Analyse statistique des processus de marche aléatoire multifractale

Laurent Duvernet

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Laurent Duvernet

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Jury

M. Emmanuel Bacry	Directeur
M. Julien Barral	Rapporteur
M. Marc Hoffmann	Directeur
Mme Carenne Ludeña	Rapporteure
...	

Résumé

On étudie certaines propriétés d'une classe de processus aléatoires réels à temps continu, les marches aléatoires multifractales. Une particularité remarquable de ces processus tient en leur propriété d'autosimilarité : la loi du processus à petite échelle est identique à celle à grande échelle moyennant un facteur *aléatoire* et indépendant du processus.

La première partie de la thèse se consacre à la question de la convergence du moment empirique de l'accroissement du processus dans une asymptotique assez générale, où le pas de l'accroissement peut tendre vers zéro en même temps que l'horizon d'observation tend vers l'infini.

La deuxième partie propose une famille de tests non-paramétriques qui distinguent entre marches aléatoires multifractales et semi-martingales d'Itō. Après avoir montré la consistance de ces tests, on étudie leur comportement sur des données simulées.

On construit dans la troisième partie un processus de marche aléatoire multifractale asymétrique tel que l'accroissement passé soit négativement corrélé avec le carré de l'accroissement futur. Ce type d'« effet levier » est notamment observé sur les prix d'actions et d'indices financiers. On compare les propriétés empiriques du processus obtenu avec des données réelles.

La quatrième partie concerne l'estimation des paramètres du processus dans un cas gaussien. On commence par montrer que sous certaines conditions, deux des trois paramètres ne peuvent être estimés. On étudie ensuite les performances théoriques et empiriques de différents estimateurs du troisième paramètre, le coefficient d'intermittence.

Mots-clés : marches aléatoires multifractales ; invariance d'échelle ; semi-martingales ; effet levier ; coefficient d'intermittence.

Abstract

We study some properties of a class of real, continuous-time random processes, multifractal random walks. A striking feature of this processes lie in their scaling property: the distribution of the process at small scale is the same as the distribution at large scale, given some *random* factor independent of the process.

The first part of the dissertation concerns the convergence of the empirical moment of the increment of the process, in a rather general asymptotic setting where the step of the increment may go to zero while the observation horizon may also go to infinity.

In the second part, we propose a family of nonparametric tests that separate multifractal random walks from Itō semi-martingales. After showing the consistency of these tests, we study their behavior on simulations.

In the third part, we build a skewed multifractal random walk process, such that the past increment is negatively correlated with the future squared increment. Such a "leverage effect" is notably seen on financial stock and index prices. We compare the empirical properties of the obtained process with real data.

The fourth part deals with the parametric estimation of the process in a Gaussian case. We first show that under certain conditions, one can not estimate two of the three parameters, even if the sample path is continuously observed on some interval. We next study the theoretical and empirical performances of some estimators of the third parameter, the intermittency coefficient.

Keywords: multifractal random walks ; scaling properties ; semi-martingales ; leverage effect ; intermittency coefficient.

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Introduction

0.1 Cadre de la thèse

Ce travail se consacre à l'étude des processus aléatoires multifractals, et plus particulièrement de certaines propriétés de la classe des marches aléatoires multifractales introduites par Bacry, Delour et Muzy [74, 11], appelées MRW pour *Multifractal Random Walks* dans le reste du texte. La thèse se compose de quatre parties, qui traitent successivement : du comportement des moments empiriques du processus MRW dans un cadre asymptotique général qui combine haute fréquence et temps d'observation grand, d'une procédure de test non-paramétrique pour reconnaître si des données ont été produites par un processus MRW ou par une classe de processus aléatoires usuelle (les semi-martingales d'Itō), de l'incorporation au sein du modèle MRW de certaines asymétries observées sur les données financières, et de la question de l'estimation statistique des paramètres du processus MRW à partir d'observations discrètes. Plus précisément, on tâche d'apporter des éléments de réponse aux questions suivantes :

1. Comment se comporte la moyenne empirique des accroissements absolus du processus pris à une certaine puissance $p > 0$? La réponse à cette question, en apparence simple, permet notamment de mettre en évidence un caractère relativement inhabituel du modèle : on montre que selon le cadre asymptotique et l'exposant p retenus, le moment empirique de l'accroissement du processus MRW peut ou bien converger vers le moment théorique, ou bien diverger, ou bien même converger vers une limite aléatoire non dégénérée. En effet, les accroissements sont stationnaires mais non

indépendants, et on est loin du domaine de validité de la loi des grands nombres. Certains des résultats que nous prouvons sont très proches notamment de ceux d'Ossiander et Waymire [75] qui ont effectué un travail similaire dans le cadre de processus à la construction semblable mais relativement plus simple que celle des MRW, en l'occurence les cascades de Mandelbrot. Malgré la proximité de nos résultats, nous sommes amenés à développer des techniques de démonstration originales du fait de la difficulté accrue posée par les processus MRW. Une étude également très proche — mais là aussi avec un objet légèrement différent et des techniques de démonstration spécifiques — se trouve dans les articles de Ludeña [62, 63]. Par ailleurs, la question de la valeur asymptotique du moment empirique de l'accroissement du processus est une question importante en analyse multifractale puisqu'elle est liée à celle de la régularité ponctuelle des trajectoires du processus par la conjecture de Frisch-Parisi [43]. Sur ce point, nous nous plaçons notamment dans la lignée du travail de Bacry, Gloter, Hoffmann et Muzy [8] qui généralisent les résultats d'Ossiander et Waymire [75] sur les cascades de Mandelbrot à un cadre asymptotique « mixte » où l'horizon d'observation croît vers $+\infty$ en même temps que le pas de l'accroissement tend vers 0. Comme le montrent ces auteurs, les régimes de convergence sont alors modifiés de manière non triviale et on voit apparaître de nouveaux niveaux de régularités ponctuelles pour les trajectoires du processus par rapport à la configuration usuelle de la conjecture de Frisch-Parisi avec horizon d'observation fixe. Nous montrons qu'on peut adapter ce cadre asymptotique mixte à l'étude des processus MRW, ce qui permet dans une certaine mesure d'obtenir des conclusions analogues à celles de Bacry, Gloter, Hoffmann et Muzy [8] concernant les cascades de Mandelbrot.

Ce travail est l'objet d'un article paru dans *Stochastic Analysis and Applications* [40].

2. Parmi les processus à temps continu, la classe des semi-martingales d'Itō est indéniablement centrale tant du fait de ses propriétés théoriques que par ses applications pratiques ; or les processus MRW ont quant à eux la propriété inhabituelle d'être des martingales continues qui ne sont pas Itō (leur processus de variation quadratique

n'est pas absolument continu). Peut-on, à l'aide de l'observation en haute fréquence d'une trajectoire, reconnaître si celle-ci relève de l'une ou l'autre de ces deux classes de processus ? Notre réponse à cette question s'appuie notamment sur le travail effectué par Aït-Sahalia et Jacod [2, 3, 4], dans lequel ces auteurs proposent des familles de tests non-paramétriques pour distinguer entre de grandes catégories de semi-martingales d'Itō (notamment continues ou à sauts). Ils utilisent en particulier les propriétés asymptotiques des moments empiriques des accroissements d'une semi-martingale d'Itō, qui diffèrent selon que les trajectoires de la semi-martingale d'Itō sont continues ou non, mais qui diffèrent également selon que le processus de semi-martingale est Itō ou MRW. Nous prolongeons donc l'approche d'Aït-Sahalia et Jacod et nous proposons une famille de tests non-paramétriques convergents qui s'appuient sur le comportement des moments empiriques d'un processus MRW établi au Chapitre 1 pour distinguer entre semi-martingales d'Itō et processus MRW.

Ce travail est l'objet d'un article écrit en collaboration avec Christian Y. Robert et Mathieu Rosenbaum et qui a été soumis pour publication à l'*Electronic Journal of Statistics* [41].

3. Une des caractéristiques remarquables des processus MRW consiste en ce qu'ils permettent de reproduire de manière fine l'essentiel des régularités statistiques observées sur les cours d'actifs financiers. Cependant, une de ces régularités leur échappe : en l'occurence un type spécifique d'asymétrie appelé « effet levier » et observé principalement sur les séries de rendements d'actions ou d'indices. Cette asymétrie est double puisqu'il s'agit d'une part de la *skewness* négative observée sur la distribution d'un rendement à une date donnée, et d'autre part d'une asymétrie dans le temps : la volatilité (définie comme le carré des rendements) est négativement corrélée avec le rendement d'hier, mais décorrélée du rendement de demain. À la suite de Pochart et Bouchaud [79] qui ont proposé un modèle multifractal asymétrique à temps discret, on construit un modèle à temps continu asymétrique qui incorpore les propriétés saillantes des processus MRW.

   Ce travail est l'objet d'un article rédigé avec Emmanuel Bacry et Jean-François

Muzy et qui a été soumis pour publication à l'*International Journal of Theoretical and Applied Finance* [7].

4. On parle de processus MRW log-normal dans le cas particulier où la construction du processus fait intervenir une loi gaussienne ; ce cas est privilégiée dans les applications, comme la finance ou la turbulence, notamment du fait de sa relative simplicité. La loi du processus est alors caractérisée par trois paramètres : comment les estimer dans la pratique sur un jeu de données ? Cette question a déjà été abordée par certains auteurs dans un cadre identique ou analogue : citons d'une part les travaux de Jaffard, Lashermes, Abry et Wendt [53, 94, 95, 93], d'autre part ceux de Ludeña [62, 63], et enfin ceux de Bacry, Kozhemyak et Muzy [10] — chacun proposant des procédures d'estimation spécifiques. On cherche par conséquent à prolonger ces différentes approches et à les comparer entre elles. Nous montrons ainsi que même dans le cas idéal où l'on observerait continûment une trajectoire du processus sur un intervalle de temps de longueur fixée (inférieure à un certain seuil), deux des paramètres ne peuvent être identifiés de manière certaine. On ne peut alors construire de procédure d'estimation convergente à l'aide d'une observation du processus sur un horizon borné, même avec un pas de discrétisation qui tendrait vers zéro. On se consacre donc par la suite à la question de l'estimation du troisième paramètre, dit *coefficient d'intermittence*, et on envisage plusieurs estimateurs, dont trois sont définis dans les travaux mentionnés ci-dessus, et dont un est à notre connaissance utilisé pour la première fois dans ce contexte. On étudie les vitesses de convergences théoriques de ces différents estimateurs ainsi que leurs performances sur des simulations.

   Plusieurs questions restent encore ouvertes à l'issue de cette partie et seront, on le souhaite, l'objet de travaux futurs : en particulier savoir si l'on a bien construit une procédure d'estimation optimale du coefficient d'intermittence, ou étendre les résultats obtenus à des processus MRW log-Poisson, voire au cas général log-infiniment divisible.

À ces quatre chapitres s'ajoute la présente introduction. Celle-ci se poursuit ci-dessous en deux parties : on expose d'abord le cadre de la modélisation aléatoire multifractale et des processus MRW, avant de détailler plus en avant, pour chacun des problèmes étudiés, les questions que l'on s'est posées et les résultats obtenus.

0.2 Les marches aléatoires multifractales

Nous essayons ici de donner quelques éléments introductifs à la modélisation aléatoire par des processus MRW. Ceux-ci s'inscrivent dans la famille plus large des cascades aléatoires, aussi nous commençons par détailler la construction relativement simple des cascades discrètes, avant de nous tourner vers la classe des cascades continues à laquelle appartiennent les processus MRW. Nous en donnons alors la définition précise. Dans un second temps, nous essayons de présenter brièvement le cadre théorique de l'analyse multifractale et la place particulière que celle-ci attribue aux moments des processus de cascade, place dont il sera question plus loin dans la thèse. Pour terminer, nous nous consacrons à des questions liées à l'application des processus MRW à des données réelles : en particulier l'estimation statistique du modèle et son utilisation pour le traitement de données financières.

Citons des travaux de synthèse précieux qui portent sur le même sujet ou sur des sujets proches, et dont on s'est inspiré dans ce qui suit : le texte de Riedi [83] sur l'analyse multifractale dans un cadre aléatoire, celui de Borland, Bouchaud, Muzy et Zumbach [23] sur les modèles de cascade en finance, la thèse de Kozhemyak sur les processus MRW [60], et les notes et transparents du cours de Vargas sur l'application du modèle MRW à des données financières [90].

0.2.1 La famille des cascades aléatoires

La notion de cascade aléatoire — dont les processus MRW sont un des exemples les plus récemment construits — a été introduite dès les années 1960 par l'école russe dans le cadre de l'étude des écoulements turbulents, *cf.* Yaglom [96]. Cependant elle ne devient un modèle central dans ce domaine qu'à partir des années 1970, suite aux travaux de Mandelbrot [66, 67] et de Kahane et Peyrière [57], et plus encore au début des années 1980. En effet, c'est alors que des résultats expérimentaux (en particulier ceux d'Anselmet, Gagne, Hopfinger et Antonia [5]) entrent en contradiction avec les hypothèses d'invariance d'échelle globale et d'homogénéité qui sont au coeur de la célèbre théorie formulée par Kolmogorov en 1941 [59]. Les notions de cascade d'énergie et de cascade aléatoire ouvrent alors la voie à un nouveau paradigme initié par Frisch et Parisi [43, 42], l'analyse multifractale, qui permet de mieux rendre compte de la manière dont l'énergie cinétique se transmet au sein d'un écoulement turbulent depuis les grandes échelles jusqu'aux échelles dissipatives. Depuis les années 1980, la modélisation par cascade aléatoire a connu un succès croissant non seulement en turbulence, mais également en finance (*cf. infra*), ainsi que dans de nombreux domaines comme la géophysique [61], le traitement d'images [32], l'étude de l'ADN [97] ou du trafic Internet [76].

Les cascades discrètes

Noux présentons dans un premier temps la définition d'une des sous-familles de ces processus de cascades qui est sans doute l'une des plus connues et l'une des plus usitées, celle proposée par Mandelbrot dans [68] et dont Kahane et Peyrière ont démontré l'existence et étudié certaines propriétés dans [57]. Les cascades de Mandelbrot se définissent à l'aide d'un arbre b-adique pour un entier $b \geq 2$; pour alléger un peu les notations nous nous restreignons ici au cas d'un arbre dyadique.

Définition 1 (Cascade dyadique de Mandelbrot)**.** *On se donne un réel strictement positif T et une variable aléatoire W à valeurs strictement positives et d'espérance 1, ainsi qu'une suite*

$$\Big(W_k,\, k = (k_1, \ldots, k_j) \in \{0,1\}^j,\, j \in \mathbb{N}\Big)$$

de copies indépendantes de W indicées par les entiers dyadiques. Pour un réel $u \in [0,T]$, notons $u_1, \ldots, u_j, \ldots$ les coefficients à valeurs dans $\{0,1\}$ de son développement dyadique sur $[0,T]$:

$$u = T \sum_{j=1}^{+\infty} 2^{-j} u_j,$$

et $m_j(u)$ la fonction qui à u associe le $j-$uplet $(u_1, \ldots, u_j)$. Alors si la limite ci-dessous est bien définie, la cascade dyadique de Mandelbrot sur $[0,T]$ associée à la suite (W_k) est *le processus* $\Big(M(t),\, 0 \le t \le T\Big)$ *tel que pour* $t \in [0,T]$

$$\int_0^t \prod_{j=1}^n W_{m_j(u)} du \to M(t) \text{ pour } n \to +\infty. \tag{0.1}$$

Comme on l'a indiqué, ce sont Kahane et Peyrière [57] qui ont donné la première étude rigoureuse du processus, en particulier les conditions nécessaires et suffisantes de non-dégénerescence dans la limite (0.1) et de finitude des moments du processus. Notons que s'il est relativement facile de montrer par un argument martingale que la convergence (0.1) est vraie presque sûrement, la question de la non-dégénerescence de la limite $M(t)$ est en revanche loin d'être triviale : en particulier pour u fixé et lorsque n tend vers l'infini, la suite $\prod_{j=1}^n W_{m_j(u)}$ tend vers 0 presque sûrement mais diverge dans L^p pour $p > 1$.

Pour $p \ge 0$, notons $\psi(p) = \log_2 \mathbb{E}\Big[W^p\Big]$ (on s'autorise la notation $\psi(p) = +\infty$). On a :

Propriété 1 (Existence d'une cascade de Mandelbrot et finitude des moments [57])**.** *Supposons* $\psi'(1) < 1$. *Alors la convergence* (0.1) *a lieu presque sûrement et dans* L^1, *si bien que la cascade* M *existe et est non-dégénérée. De plus, la suite de mesures* $\prod_{j=1}^n W_{m_j(u)} du$ *converge presque sûrement au sens de la convergence étroite vers une limite non absolument continue par rapport à la mesure de Lebesgue.*

Supposons de plus $p - \psi(p) - 1 > 0$ *pour un* $p > 1$. *Alors la convergence* (0.1) *a lieu dans* L^p. *À l'inverse, si* $p - \psi(p) - 1 \le 0$ *et* $\psi'(1) < 1$ *alors* $\mathbb{E}\Big[M(t)^p\Big] = +\infty$.

Une propriété remarquable des cascades de Mandelbrot est celle de l'autosimilarité qu'on observe sur les accroissements. Celle-ci est d'un type relativement inhabituel puisqu'à la différence de la propriété d'autosimilarité usuelle de processus comme le mouvement brownien fractionnaire ou les processus strictement stables, des poids aléatoires W_k interviennent.

Propriété 2 (Autosimilarité des cascades de Mandelbrot)**.** *Soit* $n \in \mathbb{N}$ *et* $0 \le k \le 2^n - 1$. *Alors la cascade dyadique de Mandelbrot* M *sur* $[0,T]$ *associée à la famille* (W_k) *vérifie*

$$M\Big((k+1)2^{-n}T\Big) - M\Big(k2^{-n}T\Big) \overset{loi}{=} 2^{-n}\tilde{W}_1 \ldots \tilde{W}_n\Big(M(T) - M(0)\Big),$$

où les variables $\tilde{W}_k$ sont des copies indépendantes de W_0, et sont également indépendantes de M.

Cette propriété se démontre par une application directe de la définition des cascades de Mandelbrot.

La construction ci-dessus a été étendue de bien des manières : sans prétendre à l'exhaustivité, signalons les travaux où les poids $W_{m_j(u)}$ ne sont plus forcément indépendants ou n'ont plus la même loi, que ce soit lorsque u varie (c'est par exemple le cas des cascades dites « conservatives » pour lesquelles $M(T)$ est déterministe, *cf.* par exemple l'étude que leur consacrent Resnick, Samorodnitsky, Gilbert et Willinger [81]) ou lorsque j varie (*cf.* Waymire et Williams [92] qui proposent une dynamique markovienne pour la suite $(W_{m_j(u)})_{j\in\mathbb{N}}$), l'étude d'Arneodo, Bacry et Muzy [6] où les produits $\prod_{j=1}^{n} W_{m_j(u)}$ sont envisagés comme coefficients d'une fonction aléatoire dans une base d'ondelette orthonormée, ou encore la récente généralisation de Barral, Jin et Mandelbrot [15] à des variables W_k à valeurs dans $\mathbb{C}$. On parle ici de *cascade discrète* pour désigner les processus construits par ces approches dans la mesure où elles font appel à une grille discrète $b-$adique. Cependant, selon le type d'application en vue, cette construction sur une grille peut apparaître comme artificielle, si ce n'est problématique. En particulier, les cascades de Mandelbrot décrites plus haut n'ont pas la propriété de stationnarité des accroissements : la Propriété 2 n'est ainsi vraie que pour des accroissements dyadiques. Cela convient mal à un domaine d'application comme la finance où il semble difficile de justifier cette place privilégiée qu'accorde le modèle aux instants dyadiques. Certains auteurs ont donc cherché à construire des processus de cascades aléatoires qui ne soient pas par nature liés à une grille discrète : c'est ainsi qu'ont été proposés au tournant des années 2000 les modèles de *cascades continues* que nous évoquons maintenant.

Les cascades continues et les processus MRW

Il semble malheureusement ardu de fournir une intuition simple de comment modifier la définition des cascades de Mandelbrot pour se ramener à des processus proches mais qui aient des accroissements stationnaires. On va cependant tâcher de donner quelques éléments dans ce sens, avant de préciser la définition exacte des processus MRW de Bacry et Muzy et d'en énoncer les premières propriétés.

On peut commencer par remarquer que pour substituer une construction *continue* à une construction *discrète* dans la limite (0.1), on peut s'attendre à devoir remplacer la multiplication itérée de variables aléatoire positives, indépendantes et de même loi par l'exponentielle d'un processus de Lévy. C'est effectivement ce que l'on va être amené à faire : pour construire un processus MRW, on se donne un processus infiniment divisible $\left(w_l(u),\, 0 < l < T,\, u \geq 0\right)$ tel que pour u fixé, $\mathrm{e}^{w_{2^{-n}}(u)}$ joue le même rôle que $\prod_{j=1}^{n} W_{m_j(u)}$ dans (0.1). En particulier, on a que $w_l(u) - w_{l'}(u)$ est indépendant de $w_l(u)$ pour $0 < l < l' < T$ et $u \geq 0$, et $\mathbb{E}\left[\mathrm{e}^{w_l(u)}\right] = 1$ pour tout l et u.

On a ainsi fixé la structure de dépendance du processus $\left(w_l(u),\, 0 < l < T,\, u \geq 0\right)$ en la variable l mais pas en la variable u. Pour ce faire, on peut d'abord considérer le cas simple où la loi du processus infiniment divisible w est gaussienne. Toujours en se référant à la définition des cascades de Mandelbrot ci-dessus, on peut alors remarquer que pour n grand et pour $t \in [0, T]$ petit, la covariance

$$u \mapsto \mathbb{C}\mathrm{ov}\left[\sum_{j=1}^{n} \log_2 W_{m_j(u)}, \sum_{j=1}^{n} \log_2 W_{m_j(u+t)}\right]$$

vaut « en moyenne » $\lambda^2 \log_2(T/t)$ avec $\lambda^2 = \mathbb{V}\mathrm{ar}[\log_2 W] = \psi''(0)$, la moyenne s'entendant comme une intégrale sur les valeurs possibles de u. On se réfèrera à Arneodo, Bacry et Muzy [6] pour une démonstration rigoureuse de cette affirmation. Ainsi les premiers travaux où le concept de MRW est introduit [12, 74] font appel à un processus gaussien $\left(w_l(u),\, 0 < l < T,\, u \geq 0\right)$ tel que

$$\mathbb{C}\mathrm{ov}[w_l(u), w_{l'}(u')] = \lambda^2 \log(T/|u - u'|)$$

pour $\max(l, l') \leq |u - u'| \leq T$, et $\lambda^2 > 0$.

C'est une représentation géométrique proposée par Schmitt et Marsan [88] qui va permettre de généraliser cette approche gaussienne au cas infiniment divisible. On se donne une loi de probabilité infiniment divisible $\pi(dx)$ sur $\mathbb{R}$ d'exposant de Laplace ψ : $\int e^{px}\pi(dx) = e^{\psi(p)}$, et on considère le demi-plan $\mathbb{R} \times (0, +\infty)$ muni de la mesure $\mu(dt, dl) = l^{-2}dt \times dl$ sur ses boréliens. On définit maintenant un champ aléatoire infiniment divisible P sur les boréliens du demi-plan (*cf.* Rajput et Rosinski [80]) : pour toute suite de boréliens (A_n) disjoints, les $P(A_n)$ sont des variables aléatoires telles que

$$\mathbb{E}\Big[e^{pP(\cup_i A_i)}\Big] = e^{\psi(p)\sum_i \mu(A_i)},\ p \geq 0.$$

On s'apprête maintenant à définir le processus infiniment divisible w par $w_l(u) = P\Big(\mathcal{A}_l(u)\Big)$, si bien qu'on est ramené au choix d'un domaine $\mathcal{A}_l(u)$ de $\mathbb{R} \times (0, +\infty)$. Bien qu'ils ne construisent pas d'objet limite, Schmitt et Marsan [88] proposent d'utiliser un domaine cônique : ainsi on peut remarquer que dans la Figure 1, chacune des variables aléatoires associées par P aux bandes horizontales sont indépendantes et de même loi. L'intuition de Schmitt et Marsan est alors que l'exponentielle de ces variables aléatoires doit pouvoir tenir un rôle analogue à celui des variables W_k dans la construction des cascades de Mandelbrot.

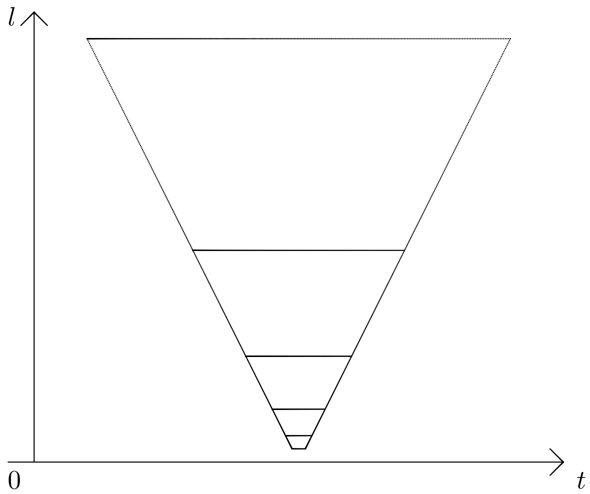

FIG. 1 – *La surface de chaque bande horizontale telle que donnée par la mesure* $\mu(dt, dl) = l^{-2}dt \times dl$ *est identique.*

Dans [11], Bacry et Muzy reprennent cette idée de Schmitt et Marsan pour énoncer une définition générale des processus MRW. Soit $T > 0$ un paramètre qui joue le rôle d'un temps de décorrélation pour le processus MRW. Bacry et Muzy définissent $\mathcal{A}_l(t)$ comme

$$\mathcal{A}_l(t) = \Big\{(t', l') \in \mathbb{R} \times (0, \infty),\, l \le l' \text{ et } |t - t'| \le \frac{1}{2}\min(l', T)\Big\}$$

(on rajoute une bande verticale au-dessus de l'ordonnée T, *cf.* Figure 2). Remarquons

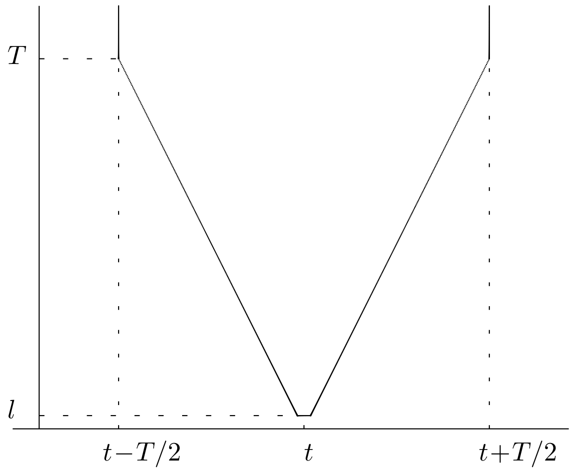

FIG. 2 – *Le cône* $\mathcal{A}_l(t)$

qu'avec cette définition, on vérifie en particulier que pour $\max(l, l') \le |t - t'| \le T$,

$$\mu\Big(\mathcal{A}_l(t) \cap \mathcal{A}_{l'}(t')\Big) = \log(T/|t - t'|).$$

La définition générale des processus MRW est alors comme suit :

Définition 2 (Processus MRM et MRW [11])**.** *Soit* $T > 0$*,* $\sigma > 0$ *et* ψ *l'exposant de Laplace d'une loi de probabilité infiniment divisible sur* $\mathbb{R}$ *tel que* $\psi(1) = 0$*. Soient* P *et* $\mathcal{A}_l(u)$ *comme ci-dessus pour* $0 < l < T$ *et* $u \ge 0$ *; on pose* $w_l(u) = P(\mathcal{A}_l(u))$*. Lorsque la limite ci-dessous est valide, le processus MRM (*Multifractal Random Measure*)* $M = \Big(M(t),\, t \ge 0\Big)$ *de paramètres* (ψ, T, σ^2) *est défini par*

$$\sigma \int_0^t \mathrm{e}^{w_l(u)} du \to M(t) \text{ pour } l \to 0 \text{ et pour } t \ge 0. \tag{0.2}$$

*On peut alors définir le processus MRW (*Multifractal Random Walk*)* $X = \Big(X(t),\, t \ge 0\Big)$

de paramètres (ψ, T, σ^2) *par*

$$X(t) = B\Big(M(t)\Big),\ t \geq 0,$$

où $B = \Big(B(t),\ t \geq 0\Big)$ *est un mouvement brownien standard indépendant de* M*.*

Comme dans le cas des cascades de Mandelbrot ci-dessus, on peut vérifier que la convergence (0.2) est toujours vérifiée presque sûrement, la difficulté étant d'obtenir une convergence L^1 pour montrer la non-dégénerescence. C'est ce qu'énonce la propriété suivante — comme le soulignent Bacry et Muzy, il s'agit d'une conséquence de la théorie générale développée par Kahane [56] :

Propriété 3 (Existence du processus MRM et finitude des moments [11])**.** *Supposons* $\psi'(1) < 1$*. Alors la convergence* (0.2) *est vraie presque sûrement et dans* L^1*.*

Supposons de plus $p - \psi(p) - 1 > 0$ *pour un* $p > 1$*. Alors pour* $t > 0$*, la convergence* (0.2) *a lieu dans* L^p*. À l'inverse, si* $p - \psi(p) - 1 \leq 0$ *et* $\psi'(1) < 1$ *alors* $\mathbb{E}\Big[M(t)^p\Big] = +\infty$.

On déduit facilement, via l'autosimilarité (au sens usuel) du mouvement brownien, l'analogue de cette propriété pour le processus MRW X.

Notons que la subordination par un mouvement brownien et son utilisation pour modéliser par exemple des données financières est une idée ancienne, préconisée en particulier par Mandelbrot dans le cadre des cascades aléatoires [69]. On renvoie par ailleurs à Barral et Seuret [19] pour une étude théorique de la subordination par un processus de cascade.

Parmi les autres constructions de cascade continues, mentionnons les suivantes : Calvet et Fisher [26, 28] donnent la construction d'un processus de cascade à accroissements stationnaires à l'aide de multiplicateurs dont les changements de valeurs sont markoviens. Barral et Mandelbrot [17] définissent une classe de processus voisine de celle des MRM de Bacry et Muzy à l'aide d'exponentielles de processus de Poisson composés, avant que Chainais, Riedi et Abry [33] n'envisagent des classes de cascades log-infiniment divisibles qui englobent les processus MRM. Des classes de cascades plus générales encore sont construites et étudiées dans les travaux Barral, Jin et Mandelbrot : *cf.* [18], ainsi que [16] pour une extension à des processus w à valeurs complexes.

Bacry et Muzy [11] montrent également :

Propriété 4 (Autosimilarité des processus MRM et MRW [11])**.** *Soit $r \in (0,1)$. Alors les processus MRM M et MRW X de paramètres (ψ, T, σ^2) vérifient*

$$\Big(M(rt),\, 0 \le t \le T\Big) \overset{loi}{=} r\mathrm{e}^{w_r}\Big(M(t),\, 0 \le t \le T\Big)$$

et

$$\Big(X(rt),\, 0 \le t \le T\Big) \overset{loi}{=} r^{1/2}\mathrm{e}^{w_r/2}\Big(X(t),\, 0 \le t \le T\Big)$$

où la variable w_r est indépendante de M et de X et satisfait pour $p \ge 0$ $\mathbb{E}\Big[\mathrm{e}^{pw_r}\Big] = r^{-\psi(p)}$.

Les processus MRM et MRW sont les seuls processus de cascades connus qui vérifient la Propriété 4 — et, peut-être, les seuls processus à temps continu non triviaux : il s'agit en effet là d'un problème ouvert formulé notamment par Rhodes et Vargas [82]. Signalons ce même travail de Rhodes et Vargas [82] qui s'inspire de la construction des MRM de Bacry et Muzy pour définir une mesure aléatoire sur $\mathbb{R}^d$, $d \ge 2$ qui satisfait le même type d'autosimilarité que celle énoncée ci-dessus.

Par ailleurs, on obtient facilement que si le processus MRM M est défini comme ci-dessus, alors le processus MRW X s'obtient également par la limite suivante :

$$\Big(\int_0^t \mathrm{e}^{w_l(u)/2} dB(u),\, t \ge 0\Big) \to \Big(X(t),\, t \ge 0\Big) \text{ en loi, pour } l \to 0,$$

avec B comme ci-dessus un mouvement brownien standard indépendant de w. Le cas où B est remplacé par un mouvement brownien fractionnaire indépendant de w a été envisagé par Muzy et Bacry [73] ; leur étude a par la suite été approfondie d'une part par Ludeña [62], et d'autre part par Abry, Chainais, Coutin et Pipiras [1]. Cette question sera à nouveau évoquée ci-dessous, lorsque nous présenterons notre construction d'un processus de type MRW qui possède des propriétés d'asymétrie.

0.2.2 Formalisme multifractal et moments d'une cascade aléatoire

Les processus de cascade constituent l'exemple type des processus aléatoires à trajectoires multifractales. Cette notion de multifractalité a été avancée dans les années 1980 pour rendre compte du caractère intermittent d'un écoulement turbulent notamment dans les petites échelles : on entend par là que pour une gamme d'exposants q, les accroissements du signal $x = (x_t,\, 0 \le t \le 1)$ satisfont

$$n^{-1} \sum_{k=0}^{n-1} |x_{(k+1)/n} - x_{k/n}|^q \approx c_q n^{-\zeta(q)} \quad \text{pour } n \text{ grand,} \tag{0.3}$$

avec c_q une constante positive qui ne dépend pas de n et $q \mapsto \zeta(q)$ une fonction *strictement* concave. Le caractère strictement concave de cette fonction pose alors problème : dans le cas de la modélisation de la turbulence, il vient en particulier contredire la théorie de Kolmogorov [59] déjà évoquée plus haut qui prédit un exposant linéaire. Remarquons cependant qu'une application de l'inégalité de Hölder permet de démontrer que si la relation ci-dessus est effectivement vérifiée pour n grand, alors ζ est bien concave au sens large.

Frisch et Parisi [43] proposent une heuristique pour lier les propriétés de l'exposant ζ à la régularité ponctuelle du signal x. Rappelons comment on définit cette notion : on dit que x est dans la classe $C^\alpha(t_0)$ s'il existe un polynôme P de degré au plus $\lfloor\alpha\rfloor$ (ici $\lfloor\alpha\rfloor$ désigne la partie entière de α) et une constante $c > 0$ tels que

$$|x_t - P(t - t_0)| \le c|t - t_0|^\alpha, \quad t \in [0, 1].$$

L'exposant de Hölder $h_x(t_0)$ de x en t_0 est défini comme $\sup\{\alpha,\, x \in C^\alpha(t_0)\}$. Afin de décrire des trajectoires qui peuvent posséder des structures de régularité ponctuelle riches, Frisch et Parisi proposent de considérer alors le spectre de singularités du signal x, c'est-à-dire la fonction $\alpha \mapsto D(\alpha)$, où $D(\alpha)$ est la dimension de Hausdorff de l'ensemble des points $t \in [0, 1]$ pour lesquels $h_x(t) = \alpha$. Selon l'argument de Frisch et Parisi, dans la somme (0.3), chaque accroissement $|x_{(k+1)/n} - x_{k/n}|$ est d'ordre $n^{-\alpha}$ si $h_x(k/n) \approx \alpha$; or par définition de la dimension de Hausdorff, on doit trouver environ $n^{D(\alpha)}$ tels accroissements parmi les n accroissements $|x_{1/n} - x_0|, \ldots, |x_1 - x_{(n-1)/n}|$. Au final, on doit donc obtenir

$$\sum_{k=0}^{n-1} |x_{(k+1)/n} - x_{k/n}|^q \approx \sum_{\alpha} n^{D(\alpha)-\alpha q},$$

et bien que cette dernière somme soit mal définie, on peut intuitivement s'attendre à ce qu'elle se comporte comme $n^{\sup_\alpha\{D(\alpha)-\alpha q\}}$. On en déduit :

$$\zeta(q) = 1 + \inf_{\alpha}\{q\alpha - D(\alpha)\},$$

ce qui est une transfomée de Legendre.

On a plus généralement coutume de dire que la *conjecture de Frisch-Parisi* est satisfaite, ou que le *formalisme multifractal* est vérifié, si $q \mapsto \zeta(q) - 1$ et $\alpha \mapsto D(\alpha)$ sont des fonctions concaves conjuguées. Si c'est bien le cas, on voit que l'on peut déduire de strictement concave de ζ le fait que le spectre D soit supporté par un intervalle non-dégénéré : on parle d'un signal x de régularité ponctuelle multifractale dans ce cas (et d'une régularité monofractale si le support de D est réduit à un point). Si de plus on remplace l'accroissement $x_{(k+1)/n} - x_{k/n}$ par un coefficient d'ondelettes, on peut mettre alors en relation l'expression (0.3) avec une norme de Besov, si bien que la fonction $p \mapsto \zeta(p)$ peut alors être définie au sens d'une régularité globale du signal x sur $[0, 1]$. C'est notamment l'approche retenue par Jaffard [52] pour la démonstration rigoureuse de la conjecture de Frisch-Parisi sur de vastes classes de signaux déterministes.

Dans le cadre d'un modèle aléatoire, on peut vouloir chercher à reproduire la propriété empirique (0.3) avec un exposant ζ strictement concave en utilisant un processus aléatoire X à temps continu — ce à quoi ne conviennent pas des processus usuels membres de vastes classes, comme les processus auto-similaires (au sens habituel) ou encore les semi-martingales d'Itō, *cf. infra*.

Il est alors en particulier souhaitable que les moments théoriques des accroissements du processus X possèdent une propriété analogue. En l'occurence, on cherche à disposer d'un processus X à accroissements stationnaires tel que

$$s^{-\zeta(q)}\mathbb{E}\Big[|X(t+s)-X(t)|^q\Big] \to c_q \quad \text{pour } s \to 0, \tag{0.4}$$

où les valeurs c_q sont des constantes strictement positives, et la fonction $q \mapsto \zeta(q)$ est strictement concave. Les processus aléatoires connus qui satisfont ce type de propriété sont de fait en petit nombre, et ils appartiennent pour la plupart à la famille des cascades aléatoires. Il est en effet facile de vérifier à partir des propriétés d'autosimilarités énoncées ci-dessus que si M est une cascade de Mandelbrot ou un processus MRM, alors on a la propriété suivante :

Propriété 5 (Moments théoriques d'une cascade de Mandelbrot et d'un processus MRM)**.** *Si M est une cascade dyadique de Mandelbrot ou un processus MRM associé aux paramètres ψ et T, alors pour $q \geq 0$ tel que $q - \psi(q) - 1 > 0$*

$$\mathbb{E}\Big[\Big(M(t+sT)-M(t)\Big)^q\Big] = c_q s^{q-\psi(q)},$$

où $c_q = \mathbb{E}\Big[M(T)^q\Big]$, et où $s \in [0,1]$ et $t \geq 0$ sont quelconques si M est un processus MRM, $s = 2^{-n}$ et $t = Tk2^{-n}$ pour $n \in \mathbb{N}$ et $0 \leq k \leq 2^n - 1$ si M est une cascade dyadique de Mandelbrot.

La fonction ψ est bien strictement convexe en tant que log-transformée de Laplace. Pour $q \in \mathbb{R}$ tel que $q - \psi(q) - 1 > 0$, posons donc $\zeta(q) = q - \psi(q)$. Molchan [71] démontre que le formalisme multifractal est effectivement vérifié par les trajectoires des cascades de Mandelbrot : presque sûrement, leur spectre de singularités est bien donné par la partie positive de $\alpha \mapsto 1 + \inf_q\{q\alpha - \zeta(q)\}$. Dans le cadre des processus MRM et MRW, le formalisme multifractal n'a pas été pleinement démontré — on se réfèrera à Barral et Mandelbrot [18] pour un état de l'art des démonstrations du formalisme multifractal sur les processus de cascades continues.

Une question importante dans les applications de l'analyse multifractale est de décider si un signal qu'on observe à des instants discrets relève des classes de régularité monofractale ou multifractale — et le cas échéant, d'obtenir le spectre D. Si la conjecture de Frisch-Parisi est satisfaite, il s'agit donc d'obtenir une approximation de ζ suffisament précise. Dans le cadre d'un modèle aléatoire, on est donc amené à poser le problème suivant :

Problème 1. Comment, à l'aide de l'observation discrète de la trajectoire de la réalisation d'un processus X à accroissements stationnaires qui satisfait (0.4), peut-on *estimer* la fonction ζ ? Peut-on de plus *tester* si X satisfait (0.4) avec une fonction ζ strictement concave sur un intervalle donné, ou bien s'il appartient à d'autres grandes classes de processus comme les processus autosimilaires (au sens usuel) ou les semi-martingales d'Itō ?

Une méthode naturelle pour répondre à ce problème est bien sûr de considérer le moment empirique d'ordre q des accroissements du processus, et de s'en servir pour estimer les moments théoriques (0.4) et par conséquent les valeurs de la fonction ζ. C'est l'approche que nous suivons aux Chapitres 1 et 2 dans le cas où X est un processus MRW ; on étudiera par ailleurs d'autres procédures d'estimation au Chapitre 4. La question du comportement asymptotique du moment empirique a été en outre traitée dans le cas relativement plus simple des cascades de Mandelbrot par un certain nombre d'auteurs, parmi lesquels Collet et Koukiou [36], Molchan [71], Resnick, Samorodnitsky, Gilbert et Willinger [81], Ossiander et Waymire [75], ainsi que Bacry, Gloter, Hoffmann et Muzy [8]. Ces auteurs montrent en particulier qu'il est possible que le moment empirique normalisé converge vers une variable aléatoire non dégénérée, ou diverge y compris dans des cas où le moment théorique est fini.

Plus précisément, on définit

$$q^* = \sup\{q > 1,\ q > \psi(q) + 1\};$$

d'après la Propriété 1 ci-dessus, le moment d'ordre $q \geq 0$ d'une cascade de Mandelbrot associée à la loi de W est fini pour $q \in [0, q^*)$. Définissons à présent, pour $\chi \geq 0$, le seuil

q_χ comme

$$q_\chi = \sup\{q > 1,\, \exists \epsilon > 0,\, \psi\Big(q(1+\epsilon)\Big) - (1+\epsilon)\psi(q) < \epsilon(1+\chi)\}.$$

En se fondant sur la convexité de ψ et l'hypothèse $\psi(0) = \psi(1) = 0$, on peut voir que d'une part $1 < q_0 \leq q^*$, et d'autre part $q_\chi \uparrow +\infty$ pour $\chi \to +\infty$. On a alors :

Propriété 6 (Convergence du moment empirique d'une cascade de Mandelbrot [71, 75, 8])**.** *Soit M une cascade de Mandelbrot obtenue par la limite* (0.1)*, et soient $\chi \geq 0$ et $q \in \Big[0, \min(q_\chi, q^*)\Big)$. Alors*

$$\frac{1}{\lfloor T2^{(\chi+1)n} \rfloor} \sum_{k=0}^{\lfloor T2^{(\chi+1)n} \rfloor - 1} \frac{\Big\{M\Big((k+1)2^{-n}\Big) - M\Big(k2^{-n}\Big)\Big\}^q}{\mathbb{E}\Big[M(2^{-n})^q\Big]}$$

converge presque sûrement et dans L^1 quand $n \to +\infty$ vers une limite qui vaut 1 si $\chi > 0$, et $M^{(q)}(T)$ si $\chi = 0$, où $M^{(q)}$ est la cascade de Mandelbrot obtenue en substituant $W^q_{m_j(u)}/\mathbb{E}\Big[W^q\Big]$ à $W_{m_j(u)}$ dans l'expression (0.1)*. Pour $\chi \geq 0$ tel que $q_\chi < q^*$ et $q \in [q_\chi, q^*)$,*

$$-\frac{1}{n}\log_2 \sum_{k=0}^{\lfloor T2^{-n(\chi+1)} \rfloor} \Big\{M\Big((k+1)2^{-n}\Big) - M\Big(k2^{-n}\Big)\Big\}^q \to q(1-\psi'(q_\chi)) \text{ quand } n \to +\infty$$

presque sûrement.

Une relation analogue est par ailleurs établie pour les exposants $q < 0$. Notons que là aussi, on peut faire le lien entre cette convergence des moments empiriques et la norme de Besov des trajectoires, *cf.* Rosenbaum [87] et les références qui y sont mentionnées.

L'introduction du paramètre χ est proposée par Bacry, Gloter, Hoffmann et Muzy [8] : comme on le voit à l'énoncé de la propriété ci-dessus, dans le cas des cascades de Mandelbrot on ne peut estimer les valeurs $\zeta(q)$ à partir du moment empirique ci-dessus que pour $q < q_\chi$. On voit donc que plus χ est grand, plus l'horizon $[0, T2^{(1+\chi)n}]$ sur lequel on observe le processus augmente rapidement par rapport à la fréquence d'échantillonnage 2^n, et plus on dispose d'exposants $q \geq 0$ pour lesquels le moment empirique d'ordre q

permet d'estimer effectivement les valeurs théoriques $\zeta(q)$. Bacry, Gloter, Hoffmann et Muzy montrent en outre que les nouvelles valeurs de cet exposant quand χ croît sont liées à de nouvelles valeurs α de la régularité ponctuelle qu'on voit apparaître sur les trajectoires du processus lorsque l'horizon d'observation tend vers l'infini. Ils démontrent en effet la version suivante du formalisme multifractal en asymptotique « mixte » :

Propriété 7 (Formalisme multifractal en asymptotique « mixte » [8])**.** *Soit M une cascade de Mandelbrot — avec $T = 1$ pour simplifier, et soit pour $n \in \mathbb{N}$, $h > 0$, $\chi > 0$ et $\varepsilon > 0$*

$$N_{j,\varepsilon}(h) = \#\Big\{k \in \{0, \dots, 2^{n(\chi+1)}\},\ 2^{-n(h+\varepsilon)} \leq M\Big((k+1)2^{-n}\Big) - M\Big(k2^{-n}\Big) \leq 2^{-n(h-\varepsilon)}\Big\}$$

Soit par ailleurs $q \mapsto \zeta_\chi(q)$, $q \in \mathbb{R}$ défini par :

$$-\frac{\log_2 \sum_{k=0}^{\lfloor 2^{-n(\chi+1)} \rfloor} \Big\{M\Big((k+1)2^{-n}\Big) - M\Big(k2^{-n}\Big)\Big\}^q}{n} \to \zeta_\chi(q) - 1 \text{ quand } n \to +\infty,$$

la convergence étant presque sûre d'après la propriété ci-dessus. Alors presque sûrement,

$$\lim_{\varepsilon \to 0} \limsup_{j \to +\infty} N_{j,\varepsilon}(h) = \lim_{\varepsilon \to 0} \liminf_{j \to +\infty} N_{j,\varepsilon}(h) = D_\chi(h),$$

où

$$D_\chi(h) = 1 + \chi + \inf_q \{\alpha q - \zeta_\chi(q)\}.$$

En particulier, on se convainc aisément que le support du nouveau spectre D_χ croît avec χ, ce qui correspond à l'apparition de nouvelles singularités par rapport à la configuration usuelle du formalisme multifractal avec $\chi = 0$.

Dans le premier chapitre de la thèse, dont les résultats sont détaillés ci-dessous, on s'efforce d'appliquer aux processus MRM et MRW la même démarche que celle qu'on vient de décrire dans le cas des cascades de Mandelbrot.

0.2.3 Applications du modèle MRW

Processus MRM et MRW log-normaux

Nous commençons par énoncer ici certaines propriétés particulières de la sous-classe des MRM et MRW log-normaux, c'est-à-dire associés à un processus générateur w gaussien. En effet cette sous-classe est privilégiée dans la littérature appliquée, notamment en raison de sa relative simplicité. Comme on l'a signalé plus haut, sa définition spécifique est également plus simple que celle de la classe générale des MRM et MRW log-infiniment divisibles, puisqu'il suffit de spécifier la fonction d'autocovariance du processus w (on note $x \vee y = \max(x, y)$) :

Définition 3 (Processus MRM et MRW log-normaux [74])**.** *Soient $\lambda \in (0, \sqrt{2})$, $T > 0$ et $\sigma > 0$, et soit $\Big(w_l(t), t \geq 0, 0 < l < T\Big)$ un processus gaussien tel que pour t, $t' \geq 0$ et $0 < l$, $l' < T$,*

$$\mathbb{C}\mathrm{ov}[w_l(t), w_{l'}(t')] = \begin{cases} \lambda^2\Big(\log\big(T/(l \vee l')\big) + 1 - |t-t'|/(l \vee l')\Big) & si\ |t-t'| \geq l \vee l' \\ \lambda^2 \log\big(T/|t-t'|\big) & si\ l \vee l' \leq |t-t'| \leq T \\ 0 & si\ |t-t'| \geq T, \end{cases}$$

et tel que $\mathbb{E}\big[w_l(t)\big] = -\mathbb{V}\mathrm{ar}[w_l(t)]/2$. Alors le processus MRM M de paramètres (λ^2, T, σ^2) est défini par

$$M(t) = \lim_{l \to 0} \int_0^t \mathrm{e}^{w_l(u)} du,\ t \geq 0,$$

la limite étant valide presque sûrement et dans L^1. Le processus MRW X de paramètres (λ^2, T, σ^2) est défini par

$$X(t) = B\big(M(t)\big),\ t \geq 0,$$

où B est un mouvement brownien standard indépendant de M.

On est bien dans un cas particulier de la Définition 2. En particulier, il existe un processus gaussien w avec la fonction d'autocovariance ci-dessus : il suffit en effet de le définir directement en posant $w_l(t) = P\big(\mathcal{A}_l(t)\big)$, avec P un bruit blanc gaussien sur

$\mathbb{R} \times (0, +\infty)$ d'espérance $-\lambda^2\mu(dt, dl)/2$ et de variance $\lambda^2\mu(dt, dl)$, et $\mu(dt, dl)$ et $\mathcal{A}_l(t)$ comme dans la Définition 2. Par ailleurs, la condition $\lambda^2 \in (0, 2)$ est l'équivalent de la condition $\psi'(1) < 1$ de la Propriété 3 qui assure la non-dégénérescence de M. On notera en outre que les processus MRW log-normaux constituent un cas particulier de la théorie du chaos multiplicatif gaussien de Kahane [55].

Il est relativement aisé de mener des calculs sur les processus MRM et MRW log-normaux, en particulier des calculs de moments : ainsi on peut déduire la propriété suivante de la Définition 3 et de la convergence L^q donnée par la Propriété 3.

Propriété 8 (Moments d'un processus MRM log-normal)**.** *Soit M un processus MRM de paramètres (λ^2, T, σ^2). Alors pour $q \geq 1$, $\mathbb{E}\Big[M(t)^q\Big] < +\infty$ si et seulement si $q < 2/\lambda^2$. Si on a de plus $\lambda^2 < 1$ et $2 \leq q < 2/\lambda^2$ avec q entier, alors pour $a_1, \ldots, a_q$, $b_1, \ldots, b_q \in [0, T]$,*

$$\mathbb{E}\Bigg[\Big(M(b_1) - M(a_1)\Big) \ldots \Big(M(b_q) - M(a_q)\Big)\Bigg] = \sigma^q \int_{a_1}^{b_1} du_1 \ldots \int_{a_q}^{b_q} du_q \prod_{1 \leq i_1 < i_2 \leq q} \left(\frac{T}{|u_{i_1} - u_{i_2}|}\right)^{\lambda^2}.$$

Cette dernière intégrale est une intégrale de Selberg dans le cas où $a_1 = \cdots = a_q$ et $b_1 = \cdots = b_q$, et on en connaît alors la valeur explicite (*cf.* [74]).

Bacry, Kozhemyak et Muzy [10, 60] se fondent sur ce type de calcul de moments pour justifier que le logarithme du processus MRM peut être approximé par un processus gaussien pour λ petit. Pour w le processus gaussien de la Définition 3, on peut démontrer (*cf.* par exemple Kozhemyak [60]) que le processus

$$\int_0^t \Big(w_l(u) - \mathbb{E}\big[w_l(u)\big]\Big) du,\ t \geq 0$$

converge en loi quand $l \to 0$ vers un processus gaussien $\Big(\Omega(t),\ t \geq 0\Big)$. On a alors :

Propriété 9 (Approximation gaussienne du logarithme de la MRM [10, 60])**.** *Soit n entier, et $a_1, \ldots, a_n$, $b_1, \ldots, b_n \geq 0$. Alors*

$$\frac{1}{\lambda}\left(\log \frac{M(b_1)}{M(a_1)}, \ldots, \log \frac{M(b_n)}{M(a_n)}\right) \to \Big(\Omega(b_1) - \Omega(a_1), \ldots, \Omega(b_n) - \Omega(a_n)\Big) \ \textit{pour}\ \lambda \to 0,$$

où la convergence a lieu en loi.

Le coefficient λ^2 étant évalué à une valeur très faible — au plus un ou deux dixièmes — dans le traitement de données financières, Bacry, Kozhemyak et Muzy [10] ont proposé des applications en finance de cette approximation sur lesquelles nous revenons dans les lignes qui suivent. Une autre application liée au caractère gaussien du processus w est donnée par le travail de Duchon, Robert et Vargas [39] concernant le conditionnement de processus gaussiens généralisés afin d'établir des formules de prédiction de M.

Application à la modélisation de données financières

Un des intérêts de la notion de processus MRW réside en son application à la modélisation de cours de prix d'actifs financiers. Il est en effet bien connu que les prix d'actifs possèdent un certain nombre de régularités statistiques, ou faits stylisés, dont on peut dire qu'ils présentent un certain caractère universel. C'est-à-dire qu'il semble que quel que soit l'actif, la période de temps, et le pas d'échantillonage considérés — de quelques minutes à quelques années — on constate que

- les accroissements du logarithme du prix, ou rendements logarithmiques, forment une série centrée et décorrélée ;
- les carrés de ces rendements, ainsi que leurs valeurs absolues, constituent des séries dont la corrélation décroît très lentement (ce sont les phénomènes de persistence et de *clusters* de volatilité) ;
- la distribution des rendements est fortement non-gaussienne et présente des queues lourdes.

Si on pose que le logarithme du prix suit la dynamique d'un processus MRW X, alors le premier point est clairement vérifié (on a même la propriété plus forte que X est une martingale par rapport à sa filtration naturelle). Le deuxième point est également valide, ce qu'on peut par exemple déduire de la Propriété 8 dans le cas d'un processus MRW log-normal. Concernant le troisième point, on a vu que la distribution de X_t n'a généralement pas de moment à tout ordre : ainsi dans le cas d'un MRW log-normal de paramètre λ^2, les moments d'ordre $p \geq 2/\lambda^2$ sont infinis. Cependant, les procédures d'estimation décrites ci-dessous donnent des ordres de grandeur de λ^2 au plus du dixième ; or les estimateurs usuels des exposants de queue indiquent généralement que les moments d'ordre p de la distribution d'un rendement financier explosent pour p de l'ordre de 3 à 5. Nous renvoyons à Muzy, Bacry et Kozhemyak [72] pour une discussion de ce point : il en ressort en particulier que ces estimateurs usuels ne sont pas convergents dans le cadre fortement dépendant des processus MRW, et qu'ils sous-estiment grandement les vrais exposants de queue.

D'autres faits stylisés sur les données financières peuvent être mis en évidence (*cf.* notamment Cont [37] ou Bouchaud et Potters [25], ainsi que Kozhemyak [60] et Bacry, Kozhemyak et Muzy [9] pour la manière dont le modèle MRW en rend compte). Un intérêt de l'approche MRW — en particulier MRW log-normale — réside dans sa « concision » : on retrouve de manière fine les propriétés statistiques des données réelles à l'aide d'un petit nombre de paramètres scalaires (trois paramètres réels dans le cas du MRW log-normal). Cette qualité est partagée par d'autres modèles de cascade, comme le montrent en particulier Calvet et Fisher concernant leur modèle de cascade continue [28] (*cf.* également Lux sur le même point [65]).

Un autre intérêt de l'utilisation de modèles de cascade en finance est bien sûr de parvenir à reproduire le type de relations (0.3) lorsqu'on les constate dans la pratique sur des données de prix. Néanmoins, il convient d'être prudent, et si certains auteurs se fondent sur ce type de relations empiriques pour promouvoir l'usage de modèles multifractals en finance — *cf.* Ghasghaie *et al.* [46], Mandelbrot [69], ou Calvet et Fisher [27], on peut se demander si des modèles plus usuels ne peuvent également vérifier la relation (0.3) pour n de l'ordre de grandeur des tailles de bases de données financières. Par exemple, on trouvera chez Veneziano *et al.* [91], Cont [37] ou bien Lux [64] des interrogations sur l'utilité en finance de la relation (0.3) pour caractériser la multifractalité des données. On souhaite ainsi disposer de tests statistiques qui permettraient de trancher ce débat ; le lecteur trouvera dans la deuxième partie de la thèse des éléments dans ce sens.

Un fait stylisé qui n'est cependant pas pris en compte par aucun des modèles de cascades mentionnés jusqu'ici est le type d'asymétrie observé dans les données actions et indices : il s'agit une corrélation faible mais significativement négative entre les rendements passés et les volatilités futures. On peut également relier cette corrélation négative à la *skewness* négative de la distribution de ces rendements. De là le problème suivant :

Problème 2. Comment modifier la définition du processus MRW X afin d'obtenir une corrélation négative entre les accroissements passés et le carré des accroissements futurs, tout en conservant l'ensemble des propriétés qui font des processus MRW log-normaux un modèle aux propriétés empiriques proches des données financières ?

On traite de cette question dans la troisième partie de la thèse. Notons qu'elle a déjà été évoquée par Pochart et Bouchaud [79] dans le cadre d'une approche à temps discret. Comme on le verra, ce sera également l'occasion de nous pencher sur un cas particulier du problème plus abstrait :

Problème 3. Sous quelles conditions et dans quel sens la limite

$$\lim_{l\to 0}\int_0^t \mathrm{e}^{w_l(u)/2} dB^H(u),$$

est-elle valide et non-dégénérée, où w est défini comme ci-dessus et B^H est un mouvement brownien fractionnaire d'exposant $H \in (0,1)$?

Comme on l'a indiqué ci-dessus, cette question a été soulevée par Muzy et Bacry [73] dans le cas d'indépendance entre w et le mouvement brownien fractionnaire et elle a été également considérée par Ludeña [62] et par Abry, Chainais, Coutin et Pipiras [1], toujours dans le cas d'indépendance. La troisième partie de la présente thèse traite d'un cas de non-indépendance.

Nous terminons cette courte présentation du modèle MRW en finance par deux exemples d'application. Le premier concerne la prédiction de volatilité ou d'un niveau de risque. Compte tenu de l'observation d'une trajectoire de prix $\big(X(s), s \leq t\big)$ jusqu'à l'instant t, on souhaite former une prédiction par exemple de la volatilité $\big(X(t+h+\delta t) - X(t+h)\big)^2$ sur un pas de temps δt et à un horizon h. Bacry, Kozhemyak et Muzy [10] proposent de résoudre ce type de problème par l'utilisation de l'approximation de la Propriété 9 : ils procèdent à une étude numérique approfondie et concluent que cette approche conduit à des performances supérieures à celles de modèles plus usuels pour la modélisation de la volatilité comme les modèles GARCH et leurs dérivés. Sur le même sujet, mentionnons les travaux de Calvet et Fisher [28], qui obtiennent avec leur modèle de cascade continue des résultats comparables à ceux obtenus sur les processus MRW log-normaux, et ceux de Duchon, Robert et Vargas [39], où les auteurs montrent comment utiliser des formules de prédiction de processus gaussiens généralisés pour répondre à cette question de la prédiction de volatilité au sein du modèle MRW log-normal. Un autre exemple d'application, cette fois à la théorie économique, est donné par le modèle de prix d'indifférence développé par Calvet et Fisher dans [30, 29].

Estimation statistique

Le problème dont il est question ici est le suivant :

Problème 4. Comment, à l'aide de l'observation en haute fréquence de la réalisation d'une trajectoire, estimer les paramètres d'un processus MRM ou MRW ?

Notons que ce problème est lié au Problème 1, qui concerne cependant plus spécifiquement l'estimation de la fonction ζ, et donc de λ^2 dans le cas du processus MRW log-normal. Il n'existe pas d'étude exhaustive de cette question ; cependant, on peut citer quatre types d'approches dans la littérature concernant le problème — encore plus général — de l'estimation des paramètres d'une cascade aléatoire. Il s'agit de :

1. L'utilisation d'estimateurs du type GMM dans le cas où l'horizon d'observation tend vers $+\infty$. Il s'agit d'un cas « facile » dans la mesure où les observations deviennent indépendantes dès qu'elles sont prises à un écart de temps plus grand que le paramètre T. Nous renvoyons notamment à Bacry, Kozhemyak et Muzy [10] pour le traitement de ce cas qui n'est pas développé dans la thèse ; on se concentre par la suite sur le cas d'un horizon fixe alors que le pas de discrétisation des observations tend vers zéro. D'autres cadres possibles que nous ne traitons pas non plus serait d'une part l'asymptotique mixte du cas $\chi > 0$ — comme dans l'énoncé de la Propriété 6

ci-dessus — et d'autre part la configuration envisagée en particulier par Ludeña [62] d'un paramètre T du même ordre de grandeur que le nombre d'observations.

2. L'utilisation de statistiques fondées sur la convergence du moment empirique (*cf.* la Propriété 6) pour construire des estimateurs de $\psi(p)$. C'est notamment la démarche des travaux d'Ossiander et Waymire [75] dans le cadre des cascades de Mandelbrot, ou de Ludeña [62, 63] dans le cadre du processus MRW avec intégrant brownien standard ou brownien fractionnaire (comme évoqué au Problème 3).

3. L'utilisation de moments empiriques ou de cumulants empiriques du logarithme de l'accroissement du processus de cascade pour estimer les dérivées successives de la fonction ζ ou ψ en 0. En effet, on peut utiliser par exemple les Propriétés 2 et 4 pour voir qu'on a la relation

$$\mathbb{V}\mathrm{ar}\left[\log M(sT)\right] = \psi''(0)\log s^{-1} + c,$$

avec $c > 0$ une constante qui ne dépend pas de s, et $s \in (0,1)$ quelconque si M est un processus MRM, $s = 2^{-n}$ pour un $n \in \mathbb{N}$ si M est une cascade de Mandelbrot. Les cumulants d'ordre supérieur permettent de plus d'obtenir les valeurs des dérivées d'ordre plus élevé de ψ en 0, ce qui permet en particulier d'identifier les cascades log-normales pour lesquelles les cumulants d'ordre supérieur à 3 sont nuls. Suivant cette idée, on va donc chercher à estimer les valeurs $\psi^{(p)}(0)$ par une régression du cumulant empirique d'ordre p du logarithme des accroissements absolus sur le logarithme du pas des accroissements. C'est notamment l'approche retenue dans les études numériques approfondies de Jaffard, Lashermes, Abry, et Wendt [53, 94, 95, 93] — à ceci près que leur procédure d'estimation est cependant plus élaborée que ce qu'on vient de décrire dans le mesure où elle repose sur des *suprema* de coefficients d'ondelettes et non des accroissements, et qu'elle fait en outre appel à des techniques de rééchantillonnage.

4. L'utilisation de la covariance empirique du logarithme des accroissements absolus du processus pour estimer le coefficient λ^2. Bacry, Kozhemyak et Muzy [10] utilisent en

effet l'approximation de la Propriété 9 pour étudier les propriétés d'un tel estimateur dans le cas spécifique d'un processus MRW log-normal. Notons cependant que le recours à cette Propriété 9 pose l'inconvénient de devoir se placer dans une asymptotique $\lambda \to 0$.

On reviendra dans la quatrième partie de la thèse sur ces différentes stratégies d'estimation, dont on proposera une comparaison. Notons qu'il n'existe pas de travaux qui proposent d'estimateurs des paramètres T ou σ^2 sur un horizon d'observation borné. Bacry, Kozhemyak et Muzy [10] donnent un argument intuitif pour justifier qu'on ne peut estimer ces paramètres si l'on dispose d'observations sur un intervalle de temps inférieur au paramètre T ; argument qui se fonde sur la propriété suivante :

Propriété 10. *Soit M un processus MRM associé aux paramètres (ψ, T, σ^2). Alors pour $r \in (0, 1)$,*

$$\Big(M(t),\, 0 \leq t \leq rT\Big) \overset{loi}{=} \mathrm{e}^{w_r} \Big(\tilde{M}(t),\, 0 \leq t \leq rT\Big),$$

avec w_r une variable aléatoire telle que $\mathbb{E}\Big[\mathrm{e}^{pw_r}\Big] = r^{-\psi(p)}$ pour $p \geq 0$ et $\tilde{M}$ un processus MRM de paramètres (ψ, rT, σ^2) indépendant de w_r.

Nous verrons également dans la quatrième partie de la thèse comment formaliser cet argument.

0.3 Présentation des résultats de la thèse

Nous décrivons ici brièvement l'ensemble des principaux résultats démontrés dans le reste du texte.

0.3.1 Première partie : convergence des moments empiriques des accroissements des processus MRM et MRW en asymptotique mixte

Dans ce chapitre, qui est adapté d'un article paru dans *Stochastic Analysis and Applications* [40], on traite du Problème 1, et particulièrement de la question de la convergence du moment empirique des accroissements des processus MRM et MRW. On s'intéresse donc au comportement asymptotique de la quantité $S_N(q,t,\chi)/\mathbb{E}\Big[S_N(q,t,\chi)\Big]$, où S_N est définie par

$$S_N(q,t,\chi) = \frac{1}{\lfloor tN^{\chi+1}\rfloor} \sum_{k=0}^{\lfloor tN^{\chi+1}\rfloor - 1} |X((k+1)/N) - X(k/N)|^q \tag{0.5}$$

lorsque N tend vers l'infini, avec X un processus MRW de paramètres (ψ, T, σ^2) comme dans la Définition 2, t un réel positif quelconque, $q > 0$ tel que l'espérance de S_N soit finie, et $\chi \geq 0$ un paramètre qui définit le type d'asymptotique considérée. On s'autorisera à désigner la quantité S_N par les termes interchangeables de moment empirique des accroissements à l'ordre q, de q-variation, voire de fonction de structure ou de fonction de partition de X.

Comme on le montre, on se ramène au comportement asymptotique de

$$\Sigma_N(q/2,t,\chi)/\mathbb{E}\Big[\Sigma_N(q/2,t,\chi)\Big],$$

où Σ est défini comme suit :

$$\Sigma_N(q/2,t,\chi) = \frac{1}{\lfloor tN^{\chi+1}\rfloor} \sum_{k=0}^{\lfloor tN^{\chi+1}\rfloor - 1} |M((k+1)/N) - M(k/N)|^{q/2}. \tag{0.6}$$

On établit en particulier que le type de comportement asymptotique dépend du choix de q et de χ ; on montre qu'il existe en particulier des cas où, bien que le moment théorique d'ordre q des accroissements du processus existe, le moment empirique renormalisé $S_N(t,q,\chi)$ dégénère pour $N \to +\infty$.

Rappelons les notations

$$q^* = \sup\{q > 1,\, q > \psi(q) + 1\}$$

et

$$q_\chi = \sup\Big\{q > 1,\, \exists \epsilon > 0,\, \psi\Big(q(1+\epsilon)\Big) - (1+\epsilon)\psi(q) < \epsilon(1+\chi)\Big\}.$$

On montre alors :

Résultat 1. *Pour* $0 \leq q < q_0$*, notons*

$$M^{(q)}(t) = \frac{1}{t} \lim_{l \to 0} \int_0^t \frac{\mathrm{e}^{q w_l(u)}}{\mathbb{E}\Big[\mathrm{e}^{q w_l(u)}\Big]} du,$$

avec w *le processus infiniment divisible de la Définition 2. Soit* (u_N) *une suite à valeurs entières qui tend vers* $+\infty$*. Alors pour* $t > 0$*,*

$$\frac{S_{u_N}(2q,t,0)}{\mathbb{E}\Big[S_{u_N}(2q,t,0)\Big]} \to M^{(q)}(t) \quad \textit{et} \quad \frac{\Sigma_{u_N}(q,t,0)}{\mathbb{E}\Big[\Sigma_{u_N}(q,t,0)\Big]} \to M^{(q)}(t),$$

la convergence ayant lieu dans L^1*, et presque sûrement si* $u_N \geq c_1 \mathrm{e}^{c_2 N}$ *pour des* $c_1,\, c_2 > 0$*.*

Résultat 2. *Soit* (u_N) *une suite à valeurs entières qui tend vers* $+\infty$*. Alors pour* $t > 0$*,* $0 \leq q < \min(q_\chi, q^*)$ *et* $\chi > 0$*,*

$$S_{u_N}(2q,t,\chi) \to 1 \ \textit{et}\ \Sigma_{u_N}(q,t,\chi) \to 1,$$

la convergence ayant lieu dans L^1*, et presque sûrement si* $u_N \geq c_1 \mathrm{e}^{c_2 N}$ *pour des* $c_1,\, c_2 > 0$*.*

Le résultat suivant montre que le domaine $[0, \min(q_\chi, q^*))$ est bien maximal :

Résultat 3. *Supposons $q_\chi < q^*$, et soit $q \in [q_\chi, q^*)$. Alors*

$$-\frac{\log(S_{u_N}(2q,t,\chi))}{\log(u_N)} \to q(1-\psi'(q_\chi)) \text{ pour } N \to +\infty$$

où la convergence a lieu en probabilité, et presque sûrement si $u_N \geq c_1 \mathrm{e}^{c_2 N}$ pour des c_1, $c_2 > 0$. Le même résultat reste vrai en remplaçant S_{u_N} par Σ_{u_N}.

On en déduit en particulier comme conséquence directe que l'ensemble des valeurs $\zeta(q) = q - \psi(q)$ qu'on peut obtenir à partir de l'équation (0.3) dépend du régime aymptotique dans lequel on se trouve : si l'on dispose d'un horizon d'observation $[0, tu_N^\chi]$ fixé ($\chi = 0$) alors que le pas d'observation $1/u_N$ tend vers zéro, on peut estimer ζ sur un intervalle $[0, 2q_0)$; cependant si l'horizon d'observation croît ($\chi > 0$), alors on peut voir apparaître un intervalle strictement plus grand $[0, 2q_\chi)$ sur lequel le moment empirique S_{u_N} permet d'estimer ζ (rappelons en effet que q_χ croît avec χ). En revanche, si χ et q sont tels que $q_\chi < q^*$ et $q \in (q_\chi, q^*)$, alors $S_{u_N}(2q, t, \chi)$ n'est pas asymptotiquement du même ordre de grandeur que son moment théorique, bien que celui-ci soit fini.

Ces résultats obtenus sur les processus MRW et MRM sont proches de résultats obtenus par Ludeña [62] sur les moments empiriques des accroissements d'une MRM intégrée contre un mouvement Brownien fractionnaire d'exposant de Hurst $H \geq 1/2$ — Ludeña n'envisage cependant pas le même cadre asymptotique ni la même gamme d'exposants q que le présent travail. Comme on l'a déjà indiqué, ces résultats s'inscrivent également dans une série d'autres travaux de même nature sur les cascades de Mandelbrot : citons Collet et Koukiou [36] qui ont les premiers mis en évidence l' « effet de linéarisation » (le fait que si l'on estime $\zeta(q) = q/2 - \psi(q/2)$ par l'équation (0.3), on obtienne une fonction $q \mapsto \zeta(q)$ affine quand q dépasse un certain seuil), Molchan [71] ainsi qu'Ossiander et Waymire [75] qui ont donné des démonstrations rigoureuses du Résultat 1 dans le cadre des cascades de Mandelbrot, et pour finir Bacry, Gloter, Hoffmann et Muzy [8] qui ont proposé le cadre asymptotique que nous utilisons et qui ont démontré l'analogue du Résultat 2, toujours sur les cascades de Mandelbrot. Ces derniers auteurs démontrent ensuite la Propriété 7 à l'aide de la convergence du moment empirique des accroissements ; étendre cette démarche au cadre MRW nécessiterait des complications techniques qui ne sont pas abordées dans la thèse, notamment de travailller avec des exposants *q réels*, alors qu'on envisage dans le présent document uniquement des exposants *positifs*.

0.3.2 Deuxième partie : construction d'un test non-paramétrique — semi-martingale d'Itō contre MRW

On traite dans le deuxième chapitre de la thèse de la construction de tests non-paramétriques en lien avec le Problème 1. Il s'agit d'un article rédigé avec Christian Y. Robert et Mathieu Rosenbaum et soumis à l'*Electronic Journal of Statistics* [41].

Une classe de modèles très vaste et très fréquemment utilisée en mathématiques financières est celle des semi-martingales d'Itō. Rappelons qu'une semi-martingale peut être définie comme la somme d'un processus prévisible à variation finie, d'une martingale locale continue, et d'un processus de sauts compensés, et qu'elle est dite d'Itō si le processus à variation finie, la variation quadratique de la martingale locale continue, et le compensateur du processus de saut sont tous absolument continus par rapport à la mesure de Lebesgue. Si on considère cependant un processus X qui est une MRW, alors il s'agit d'une martingale continue qui n'entre pas dans la classe des semi-martingales d'Itō puisque sa variation quadratique, le processus MRM M, est un processus croissant qui n'est pas absolument continu par rapport à la mesure de Lebesgue. En particulier, un processus MRW n'est pas une diffusion brownienne : du point de vue des applications en finance, on dira qu'il n'existe pas de processus de volatilité *spot*.

L'idée de construire un test statistique qui permettrait d'affirmer au vu d'un jeu de données si celles-ci ont été générées par une semi-martingale d'Itō ou par un processus MRW semble ainsi revêtue d'un certain intérêt : sur le plan pratique, on souhaiterait départager des modèles aléatoires et savoir lequel est le mieux en adéquation avec les données. Sur le plan théorique, la question de tester si les caractéristiques d'une semi-martingale sont absolument continues ou non semble nouvelle. Signalons cependant sur un sujet similaire les études de Wendt, Abry et Jaffard [94, 95], qui étudient les performances numériques de critères élaborés qui visent à distinguer entre signaux monofractals et multifractals — mais la question est un peu différente de la nôtre, puisque la classe des semi-martingales d'Itō contient elle-même des processus dont les trajectoires sont multifractales au sens strict, dans la mesure où le spectre de singularités de leurs trajectoires a un support non dégénéré, *cf.* Jaffard [51].

Ajoutons qu'on dispose de travaux déjà existants sur la construction de familles de tests pour les semi-martingales d'Itō, notamment ceux d'Aït-Sahalia et Jacod [2, 3, 4]. À partir d'hypothèses très générales (notamment le fait que le processus de volatilité soit lui-même une semi-martingale d'Itō), ces derniers construisent notamment des tests de continuité des trajectoires à partir du comportement suivant des semi-martingales d'Itō : pour $q \geq 2$,

$$\sum_{k=1}^{n} |X((k+1)/n) - X(k/n)|^q = O(n^{-q/2+1})$$

si la trajectoire $t \mapsto X(t)$ est continue sur $[0,1]$, et

$$\sum_{k=1}^{n} |X((k+1)/n) - X(k/n)|^q = O(1)$$

si la trajectoire $t \mapsto X(t)$ a des sauts sur $[0,1]$. Il est ainsi tentant d'enrichir la démarche d'Aït-Sahalia et Jacod en y intégrant le cadre des processus MRW et du Résultat 1 obtenu auparavant.

Nous posons $n = 2^N$, et nous supposons donc que nous disposons d'observations $X(0), X(2^{-N}), \ldots, X(1)$, où X est soit une semi-martingale d'Itō, soit un processus MRW. En nous fondant sur les propriétés asymptotiques de $S(q, 2^{-N})$, où

$$S(q, 2^{-N}) = 2^N S_{2^N}(q, 1, 0)$$

— *cf.* la notation (0.5), nous construisons deux familles de statistiques de test, selon que l'hypothèse nulle à tester est

$$H_0 : \text{les observations ont été générées par une semi-martingale d'Itō}$$

contre l'alternative

$$H_a \text{ : les observations ont été générées par un processus MRW}$$

ou inversement. Dans la première des deux configurations, nous définissons

$$T_1^N = m(q) 2^{(q/2-1)(\lfloor k_N N \rfloor - N)} \frac{S(q, 2^{-\lfloor k_N N \rfloor})}{(S(2q, 2^{-N}))^{1/2}} \Big(\frac{S(q, 2^{-(N-1)})}{S(q, 2^{-N})} - 2^{q/2-1} \Big),$$

avec (k_N) une suite de réels positifs telle qu'on a $0 \leq k_N \leq 1$ pour $N \geq 0$ et

$$k_N \to 1 \text{ et } (1 - k_N) N \to +\infty \text{ pour } N \to +\infty,$$

et avec $m(q)$ une constante strictement positive explicite. Nous avons alors :

Résultat 4. *Soit $q > 2$. Pour $N \to +\infty$, si X est une semi-martingale d'Itō qui satisfait les hypothèse d'Aït-Sahalia et Jacod [2], alors en restriction à l'évènement $\{t \mapsto X_t$ est continu sur $[0,1]\}$, $(T_1^N)^2$ converge en loi stablement[1] vers un $\chi^2(1)$, et en restriction à l'évènement $\{t \mapsto X_t$ a au moins un saut sur $[0,1]\}$ $(T_1^N)^2$ converge en probabilité vers 0. En revanche, sous l'alternative, c'est-à-dire si X est un processus MRW, et si on a $q < q_0$, alors (T_1^N) tend vers $+\infty$ en probabilité.*

Si, pour $0 \leq \alpha \leq 1$, on définit z_α comme l'α-quantile de la loi du $\chi^2(1)$ (une variable aléatoire qui suit une telle loi a donc une probabilité α d'être inférieure à z_α), il en résulte donc que $\{(T_1^N)^2 > z_\alpha\}$ est une zone de rejet asymptotique de l'hypothèse nulle H_0 : « X est une semi-martingale d'Itō » contre l'hypothèse H_a : « X est un MRW » au niveau α.

[1] La convergence en loi stable est une notion plus forte que la simple convergence en loi : on dit que la suite (X_n) tend stablement en loi vers la loi π si pour toute variable aléatoire bornée Y et pour toute fonction continue bornée $f : \mathbb{R} \to \mathbb{R}$,

$$\mathbb{E}\big[f(X_n)Y\big] \to \mathbb{E}\big[f(U)\big]\mathbb{E}\big[Y\big] \text{ quand } n \to +\infty$$

avec U une variable aléatoire de loi π.

Nous exposons maitenant nos résultats pour le cas inverse, où l'hypothèse nulle est

$$H_0 : \text{les observations ont été générées par un processus MRW}$$

contre l'alternative

$$H_a : \text{les observations ont été générées par une semi-martingale d'Itō}$$

Soit $0 < k < 1$. Nous définissons T_2^N :

$$T_2^N = \frac{\sqrt{3}}{\sqrt{2(2^{1-\psi(2)}-1)}} 2^{(N-\lfloor kN \rfloor)(1-\psi(2))/2} \frac{\{S(2, 2^{-N}) - S(2, 2^{-(N-1)})\}}{\sqrt{S(4, 2^{-\lfloor kN \rfloor})}}.$$

où comme précédemment ψ est l'exposant de la transformée de Laplace de la loi infiniment divisible associée au processus MRW X. On montre alors :

Résultat 5. *Pour $N \to +\infty$, si X est un processus MRW associé à l'exposant de Laplace ψ, et avec $q_0 > 2$, alors $(T_2^N)^2$ converge stablement en loi vers un $\chi^2(1)$. Si X est une semi-martingale d'Itō qui satisfait les hypothèses d'Aït-Sahalia et Jacod [2], alors $(T_2^N)^2$ converge en probabilité vers 0.*

Si, pour $0 \leq \alpha \leq 1$, on définit z_α comme l'α-quantile de la loi du $\chi^2(1)$, on en déduit donc que $\{(T_2^N)^2 < z_\alpha\}$ est une zone de rejet asymptotique de l'hypothèse nulle H_0 : « X est un MRW » contre l'hypothèse H_a : « X est une semi-martingale d'Itō » au niveau α.

On trouvera dans la thèse, outre la démonstration de ces résultats, des études numériques de la puissance de ces tests sur des simulations informatiques. Comme on le verra (*cf.* également les Tables 1 et 2), on obtient des résultats relativement satisfaisants, en particulier dans le cas où il s'agit d'opposer semi-martingale d'Itō avec sauts et processus MRW. Dans la pratique, remarquons que la valeur $\psi(2)$ n'est pas connue, ce qui pose problème puisqu'elle intervient dans la construction de la statistique de test T_2^N. Nous étudierons ainsi sur les plan théorique et pratique le comportement d'une statistique modifiée où la valeur $\psi(2)$ est remplacée par un estimateur de cette même quantité. On se réfèrera au

Chapitre 2 pour une présentation plus complète de nos résultats de simulation.

Processus simulé	Mouvement brownien		Mouvement brownien + Poisson composé	
Nombre n de données	32 768	1 048 576	32 768	1 048 576
Niveau du test				
90%	31	62	68	100
95%	16	26	34	89
99%	5	4	6	29

TAB. 1 – *Nombre de rejets de* H_0 *:* $X =$ *MRW,* $\psi(2) = 0.7$ *connu, pour 100 simulations d'une semi-martingale d'Itō.*

Processus simulé	MRW, $\lambda^2 = 0.1$		MRW, $\lambda^2 = 0.7$	
Nombre n de données	32 768	1 048 576	32 768	1 048 576
Niveau du test				
90%	11	11	10	9
95%	8	5	3	3
99%	1	2	2	1

TAB. 2 – *Nombre de rejets de* H_0 *:* $X =$ *MRW,* $\psi(2) = \lambda^2$ *connu, pour 100 simulations d'un processus MRW log-normal.*

0.3.3 Troisième partie : construction d'un processus de marche aléatoire multifractale asymétrique

On cherche dans le troisième chapitre de la thèse à traiter le Problème 2 ci-dessus, et pour ce faire, on est également amené à apporter des éléments de réponse au Problème 3. C'est un travail qui a donné lieu à un article rédigé avec Emmanuel Bacry et Jean-François Muzy et soumis à l'*International Journal of Theoretical and Applied Finance* [7].

L'une des motivations de la modélisation du prix d'un actif financier par un processus MRW log-normal est, comme on l'a évoqué, de parvenir à restituer l'essentiel des faits stylisés observés sur la série des logarithmes du prix. Cependant, par construction, le processus MRW ne peut refléter le type d'asymétries constatées dans le cas particulier de prix d'actions et d'indices financiers. Notons p_t le logarithme du prix à l'instant t, et fixons un pas d'échantillonage τ : on observe donc $p_0, \dots, p_{k\tau}, \dots$ Les asymétries dont il est question ici peuvent se décrire selon deux aspects :

– premièrement, une asymétrie dans la distribution du rendement $p_{(k+1)\tau} - p_{k\tau}$: la *skewness* empirique de cette quantité semble généralement négative, et sa distribution semble légèrement plus épaisse pour les valeurs négatives que pour les valeurs positives ;
– deuxièmement, une asymétrie temporelle dans la structure de corrélation entre les rendements et les volatilités. Définissons par exemple ces dernières comme le carré ou la valeur absolue des rendements ; on constate alors une corrélation légèrement négative, mais significativement non nulle, entre les rendements passés et les volatilités futures. Cependant, l'inverse n'est pas vérifié. L'interprétation intuitive qu'on peut en faire est la suivante : une forte baisse augmente l'incertitude et donc la variabilité du cours sur la période qui suit la baisse, mais l'effet ne joue pas dans le sens opposé ; on ne prédit pas la variation du cours demain grâce aux variations passées de la volatilité.

Ce deuxième type d'asymétrie est connu sous le nom d'effet levier (*leverage effect*), et parmi un certain nombre d'études qui y sont consacrées on peut citer notamment le texte fondateur de Black [22] et le travail de Bouchaud, Potters et Matacz [24]. Pour finir cette rapide présentation de l'asymétrie des données, remarquons que la *skewness* du rendement $p_{(k+1)\tau} - p_{k\tau}$ n'est rien d'autre que la corrélation entre le rendement $p_{(k+1)\tau} - p_{k\tau}$ et la volatilité $(p_{(k+j+1)\tau} - p_{(k+j)\tau})^2$ pour un décalage $j = 0$, si bien que la *skewness* de la distribution du rendement peut s'interpréter comme un cas particulier de l'effet levier.

Afin de fournir un modèle réaliste de la dynamique du prix d'une action ou d'un indice financier, il est donc souhaitable de modifier la définition du processus MRW (supposé log-normal pour simplifier) afin d'incorporer ces asymétries. Notons que ce problème a déjà été envisagé par Pochart et Bouchaud [79] qui ont proposé un modèle multifractal à temps discret. Nous définissons un modèle à temps continu dont nous exposons ici brièvement le principe. Parmi les modèles non multifractals qui comportent cet effet levier, citons les travaux de Bouchaud, Matacz et Potters [24], Perello et Masoliver [77], Perello, Masoliver et Bouchaud [78] ou Ciliberti, Bouchaud et Potters [35].

Informellement, après avoir remarqué que le processus MRW log-normal X se représente comme

$$X(t) = \lim_{l \to 0} \int_0^t \mathrm{e}^{w_l(u)/2} dB(u),\ t \geq 0,$$

la limite étant valide en loi, le processus gaussien stationnaire $\Big(w_l(t),\ 0 < l < T,\ t \geq 0\Big)$ étant comme ci-dessus, et B étant un mouvement brownien standard indépendant de w), on peut souhaiter vouloir modifier le modèle de sorte à faire apparaître une corrélation non nulle entre le « rendement présent » $B(u + du) - B(u)$ et la « volatilité future » $\mathrm{e}^{w_l(u+du)/2}$. C'est essentiellement la démarche qu'on a suivie, à deux nuances près : premièrement, afin de disposer d'un modèle bien défini mathématiquement, on a dû utiliser un mouvement brownien fractionnaire et non un mouvement brownien standard ; et deuxièmement, plutôt que de travailler directement sur le bruit gaussien fractionnaire $B^H(u + du) - B^H(u)$ d'exposant H, on fait appel à une famille de processus gaussiens stationnaires $\Big(\varepsilon_l^H(t),\ 0 < l < T,\ t \geq 0\Big)$ qui est telle que $\Big(\int_0^t \varepsilon_l^H(u)du,\ t \geq 0\Big)$ converge (en loi) vers un mouvement brownien fractionnaire d'exposant de Hurst H. Ainsi, on retrouve une situation analogue au cas abordé par Muzy et Bacry [73], Ludeña [62] et Abry *et al.* [1] qui étudient l'intégrabilité du processus $\mathrm{e}^{w_l(u)}$, $u \geq 0$ contre un bruit brownien fractionnaire $dB^H(u)$. Le cadre n'est cependant pas identique puisque dans ces travaux, contrairement au cas que nous envisageons, le mouvement brownien fractionnaire est *indépendant* de w.

On définit ainsi une famille gaussienne $\Big((\varepsilon_l^H(t),\ w_l(t)),\ 0 < l < T,\ t \geq 0\Big)$ telle que pour $0 < l' < l < T$ et $0 \leq t,\ t'$, on a quand $l \to 0$

$$\mathbb{C}\mathrm{ov}\Big[w_l(t), w_{l'}(t')\Big] = \gamma_l^w(|t - t'|) \uparrow \lambda^2 \max\{\log(T/|t - t'|), 0\},$$

$$\mathbb{C}\mathrm{ov}\Big[\varepsilon_l^H(t), \varepsilon_{l'}^H(t')\Big] = \gamma_l^\varepsilon(|t - t'|) \uparrow c^\varepsilon \sigma^2 |t - t'|^{-2+2H},$$

$$\mathbb{C}\mathrm{ov}\Big[\varepsilon_l^H(t), w_{l'}(t')\Big] = \gamma_l^{\varepsilon w}(t - t') \uparrow \begin{cases} c^{w\omega} \sigma \lambda (t' - t)^{-1+2H}, & \text{si } t' > t \\ 0 & \text{sinon,} \end{cases}$$

avec c^w, c^ε et $c^{w\omega}$ des constantes strictement positives données. On montre alors les résultats suivants :

Résultat 6. *Supposons* $0 < \lambda^2 < 1/4$ *et* $1/2 + \lambda^2/2 < H < 1$*, alors la limite suivante*

$$X^H(t) = \lim_{l \to 0} \int_0^t \varepsilon_l^H(u) \mathrm{e}^{w_l(u)} du,\ t \geq 0$$

est non dégénérée ; elle est valide presque sûrement, et le processus ainsi obtenu admet une modification continue.

Remarquons qu'on intègre ici le processus $\mathrm{e}^{w_l(u)}$ contre un bruit gaussien $\varepsilon_l(u)du$, et non $\mathrm{e}^{w_l(u)/2}$ contre un bruit blanc gaussien $B(du)$ comme dans le cas du MRW symétrique évoqué plus haut. Ainsi la condition $\lambda^2 < 1/4$ correspond ici à la condition $\lambda^2 < 1$ pour le MRW lognormal de la Définition 3, condition qui donne alors la finitude du moment d'ordre deux de ce processus d'après la Propriété 8.

Résultat 7. *Le processus* X^H *est à accroissements stationnaires, et il satisfait au principe d'autosimilarité stochastique : pour* $r \in (0,1)$

$$\Big(X^H(rt),\ 0 \leq t \leq T\Big) \stackrel{law}{=} r^H \mathrm{e}^{w_r} \Big(X^H(t),\ 0 \leq t \leq T\Big),$$

où w_r *est une variable aléatoire* $N(\lambda^2 \log(r)/2, -\lambda^2 \log(r))$ *qui est indépendante du processus* X^H*.*

De plus, pour $q > 0$*, s'il existe un entier pair* $p > q$ *tel que*

$$Hp - \lambda^2 p(p-1)/2 - 1 > 0, \tag{0.7}$$

alors X *vérifie :*

$$\mathbb{E}\Big[X(t)^q\Big] = c(q) t^{qH - \lambda^2 q(q-1)/2}.$$

Ici $c(q)$ *est une constante strictement positive, qui s'écrit sous forme intégrale si* q *est entier.*

Réciproquement, si $p > 2$ *ne satisfait pas* (0.7)*, alors* $\mathbb{E}\Big[|X(t)|^q\Big] = +\infty$.

Ce dernier résultat montre qu'on a bien défini un processus « multifractal », au sens de la relation d'autosimilarité satisfaite également par les processus MRW et du « scaling » du moment théorique en $\mathbb{E}\Big[X^q(t)\Big] = c(q)t^{\zeta(q)}$ avec ζ strictement concave.

Pour finir, on étudie dans quelle mesure le processus X^H ainsi défini peut être utilisé pour représenter la dynamique d'un actif financier présentant un effet levier. On remarque que du fait de l'introduction des corrélations du processus ε, le processus X^H a des accroissements qui sont eux-mêmes corrélés et présente donc de la mémoire longue, ce qui n'est pas souhaitable dans le cadre de la modélisation financière : pour avoir disposer d'un modèle satisfaisant, il faudrait en effet que la corrélation des accroissements décroissent très rapidement, voire, si possible, soit nulle. Lorsque H est petit (le régime critique est donné par $H \downarrow 1/2 + \lambda^2/2$, et non $H \downarrow 1/2$ comme il est plus courant), on obtient que ces corrélations entre les accroissements disparaissent, mais le processus n'est alors plus défini (son moment d'ordre deux explose). On est donc amené à examiner dans ce régime $H \downarrow 1/2 + \lambda^2/2$ les propriétés (et en particulier les corrélations des accroissements et l'effet levier) du processus renormalisé :

$$Y^H(t) = -\frac{X^H(t)}{\mathbb{E}\Big[\big(X^H(1)\big)^2\Big]}, \quad t \geq 0$$

le signe négatif visant à rendre compte des asymétries négatives qu'on souhaite modéliser.

On obtient les propriétés suivantes :

Résultat 8. *Fixons λ^2, notons $d = 2(H - 1/2 - \lambda^2/2)$, et soient $\tau > 0$ et $k \in \mathbb{Z}$ tels que $(|k| + 1)\tau < T$. Alors quand $H \downarrow 1/2 + \lambda^2/2$, on a*

$$\mathbb{C}\mathrm{orr}\Big[Y^H(\tau) - Y^H(0), Y^H\Big((k+1)\tau\Big) - Y^H(k\tau)\Big] = O(d) \ \textit{si}\ k \neq 0$$

$$\mathbb{C}\mathrm{orr}\Big[Y^H(\tau) - Y^H(0), \Big\{Y^H\Big((k+1)\tau\Big) - Y^H(k\tau)\Big\}^2\Big] = O(d^{1/2}) \ \textit{si}\ k \geq 0$$

$$\mathbb{C}\mathrm{orr}\Big[Y^H(\tau) - Y^H(0), \Big\{Y^H\Big((k+1)\tau\Big) - Y^H(k\tau)\Big\}^2\Big] = O(d^{3/2}) \ \textit{si}\ k < 0.$$

On en déduit donc que pour *d petit mais non nul*, les propriétés théoriques des accroissements du processus Y^H sont proches de celles qu'on observe sur les données de rendements de prix d'actions ou d'indices financiers : la corrélation des accroissements est indiscernable de celle d'un bruit blanc, mais on trouve un effet levier qui est juste au seuil de la significativité. On vérifie pour terminer, en comparant des résultats de simulations à des données réelles, qu'on a effectivement défini un processus aléatoire qui semble satisfaisant pour la modélisation de prix d'actions et d'indices financiers (*cf.* Figures 3 et 4).

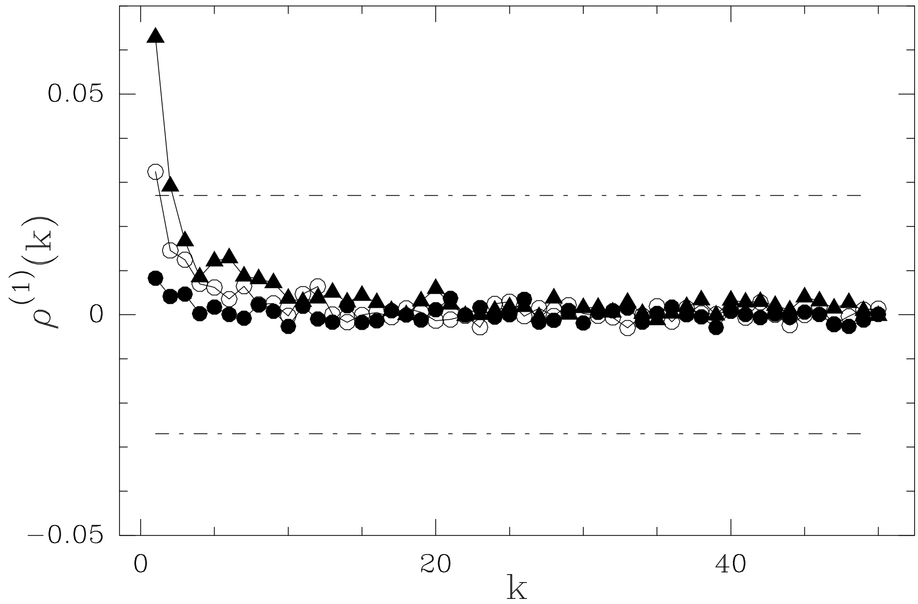

FIG. 3 – *Corrélations entre les accroissements de trajectoires simulées de Y^H, pour des décalages k tels que $1 \leq k \leq 50$, et avec $d = 0.01$ (•), $d = 0.05$ (∘) et $d = 0.10$ (▲). Les pointillés marquent l'intervalle de confiance à 95% pour une série décorrélée.*

0.3.4 Quatrième partie : questions liées à l'estimation paramétrique du modèle MRW log-normal

Nous nous penchons dans cette partie — encore exploratoire — sur le Problème 4 : nous essayons de répondre à la question de l'estimation des paramètres des processus MRM et MRW log-normaux. Rappelons que ces paramètres sont au nombre de trois : la

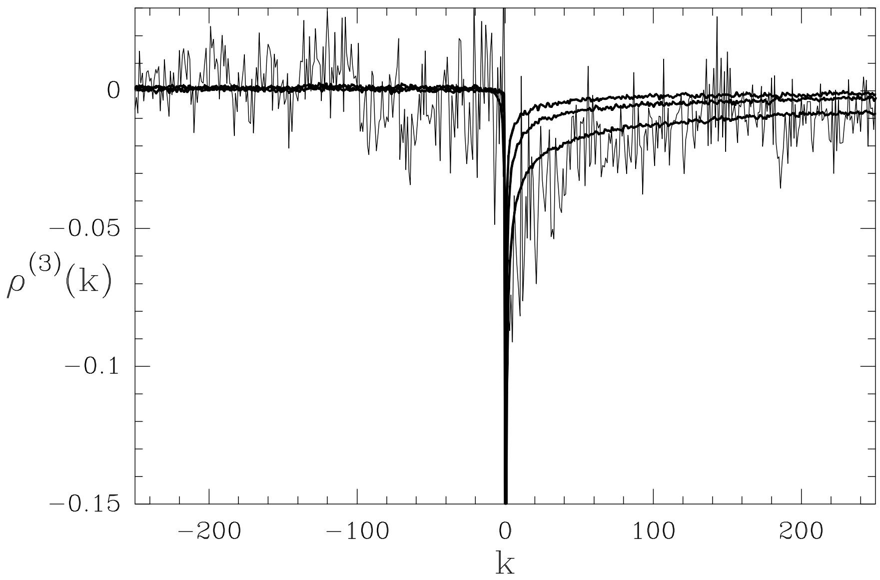

FIG. 4 – *Effet levier : corrélations entre les accroissements et le carré des accroissements de trajectoires simulées de Y^H pour des décalages k tels que $|k| \leq 250$. Les lignes épaisses représentent, de haut en bas, $d = 0.03$, $d = 0.1$ et $d = 0.30$. La courbe bruitée représente une moyenne obtenue à partir de données réelles sur cinq indices financiers.*

variance λ^2 (le coefficient d'intermittence) de la loi gaussienne utilisée dans la construction du processus, l'échelle intégrale $T > 0$, et le niveau moyen de volatilité $\sigma > 0$. À nouveau, précisons quelques notations. On considère I un intervalle inclus dans $\mathbb{R}^+$. Il sera commode ici de noter $P_I^{\lambda^2,T,\sigma^2}$ la loi d'un processus MRW $\big(X(t),\, t \in I\big)$ associée aux paramètres correspondant, où la loi $P_I^{\lambda^2,T,\sigma^2}$ est définie sur l'espace des trajectoires continues sur I muni de sa tribu borélienne pour la norme uniforme.

Résultat 9. *Soient T et σ deux réels strictement positifs, soit $\lambda \in [0, \sqrt{2})$, et soit I un intervalle de $\mathbb{R}^+$ tel que $|I| < +\infty$, où $|I|$ désigne la longueur de l'intervalle I.*

Soit $T' > 0$, $T \neq T'$, et supposons que $|I| < \min(T, T')$. Alors les lois $P_I^{\lambda^2,T,\sigma^2}$ et $P_I^{\lambda^2,T',\sigma^2}$ sont équivalentes.

Soit $\sigma' > 0$, $\sigma \neq \sigma'$, et supposons que $|I| < T$. Alors les lois $P_I^{\lambda^2,T,\sigma^2}$ et $P_I^{\lambda^2,T,\sigma'^2}$ sont équivalentes.

Soient T et $|I|$ quelconques, et soit $\lambda' \in [0, \sqrt{2})$, $\lambda \neq \lambda'$. Alors les lois $P_I^{\lambda^2,T,\sigma^2}$ et

$P_I^{\lambda'^2,T,\sigma^2}$ *sont étrangères.*

Ce résultat est d'abord un résultat négatif, dans le sens où il montre qu'on ne peut définir une asymptotique dans laquelle on pourrait estimer les paramètres T et σ à partir d'un grand nombre d'observations $X(t_0)$, $X(t_1)$, $\ldots$, X_{t_N} prises aux instants $t_i \in I$, $N \to +\infty$. En effet, même dans le cas où l'on observerait la totalité de la trajectoire $\big(X(t),\, t \in I\big)$, on voit qu'il n'existe pas d'événement qui « discrimine » les lois pour deux valeurs différentes des paramètres (T, σ) — au sens par exemple où un tel événement serait réalisé avec probabilité 1 pour une certaine valeur des paramètres et probabilité 0 pour une autre valeur. Ce résultat d'absolue continuité est une conséquence directe de la Propriété 10. Comme on l'a indiqué, Bacry, Kozhemyak et Muzy [10] fournissent un argument intuitif qui va dans le même sens que le raisonnement plus formel qu'on propose ici : si $|I| \leq \min(T, T')$, alors l'estimation de T ou de de σ est impossible faute de connaître la constante aléatoire w_r de la Propriété 10.

En revanche, le fait que les lois soient étrangères pour deux valeurs différentes de λ montre qu'il est en principe possible d'estimer le coefficient d'intermittence, pourvu qu'on dispose d'observations suffisament nombreuses. Le reste de cette partie de la thèse est de fait consacré à cette question de l'estimation de λ^2 à partir d'observations ou bien $X(0), X(1/n), \ldots, X(1)$ du processus MRW log-normal, ou bien $M(0), M(1/n), \ldots, M(1)$ du processus MRM log-normal.

On étudier et on compare quatre estimateurs différents du paramètre λ^2. Dans le cas d'observations MRW, on pose

$$x_{n,k} = \log |X((k+1)/n) - X(k/n)|$$

et $\bar{x}_n = 1/n \sum_{k=0}^{n-1} x_{n,k}$, et on définit :

1. L'estimateur $\hat{\lambda}_n^{2,p}$ pour $p > 0$, $p \neq 0$, 2, par

$$\hat{\lambda}_n^{2,p} = \frac{8(\hat{\tau}_n(p) + 1 - p/2)}{p(p-2)}$$

avec

$$\hat{\tau}_n(p) = \frac{1}{\log 2} \log\left(\frac{S_{\lfloor n/2 \rfloor}(p,1,0)}{S_n(p,1,0)}\right)$$

où S_n est comme en (0.5). Il s'agit donc d'un estimateur dont la convergence découle de celle du moment empirique. On démontre :

Résultat 10. *Pour* $4 \leq p < \min(2/\lambda^2, 2/\lambda)$, *p entier pair, la suite*

$$n^{1/2-\lambda^2 p^2/8}(\hat{\lambda}_n^{2,p} - \lambda^2)^2$$

est tendue et asymptotiquement non nulle.

Remarquons que cette vitesse en $n^{1/2-\lambda^2p^2/8}$ est analogue aux vitesses qui figurent dans les théorèmes centraux limites qu'Ossiander et Waymire [75] ou Ludeña [63] démontrent sur des statistiques voisines.

2. L'estimateur $\lambda_{J,j}^{2,var}$ par

$$\hat{\lambda}_{J,j}^{2,var} = 4\frac{\frac{1}{2^{J+j}-1}\sum_{k=0}^{2^{J+j}-1}\left(x_{2^{J+j},k} - \bar{x}_{2^{J+j}}\right)^2 - \frac{1}{2^J-1}\sum_{k=0}^{2^J-1}\left(x_{2^J,k} - \bar{x}_{2^J}\right)^2}{h(2^{j+J}) - h(2^J)}$$

avec

$$h(n) = \frac{2}{n(n-1)}\sum_{k=1}^{n-1}(n-k)\log(k+1).$$

(on suppose ici pour simplifier que $n = 2^{J+j}$ avec J et j entiers). L'idée de régresser ainsi la variance empirique du logarithme de l'accroissement du processus sur le logarithme de l'échelle repose par exemple sur la Propriété 4 dont on peut déduire qu'il existe bien une relation affine entre la variance théorique et le logarithme de l'échelle, relation dont la pente est donnée par λ^2. Comme on l'a indiqué, cet estimateur est voisin de certaines procédures d'estimation envisagées par Jaffard, Lashermes, Abry et Wendt [53, 94, 95, 93]. On montre :

Résultat 11. *On a pour* $0 \leq j \leq J$, *j fixé,*

$$\mathbb{E}\left[(\hat{\lambda}_{J,j}^{2,var} - \lambda^2)^2\right] \leq O(J^2 2^{-J}).$$

3. L'estimateur $\lambda_n^{2,cov}(h,h')$ pour h, $h' \in \mathbb{N}$, $h \neq h'$, par

$$\hat{\rho}_n(h) = \frac{1}{n-h} \sum_{k=0}^{n-h-1} (x_{n,k} - \bar{x}_n)(x_{n,k+h} - \bar{x}_n)$$

et

$$\hat{\lambda}_n^{2,cov}(h,h') = \frac{4\Big(\hat{\rho}_n(h) - \hat{\rho}_n(h')}{\gamma(h') - \gamma(h)},$$

où γ est une fonction explicite donnée par Bacry, Kozhemyak et Muzy [10]. Cet estimateur est fondé sur l'approximation gaussienne de la Propriété 9, et bien qu'il donne de très bons résultats empiriques sur les simulations, il est asymptotiquement biaisé pour $\lambda^2 > 0$. Bacry, Kozhemyak et Muzy montrent que pour $n \to +\infty$ et $\lambda^2 \to 0$,

$$\left| \mathbb{E}\Big[\hat{\lambda}_n^{2,cov}(h,h') - \lambda^2\Big] \right| \leq O(1/n) + O(\lambda^2)$$

et

$$\mathbb{V}\text{ar}\,[\hat{\lambda}_n^{2,cov}(h,h')] \leq O(\log(n)/n).$$

4. L'estimateur

$$\hat{\lambda}_n^{2,\Delta} = (m^2)^{-1}\Big(\hat{m}_n^2(\lambda^2)\Big)$$

avec

$$\hat{m}_n^2(\lambda^2) = \frac{1}{n-1} \sum_{k=0}^{n-2} (x_{n,k+1} - x_{n,k})^2$$

et $m^2(\lambda^2) = \mathbb{E}\Big[(x_{n,1} - x_{n,0})^2\Big]$. Comme on le démontre, la série stationnaire $(x_{n,k+1} - x_{n,k})_k$ a une autocorrélation qui décroît très rapidement, contrairement par exemple aux séries $(x_{n,k})_k$ ou $\Big(X((k+1)/n) - X(k/n)\Big)_k$. Cela en fait une série « agréable » à manier dans le cadre de problèmes d'estimation statistique. De plus, sa loi marginale, pour k fixé, ne dépend ni de n, ni de T, ni de σ^2. On démontre :

Résultat 12. *On a*

$$\mathbb{E}\Big[(\hat{\lambda}_n^{2,\Delta} - \lambda^2)^2\Big] \leq O(1/n).$$

Ce dernier résultat montre en particulier qu'on peut estimer λ^2 à la vitesse usuelle paramétrique $n^{-1/2}$, ce qui à notre connaissance est un résultat nouveau dans le contexte des processus MRW, ou plus généralement des processus de cascades. On notera cependant que ce dernier estimateur $\hat{\lambda}_n^{2,\Delta}$ pourra difficilement servir à caractériser l'appartenance de X à la classe des processus de cascades : en effet, et contrairement aux trois précédents estimateurs, on ne s'attend en particulier pas à ce qu'il admette une limite nulle ou infinie si on l'utilise dans le contexte de processus plus usuels comme par exemple le mouvement brownien fractionnaire.

Toutes ces définitions et tous ces résultats peuvent de plus être étendus de manière immédiate au cas d'observations MRM.

On étudie pour terminer les performances pratiques de ces estimateurs sur des jeux de données simulées par ordinateur. On considère les cas où les observations sont produites par un processus MRW ou par un processus MRM. On trouve que les quatre estimateurs se comportent de manière assez satisfaisante sur des observations MRM, bien que $\hat{\lambda}_n^{2,\Delta}$ donne des performances légèrement meilleures que celles des trois autres : *cf.* par exemple Figure 5. On trouve également que l'estimateur $\hat{\lambda}_n^{2,p}$ donne de moins bonnes performances lorsque la vraie valeur de λ^2 est grande que lorsque cette valeur est petite, ce qui est en accord avec le résultat théorique ci-dessus. Dans le cas d'observations MRW, $\hat{\lambda}_n^{2,p}$ et $\hat{\lambda}_{J,j}^{2,var}$ donnent en revanche des résultats très mauvais, y compris pour des valeurs de $n = 2^J$ assez grandes, et $\hat{\lambda}_n^{2,cov}$ produit nettement les meilleures performances, suivi par $\hat{\lambda}_n^{2,\Delta}$, *cf.* Figure 6. Une interprétation de ce phénomène est que $\hat{\lambda}_n^{2,cov}$ est un estimateur fondé sur une *covariance* empirique, si bien qu'il n'est que peu perturbé par la composition par un mouvement brownien, opération qui pour la série du logarithme des accroissements correspond à l'ajout d'un bruit blanc. En revanche, les estimateurs $\hat{\lambda}_{J,j}^{2,var}$ et $\hat{\lambda}_n^{2,\Delta}$ sont fondés sur la *variance* empirique du logarithme des accroissements du processus, et leurs performances sont ainsi très largement affectées par l'ajout d'un bruit blanc qui ne contient pas d'information sur le paramètre à estimer.

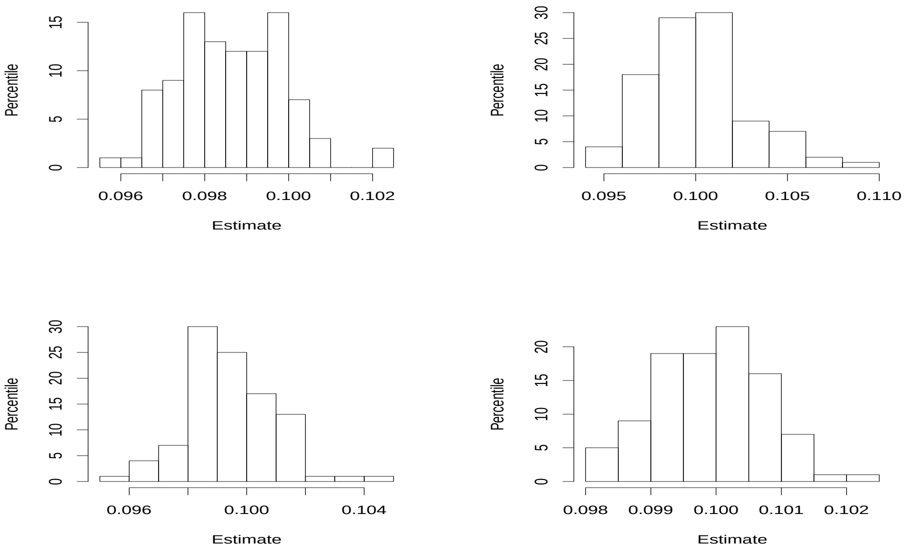

FIG. 5 – Distribution empirique des estimateurs pour 100 simulations d'un processus MRM ; $\lambda^2 = 0.1$; $n = 32\ 768$. En haut à gauche : $\hat{\lambda}_n^{2,p}$, en haut à droite : $\hat{\lambda}_n^{2,cov}$; en bas à gauche : $\hat{\lambda}_n^{2,var}$; en bas à droite : $\hat{\lambda}_n^{2,\Delta}$.

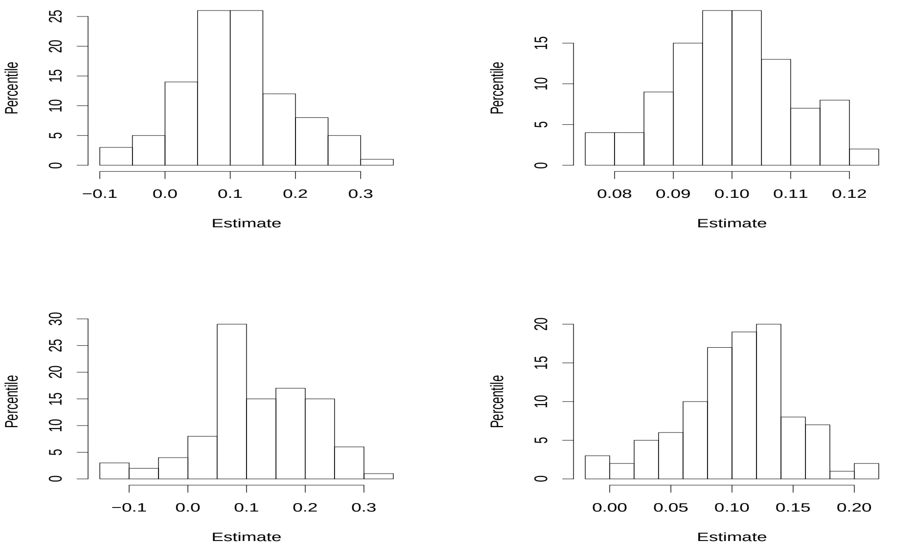

FIG. 6 – Distribution empirique des estimateurs pour 100 simulations d'un processus MRM ; $\lambda^2 = 0.1$; $n = 32\ 768$. En haut à gauche : $\hat{\lambda}_n^{2,p}$, en haut à droite : $\hat{\lambda}_n^{2,cov}$; en bas à gauche : $\hat{\lambda}_n^{2,var}$; en bas à droite : $\hat{\lambda}_n^{2,\Delta}$.

Une question de notation

On s'intéresse dans la thèse au processus MRM qui est défini par

$$M(t) = \lim_{l \to 0} \int_0^t \mathrm{e}^{w_l(u)} du, \quad t \geq 0$$

et au processus MRW défini par

$$X(t) = B\Big(M(t)\Big) \overset{loi}{=} \lim_{l \to 0} \int_0^t \mathrm{e}^{w_l(u)/2} dB(u), \quad t \geq 0.$$

Il aurait bien sûr été tout aussi naturel d'écrire

$$M(t) = \lim_{l \to 0} \int_0^t \mathrm{e}^{2w_l(u)} du \quad \text{et} \quad X(t) \overset{loi}{=} \lim_{l \to 0} \int_0^t \mathrm{e}^{w_l(u)} dB(u).$$

Dans le premier chapitre de la thèse, il est un en sens davantage question de certaines propriétés de M que de propriétés de X, aussi nous avons retenu la première notation. Nous la conservons dans les deuxième et quatrième chapitres qui sont dans la lignée du premier.

En particulier dans les cas où w est un processus gaussien, on peut le caractériser par sa variance : $\mathbb{V}\mathrm{ar}[w_l(u)] = \lambda^2\Big(\log(T/l) + 1\Big)$. Alors selon Bacry, Kozhemyak et Muzy [10], une valeur typique de λ^2 pour un modèle financier serait de $0,025$. Selon les notations que nous utilisons dans les premier, deuxième et quatrième chapitres, cela correspond donc à une valeur de 0,1.

En revanche, il est question dans la troisième partie de la thèse de la construction d'un processus X^H :

$$X^H(t) = \lim_{l \to 0} \int_0^t \mathrm{e}^{w_l(u)} \varepsilon_l^H(u) du,$$

avec ε_l^H un processus gaussien qui tend vers un bruit gaussien fractionnaire. On est donc plus proche dans cette partie de la seconde notation.

Chapter 1

Convergence of the structure function of a multifractal random walk in a mixed asymptotic setting

1.1 Introduction

Multifractal random processes have become quite popular since the last two decades, notably in fully developed turbulence (see Frisch and Parisi [43, 42], or Gagne, Marchand and Castaing [44, 45]) or finance (see Mandelbrot [69], Bouchaud and Potters [25], or Calvet and Fisher [28]) among other fields. This popularity comes from the observation of what is often called a multifractal scaling behavior, or multifractal scale invariance, in the data: Given some observation horizon $t > 0$ and some real-valued data $\Big(f(x),\, x \in [0,t]\Big)$, the structure function of the data simply refers to the empirical p-th moments of the fluctuations $|f(x+l) - f(x)|$ at a small scale $l > 0$. Then the scaling property of the data can be defined as the power-law behavior of this structure function, which means that the relation

$$\frac{1}{\lfloor t/l \rfloor} \sum_{k=1}^{\lfloor t/l \rfloor} \Big| f\Big((k+1)l\Big) - f\Big(kl\Big) \Big|^p \approx c(p) l^{\zeta(p)} \text{ as } l \to 0$$

holds for a variety of exponents $p > 0$. Here, $\lfloor a \rfloor$ is the integer part of the positive real number a. When the scaling exponent ζ is nonlinear, one speaks of multifractal scaling.

Numerous observations of multifractal scaling have motivated the mathematical study of functions that satisfy this property. In particular, a large amount of work (see notably Jaffard [52]) has been devoted to the so-called Frisch-Parisi conjecture [43] which establishes a link between the scaling exponent ζ and the regularity of the signal f taken as a function on the interval $[0, t]$: according to this conjecture, if $D(h)$ is the Hausdorff dimension of the level set of the points x where f exhibits a given pointwise Hölder exponent h, then D and ζ are related to one another by a Legendre transform.

If we now wish to model such data by a real-valued random process $X = \Big(X(t),\, t \geq 0\Big)$ with stationary increments, the moments of this process should have a multifractal scaling. That is:

Property 1.1 (Scaling of the moments)**.** *There exists a real-valued nonlinear function ζ defined on a nonempty subset $E_1 \subseteq (0, +\infty)$ such that*

$$l^{-\zeta(p)} \mathbb{E}\Big[|X(l) - X(0)|^p\Big] \to c(p) \text{ as } l \to 0$$

for all $p \in E_1$ and some positive numbers $c(p)$.

Moreover, the structure function (the empirical moments) should have the same scaling property. In this chapter, we consider the structure function taken defined with some sampling frequency $u_n \in \mathbb{N}$, $u_n \to +\infty$ for large n. We also place ourselves in a mixed asymptotic setting where the observation horizon may be fixed or may grow as tu_n^χ for some fixed numbers $\chi \geq 0$ and $t > 0$; we give incentives to do so below. Thus, we wish that X has the following property:

Property 1.2 (Scaling of the structure function)**.** *Assume that Property 1.1 holds for ζ defined on E_1. For $\chi \geq 0$, there exists a nonempty subset $E_2 \subseteq E_1$, which possibly depends* *on χ, such that for $t > 0$ and $p \in E_2$, the renormalized structure function*

$$u_n^{\zeta(p)-1-\chi} \sum_{k=1}^{\lfloor t u_n^{1+\chi} \rfloor} |X\Big((k+1)u_n^{-1}\Big) - X\Big(k u_n^{-1}\Big)|^p$$

converges to a positive finite limit as n goes to $+\infty$.

Finally, the logarithm of this structure function should provide a consistent estimator of the exponent ζ. Indeed, when dealing with real data, the multifractal nature of the data is generally characterized through a nonlinear behavior of this logarithm. This gives the following property:

Property 1.3 (Estimation of the scaling exponent)**.** *Assume that Property 1.1 holds for ζ defined on E_1. For $\chi \geq 0$, there exists a nonempty subset $E_3 \subseteq E_1$, which possibly depends on χ, such that for $t > 0$ and $p \in E_3$,*

$$\frac{\log \sum_{k=1}^{\lfloor t u_n^{1+\chi} \rfloor} \Big|X\Big((k+1)u_n^{-1}\Big) - X\Big(k u_n^{-1}\Big)\Big|^p}{\log(u_n^{-1})} \to \zeta(p) - 1 - \chi \text{ as } n \to +\infty.$$

Remark that if Property 1.2 holds with almost sure convergence and a set E_2, then clearly Property 1.3 holds with almost sure convergence and a set E_3 such that $E_2 \subseteq E_3$. However, it may be the case that the reverse inclusion $E_3 \subseteq E_2$ is not true.

This chapter is devoted to the study of Properties 1.2 and 1.3 when X belongs to the class of Multifractal Random Walks (MRW) defined by Bacry and Muzy in [11]. We give the modes of convergence and define below the sets E_1, E_2 and E_3 mentionned in this properties; they will be open intervals in $(0, +\infty)$ with $E_2 = E_3$. We also prove that they are almost maximal in the sense that if p is larger than the supremum of the interval, then the properties do not hold.

By an MRW, we mean a continuous time random process of the form

$$X(t) = B\Big(M(t)\Big), \quad t \geq 0,$$

where $B = \Big(B(t),\, t \geq 0\Big)$ is a standard Browian motion, $M = \Big(M(t),\, t \geq 0\Big)$ is a cascade process in the sense of Bacry and Muzy in [11], and B and M are independent. The process M is positive, nondecreasing, possesses stationary increments; it is also called Multifractal Random Measure (MRM) by Bacry and Muzy. Its moment of order $p > 0$ satisfies Property 1.1 whenever the moment is finite, from which we see that the process X also satifies Property 1.1. By an argument based on the scaling property of the Brownian motion B, we will see that the convergence of the structure function of X is directly connected to the convergence of the structure function of M.

Let us describe the connections between this chapter and the work of other authors. The best known examples of processes that satisfy Property 1.1 are Mandelbrot cascades (see Mandelbrot [68] and Kahane and Peyrière [57]) which are constructed by iterated multiplication of positive i.i.d. random variables on a b-adic grid for some fixed integer b. Such processes also satisfy Properties 1.2 and 1.3 as was shown by Molchan [71] (for convergence in probability) and Ossiander and Waymire [75] (for almost sure convergence); however both properties only hold when the structure function is taken on b-adic increments with the same b that is used in the definition of the process. The simplicity of the construction of these cascades indeed has the drawback that b-adic and non b-adic increments have fundamentally different properties. The MRM of Bacry and Muzy is based on one of the continuous analogues of the construction of Mandelbrot cascades, where the product of i.i.d. random variables is replaced by the exponential of a Lévy process, so that the increments are indeed stationary. To this extent, our results give a generalization of the convergence obtained by Ossiander and Waymire.

The results of Ossiander and Waymire were proved in a "fine resolution" setting where the discretization step $u_n^{-1} = b^{-n}$ goes to zero whereas the observation horizon is fixed (i.e. $\chi = 0$ with the notations of Property 1.2). However, it is not obvious that this asymptotic setting should always be the best for handling a large number of data. Indeed, an important feature of Mandelbrot cascades and Multifractal Random Walks is the parameter $T > 0$ called integral scale, which plays the role of a decorrelation time: two increments of the process are independent as soon as they are taken on intervals which lie at a distance greater than T. The behavior of the structure function will then be clearly different depending on the fact that the observation horizon tu_n^χ is less than T or much greater. The latter can notably happen in the case of turbulence study where many integral scales may be observed. A work by Bacry, Gloter, Hoffmann and Muzy [8] revisits the convergence of the structure function of a Mandelbrot cascade in a "mixed asymptotic" setting where χ is positive. Then the sets E_2 and E_3 in properties 1.2 and 1.3 nontrivially depend on the parameter $\chi \in [0, \infty)$. In particular, the authors show that the set E_3 is nondecrasing with χ, so that by averaging over tu_n^χ independent integral scales with a large χ, one may recover more exponents $\zeta(p)$ through the convergence stated in Property 1.3. We extend these results to the MRW framework: we prove Property 1.2 in this mixed asymptotic setting and show that the regimes for recovering the exponent ζ in Property 1.3 are the same for MRW's and Mandelbrot cascades.

Whereas Property 1.1 was already shown by Bacry and Muzy in [11] (actually the relation in Property 1.1 is an exact equality for all $l \leq T$), Property 1.2 has not yet been studied in the case of MRW's, with the exception of a recent work by Ludeña [62] which investigates the case of integer values of the exponent p in a slightly different framework than ours, since M is integrated with respect to a fractional Brownian motion with Hurst parameter $H \in [1/2, 3/4)$. In particular, she proves some special cases of the convergence stated in Theorem 1.1 below.

The chapter is organized as follows. In Section 1.2, we recall the construction of an MRW, state and discuss our results. Sections 1.3 and 1.4 are respectively devoted to the proofs of Theorems 1.1 and 1.2 that state Property 1.2 in the fine resolution ($\chi = 0$) and mixed asymptotic ($\chi > 0$) settings. The limit is a nondegenerate random variable in the first case and a deterministic value in the second case. Section 1.5 consists in the proof of Theorem 1.3 that states Property 1.3 and the maximality of the sets E_2 and E_3. Some technical proofs are presented in the appendices.

1.2 Definitions and results

1.2.1 Construction of M and X

Let us recall the construction of the MRM as it is described by Bacry and Muzy in [11]. We first fix a number $T > 0$ that is the integral scale of the process and an infinitely divisible distribution $\pi(dx)$ on $\mathbb{R}$. Let ψ be the Laplace exponent of π:

$$e^{\psi(q)} = \int_{\mathbb{R}} e^{qx}\pi(dx)$$

for $q \geq 0$ (possibly $\psi(q) = \infty$). For $q \geq 1$, we define the following condition on $\pi(dx)$:

Assumption 1.1 ($\mathbf{A}_q$)**.** *The positive number q and the infinitely divisible distribution $\pi(dx)$ are such that*

$$\psi(1) = 0,$$

and

$$\psi\Big(q(1+\epsilon)\Big) < q(1+\epsilon) - 1 \textit{ for some } \epsilon > 0.$$

Note that since ψ is convex and satisfies $\psi(0) = \psi(1) = 0$ under $\mathbf{A}_q$, it is an increasing function on $[1, +\infty)$, so that for $1 < q_1 < q_2$, $\mathbf{A}_{q_2} \Rightarrow \mathbf{A}_{q_1} \Rightarrow \mathbf{A}_1$.

Let μ be the measure on the open half-plane $\mathbb{R} \times (0, \infty)$ given by $\mu(dt, dl) = l^{-2}dt \otimes dl$. One can define (see Rajput and Rosinski [80]) an infinitely divisible, independently scattered random measure P on $\mathbb{R} \times (0, \infty)$ that has an intensity μ and a Laplace exponent ψ, that is:

- for every Borel set B in $\mathbb{R} \times (0, \infty)$, $P(B)$ is an infinitely divisible random variable such that:
$$\mathbb{E}\left[e^{qP(B)}\right] = e^{\mu(B)\psi(q)} \tag{1.1}$$
for every $q \geq 0$ such that $\psi(q) < \infty$,

- for every sequence $\{B_k\}_{k\in\mathbb{N}}$ of disjoint Borel sets in $\mathbb{R} \times (0, \infty)$, the variables $P(B_k)$

are independent and

$$P(\cup_{k\in\mathbb{N}} B_k) = \sum_{k\in\mathbb{N}} P(B_k) \text{ almost surely.}$$

The process $w = \big(w_l(t),\, (t,l) \in \mathbb{R}\times(0,\infty)\big)$ is defined by $w_l(t) = P(\mathcal{A}_l(t))$ for $(t,l) \in \mathbb{R}\times(0,\infty)$, where

$$\mathcal{A}_l(t) = \left\{(t',l') \in \mathbb{R}\times(0,\infty),\, l \le l' \text{ and } |t-t'| \le \frac{1}{2}\min(l',T)\right\}.$$

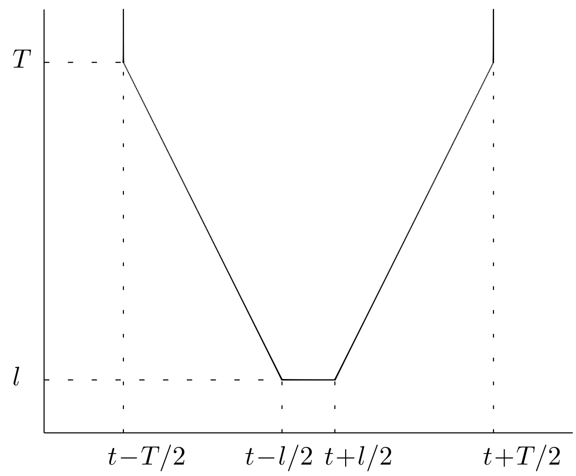

Figure 1.1: The cone $\mathcal{A}_l(t)$

Let $\sigma > 0$. The following essential result is borrowed from [11]:

Proposition 1.1 (Existence of the MRM [11])**.** *If Assumption* $\mathbf{A}_1$ *holds, then* $M_l(t) = \sigma^2 \int_0^t e^{w_l(u)} du$ *converges as* $l \to 0$ *almost surely and in* L^1 *to a random variable* $M(t)$*.*

It is then clear from the L^1 convergence and the condition $\psi(1) = 0$ that $\mathbb{E}[M(t)] = t$. Under Assumption $\mathbf{A}_1$, we can define an MRW by setting $X(t) = B\big(M(t)\big)$ where B is a standard Brownian motion independent of M. Then M and X are random processes with continuous paths and stationary increments. Let us stress that by construction, two increments $M(b) - M(a)$ and $M(d) - M(c)$ with $a < b < c < d$ are independent as soon as $|b - c| \ge T$. Obviously, the same also holds for the increments of X.

1.2.2 The moments of order $q \geq 1$ of X and M

Let us give the criterion for the existence of the moments of M and X, also borrowed from [11]:

Proposition 1.2 (Moments of the MRM [11])**.** *If Assumption $\mathbf{A}_q$ holds for some $q > 1$, then for $\epsilon > 0$ such that $\psi(q(1+\epsilon)) < q(1+\epsilon) - 1$ and $r \in \Big[0, q(1+\epsilon)\Big)$, $M_l(t)$ converges in L^r as $l \to 0$. In this case, the r-th moment of $M(t)$ for $t \in [0,T]$ and is given by*

$$\mathbb{E}\Big[M(t)^r\Big] = \gamma(r)\sigma^{2r}T^{\psi(r)}t^{r-\psi(r)}$$

where $\gamma(r)$ is positive and does not depend on t ($\gamma(0) = \gamma(1) = 1$).

Conversely, if $\mathbf{A}_1$ holds and $\psi(q) > q - 1$ for some $q > 1$ (so that $\mathbf{A}_q$ does not hold), then $\mathbb{E}\Big[M(t)^q\Big] = +\infty$.

Remark 1.1. Under Assumption $\mathbf{A}_1$, let us define

$$q^* = \sup\{q \geq 1,\ \mathbf{A}_q \text{ holds}\}. \tag{1.2}$$

From the scaling property of the Brownian motion B, the process X then satisfies Property 1.1 with a scaling exponent $\zeta^X : p \mapsto p/2 - \psi(p/2)$ defined on the set $E_1 = (0, 2q^*)$. This function is non linear as soon as the infinitely divisible distribution $\pi(dx)$ is non degenerate. Note that depending on $\pi(dx)$, it may be the case that $q^* = +\infty$ (for instance if $\pi(dx)$ is a Poisson distribution, but not if $\pi(dx)$ is a Gaussian distribution, see [11]).

1.2.3 The structure functions S_n and Σ_n

Let $(u_n,\, n \geq 0)$ be an non-decreasing sequence of positive integers such that $u_n \to +\infty$ for large n. For $t > 0$ and $\chi \geq 0$, we define $t_n = u_n^\chi t$ that is our observation horizon. Our object of interest is the structure function of X which we define with a sampling step

u_n^{-1}, $n \in \mathbb{N}$, that is:

$$S_n(2q,t,\chi) = \sum_{k=0}^{\lfloor t u_n^{1+\chi} \rfloor - 1} \Big| X\Big((k+1)u_n^{-1}\Big) - X\Big(k u_n^{-1}\Big)\Big|^{2q},\ t > 0,\ q \in [0,+\infty).$$

We define $b_{n,k}$ for $0 \le k \le \lfloor t u_n^{1+\chi} \rfloor - 1$ and $n \in \mathbb{N}$ as

$$b_{n,k} = M\Big((k+1)u_n^{-1}\Big) - M\Big(k u_n^{-1}\Big). \tag{1.3}$$

Then, using the scaling property of the Brownian motion, we see that $S_n(2q,t,\chi)$ has the same law as

$$\sum_{k=0}^{\lfloor t u_n^{1+\chi} \rfloor - 1} |\xi_k|^{2q} b_{n,k}^q$$

where the ξ_k's are i.i.d. standard normal random variables independent of M. In particular, if we define $\Sigma_n(q,t,\chi)$ as

$$\Sigma_n(q,t,\chi) = \sum_{k=0}^{\lfloor t u_n^{1+\chi} \rfloor - 1} b_{n,k}^q,$$

then under Assumption $\mathbf{A}_q$ and as soon as $u_n^{-1} \le T$, we have from Proposition 1.2:

$$\mathbb{E}\Big[\Sigma_n(q,t,\chi)\Big] = \gamma(q)\sigma^{2q} T^{\psi(q)} \lfloor t u_n^{\chi+1} \rfloor u_n^{\psi(q)-q} \tag{1.4}$$

and

$$\mathbb{E}\Big[S_n(2q,t,\chi)\Big] = \mathbb{E}\Big[|\xi_0|^{2q}\Big]\mathbb{E}\Big[\Sigma_n(q,t,\chi)\Big]. \tag{1.5}$$

We will study the behavior of $S_n(2q,t,\chi)$ and $\Sigma_n(q,t,\chi)$ in different asymptotics. In the "fine resolution“ setting, $\chi = 0$ so that the observation horizon is fixed, while the case $\chi > 0$ defines what we call the "mixed asymptotic“ setting.

With no loss of generality, we set $\sigma = 1$.

1.2.4 Asymptotic values and regimes

For $q > 0$ such that $\psi(q) < +\infty$, we introduce a new sequence of processes $M^{(q)}_{1/u_n}(t)$ that will be shown to be an asymptotic value of S_n and Σ_n. Its definition is similar to the above definition of $M_l(t)$. We write

$$P^{(q)}(B) = qP(B) - \log \mathbb{E}[e^{qP(B)}]$$

for every Borel set B of $\mathbb{R} \times (0, \infty)$. The function $\psi^{(q)} : r \mapsto \psi(qr) - r\psi(q)$, defined for nonnegative r's such that $\psi(qr) < +\infty$, is then the Laplace exponent of $P^{(q)}$. In particular, if we set

$$w^{(q)}_{1/u_n}(t) = P^{(q)}(\mathcal{A}_{1/u_n}(t)) \quad \text{for } t > 0,\, n \in \mathbb{N},$$

then the process $w^{(q)}$ has the following property:

$$\mathbb{E}\Big[e^{rw^{(q)}_{1/u_n}(t)}\Big] = e^{\mu(\mathcal{A}_{1/u_n})\psi^{(q)}(r)} = \frac{\mathbb{E}\Big[e^{qrw_{1/u_n}(t)}\Big]}{\Big(\mathbb{E}\Big[e^{q1/w_{u_n}(t)}\Big]\Big)^r}$$

for r and $q \geq 0$ such that $\psi(qr) < +\infty$. We now define $M^{(q)}_{1/u_n}(t) = \int_0^t e^{w^{(q)}_{1/u_n}(u)} du$; in particular we have $\mathbb{E}[M^{(q)}_{1/u_n}(t)] = t$. We finally introduce a condition on q and χ that will define the asymptotic regimes of S_n and Σ_n:

Assumption 1.2 ($\mathbf{B}^{(q)}(\chi)$). *The infinitely divisible distribution $\pi(dx)$ and the real numbers $q > 0$ and $\chi \geq 0$ are such that*

$$\psi\Big(q(1+\epsilon)\Big) < +\infty \text{ and } \psi^{(q)}(1+\epsilon) < \epsilon(1+\chi) \text{ for some } \epsilon > 0.$$

It is straightforward to show from the convexity of the Laplace exponent ψ that for $\epsilon > 0$, $\psi^{(q)}(1+\epsilon)$ increases with q. Thus for $\chi \geq 0$ and $0 < q_1 < q_2$, $\mathbf{B}^{(q_2)}(\chi) \Rightarrow \mathbf{B}^{(q_1)}(\chi)$. Conversely, if $0 < \chi_1 < \chi_2$ and $q > 0$, we clearly have $\mathbf{B}^{(q)}(0) \Rightarrow \mathbf{B}^{(q)}(\chi_1) \Rightarrow \mathbf{B}^{(q)}(\chi_2)$.

Note that under Assumptions $\mathbf{A}_1$ and $\mathbf{B}^{(q)}(0)$, Proposition 1.1 gives the existence of the limit $M^{(q)}(t) = \lim M^{(q)}_{1/u_n}(t)$ for $q > 0$ and $t > 0$, where the convergence is almost sure and in L^1. However, if only Assumptions $\mathbf{A}_1$ and $\mathbf{B}^{(q)}(\chi)$ hold for $\chi > 0$, then $M^{(q)}_{1/u_n}(t)$ does not necessarily have a nondegenerate limit.

1.2.5 Statement of the main results

Theorem 1.1 (Convergence of S_n and Σ_n in the fine resolution setting)**.** *Suppose that either $q \in (0, 1]$ and Assumption $\mathbf{A}_1$ holds, or $q > 1$ and Assumptions $\mathbf{A}_q$ and $\mathbf{B}^{(q)}(0)$ hold, then for $t > 0$*

$$\frac{S_n(2q, t, 0)}{\mathbb{E}\Big[S_n(2q, t, 0)\Big]} \to \frac{M^{(q)}(t)}{t} \quad as \quad n \to +\infty$$

in L^1, and almost surely if $u_n \geq c_1 e^{c_2 n}$ for some c_1, $c_2 > 0$. Moreover, the same result also holds if one replaces $S_n(2q, t, 0)$ with $\Sigma_n(q, t, 0)$.

Theorem 1.2 (Convergence of S_n and Σ_n in the mixed asymptotic setting)**.** *For $\chi > 0$, suppose that either $q \in (0, 1]$ and Assumption $\mathbf{A}_1$ holds, or $q > 1$ and Assumptions $\mathbf{A}_q$ and $\mathbf{B}^{(q)}(\chi)$ hold, then for $t > 0$*

$$\frac{S_n(2q, t, \chi)}{\mathbb{E}\Big[S_n(2q, t, \chi)\Big]} \to 1 \quad as \quad n \to +\infty$$

in L^1, and almost surely if $u_n \geq c_1 e^{c_2 n}$ for some c_1, $c_2 > 0$. Moreover, the same result also holds if one replaces $S_n(2q, t, \chi)$ with $\Sigma_n(q, t, \chi)$.

Remark 1.2. For $q > 1$, Proposition 1.1 shows that if Assumptions $\mathbf{A}_1$ (or $\mathbf{A}_q$) and $\mathbf{B}^{(q)}(0)$ hold, then $M^{(q)}(t)$ is well defined and $\mathbb{E}[M^{(q)}(t)] = t$. In particular, the strong law of large numbers proves that for $\chi > 0$

$$\frac{M^{(q)}(u_n^{\chi} t)}{\mathbb{E}[M^{(q)}(u_n^{\chi} t)]} \to 1 \quad \text{as} \quad n \to +\infty$$

almost surely and in L^1. However, if only Assumptions $\mathbf{A}_q$ and $\mathbf{B}^{(q)}(\chi)$ hold, then $M^{(q)}(t)$ is not necessarily well defined, so that Theorem 1.2 is not an immediate consequence of Theorem 1.1.

The case $q \in (0,1]$ is simpler. Indeed, one may check that Assumption $\mathbf{A}_1$ is the same as Assumption $\mathbf{B}^{(1)}(0)$, which implies Assumption $\mathbf{B}^{(q)}(0)$, so that $M^{(q)}(t)$ is always well defined in this case.

For some given $\chi \geq 0$ and infinitely divisible distribution $\pi(dx)$, we define q_χ as:

$$q_\chi = \sup\{q > 0, \, \mathbf{B}^{(q)}(\chi) \text{ holds}\}. \tag{1.6}$$

Under Assumption $\mathbf{A}_1$, it is clear that $q_\chi \geq 1$. Depending on the distribution $\pi(dx)$, it may be the case that $q_\chi = +\infty$.

Theorem 1.3 (Estimation of the scaling exponent)**.** *If $q > 0$ and the infinitely divisible distribution $\pi(dx)$ are such that Assumption $\mathbf{A}_q$ holds, then for $t > 0$ and $\chi \geq 0$, the following convergences hold in probability, and almost surely if $u_n \geq c_1 e^{c_2 n}$ for some c_1, $c_2 > 0$: if $q < q_\chi$*

$$\frac{\log(S_n(2q,t,\chi))}{\log(u_n^{-1})} \to q - \psi(q) - 1 - \chi \quad \textit{as} \quad n \to +\infty,$$

and if $q_\chi < +\infty$ and $q \geq q_\chi$

$$\frac{\log(S_n(2q,t,\chi))}{\log(u_n^{-1})} \to q(1 - \psi'(q_\chi)) \quad \textit{as} \quad n \to +\infty.$$

Moreover, the same results also hold if one replaces $S_n(2q,t,\chi)$ with $\Sigma_n(q,t,\chi)$.

The reader will find the proofs of these theorems in the remaining sections of the chapter. Recall now that q^* has been defined as the supremum of the $q \geq 1$ such that $\mathbf{A}_q$ holds. The theorems above allow us to state Properties 1.1, 1.2, and 1.3 for MRM's and MRW's. We define $\zeta^X : p \mapsto p/2 - \psi(p/2)$, $E_1^X = (0, 2q^*)$, and $E_2^X = E_3^X = (0, 2\min\{q^*, q_\chi\})$, and also $\zeta^M : q \mapsto q - \psi(q)$, $E_1^M = (0, q^*)$, and $E_2^M = E_3^M = (0, \min\{q^*, q_\chi\})$.

Corollary 1.1. *If the infinitely divisible distribution $\pi(dx)$ is such that Assumption $\mathbf{A}_1$ holds, then X and M satisfy Properties 1.1, 1.2, and 1.3, the convergence being almost sure and in L^1 for Property 1.2 and almost sure for Property 1.3. The function ζ in this* *properties is* ζ^X *for* X *and* ζ^M *for* M*, and the sets* E_1*,* E_2 *and* E_3 *are respectively* E_1^X*,* E_2^X *and* E_3^X*, and* E_1^M*,* E_2^M *and* E_3^M*. Moreover, these sets are almost maximal, in the sense that the properties do not hold for any strictly larger open sets in* $(0, +\infty)$*.*

Proof of Corollary 1.1. As we already mentioned (see Remark 1.1), Property 1 and the almost maximality of the set E_1 has been proved by Bacry and Muzy in [11]. From the definition of q_χ, it is straightforward to check that if $q_\chi < q$, then

$$q - \psi(q) - 1 - \chi \neq q(1 - \psi'(q_\chi)).$$

Thus, Theorem 1.3 clearly implies the statement of Corollary 1.1 concerning Property 1.3. Moreover, Theorems 1.1 and 1.2 state that Property 1.2 holds for the set E_2, while Theorem 1.3 also proves that Property 1.2 does not hold for an open set E such that $E_2 \subset E \subseteq E_1$.

□

Remark 1.3. The same results in the framework of Mandelbrot cascades were obtained by Ossiander and Waymire [75] (in the fine resolution setting) and Bacry *et al.* [8] (in the mixed asymptotic setting). This could be interpreted in the following way: eventhough MRW's and MRM's are quite more elaborate objects than Mandelbrot cascades, they share some essential properties.

Remark 1.4. Theorem 1.3 shows that from $\log(S_n(2q, t, \chi))/\log(u_n)$, we can easily obtain a consistent estimator of $\psi(q)$, $q \in (0, \min\{q^*, q_\chi\})$. Note that in the case $\chi = 0$ and $q > 1$, one may show with simple arguments based on the convexity of ψ that if $\mathbf{A}_1$ and $\mathbf{B}^{(q)}(0)$ both hold, then Assumption $\mathbf{A}_q$ also holds. Thus the former condition is sufficient for the convergence of S_n in Theorem 1.1. Under $\mathbf{A}_1$, we have in particular that if $q_0 < +\infty$, then $q_0 < q^*$, so that the set E_2 in Corollary 1.1 increases with χ: one is able to recover more and more values $\psi(q)$ when χ grows.

In the case $\chi = \infty$ (that is, the resolution scale u_n^{-1} is fixed while the observation horizon t goes to infinity), one will then be able to estimate $\psi(q)$ for all $q \in (0, q^*)$. Indeed, if we define for $q > 0$, $t > 0$, and $n \in \mathbb{N}$:

$$S_n(2q,t,+\infty) = \sum_{k=0}^{\lfloor tu_n \rfloor -1} \left| X\Big((k+1)u_n^{-1}\Big) - X\Big(ku_n^{-1}\Big)\right|^{2q},$$

then two increments $|X\Big((k+1)u_n^{-1}\Big) - X\Big(ku_n^{-1}\Big)\Big|$ and $|X\Big((k'+1)u_n^{-1}\Big) - X\Big(k'u_n^{-1}\Big)\Big|$ are independent as soon as $|k-k'-1|u_n^{-1} > T$. Thus, we may apply the strong law of large numbers, since for $0 < q < q^*$, Proposition 1.2 implies that $S_n(2q,t,\infty)$ has a finite expectation. This gives: almost surely,

$$\frac{1}{\lfloor tu_n \rfloor} S_n(2q,t,\infty) \to \mathbb{E}\Big[|X(u_n^{-1}) - X(0)|^{2q}\Big] \quad \text{as} \quad t \to +\infty.$$

From the scaling property of the Brownian motion and Proposition 1.2 (assuming that n is such that $u_n^{-1} \leq T$), this limit is

$$\mathbb{E}\Big[|X(u_n^{-1}) - X(0)|^{2q}\Big] = a(2q)\gamma(q)T^{\psi(q)}u_n^{n\psi(q)-q},$$

where $a(2q)$ is the absolute moment of order $2q$ of a standard normal random variable. Therefore, if we choose two different values n_1 and n_2 in $\mathbb{N}$, then almost surely

$$\frac{\log\Big(S_{n_1}(2q,t,+\infty)\Big) - \log\Big(S_{n_2}(2q,t,+\infty)\Big)}{\log(u_{n_2}) - \log(u_{n_1})} \to q - \psi(q) - 1 \quad \text{as} \quad t \to +\infty.$$

Remark 1.5. It would be interesting to obtain convergence rates of this estimator in the case $\chi \in [0,+\infty)$. This will be done in some particular cases and for $\chi = 0$ in Chapters 2 and 4 of this dissertation. However, empirical evidence (see Jaffard, Lashermes and Abry [53] and Bacry, Kozhemyak and Muzy [10] – see also Chapter 4) suggests that more elaborate estimators attain faster rates and should be used in practice.

Remark 1.6. A signal is of multifractal regularity if it exhibits several pointwise Hölder exponents h on sets of positive Hausdorff dimension $D(h)$. The so-called "multifractal formalism" claims that $D(h)$ is the Legendre transform of the exponent $q \mapsto \zeta^M(q) + 1 = \psi(q) - q + 1$ of the structure function $\Sigma_n(q,t,0)$. In the framework of Mandelbrot cascades, Bacry, Gloter, Hoffmann and Muzy [8] define $D(h)$ as a box-counting dimension. Applying a large deviations principle and an analogue of Theorem 1.3 that is also valid for negative q's, these authors show that in the setting of the mixed asymptotic, $D(h)$ is the Legendre transform of $q \mapsto \psi(q) - q + 1 + \chi$. If one could show that Theorem 1.3 remains valid in the case of negative values of q, then it would be straightforward to observe that the arguments of Bacry, Gloter, Hoffmann and Muzy [8] could be also applied to MRM's and MRW's, so that this "large deviations multifractal formalism" would hold – but this is beyond the scope of what we prove here.

We write $a_n \lesssim b_n$ if there exists some real (non-random) number $c > 0$ such that

$$\forall\, n,\ a_n \leq c b_n$$

and $a_n \asymp b_n$ if there exist some real (non-random) numbers c_1, $c_2 > 0$ such that

$$\forall\, n,\ c_1 b_n \leq a_n \leq c_2 b_n.$$

The symbol $\overset{d}{=}$ denotes equality in distribution.

1.3 Proof of Theorem 1.1

1.3.1 Outline of the proof

The proof is separated in two steps. We first prove Proposition 1.3 which states that $S_n(2q, t, \chi)$ and $\Sigma_n(q, t, \chi)$ are asymptotically equal. Next, we prove Proposition 1.4 which gives a precise upper bound for the term

$$\left| \frac{\Sigma_n(q,t,0)}{\mathbb{E}[\Sigma_n(q,t,0)]} - \frac{M^{(q)}_{1/u_n}(t)}{\mathbb{E}[M^{(q)}_{1/u_n}(t)]} \right|.$$

Under Assumptions $\mathbf{A}_1$ and $\mathbf{B}^{(q)}(0)$, we finally apply Proposition 1.1 to see that $M^{(q)}_{1/u_n}(t) \to M^{(q)}(t)$ almost surely and in L^1. This shows Theorem 1.1.

Note that the statements of these propositions remain valid under broader assumptions than those of Theorem 1.1 (for instance, they do not require Assumption $\mathbf{B}^{(q)}(0)$); this enables us to use these two propositions during the proof of Theorem 1.2.

Proposition 1.3. *If $q > 0$, $\chi \geq 0$, and the infinitely divisible distribution $\pi(dx)$ are such that Assumptions $\mathbf{A}_q$ and $\mathbf{B}^{(q)}(\chi)$ hold, then for $t > 0$*

$$\left| \frac{S_n(2q,t,\chi)}{\mathbb{E}[S_n(2q,t,\chi)]} - \frac{\Sigma_n(q,t,\chi)}{\mathbb{E}[\Sigma_n(q,t,\chi)]} \right| \to 0 \quad as \quad n \to +\infty$$

in L^1, and almost surely if $u_n \geq c_1 e^{c_2 n}$ for some c_1, $c_2 > 0$..

Proposition 1.4. *Let $q > 0$ and the infinitely divisible distribution $\pi(dx)$ be such that Assumption $\mathbf{A}_q$ holds. Then there exist some processes A_n and B_n such that for $t > 0$*

$$\frac{\Sigma_n(q,t,0)}{\mathbb{E}[\Sigma_n(q,t,0)]} - \frac{M^{(q)}_{1/u_n}(t)}{\mathbb{E}[M^{(q)}_{1/u_n}(t)]} \asymp A_n(t) + B_n(t),$$

where the processes A_n and B_n satisfy the following properties: the sequences $\Big(A_n(ku_n^{-1}),\, k \in \mathbb{N}\Big)$ and $\Big(B_n(ku_n^{-1}),\, k \in \mathbb{N}\Big)$ have stationary increments, these increments are independent as soon as they are taken on intervals that lie at a distance greater than T,

$$\mathbb{E}\Big[|A_n(t)|\Big] \lesssim u_n^{-\alpha}$$

for some $\alpha > 0$, and

$$\mathbb{E}[B_n(t)] = 0 \quad and \quad \mathbb{E}[\,|B_n(t)|^{1+\epsilon}] \lesssim u_n^{\psi(q)(1+\epsilon)-\epsilon}$$

for $\epsilon \in (0,1)$ such that $\mathbb{E}[M(t)^{q(1+\epsilon)}] < +\infty$.

1.3.2 Proof of Proposition 1.3

Recall that from the scaling property of the Brownian motion,

$$S_n(2q,t,\chi) \stackrel{d}{=} \sum_{k=0}^{\lfloor u_n^{\chi+1} t\rfloor -1} |\xi_k|^{2q} b_{n,k}^q$$

where the ξ_k's are i.i.d. standard normal random variables independent of M. From Assumption $\mathbf{B}^{(q)}(\chi)$, we may choose $\epsilon > 0$ such that

$$-\psi\Big(q(1+\epsilon)\Big) + (1+\epsilon)\psi(q) + \epsilon(1+\chi) > 0. \tag{1.7}$$

We write $a(2q)$ for the absolute moment of order $2q$ of the ξ_k's, so that

$$\mathbb{E}[S_n(2q,t,\chi)] = a(2q)\mathbb{E}[\Sigma_n(q,t,\chi)].$$

We now study the moment of order $1+\epsilon$ of

$$\left|\frac{S_n(2q,t,\chi) - a(2q)\Sigma_n(q,t,\chi)}{\mathbb{E}[S_n(2q,t,\chi)]}\right|.$$

Factorizing by the increments of M, we have:

$$\mathbb{E}\left[\left|S_n(2q,t,\chi) - a(2q)\Sigma_n(q,t,\chi)\right|^{1+\epsilon}\right] = \mathbb{E}\left[\left|\sum_{k=0}^{\lfloor u_n^{1+\chi} t\rfloor -1} \Big(|\xi_k|^{2q} - a(2q)\Big) b_{n,k}^q\right|^{1+\epsilon}\right].$$

We will use several times the following inequality: let $Y_1, ..., Y_n$ be a sequence of martingale increments and fix $\epsilon \in [0,1]$. Then

$$\mathbb{E}\left[\left|\sum_{k=1}^{n} Y_k\right|^{1+\epsilon}\right] \lesssim \sum_{k=1}^{n} \mathbb{E}\left[|Y_k|^{1+\epsilon}\right] \tag{1.8}$$

(a proof can be found in [13]). If we take

$$Y_k = \Big(|\xi_k|^{2q} - a(2q)\Big) b_{n,k}^q,$$

then conditionally on the sigma-field generated by the $b_{n,k}$'s, $k = 0, \dots, \lfloor u_n^{1+\chi} t \rfloor - 1$, it is clear that the Y_k's are i.i.d. and centered. Inequality (1.8) therefore applies:

$$\mathbb{E}\Big[|S_n(2q,t,\chi) - a(2q)\Sigma_n(q,t,\chi)|^{1+\epsilon}\Big] \lesssim \mathbb{E}[\Sigma_n(q(1+\epsilon),t,\chi)].$$

From (1.4), one has:

$$\mathbb{E}[S_n(2q,t,\chi)] \asymp \mathbb{E}[\Sigma_n(q,t,\chi)] \asymp u_n^{-(q-\psi(q)-1-\chi)},$$

so that

$$\mathbb{E}\left[\left|\frac{S_n(2q,t,\chi) - a(2q)\Sigma_n(q,t,\chi)}{\mathbb{E}S_n(2q,t,\chi)}\right|^{1+\epsilon}\right] \lesssim u_n^{\psi[q(1+\epsilon)]-(1+\epsilon)\psi(q)-\epsilon(1+\chi)}.$$

Inequality (1.7) and the Borel-Cantelli lemma end the proof.

1.3.3 Proof of Proposition 1.4

Outline of the proof

First note that if tu_n is an integer, we have from (1.4):

$$\frac{\Sigma_n(q,t,0)}{\mathbb{E}[\Sigma_n(q,t,0)]} - \frac{M_{1/u_n}^{(q)}(t)}{\mathbb{E}[M_{1/u_n}^{(q)}(t)]} = t^{-1}\left(u_n^{q-\psi(q)-1}\frac{\Sigma_n(q,t,0)}{\gamma(q)T^{\psi(q)}} - M_{1/u_n}^{(q)}(t)\right). \tag{1.9}$$

We restrict ourselves to this case. Indeed, if tu_n is not an integer, we define $B_n(t)$ to be the same as $B_n(\lfloor tu_n \rfloor u_n^{-1})$ and

$$A_n(t) = A_n(\lfloor tu_n \rfloor u_n^{-1}) + \frac{M_{1/u_n}^{(q)}(t)}{\mathbb{E}\Big[M_{1/u_n}^{(q)}(t)\Big]} - \frac{M_{1/u_n}^{(q)}(\lfloor tu_n \rfloor u_n^{-1})}{\mathbb{E}\Big[M_{1/u_n}^{(q)}(\lfloor tu_n \rfloor u_n^{-1})\Big]}.$$

Since $\mathbb{E}\Big[M_{1/u_n}^{(q)}(t)\Big] = t$ for $t > 0$, we clearly have that

$$\mathbb{E}\left[\left|\frac{M_{1/u_n}^{(q)}(t)}{\mathbb{E}\Big[M_{1/u_n}^{(q)}(t)\Big]} - \frac{M_{1/u_n}^{(q)}(\lfloor tu_n \rfloor u_n^{-1})}{\mathbb{E}\Big[M_{1/u_n}^{(q)}(\lfloor tu_n \rfloor u_n^{-1})\Big]}\right|\right] \lesssim u_n^{-1}.$$

Our proof relies on a partition of the cones $\mathcal{A}_l(s)$ that are used in the definition of the process $w_l(s)$. We fix $\delta \in (0,1)$ and $\epsilon \in (0,1)$. From Assumption $\mathbf{A}_q$, we can choose ϵ such that $\mathbb{E}\Big[M(t)^{q(1+\epsilon)}\Big] < +\infty$. For fixed n, s in the interval $[ku_n^{-1},(k+1)u_n^{-1})$, $l \leq u_n^{-1}$, and $v_n = \lfloor u_n^{1-\delta} \rfloor$, we write:

$$\mathcal{A}_l(s) = \tilde{\mathcal{A}}_{1/u_n}(k) \cup \mathcal{B}_{l,1/u_n}(s) \cup \mathcal{T}_{1/u_n}(s)$$

where:

$$\tilde{\mathcal{A}}_{1/u_n}(k) = \bigcap_{s' \in [ku_n^{-1},(k+1)u_n^{-1}]} \mathcal{A}_l(s'),$$

$\mathcal{T}_{1/u_n}(s)$ is the subset $\mathcal{A}_l(s) \setminus \tilde{\mathcal{A}}_{1/u_n}(k)$ that lies above the horizontal line of y-coordinate $v_n^{-1} - u_n^{-1}$, and $\mathcal{B}_{l,1/u_n}(s)$ is the subset that lies below. Remark in particular that $\tilde{\mathcal{A}}_{1/u_n}(k)$ does not depend on l (see figure 1.2).

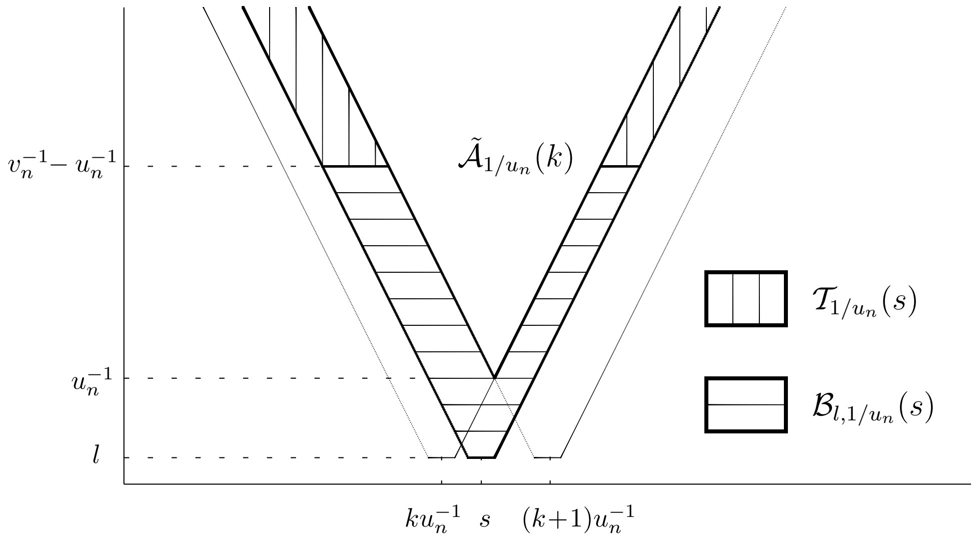

Figure 1.2: Partition of $\mathcal{A}_l(s)$

The images of these subsets by P define new random processes: $\tilde{w}_{1/u_n}(k) = P(\tilde{\mathcal{A}}_{1/u_n}(k))$, $\theta_{1/u_n}(s) = P(\mathcal{T}_{1/u_n}(s))$, and $\beta_{l,1/u_n}(s) = P(\mathcal{B}_{l,1/u_n}(s))$. Likewise, $\tilde{w}^{(q)}_{1/u_n}(k)$, $\theta^{(q)}_{1/u_n}(s)$, and $\beta^{(q)}_{l,1/u_n}(s)$ are defined by replacing P by $P^{(q)}$. It is straightforward to compute the surface of $\tilde{\mathcal{A}}_{1/u_n}(k)$ as measured by $\mu(dt,dl) = l^{-2}dt \otimes dl$:

$$\mu(\tilde{\mathcal{A}}_{1/u_n}(k)) = \log(Tu_n)$$

so that

$$\mathbb{E}\Big[e^{q\tilde{w}_{1/u_n}(0)}\Big] = T^{\psi(q)} u_n^{\psi(q)}. \tag{1.10}$$

We now justify this partition and our approach. Let us remark that from (1.10) and from the definition of the Laplace exponents ψ and $\psi^{(q)}$, one has:

$$T^{-\psi(q)} u_n^{q-\psi(q)-1} \left(u_n^{-1} e^{\tilde{w}_{1/u_n}(k)}\right)^q = u_n^{-1} e^{\tilde{w}^{(q)}_{1/u_n}(k)}. \tag{1.11}$$

This identity plays a key role in our proof. Indeed, we would like to justify the following approximations:

$$e^{qw_{1/u_n}(k)} \approx e^{q\tilde{w}_{1/u_n}(k)}$$

and

$$\Sigma_n(q,t,0) \approx \sum_k \Big(u_n^{-1} e^{w_{1/u_n}(k)}\Big)^q.$$

If we were able to do this, we could probably as well justify the following:

$$e^{w^{(q)}_{1/u_n}(k)} \approx e^{\tilde{w}^{(q)}_{1/u_n}(k)}$$

and

$$M^{(q)}_{1/u_n}(t) \approx \sum_k u_n^{-1} e^{w^{(q)}_{1/u_n}(k)}.$$

Renormalizing every quantity above by their respective expectations and using (1.11), these approximations would thus provide a link between $\Sigma_n(q,t,0)$ and $M^{(q)}_{1/u_n}(t)$. We however have no easy method to justify these approximations; part of the difficulty comes from the fact that the dependence between the variables which are summed over k decays very slowly. Therefore we introduce the decomposition

$$\mathcal{A}_l(s) \setminus \tilde{\mathcal{A}}_{1/u_n}(k) = \mathcal{B}_{l,1/u_n}(s) \cup \mathcal{T}_{1/u_n}(s)$$

in order to obtain independence for some of the $P(\mathcal{B}_{l,1/u_n}(s))$'s, which in particular enables us to apply martingale inequalities. We find that some of the terms are more easily handled in L^1 norm, while the others are better handled in $L^{1+\epsilon}$ norm, thus leading to the terms A_n and B_n in the statement of the proposition.

Let us define

$$c_{n,k} = \lim_{l\to 0} \int_{ku_n^{-1}}^{(k+1)u_n^{-1}} e^{\beta_{l,1/u_n}(s)} ds \tag{1.12}$$

and

$$d_{n,k} = \lim_{l\to 0} \int_{ku_n^{-1}}^{(k+1)u_n^{-1}} e^{\beta_{l,1/u_n}(s)+\theta_{1/u_n}(s)} ds, \tag{1.13}$$

where, according to the results of Bacry and Muzy in [11], the limits hold almost surely and in L^1 under $\mathbf{A}_1$. We will also need the term $\gamma_n(q)$:

$$\gamma_n(q) = u_n^q \mathbb{E}\left[c_{n,k}^q\right] \tag{1.14}$$

We will prove that $\gamma_n(q) \to \gamma(q)$, where $\gamma(q)$ has been defined in (1.4). It can be seen directly that $\gamma(1) = \gamma_n(1) = 1$.

We now write from (1.9):

$$\frac{\Sigma_n(q,t,0)}{\mathbb{E}[\Sigma_n(q,t,0)]} - \frac{M^{(q)}_{1/u_n}(t)}{\mathbb{E}[M^{(q)}_{1/u_n}(t)]} = t^{-1}(A_1 + B_1 + B_2 + A_2)$$

with:

$$A_1 = T^{-\psi(q)} u_n^{q-\psi(q)-1} \left(\gamma(q)^{-1}\Sigma_n(q,t,0) - \sum_{k=0}^{tu_n-1} \gamma_n(q)^{-1} c_{n,k}^q e^{q\tilde{w}_{1/u_n}(k)} \right),$$

$$B_1 = T^{-\psi(q)} u_n^{q-\psi(q)-1} \left(\sum_{k=0}^{tu_n-1} \gamma_n(q)^{-1} c_{n,k}^q e^{q\tilde{w}_{1/u_n}(k)} - (u_n^{-1} e^{\tilde{w}_{1/u_n}(k)})^q \right),$$

$$B_2 = \sum_{k=0}^{tu_n-1} \left(u_n^{-1} e^{\tilde{w}^{(q)}_{1/u_n}(k)} - \int_{ku_n^{-1}}^{(k+1)u_n^{-1}} e^{\beta^{(q)}_{1/u_n,1/u_n}(s)+\tilde{w}^{(q)}_{1/u_n}(k)} ds \right),$$

$$A_2 = \sum_{k=0}^{tu_n-1} \int_{ku_n^{-1}}^{(k+1)u_n^{-1}} e^{\beta^{(q)}_{1/u_n,1/u_n}(s)+\tilde{w}^{(q)}_{1/u_n}(k)} ds - M^{(q)}_{1/u_n}(t)$$

(recall from (1.11) that the difference C

$$C = T^{-\psi(q)} u_n^{q-\psi(q)-1} \sum_{k=0}^{tu_n-1} (u_n^{-1} e^{\tilde{w}_{1/u_n}(k)})^q - \sum_{k=0}^{tu_n-1} u_n^{-1} e^{\tilde{w}^{(q)}_{1/u_n}(k)}$$

is exactly zero.) The terms $A_n(t)$ and $B_n(t)$ in the statement of Proposition 1.4 correspond to $A_1 + A_2$ and $B_1 + B_2$. The properties stated in the proposition are easily verified, except for the upper bounds, which we prove below. The upper bound for $\mathbb{E}\Big[|A_1|\Big]$ and $\mathbb{E}\Big[|A_2|\Big]$ that we obtain is respectively $u_n^{-n\alpha_1}$ and $u_n^{-\alpha_2}$ for some α_1, $\alpha_2 > 0$; this bound is established mainly from the fact that θ_{1/u_n} becomes zero when $v_n = \lfloor u_n^{1-\delta} \rfloor \to +\infty$. The upper bound of $\mathbb{E}\Big[|B_1|^{1+\epsilon}\Big]$ and $\mathbb{E}\Big[|B_2|^{1+\epsilon}\Big]$ that we obtain is $2^{n(\psi^{(q)}(1+\epsilon)-\epsilon)}$; this bound is established as a consequence of the martingale inequality (1.8).

The following technical Lemma 1.1 will be useful:

Lemma 1.1. *Let $q > 0$ and the infinitely divisible distribution $\pi(dx)$ be such that Assumption $\mathbf{A}_q$ holds, so that we may choose $\epsilon \in (0,1)$ such that $\mathbb{E}[M(t)^{q(1+\epsilon)}] < +\infty$, and let r be a real number in $(0, q(1+\epsilon)]$. Then:*

(i) *Let $\mathcal{C}$ be a Borel set in $\mathbb{R} \times (0, +\infty)$ such that $\mu(\mathcal{C}) < +\infty$, and let $\mathcal{C} + s$ be the set $\Big\{(t,l) \in \mathbb{R} \times (0,+\infty), (t-s,l) \in \mathcal{C}\Big\}$ for $s \in \mathbb{R}$. Then for $t > 0$ the moments $\mathbb{E}\Big[\sup_{0 \le u \le t} e^{rP(\mathcal{C}+u)}\Big]$ and $\mathbb{E}\Big[\sup_{0 \le u \le t} e^{(1+\epsilon)P^{(q)}(\mathcal{C}+u)}\Big]$ are finite.*

(ii) *There exist α_1, $\alpha_2 > 0$ such that*

$$\mathbb{E}\left[\sup_{s \in [0, u_n^{-1}]} |1 - e^{r\theta_{1/u_n}(s)}|\right] \lesssim u_n^{-\alpha_1} \text{ and } \mathbb{E}\left[\sup_{s \in [0, u_n^{-1}]} |1 - e^{\theta^{(q)}_{1/u_n}(s)}|\right] \lesssim u_n^{-\alpha_2}.$$

(iii) *This value α_1 also satisfies*

$$|\gamma_n(r) - \gamma(r)| \lesssim u_n^{-\alpha_1}.$$

(iv) *We have*

$$\mathbb{E}\Big[|\gamma_n(q) - c^q_{n,k}|^{1+\epsilon}\Big] \lesssim 1.$$

(v) *We have*

$$\mathbb{E}\left[\left|1 - u_n \int_0^{u_n^{-1}} e^{\beta^{(q)}_{1/u_n,1/u_n}(s)} ds\right|^{1+\epsilon}\right] \lesssim 1.$$

The reader will find the proof of this lemma in the appendix.

Upper bound for $\mathbb{E}\Big[|A_1|\Big]$

Let us recall that

$$\mathbb{E}[\Sigma_n(q,t,0)] \asymp u_n^{-(q-\psi(q)-1)}.$$

Then statement **(iii)** of Lemma 1.1 shows that

$$u_n^{q-\psi(q)-1}\Big|(\gamma(q)^{-1} - \gamma_n(q)^{-1})\mathbb{E}\Big[\Sigma_n(q,t,0)\Big]\Big| \lesssim u_n^{-\alpha_1}.$$

We therefore only have to give an upper bound for the expectation of:

$$u_n^{q-\psi(q)-1}\Big|\Sigma_n(q,t,0) - \sum_{k=0}^{u_n t-1} c^q_{n,k} e^{q\tilde{w}_{1/u_n}(ku_n^{-1})}\Big|.$$

We begin with the triangle inequality:

$$\begin{aligned}\mathbb{E}\left[\left|\Sigma_n(q,t,0) - \sum_{k=0}^{u_n t-1} c^q_{n,k} e^{q\tilde{w}_{1/u_n}(ku_n^{-1})}\right|\right] &= \mathbb{E}\left[\left|\sum_{k=0}^{u_n t-1} b^q_{n,k} - c^q_{n,k} e^{q\tilde{w}_{1/u_n}(ku_n^{-1})}\right|\right] \\ &\leq t u_n \,\mathbb{E}\left[\left|b^q_{n,0} - c^q_{n,0} e^{q\tilde{w}_{1/u_n}(0)}\right|\right]\end{aligned}$$

and

$$\mathbb{E}\left[\left|b^q_{n,0} - c^q_{n,0} e^{q\tilde{w}_{1/u_n}(0)}\right|\right] = \mathbb{E}\left[c^q_{n,0} e^{q\tilde{w}_{1/u_n}(0)}\left|\frac{b^q_{n,0}}{c^q_{n,0} e^{q\tilde{w}_{1/u_n}(0)}} - 1\right|\right].$$

The term in the absolute value on the right is dominated by

$$\sup_{u\in[0,u_n^{-1}]} |e^{r\theta_{1/u_n}(u)} - 1|$$

which is independent of $c_{n,0}$ and of $\tilde{w}_{1/u_n}(0)$. Moreover:

$$\begin{aligned}\mathbb{E}\Big[c_{n,0}^q e^{q\tilde{w}_{1/u_n}(0)}\Big] &\asymp u_n^{\psi(q)}\mathbb{E}\Big[c_{n,0}^q\Big] \\ &\asymp u_n^{-q+\psi(q)}\gamma_n(q)\end{aligned}$$

where (1.10) and (1.14) have been used. We finally use statement **(ii)** of Lemma 1.1 to show that

$$\mathbb{E}\Big[|A_1|\Big] \lesssim u_n^{-\alpha_1}.$$

Upper bound for $\mathbb{E}\Big[|A_2|\Big]$

The proof here is very similar to the previous one. We write

$$M_{1/u_n}^{(q)}(t) = \sum_{k=0}^{u_n t-1} M_{1/u_n}^{(q)}\Big((k+1)u_n^{-1}\Big) - M_{1/u_n}^{(q)}\Big(ku_n^{-1}\Big)$$

and apply the triangle inequality:

$$\mathbb{E}\Big[|A_2|\Big] \lesssim u_n\mathbb{E}\left[\left|\int_0^{u_n^{-1}} e^{\beta_{1/u_n,1/u_n}^{(q)}(u)+\tilde{w}_{1/u_n}^{(q)}(0)}du - M_{1/u_n}^{(q)}(u_n^{-1})\right|\right].$$

Using the same arguments as in the previous section, we have:

$$\mathbb{E}\Big[|A_2|\Big] \lesssim u_n\mathbb{E}\left[e^{\tilde{w}_{1/u_n}^{(q)}(0)}\int_0^{u_n^{-1}} e^{\beta_{1/u_n,1/u_n}^{(q)}(u)}du \sup_{u\in[0,u_n^{-1}]}|1-e^{\theta_{1/u_n}^{(q)}(u)}|\right].$$

Each of the three terms in the expectation of the right hand side is independent of the other two. Moreover, the expectation of the exponential term is 1, and the expectation of the integral term is u_n^{-1}. Applying **(ii)** of Lemma 1.1 gives the result.

Upper bound for $\mathbb{E}\Big[|B_1|^{1+\epsilon}\Big]$

So as to use slightly less cumbersome notations, we suppose here that $u_n/v_n = u_n/\lfloor u_n^{1-\delta}\rfloor$ is an integer. We write

$$Z_{k/u_n} = e^{q\tilde{w}_{1/u_n}(ku_n^{-1})}\Big(\gamma_n(q) - (u_n c_{n,k})^q\Big),$$

so that

$$B_1 \asymp u_n^{-(\psi(q)+1)} \sum_{k=1}^{u_n t} Z_{k/u_n}. \tag{1.15}$$

We first apply the convexity inequality:

$$\left|\sum_{k=1}^{N} x_k\right|^{1+\epsilon} \leq N^\epsilon \sum_{k=1}^{N} |x_k|^{1+\epsilon}.$$

This gives:

$$\begin{aligned}
\mathbb{E}\left[\left|\sum_{k=0}^{tu_n-1} Z_{k/u_n}\right|^{1+\epsilon}\right] &= \mathbb{E}\left[\left|\sum_{i=0}^{v_nt-1}\sum_{j=0}^{(u_n/v_n)-1} Z_{i/v_n+j/u_n}\right|^{1+\epsilon}\right] \\
&\leq (u_n/v_n)^\epsilon \sum_{j=0}^{(u_n/v_n)-1} \mathbb{E}\left[\left|\sum_{i=0}^{v_nt-1} Z_{i/v_n+j/u_n}\right|^{1+\epsilon}\right].
\end{aligned}$$

From the stationarity of the Z_{k/u_n}'s, $\mathbb{E}\left[\left|\sum_{i=0}^{v_nt-1} Z_{i/v_n+j/u_n}\right|^{1+\epsilon}\right]$ does not depend on j. We now show that inequality (1.8) can be applied to this term. For $j=0$ and $\bar{\imath} \leq v_n t - 1$, one has:

$$\mathbb{E}\left[Z_{\bar{\imath}/v_n}\Big|\sum_{i=0}^{\bar{\imath}-1} Z_{i/v_n}\right] = \mathbb{E}\left[\mathbb{E}\left[Z_{\bar{\imath}/v_n}\Big|e^{q\tilde{w}_{1/u_n}(\bar{\imath}/v_n)}, \sum_{i=0}^{\bar{\imath}-1} Z_{i/v_n}\right]\Big|\sum_{i=0}^{\bar{\imath}-1} Z_{i/v_n}\right].$$

By factorizing, the term $\mathbb{E}\left[Z_{\bar{\imath}/v_n}\Big|e^{q\tilde{w}_{1/u_n}(\bar{\imath}/v_n)}, \sum_{i=0}^{\bar{\imath}-1} Z_{i/v_n}\right]$ becomes:

$$e^{q\tilde{w}_{1/u_n}(\bar{\imath}/v_n)}\mathbb{E}\left[\gamma_n(q) - (u_n c_{n,\bar{\imath}u_n/v_n})^q\Big|e^{q\tilde{w}_{1/u_n}(\bar{\imath}/v_n)}, \sum_{i=0}^{\bar{\imath}-1} Z_{i/v_n}\right].$$

Let us now recall that $\beta_{l,1/u_n}(s) = P(\mathcal{B}_{l,1/u_n}(s))$ and observe that if s lies between $\bar{\imath}v_n^{-1}$ and $\bar{\imath}v_n^{-1} + u_n^{-1}$, then $\mathcal{B}_{l,1/u_n}(s)$ is of empty intersection with $\tilde{\mathcal{A}}_{1/u_n}(\bar{\imath}v_n^{-1})$, the $\tilde{\mathcal{A}}_{1/u_n}(iv_n^{-1})$'s for $i \leq \bar{\imath} - 1$, and the $\mathcal{B}_{l,1/u_n}(s')$'s for $s' \leq (\bar{\imath} - 1)v_n^{-1} + u_n^{-1}$.

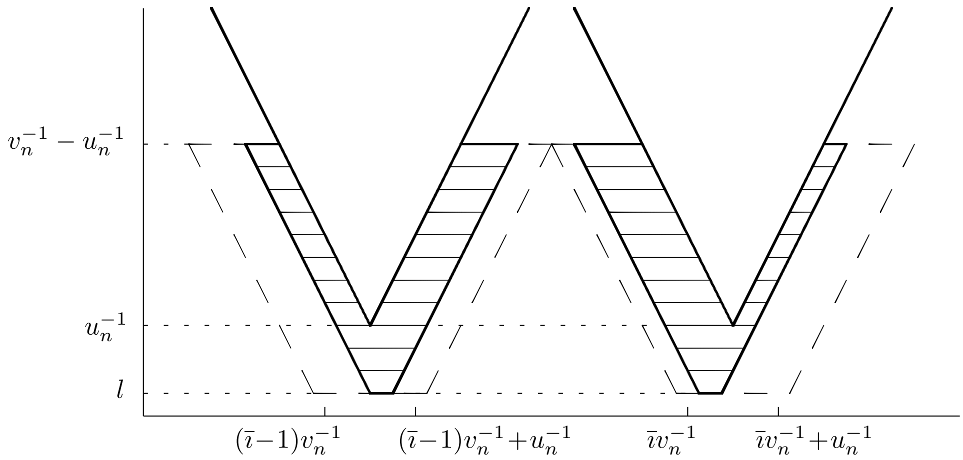

Figure 1.3: $\mathcal{B}_{l,1/u_n}(s) \cap \mathcal{B}_{l,1/u_n}(s')$ is empty if $|s - s'| > v_n^{-1} - u_n^{-1}$.

The random variables generated by P and these subsets of the halfplane are therefore independent, so that the conditional expectation above is non-random, and even zero from the definition (1.14) of $\gamma_n(q)$. Then $\left(\sum_{i=0}^{\bar{\imath}} Z_{i/v_n}\, 0 \leq \bar{\imath} \leq v_n t\rfloor - 1\right)$ is indeed a sequence of martingale increments, and inequality (1.8) applies:

$$\mathbb{E}\Big[\Big|\sum_{i=0}^{v_n t-1} Z_{i/v_n}\Big|^{1+\epsilon}\Big] \lesssim \sum_{i=0}^{v_n t-1} \mathbb{E}\Big[\Big|Z_{i/v_n}\Big|^{1+\epsilon}\Big].$$

Going back to B_1, we obtain:

$$\mathbb{E}\Big[|B_1|^{1+\epsilon}\Big] \lesssim u_n^{-(1+\epsilon)(1+\psi(q))}(u_n/v_n)^{1+\epsilon} v_n \mathbb{E}\Big[|Z_0|^{1+\epsilon}\Big].$$

Let us now give orders of magnitude for $E\Big[|Z_0|^{1+\epsilon}\Big]$:

$$\begin{aligned} E\Big[|Z_0|^{1+\epsilon}\Big] &\asymp \mathbb{E}\Big[e^{q(1+\epsilon)\tilde{w}_{1/u_n}(0)}\Big] \text{ (by \textbf{(iv)} of Lemma 1.1)} \\ &\asymp u_n^{\psi(q(1+\epsilon))} \text{ (by (1.10)).} \end{aligned}$$

Since we defined $v_n = \lfloor u_n^{1-\delta} \rfloor$, we get

$$\mathbb{E}\Big[|B_1|^{1+\epsilon}\Big] \lesssim u_n^{\psi(q(1+\epsilon))-(1+\epsilon)\psi(q)}(\lfloor u_n^{1-\delta} \rfloor)^{-\epsilon}.$$

As δ can be chosen arbitrarily small, the result follows.

Upper bound for $\mathbb{E}\Big[|B_2|^{1+\epsilon}\Big]$

We now write:

$$Z_{k/u_n} = e^{\tilde{w}^{(q)}_{1/u_n}(ku_n^{-1})}\left(1 - u_n \int_{ku_n^{-1}}^{(k+1)u_n^{-1}} e^{\beta^{(q)}_{1/u_n,1/u_n}(s)} ds\right)$$

so that

$$B_2 = u_n^{-1} \sum_{k=0}^{u_n t-1} Z_k.$$

Going along the same lines as the previous section, we find:

$$\mathbb{E}\Big[|B_2|^{1+\epsilon}\Big] \lesssim v_n^{-\epsilon}\mathbb{E}\Big[|Z_0|^{1+\epsilon}\Big].$$

From (1.10) and **(v)** of Lemma 1.1, we have

$$\mathbb{E}\Big[|Z_0|^{1+\epsilon}\Big] \lesssim u_n^{\psi(q)(1+\epsilon)}.$$

Letting $\delta \to 0$ achieves the proof.

1.4 Proof of Theorem 1.2

Note that Proposition 1.3 shows that if Theorem 1.2 holds for Σ_n, then it holds for S_n. In order to show that the theorem holds for Σ_n, we proceed in two steps. First we show that one can use Proposition 1.4 so as to bound

$$\mathbb{E}\left[\left|\frac{\Sigma_n(q,t,\chi)}{\mathbb{E}[\Sigma_n(q,t,\chi)]}-1\right|\right]$$

by the sum of a term that goes to zero faster than $u_n^{-\xi}$ for some $\xi > 0$ and the the term

$$u_n^{-\chi\epsilon}\mathbb{E}\Big[|M^{(q)}_{1/u_n}(T)-T|^{1+\epsilon}\Big].$$

Then the proof is over if Assumption $\mathbf{B}^{(q)}(0)$ holds, since in this case $M^{(q)}_{1/u_n}(T)$ converges in $L^{1+\epsilon}$ (see Proposition 1.2). However, in the case where only $\mathbf{B}^{(q)}(\chi)$ holds, a bit more work is required to show that this last term indeed goes to zero: thus our second step consists in showing

$$u_n^{-\chi\epsilon}\mathbb{E}\Big[|M^{(q)}_{1/u_n}(T)-T|^{1+\epsilon}\Big] \lesssim u_n^{-\xi'}$$

for some $\xi' > 0$. We finally apply the Borel-Cantelli lemma to obtain almost sure convergence.

1.4.1 First step

Let us define:

$$J = J(t,n,\chi,T) = \lfloor u_n^{\chi} t/T\rfloor - 1.$$

Then for $0 \leq j \leq J-1$, we set

$$\Delta_{u_n}(j) = \frac{\Sigma_n\Big(q,(j+1)T,0\Big)-\Sigma_n\Big(q,jT,0\Big)}{\mathbb{E}\Big[\Sigma_n\Big(q,T,0\Big)\Big]} - 1,$$

and

$$\Delta_{u_n}(J) = \frac{\Sigma_n\Big(q,t,\chi\Big) - \Sigma_n\Big(q,JT,0\Big)}{\mathbb{E}\Big[\Sigma_n\Big(q,T,0\Big)\Big]} - 1,$$

so that from (1.4)

$$\frac{\Sigma_n(q,t,\chi)}{\mathbb{E}[\Sigma_n(q,t,\chi)]} - 1 \asymp u_n^{-\chi} \sum_{j=0}^{J} \Delta_{u_n}(j).$$

Note that

$$0 \leq \mathbb{E}\Big[\Sigma_n\Big(q,t,\chi\Big) - \Sigma_n\Big(q,JT,0\Big)\Big] \leq \mathbb{E}\Big[\Sigma_n\Big(q,T,0\Big)\Big],$$

so that $\Delta_{u_n}(J)$ is bounded in L^1. Therefore

$$u_n^{-\chi}\mathbb{E}\Big[|\Delta_{u_n}(J)|\Big] \lesssim u_n^{-\chi}.$$

We now examine upper bounds for

$$\mathbb{E}\left[u_n^{-\chi}\left|\sum_{j=0}^{J-1} \Delta_{u_n}(j)\right|\right].$$

We introduce the process $M^{(q)}_{1/u_n}$ (let us recall that $\mathbb{E}[M^{(q)}_{1/u_n}(T)] = T$). For $0 \leq j \leq J-1$

$$\begin{aligned}\Delta_{u_n}(j) =& \frac{\Sigma_n\Big(q,(j+1)T,0\Big) - \Sigma_n\Big(q,jT,0)\Big)}{\mathbb{E}\Big[\Sigma_n\Big(q,T,0)\Big)\Big]} - \frac{M^{(q)}_{1/u_n}\Big((j+1)T\Big) - M^{(q)}_{1/u_n}\Big(jT\Big)}{T} \\ &+ \frac{M^{(q)}_{1/u_n}\Big((j+1)T\Big) - M^{(q)}_{1/u_n}\Big(jT\Big)}{T} - 1,\end{aligned}$$

From this, we write $\Delta_{u_n}(j) \asymp \Delta_{n,1}(j) + \Delta_{n,2}(j)$ with

$$\Delta_{n,1}(j) = A_n\Big((j+1)T\Big) - A_n(jT)$$

and

$$\Delta_{n,2}(j) = B_n\Big((j+1)T\Big) - B_n(jT) + \frac{M^{(q)}_{1/u_n}\Big((j+1)T\Big) - M^{(q)}_{1/u_n}\Big(jT\Big)}{T} - 1,$$

where the terms A_n and B_n have been introduced in Proposition 1.4. Thus,

$$\mathbb{E}\left[u_n^{-\chi}\left|\sum_{j=0}^{J-1}\Delta_{u_n}(j)\right|\right] \leq \mathbb{E}\left[u_n^{-\chi}\left|\sum_{j=0}^{J-1}\Delta_{n,1}(j)\right|\right] + \mathbb{E}\left[u_n^{-\chi}\left|\sum_{j=0}^{J-1}\Delta_{n,2}(j)\right|\right].$$

The triangle inequality shows that

$$\mathbb{E}\left[u_n^{-\chi}\left|\sum_{j=0}^{J-1}\Delta_{n,1}(j)\right|\right] \lesssim \mathbb{E}\Big[\Delta_{n,1}(0)\Big] = \mathbb{E}\Big[|A_n(T)|\Big].$$

According to Proposition 1.4, this term goes as $u_n^{-\alpha}$ for some $\alpha > 0$. Let us now deal with the terms $\Delta_{n,2}(j)$. From Assumptions $\mathbf{A}_q$ and $\mathbf{B}^{(q)}(\chi)$, we may choose $\epsilon \in (0,1)$ such that $\mathbb{E}\Big[M(t)^{q(1+\epsilon)}\Big] < +\infty$ and $\psi^{(q)}(1+\epsilon) - \epsilon - \chi\epsilon < 0$. For this ϵ, we have:

$$\mathbb{E}\left[u_n^{-\chi}\left|\sum_{j=0}^{J-1}\Delta_{n,2}(j)\right|\right] \leq \left(\mathbb{E}\left[u_n^{-\chi(1+\epsilon)}\left|\sum_{j=0}^{J-1}\Delta_{n,2}(j)\right|^{1+\epsilon}\right]\right)^{1/(1+\epsilon)}. \tag{1.16}$$

Moreover,

$$\begin{aligned}\mathbb{E}\left[\left|\sum_{j=0}^{J-1}\Delta_{n,2}(j)\right|^{1+\epsilon}\right] &= \mathbb{E}\left[\left|\sum_{j=0}^{\lfloor (J-1)/2\rfloor}\Delta_{n,2}(2j) + \sum_{j=0}^{\lfloor J/2\rfloor -1}\Delta_{n,2}(2j+1)\right|^{1+\epsilon}\right]\\ &\lesssim \mathbb{E}\left[\left|\sum_{j=0}^{\lfloor (J-1)/2\rfloor}\Delta_{n,2}(2j)\right|^{1+\epsilon}\right].\end{aligned}$$

Since the increments of B_n and $M_{1/u_n}^{(q)}$ are stationary and independent as soon as they are taken on intervals that lie at a distance larger than T, the $\Delta_{n,2}(2j)$'s are i.i.d. random variables. From Proposition 1.4, $\mathbb{E}[B_n(T)] = 0$, so that these variables are also centered. Therefore, inequality (1.8) can be applied, which gives:

$$\mathbb{E}\left[\left|\sum_{j=0}^{\lfloor (J-1)/2\rfloor}\Delta_{n,2}(2j)\right|^{1+\epsilon}\right] \lesssim u_n^{\chi}\mathbb{E}\Big[|\Delta_{n,2}(0)|^{1+\epsilon}\Big].$$

From the definition of $\Delta_{n,2}(0)$, we have:

$$\mathbb{E}\Big[|\Delta_{n,2}(0)|^{1+\epsilon}\Big] \lesssim \mathbb{E}\Big[|B_n(T)|^{1+\epsilon}\Big] + \mathbb{E}\Big[|M^{(q)}_{1/u_n}(T) - T|^{1+\epsilon}\Big]$$

and from the upper bound for $\mathbb{E}\Big[|B_n(T)|^{1+\epsilon}\Big]$ in Proposition 1.4, we have

$$\mathbb{E}\left[u_n^{-\chi(1+\epsilon)}\Big|\sum_{j=0}^{J-1}\Delta_{n,2}(j)\Big|^{1+\epsilon}\right] \lesssim u_n^{\psi^{(q)}(\epsilon)-\epsilon-\chi\epsilon} + u_n^{-\chi\epsilon}\mathbb{E}\Big[|M^{(q)}_{1/u_n}(T) - T|^{1+\epsilon}\Big].$$

Recall that we have chosen ϵ so that $\psi^{(q)}(\epsilon) - \epsilon - \chi\epsilon < 0$. Going back to (1.16), we have therefore proved that there exists some $\xi > 0$ such that

$$\mathbb{E}\left[\Big|\frac{\Sigma_n(q,t,\chi)}{\mathbb{E}[\Sigma_n(q,t,\chi)]} - 1\Big|\right] \lesssim u_n^{-\xi} + u_n^{-\chi\epsilon}\mathbb{E}\Big[|M^{(q)}_{1/u_n}(T) - T|^{1+\epsilon}\Big].$$

1.4.2 Second step

We show here that $u_n^{-\chi\epsilon}\mathbb{E}\Big[|M^{(q)}_{1/u_n}(T) - T|^{1+\epsilon}\Big]$ goes to zero faster than $u_n^{-\xi'}$, $\xi' > 0$. It will be enough to show that this result holds for $u_n^{-\chi\epsilon}\mathbb{E}[M^{(q)}_{1/u_n}(T)^{1+\epsilon}]$.

We define $w^{(q,\epsilon)}_{1/u_n} = w^{(q)}_{1/u_n} - \log(u_n)\chi\epsilon/(1+\epsilon)$ and $M^{(q,\epsilon)}_{1/u_n}(t) = \int_0^t e^{w^{(q,\epsilon)}_{1/u_n}(u)}du$ so that

$$u_n^{-\chi\epsilon}\mathbb{E}\left[M^{(q)}_{1/u_n}(T)^{1+\epsilon}\right] = \mathbb{E}\left[M^{(q,\epsilon)}_{1/u_n}(T)^{1+\epsilon}\right].$$

We now give two lemmas that are directly inspired by the proofs used by Bacry and Muzy in [11].

Lemma 1.2. *Under Assumption* $\mathbf{A}_1$*, for* $m \in \mathbb{N}$ *such that* $u_n^{-1} \leq T/m$*,*

$$\mathbb{E}\left[M^{(q,\epsilon)}_{1/u_n}(T)^{1+\epsilon}\right] \leq m2^{\epsilon}\mathbb{E}\left[M^{(q,\epsilon)}_{1/u_n}(T/m)^{1+\epsilon}\right] + cu_n^{-\chi\epsilon}$$

where $c > 0$ *depends on* m *but not on* n*.*

Lemma 1.3. *Under Assumption* $\mathbf{A}_1$*, for* $\lambda \in (0,1)$*,* $l \in (0,T]$*, and* $t \in (0,T]$*,*

$$M^{(q)}_{\lambda l}(\lambda t) \overset{d}{=} \lambda e^{w_\lambda} M^{(q)}_l(t)$$

where w_λ *is an infinitely divisible random variable independent of* $M^{(q)}$ *that satisfies*

$$\mathbb{E}[e^{(1+\epsilon)w_\lambda}] = \lambda^{-\psi^{(q)}(1+\epsilon)}.$$

We give a proof for Lemma 1.2 in the appendix. Lemma 1.3 is a less general statement of Lemma 2 of Bacry and Muzy in [11]. We do not reproduce its proof; it involves the computation of the characteristic function of the random vector $\big(w_l(t_1), \ldots, w_l(t_k)\big)$ through some elaborate combinatorial arguments.

Let us now remark that:

$$\begin{aligned}
\mathbb{E}\Big[M^{(q,\epsilon)}_{1/u_n}(T/m)^{1+\epsilon}\Big] &= u_n^{-\chi\epsilon}\mathbb{E}\Big[M^{(q)}_{1/u_n}(T/m)^{1+\epsilon}\Big] \\
&= u_n^{-\chi\epsilon} m^{-1-\epsilon+\psi^{(q)}(1+\epsilon)}\mathbb{E}\Big[M^{(q)}_{m/u_n}(T)^{1+\epsilon}\Big] \\
&= m^{-[1+(1+\chi)\epsilon-\psi^{(q)}(1+\epsilon)]}\mathbb{E}\Big[M^{(q,\epsilon)}_{m/u_n}(T)^{1+\epsilon}\Big],
\end{aligned}$$

where we used Lemma 1.3. Then from Lemma 1.2 we see that:

$$x_{u_n} \leq a\, x_{u_n/m} + c\, u_n^{-\chi\epsilon}$$

with

$$x_n = u_n^{-\chi\epsilon}\mathbb{E}\Big[M^{(q)}_{1/u_n}(T)^{1+\epsilon}\Big],$$

and

$$a = 2^\epsilon m^{-[(1+\chi)\epsilon-\psi^{(q)}(1+\epsilon)]}.$$

For a fixed m large enough, we will have $0 < a < 1$ since ϵ has been chosen such that

$$(1+\chi)\epsilon - \psi^{(q)}(1+\epsilon) > 0.$$

This achieves the proof.

1.5 Proof of Theorem 1.3

We follow closely the proof that is given by Ossiander and Waymire in [75] or Bacry *et al.* in [8] concerning Mandelbrot cascades. We reproduce it here for the sake of completeness. Note that it follows exactly the same pattern for S_n or Σ_n.

The case $0 < q < q_\chi$ is a direct consequence of Theorems 1.1 and 1.2 and of the relation:

$$\mathbb{E}\Big[S_n(2q, t, \chi)\Big] \asymp \mathbb{E}\Big[\Sigma_n(q; t, \chi)\Big] \asymp u_n^{-[q-\psi(q)-1-\chi]}.$$

We now consider the case $q \geq q_\chi$. All limits here are valid either in probability, or almost surely if u_n goes to zero exponentially fast. Let us take $q \geq q_\chi$.

Let ρ be in $(0, 1)$. From the sub-additivity of $x \mapsto x^\rho$,

$$\Sigma_n(q, t, \chi)^\rho \leq \Sigma_n(\rho q, t, \chi)$$

so that

$$\liminf_{n\to+\infty} \frac{\log(\Sigma_n(q, t, \chi))}{\log(1/u_n)} \geq \liminf_{n\to+\infty} \frac{\log(\Sigma_n(\rho q, t, \chi))}{\rho \log(1/u_n)}.$$

Letting $\rho \to q_\chi/q$, we obtain that

$$\liminf_{n\to+\infty} \frac{\log(\Sigma_n(q, t, \chi))}{\log(1/u_n)} \geq q \frac{q_\chi - \psi(q_\chi) - 1 - \chi}{q_\chi}.$$

Then, one can check from the definition of q_χ that the right hand side is equal to $q(1 - \psi'(q_\chi))$.

Next, let us choose q_1, q_2 so that $0 < q_1 < q_2 < q_\chi$. Then

$$\begin{aligned} \Sigma_n(q_2, t, \chi) &\leq \sup_{0\leq k\leq \lfloor tu_n \rfloor -1} b_{n,k}^{q_2-q_1} \Sigma_n(q_1, t, \chi) \\ &\leq \Sigma_n(q, t, \chi)^{\frac{q_2-q_1}{q}} \Sigma_n(q_1, t, \chi). \end{aligned}$$

From this it follows:

$$\limsup_{n\to+\infty} \frac{\log(\Sigma_n(q_2,t,\chi))}{\log(1/u_n)} \geq \frac{q_2-q_1}{q} \limsup_{n\to+\infty} \frac{\log(\Sigma_n(q,t,\chi))}{\log(1/u_n)} + \limsup_{n\to+\infty} \frac{\log(\Sigma_n(q_1,t,\chi))}{\log(1/u_n)}$$

which gives:

$$\limsup_{n\to+\infty} \frac{\log(\Sigma_n(q,t,\chi))}{\log(1/u_n)} \leq q\frac{q_2-q_1-\psi(q_2)+\psi(q_1)}{q_2-q_1}.$$

We now just have to take the limit $q_1 \to q_\chi$ and $q_2 \to q_\chi$ to obtain the result.

1.A Proof of Lemma 1.1

We first show **(v)**. By setting $s' = u_n s$, we have

$$u_n \int_0^{u_n^{-1}} e^{\beta^{(q)}_{1/u_n,1/u_n}(s)} ds = \int_0^1 e^{\beta^{(q)}_{1/u_n,1/u_n}(u_n^{-1}s')} ds'.$$

It is straightforward to check that

$$\mu(\mathcal{B}_{u_n^{-1},u_n^{-1}}(s)) = 1 - \frac{v_n/u_n}{1-v_n/u_n}.$$

We denote this quantity by ν_n. Then if $\lambda^{(q)} = \left(\lambda^{(q)}(s),\, s\geq 0\right)$ is a Lévy process such that $\mathbb{E}[e^{r\lambda^{(q)}(s)}] = e^{\psi^{(q)}(r)s}$ for $r \geq 0$, we have the equality in distribution

$$\left(\beta^{(q)}_{1/u_n,1/u_n}(u_n^{-1}s'),\, 0\leq s'\leq 1\right) \overset{d}{=} \left(\lambda^{(q)}(\nu_n s' + \nu_n) - \lambda^{(q)}(\nu_n s'),\, 0\leq s'\leq 1\right).$$

Observe that $\nu_n \to 1^-$ as $n \to +\infty$, and that we may apply the dominated convergence theorem from statement **(i)** of the lemma, so that

$$\mathbb{E}\left[\left|1 - u_n\int_0^{u_n^{-1}} e^{\beta^{(q)}_{1/u_n,1/u_n}(s)} ds\right|^{1+\epsilon}\right] \to \mathbb{E}\left[\left|1 - \int_0^1 e^{\lambda^{(q)}_-(s'+1)-\lambda^{(q)}_-(s')} ds'\right|^{1+\epsilon}\right]$$

as $n \to +\infty$, where $\lambda_-^{(q)}(s')$ is the left limit of $\lambda^{(q)}$ at time s'. Since from Assumption $\mathbf{A}_q$, $\psi^{(q)}(1+\epsilon)$ is finite, the moment on the right hand side is as well finite.

The assertion **(iv)** follows directly from **(iii)** with $r = q(1+\epsilon)$.

So as to show **(iii)**, note that from the definition of the $b_{n,k}$'s in equation (1.3) and the $d_{n,k}$'s in equation (1.13), and from the definition of the constant $\gamma(r)$ in proposition 1.2:

$$\begin{aligned}\gamma(r) &= u_n^{r-\psi(r)} T^{-\psi(r)} \mathbb{E}[b_{n,0}^r] \\ &= u_n^{r-\psi(r)} T^{-\psi(r)} \mathbb{E}[e^{r\tilde{\omega}_{1/u_n}(0)} d_{n,0}] \\ &= \mathbb{E}\Big[(u_n d_{n,0})^r\Big] \end{aligned} \tag{1.17}$$

from (1.10). Using the definition of $c_{n,0}$ in equation (1.12), we see that we can bound the difference $|\gamma_n(r) - \gamma(r)|$ as:

$$|\gamma_n(r) - \gamma(r)| \leq \mathbb{E}\left[c_{n,0}^r \left|1 - \frac{d_{n,0}^r}{c_{n,0}^r}\right|\right].$$

The term in the absolute value on the right is dominated by

$$\sup_{u \in [0, u_n^{-1}]} |1 - e^{r\theta_{1/u_n}(u)}|$$

which is independent of $c_{n,0}$. Therefore, statement **(ii)** of the lemma together with definition (1.14) of $\gamma_n(r)$ show that:

$$|\gamma_n(r) - \gamma(r)| \lesssim \gamma_n(r) u_n^{-\alpha}.$$

Moreover, one has from (1.17):

$$\gamma(r) \quad \geq \quad \gamma_n(r) \mathbb{E}\left[\inf_{u \in [0, u_n^{-1}]} e^{r\theta_{1/u_n}(u)}\right],$$

and $\mathbb{E}\Big[\inf_{u \in [0, u_n^{-1}]} e^{r\theta_{1/u_n}(u)}\Big]$ goes to 1 when n goes to ∞, again from statement **(ii)** of the lemma. Therefore

$$|\gamma_n(r) - \gamma(r)| \lesssim \gamma(r) u_n^{-\alpha},$$

hence **(iii)**.

We now show **(ii)**. The proof is the same for θ and $\theta^{(q)}$. For $s \in [0, u_n^{-1}]$, we have:

$$\mu(\mathcal{T}_{1/u_n}(s)) = (v_n/u_n)/(1 - v_n/u_n).$$

We denote by κ_n this quantity. Since $v_n = \lfloor u_n^{1-\delta} \rfloor$ for a fixed $\delta \in (0,1)$, κ_n behaves as u_n to a positive exponent as $n \to +\infty$.

If $\lambda = \big(\lambda(s),\, s \geq 0\big)$ is a Lévy process such that $\mathbb{E}[e^{r\lambda(s)}] = e^{\psi(r)s}$, it can be seen that we have the equality in distribution

$$\Big(\theta_{1/u_n}(u_n^{-1}s),\, 0 \leq s \leq 1\Big) \stackrel{d}{=} \Big(\lambda(\kappa_n s + \kappa_n) - \lambda(\kappa_n s),\, 0 \leq s \leq 1\Big),$$

so that we may show the result on λ rather than on θ_{1/u_n}.

From the decomposition

$$\lambda(\kappa_n s + \kappa_n) - \lambda(\kappa_n s) = \lambda(\kappa_n s + \kappa_n) - \lambda(\kappa_n) + \lambda(\kappa_n) - \lambda(\kappa_n s),$$

we have that

$$\mathbb{E}\Big[\sup_{s' \in [0,\kappa_n]} |1 - e^{r(\lambda(s'+\kappa_n) - \lambda(s'))}|\Big] \leq \mathbb{E}\Big[\sup_{s_1,\, s_2 \in [0,\kappa_n]} |1 - e^{r(\lambda_1(s_1)+\lambda_2(s_2))}|\Big]$$

where λ_1 and λ_2 are independent copies of λ. From the triangle inequality,

$$\begin{aligned}\sup_{s_1,\, s_2 \in [0,\kappa_n]} |1 - e^{r(\lambda_1(s_1)+\lambda_2(s_2))}| \leq & \sup_{s_1,\, s_2 \in [0,\kappa_n]} |1 - e^{r(\lambda_1(s_1)+\lambda_2(s_2)) - (s_1+s_2)\psi(r)}| \\ & + \sup_{s_1,\, s_2 \in [0,\kappa_n]} e^{r(\lambda_1(s_1)+\lambda_2(s_2))} |1 - e^{(s_1+s_2)\psi(r)}|\end{aligned}$$

The term $\mathbb{E} \sup_{s_1,\, s_2 \in [0,\kappa_n]} e^{r(\lambda_1(s_1)+\lambda_2(s_2))}$ is finite from **(i)**, and since λ is right-continuous, it goes to 1 as $n \to \infty$, so that

$$\mathbb{E}\Bigg[\sup_{s_1,\, s_2 \in [0,\kappa_n]} e^{r(\lambda_1(s_1)+\lambda_2(s_2))} |1 - e^{(s_1+s_2)\psi(r)}|\Bigg] \lesssim \kappa_n.$$

For $x > 0$, $\Big(|1 - xe^{r\lambda_1(s_1) - s_1\psi(r)}|,\, s_1 \geq 0\Big)$ and $\Big(|1 - xe^{r\lambda_2(s_2) - s_2\psi(r)}|,\, s_2 \geq 0\Big)$ are right-continuous submartingales. Thus, we may apply twice Doob's inequality, which gives

$$\mathbb{E}\Big[\sup_{s_1,\, s_2 \in [0,\kappa_n]} |1 - e^{r(\lambda_1(s_1)+\lambda_2(s_2)) - (s_1+s_2)\psi(r)}|\Big] \leq \mathbb{E}\Big[|1 - e^{r(\lambda_1(\kappa_n)+\lambda_2(\kappa_n)) - 2\kappa_n\psi(r)}|\Big].$$

Since $\Big(r(\lambda_1(s) + \lambda_2(s)) - 2v\psi(r),\, v \geq 0\Big)$ is a Lévy process, the Lévy-Khintchine formula (see for instance the first chapter of Bertoin [20]) states that it can be written as

$$r(\lambda_1(s) + \lambda_2(s)) - 2s\psi(r) = as + \sigma b(s) + x_1(s) + x_2(s),\; s \geq 0$$

where b is a standard Brownian motion, x_1 is a compound Poisson process with jumps of size greater than 1, x_2 is a pure-jump martingale with jumps of size less than 1, and b, x_1 and x_2 are independent. We first deal with the large jumps:

$$\begin{aligned}\mathbb{E}\Big[|1 - e^{a\kappa_n + \sigma b(\kappa_n) + x_1(\kappa_n) + x_2(\kappa_n)}|\Big] &= \mathbb{E}\Big[|1 - e^{a\kappa_n + \sigma b(\kappa_n) + x_1(\kappa_n) + x_2(\kappa_n)}|1_{\{x_1(\kappa_n) \neq 0\}}\Big] \\ &\quad + \mathbb{P}\Big[x_1(\kappa_n) = 0\Big]\mathbb{E}\Big[|1 - e^{a\kappa_n + \sigma b(\kappa_n) + x_2(\kappa_n)}|\Big]\end{aligned}$$

and

$$\begin{aligned}&\mathbb{E}\Big[|1 - e^{a\kappa_n + \sigma b(\kappa_n) + x_1(\kappa_n) + x_2(\kappa_n)}|1_{\{x_1(\kappa_n) \neq 0\}}\Big] \\ &\lesssim \mathbb{P}\Big[x_1(\kappa_n) \neq 0\Big]\mathbb{E}\Big[|1 - e^{a\kappa_n + \sigma b(\kappa_n) + x_1(\kappa_n) + x_2(\kappa_n)}|\Big] \\ &\lesssim 1 - e^{-\rho\kappa_n} \\ &\lesssim \kappa_n\end{aligned}$$

where ρ is the intensity of the compound Poisson process x_1. Finally, the moment of order 2 of $e^{\sigma b(\kappa_n) + x_2(\kappa_n)}$ is finite, and again from the Lévy-Khintchine formula we have

$$\mathbb{E}\Big[e^{a\kappa_n + \sigma b(\kappa_n) + x_2(\kappa_n)}\Big] = e^{\phi(1)\kappa_n}$$

and

$$\mathbb{E}\Big[e^{2(a\kappa_n+\sigma b(\kappa_n)+x_2(\kappa_n))}\Big] = e^{\phi(2)\kappa_n}$$

for some real numbers $\phi(1)$ and $\phi(2)$. Thus

$$\begin{aligned}\mathbb{E}\Big[|1-e^{a\kappa_n+\sigma b(\kappa_n)+x_2(\kappa_n)}|\Big] &\leq \Big(\mathbb{E}\Big[|1-e^{a\kappa_n+\sigma b(\kappa_n)+x_2(\kappa_n)}|^2\Big]\Big)^{1/2}\\ &\leq \Big(1-2e^{\kappa_n\phi(1)}+e^{\kappa_n\phi(2)}\Big)^{1/2}\\ &\lesssim \kappa_n^{1/2}.\end{aligned}$$

Let us show **(i)**. The proof is the same for P and $P^{(q)}$. We first suppose that $\mathcal{C}^m = \cap_{0\leq u\leq t}\mathcal{C}+u$ is not empty. Then for $u\in[0,t]$, we decompose $\mathcal{C}+u$ into three disjoints sets:

$$\mathcal{C}+u = \mathcal{C}^m \cup \mathcal{C}^l(u) \cup \mathcal{C}^r(u),$$

where $\mathcal{C}^l(u)$ is the part of $\mathcal{C}+u$ that is on the left of $\mathcal{C}^m$ and $\mathcal{C}^r(u)$ the part that is on the right. Then

$$\Big(P(\mathcal{C}^l(t-u)),\, 0\leq u\leq t\Big) \quad\text{and}\quad \Big(P(\mathcal{C}^r(u)),\, 0\leq u\leq t\Big)$$

are independent martingales, and they are also independent of $P(\mathcal{C}^m)$. Thus, applying Doob's inequality,

$$\mathbb{E}\Big[\sup_{0\leq u\leq t} e^{rP(\mathcal{C}+u)}\Big] \lesssim \mathbb{E}\Big[e^{rP(\mathcal{C}^m)}\Big]\mathbb{E}\Big[e^{rP(\mathcal{C}^l(0))}\Big]\mathbb{E}\Big[e^{rP(\mathcal{C}^r(t))}\Big].$$

Now recall from Assumption $\mathbf{A}_q$ that $\psi(r) < +\infty$ for $r\leq q(1+\epsilon)$, so that the last expression is indeed finite. Finally, in the case where t is large enough so that $\mathcal{C}^m$ is empty, we choose an integer j so that $\cap_{0\leq u\leq t/j}\mathcal{C}+u\neq\emptyset$, and we get

$$\mathbb{E}\Big[\sup_{0\leq u\leq t} e^{rP(\mathcal{C}+u)}\Big] \leq j\mathbb{E}\Big[\sup_{0\leq u\leq t/j} e^{rP(\mathcal{C}+u)}\Big].$$

1.B Proof of Lemma 1.2

We follow here closely a proof given in [11]. Let us assume that m is even. Let us decompose $M^{(q,\epsilon)}_{1/u_n}(T)$ as

$$M^{(q,\epsilon)}_{1/u_n}(T) = \sum_{k=0}^{m/2-1} e_{2k} + \sum_{k=0}^{m/2-1} e_{2k+1}$$

where

$$e_k = M^{(q,\epsilon)}_{1/u_n}\Big((k+1)T/m\Big) - M^{(q,\epsilon)}_{1/u_n}\Big(kT/m\Big).$$

Thus,

$$\mathbb{E}\Big[M^{(q,\epsilon)}_{1/u_n}(T)^{1+\epsilon}\Big] \leq 2^{1+\epsilon}\mathbb{E}\Bigg[\Big(\sum_{k=0}^{m/2-1} e_{2k}\Big)^{1+\epsilon}\Bigg].$$

We next apply the sub-additivity of the function $x \mapsto x^{(1+\epsilon)/2}$:

$$\begin{aligned}\mathbb{E}\Big[M^{(q,\epsilon)}_{1/u_n}(T)^{1+\epsilon}\Big] &\leq 2^{1+\epsilon}\mathbb{E}\Bigg[\Big(\sum_{k=0}^{m/2-1} e_{2k}^{(1+\epsilon)/2}\Big)^2\Bigg] \\ &= 2^{\epsilon} m \mathbb{E}\Big[e_0^{1+\epsilon}\Big] + 2^{1+\epsilon}\sum_{k\neq k'} \mathbb{E}[e_{2k}^{(1+\epsilon)/2} e_{2k'}^{(1+\epsilon)/2}].\end{aligned}$$

If we now define $w^{(q)}_{l,L}(u) = w^{(q)}_l(u) - w^{(q)}_L(u)$ for $0 < l < L$, then we can write:

$$e_{2k} = u_n^{-\chi\epsilon/(1+\epsilon)} \int_{2kT/m}^{2(k+1)T/m} e^{w^{(q)}_{1/u_n,T/m}(u) + w^{(q)}_{T/m}(s)} ds \tag{1.18}$$

so that

$$e_{2k} \leq u_n^{-\chi\epsilon/(1+\epsilon)} \left(\sup_{s'\in[0,T]} e^{w^{(q)}_{T/m}(s')}\right) \int_{2kT/m}^{2(k+1)T/m} e^{w^{(q)}_{1/u_n,T/m}(s)} ds.$$

In this last inequality, the sup term is independent of the integral. Moreover, two integral terms for different values of k are also independent. Let us now remark that from statement **(i)** of Lemma 1.1, there exists a constant $C > 0$ (that depends on m) such that

$$\mathbb{E}\left[\sup_{s'\in[0,T]} e^{(1+\epsilon) w^{(q)}_{T/m}(s')}\right] \leq C. \tag{1.19}$$

Using Jensen's inequality, we can therefore write for $k \neq k'$:

$$\begin{aligned}\mathbb{E}[e_{2k}^{(1+\epsilon)/2} e_{2k'}^{(1+\epsilon)/2}] &\leq u_n^{-\chi\epsilon} C\bigg(\mathbb{E}\Big[\Big(\int_0^{T/m} e^{w^{(q)}_{1/u_n,T/m}(s)} ds\Big)^{(1+\epsilon)/2}\Big]\bigg)^2 \\ &\leq u_n^{-\chi\epsilon} C\bigg(\mathbb{E}\Big[\int_0^{T/m} e^{w^{(q)}_{1/u_n,T/m}(u)} du\Big]\bigg)^{1+\epsilon} \\ &\leq u_n^{-\chi\epsilon} m^{-1-\epsilon} C T^{1+\epsilon} \\ &\lesssim u_n^{-\chi\epsilon}.\end{aligned}$$

This proves the lemma.

1.C Discrete construction of the MRM process

We now present a result that will be needed in Chapter 4 of this dissertation.

Theorem 1.4. *For $t > 0$, under Assumption $\mathbf{A}_1$,*

$$\frac{1}{u_n} \sum_{k=1}^{\lfloor tu_n \rfloor} e^{w_{1/u_n}(k/u_n)} \to M_t \qquad \text{as } n \to +\infty,$$

where the convergence holds in L^1, and almost surely if $u_n \geq c_1 e^{c_2 n}$ for some $c_1, c_2 > 0$.

Note that we do not claim originality here: indeed, the same result can be found in Bacry and Muzy [11]. However, they give an L^2 convergence, which require some slightly less general conditions than Assumption $\mathbf{A}_1$.

Proof. The proof is a simpler version of the proof of Proposition 1.4: We factorize

$$\left|e^{w_{1/u_n}(k/u_n)} - M\Big((k+1)/u_n\Big) - M\Big(k/u_n\Big)\right| = e^{\tilde{w}_{1/u_n}(k)} |1 - d_{n,k}|$$

where $d_{n,k}$ is as in (1.13). We then follow the proof of Proposition 1.4, giving an upper bound for

$$\sum_k e^{\tilde{w}_{1/n}(k)} |d_{n_k} - c_{n,k}|$$

in L^1 and for

$$\sum_k e^{\tilde{w}_{1/n}(k)}|1 - c_{n,k}|$$

in $L^{1+\epsilon}$.

□

Chapter 2

Testing the type of a semi-martingale: Itō against multifractal

2.1 Introduction

Semi-martingales are mathematically appealing quantities because they are adapted stochastic processes which can be used as integrators in the general theory of stochastic integration. They are also natural modeling objects in various fields, especially in finance (for absence of arbitrage opportunities) and turbulence, see for example Delbaen and Schachermayer [38] and Barndorff-Nielsen and Schmiegel [14]. Recall that a semi-martingale can be written as the sum of a predictable process of finite variation, a continuous local martingale, and a compensated pure jump process (the rigourous definition will be given below, as well as some useful notations). The very widely used notion of *Itō semi-martingale* refers to the case where each of the following objects is absolutely continuous with respect to the Lebesgue measure: the finite variation process, the quadratic variation process of the continuous local martingale, and the compensator of the jump measure.

A very large number of studies has been devoted to the statistical properties of Itō semi-martingales. Let us mention in particular a series of recent papers by Aït-Sahalia and Jacod [2, 3, 4] which will be of particular interest here. In these papers, the authors provide test statistics that address the following questions for Itō semi-martingales: Is the jump part of the semi-martingale equal to zero? Do the jumps have finite or infinite activity? Is the Brownian part equal to zero? A key element for the results of Aït-Sahalia and Jacod is the asymptotic behavior of the p-variation – the structure function – of the semi-martingale X, by which we mean the following quantity for $p > 0$ and some $n \in \mathbb{N}$ that goes to $+\infty$:

$$S(p, n^{-1}) = \sum_{i=1}^{n} \Big| X\Big(i/n\Big) - X\Big((i-1)/n\Big) \Big|^{p}. \tag{2.1}$$

In this chapter, our goal is to build test statistics aiming at answering questions that could be asked before the preceding ones. More precisely, we are looking for some statistical procedures allowing to say whether the data generating process is an Itō semi-martingale, against the alternative hypothesis that the data generating process belongs to a specific class of non-Itō semi-martingales, namely the Multifractal Random Walks of Bacry, Delour and Muzy [74, 11] – and conversely. As is explained below, the behavior of the p-variations will play a key role in the present study.

This problem might appear surprising. Indeed, the class of Itō semi-martingales already yields a very large collection of models. However, in the past two decades, some authors have proposed a new class of models of non-Itō semi-martingales, namely multifractal processes. These processes have indeed the nice feature of well reproducing most major stylized facts observed in finance or fully developped turbulence (in particular heavy-tail behavior, persistence and clustering of volatility, and intermittency of fluctuations), while remaining "simple" in the sense that they rely only on a small number of scalar parameters. For the introduction and pertinence of multifractal random models in turbulence and finance, we notably refer among others works to Frisch [42], Mandelbrot [69], Bouchaud and Potters [25], Bacry, Kozhemyak and Muzy [9], Calvet and Fisher [28].

In particular, Bacry, Kozhemyak and Muzy [10], Calvet and Fisher [28], and Duchon, Robert and Vargas [39] provide a thorough discussion of the multifractal approach to volatility modeling and pricing at various time scales. These authors notably show that multifractal models lead to quite superior volatility or VaR forecasts than the more usual methods based on GARCH, MS-GARCH and FIGARCH models – even when the former are calibrated out of sample and the latter are calibrated in sample. This indeed suggests that the multifractal setting should be of high interest for providing an accurate mathematical model of the dynamic of financial assets.

A distinctive property of these multifractal processes is the scaling behavior of their moments: for all p's in some real interval $I \supset [0,2]$ and $t \geq 0$,

$$\mathbb{E}[|X(t+s) - X(t)|^p] \sim c(p) s^{\tau(p)+1} \text{ as } s \to 0, \tag{2.2}$$

where $p \mapsto \tau(p)$ is a *strictly concave* function and the $c(p)$'s are some positive constants. The term multifractal, or multifractal scaling behavior, refers to the nonlinearity of the scaling exponent $\tau(\cdot)$. Therefore one would expect that the relation

$$n^{-1} S(p, n^{-1}) \approx c(p) n^{-(\tau(p)+1)} \text{ for large } n \tag{2.3}$$

holds for this class of processes, where the p-variation $S(p, n^{-1})$ is as in (2.1). Note that if X is a continuous Itō semi-martingale, one would obtain a *linear* exponent $\tau(p) = p/2 - 1$ for $p \geq 0$. Thus, when confronted to observations, it is natural to consider p-variations in order to assert whether the exponent τ is linear or not.

It should also be mentioned that the interest for the nonlinear nature $p \mapsto \tau(p)$ has rapidly grown since the seminal paper by Frisch and Parisi [43] who conjectured that this function $\tau(\cdot)$ in (2.3) characterizes the wildly varying pointwise Hölder regularity of the underlying function $t \mapsto X(t, \omega)$. Following this initial definition of the multifractal paradigm by Frisch and Parisi, the past two decades have then seen a large production of empirical studies in turbulence and finance, but also in DNA analysis or internet traffic among other fields, which base themselves notably on (2.3) to investigate the multifractal nature of the data – see for instance respectively for each of these four fields Gagne *et al.* [45], Ghashghaie *et al.* [46], Yu *et al.* [97] and Park and Willinger [76].

Nevertheless, it is important to note that these empirical works rarely rely on explicit random models; indeed, only a few research papers have directly addressed the issue of detecting the nonlinear nature of τ. This is notably the case of the works by Wendt, Abry and Jaffard [94, 95], who examine the performances on simulations of some specific algorithms that attempt to state whether some given signal is of "monofractal" or "multifractal" regularity. Note however that while it is a similar issue to the one we address here, the classes of signal that these authors consider do not coincide with the ones that we study in the present work: for instance their "monofractal" regularity class contains fractionary Brownian motions (which we do not consider here), but not all Itō type semi-martingales.

Some previous theoretical studies have already been devoted to the statistical properties of multifractal processes, see notably Ossiander and Waymire [75], Gloter and Hoffmann [47], Ludeña [62], Bacry, Kozhemyak and Muzy [10]. However, the elaboration of a probabilistic test of multifractal scaling behavior in the sense of (2.2) (assuming this informal relation is given a rigorous meaning) has apparently not been explicitly considered yet, with the exception of the studies [94] and [95] already mentionned. We give here a solution to this problem in the limited setting of multifractal processes that belong to the class of Multifractal Random Walks introduced by Bacry, Delour and Muzy [74, 11]. These processes have the nice theoretical property that the scaling relation (2.2) is satisfied with an exact equality for all s in some real interval $[0, T]$ (see Proposition 1.2 of Chapter 1, and more generally Section 1.2 for all definitions and notations concerning MRW's).

In this chapter, we are looking for simple quantities which have "opposite" behaviors when the data generating process X is an Itō semi-martingale or an MRW. More precisely, we are looking for two statistics, say T_1 and T_2, associated to the null assumption H_0 that X is an Itō semi-martingale ($X =$ Itō for short) and to the null assumption H_0 that X is an MRW ($X =$ MRW for short) such that, when $X =$ Itō (resp. $X =$ MRW), the asymptotic law of T_1 (resp. T_2) is non degenerate and known and T_2 (resp. T_1) goes to a degenerate limit.

The chapter is organized as follows. We present some definitions, build our test statistics and give their asymptotic behaviors in Section 2.2. These statistics are based on the approximate p-variations of order $p > 2$ of the process. A simulation study can be found in Section 2.3. The proofs are relegated to Section 2.4. This chapter is not technically very innovative and the results could probably be improved, for example using approximate p-variations of higher orders. However, we believe it is a first step in order to solve this new problem.

2.2 Statistical problem and results

2.2.1 Two classes of semi-martingales

A real valued process X defined on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ is called a semi-martingale if it can be decomposed as $X = X(0) + M + A$ where $X(0)$ is finite valued and $\mathcal{F}_0$-mesurable, M is a local martingale on this space and A is an adapted process of finite variation. Any semi-martingale can be written as

$$X(t) = X(0) + A^p(t) + X^c(t) + \int_0^t \int_{\mathbb{R}} \kappa(x)(\mu - \nu)(\mathrm{d}s, \mathrm{d}x) + \int_0^t \int_{\mathbb{R}} \kappa'(x)\mu(\mathrm{d}s, \mathrm{d}x),$$

where

- A^p is a predictable process of finite variation;

- X^c is a continuous local martingale with $X^c(0) = 0$, called the "continuous martingale part" of X;

- μ is the "jump measure" of X;
- ν is the "compensator" of μ;
- κ is a continuous function with compact supports such that $\kappa(x) = x$ for all x in a neighborhood of 0 and $\kappa'(x) = x - \kappa(x)$.

With this notation, the decomposition is unique (up to null sets), but the process A^p depends on the choice of the truncation function κ. Let us denote by Σ^2 the quadratic variation of the "continuous martingale part" X^c. The triple (A^p, Σ^2, ν) is called the triple of characteristics of X because, in "good cases" (see [50]), it completely determines the law of X.

An *Itō semi-martingale* is a semi-martingale with characteristics that are absolutely continuous with respect to the Lebesgue measure in the following sense

$$A^p(t,\omega) = \int_0^t a(s,\omega)\mathrm{d}s, \quad \Sigma^2(t,\omega) = \int_0^t \sigma^2(s,\omega)\mathrm{d}s, \quad \nu(\mathrm{d}t, \mathrm{d}x, \omega) = \mathrm{d}t\ F(t, \mathrm{d}x, \omega),$$

where a, σ are optional and $F(t, C)$ is optional for all Borel subsets C of $\mathbb{R}$. Itō semi-martingales have a nice representaton in terms of a Wiener process and a Poisson random measure

$$\begin{gathered} X(t) = X(0) + \int_0^t a(s)\mathrm{d}s + \int_0^t \sigma(s)\mathrm{d}W(s) + \\ \int_0^t \int_{\mathbb{R}} \kappa \circ \delta(s,x)(\underline{\mu} - \underline{\nu})(\mathrm{d}s, \mathrm{d}x) + \int_0^t \int_{\mathbb{R}} \kappa' \circ \delta(s,x)\underline{\mu}(\mathrm{d}s, \mathrm{d}x), \end{gathered} \tag{2.4}$$

where W denotes a $(\mathcal{F}_t)$-standard Wiener process and $\underline{\mu}$ is a $(\mathcal{F}_t)$-Poisson random measure on $(0, \infty) \times \mathbb{R}$ with intensity measure $\underline{\nu}(\mathrm{d}t,\mathrm{d}x)$ $=\mathrm{d}t \otimes \lambda(\mathrm{d}x)$, where λ is a σ-finite and infinite measure without atom.

Following Aït-Sahalia and Jacod [2], we restrict ourselves to the following subclass of Itō semi-martingales: the coefficients $a(t,\omega)$, $\sigma(t,\omega)$ and $\delta(t,x,\omega)$ are such that the various integrals in Equation (2.4) make sense and

$$\sigma(t) = \sigma_0 + \int_0^t \tilde{a}(s)\mathrm{d}s + \int_0^t \tilde{\sigma}(s)\mathrm{d}W(s) + \int_0^t \tilde{\sigma}'(s)\mathrm{d}W'(s) \\ + \int_0^t \int_E \kappa \circ \tilde{\delta}(s,x)(\underline{\mu} - \underline{\nu})(\mathrm{d}s,\mathrm{d}x) + \int_0^t \int_E \kappa' \circ \tilde{\delta}(s,x)\underline{\mu}(\mathrm{d}s,\mathrm{d}x), \qquad (2.5)$$

where W' is another Wiener process independent of $(W,\underline{\mu})$. Let

$$\delta^\kappa(t,\omega) = \int_0^t \kappa \circ \delta(t,x,\omega)\lambda(\mathrm{d}x)$$

if the integral makes sense and $+\infty$ otherwise. Finally, set

$$t_{inf} = \inf\{t, \Delta X(t) \neq 0\}.$$

As in [2], we will systematically consider the following assumption for X when X is an Itō semi-martingale:

Assumption 2.1.

- *All paths $t \to a(t,\omega)$, $t \to \tilde{\sigma}(t,\omega)$, $t \to \tilde{\sigma}'(t,\omega)$, $t \to \delta(t,x,\omega)$, $t \to \tilde{\delta}(t,x,\omega)$ are left-continuous with right limits.*
- *All paths $t \to \tilde{a}(t,\omega)$, $t \to sup_{x\in E}\frac{|\delta(t,x,\omega)|}{\gamma(x)}$ and $t \to sup_{x\in E}\frac{|\tilde{\delta}(t,x,\omega)|}{\tilde{\gamma}(x)}$ are locally bounded, where γ and $\tilde{\gamma}$ are (non random) nonnegative functions satisfying $\int_E(\gamma(x)^2 \wedge 1)\lambda(dx) < \infty$, $\int_E(\tilde{\gamma}(x)^2 \wedge 1)\lambda(dx) < \infty$.*
- *All paths $t \to \delta^\kappa(t,\omega)$ are left-continuous with right limits on the semi open set $[0, t_{inf}(\omega))$.*
- *We have $\int_0^t \sigma^2(s)ds > 0$, a.s., for all $t > 0$.*

Concerning MRW processeses, we refer to Section 1.2 of Chapter 1 for their definition and some of their properties. We write in this chapter

$$\tau(p) = p/2 - \psi(p/2) - 1,$$

and $\mu(p)$ for the absolute moment of order p of a standard Gaussian random variable. Recall that q_0 is defined as

$$q_0 = \sup\Big\{q > 1,\, \exists \epsilon > 0,\, \tau\Big(2q(1+\epsilon)\Big) - (1+\epsilon)\tau(2q) > 0\Big\},$$

and is also the supremum of the $q > 1$ such that Theorem 1.1 holds. It will be convenient to rewrite this Theorem 1.1 in the following weaker form:

Lemma 2.1. *For $t > 0$ and $0 \le p < 2q_0$, almost surely:*

$$2^{N\tau(p)} \sum_{i=1}^{\lfloor 2^N t \rfloor} \Big| X\Big(i2^{-N}\Big) - X\Big((i-1)2^{-N}\Big)\Big|^p \to \mu(p)\widetilde{M}^{(p)}(t),$$
$$2^{N\tau(2p)} \sum_{i=1}^{\lfloor 2^N t \rfloor} \Big| M\Big(i2^{-N}\Big) - M\Big((i-1)2^{-N}\Big)\Big|^p \to \widetilde{M}^{(2p)}(t),$$

as $N \to +\infty$, where $\widetilde{M}^{(p)}(t)$ and $\widetilde{M}^{(2p)}(t)$ are some positive random variables, independent of the Wiener process B.

(Here, $\widetilde{M}_t^{(p)}$ is the same random variable as the process $M^{(p/2)}$ of Chapter 1, up to a multiplicative positive constant.)

2.2.2 The case H_0: $X =$ Itō

We are looking for a statistic whose behavior is different when $X =$ Itō and when $X =$ MRW. To build this statistic, we will use p-variations of the form (2.1). Before explaining why such quantities are natural in our problem, we need to define the two following sets:

$$\Omega^j = \{\omega,\ s \to X(s,\omega) \text{ is discontinuous on } [0,1]\},$$

$$\Omega^c = \{\omega,\ s \to X(s,\omega) \text{ is continuous on } [0,1]\}.$$

Remark that the sample path of an Itō semi-martingale can be in Ω^c even if this semi-martingale is not continuous (if no jump occurs before $t = 1$).

Let us consider $p > 2$. From Jacod [49], we know that if $X = \text{Itō}$, in restriction to Ω^j, then the jumps dominate and $S(p, 2^{-N})$ goes to $\sum_{t \le 1} |\Delta X(t)|^p$ as $N \to \infty$. In restriction to Ω^c, then

$$2^{N(p/2-1)} S(p, 2^{-N}) \xrightarrow{\mathbb{P}} \mu(p) \int_0^1 |\sigma(t)|^p \mathrm{d}t, \qquad \text{as } N \to \infty.$$

On the other hand, if $X = \text{MRW}$,

$$2^{N\tau(p)} S(p, 2^{-N}) \xrightarrow{\mathbb{P}} \mu(p)\theta^{(p)}(1),$$

with $\mu(p)$ being the absolute moment of order p of a standard Gaussian random variable.

Thus, in the spirit of Aït-Sahalia and Jacod [2] and Rosenbaum [86], we naturally consider for some $p > 2$ the ratio

$$\frac{S(p, 2^{-(N-1)})}{S(p, 2^{-N})}. \tag{2.6}$$

If $X = \text{Itō}$, this tends to 1 in restriction to Ω^j and to $2^{p/2-1}$ in restriction to Ω^c. When X is a MRW, it goes to $2^{\tau(p)}$. Now, to have a feasible test, we need a central limit theorem (CLT) associated to this quantity. Before stating the results, we need to recall the definition of stable convergence in law. We say that a sequence T_n on $(\Omega, \mathcal{F}, \mathbb{P})$ converges stably in law to the law ϕ ($T_n \xrightarrow{\mathcal{L}_s} \phi$), in restriction to $A \in \mathcal{F}$, if for all bounded continuous functions f

and all $\mathcal{F}-$measurable bounded variables Y vanishing outside A,

$$\mathbb{E}[f(T_n)Y] \to \mathbb{E}[Y]\mathbb{E}[f(U)], \text{ with } U \text{ a random variable with law } \phi.$$

Let us also define the constant $m(p)$ as

$$m(p) = \left(\frac{\mu^2(p)}{2^{p-2}(3\mu(2p) + \mu^2(p)) - 2^{p/2}\tilde{\mu}_p}\right)^{1/2},$$

with

$$\tilde{\mu}_p = \mathbb{E}\Big[|U|^p|U+V|^p\Big]$$

for U and V some independent standard $\mathcal{N}(0,1)$ random variables.

Then from Aït-Sahalia and Jacod [2], if $X =$ Itō, in restriction to the set Ω^c, we have for $p > 2$

$$m(p)\frac{S(p,2^{-N})}{(S(2p,2^{-N}))^{1/2}}\Big(\frac{S(p,2^{-(N-1)})}{S(p,2^{-N})} - 2^{p/2-1}\Big) \xrightarrow{\mathcal{L}_s} \mathcal{N}(0,1).$$

The term

$$m(p)\frac{S(p,2^{-N})}{(S(2p,2^{-N}))^{1/2}}2^{-N/2}$$

corresponds to an estimator of the asymptotical standard deviation of the ratio in Theorem 3b) in [2].

However in restriction to the set Ω^j, we have the following convergence in probability

$$m(p)\frac{S(p,2^{-N})}{(S(2p,2^{-N}))^{1/2}}\Big(\frac{S(p,2^{-(N-1)})}{S(p,2^{-N})} - 2^{p/2-1}\Big) \xrightarrow{\mathbb{P}} -m(p)\frac{\sum_{t\leq 1}|\Delta X(t)|^p}{(\sum_{t\leq 1}|\Delta X(t)|^{2p})^{1/2}}(2^{p/2-1}-1).$$

This result can not be used to build a convenient test statistic. So, we finally choose the following slightly modified test statistic:

$$T_1^N = m(p)2^{(p/2-1)(\lfloor k_N N\rfloor - N)}\frac{S(p,2^{-\lfloor k_N N\rfloor})}{(S(2p,2^{-N}))^{1/2}}\Big(\frac{S(p,2^{-(N-1)})}{S(p,2^{-N})} - 2^{p/2-1}\Big),$$

where (k_N) is a positive sequence such that $k_N \leq 1$ for all N, $k_N \to 1$ and $(1-k_N)N \to \infty$

as $N \to \infty$. The following theorem shows that T_1^N is a suitable test statistic when the null hypothesis is H_0: $X =$ Itō. Its proof is just a direct application of the results of Aït-Sahalia and Jacod [2] for the Itō case. In the MRW case, it follows from Lemma 2.1 together with the fact that $\tau(2p)/2 > \tau(p)$ provided $q_0 > p$.

Theorem 2.1. *Let $p > 2$.*

- *Assume X is an Itō semi-martingale such that $\big(\sigma(t),\, t \geq 0\big)$ is also an Itō semi-martingale of the form* (2.5)*.*
  - *In restriction to the set Ω^c, $(T_1^N)^2 \xrightarrow{\mathcal{L}_s} \chi^2(1)$.*
  - *In restriction to the set Ω^j, $(T_1^N)^2 \xrightarrow{\mathbb{P}} 0$.*
- *Assume X is an MRW. If $q_0 > p$, then $(T_1^N)^2 \xrightarrow{\mathbb{P}} +\infty$.*

Remark 1: If we restrict ourself to continuous Itō semi-martingales in H_0, we can choose $k_N = 1$.

Eventually, we can suggest the following rejection area for the test of asymptotic level α in the case where H_0 is $X =$ Itō: $\{(T_1^N)^2 \geq z_\alpha\}$, where z_α is the α quantile of a $\chi^2(1)$ distribution.

2.2.3 The case H_0: $X =$ MRW

We now need to find a test statistic in the case H_0: $X =$ MRW. This statistic should satisfy a CLT under the MRW assumption and go to some degenerate limit under the Itō assumption. One idea would be to use once again quantities of the form (2.6) to estimate $\tau(p)$ for $p \neq 2$ (since for $p = 2$, the ratio goes to 1 under the Itō and the MRW assumption). However, to our knowledge, the available results on the asymptotic behavior of the approximate p-variations under the MRW assumption, see [63], do not enable to obtain CLTs for quantities of the form

$$\frac{S(p, 2^{-(N-1)})}{S(p, 2^{-N})} - 2^{\tau(p)}.$$

We suggest another strategy which is based on quadratic variations. What we use is the difference between the rates of convergence of the quadratic variations under the MRW and the Itō assumption. We first consider the case where the null asumption is that $X =$ MRW with a given value for $\tau(4)$.

The case H_0: $X =$ MRW and $\tau(4) = \tau_4$

Here we assume that under the null, X is an MRW where $\tau(4) = \tau_4$ is a given value smaller than 1. Let us first consider the following statistic

$$V^N = \frac{\sqrt{3}}{\sqrt{2(2^{\tau_4} - 1)}} \frac{\{S(2, 2^{-N}) - S(2, 2^{-(N-1)})\}}{\sqrt{S(4, 2^{-N})}}.$$

The following proposition is proved in Section 2.4.1

Proposition 2.1. *If X is an MRW with $q_0 > 2$, then $V^N \xrightarrow{\mathcal{L}} \mathcal{N}(0, 1)$, as $N \to \infty$.*

Now, to assess if V^N is a suitable test statistic, we have to look at its behavior when X is an Itō semi-martingale. From Jacod [49] (see also [2]), we know that in restriction to the set Ω^c, V^N is of order 1 and in restriction to the set Ω^j, V^N goes to zero in probability. Therefore, the preceding statistic is suitable only if, under the alternative, the sample path has jumps on $[0, 1]$. To solve this issue, we use an alternate estimate for the asymptotic variance in the CLT for the difference of quadratic variations (see Proposition 2.3 in Section 2.4.1). Indeed, this new estimator has different rates of convergence when $X =$ MRW and when $X =$ Itō. More precisely, we estimate this variance using only $2^{\lfloor kN \rfloor}$ data instead of 2^N for some $k \in (0, 1)$. Hence we consider the following statistic

$$T_2^N = \frac{\sqrt{3}}{\sqrt{2(2^{\tau_4} - 1)}} 2^{(N - \lfloor kN \rfloor)\tau_4/2} \frac{\{S(2, 2^{-N}) - S(2, 2^{-(N-1)})\}}{\sqrt{S(4, 2^{-\lfloor kN \rfloor})}}.$$

If X is an Itō semi-martingale, in restriction to the set Ω^c, the order of magnitude of T_2^N is $2^{(N-\lfloor kN \rfloor)(\tau_4-1)/2}$ and restriction to the set Ω^j, this order is $2^{(N-\lfloor kN \rfloor)(\tau_4-1)/2}2^{-\lfloor kN \rfloor/2}$. Thus, we finally get the following result, which is easily derived from Proposition 2.1.

Theorem 2.2.

- *If X is an MRW with $\tau(4) = \tau_4$ and $q_0 > 2$, then $(T_2^N)^2 \xrightarrow{\mathcal{L}} \chi^2(1)$.*
- *For $k \in (0,1)$, if X is an Itō semi-martingale, then $(T_2^N)^2 \xrightarrow{\mathbb{P}} 0$.*

Remark 3: If we restrict ourself to sample paths with jumps in the alternative, we can take $k = 1$.

We can suggest the following rejection area for the test of asymptotic level α in the case where H_0 is $X =$ MRW and $\tau(4) = \tau_4$: $\{(T_2^N)^2 \leq z_{1-\alpha}\}$.

The case H_0: $X =$ MRW with unknown $\tau(4)$

We now want to build a test statistic without assuming that $\tau(4)$ is known. When $X =$ MRW, a natural convergent estimator of $\tau(4)$, providing an immediate equivalent for $2^{N\tau(4)}$, is given by

$$\hat{\tau}(4) = \frac{2}{N\log(2)}\Big\{ \log\Big(S(4, 2^{-\lfloor N/2 \rfloor})\Big) - \log\Big(S(4, 2^{-N})\Big)\Big\}.$$

However, this estimator is not really convenient since it might tend to 0 or 1 if $X =$ Itō. Thus we use the following modification of $\hat{\tau}(4)$

$$\tau^*(4) = (\hat{\tau}(4) \wedge (1 - v_N)) \vee u_N,$$

with u_N and v_N two positive sequences tending to 0 such that $2^N u_N \to +\infty$, Nu_N is bounded and $Nv_N \to +\infty$. Thanks to the sequences u_N and v_N, $\tau^*(4)$ can not tend to 0 or 1 too rapidly. We have the following proposition which is proved in Section 2.4.2.

Proposition 2.2. *If X is an MRW with $q_0 > 2$, we have as $N \to +\infty$*

$$2^{N(\tau^*(4)-\tau(4))/2} \to 1.$$

If X is an Itō semi-martingale, we have as $N \to +\infty$

$$\frac{2^{N(\tau^*(4)-1)/2}}{\sqrt{2(2^{\tau^*(4)}-1)}} \to 0.$$

Note that other estimation procedures for $\tau(4)$ are be studied in Chapter 4 of this dissertation. However, this is under the hypothesis that the data has been generated by a *log-normal* MRW process. Using these estimation procedures here would require to study their consistency both in the *general MRW case* and the *Itō semi-martingale case*.

We now naturally define our last test statistic the following way:

$$\widetilde{T}_2^N = \frac{\sqrt{3}}{\sqrt{2(2^{\tau^*(4)}-1)}} 2^{(N-\lfloor kN \rfloor)\tau^*(4)/2} \frac{\{S(2, 2^{-N}) - S(2, 2^{-(N-1)})\}}{\sqrt{S(4, 2^{-\lfloor kN \rfloor})}}.$$

We can now state our last result which is easily deduced from Proposition 2.1 and Proposition 2.2.

Theorem 2.3.

- *If X is an MRW with $q_0 > 2$, then $(\widetilde{T}_2^N)^2 \xrightarrow{\mathcal{L}} \chi^2(1)$.*

- *For $k \in (0,1)$, if X is an Itō semi-martingale, then $(\widetilde{T}_2^N)^2 \xrightarrow{\mathbb{P}} 0$.*

Eventually, we can suggest the following rejection area for the test of asymptotic level α in the case where H_0 is $X =$ MRW: $\{(\widetilde{T}_2^N)^2 \leq z_{1-\alpha}\}$.

2.3 A simulation study

2.3.1 The setting

We begin here with some illustrations of our test procedures. For some integer $N \geq 1$, we simulated 100 times a sequence $X(2^{-N}) - X(0), \ldots, X(1) - X(1 - 2^{-N})$ where X is one of the following:

- A standard Brownian motion on $[0, 1]$
- An Itō semi-martingale which is the sum of a standard Brownian motion and a compound Poisson process: $X(t) = W(t) + \sum_{k=0}^{N(t)} A_k$, given that there is at least one jump. The Poisson process N has an intensity of 30 ($N(t)$ jumps 30 times on average for $0 \leq t \leq 1$) and the A_k's are uniformly distributed on the interval $[-1/2, 1/2]$. The values 30 and 1/2 have been chosen such that the sample path of the Itō semi-martingale seems at least visually hard to discern from the sample paths of the MRW, see Figures 2.1 and 2.3.
- An MRW as described in Chapter 1, with $\psi(p) = \lambda^2 p(p-1)/2$ for some $\lambda^2 \in (0, 2)$ (thus the random field $P(\mathrm{d}t, \mathrm{d}l)$ is a 2D Gaussian white noise with expectation $-\lambda^2 l^{-2}/2\mathrm{d}t \otimes \mathrm{d}l$ and variance $\lambda^2 l^{-2}\mathrm{d}t \otimes \mathrm{d}l$). We consider three possible values for λ^2: 0.02, 0.1, or 0.7. When modelling financial data, a common range for λ^2 would roughly be $[0.08, 0.20]$, see Bacry *et al.* [10]. The parameter T and σ are both set to 1, which is of little consequence here, see again [10]. For this choice of MRW, we have $\tau(p) = p/2 - 1 - \lambda^2 p(p-2)/8$ and $q_0 = \sqrt{2}/\lambda$. We refer to Bacry and Muzy [11] for the simulation procedure.

2.3.2 Case H_0: $X =$ Itō

When using the test procedure in practice, it is of first importance to have some idea of its power, that is how fast the test statistic does manifest a degenerate behavior under the

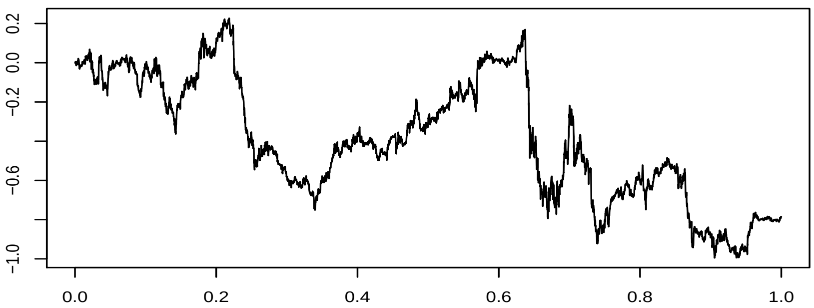

Figure 2.1: *A sample path of the log-normal MRW process on [0,1],* $\lambda^2 = 0.1$.

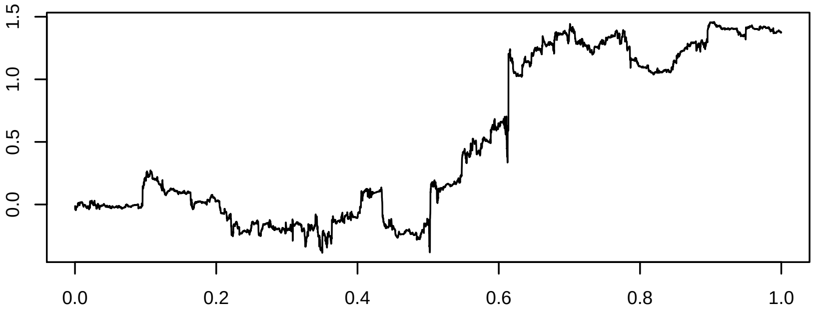

Figure 2.2: *A sample path of the log-normal MRW process on [0,1],* $\lambda^2 = 0.7$.

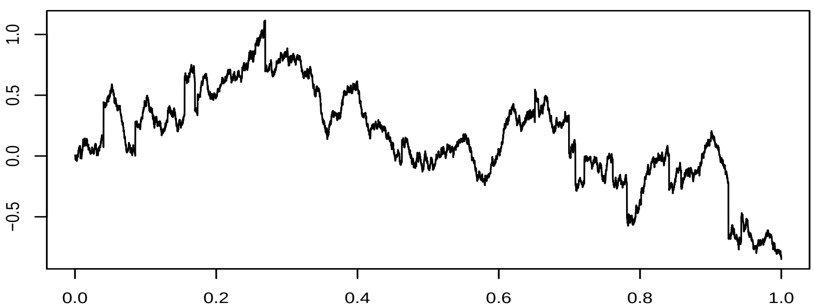

Figure 2.3: *A sample path of the Brownian motion with Poissonian jumps on [0,1].*

alternate hypothesis. Let us therefore give some orders of magnitude for the statistic T_1^N in the case where the null hypothesis is false, that is the data generating process X is an MRW.

From Proposition 1.2, we have that

$$\mathbb{E}\Big[S(p, 2^{-\lfloor k_N N \rfloor})\Big] = c(p) 2^{\lfloor k_N N \rfloor (p/2 - \psi(p/2) - 1)}$$

so that

$$T_1^N = O_P\Big(2^{N(1/2 + k_N \psi(p/2) - \psi(p)/2)} (2^{-\psi(p/2)} - 1)\Big).$$

The statistic $(T_1^N)^2$ will therefore be large if $1/2 + k_N \psi(p/2) \gg \psi(p)/2$ and $\psi(p/2) \gg 0$. In the case of the log-normal MRW, we have $\psi(p) = \lambda^2 p(p-1)/2$. Hence, supposing that $k_N \approx 1$, we have

$$T_1^N = O_P\Big(\lambda^2 2^{N(1/2 - \lambda^2 p^2/8)}\Big) \quad \text{for small } \lambda^2.$$

Therefore, the value of $(T_1^N)^2$ may be small if either λ^2 is too small (the MRW process is "close" to a Brownian motion) or too large (indeed, Lemma 2.1 doesn't hold for large λ^2 or large p, and $S(2p, 2^{-N})$ becomes degenerate in such cases).

Tables 2.1 and 2.2 show the number of simulations for which of H_0: $X =$ Itō is rejected (out of 100 simulations of each process), that is the proportion of simulated sample paths for which the statistic T_2^N is below $z_{1-\alpha}$, where $z_{1-\alpha}$ is the $1-\alpha$ quantile of a $\chi^2(1)$ distribution and $\alpha = 90\%$, 95% or 99% is the asymptotic level of the test. These simulations were obtained with $p = 3$ and $k_N = 1$. We see that for $\lambda^2 = 0.7$, the test statistic is very close to zero – indeed, Theorem 2.1 does not hold in this case. Also, we see that for the number of data we considered, our test statistic does not allow to recognize an log-normal MRW process from a Brownian motion if the value of λ^2 is too small. However, for a more reasonable value of λ^2 in the range of what can be estimated from financial data [10], we find that our test performs reasonably well, provided that the number of data is large enough.

Simulated process	Itō with no jumps		Itō with jumps	
Number n of data	32 768	1 048 576	32 768	1 048 576
Level of the test				
90%	11	11	6	0
95%	3	5	2	0
99%	2	2	0	0

Table 2.1: *Number of rejections of H_0: X = Itō for 100 simulations of an Itō semi-martingale ($p = 3$, $k_N = 1$).*

Simulated process	MRW, $\lambda^2 = 0.02$		MRW, $\lambda^2 = 0.1$		MRW, $\lambda^2 = 0.7$	
Number n of data	32 768	1 048 576	32 768	1 048 576	32 768	1 048 576
Level of the test						
90%	12	13	15	66	7	7
95%	6	6	8	58	3	3
99%	1	2	1	30	0	1

Table 2.2: *Number of rejections of H_0: X = Itō for 100 simulations of a log-normal MRW ($p = 3$, $k_N = 1$).*

2.3.3 Case H_0: X= MRW, $\tau(4)$ known

Tables 2.3 and 2.4 present the test results of H_0: X = MRW in the case where τ_4 is known. If Itō semi-martingales are simulated, we consider two configurations: either $\tau_4 = 0.9$ (that is, $\lambda^2 = 0.1$ in the case of a log-normal MRW), or $\tau_4 = 0.3$ ($\lambda^2 = 0.7$). Let us recall that if X is an Itō semi-martingale, then in restriction to the set Ω^c, the order of magnitude of T_2^N is $2^{(N-\lfloor kN \rfloor)(\tau_4-1)/2}$ and restriction to the set Ω^j, this order is $2^{(N-\lfloor kN \rfloor)(\tau_4-1)/2}2^{-\lfloor kN \rfloor/2}$, which yields a first approximation for the power of the test in this case.

When MRW's are simulated, the rejection rates are close to the theoretical rates 10%, 5%, 1%. This is in agreement with the Gaussian fit we obtain for T_2^N (see Figure 2.4). When Itō semi-martingales are simulated, we note that the test is not very powerful for low n: the probability of correctly rejecting H_0 is rather low. However, this probability becomes quite high for $N \geq 20$, especially in the case of a Brownian motion with Poissonian jumps.

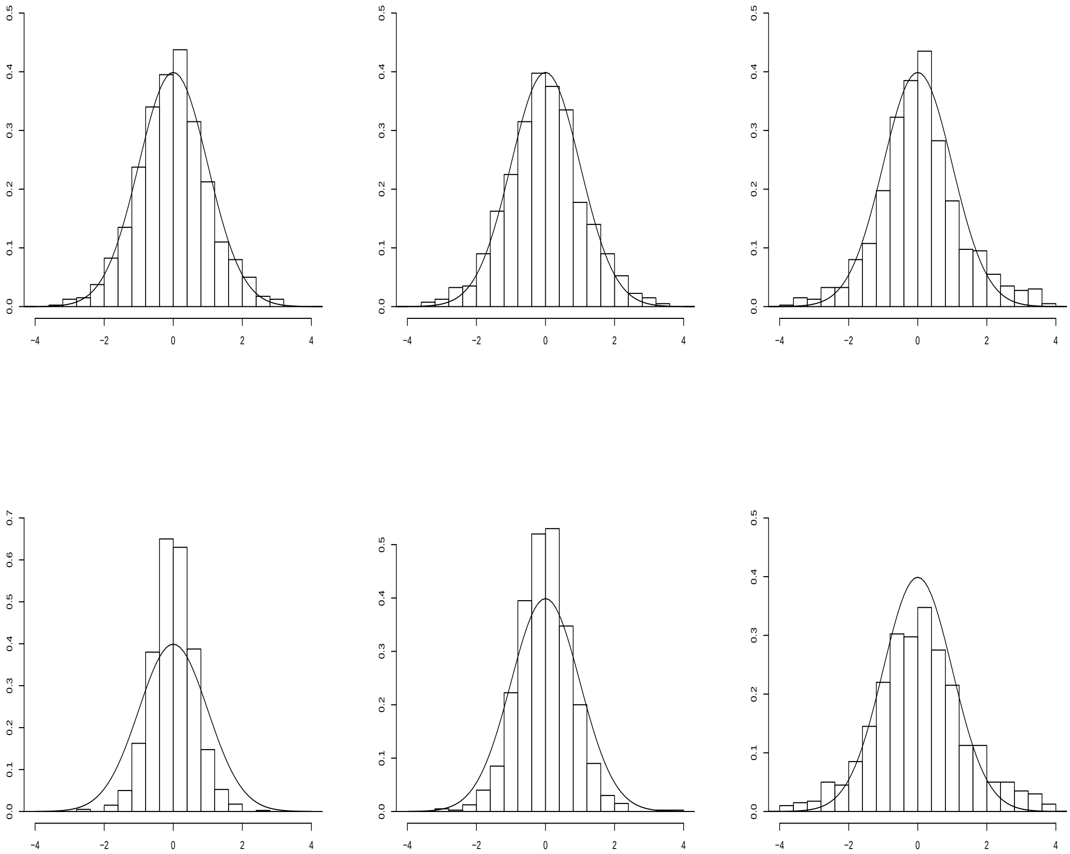

Figure 2.4: *Empirical distribution of* T_2^N *and* $\widetilde{T}_2^N$ *when MRW's are simulated, and fit with a standard Gaussian distribution* ($n = 2^N$, $N = 15$, $k = 1/2$, $v_N = 1/\sqrt{N}$, $u_N = 1/N$). *Left:* $\lambda^2 = 0.02$, *middle:* $\lambda^2 = 0.1$, *and right:* $\lambda^2 = 0.7$. *Top:* T_2^N *(case* $\tau(4)$ *known), bottom:* $\widetilde{T}_2^N$ *(case* $\tau(4)$ *unknown).*

)

Simulated process	Itō with no jumps				Itō with jumps			
Number n of data	32 768		1 048 576		32 768		1 048 576	
Value of τ_4	0.9	0.3	0.9	0.3	0.9	0.3	0.9	0.3
Level of the test								
90%	19	31	24	62	57	68	66	100
95%	4	16	5	26	19	34	40	89
99%	1	5	0	4	1	6	7	29

Table 2.3: *Number of rejections of* H_0: $X =$ *MRW,* $\tau(4)$ *known, for 100 simulations of an Itō semi-martingale* ($k = 1/2$).

Simulated process	MRW, $\lambda^2 = 0.02$		MRW, $\lambda^2 = 0.1$		MRW, $\lambda^2 = 0.7$	
Number n of data	32 768	1 048 576	32 768	1 048 576	32 768	1 048 576
Level of the test						
90%	10	12	11	11	10	9
95%	5	6	8	5	3	3
99%	1	8	1	2	2	1

Table 2.4: *Number of rejections of H_0: X = MRW, $\tau(4)$ known, for 100 simulations of a log-normal MRW ($k = 1/2$).*

2.3.4 Case H_0: X= MRW, $\tau(4)$ unknown

Next, we consider the case where $\tau(4)$ is unknown. The results we find are very similar to the previous case. However, one can see that the Gaussian fit we obtain when MRW processes are simulated is somewhat less exact than in the case where $\tau(4)$ is known: the estimation of the variance of the Gaussian limit is less accurate. Hence, we find that the rejection rates are less close to the theoretical ones in this case.

In particular, this fit seems to be slightly worse for large $n = 2^{20}$ than for $n = 2^{15}$, which might appear as surprising. This comes from the fact that the estimator $\tau^*(4)$ achieves a very slow convergence rate on our simulations, so that the estimation

$$\frac{\sqrt{3}}{\sqrt{2(2^{\tau^*(4)} - 1)}} 2^{(N - \lfloor kN \rfloor)\tau^*(4)/2}$$

of the variance used in the statistic $\widetilde{T}_2^N$ is actually less accurate for $N = 20$ than for $N = 15$.

Finally, the rejection rates we obtain for Itō semi-martingales simulations are still satisfactory, especially for processes with jumps.

Simulated process	Itō with no jumps		Itō with jumps	
Number n of data	32 768	1 048 576	32 768	1 048 576
Level of the test				
90%	15	23	67	100
95%	6	13	35	90
99%	2	2	8	34

Table 2.5: *Number of rejections of H_0: $X = MRW$, $\tau(4)$ unknown, for 100 simulations of an Itō semi-martingale ($k = 1/2$, $v_N = 1/\sqrt{N}$, $u_N = 1/N$).*

Simulated process	MRW, $\lambda^2 = 0.02$		MRW, $\lambda^2 = 0.1$		MRW, $\lambda^2 = 0.7$	
Number n of data	32 768	1 048 576	32 768	1 048 576	32 768	1 048 576
Level of the test						
90%	17	20	11	18	8	5
95%	8	12	6	9	5	2
99%	1	2	1	2	1	2

Table 2.6: *Number of rejections of H_0: $X = MRW$, $\tau(4)$ unknown, for 100 simulations of a log-normal MRW ($k = 1/2$, $v_N = 1/\sqrt{N}$, $u_N = 1/N$).*

2.4 Proofs

2.4.1 Proof of Proposition 2.1

We in fact prove a slightly more general result, Proposition 2.3, from which Proposition 2.1 is easily deduced. We consider an MRW of the form $X(t) = B\Big(M(t)\Big)$, such that $q_0 > 2$, where B is a Brownian motion with respect to some filtration $\mathcal{F}'$. We fix here a path of M (we will use afterwards the independence between B and M). Note that this also defines $\widetilde{M}^{(4)}(t)$ in Lemma 2.1. We set $\mathcal{G}_t = \mathcal{F}'_{M(t)}$ and define the $\mathcal{G}_{2i/n}$-measurable random vector $\xi_i^n = (\xi_i^{n,1}, \xi_i^{n,2})$ by

$$\begin{aligned}
\xi_i^{n,1} =& n^{\tau(4)/2}\Big\{\Big(X(2i/n) - X((2i-1)/n)\Big)^2 \\
&+ \Big(X((2i-1)/n) - X((2i-2)/n)\Big)^2 - \Big(M(2i/n) - M((2i-2)/n)\Big)\Big\} \\
\xi_i^{n,2} =& n^{\tau(4)/2}\Big\{\Big(X(2i/n) - X((2i-2)/n)\Big)^2 - \Big(M(2i/n) - M((2i-2)/n)\Big)\Big\}.
\end{aligned}$$

For $t \in [0,1]$, let C_t be the 2×2 matrix defined by

$$C_t = 2\widetilde{M}^{(4)}(t) \begin{pmatrix} 1 & 1 \\ 1 & 2^{\tau(4)} \end{pmatrix}$$

and defined the process $M^n = \{M_t^n, t \in [0,1]\}$ by

$$M_t^n = \sum_{i=1}^{\lfloor 2^N t/2 \rfloor} \xi_i^n.$$

We have the following result.

Proposition 2.3. *If $q_0 > 2$, then for a given path M, the process M^n converges in law towards a continuous centered $\mathbb{R}^2$-valued Gaussian process Z, with independent increments such that $\mathbb{E}\Big[Z_t^j Z_t^k\Big] = C_t^{jk}$. This entirely characterizes the law of the limiting process.*

Remark 4: In Proposition 2.3, we retrieve in particular the result obtained by Ludeña [63] in the one dimensional case.

For the proof of Proposition 2.3, we will consider the four following lemmas:

Lemma 2.2. *We have*

$$\mathbb{E}_{\mathcal{G}_{(2i-2)/n}}[(\xi_i^{n,1})] = 0, \ \mathbb{E}_{\mathcal{G}_{(2i-2)/n}}[(\xi_i^{n,2})] = 0.$$

Lemma 2.3. *We have*

$$\sum_{i=1}^{\lfloor 2^N t/2 \rfloor} \mathbb{E}_{\mathcal{G}_{(2i-2)/n}}[(\xi_i^{n,1})^2] \xrightarrow{\mathbb{P}} 2M^{(4)}(t), \quad \sum_{i=1}^{\lfloor 2^N t/2 \rfloor} \mathbb{E}_{\mathcal{G}_{(2i-2)/n}}[(\xi_i^{n,2})^2] \xrightarrow{\mathbb{P}} 2^{1+\tau(4)} M^{(4)}(t).$$

Lemma 2.4. *We have*

$$\sum_{i=1}^{\lfloor 2^N t/2 \rfloor} \mathbb{E}_{\mathcal{G}_{(2i-2)/n}}[\xi_i^{n,1} \xi_i^{n,2}] \xrightarrow{\mathbb{P}} 2M^{(4)}(t).$$

Lemma 2.5. *For some $\varepsilon > 0$*

$$\sum_{i=1}^{n/2} \mathbb{E}_{\mathcal{G}_{(2i-2)/n}}[(\xi_i^{n,1})^{2+\varepsilon}] \xrightarrow{\mathbb{P}} 0, \ \sum_{i=1}^{n/2} \mathbb{E}_{\mathcal{G}_{(2i-2)/n}}[(\xi_i^{n,2})^{2+\varepsilon}] \xrightarrow{\mathbb{P}} 0.$$

Since the time change $M(t)$ is fixed, $M^{(4)}(t)$ is deterministic. Thus, Proposition 2.3 follows from Lemmas 2.2-2.5 using a standard convergence result for triangular arrays of semi-martingales, see for example [50]. We now turn to the proofs of Lemmas 2.2-2.5.

Proof of Lemma 2.2

The result comes directly from the fact that

$$\begin{aligned}
&\mathbb{E}_{\mathcal{G}_{(2i-2)/n}}\Big[\Big(X(2i/n) - X((2i-1)/n)\Big)^2\Big] = M(2i/n) - M((2i-1)/n),\\
&\mathbb{E}_{\mathcal{G}_{(2i-2)/n}}\Big[\Big(X((2i-1)/n)) - X((2i-2)/n)\Big)^2\Big] = M((2i-1)/n) - M((2i-2)/n),\\
&\mathbb{E}_{\mathcal{G}_{(2i-2)/n}}\Big[\Big(X(2i/n) - X((2i-2)/n)\Big)^2\Big] = M(2i/n) - M((2i-2)/n).
\end{aligned}$$

Proof of Lemma 2.3

For simplicity, we just give the proof for $\xi^{n,2}$, the result for $\xi^{n,1}$ being obviously deduced. We easily get

$$\mathbb{E}_{\mathcal{G}_{(2i-2)/n}}[(\xi_i^{n,2})^2] = 2^{1+\tau(4)}(n/2)^{\tau(4)}\Big(M(2i/n) - M((2i-2)/n)\Big)^2.$$

We conclude using Lemma 2.1.

Proof of Lemma 2.4

Conditional on $\mathcal{G}_{(2i-2)/n}$, $\Big(X(2i/n)-X((2i-1)/n),\ X(2i/n)-X((2i-2)/n)\Big)$ is a centered Gaussian vector with variance-covariance matrix equal to

$$\begin{pmatrix} M(2i/n)-M((2i-1)/n) & M(2i/n)-M((2i-1)/n) \\ M(2i/n)-M((2i-1)/n) & M(2i/n)-M((2i-2)/n) \end{pmatrix}.$$

Thus, it has the same law as

$$\Big(\sqrt{M(2i/n)-M((2i-1)/n)}Z_1,\ \sqrt{M(2i/n)-M((2i-2)/n)}Z_2\Big),$$

where (Z_1, Z_2) is a centered Gaussian vector with variance-covariance matrix equal to

$$\begin{pmatrix} 1 & \Big(\frac{M(2i/n)-M((2i-1)/n)}{M(2i/n)-M((2i-2)/n)}\Big)^{1/2} \\ \Big(\frac{M(2i/n)-M((2i-1)/n)}{M(2i/n)-M((2i-2)/n)}\Big)^{1/2} & 1 \end{pmatrix}.$$

Then, note that

$$\mathbb{E}_{\mathcal{G}_{(2i-2)/n}}\Bigg[\Big\{\Big(X(2i/n)-X((2i-1)/n)\Big)^2-\Big(M(2i/n)-M((2i-1)/n)\Big)\Big\}$$
$$\Big\{\Big(X(2i/n)-X((2i-2)/n)\Big)^2-\Big(M(2i/n)-M((2i-2)/n)\Big)\Big\}\Bigg]$$

is equal to

$$\Big(M(2i/n)-M((2i-1)/n)\Big)\Big(M(2i/n)-M((2i-2)/n)\Big)\mathbb{E}_{\mathcal{G}_{(2i-2)/n}}[(Z_1^2-1)(Z_2^2-1)].$$

Using Mehler's formula, the preceding conditional expectation is finally equal to

$$2\Big(M(2i/n)-M((2i-1)/n)\Big)^2.$$

In the same way, we get

$$\mathbb{E}_{\mathcal{G}_{(2i-2)/n}}\Big[\xi_i^{n,2}\Big\{\Big(X((2i-1)/n)-X((2i-2)/n)\Big)^2-\Big(M((2i-1)/n)-M((2i-2)/n)\Big)\Big\}\Big]$$
$$=2n^{\tau(4)/2}\Big(M((2i-1)/n)-M((2i-2)/n)\Big)^2.$$

Eventually, we obtain

$$\mathbb{E}_{\mathcal{G}_{(2i-2)/n}}\Big[\xi_i^{n,1}\xi_i^{n,2}\Big]=$$
$$2n^{\tau(4)}\Big\{\Big(M(2i/n)-M((2i-1)/n)\Big)^2+\Big(M((2i-2)/n)-M((2i-2)/n)\Big)^2\Big\}.$$

We conclude using Lemma 2.1.

Proof of Lemma 2.5

Here again, we just give the proof for $\xi^{n,2}$. It is clear that conditional on $\mathcal{G}_{(2i-2)/n}$ the law of $(\xi_i^{n,2})$ is the same as the law of $n^{\tau(4)/2}\Big(M(2i/n)-M((2i-2)/n)\Big)(Z^2-1)$, with Z a standard centered Gaussian variable. Since we are in the case $q_0>2$, there exists some $\varepsilon>0$ such that $2(1+\varepsilon)<q_0$ and $\tau(4(1+\varepsilon))>(1+\varepsilon)\tau(4)$. Thus,

$$\mathbb{E}_{\mathcal{G}_{(2i-2)/n}}\Big[(\xi_i^{n,2})^{2(1+\varepsilon)}\Big]\leq cn^{(1+\varepsilon)\tau(4)}\Big(M(2i/n)-M((2i-2)/n)\Big)^{2(1+\varepsilon)}$$
$$\leq cn^{(1+\varepsilon)\tau(4)-\tau(4(1+\varepsilon))}n^{\tau(4(1+\varepsilon))}\Big(M(2i/n)-M((2i-2)/n)\Big)^{2(1+\varepsilon)}.$$

We conclude using Lemma 2.1.

2.4.2 Proof of Proposition 2.2

Assume first that X is an MRW. We have

$$N(\hat{\tau}(4)-\tau(4))=\frac{2}{\log(2)}\Big\{\log\Big(2^{N\tau(4)/2}S(4,2^{-\lfloor N/2\rfloor})\Big)-\log\Big(2^{N\tau(4)}S(4,2^{-N})\Big)\Big\}.$$

Using Lemma 2.1, we get that

$$\frac{2^{\lfloor N/2 \rfloor \tau(4)} S(4, 2^{-\lfloor N/2 \rfloor})}{2^{N\tau(4)} S(4, 2^{-N})} \to 1.$$

Since $S(4, 2^{-\lfloor N/2 \rfloor})$ tends to zero, we can replace $\lfloor N/2 \rfloor$ by $N/2$ and so we obtain that $N(\hat{\tau}(4) - \tau(4))$ tends to 0 almost surely. Since $0 < \tau(4) < 1$, the first assertion of Proposition 2.2 follows.

We now turn to the second assertion. In restriction to Ω^c, we get the result using that $\tau^*(4)$ tends to 1 together with the inequality

$$2^{N(\tau^*(4)-1)} \leq 2^{-Nv_N} + 2^{N(u_N-1)}.$$

In restriction to Ω^j, we use the inequality

$$2^{N(\tau^*(4)-1)} \leq 2^{N(\hat{\tau}(4)-1)} + 2^{N(u_N-1)},$$

and the facts that $N\hat{\tau}(4)$ goes to zero and that $2^{\tau^*(4)} - 1$ is of the same order than $\tau^*(4)$ which is greater than u_N.

Chapter 3

Continuous-time skewed multifractal processes as a model for financial returns

3.1 Introduction

A striking feature of the prices of financial assets resides in that a certain amount of statistical properties, generally referred to as "stylized facts", appear to be universal. Indeed, it is for instance well known that for any considered asset, and for any time scale from few minutes to few months or years, the log-returns (*ie.*, the logarithms of price increments) are a centered and uncorrelated time series with a heavy-tailed distribution, and that the absolute or squared log-returns time series presents long memory and persistence. We refer to Cont [37] and Bouchaud and Potters [25] for a thorough review of the various statistical properties that can be observed on financial data. Formulating a probabilistic, continuous-time model that reflects most of these stylized facts is naturally of first importance, both from a theoretical and practical point of view, and has been the motivation of a considerable number of research papers.

Many recent empirical studies have also suggested that financial data share statistical properties with turbulent intermittent velocity fields (Mandelbrot [69], Calvet and Fisher [28], Bacry, Kozhemyak and Muzy [10], Bouchaud and Potters [25], Duchon, Robert and Vargs DRV08): areas of rapid and violent activities alternate with more peaceful ones, and this phenomenon repeats itself at any time-scale in the "same" way. So as to take this elaborate scale invariance into account, a natural approach is then to suppose that the local Hölder regularity of the underlying continuous signal is itself random, which also translates into a "multiscaling" or a "multifractal scaling" of price return fluctuations, see Frisch and Parisi [43, 42] . A process X with stationary increments is said to have a multifractal scaling if it satisfies

$$\mathbb{E}\Big[|X(t)|^q\Big] \sim c_q t^{\zeta_q} \text{ as } t \to 0, \tag{3.1}$$

for all q's in some real interval, some positive constants c_q, and a scaling exponent $q \mapsto \zeta_q$ that is *nonlinear*. Since the pioneering work of Mandelbrot [69], the phenomenology of such multifractal models has provided new concepts and tools to analyze market fluctuations. It notably inspired the family of so-called "cascade" random processes that account for the main statistical properties of financial prices in an elegant and parsimonious way Calvet and Fisher [26] or Bacry, Muzy and Delour [12] (see also Barral and Mandelbrot [17] for a non financial approach). Moreover, these models are amenable to many analytical computations: they are easy to estimate and they lead to simple and yet very competitive solutions to the problem of conditional risk (volatility or VaR) forecasting (see Calvet and Fisher [28], Bacry, Kozhemyak and Muzy [10], and Duchon, Robert and Vargas [39]). However, although these multifractal models reproduce all the stylized facts we have already mentioned, we would like to point out one stylized fact that is not taken into account by them and actually neither by most mathematical models, namely the *leverage effect*.

The so-called leverage effect is a feature that is mostly present on stock and index prices: The variation of the log-return in the past is found to be negatively correlated to the volatility in the future (see Bouchaud, Matacz and Potters [24]). Here, the volatility may be defined for instance as the squared or absolute log-return. So, basically, this effect quantifies the "panic" effect that takes place after a large downward move of the price which tends to increase the volatility much more than a large upward move would. Let us note that this effect induces two types of asymmetry in the price process. The first one (*time asymmetry*) is that the price process is not invariant under reversion of time. Indeed, the variation of the volatility in the past is not correlated to the variation of the log-return in the future (if it were true, then a simple arbitrage could actually be performed). The second one (*return asymmetry*) is that this effect implies a negative skewness of the distribution of log-returns. The higher the leverage effect, the higher the skewness, the higher the asymmetry of the implied volatility smile (see Bouchaud and Potters [25]).

Thus, it appears clearly important to incorporate these asymmetries in a probabilistic model of log-returns. Let us note that it has already been done in a non multifractal setting for instance by Bouchaud, Matacz and Potters [24], Perello and Masoliver [77], Perello, Masoliver and Bouchaud [78] or Ciliberti, Bouchaud and Potters [35]. However, explicitly constructing a skewed multifractal process with leverage effect has not yet been done, though we should mention two very interesting works. In the first one by Pochart and Bouchaud [79], the authors built a skewed model with some multiscaling properties and leverage effect in discrete-time. However, when the sampling pace goes to zero, eventhough the multiscaling properties converge to standard multifractal behavior (3.1), the skewness tends to zero so that any leverage effect unfortunately disappears. The second work by Robert and Vargas [85] takes place in the setting of turbulence study and not of finance, and as such is not interested in this leverage effect.

In the present chapter, we try to fill this gap, and show how one can obtain a skewed, continuous time, multifractal model for log-returns that reproduces all stylized facts mentioned in this introduction, including the leverage effect. Moreover, this modelization may be described as "parsimonious" insofar as it relies only on a very small number of scalar parameters. So as to briefly summarize our approach, let us say that we extend the so-called Multifractal Random Walk (MRW) model, which is one of the simplest multifractal, continuous "cascade" model [12, 10], by introducing some explicit correlations between returns and volatility.

The chapter is organized as follows. In Section 3.2, we present a brief overview of the (symmetrical) multifractal log-normal MRW model for financial data and discuss the first attempt by Pochart and Bouchaud to input asymmetry in this model [79]. In Section 3.3, we propose a new construction of a continuous-time, skewed, multifractal process that depends on a Hurst exponent $H > 1/2$. In Section 3.4, we investigate its scaling properties and the behaviour of all q-order moments (including the moments of order 3 and the skewness). In Section 3.5, we explicitly compute the leverage effect and discuss the choice of the parameter H, which affects both the correlation of the increments (which should be close to zero, as in the case of financial data) and the skewness and the leverage effect (which should be significantly non zero). A simulation scheme and some numerical simulations are presented in Section 3.6, which also contains a comparison with real data. Some computations and proofs that are used in the rest of the chapter are finally postponed in the Appendix.

3.2 Multifractal processes

Let $X = \Big(X(t),\, t \geq 0\Big)$ be a real-valued stochastic process with stationary increments. For $t \geq 0$ and $\tau > 0$, we write $\delta_\tau X(t)$ for the increment $X(t+\tau) - X(t)$. When the moments of order p of X satisfy:

$$\mathbb{E}\Big[|\delta_\tau X(t)|^q\Big] \approx c_q \tau^{\zeta_q} \tag{3.2}$$

for (small) $\tau > 0$, it is usual to speak of either a *monofractal* scaling if the exponent ζ_q is a linear function of q, or a *multifractal* scaling if it is nonlinear.

3.2.1 The Multifractal Random Walk

In [12], Bacry, Muzy and Delour proposed the construction of a continuous-time stochastic random process that exhibits features quite similar to most stylized facts observed on the returns of financial assets (see [10]), including multifractal scaling, but excluding leverage effect. The Multifractal Random Walk (MRW) $X(t)$ can be defined as the continuous limit as n goes to $+\infty$ of the following discretized process:

$$X_n(t) = \sum_{k=0}^{\lfloor nt \rfloor} \varepsilon_n(k/n) e^{w_n(k/n)} \tag{3.3}$$

where the $\varepsilon_n(k/n)$'s are independent Gaussian random variables with mean equal to zero and variance equal to σ^2/n for some $\sigma > 0$, and w_n is a Gaussian stationary process independent of the $\varepsilon_n(k)$'s. (Alternatively, one can consider a continuous construction

$$X(t) = \sigma \lim \int_0^t e^{w_n(u)} dB(u)$$

where B is a standard Brownian motion independent of w_n, see [11].) The autocovariance of w_n is the following:

$$\mathbb{C}\mathrm{ov}\Big[w_n(j/n), w_n(k/n)\Big] = \begin{cases} \lambda^2 \log \frac{nT}{|j-k|+1} & \text{if } \frac{|j-k|+1}{n} \leq T \\ 0 & \text{else,} \end{cases} \tag{3.4}$$

and the expectation of w_n is such that $\mathbb{E}\Big[e^{2w_n(\cdot)}\Big] = 1$. Here, $\lambda^2 > 0$ is a parameter called intermittency coefficient, and $T > 0$ is a parameter (called integral scale) such that the scaling (3.2) holds exactly for all $\tau \in [0, T]$. More precisely, as shown in [11], the following relation holds for all $r \in [0, 1]$:

$$\Big(X(rt),\, 0 \leq t \leq T\Big) \overset{law}{=} r^{1/2} e^{w_r} \Big(X(t),\, 0 \leq t \leq T\Big)$$

where w_r is a Gaussian random variable independent of X, and with expectation $-\lambda^2 \log(r^{-1})$ and variance $\lambda^2 \log(r^{-1})$. One can then deduce that the following multifractal scaling is satisfied for $\tau \in [0, T]$:

$$\mathbb{E}\Big[|\delta_\tau X(t)|^q\Big] = c_q \tau^{\zeta_q}$$

with $\zeta_q = q/2 + \lambda^2(q - q^2/2)$ and $c_q = \mathbb{E}\Big[|X(T)|^q\Big] T^{-\lambda^2(q-q^2/2)}$ for all q's such that c_q is finite. It is furthermore shown in [11] that c_q is finite for all $q \in (0, 4\lambda^{-2})$ and $c_q = +\infty$ for all $q > 4\lambda^{-2}$, so that the process X does not have moments of all orders and the distribution of $X(t)$ is heavy-tailed.

3.2.2 Further extensions : towards a skewed model with leverage effect

The MRW model has been shown to have interesting applications to financial data, in particular in terms of volatility and Value at Risk forecasting, see Calvet and Fisher [28], Bacry, Kozhemyak and Muzy [10], and Duchon, Robert and Vargas [39]. However, one significant drawback of the MRW approach is that the distribution of $X(t)$ is symmetric, so that it does not reflect the skewness empirically observed on financial data. Moreover, since the two processes ε and w are independent, it follows that the increments of the process X and the square of the increments are uncorrelated. Therefore, the model does not present the "leverage effect" observed on stocks and financial indexes prices.

Pochart and Bouchaud [79] therefore proposed to modify the construction (3.3) in the following way:

$$\tilde{X}_n(t) = \sum_{k=0}^{\lfloor nt \rfloor} \varepsilon_n(k/n) e^{w_n(k/n) - n^{\alpha} \sum_{i<k} K(i/n,k/n)\varepsilon(i/n)}, \tag{3.5}$$

where K is a positive kernel. As they show, this enables to fairly well reproduce the leverage effect observed on the data, as well as retaining most of the nice properties of the original MRW. However, the authors note that this holds only for $n < +\infty$: indeed, the odd moments of the process vanish when n goes to $+\infty$, so that their approach only gives a discrete-time model for $t = 0,\ 1/n,\ \ldots, i/n,\ \ldots$ Another related work can be found in Robert and Vargas [85] where the authors study a skewed 3D generalization of the MRW model with applications to hydrodynamics.

In what follows, we propose an alternate, continuous-time, construction for a multifractal random walk with skewness and leverage effect. In particular, we modify the construction (3.3) by defining the noise ε as a *fractional* Gaussian noise with Hurst exponent H, where H is chosen in a regime where the increments of X are almost uncorrelated. Thus, our approach shares some connections with the previous works of Ludeña [62] and Abry, Chainais, Coutin and Pipiras [1] who also considered the question of constructing an MRW with a fractional noise. However, in these papers the noise ε is independent of the volatility process w, whereas we will define them as correlated processes.

3.3 Construction of a continuous-time skewed MRW

3.3.1 Definition of the skewed process

Fix the following parameters: $\lambda \in (0, 1/2)$, $T > 0$, $\sigma > 0$, $H \in (1/2 + \lambda^2/2, 1)$. The parameters λ, T and σ are of similar nature as above, while H can be seen as a Hurst exponent as in Ludeña [62] and Abry, Chainais, Coutin and Pipiras [1]. We define a skewed multifractal random walk by:

$$X(t) = \lim X_l(t) \quad \text{as} \quad l \to 0 \tag{3.6}$$

where

$$X_l(t) = \int_0^t \varepsilon_l(u) e^{w_l(u)} du$$

and $(\varepsilon, w) = \Big(\big(\varepsilon_l(u), w_l(u)\big),\ u \in \mathbb{R},\ l \in (0, T)\Big)$ is a Gaussian processes with values in $\mathbb{R}^2$ that satisfies the following properties:

Property 3.1. $\big(\varepsilon_l(u), w_l(u)\big)$ *is stationary in* u, *that is: for* $u_1, \ldots, u_n, \tau \in \mathbb{R}$

$$\Big(\big(\varepsilon_l(u_1), w_l(u_1)\big), \ldots, \big(\varepsilon_l(u_n), w_l(u_n)\big),\ l \in (0, T)\Big) \quad \textit{and}$$

$$\Big(\big(\varepsilon_l(u_1 + \tau), w_l(u_1 + \tau)\big), \ldots, \big(\varepsilon_l(u_n + \tau), w_l(u_n + \tau)\big),\ l \in (0, T)\Big)$$

have the same law.

Property 3.2. *$\big(\varepsilon_l(u), w_l(u)\big)$ has independent increments in l, that is: for $l' < l$*

$$\Big(\big(\varepsilon_{l'}(u) - \varepsilon_l(u), w_{l'}(u) - w_l(u)\big),\ u \in \mathbb{R}\Big) \text{ and } \Big(\big(\varepsilon_l(u), w_l(u)\big),\ u \in \mathbb{R}\Big)$$

are independent.

Property 3.3. *The expectation of $w_l(u)$ is $-1/2\mathbb{V}ar\big[w_l(u)\big]$ (so that $\mathbb{E}\big[e^{w_l(u)}\big] = 1$). The expectation of $\varepsilon_l(u)$ is zero.*

Property 3.4. *For $\tau \in \mathbb{R}$, let us write $\gamma_l^w(\tau)$ (resp. $\gamma_l^\varepsilon(\tau)$, $\gamma_l^{w\varepsilon}(\tau)$) for $\mathbb{C}ov[w_l(u), w_l(u+\tau)]$ (resp. $\mathbb{C}ov[\varepsilon_l(u), \varepsilon_l(u+\tau)]$, $\mathbb{C}ov[\varepsilon_l(u), w_l(u+\tau)]$), and let us define*

$$\gamma^w(\tau) = \lambda^2 \max\Big(\log(T/|\tau|), 0\Big) \tag{3.7}$$

$$\gamma^\varepsilon(\tau) = c^\varepsilon \sigma^2 |\tau|^{-2+2H} \tag{3.8}$$

$$\gamma^{w\varepsilon}(\tau) = c^{w\varepsilon} \sigma \lambda \Big(\max(\tau, 0)\Big)^{-1+H} \tag{3.9}$$

for some positive constants c^ε and $c^{w\varepsilon}$ (we use the convention $0^{-1} = +\infty$). Then for fixed τ, $\gamma_l^w(\tau) \uparrow \gamma^w(\tau)$ (resp. $\gamma_l^\varepsilon(\tau) \uparrow \gamma^\varepsilon(\tau)$, $\gamma_l^{w\varepsilon}(\tau) \uparrow \gamma^{w\varepsilon}(\tau)$) as l goes to zero. Moreover, $\gamma_l^{w\varepsilon}(\tau)$ is zero for $l \in (0, T)$ and $\tau \leq 0$.

Property 3.5. *We have the following scaling equations for $l \in (0, T)$, $\tau \in [-T, T]$ and $r \in (0, 1]$:*

$$\gamma_{rl}^w(r\tau) = -\lambda^2 \log(r) + \gamma_l^w(\tau),$$

$$\gamma_{rl}^\varepsilon(r\tau) = r^{-2+2H} \gamma_l^\varepsilon(\tau),$$

and

$$\gamma_{rl}^{w\varepsilon}(r\tau) = r^{-1+H} \gamma_l^{w\varepsilon}(\tau).$$

As explained below, these properties are sufficient to prove the convergence $X_l \to X$ and study the properties of X. However, we still need to justify the existence of such a process (ε, w); this is done in Subsection 3.3.3 where we explicitly construct an example of

(ε, w) that satisfies the above properties. The exact values $\gamma_l^w(\tau)$, $\gamma_l^\varepsilon(\tau)$ and $\gamma_l^{w\varepsilon}(\tau)$ for $\tau \in \mathbb{R}$ and $l \in (0, T)$ as well as the constants c^ε and $c^{w\varepsilon}$ corresponding to this construction can all be found in Appendix 3.A.

Remark: It can be immediately seen that the function γ^ε in Property 3.4 is the covariance of a fractional Brownian noise with Hurst exponent H, so that the process $\left(\int_0^t \varepsilon_l(u)du,\, t \geq 0\right)$ converges in law to a fractional Brownian motion as l goes to 0. (One could easily prove that the convergence also holds under stronger modes, however this is of little interest for our purpose here.)

3.3.2 Existence of X

We here prove the existence and nondegeneracy of the process X. Let us define the following condition $\mathcal{H}(p)$ on $p \geq 2$

$$\mathcal{H}(p): \quad pH - \frac{\lambda^2}{2}p(p-1) - 1 > 0. \tag{3.10}$$

Note that since we chose $H \in (1/2 + \lambda^2/2, 1)$, then $\mathcal{H}(2)$ is always satisfied.

We first state two useful results:

Proposition 3.1. *Let $(\mathcal{F}_l)_{l>0}$ be the following filtration:*

$$\mathcal{F}_l = \sigma\Big\{\Big(\varepsilon_{l'}(u), w_{l'}(u)\Big),\, u \in \mathbb{R},\, l' \geq l\Big\}.$$

Then for fixed $t > 0$, $\Big(X_l(t),\, l > 0\Big)$ is an $\mathcal{F}_l$-martingale.

Proof. This is a straightforward application of Properties 3.2 and 3.3.

□

Proposition 3.2. *Let $p \geq 2$ be an integer. Then for $t \in [0, T]$,*

$$\lim_{l \to 0} \mathbb{E}\Big[X_l(t)^p\Big] = K(p)t^{pH - \frac{\lambda^2}{2}p(p-1)}, \tag{3.11}$$

where $K(p) \in (0, +\infty]$ *is a constant that depends on the parameters* σ^2*,* T*,* λ^2*, and* H*, but not on* t*. Moreover,* $K(p)$ *is finite if and only if* p *satisfies* $\mathcal{H}(p)$*.*

The proof of this proposition is postponed in Appendix 3.B, where the reader will also find the value of the constant $K(p)$.

It is then easy to prove the following:

Theorem 3.1. *For fixed* $t \geq 0$*,* $X_l(t)$ *goes to a nondegenerate limit* $X(t)$ *as* l *goes to 0, and the convergence holds almost surely and in* L^3*. Moreover, the process* $X = \Big(X(t),\, t \geq 0\Big)$ *admits a continuous modification.*

Proof. Applying Proposition 3.1 and a classical result the theory of martingales, if for some fixed $t > 0$, the moments $\mathbb{E}\Big[|X_l(t)|^p\Big]$ remain bounded for some $p > 1$, we have:

$$X(t) = \lim X_l(t) \quad \text{as} \quad l \to 0$$

almost surely and in L^q for all $q \in [1, p)$. Since we chose $H > 1/2 + \lambda^2/2$ and $\lambda^2 < 1/4$, $\mathcal{H}(4)$ holds. Proposition 3.2 then proves the first half of the statement of the theorem.

Moreover, the same Proposition 3.2 shows that the Kolmogorov criterion for convergence and regularity of stochastic processes is satisfied: there exists some $a > 0$, $b > 0$ and $c > 0$ such that for any $t \in [0, T]$ and any $l > 0$ small enough,

$$\mathbb{E}\Big[|X_l(t)|^a\Big] \leq ct^{1+b}.$$

Indeed, we can choose $a = 4$ and $b = 4H - 6\lambda^2 - 1$. The rest of the theorem follows from a standard application of this criterion.

□

Remark: From Properties 3.1 and 3.3, X is a process with stationary increments and zero expectation.

3.3.3 Explicit construction of (ε, w)

The the construction of the process w is almost the same as the one used in the definition of the symmetrical MRW in [11]. The process ε is also constructed in a similar fashion. Let us consider the "time-scale" half-plane $\mathbb{R} \times (0, +\infty)$, and define on it a 2D Gaussian white noise $P(dt', dl')$ with variance $l'^{-2}dt' \times dl'$. Then w is obtained as:

$$w_l(t) = -\lambda^2/2\Big(\log(T/l) + 1\Big) + \lambda \int_{\mathcal{A}_l(t)} P(dt', dl'),$$

where $\mathcal{A}_l(t)$ is the conical domain

$$\mathcal{A}_l(t) = \Big\{(t', l') \in \mathbb{R} \times (0, +\infty),\ l' \geq l, 0 \leq t - t' \leq \min(l', T)\Big\}.$$

So as to construct ε, we now consider the domain $\mathcal{B}_l(t)$:

$$\mathcal{B}_l(t) = \Big\{(t', l') \in \mathbb{R} \times (0, +\infty),\ l' \geq l, 0 \leq t' - t \leq l'\Big\}$$

and define

$$\varepsilon_l(t) = \sigma \int_{\mathcal{B}_l(t)} l'^{-1+H} P(dt', dl').$$

We refer to Figure 3.1 for a graphical representation of $\mathcal{A}_l(t)$ and $\mathcal{B}_l(t)$. In Appendix 3.A we give the exact first and second moments of (ε, w), which are obtained through straightforward computations. It is then easy to check that (ε, w) satisfies Properties 1 to 5. In particular, the constants c^{ε} and $c^{w\varepsilon}$ of Property 3.4 are respectively equal to $\frac{1}{(2-2H)(3-2H)}$, and $\frac{2^{2-H}-2}{(1-H)(2-H)}$.

3.4 Scaling and moments of the skewed process X

The following theorem characterizes the scaling behavior of the distribution of $X(t)$:

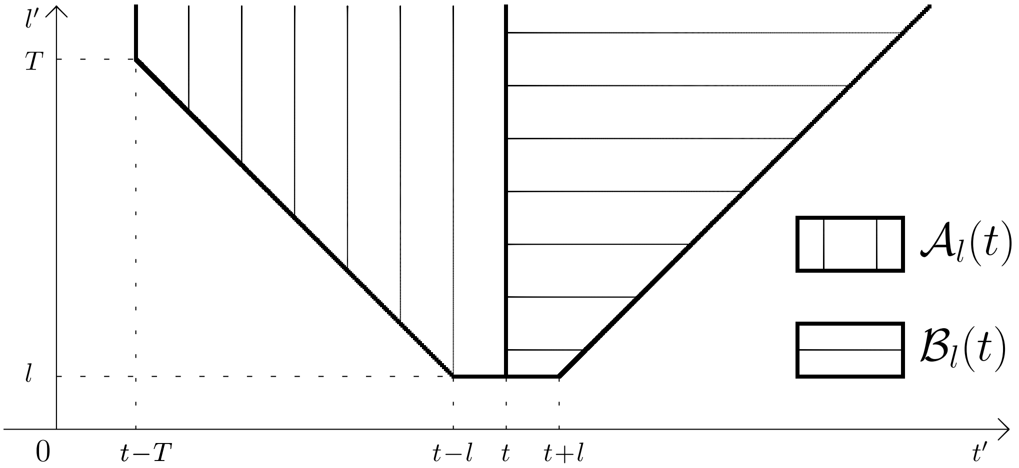

Figure 3.1: *The cones $\mathcal{A}_l(t)$ and $\mathcal{B}_l(t)$*

Theorem 3.2. *For $r \in (0, 1]$,*

$$\Big(X(rt),\, 0 \leq t \leq T\Big) \overset{law}{=} e^{w_r} r^H \Big(X(t),\, 0 \leq t \leq T\Big) \tag{3.12}$$

where $w_r \sim N(-\lambda^2 \ln(r^{-1})/2, \lambda^2 \ln(r^{-1}))$ is a Gaussian constant independent of X.

Proof. First note that it follows from Properties 3.3 and 3.5 that for fixed $l > 0$ and $r \in (0, 1]$

$$\Big(\Big(\varepsilon_{rl}(ru), w_{rl}(ru)\Big),\, u \in [0, T]\Big) \overset{law}{=} \Big(\Big(r^{-1+H}\varepsilon_l(ru), w_r + w_l(ru)\Big),\, u \in [0, T]\Big),$$

where w_r is a $N(-\lambda^2 \ln(r^{-1})/2, \lambda^2 \ln(r^{-1}))$ random variable, which is furthermore independent of (ε, w). From this we deduce:

$$\Big(\varepsilon_{rl}(ru) \exp\Big(w_{rl}(ru)\Big),\, u \in [0, T]\Big) \overset{law}{=} r^{-1+H} \exp(w_r)\Big(\varepsilon_l(u) \exp\Big(w_l(u)\Big),\, u \in [0, T]\Big)$$

We now consider the process $X_{rl}(rt)$:

$$\begin{aligned}\Big(X_{rl}(rt),\, t \in [0,T]\Big) &= \Big(\int_0^{rt} \varepsilon_{rl}(u) \exp\Big(w_{rl}(u)\Big),\, t \in [0,T]\Big)\\ &= \Big(r\int_0^{t} \varepsilon_{rl}(ru) \exp\Big(w_{rl}(ru)\Big),\, t \in [0,T]\Big)\\ &\overset{law}{=} r^H \exp(w_r)\Big(\int_0^{t} \varepsilon_{l}(u) \exp\Big(w_{l}(u)\Big),\, t \in [0,T]\Big)\\ &= r^H \exp(w_r)\Big(X_l(t),\, t \in [0,T]\Big).\end{aligned}$$

Taking the limit $l \to 0$ gives (3.12).

□

We now turn on the absolute moments of X, and show that they satisfy (3.2) with an exact equality.

Theorem 3.3. *If for some $q > 0$, there is an even integer $p > q$ such that $\mathcal{H}(p)$ is satisfied, then for $t \in [0,T]$*

$$\mathbb{E}\Big[|X(t)|^q\Big] = C(q)t^{qH-\frac{\lambda^2}{2}q(q-1)}, \tag{3.13}$$

where $C(q)$ is the positive finite constant

$$C(q) = T^{-qH+q(q-1)\lambda^2/2}\mathbb{E}\Big[|X(T)|^q\Big].$$

If q is moreover an integer, then

$$\mathbb{E}\Big[X(t)^q\Big] = K(q)t^{qH-\frac{\lambda^2}{2}q(q-1)}, \tag{3.14}$$

where $K(q)$ is the same as in Proposition 3.2. Conversely, if $\mathcal{H}(q)$ is not satisfied for some $q > 2$, $\mathbb{E}\Big[|X(t)|^q\Big] = +\infty$ for $t > 0$.

Proof. Propositions 3.1 and 3.2 yield that $X_l(T)$ converges in L^q to $X(T)$, so that $\mathbb{E}\Big[|X(T)|^q\Big]$ is finite. In the case where q is an integer, (3.14) is also a direct consequence of these two propositions. In the general case, we apply Theorem 3.2: by setting $r = t/T$, we have:

$$\mathbb{E}\Big[|X(t)|^q\Big] = T^{-qH+q(q-1)\lambda^2/2}\mathbb{E}\Big[|X(T)|^q\Big]t^{qH-\frac{\lambda^2}{2}q(q-1)}.$$

Conversely, let us suppose that $\mathbb{E}\Big[|X(t)|^q\Big]$ is finite for some $t \in (0,T]$ and $q > 1$. Then from the stationarity of the increments of X and a basic convexity inequality, we obtain:

$$\mathbb{E}\Big[|X(t)|^q\Big] > 2\mathbb{E}\Big[|X(t/2)|^q\Big].$$

Then we have from Theorem $2^{1-qH+\lambda^2 q(q-1)/2} < 0$ so that $\mathcal{H}(q)$ is satisfied. This proves the result.

□

3.5 Modeling the asymmetry of financial data

3.5.1 Preliminaries

In this section, we focus on the second and third order properties of the increments of the process X, and show how, depending on the value of the parameter H, they can reflect the following stylized facts: First, there is no statistically significant correlation between two log-returns at different times. Second, there is a negative, slightly significant correlation between the past log-returns and the future squared log-returns, while the converse is false: past volatilities and future returns appear to be uncorrelated. (Note that while the former fact is universally observed on financial assets prices, the latter is mainly observed on stocks and indices prices, see Bouchaud and Potters [25].)

Let us examine the moment of second order of X:

Proposition 3.3. *For $t \in [0,T]$,*

$$\mathbb{E}\Big[X(t)^2\Big] = \frac{2\sigma^2 T^{\lambda^2} c^{\varepsilon}}{(2H-1-\lambda^2)(2H-\lambda^2)}t^{2H-\lambda^2}.$$

Proof. This is a simple application of Proposition 3.2 and of the value of $K(2)$ given in Appendix 3.B.

□

Let us now recall that X is properly defined only in the case $H \in (1/2 + \lambda^2/2, 1)$ (so that $\mathcal{H}(4)$ holds). Thus we can write $\mathbb{E}\Big[X(t)^2\Big] = K(2)t^{1+d}$ with $K(2)$ being the above fraction and

$$d = 2H - 1 - \lambda^2 > 0.$$

So as to obtain a satisfying model of financial data, we clearly have to place ourselves in a regime where d is small, so that $\mathbb{E}\Big[X(t)^2\Big]$ scales approximately as a linear function of t, and the covariance between the increments of X at different times vanishes. However, as can be seen from Proposition 3.3, $K(2)$ goes to $+\infty$ as d goes to 0.

In this section, we therefore study in the regime of small d the second and third order properties of the normalized process:

$$Y_d(t) = -\sigma \frac{X(t)}{\Big(\mathbb{E}\Big[X(1)^2\Big]\Big)^{1/2}}.$$

Note that we introduced a minus sign so as to reproduce the negative skewness empirically observed, and we added a d subscript to emphasize the dependence on this parameter (we will continue to use the notation $d \in (0, 1 - \lambda^2)$ instead of $H \in (1/2 + \lambda^2/2, 1)$). We intend to here that for a well chosen value of d, the process Y_d reproduces the type of asymmetry observed on stocks and indices prices.

Recall that the notation $\delta_\tau Y_d(t)$ refers to the increment $Y_d(t+\tau) - Y_d(t)$. For $\tau > 0$, $k \in \mathbb{Z}$, we are interested in the following functions:

$$\rho_d^{(1)}(\tau, k) = \frac{\mathbb{E}\Big[\delta_\tau Y_d(0)\delta_\tau Y_d(k\tau)\Big]}{\mathbb{E}\Big[\delta_\tau Y_d(0)^2\Big]} \tag{3.15}$$

and

$$\rho_d^{(2)}(\tau, k) = \frac{\mathbb{E}\Big[\delta_\tau Y_d(0)\delta_\tau Y_d(k\tau)^2\Big]}{\Big(\mathbb{E}\Big[\delta_\tau Y_d(0)^2\Big]\Big)^2}, \tag{3.16}$$

where the normalization of $\rho^{(2)}$ has been introduced in Bouchaud, Matacz and Potters [24] and further used in the literature, for instance in Ciliberti, Bouchaud and Potters [35] or Perello and Masoliver [77]. Alternatively, one could wish to examine the proper linear correlation between $\delta_\tau Y_d(0)$ and $\delta_\tau Y_d(k\tau)^2$, that is:

$$\rho_d^{(3)}(\tau, k) = \frac{\mathbb{E}\Big[\delta_\tau Y_d(0)\delta_\tau Y_d(k\tau)^2\Big]}{\Big(\mathbb{E}\Big[\delta_\tau Y_d(0)^2\Big]\Big)^{1/2}\Big(\mathbb{E}\Big[\delta_\tau Y_d(0)^4\Big]\Big)^{1/2}}. \tag{3.17}$$

Since we are dealing with correlations that empirically decays to zero after a few lags, we will restrict ourselves to the case $(|k| + 1)\tau \leq T$.

3.5.2 Behavior of Y_d in the regime of small d

We first examine the moments of Y_d as d goes to 0:

Proposition 3.4. *For $t \in [0, T]$, and $p \geq 2$ an even integer such that $\mathbb{E}\Big[X(t)^p\Big] < +\infty$, $\mathbb{E}\Big[Y_d(t)^p\Big]$ remains positive and bounded as d goes to 0. However, if p is odd, $\mathbb{E}\Big[Y_d(t)^p\Big]$ goes to zero.*

Proof. Again, this is a direct consequence of Proposition 3.2 and of the value of $K(p)$ given in Appendix 3.B.

□

Remark: This suggests that a limiting process

$$Y(t) = \lim Y_d(t) \quad \text{as} \quad d \to 0$$

may exist, where the convergence is understood as a convergence in distribution. This is quite reminiscent of the study of [1] who investigated the validity of the limit

$$\lim \frac{\int_0^t e^{w_l}(u)dB^H(u)}{\mathbb{E}\Big[\Big(\int_0^t e^{w_l}(u)dB^H(u)\Big)^2\Big]^{1/2}} \quad \text{as} \quad l \to 0,$$

where B^H is a fractionary Brownian motion with Hurst exponent H and which is (in contrast with our setting) independent of w. In the case $H = 1/2 + \lambda^2/2$ (that is, $d = 0$ in our notations), these authors obtained only the convergence of the moments of integer order and postulated the convergence in law. Note however that in the present work, we are not chiefly interested in the validity of the convergence $Y_d \to Y$, since the moments of order 3 of Y_d vanish, so that the limiting process Y_0 (if it exists) has a skewness equal to 0.

We now place ourselves in the regime of small but nonzero values of d, and examine the magnitude of the correlation functions $\rho_d^{(i)}$, $i = 1, 2, 3$.

Theorem 3.4. *For $0 < \tau < T$ and $(|k| + 1)\tau \leq T$, we have:*

$$\rho_d^{(1)}(\tau, k) = O(d) \text{ if } |k| \geq 1$$
$$|\rho_d^{(2)}(\tau, k)| = O(d^{1/2}) \text{ and } |\rho_d^{(3)}(\tau, k)| = O(d^{1/2}) \text{ if } k \geq 0$$
$$|\rho_d^{(2)}(\tau, k)| = O(d^{3/2}) \text{ and } |\rho_d^{(3)}(\tau, k)| = O(d^{3/2}) \text{ if } k < 0$$

as $d \to 0$.

The proof of this theorem can be found in Appendix 3.C.

This therefore suggests that when d is of order roughly 0.01 to 0.1, $\rho_d^{(2)}(\tau, k)$ for $k \geq 0$ is significantly non zero (as it is of order $d^{1/2}$), while $\rho_d^{(2)}(\tau, k)$ for $k < 0$ and $\rho_d^{(1)}(\tau, k)$ for $k \neq 0$ are much smaller, and in practice indiscernible from the noise. We refer to Section 3.6.2 for a empirical discussion concerning the choice of d.

3.6 Numerical simulation and comparison to real data

In this section we present a numerical method for simulating the process we introduce and report some comparisons of the leverage effect observed on simulated realizations with empirical data. Since the main objective of this chapter was to define and study the mathematical properties of the model, we do not discuss any parameter estimation issue. This problem and a more exhaustive comparison to financial data will be the subject of a forthcoming work.

3.6.1 The simulation scheme

We propose in this section to approximate the increments

$$\delta_\tau X(k\tau) = \lim_{l \to 0} \int_{k\tau}^{(k+1)\tau} \varepsilon_l(u) e^{w_l(u) du}$$

of the process X (for $k \in \mathbb{N}$ and $\tau > 0$) by Riemann sums. If the parameter $d = 2H - 1 - \lambda^2$ defined in the previous section is large enough, this is easily done, however if this parameter is small, then some extra difficulties must be taken care of. This comes mainly from the fact that the approximation

$$1/n \sum_{k=1}^{n} (k/n)^{-1+d} \approx \int_0^1 u^{-1+d} du$$

is valid only in a regime $n \gg e^{1/d}$ which might be unfeasible in practice.

We set $(l_n,\, n \in \mathbb{N})$ as:

$$l_n = \Big(d(1 - d/2 - \lambda^2/2)\Big)^{1/(1-d)} n^{-1},$$

and

$$\delta_\tau \tilde{X}_{1/n}(k\tau) = n^{-1} \sum_{j=\lfloor k\tau n \rfloor}^{\lfloor n(k+1)\tau \rfloor - 1} \varepsilon_{l_n}(j/n) e^{w_{l_n}(j/n)},$$

where (ε, w) are as in Section 3.3.3. We then have the following result:

Theorem 3.5. *For $k \in \mathbb{N}$ and $\tau > 0$, $\delta_\tau \tilde{X}_{1/n}(k\tau)$ converges to $\delta_\tau \tilde{X}(k\tau)$ in L^2 as $n \to +\infty$. Moreover, let us write r_n for*

$$r_n = \frac{\left|\mathbb{E}\Big[(\delta_\tau X(k\tau))^2\Big] - \mathbb{E}\Big[(\delta_\tau \tilde{X}_{1/n}(k\tau))^2\Big]\right|}{\mathbb{E}\Big[(\delta_\tau X(k\tau))^2\Big]}.$$

Then r_n is of order dn^{-d}: that is for fixed n, r_n/d is bounded as $d \to 0$, and for fixed d, $n^d r_n$ is bounded as $d \to +\infty$.

Proof. With no loss of generality, we suppose that $\tau = 1$. The exact value of $\mathbb{E}\Big[(\delta_1 X(k))^2\Big]$ can be found in Proposition 3.3; we rewrite it as

$$\mathbb{E}\Big[(\delta_1 X(k))^2\Big] = 2\sigma^2 T^{\lambda^2} c^\varepsilon \Big(n^{-d}/d + \int_{1/n}^1 u^{-1+d} du - \int_0^1 u^d du\Big).$$

In order to compute $\mathbb{E}\Big[(\delta_1 \tilde{X}_{1/n}(k))^2\Big]$, we use the relation

$$\varepsilon_{l_n}(k_1/n)\varepsilon_{l_n}(k_2/n)e^{w_{l_n}(k_1/n)+w_{l_n}(k_2/n)} = \frac{\partial^2}{\partial x_1 \partial x_2}\bigg|_{x_1=x_2=0} e^{w_{l_n}(k_1/n)+w_{l_n}(k_2/n)+x_1\varepsilon_{l_n}(k_1/n)+x_2\varepsilon_{l_n}(k_2/n)}.$$

This and the values of the covariance functions given in Appendix 3.A yield

$$\mathbb{E}\Big[(\delta_1 \tilde{X}_{1/n}(k))^2\Big] = \frac{\sigma^2 T^{\lambda^2} c^\varepsilon}{n}\Big((2-d-\lambda^2)l_n^{-1+d} + 2\sum_{k=1}^{n-1}(1-k/n)(k/n)^{-1+d}\Big).$$

Using a first order Taylor expansion gives the result for r_n.

Moreover, going along the same lines, one easily obtains the exact value of $\mathbb{E}\Big[\delta_1 X(k)\delta_1 \tilde{X}_{1/n}(k)\Big]$. This allows to check that $\mathbb{E}\Big[(\delta_1 X(k) - \delta_1 \tilde{X}_{1/n}(k))^2\Big]$ goes to 0 as n goes to $+\infty$.

□

Remark: If l_n is not chosen as the value that we specify above, but instead as a more generic value like $l_n = 1/n$, then r_n will be of order n^{-d} which may decrease very slowly to 0 for small d.

Theorem 3.5 shows that we can well approximate the increments of the process X through the discrete process $\tilde{X}_{1/n}$. This last process is easily simulated with the help of some efficient procedures for the simulation of stationary Gaussian random fields like the one proposed by Chan and Wood [34], which is based on Fast Fourier Transforms.

3.6.2 Numerical results and comparisons to empirical data

Let us illustrate previous results on some numerical simulations. We chose the following values for the parameters: $\sigma^2 = 1$, $\lambda^2 = 0.04$ and $T = 200$, which are usual values for modeling financial data with the MRW model (see [10]). The parameter d has been chosen to vary from 0.01 to 0.3, i.e., H varies from 0.525 to 0.67. In all the reported results, we have set $\tau = 1$, $N = 5000$, $n = 500$ and we performed averages over 100 realizations of the process. For each value of d, each realization of the sequence $\delta_1\tilde{X}_{1/n}(0), \ldots, \delta_1\tilde{X}_{1/n}(N-1)$ has been simulated using the techniques described above. We then approximate $\delta_1 Y_d(k)$ by

$$\delta_1 Y_d(k) \approx -\frac{\delta_1\tilde{X}_{1/n}(k)}{\left(\overline{\left(\delta_1\tilde{X}_{1/n}\right)^2}\right)^{1/2}},$$

$\overline{\left(\delta_1\tilde{X}_{1/n}\right)^2}$ being the empirical means of the squared increments $\delta_1\tilde{X}_{1/n}(0), \ldots, \delta_1\tilde{X}_{1/n}(N-1)$.

In Figure 3.2, we check the dependency of the correlation functions $\rho_d^{(1)}$ and $\rho_d^{(3)}$ (defined in (3.15) and (3.17)) on the parameter d. Recall that we obtained $\rho_d^{(1)}(k) = O(d)$ for all $|k| \geq 1$, $\rho_d^{(3)}(k) = O(d^{1/2})$ for all $k \geq 0$, and $\rho_d^{(3)}(k) = O(d^{3/2})$ for all $k \leq -1$. This is well confirmed by our simulations.

In Figure 3.3 we plot the auto-correlations $\rho_d^{(1)}(k)$ of the return series as a function of the lag k for $d = 0.01$, 0.05 and 0.1. We see that after a few lags all series are almost uncorrelated; but it is only for d small enough ($d \leq 0.05$) that the first lag correlation is inside the 95% confidence interval of a series of $N = 5000$ uncorrelated random variables. Since financial returns are well known to be uncorrelated (or very weakly correlated), the parameter d should probably be chosen below the value 0.05.

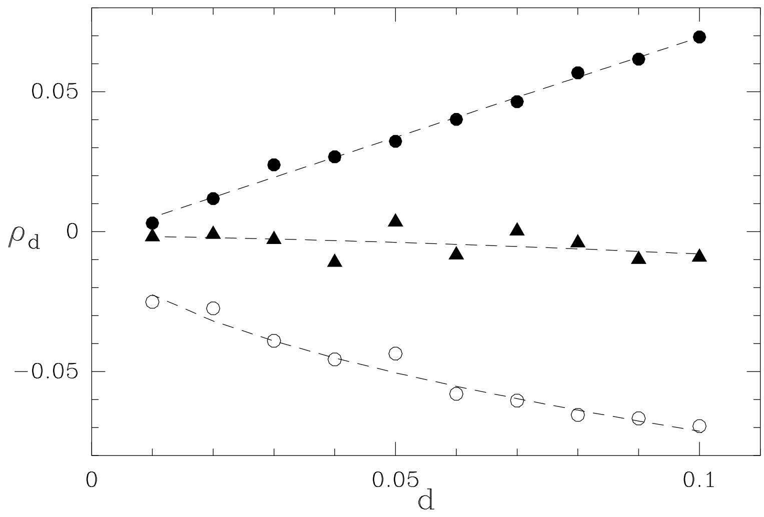

Figure 3.2: *Values* $\rho_d^{(1)}(1)$ *(•),* $\rho_d^{(3)}(1)$ *(△), and* $\rho_d^{(2)}(-1)$ *(○) for* $d = 0.01$ *to* $d = 0.10$*, and their respective adjustment to a fit* $c_1 d$*,* $c_2 d^{1/2}$*, and* $c_3 d^{3/2}$ *(dashed lines).*

In Figure 3.4 we report the estimation of the leverage effect on our simulated series. We estimate $\rho_d^{(3)}(k)$ as a function of k for 3 values of d. For comparison purpose, we also plot the correlation that we measure on real data, namely the daily quotation of 5 stock indices. More precisely we considered the CAC40, DAX, FTSE100, S&P500, and Dow-Jones index daily series from 1990/12/03 to 2010/02/15 and averaged the empirical correlations over the 5 indices, so as to reduce the noise. We confirm our previous computations: the estimated function $\rho_d^{(3)}(k)$ on our simulation exhibits a strong asymmetry and is clearly negative for positive lags k. Moreover, we see that as d increases the leverage effect indeed becomes stronger. The curves we obtain seem quite similar to the effect observed on stock index returns.

Finally, in Figure 3.5, we present another way to assess the leverage effect. Indeed, the construction of our model directly suggests that there should exist a negative correlation between past returns and the *logarithm* of future volatilities, and this correlation should behave as a power-law of the time lag. That is, if we denote by $p(t)$ the log-price of an

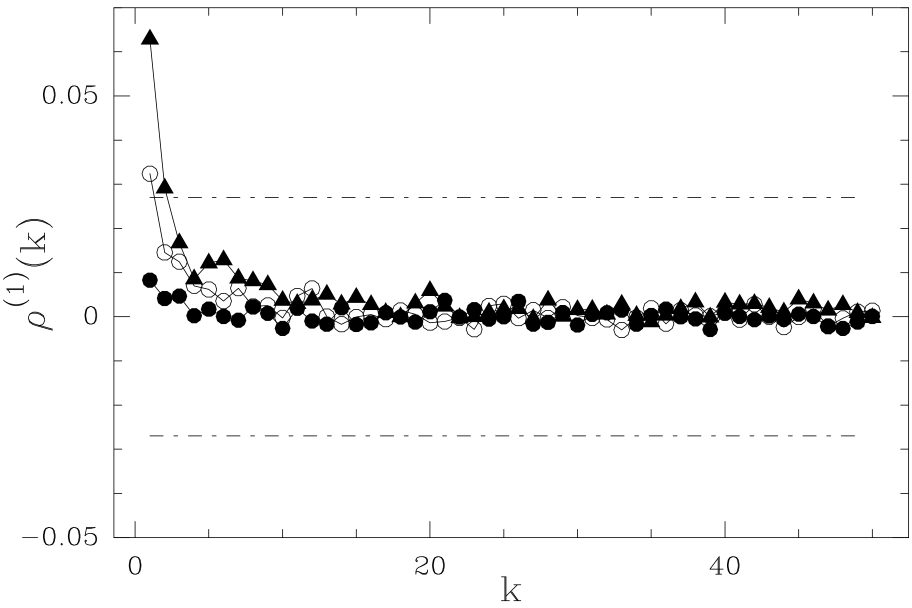

Figure 3.3: *Correlations $\rho_d^{(1)}(k)$ for $1 \leq k \leq 50$ with $d = 0.01$ (●), $d = 0.05$ (○) and $d = 0.10$ (▲). The dashed lines represent the 95% interval for uncorrelated random variables for a series of size 5000.*

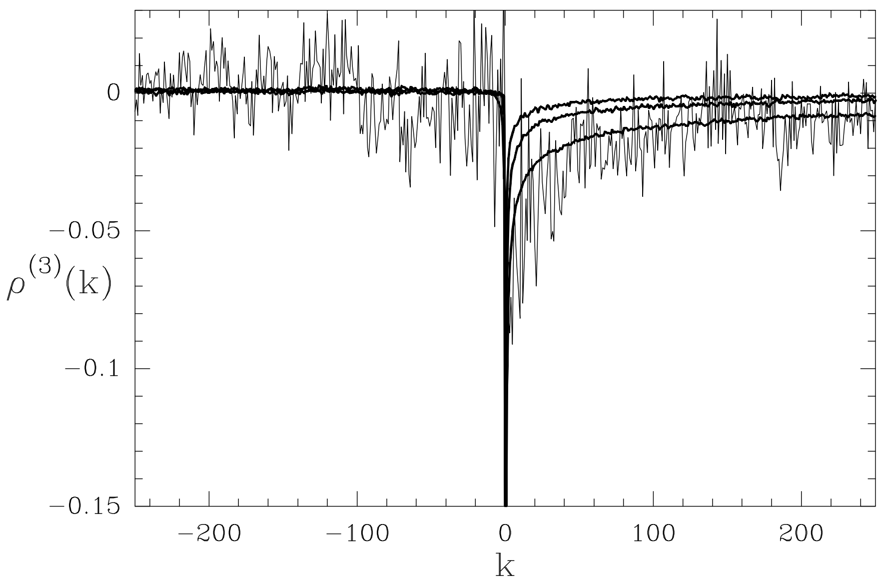

Figure 3.4: *Correlations $\rho_d^{(3)}(k)$ for $|k| \leq 250$. Thin solid lines represent, from top to bottom, $d = 0.03$, $d = 0.1$ and $d = 0.30$. The noisy curve corresponds to real data estimated on a basket of 5 indices and is shown for comparison.*

asset, then the following relation is expected:

$$C^{w\varepsilon}(k) \equiv -\mathbb{C}\mathrm{orr}[\delta_\tau p(0), 2\log(|\delta_\tau p(k\tau)|)] \sim ck^{-\alpha}$$

for $k \geq 1$, some constant $c > 0$, and some exponent $\alpha \in (0, 1)$. From the definition of $\gamma^{w\varepsilon}$, we expect to find in our model $\alpha \approx -1 + H = (-1 + \lambda^2 + d)/2$ which is close to 1/2. Figure 3.5(a) shows that this is indeed the case. We have

$$C^{w\varepsilon}(k) \simeq \gamma^{w\varepsilon}(k) \sim ck^{(-1+\lambda^2+d)/2}.$$

In the case of real data, as illustrated in Figure 3.5(b) we observed that a power-law with an exponent $\alpha \simeq 0.48$ provides of good fit of the data.

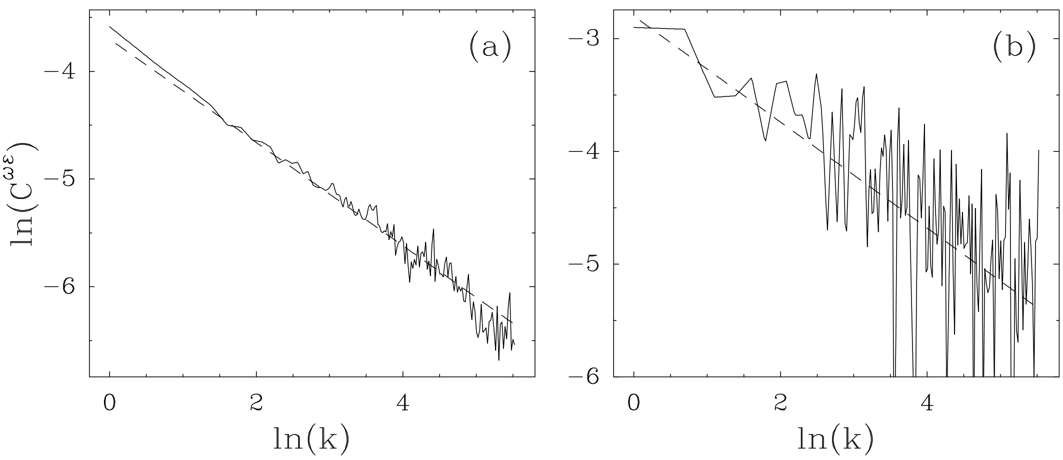

Figure 3.5: *Plot of* $C^{w\varepsilon} = -\mathbb{C}orr[\delta_1 p(0), 2\log(|\delta_1 p(k\tau)|)]$ *for* $|k| \leq 50$ *in log-log scale. (a) Estimated correlation for 100 realizations of a Skewed MRW with* $\lambda^2 = 0.04$, $T = 250$ *and* $d = 0.03$. *In dashed line the expected power-law behavior* $ck^{(-1+\lambda^2+d)/2} \sim k^{-0.48}$ *is represented. (b) Same graph for the mean correlations over 5 indices. The same power-law behavior as in (a) has been plotted for comparison purpose.*

3.A Covariances of w and ε

Through straightforward computations, it is possible to obtain the following covariance functions for (ε, w): Fix $0 < l' < l < T$ and $u, \tau \in \mathbb{R}$. The process w has following expectation:

$$\mathbb{E}\Big[w_l(u)\Big] = -\frac{\lambda^2}{2}\left(\log\left(\frac{T}{l}\right)+1\right)$$

and covariance:

$$\mathbb{C}\text{ov}[w_l(u), w_{l'}(u+\tau)] = \gamma_l^w(\tau)$$

with

$$\begin{cases} \gamma_l^w(\tau) &= \lambda^2\left(\log\left(\frac{T}{l}\right)+1-\frac{\tau}{l}\right) \text{ if } |\tau| \le l; \\ \gamma_l^w(\tau) &= \lambda^2\log\left(\frac{T}{\tau}\right) \text{ if } l \le |\tau| \le T; \\ \gamma_l^w(\tau) &= 0 \text{ if } T \le |\tau|. \end{cases}$$

The process ε has zero expectation and satisfies

$$\mathbb{C}\text{ov}[\varepsilon_l(u), \varepsilon_{l'}(u+\tau)] = \gamma_l^\varepsilon(\tau),$$

where $\gamma_l^\varepsilon(\tau)$ is defined as:

$$\begin{cases} \gamma_l^\varepsilon(\tau) &= \frac{\sigma^2}{(2-2H)(3-2H)}|\tau|^{-2+2H} \text{ if } l \le |\tau|; \\ \gamma_l^\varepsilon(\tau) &= \sigma^2\left(\frac{1}{2-2H}-\frac{1}{3-2H}\frac{|\tau|}{l}\right)l^{-2+2H} \text{ if } 0 \le |\tau| \le l. \end{cases}$$

Finally, the covariance between ε and w is given by

$$\mathbb{C}\text{ov}[\varepsilon_l(u), w_{l'}(u+\tau)] = \mathbb{C}\text{ov}[\varepsilon_{l'}(u), w_l(u+\tau)] = \gamma_l^{w\varepsilon}(\tau),$$

where $\gamma_l^{w\varepsilon}(\tau)$ is defined as:

$$\begin{cases} \gamma_l^{w\varepsilon}(\tau) &= 0 \quad \text{if} \quad \tau < 0; \\ \gamma_l^{w\varepsilon}(\tau) &= \frac{\lambda\sigma}{2-H}\tau l^{-2+H} \quad \text{if} \quad 0 \leq \tau \leq l; \\ \gamma_l^{w\varepsilon}(\tau) &= \lambda\sigma\Big(\frac{2}{1-H}l^{-1+H} - \frac{1}{2-H}\frac{\tau}{l}l^{-1+H} - \frac{2}{(1-H)(2-H)}\tau^{-1+H}\Big) \quad \text{if} \quad l \leq \tau \leq 2l; \\ \gamma_l^{w\varepsilon}(\tau) &= \frac{\lambda\sigma}{(1-H)(2-H)}\Big(2^{2-H} - 2\Big)\tau^{-1+H} \quad \text{if} \quad 2l \leq \tau \leq T; \\ \gamma_l^{w\varepsilon}(\tau) &= \frac{\lambda\sigma}{(1-H)(2-H)}\Big((2^{2-H} - 1)\tau^{-1+H} - T^{-1+H}\Big) \quad \text{if} \quad T \leq \tau \leq 2T; \\ \gamma_l^{w\varepsilon}(\tau) &= \frac{\lambda\sigma}{(1-H)(2-H)}\Big((\tau - T)^{-1+H} - \tau^{-1+H}\Big) \quad \text{if} \quad \tau \geq 2T. \end{cases}$$

3.B Proof of Proposition 3.2

We begin by evaluating the moment $\mathbb{E}\Big[X_l(t)^p\Big]$, for $l \in (0,T)$, $t \geq 0$, $p \geq 2$. From Fubini's theorem, we have

$$\mathbb{E}\bigg[\Big(\int_0^t \varepsilon_l(u)e^{w_l(u)}du\Big)^p\bigg] = \int_0^t \dots \int_0^t du_1 \dots du_p \mathbb{E}\Big[\varepsilon_l(u_1)\dots\varepsilon_l(u_p)e^{w_l(u_1)+\dots+w_l(u_p)}\Big]. \quad (3.18)$$

We are going to compute the right-hand side using the following relation:

$$\varepsilon_l(u_1)\dots\varepsilon_l(u_p)e^{w_l(u_1)+\dots+w_l(u_p)} = \frac{\partial^p}{\partial x_1 \dots \partial x_p}\bigg|_{x_1=\dots=x_p=0} e^{w_l(u_1)+\dots+w_l(u_p)+x_1\varepsilon_l(u_1)+\dots+x_p\varepsilon_l(u_p)}.$$

Permuting expectation and differentiation, we have to differentiate p times

$$\mathbb{E}\Big[e^{w_l(u_1)+\dots+w_l(u_p)+x_1\varepsilon_l(u_1)+\dots+x_p\varepsilon_l(u_p)}\Big] = \exp(S_p(x_1,\dots,x_p)).$$

The term $S_p = S_p(x_1,\dots,x_p)$ can be evaluated as

$$S_p = \sum_{1\leq i<j\leq p}\gamma_l^w(u_i - u_j) + \sum_{1\leq i,j\leq p} x_j\gamma_l^{w\varepsilon}(u_i - u_j) + \sum_{1\leq i<j\leq p} x_i x_j \gamma_l^\varepsilon(u_i - u_j) + \frac{1}{2}\sum_{i=1}^p x_i^2\gamma_l^\varepsilon(0)$$

where we used Property 3.3. From Property 3.4 $\gamma_l^{w\varepsilon}(u_i - u_j)$ is non zero if and only if $u_i > u_j$; however this will not be used in what follows: we do not keep track of the order of the u_i's in order to avoid introducing notations that would be of no use to this proof.

We will however need the following definitions: for $i,\ j = 1, \dots, p$

$$\begin{aligned} D_i &= D_i(x_1, \dots, x_p) \\ &= \frac{\partial}{\partial x_i} S_p(x_1, \dots, x_p), \end{aligned}$$

$$\begin{aligned} D_{i,j} &= D_{i,j}(x_1, \dots, x_p) \\ &= \frac{\partial}{\partial x_j} D_i(x_1, \dots, x_p), \end{aligned}$$

and for $1 \leq n \leq p$

$$\begin{aligned} R_n &= R_n(x_1, \dots, x_p) \\ &= \frac{\partial^n}{\partial x_1 \dots x_n} e^{S_p(x_1, \dots, x_p)}. \end{aligned}$$

Also, for $1 \leq n \leq p$ and $0 \leq m \leq \lfloor n/2 \rfloor$, we define $E_{m,n}$ to be the set of all partitions P of $\{1, \dots, n\}$ into $n - m$ subsets such that m of these subsets have two elements and the other $n - 2m$ subsets have one element:

$$E_{m,n} = \Big\{ P = \Big\{ \{a_1, a_2\}, \dots, \{a_{2m-1}, a_{2m}\}, \{a_{2m+1}\}, \dots \{a_n\} \Big\},\ \{a_1, \dots, a_n\} = \{1, \dots, n\} \Big\}.$$

Then by differentiating iteratively, one can see that

$$R_n = \sum_{m=0}^{\lfloor n/2 \rfloor} \sum_{P \in E_{m,n}} D_{a_1,a_2} \dots D_{a_{2m-1},a_{2m}} D_{a_{2m+1}} \dots D_{a_n} e^{S_p}. \tag{3.19}$$

Indeed, the formula is clearly true for $n = 1$. Moreover, using the fact that for $1 \leq a_i, a_{i+1} \leq n$

$$\frac{\partial}{\partial x_{n+1}} D_{a_i,a_{i+1}} = 0,$$

we have for each $P = \Big\{\{a_1, a_2\}, \dots, \{a_{2m-1}, a_{2m}\}, \{a_{2m+1}\}, \dots \{a_n\}\Big\}$ in $E_{m,n}$:

$$
\begin{aligned}
&\frac{\partial}{\partial x_{n+1}} D_{a_1,a_2} \dots D_{a_{2m-1},a_{2m}} D_{a_{2m+1}} \dots D_{a_n} e^{S_p} \\
&= \sum_{k=2m+1}^{p} (D_{a_1,a_2} \dots D_{a_{2m-1},a_{2m}} D_{a_{2m+1}} \dots D_{a_{k-1}} D_{a_k,n+1} D_{a_{k+1}} \dots D_{a_n} \\
&\quad + D_{a_1,a_2} \dots D_{a_{2m-1},a_{2m}} D_{a_{2m+1}} \dots D_{a_n} D_{n+1}) e^{S_p}.
\end{aligned}
$$

We therefore obtain (3.19) after summing over P and m.

Finally, by taking $n = p$, we have

$$
\frac{\partial^p}{\partial x_1 \dots \partial x_p} e^{S_p} = \sum_{m=0}^{\lfloor p/2 \rfloor} \sum_{P \in E_{m,p}} D_{a_1,a_2} \dots D_{a_{2m-1},a_{2m}} D_{a_{2m+1}} \dots D_{a_p} e^{S_p}. \tag{3.20}
$$

We now evaluate this expression for $x_1 = \dots = x_p = 0$. Since

$$
D_{a_i}(0, \dots, 0) = \sum_{b=1}^{p} \gamma_l^{w\varepsilon}(u_b - u_{a_i}),
$$

$$
D_{a_i,a_{i+1}}(0, \dots, 0) = \gamma_l^{\varepsilon}(u_{a_i} - u_{a_{i+1}}),
$$

and

$$
e^{S_p(0,\dots,0)} = e^{\sum_{1 \le i < j \le p} \gamma_l^{w}(u_i - u_j)},
$$

we can express the moment $\mathbb{E}\Big[X_l(t)^p\Big]$ as:

$$
\sum_{m=0}^{\lfloor p/2 \rfloor} \sum_{P \in E_{m,p}} \sum_{b_1=1}^{p} \dots \sum_{b_{p-2m}=1}^{p} \int_0^t \dots \int_0^t du_1 \dots du_p \, f_l^{m,P,b_1,\dots,b_{p-2m}}(u_1, \dots, u_p) \tag{3.21}
$$

where

$$
\begin{aligned}
f_l^{m,P,b_1,\dots,b_{p-2m}}(u_1, \dots, u_p) =& \gamma_l^{\varepsilon}(u_{a_1} - u_{a_2}) \dots \gamma_l^{\varepsilon}(u_{a_{2m-1}} - u_{a_{2m}}) \\
& \times \gamma_l^{w\varepsilon}(u_{b_1} - u_{a_{2m+1}}) \dots \gamma_l^{w\varepsilon}(u_{b_{p-2m}} - u_{a_p}) e^{\sum_{1 \le i < j \le p}^{p} \gamma_l^{w}(u_i - u_j)}.
\end{aligned}
$$

Define $\Gamma_p(m)$ as:

$$\Gamma_p(m) = \sigma^p \lambda^{p-2m} T^{\lambda^2 p(p-1)/2} (c^\varepsilon)^m (c^{w\varepsilon})^{p-2m}.$$

Then from Property 3.4, each $f_l^{m,P,b_1,\ldots,b_{p-2m}} \uparrow f^{m,P,b_1,\ldots,b_{p-2m}}$ as $l \to 0$, where $f^{m,P,b_1,\ldots,b_{p-2m}}$ is the following:

$$\begin{aligned} f^{m,P,b_1,\ldots,b_{p-2m}}(u_1,\ldots,u_p) &= \Gamma_p(m)|u_{a_1} - u_{a_2}|^{-2+2H} \ldots |u_{a_{2m-1}} - u_{a_{2m}}|^{-2+2H} \\ &\times (u_{b_1} - u_{a_{2m+1}})_+^{-1+H} \ldots (u_{b_{p-2m}} - u_{a_p})_+^{-1+H} \prod_{1 \le i < j \le p} |u_i - u_j|^{-\lambda^2}, \end{aligned}$$

which is integrable if and only if

$$-1 + pH - \frac{p(p-1)}{2}\lambda^2 > 0.$$

Applying the monotone convergence theorem gives the result, the constant $K(p)$ being:

$$\begin{aligned} &\sum_{m=0}^{\lfloor p/2 \rfloor} \Gamma_m(p) \sum_{P \in E_{m,p}} \sum_{b_1=1}^{p} \cdots \sum_{b_{p-2m}=1}^{p} \int_0^1 \cdots \int_0^1 du_1 \ldots du_p |u_{a_1} - u_{a_2}|^{-2+2H} \ldots \\ &\times |u_{a_{2m-1}} - u_{a_{2m}}|^{-2+2H} (u_{b_1} - u_{a_{2m+1}})_+^{-1+H} \ldots (u_{b_{p-2m}} - u_{a_p})_+^{-1+H} \prod_{1 \le i < j \le p} |u_i - u_j|^{-\lambda^2}. \end{aligned}$$

3.C Proof of Theorem 3.4

3.C.1 Behavior of $\rho^{(1)}$

Here and in the following, we will use the identity:

$$\varepsilon_l(u_1) \ldots \varepsilon_l(u_p) e^{w_l(u_1)+\cdots+w_l(u_p)} = \left.\frac{\partial^p}{\partial x_1 \ldots \partial x_p}\right|_{x_1=\cdots=x_p=0} e^{w_l(u_1)+\cdots+w_l(u_p)+x_1\varepsilon_l(u_1)+\cdots+x_p\varepsilon_l(u_p)}. \tag{3.22}$$

Applying this identity for $p = 2$, Property 3.4, and the monotone convergence theorem yields:

$$\mathbb{E}\Big[\delta_\tau X(0)\delta_\tau X(k\tau)\Big] = c^\varepsilon \sigma^2 T^{\lambda^2} \int_0^\tau du_1 \int_{k\tau}^{(k+1)\tau} du_2 |u_2 - u_1|^{-1+d}.$$

Note that from the value of c^ε given in Section 3.3.3, c^ε does depend on d but is approximately $1/2$ for small d. It is easy enough to compute the integral above, which gives: for $k = 0$

$$\mathbb{E}\Big[\delta_\tau X(0)^2\Big] = \frac{2c^\varepsilon \sigma^2 T^{\lambda^2}}{d(1+d)} \tau^{1+d}, \tag{3.23}$$

and for $|k| > 0$

$$\mathbb{E}\Big[\delta_\tau X(0)\delta_\tau X(k\tau)\Big] = \frac{c^\varepsilon \sigma^2 T^{\lambda^2}}{d(1+d)} \Big(|k+1|^{1+d} + |k-1|^{1+d} - 2|k|^{1+d}\Big)\tau^{1+d}.$$

It follows that for $|k| > 0$, the correlation $\rho_d^{(1)}(\tau, k)$ is of order d when d is small. More precisely, for $|k| = 1$:

$$\rho_d^{(1)}(\tau, k) \sim d\log(2) \quad \text{ as } d \to 0.$$

and for $|k| > 1$:

$$\rho_d^{(1)}(\tau, k) \sim \frac{d}{2}\Big(|k|\log(1 - 1/k^2) + \log(1 + 2/(|k|-1))\Big) \quad \text{ as } d \to 0.$$

3.C.2 Behavior of $\rho^{(2)}$ and $\rho^{(3)}$

From Proposition 3.4, it is enough to prove the result for $\rho^{(2)}$. Going along the same line as above, we get

$$\mathbb{E}\Big[\delta_\tau X(0)\delta_\tau X(k\tau)^2\Big] = c^\varepsilon c^{w\varepsilon} \sigma^3 \lambda T^{3\lambda^2} \tau^{(3-3\lambda^2+3d)/2} \sum_{i_1,i_2,i_3} \int_0^1 du_1 \int_k^{k+1} du_2 \int_k^{k+1} du_3 |u_{i_1} - u_{i_2}|^{-1+d} (u_{i_2} - u_{i_3})_+^{(-1+d+\lambda^2)/2} |u_{i_3} - u_{i_1}|^{-\lambda^2},$$

the sum being taken on all permutations i_1, i_2, i_3 of the set $\{1, 2, 3\}$. Note that depending on the sign of k and the permutation, it may be the case that u_{i_2} lies in an interval lower than u_{i_3}, so that the corresponding integral is zero. Also note that from the value of c^ε and $c^{w\varepsilon}$ given in Section 3.3.3, the product $c^\varepsilon c^{w\varepsilon}$ is approximately 0.55 for small d.

Taking into account the range of possible values for $d > 0$ and λ^2, the integrals above are clearly finite. Moreover, as d goes to zero, only the integral $I_d(k)$

$$I_d(k) = \int_0^1 du_1 \int_k^{k+1} du_2 \int_k^{k+1} du_3 |u_2 - u_3|^{-1+d} (u_3 - u_1)_+^{(-1+d+\lambda^2)/2} |u_1 - u_2|^{-\lambda^2}$$

(and the one where we permute u_2 and u_3, which is much obviously the same) does explode for $k \geq 0$, while in the case $k < 0$, $I_d(k)$ is exactly zero so that the moment $\mathbb{E}\Big[\delta_\tau X(0)\delta_\tau X(k\tau)^2\Big]$ remains bounded. For $k \geq 2$, we have the following bounds:

$$(k+1)^{(-1+d-\lambda^2)/2} \int_0^1 du \int_0^1 dv |u-v|^{-1+d} \leq I_d(k) \leq (k+1)^{(-1+d-\lambda^2)/2} \int_0^1 du \int_0^1 dv |u-v|^{-1+d}$$

from which we get

$$I_d(k) \sim c(k) d^{-1} \quad \text{as} \quad d \to 0$$

where $c(k)$ are some positive constants such that

$$2(k+1)^{(-1-\lambda^2)/2} \leq c(k) \leq 2(k-1)^{(-1-\lambda^2)/2}.$$

For $k = 0$, 1 it can be similarly shown that

$$I_d(k) \sim c(k) d^{-1} \quad \text{as} \quad d \to 0$$

where $c(0)$, $c(1)$ are some positive constants. From this, we may write: for $k < 0$

$$|\rho_d^{(2)}(\tau, k)| = O(d^{3/2}) \quad \text{as} \quad d \to 0$$

and for $k \geq 0$,

$$\rho_d^{(2)}(\tau, k) \sim -\left(\frac{c^\varepsilon T^{3\lambda^2}}{2}\right)^{1/2} c^{w\varepsilon} \lambda\, c(k) d^{1/2} \quad \text{as} \quad d \to 0.$$

Chapter 4

Some statistical properties of the log-normal multifractal random walk

4.1 Introduction

In this exploratory chapter, we present a few statistical properties of the log-normal MRM and MRW processes. The first series of our results notably give some conditions under which the distribution of a log-normal MRW (or MRM) process on some finite length interval I is singular or equivalent for different values of the parameters λ^2, T and σ^2. Here, λ^2, T and σ^2 are respectively the intermittency coefficient, the integral scale (*ie.*, the decorrelation time), and the mean volatility level of the process. As is outlined below, we show in particular that if the observation horizon is finite and less than T, then this parameter T can not be estimated, even in the case of a continuous-time observation. The same result holds for σ. Our results here can notably be seen as a more formal version of the intuitive argument given by Bacry, Kozhemyak and Muzy in [10] concerning the same question.

We next investigate the problem of estimating the parameter λ^2. In the case of a general, log-infinitely divisible process, this parameter is an infinitely divisible distribution $\pi(dx)$ on $\mathbb{R}$ with the constraint that $\int e^x \pi(dx) = 1$. In the log-normal case, the statistical problem reduces to the estimation of the variance $\lambda^2 \in [0, 2)$ of the Gaussian distribution π. We consider four different estimators, with three of them being inspired from earlier works on MRW and multifractal processes [53, 94, 95, 93, 62, 10], and one of them being new. We study their theoretical properties (their quadratic risks) and their performances on some simulated data. In particular, we prove that this new estimator achieves the usual parametric convergence rate $n^{-1/2}$. This last proof relies on some specific properties of Gaussian vectors, notably Stein's integration by parts formula [89].

We refer to Section 1.2 of Chapter 1 for the proper definition MRM and MRW processes. By log-normal processes, we denote such MRM and MRW processes for which

$$\psi(q) = \frac{\lambda^2}{2} q(q-1), \quad q \in \mathbb{R}$$

for some $\lambda \in (0, \sqrt{2})$, in which case the generating process w is clearly Gaussian. Note that Proposition 1.1 in Chapter 1 states that the MRM process is indeed well defined for $0 < \lambda^2 < 2$. These log-normal MRW and MRM processes have received some special attention in the applications, especially in finance: see Bouchaud and Potters [25], Kozhemyak [60], Bacry, Kozhemyak and Muzy [9] and Duchon, Robert and Vargas [39].

Most of our results here will be proved with the help of some special properties of Gaussian random variables and processes. However, in the few cases where our results can be generalized to any MRM or MRW processes with no further difficulty, we will explicitly mention so. We speak of a (λ^2, T, σ^2)-MRM or -MRW process when the log-normal MRM M or MRW X are constructed with these parameters. When we return to the general case of a log-infinitely process, we will then speak of a (ψ, T, σ^2)-MRM or -MRW process.

4.2 Absolute continuity of the law of a log-normal MRW process

We are going to study the question of the absolute continuity or singularity of the laws of two log-normal MRW processes on the interval $[0, 1]$ with different values of the paramers

(λ^2, T, σ^2). We consider here MRW processes as random variables on the space $(C_{[0,1]}, \mathcal{C}_{[0,1]})$ of real-valued, continuous functions on $[0,1]$, equipped with the uniform norm $\|\cdot\|_\infty$ and its Borel σ-algebra $\mathcal{C}_{[0,1]}$. MRM processes are defined on the space $(C^+_{[0,1]}, \mathcal{C}^+_{[0,1]})$, which is simply the restriction of $(C_{[0,1]}, \mathcal{C}_{[0,1]})$ to non-negative functions. We fix two values $(\lambda_0^2, T_0, \sigma_0^2)$ and $(\lambda_1^2, T_1, \sigma_1^2)$ of the parameters, and write respectively $\mathbf{P}^i_X$ and $\mathbf{P}^i_M$, $i = 0, 1$, for the law of the $(\lambda_i^2, T_i, \sigma_i^2)$-MRW and $(\lambda_i^2, T_i, \sigma_i^2)$-MRM. Thus, writing $Y = (Y_t,\ t \in [0,1])$ for the coordinate process on $C_{[0,1]}$ (defined as $Y_t(\omega) = \omega_t$ for $\omega \in C_{[0,1]}$ and $t \in [0,1]$), we have that Y is a $(\lambda_i, T_i, \sigma_i^2)$-MRW under $\mathbf{P}^i_X$. The same way, we write $Y^+ = (Y^+_t,\ t \in [0,1])$ for the coordinate process on $C^+_{[0,1]}$, so that Y^+ is a $(\lambda_i, T_i, \sigma_i^2)$-MRM under $\mathbf{P}^i_M$.

Let us recall that two probability measures P and Q on some measurable space $(\Omega, \mathcal{F})$ are said to be singular if there is some $A \in \mathcal{F}$ such that $P(A) = 0$ and $Q(A) = 1$. On the other hand, P is said to dominate Q if for any $A \in \mathcal{F}$ such that $P(A) = 0$, we also have $Q(A) = 0$. Equivalently, one also says in that case that Q is absolutely continuous with respect to P. Finally, P and Q are said to be equivalent if they dominate each other.

We first state a result that establishes some connections between the distribution of an MRM and the distribution of an MRW for some values of the parameters λ^2, T, and σ^2, namely that the distributions $\mathbf{P}^0_M$ and $\mathbf{P}^1_M$ are singular (resp. equivalent) if and only if the same holds for the MRW distributions $\mathbf{P}^0_X$ and $\mathbf{P}^1_X$. This is Proposition 4.1. Next, we give some conditions on $(\lambda_0^2, T_0, \sigma_0^2)$ and $(\lambda_1^2, T_1, \sigma_1^2)$ under which $\mathbf{P}^0_X$ and $\mathbf{P}^1_X$ are singular or equivalent. We show that these two distributions are always singular for $\lambda_0 \neq \lambda_1$: see Proposition 4.2. In Proposition 4.3, we prove that if $\lambda_0 = \lambda_1$, $\sigma_0 = \sigma_1$, and $\min(T_0, T_1) > 1$, then $\mathbf{P}^0_X$ and $\mathbf{P}^1_X$ are equivalent. Here, the value 1 would be replaced by some $t > 0$ if we would consider the distribution of $\big(X(s),\ 0 \le s \le t\big)$ instead of considering the distribution of $\big(X(s),\ 0 \le s \le 1\big)$. The question of whether the condition $\min(T_0, T_1) > 1$ is necessary is open. Finally, Proposition 4.4 states that if $\lambda_0 = \lambda_1$, $T_0 = T_1$, and $T_0 < 1$ then $\mathbf{P}^0_X$ and $\mathbf{P}^1_X$ are equivalent.

Proposition 4.1. *The distributions $\mathbf{P}^0_X$ and $\mathbf{P}^1_X$ are singular if and only if the distributions $\mathbf{P}^0_M$ and $\mathbf{P}^1_M$ are singular. The distributions $\mathbf{P}^0_X$ and $\mathbf{P}^1_X$ are equivalent if and only if the distributions $\mathbf{P}^0_M$ and $\mathbf{P}^1_M$ are equivalent.*

Proof. Let us first suppose that $\mathbf{P}_X^0$ and $\mathbf{P}_X^1$ are singular. Then there is some $A \in \mathcal{C}_{[0,1]}$ such that $\mathbf{P}_X^0(A) = 0$ and $\mathbf{P}_X^1(A) = 1$. We write $\mathbb{W}$ for the Wiener measure on the canonical space $(C_{\mathbb{R}^+}, \mathcal{C}_{\mathbb{R}^+})$ of continuous functions on $\mathbb{R}^+$ (see for example Karatzas and Shreve [58]). Then for $w \in C_{\mathbb{R}^+}$ the mapping $K_w : (C^+_{[0,1]}, \mathcal{C}^+_{[0,1]}) \to (C_{[0,1]}, \mathcal{C}_{[0,1]})$ defined by $K_w y = w \circ y$ is continuous, hence measurable. Indeed, let $\varepsilon > 0$, consider sequence $(y^n) \in C^+_{[0,1]}$ that goes uniformly to some $y \in C^+_{[0,1]}$, and let us choose n such that $\|y^n - y\|_\infty \leq \varepsilon$. Then for $t \in [0,1]$

$$
\begin{aligned}
|K_w y_t^n - K_w y_t| &\leq \sup\{|w_s - w_{y_t}|,\ s \in [y_t - \varepsilon, y_t + \varepsilon]\} \\
&\leq \sup\{|w_{s+\varepsilon} - w_s|,\ s \leq \|y\|_\infty\},
\end{aligned}
$$

which can be made arbitrarily small for $\varepsilon \to 0$ since w is uniformly continuous on $[0, \|y\|_\infty]$.

Hence $K_w^{-1}(A)$ lies in $\mathcal{C}^+_{[0,1]}$. But since for $i = 0, 1$,

$$
i = \mathbf{P}_X^i(A) = \int d\mathbb{W}(w) \mathbf{P}_M^i(w \circ Y^+ \in A), \tag{4.1}
$$

we obtain $\mathbf{P}_M^i(Y^+ \in K_w^{-1}(A)) = i$ for any w in some $\mathbb{W}$-almost sure subset of $\mathcal{C}_{\mathbb{R}^+}$. This proves that the distributions $\mathbf{P}_M^0$ and $\mathbf{P}_M^1$ are singular.

Let us now suppose that the distributions $\mathbf{P}_M^i$ are equivalent. Then for either $i = 0$ or $i = 1$, if for some $A \in \mathcal{C}_{[0,1]}$ $\mathbf{P}_X^i(A) = 0$, then for $\mathbb{W}$-almost all $w \in C$, $\mathbf{P}_M^i(Y^+ \in K_w^{-1}(A)) = 0$, so that $\mathbf{P}_M^{1-i}(Y^+ \in K_w^{-1}(A)) = 0$. Applying (4.1), we thus obtain $\mathbf{P}_X^{1-i}(A) = 0$. Thus $\mathbf{P}_X^0$ and $\mathbf{P}_X^1$ dominate each other and are therefore equivalent.

Next, let us suppose that $\mathbf{P}_M^0$ and $\mathbf{P}_M^1$ are singular. Then there exists some $A \in \mathcal{C}^+_{[0,1]}$ such that for $i = 0$, 1, $\mathbf{P}_M^i(A) = i$. We are going to use Theorem 1.1 of Chapter 1, but we first need some extra definitions. Let us define $(\bar{\mathbb{R}}_+^{[0,1]}, \bar{\mathcal{R}}_+^{[0,1]})$ as the space of positive functions on $[0,1]$ with values in $\bar{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{+\infty\}$, and equipped with the product $\sigma-$algebra $\bar{\mathcal{R}}_+^{[0,1]}$. That is, $\bar{\mathcal{R}}_+^{[0,1]}$ is generated by the class of sets

$$
\{x \in \bar{\mathbb{R}}_+^{[0,1]}, (x_{t_1}, \ldots, x_{t_n}) \in B \text{ for some } B \in \bar{\mathcal{R}}_+^n,\ t_1, \ldots, t_n \in [0,1], \text{ and } n \geq 0\},
$$

where $\bar{\mathcal{R}}_+^n$ is the Borel σ-algebra of $\bar{\mathbb{R}}_+^n$. Then there is some $A' \in \bar{\mathcal{R}}_+^{[0,1]}$ such that $A = A' \cap C_{[0,1]}^+$ (see for instance Billingsley [21], p. 57). Let us now define the following mapping $F : (C_{[0,1]}, \mathcal{C}_{[0,1]}) \to (\bar{\mathbb{R}}_+^{[0,1]}, \bar{\mathcal{R}}_+^{[0,1]})$:

$$F : x \mapsto \Big(\limsup_{n\to+\infty} \sum_{k=0}^{\lfloor t2^n \rfloor -1} (x_{(k+1)2^{-n}} - x_{k2^{-n}})^2,\, t \in [0,1] \Big),$$

and note that F is measurable. Then from Theorem 1.1 of Chapter 1, we have

$$\mathbf{P}_M^i(Y^+ \in A) = \mathbf{P}_X^i(Y \in F^{-1}(A')) = i,$$

which shows that $\mathbf{P}_X^0$ and $\mathbf{P}_X^1$ are singular.

We finally suppose that $\mathbf{P}_X^0$ and $\mathbf{P}_X^1$ are equivalent. Let $A \in \mathcal{C}_{[0,1]}^+$ be such that $\mathbf{P}_M^0(A) = 0$, and let $A' \in \bar{\mathcal{R}}_+^{[0,1]}$ be such that $A = A' \cap C_{[0,1]}^+$. Then

$$\mathbf{P}_X^0(Y \in F^{-1}(A')) = \mathbf{P}_M^0(Y^+ \in A) = 0,$$

hence

$$\mathbf{P}_M^1(Y^+ \in A) = \mathbf{P}_X^1(Y \in F^{-1}(A')) = 0.$$

□

Remark 4.1. The same result can be obtained in the case of general log-infinitely divisible MRM and MRW distributions.

Proposition 4.2. *Assume that $T_0 = T_1$, $\sigma_0 = \sigma_1$, but $\lambda_0 \neq \lambda_1$. Then $\mathbf{P}_X^0$ and $\mathbf{P}_X^1$ are singular.*

Proof. This is a simple application of Theorem 1.1 and Proposition 1.2. Indeed

$$2^{n(1/2+\lambda_0^2/8)} \sum_{k=0}^{2^n-1} \left| Y_{\frac{k+1}{2^n}} - Y_{\frac{k}{2^n}} \right|^{1/2}$$

goes $\mathbf{P}_X^1$-almost surely to either 0 or $+\infty$, while under $\mathbf{P}_X^0$ this quantity goes to a nondegenerate limit.

□

Remark 4.2. Again, the same result holds in the case of general log-infinitely divisible MRM and MRW distributions with different log-Laplace exponent ψ_0, ψ_1. It shows that one can discern between the distributions $\mathbf{P}_X^0$ and $\mathbf{P}_X^1$ (and hence, according to the previous proposition, $\mathbf{P}_M^0$ and $\mathbf{P}_M^1$) on the set

$$\Big\{2^{n(q-\psi_0(q)-1)} \limsup_{n\to+\infty} \sum_{k=0}^{2^n-1} \Big| Y_{\frac{k+1}{2^n}} - Y_{\frac{k}{2^n}} \Big|^q \in \{0, +\infty\}\Big\}.$$

The following property will be used several times in what follows:

Property 4.1 (Scaling relations satisfied by the MRM process)**.** *Let M be a (λ^2, T, σ^2)-MRM process. Then for $r \in (0,1)$,*

$$\Big(M(rt),\, 0 \le t \le T\Big) \overset{law}{=} r\mathrm{e}^{w_r}\Big(M(t),\, 0 \le t \le T\Big),$$

where w_r is a $\mathcal{N}(\lambda^2 \log(r)/2, -\lambda^2 \log(r))$ random variable independent of M. Moreover,

$$\Big(M(t),\, 0 \le t \le rT\Big) \overset{law}{=} \mathrm{e}^{w_r}\Big(\tilde{M}(t),\, 0 \le t \le rT\Big),$$

where w_r is as above, and where $\tilde{M}$ is a $(\lambda^2, rT, \sigma^2)$-MRM process independent of w_r. Finally

$$\Big(\tilde{M}(rt),\, 0 \le t \le T\Big) \overset{law}{=} r\Big(M(t),\, 0 \le t \le T\Big),$$

where $\tilde{M}$ is a $(\lambda^2, rT, \sigma^2)$-MRM.

A proof can be found in Bacry and Muzy [11].

Proposition 4.3. *Assume that $\lambda_0 = \lambda_1 = \lambda$, $\sigma_0 = \sigma_1 = \sigma$, but $T_0 \neq T_1$. Assume furthermore that $T_0 > 1$ and $T_1 > 1$. Then $\mathbf{P}_X^0$ and $\mathbf{P}_X^1$ are equivalent.*

Proof. From the second assertion of Property 4.1 with $r = 1/T_0$, we have that if X is a $(\lambda^2, T_0, \sigma^2)$-MRW and $f : (C_{[0,1]}, \mathcal{C}_{[0,1]}) \to (\mathbb{R}, \mathcal{R})$ is a measurable, continuous and bounded functional, then

$$\mathbb{E}\Big[f\Big(X(t),\, t \in [0,1]\Big)\Big] = \mathbb{E}\Big[f\Big(\mathrm{e}^{\frac{1}{2}w_{1/T_0}}\tilde{X}(t),\, t \in [0,1]\Big)\Big],$$

where $\tilde{X}$ is a $(\lambda^2, 1, \sigma^2)$-MRW. We write this last equation

$$\int f(\omega) d\mathbf{P}_X^0(\omega) = \int_{\mathbb{R}} \varphi_{T_0^{-1}}(x) dx \int d\mathbf{Q}(\omega) f(Z(\omega, x)) \tag{4.2}$$

for φ_r the density on $\mathbb{R}$ of an $\mathcal{N}(\lambda^2 \log(r)/4, -\lambda^2 \log(r)/4)$ distribution with respect to the Lebesgue measure, $\mathbf{Q}$ the distribution of a $(\lambda^2, 1, 1)$-MRW on $[0,1]$, and Z : $(C_{[0,1]} \times \mathbb{R}, \mathcal{C}_{[0,1]} \times \mathcal{R}) \to (C_{[0,1]}, \mathcal{C}_{[0,1]})$ the measurable mapping $Z(\omega, x) = e^{x/2}\omega$. Using the notation $\phi_{r,r'} = \varphi_r/\varphi_{r'}$, we get from (4.2)

$$\int f(\omega) d\mathbf{P}_X^0(\omega) = \int_{\mathbb{R}} \varphi_{T_1^{-1}}(x) dx \int d\mathbf{Q}(\omega) \phi_{T_0^{-1}, T_1^{-1}}(x) f(Z(\omega, x)).$$

Let us now define

$$g_{T_0,T_1}(Z) = \mathbb{E}_{\mathbf{Q}\otimes\varphi_{T_1^{-1}}(x)dx}\left[\phi_{T_0^{-1},T_1^{-1}}(x) \Big| Z(\omega, x)\right],$$

$\mathbb{E}_{\mathbf{Q}\otimes\varphi_{T_1^{-1}}(x)dx}[A|B]$ being the conditional expectation of A given B, defined on the product space $(C_{[0,1]} \times \mathbb{R}, \mathcal{C}_{[0,1]} \times \mathcal{R})$ under the probability $\mathbf{Q}\otimes\varphi_{T_1^{-1}}(x)dx$. Note that since $\phi_{T_0^{-1},T_1^{-1}}(x)$ is strictly positive for almost all $x \in \mathbb{R}$, so is $g_{T_0,T_1}(Z(\omega, x))$ for $\mathbf{Q} \otimes \varphi_{T_1^{-1}}(x)dx$-almost all (ω, x). We finally obtain

$$\int f(\omega) d\mathbf{P}_X^0(\omega) = \int_{\mathbb{R}} \varphi_{T_1^{-1}}(x) dx \int d\mathbf{Q}(\omega) g_{T_0,T_1}(Z(\omega, x)) f(Z(\omega, x))$$

from which we deduce, using once again the second assertion of Property 4.1,

$$\int f(\omega) d\mathbf{P}_X^0(\omega) = \int f(\omega) g_{T_0,T_1}(\omega) d\mathbf{P}_X^1(\omega)$$

It follows that $d\mathbf{P}_X^0(\omega) = g_{T_0,T_1}(\omega) d\mathbf{P}_X^1(\omega)$, and that the two distributions are equivalent.

□

Proposition 4.4. *Assume that* $\lambda_0 = \lambda_1 = \lambda$, $T_0 = T_1 = T > 1$, *but* $\sigma_0 \neq \sigma_1$. *Then* $\mathbf{P}_X^0$ *and* $\mathbf{P}_X^1$ *are equivalent.*

Proof. Going along the same lines as the previous proof, we find that

$$d\mathbf{P}_X^0(\omega) = h_{\sigma_0,\sigma_1}(\omega) d\mathbf{P}_X^1(\omega),$$

where

$$h_{\sigma_0,\sigma_1}(Z) = \mathbb{E}_{\mathbf{Q}\otimes\varphi_{1/T}(x-\log\sigma_1)dx}\left[\frac{\varphi_{1/T}(x-\log\sigma_0)}{\varphi_{1/T}(x-\log\sigma_1)}\middle| Z(\omega,x)\right],$$

and $\varphi_{1/T}$ is the density on $\mathbb{R}$ of an $\mathcal{N}(-\lambda^2\log(T)/4, \lambda^2\log(T)/4)$ distribution with respect to the Lebesgue measure.

□

Remark 4.3. In the last two propositions, the condition $T_i > 1$ would of course be replaced by $T_i > t$ if we would consider processes on $[0,t]$ instead of $[0,1]$. These propositions show essentially that, under the condition that the process is observed on an interval of length less than T, then neither T nor σ can be given with probability 1, even when observing a continuous path. Hence, we do not further consider the question of estimating these parameters.

4.3 Some estimation procedures for the intermittency coefficient λ^2 of the log-normal process

We suppose here that we have some observations $Y_{1/n} - Y_1, \ldots, Y_1 - Y_{1-1/n}$, where Y is either a log-normal (λ^2, T, σ^2)-MRW X or (λ^2, T, σ^2)-MRM M for some large n. Both settings are actually quite similar, and all theoretical definitions and properties below are only given for MRW observations, so as to avoid unnecessary notations.

We place ourselves in the situation where the three parameters (λ^2, T, σ^2) are fixed and unknown, and we consider the problem of estimating λ^2 when n goes to $+\infty$ – which, according to the previous section, is the only estimation issue that is actually solvable in this setting. Following the notations of Section 1.2 in Chapter 1, this is the "fine resolution asymptotic" framework denoted by $\chi = 0$. Other possibilities that are not considered here include the case $\chi = +\infty$, that is: one observes $Y_1 - Y_0, \ldots, Y_n - Y_1$ (see Bacry, Kozhemyak and Muzy [10] for a GMM procedure in this setting, and also Ludeña [62] for the special case where $T = O(n)$), and also the "mixed asymptotic case" $\chi \in (0, +\infty)$ of Chapter 1.

4.3.1 Estimation based on p-variations of X

Here we describe an estimation procedure that relies on the convergence of the empirical moment stated by Theorem 1.1 of Chapter 1. Let $S_n(q)$ be

$$S_n(q) = \sum_{k=0}^{n-1} |X_{(k+1)/n} - X_{k/n}|^q.$$

Then from Theorem 1.1, for $q \in [0, 2\sqrt{2}/\lambda)$,

$$n^{q/2-\psi(q/2)-1} S_n(q) \to c_q \gamma(q/2) \sigma^q T^{\psi(q/2)} M^{(q/2)}(1) \text{ as } n \to +\infty \tag{4.3}$$

where the convergence holds in probability and in L^1, where $\gamma(q/2)$ is as in Proposition 1.2, and where c_q is the absolute moment of order q of a standard Gaussian random variable.

Let us also mention two other important thresholds for the exponent q that will be used here: we have $\mathbb{E}\Big[S_n(q)^2\Big] < +\infty$ for $q \in [0, 2\lambda^{-2})$ (this follows for instance from Proposition 1.2 in Chapter 1). Moreover, one expects the convergence (4.3) to hold in L^2 for all $q \in [0, 2\lambda^{-1})$ (indeed, this is the range of exponents q for wich $M^{(q/2)}$ has a finite moment of order 2).

Since the parameter λ^2 affects the rate at which the empirical moment $S_n(q)$ converges (through the function ψ), one may use this empirical moment to produce an estimator of λ^2. In this section, we assume that n is even. We define:

$$\hat{\tau}_n(q) = \frac{1}{\log 2} \log\Big(\frac{S_{n/2}(q)}{S_n(q)}\Big)$$

and for $q \neq 0,\, 2$,

$$\hat{\lambda}_n^{2,q} = -\frac{8(\hat{\tau}_n(q) + 1 - q/2)}{q(q-2)}.$$

Then we have the following result that shows the consistency of the estimator $\hat{\lambda}_n^{2,q}$:

Theorem 4.1. *For $q \in (0,2) \cup (2, 2\sqrt{2}/\lambda)$,*

$$\hat{\lambda}_n^{2,q} \to \lambda^2 \text{ as } n \to +\infty,$$

where the convergences holds in probability.

Proof. Applying (4.3), we have for $q \in [0, 2\sqrt{2}/\lambda)$ the following convergence in probability

$$\hat{\tau}_n(q) \to q/2 - \psi(q/2) - 1 \text{ as } n \to +\infty.$$

The result follows since $\psi(q/2) = \lambda^2 q(q-2)/8$.

□

The following lemma will be useful.

Lemma 4.1. *Assume that q is an even integer, $0 \leq q < 2/\lambda^2$. Then the sequence*

$$n^{1-\lambda^2 q^2/4} \mathbb{E}\Big[\Big((n/2)^{q/2-\psi(q/2)-1} S_{n/2}(q) - n^{q/2-\psi(q/2)-1} S_n(q)\Big)^2\Big],\ n \geq 0$$

converges as n goes to $+\infty$ to a non zero limit.

Proof. In this proof, we write $\Delta_s X_t$ for $X_{t+s} - X_t$. Let us write $p = q/2$ and study the term

$$\mathbb{E}\Big[\Big((n/2)^{p-\psi(p)-1} S_{n/2}(2p) - n^{p-\psi(p)-1} S_n(2p)\Big)^2\Big]$$

With no loss of generality, let us suppose that n is even, and larger than $2/T$. Taking the stationarity into account, and developping the square, we rewrite this term as:

$$2\sum_{k=0}^{n/2-1}(n/2-k)\Big(A_1(n,k)+A_2(n,k)-2A_3(n,k)\Big) \tag{4.4}$$

with

$$A_1(n,k)=(n/2)^{2p-2\psi(p)-2}\mathbb{E}\Big[(\Delta_{2/n}X_0)^{2p}(\Delta_{2/n}X_{2k/n})^{2p}\Big]$$

$$A_2(n,k)=n^{2p-2\psi(p)-2}\mathbb{E}\Big[\Big((\Delta_{1/n}X_0)^{2p}+(\Delta_{1/n}X_{1/n})^{2p}\Big)\Big((\Delta_{1/n}X_{2k/n})^{2p}+(\Delta_{1/n}X_{(2k+1)/n})^{2p}\Big)\Big]$$

and

$$A_3(n,k)=n^{2p-2\psi(p)-2}2^{-p+\psi(p)+1}\mathbb{E}\Big[\Big((\Delta_{1/n}X_0)^{2p}+(\Delta_{1/n}X_{1/n})^{2p}\Big)(\Delta_{2/n}X_{2k/n})^{2p}\Big].$$

We examine separately the above terms for $k=0$, $1\le k\le nT/2$, and (in the case $T<1$) $k\ge nT/2+1$.

For $k=0$, we can apply Property 4.1 with $r=2/(nT)$ to obtain that

$$A_1(n,0)+A_2(n,0)-2A_3(n,0)=(2/T)^{2p-\psi(2p)}n^{-2\psi(p)+\psi(2p)-2}\times$$
$$\mathbb{E}\Big[\Big(2^{-p+\psi(p)+1}(\Delta_T X_0)^{2p}-(\Delta_{T/2}X_0)^{2p}-(\Delta_{T/2}X_{T/2})^{2p}\Big)^2\Big],$$

the expectation in the right hand side being finite from Proposition 1.2. Since

$$-2\psi(p)+\psi(2p)=\lambda^2p^2,$$

we have that

$$n^{2-\lambda^2p^2}\Big(A_1(n,0)+A_2(n,0)-2A_3(n,0)\Big)$$

is positive, finite and does not depend on n.

For $k\ge nT/2+1$, the values $(X_t,\,0\le t\le 2/n)$ and $(X_t,\,2k/n\le t\le(2k+2)/n)$ are independent. A straightforward application of Proposition 1.2 in Chapter 1 then shows that

$$A_1(n,k) + A_2(n,k) - 2A_3(n,k) = 0.$$

For $1 \leq k \leq nT/2 - 1$, we use the following relation (a proof can be found in Bacry, Muzy and Delour [12]): for $t_1 < t_2 < t_3 < t_4$ such that $t_4 - t_1 \leq T$, then

$$\mathbb{E}\Big[\Big(X(t_2) - X(t_1)\Big)^{2p}\Big(X(t_4) - X(t_3)\Big)^{2p}\Big] =$$
$$K \int_{t_1}^{t_2} du_1 \dots \int_{t_1}^{t_2} du_p \int_{t_3}^{t_4} du_{p+1} \dots \int_{t_3}^{t_4} du_{2p} \prod_{1\leq i<j\leq 2p} |u_i - u_j|^{-\lambda^2},$$

where the constant K does not depend on the t_i's. Applying this to $A_1(n,k)$, we obtain through a change of variables:

$$\begin{aligned} A_1(n,k) &= K(n/2)^{2p-2\psi(p)-2} n^{-2p+\lambda^2 p(2p-1)} 2^{2p-\lambda^2 p(p-1)} I_k(0,2,2) \\ &= 4Kn^{-2+\lambda^2p^2} I_k(0,2,2), \end{aligned}$$

where $I_k(a,b,c)$ is the integral

$$I_k(a,b,c) =$$
$$\int_0^1 du_1 \dots \int_0^1 du_{2p} \prod_{1\leq i_1<i_2\leq p} |u_{i_1} - u_{i_2}|^{-\lambda^2} \prod_{p+1\leq j_1<j_2\leq 2p} |u_{j_1} - u_{j_2}|^{-\lambda^2} \prod_{\substack{1\leq i\leq p \\ p+1\leq j\leq 2p}} |2k + a + bu_j - cu_i|^{-\lambda^2}.$$

Going along the same lines, we find

$$A_2(n,k) = Kn^{-2+\lambda^2p^2}\Big(2I_k(0,1,1) + I_k(-1,1,1) + I_k(1,1,1)\Big)$$

and

$$A_3(n,k) = 2Kn^{-2+\lambda^2p^2}\Big(I_k(0,2,1) + I_k(1,2,1)\Big).$$

Using a first order Taylor expansion, we have that if $(a,b,c) \neq (a',b',c')$

$$|I_k(a,b,c) - I_k(a',b',c')| = O(k^{-1-\lambda^2 p^2})$$

for large k. This proves that the series

$$n^{1-p^2\lambda^2} \sum_{k=1}^{n/2-1} (n/2-k)\Big(A_1(n,k) + A_2(n,k) - 2A_3(n,k)\Big)$$

is convergent. Since

$$I_k(0,2,2) \geq I_k(0,2,1)$$

and

$$\min(I_k(0,1,1),\, I_k(-1,1,1),\, I_k(1,1,1)) \geq I_k(1,2,1),$$

the limit of this series is non negative.

□

This lemma enables us to exhibit a rate of convergence for $\hat{\lambda}_n^2(q)$. Let us first recall that a sequence of real random variables X_n is said to be tight if for any $\epsilon > 0$, there is some $K_\epsilon > 0$ such that

$$\forall n, \quad \mathbb{P}\Big[|X_n| \geq K_\epsilon\Big] \leq \epsilon.$$

Recall that from Prokhorov's theorem, this is equivalent to the sequential compacity of the marginal distributions. Thus, by a tight and asymptotically non zero sequence, we mean a tight sequence (X_n) such that no subsequence goes to zero in distribution. We can now state:

Theorem 4.2. *Assume that q is an even integer in $\Big[4, \min(2/\lambda^2, 2/\lambda)\Big)$. Then the sequence*

$$n^{1/2-\lambda^2 q^2/8}(\hat{\lambda}_n^{2,q} - \lambda^2)$$

is tight and asymptotically non zero.

Proof. From Lemma 4.1, we know that the L^2 norm of the sequence

$$n^{1/2-\lambda^2 q^2/8}\Big|(n/2)^{q/2-\psi(q/2)-1}S_{n/2}(q) - n^{q/2-\psi(q/2)-1}S_n(q)\Big|$$

converges to a positive limit. Therefore this sequence is tight and does not converge to 0. We write $\tau = q/2 - \psi(q/2) - 1$. Using a first order Taylor expansion, we have

$$\begin{aligned}\Big|(n/2)^\tau S_{n/2}(q) - n^\tau S_n(q)\Big| &= (n/2)^\tau S_{n/2}(q)\Big|1 - 2^\tau S_n(q)/S_{n/2}(q)\Big| \\ &= (n/2)^\tau S_{n/2}(q)|Z_n|\Big|\log\Big(\frac{S_{n/2}(q)}{S_n(q)}\Big) - \tau\log(2)\Big|\end{aligned}$$

for some random variable Z_n that lies in the interval between 1 and $2^\tau S_n(q)/S_{n/2}(q)$. Applying (4.3), we obtain that the sequence

$$n^{1/2-\lambda^2 q^2/8}|\hat{\tau}_n - \tau|,\ n \geq 0$$

is tight and asymptotically non zero. The result follows.

□

Remark 4.4. Hence, we see that the convergence rate of the estimator $\hat{\lambda}_n^{2,q}$ is strictly lower than the usual parametric rate $n^{-1/2}$; moreover this rate becomes slower for large q and λ^2. From the results of Ossiander and Waymire [75] and Ludeña [62], we expect Theorem 4.2 to hold also for non-integer $q > 0$, and general log-infinitely divisible MRW process.

4.3.2 Estimators based on the logarithm of the increments of X

Preliminary considerations

For $n \in \mathbb{N}$, $k = 0, \ldots, n$, let us write

$$x_{n,k} = \log\Big|X((k+1)/n) - X(k/n)\Big|$$

and

$$m_{n,k} = \log\Big(M((k+1)/n) - M(k/n)\Big).$$

We first state some properties of the series $(x_{n,k})$ and $(m_{n,k})$ that will be useful.

It will be convenient in this section to use the explicit construction of the process $\big(w_l(u)\big)$ via the cones $\big(\mathcal{A}_l(u)\big)$, see Section 1.2 in Chapter 1. Let us further define for $n \geq 1$ and $0 \leq k \leq n-1$

$$\mathcal{A}^c_{n,k} = \bigcap_{k/n \leq u \leq (k+1)/n} \mathcal{A}_{1/n}(u)$$

and

$$w_{n,k} = P\Big(\mathcal{A}^c_{n,k}\Big).$$

From the hypothesis $\psi(p) = \lambda^2 p(p-1)/2$, $p \in \mathbb{R}$, it is easy to check that $w_{n,k}$ is a $\mathcal{N}(-\lambda^2 \log(nT)/2), \lambda^2 \log(nT))$ random variable.

The following representation of the sequence $(m_{n,k})_k$ will guide the estimation procedures below. We write $\log_+(x)$ for $\log\Big(\max(x,0)\Big)$.

Proposition 4.5. *For $n \geq 1$, $0 \leq k \leq n-1$, we can decompose $m_{n,k}$ as*

$$m_{n,k} = 2\log(\sigma) - \log(n) + w_{n,k} + z_{n,k},$$

where (i) for each k, $w_{n,k}$ and $z_{n,k}$ are independent; (ii) $(w_{n,k})_k$ is a Gaussian stationary process with expectation $-\lambda^2 \log(nT)/2$ and covariance

$$\mathbb{C}ov[w_{n,k}, w_{n,k'}] = \lambda^2 \log_+ \frac{nT}{|k-k'|+1};$$

(iii) $(z_{n,k})_k$ is a stationary random sequence such that the marginal distribution of $z_{n,0}$ is the same as the distribution of $\log(\tilde{M}_1)$, where $\tilde{M}$ is a $(\lambda^2, 1, 1)$-MRM; (iv) for each $p \in (0, +\infty)$, $(z_{n,k})$ has a finite moment of order p; and finally (v) there exist some constants c, $c' > 0$ such that for $n \geq 1$ and $0 \leq k$, $k' \leq n-1$,$0 \leq \mathbb{C}ov\Big[z_{n,k}, z_{n,k'}\Big] \leq c|k-k'|^{-2}$and $0 \leq \mathbb{C}ov\Big[w_{n,k}, z_{n,k'}\Big] \leq c'|k-k'|^{-1}$.

Note that the corresponding representation of $(x_{n,k})_k$ trivially follows from the equality in law

$$(x_{n,k})_k \stackrel{law}{=} \Big(\log|\varepsilon_{n,k}|\Big)_k + \Big(m_{n,k}/2\Big)_k$$

where $(\varepsilon_{n,k})_k$ is a Gaussian white noise independent of $(m_{n,k})_k$.

Proof. Going back to the definition of an MRM, it is straightforward to check that

$$e^{-w_{n,k}}\Big(M((k+1)/n) - M(k/n)\Big) = \sigma^2 \bar{M}(1/n),$$

where $\bar{M}$ is a $(\lambda^2, 1/n, 1)$ MRM. Applying Property 4.1 yields that the distribution of $z_{n,k} = \log \bar{M}(1/n) + \log(n)$ is the same as the distribution of the logarithm of a $(\lambda^2, 1, 1)$-MRM. We thus have (i), (ii) and (iii). The item (iv) follows from Theorem 5.5 in Kozhemyak [60], the proof of which is somewhat involved and requires studying the condition of existence of negative moments of X. Finally, we will need the statement of Lemma 4.2 to prove (v):

Lemma 4.2 (An upper bound for the covariance of nonlinear functionals of Gaussian random vectors [84])**.** *For p, $q \in \mathbb{N}$, let $a = (a_1, \ldots, a_p)$ and $b = (b_1, \ldots, b_q)$ be two Gaussian random vectors, each with i.i.d. $N(0,1)$ components, and let $f : \mathbb{R}^p \to \mathbb{R}$ and $g : \mathbb{R}^q \to \mathbb{R}$ be two functionals such that*

$$\mathbb{E}\Big[f(a)^2\Big] + \mathbb{E}\Big[g(b)^2\Big] < +\infty.$$

Then

$$\mathbb{C}ov[f(a), g(b)] \leq 2\sqrt{\mathbb{E}\Big[f(a)^2\Big]\mathbb{E}\Big[g(b)^2\Big]} \sum_{i=1}^{p}\sum_{j=1}^{q} |\mathbb{C}ov[a_i, b_j]|.$$

We refer to Robinson [84] for a proof of this lemma. An earlier use of the same result in the setting of MRW processes is given by Bacry, Kozhemyak and Muzy [10].

We define for $m \in \mathbb{N}$

$$z_{n,k}^m = \log\left(\frac{n}{m}\sum_{j=0}^{2^m-1} \exp\Big[P\Big(\mathcal{A}_{2^{-m}/n}\Big((k + j2^{-m})/n\Big) \setminus \mathcal{A}_{n,k}^c\Big)\Big] du\right).$$

Following Theorem 1.4 in Chapter 1, we have that $e^{z^m_{n,k}}$ goes to $e^{z_{n,k}}$ almsot surely as $m \to +\infty$. Moreover, since $z^m_{n,k}$ is bounded in L^p as $m \to +\infty$ for $p > 2$, we have that $z^m_{n,k} \to z_{n,k}$ in L^2. Therefore, it is enough to to give an upper bound for

$$\mathbb{C}\mathrm{ov}[z^m_{n,0}, z^m_{n,k}] \quad \text{and} \quad \mathbb{C}\mathrm{ov}[z^m_{n,0}, w_{n,k}], \quad 1 \leq k \leq n-1.$$

To do so, we write $z^m_{n,0}$ as a nonlinear function of the Gaussian vector $(a_j)_{0 \leq j \leq 2^m-1}$ with

$$a_0 = P\Big(\mathcal{A}_{2^{-m}/n}\Big(0\Big) \setminus \mathcal{A}^c_{n,0}\Big),$$

and for $1 \leq j \leq 2^m - 1$

$$a_j = P\bigg(\mathcal{A}_{2^{-m}/n}\Big(j2^{-m}/n\Big) \setminus \mathcal{A}_{2^{-m}/n}\Big((j-1)2^{-m}/n\Big)\bigg).$$

The same way, we write $z^m_{n,k}$ for $k \geq 1$ as a nonlinear function of the Gaussian vector $(b^k_j)_{0 \leq j \leq 2^m-1}$ with

$$b^k_{2^m-1} = P\Big(\mathcal{A}_{2^{-m}/n}\Big((k+1-2^{-m})/n\Big) \setminus \mathcal{A}^c_{n,k}\Big),$$

and for $0 \leq j \leq 2^m - 2$

$$b^k_j = P\bigg(\mathcal{A}_{2^{-m}/n}\Big((k+j2^{-m})/n\Big) \setminus \mathcal{A}_{2^{-m}/n}\Big((k+(j+1)2^{-m})/n\Big)\bigg).$$

The following variances and covariances are then straightforward to obtain for $1 \leq i \leq 2^m - 1$, $0 \leq j \leq 2^m - 2$:

$$\mathbb{V}\mathrm{ar}\Big[a_0\Big] = \mathbb{V}\mathrm{ar}\Big[b^k_{2^m-1}\Big] = \lambda^2(\log(2^m) + 1),$$

$$\mathbb{V}\mathrm{ar}\Big[a_i\Big] = \mathbb{V}\mathrm{ar}\Big[b^k_j\Big] = \lambda^2,$$

$$\mathbb{C}\mathrm{ov}\Big[a_0, w_{n,k}\Big] = \mathbb{C}\mathrm{ov}\Big[a_0, b^k_j\Big] = \mathbb{C}\mathrm{ov}\Big[a_i, b^k_{2^m-1}\Big] = \mathbb{C}\mathrm{ov}\Big[a_0, b^k_{2^m-1}\Big] = 0,$$

$$\mathbb{C}\mathrm{ov}\Big[a_i, b_j^k\Big] = \lambda^2 \log\Bigg(\frac{\big(k+(j-i+1)2^{-m}\big)^2}{\big(k+(j-i)2^{-m}\big)\big(k+(j-i+2)2^{-m}\big)}\Bigg) \leq c2^{-2m}/k^2,$$

$$\mathbb{C}\mathrm{ov}\Big[w_{n,0}, b_j^k\Big] = \mathbb{C}\mathrm{ov}\Big[a_i, w_{n,k}\Big] = \lambda^2 \log\Big(\frac{k+1-(i-1)2^{-m}}{k+1-i2^{-m}}\Big) \leq c'2^{-m}/k$$

for some c, $c' > 0$ that do not depend on i, j, k, m, n.

Applying Lemma 4.2, we obtain:

$$\mathbb{C}\mathrm{ov}\Big[z_{n,0}^m, z_{n,k}^m\Big] \leq c/k^2$$

and

$$\mathbb{C}\mathrm{ov}\Big[z_{n,0}^m, w_{n,k}\Big] \leq c'/k,$$

which proves the result.

□

Estimator based on the covariance of the $x_{n,k}$'s

This section is an adaptation from the work of Bacry, Kozhemyak and Muzy [10]. Proposition 4.5 shows that the series $(x_{n,k})_k$ can be thought of as a the sum of a Gaussian stationary sequence $(w_{n,k}/2)_k$, the logarithm of a Gaussian white noise $(\log|\varepsilon_{n,k}|)_k$, and finally a stationary random sequence $(z_{n,k}/2)_k$ which slowly becomes negligible as n grows. The theoretical autocovariance of $(x_{n,k})_k$ in particular behaves for large n as $k \mapsto \lambda^2 \log\Big(Tn/(|k|+1)\Big)/4$, so that one can hope that a consistent estimator of λ^2 can be obtained from the slope of the empirical covariance of this series. Bacry, Kozhemyak and Muzy [10] propose such an estimator. We briefly sum up their approach. Let us write

$$g(n) = \log(n) - \frac{(n+1)^2}{2} \log\frac{n+1}{n} - \frac{(n-1)^2}{2} \log\frac{n-1}{n}$$

for $n \geq 2$, and

$$g(1) = -2\log(2).$$

The function g can actually be obtained from the autocovariance function of the Gaussian stationary process:

$$\Omega(t) = \lim_{l \to 0} \int_0^t \frac{1}{\lambda}(w_l(u) - \mathbb{E}\Big[w_l(u)\Big])du,$$

the limit here being valid in law. Moreover, Bacry, Kozhemyak and Muzy show that the value Ω_t/t is also the limit in distribution of $\lambda^{-1}\log(M(t)/t)$ for $\lambda \to 0$. This enables them to prove the following result: For $n \geq 1$, let $\bar{x}_n = n^{-1}\sum_{k=0}^{n-1} x_{n,k}$. Then we have

Theorem 4.3 (Bacry *et al.* [10])**.** *For $n \geq 3$, and $1 \leq h, h' \leq n-1$, $h \neq h'$, define*

$$\hat{\rho}_n(h) = \frac{1}{n-h}\sum_{k=0}^{n-h-1}(x_{n,k} - \bar{x}_n)(x_{n,k+h} - \bar{x}_n)$$

and

$$\hat{\lambda}_n^{2,cov}(h, h') = \frac{4\Big(\hat{\rho}_n(h) - \hat{\rho}_n(h')\Big)}{g(h) - g(h')}.$$

Then

$$\left|\mathbb{E}\Big[\hat{\lambda}_n^{2,cov}(h, h')\Big] - \lambda^2\right| \leq f_1(n) + f_2(\lambda^2)$$

with $0 \leq f_1(n) \leq O(1/n)$ for large n and $0 \leq f_2(\lambda^2) \leq O(\lambda^2)$ for small λ^2. Also,

$$\mathbb{V}\mathrm{ar}\Big[\hat{\lambda}_n^{2,cov}(h, h')\Big] = f_3(n)$$

with $0 \leq f_3(n) \leq O(\log(n)/n)$ for large n.

Estimator based on the scaling properties of the $x_{n,k}$'s

A common approach in the literature to characterize the multifractal behaviour of the data (and hence, in the setting of the log-normal MRW, the coefficient λ^2) is to study the dependence of the moments or cumulants of the random variable $x_{n,k}$ on the scale $1/n$. However, we do not know if the theoretical properties of the empirical moments of $x_{n,k}$ have been explicitly studied by a previous mathematical paper in the setting of random cascades.

The notations in this part will be much easier to handle if we suppose that we have two dyadic observation steps $1/n$ and $1/n'$ with: $n = 2^{J+j}$ and $n' = 2^J$ for some $J \in \mathbb{N}$ that goes to $+\infty$ ans some fixed $j \in \mathbb{N}$. Then, using for instance Property 4.1, we see that in the MRW setting

$$\mathbb{V}\text{ar}\Big[x_{n,k}\Big] - \mathbb{V}\text{ar}\Big[x_{n',k}\Big] = \frac{\lambda^2 \log(2) j}{4}.$$

Regressing the empirical variance of $x_{n,k}$ on the logarithm of the sampling scale $1/n$ is thus a statistical technique that has already been considered in the setting of random cascades (see notably the works by Jaffard, Lashermes, Abry and Wendt [53, 94, 95, 93] for an elaborate application of this technique) to which the class of MRW belongs. We prove below that the empirical variance, or the difference of empirical variances at two different scales, does indeed provide a consistent estimator of λ^2 in the case of a log-normal MRW.

Theorem 4.4. *Define for j, $J \in \mathbb{N}$*

$$\hat{\lambda}^{2,var}_{J,j} = 4 \frac{\frac{1}{2^{J+j}-1} \sum_{k=0}^{2^{J+j}-1} \Big(x_{2^{J+j},k} - \bar{x}_{2^{J+j}}\Big)^2 - \frac{1}{2^J-1} \sum_{k=0}^{2^J-1} \Big(x_{2^J,k} - \bar{x}_{2^J}\Big)^2}{h(2^{j+J}) - h(2^J)}$$

where

$$h(n) = \frac{2}{n(n-1)} \sum_{k=1}^{n-1} (n-k) \log(k+1).$$

Then

$$\mathbb{E}\Big[\Big(\hat{\lambda}^{2,var}_{J,j} - \lambda^2\Big)^2\Big] \leq cJ^2 2^{-J} \qquad \text{as } J \to +\infty$$

for some $c > 0$.

Proof. Let us first study the bias of $\hat{\lambda}^{2,var}_{J,j}$. For $(y_1, \ldots, y_n)$ a stationary random sequence with empirical mean $\bar{y}_n = n^{-1} \sum_{k=1}^n y_k$, a standard computation yields

$$\mathbb{E}\Bigg[\frac{1}{n-1} \sum_{k=1}^n (y_k - \bar{y}_n)^2\Bigg] = \mathbb{V}\text{ar}[y_1] - \frac{2}{n(n-1)} \sum_{k=1}^{n-1} (n-k)\, \mathbb{C}\text{ov}[y_0, y_k].$$

We apply this equation to the sequence $(x_{n,k})$. Taking Property 4.5 into account, we obtain:

$$\mathbb{E}\left[\frac{1}{n-1}\sum_{k=1}^{n}(x_{n,k}-\bar{x}_{n,k})^2\right] = \gamma + \frac{h(n)\lambda^2}{4} + \tilde{h}(n),$$

where $\gamma \in \mathbb{R}$ is a constant that does not depend on n, and $0 \leq h(n) \leq O\Big(\log(n)/n\Big)$. Thus, the bias of $\hat{\lambda}^{2,var}_{J,j}$ is at most of order $\log(n)/n$.

We now study the variance of the estimator. First note that since

$$h(n) = \frac{2}{n-1}\sum_{k=1}^{n-1}(1-k/n)\log(k/n) + \log(n)\frac{2}{n-1}\sum_{k=1}^{n-1}(1-k/n)$$

we clearly have

$$h(2^{j+J}) - h(2^J) \to j\log(2)\int_0^1(1-u)du = j\log(2)/2$$

for $J \to +\infty$, and thus the denominator in the definition of $\hat{\lambda}^{2,var}_{J,j}$ does not asymptotically depend on J.

Let us now consider the numerator. We will write here y^* for the random variable $y - \mathbb{E}\Big[y\Big]$. After some algebra, we obtain:

$$\begin{aligned} &2^{-J-j}\sum_{k=0}^{2^{J+j}-1}\Big(x_{2^{J+j},k}-\bar{x}_{2^{J+j}}\Big)^2 - 2^{-J}\sum_{k=0}^{2^J-1}\Big(x_{2^J,k}-\bar{x}_{2^J}\Big)^2 \\ &= 2^{-J-j}\sum_{k=0}^{2^{J+j}-1}\Big(x^*_{2^{J+j},k}\Big)^2 - 2^{-J}\sum_{k=0}^{2^J-1}\Big(x^*_{2^J,k}\Big)^2 - \left(2^{-J-j}\sum_{k=0}^{2^{J+j}-1}x^*_{2^{J+j},k}\right)^2 + \left(2^{-J}\sum_{k=0}^{2^J-1}x^*_{2^J,k}\right)^2. \end{aligned}$$

We handle separately the upper bounds for

$$\mathbb{V}\mathrm{ar}\left[2^{-J-j}\sum_{k=0}^{2^{J+j}-1}\Big(x^*_{2^{J+j},k}\Big)^2 - 2^{-J}\sum_{k=0}^{2^J-1}\Big(x^*_{2^J,k}\Big)^2\right]$$

and

$$\mathbb{V}\mathrm{ar}\left[\left(2^{-J-j}\sum_{k=0}^{2^{J+j}-1}x^*_{2^{J+j},k}\right)^2 - \left(2^{-J}\sum_{k=0}^{2^J-1}x^*_{2^J,k}\right)^2\right].$$

We first consider

$$\mathbb{V}\text{ar}\left[2^{-J-j}\sum_{k=0}^{2^{J+j}-1}\left(x^*_{2^{J+j},k}\right)^2 - 2^{-J}\sum_{k=0}^{2^J-1}\left(x^*_{2^J,k}\right)^2\right]$$
$$= 2^{-2J-2j+1}\sum_{k=0}^{2^J-1}\sum_{l=0}^{2^j-1}(2^J-k)\,\mathbb{C}\text{ov}\left[\left(x^*_{2^{J+j},l}\right)^2 - \left(x^*_{2^J,0}\right)^2, \left(x^*_{2^{J+j},2^jk+l}\right)^2 - \left(x^*_{2^J,k}\right)^2\right].$$

Using Lemma 4.4 below, this variance is at most of order $O(J^2 2^{-J})$ for $J \to +\infty$.

We now turn to the upper bound for

$$\mathbb{V}\text{ar}\left[\left(2^{-J-j}\sum_{k=0}^{2^{J+j}-1} x^*_{2^{J+j},k}\right)^2 - \left(2^{-J}\sum_{k=0}^{2^J-1} x^*_{2^J,k}\right)^2\right].$$

From

$$\left(2^{-J-j}\sum_{k=0}^{2^{J+j}-1} x^*_{2^{J+j},k}\right)^2 - \left(2^{-J}\sum_{k=0}^{2^J-1} x^*_{2^J,k}\right)^2 =$$
$$2^{-2j}\sum_{l_1,l_2=0}^{2^j-1}\left(2^{-J}\sum_{k_1=0}^{2^J-1} x^*_{2^{J+j},k_12^j+l_1} - x^*_{2^J,k_1}\right)\left(2^{-J}\sum_{k_2=0}^{2^J-1} x^*_{2^{J+j},k_22^j+l_2} + x^*_{2^J,k_2}\right).$$

we have :

$$\mathbb{V}\text{ar}\left[\left(2^{-J-j}\sum_{k=0}^{2^{J+j}-1} x^*_{2^{J+j},k}\right)^2 - \left(2^{-J}\sum_{k=0}^{2^J-1} x^*_{2^J,k}\right)^2\right]$$
$$= c2^{-4J-4j}\sum_{0\le k_1,k_2,k_3,k_4\le 2^J-1}\;\sum_{0\le l_1,l_2,l_3,l_4\le 2^j-1} f_{J,j}(k_1,k_2,k_3,k_4,l_1,l_2,l_3,l_4)$$

with

$$f_{J,j}(k_1,k_2,k_3,k_4,l_1,l_2,l_3,l_4) = \tag{4.5}$$
$$\mathbb{C}\text{ov}\Big[(x^*_{2^{J+j},k_12^j+l_1} + x^*_{2^J,k_1})(x^*_{2^{J+j},k_22^j+l_2} - x^*_{2^J,k_2}), (x^*_{2^{J+j},k_32^j+l_3} + x^*_{2^J,k_3})(x^*_{2^{J+j},k_42^j+l_4} - x^*_{2^J,k_4})\Big].$$

We now apply Lemma 4.4 to obtain the result.

□

Lemma 4.3. *Let $n \in \mathbb{N}$, and $0 \leq k_1, k_2, k_3, k_4 \leq n-1$ with $k_{min} = \min\{|k_i - k_j|,\ i \neq j\}$. Then for fixed n, λ^2:*

$$\mathbb{C}\mathrm{ov}[x^*_{n,k_1} x^*_{n,k_2}, x^*_{n,k_3} x^*_{n,k_4}] = h_1(k_1,k_2,k_3,k_4) + h_2(k_1,k_2,k_3,k_4) + \log(n) h_3(k_1,k_2,k_3,k_4)$$

where

$$h_1(k_1,k_2,k_3,k_4) = \frac{\lambda^4}{16} \log \frac{Tn}{|k_1 - k_3| + 1} \log \frac{Tn}{|k_2 - k_4| + 1}$$

$$h_2(k_1,k_2,k_3,k_4) = \frac{\lambda^4}{16} \log \frac{Tn}{|k_1 - k_4| + 1} \log \frac{Tn}{|k_2 - k_3| + 1}$$

$$0 \leq h_3(k_1,k_2,k_3,k_4) \leq \frac{c}{k_{min} + 1}$$

for some constant $c > 0$ that does not depend on n nor on the k_i's.

Proof. We apply the decomposition of Proposition 4.5. Using some well-known properties of Gaussian random vectors, it is easy to obtain

$$\mathbb{C}\mathrm{ov}[w^*_{n,k_1} w^*_{n,k_2}/4, w^*_{n,k_3} w^*_{n,k_4}/4] = h_1(k_1,k_2,k_3,k_4) + h_2(k_1,k_2,k_3,k_4).$$

Next, we show how to obtain an upper bound for

$$\mathbb{C}\mathrm{ov}[w^*_{n,k_1} w^*_{n,k_2}, w^*_{n,k_3} z^*_{n,k_4}].$$

The idea is to apply Lemma 4.2 with

$$a_i = \frac{w^*_{n,k_i}}{\sqrt{\mathbb{E}\Big[(w^*_{n,k_i})^2\Big]}},$$

$$f(a) = w^*_{n,k_1} w^*_{n,k_2} w^*_{n,k_3},$$

and

$$g(b^{k_4}) = z^m_{n,k_4} - \mathbb{E}\Big[z^m_{n,k_4}\Big]$$

where b^{k_4} and z^m_{n,k_4} are as in the proof of Proposition 4.5. From the upper bound

$$0 \le \mathbb{C}\mathrm{ov}[w_{n,0}, b_j^{k_4}] \le 2^{-m}/|k_4|,$$

we thus get

$$\mathbb{C}\mathrm{ov}[w^*_{n,k_1} w^*_{n,k_2} w^*_{n,k_3}, z^*_{n,k_4}] \le c \frac{\log(n)}{1 + \min(|k_1 - k_4|, |k_2 - k_4|, |k_3 - k_4|)}.$$

Also, we have

$$\mathbb{E}\Big[w^*_{n,k_1} w^*_{n,k_2} w^*_{n,k_3}\Big]\mathbb{E}\Big[z^*_{n,k_4}\Big] = 0,$$

so that

$$\mathbb{C}\mathrm{ov}[w^*_{n,k_1} w^*_{n,k_2} w^*_{n,k_3}, z^*_{n,k_4}] = \mathbb{E}\Big[w^*_{n,k_1} w^*_{n,k_2} w^*_{n,k_3} z^*_{n,k_4}\Big].$$

Finally, from Proposition 4.5, we have:

$$\mathbb{E}\Big[w^*_{n,k_1} w^*_{n,k_2}\Big]\mathbb{E}\Big[w^*_{n,k_3} z^*_{n,k_4}\Big] \le c \frac{\lambda^2}{|k_4 - k_3| + 1} \log \frac{nT}{|k_1 - k_2| + 1}.$$

The result follows by applying the same approach to the other terms in

$$\mathbb{C}\mathrm{ov}[(w^*_{n,k_1} + z^*_{n,k_1})(w^*_{n,k_2} + z^*_{n,k_2}), (w^*_{n,k_3} + z^*_{n,k_3})(w^*_{n,k_4} + z^*_{n,k_4})].$$

□

Lemma 4.4. *Let $j, J \in \mathbb{N}$, $0 \le k_1, k_2, k_3', k_4 \le 2^J - 1$ and $0 \le l_1, l_2, l_3, l_4 \le 2^j - 1$. Define $k_{min} = \min\{|k_i - k_j|, i \ne j\}$. and let $f_{J,j}$ be as in (4.5). Then*

$$\Big| f_{J,j}(k_1, k_2, k_3, k_4, l_1, l_2, l_3, l_4) \Big| \le \log(2^{J+j}) h_4(k_1, \dots, k_4, l_1 \dots, l_4)$$

for some function h_4 such that

$$0 \le h_4(k_1, \dots, k_4, l_1 \dots, l_4) \le \frac{c}{k_{min} + 1}$$

for some constant $c > 0$ that does not depend on J, j nor on the k_i's and l_i's.

Proof. This is a straightforward application of Lemma 4.3.

□

Estimator based on the difference $x_{n,k+1} - x_{n,k}$

We first give some incentive for using the series $(x_{n,k+1} - x_{n,k})_k$ when estimating λ^2 rather than the series $(x_{n,k})_k$ or even $\Big(X((k+1)/n) - X(k/n)\Big)_k$.

Recall that MRW and MRM processes are very similar in nature to Mandelbrot $b-$adic cascade processes [67]. Such a cascade constructed on a dyadic tree with some random weight $W > 0$, is defined in the following way: Let

$$\Big(W_k,\, k = (k_1, \ldots, k_j) \in \{0,1\}^j,\, j \in \mathbb{N}\Big)$$

be some independent copies of W that are indexed by dyadic integers. Also, for $u \in [0, T]$, let $u_1, \ldots, u_j, \ldots$ be the coefficients with value either 0 or 1 such that :

$$u = T \sum_{j=1}^{+\infty} 2^{-j} u_j,$$

and let $m_j : (u) = (u_1, \ldots, u_j)$ for $j \in \mathbb{N}$. Then, under some condition on the distribution of W, Kahane and Peyrière [57] show that for $t \in [0, T]$,

$$\int_0^t \prod_{j=1}^{n} W_{m_j(u)} du \to M^{\mathcal{M}}(t) \text{ as } n \to +\infty.$$

almost surely and in L^1, where $M^{\mathcal{M}}(t)$ is nondegenerate.

The following property of Mandelbrot dyadic cascades are direct consequences of this definition:

Property 4.2. *Let us write*

$$m^{\mathcal{M}}_{n,k} = \log\Big(M^{\mathcal{M}}((k+1)2^{-n}T) - M^{\mathcal{M}}(k2^{-n}T)\Big)$$

for $n \in \mathbb{N}$ *and* $k = 0, \dots, 2^n - 1$. *Then*

$$m^{\mathcal{M}}_{n,k} \stackrel{law}{=} -n \log 2 + w_1 + \cdots + w_n + m^{\mathcal{M}}_{0,0},$$

where the w_i*'s are independent copies of* $\log(W)$, *and are independent of* $m^{\mathcal{M}}_{0,0} = \log\Big(M^{\mathcal{M}}(T)\Big)$.

Moreover, let us write

$$d_{n,k} = m^{\mathcal{M}}_{n,2k+1} - m^{\mathcal{M}}_{n,2k}$$

for $n \geq 2$ *and* $k = 0, \dots, 2^{n-1} - 1$. *Then for* $k \neq k'$, $d_{n,k}$ *and* $d_{n,k'}$ *are independent and have the same distribution. They satisfy*

$$d_{n,k} \stackrel{law}{=} d_{n,k'} \stackrel{law}{=} w + m - w' - m',$$

where w *and* w' *are copies of* $\log W$, m *and* m' *are copies of* $m^{\mathcal{M}}_{0,0}$, *and these four random variables are independent.*

Hence, from the series $\Big(M^{\mathcal{M}}((k+1)2^{-n}T) - M^{\mathcal{M}}(k2^{-n}T)\Big)$, which has an elaborate dependence structure, one may obtain the series $(d_{n,k})$, which is i.i.d. with a marginal distribution that does not depend on n. One may thus wish to work on the latter series rather than on the former, notably when trying to build estimation procedures.

We now return to the case of MRM and MRW, and show how to adapt the above approach. The following proposition may be seen as an analogue of Property 4.2:

Proposition 4.6. *Consider the stationary sequence* $(\Delta_{1/n} x_{n,k})_k = (x_{n,k+1} - x_{n,k})_k$. *The following holds: first, the marginal distribution of* $\Delta_{1/n} x_{n,0}$ *does not depend on* σ, T *or* n. *Second,* $\mathbb{C}ov[|\Delta_{1/n} x_{n,k}|^p, |\Delta_{1/n} x_{n,k'}|^p] \leq c_p |k - k'|^{-2}$ *for some constant* $c_p > 0$.

Proof. Applying Property 4.1, we have

$$\Big(X(2u/n),\, 0 \le u \le 1\Big) \overset{law}{=} (2/n)^{1/2}\Big(\sigma e^{w_{2/n}/2}\tilde{X}(u),\, 0 \le u \le 1\Big),$$

where $\tilde{X}$ is a $(\lambda^2, 1, 1)$-MRW independent of the normal random variable $w_{2/n}$. It follows that

$$\Delta_{1/n} x_{n,0} \overset{law}{=} \log\left(\frac{\left|\tilde{X}(1) - \tilde{X}(1/2)\right|}{\left|\tilde{X}(1/2 - \tilde{X}_0)\right|}\right),$$

and the distribution of this last term does not depend neither on σ, T, nor n.

We go along the same lines as the proof of Proposition 4.5 for giving an upper bound of $\mathbb{C}\text{ov}[|\Delta_{1/n} x_{n,k}|^p, |\Delta_{1/n} x_{n,k'}|^p]$. Note that we have

$$\mathbb{V}\text{ar}\Big[w_{n,1} - w_{n,0}\Big] = 2\lambda^2 \log(2)$$

$$\mathbb{V}\text{ar}\Big[z_{n,1} - z_{n,0}\Big] \le 2\mathbb{V}\text{ar}\Big[z_{n,0}\Big]$$

and for $2 \le k \le n-1$, $m \in \mathbb{N}$, $1 \le i \le 2^m - 1$

$$\mathbb{C}\text{ov}\Big[w_{n,1} - w_{n,0}, w_{n,k+1} - w_{n,k}\Big] = \lambda^2 \log\Big(1 - (k+1)^{-2}\Big)$$

$$\mathbb{C}\text{ov}\Big[a_i, w_{n,k+1} - w_{n,k}\Big] = \lambda^2 \log\left(1 - \frac{2^{-m}}{(k+2-i2^{-m})(k+1-(i-1)2^{-m})}\right).$$

Applying Lemma 4.2 achieves the proof.

□

Theorem 4.5. *Consider the function* $m^p : \lambda^2 \mapsto \mathbb{E}\Big[|\Delta_{1/n} x_{n,0}|^p\Big]$ *for* $p \ge 0$*, and the empirical moment estimator*

$$\hat{m}_n^p(\lambda^2) = \frac{1}{n-1}\sum_{k=0}^{n-2} |\Delta_{1/n} x_{n,k/n}|^p.$$

Then $\mathbb{E}\Big[(\hat{m}_n^p(\lambda^2) - m^p(\lambda^2))^2\Big] = O(1/n)$ *as* $n \to +\infty$*. Moreover, we have that the mapping*

$\lambda^2 \mapsto m^2(\lambda^2)$ *is monotone and differentiable with non zero derivative on* $(0,2)$. *We can thus define*

$$\hat{\lambda}_n^{2,\Delta} = (m^2)^{-1}\Big(\hat{m}_n^2(\lambda^2)\Big).$$

Then $\mathbb{E}\Big[\Big(\hat{\lambda}_n^{2,\Delta} - \lambda^2\Big)^2\Big] = O(1/n)$ *as* $n \to +\infty$.

Proof. The first part of the theorem is a simple application of the previous proposition. Concerning the second part, it is clear that $m^2(\cdot)$ is C^∞ on $(0,2)$, so that we only have to prove that its derivative is (strictly) positive for $\lambda \in (0,2)$. It will be enough to show that

$$\frac{\partial f_w}{\partial \lambda}(\lambda) > 0 \tag{4.6}$$

for some $c > 0$, where

$$f_w(\lambda) = \mathbb{E}\left[\left(\log\Big(\frac{\int_0^1 \exp(\lambda w_u)du}{\int_1^2 \exp(\lambda w_u)du}\Big)\right)^2\right],$$

$w = \Big(w_u,\ 0 \leq u \leq 2\Big)$ here being some centered stationary Gaussian process with a decreasing and non-negative autocovariance function $\gamma : u \mapsto \mathbb{C}\mathrm{ov}[w_0, w_u]$ that is moreover continuous on $[0,2]$ with a bounded derivative on $(0,2)$ – so that w has continuous sample paths, see Ibragimov and Rozanov [48]. After differentiating, we find

$$\frac{\partial f_w}{\partial \lambda}(\lambda) = 2\mathbb{E}\left[\frac{\int_0^1 w \exp(\lambda w_u)du}{\int_0^1 \exp(\lambda w_u)du} \log\Big(\frac{\int_0^1 \exp(\lambda w_u)du}{\int_1^2 \exp(\lambda w_u)du}\Big)\right]. \tag{4.7}$$

This last term can be obtained as the limit

$$\mathbb{E}\left[2\sum_{k=1}^m \phi_m^k(w_{1/m}, w_{2/m}, \ldots, w_2)\right] \to \frac{\partial f_w}{\partial \lambda}(\lambda), \quad m \to +\infty$$

where

$$\phi_m^k(x_1, \ldots, x_{2m}) = \frac{x_k \mathrm{e}^{\lambda x_k}}{\sum_{i=1}^m \mathrm{e}^{\lambda x_i}} \log\Big(\frac{\sum_{i=1}^m \mathrm{e}^{\lambda x_i}}{\sum_{i=1}^m \mathrm{e}^{\lambda x_{i+m}}}\Big).$$

We now apply the following integration by parts formula (known as Stein's lemma [89]):

Lemma 4.5 (Integration by parts for Gaussian random vectors)**.** *Let* $(X, Z_1, \ldots, Z_d)$ *be a centered Gaussian random vector, and let* $F : \mathbb{R}^d \to \mathbb{R}$ *be continuously differentiable. Then*

$$\mathbb{E}\Big[XF(Z_1, \ldots, Z_d)\Big] = \sum_{j=1}^{d} \mathbb{E}\Big[XZ_j\Big]\mathbb{E}\Big[\frac{\partial F}{\partial Z_j}(Z_1, \ldots, Z_d)\Big],$$

under the condition that all the above expectations are finite.

This formula thus yields

$$\begin{aligned}\mathbb{E}\Big[\phi_m^k(w_0, w_{1/m}, \ldots, w_{2-1/m})\Big] =& \lambda\mathbb{E}\Bigg[\frac{\sum_{i=1}^m(\gamma(0) - \gamma(i/m))\mathrm{e}^{\lambda w_{k/m}+\lambda w_{i/m}}}{(\sum_{i=1}^m \mathrm{e}^{\lambda w_{i/m}})^2}\log\Big(\frac{\sum_{i=1}^m \mathrm{e}^{\lambda w_{i/m}}}{\sum_{i=1}^m \mathrm{e}^{\lambda w_{i/m+1}}}\Big)\Bigg]\\ &+ \lambda\sum_{i=1}^{m}\gamma(i/m)\mathbb{E}\Bigg[\frac{\mathrm{e}^{\lambda w_{k/m}+\lambda w_{i/m}}}{(\sum_{i=1}^m \mathrm{e}^{\lambda w_{i/m}})^2}\Bigg]\\ &- \lambda\sum_{i=1}^{m}\gamma(1 + i/m)\mathbb{E}\Bigg[\frac{\mathrm{e}^{\lambda w_{k/m}+\lambda w_{i/m+1}}}{(\sum_{i=1}^m \mathrm{e}^{\lambda w_{i/m}})(\sum_{i=1}^m \mathrm{e}^{\lambda w_{i/m+1}})}\Bigg].\end{aligned}$$

Let us now state another useful property of Gaussian random vectors:

Lemma 4.6. *Let* $f : \mathbb{R}^d \to \mathbb{R}$ *and* $g : \mathbb{R}^{d'} \to \mathbb{R}$ *be increasing functions, in the sense that each function* $x_i \mapsto f(\ldots, x_i, \ldots)$ *and* $x_j \mapsto g(\ldots, x_j, \ldots)$ *is increasing, and let* $(X_1, \ldots, X_d, X_{d+1}, \ldots, X_{d+d'})$ *be a centerd Gaussian vector with covariance matrix* $\Sigma = (\sigma_{ij})$*. Then for* $1 \le i \le d < j \le d + d'$*, the function*

$$\sigma_{ij} \mapsto \mathbb{E}\Big[f(X_1, \ldots, X_d)g(X_{d+1}, \ldots, X_{d+d'})\Big]$$

is increasing.

A proof of this assertion can be found in [54]. Remark now that the following functions are increasing in the sense of Lemma 4.6:

$$(x_1, \ldots, x_m) \mapsto \log\Big(\sum_{i=1}^{m} \mathrm{e}^{\lambda x_i}\Big),$$

$$(x_1, \ldots, x_m) \mapsto \frac{\mathrm{e}^{\lambda x_k}}{\sum_{i=1}^m \mathrm{e}^{\lambda x_i}}$$

and also (since the previous function is non negative and increasing)

$$(x_1, \dots, x_m) \mapsto \frac{\mathrm{e}^{\lambda x_k + \lambda x_j}}{(\sum_{i=1}^m \mathrm{e}^{\lambda x_i})^2}.$$

Since the autocovariance function γ of the process w has been supposed to be decreasing, we obtain from Lemma 4.6

$$\mathbb{E}\Big[\frac{\sum_{i=1}^m \mathrm{e}^{\lambda w_{k/m} + \lambda w_{i/m}}}{(\sum_{i=1}^m \mathrm{e}^{\lambda w_{i/m}})^2} \log\Big(\frac{\sum_{i=1}^m \mathrm{e}^{\lambda w_{i/m}}}{\sum_{i=1}^m \mathrm{e}^{\lambda w_{i/m+1}}}\Big)\Big] > 0$$

and

$$\mathbb{E}\Big[\frac{\mathrm{e}^{\lambda w_{k/m} + \lambda w_{i/m}}}{(\sum_{i=1}^m \mathrm{e}^{\lambda w_{i/m}})^2}\Big] > \mathbb{E}\Big[\frac{\mathrm{e}^{\lambda w_{k/m} + \lambda w_{i/m+1}}}{(\sum_{i=1}^m \mathrm{e}^{\lambda w_{i/m}})(\sum_{i=1}^m \mathrm{e}^{\lambda w_{i/m+1}})}\Big].$$

This gives $\mathbb{E}\Big[\phi_m^k(w_0, w_{1/m}, \dots, w_{2-1/m})\Big] > 0$. Hence, we have proved that

$$\frac{\partial f_w}{\partial \lambda}(\lambda) \geq 0.$$

So as to show (4.6), let us remark that following exactly the same steps as above, we have

$$\frac{\partial f_w}{\partial \lambda}(\lambda) \geq 2\mathbb{E}\Big[\frac{\int_0^1 w \exp(\lambda w_u) du}{\int_0^1 \exp(\lambda w_u) du} \log\Big(\frac{\int_0^{1/2} \exp(\lambda w_u) du}{\int_{3/2}^2 \exp(\lambda w_u) du}\Big)\Big],$$

and

$$\mathbb{E}\Big[\frac{\int_0^1 w \exp(\lambda w_u) du}{\int_0^1 \exp(\lambda w_u) du} \log\Big(\frac{\int_0^{1/2} \exp(\lambda w_u) du}{\int_{3/2}^2 \exp(\lambda w_u) du}\Big)\Big] \geq (\gamma(1/2) - \gamma(3/2))\mathbb{E}\Big[\frac{\int_{3/2}^2 \exp(\lambda w_u) du}{\int_0^{1/2} \exp(\lambda w_u) du}\Big].$$

From our hypothesis on γ, $\gamma(1/2) - \gamma(3/2) > 0$. The result follows.

□

Remark 4.5. Hence, we have proved that this last estimator achieves the usual parametric convergence rate $n^{-1/2}$, which, to our knowledge, is a new result in the setting of MRW processes, and more generally of random cascades. However, this last estimator may not be very useful to characterize the fact that X is a random cascade. Indeed, remark that if we apply the three preceding estimation procedures to some usual, non cascade random process like fractionary Brownian motion, we expect them to converge to either 0 or $+\infty$ instead of some $\lambda^2 \in (0, 2)$. However, if the data were a (fractionary) Brownian motion, we would expect $\hat{\lambda}_n^{2,\Delta}$ to be close to some given positive finite value.

Note that we do not have explicit formulas for the function $m^2(\cdot)$. However, it is easily approximated via Monte-Carlo simulations. Figure 4.1 presents the Monte-Carlo approximation of $m^2(\lambda^2)$ through 100 000 independent simulations, and its fit with a quadratic spline.

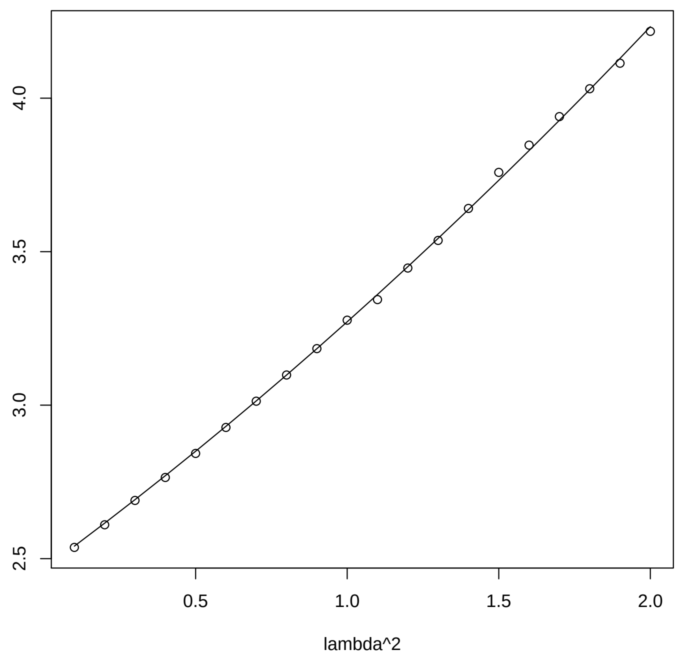

Figure 4.1: Monte-Carlo approximation for $m^2(\lambda^2)$, $0 \leq \lambda^2 \leq 2$, and fit with the quadratic function $\lambda^2 \mapsto \pi^2/4 + 0.725\lambda^2 + 0.079\lambda^4$.

4.3.3 Numerical results

We investigate here the empirical performances on the estimators of the parameter λ^2 that are defined above. As we explain below, the performances of the estimators are drastically different whether one observes an MRM or an MRW with the same parameters; or if we use the notations of the previous section, whether we observe the $d_{n,k}$'s or the values $x_{n,k} \overset{law}{=} m_{n,k}/2 + \log(|\varepsilon_{n,k}|)$, where the $\varepsilon_{n,k}$'s are a standard Gaussian white noise independent of the $(m_{n,k})$ series. While the estimators described above are consistent in both cases, with convergence rates that are the same, it is obvious that the second case (that is, the observation of an MRW) is the harder case since the white noise ε contains no statistical information on the parameter λ^2.

Whether the data generating process is an MRM or an MRW, we selected three possible values for the number of data : $n = 1\ 024$, $n = 32\ 768$ or $n = 1\ 048\ 756$, and also two values for the true parameter λ^2: a small value $\lambda^2 = 0.1$ which is for instance recommended by Bacry *et al.* for modelling financial data [10], and a quite larger value $\lambda^2 = 0.7$. Indeed, we expect for instance that the larger λ^2 is, the worse the estimator $\lambda_n^{2,p}$ performs.

The value we report for the estimator $\hat{\lambda}_n^{2,p}$ is actually an average for different values of p between 0 and 2: indeed, though we provide theoretical results only for even integer values of $p \notin \{0, 2\}$, numerical experiments show that the best performances are attained when averaging several estimates for non integer p between 0 and 2. Similarly, the estimator $\hat{\lambda}_n^{2,cov}(h, h')$ was averaged by regressing the empirical covariance over 15 lags $h = 1$ to $h = 12$, and the estimator $\hat{\lambda}_{J,j}^{2,var}$ was averaged by regressing the empirical variance over the five finest available scales j. The simulations were done using the scheme described for instance in Kozhemyak [60].

Case of the observation of an MRM process

Table 4.1 contains the empirical means and standard deviations of the estimators over 100 realization of the MRM process, while Figures 4.2 to 4.7 present the empirical distributions of the estimators. The $\hat{\lambda}_n^{2,\Delta}$ estimator performs clearly the best, and achieves the smallest

$\lambda^2 = 0.1$

	$n = 1\ 024$		$n = 32\ 768$		$n = 1\ 048\ 576$	
	Bias	St. dev.	Bias	St. dev.	Bias	St. dev.
$\hat{\lambda}_n^{2,p}$	**-1.4e-4**	6.7e-3	-1.4e-3	1.2e-3	-2.4e-3	2.4e-4
$\hat{\lambda}_n^{2,cov}$	-4.7e-4	1.5e-2	9.9e-5	2.7e-3	4.9e-5	4.3e-4
$\hat{\lambda}_n^{2,var}$	3.0e-3	8.6e-3	-5.6e-4	1.5e-3	-1.2e-3	2.9e-4
$\hat{\lambda}_n^{2,\Delta}$	-4.3e-4	**4.5e-3**	**8.6e-5**	**8.3e-4**	**-2.9e-5**	**1.5e-4**

$\lambda^2 = 0.7$

	$n = 1\ 024$		$n = 32\ 768$		$n = 1\ 048\ 576$	
	Bias	St. dev.	Bias	St. dev.	Bias	St. dev.
$\hat{\lambda}_n^{2,p}$	-1.7e-2	6.2e-2	-3.8e-2	2.1e-2	-5.5e-2	6.9e-3
$\hat{\lambda}_n^{2,cov}$	2.4e-2	1.2e-1	**5.8e-4**	1.7e-2	2.1e-3	3.1e-3
$\hat{\lambda}_n^{2,var}$	2.2e-2	8.8e-2	-1.9e-2	1.9e-2	-2.5e-2	3.8e-3
$\hat{\lambda}_n^{2,\Delta}$	**3.8e-3**	**3.3e-2**	-8.7e-4	**6.7e-3**	**-3.0e-5**	**9.9e-4**

Table 4.1: Simulation results when MRM processes are simulated : empirical mean and empirical standard deviations of the estimators over 100 realizations. Bolded values correspond to the smallest bias or standard deviation.

bias and variance in almost each configuration, eventhough all estimators behave relatively well. Note the increase in variance for all estimators when the true value of λ^2 increases; this is especially true for the estimator $\hat{\lambda}_n^{2,p}$. This drop in performance is in agreement with the theoretical result stated in Theorem 4.2.

Case of the observation of an MRW process

Table 4.2 shows that the estimator $\hat{\lambda}_n^{2,cov}$ performs the best of all four estimators when one observes an MRW process. Indeed, this estimator is based on the covariance structure of the $m_{n,k}$'s, so that it behaves roughly the same whether some white noise is added to the $m_{n,k}$'s (which is the case in the MRW setting) or not (MRM setting). This contrasts with the behaviour of the other estimators which all behave far more poorly in the MRW setting than in the MRM setting: depending on the configuration, the standard estimation of the estimator may be multiplied by a factor larger than 10. This can be explained in the following way: in the case of of the estimators $\hat{\lambda}_n^{2,var}$ and $\hat{\lambda}_n^{2,\Delta}$, the procedure is based upon empirical variances of the $x_{n,k}$'s to which a fairly large contribution is given by

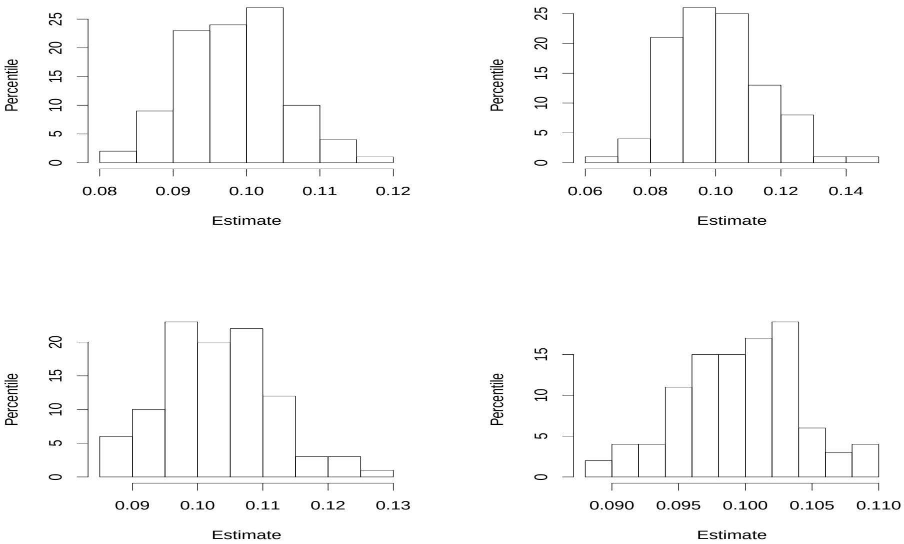

Figure 4.2: Empirical distribution of the estimators over 100 independent simulation of an MRM process; $\lambda^2 = 0.1$; $n = 1\ 024$. Top left corner: $\hat{\lambda}_n^{2,p}$, top right corner: $\hat{\lambda}_n^{2,cov}$; bottom left corner: $\hat{\lambda}_n^{2,var}$; bottom right corner: $\hat{\lambda}_n^{2,\Delta}$.

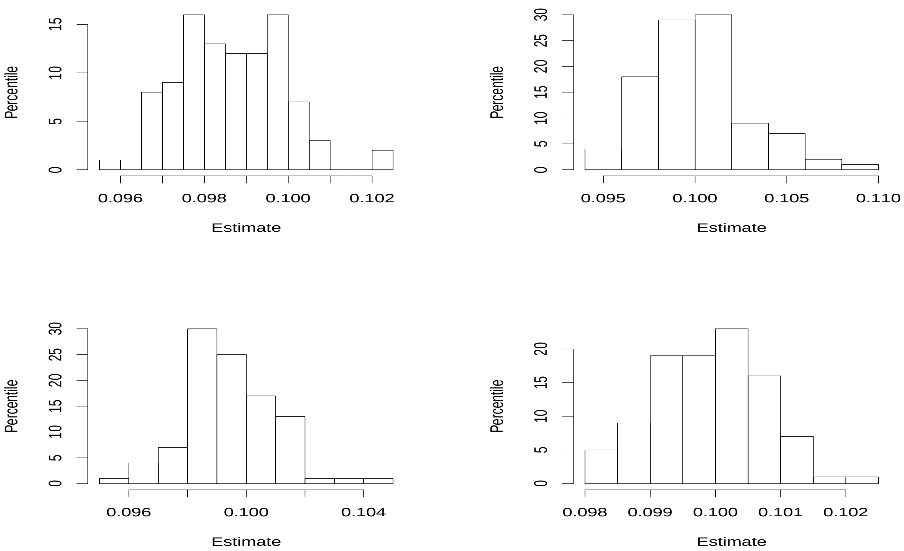

Figure 4.3: Empirical distribution of the estimators over 100 independent simulation of an MRM process; $\lambda^2 = 0.1$; $n = 32\ 768$. Top left corner: $\hat{\lambda}_n^{2,p}$, top right corner: $\hat{\lambda}_n^{2,cov}$; bottom left corner: $\hat{\lambda}_n^{2,var}$; bottom right corner: $\hat{\lambda}_n^{2,\Delta}$.

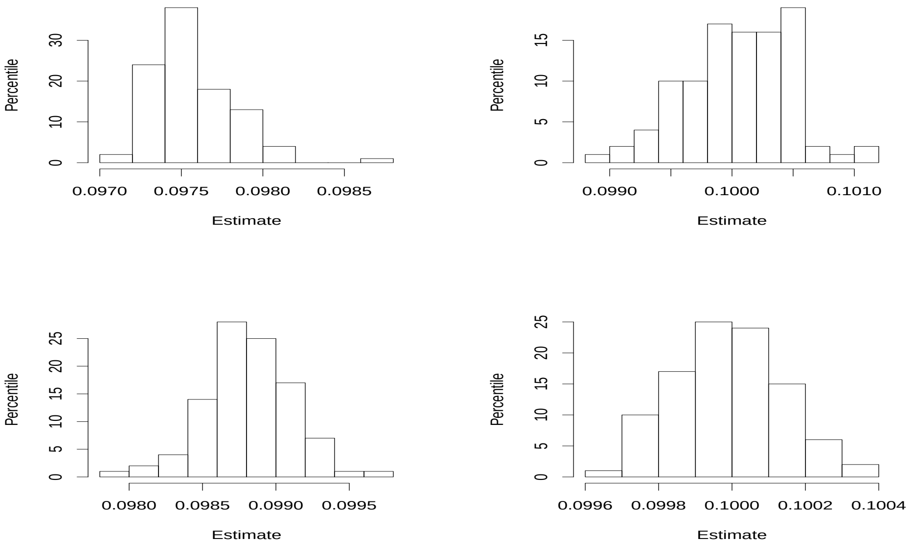

Figure 4.4: Empirical distribution of the estimators over 100 independent simulation of an MRM process; $\lambda^2 = 0.1$; $n = 1\ 048\ 576$. Top left corner: $\hat{\lambda}_n^{2,p}$, top right corner: $\hat{\lambda}_n^{2,cov}$; bottom left corner: $\hat{\lambda}_n^{2,var}$; bottom right corner: $\hat{\lambda}_n^{2,\Delta}$.

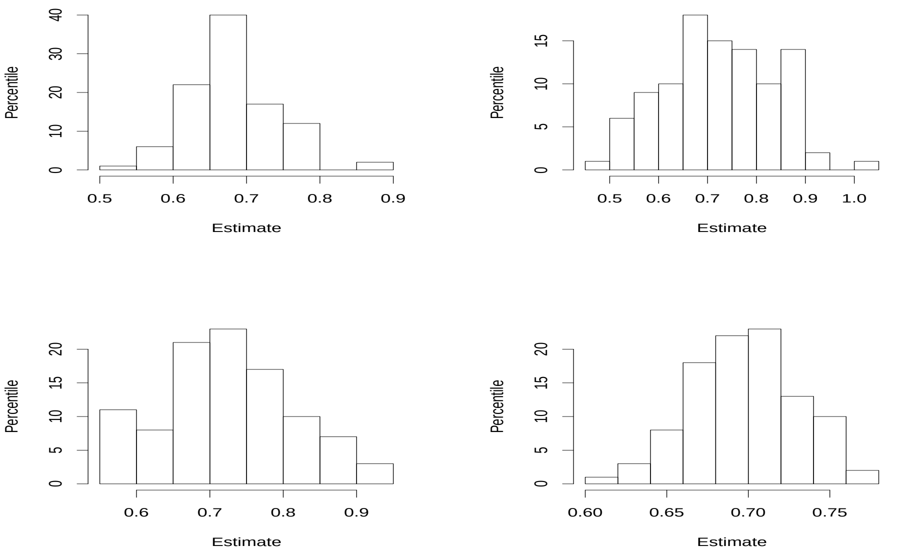

Figure 4.5: Empirical distribution of the estimators over 100 independent simulation of an MRM process; $\lambda^2 = 0.7$; $n = 1\ 024$. Top left corner: $\hat{\lambda}_n^{2,p}$, top right corner: $\hat{\lambda}_n^{2,cov}$; bottom left corner: $\hat{\lambda}_n^{2,var}$; bottom right corner: $\hat{\lambda}_n^{2,\Delta}$.

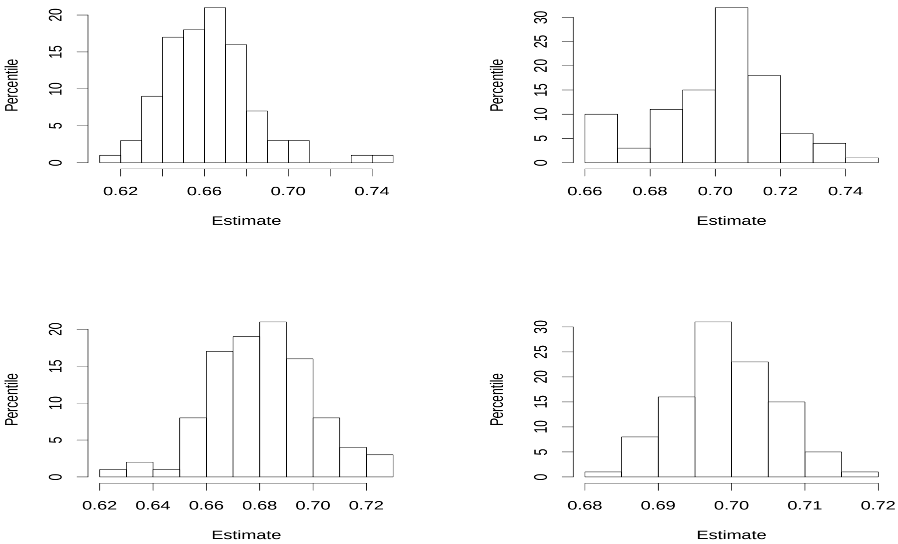

Figure 4.6: Empirical distribution of the estimators over 100 independent simulation of an MRM process; $\lambda^2 = 0.7$; $n = 32\ 768$. Top left corner: $\hat{\lambda}_n^{2,p}$, top right corner: $\hat{\lambda}_n^{2,cov}$; bottom left corner: $\hat{\lambda}_n^{2,var}$; bottom right corner: $\hat{\lambda}_n^{2,\Delta}$.

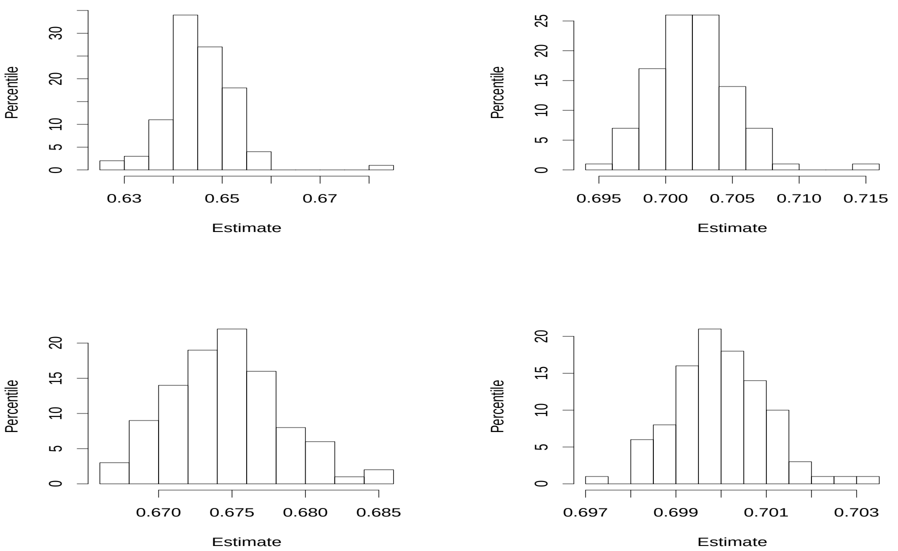

Figure 4.7: Empirical distribution of the estimators over 100 independent simulation of an MRM process; $\lambda^2 = 0.7$; $n = 1\ 048\ 576$. Top left corner: $\hat{\lambda}_n^{2,p}$, top right corner: $\hat{\lambda}_n^{2,cov}$; bottom left corner: $\hat{\lambda}_n^{2,var}$; bottom right corner: $\hat{\lambda}_n^{2,\Delta}$.

the empirical variance of the $\log(|\varepsilon_{n,k}|)$'s, where $(\varepsilon_{n,k})$ is a standard Gaussian white noise that does not depend on λ^2. When λ^2 grows, the variance of $\log(|\varepsilon_{n,k}|)$ becomes relatively smaller, so that the performances of these estimators improve, to point where $\hat{\lambda}_n^{2,\Delta}$ become almost as good as $\hat{\lambda}_n^{2,cov}$. Finally, note that under any configuration of our simulations, the estimator $\hat{\lambda}_n^{2,p}$ performs extremely poorly when the observations are generated by an MRW.

$\lambda^2 = 0.1$

	$n = 1\ 024$		$n = 32\ 768$		$n = 1\ 048\ 576$	
	Bias	St. dev.	Bias	St. dev.	Bias	St. dev.
$\hat{\lambda}_n^{2,p}$	1.3e-2	5.1e-1	7.4e-3	8.2e-2	-3.2e-2	1.7e-2
$\hat{\lambda}_n^{2,cov}$	**-1.5e-3**	**7.2e-2**	**7.9e-5**	**1.1e-2**	1.4e-3	**1.8e-3**
$\hat{\lambda}_n^{2,var}$	-4.6e-2	5.0e-1	2.3e-2	9.3e-2	-1.7e-3	1.5e-2
$\hat{\lambda}_n^{2,\Delta}$	-2.1e-2	2.6e-1	8.5e-3	4.7e-2	**1.8e-4**	9.0e-3

$\lambda^2 = 0.7$

	$n = 1\ 024$		$n = 32\ 768$		$n = 1\ 048\ 576$	
	Bias	St. dev.	Bias	St. dev.	Bias	St. dev.
$\hat{\lambda}_n^{2,p}$	-9.3e-2	8.4e-1	-3.0e-2	3.0e-1	-8.3e-2	1.2e-1
$\hat{\lambda}_n^{2,cov}$	**6.8e-3**	**1.6e-1**	**2.0e-3**	**2.8e-2**	3.6e-3	**4.6e-3**
$\hat{\lambda}_n^{2,var}$	6.6e-2	5.0e-1	-3.2e-2	1.1e-1	-2.4e-2	2.6e-2
$\hat{\lambda}_n^{2,\Delta}$	-9.0e-3	2.6e-1	-7.6e-3	4.5e-2	**-1.1e-3**	7.7e-3

Table 4.2: Simulation results when MRW processes are simulated : empirical mean and empirical standard deviations of the estimators over 100 realizations. Bolded values correspond to the smallest bias or standard deviation.

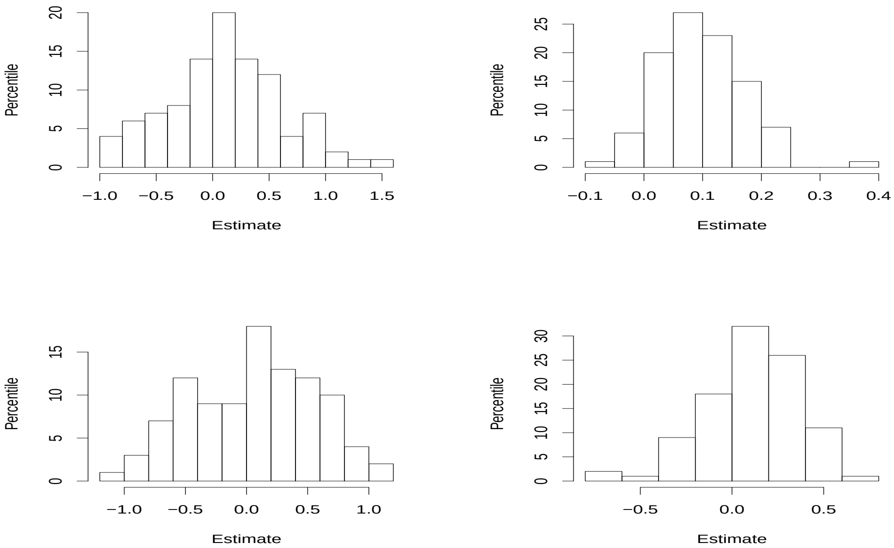

Figure 4.8: Empirical distribution of the estimators over 100 independent simulation of an MRW process; $\lambda^2 = 0.1$; $n = 1\ 024$. Top left corner: $\hat{\lambda}_n^{2,p}$, top right corner: $\hat{\lambda}_n^{2,cov}$; bottom left corner: $\hat{\lambda}_n^{2,var}$; bottom right corner: $\hat{\lambda}_n^{2,\Delta}$.

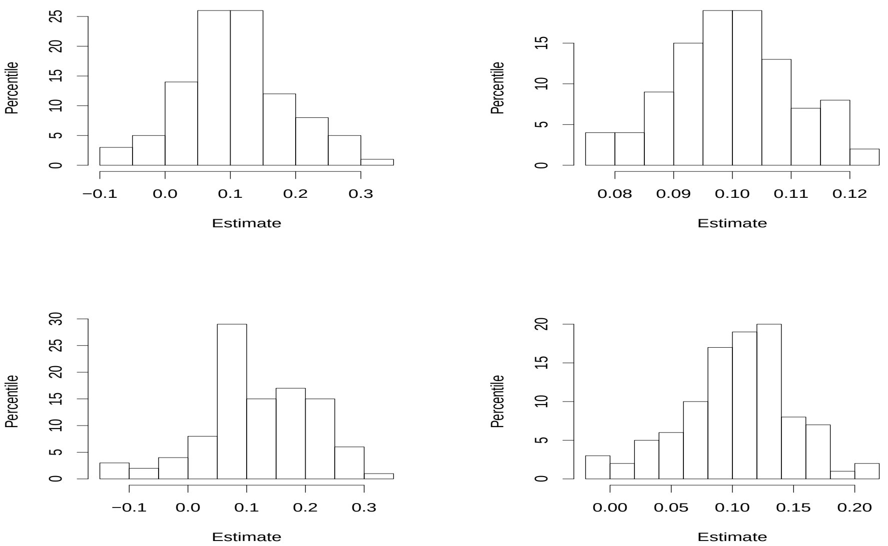

Figure 4.9: Empirical distribution of the estimators over 100 independent simulation of an MRW process; $\lambda^2 = 0.1$; $n = 32\ 768$. Top left corner: $\hat{\lambda}_n^{2,p}$, top right corner: $\hat{\lambda}_n^{2,cov}$; bottom left corner: $\hat{\lambda}_n^{2,var}$; bottom right corner: $\hat{\lambda}_n^{2,\Delta}$.

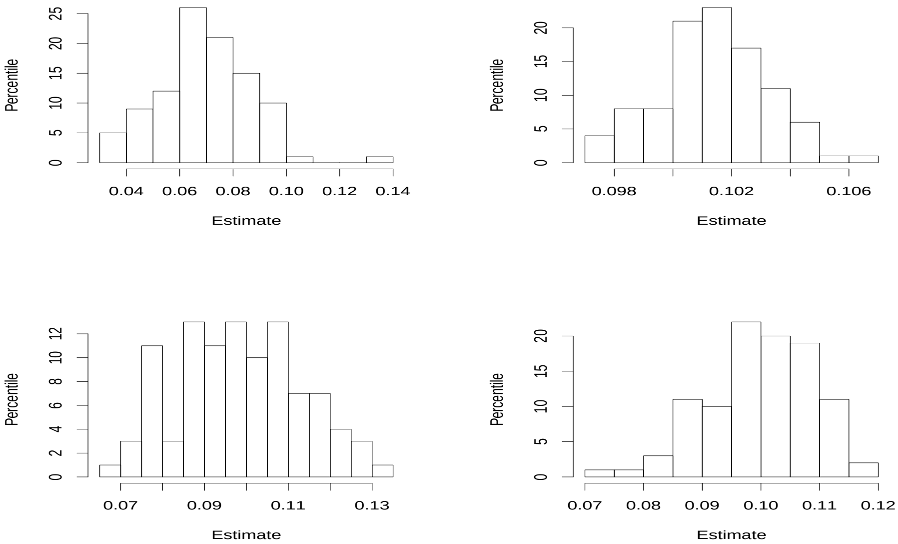

Figure 4.10: Empirical distribution of the estimators over 100 independent simulation of an MRW process; $\lambda^2 = 0.1$; $n = 1\ 048\ 576$. Top left corner: $\hat{\lambda}_n^{2,p}$, top right corner: $\hat{\lambda}_n^{2,cov}$; bottom left corner: $\hat{\lambda}_n^{2,var}$; bottom right corner: $\hat{\lambda}_n^{2,\Delta}$.

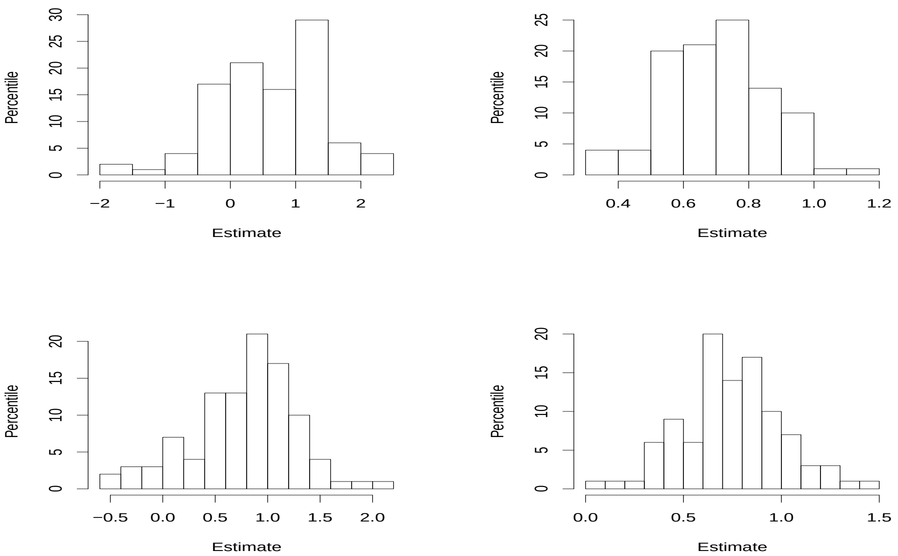

Figure 4.11: Empirical distribution of the estimators over 100 independent simulation of an MRW process; $\lambda^2 = 0.7$; $n = 1\ 024$. Top left corner: $\hat{\lambda}_n^{2,p}$, top right corner: $\hat{\lambda}_n^{2,cov}$; bottom left corner: $\hat{\lambda}_n^{2,var}$; bottom right corner: $\hat{\lambda}_n^{2,\Delta}$.

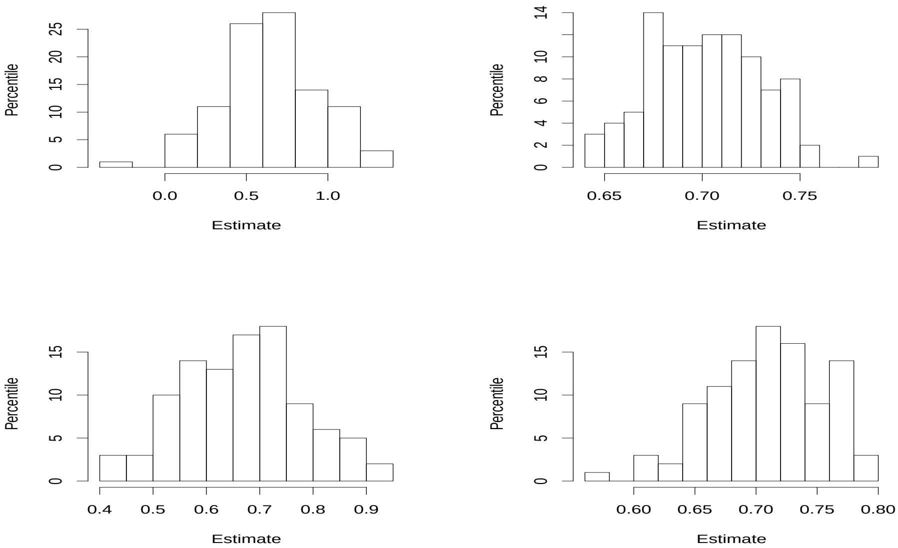

Figure 4.12: Empirical distribution of the estimators over 100 independent simulation of an MRW process; $\lambda^2 = 0.7$; $n = 32\ 768$. Top left corner: $\hat{\lambda}_n^{2,p}$, top right corner: $\hat{\lambda}_n^{2,cov}$; bottom left corner: $\hat{\lambda}_n^{2,var}$; bottom right corner: $\hat{\lambda}_n^{2,\Delta}$.

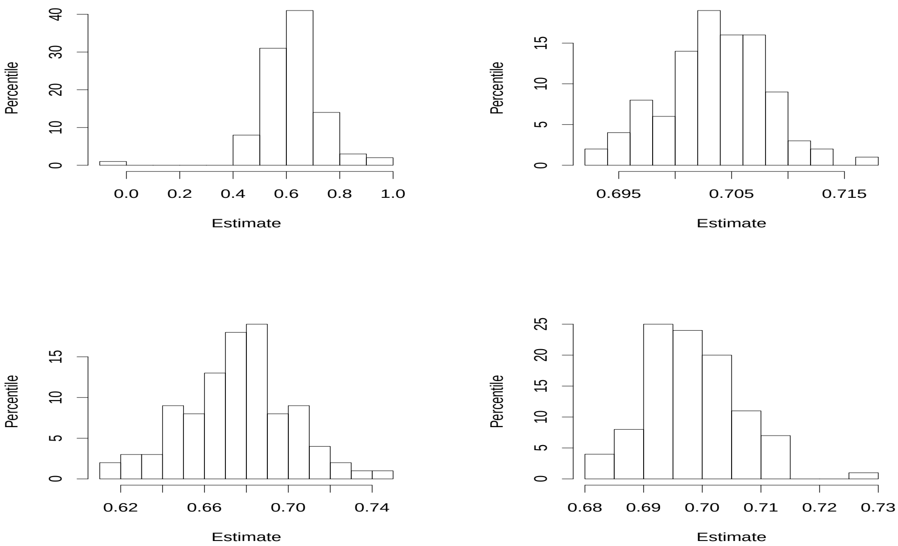

Figure 4.13: Empirical distribution of the estimators over 100 independent simulation of an MRW process; $\lambda^2 = 0.7$; $n = 1\ 048\ 576$. Top left corner: $\hat{\lambda}_n^{2,p}$, top right corner: $\hat{\lambda}_n^{2,cov}$; bottom left corner: $\hat{\lambda}_n^{2,var}$; bottom right corner: $\hat{\lambda}_n^{2,\Delta}$.

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