collection of all* $u \in L^2(\mathbb{P}, L^2(\mu))$ *having the above chaotic expansion (1.37) satisfied the condition:*

$$\sum_{k\geq 0}(k+1)!\|f_k\|^2_{L^2(\mu^(k+1))} < \infty$$

2. For $u \in$ Domδ, *the Skorohod integral of* u *is a random variable* $\delta(u)$ *such that*

$$\delta(u) = \sum_{k\geq 0} I_{k+1}(\tilde{f}_k)$$

where $\tilde{f}_k$ *is the canonical symmetrization of* $k+1$ *variable function* f_k.

The Skorohod integral δ can be seen as the adjoint operator of derivative operator D thanks to the following elegant result.

Lemma 1.2.9 (Integration by parts, Proposition 4.2 in [41]) *For every* $F \in$ Dom$D$ *and* $u \in$ Domδ, *we have*

$$\mathbb{E}[F\delta(u)] = \mathbb{E}[\langle D_z F, u_z\rangle_{L^2(\mu)}]$$

The proof of this lemma is given at page 158 in [41] Nualart and Vives's paper [41].

III) The Ornstein-Uhlenbeck generator L.

Definition 1.2.10 *1. The domain of the Ornstein-Uhlenbeck generator, denoted by* Dom$L$, *is the collection of all* $F \in L^2(\mathbb{P})$ *admitting the chaotic representation 1.8 verifies the condition:*

$$\sum_{k\geq 1} k^2 k!\|f_k\|^2_{L^2(\mu^k)} < \infty$$

2. The Ornstein-Uhlenbeck generator L *acts on random variable* $F \in$ Dom$L$ *as follows*

$$LF = -\sum_{k\geq 1} kI_k(f_k).$$

The following lemma is useful for our forthcoming calculations.

Lemma 1.2.11 $F \in$ Dom$L$ *iff* $F \in$ Dom$D$ *and* $DF \in$ Domδ. *In addition, we have*

$$\delta DF = -LF$$

The proof of the lemma comes directly from the definition of operators D, δ and L. The readers can find a proof in [46, Lemma 2.10].

IV) The pseudo-inverse of L.

Definition 1.2.12 *1. The domain of the pseudo-inverse Ornstein-Uhlenbeck generator, denoted by $L_0^2(\mathbb{P})$, is the space of* centered *random variables in $L^2(\mathbb{P})$.*

2. For $F = \sum_{k\geq 1} I_k(f_k) \in L_0^2(\mathbb{P})$, we set

$$L^{-1}F = -\sum_{k\geq 1} \frac{1}{k} I_k(f_k)$$

the pseudo-inverse Ornstein-Uhlenbeck generator of F.

Remark 1.2.13 $\forall F \in L_0^2(\mathbb{P})$, one has that $L^{-1}F \in \mathrm{Dom}L$ and

$$F = LL^{-1}F = \delta(-DL^{-1}F).$$

1.2.3 Malliavin calculus on the Gaussian space

Though this thesis is focused on the functionals on the Poisson space, the Malliavin calculus on the Gaussian spaces takes part in some calculations (e.g. in the proof of multi-dimensional Stein's lemma). We give below the definitions and preperties introduced in Chapter 1 of Nualart's book [38]. Among all the properties, Mehler's formula and the "differential characterization of L" are especially interesting and important.

Derivatives

We start by defining the class $S(G) \subset L^2(\sigma(G))$ of smooth functionals of Gaussian measure G, as the collection of random variables of the type

$$F = f(G(h_1), \dots, G(h_m)),$$

where $h_i \in \mathfrak{H}$, and f is in the class of infinitely differentiable functions on $\mathbb{R}^d$ such that f and its derivatives have polynomial growth.

Definition 1.2.14 *Let $F \in S(G)$ be as shown above.*

1. *The derivative DF of F is the $\mathfrak{H}$-valued random element given by*

$$DF = \sum_{i=1}^{m} \frac{\partial}{\partial x_i} f(G(h_1), \dots, G(h_m))h_i.$$

2. *For $k \geq 2$, the kth derivative of F, denoted by D^kF, is the element of $L^2(\sigma(G); \mathfrak{H}^{\otimes k})$ given by*

$$D^kF = \sum_{i_1,\dots,i_k=1}^{m} \frac{\partial^k}{\partial x_{i_1} \cdots x_{i_k}} f(G(h_1), \dots, G(h_m))h_{i_1} \otimes \cdots \otimes h_{i_k}.$$

Definition 1.2.15 *For every $k \geq 1$, the **domain** of the operator D^k in $L^2(\sigma(G))$, customarily denoted by $\mathbb{D}^{k,2}$, is the closure of the class $S(G)$ with respect to the seminorm*

$$\|F\|_{k,2} = \left[\mathbb{E}[F^2] + \sum_{j=1}^{k} \|D^j F\|^2_{\mathfrak{H}^{\otimes j}}\right]^{\frac{1}{2}}.$$

We also set

$$\mathbb{D}^{\infty,2} = \bigcap_{k=1}^{\infty} \mathbb{D}^{k,2}$$

Proposition 1.2.16 *Fix $k \geq 1$. A random variable $F \in L^2(\sigma(G))$ with a chaotic representation 1.29 is an element of $\mathbb{D}^{k,2}$ if and only if the kernels $\{f_q\}$ verify*

$$\sum_{q=1}^{\infty} q^k q! \|f_q\|^2_{\mathfrak{H}^{\otimes q}} < \infty, \tag{1.38}$$

and in this case

$$\mathbb{E}\|D^k F\|^2_{\mathfrak{H}^{\otimes k}} = \sum_{q=k}^{\infty} (q)_k \times q! \|f_q\|^2_{\mathfrak{H}^{\otimes q}}$$

where $(q)_k = q(q-1)\cdots(q-k+1)$ is the Pochammer symbol.

Chain rules:

Proposition 1.2.17 *Let $\varphi : \mathbb{R}^m \to \mathbb{R}$ be a continuously differentiable function with bounded partial derivatives. Assume that $F = (F_1, \dots, F_m)$ is a vector of elements of $\mathbb{D}^{1,2}$. Then, $\varphi(F) \in \mathbb{D}^{1,2}$, and*

$$D\varphi(F) = \sum_{i=1}^{m} \frac{\partial}{\partial x_i} \varphi(F) DF_i. \tag{1.39}$$

Proposition 1.2.18 *Suppose that $\mathfrak{H} = L^2(Z, \mathcal{Z}, \mu)$, and assume that $F \in \mathbb{D}^{1,2}$ admits the chaotic expansion (1.29). Then, a version of the derivative $DF = \{D_z F : z \in Z\}$ is given by*

$$D_z F = \sum_{n=1}^{\infty} n I^G_{n-1}(f_n(\cdot, z)), \quad z \in Z,$$

where, for each n and z, the integral $I^G_{n-1}(f_n(\cdot, z))$ is obtained by integrating the function on Z^{n-1} given by $(z_1, \dots, z_{n-1}) \mapsto f_n(z_1, \dots, z_{n-1}, z)$.

Divergences

Definition 1.2.19 *The **domain** of the divergence operator δ, denoted by $\mathrm{Dom}\delta$, is the collection of all random elements $u \in L^2(\sigma(G); \mathfrak{H})$ such that, for every $F \in \mathbb{D}^{1,2}$,*

$$|\mathbb{E}[\langle u, DF \rangle_{\mathfrak{H}}]| \leq c\mathbb{E}[F^2]^{1/2}, \tag{1.40}$$

where c is a constant depending on u (and not on F). For every $u \in \mathrm{Dom}\delta$, the random variable $\delta(u)$ is therefore defined as the unique element of $L^2(\sigma(G))$ verifying

$$\mathbb{E}[\langle u, DF\rangle_{\mathfrak{H}}] = \mathbb{E}[F\delta(u)], \tag{1.41}$$

for every $F \in \mathbb{D}^{1,2}$ (note that the existence of $\delta(u)$ is ensured by (1.40) and by the Rietz Representation Theorem). Relation (1.41) is called ***integration by parts formula****.*

Gaussian measure: We now consider the case $\mathfrak{H} = (Z, \mathcal{Z}, \mu)$, then the space $L^2(\sigma(G); \mathfrak{H})$ can be identified with the class of stochastic processes $u(z, \omega)$ that are $\mathcal{Z} \otimes \sigma(G)$-measurable, and verify the integrability condition

$$\mathbb{E}[\int_{Z^k} u(z)^2 \mu(dz)] < \infty. \tag{1.42}$$

We thus infer that every $u \in L^2(\sigma(G); \mathfrak{H})$ admits a representation of the type

$$u(z) = h_0(z) + \sum_{n=1}^{\infty} I_n^G(h_n(\cdot, z)), \tag{1.43}$$

where $h_0 \in L^2(\mu)$ and, for every $n \geq 1$, h_n is a function on Z^{n+1} which is symmetric in the first n variables, and moreover

$$\mathbb{E}[\int_{Z^k} u(z)^2 \mu(dz)] = \sum_{n=0}^{\infty} n! \|h_n\|^2_{L^2(\mu^{n+1})} < \infty. \tag{1.44}$$

Proposition 1.2.20 *Let $\mathfrak{H} = (Z, \mathcal{Z}, \mu)$ as above, and let $u \in L^2(\sigma(G); \mathfrak{H})$ verify (1.42), (1.47) and (1.44). Then, $u \in \mathrm{Dom}(\delta)$ if and only if*

$$\sum_{n=0}^{\infty} (n+1)! \|\tilde{h}_n\|^2_{L^2(\mu^{n+1})} < \infty. \tag{1.45}$$

where $\tilde{h}_n$ indicates the canonical symmetrization of h_n. In this case, one has moreover that

$$\delta(u) = \sum_{n=0}^{\infty} I_{n+1}(\tilde{h}_n),$$

where the series converges in $L^2(\mathbb{P})$.

Ornstein-Uhlenbeck semigroup

Definition 1.2.21 *The* ***Ornstein-Uhlenbeck semigroup*** $\{T_t : t \geq 0\}$ *is the set of contraction operators defined as*

$$T_t(F) = \mathbb{E}[F] + \sum_{q=1}^{\infty} e^{-qt} I_q^G(f_q) = \sum_{q=0}^{\infty} e^{-qt} I_q^G(f_q), \tag{1.46}$$

for every $t \geq 0$ and every $F \in L^2(\sigma(G))$ as in (1.29).

The generator of the Ornstein-Uhlenbeck semigroup and its pseudo-inverse.

Definition 1.2.22 *Let $F \in L^2(\sigma(G))$ admit the representation (1.29). We define the operator L as*

$$LF = -\sum_{q=0}^{\infty} q I_q^G(f_q), \qquad (1.47)$$

provided the previous series converges in $L^2(\mathbb{P})$. This implies that the domain of L, denoted by $\mathrm{Dom}L$*, is given by*

$$\mathrm{Dom}L = \{F \in L^2(\sigma(G)), F = \sum_{q=0}^{\infty} I_q^G(f_q) : \sum_{q=1}^{\infty} q^2 q! \|f_q\|^2_{L^2(\mathfrak{H}^{\otimes q})} < \infty\} = \mathbb{D}^{2,2}. \qquad (1.48)$$

Note that the image of L coincides with the set

$$L_0^2(\sigma(G)) = \{F \in L^2(\sigma(G)) : \mathbb{E}[F] = 0\},$$

and also that $LF = L(F - \mathbb{E}[F])$.

Definition 1.2.23 *Let $L_0^2(\sigma(G))$ admit the representation (1.29), with $\mathbb{E}[F] = I_0^G(f_0) = 0$. We define the operator L^{-1} (pseudo-inverse of L) as*

$$L^{-1} = -\sum_{q=1}^{\infty} \frac{1}{q} I_q^G(f_q). \qquad (1.49)$$

Remark 1.2.24

1. There is an important relation between the operator D, δ and L: a random variable F belongs to $\mathbb{D}^{2,2}$ if and only if $F \in \mathrm{Dom}(\delta D)$ and, in this case,

$$\delta D F = -LF. \qquad (1.50)$$

2. For any $F \in L^2(\sigma(G))$, we have that $L^{-1}F \in \mathrm{Dom}L$, and

$$LL^{-1}F = F - \mathbb{E}[F]. \qquad (1.51)$$

i) **Mehler's formula**.

Let F be an element of $L^2(\sigma(G))$, so that F can be represented as an application from $\mathbb{R}^{\mathfrak{H}}$ into $\mathbb{R}$. Then, an alternative representation (due to Mehler) of the action of the Ornstein-Uhlenbeck semigroup T (as defined in equation (1.46)) on F, is the following:

$$T_t(F) = \mathbb{E}[F(e^{-t}a + \sqrt{1-e^{-2t}}X)]|_{a=X}, \quad t \geq 0, \qquad (1.52)$$

where a designs a generic element of $\mathbb{R}^{\mathfrak{H}}$. See Section 1.4.1 in Nualart [38] for more details on this and other characterizations of T.

ii) **Differential characterization of L.**

Let $F \in L^2(\sigma(G))$ have the form $F = f(G(h_1), \ldots, G(h_d))$, where $f \in \mathcal{C}^2(\mathbb{R}^d)$ has bounded first and second derivatives, and $h_i \in \mathfrak{H}$, $i = 1, \ldots, d$. Then,

$$\begin{aligned} LF &= \sum_{i,j=1}^{d} \frac{\partial^2 f}{\partial x_i \partial x_j}(G(h_1), \ldots, G(h_d))\langle h_i, h_j\rangle_{\mathfrak{H}} - \sum_{i=1}^{d} \frac{\partial f}{\partial x_i}(G(h_1), \ldots, G(h_d))G(h_i) \qquad (1.53) \\ &= \langle C, \mathrm{Hess}\, f(X_C)\rangle_{H.S.} - \langle X_C, \nabla f(X_C)\rangle_{\mathbb{R}^d}, \qquad (1.54) \end{aligned}$$

where $X_C = (G(h_1), \ldots, G(h_d))$, and $C = \{C(i,j) : i,j = 1, \ldots, d\}$ is the $d \times d$ covariance matrix such that $C(i,j) = \mathbb{E}[G(h_i)G(h_j)] = \langle h_i, h_j\rangle_{\mathfrak{H}}$. Note that $\langle ; \rangle_{H.S.}$ is the Hilbert-Schmidt inner product. (See Definition 1.3.1.)

1.3 Introduction to Stein's method

1.3.1 The basic idea of Stein's method

Stein's method is a powerful probability tool developed by C. Stein [62, 63, 64, 65], L. H. Y. Chen [12] and many other mathematicians. A wide range of researches have been made in this topic, see e.g. the two volumes book of Barbour and Chen(2005) [1, 2] and the recent book by Chen, Goldstein and Shao [13].

In researches in probability theory, one often needs to show a random variable W has a distribution close to that of a target distribution of the random variable X. To this end, one can compare the values of the expectations of the two distributions on some class of test functions. There are several choices of test function classes. For example, one can compare the characteristic function $\phi(u) = \mathbb{E}[e^{iuW}]$ of W to that of X, thus encapsulating all expectations of the family of functions e^{iuz} for $u \in \mathbb{R}$. As this family of functions is rich enough, closeness of the characteristic functions implies closeness of the distributions. Since convolution in the space of measures become products in the realm of characteristic functions, the characteristic function is indeed a convenient choice to treat the sum of independent random variables. Another choice is to consider the family of power functions $z^k, k = 1, 2, \ldots$, which leads to so-called "method of moments". Nevertheless, all these classical methods have their own limitations. (See, for instance, the example in Section 2 of Peccati's Lecture Notes [43].) Stein's method, which based directly on a random variable characterization of a distribution, allows the manipulation of the distribution through constructions involving the basic random quantities of which W is composed, and coupling can begin to play a large role.

Consider, then, testing for the closeness of the distributions of W and X by evaluating the difference between the expectations $\mathbb{E}[h(W)]$ and $\mathbb{E}[h(X)]$ over some collection of function h. It seems clear that if the distribution of W is close to the distribution of X then the difference $\mathbb{E}[h(W)] - \mathbb{E}[h(X)]$ should be small for many functions h. Specializing the problem, for a specific distribution, we may evaluate the difference by relying on a characterization of X. For instance, we can prove that the distribution of a random variable X is $\mathcal{N}(0, 1)$ if and only if

$$\mathbb{E}[f'(X) - Xf(X)] = 0 \tag{1.55}$$

for all absolutely continuous functions f for which the expectation above exists. Again, if the distribution of W is close to that of X, then evaluating the left hand side of (1.55) when X is replaced by W should result in something small. Putting these two differences together, from the Stein characterization (1.55) we arrive at the Stein equation

$$f'(w) - wf(w) = h(w) - \mathbb{E}[h(X)] \tag{1.56}$$

Now given h, one solves (1.56) for f, evaluates the left hand side of (1.56) at W and takes the expectation, obtaining $\mathbb{E}[h(W)] - \mathbb{E}[h(X)]$.

To prepare the introduction of Stein's method, we define below the norms and distances between probability measures.

1.3.2 Norms and distances between probability measures

Definition 1.3.1 *1. The* **Hilbert-Schmidt inner product** *and the* **Hilbert - Schmidt norm** *on the class of $d \times d$ real matrices, denoted respectively by $\langle \cdot, \cdot \rangle_{H.S.}$ and $\| \cdot \|_{H.S.}$, are defined as follows: for every pair of matrices A and B, $\langle A, B \rangle_{H.S.} := Tr(AB^T)$ and $\|A\|_{H.S.} = \sqrt{\langle A, A \rangle_{H.S.}}$, where $Tr(\cdot)$ indicates the usual trace operator.*

2. *The* ***operator norm*** *of a $d \times d$ real matrix A is given by $\|A\|_{op} := \sup_{\|x\|_{\mathbb{R}^d}=1} \|Ax\|_{\mathbb{R}^d}$.*

3. *For every function $g : \mathbb{R}^d \to \mathbb{R}$, let*

$$\|g\|_{Lip} := \sup_{x \neq y} \frac{|g(x) - g(y)|}{\|x - y\|_{\mathbb{R}^d}},$$

where $\| \cdot \|_{\mathbb{R}^d}$ is the usual Euclidian norm on $\mathbb{R}^d$. If $g \in \mathbb{C}^1(\mathbb{R}^d)$, we also write

$$M_2(g) := \sup_{x \neq y} \frac{\|\nabla g(x) - \nabla g(y)\|_{\mathbb{R}^d}}{\|x - y\|_{\mathbb{R}^d}},$$

If $g \in \mathbb{C}^2(\mathbb{R}^d)$,

$$M_3(g) := \sup_{x \neq y} \frac{\|\mathrm{Hess}\, g(x) - \mathrm{Hess}\, g(y)\|_{op}}{\|x - y\|_{\mathbb{R}^d}},$$

where $\mathrm{Hess}\, g(z) = \left(\frac{\partial^2 g}{\partial x_i \partial x_j}(z) \right)_{1 \leq i \leq d, 1 \leq j \leq d}$ stands for the Hessian matrix of g evaluated at a point z.

4. *For a positive integer k and a function $g \in \mathbb{C}^k(\mathbb{R}^d)$, we set*

$$\|g^{(k)}\|_\infty = \max_{1 \leq i_1 \leq \ldots \leq i_k \leq d} \sup_{x \in \mathbb{R}^d} \left| \frac{\partial^k}{\partial x_{i_1} \ldots \partial x_{i_k}} g(x) \right|.$$

In particular, by specializing this definition to $g^{(2)} = g''$ and $g^{(3)} = g'''$, we obtain

$$\|g''\|_\infty = \max_{1 \leq i_1 \leq i_2 \leq d} \sup_{x \in \mathbb{R}^d} \left| \frac{\partial^2}{\partial x_{i_1} \partial x_{i_2}} g(x) \right|.$$

$$\|g'''\|_\infty = \max_{1 \leq i_1 \leq i_2 \leq i_3 \leq d} \sup_{x \in \mathbb{R}^d} \left| \frac{\partial^3}{\partial x_{i_1} \partial x_{i_2} \partial x_{i_3}} g(x) \right|.$$

Remark 1.3.2 1. The norm $\|g\|_{Lip}$ is written $M_1(g)$ in [11].

2. If $g \in \mathbb{C}^1(\mathbb{R}^d)$, then $\|g\|_{Lip} = \sup_{x \in \mathbb{R}^d} \|\nabla g(x)\|_{\mathbb{R}^d}$. If $g \in \mathbb{C}^2(\mathbb{R}^d)$, then

$$M_2(g) = \sup_{x \in \mathbb{R}^d} \|\mathrm{Hess}\, g(x)\|_{op}.$$

Definition 1.3.3 *The distance between the laws of two $\mathbb{R}^d$-valued random vectors X and Y such that $\mathbb{E}\|X\|_{\mathbb{R}^d}$, $\mathbb{E}\|Y\|_{\mathbb{R}^d} < \infty$, written $d_{\mathcal{G}}$, is given by*

$$d_{\mathcal{G}}(X,Y) = \sup_{g\in\mathcal{G}} |\mathbb{E}[g(X)] - \mathbb{E}[g(Y)]|,$$

where $\mathcal{G}$ indicates some class of functions. In particular,

1. *by taking $\mathcal{G} = \{g : \|g\|_{Lip} \leq 1\}$, one obtains the Wasserstein (or Kantorovich-Wasserstein) distance, denoted by d_W;*
2. *by taking $\mathcal{G}$ equal to the collection of all indicators $\mathbf{1}_B$ of Borel sets, one obtains the total variation distance, denoted by d_{TV};*
3. *by taking $\mathcal{G}$ equal to the class of all indicators functions $\mathbf{1}_{(-\infty;z_1]}\cdots\mathbf{1}_{(-\infty;z_d]}$, $(z_1,\dots,z_d)\in\mathbb{R}^d$, one has the Kolmogorov distance, denoted by d_{Kol};*
4. *the distance obtained by taking $\mathcal{G} = \{g : g \in \mathbb{C}^2(\mathbb{R}^d), \|g\|_{Lip} \leq 1, M_2(g) \leq 1\}$will be denoted by d_2;*
5. *the distance obtained by taking $\mathcal{G} = \{g : g \in \mathbb{C}^3(\mathbb{R}^d), \|g''\|_\infty \leq 1\ ,\ \|g'''\|_\infty \leq 1\}$ will be denoted by d_3.*

It is well acknowledged that the topologies induced by d_W,d_{Kol}, d_{TV} are stronger than the topology of convergence in distribution. (See e.g. chapter 11 of [16] for details). Now we show that d_2 and d_3 have similar properties.

Lemma 1.3.4 *Let $d \geq 1$ be a positive integer. We consider the distance between two random variables with the following form:*

$$d_{\mathcal{G}}(X,Y) = \sup_{g\in\mathcal{G}} |\mathbb{E}[g(X)] - \mathbb{E}[g(Y)]|$$

where $\mathcal{G}$ is a suitable collection of functions $f : \mathbb{R}^d \to \mathbb{R}$. Suppose that for any real vector $(\lambda_1,\dots,\lambda_d) \in \mathbb{R}^d$, there exists two non-zero constants $C_1(\lambda_1,\dots,\lambda_d)$ and $C_2(\lambda_1,\dots,\lambda_d)$, such that both functions

$$C_1(\lambda_1,\dots,\lambda_d)\sin(\lambda_1 x_1 + \ldots + \lambda_d x_d)$$

and

$$C_2(\lambda_1,\dots,\lambda_d)\cos(\lambda_1 x_1 + \ldots + \lambda_d x_d)$$

belong to $\mathcal{G}$. Then, for any collection of random variables $(F, F_1, F_2, \dots)$, the condition

$$d_{\mathcal{G}}(F_n, F) \to 0, \qquad \text{as } n \to \infty$$

implies the convergence in law

$$F_n \xrightarrow{law} F, \qquad \text{as } n \to \infty.$$

Proof. For d-dimension real vectors $\lambda = (\lambda_1,\dots,\lambda_d)$ and $x = (x_1,\dots,x_d)$, we define

$$A_\lambda(x) = C_1(\lambda)\sin(\lambda x) = C_1(\lambda_1,\dots,\lambda_d)\sin(\lambda_1 x_1 + \ldots + \lambda_d x_d)$$

$$B_\lambda(x) = C_2(\lambda)\cos(\lambda x) = C_2(\lambda_1, \ldots, \lambda_d)\cos(\lambda_1 x_1 + \ldots + \lambda_d x_d)$$

where

$$\lambda x = \lambda_1 x_1 + \ldots + \lambda_d x_d, \qquad \lambda x = \lambda_1 x_1 + \ldots + \lambda_d x_d.$$

For any $\lambda \in \mathbb{R}^d$, both A_λ and B_λ are elements of $\mathcal{G}$. By the definition of distance $d_\mathcal{G}$, we have, for all $\lambda \in \mathbb{R}^d$,

$$\lim_{n\to\infty} \mathbb{E}[A_\lambda(F_n)] = \mathbb{E}[A_\lambda(F)], \qquad \lim_{n\to\infty} \mathbb{E}[B_\lambda(F_n)] = \mathbb{E}[B_\lambda(F)],$$

which leads to

$$\lim_{n\to\infty} \mathbb{E}[\sin(\lambda F_n)] = \mathbb{E}[\sin(\lambda F)], \qquad \lim_{n\to\infty} \mathbb{E}[\cos(\lambda F_n)] = \mathbb{E}[\cos(\lambda F)], \quad \forall \lambda \in \mathbb{R}^d$$

or

$$\lim_{n\to\infty} \mathbb{E}[\exp(i\lambda F_n)] = \mathbb{E}[\exp(i\lambda F)].$$

The convergence in law is immediate. ■

Remark 1.3.5 *Now note that the classes $\mathcal{G}$ over which d_2 and d_3 are built satisfy the assumptions of Lemma 1.3.4, this immediately yields that the topology induced by d_2 and d_3 are stronger than that of the convergence in distribution.*

Remark 1.3.6 The distances d_2 and d_3 are related, respectively, to the estimates of Section 3.2 and Section 3.3. Let $j = 2, 3$. It is easily seen that, if $d_j(F_n, F) \to 0$, where F_n, F are random vectors in $\mathbb{R}^d$, then necessarily F_n converges in distribution to F. It will also become clear later on that, in the definition of d_2 and d_3, the choice of the constant 1 as a bound for $\|g\|_{Lip}$, $M_2(g)$, $\|g''\|_\infty$, $\|g'''\|_\infty$ is arbitrary and immaterial for the derivation of our main results (indeed, we defined d_2 and d_3 in order to obtain bounds as simple as possible). See the two tables in Section 3.3.2 for a list of available bounds involving more general test functions.

1.3.3 The case of dimension 1

Let $N \sim \mathcal{N}(0, 1)$. Consider a real-valued function $h : \mathbb{R} \to \mathbb{R}$ such that the expectation $\mathbb{E}[h(N)]$ is well-defined. The *Stein's equation* associated with h and N is classically given by

$$h(x) - \mathbb{E}[h(N)] = f'(x) - xf(x), \quad x \in \mathbb{R}. \tag{1.57}$$

A solution to (1.57) is a function f which is Lebesgue a.e.-differentiable, and such that there exists a version of f' verifying (1.57) for every $x \in \mathbb{R}$. The following result, which is the basis of the whole Stein's theory, is due to Stein ([63], [65]). The proof of point (i) (which is often referred as *Stein's lemma*) involves a standard use of the Fubini theorem. Point (ii) is proved e.g. Lemma 2.1 in [14]; point (iii) can be obtained by combining the arguments in page 25 of [65] and Lemma 5.1 in [10] ; a proof of point (iv) is provided in Lemma 3 of [65]) ; and point (v) is proved in Lemma 4.3 of [8]. Note that, the inequality $\|f'\|_\infty \le \|h'\|_\infty$ in (v) can be reinforced by $\|f'\|_\infty \le \sqrt{2/\pi}\|h'\|_\infty$. We shall present a short proof of some of the results in (iv) and (v) in the next section.

Lemma 1.3.7 *(i) Let W be a random variable. Then, $W \overset{(law)}{=} N \sim \mathcal{N}(0,1)$ if, and only if,*

$$\mathbb{E}[f'(W) - Wf(W)] = 0, \tag{1.58}$$

for every continuous and piecewise continuously differentiable function f verifying the relation $\mathbb{E}|f'(N)| < \infty$.

(ii) If $h(x) = \mathbf{1}_{(-\infty,z]}(x), z \in \mathbb{R}$, then (1.57) admits a solution f which is bounded by $\sqrt{2\pi}/4$, piecewise continuously differentiable and such that $\|f'\|_\infty \leq 1$.

(iii) If h is bounded by $1/2$, then (1.57) admits a solution f which is bounded by $\sqrt{\pi/2}$, Lebesgue a.e. differentiable and such that $\|f'\|_\infty \leq 2$.

(iv) If h is bounded and absolutely continuous, then (1.57) has a solution f which is bounded and twice differentiable, and such that $\|f\|_\infty \leq \sqrt{\pi/2}\|h - \mathbb{E}[h(N)]\|_\infty$, $\|f'\|_\infty \leq 2\|h - \mathbb{E}[h(N)]\|_\infty$.

(v) If h is absolutely continuous with bounded derivative, then (1.57) has a solution f which is twice differentiable and such that $\|f'\|_\infty \leq \|h'\|_\infty$ and $\|f''\|_\infty \leq 2\|h'\|_\infty$.

Recall the relation:

$$2d_{TV}(X,Y) = \sup\{|\mathbb{E}[u(X)] - E[u(Y)]| : \|u\|_\infty \leq 1\}, \tag{1.59}$$

we have

Corollary 1.3.8 *1.*

$$d_{Kol}(X,N) \leq \sup_{f \in \mathcal{F}_{Kol}} |\mathbb{E}[Xf(X) - f'(X)]|, \tag{1.60}$$

where $\mathcal{F}_{Kol}$ is the class of piecewise continuously differentiable functions that are bounded by $\sqrt{2\pi}/4$ and such that their derivative is bounded by 1.

2.

$$d_{TV}(X,N) \leq \sup_{f \in \mathcal{F}_{TV}} |\mathbb{E}[Xf(X) - f'(X)]|, \tag{1.61}$$

where $\mathcal{F}_{TV}$ is the class of piecewise continuously differentiable functions that are bounded by $\sqrt{\pi/2}$ and such that their derivatives are bounded by 2.

3.

$$d_W(X,N) \leq \sup_{f \in \mathcal{F}_W} |\mathbb{E}[Xf(X) - f'(X)]|, \tag{1.62}$$

where F_W is the class of twice differentiable functions, whose first derivative is bounded by 1 and whose second derivative is bounded by 2.

1.3.4 Proofs

Here we present some of the proof on point (iv) and (v) in Lemma 1.3.7. The simple proofs we give below are selected from Chatterjee's Lecture Notes [7]. All proofs are based on the two lemmas below which characterize the solution structure of Stein's equation (1.57).

Lemma 1.3.9 *For any given function $h : \mathbb{R} \to \mathbb{R}$ such that $\mathbb{E}|h(N)| < \infty$ with $N \sim \mathcal{N}(0,1)$,*

$$f(x) = e^{x^2/2} \int_{-\infty}^{x} e^{-y^2/2} \left(h(y) - \mathbb{E}[h(N)]\right) dy \tag{1.63}$$

$$= -e^{x^2/2} \int_{x}^{\infty} e^{-y^2/2} \left(h(y) - \mathbb{E}[h(N)]\right) dy. \tag{1.64}$$

is an absolutely continuous solution of (1.57).
Moreover, any absolutely continuous solution $\tilde{f}$ of (1.57) is of the form

$$\tilde{f}(x) = f(x) + ce^{x^2/2} \quad , \quad c \in \mathbb{R}.$$

Finally, f is the only solution that satisfies $\lim_{|x|\to\infty} f(x)e^{-x^2/2} = 0$.

Proof. Equation (1.57) (with unknown function f) is a standard linear Ordinary Differential Equation. We need only to apply the First Integral method:

$$\frac{d}{dx}\big(e^{-x^2/2} f(x)\big) = e^{-x^2/2}\big(f'(x) - xf(x)\big) = e^{-x^2/2}\big(h(x) - \mathbb{E}[h(X)]\big),$$

which leads to a solution candidate (1.63). On the other hand, a simple verification shows that (1.63) indeed satisfies (1.57).

The equivalence between (1.63) and (1.64) is given by the fact

$$\int_{-\infty}^{\infty} e^{-y^2/2}\big(h(y) - \mathbb{E}[h(N)]\big) dy = 0.$$

If $\tilde{f}$ is any other solution of (1.57), then

$$\frac{d}{dx}\Big(e^{-x^2/2}\big(f(x) - \tilde{f}(x)\big)\Big) = 0,$$

which means $\tilde{f}(x) = f(x) + ce^{x^2/2}$ for some $c \in \mathbb{R}$.

The last assertion is justified by applying Dominated Convergence Theorem on both (1.63) and (1.64). ■

Now we give an alternative expression of solution of of (1.57). We shall see later that this expression can be generalized in multi-dimensional case (Lemma 1.3.12).

Lemma 1.3.10 *Let h be an absolutely continuous function with bounded derivative. Then*

$$f(x) = -\int_0^1 \frac{1}{2\sqrt{t(1-t)}} \mathbb{E}\left[Nh(\sqrt{t}x + \sqrt{1-t}N)\right] dt \quad , \qquad N \sim \mathcal{N}(0,1) \tag{1.65}$$

is a solution of (1.57). Moreover, it is the same as (1.63), because $\lim_{|x|\to+\infty} f(x)e^{-x^2/2} = 0$.

Proof. We shall verify that (1.65) is indeed a solution of (1.57). Thanks to Dominated Convergence Theorem, we are able to carry the derivative inside the integral and expectation, and we have

$$f'(x) = -\int_0^1 \frac{1}{2\sqrt{(1-t)}}\mathbb{E}\left[Nh'(\sqrt{t}x+\sqrt{1-t}N)\right]dt. \tag{1.66}$$

Thus,

$$\begin{aligned} & f'(x) - xf(x) \\ = & \int_0^1 \mathbb{E}\left[\left(-\frac{N}{2\sqrt{1-t}}+\frac{x}{2\sqrt{t}}\right)h'(\sqrt{t}x+\sqrt{1-t}N)\right]dt \\ = & \int_0^1 \mathbb{E}\left[\frac{d}{dt}h'(\sqrt{t}x+\sqrt{1-t}N)\right]dt \\ = & \ h(x) - \mathbb{E}[h(N)], \end{aligned}$$

where the first line of the equation comes from the following relation which is deduced from the Stein identity (1.58):

$$\mathbb{E}\left[Nh(\sqrt{t}x+\sqrt{1-t}N)\right] = \sqrt{1-t}\mathbb{E}\left[h'(\sqrt{t}x+\sqrt{1-t}N)\right].$$

■

Now we give simple proofs of three inequalities in (iv) and (v) of Lemma 1.3.7. The proof of the second assertion in (v) is too lengthy to be put here.

(I) If $h:\mathbb{R}\to\mathbb{R}$ is bounded and absolutely continuous, then there exists a solution f of (1.57) which is bounded and twice differentiable and satisfy

$$\|f\|_\infty \le \sqrt{\pi/2}\|h-\mathbb{E}[h(N)]\|_\infty;$$

(II) the same f also satisfy

$$\|f'\|_\infty \le 2\|h-\mathbb{E}[h(N)]\|_\infty;$$

(III) If h is absolutely continuous with bounded derivative, then (1.57) has a solution f which is twice differentiable and such that

$$\|f'\|_\infty \le \sqrt{2/\pi}\|h'\|_\infty.$$

Notice that this is stronger than the result in (v) of Lemma 1.3.7.

Proof. (I): We adopt a solution f in form of (1.63). Suppose $x>0$. Using the representation in (1.64), we have

$$|f(x)| \le \|h-\mathbb{E}[h(N)]\|_\infty \left(e^{x^2/2}\int_x^\infty e^{-y^2/2}dy\right).$$

Since

$$\begin{aligned} & \frac{d}{dx} e^{x^2/2} \int_x^\infty e^{-y^2/2} dy \\ = & -1 + x e^{x^2/2} \int_x^\infty e^{-y^2/2} dy \\ \leq & -1 + x e^{x^2/2} \int_x^\infty \frac{y}{x} e^{-y^2/2} dy \\ = & -1 + x e^{x^2/2} \times \frac{1}{x} e^{-x^2/2} \\ = & \ 0, \end{aligned}$$

we see that $e^{x^2/2} \int_x^\infty e^{-y^2/2} dy$ is maximized at $x = 0$ on $[0, \infty)$ with value $\sqrt{\pi/2}$. Hence,

$$|f(x)| \leq \sqrt{\frac{\pi}{2}} \|h - \mathbb{E}[h(N)]\|_\infty, \quad \forall x > 0.$$

For $x < 0$, use the form (1.65), and proceed in the same way.

(II): Again, we will only consider $x > 0$ case. The other case will be similar.
Note that

$$f'(x) = h(x) - \mathbb{E}[h(N)] - x e^{x^2/2} \int_x^\infty e^{-y^2/2} (h(y) - \mathbb{E}[h(N)]) dy.$$

Therefore,

$$\begin{aligned} |f'(x)| & \leq \|h(x) - \mathbb{E}[h(N)]\|_\infty \left(1 + x e^{x^2/2} \int_x^\infty e^{-y^2/2} dy \right) \\ & \leq 2 \|h(x) - \mathbb{E}[h(N)]\|_\infty \text{ (By the same way as in the proof of (I)).} \end{aligned}$$

(III): From (1.66), it follows that

$$\|f'\|_\infty \leq (\mathbb{E}|N|) \|h'\|_\infty \int_0^1 \frac{1}{2\sqrt{1-t}} = \sqrt{\frac{2}{\pi}} \|h'\|_\infty$$

■

1.3.5 An interesting example

We give here an example which is taken from Lecture 3 in Chatterjee's Lecture Notes [7] on Stein's method. Indeed, this simple example illustrate how Stein's original "leave me out" idea works (See Stein's 1972 paper [63]).

We let $W = \frac{1}{\sqrt{n}} \sum_{i=1}^n X_i$, where $X_1, X_2, \ldots, X_n$ are independent variables such that $\mathbb{E}[X_i] = 0$, $\mathbf{Var}[X_i] = 1$, $\mathbb{E}|X_i|^3 < \infty$. It is easy to see that W is "close to" a standard normal distributed variable $N \sim \mathcal{N}(0, 1)$ and we wish to evaluate the Wasserstein distance $d_W(W, N)$.

Theorem 1.3.11 *Suppose $X_1, \ldots, X_n$ are independent with mean 0, variance 1, and finite third moments. Then*

$$d_W \left(\frac{\sum_1^n X_i}{\sqrt{n}}, N \right) \leq \frac{3}{n^{3/2}} \sum_1^n \mathbb{E} |X_i|^3,$$

where $N \sim \mathcal{N}(0, 1)$.

Proof. By inequality (1.62), we estimate

$$|\mathbb{E}[Wf(W) - f'(W)]|$$

by taking any $f \in C^1$ with f' absolutely continuous, and satisfying $|f'| \leq 1$ and $|f''| \leq 2$. Note that

$$\mathbb{E}[Wf(W)] = \frac{1}{\sqrt{n}} \sum \mathbb{E}\left[X_i f(W)\right]. \tag{1.67}$$

From W, we single out the part that is independent of X_i, by taking

$$W_i = W - \frac{X_i}{\sqrt{n}}.$$

Then we must have $\mathbb{E}[X_i f(W_i)] = 0$. Therefore,

$$\begin{aligned}
& \mathbb{E}\left[Wf(W)\right] - \mathbb{E}[f'(W)] \\
= & \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \mathbb{E}\left[X_i \left(f(W) - f(W_i)\right)\right] - \mathbb{E}[f'(W)] \\
= & \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \mathbb{E}\left[X_i \left(f(W) - f(W_i) - (W - W_i) f'(W_i)\right)\right] & (1.68) \\
& + \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \mathbb{E}\left[X_i (W - W_i) f'(W_i)\right] - \mathbb{E}[f'(W)]. & (1.69)
\end{aligned}$$

By applying Taylor's formula

$$\left|f(b) - f(a) - (b-a) f'(a)\right| \leq \frac{1}{2}(b-a)^2 \left|f''\right|_\infty$$

on (1.68), we deduce that

$$\begin{aligned}
& \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left|\mathbb{E}\left[X_i \left(f(W) - f(W_i) - (W - W_i) f'(W_i)\right)\right]\right| \\
\leq & \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{1}{2} \mathbb{E}\left|X_i \frac{X_i^2}{n}\right| \cdot \left|f''\right|_\infty \\
\leq & \frac{1}{n^{3/2}} \sum_{i=1}^{n} \mathbb{E}\left|X_i\right|^3.
\end{aligned}$$

On the other hand, since $\mathbb{E}[X_i^2] = 1$ and X_i is independent of W_i, we may rewrite (1.69) as $\frac{1}{n}\sum_{i=1}^n \mathbb{E}[f'(W_i)] - \mathbb{E}[f'(W)]$, which can be bounded by $\sum_{i=1}^n \mathbb{E}|X_i|$ as shown below:

$$\left|\frac{1}{n} \sum_{i=1}^{n} \mathbb{E}[f'(W_i)] - \mathbb{E}[f'(W)]\right| \leq \frac{|f''|_\infty}{n} \sum_{i=1}^{n} \mathbb{E}\left|W - W_i\right| \leq \frac{2}{n^{3/2}} \sum_{i=1}^{n} \mathbb{E}\left|X_i\right|.$$

By combining the estimations on (1.68) and (1.69) and the fact that $\mathbb{E}|X_i| \leq \left(\mathbb{E}|X_i|^3\right)^{1/3} \leq \mathbb{E}|X_i|^3$, we obtain the desired conclusion. ■

1.3.6 Multivariate case

From now on, given a $d \times d$ nonnegative definite matrix C, we write $\mathcal{N}_d(0, C)$ to indicate the law of a centered d-dimensional Gaussian vector with covariance C. Most of the results in the following Lemma is proved in [37], and the inequality (1.74) is newly obtained by Peccati and Zheng [50], which will play a important role in the estimations in the next chapter.

Lemma 1.3.12 (Stein's Lemma and estimates) *Fix an integer $d \geq 2$ and let $C = \{C(i,j) : i,j = 1, \ldots, d\}$ be a $d \times d$ nonnegative definite symmetric real matrix.*

1. *Let Y be a random variable with values in $\mathbb{R}^d$. Then $Y \sim \mathcal{N}_d(0, C)$ if and only if, for every twice differentiable function $f : \mathbb{R}^d \to \mathbb{R}$ such that $\mathbb{E}|\langle C, \operatorname{Hess} f(Y)\rangle_{H.S.}| + \mathbb{E}|\langle Y, \nabla f(Y)\rangle_{\mathbb{R}^d}| < \infty$, it holds that*

$$\mathbb{E}[\langle Y, \nabla f(Y)\rangle_{\mathbb{R}^d} - \langle C, \operatorname{Hess} f(Y)\rangle_{H.S.}] = 0 \tag{1.70}$$

2. *Assume in addition that C is positive definite and consider a Gaussian random vector $X_C \sim \mathcal{N}_d(0, C)$. Let $g : \mathbb{R}^d \to \mathbb{R}$ belong to $\mathbb{C}^2(\mathbb{R}^d)$ with first and second bounded derivatives. Then, the function $U_0(g)$ defined by*

$$U_0 g(x) := \int_0^1 \frac{1}{2t}\mathbb{E}[g(\sqrt{t}x + \sqrt{1-t}X_C) - g(X_C)]dt \tag{1.71}$$

is a solution to the following partial differential equation (with unknown function f):

$$g(x) - \mathbb{E}[g(X_C)] = \langle x, \nabla f(x)\rangle_{\mathbb{R}^d} - \langle C, \operatorname{Hess} f(x)\rangle_{H.S.},\ x \in \mathbb{R}^d. \tag{1.72}$$

3. *Moreover, one has that*

$$\sup_{x\in\mathbb{R}^d} \|\operatorname{Hess} U_0 g(x)\|_{H.S.} \leq \|C^{-1}\|_{op}\, \|C\|_{op}^{1/2} \|g\|_{Lip}, \tag{1.73}$$

and

$$M_3(U_0 g) \leq \frac{\sqrt{2\pi}}{4}\|C^{-1}\|_{op}^{3/2}\, \|C\|_{op} M_2(g). \tag{1.74}$$

Proof. We shall prove the results in 4 steps.

Step 1: We prove that (1.71) is indeed the solution of Stein's equaiton (1.72). Notice that, we can suppose without loss of generality that $X_C = (X_1, \ldots, X_d) := (G(h_1), \ldots, G(h_d))$, where G is an isonormal Gaussian process over $\mathfrak{H} = \mathbb{R}^d$, the kernels h_i belong to $\mathfrak{H}$ ($i = 1, \ldots, d$), and $\langle h_i, h_j\rangle_{\mathfrak{H}} = \mathbb{E}[G(h_i)G(h_j)] = \mathbb{E}[X_i X_j] = C(i,j)$.

$U_0 g(x)$ can be rewritten as follows:

$$U_0 g(x) = \int_0^\infty \Big(\mathbb{E}[g(e^{-u}x + \sqrt{1-e^{-2u}}N)] - \mathbb{E}[g(X_C)]\Big)du,$$

by using a simple change of variable $2u = -\log t$. This new expression of $U_0 g(x)$ reminds us the Mehler's formula (1.52) introduced before. Indeed, by defining $\tilde{g}(X_C) = g(X_C) - \mathbb{E}[g(X_C)]$, we see that, by Mehler's formula (1.52):

$$\mathbb{E}[g(e^{-u}x+\sqrt{1-e^{-2u}}X_C)]|_{x=X_C} - \mathbb{E}[g(X_C)] = \mathbb{E}[\tilde{g}(e^{-u}x+\sqrt{1-e^{-2u}}X_C)]|_{x=X_C} = T_u(\tilde{g}(X_C)),$$

where T is the Ornstein-Uhlenbeck semigroup and $J_q(\tilde{g}(X_C))$ is the projection of $\tilde{g}(X_C)$ on the qth Wiener chaos. Then we have

$$U_0 g(X_C) = \int_0^\infty T_u(\tilde{g}(X_C))du = \int_0^\infty \sum_{q\geq 1} e^{-qu} J_q(\tilde{g}(X_C))du = \sum_{q\geq 1} \frac{1}{q} J_q(\tilde{g}(X_C)) = -L^{-1}\tilde{g}(X_C),$$

by using (1.46). It is easy to see that that $U_0 g$ belongs to $\mathcal{C}^2(\mathbb{R}^d)$. By exploiting the differential representation (1.54), we deduce that

$$\langle X_C, \nabla U_0 g(X_C)\rangle_{\mathbb{R}^d} - \langle C, \mathrm{Hess}\, U_0 g(X_C)\rangle_{H.S.} = g(x) - \mathbb{E}[g(X_C)],$$

for every $x \in \mathbb{R}^d$. As a consequence, the function $U_0 g$ solves the Stein's equation (1.72).

Step 2: To prove the estimate (1.73), we first notice that there exists a non-singular symmetric matrix A such that $A^2 = C$, for positive definite matrix C, and $A^{-1}X_C \sim \mathcal{N}_d(0, I_d)$. The simple case for $C = I_d$ is provided by Lemma 3 in [11]. Now we let $U_0 g(x) = h(A^{-1}x)$, where

$$h(x) = \int_0^1 \frac{1}{2t}\mathbb{E}[g_A(\sqrt{t}x + \sqrt{1-t}A^{-1}X_C) - g_A(A^{-1}X_C)]dt$$

and $g_A(x) = g(Ax)$. As $A^{-1}X_C \sim \mathcal{N}_d(0, I_d)$, the function h solves the Stein's equation

$$\langle x, \nabla h(x)\rangle_{\mathbb{R}^d} - \Delta h(x) = g_A(x) - \mathbb{E}[g_A(Y)],$$

where $Y \sim \mathcal{N}_d(0, I_d)$ and Δ is the Laplacian. We may use the same arguments as in the proof of Lemma 3 in [11] to deduce that

$$\sup_{x\in\mathbb{R}^d} \|\mathrm{Hess}\, h(x)\|_{H.S.} \leq \|g_A\|_{Lip} \leq \|A\|_{op}\|g\|_{Lip}.$$

On the other hand, by writing $h_{A^{-1}}(x) = h(A^{-1}x)$, one obtains by standard computations (notice that A is symmetric) that

$$\mathrm{Hess}\, U_0 g(x) = \mathrm{Hess}\, h_{A^{-1}}(x) = A^{-1}\mathrm{Hess}\, h(A^{-1}x)A^{-1},$$

therefore

$$\begin{aligned}
\sup_{x\in\mathbb{R}^d} \|\mathrm{Hess}\, U_0 g(x)\|_{H.S.} &= \sup_{x\in\mathbb{R}^d} \|A^{-1}\mathrm{Hess}\, h(A^{-1}x)A^{-1}\|_{H.S.} \\
&= \sup_{x\in\mathbb{R}^d} \|A^{-1}\mathrm{Hess}\, h(x)A^{-1}\|_{H.S.} \\
&\leq \|A^{-1}\|_{op}^2 \sup_{x\in\mathbb{R}^d} \|\mathrm{Hess}\, h(x)\|_{H.S.} \\
&\leq \|A^{-1}\|_{op}^2 \|A\|_{op}\|g\|_{Lip} \\
&\leq \|C^{-1}\|_{op}\|C\|_{op}^{1/2}\|g\|_{Lip}.
\end{aligned}$$

In the chain of inequalities above, we used the relations

$$\begin{aligned}
\|A^{-1}\mathrm{Hess}\, h(x)A^{-1}\|_{H.S.} &\leq \|A^{-1}\|_{op}\|\mathrm{Hess}\, h(x)A^{-1}\|_{H.S.} \\
&\leq \|A^{-1}\|_{op}\|\mathrm{Hess}\, h(x)\|_{H.S.}\|A^{-1}\|_{op},
\end{aligned}$$

as well as the fact that

$$\|A^{-1}\|_{op} \le \sqrt{\|C^{-1}\|_{op}} \qquad \text{and} \qquad \|A\|_{op} \le \sqrt{\|C\|_{op}}.$$

Step 3: Now we may work on Point 1. The fact that a vector $Y \sim \mathcal{N}_d(0, C)$ necessarily verifies (1.70) can be easily proved. On the other hand, suppose that Y verifies (1.70). Then, according to Point 2, for every $g \in \mathcal{C}^2(\mathbb{R}^d)$ with bounded first and second derivatives,

$$\mathbb{E}[g(Y)] - E[g(X_C)] = E[\langle Y, \nabla U_0 g(Y)\rangle_{\mathbb{R}^d} - \langle C, \mathrm{Hess}\, U_0 g(Y)\rangle_{H.S.}] = 0,$$

where $X_C \sim \mathcal{N}_d(0, C)$. Since the collection of all such functions g generates the Borel σ-field on $\mathbb{R}^d$, this implies that $Y \overset{(law)}{=} X_C$, and the the Point 1 follows.
Step 4: Now we show inequality (1.74).

On the one hand, as $\mathrm{Hess}\, g_A(x) = A\, \mathrm{Hess}\, g(Ax) A$ (recall that A is symmetric), we have

$$\begin{aligned} M_2(g_A) &= \sup_{x \in \mathbb{R}^d} \|\mathrm{Hess}\, g_A(x)\|_{op} = \sup_{x \in \mathbb{R}^d} \|A \mathrm{Hess}\, g(Ax) A\|_{op} \\ &= \sup_{x \in \mathbb{R}^d} \|A \mathrm{Hess}\, g(x) A\|_{op} \le \|A\|_{op}^2 M_2(g) \\ &= \|C\|_{op} M_2(g), \end{aligned}$$

where the inequality above follows from the well-known relation $\|AB\|_{op} \le \|A\|_{op}\|B\|_{op}$. It is easily seen that

$$\mathrm{Hess}\, U_0 g(x) = \mathrm{Hess}\, h_{A^{-1}}(x) = A^{-1} \mathrm{Hess}\, h(A^{-1}x) A^{-1}.$$

It follows that

$$\begin{aligned} M_3(U_0 g) &= M_3(h_{A^{-1}}) \\ &= \sup_{x \neq y} \frac{\|\mathrm{Hess}\, h_{A^{-1}}(x) - \mathrm{Hess}\, h_{A^{-1}}(y)\|_{op}}{\|x - y\|} \\ &= \sup_{x \neq y} \frac{\|A^{-1} \mathrm{Hess}\, h(A^{-1}x) A^{-1} - A^{-1} \mathrm{Hess}\, h(A^{-1}y) A^{-1}\|_{op}}{\|x - y\|} \\ &\le \|A^{-1}\|_{op}^2 \times \sup_{x \neq y} \frac{\|\mathrm{Hess}\, h(A^{-1}x) - \mathrm{Hess}\, h(A^{-1}y)\|_{op}}{\|x - y\|} \times \frac{\|A^{-1}x - A^{-1}y\|}{\|A^{-1}x - A^{-1}y\|} \\ &\le \|A^{-1}\|_{op}^2 \times \sup_{x \neq y} \frac{\|\mathrm{Hess}\, h(A^{-1}x) - \mathrm{Hess}\, h(A^{-1}y)\|_{op}}{\|A^{-1}x - A^{-1}y\|} \times \|A^{-1}\|_{op} \\ &= \|C^{-1}\|_{op}^{3/2} M_3(h). \end{aligned}$$

Since $M_3(h) \le \frac{\sqrt{2\pi}}{4} M_2(g_A)$ (according to [11, Lemma 3]), relation (1.74) follows immediately.
■

Remark 1.3.13 *In the one-dimensional Stein's Lemma, the above inequality (1.74) can be replaced by an inequality of type $M_3(U_0 g) \le Const \times M_1(g)$, and therefore distance d_2 is replaced by Wasserstein distance d_w as shown in [31] [37] [46].*

It is natural to ask whether this kind of improvement takes place on multi-dimensional version. Unfortunately, the answer is negative. We take, for example,

$$g(x, y) = \max\{\min\{x, y\}, 0\},$$

$U_0 g$ *is then twice differentiable but* $\dfrac{\partial^2 (U_0 g)}{\partial x^2}$ *is not Lipschitz. See [11] for details.*

Chapter 2

Malliavin-Stein method and the main contributions of the dissertation

2.1 Malliavin-Stein method

The techniques developed by Stein, Chen and other authors (that are introduced in the previous section) allow us to measure the distance between the laws of a generic random variable F and $N \sim \mathcal{N}(0,1)$, by assessing the distance from zero of the quantity $\mathbb{E}[Ff(F) - f'(F)]$, for every f belonging to a "sufficiently large" class of smooth functions. Indeed, the bounds of the following type hold in great generality:

$$d(F, N) \leq C \times \sup_{f \in \mathcal{F}} |\mathbb{E}[Ff(F) - f'(F)]|. \tag{2.1}$$

(See Corollary 1.3.8.)

However, to assess quantities having the form of RHS of (2.1) is always a challenging task, towards which a great number of efforts have been directed in the last 30 years. An impressive number of approaches has been developed in this direction: the reader is referred to the two surveys by Chen and Shao [14] and Reinert [57] for a detailed discussion of these contributions. Among these developments, Peccati, Nourdin, Nualart and other authors have found an creative way to effectively estimate a quantity such as the RHS of (2.1) by using Malliavin calculus, whenever the random variable F can be represented as regular functional of a generic and possibly infinite-dimensional Gaussian or Poisson field.

Before the introduction of the "Malliavin-Stein method", we list here some important works in the development of the method. In the survey [44] by Peccati and Nourdin, the readers may find an overview of the "Malliavin-Stein method" on the Gaussian space.

- The paper [40], by Nualart and Peccati, discovered and proved the striking Theorem 2.1.4 (without condition (iv)). In this paper, the equivalence of Conditions (i), (ii), (iii) is proved by using techniques based on stochastic time-changes.
- The paper [39], by Nualart and Ortiz-Latorre, revisited Theorem 2.1.4. The authors proved the equivalence between Condition (iv) and other conditions, by means of Malliavin calculus, but not with Stein's method.

- The paper [49], by Peccati and Tudor, studied the CLT on vectors of multiple stochastic Wiener-Itô integrals on the Gaussian space, and gives Theorem 2.1.7. Their calculations rely on standard stochastic calculus.

- The paper [31], by Peccati and Nourdin, generalized and unified the results and methods in the previous papers. It is in this paper that "Malliavin-Stein method" made its first appearance. The authors combined Malliavin calculus with Stein's method, to derive explicit bounds in the Gaussian and Gamma approximation of random variables in a fixed Gaussian Wiener chaos.

- The paper [37], by Nourdin, Peccati and Réveillac, generalized the "Malliavin-Stein method" to the multi-dimensional normal approximation of functionals of Gaussian fields. The authors combined Malliavin calculus with multi-dimensional Stein's lemma, to build explicit bounds for Gaussian approximation, and re-established Theorem 2.1.7. It is worth noting that they provided a new proof of the multi-dimensional Stein's lemma, by means of Malliavin calculus.

- The paper [47], by Peccati and Taqqu, studied the normal approximation of multiple stochastic Wiener-Itô integrals of order 2 ($I_2(f^{(n)})$) on the Poisson space. The authors built an parallel of Theorem 2.1.4 for $I_2(f^{(n)})$, by providing conditions on fourth moment (as in the Gaussian case) and on star contraction operator $\star_r^l$ (instead of $\otimes_r$ in the Gaussian case).

- The paper [46], by Peccati, Solé, Taqqu and Utzet, generalized the "Malliavin-Stein method" on the Poisson space. The authors combined the standard Stein's method with a version of Malliavin calculus on the Poisson space as defined by Nualart and Vives in [41]. They generalized the work of Peccati and Taqqu [47] and built a CLT (Theorem 2.1.10) for arbitrary fixed order multiple stochastic Wiener-Itô integrals on the Poisson space. It is worth noting that the conditions in Theorem 2.1.10 are merely sufficient for integrals of order $q \geq 3$, which shows the complexity of Poisson space compared with Gaussian space.

- The paper [36], by Nourdin, Peccati and Reinert, contains an application of Stein's method, Malliavin calculus and the "Lindeberg invariance principle", in order to unify former results on normal approximation in different spaces (Gausian space, Poisson space, etc.), and study universailty for sequences of homogenous sums associated with general collections of independent random variables.

2.1.1 One-dimensional Gaussian case

This section is based on the paper [31] by Nourdin and Peccati. We study the normal approximation of functionals of a general Gaussian field in 3 steps:

1. For any centered functional F, we evaluate the distance $d(F,N)$ (where $N \sim \mathcal{N}(0,1)$) by Stein's method and Malliavin calculus, and obtain an explicit bound using Malliavin operators.

2. In the particular case of multiple stochastic Wiener-Itô integrals: $F = I_q^G(f)$, we evaluate the bound expressed by contraction operator $\otimes_r$.

3. We build a CLT for sequences $\{I_q^G(f^{(n)}); n \geq 1\}$.

Step 1:
Let $G(\mathfrak{H}) = \{G(h) : h \in \mathfrak{H}\}$ be an isonormal Gaussian process over some real separable Hilbert space $\mathfrak{H}$. Suppose F is a centered functional of G, such that $\mathbb{E}[F] = 0$ and F is differentiable in the sense of Malliavin calculus. According to the Stein-type bound (2.1), we need to assess the distance between the two quantities $\mathbb{E}[Ff(F)]$ and $\mathbb{E}[f'(F)]$ in order to evaluate the distance between the law of F and the law of a Gaussian random variable $N \sim \mathcal{N}(0,1)$. According to the methods that introduced in [31], and then further developed in the references [30, 34, 36, 37], is that the needed estimate can be realized by the following calculations: Let $f : \mathbb{R} \to \mathbb{R}$ be a $\mathcal{C}^1$ function with bounded derivative, and let $F, \tilde{F} \in \mathbb{D}^{1,2}$. By using successively equation (1.51), equation (1.50), duality relationship (1.41), and "chain rules" (1.39), one deduce that

$$\begin{aligned}
\mathbb{E}[\tilde{F}f(F)] &= \mathbb{E}[LL^{-1}\tilde{F} \times f(F)] \\
&= \mathbb{E}[\delta D(-L^{-1}\tilde{F}) \times f(F)] \\
&= \mathbb{E}[\langle Df(F), -DL^{-1}\tilde{F}\rangle_{\mathfrak{H}}] \\
&= \mathbb{E}[f'(F)\langle DF, -DL^{-1}\tilde{F}\rangle_{\mathfrak{H}}]
\end{aligned}$$

We will shortly see that the fact

$$\mathbb{E}[\tilde{F}f(F)] = \mathbb{E}[f'(F)\langle DF, -DL^{-1}\tilde{F}\rangle_{\mathfrak{H}}] \tag{2.2}$$

constitutes a fundamental element in the connection between Malliavin calculus and Stein's method. By taking $\tilde{F} = F$ in (2.2), we deduce that, if the derivative f' is bounded, then the distance between $\mathbb{E}[Ff(F)]$ and $\mathbb{E}[f'(F)]$ is controlled by the L^1-norm of the random variable $1 - \langle DF, -DL^{-1}F\rangle_{\mathfrak{H}}$. For instance, in the case of the Kolmogorov distance, one obtains that, for every centered and Malliavin differentiable random variable F,

$$d_{Kol}(F, N) \leq \mathbb{E}|1 - \langle DF, -DL^{-1}F\rangle_{\mathfrak{H}}| \tag{2.3}$$

We will see later that, in the particular case where $F = I_q^G(f)$ is a multiple Wiener-Itô integral of order $q \geq 2$ with unit variance, relation (2.3) yields the neat estimate

$$d_{Kol}(F, N) \leq \sqrt{\frac{q-1}{3q} \times |\mathbb{E}[F^4] - 3|}. \tag{2.4}$$

Note that $\mathbb{E}[F^4] - 3$ is just the fourth cumulant of F, and that the fourth cumulant of N equals to zero. We will also show that the combination of (2.3) and (2.4) allows to recover several characterizations of CLTs on a fixed Wiener chaos - as proved in [39, 40].

Theorem 2.1.1 *Let $F \in \mathbb{D}^{1,2}$ be such that $\mathbb{E}[F] = 0$ and $\mathbf{Var}[F] = 1$. Then for $N \sim \mathcal{N}(0,1)$,*

$$d_{Kol}(F, N) \leq \mathbb{E}|1 - \langle DF, -DL^{-1}F\rangle_{\mathfrak{H}}| \leq \sqrt{\mathbb{E}[(1 - \langle DF, -DL^{-1}F\rangle_{\mathfrak{H}})^2]} = \sqrt{\mathbf{Var}[\langle DF, -DL^{-1}F\rangle_{\mathfrak{H}}]} \tag{2.5}$$

$$d_W(F, N) \leq \mathbb{E}|1 - \langle DF, -DL^{-1}F\rangle_{\mathfrak{H}}| \leq \sqrt{\mathbf{Var}[\langle DF, -DL^{-1}F\rangle_{\mathfrak{H}}]} \tag{2.6}$$

$$d_{TV}(F, N) \leq 2\mathbb{E}|1 - \langle DF, -DL^{-1}F\rangle_{\mathfrak{H}}| \leq 2\sqrt{\mathbf{Var}[\langle DF, -DL^{-1}F\rangle_{\mathfrak{H}}]} \tag{2.7}$$

Proof. We give the proof for Kolmogorov distance here, and the proofs for Total Variation distance and the Wasserstein distance are similar. In view of (1.60), it is enough to prove that, for every f satisfying $\|f'\|_\infty \leq 1$, one has that the quantity $|\mathbb{E}[Ff(F) - f'(F)]|$ is less or equal to the RHS of (2.5). For such a f, relation (2.2) yields

$$\mathbb{E}[Ff(F)] = \mathbb{E}[f'(F)\langle DF, -DL^{-1}F\rangle_{\mathfrak{H}}],$$

so that

$$|\mathbb{E}[f'(F)] - \mathbb{E}[Ff(F)]| = |\mathbb{E}[f'(F)(1 - \langle DF, -DL^{-1}F\rangle_{\mathfrak{H}})]| \leq \mathbb{E}|1 - \langle DF, -DL^{-1}F\rangle_{\mathfrak{H}}|$$

Therefore, by using (1.60) and Cauchy-Schwarz inequality, we infer that

$$d_{Kol}(F, N) \leq \mathbb{E}|1 - \langle DF, -DL^{-1}F\rangle_{\mathfrak{H}}| \leq \sqrt{\mathbb{E}[(1 - \langle DF, -DL^{-1}F\rangle_{\mathfrak{H}})^2]}$$

To conclude, we choose $f(z) = z$ in (2.2), then we obtain

$$\mathbb{E}[\langle DF, -DL^{-1}F\rangle_{\mathfrak{H}}] = \mathbb{E}[F^2] = 1,$$

so that

$$\mathbb{E}[(1 - \langle DF, -DL^{-1}F\rangle_{\mathfrak{H}})^2] = \mathbf{Var}[\langle DF, -DL^{-1}F\rangle_{\mathfrak{H}}].$$

■

Step 2:
We evaluate the above bound in the particular case $F = I_q^G(f)$.

Lemma 2.1.2 *Fix an integer $q \geq 1$, and let $F = I_q^G(f)$ (with $f \in \mathfrak{H}^{\odot q}$) be such that $\mathbf{Var}[F] = \mathbb{E}[F^2] = 1$. The following three identities are in order:*

$$\frac{1}{q}\|DZ\|_{\mathfrak{H}}^2 - 1 = q\sum_{r=1}^{q-1}(r-1)!\begin{pmatrix} q-1 \\ r-1 \end{pmatrix}^2 I_{2q-2r}(f\tilde{\otimes}_r f), \tag{2.8}$$

$$\mathbf{Var}\left[\frac{1}{q}\|DZ\|_{\mathfrak{H}}^2\right] = \sum_{r=1}^{q-1}\frac{r^2}{q^2}(r!)^2\begin{pmatrix} q \\ r \end{pmatrix}^4 (2q-2r)!\|f\tilde{\otimes}_r f\|_{\mathfrak{H}^{\otimes 2q-2r}}^2 \tag{2.9}$$

and

$$\mathbb{E}[F^4] - 3 = \frac{3}{q}\sum_{r=1}^{q-1} r(r!)^2\begin{pmatrix} q \\ r \end{pmatrix}^4 (2q-2r)!\|f\tilde{\otimes}_r f\|_{\mathfrak{H}^{\otimes 2q-2r}}^2 \tag{2.10}$$

In particular,

$$\mathbf{Var}\left[\frac{1}{q}\|DZ\|_{\mathfrak{H}}^2\right] \leq \frac{q-1}{3q}(\mathbb{E}[F^4] - 3) \tag{2.11}$$

As a consequence of Lemma 2.1.2, we deduce the following "fourth moment" bound on the Kolmogorov distance (see [36]).

Theorem 2.1.3 *Let $F = I_q^G(f)$ for some $q \geq 2$ and function $f \in \mathfrak{H}^{\odot q}$. Suppose moreover that $\mathbf{Var}[F] = \mathbb{E}[F^2] = 1$. Then*

$$d_{Kol}(F, N) \leq \sqrt{\frac{q-1}{3q}(\mathbb{E}[F^4] - 3)} \tag{2.12}$$

Proof. Since $L^{-1}F = -\frac{1}{q}F$, we have

$$\langle DF, -DL^{-1}F\rangle_{\mathfrak{H}} = \frac{1}{q}\|DF\|^2_{\mathfrak{H}}.$$

So, we only need to apply Theorem 2.1.1 and formula (2.11). ■

Step 3:

Now we have all the ingredients to establish a fundamental limit theorem.

Theorem 2.1.4 (see [31],[39], [40]) *Let $\{F^{(n)}; n \geq 1\}$ be a sequence of random variables belonging to the qth Gaussian Wiener-Itô chaos, for some fixed integer $q \geq 2$. Assume that* $\mathbf{Var}(F^{(n)}) = \mathbb{E}[(F^{(n)})^2] = 1$ *for all n. Then, as $n \to \infty$, the following four assertions are equivalent:*

- *i)* $F^{(n)} \xrightarrow{law} \mathcal{N}(0,1)$, *as* $n \to \infty$;
- *ii)* $\mathbb{E}[(F^{(n)})^4] \to 3$, *as* $n \to \infty$;
- *iii)* $\forall r = 1, \ldots, q-1$, $\|f^{(n)}\tilde{\otimes}_r f^{(n)}\|_{\mathfrak{H}^{\otimes 2q-2r}} \to 0$, *as* $n \to \infty$;
- *iv)* $\|DF^{(n)}\|^2_{\mathfrak{H}} \to q$ *in* L^2 , *as* $n \to \infty$.

Proof. Let $F^{(n)} = I_n^G(f^{(n)})$, with $f^{(n)} \in \mathfrak{H}^{\odot q}$. The implication (iii) to (i) is a direct application of Theorem 2.1.3 and identity (2.10), and of the fact that the Kolmogorov distance is stronger than the topology of the convergence in law. The implication (i) to (ii) comes from a bounded convergence argument, since $\sum_{n\geq 1}\mathbb{E}[(F^{(n)})^4] < \infty$ by the hypercontractivity relation

$$\mathbb{E}[\left|F^{(n)}\right|^p] \leq (p-1)^{pn/2}\mathbb{E}[(F^{(n)})^2]^{p/2}, \quad \forall p \geq 2.$$

Now let us suppose that (ii) is in order. Then, by virtue of (2.10), we have that $\|f^{(n)}\tilde{\otimes}_r f^{(n)}\|_{\mathfrak{H}^{\otimes 2q-2r}} \to 0$, as $n \to \infty$, for all (fixed) $r = 1, \ldots, q-1$. Finally, the equivalence of iii) and iv) comes from (2.8). ■

2.1.2 Multi-dimensional Gaussian case

The method presented in this section is introduced by Nourdin, Peccati and Réveillac in their work [37]. They showed that a relation similar to (2.3) continues to hold when the random variable F is replaced by a d-dimensional ($d \geq 2$) Gaussian vector $(F_1, \ldots, F_d)$ of smooth functionals of a Gaussian field. Their results apply to Gaussian approximations by means of Gaussian vectors with arbitrary positive definite covariance matrices, therefore rebuilt a multi-dimensional CLT which was firstly introduced by Peccati and Tudor in [49]. We repeat the 3 steps scenario here:

1. For any vector $F = (F_1, \cdots, F_d)$ such that $\mathbb{E}[F_i] = 0$ and $F_i \in \mathbb{D}^{1,2}$ for every $i = 1, \cdots, d$, we evaluate the distance $d(F, X_C)$ (where $X_C \sim \mathcal{N}(0, C)$) by multi-dimensional Stein's lemma and Malliavin calculus, and obtain an explicit bound using Malliavin operators.

2. In the particular case of multiple stochastic Wiener-Itô integrals: $F_i = I^G_{q_i}(f_i)$ for every $i = 1, \cdots, d$, we evaluate the bound expressed by contraction operator $\otimes_r$.

3. We build a CLT for sequences of vectors of multiple stochastic integrals.

Step 1:
The following theorem and its proof are taken from [37].

Theorem 2.1.5 *Fix $d \geq 2$ and let $C = \{C(i,j) : i,j = 1, \cdots, d\}$ be a $d \times d$ positive definite matrix. Suppose that $X_C \sim \mathcal{N}_d(0, C)$ and that $F = (F_1, \cdots, F_d)$ is a $\mathbb{R}^d$-valued random vector such that $\mathbb{E}[F_i] = 0$ and $F_i \in \mathbb{D}^{1,2}$ for every $i = 1, \cdots, d$. Then,*

$$\begin{aligned} d_W(F, X_C) &\leq \|C^{-1}\|_{op}\|C\|_{op}^{1/2}\sqrt{\mathbb{E}[\|C - \Phi(DF)\|_{H.S}^2]} && (2.13) \\ &= \|C^{-1}\|_{op}\|C\|_{op}^{1/2}\sqrt{\sum_{i,j=1}^{d}\mathbb{E}[(C(i,j) - \langle DF_i, -DL^{-1}F_j\rangle_{\mathfrak{H}})^2]}, && (2.14) \end{aligned}$$

where we write $\Phi(DF)$ to indicate the matrix $\Phi(DF) := \{\langle DF_i, -DL^{-1}F_j\rangle_{\mathfrak{H}} : 1 \leq i,j \leq d\}$.

Proof. As any globally Lipschitz function g such that $\|g\|_{Lip} \leq 1$ can be approximated by a family $\{g_\epsilon : \epsilon > 0\}$ with bounded first and second derivatives, (e.g. we can choose $g_\epsilon(x) = \mathbb{E}[g(x + \sqrt{\epsilon}N)]$ with $N \sim \mathcal{N}_d(0, I_d)$), we need only prove that, for every $g \in \mathcal{C}^2(\mathbb{R}^d)$ with bounded first and second derivatives,

$$|E[g(F)] - E[g(X_C)]| \leq \|C^{-1}\|_{op}\|C\|_{op}^{1/2}\|g\|_{Lip}\sqrt{\mathbb{E}[\|C - \Phi(DF)\|_{H.S}^2]}$$

Now we may follow the similar deductions used to obtain (2.2). Observe that, according to Point 2 in Lemma 1.3.12,

$$\mathbb{E}[g(F)] - \mathbb{E}[g(X_C)] = \mathbb{E}[\langle F, \nabla U_0 g(F)\rangle_{\mathbb{R}_d} - \langle C, \operatorname{Hess} U_0 g(F)\rangle_{H.S.}].$$

Moreover,

$$
\begin{aligned}
& |\mathbb{E}[\langle C, \operatorname{Hess} U_0 g(F)\rangle_{H.S.} - \langle F, \nabla U_0 g(F)\rangle_{\mathbb{R}^d}]| \\
= & \left| \mathbb{E}\left[\sum_{i,j=1}^{d} C(i,j)\partial^2_{ij} U_0 g(F) - \sum_{i=1}^{d} F_i \partial_i U_0 g(F) \right] \right| \\
= & \left| \sum_{i,j=1}^{d} \mathbb{E}[C(i,j)\partial^2_{ij} U_0 g(F)] - \sum_{i=1}^{d} \mathbb{E}[(LL^{-1}F_i)\partial_i U_0 g(F)] \right| \qquad (\text{since } \mathbb{E}[F_i]=0) \\
= & \left| \sum_{i,j=1}^{d} \mathbb{E}[C(i,j)\partial^2_{ij} U_0 g(F)] - \sum_{i=1}^{d} \mathbb{E}[\delta(DL^{-1}F_i)\partial_i U_0 g(F)] \right| \qquad (\text{since } \delta D = -L) \\
= & \left| \sum_{i,j=1}^{d} \mathbb{E}[C(i,j)\partial^2_{ij} U_0 g(F)] - \sum_{i=1}^{d} \mathbb{E}[\langle D(\partial_i U_0 g(F)), -DL^{-1}F_i\rangle_{\mathfrak{H}}] \right| \qquad (\text{ by duality (1.41)}) \\
= & \left| \sum_{i,j=1}^{d} \mathbb{E}[C(i,j)\partial^2_{ij} U_0 g(F)] - \sum_{i,j=1}^{d} \mathbb{E}[\partial^2_{ji} U_0 g(F)\langle DF_j, -DL^{-1}F_i\rangle_{\mathfrak{H}}] \right| \qquad (\text{ by ``chain rules'' (1.39)}) \\
= & \left| \sum_{i,j=1}^{d} \mathbb{E}[\partial^2_{ij} U_0 g(F)\big(C(i,j) - \langle DF_j, -DL^{-1}F_i\rangle_{\mathfrak{H}}\big)] \right| \\
= & |\mathbb{E}[\langle \operatorname{Hess} U_0 g(F), C - \Phi(DF)\rangle_{H.S.}]| \\
\leq & \sqrt{\mathbb{E}[\|\operatorname{Hess} U_0 g(F)\|^2_{H.S.}]}\sqrt{\mathbb{E}[\|C-\Phi(DF)\|^2_{H.S.}} \qquad (\text{by the Cauchy-Schwarz inequality}) \\
\leq & \|C^{-1}\|_{op}\|C\|^{1/2}_{op}\|g\|_{Lip}\sqrt{\mathbb{E}[\|C-\Phi(DF)\|^2_{H.S.}} \qquad (\text{by inequality (1.73)})
\end{aligned}
$$

■

Step 2:
The following lemma is used to evaluate the bound (2.14) for multiple stochastic Wiener-Itô integrals.

Lemma 2.1.6 *Let $F = I^G_p(f)$ and $G = I^G_q(g)$, with $f \in \mathfrak{H}^{\odot p}$ and $g \in \mathfrak{H}^{\odot q}$ ($p, q \geq 1$). Let a be a real constant. If $p = q$, one has the estimate:*

$$
\begin{aligned}
\mathbb{E}\left[\left(a - \frac{1}{p}\langle DF, DG\rangle_{\mathfrak{H}}\right)^2\right] \leq & (a - p!\langle f, g\rangle_{\mathfrak{H}^{\otimes p}})^2 \\
& + \sum_{r=1}^{p-1} C(r,p)\big(\|f\otimes_{p-r} f\|^2_{\mathfrak{H}^{\otimes 2r}} + \|g\otimes_{p-r} g\|^2_{\mathfrak{H}^{\otimes 2r}}\big).
\end{aligned}
$$

for some positive constants $C(r,p)$ depending on r, p. On the other hand, if $p < q$, one has that

$$
\begin{aligned}
\mathbb{E}\left[\left(a - \frac{1}{q}\langle DF, DG\rangle_{\mathfrak{H}}\right)^2\right] \leq & a^2 + A(p,q)\|f\|^2_{\mathfrak{H}^{\otimes p}}\|g\otimes_{q-p} g\|_{\mathfrak{H}^{\otimes 2p}} \\
& + \sum_{r=1}^{p-1} D(r,p,q)\big(\|f\otimes_{p-r} f\|^2_{\mathfrak{H}^{\otimes 2r}} + \|g\otimes_{q-r} g\|^2_{\mathfrak{H}^{\otimes 2r}}\big).
\end{aligned}
$$

for a positive constant $A(p,q)$ depending on p,q, and positive constants $D(r,p,q)$ depending on r,p,q.

Step 3:
The following CLT is a direct consequence of all the estimations above.

Theorem 2.1.7 (Theorem 3.9 in [37], or see [49]) *Fix $d \geq 2$ and let $C = \{C(i,j) : i,j = 1,\dots,d\}$ be a $d \times d$ positive definite matrix. Fix integers $1 \leq q_1 \leq \dots \leq q_d$. For any $n \geq 1$ and $i = 1,\dots,d$, let $f_i^{(n)}$ belong to $L^2(\mu^{q_i})$. Assume that*

$$F^{(n)} = (F_1^{(n)},\dots,F_d^{(n)}) := (I_{q_1}^G(f_1^{(n)}),\dots,I_{q_d}^G(f_d^{(n)})) \qquad n \geq 1,$$

is such that

$$\lim_{n\to\infty} \mathbb{E}[F_i^{(n)} F_j^{(n)}] = C(i,j), \quad 1 \leq i,j \leq d.$$

Then, as $n \to \infty$, the following four assertions are equivalent:

- *(i) The vector $F^{(n)}$ converges in distribution to a d-dimensional Gaussian vector $\mathcal{N}_d(0,C)$.*
- *(ii) For every $1 \leq i \leq d$, $\mathbb{E}\left[(F_i^{(n)})^4\right] \to 3C(i,i)^2$.*
- *(iii) For every $1 \leq i \leq d$ and every $1 \leq r \leq q_i - 1$, $\|f_i^{(n)} \otimes_r f_i^{(n)}\|_{L^2} \to 0$.*
- *(iv) For every $1 \leq i \leq d$, $F_i^{(n)}$ converges in distribution to a centered Gaussian random variable with variance $C(i,i)$.*

2.1.3 One-dimensional Poisson case

This section introduces the method presented in [46]. The main idea of normal approximation on the Poisson space is similar to that on the Gaussian space. But the "chain rules" are slightly different, so the statement has a more complicated form.

We adopt the notations introduced before. Suppose F is a centered functional of measure G, such that $\mathbb{E}[F] = 0$ and F is differentiable in the sense of Malliavin calculus. Similar to the Gaussian case, we aim at evaluating the distance between the law of F and the law of a Gaussian random variable $N \sim \mathcal{N}(0,1)$, with the help of the Stein-type bound

$$d(F,N) \leq C \times \sup_{f\in\mathcal{F}} |\mathbb{E}[Ff(F) - f'(F)]|.$$

We need to assess the distance between the two quantities $\mathbb{E}[Ff(F)]$ and $\mathbb{E}[f'(F)]$ by the following calculations: Let $f : \mathbb{R} \to \mathbb{R}$ belong to some class of smooth functions, and let $F \in DomD$. By using respectively Remark 1.2.13 and duality relationship Lemma 1.2.9, one deduce that

$$\begin{aligned}
\mathbb{E}[Ff(F)] &= \mathbb{E}[LL^{-1}F \times f(F)] \\
&= \mathbb{E}[\delta D(-L^{-1}F) \times f(F)] \\
&= \mathbb{E}[\langle Df(F), -DL^{-1}F\rangle_{L^2(\mu)}]
\end{aligned}$$

Notice that the "chain rules" (1.35) on the Poisson space differs from that on the Gaussian space (1.39). We deduce, by (1.35), that

$$\mathbb{E}[Ff(F)] = \mathbb{E}[f'(F)\langle DF, -DL^{-1}F\rangle_{L^2(\mu)}] + \mathbb{E}[\langle R(DF), -DL^{-1}F\rangle_{L^2(\mu)}] \tag{2.15}$$

Therefore, if the derivative f' and f'' are bounded, then we may deduce that the Wasserstein distance between $\mathbb{E}[Ff(F)]$ and $\mathbb{E}[f'(F)]$ is controlled by the sum of $\mathbb{E}|1-\langle DF, -DL^{-1}F\rangle_{L^2(\mu)}|$ and $\int_Z \mathbb{E}[|D_zF|^2|D_zL^{-1}F|]\mu(dz)$. (See Theorem 2.1.8) We will see later that, in the particular case where $F = I_q(f)$ is a multiple Wiener-Itô integral of order $q \geq 2$ with unit variance, the estimate of the Wasserstein distance between $\mathbb{E}[Ff(F)]$ and $\mathbb{E}[f'(F)]$ has an explicite bound, by using contraction star operators $\star_t^s$. (See Theorem 2.1.9) (It is nature to repalce operator $\otimes_r$ by operator $\star_t^s$, if we have noticed the different forms of "product formula" of multiple integrals in Poisson case (1.12) and Gaussian case(1.24).)

Finally, a CLT (Theorem 2.1.10) for sequence of (fixed order) multiple Wiener-Itô integral is formed, as an analogue of Theorem 2.1.4.

Theorem 2.1.8 *Let $F \in DomD$ be such that $\mathbb{E}[F] = 0$. Let $N \sim \mathcal{N}(0,1)$. Then,*

$$\begin{aligned} d_W(F,N) &\leq \mathbb{E}|1 - \langle DF, -DL^{-1}F\rangle_{L^2(\mu)}| + \int_Z \mathbb{E}[|D_zF|^2|D_zL^{-1}F|]\mu(dz) &(2.16)\\ &\leq \sqrt{\mathbb{E}[(1 - \langle DF, -DL^{-1}F\rangle_{L^2(\mu)})^2]} + \int_Z \mathbb{E}[|D_zF|^2|D_zL^{-1}F|]\mu(dz) &(2.17)\end{aligned}$$

Proof. By virtue of Stein-type bound (1.62), it is sufficient to prove that, for every function f such that $\|f'\|_\infty \leq 1$ and $\|f''\|_\infty \leq 2$, the quantity $|\mathbb{E}[f'(F) - Ff(F)]|$ is smaller than the right-hand side of (2.16). Following the deductions that we have made above, we have

$$\mathbb{E}[Ff(F)] = \mathbb{E}[f'(F)\langle DF, -DL^{-1}F\rangle_{L^2(\mu)}] + \mathbb{E}[\langle R(DF), -DL^{-1}F\rangle_{L^2(\mu)}].$$

It follows that,

$$|\mathbb{E}[f'(F) - Ff(F)]| \leq |\mathbb{E}[f'(F)(1 - \langle DF, -DL^{-1}F\rangle_{L^2(\mu)})]| + |\mathbb{E}[\langle R(DF), -DL^{-1}F\rangle_{L^2(\mu)}]|.$$

By the fact that $\|f'\|_\infty \leq 1$ and by Cauchy-Schwarz inequality,

$$\begin{aligned} |\mathbb{E}[f'(F)(1 - \langle DF, -DL^{-1}F\rangle_{L^2(\mu)})]| &\leq \mathbb{E}|1 - \langle DF, -DL^{-1}F\rangle_{L^2(\mu)}| \\ &\leq \sqrt{\mathbb{E}[(1 - \langle DF, -DL^{-1}F\rangle_{L^2(\mu)})^2]}.\end{aligned}$$

On the other hand, one sees immediately that

$$\begin{aligned} |\mathbb{E}[\langle R(DF), -DL^{-1}F\rangle_{L^2(\mu)}]| &\leq \int_Z \mathbb{E}[|R(D_zF)D_zL^{-1}F|]\mu(dz) \\ &\leq \int_Z \mathbb{E}[|D_zF|^2|D_zL^{-1}F|]\mu(dz)\end{aligned}$$

The proof is complete. ■

The following Theorem is served to evaluate the bound for multiple stochastic integrals in Poisson Wiener chaos.

Theorem 2.1.9 *Fix $q \geq 2$ and let $N \sim \mathcal{N}(0,1)$. Let $f \in L_s^2(\mu^q)$ be such that:*

1. *whenever $\mu(Z) = \infty$, the following technical condition (see Section 3.4 for the definition of $G_a^{b,c}$) is satisfied for every $p = 1, \dots, 2(q-1)$:*

$$\int_Z \left[\int_{Z^p} \left(G_p^{q-1,q-1}(f(z,\cdot), f(z,\cdot)) \right)^2 \sqrt{d\mu^p} \right] \mu(dz) < \infty; \tag{2.18}$$

2. *for $d\mu$-almost every $z \in Z$, every $r = 1, \dots, q-1$ and every $l = 0, \dots, r-1$, the kernel $f(z,\cdot) \star_r^l f(z,\cdot)$ is an element of $L_s^2(\mu^{2(q-1)-r-l})$.*

Denote by $I_q(f)$ the multiple Wiener-Itô integral, of order q, of f with respect to $\hat{N}$. Then the following bound hold:

$$\begin{aligned} d_W(I_q(f), N) &\leq |1 - q!\|f\|^2_{L^2(\mu^q)}| + \sum_{t=1}^{q} \sum_{s=1}^{t\wedge(q-1)} C(s,t,q) \|f \star_t^s f\|_{L^2(\mu^{2q-t-s})} \\ &+ \|f\|_{L^2(\mu^q)} \times \sum_{b=1}^{q} \sum_{a=0}^{b-1} D(a,b,q) \|f \star_b^a f\|_{L^2(\mu^{2q-a-b})}, \end{aligned}$$

where $C(x,y,q)$ and $D(x,y,q)$ are positive constants depending on x, y, q. The explicit form of $C(x,y,q)$ and $D(x,y,q)$ can be found in Theorem 4.2 in [46].

At last, we are able to present a CLT of sequences of multiple stochastic integrals. However, the conditions are no longer necessary and sufficient, as in Gaussian case.

Theorem 2.1.10 (Theorem 5.1 in [46]) *Let $N \sim \mathcal{N}(0,1)$. Suppose that $\mu(Z) = \infty$, fix $q \geq 2$, and let $\{F_k = I_q(f_k); k \geq 1\}$, be a sequence of multiple stochastic Wiener-Itô integrals of order q. Suppose that, as $k \to \infty$, the normalization condition $\mathbb{E}[F_k^2] = q!\|\|^2_{L^2(\mu^q)} \to 1$ takes place. Assume moreover that the following three conditions hold:*

- *(I) For every $k \geq 1$, the kernel f_k verifies condition (2.18) for every $p = 1, \dots, 2(q-1)$.*
- *(II) For every $r = 1, \dots, q$ and every $l = 1, \dots, r \wedge (q-1)$, one has that $f_k \star_r^l f_k \in L^2(\mu^{2q-r-l})$ and also $\|f_k \star_r^l f_k\|_{L^2(\mu^{2q-r-l})} \to 0$ as $k \to \infty$.*
- *(III) For every $k \geq 1$, one has that $\int_{Z^q} f_k^4 d\mu^q < \infty$ and, as $k \to \infty$,*

$$\int_{Z^q} f_k^4 d\mu^q \to 0.$$

Then, $F_k \xrightarrow{law} N$, as $k \to \infty$.

2.2 Some extensions of the "Malliavin-Stein" method

2.2.1 The "smart path" technique

In the previous section, we have seen that Stein's method plays a crucial role in order to assess the distance between the laws of two random variables (or vectors). Now we present an alternative way for evaluating the distances. This technique is close to the "smart path" technique introduced by Talagrand in the context of spin glass (see[68]).

To see how it works, we revisit the one-dimensional Gaussian case discussed in Section 2.1.1.

Let $G = \{G(h) : h \in \mathfrak{H}\}$ be an isonormal Gaussian process over a real separable Hilbert space $\mathfrak{H}$, and $F \in \mathbb{D}^{1,2}$ be a functional of G, $N \sim \mathcal{N}(0,1)$. We shall evaluate

$$\delta = |\mathbb{E}[g(F)] - \mathbb{E}[g(N)]|$$

for C^2 test function g with bounded second derivative.

Let $\Psi(t) = \mathbb{E}[g(\sqrt{1-t}F + \sqrt{t}N)]$, we easily see that Ψ is differentiable on $(0,1)$, therefore

$$\delta = |\Psi(1) - \Psi(0)| \leq \sup_{t \in (0,1)} |\Psi'(t)|,$$

with

$$\Psi'(t) = \mathbb{E}\left[\frac{\partial g}{\partial x}\left(\sqrt{1-t}F + \sqrt{t}N\right) \times \left(\frac{1}{2\sqrt{t}}N - \frac{1}{2\sqrt{1-t}}F\right)\right].$$

By integrating by parts and Stein's characterization of normal variable (1.58) in Lemma 1.3.7 (i), we write

$$\begin{aligned} & \mathbb{E}\left[\frac{\partial g}{\partial x}\left(\sqrt{1-t}F + \sqrt{t}N\right) \times N\right] \\ = \; & \mathbb{E}\left[\mathbb{E}\left[\frac{\partial g}{\partial x}\left(\sqrt{1-t}z + \sqrt{t}N\right) \times N\right] \Big|_{z=F}\right] \\ = \; & \mathbb{E}\left[\mathbb{E}\left[\sqrt{t}\frac{\partial^2 g}{\partial x^2}\left(\sqrt{1-t}z + \sqrt{t}N\right)\right] \Big|_{z=F}\right] \\ = \; & \sqrt{t}\mathbb{E}\left[\frac{\partial^2 g}{\partial x^2}\left(\sqrt{1-t}F + \sqrt{t}N\right)\right]. \end{aligned}$$

Similarly, by relation (2.2) and integrating by parts, we write

$$\begin{aligned} & \mathbb{E}\left[\frac{\partial g}{\partial x}\left(\sqrt{1-t}F + \sqrt{t}N\right) \times F\right] \\ = \; & \mathbb{E}\left[\mathbb{E}\left[\frac{\partial g}{\partial x}\left(\sqrt{1-t}F + \sqrt{t}z\right) \times F\right] \Big|_{z=N}\right] \\ = \; & \sqrt{1-t}\mathbb{E}\left[\mathbb{E}\left[\frac{\partial^2 g}{\partial x^2}\left(\sqrt{1-t}F + \sqrt{t}z\right) \langle DF, -DL^{-1}F\rangle_{\mathfrak{H}}\right] \Big|_{z=N}\right] \\ = \; & \sqrt{1-t}\mathbb{E}\left[\frac{\partial^2 g}{\partial x^2}\left(\sqrt{1-t}F + \sqrt{t}N\right) \langle DF, -DL^{-1}F\rangle_{\mathfrak{H}}\right]. \end{aligned}$$

Combining these two parts, we conclude that

$$\delta \leq \frac{1}{2}\|g''\|_\infty \mathbb{E}[|1 - \langle DF, -DL^{-1}F\rangle_{\mathfrak{H}}|],$$

which is of course the reminiscent of inequality (2.3).

The "smart path" technique is simple and direct, thus adapts to a wide range of situations. For instance, it has been adopted in some recent developments of the "Malliavin-Stein" method, see e.g. [3, 36]. However, the "smart path" technique usually has more stringent requirements on the smoothness and the regularity of the test function g, with respect to Stein's method. In Chapter 3 of this dissertation, we draw two tables in order to illustrate a comparison between these two methods.

2.2.2 Universality of Gaussian Wiener chaos

In the process of studying CLTs by means of the "Malliavin-Stein" method, one observes a variety of similarities between the Poisson and Gaussian spaces. It is not surprising to see that Gaussian Wiener chaos enjoys some kind of "Universality" property. In the paper [36], written by Nourdin, Peccati and Reinert, the authors discovered a "Universality" result for homogeneous sums. [1]

Their study concentrates on "homogeneous sums" of the type:

$$\mathcal{Q}_q(N, f, \mathbf{X}) = \sum_{i_1,\cdots,i_q}^{N} f(i_1,\cdots,i_q) X_{i_1}\cdots X_{i_q}$$

where N, $q \geq 2$ are two positive integers, $\mathbf{X} = \{X_i : i \geq 1\}$ is a collection of centered independent random variables, and $f : \{1,\ldots,N\}^q \to \mathbb{R}$ is a symmetric function vanishing on diagonals.

We present here one of their main results (see Theorem 1.2 in [36]), which is about CLTs of homogeneous sums. Let $\mathbf{G} = \{G_i : i \geq 1\}$ be a collection of independent standard Gaussian variables, and fix integers $d \geq 1$ and $q_1,\ldots,q_d \geq 2$. For every $j = 1,\ldots,d$, let $\{(N_j^{(n)}, f_j^{(n)}); n \geq 1\}$ be a sequence such that $\{N_j^{(n)}; n \geq 1\}$ is a sequence of integers going to infinity, and each function $f_j^{(n)} : \{1,\ldots,N_j^{(n)}\}^{q_j} \to \mathbb{R}$ is symmetric and vanishes on diagonals. Assume that, for every $j = 1,\ldots,d$, the sequence $\mathbb{E}[\mathcal{Q}_{q_j}(N_j^{(n)}, f_j^{(n)}, \mathbf{G})^2], n \geq 1$, is bounded. Let V be a $d \times d$ nonnegative symmetric matrix whose diagonal elements are different from zero. Then, as $n \to \infty$, the following two statements are equivalent:

1. The vector $F_G^{(n)} = \left(\mathcal{Q}_{q_j}(N_j^{(n)}, f_j^{(n)}, \mathbf{G}), j = 1,\ldots,d\right)$ converges in law to $Z_V = \mathcal{N}(0,V)$.
2. For every sequence $\mathbf{X} = \{X_i; i \geq 1\}$ of independent centered random variables, with unit variance and such that $\sup_i \mathbb{E}|X_i|^3 < \infty$, the law of the vector

$$F_X^{(n)} = \left(\mathcal{Q}_{q_j}(N_j^{(n)}, f_j^{(n)}, \mathbf{X}), j = 1,\ldots,d\right)$$

converges to $Z_V = \mathcal{N}(0,V)$ under the topology induced by Kolmogorov distance.

We give a sketch of the proof of the above universality statement. The crucial point is the triangle inequality:

$$d_K(F_X^{(n)}, Z_V) \leq d_K(F_X^{(n)}, F_G^{(n)}) + d_K(F_G^{(n)}, Z_V).$$

[1] In general, a "Universality" result is any mathematical statement implying that the asymptotic behavior of a large random system does not depend on the distribution of the components. Examples are the CLT, the Donsker theorem and the semicircular law in Random Matrix Theory.

Suppose that $d_K(F_G^{(n)}, Z_V) \to 0$ as $n \to \infty$, then according to the "Malliavin-Stein" theory (see Section 2.1), we have in particular that $\|f_j^{(n)} \otimes_{q_j-1} f_j^{(n)}\|_{\mathfrak{H}^{\otimes 2}} \to 0$. One then use the inequality:

$$\begin{aligned} \|f_j^{(n)} \otimes_{q_j-1} f_j^{(n)}\|^2_{\mathfrak{H}^{\otimes 2}} &\geq \sum_{1 \leq i \leq N_j^{(n)}} \left[\sum_{1 \leq i_2, \ldots, i_d \leq N_j^{(n)}} f_j^{(n)}(i, i_2, \ldots, i_d)^2 \right]^2 \\ &\geq \left[(q_j - 1)! \max_{1 \leq i \leq N_j^{(n)}} \mathbf{Inf}_i(f_j^{(n)}) \right]^2, \end{aligned}$$

where $\mathbf{Inf}_i(f) := \dfrac{1}{(d-1)!} \sum_{1 \leq i_2, \ldots, i_d \leq N} f^2(i, i_2, \ldots, i_d)$ is called the *influence* of the variable i. Finally, since

$$\max_{1 \leq i \leq N_j^{(n)}} \mathbf{Inf}_i(f_j^{(n)}) \to 0, \qquad \forall j = 1, \ldots, d,$$

one deduces the conclusion by means of the results established in Mossel's paper [28] and Mossel, O'Donnell and Oleszkiewicz's paper [29].

2.2.3 Almost Sure Central Limit Theorem (ASCLT)

The Almost Sure Central Limit Theorem is a generalization of the classic Central Limit Theorems. For example, let $\{X_n; n \geq 1\}$ be a sequence of i.i.d. random variables such that $\mathbb{E}[X_n] = 0$ and $\mathbb{E}[X_n^2] = 1$. We denote $S_n = \frac{1}{\sqrt{n}} \sum_{k=1}^n X_k$. Then, the CLT tells us that S_n converges to $N \sim \mathcal{N}(0,1)$ in law. The ASCLT shows that the sequence of random empirical measure defined by

$$\frac{1}{\log n} \sum_{k=1}^n \frac{1}{k} \delta_{S_k}$$

weakly converges almost surely to N as $n \to \infty$. In other words, one has that, the convergence

$$\frac{1}{\log n} \sum_{k=1}^n \frac{1}{k} \varphi(S_k) \to \mathbb{E}[\varphi(N)], \text{ as } n \to \infty,$$

takes place almost surely, for any bounded and continuous function $\varphi : \mathbb{R} \to \mathbb{R}$. Generally speaking, if $\{F_n\}$ converges in law towards a random variable F_∞, then we say that ASCLT holds if for any bounded and continuous function $\varphi : \mathbb{R} \to \mathbb{R}$,

$$\frac{1}{\log n} \sum_{k=1}^n \frac{1}{k} \varphi(F_k) \to \mathbb{E}[\varphi(F_\infty)], \text{ as } n \to \infty.$$

For a history of this result, see [18], or Chapter 5.

Having seen the power of the "Malliavin-Stein" method in deducing CLTs, we may ask whether this method can be applied to ASCLTs. In the paper [3] by Bercu, Nourdin and Taqqu, the authors give a positive answer. They prove that, under some suitable criteria involving the star contractions of kernels, a sequence of multiple integral functionals that satisfies a CLT must also satisfy an ASCLT.

Technically, their theory is based on Ibragimov and Lifshits's criterion. (See Theorem 3.1 in [3], or Theorem 5.1 in [32]) In order to show an ASCLT for a sequence $\{F_n\}$, one needs only to prove that

$$\sup_{|t|\le r}\sum_n \frac{\mathbb{E}|\Delta_n(t)|^2}{n\log n}<\infty,$$

for all $r>0$, where

$$\Delta_n(t)=\frac{1}{\log n}\sum_{k=1}^n\frac{1}{k}\left(e^{itF_k}-\mathbb{E}\left[e^{itF_\infty}\right]\right).$$

On the Gaussian space, we may apply the "smart path" technique, together with Malliavin calculus and Poincaré inequality, to obtain an upper bound for $\Delta_n(t)$ that involves the cross products of second derivatives $\{D^2F_k, k=1,2,\ldots\}$. If we restrict the problem on sequence of multiple integrals $F_k=I_q^G(f_k)$, the upper bound can in turn be reduced to some sum of contractions of the kernels $\{f_k\}$. Below is one of their main results.

Proposition 2.2.1 (Corollary 3.6 in [3]) *Fix $q\ge 2$, and let $\{F_n\}$ be a sequence of the form $F_n=I_q^G(f_n)$, with $f_n\in\mathfrak{H}^{\odot q}$. Assume that $\mathbb{E}[F_n^2]=q!\|f_n\|^2_{\mathfrak{H}^{\otimes q}}=1$ for all n, and that*

$$\forall r=1,\ldots,q-1,\qquad \|f_n\otimes_r f_n\|_{\mathfrak{H}^{\otimes 2(q-r)}}\to 0,\ \text{as } n\to\infty.$$

Then, $F_n\xrightarrow{law} N\sim\mathcal{N}(0,1)$ as $n\to\infty$. Moreover, if the two following conditions are satisfied

(A_1) $\sum_{n\ge 2}\frac{1}{n\log^2 n}\sum_{k=1}^n\frac{1}{k}\|f_n\otimes_r f_n\|_{\mathfrak{H}^{\otimes 2(q-r)}}<\infty$, *for every $r=1,\ldots,q-1$;*

(A_2) $\sum_{n\ge 2}\frac{1}{n\log^3 n}\sum_{k,l=1}^n\frac{|\langle f_k,f_l\rangle_{\mathfrak{H}^{\otimes q}}|}{kl}<\infty$,

then $\{F_n\}$ satisfies an ASCLT. In other words, almost surely, for all continuous and bounded function $\varphi:\mathbb{R}\to\mathbb{R}$,

$$\frac{1}{\log n}\sum_{k=1}^n\frac{1}{k}\varphi(F_k)\to\mathbb{E}[\varphi(N)],\ \text{as } n\to\infty.$$

2.3 Main contributions

The principal aim of this dissertation is to generalize the "Malliavin-Stein" method on the Poisson space, in order to study CLTs and associated convergence result. My research focus on three different but relevant directions: Central Limit Theorem(CLT) for random vectors on the Poisson space, universality results on the Poisson space, and Almost Sure Central Limit Theorems (ASCLTs) on the Poisson space. The Chapter 3, 4, 5 are respectively dedicated to these three subjects.

2.3.1 Multi-dimensional CLTs on the Poisson space

This part is based on the published paper [50] by G. Peccati and C. Zheng. See Chapter 3.

The findings of this part is a multi-dimensional generalization of the theory described in Section 2.1.1, 2.1.2 and 2.1.3, which are in turn based on the results developed in [31, 37, 46]. Our main tasks will be:

I. Let $F = (F_1, \dots, F_d)$ be a random vector on the Poisson space, and $X_C \sim \mathcal{N}(0, C)$ be a Gaussian vector with positive definite covariance matrix C. We shall develop new techniques in order to evaluate the distance between F and C, by using Stein's method and Malliavin operators.

II. Let $F^{(n)} = (F_1^{(n)}, \dots, F_d^{(n)}) = (I_{q_1}(f_1^{(n)}), \dots, I_{q_d}(f_d^{(n)}))$ be a sequence of vectors of multiple integrals with fixed orders $q_1, \dots, q_d$. We wish to build a CLT similar to Theorem 2.1.7. In particular, the conditions of such a CLT should involve only the covariance of F_n and thge component-wise convergence of star contractions of kernels.

To accomplish Task I, we follow the procedures shown in the previous section, and apply the multi-dimensional Stein's Lemma 1.3.12 in order to evaluate the distance between a random vector and a given Gaussian vector. The known distances such as Wasserstein distance, total variation distance no longer fulfill our needs. In view of this, we establish inequality (1.74) and define a distance d_2 in order to build an analogue of inequality (2.13).

We will revisit the above inequalities by using the "smart path" technique instead of Stein's method, in the Gaussian and the Poisson cases, both for random variables and random vector sequences. We will draw two tables in order to analyze the specificities of these two methods.

Task II requires an explicit expression for the upper bound obtained by using the Malliavin operators. The multiplication formula (1.12) plays a central role in the process of evaluation. Though the main procedures are similar to those in Section 2.1.1, 2.1.2, 2.1.3, the calculations turn out to be much more complicated.

It is worth highlighting that the Cauchy-Schwartz type inequality (1.9) in Lemma 1.1.8 paves the way for an upper bound composed only of component-wise star product contractions $f \star_r^l f$. Finally, we achieve a CLT with conditions similar to that of Theorem 2.1.7. Nevertheless, in this new CLT the conditions are no longer sufficient and necessary as in its Gaussian case.

2.3.2 Universality of Poisson Wiener chaos

This part is based on the paper in preparation [51] by G. Peccati and C. Zheng. See Chapter 4.

The study on this subject is based on the research of CLTs on the Poisson space and the "Universality" of the Gaussian Wiener chaos. We focus on homogeneous sums involving Poisson variables.

Let $\{\lambda_i, i \geq 1\}$ be a collection of positive real numbers, under the assumption $\inf_i \lambda_i = \eta > 0$. Let $\mathbf{P} = \{P_i, i \geq 1\}$ be a collection of independent random variables such that $\forall i$,

P_i is a centered Poisson variable with parameter λ_i. Fix integers $d \geq 1, q_d \geq \cdots q_1 \geq 1$. Let $\{N_j^{(n)}, f_j^{(n)} : j = 1, \cdots, d, n \geq 1\}$ be such that for every fixed j, $\{N_j^{(n)} : n \geq 1\}$ is a sequence of integers going to infinity, and each $f_j^{(n)} : [N_j^{(n)}]^{q_j} \to \mathbb{R}$ is symmetric and vanishes on diagonals. We consider random vectors $F^{(n)} = (F_1^{(n)}, \cdots, F_d^{(n)})$, where for every $1 \leq j \leq d$,

$$F_j^{(n)} = \mathcal{Q}_{q_j}(N_j^{(n)}, f_j^{(n)}, \mathbf{P}) = \sum_{i_1, \cdots, i_{q_j}}^{N_j^{(n)}} f_j^{(n)}(i_1, \cdots, i_{q_j}) P_{i_1} \cdots P_{i_{q_j}} = I_{q_j}(h_j^{(n)})$$

with

$$h_j^{(n)} = \sum_{i_1, \cdots, i_{q_j}}^{N_j^{(n)}} f_j^{(n)}(i_1, \cdots, i_{q_j}) g_{i_1} \otimes \cdots \otimes g_{i_{q_j}}.$$

We are interested in the following two questions:

1. We have succeeded in building the "Malliavin-Stein" CLT theory on the Poisson space, but conditions are no longer equivalent as they are in the Gaussian case. Can we build a CLT for homogeneous sums inside the Poisson Wiener chaos with necessary and sufficient conditions?

2. We know that the Gaussian Wiener chaos has some beautiful "Universality" properties. Is Poisson Wiener chaos universal? That is, does the convergence to Gaussian of a Poisson homogeneous sum vector $F^{(n)}$ imply the convergence of any homogeneous sum vector with same parameters and kernels?

The answers of these two questions are in fact relevant. Suppose that, under certain technical assumptions, we are able to build a CLT with necessary and sufficient conditions for Poisson homogeneous sum vector $F^{(n)}$. Therefore, the convergence of $F^{(n)}$ to a normal vector implies that, for each $j = 1, \cdots, d$, $\int (h_j^{(n)})^4 \to 0$ and $\forall r = 1, \cdots, q_j$, $\forall l = 1, \cdots, r \wedge (q_j - 1)$, $\|h_j^{(n)} \star_r^l h_j^{(n)}\|_{L^2} \to 0$. Then for $r = 1, \cdots, q_j$, the convergence $\|h_j^{(n)} \otimes_r h_j^{(n)}\|_{L^2} \to 0$ immediately follows, which gives the convergence to normal vector of the Gaussian counterpart of $F^{(n)}$ due to Theorem 2.1.4. Consequently, "Universality" of Poisson Wiener chaos is linked to Gaussian "Universality".

The key to the expected "equivalence version" CLT, is a technical proposition, which says that: if $\|h_n \star_q^q h_n\|_{L^2} \to 0$, then: A) $\int h_n^4 \to 0$. B) $\forall r = 1, \cdots, q$, $\forall l = 1, \cdots, r \wedge (q - 1)$, $\|h_n \star_r^l h_n\|_{L^2} \to 0$. The proof of this proposition involves some technical computations on star product contraction.

2.3.3 ASCLT on the Poisson space

This part is based on the paper in preparation [69] by C. Zheng. See Chapter 5.

Here we want to study ASCLTs on the Poisson space, in both the one-dimensional case and the multi-dimensional case.

The research in [3] set a roadmap for us, but the study on the Poisson space always requires more efforts and techniques. Based on Ibragimov and Lifshits's criterion and the "smart path" technique, the estimation of $\Delta_n(t)$ relies on the following tools:

1) Differentiation rule:

$$D\langle DF, DF^*\rangle_{L^2(\mu)} = \langle DF, D^2F^*\rangle_{L^2(\mu)} + \langle D^2F, DF^*\rangle_{L^2(\mu)} + \langle D^2F, D^2F^*\rangle_{L^2(\mu)},$$

for regular Poisson functionals F,F^*.

2) Generalized chain rules:

$$\mathbb{E}[Fg(F^*)] = \mathbb{E}[g'(F^*)\langle DF^*, -DL^{-1}F\rangle_{L^2(\mu)}] + \mathbb{E}[\langle R, -DL^{-1}F\rangle_{L^2(\mu)}],$$

for regular Poisson functionals F,F^* and function $g \in \mathcal{C}^2(\mathbb{R})$, where R is a functional satisfying

$$\left|\mathbb{E}[\langle R, -DL^{-1}F\rangle_{L^2(\mu)}]\right| \leq \frac{1}{2}\sup_{y\in\mathbb{R}}|g''(y)| \int_Z \mu(dz)\mathbb{E}\left[|D_zF^*|^2|DL^{-1}F|\right].$$

3) Poincaré inequality:

$$\mathbf{Var}(F) \leq \mathbb{E}\|DF\|^2_{L^2(\mu)},$$

for regular Poisson functional F.

Indeed, 1) and 2) have forms that are more complex than their Gaussian counterpart. However, the estimation remains feasible (though very technical) since we adopt the orthogonal decomposition technique by means of functional $G_k^{p,q}(\cdot,\cdot)$.

Using these results, we shall build ASCLTs on the Poisson space with criteria that parallel to their Gaussian counterparts. We also provide simple versions of these criteria, which are easier to verify and apply.

We further generalize our research to the multi-dimensional case, that is, the ASCLT of random vectors on the Poisson space. Here we apply some estimations obtained in the CLT research, as well as the Cauchy-Schwartz type inequality (1.9) in Lemma 1.1.8. We succeed in building criteria involving only component-wise kernel star product contractions.

At the end of the chapter, we revisit all the examples concerning Ornstein-Uhlenbeck functionals, and build the corresponding ASCLTs.

Chapter 3

Central Limit Theorems on the Poisson space

This chapter is based on the published paper [50] by G. Peccati and C. Zheng.

3.1 Introduction of the chapter

We fix $d \geq 2$, let $F = (F_1, \ldots, F_d) \subset L^2(\sigma(\hat{N}), \mathbb{P})$ be a vector of square-integrable functionals of $\hat{N}$, and let $X = (X_1, \ldots, X_d)$ be a centered Gaussian vector. The aim of this chapter is to develop several techniques, allowing to assess quantities of the type

$$d_{\mathcal{H}}(F, X) = \sup_{g \in \mathcal{H}} |\mathbb{E}[g(F)] - \mathbb{E}[g(X)]|, \tag{3.1}$$

where $\mathcal{H}$ is a suitable class of real-valued test functions on $\mathbb{R}^d$. As discussed below, our principal aim is the derivation of explicit upper bounds in multi-dimensional Central limit theorems (CLTs) involving vectors of general functionals of $\hat{N}$. Our techniques rely on a powerful combination of Malliavin calculus (in a form close to Nualart and Vives [41]), Stein's method for multivariate normal approximations (see e.g. [11, 37, 58] and the references therein), as well as some interpolation techniques reminiscent of Talagrand's "smart path method" (see [68], and also [9, 36]). As such, our findings can be seen as substantial extensions of the results and techniques developed e.g. in [31, 37, 46], where Stein's method for normal approximation is successfully combined with infinite-dimensional stochastic analytic procedures (in particular, with infinite-dimensional integration by parts formulae).

The main findings of the present chapter are the following:

(I) We shall use both Stein's method and interpolation procedures in order to obtain explicit upper bounds for distances such as (3.1). Our bounds will involve Malliavin derivatives and infinite-dimensional Ornstein-Uhlenbeck operators. A careful use of interpolation techniques also allows to consider Gaussian vectors with a non-positive definite covariance matrix. As seen below, our estimates are the exact Poisson counterpart of the bounds deduced in a Gaussian framework in Nourdin, Peccati and Réveillac [37] and Nourdin, Peccati and Reinert [36].

(II) The results at point **(I)** are applied in order to derive explicit sufficient conditions for multivariate CLTs involving vectors of multiple Wiener-Itô integrals with respect to $\hat{N}$. These results extend to arbitrary orders of integration and arbitrary dimensions the CLTs deduced by Peccati and Taqqu [47] in the case of single and double Poisson integrals (note that the techniques developed in [47] are based on decoupling). Moreover, our findings partially generalize to a Poisson framework the main result by Peccati and Tudor [49], where it is proved that, on a Gaussian Wiener chaos (and under adequate conditions), componentwise convergence to a Gaussian vector is always equivalent to joint convergence. (See also [37].) As demonstrated in Section 6, this property is particularly useful for applications.

The rest of the chapter is organized as follows. In Section 3.2, we use Malliavin-Stein techniques to deduce explicit upper bounds for the Gaussian approximation of a vector of functionals of a Poisson measure. In Section 3.3, we use an interpolation method (close to the one developed in [36]) to deduce some variants of the inequalities of Section 3.2. Section 3.4 is devoted to CLTs for vectors of multiple Wiener-Itô integrals. Section 3.5 focuses on examples, involving in particular functionals of Ornstein-Uhlenbeck Lévy processes.

The main results in the current chapter are included in the published paper [50].

3.2 Upper bounds obtained by Malliavin-Stein methods

We will now deduce one of the main findings of the present chapter, namely Theorem 3.2.3. This result allows to estimate the distance between the law of a vector of Poisson functionals and the law of a Gaussian vector, by combining the multi-dimensional Stein's Lemma 1.3.12 with the algebra of the Malliavin operators. Note that, in this section, all Gaussian vectors are supposed to have a positive definite covariance matrix.

We start by proving a technical lemma, which is a crucial element in most of our proofs.

Lemma 3.2.1 *Fix $d \geq 1$ and consider a vector of random variables $F := (F_1, \ldots, F_d) \subset L^2(\mathbb{P})$. Assume that, for all $1 \leq i \leq d$, $F_i \in \operatorname{Dom} D$, and $\mathbb{E}[F_i] = 0$. For all $\phi \in \mathbb{C}^2(\mathbb{R}^d)$ with bounded derivatives, one has that*

$$D_z\phi(F_1, \ldots, F_d) = \sum_{i=1}^{d} \frac{\partial}{\partial x_i}\phi(F)(D_zF_i) + \sum_{i,j=1}^{d} R_{ij}(D_zF_i, D_zF_j), \ z \in Z,$$

where the mappings R_{ij} satisfy

$$|R_{ij}(y_1, y_2)| \leq \frac{1}{2} \sup_{x\in\mathbb{R}^d} \left|\frac{\partial^2}{\partial x_i \partial x_j}\phi(x)\right| \times |y_1y_2| \leq \frac{1}{2}\|\phi''\|_\infty |y_1y_2|. \tag{3.2}$$

Proof. By the multivariate Taylor theorem and Lemma 1.2.4,

$$\begin{aligned}D_z\phi(F_1,\ldots,F_d) &= \phi(F_1,\ldots,F_d)(\omega+\delta_z)-\phi(F_1,\ldots,F_d)(\omega)\\ &= \phi(F_1(\omega+\delta_z),\ldots,F_d(\omega+\delta_z))-\phi(F_1(\omega),\ldots,F_d(\omega))\\ &= \sum_{i=1}^d \frac{\partial}{\partial x_i}\phi(F_1(\omega),\ldots,F_d(\omega))(F_i(\omega+\delta_z)-F_i(\omega))+R\\ &= \sum_{i=1}^d \frac{\partial}{\partial x_i}\phi(D_zF_i)+R,\end{aligned}$$

where the term R represents the residue:

$$R=R(D_zF_1,\ldots,D_zF_d)=\sum_{i,j=1}^d R_{ij}(D_zF_i,D_zF_j),$$

and the mapping $(y_1,y_2)\mapsto R_{ij}(y_1,y_2)$ verifies (3.2). ∎

Remark 3.2.2 Lemma 3.2.1 is the Poisson counterpart of the multi-dimensional "chain rules" verified by the Malliavin derivative on a Gaussian space (see [31, 37]). Notice that the term R does not appear in the Gaussian framework.

The following result uses the two Lemmas 1.3.12 and 3.2.1, in order to compute explicit bounds on the distance between the laws of a vector of Poisson functionals and the law of a Gaussian vector.

Theorem 3.2.3 (Malliavin-Stein inequalities on the Poisson space) *Fix $d\geq 2$ and let $C=\{C(i,j):i,j=1,\ldots,d\}$ be a $d\times d$ positive definite matrix. Suppose that $X\sim\mathcal{N}_d(0,C)$ and that $F=(F_1,\ldots,F_d)$ is a $\mathbb{R}^d$-valued random vector such that $\mathbb{E}[F_i]=0$ and $F_i\in \operatorname{Dom} D$, $i=1,\ldots,d$. Then,*

$$d_2(F,X)\leq \|C^{-1}\|_{op}\|C\|_{op}^{1/2}\sqrt{\sum_{i,j=1}^d \mathbb{E}[(C(i,j)-\langle DF_i,-DL^{-1}F_j\rangle_{L^2(\mu)})^2]} \tag{3.3}$$

$$+\frac{\sqrt{2\pi}}{8}\|C^{-1}\|_{op}^{3/2}\|C\|_{op}\int_Z\mu(dz)\mathbb{E}\left[\left(\sum_{i=1}^d|D_zF_i|\right)^2\left(\sum_{i=1}^d|D_zL^{-1}F_i|\right)\right]. \tag{3.4}$$

Proof. If either one of the expectations in (3.3) and (3.4) are infinite, there is nothing to prove: we shall therefore work under the assumption that both expressions (3.3)–(3.4) are finite. By the definition of the distance d_2, and by using an interpolation argument (identical to the one used at the beginning of the proof of Theorem 4 in [11]), we need only show the following inequality:

$$\begin{aligned}&|\mathbb{E}[g(X)]-\mathbb{E}[g(F)]|\\ &\leq A\|C^{-1}\|_{op}\|C\|_{op}^{1/2}\sqrt{\sum_{i,j=1}^d \mathbb{E}[(C(i,j)-\langle DF_i,-DL^{-1}F_j\rangle_{L^2(\mu)})^2]} \qquad (3.5)\\ &+\frac{\sqrt{2\pi}}{8}B\|C^{-1}\|_{op}^{3/2}\|C\|_{op}\int_Z\mu(dz)\mathbb{E}\left[\left(\sum_{i=1}^d|D_zF_i|\right)^2\left(\sum_{i=1}^d|D_zL^{-1}F_i|\right)\right]\end{aligned}$$

for any $g \in \mathbb{C}^\infty(\mathbb{R}^d)$ with first and second bounded derivatives, such that $\|g\|_{Lip} \le A$ and $M_2(g) \le B$. To prove (3.5), we use Point (ii) in Lemma 1.3.12 to deduce that

$$\begin{aligned}
&|\mathbb{E}[g(X)] - \mathbb{E}[g(F)]| \\
&= |\mathbb{E}[\langle C, \operatorname{Hess} U_0 g(F)\rangle_{H.S.} - \langle F, \nabla U_0 g(F)\rangle_{\mathbb{R}^d}]| \\
&= \left| \mathbb{E}\left[\sum_{i,j=1}^d C(i,j) \frac{\partial^2}{\partial x_i \partial x_j} U_0 g(F) - \sum_{k=1}^d F_k \frac{\partial}{\partial x_k} U_0 g(F) \right] \right| \\
&= \left| \sum_{i,j=1}^d \mathbb{E}\left[C(i,j) \frac{\partial^2}{\partial x_i \partial x_j} U_0 g(F) \right] + \sum_{k=1}^d \mathbb{E}\left[\delta(DL^{-1}F_k) \frac{\partial}{\partial x_k} U_0 g(F) \right] \right| \\
&= \left| \sum_{i,j=1}^d \mathbb{E}\left[C(i,j) \frac{\partial^2}{\partial x_i \partial x_j} U_0 g(F) \right] - \sum_{k=1}^d \mathbb{E}\left[\left\langle D\left(\frac{\partial}{\partial x_k} U_0 g(F) \right), -DL^{-1}F_k \right\rangle_{L^2(\mu)} \right] \right|.
\end{aligned}$$

We write $\frac{\partial}{\partial x_k} U_0 g(F) := \phi_k(F_1, \ldots, F_d) = \phi_k(F)$. By using Lemma 3.2.1, we infer

$$D_z \phi_k(F_1, \ldots, F_d) \quad = \quad \sum_{i=1}^d \frac{\partial}{\partial x_i} \phi_k(F)(D_z F_i) + R_k,$$

with $R_k = \sum_{i,j=1}^d R_{i,j,k}(D_z F_i, D_z F_j)$, and

$$|R_{i,j,k}(y_1, y_2)| \le \frac{1}{2} \sup_{x \in \mathbb{R}^d} \left| \frac{\partial^2}{\partial x_i \partial x_j} \phi_k(x) \right| \times |y_1 y_2|.$$

It follows that

$$\begin{aligned}
& |\mathbb{E}[g(X)] - \mathbb{E}[g(F)]| \\
= \;& \Bigg| \sum_{i,j=1}^d \mathbb{E}\left[C(i,j) \frac{\partial^2}{\partial x_i \partial x_j} U_0 g(F) \right] - \sum_{i,k=1}^d \mathbb{E}\left[\frac{\partial^2}{\partial x_i \partial x_k} (U_0 g(F)) \langle DF_i, -DL^{-1}F_k \rangle_{L^2(\mu)} \right] \\
& + \sum_{i,j,k=1}^d \mathbb{E}\left[\langle R_{i,j,k}(DF_i, DF_j), -DL^{-1}F_k \rangle_{L^2(\mu)} \right] \Bigg| \\
\le \;& \sqrt{\mathbb{E}[\|\operatorname{Hess} U_0 g(F)\|_{H.S.}^2]} \times \sqrt{\sum_{i,j=1}^d \mathbb{E}\left[\left(C(i,j) - \langle DF_i, -DL^{-1}F_j \rangle_{L^2(\mu)} \right)^2 \right]} + |R_2|,
\end{aligned}$$

where

$$R_2 = \sum_{i,j,k=1}^d \mathbb{E}[\langle R_{i,j,k}(DF_i, DF_j), -DL^{-1}F_k \rangle_{L^2(\mu)}].$$

Note that (1.73) implies that $\|\text{Hess}\, U_0 g(F)\|_{H.S.} \leq \|C^{-1}\|_{op} \|C\|_{op}^{1/2} \|g\|_{Lip}$. By using (1.74) and the fact $\|g'''\|_\infty \leq M_3(g)$, we have

$$|R_{i,j,k}(y_1, y_2)| \leq \frac{1}{2} \sup_{x \in \mathbb{R}^d} \left| \frac{\partial^3}{\partial x_i \partial x_j \partial x_k} U_0(g(y)) \right| \times |y_1 y_2|$$

$$\leq \frac{\sqrt{2\pi}}{8} M_2(g) \|C^{-1}\|_{op}^{3/2} \|C\|_{op} \times |y_1 y_2| \leq \frac{\sqrt{2\pi}}{8} B \|C^{-1}\|_{op}^{3/2} \|C\|_{op} \times |y_1 y_2|,$$

from which we deduce the desired conclusion. ■

Now recall that, for a random variable $F = \hat{N}(h) = I_1(h)$ in the first Wiener chaos of $\hat{N}$, one has that $DF = h$ and $L^{-1}F = -F$. By virtue of Remark 1.3.6, we immediately deduce the following consequence of Theorem 3.2.3.

Corollary 3.2.4 *For a fixed $d \geq 2$, let $X \sim \mathcal{N}_d(0, C)$, with C positive definite, and let*

$$F_n = (F_{n,1}, \ldots, F_{n,d}) = (\hat{N}(h_{n,1}), \ldots, \hat{N}(h_{n,d})), \ n \geq 1,$$

be a collection of d-dimensional random vectors living in the first Wiener chaos of $\hat{N}$. Call K_n the covariance matrix of F_n, that is: $K_n(i,j) = \mathbb{E}[\hat{N}(h_{n,i})\hat{N}(h_{n,j})] = \langle h_{n,i}, h_{n,j} \rangle_{L^2(\mu)}$. Then,

$$d_2(F_n, X) \leq \|C^{-1}\|_{op} \|C\|_{op}^{1/2} \|C - K_n\|_{H.S.} + \frac{d^2\sqrt{2\pi}}{8} \|C^{-1}\|_{op}^{3/2} \|C\|_{op} \sum_{i=1}^{d} \int_Z |h_{n,i}(z)|^3 \mu(dz).$$

In particular, if

$$K_n(i,j) \to C(i,j) \quad \text{and} \quad \int_Z |h_{n,i}(z)|^3 \mu(dz) \to 0 \tag{3.6}$$

(as $n \to \infty$ and for every $i, j = 1, \ldots, d$), then $d_2(F_n, X) \to 0$ and F_n converges in distribution to X.

Remark 3.2.5

1. The conclusion of Corollary 3.2.4 is by no means trivial. Indeed, apart from the requirement on the asymptotic behavior of covariances, the statement of Corollary 3.2.4 does not contain *any* assumption on the joint distribution of the components of the random vectors F_n. We will see in Section 3.4 that analogous results can be deduced for vectors of multiple integrals of arbitrary orders. We will also see in Corollary 3.3.3 that one can relax the assumption that C is positive definite.

2. The inequality appearing in the statement of Corollary 3.2.4 should also be compared with the following result, proved in [37], yielding a bound on the Wasserstein distance between the laws of two Gaussian vectors of dimension $d \geq 2$. Let $Y \sim \mathcal{N}_d(0, K)$ and $X \sim \mathcal{N}_d(0, C)$, where K and C are two positive definite covariance matrices. Then, $d_W(Y, X) \leq Q(C, K) \times \|C - K\|_{H.S.}$, where

$$Q(C, K) := \min\{\|C^{-1}\|_{op} \|C\|_{op}^{1/2}, \|K^{-1}\|_{op} \|K\|_{op}^{1/2}\},$$

and d_W denotes the Wasserstein distance between the laws of random variables with values in $\mathbb{R}^d$.

3.3 Upper bounds obtained by interpolation methods

3.3.1 Main estimates

In this section, we deduce an alternate upper bound (similar to the ones proved in the previous section) by adopting an approach based on interpolations. We first prove a result involving Malliavin operators.

Lemma 3.3.1 *Fix $d \geq 1$. Consider $d+1$ random variables $F_i \in L^2(\mathbb{P})$, $0 \leq i \leq d$, such that $F_i \in \operatorname{Dom} D$ and $\mathbb{E}[F_i] = 0$. For all $g \in \mathbb{C}^2(\mathbb{R}^d)$ with bounded derivatives,*

$$\mathbb{E}[g(F_1,\dots,F_d)F_0]=\mathbb{E}\left[\sum_{i=1}^{d}\frac{\partial}{\partial x_i}g(F_1,\dots,F_d)\langle DF_i,-DL^{-1}F_0\rangle_{L^2(\mu)}\right]+\mathbb{E}\left[\langle R,-DL^{-1}F_0\rangle_{L^2(\mu)}\right],$$

where

$$\begin{aligned}&|\mathbb{E}[\langle R,-DL^{-1}F_0\rangle_{L^2(\mu)}]| \\ &\leq \frac{1}{2}\max_{i,j}\sup_{x\in\mathbb{R}^d}\left|\frac{\partial^2}{\partial x_i\partial x_j}g(x)\right| \times \int_Z \mu(dz)\mathbb{E}\left[\left(\sum_{k=1}^{d}|D_zF_k|\right)^2 |D_zL^{-1}F_0|\right].\end{aligned} \quad (3.7)$$

Proof. By applying Lemma 3.2.1,

$$\begin{aligned}&\mathbb{E}[g(F_1,\dots,F_d)F_0] \\ =\;&\mathbb{E}[(LL^{-1}F_0)g(F_1,\dots,F_d)] \\ =\;&-\mathbb{E}[\delta(DL^{-1}F_0)g(F_1,\dots,F_d)] \\ =\;&\mathbb{E}[\langle Dg(F_1,\dots,F_d),-DL^{-1}F_0\rangle_{L^2(\mu)}] \\ =\;&\mathbb{E}\left[\sum_{i=1}^{d}\frac{\partial}{\partial x_i}g(F_1,\dots,F_d)\langle DF_i,-DL^{-1}F_0\rangle_{L^2(\mu)}\right]+\mathbb{E}[\langle R,-DL^{-1}F_0\rangle_{L^2(\mu)}],\end{aligned}$$

and $\mathbb{E}[\langle R,-DL^{-1}F_0\rangle_{L^2(\mu)}]$ verifies the inequality (3.7). ■

As anticipated, we will now use an interpolation technique inspired by the so-called "smart path method", which is sometimes used in the framework of approximation results for spin glasses (see [68]). Note that the computations developed below are very close to the ones used in the proof of Theorem 7.2 in [36].

Theorem 3.3.2 *Fix $d \geq 1$ and let $C = \{C(i,j) : i,j = 1,\dots,d\}$ be a $d \times d$ covariance matrix (not necessarily positive definite). Suppose that $X = (X_1,\dots,X_d) \sim \mathcal{N}_d(0,C)$ and that $F = (F_1,\dots,F_d)$ is a $\mathbb{R}^d$-valued random vector such that $\mathbb{E}[F_i] = 0$ and $F_i \in \operatorname{Dom} D$, $i = 1,\dots,d$. Then,*

$$d_3(F,X) \leq \frac{d}{2}\sqrt{\sum_{i,j=1}^{d}\mathbb{E}[(C(i,j)-\langle DF_i,-DL^{-1}F_j\rangle_{L^2(\mu)})^2]} \quad (3.8)$$

$$+\frac{1}{4}\int_Z \mu(dz)\mathbb{E}\left[\left(\sum_{i=1}^{d}|D_zF_i|\right)^2\left(\sum_{i=1}^{d}|D_zL^{-1}F_i|\right)\right]. \quad (3.9)$$

Proof. We will work under the assumption that both expectations in (3.8) and (3.9) are finite. By the definition of distance d_3, we need only to show the following inequality:

$$\begin{aligned}|\mathbb{E}[\phi(X)] - \mathbb{E}[\phi(F)]| &\leq \frac{1}{2}\|\phi''\|_\infty \sum_{i,j=1}^{d} \mathbb{E}[|C(i,j) - \langle DF_i, -DL^{-1}F_j\rangle_{L^2(\mu)}|] \\ &\quad + \frac{1}{4}\|\phi'''\|_\infty \int_Z \mu(dz)\mathbb{E}\left[\left(\sum_{i=1}^{d}|D_zF_i|\right)^2\left(\sum_{i=1}^{d}|D_zL^{-1}F_i|\right)\right]\end{aligned}$$

for any $\phi \in \mathbb{C}^3(\mathbb{R}^d)$ with second and third bounded derivatives. Without loss of generality, we may assume that F and X are independent. For $t \in [0,1]$, we set

$$\Psi(t) = \mathbb{E}[\phi(\sqrt{1-t}(F_1,\ldots,F_d) + \sqrt{t}X)]$$

We have immediately

$$|\Psi(1) - \Psi(0)| \leq \sup_{t\in(0,1)} |\Psi'(t)|.$$

Indeed, due to the assumptions on ϕ, the function $t \mapsto \Psi(t)$ is differentiable on $(0,1)$, and one has also

$$\begin{aligned}\Psi'(t) &= \sum_{i=1}^{d} \mathbb{E}\left[\frac{\partial}{\partial x_i}\phi\left(\sqrt{1-t}(F_1,\ldots,F_d) + \sqrt{t}X\right)\left(\frac{1}{2\sqrt{t}}X_i - \frac{1}{2\sqrt{1-t}}F_i\right)\right] \\ &:= \frac{1}{2\sqrt{t}}\mathfrak{A} - \frac{1}{2\sqrt{1-t}}\mathfrak{B}.\end{aligned}$$

On the one hand, we have

$$\begin{aligned}\mathfrak{A} &= \sum_{i=1}^{d} \mathbb{E}\left[\frac{\partial}{\partial x_i}\phi(\sqrt{1-t}(F_1,\ldots,F_d) + \sqrt{t}X)X_i\right] \\ &= \sum_{i=1}^{d} \mathbb{E}\left[\mathbb{E}\left[\frac{\partial}{\partial x_i}\phi(\sqrt{1-t}a + \sqrt{t}X)X_i\right]_{|a=(F_1,\ldots,F_d)}\right] \\ &= \sqrt{t}\sum_{i,j=1}^{d} C(i,j)\mathbb{E}\left[\mathbb{E}\left[\frac{\partial^2}{\partial x_i\partial x_j}\phi(\sqrt{1-t}a + \sqrt{t}X)\right]_{|a=(F_1,\ldots,F_d)}\right] \\ &= \sqrt{t}\sum_{i,j=1}^{d} C(i,j)\mathbb{E}\left[\frac{\partial^2}{\partial x_i\partial x_j}\phi(\sqrt{1-t}(F_1,\ldots,F_d) + \sqrt{t}X)\right].\end{aligned}$$

On the other hand,

$$\begin{aligned}\mathfrak{B} &= \sum_{i=1}^{d} \mathbb{E}\left[\frac{\partial}{\partial x_i}\phi(\sqrt{1-t}(F_1,\ldots,F_d) + \sqrt{t}X)F_i\right] \\ &= \sum_{i=1}^{d} \mathbb{E}\left[\mathbb{E}\left[\frac{\partial}{\partial x_i}\phi(\sqrt{1-t}(F_1,\ldots,F_d) + \sqrt{t}b)F_i\right]_{|b=X}\right].\end{aligned}$$

We now write $\phi_i^{t,b}(\cdot)$ to indicate the function on $\mathbb{R}^d$ defined by

$$\phi_i^{t,b}(F_1,\dots,F_d) = \frac{\partial}{\partial x_i}\phi(\sqrt{1-t}(F_1,\dots,F_d)+\sqrt{t}b)$$

By using Lemma 3.3.1, we deduce that

$$\begin{aligned}&\mathbb{E}[\phi_i^{t,b}(F_1,\dots,F_d)F_i]\\ &= \mathbb{E}\left[\sum_{j=1}^d \frac{\partial}{\partial x_j}\phi_i^{t,b}(F_1,\dots,F_d)\langle DF_j, -DL^{-1}F_i\rangle_{L^2(\mu)}\right] + \mathbb{E}\left[\langle R_b^i, -DL^{-1}F_i\rangle_{L^2(\mu)}\right],\end{aligned}$$

where R_b^i is a residue verifying

$$\begin{aligned}&|\mathbb{E}[\langle R_b^i, -DL^{-1}F_i\rangle_{L^2(\mu)}]| \\ &\le \frac{1}{2}\left(\max_{k,l}\sup_{x\in\mathbb{R}^d}\left|\frac{\partial}{\partial x_k\partial x_l}\phi_i^{t,b}(x)\right|\right)\int_Z \mu(dz)\mathbb{E}\left[\left(\sum_{j=1}^d |D_zF_j|\right)^2 |D_zL^{-1}F_i|\right].\end{aligned} \tag{3.10}$$

Thus,

$$\begin{aligned}\mathfrak{B} &= \sqrt{1-t}\sum_{i,j=1}^d \mathbb{E}\left[\mathbb{E}\left[\frac{\partial^2}{\partial x_i\partial x_j}\phi(\sqrt{1-t}(F_1,\dots,F_d)+\sqrt{t}b)\langle DF_i, -DL^{-1}F_j\rangle_{L^2(\mu)}\right]_{|b=X}\right]\\ &\quad +\sum_{i=1}^d \mathbb{E}\left[\mathbb{E}\left[\langle R_b^i, -DL^{-1}F_i\rangle_{L^2(\mu)}\right]_{|b=X}\right]\\ &= \sqrt{1-t}\sum_{i,j=1}^d \mathbb{E}\left[\frac{\partial^2}{\partial x_i\partial x_j}\phi(\sqrt{1-t}(F_1,\dots,F_d)+\sqrt{t}X)\langle DF_i, -DL^{-1}F_j\rangle_{L^2(\mu)}\right]\\ &\quad +\sum_{i=1}^d \mathbb{E}\left[\mathbb{E}\left[\langle R_b^i, -DL^{-1}F_i\rangle_{L^2(\mu)}\right]_{|b=X}\right].\end{aligned}$$

Putting the estimates on $\mathfrak{A}$ and $\mathfrak{B}$ together, we infer

$$\begin{aligned}\Psi'(t) &= \frac{1}{2}\sum_{i,j=1}^d \mathbb{E}\left[\frac{\partial^2}{\partial x_i\partial x_j}\phi(\sqrt{1-t}(F_1,\dots,F_d)+\sqrt{t}X)(C(i,j)-\langle DF_i, -DL^{-1}F_j\rangle_{L^2(\mu)})\right]\\ &\quad -\frac{1}{2\sqrt{1-t}}\sum_{i=1}^d \mathbb{E}\left[\mathbb{E}\left[\langle R_b^i, -DL^{-1}F_i\rangle_{L^2(\mu)}\right]_{|b=X}\right].\end{aligned}$$

We notice that

$$\left|\frac{\partial^2}{\partial x_i\partial x_j}\phi(\sqrt{1-t}(F_1,\dots,F_d)+\sqrt{t}b)\right| \le \|\phi''\|_\infty,$$

and also

$$\begin{aligned}\left|\frac{\partial^2}{\partial x_k\partial x_l}\phi_i^{t,b}(F_1,\dots,F_d)\right| &= (1-t)\times\left|\frac{\partial^3}{\partial x_i\partial x_k\partial x_l}\phi(\sqrt{1-t}(F_1,\dots,F_d)+\sqrt{t}b)\right|\\ &\le (1-t)\|\phi'''\|_\infty.\end{aligned}$$

To conclude, we can apply inequality (3.10) as well as Cauchy-Schwartz inequality and deduce the estimates

$$
\begin{aligned}
& |\mathbb{E}[\phi(X)] - \mathbb{E}[\phi(F)]| \\
\leq\ & \sup_{t\in(0,1)} |\Psi'(t)| \\
\leq\ & \frac{1}{2}\|\phi''\|_\infty \sum_{i,j=1}^{d} \mathbb{E}[|C(i,j) - \langle DF_i, -DL^{-1}F_j\rangle_{L^2(\mu)}|] \\
& + \frac{1-t}{4\sqrt{1-t}}\|\phi'''\|_\infty \int_Z \mu(dz)\mathbb{E}\left[\left(\sum_{i=1}^{d}|D_zF_i|\right)^2\left(\sum_{i=1}^{d}|D_zL^{-1}F_i|\right)\right] \\
\leq\ & \frac{d}{2}\|\phi''\|_\infty \sqrt{\sum_{i,j=1}^{d} \mathbb{E}[(C(i,j) - \langle DF_i, -DL^{-1}F_j\rangle_{L^2(\mu)})^2]} \\
& + \frac{1}{4}\|\phi'''\|_\infty \int_z \mu(dz)\mathbb{E}\left[\left(\sum_{i=1}^{d}|D_zF_i|\right)^2\left(\sum_{i=1}^{d}|D_zL^{-1}F_i|\right)\right],
\end{aligned}
$$

thus concluding the proof. ■

The following statement is a direct consequence of Theorem 3.3.2, as well as a natural generalization of Corollary 3.2.4.

Corollary 3.3.3 *For a fixed $d \geq 2$, let $X \sim \mathcal{N}_d(0,C)$, with C a generic covariance matrix. Let*

$$F_n = (F_{n,1},\dots,F_{n,d}) = (\hat{N}(h_{n,1}),\dots,\hat{N}(h_{n,d})),\ n \geq 1,$$

be a collection of d-dimensional random vectors in the first Wiener chaos of $\hat{N}$, and denote by K_n the covariance matrix of F_n. Then,

$$d_3(F_n, X) \leq \frac{d}{2}\|C - K_n\|_{H.S.} + \frac{d^2}{4}\sum_{i=1}^{d}\int_Z |h_{n,i}(z)|^3\mu(dz).$$

In particular, if relation (3.6) is verified for every $i,j = 1,\dots,d$ (as $n \to \infty$), then $d_3(F_n,X) \to 0$ and F_n converges in distribution to X.

3.3.2 Stein's method versus smart paths: two tables

In the two tables below, we compare the estimations obtained by the Malliavin-Stein method with those deduced by interpolation techniques, both in a Gaussian and Poisson setting. Note that the test functions considered below have (partial) derivatives that are not necessarily bounded by 1 (as it is indeed the case in the definition of the distances d_2 and d_3) so that the L^∞ norms of various derivatives appear in the estimates. In both tables, $d \geq 2$ is a given positive integer. We write $(G, G_1,\dots,G_d)$ to indicate a vector of centered Malliavin differentiable functionals of an isonormal Gaussian process over some separable real Hilbert space $\mathfrak{H}$ (see [38] for definitions). We write $(F, F_1,\dots,F_d)$ to indicate a vector of centered functionals of $\hat{N}$, each belonging to Dom$D$. The symbols D and L^{-1} stand for the Malliavin derivative and the inverse of the Ornstein-Uhlenbeck generator: plainly, both are to be regarded as defined either

Table 3.1: Estimates proved by means of Malliavin-Stein techniques

Regularity of the test function h	**Upper bound**
$\|h\|_{Lip}$ is finite	$\|\mathbb{E}[h(G)] - \mathbb{E}[h(X)]\| \leq$ $\|h\|_{Lip}\sqrt{\mathbb{E}[(1-\langle DG, -DL^{-1}G\rangle_{\mathfrak{H}})^2]}$
$\|h\|_{Lip}$ is finite	$\|\mathbb{E}[h(G_1,\ldots,G_d)] - \mathbb{E}[h(X_C)]\| \leq$ $\|h\|_{Lip}\|C^{-1}\|_{op}\|C\|_{op}^{1/2}\sqrt{\sum_{i,j=1}^d \mathbb{E}[(C(i,j)-\langle DG_i, -DL^{-1}G_j\rangle_{\mathfrak{H}})^2]}$
$\|h\|_{Lip}$ is finite	$\|\mathbb{E}[h(F)] - \mathbb{E}[h(X)]\| \leq$ $\|h\|_{Lip}(\sqrt{\mathbb{E}[(1-\langle DF, -DL^{-1}F\rangle_{L^2(\mu)})^2]}$ $+\int_Z \mu(dz)\mathbb{E}[\|D_zF\|^2\|D_zL^{-1}F\|])$
$h \in \mathbb{C}^2(\mathbb{R}^d)$ $\|h\|_{Lip}$ is finite $M_2(h)$ is finite	$\|\mathbb{E}[h(F_1,\ldots,F_d)] - \mathbb{E}[h(X_C)]\| \leq$ $\|h\|_{Lip}\|C^{-1}\|_{op}\|C\|_{op}^{1/2}\sqrt{\sum_{i,j=1}^d \mathbb{E}[(C(i,j)-\langle DF_i, -DL^{-1}F_j\rangle_{L^2(\mu)})^2]}$ $+M_2(h)\frac{\sqrt{2\pi}}{8}\|C^{-1}\|_{op}^{3/2}\|C\|_{op}\int_Z \mu(dz)\mathbb{E}\left[\left(\sum_{i=1}^d \|D_zF_i\|\right)^2\left(\sum_{i=1}^d \|D_zL^{-1}F_i\|\right)\right]$

on a Gaussian space or on a Poisson space, according to the framework. We also consider the following Gaussian random elements: $X \sim \mathcal{N}(0,1)$, $X_C \sim \mathcal{N}_d(0,C)$ and $X_M \sim \mathcal{N}_d(0,M)$, where C is a $d\times d$ positive definite covariance matrix and M is a $d\times d$ covariance matrix (not necessarily positive definite).

In Table 1, we present all estimates on distances involving Malliavin differentiable random variables (in both cases of an underlying Gaussian and Poisson space), that have been obtained by means of Malliavin-Stein techniques. These results are taken from: [31] (Line 1), [37] (Line 2), [46] (Line 3) and Theorem 3.2.3 and its proof (Line 4).

In Table 2, we list the parallel results obtained by interpolation methods. The bounds involving functionals of a Gaussian process come from [36], whereas those for Poisson functionals are taken from Theorem 3.3.2 and its proof.

Observe that:

- in contrast to the Malliavin-Stein method, the covariance matrix M is not required to be positive definite when using the interpolation technique,

- in general, the interpolation technique requires more regularity on test functions than the Malliavin-Stein method.

Table 3.2: Estimates proved by means of interpolations

Regularity of the test function ϕ	Upper bound
$\phi \in \mathbb{C}^2(\mathbb{R})$ $\\|\phi''\\|_\infty$ is finite	$\|\mathbb{E}[\phi(G)] - \mathbb{E}[\phi(X)]\| \leq$ $\frac{1}{2}\\|\phi''\\|_\infty \sqrt{\mathbb{E}[(1 - \langle DG, -DL^{-1}G\rangle_{\mathfrak{H}})^2]}$
$\phi \in \mathbb{C}^2(\mathbb{R}^d)$ $\\|\phi''\\|_\infty$ is finite	$\|\mathbb{E}[\phi(G_1, \dots, G_d)] - \mathbb{E}[\phi(X_M)]\| \leq$ $\frac{d}{2}\\|\phi''\\|_\infty \sqrt{\sum_{i,j=1}^d \mathbb{E}[(M(i,j) - \langle DG_i, -DL^{-1}G_j\rangle_{\mathfrak{H}})^2]}$
$\phi \in \mathbb{C}^3(\mathbb{R})$ $\\|\phi''\\|_\infty$ is finite $\\|\phi'''\\|_\infty$ is finite	$\|\mathbb{E}[\phi(F)] - \mathbb{E}[\phi(X)]\| \leq$ $\frac{1}{2}\\|\phi''\\|_\infty \sqrt{\mathbb{E}[(1 - \langle DF, -DL^{-1}F\rangle_{L^2(\mu)})^2]}$ $+\frac{1}{4}\\|\phi'''\\|_\infty \int_Z \mu(dz)\mathbb{E}[\|D_zF\|^2(\|D_zL^{-1}F\|)]$
$\phi \in \mathbb{C}^3(\mathbb{R}^d)$ $\\|\phi''\\|_\infty$ is finite $\\|\phi'''\\|_\infty$ is finite	$\|\mathbb{E}[\phi(F_1, \dots, F_d)] - \mathbb{E}[\phi(X_M)]\| \leq$ $\frac{d}{2}\\|\phi''\\|_\infty \sqrt{\sum_{i,j=1}^d \mathbb{E}[(M(i,j) - \langle DF_i, -DL^{-1}F_j\rangle_{L^2(\mu)})^2]}$ $+\frac{1}{4}\\|\phi'''\\|_\infty \int_Z \mu(dz)\mathbb{E}\left[\left(\sum_{i=1}^d \|D_zF_i\|\right)^2 \left(\sum_{i=1}^d \|D_zL^{-1}F_i\|\right)\right]$

3.4 CLTs for Poisson multiple integrals

In this section, we study the Gaussian approximation of vectors of Poisson multiple stochastic integrals by an application of Theorem 3.2.3 and Theorem 3.3.2. To this end, we shall explicitly evaluate the quantities appearing in formulae (3.3)–(3.4) and (3.8)–(3.9).

Remark 3.4.1 (Regularity conventions) From now on, every kernel $f \in L_s^2(\mu^p)$ is supposed to verify both Assumptions A and B of Definition 1.1.9. As before, given $f \in L_s^2(\mu^p)$, and for a fixed $z \in Z$, we write $f(z, \cdot)$ to indicate the function defined on Z^{p-1} as $(z_1, \dots, z_{p-1}) \mapsto f(z, z_1, \dots, z_{p-1})$. The following convention will be also in order: given a vector of kernels $(f_1, \dots, f_d)$ such that $f_i \in L_s^2(\mu^{p_i})$, $i = 1, \dots, d$, we will implicitly set

$$f_i(z, \cdot) \equiv 0, \quad i = 1, \dots, d,$$

for every $z \in Z$ belonging to the exceptional set (of μ measure 0) such that

$$f_i(z, \cdot) \star_r^l f_j(z, \cdot) \notin L^2(\mu^{p_i + p_j - r - l - 2})$$

for at least one pair (i, j) and some $r = 0, \dots, p_i \wedge p_j - 1$ and $l = 0, \dots, r$. See Point 3 of Remark 1.1.9.

3.4.1 The operators $G_k^{p,q}$ and $\widehat{G_k^{p,q}}$

Fix integers $p, q \geq 0$ and $|q - p| \leq k \leq p + q$, consider two kernels $f \in L_s^2(\mu^p)$ and $g \in L_s^2(\mu^q)$, and recall the multiplication formula (1.12). We will now introduce an operator $G_k^{p,q}$, transforming the function f, of p variables, and the function g, of q variables, into a “hybrid” function $G_k^{p,q}(f,g)$, of k variables. More precisely, for p,q,k as above, we define the function $(z_1,\dots,z_k)\mapsto G_k^{p,q}(f,g)(z_1,\dots,z_k)$, from Z^k into $\mathbb{R}$, as follows:

$$G_k^{p,q}(f,g)(z_1,\dots,z_k)=\sum_{r=0}^{p\wedge q}\sum_{l=0}^{r}\mathbf{1}_{(p+q-r-l=k)}r!\binom{p}{r}\binom{q}{r}\binom{r}{l}\widetilde{f\star_r^l g}, \tag{3.11}$$

where the tilde $\sim$ means symmetrization, and the star contractions are defined in formula (1.9) and the subsequent discussion. Observe the following three special cases: (i) when $p=q=k=0$, then f and g are both real constants, and $G_0^{0,0}(f,g)=f\times g$, (ii) when $p=q\geq 1$ and $k=0$, then $G_0^{p,p}(f,g)=p!\langle f,g\rangle_{L^2(\mu^p)}$, (iii) when $p=k=0$ and $q>0$ (then, f is a constant), $G_0^{0,p}(f,g)(z_1,\dots,z_q)=f\times g(z_1,\dots,z_q)$. By using this notation, (1.12) becomes

$$I_p(f)I_q(g)=\sum_{k=|q-p|}^{p+q}I_k(G_k^{p,q}(f,g)). \tag{3.12}$$

The advantage of representation (3.12) (as opposed to (1.12)) is that the RHS of (3.12) is an *orthogonal sum*, a feature that will greatly simplify our forthcoming computations.

For two functions $f\in L_s^2(\mu^p)$ and $g\in L_s^2(\mu^q)$, we define the function $(z_1,\dots,z_k)\mapsto \widehat{G_k^{p,q}}(f,g)(z_1,\dots,z_k)$, from Z^k into $\mathbb{R}$, as follows:

$$\widehat{G_k^{p,q}}(f,g)(\cdot)=\int_Z\mu(dz)G_k^{p-1,q-1}(f(z,\cdot),g(z,\cdot)),$$

or, more precisely,

$$\begin{aligned}
&\widehat{G_k^{p,q}}(f,g)(z_1,\dots,z_k)\\
&=\int_Z\mu(dz)\sum_{r=0}^{p\wedge q-1}\sum_{l=0}^{r}\mathbf{1}_{(p+q-r-l-2=k)}r!\\
&\qquad\times\binom{p-1}{r}\binom{q-1}{r}\binom{r}{l}\widetilde{f(z,\cdot)\star_r^l g(z,\cdot)}(z_1,\dots,z_k)\\
&=\sum_{t=1}^{p\wedge q}\sum_{s=1}^{t}\mathbf{1}_{(p+q-t-s=k)}(t-1)!\binom{p-1}{t-1}\binom{q-1}{t-1}\binom{t-1}{s-1}\widetilde{f\star_t^s g}(z_1,\dots,z_k).
\end{aligned} \tag{3.13}$$

Note that the implicit use of a Fubini theorem in the relation (3.13) is justified by Assumption B – see again Point 3 of Remark 1.1.9.

The following technical lemma will be applied in the next subsection.

Lemma 3.4.2 *Consider three positive integers p,q,k such that $p,q\geq 1$ and $|q-p|\vee 1\leq k\leq p+q-2$ (note that this excludes the case $p=q=1$). For any two kernels $f\in L_s^2(\mu^p)$ and $g\in L_s^2(\mu^q)$, both verifying Assumptions A and B, we have*

$$\int_{Z^k}d\mu^k(\widehat{G_k^{p,q}}(f,g)(z_1,\dots,z_k))^2\leq C\sum_{t=1}^{p\wedge q}\mathbf{1}_{1\leq s(t,k)\leq t}\|f\,\widetilde{\star_t^{s(t,k)}}\,g\|^2_{L^2(\mu^k)} \tag{3.14}$$

where $s(t,k) = p+q-k-t$ *for* $t = 1, \dots, p \wedge q$. *Also,* C *is the constant given by*

$$C = \sum_{t=1}^{p\wedge q} \left[(t-1)! \binom{p-1}{t-1} \binom{q-1}{t-1} \binom{t-1}{s(t,k)-1} \right]^2 .$$

Proof. We rewrite the sum in (3.13) as

$$\widehat{G_k^{p,q}}(f,g)(z_1,\dots,z_k) = \sum_{t=1}^{p\wedge q} a_t \mathbf{1}_{1\le s(t,k)\le t} f \widetilde{\star_t^{s(t,k)}} g(z_1,\dots,z_k), \tag{3.15}$$

with $a_t = (t-1)! \binom{p-1}{t-1} \binom{q-1}{t-1} \binom{t-1}{s(t,k)-1}$, $\quad 1 \le t \le p \wedge q$. Thus,

$$\begin{aligned}
&\int_{Z^k} d\mu^k (\widehat{G_k^{p,q}}(f,g)(z_1,\dots,z_k))^2 \\
=& \int_{Z^k} d\mu^k \left(\sum_{t=1}^{p\wedge q} a_t \mathbf{1}_{1\le s(t,k)\le t} f \widetilde{\star_t^{s(t,k)}} g(z_1,\dots,z_k) \right)^2 \\
\le& \left(\sum_{t=1}^{p\wedge q} a_t^2 \right) \int_{Z^k} d\mu^k \left(\sum_{t=1}^{p\wedge q} (\mathbf{1}_{1\le s(t,k)\le t} f \widetilde{\star_t^{s(t,k)}} g(z_1,\dots,z_k))^2 \right) \\
=& C \sum_{t=1}^{p\wedge q} \int_{Z^k} d\mu^k \mathbf{1}_{1\le s(t,k)\le t} (f \widetilde{\star_t^{s(t,k)}} g(z_1,\dots,z_k))^2 \\
=& C \sum_{t=1}^{p\wedge q} \mathbf{1}_{1\le s(t,k)\le t} \| f \widetilde{\star_t^{s(t,k)}} g \|_{L^2(\mu^k)}^2,
\end{aligned}$$

with

$$C = \sum_{t=1}^{p\wedge q} a_t^2 = \sum_{t=1}^{p\wedge q} \left[(t-1)! \binom{p-1}{t-1} \binom{q-1}{t-1} \binom{t-1}{s(t,k)-1} \right]^2$$

Note that the Cauchy-Schwarz inequality

$$\left(\sum_{i=1}^n a_i x_i \right)^2 \le \left(\sum_{i=1}^n a_i^2 \right) \left(\sum_{i=1}^n x_i^2 \right)$$

has been used in the above deduction. ■

Remark 3.4.3 *The constant* $C(p,q,k,t)$ *may be replaced by smaller constant. Indeed, in the sum (3.15), the number of non-zero elements is less than* $p \wedge q$. *In order that* $s(t,k) = p+q-k-t$ *satisfies* $1 \le s(t,k) \le t$, *we need*

$$t+1 \le t+s = p+q-k \le 2t,$$

or

$$\frac{1}{2}(p+q-k) \le t \le p+q-k-1.$$

But t *also satisfies* $1 \le t \le p \wedge q$, *we have*

$$t \in \begin{cases} [\frac{1}{2}(p+q-k), p \wedge q], & \text{if } |q-p| \le k \le p \vee q - 1; \\ [\frac{1}{2}(p+q-k), p+q-k-1], & \text{if } p \vee q \le k \le p+q-2. \end{cases}$$

By repeating the same deduction in the lemma, we may get the same inequality by replacing $C(p,q,k,t)$ *by*

$$C_1(p,q,k,t) = \sum_{t=\frac{1}{2}(p+q-k)}^{p\wedge q} [(t-1)! \begin{pmatrix} p-1 \\ t-1 \end{pmatrix} \begin{pmatrix} q-1 \\ t-1 \end{pmatrix} \begin{pmatrix} t-1 \\ s(t,k)-1 \end{pmatrix}]^2,$$

if $|q-p| \le k \le p \vee q - 1$, *or by*

$$C_2(p,q,k,t) = \sum_{t=\frac{1}{2}(p+q-k)}^{p+q-k-1} [(t-1)! \begin{pmatrix} p-1 \\ t-1 \end{pmatrix} \begin{pmatrix} q-1 \\ t-1 \end{pmatrix} \begin{pmatrix} t-1 \\ s(t,k)-1 \end{pmatrix}]^2,$$

if $p \vee q \le k \le p+q-2$.

3.4.2 Some technical estimates

As anticipated, in order to prove the multivariate CLTs of the forthcoming Section 3.4.3, we need to establish explicit bounds on the quantities appearing in (3.3)–(3.4) and (3.8)–(3.9), in the special case of chaotic random variables.

Definition 3.4.4 *The kernels* $f \in L^2_s(\mu^p)$, $g \in L^2_s(\mu^q)$ *are said to satisfy* **Assumption C** *if either* $p = q = 1$, *or* $\max(p,q) > 1$ *and, for every* $k = |q-p| \vee 1, \ldots, p+q-2$,

$$\int_Z \left[\sqrt{\int_{Z^k} (G_k^{p-1,q-1}(f(z,\cdot), g(z,\cdot)))^2 d\mu^k} \right] \mu(dz) < \infty. \tag{3.16}$$

Remark 3.4.5 By using (3.11), one sees that (3.16) is implied by the following stronger condition: for every $k = |q-p| \vee 1, \ldots, p+q-2$, and every (r,l) satisfying $p+q-2-r-l = k$, one has

$$\int_Z \left[\sqrt{\int_{Z^k} (f(z,\cdot) \star_r^l g(z,\cdot))^2 d\mu^k} \right] \mu(dz) < \infty. \tag{3.17}$$

One can easily write down sufficient conditions, on f and g, ensuring that (3.17) is satisfied. For instance, in the examples of Section 3.5, we will use repeatedly the following fact: if both f and g verify Assumption A, and if their supports are contained in some rectangle of the type $B \times \ldots \times B$, with $\mu(B) < \infty$, then (3.17) is automatically satisfied.

Proposition 3.4.6 *Denote by* L^{-1} *the pseudo-inverse of the Ornstein-Uhlenbeck generator , and, for* $p, q \ge 1$, *let* $F = I_p(f)$ *and* $G = I_q(g)$ *be such that the kernels* $f \in L^2_s(\mu^p)$ *and*

$g \in L_s^2(\mu^q)$ verify Assumptions A, B and C. If $p \neq q$, then

$$\begin{aligned}
& \mathbb{E}[(a - \langle DF, -DL^{-1}G\rangle_{L^2(\mu)})^2] \\
\leq\ & a^2 + p^2 \sum_{k=|q-p|}^{p+q-2} k! \int_{Z^k} d\mu^k (\widehat{G_k^{p,q}}(f,g))^2 \\
\leq\ & a^2 + Cp^2 \sum_{k=|q-p|}^{p+q-2} k! \sum_{t=1}^{p\wedge q} \mathbf{1}_{1\leq s(t,k)\leq t} \|\widetilde{f \star_t^{s(t,k)} g}\|^2_{L^2(\mu^k)} \\
\leq\ & a^2 + \frac{1}{2}Cp^2 \sum_{k=|q-p|}^{p+q-2} k! \sum_{t=1}^{p\wedge q} \mathbf{1}_{1\leq s(t,k)\leq t} (\|f \star_{p-s(t,k)}^{p-t} f\|_{L^2(\mu^{t+s(t,k)})} \times \|g \star_{q-s(t,k)}^{q-t} g\|_{L^2(\mu^{t+s(t,k)})})
\end{aligned}$$

If $p = q \geq 2$, then

$$\begin{aligned}
& \mathbb{E}[(a - \langle DF, -DL^{-1}G\rangle_{L^2(\mu)})^2] \\
\leq\ & (p!\langle f,g\rangle_{L^2(\mu^p)} - a)^2 + p^2 \sum_{k=1}^{2p-2} k! \int_{Z^k} d\mu^k (\widehat{G_k^{p,q}}(f,g))^2 \\
\leq\ & (p!\langle f,g\rangle_{L^2(\mu^p)} - a)^2 + Cp^2 \sum_{k=1}^{2p-2} k! \sum_{t=1}^{p\wedge q} \mathbf{1}_{1\leq s(t,k)\leq t} \|\widetilde{f \star_t^{s(t,k)} g}\|^2_{L^2(\mu^k)} \\
\leq\ & (p!\langle f,g\rangle_{L^2(\mu^p)} - a)^2 \\
& + \frac{1}{2}Cp^2 \sum_{k=1}^{2p-2} k! \sum_{t=1}^{p\wedge q} \mathbf{1}_{1\leq s(t,k)\leq t} (\|f \star_{p-s(t,k)}^{p-t} f\|_{L^2(\mu^{t+s(t,k)})} \times \|g \star_{q-s(t,k)}^{q-t} g\|_{L^2(\mu^{t+s(t,k)})})
\end{aligned}$$

where $s(t,k) = p + q - k - t$ for $t = 1, \ldots, p\wedge q$, and the constant C is given by

$$C = \sum_{t=1}^{p\wedge q} \left[(t-1)! \begin{pmatrix} p-1 \\ t-1 \end{pmatrix} \begin{pmatrix} q-1 \\ t-1 \end{pmatrix} \begin{pmatrix} t-1 \\ s(t,k)-1 \end{pmatrix} \right]^2 .$$

If $p = q = 1$, then

$$(a - \langle DF, -DL^{-1}G\rangle_{L^2(\mu)})^2 = (a - \langle f,g\rangle_{L^2(\mu)})^2.$$

Proof. The case $p = q = 1$ is trivial, so that we can assume that either p or q is strictly greater than 1. We select two versions of the derivatives $D_zF = pI_{p-1}(f(z,\cdot))$ and $D_zG = qI_{q-1}(g(z,\cdot))$, in such a way that the conventions pointed out in Remark **??** are satisfied. By using the definition of L^{-1} and (3.12), we have

$$\begin{aligned}
\langle DF, -DL^{-1}G\rangle_{L^2(\mu)} &= \langle DI_p(f), q^{-1}DI_q(g)\rangle_{L^2(\mu)} \\
&= p\int_Z \mu(dz) I_{p-1}(f(z,\cdot)) I_{q-1}(g(z,\cdot)) \\
&= p\int_Z \mu(dz) \sum_{k=|q-p|}^{p+q-2} I_k(G_k^{p-1,q-1}(f(z,\cdot), g(z,\cdot)))
\end{aligned}$$

Notice that for $i \neq j$, the two random variables

$$\int_Z \mu(dz) I_i(G_i^{p-1,q-1}(f(z,\cdot), g(z,\cdot)) \quad \text{and} \quad \int_Z \mu(dz) I_j(G_j^{p-1,q-1}(f(z,\cdot), g(z,\cdot)))$$

are orthogonal in $L^2(\mathbb{P})$. It follows that

$$\mathbb{E}[(a - \langle DF, -DL^{-1}G\rangle_{L^2(\mu)})^2] = a^2 + p^2 \sum_{k=|q-p|}^{p+q-2} \mathbb{E}\left[\left(\int_Z \mu(dz) I_k(G_k^{p-1,q-1}(f(z,\cdot), g(z,\cdot)))\right)^2\right] \tag{3.18}$$

for $p \neq q$, and, for $p = q$,

$$\mathbb{E}[(a - \langle DF, -DL^{-1}G\rangle_{L^2(\mu)})^2] = (p!\langle f, g\rangle_{L^2(\mu^p)} - a)^2 + p^2 \sum_{k=1}^{2p-2} \mathbb{E}\left[\left(\int_Z \mu(dz) I_k(G_k^{p-1,q-1}(f(z,\cdot), g(z,\cdot)))\right)^2\right]. \tag{3.19}$$

We shall now assess the expectations appearing on the RHS of (3.18) and (3.19). To do this, fix an integer k and use the Cauchy-Schwartz inequality together with (3.16) to deduce that

$$\begin{aligned}
& \int_Z \mu(dz) \int_Z \mu(dz') \mathbb{E}\left[\left| I_k(G_k^{p-1,q-1}(f(z,\cdot), g(z,\cdot))) I_k(G_k^{p-1,q-1}(f(z',\cdot), g(z',\cdot)))\right|\right] \\
\leq\ & \int_Z \mu(dz) \int_Z \mu(dz') \sqrt{\mathbb{E}[I_k^2(G_k^{p-1,q-1}(f(z,\cdot), g(z,\cdot)))]} \sqrt{\mathbb{E}[I_k^2(G_k^{p-1,q-1}(f(z',\cdot), g(z',\cdot)))]} \\
=\ & k! \left[\int_Z \mu(dz) \sqrt{\int_{Z^k} d\mu^k (G_k^{p-1,q-1}(f(z,\cdot), g(z,\cdot)))^2}\ \right] \\
& \qquad \times \left[\int_Z \mu(dz') \sqrt{\int_{Z^k} d\mu^k (G_k^{p-1,q-1}(f(z',\cdot), g(z',\cdot)))^2}\ \right] \\
=\ & k! \left[\int_Z \mu(dz) \sqrt{\int_{Z^k} d\mu^k (G_k^{p-1,q-1}(f(z,\cdot), g(z,\cdot)))^2}\ \right]^2 < \infty.
\end{aligned} \tag{3.20}$$

Relation (3.20) justifies the use of a Fubini theorem, and we can consequently infer that

$$\begin{aligned}
& \mathbb{E}\left[\left(\int_Z \mu(dz) I_k(G_k^{p-1,q-1}(f(z,\cdot), g(z,\cdot)))\right)^2\right] \\
=\ & \int_Z \mu(dz) \int_Z \mu(dz') \mathbb{E}[I_k(G_k^{p-1,q-1}(f(z,\cdot), g(z,\cdot))) I_k(G_k^{p-1,q-1}(f(z',\cdot), g(z',\cdot)))] \\
=\ & k! \int_Z \mu(dz) \int_Z \mu(dz') \left[\int_{Z^k} d\mu^k G_k^{p-1,q-1}(f(z,\cdot), g(z,\cdot)) G_k^{p-1,q-1}(f(z',\cdot), g(z',\cdot))\right] \\
=\ & k! \int_{Z^k} d\mu^k \left[\int_Z \mu(dz) G_k^{p-1,q-1}(f(z,\cdot), g(z,\cdot))\right]^2 \\
=\ & k! \int_{Z^k} d\mu^k (\widehat{G_k^{p,q}}(f,g))^2.
\end{aligned}$$

The remaining estimates in the statement follow (in order) from Lemma 3.4.2 and Lemma 1.1.8, as well as from the fact that $\|\widetilde{f}\|_{L^2(\mu^n)} \leq \|f\|_{L^2(\mu^n)}$, for all $n \geq 2$. ■

The next statement will be used in the subsequent section.

Proposition 3.4.7 *Let $F = (F_1, \dots, F_d) := (I_{q_1}(f_1), \dots, I_{q_d}(f_d))$ be a vector of Poisson functionals, such that the kernels f_j verify Assumptions A and B. Then, writing $q_* := \min\{q_1, \dots, q_d\}$,*

$$\int_Z \mu(dz)\mathbb{E}\left[\left(\sum_{i=1}^d |D_z F_i|\right)^2 \left(\sum_{i=1}^d |D_z L^{-1} F_i|\right)\right]$$
$$\leq \frac{d^2}{q_*}\sum_{i=1}^d \Bigg(q_i^3 \sqrt{(q_i-1)!\|f\|^2_{L^2(\mu^{q_i})}} \times \sum_{b=1}^{q_i}\sum_{a=0}^{b-1} \mathbf{1}_{1\leq a+b\leq 2q_i-1}(a+b-1)!^{1/2}(q_i-a-1)!$$
$$\times \begin{pmatrix} q_i-1 \\ q_i-1-a \end{pmatrix}^2 \begin{pmatrix} q_i-1-a \\ q_i-b \end{pmatrix} \|f \star_b^a f\|_{L^2(\mu^{2q_i-a-b})}\Bigg).$$

Remark 3.4.8 When $q = 1$, one has that

$$q^3 \sqrt{(q-1)!\|f\|^2_{L^2(\mu^q)}} \times \sum_{b=1}^{q}\sum_{a=0}^{b-1} \mathbf{1}_{1\leq a+b\leq 2q-1}(a+b-1)!^{1/2}(q-a-1)!$$
$$\times \begin{pmatrix} q-1 \\ q-1-a \end{pmatrix}^2 \begin{pmatrix} q-1-a \\ q-b \end{pmatrix} \|f \star_b^a f\|_{L^2(\mu^{2q-a-b})}$$
$$= \|f\|_{L^2(\mu)} \times \|f\|^2_{L^4(\mu)}.$$

Proof of Proposition 3.4.7. One has that

$$\int_Z \mu(dz)\mathbb{E}\left[\left(\sum_{i=1}^d |D_z F_i|\right)^2 \left(\sum_{i=1}^d |D_z L^{-1} F_i|\right)\right]$$
$$= \int_Z \mu(dz)\mathbb{E}\left[\left(\sum_{i=1}^d |D_z F_i|\right)^2 \left(\sum_{i=1}^d \frac{1}{q_i}|D_z F_i|\right)\right]$$
$$\leq \frac{1}{q_*}\int_Z \mu(dz)\mathbb{E}\left[\left(\sum_{i=1}^d |D_z F_i|\right)^3\right]$$
$$\leq \frac{d^2}{q_*}\sum_{i=1}^d \int_Z \mu(dz)\mathbb{E}[|D_z F_i|^3].$$

To conclude, use the inequality

$$\int_Z \mu(dz)\mathbb{E}[|D_z I_q(f)|^3]$$
$$\leq q^3 \sqrt{(q-1)!\|f\|^2_{L^2(\mu^q)}} \times \sum_{b=1}^{q}\sum_{a=0}^{b-1} \mathbf{1}_{1\leq a+b\leq 2q-1}(a+b-1)!^{1/2}(q-a-1)!$$
$$\times \begin{pmatrix} q-1 \\ q-1-a \end{pmatrix}^2 \begin{pmatrix} q-1-a \\ q-b \end{pmatrix} \|f \star_b^a f\|_{L^2(\mu^{2q-a-b})}$$

which is proved in [46, Theorem 4.2] for the case $q \geq 2$ (see in particular formulae (4.13) and (4.18) therein), and follows from the Cauchy-Schwarz inequality when $q = 1$. ■

3.4.3 Central limit theorems with contraction conditions

We will now deduce the announced CLTs for sequences of vectors of the type

$$F^{(n)} = (F_1^{(n)}, \dots, F_d^{(n)}) := (I_{q_1}(f_1^{(n)}), \dots, I_{q_d}(f_d^{(n)})), \quad n \geq 1. \tag{3.21}$$

As already discussed, our results should be compared with other central limit results for multiple stochastic integrals in a Gaussian or Poisson setting – see e.g. [31, 37, 39, **?**, 47, 49]. The following statement, which is a genuine multi-dimensional generalization of Theorem 5.1 in [46], is indeed one of the main achievements of the present chapter.

Theorem 3.4.9 (CLT for chaotic vectors) *Fix $d \geq 2$, let $X \sim \mathcal{N}(0, C)$, with*

$$C = \{C(i,j) : i,j = 1, \dots, d\}$$

a $d \times d$ nonnegative definite matrix, and fix integers $q_1, \dots, q_d \geq 1$. For any $n \geq 1$ and $i = 1, \dots, d$, let $f_i^{(n)}$ belong to $L_s^2(\mu^{q_i})$. Define the sequence $\{F^{(n)}; n \geq 1\}$, according to (3.21) and suppose that

$$\lim_{n\to\infty} \mathbb{E}[F_i^{(n)} F_j^{(n)}] = \mathbf{1}_{(q_j=q_i)} q_j! \times \lim_{n\to\infty} \langle f_i^{(n)}, f_j^{(n)} \rangle_{L^2(\mu^{q_i})} = C(i,j), \qquad 1 \leq i,j \leq d. \tag{3.22}$$

Assume moreover that the following Conditions 1–4 hold for every $k = 1, \dots, d$:

1. *For every n, the kernel $f_k^{(n)}$ satisfies Assumptions A and B.*
2. *For every $l = 1, \dots, d$ and every n, the kernels $f_k^{(n)}$ and $f_l^{(n)}$ satisfy Assumption C.*
3. *For every $r = 1, \dots, q_k$ and every $l = 1, \dots, r \wedge (q_k - 1)$, one has that*

$$\|f_k^{(n)} \star_r^l f_k^{(n)}\|_{L^2(\mu^{2q_k - r - l})} \to 0,$$

   as $n \to \infty$.

4. *As $n \to \infty$, $\int_{Z^{q_k}} d\mu^{q_k} \left(f_k^{(n)}\right)^4 \to 0$.*

Then, $F^{(n)}$ converges to X in distribution as $n \to \infty$. The speed of convergence can be assessed by combining the estimates of Proposition 3.4.6 and Proposition 3.4.7 either with Theorem 3.2.3 (when C is positive definite) or with Theorem 3.3.2 (when C is merely nonnegative definite).

Remark 3.4.10 1. For every $f \in L_s^2(\mu^q)$, $q \geq 1$, one has that

$$\|f \star_q^0 f\|_{L^2(\mu^q)}^2 = \int_{Z^q} d\mu^q f^4.$$

2. When $q_i \neq q_j$, then $F_i^{(n)}$ and $F_j^{(n)}$ are not in the same chaos, yielding that $C(i,j) = 0$ in formula (5.20). In particular, if Conditions 1-4 of Theorem 3.4.9 are verified, then $F_i^{(n)}$ and $F_j^{(n)}$ are asymptotically independent.

3. When specializing Theorem 3.4.9 to the case $q_1 = \ldots = q_d = 1$, one obtains a set of conditions that are different from the ones implied by Corollary 3.3.3. First observe that, if $q_1 = \ldots = q_d = 1$, then Condition 3 in the statement of Theorem 3.4.9 is immaterial. As a consequence, one deduces that $F^{(n)}$ converges in distribution to X, provided that (5.20) is verified and $\|f^{(n)}\|_{L^4(\mu)} \to 0$. The L^4 norms of the functions $f^{(n)}$ appear due to the use of Cauchy-Schwarz inequality in the proof of Proposition 3.4.7.

Proof of Theorem 3.4.9. By Theorem 3.3.2,

$$d_3(F^{(n)}, X) \leq \frac{d}{2}\sqrt{\sum_{i,j=1}^{d} \mathbb{E}[(C(i,j) - \langle DF_i^{(n)}, -DL^{-1}F_j^{(n)}\rangle_{L^2(\mu)})^2]} \tag{3.23}$$

$$+\frac{1}{4}\int_Z \mu(dz)\mathbb{E}\left[\left(\sum_{i=1}^{d} |D_z F_i^{(n)}|\right)^2 \left(\sum_{i=1}^{d} |D_z L^{-1} F_i^{(n)}|\right)\right], \tag{3.24}$$

so that we need only show that, under the assumptions in the statement, both (3.23) and (3.24) tend to 0 as $n \to \infty$.

On the one hand, we take $a = C(i,j)$ in Proposition 3.4.6. In particular, we take $a = 0$ when $q_i \neq q_j$. Admitting Condition 3 , 4 and (5.20), line (3.23) tends to 0 is a direct consequence of Proposition 3.4.6.

On the other hand, under Condition 3 and 4, Proposition 3.4.7 shows that (3.24) converges to 0. This concludes the proof and the above inequality gives the speed of convergence.

If the matrix C is positive definite, then one can alternatively use Theorem 3.2.3 instead of Theorem 3.3.2 while the deduction remains the same. ■

Remark 3.4.11 Apart from the asymptotic behavior of the covariances (5.20) and the presence of Assumption C, the statement of Theorem 3.4.9 does not contain any requirements on the joint distribution of the components of $F^{(n)}$. Besides the technical requirements in Condition 1 and Condition 2, the joint convergence of the random vectors $F^{(n)}$ only relies on the 'one-dimensional' Conditions 3 and 4, which are the same as condition (II) and (III) in the statement of Theorem 5.1 in [46]. See also Remark 3.2.5.

3.5 Examples

In what follows, we provide several explicit applications of the main estimates proved in the chapter. In particular:

- Section 3.5.1 focuses on vectors of single and double integrals.

- Section 3.5.2 deals with three examples of continuous-time functionals of Ornstein-Uhlenbeck Lévy processes.

3.5.1 Vectors of single and double integrals

The following statement corresponds to Theorem 3.2.3, in the special case

$$F = (F_1, \dots, F_d) = (I_1(g_1), \dots, I_1(g_m), I_2(h_1), \dots, I_2(h_n)). \tag{3.25}$$

The proof, which is based on a direct computation of the general bounds proved in Theorem 3.2.3, serves as a further illustration (in a simpler setting) of the techniques used throughout the chapter. Some of its applications will be illustrated in Section 3.5.2.

Proposition 3.5.1 *Fix integers $n, m \geq 1$, let $d = n + m$, and let C be a $d \times d$ nonnegative definite matrix. Let $X \sim \mathcal{N}_d(0, C)$. Assume that the vector in (3.25) is such that*

1. *the function g_i belongs to $L^2(\mu) \cap L^3(\mu)$, for every $1 \leq i \leq m$,*
2. *the kernel $h_i \in L^2_s(\mu^2)$ $(1 \leq i \leq n)$ is such that:* (a) *$h_{i_1} \star_2^1 h_{i_2} \in L^2(\mu^1)$, for $1 \leq i_1, i_2 \leq n$,* (b) *$h_i \in L^4(\mu^2)$ and* (c) *the functions $|h_{i_1}| \star_2^1 |h_{i_2}|$, $|h_{i_1}| \star_2^0 |h_{i_2}|$ and $|h_{i_1}| \star_1^0 |h_{i_2}|$ are well defined and finite for every value of their arguments and for every $1 \leq i_1, i_2 \leq n$,* (d) *every pair $(h_i,\, h_j)$ verifies Assumption C, that in this case is equivalent to requiring that*

$$\int_Z \sqrt{\int_Z \mu(da) h_i^2(z,a) h_j^2(z,a)}\mu(dz) < \infty.$$

Then,

$$\begin{aligned} d_3(F, X) &\leq \frac{1}{2}\sqrt{S_1 + S_2 + S_3} + S_4 \\ &\leq \frac{1}{2}\sqrt{S_1 + S_5 + S_6} + S_4 \end{aligned}$$

where

$$\begin{aligned}
S_1 &= \sum_{i_1, i_2 = 1}^{m} (C(i_1, i_2) - \langle g_{i_1}, g_{i_2} \rangle_{L^2(\mu)})^2 \\
S_2 &= \sum_{j_1, j_2 = 1}^{n} (C(m + j_1, m + j_2) - 2\langle h_{j_1}, h_{j_2} \rangle_{L^2(\mu^2)})^2 + 4\|h_{j_1} \star_2^1 h_{j_2}\|^2_{L^2(\mu)} + 8\|h_{j_1} \star_1^1 h_{j_2}\|^2_{L^2(\mu^2)} \\
S_3 &= \sum_{i=1}^{m} \sum_{j=1}^{n} 2C(i, m + j)^2 + 5\|g_i \star_1^1 h_j\|^2_{L^2(\mu)} \\
S_4 &= m^2 \sum_{i=1}^{m} \|g_i\|^3_{L^3(\mu)} + 8n^2 \sum_{j=1}^{n} \|h_j\|_{L^2(\mu^2)} (\|h_j\|^2_{L^4(\mu^2)} + \sqrt{2}\|h_{j_1} \star_1^0 h_{j_1}\|_{L^2(\mu^3)}) \\
S_5 &= \sum_{j_1, j_2 = 1}^{n} (C(m + j_1, m + j_2) - 2\langle h_{j_1}, h_{j_2} \rangle_{L^2(\mu^2)})^2 + 4\|h_{j_1} \star_1^0 h_{j_1}\|_{L^2(\mu^3)} \times \|h_{j_2} \star_1^0 h_{j_2}\|_{L^2(\mu^3)} \\
&\quad + 8\|h_{j_1} \star_1^1 h_{j_1}\|_{L^2(\mu^2)} \times \|h_{j_2} \star_1^1 h_{j_2}\|_{L^2(\mu^2)} \\
S_6 &= \sum_{i=1}^{m} \sum_{j=1}^{n} 2C(i, m + j)^2 + 5\|g_i\|^2_{L^2(\mu)} \times \|h_j \star_1^1 h_j\|_{L^2(\mu^2)}
\end{aligned}$$

Proof. Assumptions 1 and 2 in the statement ensure that each integral appearing in the proof is well-defined, and that the use of Fubini arguments is justified. In view of Theorem 3.3.2, our strategy is to study the quantities in line (3.8) and line (3.9) separately. On the one hand, we know that: for $1 \leq i \leq m, 1 \leq j \leq n$,

$$D_z I_1(g_i(\cdot)) = g_i(z), \qquad -D_z L^{-1} I_1(g_i(\cdot)) = g_i(z)$$

$$D_z I_2(h_j(\cdot,\cdot)) = 2I_1(h_j(z,\cdot)), \qquad -D_z L^{-1} I_2(h_j(\cdot,\cdot)) = I_1(h_j(z,\cdot))$$

Then, for any given constant a, we have:

– for $1 \leq i \leq m, 1 \leq j \leq n$,

$$\mathbb{E}[(a - \langle D_z I_1(g_{i_1}), -D_z L^{-1} I_1(g_{i_2})\rangle)^2] = (a - \langle g_{i_1}, g_{i_2}\rangle_{L^2(\mu)})^2;$$

– for $1 \leq j_1, j_2 \leq n$,

$$\begin{aligned} &\mathbb{E}[(a - \langle D_z I_2(h_{j_1}), -D_z L^{-1} I_2(h_{j_2})\rangle)^2] \\ =\ &(a - 2\langle h_{j_1}, h_{j_2}\rangle_{L^2(\mu^2)})^2 + 4\|h_{j_1} \star_2^1 h_{j_2}\|^2_{L^2(\mu)} + 8\|h_{j_1} \star_1^1 h_{j_2}\|^2_{L^2(\mu^2)}; \end{aligned}$$

– for $1 \leq i \leq m, 1 \leq j \leq n$,

$$\mathbb{E}[(a - \langle D_z I_2(h_j), -D_z L^{-1} I_1(g_i)\rangle)^2] = a^2 + 4\|g_i \star_1^1 h_j\|^2_{L^2(\mu)}$$

$$\mathbb{E}[(a - \langle D_z I_1(g_i), -D_z L^{-1} I_2(h_j)\rangle)^2] = a^2 + \|g_i \star_1^1 h_j\|^2_{L^2(\mu)}.$$

So

$$(3.8) = \frac{1}{2}\sqrt{S_1 + S_2 + S_3}$$

where S_1, S_2, S_3 are defined as in the statement of proposition.

On the other hand,

$$\left(\sum_{i=1}^{2} |D_z F_i|\right)^2 = \left(\sum_{i=1}^{m} |g_i(z)| + 2\sum_{j=1}^{n} |I_1(h_j(z,\cdot))|\right)^2,$$

$$\sum_{i=1}^{d} |D_z L^{-1} F_i| = \sum_{i=1}^{m} |g_i(z)| + \sum_{j=1}^{n} |I_1(h_j(z,\cdot))|.$$

As the following inequality holds for all positive reals a, b:

$$(a + 2b)^2(a + b) \leq (a + 2b)^3 \leq 4a^3 + 32b^3,$$

we have,

$$\begin{aligned}
&\mathbb{E}\left[\left(\sum_{i=1}^{d}|D_zF_i|\right)^2\left(\sum_{i=1}^{d}|D_zL^{-1}F_i|\right)\right]\\
=\ &\mathbb{E}\left[\left(\sum_{i=1}^{m}|g_i(z)|+2\sum_{j=1}^{n}|I_1(h_j(z,\cdot))|\right)^2\left(\sum_{i=1}^{m}|g_i(z)|+\sum_{j=1}^{n}|I_1(h_j(z,\cdot))|\right)\right]\\
\leq\ &\mathbb{E}\left[4\left(\sum_{i=1}^{m}|g_i(z)|\right)^3+32\left(\sum_{j=1}^{n}|I_1(h_j(z,\cdot))|\right)^3\right]\\
\leq\ &\mathbb{E}[4m^2\sum_{i=1}^{m}|g_i(z)|^3+32n^2\sum_{j=1}^{n}|I_1(h_j(z,\cdot))|^3].
\end{aligned}$$

By applying the Cauchy-Schwarz inequality, one infers that

$$\int_Z \mu(dz)\mathbb{E}[|I_1(h(z,\cdot))|^3]\leq\sqrt{\mathbb{E}\left[\int_Z\mu(dz)|I_1(h(z,\cdot))|^4\right]}\times\|h\|_{L^2(\mu^2)}.$$

Notice that

$$\mathbb{E}\left[\int_Z\mu(dz)|I_1(h(z,\cdot))|^4\right]=2\|h\star_2^1 h\|^2_{L^2(\mu)}+\|h\|^4_{L^4(\mu^2)}$$

We have

$$\begin{aligned}
(3.9)\ =\ &\frac{1}{4}m^2\|C^{-1}\|^{3/2}_{op}\|C\|_{op}\int_Z\mu(dz)\mathbb{E}\left[\left(\sum_{i=1}^{d}|D_zF_i|\right)^2\left(\sum_{i=1}^{d}|D_zL^{-1}F_i|\right)\right]\\
\leq\ &\|C^{-1}\|^{3/2}_{op}\|C\|_{op}\Big(m^2\sum_{i=1}^{m}\|g_i\|^3_{L^3(\mu)}\\
&+8n^2\sum_{j=1}^{n}\|h_j\|_{L^2(\mu^2)}(\|h_j\|^2_{L^4(\mu^2)}+\sqrt{2}\|h_j\star_2^1 h_j\|_{L^2(\mu)})\Big)\\
=\ &\|C^{-1}\|^{3/2}_{op}\|C\|_{op}S_4
\end{aligned}$$

We will now apply Lemma 1.1.8 to further assess some of the summands appearing the definition of S_2,S_3. Indeed,

– for $1\leq j_1,j_2\leq n$,

$$\|h_{j_1}\star_2^1 h_{j_2}\|^2_{L^2(\mu)}\leq\|h_{j_1}\star_1^0 h_{j_1}\|_{L^2(\mu^3)}\times\|h_{j_2}\star_1^0 h_{j_2}\|_{L^2(\mu^3)}$$

$$\|h_{j_1}\star_1^1 h_{j_2}\|^2_{L^2(\mu^2)}\leq\|h_{j_1}\star_1^1 h_{j_1}\|_{L^2(\mu^2)}\times\|h_{j_2}\star_1^1 h_{j_2}\|_{L^2(\mu^2)};$$

– for $1\leq i\leq m, 1\leq j\leq n$,

$$\|g_i\star_1^1 h_j\|^2_{L^2(\mu)}\leq\|g_i\|^2_{L^2(\mu)}\times\|h_j\star_1^1 h_j\|_{L^2(\mu^2)}$$

by using the relation $\|g_i^{(k)}\star_0^0 g_i^{(k)}\|^2_{L^2(\mu^2)}=\|g_i^{(k)}\|^4_{L^2(\mu)}$.

Consequently,

$$\begin{aligned}
S_2 &\le \sum_{j_1,j_2=1}^{n} (C(m+j_1,m+j_2) - 2\langle h_{j_1}, h_{j_2}\rangle_{L^2(\mu^2)})^2 + 4\|h_{j_1} \star_1^0 h_{j_1}\|_{L^2(\mu^3)} \times \|h_{j_2} \star_1^0 h_{j_2}\|_{L^2(\mu^3)} \\
&\quad +8\|h_{j_1} \star_1^1 h_{j_1}\|_{L^2(\mu^2)} \times \|h_{j_2} \star_1^1 h_{j_2}\|_{L^2(\mu^2)} \\
&= S_5, \\
S_3 &\le \sum_{i=1}^{m}\sum_{j=1}^{n} 2C(i,m+j)^2 + 5\|g_i\|^2_{L^2(\mu)} \times \|h_j \star_1^1 h_j\|_{L^2(\mu^2)} \\
&= S_6
\end{aligned}$$

■

Remark 3.5.2 If the matrix C is positive definite, then we have

$$\begin{aligned}
d_2(F,X) &\le \|C^{-1}\|_{op}\|C\|_{op}^{1/2}\sqrt{S_1+S_2+S_3} + \frac{\sqrt{2\pi}}{2}\|C^{-1}\|_{op}^{3/2}\|C\|_{op}S_4 \\
&\le \|C^{-1}\|_{op}\|C\|_{op}^{1/2}\sqrt{S_1+S_5+S_6} + \frac{\sqrt{2\pi}}{2}\|C^{-1}\|_{op}^{3/2}\|C\|_{op}S_4
\end{aligned}$$

by using Theorem 3.2.3.

The following result can be proved by means of Proposition 3.5.1.

Corollary 3.5.3 *Let $d = m+n$, with $m, n \ge 1$ two integers . Let $X_C \sim \mathcal{N}_d(0, C)$ be a centered d-dimensional Gaussian vector, where $C = \{C(s,t) : s,t = 1,\dots,d\}$ is a $d \times d$ nonnegative definite matrix such that*

$$C(i, j+m) = 0, \qquad \forall 1 \le i \le m, 1 \le j \le n.$$

Assume that

$$F^{(k)} = (F_1^{(k)}, \dots, F_d^{(k)}) := (I_1(g_1^{(k)}), \dots, I_1(g_m^{(k)}), I_2(h_1^{(k)}), \dots, I_2(h_n^{(k)}))$$

where for all k, the kernels $g_1^{(k)}, \dots, g_m^{(k)}$ and $h_1^{(k)}, \dots, h_n^{(k)}$ satisfy respectively the technical Conditions 1 and 2 in Proposition 3.5.1 . Assume moreover that the following conditions hold for each $k \ge 1$:

1.

$$\lim_{k\to\infty} \mathbb{E}[F_s^{(k)} F_t^{(k)}] = C(s,t), \qquad 1 \le s,t \le d.$$

or equivalently

$$\lim_{k\to\infty} \langle g_{i_1}^{(k)}, g_{i_2}^{(k)}\rangle_{L^2(\mu)} = C(i_1,i_2), \qquad 1 \le i_1, i_2 \le m,$$

$$\lim_{k\to\infty} 2\langle h_{j_1}^{(k)}, h_{j_2}^{(k)}\rangle_{L^2(\mu^2)} = C(m+j_1,m+j_2), \qquad 1 \le j_1, j_2 \le n.$$

2. For every $i = 1, \dots, m$ and every $j = 1, \dots, n$, one has the following conditions are satisfied as $k \to \infty$:
(a) $\|g_i^{(k)}\|^3_{L^3(\mu)} \to 0$; *(b)* $\|h_j^{(k)}\|^2_{L^4(\mu^2)} \to 0$;
(c) $\|h_j^{(k)} \star_2^1 h_j^{(k)}\|_{L^2(\mu)} = \|h_j^{(k)} \star_1^0 h_j^{(k)}\|_{L^2(\mu^3)} \to 0$;
(d) $\|h_j^{(k)} \star_1^1 h_j^{(k)}\|^2_{L^2(\mu^2)} \to 0$.

Then $F^{(k)} \to X$ in law, as $k \to \infty$. An explicit bound on the speed of convergence in the distance d_3 is provided by Proposition 3.5.1.

3.5.2 Vector of functionals of Ornstein-Uhlenbeck processes

In this section, we study CLTs for some functionals of Ornstein-Uhlenbeck Lévy process. These processes have been intensively studied in recent years, and applied to various domains such as e.g. mathematical finance (see [42]) and non-parametric Bayesian survival analysis (see e.g. [5, 45]). Our results are multi-dimensional generalizations of the content of [46, Section 7] and [47, Section 4].

We denote by $\hat{N}$ a centered Poisson measure over $\mathbb{R} \times \mathbb{R}$, with control measure given by $\nu(du)$, where $\nu(\cdot)$ is positive, non-atomic and σ-finite. For all positive real number λ, we define the stationary *Ornstein-Uhlenbeck Lévy process* with parameter λ as

$$Y_t^\lambda = I_1(f_t^\lambda) = \sqrt{2\lambda} \int_{-\infty}^{t} \int_{\mathbb{R}} u \exp(-\lambda(t-x)) \hat{N}(du, dx), \quad t \geq 0$$

where $f_t^\lambda(u, x) = \sqrt{2\lambda} \mathbf{1}_{(-\infty,t]}(x) u \exp(-\lambda(t-x))$. We make the following technical assumptions on the measure ν: $\int_{\mathbb{R}} u^j \nu(du) < \infty$ for $j = 2, 3, 4, 6$, and $\int_{\mathbb{R}} u^2 \nu(du) = 1$, to ensure among other things that Y_t^λ is well-defined. These assumptions yield in particular that

$$\mathbf{Var}(Y_t^\lambda) = \mathbb{E}[(Y_t^\lambda)^2] = 2\lambda \int_{-\infty}^{t} \int_{\mathbb{R}} u^2 \exp(-2\lambda(t-x)) \nu(du) dx = 1$$

We shall obtain Central Limit Theorems for three kind of functionals of Ornstein-Uhlenbeck Lévy processes. In particular, each of the forthcoming examples corresponds to a "realized empirical moment" (in continuous time) associated with Y^λ, namely: Example 1 corresponds to an asymptotic study of the mean, Example 2 concerns second moments, whereas Example 3 focuses on joint second moments of shifted processes.

Observe that all kernels considered in the rest of this section automatically satisfy our Assumptions A, B and C.

Example 1 (Empirical means)

We define the functional $A(T, \lambda)$ by $A(T, \lambda) = \frac{1}{\sqrt{T}} \int_0^T Y_t^\lambda dt$. We recall the following limit theorem for $A(T, \lambda)$, taken from Example 3.6 in [46].

Theorem 3.5.4 *As $T \to \infty$,*

$$\frac{A(T, \lambda)}{\sqrt{2/\lambda}} = \frac{1}{\sqrt{2T/\lambda}} \int_0^T Y_t^\lambda dt \overset{(law)}{\longrightarrow} X \sim \mathcal{N}(0, 1),$$

and there exists a constant $0 < \alpha(\lambda) < \infty$*, independent of* T *and such that*

$$d_w\left(\frac{A(T,\lambda)}{\sqrt{2/\lambda}}, X\right) \le \frac{\alpha(\lambda)}{\sqrt{T}}.$$

Here, we present a multi-dimensional generalization of the above result.

Theorem 3.5.5 *For* $\lambda_1, \ldots, \lambda_d > 0$*, as* $T \to \infty$*,*

$$\bar{A}(T) = (A(T,\lambda_1), \ldots, A(T,\lambda_d)) \stackrel{(law)}{\longrightarrow} X_B, \tag{3.26}$$

where X_B *is a centered* d*-dimensional Gaussian vector with covariance matrix* $B = (B_{ij})_{d\times d}$*, with* $B_{ij} = 2/\sqrt{\lambda_i\lambda_j}$*,* $1 \le i,j \le d$*. Moreover, there exists a constant* $0 < \alpha = \alpha(\bar{\lambda}) = \alpha(\lambda_1, \ldots, \lambda_d) < \infty$*, independent of* T *and such that*

$$d_3(\bar{A}(T), X_B) \le \frac{\alpha(\bar{\lambda})}{\sqrt{T}}.$$

Proof. By applying Fubini theorem on $A(T,\lambda)$, we have

$$\frac{1}{\sqrt{T}}\int_0^T Y_t^\lambda dt = I_1(g_{\lambda,T})$$

where

$$g_{\lambda,T} = \mathbf{1}_{(-\infty,T]}(x)u\sqrt{\frac{2\lambda}{T}}\int_{x\vee 0}^T \exp(-\lambda(t-x))dt$$

$$\begin{aligned}
&\mathbb{E}[A(T,\lambda_i)A(T,\lambda_j)]\\
= &\int_{\mathbb{R}} u^2\nu(du)\Big(\int_{-\infty}^0 dx\frac{2}{T\sqrt{\lambda_i\lambda_j}}\exp\big((\lambda_i+\lambda_j)x\big)\times\big(1-\exp(-\lambda_i T)\big)\times\big(1-\exp(-\lambda_j T)\big)\\
&+\int_0^T dx\frac{2}{T\sqrt{\lambda_i\lambda_j}}\exp\big((\lambda_i+\lambda_j)x\big)\times\big(\exp(-\lambda_i x)-\exp(-\lambda_i T)\big)\times\big(\exp(-\lambda_j x)-\exp(-\lambda_j T)\big)\Big)\\
= &\frac{2}{T\sqrt{\lambda_i\lambda_j}}\Big(\frac{1}{\lambda_i+\lambda_j}\times\big(1-\exp(-\lambda_i T)\big)\times\big(1-\exp(-\lambda_j T)\big)+T-\frac{1}{\lambda_i}\times(1-\exp(-\lambda_i T))\\
&-\frac{1}{\lambda_j}\big(1-\exp(-\lambda_j T)\big)+\frac{1}{\lambda_i+\lambda_j}\big(1-\exp(-(\lambda_i+\lambda_j)T)\big)\Big)\\
= &\frac{2}{\sqrt{\lambda_i\lambda_j}}+O\left(\frac{1}{T}\right) \qquad \text{as } T\to\infty.
\end{aligned}$$

And we may verify that $\|g_{\lambda,T}\|^3_{L^3(d\nu dx)} \sim \frac{1}{\sqrt{T}}$. for all $\lambda \in \mathbb{R}$. (See [46] and [47] for details.) Finally, we deduce the conclusion by using Corollary 3.3.3. ■

Example 2 (Empirical second moments)
We are interested in the quadratic functional $Q(T,\lambda)$ given by:

$$Q(T,\lambda) := \sqrt{T}\left(\frac{1}{T}\int_0^T (Y_t^\lambda)^2 dt - 1\right), \qquad T>0, \lambda>0$$

In [46] and [47], the authors have proved the following limit theorem for $Q(T,\lambda)$. (See Theorem 7.1 in [46] and Proposition 7 in [47])

Theorem 3.5.6 *For every $\lambda > 0$, as $T \to \infty$,*

$$Q(T,\lambda) := \sqrt{T}\left(\frac{1}{T}\int_0^T (Y_t^\lambda)^2 dt - 1\right) \overset{(law)}{\longrightarrow} \sqrt{\frac{2}{\lambda} + c_\nu^2} \times X$$

where $X \sim \mathcal{N}(0,1)$ is a standard Gaussian random variable and $c_\nu^2 = \int_{\mathbb{R}} u^4 \nu(du)$ is a constant. And there exists a constant $0 < \beta(\lambda) < \infty$, independent of T and such that

$$d_w\left(\frac{Q(T,\lambda)}{\sqrt{\frac{2}{\lambda} + c_\nu^2}}, X\right) \leq \frac{\beta(\lambda)}{\sqrt{T}}$$

We introduce here a multi-dimensional generalization of the above result.

Theorem 3.5.7 *Given an integer $d \geq 2$. For $\lambda_1, \ldots, \lambda_d > 0$, as $T \to \infty$,*

$$\bar{Q}(T) = (Q(T,\lambda_1), \ldots, Q(T,\lambda_d)) \overset{(law)}{\longrightarrow} X_C, \tag{3.27}$$

where X_C is a centered d-dimensional Gaussian vector with covariance matrix $C = (C_{ij})_{d\times d}$, defined by

$$C_{ij} = \frac{4}{\lambda_i + \lambda_j} + c_\nu^2, \qquad 1 \leq i,j \leq d,$$

and $c_\nu^2 = \int_{\mathbb{R}} u^4 \nu(du)$. And there exists a constant $0 < \beta(\bar{\lambda}) = \beta(\lambda_1, \ldots, \lambda_d) < \infty$, independent of T and such that

$$d_3(\bar{Q}(T), X_C) \leq \frac{\beta(\bar{\lambda})}{\sqrt{T}}$$

Proof. For every $T > 0$ and $\lambda > 0$, we introduce the notations

$$\begin{aligned} H_{\lambda,T}(u,x;u',x') &= (u \times u')\frac{\mathbf{1}_{(-\infty,T)^2}(x,x')}{T}\Big(\exp\big(\lambda(x+x')\big) \times \big(1 - \exp(-2\lambda T)\big) \times \mathbf{1}_{(x\vee x' \leq 0)} \\ &\quad + \exp\big(\lambda(x+x')\big) \times \big(\exp(-2\lambda(x \vee x')) - \exp(-2\lambda T)\big) \times \mathbf{1}_{(x\vee x' > 0)}\Big) \end{aligned}$$

$$\begin{aligned} H^\star_{\lambda,T}(u,x) &= u^2\frac{\mathbf{1}_{(-\infty,T)}(x)}{T}\Big(\exp(2\lambda x) \times \big(1 - \exp(-2\lambda T)\big) \times \mathbf{1}_{(x \leq 0)} \\ &\quad + \exp(2\lambda x) \times \big(\exp(-2\lambda x) - \exp(-2\lambda T)\big) \times \mathbf{1}_{(x>0)}\Big) \end{aligned}$$

By applying the multiplication formula (1.12) and a Fubini argument, we deduce that

$$Q(T,\lambda) = I_1(\sqrt{T}H^\star_{\lambda,T}) + I_2(\sqrt{T}H_{\lambda,T}),$$

which is the sum of a single and a double Wiener-Itô integral. Instead of deducing the convergence for $(Q(T,\lambda_1), \ldots, Q(T,\lambda_d))$, we prove the stronger result:

$$(I_1(\sqrt{T}H^\star_{\lambda_1,T}), \ldots, I_1(\sqrt{T}H^\star_{\lambda_d,T}), I_2(\sqrt{T}H_{\lambda_1,T}), \ldots, I_2(\sqrt{T}H_{\lambda_d,T})) \overset{(law)}{\longrightarrow} X_D \tag{3.28}$$

as $T \to \infty$. Here, X_D is a centered $2d$-dimensional Gaussian vector with covariance matrix D defined as:

$$D(i,j) = \begin{cases} c_\nu^2, & \text{if } 1 \le i,j \le d \\ \dfrac{4}{\lambda_i + \lambda_j}, & \text{if } d+1 \le i,j \le 2d \\ 0, & \text{otherwise.} \end{cases}$$

We prove (3.28) in two steps (by using Corollary 3.5.3). Firstly, we aim at verifying

$$\lim_{T\to\infty} \mathbb{E}[F_i^{(T)} F_j^{(T)}] = D(i,j), \qquad 1 \le i,j \le 2d,$$

for

$$F_k^{(T)} = \begin{cases} I_1(\sqrt{T} H^\star_{\lambda_k,T}), & \text{if } 1 \le k \le d \\ I_2(\sqrt{T} H_{\lambda_k,T}), & \text{if } d+1 \le k \le 2d \end{cases}$$

Indeed, by standard calculations, we have

$$\begin{aligned} & T \int_{\mathbb{R}\times\mathbb{R}} H^\star_{\lambda_i,T}(u,x) H^\star_{\lambda_j,T}(u,x) \nu(du) dx \\ = \; & \frac{1}{T} c_\nu^2 \Big(\frac{1}{2(\lambda_i+\lambda_j)} \times \big(1 - \exp(-2\lambda_i T)\big) \times \big(1 - \exp(-2\lambda_j T)\big) + T - \frac{1}{2\lambda_i} \times \big(1 - \exp(-2\lambda_i T)\big) \\ & - \frac{1}{2\lambda_j} \times \big(1 - \exp(-2\lambda_j T)\big) + \frac{1}{2(\lambda_i+\lambda_j)} \times \big(1 - \exp(-2(\lambda_i+\lambda_j)T)\big)\Big) \\ = \; & c_\nu^2 + O\left(\frac{1}{T}\right), \qquad \text{as } T \to \infty, \end{aligned}$$

and

$$\begin{aligned} & 2T \int_{\mathbb{R}^4} H_{\lambda_i,T}(u,x;u',x') H_{\lambda_j,T}(u,x;u',x') \nu(du)\nu(du') dx dx' \\ = \; & \frac{2}{T} \Big(\frac{(1 - \exp(-2\lambda_i T)) \times (1 - \exp(-2\lambda_j T))}{(\lambda_i+\lambda_j)^2} + \frac{2}{\lambda_i+\lambda_j} \times \big(T - \frac{1}{2\lambda_i}\big(1 - \exp(-2\lambda_i T)\big) \\ & - \frac{1}{2\lambda_j} \times \big(1 - \exp(-2\lambda_j T)\big) + \frac{1}{2(\lambda_i+\lambda_j)} \times \big(1 - \exp(-2(\lambda_i+\lambda_j)T)\big)\big)\Big) \\ = \; & \frac{4}{\lambda_i+\lambda_j} + O\left(\frac{1}{T}\right), \qquad \text{as } T \to \infty. \end{aligned}$$

Secondly, we use the fact that for $\lambda = \lambda_1, \ldots, \lambda_d$, the following asymptotic relations holds as $T \to \infty$:

(a) $\quad \|\sqrt{T} H^\star_{\lambda,T}\|^3_{L^3(d\nu dx)} \sim \dfrac{1}{\sqrt{T}}$;

(b) $\quad \|\sqrt{T} H_{\lambda,T}\|^2_{L^4((d\nu dx)^2)} \sim \dfrac{1}{\sqrt{T}}$;

(c) $\quad \|(\sqrt{T} H_{\lambda,T}) \star_2^1 (\sqrt{T} H_{\lambda,T})\|_{L^2(d\nu dx)} = \|(\sqrt{T} H_{\lambda,T}) \star_1^0 (\sqrt{T} H_{\lambda,T})\|_{L^2((d\nu dx)^3)} \sim \dfrac{1}{\sqrt{T}}$;

(d) $\quad \|(\sqrt{T} H_{\lambda,T}) \star_1^1 (\sqrt{T} H_{\lambda,T})\|_{L^2((d\nu dx)^2)} \sim \dfrac{1}{\sqrt{T}}$;

(e) $\quad \|(\sqrt{T}H^{\star}_{\lambda,T}) \star_1^1 (\sqrt{T}H_{\lambda,T})\|_{L^2(d\nu dx)} \sim \frac{1}{\sqrt{T}}.$

The reader is referred to [46, Section 7] and [47, Section 4] for a proof of the above asymptotic relations. ■

Example 3 (Empirical joint moments of shifted processes)
We are now able to study a generalization of Example 2. We define

$$Q_h(T,\lambda) := \sqrt{T}\left(\frac{1}{T}\int_0^T Y_t^\lambda Y_{t+h}^\lambda dt - \exp(-\lambda h)\right), \qquad h > 0, T > 0, \lambda > 0.$$

The theorem below is a multi-dimensional CLT for $Q_h(T,\lambda)$.

Theorem 3.5.8 *For $\lambda_1,\ldots,\lambda_d > 0$ and $h \geq 0$, as $T \to \infty$,*

$$\bar{Q}_h(T) = (Q_h(T,\lambda_1),\ldots,Q_h(T,\lambda_d)) \stackrel{(law)}{\longrightarrow} X_E, \tag{3.29}$$

where X_E is a centered d-dimensional Gaussian vector with covariance matrix $E = (E_{ij})_{d\times d}$, with

$$E_{ij} = \frac{4}{\lambda_i + \lambda_j} + c_\nu^2 \exp\Big(-(\lambda_i+\lambda_j)h\Big), \qquad 1 \leq i,j \leq d$$

and $c_\nu^2 = \int_{\mathbb{R}} u^4 \nu(du)$. Moreover, there exists a constant $0 < \gamma(h,\bar{\lambda}) = \gamma(h,\lambda_1,\ldots,\lambda_d) < \infty$, independent of T and such that

$$d_3(\bar{Q}_h(T), X_E) \leq \frac{\gamma(h,\bar{\lambda})}{\sqrt{T}}$$

Proof. We have

$$\begin{aligned}
\int_0^T Y_t^\lambda Y_{t+h}^\lambda dt &= \int_0^T I_1(f_t^\lambda) I_1(f_{t+h}^\lambda) dt \\
&= \int_0^T \Big(I_2(f_t^\lambda \star_0^0 f_{t+h}^\lambda) + I_1(f_t^\lambda \star_1^0 f_{t+h}^\lambda) + f_t^\lambda \star_1^1 f_{t+h}^\lambda\Big) dt \\
&= \int_0^T \Big(I_2(\hat{h}_{t,h}^\lambda) + I_1(\hat{h}_{t,h}^{*,\lambda}) + \exp(-\lambda h)\Big) dt \\
&= I_2(TH_{\lambda,T}^h) + I_1(TH_{\lambda,T}^{*,h}) + \exp(-\lambda h) T
\end{aligned}$$

and

$$Q_h(T,\lambda) = I_2(\sqrt{T}H_{\lambda,T}^h) + I_1(\sqrt{T}H_{\lambda,T}^{*,h})$$

by using multiplication formula (1.12) and Fubini theorem. By simple calculations, we obtain that

$$\hat{h}_{t,h}^\lambda(u,x;u',x') = 2\lambda \mathbf{1}_{(-\infty,t]\times(-\infty,t+h]}(x,x') \times uu' \exp(-\lambda(2t+h-x-x'))$$

$$\hat{h}_{t,h}^{*,\lambda}(u,x) = 2\lambda \mathbf{1}_{(-\infty,t]}(x) \times u^2 \exp(-\lambda(2t+h-2x))$$

as well as

$$\begin{aligned} H^{*,h}_{\lambda,T}(u,x) &= \frac{1}{T}\int_0^T \hat{h}^{*,\lambda}_{t,h}(u,x)dt \\ &= u^2\frac{\mathbf{1}_{(-\infty,T]}(x)}{T}\times\exp(\lambda(2x-h))\times\Big(\mathbf{1}_{(x>0)}\times(\exp(-2\lambda x)-\exp(-2\lambda T)) \\ &\quad +\mathbf{1}_{(x\le 0)}\times(1-\exp(-2\lambda T))\Big) \end{aligned}$$

$$\begin{aligned} H^{h}_{\lambda,T}(u,x;u',x') &= \frac{1}{T}\int_0^T \hat{h}^{\lambda}_{t,h}(u,x;u',x')dt \\ &= uu'\frac{\mathbf{1}_{(-\infty,T]}(x)\mathbf{1}_{(-\infty,T+h]}(x')}{T}\times\exp(\lambda(x+x'-h)) \\ &\quad \times\Big(\mathbf{1}_{(x\vee(x'-h)>0)}\times\big(\exp(-2\lambda(x\vee(x'-h)))-\exp(-2\lambda T)\big) \\ &\quad +\mathbf{1}_{(x\vee(x'-h)\le 0)}\times(1-\exp(-2\lambda T))\Big) \end{aligned}$$

Similar to the procedures in the precedent example, we prove the stronger result:

$$(I_1(\sqrt{T}H^{\star,h}_{\lambda_1,T}),\ldots,I_1(\sqrt{T}H^{\star,h}_{\lambda_d,T}),I_2(\sqrt{T}H^{h}_{\lambda_1,T}),\ldots,I_2(\sqrt{T}H^{h}_{\lambda_d,T}))\overset{(law)}{\longrightarrow}X_{D^h} \tag{3.30}$$

as $T\to\infty$. Here, X_{D^h} is a centered $2d$-dimensional Gaussian vector with covariance matrix D^h defined as:

$$D^h(i,j)=\begin{cases} c_\nu^2\exp(-(\lambda_i+\lambda_j)h), & \text{if } 1\le i,j\le d \\ \dfrac{4}{\lambda_i+\lambda_j}, & \text{if } d+1\le i,j\le 2d \\ 0, & \text{otherwise.} \end{cases}$$

We have

$$\begin{aligned} & T\int_{\mathbb{R}\times\mathbb{R}} H^{*,h}_{\lambda,T}(u,x)H^{*,h}_{\lambda,T}(u,x)\nu(du)dx \\ = \ & \frac{1}{T}c_\nu^2\Big(\int_{-\infty}^0 dx\exp\big((\lambda_i+\lambda_j)(2x-h)\big)\times\big(1-\exp(-2\lambda_i T)\big)\times\big(1-\exp(-2\lambda_j T)\big) \\ + \ & \int_0^T dx\exp\big((\lambda_i+\lambda_j)(2x-h)\big)\times\big(\exp(-2\lambda_i x)-\exp(-2\lambda_i T)\big)\times\big(\exp(-2\lambda_j x)-\exp(-2\lambda_j T)\big)\Big) \\ = \ & c_\nu^2\exp(-(\lambda_i+\lambda_j)h)+O\left(\frac{1}{T}\right). \qquad \text{as } T\to\infty, \end{aligned}$$

We notice that

$$H^h_{\lambda,T}(u,x;u',x')=H_{\lambda,T}(u,x;u',x'-h)$$

Then, as shown in the proof of Theorem 3.5.7, we have

$$2T\int_{\mathbb{R}\times\mathbb{R}} H^h_{\lambda,T}(u,x)H^h_{\lambda,T}(u,x)\nu(du)dx=\frac{4}{\lambda_i+\lambda_j}+O\left(\frac{1}{T}\right). \qquad \text{as } T\to\infty.$$

Just as the precedent example, we may verify that for $\lambda = \lambda_1, \ldots, \lambda_d$ and $h \geq 0$, the following asymptotic relations holds as $T \to \infty$:

(a) $\|\sqrt{T} H^{*,h}_{\lambda,T}\|^3_{L^3(d\nu dx)} \sim \frac{1}{\sqrt{T}}$;

(b) $\|\sqrt{T} H^{h}_{\lambda,T}\|^2_{L^4((d\nu dx)^2)} \sim \frac{1}{\sqrt{T}}$;

(c) $\|(\sqrt{T} H^{h}_{\lambda,T}) \star_2^1 (\sqrt{T} H^{h}_{\lambda,T}\|_{L^2(d\nu dx)} = \|(\sqrt{T} H_{\lambda,T}) \star_1^0 (\sqrt{T} H^{h}_{\lambda,T})\|_{L^2((d\nu dx)^3)} \sim \frac{1}{\sqrt{T}}$;

(d) $\|(\sqrt{T} H^{h}_{\lambda,T}) \star_1^1 (\sqrt{T} H^{h}_{\lambda,T})\|_{L^2((d\nu dx)^2)} \sim \frac{1}{\sqrt{T}}$;

(e) $\|(\sqrt{T} H^{*,h}_{\lambda,T}) \star_1^1 (\sqrt{T} H^{h}_{\lambda,T})\|_{L^2(d\nu dx)} \sim \frac{1}{\sqrt{T}}$.

We conclude the proof by analogous arguments as in the proof of (3.27). ■

The calculations above enable us to derive immediately the following new one-dimensional result, which is a direct generalization of Theorem 5.1 in [46].

Corollary 3.5.9 *For every $\lambda > 0$, as $T \to \infty$,*

$$Q_h(T, \lambda) \xrightarrow{(law)} \sqrt{\frac{2}{\lambda} + c_\nu^2 \exp(-2\lambda h)} \times X$$

where $X \sim \mathcal{N}(0,1)$ is a standard Gaussian random variable. Moreover, there exists a constant $0 < \gamma(h, \lambda) < \infty$, independent of T and such that

$$d_w \left(\frac{Q_h(T, \lambda)}{\sqrt{2/\lambda + c_\nu^2 \exp(-2\lambda h)}}, X \right) \leq \frac{\gamma(h, \lambda)}{\sqrt{T}}$$

Chapter 4

Universality of the Poisson Wiener chaos

This chapter is based on the paper in preparation [51] by G. Peccati and C. Zheng.

4.1 Introduction of the chapter

In a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we shall consider the following objects.

- $\mathbf{G} = \{G_i : i \geq 1\}$, where $G_i \sim \mathcal{N}(0,1)$ are independent Gaussian variables;
- $\mathbf{E} = \{e_i : i \geq 1\}$ is a Rademacher sequence. By this expression we simply mean that the random variables e_i are i.i.d. and $\mathbb{P}(e_i = 1) = \mathbb{P}(e_i = -1) = \frac{1}{2}$ for $i = 1, 2, \ldots$;
- $\mathbf{P} = \{P_i : i \geq 1\}$, where P_i are independent Poisson random variables, distributed as $P(\lambda_i) - \lambda_i$. Here, $P(\lambda_i)$ indicates a Poisson variable with parameter $\lambda_i > 0$, $i \geq 1$.

We introduce the notion of *homogeneous sum*.

Definition 4.1.1 (Homogeneous sums) *Fix some integers $N, q \geq 2$. Let $[N]$ indicate the set $\{1, 2, \cdots, N\}$. Let $\boldsymbol{X} = \{X_i : i \geq 1\}$ be a collection of centered independent random variables, and let $f : [N]^q \to \mathbb{R}$ be a* symmetric function vanishing on diagonals *(i.e. $f(i_1, \cdots, i_q) = 0$ if $\exists k \neq l : i_k = i_l$). The random variable*

$$\mathcal{Q}_q(N, f, \boldsymbol{X}) = \sum_{1 \leq i_1, \cdots, i_q \leq N} f(i_1, \cdots, i_q) X_{i_1} \cdots X_{i_q}$$

is called the multilinear homogeneous sum, of order q, based on f and on the first N elements of $\boldsymbol{X}$.

Remark 4.1.2 If, for $i = 1, 2, \ldots$, $\mathbb{E}[X_i] = 0$ and $\mathbb{E}[X_i^2] = \lambda_i$, as in the examples of $\mathbf{G}$ and $\mathbf{E}$ (where $\lambda_i = 1$) and $\mathbf{P}$, then we deduce immediately that the mean and variance of $Q_q = \mathcal{Q}_q(N, f, \mathbf{X})$ are given by:

$$\mathbb{E}[Q_q] = 0, \qquad \mathbb{E}[Q_q^2] = q! \sum_{1 \leq i_1, \cdots, i_q \leq N} f^2(i_1, \cdots, i_q) \lambda_{i_1} \cdots \lambda_{i_q}$$

The next three examples show that homogeneous sums based on $\mathbf{G}$, $\mathbf{E}$, $\mathbf{P}$ can always be represented as "chaotic random variables", such as the ones that are at the core of this dissertation.

Example 4.1.3 (Homogeneous sums based on G) Let $\mathbf{G} = \{G_i : i \geq 1\}$ be defined as above. Without loss of generality, we can always assume that $G_i = I_1^G(h_i) = G(h_i)$, for some isonormal Gaussian process $G = \{G(h) : h \in \mathfrak{H}\}$ based on a real separable Hilbert space $\mathfrak{H}$, where $\{h_i : i \geq 1\}$ is an orthonormal basis of $\mathfrak{H}$. (See Section 1.1.3 for the notations.) With this representation, one has that $\mathcal{Q}_q(N, f, \mathbf{G})$ belongs to q-th Gaussian Wiener chaos of G. Indeed, we may write

$$\mathcal{Q}_q(N, f, \mathbf{G}) = I_q^G(g),$$

where

$$g = \sum_{i_1,\cdots,i_q}^{N} f(i_1, \cdots, i_q) h_{i_1} \otimes \cdots \otimes h_{i_q}.$$

Example 4.1.4 (Homogeneous sums based on P) Let $\mathbf{P} = \{P_i : i \geq 1\}$ be defined as above. Without loss of generality, we can always assume that for every $i \geq 1$, $P_i = I_1^{\hat{N}}(g_i)$, where $\hat{N}$ indicates a compensated Poisson measure on $(Z, \mathcal{Z})$, with control μ, and $\mathbf{g} = \{g_i : i \geq 1\}$ is a collection of functions in $L^2(\mu)$ with disjoint finite supports, such that for each i, $\|g_i\|^2_{L^2(\mu)} = \lambda_i$. (See Section 1.1.1 for the notations.) For instance, one may take $Z = \mathbb{R}_+$, μ =Lebesgue measure and $g_i = \mathbf{1}_{(\lambda_1+\cdots+\lambda_{i-1}, \lambda_1+\cdots+\lambda_i]}$ for $i \geq 2$, $g_1 = \mathbf{1}_{(0,\lambda_1]}$. Therefore, $\mathcal{Q}_q(N, f, \mathbf{P})$ belongs to q-th Poisson Wiener chaos of $\hat{N}$ since

$$\mathcal{Q}_q(N, f, \mathbf{P}) = I_q(h),$$

where

$$h = \sum_{i_1,\cdots,i_q}^{N} f(i_1, \cdots, i_q) g_{i_1} \otimes \cdots \otimes g_{i_q}.$$

Example 4.1.5 (Homogeneous sums based on E) Fix $q \geq 1$, let $f : \mathbb{N}^q \to \mathbb{R}$ be a symmetric function vanishing on diagonals. We consider the Rademacher sequence $\mathbf{E} = \{e_i : i \geq 1\}$ defined above. A random variable with the form,

$$J_q = \sum_{i_1,\cdots,i_q} f(i_1, \cdots, i_q) e_{i_1} \cdots e_{i_q},$$

where the series converge in $L^2(\mathbb{P})$, compose the so-called *q-th Walsh chaos* of $\mathbf{E}$. (See [25, Chapter IV], or Remark 2.7 in [35].) In particular, let $f : [N]^q \to \mathbb{R}$ be a symmetric function vanishing on diagonals, then the homogeneous sums of the type

$$\mathcal{Q}_q(N, f, \mathbf{E}) = \sum_{i_1,\cdots,i_q}^{N} f(i_1, \cdots, i_q) e_{i_1} \cdots e_{i_q}$$

are elements of q-th Walsh chaos of $\mathbf{E}$.

Recall (see Remark 2.7 in [35]) that the Walsh chaos has the following chaotic decomposition property: for every $F \in L^2(\sigma(\mathbf{E}))$ (that is, the set of square integrable functional of the sequence $\mathbf{E}$), there exists a unique sequence of symmetric functions vanishing on diagonals $f_q \in L^2_s(\mu^q)$, $q \geq 1$, such that

$$F = E[F] + \sum_{q\geq 1} q! \sum_{i_1<i_2<\ldots<i_q} f_q(i_1,\ldots,i_q) e_{i_1}\cdots e_{i_q},$$

where the double series converges in L^2.

Concerning the "Universality" of these chaotic variables, the following two results have been proved in [36].

Theorem 4.1.6 (Theorem 1.10 in [36]) *Homogeneous sums inside the Gaussian Wiener chaos are* **universal** *with respect to normal approximations, in the following sense: fix $q \geq 2$, let $\{N^{(n)} : n \geq 1\}$ be a sequence of integers going to infinity, and let $\{f^{(n)}; n \geq 1\}$ be a sequence of mappings, such that each function $f^{(n)} : [N^{(n)}]^q \to \mathbb{R}$ is symmetric and vanishes on diagonals. Assume that $\mathbb{E}[\mathcal{Q}_q(N^{(n)}, f^{(n)}, \boldsymbol{G})^2] \to 1$ as $n \to \infty$. Then, the following three properties are equivalent as $n \to \infty$.*

1. *The sequence $\{\mathcal{Q}_q(N^{(n)}, f^{(n)}, \boldsymbol{G}); n \geq 1\}$ converges in law to $N \sim \mathcal{N}(0,1)$;*
2. *$\mathbb{E}[\mathcal{Q}_q(N^{(n)}, f^{(n)}, \boldsymbol{G})^4] \to 3$;*
3. *for every sequence $\boldsymbol{X} = \{X_i; i \geq 1\}$ of independent and identically distributed centered random variables with unit variance, the sequence $\{\mathcal{Q}_q(N^{(n)}, f^{(n)}, \boldsymbol{X}); n \geq 1\}$ converge in law to $\mathcal{N}(0,\sigma^2)$.*

Remark 4.1.7 We do not list the property (3) in the original version of Theorem 1.10 in [36]. Indeed, according to Proposition C.3.2 in [33], the Kolmogorov distance is exactly the distance induced by convergence in law, under the assumptions of Theorem 1.10 in [36]. Consequently, the original property (3) is practically the same as property (4) (or the third property in the above statement).

We also have the following negative result.

Proposition 4.1.8 *Homogeneous sums inside the Walsh chaos are* **not universal** *with respect to normal approximations.*

Proof. To show this assertion, we present the counter-example introduced at page 1956 in [36]. Let $\mathbf{G}$, $\mathbf{E}$ be defined above. Fix $q \geq 2$. For each $N \geq q$, we define

$$f_N(i_1,i_2,\ldots,i_q) = \begin{cases} 1/(q!\sqrt{N-q+1}), & \text{if } \{i_1,i_2,\ldots,i_q\} = \{1,2,\ldots,q-1,s\} \text{ for } q \leq s \leq N; \\ 0, & \text{else.} \end{cases}$$

The homogeneous sum thus defined is

$$\mathcal{Q}_q(N, f_N, \mathbf{E}) = e_1 e_2 \cdots e_{q-1} \sum_{i=q}^{N} \frac{e_i}{\sqrt{N-q+1}},$$

with $E[\mathcal{Q}_q(N, f_N, \mathbf{E})] = 0$ and $\mathbf{Var}[\mathcal{Q}_q(N, f_N, \mathbf{E})] = 1$. Since $e_1 e_2 \cdots e_{q-1}$ is a random sign independent of $\{e_i : i \geq q\}$, we have that $\mathcal{Q}_q(N, f_N, \mathbf{E}) \xrightarrow{law} \mathcal{N}(0,1)$, as $N \to \infty$, by using the

Central Limit Theorem. However, for every $N \geq 2$, $\mathcal{Q}_q(N, f_N, \mathbf{G}) \overset{(law)}{=} G_1 G_2 \cdots G_q$, which is not Gaussian (for $q \geq 2$). Consequently, $\mathcal{Q}_q(N, f_N, \mathbf{G})$ does not converge to a normal variable. ■

In the present chapter, we are going to address the following question.

Problem 1 *Are homogeneous sums inside the Poisson Wiener chaos universal with respect to normal approximations?*

We will see in section 4.2 that the answer is positive both in the one-dimensional and multi-dimensional cases. Our techniques are based on the tools developped in [PSTU] and [PZ], that are in turn recent developments of "Malliavin-Stein method" – the combination of Stein's method and Malliavin calculus. (See Section 2.1)

As a by-product of our achievements, we will also prove some refinements of the Central Limit Theorem on the Poisson Wiener chaos. Indeed, in the forthcoming Theorem 4.2.2 and Theorem 4.2.6, we shall show that, in the special case of elements of the Poisson Wiener chaos that are also homogeneous sums, the sufficient conditions for the CLTs established in the precedent chapters (in particular Theorem 2.1.10 and Theorem 3.4.9) turn out to be also necessary.

The chapter is organized as follows. In Section 4.2 we present the main results, in both the one-dimensional and multi-dimensional cases, and demonstrate the “Universality” of the Poisson Wiener chaos. In Section 4.3, we introduce an important technical proposition as well as the proofs of the main theorems.

4.2 Theorems

In the forthcoming discussion, we shall use several **abbreviations:**

- We write $\sum\limits_{i_1,\cdots,i_q}^{N}$ for $\sum\limits_{1\leq i_1,\cdots,i_q\leq N}$.
- For any function f in q variables, we write $\|f\|_{L^k}$ for $\|f\|_{L^k(\mu^q)}$.
- For any function f in q variables, we write $\|f\|$ for $\|f\|_{L^2} = \|f\|_{L^2(\mu^q)}$.
- For any positive integer N, the symbol $[N]$ indicates the set $\{1, 2, \cdots, N\}$.
- The operator $\otimes$ is used to denote a tensor product. In particular, for two functions $f, g \in L^2(\mu)$, $f \otimes g$ is the tensor product of f and g. That is, $f \otimes g(x, y) = f(x)g(y)$.

We recall that the definition of the contraction operators $f \star_r^l g$ and $f \otimes_r g$ were given in Definition 1.1.6 and Definition 1.1.17. It is worth mentioning that $\otimes_r$ is a particular case of $\star_r^l$, in the following sense:

$$\begin{aligned} & f \otimes_r f = f \star_r^r g(t_1, , \ldots, t_{p-r}, s_1, , \ldots, s_{q-r}) \qquad (4.1)\\ = & \int_{Z^r} \mu^r(dz_1, ..., dz_r) f(z_1, , \ldots, z_r, t_1, \ldots, t_{p-r}) \times g(z_1, , \ldots, z_r, s_1, \ldots, s_{q-r}). \end{aligned}$$

Fix integers $p, q \geq 0$ and $|q-p| \leq k \leq p+q$, and consider two kernels $f \in L_s^2(\mu^p)$ and $g \in L_s^2(\mu^q)$. We recall that the operator $G_k^{p,q}$ is defined as follows (see formula (3.11)):

$$G_k^{p,q}(f,g)(z_1,\ldots,z_k) = \sum_{r=0}^{p\wedge q}\sum_{l=0}^{r} \mathbf{1}_{(p+q-r-l=k)} r! \binom{p}{r}\binom{q}{r}\binom{r}{l} \widetilde{f \star_r^l g}, \tag{4.2}$$

where the tilde $\sim$ means symmetrization. By using this notation, the multiplication formula (1.12) becomes the following orthogonal sum (see formula (3.12)):

$$I_p(f)I_q(g) = \sum_{k=|q-p|}^{p+q} I_k(G_k^{p,q}(f,g)). \tag{4.3}$$

4.2.1 One-dimensional case: fourth moments and Universality

We recall the following theorem, which is the starting point of the "Malliavin-Stein method". (See Section 2.1.) This theorem states the simple fact: the convergence in law of a sequence of Gaussian Wiener-Itô integrals towards a normal distribution can be characterized by their variances and fourth moments.

Theorem 4.2.1 (see [39], [40], or Theorem 2.1.4 in this dissertation) *Let*

$$\{Z^{(n)} = I_q^G(h^{(n)}); n \geq 1\}$$

be a sequence of random variables belonging to the qth Gaussian Wiener-Itô chaos, for some fixed integer $q \geq 2$. *Assume that* $\mathbf{Var}(Z^{(n)}) = \mathbb{E}\left[\left(Z^{(n)}\right)^2\right] = 1$ *for all n. Then, as* $n \to \infty$, *the following three assertions are equivalent:*

- *i)* $Z^{(n)} \xrightarrow{law} \mathcal{N}(0,1)$;
- *ii)* $\mathbb{E}\left[\left(Z^{(n)}\right)^4\right] \to 3$;
- *iii)* $\forall r = 1, \ldots, q-1$, $\|h^{(n)} \tilde{\otimes}_r h^{(n)}\|_{L^2} \to 0$.

Recall that $f\tilde{\otimes}_r g$ *is the canonical symmetrization of* $f \otimes_r g$.

Now a natural question can be raised: on the Poisson space, can the convergence in law of a sequence of Wiener-Itô integrals be characterized by their variances and fourth moments? Theorem 2 in [47] gives a partial answer to this question, in the simple case of double Poisson integrals. As a Poisson counterpart of Theorem 4.2.1, the theorem below provides a satisfactory answer for homogeneous sums inside a fixed order Poisson chaos.

Theorem 4.2.2 *Let* $\{\lambda_i, i \geq 1\}$ *be a collection of positive real numbers, under the assumption:* $\inf_{i\geq 1} \lambda_i = \eta > 0$. *Let* $\boldsymbol{P} = \{P_i, i \geq 1\}$ *be a collection of independent random variables such that* $\forall i$, P_i *is a centered Poisson variable with parameter* λ_i. *Fix an integer* $q \geq 1$. *Let* $\{N^{(n)}, f^{(n)} : n \geq 1\}$ *be such that* $\{N^{(n)}; n \geq 1\}$ *is a sequence of integers going to infinity, and each* $f^{(n)} : [N^{(n)}]^q \to \mathbb{R}$ *is symmetric and vanishes on diagonals. We set*

$$F^{(n)} = \mathcal{Q}_q(N^{(n)}, f^{(n)}, \boldsymbol{P}) = \sum_{i_1,\cdots,i_q}^{N^{(n)}} f^{(n)}(i_1,\cdots,i_q)P_{i_1}\cdots P_{i_q} = I_q(h^{(n)}),$$

with

$$h^{(n)} = \sum_{i_1,\cdots,i_q}^{N^{(n)}} f^{(n)}(i_1,\cdots,i_q) g_{i_1} \otimes \cdots \otimes g_{i_q},$$

where the representation of $F^{(n)}$ as a multiple integral is the same as in Example 4.1.4. Suppose that $\mathbb{E}\left[\left(F^{(n)}\right)^2\right] \to \sigma^2$, for a given fixed constant σ. For the three statements:

- *I) $F^{(n)} \xrightarrow{law} \mathcal{N}(0,\sigma^2)$, as $n \to \infty$;*
- *II) $\mathbb{E}\left[\left(F^{(n)}\right)^4\right] \to 3\sigma^4$, as $n \to \infty$;*
- *III) $\int \left(h^{(n)}\right)^4 \to 0$ and $\forall r = 1,\cdots,q$, $\forall l = 1,\cdots, r\wedge(q-1)$, $\|h^{(n)} \star_r^l h^{(n)}\|_{L^2} \to 0$.*

One has the following implications:

- *III) implies I);*
- *II) is equivalent to III);*
- *I) implies II), whenever $\{\left(F^{(n)}\right)^4\}$ is uniformly integrable.*

The following theorem and the associated remark respond to the "Universality" question raised in Introduction. Indeed, we show that homogeneous sums inside the Poisson Wiener chaos possess the "Universality" property introduced in [36]

Theorem 4.2.3 (Universality of Poisson homogeneous sums) *Let the notations of Theorem 4.2.2 prevail. If either one of conditions II) and III) is verified, then*

$$\mathcal{Q}_q(N^{(n)}, f^{(n)}, \widetilde{\boldsymbol{G}}) = \sum_{i_1,\cdots,i_q}^{N^{(n)}} f^{(n)}(i_1,\cdots,i_q)\widetilde{G_{i_1}}\cdots\widetilde{G_{i_q}} = I_q^G(h^{(n)}) \xrightarrow{law} \mathcal{N}(0,\sigma^2),$$

with $\widetilde{\boldsymbol{G}} = \{\widetilde{G_i} : i \geq 1\}$, where $\widetilde{G_1}, \widetilde{G_2}, \cdots$ are independent Gaussian variables such that $\widetilde{G_i} \sim \mathcal{N}(0,\lambda_i)$. Here, the representation of $\mathcal{Q}_q(N^{(n)}, f^{(n)}, \widetilde{\boldsymbol{G}})$ as a multiple integral is the same as in Example 4.1.3.

Remark 4.2.4 As a consequence of Theorem 4.2.3 and Theorem 1.10 in [36], we deduce that, the Poisson Wiener chaos is *universal* in the following sense: if

$$F^{(n)} = \mathcal{Q}_q(N^{(n)}, f^{(n)}, \mathbf{P}) \xrightarrow{law} \mathcal{N}(0,\sigma^2),$$

as $n \to \infty$, and $\{\left(F^{(n)}\right)^4\}$ is uniformly integrable, then the following universal phenomenon takes place:

- For every sequence $\tilde{\mathbf{X}} = \{\tilde{X}_i; i \geq 1\}$ of independent and identically distributed centered random variables with variance $\mathbf{Var}[\tilde{X}_i] = \lambda_i$, the sequence $\{\mathcal{Q}_q(N^{(n)}, f^{(n)}, \tilde{\mathbf{X}}); n \geq 1\}$ converges in law to $\mathcal{N}(0,\sigma^2)$.

4.2.2 Multi-dimensional case

The following theorem, which is a multi-dimensional generalization of Theorem 4.2.1, is introduced in [37]. For details of this theorem and its proof, please see [37, Theorem 3.9], [39, Theorem 7], [49, Theorem 1], as well as Theorem 2.1.7 in this dissertation.

Theorem 4.2.5 (Theorem 3.9 in [37]) *Fix $d \geq 2$ and let $C = \{C(i,j) : i,j = 1,\ldots,d\}$ be a $d \times d$ positive definite matrix. Fix integers $1 \leq q_1 \leq \cdots \leq q_d$. For any $n \geq 1$ and $i = 1,\ldots,d$, let $h_i^{(n)}$ belong to $L^2(\mu^{q_i})$. Assume that*

$$F^{(n)} = (F_1^{(n)},\ldots,F_d^{(n)}) := (I_{q_1}^G(h_1^{(n)}),\ldots,I_{q_d}^G(h_d^{(n)})) \qquad n \geq 1,$$

is such that

$$\lim_{n\to\infty} \mathbb{E}[F_i^{(n)} F_j^{(n)}] = C(i,j), \quad 1 \leq i,j \leq d.$$

Then, as $n \to \infty$, the following four assertions are equivalent:

- *(i) The vector $F^{(n)}$ converges in distribution to a d-dimensional Gaussian vector $\mathcal{N}_d(0,C)$;*
- *(ii) for every $1 \leq i \leq d$, $\mathbb{E}\left[(F_i^{(n)})^4\right] \to 3C(i,i)^2$;*
- *(iii) for every $1 \leq i \leq d$ and every $1 \leq r \leq q_i - 1$, $\|h_i^{(n)} \otimes_r h_i^{(n)}\|_{L^2} \to 0$;*
- *(iv) for every $1 \leq i \leq d$, $F_i^{(n)}$ converges in distribution to a centered Gaussian random variable with variance $C(i,i)$.*

As in the one-dimensional case, we are looking for an equivalence of Theorem 4.2.5 on the Poisson space. In a recent article together with G. Peccati [50] (see Chapter 3 in this dissertation), we have found a partial answer by proving a CLT for vectors of multiple Wiener-Itô integrals. (See Theorem 5.8 in [50] or Theorem 3.4.9 in this dissertation). Now we present an actual analogue of Theorem 4.2.5 for homogeneous sums inside the Poisson Wiener chaos.

Theorem 4.2.6 *Let $\{\lambda_i, i \geq 1\}$ be a collection of positive real numbers, under the assumption $\inf_i \lambda_i = \eta > 0$. Let $\boldsymbol{P} = \{P_i, i \geq 1\}$ be a collection of independent random variables such that $\forall i$, P_i is a centered Poisson variable with parameter λ_i. Fix integers $d \geq 1, q_d \geq \cdots q_1 \geq 1$. Let $\{N_j^{(n)}, f_j^{(n)} : j = 1,\cdots,d, n \geq 1\}$ be such that for every fixed j, $\{N_j^{(n)}; n \geq 1\}$ is a sequence of integers going to infinity, and each $f_j^{(n)} : [N_j^{(n)}]^{q_j} \to \mathbb{R}$ is symmetric and vanishes on diagonals. We consider $F^{(n)} = (F_1^{(n)},\cdots,F_d^{(n)})$, where for every $1 \leq j \leq d$,*

$$F_j^{(n)} = \mathcal{Q}_{q_j}(N_j^{(n)}, f_j^{(n)}, \boldsymbol{P}) = \sum_{i_1,\cdots,i_{q_j}}^{N_j^{(n)}} f_j^{(n)}(i_1,\cdots,i_{q_j}) P_{i_1} \cdots P_{i_{q_j}} = I_{q_j}(h_j^{(n)})$$

with

$$h_j^{(n)} = \sum_{i_1,\cdots,i_{q_j}}^{N_j^{(n)}} f_j^{(n)}(i_1,\cdots,i_{q_j}) g_{i_1} \otimes \cdots \otimes g_{i_{q_j}},$$

and the representation of $F_j^{(n)}$ as a multiple integral is the same as in Example 4.1.4. Given a positive definite matrix $C = \Big(C(i,j)\Big)_{d\times d}$, suppose that $\mathbb{E}[F_i^{(n)}F_j^{(n)}] \to C(i,j)$, then, for the four statements:

- *I) $F^{(n)} \xrightarrow{law} \mathcal{N}(0,C)$, as $n\to\infty$;*
- *II) for each $j=1,\cdots,d$, $\mathbb{E}[(F_j^{(n)})^4] \to 3C(j,j)^2$, as $n\to\infty$;*
- *III) for each $j=1,\cdots,d$, $\int (h_j^{(n)})^4 \to 0$ and $\forall r=1,\cdots,q_j$, $\forall l=1,\cdots,r\wedge(q_j-1)$, $\|h_j^{(n)}\star_r^l h_j^{(n)}\|_{L^2}\to 0$;*
- *IV) for each $j=1,\cdots,d$, $F_j^{(n)} \xrightarrow{law} \mathcal{N}(0,C(j,j))$, as $n\to\infty$.*

One has the following implications:

- *II) is equivalent to III);*
- *III) implies I);*
- *I) implies IV);*
- *and IV) implies II), whenever $\{\Big(F_j^{(n)}\Big)^4 : n=1,2,\cdots\}$ is uniformly integrable for each fixed $j=1,\cdots,d$.*

Theorem 4.2.7 (Multi-dimensional Universality of Poisson homogeneous sum) *Let all the notations in the preceeding theorem prevail. If either one of condition II) and III) is true, then*

$$H^{(n)} = (H_1^{(n)},\cdots,H_d^{(n)}) \xrightarrow{law} \mathcal{N}(0,C), \qquad n\to\infty$$

where

$$H_j^{(n)} = \mathcal{Q}_{q_j}(N_j^{(n)}, f_j^{(n)}, \widetilde{\boldsymbol{G}}) = \sum_{i_1,\cdots,i_{q_j}}^{N_j^{(n)}} f_j^{(n)}(i_1,\cdots,i_{q_j})\widetilde{G_{i_1}}\cdots\widetilde{G_{i_{q_j}}} = I_{q_j}^G(h_j^{(n)}),$$

with $\widetilde{\boldsymbol{G}} = \{\widetilde{G_i} : i\geq 1\}$, with $\widetilde{G_1},\widetilde{G_2},\cdots$ independent Gaussian variables such that $\widetilde{G_i}\sim\mathcal{N}(0,\lambda_i)$. Here, the representation of $H_j^{(n)}$ as a multiple integral is the same as in Example 4.1.3.

Remark 4.2.8 As a consequence of Theorem 4.2.7 and Theorem 7.5 in [36], the vectors of homogeneous sums inside the Poisson Wiener chaos are *universal* in the following sense: if, for sequence of vectors $F^{(n)} = (F_1^{(n)},\cdots,F_d^{(n)})$, one has:

$$F^{(n)} \xrightarrow{law} \mathcal{N}(0,C),$$

as $n\to\infty$, where for every $1\leq j\leq d$,

$$F_j^{(n)} = \mathcal{Q}_{q_j}(N_j^{(n)}, f_j^{(n)}, \mathbf{P}) = \sum_{i_1,\cdots,i_{q_j}}^{N_j^{(n)}} f_j^{(n)}(i_1,\cdots,i_{q_j})P_{i_1}\cdots P_{i_{q_j}},$$

and $\{\Big(F_j^{(n)}\Big)^4 : n=1,2,\cdots\}$ is uniformly integrable for each fixed $j=1,\cdots,d$, then:

- For every sequence $\tilde{\mathbf{X}} = \{\tilde{X}_i; i \geq 1\}$ of independent centered random variables with variance $\mathbf{Var}[\tilde{X}_i] = \lambda_i$ and such that $\sup_i \mathbb{E}|\tilde{X}_i|^3 < \infty$, we have that the sequence $A^{(n)} = (A_1^{(n)}, \cdots, A_d^{(n)})$ converges in law to $\mathcal{N}(0, C)$, as $n \to \infty$. Here, for each $j = 1, \ldots, d$, $A_j^{(n)} = \mathcal{Q}_{q_j}(N_j^{(n)}, f_j^{(n)}, \tilde{\mathbf{X}})$.

4.3 Proofs

4.3.1 Technical results

We recall the ideas appearing in Example 4.1.4. Let $\mathbf{P} = \{P_i : i \geq 1\}$ be defined as above, without loss of generality, we set for every $i \geq 1$, $P_i = I_1^{\hat{N}}(g_i)$, where $\hat{N}$ indicates a compensated Poisson measure on $(Z, \mathcal{Z})$ with control μ, and $\mathbf{g} = \{g_i : i \geq 1\}$ is a collection of indicator functions in $L^2(\mu)$ with disjoint finite supports. Precisely, we take $g_i = \mathbf{1}_{A_i}$ with pairwise disjoint sets A_i on Z, such that $\mu(A_i) = \|g_i\|_{L^2}^2 = \lambda_i$. Of course,for every integer $p \geq 2$, we also know that $\|g_i\|_{L^p}^p = \lambda_i$. As in the statements of Theorem 4.2.2 and Theorem 4.2.6, we assume that

$$\inf_i \lambda_i = \eta > 0.$$

We fix $q \geq 2$. Now we consider a sequence $\{F^{(n)}; n \geq 1\}$ such that

$$F^{(n)} = \mathcal{Q}_q(N^{(n)}, f^{(n)}, \mathbf{P}) = \sum_{i_1, \cdots, i_q}^{N^{(n)}} f^{(n)}(i_1, \cdots, i_q) I_1(g_{i_1}) \cdots I_1(g_{i_q}),$$

where $\{N_j^{(n)}, f_j^{(n)} : j = 1, \cdots, d, n \geq 1\}$ is such that for every fixed j, $\{N_j^{(n)}; n \geq 1\}$ is a sequence of integers going to infinity, and each $f_j^{(n)} : [N_j^{(n)}]^{q_j} \to \mathbb{R}$ is symmetric and vanishes on diagonals. Thanks to the above discussion, each $F^{(n)}$ can be written as:

$$F^{(n)} = I_q(h^{(n)}), \qquad h^{(n)} = \sum_{i_1, \cdots, i_q}^{N^{(n)}} f^{(n)}(i_1, \cdots, i_q) g_{i_1} \otimes \cdots \otimes g_{i_q}.$$

The following proposition is the key to the proof of our results in this chapter.

Proposition 4.3.1 *We adopt the above notations and fix $q \geq 2$. If $\forall p = 1, 2, \ldots, q-1$, one has $\|h^{(n)} \star_p^p h^{(n)}\|_{L^2} \to 0$, as $n \to \infty$, then:*

- *A) $\int \left(h^{(n)}\right)^4 \to 0$, as $n \to \infty$.*

- *B) $\forall r = 1, \cdots, q$, $\forall l = 1, \cdots, r \wedge (q-1)$, $\|h^{(n)} \star_r^l h^{(n)}\|_{L^2} \to 0$, as $n \to \infty$.*

Proof. We have that, for $p = 1, 2, \ldots, q-1$

$$\begin{aligned}
h^{(n)} \star_p^p h^{(n)} &= \sum_{i_1,\cdots,i_q} \sum_{j_1,\cdots,j_q} f^{(n)}(i_1,\cdots,i_q) f^{(n)}(j_1,\cdots,j_q) \\
&\quad \times (g_{i_1} \otimes \cdots \otimes g_{i_q}) \star_p^p (g_{j_1} \otimes \cdots \otimes g_{j_q}) \\
&= \sum_{a_1,\cdots,a_p} \Big(\sum_{i_1,\cdots,i_{q-p}} \sum_{j_1,\cdots,j_{q-p}} \prod_{l=1}^{p} \|g_{a_l}\|_{L^2}^2 f^{(n)}(a_1,\cdots,a_p,i_1,\cdots,i_{q-p}) \\
&\quad \times f^{(n)}(a_1,\cdots,a_p,j_1,\cdots,j_{q-p})\, g_{i_1} \otimes \cdots \otimes g_{i_{q-p}} \otimes g_{j_1} \otimes \cdots \otimes g_{j_{q-p}} \Big) \\
&= \sum_{k_1,\cdots,k_{2q-2p}} \sum_{a_1,\cdots,a_p} f^{(n)}(a_1,\cdots,a_p,k_1,\cdots,k_{q-p}) f^{(n)}(a_1,\cdots,a_p,k_{q-p+1},\cdots,k_{2q-2p}) \\
&\quad \times \prod_{l=1}^{p} \|g_{a_l}\|_{L^2}^2 g_{k_1} \otimes \cdots \otimes g_{k_{2q-2p}} \\
&= \sum_{k_1,\cdots,k_{2q-2p}} \sum_{a_1,\cdots,a_p} f^{(n)}(a_1,\cdots,a_p,k_1,\cdots,k_{q-p}) f^{(n)}(a_1,\cdots,a_p,k_{q-p+1},\cdots,k_{2q-2p}) \\
&\quad \times \left(\prod_{l=1}^{p} \lambda_{a_l} \right) g_{k_1} \otimes \cdots \otimes g_{k_{2q-2p}} \quad ,
\end{aligned}$$

therefore

$$\begin{aligned}
\|h^{(n)} \star_p^p h^{(n)}\|_{L^2}^2 &= \sum_{k_1,\cdots,k_{2q-2p}} \Big(\sum_{a_1,\cdots,a_p} \prod_{l=1}^{p} \lambda_{a_l} f^{(n)}(a_1,\cdots,a_p,k_1,\cdots,k_{q-p}) \\
&\quad \times f^{(n)}(a_1,\cdots,a_p,k_{q-p+1},\cdots,k_{2q-2p}) \Big)^2 \prod_{m=1}^{2q-2p} \lambda_{k_m}. \qquad (4.4)
\end{aligned}$$

Firstly, we prove A).

$$\begin{aligned}
\left(h^{(n)}\right)^4 &= \sum_{i_1,\cdots,i_q} \sum_{j_1,\cdots,j_q} \sum_{k_1,\cdots,k_q} \sum_{s_1,\cdots,s_q} f^{(n)}(i_1,\cdots,i_q) f^{(n)}(j_1,\cdots,j_q) f^{(n)}(k_1,\cdots,k_q) f^{(n)}(s_1,\cdots,s_q) \\
&\quad \times (g_{i_1} \otimes \cdots \otimes g_{i_q}) \times (g_{j_1} \otimes \cdots \otimes g_{j_q}) \times (g_{k_1} \otimes \cdots \otimes g_{k_q}) \times (g_{s_1} \otimes \cdots \otimes g_{s_q}) \\
&= \sum_{i_1,\cdots,i_q} \left(f^{(n)}\right)^4 (i_1,\cdots,i_q) g_{i_1} \otimes \cdots \otimes g_{i_q},
\end{aligned}$$

therefore,

$$\begin{aligned}
\int \left(h^{(n)}\right)^4 d\mu^q &= \sum_{i_1,\cdots,i_q} \left(f^{(n)}\right)^4 (i_1,\cdots,i_q) \prod_{l=1}^{q} \|g_{i_l}\|_{L^2}^2 \\
&= \sum_{i_1,\cdots,i_q} \left(f^{(n)}\right)^4 (i_1,\cdots,i_q) \prod_{l=1}^{q} \lambda_{i_l}.
\end{aligned}$$

Formula (4.4) in the case $p = q - 1$ yields,

$$
\begin{aligned}
\|h^{(n)} \star_{q-1}^{q-1} h^{(n)}\|^2 &= \sum_{k_1,k_2} \Big(\sum_{a_1,\cdots,a_{q-1}} \prod_{l=1}^{q-1} \lambda_{a_l} f^{(n)}(a_1,\cdots,a_{q-1},k_1) \\
&\qquad \times f^{(n)}(a_1,\cdots,a_{q-1},k_2)\Big)^2 \lambda_{k_1} \times \lambda_{k_2} \\
&\geq \sum_k \Big(\sum_{a_1,\cdots,a_{q-1}} \prod_{l=1}^{q-1} \|g_{a_l}\|_{L^2}^2 \big(f^{(n)}\big)^2 (a_1,\cdots,a_{q-1},k)\Big)^2 \lambda_k^2 \\
&= \sum_{a_1,\cdots,a_{q-1}} \sum_{b_1,\cdots,b_{q-1}} \Big(\prod_{l=1}^{q-1} \lambda_{a_l}\lambda_{b_l} \\
&\quad \times \sum_k \big(f^{(n)}\big)^2(a_1,\cdots,a_{q-1},k)\big(f^{(n)}\big)^2(b_1,\cdots,b_{q-1},k)\lambda_k^2\Big) \\
&\geq \sum_{a_1,\cdots,a_{q-1}} \big(f^{(n)}\big)^4(a_1,\cdots,a_q) \prod_{l=1}^{q} \lambda_{a_l}^2 \\
&\geq \int \Big(h^{(n)}\Big)^4 d\mu^q \times \eta^q,
\end{aligned}
$$

which proves statement A), since $\eta = \inf_i \{\mu(A_i)\} > 0$.

The proof of B) consists of two steps.
B1) Let $r = q$, for any $l \in \{1,\cdots,q-1\}$, we have,

$$
\begin{aligned}
& h^{(n)} \star_q^l h^{(n)} \\
= & \sum_{i_1,\cdots,i_q} \sum_{j_1,\cdots,j_q} f^{(n)}(i_1,\cdots,i_q) f^{(n)}(j_1,\cdots,j_q)[g_{i_1} \otimes \cdots \otimes g_{i_q}] \star_r^l [g_{j_1} \otimes \cdots \otimes g_{j_q}] \\
= & \sum_{a_1,\cdots,a_l} \prod_{s=1}^{l} \lambda_{a_s} \sum_{b_1,\cdots,b_{q-l}} g_{b_1} \otimes \cdots \otimes g_{b_{q-l}} \times f^2(a_1,\cdots,a_l,b_1,\cdots,b_{q-l}) \\
= & \sum_{b_1,\cdots,b_{q-l}} g_{b_1} \otimes \cdots \otimes g_{b_{q-l}} \Big(\sum_{a_1,\cdots,a_l} \prod_{s=1}^{l} \lambda_{a_s} f^2(a_1,\cdots,a_l,b_1,\cdots,b_{q-l})\Big),
\end{aligned}
$$

which leads to

$$
\begin{aligned}
& \|h^{(n)} \star_q^l h^{(n)}\|^2 \\
= & \sum_{b_1,\cdots,b_{q-l}} \prod_{t=1}^{q-l} \lambda_{b_t} \Big(\sum_{a_1,\cdots,a_l} \prod_{s=1}^{l} \lambda_{a_s} f^2(a_1,\cdots,a_l,b_1,\cdots,b_{q-l})\Big)^2 \\
\leq & \frac{1}{\eta^{q-l}} \|h^{(n)} \star_l^l h^{(n)}\|,
\end{aligned}
$$

which yields the desired conclusion.

B2) For any $r = 1, \cdots, q-1$, and $l = 1, \cdots, r$, we see that

$$\begin{aligned} & h^{(n)} \star_r^l h^{(n)} \\ = & \sum_{a_1,\cdots,a_l} \prod_{s=1}^{l} \lambda_{a_s} \left[\sum_{b_1,\cdots,b_{r-l}} g_{b_1} \otimes \cdots \otimes g_{b_{r-l}} \right] \sum_{i_1,\cdots,i_{q-r}} \sum_{j_1,\cdots,j_{q-r}} g_{i_1} \otimes \cdots \otimes g_{i_{q-r}} \otimes g_{j_1} \otimes \cdots \otimes g_{j_{q-r}} \\ & \times f(a_1,\cdots,a_l,b_1,\cdots,b_{r-l},i_1,\cdots,i_{q-r}) f(a_1,\cdots,a_l,b_1,\cdots,b_{r-l},j_1,\cdots,j_{q-r}) \\ = & \sum_{b_1,\cdots,b_{r-l}} \sum_{i_1,\cdots,i_{q-r}} \sum_{j_1,\cdots,j_{q-r}} g_{b_1} \otimes \cdots \otimes g_{b_{r-l}} \otimes g_{i_1} \otimes \cdots \otimes g_{i_{q-r}} \otimes g_{j_1} \otimes \cdots \otimes g_{j_{q-r}} \\ & \times \sum_{a_1,\cdots,a_l} \prod_{s=1}^{l} \lambda_{a_s} f(a_1,\cdots,a_l,b_1,\cdots,b_{r-l},i_1,\cdots,i_{q-r}) f(a_1,\cdots,a_l,b_1,\cdots,b_{r-l},j_1,\cdots,j_{q-r}). \end{aligned}$$

Consequently,

$$\begin{aligned} \|h^{(n)} \star_r^l h^{(n)}\|^2 = & \sum_{b_1,\cdots,b_{r-l}} \sum_{i_1,\cdots,i_{q-r}} \sum_{j_1,\cdots,j_{q-r}} \prod_{u=1}^{r-l} \lambda_{b_u} \prod_{v=1}^{q-r} \lambda_{i_v} \lambda_{j_v} \\ \times & \left[\sum_{a_1,\cdots,a_l} \prod_{s=1}^{l} \lambda_{a_s} f(a_1,\cdots,a_l,b_1,\cdots,b_{r-l},i_1,\cdots,i_{q-r}) f(a_1,\cdots,a_l,b_1,\cdots,b_{r-l},j_1,\cdots,j_{q-r}) \right]^2 \\ \leq & \frac{1}{\eta^{r-l}} \|h^{(n)} \star_l^l h^{(n)}\|. \end{aligned}$$

which concludes the proof, as $\eta = \inf_i \{\mu(A_i)\} > 0$. ■

4.3.2 Proofs of Theorems

We give below the proof of Theorem 4.2.2.

Proof of Theorem 4.2.2.
I) to II) is a consequence of U.I. of $\{(F^{(n)})^4\}$. And III) to I) is given by Theorem 5.1 in [46] (or Theorem 2.1.10 in this dissertation). We need only to show the equivalence between II) and III).

For any function $f \in L^2(\mu^q)$, the orthogonal sum of $I_q(f)$ in formula (4.3) implies that

$$I_q(f)^2 = \sum_{k=0}^{2q} I_k\big(G_k^{q,q}(f,f)\big).$$

As a consequence, by exploiting the orthogonality of multiple integrals with different orders,

$$\begin{aligned} \mathbb{E}[I_q(f)^4] &= \sum_{k=0}^{2q} k! \|G_k^{q,q}(f,f)\|^2 \\ &= \|G_0^{q,q}(f,f)\|^2 + (2q)! \|G_{2q}^{q,q}(f,f)\|^2 + \sum_{k=1}^{2q-1} k! \|G_k^{q,q}(f,f)\|^2, \qquad (4.5) \end{aligned}$$

where

$$\|G_0^{q,q}(f,f)\|^2 = q!\|f\|^4,$$

and

$$(2q)!\|G_{2q}^{q,q}(f,f)\|^2 = (2q)!\|\widetilde{f\star_0^0 f}\|^2 = 2q!\|f\|^4 + \sum_{p=1}^{q-1}\frac{(q!)^4}{(p!(q-p)!)^2}\|f\star_p^p f\|^2, \tag{4.6}$$

according to (11.6.30) in [48].

If condition III) holds, we know that $\|h^{(n)}\star_r^l h^{(n)}\|\to 0$, as $n\to\infty$, $\forall r=1,\cdots,q$, $\forall l=1,\cdots,r\wedge(q-1)$. We take $f=h^{(n)}$ in the above relation (4.5). In view of the definition of $G_k^{a,b}$ in formula (4.2), and the fact that $\|h^{(n)}\|^2\to\sigma^2$, we deduce immediately that $\mathbb{E}[(F^{(n)})^4]\to 3\sigma^4$. So condition III) implies condition II).

On the other hand, by relation (4.5) and (4.6), $\mathbb{E}[(F^{(n)})^4]\to 3\sigma^4$ implies $\|h^{(n)}\star_p^p h^{(n)}\|\to 0$, $\forall p=1,\cdots,q-1$. By taking $f=h^{(n)}$ in Proposition 4.3.1, we have immediately that $\int(h^{(n)})^4\to 0$ and $\forall r=1,\cdots,q$, $\forall l=1,\cdots,r\wedge(q-1)$, $\|h^{(n)}\star_r^l h^{(n)}\|_{L^2}^2\to 0$. So condition III) is implied by condition II).

■

Next, we show Theorem 4.2.3.

Proof of Theorem 4.2.3.

Since

$$F^{(n)} = \sum_{i_1,\cdots,i_q}^{N^{(n)}} f^{(n)}(i_1,\cdots,i_q)P_{i_1}\cdots P_{i_q} = I_q^{\hat{N}}(h^{(n)})$$

and

$$\mathcal{Q}_q(N^{(n)},f^{(n)},\widetilde{\mathbf{G}}) = \sum_{i_1,\cdots,i_q}^{N^{(n)}} f^{(n)}(i_1,\cdots,i_q)\widetilde{G_{i_1}}\cdots\widetilde{G_{i_q}} = I_q^G(h^{(n)})$$

share the kernel

$$h^{(n)} = \sum_{i_1,\cdots,i_q} f^{(n)}(i_1,\cdots,i_q)g_{i_1}\otimes\cdots\otimes g_{i_q},$$

we need only to verify condition iii) in Theorem 4.2.1 for kernel $h^{(n)}$, in order to obtain the convergence of $\mathcal{Q}_q(N^{(n)},f^{(n)},\widetilde{\mathbf{G}})$. In view of Jensen's inequality,

$$\|h^{(n)}\tilde{\otimes}_r h^{(n)}\|_{L^2} = \|h^{(n)}\widetilde{\star_r^r}h^{(n)}\|_{L^2}\le\|h^{(n)}\star_r^r h^{(n)}\|_{L^2}.$$

condition III) implies immediately condition iii) in Theorem 4.2.1, and we conclude the proof.

■

Since we have shown Theorem 4.2.2, the proof of Theorem 4.2.6 is easy.

Proof of Theorem 4.2.6.

The equivalence of II) and III) is given by Theorem 4.2.2 above. By Theorem 5.8 in [50] (or Theorem 3.4.9 in this dissertation), III) implies I). From I) to IV) is a property of Gaussian vectors. By using again Theorem 4.2.2, we may deduce II) from VI) under the U.I. condition.

■

Finally, the proof of Theorem 4.2.7 is analogous to that of Theorem 4.2.3.

Proof of Theorem 4.2.7. By Proposition 4.3.1,

$$\|h^{(n)} \star_r^r h^{(n)}\|_{L^2} \to 0$$

implies

$$\|h^{(n)} \tilde{\otimes}_r h^{(n)}\|_{L^2} \to 0,$$

as $n \to \infty$. Since $\mathcal{Q}_{q_j}(N_j^{(n)}, f_j^{(n)}, \mathbf{P})$ and $\mathcal{Q}_{q_j}(N_j^{(n)}, f_j^{(n)}, \widetilde{\mathbf{G}})$ have the same kernel when represented as multiple integrals, if condition III) is true for $\mathcal{Q}_{q_j}(N_j^{(n)}, f_j^{(n)}, \mathbf{P})$, condition iii) in Theorem 4.2.5 also holds for $\mathcal{Q}_{q_j}(N_j^{(n)}, f_j^{(n)}, \widetilde{\mathbf{G}})$. ■

Chapter 5

Almost Sure Central Limit Theorems on the Poisson space

This chapter is based on the paper in preparation [69] by C. Zheng.

5.1 Introduction of the chapter

In a recent paper, Bercu, Nourdin and Taqqu [3] have studied Almost Sure Central Limit Theorems (ASCLTs in the sequel) for sequences of functionals of general Gaussian fields, by combining Malliavin calculus and some probabilistic estimate techniques (namely Stein-type techniques and the "smart path" technique, see Chapter 2).

The aim of the present chapter is to extend the analysis initiated in [3] to the framework of the normal approximation of regular functionals of Poisson measures defined on abstract Borel spaces. As the main result of our study, we obtain ASCLTs for functionals of Poisson measures, in both the one-dimensional and the multi-dimensional settings. Concretely, we prove ASCLTs for sequences of multiple Wiener-Itô stochastic integrals of arbitrary fixed order with respect to a general Poisson measure, as well as ASCLTs for sequences of vectors of Wiener -Itô integrals. We have established a set of conditions which are expressed in terms of the "star contraction operators", introduced in Section 2.1.3 (see also [46]) and in Chapter 3.

Definition 5.1.1 (ASCLT) *Fix an integer $m \geq 1$. Let $\{G_n\}$ be a sequence of $\mathbb{R}^m$-valued random elements converging in distribution towards a $\mathbb{R}^m$-valued random element G_∞. We say that an ASCLT holds for $\{G_n\}$, if, almost surely, for all continuous and bounded function $\varphi : \mathbb{R}^m \to \mathbb{R}$, we have that*

$$\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}\varphi(G_k) \longrightarrow \mathbb{E}[\varphi(G_\infty)], \qquad \textit{as } n \to \infty.$$

Remark 5.1.2 Let

$$S_k = \frac{1}{\sqrt{k}}\sum_{i=1}^{k} X_i, \qquad k \geq 1,$$

with $\{X_i; i \geq 1\}$ a sequence of real-valued independent identical distribution random variables, with $\mathbb{E}[X_i] = 0$, $\mathbb{E}[X_i^2] = 1$. Then, a classic result states that the sequence S_k verifies an ASCLT in the sense of Definition 5.1.1, where G_∞ is a standard Gaussian variable. In other words, the sequence of weighted random empirical measures, given by

$$\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}\delta_{S_k}$$

almost surely weakly converges to the $\mathcal{N}(0,1)$ distribution as $n \to \infty$. This ASCLT was first stated by P. Lévy in [26] without proof. It was rediscovered by Brosamler [6] and Schatte [60] half a century later. After that, Lacey and Philipp [23], and other researchers (e.g. [4, 17]) studied the ASCLT for partial sums both in the case of independent random variables and in the case of weak dependence, for instance, in the context of strong mixing or ρ-mixing. (See [18] for a survey of ASCLT.) Among these authors, Ibragimov and Lifshits [20] have provided a criterion for general ASCLTs.

We present here the important result by Ibragimov and Lifshits [20], which is the basis of our forthcoming discussion.

For $x, y \in \mathbb{R}^m$, we write $\langle x, y\rangle = x_1y_1 + \ldots + x_my_m$ (resp. $|x| = \sqrt{\langle x, x\rangle}$) to indicate the inner product of x and y (resp. the norm of x).

Theorem 5.1.3 (Ibragimov and Lifshits, [20]) *Fix an integer $m \geq 1$. Let $\{G_n\}$ be a sequence of $\mathbb{R}^m$-valued random elements converging in distribution towards a $\mathbb{R}^m$-valued random element G_∞, and set*

$$\Delta_n(t) = \frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}\Big(\exp(i\langle t, G_k\rangle) - \mathbb{E}[\exp(i\langle t, G_\infty\rangle)]\Big), \qquad t \in \mathbb{R}^m.$$

If, for all $r > 0$,

$$\sup_{|t|\leq r}\sum_{n}\frac{\mathbb{E}[|\Delta_n(t)|^2]}{n\log n} < \infty,$$

then the ASCLT holds for $\{G_n\}$.

The main technique used in this chapter is the "Malliavin-Stein" method introduced in Chapter 2 and Chapter 3. Once again, the main idea, is to assess the distance between the law of a functional F in the Poisson space and that of a normally distributed element Z:

$$\sup|\mathbb{E}[\phi(F)] - \mathbb{E}[\phi(Z)]|$$

by means of estimates involving Malliavin operators. As shown in Chapter 3, the estimation can be done either by Stein's method or by the "smart path" interpolation technique.

In the present chapter, we shall consider the particular case where $\phi(x) = \exp(itx)$ (or taking $\phi(x) = \exp(i\langle t, x\rangle)$ in the multivariate case), and then derive an upper bound (see

Proposition 5.2.5) of the type:

$$\begin{aligned} & \left|\mathbb{E}[\exp(itF)] - \exp(-t^2/2)\right| \\ \leq \; & |t|^2 \left|1 - \mathbb{E}[F^2]\right| + \frac{1}{2}|t|^4 \int_Z \mu(dz)\mathbb{E}\left[|D_zF|^2 |D_zL^{-1}F|\right] \\ + \; & \sqrt{3}|t|^2 \Big(\sqrt{\mathbb{E}\|\langle D^2F, -DL^{-1}F\rangle_{L^2(\mu)}\|^2_{L^2(\mu)}} + \sqrt{\mathbb{E}\|\langle DF, -D^2L^{-1}F\rangle_{L^2(\mu)}\|^2_{L^2(\mu)}} \\ + \; & \sqrt{\mathbb{E}\|\langle D^2F, -D^2L^{-1}F\rangle_{L^2(\mu)}\|^2_{L^2(\mu)}} \Big), \end{aligned}$$

with the help of the Poincaré inequality stated in Lemma 5.2.1. A multi-dimensional counterpart of this estimates(Proposition 5.5.1) will be developed along the same route.

Based on this assessment, we may deduce an upper bound for $\mathbb{E}|\Delta_n(t)|^2$ in Proposition 5.2.6. In the particular case of Wiener-Itô multiple integrals, one can obtain an explicit evaluation of the above upper bound, which leads to ASCLTs through Ibragimov and Lifshits's findings. Consequently, for each CLT built in [46] and in Chapter 3, we establish its corresponding ASCLT with practical and simple criteria.

The chapter is organized as follows. In Section 5.2, we apply the the "smart path" interpolation techniques in order to deduce an upper bound for the functional $\mathbb{E}|\Delta_n(t)|^2$. In Section 5.3 we develop several tools crucial for the derivation of an upper bound for $\mathbb{E}|\Delta_n(t)|^2$, in the case of Wiener-Itô multiple integrals. In Section 5.4 we find the ASCLT counterparts of the one-dimensional CLTs built in Section 2.1.3 (or in [46]), and in Section 5.5 we study multi-dimensional ASCLTs. To conclude the chapter, in Section 5.6, we revisit all the examples concerning functionals of Ornstein-Uhlenbeck Lévy processes from Chapter 3.

5.2 Estimation

In order to study the ASCLTs on the Poisson space, we shall verify the conditions in Theorem 5.1.3. In this section, we present a series of estimate techniques in the framework of Malliavin calculus leading to an explicit upper bound for $\mathbb{E}[|\Delta_n(t)|^2]$ in Proposition 5.2.6.

Lemma 5.2.1 (Poincaré inequality — Houdré, Pérez-Abreu, [19]) *Let $F \in \mathrm{Dom}D$ be a Poisson functional, then*

$$\mathbf{Var}(F) \leq \mathbb{E}\|DF\|^2_{L^2(\mu)}.$$

Lemma 5.2.2 (Nualart and Vives, Lemma 6.1 and Theorem 6.2 in [41]) *Let F,F^* be two Poisson functionals such that $F, F^*, FF^* \in \mathrm{Dom}D$. Then we have*

$$D(FF^*) = DF \times F^* + F \times DF^* + DF \times DF^*.$$

The following lemma is a direct application of Lemma 5.2.2.

Lemma 5.2.3 *Let F,F^* be two Poisson functionals such that $F, F^* \in \mathbb{D}^{2,2}$. Suppose that $\forall z \in Z$, $D_zF \times D_zF^* \in \mathrm{Dom}D$, and $\langle DF, DF^*\rangle_{L^2(\mu)} \in \mathrm{Dom}D$. Then we have*

$$D_t\langle DF, DF^*\rangle_{L^2(\mu)} = \langle DF, DD_tF^*\rangle_{L^2(\mu)} + \langle DD_tF, DF^*\rangle_{L^2(\mu)} + \langle DD_tF, DD_tF^*\rangle_{L^2(\mu)},$$

$\mathbb{P}$-a.s. , for μ-a.e. $t \in Z$.

Proof. We shall show that, $\mathbb{P}$-a.s. , for μ-a.e. $t \in Z$:

$$
\begin{aligned}
D_t\langle DF, DF^*\rangle_{L^2(\mu)} &= D_t\left(\int_Z \mu(dz) D_zF \times D_zF^*\right) \\
&\overset{(1)}{=} \Psi_t\left(\int_Z \mu(dz) D_zF \times D_zF^*\right) \\
&\overset{(2)}{=} \int_Z \mu(dz)\Psi_t(D_zF \times D_zF^*) \\
&\overset{(3)}{=} \int_Z \mu(dz)\Big(D_zF \times \Psi_t D_zF^* + \Psi_t D_zF \times D_zF^* + \Psi_t D_zF \times \Psi_t D_zF^*\Big) \\
&\overset{(4)}{=} \langle DF, \Psi_t DF^*\rangle_{L^2(\mu)} + \langle \Psi_t DF, DF^*\rangle_{L^2(\mu)} + \langle \Psi_t DF, \Psi_t DF^*\rangle_{L^2(\mu)} \\
&\overset{(5)}{=} \langle DF, D_t DF^*\rangle_{L^2(\mu)} + \langle D_t DF, DF^*\rangle_{L^2(\mu)} + \langle D_t DF, D_t DF^*\rangle_{L^2(\mu)},
\end{aligned}
$$

where $\Psi_t F = F(\omega + \delta_t) - F(\omega)$ with δ_t a Dirac operator on t. Now we justify relation (1)-(5).

Relation (1) and (5) are implied by Lemma 1.2.4. Relation (3) is obtained by using Lemma 5.2.2 and Lemma 1.2.4. In order to justify (2) and (4), we need the following facts.

By definition, $F, F^* \in \mathbb{D}^{2,2}$ imply that D_zF and D_zF^* are both elements of $L^2(\mathbb{P}; L^2(\mu))$. Without loss of generality, one can alway select versions of DF and DF^* such that

$$\{z \mapsto D_zF(\omega)\} \in L^2(\mu), \qquad \forall \omega \in \Omega,$$

$$\{z \mapsto D_zF^*(\omega')\} \in L^2(\mu), \qquad \forall \omega' \in \Omega.$$

So, for any pair $(\omega, \omega') \in \Omega \times \Omega$, we have, by Cauchy-Schwarz inequality, that

$$\{z \mapsto D_zF(\omega) \times D_zF^*(\omega')\} \in L^1(\mu). \tag{5.1}$$

To justify (2), we set $\phi(\omega, z) = D_zF(\omega) \times D_zF^*(\omega)$, Since $\phi \in \mathrm{Dom}D$, $\int_Z \phi(\omega, z)\mu(dz) \in \mathrm{Dom}D$, and $\{z \mapsto \phi(\omega, z)\} \in L^1(\mu), \forall \omega$, we have that,

$$
\begin{aligned}
&\Psi_t \int_Z \phi(\omega, z)\mu(dz) \\
&= \int_Z \phi(\omega + \delta_t, z)\mu(dz) - \int_Z \phi(\omega, z)\mu(dz) \\
&= \int_Z \Psi_t \phi(\omega, z)\mu(dz)
\end{aligned}
$$

, which justifies (2).

To justify (4), we need to show that $\forall \omega \in \Omega$, the following applications:

$$\{z \mapsto D_zF(\omega) \times \Psi_t D_zF^*(\omega)\}$$

$$\{z \mapsto \Psi_t D_zF(\omega) \times D_zF^*(\omega)\}$$

$$\{z \mapsto \Psi_t D_zF(\omega) \times \Psi_t D_zF^*(\omega)\}$$

are elements in $L^1(\mu)$.

Indeed, since $D_zF^*(\omega) \in \text{Dom}D$, by Lemma 1.2.4 and relation (5.1), we have that, for any $\omega \in \Omega$, the following application (with respect to z):

$$\begin{aligned} & D_zF(\omega) \times (\Psi_t D_zF^*(\omega)) \\ = \;& D_zF(\omega) \times (D_zF^*(\omega+\delta_t) - D_zF^*(\omega)) \\ = \;& D_zF(\omega) \times D_zF^*(\omega+\delta_t) - D_zF(\omega) \times D_zF^*(\omega)) \end{aligned}$$

belongs to $L^1(\mu)$.

The proofs that the other two elements belong to $L^1(\mu)$ is completely analogous. ■

Remark 5.2.4 Let $F = I_p(f)$ and $F^* = I_p(f^*)$ for two kernel functions $f \in L_s^2(\mu^p)$, $f^* \in L_s^2(\mu^q)$. Suppose that f, f^* satisfy Assumption A and B defined in Definition 1.1.7. We look for appropriate conditions on f and f^* in order to apply Lemma 5.2.3. We need only to verify the following two conditions: (1) $\langle DF, DF^*\rangle_{L^2(\mu)} \in \text{Dom}D$; (2) $D_zF \times D_zF^* \in \text{Dom}D$ for all $z \in Z$.

Let $f_z(\cdot)$ denote $f(z,\cdot)$. By using relation (3.12), we have,

$$\begin{aligned} \langle DF, DF^*\rangle_{L^2(\mu)} &= pq\int_Z \mu(dz) I_{p-1}(f_z) I_{q-1}(f_z^*) \\ &= pq\int_Z \mu(dz) \sum_{k=|q-p|}^{p+q-2} I_k(G_k^{p-1,q-1}(f_z, f_z^*)). \end{aligned}$$

Therefore, by conducting the same deductions shown in the proof of Proposition 3.4.6, Condition (1) is justified under the Assumption C in Definition 3.4.4 in this dissertation.

In practice, Assumption C can be replaced by simpler (but stronger) assumptions. Indeed, according to Remark 3.4.5, Assumption C is implied by the following stronger condition: for every $k = |q-p| \vee 1, \ldots, p+q-2$, and every (r,l) satisfying $p+q-2-r-l = k$, one has

$$\int_Z \left[\sqrt{\int_{Z^k} (f(z,\cdot) \star_r^l f^*(z,\cdot))^2 d\mu^k}\;\right] \mu(dz) < \infty. \tag{5.2}$$

Then, by inequality (1.9), we have

$$\begin{aligned} & \int_Z \left[\sqrt{\int_{Z^k} (f(z,\cdot) \star_r^l f^*(z,\cdot))^2 d\mu^k}\;\right] \mu(dz) \\ = \;& \int_Z \left[\sqrt{\|f_z \star_r^l f_z^*\|^2_{L^2(\mu^k)}}\;\right] \mu(dz) \\ \le \;& \int_Z \left[\sqrt{\frac{1}{2}\|f_z \star_r^l f_z\|^2_{L^2(\mu^{2p-r-l-2})} + \frac{1}{2}\|f_z^* \star_r^l f_z^*\|^2_{L^2(\mu^{2q-r-l-2})}}\;\right] \mu(dz) \\ \le \;& \frac{\sqrt{2}}{2}\left(\int_Z \|f_z \star_r^l f_z\|_{L^2(\mu^{2p-r-l-2})}\mu(dz) + \int_Z \|f_z^* \star_r^l f_z^*\|_{L^2(\mu^{2q-r-l-2})}\mu(dz)\right). \end{aligned}$$

From now on, a kernel $f \in L^2_s(\mu^p)$ is said to satisfy **Assumption D** if for any integers s,t satisfying $1 \le s \le t \le p-1$,

$$\int_Z \|f_z \star_t^s f_z\|_{L^2(\mu^{2p-s-t-2})}\mu(dz) < \infty. \tag{5.3}$$

So Assumption C is verified whenever both f and f^* satisfy Assumption D.

In order to satisfy Condition (2), we need to verify that for each $z \in Z$,

$$pqI_{p-1}(f_z)I_{q-1}(f_z^*) \in \mathrm{Dom}D.$$

We know from relation (3.12) that

$$I_{p-1}(f_z)I_{q-1}(f_z^*) = \sum_{k=|q-p|}^{p+q-2} I_k(G_k^{p-1,q-1}(f_z, f_z^*)).$$

Therefore, Condition (2) is implied by the following
Assumption E: $\forall z \in Z$, for each $k = |q-p| \vee 1, \cdots, p+q-2$,

$$\int_{Z^k} G_k^{p-1,q-1}(f(z,\cdot), f^*(z,\cdot))d\mu^k < \infty. \tag{5.4}$$

Consequently, multiple integrals satisfying Assumption D and E are eligible for Lemma 5.2.3.

To study the functional Δ_n in Theorem 5.1.3, we need the estimate below.

Proposition 5.2.5 *Let F be a Poisson functional such that $\mathbb{E}[F] = 0$, $F \in \mathbb{D}^{2,2}$, $DF \times DL^{-1}F \in \mathrm{Dom}D$ and $\langle DF, DL^{-1}F\rangle_{L^2(\mu)} \in \mathrm{Dom}D$. Then, $\forall t \in \mathbb{R}$,*

$$\begin{aligned}
&\left|\mathbb{E}[\exp(itF)] - \exp(-t^2/2)\right| \\
\le\ & |t|^2\left|1-\mathbb{E}[F^2]\right| + |t|^2\sqrt{\mathbf{Var}(W)} + \frac{1}{2}|t|^4\int_Z \mu(dz)\mathbb{E}\left[|D_zF|^2|D_zL^{-1}F|\right] \\
\le\ & |t|^2\left|1-\mathbb{E}[F^2]\right| + \frac{1}{2}|t|^4\int_Z \mu(dz)\mathbb{E}\left[|D_zF|^2|D_zL^{-1}F|\right] \\
+\ & \sqrt{3}|t|^2\Big(\sqrt{\mathbb{E}\|\langle D^2F, -DL^{-1}F\rangle_{L^2(\mu)}\|^2_{L^2(\mu)}} + \sqrt{\mathbb{E}\|\langle DF, -D^2L^{-1}F\rangle_{L^2(\mu)}\|^2_{L^2(\mu)}} \\
+\ & \sqrt{\mathbb{E}\|\langle D^2F, -D^2L^{-1}F\rangle_{L^2(\mu)}\|^2_{L^2(\mu)}}\Big),
\end{aligned}$$

where $W = \langle DF, -DL^{-1}F\rangle_{L^2(\mu)}$.

Proof. Let $\varphi(t) = \exp(t^2/2)\mathbb{E}[\exp(itF)]$, then by using the generalized chain rules Lemma 3.2.1,

$$\begin{aligned}
\varphi'(t) &= t\exp(t^2/2)\mathbb{E}[\exp(itF)] + i\exp(t^2/2)\mathbb{E}[F\exp(itF)] \\
&= t\exp(t^2/2)\mathbb{E}[\exp(itF)] - t\exp(t^2/2)\mathbb{E}[\exp(itF)\langle DF, -DL^{-1}F\rangle_{L^2(\mu)}] \\
&+ it\exp(t^2/2)\mathbb{E}[\langle R, -DL^{-1}F\rangle_{L^2(\mu)}],
\end{aligned}$$

with

$$\left|\mathbb{E}[\langle R, -DL^{-1}F\rangle_{L^2(\mu)}]\right| \leq \frac{1}{2}\|\big(\exp(itx)\big)''_{xx}\|_\infty \times \int_Z \mu(dz)\mathbb{E}\left[|D_zF|^2|D_zL^{-1}F|\right].$$

Then,

$$\begin{aligned}
|\varphi(t)-\varphi(0)| &\leq \sup_{u\in[0,t]} |\varphi'(u)| \times |t| \\
&\leq |t|^2 \exp(t^2/2)\mathbb{E}\left[|1-\langle DF, -DL^{-1}F\rangle_{L^2(\mu)}|\right] \\
&+ \frac{1}{2}|t|^2 \exp(t^2/2) \times \|\big(\exp(itx)\big)''_{xx}\|_\infty \times \int_Z \mu(dz)\mathbb{E}\left[|D_zF|^2|D_zL^{-1}F|\right] \\
&\leq |t|^2 \exp(t^2/2)\mathbb{E}\left[|1-\langle DF, -DL^{-1}F\rangle_{L^2(\mu)}|\right] \\
&+ \frac{1}{2}|t|^4 \exp(t^2/2) \int_Z \mu(dz)\mathbb{E}\left[|D_zF|^2|D_zL^{-1}F|\right].
\end{aligned}$$

What remains to us is to evaluate the upper bound of the above inequality.
Since

$$\mathbf{Var}(F) = \mathbb{E}[F^2] = \mathbb{E}[\langle DF, -DL^{-1}F\rangle_{L^2(\mu)}],$$

we apply Cauchy-Schwarz inequality, and inequality

$$\mathbf{Var}(W) \leq \mathbb{E}[\|DW\|^2_{L^2(\mu)}] \qquad \text{(by Poincaré inequality Lemma 5.2.1)},$$

then we have

$$\begin{aligned}
\mathbb{E}\left[|1-\langle DF, -DL^{-1}F\rangle_{L^2(\mu)}|\right] &\leq \mathbb{E}\left[|1-\mathbb{E}[F^2]|\right] + \sqrt{\mathbf{Var}(W)} \\
&\leq \mathbb{E}\left[|1-\mathbb{E}[F^2]|\right] + \sqrt{\mathbb{E}[\|DW\|^2_{L^2(\mu)}]}
\end{aligned}$$

By the definition of D and L^{-1}, it is easy to deduce that $F \in \mathbb{D}^{2,2}$ implies $L^{-1}F \in \mathbb{D}^{2,2}$, so all the conditions to apply Lemma 5.2.3 on F and $L^{-1}F$ are satisfied. Note that $(a+b+c)^2 \leq 3(a^2+b^2+c^2)$, $\forall a,b,c \in \mathbb{R}$, we have:

$$\begin{aligned}
\|DW\|^2_{L^2(\mu)} &= \|\langle D^2F, -DL^{-1}F\rangle_{L^2(\mu)} + \langle DF, -D^2L^{-1}F\rangle_{L^2(\mu)} + \langle D^2F, -D^2L^{-1}F\rangle_{L^2(\mu)}\|^2_{L^2(\mu)} \\
&\leq 3\|\langle D^2F, -DL^{-1}F\rangle_{L^2(\mu)}\|^2_{L^2(\mu)} + 3\|\langle DF, -D^2L^{-1}F\rangle_{L^2(\mu)}\|^2_{L^2(\mu)} \\
&+ 3\|\langle D^2F, -D^2L^{-1}F\rangle_{L^2(\mu)}\|^2_{L^2(\mu)}.
\end{aligned}$$

Finally, the conclusion is obtained by using the inequality

$$\sqrt{a^2+b^2+c^2} \leq |a|+|b|+|c|, \qquad \forall a,b,c \in \mathbb{R}.$$

■

To conclude the section, we present an explicit upper bound for $\mathbb{E}[|\Delta_n(t)|^2]$, with the help of Malliavin calculus.

Proposition 5.2.6 *Let $\{F^{(k)}; k=1,\ldots,n\}$ be a sequence of Poisson functionals such that $F^{(k)} \in \mathbb{D}^{2,2}$, $\mathbb{E}[F^{(k)}] = 0$ and $\mathbf{Var}[F^{(k)}] = 1$, for all $k=1,\ldots,n$. We also suppose that*

$DF^{(k)} \times DL^{-1}F^{(l)} \in \mathrm{Dom}D$ and $\langle DF^{(k)}, DL^{-1}F^{(l)}\rangle_{L^2(\mu)} \in \mathrm{Dom}D$ for all $k,l = 1,\dots,n$. Then, for any fixed positive real number r and $\forall t \le r$, the functional $\Delta_n(t)$ defined as

$$\Delta_n(t) = \frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}\big(\exp(itF^{(k)}) - \mathbb{E}[\exp(itZ)]\big), \qquad Z \sim \mathcal{N}(0,1)$$

satisfies:

$$\begin{aligned}
\mathbb{E}|\Delta_n(t)|^2 &\le \frac{1}{\log^2 n}\sum_{k,l=1}^{n}\frac{1}{kl}\Big(2r^2|\mathbb{E}[F^{(k)}F^{(l)}]| + 4\sqrt{3}r^2 S(F^{(k)},F^{(l)})\\
&+ 2r^4\int_Z \mu(dz)\mathbb{E}\left[|D_z(F^{(k)}-F^{(l)})|^2|D_zL^{-1}(F^{(k)}-F^{(l)})|\right]\Big)\\
&+ \frac{2L_n}{\log^2 n}\sum_{k=1}^{n}\frac{1}{k}\Big(3\sqrt{3}r^2 S(F^{(k)},F^{(k)}) + \frac{1}{2}r^4\int_Z \mu(dz)\mathbb{E}\left[|D_zF^{(k)}|^2|D_zL^{-1}F^{(k)}|\right]\Big)
\end{aligned}$$

with

$$\begin{aligned}
S(F,F^*) &= \sqrt{\mathbb{E}\|\langle D^2F, -DL^{-1}F^*\rangle_{L^2(\mu)}\|^2_{L^2(\mu)}} + \sqrt{\mathbb{E}\|\langle DF, -D^2L^{-1}F^*\rangle_{L^2(\mu)}\|^2_{L^2(\mu)}}\\
&+ \sqrt{\mathbb{E}\|\langle D^2F, -D^2L^{-1}F^*\rangle_{L^2(\mu)}\|^2_{L^2(\mu)}} \qquad (5.5)
\end{aligned}$$

and $L_n = \sum_{k=1}^{n}\frac{1}{k}$.

Proof. Given a positive real number r. We note $g(t) = \mathbb{E}[\exp(itZ)] = \exp(-t^2/2)$, $|t| \le r$. Then

$$\begin{aligned}
\mathbb{E}|\Delta_n(t)|^2 &= \frac{1}{\log^2 n}\sum_{k,l=1}^{n}\frac{1}{kl}\mathbb{E}[\big(\exp(itF^{(k)}) - g(t)\big)\big(\exp(-itF^{(l)}) - g(t)\big)]\\
&= \frac{1}{\log^2 n}\sum_{k,l=1}^{n}\frac{1}{kl}\Big(\big(\mathbb{E}\Big[\exp\big(it(F^{(k)}-F^{(l)})\big)\Big] - g^2(t)\big) - g(t)\big(\mathbb{E}[\exp(itF^{(k)})] - g(t)\big)\\
&\quad -g(t)\big(\mathbb{E}[\exp(-itF^{(l)})] - g(t)\big)\Big).
\end{aligned}$$

Now we apply Proposition 5.2.5 to Poisson functional $\frac{F^{(k)}-F^{(l)}}{\sqrt{2}}$, and we have

$$\begin{aligned}
&\left|\mathbb{E}\Big[\exp\big(it(F^{(k)}-F^{(l)})\big)\Big] - g^2(t)\right|\\
=& \left|\mathbb{E}[\exp(it\sqrt{2}\frac{F^{(k)}-F^{(l)}}{\sqrt{2}})] - g(\sqrt{2}t)\right|\\
\le& 2r^2\left|1 - \frac{1}{2}\mathbb{E}[(F^{(k)}-F^{(l)})^2]\right| + 2r^2\sqrt{\mathbf{Var}\big(\langle D(F^{(k)}-F^{(l)}), -DL^{-1}(F^{(k)}-F^{(l)})\rangle_{L^2(\mu)}\big)}\\
+& \frac{1}{2}\times 4r^4\int_Z \mu(dz)\mathbb{E}\left[|D_z(F^{(k)}-F^{(l)})|^2|D_zL^{-1}(F^{(k)}-F^{(l)})|\right].
\end{aligned}$$

Notice that

$$\left|1 - \frac{1}{2}\mathbb{E}[(F^{(k)}-F^{(l)})^2]\right| = \left|1 - \frac{1}{2}\mathbb{E}[(F^{(k)})^2] - \frac{1}{2}\mathbb{E}[(F^{(l)})^2] + \mathbb{E}[F^{(k)}F^{(l)}]\right| = |\mathbb{E}[F^{(k)}F^{(l)}]|,$$

and

$$
\begin{aligned}
& \sqrt{\mathbf{Var}\big(\langle D(F^{(k)}-F^{(l)}), -DL^{-1}(F^{(k)}-F^{(l)})\rangle_{L^2(\mu)}\big)} \\
= & \sqrt{\mathbf{Var}\big(\langle DF^{(k)}, -DL^{-1}F^{(k)}\rangle_{L^2(\mu)} - \langle DF^{(k)}, -DL^{-1}F^{(l)}\rangle_{L^2(\mu)} - \langle DF^{(l)}, -DL^{-1}F^{(k)}\rangle_{L^2(\mu)} + \langle DF^{(l)}, -DL^{-1}F^{(l)}\rangle_{L^2(\mu)}\big)} \\
\leq & \sqrt{\mathbf{Var}\big(\langle DF^{(k)}, -DL^{-1}F^{(k)}\rangle_{L^2(\mu)}\big)} + \sqrt{\mathbf{Var}\big(\langle DF^{(k)}, -DL^{-1}F^{(l)}\rangle_{L^2(\mu)}\big)} \\
+ & \sqrt{\mathbf{Var}\big(\langle DF^{(l)}, -DL^{-1}F^{(k)}\rangle_{L^2(\mu)}\big)} + \sqrt{\mathbf{Var}\big(\langle DF^{(l)}, -DL^{-1}F^{(l)}\rangle_{L^2(\mu)}\big)}.
\end{aligned}
$$

By similar arguments in the proof of Proposition 5.2.5, we get

$$
\begin{aligned}
& \Big|\mathbb{E}\Big[\exp\big(it(F^{(k)}-F^{(l)})\big)\Big] - g^2(t)\Big| \\
\leq & 2r^2|\mathbb{E}[F^{(k)}F^{(l)}]| + \frac{1}{2}\times 4r^4\int_Z \mu(dz)\mathbb{E}\Big[|D_z(F^{(k)}-F^{(l)})|^2|D_zL^{-1}(F^{(k)}-F^{(l)})|\Big] \\
+ & 2\sqrt{3}r^2\Big(S(F^{(k)},F^{(k)}) + S(F^{(k)},F^{(l)}) + S(F^{(l)},F^{(k)}) + S(F^{(l)},F^{(l)})\Big),
\end{aligned}
$$

with $S(\cdot,\cdot)$ defined in (5.5).

Then, by Proposition 5.2.5 and the fact that $|g(t)|\leq 1$,

$$
\begin{aligned}
\mathbb{E}|\Delta_n(t)|^2 \leq & \frac{1}{\log^2 n}\sum_{k,l=1}^{n}\frac{1}{kl}\Big(2r^2|\mathbb{E}[F^{(k)}F^{(l)}]| + 2\sqrt{3}r^2\big(S(F^{(k)},F^{(l)}) + S(F^{(l)},F^{(k)})\big) \\
+ & (2\sqrt{3}+\sqrt{3})r^2\big(S(F^{(k)},F^{(k)}) + S(F^{(l)},F^{(l)})\big) \\
+ & \frac{1}{2}r^4\int_Z\Big(\mathbb{E}\Big[|D_zF^{(k)}|^2|D_zL^{-1}F^{(k)}|\Big] + \mathbb{E}\Big[|D_zF^{(l)}|^2|D_zL^{-1}F^{(l)}|\Big] \\
+ & 4\mathbb{E}\Big[|D_z(F^{(k)}-F^{(l)})|^2|D_zL^{-1}(F^{(k)}-F^{(l)})|\Big]\Big)\mu(dz)\Big) \\
= & \frac{1}{\log^2 n}\sum_{k,l=1}^{n}\frac{1}{kl}\Big(2r^2|\mathbb{E}[F^{(k)}F^{(l)}]| + 4\sqrt{3}r^2S(F^{(k)},F^{(l)}) \\
+ & 2r^4\int_Z\mu(dz)\mathbb{E}\Big[|D_z(F^{(k)}-F^{(l)})|^2|D_zL^{-1}(F^{(k)}-F^{(l)})|\Big]\Big) \\
+ & \frac{2L_n}{\log^2 n}\sum_{k=1}^{n}\frac{1}{k}\Big(3\sqrt{3}r^2S(F^{(k)},F^{(k)}) + \frac{1}{2}r^4\int_Z\mu(dz)\big(\mathbb{E}\Big[|D_zF^{(k)}|^2|D_zL^{-1}F^{(k)}|\Big]\Big).
\end{aligned}
$$

■

5.3 The case of the Poisson Wiener chaos

In the previous section, we have obtained an explicit bound for $\mathbb{E}[|\Delta_n(t)|^2]$, but the explicit assessment of Malliavin operators remains a obstacle. From now on, we study the Poisson multiple stochastic integrals, for which we are able to evaluate these Malliavin functionals. We shall establish ASCLTs for them.

5.3.1 Two properties of the star contraction

We present here two properties of the star contraction. By convention, we denote by $f_u(\cdot)$ the function $f(u, \cdot)$ with a fixed parameter u.

Lemma 5.3.1 *Given integers s, t, p, q such that $0 \le s \le t \le p \wedge q$, we define $k = p + q - s - t$. Let $f \in L^2_s(\mu^{p+1})$ and $g \in L^2_s(\mu^q)$ be two symmetric functions. Then*

$$\int_Z \mu(du) \|f_u \star_t^s g\|^2_{L^2(\mu^k)} = \|f \star_t^s g\|^2_{L^2(\mu^{k+1})}.$$

Proof. Let $\mathbf{x}^n$, $\mathbf{y}^n$, $\mathbf{z}^n$, $\mathbf{w}^n$ be shorthand for $(x_1, \ldots, x_n)$, $(y_1, \ldots, y_n)$, $(z_1, \ldots, z_n)$ and $(w_1, \ldots, w_n)$ and let $d\mathbf{x}^n$, $d\mathbf{y}^n$, $d\mathbf{z}^n$, $d\mathbf{w}^n$ be shorthand for $\mu(dx_1)\mu(dx_2)\ldots\mu(dx_n)$, $\mu(dy_1)\mu(dy_2)\ldots\mu(dy_n)$, $\mu(dz_1)\mu(dz_2)\ldots\mu(dz_n)$ and $\mu(dw_1)\mu(dw_2)\ldots\mu(dw_n)$ respectively.

By the definition of the star contraction operator, we know that

$$f_u \star_t^s g(\mathbf{x}^{t-s}, \mathbf{y}^{p-t}, \mathbf{z}^{q-t}) = \int_{Z^s} d\mathbf{w}^s f_u(\mathbf{w}^s, \mathbf{x}^{t-s}, \mathbf{y}^{p-t}) g(\mathbf{w}^s, \mathbf{x}^{t-s}, \mathbf{z}^{q-t}).$$

Then we have,

$$\begin{aligned}
& \int_Z \mu(du) ||f_u \star_t^s g||^2_{L^2(\mu^k)} \\
= & \int_Z \mu(du) \int_{Z^k} d\mathbf{x}^{t-s} d\mathbf{y}^{p-t} d\mathbf{z}^{q-t} \int_{Z^s} d\mathbf{w}^s \int_{Z^s} d\mathbf{w}'^s f_u(\mathbf{w}^s, \mathbf{x}^{t-s}, \mathbf{y}^{p-t}) g(\mathbf{w}^s, \mathbf{x}^{t-s}, \mathbf{z}^{q-t}) \\
& \times f_u(\mathbf{w}'^s, \mathbf{x}^{t-s}, \mathbf{y}^{p-t}) g(\mathbf{w}'^s, \mathbf{x}^{t-s}, \mathbf{z}^{q-t}) \\
= & \int_{Z^{k+1}} d\mathbf{x}^{t-s} d\bar{\mathbf{y}}^{p+1-t} d\mathbf{z}^{q-t} \int_{Z^s} d\mathbf{w}^s \int_{Z^s} d\mathbf{w}'^s f(\mathbf{w}^s, \mathbf{x}^{t-s}, \bar{\mathbf{y}}^{p+1-t}) g(\mathbf{w}^s, \mathbf{x}^{t-s}, \mathbf{z}^{q-t}) \\
& \times f(\mathbf{w}'^s, \mathbf{x}^{t-s}, \bar{\mathbf{y}}^{p+1-t}) g(\mathbf{w}'^s, \mathbf{x}^{t-s}, \mathbf{z}^{q-t}) \\
= & \|f \star_t^s g\|^2_{L^2(\mu^{k+1})}.
\end{aligned}$$

In the preceding calculations, the notation $\bar{\mathbf{y}}^{p+1-t}$ is defined by

$$\bar{\mathbf{y}}^{p+1-t} = (u, \mathbf{y}^{p-t}) = (u, y_1, \ldots, y_{p-t}).$$

■

By the same deduction, one also proves the following statement.

Lemma 5.3.2 *Given integers s, t, p, q such that $0 \le s \le t \le p \wedge q$, we define $k = p + q - s - t$. Let $f \in L^2_s(\mu^{p+1})$ and $g \in L^2_s(\mu^{q+1})$ be two symmetric functions. Then*

$$\int_Z \mu(du) \|f_u \star_t^s g_u\|^2_{L^2(\mu^k)} = \|f \star_{t+1}^s g\|^2_{L^2(\mu^{k+1})}.$$

5.3.2 Upper bounds

While studying Poisson multiple stochastic integrals, we are able to provide deterministic upper bounds to replace the Malliavin calculus involved in the estimation in Proposition 5.2.6.

ATTENTION: For the rest of this chapter, we suppose that all the kernel functions $f \in L_s^2(\mu^p)$ satisfy Assumption A, B, D, E.

We shall prove three technical lemmas.

Lemma 5.3.3 *Fix two integers $p, q \geq 2$. Let $F = I_p(f)$, $F^* = I_q(f^*)$, with $f \in L_s^2(\mu^p)$ and $f^* \in L_s^2(\mu^q)$. Then*

$$\begin{aligned} &\mathbb{E}\|\langle D^2F, -DL^{-1}F^*\rangle_{L^2(\mu)}\|^2_{L^2(\mu)} \\ \leq\ & p^2(p-1)^2 \sum_{k=|q-p+1|}^{p+q-3} \sum_{t=1}^{(p-1)\wedge q} C(p-1,q,k,t)\mathbf{1}_{1\leq s(t,k)\leq t}\|f \star_t^{s(t,k)} f^*\|^2_{L^2(\mu^{k+1})}, \end{aligned}$$

with $s(t,k) = p+q-1-k-t$ and $C(p-1,q,k,t)$ given by Lemma 3.4.2 or Remark 3.4.3 in this dissertation.

Proof.

$$D^2_{u,z}F = p(p-1)I_{p-2}\big(f(u,z,\cdot)\big) \quad , \qquad -D_zL^{-1}F^* = I_{q-1}\big(f^*(z,\cdot)\big).$$

Then, by the "product formula" equation (1.12), we have

$$\begin{aligned} &\langle D^2F, -DL^{-1}F^*\rangle_{L^2(\mu)}(u) \\ =\ & \int_Z \mu(dz)p(p-1)I_{p-2}\big(f(u,z,\cdot)\big) \times I_{q-1}\big(f^*(z,\cdot)\big) \\ =\ & p(p-1)\int_Z \mu(dz) \sum_{k=|q-p+1|}^{p+q-3} I_k\left(G_k^{p-2,q-1}\big(f_u(z,\cdot), f^*(z,\cdot)\big)\right). \end{aligned} \tag{5.6}$$

The above expression (5.6) is similar to that in the proof of Proposition 3.4.6, so we need the following assumption (analogous to Assumption C): for every $k = |q-p| \vee 1, \dots, p+q-3$,

$$\int_Z \left[\sqrt{\int_{Z^k} (G_k^{p-2,q-1}(f(z,\cdot), g(z,\cdot)))^2 d\mu^k}\ \right]\mu(dz) < \infty. \tag{5.7}$$

As shown in Remark 5.2.4, Assumption D is stronger than the above assumption, thus permits us to do the forthcoming calculations.

We know that line (5.6) equals

$$p(p-1)\sum_{k=|q-p+1|}^{p+q-3} I_k\left(\widehat{G_k^{p-1,q}}(f_u, f^*)\right).$$

Consequently, by Lemma 5.3.1,

$$\begin{aligned}
&\mathbb{E}\|\langle D^2F, -DL^{-1}F^*\rangle_{L^2(\mu)}\|^2_{L^2(\mu)} \\
= \ & p^2(p-1)^2 \int_Z \mu(dz) \sum_{k=|q-p+1|}^{p+q-3} \mathbb{E}\left[I_k^2\Big(\widehat{G_k^{p-1,q}}(f_u, f^*)\Big)\right] \\
= \ & p^2(p-1)^2 \sum_{k=|q-p+1|}^{p+q-3} \int_Z \mu(dz) \left\| \Big(\widehat{G_k^{p-1,q}}(f_u, f^*)\Big)\right\|^2_{L^2(\mu^k)} \\
\leq \ & p^2(p-1)^2 \sum_{k=|q-p+1|}^{p+q-3} \int_Z \mu(dz) \sum_{t=1}^{(p-1)\wedge q} C(p-1,q,k,t)\mathbf{1}_{1\leq s(t,k)\leq t}\|f_u \star_t^{s(t,k)} f^*\|^2_{L^2(\mu^k)} \\
\leq \ & p^2(p-1)^2 \sum_{k=|q-p+1|}^{p+q-3} \sum_{t=1}^{(p-1)\wedge q} C(p-1,q,k,t)\mathbf{1}_{1\leq s(t,k)\leq t}\|f \star_t^{s(t,k)} f^*\|^2_{L^2(\mu^{k+1})},
\end{aligned}$$

where $s(t,k) = p+q-1-k-t$. ■

Remark 5.3.4 Note that, the upper bounds appeared in Lemma 5.3.3 and in Lemma 5.3.5, Lemma 5.3.6 below can be infinite. But in the forthcoming discussions of almost sure central limit theorems, we will only deal with the cases with finite bounds.

Lemma 5.3.5 *Fix two integers $p, q \geq 2$. Let $F = I_p(f)$, $F^* = I_q(f^*)$, with $f \in L_s^2(\mu^p)$ and $f^* \in L_s^2(\mu^q)$. Then*

$$\begin{aligned}
&\mathbb{E}\|\langle DF, -D^2L^{-1}F^*\rangle_{L^2(\mu)}\|^2_{L^2(\mu)} \\
\leq \ & p^2(q-1)^2 \sum_{k=|p-q+1|}^{p+q-3} \sum_{t=1}^{p\wedge(q-1)} C(p,q-1,k,t)\mathbf{1}_{1\leq s(t,k)\leq t}\|f \star_t^{s(t,k)} f^*\|^2_{L^2(\mu^{k+1})},
\end{aligned}$$

with $s(t,k) = p+q-1-k-t$ and $C(p,q-1,k,t)$ defined in Lemma 3.4.2 or Remark 3.4.3 in this dissertation.

Proof. Under Assumption A, B, D, E on function f and f^*, we are allowed to do the following calculations. Since

$$D_zF = pI_{p-1}\big(f(z,\cdot)\big) \quad , \qquad -D^2_{u,z}L^{-1}F^* = (q-1)I_{q-2}\big(f^*(u,z,\cdot)\big),$$

then, by the "product formula" equation (1.12), we have

$$\begin{aligned}
&\langle DF, -D^2L^{-1}F^*\rangle_{L^2(\mu)}(u) \\
= \ & \int_Z \mu(dz)pI_{p-1}\big(f(z,\cdot)\big) \times (q-1)I_{q-2}\big(f^*(u,z,\cdot)\big) \\
= \ & p(q-1)\int_Z \mu(dz) \sum_{k=|p-q+1|}^{p+q-3} I_k\left(G_k^{p-1,q-2}(f(z,\cdot), f_u^*(z,\cdot))\right) \\
= \ & p(q-1) \sum_{k=|p-q+1|}^{p+q-3} I_k\left(\widehat{G_k^{p,q-1}}(f, f_u^*)\right).
\end{aligned}$$

Similarly, by Lemma 5.3.1,

$$\begin{aligned}
&\mathbb{E}\|\langle DF, -D^2L^{-1}F^*\rangle_{L^2(\mu)}\|^2_{L^2(\mu)} \\
= \;& p^2(q-1)^2 \int_Z \mu(dz) \sum_{k=|p-q+1|}^{p+q-3} \mathbb{E}\left[I_k^2\Big(\widehat{G_k^{p,q-1}}(f, f_u^*)\Big)\right] \\
= \;& p^2(q-1)^2 \sum_{k=|p-q+1|}^{p+q-3} \int_Z \mu(dz) \left\| \Big(\widehat{G_k^{p,q-1}}(f, f_u^*)\Big) \right\|^2_{L^2(\mu^k)} \\
\le \;& p^2(q-1)^2 \sum_{k=|p-q+1|}^{p+q-3} \int_Z \mu(dz) \sum_{t=1}^{p\wedge(q-1)} C(p, q-1, k, t) \mathbf{1}_{1\le s(t,k)\le t} \|f \star_t^{s(t,k)} f_u^*\|^2_{L^2(\mu^k)} \\
\le \;& p^2(q-1)^2 \sum_{k=|p-q+1|}^{p+q-3} \sum_{t=1}^{p\wedge(q-1)} C(p, q-1, k, t) \mathbf{1}_{1\le s(t,k)\le t} \|f \star_t^{s(t,k)} f^*\|^2_{L^2(\mu^{k+1})},
\end{aligned}$$

where $s(t,k) = p + q - 1 - k - t$. ■

Lemma 5.3.6 *Fix two integers $p, q \ge 2$. Let $F = I_p(f)$, $F^* = I_q(f^*)$, with $f \in L_s^2(\mu^p)$ and $f^* \in L_s^2(\mu^q)$. Then*

$$\begin{aligned}
&\mathbb{E}\|\langle D^2F, -D^2L^{-1}F^*\rangle_{L^2(\mu)}\|^2_{L^2(\mu)} \\
\le \;& p^2(p-1)^2(q-1)^2 \sum_{k=|p-q|}^{p+q-4} \sum_{t=1}^{(p-1)\wedge(q-1)} C(p-1, q-1, k, t) \mathbf{1}_{1\le s(t,k)\le t} \|f \star_{t+1}^{s(t,k)} f^*\|^2_{L^2(\mu^{k+1})},
\end{aligned}$$

with $s(t,k) = p + q - 2 - k - t$ and $C(p-1, q-1, k, t)$ defined in Lemma 3.4.2 or Remark 3.4.3 in this dissertation.

Proof. Under Assumption A, B, D, E on function f and f^*, we are allowed to do the following calculations. Since

$$D^2_{u,z}F = p(p-1)I_{p-2}\big(f(u, z, \cdot)\big) \quad , \qquad -D^2_{u,z}L^{-1}F^* = (q-1)I_{q-2}\big(f^*(u, z, \cdot)\big),$$

then, by the "product formula" equation (1.12), we have

$$\begin{aligned}
&\langle D^2F, -D^2L^{-1}F^*\rangle_{L^2(\mu)}(u) \\
= \;& \int_Z \mu(dz) p(p-1) I_{p-2}\big(f(u, z, \cdot)\big) \times (q-1) I_{q-2}\big(f^*(u, z, \cdot)\big) \\
= \;& p(p-1)(q-1) \int_Z \mu(dz) \sum_{k=|p-q|}^{p+q-4} I_k\left(G_k^{p-2,q-2}\big(f_u(z, \cdot), f_u^*(z, \cdot)\big)\right) \\
= \;& p(p-1)(q-1) \sum_{k=|p-q|}^{p+q-4} I_k\left(\widehat{G_k^{p-1,q-1}}(f_u, f_u^*)\right).
\end{aligned}$$

In consequence, by Lemma 5.3.2,

$$\begin{aligned}
&\mathbb{E}\|\langle D^2F, -D^2L^{-1}F^*\rangle_{L^2(\mu)}\|^2_{L^2(\mu)}\\
= \;& p^2(p-1)^2(q-1)^2 \int_Z \mu(dz) \sum_{k=|p-q|}^{p+q-4} \mathbb{E}\left[I_k^2\left(\widehat{G_k^{p-1,q-1}}(f_u, f_u^*)\right)\right]\\
= \;& p^2(p-1)^2(q-1)^2 \sum_{k=|p-q|}^{p+q-4} \int_Z \mu(dz) \left\|\widehat{G_k^{p-1,q-1}}(f_u, f_u^*)\right\|^2_{L^2(\mu^k)}\\
\leq \;& p^2(p-1)^2(q-1)^2 \sum_{k=|p-q|}^{p+q-4} \int_Z \mu(dz) \sum_{t=1}^{(p-1)\wedge(q-1)} C(p-1,q-1,k,t)\mathbf{1}_{1\leq s(t,k)\leq t}\|f_u \star_t^{s(t,k)} f_u^*\|^2_{L^2(\mu^k)}\\
\leq \;& p^2(p-1)^2(q-1)^2 \sum_{k=|p-q|}^{p+q-4} \sum_{t=1}^{(p-1)\wedge(q-1)} C(p-1,q-1,k,t)\mathbf{1}_{1\leq s(t,k)\leq t}\|f \star_{t+1}^{s(t,k)} f^*\|^2_{L^2(\mu^{k+1})},
\end{aligned}$$

where $s(t,k) = p+q-2-k-t$. ■

From these three results we obtain immediately the following lemma:

Lemma 5.3.7 *Fix two integers $p, q \geq 2$. Let $F = I_p(f)$, $F^* = I_q(f^*)$, with $f \in L_s^2(\mu^p)$ and $f^* \in L_s^2(\mu^q)$. Then there exist nonnegative constants $c_{s,t}, 0 \leq s \leq t \leq p\wedge q$, such that*

$$\begin{aligned}
S(F,F^*) &\leq \sum_{t=1}^{p\wedge q}\sum_{s=0}^{t} c_{s,t}\|f \star_t^s f^*\|_{L^2(\mu^{p+q-t-s})}\\
&\leq \frac{\sqrt{2}}{2}\sum_{t=1}^{p\wedge q}\sum_{s=0}^{t} c_{s,t}(\|f \star_t^s f\|_{L^2(\mu^{2p-t-s})} + \|f^* \star_t^s f^*\|_{L^2(\mu^{2q-t-s})}).
\end{aligned}$$

5.4 ASCLT

Using the explicit estimates of the precedent sections, we are now able to deduce ASCLTs for sequences of random variables with the form of Poisson multiple integrals, by providing adequate conditions.

5.4.1 Integrals of order one

We note $\|f\|_{L^3(\mu)} = (\int_Z \mu(dz)|f(z)|^3)^{1/3}$. We now consider a sequence of random variables $\{F^{(k)}; k = 1, 2, \ldots\}$ such that $F^{(k)} = I_1(f^{(k)})$, where $f^{(k)}(\cdot) \in L_s^2(\mu), k = 1, 2, \ldots$ are functions in one variable.

The following theorem gives an ASCLT for Poisson multiple stochastic integrals of fixed order $q = 1$, by providing a criterion on the kernels of these integrals.

Theorem 5.4.1 *Let $\{F^{(n)}\}$ be a sequence of random variables of the form $F^{(n)} = I_1(f^{(n)})$, with $f^{(n)} \in L_s^2(\mu) \cap L_s^3(\mu)$. Assume that*

$$\mathbf{Var}[F^{(n)}] = \mathbb{E}[(F^{(n)})^2] = \|f^{(n)}\|^2_{L^2(\mu)} = 1,$$

and that

$$\|f^{(n)}\|^3_{L^3(\mu)} \to 0 \qquad as \quad n \to \infty.$$

Then, $F^{(n)} \xrightarrow{law} N \sim \mathcal{N}(0,1)$ *as* $n \to \infty$*. Moreover, if the two conditions (A1) and (A2) below are satisfied :*

- *(A1)*

$$\sum_{n\geq 2} \frac{1}{n\log^3 n} \sum_{k,l=1}^{n} \frac{|\mathbb{E}[F^{(k)}F^{(l)}]|}{kl} = \sum_{n\geq 2} \frac{1}{n\log^3 n} \sum_{k,l=1}^{n} \frac{|\langle f^{(k)}, f^{(l)}\rangle_{L^2(\mu)}|}{kl} < \infty;$$

- *(A2)*

$$\sum_{n\geq 2} \frac{1}{n\log^2 n} \sum_{k=1}^{n} \frac{1}{k} \|f^{(k)}\|^3_{L^3(\mu)} < \infty,$$

then $\{F^{(n)}\}$ *satisfies an ASCLT. In other words, almost surely, for all continuous and bounded function* $\varphi : \mathbb{R} \to \mathbb{R}$*, we have*

$$\frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \varphi(F^{(k)}) \longrightarrow \mathbb{E}[\varphi(N)], \qquad as\ n \to \infty.$$

Proof. The CLT part:

$$F^{(n)} \xrightarrow{law} N$$

follows immediately from Corollary 3.4 in [46] . In order to apply Theorem 5.1.3, we use an argument similar to that in the proof of Proposition 5.2.6.
We fix $r > 0$, for every t such that $|t| \leq r$,

$$\begin{aligned} & \left|\mathbb{E}\Big[\exp\big(it(F^{(k)} - F^{(l)})\big)\Big] - g^2(t)\right| \\ = \ & 2r^2|\mathbb{E}[F^{(k)}F^{(l)}]| + 2r^4 \int_Z \mu(dz)|f^{(k)}(z) - f^{(l)}(z)|^3 \\ \leq \ & 2r^2|\mathbb{E}[F^{(k)}F^{(l)}]| + 8r^4(\|f^{(k)}\|^3_{L^3(\mu)} + \|f^{(l)}\|^3_{L^3(\mu)}). \end{aligned}$$

On the other hand, by Proposition 5.2.5, we know that

$$|\mathbb{E}[\exp(F^{(k)})] - g(t)| \leq \frac{1}{2} r^4 \int_Z \mu(dz)|f^{(k)}(z)|^3 = \frac{1}{2} r^4 \|f^{(k)}(z)\|^3_{L^3(\mu)}$$

Therefore,

$$\begin{aligned} \mathbb{E}[|\Delta_n(t)|^2] \ \leq \ & \frac{1}{\log^2 n} \sum_{k,l=1}^{n} \frac{1}{kl} \Big(2r^2|\mathbb{E}[F^{(k)}F^{(l)}]| \\ & + \frac{17}{2} r^4 (\|f^{(k)}\|^3_{L^3(\mu)} + \|f^{(l)}\|^3_{L^3(\mu)}) \Big). \end{aligned}$$

This shows that the condition of Theorem 5.1.3 are verified when conditions (A1) and (A2) in the statement are satisfied. ■

Remark 5.4.2 The following conditions are sufficient conditions for (A1) and (A2) respectively:

- (A1') $\exists \alpha > 0, C > 0$, such that

$$|\mathbb{E}[F^{(l)}F^{(k)}]| \leq C\left(\frac{l}{k}\right)^{\alpha}$$

  for all $1 \leq l \leq k$;

- (A2') $\exists \beta > 0$, such that

$$\|f^{(k)}\|^3_{L^3(\mu)} = O(k^{-\beta}), \qquad k \to \infty.$$

(A2')⇒(A2):
By standard calculation, we know that there exists some constant C_0, such that

$$\sum_{k=1}^{n} \frac{1}{k}\|f^{(k)}\|^3_{L^3(\mu)} < C_0$$

for all $n \in \mathbb{N}$.
Therefore,

$$\sum_{n\geq 2} \frac{1}{n\log^2 n}\sum_{k=1}^{n}\frac{1}{k}\|f^{(k)}\|^3_{L^3(\mu)} < C_0 \sum_{n\geq 2}\frac{1}{n\log^2 n} < \infty.$$

(A1')⇒(A1):
For some constants C_1 and C_2, we have

$$\begin{aligned}
\sum_{n\geq 2}\frac{1}{n\log^3 n}\sum_{k,l=1}^{n}\frac{|\mathbb{E}[F^{(l)}F^{(k)}]|}{kl} &\leq C\sum_{n\geq 2}\frac{1}{n\log^3 n}\sum_{k=1}^{n}k^{-\alpha-1}\sum_{l=1}^{k}l^{\alpha-1} \\
&\leq C_1\sum_{n\geq 2}\frac{1}{n\log^3 n}\sum_{k=1}^{n}\frac{1}{k} \\
&\leq C_2\sum_{n\geq 2}\frac{1}{n\log^3 n}\times \log n \\
&< \infty.
\end{aligned}$$

We point out that Condition (A1') and (A2') are easier to verify in practice.

We present here an example of application of the Theorem 5.4.1. Let $\hat{N}$ be a centered Poisson measure on $\mathbb{R}$.

Proposition 5.4.3 *Let $\{F^{(n)}, n = 1, 2, \ldots\}$ be a sequence of random variables on the Poisson space, defined by*

$$F^{(n)} = I_1(h_n) = \int_{\mathbb{R}} h_n(x)d\hat{N}(x)$$

where $h_n = n^{-1/2}\mathbf{1}_{[0,n]}$ is an element of the first Wiener chaos associated with a centered Poisson measure $\hat{N}$. Then $F^{(n)} \stackrel{law}{\longrightarrow} N \sim \mathcal{N}(0,1)$, as $n \to \infty$. Moreover, $\{F^{(n)}\}$ satisfies an

ASCLT. In other words, almost surely, for all continuous and bounded function $\varphi : \mathbb{R} \to \mathbb{R}$, *we have*

$$\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}\varphi(F^{(k)}) \longrightarrow \mathbb{E}[\varphi(N)], \qquad \text{as } n \to \infty.$$

Proof. The fact $F^{(n)} \xrightarrow{law} N$ comes from a simple application of the standard CLT.

As an application of Theorem 5.4.1 and Remark 5.4.2, we now verify Condition (A1') and (A2').

Firstly, we notice that, for positive integers $1 \le l \le k$,

$$\mathbb{E}[F^{(l)}F^{(k)}] = \int_0^l \frac{1}{\sqrt{kl}}dx = \sqrt{\frac{l}{k}}.$$

On the other hand, we have

$$\|h_k\|_{L^3}^3 = \frac{1}{\sqrt{k}}.$$

As shown in Remark 5.4.2, both (A1') and (A2') are satisfied, which leads to ASCLT. ■

5.4.2 Multiple integrals of order $q \ge 2$

For a fixed integer $q \ge 2$, we consider a sequence of random variables $\{F^{(k)}, k = 1, 2, \ldots\}$ such that $F^{(k)} = I_q(f^{(k)})$, where $f^{(k)}(\cdot) \in L_s^2(\mu^q), k = 1, 2, \ldots$ are kernel functions with q variables.

Theorem 5.4.4 *We fix an integer* $q \ge 2$. *Let* $\{F^{(k)}\}$ *be a sequence of the form* $F^{(k)} = I_q(f^{(k)})$, *with* $f^{(k)} \in L_s^2(\mu^q)$. *Assume that*

$$\mathbb{E}[(F^{(k)})^2] = q!\|f^{(k)}\|_{L^2(\mu)}^2 = 1 \quad \text{for} \quad k = 1, 2, \ldots,$$

and that the following conditions hold:

- *for every* $k \ge 1$, *the kernel* $f^{(k)}$ *verifies Assumption A, B, D, E for every* $p = 1, \ldots, 2(q-1)$;
- *for every* $r = 1, \ldots, q$ *and every* $l = 1, \ldots, r \wedge (q-1)$, *one has that* $\|f^{(k)} \star_r^l f^{(k)}\|_{L^2(\mu^{2q-r-l})} \to 0$ *as* $k \to \infty$;
- $\int_{Z^q} (f^{(k)})^4 d\mu^q \to 0$ *as* $k \to \infty$,

then, $F^{(n)} \xrightarrow{law} N \sim \mathcal{N}(0,1)$ *as* $n \to \infty$. *Moreover, if the three conditions (B1), (B2) and (B3) below are satisfied :*

- *(B1)*

$$\sum_{n\ge2}\frac{1}{n\log^3 n}\sum_{k,l=1}^{n}\frac{|\mathbb{E}[F^{(k)}F^{(l)}]|}{kl} = \sum_{n\ge2}\frac{1}{n\log^3 n}\sum_{k,l=1}^{n}\frac{|\langle f^{(k)}, f^{(l)}\rangle_{L^2(\mu^n)}|}{kl} < \infty;$$

- *(B2)*
$$\sum_{n\geq 2}\frac{1}{n\log^2 n}\sum_{k=1}^{n}\frac{1}{k}\|f^{(k)}\star_r^l f^{(k)}\|_{L^2(\mu^{2q-r-l})}<\infty$$
for every $r=1,\ldots,q,\quad l=1,\ldots,r\wedge(q-1)$*;*

- *(B3)*
$$\sum_{n\geq 2}\frac{1}{n\log^2 n}\sum_{k=1}^{n}\frac{1}{k}\int_{Z^q}(f^{(k)})^4 d\mu^q<\infty,$$

then $\{F^{(n)}\}$ *satisfies an ASCLT. In other words, almost surely, for all continuous and bounded function* $\varphi:\mathbb{R}\to\mathbb{R}$*, we have*

$$\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}\varphi(F^{(k)})\longrightarrow\mathbb{E}[\varphi(N)],\qquad \text{as } n\to\infty.$$

Remark 5.4.5 Note that
$$\|f\star_q^0 f\|_{L^2(\mu^q)}=\sqrt{\int_{Z^q}f^4 d\mu^q}.$$
Condition (A3) can be seen as a complement to Condition (A2) for $r=q, l=0$.

Remark 5.4.6 We may use the following sufficient conditions to replace (B1), (B2) and (B3):

- (B1') $\exists\alpha>0, C>0$, such that
$$|\mathbb{E}[F^{(l)}F^{(k)}]|\leq C\left(\frac{l}{k}\right)^{\alpha}$$
for all $1\leq l\leq k$;

- (B2') $\exists\beta>0$, such that
$$\|f^{(k)}\star_r^l f^{(k)}\|_{L^2(\mu^{2q-r-l})}=O(k^{-\beta}),\qquad k\to\infty$$
for every $r=1,\ldots,q,\quad l=1,\ldots,r\wedge(q-1)$;

- (B3') $\exists\gamma>0$, such that
$$\|f^{(k)}\|_{L^4(\mu^q)}^4=\int_{Z^q}(f^{(k)})^4 d\mu^q=O(k^{-\gamma}),\qquad k\to\infty.$$

See Remark 5.4.2 for details.

Proof. The CLT part of Theorem 5.4.4 is an application of Theorem 2.1.10 (or Theorem 5.1 in [46]).

In order to show the ASCLT, we apply Proposition 5.2.6 and Theorem 5.1.3.

According to Proposition 5.2.6,

$$\mathbb{E}|\Delta_n(t)|^2 \leq \frac{1}{\log^2 n}\sum_{k,l=1}^{n}\frac{1}{kl}\times 2r^2|\mathbb{E}[F^{(k)}F^{(l)}]| \tag{5.8}$$

$$+ \quad \frac{1}{\log^2 n}\sum_{k,l=1}^{n}\frac{1}{kl}\times 4\sqrt{3}r^2 S(F^{(k)},F^{(l)}) + \frac{2L_n}{\log^2 n}\sum_{k=1}^{n}\frac{1}{k}\times 3\sqrt{3}r^2 S(F^{(k)},F^{(k)}) \tag{5.9}$$

$$+ \quad \frac{1}{\log^2 n}\sum_{k,l=1}^{n}\frac{1}{kl}\times 2r^4\int_Z \mu(dz)\mathbb{E}\left[|D_z(F^{(k)}-F^{(l)})|^2|D_zL^{-1}(F^{(k)}-F^{(l)})|\right] \tag{5.10}$$

$$+ \quad \frac{2L_n}{\log^2 n}\sum_{k=1}^{n}\frac{1}{k}\times\frac{1}{2}r^4\int_Z \mu(dz)\mathbb{E}\left[|D_zF^{(k)}|^2|D_zL^{-1}F^{(k)}|\right] \tag{5.11}$$

with $L_n = \sum_{k=1}^n 1/k$.
So, the RHS of line (5.8) is finite whenever Condition (B1) is satisfied.

As for line (5.9), we know that, by virtue of Lemma 5.3.7, for every $k \geq 1$,

$$S(F^{(k)},F^{(k)}) \quad \leq \quad \sum_{t=1}^{q}\sum_{s=0}^{t} c_{s,t}\|f^{(k)}\star_t^s f^{(k)}\|_{L^2(\mu^{2q-t-s})}$$

with $c_{s,t}$ nonnegative constants, $0 \leq s \leq t \leq q$. On the other hand, by Lemma 5.3.7, for every $k,l \geq 1, k \neq l$, we have

$$S(F^{(k)},F^{(l)}) \leq \frac{\sqrt{2}}{2}\sum_{t=1}^{q}\sum_{s=0}^{t} c_{s,t}(\|f^{(k)}\star_t^s f^{(k)}\|_{L^2(\mu^{2q-t-s})} + \|f^{(l)}\star_t^s f^{(l)}\|_{L^2(\mu^{2q-t-s})})$$

with $c_{s,t}$ nonnegative constants, $0 \leq s \leq t \leq q$. Since $L_n \sim \log n$, the finiteness of line (5.9) is a consequence of Condition (B2) and (B3).

For the remaining lines, we take $F^{(k)} = I_q(f^{(k)})$ and $F^{(l)} = I_q(f^{(l)})$, then

$$\begin{aligned} & \mathbb{E}\left[|D_z(F^{(k)}-F^{(l)})|^2|D_zL^{-1}(F^{(k)}-F^{(l)})|\right] \\ = \quad & \frac{1}{q}\mathbb{E}[|D_zI_q(f^{(k)}) - D_zI_q(f^{(l)})|^3] \\ \leq \quad & \frac{4}{q}(\mathbb{E}[|D_zI_q(f^{(k)})|^3] + \mathbb{E}[|D_zI_q(f^{(l)})|^3]). \end{aligned}$$

By using the inequality

$$\begin{aligned}
&\int_Z \mu(dz)\mathbb{E}[|D_z I_q(f)|^3] \\
\leq\ & q^3\sqrt{(q-1)!\|f\|^2_{L^2(\mu^q)}} \times \sum_{b=1}^{q}\sum_{a=0}^{b-1} \mathbf{1}_{1\leq a+b\leq 2q-1}(a+b-1)!^{1/2}(q-a-1)! \\
&\times \binom{q-1}{q-1-a}^2 \binom{q-1-a}{q-b} \|f \star_b^a f\|_{L^2(\mu^{2q-a-b})}
\end{aligned}$$

which is proved in [46, Theorem 4.2] (see in particular formulae (4.13) and (4.18) therein), we see immediately that line (5.9) and (5.10) are finite whenever Condition (B2) and (B3) are satisfied. ■

5.5 Multivariate ASCLT

In this section, we introduce the multivariate ASCLTs for sequences of vectors of Poisson multiple Wiener-Itô integrals, as a natural generalization of the results in section 5.4.

5.5.1 Estimation

We shall repeat the estimate procedures in section 5.2 and 5.3. As an analogue to Proposition 5.2.5, the following result is the starting point of our discussion in the multivariate case.

Proposition 5.5.1 *Let $F = (F_1, \ldots, F_d)$ be a vector of Poisson functional, such that for each $k = 1, \ldots, d$, $\mathbb{E}[F_k] = 0$, $F_k \in \mathbb{D}^{2,2}$, and for each $k, l = 1, \ldots, d$, $DF_k \times DL^{-1}F_l \in \mathrm{Dom}D$, $\langle DF_k, DL^{-1}F_l\rangle_{L^2(\mu)} \in \mathrm{Dom}D$. Let $X_C \sim \mathcal{N}(0, C)$ be a Gaussian vector, then $\forall t = (t_1, \ldots, t_d) \in \mathbb{R}^d$, we have*

$$\begin{aligned}
&|\mathbb{E}[\exp(i\langle t, F\rangle)] - \mathbb{E}[\exp(i\langle t, X_C\rangle)]| \\
\leq\ & \frac{d}{2}t_*^2 \sum_{i,j=1}^{d} \Big(S(F_i, F_j) + |\mathbb{E}[F_iF_j] - C(i,j)|\Big) \\
+\ & \frac{1}{4}t_*^3 \int_Z \mu(dz)\mathbb{E}\left[\left(\sum_{i=1}^{d}|D_zF_i|\right)^2 \left(\sum_{i=1}^{d}|D_zL^{-1}F_i|\right)\right]
\end{aligned}$$

with $t_* = \max_{i=1,\ldots,d}|t_i|$ and $S(\cdot,\cdot)$ defined in (5.5).

Proof. Recall that, in the proof of Theorem 3.3.2 in Chapter 3, we have shown the following inequality:

$$\begin{aligned}
|\mathbb{E}[\phi(X_C)] - \mathbb{E}[\phi(F)]| \leq\ & \frac{d}{2}\|\phi''\|_\infty \sum_{i,j=1}^{d} \mathbb{E}[|C(i,j) - \langle DF_i, -DL^{-1}F_j\rangle_{L^2(\mu)}|] \\
& + \frac{1}{4}\|\phi'''\|_\infty \int_Z \mu(dz)\mathbb{E}\left[\left(\sum_{i=1}^{d}|D_zF_i|\right)^2 \left(\sum_{i=1}^{d}|D_zL^{-1}F_i|\right)\right]
\end{aligned}$$

for any $\phi \in \mathbb{C}^3(\mathbb{R}^d)$ with second and third bounded derivatives, where $F = (F_1, \dots, F_d)$ is a vector of Poisson functional, and $X_C \sim \mathcal{N}(0, C)$ is a Gaussian vector with nonnegative definite covariance matrix C.

Given a fixed d-dimensional real vector $t = (t_1, \dots, t_d)$, we take

$$\phi(F_1, \dots, F_d) = \exp(i\langle t, F\rangle) = \exp\big(i(t_1F_1 + \dots + t_dF_d)\big).$$

And we note

$$g_C(t) = \mathbb{E}[\exp(i\langle t, X_C\rangle)] = \exp(-\frac{1}{2}t'Ct) = \exp(-\frac{1}{2}\sum_{i,j=1}^{d} C(i,j)t_it_j).$$

Notice that $\|\phi''\|_\infty = t_*^2$ and $\|\phi'''\|_\infty = t_*^3$, with $t_* = \max_{i=1,\dots,d} |t_i|$. Therefore, the above inequality gives

$$\begin{aligned}
|\mathbb{E}[\exp(i\langle t, F\rangle)] - g_C(t)| &\leq \frac{d}{2}t_*^2 \sum_{i,j=1}^{d} \mathbb{E}[|C(i,j) - \langle DF_i, -DL^{-1}F_j\rangle_{L^2(\mu)}|] \\
&\quad + \frac{1}{4}t_*^3 \int_Z \mu(dz)\mathbb{E}\left[\left(\sum_{i=1}^{d}|D_zF_i|\right)^2\left(\sum_{i=1}^{d}|D_zL^{-1}F_i|\right)\right] \\
&\leq \frac{d}{2}t_*^2 \sum_{i,j=1}^{d} \big(|\mathbb{E}[F_iF_j] - C(i,j)| + \mathbb{E}[|\mathbb{E}[F_iF_j] - \langle DF_i, -DL^{-1}F_j\rangle_{L^2(\mu)}|]\big) \\
&\quad + \frac{1}{4}t_*^3 \int_Z \mu(dz)\mathbb{E}\left[\left(\sum_{i=1}^{d}|D_zF_i|\right)^2\left(\sum_{i=1}^{d}|D_zL^{-1}F_i|\right)\right].
\end{aligned}$$

By taking $g(x) = x$ in Lemma 3.2.1, we have immediately

$$\mathbb{E}[F_iF_j] = \mathbb{E}[\langle DF_i, -DL^{-1}F_j\rangle_{L^2(\mu)}],$$

for $1 \leq i, j \leq d$. Then, by using Cauchy-Schwartz inequality and Poincaré inequality (Lemma 5.2.1), we have

$$\begin{aligned}
&\mathbb{E}[|\mathbb{E}[F_iF_j] - \langle DF_i, -DL^{-1}F_j\rangle_{L^2(\mu)}|] \\
\leq\ &\sqrt{\mathbf{Var}(W^{i,j})}\ , \qquad \text{with} \quad W^{i,j} = \langle DF_i, -DL^{-1}F_j\rangle_{L^2(\mu)}, \\
\leq\ &\sqrt{\mathbb{E}[\|DW^{i,j}\|^2_{L^2(\mu)}]}.
\end{aligned}$$

Since

$$\begin{aligned}
\|DW^{i,j}\|^2_{L^2(\mu)} &= \|\langle D^2F_i, -DL^{-1}F_j\rangle_{L^2(\mu)} + \langle DF_i, -D^2L^{-1}F_j\rangle_{L^2(\mu)} + \langle D^2F_i, -D^2L^{-1}F_j\rangle_{L^2(\mu)}\|^2_{L^2(\mu)} \\
&\leq 3\|\langle D^2F_i, -DL^{-1}F_j\rangle_{L^2(\mu)}\|^2_{L^2(\mu)} + 3\|\langle DF_i, -D^2L^{-1}F_j\rangle_{L^2(\mu)}\|^2_{L^2(\mu)} \\
&+ 3\|\langle D^2F_i, -D^2L^{-1}F_j\rangle_{L^2(\mu)}\|^2_{L^2(\mu)}.
\end{aligned}$$

Then,

$$\mathbb{E}[|\mathbb{E}[F_iF_j] - \langle DF_i, -DL^{-1}F_j\rangle_{L^2(\mu)}|] \leq \sqrt{3}S(F_i, F_j),$$

from which the conclusion follows. ■

The proposition below is a generalization of Proposition 5.2.6 to the multivariate case. Notice that we no longer make assumption on the covariances of $(F^{(k)}, k = 1, 2, \ldots)$.

Proposition 5.5.2 *Fix an integer $d \geq 2$. Let $\{F^{(k)}, k = 1, 2, \ldots, n\}$ be a sequence of d-dimensional Poisson functional vectors, where $F^{(k)} = (F_1^{(k)}, \ldots, F_d^{(k)})$. Suppose that $\mathbb{E}[F_i^{(k)}] = 0$, $F_i^{(k)} \in \mathbb{D}^{2,2}$, for all $i = 1, 2, \ldots, d$, $k = 1, 2, \ldots, n$, and $DF_i^{(k)} \times DL^{-1}F_j^{(l)} \in \text{Dom}D$, $\langle DF_i^{(k)}, DL^{-1}F_j^{(l)} \rangle_{L^2(\mu)} \in \text{Dom}D$ for all $i, j = 1, 2, \ldots, d$, $k, l = 1, 2, \ldots, n$, then*

$$\begin{aligned}
\mathbb{E}[|\Delta_n(t)|^2] &\leq \frac{1}{\log^2 n} \sum_{k,l=1}^{n} \frac{1}{kl} \Big(r^2 d \sum_{i,j=1}^{d} \big(|\mathbb{E}[F_i^{(k)} F_j^{(l)}]| + 2S(F_i^{(l)}, F_j^{(k)}) \big) \\
&+ \frac{1}{4} r^3 \int_Z \mu(dz) \mathbb{E}\left[\big(\sum_{i=1}^{d} |D_z(F_i^{(k)} - F_i^{(l)})| \big)^2 \big(\sum_{i=1}^{d} |D_z L^{-1}(F_i^{(k)} - F_i^{(l)})| \big) \right] \Big) \\
&+ \frac{L_n}{\log^2 n} \sum_{k=1}^{n} \frac{1}{k} \Big(r^2 d \sum_{i,j=1}^{d} \big(2|C(i,j) - \mathbb{E}[F_i^{(k)} F_j^{(k)}]| + 4S(F_i^{(k)}, F_j^{(k)}) \big) \\
&+ \frac{1}{2} r^3 \int_Z \mu(dz) \mathbb{E}\left[\big(\sum_{i=1}^{d} |D_z F_i^{(k)}| \big)^2 \big(\sum_{i=1}^{d} |D_z L^{-1} F_i^{(k)}| \big) \right] \Big)
\end{aligned}$$

with $S(\cdot, \cdot)$ defined in (5.5) , and $L_n = \sum_{i=1}^{n} \frac{1}{i}$.

Proof. Given a positive real number r. We note $g_C(t) = \mathbb{E}[\exp(i\langle t, X_C \rangle)]$, for $t = (t_1, \ldots, t_d)$ such that $|t_i| \leq r, i = 1, \ldots, d$. Then

$$\begin{aligned}
\mathbb{E}|\Delta_n(t)|^2 &= \frac{1}{\log^2 n} \sum_{k,l=1}^{n} \frac{1}{kl} \mathbb{E}[\big(\exp(i\langle t, F^{(k)} \rangle) - g_C(t)\big)\big(\exp(-i\langle t, F^{(l)} \rangle) - g(t)\big)] \\
&= \frac{1}{\log^2 n} \sum_{k,l=1}^{n} \frac{1}{kl} \Big(\big(\mathbb{E}[\exp(i\langle t, F^{(k)} - F^{(l)} \rangle)] - g_C^2(t)\big) \\
&\quad - g_C(t)\big(\mathbb{E}[\exp(i\langle t, F^{(k)} \rangle)] - g_C(t)\big) - g_C(t)\big(\mathbb{E}[\exp(-i\langle t, F^{(l)} \rangle)] - g_C(t)\big) \Big).
\end{aligned}$$

On the one hand, we apply Proposition 5.5.1 and obtain

$$\begin{aligned}
&\left| \mathbb{E}[\exp(i\langle t, F^{(k)} - F^{(l)} \rangle)] - g_C^2(t) \right| \\
= &\left| \mathbb{E}[\exp(i\langle \sqrt{2}t, \frac{F^{(k)} - F^{(l)}}{\sqrt{2}} \rangle)] - g_C(\sqrt{2}t) \right| \\
\leq &\ r^2 d \sum_{i,j=1}^{d} \Big(|C(i,j) - \frac{1}{2}\mathbb{E}[(F_i^{(k)} - F_i^{(l)})(F_j^{(k)} - F_j^{(l)})]| + S(\frac{F_i^{(k)} - F_i^{(l)}}{\sqrt{2}}, \frac{F_j^{(k)} - F_j^{(l)}}{\sqrt{2}}) \Big) \\
+ &\ \frac{1}{4} r^3 \int_Z \mu(dz) \mathbb{E}\left[\Big(\sum_{i=1}^{d} |D_z(F_i^{(k)} - F_i^{(l)})| \Big)^2 \Big(\sum_{i=1}^{d} |D_z L^{-1}(F_i^{(k)} - F_i^{(l)})| \Big) \right].
\end{aligned}$$

By the definition of $S(\cdot,\cdot)$, we know that

$$\begin{aligned} & S(F_i^{(k)}-F_i^{(l)}, F_j^{(k)}-F_j^{(l)}) \\ = & \sqrt{\mathbb{E}\|\langle D^2(F_i^{(k)}-F_i^{(l)}), -DL^{-1}(F_j^{(k)}-F_j^{(l)})\rangle_{L^2(\mu)}\|^2_{L^2(\mu)}} \\ + & \sqrt{\mathbb{E}\|\langle D(F_i^{(k)}-F_i^{(l)}), -D^2L^{-1}(F_j^{(k)}-F_j^{(l)})\rangle_{L^2(\mu)}\|^2_{L^2(\mu)}} \\ + & \sqrt{\mathbb{E}\|\langle D^2(F_i^{(k)}-F_i^{(l)}), -D^2L^{-1}(F_j^{(k)}-F_j^{(l)})\rangle_{L^2(\mu)}\|^2_{L^2(\mu)}}. \end{aligned}$$

By writing

$$\mathfrak{A} = \langle D^2F_i^{(k)}, -DL^{-1}F_j^{(k)}\rangle_{L^2(\mu)}, \qquad \mathfrak{B} = \langle D^2F_i^{(k)}, -DL^{-1}F_j^{(l)}\rangle_{L^2(\mu)},$$

$$\mathfrak{C} = \langle D^2F_i^{(l)}, -DL^{-1}F_j^{(k)}\rangle_{L^2(\mu)}, \qquad \mathfrak{D} = \langle D^2F_i^{(l)}, -DL^{-1}F_j^{(l)}\rangle_{L^2(\mu)},$$

we obtain that

$$\begin{aligned} & \sqrt{\mathbb{E}\|\langle D^2(F_i^{(k)}-F_i^{(l)}), -DL^{-1}(F_j^{(k)}-F_j^{(l)})\rangle_{L^2(\mu)}\|^2_{L^2(\mu)}} \\ = & \sqrt{\mathbb{E}\|\mathfrak{A}-\mathfrak{B}-\mathfrak{C}+\mathfrak{D}\|^2_{L^2(\mu)}} \\ \leq & \sqrt{4\mathbb{E}[\|\mathfrak{A}\|^2_{L^2(\mu)}+\|\mathfrak{B}\|^2_{L^2(\mu)}+\|\mathfrak{C}\|^2_{L^2(\mu)}+\|\mathfrak{D}\|^2_{L^2(\mu)}]} \\ \leq & 2\left(\sqrt{\mathbb{E}[\|\mathfrak{A}\|^2_{L^2(\mu)}]}+\sqrt{\mathbb{E}[\|\mathfrak{B}\|^2_{L^2(\mu)}]}+\sqrt{\mathbb{E}[\|\mathfrak{C}\|^2_{L^2(\mu)}]}+\sqrt{\mathbb{E}[\|\mathfrak{D}\|^2_{L^2(\mu)}]}\right). \end{aligned}$$

We may do similar calculations for

$$\sqrt{\mathbb{E}\|\langle D(F_i^{(k)}-F_i^{(l)}), -D^2L^{-1}(F_j^{(k)}-F_j^{(l)})\rangle_{L^2(\mu)}\|^2_{L^2(\mu)}}$$

and

$$\sqrt{\mathbb{E}\|\langle D^2(F_i^{(k)}-F_i^{(l)}), -D^2L^{-1}(F_j^{(k)}-F_j^{(l)})\rangle_{L^2(\mu)}\|^2_{L^2(\mu)}} \qquad ,$$

then deduce that

$$\begin{aligned} & S\Big(\frac{F_i^{(k)}-F_i^{(l)}}{\sqrt{2}}, \frac{F_j^{(k)}-F_j^{(l)}}{\sqrt{2}}\Big) \\ = & \frac{1}{2}S(F_i^{(k)}-F_i^{(l)}, F_j^{(k)}-F_j^{(l)}) \\ \leq & \left(S(F_i^{(k)},F_j^{(k)})+S(F_i^{(k)},F_j^{(l)})+S(F_i^{(l)},F_j^{(k)})+S(F_i^{(l)},F_j^{(l)})\right). \end{aligned}$$

Notice that,

$$\begin{aligned} & |C(i,j)-\frac{1}{2}\mathbb{E}[(F_i^{(k)}-F_i^{(l)})(F_j^{(k)}-F_j^{(l)})]| \\ \leq & \frac{1}{2}|C(i,j)-\mathbb{E}[F_i^{(k)}F_j^{(k)}]|+\frac{1}{2}|C(i,j)-\mathbb{E}[F_i^{(l)}F_j^{(l)}]|+\frac{1}{2}|\mathbb{E}[F_i^{(k)}F_j^{(l)}]|+\frac{1}{2}|\mathbb{E}[F_i^{(l)}F_j^{(k)}]|, \end{aligned}$$

therefore,

$$\begin{aligned}
&\Big|\mathbb{E}[\exp(i\langle t, F^{(k)} - F^{(l)}\rangle)] - g_C^2(t)\Big| \\
\leq\ & r^2 d \sum_{i,j=1}^{d} \Big(\frac{1}{2}|C(i,j) - \mathbb{E}[F_i^{(k)}F_j^{(k)}]| + \frac{1}{2}|C(i,j) - \mathbb{E}[F_i^{(l)}F_j^{(l)}]| + \frac{1}{2}|\mathbb{E}[F_i^{(k)}F_j^{(l)}]| + \frac{1}{2}|\mathbb{E}[F_i^{(l)}F_j^{(k)}]| \\
+\ & S(F_i^{(k)}, F_j^{(k)}) + S(F_i^{(k)}, F_j^{(l)}) + S(F_i^{(l)}, F_j^{(k)}) + S(F_i^{(l)}, F_j^{(l)})\Big) \\
+\ & \frac{1}{4} r^3 \int_Z \mu(dz)\mathbb{E}\left[\Big(\sum_{i=1}^{d} |D_z(F_i^{(k)} - F_i^{(l)})|\Big)^2 \Big(\sum_{i=1}^{d} |D_z L^{-1}(F_i^{(k)} - F_i^{(l)})|\Big)\right].
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&\Big|\mathbb{E}[\exp(i\langle t, F^{(k)}\rangle)] - g_C(t)\Big| \\
\leq\ & \frac{1}{2} r^2 d \sum_{i,j=1}^{d} \Big(|C(i,j) - \mathbb{E}[F_i^{(k)}F_j^{(k)}]| + S(F_i^{(k)}, F_j^{(k)})\Big) \\
+\ & \frac{1}{4} r^3 \int_Z \mu(dz)\mathbb{E}\left[\left(\sum_{i=1}^{d} |D_z F_i^{(k)}|\right)^2 \left(\sum_{i=1}^{d} |D_z L^{-1} F_i^{(k)}|\right)\right].
\end{aligned}$$

Putting these two parts together, we get

$$\begin{aligned}
\mathbb{E}[|\Delta_n(t)|^2] \leq\ & \frac{1}{\log^2 n} \sum_{k,l=1}^{n} \frac{1}{kl} \Big(r^2 d \sum_{i,j=1}^{d} \big(|\mathbb{E}[F_i^{(k)}F_j^{(l)}]| + 2S(F_i^{(l)}, F_j^{(k)})\big) \\
+\ & \frac{1}{4} r^3 \int_Z \mu(dz)\mathbb{E}\left[\big(\sum_{i=1}^{d} |D_z(F_i^{(k)} - F_i^{(l)})|\big)^2 \big(\sum_{i=1}^{d} |D_z L^{-1}(F_i^{(k)} - F_i^{(l)})|\big)\right]\Big) \\
+\ & \frac{L_n}{\log^2 n} \sum_{k=1}^{n} \frac{1}{k} \Big(r^2 d \sum_{i,j=1}^{d} \big(2|C(i,j) - \mathbb{E}[F_i^{(k)}F_j^{(k)}]| + 4S(F_i^{(k)}, F_j^{(k)})\big) \\
+\ & \frac{1}{2} r^3 \int_Z \mu(dz)\mathbb{E}\left[\big(\sum_{i=1}^{d} |D_z F_i^{(k)}|\big)^2 \big(\sum_{i=1}^{d} |D_z L^{-1} F_i^{(k)}|\big)\right]\Big).
\end{aligned}$$

■

5.5.2 ASCLT

Fix an positive integer $d \geq 2$, and positive integers $q_1, q_2, \ldots, q_d$. We consider sequences of vectors of the type

$$F^{(k)} = (F_1^{(k)}, \ldots, F_d^{(k)}) = \big(I_{q_1}(f_1^{(k)}), \ldots, I_{q_d}(f_d^{(k)})\big), \quad k = 1, 2, \ldots$$

with $f_i^{(k)} \in L_s^2(\mu^{q_i})$. In Chapter 3 we have studied the CLTs for these vectors of Poisson multiple integrals, now we build ASCLTs for them.

Integral of order 1

We first suppose that $q_1 = q_2 = \ldots = q_d = 1$.

Theorem 5.5.3 *Fix $d \geq 2$, let $X \sim \mathcal{N}(0, C)$, with*

$$C = \{C(i,j) : i,j = 1, \ldots, d\}$$

a $d \times d$ nonnegative definite matrix. Let $(F^{(k)}, k = 1, 2, \ldots)$ defined by

$$F^{(k)} = (I_1(g_1^{(k)}), \ldots, I_1(g_d^{(k)})) = (\hat{N}(g_1^{(k)}), \ldots, \hat{N}(g_d^{(k)}))$$

be a collection of d-dimensional random vectors living in the first Wiener chaos of the compensated Poisson measure $\hat{N}$, where $g_i^{(k)} \in L_s^2(\mu) \cap L_s^3(\mu)$ for $i = 1, \ldots, d, k = 1, 2, \ldots$. Suppose that

$$\lim_{k\to\infty} \mathbb{E}[F_i^{(k)} F_j^{(k)}] = \lim_{k\to\infty} \langle g_i^{(k)}, g_j^{(k)} \rangle_{L^2(\mu)} = C(i,j) \qquad , 1 \leq i, j \leq d.$$

If for every i, $1 \leq i \leq d$, $\|g_i^{(k)}\|^3_{L^3(\mu)} \to 0$ as $k \to \infty$, then $F^{(k)} \xrightarrow{law} X$. Moreover, if the three conditions (C0), (C1) and (C2) below are satisfied :

- *(C0) for every pair (i,j), $1 \leq i, j \leq d$,*

$$\sum_{n\geq 2} \frac{1}{n \log^2 n} \sum_{k=1}^{n} \frac{1}{k} \left| C(i,j) - \mathbb{E}[F_i^{(k)} F_j^{(k)}] \right| = \sum_{n\geq 2} \frac{1}{n \log^2 n} \sum_{k=1}^{n} \frac{1}{k} \left| C(i,j) - \langle g_i^{(k)}, g_j^{(k)} \rangle_{L^2(\mu)} \right| < \infty;$$

- *(C1) for every pair (i,j), $1 \leq i, j \leq d$,*

$$\sum_{n\geq 2} \frac{1}{n \log^3 n} \sum_{k,l=1}^{n} \frac{|\mathbb{E}[F_i^{(k)} F_j^{(l)}]|}{kl} = \sum_{n\geq 2} \frac{1}{n \log^3 n} \sum_{k,l=1}^{n} \frac{|\langle g_i^{(k)}, g_j^{(l)} \rangle_{L^2(\mu)}|}{kl} < \infty;$$

- *(C2) for every i, $1 \leq i \leq d$,*

$$\sum_{n\geq 2} \frac{1}{n \log^2 n} \sum_{k=1}^{n} \frac{1}{k} \|g_i^{(k)}\|^3_{L^3(\mu)} < \infty,$$

then $\{F^{(n)}\}$ satisfies an ASCLT. In other words, almost surely, for all continuous and bounded function $\varphi : \mathbb{R}^d \to \mathbb{R}$, we have

$$\frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \varphi(F^{(k)}) \longrightarrow \mathbb{E}[\varphi(X)], \qquad \text{as } n \to \infty.$$

Proof. The CLT part is an application of Theorem 3.4.9 and Remark 3.4.10 in Chapter 3.

The ASCLT part is a direct consequence of Proposition 5.5.2 and Theorem 5.1.3. By Proposition 5.5.2,

$$
\begin{aligned}
& \mathbb{E}[|\Delta_n(t)|^2] && (5.12)\\
\leq\ & \frac{1}{\log^2 n}\sum_{k,l=1}^{n}\frac{1}{kl}\times r^2 d\sum_{i,j=1}^{d}\big(|\mathbb{E}[F_i^{(k)}F_j^{(l)}]| && (5.13)\\
+\ & \frac{L_n}{\log^2 n}\sum_{k=1}^{n}\frac{1}{k}\times r^2 d\sum_{i,j=1}^{d}\big(2|C(i,j)-\mathbb{E}[F_i^{(k)}F_j^{(k)}]| && (5.14)\\
+\ & \frac{1}{\log^2 n}\sum_{k,l=1}^{n}\frac{1}{kl}\times 2S(F_i^{(l)},F_j^{(k)})+\frac{L_n}{\log^2 n}\sum_{k=1}^{n}\frac{1}{k}4S(F_i^{(k)},F_j^{(k)})\big) && (5.15)\\
+\ & \frac{L_n}{\log^2 n}\sum_{k=1}^{n}\frac{1}{4k}\times\frac{1}{4}r^3\int_Z\mu(dz)\mathbb{E}\Big[\big(\sum_{i=1}^{d}|D_z(F_i^{(k)}-F_i^{(l)})|\big)^2 && (5.16)\\
& \qquad\times\big(\sum_{i=1}^{d}|D_zL^{-1}(F_i^{(k)}-F_i^{(l)})|\big)\Big] && (5.17)\\
+\ & \frac{1}{\log^2 n}\sum_{k,l=1}^{n}\frac{1}{kl}\times\frac{1}{2}r^3\int_Z\mu(dz)\mathbb{E}\left[\big(\sum_{i=1}^{d}|D_zF_i^{(k)}|\big)^2\big(\sum_{i=1}^{d}|D_zL^{-1}F_i^{(k)}|\big)\right] && (5.18)
\end{aligned}
$$

Indeed, we need only to focus on the parts having Malliavin operators (that is, line (5.16), (5.17) and (5.18), while the rest may be treated by similar arguments as in the proof of Theorem 5.4.1 and Theorem 5.4.4, with the help of Lemma 5.3.7. We have

$$
\begin{aligned}
& \int_Z\mu(dz)\mathbb{E}\left[\left(\sum_{i=1}^{d}|D_z(F_i^{(k)}-F_i^{(l)})|\right)^2\left(\sum_{i=1}^{d}|D_zL^{-1}(F_i^{(k)}-F_i^{(l)})|\right)\right]\\
=\ & \int_Z\mu(dz)\mathbb{E}\left[\left(\sum_{i=1}^{d}|g_i^{(k)}-g_i^{(l)}|\right)^3\right]\\
\leq\ & d^2\sum_{i=1}^{d}\int_Z\mu(dz)|g_i^{(k)}-g_i^{(l)}|^3\\
\leq\ & 4d^2\sum_{i=1}^{d}\left(\|g_i^{(k)}\|^3_{L^3(\mu)}+\|g_i^{(l)}\|^3_{L^3(\mu)}\right)\quad .
\end{aligned}
$$

and

$$
\int_Z\mu(dz)\mathbb{E}\left[\left(\sum_{i=1}^{d}|D_zF_i^{(k)}|\right)^2\left(\sum_{i=1}^{d}|D_zL^{-1}F_i^{(k)}|\right)\right]=\int_Z\mu(dz)\left(\sum_{i=1}^{d}|g_i^{(k)}|\right)^3\leq d^2\sum_{i=1}^{d}\|g_i^{(k)}\|^3_{L^3(\mu)}.
$$

So the convergence of the sum of these parts (with a weight $\frac{1}{n\log n}$) is justified by Condition (C2). ■

Remark 5.5.4 Notice that Condition (C0) is new with respect to Theorem 5.4.1. Indeed, we make the assumption that $\mathbf{Var}[F^{(n)}] = 1$ in the statement of Theorem 5.4.1, while in Theorem 5.5.3 we no longer make analogous assumptions. Consequently, Theorem 5.5.3 is a genuine generalization of Theorem 5.4.1.

Remark 5.5.5 By the same arguments as in Remark 5.4.2, the following are sufficient conditions for (C0), (C1) and (C2):

- (C0') for every pair (i,j), $1 \le i,j \le d$, there exists $\delta > 0$, such that

$$\left| C(i,j) - \mathbb{E}[F_i^{(k)} F_j^{(k)}] \right| = O(k^{-\delta}), \qquad k \to \infty;$$

- (C1') for every pair (i,j), $1 \le i,j \le d$, there exists $\alpha > 0$, such that

$$|\mathbb{E}[F_i^{(k)} F_j^{(l)}]| < C \left(\frac{l}{k} \right)^{\alpha}, \qquad \forall 1 \le l \le k;$$

- (C2') for every i, $1 \le i \le d$, there exists $\beta > 0$, such that

$$\|g_i^{(k)}\|^3_{L^3(\mu)} = O(k^{-\beta}), \qquad k \to \infty.$$

Multiple integrals of order $q \ge 2$

Now we suppose that $q_1, \dots, q_d \ge 2$.

Theorem 5.5.6 *Fix $d \ge 2$, let $X \sim \mathcal{N}(0, C)$, with*

$$C = \{C(i,j) : i,j = 1, \dots, d\}$$

a $d \times d$ nonnegative definite matrix, and fix integers $q_1, \dots, q_d \ge 2$. For any $n \ge 1$ and $i = 1, \dots, d$, let $f_i^{(n)}$ belong to $L^2_s(\mu^{q_i})$. Define the sequence $\{F^{(n)}; n \ge 1\}$ by

$$F^{(n)} = (F_1^{(n)}, \dots, F_d^{(n)}) = \left(I_{q_1}(f_1^{(n)}), \dots, I_{q_d}(f_d^{(n)}) \right), \quad n = 1, 2, \dots$$

with $f_i^{(n)} \in L^2_s(\mu^{q_i})$ and suppose that

$$\lim_{n\to\infty} \mathbb{E}[F_i^{(n)} F_j^{(n)}] = \mathbf{1}_{(q_j = q_i)} q_i! \times \lim_{n\to\infty} \langle f_i^{(n)}, f_j^{(n)} \rangle_{L^2(\mu^{q_i})} = C(i,j), \qquad 1 \le i,j \le d. \tag{5.19}$$

Assume that the following Conditions hold for every $k = 1, \dots, d$:

1. *For every n, the kernel $f_k^{(n)}$ satisfies Assumptions A , B , D , E.*
2. *For every $r = 1, \dots, q_k$ and every $l = 1, \dots, r \wedge (q_k - 1)$, one has that*

$$\|f_k^{(n)} \star_r^l f_k^{(n)}\|_{L^2(\mu^{2q_k - r - l})} \to 0,$$

as $n \to \infty$.

3. *As $n \to \infty$, $\int_{Z^{q_k}} d\mu^{q_k} \left(f_k^{(n)} \right)^4 \to 0$.*

Then, $F^{(n)} \xrightarrow{law} N \sim \mathcal{N}(0,1)$ *as* $n \to \infty$.

Moreover, if the four conditions (D0), (D1), (D2) and (D3) below are satisfied :

- *(D0) for every pair* (i,j), $1 \le i,j \le d$, *with* $q_i = q_j$,

$$\sum_{n\ge 2} \frac{1}{n\log^2 n} \sum_{k=1}^{n} \frac{1}{k} \left| C(i,j) - \mathbb{E}[F_i^{(k)} F_j^{(k)}] \right| < \infty;$$

- *(D1) for every pair* (i,j), $1 \le i,j \le d$, *with* $q_i = q_j$,

$$\sum_{n\ge 2} \frac{1}{n\log^3 n} \sum_{k,l=1}^{n} \frac{|\mathbb{E}[F_i^{(k)} F_j^{(l)}]|}{kl} = \sum_{n\ge 2} \frac{1}{n\log^3 n} \sum_{k,l=1}^{n} \frac{|\langle f_i^{(k)}, f_j^{(l)} \rangle_{L^2(\mu^{q_i})}|}{kl} < \infty;$$

- *(D2) for every pair* (i,j), $1 \le i,j \le d$ *and for every* $r = 1,\dots,q$, $\quad l = 1,\dots,r\wedge(q-1)$,

$$\sum_{n\ge 2} \frac{1}{n\log^2 n} \sum_{k=1}^{n} \frac{1}{k} \| f_i^{(k)} \star_r^l f_j^{(k)} \|_{L^2(\mu^{2q-r-l})} < \infty;$$

- *(D3) for every* i, $1 \le i \le d$,

$$\sum_{n\ge 2} \frac{1}{n\log^2 n} \sum_{k=1}^{n} \frac{1}{k} \int_{Z^q} (f_i^{(k)})^4 d\mu^q < \infty,$$

then $\{F^{(n)}\}$ *satisfies an ASCLT. In other words, almost surely, for all continuous and bounded function* $\varphi : \mathbb{R}^d \to \mathbb{R}$, *we have*

$$\frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \varphi(F^{(k)}) \longrightarrow \mathbb{E}[\varphi(X)], \qquad \text{as } n \to \infty.$$

Proof. The CLT part is an application of Theorem 3.4.9 in Chapter 3.

As for the ASCLT part, we shall use the same arguments as in the proof of Theorem 5.4.1 and Theorem 5.4.4, by applying Proposition 5.5.2, Theorem 5.1.3 and Lemma 5.3.7. Now we need only to focus on the parts having Malliavin operators.

Denote q_* by $q_* = \min\{q_i, i = 1, \ldots, d\}$. Since $F_i^{(k)} = I_{q_i}(f_i^{(k)})$, we have,

$$\begin{aligned}
&\int_Z \mu(dz)\mathbb{E}\left[\left(\sum_{i=1}^d |D_z(F_i^{(k)} - F_i^{(l)})|\right)^2 \left(\sum_{i=1}^d |D_z L^{-1}(F_i^{(k)} - F_i^{(l)})|\right)\right] \\
= &\int_Z \mu(dz)\mathbb{E}\left[\left(\sum_{i=1}^d |D_z(F_i^{(k)} - F_i^{(l)})|\right)^2 \left(\sum_{i=1}^d \frac{1}{q_i}|D_z(F_i^{(k)} - F_i^{(l)})|\right)\right] \\
\leq &\frac{1}{q_*}\int_Z \mu(dz)\mathbb{E}\left[\left(\sum_{i=1}^d |D_z(F_i^{(k)} - F_i^{(l)})|\right)^3\right] \\
\leq &\frac{d^2}{q_*}\int_Z \mu(dz)\mathbb{E}\left[\sum_{i=1}^d |D_z(F_i^{(k)} - F_i^{(l)})|^3\right] \\
\leq &\frac{4d^2}{q_*}\sum_{i=1}^d \int_Z \mu(dz)\left(\mathbb{E}[|D_z F_i^{(k)}|^3] + \mathbb{E}[|D_z F_i^{(l)}|^3]\right),
\end{aligned}$$

and

$$\begin{aligned}
&\int_Z \mu(dz)\mathbb{E}\left[\left(\sum_{i=1}^d |D_z F_i^{(k)}|\right)^2 \left(\sum_{i=1}^d |D_z L^{-1} F_i^{(k)}|\right)\right] \\
= &\int_Z \mu(dz)\mathbb{E}\left[\left(\sum_{i=1}^d |D_z F_i^{(k)}|\right)^2 \left(\sum_{i=1}^d \frac{1}{q_i}|D_z F_i^{(k)}|\right)\right] \\
\leq &\frac{1}{q_*}\int_Z \mu(dz)\mathbb{E}\left[\left(\sum_{i=1}^d |D_z F_i^{(k)}|\right)^3\right] \\
\leq &\frac{d^2}{q_*}\sum_{i=1}^d \int_Z \mu(dz)\mathbb{E}[|D_z F_i^{(k)}|^3].
\end{aligned}$$

Finally, by using the inequality in [46, Theorem 4.2] below:

$$\begin{aligned}
&\int_Z \mu(dz)\mathbb{E}[|D_z I_q(f)|^3] \\
\leq\; &q^3\sqrt{(q-1)!\|f\|^2_{L^2(\mu^q)}} \times \sum_{b=1}^q \sum_{a=0}^{b-1} \mathbf{1}_{1\leq a+b\leq 2q-1}(a+b-1)!^{1/2}(q-a-1)! \\
&\times \binom{q-1}{q-1-a}^2 \binom{q-1-a}{q-b} \|f \star_b^a f\|_{L^2(\mu^{2q-a-b})},
\end{aligned}$$

we know that Condition (D2) and (D3) are enough to let the sum of the parts containing Malliavin operators converge, therefore, the conclusion follows immediately. ■

Remark 5.5.7 By the same arguments in Remark 5.4.2, the following conditions are sufficient conditions for (D0), (D1) (D2) and (D3):

- (D0') for every pair (i,j), $1 \le i,j \le d$, if $q_i = q_j$, there exists $\delta > 0$, such that
$$\left|C(i,j) - \mathbb{E}[F_i^{(k)} F_j^{(k)}]\right| = O(k^{-\delta}), \qquad k \to \infty,$$
otherwise, $C(i,j) = 0$;

- (D1') for every pair (i,j), $1 \le i,j \le d$, with $q_i = q_j$, there exists $\alpha > 0$, such that
$$|\mathbb{E}[F_i^{(k)} F_j^{(l)}]| < C \left(\frac{l}{k}\right)^{\alpha}, \qquad \forall 1 \le l \le k;$$

- (D2') for every pair (i,j), $1 \le i,j \le d$ and for every $r = 1, \dots, q$, $\quad l = 1, \dots, r \wedge (q-1)$, there exists $\beta > 0$, such that
$$\|f_i^{(k)} \star_r^l f_j^{(k)}\|_{L^2(\mu^{2q-r-l})} = O(k^{-\beta}), \qquad k \to \infty;$$

- (D3') for every i, $1 \le i \le d$, there exists $\gamma > 0$, such that
$$\int_{Z^q} (f_i^{(k)})^4 d\mu^q = O(k^{-\gamma}), \qquad k \to \infty.$$

Mixed case

In this section, we study the ASCLT for vectors that have multiple integral components of both order 1 and order $q \ge 2$. The following theorem summarizes the results in the last two sections.

Theorem 5.5.8 *Let $d = a + b$, with a, b two fixed positive integers. Let $X \sim \mathcal{N}(0, C)$, with*
$$C = \{C(i,j) : i,j = 1, \dots, d\}$$
a $d \times d$ nonnegative definite matrix, such that
$$C(i, j+a) = 0, \qquad \forall 1 \le i \le a, 1 \le j \le b.$$
Given fixed integers $q_1, \dots, q_b \ge 2$. For any $k \ge 1$ and $i = 1, \dots, a$, $j = 1, \dots, b$, let $g_i^{(k)}$ belongs to $L_s^2(\mu)$ and let $f_j^{(k)}$ belongs to $L_s^2(\mu^{q_j})$. Define the sequence $\{F^{(k)}; k \ge 1\}$ by
$$F^{(k)} = (F_1^{(k)}, \dots, F_d^{(k)}) = \big(I_1(g_1^{(k)}), \dots, I_1(g_a^{(k)}), I_{q_1}(f_1^{(k)}), \dots, I_{q_b}(f_b^{(k)})\big), \quad k = 1, 2, \dots$$
and suppose that
$$\lim_{n\to\infty} \mathbb{E}[F_i^{(n)} F_j^{(n)}] = C(i,j), \qquad 1 \le i,j \le d. \tag{5.20}$$
Assume that $\{f_i^{(k)}, i = 1, \dots, b, k = 1, 2, \dots\}$ satisfy Conditions 1,2,3 in the statement of Theorem 5.5.6, and for $i = 1, \dots, a$, $\|g_i^{(k)}\|_{L^3(\mu)}^3 \to 0$ as $k \to \infty$, Then, $F^{(n)} \xrightarrow{law} N \sim \mathcal{N}(0,1)$ as $n \to \infty$.

Moreover, if the five conditions (E0), (E1), (E2), (E3) and (E4) below are satisfied:

- *(E0) for every pair (i, j), $1 \leq i, j \leq a$, or $a + 1 \leq i, j \leq d$ with $q_{i-a} = q_{j-a}$,*

$$\sum_{n \geq 2} \frac{1}{n \log^2 n} \sum_{k=1}^{n} \frac{1}{k} \left| C(i,j) - \mathbb{E}[F_i^{(k)} F_j^{(k)}] \right| < \infty;$$

- *(E1) for every pair (i, j), $1 \leq i, j \leq a$, or $a + 1 \leq i, j \leq d$ with $q_{i-a} = q_{j-a}$,*

$$\sum_{n \geq 2} \frac{1}{n \log^3 n} \sum_{k,l=1}^{n} \frac{|\mathbb{E}[F_i^{(k)} F_j^{(l)}]|}{kl} < \infty;$$

- *(E2) for every pair (i, j), $1 \leq i, j \leq b$ and for every $r = 1, \ldots, q, \quad l = 1, \ldots, r \wedge (q-1)$,*

$$\sum_{n \geq 2} \frac{1}{n \log^2 n} \sum_{k=1}^{n} \frac{1}{k} \| f_i^{(k)} \star_r^l f_j^{(k)} \|_{L^2(\mu^{2q-r-l})} < \infty;$$

- *(E3) for every i, $1 \leq i \leq b$,*

$$\sum_{n \geq 2} \frac{1}{n \log^2 n} \sum_{k=1}^{n} \frac{1}{k} \int_{Z^q} (f_i^{(k)})^4 d\mu^q < \infty;$$

- *(E4) for every i, $1 \leq i \leq a$,*

$$\sum_{n \geq 2} \frac{1}{n \log^2 n} \sum_{k=1}^{n} \frac{1}{k} \| g_i^{(k)} \|^3_{L^3(\mu)} < \infty,$$

then $\{F^{(n)}\}$ satisfies an ASCLT. In other words, almost surely, for all continuous and bounded function $\varphi : \mathbb{R}^d \to \mathbb{R}$, we have

$$\frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \varphi(F^{(k)}) \longrightarrow \mathbb{E}[\varphi(X)], \qquad \textit{as } n \to \infty.$$

Remark 5.5.9 The above conditions can be replaced by the following sufficient conditions.

- (E0') for every pair (i, j), $1 \leq i, j \leq a$, or $a + 1 \leq i, j \leq d$ with $q_{i-a} = q_{j-a}$, there exists $\delta > 0$, such that

$$\left| C(i,j) - \mathbb{E}[F_i^{(k)} F_j^{(k)}] \right| = O(k^{-\delta}), \qquad k \to \infty;$$

- (E1') for every pair (i, j), $1 \leq i, j \leq a$, or $a + 1 \leq i, j \leq d$ with $q_{i-a} = q_{j-a}$, there exists $\alpha > 0$, such that

$$|\mathbb{E}[F_i^{(k)} F_j^{(l)}]| < C \left(\frac{l}{k} \right)^{\alpha}, \qquad \forall 1 \leq l \leq k;$$

- (E2') for every pair (i, j), $1 \leq i, j \leq b$ and for every $r = 1, \ldots, q, \quad l = 1, \ldots, r \wedge (q-1)$, there exists $\beta > 0$, such that

$$\| f_i^{(k)} \star_r^l f_j^{(k)} \|_{L^2(\mu^{2q-r-l})} = O(k^{-\beta}), \qquad k \to \infty;$$

- (E3') for every i, $1 \le i \le b$, there exists $\gamma > 0$, such that

$$\int_{Z^q} (f_i^{(k)})^4 d\mu^q = O(k^{-\gamma}), \qquad k \to \infty;$$

- (E4') for every i, $1 \le i \le a$, there exists $\eta > 0$, such that

$$\|g_i^{(k)}\|^3_{L^3(\mu)} = O(k^{-\eta}), \qquad k \to \infty.$$

5.6 ASCLTs for functionals of Ornstein-Uhlenbeck processes

In Section 3.5 in Chapter 3 (or Section 6 in Chapter 3), we have studied CLTs for some functionals of Ornstein-Uhlenbeck Lévy process. We may extend our investigations and study ASCLTs to these examples.

We keep the notations in Section 3.5 in Chapter 3. Recall that $\hat{N}$ is a centered Poisson measure over $\mathbb{R} \times \mathbb{R}$, with control measure $\nu(du)$. We consider the stationary *Ornstein-Uhlenbeck Lévy process* with parameter $\lambda > 0$. defined by

$$Y_t^\lambda = I_1(f_t^\lambda) = \sqrt{2\lambda} \int_{-\infty}^{t} \int_{\mathbb{R}} u \exp(-\lambda(t-x)) \hat{N}(du, dx), \quad t \ge 0$$

where $f_t^\lambda(u,x) = \sqrt{2\lambda}\mathbf{1}_{(-\infty,t]}(x) u \exp(-\lambda(t-x))$. And we make some following technical assumptions on the measure ν in order that $\mathbf{Var}(Y_t^\lambda) = 1$.

We are interested in the multivariate ASCLT for the vectors of the functionals of (Y_t^λ).

Example 1 (Empirical means)

We define the functional $A(k,\lambda)$ by $A(k,\lambda) = \dfrac{1}{\sqrt{k}} \int_0^k Y_t^\lambda dt$. The first half part of the following Theorem is an application of Theorem 3.5.5 in Chapter 3, while the second half part provides a multivariate ASCLT.

Theorem 5.6.1 *For $\lambda_1, \ldots, \lambda_d > 0$, as $k \to \infty$,*

$$\bar{A}(k) = (A(k,\lambda_1), \ldots, A(k,\lambda_d)) \xrightarrow{(law)} X_B, \tag{5.21}$$

where X_B is a centered d-dimensional Gaussian vector with covariance matrix $B = (B_{ij})_{d\times d}$, with $B_{ij} = 2/\sqrt{\lambda_i \lambda_j}$, $1 \le i,j \le d$. Moreover, $\{\bar{A}(n)\}$ satisfies an ASCLT. In other words, almost surely, for all continuous and bounded function $\varphi : \mathbb{R}^d \to \mathbb{R}$, we have

$$\frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \varphi(\bar{A}(k)) \longrightarrow \mathbb{E}[\varphi(X_B)], \qquad \text{as } n \to \infty.$$

Proof. The CLT part is an application of Theorem 3.5.5 in Chapter 3, we now work on the ASCLT part. By Theorem 5.5.3 and Remark 5.5.5, we need only to verify Condition (C0'), (C1') and (C2').

By applying Fubini theorem on $A(k,\lambda)$, we have

$$\frac{1}{\sqrt{k}}\int_0^k Y_t^\lambda dt = I_1(g_{\lambda,k}),$$

where

$$g_{\lambda,k} = \mathbf{1}_{(-\infty,k]}(x)u\sqrt{\frac{2\lambda}{k}}\int_{x\vee 0}^k \exp(-\lambda(t-x))dt.$$

Firstly, let $1 \le l \le k$ be two integers, then

$$\begin{aligned}
&\mathbb{E}[A(l,\lambda_i)A(k,\lambda_j)]\\
=\ & \int_{\mathbb{R}} u^2\nu(du)\Big(\int_{-\infty}^0 dx\frac{2}{\sqrt{lk}\times\sqrt{\lambda_i\lambda_j}}\exp\big((\lambda_i+\lambda_j)x\big)\times\big(1-\exp(-\lambda_i l)\big)\times\big(1-\exp(-\lambda_j k)\big)\\
&+\int_0^l dx\frac{2}{\sqrt{lk}\times\sqrt{\lambda_i\lambda_j}}\exp\big((\lambda_i+\lambda_j)x\big)\times\big(\exp(-\lambda_i x)-\exp(-\lambda_i l)\big)\times\big(\exp(-\lambda_j x)-\exp(-\lambda_j k)\big)\Big)\\
=\ & \frac{2}{\sqrt{lk}\times\sqrt{\lambda_i\lambda_j}}\Big(\frac{1}{\lambda_i+\lambda_j}\times\big(1-\exp(-\lambda_i l)\big)\times\big(1-\exp(-\lambda_j k)\big)+l-\frac{1}{\lambda_i}\times(1-\exp(-\lambda_i l))\\
&-\frac{1}{\lambda_j}\big(\exp(\lambda_j l)-1\big)\exp(-\lambda_j k)+\frac{1}{\lambda_i+\lambda_j}\big(\exp(\lambda_j l-\lambda_j k)-\exp(-\lambda_i l-\lambda_j k)\big)\Big)\\
\le\ & \frac{2}{\sqrt{lk}\times\sqrt{\lambda_i\lambda_j}}\times(\frac{2}{\lambda_i+\lambda_j}+l)\\
\le\ & \frac{2}{\sqrt{\lambda_i\lambda_j}}\times(\frac{2}{\lambda_i+\lambda_j}+1)\times\frac{\sqrt{l}}{\sqrt{k}}.
\end{aligned}$$

Secondly, we know from the proof of Theorem 3.5.5 in Chapter 3 that, for every k, and $1 \le i,j \le d$,

$$\begin{aligned}
&|\mathbb{E}[A(k,\lambda_i)A(k,\lambda_j)]-B(i,j)|\\
=\ & |\mathbb{E}[A(k,\lambda_i)A(k,\lambda_j)]-\frac{2}{\sqrt{\lambda_i\lambda_i}}|\\
=\ & O(1/k),
\end{aligned}$$

and

$$\|g_{\lambda,k}\|^3_{L^3(d\nu dx)} \sim \frac{1}{\sqrt{k}}, \qquad \forall\lambda\in\mathbb{R},$$

as $k\to\infty$.

In conclusion, Condition (C0'), (C1') and (C2') are successfully verified, which leads to the ASCLT. ■

Example 2 (Empirical second moments)
We are interested in the quadratic functional $Q(k,\lambda)$ given by:

$$Q(k,\lambda) := \sqrt{k}\left(\frac{1}{k}\int_0^k (Y_t^\lambda)^2 dt - 1\right), \qquad k>0, \lambda>0.$$

Theorem 5.6.2 *Given an integer $d \geq 2$. For $\lambda_1, \dots, \lambda_d > 0$, as $k \to \infty$,*

$$\bar{Q}(k) = (Q(k,\lambda_1), \dots, Q(k,\lambda_d)) \stackrel{(law)}{\longrightarrow} X_C, \tag{5.22}$$

where X_C is a centered d-dimensional Gaussian vector with covariance matrix $C = (C_{ij})_{d\times d}$, defined by

$$C_{ij} = \frac{4}{\lambda_i + \lambda_j} + c_\nu^2, \qquad 1 \leq i, j \leq d,$$

with $c_\nu^2 = \int_{\mathbb{R}} u^4 \nu(du)$. Moreover, $\{\bar{Q}(n)\}$ satisfies an ASCLT. In other words, almost surely, for all continuous and bounded function $\varphi : \mathbb{R}^d \to \mathbb{R}$, we have

$$\frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \varphi(\bar{Q}(k)) \longrightarrow \mathbb{E}[\varphi(X_C)], \qquad \text{as } n \to \infty.$$

Proof. For every $k > 0$ and $\lambda > 0$, we introduce the notations

$$\begin{aligned} H_{\lambda,k}(u,x;u',x') &= (u \times u') \frac{\mathbf{1}_{(-\infty,k)^2}(x,x')}{k} \Big(\exp\big(\lambda(x+x')\big) \times \big(1 - \exp(-2\lambda k)\big) \times \mathbf{1}_{(x \vee x' \leq 0)} \\ &\quad + \exp\big(\lambda(x+x')\big) \times \big(\exp(-2\lambda(x \vee x')) - \exp(-2\lambda k)\big) \times \mathbf{1}_{(x\vee x' > 0)} \Big), \end{aligned}$$

and

$$\begin{aligned} H^\star_{\lambda,k}(u,x) &= u^2 \frac{\mathbf{1}_{(-\infty,k)}(x)}{k} \Big(\exp(2\lambda x) \times \big(1 - \exp(-2\lambda k)\big) \times \mathbf{1}_{(x \leq 0)} \\ &\quad + \exp(2\lambda x) \times \big(\exp(-2\lambda x) - \exp(-2\lambda k)\big) \times \mathbf{1}_{(x > 0)} \Big). \end{aligned}$$

As shown in the proof of Theorem 3.5.7 in Chapter 3, we know that

$$Q(k,\lambda) = I_1(\sqrt{k} H^\star_{\lambda,k}) + I_2(\sqrt{k} H_{\lambda,k}),$$

and we have proved the stronger result:

$$(I_1(\sqrt{k} H^\star_{\lambda_1,k}), \dots, I_1(\sqrt{k} H^\star_{\lambda_d,k}), I_2(\sqrt{k} H_{\lambda_1,k}), \dots, I_2(\sqrt{k} H_{\lambda_d,k})) \stackrel{(law)}{\longrightarrow} X_D \tag{5.23}$$

as $k \to \infty$, where X_D is a centered $2d$-dimensional Gaussian vector with covariance matrix D defined as:

$$D(i,j) = \begin{cases} c_\nu^2, & \text{if } 1 \leq i,j \leq d \\ \dfrac{4}{\lambda_i + \lambda_j}, & \text{if } d+1 \leq i,j \leq 2d \\ 0, & \text{otherwise.} \end{cases}$$

We now verify Condition (E0') – (E4') for LHS of (5.23). Firstly, we concentrate on Condition

(E1'). Indeed, by standard calculations, we have: For $1 \le l \le k$,

$$\begin{aligned}
&\sqrt{lk}\int_{\mathbb{R}\times\mathbb{R}} H^\star_{\lambda_i,l}(u,x)H^\star_{\lambda_j,k}(u,x)\nu(du)dx\\
=\;& \frac{1}{\sqrt{lk}}c_\nu^2\Big(\frac{1}{2(\lambda_i+\lambda_j)}\times\big(1-\exp(-2\lambda_i l)\big)\times\big(1-\exp(-2\lambda_j k)\big)+l-\frac{1}{2\lambda_i}\times(1-\exp(-2\lambda_i l))\\
&-\frac{1}{2\lambda_j}\big(\exp(2\lambda_j l)-1\big)\exp(-2\lambda_j k)+\frac{1}{2(\lambda_i+\lambda_j)}\big(\exp(2\lambda_j(l-k))-\exp(-\lambda_i l-\lambda_j k)\big)\Big)\\
\le\;& \frac{c_\nu^2}{\sqrt{lk}}\times(\frac{1}{\lambda_i+\lambda_j}+l)\\
\le\;& c_\nu^2\times(\frac{1}{\lambda_i+\lambda_j}+1)\times\frac{\sqrt{l}}{\sqrt{k}},
\end{aligned}$$

and

$$\begin{aligned}
&2\sqrt{lk}\int_{\mathbb{R}^4} H_{\lambda_i,l}(u,x;u',x')H_{\lambda_j,k}(u,x;u',x')\nu(du)\nu(du')dxdx'\\
=\;& \frac{2}{\sqrt{lk}}(\int_{\mathbb{R}}u^2\nu(du))^2\int_{-\infty}^{l}dx\int_{-\infty}^{l}dx'\exp\big((\lambda_i+\lambda_j)(x+x')\big)\times\big(1-\exp(-2\lambda_i l)\big)\\
&\times\big(1-\exp(-2\lambda_j k)\big)\times\mathbf{1}_{(x\vee x'\le 0)}+\exp\big((\lambda_i+\lambda_j)(x+x')\big)\times\Big(\exp\big(-2\lambda_i(x\vee x')\big)-\exp(-2\lambda_i l)\Big)\\
&\times\Big(\exp\big(-2\lambda_j(x\vee x')\big)-\exp(-2\lambda_j k)\Big)\times\mathbf{1}_{(x\vee x'>0)}\\
=\;& \frac{2}{\sqrt{lk}}\Big(\int_{-\infty}^{0}dx\int_{-\infty}^{0}dx'\exp\big((\lambda_i+\lambda_j)(x+x')\big)\big(1-\exp(-2\lambda_i l)\big)\big(1-\exp(-2\lambda_j k)\big)\\
&+2\int_0^l dx\int_{-\infty}^{x}dx'\exp\big((\lambda_i+\lambda_j)(x+x')\big)\big(\exp(-2\lambda_i x)-\exp(-2\lambda_i l)\big)\\
&\times\big(\exp(-2\lambda_j x)-\exp(-2\lambda_j k)\big)\Big)\\
=\;& \frac{2}{\sqrt{lk}}\Big(\frac{(1-\exp(-2\lambda_i l))\times(1-\exp(-2\lambda_j k))}{(\lambda_i+\lambda_j)^2}+\frac{2}{\lambda_i+\lambda_j}\times\Big(l-\frac{1}{2\lambda_i}\big(1-\exp(-2\lambda_i l)\big)\\
&-\frac{1}{2\lambda_j}\times\big(\exp(2\lambda_j(l-k))-\exp(-2\lambda_j k)\big)+\frac{1}{2(\lambda_i+\lambda_j)}\times\big(\exp(2\lambda_j(l-k))-\exp(-2\lambda_j(l+k))\big)\Big)\Big)\\
\le\;& \frac{4}{\sqrt{lk}(\lambda_i+\lambda_j)}\times(\frac{1}{\lambda_i+\lambda_j}+l)\\
\le\;& \frac{4}{\lambda_i+\lambda_j}\times(\frac{1}{\lambda_i+\lambda_j}+1)\times\frac{\sqrt{l}}{\sqrt{k}}.
\end{aligned}$$

Secondly, we use the fact that for $\lambda=\lambda_1,\ldots,\lambda_d$, the following asymptotic relations holds as $k\to\infty$:

(a) $\qquad \|\sqrt{k}H^\star_{\lambda,k}\|^3_{L^3(d\nu dx)}\sim\frac{1}{\sqrt{k}}$;

(b) $\qquad \|\sqrt{k}H_{\lambda,k}\|^2_{L^4((d\nu dx)^2)}\sim\frac{1}{\sqrt{k}}$;

(c) $\qquad \|(\sqrt{k}H_{\lambda,k})\star_2^1(\sqrt{k}H_{\lambda,k})\|_{L^2(d\nu dx)}=\|(\sqrt{k}H_{\lambda,k})\star_1^0(\sqrt{k}H_{\lambda,k})\|_{L^2((d\nu dx)^3)}\sim\frac{1}{\sqrt{k}}$;

(d) $\quad \|(\sqrt{k}H_{\lambda,k}) \star_1^1 (\sqrt{k}H_{\lambda,k})\|_{L^2((d\nu dx)^2)} \sim \dfrac{1}{\sqrt{k}};$

(e) $\quad \|(\sqrt{k}H^\star_{\lambda,k}) \star_1^1 (\sqrt{k}H_{\lambda,k})\|_{L^2(d\nu dx)} \sim \dfrac{1}{\sqrt{k}};$

(f) $\quad \left|k\int_{\mathbb{R}\times\mathbb{R}} H^\star_{\lambda_i,k}(u,x)H^\star_{\lambda_j,k}(u,x)\nu(du)dx - c_\nu^2\right| = O\left(\dfrac{1}{k}\right);$

(g) $\quad \left|2k\int_{\mathbb{R}^4} H_{\lambda_i,k}(u,x;u',x')H_{\lambda_j,k}(u,x;u',x')\nu(du)\nu(du')dxdx' - \frac{4}{\lambda_i+\lambda_j}\right| = O\left(\dfrac{1}{k}\right).$

The reader is referred to [46, Section 7] , [47, Section 4] and the proof of Theorem 3.5.7 in Chapter 3 for a proof of the above asymptotic relations.

In summary, all the conditions are verified and the conclusion follows. ■

Example 3 (Empirical joint moments of shifted processes)
We now study a generalization of Example 2. We define

$$Q_h(k,\lambda) := \sqrt{k}\left(\frac{1}{k}\int_0^k Y_t^\lambda Y_{t+h}^\lambda dt - \exp(-\lambda h)\right), \qquad h>0, k>0, \lambda>0.$$

The theorem below is a multivariate CLT and ASCLT for $Q_h(k,\lambda)$.

Theorem 5.6.3 *For $\lambda_1,\dots,\lambda_d>0$ and $h\geq 0$, as $k\to\infty$,*

$$\bar{Q}_h(k) = (Q_h(k,\lambda_1),\dots,Q_h(k,\lambda_d)) \stackrel{(law)}{\longrightarrow} X_E, \tag{5.24}$$

where X_E is a centered d-dimensional Gaussian vector with covariance matrix $E=(E_{ij})_{d\times d}$, with

$$E_{ij} = \frac{4}{\lambda_i+\lambda_j} + c_\nu^2\exp\Big(-(\lambda_i+\lambda_j)h\Big), \qquad 1\leq i,j\leq d$$

and $c_\nu^2 = \int_{\mathbb{R}} u^4\nu(du)$. Moreover, there exists a constant $0<\gamma(h,\bar{\lambda}) = \gamma(h,\lambda_1,\dots,\lambda_d)<\infty$, independent of k and such that

$$d_3(\bar{Q}_h(k), X_E) \leq \frac{\gamma(h,\bar{\lambda})}{\sqrt{k}}.$$

Proof. From the proof of Theorem 3.5.8 in Chapter 3, we know that

$$Q_h(k,\lambda) = I_2(\sqrt{k}H^h_{\lambda,k}) + I_1(\sqrt{k}H^{*,h}_{\lambda,k}),$$

where

$$\begin{aligned} H^{*,h}_{\lambda,k}(u,x) &= u^2\frac{\mathbf{1}_{(-\infty,k]}(x)}{k}\times\exp(\lambda(2x-h))\times\Big(\mathbf{1}_{(x>0)}\times(\exp(-2\lambda x)-\exp(-2\lambda k)) \\ &\quad +\mathbf{1}_{(x\leq 0)}\times(1-\exp(-2\lambda k))\Big), \end{aligned}$$

and

$$\begin{aligned} H^h_{\lambda,k}(u,x;u',x') &= uu'\frac{\mathbf{1}_{(-\infty,k]}(x)\mathbf{1}_{(-\infty,k+h]}(x')}{k}\times\exp(\lambda(x+x'-h)) \\ &\times\Big(\mathbf{1}_{(x\vee(x'-h)>0)}\times\big(\exp(-2\lambda(x\vee(x'-h)))-\exp(-2\lambda k)\big) \\ &+\mathbf{1}_{(x\vee(x'-h)\le 0)}\times(1-\exp(-2\lambda k))\Big). \end{aligned}$$

And we have proved the stronger CLT result:

$$(I_1(\sqrt{k}H^{\star,h}_{\lambda_1,k}),\ldots,I_1(\sqrt{k}H^{\star,h}_{\lambda_d,k}),I_2(\sqrt{k}H^h_{\lambda_1,k}),\ldots,I_2(\sqrt{k}H^h_{\lambda_d,k}))\overset{(law)}{\longrightarrow}X_{D^h} \tag{5.25}$$

as $k\to\infty$. Here, X_{D^h} is a centered $2d$-dimensional Gaussian vector with covariance matrix D^h defined as:

$$D^h(i,j)=\begin{cases} c_\nu^2\exp(-(\lambda_i+\lambda_j)h), & \text{if } 1\le i,j\le d \\ \dfrac{4}{\lambda_i+\lambda_j}, & \text{if } d+1\le i,j\le 2d \\ 0, & \text{otherwise.}\end{cases}$$

Now we work on the ASCLT part. It suffices to check Condition (E0) – (E4). Firstly, we look at Condition (E1):
For $1\le l\le k$, we have

$$\begin{aligned} &\sqrt{lk}\int_{\mathbb{R}\times\mathbb{R}}H^{*,h}_{\lambda,l}(u,x)H^{*,h}_{\lambda,k}(u,x)\nu(du)dx \\ =\ & \frac{1}{\sqrt{lk}}c_\nu^2\Big(\int_{-\infty}^0 dx\exp\big((\lambda_i+\lambda_j)(2x-h)\big)\times\big(1-\exp(-2\lambda_i l)\big)\times\big(1-\exp(-2\lambda_j k)\big) \\ +\ & \int_0^l dx\exp\big((\lambda_i+\lambda_j)(2x-h)\big)\times\big(\exp(-2\lambda_i x)-\exp(-2\lambda_i l)\big)\times\big(\exp(-2\lambda_j x)-\exp(-2\lambda_j k)\big)\Big) \\ =\ & \frac{1}{\sqrt{lk}}c_\nu^2\exp(-(\lambda_i+\lambda_j)h)\Big(\frac{1}{2(\lambda_i+\lambda_j)}\times\big(1-\exp(-2\lambda_i l)\big)\times\big(1-\exp(-2\lambda_j k)\big)+l \\ & -\frac{1}{2\lambda_i}\times(1-\exp(-2\lambda_i l))-\frac{1}{2\lambda_j}\big(\exp(2\lambda_j l)-1\big)\exp(-2\lambda_j k) \\ & +\frac{1}{2(\lambda_i+\lambda_j)}\big(\exp(2\lambda_j(l-k))-\exp(-\lambda_i l-\lambda_j k)\big)\Big) \\ \le\ & \frac{c_\nu^2}{\sqrt{lk}}\exp(-(\lambda_i+\lambda_j)h)\times(\frac{4}{\lambda_i+\lambda_j}+l) \\ \le\ & c_\nu^2\exp(-(\lambda_i+\lambda_j)h)\times(\frac{1}{\lambda_i+\lambda_j}+1)\times\frac{\sqrt{l}}{\sqrt{k}}. \end{aligned}$$

We notice that

$$H^h_{\lambda,k}(u,x;u',x')=H_{\lambda,k}(u,x;u',x'-h)$$

Then, as shown in the proof of Theorem 3.5.8 in Chapter 3, we have

$$\begin{aligned} &2\sqrt{lk}\int_{\mathbb{R}^4}H_{\lambda_i,l}(u,x;u',x')H_{\lambda_j,k}(u,x;u',x')\nu(du)\nu(du')dxdx' \\ \le\ & \frac{4}{\lambda_i+\lambda_j}\times(\frac{1}{\lambda_i+\lambda_j}+1)\times\frac{\sqrt{l}}{\sqrt{k}}. \end{aligned}$$

Secondly, we verify the rest of conditions.

Indeed, we know from the proof of Theorem 3.5.8 in Chapter 3 that: For $\lambda = \lambda_1, \dots, \lambda_d$ and $h \geq 0$, the following asymptotic relations holds as $k \to \infty$:

(a) $\|\sqrt{k} H^{*,h}_{\lambda,k}\|^3_{L^3(d\nu dx)} \sim \frac{1}{\sqrt{k}}$;

(b) $\|\sqrt{k} H^{h}_{\lambda,k}\|^2_{L^4((d\nu dx)^2)} \sim \frac{1}{\sqrt{k}}$;

(c) $\|(\sqrt{k} H^{h}_{\lambda,k}) \star_2^1 (\sqrt{k} H^{h}_{\lambda,k}\|_{L^2(d\nu dx)} = \|(\sqrt{k} H_{\lambda,k}) \star_1^0 (\sqrt{k} H^{h}_{\lambda,k})\|_{L^2((d\nu dx)^3)} \sim \frac{1}{\sqrt{k}}$;

(d) $\|(\sqrt{k} H^{h}_{\lambda,k}) \star_1^1 (\sqrt{k} H^{h}_{\lambda,k})\|_{L^2((d\nu dx)^2)} \sim \frac{1}{\sqrt{k}}$;

(e) $\|(\sqrt{k} H^{*,h}_{\lambda,k}) \star_1^1 (\sqrt{k}^h H_{\lambda,k})\|_{L^2(d\nu dx)} \sim \frac{1}{\sqrt{k}}$;

(f) $\left| 2k \int_{\mathbb{R}\times\mathbb{R}} H^{h}_{\lambda,k}(u,x,u',x') H^{h}_{\lambda,k}(u,x,u',x') \nu(du)\nu(du') dx dx' - \frac{4}{\lambda_i + \lambda_j} \right| = O\left(\frac{1}{k}\right)$;

(g) $\left| k \int_{\mathbb{R}\times\mathbb{R}} H^{*,h}_{\lambda,l}(u,x) H^{*,h}_{\lambda,k}(u,x) \nu(du) dx - c_\nu^2 \exp(-(\lambda_i + \lambda_j)h) \right| = O\left(\frac{1}{k}\right)$.

In conclusion, all the conditions are justified and the ASCLT holds. ■

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