Habilitation Thesis
From disordered elastic systems to the statistics of rare events

G. Schehr

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CNRS-Université Paris Sud-XI, Orsay

February 1, 2011

Overview of the main topics

1. Disordered elastic systems
   - Dynamics at the depinning transition of an elastic line
   - Solid on Solid model on a disordered substrate
2. Persistence properties
   - Persistence of the global magnetization at a critical point
   - Persistence and random polynomials
   - Beyond persistence : statistics of the largest excursion
3. Extreme value statistics
   - Extreme statistics of random walks
   - Fluctuating interfaces and the Airy distribution
   - Vicious walkers and random matrices

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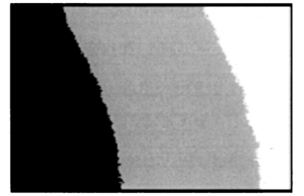
Disordered elastic systems

- Relevant in many physical situations
  - Domain walls in magnets
  - Contact lines in wetting
  - Fracture experiments
  - Vortex lattice in superconductors
  - . . .



Lemerle *et al.* PRL 1998

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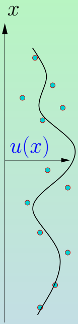

- Models : elasticity + disorder

$$H[u] = \underbrace{\frac{c}{2}\int [\nabla u(r)]^2 d^d r}_{\text{elasticity}} + \underbrace{\int V_{\rm dis}(u(r), r) d^d r}_{\text{disorder}}$$

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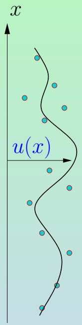

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$$H[u] = \underbrace{\frac{c}{2}\int [\nabla u(r)]^2 d^d r}_{\text{elasticity}} + \underbrace{\int V_{\text{dis}}(u(r), r) d^d r}_{\text{disorder}} \Longrightarrow \text{complex energy landscape}$$

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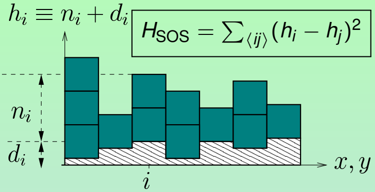
Solid on solid (SOS) model on a disordered substrate



$$H_{SOS} = \sum_{\langle ij \rangle} (h_i - h_j)^2$$

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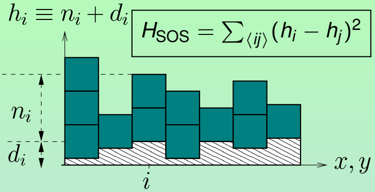

Sine Gordon + random phases

$$H_{SG} = \int d^2 r (\nabla h(r))^2 - \lambda \cos\left[2\pi(h(r) - d(r))\right]$$

- $h(r) \in]-\infty, +\infty[$
- $d(r) \in [0, 1]$ quenched var.

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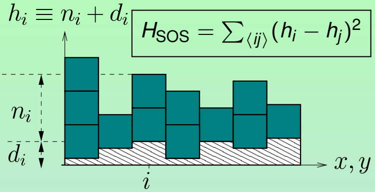

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Ground state ($T = 0$) properties

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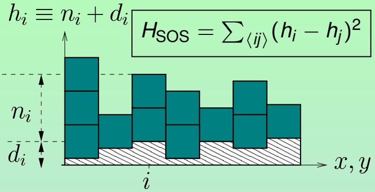

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Ground state ($T = 0$) properties

- Superrough interface — Le Doussal, G.S., PRB 07
  - $\overline{(h(r) - h(0))^2} \sim A\,(\log r)^2$
  - via Functional Renormalization Group in $d = 4 - \epsilon$
  - good agreement with exact numerics

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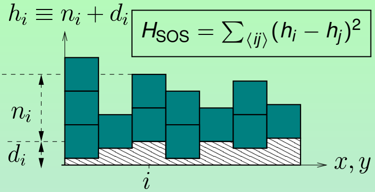

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  - good agreement with exact numerics
- Disorder chaos — K. Schwarz, A. Karrenbauer, G.S., H. Rieger, JSTAT 09
  - extreme sensitivity when $d_i \to d_i + \delta d_i$, $\delta \ll 1$
  - numerical study via exact ground state calculations

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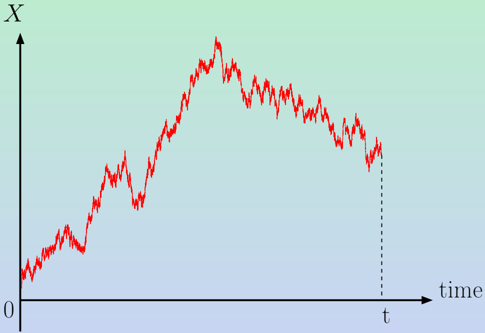
Persistence properties

Persistence probability $p_0(t)$

- $X(t) \equiv$ stochastic random var. evolving in time t, $\langle X(t) \rangle = 0$
- Persistence probability: $p_0(t) \equiv$ Proba. that $X(\cdot)$ has not changed sign up to time t



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Persistence in physical systems

- coarsening spin field at low T
- global mag. of ferro. at T_c
- height of a fluctuating interfaces
- diffusion field
- ...

$$p_0(t) \propto t^{-\theta_p}$$

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Persistence for the diffusion equation

Diffusion equation with random initial conditions

$$\partial_t \phi(x,t) = \nabla^2 \phi(x,t)$$
$$\langle \phi(x,0)\phi(x',0)\rangle = \delta^d(x-x')$$

Single length scale
$$\ell(t) \propto t^{1/2}$$

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Persistence $p_0(t,L)$ for a d-dim. system of linear size L

$p_0(t,L) \equiv$ Proba. that $\phi(x,t)$ has not changed sign up to t

S. N. Majumdar, C. Sire, A. J. Bray and S. J. Cornell, PRL 96

B. Derrida, V. Hakim and R. Zeitak, PRL 96

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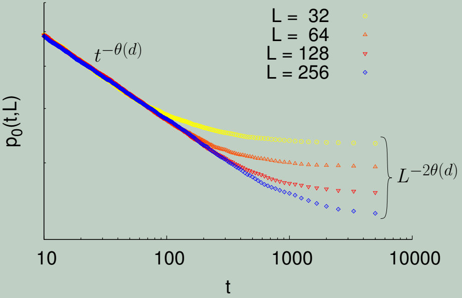

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Persistence $p_0(t,L)$ for a d-dim. system of linear size L

$p_0(t,L) \equiv$ Proba. that $\phi(x,\cdot)$ has not changed sign up to t

$$p_0(t,L) \propto L^{-2\theta(d)} h(t/L^2)$$

$$\begin{aligned} \theta(1) &= 0.1207 \\ \theta(2) &= 0.1875 \quad , \quad \text{Numerics} \end{aligned}$$

Real Kac's polynomials

$$K_n(x) = \sum_{i=0}^{n} a_i \, x^i$$

$a_i \equiv$ Gaussian random variables, $\langle a_i \rangle = 0$, $\langle a_i a_j \rangle = \delta_{ij}$

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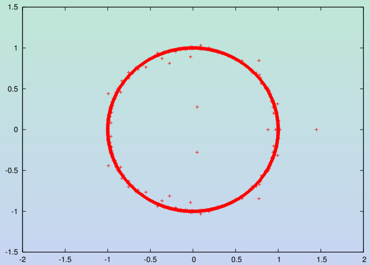
Complex roots



Real random polynomials

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Real roots

$\mathcal{N}_n \equiv$ mean number of roots on the real axis M. Kac 43

$$\mathcal{N}_n \sim \frac{2}{\pi} \log n$$

Real roots of Kac's polynomials

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AMERICAN MATHEMATICAL SOCIETY
Volume 15, Number 4, Pages 857–892
S 0894-0347(02)00386-7
Article electronically published on May 16, 2002

RANDOM POLYNOMIALS HAVING FEW OR NO REAL ZEROS

AMIR DEMBO, BJORN POONEN, QI-MAN SHAO, AND OFER ZEITOUNI

$q_0(n) \equiv$ Probability that $K_n(x)$ has no real root in $[0, 1]$

$$q_0(n) \propto n^{-\gamma}$$

with $\gamma = 0.19(1)$ (Numerics)

Connection between random polynomials & diffusion equation

Generalized Kac's polynomials

$$K_n(x) = a_0 + \sum_{i=1}^{n} a_i \, i^{(d-2)/4} x^i$$

$a_i \equiv$ Gaussian random variables, $\langle a_i \rangle = 0$, $\langle a_i a_j \rangle = \delta_{ij}$

Proba. of no real root

$$q_0(n) \propto n^{-b(d)}$$

Persistence of diffusion

$$p_0(t \gg 1, L) \propto L^{-2\theta(d)}$$

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$$b(d) = \theta(d)$$

G. S., S. N. Majumdar, PRL 07 & J. Stat. Phys. 08

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Statement of the problem

- $\{X_1, X_2, ..., X_N\} \equiv N$ random variables, $P_{\rm joint}(X_1, ..., X_N)$
- $X_{\max} = \max\{X_i\}_{1 \leq i \leq N}$
- Q: $F_N(\xi) \equiv {\rm Probability}[X_{\max} \leq \xi] = ?$

Extreme Value Statistics (EVS)

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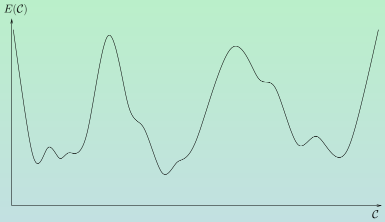
Well known problem in meteorology, engineering,...

- Disordered and complex systems : "rough" energy landscape



  * Low T statics : thermodynamics is dominated by the minima of free energy
  * Low T dynamics : long time dynamics is dominated by the largest free energy barriers

I.i.d random variables : 3 universality classes

Statement of the problem

- $\{X_1, X_2, ..., X_N\} \equiv N$ i.i.d. random variables, $P_0(X)$
- Q: $F_N(\xi) \equiv \mathrm{Probability}[X_{\max} \leq \xi] = ?$

Three universality classes depending on $P_0(X \to \infty)$

i) Faster than any power law (Gumbel)

$$F^*(\xi) = e^{e^{-\xi}}$$

ii) Power law (Fréchet)

$$F^*(\xi) = \begin{cases} 0 & \xi \leq 0 \\ e^{-\xi^{-\alpha}} & \xi > 0 \end{cases}$$

iii) Finite cutoff (Weibull)

$$F^*(\xi) = \begin{cases} e^{-(-\xi)^{-\alpha}} & \xi \leq 0 \\ 1 & \xi > 0 \end{cases}$$

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Beyond the paradigm of uncorrelated and indentical variables

$\rightarrow$ Emergence of universality classes ?

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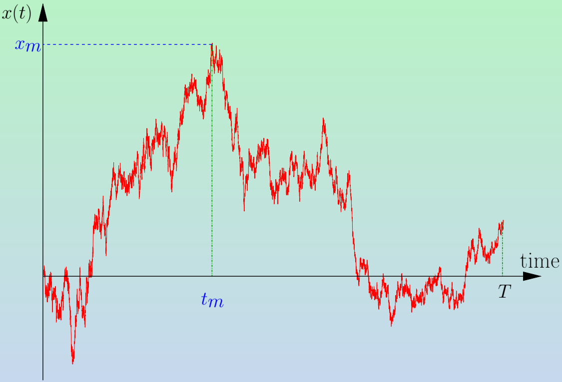
EVS of random walks



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$$x(t_m) = x_m$$

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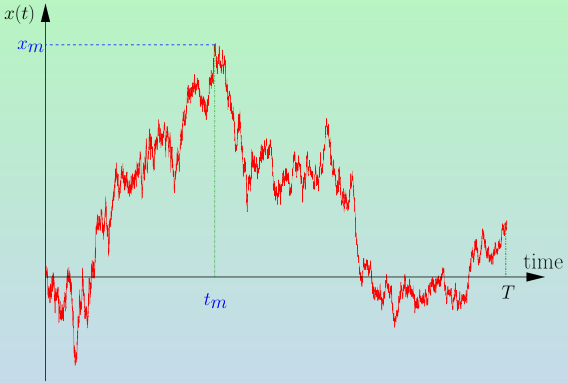

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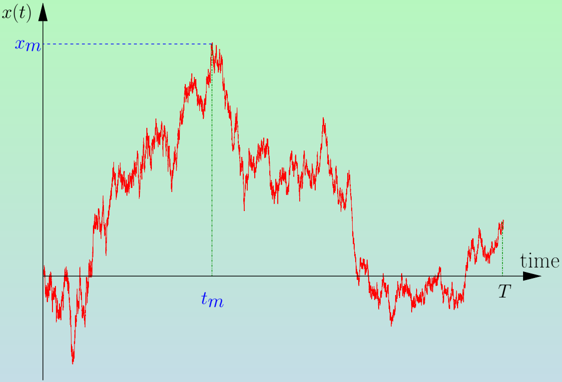

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What are the probability distribution functions of x_m, t_m ?

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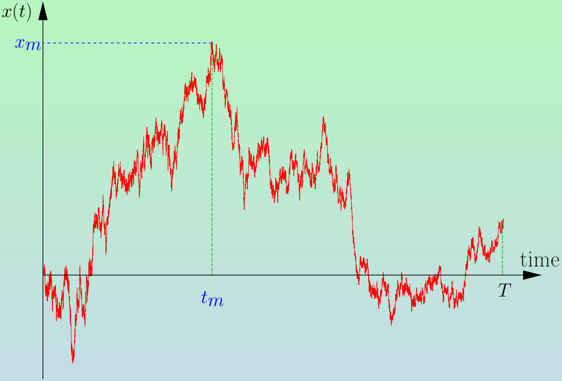

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Exact results for the joint p.d.f. of x_m, t_m via Real Space RG for:

- Brownian motion
- Bessel process (radius of the *d*-dim. Brownian motion)
- Continuous Time Random Walks

G. S., P. Le Doussal, JSTAT 10

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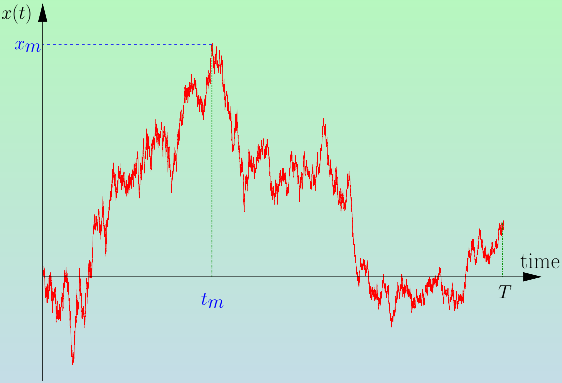

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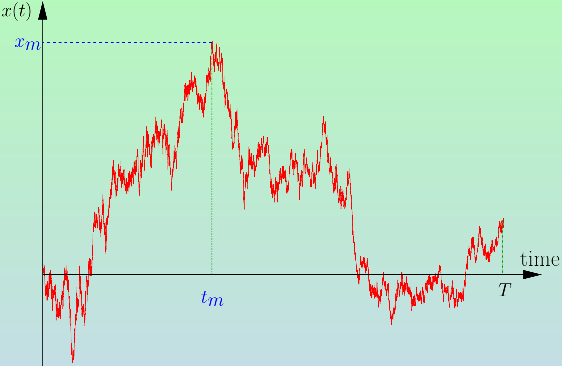

$$x_m = \max_{0 \leq t \leq T} x(t)$$

$$x(t_m) = x_m$$

Exact results for the joint p.d.f. of x_m, t_m for a Bessel bridge:

$$P_T(x_M, t_M) = \frac{8T^{\nu+1}}{\Gamma(1+\nu) x_M^{5+2\nu}} \sum_{k,l=1}^{\infty} \frac{j_{\nu,k}^{\nu+1} j_{\nu,l}^{\nu+1}}{J_{\nu+1}(j_{\nu,k}) J_{\nu+1}(j_{\nu,l})} e^{-\frac{j_{\nu,k}^2 t_M}{x_M^2}} e^{-\frac{j_{\nu,l}^2 (T-t_M)}{x_M^2}}$$

with $\nu = -1 + d/2$

G. S., P. Le Doussal, JSTAT 10

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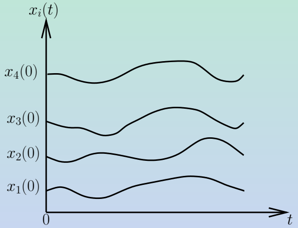
Non-intersecting Brownian motions in $1d$

- N Brownian motions in one-dimension

$$\dot{x}_i(t) = \zeta_i(t)\ , \ \langle \zeta_i(t) \zeta_j(t') \rangle = \delta_{i,j} \delta(t-t')$$
$$x_1(0) < x_2(0) < ... < x_N(0)$$

- Non-intersecting condition

$$x_1(t) < x_2(t) < ... < x_N(t)\ , \quad \forall t \geq 0$$



Vicious walkers in physics

THE JOURNAL OF CHEMICAL PHYSICS VOLUME 48, NUMBER 5 1 MARCH 1968

Soluble Model for Fibrous Structures with Steric Constraints

P.-G. de Gennes

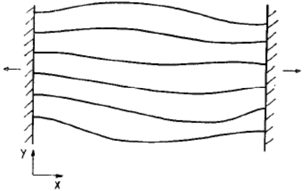


Fig. 1. Model for a two-dimensional fiber structure. The component chains are assumed to be attached to two plates I and F and placed under tension. The chains are bent by thermal fluctuations. Different chains cannot intersect each other.

Watermelons configurations

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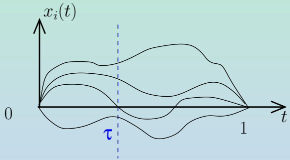


watermelons

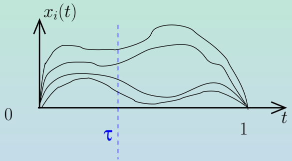
Watermelons configurations

- N Brownian motions in one-dimension

$$\dot{x}_i(t) = \zeta_i(t) \ , \ \langle \zeta_i(t) \zeta_j(t') \rangle = \delta_{i,j} \delta(t - t')$$
$$x_1(0) < x_2(0) < ... < x_N(0)$$

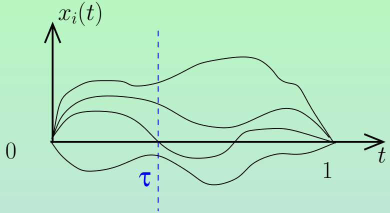
- Non-intersecting condition

$$x_1(t) < x_2(t) < ... < x_N(t) \ , \ \forall t \geq 0$$



watermelons
"with a wall"

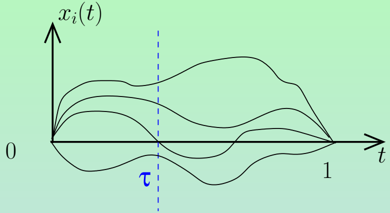
Watermelons and random matrices



- Joint probability of $x_1(\tau), x_2(\tau), \cdots, x_N(\tau)$ at fixed time τ

Watermelons and random matrices

G. S., S. N. Majumdar, A. Comtet, J. Randon-Furling PRL 08



- Joint probability of $x_1(\tau), x_2(\tau), \cdots, x_N(\tau)$ at fixed time τ

$$P_{\rm joint}(x_1, x_2, \cdots, x_N, \tau) \propto \sigma(\tau)^{-N^2} \prod_{i<j=1}^{N} (x_i - x_j)^2 e^{-\frac{1}{\sigma^2(\tau)} \sum_{i=1}^N x_i^2}$$

$$\sigma(\tau) = \sqrt{2\tau(1-\tau)}$$

- The rescaled positions $\frac{x_i}{\sigma(\tau)}$ are distributed like the eigenvalues of random matrices of Gaussian Unitary Ensemble (GUE, $\beta = 2$)

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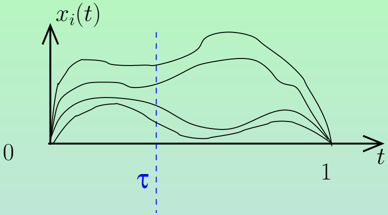


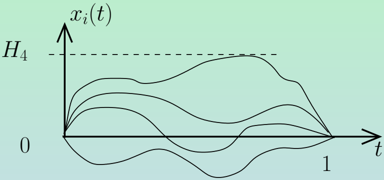
- At fixed time τ

$$P_{\text{joint}}(\mathbf{x}, \tau) \propto \sigma(\tau)^{-N(2N+1)} \prod_{i=1}^{N} x_i^2 \prod_{1 \leq i < j \leq N} (x_i^2 - x_j^2)^2 e^{-\frac{\mathbf{x}^2}{\sigma^2(\tau)}}$$

$$\sigma(\tau) = \sqrt{2\tau(1-\tau)}$$

- $y_i = \frac{x_i^2}{2\sigma^2(\tau)}$ are distributed like the eigenvalues of Wishart matrices, with $\beta = 2$ and $M - N = \frac{1}{2}$

Extreme statistics of vicious walkers



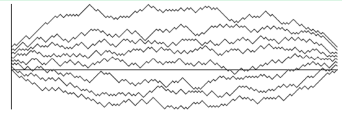
Maximal height of watermelons

$$x_1(t) < x_2(t) < ... < x_N(t)$$
$$H_N = \max_\tau [x_N(\tau), 0 \leq \tau \leq 1]$$
$$\langle H_N \rangle \ = ?$$

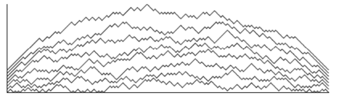
Main results for $\langle H_N \rangle$

- Connection between watermelons and random matrices

$\Longrightarrow$ exact asymptotic results for $\langle H_N \rangle$ for $N \gg 1$



$$\langle H_N \rangle \sim \sqrt{N}$$



$$\langle H_N \rangle \sim \sqrt{2N}$$

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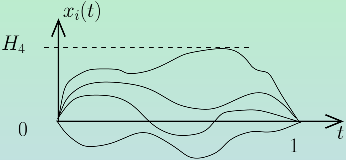
Extreme statistics of vicious walkers



Cumulative distribution

$$x_1(t) < x_2(t) < \ldots < x_N(t)$$
$$H_N = \max_{\tau}[x_N(\tau), 0 \leq \tau \leq 1]$$
$$F_N(M) = \mathrm{Proba}[H_N \leq M] \ = ?$$

Distribution of the maximal height : with a wall

- Using a path integral approach for free fermions

  G. S., S. N. Majumdar, A. Comtet, J. Randon-Furling PRL 08

$$F_N(M) = \frac{A_N}{M^{2N^2+N}} \sum_{n_1,\cdots,n_N=0}^{+\infty} \prod_{i=1}^{N} n_i^2 \prod_{1\leq j<k\leq N} (n_j^2 - n_k^2)^2 e^{-\frac{\pi^2}{2M^2}\sum_{i=1}^N n_i^2}$$

$$A_N = \frac{\pi^{2N^2+N}}{2^{N^2-N/2}\prod_{j=0}^{N-1}\Gamma(2+j)\Gamma(\frac{3}{2}+j)} \text{ cf Selberg integral}$$

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What about the asymptotic behavior of $F_N(M)$ for $N \to \infty$?

- Using a path integral approach for free fermions

G. S., S. N. Majumdar, A. Comtet, J. Randon-Furling PRL 08

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What about the asymptotic behavior of $F_N(M)$ for $N \to \infty$?

$\Longrightarrow$ Connection with Yang-Mills theory on the sphere

P. J. Forrester, S. N. Majumdar, G. S., Nucl. Phys. B 11

Partition function of YM_2 in 2d

- Partition function of Yang-Mills theory on a 2*d* manifold $\mathcal{M}$ with a gauge group G, described by a gauge field $A_\mu(x) \equiv A_\mu^a(x) T^a$

$$\mathcal{Z}_\mathcal{M} = \int [\mathcal{D}A_\mu] e^{-\frac{1}{4\lambda^2} \int \mathrm{Tr}[F^{\mu\nu} F_{\mu\nu}] \sqrt{g} d^2 x}$$

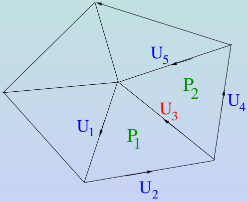
$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu]$$

Ex: $G = SU(2)$: electro-weak interaction, $G = SU(3)$: chromodynamics

- Regularization on the lattice, *e.g.* $G = \mathrm{U}(N)$

$$\mathcal{Z}_\mathcal{M} = \int \prod_L dU_L \prod_{\text{plaquettes}} Z_P[U_P]$$

$$U_P = \prod_{L \in \text{plaquette}} U_L$$



Heat-kernel action

$$\mathcal{Z}_{\mathcal{M}} = \int \prod_L dU_L \prod_{\text{plaquettes}} Z_P[U_P]$$

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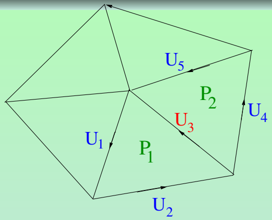

- A common choice : Wilson's action Wilson'74

$$Z_P(U_P) = \exp\left[bN\,\mathrm{Tr}(U_P + U_P^\dagger)\right]$$

- Alternative choice : invariance under decimation $\Rightarrow$ Migdal's recursion relation

$$\int dU_3\, Z_{P_1}(U_1 U_2 U_3) Z_{P_2}(U_4 U_5 U_3^\dagger) = Z_{P_1+P_2}(U_1 U_2 U_4 U_5)$$

Heat-kernel action

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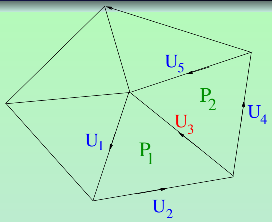

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$$Z_P = \sum_R d_R \chi_R(U_P) \exp\left[-\frac{A_P}{2N} C_2(R)\right]$$

Migdal'75, Rusakov'90

Partition function of YM_2 on the sphere

- Exact formula for the partition function computed with the heat-kernel action

$$\mathcal{Z}_{\mathcal{M}} = \sum_R d_R^{2-2g} \exp\left[-\frac{A}{2N} C_2(R)\right]$$

- If the gauge group $G = \mathrm{Sp(2N)}$ and $\mathcal{M} = S^2$ (sphere in $\mathbb{R}^3$)

$$\mathcal{Z}_{\mathcal{M}} = \hat{c}_N \, e^{A\,(N+\frac{1}{2})\,\frac{N+1}{12}} \sum_{n_1,\ldots,n_N=0}^{\infty} \left(\prod_{j=1}^{N} n_j^2\right) \prod_{i<j} (n_i^2 - n_j^2)^2 \, e^{-\frac{A}{4N}\sum_{j=1}^{N} n_j^2}$$

Correspondence between YM_2 on the sphere and watermelons

- Partition function of YM_2 on the sphere with gauge group $\mathrm{Sp}(2N)$

$$\mathcal{Z}_{\mathcal{M}} = \mathcal{Z}(A; \mathrm{Sp}(2N))$$

$$\mathcal{Z}(A; \mathrm{Sp}(2N)) = \hat{c}_N \, e^{A\,(N+\frac{1}{2})\,\frac{N+1}{12}} \sum_{n_1,\ldots,n_N=0}^{\infty} \left(\prod_{j=1}^{N} n_j^2 \right) \prod_{i<j} (n_i^2 - n_j^2)^2 \, e^{-\frac{A}{4N}\sum_{j=1}^{N} n_j^2}$$

- Cumulative distribution of the maximal height of watermelons with a wall

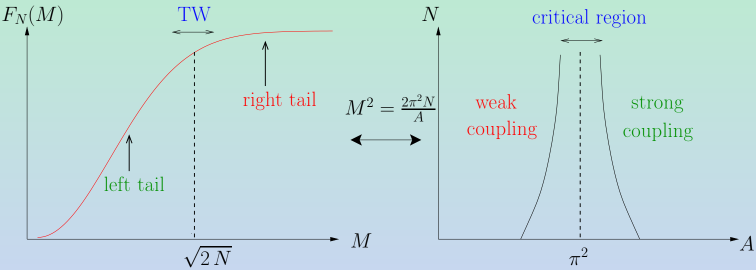
$$F_N(M) = \frac{A_N}{M^{2N^2+N}} \sum_{n_1,\cdots,n_N=0}^{+\infty} \left(\prod_{j=1}^{N} n_j^2 \right) \prod_{i<j} (n_i^2 - n_j^2)^2 \, e^{-\frac{\pi^2}{2M^2}\sum_{j=1}^{N} n_j^2}$$

$$\propto \; \mathcal{Z}\left(A = \frac{2\pi^2 N}{M^2}; \mathrm{Sp}(2N) \right)$$

P. J. Forrester, S. N. Majumdar, G.S. '10

Large N limit of YM_2 and consequences for $F_N(M)$

- Weak-strong coupling transition in YM_2 — Durhuus-Olesen '81, Douglas-Kazakov '93



Large N limit of YM_2 and consequences for $F_N(M)$

- In the critical regime, "double-scaling limit", one shows

$$\frac{d^2}{dt^2} \log F_N\left(\sqrt{2N}(1 + t/(2^{7/3}N^{2/3}))\right) = -\frac{1}{2}\left(q^2(t) + q'(t)\right)$$
$$q''(t) = 2q^3(t) + tq(t) \; , \; q(t) \sim \mathrm{Ai}(t) \; , \; t \to \infty$$

i.e.

$$F_N(M) \to \mathcal{F}_1\left(2^{11/6} N^{1/6} \left|M - \sqrt{2N}\right|\right)$$
$$\mathcal{F}_1(t) = \exp\left(-\frac{1}{2}\int_t^\infty \left((s-t)\, q^2(s) - q(s)\right)\, ds\right)$$
$$\equiv$$ Tracy-Widom distribution for $\beta = 1$

Conclusion

1. Disordered elastic systems
   - Dynamics at the depinning transition of an elastic line
   - Solid on Solid model on a disordered substrate
2. Persistence properties
   - Persistence of the global magnetization at a critical point
   - Persistence and random polynomials
   - Beyond persistence : statistics of the largest excursion
3. Extreme value statistics
   - Extreme statistics of random walks via real space RG
   - Fluctuating interfaces and the Airy distribution
   - Vicious walkers, random matrices and Yang-Mills theories

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     $\Longrightarrow$ Application to a toy model of disordered elastic system directed polymer in random media

J. Rambeau, G. S., EPL 10