Scaling Limit of Arbitrary Genus Random Maps

Jérémie Bettinelli

UNIVERSITÉ PARIS-SUD 11

Faculté des sciences d'Orsay

THÈSE

présentée pour obtenir

LE GRADE DE DOCTEUR EN SCIENCES DE L'UNIVERSITÉ PARIS-SUD 11

Spécialité : Mathématiques

par

Jérémie BETTINELLI

Limite d'échelle de cartes aléatoires en genre quelconque

Directeur de thèse	M. Grégory MIERMONT
Rapporteurs	M. Omer ANGEL M. Jean-François MARCKERT

soutenue le 26 octobre 2011 devant la commission d'examen :

M. Jérémie BOUTTIER	*Examinateur*
M. Jean-François LE GALL	*Examinateur*
M. Jean-François MARCKERT	*Rapporteur*
M. Grégory MIERMONT	*Directeur de thèse*
M. Gilles SCHAEFFER	*Examinateur*
M. Marc YOR	*Examinateur*

Jérémie BETTINELLI
jeremie.bettinelli@normalesup.org
www.normalesup.org/~bettinel

Thèse préparée au

Département de Mathématiques d'Orsay
Laboratoire de Mathématiques (UMR 8628), Bât. 425
Université Paris-Sud XI
F-91 405 Orsay CEDEX

À l'échelle d'une carte, le monde est un jeu d'enfant.

Laurent GRAFF, dans *Voyage, voyages*.

Limite d'échelle de cartes aléatoires en genre quelconque

AU COURS DE CE TRAVAIL, nous nous intéressons aux limites d'échelle de deux classes de cartes. Dans un premier temps, nous regardons les quadrangulations biparties de genre $g \geq 1$ fixé et, dans un second temps, les quadrangulations planaires à bord dont la longueur du bord est de l'ordre de la racine carrée du nombre de faces. Nous voyons ces objets comme des espaces métriques, en munissant leurs ensembles de sommets de la distance de graphe, convenablement renormalisée.

Nous montrons qu'une carte prise uniformément parmi les cartes ayant n faces dans l'une de ces deux classes tend en loi, au moins à extraction près, vers un espace métrique limite aléatoire lorsque n tend vers l'infini. Cette convergence s'entend au sens de la topologie de Gromov–Hausdorff. On dispose de plus des informations suivantes sur l'espace limite que l'on obtient. Dans le premier cas, c'est presque sûrement un espace de dimension de Hausdorff 4 homéomorphe à la surface de genre g. Dans le second cas, c'est presque sûrement un espace de dimension 4 avec une frontière de dimension 2, homéomorphe au disque unité de $\mathbb{R}^2$. Nous montrons en outre que, dans le second cas, si la longueur du bord est un petit o de la racine carrée du nombre de faces, on obtient la même limite que pour les quadrangulations sans bord, c'est-à-dire la *carte brownienne*, et l'extraction n'est plus requise.

Mots-clefs : cartes aléatoires, arbres aléatoires, limite d'échelle, processus conditionnés, convergence régulière, topologie de Gromov, dimension de Hausdorff, arbre continu brownien, espaces métriques aléatoires.
Classification AMS : 60F17, 60D05, 57N05, 60C05.

Scaling Limit of Arbitrary Genus Random Maps

IN THIS WORK, we discuss the scaling limits of two particular classes of maps. In a first time, we address bipartite quadrangulations of fixed genus $g \geq 1$, and, in a second time, planar quadrangulations with a boundary whose length is of order the square root of the number of faces. We view these objects as metric spaces by endowing their sets of vertices with the graph metric, suitably rescaled.

We show that a map uniformly chosen among the maps having n faces in one of these two classes converges in distribution, at least along some subsequence, toward a limiting random metric space as n tends to infinity. This convergence holds in the sense of the Gromov–Hausdorff topology on compact metric spaces. We moreover have the following information on the limiting space. In the first case, it is almost surely a space of Hausdorff dimension 4 that is homeomorphic to the genus g surface. In the second case, it is almost surely a space of Hausdorff dimension 4 with a boundary of Hausdorff dimension 2 that is homeomorphic to the unit disc of $\mathbb{R}^2$. We also show that in the second case, if the length of the boundary is little-o of the square root of the number of faces, the same convergence holds without extraction and the limit is the same as for quadrangulations without boundary, that is the *Brownian map*.

Key words and phrases: random maps, random trees, scaling limits, conditioned processes, regular convergence, Gromov topology, Hausdorff dimension, Brownian CRT, random metric spaces.
AMS classification: 60F17, 60D05, 57N05, 60C05.

Remerciements

Je tiens tout d'abord à remercier Grégory Miermont, qui m'a spontanément proposé un sujet de thèse passionnant. Il a été pour moi un directeur exemplaire, toujours disponible malgré un emploi du temps bien chargé. Il a su m'encadrer, me guider, me conseiller et me faire partager sa culture mathématique et son enthousiasme, toujours dans la bonne humeur. Pour toutes ces choses, je lui suis très reconnaissant.

Je suis très honoré que Jean-François Marckert et Omer Angel, dont j'admire les travaux mathématiques, aient accepté de rapporter cette thèse. Je les remercie pour toute l'attention qu'ils ont portée à la lecture de ce manuscrit, ainsi que pour leurs remarques pertinentes. Je remercie également Jérémie Bouttier, Jean-François Le Gall, Gilles Schaeffer et Marc Yor d'avoir montré leur intérêt pour cette thèse en acceptant de faire partie du jury. Je remercie plus particulièrement Jean-François Le Gall qui m'a fait découvrir les probabilités à travers d'excellents cours et m'a conseillé sur mon cursus universitaire, en m'orientant notamment vers Grégory.

Merci à Olivier Garet d'avoir encadré mes premiers pas dans la recherche en me proposant un mémoire de Master 2 très intéressant. Merci également à tous les enseignants qui ont influencé mon parcours et m'ont fait prendre goût aux mathématiques en général et aux probabilités en particulier. Un grand merci aussi à toutes les personnes qui ont facilité mes démarches administrative : Bénédicte, Catherine, Christine, Laurence, Valérie et Zaïna.

Il est maintenant temps de remercier mes collègues et amis qui m'ont supporté, aidé, distrait tout au long de mes études supérieures, en commençant par mes camarades de promo de l'ENS : Adrien, Amic, Cyrille, Danié, Hugo, J.-P., Nico, Tibo, et tous les autres dont je ne cite pas les noms tant la liste serait longue. Merci à Patricia, dont la mutation à Nice m'a permit de bénéficier d'une place de choix au sein du DMA, en compagnie de Marie, Amandine et Vincent, où j'ai été très bien accueilli. Merci également à tous les occupants du passage vert et assimilés (dont le bureau 304 fait partie) qui ont contribué à une bonne ambiance quotidienne dans ces bureaux. Merci enfin aux habitants (à temps plein ou partiel) du bâtiment 430 à Orsay qui ont réussi à m'intégrer malgré mon désintérêt total pour le tennis ou le foot (mais quand même, allez Lens). Merci particulièrement à Sébastien qui rentre dans les deux dernières catégories et dont la présence au bureau 108, à défaut de présager une journée studieuse, présage toujours une journée agréable...

Mes derniers remerciements vont à mes amis, à mes proches et aux membres de ma famille. Merci en particulier à mes parents, Bernard & Bernadette sans qui cette thèse ne serait pas, et qui font partie des rares personnes à en avoir intégralement lu l'introduction. L'effort est particulièrement admirable lorsque l'on sait qu'ils l'ont fait à la recherche des fautes d'orthographe qui m'ont résisté. Merci à Hélène, Aymeric et Marie qui me font vivre des moments inoubliables (et me font parfois revivre des moments que j'aimerais oublier), ainsi que pour leur soutien fraternel. Enfin, merci à Nina pour avoir su me soutenir et m'encourager au cours de ces dernières années et sans qui me vie serait certainement moins plaisante.

Table des matières

I

Introduction

Ce qui compte, ce n'est pas de disposer de bonnes ou de mauvaises cartes mais de savoir jouer avec les mauvaises.

Bernard WERBER, dans *L'Empire des anges*.

Présentation générale 1

Les résultats originaux de cette thèse sont présentés à partir de la section 1.4, et sont signalés à l'aide d'un fond grisé. Ils sont extraits des références [Bet10a, Bet10b, Bet11]. Sauf mention contraire, toutes les variables aléatoires seront définies sur un espace de probabilité commun $(\Omega, F, \mathbb{P})$.

1.1 Premières définitions

Nous commençons par présenter les notions dont nous aurons besoin tout au long de ce mémoire. Nous renvoyons le lecteur à l'article de survol de Grégory Miermont [Mie09a] pour plus de détails, ainsi qu'aux notes de cours de l'école d'été de Buzios par Grégory Miermont et Jean-François Le Gall [LGM11b] pour une présentation plus complète.

1.1.1 Cartes

Nous définissons ici les cartes de la façon la plus géométrique. Pour d'autres définitions, ainsi que la preuve de l'équivalence de ces définitions, nous référons le lecteur au livre de Bojan Mohar et Carsten Thomassen [MT01]. Bien que plutôt intuitive, la notion de carte demande un peu de temps pour être définie proprement. De façon informelle, une carte est simplement le dessin d'un graphe sur une surface, considérée à homéomorphisme direct près. Soyons un peu plus précis.

Un **graphe** est la donnée d'un triplet (V, E, i), où V et E sont des ensembles finis dont les éléments sont appelés respectivement **sommets** et **arêtes**, et i associe à chaque arête un ou deux sommets du graphe, appelés ses **extrémités**. Les arêtes n'ayant qu'une seule extrémité sont appelées **boucles**. Notons que deux arêtes distinctes peuvent avoir les mêmes extrémités : on parle dans ce cas d'arêtes multiples. Attention au fait que cette notion diffère de la notion de graphe que l'on rencontre traditionnellement en théorie des graphes, elle correspond plutôt à la notion de multigraphe fini.

Le théorème de classification[1] des surfaces affirme que les surfaces compactes connexes orientables sans bord sont caractérisées à homéomorphisme près par un entier $g \geq 0$ que l'on appelle son **genre**. La surface de genre 0 est la sphère $\mathbb{S}^2$ de $\mathbb{R}^3$, et pour $g \geq 1$, la surface de genre g, que l'on notera $\mathbb{T}_g$ et nommera tore à g trous, s'obtient par somme connexe de g tores de dimension 2. On peut aussi définir $\mathbb{T}_g$ comme la sphère $\mathbb{S}^2$ à laquelle on rajoute g anses.

Un graphe (V, E, i) est **plongé** dans la surface $\mathcal{S}$ si $V \subseteq \mathcal{S}$ et, pour tout $e \in E$, il existe une application continue $f_e : [0,1] \to \mathcal{S}$ telle que sa restriction à $[0,1[$ soit injective, $i(e) = \{f_e(0), f_e(1)\}$, $f_e([0,1]) = e$, et $f_e(]0,1[)$ n'intersecte ni V ni les autres arêtes. Autrement dit, le graphe est dessiné sur la surface, de telle façon que les arêtes ne s'intersectent éventuellement qu'en les sommets du graphe. Les **faces** d'un graphe plongé (V, E, i) sont les composantes connexes du complémentaire de l'ensemble

$$V \cup \bigcup_{e \in E} e.$$

Un graphe est plongé de façon **cellulaire** si toutes ses faces sont homéomorphes à des disques ouverts de $\mathbb{R}^2$. Notons que cela présuppose que le graphe soit connexe, et, dans le cas où la surface est la sphère, tout graphe connexe plongé l'est automatiquement de façon cellulaire, par le théorème de Jordan–Schoenflies. On dit que deux graphes plongés de façon cellulaire sont **équivalents** s'il existe un homéomorphisme des surfaces sous-jacentes qui préserve l'orientation et induit un isomorphisme des graphes.

Définition 1.1. *Une **carte** est une classe d'équivalence de graphes plongés de façon cellulaire.*

Remarquons en particulier que les surfaces sous-jacentes de deux plongements cellulaires homéomorphes sont elles-même homéomorphes, et ont donc même genre. Le **genre** d'une carte est alors défini comme le genre de la surface sous-jacente d'un de ses représentants. Dans le cas où le genre est 0, on peut plonger les cartes dans le plan plutôt que dans la sphère. Cela revient à distinguer une face particulière qui correspond à la composante infinie du plan. Pour cette raison, on appelle **cartes planaires** les cartes de genre 0.

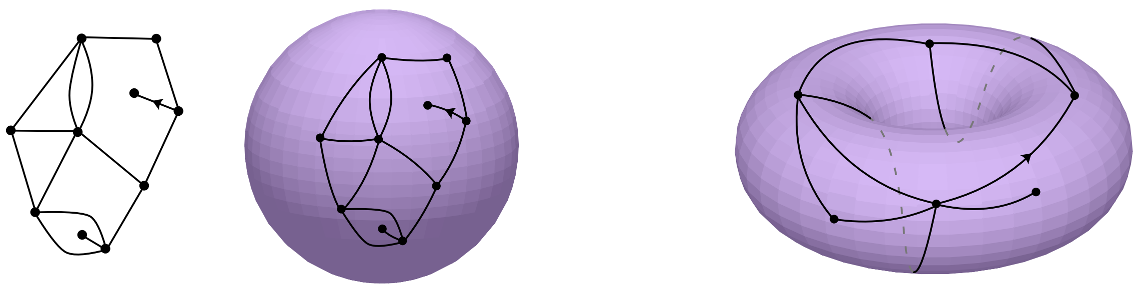

FIGURE 1.1. ***Gauche.*** *Exemple de carte planaire plongée dans le plan ou* $\mathbb{S}^2$. ***Droite.*** *Exemple de carte de genre* 1.

[1] Voir par exemple [MT01, Chapitre 3].

L'information contenue dans le graphe sous-jacent d'une carte ne détermine pas la carte. En effet, il est possible de plonger de façon cellulaire un même graphe de sorte à obtenir des cartes différentes (éventuellement même de genres différents), comme on peut le voir sur la figure 1.2.

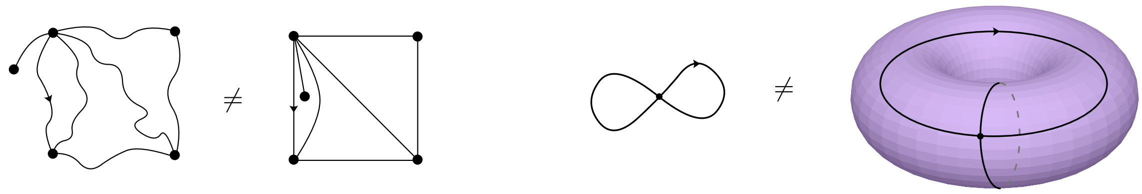

FIGURE 1.2. *Deux exemples de cartes différentes ayant le même graphe sous-jacent.*

Il faut aussi garder à l'esprit qu'un homéomorphisme entre deux graphes plongés n'est pas nécessairement homotope à l'identité. La déformation des arêtes peut être plus subtile que l'on pense. L'exemple classique est celui d'un *twist de Dehn*, homéomorphisme qui consiste à couper $\mathbb{T}_1$ selon l'un de ses cycles générateurs, faire un tour, et recoller. Cela se voit mieux dans le revêtement universel du tore, où l'on fait simplement une translation selon un des axes.

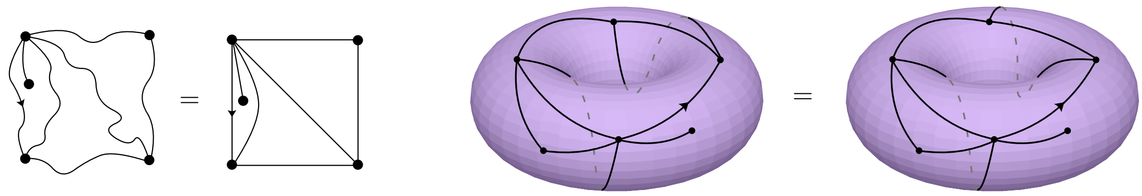

FIGURE 1.3. *Deux exemples de représentants de la même carte.*

Toutes les arêtes possèdent naturellement deux orientations. On appelle **demi-arête** une arête munie d'une de ces deux orientations. Implicitement, toutes les cartes que l'on considère seront enracinées : cela signifie que l'on distingue une demi-arête que l'on appelle **racine**. Sur les figures, on représentera toujours la racine à l'aide d'une pointe de flèche. L'enracinement des cartes permet de supprimer les symétries non triviales : un homéomorphisme d'un graphe plongé de façon cellulaire sur lui-même qui fixe une demi-arête fixe toutes les demi-arêtes. Par convention, on notera généralement $\mathfrak{e}_*$ la racine d'une carte. De plus, si $\mathfrak{e}$ est une demi-arête, on note $\bar{\mathfrak{e}}$ la demi-arête correspondant à la même arête que $\mathfrak{e}$, mais avec l'orientation inverse, et on désigne par $\mathfrak{e}^-$ et $\mathfrak{e}^+$ son origine et sa cible. Ainsi, $\mathfrak{e}$ est la courbe orientée allant de $\mathfrak{e}^-$ à $\mathfrak{e}^+$, alors que $\bar{\mathfrak{e}}$ est la courbe orientée de $\mathfrak{e}^+$ à $\mathfrak{e}^-$.

1.1.2 Quadrangulations biparties

On dit qu'une demi-arête $\mathfrak{e}$ est **incidente** à un face f (ou indifféremment que f est incidente à $\mathfrak{e}$) si $\mathfrak{e}$ borde f et est orientée de telle sorte que f se situe à la gauche de $\mathfrak{e}$. On appelle **degré** d'une face le nombre de demi-arêtes qui lui sont incidentes. Remarquons notamment que les deux demi-arêtes correspondant à une arête peuvent très bien être incidentes à la même face. Dans ce cas, il faudra veiller à compter deux fois cette arête pour déterminer le degré de la face qui leur est incidente. Dans la suite, les faces de degré 4, que l'on appelle **quadrangles** joueront un rôle particulier.

Définition 1.2. *On appelle **quadrangulation** une carte dont toutes les faces sont des quadrangles.*

Ces cartes, bien que moins communes dans la littérature que, par exemple, les triangulations (cartes dont toutes les faces sont de degré 3) sont les objets qui apparaissent naturellement dans les études bijectives que nous décriront plus tard. Plus particulièrement, nous serons amenés à travailler avec les quadrangulations biparties : une carte est **bipartie** s'il existe un 2-coloriage de ses sommets tel que deux sommets adjacents quelconques sont toujours de couleurs différentes. Dans le cas planaire, il n'est pas difficile de voir que toutes les quadrangulations sont biparties. En revanche, cela n'est plus vrai en genre supérieur à 1 : par exemple, la carte de genre 1 tout à droite de la figure 1.2 est bien une quadrangulation, mais elle n'est pas bipartie, puisqu'elle contient des boucles. En fait, on peut montrer qu'en genre g, la proportion de quadrangulations à n faces qui sont biparties tend vers $1/4^g$ lorsque n tend vers l'infini [Ben91, Équation (2.2)]. Dans la suite, on note $\mathcal{Q}_n^g$ l'ensemble des quadrangulations biparties de genre g à n faces. Souvent, nous travaillerons à genre fixé et, dans le but d'alléger les notations, nous noterons simplement $\mathcal{Q}_n$ cet ensemble.

Une des raisons pour lesquelles les quadrangulations biparties sont particulièrement intéressantes est la bijection illustrée sur la figure 1.4, due à William Tutte, entre les cartes de genre $g \geq 0$ fixé à n arêtes et les quadrangulations biparties de même genre à n faces. Néanmoins, la raison qui nous pousse à étudier ces objets dans cette thèse est une autre bijection que l'on exposera en détails dans la section 1.3.1.

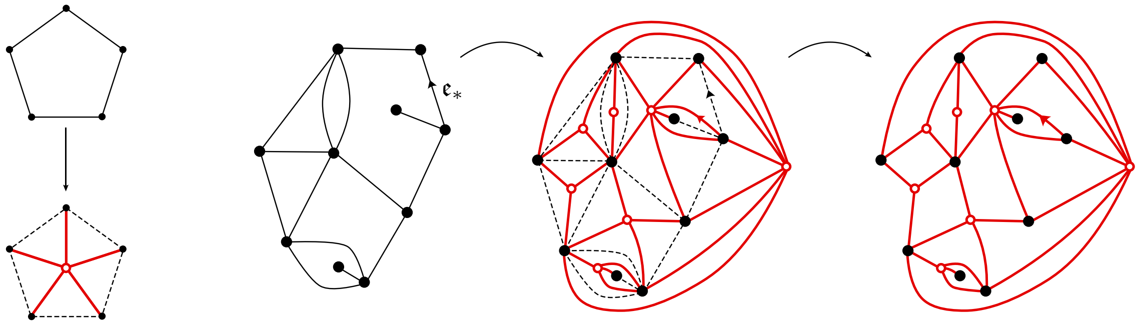

FIGURE 1.4. *Bijection entre les cartes générales et les quadrangulations biparties. Partant d'une carte générale enracinée en $\mathfrak{e}_*$, on ajoute dans chaque face un nouveau sommet que l'on relie aux sommets de la face, puis on efface les anciennes arêtes. Enfin, la racine de cette nouvelle carte est la demi-arête partant de $\mathfrak{e}_*^-$ la plus proche de $\mathfrak{e}_*$, dans la face qui lui est incidente.*

Soit $\mathfrak{m}$ une carte de genre g. On note respectivement $V(\mathfrak{m})$, $E(\mathfrak{m})$, $\vec{E}(\mathfrak{m})$, et $F(\mathfrak{m})$ les ensembles de ses sommets, arêtes, demi-arêtes, et faces. Il est alors évident que $|\vec{E}(\mathfrak{m})| = 2\,|E(\mathfrak{m})|$ (où l'on note $|A|$ le cardinal d'un ensemble fini A). De plus, les cardinaux de ces ensembles sont reliés par la célèbre formule d'Euler :

$$|V(\mathfrak{m})| - |E(\mathfrak{m})| + |F(\mathfrak{m})| = 2 - 2g.$$

Ainsi, il suffit d'une relation supplémentaire pour pouvoir déterminer toutes ces quantités dès lors que l'on en connaît une. Si, par exemple, $\mathfrak{q}$ est une quadrangulation, alors on sait que $4\,|F(\mathfrak{q})| = |\vec{E}(\mathfrak{m})|$ puisque, par définition, chaque face est incidente à exactement 4 demi-arêtes. Les éléments de $\mathcal{Q}_n^g$ ont donc n faces, $2n$ arêtes, $4n$ demi-arêtes, et $n + 2 - 2g$ sommets. Cette remarque sera utile dans la suite.

1.1.3 Limite d'échelle

Nous allons nous intéresser dans cette thèse à la limite d'échelle des cartes aléatoires. Le concept de limite d'échelle est bien connu en théorie des probabilités. Le principe général en est le suivant. On se donne une certaine classe d'objets combinatoires pour lesquels on dispose d'une notion de *volume* et d'une notion de *taille*. Lorsque le volume tends vers l'infini, on cherche à renormaliser la taille de manière à obtenir une limite intéressante. Plus précisément, on choisit un objet au hasard parmi les objets de volume n appartenant à cette classe. Il se peut que, une fois la taille renormalisée convenablement, cet objet tende en loi vers un objet limite continu lorsque $n \to \infty$. Dans le cas de la marche aléatoire standard, par exemple, si l'on appelle volume le nombre de pas et taille la valeur de la marche, alors cet objet admet comme limite d'échelle le mouvement brownien : on choisit un chemin uniformément au hasard parmi les chemins constitués de n pas valant ± 1 et, après avoir renormalisé le temps par n et l'espace par $\sqrt{n}$, ce chemin tend en loi vers un mouvement brownien défini sur $[0,1]$, d'après le théorème de Donsker. On peut aussi penser à divers modèles d'arbres, pour lesquels le volume est par exemple le nombre de sommets et la taille est la hauteur.

L'intérêt de cette approche est double. D'une part, l'objet que l'on obtient à la limite est un objet continu qui est souvent intéressant en soi, et cela indépendamment du fait qu'il apparaisse comme limite d'échelle du modèle discret. D'autre part, l'étude du modèle continu peut faire apparaître des propriétés asymptotiques du modèle discret difficiles à obtenir directement.

Par ailleurs, l'objet limite possède souvent une propriété d'*universalité* : on obtient la même limite d'échelle pour plusieurs classes différentes (mais similaires) d'objets. C'est le cas par exemple du mouvement brownien qui apparaît comme limite d'échelle de n'importe quelle marche aléatoire convenablement renormalisée, à condition que la loi de ses pas soit centrée et de variance finie. On peut aussi donner l'exemple du CRT (voir section 1.3.2.1) de David Aldous [Ald91, Ald93] qui est la limite d'échelle de nombreux modèles d'arbres aléatoires [DLG02].

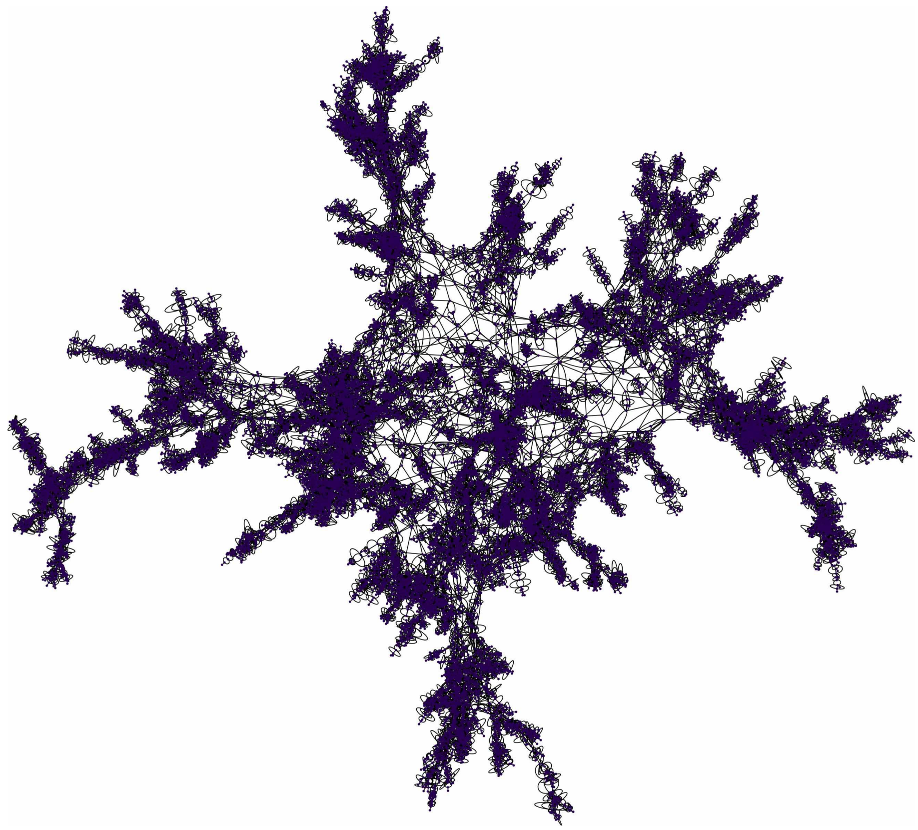

FIGURE 1.5. *Simulation d'une quadrangulation planaire uniforme à* 30 000 *faces réalisée par Jean-François Marckert. Cette représentation ne prend pas en compte le plongement, mais uniquement la structure de graphe sous-jacente.*

Cette approche nécessite de définir divers paramètres. Dans notre cas, la classe que l'on va considérer sera la classe des quadrangulations biparties de genre $g \geq 0$ fixé, et la taille d'une quadrangulation sera son nombre de faces. La loi que l'on choisit généralement est la loi uniforme, lorsque les ensembles considérés sont finis. Même si, au vu de la définition que l'on a donnée, il n'est pas immédiat que $\mathcal{Q}_n^g$ soit fini, c'est bien le cas. Pour le voir, remarquons déjà que le nombre de graphes connexes à $2n$ arêtes est fini. Ensuite, on peut se convaincre qu'il n'y a qu'un nombre fini de manière de plonger un graphe dans une surface, à homéomorphisme près. Ceci provient du fait qu'il y a au plus $k!$ façons d'ordonner les arêtes incidentes à un sommet de degré k. On peut donc choisir une quadrangulation $\mathfrak{q}_n$ uniformément au hasard dans $\mathcal{Q}_n^g$.

Il faut ensuite préciser l'espace dans lequel on se place, ainsi que la topologie que l'on considère. Nous adopterons le point de vue qui consiste à voir une carte $\mathfrak{m}$ comme un espace métrique fini, en munissant l'ensemble de ses sommets $V(\mathfrak{m})$ de la **distance de graphe** $d_\mathfrak{m}$: pour a et $b \in V(\mathfrak{m})$, on définit $d_\mathfrak{m}(a,b)$ comme le nombre minimal d'arêtes qu'il faut traverser pour joindre a à b dans $\mathfrak{m}$. Notons que, avec ce point de vue, seule la structure de graphe est prise en compte. L'espace sur lequel on travaille est alors l'ensemble $\mathbb{M}$ des classes d'isométrie d'espaces métriques compacts. La topologie naturelle à mettre sur $\mathbb{M}$ est la **topologie de Gromov–Hausdorff** [Gro99]. Rappelons tout d'abord que la distance de Hausdorff entre deux compacts A et B d'un espace métrique $(\mathcal{X}, \delta)$ est donnée par

$$\delta_\mathcal{H}(A,B) \coloneqq \inf\left\{\varepsilon > 0 \,:\, A \subseteq B^\varepsilon \text{ et } B \subseteq A^\varepsilon\right\},$$

où, pour toute partie $X \subseteq \mathcal{X}$, on désigne par $X^\varepsilon \coloneqq \{x \in \mathcal{X} \,:\, \delta(x,X) \leq \varepsilon\}$ son ε-voisinage. Ensuite, pour deux espaces métriques compacts $(\mathcal{X}, \delta)$ et $(\mathcal{X}', \delta')$, on définit la **distance de Gromov–Hausdorff** entre eux par

$$d_{GH}\Big((\mathcal{X},\delta),(\mathcal{X}',\delta')\Big) \coloneqq \inf\Big\{\delta_\mathcal{H}\Big(\varphi(\mathcal{X}),\varphi'(\mathcal{X}')\Big)\Big\},$$

où la borne inférieure est prise sur tous les plongements isométriques $\varphi : \mathcal{X} \to \mathcal{X}''$ et $\varphi' : \mathcal{X}' \to \mathcal{X}''$ de $\mathcal{X}$ et $\mathcal{X}'$ dans un espace métrique commun $(\mathcal{X}'', \delta'')$. On définit ainsi une pseudo-distance qui passe au quotient et devient une distance sur $\mathbb{M}$ ([BBI01, Théorème 7.3.30]), qui le rend polonais[2]. Comme on le verra par la suite, il existe une interprétation plus agréable de cette distance en termes de correspondances.

On est alors amenés à se demander s'il est possible de renormaliser l'espace métrique $(V(\mathfrak{q}_n), d_{\mathfrak{q}_n})$ de sorte qu'il admette une limite en loi, au sens de la topologie de Gromov–Hausdorff. Cette approche a été formulée pour la première fois par Oded Schramm [Sch07] dans le contexte légèrement différent des triangulations. Avant de tenter d'apporter une réponse à cette question très compliquée qui va nous occuper tout au long de cette thèse, donnons un petit aperçu historique du domaine.

1.2 Bref historique

Dans les années 1960, William Tutte s'intéresse à divers problèmes de comptage des cartes en relation avec son travail sur la fameuse conjecture (devenue théorème depuis) des quatre couleurs. En résolvant les équations satisfaites par certaines fonctions génératrices, il obtient de nombreuses formules combinatoires pour l'énumération des cartes planaires [Tut62a, Tut62b, Tut62c, Tut63], et développe des méthodes très robustes pour la résolution de ce genre de problèmes. En genre quelconque, Timothy Walsh et Alfred Lehman obtiennent, à l'aide de décompositions récursives, les premières formules énumératives [WL72a, WL72b, WL75]. Par la suite, Edward Bender et Rodney Canfield adaptent l'approche de William Tutte et dérivent des formules asymptotiques pour les nombres de cartes de genre fixé [BC86, BC91, BCR93].

Par ailleurs, on retrouve les cartes dans diverses branches de la physique théorique. Dans les années 1970, Gerardus 't Hooft [tH74], puis Édouard Brézin, Claude Itzykson, Giorgio Parisi et Jean-Bernard Zuber [BIPZ78], mettent en relation les problèmes d'énumération de cartes en genre quelconque avec des modèles d'intégrales de matrices. Ceci généra par la suite de nombreux travaux sur le sujet, en raison des liens existant avec notamment la physique statistique, la géométrie algébrique, et la théorie des représentations ; voir par exemple le livre [LZ04]. Plus récemment, les cartes ont été utilisées en physique théorique comme un moyen de discretiser une surface. Cette approche intervient en théorie de la gravité quantique 2-dimensionnelle. On renvoie par exemple au livre [ADJ97] pour plus d'informations à ce sujet. Il existe en outre une approche mathématique à la gravité quantique, qui repose sur le champ libre gaussien ; voir par exemple [DS08] pour plus de détails. On renvoie le lecteur à la thèse

[2]C'est une simple conséquence du théorème de compacité de Gromov [BBI01, Théorème 7.4.15].

de Jérémie Bouttier [Bou05] pour une présentation des liens entre les cartes planaires et la physique statistique des surfaces aléatoires.

MALGRÉ LA DIFFICULTÉ CALCULATOIRE de la méthode de William Tutte, les formules énumératives obtenues sont étonament simples, et certaines font intervenir les nombres de Catalan qui comptent des arbres. Cette observation amène Robert Cori et Bernard Vauquelin, dans les années 1980, à développer des approches bijectives pour compter les cartes [CV81]. Par la suite, Gilles Schaeffer [Sch98] améliore ces méthodes et fait apparaître une information géométrique (distance à un sommet distingué dans la carte) dans le codage, rendant ainsi ces techniques plus exploitables dans les problèmes probabilistes. Plus précisément, les travaux de Gilles Schaeffer font apparaître une bijection naturelle entre les quadrangulations planaires (qui sont des objets relativement compliqués) et les arbres bien étiquetés (qui s'avèrent beaucoup plus simples à étudier). À l'aide de cette bijection, Philippe Chassaing et Gilles Schaeffer [CS04] donnent la limite d'échelle de certaines fonctionnelles des quadrangulations planaires uniformes. En particulier, ils étudient le *profil* de la carte, qui donne le nombre de sommets en fonction de leur distance à un sommet distingué, ainsi que son *rayon*, défini comme la distance maximale d'un sommet au sommet distingué. Ils montrent notamment que, si $\mathfrak{q}_n$ est choisie uniformément parmi les quadrangulations planaires à n faces, alors les distances dans $\mathfrak{q}_n$ sont de l'ordre de $n^{1/4}$, et le profil et rayon de $\mathfrak{q}_n$, une fois que les distances sont renormalisées par le facteur $n^{-1/4}$, admettent une limite en loi.

Jean-François Marckert et Abdelkader Mokkadem [MM06] se sont ensuite interessés au problème de la convergence des quadrangulations planaires uniformes dans leur ensemble, en considérant les cartes comme des espaces métriques munies de leur distance de graphe, renormalisée par $n^{-1/4}$. Ils construisent un espace métrique limite, qu'ils appellent *carte brownienne*, et montrent la convergence, dans un certain sens, des espaces discrets vers ce dernier. Le problème naturel de convergence au sens de la topologie de Gromov–Hausdorff [Gro99] reste alors encore mal compris. Peu de temps après, Jean-François Le Gall [LG07] montre la convergence des espaces métriques discrets vers un espace métrique aléatoire, mais seulement à extraction près. Plus précisément, il montre que la suite des lois de ces espaces métriques est tendue, ce qui implique qu'elle admet des valeurs d'adhérence, et conjecture que l'extraction n'est pas requise, c'est-à-dire qu'il n'existe qu'une seule valeur d'adhérence. En référence à cette conjecture et à [MM06], il est alors courant de parler, par abus de langage, de *la* carte brownienne pour désigner indifféremment n'importe quelle valeur d'adhérence potentielle de cette suite. Cette conjecture, parfois appelé problème de l'*unicité de la carte brownienne* n'a été validée que récemment, à l'aide de deux méthodes différentes, par Grégory Miermont [Mie11] et par Jean-François Le Gall [LG11].

Il n'est néanmoins pas nécessaire de connaitre ce dernier résultat pour donner des informations sur la carte brownienne (ou plutôt, dans le contexte antérieur à 2011, de n'importe quelle espace limite possible). Jean-François Le Gall [LG07] montre que la dimension de Hausdorff de la carte brownienne est presque sûrement égale à 4. Il montre ensuite, avec Frédéric Paulin [LGP08], que la carte brownienne est presque sûrement homéomorphe à la sphère $\mathbb{S}^2$ de $\mathbb{R}^3$, en utilisant un théorème dû à Robert Moore [Moo25], qui donne une condition suffisante pour qu'un quotient de $\mathbb{S}^2$ reste homéomorphe à $\mathbb{S}^2$. En corollaire de ce résultat, ils obtiennent que les grandes quadrangulations planaires ne possèdent pas de « goulets d'étranglement », c'est-à-dire de petits lacets dans la carte qui la séparent en deux grosses composantes. Procédant dans l'autre sens, Grégory Miermont [Mie08] montre l'absence de ces goulets d'étranglement par une autre méthode, et en re-déduit que la carte brownienne est presque sûrement homéomorphe à $\mathbb{S}^2$, à l'aide d'un théorème dû à Edward Begle [Beg44] basé sur la notion de 1-régularité développée par Gordon Whyburn [Why35a, Why35b].

IL EST ALORS NATUREL d'essayer de généraliser cette approche à d'autres classes de cartes. Pour cela, la première chose à faire est d'adapter les méthodes bijectives de Cori–Vauquelin–Schaeffer. Cela a été fait à plusieurs reprises. Jérémie Bouttier, Philippe Di Francesco et Emmanuel Guitter [BDG04] ont étendu la bijection de Cori–Vauquelin–Schaeffer en une bijection plus générale permettant entre autre de coder toutes les cartes planaires. Cette bjection a surtout été utilisée dans le cas où les cartes sont en plus biparties, car les objets intervenant dans ce cas sont plus simples à étudier.

Pour être un peu plus précis, Jean-François Le Gall utilise en fait dans [LG07, LG11] cette généralisation (dans le cas biparti) et traite les limites d'échelle des $2p$-angulations, pour $p \geq 2$. Dans [LG11], il inclut en plus, à l'aide d'un très bel argument d'invariance de la carte brownienne par réenracinement uniforme, la classe des triangulations, ainsi que des modèles de cartes biparties prises selon une loi de Boltzmann, conditionnées par leur nombre de sommets. Il montre en outre qu'à une constante multiplicative près, la limite de toutes ces classes est toujours la même. En d'autres termes, il montre une propriété d'universalité de la carte brownienne.

En genre $g \geq 1$, Guillaume Chapuy, Michel Marcus et Gilles Schaeffer [CMS09] étendent la bijection de Cori–Vauquelin–Schaeffer en une bijection entre les quadrangulations biparties de genre g et les cartes à une face de genre g. Les cartes à une face de genre g, aussi appelées *cartes unicellulaires* ou encore g-arbres (cette dernière terminologie provient du fait qu'une carte de genre 0 à une face est simplement un arbre planaire) ont été étudiées notamment par Guillaume Chapuy [Cha10], qui établit la limite d'échelle de leur profil via des arguments combinatoires et une bijection permettant de les décomposer en arbres plans munis de triplets de points distingués.

Une autre généralisation a été obtenue par Grégory Miermont [Mie09b] entre les quadrangulations biparties de genre $g \geq 0$ ayant un nombre fixé $k \geq 1$ de sommets distingués et les cartes de même genre à k faces, le cas $k = 1$ de cette bijection correspondant à la bijection de Chapuy-Marcus-Schaeffer. Cette bijection permet notamment de montrer la compacité relative de la famille des distributions de Boltzmann-Gibbs des quadrangulations biparties, prise dans la bonne échelle, ainsi que l'unicité des géodésiques typiques dans les espaces limites. Une autre importante application de cette bijection est le calcul par Jérémie Bouttier et Emmanuel Guitter [BG08] de la fonction à trois points dans la carte brownienne, c'est-à-dire de la distribution des distances entre 3 points pris uniformément dans la carte brownienne. Il est naturel de se demander si leur approche se généralise à plus de 3 points, mais il ne semble pas que ce soit le cas. En fait, si c'était le cas, cela donnerait une troisième preuve de l'unicité de la carte brownienne. Mentionnons aussi au passage que cette bijection constitue un ingrédient important dans la preuve de Grégory Miermont [Mie11] de l'unicité de la carte brownienne.

À LA LUMIÈRE DES RÉCENTS TRAVAUX de Jean-François Le Gall [LG11], il semblerait que la carte brownienne apparaisse comme limite d'échelle de n'importe quelle classe « raisonnable » de cartes planaires uniformes. On peut alors se demander s'il est possible de construire une limite d'échelle différente, en choisissant convenablement la loi des cartes. Jean-François Le Gall et Grégory Miermont [LGM11a] ont montré que cela était possible lorsque l'on favorise les « grandes » faces en imposant que le degré d'une face typique soit dans le domaine d'attraction d'une loi stable d'indice $\alpha \in (1, 2)$. Dans ce cas, le facteur d'échelle qui apparaît est $n^{-1/2\alpha}$, et la limite d'échelle est un espace métrique aléatoire presque sûrement de dimension de Hausdorff 2α.

Mentionnons pour finir qu'il existe un autre point de vue pour comprendre les grandes cartes aléatoires. Plutôt que de regarder des limites d'échelle qui font apparaître un objet continu à la limite, on peut regarder des *limites locales*. Dans cette approche, les distances ne sont pas renormalisées, et on s'intéresse à un voisinage arbitrairement grand de la racine. Cela fait apparaître à la limite une structure infinie, mais qui reste discrète. Les premiers à s'intéresser à cette question sont Omer Angel et Oded Schramm [AS03, Ang03] dans le cas des triangulations, donnant ainsi naissance à la *triangulation infinie uniforme du plan*. Dans le cas des quadrangulations, Maxim Krikun [Kri05], ainsi que Philippe Chassaing et Bergfinnur Durhuus [CD06], définissent par deux approches différentes la *quadrangulation infinie uniforme du plan*. Laurent Ménard [Mén10] montre par la suite que ces deux approches définissent bien le même objet.

1.3 Cas planaire

Pour cette section, nous nous plaçons dans le cas planaire, c'est-à-dire $g = 0$. Dans ce contexte, $\mathcal{Q}_n$ désigne alors l'ensemble des quadrangulations planaires à n faces. L'étude de la limite d'échelle des quadrangulations planaires commence par la bijection de Cori–Vauquelin–Schaeffer, que nous présentons maintenant.

1.3.1 Quadrangulations pointées et arbres bien étiquetés

Bien que cela ne soit pas la définition la plus classique, il est possible de définir un **arbre** (plan) comme une carte planaire à une seule face.

Définition 1.3. *Un **arbre bien étiqueté** est un couple* $(\mathfrak{t}, \mathfrak{l})$ *formé d'un arbre* $\mathfrak{t}$ *et d'une fonction d'étiquettes* $\mathfrak{l} : V(\mathfrak{t}) \to \mathbb{Z}$ *vérifiant :*

- ✧ $\mathfrak{l}(\mathfrak{e}_*^-) = 0$*, où l'on note* $\mathfrak{e}_*$ *la racine de* $\mathfrak{t}$*,*
- ✧ $|\mathfrak{l}(u) - \mathfrak{l}(v)| \leq 1$ *dès que* u *et* v *sont reliés par une arête de* $\mathfrak{t}$*.*

On note $\mathcal{T}_n$ *l'ensemble des arbres bien étiquetés ayant* n *arêtes.*

En d'autres termes, un arbre bien étiqueté est un arbre muni d'étiquettes entières qui varient d'au plus 1 le long des arêtes, et telles que l'étiquette de l'origine de la racine vaille 0. En particulier, on voit tout de suite qu'il y a exactement 3^n façons différentes de bien étiqueter un arbre à n arêtes. Comme les arbres sont comptés par les nombres de Catalan, on obtient que

$$|\mathcal{T}_n| = 3^n \, \frac{1}{n+1} \binom{2n}{n}. \tag{1.1}$$

Définissons l'ensemble

$$\mathcal{Q}_n^\bullet = \left\{(\mathfrak{q}, v^\bullet) \;:\; \mathfrak{q} \in \mathcal{Q}_n,\, v^\bullet \in V(\mathfrak{q})\right\}$$

des quadrangulations **pointées** à n faces. La bijection de Cori–Vauquelin–Schaeffer s'établit entre les ensembles $\mathcal{Q}_n^\bullet$ et $\mathcal{T}_n \times \{-1, 1\}$:

Théorème 1.1 ([CS04, Théorème 4])**.** *Il est possible de construire une bijection entre les ensembles* $\mathcal{Q}_n^\bullet$ *et* $\mathcal{T}_n \times \{-1, 1\}$ *telle que, lorsque* $(\mathfrak{q}, v^\bullet)$ *correspond à* $((\mathfrak{t}, \mathfrak{l}), \varepsilon_\pm)$*, les ensembles* $V(\mathfrak{q})\backslash\{v^\bullet\}$ *et* $V(\mathfrak{t})$ *sont identifiés, et, pour tout* $v \in V(\mathfrak{t})$*,*

$$d_\mathfrak{q}(v, v^\bullet) = \mathfrak{l}(v) - \min_{V(\mathfrak{t})} \mathfrak{l} + 1.$$

Si l'on utilise (1.1) et la formule d'Euler, on en déduit que

$$|\mathcal{Q}_n| = \frac{2}{n+2}\, 3^n \, \frac{1}{n+1} \binom{2n}{n}.$$

À l'aide de la bijection de Tutte, on voit que cette formule compte aussi les cartes planaires à n arêtes. En fait, c'est ce genre d'identités combinatoires simples qui a initié le développement des approches bijectives de Robert Cori et Bernard Vauquelin. Notons aussi que, outre l'aspect combinatoire, cette bijection donne une interprétation géométrique au codage, qui sera très utile dans la suite. Il est important de bien comprendre le mécanisme de cette bijection, c'est pourquoi nous exposons brièvement sa construction.

Des quadrangulations aux arbres bien étiquetés. Partant d'une quadrangulation pointée $(\mathfrak{q}, v^\bullet) \in \mathcal{Q}_n^\bullet$, on commence par associer à chaque sommet v une étiquette $\hat{\mathfrak{l}}(v) := d_\mathfrak{q}(v, v^\bullet)$. Le caractère biparti de la carte assure que les étiquettes de deux points reliés par une arête diffèrent d'exactement 1. Ainsi, lorsque l'on fait le tour d'une face, les étiquettes ne peuvent être que d, $d+1$, $d+2$, $d+1$, ou bien d, $d+1$, d, $d+1$, pour un certain entier d. On trace sur chaque face une nouvelle arête, comme décrit sur la figure 1.6, et on ne garde que ces nouvelles arêtes, ainsi que les sommets de $V(\mathfrak{q})\backslash\{v^\bullet\}$. On obtient alors un arbre $\mathfrak{t}$ que l'on enracine selon la racine $\mathfrak{e}_*$ de $\mathfrak{q}$, à l'aide de la convention suivante. Si $\hat{\mathfrak{l}}(\mathfrak{e}_*^-) = \hat{\mathfrak{l}}(\mathfrak{e}_*^+) - 1$, la racine de $\mathfrak{t}$ est l'arête tracée dans la face incidente à $\mathfrak{e}_*$, orientée en partant de $\mathfrak{e}_*^+$. Sinon, c'est l'arête tracée dans la face incidente à $\bar{\mathfrak{e}}_*$, orientée en partant de $\mathfrak{e}_*^-$. Dans le premier cas, on définit $\varepsilon_\pm := 1$, et dans le second, $\varepsilon_\pm := -1$. Enfin, on définit la fonction d'étiquetage $\mathfrak{l}$ en translatant $\hat{\mathfrak{l}}$ de sorte que la valeur de $\mathfrak{l}$ en l'origine de la racine de $\mathfrak{t}$ vaille 0. Voir la figure 1.6.

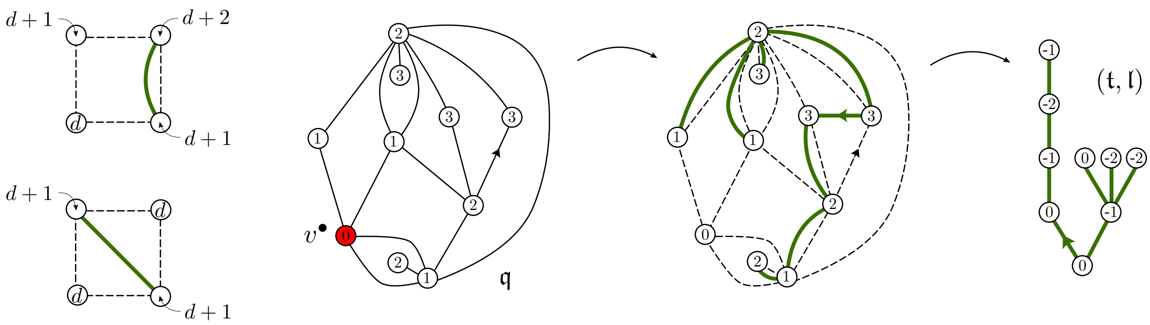

FIGURE 1.6. ***Gauche.*** *Les deux types de faces possibles et l'arête à ajouter.* ***Droite.*** *Construction de l'arbre bien étiqueté* $(\mathfrak{t}, \mathfrak{l})$ *correspondant à une quadrangulation pointée* $(\mathfrak{q}, v^\bullet)$*. Dans cet exemple,* $\varepsilon_\pm = 1$*.*

Des arbres bien étiquetés aux quadrangulations. Soit $(\mathfrak{t}, \mathfrak{l})$ un arbre bien étiqueté et $\varepsilon_\pm \in \{-1, 1\}$. On translate dans un premier temps les étiquettes de manière à ce que la plus petite soit égale à 1 : pour $v \in V(\mathfrak{t})$, on définit $\hat{\mathfrak{l}}(v) := \mathfrak{l}(v) - \min_{V(\mathfrak{t})} \mathfrak{l} + 1$. On rajoute un point $v^\bullet$ d'étiquette $\hat{\mathfrak{l}}(v^\bullet) := 0$ dans l'unique face de $\mathfrak{t}$. Ensuite, en faisant le tour de l'arbre dans le sens direct, on relie d'une arche chaque coin au premier coin suivant d'étiquette strictement inférieure, et ce, sans croiser aucune arête ni arche déjà dessinée. Si c'est un coin d'étiquette 1, on le relie simplement au sommet ajouté $v^\bullet$. Enfin, on efface les arêtes de $\mathfrak{t}$ et on enracine la carte obtenue en l'arche issue du coin racine de $\mathfrak{t}$, en direction du sommet racine si $\varepsilon_\pm = 1$, et dans l'autre sens si $\varepsilon_\pm = -1$.

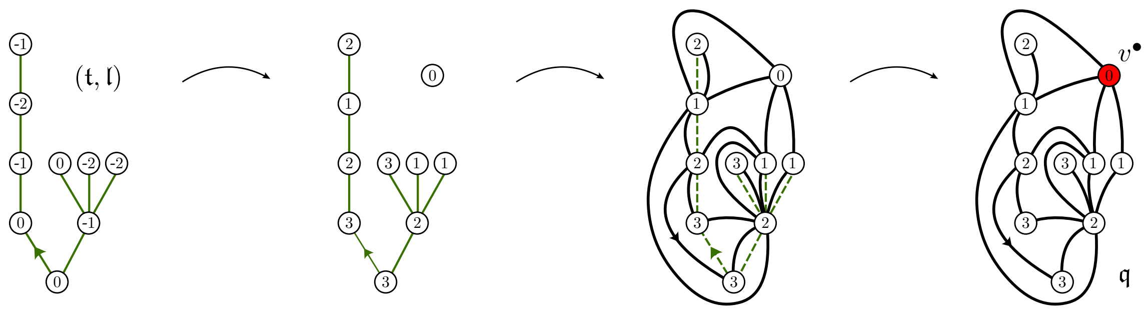

FIGURE 1.7. *Construction inverse de celle de la figure 1.6.*

1.3.2 Limite d'échelle des arbres bien étiquetés uniformes

Rappelons que nous avons choisi $\mathfrak{q}_n$ uniformément dans l'ensemble $\mathcal{Q}_n$ des quadrangulations planaires à n faces. Soit $v_n^\bullet$ un sommet uniforme de $\mathfrak{q}_n$. Comme, d'après la formule d'Euler, toutes les quadrangulations de $\mathcal{Q}_n$ ont le même nombre de sommets, la quadrangulation pointée $(\mathfrak{q}_n, v_n^\bullet)$ est uniformément distribuée dans $\mathcal{Q}_n^\bullet$. L'arbre bien étiqueté $(\mathfrak{t}_n, \mathfrak{l}_n)$ correspondant via la bijection de Cori–Vauquelin–Schaeffer est alors uniformément distribué dans $\mathcal{T}_n$. Regardons sa limite d'échelle.

1.3.2.1 L'arbre continu brownien

On oublie dans un premier temps les étiquettes, et on ne s'intéresse qu'à l'arbre $\mathfrak{t}_n$. Comme le nombre de façons de bien étiqueter un arbre ne dépend que de son nombre d'arêtes, on voit que $\mathfrak{t}_n$ est uniforme sur l'ensemble des arbres à n arêtes.

Un arbre $\mathfrak{t}$ à n arêtes est naturellement codé par sa **fonction de contour** $C_\mathfrak{t} : [0, 2n] \to \mathbb{R}_+$, définie de la manière suivante. On appelle $\mathfrak{e}_1 = \mathfrak{e}_*, \mathfrak{e}_2, \ldots, \mathfrak{e}_{2n}$ les demi-arêtes de $\mathfrak{t}$, rangées suivant l'ordre dans lequel elles apparaissent lorsque l'on en fait le tour dans le sens direct, en commençant par la racine. Pour plus de simplicité, on convient que $\mathfrak{e}_{2n+1} := \mathfrak{e}_*$. On définit alors

$$C_{\mathfrak{t}}(i) := d_{\mathfrak{t}_n}(\mathfrak{e}_*^-, \mathfrak{e}_{i+1}^-), \qquad 0 \le i \le 2n,$$

et on interpole linéairement $C_{\mathfrak{t}}$ entre les valeurs entières. On voit que $C_{\mathfrak{t}}$ est une fonction positive obtenue comme la concaténation de $2n$ pas de ± 1, et telle que $C_{\mathfrak{t}}(0) = C_{\mathfrak{t}}(2n) = 0$. Réciproquement, il est facile de voir que toute fonction vérifiant ces conditions est la fonction de contour d'un unique arbre à n arêtes (c.f. paragraphe suivant). Ainsi, $C_n := C_{\mathfrak{t}_n}$ est une marche aléatoire simple à $2n$ pas, conditionnée à rester positive et à finir en 0 au temps $2n$. Une version conditionnée du théorème de Donsker due à William Kaigh [Kai76, Théorème 2.6] nous donne alors la convergence suivante :

$$\left(\frac{C_n(2ns)}{\sqrt{2n}}\right)_{0 \le s \le 1} \xrightarrow[n\to\infty]{(d)} (\mathbb{e}_s)_{0 \le s \le 1}, \tag{1.2}$$

pour la topologie uniforme sur l'espace $\mathscr{C}([0,1],\mathbb{R})$ des fonctions réelles continues sur $[0,1]$, où $\mathbb{e}$ désigne l'**excursion brownienne normalisée**, dont on peut définir la loi par renormalisation d'une excursion brownienne de la façon suivante [RY99, Exercice XII.2.13]. On se donne un mouvement brownien standard $(\beta_s)_{s\ge 0}$ issu de 0, et on note $g := \sup\{s \le 1 : \beta_s = 0\}$ et $d := \inf\{s \ge 1 : \beta_s = 0\}$. Le processus $\mathbb{e}$ a alors la même loi que

$$\left(\frac{|\beta_{g+(d-g)s}|}{\sqrt{d-g}}\right)_{0 \le s \le 1}.$$

Pour retrouver un arbre $\mathfrak{t}$ à partir de sa fonction de contour $C_{\mathfrak{t}}$, l'idée est d'identifier les points du graphe de $C_{\mathfrak{t}}$ qui se font face en-dessous du graphe. Formellement, pour une fonction continue $g : [0,\ell] \to \mathbb{R}_+$ vérifiant $g(0) = g(\ell) = 0$, on définit la fonction δ_g par

$$\delta_g(s,t) := g(s) + g(t) - 2 \min_{[s\wedge t, s\vee t]} g, \qquad 0 \le s,t \le \ell.$$

Il est alors facile de voir que la fonction δ_g définit sur $[0,\ell]$ une pseudo-distance. Ainsi, on peut définir une relation d'équivalence sur $[0,\ell]$ par $s \sim_g t$ si et seulement si $\delta_g(s,t) = 0$, et munir le quotient $\mathscr{T}_g := [0,\ell]_{/\sim_g}$ par la distance quotient, que l'on note encore δ_g. Avec ces définitions, il est facile de voir que l'espace $(\mathscr{T}_{C_{\mathfrak{t}}}, \delta_{C_{\mathfrak{t}}})$ est à d_{GH}-distance au plus 1 de $(V(\mathfrak{t}), d_{\mathfrak{t}})$: en fait, si l'on remplace les arêtes de $\mathfrak{t}$ par des segments de longueur 1, on obtient un espace isométrique à $(\mathscr{T}_{C_{\mathfrak{t}}}, \delta_{C_{\mathfrak{t}}})$.

L'avantage de cette construction est qu'elle permet d'atteindre une classe beaucoup plus grande que la classe des arbres plans. Les espaces métriques que l'on obtient sont appelés *arbres réels*. On renvoie le lecteur à [LG05] pour plus d'information sur ces objets, notamment pour une définition plus géométrique où la structure d'arbre apparaît plus clairement. Lorsque l'on prend pour g la fonction aléatoire $\mathbb{e}$, on obtient l'objet suivant, introduit par David Aldous [Ald91, Ald93] :

Définition 1.4. *L'espace métrique aléatoire $(\mathscr{T}_{\mathbb{e}}, \delta_{\mathbb{e}})$ est appelé **arbre continu brownien** ou, en abrégé, **CRT** pour Continuum Random Tree.*

Cette définition n'est pas la construction originale de David Aldous. Dans [Ald91, Ald93], il définit le CRT par « stick-breaking » comme l'adhérence d'une union dénombrable de segments. La définition que nous avons donnée ici correspond (à un facteur sans importance 2 près) à [Ald93, Corollaire 22]. En utilisant la convergence (1.2), il n'est alors pas très compliqué de montrer que l'espace métrique

$$\left(V(\mathfrak{t}_n), (2n)^{-1/2} d_{\mathfrak{t}_n}\right) \xrightarrow[n\to\infty]{(d)} (\mathscr{T}_{\mathbb{e}}, \delta_{\mathbb{e}}),$$

pour la topologie de Gromov–Hausdorff. Pour finir cette section, mentionnons que la dimension de Hausdorff du CRT est presque sûrement égale à 2 (voir par exemple [Mie09a, Proposition 3.4]). Cette propriété peut surprendre au vu de la construction du CRT mentionnée précédemment, puisque l'on recolle un nombre dénombrable d'objets de dimension 1. Notons que l'on observe un phénomène similaire dans le cas du mouvement brownien, et cela se produira aussi dans le cas des cartes, comme on le verra par la suite.

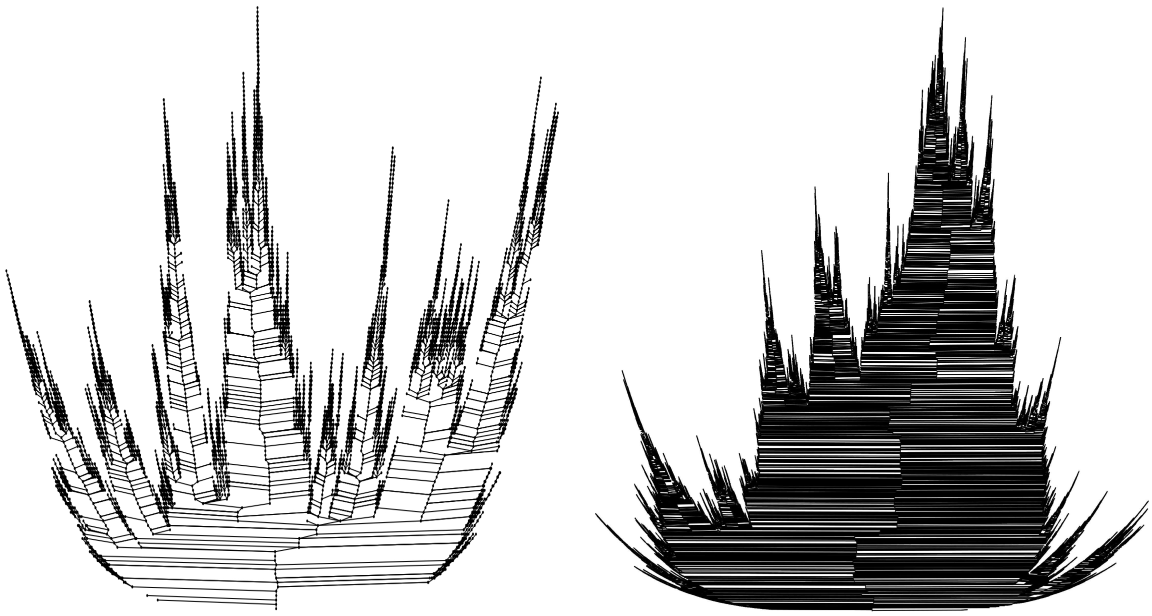

FIGURE 1.8. *Simulation d'un arbre uniforme à* 12 500 *arêtes à gauche,* 500 000 *arêtes à droite.*

1.3.2.2 La tête du serpent brownien

On définit sur $\mathscr{C}([0,1],\mathbb{R})^2$ le processus

$$\Big((\mathbb{e}_s)_{0\le s\le 1},\ (Z_s)_{0\le s\le 1}\Big)$$

de la façon suivante. Le processus $\mathbb{e}$ est l'excursion brownienne normalisée et, conditionnellement à $\mathbb{e}$, le processus Z est un processus gaussien centré dont la covariance est donnée par $Z_0 = 0$ et

$$\mathbb{E}\left[(Z_s - Z_t)^2\right] = \delta_{\mathbb{e}}(s,t), \qquad 0 \le s,t \le 1.$$

Ce processus est appelé **tête du serpent brownien** et peut être défini à partir d'un processus markovien appelé *serpent brownien* que l'on utilisera au cours du chapitre 6. On renvoie le lecteur à [LG99] pour plus d'informations sur le sujet.

Revenons maintenant aux arbres bien étiquetés. De la même façon que la fonction de contour d'un arbre code les distances dans l'arbre à la racine, la **fonction de contour spatial** enregistre les étiquettes d'un arbre bien étiqueté. Soit $(\mathfrak{t},\mathfrak{l}) \in \mathcal{T}_n$. On nomme ici encore $\mathfrak{e}_1 = \mathfrak{e}_*$, $\mathfrak{e}_2$, …, $\mathfrak{e}_{2n}$, $\mathfrak{e}_{2n+1} = \mathfrak{e}_*$ les demi-arêtes de $\mathfrak{t}$, ordonnées comme dans la section précédente, et on définit

$$L_{\mathfrak{t},\mathfrak{l}}(i) := \mathfrak{l}(\mathfrak{e}_{i+1}^-), \qquad 0 \le i \le 2n,$$

que l'on interpole linéairement entre les valeurs entières. On note $L_n := L_{\mathfrak{t}_n,\mathfrak{l}_n}$ la fonction de contour spatial de notre arbre bien étiqueté uniforme. La limite d'échelle du couple (C_n, L_n) a été exhibée par Philippe Chassaing et Gilles Schaeffer dans leur article pionnier [CS04] :

Théorème 1.2 ([CS04, Théorème 5])**.** *On a*

$$\left(\left(\frac{C_n(2ns)}{\sqrt{2n}}\right)_{0\le s\le 1}, \left(\frac{L_n(2ns)}{\gamma n^{1/4}}\right)_{0\le s\le 1}\right) \xrightarrow[n\to\infty]{(d)} \Big((\mathbb{e}_s)_{0\le s\le 1},\ (Z_s)_{0\le s\le 1}\Big),$$

pour la topologie uniforme sur $\mathscr{C}([0,1],\mathbb{R})^2$*, où* $\gamma := (8/9)^{1/4}$.

L'idée de ce théorème est la suivante. Premièrement, la convergence de la première composante est simplement (1.2). Ensuite, conditionnellement à $\mathfrak{t}_n$, les étiquettes sont données par des variables i.i.d. uniformes sur $\{-1, 0, 1\}$, dont la variance est $2/3$. Ainsi, la variation d'étiquettes entre deux points $\mathfrak{c}_{i+1}^-$ et $\mathfrak{c}_{j+1}^-$ est donnée par une marche aléatoire de longueur $d_{\mathfrak{t}_n}(\mathfrak{c}_{i+1}^-, \mathfrak{c}_{j+1}^-)$, qui est de l'ordre de $\sqrt{2n}\,\delta_{\mathbb{e}}(i/2n, j/2n)$ via la convergence de la première composante. Par le théorème de Donsker, cette marche approche alors dans l'échelle $\gamma n^{1/4}$ un mouvement brownien de longueur $\delta_{\mathbb{e}}(i/2n, j/2n)$.

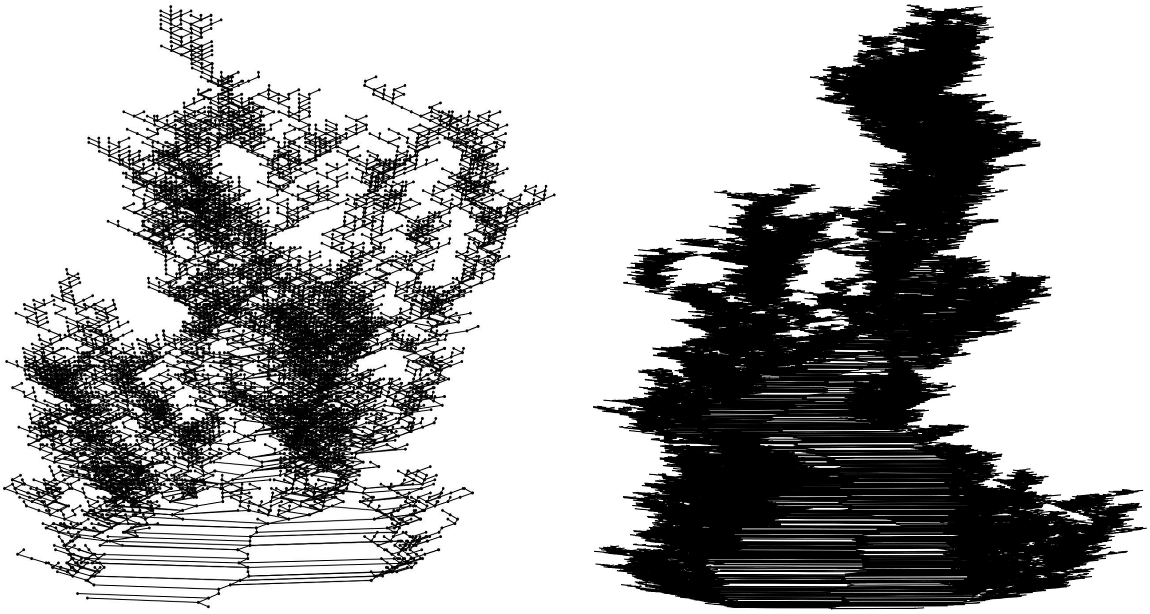

FIGURE 1.9. *Simulations d'arbres bien étiquetés uniformes. Les arbres sont ceux de la figure 1.8, ils sont représentés en deux dimensions et les étiquettes sont représentées dans la troisième dimension. L'angle par rapport au plan de l'arbre est de 30 degrés à gauche, et 20 degrés à droite.*

1.3.3 Limite d'échelle des quadrangulations

1.3.3.1 Rayon et profil d'une grande quadrangulation

Soit $(\mathfrak{q}, v^\bullet) \in \mathcal{Q}_n^\bullet$ une quadrangulation pointée. Son **rayon** (vu de $v^\bullet$) est défini par

$$\mathcal{R}_{\mathfrak{q},v^\bullet} := \max_{v \in V(\mathfrak{q})} d_{\mathfrak{q}}(v, v^\bullet),$$

et son **profil** (vu de $v^\bullet$) par

$$\mathcal{I}_{\mathfrak{q},v^\bullet}(k) := |\{v \in V(\mathfrak{q}) \ : \ d_{\mathfrak{q}}(v, v^\bullet) = k\}|, \qquad k \geq 0.$$

En particulier, $(n+2)^{-1}\,\mathcal{I}_{\mathfrak{q},v^\bullet}$ est la loi de la distance à $v^\bullet$ d'un sommet pris uniformément dans $V(\mathfrak{q})$. Rappelons que $(\mathfrak{q}_n, v_n^\bullet)$ est uniformément distribuée dans $\mathcal{Q}_n^\bullet$. On a alors le résultat suivant :

Théorème 1.3 ([CS04, Corollaires 3 et 4])**.** *La convergence du rayon renormalisé est donnée par*

$$\frac{\mathcal{R}_{\mathfrak{q}_n,v_n^\bullet}}{\gamma n^{1/4}} \xrightarrow[n\to\infty]{(d)} \sup_{[0,1]} Z - \inf_{[0,1]} Z.$$

Le profil renormalisé

$$\frac{\mathcal{I}_{\mathfrak{q}_n,v_n^\bullet}(\gamma n^{1/4}\cdot)}{n+2} \xrightarrow[n\to\infty]{(d)} \overline{\mathcal{I}},$$

pour la topologie de la convergence faible sur l'espace des mesures de probabilité sur $\mathbb{R}_+$, où $\overline{\mathcal{I}}$ est la mesure aléatoire que l'on définit, pour toute fonction mesurable $h : \mathbb{R}_+ \to \mathbb{R}_+$, par

$$\langle \overline{\mathcal{I}}, h \rangle := \int_0^1 ds\, h\Big(Z_s - \inf_{[0,1]} Z\Big).$$

Ces deux résultats sont des conséquences assez directes des théorèmes 1.1 et 1.2. La première assertion s'obtient immédiatement en remarquant que $\mathcal{R}_{\mathfrak{q}_n,v_n^\bullet} = \sup L_n - \inf L_n + 1$ d'après le théorème 1.1, et en appliquant la convergence du théorème 1.2. Pour obtenir la seconde assertion, on utilise le fait que, d'après le théorème 1.1, pour $k \geq 1$,

$$\mathcal{I}_{\mathfrak{q}_n,v_n^\bullet} = |\{v \in V(\mathfrak{t}_n) \,:\, \mathfrak{l}_n(v) - \inf \mathfrak{l}_n + 1 = k\}.$$

Ce théorème montre en particulier que le bon facteur d'échelle de notre problème est $n^{1/4}$.

1.3.3.2 La carte brownienne

Une des avancées majeures en vue de la résolution de notre problème à été faite par Jean-François Le Gall dans l'article [LG07], où il montre par un argument de compacité l'existence d'une limite d'échelle pour les quadrangulations uniformes, le long de sous-suites.

Via la bijection de Cori–Vauquelin–Schaeffer, on peut voir la quadrangulation $\mathfrak{q}_n$ comme l'arbre $\mathfrak{t}_n$ muni d'une certaine distance, qui s'exprime en fonction des étiquettes $\mathfrak{l}_n$ (mais pas par une expression simple). À la limite, on s'attend donc à obtenir un quotient de $\mathscr{T}_{\mathbb{e}}$, défini en termes d'étiquettes browniennes. Pour $s, t \in [0,1]^2$, définissons la pseudo-distance

$$D^\circ(s,t) := Z_s + Z_t - 2\max\left(\min_{r\in\overrightarrow{[s,t]}} Z_r,\ \min_{r\in\overrightarrow{[t,s]}} Z_r\right), \qquad \text{où} \quad \overrightarrow{[s,t]} := \begin{cases} [s,t] & \text{si} \quad s \leq t, \\ [s,1]\cup[0,t] & \text{si} \quad t < s. \end{cases}$$

Ensuite, pour a et $b \in \mathscr{T}_{\mathbb{e}}$, on définit

$$D^\circ(a,b) := \inf\{D^\circ(s,t) \,:\, \boldsymbol{p}_{\mathbb{e}}(s) = a,\, \boldsymbol{p}_{\mathbb{e}}(t) = b\},$$

où l'on note $\boldsymbol{p}_{\mathbb{e}} : [0,1] \to \mathscr{T}_{\mathbb{e}}$ la projection canonique. Comme cette fonction ne satisfait plus l'inégalité triangulaire, on est amenés à introduire

$$D^*(a,b) := \inf\Big\{\sum_{i=0}^{k-1} D^\circ(a_i,a_{i+1}) \,:\, k\geq 1,\, a_0 = a,\, a_k = b\Big\}, \qquad a,b \in \mathscr{T}_{\mathbb{e}}.$$

Cette fonction est alors une pseudo-distance sur $\mathscr{T}_{\mathbb{e}}$. On définit enfin $\mathfrak{m}_\infty$ comme le quotient de $\mathscr{T}_{\mathbb{e}}$ par la relation d'équivalence associée $\{D^* = 0\}$.

Théorème 1.4 ([LG07]). *De toute suite strictement croissante d'entiers, il est possible d'extraire une sous-suite $(n_k)_{k\geq 0}$ selon laquelle on a la convergence suivante, pour la topologie de Gromov–Hausdorff :*

$$\Big(V(\mathfrak{q}_{n_k}), \gamma^{-1}\, n_k^{-1/4}\, d_{\mathfrak{q}_{n_k}}\Big) \xrightarrow[k\to\infty]{(d)} (\mathfrak{m}_\infty, D),$$

où D est une distance aléatoire sur $\mathfrak{m}_\infty$ qui induit la topologie quotient sur $\mathfrak{m}_\infty$.

De plus, l'égalité suivante est vraie presque sûrement :

$$\{D = 0\} = \{D^* = 0\} = \{D^\circ = 0\}.$$

Puisque $\mathfrak{m}_\infty = \mathscr{T}_{\mathbb{e}}/\{D^\circ = 0\}$, la construction de ce quotient est assez intuitive, c'est en quelque sorte une version continue de la bijection de Cori–Vauquelin–Schaeffer. Deux points de $\mathscr{T}_{\mathbb{e}}$ sont identifiés s'ils ont même étiquette, et, lorsque l'on fait le « tour » de $\mathscr{T}_{\mathbb{e}}$ d'un point à l'autre (dans l'un des deux sens possibles), les étiquettes rencontrées sont toutes plus grandes.

La pseudo-distance D^* définit une distance sur $\mathfrak{m}_\infty$ que l'on note encore D^* par abus de notation. À ce stade, la distance D du théorème précédent dépend a priori de la sous-suite $(n_k)_{k\geq 0}$ considérée. Jean-François Le Gall a conjecturé que ce n'était pas le cas, et que $D = D^*$. Cela motive la définition suivante.

Définition 1.5. *L'espace métrique aléatoire* $(\mathfrak{m}_\infty, D^*)$ *est appelé* ***carte brownienne****.*

En pratique, il était courant d'appeler par abus de langage carte brownienne n'importe quelle limite $(\mathfrak{m}_\infty, D)$ du théorème précédent. Le résultat suivant lève l'ambiguité de cette définition. Via une étude très fine des géodésiques, Jean-François Le Gall et Grégory Miermont ont montré récemment, par deux méthodes indépendantes, que cet espace est bien la limite d'échelle des quadrangulations uniformes planaires.

Théorème 1.5 ([LG11, Théorème 1.1] ou [Mie11, Théorème 1])**.** *L'espace métrique* $(V(\mathfrak{q}_n), \gamma^{-1}\, n^{-1/4}\, d_{\mathfrak{q}_n})$ *tend en loi, au sens de la topologie de Gromov–Hausdorff, vers la carte brownienne* $(\mathfrak{m}_\infty, D^*)$*.*

1.3.3.3 Propriétés principales de la carte brownienne

Les théorèmes de cette section ont été montrés avant le théorème 1.5, car le théorème 1.4 suffit à donner des informations sur la carte brownienne. Commençons par sa dimension.

Théorème 1.6 ([LG07, Théorème 6.1])**.** *La dimension de Hausdorff de la carte brownienne* $(\mathfrak{m}_\infty, D^*)$ *est presque sûrement égale à* 4*.*

Sa topologie est aussi connue.

Théorème 1.7 ([LGP08, Théorème 1.1] ou [Mie08, Théorème 1])**.** *La carte brownienne* $(\mathfrak{m}_\infty, D^*)$ *est presque sûrement homéomorphe à la sphère* $\mathbb{S}^2$ *de* $\mathbb{R}^3$*.*

Ces deux théorèmes mis côte à côte peuvent sembler surprenants mais, si on se rappelle l'analogie avec le mouvement brownien esquissée lors de la section 1.1.3, on voit apparaître un phénomène similaire. En effet, il est connu que le graphe du mouvement brownien est presque sûrement de dimension de Hausdorff égale à $3/2$ (voir par exemple [MP10, Théorème 4.29]) et, en tant que graphe d'une fonction continue, il est homéomorphe à $\mathbb{R}$.

On peut aussi noter les faits remarquables suivants concernant les géodésiques dans la carte brownienne. Jean-François Le Gall [LG10] a montré qu'il était possible d'identifier toutes les géodésiques à la racine $\partial := \boldsymbol{p}_{\mathbb{e}}(0)$ de $\mathscr{T}_{\mathbb{e}}$. Une des conséquences principales est la suivante. On note $\mathrm{Sk}(\mathscr{T}_{\mathbb{e}})$ le *squelette* de $\mathscr{T}_{\mathbb{e}}$, défini comme l'ensemble des points $a \in \mathscr{T}_{\mathbb{e}}$ tels que $\mathscr{T}_{\mathbb{e}}\backslash\{a\}$ n'est pas connexe. Grossièrement, $\mathrm{Sk}(\mathscr{T}_{\mathbb{e}})$ est obtenu en « enlevant les feuilles » de $\mathscr{T}_{\mathbb{e}}$. Alors, d'une part, la projection canonique $\pi_\infty : \mathscr{T}_{\mathbb{e}} \to \mathfrak{m}_\infty$ restreinte à $\mathrm{Sk}(\mathscr{T}_{\mathbb{e}})$ est un homéomorphisme sur son image $\mathrm{Sk}_\infty := \pi_\infty(\mathrm{Sk}(\mathscr{T}_{\mathbb{e}}))$, qui est de dimension de Hausdorff presque sûrement égale à 2. D'autre part, on a la caractérisation suivante :

Théorème 1.8 ([LG10, Théorème 1.4])**.** *Ceci est vrai presque sûrement. Pour tout* $x \in \mathfrak{m}_\infty \backslash \mathrm{Sk}_\infty$*, il existe une unique géodésique de* x *à* ∂*, et, pour tout* $y \in \mathrm{Sk}_\infty$*, le nombre de géodésiques distinctes de* y *à* ∂ *est donné par le nombre de composantes connexes de* $\mathrm{Sk}_\infty \backslash\{y\}$*. En particulier, ce nombre est majoré par* 3*, et il existe une quantité dénombrable de points pour lesquels cette borne est atteinte.*

On a aussi une propriété assez surprenante de confluence des géodésiques vers la racine.

Théorème 1.9 ([LG10, Corollaire 7.7])**.** *Presque sûrement, pour tout* $\eta > 0$*, il existe* $\alpha \in]0, \eta[$ *vérifiant ceci. Soit* $x, x' \in \mathfrak{m}_\infty$ *tels que* $D^*(x, \partial) \geq \eta$ *et* $D^*(x', \partial) \geq \eta$*, et soient* $\wp$ *et* $\wp'$ *des géodésiques de* ∂ *à* x *et* x'*. Alors* $\wp(s) = \wp'(s)$ *pour tout* $s \in [0, \alpha]$*.*

1.3.3.4 Universalité de la carte brownienne

À l'instar du mouvement brownien, la carte brownienne possède une propriété d'universalité. On fixe un entier q soit égal à 3, soit pair et supérieur à 4. On définit $c_q := 6^{1/4}$ si $q = 3$, et sinon

$$c_q := \left(\frac{9}{q(q-2)}\right)^{1/4}.$$

Théorème 1.10 ([LG11, Théorème 1.1])**.** *Soit* $\mathfrak{m}_n$ *une variable uniformément distribuée parmi l'ensemble des* q*-angulations*[3] *à* n *faces. Alors l'espace métrique* $(V(\mathfrak{m}_n), c_q\, n^{-1/4}\, d_{\mathfrak{m}_n})$ *tend en loi, au sens de la topologie de*

[3] Une q-angulation est une carte dont toutes les faces sont de degré q.

Gromov–Hausdorff, vers la carte brownienne $(\mathfrak{m}_\infty, D^*)$.

Un des arguments clé de ce résultat est qu'il est possible de coder les q-angulations par des objets un peu plus compliqués que les arbres bien étiquetés, mais qui ont la même limite d'échelle, à la constante c_q près. Ensuite, une propriété d'invariance par réenracinement uniforme de la carte brownienne suffit à montrer ce théorème. Malheureusement, la seule preuve de cette propriété qui existe utilise le théorème 1.5 dont les deux preuves connues sont très compliquées, bien que son analogue discret soit beaucoup plus simple à montrer.

L'argument mentionné ci-dessus est dû à Jean-François Le Gall [LG11] : il s'en sert pour traiter le cas des triangulations. On renvoie le lecteur intéressé à [LG11, Section 8.3] pour plus de détails. Voir aussi la remarque à la fin de [LG11, Section 9.3] qui explique entre autre qu'une preuve directe de cette propriété d'invariance donnerait une approche sans doute plus simple pour montrer le théorème 1.5.

1.4 Cas du genre quelconque

À partir de maintenant, nous fixons un entier $g \geq 1$ et travaillons en genre g. Dans ce contexte, rappelons que $\mathfrak{q}_n$ désigne une carte uniformément distribuée sur l'ensemble $\mathcal{Q}_n$ des quadrangulations biparties de genre g à n faces. La partie II de ce manuscrit est une recompilation des références [Bet10a, Bet10b] dont l'objet est d'étendre les résultats de la section 1.3.3 à ce cadre.

1.4.1 Existence d'une limite à extraction près

Notre premier résultat généralise les théorèmes 1.4 et 1.6.

Théorème 1.11 ([Bet10a, Théorème 1]). *De toute suite strictement croissante d'entiers, on peut extraire une sous-suite* $(n_k)_{k\geq 0}$ *pour laquelle il existe un espace métrique aléatoire* $(\mathfrak{q}_\infty, d_\infty)$ *tel que*

$$\Big(V(\mathfrak{q}_{n_k}), \gamma^{-1}\, n_k^{-1/4}\, d_{\mathfrak{q}_{n_k}}\Big) \xrightarrow[k\to\infty]{(d)} (\mathfrak{q}_\infty, d_\infty)$$

au sens de la topologie de Gromov–Hausdorff, où l'on rappelle que $\gamma = (8/9)^{1/4}$.

De plus, la dimension de Hausdorff de l'espace limite $(\mathfrak{q}_\infty, d_\infty)$ *est presque sûrement égale à* 4*, quelle que soit la suite considérée.*

Prenant exemple sur le cas planaire, on conjecture que l'extraction n'est pas nécessaire dans ce théorème, et que la limite est universelle dans un sens similaire au théorème 1.10. On pense que la limite est l'espace métrique $(\mathfrak{q}_\infty, d_\infty^*)$, où la distance d_∞^* se définit par une expression analogue à celle du cas planaire (voir (3.8)). Nous commettrons aussi le même abus de langage et appellerons **carte brownienne de genre *g*** n'importe quelle limite $(\mathfrak{q}_\infty, d_\infty)$ possible, tout en gardant à l'esprit que cette terminologie est légèrement ambigüe en l'absence d'une preuve de son unicité.

Pour montrer ce théorème, on se sert d'une généralisation de la bijection de Cori–Vauquelin–Schaeffer due à Guillaume Chapuy, Michel Marcus et Gilles Schaeffer [CMS09], que l'on expose dans la section 2.1. Cette bijection fait apparaître des g-arbres (cartes de genre g à une unique face) bien étiquetés, que l'on décompose au cours de la section 2.2 en objets plus simples, à savoir des schémas de genre g (g-arbres dont tous les sommets ont degré supérieur à 3) et des forêts bien étiquetées. La fin du chapitre 2 est consacrée à l'étude de la limite d'échelle de ces objets, et la preuve du théorème 1.11 est donnée dans la section 3.1.

Notre étude permet en outre de retrouver une expression asymptotique du cardinal de $\mathcal{Q}_n$, qui apparaissait déjà dans [CMS09]. On appelle **schéma dominant** de genre g un g-arbre dont tous les sommets sont de degré exactement 3, et on note $\mathfrak{S}^*$ leur ensemble (fini). Il est connu (voir par exemple [BC91, CMS09, Mie09b]) qu'il existe une constante t_g telle que

$$|\mathcal{Q}_n| \sim t_g\, n^{\frac{5}{2}(g-1)}\, 12^n.$$

Cette constante joue un rôle important en énumération des cartes : de même que l'exposant $5(g-1)/2$, elle intervient dans de nombreuse formules énumératives [BC91, Gao93]. Le résultat qui suit est montré au cours de la section 3.2.

Théorème 1.12 ([CMS09, Corollaire 10] ou [Bet10a, Théorème 2]). *On a l'expression suivante :*

$$t_g = \frac{3^g}{2^{11g-7}\,(6g-3)\Gamma\left(\frac{5g-3}{2}\right)} \sum_{\mathfrak{s}\in\mathfrak{S}^*} \sum_{\lambda\in\mathcal{O}_\mathfrak{s}} \prod_{i=1}^{4g-3} \frac{1}{d(\lambda,i)}, \tag{1.3}$$

où la seconde somme est prise sur toutes les $(4g-2)!$ façons d'ordonner les sommets d'un schéma dominant $\mathfrak{s}\in\mathfrak{S}^*$, c'est-à-dire sur les bijections de $[\![0,4g-3]\!]$ sur $V(\mathfrak{s})$, et

$$d(\lambda,k) := \left|\left\{\mathfrak{e}\in\vec{E}(\mathfrak{s}),\ \lambda^{-1}_{\mathfrak{e}^-} < k \leq \lambda^{-1}_{\mathfrak{e}^+}\right\}\right|. \tag{1.4}$$

1.4.2 Topologie de la carte brownienne de genre g

Le résultat principal de la partie II est l'analogue suivant du théorème 1.7, qui fera l'objet du chapitre 4.

Théorème 1.13 ([Bet10b, Théorème 2]). *Soit $(\mathfrak{q}_\infty, d_\infty)$ une limite possible du théorème 1.11. Alors, cet espace est presque sûrement homéomorphe à la surface $\mathbb{T}_g$ de genre g.*

La méthode que l'on utilise consiste à adapter l'approche de Jean-François Le Gall [LG07], puis celle de Grégory Miermont [Mie08]. En premier lieu, on définit une sorte de g-arbre continu brownien qui apparaît comme limite des g-arbres discrets, et on voit $(\mathfrak{q}_\infty, d_\infty)$ comme un quotient de cet espace via une relation d'équivalence définie en termes d'étiquettes browniennes. Ensuite, on utilise la notion de 1-régularité développée par Gordon Whyburn [Why35b] afin de voir que la convergence du théorème 1.11 est suffisamment « régulière » pour que le genre soit conservé à la limite.

Les estimées techniques dont nous avons besoin seront déduites d'estimées connues dans le cas planaire à l'aide d'une bijection due à Guillaume Chapuy [Cha10] qui consiste à « ouvrir » les g-arbres afin de les rendre planaires. Cela nous donne une relation d'absolue continuité des fonctions de codage du g-arbre limite muni de ses étiquettes browniennes par rapport à la tête du serpent brownien $(\mathbb{e}, Z)$, d'où les estimées souhaitées découlent facilement.

1.5 Quadrangulations planaires à bord

Nous présentons maintenant les résultats de la partie III, qui correspond à la référence [Bet11]. Nous nous plaçons à nouveau dans le cas planaire, et nous étudions les cartes dont toutes les faces sauf éventuellement la face incidente à la racine sont des quadrangles. De telles cartes sont appelées **quadrangulations à bord**. La face incidente à la racine s'appelle **face externe**, et les autres faces s'appellent **faces internes**. Remarquons qu'il y a nécessairement un nombre pair d'arêtes incidentes à la face externe ; ces arêtes constituent le **bord** de la quadrangulation à bord. Attention au fait que ce bord n'est en général pas simple (au sens où il peut contenir des sommets séparants) dans ce modèle. Pour $n\geq 0$ et $\sigma\geq 1$, on note $\mathcal{Q}_{n,\sigma}$ l'ensemble des quadrangulations à bord ayant n faces internes et 2σ arêtes sur le bord. Remarquons que, puisqu'on ne demande pas que le bord soit simple, cet ensemble n'est jamais vide. Par exemple, $\mathcal{Q}_{0,\sigma}$ est l'ensemble des arbres à σ arêtes. Avant d'énoncer nos résultats, mentionnons que les quadrangulations à bord ont été étudiées par Jérémie Bouttier et Emmanuel Guitter [BG09]. Ils s'intéressent dans cette référence à des limites d'échelle de certaines fonctionelles à l'aide d'arguments combinatoires, mais ne considèrent pas le problème de la convergence au sens Gromov-Hausdorff.

1.5.1 Cas générique

Comme les quadrangulations à bord sont des cartes planaires biparties, on peut leur appliquer la bijection de Bouttier-Di Francesco-Guitter. Dans ce cas particulier, le codage est relativement simple et fait intervenir uniquement des forêts portant des étiquettes qui varient comme des marches aléatoires le long des branches. Nous présentons cette bijection au cours de la section 5.1. À travers cette bijection, le bord d'une quadrangulation à bord correspond essentiellement au *plancher* de la forêt qui la code, c'est-à-dire aux racines des arbres qui composent la forêt. On a vu précédemment que, dans un grand arbre, les distances sont de l'ordre de $n^{1/2}$, alors que dans une grande carte, elles sont de l'ordre de $n^{1/4}$. Il est alors naturel de s'intéresser au cas où le bord de la carte grandit comme la racine carrée du nombre de faces. On obtient un théorème dans la lignée des théorèmes précédents :

Théorème 1.14 ([Bet11, Théorème 1]). *Soit $\sigma > 0$, et $(\sigma_n)_{n\geq 1}$ une suite d'entiers strictement positifs telle que $\sigma_n/\sqrt{2n} \to \sigma$ quand $n \to \infty$. Soit $\mathfrak{q}_n$ uniformément distribuée sur l'ensemble $\mathcal{Q}_{n,\sigma_n}$ des quadrangulations à bord ayant n faces internes et $2\sigma_n$ arêtes sur le bord. Alors, de toute suite strictement croissante d'entiers, on peut extraire une sous-suite $(n_k)_{k\geq 0}$ pour laquelle il existe un espace métrique aléatoire $(\mathfrak{q}_\infty^\sigma, d_\infty^\sigma)$ tel que*

$$\Big(V(\mathfrak{q}_{n_k}), \gamma^{-1}\, n_k^{-1/4}\, d_{\mathfrak{q}_{n_k}}\Big) \xrightarrow[k\to\infty]{(d)} (\mathfrak{q}_\infty^\sigma, d_\infty^\sigma)$$

au sens de la topologie de Gromov–Hausdorff, où l'on note toujours $\gamma = (8/9)^{1/4}$.

De plus, la dimension de Hausdorff de l'espace limite $(\mathfrak{q}_\infty^\sigma, d_\infty^\sigma)$ est presque sûrement égale à 4, quelles que soient les suites considérées ci-dessus.

Comme précédemment, on s'attend à ce que la limite soit unique est universelle, et ne dépende que de σ. La preuve de ce théorème, donnée en section 5.2, utilise des techniques similaires aux preuves précédentes. On a insisté sur le fait que le bord n'avait aucune raison d'être simple. Il s'avère que, à la limite, il le devient naturellement :

Théorème 1.15 ([Bet11, Théorème 2]). *Pour $\sigma > 0$, toute limite possible $(\mathfrak{q}_\infty^\sigma, d_\infty^\sigma)$ du théorème 1.14 est presque sûrement homéomorphe au disque 2-dimensionnel $\mathbb{D}_2$.*

En particulier, le bord d'une grande quadrangulation à bord, bien qu'il se replie beaucoup sur lui-même, ne forme qu'une seule grande composante, toutes les autres disparaissant à la limite. En pratique, on utilise ce fait dans la preuve du théorème. La méthode que l'on utilise repose encore sur la notion de régularité de Gordon Whyburn [Why35b], mais le bord pose de nouveaux problèmes. En plus de montrer qu'il n'y a pas de goulet d'étranglement, il s'agit de voir que le bord ne se recolle pas en séparant deux portions macroscopiques. Cela se formalise à l'aide de la notion de 0-régularité.

Il est possible de calculer la dimension du bord de l'espace limite : on définit $\partial\mathfrak{q}_\infty^\sigma \subseteq \mathfrak{q}_\infty^\sigma$ comme l'ensemble des points dont aucun voisinage n'est homéomorphe à un disque.

Théorème 1.16 ([Bet11, Théorème 3]). *Pour tout $\sigma > 0$, le bord $\partial\mathfrak{q}_\infty^\sigma$ est un sous-ensemble de $(\mathfrak{q}_\infty^\sigma, d_\infty^\sigma)$ dont la dimension de Hausdorff est presque sûrement égale à 2.*

Ce résultat n'est pas surprenant si l'on se convainc que le bord correspond au plancher de la forêt limite, et que l'on se souvient que, dans le cas planaire, le squelette Sk_∞ de l'arbre limite est p.s. de dimension 2. Les sections 5.3 et 5.4 sont dédiées aux preuves de ces deux derniers théorèmes.

On pourrait penser qu'une quadrangulation à bord est essentiellement une moitié de quadrangulation planaire, en ce sens qu'on pourrait s'attendre à ce qu'il soit possible de créer une quadrangulation planaire en recollant d'une certaine façon deux quadrangulations à bord. En d'autres termes, on pourrait s'attendre à ce que la forêt limite soit à peu de choses près un demi-CRT muni d'étiquettes browniennes. Mais ce qu'on observe est en fait plus compliqué que ça. En particulier, les étiquettes browniennes ne se comportent pas de la même manière le long du plancher que dans les arbres de la forêt : alors que le facteur d'échelle vaut 1 dans les arbres (les étiquettes varient comme des mouvements browniens standards), il vaut $\sqrt{3}$ le long du plancher.

Ce facteur génère certaines difficultés. Entre autres, il nous empêche d'utiliser directement certaines estimées techniques du papier [LG07] de Jean-François Le Gall et nous pousse à trouver d'autres preuves, que l'on regroupe dans le chapitre 6. On utilise en particulier une preuve par « recouvrements » basée sur un théorème de Larry Shepp [She72] pour étudier les *points de croissance* de notre processus de codage limite. Notre approche permet de retrouver le lemme clé [LGP08, Lemme 3.2] concernant les points de croissance du processus $(\mathbb{e}, Z)$.

1.5.2 Cas où $\sigma = 0$

Dans le cas où $\sigma = 0$, on peut donner un résultat plus précis. En particulier, on obtient une convergence sans extraction. À la limite, le bord « disparaît » et l'on obtient la même limite que dans le cas sans bord, c'est-à-dire la carte brownienne $(\mathfrak{m}_\infty, D^*)$.

Théorème 1.17 ([Bet11, Théorème 4]). *Soit $(\sigma_n)_{n\geq 1}$ une suite d'entiers strictement positifs telle que $\sigma_n/\sqrt{2n} \to 0$ lorsque $n \to \infty$. Soit $\mathfrak{q}_n$ uniformément distribuée sur l'ensemble $\mathcal{Q}_{n,\sigma_n}$ des quadrangulations à bord ayant n faces internes et $2\sigma_n$ arêtes sur le bord. Alors,*

$$\Big(V(\mathfrak{q}_n), \gamma^{-1}\, n^{-1/4}\, d_{\mathfrak{q}_n}\Big) \xrightarrow[n\to\infty]{(d)} (\mathfrak{m}_\infty, D^*)$$

au sens de la topologie de Gromov–Hausdorff.

L'idée qui se cache derrière ce théorème est qu'une quadrangulation avec un « petit » bord n'est pas très loin d'une quadrangulation sans bord. Cela se voit dans le codage : à la limite, un seul arbre de la forêt survit. Ce théorème est prouvé au cours de la section 5.5.

1.5.3 Développements

Plusieurs questions naturelles se posent. On peut par exemple se demander ce qu'il se passe si on impose au bord d'être simple. Cela se traduit par un conditionnement de la forêt codante qui pose quelques difficultés, et nos résultats ne s'adaptent pas immédiatement à ce cas. On s'attend néanmoins à obtenir les mêmes limites d'échelle, à un facteur modifiant la longueur du bord près.

On peut aussi regarder le modèle suivant de marches auto-évitantes, inspiré de [BBG11]. Dans cette référence, Jérémie Bouttier et Emmanuel Guitter étudient un modèle de boucles tracées sur le dual d'une quadrangulation. En considérant un chemin auto-évitant plutôt que des boucles, on se ramène à l'étude des quadrangulations à bord. En quelques mots, pour $n \geq 0$ et $\sigma \geq 1$, on considère une carte dont les faces sont de plusieurs types possibles (voir la figure 1.10) :

- ✧ la face incidente à la racine est de degré 2, et un chemin relie son centre au milieu de la demi-arête incidente qui n'est pas la racine,
- ✧ $\sigma-1$ faces sont des quadrangles dont les milieux de deux cotés opposés sont reliés par un chemin,
- ✧ une autre face est de degré 2, et un chemin relie son centre au milieu d'une des demi-arêtes qui lui sont incidentes,
- ✧ les n faces restantes sont des quadrangles.

On demande de plus que la réunion des chemins tracées dans les faces forme un chemin connexe auto-évitant.

Il n'est alors pas très difficile de voir que les cartes qui vérifient ces conditions sont en bijection avec les cartes de $\mathcal{Q}_{n,\sigma}$, via une opération de « recollement du bord ». Partant d'une telle carte, on supprime les σ arêtes traversées par le chemin, et l'on obtient une carte dont les faces sont n quadrangles ainsi qu'une autre face de degré $1 + 2(\sigma - 1) + 1 = 2\sigma$ qui est incidente à la racine. Cela est donc bien une quadrangulation à bord de $\mathcal{Q}_{n,\sigma}$. Dans l'autre sens, considérons une quadrangulation de $\mathcal{Q}_{n,\sigma}$, et appelons $\mathfrak{e}_1 = \mathfrak{e}_*, \mathfrak{e}_2, \ldots, \mathfrak{e}_{2\sigma}$ les demi-arêtes incidentes à sa face externe, classées dans l'ordre indirect. On ajoute des arêtes (sans qu'elles se croisent) reliant $\mathfrak{e}_{2\sigma-i}^+$ à $\mathfrak{e}_{i+1}^+$ pour $0 \leq i \leq \sigma - 1$, et on relie les

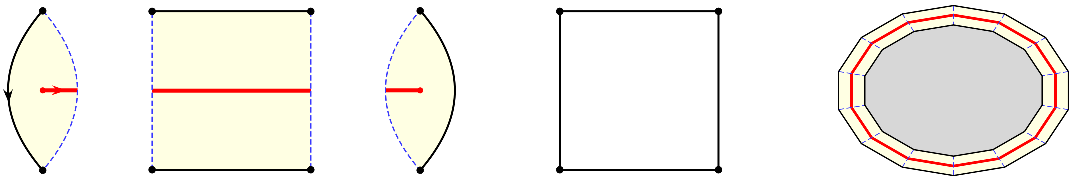

FIGURE 1.10. *Modèle de chemins auto-évitants de Jérémie Bouttier et Emmanuel Guitter. Les chemins sont représentés d'un trait (rouge) épais, et les arêtes qu'ils coupent sont en pointillés fins (bleus). Le chemin part de la face de degré* 2 *incidente à la racine et finit dans l'autre face de degré* 2*. La pointe de flèche sur le chemin symbolise ce fait.*

milieux de ces arêtes par des chemins. On crée ainsi une carte qui vérifie les conditions précédentes (voir la figure 1.11).

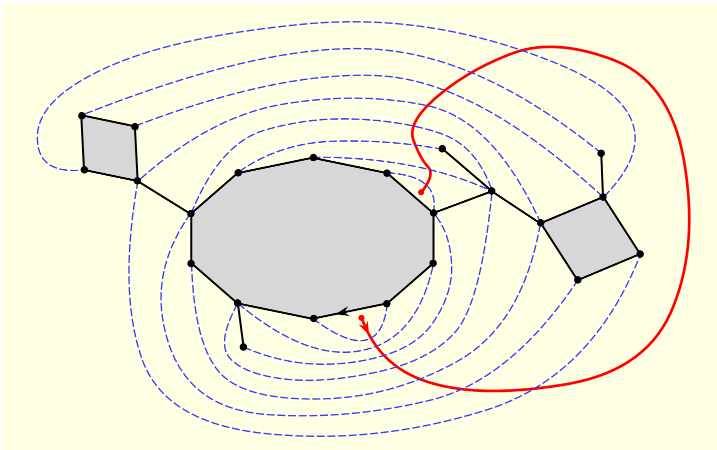

FIGURE 1.11. *Une quadrangulation à bord et la carte correspondante. Sur cette figure,* $\sigma = 15$.

En fait, les cartes définies ici correspondent au cas *rigide* de [BBG11]. On pourrait tout aussi bien autoriser les chemins à « tourner », en enlevant l'hypothèse que les arêtes d'un quadrangle qu'ils relient sont opposées. À la limite, on s'attend à ce que cela ne change rien et à obtenir les mêmes objets.

Afin de traiter ce modèle, il faudrait bien comprendre l'opération de recollement du bord, et voir comment nos résultats peuvent s'interpréter. Par exemple, il semble naturel de penser que le chemin tende vers un chemin limite injectif de dimension 2. Dans le continu, on peut facilement définir une opération de recollement du bord similaire, en quotientant l'espace limite par la relation d'équivalence la plus grossière qui identifie les points du bord qui sont à même distance (dans la forêt) du sommet racine. Malheureusement, cette opération pose quelques problèmes du point de vue métrique et, même si l'espace discret tend vers l'espace continu, il n'est pas si clair que l'espace discret quotient tende vers l'espace continu quotient.

Le même genre de questions se pose si l'on cherche à étudier les marches auto-évitantes au sens classique sur les quadrangulations, qui correspondent aux quadrangulations à bord simple, via une opération de découpage et recollement similaire.

Les travaux [Bet10a, Bet10b] et [Bet11] utilisent des techniques similaires et ont été élaborés dans le but d'être publiés indépendamment, c'est pourquoi certaines répétitions peuvent apparaître entre les parties II et III. Certaines notations peuvent aussi désigner des objets différents entre la partie II et la partie III. Par exemple, $\mathfrak{q}_n$ *désigne dans la partie II une quadrangulation de genre* g *et, dans la partie III, une quadrangulation à bord. Afin de ne pas trop déstructurer le document et risquer d'introduire des incohérences, nous avons préféré laisser les choses ainsi, et nous nous excusons par avance du désagrément occasionné.*

II

Limite d'échelle de cartes aléatoires en genre quelconque

Scaling limit of uniform well-labeled g-trees 2

The first step to our problem is to understand the scaling limit of uniform random well-labeled g-trees. These are much simpler objects than quadrangulations that are related to them through the Chapuy–Marcus–Schaeffer bijection [CMS09]—which we briefly present in the first section. In the following sections, we investigate the structure of these objects, and exhibit their scaling limit. In a nutshell, this scaling limit can be viewed as a Brownian g-tree (a so-called scheme on which Brownian forests are grafted) with Brownian labels on it. In other words, we find a genus g analog to the Brownian snake on Aldous's CRT [Ald91, Ald93].

2.1 The Chapuy–Marcus–Schaeffer bijection

The first main tool we use consists in the Chapuy–Marcus–Schaeffer bijection [CMS09, Corollary 2 to Theorem 1], which allows us to code (rooted) quadrangulations by so-called well-labeled (rooted) g-trees.

It may be convenient to represent a g-tree $\mathfrak{t}$ with n edges by a $2n$-gon whose edges are pairwise identified (see Figure 2.1). We note $\mathfrak{e}_1 := \mathfrak{e}_*$, $\mathfrak{e}_2$, ..., $\mathfrak{e}_{2n}$ the half-edges of $\mathfrak{t}$ arranged according to the clockwise order around this $2n$-gon. The half-edges are said to be arranged according to the **facial order** of $\mathfrak{t}$. Informally, for $2 \leq i \leq 2n$, $\mathfrak{e}_i$ is the "first half-edge to the left after $\mathfrak{e}_{i-1}$." We call **facial sequence** of $\mathfrak{t}$ the sequence $\mathfrak{t}(0), \mathfrak{t}(1), \dots, \mathfrak{t}(2n)$ defined by $\mathfrak{t}(0) = \mathfrak{t}(2n) = \mathfrak{e}_1^- = \mathfrak{e}_{2n}^+$ and for $1 \leq i \leq 2n-1$, $\mathfrak{t}(i) = \mathfrak{e}_i^+ = \mathfrak{e}_{i+1}^-$. Imagine a fly flying along the boundary of the unique face of $\mathfrak{t}$. Let it start at time 0

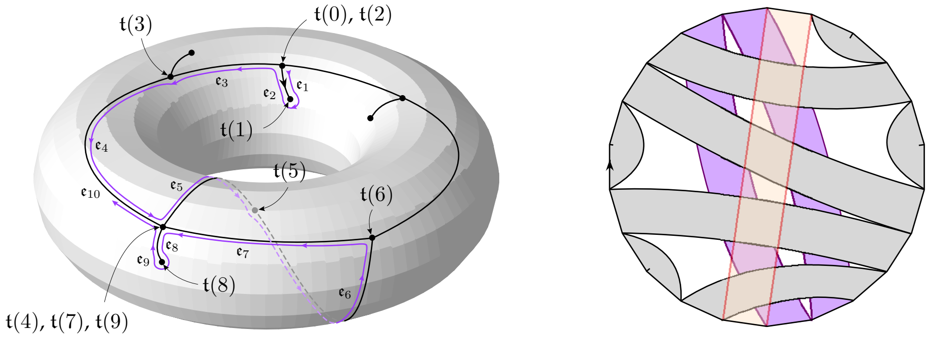

Figure 2.1. ***Left.*** *The facial order and facial sequence of a g-tree.* ***Right.*** *Its representation as a polygon whose edges are pairwise identified.*

by following the root $\mathfrak{e}_*$ and let it take one unit of time to follow each half-edge, then $\mathfrak{t}(i)$ is the vertex where the fly is at time i.

Let $\mathfrak{t}$ be a g-tree. Two vertices $u, v \in V(\mathfrak{t})$ are said to be **neighbors**, and we write $u \sim v$, if there is an edge linking them.

Definition 2.1. *A* ***well-labeled g-tree*** *is a pair $(\mathfrak{t}, \mathfrak{l})$ where $\mathfrak{t}$ is a g-tree and $\mathfrak{l} : V(\mathfrak{t}) \to \mathbb{Z}$ is a function (thereafter called* ***labeling function****) satisfying:*

i. $\mathfrak{l}(\mathfrak{e}_*^-) = 0$, *where $\mathfrak{e}_*$ is the root of $\mathfrak{t}$,*

ii. *if $u \sim v$, then $|\mathfrak{l}(u) - \mathfrak{l}(v)| \leq 1$.*

We call $\mathcal{T}_n$ the set of all well-labeled g-trees with n edges.

A **pointed quadrangulation** is a pair $(\mathfrak{q}, v^\bullet)$ consisting in a quadrangulation $\mathfrak{q}$ together with a distinguished vertex $v^\bullet \in V(\mathfrak{q})$. We call

$$\mathcal{Q}_n^\bullet := \{(\mathfrak{q}, v^\bullet) : \mathfrak{q} \in \mathcal{Q}_n,\ v^\bullet \in V(\mathfrak{q})\}$$

the set of all pointed bipartite quadrangulations of genus g with n faces.

The Chapuy–Marcus–Schaeffer bijection is a bijection between the sets $\mathcal{T}_n \times \{-1, +1\}$ and $\mathcal{Q}_n^\bullet$. As a result, because every quadrangulation $\mathfrak{q} \in \mathcal{Q}_n$ has exactly $n + 2 - 2g$ vertices, we obtain the relation

$$(n + 2 - 2g)\, |\mathcal{Q}_n| = 2\, |\mathcal{T}_n|. \tag{2.1}$$

We briefly describe here the mapping from $\mathcal{T}_n \times \{-1, +1\}$ onto $\mathcal{Q}_n^\bullet$, and we refer the reader to [CMS09] for a more precise description. Let $(\mathfrak{t}, \mathfrak{l}) \in \mathcal{T}_n$ be a well-labeled g-tree with n edges and $\varepsilon_\pm \in \{-1, +1\}$. As above, we write $\mathfrak{t}(0)$, $\mathfrak{t}(1)$, …, $\mathfrak{t}(2n)$ its facial sequence. The pointed quadrangulation $(\mathfrak{q}, v^\bullet)$ corresponding to $((\mathfrak{t}, \mathfrak{l}), \varepsilon_\pm)$ is then constructed as follows. First, shift all the labels in such a way that the minimal label is 1. Let us call $\tilde{\mathfrak{l}} := \mathfrak{l} - \min \mathfrak{l} + 1$ this shifted labeling function. Then, add an extra vertex $v^\bullet$ carrying the label $\tilde{\mathfrak{l}}(v^\bullet) := 0$ inside the only face of $\mathfrak{t}$. Finally, following the facial sequence, for every $0 \leq i \leq 2n - 1$, draw an arc—without crossing any edge of $\mathfrak{t}$ or arc already drawn—between $\mathfrak{t}(i)$ and $\mathfrak{t}(\mathrm{succ}(i))$, where $\mathrm{succ}(i)$ is the **successor** of i, defined by

$$\mathrm{succ}(i) := \begin{cases} \inf\{k \geq i : \tilde{\mathfrak{l}}(\mathfrak{t}(k)) = \tilde{\mathfrak{l}}(\mathfrak{t}(i)) - 1\} & \text{if } \{k \geq i : \tilde{\mathfrak{l}}(\mathfrak{t}(k)) = \tilde{\mathfrak{l}}(\mathfrak{t}(i)) - 1\} \neq \varnothing, \\ \inf\{k \geq 1 : \tilde{\mathfrak{l}}(\mathfrak{t}(k)) = \tilde{\mathfrak{l}}(\mathfrak{t}(i)) - 1\} & \text{otherwise,} \end{cases} \tag{2.2}$$

with the conventions $\inf \varnothing = \infty$, and $\mathfrak{t}(\infty) = v^\bullet$.

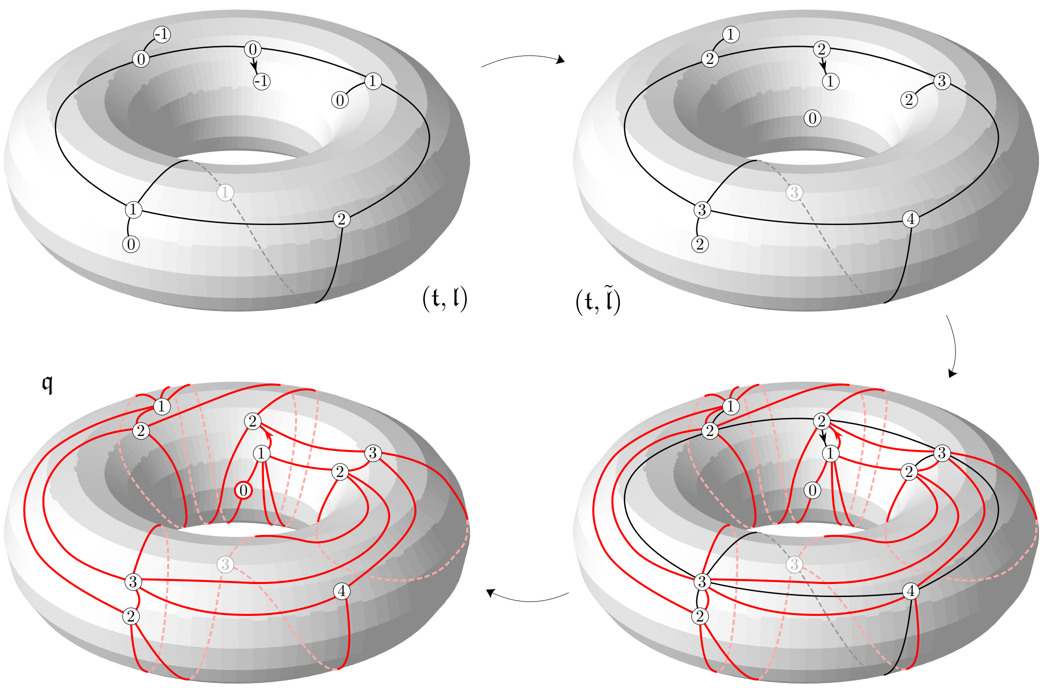

Figure 2.2. *The Chapuy–Marcus–Schaeffer bijection. In this example, $\varepsilon_\pm = +1$. On the bottom-left picture, the vertex $v^\bullet$ has a thicker (red) borderline.*

The quadrangulation $\mathfrak{q}$ is then defined as the map whose set of vertices is $V(\mathfrak{t}) \cup \{v^\bullet\}$, whose edges are the arcs we drew and whose root is the first arc drawn, oriented *from* $\mathfrak{t}(0)$ if $\varepsilon_\pm = -1$ or *toward* $\mathfrak{t}(0)$ if $\varepsilon_\pm = +1$ (see Figure 2.2).

Because of the way we drew the arcs of $\mathfrak{q}$, we see that for any vertex $v \in V(\mathfrak{q})$, $\tilde{\mathfrak{l}}(v) = d_{\mathfrak{q}}(v^\bullet, v)$. When seen as a vertex in $V(\mathfrak{q})$, we write $\mathfrak{q}(i)$ instead of $\mathfrak{t}(i)$. In particular,

$$\{\mathfrak{q}(i), 0 \le i \le 2n\} = V(\mathfrak{q}) \backslash \{v^\bullet\}.$$

We end this section by giving an upper bound for the distance between two vertices $\mathfrak{q}(i)$ and $\mathfrak{q}(j)$, in terms of the labeling function $\mathfrak{l}$:

$$d_{\mathfrak{q}}(\mathfrak{q}(i), \mathfrak{q}(j)) \le \mathfrak{l}(\mathfrak{t}(i)) + \mathfrak{l}(\mathfrak{t}(j)) - 2\max\left(\min_{k \in \overrightarrow{[\![i,j]\!]}} \mathfrak{l}(\mathfrak{t}(k)), \min_{k \in \overrightarrow{[\![j,i]\!]}} \mathfrak{l}(\mathfrak{t}(k))\right) + 2 \tag{2.3}$$

where we note, for $i \le j$, $[\![i,j]\!] := [i,j] \cap \mathbb{Z} = \{i, i+1, \ldots, j\}$, and

$$\overrightarrow{[\![i,j]\!]} := \begin{cases} [\![i,j]\!] & \text{if} \quad i \le j, \\ [\![i,2n]\!] \cup [\![0,j]\!] & \text{if} \quad j < i. \end{cases} \tag{2.4}$$

We refer the reader to [Mie09b, Lemma 4] for a detailed proof of this bound. The idea is the following: we consider the paths starting from $\mathfrak{t}(i)$ and $\mathfrak{t}(j)$ and made of the successive arcs going from vertices to their successors without crossing the g-tree. They are bound to meet at a vertex with label $m-1$, where

$$m := \min_{k \in \overrightarrow{[\![i,j]\!]}} \mathfrak{l}(\mathfrak{t}(k)).$$

On Figure 2.3, we see that the (red) plain path has length $\mathfrak{l}(\mathfrak{t}(i)) - m + 1$ and that the (purple) dashed one has length $\mathfrak{l}(\mathfrak{t}(j)) - m + 1$.

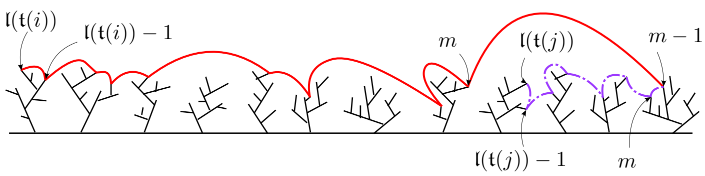

Figure 2.3. *Visual proof for* (2.3)*. Both paths are made of arcs constructed as explained above.*

2.2 Decomposition of a g-tree

We investigate here more closely the structure of a g-tree $\mathfrak{t}$. We call **scheme** a g-tree with no vertices of degree 1 or 2. Roughly speaking, a g-tree is a scheme in which every half-edge is replaced by a forest.

2.2.1 Forests

2.2.1.1 Formal definitions

We adapt the standard formalism for plane trees—as found in [Nev86] for instance—to forests. Let

$$\mathcal{U} := \bigcup_{n=1}^{\infty} \mathbb{N}^n$$

where $\mathbb{N} := \{1, 2, \ldots\}$. If $u \in \mathbb{N}^n$, we write $\|u\| := n$. For $u = (u_1, \ldots, u_n)$, $v = (v_1, \ldots, v_p) \in \mathcal{U}$, we let $uv := (u_1, \ldots, u_n, v_1, \ldots, v_p)$ be the concatenation of u and v. If $w = uv$ for some $u, v \in \mathcal{U}$, we say that u is an **ancestor** of w and that w is a **descendant** of u. In the case where $v \in \mathbb{N}$, we may also use the terms **parent** and **child** instead.

Definition 2.2. *A **forest** is a finite subset $\mathfrak{f} \subset \mathcal{U}$ satisfying the following:*

i. there is an integer $t(\mathfrak{f}) \geq 1$ such that $\mathfrak{f} \cap \mathbb{N} = [\![1, t(\mathfrak{f}) + 1]\!]$,

ii. if $u \in \mathfrak{f} \backslash \mathbb{N}$, then its parent belongs to $\mathfrak{f}$,

iii. for every $u \in \mathfrak{f}$, there is an integer $c_u(\mathfrak{f}) \geq 0$ such that $ui \in \mathfrak{f}$ if and only if $1 \leq i \leq c_u(\mathfrak{f})$,

iv. $c_{t(\mathfrak{f})+1}(\mathfrak{f}) = 0$.

*The integer $t(\mathfrak{f})$ is called the **number of trees** of $\mathfrak{f}$.*

We will see in a moment why we require $t(\mathfrak{f}) + 1$ to lie in $\mathfrak{f}$. For $u = (u_1, \ldots, u_p) \in \mathfrak{f}$, we call $\mathfrak{a}(u) := u_1$ its oldest ancestor. A **tree** of $\mathfrak{f}$ is a level set for $\mathfrak{a}$: for $1 \leq j \leq t(\mathfrak{f})$, the j-th tree of $\mathfrak{f}$ is the set $\{u \in \mathfrak{f} : \mathfrak{a}(u) = j\}$. The integer $\mathfrak{a}(u)$ hence records which tree u belongs to. We call the set $\mathit{fl} := \mathfrak{f} \cap \mathbb{N} = \{\mathfrak{a}(u),\ u \in \mathfrak{f}\}$ the **floor** of the forest $\mathfrak{f}$. When $u \in \mathit{fl}$, we sometime note it (u) to avoid confusion between the integer u and the point $(u) \in \mathfrak{f}$.

For $u, v \in \mathfrak{f}$, we write $u \sim v$ if either

- u is a parent or child of v, or
- $u, v \in \mathit{fl}$ and $|u - v| = 1$.

It is convenient, when representing a forest, to draw edges between u's and v's such that $u \sim v$ (see Figure 2.5). We say that an edge drawn between a parent and its child is a **tree edge** whereas an edge drawn between two consecutive tree roots, i.e. between some i and $i + 1$, will be called a **floor edge**.

We call $\mathcal{F}_\sigma^m$ the set of all forests with σ trees and m tree edges, that is

$$\mathcal{F}_\sigma^m := \{\mathfrak{f} : t(\mathfrak{f}) = \sigma,\ |\mathfrak{f}| = m + \sigma + 1\}.$$

Definition 2.3. *A **well-labeled forest** is a pair* $(\mathfrak{f}, \mathfrak{l})$ *where* $\mathfrak{f}$ *is a forest and* $\mathfrak{l} : \mathfrak{f} \to \mathbb{Z}$ *is a function satisfying:*

i. for all $u \in \mathfrak{fl}$*, we have* $\mathfrak{l}(u) = 0$*,*

ii. if $u \sim v$*, then* $|\mathfrak{l}(u) - \mathfrak{l}(v)| \leq 1$*.*

Let

$$\mathfrak{F}_\sigma^m := \{(\mathfrak{f}, \mathfrak{l}) : \mathfrak{f} \in \mathcal{F}_\sigma^m\}$$

be the set of well-labeled forests with σ trees and m tree edges.

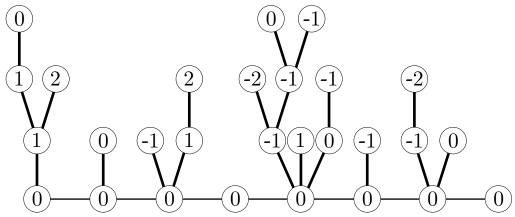

Figure 2.4. *An example of well-labeled forest from* $\mathfrak{F}_7^{20}$*.*

Remark. For every forest in $\mathcal{F}_\sigma^m$, there are exactly 3^m admissible ways to label it: for all tree edges, one may choose any increment in $\{-1, 0, 1\}$. As a result, $|\mathfrak{F}_\sigma^m| = 3^m |\mathcal{F}_\sigma^m|$.

2.2.1.2 Encoding by contour and spatial contour functions

There is a very convenient way to code forests and well-labeled forests. Let $\mathfrak{f} \in \mathcal{F}_\sigma^m$ be a forest. Let us begin by defining its **facial sequence** $\mathfrak{f}(0), \mathfrak{f}(1), \ldots, \mathfrak{f}(2m+\sigma)$ as follows (see Figure 2.5): $\mathfrak{f}(0) := (1)$, and for $0 \leq i \leq 2m + \sigma - 1$,

- if $\mathfrak{f}(i)$ has children that do not appear in the sequence $\mathfrak{f}(0), \mathfrak{f}(1), \ldots, \mathfrak{f}(i)$, then $\mathfrak{f}(i+1)$ is the first of these children, that is $\mathfrak{f}(i+1) := \mathfrak{f}(i)j_0$ where

$$j_0 = \min \{j \geq 1 \,:\, \mathfrak{f}(i)j \notin \{\mathfrak{f}(0), \mathfrak{f}(1), \ldots, \mathfrak{f}(i)\}\},$$

- otherwise, if $\mathfrak{f}(i)$ has a parent (that is $\mathfrak{f}(i) \notin \mathfrak{fl}$), then $\mathfrak{f}(i+1)$ is this parent,
- if neither of these cases occur, which implies that $\mathfrak{f}(i) \in \mathfrak{fl}$, then $\mathfrak{f}(i+1) := \mathfrak{f}(i) + 1$.

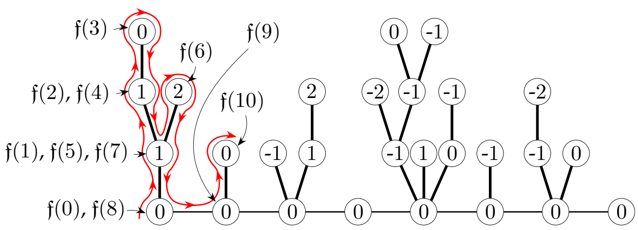

Figure 2.5. *The facial sequence associated with the well-labeled forest from Figure 2.4.*

It is easy to see that each tree edge is visited exactly twice—once going from the parent to the child, once going the other way around—whereas each floor edge is visited only once—from some i to $i + 1$. As a result, $\mathfrak{f}(2m + \sigma) = t(\mathfrak{f}) + 1$.

The **contour function** of $\mathfrak{f}$ is the function $C_{\mathfrak{f}} : [0, 2m+\sigma] \to \mathbb{R}_+$ defined, for $0 \leq i \leq 2m+\sigma$, by

$$C_{\mathfrak{f}}(i) := \|\mathfrak{f}(i)\| + t(\mathfrak{f}) - \mathfrak{a}\left(\mathfrak{f}(i)\right)$$

and linearly interpolated between integer values (see Figure 2.6).

We can easily check that the function $C_{\mathfrak{f}}$ entirely determines the forest $\mathfrak{f}$. We see that $C_{\mathfrak{f}}$ ranges in the set of paths of a simple random walk starting from $t(\mathfrak{f})$ and conditioned to hit 0 for the first time at $2m+\sigma$. This allows us to compute the cardinality of $\mathcal{F}_\sigma^m$:

Lemma 2.1. *Let $\sigma \geq 1$ and $m \geq 0$ be two integers. The number of forests with σ trees and m tree edges is:*

$$\left|\mathcal{F}_\sigma^m\right| = \frac{\sigma}{2m+\sigma}\, 2^{2m+\sigma}\, \mathbb{P}(S_{2m+\sigma} = \sigma) = \frac{\sigma}{2m+\sigma}\binom{2m+\sigma}{m},$$

where $(S_i)_{i\geq 0}$ is a simple random walk on $\mathbb{Z}$.

Proof. Shifting the contour functions, we see that $|\mathcal{F}_\sigma^m|$ is the number of different paths of a simple random walk starting from 0 and conditioned to hit $-\sigma$ for the first time at $2m+\sigma$. We have

$$\begin{aligned}\left|\mathcal{F}_\sigma^m\right| &= 2^{2m+\sigma}\, \mathbb{P}\left(S_{2m+\sigma} = -\sigma\,;\, \forall i \in [\![0, 2m+\sigma-1]\!],\, S_i > -\sigma\right)\\ &= \frac{\sigma}{2m+\sigma}\, 2^{2m+\sigma}\, \mathbb{P}(S_{2m+\sigma} = \sigma),\end{aligned}$$

where the second equality is an application of the so-called cycle lemma (see for example [BCP03, Lemma 2]). The second equality of the lemma is obtained by seeing that $S_{2m+\sigma} = \sigma$ if and only if the walk goes exactly $m+\sigma$ times up and m times down. □

Now, if we have a well-labeled forest $(\mathfrak{f}, \mathfrak{l})$, the contour function $C_{\mathfrak{f}}$ enables us to recover $\mathfrak{f}$. To record the labels, we use the **spatial contour function** $L_{\mathfrak{f},\mathfrak{l}} : [0, 2m+\sigma] \to \mathbb{R}$ defined, for $0 \leq i \leq 2m+\sigma$, by

$$L_{\mathfrak{f},\mathfrak{l}}(i) := \mathfrak{l}(\mathfrak{f}(i))$$

and linearly interpolated between integer values (see Figure 2.6). The **contour pair** $(C_{\mathfrak{f}}, L_{\mathfrak{f},\mathfrak{l}})$ then entirely determines $(\mathfrak{f}, \mathfrak{l})$.

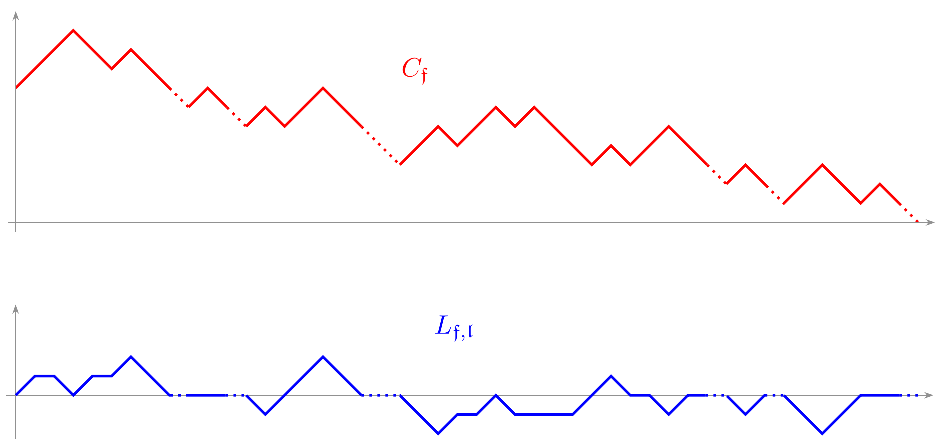

Figure 2.6. *The contour pair of the well-labeled forest appearing in Figures 2.4 and 2.5. The paths are dashed on the intervals corresponding to floor edges.*

2.2.2 Scheme

2.2.2.1 Extraction of the scheme out of a g-tree

Definition 2.4. *We call* ***scheme*** *of genus g a g-tree with no vertices of degree one or two. A scheme is said to be* ***dominant*** *when it only has vertices of degree exactly three.*

Remark. The Euler characteristic formula easily shows that the number of schemes of genus g is finite. We call $\mathfrak{S}$ the set of all schemes of genus g and $\mathfrak{S}^*$ the set of all dominant schemes of genus g.

It was explained in [CMS09] how to extract the scheme out of a g-tree $\mathfrak{t}$. Let us recall now this operation. By iteratively deleting all its vertices of degree 1, we are left with a—non-necessarily rooted—g-tree. If the root has been removed, we root this new g-tree on the first remaining half-edge following the actual root in the facial order of $\mathfrak{t}$.

The vertices of degree 2 in the new g-tree are organized into maximal chains connected together at vertices of degree at least 3. We replace each of these maximal chains by a single new edge. The edge replacing the chain containing the root is chosen to be the final root (with the same orientation).

By construction, the map $\mathfrak{s}$ we obtain is a scheme of genus g, which we call the scheme of the g-tree $\mathfrak{t}$. See Figure 2.7.

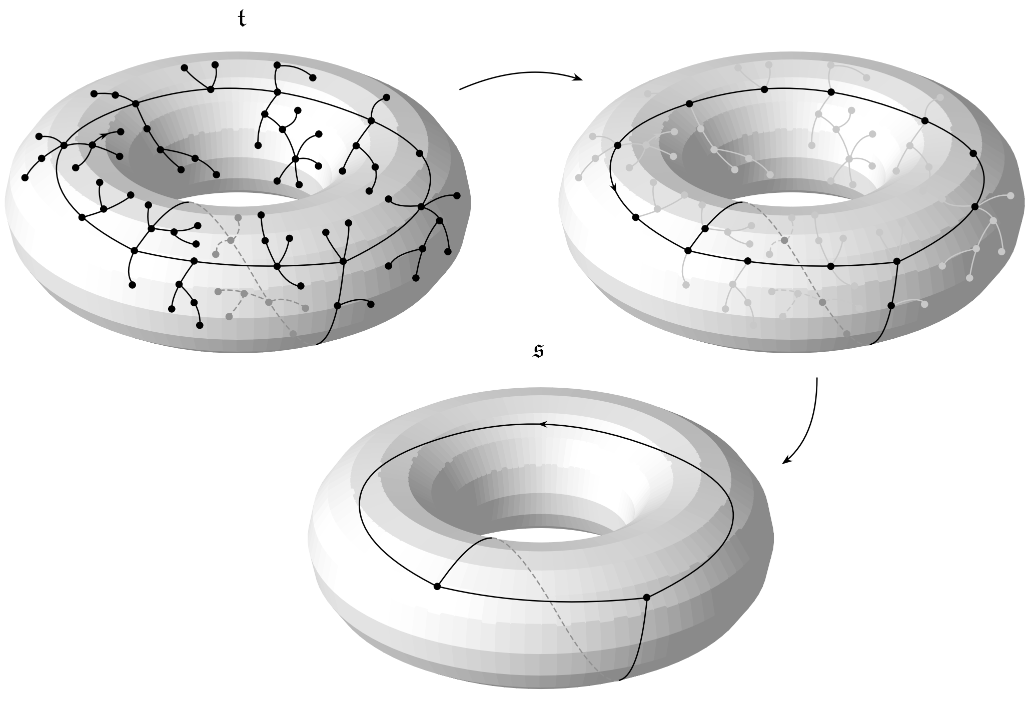

Figure 2.7. *Extraction of the scheme $\mathfrak{s}$ out of the g-tree $\mathfrak{t}$.*

2.2.2.2 Decomposition of a g-tree

When iteratively removing vertices of degree 1, we actually remove whole trees. Let $\mathfrak{c}_1, \mathfrak{c}_2, \ldots, \mathfrak{c}_k$ be one of the maximal chains of half-edges linking two nodes. The trees that we remove, appearing on the left side of this chain, connected to one of the $\mathfrak{c}_i^-$'s, form a forest—with k trees—as defined in Section 2.2.1. Beware that the tree connected to $\mathfrak{c}_k^+$ is not a part of this forest; it will be the first tree of some other forest. Remember that the forests we consider always end by a single vertex not considered to be a tree. This chain being later replaced by a single half-edge of the scheme, we see that a g-tree $\mathfrak{t}$

can be decomposed into its scheme $\mathfrak{s}$ and a collection of forests $(\mathfrak{f}^{\mathfrak{e}})_{\mathfrak{e}\in\vec{E}(\mathfrak{s})}$. Recall that $\vec{E}(\mathfrak{s})$ is the set of all half-edges of $\mathfrak{s}$.

For $\mathfrak{e} \in \vec{E}(\mathfrak{s})$, let us define the integers $m^{\mathfrak{e}} \geq 0$ and $\sigma^{\mathfrak{e}} \geq 1$ by

$$\mathfrak{f}^{\mathfrak{e}} \in \mathcal{F}_{\sigma^{\mathfrak{e}}}^{m^{\mathfrak{e}}}, \tag{2.5}$$

so that $m^{\mathfrak{e}}$ records the "size" of the forest attached on the half-edge $\mathfrak{e}$ and $\sigma^{\mathfrak{e}}$ its "length."

In order to recover $\mathfrak{t}$ from $\mathfrak{s}$ and these forests, we need to record the position its root. It may be seen as a half-edge of the forest $\mathfrak{f}^{\mathfrak{e}_*}$ corresponding to the root $\mathfrak{e}_*$ of $\mathfrak{s}$. We code it by the integer

$$u \in [\![0,\, 2m^{\mathfrak{e}_*} + \sigma^{\mathfrak{e}_*}[\![\tag{2.6}$$

for which this half-edge links $\mathfrak{f}^{\mathfrak{e}_*}(u)$ to $\mathfrak{f}^{\mathfrak{e}_*}(u+1)$.

For every half-edge $\mathfrak{e} \in \vec{E}(\mathfrak{s})$, if we call $\bar{\mathfrak{e}}$ its reverse, we readily obtain the relation:

$$\sigma^{\bar{\mathfrak{e}}} = \sigma^{\mathfrak{e}}. \tag{2.7}$$

This decomposition may be inverted. Let us suppose that we have a scheme $\mathfrak{s}$ and a collection of forests $(\mathfrak{f}^{\mathfrak{e}})_{\mathfrak{e}\in\vec{E}(\mathfrak{s})}$. Let us define the integers $m^{\mathfrak{e}}$'s and $\sigma^{\mathfrak{e}}$'s by (2.5) and suppose they satisfy (2.7). Let again $0 \leq u < 2m^{\mathfrak{e}_*} + \sigma^{\mathfrak{e}_*}$ be an integer. Then we may construct a g-tree as follows. First, we replace every edge $\{\mathfrak{e}, \bar{\mathfrak{e}}\}$ by a chain of $\sigma^{\mathfrak{e}} = \sigma^{\bar{\mathfrak{e}}}$ edges. Then, for every half-edge $\mathfrak{e} \in \vec{E}(\mathfrak{s})$, we replace the chain of half-edges corresponding to it by the forest $\mathfrak{f}^{\mathfrak{e}}$, in such a way that its floor[1] matches with the chain. Finally, we find the root inside $\mathfrak{f}^{\mathfrak{e}_*}$ thanks to the integer u.

This discussion is summed up by the following proposition. The factor $1/2$ in the last statement comes from the fact that the floor of $\mathfrak{f}^{\mathfrak{e}}$ and that of $\mathfrak{f}^{\bar{\mathfrak{e}}}$ are overlapping in the g-tree, thus their edges should be counted only once.

Proposition 2.2. *The above construction provides us with a bijection between the set of all g-trees and the set of all triples $\left(\mathfrak{s}, (\mathfrak{f}^{\mathfrak{e}})_{\mathfrak{e}\in\vec{E}(\mathfrak{s})}, u\right)$ where $\mathfrak{s} \in \mathfrak{S}$ is a scheme (of genus g), the forests $\mathfrak{f}^{\mathfrak{e}} \in \mathcal{F}_{\sigma^{\mathfrak{e}}}^{m^{\mathfrak{e}}}$ satisfy* (2.7) *and u satisfies* (2.6).

Moreover, g-trees with n edges correspond to triples satisfying $\sum_{\mathfrak{e}\in\vec{E}(\mathfrak{s})} \left(m^{\mathfrak{e}} + \frac{1}{2}\sigma^{\mathfrak{e}}\right) = n$.

Let $\mathfrak{t}$ be a g-tree and $\left(\mathfrak{s}, (\mathfrak{f}^{\mathfrak{e}})_{\mathfrak{e}\in\vec{E}(\mathfrak{s})}, u\right)$ be the corresponding triple. We say that $\mathfrak{s}$ is the scheme of $\mathfrak{t}$ and that the forests $\mathfrak{f}^{\mathfrak{e}}$, $\mathfrak{e} \in \vec{E}(\mathfrak{s})$, are its forests. The set $V(\mathfrak{s})$ may be seen as a subset of $\mathfrak{t}$; we call **nodes** its elements. Finally, we call **floor** of $\mathfrak{t}$ the set fl of vertices we obtain after replacing the edges of $\mathfrak{s}$ by chains of edges (see Figure 2.8).

2.2.2.3 Decomposition of a well-labeled g-tree

We now deal with a well-labeled g-tree. We will need the following definitions:

Definition 2.5. *We call* **Motzkin path** *a sequence of the form $(M_n)_{0\leq n\leq\sigma}$ for some $\sigma \geq 0$ such that $M_0 = 0$ and for $0 \leq i \leq \sigma - 1$, $M_{i+1} - M_i \in \{-1, 0, 1\}$. We write $\sigma(M) := \sigma$ its lifetime, and $\hat{M} := M_{\sigma(M)}$ its final value.*

A **Motzkin bridge** of lifetime σ from $l_1 \in \mathbb{Z}$ to $l_2 \in \mathbb{Z}$ is an element of the set

$$\mathcal{M}_{[0,\sigma]}^{l_1\to l_2} := \left\{ l_1 + M \;:\; M \text{ Motzkin path such that } \sigma(M) = \sigma,\ \hat{M} = l_2 - l_1 \right\}.$$

We say that $(M_n)_{n\geq 0}$ is a **simple Motzkin walk** if it is defined as the sum of i.i.d. random variables with law $\frac{1}{3}(\delta_{-1} + \delta_0 + \delta_1)$.

[1] The floor of a forest $\mathfrak{f}$ is naturally oriented from 1 to $t(\mathfrak{f}) + 1$. The forest $\mathfrak{f}^{\mathfrak{e}}$ is then grafted "to the left side" of $\mathfrak{e}$.

Remark. We then have

$$\left|\mathcal{M}_{[0,\sigma]}^{l_1 \to l_2}\right| = 3^{\sigma}\, \mathbb{P}(M_{\sigma} = l_2 - l_1)$$

where $(M_i)_{i \geq 0}$ is a simple Motzkin walk.

When decomposing a well-labeled g-tree $(\mathfrak{t}, \mathfrak{l})$ into a triple $(\mathfrak{s}, (\mathfrak{f}^{\mathfrak{e}}), u)$ according to Proposition 2.2, every forest $\mathfrak{f}^{\mathfrak{e}}$ naturally inherits a labeling function noted $\tilde{\mathfrak{l}}^{\mathfrak{e}}$ from $\mathfrak{l}$. In general, the forest $(\mathfrak{f}^{\mathfrak{e}}, \tilde{\mathfrak{l}}^{\mathfrak{e}})$ is not well-labeled, because the labels of its floor have no reason to be equal to 0. We will transform it into a Motzkin bridge $\mathfrak{M}^{\mathfrak{e}}$ starting from 0 and a well-labeled forest $(\mathfrak{f}^{\mathfrak{e}}, \mathfrak{l}^{\mathfrak{e}})$. The Motzkin bridge records the floor labels shifted in order to start from 0: for $0 \leq i \leq t(\mathfrak{f}^{\mathfrak{e}})$, $\mathfrak{M}^{\mathfrak{e}}(i) := \tilde{\mathfrak{l}}^{\mathfrak{e}}(i+1) - \tilde{\mathfrak{l}}^{\mathfrak{e}}(1)$, where, on the right-hand side, we used the notation $\{1, 2, \ldots, t(\mathfrak{f}^{\mathfrak{e}}) + 1\}$ for the floor of $\mathfrak{f}^{\mathfrak{e}}$. The well-labeled forest is obtained by shifting all the labels tree by tree in such a way that the root label of any tree is 0: for all $w \in \mathfrak{f}^{\mathfrak{e}}$, $\mathfrak{l}^{\mathfrak{e}}(w) := \tilde{\mathfrak{l}}^{\mathfrak{e}}(w) - \tilde{\mathfrak{l}}^{\mathfrak{e}}(\mathfrak{a}(w))$.

We thus decompose the well-labeled g-tree $(\mathfrak{t}, \mathfrak{l})$ into its scheme $\mathfrak{s}$, a collection $(\mathfrak{M}^{\mathfrak{e}})_{\mathfrak{e} \in \vec{E}(\mathfrak{s})}$ of Motzkin bridges started at 0, a collection $(\mathfrak{f}^{\mathfrak{e}}, \mathfrak{l}^{\mathfrak{e}})_{\mathfrak{e} \in \vec{E}(\mathfrak{s})}$ of well-labeled forests and an integer u, as shown on Figure 2.8.

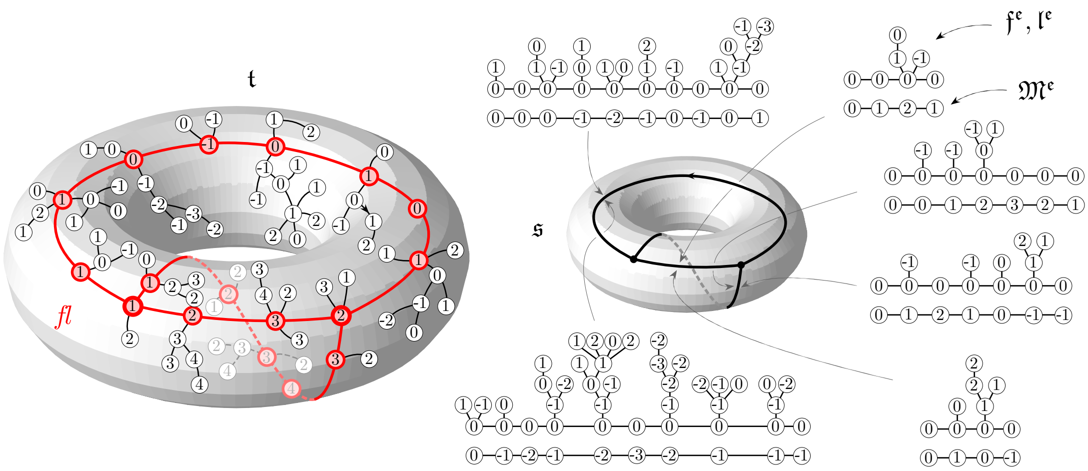

Figure 2.8. *Decomposition of a well-labeled g-tree* $\mathfrak{t}$ *into its scheme* $\mathfrak{s}$*, the collection of its Motzkin bridges* $(\mathfrak{M}^{\mathfrak{e}})_{\mathfrak{e} \in \vec{E}(\mathfrak{s})}$*, and the collection of its well-labeled forests* $(\mathfrak{f}^{\mathfrak{e}}, \mathfrak{l}^{\mathfrak{e}})_{\mathfrak{e} \in \vec{E}(\mathfrak{s})}$*. In this example, the integer* $u = 10$*. The floor of* $\mathfrak{t}$ *is more thickly outlined, and its two nodes are even more thickly outlined.*

For $\mathfrak{e} \in \vec{E}(\mathfrak{s})$, we define the integer $l^{\mathfrak{e}} \in \mathbb{Z}$ to be such that

$$\mathfrak{M}^{\mathfrak{e}} \in \mathcal{M}_{[0,\sigma^{\mathfrak{e}}]}^{0 \to l^{\mathfrak{e}}}. \tag{2.8}$$

It records the spatial displacement made along the half-edge $\mathfrak{e}$. Because the floor of $\mathfrak{f}^{\mathfrak{e}}$ overlaps the floor of $\mathfrak{f}^{\bar{\mathfrak{e}}}$ in the g-tree, $\mathfrak{M}^{\mathfrak{e}}$ and $\mathfrak{M}^{\bar{\mathfrak{e}}}$ read the same labels in opposite direction:

$$\mathfrak{M}^{\bar{\mathfrak{e}}}(i) = \mathfrak{M}^{\mathfrak{e}}(\sigma^{\mathfrak{e}} - i) - l^{\mathfrak{e}}. \tag{2.9}$$

In particular, $l^{\bar{\mathfrak{e}}} = -l^{\mathfrak{e}}$. But this is not the only constraints on the family $(l^{\mathfrak{e}})_{\mathfrak{e} \in \vec{E}(\mathfrak{s})}$. These will be easier to understand while looking at vertices instead of edges. For every vertex $v \in V(\mathfrak{s})$, we let l^v be the label of the corresponding node shifted in such a way that $l^{\mathfrak{e}_*^-} = 0$. We have the following relation between $(l^{\mathfrak{e}})_{\mathfrak{e} \in \vec{E}(\mathfrak{s})}$ and $(l^v)_{v \in V(\mathfrak{s})}$: for all $\mathfrak{e} \in \vec{E}(\mathfrak{s})$,

$$l^{\mathfrak{e}} = l^{\mathfrak{e}^+} - l^{\mathfrak{e}^-}, \tag{2.10}$$

so that the family $(l^v)_{v\in V(\mathfrak{s})}$ entirely determines $(l^{\mathfrak{e}})_{\mathfrak{e}\in\vec{E}(\mathfrak{s})}$. Because of the choice we made, $l^{\mathfrak{e}_*^-}=0$, it is easy to see that $(l^{\mathfrak{e}})_{\mathfrak{e}\in\vec{E}(\mathfrak{s})}$ determines $(l^v)_{v\in V(\mathfrak{s})}$ as well.

It now becomes clear that the only constraint on $(l^{\mathfrak{e}})_{\mathfrak{e}\in\vec{E}(\mathfrak{s})}$ is to be obtained from a family $(l^v)_{v\in V(\mathfrak{s})}$ by the relations (2.10).

Let $\mathfrak{s}$ be a scheme, $(\mathfrak{M}^{\mathfrak{e}})_{\mathfrak{e}\in\vec{E}(\mathfrak{s})}$ be a family of Motzkin bridges started from 0, $(\mathfrak{f}^{\mathfrak{e}},\mathfrak{l}^{\mathfrak{e}})_{\mathfrak{e}\in\vec{E}(\mathfrak{s})}$ be a family of well-labeled forests, and u be an integer. Let the integers $m^{\mathfrak{e}}$'s, $\sigma^{\mathfrak{e}}$'s and $l^{\mathfrak{e}}$'s be defined by (2.5) and (2.8). We will say that the quadruple $\big(\mathfrak{s},(\mathfrak{M}^{\mathfrak{e}})_{\mathfrak{e}\in\vec{E}(\mathfrak{s})},(\mathfrak{f}^{\mathfrak{e}},\mathfrak{l}^{\mathfrak{e}})_{\mathfrak{e}\in\vec{E}(\mathfrak{s})},u\big)$ is *compatible* if the integers $\sigma^{\mathfrak{e}}$'s satisfy the constraints (2.7), the Motzkin bridges $\mathfrak{M}^{\mathfrak{e}}$'s satisfy (2.9), the integers $l^{\mathfrak{e}}$'s can be obtained from a family $(l^v)_{v\in V(\mathfrak{s})}$ by the relations (2.10), and u satisfies (2.6).

Let suppose now that we have a compatible quadruple $\big(\mathfrak{s},(\mathfrak{M}^{\mathfrak{e}})_{\mathfrak{e}\in\vec{E}(\mathfrak{s})},(\mathfrak{f}^{\mathfrak{e}},\mathfrak{l}^{\mathfrak{e}})_{\mathfrak{e}\in\vec{E}(\mathfrak{s})},u\big)$. We may reconstruct a well-labeled g-tree as follows. We begin by suitably relabeling the forests. For every half-edge $\mathfrak{e}$, first, we shift the labels of $\mathfrak{M}^{\mathfrak{e}}$ by $l^{\mathfrak{e}^-}$ so that it becomes a bridge from $l^{\mathfrak{e}^-}$ to $l^{\mathfrak{e}^+}$. Then, we shift all the labels of $(\mathfrak{f}^{\mathfrak{e}},\mathfrak{l}^{\mathfrak{e}})$ tree by tree according to the Motzkin bridge: precisely, we change $\mathfrak{l}^{\mathfrak{e}}$ into $w\in\mathfrak{f}^{\mathfrak{e}}\mapsto l^{\mathfrak{e}^-}+\mathfrak{M}^{\mathfrak{e}}(\mathfrak{a}(w)-1)+\mathfrak{l}^{\mathfrak{e}}(w)$. Then, we replace the half-edge $\mathfrak{e}$ by this forest, as in the previous section. As before, we find the root thanks to u. Finally, we shift all the labels for the root label to be equal to 0. This discussion is summed up by the following proposition.

Proposition 2.3. *The above construction provides a bijection between the set of all well-labeled g-trees and the set of all compatible quadruples $\big(\mathfrak{s},(\mathfrak{M}^{\mathfrak{e}})_{\mathfrak{e}\in\vec{E}(\mathfrak{s})},(\mathfrak{f}^{\mathfrak{e}},\mathfrak{l}^{\mathfrak{e}})_{\mathfrak{e}\in\vec{E}(\mathfrak{s})},u\big)$.*

Moreover, g-trees with n edges correspond to quadruples satisfying $\sum_{\mathfrak{e}\in\vec{E}(\mathfrak{s})}\big(m^{\mathfrak{e}}+\frac{1}{2}\sigma^{\mathfrak{e}}\big)=n$.

If we call $(C^{\mathfrak{e}},L^{\mathfrak{e}})$ the contour pair of $(\mathfrak{f}^{\mathfrak{e}},\mathfrak{l}^{\mathfrak{e}})$, then we may retrieve the oldest ancestor of $\mathfrak{f}^{\mathfrak{e}}(i)$ thanks to $C^{\mathfrak{e}}$ by the relation

$$\mathfrak{a}\big(\mathfrak{f}^{\mathfrak{e}}(i)\big)-1=\sigma^{\mathfrak{e}}-\underline{C}^{\mathfrak{e}}(i),$$

where we use the notation

$$\underline{X}_s:=\inf_{[0,s]}X$$

for any process $(X_s)_{s\geq 0}$. The function

$$\mathfrak{L}^{\mathfrak{e}}:=\Big(L^{\mathfrak{e}}(t)+\mathfrak{M}^{\mathfrak{e}}\big(\sigma^{\mathfrak{e}}-\underline{C}^{\mathfrak{e}}(t)\big)\Big)_{0\leq t\leq 2m^{\mathfrak{e}}+\sigma^{\mathfrak{e}}} \tag{2.11}$$

then records the labels of the forest $\mathfrak{f}^{\mathfrak{e}}$, once shifted tree by tree according to the Motzkin bridge $\mathfrak{M}^{\mathfrak{e}}$. This function will play an important part in Section 3.1.

Through the Chapuy–Marcus–Schaeffer bijection, a uniform random quadrangulation corresponds to a uniform random well-labeled g-tree. In order to investigate the scaling limit of the latter, we will proceed in two steps. First, we consider the scaling limit of its *structure*, consisting in its scheme along with the integers $m^{\mathfrak{e}}$'s, $\sigma^{\mathfrak{e}}$'s, l^v's and u previously defined. Then, we deal with its Motzkin bridges and forests conditionally given its structure.

2.3 Convergence of the structure of a uniform well-labeled g-tree

2.3.1 Preliminaries

We investigate here the convergence of the integers previously defined, suitably rescaled, in the case of a uniform random well-labeled g-tree with n vertices. Let $(\mathfrak{t}_n,\mathfrak{l}_n)$ be uniformly distributed over the set $\mathcal{T}_n$ of well-labeled g-trees with n vertices. We call its scheme $\mathfrak{s}_n$ and we define

$$(\mathfrak{M}^{\mathfrak{e}}_n)_{\mathfrak{e}\in\vec{E}(\mathfrak{s}_n)},\ (\mathfrak{f}^{\mathfrak{e}}_n,\mathfrak{l}^{\mathfrak{e}}_n)_{\mathfrak{e}\in\vec{E}(\mathfrak{s}_n)},\ (m^{\mathfrak{e}}_n)_{\mathfrak{e}\in\vec{E}(\mathfrak{s}_n)},\ (\sigma^{\mathfrak{e}}_n)_{\mathfrak{e}\in\vec{E}(\mathfrak{s}_n)},\ (l^{\mathfrak{e}}_n)_{\mathfrak{e}\in\vec{E}(\mathfrak{s}_n)},\ (l^v_n)_{v\in V(\mathfrak{s}_n)},\ \text{and } u_n$$

as in the previous section. We know that the right scalings are $2n$ for sizes, $\sqrt{2n}$ for distances in the g-tree, and $\gamma\, n^{1/4}$ for spatial displacements[2], so we set:

$$m^{\mathfrak{e}}_{(n)} := \frac{2m^{\mathfrak{e}}_n + \sigma^{\mathfrak{e}}_n}{2n}, \qquad \sigma^{\mathfrak{e}}_{(n)} := \frac{\sigma^{\mathfrak{e}}_n}{\sqrt{2n}}, \qquad l^{\mathfrak{e}}_{(n)} := \frac{l^{\mathfrak{e}}_n}{\gamma\, n^{1/4}}, \qquad l^{v}_{(n)} := \frac{l^{v}_n}{\gamma\, n^{1/4}} \qquad \text{and} \qquad u_{(n)} := \frac{u_n}{2n}.$$

Remark. Throughout this paper, the notation with a parenthesized n will always refer to suitably rescaled objects—as in the definitions above.

As sensed in the previous section, it will be more convenient to work with l^v's instead of $l^{\mathfrak{e}}$'s. We use the notation $\mathbb{Z}_+ := \{0,1,\ldots\}$ for the set of nonnegative integers. For any scheme $\mathfrak{s} \in \mathfrak{S}$, we define the set $\mathfrak{C}_n(\mathfrak{s})$ of quadruples (m,σ,l,u) lying in $\mathbb{Z}_+^{\vec{E}(\mathfrak{s})} \times \mathbb{N}^{\vec{E}(\mathfrak{s})} \times \mathbb{Z}^{V(\mathfrak{s})} \times \mathbb{Z}_+$ such that:

- $\forall \mathfrak{e} \in \vec{E}(\mathfrak{s})$, $\sigma^{\bar{\mathfrak{e}}} = \sigma^{\mathfrak{e}}$,
- $l^{\mathfrak{e}_*^-} = 0$,
- $0 \le u \le 2m^{\mathfrak{e}_*} + \sigma^{\mathfrak{e}_*} - 1$,
- $\sum_{\mathfrak{e}\in\vec{E}(\mathfrak{s})} \left(m^{\mathfrak{e}} + \frac{1}{2}\sigma^{\mathfrak{e}}\right) = n$.

This is the set of integers satisfying the constraints discussed in the previous section for a well-labeled g-tree with n edges. For $(m,\sigma,l,u) \in \mathfrak{C}_n(\mathfrak{s})$, we will compute the probability that $\mathfrak{s}_n = \mathfrak{s}$ and $(m_n,\sigma_n,l_n,u_n) = (m,\sigma,l,u)$. A g-tree has such features if and only if its scheme is $\mathfrak{s}$ and, for every $\mathfrak{e} \in \vec{E}(\mathfrak{s})$, the path $\mathfrak{M}^{\mathfrak{e}}$ is a Motzkin bridge from 0 to $l^{\mathfrak{e}} = l^{\mathfrak{e}^+} - l^{\mathfrak{e}^-}$ on $[0,\sigma^{\mathfrak{e}}]$, and the well-labeled forest $(\mathfrak{f}^{\mathfrak{e}},\mathfrak{l}^{\mathfrak{e}})$ lies in $\mathfrak{F}^{m^{\mathfrak{e}}}_{\sigma^{\mathfrak{e}}}$.

Moreover, because of the relation (2.9), the Motzkin bridges $(\mathfrak{M}^{\mathfrak{e}})_{\mathfrak{e}\in\vec{E}(\mathfrak{s})}$ are entirely determined by $(\mathfrak{M}^{\mathfrak{e}})_{\mathfrak{e}\in\check{E}(\mathfrak{s})}$, where $\check{E}(\mathfrak{s})$ is any orientation of $\vec{E}(\mathfrak{s})$. Using Lemma 2.1, we obtain

$$\begin{aligned} &\mathbb{P}\left(\mathfrak{s}_n = \mathfrak{s},\, (m_n,\sigma_n,l_n,u_n) = (m,\sigma,l,u)\right) \\ &\qquad = \frac{1}{|\mathcal{T}_n|} \prod_{\mathfrak{e}\in\check{E}(\mathfrak{s})} \left|\mathcal{M}^{0\to l^{\mathfrak{e}}}_{[0,\sigma^{\mathfrak{e}}]}\right| \left|\mathfrak{F}^{m^{\mathfrak{e}}}_{\sigma^{\mathfrak{e}}}\right| \left|\mathfrak{F}^{m^{\bar{\mathfrak{e}}}}_{\sigma^{\bar{\mathfrak{e}}}}\right| \\ &\qquad = \frac{12^n}{|\mathcal{T}_n|} \prod_{\mathfrak{e}\in\vec{E}(\mathfrak{s})} \frac{\sigma^{\mathfrak{e}}}{2m^{\mathfrak{e}}+\sigma^{\mathfrak{e}}}\, \mathbb{P}(S_{2m^{\mathfrak{e}}+\sigma^{\mathfrak{e}}} = \sigma^{\mathfrak{e}}) \prod_{\mathfrak{e}\in\check{E}(\mathfrak{s})} \mathbb{P}(M_{\sigma^{\mathfrak{e}}} = l^{\mathfrak{e}}) \end{aligned} \tag{2.12}$$

where $(S_i)_{i\ge 0}$ is a simple random walk on $\mathbb{Z}$ and $(M_i)_{i\ge0}$ is a simple Motzkin walk.

We will need the following local limit theorem (see [Pet75, Theorems VII.1.6 and VII.3.16]) to estimate the probabilities above. We call p the density of a standard Gaussian random variable: $p(x) = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{x^2}{2}}$.

Proposition 2.4. *Let $(X_i)_{i\ge0}$ be a sequence of i.i.d. integer-valued centered random variables with a moment of order r_0 for some $r_0 \ge 3$. Let $\eta^2 := \mathrm{Var}(X_1)$, h be the maximal span[3] of X_1 and the integer a be such that a.s. $X_1 \in a + h\mathbb{Z}$. We define $\Sigma_k := \sum_{i=0}^k X_i$, and we write $Q^\Sigma_k(i) := \mathbb{P}(\Sigma_k = i)$.*

1. *We have*

$$\sup_{i\in ka+h\mathbb{Z}} \left| \frac{\eta}{h}\sqrt{k}\, Q^\Sigma_k(i) - p\left(\frac{i}{\eta\sqrt{k}}\right)\right| = o\left(k^{-1/2}\right).$$

[2]Recall that $\gamma := \left(\frac{8}{9}\right)^{\frac{1}{4}}$.

[3]We call **maximal span** of an integer-valued random variable X the greatest $h \in \mathbb{N}$ for which there exists an integer a such that a.s. $X \in a + h\mathbb{Z}$.

2. *For all $2 \leq r \leq r_0$, there exists a constant C such that for all $i \in \mathbb{Z}$ and $k \geq 1$,*

$$\left|\frac{\eta}{h}\sqrt{k}\, Q_k^{\Sigma}(i)\right| \leq \frac{C}{1+\left|\frac{i}{\eta\sqrt{k}}\right|^r}.$$

Proof. The first part of this theorem is merely [Pet75, Theorem VII.1.6] applied to the variables $\frac{1}{h}(X_k - a)$, which have 1 as maximal span. The second part is an easy consequence of [Pet75, Theorem VII.3.16]. □

In what follows, we will always use the notation S for simple random walks, M for simple Motzkin walks, and Σ for any other random walks. We will use this theorem with S and M: we find $(\eta, h) = (1, 2)$ for S and $(\eta, h) = (\sqrt{2/3}, 1)$ for M. In both cases, we may take r as large as we want.

2.3.2 Result

Recall that $\mathfrak{S}^*$ is the set of all dominant schemes of genus g, that is schemes with only vertices of degree 3. We call p_a the density of a centered Gaussian variable with variance a, as well as p'_a its derivative:

$$p_a(x) := \frac{1}{\sqrt{a}}\, p\left(\frac{x}{\sqrt{a}}\right) \qquad \text{and} \qquad p'_a(x) = -\frac{x}{a^{3/2}}\, p\left(\frac{x}{\sqrt{a}}\right).$$

For any $\mathfrak{s} \in \mathfrak{S}$, we identify an element $(m, \sigma, l, u) \in \mathbb{R}_+^{\vec{E}(\mathfrak{s})\setminus\{\mathfrak{e}_*\}} \times (\mathbb{R}_+^*)^{\check{E}(\mathfrak{s})} \times \mathbb{R}^{V(\mathfrak{s})\setminus\{\mathfrak{e}_*^-\}} \times \mathbb{R}_+$ with an element of $\mathbb{R}_+^{\vec{E}(\mathfrak{s})} \times (\mathbb{R}_+^*)^{\vec{E}(\mathfrak{s})} \times \mathbb{R}^{V(\mathfrak{s})} \times \mathbb{R}_+$ by setting:

- ✧ $m^{\mathfrak{e}_*} := 1 - \sum_{\mathfrak{e}\in\vec{E}(\mathfrak{s})\setminus\{\mathfrak{e}_*\}} m^{\mathfrak{e}}$, (2.13a)
- ✧ for every $\mathfrak{e} \in \check{E}(\mathfrak{s})$, $\sigma^{\bar{\mathfrak{e}}} := \sigma^{\mathfrak{e}}$, (2.13b)
- ✧ $l^{\mathfrak{e}_*^-} := 0.$ (2.13c)

We write

$$\Delta_{\mathfrak{s}} := \left\{(x_{\mathfrak{e}})_{\mathfrak{e}\in\vec{E}(\mathfrak{s})} \in [0,1]^{\vec{E}(\mathfrak{s})},\ \textstyle\sum_{\mathfrak{e}\in\vec{E}(\mathfrak{s})} x_{\mathfrak{e}} = 1\right\}$$

the simplex of dimension $|\vec{E}(\mathfrak{s})| - 1$. Note that the vector m lies in the simplex $\Delta_{\mathfrak{s}}$ as long as $m^{\mathfrak{e}_*} \geq 0$. We define the probability μ by, for all nonnegative measurable function φ on $\bigcup_{\mathfrak{s}\in\mathfrak{S}} \{\mathfrak{s}\}\times\Delta_{\mathfrak{s}}\times(\mathbb{R}_+^*)^{\vec{E}(\mathfrak{s})}\times \mathbb{R}^{V(\mathfrak{s})} \times [0,1]$,

$$\mu(\varphi) = \frac{1}{\Upsilon} \sum_{\mathfrak{s}\in\mathfrak{S}^*} \int_{\mathcal{S}^{\mathfrak{s}}} d\mathcal{L}^{\mathfrak{s}}\ \mathbb{1}_{\{m^{\mathfrak{e}_*}\geq 0,\, u<m^{\mathfrak{e}_*}\}}\ \varphi(\mathfrak{s}, m, \sigma, l, u) \prod_{\mathfrak{e}\in\vec{E}(\mathfrak{s})} -p'_{m^{\mathfrak{e}}}(\sigma^{\mathfrak{e}}) \prod_{\mathfrak{e}\in\check{E}(\mathfrak{s})} p_{\sigma^{\mathfrak{e}}}(l^{\mathfrak{e}}),$$

where $d\mathcal{L}^{\mathfrak{s}} = d(m^{\mathfrak{e}})\, d(\sigma^{\mathfrak{e}})\, d(l^v)\, du$ is the Lebesgue measure on the set

$$\mathcal{S}^{\mathfrak{s}} := [0,1]^{\vec{E}(\mathfrak{s})\setminus\{\mathfrak{e}_*\}} \times (\mathbb{R}_+^*)^{\check{E}(\mathfrak{s})} \times \mathbb{R}^{V(\mathfrak{s})\setminus\{\mathfrak{e}_*^-\}} \times [0,1]$$

and

$$\Upsilon = \sum_{\mathfrak{s}\in\mathfrak{S}^*} \int_{\mathcal{S}^{\mathfrak{s}}} d\mathcal{L}^{\mathfrak{s}}\ \mathbb{1}_{\{m^{\mathfrak{e}_*}\geq 0,\, u<m^{\mathfrak{e}_*}\}} \prod_{\mathfrak{e}\in\vec{E}(\mathfrak{s})} -p'_{m^{\mathfrak{e}}}(\sigma^{\mathfrak{e}}) \prod_{\mathfrak{e}\in\check{E}(\mathfrak{s})} p_{\sigma^{\mathfrak{e}}}(l^{\mathfrak{e}}) \tag{2.14}$$

is a normalization constant. We may now state the main result of this section.

Proposition 2.5. *The law of the random variable*

$$\left(\mathfrak{s}_n, \left(m_{(n)}^{\mathfrak{e}}\right)_{\mathfrak{e}\in\vec{E}(\mathfrak{s}_n)}, \left(\sigma_{(n)}^{\mathfrak{e}}\right)_{\mathfrak{e}\in\vec{E}(\mathfrak{s}_n)}, \left(l_{(n)}^{v}\right)_{v\in V(\mathfrak{s}_n)}, u_{(n)}\right)$$

converges weakly toward the probability μ.

Proof. Let φ be a bounded continuous function on the set

$$\bigcup_{\mathfrak{s}\in\mathfrak{S}} \{\mathfrak{s}\} \times \Delta_{\mathfrak{s}} \times (\mathbb{R}_+^*)^{\vec{E}(\mathfrak{s})} \times \mathbb{R}^{V(\mathfrak{s})} \times [0,1].$$

We need to look at the convergence of

$$\mathbb{E}_n := \mathbb{E}\left[\varphi\left(\mathfrak{s}_n, \left(m^{\mathfrak{e}}_{(n)}\right)_{\mathfrak{e}\in\vec{E}(\mathfrak{s}_n)}, \left(\sigma^{\mathfrak{e}}_{(n)}\right)_{\mathfrak{e}\in\vec{E}(\mathfrak{s}_n)}, \left(l^{v}_{(n)}\right)_{v\in V(\mathfrak{s}_n)}, u_{(n)}\right)\right].$$

Step 1. Let $n \in \mathbb{N}$. For the time being, we identify $(m,\sigma,l,u) \in \mathbb{Z}_+^{\vec{E}(\mathfrak{s})\setminus\{\mathfrak{e}_*\}} \times \mathbb{N}^{\check{E}(\mathfrak{s})} \times \mathbb{Z}^{V(\mathfrak{s})\setminus\{\mathfrak{e}_*^-\}} \times \mathbb{Z}_+$ with an element of $\mathbb{Z}_+^{\vec{E}(\mathfrak{s})} \times \mathbb{N}^{\check{E}(\mathfrak{s})} \times \mathbb{Z}^{V(\mathfrak{s})} \times \mathbb{Z}_+$ by (2.13b), (2.13c), and

$$\diamond\ m^{\mathfrak{e}_*}(n) := n - \textstyle\sum_{\mathfrak{e}\in\vec{E}(\mathfrak{s})\setminus\{\mathfrak{e}_*\}} m^{\mathfrak{e}} - \sum_{\mathfrak{e}\in\check{E}(\mathfrak{s})} \sigma^{\mathfrak{e}}, \tag{2.13a'}$$

instead of (2.13a), which may be seen as its discrete counterpart. This is an element of $\mathfrak{C}_n(\mathfrak{s})$ provided that $m^{\mathfrak{e}_*}(n) \geq 0$ and $0 \leq u < 2m^{\mathfrak{e}_*}(n)+\sigma^{\mathfrak{e}_*}$. Beware that here the definition of $m^{\mathfrak{e}_*}(n)$ actually depends on n. It also depends on σ but we chose not to let it figure in the notation for space reasons.

For any vector $x = (x_1, x_2, \ldots, x_k) \in \mathbb{R}^k$, we note $\lfloor x \rfloor$ the vector $(\lfloor x_1 \rfloor, \lfloor x_2 \rfloor, \ldots, \lfloor x_k \rfloor) \in \mathbb{Z}^k$. Note that for $m \in \mathbb{R}_+^{\vec{E}(\mathfrak{s})\setminus\{\mathfrak{e}_*\}}$, $\lfloor m \rfloor^{\mathfrak{e}_*}(n)$ is well defined through (2.13a'). Until further notice, we will write $\lfloor m \rfloor^{\mathfrak{e}_*}$ for $\lfloor m \rfloor^{\mathfrak{e}_*}(n)$, which n being implicit. So when we write $\lfloor m \rfloor$, we mean the vector such that $\lfloor m \rfloor^{\mathfrak{e}} = \lfloor m^{\mathfrak{e}} \rfloor$ for $\mathfrak{e} \neq \mathfrak{e}_*$ and $\lfloor m \rfloor^{\mathfrak{e}_*} = \lfloor m \rfloor^{\mathfrak{e}_*}(n)$. Using (2.12), we find

$$\mathbb{E}_n = \frac{12^n}{|\mathcal{T}_n|} \sum_{\mathfrak{s}\in\mathfrak{S}} \sum_{(m,\sigma,l,u)\in\mathfrak{C}_n(\mathfrak{s})} \varphi\left(\mathfrak{s}, \frac{2m+\sigma}{2n}, \frac{\sigma}{\sqrt{2n}}, \frac{l}{\gamma\, n^{1/4}}, \frac{u}{2n}\right) \prod_{\mathfrak{e}\in\vec{E}(\mathfrak{s})} h_1(m^{\mathfrak{e}}, \sigma^{\mathfrak{e}}) \prod_{\mathfrak{e}\in\check{E}(\mathfrak{s})} h_2(\sigma^{\mathfrak{e}}, l^{\mathfrak{e}}),$$

where

$$h_1(m^{\mathfrak{e}}, \sigma^{\mathfrak{e}}) := \frac{\sigma^{\mathfrak{e}}}{2m^{\mathfrak{e}}+\sigma^{\mathfrak{e}}}\, Q^S_{2m^{\mathfrak{e}}+\sigma^{\mathfrak{e}}}(\sigma^{\mathfrak{e}})\, \mathbb{1}_{\{\sigma^{\mathfrak{e}}\geq 1\}} \qquad \text{and} \qquad h_2(\sigma^{\mathfrak{e}}, l^{\mathfrak{e}}) := Q^M_{\sigma^{\mathfrak{e}}}(l^{\mathfrak{e}})\, \mathbb{1}_{\{\sigma^{\mathfrak{e}}\geq 1\}}.$$

Writing the sum over $\mathfrak{C}_n(\mathfrak{s})$ in the form of an integral, we obtain

$$\mathbb{E}_n = \frac{12^n}{|\mathcal{T}_n|} \sum_{\mathfrak{s}\in\mathfrak{S}} \int_{\tilde{\mathcal{S}}^{\mathfrak{s}}} d\tilde{\mathcal{L}}^{\mathfrak{s}}\, \mathbb{1}_{\mathcal{E}_n^{\mathfrak{s}}}(m,\sigma,u)\, \varphi_{\lfloor\cdot\rfloor} \prod_{\mathfrak{e}\in\vec{E}(\mathfrak{s})} h_1(\lfloor m \rfloor^{\mathfrak{e}}, \lfloor \sigma \rfloor^{\mathfrak{e}}) \prod_{\mathfrak{e}\in\check{E}(\mathfrak{s})} h_2\Big(\lfloor \sigma \rfloor^{\mathfrak{e}}, \lfloor l \rfloor^{\mathfrak{e}^+} - \lfloor l \rfloor^{\mathfrak{e}^-}\Big),$$

where $\varphi_{\lfloor\cdot\rfloor}$ stands for

$$\varphi\Big(\mathfrak{s}, \frac{2\lfloor m \rfloor + \lfloor \sigma \rfloor}{2n}, \frac{\lfloor \sigma \rfloor}{\sqrt{2n}}, \frac{\lfloor l \rfloor}{\gamma\, n^{1/4}}, \frac{\lfloor u \rfloor}{2n}\Big),$$

$d\tilde{\mathcal{L}}^{\mathfrak{s}}$ is the Lebesgue measure on the set $\tilde{\mathcal{S}}^{\mathfrak{s}} := \mathbb{R}_+^{\vec{E}(\mathfrak{s})\setminus\{\mathfrak{e}_*\}} \times (\mathbb{R}_+^*)^{\check{E}(\mathfrak{s})} \times \mathbb{R}^{V(\mathfrak{s})\setminus\{\mathfrak{e}_*^-\}} \times \mathbb{R}_+$ and

$$\mathcal{E}_n^{\mathfrak{s}} := \Big\{(m,\sigma,u) \in \mathbb{R}_+^{\vec{E}(\mathfrak{s})\setminus\{\mathfrak{e}_*\}} \times (\mathbb{R}_+^*)^{\check{E}(\mathfrak{s})} \times \mathbb{R}_+ \,:\, \lfloor m \rfloor^{\mathfrak{e}_*}(n) \geq 0,\ u < 2\lfloor m \rfloor^{\mathfrak{e}_*}(n) + \lfloor \sigma \rfloor^{\mathfrak{e}_*}\Big\}.$$

Finally, the changes of variables $m \mapsto nm$, $\sigma \mapsto \sqrt{2n}\,\sigma$, $l \mapsto \gamma\, n^{1/4}\, l$, and $u \mapsto 2nu$ yields

$$\mathbb{E}_n = \frac{12^n}{|\mathcal{T}_n|} \sum_{\mathfrak{s}\in\mathfrak{S}} n^{\frac{|E(\mathfrak{s})|-g}{2}}\, 2^{\frac{|E(\mathfrak{s})|-3g+2}{2}}\, 3^g \int_{\mathcal{S}^{\mathfrak{s}}} d\mathcal{L}^{\mathfrak{s}}\, A_n^{\mathfrak{s}} \prod_{\mathfrak{e}\in\vec{E}(\mathfrak{s})} B_n^{\mathfrak{s},\mathfrak{e}} \prod_{\mathfrak{e}\in\check{E}(\mathfrak{s})} C_n^{\mathfrak{s},\mathfrak{e}} \tag{2.15}$$

where

$$\begin{aligned}
A_n^{\mathfrak{s}} &= \mathbb{1}_{\mathcal{E}_n^{\mathfrak{s}}}(nm, \sqrt{2n}\,\sigma, 2n\,u)\, \varphi\Big(\mathfrak{s}, \frac{2\lfloor nm \rfloor + \lfloor \sqrt{2n}\,\sigma \rfloor}{2n}, \frac{\lfloor \sqrt{2n}\,\sigma \rfloor}{\sqrt{2n}}, \frac{\lfloor \gamma\, n^{1/4} l \rfloor}{\gamma\, n^{1/4}}, \frac{\lfloor 2nu \rfloor}{2n}\Big),\\
B_n^{\mathfrak{s},\mathfrak{e}} &= n\, h_1\left(\lfloor nm \rfloor^{\mathfrak{e}}, \lfloor \sqrt{2n}\,\sigma \rfloor^{\mathfrak{e}}\right),\\
C_n^{\mathfrak{s},\mathfrak{e}} &= \gamma\, n^{1/4}\, h_2\Big(\lfloor \sqrt{2n}\,\sigma \rfloor^{\mathfrak{e}}, \lfloor \gamma\, n^{1/4} l \rfloor^{\mathfrak{e}^+} - \lfloor \gamma\, n^{1/4} l \rfloor^{\mathfrak{e}^-}\Big).
\end{aligned}$$

Step 2. We are now going to see that every integral term of the sum appearing in the equation (2.15) converges, by dominated convergence. We no longer use (2.13a) but (2.13a') in the identification (2.13). Because

$$\frac{2\lfloor nm\rfloor^{\mathfrak{e}_*}(n)+\lfloor\sqrt{2n}\,\sigma^{\mathfrak{e}}\rfloor}{2n}=1-\sum_{\mathfrak{e}\in\vec{E}(\mathfrak{s})\setminus\{\mathfrak{e}_*\}}\frac{2\lfloor nm^{\mathfrak{e}}\rfloor+\lfloor\sqrt{2n}\,\sigma^{\mathfrak{e}}\rfloor}{2n}\xrightarrow[n\to\infty]{}1-\sum_{\mathfrak{e}\in\vec{E}(\mathfrak{s})\setminus\{\mathfrak{e}_*\}}m^{\mathfrak{e}}=m^{\mathfrak{e}_*},$$

we see that $A_n^{\mathfrak{s}}\to\mathbb{1}_{\{m^{\mathfrak{e}_*}\geq 0,\,u<m^{\mathfrak{e}_*}\}}\,\varphi(\mathfrak{s},m,\sigma,l,u)$. Thanks to Proposition 2.4, we then obtain

$$B_n^{\mathfrak{s},\mathfrak{e}}\to -p'_{m^{\mathfrak{e}}}(\sigma^{\mathfrak{e}})\qquad\text{and}\qquad C_n^{\mathfrak{s},\mathfrak{e}}\to p_{\sigma^{\mathfrak{e}}}(l^{\mathfrak{e}}).$$

It remains to prove that the convergence is dominated. To that end, we use the second part of Proposition 2.4. In the remainder of the proof, C will denote a constant in $(0,\infty)$, the value of which may differ from line to line. First, notice that

$$A_n^{\mathfrak{s}}\leq\|\varphi\|_\infty.$$

Then, applying Proposition 2.4 with $r=3$, we obtain, for $n\geq 2$,

$$\begin{aligned}
B_n^{\mathfrak{s},\mathfrak{e}}&=\frac{2n\,\lfloor\sqrt{2n}\,\sigma\rfloor^{\mathfrak{e}}}{\left(2\lfloor nm\rfloor^{\mathfrak{e}}+\lfloor\sqrt{2n}\,\sigma\rfloor^{\mathfrak{e}}\right)^{\frac{3}{2}}}\,\frac{1}{2}\left(2\lfloor nm\rfloor^{\mathfrak{e}}+\lfloor\sqrt{2n}\,\sigma\rfloor^{\mathfrak{e}}\right)^{\frac{1}{2}}Q^{S}_{2\lfloor nm\rfloor^{\mathfrak{e}}+\lfloor\sqrt{2n}\,\sigma\rfloor^{\mathfrak{e}}}\left(\lfloor\sqrt{2n}\,\sigma\rfloor^{\mathfrak{e}}\right)\mathbb{1}_{\{\sqrt{2n}\,\sigma^{\mathfrak{e}}\geq 1\}}\\
&\leq C\left(\frac{\lfloor\sqrt{2n}\sigma\rfloor^{\mathfrak{e}}}{\sqrt{2n}}\right)^{-2}\frac{\left(\frac{(\lfloor\sqrt{2n}\sigma\rfloor^{\mathfrak{e}})^2}{2\lfloor nm\rfloor^{\mathfrak{e}}+\lfloor\sqrt{2n}\sigma\rfloor^{\mathfrak{e}}}\right)^{3/2}}{1+\left(\frac{(\lfloor\sqrt{2n}\sigma\rfloor^{\mathfrak{e}})^2}{2\lfloor nm\rfloor^{\mathfrak{e}}+\lfloor\sqrt{2n}\sigma\rfloor^{\mathfrak{e}}}\right)^{3/2}}\mathbb{1}_{\{\sqrt{2n}\,\sigma^{\mathfrak{e}}\geq 1\}}\\
&\leq C\,(m^{\mathfrak{e}})^{-1}\wedge(\sigma^{\mathfrak{e}})^{-2},
\end{aligned}$$

where we used the fact that for $x\geq 1$, $\lfloor x\rfloor^{-1}\leq 2/x$. The case $\lfloor nm\rfloor=0$ is to be treated separately, and is left to the reader. Applying now Proposition 2.4 with $r=2$, we find that, for $n\geq 2$,

$$\begin{aligned}
C_n^{\mathfrak{s},\mathfrak{e}}&=\frac{(2n)^{\frac{1}{4}}}{\left(\lfloor\sqrt{2n}\sigma\rfloor^{\mathfrak{e}}\right)^{\frac{1}{2}}}\sqrt{\frac{2}{3}}\left(\lfloor\sqrt{2n}\sigma\rfloor^{\mathfrak{e}}\right)^{\frac{1}{2}}Q^{M}_{\lfloor\sqrt{2n}\sigma\rfloor^{\mathfrak{e}}}\left(\lfloor\gamma\,n^{1/4}l\rfloor^{\mathfrak{e}^+}-\lfloor\gamma\,n^{1/4}l\rfloor^{\mathfrak{e}^-}\right)\mathbb{1}_{\{\sqrt{2n}\,\sigma^{\mathfrak{e}}\geq 1\}}\\
&\leq\frac{C}{\sqrt{\sigma^{\mathfrak{e}}}}\left(1+\frac{3}{2}\frac{\left(\lfloor\gamma\,n^{1/4}l\rfloor^{\mathfrak{e}^+}-\lfloor\gamma\,n^{1/4}l\rfloor^{\mathfrak{e}^-}\right)^2}{\lfloor\sqrt{2n}\sigma\rfloor^{\mathfrak{e}}}\right)^{-1}\\
&\leq\frac{C}{\sqrt{\sigma^{\mathfrak{e}}}}\left(1+\frac{\left(\left|l^{\mathfrak{e}^+}-l^{\mathfrak{e}^-}\right|-1\right)^2}{\sigma^{\mathfrak{e}}}\mathbb{1}_{\{|l^{\mathfrak{e}^+}-l^{\mathfrak{e}^-}|>1\}}\right)^{-1}.
\end{aligned}$$

Any integrand in the equation (2.15) is then bounded by

$$C\prod_{\mathfrak{e}\in\vec{E}(\mathfrak{s})}(m^{\mathfrak{e}})^{-1}\wedge(\sigma^{\mathfrak{e}})^{-2}\prod_{\mathfrak{e}\in\check{E}(\mathfrak{s})}(\sigma^{\mathfrak{e}})^{-1/2}\left(1+\frac{\left(\left|l^{\mathfrak{e}^+}-l^{\mathfrak{e}^-}\right|-1\right)^2}{\sigma^{\mathfrak{e}}}\mathbb{1}_{\{|l^{\mathfrak{e}^+}-l^{\mathfrak{e}^-}|>1\}}\right)^{-1}.\tag{2.16}$$

We have to see that this expression is integrable. First, note that we integrate with respect to u on a compact set. Moreover,

$$\begin{aligned}
\int_{\mathbb{R}}dl^{\mathfrak{e}^-}\left(1+\frac{\left(\left|l^{\mathfrak{e}^+}-l^{\mathfrak{e}^-}\right|-1\right)^2}{\sigma^{\mathfrak{e}}}\mathbb{1}_{\{|l^{\mathfrak{e}^+}-l^{\mathfrak{e}^-}|>1\}}\right)^{-1}&=2+\pi\sqrt{\sigma^{\mathfrak{e}}}\\
&\leq C\,1\vee\sqrt{\sigma^{\mathfrak{e}}},
\end{aligned}$$

and we have the same bound if we integrate with respect to $l^{\mathfrak{e}^+}$ instead of $l^{\mathfrak{e}^-}$.

It is possible to injectively associate with every vertex $v \in V(\mathfrak{s})\backslash\{\mathfrak{e}_*^-\}$ a half-edge $\mathfrak{e}_v \in \check{E}(\mathfrak{s})$ such that v is an extremity of $\mathfrak{e}_v$. Let us call E_V the range of such an injection. The integral of the expression (2.16) with respect to u and l is then bounded by

$$C \prod_{\mathfrak{e}\in\vec{E}(\mathfrak{s})} (m^{\mathfrak{e}})^{-1} \wedge (\sigma^{\mathfrak{e}})^{-2} \prod_{\mathfrak{e}\in E_V} 1 \vee (\sigma^{\mathfrak{e}})^{-1/2} \prod_{\mathfrak{e}\in\check{E}(\mathfrak{s})\backslash E_V} (\sigma^{\mathfrak{e}})^{-1/2}.$$

Finally, it is easy to see that this expression, once integrated[4] with respect to σ, is bounded by

$$C \prod_{\mathfrak{e}\in\vec{E}(\mathfrak{s})} (m^{\mathfrak{e}})^{-7/8},$$

which is integrable with respect to m.

Step 3. We just saw that the integral expression in (2.15) converges toward

$$\int_{\mathcal{S}^{\mathfrak{s}}} d\mathcal{L}^{\mathfrak{s}}\, \mathbb{1}_{\{m^{\mathfrak{e}_*}\geq 0,\, u<m^{\mathfrak{e}_*}\}}\, \varphi(\mathfrak{s}, m, \sigma, l, u) \prod_{\mathfrak{e}\in\vec{E}(\mathfrak{s})} -p'_{m^{\mathfrak{e}}}(\sigma^{\mathfrak{e}}) \prod_{\mathfrak{e}\in\check{E}(\mathfrak{s})} p_{\sigma^{\mathfrak{e}}}(l^{\mathfrak{e}}).$$

The dominant terms in the equation (2.15) are the ones for which $|E(\mathfrak{s})|$ is the largest. The corresponding schemes are exactly the dominant ones: for a scheme, $2\,|E(\mathfrak{s})| = \sum_{v\in V(\mathfrak{s})} \deg(v) \geq 3\,|V(\mathfrak{s})|$ and the Euler characteristic formula gives $|E(\mathfrak{s})| \leq 6g-3$, the equality being reached when $2\,|E(\mathfrak{s})| = 3\,|V(\mathfrak{s})|$, that is when $\mathfrak{s}$ is dominant. Note that this situation is exactly the same as the one encountered in [Cha10, CMS09, Mie09b].

Hence, if φ is momentarily chosen to be constantly equal to 1, we obtain that

$$|\mathcal{T}_n| \sim 12^n\, n^{\frac{5g-3}{2}}\, 2^{\frac{3g-1}{2}}\, 3^g\, \Upsilon \tag{2.17}$$

where Υ is defined by (2.14). Finally,

$$\mathbb{E}_n \xrightarrow[n\to\infty]{} \frac{1}{\Upsilon} \sum_{\mathfrak{s}\in\mathfrak{S}^*} \int_{\mathcal{S}^{\mathfrak{s}}} d\mathcal{L}^{\mathfrak{s}}\, \mathbb{1}_{\{m^{\mathfrak{e}_*}\geq 0,\, u<m^{\mathfrak{e}_*}\}}\, \varphi(\mathfrak{s}, m, \sigma, l, u) \prod_{\mathfrak{e}\in\vec{E}(\mathfrak{s})} -p'_{m^{\mathfrak{e}}}(\sigma^{\mathfrak{e}}) \prod_{\mathfrak{e}\in\check{E}(\mathfrak{s})} p_{\sigma^{\mathfrak{e}}}(l^{\mathfrak{e}}),$$

which is the result we sought. □

2.4 Convergence of the Motzkin bridges and the forests

Conditionally given the vector

$$\left(\mathfrak{s}_n, (m_n^{\mathfrak{e}})_{\mathfrak{e}\in\vec{E}(\mathfrak{s}_n)}, (\sigma_n^{\mathfrak{e}})_{\mathfrak{e}\in\vec{E}(\mathfrak{s}_n)}, (l_n^v)_{v\in V(\mathfrak{s}_n)}\right),$$

the Motzkin bridges $\mathfrak{M}_n^{\mathfrak{e}}$, $\mathfrak{e} \in \check{E}(\mathfrak{s}_n)$ and the well-labeled forests $(\mathfrak{f}_n^{\mathfrak{e}}, \mathfrak{l}_n^{\mathfrak{e}})$, $\mathfrak{e} \in \vec{E}(\mathfrak{s}_n)$ are independent and

- for every $\mathfrak{e} \in \check{E}(\mathfrak{s}_n)$, $\mathfrak{M}_n^{\mathfrak{e}}$ is uniformly distributed over the set $\mathcal{M}_{[0,\sigma_n^{\mathfrak{e}}]}^{0\to l_n^{\mathfrak{e}}}$ of Motzkin bridges on $[0, \sigma_n^{\mathfrak{e}}]$ from 0 to $l_n^{\mathfrak{e}} = l_n^{\mathfrak{e}^+} - l_n^{\mathfrak{e}^-}$,
- for every $\mathfrak{e} \in \vec{E}(\mathfrak{s}_n)$, $(\mathfrak{f}_n^{\mathfrak{e}}, \mathfrak{l}_n^{\mathfrak{e}})$ is uniformly distributed over the set $\mathfrak{F}_{\sigma_n^{\mathfrak{e}}}^{m_n^{\mathfrak{e}}}$ of well-labeled forests with $\sigma_n^{\mathfrak{e}}$ trees and $m_n^{\mathfrak{e}}$ tree edges.

[4] Be careful that, when integrating with respect to $\sigma^{\mathfrak{e}}$ for some $\mathfrak{e} \in \check{E}(\mathfrak{s})$, both half-edges $\mathfrak{e}$ and $\bar{\mathfrak{e}}$ are to be considered.

The convergence of Motzkin bridges is already known. We will properly state the result we need in Lemma 2.8.

The convergence of a uniform well-labeled tree with n edges is well-known, see [CS04], for example. We will need a conditioned version of this result: roughly speaking, instead of looking at one large tree with n edges uniformly labeled such that the root label is 0, we look at a forest with n edges, a number of trees growing like $\sqrt{n}$, that are uniformly labeled provided the root label of every tree is 0. For that matter, we will adapt the arguments provided in [LG94, Chapter 6].

Let us define the space $\mathcal{K}$ of continuous real-valued functions on $\mathbb{R}_+$ killed at some time:

$$\mathcal{K} := \bigcup_{x\in\mathbb{R}_+} \mathcal{C}([0,x],\mathbb{R}).$$

For an element $f \in \mathcal{K}$, we will define its lifetime $\sigma(f)$ as the only x such that $f \in \mathcal{C}([0,x],\mathbb{R})$. We endow this space with the following metric:

$$d_{\mathcal{K}}(f,g) := |\sigma(f)-\sigma(g)| + \sup_{y\geq 0}\left| f\big(y\wedge\sigma(f)\big) - g\big(y\wedge\sigma(g)\big)\right|.$$

Recall that we use the notation $\underline{X}(s)$ for the infimum up to time s of any process $X \in \mathcal{K}$. Throughout this section, m and σ will denote positive real numbers and l will be any real number.

2.4.1 Brownian bridge and first-passage Brownian bridge

The results we show in this section are part of the probabilistic folklore. Because of the scarceness of the references, we will give complete proofs for the sake of self-containment.

We define here the Brownian bridge $B^{0\to l}_{[0,m]}$ on $[0,m]$ from 0 to l and the first-passage Brownian bridge $F^{0\to -\sigma}_{[0,m]}$ on $[0,m]$ from 0 to $-\sigma$. Informally, $B^{0\to l}_{[0,m]}$ and $F^{0\to -\sigma}_{[0,m]}$ are a standard Brownian motion β on $[0,m]$ conditioned respectively on the events $\{\beta_m = l\}$ and $\{\inf\{s\geq 0 \,:\, \beta_s = -\sigma\} = m\}$. Because both theses events occur with probability 0, we need to define these objects properly. There are several equivalent ways to do so (see for example [BCP03, Bil68, RY99]).

Remember that we call p_a the Gaussian density with variance a and mean 0, as well as p'_a its derivative. Let $(\beta_t)_{0\leq t\leq m}$ be a standard Brownian motion. As explained in [FPY93, Proposition 1], the law of the Brownian bridge is characterized by $B^{0\to l}_{[0,m]}(m) = l$ and the formula

$$\mathbb{E}\left[f\left(\left(B^{0\to l}_{[0,m]}(t)\right)_{0\leq t\leq m'}\right)\right] = \mathbb{E}\left[f\left((\beta_t)_{0\leq t\leq m'}\right)\frac{p_{m-m'}(l-\beta_{m'})}{p_m(l)}\right] \tag{2.18}$$

for all bounded measurable functions f on $\mathcal{K}$, for all $0\leq m' < m$.

We define the law of the first-passage Brownian bridge in a similar way, by letting

$$\mathbb{E}\left[f\left(\left(F^{0\to -\sigma}_{[0,m]}(t)\right)_{0\leq t\leq m'}\right)\right] = \mathbb{E}\left[f\left((\beta_t)_{0\leq t\leq m'}\right)\frac{p'_{m-m'}(-\sigma-\beta_{m'})}{p'_m(-\sigma)}\mathbb{1}_{\{\underline{\beta}_{m'}>-\sigma\}}\right] \tag{2.19}$$

for all bounded measurable functions f on $\mathcal{K}$, for all $0\leq m' < m$, and

$$F^{0\to -\sigma}_{[0,m]}(m) = -\sigma.$$

These formulas set the finite-dimensional laws of the first-passage Brownian bridge. It remains to see that it admits a continuous version. Because its law is absolutely continuous with respect to the Wiener measure on every $[0,m']$, $m' < m$, the only problem arises at time m. We will, however, use Kolmogorov's lemma [RY99, Theorem I.1.8] to obtain the continuity of the whole trajectory. We will see during the proof of Lemma 2.12 that, as for the Brownian motion, the trajectories of the first-passage bridge are α-Hölder for every $\alpha < 1/2$.

The motivation of these definitions may be found in the following lemma:

Lemma 2.6. *Let $(\beta_t)_{0\le t\le m}$ be a standard Brownian motion. Let $(B^\varepsilon_t)_{0\le t\le m}$ and $(F^\varepsilon_t)_{0\le t\le m}$ have the law of β conditioned respectively on the events*

$$\{|\beta_m - l| < \varepsilon\} \qquad \textit{and} \qquad \{\beta_m < -\sigma+\varepsilon,\, \underline{\beta}_m > -\sigma-\varepsilon\}\,.$$

Then, as ε goes to 0,

$$B^\varepsilon \to B^{0\to l}_{[0,m]} \qquad \textit{and} \qquad F^\varepsilon \to F^{0\to -\sigma}_{[0,m]}$$

in law in the space $(\mathcal{C}([0,m],\mathbb{R}), \|\cdot\|_\infty)$.

The proof of this lemma uses similar methods as those we will use for Lemma 2.8 so we let the details to the reader. In what follows, we will use the following lemma, which is a consequence of the Rosenthal Inequality [Pet95, Theorem 2.9 and 2.10]:

Lemma 2.7. *Let X_1, X_2,... X_k be independent centered random variables and $q \ge 2$. Then, there exists a constant $c(q)$ depending only on q such that*

$$\mathbb{E}\left[\left|\sum_{i=1}^k X_i\right|^q\right] \le c(q)\, k^{\frac{q}{2}-1} \sum_{i=1}^k \mathbb{E}\left[|X_i|^q\right].$$

In particular, if X_1, X_2,... X_k are i.i.d.,

$$\mathbb{E}\left[\left|\sum_{i=1}^k X_i\right|^q\right] \le c(q)\, k^{\frac{q}{2}}\, \mathbb{E}\left[|X_1|^q\right].$$

Discrete bridges

We will see in this paragraph two lemmas showing that these two objects are the limits of their discrete analogs. These lemmas, in themselves, motivate our definitions of bridges and first-passage bridges. Let us begin with bridges.

We consider a sequence $(X_k)_{k\ge 0}$ of i.i.d. centered integer-valued random variables with a moment of order q_0 for some $q_0 \ge 3$. We write $\eta^2 := \mathrm{Var}(X_1)$ its variance and h its maximal span. We define $\Sigma_i := \sum_{k=0}^i X_k$ and still write Σ its linearly interpolated version. Let $(m_n) \in \mathbb{Z}_+^{\mathbb{N}}$ and $(l_n) \in \mathbb{Z}^{\mathbb{N}}$ be two sequences of integers such that

$$m_{(n)} := \frac{m_n}{n} \xrightarrow[n\to\infty]{} m \qquad \text{and} \qquad l_{(n)} := \frac{l_n}{\eta\sqrt{n}} \xrightarrow[n\to\infty]{} l.$$

Let $(B_n(i))_{0\le i\le m_n}$ be the process whose law is the law of $(\Sigma_i)_{0\le i\le m_n}$ conditioned on the event

$$\{\Sigma_{m_n} = l_n\},$$

which we suppose occurs with positive probability. We let

$$B_{(n)} := \left(\frac{B_n(ns)}{\eta\sqrt{n}}\right)_{0\le s\le m_{(n)}}$$

be its rescaled version.

Lemma 2.8. *As n goes to infinity, the process $B_{(n)}$ converges in distribution toward the process $B^{0\to l}_{[0,m]}$, in the space $(\mathcal{K},\, d_{\mathcal{K}})$.*

Proof. We note $\mathcal{F}_i := \sigma(\Sigma_k, 0\le k\le i)$ the natural filtration associated with Σ. Applying Skorokhod's representation theorem (see e.g. [EK86, Theorem 3.1.8]), we may and will assume that

$$\left(\frac{\Sigma_{ns}}{\eta\sqrt{n}}\right)_{0\le s\le m}$$

converges a.s. toward a standard Brownian motion $(\beta_s)_{0\le s\le m}$ for the uniform topology.

Step 1. Let $m' < m$. We begin by looking at $B_{(n)}$ on $[0, m']$. For n large enough, $\lceil nm' \rceil < m_n$. Let f be continuous bounded from $\mathcal{K}$ to $\mathbb{R}$. We have

$$\begin{aligned}\mathbb{E}\left[f\left((B_{(n)}(s))_{0\leq s\leq m'}\right)\right] &= \mathbb{E}\left[f\left(\left(\frac{\Sigma_{ns}}{\eta\sqrt{n}}\right)_{0\leq s\leq m'}\right) \,\Big|\, \Sigma_{m_n} = l_n\right] \\ &= \mathbb{E}\left[f\left(\left(\frac{\Sigma_{ns}}{\eta\sqrt{n}}\right)_{0\leq s\leq m'}\right) \frac{\mathbb{P}\left(\Sigma_{m_n} = l_n \mid \mathcal{F}_{\lceil nm' \rceil}\right)}{\mathbb{P}\left(\Sigma_{m_n} = l_n\right)}\right]. \qquad (2.20)\end{aligned}$$

Recall the notation $Q_k^{\Sigma}(i) = \mathbb{P}(\Sigma_k = i)$. Using the Markov property, we obtain

$$\begin{aligned}\mathbb{P}\left(\Sigma_{m_n} = l_n \mid \mathcal{F}_{\lceil nm' \rceil}\right) &= Q^{\Sigma}_{m_n - \lceil nm' \rceil}(l_n - \Sigma_{\lceil nm' \rceil}) \\ &\sim \frac{h}{\eta\sqrt{n}}\, p_{m-m'}(l - \beta_{m'}). \qquad (2.21)\end{aligned}$$

where the second line comes from Proposition 2.4. Note that the denominator of the fractional term in (2.20) is the same as the numerator when m' is chosen to be 0. So the fractional term in (2.20) converges a.s. toward

$$\frac{p_{m-m'}(l - \beta_{m'})}{p_m(l)},$$

the convergence being dominated—by Proposition 2.4. Finally,

$$\begin{aligned}\mathbb{E}\left[f\left((B_{(n)}(s))_{0\leq s\leq m'}\right)\right] &\xrightarrow[n\to\infty]{} \mathbb{E}\left[f\left((\beta_s)_{0\leq s\leq m'}\right) \frac{p_{m-m'}(l-\beta_{m'})}{p_m(l)}\right] \\ &= \mathbb{E}\left[f\left((B^{0\to l}_{[0,m]}(s))_{0\leq s\leq m'}\right)\right].\end{aligned}$$

Step 2. We will use the following lemmas, the proofs of which we postpone right after the end of this proof.

Lemma 2.9. *There exists an integer $n_0 \in \mathbb{N}$ such that, for every $2 \leq q \leq q_0$, there exists a constant C_q satisfying, for all $n \geq n_0$ and $0 \leq s \leq t \leq m_{(n)}$,*

$$\mathbb{E}\left[\left|B_{(n)}(t) - B_{(n)}(s)\right|^q\right] \leq C_q \left|t - s\right|^{\frac{q}{2}}.$$

Lemma 2.10. *We note $B := B^{0\to l}_{[0,m]}$. For any $q \geq 2$, there exists a constant C_q such that, for all $0 \leq s \leq t \leq m$,*

$$\mathbb{E}\left[\left|B(t) - B(s)\right|^q\right] \leq C_q \left|t - s\right|^{\frac{q}{2}}.$$

By the Portmanteau theorem [Bil68, Theorem 2.1], we can restrict ourselves to bounded uniformly continuous functions from $\mathcal{K}$ to $\mathbb{R}$. Let f be such a function. Let $\varepsilon > 0$, and $\delta > 0$ be such that $d_{\mathcal{K}}(X, Y) < \delta$ implies $|f(X) - f(Y)| < \varepsilon$.

Let $0 < \alpha < 1/2 - 1/q_0$. Thanks to Lemmas 2.9 and 2.10, Kolmogorov's criterion [Str99, Theorem 3.3.16] provides us with some constant C such that

$$\sup_n \mathbb{P}\left(B_{(n)} \notin K\right) \vee \mathbb{P}\left(B \notin K\right) < \frac{\varepsilon}{\|f\|_\infty},$$

where

$$K := \left\{X \in \mathcal{K} \,:\, \sup_{s\neq t} \frac{|X(t) - X(s)|}{|t - s|^{\alpha}} \leq C\right\}.$$

We take m' satisfying

$$|m - m'| + C\,|m - m'|^{\alpha} < \frac{\delta}{2},$$

so that, for n sufficiently large,

$$|m_{(n)} - m'| + C\,|m_{(n)} - m'|^{\alpha} < \delta.$$

For any function $X = (X(s))_{0\le s\le x} \in \mathcal{K}$, we define $X_{|y} := (X(s))_{0\le s\le y} \in \mathcal{K}$. Hence

$$\mathbb{E}\left[\left|f\left(B_{(n)}\right) - f\left(B\right)\right|\right] \le \mathbb{E}\left[\left|f\left(B_{(n)}\right) - f\left(B_{(n)_{|m'}}\right)\right|\right] + \mathbb{E}\left[\left|f\left(B_{(n)_{|m'}}\right) - f\left(B_{|m'}\right)\right|\right] + \mathbb{E}\left[\left|f\left(B_{|m'}\right) - f\left(B\right)\right|\right]. \tag{2.22}$$

Thanks to point 1), for n large enough, the second term of the right-hand side of (2.22) is less than ε. The first and third terms are treated in the same way (for the third term, just remove the (n)'s): on the set $\left\{B_{(n)} \in K\right\}$,

$$\begin{aligned} d_{\mathcal{K}}\left(B_{(n)}, B_{(n)_{|m'}}\right) &= |m_{(n)} - m'| + \sup_{m'\le t\le m_{(n)}} \left|B_{(n)}(t) - B_{(n)}(m')\right| \\ &\le |m_{(n)} - m'| + C\,|m_{(n)} - m'|^{\alpha} \\ &< \delta, \end{aligned}$$

and

$$\begin{aligned} \mathbb{E}\left[\left|f\left(B_{(n)}\right) - f\left(B_{(n)_{|m'}}\right)\right|\right] &\le \mathbb{E}\left[\left|f\left(B_{(n)}\right) - f\left(B_{(n)_{|m'}}\right)\right| \mathbb{1}_{\left\{B_{(n)}\in K\right\}}\right] + 2\|f\|_{\infty}\,\mathbb{P}\left(B_{(n)} \notin K\right) \\ &< 3\varepsilon. \end{aligned}$$

All in all, for n large enough

$$\mathbb{E}\left[\left|f\left(B_{(n)}\right) - f\left(B\right)\right|\right] \le 7\varepsilon,$$

and $B_{(n)}$ converges weakly toward B. □

It remains to prove Lemmas 2.9 and 2.10.

Proof of Lemma 2.9. If $|t-s| < 1/n$, the fact that B_n is linear on every interval $[i, i+1]$ implies that

$$|B_{(n)}(t) - B_{(n)}(s)| \le \frac{n(t-s)}{\eta\sqrt{n}} \le \frac{1}{\eta}\sqrt{t-s},$$

which gives the desired result. By the triangle inequality, we can restrict ourselves to the cases where ns and nt are integers, and either $t \le m_{(n)}/2$ or $m_{(n)}/2 \le s$.

First, let us suppose that $0 \le s \le t \le m_{(n)}/2$. Applying (2.20) with $m' = t$ and the proper function f, we obtain

$$\mathbb{E}\left[\left|B_{(n)}(t) - B_{(n)}(s)\right|^{q}\right] = \eta^{-q} n^{-\frac{q}{2}}\,\mathbb{E}\left[\left|\Sigma_{nt} - \Sigma_{ns}\right|^{q} \frac{Q^{\Sigma}_{m_n - nt}(l_n - \Sigma_{nt})}{Q^{\Sigma}_{m_n}(l_n)}\right]. \tag{2.23}$$

The asymptotic formula (2.21) and the fact that $m_{(n)} \to m$ yield the existence of a positive constant c and an integer n_0 such that for $n \ge n_0$,

$$\sqrt{n}\,Q^{\Sigma}_{m_n}(l_n) \ge c \qquad \text{and} \qquad m_{(n)} > \frac{m}{2}.$$

Then Proposition 2.4 ensures us that for $n \ge n_0$,

$$\begin{aligned} \sqrt{n}\,Q^{\Sigma}_{m_n - nt}(l_n - \Sigma_{nt}) &\le \sqrt{n} \sup_{x\in\mathbb{R}} \sup_{y > \frac{m}{4}} Q^{\Sigma}_{ny}\left(x\sqrt{n}\right) \\ &\le \frac{2h}{\eta\sqrt{m}} \sup_{x\in\mathbb{R}} \sup_{y>0} \sup_{n\in\mathbb{N}} \left(\frac{\eta}{h}\sqrt{ny}\,Q^{\Sigma}_{ny}\left(x\sqrt{n}\right)\right) < \infty. \end{aligned}$$

Thus, the fractional term in the equation (2.23) is uniformly bounded as soon as $n \geq n_0$, and

$$\begin{aligned}\mathbb{E}\left[\left|B_{(n)}(t) - B_{(n)}(s)\right|^q\right] &\leq C\, n^{-\frac{q}{2}}\, \mathbb{E}\left[|\Sigma_{nt} - \Sigma_{ns}|^q\right] \\ &\leq C\, n^{-\frac{q}{2}}\, \mathbb{E}\left[\left|\Sigma_{n(t-s)}\right|^q\right] \\ &\leq C_q\, |t-s|^{\frac{q}{2}}\end{aligned}$$

by means of the Rosenthal Inequality (Lemma 2.7).

Now, if $m_{(n)}/2 \leq s \leq t \leq m_{(n)}$, we use the following time reversal invariance:

$$\left(B_{(n)}(s)\right)_{0 \leq s \leq m_{(n)}} \overset{(d)}{=} \left(l_{(n)} - B_{(n)}(m_{(n)} - s)\right)_{0 \leq s \leq m_{(n)}}. \tag{2.24}$$

We have

$$\mathbb{E}\left[\left|B_{(n)}(t) - B_{(n)}(s)\right|^q\right] = \mathbb{E}\left[\left|B_{(n)}(m_{(n)} - s) - B_{(n)}(m_{(n)} - t)\right|^q\right]$$

and we are back in the case we just treated. Note that it is important that $m_{(n)}$ be a deterministic time. □

Proof of Lemma 2.10. We show the inequality for $2 \leq q \leq q_0$. As B appears as the limit of $B_{(n)}$ (in a certain sense), we may choose the X_k's to have arbitrarily large moments, and we see that it actually holds for any value of $q \geq 2$. For $0 \leq s \leq t < m$, point 1) in the proof of Lemma 2.8 shows that

$$\left(B_{(n)}(s), B_{(n)}(t)\right) \xrightarrow[n\to\infty]{(d)} (B(s), B(t)),$$

and

$$\begin{aligned}\mathbb{E}\left[|B(t) - B(s)|^q\right] &= \lim_{M\to\infty} \mathbb{E}\left[|B(t) - B(s)|^q \wedge M\right] \\ &= \lim_{M\to\infty} \lim_{n\to\infty} \mathbb{E}\left[\left|B_{(n)}(t) - B_{(n)}(s)\right|^q \wedge M\right] \\ &\leq C_q\, |t-s|^{\frac{q}{2}},\end{aligned}$$

where C_q is the constant of Lemma 2.9. It only remains to see that $B_{(n)}(m \wedge m_{(n)}) \to B(m)$ in probability in order to obtain the same inequality for $t = m$. The time reversal invariance (2.24) implies that

$$B_{(n)}(m \wedge m_{(n)}) \overset{(d)}{=} l_{(n)} - B_{(n)}\left((m_{(n)} - m) \vee 0\right),$$

and, thanks to 1),

$$\begin{aligned}\left|B_{(n)}\left((m_{(n)} - m) \vee 0\right)\right| &\leq \left|B_{(n)}\left((m_{(n)} - m) \vee 0\right) - B\left((m_{(n)} - m) \vee 0\right)\right| + \left|B\left((m_{(n)} - m) \vee 0\right)\right| \\ &\to 0\end{aligned}$$

in probability, so that $B_{(n)}(m \wedge m_{(n)}) \to l = B(m)$ in probability. □

Discrete first-passage bridges

We now see a lemma similar to Lemma 2.8 for first-passage bridges, in which we will only consider simple random walks. Let $(m_n) \in \mathbb{Z}_+^{\mathbb{N}}$ and $(\sigma_n) \in \mathbb{N}^{\mathbb{N}}$ be two sequences of integers such that

$$m_{(n)} := \frac{m_n}{n} \xrightarrow[n\to\infty]{} m \qquad \text{and} \qquad \sigma_{(n)} := \frac{\sigma_n}{\sqrt{n}} \xrightarrow[n\to\infty]{} \sigma.$$

We consider a sequence $(X_k)_{k \geq 1}$ of i.i.d. random variables with law $(\delta_{-1} + \delta_1)/2$ and define $S_i := \sum_{k=1}^{i} X_k$ (and, by convention, $S_0 = 0$). We still write S its linearly interpolated version. We call

$(B_n(i))_{0\le i\le m_n}$ and $(F_n(i))_{0\le i\le m_n}$ the two processes whose laws are the law of $(S_i)_{0\le i\le m_n}$ conditioned respectively on the events

$$\{S_{m_n} = -\sigma_n\} \qquad \text{and} \qquad \{S_{m_n} = -\sigma_n,\, \underline{S}_{m_n-1} > -\sigma_n\},$$

which we suppose occur with positive probability. Finally, we define

$$B_{(n)} := \left(\frac{B_n(ns)}{\sqrt{n}}\right)_{0\le s\le m_{(n)}} \qquad \text{and} \qquad F_{(n)} := \left(\frac{F_n(ns)}{\sqrt{n}}\right)_{0\le s\le m_{(n)}}$$

their rescaled versions.

There is actually a very convenient way to construct F_n from B_n. For $0\le k\le m_n$, the shifted path of B_n is defined by

$$\Theta_k(B_n)(x) = \begin{cases} B_n(k+x) - B_n(k) & \text{if } 0\le x\le m_n - k, \\ B_n(k+x-m_n) + B_n(m_n) - B_n(k) & \text{if } m_n - k\le x\le m_n. \end{cases}$$

For $0\le k\le \sigma_n - 1$, the first time at which B_n reaches its minimum plus k is noted

$$r_k(B_n) := \inf\left\{i \,:\, B_n(i) = \inf_{0\le j\le m_n} B_n(j) + k\right\}.$$

The following proposition [BCP03, Theorem 1] gives a construction of F_n from B_n.

Proposition 2.11 (Bertoin - Chaumont - Pitman). *Let ν_n be a random variable independent of S and uniformly distributed on $\{0,1,\dots,\sigma_n - 1\}$. Then, the process $\Theta_{r_{\nu_n}(B_n)}(B_n)$ has the same law as F_n.*

Using this construction, we may show that the first-passage Brownian bridge is the limit of its discrete analog:

Lemma 2.12. *As n goes to infinity, the process $F_{(n)}$ converges in law toward the process $F^{0\to-\sigma}_{[0,m]}$, in the space $(\mathcal{K},\, d_{\mathcal{K}})$.*

Proof. We begin as in the proof of Lemma 2.8. We note $\mathcal{F}_i := \sigma(S_k, 0\le k\le i)$ the natural filtration associated with S, and by Skorokhod's representation theorem, we may and will assume that

$$\left(\frac{S_{ns}}{\sqrt{n}}\right)_{0\le s\le m}$$

converges a.s. toward a standard Brownian motion $(\beta_s)_{0\le s\le m}$ for the uniform topology.

Step 1. Let $m' < m$. For n large enough, $\lceil nm'\rceil < m_n$. Let f be continuous bounded from $\mathcal{K}$ to $\mathbb{R}$. We have

$$\begin{aligned} &\mathbb{E}\left[f\left((F_{(n)}(s))_{0\le s\le m'}\right)\right] \\ &\qquad = \mathbb{E}\left[f\left(\left(\frac{S_{ns}}{\sqrt{n}}\right)_{0\le s\le m'}\right) \,\middle|\, S_{m_n} = -\sigma_n,\, \underline{S}_{m_n-1} > -\sigma_n\right] \\ &\qquad = \mathbb{E}\left[f\left(\left(\frac{S_{ns}}{\sqrt{n}}\right)_{0\le s\le m'}\right) \frac{\mathbb{P}\left(S_{m_n} = -\sigma_n,\, \underline{S}_{m_n-1} > -\sigma_n \mid \mathcal{F}_{\lceil nm'\rceil}\right)}{\mathbb{P}\left(S_{m_n} = -\sigma_n,\, \underline{S}_{m_n-1} > -\sigma_n\right)}\right]. \end{aligned} \tag{2.25}$$

Recall the notation $Q^S_k(i) = \mathbb{P}(S_k = i)$. We have to deal with terms of the form

$$\mathbb{P}(S_k = -i,\, \underline{S}_{k-1} > -i) = \frac{i}{k}\,\mathbb{P}(S_k = -i) = \frac{i}{k}\,Q^S_k(-i),$$

where the first equality is an application of the so-called cycle lemma (see e.g. [BCP03, Lemma 2]). Using the Markov property, we obtain

$$\mathbb{P}\left(S_{m_n} = -\sigma_n,\, \underline{S}_{m_n-1} > -\sigma_n \;\middle|\; \mathcal{F}_{\lceil nm' \rceil}\right) \\ = \frac{\sigma_n + S_{\lceil nm' \rceil}}{m_n - \lceil nm' \rceil}\, Q^S_{m_n - \lceil nm' \rceil}\left(-\sigma_n - S_{\lceil nm' \rceil}\right) \mathbb{1}_{\left\{\underline{S}_{\lceil nm' \rceil} > -\sigma_n\right\}}.$$

Here again, the denominator of the fractional term in (2.25) is the same as the numerator when m' is chosen to be 0. The fractional term in (2.25) converges a.s. toward

$$\frac{p'_{m-m'}(-\sigma - \beta_{m'})}{p'_m(-\sigma)}\, \mathbb{1}_{\{\underline{\beta}_{m'} > -\sigma\}},$$

and Proposition 2.4 ensures that this convergence is dominated. So,

$$\mathbb{E}\left[f\left((F_{(n)}(s))_{0\le s\le m'}\right)\right] \xrightarrow[n\to\infty]{} \mathbb{E}\left[f\left((\beta_s)_{0\le s\le m'}\right) \frac{p'_{m-m'}(-\sigma - \beta_{m'})}{p'_m(-\sigma)}\, \mathbb{1}_{\{\underline{\beta}_{m'} > -\sigma\}}\right] \\ = \mathbb{E}\left[f\left((F^{0\to-\sigma}_{[0,m]}(s))_{0\le s\le m'}\right)\right].$$

Step 2. For any $\alpha > 0$ and $X = (X(s))_{0\le s\le x} \in \mathcal{K}$, we write

$$\|X\|_\alpha := \sup_{0\le s<t\le x} \frac{|X(t) - X(s)|}{|t-s|^\alpha}$$

its α-Hölder norm. Proposition 2.11 gives a stochastic domination of the α-Hölder norm of $F_{(n)}$ by that of $B_{(n)}$: we may assume that $F_n = \Theta_{r_{\nu_n}(B_n)}(B_n)$. If $0 \le s < t \le m_{(n)} - r_{\nu_n}(B_n)$,

$$\begin{aligned}\left|F_{(n)}(t) - F_{(n)}(s)\right| &= \frac{1}{\sqrt{n}}\left|\Theta_{r_{\nu_n}(B_n)}(B_n)(nt) - \Theta_{r_{\nu_n}(B_n)}(B_n)(ns)\right| \\ &= \frac{1}{\sqrt{n}}\left|B_n\left(r_{\nu_n}(B_n) + nt\right) - B_n\left(r_{\nu_n}(B_n) + ns\right)\right| \\ &= \left|B_{(n)}\left(\frac{r_{\nu_n}(B_n)}{n} + t\right) - B_{(n)}\left(\frac{r_{\nu_n}(B_n)}{n} + s\right)\right| \\ &\le \left\|B_{(n)}\right\|_\alpha |t-s|^\alpha.\end{aligned}$$

We obtain the same inequality when $m_{(n)} - r_{\nu_n}(B_n) \le s < t \le m_{(n)}$, and by the triangle inequality, we find

$$\left\|F_{(n)}\right\|_\alpha \le 2\left\|B_{(n)}\right\|_\alpha.$$

Step 3. We now suppose that $0 < \alpha < 1/2$. Let $\varepsilon > 0$. Thanks to Lemma 2.9—for which we now have q_0 arbitrarily large—and Kolmogorov's criterion, we can find some constant C such that

$$\sup_n \mathbb{P}\left(F_{(n)} \notin K\right) < \varepsilon \quad \text{with} \quad K := \{X \in \mathcal{K} : \|X\|_\alpha \le C\}. \tag{2.26}$$

Ascoli's theorem [Sch70, Chapter XX] shows that K is a compact set, so that the laws of the $F_{(n)}$'s are tight.

Step 4. We almost have the convergence of the finite-dimensional marginals of $F_{(n)}$ toward those of $F := F^{0\to-\sigma}_{[0,m]}$. Point 1) shows that for any $p \ge 1$, $0 \le s_1 < s_2 < \cdots < s_p < m$,

$$\left(F_{(n)}(s_1), F_{(n)}(s_2), \ldots, F_{(n)}(s_p)\right) \to \left(F(s_1), F(s_2), \ldots, F(s_p)\right).$$

It only remains to deal with the point m. Let $\delta > 0$. For n large enough, on $\left\{F_{(n)} \in K\right\}$,

$$\left|F_{(n)}(m \wedge m_{(n)}) + \sigma\right| \leq \left|\sigma_{(n)} - \sigma\right| + C\left|m_{(n)} - m\right|^{\alpha} < \delta,$$

therefore

$$\mathbb{P}\left(\left|F_{(n)}\left(m \wedge m_{(n)}\right) + \sigma\right| > \delta\right) \leq \mathbb{P}\left(F_{(n)} \notin K\right) < \varepsilon.$$

We have just shown that $F_{(n)}\left(m \wedge m_{(n)}\right)$ converges in law toward the deterministic value $-\sigma$. Slutzky's lemma then allows us to conclude that the finite-dimensional marginals of $F_{(n)}$ converge toward those of F. This, together with the tightness of the laws of the $F_{(n)}$'s, yields the result thanks to Prokhorov's lemma. □

For any real numbers m_1, m_2, l_1, l_2, we define the bridge on $[m_1, m_2]$ from l_1 to l_2 by

$$\left(B^{l_1 \to l_2}_{[m_1,m_2]}(s)\right)_{m_1 \leq s \leq m_2} := l_1 + \left(B^{0 \to l_2 - l_1}_{[0,m_2-m_1]}(s - m_1)\right)_{m_1 \leq s \leq m_2},$$

and for $\sigma_1 > \sigma_2$, we define the first-passage bridge on $[m_1, m_2]$ from σ_1 to σ_2 by

$$\left(F^{\sigma_1 \to \sigma_2}_{[m_1,m_2]}(s)\right)_{m_1 \leq s \leq m_2} := \sigma_1 + \left(F^{0 \to \sigma_2 - \sigma_1}_{[0,m_2-m_1]}(s - m_1)\right)_{m_1 \leq s \leq m_2}.$$

2.4.2 The Brownian snake

We need a version of the Brownian snake's head driven by a first-passage Brownian bridge. There are several ways to define such an object.

We may define it as a the head of a Brownian snake with lifetime process a first-passage Brownian bridge $F^{\sigma \to 0}_{[0,m]}$ and starting from the path $0_\sigma := t \in [0, \sigma] \mapsto 0$ (see [LG99, Chapter IV] or [DLG02, Chapter 4] for a proper definition).

Let $(F_s)_{0 \leq s \leq m}$ be a first-passage Brownian bridge from σ to 0. The Brownian snake driven by F and started at 0_σ is the path-valued process $(F_s, (W(s,t),\, 0 \leq t \leq F_s))_{0 \leq s \leq m}$ whose law is defined by:

- for all $0 \leq t \leq \sigma$, $W(0,t) = 0$,
- for all $0 \leq s \leq m$, $W(s,0) = 0$,
- the conditional law of $W(s,\cdot)$ given F is the law of an inhomogeneous Markov process whose transition kernel is described as follows: for $0 \leq s \leq s' \leq m$,
  - $W(s',t) = W(s,t)$ for all $0 \leq t \leq \inf_{[s,s']} F$,
  - $\left(W(s', \inf_{[s,s']} F + t)\right)_{0 \leq t \leq F_{s'} - \inf_{[s,s']} F}$ is independent of $W(s,\cdot)$ and distributed as a real Brownian motion started from $W(s, \inf_{[s,s']} F)$ and stopped at time $F_{s'} - \inf_{[s,s']} F$.

The head of this process is then defined by

$$\left(F^{\sigma \to 0}_{[0,m]}, Z_{[0,m]}\right) := \left((F_s)_{0 \leq s \leq m}, (W(s, F_s))_{0 \leq s \leq m}\right).$$

This description has the advantage of being very visual: $W(0,\cdot)$ is the function 0_σ. Then, every time F decreases, we erase the tip of the previous path, and when F increases, we glue a part of an independent Brownian motion (see Figure 2.9).

In the following, we will only need the head and not the whole process. The following description gives a direct construction of this head. Conditionally given $F = F^{\sigma \to 0}_{[0,m]}$, we define a Gaussian process $(\Gamma_s)_{0 \leq s \leq m}$ with covariance function

$$\operatorname{cov}(\Gamma_s, \Gamma_{s'}) = \inf_{[s,s']}(F - \underline{F}).$$

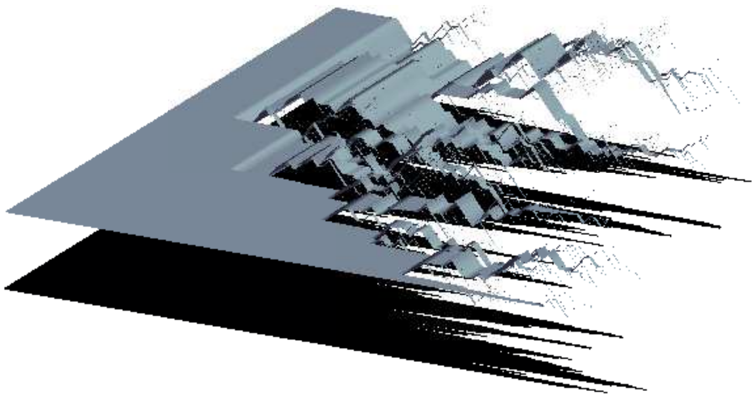

Figure 2.9. *An approximation of the conditioned Brownian snake. The first-passage bridge from σ to 0 is represented by the shadowy part of the figure. In order to see $W(s,\cdot)$, one must "cut" the surface at s and look at the edge of the cut piece.*

The processes (F,Γ) then has the same law as the process $\left(F^{\sigma\to 0}_{[0,m]}, Z_{[0,m]}\right)$ defined above.

We easily see that we can derive the law of the head from the law of the snake, and it is actually also possible to recover the whole snake from its head (see [MM03, Section 2]): starting from the process $(F,Z)=\left(F^{\sigma\to 0}_{[0,m]}, Z_{[0,m]}\right)$, we define

$$W(s,t) := Z\Big(\inf\{r\geq s,\, F(r)=t\}\Big), \qquad 0\leq t\leq F(s),\, 0\leq s\leq m.$$

The process $\Big(F(s),(W(s,t),\, 0\leq t\leq F(s))\Big)_{0\leq s\leq m}$ then has the law of the Brownian snake defined above. In particular, for $s\in[0,m]$ fixed, the process

$$\Big(Z\Big(\inf\{r\geq s,\, F(r)=t\}\Big)\Big)_{\underline{F}(s)\leq t\leq F(s)}$$

has the law of a real Brownian motion started from 0. Using time reversal invariance, we see that the process

$$\Big(Z\Big(\inf\{r\geq s,\, F(r)=F(s)-x\}\Big)-Z(s)\Big)_{0\leq x\leq F(s)-\underline{F}(s)}$$

has the same law. This fact will be used in Section 3.1.

2.4.3 The discrete snake

We will describe here an analog of the Brownian snake in the discrete setting. Let us first consider three sequences of integers (σ_n), (m_n) and (l_n) such that

$$\sigma_{(n)} := \frac{\sigma_n}{\sqrt{2n}} \to \sigma, \qquad m_{(n)} := \frac{2m_n+\sigma_n}{2n} \to m \qquad \text{and} \qquad l_{(n)} := \frac{l_n}{\gamma n^{\frac14}} \to l.$$

We call (C_n,L_n) the contour pair of a random forest uniformly distributed over the set $\mathfrak{F}^{m_n}_{\sigma_n}$ of well-labeled forests with σ_n trees and m_n tree edges. We define

$$C_{(n)} := \left(\frac{C_n(2nt)}{\sqrt{2n}}\right)_{0\leq t\leq m_{(n)}} \qquad \text{and} \qquad L_{(n)} := \left(\frac{L_n(2nt)}{\gamma\, n^{1/4}}\right)_{0\leq t\leq m_{(n)}}$$

their scaled versions.

We define the discrete snake $(W_n(i,j), 0 \le j \le C_n(i))_{0 \le i \le 2m_n+\sigma_n}$ by (see Figure 2.10)

$$W_n(i,j) := L_n\left(\sup\{k \le i \,:\, C_n(k) = j\}\right) = L_n\left(\inf\{k \ge i \,:\, C_n(k) = j\}\right).$$

Let $(\mathfrak{f}, \mathfrak{l})$ be the well-labeled forest coded by (C_n, L_n). Then for $0 \le i \le 2m_n + \sigma_n$,

$$(W_n(i,j))_{0 \le j \le C_n(i)}$$

records the labels of the unique path going from $t(\mathfrak{f}) + 1$ to $\mathfrak{f}(i)$. As a result, $W_n(i,j) = 0$ for $0 \le j \le t(\mathfrak{f}) + 1 - \mathfrak{a}(\mathfrak{f}(i))$.

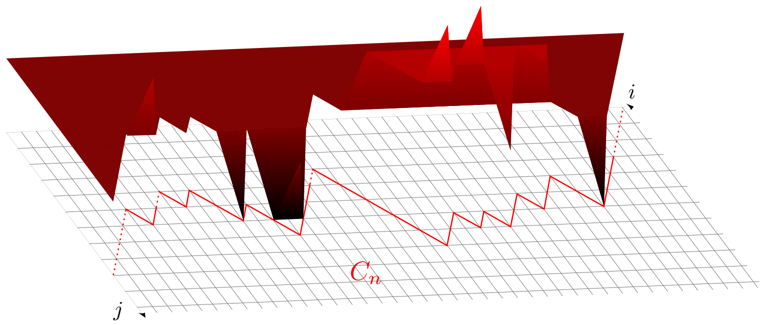

Figure 2.10. *Discrete snake.*

We then extend W_n to $\{(s,t) \,:\, s \in [0, 2m_n + \sigma_n],\ t \in [0, C_n(s)]\}$ by linear interpolation and we let, for $0 \le s \le m_{(n)}$, $0 \le t \le C_{(n)}(s)$,

$$W_{(n)}(s,t) := \frac{W_n(2ns, \sqrt{2n}\,t)}{\gamma\, n^{1/4}}.$$

For each $0 \le s \le m_{(n)}$, $W_{(n)}(s, \cdot)$ is a path lying in

$$\mathcal{K}_0 := \{f \in \mathcal{K} \mid f(0) = 0\},$$

so that we can see $W_{(n)}$ as an element of

$$\mathcal{W}_0 := \bigcup_{x \in \mathbb{R}_+} \mathcal{C}([0,x], \mathcal{K}_0).$$

For $X \in \mathcal{W}_0$, we call $\xi(X)$ the real number such that $X \in \mathcal{C}([0, \xi(X)], \mathcal{K}_0)$, and we endow $\mathcal{W}_0$ with the metric

$$d_{\mathcal{W}_0}(X,Y) := |\xi(X) - \xi(Y)| + \sup_{s \ge 0} d_{\mathcal{K}}\left(X(s \wedge \xi(X), \cdot), Y(s \wedge \xi(Y), \cdot)\right).$$

2.4.4 Convergence of a uniform well-labeled forest

We will prove the following result.

Proposition 2.13. *The pair $(C_{(n)}, W_{(n)})$ converges weakly toward the pair $\left(F^{\sigma\to 0}_{[0,m]}, W\right)$, in the space $(\mathcal{K},\ d_{\mathcal{K}}) \times (\mathcal{W}_0,\ d_{\mathcal{W}_0})$.*

We readily obtain the following corollary:

Corollary 2.14. *The pair $(C_{(n)}, L_{(n)})$ converges weakly toward the pair $\left(F^{\sigma\to 0}_{[0,m]}, Z_{[0,m]}\right)$, in the space $(\mathcal{K},\ d_{\mathcal{K}})^2$.*

Proposition 2.13 may appear stronger than Corollary 2.14, but is actually not, because of the strong link between the whole snake and its head [MM03]. We begin by a lemma.

Lemma 2.15. *For all $0 < \delta < 1/4$, for all $\varepsilon > 0$, there exist a constant C and an integer n_0 such that, as soon as $n \geq n_0$, $\mathbb{P}(W_{(n)} \notin A) < \varepsilon$, where*

$$A := \left\{ X \in \mathcal{W}_0 \;:\; \sup_{s \neq s'} \frac{d_{\mathcal{K}}\left(X(s,\cdot) - X(s',\cdot)\right)}{|s-s'|^{\delta}} \leq C \right\}.$$

Proof. It is based on (2.26) and a similar inequality for Motzkin paths (which is merely Rosenthal Inequality). The fact that the steps of the random walks we consider are bounded allows us to take the q of Lemma 2.7 arbitrary large.

Let $0 \leq s < s' \leq m_{(n)}$. Conditionally given $C_{(n)}$,

$$\begin{aligned} d_{\mathcal{K}}&\left(W_{(n)}(s,\cdot), W_{(n)}(s',\cdot)\right) \\ &= \left|C_{(n)}(s) - C_{(n)}(s')\right| + \sup_{t \geq a_n} \left| W_{(n)}\left(s, t \wedge C_{(n)}(s)\right) - W_{(n)}\left(s', t \wedge C_{(n)}(s')\right) \right|, \end{aligned}$$

where $a_n := \inf_{[s,s']} C_{(n)}$.

We need to distinguish two cases:

✧ if $b_n := \inf_{[0,s]} C_{(n)} \leq a_n$, then

$$\left(W_{(n)}(s,t) - W_{(n)}(s,a_n)\right)_{a_n \leq t \leq C_{(n)}(s)}$$

is merely a rescaled Motzkin path.

✧ if $b_n > a_n$, then $W_{(n)}(s,t) = 0$ for $a_n \leq t \leq b_n$ and

$$\left(W_{(n)}(s,t) - W_{(n)}(s,b_n)\right)_{b_n \leq t \leq C_{(n)}(s)}$$

is a rescaled Motzkin path.

In both cases,

$$\left(W_{(n)}(s',t) - W_{(n)}(s',a_n)\right)_{a_n \leq t \leq C_{(n)}(s')}$$

is also a rescaled Motzkin path—independent from $\left(W_{(n)}(s,t) - W_{(n)}(s,a_n)\right)_{a_n \leq t \leq C_{(n)}(s)}$.

Treating both cases separately, we obtain that there exists a constant M, independent of s, such that for n large enough,

$$\mathbb{E}\left[\sup_{a_n \leq t \leq C_{(n)}(s)} \left|W_{(n)}(s,t) - W_{(n)}(s,a_n)\right|^q \;\middle|\; C_{(n)}\right] \leq M \,\left|C_{(n)}(s) - a_n\right|^{\frac{q}{2}},$$

by Lemma 2.7. The same inequality holds with s' instead of s. We have

$$\begin{aligned} \mathbb{E}\left[d_{\mathcal{K}}\left(W_{(n)}(s,\cdot), W_{(n)}(s',\cdot)\right)^q \;\middle|\; C_{(n)}\right] &\leq M' \left(\left\|C_{(n)}\right\|_{\alpha}^{q} |s-s'|^{\alpha q} + \left\|C_{(n)}\right\|_{\alpha}^{\frac{q}{2}} |s-s'|^{\alpha \frac{q}{2}}\right) \\ &\leq M_q \left(\left\|C_{(n)}\right\|_{\alpha}^{q} \vee 1\right) |s-s'|^{\alpha \frac{q}{2}}. \end{aligned}$$

For $C \geq 1$,

$$\mathbb{E}\left[d_{\mathcal{K}}\left(W_{(n)}(s,\cdot), W_{(n)}(s',\cdot)\right)^q \;\middle|\; \left\|C_{(n)}\right\|_{\alpha} \leq C\right] \leq M_q\, C^q |s-s'|^{\alpha \frac{q}{2}}. \tag{2.27}$$

Let $0 < \delta < \frac{1}{4}$. Then, let $0 < \alpha < 1/2$ be such that $\delta < \alpha/2$, and $\varepsilon > 0$. Thanks to (2.26), we may find a constant C such that, for n sufficiently large,

$$\mathbb{P}\left(\left\|C_{(n)}\right\|_{\alpha} > C\right) < \varepsilon.$$

For this C, the inequality (2.27) allows us to apply Kolmogorov's criterion [Str99, Theorem 3.3.16]: we find a constant C' such that, for n large enough,

$$\mathbb{P}\left(\sup_{s\neq s'} \frac{d_{\mathcal{K}}\left(W_{(n)}(s,\cdot) - W_{(n)}(s',\cdot)\right)}{|s-s'|^{\delta}} > C' \;\middle|\; \left\|C_{(n)}\right\|_{\alpha} \leq C\right) < \varepsilon.$$

Finally,

$$\mathbb{P}\left(\sup_{s\neq s'} \frac{d_{\mathcal{K}}\left(W_{(n)}(s,\cdot) - W_{(n)}(s',\cdot)\right)}{|s-s'|^{\delta}} > C'\right) < \frac{\varepsilon}{1-\varepsilon} + \varepsilon,$$

which is what we needed. □

Proof of Proposition 2.13. We begin by showing the convergence of a finite number of trajectories, together with the whole contour process, and then conclude by a tightness argument using Lemma 2.15.

Convergence of the finite-dimensional laws. Let $p \geq 1$ and $0 \leq s_1 < \cdots < s_p < m$. We will show by induction on p that

$$\left((C_{(n)}(s))_{0\leq s\leq m_{(n)}}, W_{(n)}(s_1,\cdot), \ldots, W_{(n)}(s_p,\cdot)\right) \xrightarrow[n\to\infty]{(d)} \left(F^{\sigma\to 0}_{[0,m]}, W(s_1,\cdot), \ldots, W(s_p,\cdot)\right). \tag{2.28}$$

Because $m_{(n)} \to m$, for n sufficiently large, $s_p \leq m_{(n)}$ and the vector we consider is well defined.

Step 1. For $p = 1$, we may only consider the case $s_1 = 0$. $(C_n(i))_{0\leq i\leq 2m_n+\sigma_n}$ is a discrete first-passage bridge on $[0, 2m_n + \sigma_n]$ from σ_n to 0 and $W_n(0, j) = 0$ for $0 \leq j \leq \sigma_n$. Lemma 2.12 thus ensures us that

$$\left((C_{(n)}(s))_{0\leq s\leq m_{(n)}}, (W_{(n)}(0,t))_{0\leq t\leq \sigma_{(n)}}\right) \xrightarrow[n\to\infty]{(d)} \left((F^{\sigma\to 0}_{[0,m]}(s))_{0\leq s\leq m}, (W(0,t))_{0\leq t\leq \sigma}\right).$$

Step 2. Let us assume (2.28) with $p-1$ instead of p. There exists a Motzkin path M, independent of $C_{(n)}$ and $W_{(n)}(s_i,\cdot)$, $1 \leq i \leq p-1$, such that conditionally given

$$\left((C_{(n)}(s))_{0\leq s\leq m_{(n)}}, W_{(n)}(s_1,\cdot), \ldots, W_{(n)}(s_{p-1},\cdot)\right),$$

for $0 \leq t \leq C_{(n)}(s_p)$,

$$W_{(n)}(s_p,t) = W_{(n)}(s_{p-1}, t\wedge a_n) + \frac{M_{\sqrt{2n}(t-a_n)^+}}{\gamma\, n^{1/4}}$$

where $a_n := \inf_{[s_{p-1},s_p]} C_{(n)}$ and $x^+ := x.\mathbb{1}_{\{x\geq 0\}}$ stands for the positive part of x. The Donsker Invariance Principle [Bil68] ensures that

$$\left(\frac{M_{\sqrt{2n}t}}{\gamma\, n^{1/4}}\right)_{t\geq 0}$$

converges weakly toward a Brownian motion β for the uniform topology on every compact sets.

By means of Skorokhod's representation theorem, we may and will assume that this convergence holds almost surely. We also suppose that (2.28) holds for $p-1$. Then, a.s.,

$$\left(W_{(n)}(s_p,t)\right)_{0\leq t\leq C_{(n)}(s_p)} \to \left(W(s_{p-1}, t\wedge a) + \beta_{(t-a)^+}\right)_{0\leq t\leq F^{\sigma\to 0}_{[0,m]}(s_p)}$$

where $a := \inf_{[s_{p-1},s_p]} F_{[0,m]}^{\sigma\to 0}$. To see this, observe that

$$\left| C_{(n)}(s_p) - F_{[0,m]}^{\sigma\to 0}(s_p) \right| \to 0$$

and

$$\begin{aligned} \sup_t \left| W_{(n)}(s_{p-1}, t\wedge a_n) - W(s_{p-1}, t\wedge a) \right| &\leq \sup_{0\leq t\leq a_n} \left| W_{(n)}(s_{p-1},t) - W(s_{p-1},t) \right| \\ &\quad + \sup_{a_n\wedge a\leq t\leq a_n\vee a} \left| W(s_{p-1},t) - W(s_{p-1}, a_n\wedge a) \right| \\ &\to 0, \end{aligned}$$

by continuity of $W(s_{p-1},\cdot)$. A similar inequality holds for M.

Finally, the law of

$$\Big(W(s_{p-1}, t\wedge a) + \beta_{(t-a)^+} \Big)_{0\leq t\leq F_{[0,m]}^{\sigma\to 0}(s_p)}$$

is that of $W(s_p,\cdot)$, conditionally given

$$\Big((F_{[0,m]}^{\sigma\to 0}(s))_{0\leq s\leq m'}, W(s_1,\cdot), \ldots, W(s_{p-1},\cdot) \Big),$$

which is precisely what we wanted.

Tightness. Let $0<\delta<1/4$ and $\varepsilon>0$. Lemma 2.15 provides us with a constant C and an integer n_0 such that for all $n\geq n_0$, $\mathbb{P}(W_{(n)}\notin A)<\varepsilon$, where

$$A := \left\{ X\in\mathcal{W}_0 \,:\, \sup_{s\neq s'} \frac{d_{\mathcal{K}}\left(X(s,\cdot)-X(s',\cdot)\right)}{|s-s'|^{\delta}} \leq C \right\}.$$

Let $(s_k)_{k\geq 1}$ be a countable dense subset of $[0,m)$. As for every $k\geq 1$, $\Big(W_{(n)}(s_k,\cdot)\Big)_n$ is tight, we can find compact sets $K_k\subseteq\mathcal{W}_0$ such that for all $k\geq 1$, for all $n\geq n_0$,

$$\mathbb{P}\left(W_{(n)}(s_k,\cdot)\notin K_k \right) < \frac{\varepsilon}{2^k}.$$

The set

$$\mathbb{K} := A\cap\{X\in\mathcal{W}_0 \,:\, \forall k\geq 1,\ X(s_k,\cdot)\in K_k\}.$$

is a compact subset of $\mathcal{W}_0$ by Ascoli's theorem [Sch70, XX] and for $n\geq n_0$, $\mathbb{P}\left(W_{(n)}\notin\mathbb{K}\right)<2\varepsilon$, hence the tightness of the sequence of $W_{(n)}$'s laws. □

Scaling limit of positive genus random quadrangulations

3

In this chapter, we use the results of Chapter 2 in order to prove Theorem 1.11. We adapt the approach from [LG07] for the case $g = 0$ to our case $g \geq 1$. In a second time, we give a nicer expression for the constant Υ appearing in (2.14), and show Theorem 1.12.

3.1 Proof of Theorem 1.11

3.1.1 Setting

Let $\mathfrak{q}_n$ be uniformly distributed over the set $\mathcal{Q}_n$ of bipartite quadrangulations of genus g with n faces. Conditionally given $\mathfrak{q}_n$, we take $v_n^\bullet$ uniformly over $V(\mathfrak{q}_n)$ so that $(\mathfrak{q}_n, v_n^\bullet)$ is uniform over the set $\mathcal{Q}_n^\bullet$ of pointed bipartite quadrangulations of genus g with n faces. Recall that every element of $\mathcal{Q}_n$ has the same number of vertices: $n + 2 - 2g$. Through the Chapuy–Marcus–Schaeffer bijection, $(\mathfrak{q}_n, v_n^\bullet)$ corresponds to a uniform well-labeled g-tree with n edges $(\mathfrak{t}_n, \mathfrak{l}_n)$. The parameter $\varepsilon_\pm \in \{-1, 1\}$ appearing in the bijection will be irrelevant to what follows.

Recall the notation $\mathfrak{t}_n(0)$, $\mathfrak{t}_n(1)$, ..., $\mathfrak{t}_n(2n)$ and $\mathfrak{q}_n(0)$, $\mathfrak{q}_n(1)$, ..., $\mathfrak{q}_n(2n)$ from Section 2.1. For technical reasons, it will be more convenient, when traveling along the g-tree, not to begin by its root but rather by the first edge of the first forest. Precisely, we define

$$\dot{\mathfrak{t}}_n(i) := \begin{cases} \mathfrak{t}_n(i - u_n + 2n) & \text{if} \quad 0 \leq i \leq u_n, \\ \mathfrak{t}_n(i - u_n) & \text{if} \quad u_n \leq i \leq 2n, \end{cases}$$

and

$$\dot{\mathfrak{q}}_n(i) := \begin{cases} \mathfrak{q}_n(i - u_n + 2n) & \text{if} \quad 0 \leq i \leq u_n, \\ \mathfrak{q}_n(i - u_n) & \text{if} \quad u_n \leq i \leq 2n, \end{cases}$$

where u_n is the integer recording the position of the root in the first forest of $\mathfrak{t}_n$. We endow $[\![0, 2n]\!]$ with the pseudo-metric d_n defined by

$$d_n(i,j) := d_{\mathfrak{q}_n}\left(\dot{\mathfrak{q}}_n(i), \dot{\mathfrak{q}}_n(j)\right).$$

We define the equivalence relation $\sim_n$ on $[\![0, 2n]\!]$ by declaring that $i \sim_n j$ if $\dot{\mathfrak{q}}_n(i) = \dot{\mathfrak{q}}_n(j)$, that is, if $d_n(i,j) = 0$. We call π_n the canonical projection from $[\![0, 2n]\!]$ to $[\![0, 2n]\!]_{/\sim_n}$ and we slightly abuse notation by seeing d_n as a metric on $[\![0, 2n]\!]_{/\sim_n}$ defined by $d_n(\pi_n(i), \pi_n(j)) := d_n(i,j)$. In what follows, we will always make the same abuse with every pseudo-metric. The metric space $\left([\![0, 2n]\!]_{/\sim_n}, d_n\right)$ is then isometric to $(V(\mathfrak{q}_n)\backslash\{v_n^\bullet\}, d_{\mathfrak{q}_n})$, which is at d_{GH}-distance at most 1 from the space $(V(\mathfrak{q}_n), d_{\mathfrak{q}_n})$.

We extend the definition of d_n to noninteger values by linear interpolation: for $s, t \in [0, 2n]$,

$$d_n(s,t) := \underline{s}\,\underline{t}\, d_n(\lceil s \rceil, \lceil t \rceil) + \underline{s}\,\overline{t}\, d_n(\lceil s \rceil, \lfloor t \rfloor) + \overline{s}\,\underline{t}\, d_n(\lfloor s \rfloor, \lceil t \rceil) + \overline{s}\,\overline{t}\, d_n(\lfloor s \rfloor, \lfloor t \rfloor), \tag{3.1}$$

where $\lfloor s \rfloor := \sup\{k \in \mathbb{Z},\, k \leq s\}$, $\lceil s \rceil := \lfloor s \rfloor + 1$, $\underline{s} := s - \lfloor s \rfloor$ and $\overline{s} := \lceil s \rceil - s$. Beware that d_n is no longer a pseudo-metric on $[0, 2n]$: indeed, $d_n(s,s) = 2\,\underline{s}\,\overline{s}\, d_n(\lceil s \rceil, \lfloor s \rfloor) > 0$ as soon as $s \notin \mathbb{Z}$. The triangle inequality, however, remains valid for all $s, t \in [0, 2n]$. Using the Chapuy–Marcus–Schaeffer bijection, it is easy to see that $d_n(\lceil s \rceil, \lfloor s \rfloor)$ is equal to either 1 or 2, so that $d_n(s,s) \leq 1/2$.

As usual, we define the rescaled version: for $s, t \in [0, 1]$, we let

$$d_{(n)}(s,t) := \frac{1}{\gamma\, n^{1/4}}\, d_n(2ns, 2nt), \tag{3.2}$$

so that

$$d_{GH}\left(\left(\frac{1}{2n}[\![0, 2n]\!]_{/\sim_n}, d_{(n)}\right), \left(V(\mathfrak{q}_n), \frac{1}{\gamma\, n^{1/4}}\, d_{\mathfrak{q}_n}\right)\right) \leq \frac{1}{\gamma\, n^{1/4}}. \tag{3.3}$$

3.1.2 Tightness of the distance processes

The first step is to show the tightness of the processes $d_{(n)}$'s laws. For that matter, we use the bound (2.3). We define

$$d_n^\circ(i,j) := \mathfrak{l}_n\left(\dot{\mathfrak{t}}_n(i)\right) + \mathfrak{l}_n\left(\dot{\mathfrak{t}}_n(j)\right) - 2\max\left(\min_{k \in \overrightarrow{[\![i,j]\!]}} \mathfrak{l}_n\left(\dot{\mathfrak{t}}_n(k)\right), \min_{k \in \overrightarrow{[\![j,i]\!]}} \mathfrak{l}_n\left(\dot{\mathfrak{t}}_n(k)\right)\right) + 2,$$

we extend it to $[0, 2n]$ as we did for d_n by (3.1), and we define its rescaled version $d_{(n)}^\circ$ as we did for d_n by (3.2). We readily obtain the following bound,

$$d_{(n)}(s,t) \leq d_{(n)}^\circ(s,t). \tag{3.4}$$

Expression of $d_{(n)}^\circ$ in terms of the spatial contour function of the g-tree

Although it is not straightforward to define a contour function for the whole g-tree, we may define its spatial contour function $\mathfrak{L}_n : [0, 2n] \to \mathbb{R}$ by,

$$\mathfrak{L}_n(i) := \mathfrak{l}_n\left(\dot{\mathfrak{t}}_n(i)\right) - \mathfrak{l}_n\left(\dot{\mathfrak{t}}_n(0)\right), \qquad 0 \leq i \leq 2n,$$

and by linearly interpolating it between integer values. The rescaled version is then defined by

$$\mathfrak{L}_{(n)} := \left(\frac{\mathfrak{L}_n(2nt)}{\gamma\, n^{1/4}}\right)_{0 \leq t \leq 1},$$

and we easily see that

$$d_{(n)}^\circ(s,t) = \mathfrak{L}_{(n)}(s) + \mathfrak{L}_{(n)}(t) - 2\max\left(\min_{x \in \overrightarrow{[s,t]}} \mathfrak{L}_{(n)}(x), \min_{x \in \overrightarrow{[t,s]}} \mathfrak{L}_{(n)}(x)\right) + O\left(n^{\frac{1}{4}}\right)$$

where

$$\overrightarrow{[s,t]} := \begin{cases} [s,t] & \text{if} \quad s \leq t, \\ [s,1] \cup [0,t] & \text{if} \quad t < s. \end{cases} \tag{3.5}$$

Convergence results

As in Section 2.2, we call $\mathfrak{s}_n$ the scheme of $\mathfrak{t}_n$, $(\mathfrak{f}_n^{\mathfrak{e}}, \mathfrak{l}_n^{\mathfrak{e}})_{\mathfrak{e}\in\vec{E}(\mathfrak{s}_n)}$ its well-labeled forests, $(m_n^{\mathfrak{e}})_{\mathfrak{e}\in\vec{E}(\mathfrak{s}_n)}$ and $(\sigma_n^{\mathfrak{e}})_{\mathfrak{e}\in\vec{E}(\mathfrak{s}_n)}$ respectively their sizes and lengths, $(l_n^v)_{v\in V(\mathfrak{s}_n)}$ the shifted labels of its nodes, $(\mathfrak{M}_n^{\mathfrak{e}})_{\mathfrak{e}\in\vec{E}(\mathfrak{s}_n)}$ its Motzkin bridges, and u_n the integer recording the position of the root in the first forest $\mathfrak{f}_n^{\mathfrak{e}_*}$. We call $(C_n^{\mathfrak{e}}, L_n^{\mathfrak{e}})$ the contour pair of the well-labeled forest $(\mathfrak{f}_n^{\mathfrak{e}}, \mathfrak{l}_n^{\mathfrak{e}})$ and we extend the definition of $\mathfrak{M}_n^{\mathfrak{e}}$ to $[0,\sigma_n^{\mathfrak{e}}]$ by linear interpolation. As usual, we define the rescaled versions of these objects

$$m_{(n)}^{\mathfrak{e}} := \frac{2m_n^{\mathfrak{e}} + \sigma_n^{\mathfrak{e}}}{2n}, \qquad \sigma_{(n)}^{\mathfrak{e}} := \frac{\sigma_n^{\mathfrak{e}}}{\sqrt{2n}}, \qquad l_{(n)}^{v} := \frac{l_n^v}{\gamma\, n^{1/4}}, \qquad u_{(n)} := \frac{u_n}{2n}$$

and

$$C_{(n)}^{\mathfrak{e}} := \left(\frac{C_n^{\mathfrak{e}}(2nt)}{\sqrt{2n}}\right)_{0\le t\le m_{(n)}^{\mathfrak{e}}}, \; L_{(n)}^{\mathfrak{e}} := \left(\frac{L_n^{\mathfrak{e}}(2nt)}{\gamma\, n^{1/4}}\right)_{0\le t\le m_{(n)}^{\mathfrak{e}}}, \; \mathfrak{M}_{(n)}^{\mathfrak{e}} := \left(\frac{\mathfrak{M}_n^{\mathfrak{e}}(\sqrt{2n}\,t)}{\gamma\, n^{1/4}}\right)_{0\le t\le \sigma_{(n)}^{\mathfrak{e}}}.$$

Combining the results of Proposition 2.5, Lemma[1] 2.8 and Corollary 2.14, we obtain the following proposition.

Proposition 3.1. *The random vector*

$$\left(\mathfrak{s}_n, \left(m_{(n)}^{\mathfrak{e}}\right)_{\mathfrak{e}\in\vec{E}(\mathfrak{s}_n)}, \left(\sigma_{(n)}^{\mathfrak{e}}\right)_{\mathfrak{e}\in\vec{E}(\mathfrak{s}_n)}, \left(l_{(n)}^{v}\right)_{v\in V(\mathfrak{s}_n)}, u_{(n)}, \left(C_{(n)}^{\mathfrak{e}}, L_{(n)}^{\mathfrak{e}}\right)_{\mathfrak{e}\in\vec{E}(\mathfrak{s}_n)}, \left(\mathfrak{M}_{(n)}^{\mathfrak{e}}\right)_{\mathfrak{e}\in\vec{E}(\mathfrak{s}_n)}\right)$$

converges in law toward the random vector

$$\left(\mathfrak{s}_\infty, (m_\infty^{\mathfrak{e}})_{\mathfrak{e}\in\vec{E}(\mathfrak{s}_\infty)}, (\sigma_\infty^{\mathfrak{e}})_{\mathfrak{e}\in\vec{E}(\mathfrak{s}_\infty)}, (l_\infty^{v})_{v\in V(\mathfrak{s}_\infty)}, u_\infty, (C_\infty^{\mathfrak{e}}, L_\infty^{\mathfrak{e}})_{\mathfrak{e}\in\vec{E}(\mathfrak{s}_\infty)}, (\mathfrak{M}_\infty^{\mathfrak{e}})_{\mathfrak{e}\in\vec{E}(\mathfrak{s}_\infty)}\right)$$

whose law is defined as follows:

- ✧ *the law of the vector*

$$\mathfrak{I}_\infty := \left(\mathfrak{s}_\infty, (m_\infty^{\mathfrak{e}})_{\mathfrak{e}\in\vec{E}(\mathfrak{s}_\infty)}, (\sigma_\infty^{\mathfrak{e}})_{\mathfrak{e}\in\vec{E}(\mathfrak{s}_\infty)}, (l_\infty^{v})_{v\in V(\mathfrak{s}_\infty)}, u_\infty\right)$$

  is the probability μ defined before Proposition 2.5,

- ✧ *conditionally given $\mathfrak{I}_\infty$,*
  - *the processes $(C_\infty^{\mathfrak{e}}, L_\infty^{\mathfrak{e}})$, $\mathfrak{e}\in\vec{E}(\mathfrak{s}_\infty)$ and $(\mathfrak{M}_\infty^{\mathfrak{e}})$, $\mathfrak{e}\in\check{E}(\mathfrak{s}_\infty)$ are independent,*
  - *the process $(C_\infty^{\mathfrak{e}}, L_\infty^{\mathfrak{e}})$ has the law of a Brownian snake's head on $[0, m_\infty^{\mathfrak{e}}]$ going from $\sigma_\infty^{\mathfrak{e}}$ to 0:*

$$(C_\infty^{\mathfrak{e}}, L_\infty^{\mathfrak{e}}) \overset{(d)}{=} \left(F_{[0,m_\infty^{\mathfrak{e}}]}^{\sigma_\infty^{\mathfrak{e}}\to 0}, Z_{[0,m_\infty^{\mathfrak{e}}]}\right),$$

  - *the process $(\mathfrak{M}_\infty^{\mathfrak{e}})$ has the law of a Brownian bridge on $[0,\sigma_\infty^{\mathfrak{e}}]$ from 0 to $l_\infty^{\mathfrak{e}} := l_\infty^{\mathfrak{e}^+} - l_\infty^{\mathfrak{e}^-}$:*

$$(\mathfrak{M}_\infty^{\mathfrak{e}}) \overset{(d)}{=} B_{[0,\sigma_\infty^{\mathfrak{e}}]}^{0\to l_\infty^{\mathfrak{e}}},$$

  - *the Motzkin paths are linked through the relation*

$$\mathfrak{M}_\infty^{\bar{\mathfrak{e}}}(s) = \mathfrak{M}_\infty^{\mathfrak{e}}(\sigma_\infty^{\mathfrak{e}} - s) - l_\infty^{\mathfrak{e}}.$$

Applying Skorokhod's representation theorem, we may and will assume that this convergence holds almost surely. As a result, note that for n large enough, $\mathfrak{s}_n = \mathfrak{s}_\infty$.

[1]Remark that $\gamma\, n^{1/4} = \sqrt{\frac{2}{3}}\sqrt{\sqrt{2n}}$.

Decomposition of $\mathfrak{L}_{(n)}$ along the forests

In order to study the convergence of $\mathfrak{L}_{(n)}$, we will express it in terms of the $L^{\mathfrak{e}}_{(n)}$'s and $\mathfrak{M}^{\mathfrak{e}}_{(n)}$'s. First, the labels in the forest $(\mathfrak{f}^{\mathfrak{e}}_n, \mathfrak{l}^{\mathfrak{e}}_n)$ are to be shifted by the value of the Motzkin path $\mathfrak{M}^{\mathfrak{e}}_n$ at the time telling which subtree is visited: recall the definition (2.11) of the process

$$\mathfrak{L}^{\mathfrak{e}}_n := \Big(L^{\mathfrak{e}}_n(t) + \mathfrak{M}^{\mathfrak{e}}_n\big(\sigma^{\mathfrak{e}}_n - \underline{C}^{\mathfrak{e}}_n(t)\big)\Big)_{0\le t\le 2m^{\mathfrak{e}}_n+\sigma^{\mathfrak{e}}_n}.$$

We define its rescaled version

$$\mathfrak{L}^{\mathfrak{e}}_{(n)} := \left(\frac{\mathfrak{L}^{\mathfrak{e}}_n(2nt)}{\gamma\, n^{1/4}}\right)_{0\le t\le m^{\mathfrak{e}}_{(n)}} = \Big(L^{\mathfrak{e}}_{(n)}(t) + \mathfrak{M}^{\mathfrak{e}}_{(n)}\big(\sigma^{\mathfrak{e}}_{(n)} - \underline{C}^{\mathfrak{e}}_{(n)}(t)\big)\Big)_{0\le t\le m^{\mathfrak{e}}_{(n)}},$$

as well as its limit in the space $(\mathcal{K}, d_{\mathcal{K}})$,

$$\mathfrak{L}^{\mathfrak{e}}_{(n)} \xrightarrow[n\to\infty]{} \mathfrak{L}^{\mathfrak{e}}_{\infty} := \Big(L^{\mathfrak{e}}_{\infty}(t) + \mathfrak{M}^{\mathfrak{e}}_{\infty}\big(\sigma^{\mathfrak{e}}_{\infty} - \underline{C}^{\mathfrak{e}}_{\infty}(t)\big)\Big)_{0\le t\le m^{\mathfrak{e}}_{\infty}}.$$

We then need to concatenate these processes. For $f, g \in \mathcal{K}_0$ two functions started at 0, we call $f \bullet g \in \mathcal{K}_0$ their concatenation defined by $\sigma(f \bullet g) := \sigma(f) + \sigma(g)$ and, for $0 \le t \le \sigma(f \bullet g)$,

$$f \bullet g(t) := \begin{cases} f(t) & \text{if} \quad 0 \le t \le \sigma(f), \\ f(\sigma(f)) + g(t - \sigma(f)) & \text{if} \quad \sigma(f) \le t \le \sigma(f) + \sigma(g). \end{cases}$$

We sort the half-edges of $\mathfrak{s}_n$ according to its facial order, beginning with the root: $\mathfrak{e}_1 = \mathfrak{e}_*, \dots, \mathfrak{e}_{2(6g-3)}$ and we see that

$$\mathfrak{L}_{(n)} = \mathfrak{L}^{\mathfrak{e}_1}_{(n)} \bullet \mathfrak{L}^{\mathfrak{e}_2}_{(n)} \bullet \cdots \bullet \mathfrak{L}^{\mathfrak{e}_{2(6g-3)}}_{(n)}.$$

We also sort the half-edges of $\mathfrak{s}_\infty$ in the same way and define $\mathfrak{L}_\infty := \mathfrak{L}^{\mathfrak{e}_1}_{\infty} \bullet \mathfrak{L}^{\mathfrak{e}_2}_{\infty} \bullet \cdots \bullet \mathfrak{L}^{\mathfrak{e}_{2(6g-3)}}_{\infty}$.

Lemma 3.2. *The concatenation is continuous from $(\mathcal{K}_0, d_{\mathcal{K}})^2$ to $(\mathcal{K}_0, d_{\mathcal{K}})$.*

Proof. Let (f_n, g_n) be a sequence of functions in $\mathcal{K}_0^2$ converging toward $(f, g) \in \mathcal{K}_0^2$ and $\varepsilon > 0$. There exist an $0 < \eta < \varepsilon$ and an n_0 such that

$$|s - t| < \eta \Rightarrow |f \bullet g(s) - f \bullet g(t)| < \varepsilon \qquad \text{and} \qquad n \ge n_0 \Rightarrow d_{\mathcal{K}}(f_n, f) \vee d_{\mathcal{K}}(g_n, g) < \eta.$$

Let $0 \le t \le \sigma(f \bullet g) \wedge \sigma(f_n \bullet g_n)$ and $n \ge n_0$ be fixed. If $t \le \sigma(f_n)$, we call $\tilde{t} := t \wedge \sigma(f)$. In that case,

$$|f_n \bullet g_n(t) - f \bullet g(\tilde{t})| = |f_n(t) - f(t \wedge \sigma(f))| \le d_{\mathcal{K}}(f_n, f) < \varepsilon.$$

If $\sigma(f_n) < t$, we call $\tilde{t} := \sigma(f) + (t - \sigma(f_n)) \wedge \sigma(g)$ and we have

$$|f_n \bullet g_n(t) - f \bullet g(\tilde{t})| = |g_n((t - \sigma(f_n)) \wedge \sigma(g_n)) - g((t - \sigma(f_n)) \wedge \sigma(g))| \le d_{\mathcal{K}}(g_n, g) < \varepsilon.$$

In both cases, $|t - \tilde{t}| < \eta$, so that $|f \bullet g(\tilde{t}) - f \bullet g(t)| < \varepsilon$. Hence

$$d_{\mathcal{K}}(f_n \bullet g_n, f \bullet g) < |\sigma(f_n) - \sigma(f)| + |\sigma(g_n) - \sigma(g)| + 2\varepsilon < 4\varepsilon. \qquad \square$$

This ensures us that $\mathfrak{L}_{(n)}$ converges in $(\mathcal{K}, d_{\mathcal{K}})$ toward $\mathfrak{L}_\infty$, so that $\big(d^{\circ}_{(n)}(s,t)\big)_{0\le s,t\le 1}$ converges in $\big(\mathcal{C}([0,1]^2, \mathbb{R}), \|\cdot\|_\infty\big)$ toward $(d^{\circ}_{\infty}(s,t))_{0\le s,t\le 1}$ defined by

$$d^{\circ}_{\infty}(s,t) := \mathfrak{L}_\infty(s) + \mathfrak{L}_\infty(t) - 2\max\left(\min_{x\in\overrightarrow{[s,t]}} \mathfrak{L}_\infty(x),\ \min_{x\in\overrightarrow{[t,s]}} \mathfrak{L}_\infty(x)\right).$$

Tightness

Lemma 3.3. *The sequence of the laws of the processes*

$$\left(d_{(n)}(s,t)\right)_{0\le s,t\le 1}$$

is tight in the space of probability measure on $\mathcal{C}([0,1]^2,\mathbb{R})$.

Proof. First observe that, for every s, s', t, $t' \in [0,1]$,

$$\left|d_{(n)}(s,t)-d_{(n)}(s',t')\right| \le d_{(n)}(s,s')+d_{(n)}(t,t') \le d^\circ_{(n)}(s,s')+d^\circ_{(n)}(t,t').$$

By Fatou's lemma, we have for every $k\in\mathbb{N}$ and $\delta>0$,

$$\limsup_{n\to\infty}\mathbb{P}\left(\sup_{|s-s'|\le\delta} d^\circ_{(n)}(s,s')\ge 2^{-k}\right)\le \mathbb{P}\left(\sup_{|s-s'|\le\delta} d^\circ_{\infty}(s,s')\ge 2^{-k}\right).$$

Since d°_∞ is continuous and null on the diagonal, for $\varepsilon>0$, we may find $\delta_k>0$ such that, for n sufficiently large,

$$\mathbb{P}\left(\sup_{|s-s'|\le\delta_k} d^\circ_{(n)}(s,s')\ge 2^{-k}\right)\le 2^{-k}\varepsilon. \tag{3.6}$$

By taking δ_k even smaller if necessary, we may assume that the inequality (3.6) holds for all $n\ge 1$. Summing over $k\in\mathbb{N}$, we find that for every $n\ge 1$,

$$\mathbb{P}\left(d_{(n)}\in\mathcal{K}_\varepsilon\right)\ge 1-\varepsilon,$$

where

$$\mathcal{K}_\varepsilon := \left\{f\in\mathcal{C}([0,1]^2,\mathbb{R})\,:\,f(0,0)=0,\ \forall k\in\mathbb{N},\ \sup_{|s-s'|\wedge|t-t'|\le\delta_k}|f(s,t)-f(s',t')|\le 2^{1-k}\right\}$$

is a compact set. □

3.1.3 The genus g Brownian map

Proof of the first assertion of Theorem 1.11

Thanks to Lemma 3.3, there exist a subsequence $(n_k)_{k\ge0}$ and a function $d_\infty\in\mathcal{C}([0,1]^2,\mathbb{R})$ such that

$$\left(d_{(n_k)}(s,t)\right)_{0\le s,t\le1}\xrightarrow[k\to\infty]{(d)}\left(d_\infty(s,t)\right)_{0\le s,t\le1}. \tag{3.7}$$

By Skorokhod's representation theorem, we will assume that this convergence holds almost surely. As the $d_{(n)}$ functions, the function d_∞ obeys the triangle inequality. And because $d_{(n)}(s,s)=O(n^{-1/4})$ for all $s\in[0,1]$, the function d_∞ is actually a pseudo-metric. We define the equivalence relation associated with it by saying that $s\sim_\infty t$ if $d_\infty(s,t)=0$, and we call $\mathfrak{q}_\infty := [0,1]_{/\sim_\infty}$.

We will show the convergence claimed in Theorem 1.11 along the same subsequence $(n_k)_{k\ge0}$. Because of (3.3), we only need to see that

$$d_{GH}\left(\left((2n_k)^{-1}[\![0,2n_k]\!]_{/\sim_{n_k}},d_{(n_k)}\right),(\mathfrak{q}_\infty,d_\infty)\right)\xrightarrow[k\to\infty]{}0.$$

For that matter, we will use the characterization of the Gromov–Hausdorff distance via correspondences. Recall that a correspondence between two metric spaces $(\mathcal{X},\delta)$ and $(\mathcal{X}',\delta')$ is a subset $\mathcal{R}\subseteq\mathcal{X}\times\mathcal{X}'$ such that for all $x\in\mathcal{X}$, there is at least one $x'\in\mathcal{X}'$ for which $(x,x')\in\mathcal{R}$ and vice versa. The distortion of the correspondence $\mathcal{R}$ is defined by

$$\mathrm{dis}(\mathcal{R}) := \sup\left\{|\delta(x,y)-\delta(x',y')|\,:\,(x,x'),(y,y')\in\mathcal{R}\right\}.$$

Then we have [BBI01, Theorem 7.3.25]

$$d_{GH}(\mathcal{X},\mathcal{X}') = \frac{1}{2}\inf_{\mathcal{R}} \operatorname{dis}(\mathcal{R})$$

where the infimum is taken over all correspondences between $\mathcal{X}$ and $\mathcal{X}'$.

Recall that π_n denotes the canonical projection from $[\![0,2n]\!]$ to $[\![0,2n]\!]_{/\sim_n}$. For $t \in [0,1]$, we call $\mathfrak{q}_\infty(t)$ the equivalence class of t in $\mathfrak{q}_\infty$. We define the correspondence $\mathcal{R}_n$ between $\big((2n)^{-1}[\![0,2n]\!]_{/\sim_n}, d_{(n)}\big)$ and $(\mathfrak{q}_\infty, d_\infty)$ as the set

$$\mathcal{R}_n \coloneqq \left\{ \left((2n)^{-1}\,\pi_n(\lfloor 2nt\rfloor), \mathfrak{q}_\infty(t)\right),\, t\in[0,1]\right\}.$$

Its distortion is

$$\operatorname{dis}(\mathcal{R}_n) = \sup_{0\le s,t\le 1}\left| d_{(n)}\left(\frac{\lfloor 2ns\rfloor}{2n}, \frac{\lfloor 2nt\rfloor}{2n}\right) - d_\infty(s,t)\right|,$$

and, thanks to (3.7),

$$d_{GH}\left(\left((2n_k)^{-1}[\![0,2n_k]\!]_{/\sim_{n_k}}, d_{(n_k)}\right), (\mathfrak{q}_\infty, d_\infty)\right) \le \frac{1}{2}\operatorname{dis}(\mathcal{R}_{n_k}) \xrightarrow[k\to\infty]{} 0.$$

A bound on d_∞

If we take the limit of the inequality (3.4) along the subsequence $(n_k)_{k\ge 0}$, we find $d_\infty(s,t) \le d^\circ_\infty(s,t)$. Because d°_∞ does not satisfy the triangle inequality, we may improve this bound by considering the largest metric on $\mathfrak{q}_\infty$ that is smaller than d°_∞: for all a and $b\in\mathfrak{q}_\infty$, we have

$$d_\infty(a,b) \le d^*_\infty(a,b) \coloneqq \inf\left\{\sum_{i=0}^{k} d^\circ_\infty(s_i,t_i)\right\} \tag{3.8}$$

where the infimum is taken over all integer $k\ge 0$ and all sequences s_0, t_0, s_1, t_1,…, s_k, t_k satisfying $a=\mathfrak{q}_\infty(s_0)$, for all $0\le i\le k-1$, $t_i\sim_\infty s_{i+1}$, and $b=\mathfrak{q}_\infty(t_k)$.

We conjecture that this inequality is actually an equality. In the planar case, Miermont [Mie11] as well as Le Gall [LG11] very recently showed that a similar equality holds.

3.1.4 Hausdorff dimension of the genus g Brownian map

We now prove the second assertion of Theorem 1.11. We follow the method provided by Le Gall and Miermont [LGM11a]. As usual, we proceed in two steps.

Upper bound

Let $0<\alpha<\frac{1}{4}$. For every $\mathfrak{e}\in\vec{E}(\mathfrak{s}_\infty)$, Lemmas 2.10 and 2.15, together with (2.26), imply that $\mathfrak{L}^{\mathfrak{e}}_\infty$ is α-Hölder. The same goes for $\mathfrak{L}_\infty$ by finite concatenation. This yields that the canonical projection from $([0,1],|\cdot|)$ to $(\mathfrak{q}_\infty,d_\infty)$ is α-Hölder as well: for $0\le s,t\le 1$,

$$d_\infty(\mathfrak{q}_\infty(s),\mathfrak{q}_\infty(t)) = d_\infty(s,t) \le d^\circ_\infty(s,t) \le 2\|\mathfrak{L}_\infty\|_\alpha\,|s-t|^\alpha.$$

It follows that $\dim_{\mathcal{H}}(\mathfrak{q}_\infty,d_\infty)\le\frac{1}{\alpha}\dim_{\mathcal{H}}([0,1])$. Taking the infimum over $\alpha\in(0,1/4)$, we have

$$\dim_{\mathcal{H}}(\mathfrak{q}_\infty,d_\infty)\le 4.$$

Lower bound

We start with a lemma giving a lower bound on $d_\infty(s,t)$. Let us first define a contour function $\mathfrak{C}_n : [0,2n] \to \mathbb{R}_+$ for the g-tree $\mathfrak{t}_n$ by

$$\mathfrak{C}_n := \left(C_n^{\mathfrak{e}_1} - \sigma_n^{\mathfrak{e}_1}\right) \bullet \left(C_n^{\mathfrak{e}_1} - \sigma_n^{\mathfrak{e}_2}\right) \bullet \dots \bullet \left(C_n^{\mathfrak{e}_1} - \sigma_n^{\mathfrak{e}_{2(6g-3)}}\right) + \sum_{i=1}^{2(6g-3)} \sigma_n^{\mathfrak{e}_i}$$

where the half-edges $\mathfrak{e}_1 = \mathfrak{e}_*, \dots, \mathfrak{e}_{2(6g-3)}$ are sorted according to the facial order of $\mathfrak{s}_n$. This function is actually the contour function of the "large" forest consisting in the concatenation of $\mathfrak{f}_n^{\mathfrak{e}_1}, \mathfrak{f}_n^{\mathfrak{e}_2}, \dots, \mathfrak{f}_n^{\mathfrak{e}_{2(6g-3)}}$. As usual, we define its rescaled version $\mathfrak{C}_{(n)}$, as well as its limit

$$\mathfrak{C}_{(n)} \xrightarrow[n\to\infty]{} \mathfrak{C}_\infty := \left(C_\infty^{\mathfrak{e}_1} - \sigma_\infty^{\mathfrak{e}_1}\right) \bullet \left(C_\infty^{\mathfrak{e}_1} - \sigma_\infty^{\mathfrak{e}_2}\right) \bullet \dots \bullet \left(C_\infty^{\mathfrak{e}_1} - \sigma_\infty^{\mathfrak{e}_{2(6g-3)}}\right) + \sum_{i=1}^{2(6g-3)} \sigma_\infty^{\mathfrak{e}_i}$$

where, this time, the half-edges are sorted according to the facial order of $\mathfrak{s}_\infty$.

For $0 \le s,t \le 1$, we define the set

$$\mathscr{L}_\infty(s,t) := \left\{ s \wedge t \le x \le s \vee t \,:\, \underline{\mathfrak{C}}_\infty(x) = \underline{\mathfrak{C}}_\infty(s),\ \mathfrak{C}_\infty(x) = \inf_{[x\wedge s,\, x\vee s]} \mathfrak{C}_\infty \right\}.$$

It will become clearer in a moment what this set represents, while looking at its discrete analog.

Lemma 3.4. *The following bound holds*

$$d_\infty(s,t) \ge \mathfrak{L}_\infty(s) - \min_{\mathscr{L}_\infty(s,t)} \mathfrak{L}_\infty$$

Proof. This inequality follows easily by approximation, once we have shown its discrete analog:

$$d_n(i,j) \ge \mathfrak{L}_n(i) - \min_{\mathscr{L}_n(i,j)} \mathfrak{L}_n \tag{3.9}$$

where the set

$$\mathscr{L}_n(i,j) := \left\{ i \wedge j \le k \le i \vee j \,:\, \underline{\mathfrak{C}}_n(k) = \underline{\mathfrak{C}}_n(i),\ \mathfrak{C}_n(k) = \inf_{[k\wedge i,\, k\vee i]} \mathfrak{C}_n \right\}$$

represents the ancestral lineage of $\dot{\mathfrak{t}}_n(i)$ between i and j. An integer k belongs to $\mathscr{L}_n(i,j)$ if and only if k is between i and j (first constraint), $\dot{\mathfrak{t}}_n(k)$ lies in the same subtree as $\dot{\mathfrak{t}}_n(i)$ (second constraint), and $\dot{\mathfrak{t}}_n(k)$ is an ancestor of $\dot{\mathfrak{t}}_n(i)$ (third constraint). Beware that $\mathscr{L}_n(j,i)$ is in general a totally different set.

We can suppose $i \ne j$. In order to show (3.9), we consider a geodesic path $\wp(0), \wp(1), \dots, \wp(d_n(i,j))$ from $\dot{\mathfrak{t}}_n(i)$ to $\dot{\mathfrak{t}}_n(j)$ and call $k \in \mathscr{L}_n(i,j)$ an integer for which $\mathfrak{L}_n(k) = \min_{\mathscr{L}_n(i,j)} \mathfrak{L}_n$. Removing the edges incident to $\dot{\mathfrak{t}}_n(k)$ breaks $\mathfrak{t}_n$ into some connected components. One of these components contains $\dot{\mathfrak{t}}_n(i)$ and another one contains $\dot{\mathfrak{t}}_n(j)$. Say that $\wp(r)$, $r < d_n(i,j)$ is the last vertex of the geodesic path lying in the same component as $\dot{\mathfrak{t}}_n(i)$. Then $\wp(r)$ is linked by an edge of $\mathfrak{q}_n$ to $\wp(r+1)$, which lies in another component. Moreover, the facial sequence of $\mathfrak{t}_n$ must visit $\dot{\mathfrak{t}}_n(k)$ between any time it visits $\wp(r)$ and any time it visits $\wp(r+1)$ (in that order or the other). The way we construct edges in the Chapuy–Marcus–Schaeffer bijection thus imposes $\mathfrak{l}_n(\dot{\mathfrak{t}}_n(k)) \ge \mathfrak{l}_n(\wp(r)) \vee \mathfrak{l}_n(\wp(r+1))$. Finally,

$$d_n(i,j) \ge d_{\mathfrak{q}_n}(\dot{\mathfrak{q}}_n(i), \wp(r)) \ge d_{\mathfrak{q}_n}(\dot{\mathfrak{q}}_n(i), v_n^\bullet) - d_{\mathfrak{q}_n}(v_n^\bullet, \wp(r)) = \mathfrak{l}_n(\dot{\mathfrak{t}}_n(i)) - \mathfrak{l}_n(\wp(r)),$$

and the same holds with $r+1$ instead of r, yielding

$$d_n(i,j) \ge \mathfrak{l}_n(\dot{\mathfrak{t}}_n(i)) - \mathfrak{l}_n(\dot{\mathfrak{t}}_n(k)) = \mathfrak{L}_n(i) - \min_{\mathscr{L}_n(i,j)} \mathfrak{L}_n. \qquad \square$$

Let us define the measure λ on $\mathfrak{q}_\infty$ as the image of the Lebesgue measure on $[0,1]$ by the canonical projection from $[0,1]$ to $\mathfrak{q}_\infty$. From now on, we work conditionally given the parameters vector $\mathfrak{I}_\infty$. Let $0 \le s \le 1$ be a point that is not of the form $\sum_{i=1}^{k} m_\infty^{\mathfrak{e}_i}$ for some $k = 0, \dots, 2(6g-3)$. This means that it is not 0, 1, or a point at which two functions are being concatenated. Such points will thereafter be called *junction points*.

Suppose that for some $\delta > 0$, we can find two positive numbers r_- and r_+ such that

$$\mathfrak{L}_\infty(s) - \min_{\mathscr{L}_\infty(s,s-r_-)} \mathfrak{L}_\infty > \delta \qquad \text{and} \qquad \mathfrak{L}_\infty(s) - \min_{\mathscr{L}_\infty(s,s+r_+)} \mathfrak{L}_\infty > \delta. \tag{3.10}$$

For $a \in \mathfrak{q}_\infty$ and $r > 0$, we call $B_\infty(a,r)$ the open ball centered at a with radius r for the metric d_∞. Using Lemma 3.4 and the elementary fact that $\mathscr{L}_\infty(s,t) \subseteq \mathscr{L}_\infty(s,t')$ as soon as $|t-s| \le |t'-s|$, we find that $B_\infty(\mathfrak{q}_\infty(s),\delta) \subseteq \mathfrak{q}_\infty\big((s-r_-, s+r_+)\big)$. As a result, we would have $\lambda(B_\infty(\mathfrak{q}_\infty(s),\delta)) \le r_- + r_+$.

For all $0 \le x \le \mathfrak{C}_\infty(s) - \underline{\mathfrak{C}}_\infty(s)$, we define

$$\tau_x \coloneqq \inf\{r \ge s,\, \mathfrak{C}_\infty(r) = \mathfrak{C}_\infty(s) - x\}$$

and we see that $\mathscr{L}_\infty(s,\tau_x) = \{\tau_y,\, 0 \le y \le x\}$. The discussion preceding Section 2.4.3 shows that the process

$$\Big(\mathfrak{L}_\infty(\tau_x) - \mathfrak{L}_\infty(s)\Big)_{0 \le x \le \mathfrak{C}_\infty(s) - \underline{\mathfrak{C}}_\infty(s)}$$

has the law of a real Brownian motion started from 0. Let $\eta > 0$. Almost surely, provided that $\mathfrak{C}_\infty(s) - \underline{\mathfrak{C}}_\infty(s) > 0$, the law of the iterated logarithm ensures us that for x small enough,

$$\inf_{0 \le y \le x} (\mathfrak{L}_\infty(\tau_y) - \mathfrak{L}_\infty(s)) < -x^{\frac{1}{2}+\eta},$$

so that

$$\mathfrak{L}_\infty(s) - \min_{\mathscr{L}_\infty(s,\tau_x)} \mathfrak{L}_\infty = \mathfrak{L}_\infty(s) - \inf_{0 \le y \le x} \mathfrak{L}_\infty(\tau_y) > x^{\frac{1}{2}+\eta}.$$

We choose $\delta = x^{\frac{1}{2}+\eta}$ and $r_+ = \tau_x - s$ so that the second part of (3.10) holds. Moreover, because s is not a junction point, on one of its neighborhoods, the function $\mathfrak{C}_\infty$ is a first-passage Brownian bridge, and is then absolutely continuous with respect to the Wiener measure on this neighborhood. It therefore obeys the law of the iterated logarithm as well. So, a.s., for r small enough,

$$\inf_{0 \le t \le r} (\mathfrak{C}_\infty(s+t) - \mathfrak{C}_\infty(s)) < -r^{\frac{1}{2}+\eta}.$$

It follows that $r_+ \le x^{(\frac{1}{2}+\eta)^{-1}} = \delta^{(\frac{1}{2}+\eta)^{-2}} = \delta^{4-\eta'}$ for some $\eta' > 0$. In a similar way, we can find an $r_- < \delta^{4-\eta'}$ satisfying the first part of (3.10). This yields, for all $\delta > 0$ small enough,

$$\lambda(B_\infty(\mathfrak{q}_\infty(s),\delta)) \le 2\delta^{4-\eta'},$$

which implies that, for all $\eta' > 0$,

$$\limsup_{\delta \to 0} \frac{\lambda(B_\infty(\mathfrak{q}_\infty(s),\delta))}{\delta^{4-\eta'}} \le 2. \tag{3.11}$$

Once again, because $\mathfrak{C}_\infty$ is absolutely continuous with respect to the Wiener measure on a neighborhood of s, a.s. $\mathfrak{C}_\infty(s) - \underline{\mathfrak{C}}_\infty(s) > 0$. For the record, note that if s was a junction point, we would always have $\mathfrak{C}_\infty(s) = \underline{\mathfrak{C}}_\infty(s)$ by definition of a first-passage bridge. We obtain that for every s that is not a junction point, (3.11) holds almost surely. Finally, as there are only $2(6g-3)+1$ junction points, Fubini-Tonelli's theorem shows that a.s., for λ-almost every a,

$$\limsup_{\delta \to 0} \frac{\lambda(B_\infty(a,\delta))}{\delta^{4-\eta'}} \le 2.$$

We then conclude that $\dim_{\mathcal{H}}(\mathfrak{q}_\infty, d_\infty) \ge 4-\eta'$ for all $\eta' > 0$ by standard density theorems for Hausdorff measures ([Fed69, Theorem 2.10.19]).

3.2 An expression of the constant t_g

This section is dedicated to the proof of Theorem 1.12. Recall that the constant t_g is defined by: $|\mathcal{Q}_n| \sim t_g\, n^{\frac{5}{2}(g-1)}\, 12^n$. The relation (2.1) gives that $|\mathcal{T}_n| \sim \frac{1}{2}\, t_g\, n^{\frac{5g-3}{2}}\, 12^n$, so that, thanks to (2.17),

$$t_g = 2^{\frac{3g+1}{2}}\, 3^g\, \Upsilon$$

where Υ was defined by (2.14). For a given $\mathfrak{s} \in \mathfrak{S}^*$, we will concentrate on

$$\int_{\mathcal{S}^{\mathfrak{s}}} d\mathcal{L}^{\mathfrak{s}}\, \mathbb{1}_{\{m^{\mathfrak{e}_*} \geq 0,\, u < m^{\mathfrak{e}_*}\}} \prod_{\mathfrak{e} \in \vec{E}(\mathfrak{s})} -p'_{m^{\mathfrak{e}}}\left(\sigma^{\mathfrak{e}}\right) \prod_{\mathfrak{e} \in \check{E}(\mathfrak{s})} p_{\sigma^{\mathfrak{e}}}\left(l^{\mathfrak{e}}\right). \tag{3.12}$$

First, notice that by integrating with respect to u, only a factor $m^{\mathfrak{e}_*}$ appears.

3.2.1 Integrating with respect to $(m^{\mathfrak{e}})_{\mathfrak{e} \in \vec{E}(\mathfrak{s}) \setminus \{\mathfrak{e}_*\}}$

For $\mathfrak{e} \neq \mathfrak{e}_*$, $m^{\mathfrak{e}}$ is only present in the factor

$$m^{\mathfrak{e}_*}\left(-p'_{m^{\mathfrak{e}_*}}(\sigma^{\mathfrak{e}_*})\right)\left(-p'_{m^{\mathfrak{e}}}(\sigma^{\mathfrak{e}})\right) = \sigma^{\mathfrak{e}_*}\, p_{m^{\mathfrak{e}_*}}(\sigma^{\mathfrak{e}_*})\left(-p'_{m^{\mathfrak{e}}}(\sigma^{\mathfrak{e}})\right), \tag{3.13}$$

so we have to compute an integral of the form given in the following lemma:

Lemma 3.5. *Let a, b, and t be three positive numbers. Then*

$$\int_0^t p_{t-m}(a)\left(-p'_m(b)\right) dm = p_t(a+b).$$

Proof. Let us call $f_t(a,b)$ the integral we have to compute, that is

$$f_t(a,b) = \frac{b}{2\pi} \int_0^t (t-m)^{-\frac{1}{2}}\, m^{-\frac{3}{2}}\, e^{-\frac{1}{2}\left(\frac{a^2}{t-m} + \frac{b^2}{m}\right)} dm.$$

By doing the change of variable $m \mapsto \frac{m}{t-m}$, we find

$$f_t(a,b) = \frac{b}{2\pi t} \int_0^\infty x^{-\frac{3}{2}}\, e^{-\frac{1}{2t}\left(a^2(1+x) + b^2\left(1+\frac{1}{x}\right)\right)} dx$$

The change of variable $x \mapsto \frac{a^2}{b^2}\, x$ in this integral yields the identity

$$f_t(a,b) = f_t(b,a). \tag{3.14}$$

When differentiating with respect to a, a factor $-\frac{a}{t}(1+x)$ appears inside the integral. We may split it into two terms, the first one being merely $-\frac{a}{t} f_t(a,b)$ and the second one being

$$-\frac{b}{t}\frac{a}{2\pi t} \int_0^\infty x^{-\frac{1}{2}}\, e^{-\frac{1}{2t}\left(a^2(1+x) + b^2\left(1+\frac{1}{x}\right)\right)} dx = -\frac{b}{t} f_t(b,a) = -\frac{b}{t} f_t(a,b),$$

thanks to the change of variable $x \mapsto \frac{1}{x}$. All in all, we obtain

$$\partial_a f_t(a,b) = -\frac{a+b}{t} f_t(a,b),$$

so that there exists a function g_t satisfying

$$f_t(a,b) = e^{-\frac{1}{2t}(a+b)^2}\, g_t(b).$$

Because of (3.14), the function g_t is actually constant and

$$g_t(b) = e^{\frac{1}{2t}}\, f_t(0,1) = \frac{1}{2\pi t} \int_0^\infty x^{-\frac{3}{2}}\, e^{-\frac{1}{2tx}} dx = \frac{1}{2\pi t} \int_0^\infty e^{-\frac{1}{2t} y^2} dy = \frac{1}{\sqrt{2\pi t}}.$$

Putting all this together, we obtain the result. □

The first time we integrate with respect to an $m^{\mathfrak{e}}$, for an $\mathfrak{e} \neq \mathfrak{e}_*$, we apply Lemma 3.5 with $a = \sigma^{\mathfrak{e}_*}$, $b = \sigma^{\mathfrak{e}}$ and $t = m^{\mathfrak{e}_*} + m^{\mathfrak{e}}$ (t does not depend on $m^{\mathfrak{e}}$) and the factor (3.13) is changed into

$$\sigma^{\mathfrak{e}_*}\, p_{m^{\mathfrak{e}_*}+m^{\mathfrak{e}}}(\sigma^{\mathfrak{e}_*} + \sigma^{\mathfrak{e}}).$$

We may then apply Lemma 3.5 again, with $a = \sigma^{\mathfrak{e}_*} + \sigma^{\mathfrak{e}}$, $b = \sigma^{\mathfrak{e}'}$ and $t = m^{\mathfrak{e}_*} + m^{\mathfrak{e}} + m^{\mathfrak{e}'}$ when integrating with respect to $m_{\mathfrak{e}'}$ and so on. In the end, after integrating with respect to u and $(m^{\mathfrak{e}})_{\mathfrak{e}\neq\mathfrak{e}_*}$, the

$$\mathbb{1}_{\{m^{\mathfrak{e}_*}\geq 0,\, u<m^{\mathfrak{e}_*}\}} \prod_{\mathfrak{e}\in\vec{E}(\mathfrak{s})} -p'_{m^{\mathfrak{e}}}(\sigma^{\mathfrak{e}})$$

part in the integrand of (3.12) merely becomes

$$\sigma^{\mathfrak{e}_*}\, p_1\left(\textstyle\sum_{\mathfrak{e}\in\vec{E}(\mathfrak{s})} \sigma^{\mathfrak{e}}\right) = \sigma^{\mathfrak{e}_*}\, p\left(2\textstyle\sum_{\mathfrak{e}\in\check{E}(\mathfrak{s})} \sigma^{\mathfrak{e}}\right) = \frac{\sigma^{\mathfrak{e}_*}}{\sqrt{2\pi}}\, e^{-2\left(\sum_{\mathfrak{e}\in\check{E}(\mathfrak{s})}\sigma^{\mathfrak{e}}\right)^2}.$$

3.2.2 Integrating with respect to $(\sigma^{\mathfrak{e}})_{\mathfrak{e}\in\check{E}(\mathfrak{s})}$

We call $s = \sum_{\mathfrak{e}\in\check{E}(\mathfrak{s})} \sigma^{\mathfrak{e}}$. In order to integrate with respect to $(\sigma^{\mathfrak{e}})_{\mathfrak{e}\in\check{E}(\mathfrak{s})}$, we will integrate with respect to s and with respect to $(\sigma^{\mathfrak{e}})_{\mathfrak{e}\in\check{E}(\mathfrak{s})}$ on the simplex, precisely,

$$d(\sigma^{\mathfrak{e}})_{\mathfrak{e}\in\check{E}(\mathfrak{s})} = ds\, \mathbb{1}_{\{\sigma^{\mathfrak{e}_*}>0\}}\, d(\sigma^{\mathfrak{e}})_{\mathfrak{e}\in\check{E}(\mathfrak{s})\setminus\{\mathfrak{e}_*\}},$$

where $\sigma^{\mathfrak{e}_*} = s - \sum_{\mathfrak{e}\in\check{E}(\mathfrak{s})\setminus\{\mathfrak{e}_*\}} \sigma^{\mathfrak{e}}$.

We then do the changes of variables $\sigma^{\mathfrak{e}} \mapsto s\,\sigma^{\mathfrak{e}}$ and $l^v \mapsto \sqrt{s}\, l^v$ for all $\mathfrak{e} \neq \mathfrak{e}_*$ and $v \neq \mathfrak{e}_*^-$, so that $\sigma^{\mathfrak{e}_*}$ becomes $1 - \sum_{\mathfrak{e}\in\check{E}(\mathfrak{s})\setminus\{\mathfrak{e}*\}} \sigma^{\mathfrak{e}}$ and the integral (3.12) becomes

$$\int d(l^v)\, \frac{1}{\sqrt{2\pi}} \int_0^\infty ds\, s^{5g-3}\, e^{-2s^2} \int d(\sigma^{\mathfrak{e}})_{\mathfrak{e}\neq\mathfrak{e}_*} \mathbb{1}_{\{\sigma^{\mathfrak{e}_*}>0\}}\, \sigma^{\mathfrak{e}_*} \prod_{\mathfrak{e}\in\check{E}(\mathfrak{s})} p_{\sigma^{\mathfrak{e}}}(l^{\mathfrak{e}}).$$

The first part is easily enough dealt with,

$$\int_0^\infty ds\, s^{5g-3}\, e^{-2s^2} = 2^{-\frac{5g}{2}}\, \Gamma\left(\frac{5g}{2} - 1\right).$$

We then focus on

$$\int d(\sigma^{\mathfrak{e}})_{\mathfrak{e}\neq\mathfrak{e}_*} \mathbb{1}_{\{\sigma^{\mathfrak{e}_*}>0\}}\, \sigma^{\mathfrak{e}_*} \prod_{\mathfrak{e}\in\check{E}(\mathfrak{s})} p_{\sigma^{\mathfrak{e}}}(l^{\mathfrak{e}}) = \varphi\left((|l^{\mathfrak{e}}|)_{\mathfrak{e}\in\check{E}(\mathfrak{s})}\right),$$

where the function φ is defined, for $x^{\mathfrak{e}} > 0$, $\mathfrak{e} \in \check{E}(\mathfrak{s})$ by

$$\varphi\left((x^{\mathfrak{e}})_{\mathfrak{e}\in\check{E}(\mathfrak{s})}\right) := \int d(\sigma^{\mathfrak{e}})_{\mathfrak{e}\neq\mathfrak{e}_*} \mathbb{1}_{\{\sigma^{\mathfrak{e}_*}>0\}}\, \sigma^{\mathfrak{e}_*} \prod_{\mathfrak{e}\in\check{E}(\mathfrak{s})} p_{\sigma^{\mathfrak{e}}}(x^{\mathfrak{e}}).$$

If we differentiate this function φ with respect to every variables $x^{\mathfrak{e}}$, we recognize the same integral we treated while integrating with respect to $(m^{\mathfrak{e}})$,

$$\begin{aligned}\prod_{\mathfrak{e}\in\check{E}(\mathfrak{s})} (-\partial_{x^{\mathfrak{e}}})\, \varphi\left((x^{\mathfrak{e}})_{\mathfrak{e}\in\check{E}(\mathfrak{s})}\right) &= \int d(\sigma^{\mathfrak{e}})_{\mathfrak{e}\neq\mathfrak{e}_*} \mathbb{1}_{\{\sigma^{\mathfrak{e}_*}>0\}}\, \sigma^{\mathfrak{e}_*} \prod_{\mathfrak{e}\in\check{E}(\mathfrak{s})} (-p'_{\sigma^{\mathfrak{e}}}(x^{\mathfrak{e}}))\\ &= x^{\mathfrak{e}_*}\, p_1\left(\textstyle\sum_{\mathfrak{e}\in\check{E}(\mathfrak{s})} x^{\mathfrak{e}}\right).\end{aligned}$$

Integrating back, we obtain

$$\varphi\left((x^{\mathfrak{e}})_{\mathfrak{e}\in\check{E}(\mathfrak{s})}\right) = p^{[6g-1]}\left(\textstyle\sum_{\mathfrak{e}\in\check{E}(\mathfrak{s})} x^{\mathfrak{e}}\right) + x^{\mathfrak{e}_*}\, p^{[6g-2]}\left(\textstyle\sum_{\mathfrak{e}\in\check{E}(\mathfrak{s})} x^{\mathfrak{e}}\right),$$

where, for all $n \geq 1$, the functions $p^{[n]}$ are defined by

$$p^{[n]}(y) := \int_y^\infty dy_{n-1} \int_{y_{n-1}}^\infty dy_{n-2} \dots \int_{y_2}^\infty dy_1\, p_1(y_1). \tag{3.15}$$

The integral (3.12) is now equal to some constant times

$$\int d(l^v)_{v\neq \mathfrak{e}_*^-}\, p^{[6g-1]}\left(\textstyle\sum_{\mathfrak{e}\in \check{E}(\mathfrak{s})} |l^{\mathfrak{e}}|\right) + |l^{\mathfrak{e}_*}|\, p^{[6g-2]}\left(\textstyle\sum_{\mathfrak{e}\in \check{E}(\mathfrak{s})} |l^{\mathfrak{e}}|\right). \tag{3.16}$$

3.2.3 Integrating with respect to $(l^v)_{v\in V(\mathfrak{s})\setminus\{\mathfrak{e}_*^-\}}$

We follow here the ideas of [CMS09]. The term $\sum_{\mathfrak{e}\in \check{E}(\mathfrak{s})} |l^{\mathfrak{e}}|$ is a linear combination of l^v's. We will break the integral (3.16) into parts on which these coefficients are constant. This happens when the vertex labels are sorted according to a given ordering: we call $\mathcal{O}_{\mathfrak{s}}$ the set of bijections from $[\![0, 4g-3]\!]$ into $V(\mathfrak{s})$.

Let $\lambda \in \mathcal{O}_{\mathfrak{s}}$ be an ordering and $v \in V(\mathfrak{s})$. Because $\mathfrak{s}$ is dominant, v is connected to exactly three other vertices—not necessarily distinct—that we call v', v'', and v'''. When the labels are sorted according to λ, that is, when $l^{\lambda_0} < l^{\lambda_1} < \dots < l^{\lambda_{4g-3}}$, the coefficient of l^v in the sum $\sum_{\mathfrak{e}\in \check{E}(\mathfrak{s})} |l^{\mathfrak{e}}|$ is

$$c(\lambda, v) := 2\left(\mathbb{1}_{\{\lambda_{v'}^{-1} < \lambda_v^{-1}\}} + \mathbb{1}_{\{\lambda_{v''}^{-1} < \lambda_v^{-1}\}} + \mathbb{1}_{\{\lambda_{v'''}^{-1} < \lambda_v^{-1}\}}\right) - 3.$$

For $0 \leq k \leq 4g-3$, we let

$$d(\lambda, k) := \sum_{i=k}^{4g-3} c(\lambda, \lambda_i).$$

Let $\mathfrak{e} \in \check{E}(\mathfrak{s})$ be a half-edge and i (resp. j) be the smaller (resp. larger) of $\lambda_{\mathfrak{e}^-}^{-1}$ and $\lambda_{\mathfrak{e}^+}^{-1}$. Then $|l^{\mathfrak{e}}| = l^{\lambda_j} - l^{\lambda_i}$ and $\mathfrak{e}$ will contribute to the sum by a factor $+1$ for l^{λ_j} and -1 for l^{λ_i}. So $\mathfrak{e}$ will contribute to $d(\lambda, k)$ by a factor $+1$ for $k \leq j$ plus a factor -1 for $k \leq i$. Thus the definition we just gave for $d(\lambda, k)$ is consistent with (1.4). This, by the way, also prove that $d(\lambda, k) > 0$ for $k \neq 0$.

We have

$$\sum_{\mathfrak{e}\in \check{E}(\mathfrak{s})} |l^{\mathfrak{e}}| = \sum_{v\in V(\mathfrak{s})} c(\lambda, v)\, l^v = \sum_{i=1}^{4g-3} d(\lambda, i)\left(l^{\lambda_i} - l^{\lambda_{i-1}}\right).$$

Let us call $k = \lambda_{\mathfrak{e}_*^-}^{-1}$. We will write $\mathbb{1}_\lambda := \mathbb{1}_{\{l^{\lambda_0} < l^{\lambda_1} < \dots < l^{\lambda_{4g-3}}\}}$ for short. We integrate

$$\mathbb{1}_\lambda\, p^{[6g-1]}\left(\sum_{i=1}^{4g-3} d(\lambda, i)\left(l^{\lambda_i} - l^{\lambda_{i-1}}\right)\right)$$

with respect to $l^{\lambda_{4g-3}}$, then $l^{\lambda_{4g-4}}$, and so on up to $l^{\lambda_{k+1}}$. We then integrate with respect to l^{λ_0}, l^{λ_1}, …, $l^{\lambda_{k-1}}$. By doing so, factors $(d(\lambda, 4g-3))^{-1}$, $(d(\lambda, 4g-4))^{-1}$, …, $(d(\lambda, k+1))^{-1}$ then $(d(\lambda, 1))^{-1}$, $(d(\lambda, 2))^{-1}$, …, $(d(\lambda, k))^{-1}$ successively appear and every time we integrate, $p^{[n]}$ is changed into $p^{[n+1]}$. All in all,

$$\int d(l^v)_{v\neq \mathfrak{e}_*^-}\, \mathbb{1}_\lambda\, p^{[6g-1]}\left(\textstyle\sum_{\mathfrak{e}\in \check{E}(\mathfrak{s})} |l^{\mathfrak{e}}|\right) = p^{[10g-4]}(0) \prod_{i=1}^{4g-3} \frac{1}{d(\lambda, i)}.$$

The second part of (3.16) is a little bit trickier because it distinguishes the root from the other vertices. In order to circumvent this, we will consider the sum over all scheme with the same "unrooted" structure (we do not consider an ordering λ at this time). For any scheme $s \in \mathfrak{S}$, we note $\vec{\mathfrak{s}}$ the non-rooted scheme corresponding to $\mathfrak{s}$, and for any non-rooted scheme $\mathfrak{u}$, we note $\mathfrak{u}_{\mathfrak{e}}$ the scheme $\mathfrak{u}$ rooted at the half-edge $\mathfrak{e}$.

Let $\mathfrak{u}$ be a non-rooted scheme. We look at $\sum_{\mathfrak{s},\,\vec{\mathfrak{s}}=\mathfrak{u}} \psi(\mathfrak{s})$ where

$$\psi(\mathfrak{s}) := \int d(l^v)_{v\neq \mathfrak{e}_*^-}\, |l^{\mathfrak{e}_*}|\, p^{[6g-2]}\left(\sum\nolimits_{\mathfrak{e}\in\check{E}(\mathfrak{s})} |l^{\mathfrak{e}}|\right).$$

This is

$$\begin{aligned}\sum_{\mathfrak{s},\,\vec{\mathfrak{s}}=\mathfrak{u}} \psi(\mathfrak{s}) &= \frac{1}{\mathrm{Aut}(\mathfrak{u})} \sum_{\mathfrak{e}\in\vec{E}(\mathfrak{u})} \psi(\mathfrak{u}_{\mathfrak{e}})\\ &= \frac{1}{|\{\mathfrak{s},\,\vec{\mathfrak{s}}=\mathfrak{u}\}|} \sum_{\mathfrak{s},\,\vec{\mathfrak{s}}=\mathfrak{u}} \frac{1}{\mathrm{Aut}(\mathfrak{u})} \sum_{\mathfrak{e}\in\vec{E}(\mathfrak{u})} \psi(\mathfrak{u}_{\mathfrak{e}})\\ &= \sum_{\mathfrak{s},\,\vec{\mathfrak{s}}=\mathfrak{u}} \frac{1}{6g-3} \sum_{\mathfrak{e}\in\vec{E}(\mathfrak{u})} \psi(\mathfrak{u}_{\mathfrak{e}}).\end{aligned}$$

We chose the convention to fix $l^{\mathfrak{e}_*^-}$ to be 0 because we needed one of the l^v's to be 0 and $\mathfrak{e}_*^-$ was already distinguished as the root. This choice was totally arbitrary and we could have taken any other vertex v_0. This translates in the fact that, for any function χ,

$$\int d(l^v)_{v\neq \mathfrak{e}_*^-}\, \chi\left((l^{\mathfrak{e}})_{\mathfrak{e}\in\check{E}(\mathfrak{s})}\right)$$

does not actually depend on $\mathfrak{e}_*^-$. In order to see this properly, we do the following change of variables:

$$\text{for every } v\notin\{v_0,\mathfrak{e}_*^-\},\ \tilde{l}^v := l^v - l^{v_0},\qquad \tilde{l}^{\mathfrak{e}_*^-} := -l^{v_0}\qquad\text{and}\qquad \tilde{l}^{v_0} := 0,$$

so that $\tilde{l}^{\mathfrak{e}} = l^{\mathfrak{e}}$; and

$$\int d(l^v)_{v\neq \mathfrak{e}_*^-}\, \chi\left((l^{\mathfrak{e}})_{\mathfrak{e}\in\check{E}(\mathfrak{s})}\right) = \int d(l^v)_{v\neq v_0}\, \chi\left((l^{\mathfrak{e}})_{\mathfrak{e}\in\check{E}(\mathfrak{s})}\right).$$

Using this fact, we see that

$$\psi(\mathfrak{u}_{\mathfrak{e}}) = \int d(l^v)_{v\neq \mathfrak{e}^-}\, |l^{\mathfrak{e}}|\, p^{[6g-2]}\left(\sum\nolimits_{\mathfrak{e}'\in\check{E}(\mathfrak{s})} |l^{\mathfrak{e}'}|\right) = \int d(l^v)_{v\neq \mathfrak{e}_*^-}\, |l^{\mathfrak{e}}|\, p^{[6g-2]}\left(\sum\nolimits_{\mathfrak{e}'\in\check{E}(\mathfrak{s})} |l^{\mathfrak{e}'}|\right),$$

and

$$\sum_{\mathfrak{s},\,\vec{\mathfrak{s}}=\mathfrak{u}} \psi(\mathfrak{s}) = \sum_{\mathfrak{s},\,\vec{\mathfrak{s}}=\mathfrak{u}} \frac{1}{6g-3} \int d(l^v)_{v\neq \mathfrak{e}_*^-} \left(\sum\nolimits_{\mathfrak{e}\in\check{E}(\mathfrak{s})} |l^{\mathfrak{e}}|\right) p^{[6g-2]}\left(\sum\nolimits_{\mathfrak{e}\in\check{E}(\mathfrak{s})} |l^{\mathfrak{e}}|\right).$$

We now consider an ordering $\lambda\in\mathcal{O}_{\mathfrak{s}}$. A computation very similar to the one we conducted above (just change $p^{[6g-1]}$ into $x\mapsto x\,p^{[6g-2]}(x)$, which becomes, after $4g-3$ successive integrations, $x\mapsto x\,p^{[10g-5]}(x) + (4g-3)p^{[10g-4]}(x)$) yields

$$\frac{1}{6g-3}\int d(l^v)_{v\neq \mathfrak{e}_*^-}\, \mathbb{1}_{\lambda} \sum_{\mathfrak{e}\in\check{E}(\mathfrak{s})} |l^{\mathfrak{e}}|\, p^{[6g-2]}\left(\sum\nolimits_{\mathfrak{e}\in\check{E}(\mathfrak{s})} |l^{\mathfrak{e}}|\right) = \frac{4g-3}{6g-3}\, p^{[10g-4]}(0) \prod_{i=1}^{4g-3} \frac{1}{d(\lambda,i)}.$$

The sum over all dominant schemes of (3.16) then becomes

$$\frac{2(5g-3)}{6g-3}\, p^{[10g-4]}(0) \sum_{\mathfrak{s}\in\mathfrak{S}} \sum_{\lambda\in\mathcal{O}_{\mathfrak{s}}} \prod_{i=1}^{4g-3} \frac{1}{d(\lambda,i)}.$$

3.2.4 Conclusion

We still have to compute $p^{[10g-4]}(0)$. For that matter, we may use Fubini-Tonnelli's theorem and rewrite (3.15), for $n \geq 4$, as

$$\begin{aligned} p^{[n]}(0) &= \int_0^\infty dy_1 \int_0^{y_1} dy_2 \ldots \int_0^{y_{n-2}} dy_{n-1}\, p_1(y_1) = \int_0^\infty dy_1\, \frac{y_1^{n-2}}{(n-2)!}\, p_1(y_1) \\ &= \frac{1}{n-2} \int_0^\infty dy_1\, \frac{y_1^{n-4}}{(n-4)!}\, p_1(y_1) = \frac{1}{n-2}\, p^{[n-2]}(0), \end{aligned}$$

where the second line is obtained from an integration by parts (we differentiate $y \mapsto y^{n-3}$ and integrate $y \mapsto y\, p_1(y)$). As $p^{[2]}(0) = \frac{1}{2}$, we find that

$$p^{[10g-4]}(0) = \left(2^{5g-2}(5g-3)!\right)^{-1}.$$

Taking into account everything we have done so far, we find

$$t_g = \frac{1}{\sqrt{\pi}} \frac{3^g\, \Gamma\left(\frac{5g}{2}-1\right)}{2^{6g-3}\,(6g-3)\,(5g-2)!} \sum_{\mathfrak{s} \in \mathfrak{S}^*} \sum_{\lambda \in \mathcal{O}_{\mathfrak{s}}} \prod_{i=1}^{4g-3} \frac{1}{d(\lambda, i)}.$$

The expression we claimed in (1.3) is then obtained by using the identity

$$\Gamma\left(\frac{5g}{2}-1\right)\Gamma\left(\frac{5g-3}{2}\right) = \frac{(5g-2)!}{2^{5g-4}}\sqrt{\pi}.$$

4 The topology of the scaling limit of positive genus random quadrangulations

The object of this chapter is Theorem 1.13. In the general picture, we rely on the same techniques as in the planar case. The study of Chapter 3 leads to the construction of a continuum random g-tree, which generalizes Aldous's CRT [Ald91, Ald93]. The first step of our proof is to carry out the analysis of Le Gall [LG07] in the non-planar case and see the space $(\mathfrak{q}_\infty, d_\infty)$ as a quotient of this continuum random g-tree via an equivalence relation defined in terms of Brownian labels on it. We then adapt Miermont's approach [Mie08], and use the notion of 1-regularity introduced by Whyburn [Why35b] and studied by Whyburn and Begle [Beg44, Why35b] in order to see that the genus remains the same in the limit. Finally, we deduce in Section 4.4 the technical estimates we need from the planar case thanks to a bijection due to Chapuy [Cha10] between well-labeled g-trees and well-labeled plane trees with g distinguished triples of vertices.

4.1 Real g-trees

In the discrete setting, it is sometimes convenient to work directly with the space $\mathfrak{t}_n$ instead of $[\![0, 2n]\!]$. In the continuous setting, we will see $\mathfrak{q}_\infty$ as a quotient of a continuous version of a g-tree, which we will call real g-tree. In other words, we will see the identifications $s \sim_\infty t$ as of two different kinds: some are inherited "from the g-tree structure," whereas the others come "from the map structure."

4.1.1 Definitions

As g-trees generalize plane trees in genus g, real g-trees are the objects that naturally generalize real trees. We will only use basic facts on real trees in this work. See, for example, [LG05] for more detail.

We consider a fixed dominant scheme $\mathfrak{s} \in \mathfrak{S}^*$. Let $(m^{\mathfrak{e}})_{\mathfrak{e}\in\vec{E}(\mathfrak{s})}$ and $(\sigma^{\mathfrak{e}})_{\mathfrak{e}\in\vec{E}(\mathfrak{s})}$ be two families of positive numbers satisfying $\sum_{\mathfrak{e}} m^{\mathfrak{e}} = 1$ and $\sigma^{\mathfrak{e}} = \sigma^{\bar{\mathfrak{e}}}$ for all $\mathfrak{e}$. As usual, we arrange the half-edges of $\mathfrak{s}$ according to its facial order: $\mathfrak{e}_1 = \mathfrak{e}_*, \dots, \mathfrak{e}_{2(6g-3)}$. For every $s \in [0,1)$, there exists a unique $1 \le k \le 2(6g-3)$ such that

$$\sum_{i=1}^{k-1} m^{\mathfrak{e}_i} \le s < \sum_{i=1}^{k} m^{\mathfrak{e}_i}.$$

We let $\mathfrak{e}(s) := \mathfrak{e}_k$ and $\langle s\rangle := s - \sum_{i=1}^{k-1} m^{\mathfrak{e}_i} \in [0, m^{\mathfrak{e}(s)})$. By convention, we set $\mathfrak{e}(1) = \mathfrak{e}_1$ and $\langle 1\rangle = 0$. Beware that these notions depend on the family $(m^{\mathfrak{e}})_{\mathfrak{e}\in\vec{E}(\mathfrak{s})}$. There should be no ambiguity in what follows.

Let us suppose we have a family $(h^{\mathfrak{e}})_{\mathfrak{e}\in\vec{E}(\mathfrak{s})}$ of continuous functions $h^{\mathfrak{e}} : [0, m^{\mathfrak{e}}] \to \mathbb{R}_+$ such that $h^{\mathfrak{e}}(0) = \sigma^{\mathfrak{e}}$ and $h^{\mathfrak{e}}(m^{\mathfrak{e}}) = 0$. It will be useful to consider their concatenation: we define the continuous function $h : [0,1] \to \mathbb{R}_+$ going from $\sum_{\mathfrak{e}} \sigma^{\mathfrak{e}}$ to 0 by

$$h := \left(h^{\mathfrak{e}_1} - \sigma^{\mathfrak{e}_1}\right) \bullet \left(h^{\mathfrak{e}_2} - \sigma^{\mathfrak{e}_2}\right) \bullet \dots \bullet \left(h^{\mathfrak{e}_{2(6g-3)}} - \sigma^{\mathfrak{e}_{2(6g-3)}}\right) + \sum_{i=1}^{2(6g-3)} \sigma^{\mathfrak{e}_i}. \tag{4.1}$$

We define the relation $\simeq$ on $[0,1]$ as the coarsest equivalence relation for which $s \simeq t$ if one of the following occurs:

✧ $h(s) = h(t) = \inf_{[s\wedge t, s\vee t]} h$, (4.2a)

✧ $h(s) = \underline{h}(s)$, $h(t) = \underline{h}(t)$, $\mathfrak{e}(s) = \overline{\mathfrak{e}(t)}$, and $h^{\mathfrak{e}(s)}(\langle s\rangle) = \sigma^{\mathfrak{e}(t)} - h^{\mathfrak{e}(t)}(\langle t\rangle)$, (4.2b)

✧ $\langle s\rangle = \langle t\rangle = 0$ and $\mathfrak{e}(s)^- = \mathfrak{e}(t)^-$. (4.2c)

If we see the $h^{\mathfrak{e}}$'s as contour functions (in a continuous setting), the first item identifies numbers coding the same point in one of the forests. The second item identifies the floors of forests "facing each other": the numbers s and t should code floor points (two first equalities) of forests facing each other (third equality) and correspond to the same point (fourth equality). Finally, the third item identifies the nodes. We call **real g-tree** any space $\mathscr{T} := [0,1]_{/\simeq}$ obtained by such a construction[1].

We now define the notions we will use throughout this work (see Figure 4.1). For $s \in [0,1]$, we write $\mathscr{T}(s)$ its equivalence class in the quotient $\mathscr{T} = [0,1]_{/\simeq}$. Similarly to the discrete case, the floor of $\mathscr{T}$ is defined as follows.

Definition 4.1. *We call **floor** of $\mathscr{T}$ the set $fl := \mathscr{T}(\{s : h(s) = \underline{h}(s)\})$.*

For $a = \mathscr{T}(s) \in \mathscr{T}\backslash fl$, let $l := \inf\{t \le s : \underline{h}(t) = \underline{h}(s)\}$ and $r := \sup\{t \ge s : \underline{h}(t) = \underline{h}(s)\}$. The set $\tau_a := \mathscr{T}([l,r])$ is a real tree rooted at $\rho_a := \mathscr{T}(l) = \mathscr{T}(r) \in fl$.

Definition 4.2. *We call **tree** of $\mathscr{T}$ a set of the form τ_a for any $a \in \mathscr{T}\backslash fl$.*

[1] There should be a more intrinsic definition for these spaces in terms of compact metric spaces that are locally real trees. As we will need to use this construction in what follows, we chose to define them as such for simplicity.

If $a \in \mathit{fl}$, we simply set $\rho_a := a$. Let τ be a tree of $\mathscr{T}$ rooted at ρ, and $a, b \in \tau$. We call $[[a,b]]$ the range of the unique injective path linking a to b. In particular, the set $[[\rho, a]]$ will be of interest. It represents the ancestral lineage of a in the tree τ. We say that a is an **ancestor** of b, and we write $a \preceq b$, if $a \in [[\rho, b]]$. We write $a \prec b$ if $a \preceq b$ and $a \neq b$.

Definition 4.3. *Let $b = \mathscr{T}(t) \in \mathscr{T} \backslash \mathit{fl}$ and $\rho \in [[\rho_b, b]] \backslash \{\rho_b, b\}$. Let $l' := \inf\{s \le t : \mathscr{T}(s) = \rho\}$ and $r' := \sup\{s \le t : \mathscr{T}(s) = \rho\}$. Then, provided $l' \neq r'$, we call* ***tree to the left*** *of $[[\rho_b, b]]$ rooted at ρ the set $\mathscr{T}([l', r'])$.*

We define the ***tree to the right*** *of $[[\rho_b, b]]$ rooted at ρ in a similar way, by replacing "$\le$" with "$\ge$" in the definitions of l' and r'.*

Definition 4.4. *We call* ***subtree*** *of $\mathscr{T}$ any tree of $\mathscr{T}$, or any tree to the left or right of $[[\rho_b, b]]$ for some $b \in \mathscr{T} \backslash \mathit{fl}$.*

Note that subtrees of $\mathscr{T}$ are real trees, and that trees of $\mathscr{T}$ are also subtrees of $\mathscr{T}$. For a subtree τ, the maximal interval $[s, t]$ such that $\tau = \mathscr{T}([s,t])$ is called the **interval coding** the subtree τ.

Definition 4.5. *For $\mathfrak{e} \in \vec{E}(\mathfrak{s})$, we call* ***forest to the left of*** *$\mathfrak{e}$ the set $\mathfrak{f}^{\mathfrak{e}} := \mathscr{T}\left(\overline{\{s : \mathfrak{e}(s) = \mathfrak{e}\}}\right)$.*

The **nodes** of $\mathscr{T}$ are the elements of $\mathscr{T}(\{s : \langle s \rangle = 0\})$. In what follows, we will identify the nodes of $\mathscr{T}$ with the vertices of $\mathfrak{s}$. In particular, the two nodes $\mathfrak{e}^-$ and $\mathfrak{e}^+$ lie in $\mathfrak{f}^{\mathfrak{e}}$. We extend the definition of $[[a,b]]$ to the floor of $\mathfrak{f}^{\mathfrak{e}}$: for $a, b \in \mathfrak{f}^{\mathfrak{e}} \cap \mathit{fl}$, let $s, t \in \overline{\{r : \mathfrak{e}(r) = \mathfrak{e}\}}$ be such that $a = \mathscr{T}(s)$ and $b = \mathscr{T}(t)$. We define

$$[[a,b]] := \mathscr{T}([s \wedge t, s \vee t]) \cap \mathit{fl}$$

the range of the unique[2] injective path from a to b that stays inside $\mathfrak{f}^{\mathfrak{e}}$. For clarity, we write the set $[[\mathfrak{e}^-, \mathfrak{e}^+]]$ simply as $[[\mathfrak{e}]]$. Note that, in particular, $[[\mathfrak{e}]] = \mathfrak{f}^{\mathfrak{e}} \cap \mathfrak{f}^{\bar{\mathfrak{e}}} = \mathfrak{f}^{\mathfrak{e}} \cap \mathit{fl}$.

Let $a, b \in \mathscr{T}$. There is a natural way[3] to explore $\mathscr{T}$ from a to b. If $\inf \mathscr{T}^{-1}(a) \le \sup \mathscr{T}^{-1}(b)$, then let $t := \inf\{r \ge \inf \mathscr{T}^{-1}(a) : b = \mathscr{T}(r)\}$ and $s := \sup\{r \le t : a = \mathscr{T}(r)\}$. If $\sup \mathscr{T}^{-1}(b) < \inf \mathscr{T}^{-1}(a)$, then let $t := \inf \mathscr{T}^{-1}(b)$ and $s := \sup \mathscr{T}^{-1}(a)$. We define

$$[a,b] := \mathscr{T}\left(\overrightarrow{[s,t]}\right), \tag{4.3}$$

where $\overrightarrow{[s,t]}$ is defined by (3.5).

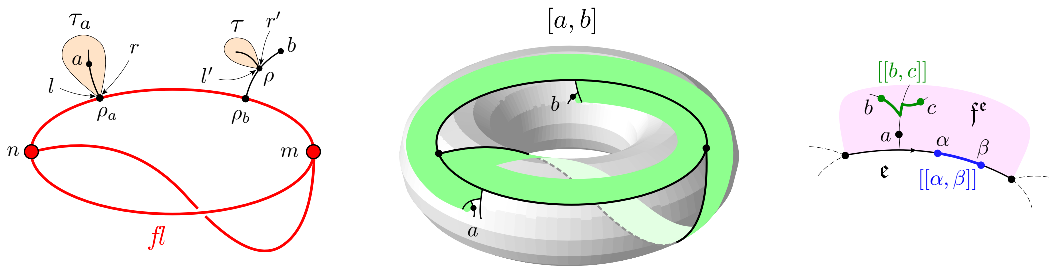

Figure 4.1. ***Left.*** *On this picture, we can see the floor fl, the two nodes n and m, an example of tree τ_a, and an example of tree τ to the left of $[[\rho_b, b]]$ rooted at ρ.* ***Middle.*** *The set $[a,b]$.* ***Right.*** *On this picture, a is an ancestor of b and c, and we can see the sets $[[b,c]]$, $[[\alpha,\beta]]$, and $\mathfrak{f}^{\mathfrak{e}}$.*

We call $\mathscr{T}_n$ (resp. $\mathscr{T}_\infty$) the real g-tree obtained from the scheme $\mathfrak{s}_n$ (resp. $\mathfrak{s}_\infty$) and the family $(C^{\mathfrak{e}}_{(n)})_{\mathfrak{e} \in \vec{E}(\mathfrak{s}_n)}$ (resp. $(C^{\mathfrak{e}}_\infty)_{\mathfrak{e} \in \vec{E}(\mathfrak{s}_\infty)}$). For the sake of consistency with Chapter 3, we call $\mathfrak{C}_{(n)}$ and $\mathfrak{C}_\infty$ the

[2] Note that $\mathfrak{e}^+ \neq \mathfrak{e}^-$ because $\mathfrak{s}$ is a dominant scheme.

[3] Note that, if $a, b \in \mathit{fl}$, there are other possible ways to explore the g-tree between them. Indeed, a point of fl is visited twice—or three times if it is a node—when we travel around fl. In particular, this definition depends on the position of the root in $\mathfrak{s}$ for such points. In what follows, we never use this definition for such points, so there will be no confusion.

functions obtained by (4.1) in this construction. We also call $\simeq_{(n)}$ and $\simeq_\infty$ the corresponding equivalence relations. When dealing with $\mathscr{T}_\infty$, we add an ∞ symbol to the notation defined above: for example, the floor of $\mathscr{T}_\infty$ will be noted fl_∞, and its forest to the left of $\mathfrak{e}$ will be noted $\mathfrak{f}^{\mathfrak{e}}_\infty$. It is more natural to use $\mathfrak{t}_n$ rather than $\mathscr{T}_n$ in the discrete setting. As $\mathfrak{t}_n$ may be viewed as a subset of $\mathscr{T}_n$, we will use for $\mathfrak{t}_n$ the formalism we defined above simply by restriction. Note that the notions of floor, forests, trees, and nodes are consistent with the definitions we gave in Section 2.2 in that case.

Note that, because the functions $C^{\mathfrak{e}}_\infty$'s are first-passage Brownian bridges, the probability that there exists $\varepsilon > 0$ such that $C^{\mathfrak{e}}_\infty(s) > C^{\mathfrak{e}}_\infty(0)$ for all $s \in (0, \varepsilon)$ is equal to 0. As a result, there are almost surely no trees rooted at the nodes of $\mathscr{T}_\infty$. Moreover, the fact that the forests $\mathfrak{f}^{\mathfrak{e}}$ and $\mathfrak{f}^{\bar{\mathfrak{e}}}$ are independent yields that, almost surely, we cannot have a tree in $\mathfrak{f}^{\mathfrak{e}}$ and a tree in $\mathfrak{f}^{\bar{\mathfrak{e}}}$ rooted at the same point. As a consequence, we see that, almost surely, all the points of $\mathscr{T}_\infty$ are of order less than 3.

4.1.2 Maps seen as quotients of real g-trees

Consistently with the notation $\mathfrak{t}_n(i)$ and $\mathfrak{q}_n(i)$ in the discrete setting, we call $\mathscr{T}_\infty(s)$ (resp. $\mathfrak{q}_\infty(s)$) the equivalence class of $s \in [0,1]$ in $\mathscr{T}_\infty = [0,1]_{/\simeq_\infty}$ (resp. in $\mathfrak{q}_\infty = [0,1]_{/\sim_\infty}$).

Lemma 4.1. *The equivalence relation $\simeq_\infty$ is coarser than $\sim_\infty$, so that we can see $\mathfrak{q}_\infty$ as the quotient of $\mathscr{T}_\infty$ by the equivalence relation on $\mathscr{T}_\infty$ induced from $\sim_\infty$.*

Proof. By definition of $\simeq_\infty$, it suffices to show that if $s < t$ satisfy (4.2a), (4.2b), or (4.2c), then $s \sim_\infty t$. Let us first suppose that s and t satisfy (4.2a), that is

$$\mathfrak{C}_\infty(s) = \mathfrak{C}_\infty(t) = \inf_{[s,t]} \mathfrak{C}_\infty.$$

In a first time, we moreover suppose that $\mathfrak{C}_\infty(r) > \mathfrak{C}_\infty(s)$ for all $r \in (s,t)$. Using Proposition 3.1, we can find integers $0 \le s_n < t_n \le 2n$ such that $(s_{(n)}, t_{(n)}) := (s_n/2n, t_n/2n) \to (s,t)$ and $\mathfrak{C}_{(n)}(s_{(n)}) = \mathfrak{C}_{(n)}(t_{(n)}) = \inf_{[s_{(n)}, t_{(n)}]} \mathfrak{C}_{(n)}$. The latter condition imposes $\dot{\mathfrak{t}}_n(s_n) = \dot{\mathfrak{t}}_n(t_n)$ so that $d_n(s_n, t_n) = 0$ and $s \sim_\infty t$ by (3.7).

Equation (2.19) shows that, for every $\mathfrak{e}$, the law of $C^{\mathfrak{e}}_\infty$ is absolutely continuous with respect to the Wiener measure on any interval $[0, m^{\mathfrak{e}}_\infty - \varepsilon]$, for $\varepsilon > 0$. Because local minimums of Brownian motion are pairwise distinct, this is also true for any $C^{\mathfrak{e}}_\infty$, and thus for the whole process $\mathfrak{C}_\infty$ by construction. If there exists $r \in (s,t)$ for which $\mathfrak{C}_\infty(r) = \mathfrak{C}_\infty(s)$, it is thus unique. We may then apply the previous reasoning to (s,r) and (r,t) and find that $s \sim_\infty r$ and $r \sim_\infty t$, so that $s \sim_\infty t$.

Let us now suppose that s and t satisfy (4.2b). If there is $0 \le r < s$ such that $\mathfrak{C}_\infty(r) = \mathfrak{C}_\infty(s)$ then $r \simeq_\infty s$ by (4.2a). The same holds with t instead of s. We may thus restrict our attention to s and t for which $\mathfrak{C}_\infty(r) > \mathfrak{C}_\infty(s)$ for all $r \in [0,s)$ and $\mathfrak{C}_\infty(r) > \mathfrak{C}_\infty(t)$ for all $r \in [0,t)$. Let us call $\mathfrak{e} = \mathfrak{e}(s) = \overline{\mathfrak{e}(t)}$. In order to avoid confusion, we use the notation $\langle \cdot \rangle_n$ and $\mathfrak{e}_n(\cdot)$ when dealing with the functions $\mathfrak{C}^{\mathfrak{e}}_{(n)}$'s. We know that for n large enough, we have $\mathfrak{s}_n = \mathfrak{s}_\infty$. We only consider such n's in the following. We first find $0 \le s_n \le 2n$ such that $s_{(n)} := s_n/2n \to s$, $\mathfrak{e}_n(s_{(n)}) = \mathfrak{e}$, and $\mathfrak{C}_{(n)}(s_{(n)}) = \underline{\mathfrak{C}}_{(n)}(s_{(n)})$. We define

$$t_{(n)} := \inf \left\{ r \in \frac{1}{2n}[\![0, 2n]\!] \,:\, \mathfrak{e}_n(r) = \bar{\mathfrak{e}},\; \mathfrak{C}^{\bar{\mathfrak{e}}}_{(n)}\Big(\langle r \rangle_n\Big) = \sigma^{\mathfrak{e}}_{(n)} - \mathfrak{C}^{\mathfrak{e}}_{(n)}\Big(\langle s_{(n)} \rangle_n\Big) \right\},$$

so that $t_{(n)} \simeq_{(n)} s_{(n)}$, and then $d_{(n)}\Big(s_{(n)}, t_{(n)}\Big) = 0$. Taking an extraction if needed, we may suppose that $t_{(n)} \to t' \sim_\infty s$. By construction, $\mathfrak{e}(t') = \mathfrak{e}(t)$ and $\mathfrak{C}_\infty(t') = \underline{\mathfrak{C}}_\infty(t') = \mathfrak{C}_\infty(t)$. So t' and t fulfill requirement (4.2a) and $t' \sim_\infty t$ by the above argument. The case of (4.2c) is easier and may be treated in a similar way. □

This lemma allows us to define a pseudo-metric and an equivalence relation on $\mathscr{T}_\infty$, still denoted by d_∞ and $\sim_\infty$, by setting $d_\infty\Big(\mathscr{T}_\infty(s), \mathscr{T}_\infty(t)\Big) := d_\infty(s,t)$ and declaring $\mathscr{T}_\infty(s) \sim_\infty \mathscr{T}_\infty(t)$ if $s \sim_\infty t$. The metric space $(\mathfrak{q}_\infty, d_\infty)$ is then isometric to $\Big(\mathscr{T}_{\infty/\sim_\infty}, d_\infty\Big)$. We define d°_∞ on $\mathscr{T}_\infty$ by letting

$$d^\circ_\infty(a,b) := \inf \left\{ d^\circ_\infty(s,t) \,:\, a = \mathscr{T}_\infty(s),\, b = \mathscr{T}_\infty(t) \right\}.$$

We will see in Lemma 4.4 that there is a.s. only one point where the function $\mathfrak{L}_\infty$ reaches its minimum. On this event, the following lemma holds.

Lemma 4.2. *Let $s^\bullet$ be the unique point where $\mathfrak{L}_\infty$ reaches its minimum. Then*

$$d_\infty(s, s^\bullet) = \mathfrak{L}_\infty(s) - \mathfrak{L}_\infty(s^\bullet).$$

Moreover, $s \sim_\infty t$ implies $\mathfrak{L}_\infty(s) = \mathfrak{L}_\infty(t)$.

Proof. This readily comes from the discrete setting. Let $0 \le s_n^\bullet \le 2n$ be an integer where $\mathfrak{L}_n$ reaches its minimum. By extracting if necessary, we may suppose that $s_n^\bullet/2n$ converges, necessarily toward $s^\bullet$. Let $0 \le s_n \le 2n$ be such that $s_n/2n \to s$. From the Chapuy–Marcus–Schaeffer bijection, $d_n(s_n, s_n^\bullet) = \mathfrak{L}_n(s_n) - \mathfrak{L}_n(s_n^\bullet) + 1$. Letting $n \to \infty$ after renormalizing yields the first assertion. The second one follows from the first one and the triangle inequality. ☐

As a result of Lemmas 4.1 and 4.2, we can define $\mathfrak{L}_\infty$ on $\mathscr{T}_\infty$ by $\mathfrak{L}_\infty\big(\mathscr{T}_\infty(s)\big) := \mathfrak{L}_\infty(s)$. When $(a, b) \notin (\mathit{fl}_\infty)^2$, we have

$$d^\circ_\infty(a, b) = \mathfrak{L}_\infty(a) + \mathfrak{L}_\infty(b) - 2\max\left(\min_{x\in[a,b]} \mathfrak{L}_\infty(x), \min_{x\in[b,a]} \mathfrak{L}_\infty(x)\right), \tag{4.4}$$

where $[a, b]$ was defined by (4.3).

4.2 Points identifications

This section is dedicated to the proof of the following theorem:

Theorem 4.3. *A.s., for every $a, b \in \mathscr{T}_\infty$, $a \sim_\infty b$ is equivalent to $d^\circ_\infty(a, b) = 0$.*

We already know that $d^\circ_\infty(a, b) = 0$ implies $a \sim_\infty b$ from the bound $d_\infty \le d^\circ_\infty$. We will show the converse through a series of lemmas. We adapt the approach of Le Gall [LG07] to our setting.

4.2.1 Preliminary lemmas

Let us begin by giving some information on the process $(\mathfrak{C}_\infty, \mathfrak{L}_\infty)$.

Lemma 4.4. *The set of points where $\mathfrak{L}_\infty$ reaches its minimum is a.s. a singleton.*

Let $f : [0, 1] \to \mathbb{R}$ be a continuous function. We say that $s \in [0, 1)$ is a **right-increase point** of f if there exists $t \in (s, 1]$ such that $f(r) \ge f(s)$ for all $s \le r \le t$. A **left-increase point** is defined in a symmetric way. We call $\mathrm{IP}(f)$ the set of all (left or right) increase points of f.

Lemma 4.5. *A.s., $\mathrm{IP}(\mathfrak{C}_\infty)$ and $\mathrm{IP}(\mathfrak{L}_\infty)$ are disjoint sets.*

As the proofs of these lemmas are rather technical and unrelated to what follows, we postpone them to Section 4.4.

4.2.2 Key lemma

Remark. In what follows, every discrete path denoted by the letter "$\wp$" will always be a path in the *map*, never in the tree, i.e. a path using the edges of the map.

Let τ be a subtree of $\mathfrak{t}_n$ and $\wp = (\wp(0), \wp(1), \ldots, \wp(r))$ be a path in $\mathfrak{q}_n$ that avoids the base point $v_n^\bullet$. We say that the arc $(\wp(0), \wp(1))$ enters the subtree τ from the left (resp. from the right) if $\wp(0) \notin \tau$, $\wp(1) \in \tau$ and $\mathfrak{l}_n(\wp(1)) - \mathfrak{l}_n(\wp(0)) = -1$ (resp. $\mathfrak{l}_n(\wp(1)) - \mathfrak{l}_n(\wp(0)) = 1$). We say that the path $\wp$ **passes through** the subtree τ between times i and j, where $0 < i \le j < r$, if

✧ $\wp(i-1) \notin \tau$; $\wp([\![i,j]\!]) \subseteq \tau$; $\wp(j+1) \notin \tau$,

✧ $\mathfrak{l}_n(\wp(i)) - \mathfrak{l}_n(\wp(i-1)) = \mathfrak{l}_n(\wp(j+1)) - \mathfrak{l}_n(\wp(j))$.

The first condition states that $\wp$ "visits" τ, whereas the second one ensures that it really goes "through." It enters and exits τ going "in the same direction."

We say that a vertex $a_n \in \mathfrak{t}_n$ converges toward a point $a \in \mathscr{T}_\infty$ if there exists a sequence of integers $s_n \in [\![0, 2n]\!]$ coding a_n (i.e. $a_n = \dot{\mathfrak{t}}_n(s_n)$) such that $s_n/2n$ admits a limit s satisfying $a = \mathscr{T}_\infty(s)$. Let $[\![l_n, r_n]\!]$ be the intervals coding subtrees $\tau_n \subseteq \mathfrak{t}_n$. We say that the subtree τ_n converges toward a subtree $\tau \subseteq \mathscr{T}_\infty$ if the sequences $l_n/2n$ and $r_n/2n$ admit limits l and r such that the interval coding τ is $[l, r]$. The following lemma is adapted from Le Gall [LG07, End of Proposition 4.2].

Lemma 4.6. *With full probability, the following occurs. Let $a, b \in \mathscr{T}_\infty$ be such that $\mathfrak{L}_\infty(a) = \mathfrak{L}_\infty(b)$. We suppose that there exists a subtree τ rooted at ρ such that $\inf_\tau \mathfrak{L}_\infty < \mathfrak{L}_\infty(a) < \mathfrak{L}_\infty(\rho)$. We further suppose that we can find vertices a_n, $b_n \in \mathfrak{t}_n$ and subtrees τ_n in $\mathfrak{t}_n$ converging respectively toward a, b, τ and satisfying the following property: for infinitely many n's, there exists a geodesic path $\wp_n$ in $\mathfrak{q}_n$ from a_n to b_n that avoids the base point $v_n^\bullet$ and passes through the subtree τ_n.*

Then, $a \not\sim_\infty b$.

Proof. The idea is that if a and b were identified, then all the points in the discrete subtrees close (in a certain sense) to the geodesic path would be close to a in the limit. Fine estimates on the sizes of balls yield the result. We proceed to the rigorous proof.

We reason by contradiction and suppose that $a \sim_\infty b$. We only consider integers n for which the hypothesis holds. We call ρ_n the root of τ_n, and we set, for $\varepsilon > 0$,

$$\mathcal{U}_\infty^\varepsilon := \left\{ y \in \tau \,:\, \mathfrak{L}_\infty(y) < \mathfrak{L}_\infty(a) + \varepsilon\,;\, \forall x \in [[\rho, y]],\, \mathfrak{L}_\infty(x) > \mathfrak{L}_\infty(a) + \frac{\varepsilon}{8} \right\}.$$

We first show that $\mathcal{U}_\infty^\varepsilon \subseteq B_\infty(a, 2\varepsilon)$, where $B_\infty(a, 2\varepsilon)$ denotes the closed ball of radius 2ε centered at a in the metric space $(\mathfrak{q}_\infty, d_\infty)$. Let $y \in \mathcal{U}_\infty^\varepsilon$. We can find $y_n \in \tau_n \backslash \{\rho_n\}$ converging toward y. For n large enough, we have

$$d_{\mathfrak{q}_n}(a_n, b_n) \le \frac{\varepsilon}{32}\, n^{1/4}, \qquad \sup_{c \in \wp_n} |\mathfrak{l}_n(c) - \mathfrak{l}_n(a_n)| \le \frac{\varepsilon}{32}\, n^{1/4},$$
$$\mathfrak{l}_n(y_n) \le \mathfrak{l}_n(a_n) + \frac{3}{2}\,\varepsilon\, n^{1/4}, \qquad \forall x \in [[\rho_n, y_n]],\, \mathfrak{l}_n(x) \ge \mathfrak{l}_n(a_n) + \frac{\varepsilon}{16}\, n^{1/4}.$$

The first inequality comes from the fact that $a \sim_\infty b$. The second inequality is a consequence of the first one. The third inequality holds because $(\mathfrak{l}_n(y_n) - \mathfrak{l}_n(v_n^\bullet))/\gamma\, n^{1/4} \to \mathfrak{L}_\infty(y)$ and $(\mathfrak{l}_n(a_n) - \mathfrak{l}_n(v_n^\bullet))/\gamma\, n^{1/4} \to \mathfrak{L}_\infty(a)$. Finally, the fourth inequality follows by compactness of $[[\rho, y]]$.

From now on, we only consider such n's. We call $t_n := \sup\{t \,:\, y_n = \dot{\mathfrak{t}}_n(t)\}$ the last integer coding y_n, and $[\![l_n, r_n]\!]$ the interval coding τ_n. We also call $i \le j$ two integers such that $\wp_n$ passes through τ_n between times i and j. For the sake of simplicity, we suppose that $\wp_n$ enters τ_n from the left[4]. Notice that the path $\wp_n$ does not intersect $[[\rho_n, y_n]]$, because the labels on $[[\rho_n, y_n]]$ are strictly greater than the labels on $\wp_n$. Let k be the largest integer in $[\![i-1, j]\!]$ such that $\wp_n(k)$ belongs to the set $\{\wp_n(i-1)\} \cup \dot{\mathfrak{t}}_n([\![l_n, t_n]\!])$. Then $\wp_n(k+1) \in \{\wp_n(j+1)\} \cup \dot{\mathfrak{t}}_n([\![t_n, r_n]\!])$. Moreover, $\mathfrak{l}_n(\wp_n(k+1)) = \mathfrak{l}_n(\wp_n(k)) - 1$: otherwise, all the vertices in $[\wp_n(k+1), \wp_n(k)]$ would have labels greater than $\mathfrak{l}_n(\wp_n(k))$, and it is easy to see that this would prohibit $\wp_n$ from exiting τ_n by going "to the right," in the sense that we would not have $\mathfrak{l}_n(\wp_n(j+1)) = \mathfrak{l}_n(\wp_n(j)) - 1$. As a result, when performing the Chapuy–Marcus–Schaeffer bijection for the arc linking $\wp_n(k)$ to $\wp_n(k+1)$, we have to visit y_n. Then, going through consecutive successors of t_n, we are bound to hit $\wp_n(k+1)$, so that $d_{\mathfrak{q}_n}(y_n, \wp_n) \le \mathfrak{l}_n(y_n) - \mathfrak{l}_n(\wp_n(k+1))$. This yields that $d_{\mathfrak{q}_n}(a_n, y_n) \le d_{\mathfrak{q}_n}(a_n, b_n) + d_{\mathfrak{q}_n}(y_n, \wp_n) \le 2\varepsilon\, \gamma\, n^{1/4}$, and, by taking the limit, $d_\infty(a, y) \le 2\varepsilon$.

[4] The case where $\wp_n$ enters τ_n from the right may be treated by considering the path $h \mapsto \wp_n(d_{\mathfrak{q}_n}(a_n, b_n) - h)$ instead of $\wp_n$.

We conclude thanks to two lemmas, whose proofs are postponed to Section 4.4. They are derived from similar results in the planar case: [LG07, Lemma 2.4] and [LG10, Corollary 6.2]. We call λ the volume measure on $\mathfrak{q}_\infty$, that is, the image of the Lebesgue measure on $[0,1]$ by the canonical projection from $[0,1]$ to $\mathfrak{q}_\infty$.

Lemma 4.7. *Almost surely, for every $\eta > 0$ and every subtree τ rooted at ρ, the condition $\inf_\tau \mathfrak{L}_\infty < \mathfrak{L}_\infty(\rho) - \eta$ implies that*

$$\liminf_{\varepsilon\to 0} \varepsilon^{-2}\lambda\left(\left\{y \in \tau \,:\, \mathfrak{L}_\infty(y) < \mathfrak{L}_\infty(\rho) - \eta + \varepsilon\,;\ \forall x \in [[\rho, y]],\ \mathfrak{L}_\infty(x) > \mathfrak{L}_\infty(\rho) - \eta + \frac{\varepsilon}{8}\right\}\right) > 0.$$

Lemma 4.8. *Let $\delta \in (0,1]$. For every $p \geq 1$,*

$$\mathbb{E}\left[\left(\sup_{\varepsilon>0}\left(\sup_{x\in\mathfrak{q}_\infty}\frac{\lambda(B_\infty(x,\varepsilon))}{\varepsilon^{4-\delta}}\right)\right)^p\right] < \infty.$$

We apply Lemma 4.7 to τ and $\eta = \mathfrak{L}_\infty(\rho) - \mathfrak{L}_\infty(a) > 0$, and we find that, for ε small enough,

$$\lambda(\mathcal{U}_\infty^\varepsilon) \geq \varepsilon^{5/2}.$$

The inclusion $\mathcal{U}_\infty^\varepsilon \subseteq B_\infty(a, 2\varepsilon)$ yields that

$$S := \sup_{\varepsilon>0}\left(\sup_{x\in\mathfrak{q}_\infty}\frac{\lambda(B_\infty(x,\varepsilon))}{\varepsilon^{7/2}}\right) = \infty.$$

Lemma 4.8 applied to $\delta = 1/2$ and $p = 1$ yields that S is integrable, so that $S < \infty$ a.s. This is a contradiction. □

4.2.3 Set overflown by a path

We call fl_n the floor of $\mathfrak{t}_n$. Let $i \in [\![0, 2n]\!]$, and let $\mathrm{succ}(i)$ be its successor in $(\mathfrak{t}_n, \mathfrak{l}_n)$, defined by (2.2). We moreover suppose that $\mathrm{succ}(i) \neq \infty$. We say that the arc linking $\mathfrak{t}_n(i)$ to $\mathfrak{t}_n(\mathrm{succ}(i))$ **overflies** the set

$$\mathfrak{t}_n\left(\overrightarrow{[\![i, \mathrm{succ}(i)]\!]}\right) \cap \mathit{fl}_n,$$

where $\overrightarrow{[\![i, \mathrm{succ}(i)]\!]}$ was defined by (2.4). We define the set overflown by a path $\wp$ in $\mathfrak{q}_n$ that avoids the base point $v_n^\bullet$ as the union of the sets its arcs overfly. We denote it by $\mathit{of}(\wp) \subseteq \mathit{fl}_n$.

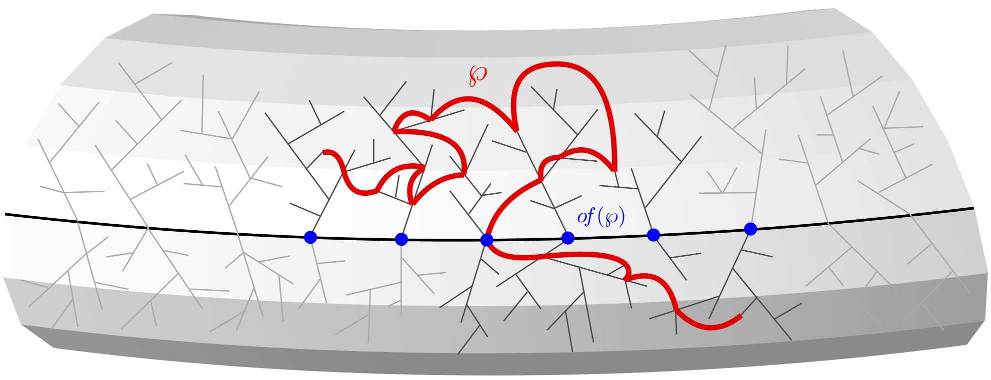

Figure 4.2. *The set overflown by the path $\wp$ is the set of (blue) large dots.*

Lemma 4.9. *Let $a \sim_\infty b \in \mathscr{T}_\infty$ and $\alpha, \beta \in \mathfrak{f}_\infty^{\mathfrak{e}} \cap fl_\infty$. We suppose that, for n sufficiently large, there exist vertices $\alpha_n, \beta_n \in \mathfrak{f}_n^{\mathfrak{e}} \cap fl_n$ and $a_n, b_n \in \mathfrak{t}_n$ converging respectively toward α, β, a and b. If, for infinitely many n's, there exists a geodesic path $\wp_n$ from a_n to b_n that overflies $[[\alpha_n, \beta_n]]$, then for all $c \in [[\alpha, \beta]]$,*

$$\mathfrak{L}_\infty(c) \geq \mathfrak{L}_\infty(a) = \mathfrak{L}_\infty(b).$$

Moreover, if there exists $c \in [[\alpha, \beta]]$ for which $\mathfrak{L}_\infty(c) = \mathfrak{L}_\infty(a)$, then $a \sim_\infty c$.

Proof. Let $c \in [[\alpha, \beta]]$. We can find vertices $c_n \in [[\alpha_n, \beta_n]]$ converging to c. By definition, there is an arc of $\wp_n$ that overflies c_n. Say it links a vertex labeled l to a vertex v labeled $l-1$. From the Chapuy–Marcus–Schaeffer construction, we readily obtain that $\mathfrak{l}_n(c_n) \geq l$. Using the fact that $\mathfrak{l}_n(a_n) - l \leq d_{\mathfrak{q}_n}(a_n, b_n)$, we find

$$\mathfrak{l}_n(c_n) \geq \mathfrak{l}_n(a_n) - d_{\mathfrak{q}_n}(a_n, b_n).$$

Moreover, we can construct a path from c_n to v going through consecutive successors of c_n. As a result, $d_{\mathfrak{q}_n}(c_n, \wp_n) \leq \mathfrak{l}_n(c_n) - l + 1$, so that

$$d_{\mathfrak{q}_n}(c_n, a_n) \leq \mathfrak{l}_n(c_n) - \mathfrak{l}_n(a_n) + 2 d_{\mathfrak{q}_n}(a_n, b_n) + 1.$$

Both claims follow by taking limits in these inequalities after renormalization, and by using the fact that $d_{\mathfrak{q}_n}(a_n, b_n) = o(n^{1/4})$. □

4.2.4 Points identifications

We proceed in three steps. We first show that points of fl_∞ are not identified with any other points, then that points cannot be identified with their strict ancestors, and finally Theorem 4.3.

4.2.4.1 Floor points are not identified with any other points

Lemma 4.10. *A.s., if $a \in fl_\infty$ and $b \in \mathscr{T}_\infty$ are such that $a \sim_\infty b$, then $a = b$.*

Proof. Let $a \in fl_\infty$ and $b \in \mathscr{T}_\infty \backslash \{a\}$ be such that $a \sim_\infty b$. We first suppose that a is not a node. There exists $\mathfrak{e} \in \vec{E}(\mathfrak{s}_\infty)$ such that $a \in \mathfrak{f}_\infty^{\mathfrak{e}} \cap \mathfrak{f}_\infty^{\bar{\mathfrak{e}}}$, and we can find s, t satisfying $a = \mathscr{T}_\infty(s) = \mathscr{T}_\infty(t)$, $\mathfrak{e}(s) = \mathfrak{e}$ and $\mathfrak{e}(t) = \bar{\mathfrak{e}}$. Without loss of generality, we may suppose that $s < t$. Until further notice, we will moreover suppose that $\rho_b \notin [[\mathfrak{e}]]$.

We restrict ourselves to the case $\mathfrak{s}_n = \mathfrak{s}_\infty$, which happens for n sufficiently large. We can find $a_n \in fl_n$ and $b_n \in \mathfrak{t}_n$ converging toward a and b and satisfying $\rho_{b_n} \notin [[\mathfrak{e}]]$. Let $\wp_n$ be a geodesic path (in $\mathfrak{q}_n$, for $d_{\mathfrak{q}_n}$) from a_n to b_n. It has to overfly at least $[[a_n, \mathfrak{e}^-]]$ or $[[a_n, \mathfrak{e}^+]]$. Indeed, every pair $(x, y) \in [[a_n, \mathfrak{e}^-]] \times [[a_n, \mathfrak{e}^+]]$ breaks $\mathfrak{t}_n$ into connected components, and the points a_n and b_n do not belong to the same of these components. There has to be an arc of $\wp_n$ that links a point belonging to the component containing a_n to one of the other components. Such an arc overflies x or y.

Let us suppose that, for infinitely many n's, $\wp_n$ overflies $[[a_n, \mathfrak{e}^-]]$. Lemma 4.9 then ensures that $\mathfrak{L}_\infty(c) \geq \mathfrak{L}_\infty(a) = \mathfrak{L}_\infty(b)$ for all $c \in [[a, \mathfrak{e}^-]]$. Properties of Brownian snakes show that the labels on $[[a, \mathfrak{e}^-]]$ are Brownian. Precisely, we may code $[[\mathfrak{e}]]$ by the interval $[0, \sigma^{\mathfrak{e}}]$ as follows. For $x \in [0, \sigma^{\mathfrak{e}}]$, we define $T_x := \inf\{r \geq \langle s \rangle : \mathfrak{C}_\infty(r) = \mathfrak{C}_\infty(\langle s \rangle) - x\}$. Then $[[\mathfrak{e}]] = \mathscr{T}_\infty(\{T_x, 0 \leq x \leq \sigma_{\mathfrak{e}}\})$, and

$$\Big(\mathfrak{L}_\infty(T_x) - \mathfrak{L}_\infty(\langle s \rangle)\Big)_{0 \leq x \leq \sigma_{\mathfrak{e}}} = \Big(\mathfrak{M}_\infty^{\mathfrak{e}}(x)\Big)_{0 \leq x \leq \sigma_{\mathfrak{e}}},$$

where, conditionally given $\mathfrak{I}_\infty$, the process $\mathfrak{M}_\infty^{\mathfrak{e}}$ (defined during Proposition 3.1) has the law of a certain Brownian bridge. Using the fact that local minimums of Brownian motion are distinct, we can find $d \in [[a, \mathfrak{e}^-]] \backslash \{a\}$ such that $\mathfrak{L}_\infty(c) > \mathfrak{L}_\infty(a)$ for all $c \in [[a, d]] \backslash \{a\}$.

Because $a \in fl_\infty$, s and t are both increase points of $\mathfrak{C}_\infty$ and thus are not increase points of $\mathfrak{L}_\infty$, by Lemma 4.5. As a result, there exist two trees $\tau^1 \subseteq \mathfrak{f}_\infty^{\mathfrak{e}}$ and $\tau^2 \subseteq \mathfrak{f}_\infty^{\bar{\mathfrak{e}}}$ rooted at $\rho^1, \rho^2 \in [[a, d]] \backslash \{a\}$ satisfying $\inf_{\tau^i} \mathfrak{L}_\infty < \mathfrak{L}_\infty(a) < \mathfrak{L}_\infty(\rho^i)$ (see Figure 4.3).

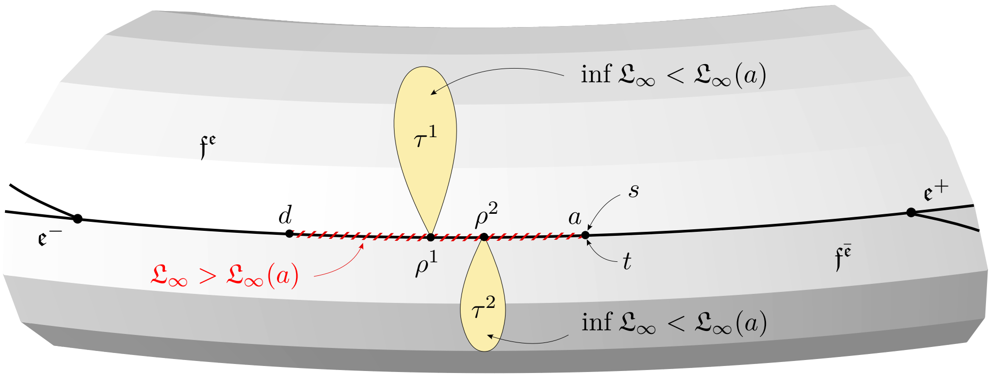

Figure 4.3. *The trees τ^1 and τ^2.*

Similarly, if for infinitely many n's, $\wp_n$ overflies $[[a_n, \mathfrak{e}^+]]$, then we can find two trees $\tau^3 \subseteq \mathfrak{f}^{\mathfrak{e}}_\infty$ and $\tau^4 \subseteq \mathfrak{f}^{\bar{\mathfrak{e}}}_\infty$ rooted at ρ^3, $\rho^4 \in [[a, \mathfrak{e}^+]]\backslash\{a\}$ satisfying $\inf_{\tau^i} \mathfrak{L}_\infty < \mathfrak{L}_\infty(a) < \mathfrak{L}_\infty(\rho^i)$, and $\mathfrak{L}_\infty(c) > \mathfrak{L}_\infty(a)$ for all $c \in [[\rho^3, \rho^4]]$. Three cases may occur:

(i) for n large enough, $\wp_n$ does not overfly $[[a_n, \mathfrak{e}^+]]$ (and therefore overflies $[[a_n, \mathfrak{e}^-]]$),

(ii) for n large enough, $\wp_n$ does not overfly $[[a_n, \mathfrak{e}^-]]$ (and therefore overflies $[[a_n, \mathfrak{e}^+]]$),

(iii) for infinitely many n's, $\wp_n$ overflies $[[a_n, \mathfrak{e}^+]]$, and for infinitely many n's, $\wp_n$ overflies $[[a_n, \mathfrak{e}^-]]$.

In case (i), the trees τ^1 and τ^2 are well defined. Let $\tau^1_n \subseteq \mathfrak{f}^{\mathfrak{e}}_n$, $\tau^2_n \subseteq \mathfrak{f}^{\bar{\mathfrak{e}}}_n$ be trees rooted at ρ^1_n, $\rho^2_n \in [[a_n, \mathfrak{e}^-]]$ converging to τ^1 and τ^2. We claim that, for n sufficiently large, $\wp_n$ passes through τ^1_n or τ^2_n. First, notice that, for n large enough, $\wp_n \cap [[\rho^1_n, \rho^2_n]] = \varnothing$. Otherwise, for infinitely many n's, we could find $\alpha_n \in \wp_n \cap [[\rho^1_n, \rho^2_n]]$, and, up to extraction, we would have $\alpha_n \to \alpha \in [[\rho^1, \rho^2]] \subseteq [[a, d]]\backslash\{a\}$. Furthermore, $d_{\mathfrak{q}_n}(a_n, \alpha_n) \le d_{\mathfrak{q}_n}(a_n, b_n)$ so that $a \sim_\infty \alpha$, and $\mathfrak{L}_\infty(a) = \mathfrak{L}_\infty(\alpha)$ by Lemma 4.2, which is impossible. For n even larger, it holds that $\inf_{\tau^i_n} \mathfrak{l}_n < \inf_{\wp_n} \mathfrak{l}_n$. Roughly speaking, $\wp_n$ cannot go from a tree located at the right of τ^1_n (resp. at the left of τ^2_n) to a tree located at its left in $\mathfrak{f}^{\mathfrak{e}}_n$ (resp. to a tree located at its right in $\mathfrak{f}^{\bar{\mathfrak{e}}}_n$) without entering it. Then $\wp_n$ has to enter τ^1_n from the right or τ^2_n from the left and pass through one of these trees (see Figure 4.4).

More precisely, we call $[\![s^1_n, t^1_n]\!]$ and $[\![s^2_n, t^2_n]\!]$ the sets coding the subtrees τ^1_n and τ^2_n. Let $\omega_n \in [[a_n, \mathfrak{e}^+]]$ be a point that is not overflown by $\wp_n$, $p_n := \inf\{t^1_n \le r \le 2n : \omega_n = \mathfrak{i}_n(r)\}$ and $q_n := \sup\{0 \le r \le s^2_n : \omega_n = \mathfrak{i}_n(r)\}$. Then, we let

$$A_n := \mathfrak{i}_n\left(\overrightarrow{[\![t^1_n, p_n]\!]} \cup \overrightarrow{[\![q_n, s^2_n]\!]}\right).$$

We call $\wp_n(i-1)$ the last point of $\wp_n$ belonging to A_n. Such a point exists because $a_n \in A_n$ and $b_n \notin A_n$. The remarks in the preceding paragraphs yield that neither $\wp_n(i-1)$ nor $\wp_n(i)$ belong to $[[\rho^1_n, \rho^2_n]]$, and, because of the way arcs are constructed in the Chapuy–Marcus–Schaeffer bijection, we see that $\wp_n(i) \in \tau^1_n \cup \tau^2_n$. Without loss of generality, we may assume that $\wp_n(i) \in \tau^1_n$. Because $\wp_n$ does not overfly ω_n, it enters τ^1_n from the right at time i, that is, $\mathfrak{l}_n(\wp_n(i)) = \mathfrak{l}_n(\wp_n(i-1)) + 1$. Let $\wp_n(j+1)$ be the first point after $\wp_n(i)$ not belonging to τ^1_n. It exists because $b_n \notin \tau^1_n$. Then, because $\wp_n(j+1) \notin A_n$ and $\wp_n$ does not overfly ω_n, we see that $\mathfrak{l}_n(\wp_n(j+1)) = \mathfrak{l}_n(\wp_n(j)) + 1$, so that $\wp_n$ passes through τ^1_n between times i and j.

In case (ii), we apply the same reasoning with τ^3 and τ^4 instead of τ^1 and τ^2. In case (iii), the four trees τ^1, τ^2, τ^3 and τ^4 are well defined and we obtain that $\wp_n$ has to pass through one of their discrete approximations. We then conclude by Lemma 4.6 that $a \not\sim_\infty b$, which contradicts our hypothesis.

We treat the case where $\rho_b \in [[\mathfrak{e}]]\backslash\{a\}$ in a similar way, simply by replacing $\mathfrak{e}^+$ (resp. $\mathfrak{e}^-$) by ρ_b if $\rho_b \in [[a, \mathfrak{e}^+]]$ (resp. $\rho_b \in [[a, \mathfrak{e}^-]]$). When a is a node, we apply the same arguments, finding up to six

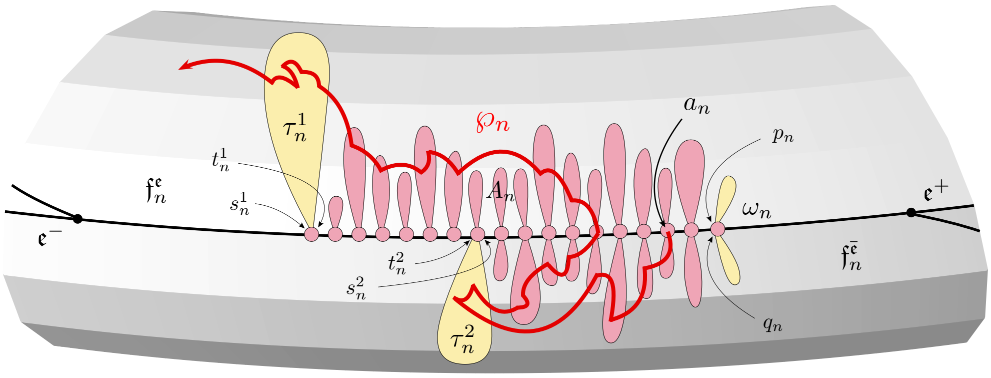

Figure 4.4. *The path $\wp_n$ passing through the tree τ_n^1.*

trees (one for each forest containing a). Finally, if $\rho_b = a$, then a is a strict ancestor of b. This will be a particular case of Lemma 4.11. □

4.2.4.2 Points are not identified with their strict ancestors

Lemma 4.11. *A.s., for every $a, b \in \mathscr{T}_\infty$ such that $\rho_a = \rho_b$ and $a \prec b$, we have $a \not\sim_\infty b$.*

The proof of this lemma uses the same kind of arguments we used in Section 4.2.4.1, is slightly easier than the proof of Lemma 4.10, and is very similar to Le Gall's proof for Proposition 4.2 in [LG07], so that we leave the details to the reader.

4.2.4.3 Points a, b are only identified when $d_\infty^\circ(a, b) = 0$

Lemma 4.12. *A.s., for every tree $\tau \subseteq \mathscr{T}_\infty$ rooted at $\rho \in \mathit{fl}_\infty$ and all $a, b \in \tau \backslash \{\rho\}$ satisfying $a \sim_\infty b$, we have $d_\infty^\circ(a, b) = 0$.*

Proof. Let $\tau \subseteq \mathscr{T}_\infty$ be a tree rooted at $\rho \in \mathit{fl}_\infty$ and $a, b \in \tau \backslash \{\rho\}$ satisfying $a \neq b$ and $a \sim_\infty b$. By Lemma 4.11, we know that $a \not\prec b$ and $b \not\prec a$. As a consequence, we have either $s < t$ for all $(s, t) \in \mathscr{T}_\infty^{-1}(a) \times \mathscr{T}_\infty^{-1}(b)$ or $s > t$ for all $(s, t) \in \mathscr{T}_\infty^{-1}(a) \times \mathscr{T}_\infty^{-1}(b)$. Without loss of generality, we will assume that the first case occurs. Let us suppose that $d_\infty^\circ(a, b) > 0$. By Lemma 4.2, we know that $\mathfrak{L}_\infty(a) = \mathfrak{L}_\infty(b)$, and by (4.4), we have both $\inf_{[a,b]} \mathfrak{L}_\infty < \mathfrak{L}_\infty(a)$ and $\inf_{[b,a]} \mathfrak{L}_\infty < \mathfrak{L}_\infty(a)$. As a result, there are two subtrees $\tau^1 \subseteq [a, b]$ and $\tau^2 \subseteq [b, a]$ rooted at $\rho^1 \in [[a, b]] \backslash \{a, b\}$ and $\rho^2 \in ([[\rho, a]] \cup [[\rho, b]] \cup \mathit{fl}_\infty) \backslash \{a, b\}$ satisfying $\inf_{\tau^i} \mathfrak{L}_\infty < \mathfrak{L}_\infty(a)$.

Let $\tau_n \subseteq \mathfrak{t}_n$ be a tree rooted at ρ_n and a_n, $b_n \in \mathfrak{t}_n$ be points converging to τ, a, and b. Let $\tau_n^1 \subseteq [a_n, b_n]$ and $\tau_n^2 \subseteq [b_n, a_n]$ be subtrees rooted at $\rho_n^1 \in [[a_n, b_n]] \backslash \{a_n, b_n\}$ and $\rho_n^2 \in ([[\rho_n, a_n]] \cup [[\rho_n, b_n]] \cup \mathit{fl}_n) \backslash \{a_n, b_n\}$ converging toward τ^1 and τ^2. We consider a geodesic path $\wp_n$ from a_n to b_n. Recall that $a \sim_\infty b$ implies that $d_{\mathfrak{q}_n}(a_n, b_n) = o(n^{1/4})$.

Because every point in $[[\rho, \rho^1]]$ is a strict ancestor to a or b, for n large enough, $\wp_n$ does not intersect $[[\rho_n, \rho_n^1]]$. Otherwise, we could find an accumulation point α identified with a and b, such that $\alpha \prec a$ or $\alpha \prec b$ (possibly both), and this would contradict Lemma 4.11. If $\rho^2 \in \tau$, for n large, $\wp_n$ does not intersect $[[\rho_n, \rho_n^2]]$ either. The same reasoning yields that $\wp_n$ does not intersect fl_n for n sufficiently large, because of Lemma 4.10.

Let $[\![s_n^1, t_n^1]\!]$ and $[\![s_n^2, t_n^2]\!]$ be the sets coding the subtrees τ_n^1 and τ_n^2. We let

$$A_n := \mathfrak{t}_n\left(\overrightarrow{[\![t_n^2, s_n^1]\!]}\right) \qquad \text{and} \qquad B_n := \mathfrak{t}_n\left(\overrightarrow{[\![t_n^1, s_n^2]\!]}\right).$$

By convention, if $\rho_n^2 \notin \mathfrak{f}_n^{\mathfrak{e}}$, we set $[[\rho_n, \rho_n^2]] := \varnothing$. It is easy to see that $a_n \in A_n$, $b_n \in B_n$, $A_n \cap B_n \subseteq [[\rho_n, \rho_n^1]] \cup [[\rho_n, \rho_n^2]] \cup \mathit{fl}_n$ and $A_n \cup B_n \cup \tau_n^1 \cup \tau_n^2 = \mathfrak{t}_n$.

We conclude as in the proof of Lemma 4.10. We call $\wp_n(i-1)$ the last point of $\wp_n$ belonging to A_n. Such a point exists because $a_n \in A_n$ and $b_n \notin A_n$. The remarks in the preceding paragraphs yield that, for n large enough, neither $\wp_n(i-1)$ nor $\wp_n(i)$ belong to $A_n \cap B_n$. For n even larger, $\inf_{\tau_n^j} \mathfrak{l}_n < \inf_{\wp_n} \mathfrak{l}_n$, and because of the way arcs are constructed in the Chapuy–Marcus–Schaeffer bijection, we see that $\wp_n(i) \in \tau_n^1 \cup \tau_n^2$. The path $\wp_n$ either enters τ_n^1 from the left or enters τ_n^2 from the right. Without loss of generality, we may suppose that $\wp_n(i) \in \tau_n^1$. Let $\wp_n(i'+1)$ be the first point after $\wp_n(i)$ not belonging to τ_n^1. Then $\wp_n(i'+1) \in B_n \cup \tau_n^2$. If $\wp_n$ passes through τ_n^1 between times i and i', we are done. Otherwise, $\wp_n(i'+1) \in \tau_n^2$ because of the condition $\inf_{\tau_n^2} \mathfrak{l}_n < \inf_{\wp_n} \mathfrak{l}_n$ (informally, $\wp_n$ cannot pass over τ_n^2 without entering it). We consider the first point $\wp_n(i''+1)$ after $\wp_n(i')$ not belonging to τ_n^2, and reiterate the argument. Because $\wp_n$ is a finite path, we see that $\wp_n$ will eventually pass through τ_n^1 or τ_n^2.

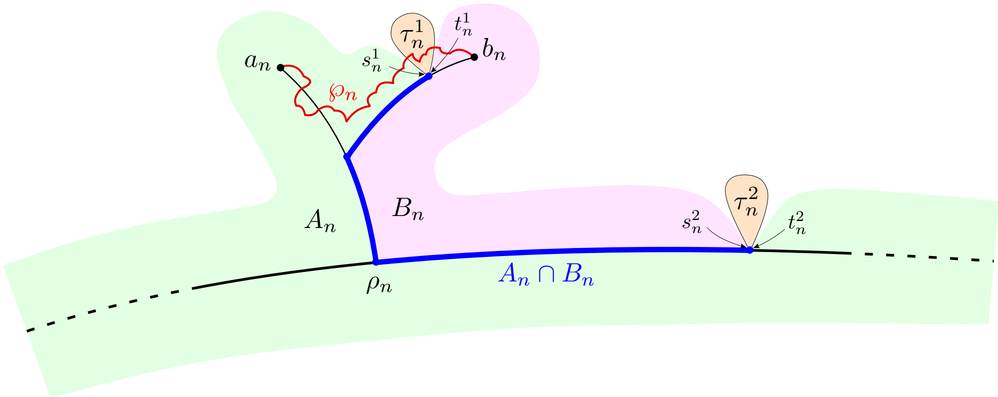

Figure 4.5. *The path $\wp_n$ passing through the subtree τ_n^1.*

If $\wp_n$ passes through τ_n^1 (resp. τ_n^2) for infinitely many n's, a reasoning similar to the one we used in the proof of Lemma 4.9 yields that $\mathfrak{L}_\infty(\rho^1) > \mathfrak{L}_\infty(a)$ (resp. $\mathfrak{L}_\infty(\rho^2) > \mathfrak{L}_\infty(a)$). We conclude by Lemma 4.6 that $a \sim_\infty b$. This is a contradiction. $\square$

Lemma 4.13. *A.s., for all $a, b \in \mathscr{T}_\infty \backslash \mathit{fl}_\infty$ such that $\rho_a \neq \rho_b$ and $a \sim_\infty b$, we have $d_\infty^\circ(a,b) = 0$.*

Proof. The proof of this lemma is very similar to that of Lemma 4.12. Let $a, b \in \mathscr{T}_\infty \backslash \mathit{fl}_\infty$ be such that $\rho_a \neq \rho_b$ and $a \sim_\infty b$. Here again, we may suppose that $s < t$ for all $(s,t) \in \mathscr{T}_\infty^{-1}(a) \times \mathscr{T}_\infty^{-1}(b)$, and we can find two subtrees $\tau^1 \subseteq [a,b]$ and $\tau^2 \subseteq [b,a]$ rooted at $\rho^1, \rho^2 \in ([[\rho_a, a]] \cup [[\rho_b, b]] \cup \mathit{fl}_\infty) \backslash \{a,b\}$ satisfying $\inf_{\tau^i} \mathfrak{L}_\infty < \mathfrak{L}_\infty(a)$. As before, we consider the discrete approximations a_n, b_n, $\tau_n^1 = \mathfrak{i}_n([\![s_n^1, t_n^1]\!])$ and $\tau_n^2 = \mathfrak{i}_n([\![s_n^2, t_n^2]\!])$ of a, b, τ^1 and τ^2. Let $\wp_n$ be a geodesic path from a_n to b_n. We still define

$$A_n := \mathfrak{i}_n\left(\overrightarrow{[\![t_n^2, s_n^1]\!]}\right) \qquad \text{and} \qquad B_n := \mathfrak{i}_n\left(\overrightarrow{[\![t_n^1, s_n^2]\!]}\right),$$

and we see by the same arguments as in Lemma 4.12 that, for n sufficiently large, $\wp_n$ does not intersect $A_n \cap B_n$. We then conclude exactly as before. $\square$

Theorem 4.3 follows from Lemmas 4.10, 4.11, 4.12 and 4.13. A straightforward consequence of Theorem 4.3 is that, if the equivalence class of $a = \mathscr{T}_\infty(s)$ for $\sim_\infty$ is not trivial, then s is an increase point of $\mathfrak{L}_\infty$. By Lemma 4.5, the equivalence class of a for $\simeq_\infty$ is then trivial. Such points may be called **leaves** by analogy with tree terminology.

4.3 1-regularity of quadrangulations

The goal of this section is to prove Theorem 1.13. To that end, we use the notion of regular convergence, introduced by Whyburn [Why35b].

4.3.1 1-regularity

Recall that $(\mathbb{M}, d_{GH})$ denotes the set of isometry classes of compact metric spaces, endowed with the Gromov–Hausdorff metric. We say that a compact metric space $(\mathcal{X}, \delta)$ is a **path metric space** if any two points $x, y \in \mathcal{X}$ may be joined by a path isometric to a real segment—necessarily of length $\delta(x, y)$. We call $\mathbb{PM}$ the set of isometry classes of path metric spaces. By [BBI01, Theorem 7.5.1], $\mathbb{PM}$ is a closed subset of $\mathbb{M}$.

Definition 4.6. *We say that a sequence $(\mathcal{X}_n)_{n\geq 1}$ of path metric spaces is **1-regular** if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for n large enough, every loop of diameter less than δ in $\mathcal{X}_n$ is homotopic to 0 in its ε-neighborhood.*

This definition is actually slightly stronger than Whyburn's original definition [Why35b]. See the discussion in the second section of [Mie08] for more details. We also chose here not to restrict the notion of 1-regularity only to converging sequences of path metric spaces, as it was done in [Mie08, Why35b], because the notion of 1-regularity (as stated here) is not directly related to the convergence of the sequence of path metric spaces. Our main tool is the following theorem, which is a simple consequence of Begle [Beg44, Theorem 7].

Proposition 4.14. *Let $(\mathcal{X}_n)_{n\geq 1}$ be a sequence of path metric spaces all homeomorphic to the g-torus $\mathbb{T}_g$. Suppose that $\mathcal{X}_n$ converges toward $\mathcal{X}$ for the Gromov–Hausdorff topology, and that the sequence $(\mathcal{X}_n)_{n\geq 1}$ is 1-regular. Then $\mathcal{X}$ is homeomorphic to $\mathbb{T}_g$ as well.*

4.3.2 Representation as metric surfaces

In order to apply Proposition 4.14, we construct a path metric space $(\mathcal{S}_n, \delta_n)$ homeomorphic to $\mathbb{T}_g$, and an embedded graph that is a representative of the map $\mathfrak{q}_n$, such that the restriction of $(\mathcal{S}_n, \delta_n)$ to the embedded graph is isometric to $\big(V(\mathfrak{q}_n), d_{\mathfrak{q}_n}\big)$. We use the method provided by Miermont in [Mie08, Section 3.1].

We write $F(\mathfrak{q}_n)$ the set of faces of $\mathfrak{q}_n$. Let (X_f, D_f), $f \in F(\mathfrak{q}_n)$ be n copies of the hollow bottomless unit cube

$$X_f := [0,1]^3 \backslash \big((0,1)^2 \times [0,1)\big)$$

endowed with the intrinsic metric D_f inherited from the Euclidean metric. (The distance between two points x and y is the Euclidean length of a minimal path in X_f linking x to y.)

Let $f \in F(\mathfrak{q}_n)$, and let e_1, e_2, e_3, and e_4 be the four half-edges incident to f, ordered according to the counterclockwise order. For $0 \leq t \leq 1$, we define

$$\begin{aligned} c_{e_1}(t) &= (t, 0, 0) \in X_f, \\ c_{e_2}(t) &= (1, t, 0) \in X_f, \\ c_{e_3}(t) &= (1-t, 1, 0) \in X_f, \\ c_{e_4}(t) &= (0, 1-t, 0) \in X_f. \end{aligned}$$

In this way, we associate with every half-edge $e \in \vec{E}(\mathfrak{q}_n)$ a path along one of the four edges of the square ∂X_f, where f is the face located to the left of e.

We then define the relation $\approx$ as the coarsest equivalence relation for which $c_e(t) \approx c_{\bar{e}}(1-t)$ for all $e \in \vec{E}(\mathfrak{q}_n)$ and $t \in [0,1]$. This corresponds to gluing the spaces X_f's along their boundaries according to the map structure of $\mathfrak{q}_n$. The topological quotient $\mathcal{S}_n := (\coprod_{f\in F(\mathfrak{q}_n)} X_f)_{/\approx}$ is a 2-dimensional CW-complex satisfying the following. Its 1-skeleton $\mathcal{E}_n = (\coprod_{f\in F(\mathfrak{q}_n)} \partial X_f)_{/\approx}$ is an embedding of $\mathfrak{q}_n$ with faces $X_f \backslash \partial X_f$. To the edge $\{e, \bar{e}\} \in E(\mathfrak{q}_n)$ corresponds the edge of $\mathcal{S}_n$ made of the equivalence class of the points in $c_e([0,1])$. Its 0-skeleton $\mathcal{V}_n$ is in one-to-one correspondence with $V(\mathfrak{q}_n)$. Its vertices are the equivalence classes of the corners of the squares ∂X_f.

We endow the space $\coprod_{f\in F(\mathfrak{q}_n)} X_f$ with the largest pseudo-metric δ_n compatible with D_f, $f \in F(\mathfrak{q}_n)$ and $\approx$, in the sense that $\delta_n(x,y) \leq D_f(x,y)$ for $x, y \in X_f$ and $\delta_n(x,y) = 0$ whenever $x \approx y$. Its quotient—still noted δ_n—then defines a pseudo-metric on $\mathcal{S}_n$ (which actually is a true metric, as we will see in Proposition 4.15). As usual, we define $\delta_{(n)} := \delta_n/\gamma\, n^{1/4}$ its rescaled version.

We rely on the following proposition. It was actually stated in [Mie08] for the 2-dimensional sphere but readily extends to the g-torus.

Proposition 4.15 ([Mie08, Proposition 1]). *The space $(\mathcal{S}_n, \delta_n)$ is a path metric space homeomorphic to $\mathbb{T}_g$. Moreover, the restriction of $\mathcal{S}_n$ to $\mathcal{V}_n$ is isometric to $(V(\mathfrak{q}_n), d_{\mathfrak{q}_n})$, and any geodesic path in $\mathcal{S}_n$ between points in $\mathcal{V}_n$ is a concatenation of edges of $\mathcal{S}_n$. Finally, $d_{GH}\Big((V(\mathfrak{q}_n), d_{\mathfrak{q}_n}), (\mathcal{S}_n, \delta_n)\Big) \le 3$, so that, by Theorem 1.11,*

$$\Big(\mathcal{S}_{n_k}, \delta_{(n_k)}\Big) \xrightarrow[k\to\infty]{(d)} (\mathfrak{q}_\infty, d_\infty)$$

in the sense of the Gromov–Hausdorff topology.

4.3.3 Proof of Theorem 1.13

We prove here that $(\mathfrak{q}_\infty, d_\infty)$ is a.s. homeomorphic to $\mathbb{T}_g$ by means of Propositions 4.14 and 4.15. To this end, we only need to show that the sequence $\Big(\mathcal{S}_{n_k}, \delta_{(n_k)}\Big)_k$ is 1-regular. At first, we only consider simple loops made of edges. We proceed in two steps: Lemma 4.16 shows that there are no noncontractible "small" loops; then Lemma 4.17 states that the small loops are homotopic to 0 in their ε-neighborhood.

Lemma 4.16. *A.s., there exists $\varepsilon_0 > 0$ such that for all k large enough, any noncontractible simple loop made of edges in $\mathcal{S}_{n_k}$ has diameter greater than ε_0.*

Proof. The basic idea is that a noncontractible loop in $\mathcal{S}_n$ has to intersect fl_n and to "jump" from a forest to another one. At the limit, the loop transits from a forest to another by visiting two points that $\sim_\infty$ identifies. If the loops vanish at the limit, then these two identified points become identified with a point in fl_∞, creating an increase point for both $\mathfrak{L}_\infty$ et $\mathfrak{C}_\infty$. We proceed to the rigorous proof.

We argue by contradiction and assume that, with positive probability, along some (random) subsequence of the sequence $(n_k)_{k\ge 0}$, there exist noncontractible simple loops $\wp_n$ made of edges in $\mathcal{S}_n$ with diameter tending to 0 (with respect to the rescaled metric $\delta_{(n)}$). We reason on this event.

Because $\wp_n$ is noncontractible, it has to intersect fl_n: if not, $\wp_n$ would entirely be drawn in the unique face of $\mathfrak{s}_n$, which is homeomorphic to a disk, by definition of a map. It would thus be contractible, by the Jordan curve theorem. Let $a_n \in \wp_n \cap \mathit{fl}_n$. Up to further extraction, we may suppose that $a_n \to a \in \mathit{fl}_\infty$. Notice that every time $\wp_n$ intersects fl_n, it has to be "close" to a_n. Precisely, if $b_n \in \wp_n \cap \mathit{fl}_n$ tends to b, then $\delta_{(n)}(a_n, b_n) \le \operatorname{diam}(\wp_n) \to 0$, which yields $a \sim_\infty b$, and $a = b$ by Lemma 4.10. Moreover, for n sufficiently large, the base point $v_n^\bullet \notin \wp_n$: otherwise, for infinitely many n's, $(\mathfrak{l}_n(a_n) - \min \mathfrak{l}_n + 1)/\gamma\, n^{1/4} \le \operatorname{diam}(\wp_n) \to 0$, so that $\mathfrak{L}_\infty$ would reach its minimum at a, and we know by Lemma 4.4 that this is not the case.

Let us first suppose that a is not a node of $\mathscr{T}_\infty$. There exists $\mathfrak{e} \in \vec{E}(\mathfrak{s}_\infty)$ such that $a \in \mathfrak{f}_\infty^{\mathfrak{e}} \cap \mathfrak{f}_\infty^{\bar{\mathfrak{e}}}$ and for n large enough, $a_n \in \mathfrak{f}_n^{\mathfrak{e}} \cap \mathfrak{f}_n^{\bar{\mathfrak{e}}}$. For n even larger, the whole loop $\wp_n$ "stays in $\mathfrak{f}_n^{\mathfrak{e}} \cup \mathfrak{f}_n^{\bar{\mathfrak{e}}}$." Precisely, for all $\mathfrak{e}' \in \vec{E}(\mathfrak{s}_\infty)\backslash\{\mathfrak{e}, \bar{\mathfrak{e}}\}$, we have $\wp_n \cap \mathfrak{f}_n^{\mathfrak{e}'} = \varnothing$. Otherwise, since $\vec{E}(\mathfrak{s}_\infty)$ is finite, there would exist $\mathfrak{e}' \notin \{\mathfrak{e}, \bar{\mathfrak{e}}\}$ such that for infinitely many n's, we can find $c_n \in \wp_n \cap \mathfrak{f}_n^{\mathfrak{e}'}$. Up to extraction, $c_n \to c \in \mathfrak{f}_\infty^{\mathfrak{e}'}$, so that $c \ne a$ (a is not a node) and $c \sim_\infty a$, which is impossible, by Lemma 4.10.

We claim that there exists an arc of $\wp_n$ linking a point $b_n \in \mathfrak{f}_n^{\mathfrak{e}}$ to some point in $\mathfrak{f}_n^{\bar{\mathfrak{e}}}$ that overflies either $[[\rho_{b_n}, \mathfrak{e}^+]]$ or $[[\mathfrak{e}^-, \rho_{b_n}]]$ (see Figure 4.6). Let suppose for a moment that this does not hold. In particular, there is no arc linking a point in $\mathfrak{f}_n^{\mathfrak{e}}\backslash\mathit{fl}_n$ to a point in $\mathfrak{f}_n^{\bar{\mathfrak{e}}}\backslash\mathit{fl}_n$. It will be more convenient here to write $\wp_n$ as $(a_n = v_1, \alpha_1, v_2, \alpha_2, \dots, v_{r-1}, \alpha_{r-1}, v_r = a_n)$ where the v_i's are vertices and the α_i's are arcs. Let $i := \inf\{j \in [\![2, r]\!] : v_j \in \mathit{fl}_n\}$ be the index of the first time $\wp_n$ returns to fl_n. Then $v_2, \dots, v_{i-1}$ belong to the same set $\mathfrak{f}_n^{\mathfrak{e}}\backslash\mathit{fl}_n$ or $\mathfrak{f}_n^{\bar{\mathfrak{e}}}\backslash\mathit{fl}_n$, and $(\alpha_1, v_2, \alpha_2, \dots, v_{i-1}, \alpha_{i-1})$ is thus drawn inside the face of $\mathfrak{s}_n$. As a result, the path $(v_1, \alpha_1, v_2, \dots, v_{i-1}, \alpha_{i-1}, v_i)$ is homotopic to the segment $[[v_1, v_i]]$. Repeating the argument for every "excursion" away from fl_n, we see that $\wp_n$ is homotopic to a finite concatenation of

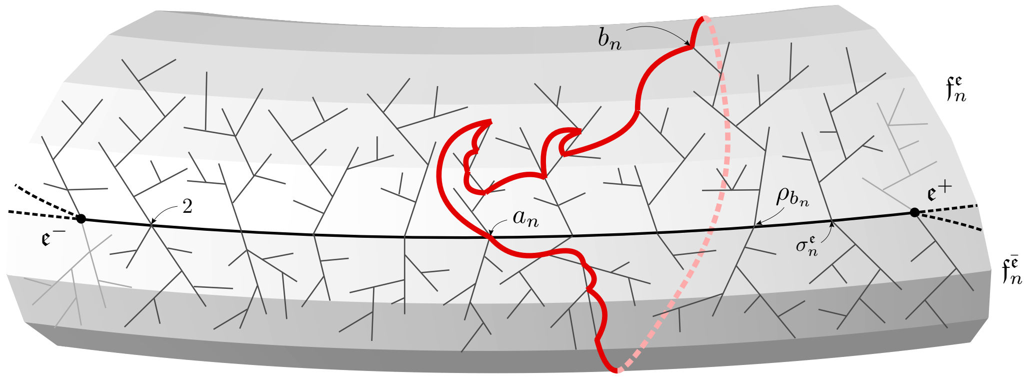

Figure 4.6. *A noncontractible loop intersecting $\wp_n$ at a_n and "jumping" from $\mathfrak{f}_n^{\mathfrak{e}}$ to $\mathfrak{f}_n^{\bar{\mathfrak{e}}}$ at b_n.*

segments all included in the topological segment $[[2, \sigma_n^{\mathfrak{e}}]]$, where we used the notation of Section 2.2.1 for the forest $\mathfrak{f}_n^{\mathfrak{e}}$ (see Figure 4.6). It follows that $\wp_n$ is contractible, which is a contradiction.

We consider the case where the arc from the previous paragraph overflies $[[\rho_{b_n}, \mathfrak{e}^+]]$. The other case is treated in a similar way. From the construction of the Chapuy–Marcus–Schaeffer bijection, we can find integers $s_n \le t_n$ such that $b_n = \mathfrak{i}_n(s_n)$, $\mathfrak{e}^+ = \mathfrak{i}_n(t_n)$ and for all $s_n \le r \le t_n$, $\mathfrak{L}_n(r) \ge \mathfrak{L}_n(s_n)$. Up to further extraction, we may suppose that $s_n/2n \to s$ and $t_n/2n \to t$. Therefore, for all $s \le r \le t$, $\mathfrak{L}_\infty(r) \ge \mathfrak{L}_\infty(s)$. Moreover, the fact that $b_n \to a \neq \mathfrak{e}^+$ yields $s < t$, so that s is an increase point for $\mathfrak{L}_\infty$. But $\mathscr{T}_\infty(s) = a$ and s has to be an increase point for $\mathfrak{C}_\infty$. By Lemma 4.5, this cannot happen.

If a is a node, there are three half-edges $\mathfrak{e}_1$, $\mathfrak{e}_2$ and $\mathfrak{e}_3$ such that $a = \mathfrak{e}_1^+ = \mathfrak{e}_2^+ = \mathfrak{e}_3^+$. A reasoning similar to what precedes yields the existence of an arc of $\wp_n$ linking a point b_n in one of the three sets $\mathfrak{f}^{\mathfrak{e}_i} \cup \mathfrak{f}^{\bar{\mathfrak{e}}_{i+1}}$, $i = 1, 2, 3$ (where we use the convention $\mathfrak{e}_4 = \mathfrak{e}_1$) to a point lying in another one of these three sets that overflies either, if $b_n \in \mathfrak{f}_\infty^{\mathfrak{e}_i}$, $[[\rho_{b_n}, a]] \cup [[\mathfrak{e}_{i+1}]]$ or $[[\mathfrak{e}_i^-, \rho_{b_n}]]$, or, if $b_n \in \mathfrak{f}_\infty^{\mathfrak{e}_{i+1}}$, $[[\rho_{b_n}, \mathfrak{e}_{i+1}^+]]$ or $[[\mathfrak{e}_i]] \cup [[a, \rho_{b_n}]]$. We conclude by similar arguments. □

We now turn our attention to contractible loops. Let $\wp$ be a contractible simple loop in $\mathcal{S}_n$ made of edges. Then $\wp$ splits $\mathcal{S}_n$ into two domains. Only one of these is homeomorphic to a disk[5]. We call it the **inner domain** of $\wp$, and we call the other one the **outer domain** of $\wp$. In particular, these domains are well defined for loops whose diameter is smaller than ε_0, when n is large enough.

Lemma 4.17. *A.s., for all $\varepsilon > 0$, there exists $0 < \delta < \varepsilon \wedge \varepsilon_0$ such that for all k sufficiently large, the inner domain of any simple loop made of edges in $\mathcal{S}_{n_k}$ with diameter less than δ has diameter less than ε.*

Proof. We adapt the method used by Miermont in [Mie08]. The idea is that a contractible loop separates a whole part of the map from the base point. Then the labels in one of the two domains it separates are larger than the labels on the loop. In the g-tree, this corresponds to having a part with labels larger than the labels on the "border." In the continuous limit, this creates an increase point for both $\mathfrak{C}_\infty$ and $\mathfrak{L}_\infty$.

Suppose that, with positive probability, there exists $0 < \varepsilon < \varepsilon_0$ for which, along some (random) subsequence of the sequence $(n_k)_{k\ge 0}$, there exist contractible simple loops $\wp_n$ made of edges in $\mathcal{S}_n$ with diameter tending to 0 (with respect to the rescaled metric $\delta_{(n)}$) and whose inner domains are of diameter larger than ε. Let us reason on this event. First, notice that, because $g \ge 1$, the outer domain of $\wp_n$ contains at least one noncontractible loop, so that its diameter is larger than $\varepsilon_0 > \varepsilon$ by Lemma 4.16.

Let $s^\bullet$ be the unique point where $\mathfrak{L}_\infty$ reaches its minimum, and $s_n^\bullet$ be an integer where $\mathfrak{L}_n$ reaches its minimum. We call $w_n^\bullet := \mathfrak{i}_n(s_n^\bullet)$ the corresponding point in the g-tree. This is a vertex at δ_n-distance 1 from $v_n^\bullet$. Let us take x_n in the domain that does not contain $w_n^\bullet$, such that the distance

[5] This is a consequence of the Jordan–Schönflies theorem, applied in the universal cover of $\mathcal{S}_n$, which is either the plane when $g = 1$, or the unit disk when $g \ge 2$; see, for example, [Eps66, Theorem 1.7].

between x_n and $\wp_n$ is maximal. (If $w_n^\bullet \in \wp_n$, we take x_n in either of the two domains according to some convention.) Let $y_n \in \wp_n \cap ([[\rho_{w_n^\bullet}, w_n^\bullet]] \cup \mathit{fl}_n \cup [[\rho_{x_n}, x_n]])$ be such that there exists an injective path[6] $\mathfrak{p}_n$ in $\mathfrak{t}_n$ from x_n to y_n that intersects $\wp_n$ only at y_n. In other words, when going from x_n to $w_n^\bullet$ along some injective path, y_n is the first vertex belonging to $\wp_n$ we meet (see Figure 4.7). Such a point exists, because x_n and $w_n^\bullet$ do not belong to the same of the two components delimited by $\wp_n$. Up to further extraction, we suppose that $s_n^\bullet/2n \to s^\bullet$, $x_n \to x$, and $y_n \to y$. We call $\mathfrak{p} \subseteq [[\rho_{w^\bullet}, w^\bullet]] \cup \mathit{fl}_\infty \cup [[\rho_x, x]]$ the injective path corresponding to $\mathfrak{p}_n$ in the limit, that is, the path defined as $\mathfrak{p}_n$ "without the subscripts n." Because the distance between two points in the same domain as x_n is smaller than $2\delta_{(n)}(x_n, \wp_n) + \mathrm{diam}(\wp_n)$, we obtain that $\delta_{(n)}(x_n, y_n) \geq \varepsilon/4$, as soon as $\mathrm{diam}(\wp_n) \leq \varepsilon/2$. In particular, we see that $x \neq y$, and that the path $\mathfrak{p}$ is not reduced to a single point.

Let us first suppose that $y \neq w^\bullet := \mathscr{T}_\infty(s^\bullet)$. (In particular, $w_n^\bullet \notin \wp_n$ for n large, so that there is no ambiguity on which domain to chose x_n.) In that case, $y \in ([[\rho_{w^\bullet}, w^\bullet]] \cup \mathit{fl}_\infty \cup [[\rho_x, x]]) \backslash \{x, w^\bullet\}$, so that the points in $\mathscr{T}_\infty^{-1}(y)$ are increase points of $\mathfrak{C}_\infty$. By Lemma 4.5, we can find a subtree[7] τ, not containing y, satisfying $\inf_\tau \mathfrak{L}_\infty < \mathfrak{L}_\infty(y)$ and rooted on the path $\mathfrak{p}$.

We consider a discrete approximation τ_n rooted on $\mathfrak{p}_n$. Because the loop $\wp_n$ is contractible, all the labels of the points in the same domain as x_n are larger than $\inf_{\wp_n} \mathfrak{l}_n$. Indeed, the labels represent the distances (up to an additive constant) in $\mathfrak{q}_n$ to the base point, and every geodesic path from such a point to the base point has to intersect $\wp_n$. For n large enough, it holds that $\inf_{\tau_n} \mathfrak{l}_n < \inf_{\wp_n} \mathfrak{l}_n$. As a consequence, τ_n cannot entirely be included in the domain containing x_n. Therefore, the set $\wp_n \cap \tau_n$ is not empty, so that we can find $z_n \in \wp_n \cap \tau_n$. Up to extraction, we may suppose that $z_n \to z$.

On the one hand, $\delta_{(n)}(y_n, z_n) \leq \mathrm{diam}(\wp_n)$, so that $y \sim_\infty z$. On the other hand, $z \in \tau$ and $y \notin \tau$, so that $y \neq z$. Because y is not a leaf, this contradicts Theorem 4.3.

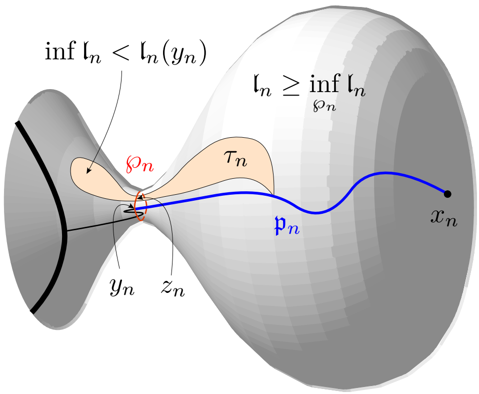

Figure 4.7. *The path $\wp_n$ intersects τ_n. This figure represents the case where $y_n \in [[\rho_{x_n}, x_n]]$.*

[6]Depending on the case, the path $\mathfrak{p}_n$ will be of one of the following forms

- $[[x_n, y_n]]$, with $y_n \in [[\rho_{x_n}, x_n]]$,
- $[[x_n, \rho_{x_n}]] \cup [[\rho_{x_n}, y_n]]$, with $y_n \in \mathit{fl}_n$,
- $[[x_n, \rho_{x_n}]] \cup [[\rho_{x_n}, e_1^+]] \cup [[e_2]] \cup \cdots \cup [[e_k]] \cup [[e_k^+, y_n]]$ for some half-edges $e_1, e_2, \ldots, e_k$ of $\mathfrak{s}_n$ satisfying $e_i^+ = e_{i+1}^-$, with $y_n \in \mathit{fl}_n$,
- $[[x_n, \rho_{x_n}]] \cup [[\rho_{x_n}, e_1^+]] \cup [[e_2]] \cup \cdots \cup [[e_k]] \cup [[e_k^+, \rho_{w_n^\bullet}]] \cup [[\rho_{w_n^\bullet}, y_n]]$ for some half-edges $e_1, e_2, \ldots, e_k$ of $\mathfrak{s}_n$ satisfying $e_i^+ = e_{i+1}^-$, with $y_n \in [[\rho_{w_n^\bullet}, w_n^\bullet]]$.

[7]Here again, we need to distinguish between some cases:

- if $y \in [[\rho_x, x]]$, then $\mathfrak{p} = [[x, y]]$ and τ is a tree to the left or right of $[[\rho_x, x]]$ rooted at some point in $[[x, y]] \backslash \{x, y\}$,
- if $y \in \mathit{fl}_\infty \backslash \{\rho_x\}$, then τ is a tree of $\mathscr{T}_\infty$ rooted on $(\mathfrak{p} \cap \mathit{fl}_\infty) \backslash \{y\}$,
- if $y \in [[\rho_{w^\bullet}, w^\bullet]] \backslash \{\rho_{w^\bullet}\}$, then τ is a tree to the left or right of $[[\rho_{w^\bullet}, y]]$.

When $y = w^\bullet$, we use a different argument. Let $a_n = \mathfrak{t}_n(\alpha_n)$ and $b_n = \mathfrak{t}_n(\beta_n)$ be respectively in the inner and outer domains of $\wp_n$, such that their distance to $\wp_n$ is maximal. Because a_n and b_n do not belong to the same domain, we can find

$$t_n^1 \in \overrightarrow{[\![\alpha_n, \beta_n]\!]} \qquad \text{and} \qquad t_n^2 \in \overrightarrow{[\![\beta_n, \alpha_n]\!]}$$

such that $\mathfrak{t}_n(t_n^1), \mathfrak{t}_n(t_n^2) \in \wp_n$. Up to extraction, we suppose that

$$\frac{\alpha_n}{2n} \to \alpha, \qquad \frac{\beta_n}{2n} \to \beta, \qquad \frac{t_n^1}{2n} \to t^1 \in \overrightarrow{[\alpha, \beta]} \qquad \text{and} \qquad \frac{t_n^2}{2n} \to t^2 \in \overrightarrow{[\beta, \alpha]}.$$

Because $\mathrm{diam}(\wp_n) \to 0$, we have $\mathscr{T}_\infty(t^1) = \mathscr{T}_\infty(t^2) = w^\bullet$. Moreover, the argument we used to prove that $x \neq y$ yields that $\mathscr{T}_\infty(\alpha) \neq w^\bullet$ and $\mathscr{T}_\infty(\beta) \neq w^\bullet$. As a result, we obtain that $t^1 \neq t^2$. This contradicts Lemma 4.4. □

It remains to deal with general loops that are not necessarily made of edges. We reason on the set of full probability where Lemmas 4.16 and 4.17 hold. We fix $0 < \varepsilon < \mathrm{diam}(\mathfrak{q}_\infty)/4$. Let ε_0 be as in Lemma 4.16 and δ as in Lemma 4.17. For k sufficiently large, the conclusions of both lemmas hold, together with the inequality $\delta\, \gamma\, n_k^{1/4} \geq 12$. Now, take any loop $\mathscr{L}$ drawn in $\mathcal{S}_{n_k}$ with diameter less than $\delta/2$. Consider the union of the closed faces[8] visited by $\mathscr{L}$. The boundary of this union consists in simple loops made of edges in $\mathcal{S}_{n_k}$. Let us call Λ the set of these simple loops.

Because every face of $\mathcal{S}_{n_k}$ has a diameter smaller than $3/\gamma\, n_k^{1/4}$, we see that for all $\lambda \in \Lambda$, $\mathrm{diam}(\lambda) \leq \mathrm{diam}(\mathscr{L}) + 6/\gamma\, n_k^{1/4} \leq \delta$. Then, by Lemma 4.16, λ is contractible and, by Lemma 4.17, its inner domain is of diameter less than ε. By definition, for all $\lambda \in \Lambda$, $\mathscr{L}$ entirely lies either inside the inner domain of λ, or inside its outer domain. We claim that there exists one loop in Λ such that $\mathscr{L}$ lies in its inner domain. Then, it will be obvious that $\mathscr{L}$ is homotopic to 0 in its ε-neighborhood.

Let us suppose that $\mathscr{L}$ lies in the outer domain of every loop $\lambda \in \Lambda$. Then, every face of $\mathcal{S}_{n_k}$ is either visited by $\mathscr{L}$, or included in the inner domain of some loop $\lambda \in \Lambda$. As a result, we obtain that $\mathrm{diam}(\mathfrak{q}_\infty) \leq \mathrm{diam}(\mathscr{L}) + 2 \sup_{\lambda \in \Lambda} \mathrm{diam}(\lambda) + 6/\gamma\, n_k^{1/4} \leq 3\delta$. This is in contradiction with our choice of δ.

4.4 Transfering results from the planar case through Chapuy's bijection

In order to prove Lemmas 4.4, 4.5, 4.7, and 4.8, we rely on similar results for the Brownian snake driven by a normalized excursion $(\mathbb{e}, Z)$. This means that $\mathbb{e}$ has the law of a normalized Brownian excursion, and, conditionally given $\mathbb{e}$, the process Z is a Gaussian process with covariance

$$\mathrm{cov}(Z_x, Z_y) = \inf_{[x \wedge y, x \vee y]} \mathbb{e}.$$

We first focus on the proofs of Lemmas 4.4 and 4.5. Lemmas 3.1 and 3.2 in [LGP08] state that, a.s., Z reaches its minimum at a unique point, and that, a.s., $\mathrm{IP}(\mathbb{e})$ and $\mathrm{IP}(Z)$ are disjoint sets. We will use a bijection due to Chapuy [Cha10] to transfer these results to our case.

4.4.1 Chapuy's bijection

Chapuy's bijection consists in "opening" g-trees into plane trees. We briefly describe it here, and refer to [Cha10] for more details. Let $\mathfrak{t}$ be a g-tree whose scheme $\mathfrak{s}$ is dominant. Such a g-tree will be called **dominant** in the following. As usual, we arrange the half-edges of $\mathfrak{s}$ according to its facial order: $\mathfrak{e}_1 = \mathfrak{e}_*, \dots, \mathfrak{e}_{2(6g-3)}$. Let v be one of the nodes of $\mathfrak{t}$. We can see it as a vertex of $\mathfrak{s}$. Let us call $\mathfrak{e}_{i_1}$, $\mathfrak{e}_{i_2}$, and $\mathfrak{e}_{i_3}$ the three half-edges starting from v (i.e. $v = \mathfrak{e}_{i_1}^- = \mathfrak{e}_{i_2}^- = \mathfrak{e}_{i_3}^-$), where $i_1 < i_2 < i_3$. We say that v is **intertwined** if the half-edges $\mathfrak{e}_{i_1}$, $\mathfrak{e}_{i_2}$, $\mathfrak{e}_{i_3}$ are arranged according to the counterclockwise order

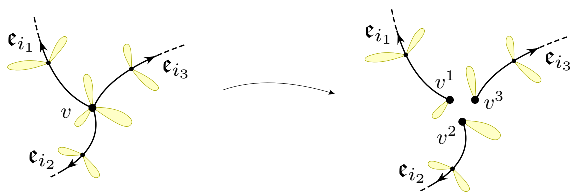

Figure 4.8. *Slicing an intertwined node v.*

around v (see Figure 4.8). When v is intertwined, we may **slice** it: we define a new map, denoted by $\mathfrak{t} \setminus\!\!\setminus v$, by slicing the node v into three new vertices v^1, v^2, and v^3 (see Figure 4.8).

The map obtained by such an operation turns out to be a dominant $(g-1)$-tree. After repeating g times this operation, we are left with a plane tree. In that regard, we call **opening sequence** of $\mathfrak{t}$ a g-uple $(v_1, \ldots, v_g)$ such that v_g is an intertwined node of $\mathfrak{t}$, and for all $1 \leq i \leq g-1$, the vertex v_i is an intertwined node of $\mathfrak{t} \setminus\!\!\setminus v_g \setminus\!\!\setminus \cdots \setminus\!\!\setminus v_{i+1}$. We can show that every g-tree has exactly $2g$ intertwined nodes, and thus $2^g g!$ opening sequences.

To reverse the slicing operation, we have to intertwine and glue back the three vertices together. We then need to record which vertices are to be glued together. This motivates the following definition: we call **tree with g triples** a pair $(\boldsymbol{t}, (c_1, \ldots, c_g))$, where

- $\boldsymbol{t}$ is a (rooted) plane tree,
- for $1 \leq i \leq g$, $c_i = \{v_i^1, v_i^2, v_i^3\} \subseteq V(\boldsymbol{t})$ is a set of three vertices of $\boldsymbol{t}$,
- the vertices v_i^j, $1 \leq i \leq g$, $1 \leq j \leq 3$, are pairwise distinct,
- the vertices of the tree
$$\bigcup_{i,i',j,j'} [[v_i^j, v_{i'}^{j'}]]$$
have degree at most 3 and the v_i^j's have degree exactly 1 in that tree. (As in the case of g-trees, the set $[[a, b]]$ represents the range of the unique path linking a and b in the tree.)

Let $\mathfrak{t}$ be a g-tree together with an opening sequence $(v_1, \ldots, v_g)$. For all $1 \leq i \leq g$, let us call c_i the triple of vertices obtained from the slicing of v_i, as well as $\boldsymbol{t} := \mathfrak{t} \setminus\!\!\setminus v_g \setminus\!\!\setminus \cdots \setminus\!\!\setminus v_1$ the plane tree. We define $\Phi(\mathfrak{t}, (v_1, \ldots, v_g)) := (\boldsymbol{t}, (c_1, \ldots, c_g))$. Then Φ is a bijection from the set of all dominant g-tree equipped with an opening sequence into the set of all trees with g triples.

Now, when the g-tree is well-labeled, we can do the same slicing operation, and the three vertices we obtain all have the same label. We call **well-labeled tree with g triples** a tree with g triples $(\boldsymbol{t}, (c_1, \ldots, c_g))$ carrying a labeling function $\boldsymbol{l} : V(\boldsymbol{t}) \to \mathbb{Z}$ such that

- $\boldsymbol{l}(e^-) = 0$, where e is the root of $\boldsymbol{t}$,
- for every pair of neighboring vertices $v \sim v'$, we have $\boldsymbol{l}(v) - \boldsymbol{l}(v') \in \{-1, 0, 1\}$,
- for all $1 \leq i \leq g$, we have $\boldsymbol{l}(v_i^1) = \boldsymbol{l}(v_i^2) = \boldsymbol{l}(v_i^3)$.

We call $\mathcal{W}_n$ the set of all well-labeled trees with g triples having n edges. The bijection Φ then extends to a bijection between dominant well-labeled g-trees equipped with an opening sequence and well-labeled trees with g triples.

[8]We call **closed face** the closure of a face.

4.4.2 Contour pair of an opened g-tree

The contour pair of an opened g-tree can be obtained from the contour pair of the g-tree itself (and vice versa). The labeling function is basically the same, but read in a different order. The contour function is slightly harder to recover, because half of the forests are to be read with the floor directed "upward" instead of "downward." Because we will deal at the same time with g-trees and plane trees in this section, we will use a Gothic font for objects related to g-trees, and a boldface font for objects related to plane trees. In the following, we use the notation of Section 2.2.

Let $(\mathfrak{t}, \mathfrak{l})$ be a well-labeled dominant g-tree with scheme $\mathfrak{s}$ and $(\boldsymbol{t}, \boldsymbol{l})$ be one of the $2^g g!$ corresponding opened well-labeled trees. The intertwined nodes of the g-tree correspond to intertwined nodes of its scheme, so that the opening sequence used to open $(\mathfrak{t}, \mathfrak{l})$ into $(\boldsymbol{t}, \boldsymbol{l})$ naturally corresponds to an opening sequence of $\mathfrak{s}$. Let $\boldsymbol{s}$ be the tree obtained by opening $\mathfrak{s}$ along this opening sequence. We identify the half-edges of $\mathfrak{s}$ with the half-edges of $\boldsymbol{s}$, and arrange them according to the facial order of $\boldsymbol{s}$: $\boldsymbol{e}_1 = \mathfrak{e}_*$, $\boldsymbol{e}_2$, ..., $\boldsymbol{e}_{2(6g-3)}$. (Beware that this is not the usual arrangement according to the facial order of $\mathfrak{s}$.) Now, the plane tree $\boldsymbol{t}$ is obtained by replacing every half-edge $\boldsymbol{e}$ of $\boldsymbol{s}$ with the corresponding forest $\mathfrak{f}^{\boldsymbol{e}}$ of Proposition 2.3, as in Section 2.2.2.3.

We call $(C^{\mathfrak{e}}, L^{\mathfrak{e}})$ the contour pair of $(\mathfrak{f}^{\mathfrak{e}}, \mathfrak{l}^{\mathfrak{e}})$, we let $\mathfrak{C}^{\mathfrak{e}} := C^{\mathfrak{e}} - \sigma^{\mathfrak{e}}$, and we define $\mathfrak{L}^{\mathfrak{e}}$ by (2.11). For any edge $\{\boldsymbol{e}_i, \boldsymbol{e}_j\} \neq \{\mathfrak{e}_*, \bar{\mathfrak{e}}_*\}$ with $i < j$, we will visit the forest $\mathfrak{f}^{\boldsymbol{e}_i}$ while "going up" and the forest $\mathfrak{f}^{\boldsymbol{e}_j}$ while "coming down" when we follow the contour of $\boldsymbol{t}$. Precisely, we define

$$\boldsymbol{C}^{\boldsymbol{e}_i} := \mathfrak{C}^{\boldsymbol{e}_i} - 2\,\underline{\mathfrak{C}}^{\boldsymbol{e}_i} \qquad \text{and} \qquad \boldsymbol{C}^{\boldsymbol{e}_j} := \mathfrak{C}^{\boldsymbol{e}_j}. \tag{4.5}$$

The first function is the concatenation of the contour functions of the trees in $\mathfrak{f}^{\boldsymbol{e}_i}$ with an extra "up step" between every consecutive trees. The second one is the concatenation of the contour functions of the trees in $\mathfrak{f}^{\boldsymbol{e}_j}$ with an extra "down step" between every consecutive trees. It is merely the contour function of $\mathfrak{f}^{\boldsymbol{e}_j}$ shifted in order to start at 0. What happens to the forests $\mathfrak{f}^{\mathfrak{e}_*}$ and $\mathfrak{f}^{\bar{\mathfrak{e}}_*}$ is a little more intricate. Let us first call (see Figure 4.9)

$$x := \inf\{s \,:\, \mathfrak{C}^{\mathfrak{e}_*}(s) = \underline{\mathfrak{C}}^{\mathfrak{e}_*}(u)\} \qquad \text{and} \qquad y := \inf\left\{s \,:\, \mathfrak{C}^{\bar{\mathfrak{e}}_*}(s) = -\sigma^{\mathfrak{e}_*} - \underline{\mathfrak{C}}^{\mathfrak{e}_*}(u)\right\}. \tag{4.6}$$

When visiting the forest $\mathfrak{f}^{\bar{\mathfrak{e}}_*}$, the floor is directed downward up to time y and then upward:

$$\boldsymbol{C}^{\bar{\mathfrak{e}}_*} := \left(\mathfrak{C}^{\bar{\mathfrak{e}}_*}(s)\right)_{0 \le s \le y} \bullet \left(\mathfrak{C}^{\bar{\mathfrak{e}}_*}(y+s) - 2 \inf_{[y, y+s]} \mathfrak{C}^{\bar{\mathfrak{e}}_*} + \mathfrak{C}^{\bar{\mathfrak{e}}_*}(y)\right)_{0 \le s \le m^{\bar{\mathfrak{e}}_*} - y}. \tag{4.7}$$

Finally, the forest $\mathfrak{f}^{\mathfrak{e}_*}$ is visited twice. The first time (when beginning the contour), it is visited between times u and $m^{\mathfrak{e}_*}$, and the floor is directed upward:

$$\boldsymbol{C}^{\mathfrak{e}_*, 1} := \left(\mathfrak{C}^{\mathfrak{e}_*}(u+s) - 2 \inf_{[u, u+s]} \mathfrak{C}^{\mathfrak{e}_*} + \mathfrak{C}^{\mathfrak{e}_*}(u)\right)_{0 \le s \le m^{\mathfrak{e}_*} - u}. \tag{4.8}$$

The second time (when finishing the contour), we visit it between times 0 and x with the floor directed downward, then we visit a part of the tree containing the root between times x and u:

$$\boldsymbol{C}^{\mathfrak{e}_*, 2} := \left(\mathfrak{C}^{\mathfrak{e}_*}(s)\right)_{0 \le s \le x} \bullet \left(\mathfrak{C}^{\mathfrak{e}_*}(x+s) - 2 \inf_{[x+s, u]} \mathfrak{C}^{\mathfrak{e}_*} + \underline{\mathfrak{C}}^{\mathfrak{e}_*}(u)\right)_{0 \le s \le u - x}. \tag{4.9}$$

The contour pair of $(\boldsymbol{t}, \boldsymbol{l})$ is then given by

$$\left\{\begin{array}{rcl} \boldsymbol{C} & := & \boldsymbol{C}^{\boldsymbol{e}_1, 1} \bullet \boldsymbol{C}^{\boldsymbol{e}_2} \bullet \boldsymbol{C}^{\boldsymbol{e}_3} \bullet \cdots \bullet \boldsymbol{C}^{\boldsymbol{e}_{2(6g-3)}} \bullet \boldsymbol{C}^{\boldsymbol{e}_1, 2}, \\ \boldsymbol{L} & := & \left(\mathfrak{L}^{\boldsymbol{e}_1}(u+s) - \mathfrak{L}^{\boldsymbol{e}_1}(u)\right)_{0 \le s \le m^{\boldsymbol{e}_1} - u} \bullet \mathfrak{L}^{\boldsymbol{e}_2} \bullet \mathfrak{L}^{\boldsymbol{e}_3} \bullet \cdots \bullet \mathfrak{L}^{\boldsymbol{e}_{2(6g-3)}} \bullet \left(\mathfrak{L}^{\boldsymbol{e}_1}(s)\right)_{0 \le s \le u}. \end{array}\right. \tag{4.10}$$

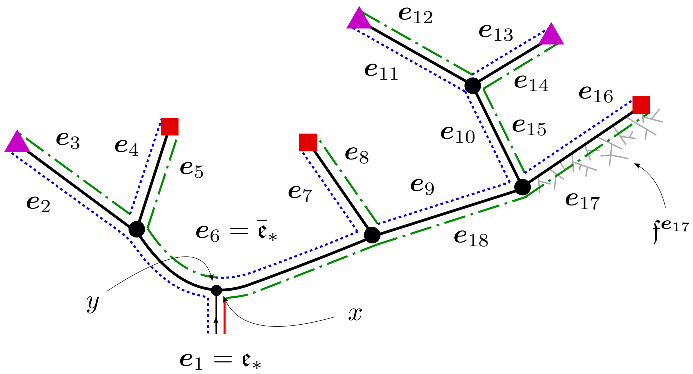

Figure 4.9. *Opening of a 2-tree. The squares form one triple and the triangles the other one. The (blue) short dashes correspond to the upward-directed floors and the (green) long dashes to the downward-directed floors. The (red) solid line on the right of the root corresponds to the part of the tree containing the root that has to be visited at the end. The forest $\mathfrak{f}^{e_{17}}$ is also represented on this figure.*

4.4.3 Opened uniform well-labeled g-tree

As before, we let $(\mathfrak{t}_n, \mathfrak{l}_n)$ be uniformly distributed over the set $\mathcal{T}_n$ of well-labeled g-trees with n edges, and, applying Skorokhod's representation theorem, we assume that the convergence of Proposition 3.1 holds almost surely. Let $(\mathfrak{i}_n)_{n\in\mathbb{N}}$ be a sequence of i.i.d. random variables uniformly distributed over $[\![1, 2^g g!]\!]$ and independent of $(\mathfrak{t}_n, \mathfrak{l}_n)_{n\in\mathbb{N}}$. With any dominant scheme $\mathfrak{s} \in \mathfrak{S}_*$ and integer $\mathfrak{i} \in [\![1, 2^g g!]\!]$, we associate a deterministic opening sequence. When $(\mathfrak{t}_n, \mathfrak{l}_n)$ is dominant, we may then define $(\boldsymbol{t}_n, \boldsymbol{l}_n)$ as the opened tree of $(\mathfrak{t}_n, \mathfrak{l}_n)$ according to the opening sequence determined by the integer $\mathfrak{i}_n$. In this case, we call $(\boldsymbol{C}_n, \boldsymbol{L}_n)$ the contour pair of $(\boldsymbol{t}_n, \boldsymbol{l}_n)$. When $(\mathfrak{t}_n, \mathfrak{l}_n)$ is not dominant, we simply set $(\boldsymbol{C}_n, \boldsymbol{L}_n) = (\mathbf{0}_{2n}, \mathbf{0}_{2n})$, where we write $\mathbf{0}_\zeta : x \in [0, \zeta] \mapsto 0$. We also let

$$\boldsymbol{C}_{(n)} := \left(\frac{\boldsymbol{C}_n(2nt)}{\sqrt{2n}}\right)_{0\le t\le 1} \qquad \text{and} \qquad \boldsymbol{L}_{(n)} := \left(\frac{\boldsymbol{L}_n(2nt)}{\gamma\, n^{1/4}}\right)_{0\le t\le 1}$$

be the rescaled versions of $\boldsymbol{C}_n$ and $\boldsymbol{L}_n$.

We now work at fixed ω for which Proposition 3.1 holds, $\mathfrak{s}_\infty \in \mathfrak{S}_*$, and such that for all $\mathfrak{i} \in [\![1, 2^g g!]\!]$, $|\{n \in \mathbb{N} : \mathfrak{i}_n = \mathfrak{i}\}| = \infty$. Note that the set of such ω's is of full probability. For n large enough, $\mathfrak{s}_n = \mathfrak{s}_\infty \in \mathfrak{S}_*$, so that $(\boldsymbol{t}_n, \boldsymbol{l}_n)$ is well defined. For all n such that $\mathfrak{s}_n = \mathfrak{s}_\infty$ and $\mathfrak{i}_n = \mathfrak{i}$, we always open the g-tree $(\mathfrak{t}_n, \mathfrak{l}_n)$ according to the same opening sequence, so that the ordering $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_{2(6g-3)}$ of the half-edges of $\mathfrak{s}_n$ is always the same. As a result, we obtain that

$$(\boldsymbol{C}_{(n)}, \boldsymbol{L}_{(n)}) \xrightarrow[n\to\infty]{n:\, \mathfrak{i}_n = \mathfrak{i}} (\boldsymbol{C}^{\mathfrak{i}}_\infty, \boldsymbol{L}^{\mathfrak{i}}_\infty),$$

where $(\boldsymbol{C}^{\mathfrak{i}}_\infty, \boldsymbol{L}^{\mathfrak{i}}_\infty)$ is defined by (4.5), (4.6), (4.7), (4.8), (4.9), and (4.10) when replacing every occurrence of $\mathfrak{C}^{\mathfrak{e}}$ by $\mathfrak{C}^{\mathfrak{e}}_\infty := C^{\mathfrak{e}}_\infty - \sigma^{\mathfrak{e}}_\infty$ and every occurrence of $\mathfrak{L}^{\mathfrak{e}}$ by $\mathfrak{L}^{\mathfrak{e}}_\infty$. Note that $(\boldsymbol{C}_{(n)}, \boldsymbol{L}_{(n)})$ has exactly $2^g g!$ a priori distinct accumulation points, each corresponding to one of the ways of opening the real g-tree $\mathscr{T}_\infty$.

Now, because every $\mathfrak{L}^{\mathfrak{e}}_\infty$ goes from 0 to 0, it is easy to see that for all $\mathfrak{i}$, the points where $\mathfrak{L}_\infty$ reaches its minimum are in one-to-one correspondence with the points where $\boldsymbol{L}^{\mathfrak{i}}_\infty$ reaches its minimum. Moreover, we can see that if $\mathfrak{C}_\infty$ and $\mathfrak{L}_\infty$ have a common increase point, then at least one of the pairs $(\boldsymbol{C}^{\mathfrak{i}}_\infty, \boldsymbol{L}^{\mathfrak{i}}_\infty)$ will also have a common increase point. Indeed, let us suppose that $\mathfrak{C}_\infty$ and $\mathfrak{L}_\infty$ have a common increase point. Then, there exists $\mathfrak{e} \in \vec{E}(\mathfrak{s}_\infty)$ such that $\mathfrak{C}^{\mathfrak{e}}_\infty$ and $\mathfrak{L}^{\mathfrak{e}}_\infty$ have a common increase point $s \in [0, m^{\mathfrak{e}}_\infty]$. We use the following lemma:

Lemma 4.18. *Let $f : [0, m] \to \mathbb{R}$ be a function.*

✧ *If $s \in [0, m)$ is an increase point of f, then s is an increase point of $f - 2\underline{f}$ as well.*

✧ *If $s \in (0, m]$ is an increase point of f, then s is an increase point of $r \mapsto f(r) - 2\inf_{[r,m]} f$.*

We postpone the proof of this lemma and finish our argument. If $s < m_\infty^{\mathfrak{e}}$, then s is a common increase point of $\boldsymbol{C}_\infty^{\mathfrak{e}}$ and $\boldsymbol{L}_\infty^{\mathfrak{e}}$ thanks to Lemma 4.18. When $\mathfrak{e} = \mathfrak{e}_*$, this fact remains true if we define $\boldsymbol{C}_\infty^{\mathfrak{e}} := \boldsymbol{C}_\infty^{\mathfrak{e},2} \bullet \boldsymbol{C}_\infty^{\mathfrak{e},1}$. Note that x is an increase point of $\boldsymbol{C}_\infty^{\mathfrak{e}}$, even if 0 is not an increase point of the second function defining $\boldsymbol{C}_\infty^{\mathfrak{e},2}$ in (4.9). In this case, for all $\mathfrak{i}$, $\boldsymbol{C}_\infty^{\mathfrak{i}}$ and $\boldsymbol{L}_\infty^{\mathfrak{i}}$ have a common increase point.

Let us now suppose that $s = m_\infty^{\mathfrak{e}}$, and let us fix $\mathfrak{i} \in [\![1, 2^g g!]\!]$. We consider the opening corresponding to $\mathfrak{i}$. If $\boldsymbol{e}_i = \mathfrak{e}$ is visited while coming down in the contour of the opened tree, then we conclude as above. If both $\boldsymbol{e}_i$ and $\boldsymbol{e}_{i+1}$ are visited while going up, then 0 will be an increase point of $\boldsymbol{C}_\infty^{\boldsymbol{e}_{i+1}}$, so that $\boldsymbol{C}_\infty^{\mathfrak{i}}$ and $\boldsymbol{L}_\infty^{\mathfrak{i}}$ will still have a common increase point. In the remaining case where $\boldsymbol{e}_i$ is visited while going up and $\boldsymbol{e}_{i+1}$ is visited while coming down (i.e. $\boldsymbol{e}_{i+1} = \bar{\boldsymbol{e}_i}$), we cannot conclude that $\boldsymbol{C}_\infty^{\mathfrak{i}}$ and $\boldsymbol{L}_\infty^{\mathfrak{i}}$ have a common increase point. This, however, only happens when the node $\mathfrak{e}^+$ belongs to the opening sequence. But when we pick an opening sequence, we can always choose not to pick a given node, because at each stage of the process, we have at least 2 intertwined nodes. This implies that at least one of the opening sequences will not contain $\mathfrak{e}^+$, and the corresponding pair $(\boldsymbol{C}_\infty^{\mathfrak{i}}, \boldsymbol{L}_\infty^{\mathfrak{i}})$ will have a common increase point.

Proof of Lemma 4.18. Let $s \in [0, m)$ be an increase point of f. If s is a right-increase point of f, then $f(r) \geq f(s)$ when $s \leq r \leq t$ for some $t > s$. For such r's, $\underline{f}(r) = \underline{f}(s)$, so that $f(r) - 2\underline{f}(r) \geq f(s) - 2\underline{f}(s)$, and s is a right-increase point of $f - 2\underline{f}$.

If s is a left-increase point of f, then $f(r) \geq f(s)$ when $t \leq r \leq s$ for some $t < s$. If $f(s) > \underline{f}(s)$, then, using the fact that $\underline{f}(s) = \underline{f}(r) \wedge \inf_{[r,s]} f$, we obtain that $\underline{f}(r) = \underline{f}(s)$ when $t \leq r \leq s$ and conclude as above that s is a left-increase point of $f - 2\underline{f}$. Finally, if $f(s) = \underline{f}(s)$, then for all $r \geq s$, we have $f(r) - 2\underline{f}(r) = \big(f(r) - \underline{f}(r)\big) - \underline{f}(r) \geq 0 - \underline{f}(s) = f(s) - 2\underline{f}(s)$, and because $s < m$, we conclude that s is a right-increase point of $f - 2\underline{f}$.

We obtain the second assertion of the lemma by applying the first one to $m - s$ and the function $x \mapsto f(m - x)$. □

4.4.4 Uniform well-labeled tree with g triples

Conditionally on the event $D_n := \{(\boldsymbol{C}_n, \boldsymbol{L}_n) \neq (\boldsymbol{0}_{2n}, \boldsymbol{0}_{2n})\}$, the distribution of $(\boldsymbol{C}_n, \boldsymbol{L}_n)$ is that of the contour pair of a uniform well-labeled tree with g triples. We use this fact to see that the law of $(\boldsymbol{C}_{(n)}, \boldsymbol{L}_{(n)})$ converges weakly toward a law absolutely continuous with respect to the law of $(\mathbb{e}, Z)$. Let (τ_n, λ_n) be uniformly distributed over the set $\mathcal{T}_n^0$ of all well-labeled plane trees with n edges. We call (Γ_n, Λ_n) the contour pair of (τ_n, λ_n), and define as usual the rescaled versions of both functions:

$$\Gamma_{(n)} := \left(\frac{\Gamma_n(2nt)}{\sqrt{2n}}\right)_{0 \leq t \leq 1} \qquad \text{and} \qquad \Lambda_{(n)} := \left(\frac{\Lambda_n(2nt)}{\gamma\, n^{1/4}}\right)_{0 \leq t \leq 1}. \tag{4.11}$$

For all $n \geq 1$, $k \in \mathbb{Z}$ and $x \in \mathbb{R}$, we define

$$X_n(k) := |\{v \in \tau_n \,:\, \lambda_n(v) = k\}| \qquad \text{and} \qquad X_{(n)}(x) := \frac{1}{n}\, \gamma\, n^{1/4}\, X_n\left(\left\lfloor \gamma\, n^{1/4} x \right\rfloor\right),$$

respectively the profile and rescaled profile of (τ_n, λ_n). We let $\mathcal{I}$ be the one-dimensional ISE (random) measure defined by

$$\langle \mathcal{I}, h \rangle := \int_0^1 dt\; h(Z_t)$$

for every nonnegative measurable function h. By [BMJ06, Theorem 2.1], it is known that $\mathcal{I}$ a.s. has a continuous density f_{ISE} with compact support. In other words, $\langle \mathcal{I}, h \rangle = \int_{\mathbb{R}} dx\; h(x) f_{\text{ISE}}(x)$ for every nonnegative measurable function h.

Proposition 4.19. *The triple* $\left(\Gamma_{(n)}, \Lambda_{(n)}, X_{(n)}\right)$ *converges weakly toward the triple* $(\mathbb{e}, Z, f_{\text{ISE}})$ *in the space* $\mathcal{C}([0,1],\mathbb{R})^2 \times \mathcal{C}_c(\mathbb{R})$ *endowed with the product topology.*

Proof. It is known that the pair $\left(\Gamma_{(n)}, \Lambda_{(n)}\right)$ converges weakly to $(\mathbb{e}, Z)$: in [CS04, Theorem 5], Chassaing and Schaeffer proved this fact with $\lfloor 2nt \rfloor$ instead of $2nt$ in the definition (4.11). The claim as stated here easily follows by using the uniform continuity of $(\mathbb{e}, Z)$. Using [BMJ06, Theorem 3.6] and the fact that f_{ISE} is a.s. uniformly continuous [BMJ06, Theorem 2.1], we also obtain that the sequence $X_{(n)}$ converges weakly to f_{ISE}. As a result, the sequences of the laws of the processes $\Gamma_{(n)}$, $\Lambda_{(n)}$ and $X_{(n)}$ are tight. The sequence (ν_n) of the laws of $\left(\Gamma_{(n)}, \Lambda_{(n)}, X_{(n)}\right)$ is then tight as well, and, by Prokhorov's lemma, the set $\{\nu_n, n \geq 0\}$ is relatively compact. Let ν be an accumulation point of the sequence (ν_n). There exists a subsequence along which $\left(\Gamma_{(n)}, \Lambda_{(n)}, X_{(n)}\right)$ converges weakly toward a random variable $(\mathbb{e}', Z', f')$ with law ν. Thanks to Skorokhod's theorem, we may and will assume that this convergence holds almost surely along this subsequence. We know that

$$(\mathbb{e}', Z') \overset{(d)}{=} (\mathbb{e}, Z) \qquad \text{and} \qquad f' \overset{(d)}{=} f_{\text{ISE}}.$$

It remains to see that f' is the density of the occupation measure of Z', that is

$$\int_0^1 dt\, h(Z'_t) = \int_{\mathbb{R}} dx\, h(x) f'(x), \tag{4.12}$$

for all h continuous with compact support. First, notice that

$$\begin{aligned}
\frac{1}{n}\sum_{k\in\mathbb{Z}} X_n(k) h\left(\gamma^{-1} n^{-\frac{1}{4}} k\right) &= \frac{1}{n}\int_{\mathbb{R}} dx\, X_n\left(\lfloor x \rfloor\right) h\left(\gamma^{-1} n^{-\frac{1}{4}} \lfloor x \rfloor\right) \\
&= \int_{\mathbb{R}} dx\, X_{(n)}(x) h\left(\gamma^{-1} n^{-\frac{1}{4}} \left\lfloor \gamma\, n^{1/4} x \right\rfloor\right) \\
&\to \int_{\mathbb{R}} dx\, f'(x) h(x)
\end{aligned}$$

by dominated convergence, a.s. as $n \to \infty$ along the subsequence we consider. It is convenient to introduce now the notation $\langle\!\langle s \rangle\!\rangle_n$ defined as follows: for $s \in [0, 2n)$, we set

$$\langle\!\langle s \rangle\!\rangle_n := \begin{cases} \lceil s \rceil & \text{if } \Gamma_n(\lceil s \rceil) - \Gamma_n(\lfloor s \rfloor) = 1, \\ \lfloor s \rfloor & \text{if } \Gamma_n(\lceil s \rceil) - \Gamma_n(\lfloor s \rfloor) = -1. \end{cases}$$

Then, if we denote by $\tau_n(i)$ the ith vertex of the facial sequence of τ_n, and by ρ_n the root of τ_n, we obtain that the time the process $\left(\tau_n(\langle\!\langle s \rangle\!\rangle_n)\right)_{s\in[0,2n)}$ spends at each vertex $v \in \tau_n \backslash \{\rho_n\}$ is exactly 2. So we have

$$\begin{aligned}
\frac{1}{n}\sum_{k\in\mathbb{Z}} X_n(k) h\left(\gamma^{-1} n^{-\frac{1}{4}} k\right) &= \frac{1}{n}\sum_{v\in\tau_n\backslash\{\rho_n\}} h\left(\gamma^{-1} n^{-\frac{1}{4}} \lambda_n(v)\right) + \frac{1}{n} h(0) \\
&= \frac{1}{2n}\int_0^{2n} ds\, h\left(\gamma^{-1} n^{-\frac{1}{4}} \Lambda_n\left(\langle\!\langle s \rangle\!\rangle_n\right)\right) + \frac{1}{n} h(0) \\
&= \int_0^1 ds\, h\left(\gamma^{-1} n^{-\frac{1}{4}} \Lambda_n\left(\langle\!\langle 2ns \rangle\!\rangle_n\right)\right) + \frac{1}{n} h(0) \\
&\to \int_0^1 dt\, h(Z'_t)
\end{aligned}$$

a.s. along the subsequence considered. We used the fact that $\gamma^{-1} n^{-\frac{1}{4}} \Lambda_n\left(\langle\!\langle 2ns \rangle\!\rangle_n\right) \to Z'_s$, which is obtained by using the uniform continuity of Z'.

This proves that $(\mathbb{e}', Z', f')$ has the same law as $(\mathbb{e}, Z, f_{\text{ISE}})$. Thus the only accumulation point ν of the sequence (ν_n) is the the law of the process $(\mathbb{e}, Z, f_{\text{ISE}})$. By relative compactness of the set $\{\nu_n, n \geq 0\}$, we obtain the weak convergence of the sequence (ν_n) towards ν. □

We define
$$W := \frac{\left(\int f_{\mathrm{ISE}}^3\right)^g}{\mathbb{E}\left[\left(\int f_{\mathrm{ISE}}^3\right)^g\right]}.$$
This quantity is well defined [Cha10, Lemma 10]. We also define the law of the pair $(\boldsymbol{C}_\infty, \boldsymbol{L}_\infty)$ by the following formula: for every bounded Borel function φ on $\mathcal{C}([0,1],\mathbb{R})^2$,
$$\mathbb{E}\left[\varphi(\boldsymbol{C}_\infty, \boldsymbol{L}_\infty)\right] = \mathbb{E}\left[W\,\varphi(\mathbb{e}, Z)\right]. \tag{4.13}$$

Proposition 4.20. *The pair $\left(\boldsymbol{C}_{(n)}, \boldsymbol{L}_{(n)}\right)$ converges weakly to $(\boldsymbol{C}_\infty, \boldsymbol{L}_\infty)$ in the space $\left(\mathcal{C}([0,1],\mathbb{R})^2, \|\cdot\|_\infty\right)$ of pair of continuous real-valued functions on $[0,1]$ endowed with the uniform topology.*

Proof. Let f be a bounded continuous function on $\mathcal{C}([0,1],\mathbb{R})^2$. We have
$$\mathbb{E}\left[f\left(\boldsymbol{C}_{(n)}, \boldsymbol{L}_{(n)}\right)\right] = \mathbb{P}(D_n) \sum_{\substack{(\tau,\lambda)\in\mathcal{T}_n^0 \\ (\tau,\lambda)\leftrightarrow(\boldsymbol{C},\boldsymbol{L})}} f(\boldsymbol{C},\boldsymbol{L})\,\mathbb{P}\left((\tau_n,\lambda_n) = (\tau,\lambda)\,|\,D_n\right) + \mathbb{P}\left(\overline{D}_n\right) f\left(\mathbf{0}_{2n}, \mathbf{0}_{2n}\right)$$

where we used the notation $(\tau,\lambda) \leftrightarrow (\boldsymbol{C},\boldsymbol{L})$ to mean that the well-labeled tree (τ,λ) is coded by the contour pair $(\boldsymbol{C},\boldsymbol{L})$. It was shown in [Cha10, Lemma 8] that the number of well-labeled trees with g triples having n edges is equivalent to the number of well-labeled plane trees having n edges, together with g triples of vertices (not necessarily distinct and not arranged) such that all the vertices of the same triple have the same label. More precisely, we have
$$\mathbb{P}\left((\tau_n,\lambda_n) = (\tau,\lambda)\,|\,D_n\right) = \frac{1}{|\mathcal{W}_n|}\left(\sum_{k\in\mathbb{Z}} \left|\{v\in\tau\,:\,\lambda(v)=k\}\right|^3\right)^g + O\left(n^{-\frac{1}{4}}\right).$$

And, because f is bounded and $\mathbb{P}(D_n)\to 1$, we obtain that
$$\mathbb{E}\left[f\left(\boldsymbol{C}_{(n)}, \boldsymbol{L}_{(n)}\right)\right] \sim \frac{|\mathcal{T}_n^0|}{|\mathcal{W}_n|}\,\mathbb{E}\left[\left(\sum_{k\in\mathbb{Z}} X_n(k)^3\right)^g f\left(\Gamma_{(n)}, \Lambda_{(n)}\right)\right].$$

Using the asymptotic formulas $|\mathcal{T}_n^0| \sim \sqrt{\pi}\,12^n\,n^{-3/2}$ and $|\mathcal{W}_n| \sim c_g\,12^n\,n^{(5g-3)/2}$ for some positive constant c_g only depending on g ([Cha10, Lemma 8]), as well as the computation
$$n^{-5/2}\sum_{k\in\mathbb{Z}} X_n(k)^3 = n^{-5/2}\int_{\mathbb{R}} dx\, X_n\left(\lfloor x\rfloor\right)^3 = \gamma^{-2}\int_{\mathbb{R}} dx\, X_{(n)}(x)^3,$$

we see that there exists a positive constant c such that
$$\mathbb{E}\left[f\left(\boldsymbol{C}_{(n)}, \boldsymbol{L}_{(n)}\right)\right] \sim c\,\mathbb{E}\left[\left(\int_{\mathbb{R}} dx\, X_{(n)}(x)^3\right)^g f\left(\Gamma_{(n)}, \Lambda_{(n)}\right)\right].$$

Now, let $\varepsilon > 0$. Thanks to [Cha10, Lemma 10], we see that both quantities $\mathbb{E}\left[\left(\int f_{\mathrm{ISE}}^3\right)^g\right]$ and $\sup_n \mathbb{E}\left[\left(\int X_{(n)}^3\right)^{g+1}\right]$ are finite. Then, using the fact that
$$\mathbb{E}\left[\left(\int X_{(n)}^3\right)^g \mathbb{1}_{\left\{\int X_{(n)}^3 > L\right\}}\right] \le \frac{1}{L}\,\mathbb{E}\left[\left(\int X_{(n)}^3\right)^{g+1}\right],$$

we obtain that, for L sufficiently large,
$$\sup_n \mathbb{E}\left[\left(\int_{\mathbb{R}} dx\, X_{(n)}(x)^3\right)^g f\left(\Gamma_{(n)}, \Lambda_{(n)}\right)\mathbb{1}_{\left\{\int X_{(n)}^3 > L\right\}}\right] < \varepsilon$$

and

$$\mathbb{E}\left[\left(\int f_{\text{ISE}}^3\right)^g f(\mathbb{e}, Z)\,\mathbb{1}_{\left\{\int f_{\text{ISE}}^3 > L\right\}}\right] < \varepsilon.$$

Thanks to the Proposition 4.19, for n sufficiently large,

$$\left|\mathbb{E}\left[\left(\int_{\mathbb{R}} dx\, X_{(n)}(x)^3\right)^g f\left(\Gamma_{(n)}, \Lambda_{(n)}\right)\mathbb{1}_{\left\{\int X_{(n)}^3 \le L\right\}}\right] - \mathbb{E}\left[\left(\int f_{\text{ISE}}^3\right)^g f(\mathbb{e}, Z)\,\mathbb{1}_{\left\{\int f_{\text{ISE}}^3 \le L\right\}}\right]\right| < \varepsilon.$$

This yields the existence of a constant C such that

$$\mathbb{E}\left[f\left(\boldsymbol{C}_{(n)}, \boldsymbol{L}_{(n)}\right)\right] \xrightarrow[n\to\infty]{} C\,\mathbb{E}\left[\left(\int f_{\text{ISE}}^3\right)^g f(\mathbb{e}, Z)\right],$$

and we compute the value of C by taking $f \equiv 1$. □

Thanks to (4.13), we see that the properties that hold almost surely for the pair $(\mathbb{e}, Z)$ also hold almost surely for $(\boldsymbol{C}_\infty, \boldsymbol{L}_\infty)$. We may now conclude thanks to [LGP08, Lemma 3.1] that

$$\begin{aligned}\mathbb{P}\Big(\exists\, s \neq t \,:\, \mathfrak{L}_\infty(s) = \mathfrak{L}_\infty(t) = \min \mathfrak{L}_\infty\Big) &\le \frac{1}{2^g g!}\sum_{\mathrm{i}=1}^{2^g g!} \mathbb{P}\Big(\exists\, s \neq t \,:\, \boldsymbol{L}_\infty^{\mathrm{i}}(s) = \boldsymbol{L}_\infty^{\mathrm{i}}(t) = \min \boldsymbol{L}_\infty^{\mathrm{i}}\Big)\\ &= \mathbb{P}\Big(\exists\, s \neq t \,:\, \boldsymbol{L}_\infty(s) = \boldsymbol{L}_\infty(t) = \min \boldsymbol{L}_\infty\Big) = 0,\end{aligned}$$

and, by [LGP08, Lemma 3.2],

$$\begin{aligned}\mathbb{P}\Big(\mathrm{IP}(\mathfrak{C}_\infty) \cap \mathrm{IP}(\mathfrak{L}_\infty) \neq \varnothing\Big) &\le \sum_{\mathrm{i}=1}^{2^g g!} \mathbb{P}\Big(\mathrm{IP}(\boldsymbol{C}_\infty^{\mathrm{i}}) \cap \mathrm{IP}(\boldsymbol{L}_\infty^{\mathrm{i}}) \neq \varnothing\Big)\\ &= 2^g g!\,\mathbb{P}\Big(\mathrm{IP}(\boldsymbol{C}_\infty) \cap \mathrm{IP}(\boldsymbol{L}_\infty) \neq \varnothing\Big) = 0.\end{aligned}$$

This concludes the proof of Lemmas 4.4 and 4.5.

4.4.5 Remaining proofs

4.4.5.1 Proof of Lemma 4.7

Chapuy's bijection may naturally be transposed in the continuous setting. Let $\mathrm{i} \in [\![1, 2^g g!]\!]$ be an integer corresponding to an opening sequence, and $\mathbf{T}_\infty^{\mathrm{i}}$ the real tree coded by $\boldsymbol{C}_\infty^{\mathrm{i}}$. The interval $[0,1]$ may be split into $2g+1$ intervals coding the two halves of $\mathfrak{f}_\infty^{\mathfrak{e}_*}$ and the other forests of $\mathscr{T}_\infty$. Through the continuous analog of Chapuy's bijection, these intervals are reordered into an order corresponding to the opening sequence. We call $\varphi^{\mathrm{i}} : [0,1] \to [0,1]$ the bijection accounting for this reordering. It is a cadlag function with derivative 1 satisfying $\mathfrak{L}_\infty(s) = \boldsymbol{L}_\infty^{\mathrm{i}}(\varphi^{\mathrm{i}}(s))$ for all $s \in [0,1]$.

In order to see that Lemma 4.7 is a consequence of [LG07, Lemma 2.4], let us first see what happens to subtrees of $\mathscr{T}_\infty$ through the continuous analog of Chapuy's bijection. It is natural to call root of $\mathscr{T}_\infty$ the point $\partial := \mathscr{T}_\infty(u_\infty)$, where the real number u_∞ was defined in Proposition 3.1 as the limit of the integer coding the root in $\mathfrak{t}_n$, properly rescaled. Using classical properties of the Brownian motion together with Proposition 3.1, it is easy to see that, almost surely, ∂ is a leaf of $\mathscr{T}_\infty$, so that τ_∂ is well defined. Any subtree of $\mathscr{T}_\infty$ not included in τ_∂ (these subtrees require extra care, we will treat them separately) is transformed through Chapuy's bijection into some subtree of the opened tree $\mathbf{T}_\infty^{\mathrm{i}}$ (that is, into some tree to the left or right of some branch of $\mathbf{T}_\infty^{\mathrm{i}}$). This is easy to see when the subtree is not rooted at a node of $\mathscr{T}_\infty$, and we saw at the end of Section 4.1.1 that, almost surely, all the subtrees are rooted outside the set of nodes of $\mathscr{T}_\infty$.

We reason by contradiction to rule out these subtrees. We call $\mathscr{L}$ the Lebesgue measure on $[0,1]$. Let us suppose that there exist $\eta > 0$, and some subtree τ, coded by $[l, r]$, not included in τ_∂, such that $\inf_{[l,r]} \mathfrak{L}_\infty < \mathfrak{L}_\infty(l) - \eta$, and

$$\liminf_{\varepsilon \to 0} \varepsilon^{-2} \mathscr{L}\Big(\Big\{s \in [l,r] \,:\, \mathfrak{L}_\infty(s) < \mathfrak{L}_\infty(l) - \eta + \varepsilon\,;\, \forall x \in [\mathfrak{C}_\infty(l), \mathfrak{C}_\infty(s)], \\ \mathfrak{L}_\infty(\sup\{t \le s \,:\, \mathfrak{C}_\infty(t) = x\}) > \mathfrak{L}_\infty(l) - \eta + \frac{\varepsilon}{8}\Big\}\Big) = 0. \quad (4.14)$$

Note that, by definition of $\boldsymbol{C}^{\mathrm{i}}_\infty$, the function $s \mapsto \mathfrak{C}_\infty(s) - \boldsymbol{C}^{\mathrm{i}}_\infty(\varphi^{\mathrm{i}}(s))$ is constant on $[l, r]$. Let us call $l' := \varphi^{\mathrm{i}}(l)$ and $r' := \varphi^{\mathrm{i}}(r)$. It is easy to see that (4.14) remains true when replacing respectively l, r, $\mathfrak{C}_\infty$ and $\mathfrak{L}_\infty$ with l', r', $\boldsymbol{C}^{\mathrm{i}}_\infty$ and $\boldsymbol{L}^{\mathrm{i}}_\infty$. Thanks to Proposition 4.20, the conclusion of [LG07, Lemma 2.4] is also true for the opened tree $\mathbf{T}^{\mathrm{i}}_\infty$, and the fact that $[l', r']$ codes a subtree of the opened tree yields a contradiction.

We then use a re-rooting argument to conclude. With positive probability, τ_∂ is no longer the tree containing the root in the uniformly re-rooted g-tree. Let us suppose that, with positive probability, there exists a subtree of $\mathscr{T}_\infty$ included in τ_∂, satisfying the hypotheses but not the conclusion of Lemma 4.7. Then, with positive probability, there will exist a subtree not included in the tree containing the root of the uniformly re-rooted g-tree, satisfying the hypotheses but not the conclusion of Lemma 4.7. The fact that the uniformly re-rooted g-tree has the same law as $\mathscr{T}_\infty$ yields a contradiction.

4.4.5.2 Proof of Lemma 4.8

Using the same arguments as in [LG10], we can see that Lemma 4.8 is a consequence of the following lemma (see [LG10, Corollary 6.2]):

Lemma 4.21. *For every $p \ge 1$ and every $\delta \in (0,1]$, there exists a constant $c_{p,\delta} < \infty$ such that, for every $\varepsilon > 0$,*

$$\mathbb{E}\left[\left(\int_0^1 \mathbb{1}_{\{\mathfrak{L}_\infty(s) \le \min \mathfrak{L}_\infty + \varepsilon\}}\, ds\right)^p\right] \le c_{p,\delta}\, \varepsilon^{4p-\delta}.$$

Proof. This readily comes from [LG10, Lemma 6.1] stating that for every $p \ge 1$ and every $\delta \in (0,1]$, there exists a constant $c'_{p,\delta} < \infty$ such that, for every $\varepsilon > 0$,

$$\mathbb{E}\left[\left(\int_0^1 \mathbb{1}_{\{Z_s \le \min Z + \varepsilon\}}\, ds\right)^p\right] \le c'_{p,\delta}\, \varepsilon^{4p-\delta}.$$

Obviously, this still holds for $\delta \in (1,2]$. Using the link between $\mathfrak{L}_\infty$ and $\boldsymbol{L}_\infty$, as well as Proposition 4.20, we see that, for $p \ge 1$ and $\delta \in (0,1]$,

$$\begin{aligned}\mathbb{E}\left[\left(\int_0^1 \mathbb{1}_{\{\mathfrak{L}_\infty(s) \le \min \mathfrak{L}_\infty + \varepsilon\}}\, ds\right)^p\right] &= \mathbb{E}\left[\left(\int_0^1 \mathbb{1}_{\{\boldsymbol{L}_\infty(s) \le \min \boldsymbol{L}_\infty + \varepsilon\}}\, ds\right)^p\right] \\ &= \mathbb{E}\left[W\left(\int_0^1 \mathbb{1}_{\{Z_s \le \min Z + \varepsilon\}}\, ds\right)^p\right] \\ &\le \left(\mathbb{E}\left[W^2\right] c'_{2p,2\delta}\right)^{\frac{1}{2}} \varepsilon^{4p-\delta} = c_{p,\delta}\, \varepsilon^{4p-\delta},\end{aligned}$$

where $c_{p,\delta} := \left(\mathbb{E}\left[W^2\right] c'_{2p,2\delta}\right)^{\frac{1}{2}} < \infty$, by [Cha10, Lemma 10]. □

III

Limite d'échelle de cartes planaires à bord

5 Scaling limit of random planar quadrangulations with a boundary

From this point on, when we speak of quadrangulations, we always mean planar quadrangulations with a boundary, and, by convention, we always draw the external face as the infinite component of the plane.

We begin by exposing in Section 5.1 the version of the Bouttier–Di Francesco–Guitter bijection that we will need. As we do not use it in its usual setting, we spend some time explaining it. In particular, we introduce a notion of bridge that is not totally standard. We then investigate in Section 5.2 the scaling limit of the objects appearing in this bijection, and deduce Theorem 1.14.

Discrete forests play an important part in the coding of quadrangulations with a boundary through the Bouttier–Di Francesco–Guitter bijection, and the analysis of Section 5.2 leads to the construction of a continuum random forest. We carry out the analysis of Le Gall [LG07] to our case in Section 5.3 and see any limiting space of Theorem 1.14 as a quotient of this continuum random forest via an equivalence relation defined in terms of Brownian labels on it.

Following Miermont [Mie08], we then prove Theorem 1.15 in Section 5.4 thanks to the notion of regularity introduced by Whyburn [Why35a, Why35b]. As we consider in this work surfaces with a boundary, the notion of 1-regularity used by Miermont in [Mie08] is no longer sufficient: we will also need here the notion of 0-regularity, which we will expose in Section 5.4.1.

Section 5.5 is dedicated to the case $\sigma = 0$ in which we use a totally different approach, consisting in comparing quadrangulations with a "small" boundary with quadrangulations without boundary.

Our general strategy is in many points similar to Part II. Although we will try to make this work as self-contained as possible, we will often refer the reader to this part when the proofs are readily adaptable, and will rather focus on the new ingredients.

5.1 The Bouttier–Di Francesco–Guitter bijection

As is often the case in this kind of problems, we start with a bijection allowing us to work with simpler objects. We use here a particular instance of the so-called Bouttier–Di Francesco–Guitter bijection [BDG04], which has already been used in [BG09]. For more convenience, we modify it a little to better fit our purpose. This will allow us to code quadrangulations with a boundary by forests whose vertices carry integer labels.

5.1.1 Forests

We use for forests the same formalism as before, which we briefly recall here. We denote by $\mathbb{N} := \{1, 2, \ldots\}$ the set of positive integers. For $u = (u_1, \ldots, u_n)$, $v = (v_1, \ldots, v_p) \in \bigcup_{n=1}^{\infty} \mathbb{N}^n$, we let $\|u\| := n$ be the height of u, and $uv := (u_1, \ldots, u_n, v_1, \ldots, v_p)$ be the concatenation of u and v. We say that u is an **ancestor** of uv and that uv is a **descendant** of u. In the case where $v \in \mathbb{N}$, we use the terms **parent** and **child** instead.

Definition 5.1. *A **forest** is a finite subset $\mathfrak{f} \subset \bigcup_{n=1}^{\infty} \mathbb{N}^n$ satisfying:*

*(i) there is an integer $t(\mathfrak{f}) \geq 1$, called the **number of trees** of $\mathfrak{f}$, such that $\mathfrak{f} \cap \mathbb{N} = [\![1, t(\mathfrak{f}) + 1]\!]$,*

(ii) if $u \in \mathfrak{f} \backslash \mathbb{N}$, then its parent belongs to $\mathfrak{f}$,

(iii) for every $u \in \mathfrak{f}$, there is an integer $c_u(\mathfrak{f}) \geq 0$ such that $ui \in \mathfrak{f}$ if and only if $1 \leq i \leq c_u(\mathfrak{f})$,

(iv) $c_{t(\mathfrak{f})+1}(\mathfrak{f}) = 0$.

The set $\mathit{fl} := \mathfrak{f} \cap \mathbb{N}$ is called the **floor** of the forest $\mathfrak{f}$. When $u \in \mathit{fl}$, we sometime note it (u) to avoid confusion between the integer u and the point $(u) \in \mathfrak{f}$. For $u = (u_1, \ldots, u_p) \in \mathfrak{f}$, we call $\mathfrak{a}(u) := u_1 \in \mathit{fl}$ its oldest ancestor. For $1 \leq j \leq t(\mathfrak{f})$, the set $\{u \in \mathfrak{f} : \mathfrak{a}(u) = j\}$ is called tree of $\mathfrak{f}$ rooted at (j). The points $u, v \in \mathfrak{f}$ are called **neighbors**, and we write $u \sim v$, if either u is a parent or child of v, or $u, v \in \mathit{fl}$ and $|u - v| = 1$. On the figures, we always draw edges between neighbors (see Figure 5.1). We say that an edge drawn between a parent and its child is a **tree edge** whereas an edge drawn between two consecutive tree roots will be called a **floor edge**.

Definition 5.2. *A **well-labeled forest** is a pair $(\mathfrak{f}, \mathfrak{l})$ where $\mathfrak{f}$ is a forest and $\mathfrak{l} : \mathfrak{f} \to \mathbb{Z}$ is a function satisfying:*

(i) for all $u \in \mathit{fl}$, $\mathfrak{l}(u) = 0$,

(ii) if $u \sim v$, then $|\mathfrak{l}(u) - \mathfrak{l}(v)| \leq 1$.

Let $\mathfrak{F}_\sigma^n := \{(\mathfrak{f}, \mathfrak{l}) : t(\mathfrak{f}) = \sigma, |\mathfrak{f}| = n + \sigma + 1\}$ be the set of well-labeled forests with σ trees and n tree edges. By a simple application (see Lemma 2.1) of the so-called cycle lemma [BCP03, Lemma 2], and the fact that to every forest with n tree edges correspond exactly 3^n labeling functions, we obtain that

$$|\mathfrak{F}_\sigma^n| = 3^n \frac{\sigma}{2n + \sigma} \binom{2n + \sigma}{n}. \tag{5.1}$$

For a forest $\mathfrak{f}$ with σ trees and n tree edges, we define its **facial sequence** $\mathfrak{f}(0), \mathfrak{f}(1), \ldots, \mathfrak{f}(2n + \sigma)$ as follows (see Figure 5.1): $\mathfrak{f}(0) := (1)$, and for $0 \leq i \leq 2n + \sigma - 1$,

- if $\mathfrak{f}(i)$ has children that do not appear in the sequence $\mathfrak{f}(0), \mathfrak{f}(1), \dots, \mathfrak{f}(i)$, then $\mathfrak{f}(i+1)$ is the first of these children, that is, $\mathfrak{f}(i+1) := \mathfrak{f}(i)j_0$ where

$$j_0 = \min\{j \geq 1 : \mathfrak{f}(i)j \notin \{\mathfrak{f}(0), \mathfrak{f}(1), \dots, \mathfrak{f}(i)\}\},$$

- otherwise, if $\mathfrak{f}(i) \notin \mathit{fl}$, then $\mathfrak{f}(i+1)$ is the parent of $\mathfrak{f}(i)$,

- if neither of these cases occur, which implies that $\mathfrak{f}(i) \in \mathit{fl}$, then $\mathfrak{f}(i+1) := \mathfrak{f}(i) + 1$.

A well-labeled forest $(\mathfrak{f}, \mathfrak{l})$ is then entirely determined by its so-called **contour pair** $(C_\mathfrak{f}, L_{\mathfrak{f},\mathfrak{l}})$ consisting in its **contour function** $C_\mathfrak{f} : [0, 2n+\sigma] \to \mathbb{R}_+$ and its **spatial contour function** $L_{\mathfrak{f},\mathfrak{l}} : [0, 2n+\sigma] \to \mathbb{R}$ defined by

$$C_\mathfrak{f}(i) := \|\mathfrak{f}(i)\| + t(\mathfrak{f}) - \mathfrak{a}(\mathfrak{f}(i)) \qquad \text{and} \qquad L_{\mathfrak{f},\mathfrak{l}}(i) := \mathfrak{l}(\mathfrak{f}(i)), \qquad 0 \leq i \leq 2n+\sigma,$$

and linearly interpolated between integer values (see Figure 5.1).

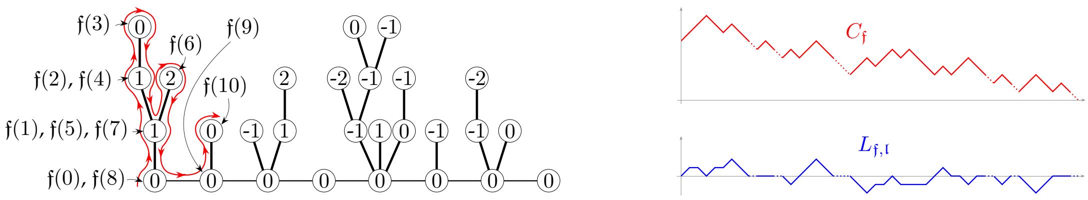

Figure 5.1. *The facial sequence and contour pair of a well-labeled forest from $\mathfrak{F}_7^{20}$. The paths are dashed on the intervals corresponding to floor edges.*

5.1.2 Bridges

Definition 5.3. *We say that a sequence of integers $(\mathfrak{b}(0), \mathfrak{b}(1), \dots, \mathfrak{b}(\sigma))$ for some $\sigma \geq 1$ is a **bridge** if $\mathfrak{b}(0) = 0$, $\mathfrak{b}(\sigma) \leq 0$, and, for all $0 \leq i \leq \sigma - 1$, we have $\mathfrak{b}(i+1) - \mathfrak{b}(i) \geq -1$. The integer σ will be called the **length** of the bridge.*

The somehow unusual condition $\mathfrak{b}(\sigma) \leq 0$ will become clear in the following section: it will be used to keep track of the position of the root in the quadrangulation. We call $\mathscr{B}_\sigma$ the set of all bridges of length σ. In the following, when we consider a bridge $\mathfrak{b} \in \mathscr{B}_\sigma$, we will always implicitly extend its definition to $[0, \sigma]$ by linear interpolation between integer values.

Lemma 5.1. *The cardinality of the set $\mathscr{B}_\sigma$ is*

$$|\mathscr{B}_\sigma| = \binom{2\sigma}{\sigma}.$$

Proof. With a bridge $(\mathfrak{b}(i))_{0 \leq i \leq \sigma} \in \mathscr{B}_\sigma$, we associate the following sequence

$$(\tilde{\mathfrak{b}}_j)_{1 \leq j \leq 2\sigma} := (\underbrace{+1, +1, \dots, +1}_{\mathfrak{b}(0) - \mathfrak{b}(\sigma) \text{ times}}, -1, \underbrace{+1, +1, \dots, +1}_{\mathfrak{b}(1) - \mathfrak{b}(0) + 1 \text{ times}}, -1, \underbrace{+1, +1, \dots, +1}_{\mathfrak{b}(2) - \mathfrak{b}(1) + 1 \text{ times}}, \dots, -1, \underbrace{+1, +1, \dots, +1}_{\mathfrak{b}(\sigma) - \mathfrak{b}(\sigma-1) + 1 \text{ times}}).$$

The set $\mathscr{B}_\sigma$ is then in one-to-one correspondence with the set of sequences in $\{-1, +1\}^{2\sigma}$ counting exactly σ times the number -1. The number of bridges of length σ is then the number of choices we have to place these σ numbers among the 2σ spots. ☐

5.1.3 The bijection

A **pointed quadrangulation** (with a boundary) is a pair $(\mathfrak{q}, v^\bullet)$ consisting in a quadrangulation (with a boundary) $\mathfrak{q}$ together with a distinguished vertex $v^\bullet \in V(\mathfrak{q})$. We call

$$\mathcal{Q}^\bullet_{n,\sigma} := \{(\mathfrak{q}, v^\bullet) : \mathfrak{q} \in \mathcal{Q}_{n,\sigma},\ v^\bullet \in V(\mathfrak{q})\}$$

the set of all pointed quadrangulations with n internal faces and 2σ edges on the boundary. The Bouttier–Di Francesco–Guitter bijection may easily be adapted into a bijection between the sets $\mathcal{Q}^\bullet_{n,\sigma}$ and $\mathfrak{F}^n_\sigma \times \mathscr{B}_\sigma$. We briefly describe it here, and refer the reader to [BDG04] for proofs and further details.

5.1.3.1 From quadrangulations to forests and bridges

Let us start with the mapping from $\mathcal{Q}^\bullet_{n,\sigma}$ onto $\mathfrak{F}^n_\sigma \times \mathscr{B}_\sigma$. Let $(\mathfrak{q}, v^\bullet) \in \mathcal{Q}^\bullet_{n,\sigma}$. We label the vertices of $\mathfrak{q}$ as follows: for every vertex $v \in V(\mathfrak{q})$, we set $\tilde{\mathfrak{l}}(v) = d_\mathfrak{q}(v^\bullet, v)$. Because $\mathfrak{q}$ is bipartite, the labels of both ends of any edge differ by exactly 1. As a result, the internal faces can be of two types: the labels around the face are either d, $d+1$, $d+2$, $d+1$, or d, $d+1$, d, $d+1$ for some d. We add a new edge to every internal face as shown on the left part of Figure 5.2.

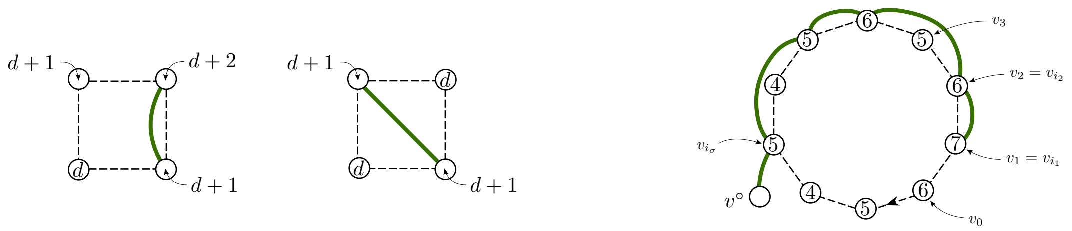

Figure 5.2. ***Left.*** *Adding the new edge to an internal face.* ***Right.*** *Example of the operation on the external face. In this example,* $\mathfrak{b} = (0, -1, -1, -2, -2, -1)$.

The operation regarding the external face is a little more intricate. We call $v_0, v_1, \ldots, v_{2\sigma-1}$ its vertices read in counterclockwise order, starting at the origin of the root, $v_0 = \mathfrak{e}_*^-$ (and we use the convention $v_{2\sigma} = v_0$). We only consider the vertices v_i such that $\tilde{\mathfrak{l}}(v_{i+1}) = \tilde{\mathfrak{l}}(v_i) - 1$. Note that, because $\tilde{\mathfrak{l}}(v_{i+1}) - \tilde{\mathfrak{l}}(v_i) \in \{-1, +1\}$, there are exactly σ such vertices. We call them $v_{i_1}, v_{i_2}, \ldots, v_{i_\sigma}$, with $0 \le i_1 < i_2 < \cdots < i_\sigma < 2\sigma$. Finally, we add a new vertex v° inside the external face, and draw extra edges linking v_{i_k} to $v_{i_{k+1}}$ for all $1 \le k \le \sigma - 1$, and v_{i_σ} to v°. See the right part of Figure 5.2.

We then only keep the new edges we added and the vertices in $(V(\mathfrak{q})\backslash\{v^\bullet\}) \cup \{v^\circ\}$. We obtain a forest $\mathfrak{f}$ whose floor is drawn in the external face: $(k) = v_{i_k}$ for $1 \le k \le \sigma$, and $(\sigma+1) = v^\circ$. To obtain the labels of $\mathfrak{f}$, we shift the labels tree by tree, in such a way that the floor labels are 0: we define $\mathfrak{l}(u) := \tilde{\mathfrak{l}}(u) - \tilde{\mathfrak{l}}(\mathfrak{a}(u))$, and $\mathfrak{l}(v^\circ) = 0$. Finally, the bridge $\mathfrak{b}$ records the labels of the floor before the shifting operation: for $0 \le k \le \sigma - 1$, we let $\mathfrak{b}(k) := \tilde{\mathfrak{l}}(v_{i_{k+1}}) - \tilde{\mathfrak{l}}(v_{i_1})$, and $\mathfrak{b}(\sigma) = \tilde{\mathfrak{l}}(v_0) - \tilde{\mathfrak{l}}(v_{i_1})$ (so that $\mathfrak{b}(\sigma)$ keeps track of the position of the root).

The pointed quadrangulation $(\mathfrak{q}, v^\bullet)$ corresponds to the pair $((\mathfrak{f}, \mathfrak{l}), \mathfrak{b})$.

5.1.3.2 From forests and bridges to quadrangulations

Let us now describe the mapping from $\mathfrak{F}^n_\sigma \times \mathscr{B}_\sigma$ onto $\mathcal{Q}^\bullet_{n,\sigma}$. Let $(\mathfrak{f}, \mathfrak{l}) \in \mathfrak{F}^n_\sigma$ be a well-labeled forest and $\mathfrak{b} \in \mathscr{B}_\sigma$ be a bridge. As above, we write $\mathfrak{f}(0), \mathfrak{f}(1), \ldots, \mathfrak{f}(2n+\sigma)$ the facial sequence of $\mathfrak{f}$. The pointed quadrangulation $(\mathfrak{q}, v^\bullet)$ corresponding to $((\mathfrak{f}, \mathfrak{l}), \mathfrak{b})$ is then constructed as follows. First, we shift all the labels of $\mathfrak{f}$ tree by tree according to the bridge $\mathfrak{b}$: precisely, we define $\hat{\mathfrak{l}}(u) := \mathfrak{l}(u) + \mathfrak{b}(\mathfrak{a}(u) - 1)$. Then, we shift all the labels in such a way that the minimal label is equal to 1: let us call $\tilde{\mathfrak{l}} := \hat{\mathfrak{l}} - \min \hat{\mathfrak{l}} + 1$ this shifted labeling function. We add an extra vertex $v^\bullet$ carrying the label $\tilde{\mathfrak{l}}(v^\bullet) := 0$ inside the only

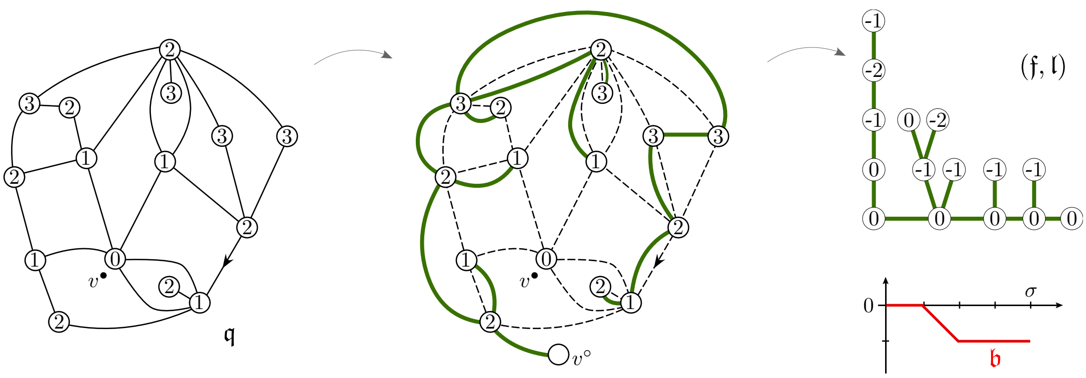

Figure 5.3. *The mapping from $\mathcal{Q}^\bullet_{n,\sigma}$ onto $\mathfrak{F}^n_\sigma \times \mathscr{B}_\sigma$. In this picture, $n = 10$ and $\sigma = 4$.*

face of $\mathfrak{f}$. Finally, following the facial sequence, for every $0 \le i \le 2n + \sigma - 1$, we draw an arc—without intersecting any edge of $\mathfrak{f}$ or arc already drawn—between $\mathfrak{f}(i)$ and $\mathfrak{f}(\mathrm{succ}(i))$, where $\mathrm{succ}(i)$ is the **successor** of i, defined by

$$\mathrm{succ}(i) := \begin{cases} \inf S_\ge & \text{if } S_\ge \neq \varnothing \\ \inf S_\le & \text{otherwise} \end{cases} \quad \text{with} \quad \begin{array}{l} S_\ge := \{k \in [\![i, 2n+\sigma-1]\!] \,:\, \tilde{\mathfrak{l}}(\mathfrak{f}(k)) = \tilde{\mathfrak{l}}(\mathfrak{f}(i)) - 1\} \\ S_\le := \{k \in [\![0, i-1]\!] \,:\, \tilde{\mathfrak{l}}(\mathfrak{f}(k)) = \tilde{\mathfrak{l}}(\mathfrak{f}(i)) - 1\} \end{array} \tag{5.2}$$

with the conventions $\inf \varnothing = \infty$, and $\mathfrak{f}(\infty) = v^\bullet$.

Because there may be more that one arc linking $\mathfrak{f}(i)$ to $\mathfrak{f}(\mathrm{succ}(i))$, we will speak of the arc linking i to $\mathrm{succ}(i)$ to avoid any confusion, and we will write it

$$i \frown \mathrm{succ}(i) \qquad \text{or} \qquad \mathrm{succ}(i) \frown i.$$

When we need an orientation, we will write $i \curvearrowright \mathrm{succ}(i)$ the arc orientated from i toward $\mathrm{succ}(i)$ and $i \curvearrowleft \mathrm{succ}(i)$ the arc orientated from $\mathrm{succ}(i)$ toward i. The quadrangulation $\mathfrak{q}$ is then defined as the map whose set of vertices is $(\mathfrak{f} \backslash \{(\sigma+1)\}) \cup \{v^\bullet\}$, whose edges are the arcs we drew, and whose root is either $\mathrm{succ}^{-\mathfrak{b}(\sigma)}(0) \curvearrowright \mathrm{succ}^{-\mathfrak{b}(\sigma)+1}(0)$ if $\mathfrak{b}(\sigma) > \mathfrak{b}(\sigma-1) - 1$, or $2n + \sigma - 1 \curvearrowleft \mathrm{succ}(2n + \sigma - 1)$ if $\mathfrak{b}(\sigma) = \mathfrak{b}(\sigma - 1) - 1$.

5.1.3.3 Some remarks

1. Because of the way we drew the arcs of $\mathfrak{q}$ in Section 5.1.3.2, it is easy to see that for any vertex $v \in V(\mathfrak{q})$, $\tilde{\mathfrak{l}}(v) = d_{\mathfrak{q}}(v^\bullet, v)$, so that both functions $\tilde{\mathfrak{l}}$ of Sections 5.1.3.1 and 5.1.3.2 coincide.

2. Note that the sequence $\tilde{\mathfrak{b}}$ from the proof of Lemma 5.1 reads the increments of the labels around the boundary: $\tilde{\mathfrak{b}}_j = \tilde{\mathfrak{l}}(v_j) - \tilde{\mathfrak{l}}(v_{j-1})$ for $1 \le j \le 2\sigma$.

3. Using Lemma 5.1, equation (5.1), and the fact that every quadrangulation in $\mathcal{Q}_{n,\sigma}$ has exactly $n+\sigma+1$ vertices, we see that

$$|\mathcal{Q}_{n,\sigma}| = \frac{|\mathfrak{F}^n_\sigma|\,|\mathscr{B}_\sigma|}{n+\sigma+1} = \frac{3^n (2\sigma)!\,(2n+\sigma-1)!}{\sigma!(\sigma-1)!\,n!\,(n+\sigma+1)!}.$$

Using generating functions techniques, Bouttier and Guitter [BG09, Equation (2.3)] already gave the same expression.

4. If we call (C, L) the contour pair of $(\mathfrak{f}, \mathfrak{l})$, then we may retrieve the oldest ancestor of $\mathfrak{f}(i)$ thanks to C by the relation

$$\mathfrak{a}\big(\mathfrak{f}(i)\big) - 1 = \sigma - \underline{C}(i),$$

where we use the notation

$$\underline{X}_s := \inf_{[0,s]} X$$

for any process $(X_s)_{s\geq 0}$. The function

$$\mathfrak{L} := \Big(L(s) + \mathfrak{b}\big(\sigma - \underline{C}(s)\big)\Big)_{0\leq s\leq 2n+\sigma}$$

then records the labels of the forest, once shifted tree by tree according to the bridge $\mathfrak{b}$. As a result, we see that $\mathfrak{L}(i) - \min \mathfrak{L} + 1$ represents the distance in $\mathfrak{q}$ between $v^\bullet$ and the point corresponding to $\mathfrak{f}(i)$.

5. This gives a natural way to explore the vertices of $\mathfrak{q}$: we call $\mathfrak{q}(i)$ the vertex corresponding to $\mathfrak{f}(i)$. In particular, $\{\mathfrak{q}(i),\, 0 \leq i \leq 2n+\sigma-1\} = V(\mathfrak{q})\backslash\{v^\bullet\}$. We end this section by giving an upper bound for the distance between two vertices $\mathfrak{q}(i)$ and $\mathfrak{q}(j)$, in terms of the function $\mathfrak{L}$:

$$d_{\mathfrak{q}}(\mathfrak{q}(i), \mathfrak{q}(j)) \leq \mathfrak{L}(i) + \mathfrak{L}(j) - 2\max\left(\min_{k\in\overrightarrow{[\![i,j]\!]}} \mathfrak{L}(k),\, \min_{k\in\overrightarrow{[\![j,i]\!]}} \mathfrak{L}(k)\right) + 2 \tag{5.3}$$

where we note, for $i \leq j$, $[\![i,j]\!] := [i,j] \cap \mathbb{Z} = \{i, i+1, \dots, j\}$, and

$$\overrightarrow{[\![i,j]\!]} := \begin{cases} [\![i,j]\!] & \text{if} \quad i \leq j, \\ [\![i, 2n+\sigma-1]\!] \cup [\![0,j]\!] & \text{if} \quad j < i. \end{cases} \tag{5.4}$$

We refer the reader to [Mie09b, Lemma 4] for a detailed proof of this bound.

5.2 Proof of Theorem 1.14

5.2.1 Convergence of the coding functions

Let $(\sigma_n)_{n\geq 1}$ be a sequence of positive integers such that

$$\sigma_{(n)} := \frac{\sigma_n}{\sqrt{2n}} \xrightarrow[n\to\infty]{} \sigma \geq 0.$$

Until further notice, we suppose that $\sigma > 0$. The remining case $\sigma = 0$ will be treated separately in Section 5.5. Let $\mathfrak{q}_n$ be uniformly distributed over the set $\mathcal{Q}_{n,\sigma_n}$ of quadrangulation with n internal faces and $2\sigma_n$ edges on the boundary. Conditionally given $\mathfrak{q}_n$, we let $v_n^\bullet$ be uniformly distributed over the set $V(\mathfrak{q}_n)$. Because every quadrangulation in $\mathcal{Q}_{n,\sigma_n}$ has exactly $n + \sigma_n + 1$ vertices (by Euler characteristic formula), we see that $(\mathfrak{q}_n, v_n^\bullet)$ is uniformly distributed over $\mathcal{Q}^\bullet_{n,\sigma_n}$, and thus correspond through the Bouttier–Di Francesco–Guitter bijection to a pair $\big((\mathfrak{f}_n, \mathfrak{l}_n), \mathfrak{b}_n\big)$ uniformly distributed over the set $\mathfrak{F}^n_{\sigma_n} \times \mathscr{B}_{\sigma_n}$.

5.2.1.1 Brownian bridges, first-passage Brownian bridges, and Brownian snake

Let us define the space

$$\mathcal{K} := \bigcup_{x\in\mathbb{R}_+} \mathcal{C}([0,x], \mathbb{R})$$

of continuous real-valued functions on $\mathbb{R}_+$ killed after some time. For an element $f \in \mathcal{K}$, we call $\zeta(f)$ its lifetime, that is, the only x such that $f \in \mathcal{C}([0,x], \mathbb{R})$. We endow this space with the following metric:

$$d_{\mathcal{K}}(f,g) := |\zeta(f) - \zeta(g)| + \sup_{y\geq 0}\Big|f\big(y\wedge\zeta(f)\big) - g\big(y\wedge\zeta(g)\big)\Big|.$$

We write $B^{0\to 0}_{[0,\sigma]}$ a Brownian bridge on $[0,\sigma]$ from 0 to 0, defined as a standard Brownian motion on $[0,\sigma]$ started at 0, conditioned on being at 0 at time σ (see for example [BCP03, Bil68, RY99] or Chapter 2). We also call $F^{\sigma\to 0}_{[0,1]}$ a first-passage Brownian bridge on $[0,1]$ from σ to 0, defined as a standard Brownian motion on $[0,1]$ started at σ, and conditioned on hitting 0 for the first time at time 1. We refer the reader to Chapter 2 for a proper definition of this conditioning, as well as for some convergence results of the discrete analogs.

The so-called Brownian snake's head may then be defined as the process $\left(F^{\sigma\to 0}_{[0,1]}, Z_{[0,1]}\right)$, where, conditionally given $F^{\sigma\to 0}_{[0,1]}$, the process $\left(Z_{[0,1]}(s)\right)_{0\leq s\leq 1}$ is a centered Gaussian process with covariance function

$$\operatorname{cov}\left(Z_{[0,1]}(s), Z_{[0,1]}(s')\right) = \inf_{[s\wedge s', s\vee s']} \left(F^{\sigma\to 0}_{[0,1]} - \underline{F}^{\sigma\to 0}_{[0,1]}\right). \tag{5.5}$$

We refer to [DLG02, LG99] or Chapter 2 for more details.

5.2.1.2 Convergence of the bridge and the contour pair of the well-labeled forest

We call (C_n, L_n) the contour pair of $(\mathfrak{f}_n, \mathfrak{l}_n)$, and define the scaled versions of C_n, L_n, and $\mathfrak{b}_n$ by

$$C_{(n)} := \left(\frac{C_n\big((2n+\sigma_n-1)s\big)}{\sqrt{2n}}\right)_{0\leq s\leq 1} \qquad L_{(n)} := \left(\frac{L_n\big((2n+\sigma_n-1)s\big)}{\gamma\, n^{1/4}}\right)_{0\leq s\leq 1}$$

$$\mathfrak{b}_{(n)} := \left(\frac{\mathfrak{b}_n(\sqrt{2n}\, s)}{\gamma\, n^{1/4}}\right)_{0\leq s\leq \sigma_{(n)}}$$

where the constant γ was defined during the statement of Theorem 1.14.

Remark. Following Part II, the notation with a parenthesized n will always refer to suitably rescaled objects, as in the definitions above.

The aim of this section is the following proposition.

Proposition 5.2. *The triple $(C_{(n)}, L_{(n)}, \mathfrak{b}_{(n)})$ converges in distribution in the space $(\mathcal{K}, d_{\mathcal{K}})^3$ toward a triple $(C_\infty, L_\infty, \mathfrak{b}_\infty)$ whose law is defined as follows:*

- ✧ *the processes (C_∞, L_∞) and $\mathfrak{b}_\infty$ are independant,*
- ✧ *the process (C_∞, L_∞) has the law of a Brownian snake's head on $[0,1]$ going from σ to 0:*

$$(C_\infty, L_\infty) \overset{(d)}{=} \left(F^{\sigma\to 0}_{[0,1]}, Z_{[0,1]}\right),$$

- ✧ *the process $\mathfrak{b}_\infty$ has the law of a Brownian bridge on $[0,\sigma]$ from 0 to 0, scaled by the factor $\sqrt{3}$:*

$$\mathfrak{b}_\infty \overset{(d)}{=} \sqrt{3}\, B^{0\to 0}_{[0,\sigma]}.$$

Proof. By Corollary 2.14, the pair $(C_{(n)}, L_{(n)})$ converges in distribution[1] toward $\left(F^{\sigma\to 0}_{[0,1]}, Z_{[0,1]}\right)$, in the space $(\mathcal{K}, d_{\mathcal{K}})^2$. Moreover, (C_n, L_n) and $\mathfrak{b}_n$ are independent, so that it only remains to show that $\mathfrak{b}_{(n)}$ converges in distribution toward $\sqrt{3}\, B^{0\to 0}_{[0,\sigma]}$. To this end, we will use Lemma 2.8.

Let $(X_i)_{i\geq 1}$ be a sequence of i.i.d. random variables with distribution given by

$$X_i \sim \sum_{k=0}^{\infty} 2^{-k-1}\, \delta_{k-1}.$$

We call $\Sigma_0 := 0$ and, for $j \geq 1$, $\Sigma_j := \sum_{i=1}^{j} X_i$. For $k \geq 0$ fixed, and n such that $\sigma_n \geq k$, we also define a process $(S_n^k(i))_{0\leq i\leq \sigma_n}$ distributed as $(\Sigma_i)_{0\leq i\leq \sigma_n}$ conditioned on the event $\{\Sigma_{\sigma_n} = -k\}$. We

[1] In Chapter 2, the processes considered were the same except that the term $(2n+\sigma_n-1)$ was replaced with $2n$. The fact that $\sigma_n/2n \to 0$ and the uniform continuity of the process $\left(F^{\sigma\to 0}_{[0,1]}, Z_{[0,1]}\right)$ yield the result as stated here.

extend its definition to $[0,\sigma_n]$ by linear interpolation between integer values. Because X_1 is centered, has moments of any order, and has variance 2, we may apply Lemma 2.8 and we see that the process

$$\left(\frac{S_n^k(\sqrt{2n}\,s)}{\sqrt{3}\,\gamma\,n^{1/4}}\right)_{0\le s\le\sigma_{(n)}} \xrightarrow[n\to\infty]{(d)} B^{0\to 0}_{[0,\sigma]}. \tag{5.6}$$

Moreover, it is easy to see that the bridge S_n^k is uniform over the set $\{\mathfrak{b}\in\mathscr{B}_{\sigma_n} : \mathfrak{b}(\sigma_n)=-k\}$. Indeed, for any $\mathfrak{b}\in\mathscr{B}_{\sigma_n}$ such that $\mathfrak{b}(\sigma_n)=-k$, we have

$$\mathbb{P}(S_n^k=\mathfrak{b}) = \frac{\mathbb{P}\Big(\forall i\in[\![1,\sigma_n]\!],\ X_i=\mathfrak{b}(i)-\mathfrak{b}(i-1)\Big)}{\mathbb{P}(\Sigma_{\sigma_n}=-k)} = \frac{2^{-2\sigma_n+k}}{\mathbb{P}(\Sigma_{\sigma_n}=-k)},$$

which does not depend on $\mathfrak{b}$ but only on n and k. For such a $\mathfrak{b}$, we call

$$c_{n,k} := \frac{\mathbb{P}(\mathfrak{b}_n=\mathfrak{b})}{\mathbb{P}(S_n^k=\mathfrak{b})} = \binom{2\sigma_n}{\sigma_n}^{-1}\binom{2\sigma_n-k-1}{\sigma_n-1}.$$

(We may use the bijection of Lemma 5.1 to compute the denominator.) We have that

$$c_{n,k} = \frac{1}{2}\frac{(2\sigma_n-k-1)!}{(2\sigma_n-1)!}\frac{\sigma_n!}{(\sigma_n-k)!} \le \frac{1}{2}\prod_{i=0}^{k-1}\frac{\sigma_n-i}{\sigma_n-i+\sigma_n-1} \le 2^{-k},$$

and that $c_{n,k}\to 2^{-k-1}$ as $n\to\infty$. Now, let $\varphi:\mathcal{K}\to\mathbb{R}$ be a bounded measurable function. Using (5.6), we obtain by dominated convergence that

$$\mathbb{E}\left[\varphi(\mathfrak{b}_{(n)})\right] = \sum_{k=0}^{\infty} c_{n,k}\,\mathbb{E}\left[\varphi\left(\left(\frac{S_n^k(\sqrt{2n}\,s)}{\gamma\,n^{1/4}}\right)_{0\le s\le\sigma_{(n)}}\right)\right] \xrightarrow[n\to\infty]{} \mathbb{E}\left[\varphi\Big(\sqrt{3}\,B^{0\to 0}_{[0,\sigma]}\Big)\right].$$

This completes the proof. ◻

Recall the notation $\mathfrak{q}_n(i)$ introduced at the end of Section 5.1 for the vertex corresponding to $\mathfrak{f}_n(i)$ through the Bouttier–Di Francesco–Guitter bijection. Remember that $d_{\mathfrak{q}_n}(v_n^\bullet,\mathfrak{q}_n(i)) = \mathfrak{L}_n(i)-\min\mathfrak{L}_n+1$, where

$$\mathfrak{L}_n := \Big(L_n(s)+\mathfrak{b}_n\Big(\sigma_n-\underline{C}_n(s)\Big)\Big)_{0\le s\le 2n+\sigma_n}. \tag{5.7}$$

The rescaled version of $\mathfrak{L}_n$ is then given by

$$\mathfrak{L}_{(n)} := \left(\frac{\mathfrak{L}_n\Big((2n+\sigma_n-1)s\Big)}{\gamma\,n^{1/4}}\right)_{0\le s\le 1} = \Big(L_{(n)}(s)+\mathfrak{b}_{(n)}\Big(\sigma_{(n)}-\underline{C}_{(n)}(s)\Big)\Big)_{0\le s\le 1}.$$

Corollary 5.3. *The process* $(C_{(n)},\mathfrak{L}_{(n)})$ *converges in distribution in the space* $(\mathcal{K},d_{\mathcal{K}})^2$ *toward the process* $(C_\infty,\mathfrak{L}_\infty)$*, where*

$$\mathfrak{L}_\infty := \Big(L_\infty(s)+\mathfrak{b}_\infty\Big(\sigma-\underline{C}_\infty(s)\Big)\Big)_{0\le s\le 1}. \tag{5.8}$$

5.2.2 Proof of Theorem 1.14

The proof of Theorem 1.14 is very similar to Section 3.1, so that we only sketch it. Our approach is adapted from Le Gall [LG07] for the first assertion, and from Le Gall and Miermont [LGM11a] for the Hausdorff dimension. In addition, we use this occasion to introduce some notation that will be usefull later.

We define on $[\![0, 2n+\sigma_n-1]\!]$ the pseudo-metric d_n by

$$d_n(i,j) := d_{\mathfrak{q}_n}\left(\mathfrak{q}_n(i), \mathfrak{q}_n(j)\right),$$

we extend its definition to non integer values by linear interpolation: for s,t in $[0, 2n+\sigma_n-1]$,

$$d_n(s,t) := \underline{s}\,\underline{t}\, d_n(\lceil s\rceil, \lceil t\rceil) + \underline{s}\,\underline{t}\, d_n(\lceil s\rceil, \lfloor t\rfloor) + \overline{s}\,\underline{t}\, d_n(\lfloor s\rfloor, \lceil t\rceil) + \overline{s}\,\overline{t}\, d_n(\lfloor s\rfloor, \lfloor t\rfloor),$$

where $\lfloor s\rfloor := \sup\{k\in\mathbb{Z},\, k\le s\}$, $\lceil s\rceil := \lfloor s\rfloor + 1$, $\underline{s} := s - \lfloor s\rfloor$ and $\overline{s} := \lceil s\rceil - s$, and we define its rescaled version: for $s,t\in[0,1]$, we let

$$d_{(n)}(s,t) := \frac{1}{\gamma\, n^{1/4}}\, d_n\Big((2n+\sigma_n-1)s, (2n+\sigma_n-1)t\Big).$$

We also define the equivalence relation $\sim_n$ on $[\![0, 2n+\sigma_n-1]\!]$ by declaring that $i\sim_n j$ when $\mathfrak{q}_n(i) = \mathfrak{q}_n(j)$, which is equivalent to $d_n(i,j)=0$. The function $d_{(n)}$ may then be seen as a metric on

$$\mathscr{Q}_n := (2n+\sigma_n-1)^{-1}\,[\![0, 2n+\sigma_n-1]\!]_{/\sim_n},$$

and, as $v_n^\bullet$ is the only point of $\mathfrak{q}_n$ that does not lie in $\{\mathfrak{q}_n(i) \,:\, 0\le i\le 2n+\sigma_n-1\}$, we have

$$d_{GH}\left(\Big(\mathscr{Q}_n, d_{(n)}\Big), \Big(V(\mathfrak{q}_n), \frac{1}{\gamma\, n^{1/4}}\, d_{\mathfrak{q}_n}\Big)\right) \le \frac{1}{\gamma\, n^{1/4}}. \tag{5.9}$$

The bound (5.3) gives us a control on the metric $d_{(n)}$, from where we can derive the following lemma (see Lemma 3.3).

Lemma 5.4. *The sequence of the laws of the processes*

$$\Big(d_{(n)}(s,t)\Big)_{0\le s,t\le 1}$$

is tight in the space of probability measure on $\mathcal{C}([0,1]^2,\mathbb{R})$.

As a result of Lemma 5.4, from any increasing sequence of integers, we may extract a (deterministic) subsequence $(n_k)_{k\ge0}$ such that there exists a random function $d^\sigma_\infty\in\mathcal{C}([0,1]^2,\mathbb{R})$ satisfying

$$\Big(d_{(n_k)}(s,t)\Big)_{0\le s,t\le1} \xrightarrow[k\to\infty]{(d)} \left(d^\sigma_\infty(s,t)\right)_{0\le s,t\le 1}. \tag{5.10}$$

By Skorokhod's representation theorem, we will assume that this convergence holds almost surely. In the limit, the bound (5.3) becomes

$$d^\sigma_\infty(s,t) \le d^\circ_\infty(s,t) := \mathfrak{L}_\infty(s) + \mathfrak{L}_\infty(t) - 2\max\left(\min_{x\in\overrightarrow{[s,t]}} \mathfrak{L}_\infty(x), \min_{x\in\overrightarrow{[t,s]}} \mathfrak{L}_\infty(x)\right), \quad 0\le s,t\le 1, \tag{5.11}$$

where

$$\overrightarrow{[s,t]} := \begin{cases} [s,t] & \text{if} \quad s\le t,\\ [s,1]\cup[0,t] & \text{if} \quad t<s. \end{cases} \tag{5.12}$$

Adding to this the fact that the functions $d_{(n)}$ obey the triangle inequality, we see that the function d^σ_∞ is a pseudo-metric. We define the equivalence relation associated with it by saying that $s\sim_\infty t$ if $d^\sigma_\infty(s,t)=0$, and we call $\mathfrak{q}^\sigma_\infty := [0,1]_{/\sim_\infty}$. The convergence claimed in Theorem 1.14 holds along the same subsequence $(n_k)_{k\ge0}$.

To see this, we use the characterization of the Gromov–Hausdorff distance via correspondences. Recall that a correspondence between two metric spaces $(\mathcal{X},\delta)$ and $(\mathcal{X}',\delta')$ is a subset $\mathcal{R}\subseteq\mathcal{X}\times\mathcal{X}'$ such that for all $x\in\mathcal{X}$, there is at least one $x'\in\mathcal{X}'$ for which $(x,x')\in\mathcal{R}$ and vice versa. The distortion of the correspondence $\mathcal{R}$ is defined by

$$\mathrm{dis}(\mathcal{R}) := \sup\left\{|\delta(x,y) - \delta(x',y')| \,:\, (x,x'),(y,y')\in\mathcal{R}\right\}.$$

Then we have [BBI01, Theorem 7.3.25]

$$d_{GH}(\mathcal{X},\mathcal{X}') = \frac{1}{2}\inf_{\mathcal{R}} \mathrm{dis}(\mathcal{R}), \tag{5.13}$$

where the infimum is taken over all correspondences between $\mathcal{X}$ and $\mathcal{X}'$.

We call $\boldsymbol{p}_n$ the canonical projection from $[\![0, 2n+\sigma_n-1]\!]$ to $[\![0, 2n+\sigma_n-1]\!]_{/\sim_n}$. For $t \in [0,1]$, we define $\boldsymbol{p}_{(n)}(t) := (2n+\sigma_n-1)^{-1}\,\boldsymbol{p}_n(\lfloor(2n+\sigma_n-1)t\rfloor)$, and we call $\mathfrak{q}^\sigma_\infty(t)$ the equivalence class of t in $\mathfrak{q}^\sigma_\infty$. We then define the correspondence $\mathcal{R}_n$ between the spaces $(\mathscr{Q}_n, d_{(n)})$ and $(\mathfrak{q}^\sigma_\infty, d^\sigma_\infty)$ as the set

$$\mathcal{R}_n := \left\{ \left(\boldsymbol{p}_{(n)}(t), \mathfrak{q}^\sigma_\infty(t)\right),\ t\in[0,1]\right\}.$$

Its distortion is

$$\mathrm{dis}(\mathcal{R}_n) = \sup_{0\le s,t\le 1} \left| d_{(n)}\left(\frac{\lfloor(2n+\sigma_n-1)s\rfloor}{2n+\sigma_n-1}, \frac{\lfloor(2n+\sigma_n-1)t\rfloor}{2n+\sigma_n-1}\right) - d^\sigma_\infty(s,t)\right|,$$

and, thanks to (5.10),

$$d_{GH}\left(\left(\mathscr{Q}_{n_k}, d_{(n_k)}\right), (\mathfrak{q}^\sigma_\infty, d^\sigma_\infty)\right) \le \frac{1}{2}\,\mathrm{dis}\left(\mathcal{R}_{n_k}\right) \xrightarrow[k\to\infty]{} 0.$$

Combining this with (5.9), we obtain the first assertion of Theorem 1.14.

The Hausdorff dimension of the limit may be computed by the technique we used in Chapter 3. Because the proof is very similar, and is not really related to our purpose here, we leave it to the reader. The idea is roughly the following. To prove that the Hausdorff dimension is less than 4, we use the fact that $\mathfrak{L}_\infty$ is almost surely α-Hölder for all $\alpha\in(0,1/4)$ [c.f. (5.7)], yielding that the canonical projection from $([0,1], |\cdot|)$ to $(\mathfrak{q}^\sigma_\infty, d^\sigma_\infty)$ is also α-Hölder for the same values of α. To prove that it is greater than 4, we show that the size of the balls of diameter δ is of order δ^4. To see this, we first bound from below the distances in terms of label variation along the branches of the forest, and then use twice the law of the iterated logarithm: this tells us that, for a fixed $s\in[0,1]$, the points outside of the set $[s-\delta^4, s+\delta^4]$ code points that are at distance at least δ^2 from $\mathfrak{q}^\sigma_\infty(s)$ in the forest, so that their distance from $\mathfrak{q}^\sigma_\infty(s)$ is at least δ in the map. See Section 3.1.4 for a complete proof. We will also use a similar approach to show Theorem 1.16 in Section 5.4.4.

From now on, we fix a subsequence $(n_k)_{k\ge 0}$ along which (5.10) holds. We will generally focus on this particular subsequence in the following, and we will often consider convergences when $n\to\infty$ to hold along this particular subsequence.

5.3 Maps seen as quotients of real forests

In the discrete setting, the metric space $(V(\mathfrak{q}_n), d_{\mathfrak{q}_n})$ may either be seen as a quotient of $[\![0, 2n+\sigma_n-1]\!]$, as in last section, or directly as the space $\mathfrak{f}_n$ endowed with the proper metric. In the continuous setting, we defined $\mathfrak{q}^\sigma_\infty$ as a quotient of $[0,1]$, but it will also be useful to see it as a quotient of a continuous analog to $\mathfrak{f}_n$. We obtain a quotient, because some points may be very close in the discrete forest, and become identified in the limit. Finding a criterion telling which points are identified in the limit will be the object of Section 5.3.3. In a first time, we define the continuous analog to forests.

5.3.1 Real forests

We define here real forests in a way convenient to our purpose, by adapting the notions used in Section 4.1. We will also need basic facts on real trees (see for example [LG05]). We dispose of a continuous function $h : [0,1] \to \mathbb{R}_+$ such that $h(1) = 0$, and we define on $[0,1]$ the relation $\simeq$ as the coarsest equivalence relation such that $0 \simeq 1$, and $s \simeq t$ if

$$h(s) = h(t) = \inf_{[s\wedge t, s\vee t]} h. \tag{5.14}$$

We call **real forest** any set $\mathscr{F} := [0,1]_{/\simeq}$ obtained by such a construction. It is possible to endow it with a natural metric, but we will not use it in this work. We now define the notions we will use throughout this work (see Figure 5.4). For $s \in [0,1]$, we write $\mathscr{F}(s)$ its equivalence class in the quotient $\mathscr{F} = [0,1]_{/\simeq}$. In a way, we see $(\mathscr{F}(s))_{0\le s\le 1}$ as the continuous facial sequence of $\mathscr{F}$. We call **root** of $\mathscr{F}$ the point $\partial := \mathscr{F}(0) = \mathscr{F}(1)$.

Definition 5.4. *The* ***floor*** *of $\mathscr{F}$ is the set $\mathit{fl} := \mathscr{F}(\{s : h(s) = \underline{h}(s)\})$.*

For $a = \mathscr{F}(s) \in \mathscr{F}\backslash \mathit{fl}$, let $l := \inf\{t \le s : \underline{h}(t) = \underline{h}(s)\}$ and $r := \sup\{t \ge s : \underline{h}(t) = \underline{h}(s)\}$. Note that, once endowed with the natural metric, the set $\tau_a := \mathscr{F}([l,r])$ is a real tree rooted at $\rho_a := \mathscr{F}(l) = \mathscr{F}(r) \in \mathit{fl}$. In the following, we will only use ensemblist notions about real trees.

Definition 5.5. *We call* ***tree*** *of $\mathscr{F}$ a set of the form τ_a for any $a \in \mathscr{F}\backslash \mathit{fl}$.*

If $a \in \mathit{fl}$, we simply set $\rho_a := a$. Let τ be a tree of $\mathscr{F}$ rooted at ρ, and $a, b \in \tau$. We call $[[a,b]]$ the range of the unique injective path linking a to b. In particular, the set $[[\rho, a]]$ represents the ancestral lineage of a in the tree τ. We say that a is an **ancestor** of b, and we write $a \preceq b$, if $a \in [[\rho, b]]$. We write $a \prec b$ if $a \preceq b$ and $a \ne b$.

Let $a, b \in \mathscr{F}$ be two points. There is a natural way to explore the forest $\mathscr{F}$ from a to b. If $\inf \mathscr{F}^{-1}(a) \le \sup \mathscr{F}^{-1}(b)$, then let $t := \inf\{r \ge \inf \mathscr{F}^{-1}(a) : b = \mathscr{F}(r)\}$ and $s := \sup\{r \le t : a = \mathscr{F}(r)\}$. If $\sup \mathscr{F}^{-1}(b) < \inf \mathscr{F}^{-1}(a)$, then let $t := \inf \mathscr{F}^{-1}(b)$ and $s := \sup \mathscr{F}^{-1}(a)$. We define

$$[a,b] := \mathscr{F}\left(\overrightarrow{[s,t]}\right), \tag{5.15}$$

where $\overrightarrow{[s,t]}$ is defined by (5.12). We may now extend the definition of $[[a,b]]$ to any two points in $\mathscr{F}$. First, for $a, b \in \mathit{fl}$, we let $[[a,b]] := [a,b] \cap \mathit{fl}$. Then, for any points $a, b \in \mathscr{F}$ such that $\rho_a \ne \rho_b$, we define

$$[[a,b]] := [[a,\rho_a]] \cup [[\rho_a,\rho_b]] \cup [[\rho_b, b]],$$

so that it is the range of the unique injective path from a to b that stays inside $[a,b]$.

Definition 5.6. *Let $b = \mathscr{F}(t) \in \mathscr{F}\backslash \mathit{fl}$ and $\rho \in [[\rho_b, b]]\backslash\{\rho_b, b\}$. Let $l' := \inf\{s \le t : \mathscr{F}(s) = \rho\}$ and $r' := \sup\{s \le t : \mathscr{F}(s) = \rho\}$. Then, provided $l' \ne r'$, we call* ***tree to the left*** *of $[[\rho_b, b]]$ rooted at ρ the set $\mathscr{F}([l', r'])$.*

We define the ***tree to the right*** *of $[[\rho_b, b]]$ rooted at ρ in a similar way, by replacing "$\le$" with "$\ge$" in the definitions of l' and r'.*

Definition 5.7. *We call* ***subtree*** *of $\mathscr{F}$ any tree of $\mathscr{F}$, or any tree to the left or right of $[[\rho_b, b]]$ for some $b \in \mathscr{F}\backslash \mathit{fl}$.*

Note that subtrees of $\mathscr{F}$ are real trees, and that trees of $\mathscr{F}$ are also subtrees of $\mathscr{F}$. The maximal interval $[s,t]$ such that $\tau = \mathscr{F}([s,t])$ is called the **interval coding** the subtree τ.

We call $\mathscr{F}_n$ the real forest obtained from the function $s \in [0,1] \mapsto C_n\Big((2n+\sigma_n)s\Big)$, as well as $\mathscr{F}_\infty$ the real forest obtained from the function C_∞. We also call $\simeq_{(n)}$ and $\simeq_\infty$ the corresponding equivalence relations. We write ∂_∞ the root of $\mathscr{F}_\infty$, and fl_∞ its floor. It is more natural to use $\mathfrak{f}_n$ rather than $\mathscr{F}_n$ in the discrete setting. As $\mathfrak{f}_n$ may be viewed as a subset of $\mathscr{F}_n$ (when identifying $(\sigma_n + 1)$ with (1)), we will use for $\mathfrak{f}_n$ the formalism we defined above simply by restriction. Note that the notions of floor and trees are consistent with the definitions we gave in Section 5.1.1 in this case.

Remark that, because the function C_∞ is a first-passage Brownian bridge, there are almost surely no trees rooted at the root ∂_∞ of $\mathscr{F}_\infty$, and all the points of $\mathscr{F}_\infty$ are of order less than 3, in the sense that for all $a \in \mathscr{F}_\infty$ and all connected subset $C \subseteq \mathscr{F}_\infty$, the number of connected components of $\mathscr{F}_\infty \cap C\backslash\{a\}$ is less than 3. We will not use this remark in the following, so that we do not go into further details.

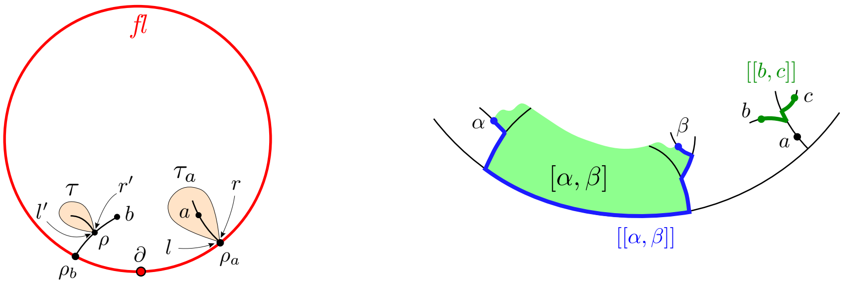

Figure 5.4. ***Left.*** *On this picture, we can see the root* ∂*, the floor fl, an example of tree* τ_a *(coded by* $[l, r]$*), and an example of tree* τ *to the left of* $[[\rho_b, b]]$ *rooted at* ρ *(coded by* $[l', r']$*).* ***Right.*** *On this picture, a is an ancestor of b and c, and we can see the sets* $[[b, c]]$*,* $[\alpha, \beta]$*, and* $[[\alpha, \beta]]$*.*

5.3.2 Quotient of real forests

Similarly to the notation $\mathfrak{f}_n(i)$ and $\mathfrak{q}_n(i)$ in the discrete setting, we call $\mathscr{F}_\infty(s)$ (resp. $\mathfrak{q}^\sigma_\infty(s)$) the equivalence class of $s \in [0, 1]$ in $\mathscr{F}_\infty = [0, 1]_{/\simeq_\infty}$ (resp. in $\mathfrak{q}^\sigma_\infty = [0, 1]_{/\sim_\infty}$).

Lemma 5.5. *The equivalence relation* $\simeq_\infty$ *is coarser than* $\sim_\infty$*.*

Proof. First, notice that, by (5.11), we have $d^\sigma_\infty(0, 1) \le d^\circ_\infty(0, 1) = 0$, so that $0 \sim_\infty 1$. The remaining is then identical to the first part of the proof of Lemma 4.1. □

This allows us to define a pseudo-metric and an equivalence relation on $\mathscr{F}_\infty$, still noted d^σ_∞ and $\sim_\infty$, by setting $d^\sigma_\infty\big(\mathscr{F}_\infty(s), \mathscr{F}_\infty(t)\big) := d^\sigma_\infty(s, t)$ and declaring $\mathscr{F}_\infty(s) \sim_\infty \mathscr{F}_\infty(t)$ if $s \sim_\infty t$. The metric space $(\mathfrak{q}^\sigma_\infty, d^\sigma_\infty)$ is thus isometric to $\big(\mathscr{F}_{\infty/\sim_\infty}, d^\sigma_\infty\big)$. We also define d°_∞ on $\mathscr{F}_\infty$ by letting

$$d^\circ_\infty(a, b) := \inf\left\{d^\circ_\infty(s, t) \,:\, a = \mathscr{F}_\infty(s),\, b = \mathscr{F}_\infty(t)\right\}.$$

We will see in Lemma 5.6 that there is a.s. only one point where the function $\mathfrak{L}_\infty$ reaches its minimum. If we call $s^\bullet \in [0, 1]$ this point, then it is not hard (see Lemma 4.2) to see from the fourth remark of Section 5.1.3.3 that

$$d^\sigma_\infty(s, s^\bullet) = \mathfrak{L}_\infty(s) - \mathfrak{L}_\infty(s^\bullet).$$

By the triangle inequality, we obtain that $s \sim_\infty t$ implies $\mathfrak{L}_\infty(s) = \mathfrak{L}_\infty(t)$, so that, in particular, $s \simeq_\infty t$ implies $\mathfrak{L}_\infty(s) = \mathfrak{L}_\infty(t)$, by Lemma 5.5. It is then licit to see $\mathfrak{L}_\infty$ as a function on $\mathscr{F}_\infty$ by setting $\mathfrak{L}_\infty\big(\mathscr{F}_\infty(s)\big) := \mathfrak{L}_\infty(s)$. This yields a more explicit expression for d°_∞:

$$d^\circ_\infty(a, b) = \mathfrak{L}_\infty(a) + \mathfrak{L}_\infty(b) - 2\max\left(\min_{x\in[a,b]} \mathfrak{L}_\infty(x), \min_{x\in[b,a]} \mathfrak{L}_\infty(x)\right), \tag{5.16}$$

where $[a, b]$ was defined by (5.15). Similarly, for $a \in \mathfrak{f}_n$, we call $\mathfrak{L}_n(a) := \mathfrak{l}_n(a) + \mathfrak{b}_n(\mathfrak{a}(a) - 1)$, so that $\mathfrak{L}_n(\mathfrak{f}_n(i)) = \mathfrak{L}_n(i)$ for all $0 \le i \le 2n + \sigma_n - 1$.

5.3.3 Point identifications

5.3.3.1 Criterion telling which points are identified

Our analysis starts with the following two observations on the process $(C_\infty(s), \mathfrak{L}_\infty(s))_{0\le s\le 1}$.

Lemma 5.6. *The set of points where* $\mathfrak{L}_\infty$ *reaches its minimum is a.s. a singleton.*

Let $f : [0, \ell] \to \mathbb{R}$ be a continuous function. We say that $s \in [0, \ell)$ is a **right-increase point** of f if there exists $t \in (s, \ell]$ such that $f(r) \ge f(s)$ for all $s \le r \le t$. A **left-increase point** is defined in a symmetric way. We call $\mathrm{IP}(f)$ the set of all (left or right) increase points of f.

Lemma 5.7. *Almost surely,* $\mathrm{IP}(C_\infty)$ *and* $\mathrm{IP}(\mathfrak{L}_\infty)$ *are disjoint sets.*

The proofs of these lemmas make intensive use of the so-called Brownian snake, so that we postpone them to Chapter 6. We have the following criterion:

Theorem 5.8. *Almost surely, for every* $a, b \in \mathscr{F}_\infty$*,* $a \sim_\infty b$ *is equivalent to* $d^\circ_\infty(a,b) = 0$*. In other words,*

$$d^\sigma_\infty(a,b) = 0 \;\Leftrightarrow\; d^\circ_\infty(a,b) = 0.$$

We call **leaves** the points of $\mathscr{F}_\infty$ whose equivalence class for $\simeq_\infty$ is trivial. It will be important in what follows to observe that, by Lemma 5.7 and Theorem 5.8, only leaves of $\mathscr{F}_\infty$ can be identified by $\sim_\infty$.

The proof of Theorem 5.8 is based on Lemma 5.6, Lemma 5.7, and Lemma 5.10 below, which we will prove in Chapter 6. Once we have these lemmas, the arguments of the proof of Theorem 4.3 (which uses the ideas of [LG07]) may readily be adapted to our case. For the sake of self-containment, we give here the main ingredients. By the bound (5.11), we already have one implication:

$$d^\circ_\infty(a,b) = 0 \;\Rightarrow\; d^\sigma_\infty(a,b) = 0.$$

The converse is shown in three steps. First, we show that the floor points are not identified (by $\sim_\infty$) with any other points, then that points are not identified with their strict ancestors, and finally the general case. The main point of the two first steps is that, if we take $a \sim_\infty b$, then a and b are not identified with any other points of the privileged paths $[[a,b]]$ and $[[b,a]]$. In the discrete setting, this translates into saying that the geodesics do not intersect these paths, and are thus more easily controlled. As an example, we will treat here the first step mentioned above. As we will see, the adaptation is almost verbatim, and is a little easier. The other steps use the same ideas and are even more straightforwardly adaptable, so that we leave them to the reader. Precisely, we are going to show the following lemma:

Lemma 5.9. *Almost surely, for every* $b \in \mathscr{F}_\infty$ *and every* $a \in \mathit{fl}_\infty \backslash \{\rho_b\}$*, we have* $a \not\sim_\infty b$*.*

5.3.3.2 Set overflown by a path and paths passing through subtrees

We give in this section the two notions we will need for discrete paths. In the following, we will never consider paths using the edges of the forest, but always paths using the edges of the map, and we will always use the letter "$\wp$" to denote these paths.

The first notion is the notion of set overflown by a path: roughly speaking, imagine a squirrel jumping from tree to tree in the forest along the edges of a path $\wp$ in the map. Then the set overflown by $\wp$ is the ground covered by the squirrel along its journey. Let us denote by fl_n the floor of $\mathfrak{f}_n$. Let $i \in [\![0, 2n + \sigma_n - 1]\!]$, and let $\mathrm{succ}(i)$ be its successor in $(\mathfrak{f}_n, \mathfrak{l}_n)$, defined by (5.2). We moreover suppose that $\mathrm{succ}(i) \neq \infty$. We say that the arc $i \frown \mathrm{succ}(i)$ linking $\mathfrak{f}_n(i)$ to $\mathfrak{f}_n(\mathrm{succ}(i))$ overflies the set

$$\mathfrak{f}_n\left(\overrightarrow{[\![i, \mathrm{succ}(i)]\!]}\right) \cap \mathit{fl}_n,$$

where $\overrightarrow{[\![i, \mathrm{succ}(i)]\!]}$ was defined by (5.4). We define the set overflown by a path $\wp$ in $\mathfrak{q}_n$ that avoids the base point $v^\bullet_n$ as the union of the sets its arcs overfly.

Remark. Note that, by the Bouttier–Di Francesco–Guitter construction, all the labels of the set overflown by a path are larger than or equal to the minimum label on the path.

The second notion is the notion of path passing through a subtree: here again, imagine a squirrel moving along the path $\wp$. The path $\wp$ passes through a subtree τ if the squirrel visits τ, and moreover enters it when going in one direction (from left to right or from right to left) and exits it while going in the same direction. Let τ be a subtree of $\mathfrak{f}_n$ and $\wp = (\wp(0), \wp(1), \dots, \wp(r))$ a path in $\mathfrak{q}_n$ that avoids the base point $v^\bullet_n$. We say that the path $\wp$ passes through the subtree τ between times i and j, where $0 < i \leq j < r$, if

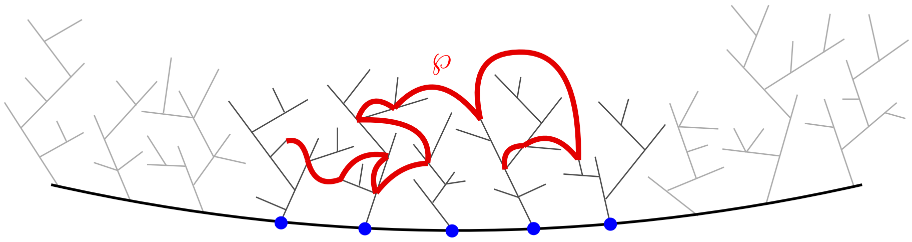

Figure 5.5. *The set overflown by the path $\wp$ is the set of (blue) large dots.*

- $\wp(i-1) \notin \tau$; $\wp([\![i,j]\!]) \subseteq \tau$; $\wp(j+1) \notin \tau$,
- $\mathfrak{L}_n(\wp(i)) - \mathfrak{L}_n(\wp(i-1)) = \mathfrak{L}_n(\wp(j+1)) - \mathfrak{L}_n(\wp(j))$.

We say that a vertex $a_n \in \mathfrak{f}_n$ converges toward a point $a \in \mathscr{F}_\infty$ if there exists a sequence of integers $s_n \in [\![0, 2n+\sigma_n-1]\!]$ coding a_n (i.e. $a_n = \mathfrak{f}_n(s_n)$) such that $s_n/(2n+\sigma_n-1)$ admits a limit s coding a, i.e. such that $a = \mathscr{F}_\infty(s)$. Let $[\![l_n, r_n]\!]$ be the intervals coding subtrees $\tau_n \subseteq \mathfrak{f}_n$. We say that the subtree τ_n converges toward a subtree $\tau \subseteq \mathscr{F}_\infty$ if the sequences $l_n/(2n+\sigma_n-1)$ and $r_n/(2n+\sigma_n-1)$ admit limits l and r such that the interval coding τ is $[l,r]$. The key lemma of our approach is the following. It is adapted from Le Gall [LG07, End of Proposition 4.2], and will be proved in Chapter 6.

Lemma 5.10. *With full probability, the following occurs. Let $a, b \in \mathscr{F}_\infty$ be such that $\mathfrak{L}_\infty(a) = \mathfrak{L}_\infty(b)$. We suppose that there exists a subtree τ rooted at ρ such that $\inf_\tau \mathfrak{L}_\infty < \mathfrak{L}_\infty(a) < \mathfrak{L}_\infty(\rho)$. We further suppose that we can find vertices a_n, $b_n \in \mathfrak{f}_n$ and subtrees τ_n in $\mathfrak{f}_n$ converging respectively toward a, b, τ and satisfying the following property: for infinitely many n's, there exists a geodesic path $\wp_n$ in $\mathfrak{q}_n$ from a_n to b_n that avoids the base point $v_n^\bullet$ and passes through the subtree τ_n.*

Then, $a \not\sim_\infty b$.

5.3.3.3 Proof of Lemma 5.9

Proof of Lemma 5.9. We argue by contradiction and suppose that we can find $b \in \mathscr{F}_\infty$ and $a \in \mathit{fl}_\infty \backslash \{\rho_b\}$ such that $a \sim_\infty b$. It is easy to find $a_n \in \mathit{fl}_n$ and $b_n \in \mathfrak{f}_n$ converging respectively toward a and b. Let $\wp_n$ be a geodesic path (in $\mathfrak{q}_n$, for $d_{\mathfrak{q}_n}$) from a_n to b_n. For n large, $\wp_n$ avoids the base-point, because otherwise, a and b would have the minimal label and this would contradict Lemma 5.6. For such an n, $\wp_n$ has to overfly at least $[[\rho_{b_n}, a_n]]$ or $[[a_n, \rho_{b_n}]]$. To see this, let $(x,y) \in [[\rho_{b_n}, a_n]] \times [[a_n, \rho_{b_n}]]$. When we remove from $\mathfrak{f}_n$ all the edges incident to x and all the edges incident to y, we obtain several connected components, and the points a_n and b_n do not belong to the same of these components. There has to be an arc of $\wp_n$ that links a point belonging to the component containing a_n to one of the other components. Such an arc overflies x or y.

Let us suppose that, for infinitely many n's, $\wp_n$ overflies $[[\rho_{b_n}, a_n]]$. By the remark concerning the labels on the set overflown by a path in the previous section, a simple argument (see Lemma 4.9) shows that $\mathfrak{L}_\infty(c) \geq \mathfrak{L}_\infty(a) = \mathfrak{L}_\infty(b)$ for all $c \in [[\rho_b, a]]$. The labels on fl_∞ are given by the process $\mathfrak{b}_\infty$, defined during Proposition 5.2: for $x \in [0,\sigma]$, we define $T_x := \inf\{r \geq 0 \,:\, C_\infty(r) = \sigma - x\}$, so that $\mathit{fl}_\infty = \mathscr{F}_\infty(\{T_x,\, 0 \leq x \leq \sigma\})$, and

$$\Big(\mathfrak{L}_\infty(T_x)\Big)_{0 \leq x \leq \sigma} = \Big(\mathfrak{b}_\infty(x)\Big)_{0 \leq x \leq \sigma}.$$

As $\mathfrak{b}_\infty$ has the law of a certain Brownian bridge (scaled by $\sqrt{3}$), and as local minimums of Brownian motion are distinct, we can find $d \in [[\rho_b, a]] \backslash \{a, \rho_b\}$ such that $\mathfrak{L}_\infty(c) > \mathfrak{L}_\infty(a)$ for all $c \in [[d,a]] \backslash \{a\}$.

Because $a \in \mathit{fl}_\infty$, every number coding it is an increase point of C_∞ and thus is not an increase point of $\mathfrak{L}_\infty$, by Lemma 5.7. As a result, there exists a tree τ^1 rooted at $\rho^1 \in [[d,a]] \backslash \{a\}$ satisfying $\inf_{\tau^1} \mathfrak{L}_\infty < \mathfrak{L}_\infty(a) < \mathfrak{L}_\infty(\rho^1)$ (see Figure 5.6).

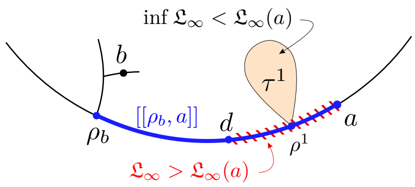

Figure 5.6. *The tree* τ^1.

Similarly, if for infinitely many n's, $\wp_n$ overflies $[[a_n, \rho_{b_n}]]$, then we can find a tree τ^2 rooted at $\rho^2 \in [[a, \rho_b]]\backslash\{a, \rho_b\}$ satisfying $\inf_{\tau^2} \mathfrak{L}_\infty < \mathfrak{L}_\infty(a) < \mathfrak{L}_\infty(\rho^2)$. Three cases may occur:

(i) for n large enough, $\wp_n$ does not overfly $[[a_n, \rho_{b_n}]]$ (and therefore overflies $[[\rho_{b_n}, a_n]]$),

(ii) for n large enough, $\wp_n$ does not overfly $[[\rho_{b_n}, a_n]]$ (and therefore overflies $[[a_n, \rho_{b_n}]]$),

(iii) $\wp_n$ overflies $[[\rho_{b_n}, a_n]]$ for infinitely many n's, and $[[a_n, \rho_{b_n}]]$ also for infinitely many n's.

In case (i), the tree τ^1 is well defined. Let $\tau_n^1 \subseteq \mathfrak{f}_n^{\mathfrak{c}}$ be a tree rooted at $\rho_n^1 \in [[\rho_{b_n}, a_n]]$ converging to τ^1. We claim that, for n sufficiently large, $\wp_n$ passes through τ_n^1. First, notice that, for n large enough, $\inf_{\tau_n^1} \mathfrak{L}_n < \inf_{\wp_n} \mathfrak{L}_n$. The idea is that, at some point, $\wp_n$ has to go from a tree located at the right of τ_n^1 to a tree located at its left, and, because it does not overfly $[[a_n, \rho_{b_n}]]$, it has no other choice than passing through τ_n^1 (see Figure 5.7).

More precisely, we call $[\![s_n^1, t_n^1]\!]$ the set coding the subtree τ_n^1, and we let $\omega_n = \mathfrak{f}_n(p_n) \in [[a_n, \rho_{b_n}]]$ be a point that is not overflown by $\wp_n$. Then, we define

$$A_n := \mathfrak{f}_n\left(\overrightarrow{[\![t_n^1 + 1, p_n]\!]}\right).$$

We call $\wp_n(i-1)$ the last point of $\wp_n$ belonging to A_n. Such a point exists because $a_n \in A_n$ and $b_n \notin A_n$. For n large, because $\wp_n$ does not overfly ω_n, and because $\inf_{\tau_n^1} \mathfrak{L}_n < \inf_{\wp_n} \mathfrak{L}_n$, we see that $\wp_n(i) \in \tau_n^1$. Let $\wp_n(j+1)$ be the first point after $\wp_n(i)$ not belonging to τ_n^1. It exists because $b_n \notin \tau_n^1$. Using the facts that $\wp_n$ does not overfly ω_n, and that $\wp_n(j+1) \notin A_n$, we see that $\wp_n$ passes through τ_n^1 between times i and j.

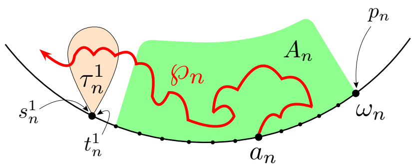

Figure 5.7. *The path* $\wp_n$ *passing through the tree* τ_n^1.

In case (ii), we apply the same reasoning with τ^2 instead of τ^1. In case (iii), both trees τ^1 and τ^2 are well defined and we obtain that $\wp_n$ has to pass through one of their discrete approximations. We then conclude by Lemma 5.10 that $a \not\simeq_\infty b$, which contradicts our hypothesis. □

5.4 Regularity of quadrangulations

Recently, the notion of regularity has been used to identify the topology of the scaling limit of random uniform planar quadrangulations in [Mie08], and then positive genus quadrangulations in Chapter 4. In both these references, it is the notion of 1-regularity that is used, roughly stating that there are no small loops separating the surface in large components. In the case of surfaces with a boundary, a new problem arises, and we also need the notion of 0-regularity for the boundary. In this section, we expose both these notions, which were introduced in a slightly different context (see the discussion in [Mie08, Section 2]) by Whyburn [Why35a, Why35b], and then use them to prove Theorem 1.15.

5.4.1 0-regularity and 1-regularity

Recall that we wrote $(\mathbb{M}, d_{GH})$ the set of isometry classes of compact metric spaces, endowed with the Gromov–Hausdorff metric. A compact metric space $(\mathcal{X}, \delta)$ is called a **path metric space** if any two points $x, y \in \mathcal{X}$ can be joined by a path isometric to the segment $[0, \delta(x, y)]$. We let $\mathbb{PM}$ be the set of isometry classes of path metric spaces, which is a closed subset of $\mathbb{M}$, by [BBI01, Theorem 7.5.1].

Definition 5.8. *We say that a sequence $(\mathcal{X}_n)_{n\geq 1}$ of compact metric spaces is **1-regular** if for every $\varepsilon > 0$, there exists $\eta > 0$ such that for n large enough, every loop of diameter less than η in $\mathcal{X}_n$ is homotopic to 0 in its ε-neighborhood.*

The theorem (derived from [Beg44, theorem 7]) that was used in [Mie08] and Chapter 4 states that the limit of a converging 1-regular sequence of path metric spaces all homeomorphic to the g-torus is either reduced to a singleton (this can only happen when $g = 0$), or homeomorphic to the g-torus as well. In other words, this gives a sufficient condition for the limit to be homeomorphic to the surface we started with. In the case of the 2-dimensional disc $\mathbb{D}_2$, this condition is no longer sufficient. For example, take for the space $\mathcal{X}_n$ the union of two unit discs whose centers are at distance $2 - 1/n$. This peanut-shaped space is homeomorphic to $\mathbb{D}_2$, and it is easy to see that $(\mathcal{X}_n)_n$ is 1-regular and converges to the wedge sum (or bouquet) of two discs. The following definition discards this kind of degeneracy.

Definition 5.9. *We say that a sequence $(\mathcal{X}_n)_{n\geq 1}$ of compact metric spaces is **0-regular** if for every $\varepsilon > 0$, there exists $\eta > 0$ such that for n large enough, every pair of points in $\mathcal{X}_n$ lying at a distance less than η from each other belong to a connected subset of $\mathcal{X}_n$ of diameter less than ε.*

We will rely on the following theorem, which is a simple consequence of [Why35a, Theorem 6.4].

Proposition 5.11 (Whyburn). *Let $(\mathcal{X}_n)_{n\geq 1}$ be a sequence of path metric spaces all homeomorphic to the 2-dimensional disc $\mathbb{D}_2$, converging for the Gromov–Hausdorff topology toward a metric space $\mathcal{X}$ not reduced to a single point. Suppose that the sequence $(\mathcal{X}_n)_{n\geq 1}$ is 1-regular, and that the sequence $(\partial\mathcal{X}_n)_n$ is 0-regular.*
Then $\mathcal{X}$ is homeomorphic to $\mathbb{D}_2$ as well.

In [Why35a], Whyburn actually considered convergence in the sense of the Hausdorff topology, and made the extra hypothesis that $\partial\mathcal{X}_n$ converges to a set B. He concluded that $\mathcal{X}$ was homeomorphic to $\mathbb{D}_2$ and that $\partial\mathcal{X}$ was equal to B. To derive the version that we state here, we proceed as follows. First, by [GPW09, Lemma A.1], we can find a compact metric space $\mathcal{Z}$, and isometric embeddings φ, $\varphi_1, \varphi_2, \ldots$ of $\mathcal{X}, \mathcal{X}_1, \mathcal{X}_2, \ldots$ into $\mathcal{Z}$ such that $\varphi_n(\mathcal{X}_n)$ converges toward $\varphi(\mathcal{X})$ for the Hausdorff topology in $\mathcal{Z}$. Then, by [BBI01, Theorem 7.3.8], the family $\{\partial(\varphi_n(\mathcal{X}_n))\}$ is relatively compact for the Hausdorff topology. Let us consider a subsequence along which $\partial(\varphi_n(\mathcal{X}_n))$ converges to a set B. Applying Whyburn's original theorem along this subsequence, we obtain that $\varphi(\mathcal{X})$ is homeomorphic to $\mathbb{D}_2$, so that $\mathcal{X}$ is homeomorphic to $\mathbb{D}_2$ as well. We moreover obtain that $\partial(\varphi(\mathcal{X})) = B$, and, using the same argument, we see that any accumulation point of the sequence $(\partial(\varphi_n(\mathcal{X}_n)))_n$ has to be $\partial(\varphi(\mathcal{X}))$, so that $\partial(\varphi_n(\mathcal{X}_n)) = \varphi_n(\partial\mathcal{X}_n)$ actually converges toward $\partial(\varphi(\mathcal{X})) = \varphi(\partial\mathcal{X})$ for the Hausdorff topology. This last observation will be used in Section 5.4.4 to identify the boundary of $\mathfrak{q}_\infty^\sigma$.

5.4.2 Representation as metric surfaces

As the space $(V(\mathfrak{q}_n), d_{\mathfrak{q}_n})$ is not a surface, we cannot directly apply Proposition 5.11. In a first time, we will construct a path metric space $(\mathcal{S}_n, \delta_n)$ homeomorphic to $\mathbb{D}_2$, and an embedded graph that is a representative of the map $\mathfrak{q}_n$, such that the restriction of $(\mathcal{S}_n, \delta_n)$ to the embedded graph is isometric to $(V(\mathfrak{q}_n), d_{\mathfrak{q}_n})$. We use the same method as Miermont in [Mie08, Section 3.1] (see also Section 4.3.2), roughly consisting in gluing hollow boxes together according to the structure of $\mathfrak{q}_n$.

Let us be a little more specific. We call f_* the external face of $\mathfrak{q}_n$, $F(\mathfrak{q}_n)$ its set of internal faces, and $F_*(\mathfrak{q}_n) \coloneqq F(\mathfrak{q}_n) \cup \{f_*\}$ the set of all its faces. We also note $\mathcal{G}$ a regular $2\sigma_n$-gon with unit length edges embedded in $\mathbb{R}^2$, and call z_k, $0 \le k \le 2\sigma_n$, its vertices (with $z_0 = z_{2\sigma_n}$). With every quadrangle $f \in F(\mathfrak{q}_n)$, we associate a copy of the "hollow bottomless unit cube," and with f_*, we associate a "hollow bottomless $2\sigma_n$-sided prism": we define

$$X_f \coloneqq [0,1]^3 \backslash \Big((0,1)^2 \times [0,1)\Big), \quad f \in F(\mathfrak{q}_n), \qquad \text{and} \qquad X_{f_*} \coloneqq \Big(\mathcal{G} \times [0,1]\Big) \backslash \Big(\mathring{\mathcal{G}} \times [0,1)\Big),$$

and we endow these spaces with the intrinsic metric D_f inherited from the Euclidean metric. This means that the distance between two points x and y is the Euclidean length of a minimal path in X_f linking x to y. Note in particular that if x and y are on the boundary, this path is entirely contained in the boundary. This will ensure that, when we will glue these spaces together, we will not alter the graph metric. Note also that, so far, the external face is not really treated differently from the other faces (except for the fact that it has a different number of edges). In the end, we will remove the "top" $\mathring{\mathcal{G}} \times \{1\}$ from X_{f_*}.

Now, we associate with every half-edge $e \in \vec{E}(\mathfrak{q}_n)$ a path along one of the edges of the polygon ∂X_f, where f is the face incident to e. We call $e_1, e_2, \ldots, e_{2\sigma_n}$ the half-edges bordering f_* ordered in the clockwise order (recall that, by convention, f_* is the infinite face of $\mathfrak{q}_n$, so that the order is reversed), and define

$$c_{e_k}(t) \coloneqq \Big((1-t)z_{k-1} + tz_k, 0\Big) \in X_{f_*}, \qquad t \in [0,1], \quad 1 \le k \le 2\sigma_n.$$

In a similar way, for every internal face $f \in F(\mathfrak{q}_n)$, and every half-edge e incident to it, we define a function $c_e : [0,1] \to \partial X_f$ parameterizing an edge of ∂X_f. We do this in such a way that the parameterization of ∂X_f is coherent with the counterclockwise order around f (see [Mie08, Section 3.1] or Section 4.3.2).

We may now glue these spaces together along their boundaries: we define the relation $\approx$ as the coarsest equivalence relation for which $c_e(t) \approx c_{\bar{e}}(1-t)$ for all $e \in \vec{E}(\mathfrak{q}_n)$ and $t \in [0,1]$, where $\bar{e}$ denotes the reverse of e. The topological quotient $\hat{\mathcal{S}}_n \coloneqq (\bigsqcup_{f \in F_*(\mathfrak{q}_n)} X_f)_{/\approx}$ is then a 2-dimensional CW-complex satisfying the following properties. Its 1-skeleton $\mathcal{E}_n = (\bigsqcup_{f \in F_*(\mathfrak{q}_n)} \partial X_f)_{/\approx}$ is an embedding of $\mathfrak{q}_n$ with faces $X_f \backslash \partial X_f$. The edge $\{e, \bar{e}\} \in E(\mathfrak{q}_n)$ corresponds to the edge of $\hat{\mathcal{S}}_n$ made of the equivalence class of the points in $c_e([0,1])$. Its 0-skeleton $\mathcal{V}_n$ is in one-to-one correspondence with $V(\mathfrak{q}_n)$, and its vertices are the equivalence classes of the vertices of the polygons ∂X_f's.

We endow the space $\bigsqcup_{f \in F_*(\mathfrak{q}_n)} X_f$ with the largest pseudo-metric δ_n compatible with D_f, $f \in F_*(\mathfrak{q}_n)$ and $\approx$, in the sense that $\delta_n(x,y) \le D_f(x,y)$ for $x, y \in X_f$ and $\delta_n(x,y) = 0$ whenever $x \approx y$. Its quotient, which we still note δ_n, then defines a pseudo-metric on $\hat{\mathcal{S}}_n$ (which is actually a true metric, as we will see in Proposition 5.12). We also define $\delta_{(n)} \coloneqq \delta_n / \gamma\, n^{1/4}$ its rescaled version. Finally, we call $\mathcal{S}_n \coloneqq (\bigsqcup_{f \in F_*(\mathfrak{q}_n)} Y_f)_{/\approx} \subseteq \hat{\mathcal{S}}_n$, where $Y_{f_*} \coloneqq X_{f_*} \backslash (\mathring{\mathcal{G}} \times \{1\})$ and $Y_f \coloneqq X_f$ when $f \ne f_*$.

Proposition 5.12 ([Mie08, Proposition 1]). *The space $(\hat{\mathcal{S}}_n, \delta_n)$ is a path metric space homeomorphic to $\mathbb{S}_2$. Moreover, the restriction of $\hat{\mathcal{S}}_n$ to $\mathcal{V}_n$ is isometric to $(V(\mathfrak{q}_n), d_{\mathfrak{q}_n})$, and any geodesic path in $\hat{\mathcal{S}}_n$ between points in $\mathcal{V}_n$ is a concatenation of edges of $\hat{\mathcal{S}}_n$.*

We readily obtain the following corollary.

Corollary 5.13. *The space $(\mathcal{S}_n, \delta_n)$ is a path metric space homeomorphic to $\mathbb{D}_2$. Moreover, the restriction of $\mathcal{S}_n$ to $\mathcal{V}_n$ is isometric to $(V(\mathfrak{q}_n), d_{\mathfrak{q}_n})$, and any geodesic path in $\mathcal{S}_n$ between points in $\mathcal{V}_n$ is a concatenation of edges of $\mathcal{S}_n$. Finally, $d_{GH}\Big((V(\mathfrak{q}_n), d_{\mathfrak{q}_n}), (\mathcal{S}_n, \delta_n)\Big) \le 3$, so that, by Theorem 1.14,*

$$\Big(\mathcal{S}_{n_k}, \delta_{(n_k)}\Big) \xrightarrow[k\to\infty]{(d)} (\mathfrak{q}^\sigma_\infty, d^\sigma_\infty)$$

in the sense of the Gromov–Hausdorff topology.

Note that, although the boundary of $\mathfrak{q}_n$ is not topologically a circle in general, $\partial\mathcal{S}_n$ (which corresponds to $\partial\mathcal{G} \times \{1\}$ in Y_{f_*}) always is. In what follows, we will see $V(\mathfrak{q}_n)$ as a subset of $\mathcal{S}_n$. In other words, we identify $\mathcal{V}_n$ with $V(\mathfrak{q}_n)$.

5.4.3 Proof of Theorem 1.15

We now prove that $(\mathfrak{q}^\sigma_\infty, d^\sigma_\infty)$ is a.s. homeomorphic to $\mathbb{D}_2$ thanks to Proposition 5.11 and Corollary 5.13. As $(\mathfrak{q}^\sigma_\infty, d^\sigma_\infty)$ is a.s. not reduced to a point[2], it is enough to show that the sequence $(\partial\mathcal{S}_{n_k})_k$ is 0-regular, and that the sequence $(\mathcal{S}_{n_k})_k$ is 1-regular. The 1-regularity of $(\mathcal{S}_{n_k})_k$ is readily adaptable from Section 4.3.3 so that we begin with the 0-regularity of the boundary. We call $\pi_\infty : \mathscr{F}_\infty \to \mathfrak{q}^\sigma_\infty$ the canonical projection.

5.4.3.1 0-regularity of the boundary

Lemma 5.14. *The sequence $(\partial\mathcal{S}_{n_k})_k$ is 0-regular.*

Proof. The idea is that $f\!l_\infty$ has no cut points in $\mathscr{F}_\infty$, and because the points in $f\!l_\infty$ are not identified with any other points, $\pi_\infty(f\!l_\infty)$ does not have any cut points either.

We argue by contradiction and assume that, with positive probability, along some (random) subsequence of the sequence $(n_k)_{k\geq 0}$, there exist $\varepsilon > 0$, x_n, $y_n \in \partial\mathcal{S}_n$ such that $\delta_{(n)}(x_n, y_n) \to 0$, and x_n and y_n do not belong to the same connected component of $B_{(n)}(x_n, \varepsilon) \cap \partial\mathcal{S}_n$, where $B_{(n)}(x_n, \varepsilon)$ denotes the open ball of radius ε centered at x_n for the metric $\delta_{(n)}$. We reason on this event.

As x_n and y_n do not belong to the same connected component of $B_{(n)}(x_n, \varepsilon) \cap \partial\mathcal{S}_n$, we can find x'_n, $y'_n \in \partial\mathcal{S}_n \backslash B_{(n)}(x_n, \varepsilon)$ such that x'_n belongs to one of the two arcs joining x_n to y_n in $\partial\mathcal{S}_n$, and such that y'_n belongs to the other one. We are going to approach these four points with points of $f\!l_n$.

We note $\partial\mathfrak{q}_n \subseteq \vec{E}(\mathfrak{q}_n)$ the set of half-edges incident to the external face of $\mathfrak{q}_n$. With every point $x \in \partial\mathcal{S}_n$ naturally corresponds a half-edge $e(x) \in \partial\mathfrak{q}_n$: if x corresponds to $((1-t)z_{k-1} + tz_k, 1) \in X_{f_*}$ for some $t \in [0, 1)$, then $e(x)$ is the half-edge e_k. We consider the first half-edge $e \in \partial\mathfrak{q}_n$ after $e(x_n)$ ($e(x_n)$ included) in the clockwise order such that $\mathfrak{L}_n(e^+) = \mathfrak{L}_n(e^-) + 1$, and we call $a_n := e^+$. By definition of the Bouttier–Di Francesco–Guitter bijection, $a_n \in f\!l_n$. Moreover, a_n is "close" to x_n, in the sense that $\delta_n(a_n, x_n) \leq 1 + \sup_{0 \leq i < 2\sigma_n} |\mathfrak{b}_n(i+1) - \mathfrak{b}_n(i) + 2|$, so that

$$\delta_{(n)}(a_n, x_n) \leq \frac{3}{\gamma\, n^{1/4}} + \sup_x \left| \mathfrak{b}_{(n)}\left(x + \frac{1}{\sqrt{2n}}\right) - \mathfrak{b}_{(n)}(x) \right| \leq \frac{3}{\gamma\, n^{1/4}} + \omega_{\mathfrak{b}_{(n)}}(\eta),$$

as soon as $n \geq 1/2\eta^2$. Here, $\omega_{\mathfrak{b}_{(n)}}$ denotes the modulus of continuity of $\mathfrak{b}_{(n)}$. Hence, we obtain that $\limsup \delta_{(n)}(a_n, x_n) \leq \omega_{\mathfrak{b}_\infty}(\eta)$, for all $\eta > 0$, so that $\delta_{(n)}(a_n, x_n) \to 0$.

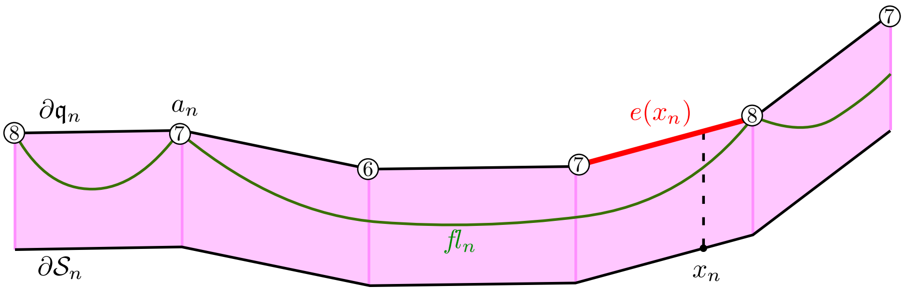

Figure 5.8. *Approaching a point $x_n \in \partial\mathcal{S}_n$ with a point $a_n \in f\!l_n$.*

We define in a similar way points b_n, a'_n, and b'_n in $f\!l_n$ corresponding to y_n, x'_n, and y'_n. Exchanging x'_n and y'_n if necessary, we may suppose that the points a_n, a'_n, b_n, b'_n are encountered in this order

[2] It is for example a.s. of Hausdorff dimension 4 by Theorem 1.14.

when traveling in the counterclockwise order around $\partial\mathfrak{q}_n$. Up to further extraction, we may suppose that $(a_n, a'_n, b_n, b'_n) \to (a, a', b, b') \in \mathit{fl}^4_\infty$, so that $a' \in [[a,b]]$ and $b' \in [[b,a]]$. Moreover, because $\delta_{(n)}(x_n, x'_n) \geq \varepsilon$, we see that $d^\sigma_\infty(a, a') \geq \varepsilon$. Similarly, we obtain that $d^\sigma_\infty(b, a') \geq \varepsilon$, $d^\sigma_\infty(a, b') \geq \varepsilon$, and $d^\sigma_\infty(b, b') \geq \varepsilon$, so that $a \neq b$. Finally, the fact that $\delta_{(n)}(x_n, y_n) \to 0$ implies that $d^\sigma_\infty(a, b) = 0$, so that $a \sim_\infty b$. This contradicts Lemma 5.9. ☐

5.4.3.2 1-regularity of $\mathcal{S}_n$

In order to show that the sequence $(\mathcal{S}_{n_k})_k$ is 1-regular, we first only consider simple loops made of edges in $\mathcal{S}_n$. A simple loop $\wp$ splits $\mathcal{S}_n$ into two domains. By the Jordan curve theorem, one of these is homeomorphic to a disc. We call it the **inner domain** of $\wp$. The other domain contains $\partial\mathcal{S}_n$ in its closure, and we call it the **outer domain** of $\wp$.

Lemma 5.15. *A.s., for all $\varepsilon > 0$, there exists $0 < \eta < \varepsilon$ such that for all k sufficiently large, the inner domain of any simple loop made of edges in $\mathcal{S}_{n_k}$ with diameter less than η has diameter less than ε.*

The proof of this Lemma is readily adaptable from the proof of Lemma 4.17, which uses the method employed by Miermont in [Mie08]. The general idea is that a loop separates a whole part of the map from the base point. As a result, the labels in one of the two domains are larger than the labels on the loop. In the forest, this corresponds to having a part with labels larger than the labels on the "border." In the continuous limit, this creates an increase point for both C_∞ and $\mathfrak{L}_\infty$. We recall now the main steps.

Proof. We argue by contradiction and suppose that, with positive probability, there exists $\varepsilon > 0$ for which, along some (random) subsequence of the sequence $(n_k)_{k\geq 0}$, there exist simple loops $\wp_n$ made of edges in $\mathcal{S}_n$ with diameter tending to 0 (with respect to the rescaled metric $\delta_{(n)}$) and whose inner domains are of diameter larger than ε. We reason on this event. We will show in the proof of Proposition 5.16 that $\partial\mathcal{S}_{n_k}$ tends, for the Gromov–Hausdorff topology, toward $\pi_\infty(\mathit{fl}_\infty)$. Because fl_∞ is not reduced to a singleton, we see by Lemma 5.9 that $\pi_\infty(\mathit{fl}_\infty)$ is not a singleton either, so that $\mathrm{diam}(\pi_\infty(\mathit{fl}_\infty)) > 0$. To avoid trivialities, we moreover suppose that $\varepsilon < \mathrm{diam}(\pi_\infty(\mathit{fl}_\infty))$. Because $\partial\mathcal{S}_n$ is entirely contained in the outer domain of $\wp_n$, we obtain that, for n large enough, the outer domain of $\wp_n$ is also of diameter larger than ε.

Let $s^\bullet_n$ be an integer where $\mathfrak{L}_n$ reaches its minimum, and $w^\bullet_n := \mathfrak{f}_n(s^\bullet_n)$ the corresponding point in the forest. Let us suppose for the moment that $w^\bullet_n \notin \wp_n$. We take x_n as far as possible from $\wp_n$ in the domain that do not contain $w^\bullet_n$, and we call y_n the first vertex of the path $[[x_n, w^\bullet_n]]$ that belongs to $\wp_n$. Up to further extraction, we suppose that $s^\bullet_n/(2n + \sigma_n - 1) \to s^\bullet := \mathrm{argmin}\,\mathfrak{L}_\infty$, $x_n \to x$, and $y_n \to y$. Because of the way x_n and y_n were chosen, it is not hard to see that $x \neq y$.

Let us first suppose that $y \neq w^\bullet := \mathscr{F}_\infty(s^\bullet)$. In particular, $w^\bullet_n \notin \wp_n$ for n large, so that x_n and y_n are well defined. In this case, $y \in [[x, w^\bullet]]\backslash\{x, w^\bullet\}$, so that the points in $\mathscr{F}^{-1}_\infty(y)$ are increase points of C_∞. By Lemma 5.7, we can find a subtree τ, not containing y, satisfying $\inf_\tau \mathfrak{L}_\infty < \mathfrak{L}_\infty(y)$ and rooted on $[[x, y]]$. We consider a discrete approximation τ_n of this subtree, rooted on $[[x_n, y_n]]$. When n is sufficiently large, we thus have $\inf_{\tau_n} \mathfrak{L}_n < \inf_{\wp_n} \mathfrak{L}_n$.

As the labels of the forest represent the distances in $\mathfrak{q}_n$ to the base point (up to some additive constant), we see that all the labels of the points in the same domain as x_n are larger than $\inf_{\wp_n} \mathfrak{L}_n$. As a consequence, τ_n cannot be entirely included in this domain, so that the set $\wp_n \cap \tau_n$ is not empty. We take $z_n \in \wp_n \cap \tau_n$, and, up to further extraction, we suppose that $z_n \to z$. On the one hand, $\delta_{(n)}(y_n, z_n) \leq \mathrm{diam}(\wp_n)$, so that $y \sim_\infty z$. On the other hand, $z \in \tau$ and $y \notin \tau$, so that $y \neq z$. Because y is not a leaf, this contradicts Theorem 5.8.

The case $y = w^\bullet$ is treated with a slightly different argument. As the argument is exactly the same as in Chapter 4, we do not treat it here. ☐

We now turn to general loops that are not necessarily made of edges. Here again, we use an argument similar to the one used in [Mie08] or Chapter 4, with some minor changes. We fix $\varepsilon > 0$, and we let η be as in Lemma 5.15. For k sufficiently large, the conclusion of Lemma 5.15 holds, together with the inequality $\eta\gamma n_k^{1/4} \geq 12$.

We call **pane** of $\mathcal{S}_n$ the projection in $\mathcal{S}_n$ of a $[z_{j-1}, z_j] \times [0,1] \subseteq X_{f_*}$ for some $1 \leq j \leq 2\sigma_n$, with the notation of Section 5.4.2. We also call **semi-edge** the projection in $\mathcal{S}_n$ of either $\{z_j\} \times [0,1] \subseteq X_{f_*}$ or $[z_{j-1}, z_j] \times \{1\} \subseteq X_{f_*}$ for some $1 \leq j \leq 2\sigma_n$. These correspond to the edges of the prism X_{f_*} that are not already edges in $\mathcal{S}_n$. Let us consider a loop $\mathscr{L}$ drawn in $\mathcal{S}_{n_k}$ with diameter less than $\eta/2$. Consider the union of the closed internal faces[3] and panes visited by $\mathscr{L}$. The boundary of this union consists in simple loops made of edges and semi-edges in $\mathcal{S}_{n_k}$. It should be clear that one of these loops entirely contains $\mathscr{L}$ in the closure of its inner domain. Let us call this loop λ.

We call $\tilde{\lambda}$ the largest (in the sense of the inclusion of the inner domains) simple loop made of edges contained in the closure of the inner domain of λ (that is, the loop obtained by removing the semi-edges of the form $\{z_j\} \times [0,1]$ and changing the ones of the form $[z_{j-1}, z_j] \times \{1\}$ by $[z_{j-1}, z_j] \times \{0\}$). Because every internal face and every pane of $\mathcal{S}_{n_k}$ has diameter less than $3/\gamma n_k^{1/4}$, we see that $\operatorname{diam}(\tilde{\lambda}) \leq \operatorname{diam}(\mathscr{L}) + 6/\gamma n_k^{1/4} \leq \eta$. Then, by Lemma 5.15, the diameter of the inner domain of $\tilde{\lambda}$ is less than ε. As a result, the diameter of the inner domain of λ is less than 2ε, so that $\mathscr{L}$ is homotopic to 0 in its 2ε-neighborhood.

5.4.4 Boundary of $\mathfrak{q}_\infty^\sigma$

We use the observation following Proposition 5.11 to show that the boundary of $\mathfrak{q}_\infty^\sigma$ is (the image in $\mathfrak{q}_\infty^\sigma$ of) the floor fl_∞ of $\mathscr{F}_\infty$, and then give a lower bound on its Hausdorff dimension. We postpone the proof of the upper bound to Section 6.5, because we will need the notation of Chapter 6.

Proposition 5.16. *The boundary of* $\mathfrak{q}_\infty^\sigma$ *is given by* $\partial\mathfrak{q}_\infty^\sigma = \pi_\infty(fl_\infty)$.

Proof. We define a pseudo-metric $\tilde{d}_{GH}$ on the set of triples $(\mathcal{X}, \delta, A)$ where $(\mathcal{X}, \delta)$ is a compact metric space and $A \subseteq \mathcal{X}$ is a closed subset of $\mathcal{X}$ by

$$\tilde{d}_{GH}\Big((\mathcal{X}, \delta, A), (\mathcal{X}', \delta', A')\Big) := \inf\Big\{\delta_{\mathcal{H}}\Big(\varphi(\mathcal{X}), \varphi'(\mathcal{X}')\Big) \vee \delta_{\mathcal{H}}\Big(\varphi(A), \varphi'(A')\Big)\Big\},$$

where the infimum is taken over all isometric embeddings $\varphi : \mathcal{X} \to \mathcal{X}''$ and $\varphi' : \mathcal{X}' \to \mathcal{X}''$ of $\mathcal{X}$ and $\mathcal{X}'$ into the same metric space $(\mathcal{X}'', \delta'')$. By slightly adapting the proof of [BBI01, Theorem 7.3.30], we can show that $\tilde{d}_{GH}((\mathcal{X}, \delta, A), (\mathcal{X}', \delta', A')) = 0$ if and only if there is an isometry from $(\mathcal{X}, \delta)$ onto $(\mathcal{X}', \delta')$ whose restriction to A maps A onto A'.

We proceed in three steps. First, note that the observation following Proposition 5.11 implies that

$$\tilde{d}_{GH}\Big((\mathcal{S}_{n_k}, \delta_{(n_k)}, \partial\mathcal{S}_{n_k}), (\mathfrak{q}_\infty^\sigma, d_\infty^\sigma, \partial\mathfrak{q}_\infty^\sigma)\Big) \xrightarrow[k\to\infty]{} 0. \tag{5.17}$$

Secondly, we show that

$$\tilde{d}_{GH}\Big((\mathcal{S}_n, \delta_{(n)}, \partial\mathcal{S}_n), (V(\mathfrak{q}_n)\backslash\{v_n^\bullet\}, \delta_{(n)}, fl_n)\Big) \xrightarrow[n\to\infty]{} 0. \tag{5.18}$$

We work here in $\mathcal{S}_n$ and see $V(\mathfrak{q}_n)\backslash\{v_n^\bullet\}$ as one of its subsets. Because of the way $\mathcal{S}_n$ is constructed, we see that $\delta_{\mathcal{H}}(\mathcal{S}_n, V(\mathfrak{q}_n)\backslash\{v_n^\bullet\}) \leq 3/\gamma n^{1/4}$. Using the technique we used in the proof of Lemma 5.14 to approach the points of $\partial\mathcal{S}_n$ by points lying in fl_n, and the fact that every point in fl_n is at distance at most $1/\gamma n^{1/4}$ from $\partial\mathcal{S}_n$, we obtain that

$$\delta_{\mathcal{H}}(\partial\mathcal{S}_n, fl_n) \leq \frac{3}{\gamma n^{1/4}} + \omega_{\mathfrak{b}_{(n)}}(\eta),$$

as soon as $n \geq 1/2\eta^2$. As a result, $\limsup \tilde{d}_{GH}((\mathcal{S}_n, \delta_{(n)}, \partial\mathcal{S}_n), (V(\mathfrak{q}_n)\backslash\{v_n^\bullet\}, \delta_{(n)}, fl_n)) \leq \omega_{\mathfrak{b}_\infty}(\eta)$ for all $\eta > 0$, and (5.18) follows by letting $\eta \to 0$.

[3] We call **closed face** the closure of a face.

Finally, we see that

$$\tilde{d}_{GH}\Big((V(\mathfrak{q}_{n_k})\backslash\{v_{n_k}^\bullet\}, \delta_{(n_k)}, \mathit{fl}_{n_k}), (\mathfrak{q}_\infty^\sigma, d_\infty^\sigma, \pi_\infty(\mathit{fl}_\infty))\Big) \xrightarrow[k\to\infty]{} 0. \tag{5.19}$$

Recall that $(V(\mathfrak{q}_n)\backslash\{v_n^\bullet\}, \delta_{(n)})$ is isometric to the space $(\mathscr{Q}_n, d_{(n)})$ defined in Section 5.2.2. We slightly abuse notation and view the floor fl_n of $\mathfrak{f}_n$ as a subset of $\mathscr{Q}_n$. We call $r_n := \mathrm{dis}(\mathcal{R}_n)/2$, where $\mathcal{R}_n$ is the correspondence between $\mathscr{Q}_n$ and $\mathfrak{q}_\infty^\sigma$ defined during Section 5.2.2, and we define the pseudo-metric Δ_n on the disjoint union $\mathscr{Q}_n \sqcup \mathfrak{q}_\infty^\sigma$ by $\Delta_n(x,y) := d_{(n)}(x,y)$ if $x, y \in \mathscr{Q}_n$, $\Delta_n(x,y) := d_\infty^\sigma(x,y)$ if $x, y \in \mathfrak{q}_\infty^\sigma$,

$$\Delta_n(x,y) := \inf\{d_{(n)}(x,x') + r_n + d_\infty^\sigma(y',y) \,:\, (x',y') \in \mathcal{R}_n\}$$

if $x \in \mathscr{Q}_n$ and $y \in \mathfrak{q}_\infty^\sigma$, and $\Delta_n(x,y) := \Delta_n(y,x)$ if $x \in \mathfrak{q}_\infty^\sigma$ and $y \in \mathscr{Q}_n$. It is a simple exercise to verify that Δ_n is indeed a pseudo-metric and that $\delta_{\mathcal{H}}(\mathscr{Q}_n, \mathfrak{q}_\infty^\sigma) \le r_n$. We showed in Section 5.2.2 that $r_{n_k} \to 0$ as $k \to 0$, so that it is sufficient to prove that $\delta_{\mathcal{H}}(\mathit{fl}_{n_k}, \pi_\infty(\mathit{fl}_\infty)) \to 0$ as well. Let us argue by contradiction and suppose that this is not the case. There exists $\varepsilon > 0$ such that one of the following occurs:

(i) for infinitely many n's, we can find a point t_n in the set $(2n+\sigma_n-1)^{-1}\,[\![0, 2n+\sigma_n-1]\!]$ such that $\boldsymbol{p}_{(n)}(t_n) \in \mathit{fl}_n$, and $\Delta_n(\boldsymbol{p}_{(n)}(t_n), \pi_\infty(\mathit{fl}_\infty)) \ge \varepsilon$,

(ii) for infinitely many n's, there is $s_n \in [0,1]$ such that $\mathscr{F}_\infty(s_n) \in \mathit{fl}_\infty$, and $\Delta_n(\mathfrak{q}_\infty^\sigma(s_n), \mathit{fl}_n) \ge \varepsilon$.

In the first case, up to extraction, we may suppose that $t_n \to t$. The fact that $\boldsymbol{p}_{(n)}(t_n) \in \mathit{fl}_n$ yields that $C_{(n)}(t_n) = \underline{C}_{(n)}(t_n)$, so that $C_\infty(t) = \underline{C}_\infty(t)$ by continuity, and $\mathscr{F}_\infty(t) \in \mathit{fl}_\infty$. We then have

$$\varepsilon \le \Delta_n\Big(\boldsymbol{p}_{(n)}(t_n), \pi_\infty(\mathit{fl}_\infty)\Big) \le \Delta_n\Big(\boldsymbol{p}_{(n)}(t_n), \mathfrak{q}_\infty^\sigma(t)\Big) \le d_\infty^\sigma(t_n, t) + r_n \to 0$$

along some subsequence. This is a contradiction. In the second case, we may also suppose that $s_n \to s$, and we have $\mathscr{F}_\infty(s) \in \mathit{fl}_\infty$. We call $t_n := \inf\{t \,:\, C_{(n)}(t) = \underline{C}_{(n)}(s)\}$, so that $\boldsymbol{p}_{(n)}(t_n) \in \mathit{fl}_n$. Up to further extraction, we have that $t_n \to t$, and because $C_\infty(t) = \underline{C}_\infty(s) = C_\infty(s)$, we see that $s \simeq_\infty t$, which yields $d_\infty^\sigma(s,t) = 0$. Finally,

$$\varepsilon \le \Delta_n(\mathfrak{q}_\infty^\sigma(s_n), \mathit{fl}_n) \le \Delta_n\Big(\mathfrak{q}_\infty^\sigma(s_n), \boldsymbol{p}_{(n)}(t_n)\Big) \le d_\infty^\sigma(s_n, t_n) + r_n \to 0$$

along some subsequence.

Now, (5.17), (5.18), and (5.19) yield that $\tilde{d}_{GH}\Big((\mathfrak{q}_\infty^\sigma, d_\infty^\sigma, \partial\mathfrak{q}_\infty^\sigma), (\mathfrak{q}_\infty^\sigma, d_\infty^\sigma, \pi_\infty(\mathit{fl}_\infty))\Big) = 0$, so that there exists an isometry $\varphi : \mathfrak{q}_\infty^\sigma \to \mathfrak{q}_\infty^\sigma$ such that $\pi_\infty(\mathit{fl}_\infty) = \varphi(\partial\mathfrak{q}_\infty^\sigma) = \partial(\varphi(\mathfrak{q}_\infty^\sigma)) = \partial\mathfrak{q}_\infty^\sigma$. □

We are now able to bound from below the Hausdorff dimension of $\partial\mathfrak{q}_\infty^\sigma$. We start with a lemma.

Lemma 5.17. *For a, $b \in \mathit{fl}_\infty$, we have*

$$d_\infty^\sigma(a,b) \ge \mathfrak{L}_\infty(a) - \max\left(\min_{[[a,b]]}\mathfrak{L}_\infty, \min_{[[b,a]]}\mathfrak{L}_\infty\right).$$

Proof. Let a_n, $b_n \in \mathit{fl}_n$ be points converging to a and b, and let $\wp_n$ be a geodesic from a_n to b_n. Reasoning as in the beginning of the proof of Lemma 5.9, we see that $\wp_n$ either overflies $[[a_n, b_n]]$ for infinitely many n's, or it overflies $[[b_n, a_n]]$ for infinitely many n's.

In the first case, let $c \in [[a,b]]$, and let $c_n \in [[a_n, b_n]]$ be a point converging toward c. For the values of n for which $\wp_n$ overflies $[[a_n, b_n]]$, we obtain by the remark of Section 5.3.3.2, and the triangle inequality, that

$$\mathfrak{L}_n(c_n) \ge \mathfrak{L}_n(a_n) - d_{\mathfrak{q}_n}(a_n, b_n).$$

Taking the limit after renormalization along these values of n, we obtain that $\mathfrak{L}_\infty(c) \ge \mathfrak{L}_\infty(a) - d_\infty^\sigma(a,b)$. Taking the infimum for c over $[[a,b]]$, we find $d_\infty^\sigma(a,b) \ge \mathfrak{L}_\infty(a) - \min_{[[a,b]]}\mathfrak{L}_\infty$. In the second case, a similar reasoning yields that $d_\infty^\sigma(a,b) \ge \mathfrak{L}_\infty(a) - \min_{[[b,a]]}\mathfrak{L}_\infty$. □

Proof of Theorem 1.16 (lower bound). Recall that, for $x \in [0,\sigma]$, we defined $T_x := \inf\{r \geq 0 \,:\, C_\infty(r) = \sigma - x\}$. We also call $\mathit{fl}(x) := \mathfrak{q}_\infty^\sigma(T_x)$, so that $\pi_\infty(\mathit{fl}_\infty) = \{\mathit{fl}(x),\, 0 \leq x \leq \sigma\}$.

To obtain the lower bound, we proceed as follows. We define the measure Λ_{fl} on $\mathfrak{q}_\infty^\sigma$ supported by $\pi_\infty(\mathit{fl}_\infty)$ as the image of the Lebesgue measure on $[0,\sigma]$ by the map $y \in [0,\sigma] \mapsto \mathit{fl}(y)$. Let us fix $x \in [0,\sigma]$. Because the process $y \in [0,\sigma] \mapsto \mathfrak{L}_\infty(T_y) = \mathfrak{b}_\infty(y)$ has the law of a Brownian bridge (up to a factor $\sqrt{3}$), the law of the iterated logarithm ensures us that, a.s., for $\eta > 0$, and δ small enough,

$$\mathfrak{L}_\infty(T_x) - \min_{\overrightarrow{y \in [x-\delta^{2-\eta},x]}} \mathfrak{L}_\infty(T_y) > \delta \qquad \text{and} \qquad \mathfrak{L}_\infty(T_x) - \min_{\overrightarrow{y \in [x,x+\delta^{2-\eta}]}} \mathfrak{L}_\infty(T_y) > \delta. \tag{5.20}$$

For $a \in \mathfrak{q}_\infty^\sigma$ and $r > 0$, we call $B_\infty(a,r) \subseteq \mathfrak{q}_\infty^\sigma$ the open ball centered at a with radius r for the metric d_∞^σ. Using Lemma 5.17, we see that, whenever (5.20) holds,

$$B_\infty(\mathit{fl}(x),\delta) \cap \pi_\infty(\mathit{fl}_\infty) \subseteq \mathit{fl}\Big((x-\delta^{2-\eta}, x+\delta^{2-\eta})\Big),$$

so that $\Lambda_{\mathit{fl}}(B_\infty(\mathit{fl}(x),\delta)) \leq 2\delta^{2-\eta}$. Finally, we obtain that, a.s., for all $a \in \pi_\infty(\mathit{fl}_\infty)$,

$$\limsup_{\delta\to 0} \frac{\Lambda_{\mathit{fl}}(B_\infty(a,\delta))}{\delta^{2-\eta}} \leq 2.$$

We then conclude that $\dim_{\mathcal{H}}(\mathfrak{q}_\infty^\sigma, d_\infty^\sigma) \geq 2-\eta$ for all $\eta > 0$ by standard density theorems for Hausdorff measures ([Fed69, Theorem 2.10.19]). □

5.5 Case $\sigma = 0$

In the case $\sigma = 0$, we could apply a reasoning similar to the one we used in Sections 5.2 through 5.4. We would obtain for the law of (C_∞, L_∞) the law of a Brownian snake driven by a normalized Brownian excursion, and we would use a result of Whyburn [Why35a, Corollary 5.21 and Theorem 6.3] treating the case where $\operatorname{diam}(\partial\mathcal{X}_n) \to 0$. Instead, we use a more direct approach, roughly consisting in saying that a uniform quadrangulation with "small" boundary is close to a uniform quadrangulation without boundary. A non-negligible advantage of this method is that it gives a more precise statement, Theorem 1.17, and completely identifies the limiting space as the Brownian map.

Let us begin with a lemma giving an upper bound on the Gromov–Hausdorff distance between a quadrangulation with a boundary and the quadrangulation obtained by applying Schaeffer's bijection to one of the trees of the forest that corresponds through the Bouttier–Di Francesco–Guitter bijection.

Lemma 5.18. *Let $(\mathfrak{f},\mathfrak{l}) \in \mathfrak{F}_\sigma^n$ be a well-labeled forest, $\mathfrak{b} \in \mathscr{B}_\sigma$ a bridge, $\mathfrak{t}$ a tree of $\mathfrak{f}$ rooted at ρ, and $\boldsymbol{b} \in \{-1,0\}$. Then $(0,\boldsymbol{b}) \in \mathscr{B}_1$, and, up to a trivial transformation, $(\mathfrak{t},\mathfrak{l}_{|\mathfrak{t}})$ may be seen as an element of $\mathfrak{F}_1^{|\mathfrak{t}|-1}$. We call $\mathfrak{q}_\mathfrak{f} \in \mathcal{Q}_{n,\sigma}$ (resp. $\mathfrak{q}_\mathfrak{t} \in \mathcal{Q}_{|\mathfrak{t}|-1,1}$) the quadrangulation corresponding to $((\mathfrak{f},\mathfrak{l}),\mathfrak{b})$ [resp. to $((\mathfrak{t},\mathfrak{l}_{|\mathfrak{t}}),(0,\boldsymbol{b}))$] through the Bouttier–Di Francesco–Guitter bijection (we omit here the distinguished vertices). Then*

$$d_{GH}\left(\left(\mathfrak{q}_\mathfrak{f}, d_{\mathfrak{q}_\mathfrak{f}}\right), \left(\mathfrak{q}_\mathfrak{t}, d_{\mathfrak{q}_\mathfrak{t}}\right)\right) \leq 2\left(\max_{\mathfrak{f}\setminus\mathring{\mathfrak{t}}} \hat{\mathfrak{l}} - \min_{\mathfrak{f}\setminus\mathring{\mathfrak{t}}} \hat{\mathfrak{l}} + 1\right),$$

where $\mathring{\mathfrak{t}} := \mathfrak{t}\setminus\{\rho\}$, and

$$\hat{\mathfrak{l}}(u) := \mathfrak{l}(u) + \mathfrak{b}(\mathfrak{a}(u)-1), \qquad u \in \mathfrak{f}$$

is the labeling function, shifted tree by tree according to the bridge, as in Section 5.1.3.2.

Proof. Before we begin, let us introduce some useful notation. For arcs $i_1 \frown i_2$, $i_2 \frown i_3$, ..., $i_{r-1} \frown i_r$, we write

$$i_1 \frown i_2 \frown \cdots \frown i_r$$

the path obtained by concatenating them. Let us call $v^\bullet$ the extra vertex we add when performing the Bouttier–Di Francesco–Guitter bijection. We call $v^\circ := (\sigma+1) \in \mathfrak{f}$ the last vertex of $\mathfrak{f}$, and we will identify the sets $\mathfrak{t} \cup \{v^\bullet\}$ with $V(\mathfrak{q}_\mathfrak{t})$, as well as $(\mathfrak{f}\setminus\{v^\circ\}) \cup \{v^\bullet\}$ with $V(\mathfrak{q}_\mathfrak{f})$. Then the set

$$\mathcal{R} := \Big\{(a,a) \,:\, a \in \mathfrak{t} \cup \{v^\bullet\}\Big\} \cup \Big\{(a,\rho) \,:\, a \in \mathfrak{f}\setminus(\mathfrak{t} \cup \{v^\circ\})\Big\}$$

is a correspondence between $\mathfrak{q}_\mathfrak{f}$ and $\mathfrak{q}_\mathfrak{t}$. Without loss of generality, we may suppose that $\mathfrak{t}$ is the first tree of $\mathfrak{f}$. This yields in particular that an integer $i \in [\![0, 2|\mathfrak{t}| - 2]\!]$ codes the same vertex in $\mathfrak{t}$ and in $\mathfrak{f}$, namely $\mathfrak{t}(i) = \mathfrak{f}(i)$. Because we will apply the Bouttier–Di Francesco–Guitter bijection at the same time to both $((\mathfrak{f}, \mathfrak{l}), \mathfrak{b})$ and $((\mathfrak{t}, \mathfrak{l}_{|\mathfrak{t}}), (0, \boldsymbol{b}))$, we will write $\mathrm{succ}_\mathfrak{f}(i)$ the successor of $i \in [\![0, 2n + \sigma - 1]\!]$ in the forest $\mathfrak{f}$, and $\mathrm{succ}_\mathfrak{t}(i)$ the successor of $i \in [\![0, 2|\mathfrak{t}| - 2]\!]$ in the tree $\mathfrak{t}$, in order to avoid confusion. We also set $l_2 := \max_{\mathfrak{f}\setminus\mathring{\mathfrak{t}}} \hat{\mathfrak{l}}$ and $l_1 := \min_{\mathfrak{f}\setminus\mathring{\mathfrak{t}}} \hat{\mathfrak{l}}$ for more clarity. Using the characterization (5.13) of the Gromov–Hausdorff distance via correspondences, we see that it suffices to show that, for all (a, a'), $(b, b') \in \mathcal{R}$, we have

$$\left|d_{\mathfrak{q}_\mathfrak{f}}(a, b) - d_{\mathfrak{q}_\mathfrak{t}}(a', b')\right| \leq 4(l_2 - l_1 + 1).$$

First case: $a, b \in \mathfrak{f}\setminus(\mathring{\mathfrak{t}} \cup \{v^\circ\})$**.** In this case, either $[a, b]$ or $[b, a]$ entirely lies inside $\mathfrak{f}\setminus\mathring{\mathfrak{t}}$. As a result, (5.3) gives

$$\left|d_{\mathfrak{q}_\mathfrak{f}}(a, b) - d_{\mathfrak{q}_\mathfrak{t}}(\rho, \rho)\right| \leq \hat{\mathfrak{l}}(a) + \hat{\mathfrak{l}}(b) - 2\min_{\mathfrak{f}\setminus\mathring{\mathfrak{t}}} \hat{\mathfrak{l}} + 2 \leq 2(l_2 - l_1 + 1).$$

Second case: $a, b \in \mathfrak{t} \cup \{v^\bullet\}$**.** We may suppose $a \neq b$. We proceed in two steps. We first claim that

$$d_{\mathfrak{q}_\mathfrak{t}}(a, b) \leq d_{\mathfrak{q}_\mathfrak{f}}(a, b).$$

To see this, let $\wp = (\wp(0), \wp(1), \ldots, \wp(k))$ be any path (not necessarily geodesic) between a and b in $\mathfrak{q}_\mathfrak{f}$. We will construct a shorter path from a to b in $\mathfrak{q}_\mathfrak{t}$, and our claim will immediately follow. Our construction is based on the simple observation that, if an arc exists in $\mathfrak{q}_\mathfrak{f}$ between two points of $\mathfrak{t} \cup \{v^\bullet\}$, then the same arc also exists in $\mathfrak{q}_\mathfrak{t}$. We then only have to replace the portions of $\wp$ that "exit" $\mathfrak{t} \cup \{v^\bullet\}$ with shorter paths in $\mathfrak{q}_\mathfrak{t}$. Precisely, we can restrict ourselves to the case where $\wp(r) \in \mathfrak{f}\setminus(\mathfrak{t} \cup \{v^\bullet\})$ for $0 < r < k$, with $k \geq 2$. We will also need to observe that a path linking two vertices of label l and l' has length at least $|l - l'|$.

Let us call i the integer such that the arc $(\wp(0), \wp(1))$ is either $i \curvearrowright \mathrm{succ}_\mathfrak{f}(i)$ or $i \curvearrowleft \mathrm{succ}_\mathfrak{f}(i)$. We will say that $(\wp(0), \wp(1))$ is oriented **to the right** in the first case, and **to the left** in the second case. We also define j in a similar way for the arc $(\wp(k), \wp(k-1))$. Four possibilities are then to be considered (see Figure 5.9):

- ✧ Both $(\wp(0), \wp(1))$ and $(\wp(k), \wp(k-1))$ are oriented to the right. Without loss of generality, we may suppose $i < j$. Properties of the Bouttier–Di Francesco–Guitter bijection then show that $\hat{\mathfrak{l}}(\mathfrak{f}(j)) \geq \hat{\mathfrak{l}}(\mathfrak{f}(i))$, and we have

$$k \geq 1 + |(\hat{\mathfrak{l}}(\mathfrak{f}(j)) - 1) - (\hat{\mathfrak{l}}(\mathfrak{f}(i)) - 1)| + 1 = \hat{\mathfrak{l}}(\mathfrak{f}(j)) - \hat{\mathfrak{l}}(\mathfrak{f}(i)) + 2.$$

The following path in $\mathfrak{q}_\mathfrak{t}$,

$$j \frown \mathrm{succ}_\mathfrak{t}(j) \frown \cdots \frown \mathrm{succ}_\mathfrak{t}^{\hat{\mathfrak{l}}(\mathfrak{f}(j)) - \hat{\mathfrak{l}}(\mathfrak{f}(i)) + 1}(j) = \mathrm{succ}_\mathfrak{t}(i) \frown i,$$

links a to b in $\mathfrak{q}_\mathfrak{t}$ and has length less than k. The equality in the last line is an easy consequence of the Bouttier–Di Francesco–Guitter construction.

- ✧ Both $(\wp(0), \wp(1))$ and $(\wp(k), \wp(k-1))$ are oriented to the left. Here again, we may suppose $i < j$. In this case, $\hat{\mathfrak{l}}(\mathfrak{f}(j)) > \hat{\mathfrak{l}}(\mathfrak{f}(i))$, and

$$\mathrm{succ}_\mathfrak{f}(j) \frown \mathrm{succ}_\mathfrak{t}(\mathrm{succ}_\mathfrak{f}(j)) \frown \cdots \frown \mathrm{succ}_\mathfrak{t}^{\hat{\mathfrak{l}}(\mathfrak{f}(j)) - \hat{\mathfrak{l}}(\mathfrak{f}(i))}(\mathrm{succ}_\mathfrak{f}(j)) = \mathrm{succ}_\mathfrak{f}(i)$$

fulfills our requirements.

- ✧ $(\wp(0), \wp(1))$ is oriented to the right, and $(\wp(k), \wp(k-1))$ is oriented to the left. Necessarily, we have $\mathrm{succ}_\mathfrak{f}(j) < i$, or $\mathrm{succ}_\mathfrak{f}(j) = \infty$. If $\hat{\mathfrak{l}}(\mathfrak{f}(i)) \geq \hat{\mathfrak{l}}(\mathfrak{f}(j))$, then we take

$$i \frown \mathrm{succ}_\mathfrak{t}(i) \frown \cdots \frown \mathrm{succ}_\mathfrak{t}^{\hat{\mathfrak{l}}(\mathfrak{f}(i)) - \hat{\mathfrak{l}}(\mathfrak{f}(j)) + 1}(i) = \mathrm{succ}_\mathfrak{f}(j),$$

otherwise, we take

$$\mathrm{succ}_{\mathfrak{f}}(j) \frown \mathrm{succ}_{\mathfrak{t}}(\mathrm{succ}_{\mathfrak{f}}(j)) \frown \cdots \frown \mathrm{succ}_{\mathfrak{t}}^{\hat{\mathfrak{l}}(\mathfrak{f}(j))-\hat{\mathfrak{l}}(\mathfrak{f}(i))}(\mathrm{succ}_{\mathfrak{f}}(j)) = \mathrm{succ}_{\mathfrak{t}}(i) \frown i,$$

✧ $(\wp(0), \wp(1))$ is oriented to the left, and $(\wp(k), \wp(k-1))$ is oriented to the right. By considering the path $\bar{\wp} := (\wp(k), \wp(k-1), \ldots, \wp(0))$ instead of $\wp$, we are back to the previous case.

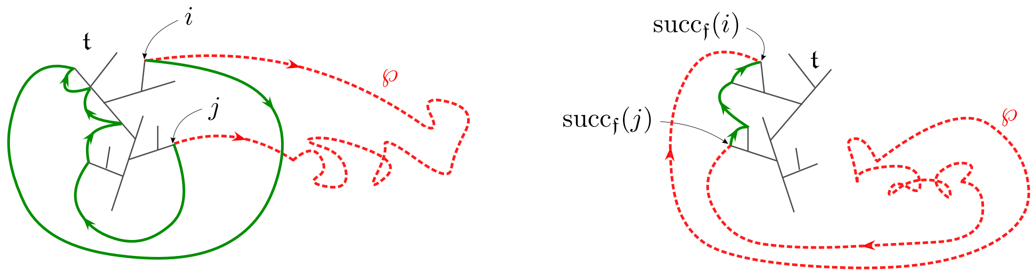

Figure 5.9. *On this picture,* $\mathfrak{t}$ *is the only part of* $\mathfrak{f}$ *represented. The dashed (red) line represents the path* $\wp$ *(in* $\mathfrak{q}_{\mathfrak{f}}$*) and the (green) solid path is the path in* $\mathfrak{q}_{\mathfrak{t}}$*. Both first cases are represented.*

We now show that

$$d_{\mathfrak{q}_{\mathfrak{f}}}(a,b) \leq d_{\mathfrak{q}_{\mathfrak{t}}}(a,b) + 2(l_2 - l_1 + 1).$$

Let us consider a path $\wp$ of length k in $\mathfrak{q}_{\mathfrak{t}}$ from a to b. We are going to construct a path in $\mathfrak{q}_{\mathfrak{f}}$ from a to b, with length less than $k + 2(l_2 - l_1 + 1)$. The only arcs present is $\mathfrak{q}_{\mathfrak{t}}$ but not in $\mathfrak{q}_{\mathfrak{f}}$ are of the form $i \frown \mathrm{succ}_{\mathfrak{t}}(i)$ with $\mathrm{succ}_{\mathfrak{t}}(i) < i$ or $\mathrm{succ}_{\mathfrak{t}}(i) = \infty$, and $l_1 + 1 \leq \hat{\mathfrak{l}}(\mathfrak{f}(i)) \leq l_2 + 1$. For convenience, let us call **pathological** such arcs. For all pathological arcs $i \frown \mathrm{succ}_{\mathfrak{t}}(i)$ and $j \frown \mathrm{succ}_{\mathfrak{t}}(j)$ with $i < j$, we can construct the path

$$j \frown \mathrm{succ}_{\mathfrak{f}}(j) \frown \cdots \frown \mathrm{succ}_{\mathfrak{f}}^{\hat{\mathfrak{l}}(\mathfrak{f}(j))-\hat{\mathfrak{l}}(\mathfrak{f}(i))+1}(j) = \mathrm{succ}_{f}(i) \frown i \tag{5.21}$$

linking $\mathfrak{f}(i)$ to $\mathfrak{f}(j)$ in $\mathfrak{q}_{\mathfrak{f}}$, its length being $\hat{\mathfrak{l}}(\mathfrak{f}(j)) - \hat{\mathfrak{l}}(\mathfrak{f}(i)) + 2$. We can also construct the path

$$j \frown \mathrm{succ}_{\mathfrak{f}}(j) \frown \cdots \frown \mathrm{succ}_{\mathfrak{f}}^{\hat{\mathfrak{l}}(\mathfrak{f}(j))-l_1+1}(j) = \mathrm{succ}_{\mathfrak{f}}^{\hat{\mathfrak{l}}(\mathfrak{f}(j))-l_1}(\mathrm{succ}_{\mathfrak{t}}(j)) \frown \cdots \frown \mathrm{succ}_{\mathfrak{t}}(j) \tag{5.22}$$

linking $\mathfrak{f}(j)$ to $\mathfrak{f}(\mathrm{succ}_{\mathfrak{t}}(j))$ in $\mathfrak{q}_{\mathfrak{f}}$, its length being $2(\hat{\mathfrak{l}}(\mathfrak{f}(j)) - l_1) + 1 \leq 2(l_2 - l_1 + 1) + 1$. Using these paths, we construct our path in $\mathfrak{q}_{\mathfrak{f}}$ as follows. If $\wp$ does not use any pathological arcs, then $\wp$ can be seen as a path in $\mathfrak{q}_{\mathfrak{f}}$. If $\wp$ uses exactly one pathological arc, we construct our new path by changing this arc into a path of the form (5.22). By doing so, we obtain a path from a to b in $\mathfrak{q}_{\mathfrak{f}}$ with length smaller than $k - 1 + 2(l_2 - l_1 + 1) + 1$. Now, if $\wp$ uses more than two pathological arcs, let $i \frown \mathrm{succ}_{\mathfrak{t}}(i)$ be the first one it uses, and $j \frown \mathrm{succ}_{\mathfrak{t}}(j)$ the last one. Let us call i_1 and i_2 the indexes at which $\wp$ uses them: $(\wp(i_1), \wp(i_1+1)) = i \curvearrowright \mathrm{succ}_{\mathfrak{t}}(i)$ or $i \curvearrowleft \mathrm{succ}_{\mathfrak{t}}(i)$ and $(\wp(i_2), \wp(i_2+1)) = j \curvearrowright \mathrm{succ}_{\mathfrak{t}}(j)$ or $j \curvearrowleft \mathrm{succ}_{\mathfrak{t}}(j)$. Changing $\wp$ into its reverse $\bar{\wp}$ if needed, we may suppose that $(\wp(i_1), \wp(i_1+1)) = i \curvearrowright \mathrm{succ}_{\mathfrak{t}}(i)$. If $(\wp(i_2), \wp(i_2+1)) = j \curvearrowleft \mathrm{succ}_{\mathfrak{t}}(j)$, we change the portion $(\wp(i_1), \wp(i_1+1), \ldots, \wp(i_2+1))$ into the path (5.21), and obtain a new path shorter than $\wp$. Finally, if $(\wp(i_2), \wp(i_2+1)) = j \curvearrowright \mathrm{succ}_{\mathfrak{t}}(j)$, we change the portion $(\wp(i_1), \wp(i_1+1), \ldots, \wp(i_2+1))$ into the path (5.21) concatenated with the path (5.22), and obtain a new path satisfying our requirements.

Third case: $a \in \mathfrak{t} \cup \{v^\bullet\}$, $b \in \mathfrak{f} \backslash (\mathfrak{t} \cup \{v^\circ\})$. We can write

$$\begin{aligned}
\left| d_{\mathfrak{q}_{\mathfrak{f}}}(a,b) - d_{\mathfrak{q}_{\mathfrak{t}}}(a,\rho) \right| &\leq \left| d_{\mathfrak{q}_{\mathfrak{f}}}(a,b) - d_{\mathfrak{q}_{\mathfrak{f}}}(a,\rho) \right| + \left| d_{\mathfrak{q}_{\mathfrak{f}}}(a,\rho) - d_{\mathfrak{q}_{\mathfrak{t}}}(a,\rho) \right| \\
&\leq d_{\mathfrak{q}_{\mathfrak{f}}}(b,\rho) + \left| d_{\mathfrak{q}_{\mathfrak{f}}}(a,\rho) - d_{\mathfrak{q}_{\mathfrak{t}}}(a,\rho) \right| \\
&\leq 4(l_2 - l_1 + 1),
\end{aligned}$$

by applying the first case to (b, ρ) and the second case to (a, ρ). This ends the proof. □

We may now proceed to the proof of Theorem 1.17. We use the same notation as in Section 5.2.1, and Corollary 5.3 remains true, if the process $(C_\infty, \mathfrak{L}_\infty)$ has the law of a Brownian snake driven by a normalized Brownian excursion. As we will not need the explicit law of the process $(C_\infty, \mathfrak{L}_\infty)$ in what follows, we do not prove this, and refer the reader to Chapter 2, in particular to Proposition 2.13 for similar results. By Skorokhod's representation theorem, we still assume that this convergence holds almost surely.

Proof of Theorem 1.17. We call $\mathfrak{t}_n$ the largest tree of $\mathfrak{f}_n$ (if there are more than one largest tree, we take $\mathfrak{t}_n$ according to some convention, for example the first one), and we consider a random variable $\boldsymbol{b}_n$ uniformly distributed over $\{-1, 0\}$, independent of $\mathfrak{q}_n$. We call $\hat{\mathfrak{q}}_n$ the quadrangulation corresponding, as in the statement of Lemma 5.18, to $((\mathfrak{t}_n, \mathfrak{l}_{n\,|\mathfrak{t}_n}), (0, \boldsymbol{b}_n))$ through the Bouttier–Di Francesco–Guitter bijection. Then, conditionally given $|\mathfrak{t}_n| = k+1$, the quadrangulation $\hat{\mathfrak{q}}_n$ is uniformly distributed over the set $\mathcal{Q}_{k,1}$.

From now on, we work on the set of full probability where the convergence $C_{(n)} \to C_\infty$ holds. Let $\varepsilon \in (0, 1/4)$, and $2\eta := \min_{[\varepsilon, 1-\varepsilon]} C_\infty > 0$. As $C_{(n)}$ tends to C_∞, for n large enough, we have $\min_{[\varepsilon,1-\varepsilon]} C_{(n)} \geq \eta$ and $\sigma_{(n)} < \eta$. As a result,

$$s_n := \inf\left\{ r \leq \frac{1}{2} \,:\, C_{(n)}(r) = \underline{C}_{(n)}\left(\frac{1}{2}\right)\right\} \leq \varepsilon,$$
$$t_n := \sup\left\{ r \geq \frac{1}{2} \,:\, C_{(n)}(r) = \underline{C}_{(n)}\left(\frac{1}{2}\right)\right\} \geq 1-\varepsilon,$$

so that $\mathfrak{t}_n$ is coded by $[(2n + \sigma_n - 1)\, s_n, (2n + \sigma_n - 1)\, t_n]$. Note that, in particular, this implies that $|\mathfrak{t}_n| \geq n\,(1 - 2\varepsilon)$. This fact will be used later. By Lemma 5.18,

$$\limsup_{n\to\infty} d_{GH}\left(\left(V(\mathfrak{q}_n), d_{\mathfrak{q}_n}/\gamma n^{1/4}\right), \left(V(\hat{\mathfrak{q}}_n), d_{\hat{\mathfrak{q}}_n}/\gamma n^{1/4}\right)\right)$$
$$\leq 2 \limsup_{n\to\infty}\left(\sup_{\overrightarrow{[1-\varepsilon,\varepsilon]}} \mathfrak{L}_{(n)} - \inf_{\overrightarrow{[1-\varepsilon,\varepsilon]}} \mathfrak{L}_{(n)} + \frac{1}{\gamma\, n^{1/4}}\right) = 2\left(\sup_{\overrightarrow{[1-\varepsilon,\varepsilon]}} \mathfrak{L}_\infty - \inf_{\overrightarrow{[1-\varepsilon,\varepsilon]}} \mathfrak{L}_\infty\right) \xrightarrow[\varepsilon\to 0]{} 0.$$

Let us call $\hat{\delta}_{(n,k)} := d_{\hat{\mathfrak{q}}_n}/\gamma k^{1/4}$. We then have to see that $(V(\hat{\mathfrak{q}}_n), \hat{\delta}_{(n,n)})$ converges toward the Brownian map $(\mathfrak{m}_\infty, D^*)$. Let $f : \mathbb{M} \to \mathbb{R}$ be uniformly continuous and bounded. By the Portmanteau theorem [Bil68, Theorem 2.1], we only need to show that

$$\mathbb{E}\left[f\left(V(\hat{\mathfrak{q}}_n), \hat{\delta}_{(n,n)}\right)\right] \xrightarrow[n\to\infty]{} \mathbb{E}\left[f\left(\mathfrak{m}_\infty, D^*\right)\right].$$

Let $\varepsilon > 0$. If we delete from $\hat{\mathfrak{q}}_n$ the only edge on the boundary that is not the root, we obtain a quadrangulation without boundary, which, conditionally given $|\mathfrak{t}_n| = k+1$, is uniformly distributed over the set of planar quadrangulations with k faces. As this operation does not affect the underlying metric space, by [Mie11, Theorem 1] or [LG11, Theorem 1.1], we obtain that the distribution of $(V(\hat{\mathfrak{q}}_n), \hat{\delta}_{(n,k)})$ conditionned on $|\mathfrak{t}_n| = k+1$ converges toward the distribution of $(\mathfrak{m}_\infty, D^*)$ as $k \to \infty$. As $(\mathfrak{m}_\infty, D^*)$ is a compact metric space, we can find large n_0 and M such that, for all $k \geq n_0/2$ and n for which $\mathbb{P}(|\mathfrak{t}_n| = k+1) > 0$,

$$\mathbb{P}\left(\operatorname{diam}\left(V(\hat{\mathfrak{q}}_n), \hat{\delta}_{(n,k)}\right) \geq M \;\middle|\; |\mathfrak{t}_n| = k+1\right) < \frac{\varepsilon}{2 \sup f}, \tag{5.23}$$

and

$$\left|\mathbb{E}\left[f\left(V(\hat{\mathfrak{q}}_n), \hat{\delta}_{(n,k)}\right) \;\middle|\; |\mathfrak{t}_n| = k+1\right] - \mathbb{E}\left[f\left(\mathfrak{m}_\infty, D^*\right)\right]\right| < \varepsilon. \tag{5.24}$$

We then choose $\eta \in (0, 1/2)$ such that

$$d_{GH}\left((\mathcal{X}, \delta), (\mathcal{X}', \delta')\right) \leq \frac{1}{2} M \left(1 - (1-\eta)^{1/4}\right) \qquad \Rightarrow \qquad \left|f\left((\mathcal{X}, \delta)\right) - f\left((\mathcal{X}', \delta')\right)\right| < \varepsilon. \tag{5.25}$$

For $n \geq n_0$, we then have

$$\Big| \mathbb{E}\Big[f\Big(V(\hat{\mathfrak{q}}_n), \hat{\delta}_{(n,n)}\Big)\Big] - \mathbb{E}\Big[f\Big(\mathfrak{m}_\infty, D^*\Big)\Big]\Big| \leq 2 \sup f \, \mathbb{P}\Big(|\mathfrak{t}_n| \leq n\,(1-\eta)\Big)$$
$$+ \sum_{k=\lceil n\,(1-\eta)\rceil}^{n} \mathbb{P}(|\mathfrak{t}_n| = k+1) \,\Big| \mathbb{E}\Big[f\Big(V(\hat{\mathfrak{q}}_n), \hat{\delta}_{(n,n)}\Big) \;\Big|\; |\mathfrak{t}_n| = k+1\Big] - \mathbb{E}\Big[f\Big(\mathfrak{m}_\infty, D^*\Big)\Big]\Big| .$$

By the observation we previously made, we see that the first term in the right-hand side tends to 0 as $n \to \infty$. To conclude, it will be sufficient to show that the term between vertical bars in the sum is smaller than 3ε. Using (5.23), (5.24), and the fact that $n\,(1-\eta) \geq n_0/2$, we obtain that it is smaller than

$$2\varepsilon + \mathbb{E}\Big[\Big(f\Big(V(\hat{\mathfrak{q}}_n), \hat{\delta}_{(n,n)}\Big) - f\Big(V(\hat{\mathfrak{q}}_n), \hat{\delta}_{(n,k)}\Big)\Big) \mathbb{1}_{\{\mathrm{diam}(V(\hat{\mathfrak{q}}_n), \hat{\delta}_{(n,k)}) < M\}} \;\Big|\; |\mathfrak{t}_n| = k+1 \Big] .$$

By taking a trivial correspondance between $(V(\hat{\mathfrak{q}}_n), \hat{\delta}_{(n,n)})$ and $(V(\hat{\mathfrak{q}}_n), \hat{\delta}_{(n,k)})$, it is not hard to see that the Gromov–Hausdorff distance between these two spaces is smaller than

$$\frac{1}{2}\,\mathrm{diam}\Big(V(\hat{\mathfrak{q}}_n), \hat{\delta}_{(n,k)}\Big)\Big(1 - (k/n)^{1/4}\Big) .$$

We finaly obtain the desired bound thanks to (5.25). □

6 Proofs using the Brownian snake

In this chapter, we proove the remaining technical results. To this end, we will need to use the Brownian snake, whose definition is briefly recalled in Section 6.1.

6.1 The Brownian snake

In this chapter, we prove Lemmas 5.6, 5.7, 5.10, and complete the proof of Theorem 1.16. To this end, we will need some notions about the Brownian snake. We refer the reader to [LG99] for a complete description of this object. Recall that we denoted by $\mathcal{K}$ the space of continuous real-valued functions on $\mathbb{R}_+$ killed at some time, and that we wrote $\zeta(w)$ the lifetime of an element $w \in \mathcal{K}$. We also use the notation $\widehat{w} := w(\zeta(w))$ for the final value of a path $w \in \mathcal{K}$. From now on, we will work on the space $\Omega' := \mathcal{C}(\mathbb{R}_+, \mathcal{K})$ of continuous functions from $\mathbb{R}_+$ into $\mathcal{K}$, equipped with the topology of uniform convergence on every compact subset of $\mathbb{R}_+$. We write $W_s := \omega(s)$ the canonical process on Ω', and call $\zeta_s := \zeta(W_s)$ its lifetime.

For $w \in \mathcal{K}$, we call $\mathbb{P}_w$ the law of the Brownian snake started from w. This means that, under $\mathbb{P}_w$, the process $(\zeta_s)_{s\geq 0}$ has the law of a reflected Brownian motion on $\mathbb{R}_+$ started from $\zeta(w)$, and that the conditional distribution of $(W_s)_{s\geq 0}$ knowing $(\zeta_s)_{s\geq 0}$, denoted by Θ_w^ζ, is characterized by

- $W_0 = w$, Θ_w^ζ a.s.
- the process $(W_s)_{s\geq 0}$ is time-inhomogeneous Markov under Θ_w^ζ and, for $0 \leq s \leq s'$,
  - $W_{s'}(t) = W_s(t)$ for all $0 \leq t \leq \zeta_r$, Θ_w^ζ a.s., where $\zeta_r := \inf_{r' \in [s,s']} \zeta_{r'}$,
  - under Θ_w^ζ, the process $(W_{s'}(\zeta_r + t))_{0 \leq t \leq \zeta_{s'} - \zeta_r}$ is independent of W_s and distributed as a real Brownian motion started from $W_s(\zeta_r)$ and stopped at time $\zeta_{s'} - \zeta_r$.

Let us call $I_a := \inf\{s : \zeta_s = a\}$ and define the probability measure on Ω'

$$\mathbb{P}_w^0 := \mathbb{P}_w(\,\cdot \mid I_0 = 1).$$

This conditioning may be properly defined by saying that, under $\mathbb{P}^0_w$, the law of $(\zeta_s)_{0\leq s\leq 1}$ is the law of a first-passage Brownian bridge on $[0,1]$ from $\zeta(w)$ to 0, the law of $(\zeta_s)_{s\geq 1}$ is the law of a reflected Brownian motion on $[1,+\infty)$ issued from 0, and the conditional distribution of $(W_s)_{s\geq 0}$ knowing $(\zeta_s)_{s\geq 0}$ is Θ^ζ_w.

We call $\mathbf{0}_\sigma \in \mathcal{K}$ the function $s\in[0,\sigma]\mapsto 0$. Under $\mathbb{P}^0_{\mathbf{0}_\sigma}$, the process $((\zeta_s)_{0\leq s\leq 1}, (\widehat{W}_s)_{0\leq s\leq 1})$ has the same law as the process $\left(F^{\sigma\to 0}_{[0,1]}, Z_{[0,1]}\right)$ defined during Section 5.2.1.1. If we call $\mathbb{B}$ the law on $\mathcal{K}$ of a Brownian bridge on $[0,\sigma]$ from 0 to 0, multiplied by the factor $\sqrt{3}$, we then obtain that, under

$$\int_{\mathcal{K}} \mathbb{B}(dw)\, \mathbb{P}^0_w(d\omega),$$

the process $((\zeta_s)_{0\leq s\leq 1}, (\widehat{W}_s)_{0\leq s\leq 1})$ has the same law as $(C_\infty, \mathfrak{L}_\infty)$ (under the common probability measure $\mathbb{P}$).

We note $n(de)$ the Itô measure of positive Brownian excursions, whose normalization is given by the relation $n(\sup e > \varepsilon) = 1/2\varepsilon$, and we call

$$\mathbb{N}_x := \int_{\mathcal{C}(\mathbb{R}_+,\mathbb{R}_+)} n(de)\, \Theta^e_{\bar{x}}$$

the excursion measure of the Brownian snake away from the path $\bar{x}: 0\mapsto x$. Let us call (α_i,β_i), $i\in I$, the excursion intervals of $s\in[0,I_0]\mapsto \zeta_s - \underline{\zeta}_s$, that is, the connected components of the open set $[0,I_0]\cap\{s : \zeta_s > \underline{\zeta}_s\}$. For $i\in I$, we define $W^{(i)}\in\mathcal{C}(\mathbb{R}_+,\mathcal{K})$ by setting, for $s\geq 0$,

$$W^{(i)}_s(t) = W_{(\alpha_i+s)\wedge\beta_i}(\zeta_{\alpha_i}+t), \qquad 0\leq t\leq \zeta^{(i)}_s := \zeta_{(\alpha_i+s)\wedge\beta_i} - \zeta_{\alpha_i}.$$

One of the main ingredient to our proofs is the following lemma.

Lemma 6.1 ([LG99, Lemma V.5]). *The point measure*

$$\sum_{i\in I} \delta_{(\zeta_{\alpha_i}, W^{(i)})}(dt\, d\omega) \tag{6.1}$$

is under $\mathbb{P}_w$ a Poisson point measure on $\mathbb{R}_+\times\mathcal{C}(\mathbb{R}_+,\mathcal{K})$ with intensity

$$\mathcal{N}_w(dt\, d\omega) := 2\,\mathbb{1}_{[0,\zeta(w)]}(t)dt\,\mathbb{N}_{w(t)}(d\omega).$$

We will also need the explicit "law" of the minimum of the Brownian snake's head under $\mathbb{N}_x$.

Lemma 6.2 ([LGW06, Lemma 2.1]). *For all $x, y\in\mathbb{R}$ with $y<x$,*

$$\mathbb{N}_x\left(\min_{s\geq 0}\widehat{W}_s < y\right) = \frac{3}{2(x-y)^2}.$$

In this setting, we have two singular conditionings: one being $I_0 = 1$, and the second one being the fact that w is under $\mathbb{B}(dw)$ a bridge, instead of a Brownian motion. The first step in our proofs will generally be to dispose of the first of these conditionings (and sometimes the second as well), making us work under $\mathbb{P}_w$ instead of $\mathbb{P}^0_w$. This will usually be done by a simple absolute continuity argument, at least for almost sure properties. Another difficulty will arise from the factor $\sqrt{3}$, and we will sometimes need to take extra care because of it.

6.2 Proof of Lemma 5.6

Thanks to Lemma 6.1, we will derive Lemma 5.6 from the following similar result under $\mathbb{N}_x$, which is due to Le Gall and Weill [LGW06].

Proposition 6.3 ([LGW06, Proposition 2.5])**.** *There exists $\mathbb{N}_x$ a.e. a unique instant where $(\widehat{W}_s)_{s\geq 0}$ reaches its minimum.*

Proof of Lemma 5.6. It is sufficient to show that, for $a \in [0,\zeta(w)]$, the process $(\widehat{W}_s)_{0\leq s\leq I_a}$ reaches its minimum only once $\mathbb{P}_w$ a.s., for w fixed in some subset of $\mathcal{K}$ of full $\mathbb{B}$-measure. Indeed, as for all $\varepsilon \in (0,1)$, the distribution of $(W_s)_{0\leq s\leq 1-\varepsilon}$ under $\mathbb{P}^0_w$ is absolutely continuous with respect to the distribution of $(W_s)_{0\leq s\leq 1-\varepsilon}$ under $\mathbb{P}_w(d\omega \mid I_0 \geq 1-\varepsilon)$, this entails that, for those w's, $\mathbb{P}^0_w$ a.s., for every rational $\varepsilon \in (0,1)$ and every rational $a \in [0,\zeta(w)]$, on the event $\{I_a \leq 1-\varepsilon\}$, the process $(\widehat{W}_s)_{0\leq s\leq I_a}$ reaches its minimum only once. Discarding a set of zero $\mathbb{B}$-measure if needed, we may only consider w's for which $\min_{x\in[0,\zeta(w)]} w(x) < 0$. Because $\min_{s\in[0,1]} \widehat{W}_s \leq \min_{x\in[0,\zeta(w)]} w(x) < 0 = \widehat{W}_1$, we see that under $\mathbb{P}^0_w$, $(\widehat{W}_s)_{0\leq s\leq 1}$ does not reach its minimum at time 1. As a consequence, if this process were reaching its minimum twice at times $t_1 < t_2$, then we could find $a \in \mathbb{Q}\cap(0,\zeta(w)]$ such that $t_2 \leq I_a$, and then we could find $\varepsilon \in \mathbb{Q}\cap(0,1)$ satisfying $I_a \leq 1-\varepsilon$. This would be a contradiction. We finally obtain that $(\widehat{W}_s)_{0\leq s\leq 1}$ reaches its minimum only once $\int_{\mathcal{K}} \mathbb{B}(dw)\,\mathbb{P}^0_w(d\omega)$ a.s. and the result follows because $(\widehat{W}_s)_{0\leq s\leq 1}$ has under $\int_{\mathcal{K}} \mathbb{B}(dw)\,\mathbb{P}^0_w(d\omega)$ the same law as $\mathfrak{L}_\infty$ under $\mathbb{P}$.

For the moment, we do not need to make any assumptions on $w \in \mathcal{K}$. We claim that, $\mathbb{P}_w$ a.s., the numbers $\min_{s\in[\alpha_i,\beta_i]} \widehat{W}_s$, $i \in I$, are pairwise distinct, because the "law" of $\min_{s\geq 0} \widehat{W}_s$ is diffuse under $\mathbb{N}_x$, by Lemma 6.2. Let us show this claim. We call $\ell := \sup\{s \geq 0 : \zeta_s > 0\}$. It is a classical result ([RY99, Proposition XII.2.8]) that $n(\ell \geq t) = (2\pi t)^{-1/2}$, so that, for $k \geq 1$,

$$\mathcal{N}_w(A_k) < \infty, \qquad \text{where } A_k := \left\{\ell \in \left[\frac{1}{k}, \frac{1}{k-1}\right)\right\}$$

with the convention $1/0 = \infty$. On $(\Omega, F, \mathbb{P})$, we consider independent random variables N_k, $k \geq 1$, and $(t_{k,i}, W^{k,i})$, $k, i \geq 1$. We suppose that N_k is Poisson with mean $\mathcal{N}_w(A_k)$, and that the law of $(t_{k,i}, W^{k,i})$ is $\mathcal{N}_w(\,\cdot \mid A_k)$. Basic facts about Poisson point measures show that the point measure

$$\sum_{k=1}^{\infty}\sum_{i=1}^{N_k} \delta_{(t_{k,i},W^{k,i})}$$

has (under $\mathbb{P}$) the same law as (6.1) under $\mathbb{P}_w$. We then have

$$\mathbb{P}_w\left(\exists i \neq j \,:\, \min_{s\in[\alpha_i,\beta_i]} \widehat{W}_s = \min_{s\in[\alpha_j,\beta_j]} \widehat{W}_s\right) \leq \sum_{(k,i)\neq(l,j)} \mathbb{P}\left(\min_{s\geq 0} \widehat{W}^{k,i}_s = \min_{s\geq 0} \widehat{W}^{l,j}_s\right) = 0,$$

because, for $(k,i) \neq (l,j)$, the variables $\min_{s\geq 0} \widehat{W}^{k,i}_s$ and $\min_{s\geq 0} \widehat{W}^{l,j}_s$ are independent, and their laws have no atoms.

A consequence of what we just showed is that, under $\mathbb{P}_w$, the process $(\widehat{W}_s)_{0\leq s\leq I_a}$ does not reach its minimum on two different intervals of the form $[\alpha_i,\beta_i]$, $i \in I$. Now, the probability that it reaches its minimum more than once on some such interval is smaller than

$$\begin{aligned}\mathbb{P}_w&\left(\exists i \in I \,:\, \exists \alpha_i \leq s < t \leq \beta_i \,:\, \widehat{W}_s = \widehat{W}_t = \min_{s\in[\alpha_i,\beta_i]} \widehat{W}_s\right)\\ &= 1 - \exp\left(-2\int_0^{\zeta(w)} dt\, \mathbb{N}_{w(t)}\left(\exists s < t \,:\, \widehat{W}_s = \widehat{W}_t = \min_{s\geq 0} \widehat{W}_s\right)\right) = 0,\end{aligned}$$

by Proposition 6.3.

We will now see see that $(\widehat{W}_s)_{0\le s\le I_a}$ does not reach its minimum on $[0,I_a]\setminus\bigcup_{i\in I}[\alpha_i,\beta_i]$, which will complete the proof. It is at this time that we make extra assumptions on w. The so-called snake property shows that

$$\left\{\widehat{W}_s\ :\ s\in[0,I_a]\setminus\bigcup_{i\in I}(\alpha_i,\beta_i)\right\}=\Big\{w(t)\ :\ a\le t\le\zeta(w)\Big\},$$

so that it will be enough to see that, $\mathbb{P}_w$ a.s., $\min_{0\le s\le I_a}\widehat{W}_s<\min_{[a,\zeta(w)]}w$. Using Lemma 6.1 then Lemma 6.2, we obtain

$$\begin{aligned}\mathbb{P}_w\left(\min_{0\le s\le I_a}\widehat{W}_s<\min_{[a,\zeta(w)]}w\right)&=1-\exp\left(-2\int_a^{\zeta(w)}dt\,\mathbb{N}_{w(t)}\left(\min_{s\ge0}\widehat{W}_s<\min_{[a,\zeta(w)]}w\right)\right)\\&=1-\exp\left(-3\int_a^{\zeta(w)}dt\,\left(w(t)-\min_{[a,\zeta(w)]}w\right)^{-2}\right)\end{aligned}$$

An easy application of Lévy's modulus of continuity (see for example [RY99, Theorem I.2.7]) shows that, $\mathbb{B}(dw)$ a.s., this quantity equals 1. This concludes the proof. □

6.3 Proof of Lemma 5.7

For a continuous function $f:[0,\ell]\to\mathbb{R}$, we write $\mathrm{IP}_{left}(f)$ (resp. $\mathrm{IP}_{right}(f)$) the set of its left-increase points (resp. right-increase points). Remember that $s\in(0,\ell]$ is a left-increase point of f if there exists $t\in[0,s)$ satisfying $f(r)\ge f(s)$ for all $t\le r\le s$, and that a right-increase point is defined in a symmetrical way. We also call $\mathrm{IP}(f)=\mathrm{IP}_{left}(f)\cup\mathrm{IP}_{right}(f)$ the set of all its increase points. Due to the fact that the points I_a, $a\in[0,\zeta(w)]$ are left-increase points of ζ and do not always lie in $\cup_{i\in I}(\alpha_i,\beta_i]$, we cannot directly apply the same strategy as in the previous section and derive Lemma 5.7 from a similar statement under $\mathbb{N}_x$. Instead, we use a technique of covering intervals inspired from [Ber91] and a theorem of Shepp [She72]. In [Ber91], Bertoin is interested with a similar problem: he characterizes the Lévy processes X for which the set $\mathrm{IP}_{right}(X)\cap\mathrm{IP}_{left}(-X)$ is almost surely empty. Our method gives, in particular, another proof to [LGP08, Lemma 3.2], which states that the set

$$\mathrm{IP}\Big((\zeta_s)_{0\le s\le\ell}\Big)\cap\mathrm{IP}\left((\widehat{W}_s)_{0\le s\le\ell}\right)$$

is $\mathbb{N}_x$ a.e. empty. (Recall that we write $\ell:=\sup\{s\ge0\ :\ \zeta_s>0\}$.) This comes very roughly from the fact that, if ζ and $\widehat{W}$ do not share any increase points on $[0,I_0]$, in particular, they do not share any increase points on any (α_i,β_i) either, and, by Lemma 6.1, the process restricted to (α_i,β_i) is then "distributed" under $\mathbb{N}_x$.

For $y\in\mathbb{R}$, we call $\boldsymbol{T}_y:=\inf\{s\ge0\ :\ w(s)=y\}$, where w is the canonical process on $\mathcal{K}$, and, for $y<a$ and $\kappa>0$, we call $P_{a,\kappa}^{(y,\infty)}$ the law on $\mathcal{K}$ of a standard Brownian motion multiplied by κ, started from a and stopped at time $\boldsymbol{T}_y$. We also call, for $x>0$, $P_{a,\kappa}^{x}$ the law of a standard Brownian motion multiplied by κ, started from a and stopped at time x. When we omit the value of κ, it will be assumed to be 1.

Although quite long to properly write in full detail, our strategy is pretty simple. One of the main difficulty comes from the two levels of randomness of the Brownian snake. In contrast to the previous proof where we worked under $\mathbb{P}_w$ for a fixed $w\in\mathcal{K}$, we will need here to work under $\mathbb{B}(dw)\mathbb{P}_w^0(d\omega)$ and see w as random. As a consequence, we will need to consider the timescale of ζ and W, as well as the timescale of w. Juggling from one to the other may also cause confusion.

In order to facilitate the reading of our proof, let us outline it now. By absolute continuity arguments, we get rid of the conditionings and work under $P_{0,\sqrt{3}}^{x}(dw)\,\mathbb{P}_w(d\omega)$ instead of $\mathbb{B}(dw)\mathbb{P}_w^0(d\omega)$. We then only consider left-increase points of $\widehat{W}$ that are also increase points of ζ, and treat later right-increase points of $\widehat{W}$ by a nice time-reversal argument. It should not be too hard to convince oneself[1] that it suffices to look at points $s \in [0, I_0]$ such that $\zeta_s = \underline{\zeta}_s$. If s is such a point and also, say, a right-increase point of $\widehat{W}$, we will first see that s is not starting an excursion. This will entail that ζ_s is a left-increase point of w. Beware that, as $\underline{\zeta}$ is non-increasing, the notions of left and right are reversed. By an argument similar as before, we only consider points s satisfying $\zeta_s = \underline{\zeta}_s = \inf\{t : w(t) = w(\zeta_s)\}$. See Figure 6.1.

Now, we will consider the excursions of $w - \underline{w}$ and look at the minimum of $\widehat{W}$ on the intervals corresponding to these excursions. Using [She72], we will see that, as close as we want before ζ_s, we can find an excursion of $w - \underline{w}$ where the minimum of $\widehat{W}$ is smaller than $w(\zeta_s)$, prohibiting ζ_s from being a left-increase point of w.

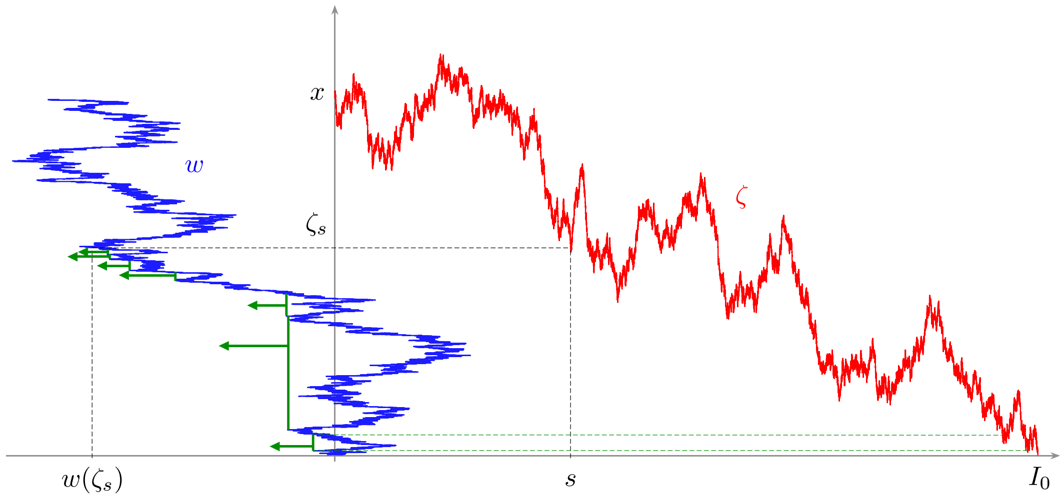

Figure 6.1. *Visual aid for the proof of Lemma 5.7. On this picture, the timescale of ζ and W is horizontal, whereas the timescale of w is vertical. The point s satisfies $\zeta_s = \underline{\zeta}_s = \inf\{t : w(t) = w(\zeta_s)\}$. We represented by (green) solid lines the seven longest excursions of $w - \underline{w}$, and the arrows represent the minimum of $\widehat{W}$ on the corresponding intervals.*

We start with a lemma stating that the extremities of any excursion interval (α_i, β_i) are not increase points of $\widehat{W}$ restricted to this excursion.

Lemma 6.4. *Let $w \in \mathcal{K}$. Then, $\mathbb{P}_w(d\omega)$ a.s., for all $i \in I$,*

$$\alpha_i \notin \mathrm{IP}_{right}\left((\widehat{W}_s)_{0 \le s \le I_0}\right) \qquad \textit{and} \qquad \beta_i \notin \mathrm{IP}_{left}\left((\widehat{W}_s)_{0 \le s \le I_0}\right).$$

Proof. It is enough to show that $\mathbb{N}_x$ a.e. $0 \notin \mathrm{IP}_{right}\left((\widehat{W}_s)_{0 \le s \le \ell}\right)$. Indeed, this entails that

$$\mathbb{P}_w\left(\exists i \in I \,:\, \alpha_i \in \mathrm{IP}_{right}\left((\widehat{W}_s)_{0 \le s \le I_0}\right)\right) = 1 - \exp\left(-2\int_0^{\zeta(w)} dt\; \mathbb{N}_{w(t)}\left(0 \in \mathrm{IP}_{right}\left((\widehat{W}_s)_{0 \le s \le \ell}\right)\right)\right) = 0.$$

Then, by the time-reversal property under $\mathbb{N}_x$ (the process $(\zeta_{\ell-s}, W_{\ell-s})_{0 \le s \le \ell}$ has under $\mathbb{N}_x$ the same distribution as $(\zeta_s, W_s)_{0 \le s \le \ell}$), we see that $\mathbb{N}_x$ a.e. $\ell \notin \mathrm{IP}_{left}\left((\widehat{W}_s)_{0 \le s \le \ell}\right)$, and we conclude in the same way.

[1] To be more accurate, if s does not satisfy this hypothesis, we apply the Markov property at some (rational) time a close enough before s so that $\zeta_s = \inf_{[a,s]} \zeta$. When doing so, we have to work under $P_0^x(dw)\,\mathbb{P}_w(d\omega)$ and not $P_{0,\sqrt{3}}^x(dw)\,\mathbb{P}_w(d\omega)$.

Let e be an excursion. By definition, under $\Theta_{\bar{x}}^{e}$, the process

$$\left(\widehat{W}(\sup\{s \le \ell/2 \,:\, e_s = y\})\right)_{0 \le y \le e_{\ell/2}}$$

has the law $P_x^{e_{\ell/2}}$. Then, because $n(de)$ a.e. $\sup\{s \le \ell/2 \,:\, e_s = 0\} = 0$, we see that $n(de)$ a.e. $\Theta_{\bar{x}}^{e}$ a.s. $0 \notin \mathrm{IP}_{right}\left((\widehat{W}_s)_{0 \le s \le \ell}\right)$. □

The following lemma will only be used for $\kappa = 1$ or $\kappa = \sqrt{3}$ in what follows, but the proof works for any $\kappa \le \sqrt{3}$, so that we consider all these values.

Lemma 6.5. *Let $\kappa \le \sqrt{3}$ and $x > 0$. The sets*

$$\mathcal{A} := \left\{ s \in \mathrm{IP}_{right}\left((\widehat{W}_s)_{0 \le s \le I_0}\right) \,:\, \zeta_s = \underline{\zeta}_s = \inf\{t \,:\, w(t) = w(\zeta_s)\} \right\}$$

and

$$\mathcal{B} := \left\{ s \in \mathrm{IP}_{left}\left((\widehat{W}_s)_{0 \le s \le I_0}\right) \,:\, \zeta_s = \underline{\zeta}_s = \sup\{t \,:\, w(t) = w(\zeta_s)\} \right\}$$

are $P_{0,\kappa}^{x}(dw)\,\mathbb{P}_w(d\omega)$ a.s. empty.

Proof. Let $y > 0$. We are going to show that the set $\mathcal{A}$ is $P_{0,\kappa}^{(-y,\infty)}(dw)\,\mathbb{P}_w(d\omega)$ a.s. empty. This will entail in particular that the set

$$\left\{ s \in \mathrm{IP}_{right}\left((\widehat{W}_s)_{I_x \le s \le I_0}\right) \,:\, \zeta_s = \underline{\zeta}_s = \inf\{t \,:\, w(t) = w(\zeta_s)\} \right\}$$

is $P_{0,\kappa}^{(-y,\infty)}(dw \mid \boldsymbol{T}_{-y} \ge x)\,\mathbb{P}_w(d\omega)$ a.s. empty. By the Markov property, under the latter measure, the distribution of

$$\left((w(s))_{0 \le s \le x}, (\zeta_{I_x + s})_{0 \le s \le I_0 - I_x}, (W_{I_x + s})_{0 \le s \le I_0 - I_x}\right)$$

is precisely $P_{0,\kappa}^{x}(dw \mid \underline{w}_x \ge -y)\,\mathbb{P}_w(d\omega)$. Letting $y \to \infty$ yields that $\mathcal{A}$ is $P_{0,\kappa}^{x}(dw)\,\mathbb{P}_w(d\omega)$ a.s. empty.

Step 1. Let us call (u_j, v_j), $j \in J$, the excursion intervals of $w - \underline{w}$, and

$$w^{(j)} := w((u_j + s) \wedge v_j) - w(u_j) \qquad j \in J.$$

We will need to find the distribution under $P_{0,\kappa}^{(-y,\infty)}(dw)\,\mathbb{P}_w(d\omega)$ of the point measure

$$\mathscr{P} := \sum_{j \in J} \delta_{(-w(u_j),\, m^{(j)})} \qquad \text{where } m^{(j)} := w(u_j) - \min_{[I_{v_j}, I_{u_j}]} \widehat{W}.$$

To this end, we adapt a computation of Miermont [Mie11, Lemma 31]. By Itô's excursion theory, under $P_{0,\kappa}^{(-y,\infty)}(dw)$, the point measure

$$\sum_{j \in J} \delta_{(-w(u_j),\, w^{(j)})}$$

is a Poisson point measure on $\mathbb{R}_+ \times \mathcal{K}$ with intensity $\kappa^{-1}\mathbb{1}_{[0,y]}(t)dt\, 2n(de/\kappa)$. Using Lemma 6.1, we may see the $m^{(j)}$, $j \in J$, as independent marks on $w^{(j)}$, $j \in J$, with law $\mathbb{P}_{w^{(j)}}(-\min \widehat{W} \in dz)$. The marking theorem of Poisson point measures [Kin93, Marking Theorem] shows that, under $P_{0,\kappa}^{(-y,\infty)}(dw)\,\mathbb{P}_w(d\omega)$, $\mathscr{P}$ is also a Poisson point measure on $\mathbb{R}_+ \times \mathbb{R}_+$ with intensity

$$\kappa^{-1}\mathbb{1}_{[0,y]}(t)dt \int_{\mathcal{K}} 2n(de)\,\mathbb{P}_{\kappa e}\left(-\min \widehat{W} \in dz\right).$$

To compute explicitly this intensity, we use Lemmas 6.1 and 6.2, and then Bismut's description of n [RY99, Theorem XII.4.7]:

$$\begin{aligned}\int_{\mathcal{K}} 2n(de)\,\mathbb{P}_{\kappa e}\left(-\min \widehat{W} \geq z\right) &= \int_{\mathcal{K}} 2n(de)\left(1-\exp\left(-\int_0^\ell \frac{3\,ds}{(\kappa e_s+z)^2}\right)\right)\\ &= \int_{\mathcal{K}} 2n(de)\int_0^\ell \frac{3\,dt}{(\kappa e_t+z)^2}\,\exp\left(-\int_t^\ell \frac{3\,ds}{(\kappa e_s+z)^2}\right)\\ &= 6\int_0^\infty \frac{da}{(\kappa a+z)^2}\,E_a^{(0,\infty)}\left[\exp\left(-\int_0^{\boldsymbol{T}_0}\frac{3\,ds}{(\kappa w(s)+z)^2}\right)\right]\\ &= 6\int_0^\infty \frac{da}{(\kappa a+z)^2}\,E_{a+z/\kappa}^{(0,\infty)}\left[\exp\left(-\frac{3}{\kappa^2}\int_0^{\boldsymbol{T}_{z/\kappa}}\frac{ds}{(w(s))^2}\right)\right].\end{aligned}$$

Using the absolute continuity relations between Bessel processes with different indexes, which is due to Yor [RY99, Exercise XI.1.22], and the fact that reflected Brownian motion is a 1-dimensional Bessel process, we see that

$$\begin{aligned}E_{a+z/\kappa}^{(0,\infty)}\left[\exp\left(-\frac{3}{\kappa^2}\int_0^{\boldsymbol{T}_{z/\kappa}}\frac{ds}{(w(s))^2}\right)\right] &= \lim_{t\to\infty} E_{a+z/\kappa}^{(0,\infty)}\left[\exp\left(-\frac{3}{\kappa^2}\int_0^{\boldsymbol{T}_{z/\kappa}\wedge t}\frac{ds}{(w(s))^2}\right)\right]\\ &= \lim_{t\to\infty} E_{a+z/\kappa}^{\langle 2+2\nu\rangle}\left[\left(\frac{w(\boldsymbol{T}_{z/\kappa}\wedge t)}{a+z/\kappa}\right)^{-\nu-1/2}\right]\\ &= \left(\frac{z}{\kappa a+z}\right)^{-\nu-1/2} P_{a+z/\kappa}^{\langle 2+2\nu\rangle}\left[\boldsymbol{T}_{z/\kappa}<\infty\right]\\ &= \left(\frac{z}{\kappa a+z}\right)^{\nu-1/2}.\end{aligned}$$

where $\nu := \sqrt{24+\kappa^2}\,/2\kappa$ and $P_a^{\langle 2+2\nu\rangle}$ denotes the distribution of a Bessel process of dimension $2+2\nu$. In the last line, we used the fact that, for $b>c$, $P_b^{\langle 2+2\nu\rangle}\,[\boldsymbol{T}_c<\infty] = (c/b)^{2\nu}$ (see [RY99, Chapter XI]). Putting all this together, we obtain that the intensity of $\mathscr{P}$ is

$$\mathbb{1}_{[0,y]}(t)dt\,\frac{\lambda}{z^2}\,dz, \qquad \text{where } \lambda := \frac{12\kappa^{-1}}{\sqrt{24+\kappa^2}+\kappa} \geq 1.$$

Step 2. Let $\sum_{k\in K}\delta_{(t_k,z_k)}$ be a Poisson random measure with intensity $dt\,\lambda z^{-2}dz$. Then, by the restriction property of Poisson random measures, for all $\varepsilon>0$, $\sum_{k\in K}\delta_{(t_k,z_k)}\mathbb{1}_{\{z_k\leq\varepsilon\}}$ is a Poisson random measure with intensity $dt\,\lambda z^{-2}\mathbb{1}_{\{z\leq\varepsilon\}}\,dz$. By a theorem of Shepp[2] [She72], we obtain that the random set

$$\bigcup_{k\in K\,:\,z_k\leq\varepsilon}(t_k,t_k+z_k)$$

is a.s. equal to $\mathbb{R}$. As a result, the set

$$\bigcup_{z_k\leq\varepsilon,\,0\leq t_k\leq y}(t_k,t_k+z_k)$$

a.s. contains $[\varepsilon,y]$. Because the point measure $\sum_{k\in K}\delta_{(t_k,z_k)}\mathbb{1}_{\{0\leq t_k\leq y\}}$ has the same law as $\mathscr{P}$ under $P_{0,\kappa}^{(-y,\infty)}(dw)\,\mathbb{P}_w(d\omega)$, we find that, $P_{0,\kappa}^{(-y,\infty)}(dw)\,\mathbb{P}_w(d\omega)$ a.s., for all rational $\varepsilon>0$, the set $[-y,-\varepsilon]$ is contained in

$$\mathrm{Cov}_\varepsilon := \bigcup_{j\in J\,:\,m^{(j)}\leq\varepsilon}\left(w(u_j)-m^{(j)},w(u_j)\right).$$

[2] We use here the fact that $\lambda\geq 1$, ensuring that $\mathbb{R}$ is covered with "small" intervals. See in particular the remark on high frequency coverings in [She72, Section 5].

Now, let us assume that $\mathcal{A}$ is not empty, and let us take $s \in \mathcal{A}$. By Lemma 6.4, $s \notin \{\alpha_i,\, i \in I\}$. As a result, there exists $\eta > 0$ such that $\widehat{W}_r \geq \widehat{W}_s$ for all $r \in [s, I_{\zeta_s - \eta}]$. In particular, for all $j \in J$ such that $[u_j, v_j] \subseteq [\zeta_s - \eta, \zeta_s]$, we have $w(u_j) - m^{(j)} \geq \widehat{W}_s = w(\zeta_s)$. As $\zeta_s = \inf\{t : w(t) = w(\zeta_s)\}$, we can find a rational $\varepsilon > 0$ satisfying $w(\zeta_s) \leq \underline{w}(\zeta_s - \eta) - \varepsilon \leq -\varepsilon$.

Let $j \in J$ be such that $m^{(j)} \leq \varepsilon$. If $u_j \leq \zeta_s - \eta$, then $w(u_j) - m^{(j)} \geq \underline{w}(\zeta_s - \eta) - \varepsilon \geq w(\zeta_s)$. If $u_j \in [\zeta_s - \eta, \zeta_s]$, we already observed that $w(u_j) - m^{(j)} \geq w(\zeta_s)$. Finally, if $u_j \geq \zeta_s$, then $w(u_j) \leq w(\zeta_s)$. In all cases, $w(\zeta_s) \notin (w(u_j) - m^{(j)}, w(u_j))$. We found a point $w(\zeta_s) \in [-y, -\varepsilon]$ that does not belong to Cov_ε. This can only happen with probability 0.

Similar arguments show that the set $\mathcal{B}$ is $\overline{P}^{(-y,\infty)}_{0,\kappa}(dw)\,\mathbb{P}_w(d\omega)$ a.s. empty, where we write $\overline{P}^{(-y,\infty)}_{0,\kappa}$ the pushforward of $P^{(-y,\infty)}_{0,\kappa}$ under $w \mapsto \overline{w} := (w(\zeta(w) - s))_{0 \leq s \leq \zeta(w)}$. This entails that $\mathcal{B}$ is also $\overline{P}^{x}_{0,\kappa}(dw)\,\mathbb{P}_w(d\omega)$ a.s. empty, and, by time-reversal, $\overline{w} - w(x)$ has under $P^x_{0,\kappa}(dw)$ the same distribution as w, so that the result also holds $P^x_{0,\kappa}(dw)\,\mathbb{P}_w(d\omega)$ a.s. We leave the details to the reader. □

We may now proceed to the proof of Lemma 5.7. We define

$$I^{(a)}_b := \inf\{s \geq a \,:\, \zeta_s = b\}.$$

Proof of Lemma 5.7. As above, we start by working under $P^x_{0,\kappa}(dw)\,\mathbb{P}_w(d\omega)$ with $\kappa \leq \sqrt{3}$.

Step 1. The first step consists in treating the left-increase points of ζ. To do so, we will use the Markov property of the Brownian snake and "insert" rational numbers in order to be able to apply the previous lemmas. Let $b \in [0, x]$. Because the process

$$\Big(\big(w(b+r) - w(b)\big)_{0 \leq r \leq x-b},\, \big((W_r(b+t))_{0 \leq t \leq \zeta_r - b}\big)_{0 \leq r \leq I_b}\Big)$$

has under $P^x_{0,\kappa}(dw)\,\mathbb{P}_w(d\omega)$ the law $P^{x-b}_{0,\kappa}(dw)\,\mathbb{P}_w(d\omega)$, we see by Lemma 6.5 that the set

$$\mathcal{A}^b := \Big\{s \in \mathrm{IP}_{right}\left((\widehat{W}_s)_{0 \leq s \leq I_b}\right) \,:\, \zeta_s = \underline{\zeta}_s = \inf\{t \geq b \,:\, w(t) = w(\zeta_s)\}\Big\}$$

is $P^x_{0,\kappa}(dw)\,\mathbb{P}_w(d\omega)$ a.s. empty. Similarly, for $b \in [0, x]$, $c > b$, and a, the Markov property shows that the process

$$\Big(\big(W_a(b+r) - W_a(b)\big)_{0 \leq r \leq c-b},\, \big((W_{I^{(a)}_c + r}(b+t))_{0 \leq t \leq \zeta_{I^{(a)}_c + r} - b}\big)_{0 \leq r \leq I^{(a)}_b - I^{(a)}_c}\Big)$$

has under $P^x_{0,\kappa}(dw)\,\mathbb{P}_w(d\omega \mid a \leq I_0,\, \underline{\zeta}_a < b,\, c < \zeta_a)$ the law $P^{c-b}_0(dw)\,\mathbb{P}_w(d\omega)$. (Beware that here the factor κ does not appear.) As a result, Lemmas 6.4 and 6.5 successively show that, on the event $\{a \leq I_0,\, \underline{\zeta}_a < b,\, c < \zeta_a\}$, the sets

$$\mathcal{C}^{b,c}_a := \left\{s \in \mathrm{IP}_{right}\left((\widehat{W}_s)_{I^{(a)}_c \leq s \leq I^{(a)}_b}\right) \cap \mathrm{IP}_{right}(\zeta) \,:\, \zeta_s = \inf_{[a,s]} \zeta\right\}$$

and

$$\mathcal{A}^{b,c}_a := \left\{s \in \mathrm{IP}_{right}\left((\widehat{W}_s)_{I^{(a)}_c \leq s \leq I^{(a)}_b}\right) \,:\, \zeta_s = \inf_{[a,s]} \zeta = \inf\{t \geq b \,:\, W_a(t) = W_a(\zeta_s)\}\right\}$$

are $P^x_{0,\kappa}(dw)\,\mathbb{P}_w(d\omega)$ a.s. empty. As a result, we obtain that $P^x_{0,\kappa}(dw)\,\mathbb{P}_w(d\omega)$ a.s., for all rational values of a, b, and c, these sets are empty.

Now, if the set $\mathrm{IP}_{left}\big((\zeta_s)_{0 \leq s \leq I_0}\big) \cap \mathrm{IP}_{right}\big((\widehat{W}_s)_{0 \leq s \leq I_0}\big)$ is not empty, let s be a point lying in it. Let us first suppose that $\zeta_s = \underline{\zeta}_s$. By Lemma 6.4, we know that $s \notin \{\alpha_i,\, i \in I\}$. This implies that $\zeta_s \in \mathrm{IP}_{left}(w)$. As local minimums of Brownian motion are distinct, we can find a rational $b \in [0, \zeta_s]$

such that $\zeta_s = \inf\{t \geq b : w(t) = w(\zeta_s)\}$, and $s \in \mathcal{A}_b$. Otherwise, $\zeta_s > \underline{\zeta}_s$. As $s \in \mathrm{IP}_{left}(\zeta)$, we can find a rational $a \in [0, s)$ such that $\zeta_a > \zeta_s$ and $\zeta_s = \inf_{[a,s]} \zeta$. If $s \in \mathrm{IP}_{right}(\zeta)$, we can find rationals $b \in (\underline{\zeta}_a, \zeta_s)$ and $c \in (\zeta_s, \zeta_a)$ so that $s \in \mathcal{C}_a^{b,c}$. If $s \notin \mathrm{IP}_{right}(\zeta)$, then $\zeta_s \in \mathrm{IP}_{left}(W_a)$. We can then find rationals b and c such that $s \in \mathcal{A}_a^{b,c}$.

Summing up, we obtain that the set $\mathrm{IP}_{left}\Big((\zeta_s)_{0\leq s\leq I_0}\Big) \cap \mathrm{IP}_{right}\Big((\widehat{W}_s)_{0\leq s\leq I_0}\Big)$ is $P^x_{0,\kappa}(dw)\,\mathbb{P}_w(d\omega)$ a.s. empty. By a similar argument, we show that the set $\mathrm{IP}_{left}\Big((\zeta_s)_{0\leq s\leq I_0}\Big) \cap \mathrm{IP}_{left}\Big((\widehat{W}_s)_{0\leq s\leq I_0}\Big)$ is also $P^x_{0,\kappa}(dw)\,\mathbb{P}_w(d\omega)$ a.s. empty.

Step 2. We now use a time-reversal argument under $\mathbb{N}_y$ to treat the right-increase points of ζ. By translation, the quantity

$$\Delta := \mathbb{N}_y\left(\mathrm{IP}_{left}\Big((\zeta_s)_{0\leq s\leq \ell}\Big) \cap \mathrm{IP}\Big((\widehat{W}_s)_{0\leq s\leq \ell}\Big) \neq \varnothing\right)$$

does not depend on y. Using the Poissonian decomposition of the excursions of $\zeta - \underline{\zeta}$, we see that, $P^x_{0,\kappa}(dw)$ a.s.,

$$0 = \mathbb{P}_w\left(\mathrm{IP}_{left}\Big((\zeta_s)_{0\leq s\leq I_0}\Big) \cap \mathrm{IP}\Big((\widehat{W}_s)_{0\leq s\leq I_0}\Big) \neq \varnothing\right) \geq 1 - e^{-2x\Delta},$$

so that $\Delta = 0$. Note that a priori we only have an inequality, because some left-increase points of $(\zeta_s)_{0\leq s\leq I_0}$ may well lie outside of the set $\cup_{i\in I}(\alpha_i, \beta_i]$. Using time-reversal under $\mathbb{N}_y$, we find that $\mathrm{IP}_{right}\Big((\zeta_s)_{0\leq s\leq \ell}\Big) \cap \mathrm{IP}\Big((\widehat{W}_s)_{0\leq s\leq \ell}\Big)$ is also $\mathbb{N}_y$ a.e. empty. As announced at the beginning of this section, we re-obtained here [LGP08, Lemma 3.2]. It is then easier to deal with right-increase points of $(\zeta_s)_{0\leq s\leq I_0}$, because they all lie in $\cup_{i\in I}[\alpha_i, \beta_i)$: using once again the Poissonian decomposition of the excursions of $\zeta - \underline{\zeta}$, we find (for any $w \in \mathcal{K}$)

$$\mathbb{P}_w\left(\mathrm{IP}_{right}\Big((\zeta_s)_{0\leq s\leq I_0}\Big) \cap \mathrm{IP}\Big((\widehat{W}_s)_{0\leq s\leq I_0}\Big) \neq \varnothing\right) = 1 - e^{-2\zeta(w)\Delta} = 0.$$

Putting it all together, we showed that $\mathrm{IP}\Big((\zeta_s)_{0\leq s\leq I_0}\Big) \cap \mathrm{IP}\Big((\widehat{W}_s)_{0\leq s\leq I_0}\Big)$ is $P^x_{0,\kappa}(dw)\,\mathbb{P}_w(d\omega)$ a.s. empty. Using the fact that the distribution of $(w(s))_{0\leq s\leq \sigma-\varepsilon}$ under $\mathbb{B}$ is absolutely continuous with respect to the distribution of $(w(s))_{0\leq s\leq \sigma-\varepsilon}$ under $P^{\sigma-\varepsilon}_{0,\sqrt{3}}(dw)$, we obtain that, $\mathbb{B}(dw)\,\mathbb{P}_w(d\omega)$ a.s., for all rational $\varepsilon \in (0, \sigma)$, we have $\mathrm{IP}\Big((\zeta_s)_{I_{\sigma-\varepsilon}\leq s\leq I_0}\Big) \cap \mathrm{IP}\Big((\widehat{W}_s)_{I_{\sigma-\varepsilon}\leq s\leq I_0}\Big) = \varnothing$. Standard properties of Brownian motion show that, $\mathbb{P}_w$ a.s. $0 \notin \mathrm{IP}(\zeta)$ and $\inf_{\varepsilon\in(0,\sigma)\cap\mathbb{Q}} I_{\sigma-\varepsilon} = 0$, so that $\mathbb{B}(dw)\,\mathbb{P}_w(d\omega)$ a.s. $\mathrm{IP}\Big((\zeta_s)_{0\leq s\leq I_0}\Big) \cap \mathrm{IP}\Big((\widehat{W}_s)_{0\leq s\leq I_0}\Big) = \varnothing$. Using another absolute continuity argument and the fact that $1 \notin \mathrm{IP}_{left}(\widehat{W})$ (because $0 \notin \mathrm{IP}(w)$), we conclude that $\int_{\mathcal{K}} \mathbb{B}(dw)\,\mathbb{P}^0_w(d\omega)$ a.s. $\mathrm{IP}\Big((\zeta_s)_{0\leq s\leq 1}\Big) \cap \mathrm{IP}\Big((\widehat{W}_s)_{0\leq s\leq 1}\Big) = \varnothing$. This completes the proof. □

6.4 Proof of Lemma 5.10

As explained in the proof of Lemma 4.6, Lemma 5.10 is a consequence of the following two lemmas, which we state here directly in terms of the Brownian snake. We will use a strategy similar to that of Section 6.2 and derive these lemmas from similar statements under $\mathbb{N}_x$, namely [LG07, Lemma 2.4] and [LG10, Lemma 6.1]. We call $\mathscr{L}$ the Lebesgue measure on $\mathbb{R}$.

Lemma 6.6. *Let $w \in \mathcal{K}$. $\mathbb{P}^0_w$ a.s., for every $\eta > 0$, for all $x \in [0, 1]$ and all $l < r$ such that,*

- *either $0 < l < r < x$ and $\zeta_l = \zeta_r = \inf_{[l,x]} \zeta$,*
- *or $x < l < r < 1$ and $\zeta_l = \zeta_r = \inf_{[x,r]} \zeta$,*

the condition $\inf_{[l,r]} \widehat{W} < \widehat{W}_l - \eta$ implies that

$$\liminf_{\varepsilon\to 0} \varepsilon^{-2} \mathscr{L}\left(\left\{s \in [l, r] \,:\, \widehat{W}_s < \widehat{W}_l - \eta + \varepsilon\,;\ \forall y \in [\zeta_l, \zeta_s], \widehat{W}_{\sup\{t\leq s\,:\,\zeta_t = y\}} > \widehat{W}_l - \eta + \frac{\varepsilon}{8}\right\}\right) > 0.$$

Lemma 6.7. *For every $p \geq 1$ and every $\delta \in (0,1]$, there exists a constant $c_{p,\delta} < \infty$ such that, for every $\varepsilon > 0$,*

$$\int_{\mathcal{K}} \mathbb{B}(dw)\, \mathbb{E}^0_w \left[\left(\int_0^1 \mathbb{1}_{\{\widehat{W}_s \leq \min_{0\leq r\leq 1} \widehat{W}_r + \varepsilon\}}\, ds \right)^p \right] \leq c_{p,\delta}\, \varepsilon^{4p-\delta}.$$

Proof of Lemma 6.6. By an absolute continuity argument, it is sufficient to show the result under $\mathbb{P}_w$ (while replacing 1 with I_0). By [LG07, Lemma 2.4], the result holds under $\mathbb{N}_0$ (while replacing 1 with ℓ), and we may extend it to the case $l = 0$, $r = \ell$ as follows. Let us suppose that there exists $\eta > 0$ such that $\inf_{[0,\ell]} \widehat{W} < -\eta$ and $\liminf_{\varepsilon\to 0} \varepsilon^{-2} \mathscr{L}(A_\varepsilon) = 0$, where

$$A_\varepsilon := \Big\{ s \in [0,\ell] \,:\, \widehat{W}_s < -\eta + \varepsilon \,;\, \forall y \in [0,\zeta_s], \widehat{W}_{\sup\{t\leq s\,:\,\zeta_t = y\}} > -\eta + \frac{\varepsilon}{8} \Big\}.$$

In terms of tree, we are going to look at the ancestral lineage of the point with minimum label. By cutting this line sufficiently close to the root, we will create a subtree that contradicts [LG07, Lemma 2.4]. We call $s^\bullet \in (0,1)$ the (unique) point where $\widehat{W}$ reaches its minimum. As $z \mapsto \widehat{W}_{\sup\{t\leq s^\bullet\,:\,\zeta_t = z\}}$ is continuous, we can find $z_0 > 0$ such that

$$\sup_{0\leq z\leq z_0} \left| \widehat{W}_{\sup\{t\leq s^\bullet\,:\,\zeta_t = z\}} \right| < \eta.$$

We take $z \in (0, z_0]$ such that $l := \sup\{t \leq s^\bullet \,:\, \zeta_t = z\}$ is a local minimum of ζ, $x \in (0,l)$ such that $\zeta_l = \inf_{[x,l]} \zeta$, and finally $r := \inf\{t \geq s^\bullet \,:\, \zeta_t = z\}$. Then this pair (l,r) satisfies the hypothesis of the lemma for $\eta' := \eta + \widehat{W}_l > 0$, but not the conclusion, as the set

$$\Big\{ s \in [l,r] \,:\, \widehat{W}_s < \widehat{W}_l - \eta' + \varepsilon \,;\, \forall y \in [\zeta_l, \zeta_s], \widehat{W}_{\sup\{t\leq s\,:\,\zeta_t = y\}} > \widehat{W}_l - \eta' + \frac{\varepsilon}{8} \Big\}$$

is contained inside A_ε as soon as

$$\inf_{0\leq y\leq z} \widehat{W}_{\sup\{t\leq s^\bullet\,:\,\zeta_t = y\}} > -\eta + \frac{\varepsilon}{8},$$

which happens for ε small enough.

We conclude by saying that, under $\mathbb{P}_w$, if the numbers l, r, x, and η satisfy the hypothesis but not the conclusion, then there exists an $i \in I$ such that, either $(l,r) = (\alpha_i, \beta_i)$, in which case 0, $\beta_i - \alpha_i$, and η also satisfy the hypothesis but not the conclusion for the process $(\zeta^{(i)}, W^{(i)})$, or $l - \alpha_i$, $r - \alpha_i$, $x - \alpha_i$, and η also satisfy the hypothesis but not the conclusion for the process $(\zeta^{(i)}, W^{(i)})$. The probability of this event is then equal to 0, by Lemma 6.1. □

We will need in the proof of Lemma 6.7 the following fact: for a stopping time T such that $T < 1$ $\mathbb{P}$ a.s., we have, for any bounded measurable function f on $\mathcal{K}$,

$$\mathbb{E}\left[f\left(\left(F^{\sigma\to 0}_{[0,1]}(t) \right)_{0\leq t\leq T} \right) \right] = E^1_\sigma \left[f\left((w(t))_{0\leq t\leq T} \right) \frac{p'_{1-T}(w(T))}{p'_1(\sigma)} \mathbb{1}_{\{\underline{w}(T)>0,\, T<1\}} \right] \tag{6.2}$$

where

$$p'_a(x) := -\frac{x}{\sqrt{2\pi a^3}} \exp\left(-\frac{x^2}{2a} \right)$$

denotes the derivative of the density of a centered Gaussian variable with variance a. This is an easy consequence of Equation (2.19): we write

$$\begin{aligned} \mathbb{E}\left[f\left(\left(F^{\sigma\to 0}_{[0,1]}(t) \right)_{0\leq t\leq T} \right) \right] &= \lim_{m\to 1} \mathbb{E}\left[f\left(\left(F^{\sigma\to 0}_{[0,1]}(t) \right)_{0\leq t\leq T\wedge m} \right) \right] \\ &= \lim_{m\to 1} \lim_{n\to\infty} \sum_{j=1}^{\lceil m2^n \rceil} \mathbb{E}\left[f\left(\left(F^{\sigma\to 0}_{[0,1]}(t) \right)_{0\leq t\leq j2^{-n}} \right) \mathbb{1}_{\{(j-1)2^{-n} < T\wedge m \leq j2^{-n}\}} \right], \end{aligned}$$

and apply Equation (2.19) inside the sum. We leave the technical details to the reader.

Proof of Lemma 6.7. We fix $p \geq 1$ and $\delta \in (0,1]$.

Step 1. As before, the first step is to dispose of the two conditionings. We notice that

$$\int_0^1 \mathbb{1}_{\{\widehat{W}_s \le \underline{\widehat{W}}_1+\varepsilon\}}\,ds = \int_0^{I_{\sigma/2}} \mathbb{1}_{\{\widehat{W}_s \le \underline{\widehat{W}}_1+\varepsilon\}}\,ds + \int_{I_{\sigma/2}}^1 \mathbb{1}_{\{\widehat{W}_s \le \underline{\widehat{W}}_1+\varepsilon\}}\,ds$$
$$\le \int_0^{I_{\sigma/2}} \mathbb{1}_{\{\widehat{W}_s \le \underline{\widehat{W}}_{I_{\sigma/2}}+\varepsilon\}}\,ds + \int_{I_{\sigma/2}}^1 \mathbb{1}_{\{\widehat{W}_s \le \min_{I_{\sigma/2}\le r\le 1}\widehat{W}_r+\varepsilon\}}\,ds.$$

As both terms in the previous line have the same law under $\int_{\mathcal{K}} \mathbb{B}(dw)\,\mathbb{P}_w^0(d\omega)$, we obtain

$$\int_{\mathcal{K}} \mathbb{B}(dw)\,\mathbb{E}_w^0\left[\left(\int_0^1 \mathbb{1}_{\{\widehat{W}_s \le \underline{\widehat{W}}_1+\varepsilon\}}\,ds\right)^p\right] \le 2^{p+1}\int_{\mathcal{K}} \mathbb{B}(dw)\,\mathbb{E}_w^0\left[\left(\int_0^{I_{\sigma/2}} \mathbb{1}_{\{\widehat{W}_s \le \underline{\widehat{W}}_{I_{\sigma/2}}+\varepsilon\}}\,ds\right)^p\right].$$

Applying (6.2), we see that

$$\mathbb{E}_w^0\left[\left(\int_0^{I_{\sigma/2}} \mathbb{1}_{\{\widehat{W}_s \le \underline{\widehat{W}}_{I_{\sigma/2}}+\varepsilon\}}\,ds\right)^p\right] = \mathbb{E}_w\left[\left(\int_0^{I_{\sigma/2}} \mathbb{1}_{\{\widehat{W}_s \le \underline{\widehat{W}}_{I_{\sigma/2}}+\varepsilon\}}\,ds\right)^p \frac{p'_{1-I_{\sigma/2}}(\sigma/2)}{p'_1(\sigma)}\,\mathbb{1}_{\{I_{\sigma/2}<1\}}\right]$$
$$\le c_\sigma\,\mathbb{E}_w\left[\left(\int_0^{I_{\sigma/2}} \mathbb{1}_{\{\widehat{W}_s \le \underline{\widehat{W}}_{I_{\sigma/2}}+\varepsilon\}}\,ds\right)^p\right],$$

where

$$c_\sigma := \sup_{a>0}\frac{p'_a(\sigma/2)}{p'_1(\sigma)} < \infty.$$

We then dispose of the second conditioning. By using Equation (2.18), it is not hard to see that

$$\int_{\mathcal{K}} \mathbb{B}(dw)\,\mathbb{E}_w\left[\left(\int_0^{I_{\sigma/2}} \mathbb{1}_{\{\widehat{W}_s \le \underline{\widehat{W}}_{I_{\sigma/2}}+\varepsilon\}}\,ds\right)^p\right]$$
$$\le \sqrt{2}\int_{\mathcal{K}} P_{0,\kappa}^{\sigma/2}(dw)\,\mathbb{E}_w\left[\left(\int_0^{I_0} \mathbb{1}_{\{\widehat{W}_s \le \underline{\widehat{W}}_{I_0}+\varepsilon\}}\,ds\right)^p\right],$$

where, from now on $\kappa = \sqrt{3}$. Putting it all together, we see that it will suffice to bound the right-hand side of this inequality.

Step 2. The second difficulty comes from the factor κ. Our strategy is the following. By stretching the time by a factor κ^2 for the process w, we obtain a standard Brownian motion, no longer rescaled. But we have to be careful that, by doing so, we change the intensity of the Poisson point measure (6.1). Precisely, let $(X_i)_{i\in I}$, be a sequence of i.i.d. random Bernoulli variables with mean $1/\kappa^2$, independent from the process (6.1). Then, the marking theorem of Poisson point measures entails that the process $\sum_{i\in I}\delta_{(\kappa^2\zeta_{\alpha_i},W^{(i)})}$ has, under $\int_{\mathcal{K}} P_{0,\kappa}^{\sigma/2}(dw)\,\mathbb{P}_w(d\omega)$, the same distribution as the process

$$\sum_{i\in I}\delta_{(\zeta_{\alpha_i},W^{(i)})}\mathbb{1}_{\{X_i=1\}}, \qquad \text{under} \qquad \int_{\mathcal{K}} P_0^{\kappa^2\sigma/2}(dw)\,\mathbb{P}_w(d\omega).$$

As a result, writing $I' := \{i \in I : X_i = 1\}$, we obtain

$$\int_{\mathcal{K}} P_{0,\kappa}^{\sigma/2}(dw)\,\mathbb{E}_w\left[\left(\int_0^{I_0} \mathbb{1}_{\{\widehat{W}_s \le \underline{\widehat{W}}_{I_0}+\varepsilon\}}\,ds\right)^p\right]$$
$$= \int_{\mathcal{K}} P_{0,\kappa}^{\sigma/2}(dw)\,\mathbb{E}_w\left[\left(\sum_{i\in I}\int_0^{\beta_i-\alpha_i} \mathbb{1}_{\{\widehat{W}_s^{(i)} \le \min\{\min \widehat{W}^{(j)},\, j\in I\}+\varepsilon\}}\,ds\right)^p\right]$$
$$= \int_{\mathcal{K}} P_0^{\kappa^2\sigma/2}(dw)\,\mathbb{E}_w\left[\left(\sum_{i\in I'}\int_0^{\beta_i-\alpha_i} \mathbb{1}_{\{\widehat{W}_s^{(i)} \le \min\{\min \widehat{W}^{(j)},\, j\in I'\}+\varepsilon\}}\,ds\right)^p\right]$$
$$\le \kappa^2\int_{\mathcal{K}} P_0^{\kappa^2\sigma/2}(dw)\,\mathbb{E}_w\left[\left(\int_0^{I_0} \mathbb{1}_{\{\widehat{W}_s \le \underline{\widehat{W}}_{I_0}+\varepsilon\}}\,ds\right)^p\right].$$

We used in the first equality the fact that $(\widehat{W}_s)_{0\le s\le I_0}$ reaches its minimum on $\bigcup_{i\in I}[\alpha_i,\beta_i]$, which was proven during the proof of Lemma 5.6. To obtain the last inequality, we conditioned by the event

$$\Big\{ \min\{\min \widehat{W}^{(j)},\, j\in I'\} = \min\{\min \widehat{W}^{(j)},\, j\in I\}\Big\},$$

which happens with probability $1/\kappa^2$.

Step 3. We will now use Bismut's description of n in order to apply [LG10, Lemma 6.1]. In some sense, the measure we have been considering so far takes into account only half of the excursion measure $\mathbb{N}_0$. We remedy to this by defining, for $a>0$, the following probability measure on $\mathcal{C}(\mathbb{R}_+,\mathcal{K})^2$:

$$\mathbb{P}^a(dW^1\,dW^2) := \int_{\mathcal{K}} P_0^a(dw)\,\mathbb{P}_w(dW^1)\mathbb{P}_w(dW^2).$$

Under this measure, we call $I_0^1 := \inf\{s : \zeta(W_s^1)=0\}$ and $I_0^2 := \inf\{s : \zeta(W_s^2)=0\}$. It will be enough to see that

$$\sup_{\varepsilon>0}\frac{\Phi_a(\varepsilon)}{\varepsilon^{4p-\delta}}<\infty, \qquad \text{where } \Phi_a(\varepsilon) := \mathbb{E}^a\left[\left(\int_0^{I_0^1}\mathbb{1}_{\{\widehat{W}_s^1\le \min\widehat{W}^1+\varepsilon\}}\,ds\right)^p\right].$$

Let us call $m := \min\widehat{W}^1\wedge\min\widehat{W}^2$. Conditioning on the event $\{\min\widehat{W}^1<\min\widehat{W}^2\}$, and then on $\{I_0^1<1,\, I_0^2<1\}$, we obtain, for $a\le A$,

$$\begin{aligned}
\Phi_a(\varepsilon) &\le 2\,\mathbb{E}^a\left[\left(\int_0^{I_0^1}\mathbb{1}_{\{\widehat{W}_s^1\le m+\varepsilon\}}\,ds+\int_0^{I_0^2}\mathbb{1}_{\{\widehat{W}_s^2\le m+\varepsilon\}}\,ds\right)^p\right]\\
&\le \frac{2}{\big(\mathbb{P}^a(I_0^1<1)\big)^2}\,\mathbb{E}^a\left[\left(\int_0^{I_0^1}\mathbb{1}_{\{\widehat{W}_s^1\le m+\varepsilon\}}\,ds+\int_0^{I_0^2}\mathbb{1}_{\{\widehat{W}_s^2\le m+\varepsilon\}}\,ds\right)^p\mathbb{1}_{\{I_0^1<1,\,I_0^2<1\}}\right]\\
&\le C_A\,\mathbb{E}^a\left[\left(\int_0^{I_0^1}\mathbb{1}_{\{\widehat{W}_s^1\le m+\varepsilon\}}\,ds+\int_0^{I_0^2}\mathbb{1}_{\{\widehat{W}_s^2\le m+\varepsilon\}}\,ds\right)^p\mathbb{1}_{\{I_0^1+I_0^2<2\}}\right],
\end{aligned}$$

where C_A is a finite constant depending only on A. Using Bismut's description of n [RY99, Theorem XII.4.7], then [LG10, Lemma 6.1], we see that

$$\int_0^A da\;\Phi_a(\varepsilon)\le C_A\;\mathbb{N}_0\left(\int_0^\ell dt\,\mathbb{1}_{\{\ell<2\}}\left(\int_0^\ell \mathbb{1}_{\{\widehat{W}_s\le\min\widehat{W}+\varepsilon\}}\,ds\right)^p\right)\le 2C_A\,c'_{p,\delta}\,\varepsilon^{4p-\delta},$$

where $c'_{p,\delta}<\infty$ depends only on p and δ. As a result, we obtain that, for Lebesgue almost every $a\in(0,A]$, we have $\sup_{\varepsilon>0}\varepsilon^{\delta-4p}\,\Phi_a(\varepsilon)<\infty$. We conclude by noticing that, for all $a,b>0$, we have $\Phi_{a+b}(\varepsilon)\le 2^p(\Phi_a(\varepsilon)+\Phi_b(\varepsilon))$. If for some A, we had $\sup_{\varepsilon>0}\varepsilon^{\delta-4p}\,\Phi_A(\varepsilon)=\infty$, then, for all $a\in(0,A)$ we would have $\sup_{\varepsilon>0}\varepsilon^{\delta-4p}\,\Phi_a(\varepsilon)=\infty$ or $\sup_{\varepsilon>0}\varepsilon^{\delta-4p}\,\Phi_{A-a}(\varepsilon)=\infty$ (possibly both), which would contradict our latest observation. □

6.5 Upper bound for the Hausdorff dimension of $\partial\mathfrak{q}_\infty^\sigma$

We may now end the proof of Theorem 1.16.

Proof of Theorem 1.16 (upper bound). Under $\mathbb{P}_w$, we will call $T_x := \inf\{r\ge 0 : \zeta_r=\zeta(w)-x\}$. For $s\in[0,I_0]$, we set $s^+ := \sup\{t : \zeta_t=\zeta_s\}$. Using the same kind of reasoning as in the previous sections, we see that it is enough to show that, for any pseudo-metric d on $[0,I_0]$ such that

$$d(s,t)\le\widehat{W}_s+\widehat{W}_t-2\min_{r\in[s^+,t]}\widehat{W}_r, \qquad 0\le s\le t\le I_0,$$

we have

$$\mathbb{P}_w(\dim_{\mathcal{H}}(\{T_x,\ 0 \le x \le \zeta(w)\}, d) \le 2) = 1, \qquad \mathbb{B}(dw) \text{ a.s.}$$

We fix $\eta \in (0,1)$. We will cover $\{T_x,\ 0 \le x \le \zeta(w)\}$ by small open balls. Let us first bound the distance between two points in $\{T_x,\ 0 \le x \le \zeta(w)\}$. Using [LG07, Lemma 5.1] and the fact that, $\mathbb{B}(dw)$ a.s., w is $1/(2+\eta)$-Hölder continuous, it is not hard to see that there exists a (random) constant $c < \infty$ such that, $\mathbb{B}(dw)$ a.s. $\mathbb{P}_w$ a.s.,

$$\left|\widehat{W}_s - \widehat{W}_t\right| \le c\ (d_\zeta(s,t))^{\frac{1}{2+\eta}}, \qquad \text{for all } s,t \in [0, I_0], \tag{6.3}$$

where

$$d_\zeta(s,t) := \zeta_s + \zeta_t - 2 \min_{r \in [s\wedge t, s \vee t]} \zeta_r, \qquad s,t \in [0, I_0].$$

Let $0 \le x \le y \le \zeta(w)$, and $m(x,y) \in [T_x^+, T_y]$ be such that $\widehat{W}_{m(x,y)} = \min_{s \in [T_x^+, T_y]} \widehat{W}_s$. When (6.3) holds, we have

$$\begin{aligned} d(T_x, T_y) &\le \widehat{W}_{T_x} + \widehat{W}_{T_y} - 2\widehat{W}_{m(x,y)} \\ &\le c\left(\Big(d_\zeta(T_x, m(x,y))\Big)^{\frac{1}{2+\eta}} + \Big(d_\zeta(T_y, m(x,y))\Big)^{\frac{1}{2+\eta}}\right) \\ &\le 2c\left(y - x + 2\Big(\zeta_{m(x,y)} - \underline{\zeta}_{m(x,y)}\Big)\right)^{\frac{1}{2+\eta}}. \end{aligned} \tag{6.4}$$

In order to control the term $(\zeta_{m(x,y)} - \underline{\zeta}_{m(x,y)})$ in the above inequality, we sort out the excursions going "too high." Namely, we fix $\varepsilon > 0$, and set

$$I^{(\varepsilon)} := \left\{ i \in I\ :\ \sup_{s \ge 0} \zeta_s^{(i)} > \varepsilon \right\}.$$

By Lemma 6.1, the cardinality of $I^{(\varepsilon)}$ is under $\mathbb{P}_w$ a Poisson random variable with mean

$$2\int_0^{\zeta(w)} dt\, \mathbb{N}_{w(t)}\left(\sup_{s\ge 0} \zeta_s > \varepsilon\right) = 2\zeta(w)\, n\left(\sup_{s \ge 0} e_s > \varepsilon\right) = \frac{\zeta(w)}{\varepsilon}.$$

In particular, $|I^{(\varepsilon)}| < \infty$, $\mathbb{P}_w$ a.s. We call $B(s,r) \subseteq [0, I_0]$ the open ball of radius r centered at s, for the pseudo-metric d, and, for $i \in I$, we call $x_i := \zeta(w) - \zeta_{\alpha_i}$. If $\delta := 2c\,(3\varepsilon)^{1/(2+\eta)}$, we claim that the set

$$\left\{B\left(T_{x_i}, \delta\right),\ i \in I^{(\varepsilon)}\right\} \cup \left\{B\left(T_{k\varepsilon}, \delta\right),\ 0 \le k \le \frac{\zeta(w)}{\varepsilon}\right\}$$

is a covering of $\{T_x,\ 0 \le x \le \zeta(w)\}$. To see this, let us take a point $y \in [0, \zeta(w)]$, and let us consider $x := \max\{s \in \{0\} \cup \{x_i,\ i \in I^{(\varepsilon)}\}\ :\ s \le y\}$. Because $T_x^+ \le m(x,y) \le T_y$, we see that $\zeta_{m(x,y)} - \underline{\zeta}_{m(x,y)} \le \varepsilon$. Then, if $y - x < \varepsilon$, by (6.4), we have $T_y \in B(T_x, \delta)$. If $y - x \ge \varepsilon$, then $y - \lfloor y/\varepsilon \rfloor \varepsilon < \varepsilon$, and $\lfloor y/\varepsilon \rfloor \varepsilon \ge x$, so that $\zeta_{m(\lfloor y/\varepsilon \rfloor \varepsilon, y)} - \underline{\zeta}_{m(\lfloor y/\varepsilon \rfloor \varepsilon, y)} \le \varepsilon$. This yields that $T_y \in B(T_{\lfloor y/\varepsilon \rfloor \varepsilon}, \delta)$, by (6.4).

The $(2+\eta)(1+\eta)$-value of this covering is less than

$$\left(|I^{(\varepsilon)}| + \frac{\zeta(w)}{\varepsilon} + 1\right)\ (2\delta)^{(2+\eta)(1+\eta)} \le c'\ \left(|I^{(\varepsilon)}| + \frac{\zeta(w)}{\varepsilon} + 1\right)\ \varepsilon^{1+\eta},$$

for some constant c', independent of ε. By Chebyshev's inequality, we see that with $\mathbb{P}_w$-probability at least $1 - \varepsilon/\zeta(w)$, we have $|I^{(\varepsilon)}| \le 2\zeta(w)/\varepsilon$. We conclude that, $\mathbb{B}(dw)$ a.s., the $(2+\eta)(1+\eta)$-Hausdorff content of $\{T_x,\ 0 \le x \le \zeta(w)\}$ is $\mathbb{P}_w$ a.s. equal to 0, so that $\dim_{\mathcal{H}}(\{T_x,\ 0 \le x \le \zeta(w)\}, d) \le (2+\eta)(1+\eta)$. Finally, letting $\eta \to 0$ yields the result. □

Liste des figures

Références

[ADJ97] Jan Ambjørn, Bergfinnur Durhuus, and Thordur Jonsson. *Quantum geometry*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, Cambridge, 1997. A statistical field theory approach.

[Ald91] David Aldous. The continuum random tree. I. *Ann. Probab.*, 19(1):1–28, 1991.

[Ald93] David Aldous. The continuum random tree. III. *Ann. Probab.*, 21(1):248–289, 1993.

[Ang03] Omer Angel. Growth and percolation on the uniform infinite planar triangulation. *Geom. Funct. Anal.*, 13(5):935–974, 2003.

[AS03] Omer Angel and Oded Schramm. Uniform infinite planar triangulations. *Comm. Math. Phys.*, 241(2-3):191–213, 2003.

[BBG11] Gaetan Borot, Jérémie Bouttier, and Emmanuel Guitter. A recursive approach to the $O(n)$ model on random maps via nested loops. *Preprint*, `arXiv:1106.0153`, 2011.

[BBI01] Dmitri Burago, Yuri Burago, and Sergei Ivanov. *A course in metric geometry*, volume 33 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2001.

[BC86] Edward A. Bender and E. Rodney Canfield. The asymptotic number of rooted maps on a surface. *J. Combin. Theory Ser. A*, 43(2):244–257, 1986.

[BC91] Edward A. Bender and E. Rodney Canfield. The number of rooted maps on an orientable surface. *J. Combin. Theory Ser. B*, 53(2):293–299, 1991.

[BCP03] Jean Bertoin, Loïc Chaumont, and Jim Pitman. Path transformations of first passage bridges. *Electron. Comm. Probab.*, 8:155–166 (electronic), 2003.

[BCR93] Edward A. Bender, E. Rodney Canfield, and L. Bruce Richmond. The asymptotic number of rooted maps on a surface. II. Enumeration by vertices and faces. *J. Combin. Theory Ser. A*, 63(2):318–329, 1993.

[BDG04] Jérémie Bouttier, Philippe Di Francesco, and Emmanuel Guitter. Planar maps as labeled mobiles. *Electron. J. Combin.*, 11(1):Research Paper 69, 27 pp. (electronic), 2004.

[Beg44] Edward G. Begle. Regular convergence. *Duke Math. J.*, 11:441–450, 1944.

[Ben91] Edward A. Bender. Some unsolved problems in map enumeration. *Bull. Inst. Combin. Appl.*, 3:51–56, 1991.

[Ber91] Jean Bertoin. Increase of a Lévy process with no positive jumps. *Stochastics Stochastics Rep.*, 37(4):247–251, 1991.

[Bet10a] Jérémie Bettinelli. Scaling limits for random quadrangulations of positive genus. *Electron. J. Probab.*, 15:no. 52, 1594–1644, 2010.

[Bet10b] Jérémie Bettinelli. The topology of scaling limits of positive genus random quadrangulations. *Preprint,* `arXiv:1012.3726`*, to appear in Ann. Probab.*, 2010.

[Bet11] Jérémie Bettinelli. Scaling limit of random planar quadrangulations with a boundary. *in preparation*, 2011.

[BG08] Jérémie Bouttier and Emmanuel Guitter. The three-point function of planar quadrangulations. *J. Stat. Mech. Theory Exp.*, (7):P07020, 39, 2008.

[BG09] Jérémie Bouttier and Emmanuel Guitter. Distance statistics in quadrangulations with a boundary, or with a self-avoiding loop. *J. Phys. A*, 42(46):465208, 44, 2009.

[Bil68] Patrick Billingsley. *Convergence of probability measures*. John Wiley & Sons Inc., New York, 1968.

[Bil95] Patrick Billingsley. Probability and measure. pages xiv+593, 1995. A Wiley-Interscience Publication.

[BIPZ78] Édouard Brézin, Claude Itzykson, Giorgio Parisi, and Jean-Bernard Zuber. Planar diagrams. *Comm. Math. Phys.*, 59(1):35–51, 1978.

[BMJ06] Mireille Bousquet-Mélou and Svante Janson. The density of the ISE and local limit laws for embedded trees. *Ann. Appl. Probab.*, 16(3):1597–1632, 2006.

[Bou05] Jérémie Bouttier. *Physique statistique des surfaces aléatoires et combinatoire bijective des cartes planaires*. PhD thesis, Université Paris VI, 2005.

[CD06] Philippe Chassaing and Bergfinnur Durhuus. Local limit of labeled trees and expected volume growth in a random quadrangulation. *Ann. Probab.*, 34(3):879–917, 2006.

[Cha10] Guillaume Chapuy. The structure of unicellular maps, and a connection between maps of positive genus and planar labeled trees. *Probab. Theory Related Fields*, 147(3-4):415–447, 2010.

[CMS09] Guillaume Chapuy, Michel Marcus, and Gilles Schaeffer. A bijection for rooted maps on orientable surfaces. *SIAM J. Discrete Math.*, 23(3):1587–1611, 2009.

[CS04] Philippe Chassaing and Gilles Schaeffer. Random planar lattices and integrated super-Brownian excursion. *Probab. Theory Related Fields*, 128(2):161–212, 2004.

[CV81] Robert Cori and Bernard Vauquelin. Planar maps are well labeled trees. *Canad. J. Math.*, 33(5):1023–1042, 1981.

[DLG02] Thomas Duquesne and Jean-François Le Gall. Random trees, Lévy processes and spatial branching processes. *Astérisque*, (281):vi+147, 2002.

[DS08] Bertrand Duplantier and Scott Sheffield. Liouville quantum gravity and KPZ. *Preprint,* `arXiv:0808.1560`, 2008.

[EK86] Stewart N. Ethier and Thomas G. Kurtz. *Markov processes*. Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics. John Wiley & Sons Inc., New York, 1986. Characterization and convergence.

[Eps66] David B. A. Epstein. Curves on 2-manifolds and isotopies. *Acta Math.*, 115:83–107, 1966.

[Fed69] Herbert Federer. *Geometric measure theory*. Die Grundlehren der mathematischen Wissenschaften, Band 153. Springer-Verlag New York Inc., New York, 1969.

[FPY93] Pat Fitzsimmons, Jim Pitman, and Marc Yor. Markovian bridges: construction, Palm interpretation, and splicing. In *Seminar on Stochastic Processes, 1992 (Seattle, WA, 1992)*, volume 33 of *Progr. Probab.*, pages 101–134. Birkhäuser Boston, Boston, MA, 1993.

[Gao93] Zhicheng Gao. The number of degree restricted maps on general surfaces. *Discrete Math.*, 123(1-3):47–63, 1993.

[GPW09] Andreas Greven, Peter Pfaffelhuber, and Anita Winter. Convergence in distribution of random metric measure spaces (Λ-coalescent measure trees). *Probab. Theory Related Fields*, 145(1-2):285–322, 2009.

[Gro99] Misha Gromov. *Metric structures for Riemannian and non-Riemannian spaces*, volume 152 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, 1999. Based on the 1981 French original [MR0682063 (85e:53051)], With appendices by M. Katz, P. Pansu and S. Semmes, Translated from the French by Sean Michael Bates.

[Kai76] William D. Kaigh. An invariance principle for random walk conditioned by a late return to zero. *Ann. Probability*, 4(1):115–121, 1976.

[Kin93] John F. C. Kingman. *Poisson processes*, volume 3 of *Oxford Studies in Probability*. The Clarendon Press Oxford University Press, New York, 1993. Oxford Science Publications.

[Kri05] Maxim Krikun. Local structure of random quadrangulations. *Preprint*, `arXiv:0512304`, 2005.

[LG91] Jean-François Le Gall. Brownian excursions, trees and measure-valued branching processes. *Ann. Probab.*, 19(4):1399–1439, 1991.

[LG93] Jean-François Le Gall. A class of path-valued Markov processes and its applications to superprocesses. *Probab. Theory Related Fields*, 95(1):25–46, 1993.

[LG94] Jean-François Le Gall. Master course: mouvement brownien, processus de branchement et superprocessus. *Available on* `http://www.math.u-psud.fr/~jflegall/`, 1994.

[LG99] Jean-François Le Gall. *Spatial branching processes, random snakes and partial differential equations*. Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 1999.

[LG05] Jean-François Le Gall. Random trees and applications. *Probab. Surv.*, 2:245–311 (electronic), 2005.

[LG07] Jean-François Le Gall. The topological structure of scaling limits of large planar maps. *Invent. Math.*, 169(3):621–670, 2007.

[LG10] Jean-François Le Gall. Geodesics in large planar maps and in the Brownian map. *Acta Math.*, 205(2):287–360, 2010.

[LG11] Jean-François Le Gall. Uniqueness and universality of the Brownian map. *Preprint*, `arXiv:1105.4842`, 2011.

[LGM11a] Jean-François Le Gall and Grégory Miermont. Scaling limits of random planar maps with large faces. *Ann. Probab.*, 39(1):1–69, 2011.

[LGM11b] Jean-François Le Gall and Grégory Miermont. Scaling limits of random trees and planar maps. *Preprint*, `arXiv:1101.4856`, *to appear in Clay Math. Proc.*, 2011.

[LGP08] Jean-François Le Gall and Frédéric Paulin. Scaling limits of bipartite planar maps are homeomorphic to the 2-sphere. *Geom. Funct. Anal.*, 18(3):893–918, 2008.

[LGW06] Jean-François Le Gall and Mathilde Weill. Conditioned Brownian trees. *Ann. Inst. H. Poincaré Probab. Statist.*, 42(4):455–489, 2006.

[Lig68] Thomas M. Liggett. An invariance principle for conditioned sums of independent random variables. *J. Math. Mech.*, 18:559–570, 1968.

[LZ04] Sergei K. Lando and Alexander K. Zvonkin. *Graphs on surfaces and their applications*, volume 141 of *Encyclopaedia of Mathematical Sciences*. Springer-Verlag, Berlin, 2004. With an appendix by Don B. Zagier, Low-Dimensional Topology, II.

[Mén10] Laurent Ménard. The two uniform infinite quadrangulations of the plane have the same law. *Ann. Inst. Henri Poincaré Probab. Stat.*, 46(1):190–208, 2010.

[Mie08] Grégory Miermont. On the sphericity of scaling limits of random planar quadrangulations. *Electron. Commun. Probab.*, 13:248–257, 2008.

[Mie09a] Grégory Miermont. Random maps and their scaling limits. In *Fractal geometry and stochastics IV*, volume 61 of *Progr. Probab.*, pages 197–224. Birkhäuser Verlag, Basel, 2009.

[Mie09b] Grégory Miermont. Tessellations of random maps of arbitrary genus. *Ann. Sci. Éc. Norm. Supér. (4)*, 42(5):725–781, 2009.

[Mie11] Grégory Miermont. The Brownian map is the scaling limit of uniform random plane quadrangulations. *Preprint*, `arXiv:1104.1606`, 2011.

[MM03] Jean-François Marckert and Abdelkader Mokkadem. States spaces of the snake and its tour—convergence of the discrete snake. *J. Theoret. Probab.*, 16(4):1015–1046 (2004), 2003.

[MM06] Jean-François Marckert and Abdelkader Mokkadem. Limit of normalized quadrangulations: the Brownian map. *Ann. Probab.*, 34(6):2144–2202, 2006.

[Moo25] Robert L. Moore. Concerning upper semi-continuous collections of continua. *Trans. Amer. Math. Soc.*, 27(4):416–428, 1925.

[MP10] Peter Mörters and Yuval Peres. *Brownian motion*. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, Cambridge, 2010. with an appendix by Oded Schramm and Wendelin Werner.

[MT01] Bojan Mohar and Carsten Thomassen. *Graphs on surfaces*. Johns Hopkins Studies in the Mathematical Sciences. Johns Hopkins University Press, Baltimore, MD, 2001.

[Nev86] Jacques Neveu. Arbres et processus de Galton-Watson. *Ann. Inst. H. Poincaré Probab. Statist.*, 22(2):199–207, 1986.

[Pet75] Valentin V. Petrov. Sums of independent random variables. pages x+346, 1975. Translated from the Russian by A. A. Brown, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 82.

[Pet95] Valentin V. Petrov. *Limit theorems of probability theory*, volume 4 of *Oxford Studies in Probability*. The Clarendon Press Oxford University Press, New York, 1995. Sequences of independent random variables, Oxford Science Publications.

[RY99] Daniel Revuz and Marc Yor. *Continuous martingales and Brownian motion*, volume 293 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, third edition, 1999.

[Sch70] Laurent Schwartz. *Analyse*. Hermann, Paris, 1970. Deuxième partie: Topologie générale et analyse fonctionnelle, Collection Enseignement des Sciences, No. 11.

[Sch98] Gilles Schaeffer. *Conjugaison d'arbres et cartes combinatoires aléatoires*. PhD thesis, Université de Bordeaux 1, 1998.

[Sch07] Oded Schramm. Conformally invariant scaling limits: an overview and a collection of problems. In *International Congress of Mathematicians. Vol. I*, pages 513–543. Eur. Math. Soc., Zürich, 2007.

[She72] Larry A. Shepp. Covering the line with random intervals. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 23:163–170, 1972.

[Str99] Daniel W. Stroock. *Probability theory, an analytic view*. Cambridge University Press, 1999.

[tH74] Gerardus 't Hooft. A planar diagram theory for strong interactions. *Nuclear Physics B*, 72(3):461–473, 1974.

[Tut62a] William T. Tutte. A census of Hamiltonian polygons. *Canad. J. Math.*, 14:402–417, 1962.

[Tut62b] William T. Tutte. A census of planar triangulations. *Canad. J. Math.*, 14:21–38, 1962.

[Tut62c] William T. Tutte. A census of slicings. *Canad. J. Math.*, 14:708–722, 1962.

[Tut63] William T. Tutte. A census of planar maps. *Canad. J. Math.*, 15:249–271, 1963.

[Ver79] Wim Vervaat. A relation between Brownian bridge and Brownian excursion. *Ann. Probab.*, 7(1):143–149, 1979.

[Why35a] Gordon T. Whyburn. On sequences and limiting sets. *Fund. Math.*, 25:408–426, 1935.

[Why35b] Gordon T. Whyburn. Regular convergence and monotone transformations. *Amer. J. Math.*, 57(4):902–906, 1935.

[WL72a] Timothy R. S. Walsh and Alfred B. Lehman. Counting rooted maps by genus. I. *J. Combinatorial Theory Ser. B*, 13:192–218, 1972.

[WL72b] Timothy R. S. Walsh and Alfred B. Lehman. Counting rooted maps by genus. II. *J. Combinatorial Theory Ser. B*, 13:122–141, 1972.

[WL75] Timothy R. S. Walsh and Alfred B. Lehman. Counting rooted maps by genus. III: Nonseparable maps. *J. Combinatorial Theory Ser. B*, 18:222–259, 1975.

Limite d'échelle de cartes aléatoires en genre quelconque

AU COURS DE CE TRAVAIL, nous nous intéressons aux limites d'échelle de deux classes de cartes. Dans un premier temps, nous regardons les quadrangulations biparties de genre $g \geq 1$ fixé et, dans un second temps, les quadrangulations planaires à bord dont la longueur du bord est de l'ordre de la racine carrée du nombre de faces. Nous voyons ces objets comme des espaces métriques, en munissant leurs ensembles de sommets de la distance de graphe, convenablement renormalisée.

Nous montrons qu'une carte prise uniformément parmi les cartes ayant n faces dans l'une de ces deux classes tend en loi, au moins à extraction près, vers un espace métrique limite aléatoire lorsque n tend vers l'infini. Cette convergence s'entend au sens de la topologie de Gromov–Hausdorff. On dispose de plus des informations suivantes sur l'espace limite que l'on obtient. Dans le premier cas, c'est presque sûrement un espace de dimension de Hausdorff 4 homéomorphe à la surface de genre g. Dans le second cas, c'est presque sûrement un espace de dimension 4 avec une frontière de dimension 2, homéomorphe au disque unité de $\mathbb{R}^2$. Nous montrons en outre que, dans le second cas, si la longueur du bord est un petit o de la racine carrée du nombre de faces, on obtient la même limite que pour les quadrangulations sans bord, c'est-à-dire la *carte brownienne*, et l'extraction n'est plus requise.

Mots-clefs: cartes aléatoires, arbres aléatoires, limite d'échelle, processus conditionnés, convergence régulière, topologie de Gromov, dimension de Hausdorff, arbre continu brownien, espaces métriques aléatoires.
Classification AMS: 60F17, 60D05, 57N05, 60C05.

Scaling Limit of Arbitrary Genus Random Maps

IN THIS WORK, we discuss the scaling limits of two particular classes of maps. In a first time, we address bipartite quadrangulations of fixed genus $g \geq 1$, and, in a second time, planar quadrangulations with a boundary whose length is of order the square root of the number of faces. We view these objects as metric spaces by endowing their sets of vertices with the graph metric, suitably rescaled.

We show that a map uniformly chosen among the maps having n faces in one of these two classes converges in distribution, at least along some subsequence, toward a limiting random metric space as n tends to infinity. This convergence holds in the sense of the Gromov–Hausdorff topology on compact metric spaces. We moreover have the following information on the limiting space. In the first case, it is almost surely a space of Hausdorff dimension 4 that is homeomorphic to the genus g surface. In the second case, it is almost surely a space of Hausdorff dimension 4 with a boundary of Hausdorff dimension 2 that is homeomorphic to the unit disc of $\mathbb{R}^2$. We also show that in the second case, if the length of the boundary is little-o of the square root of the number of faces, the same convergence holds without extraction and the limit is the same as for quadrangulations without boundary, that is the *Brownian map*.

Key words and phrases: random maps, random trees, scaling limits, conditioned processes, regular convergence, Gromov topology, Hausdorff dimension, Brownian CRT, random metric spaces.
AMS classification: 60F17, 60D05, 57N05, 60C05.