(Super)symmetries of semi-classical models in theoretical and condensed matter physics

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Résumé

L'algorithme covariant de van Holten, servant à construire des quantités conservées, est présenté avec une attention particulière portée sur les vecteurs de type Runge-Lenz. La dynamique classique des particules portant des charges isospins est passée en revue. Plusieures applications physiques sont considerées. Des champs de type monopôles non-Abéliens, générés par des mouvements nucléaires dans les molécules diatomiques, introduites par Moody, Shapere et Wilczek, sont étudiées. Dans le cas des espaces courbes, le formalisme de van Holten permet de décrire la symétrie dynamique des monopôles Kaluza-Klein généralisés. La procédure est étendue à la supersymétrie et appliquée aux monopôles supersymétriques. Une autre application, concernant l'oscillateur non-commutatif en dimension trois, est également traitée.

Abstract

Van Holten's covariant algorithm for deriving conserved quantities is presented, with particular attention paid to Runge-Lenz-type vectors. The classical dynamics of isospin-carrying particles is reviewed. Physical applications including non-Abelian monopole-type systems in diatoms, introduced by Moody, Shapere and Wilczek, are considered. Applied to curved space, the formalism of van Holten allows us to describe the dynamical symmetries of generalized Kaluza-Klein monopoles. The framework is extended to supersymmetry and applied to the SUSY of the monopoles. Yet another application concerns the three-dimensional non-commutative oscillator.

Contents

Vue d'ensemble sur les (Super)symétries dans les modèles semi-classiques

Symétries et quantités conservées en théorie non-abélienne.

Cette thèse de doctorat s'articule autour de la description des symétries et quantités conservées associées au mouvement des particules à isospin en interaction avec des champs de type Yang-Mills-Higgs.

Dans le contexte des théories Kaluza-Klein (KK), les équations de champs classiques décrivant l'évolution des particules à isospin sont obtenues suivant l'approche de Kerner [Kerner 1968]. En effet, cette approche consiste à généraliser la théorie KK à 5 dimensions ($\mathcal{M}^4 \otimes \mathcal{S}^1$) en une théorie à 7 dimensions (7D). Dans la théorie étendue, l'espace total noté, $\mathcal{M} = \mathcal{M}^4 \otimes \mathcal{S}^3$, est composé d'un espace-temps de base $\mathcal{M}^4$ dont les coordonnées s'écrivent x^μ, $\mu = 0, \cdots, 3$; et d'un espace supplémentaire, noté $\mathcal{S}^3$, de coordonnées localement géodésiques y^a, $a, b = 4, \cdots, 6$. $\mathcal{S}^3$, qui désigne une 3-sphère, a pour générateurs d'isométries,

$$\Xi_j = -i\xi_j^b(y)\partial_b \,, \tag{1}$$

reproduisant l'algèbre de Lie $su(2)$,

$$[\Xi_j, \Xi_k] = i\,\varepsilon^l_{\ jk}\,\Xi_l \,. \tag{2}$$

Par conséquent, $\mathcal{S}^3$ vue comme un groupe de Lie est isomorphe au groupe non-abélien $SU(2)$, avec ε^l_{jk} dénotant la constante de structure de $SU(2)$.

La métrique 7D généralisée de la théorie KK s'écrit,

$$\tilde{g}_{CD} = \begin{pmatrix} \gamma_{\mu\nu} + \kappa_{ab}\,A^c_\mu\,A^d_\nu\,\xi^a_c\,\xi^b_d & A^c_\mu\,\xi^b_c\,\kappa_{ba} \\ \kappa_{ab}\,A^c_\nu\,\xi^a_c & \kappa_{ab} \end{pmatrix}, \quad C, D = 0, \cdots, 6\,, \tag{3}$$

où κ_{ab} est une métrique $SU(2)$ invariante. La 1-form, A^b_μ , de l'algèbre de Lie $SU(2)$ est un champ de type Yang-Mills se transformant comme un champ de jauge non-abelien,

$$\tilde{A}^a_\mu = A^a_\mu(x) - \partial_\mu f^a + \varepsilon^a_{bc}A^b_\mu f^c = A^a_\mu(x) - D_\mu f^a \,, \tag{4}$$

où

$$D_\mu f^a = \partial_\mu f^a - \varepsilon^a_{bc}A^b_\mu f^c \tag{5}$$

est la dérivée covariante de jauge. Le champ de force du potential A^b_μ,

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - \varepsilon^a_{bc} A^b_\mu A^c_\nu \,. \tag{6}$$

se transforme selon

$$\tilde{F}^a_{\mu\nu} = F^a_{\mu\nu} - \varepsilon^a{}_{bc} f^b F^c_{\mu\nu} \,. \tag{7}$$

On concentre maintenant notre attention sur la réduction de la dynamique, d'une particule-test ponctuelle de masse unitaire, provenant du Lagrangien décrivant le mouvement géodésique dans l'espace total [Montgomery],

$$\mathcal{L} = \gamma_{\mu\nu} \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} + \kappa_{ab}\Big(\frac{dy^a}{d\tau} + A^a_\mu \frac{dx^\mu}{d\tau}\Big)\Big(\frac{dy^b}{d\tau} + A^b_\mu \frac{dx^\mu}{d\tau}\Big) \,. \tag{8}$$

Un calcul des équations d'Euler-Lagrange associées,

$$\begin{cases} \dfrac{d}{d\tau}\left(\dfrac{\partial \mathcal{L}}{\partial\big(\frac{dx^\alpha}{d\tau}\big)}\right) - \dfrac{\partial \mathcal{L}}{\partial x^\alpha} = 0\,, \quad \alpha = 0, \cdots, 3 \\ \dfrac{d}{d\tau}\left(\dfrac{\partial \mathcal{L}}{\partial\big(\frac{dy^c}{d\tau}\big)}\right) - \dfrac{\partial \mathcal{L}}{\partial y^c} = 0\,, \quad c = 4, 5, 6\,, \end{cases} \tag{9}$$

nous donne

$$\begin{cases} \dfrac{d^2 x^\beta}{d\tau^2} + \Gamma^\beta_{\mu\nu} \dfrac{dx^\mu}{d\tau}\dfrac{dx^\nu}{d\tau} + \gamma^{\nu\beta} F^b_{\mu\nu} \mathcal{I}_b \dfrac{dx^\mu}{d\tau} = 0\,, \\ \dfrac{d\mathcal{I}_c}{d\tau} - \mathcal{I}_a \varepsilon^a{}_{bc} A^b_\mu \dfrac{dx^\mu}{d\tau} = 0\,, \quad \mathcal{I}_a = \kappa_{ab}\left(\dfrac{dy^b}{d\tau} + A^b_\nu \dfrac{dx^\nu}{d\tau}\right)\,, \end{cases} \tag{10}$$

où $\mathcal{I}_a$ représente la variable isospin classique. La première équation dans (10) décrit le mouvement projeté sur l'espace-temps 4D et contient la force de Lorentz généralisée (la charge électrique est ici remplacée par l'isospin $\mathcal{I}^a$),

$$\gamma^{\nu\beta} F^b_{\mu\nu} \mathcal{I}_b \frac{dx^\mu}{d\tau} \,. \tag{11}$$

La seconde équation dans (10), connue sous le nom d'équation de Kerner-Wong, implique que l'isospin subit un transport parallèle dans l'espace interne. La dérivation de la charge isospin est analogue à celle de la charge électrique de la théorie KK 5D. En effet, les deux charges résultent de la contraction du vecteur de Killing générant les translations "verticales" avec le champ de direction de la géodésique.

On peut remarquer que quelques temps après Kerner, Wong [Wong 1970] obtint la même equation en "déquantifiant" l'équation de Dirac. Plus tard, Balachandran et al. [Balachandran 1977] déduisirent aussi les équations (2.28) en utilisant un principe variationnel. Alternativement, une approche symplectique peut également reproduire les mêmes résultats [Duval 1978, Duval 1982, Fehér 1986*].

Après avoir déterminé l'isospin classique d'une particule dans un champ de jauge non-abélien, on peut maintenant étudier les symétries et les quantités conservées associées. L'approche standard pour identifier les constantes du mouvement est atteinte par le théorème de Noether [Forgács-Manton 1980, Jackiw-Manton 1980].

Cependant, plus récemment, une approche alternative a été avancée par van Holten [van Holten 2007]. Pour la voir, on considère une particule non-relativiste portant une charge isospin dans un espace à trois dimensions. L'Hamiltonien du système s'écrit

$$\mathcal{H} = \frac{\vec{\pi}^2}{2} + V(\vec{x}, \mathcal{I}^a)\,, \quad \vec{\pi} = \vec{p} - e\vec{A}\,, \tag{12}$$

avec $\vec{p}$ et $\vec{\pi}$ définissant, respectivement, le moment canonique et le moment covariant de jauge. V est un potentiel scalaire supplémentaire tandis que $\vec{A} = \vec{A}^a\,\mathcal{I}^a$ est un potentiel de jauge non-abélien dont les indices internes $a = 1, 2, 3$ font référence à l'algèbre de Lie su(2).

En identifiant l'algèbre de Lie su(2) de la variable non-abélienne avec $\mathbb{R}^3$, on peut considérer l'espace des phases covariant $\big(\vec{x}, \vec{\pi}, \vec{\mathcal{I}}\,\big)$ où la dynamique du système

$$\dot{f} = \{f, H\}\,,$$

est définie par les crochets de Poisson covariants sur $M = T^*\mathbb{R}^3 \otimes su(2)^\star$,

$$\{f,g\} = D_j f \frac{\partial g}{\partial \pi_j} - \frac{\partial f}{\partial \pi_j} D_j g + e\,\mathcal{I}^a F^a_{jk} \frac{\partial f}{\partial \pi_j}\frac{\partial g}{\partial \pi_k} - \epsilon_{abc}\frac{\partial f}{\partial \mathcal{I}^a}\frac{\partial g}{\partial \mathcal{I}^b}\mathcal{I}^c\,. \tag{13}$$

Le champ de force et la dérivée covariante s'écrivent respectivement,

$$\begin{aligned} F_{jk} &= \partial_j A_k - \partial_k A_j - e\,\epsilon_{abc}\mathcal{I}^a A^b_j A^c_k\,,\\ D_j &= \partial_j - e\,\epsilon_{abc}\mathcal{I}^a A^b_j \frac{\partial}{\partial \mathcal{I}_c}\,, \end{aligned} \tag{14}$$

et le commutateur des dérivées covariantes est donné par

$$[D_i, D_j] = -\epsilon_{abc}\mathcal{I}^a F^b_{ij}\,\frac{\partial}{\partial \mathcal{I}^c}\,. \tag{15}$$

Un calcul direct nous donne les crochets de Poisson non-nuls,

$$\{x_i, \pi_j\} = \delta_{ij}, \qquad \{\pi_i, \pi_j\} = e\,\mathcal{I}^a F^a_{ij}, \qquad \{\mathcal{I}^a, \mathcal{I}^b\} = -\epsilon_{abc}\mathcal{I}^c. \tag{16}$$

On remarque que l'identité de Jacobi implique l'équation de champ électromagnétique,

$$\{\pi_i, \{\pi_j, \pi_k\}\} + \{\pi_j, \{\pi_k, \pi_i\}\} + \{\pi_k, \{\pi_i, \pi_j\}\} = 0 \quad \Leftrightarrow \quad D_i\big(\mathcal{I}^a F^a_{ij}\big) = 0\,. \tag{17}$$

On peut ainsi déduire les équations du mouvement de Kerner-Wong [cf. (10)],

$$\begin{cases} \dfrac{d^2x_i}{dt^2} - e\,\mathcal{I}^a F^a_{ij}\dfrac{dx^j}{dt} + D_iV & = 0\,, \\ \dfrac{d\mathcal{I}^a}{dt} - \epsilon_{abc}\mathcal{I}^b\left(\dfrac{\partial V}{\partial \mathcal{I}^c} - e\,A^c_j\,\dfrac{dx^j}{dt}\right) & = 0\,, \end{cases} \tag{18}$$

où V est un potentiel scalaire indépendant du moment.

Pour construire des quantités dynamiques $\mathcal{Q}\big(\vec{x},\vec{\pi},\vec{\mathcal{I}}\big)$ qui sont conservées au cours du mouvement, on utilise la procédure covariante de van Holten [van Holten 2007]. L'idée, ici, est de faire un développement de $\mathcal{Q}$ en série de puissance du moment covariant,

$$\mathcal{Q}\big(\vec{x},\vec{\pi},\vec{\mathcal{I}}\big) = C(\vec{x},\vec{\mathcal{I}}) + C_i(\vec{x},\vec{\mathcal{I}})\pi_i + \frac{1}{2!}C_{ij}(\vec{x},\vec{\mathcal{I}})\pi_i\pi_j + \tag{19}$$

En exigeant que le crochet de Poisson de $\mathcal{Q}$ et du Hamiltonien soit nul,

$$\{\mathcal{Q},\mathcal{H}\} = 0\,, \tag{20}$$

implique la série infinie de contraintes,

$$\begin{array}{lr} C_iD_iV + \epsilon_{abc}\mathcal{I}^a\dfrac{\partial C}{\partial \mathcal{I}^b}\dfrac{\partial V}{\partial \mathcal{I}^c} = 0, & o(0) \\ D_iC = e\mathcal{I}^aF^a_{ij}C_j + C_{ij}D_jV + \epsilon_{abc}\mathcal{I}^a\dfrac{\partial C_i}{\partial \mathcal{I}^b}\dfrac{\partial V}{\partial \mathcal{I}^c}, & o(1) \\ D_iC_j + D_jC_i = e\mathcal{I}^a(F^a_{ik}C_{kj} + F^a_{jk}C_{ki}) + C_{ijk}D_kV + \epsilon_{abc}\mathcal{I}^a\dfrac{\partial C_{ij}}{\partial \mathcal{I}^b}\dfrac{\partial V}{\partial \mathcal{I}^c}, & o(2) \\ D_iC_{jk} + D_jC_{ki} + D_kC_{ij} = e\mathcal{I}^a(F^a_{il}C_{ljk} + F^a_{jl}C_{lki} + F^a_{kl}C_{lij}) + C_{ijkl}D_lV & \\ \qquad\qquad + \epsilon_{abc}\mathcal{I}^a\dfrac{\partial C_{ijk}}{\partial \mathcal{I}^b}\dfrac{\partial V}{\partial \mathcal{I}^c}, & o(3) \\ \vdots \qquad\qquad \vdots & \vdots \end{array} \tag{21}$$

Cependant, pour des quantités conservées admettant une expansion finie en moments covariants, la série de contraintes (21) peut être tronquée à l'ordre fini (n+*1*) pourvu que l'on soit en train de chercher des constantes du mouvement d'ordre-n. Par conséquent, on prend $C_{i_1\ldots i_{n+1}\ldots} = 0$, telle que la contrainte d'ordre supérieur de (21) devient une équation de Killing,

$$D_{(i_1}C_{i_2\ldots i_{n+1})} = 0\,, \quad n \in \mathbb{N}^\star. \tag{22}$$

On peut noter qu'à part les constantes du mouvement d'ordre zéro, c'est à dire, qui ne dépendent pas du moment covariant, tous les autres invariants d'ordre-n, $n \geq 1$, sont construits à partir de la méthode systématique (21) impliquant des tenseurs de Killing de rang-n. Chaque tenseur de Killing est solution de l'équation (22) et représente également le coefficient de plus haut rang de l'expansion (4.164) et par conséquent génère la quantité conservée. Les contraintes d'ordre intermédiaire de (21) determinent les autres termes-coefficients de l'invariant tandis que l'équation à l'ordre zéro peut être interprétée comme étant une condition de cohérence entre le potentiel scalaire et la quantité conservée ainsi construite.

• Pour $n = 1$, (22) nous donne des vecteurs de Killing. Par exemple, pour un vecteur unitaire $\vec{n}$, on obtient le générateur des rotations autour de l'axe $\vec{n}$,

$$\vec{C} = \vec{n} \times \vec{x} \tag{23}$$

associé au moment angulaire conservé.

Le générateur des translations spatiales le long de l'axe $\vec{n}$,

$$\vec{C} = \alpha\vec{n}\,, \quad \alpha \in \mathbb{R}\,, \tag{24}$$

entraîne une quantité conservée identifiée en tant que "translation magnétique".

• Pour $n = 2$, alors (22) donne des tenseurs de Killing de rang-2 . De façon similaire, pour un vecteur unitaire quelconque $\vec{n}$,

$$C_{ij} = 2\delta_{ij}\,\vec{n}\cdot\vec{x} - (n_i x_j + n_j x_i) \tag{25}$$

est un tenseur de Killing de rang-2 associé au vecteur conservé de type Laplace-Runge-Lenz.

Le tenseur de rang-2 impliquant la conservation de l'énergie s'écrit

$$C_{ij} = \delta_{ij}\,. \tag{26}$$

Le tenseur de Killing de rang-2 constant, générant les symétries oscillateur $SU(3)$, est donné par

$$C_{ij} = \alpha_{ij}\,, \quad \alpha_{ij} = \text{const}\,. \tag{27}$$

• Pour $n \geq 3$, l'équation (22) nous donne des tenseurs de Killing de rang supérieur, lesquels en général, génèrent des produits de constantes du mouvement déjà connues.

Il est à signaler que nous venons de discuter de la procédure de van Holten (21) adaptée à la recherche de symétries en espace plat. On peut convenablement étendre l'algorithme à des espaces courbes pourvu que l'on remplace la dérivée partielle par une dérivée covariante de la métrique, $\partial_i \to \nabla_i$.

Notre construction (21) est une méthode alternative à celle de Forgács-Jackiw-Manton [Forgács-Manton 1980, Jackiw-Manton 1980]. Dans cette approche basée sur l'étude des champs de jauge symétriques [Forgács-Manton 1980]. On obtient la condition de symétrie équivalente impliquant la contribution du champ de jauge

$$F_{\beta\mu}\,\omega^\mu = D_\beta C^\omega_{gauge} \quad \text{avec} \quad C^\omega_{gauge} = \omega^\mu A_\mu - Q^\omega\,. \tag{28}$$

Le terme ω^μ représente le vecteur de Killing générant la transformation de symétrie, tandis que la contribution du champ de jauge, C^{ω}_{gauge} , est une fonction scalaire différentiable que l'on détermine par une intégration de l'équation (28).

Le statut physique du terme C^{ω}_{gauge} est maintenant clair. En effet, il représente la réponse du champ (symétrique) externe à un difféomorphisme de l'espace-temps. Il restore la symétrie du système et apparaît comme étant un champ scalaire à valeur dans l'algèbre de Lie contribuant à la constante du mouvement. Ainsi, la quantité conservée totale s'écrit

$$C^{\omega} = \omega^{\nu}\pi_{\nu} + \int_{\mathcal{C}} \omega^{\alpha} F_{\alpha\beta} DX^{\beta} \,. \tag{29}$$

On remarque qu'en identifiant l'algèbre de Lie du groupe de jauge $SU(2)$ à $\mathbb{R}^3$, la dérivée covariante de jauge s'exprime comme dans (14). La règle consiste simplement à remplacer les générateurs de $SU(2)$, notés τ^a $(a = 1, 2, 3)$, par les composantes du vecteur isospin, $\mathcal{I}^a$. Sous ce changement, la condition de symétrie (28) devient précisément (sans potentiel scalaire supplémentaire) la condition à l'ordre-1 dans (21), qu'une quantité conservée linéaire en moment covariant doit satisfaire.

Ainsi, l'approche de Forgács-Jackiw-Manton est, en fait, équivalente à celle de van Holten pour des transformations de symétries de l'espace-temps. Afin de généraliser la technique FMJ à des constantes du mouvement de rang supérieur, on exige que le champ de jauge tolère des tenseurs de Killing d'ordre supérieur. Comme dans le cas des quantités conservées linéaires, les invariants peuvent être séparés en deux contributions,

$$C^{\omega} = C^{\omega}_{matter} + C^{\omega}_{gauge} \,.$$

Dans ce cas, les contributions provenant de la matière et de l'interaction matière-champ de jauge donnent naissance au terme,

$$C^{\omega}_{matter} = \frac{1}{n!} \omega^{\mu_1 \cdots \mu_n} \pi_{\mu_1} \cdots \pi_{\mu_n} \,, \tag{30}$$

où $\omega^{\mu_1 \cdots \mu_n}$ est le tenseur de Killing générant la symétrie. Le champ de jauge externe apporte, cependant, les contributions C^{ω}_{gauge} satisfaisant les contraintes,

$$D^{(\mu_1} C^{\omega,\ \mu_2 \cdots \mu_{n-1})}_{gauge} = F^{(\mu_1}_{\beta}\ \omega^{\omega,\ \mu_2 \cdots \mu_n)}_{\beta} \,. \tag{31}$$

On reconnaît ici la série de contraintes de l'algorithme de van Holten algorithm (21) pour une particule, portant une charge isospin, évoluant dans un champ de jauge externe, mais en l'absence d'un potentiel scalaire supplémentaire.

Monopôles abéliens

La théorie Kaluza-Klein (KK) est une des plus anciennes idées tentant d'unifier la gravitation aux théories de jauge [Kaluza 1919, Klein 1926]. Dans le contexte KK, l'hypothèse physique est de considérer le monde comme étant constitué, en plus de l'espace-temps 4D, d'une dimension cyclique supplémentaire tellement petite qu'elle est inobservable. Ainsi, la relativité générale en dimension 5 peut être vue comme possédant une symétrie de jauge locale $U(1)$ provenant de l'isométrie de l'espace supplémentaire caché.

Plus tard, Sorkin [Sorkin 1983], et Gross et Perry [Gross 1983], introduisirent le monopôle Kaluza-Klein qui est obtenu en plongeant l'instanton gravitationnel Taub-NUT dans la théorie KK. La métrique globale obtenue s'écrit,

$$\begin{gathered} ds^2 = -dt^2 + f(r)\left(dr^2 + r^2\big(d\theta^2 + \sin^2\theta d\phi^2\big)\right) + f^{-1}(r)\big(dx^4 + A_\phi d\phi\big)^2\,, \\ \text{avec} \quad \theta \in [0,\pi]\,, \quad \phi \in [0, 2\pi]\,, \\ A_\phi \equiv \text{champ de monopôle de Dirac}\,, \quad x^4 \equiv \text{coordonnée verticale}\,, \end{gathered} \tag{32}$$

et produit des solutions exactes des équations de la gravité Euclidienne 4D approchant le vide à l'infini spatial.

Notre objectif, dans cette partie, est de présenter une analyse systématique des métriques de type Kaluza-Klein admettant des symétries kepleriennes. Pour cela, on considère la famille de métriques stationnaires,

$$dS^2 = f(\vec{x})\,\delta_{ij}(\vec{x})\,dx^i\,dx^j + h(\vec{x})\,\big(dx^4 + A_k\,dx^k\big)^2\,, \tag{33}$$

Dans cette famille de métriques, $f(\vec{x})$ et $h(\vec{x})$ sont des fonctions réelles et $A_k\,dx^k$ est un potentiel de jauge d'un monopôle de Dirac chargé.

Inspiré par l'hypothèse de Kaluza, puisque la dimension supplémentaire est cyclique, on utilise la conservation de la composante "verticale" du moment pour réduire le problème 4D en un problème 3D pour lequel nous avons de fortes présomptions sur la manière dont la symétrie dynamique peut être générée [Ngome 08/2009]. Ainsi, le problème de relèvement peut être convenablement résolu en utilisant la technique de van Holten.

On considère le mouvement géodésique d'une particule-test sans spin et de masse unitaire. Le Lagrangien du mouvement géodésique sur la variété 4D dotée de la métrique (33) est

$$\mathcal{L} = \frac{1}{2}\,f(\vec{x})\,\delta_{ij}\,\frac{dx^i}{dt}\,\frac{dx^j}{dt} + \frac{1}{2}\,h(\vec{x})\,\big(\,\frac{dx^4}{dt} + A_k\,\frac{dx^k}{dt}\,\big)^2 - U(\vec{x})\,, \tag{34}$$

où on a ajouté un potentiel scalaire additionnel, $U(\vec{x})$. Les moments canoniques conjugués aux positions $(x^j,\, x^4)$ sont donnés par

$$\begin{aligned} p_j &= \frac{\partial \mathcal{L}}{\partial\big(dx^j/dt\big)} = f(\vec{x})\,\delta_{ij}\,\frac{dx^i}{dt} + h(\vec{x})\,\big(\,\frac{dx^4}{dt} + A_k\,\frac{dx^k}{dt}\,\big)\,A_j\;, \\ p_4 &= \frac{\partial \mathcal{L}}{\partial\big(dx^4/dt\big)} = h(\vec{x})\,\big(\,\frac{dx^4}{dt} + A_k\,\frac{dx^k}{dt}\,\big) = q\;. \end{aligned} \tag{35}$$

Le moment "vertical", $p_4 = q$, associé à la variable périodique, x^4, est conservé et peut donc être interprété comme étant la charge électrique conservée. Ainsi, on introduit le moment covariant,

$$\Pi_j = f(\vec{x})\,\delta_{ij}\,\frac{dx^i}{dt} = p_j - q\,A_j\;. \tag{36}$$

Le mouvement géodésique sur la variété 4D se projette alors sur la variété courbe 3D de métrique $g_{ij}(\vec{x}) = f(\vec{x})\,\delta_{ij}$, augmenté d'un potentiel scalaire. Le hamiltonien s'écrit

$$\mathcal{H} = \frac{1}{2}\,g^{ij}(\vec{x})\Pi_i\,\Pi_j + V(\vec{x}) \quad \text{avec} \quad V(\vec{x}) = \frac{q^2}{2h(\vec{x})} + U(\vec{x})\,. \tag{37}$$

Les crochets de Poisson covariants sont donnés par [Souriau 1970]

$$\{B, D\} = \partial_k B\,\frac{\partial D}{\partial \Pi_k} - \frac{\partial B}{\partial \Pi_k}\,\partial_k D + qF_{kl}\,\frac{\partial B}{\partial \Pi_k}\,\frac{\partial D}{\partial \Pi_l}\,, \tag{38}$$

où $F_{kl} = \partial_k A_l - \partial_l A_k$ est le champ de force du monopôle. Les crochets fondamentaux non-nuls sont

$$\big\{x^i,\,\Pi_j\big\} = \delta^i_j\;, \quad \{\Pi_i,\,\Pi_j\} = q\,F_{ij}\,. \tag{39}$$

On peut maintenant déduire les équations de Hamilton impliquant le mouvement géodésique d'une particule scalaire sur la variété 3D,

$$\dot{x}^i \;=\; \big\{x^i,\,\mathcal{H}\big\} = g^{ij}(\vec{x})\,\Pi_j, \tag{40}$$

$$\dot{\Pi}_i \;=\; \{\Pi_i,\,\mathcal{H}\} = q\,F_{ij}\,\dot{x}^j - \partial_i V + \Gamma^k_{ij}\,\Pi_k\,\dot{x}^j\;. \tag{41}$$

On remarque que les équations de Lorentz (41) impliquent en plus du terme de potentiel et celui portant le champ de monopôle, un terme de courbure (quadratique en vitesse) typique pour des mouvements dans des espaces courbes.

Maintenant nous nous intéressons aux symétries du système. Pour cela, on rappelle que les quantités conservées associées, notées Q, sont dérivées à partir de l'algorithme de van Holten [van Holten 2007], basé sur les tenseurs de Killing, et prennent la forme polynomiale

$$Q = \sum_{k=0}^{p-1} \frac{1}{k!}\,C^{i_1\cdots i_k}\,\Pi_{i_1}\cdots\Pi_{i_k}\,. \tag{42}$$

Avant d'aller plus en avant, on va discuter de deux tenseurs de Killing particuliers sur la variété 3D portant la métrique,

$$g_{ij}(\vec{x}) = f(\vec{x})\,\delta_{ij}\,. \tag{43}$$

Notre stratégie consiste à relever les tenseurs de Killing générant le moment angulaire et le vecteur de Runge-Lenz conservés, d'un espace 3D vers un espace 4D de type "Kaluza-Klein".

• On considère d'abord le tenseur de Killing de rang-1 générant les rotations spatiales ordinaires,

$$C_i = g_{ij}(\vec{x})\,\epsilon^j{}_{kl}\,n^k\,x^l\,. \tag{44}$$

En exigeant que C_i soit un tenseur de Killing $\mathcal{D}_{(i}C_{j)} = 0$, on obtient le théorème suivant [Ngome 08/2009] :

Théorème 0.0.1. *Sur la variété 3D portant la métrique* $g_{ij}(\vec{x}) = f(\vec{x})\,\delta_{ij}$*, le tenseur de rang-1*

$$C_i = g_{ij}(\vec{x})\,\epsilon^j{}_{kl}\,n^k\,x^l$$

est un tenseur de Killing générant les symétries rotationnelles autour du vecteur unitaire $\vec{n}$*, si et seulement si*

$$\Big(\vec{x}\times\vec{\nabla} f(\vec{x})\Big)\cdot\vec{n} = 0\,. \tag{45}$$

Il est à noter que le Théorème 0.0.1 peut être satisfait par certains vecteurs $\vec{n}$. Dans le cas de la métrique à deux centres, par exemple, le théorème est satisfait pour des vecteurs $\vec{n}$ parallèles à l'axe des deux centres.

Un cas particulier important a lieu lorsque la fonction $f(\vec{x})$ est radiale,

$$f(\vec{x}) = f(r)\,, \tag{46}$$

ce qui inclut les métriques Taub-NUT. Dans ce cas le Théorème (0.0.1) est vérifié pour tout $\vec{n}$. Par conséquent, le Théorème 0.0.1 est toujours satisfait pour des métriques radiales.

• Ensuite, inspiré par l'expression connue en espace plat, on considère le tenseur de rang-2 associé aux quantités de type Runge-Lenz,

$$C_{ij} = 2\,g_{ij}(\vec{x})\,n_k\,x^k - g_{ik}(\vec{x})\,n_j\,x^k - g_{jk}(\vec{x})\,n_i\,x^k\,. \tag{47}$$

Dans le but d'obtenir des conditions sur les métriques admettant des symétries dynamiques de type keplerien, on impose $\mathcal{D}_{(i}C_{jl)} = 0$. On obtient ainsi le théorème [Ngome 08/2009] suivant :

Théorème 0.0.2. *Sur la variété 3D portant la métrique* $g_{ij}(\vec{x}) = f(\vec{x})\,\delta_{ij}$*, le tenseur*

$$C_{ij} = 2\,g_{ij}(\vec{x})\,n_k\,x^k - g_{ik}(\vec{x})\,n_j\,x^k - g_{jk}(\vec{x})\,n_i\,x^k$$

est un tenseur de Stäkel-Killing de rang-2 associé aux quantités conservées de type Runge-Lenz si et seulement si

$$\vec{n} \times \left(\vec{x} \times \vec{\nabla} f(\vec{x})\right) = 0\,. \tag{48}$$

Il est à signaler que les métriques radiales (46) satisfont encore le Théorème 0.0.2 telles qu'en plus de la symétrie rotationnelle, elles admettent des symétries dynamiques kepleriennes.

Notre classe de métriques satisfaisant les théorèmes 0.0.1 et 0.0.2 inclut :

1. Le cas Taub-NUT original [Sorkin 1983, Gross 1983] avec un potentiel externe nul, $U(r) = 0$,

$$f(r) = \frac{1}{h(r)} = 1 + \frac{4m}{r}, \tag{49}$$

   où m est réel [Fehér 10/1986, Gibbons 1987]. on note que le cas "monopole scattering" correspond à $m = -1/2$, voir [Gibbons 04/1986, Fehér 10/1986, Gibbons 12/1986]. On obtient alors pour

$$\gamma = q^2/2 - \mathcal{E} \quad \text{et de charge} \quad g = \pm 4m\,, \tag{50}$$

   le vecteur de Runge-Lenz conservé,

$$\vec{K} = \vec{\Pi} \times \vec{J} - 4m\left(\mathcal{E} - q^2\right)\frac{\vec{x}}{r}\,. \tag{51}$$

2. Lee et Lee [Lee 2000] ont argué que pour le "monopole scattering" avec des composantes indépendantes des valeurs de Higgs, le Lagrangien géodésique (3.60) devrait être remplacé par $L \to L - U(r)$, où le potentiel externe s'écrit

$$U(r) = \frac{1}{2}\,\frac{{a_0}^2}{1 + \dfrac{4m}{r}}\,. \tag{52}$$

   Il est maintenant clair que cette addition décale simplement l'énergie du système par une constante ${a_0}^2/2$. Par conséquent le vecteur de Runge-Lenz précédent (51) est encore valable dans ce cas.

3. La métrique associée au "winding strings" [Gibbons 1988] où

$$f(r) = 1, \qquad h(r) = \frac{1}{(1 - \dfrac{1}{r})^2}\,. \tag{53}$$

Pour des charges $g = \pm 1$, on déduit que

$$(\beta + q^2) - r\left(U(r) - \gamma + \frac{q^2}{2} - \mathcal{E}\right) = 0\,,$$

tel que pour l'énergie fixée, $\mathcal{E} = q^2/2 - \gamma + U(r)$, le vecteur de Runge-Lenz conservée est donné par

$$\vec{K} = \dot{\vec{x}} \times \vec{J} - q^2\,\frac{\vec{x}}{r}\,. \tag{54}$$

4. La métrique Taub-NUT étendue [Iwai 05/1994, Iwai 06/1994] où

$$f(r) = b + \frac{a}{r}, \quad h(r) = \frac{a\,r + b\,r^2}{1 + d\,r + c\,r^2}\,, \tag{55}$$

avec les constantes $(a,\,b,\,c,\,d\,) \in \mathbb{R}$. Les choix $U(r) = 0$ et $g = \pm 1$, impliquent que

$$-r\,f(r)\,\mathcal{E} + \frac{r\,f(r)}{h(r)}\,\frac{q^2}{2} - \frac{q^2}{2\,r} - \gamma\,r = \beta = \text{const}\,.$$

Insérant (55) donne

$$\Big(-a\,\mathcal{E} + \frac{1}{2}d\,q^2 - \beta\Big) + r\Big(-b\,\mathcal{E} + \frac{1}{2}c\,q^2 - \gamma\Big) = 0,$$

qui se produit quand

$$\beta = -a\,\mathcal{E} + \frac{1}{2}d\,q^2 \quad \text{et} \quad \gamma = -b\,\mathcal{E} + \frac{1}{2}c\,q^2\,. \tag{56}$$

Alors, on obtient le vecteur de Runge-Lenz conservé

$$\vec{K} = \vec{\Pi} \times \vec{J} - \left(a\,\mathcal{E} - \frac{1}{2}\,d\,q^2\right)\frac{\vec{x}}{r}\,. \tag{57}$$

5. Considérant la métrique de type oscillateur discutée par Iwai et Katayama [Iwai 05/1994, Iwai 06/1994], les fonctions $f(r)$ et $h(r)$ prennent les formes

$$f(r) = b + ar^2 \quad \text{et} \quad h(r) = \frac{ar^4 + br^2}{1 + cr^2 + dr^4}\,. \tag{58}$$

Un calcul direct entraîne le vecteur de type Runge-Lenz suivant,

$$\vec{K} = \left(b + ar^2\right)\,\dot{\vec{x}} \times \vec{J} + \beta\,\frac{\vec{x}}{r}\,. \tag{59}$$

qui est conservé seulement pour des potentiels scalaires de la forme

$$U(r) = \left(\frac{q^2 g^2}{2r^2} + \frac{\beta}{r} + \gamma\right)\left(b + ar^2\right)^{-1} - q^2\left(\frac{1 + cr^2 + dr^4}{ar^4 + br^2}\right)\,. \tag{60}$$

Monopôles non-abéliens

On considère les champs de type monopôles non-abéliens obtenus à l'aide de la phase de Berry dans le contexte de la molécule diatomique. Ces champs de jauge peuvent s'écrire sous la forme “hedgehog” suivante :

$$\widetilde{A}_i^a = (1-\kappa)\epsilon_{iaj}\,\frac{x^j}{r^2}\,, \quad \widetilde{F}_{ij}^{\,a} = (1-\kappa^2)\epsilon_{ijk}\frac{x^k x^a}{r^4}\,. \tag{61}$$

On peut signaler que (61) ressemble à la structure d'un champ de monopôle non-abelien ['t Hooft 1974, Polyakov 1974]. On note *la présence du paramètre constant non-quantifié* $(1-\kappa^2)$ *dans le champ magnétique ci-dessus.*

Le hamiltonien décrivant la dynamique d'une particule dans le champ de jauge (61) s'exprime comme

$$\mathcal{H} = \frac{\vec{\pi}^2}{2} - \big(g/4\big)\epsilon_{ijk}\widetilde{F}_{ij}^{\,a}\,S^k + V(\vec{x},\,\mathcal{I}^a)\,, \quad \pi_i = p_i - \widetilde{A}_i^a\,\mathcal{I}^a\,, \tag{62}$$

où le terme de couplage spin-rotation disparaît quand on considère des particules portant une charge gyromagnétique nulle, $g=0$. Dans ce cas, le hamiltonien résultant s'écrit comme celui d'une particule scalaire [1] évoluant dans le même champ magnétique.

Les équations du mouvement gouvernant la particule à isospin dans (61) s'écrivent

$$\begin{cases} \ddot{x}_i - \mathcal{I}^a\widetilde{F}_{ij}^a\,\dot{x}^j + D_iV = 0\,, \\ \dot{\mathcal{I}}^a + \epsilon_{abc}\,\mathcal{I}^b\left(\widetilde{A}_j^c\,\dot{x}^j - \dfrac{\partial V}{\partial \mathcal{I}^c}\right) = 0\,, \end{cases} \tag{63}$$

avec la dérivée covariante de jauge s'écrivant comme suit :

$$D_jf = \partial_jf - \epsilon_{abc}\mathcal{I}^a\widetilde{A}_j^b\,\frac{\partial f}{\partial \mathcal{I}^c}\,. \tag{64}$$

La première équation dans (63) décrit le mouvement réel 3D impliquant la force de Lorentz généralisée, plus une interaction avec le potentiel scalaire. La seconde équation est une équation de Kerner-Wong augmentée d'une interaction avec un champ scalaire. Cette dernière décrit, comme attendu, le mouvement de l'isospin classique dans l'espace interne.

On concentre maintenant notre attention dans la recherche des constantes du mouvement du système. La technique de van Holten implique la résolution de la série de contraintes,

[1] c'est à dire sans spin.

$$
\begin{aligned}
&C_iD_iV+\epsilon_{abc}\mathcal{I}^a\frac{\partial C}{\partial\mathcal{I}^b}\frac{\partial V}{\partial\mathcal{I}^c}=0, && o(0)\\
&D_iC=\mathcal{I}^a\widetilde{F}^a_{ij}C_j+C_{ij}D_jV+\epsilon_{abc}\mathcal{I}^a\frac{\partial C_i}{\partial\mathcal{I}^b}\frac{\partial V}{\partial\mathcal{I}^c}, && o(1)\\
&D_iC_j+D_jC_i=\mathcal{I}^a(\widetilde{F}^a_{ik}C_{kj}+\widetilde{F}^a_{jk}C_{ki})+C_{ijk}D_kV+\epsilon_{abc}\mathcal{I}^a\frac{\partial C_{ij}}{\partial\mathcal{I}^b}\frac{\partial V}{\partial\mathcal{I}^c}, && o(2)\\
&D_iC_{jk}+D_jC_{ki}+D_kC_{ij}=\mathcal{I}^a(\widetilde{F}^a_{il}C_{ljk}+\widetilde{F}^a_{jl}C_{lki}+\widetilde{F}^a_{kl}C_{lij})\\
&\qquad\qquad +C_{ijkl}D_lV+\epsilon_{abc}\mathcal{I}^a\frac{\partial C_{ijk}}{\partial\mathcal{I}^b}\frac{\partial V}{\partial\mathcal{I}^c}, && o(3)\\
&\vdots\qquad\qquad\qquad\qquad\vdots && \vdots
\end{aligned}
\tag{65}
$$

- La charge d'ordre zéro en moment covariant,

$$
Q=\frac{\vec{x}\cdot\vec{\mathcal{I}}}{r}\,, \tag{66}
$$

qui est conservée dans le cas particulier $\kappa=0$, ne l'est plus en général

$$
\{Q,\mathcal{H}\}=\vec{\pi}\cdot\vec{D}Q,\quad D_jQ=\frac{\kappa}{r}\left(\mathcal{I}^j-Q\frac{x_j}{r}\right). \tag{67}
$$

Une exception a lieu, lorsque l'isospin est aligné dans la direction radiale, comme on peut le voir dans (67). Un calcul détaillé nous montre que l'équation $D_jQ=0$ peut uniquement être résolue, pour un monopôle Abelien plongé dans $SU(2)$, quand $\kappa=0,\pm1$.

La charge Q^2, non plus, n'est pas conservée puisque

$$
\{Q^2,\mathcal{H}\}=2\kappa Q(\vec{\pi}\cdot\vec{D}Q)\,. \tag{68}
$$

Par contre la norme de l'isospin, $\mathcal{I}^2$, *est* toujours conservée,

$$
\{\mathcal{H},\mathcal{I}^2\}=0\,.
$$

Notre champ de jauge de type monopôle non-abélien (61) est à symétrie sphérique tel qu'une particule, portant une charge isospin, en mouvement dans ce fond admet un moment angulaire conservé, $\vec{J}$, [Wilczek 1986, Jackiw 1986]. Cependant sa forme est non-conventionnelle, et on le redérive suivant la procédure de van Holten [Ngome 02/2009],

$$
\vec{J}\;=\;\vec{x}\times\vec{\pi}-\vec{\Psi}\,, \tag{69}
$$

$$
\vec{\Psi}\;=\;(1-\kappa)\,Q\,\frac{\vec{x}}{r}+\kappa\vec{\mathcal{I}}\;=Q\,\frac{\vec{x}}{r}+\kappa(\frac{\vec{x}}{r}\times\vec{\mathcal{I}})\times\frac{\vec{x}}{r}. \tag{70}
$$

Ce résultat est consistant avec celui trouvé par Jackiw [Jackiw 1986, Rho 1992]. Il est à signaler que pour $\kappa=0$, on retrouve l'expression du cas Wu-Yang. Le fait d'éliminer $\vec{\pi}$

au profit de $\vec{p}-\vec{A}=\vec{\pi}$ permet de réécrire le moment angulaire conservé sous la forme

$$\vec{J}=\vec{x}\times\vec{p}-\vec{\mathcal{I}}\,, \tag{71}$$

rendant manifeste le fameux terme "spin from isospin" [Jackiw 1976].

On considère maintenant le vecteur de Killing

$$\vec{C}=2\kappa\big(\vec{x}\times\vec{\mathcal{I}}\big)\,, \tag{72}$$

qui n'existe que dans le cas réellement "non-abélien" $\kappa\neq 0$ [pour $\kappa=0$ on retombe dans le cas d'un monopôle de Dirac plongé dans $SU(2)$]. Ce vecteur de Killing implique la conservation de la charge,

$$\Gamma=J^2-L^2=\big(1-\kappa\big)^2Q^2-\kappa^2\vec{\mathcal{I}}^2-2\kappa\vec{J}\cdot\vec{\mathcal{I}}\,, \tag{73}$$

au cours du mouvement de la particule dans le champ de type monopôole induit par la phase de Berry dans le système de la molécule diatomique.

On note que la charge Γ *correspond, dans la limite abélienne* $\kappa=0$ *au carré de la charge électrique.* Tout comme les constantes du mouvement $\vec{J}$, J^2 et L^2, la charge Γ est conservée pour des potentiels à symétrie radiale, $V(r)$.

La décomposition du moment covariant, en une partie radiale et une autre transverse, avec l'identité vectorielle

$$(\vec{\pi})^2=(\vec{\pi}\cdot\frac{\vec{x}}{r})^2+(\vec{\pi}\times\frac{\vec{x}}{r})^2=\pi_r^2+\frac{L^2}{r^2}\,, \tag{74}$$

nous permet d'exprimer le hamiltonien de la molécule diatomique (62) comme

$$\mathcal{H}=\frac{1}{2}(\vec{\pi}\cdot\frac{\vec{x}}{r})^2+\frac{J^2}{2r^2}-\left\{\frac{(1-\kappa)^2Q^2-\kappa^2\mathcal{I}^2-2\kappa\vec{J}\cdot\vec{\mathcal{I}}}{2r^2}\right\}+V(r)\,. \tag{75}$$

En posant $Q^2=\mathcal{I}^2=1/4$, Jackiw a trouvé une décomposition similaire à (75) [Jackiw 1986], mais cela n'est légitime que lorsque $\kappa=0$, puisque Q^2 n'est pas conservé quand $\kappa\neq 0$.

Pour $\kappa\neq 0$, la "bonne" approche est de reconnaître la charge fixée Γ, qui nous donne la bonne décomposition,

$$\mathcal{H}=\frac{1}{2}(\vec{\pi}\cdot\frac{\vec{x}}{r})^2+\frac{J^2}{2r^2}-\frac{\Gamma}{2r^2}+V(r)\,. \tag{76}$$

On peut souligner que le champ effectif de la molécule diatomique fournit une généralisation intéressante du champ du monopôle Wu-Yang. Pour $\kappa\neq 0,\pm 1$, il est "réellement non-abélien", c'est à dire, *non réductible* à un fibré $U(1)$. Dans ce cas, il n'existe pas de champ de direction covariament constant, et, par conséquent, *pas de charge électrique conservée.*

Le champ est néanmoins à symétrie radiale, mais le moment angulaire conservé (70) a une forme non-conventionnelle.

On peut signaler que pour $\kappa \neq 0$ la configuration (61) ne satisfait pas aux équations de Yang-Mills dans le vide. Elles sont par contre satisfaites avec un convenable courant conservé [Jackiw 1986],

$$\mathcal{D}_i F_{ik} = j_k, \quad \vec{\jmath} = \frac{\kappa(1-\kappa^2)}{r^4}\, \vec{x} \times \vec{T}\,. \tag{77}$$

Ce courant peut également être produit par le champ “hedgehog” de Higgs,

$$j_k = \big[\mathcal{D}_k \Phi, \Phi\big], \quad \Phi^a = \frac{\sqrt{1-\kappa^2}}{r}\,\frac{x_a}{r}\,. \tag{78}$$

Pour $\kappa = 0$, il est facile d'avoir un vecteur de type Runge-Lenz conservé puisque ce cas est exactement équivalent au cas Wu-Yang [un monopôle de Dirac plongé dans $SU(2)$]. Pour $\kappa \neq 0, \pm 1$, on a dérivé une nouvelle charge conservée, à savoir Γ, qui a aussi une forme inhabituelle. Dans le cas limite $\kappa = 0$, cette charge se réduit à $\Gamma = Q^2$; tandis que pour $\kappa = \pm 1$, on obtient $\Gamma \sim \vec{L} \cdot \vec{\mathcal{I}}$.

Extension super-symétrique de l'algorithme de van Holten

On étudie dans cette partie, d'une part les super-symétries, et d'autre part les symétries bosoniques du hamiltonien de Pauli,

$$\mathcal{H}_g = \frac{\vec{\Pi}^2}{2} - \frac{eg}{2}\vec{S}\cdot\vec{B} + V(r)\,, \quad \vec{\Pi} = \vec{p} - e\vec{A}\,, \tag{79}$$

décrivant le mouvement d'un fermion de spin $\vec{S}$ et de charge électrique e, dans un champ magnétique, $\vec{B}$, augmenté d'un champ scalaire, $V(r)$, à symétrie sphérique pouvant inclure un terme de Coulomb. Dans le hamiltonien (79), $\vec{\Pi}$ représente le moment covariant de jauge et le paramètre g représente le facteur gyromagnétique de la particule.

Le mouvement de la particule à spin est décrit par la courbe

$$\tau \to (\,x(\tau),\,\psi(\tau))\,\in\,\mathcal{M}^{3+3}\,, \tag{80}$$

où τ est un paramètre d'évolution.

On concentre notre attention sur des particules de spin-$\frac{1}{2}$, chargées, en interaction avec un champ de monopôle de Dirac,

$$\vec{B} = \vec{\nabla}\times\vec{A} = \frac{q}{e}\frac{\vec{x}}{r^3}\,. \tag{81}$$

L'espace des phases est défini par $\left(x^j, \Pi_j, \psi^a\right)$, où les variables ψ^a se transforment comme des vecteurs tangents satisfaisant l'algèbre de Grassmann,

$$\psi^i\psi^j + \psi^j\psi^i = 0\,. \tag{82}$$

Le moment angulaire interne de la particule peut aussi être décrit en terme de vecteur de type Grassmannien,

$$S^j = -\frac{i}{2}\epsilon^j_{\;kl}\psi^k\,\psi^l\,. \tag{83}$$

On définit le crochet de Poisson covariant par [Berezin]

$$\{f,h\} \;=\; \partial_j f\,\frac{\partial h}{\partial \Pi_j} - \frac{\partial f}{\partial \Pi_j}\,\partial_j h + eF_{ij}\,\frac{\partial f}{\partial \Pi_i}\frac{\partial h}{\partial \Pi_j} + i(-1)^{a^f}\,\frac{\partial f}{\partial \psi^a}\frac{\partial h}{\partial \psi_a}\,, \tag{84}$$

où $a^f = (0,1)$ est donné par la parité des variables de Grassmann dans la fonction f.

Les équations du mouvement sont

$$\dot{\vec{x}} = \vec{\Pi}\,, \qquad \dot{\vec{\psi}} = \frac{eg}{2}\,\vec{\psi}\times\vec{B}\,, \qquad \dot{\vec{S}} = \frac{eg}{2}\,\vec{S}\times\vec{B}\,, \tag{85}$$

$$\dot{\vec{\Pi}} = e\,\vec{\Pi}\times\vec{B} - \vec{\nabla}V(r) + \frac{eg}{2}\,\vec{\nabla}\Big(\vec{S}\cdot\vec{B}\Big)\,. \tag{86}$$

Les lois de conservation du système (79), sont décrite en utilisant l'extension super-symétrique de l'algorithme de van Holten [Ngome 03/2010]. Ce qui est nouveau ici est que les générateurs de la SUSY prennent leurs valeurs dans l'algèbre de Grassmann.

En posant,

$$\mathcal{Q}\left(\vec{x},\,\vec{\Pi},\,\vec{\psi}\right)=C(\vec{x},\,\vec{\psi})+\sum_{k=1}^{n-1}\frac{1}{k!}\,C^{i_1\cdots i_k}(\vec{x},\,\vec{\psi})\,\Pi_{i_1}\cdots\Pi_{i_k}\,, \tag{87}$$

puis en exigeant que $\mathcal{Q}$ commute avec le hamiltonien $\{\mathcal{H}_g,\mathcal{Q}\}=0$, implique la série de contraintes

$$\begin{aligned}
&C_i\partial_iV+\frac{ieg}{4}\psi^l\psi^mC_j\partial_jF_{lm}-\frac{eg}{2}\psi^m\frac{\partial C}{\partial\psi^a}F_{am}=0, && o(0)\\
&\partial_jC=C_{jk}\partial_kV+eF_{jk}C_k+\frac{ieg}{4}\psi^l\psi^mC_{jk}\partial_kF_{lm}-\frac{eg}{2}\psi^m\frac{\partial C_j}{\partial\psi^a}F_{am}, && o(1)\\
&\partial_{(j}C_{k)}=C_{jkm}\partial_mV+e\left(F_{jm}C_{mk}+F_{km}C_{mj}\right)\\
&\qquad\qquad+\frac{ieg}{4}\psi^l\psi^mC_{ijk}\partial_iF_{lm}-\frac{eg}{2}\psi^m\frac{\partial C_{jk}}{\partial\psi^a}F_{am}, && o(2)\\
&\partial_{(j}C_{kl)}=C_{jklm}\partial_mV+e\left(F_{jm}C_{mkl}+F_{lm}C_{mjk}+F_{km}C_{mlj}\right)\\
&\qquad\qquad+\frac{ieg}{4}\psi^m\psi^nC_{ijkl}\partial_iF_{mn}-\frac{eg}{2}\psi^m\frac{\partial C_{jkl}}{\partial\psi^a}F_{am}\,, && o(3)\\
&\vdots\qquad\qquad\vdots && \vdots
\end{aligned} \tag{88}$$

• La forme générale de l'extension Grassmannienne des tenseurs de Killing de rang-1 s'écrit

$$C^i(\vec{x},\,\vec{\psi})=\sum_{k\geq0}\left(M^{ij}\,x^j+N^i\right)_{a_1\cdots a_k}\psi^{a_1}\cdots\psi^{a_k}\,,\quad M^{ij}=-M^{ji}\,, \tag{89}$$

où N^i et le tenseur anti-symétrique M^{ij} définissent des tenseurs constants.

• De manière analogue, la forme générale de l'extension Grassmannienne des tenseurs de Killing de rang-2 s'écrit

$$\begin{aligned}
C^{ij}(\vec{x},\,\vec{\psi})=\sum_{k\geq0}\Big(&M^{(i}_{\;ln}\widetilde{M}^{j)\,n}_{m}x^lx^m+M^{(i}_{\;ln}\widetilde{N}^{j)\,n}x^l\\
&+N^{(i}_{\;n}\widetilde{M}^{j)\,n}_{m}x^m+N^{(i}_{\;n}\widetilde{N}^{j)\,n}\Big)_{a_1\cdots a_k}\psi^{a_1}\cdots\psi^{a_k}\,,
\end{aligned} \tag{90}$$

avec $M^{ij}_{\;\;k}$, $\widetilde{M}^{ij}_{\;\;k}$, N^j_k et $\widetilde{N}^j_k$ des tenseurs constants complètement anti-symétriques.

Ayant construit les tenseurs de Killing génériques (89) et (90), on peut maintenant décrire aussi bien les super-symétries que les symétries bosoniques du hamiltonien de Pauli (79). Les principaux résultats sont consignés dans le tableau suivant :

Valeur de g	Tenseurs de Killing	quantités conservées
$g \neq 0$	pas de tenseur de Killing	$\mathcal{Q}_c = \vec{\psi}\cdot\vec{S}$
$g = 2$	$C^j = \delta^j_a \psi^a$	$\mathcal{Q} = \vec{\psi}\cdot\vec{\Pi}$
$g = 2$	$C^j = \left(\vec{\psi}\times\vec{\psi}\right)^j$	$\mathcal{Q}_1 = \vec{S}\cdot\vec{\Pi}$
$g = 2$	$C^j = \left(\vec{S}\times\vec{x}\right)^j$	$\mathcal{Q}_2 = \vec{S}\cdot(\vec{x}\times\vec{\Pi})$
quelconque	$C^j = \left(\vec{\psi}\times\vec{x}\right)^j$	$\mathcal{Q}_3 = (\vec{x}\times\vec{\Pi})\cdot\vec{\psi} + \frac{q}{2}(g-2)\frac{\vec{\psi}\cdot\vec{x}}{r}$
quelconque	$C^{ij} = \delta^{ij}$	$\mathcal{E} = \frac{\vec{\Pi}^2}{2} - \frac{eg}{2}\vec{S}\cdot\vec{B} + V(r)$
quelconque	$C^{ij} = 2\delta^{ij}(\vec{x}\times\vec{\psi}) - x^i\psi^j - x^j\psi^i$	$\mathcal{Q}_4 = \left(\vec{\Pi}\times(\vec{x}\times\vec{\Pi})\right) + \frac{q}{2}(2-g)\frac{\vec{x}\times\vec{\Pi}}{r}\cdot\vec{\psi} + \left(\frac{\alpha}{r} - eg\,\vec{S}\cdot\vec{B}\right)\vec{x}\cdot\vec{\psi}$
quelconque	$C^j = (\vec{n}\times\vec{x})^j$	$\vec{J} = \vec{x}\times\vec{\Pi} - q\frac{\vec{x}}{r} + \vec{S}$
quelconque	$C^{ij} = 2\left(\delta^{ij}\vec{x}^2 - x^i x^j\right)$	$Q_5 = \vec{J}^2 - q^2 + (g-2)\vec{J}\cdot\vec{S} - g\mathcal{Q}_2$
$g = 0$ ou $g = 4$	$C^{ij} = 2\delta^{ij}(\vec{n}\cdot\vec{x}) - n^i x^j - n^j x^i$	$\vec{K} = \vec{\Pi}\times\vec{J} + \mu\frac{\vec{x}}{r} + \left(1-\frac{g}{2}\right)\vec{S}\times\vec{\Pi} - \frac{eg}{2}(\vec{S}\cdot\vec{B})\,\vec{x} - \frac{qg}{2}\left(1-\frac{g}{2}\right)\frac{\vec{S}}{r} + \frac{\mu}{q}\vec{S}$
$g = 0$ ou $g = 4$	$C^{ij} = 2\delta^{ij}(\vec{S}\cdot\vec{n}) - \frac{g}{2}\left(S^i n^j + n^j S^i\right)$	$\Omega_0 = 2\mathcal{E}\vec{S}$ $\Omega_4 = \left(\vec{\Pi}^2 - 2V(r)\right)\vec{S} - 2(\vec{\Pi}\cdot\vec{S})\,\vec{\Pi} + 2\left(\frac{q}{r}+\frac{\mu}{q}\right)\vec{S}\times\vec{\Pi} - 4\left(\frac{\mu^2}{2q^2}-\gamma\right)\vec{S}$

Modèles non-commutatifs

Récemment, un remarquable modèle non-commutatif a été déduit dans le contexte de la physique du solide par Chang et Niu [Chang 1995]. En effet, l'analyse semi-classique de l'électron de Bloch, dans un maillage 3D d'un cristal, révèle un terme de “ phase de Berry” supplémentaire, $\vec{\Theta}$, pouvant prendre la forme d'un monopôle dans la structure de bande.

En l'absence d'un champ magnétique externe [with $\vec{B} = \vec{0}$], les équations du mouvement de ce système peuvent être déduites en utilisant la 2-forme symplectique,

$$\Omega = dp_i \wedge dx_i + \frac{1}{2}\epsilon_{ijk}\Theta^i dp_j \wedge dp_k\,, \tag{91}$$

où le terme “extra”, dans (91), induit par la phase de Berry rend les variables-positions non-commutatives [Chang 1995, Niu],

$$\{x_i, x_j\} = \epsilon_{ijk}\Theta_k = \Theta_{ij}, \qquad \{x_i, p_j\} = \delta_{ij}, \qquad \{p_i, p_j\} = 0\,. \tag{92}$$

Appliquant l'identité de Jacobi aux coordonnées, on obtient

$$\begin{cases} 0 = \{p_i, \{x_j, x_k\}\}_{cyclic} = -\epsilon_{jkm} \dfrac{\partial \Theta^m}{\partial x_i}\,, \\ 0 = \{x_i, \{x_j, x_k\}\}_{cyclic} = \dfrac{\partial \Theta^{ij}}{\partial p_k} + \dfrac{\partial \Theta^{jk}}{\partial p_i} + \dfrac{\partial \Theta^{ki}}{\partial p_j}\,. \end{cases} \tag{93}$$

Ainsi, le champ de vecteur $\vec{\Theta}$ a la propriété [2] d'être un champ dépendant du moment [Bérard 2004],

$$\Theta_i = \Theta_i(\vec{p})\,, \tag{94}$$

et aussi exige la condition de cohérence,

$$\vec{\nabla}_{\vec{p}} \cdot \vec{\Theta}(\vec{p}) = 0\,, \tag{95}$$

qui peut être interprétée comme une équation de champ de Maxwell dans l'espace dual des moments.

L'exigence d'une symétrie rotationnelle totale nous induit à considérer $\vec{\Theta}$ comme étant un "monopôle dans le $\vec{p}$-space" [Bérard 2004],

$$\Theta_i = \theta\, \frac{p_i}{p^3}\,, \qquad \theta = \text{const}\,, \tag{96}$$

où $p = |\vec{p}|$. En effet, loin de l'origine, le monopôle dual (6.9) est l'unique possibilité à symétrie sphérique cohérente avec l'identité de Jacobi. On peut mentionner que le $\vec{p}$-monopôle (96) a déjà été observé expérimentalement par Fang et al. dans le contexte de l'effet Hall anomal (AHE) dans un métal ferromagnétique SrRuO3 [Fang 2003].

On peut maintenant étudier la mécanique $3D$ non-commutative induite par (96). Le hamiltonien du système s'écrit

$$\mathcal{H} = \frac{p^2}{2} + V(\vec{x}, \vec{p})\,, \tag{97}$$

où on permet au potentiel supplémentaire d'avoir une dépendance en terme de moment [3].

Les équations du mouvement du système s'établissent comme suit :

$$\dot{x}_i = p_i + \frac{\partial V}{\partial p_i} + \theta \epsilon_{ijk} \frac{p_k}{p^3} \frac{\partial V}{\partial x_j}, \qquad \dot{p}_i = -\frac{\partial V}{\partial x_i}\,. \tag{98}$$

Dans la première relation, le "terme de vitesse anomale" est due à notre hypothèse (96).

On cherche particulièrement les quantités conservées du système non-commutatif. Pour cela on va utiliser la technique "duale de van Holten" [Ngome 06/2010], qui revient à chercher une expansion de la quantité dynamique en termes de puissance entière de la

[2] Pour une théorie plus générale incluant des champs magnétiques, regarder [Chang 1995, Niu, Duval 2000]. Par simplicité la masse est prise égale à un.

[3] Il est à noter que les potentiels dépendant du moment sont fréquement utilisés en physique nucléaire et correspondent à des interactions non-locales.

position,

$$Q = C(\vec{p}) + C_i(\vec{p})x_i + \frac{1}{2!}C_{ij}(\vec{p})x_ix_j + \frac{1}{3!}C_{ijk}(\vec{p})x_ix_jx_k \dots . \tag{99}$$

Alors, l'algorithme covariant de van Holten (21), présenté plus haut, est remplacé par

$$\begin{array}{lcl}
C_i\left(p_i + \dfrac{\partial V}{\partial p_i}\right) & = 0 & \mathrm{o}(0) \\
\dfrac{1}{r}\dfrac{\partial V}{\partial r}\left(\theta\epsilon_{ijk}\dfrac{p_k}{p^3}C_i - \dfrac{\partial C}{\partial p_j}\right) + C_{ij}\left(p_i + \dfrac{\partial V}{\partial p_i}\right) & = 0 & \mathrm{o}(1) \\
\dfrac{1}{r}\dfrac{\partial V}{\partial r}\left(\theta\dfrac{p_m}{p^3}\left(\epsilon_{ijm}C_{ik} + \epsilon_{ikm}C_{ij}\right) - \left(\dfrac{\partial C_k}{\partial p_j} + \dfrac{\partial C_j}{\partial p_k}\right)\right) + C_{ijk}\left(p_i + \dfrac{\partial V}{\partial p_i}\right) & = 0 & \mathrm{o}(2) \\
\dfrac{1}{r}\dfrac{\partial V}{\partial r}\left(\theta\dfrac{p_m}{p^3}\left(\epsilon_{lim}C_{ljk} + \epsilon_{ljm}C_{lki} + \epsilon_{lkm}C_{lij}\right) - \left(\dfrac{\partial C_{ij}}{\partial p_k} + \dfrac{\partial C_{jk}}{\partial p_i} + \dfrac{\partial C_{ki}}{\partial p_j}\right)\right) + & & \\
\qquad C_{lijk}\left(p_l + \dfrac{\partial V}{\partial p_l}\right) & = 0 & \mathrm{o}(3) \\
\vdots \qquad\qquad \vdots & \vdots &
\end{array}$$

où $r = |\vec{x}|$. L'expansion (99) peut encore être tronquée à un ordre fini n, pourvu que la contrainte de plus haut rang de la série précédente devienne une équation de Killing duale,

$$\partial_{(p_{i_1}}C_{p_{i_2}\dots p_{i_n})} = 0\,. \tag{100}$$

• Pour des quantités linéaires, $Q = C_0(\vec{p}) + C_i(\vec{p})x_i$, on peut prendre $C_{ij} = C_{ijk} = \dots 0$. L'algorithme dual se réduit donc comme

$$\begin{cases}
C_i\left(p_i + \dfrac{\partial V}{\partial p_i}\right) = 0\,, & \mathrm{o}(0) \\
\theta\epsilon_{ijk}\dfrac{p_k}{p^3}C_i - \dfrac{\partial C}{\partial p_j} = 0\,, & \mathrm{o}(1) \\
\dfrac{\partial C_k}{\partial p_j} + \dfrac{\partial C_j}{\partial p_k} = 0\,. & \mathrm{o}(2)
\end{cases} \tag{101}$$

Introduisant le vecteur de Killing dual,

$$\vec{C} = \vec{n} \times \vec{p}\,,$$

on obtient

$$C = \theta\,\vec{n}\cdot\hat{p}\,, \qquad \hat{p} = \frac{\vec{p}}{p}\,, \tag{102}$$

Et par conséquent le moment angulaire conservé

$$\vec{J} = \vec{L} - \theta\,\hat{p} = \vec{x}\times\vec{p} - \theta\,\hat{p}\,. \tag{103}$$

• L'étape suivante consiste à chercher des quantités conservées quadratiques en moment.

Dans ce cas, la série de contraintes que l'on doit résoudre s'écrit :

$$\begin{cases} C_i\left(p_i + \dfrac{\partial V}{\partial p_i}\right) = 0\,, & \mathrm{o}(0) \\ \dfrac{1}{r}\dfrac{\partial V}{\partial r}\left(\theta\epsilon_{ijk}\dfrac{p_k}{p^3}C_i - \dfrac{\partial C}{\partial p_j}\right) + C_{ij}\left(p_i + \dfrac{\partial V}{\partial p_i}\right) = 0\,, & \mathrm{o}(1) \\ \theta\dfrac{p_m}{p^3}\big(\epsilon_{ijm}C_{ik} + \epsilon_{ikm}C_{ij}\big) - \big(\dfrac{\partial C_k}{\partial p_j} + \dfrac{\partial C_j}{\partial p_k}\big) = 0\,, & \mathrm{o}(2) \\ \dfrac{\partial C_{ij}}{\partial p_k} + \dfrac{\partial C_{jk}}{\partial p_i} + \dfrac{\partial C_{ki}}{\partial p_j} = 0\,. & \mathrm{o}(3) \end{cases} \tag{104}$$

Le vecteur de type Runge-Lenz est généré par le tenseur de Killing dual de rang-2 ,

$$C_{ij} = 2\delta_{ij}\vec{n}\cdot\vec{p} - n_i p_j - n_j p_i\,, \tag{105}$$

qui vérifie l'équation de Killing dual (d'ordre-3 dans (104)). L'équation d'ordre-2 (104) entraîne que

$$\vec{C} = \theta\frac{\vec{n}\times\vec{p}}{p}\,. \tag{106}$$

Puis, en insérant le résultat précédent dans la contrainte du premier ordre dans (104) et en supposant que $\partial_r V \neq 0$, cette contrainte est satisfaite par

$$C = \alpha\,\vec{n}\cdot\hat{p}\,, \tag{107}$$

α étant une constante arbitraire, *pourvu que le potentiel dépendant du moment et le hamiltonien soient sous la forme*,

$$V = \frac{\vec{x}^2}{2} - \frac{p^2}{2} + \frac{\theta^2}{2p^2} + \frac{\alpha}{p} \quad \text{et} \quad \mathcal{H} = \frac{\vec{x}^2}{2} + \frac{\theta^2}{2p^2} + \frac{\alpha}{p}\,. \tag{108}$$

Ainsi, l'algorithme dual nous donne le vecteur de type Runge-Lenz,

$$\vec{K} = \vec{x}\times\vec{J} - \alpha\hat{p}\,. \tag{109}$$

Sa conservation peut être vérifiée, par un calcul direct, en utilisant les équations du mouvement,

$$\dot{\vec{x}} = \theta\,\frac{\vec{x}\times\vec{p}}{p^3} - \left(\frac{\theta^2}{p^4} + \frac{\alpha}{p^3}\right)\vec{p}\,, \qquad \dot{\vec{p}} = -\vec{x}\,, \tag{110}$$

où, dans la première relation, la vitesse anomale est tranverse.

On note que le terme $(-p^2/2)$ du potentiel V annihile le terme cinétique habituel. Le système décrit alors une *particule non-relativiste, non-commutative, sans terme de masse, évoluant dans un champ d'oscillateur augmenté d'une interaction dépendante du moment.*

En réécrivant le hamiltonien comme

$$\mathcal{H} = \frac{\vec{x}^2}{2} + \frac{\theta^2}{2}\left(\frac{1}{p} + \frac{\alpha}{\theta^2}\right)^2 - \frac{\alpha^2}{2\theta^2} \tag{111}$$

montre cependant que $\mathcal{H} \geq -\dfrac{\alpha^2}{2\theta^2}$ avec l'égalité atteinte seulement dans le cas $p = -\dfrac{\theta^2}{\alpha}$, qui exige $\alpha < 0$.

Les quantités conservées obtenues donnent des informations précieuses sur le mouvement de la particule. Comme dans le cas du système MICZ, on constate que

$$\vec{J} \cdot \hat{p} = -\theta \tag{112}$$

impliquant que le vecteur moment $\vec{p}$ évolue sur un cône d'angle d'ouverture, β, donné par

$$\beta = \arccos\big(-\theta/J\big)\,.$$

D'autre part, on définit le vecteur conservé

$$\vec{N} = \alpha\vec{J} - \theta\vec{K}\,, \tag{113}$$

qui nous permet de construire la constante,

$$\vec{N} \cdot \vec{p} = \theta(J^2 - \theta^2) = \theta L^2\,, \tag{114}$$

telle que les $\vec{p}$-mouvements se font sur un plan perpendiculaire à $\vec{N}$. Par conséquent, *le mouvement dans l'espace des moments décrit des sections coniques.*

Ce dernier est le résultat principal dans le cas du système MICZ; mais dans notre problème non-commutatif, l'intérêt principal se trouve dans la recherche des $\vec{x}(t)$-trajectoires dans l'espace réel. Vu les équations du mouvement (110), cela revient à trouver les *"$\vec{p}$-hodographes"* de la particule.

On note que

$$\vec{N} \cdot \vec{x} = 0\,, \tag{115}$$

ce qui implique que les trajectoires $\vec{x}(t)$ appartiennent au plan oblique, dont la normale est donnée par $\vec{N} = \alpha\vec{J} - \theta\vec{K}$. On peut ainsi étudier le problème dans un système de coordonnées adapté. On prouve en effet que

$$\begin{gathered}\Big\{\hat{\imath}, \hat{\jmath}, \hat{k}\Big\} = \Big\{\frac{1}{|\epsilon L|}\vec{K} \times \vec{J},\ \frac{1}{|\lambda\epsilon|}(2\theta\mathcal{H}\vec{J} + \alpha\vec{K}),\ \frac{1}{|\lambda L|}(\alpha\vec{J} - \theta\vec{K})\Big\} \\ \text{avec} \quad \lambda^2 = \alpha^2 + 2\mathcal{H}\theta^2\,, \quad \epsilon^2 = \alpha^2 + 2\mathcal{H}J^2 \quad \text{et} \quad L^2 = J^2 - \theta^2\,,\end{gathered} \tag{116}$$

est une base orthonormale convenable pour décrire les trajectoires $\vec{x}$.

• Premièrement, une projection sur la base orthonormale,

$$\begin{cases} p_z = \vec{p} \cdot \hat{k} = \theta L/|\lambda| = \text{const}\,, \\ p_x = \vec{p} \cdot \hat{\imath}\,, \\ p_y = \vec{p} \cdot \hat{\jmath}\,, \end{cases} \tag{117}$$

nous donne l'équation,

$$\frac{\left(p_y + \dfrac{|\epsilon|\alpha}{2|\lambda|\mathcal{H}}\right)^2}{\lambda^2/4\mathcal{H}^2} - \frac{p_x^2}{L^2/2\mathcal{H}} = 1, \tag{118}$$

qui est celle d'une hyperbole ou d'une ellipse dans l'espace des moments, selon que le signe de $\mathcal{H}$ soit positif ou négatif. Dans le cas où $\mathcal{H} = 0$ on obtient une parabole. Cela confirme les résultats déjà connus pour le problème MICZ [Mcintosh 1970, Zwanziger 1968], et cohérent avec ce que l'on a déduit géométriquement.

- Ensuite, en projetant la position $\vec{x}$ sur la base orthonormale (116), on obtient

$$X = \vec{x}\cdot\hat{\imath} = -\frac{2|L|}{|\epsilon|}(\mathcal{H} - \frac{\alpha}{2p})\,, \qquad Y = \vec{x}\cdot\hat{\jmath} = -\frac{|\lambda|}{|\epsilon|}\frac{\vec{x}\cdot\vec{p}}{p}\,, \qquad Z = \vec{x}\cdot\hat{k} = 0\,. \tag{119}$$

Un calcul simple entraîne l'équation

$$\left(X + \frac{|\epsilon|L}{J^2}\right)^2 + \frac{\alpha^2 L^2}{\lambda^2 J^2}Y^2 = \frac{L^2\alpha^2}{J^4} \tag{120}$$

qui décrit toujours des ellipses ou des arcs d' ellipse, puisque

$$\lambda^2 = \alpha^2 + 2\mathcal{H}\theta^2 \geq 0\,. \tag{121}$$

Le centre a été déplacé parallèlement le long de l'axe $\hat{\imath}$ d'une distance de $\left(-|\epsilon|L/J^2\right)$ et le demi-grand axe est dirigé vers $\hat{\jmath}$. On peut remarquer que contrairement au mouvement dans l'espace des moments $\vec{p}$, les trajectoires réelles sont toujours fermées.

Quand l'énergie devient négative, $\mathcal{H} < 0$, ce qui est possible seulement dans le cas où le potentiel Newtonien devient attractif $\alpha < 0$, les trajectoires réelles $\vec{x}$ décrivent des ellipses entières (avec l'origine se trouvant à l'intérieur de l'ellipse) :

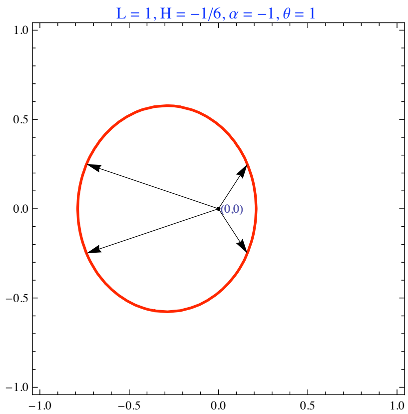

Figure 1: $\mathcal{H} < 0$ et le potentiel Newtonien est attractif $\alpha < 0$, telle que la trajectoires décrit une ellipse complète.

Quand $\mathcal{H} > 0$, qui est l'unique possibilité dans le cas répulsif $\alpha > 0$, l'origine est à l'extérieur de l'ellipse tel que seul est obtenu l'arc de droite [représenté par la ligne en rouge sur la figure de gauche de (2)] tracées entre les tangentes partant de l'origine . Cependant, un hamiltonien positif $\mathcal{H} > 0$, pour des potentiels Newtoniens attractifs $\alpha < 0$. Mais dans ce cas, l'origine se trouve à l'extérieur de l'ellipse tel que les $\vec{x}$-trajectoires réelles balayent l'arc d'ellipse gauche [représenté par la ligne en rouge sur la figure de droite de (2)] :

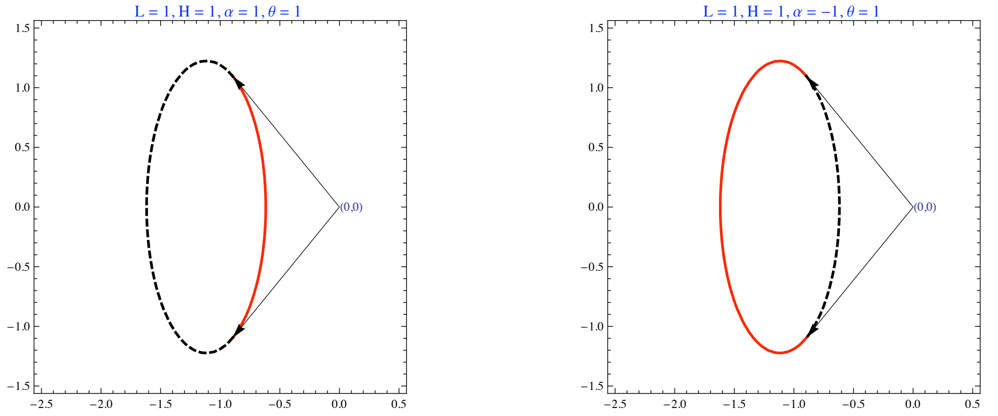

Figure 2: La figure de gauche représente l'arc de droite de l'ellipse balayé par les trajectoires $\vec{x}$ lorsque $\mathcal{H} > 0$ et $\alpha > 0$. Tandis que la figure de droite représente l'arc de gauche de l'ellipse balayé par les trajectoires $\vec{x}$, pour $\mathcal{H} > 0$ et $\alpha < 0$.

Pour $\mathcal{H} = 0$, l'origine se trouve sur l'ellipse et le "mouvement" se réduit à un point unique :

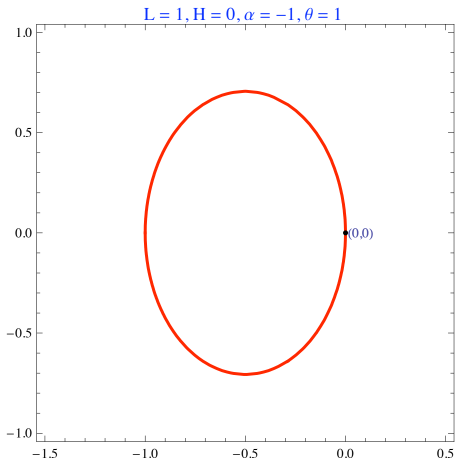

Figure 3: Les $\vec{x}$-trajectoires réelles dégénèrent en un point unique lorsque $\mathcal{H} = 0$.

Quand la non-commutativité s'éteint, $\theta \to 0$, on retrouve les hodographes circulaires du problèmes de Kepler dual. Comme $\alpha \to 0$, les mouvements ne sont plus liés mais balayent l'axe y.

Bien que notre problème soit un problème à l'orgine classique, il est posssible de passer à la qantification en utilisant des propriétés de la théorie des groupes du système MICZ dans l'espace dual. La non-commutativité, alias la charge du monopôle, θ doit être entière ou demi-entière [Souriau 1970, Kostant]. Ceci est, en effet, une première indication sur la quantification du paramètre non-commutatif. La fonction d'onde devrait être choisie dans la représentation impulsion, $\psi(\vec{p})$. Le moment angulaire, $\vec{J}$, et le vecteur de Runge-Lenz rééchelonné, $\vec{K}/\sqrt{2|\mathcal{H}|}$, se referme suivant l'algèbre o(3, 1)/o(4) selon le signe de l'énergie. Dans ce dernier cas, la théorie des representations fournit le spectre en énergie de la particule non-commutative [en unité $\hbar = 1$],

$$E_n = -\frac{\alpha^2}{2n^2}, \qquad n = n_r + \frac{1}{2} + (l + \frac{1}{2})\sqrt{1 + \frac{4\theta^2}{(2l+1)^2}}\,, \tag{122}$$

où $n = 0, 1, \ldots$, $l = 0, 1, \ldots$, avec la dégénerescence

$$n^2 - \theta^2 = \big(n - \theta\big)\big(n + \theta\big)\,.$$

On peut noter que la dégénerescence prend toujours des valeurs entières ou demi-entières, puisque n et θ sont simultanément entiers ou demi-entiers.

Le même résultat peut être obtenu directement en résolvant l'équation de Schrödinger dans l'espace dual $\vec{p}$ [Mcintosh 1970, Zwanziger 1968]. De plus, la symétrie peut être étendue au groupe conforme o(4, 2) [Cordani 1990].

On peut observer que dans bien des approches il s'agit d'étudier les propriétés (telles que les trajectoires, les symétries, etc.) d'un système physique donné. Dans notre cas, on suit la direction contraire : Après avoir posé les relations de commutation fondamentales, on s'est mis à la recherche de potentiels ayant des propriétés de symétries remarquables. Cela a conduit à des potentiels moment-dépendant largement utilisés en physique nucléaire (108), réalisant une sorte de système de McIntosh-Cisneros-Zwanziger [Mcintosh 1970, Zwanziger 1968] dans l'espace dual.

Chapter 1

Introduction

The knowledge of the symmetries is essential in theoretical and condensed matter physics. Indeed, symmetries can be exploited to obtain valuable informations on the motion of a classical system or after quantization to generate the energy spectrum algebraically.

The usual classification provides us with discrete and continuous symmetry transformations. The discrete symmetries are described by finite groups while continuous symmetries, in which we are especially interested, are described by Lie groups.

A deep basis for the understanding of global conservation laws in modern physics was given by Emmy Noether in 1918 [Noether 1918]. She established that conservation laws directly follow from the symmetry properties of a physical system. See also [Trautman 1967]. For instance, the invariance by time translation implies the conservation of the energy; the invariance by spatial translation yields the conserved momentum and the invariance under rotations provides us with the conserved angular momentum.

In this thesis, we focus our attention on a novel way of deriving conserved quantities which has been put forward recently by van Holten [van Holten 2007]. In this formalism, invariants are constructed via Killing tensors which are, indeed, the main ingredients of this technique.

Our main endeavor will be to apply van Holten's covariant recipe to various physical systems.

1. Firstly, we clarify the symmetries associated with isospin-Yang-Mills-Higgs field interactions. To this end, we review, in the context of Kaluza-Klein theories, the classical equations describing the motion of an isospin-carrying particle evolving in a non-Abelian background. Our presentation follows that of [Kerner 1968], who first introduced these equations, using a "Kaluza-Klein" approach [Kerner 1968].

   Next, we discuss the covariant van Holten formalism we use to investigate the symmetries of systems. We note that the symmetry conditions of the van Holten formulation are the same as in the Forgács-Manton-Jackiw (F-M-J) approach [Forgács-Manton 1980, Jackiw-Manton 1980] to symmetric gauge fields.

2. Most applications of the van Holten algorithm involve various (Abelian but also non-Abelian) monopoles and their symmetries.

   In detail, for a "naked" Dirac monopole, the angular momentum admits a celebrated radial term. It has been proved in turn that no globally defined conserved Runge-Lenz vector can exist [Fehér 1987]. It has, however, been found before by McIntosh and Cisneros, and by Zwanziger (MICZ) [Mcintosh 1970, Zwanziger 1968] that adding a suitable inverse-square potential can remove the obstruction such that the combined system can accommodate a conserved Runge-Lenz-type vector.

The archetype of non-Abelian monopoles corresponds to the one introduced in 1968 by Wu and Yang in pure Yang-Mills theory [Wu Yang 1968]. One can wonder if a particle in the Wu-Yang field admits a Kepler-type dynamical symmetry. Generalizing the trick of McIntosh and Cisneros, and of Zwanziger, we find below the most general scalar potential such that the combined system admits a conserved Runge-Lenz vector. This result had to be expected, since Wu and Yang monopole is in fact an imbedded Dirac monopole.

Although no monopoles were ever seen in high-energy experiments, *monopole-like effective fields* can arise in Condensed Matter Physics. It has been noted by Moody, Shapere and Wilczek, for example, that an effective non-Abelian field arises in a diatomic molecule through Berry's phase due to nuclear motion [Wilczek 1986]. For some particular value of a certain parameter, it is just a Wu-Yang monopole field. For a full range of the parameter, however, the effective field becomes "truly" non-Abelian. Electric charge is not more conserved in this case. The system has still spherical symmetry, though, and Moody, Shapere and Wilczek do derive a conserved angular momentum – but one which has an "unusual" form. But they confess not having a systematic way to obtain it. This goal has been achieved by Jackiw [Jackiw 1986] in the F-M-J framework mentioned above.

Here, after a short outline of Berry's phase, we re-derive the correct expression for the conserved angular momentum [Ngome 02/2009], using van Holten's algorithm. In addition, we constructed an "unconventional" conserved charge which reduces to the square of the electric charge in the Wu-Yang limit.

3. The next application of van Holten's approach concerns curved spaces of the Kaluza-Klein monopole type [Sorkin 1983, Gross 1983, Gibbons 04/1986, Gibbons 12/1986]. Mimicking what had been done for the MICZ system, we construct, on curved manifolds, conserved Runge-Lenz-type vectors along the geodesic motion. To this end, using the conservation of the "vertical" component of the momentum, we perform a dimensional reduction of our curved manifold. This allows us to find the conditions under which the dimensionally reduced manifold admits a Killing tensor field associated with a Kepler-type dynamical symmetry [Ngome 08/2009]. Our strategy is to lift $3D$ expressions to the extended Kaluza-Klein manifold.

   Applied to a generalized Taub-NUT metric, we find the most general external potential which can be added such that the combined system exhibits a conserved Runge-Lenz-type vector.

   In the multi-center metric case [Gibbons 12/1986], we show that, under certain conditions, a conserved scalar of Runge-Lenz-type does exist for two-centers [Ngome 08/2009]. For more than two centers no Runge-Lenz-type invariant does exist.

4. Supersymmetries arise for fermions in a three-dimensional monopole background [D'Hoker 1984, DeJonghe 1995, Plyushchay 2000, Leiva 2003, Avery 2008]. The Hamiltonian of the system then involves an additional spin-orbit coupling term, parametrized by the gyromagnetic ratio g.

   Below we construct the (super)invariants using a SUSY extension of the van Holten algorithm. Our clue here is that the symmetry generators can be enlarged to Grassmann-algebra-valued Killing tensors [Ngome 03/2010]. Conserved quantities are obtained for certain definite values of the gyromagnetic factor : $\mathcal{N} = 1$ SUSY requires $g = 2$ [Spector]; a Kepler-type dynamical symmetry only arises, however, for the anomalous values $g = 0$ and $g = 4$. The latter case has the additional bonus to contain an extra "spin" symmetry.

   We find that the two contradictory conditions, namely that of having both super and dynamical symmetry, can be conciliated by doubling the number of Grassmann variables. The anomalous systems with $g = 0$ and $g = 4$ will then become superpartners inside a unified $\mathcal{N} = 2$ SUSY system.

   For a planar fermion in any planar magnetic field, i.e. one perpendicular to the plane, an $\mathcal{N} = 2$ SUSY arises without Grassmann variable doubling.

5. We also construct a three-dimensional non-commutative oscillator with no kinetic term, but with a non-conventional momentum-dependent potential such that it admits a conserved Runge-Lenz-type vector. The latter is derived by adapting van Holten's method to a "dual" description in momentum space [Ngome 06/2010].

   Our system, with monopole-type non-commutativity has the remarkable property to confine particle's motion to bounded trajectories, namely to (arcs of) ellipses. The best way to figure the motions followed by the particle is to think of them as generalizations of the familiar circular hodographs of the Kepler problem, to which they indeed reduce when the noncommutativity is turned off.

Chapter 2

Symmetries and conserved quantities in a non-Abelian field theory

Contents

The classical equations governing isospin-carrying particle motion in a non-Abelian background are derived using Kerner's Kaluza-Klein framework. The van Holten covariant method based on Killing tensors and the Forgács-Manton-Jackiw approach based on the study of symmetric gauge fields are presented.

2.1 The "Kaluza-Klein" framework

In this section, we deal with Kerner's extension of the Kaluza-Klein (KK) approach to a non-Abelian gauge theory [Kerner 1968].

Abelian Kaluza-Klein theory

First of all, let us recall that electromagnetism can be imbedded into general relativity (GR) by adding $U(1)$ local gauge invariance to the theory [Kaluza 1919, Klein 1926]. See also [Einstein 1938, Kerner 1981]. Indeed, let us consider the five-dimensional Einstein-Hilbert action given by

$$S_5 = -\frac{1}{16\pi\tilde{G}_5}\int dx^5\sqrt{-g_5}\,\mathcal{R}_5\,, \tag{2.1}$$

where $\tilde{G}_5$ is the coupling constant and $\mathcal{R}_5$ denotes the 5D scalar curvature.

Viewing the 5D manifold as a direct product of a 4D space-time with an unobservable space-like loop, and assuming that all components of the metric are independent of the extra coordinate, y, we get the most general transformations allowed

$$x^\mu \longrightarrow x'^\mu(x^\nu)\,,\quad y \longrightarrow y + f(x^\mu)\,. \tag{2.2}$$

Putting $g_{44} = V$, the 5D metric tensor reads therefore as

$$g_{AB} = \begin{pmatrix} \gamma_{\mu\nu} + V A_\mu A_\nu & A_\mu V \\ A_\nu V & V \end{pmatrix}, \quad \mu, \nu = 0, \cdots, 3\,. \tag{2.3}$$

The transformations (2.2) imply that A_μ transforms as a gauge vector field,

$$g_{\mu 4} \longrightarrow g_{\mu 4} - V \partial_\mu f \quad \Rightarrow \quad A_\mu \longrightarrow A_\mu - \partial_\mu f\,. \tag{2.4}$$

The “vertical” translation yields hence a $U(1)$ gauge transformation for the vector field A_μ so that the theory (2.3) is locally $U(1)$ gauge invariant. The Kaluza-Klein vector A_μ can thus be identify with the electromagnetic field.

Let us now embed the metric (2.3) into the Einstein-Hilbert action defined in (2.1). We have

$$\det(g_{AB}) = \det(\gamma_{\mu\nu}) V = g_4 V\,,$$

and it is also useful to calculate the Christoffel connections. The 5D Ricci scalar $\mathcal{R}_5$ is expressed in terms of the 4D scalar curvature $\mathcal{R}_4$, the field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and the scalar field V,

$$\mathcal{R}_5 = \mathcal{R}_4 - \frac{1}{4} V F_{\mu\nu} F^{\mu\nu} - \frac{2}{\sqrt{V}} \Box \sqrt{V}\,. \tag{2.5}$$

Substituting this $\mathcal{R}_5$ into the action in (2.1) and integrating with respect to the cyclic variable y, we obtain the 4D effective action,

$$S_4 = -\frac{1}{16\pi G_5} \int_{\mathcal{M}_4} dx^4 \sqrt{-g_4 V} \big(\mathcal{R}_4 - \frac{1}{4} V F_{\mu\nu} F^{\mu\nu}\big) + \frac{1}{8\pi G_5} \int_{\mathcal{M}_4} dx^4 \sqrt{-g_4}\, \Box \sqrt{V}\,, \tag{2.6}$$

where G_5 is the 5D Newton coupling constant. The second integral term in (2.6) can be dropped since it is a surface term and does not affect therefore the equations of the motion. Thus we end up with the following 4D action,

$$S_4 = -\frac{1}{16\pi G_5} \int_{\mathcal{M}_4} dx^4 \sqrt{-g_4 V} \big(\mathcal{R}_4 - \frac{1}{4} V F_{\mu\nu} F^{\mu\nu}\big)\,, \tag{2.7}$$

wich involves GR and the Maxwell theory, coupled to an additional scalar field V.

Let us now study the dynamics of a classical point-like test particle of unit mass in our 5D space-time. Consider 5D geodesic motion,

$$\frac{d^2 x^A}{d\tau^2} + \Gamma^A_{BC} \frac{dx^B}{d\tau} \frac{dx^C}{d\tau} = 0\,, \tag{2.8}$$

where τ denotes the proper time. Using the effective theory (2.3) in (2.8), a routine calculation yields the equations of the motion,

$$\begin{aligned} &\frac{d}{d\tau}(VA_\mu \frac{dx^\mu}{d\tau} + V\frac{dy}{d\tau}) = \frac{dq}{d\tau} = 0\,, \\ &\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau} - qF^\mu_\lambda \frac{dx^\lambda}{d\tau} - \frac{q^2}{2}\frac{\partial^\mu V}{V^2} = 0\,. \end{aligned} \tag{2.9}$$

The first equation in (2.9) tells us that the “charge”,

$$q = V\left(A_\mu \frac{dx^\mu}{d\tau} + \frac{dy}{d\tau}\right)\,, \tag{2.10}$$

is conserved along the 5D geodesics. The latter can also be viewed as being associated with translation, in the “extra” direction, generated by the Killing vector ∂_y. The second equation in (2.9) is a 4D geodesic equation involving in addition to the Lorentz force an interaction with the scalar field V. See [Kerner 2000] for a point of view with $V = 1$. See also [Kibble 1961, Trautman 1970].

Non-Abelian generalization.

The non-Abelian extension of the 5D KK approach was given by Kerner in [Kerner 1968]. First, we generalize our previous 5D manifold into a (4+d)-dimensional manifold noted as $\mathcal{M} = \mathcal{M}^4 \otimes \mathcal{S}^d$. The base $\mathcal{M}^4$ denotes the usual space-time with coordinates x^μ, and $\mathcal{S}^d$ represents an unobservable d-dimensional extra space with the locally geodesic coordinates y^a, $a, b = 4, \cdots, (3+d)$. For definiteness, we fix $d = 3$ so that $\mathcal{S}^3$, viewed as a Lie group, is isomorphic to the non-Abelian group $SU(2)$. Moreover, the compact manifold $\mathcal{S}^3$ admits the isometry generators, $\Xi_j = -i\xi^b_j(y)\partial_b$, whose algebra reproduces the $SU(2)$ Lie algebra,

$$[\Xi_j, \Xi_k] = i\,\varepsilon^{\,l}_{\;jk}\,\Xi_l\,, \tag{2.11}$$

and which imply the relation,

$$\xi^b_k(y)\,\partial_b\xi^a_j(y) - \xi^b_j(y)\,\partial_b\xi^a_k(y) = \varepsilon^{\,l}_{\;jk}\,\xi^a_l(y)\,. \tag{2.12}$$

The anti-symmetric tensor $\varepsilon^{\,l}_{\;jk}$ denotes the structure constants of the $SU(2)$ non-Abelian gauge group. In the KK approach the 7D diffeomorphism symmetry is broken into 4D infinitesimal coordinates transformations augmented with translations along the extra dimensions,

$$x^\mu \longrightarrow x^\mu + \delta x^\mu\,, \quad y^a \longrightarrow y^a + f^i(x^\nu)\xi^a_i(y)\,. \tag{2.13}$$

Here the $f^i(x^\nu)$ are functions. The 7D generalized metric, invariant under (2.13), then reads

$$\tilde{g}_{CD} = \begin{pmatrix} \gamma_{\mu\nu} + \kappa_{ab}B^a_\mu B^b_\nu & B^b_\mu\kappa_{ba} \\ \kappa_{ab}B^a_\nu & \kappa_{ab} \end{pmatrix}, \quad C, D = 0, \cdots, 6\,, \tag{2.14}$$

where κ_{ab} is the $SU(2)$ invariant metric and

$$B^a_\mu = A^b_\mu \, \xi^a_b \, . \tag{2.15}$$

The $SU(2)$ Lie algebra-valued one-form A^b_μ here will be identified with the Yang-Mills field. A^a_μ transforms indeed as a non-Abelian gauge field. Under (2.13) the part κ_{ab} of the metric (2.3) is preserved. Using the formula $\xi'^a_k = \xi^a_k + \xi^a_j \varepsilon^j_{kl} f^l$ due to (2.13), the off-diagonal components $\tilde{g}_{\mu b}$ of $\tilde{g}_{CD}$ change as

$$\tilde{A}^a_\mu = A^a_\mu(x) - \partial_\mu f^a + \varepsilon^a_{bc} A^b_\mu f^c = A^a_\mu(x) - D_\mu f^a \, , \tag{2.16}$$

where

$$D_\mu f^a = \partial_\mu f^a - \varepsilon^a_{bc} A^b_\mu f^c \tag{2.17}$$

is the gauge-covariant derivative. The field strength of the potential A^b_μ ,

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - \varepsilon^a_{bc} A^b_\mu A^c_\nu \, . \tag{2.18}$$

changes in turn as

$$\tilde{F}^a_{\mu\nu} = F^a_{\mu\nu} - \varepsilon^a_{\ bc} f^b F^c_{\mu\nu} \, . \tag{2.19}$$

For Abelian groups, the structure constants vanish so that the field strength is invariant and (2.17) reduces to simple derivative.

This is exactly how an infinitesimal gauge transformation, $\delta y^a = f^i(x^\nu)\xi^a_i(y)$, acts on a non-Abelian gauge field. The result (2.16) differs from the transformation law of Abelian gauge fields by the presence of the term $\varepsilon^a_{bc} A^b_\mu f^c$.

We now discuss the reduction of the dynamics starting from the 7D Einstein-Hilbert action,

$$S_7 = -\frac{1}{16\pi \tilde{G}_7} \int_{\mathcal{M}_4} d^4x d^3y \sqrt{-g_7}\, \mathcal{R}_7 \, . \tag{2.20}$$

A tedious calculation provides us with the reduced scalar curvature so that the action (2.20) can be reduced as

$$S_4 = -\frac{1}{16\pi G_7} \int_{\mathcal{M}_4} d^4x \sqrt{\gamma} \left(\mathcal{R}_4 + \frac{1}{vol(\mathcal{S}^3)} \int_{\mathcal{S}^3} d^3y \sqrt{\kappa}\, \mathcal{R}_3 - \frac{1}{4} \kappa_{ab} F^a_{\mu\nu} F^{b\,\mu\nu} \right) . \tag{2.21}$$

The 7D Newton constant reads $G_7 = \tilde{G}_7 / vol(\mathcal{S}^3)$ while $\mathcal{R}_4$ and $\mathcal{R}_3$ are the scalar curvatures associated with the metrics $\gamma_{\mu\nu}$ and κ_{ab}, respectively. The action (2.21) describes an Einstein-like dynamics plus its coupling to the Yang-Mills fields. Note that the second term in (2.21) is given by the curvature of the extra-space.

We focus our attention on the dynamics of a classical point-like test particle of unit mass in $(4+3)$-dimensional space-time. To this end, we consider the Lagrangian for geodesic motion in total space,

$$\mathcal{L} = \tilde{g}_{CD} \frac{dx^C}{d\tau} \frac{dx^D}{d\tau} \, . \tag{2.22}$$

For our metric (2.14), the Lagrange function (2.22) corresponds to

$$\mathcal{L} = \gamma_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} + \kappa_{ab}\big(\frac{dy^a}{d\tau} + A^a_\mu\frac{dx^\mu}{d\tau}\big)\big(\frac{dy^b}{d\tau} + A^b_\mu\frac{dx^\mu}{d\tau}\big)\,, \tag{2.23}$$

and we evaluate the associated Euler-Lagrange equations,

$$\begin{cases} \dfrac{d}{d\tau}\left(\dfrac{\partial\mathcal{L}}{\partial\big(\frac{dx^\alpha}{d\tau}\big)}\right) - \dfrac{\partial\mathcal{L}}{\partial x^\alpha} = 0\,, \quad \alpha = 0,\cdots,3 \\ \dfrac{d}{d\tau}\left(\dfrac{\partial\mathcal{L}}{\partial\big(\frac{dy^c}{d\tau}\big)}\right) - \dfrac{\partial\mathcal{L}}{\partial y^c} = 0\,, \quad c = 4,5,6\,. \end{cases} \tag{2.24}$$

The first equation in (2.24) yields the motion projected into real 4D space-time, whereas the second equation describes the motion in 3D internal space. In details, we have

$$\begin{aligned} \partial_c\mathcal{L} &= 2\kappa_{ab}\left(\frac{dy^b}{d\tau} + A^b_\nu\frac{dx^\nu}{d\tau}\right)\varepsilon^a_{\ bc}A^b_\mu\frac{dx^\mu}{d\tau}\,, \\ \partial_\alpha\mathcal{L} &= \partial_\alpha\gamma_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\mu}{d\tau} + 2\kappa_{ab}\left(\frac{dy^a}{d\tau} + A^a_\mu\frac{dx^\mu}{d\tau}\right)\partial_\alpha A^b_\nu\frac{dx^\nu}{d\tau}\,. \end{aligned} \tag{2.25}$$

Let us now identify the following quantity,

$$\mathcal{I}_a = \kappa_{ab}\left(\frac{dy^b}{d\tau} + A^b_\nu\frac{dx^\nu}{d\tau}\right)\,, \tag{2.26}$$

as the classical isospin variable which describes the motion in (non-Abelian) internal space.

Next, calculating the remaining terms in (2.24), we find

$$\begin{aligned} \frac{d}{d\tau}\left(\frac{\partial\mathcal{L}}{\partial\big(\frac{dy^c}{d\tau}\big)}\right) &= 2\frac{d\mathcal{I}_c}{d\tau}\,, \\ \frac{d}{d\tau}\left(\frac{\partial\mathcal{L}}{\partial\big(\frac{dx^\alpha}{d\tau}\big)}\right) &= \big(\partial_\mu\gamma_{\alpha\nu} + \partial_\nu\gamma_{\alpha\mu}\big)\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} + 2\gamma_{\alpha\nu}\frac{d^2x^\nu}{d\tau^2} \\ &\quad + 2\mathcal{I}_b\big(\partial_\beta A^b_\alpha\frac{dx^\beta}{d\tau} + \varepsilon^b_{\ ca}A^a_\alpha A^c_\mu\frac{dx^\mu}{d\tau}\big)\,. \end{aligned} \tag{2.27}$$

Collecting the results (2.25) and (2.27), we obtain the equations of motion of an isospin-carrying particle in a curved space plus a Yang-Mills field,

$$\begin{cases} \dfrac{d^2x^\beta}{d\tau^2} + \Gamma^\beta_{\mu\nu}\dfrac{dx^\mu}{d\tau}\dfrac{dx^\nu}{d\tau} + \gamma^{\nu\beta}F^b_{\mu\nu}\mathcal{I}_b\dfrac{dx^\mu}{d\tau} = 0\,, \\ \dfrac{d\mathcal{I}_c}{d\tau} - \mathcal{I}_a\varepsilon^a_{\ bc}A^b_\mu\dfrac{dx^\mu}{d\tau} = 0\,. \end{cases} \tag{2.28}$$

The first equation in (2.28) describes the motion in 4D real space. Note here the generalized Lorentz force

$$\gamma^{\nu\beta} F^b_{\mu\nu} \mathcal{I}_b \frac{dx^\mu}{d\tau} \tag{2.29}$$

due to the Yang-Mills field, where the electric charge is replaced by the isospin $\mathcal{I}^a$. The derivation of the latter is analogous to that of the electric charge in the 5D KK theory, since it is also the contraction of the Killing vector field generating “vertical” translations with the direction field of the geodesic.

The second equation in (2.28) says that the isospin is parallel transported in the internal space. Remark that the equations (2.28) can also be obtained using the 7D geodesic equation,

$$\frac{d^2x^C}{d\tau^2} + \tilde{\Gamma}^C_{DE} \frac{dx^D}{d\tau} \frac{dx^E}{d\tau}, \quad C, D, E = 0, \cdots, 6. \tag{2.30}$$

The equations (2.28) are known as the Kerner-Wong equations. Indeed, some time after Kerner, Wong [Wong 1970] obtained the same equations by “dequantizing” the Dirac equation. Later Balachandran et al. [Balachandran 1977] also deduced the equations (2.28) using a variational principle. Alternatively, they can be studied using a symplectic approach, [Duval 1978, Duval 1982, Fehér 1986*].

2.2 van Holten's covariant Hamiltonian dynamics

The standard approach to identify the constants of the motion associated with the symmetries of a given mechanical system is achieved through Noether's theorem [Forgács-Manton 1980, Jackiw-Manton 1980], summarized in the next subsection. More recently, however, an alternative approach has been put forward by van Holten [van Holten 2007]. To present his covariant Hamiltonian dynamics, let us consider a non-relativistic charged isospin-carrying particle in three-dimensions with Hamiltonian

$$\mathcal{H} = \frac{\vec{\pi}^2}{2} + V(\vec{x}, \mathcal{I}^a)\,, \quad \vec{\pi} = \vec{p} - e\vec{A}\,. \tag{2.31}$$

Here $\vec{p}$ and $\vec{\pi}$ define the canonical and the gauge-covariant momenta, respectively, and V is an additional momentum-independent scalar potential. The gauge potential $\vec{A} = \vec{A}^a\,\mathcal{I}^a$, with the internal index $a = 1, 2, 3$ referring to the non-Abelian su(2) Lie algebra, describes a static non-Abelian gauge field. Note that all dynamical variables here are gauge invariant.

Identifying the su(2) Lie algebra of the non-Abelian variable with $\mathbb{R}^3$, we consider the covariant phase space $\big(\vec{x},\, \vec{\pi},\, \vec{\mathcal{I}}\,\big)$, where the dynamics,

$$\dot{f} = \{f, H\}\,,$$

is defined by the covariant Poisson brackets,

$$\{f, g\} = D_j f\,\frac{\partial g}{\partial \pi_j} - \frac{\partial f}{\partial \pi_j} D_j g + e\,\mathcal{I}^a F^a_{jk}\frac{\partial f}{\partial \pi_j}\frac{\partial g}{\partial \pi_k} - \epsilon_{abc}\frac{\partial f}{\partial \mathcal{I}^a}\frac{\partial g}{\partial \mathcal{I}^b}\mathcal{I}^c\,. \tag{2.32}$$

The field strength and the gauge covariant derivative read

$$\begin{aligned} F_{jk} &= \partial_j A_k - \partial_k A_j - e\,\epsilon_{abc}\mathcal{I}^a A^b_j A^c_k\,, \\ D_j &= \partial_j - e\,\epsilon_{abc}\mathcal{I}^a A^b_j \frac{\partial}{\partial \mathcal{I}_c}\,, \end{aligned} \tag{2.33}$$

respectively. The commutator of the covariant derivatives is recorded as

$$[D_i, D_j] = -\epsilon_{abc}\mathcal{I}^a F^b_{ij}\,\frac{\partial}{\partial \mathcal{I}^c}\,. \tag{2.34}$$

It is straightforward to obtain the non-vanishing fundamental Poisson-brackets,

$$\{x_i, \pi_j\} = \delta_{ij}, \qquad \{\pi_i, \pi_j\} = e\,\mathcal{I}^a F^a_{ij}, \qquad \{\mathcal{I}^a, \mathcal{I}^b\} = -\epsilon_{abc}\mathcal{I}^c. \tag{2.35}$$

Let us remark that from the Jacobi identities we can derive the electromagnetic field equation,

$$\{\pi_i, \{\pi_j, \pi_k\}\} + \{\pi_j, \{\pi_k, \pi_i\}\} + \{\pi_k, \{\pi_i, \pi_j\}\} = 0 \quad \Leftrightarrow \quad D_i\big(\mathcal{I}^a F^a_{ij}\big) = 0\,. \tag{2.36}$$

We can now derive the Kerner-Wong equations of motion [cf. (2.28)],

$$\left\{\begin{array}{lcl}\dfrac{d^2x_i}{dt^2}-e\,\mathcal{I}^aF^a_{ij}\dfrac{dx^j}{dt}+D_iV & = & 0\,,\\ \dfrac{d\mathcal{I}^a}{dt}-\epsilon_{abc}\mathcal{I}^b\left(\dfrac{\partial V}{\partial\mathcal{I}^c}-e\,A^c_j\,\dfrac{dx^j}{dt}\right) & = & 0\,.\end{array}\right. \tag{2.37}$$

To construct the dynamical quantities $\mathcal{Q}\big(\vec{x},\vec{\pi},\vec{\mathcal{I}}\big)$ which are conserved along the motion, we use the covariant van Holten recipe [van Holten 2007]. The clue here is to expand constants of the motion in powers series of the covariant momenta,

$$\mathcal{Q}\big(\vec{x},\vec{\pi},\vec{\mathcal{I}}\big)=C(\vec{x},\vec{\mathcal{I}})+C_i(\vec{x},\vec{\mathcal{I}})\pi_i+\frac{1}{2!}C_{ij}(\vec{x},\vec{\mathcal{I}})\pi_i\pi_j+\cdots \tag{2.38}$$

Requiring $\mathcal{Q}$ to Poisson-commute with the Hamiltonian,

$$\{\mathcal{Q},\mathcal{H}\}=0\,, \tag{2.39}$$

leads us with the set of constraints,

$$\begin{array}{lr}C_iD_iV+\epsilon_{abc}\mathcal{I}^a\dfrac{\partial C}{\partial\mathcal{I}^b}\dfrac{\partial V}{\partial\mathcal{I}^c}=0, & o(0)\\ D_iC=e\mathcal{I}^aF^a_{ij}C_j+C_{ij}D_jV+\epsilon_{abc}\mathcal{I}^a\dfrac{\partial C_i}{\partial\mathcal{I}^b}\dfrac{\partial V}{\partial\mathcal{I}^c}, & o(1)\\ D_iC_j+D_jC_i=e\mathcal{I}^a(F^a_{ik}C_{kj}+F^a_{jk}C_{ki})+C_{ijk}D_kV+\epsilon_{abc}\mathcal{I}^a\dfrac{\partial C_{ij}}{\partial\mathcal{I}^b}\dfrac{\partial V}{\partial\mathcal{I}^c}, & o(2)\\ D_iC_{jk}+D_jC_{ki}+D_kC_{ij}=e\mathcal{I}^a(F^a_{il}C_{ljk}+F^a_{jl}C_{lki}+F^a_{kl}C_{lij})+C_{ijkl}D_lV & \\ \qquad\qquad+\epsilon_{abc}\mathcal{I}^a\dfrac{\partial C_{ijk}}{\partial\mathcal{I}^b}\dfrac{\partial V}{\partial\mathcal{I}^c}, & o(3)\\ \vdots \qquad\qquad \vdots & \vdots\end{array} \tag{2.40}$$

The series of constraints (2.40) is a priori infinite since the expansion (2.38) is also infinite. But for conserved quantities which admit finite expansion in covariant momenta, the series of constraints (2.40) can be truncated at a finite order-($n+1$) provided we search for order-n constants of motion. Hence, we can set $C_{i_1\ldots i_{n+1}\ldots}=0$, such that the higher-order constraint of (2.40) becomes the covariant Killing equation,

$$D_{(i_1}C_{i_2\ldots i_{n+1})}=0\,,\quad n\in\mathbb{N}^\star. \tag{2.41}$$

It is worth noting that apart from the zeroth-order constants of the motion, i.e., which does not depend on the covariant momentum, all order-n invariants are deduced from the systematic method (2.40) implying rank-n Killing tensors, each Killing tensor solving the equation (2.41). These Killing tensors also represent the higher-order coefficient of the expansion (2.38) and, thus, can generate conserved quantities. The intermediate-order constraints of (2.40) determine the other coefficient-terms of the invariant whereas the zeroth-order equation can be interpreted as a consistency condition between the scalar potential and the conserved quantity constructed.

To determine the conserved quantities of the system, we can consider the point of view consisting on solving first the equation (2.41) in order to deduce Killing tensors. This task is extremely difficult due to the fact that the gauge covariant derivatives do not commute. But, in the particular case where

$$C_{i_2\ldots i_{n+1}}(\vec{x},\vec{\mathcal{I}}) \equiv C_{i_2\ldots i_{n+1}}(\vec{x})\,, \tag{2.42}$$

we can easily solve (2.41) for $n=1$. We find the general form of the Killing vectors,

$$C_i = \mathcal{C}_{i\,Y_1\cdots Y_m}\,\xi^{Y_1\cdots Y_m} = a_{ij}x^j + b_i\,, \quad a_{ij} = -a_{ji}\,, \tag{2.43}$$

where b_j and the anti-symmetric a_{ij} denote constant tensors. It is worth mentioning that from the Killing vectors (2.43), one can define the associated Killing tensors of Yano-type, $\mathcal{C}_{i\,Y_1\cdots Y_m}(\vec{x})$, which satisfy the Killing equation

$$D_i\mathcal{C}_{j\,Y_1\cdots Y_m}(\vec{x}) + D_j\mathcal{C}_{i\,Y_1\cdots Y_m}(\vec{x}) = 0\,. \tag{2.44}$$

Note that the Killing-Yano tensors are completely anti-symmetric differential forms in their indices.

To solve (2.41) for $n=2$, even under the condition (2.42), remains, however, an awkward task. The trick is to construct the rank-2 Killing tensors $C_{ij}(\vec{x})$ as a symmetrized product [Gibbons 12/1986] of Yano-type Killing tensors,

$$C_{ij}(\vec{x}) = \mathcal{C}_{i\,Y_1\cdots Y_m}\,\widetilde{\mathcal{C}}_{j\,Y_1\cdots Y_m} + \widetilde{\mathcal{C}}_{i\,Y_1\cdots Y_m}\,\mathcal{C}_{j\,Y_1\cdots Y_m}\,. \tag{2.45}$$

As an illustration, consider the two Killing-Yano tensors,

$$\begin{cases} \mathcal{C}_{i\,Y} = \epsilon_{iYl}\,n^l\,, \quad (\text{with } \vec{n} \text{ a constant unit vector})\,, \\[2mm] \widetilde{\mathcal{C}}_{j\,Y} = \epsilon_{jYk}\,x^k\,, \end{cases} \tag{2.46}$$

extracted from (2.43). The symmetrized product (2.45) of the Killing-Yano tensors (2.46) provides us with the rank-2 Killing tensor generating Kepler-type dynamical symmetry [Ngome 03/2010],

$$C_{ij}(\vec{x}) = 2\delta_{ij}\big(\vec{n}\cdot\vec{x}\big) - n_ix_j - n_jx_i\,. \tag{2.47}$$

In the following, we focus our attention at given Killing tensors.

• For $n=1$, (2.41) provides us with Killing vectors. For example, we have, for any unit vector $\vec{n}$, the generator of rotations around the axis $\vec{n}$,

$$\vec{C} = \vec{n}\times\vec{x} \tag{2.48}$$

leading to the conserved angular momentum.

The generator of space translations along an axis $\vec{n}$,

$$\vec{C} = \alpha \vec{n}\,, \quad \alpha \in \mathbb{R}\,, \tag{2.49}$$

implies a conserved quantity which is identified with the "magnetic translations".

• For $n = 2$, then (2.41) yields rank-2 Killing tensors. Similarly, for any unit vector $\vec{n}$,

$$C_{ij} = 2\delta_{ij}\,\vec{n}\cdot\vec{x} - (n_i x_j + n_j x_i) \tag{2.50}$$

is a Killing tensor of rank 2 associated with the conserved Laplace-Runge-Lenz vector.

The rank-2 Killing tensor implying the conservation of energy reads

$$C_{ij} = \delta_{ij}\,. \tag{2.51}$$

The constant rank-2 Killing tensor generating the conserved Fradkin tensor, associated with the three-dimensional $SU(3)$ oscillator symmetry, is

$$C_{ij} = \alpha_{ij}\,, \quad \alpha_{ij} = \text{const}\,. \tag{2.52}$$

• For $n \geq 3$, the equation (2.41) provides us with higher-rank Killing tensors which, in general, generate product of already known constants of motion. Thus, no new conserved quantities and therefore no new symmetries arise in general from these higher-order Killing tensors.

In this section, we outlined the van Holten procedure (2.40) to derive the symmetries of particle in flat space. This recipe can conveniently be extended to curved space provided the partial derivative is replaced by the metric covariant derivative, $\partial_i \to \nabla_i$. Moreover, the van Holten algorithm is practical and efficient to derive linear and higher-order invariants in the momenta since the only requirement is to have a Killing tensor corresponding to a symmetry transformation.

2.3 The Forgács-Manton-Jackiw approach

The invariants of a system can also be sought using the Forgács-Manton-Jackiw approach based on the study of symmetric gauge fields [Forgács-Manton 1980, Jackiw-Manton 1980, Jackiw 1980]. Let us first consider indeed, the evolution of a free matter system, in the absence of a gauge field. In this case, the dynamics is characterized by several conserved quantities associated with spacetime diffeomorphisms. For instance, invariance under temporal translation generates the conserved energy whereas the space rotational invariance generates conserved angular momentum.

Let us now assume that the matter field interacts with an external gauge field, A_α. In general, the symmetry of the system is broken by this gauge-matter field interaction. However, when A_α and the matter field are both invariant under the same three-dimensional space infinitesimal diffeomorphisms

$$\delta x^\alpha = \omega^\alpha \,, \quad \alpha = 1,2,3\,, \tag{2.53}$$

then, the constants of the motion, namely C^ω, wins an extra term and can therefore be decomposed into two contributing parts,

$$C^\omega = C^\omega_{matter} + C^\omega_{gauge}\,. \tag{2.54}$$

The first term on the right hand side of (2.54), which was the total conserved quantity in the absence of gauge field, corresponds, in the presence of an external gauge field, to the matter contribution augmented with that of the gauge field-matter interaction into the constant of the motion,

$$C^\omega_{matter} = \omega^\alpha \pi_\alpha\,, \tag{2.55}$$

where π_α represents the gauge covariant momentum.

The second term of (2.54), namely C^ω_{gauge}, is the gauge field's additional contribution restoring the full symmetry (2.53) of the system. The Forgács-Manton-Jackiw approach developped here is thus a systematic method to construct the gauge field additional contribution C^ω_{gauge} into the constant of the motion.

Let us first define the Lie derivative of a tensor dragged along the flow, $\mathcal{C}$, described by the vector field ω^α. For this, we consider the infinitesimal coordinate transformation,

$$x^\alpha \longrightarrow x'^\alpha = x^\alpha + \delta t\, \omega^\alpha\,, \quad \delta t \ll 1\,, \tag{2.56}$$

associated with the diffeomorphisms generate by the vector flow ω^α. See (2.53). (2.56) thus implies the tensor field transformations,

$$\upsilon^\beta\left(x^\alpha\right) \longrightarrow \upsilon'^\beta\left(x'^\alpha\right)\,, \quad A_\beta\left(x^\alpha\right) \longrightarrow A'_\beta\left(x'^\alpha\right)\,. \tag{2.57}$$

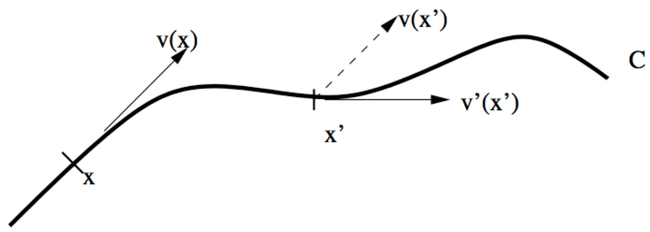

Figure 2.1:

The Lie derivative along $\mathcal{C}$ takes the standard form

$$\begin{cases} L_\omega \upsilon^\alpha = \delta\upsilon^\alpha = \omega^\mu \partial_\mu \upsilon^\alpha - \upsilon^\mu \partial_\mu \omega^\alpha \\ L_\omega A_\beta = \delta A_\alpha = \omega^\mu \partial_\mu A_\beta + A_\mu \partial_\beta \omega^\mu \\ L_\omega f = \omega^\beta \partial_\beta f \,, \end{cases} \tag{2.58}$$

and can be generalize to a (p, q)-tensor field by

$$\begin{aligned} L_\omega T^{m_1,\ldots,m_p}_{n_1,\ldots,n_q} = \omega^\mu \partial_\mu T^{m_1,\ldots,m_p}_{n_1,\ldots,n_q} + T^{m_1,\ldots,m_p}_{\mu,n_2,\ldots,n_q} \partial_{n_1} \omega^\mu + \ldots + T^{m_1,\ldots,m_p}_{n_1,\ldots,n_{q-1},\mu} \partial_{n_q} \omega^\mu \\ - T^{\mu,m_2,\ldots,m_p}_{n_1,\ldots,n_q} \partial_\mu \omega^{m_1} - \ldots - T^{m_1,\ldots,m_{p-1},\mu}_{n_1,\ldots,n_q} \partial_\mu \omega^{m_p} \,. \end{aligned} \tag{2.59}$$

Following Forgács, Jackiw and Manton, we write the symmetry condition of the gauge field, A_β, along the vector flow ω^β as

$$L_\omega A_\alpha = D_\alpha Q^\omega \,, \tag{2.60}$$

where Q^ω is a differentiable Lie algebra-valued scalar function. Geometrically, the condition (2.60) refers to the action of infinitesimal automorphism of the principal bundle. Note that the effect of the gauge freedom on A_β,

$$A_\alpha \longrightarrow \widetilde{A}_\alpha = A_\alpha + \partial_\alpha \Lambda \,, \tag{2.61}$$

does not affect the symmetry condition (2.60), but only shifts the differentiable scalar field so that

$$L_\omega \widetilde{A}_\alpha = D_\alpha \widetilde{Q}^\omega \quad \text{with} \quad \widetilde{Q}^\omega = Q^\omega + \omega^\mu \partial_\mu \Lambda \,. \tag{2.62}$$

Consequently, the gauge potential $\widetilde{A}_\beta$ also remains invariant under the symmetry transformation generated by ω^b. Thus, as expected, we can conclude that the symmetry condition defined in (2.60) is gauge invariant.

An equivalent way to describe the symmetry condition of a gauge field and therefore to obtain the gauge field contribution to the constant of the motion is to express the Lie derivative in term of the field strength $F_{\mu\nu}$,

$$L_\omega A_\beta = \omega^\mu F_{\mu\beta} + D_\beta\big(\omega^\mu A_\mu\big)\,. \tag{2.63}$$

Injecting this result into (2.60), it is straightforward to obtain the following equivalent symmetry condition implying the gauge field contribution, discussed by Jackiw [Jackiw 1980],

$$F_{\beta\mu}\,\omega^\mu = D_\beta C^\omega_{gauge} \quad \text{with} \quad C^\omega_{gauge} = \omega^\mu A_\mu - Q^\omega\,. \tag{2.64}$$

Here, the gauge field contribution to the constant of the motion is a differentiable scalar function which can be determined by an integration of the equation (2.64).

The physics status of the term C^ω_{gauge} is now clear. Indeed, it represents the response of the external (symmetric) gauge field to a spacetime diffeomorphism. It restores the symmetry of the system and appears as a Lie algebra-valued scalar field contribution to the constant of motion. The complete constant of motion reads therefore as

$$C^\omega = \omega^\nu \pi_\nu + \int \omega^\alpha(x) F_{\alpha\beta}(x)\, dx^\beta\,. \tag{2.65}$$

Let us remark that identifying the Lie algebra of the $SU(2)$ gauge group with $\mathbb{R}^3$, the usual gauge covariant derivative, which we use in this section, becomes the gauge covariant derivative defined as (2.33) in the previous section. The rule is simply to replace the generators of the Lie algebra, τ^a $(a = 1,2,3)$, by the components of the isospin vector, $\mathcal{I}^a$. Under this transformation, the symmetry condition (2.64) becomes precisely (with no scalar potential) the first-order condition in (2.40) that a linear, in the covariant momentum, conserved quantity has to satisfy.

Thus, the Forgács-Jackiw-Manton approach is, in fact, equivalent to the van Holten procedure for linear invariants. To generalize the first-cited method to higher-order contants of the motion, we require the symmetric gauge field to admitting higher-order Killing tensors. Then, as in the case of linear conserved quantities, the invariants can, in that event, be splitted into the two contributing parts (2.54),

$$C^\omega = C^\omega_{matter} + C^\omega_{gauge}\,.$$

In that event, the matter plus matter-gauge fields contributions give rise to the term

$$C^\omega_{matter} = \frac{1}{n!}\omega^{\mu_1\cdots\mu_n}\pi_{\mu_1}\cdots\pi_{\mu_n}\,, \tag{2.66}$$

where $\omega^{\mu_1\cdots\mu_n}$ denotes the Killing tensor field generating the symmetry. The external gauge field carries, however, the contribution C^ω_{gauge} satisfying the constraints,

$$D^{(\mu_1} C^{\omega,\ \mu_2\cdots\mu_{n-1})}_{gauge} = F^{(\mu_1}_{\beta}\ \omega^{\omega,\ \mu_2\cdots\mu_n)}_{\beta}\,. \tag{2.67}$$

We still here recognize the series of constraints of the van Holten algorithm (2.40) for particle evolving in an external gauge field in the absence of an additional scalar potential.

Chapter 3

Abelian monopoles

Contents

Dirac's quantization of magnetic monopole strength is obtained from the associativity of operators multiplication. (Dynamical) symmetries of the generalized Taub-NUT metric and its multi-center extension are investigated.

3.1 Dirac monopole

The concept of magnetic monopole is one of the most influential idea in modern theoretical physics. The hypothesis of particles carrying magnetic charge, g, was first made by Dirac [Dirac 1931], who observed that the phase unobservability in quantum mechanics allows singularities as sources of magnetic fields, just as point electric monopoles are sources of electric fields. These singularities define the celebrated "Dirac string" whose position is not detectable. This implies the so-called Dirac quantization condition,

$$eg = \hbar c \frac{N}{2}\,, \quad N \in \mathbb{Z}^{\star}\,. \tag{3.1}$$

Consequently, the existence of a single magnetic monopole in the universe would explain the quantization of electric charge, for which there is no alternative explanation till this day.

In work preceding Dirac by over fifty years, Maxwell established the equations describing the electromagnetism. A surprising asymmetry inside these Maxwell's equations made Poincaré and J. J. Thompson to infer that a magnetic charge has to be introduced in the theory. The Maxwell equations with this assumption then read

$$\begin{cases} \vec{\nabla}\cdot\vec{E} = 4\pi\rho_e\,, & \vec{\nabla}\cdot\vec{B} = 4\pi\rho_m \\ \vec{\nabla}\times\vec{B} = \dfrac{1}{c}\dfrac{\partial\vec{E}}{\partial t} + \dfrac{4\pi}{c}\vec{j}_e\,, & \vec{\nabla}\times\vec{E} = -\dfrac{1}{c}\dfrac{\partial\vec{B}}{\partial t} - \dfrac{4\pi}{c}\vec{j}_m\,, \end{cases} \tag{3.2}$$

where $\rho_{e,m}$ and $\vec{j}_{e,m}$ denote the electric/magnetic charge and current density, respectively. But this introduction responded to a mathematical convenience and had, at that time, no physical reality; although, at the same period, P. Curie raised the possibility of the existence of free magnetic poles [P. Curie 1894].

However, studying the motion of a charged particle in the field of an hypothetic isolated magnetic monopole, Poincaré [Poincaré 1896] observed that, as the particle is no longer deal with central forces, the angular momentum is no longer conserved and the motion is no longer necessarily planar. However, a certain amount of angular momentum resides in the magnetic field, and that a total angular momentum does exist,

$$\vec{J} = \vec{L} - q\frac{\vec{x}}{r}\,, \quad \vec{L} = \vec{x} \times \vec{\pi}\,. \tag{3.3}$$

Here $\vec{L}$ denotes the mechanical angular momentum and the term $\big(-q\vec{x}/r\big)$ represents the Poincaré momentum with q denoting the magnetic pole strength. The total angular momentum (3.3) is conserved along the motion.

Later, Wu and Yang [Wu Yang 1975] showed that the Dirac string, which was introduced as a mathematical artifact, can be totally avoided using two different choices of vector potential compatible with the monopole field strength. These two patches read

$$A_r = A_\theta = 0\,, A_\phi = \begin{cases} \dfrac{g}{r\sin\theta}\big(1-\cos\theta\big) & \text{for} \quad 0 \le \theta \le \dfrac{\pi}{2} + \delta\,, \\ \dfrac{-g}{r\sin\theta}\big(1+\cos\theta\big) & \text{for} \quad \dfrac{\pi}{2} - \delta \le \theta \le \pi\,, \end{cases} \tag{3.4}$$

for any arbitrary δ in the range $0 < \delta < \pi/2$. Each region contains a singularity if we try to extend them over the entire region around the monopole as Dirac did, but is regular in its restricted hemisphere. In the overlapping region

$$\pi/2 - \delta \le \theta \le \pi/2 + \delta\,,$$

the two patches are related by a gauge transformation,

$$\vec{A}_N = \vec{A}_S - \vec{\nabla}\big(2g\Lambda(\vec{x})\big)\,.$$

The latter transformation changes the particle wave functions as

$$\Psi(\vec{x}) \longrightarrow \exp\big(2ieg\Lambda(\vec{x})\big)\Psi(\vec{x})\,,$$

so that requiring the exponential to be single valued everywhere leads to the Dirac quantization condition [Wu Yang 1975], [cf. 3.1].

From now on, we discuss the Dirac magnetic monopole without reference to singular patches or vector potential [Jackiw 12/2002]. To this end, we define the Hamiltonian of the monopole system as

$$\mathcal{H} = \frac{\pi^2}{2m}\,, \quad \pi_j = p_j - \frac{e}{c}A_j\,, \quad p_j = -i\hbar\partial_j\,, \tag{3.5}$$

and the following fundamental commutation rules are posited

$$\left[x^i, x^j\right] = 0, \quad \left[x^i, \pi_j\right] = i\hbar\delta^i_j, \quad \left[\pi_i, \pi_j\right] = ie\frac{\hbar}{c}\epsilon_{ijk}B^k\,. \tag{3.6}$$

Taking into account (3.5) and (3.6), we derive the gauge-invariant Lorentz-Heisenberg equations specifying the motion of a massive charged particle in the external monopole field $\vec{B}$,

$$\begin{cases} \dot{\vec{x}} = \frac{i}{\hbar}\left[\mathcal{H}, \vec{x}\right] = \frac{\vec{\pi}}{m} \\ \dot{\vec{\pi}} = \frac{i}{\hbar}\left[\mathcal{H}, \vec{\pi}\right] = \frac{e}{2mc}\left(\vec{\pi}\times\vec{B} - \vec{B}\times\vec{\pi}\right). \end{cases} \tag{3.7}$$

A priori no constraints on the monopole field $\vec{B}$ are required in the previous equations of the motion. Indeed, equations (3.7) make sense both when $\vec{B}$ is source-free, $\vec{\nabla}\cdot\vec{B} = 0$, or not, $\vec{\nabla}\cdot\vec{B} \neq 0$. However, when we look the Jacobi identities for the commutators of the momenta $\vec{\pi}$, we find

$$\epsilon_{ijk}\left[\pi_i, \left[\pi_j, \pi_k\right]\right] = 2e\frac{\hbar^2}{c}\vec{\nabla}\cdot\vec{B}\,, \tag{3.8}$$

which vanishes, as it should, for a source-free magnetic fields, $\vec{B} = \vec{\nabla}\times\vec{A}$.

In order to obtain the exact form of $\vec{B}$, we study now the Lie algebra associated with the O(3) symmetry of the monopole system. We first remark that the usual angular momentum operator, $\vec{L} = \vec{x}\times\vec{\pi}$, does not satisfy the o(3) Lie algebra, since we get an obstruction term inside of the commutator,

$$\left[L_i, L_j\right] = i\hbar\epsilon_{ijk}L^k + ie\frac{\hbar}{c}\epsilon_{ijk}x^k\left(\vec{x}\cdot\vec{B}\right). \tag{3.9}$$

Following Jackiw [Jackiw 1980], we restore the spherical symmetry of the system by adding a gauge field contribution, $\vec{C}(\vec{x})$, into the angular momentum, $\vec{L}$,

$$\vec{J} = \vec{L} + \vec{C}\,, \tag{3.10}$$

so that we obtain the modified angular momentum algebra,

$$\begin{aligned} \left[x^i, J_j\right] &= i\hbar\epsilon^i{}_{jk}x^k \\ \left[\pi_i, J_j\right] &= i\hbar\epsilon_{ij}{}^k\pi_k + ie\frac{\hbar}{c}\left(x_iB_j - \delta_{ij}\left(\vec{x}\cdot\vec{B}\right)\right) - i\hbar\partial_iC_j \\ \left[J_i, J_j\right] &= i\hbar\epsilon_{ijk}L^k + ie\frac{\hbar}{c}\epsilon_{ijk}x^k\left(\vec{x}\cdot\vec{B}\right) + i\hbar\epsilon_{ijk}x^m\left(\epsilon^k{}_{pl}\epsilon^n{}_{mp}\partial_nC_l\right). \end{aligned} \tag{3.11}$$

It is now clear that the contribution $\vec{C}(\vec{x})$ restores the standard angular momentum algebra,

$$\left[x^i, J_j\right] = i\hbar\epsilon^i{}_{jk}x^k\,, \quad [\pi_i, J_j] = i\hbar\epsilon_{ijk}\pi^k\,, \quad [J_i, J_j] = i\hbar\epsilon_{ijk}J^k\,, \tag{3.12}$$

provided that the following constraints are verified,

$$\begin{cases} \partial_i C^j = \dfrac{e}{c}\left(x_i B^j - \delta_i^j\left(\vec{x}\cdot\vec{B}\right)\right) \\ C^k = \dfrac{e}{c}x^k\left(\vec{x}\cdot\vec{B}\right) - x^j\left(\epsilon_{il}{}^k\epsilon_{ij}{}^m\partial_m C^l\right) \end{cases} \Longrightarrow \quad C^k + \frac{e}{c}x^k\left(\vec{x}\cdot\vec{B}\right) = 0\,. \tag{3.13}$$

Consequently, the conserved generalized angular momentum along the motion becomes

$$\vec{J} = \vec{x}\times\vec{\pi} - \frac{e}{c}(\vec{x}\cdot\vec{B})\vec{x}\,. \tag{3.14}$$

Moreover, the integrability condition coming from the equations in the left hand side of (3.13),

$$[\partial_i, \partial_k]\, C^j = 0\,, \tag{3.15}$$

imposes that the field $\vec{B}$ satisfies the structural equation,

$$x_k\partial_i B^j - x_i\partial_k B^j + \delta_i^j\left(B_k + x_m\partial_k B^m\right) - \delta_k^j\left(B_i + x_m\partial_i B^m\right) = 0\,, \tag{3.16}$$

which can conveniently be solved with the magnetic monopole field,

$$\vec{B} = g\frac{\vec{x}}{r^3}\,, \tag{3.17}$$

where g represents the magnetic charge centered at the origin. In fact, the expression (3.17) is the only spherically symmetric possibility consistent with the Jacobi identity (except in the origin),

$$\vec{\nabla}\cdot\vec{B} = 4\pi g\delta^3(\vec{x})\,. \tag{3.18}$$

Note that the obstruction occurs only at the isolated location of the magnetic source, at origin, which has to be excluded.

Following Jackiw [Jackiw 12/2002], the violation, at the origin, of the Jacobi identity (3.18) can be better understood by examining the unitary operator,

$$U(\vec{a}) = \exp\left(-\frac{i}{\hbar}\vec{a}\cdot\vec{\pi}\right)\,, \tag{3.19}$$

which according to (3.6) implements finite translations by $\vec{a}$ on $\vec{x}$,

$$U^{-1}(\vec{a})\,\vec{x}\,U(\vec{a}) = \vec{x} + \vec{a}\,. \tag{3.20}$$

As the momenta operators do not commute according to (3.6), we obtain [1]

$$U(\vec{a})U(\vec{b}) = \exp\left(-\frac{ie}{\hbar c}\Phi\big(\vec{x},\vec{a},\vec{b}\big)\right)U(\vec{a}+\vec{b})\,. \tag{3.21}$$

Here

$$\Phi\big(\vec{x},\vec{a},\vec{b}\big) = \frac{1}{2}\big(\vec{a}\times\vec{b}\big)\cdot\vec{B}$$

represents the flux of the magnetic source through the triangle defined by the three vertices located at $\vec{x}$, $\vec{x}+\vec{a}$ and $\vec{x}+\vec{a}+\vec{b}$.

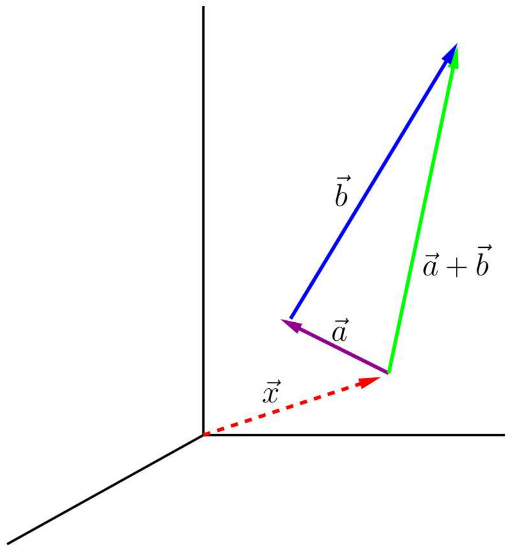

Figure 3.1: Magnetic flux through the triangle.

Using (3.21) it is straightforward to derive the following expressions,

$$\Big(U(\vec{a})U(\vec{b})\Big)\,U(\vec{c}) = \exp\left(-\frac{ie}{\hbar c}\Phi\big(\vec{x},\vec{a},\vec{b}\big)\right)\exp\left(-\frac{ie}{\hbar c}\Phi\big(\vec{x},\vec{a}+\vec{b},\vec{c}\big)\right)U(\vec{a}+\vec{b}+\vec{c})\,,$$

$$U(\vec{a})\,\Big(U(\vec{b})U(\vec{c})\Big) = \exp\left(-\frac{ie}{\hbar c}\Phi\big(\vec{x}-\vec{a},\vec{b},\vec{c}\big)\right)\exp\left(-\frac{ie}{\hbar c}\Phi\big(\vec{x},\vec{a},\vec{b}+\vec{c}\big)\right)U(\vec{a}+\vec{b}+\vec{c})\,.$$

Combining the two previous formulas, we obtain

$$\Big(U(\vec{a})U(\vec{b})\Big)\,U(\vec{c}) = \exp\left(-\frac{ie}{\hbar c}\Omega(\vec{x},\vec{a},\vec{b},\vec{c})\right)U(\vec{a})\,\Big(U(\vec{b})U(\vec{c})\Big)\,, \tag{3.22}$$

where the first term on the right-hand side of (3.22) reads

$$e^{-\frac{ie}{\hbar c}\Omega(\vec{x},\vec{a},\vec{b},\vec{c})} = e^{-\frac{ie}{\hbar c}\Phi\left(\vec{x},\vec{a},\vec{b}\right)}e^{-\frac{ie}{\hbar c}\Phi\left(\vec{x},\vec{a}+\vec{b},\vec{c}\right)}e^{\frac{ie}{\hbar c}\Phi\left(\vec{x},\vec{a},\vec{b}+\vec{c}\right)}e^{\frac{ie}{\hbar c}\Phi\left(\vec{x}-\vec{a},\vec{b},\vec{c}\right)}\,. \tag{3.23}$$

[1] we use the Baker-Campbell-Hausdorff formula: $e^Ae^B = e^{A+B+\frac{1}{2}[A,B]+\cdots}$.

Here $\Omega(\vec{x}, \vec{a}, \vec{b}, \vec{c})$ can be interpreted as the total magnetic flux,

$$\Omega(\vec{x}, \vec{a}, \vec{b}, \vec{c}) = \int d\vec{S} \cdot \vec{B} = \int d\vec{x}\, \vec{\nabla} \cdot \vec{B}\,, \tag{3.24}$$

emerging out from the tetrahedron constructed with the three vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ with one vertex at $\vec{x}$. See Figure 3.2 below.

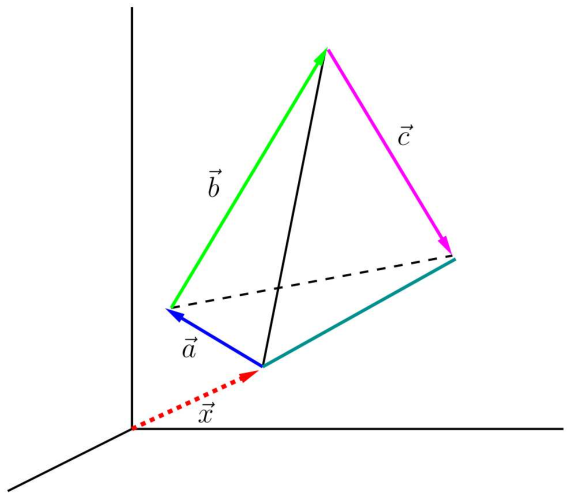

Figure 3.2: Magnetic flux through the tetrahedron.

Positing the following axiom of Quantum mechanics;

Axiom 3.1.1. *: Quantum mechanics realized by linear operators acting on a Hilbert space requires operator multiplication to be associative.*

We therefore obtain that the phase factor on the right hand side of (3.22) has to be unobservable so that the flux is quantized for arbitrary $\vec{a}$, $\vec{b}$ and $\vec{c}$,

$$\exp\left(-\frac{ie}{\hbar c}\Omega(\vec{x}, \vec{a}, \vec{b}, \vec{c})\right) = \exp\left(-i2\pi N\right) = 1\,, \quad N \in \mathbb{Z}\,. \tag{3.25}$$

Consequently we obtain

$$\int d\vec{x}\, \vec{\nabla} \cdot \vec{B} = 2\pi \frac{\hbar c}{e} N \quad \text{with} \quad \vec{\nabla} \cdot \vec{B} = 4\pi g \delta^3(\vec{x}) \neq 0\,, \tag{3.26}$$

which provides us with the Dirac's quantization relation [Dirac 1931],

$$\frac{eg}{\hbar c} = \frac{N}{2}\,, \quad N \in \mathbb{Z}^\star\,. \tag{3.27}$$

Note that the equation (3.26) saves the associativity of operators acting on Hilbert space and thus implies the quantization of the magnetic charge. The only requirement here is that the magnetic field must be a point source or a set of point sources in order to conserve the integrality of the left hand side of (3.26).

Let us now investigate the classical dynamics of a particle evolving in a magnetic monopole field, augmented with a scalar potential V. We inquire, in particular, about Kepler-type dynamical symmetries. Our investigations lead us to the well-known Mcintosh-Cisneros-Zwanziger system [Mcintosh 1970, Zwanziger 1968].

We start with searching conserved quantities associated with the system. A relevant recipe to search for constants of the motion is the van Holten algorithm, presented in Section 2.2. From now on we fix $\hbar = c = 1$ and we expand the constant of motions in terms of covariant momenta,

$$\mathcal{Q}(\vec{x}, \vec{\pi}) = C(\vec{x}) + C_i(\vec{x})\pi_i + \frac{1}{2!}C_{ij}(\vec{x})\pi_i\pi_j + \cdots \tag{3.28}$$

and we require $\mathcal{Q}$ to Poisson-commute with the Hamiltonian of the system. This therefore implies to solve the series of constraints,

$$\begin{cases} C_i\partial_i V = 0, & o(0) \\ \partial_i C = e\,F_{ij}C_j + C_{ij}\partial_j V & o(1) \\ \partial_i C_j + \partial_j C_i = e\,(F_{ik}C_{kj} + F_{jk}C_{ki}) + C_{ijk}\partial_k V & o(2) \\ \partial_i C_{jk} + \partial_j C_{ki} + \partial_k C_{ij} = e(F_{il}C_{ljk} + F_{jl}C_{lki} + F_{kl}C_{lij}) + C_{ijkl}\partial_l V & o(3) \\ \vdots \qquad\qquad \vdots & \vdots \end{cases}$$

Searching for conserved quantity linear in the momentum, we recall that introducing the Killing vector generating space rotations,

$$\vec{C} = \vec{n} \times \vec{x}\,, \tag{3.29}$$

we directly get the associated generalized angular momentum [see (3.14)],

$$\vec{J} = \vec{x} \times \vec{\pi} - eg\frac{\vec{x}}{r}\,. \tag{3.30}$$

Considering quadratic conserved quantities, we first obtain that the rank-2 Killing tensor,

$$C_{ij} = \delta_{ij}\,, \tag{3.31}$$

generates the conserved energy of the system,

$$\mathcal{E} = \frac{\vec{\pi}^2}{2} + V(r)\,. \tag{3.32}$$

On the other hand, inserting into the algorithm the rank-2 Killing tensor generating the

Kepler-type dynamical symmetry,

$$C_{ij} = 2\delta_{ij}\left(\vec{n}\cdot\vec{x}\right) - n_i x_j - n_j x_i\,, \tag{3.33}$$

we solve the second-order constraint with,

$$\vec{C} = eg\frac{\vec{n}\times\vec{x}}{r}\,. \tag{3.34}$$

Next, we insert (3.33) and (3.34) into the first-order constraint of the algorithm and we investigate the integrability condition of this equation by requiring the vanishing of the commutator,

$$[\,\partial_i\,,\,\partial_j\,]\,C = 0 \quad\Longrightarrow\quad \Delta\left(V(r) - \frac{e^2g^2}{2r^2}\right) = 0\,. \tag{3.35}$$

Thus, the bracketed quantity must satisfy a *Laplace equation* so that a Runge-Lenz-type vector does exist only for radial effective potential of the form,

$$V(r) = \frac{e^2g^2}{2r^2} + \frac{\beta}{r} + \gamma \quad \text{with} \quad \beta, \gamma \,\in\, \mathbb{R}\,. \tag{3.36}$$

Consequently, the zeroth-order constraint is identically satisfied and the solution of the first-order constraint reads,

$$C = \beta\frac{\vec{n}\cdot\vec{x}}{r}\,. \tag{3.37}$$

Collecting the results (3.33), (3.34) and (3.37), we get the Runge-Lenz vector conserved along the particle's motion,

$$\vec{K} = \frac{1}{2}\left(\vec{\pi}\times\vec{J} - \vec{J}\times\vec{\pi}\right) + \beta\frac{\vec{x}}{r}\,. \tag{3.38}$$

Note that the presence of the fine-tuned inverse-square term in (3.36) is *mandatory* for canceling the effect of the monopole. For a "naked" monopole, $V \equiv 0$, in particular, no conserved Runge-Lenz vector does exist [Fehér 1986*].

Now we can give a complete description of the classical motion of a charged particle in the Dirac monopole field, augmented with the potential (3.36). A MICZ system in what follows. Firstly, from the angular momentum (3.30) we obtain

$$\vec{J}\cdot\frac{\vec{x}(t)}{r} = -eg\,, \tag{3.39}$$

so that the trajectory followed by the particle lies on a cone with axis $\vec{J}$ and fix opening angle θ defined by

$$\theta = \arccos\left(\frac{-eg}{J}\right)\,. \tag{3.40}$$

Secondly, the conservation of the Runge-Lenz vector (3.38) allows us to construct the new conserved vector,

$$\vec{N} = \frac{\beta}{eg}\vec{J} + \vec{K} \qquad \text{such that} \qquad \vec{N}\cdot\vec{x}(t) = J^2 - e^2g^2 = \text{const} \; . \tag{3.41}$$

The result (3.41) implies that the particle motion also lies in the oblique plan perpendicular to $\vec{N}$. Consequently, combining (3.39) and (3.41) the particle motion is viewed to be confined onto conic sections [Gibbons 04/1986, Fehér 02/2009].

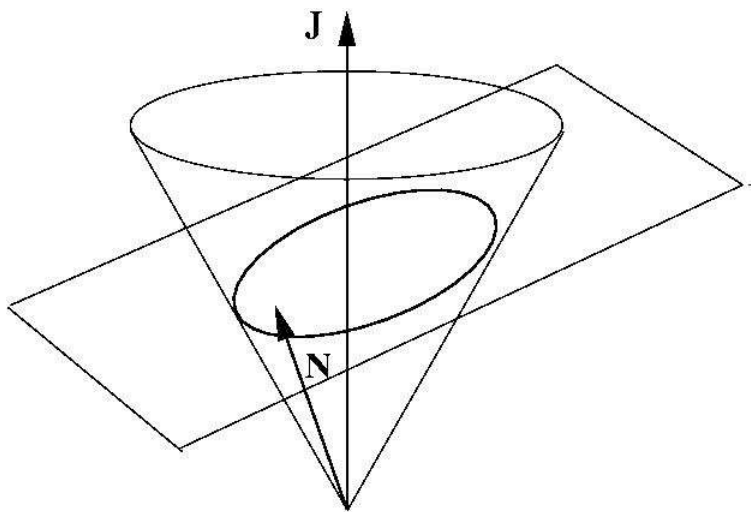

Figure 3.3: The motion lies on the conic section obtained by intersecting the cone, due to conserved angular momentum $\vec{J}$, with the oblique plane determined by the additional conserved quantity $\vec{N}$.

The particular form of the conic section depends only on the angle β (mod $[\pi]$) given by

$$\cos\beta = \frac{\vec{J}\cdot\vec{N}}{J\,N} \, , \tag{3.42}$$

and which determines the inclination of the oblique plane in comparison to the angular momentum vector.

We thus obtain the following properties :

$$\text{For} \quad \begin{cases} \beta \in \left[0,\, \frac{\pi}{2} - \alpha\right[\\ \beta = \frac{\pi}{2} - \alpha\,, \\ \beta \in \left]\frac{\pi}{2} - \alpha,\, \frac{\pi}{2}\right] \end{cases} \quad \text{the trajectories lie on} \quad \begin{cases} \text{ellipses} \\ \text{parabolae} \\ \text{hyperbolae}\,. \end{cases} \tag{3.43}$$

It is worth noting that the momentum-trajectories called the hodographs are also confined to a plane perpendicular to the conserved vector $\vec{N}$ since

$$\vec{N}\cdot\vec{\pi}(t)=0\,. \tag{3.44}$$

But in some interesting cases, the momentum trajectories can be completely determined. For example, in the Kepler problem the $\vec{\pi}$-trajectories are known to be (arcs) of circle. In the context of non-commutative oscillator mechanics (see later), we prove that the hodographs of the MICZ-system lie on (arcs) of ellipses [Ngome 06/2010].

Another illustration of using the symmetry (3.38) is to derive the energy spectrum from the dynamical symmetry [Fehér 10/1986, Horváthy 1990, Fehér 02/2009]. To this end, we return to quantum mechanics and consider the vectors $\vec{J}$ and $\vec{K}$ defined in (3.30) and (3.38), respectively, as operators of the Hilbert space satisfying the quantized commutation relations,

$$[J_i,J_j]=i\epsilon_{ijk}J^k,\quad [J_i,K_j]=i\epsilon_{ijk}K^k,\quad [K_i,K_j]=i\big(2\gamma-2\mathcal{H}\big)\epsilon_{ijk}J^k\,. \tag{3.45}$$

Let us define, on the fixed-energy eigenspace $\mathcal{H}\Psi=\mathcal{E}\Psi$, the rescaled Runge-Lenz operator,

$$\vec{\widetilde{K}}=\begin{cases}\big(2\gamma-2\mathcal{E}\big)^{-\frac{1}{2}}\,\vec{K} & \text{for}\quad \mathcal{E}<\gamma\\ \vec{K}\quad \text{for}\quad \mathcal{E}=\gamma\\ \big(2\mathcal{E}-2\gamma\big)^{-\frac{1}{2}}\,\vec{K} & \text{for}\quad \mathcal{E}>\gamma\end{cases}\,. \tag{3.46}$$

We therefore obtain the commutation relations between $\vec{J}$ and $\vec{\widetilde{K}}$,

$$[J_i,J_j]=i\epsilon_{ijk}J^k,\quad \Big[J_i,\widetilde{K}_j\Big]=i\epsilon_{ijk}\widetilde{K}^k,\quad \Big[\widetilde{K}_i,\widetilde{K}_j\Big]=\begin{cases}i\epsilon_{ijk}J^k\quad\text{for}\quad\mathcal{E}<\gamma\\ 0\quad\text{for}\quad\mathcal{E}=\gamma\\ -i\epsilon_{ijk}J^k\quad\text{for}\quad\mathcal{E}>\gamma\end{cases} \tag{3.47}$$

Thus,

$$\text{for}\quad\begin{cases}\mathcal{E}<\gamma\\ \mathcal{E}=\gamma\\ \mathcal{E}>\gamma\end{cases},\quad \vec{\widetilde{K}}\text{ and }\vec{J}\text{ generate the}\begin{cases}o(4)\text{ Lie algebra}\\ o(3)\otimes\mathbb{R}^3=e(3)\\ o(3,1)\text{ Lie algebra}\end{cases} \tag{3.48}$$

For a fixed value of the energy, $\mathcal{E}<\gamma$, we consider the more convenient commuting operators

$$\vec{A}=\frac{1}{2}\big(\vec{J}+\vec{\widetilde{K}}\big)\quad\text{and}\quad\vec{B}=\frac{1}{2}\big(\vec{J}-\vec{\widetilde{K}}\big)\,, \tag{3.49}$$

verifying the following relations

$$[A_i, A_j] = i\epsilon_{ijk}A^k, \quad [B_i, B_j] = i\epsilon_{ijk}B^k, \quad [A_i, B_j] = 0\,. \tag{3.50}$$

Then, the operators $\vec{A}$ and $\vec{B}$ extend the manifest $o(3)$ symmetry into a dynamical $o(3) \oplus o(3) = o(4)$ Lie algebra. The common eigenvector Ψ of the commuting operators, $\mathcal{H}$, $\vec{A}^2$, $\vec{B}^{\,2}$ satisfies,

$$\vec{A}^2\Psi = a\big(a+1\big)\Psi\,, \quad \vec{B}^{\,2}\Psi = b\big(b+1\big)\Psi\,, \quad \mathcal{H}\Psi = \mathcal{E}\Psi\,, \tag{3.51}$$

where a and b are half-integers. Considering the so far non-negative number,

$$n = -\frac{\beta}{\sqrt{2\gamma - 2\mathcal{E}}}\,, \tag{3.52}$$

we use the Casimir operators,

$$\vec{\tilde{K}}^{\,2} = -\vec{J}^2 + e^2g^2 - 1 + \frac{\beta^2}{2\gamma - 2\mathcal{E}} \quad \text{and} \quad \vec{J}\cdot\vec{\tilde{K}} = -\frac{eg\beta}{\sqrt{2\gamma - 2\mathcal{E}}}\,, \tag{3.53}$$

to obtain the equalities,

$$\begin{cases} a\big(a+1\big) + b\big(b+1\big) = \dfrac{1}{2}\big(e^2g^2 - 1 + n^2\big), \\ a\big(a+1\big) - b\big(b+1\big) = \big(eg\big)n\,. \end{cases} \tag{3.54}$$

Solving the equations (3.54) provide us with the relations,

$$\begin{cases} 2a + 1 = \pm\big(n + eg\big) \\ 2b + 1 = \pm\big(n - eg\big) \\ a - b = \pm eg \\ a + b + 1 = n\,. \end{cases} \tag{3.55}$$

Let us recall that from equation (3.27), the product $\big(eg\big)$ is quantized in integers or half-integers [in units $\hbar = c = 1$]. Consequently the first relation in (3.55) implies that n is integer or half-integer depending on the value of $\big(eg\big)$ being integer or half-integer.

We can now derive from (3.52) the bound-state energy spectrum,

$$\mathcal{E}_n = \gamma - \frac{\beta^2}{2n^2}\,, \quad n = \pm eg + 1\,, \pm eg + 2\,\cdots\,, \tag{3.56}$$

with the integer value of the degeneracy

$$n^2 - e^2g^2 = \big(n - eg\big)\big(n + eg\big)\,, \tag{3.57}$$

since n and eg are simultaneously integers or half-integers.

3.2 Kaluza-Klein-type monopoles

Kaluza-Klein theory is one of the oldest ideas attempting to unify gravitation and gauge theory [Kaluza 1919, Klein 1926]. The physical assumption, in this framework, is that the world contains four space-time dimensions, plus an extra cyclic dimension so small that it can not be observed. Thus, the ordinary general relativity in five dimensions is considered to possess a local $U(1)$ gauge symmetry arising from the isometry transformation of the hidden extra dimension.

Later, Sorkin [Sorkin 1983], and Gross and Perry [Gross 1983], introduced the Kaluza-Klein monopole which is obtained by imbedding the Taub-NUT gravitational instanton into Kaluza-Klein theory. The global stationary metric obtained,

$$\begin{aligned} ds^2 = -dt^2 + f(r)\left(dr^2 + r^2\big(d\theta^2 + \sin^2\theta d\phi^2\big)\right) + f^{-1}(r)\big(dx^4 + A_\phi d\phi\big)^2\,, \\ \text{with} \quad \theta \in [0,\pi]\;, \quad \phi \in [0, 2\pi]\;, \quad A_\phi \equiv \text{Dirac potential}\,, \end{aligned} \tag{3.58}$$

has lead to exact solution of the equations of the four-dimensional Euclidean gravity, approaching the vacuum solution at spatial infinity.

In 1986, Gibbons and Manton studied the hidden symmetry of Kaluza-Klein-type metrics and found, in the context of monopole scattering [Gibbons 04/1986, Gibbons 12/1986], that the geodesic motion in the Taub-NUT metric admits a Kepler-type dynamical symmetry [Fehér 10/1986, Gibbons 1987, Cordani 1988, Cordani 1990]. (See [Fehér 02/2009] for a review).

A better understanding of such hidden symmetries of Kaluza-Klein-type monopoles was achieved by various generalizations [Visinescu 01/1994, Iwai 05/1994, Iwai 06/1994, Visinescu 07/1994, van Holten 1994, Comtet, Vaman 1996, Cotaescu 1999, Cotaescu 2004, Krivonos 2006, Ballesteros 03/2008, Ballesteros 10/2008, Krivonos 2009, Visinescu 2009, Krivonos 2010, Nersessian, Visinescu 2011, Marquette 2011].

More recently, Gibbons and Warnick considered geodesic motion on hyperbolic space [Gibbons 09/2006] and found a large class of systems admitting such a dynamical symmetry.

Our aim, in this section, is to present a systematic analysis of Kaluza-Klein-type metrics admitting a conserved Runge-Lenz-type conserved quantity. To this end, we consider the stationary family of metrics,

$$dS^2 = f(\vec{x})\,\delta_{ij}\,dx^i\,dx^j + h(\vec{x})\left(dx^4 + A_k\,dx^k\right)^2\,. \tag{3.59}$$

In these metrics, $f(\vec{x})$ and $h(\vec{x})$ are real functions and the 1-form A_k is the gauge potential of a charged Dirac monopole.

Inspired by Kaluza's hypothesis, as the fourth dimension here is considered to be cyclic, we use the conservation of the "vertical" component of the momentum to reduce the four-dimensional problem to one in three dimensions, where we have strong candidates for the way these symmetries act [Ngome 08/2009]. Then, the lifting problem can be conveniently solved using the Van Holten technique [see section 2.2].

Let us first investigate the four-dimensional geodesic motion of a classical point-like test scalar particle with unit mass. The Lagrangian of geodesic motion on the 4-manifold endowed with the metric (3.59) is

$$\mathcal{L} = \frac{1}{2}\, f(\vec{x})\, \delta_{ij}\, \frac{dx^i}{dt}\, \frac{dx^j}{dt} + \frac{1}{2}\, h(\vec{x})\, \Big(\frac{dx^4}{dt} + A_k\, \frac{dx^k}{dt} \Big)^2 - U(\vec{x})\,, \tag{3.60}$$

where we also added an external scalar potential, namely $U(\vec{x})$, for later convenience. The canonical momenta conjugate to the coordinates $(x^j,\, x^4)$ read as

$$\begin{aligned} p_j &= \frac{\partial \mathcal{L}}{\partial\big(dx^j/dt\big)} = f(\vec{x})\, \delta_{ij}\, \frac{dx^i}{dt} + h(\vec{x})\, \Big(\frac{dx^4}{dt} + A_k\, \frac{dx^k}{dt} \Big)\, A_j\ , \\ p_4 &= \frac{\partial \mathcal{L}}{\partial\big(dx^4/dt\big)} = h(\vec{x})\, \Big(\frac{dx^4}{dt} + A_k\, \frac{dx^k}{dt} \Big) = q\ . \end{aligned} \tag{3.61}$$

The "vertical" momentum, $p_4 = q$, associated with the periodic variable, x^4, is conserved and can be interpreted as conserved electric charge. Thus, we can introduce the covariant momentum,

$$\Pi_j = f(\vec{x})\, \delta_{ij}\, \frac{dx^i}{dt} = p_j - q\, A_j\ . \tag{3.62}$$

The geodesic motion on the 4-manifold projects therefore onto the curved 3-manifold with metric $g_{ij}(\vec{x}) = f(\vec{x})\, \delta_{ij}$, augmented with a scalar potential. The Hamiltonian reads as

$$\mathcal{H} = \frac{1}{2}\, g^{ij}(\vec{x}) \Pi_i\, \Pi_j + V(\vec{x}) \quad \text{with} \quad V(\vec{x}) = \frac{q^2}{2h(\vec{x})} + U(\vec{x})\,. \tag{3.63}$$

For a particle without spin, the covariant Poisson brackets are given by [Souriau 1970]

$$\{B, D\} = \partial_k B\, \frac{\partial D}{\partial \Pi_k} - \frac{\partial B}{\partial \Pi_k}\, \partial_k D + qF_{kl}\, \frac{\partial B}{\partial \Pi_k}\, \frac{\partial D}{\partial \Pi_l}\,, \tag{3.64}$$

where $F_{kl} = \partial_k A_l - \partial_l A_k$ is the monopole field strength. Then, the nonvanishing fundamental brackets are

$$\big\{x^i,\, \Pi_j\big\} = \delta^i_j\ , \quad \{\Pi_i,\, \Pi_j\} = q\, F_{ij}\,. \tag{3.65}$$

We can now deduce the Hamilton equations yielding the geodesic motion of the scalar particle on the 3-manifold,

$$\dot{x}^i \;=\; \big\{x^i,\, \mathcal{H}\big\} = g^{ij}(\vec{x})\, \Pi_j, \tag{3.66}$$

$$\dot{\Pi}_i \;=\; \{\Pi_i,\, \mathcal{H}\} = q\, F_{ij}\, \dot{x}^j - \partial_i V + \Gamma^k_{ij}\, \Pi_k\, \dot{x}^j\ . \tag{3.67}$$

Note that the Lorentz equation (3.67) involves also in addition to the monopole and potential terms a curvature term, typical for motion in curved space, which is quadratic in the velocity.

We now inquire about the symmetries of the system. For our investigation, we recall that constants of the motion, noted as Q, which are polynomial in the momenta, can be derived following van Holten's algorithm [van Holten 2007]. The clue in this technique is to expand Q into a power series of the covariant momentum,

$$Q = C + C^i\,\Pi_i + \frac{1}{2!}\,C^{ij}\,\Pi_i\Pi_j + \frac{1}{3!}\,C^{ijl}\,\Pi_i\Pi_j\Pi_l + \cdots, \tag{3.68}$$

and to require Q to Poisson-commute with the Hamiltonian augmented with an effective potential, $\mathcal{H} = \dfrac{1}{2}\,\vec{\Pi}^2 + V(\vec{x})$. This yields the series of constraints,

$$\begin{cases} C^m\,\partial_m\,V(\vec{x}) = 0 & \text{o}(0) \\ \partial_n C = q\,F_{nm}\,C^m + C_n^{\;m}\partial_m\,V(\vec{x}) & \text{o}(1) \\ \mathcal{D}_i C_l + \mathcal{D}_l C_i = q\,\left(F_{im}\,C_l^{\;m} + F_{lm}\,C_i^{\;m}\right) + C_{il}^{\;\;k}\partial_k\,V(\vec{x}) & \text{o}(2) \\ \mathcal{D}_i C_{lj} + \mathcal{D}_j C_{il} + \mathcal{D}_l C_{ij} = q\,\Big(F_{im}\,C_{lj}^{\;\;m} + F_{jm}\,C_{il}^{\;\;m} + F_{lm}\,C_{ij}^{\;\;m}\Big) & \\ \qquad\qquad + \quad C_{ijl}^{\;\;\;m}\partial_m\,V(\vec{x}) & \text{o}(3) \\ \cdots\cdots. & \end{cases} \tag{3.69}$$

which have to be solved. Here the zeroth-order constraint can be interpreted as a consistency condition for the effective potential. It is worth noting that the expansion can be truncated at a finite order provided some higher-order constraint reduces to a Killing equation,

$$\mathcal{D}_{(i_1}\,C_{i_2\,\cdots\,i_n)} = 0\,, \tag{3.70}$$

where the covariant derivative is constructed with the Levi-Civita connection so that

$$\mathcal{D}_i C^j = \partial_i C^j + \Gamma^j_{\;ik}\,C^k\,. \tag{3.71}$$

Then, $C_{i_1 \cdots i_p} = 0$ for all $p \geqslant n$ and the constant of motion takes the polynomial form,

$$Q = \sum_{k=0}^{p-1} \frac{1}{k!}\,C^{i_1 \cdots i_k}\,\Pi_{i_1} \cdots \Pi_{i_k}\,. \tag{3.72}$$

The previously presented van Holten recipe is based on Killing tensors of the 3-manifold. Indeed, the conserved angular momentum is associated with a rank-1 Killing tensor (i.e. Killing vector), which generates spatial rotations. Rank-2 Killing tensors lead to conserved quantities quadratic in covariant momenta $\vec{\Pi}$'s, etc. Note that Killing tensors has been advocated by Carter in the context of the Kerr metric [Carter 1968].

Let us discuss two particular Killing tensors on the 3-manifold which carries the metric,

$$g_{ij}(\vec{x}) = f(\vec{x})\,\delta_{ij}\,. \tag{3.73}$$

Our strategy is to find conditions for lifting the Killing tensors, which generate the conserved angular momentum and the Runge-Lenz vector of planetary motion in flat space, respectively, to the "Kaluza-Klein" 4-space.

- First, we search for a rank-1 Killing tensor generating ordinary spatial rotations as

$$C_i = g_{ij}(\vec{x})\,\epsilon^j{}_{kl}\,n^k\,x^l\,. \tag{3.74}$$

We require C_i to satisfy the Killing equation $\mathcal{D}_{(i}C_{j)} = 0$, so that we obtain the following theorem [Ngome 08/2009]:

Theorem 3.2.1. *On the curved 3-manifold carrying the metric $g_{ij}(\vec{x}) = f(\vec{x})\,\delta_{ij}$, the rank-1 tensor*

$$C_i = g_{ij}(\vec{x})\,\epsilon^j{}_{kl}\,n^k\,x^l$$

is a Killing tensor generating spatial rotations around the fixed unit vector $\vec{n}$, provided

$$\Big(\vec{x}\times\vec{\nabla} f(\vec{x})\Big)\cdot\vec{n} = 0\,. \tag{3.75}$$

Note that Theorem 3.2.1 can be satisfied for some, but not all $\vec{n}$'s. In the two-center metric case, for example, it only holds for $\vec{n}$ parallel to the axis of the two centers (see the next section).

An important case to consider is when the metric is radial,

$$f(\vec{x}) = f(r)\,, \tag{3.76}$$

including the Taub-NUT metrics. In that event, the gradient is parallel to $\vec{x}$ and (3.75) holds for all $\vec{n}$'s. Thus, Theorem 3.2.1 is always satisfied for radial metrics.

- Next, inspired by the known flat-space expression, we consider the rank-2 Killing tensor associated with the Runge-Lenz-type conserved quantity

$$C_{ij} = 2\,g_{ij}(\vec{x})\,n_k\,x^k - g_{ik}(\vec{x})\,n_j\,x^k - g_{jk}(\vec{x})\,n_i\,x^k\,. \tag{3.77}$$

In order to deduce conditions on the metrics admitting a Kepler-type dynamical symmetry, we impose $\mathcal{D}_{(i}C_{jl)}$ to vanish. A tedious calculation provides us with

$$\begin{aligned}\mathcal{D}_{(i}C_{jl)} = 2\,n_k\,x^m\,\Big(g_{ij}(\vec{x})\,\Gamma^k_{lm} + g_{il}(\vec{x})\,\Gamma^k_{jm} + g_{jl}(\vec{x})\,\Gamma^k_{im}\Big) - n_i x^m\partial_m g_{jl}(\vec{x})\\ - n_j\,x^m\partial_m g_{il}(\vec{x}) - n_l\,x^m\partial_m g_{ij}(\vec{x})\,.\end{aligned} \tag{3.78}$$

Let us now calculate each term on the right hand side of (3.78). We first obtain

$$\begin{aligned} n_i\, x^m\, \partial_m g_{jl}(\vec{x}) &= f^{-1}(\vec{x})\, n_i\, g_{jl}(\vec{x})\, x^m\, \partial_m f(\vec{x}) \\ n_j\, x^m\, \partial_m g_{il}(\vec{x}) &= f^{-1}(\vec{x})\, n_j\, g_{il}(\vec{x})\, x^m\, \partial_m f(\vec{x}) \\ n_l\, x^m\, \partial_m g_{ij}(\vec{x}) &= f^{-1}(\vec{x})\, n_l\, g_{ij}(\vec{x})\, x^m\, \partial_m f(\vec{x})\,, \end{aligned} \tag{3.79}$$

and next the curvature terms yield,

$$\begin{aligned} 2\, g_{jl}(\vec{x})\, n^k\, x^m\, \Gamma^k_{im} = {} & f^{-1}(\vec{x})\, n_i\, g_{jl}(\vec{x})\, x^m \partial_m f(\vec{x}) + f^{-1}(\vec{x})\, n_m\, x^m\, g_{jl}(\vec{x})\, \partial_i f(\vec{x}) \\ & + f^{-1}(\vec{x})\, n_k\, g_{jl}(\vec{x})\, g^{nk}(\vec{x})\, g_{im}(\vec{x})\, x^m\, \partial_n f(\vec{x})\,, \end{aligned}$$

$$\begin{aligned} 2\, g_{il}(\vec{x})\, n^k\, x^m\, \Gamma^k_{jm} = {} & f^{-1}(\vec{x})\, n_j\, g_{il}(\vec{x})\, x^m \partial_m f(\vec{x}) + f^{-1}(\vec{x})\, n_m\, x^m\, g_{il}(\vec{x})\, \partial_j f(\vec{x}) \\ & + f^{-1}(\vec{x})\, n_k\, g_{il}(\vec{x})\, g^{nk}(\vec{x})\, g_{jm}(\vec{x})\, x^m\, \partial_n f(\vec{x})\,, \end{aligned}$$

$$\begin{aligned} 2\, g_{ij}(\vec{x})\, n^k\, x^m\, \Gamma^k_{lm} = {} & f^{-1}(\vec{x})\, n_l\, g_{ij}(\vec{x})\, x^m \partial_m f(\vec{x}) + f^{-1}(\vec{x})\, n_m\, x^m\, g_{ij}(\vec{x})\, \partial_l f(\vec{x}) \\ & + f^{-1}(\vec{x})\, n_k\, g_{ij}(\vec{x})\, g^{nk}(\vec{x})\, g_{lm}(\vec{x})\, x^m\, \partial_n f(\vec{x})\,. \end{aligned}$$

Inserting (3.79) and the previous curvature terms into (3.78), we get

$$\mathcal{D}_{(i}C_{jl)} = f^{-1}(\vec{x})\Big(g_{(ij}\,\partial_{l)} f(\vec{x})\, n_m\, x^m - g_{(ij}\, x_{l)}\, n^m \partial_m f(\vec{x})\Big)\,.$$

Requiring $\mathcal{D}_{(i}C_{jl)} = 0$ yields the following theorem [Ngome 08/2009]:

Theorem 3.2.2. *On the curved* 3*-manifold carrying the metric* $g_{ij}(\vec{x}) = f(\vec{x})\,\delta_{ij}$*, the tensor*

$$C_{ij} = 2\, g_{ij}(\vec{x})\, n_k\, x^k - g_{ik}(\vec{x})\, n_j\, x^k - g_{jk}(\vec{x})\, n_i\, x^k$$

*is a symmetrical rank-*2 *Killing tensor, associated with the Runge-Lenz-type vector, provided*

$$\vec{n} \times \Big(\vec{x} \times \vec{\nabla}\, f(\vec{x})\Big) = 0\,. \tag{3.80}$$

Note that the radial metrics (3.76) satisfy again the Theorem 3.2.2 so that, in addition to the rotational symmetry, they also admit a Kepler-type dynamical symmetry.

We can also remark that taking into account the compatibility condition of the metric tensor,

$$\mathcal{D}_k\, g_{ij}(\vec{x}) = 0\,, \tag{3.81}$$

the $g_{ij}(\vec{x})$ always verifies the order-2 Killing equation $\mathcal{D}_{(k}\, g_{ij)} = 0$. Hence, the metric tensor is itself a symmetrical rank-2 Killing tensor and the associated conserved quantity is the Hamiltonian [Gibbons 1987, van Holten 2007].

Having determined the generators of the symmetry which were previously the object of our considerations, we can construct the associated constants of the geodesic motion using the algorithm (3.69). We investigate the radially symmetric generalized TAUB-NUT metric so that (3.59) becomes,

$$dS^2 = f(r)\,\delta_{ij}\,dx^i\,dx^j + h(r)\,\big(dx^4 + A_k\,dx^k\big)^2\,. \tag{3.82}$$

Then, the Lagrangian (3.60) takes the form,

$$\mathcal{L} = \frac{1}{2}\,f(r)\,\dot{\vec{x}}^{\,2} + \frac{1}{2}h(r)\,\big(\,\frac{dx^4}{dt} + A_k\,\frac{dx^k}{dt}\,\big)^2 - U(r)\,, \tag{3.83}$$

where the scalar potential $U(r)$ is necessary to furnish Killing 2-tensors. Respectively associated with the cyclic variables x^4 and time t, the conserved electric charge and the energy read

$$q = h(r)\,\big(\,\frac{dx^4}{dt} + A_k\,\frac{dx^k}{dt}\,\big)\,, \quad \mathcal{E} = \frac{\vec{\Pi}^2}{2\,f(r)} + \frac{q^2}{2\,h(r)} + U(r)\,. \tag{3.84}$$

Using the relations (3.84), we can rearrange the dimensionally reduced Hamiltonian as

$$\mathcal{H} = \frac{1}{2}\,\vec{\Pi}^2 + f(r)\,W(r) \quad \text{with} \quad W(r) = U(r) + \frac{q^2}{2\,h(r)} + \frac{\mathcal{E}}{f(r)} - \mathcal{E}\,, \tag{3.85}$$

which we can now use to derive conserved quantities via the algorithm (3.69).

1) First, we look for conserved angular momentum which is linear in the covariant momentum since the 3-metric now satisfies Theorem 3.2.1. Hence, $C_{ij} = C_{ijk} = \cdots = 0$ so that (3.69) reduces to

$$\begin{cases} C^m\,\partial_m\,\big(f(r)\,W(r)\big) = 0 & \text{o}(0) \\ \partial_n C = q\,F_{nm}\,C^m & \text{o}(1) \\ \mathcal{D}_i C_l + \mathcal{D}_l C_i = 0\,. & \text{o}(2) \end{cases} \tag{3.86}$$

- The second- and the first-order constraints yield

$$C_i = g_{im}(r)\,\epsilon^m{}_{nk}\,n^n\,x^k \quad \text{and} \quad C = -qg\,n_k\,\frac{x^k}{r}\,, \tag{3.87}$$

respectively. The zeroth-order consistency condition in (3.86) is satisfied for an arbitrary radial effective potential, providing us with the conserved angular momentum,

$$\vec{J} = \vec{x} \times \vec{\Pi} - qg\,\frac{\vec{x}}{r}\,, \tag{3.88}$$

involving the typical monopole term.

2) Let us now turn to quadratic conserved quantities. In that event, we have $C_{ijk} = \cdots = 0$ which implies the series of constraints,

$$\begin{cases} C^m\,\partial_m\big(f(r)\,W(r)\big) = 0 & \text{o}(0) \\ \partial_n C = q\,F_{nm}\,C^m + C_n^{\ m}\,\partial_m\big(f(r)\,W(r)\big) & \text{o}(1) \\ \mathcal{D}_i C_l + \mathcal{D}_l C_i = q\,\left(F_{im}\,C_l^{\ m} + F_{lm}\,C_i^{\ m}\right) & \text{o}(2) \\ \mathcal{D}_i C_{lj} + \mathcal{D}_j C_{il} + \mathcal{D}_l C_{ij} = 0\,. & \text{o}(3) \end{cases} \tag{3.89}$$

• Taking $C_{ij} = g_{ij}(r)$ as a rank-2 Killing tensor, we deduce from the second-order equation of (3.89) that $C_i = 0$. As expected, the first-order and the zeroth-order consistency relation are both satisfied by any radial effective potential $C = f(r)\,W(r)$. The conserved energy associated, therefore, read as

$$\mathcal{E} = \frac{1}{2}\vec{\Pi}^2 + f(r)\,W(r)\,. \tag{3.90}$$

• Next, we search for a Runge-Lenz-type vector generating the Kepler-type dynamical symmetry of the system. Since Theorem 3.2.2 is satisfied by the considered radial 3-metric, we have to solve the constraints (3.89) using the rank-2 Killing tensor

$$C_{ij} = 2\,g_{ij}(r)\,n_k\,x^k - g_{ik}(r)\,n_j\,x^k - g_{jk}(r)\,n_i\,x^k \tag{3.91}$$

inspired by its form in the Kepler problem. We solve the second-order constraint of (3.89), and we get

$$C_i = \frac{q\,g}{r}\,g_{im}(r)\,\epsilon^m_{\ jk}\,n^j\,x^k\,. \tag{3.92}$$

Next, inserting (5.56) and (5.57) into the first-order constraint of (3.89), we obtain

$$\partial_j C = \left(\frac{\big(f(r)\,W(r)\big)'}{r} + \frac{q^2g^2}{r^4}\right) x_j\,n_k\,x^k - \left(r\big(f(r)\,W(r)\big)' + \frac{q^2g^2}{r^2}\right) n_j\,.$$

It is now easy to analyze the integrability condition of the previous equation by requiring the vanishing of the commutator,

$$[\,\partial_i\,,\,\partial_j\,]\,C = 0 \quad\Longrightarrow\quad \Delta\left(f(r)\,W(r) - \frac{q^2g^2}{2r^2}\right) = 0\,. \tag{3.93}$$

Thus, the bracketed quantity must satisfy the *Laplace equation.*

The zeroth-order equation is identically satisfied. Consequently, a Runge-Lenz-type conserved vector does exist only when the radial effective potential is

$$f(r)\,W(r) = \frac{q^2 g^2}{2r^2} + \frac{\beta}{r} + \gamma \quad \text{with} \quad \beta, \gamma \,\in\, \mathbb{R}\,. \tag{3.94}$$

Equivalent to the result of Gibbons and Warnick [Gibbons 09/2006], the formulas (3.85) and (3.94) allow us to announce the theorem [Ngome 08/2009]:

Theorem 3.2.3. *For the generalized TAUB-NUT metric (3.82), the most general potentials $U(r)$ permitting the existence of a Runge-Lenz-type conserved vector are given by*

$$U(r) = \left(\frac{q^2 g^2}{2r^2} + \frac{\beta}{r} + \gamma\right)\frac{1}{f(r)} - \frac{q^2}{2\,h(r)} + \mathcal{E}\,, \tag{3.95}$$

where q and g denote the particle and the monopole charge. And β, γ are free constants and $\mathcal{E}$ is the fixed energy [cf. (3.90)].

Inserting now (3.94) into the first-order constraint of (3.89) provides us with

$$\partial_n C = \frac{\beta}{r}\, n_n - \frac{\beta}{r^3}\, n_k\, x^k\, x_n\,, \tag{3.96}$$

which is solved by

$$C = \frac{\beta}{r}\, n_k\, x^k\,. \tag{3.97}$$

Collecting the results (5.56), (5.57) and (5.63) yield the conserved Runge-Lenz-type vector,

$$\vec{K} = \vec{\Pi} \times \vec{J} + \beta\, \frac{\vec{x}}{r}\,. \tag{3.98}$$

Due to the simultaneous existence of the conserved angular momentum (3.88) and the conserved Runge-Lenz vector (3.98), we obtain a complete description of the motion for generalized TAUB-NUT metric. Indeed, the motions of the particle are confined to conic sections [Fehér 10/1986]. Our class of metrics, which satisfy Theorem 3.2.3, includes the following.

1. The original TAUB-NUT case [Sorkin 1983, Gross 1983] with vanishing external $U(r) = 0$,

$$f(r) = \frac{1}{h(r)} = 1 + \frac{4m}{r}, \tag{3.99}$$

   where m is real [Fehér 10/1986, Gibbons 1987]. We note that the monopole scattering case corresponds to $m = -1/2$, see [Gibbons 04/1986, Fehér 10/1986, Gibbons 12/1986]. We then obtain for

$$\gamma = q^2/2 - \mathcal{E} \quad \text{and charge} \quad g = \pm 4m\,, \tag{3.100}$$

the conserved Runge-Lenz vector,

$$\vec{K} = \vec{\Pi} \times \vec{J} - 4m\left(\mathcal{E} - q^2\right)\frac{\vec{x}}{r}\,. \tag{3.101}$$

2. Lee and Lee [Lee 2000] argued that for monopole scattering with independent components of the Higgs expectation values, the geodesic Lagrangian (3.60) should be replaced by $L \to L - U(r)$, where the external potential reads

$$U(r) = \frac{1}{2}\,\frac{{a_0}^2}{1 + \dfrac{4m}{r}}\,. \tag{3.102}$$

It is now easy to see that this addition merely shifts the value in the brackets in (3.93) by a constant and corresponds to a shift of ${a_0}^2/2$ in the energy. Hence, the Laplace equation in (3.93) is still satisfied. So the previously found Runge-Lenz vector (3.101) is still valid.

3. The metric associated with winding strings [Gibbons 1988] where

$$f(r) = 1, \qquad h(r) = \frac{1}{\left(1 - \dfrac{1}{r}\right)^2}\,. \tag{3.103}$$

For charge $g = \pm 1$, we deduce from Theorem 3.2.3,

$$\left(\beta + q^2\right) - r\left(U(r) - \gamma + \frac{q^2}{2} - \mathcal{E}\right) = 0\,,$$

so that for the fixed energy, $\mathcal{E} = q^2/2 - \gamma + U(r)$, the conserved Runge-Lenz vector reads as

$$\vec{K} = \dot{\vec{x}} \times \vec{J} - q^2\,\frac{\vec{x}}{r}\,. \tag{3.104}$$

4. The extended TAUB-NUT metric [Iwai 05/1994, Iwai 06/1994] where

$$f(r) = b + \frac{a}{r}, \qquad h(r) = \frac{a\,r + b\,r^2}{1 + d\,r + c\,r^2}\,, \tag{3.105}$$

with the constants $(a,\, b,\, c,\, d\,) \in \mathbb{R}$. With the choices $U(r) = 0$ and charge $g = \pm 1$, Theorem 3.2.3 requires

$$-r\,f(r)\,\mathcal{E} + \frac{r\,f(r)}{h(r)}\,\frac{q^2}{2} - \frac{q^2}{2\,r} - \gamma\,r = \beta = \text{const}\,.$$

Inserting here (3.105) yields

$$\Big(-a\,\mathcal{E} + \frac{1}{2}d\,q^2 - \beta\Big) + r\Big(-b\,\mathcal{E} + \frac{1}{2}c\,q^2 - \gamma\Big) = 0,$$

which holds when

$$\beta = -a\,\mathcal{E} + \frac{1}{2}d\,q^2 \quad \text{and} \quad \gamma = -b\,\mathcal{E} + \frac{1}{2}c\,q^2\,. \tag{3.106}$$

Then, we get the conserved Runge-Lenz vector

$$\vec{K} = \vec{\Pi} \times \vec{J} - \left(a\,\mathcal{E} - \frac{1}{2}\,d\,q^2\right)\frac{\vec{x}}{r}\,. \tag{3.107}$$

5. Considering the oscillator-type metric discussed by Iwai and Katayama [Iwai 05/1994, Iwai 06/1994], the functions $f(r)$ and $h(r)$ take the form

$$f(r) = b + ar^2 \quad \text{and} \quad h(r) = \frac{ar^4 + br^2}{1 + cr^2 + dr^4}\,. \tag{3.108}$$

A direct calculation leads to the following Runge-Lenz-type vector [Marquette 2010],

$$\vec{K} = \left(b + ar^2\right)\,\dot{\vec{x}} \times \vec{J} + \beta\,\frac{\vec{x}}{r}\,. \tag{3.109}$$

Which is conserved only for scalar potential of the form

$$U(r) = \left(\frac{q^2g^2}{2r^2} + \frac{\beta}{r} + \gamma\right)\left(b + ar^2\right)^{-1} - q^2\left(\frac{1 + cr^2 + dr^4}{ar^4 + br^2}\right)\,. \tag{3.110}$$

Let us just conclude by outlining that the five examples treated above are shown to be particular cases deduced from the general expression (3.98), see [Ngome 08/2009]. See also [Igata 2010] for applications on the Kerr metric. The case of SUSY of the Kerr metric is investigated in [Galajinsky].

3.3 Multi-center metrics

The multi-center metrics family in which we are interested in this section are known to be Euclidean vacuum solutions of the Einstein equations, with self-dual curvature. The multi-center metrics can also be viewed as an extension of the TAUB-NUT metrics studied in the previous section [Gibbons 1987].

Let us begin by considering a scalar particle moving in the Gibbons-Hawking space [Gibbons 01/1979], which generalizes the TAUB-NUT space. The Lagrangian function associated with this dynamical system, as in (3.60), is given by

$$\mathcal{L} = \frac{1}{2}\, f(\vec{x})\, \dot{\vec{x}}^2 + \frac{1}{2} f^{-1}(\vec{x})\, \big(\frac{dx^4}{dt} + A_k\, \frac{dx^k}{dt} \big)^2 - U(\vec{x})\,.$$

But here the functions $f(\vec{x})$ obey the following "self-dual" ANSATZ [Gibbons 01/1979],

$$\vec{\nabla} f = \pm \vec{\nabla} \times \vec{A}\,. \tag{3.111}$$

Hence, $f(\vec{x})$ is an harmonic function,

$$\Delta\, f(\vec{x}) = 0\,. \tag{3.112}$$

The most general solution of (3.112) is given by

$$f(\vec{x}) = f_0 + \sum_{i=1}^{N} \frac{m_i}{|\vec{x} - \vec{a}_i|} \quad \text{with} \quad (f_0\,,\; m_i) \in \mathbb{R}^{N+1}\,. \tag{3.113}$$

Thus, the multi-center metric admits multi-NUT singularities so that the *jth* NUT singularity is characterized by the charge m_j and is located at $\vec{a}_j$. However, we can remove these singularities provided all NUT charges are equal,

$$m_1 = m_2 = \cdots = m_i = \frac{g}{2}\,. \tag{3.114}$$

In this case, the cyclic variable x^4 is periodic with the range,

$$0 \leq x^4 \leq 4\pi\, \frac{g}{N}\,. \tag{3.115}$$

We can now investigate the symmetries associated with the projection of the particle's motion onto the curved 3-manifold described by the multi-center metric tensor,

$$g_{jk}(\vec{x}) = \left(f_0 + \sum_{i=1}^{N} \frac{m_i}{|\vec{x} - \vec{a}_i|} \right) \delta_{jk}\,. \tag{3.116}$$

The projected Hamiltonian is given by

$$\mathcal{H} = \frac{1}{2}\vec{\Pi}^2 + f(\vec{x}) W(\vec{x}) \quad \text{with} \quad W(\vec{x}) = U(\vec{x}) + \frac{q^2}{2} f(\vec{x}) + \mathcal{E} f^{-1}(\vec{x}) - \mathcal{E}\,. \tag{3.117}$$

Let us first note that for multi-center metric (3.116), it is straightforward to deduce from Theorem 3.2.2 the metric condition,

$$\sum_{i=1}^{N} \frac{(\vec{n}\cdot\vec{x})\vec{a}_i - (\vec{n}\cdot\vec{a}_i)\vec{x}}{|\vec{x}-\vec{a}_i|^3} = 0\,, \tag{3.118}$$

which can not hold for more than two centers. Thus, we state the following theorem [Ngome 08/2009]:

Theorem 3.3.1. *In the curved 3-manifold carrying the N-center metric,*

$$g_{jk}(\vec{x}) = \left(f_0 + \sum_{i=1}^{N}\frac{m_i}{|\vec{x}-\vec{a}_i|}\right)\delta_{jk}\,,$$

no symmetry of the Kepler-type occurs for $N > 2$.

For simplicity, from now on we limit ourselves to a discussion of the two-center metrics,

$$f(\vec{x}) = f_0 + \frac{m_1}{|\vec{x}-\vec{a}|} + \frac{m_2}{|\vec{x}+\vec{a}|}\,, \qquad |\vec{x}\pm\vec{a}| \neq 0\,, \tag{3.119}$$

which are relevant for diatomic molecule systems and which possess some interesting symmetry properties. These metrics include, as special regular cases, those listed in Table 3.1.

f_0	N	Type of Metric
0	1	(m_1 or $m_2 = 0$) Flat space
1	1	(m_1 or $m_2 = 0$) TAUB-NUT
0	2	Eguchi-Hanson
1	2	Double TAUB-NUT .

Table 3.1: Examples of two-center metrics.

Let us now apply the van Holten algorithm (3.69) to derive the symmetry of the two-center metric.

1) First, finding conserved quantities linear in the covariant momentum require to solve the reduced series of constraints,

$$\begin{cases} C^m\,\partial_m\big(f(\vec{x})\,W(\vec{x})\big) = 0 & \text{o}(0)\\ \partial_n C = q\,F_{nm}\,C^m & \text{o}(1)\\ \mathcal{D}_i C_l + \mathcal{D}_l C_i = 0\,. & \text{o}(2)\end{cases} \tag{3.120}$$

• From Theorem 3.2.1, we deduce the rank-1 Killing tensor satisfying the second-order constraint of (3.120),

$$C_i = g_{im}(\vec{x})\,\epsilon^m{}_{lk}\,\frac{a^l}{a}\,x^k\,, \quad a = ||\vec{a}||\,. \tag{3.121}$$

The tensor (3.121) generates rotational symmetry around the axis through the two centers. Next, injecting both (3.121) and the magnetic field of the two centers,

$$\vec{B} = m_1\,\frac{\vec{x}-\vec{a}}{|\vec{x}-\vec{a}|^3} + m_2\,\frac{\vec{x}+\vec{a}}{|\vec{x}+\vec{a}|^3}\,, \tag{3.122}$$

into the first-order equation of (3.120) yield

$$C = -q\left(m_1\,\frac{\vec{x}-\vec{a}}{|\vec{x}-\vec{a}|} + m_2\,\frac{\vec{x}+\vec{a}}{|\vec{x}+\vec{a}|}\right)\cdot\frac{\vec{a}}{a}\,. \tag{3.123}$$

Finally we obtain, as conserved quantity, the projection of the angular momentum onto the axis of the two centers,

$$\mathcal{J}_a = \mathcal{L}_a - q\left(m_1\,\frac{\vec{x}-\vec{a}}{|\vec{x}-\vec{a}|} + m_2\,\frac{\vec{x}+\vec{a}}{|\vec{x}+\vec{a}|}\right)\cdot\frac{\vec{a}}{a} \quad \text{with} \quad \mathcal{L}_a = \left(\vec{x}\times\vec{\Pi}\right)\cdot\frac{\vec{a}}{a}\,, \tag{3.124}$$

which is consistent with the axial symmetry of the two-center metric.

2) Now we study quadratic conserved quantities, $Q = C + C^i\,\Pi_i + \frac{1}{2}\,C^{ij}\,\Pi_i\Pi_j$. Putting $C_{ijk} = C_{ijkl} = \cdots = 0$, leaves us with,

$$\begin{cases} C^m\,\partial_m\left(f(\vec{x})\,W(\vec{x})\right) = 0 & \text{o}(0)\\ \partial_n C = q\,F_{nm}\,C^m + C_n{}^m\,\partial_m\left(f(\vec{x})\,W(\vec{x})\right) & \text{o}(1)\\ \mathcal{D}_i C_l + \mathcal{D}_l C_i = q\,\left(F_{im}\,C_l{}^m + F_{lm}\,C_i{}^m\right) & \text{o}(2)\\ \mathcal{D}_i C_{lj} + \mathcal{D}_j C_{il} + \mathcal{D}_l C_{ij} = 0\,. & \text{o}(3) \end{cases} \tag{3.125}$$

- We consider the reducible rank-2 Killing tensor,

$$C_{ij} = \frac{2}{a^2}\,g_{im}(\vec{x})\,g_{jn}(\vec{x})\,\epsilon^m{}_{lk}\,\epsilon^n{}_{pq}\,a^l\,a^p\,x^k\,x^q + \frac{2}{a^2}\,g_{il}(\vec{x})\,g_{jm}(\vec{x})\,a^l\,a^m\,, \tag{3.126}$$

which is a symmetrized product of Killing-Yano tensors. $C_i = g_{im}(\vec{x})\,\epsilon^m{}_{lk}\,\frac{a^l}{a}\,x^k$ generates rotations around the axis of the two centers and $\widetilde{C}_j = g_{jm}(\vec{x})\,\frac{a^m}{a}$ generates spatial translation along the axis of the two centers. Injecting (3.126) into the second-order constraint of (3.125) yields

$$C_i = -\frac{2\,q}{a^2}\,g_{im}\,\epsilon^m{}_{jk}\,a^j\,x^k\,a_l\left(m_1\,\frac{x^l-a^l}{|\vec{x}-\vec{a}|} + m_2\,\frac{x^l+a^l}{|\vec{x}+\vec{a}|}\right). \tag{3.127}$$

For vanishing effective potential, we solve the first-order constraint with

$$C = \frac{q^2}{a^2}\left(m_1 \frac{(x^l - a^l)}{|\vec{x}-\vec{a}|}\, a_l + m_2 \frac{(x^l + a^l)}{|\vec{x}+\vec{a}|}\, a_l\right)^2 \tag{3.128}$$

so that we obtain the *square of the projection of the angular momentum onto the axis of the two centers, plus a squared component along the axis of the two centers of the covariant momentum*,

$$Q = \mathcal{J}_a^2 + \Pi_a^2 \,. \tag{3.129}$$

As expected, this conserved quantity is not really a new constant of the motion [Gibbons 1987, Valent 09/2003, Valent 07/2004, Duval 05/2005].

• Now we turn to the Kepler-type dynamical symmetry. Let us first check if a rank-2 Killing tensor associated with Runge-Lenz type conserved quantity does exist. To this end we apply Theorem 3.2.2 to the two-center metric,

$$g_{jk}(\vec{x}) = f(\vec{x})\delta_{jk}\,, \quad f(\vec{x}) = \left(f_0 + \frac{m_1}{|\vec{x}-\vec{a}|} + \frac{m_2}{|\vec{x}+\vec{a}|}\right). \tag{3.130}$$

We obtain

$$\vec{n}\times\left(\vec{x}\times\vec{\nabla}\, f(\vec{x})\right) = \left(\frac{m_2}{|\vec{x}+\vec{a}|^3} - \frac{m_1}{|\vec{x}-\vec{a}|^3}\right)(\vec{x}\times\vec{a})\times\vec{n} = 0\,, \tag{3.131}$$

according to Theorem 3.2.2. Consequently we get

$$\frac{m_2}{|\vec{x}+\vec{a}|^3} - \frac{m_1}{|\vec{x}-\vec{a}|^3} = 0 \qquad \text{or} \qquad \vec{x} = k\,\vec{a}\,, \quad k = \text{const}\ . \tag{3.132}$$

The right condition in (3.132) restricted to motions parallel to $\vec{a}$ and therefore implies no interesting case.

Considering the first case given by (3.132), we assume that both charges are positive $m_1 > 0\,,\ m_2 > 0\,,$ and we write $\vec{a} = (a_1,\, a_2,\, a_3)\,.$ Thus, the left Equation in (3.132) becomes

$$\begin{aligned} &(x - a_1\,\rho)^2 + (y - a_2\,\rho)^2 + (z - a_3\,\rho)^2 = a^2\,\left(\rho^2 - 1\right) \\ &\text{with} \quad \rho = \frac{m_1^{2/3} + m_2^{2/3}}{m_2^{2/3} - m_1^{2/3}}\,. \end{aligned} \tag{3.133}$$

We recognize here the equation of a 2-sphere of center $\vec{a}\,\rho$ and radius $R = a\sqrt{\rho^2-1}\,,$ noted as $\mathcal{S}^2\,.$ The latter shows that for two-center metric, a Kepler-type dynamical symmetry is only possible for motion confined onto the sphere, $\mathcal{S}^2\,.$

Before searching for the exact form of the associated Runge-Lenz conserved quantity, let us first check that the motions can be consistently confined onto this 2-sphere, $\mathcal{S}^2\,.$

To this end, we assume that the initial velocity is tangent to $\mathcal{S}^2$ and, using the equations of motion, we verify that at time $(t+\delta t)$ the velocity remains tangent to $\mathcal{S}^2$. Thus we write

$$\vec{v}(t_0+\delta t) = \vec{v}_0 + \delta t\,\dot{\vec{v}}_0 \quad \text{with } \vec{v}_0 = \vec{v}(t_0) \text{ tangent to } \mathcal{S}^2\,. \tag{3.134}$$

The equations of motion in the effective scalar potential chooses as (3.138) have the shape

$$\dot{\vec{\Pi}} = q\,\vec{v}\times\vec{B} - \vec{\nabla}\left(f(\vec{x})W(\vec{x})\right) - \frac{v^2}{2}\left(\frac{m_1}{|\vec{x}-\vec{a}|^3}+\frac{m_2}{|\vec{x}+\vec{a}|^3}\right)\vec{x}\,. \tag{3.135}$$

Injecting the expressions of the magnetic field of the two-center (3.122) and the effective potential (3.138), we obtain

$$\dot{\vec{\Pi}}_0 = \left(\frac{m_1}{|\vec{x}-\vec{a}|^3}+\frac{m_2}{|\vec{x}+\vec{a}|^3}\right)\left[q^2\left(\frac{m_1}{|\vec{x}-\vec{a}|}+\frac{m_2}{|\vec{x}+\vec{a}|}\right)-\frac{v_0^2}{2}+\beta\right]\vec{x}_0 = f(\vec{x})\,\dot{\vec{v}}_0\,.$$

Thus $\vec{v}(t_0+\delta t)$ in (3.134) becomes

$$\vec{v}_0 + \gamma\,\delta t\,\vec{x}_0\,,\quad \gamma = f^{-1}(\vec{x})\left(\frac{m_1}{|\vec{x}-\vec{a}|^3}+\frac{m_2}{|\vec{x}+\vec{a}|^3}\right)\left(q^2\left(\frac{m_1}{|\vec{x}-\vec{a}|}+\frac{m_2}{|\vec{x}+\vec{a}|}\right)-\frac{v_0^2}{2}+\beta\right),$$

where $\vec{v}_0$ and $\vec{x}_0$ are tangent to the 2-sphere $\mathcal{S}^2$. Hence, the velocity remains tangent to $\mathcal{S}^2$ along the motion.

Having shown the consistency of motions on the 2-sphere $\mathcal{S}^2$, we can state the following theorem [Ngome 08/2009]:

Theorem 3.3.2. *In the curved 3-manifold carrying the 2-center metric,*

$$g_{jk}(\vec{x}) = \left(f_0 + \frac{m_1}{|\vec{x}-\vec{a}|}+\frac{m_2}{|\vec{x}+\vec{a}|}\right)\delta_{jk}\,, \tag{3.136}$$

a scalar Runge-Lenz-type conserved quantity does exist only for a particle moving along the axis of the two centers or for motions confined on the two-sphere of radius $R = a\sqrt{\rho^2-1}$ centered at $\vec{a}\,\rho$ $(m_1,\,m_2>0)$. In the Eguchi-Hanson case $(m_1=m_2)$, the 2-sphere is replaced by the median plane of the two centers.

Our method here is particular since instead searching for Kepler-type dynamical symmetry directly, we already look at the conditions of its existence. Knowing now, for the two-center metric, that only motions confined on $\mathcal{S}^2$ allow a Runge-Lenz-tpe conserved quantity, we can solve the second-order constraint of (3.125) using (3.133). Thus, we obtain

$$C_i = \frac{q}{a}\,g_{im}\,\epsilon^m{}_{jk}\,a^j\,x^k\left(\frac{m_1}{|\vec{x}-\vec{a}|}+\frac{m_2}{|\vec{x}+\vec{a}|}\right), \tag{3.137}$$

where the only component of $\vec{n}$ is along the axis $\vec{a}/a$. The final step consists to solve the first-order constraint of (3.125). Indeed, following (3.94) the clue is to choose

$$f(\vec{x})W(\vec{x}) = \frac{q^2}{2}\left(\frac{m_1}{|\vec{x}-\vec{a}|} + \frac{m_2}{|\vec{x}+\vec{a}|}\right)^2 + \beta\left(\frac{m_1}{|\vec{x}-\vec{a}|} + \frac{m_2}{|\vec{x}+\vec{a}|}\right) + \gamma\,, \tag{3.138}$$

with $\beta, \gamma \in \mathbb{R}$. Let us precise that this potential satisfies the consistency condition given by the zeroth-order constraint of (3.125). Moreover, the leading coefficient of the effective potential cancels the obstruction due to the magnetic field of the two centers, and the remaining part on the right-hand side of (3.138) leads to

$$C = \beta\left(m_1\,\frac{\vec{x}-\vec{a}}{|\vec{x}-\vec{a}|} + m_2\,\frac{\vec{x}+\vec{a}}{|\vec{x}+\vec{a}|}\right)\cdot\frac{\vec{a}}{a}\,. \tag{3.139}$$

Collecting our results (3.77), (3.137) and (3.139) provide us with the scalar,

$$K_a = \left(\vec{\Pi}\times\vec{J}\right)\cdot\frac{\vec{a}}{a} + \frac{\beta}{q}\left(\mathcal{L}_a - \mathcal{J}_a\right), \tag{3.140}$$

which represents, in the case of two-center metrics (3.119), a conserved Runge-Lenz-type scalar for particle moving on the 2-sphere of center positioned at $\vec{a}\,\rho$ and radius $R = a\sqrt{\rho^2-1}$, combined with the effective potential (3.138).

3.4 Killing-Stäckel Tensors on extended manifolds

Having discussed, in the two previous sections, the conditions on Killing tensors which are related to the existence of constants of motion on the dimensionally reduced curved manifold. We can observe that the Killing tensor generating the Runge-Lenz-type quantity preserved by the geodesic motion can be lifted to an extended manifold.

Let us study this lifting problem in detail, by considering the geodesic motion of a particle before dimensional reduction (3.62). The particle evolves on the extended 4-manifold carrying the metric $g_{\mu\nu}(x)$ with $\mu, \nu = 1, \cdots, 4$. A rank-2 Killing-Stäckel tensor on this curved 4-manifold is a symmetric tensor, $C_{\mu\nu}$, which satisfies

$$\mathcal{D}_{(\lambda} C_{\mu\nu)} = 0\,, \quad \lambda,\, \mu,\, \nu = 1\,, \cdots, 4\,. \tag{3.141}$$

For the Killing-Stäckel tensor generating the Runge-Lenz-type conserved quantity, the degree-2 polynomial function in the canonical momenta p_μ associated with the local coordinates x^μ,

$$K = \frac{1}{2}\, C^{\mu\nu}\, p_\mu\, p_\nu \quad (\mu,\, \nu = 1, \cdots, 4)\,, \tag{3.142}$$

is preserved along the geodesics. Then, the lifted Killing-Stäckel tensor on the 4-manifold, which directly yields the Runge-Lenz-type conserved quantity is written as

$$C^{\mu\nu} = \begin{pmatrix} C^{i\,j} & C^{i\,4} \\ C^{4\,j} & C^{4\,4} \end{pmatrix}\,, \quad i,\, j = 1\,, 2\,, 3\,. \tag{3.143}$$

The tensor $C_{i\,j}$ is, therefore, a rank-2 Killing tensor on the dimensionally reduced curved 3-manifold carrying the metric $g_{ij}(\vec{x}) = f(\vec{x})\,\delta_{ij}$, which generates a Runge-Lenz-type quantity conserved along the projection of the geodesic motion onto the curved 3-manifold. The off- and the diagonal contravariant components read

$$C^{i\,4} = C^{4\,i} = \frac{1}{q} C^i - C^i{}_k\, A^k\,, \quad C^{4\,4} = \left(2/q^2\right) C - \left(2/q\right) C_k\, A^k + C_{jk}\, A^j A^k\,. \tag{3.144}$$

The term A^k represents the component of the vector potential of the magnetic field. In the case of the generalized TAUB-NUT metrics, the terms C and C_k are the results (5.63) and (5.57) of the first- and the second-order constraints of (3.89), respectively. In the case of the two-center metrics, C and C_k are given by the results (3.139) and (3.137), respectively.

• As an illustration, let us consider a particle in the gravitational potential, $V(r) = -\dfrac{m_0\, G_0}{r}$, described by the Lorentz metric [Bargmann, Duval 05/1984, Balachandran 1986, Duval 1991],

$$dS^2 = d\vec{x}^{\,2} + 2\, dx^4 dx^5 - 2\, V(r) \left(dx^5\right)^2\,. \tag{3.145}$$

The variable $x^5 = t$ is the non-relativistic time and x^4 the vertical coordinate. Rotations, time translations and "vertical" [on the fourth direction] translations generate as conserved quantities the angular momentum $\vec{L}$, the energy and the fixed mass m, respectively. The Runge-Lenz-type conserved quantity, along null geodesics of the 5-manifold described by the metric (3.145),

$$K = \frac{1}{2} C^{ab} p_a p_b \quad \text{with} \quad a,\, b,\, c = 1, \cdots, 5 \tag{3.146}$$

is derived from the trace-free rank-2 Killing-Stäckel tensor [Duval 1991],

$$C^{ab} = \left(\frac{\hat{\eta}}{g^c{}_c}\right) g^{ab} - \eta^{ab} \quad \text{with} \quad \hat{\eta} = \eta^{ab} g_{ab}\,. \tag{3.147}$$

For some $\vec{n} \in \mathbb{R}^3$, the nonvanishing contravariant components of η are given by

$$\eta^{ij} = n^i\, x^j + n^j\, x^i - \hat{\eta}\, \delta^{ij} \quad \text{and} \quad \eta^{45} = \eta^{54} = \hat{\eta} = n_i\, x^i\,, \tag{3.148}$$

where we recognize, in the left hand side of (3.148), the generator of Kepler-type symmetry in the dimensionally reduced 3-manifold.

A calculation of each matrix element of the Killing tensor (3.147) leads to C^{ab} whose only nonvanishing components are,

$$C^{ij} = 2\,\hat{\eta}\,\delta^{ij} - n^i\, x^j - n^j\, x^i \quad \text{and} \quad C^{44} = 2\,\hat{\eta}\, V(r)\,. \tag{3.149}$$

Consequently, the associated Runge-Lenz-type conserved quantity reads as

$$\vec{K} \cdot \vec{n} = \left(\vec{p} \times \vec{L} + m^2\, V(r)\, \vec{x}\right) \cdot \vec{n}\,. \tag{3.150}$$

Note, in the previous expression, that the mass "m" is preserved by the "vertical" reduction. In the original Kepler case, we thus deduce on the dimensionally reduced flat 3-manifold that the symmetric tensor

$$C^{ij} = 2\,\delta^{ij}\, n_k\, x^k - n^i\, x^j - n^j\, x^i \tag{3.151}$$

is a Killing-Stäckel tensor generating the Runge-Lenz-type conserved quantity along the projection of the null geodesic of the 5-manifold onto the 3-manifold carrying the flat Euclidean metric.

Chapter 4

Non-Abelian gauge fields and the Berry phase

Contents

Conserved quantities of an isospin-carrying particle in non-Abelian monopole-like fields are investigated. In the effective non-Abelian field for nuclear motion, obtained through the Berry phase in a diatomic molecule, due to Wilczek et al., an unusual conserved charge and angular momentum are constructed.

4.1 The Wu-Yang monopole

In 1968, T.T. Wu and C.N. Yang found the first "monopole-like" classical solution of the Yang-Mills field equations [Wu Yang 1968]. Such a solution can also be viewed as an extension of the Abelian Dirac monopole solution when usual electrodynamics, with $U(1)$ symmetry, is considered as a part of a larger theory. The generator of the electromagnetism $U(1)$ subgroup should be embedded into the non-Abelian $SU(2)$ gauge group.

In order to investigate the Wu-Yang monopole solution, we consider here a Yang-Mills theory described by the Lagrangian density,

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{a\,\mu\nu} \,, \quad \mu,\, \nu = 0, 1,\, 2,\, 3\,. \tag{4.1}$$

where $F^a_{\mu\nu}$ represents the antisymmetric Yang-Mills field strength tensor taking values in the Lie algebra of the gauge group $SU(2)$,

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - e\, \varepsilon_{abc} A^b_\mu A^c_\nu \,, \quad a = 1,\, 2,\, 3\,. \tag{4.2}$$

Here e and the antisymmetric tensor ε_{abc} denote the gauge coupling constant and the structure constant of the gauge group, respectively. As expected,

$$A_\mu = A^a_\mu \, \tau^a \,, \tag{4.3}$$

takes value in the $su(2)$ Lie algebra. The Hermitian and traceless infinitesimal generators of the $SU(2)$ gauge group verify the commutation relation,

$$\left[\tau^a, \tau^b\right] = i\varepsilon_{abc}\tau^c\,, \quad \tau^a = \frac{1}{2}\sigma^a\,, \tag{4.4}$$

where the σ^a are Pauli matrices. Our approach now is to find the equations of the motion using a variational principle with the Yang-Mills Lagrangian density (4.1). We obtain the classical source-free Yang-Mills (YM) equations

$$\partial_\beta F^{d\,\alpha\beta} - e\,\varepsilon^{adc} A^c_\beta\, F^{a\,\alpha\beta} = 0\,, \tag{4.5}$$

which can be written in a more compact way as

$$D_\beta F^{d\,\alpha\beta} = 0\,. \tag{4.6}$$

Searching for solutions of the equations of the motion (4.6) in the "temporal" gauge,

$$A^a_0 = 0\,, \tag{4.7}$$

only time-independent gauge field and local gauge invariance are permitted. We can now posit the spherically symmetric Wu-Yang ANSATZ [Wu Yang 1968],

$$\begin{aligned} A^a_i &= g\,\varepsilon_{iaj}\,x^j\,\frac{(1-\Phi\,(r))}{r^2}\,, \\ r^2 &= x_i\,x^i \quad \text{with} \quad i,\,j = 1,2,3\,, \end{aligned} \tag{4.8}$$

where a, (i,j) and g represent the color index, the space indices and the quantized Wu-Yang monopole charge, respectively. Also remark that Φ is a radial real function which is to be determined. As expected, a direct calculation implies that all time-dependent components of the field strength vanish,

$$F^a_{0\mu} = 0\,, \quad \mu = 0,1,\cdots,4\,, \tag{4.9}$$

whereas the spatial components of the 2-form curvature reads

$$\begin{aligned} F^a_{ij} = g\,\varepsilon_{ijk}&\left\{2\delta^{ak}\frac{(1-\Phi)}{r^2} - eg\,x^k x^a\left(\frac{1-\Phi}{r^2}\right)^2\right\} \\ &+\frac{g}{r}\frac{d}{dr}\left(\frac{1-\Phi}{r^2}\right)\left\{\varepsilon_{jal}\,x^i\,x^l - \varepsilon_{ial}\,x^j\,x^l\right\}\,. \end{aligned} \tag{4.10}$$

The right hand side bracket of (4.10) can be rewritten by using the relation,

$$\left(\varepsilon_{jal}\,x^i\,x^l - \varepsilon_{ial}\,x^j\,x^l\right)\tau^a = \varepsilon_{ijk}\left(r^2\delta^{ak} - x^a x^k\right)\tau^a\,, \tag{4.11}$$

so that the "Wu-Yang" field strength reduces to

$$F^a_{ij} = g\,\varepsilon_{ijk}\left[-\frac{\delta^{ak}}{r}\,\frac{d\Phi}{dr} + \frac{x^a\,x^k}{r^3}\left(\frac{d\Phi}{dr} - eg\frac{\Phi^2}{r} + 2\big(eg-1\big)\frac{\Phi}{r} + \frac{2-eg}{r}\right)\right]. \tag{4.12}$$

Injecting the relations (4.8) and (4.12) into the Yang-Mills field equations (4.6) provide us with the non-linear Wu-Yang equation,

$$r^2\,\frac{d^2\Phi}{dr^2} - eg\big(1-eg+eg\Phi\big)\big(\Phi-1\big)\left(\Phi + \frac{2-eg}{eg}\right) = 0\,. \tag{4.13}$$

Note that due to the non-linear nature of the YM equations, to search analytical solutions of (4.13) is an unconquerable task.

Hence, we investigate numerically the non-linear equation (4.13) viewed as a dynamical system. To this end, without loss of generality, we reduce ourselves to the case where $eg = 1$ [1]. Thus, the equation (4.13) takes the simple form,

$$r^2\,\frac{d^2\Phi}{dr^2} - \Phi\big(\Phi-1\big)\big(\Phi+1\big) = 0\,. \tag{4.14}$$

We first posit the variable change

$$r = \exp(\tau)\,, \quad \tau \in \mathbb{R}\,,\; r \in \mathbb{R}^+\,, \tag{4.15}$$

where τ is viewed as an evolution parameter. Next, we multiply the resulting τ-dependent equation (4.14) by $\dfrac{d\Phi(\tau)}{d\tau}$ so that we get

$$\frac{d}{d\tau}\left\{\frac{1}{2}\left(\frac{d\Phi(\tau)}{d\tau}\right)^2 - \frac{1}{4}\left(\Phi^2(\tau)-1\right)^2\right\} = \left(\frac{d\Phi(\tau)}{d\tau}\right)^2. \tag{4.16}$$

Then, the equation (4.16) can be interpreted as the equation of motion of a unit mass particle with non-conserved Hamiltonian [2],

$$\mathcal{H} = \frac{1}{2}\dot{\Phi}^2 + \mathcal{V} \quad \text{with} \quad \mathcal{V} = -\frac{1}{4}\left(\Phi^2-1\right)^2. \tag{4.17}$$

Here $\mathcal{V}$ represents an Higgs potential in which the particle evolves. Let us remark that the kinetic frictional force,

$$F_+ = \dot{\Phi}\,, \tag{4.18}$$

exerted on the particle has a positive friction coefficient and makes the energy to grow, since

$$\frac{dE}{d\tau} \geq 0\,. \tag{4.19}$$

[1] See the formula (3.27) in section 3.1.

[2] Here the dot means derivative w.r.t. the evolution parameter $\dfrac{d}{d\tau}$.

The system receives energy from the exterior so that the Rayleigh function, $\mathcal{R}$, is negative and reads

$$\mathcal{R} = -\int_0^{\dot{\Phi}} F_+ d\dot{\Phi} = -\frac{1}{2}\,\dot{\Phi}^2 \,. \tag{4.20}$$

When positing the conjugate momentum as $\Psi = \dot{\Phi}$, we can construct the canonical phase-space $\big(\Phi,\,\Psi\big)$ in which we define an extension of the equations of the motion of our non-conservative system (4.16) as

$$\begin{cases} \dot{\Phi} = \dfrac{\partial \mathcal{H}}{\partial \Psi} = \Psi \,, \\[2ex] \dot{\Psi} = -\dfrac{\partial \mathcal{H}}{\partial \Phi} - \dfrac{\partial \mathcal{R}}{\partial \Psi} = \Psi + \Phi\big(\Phi^2 - 1\big) \,. \end{cases} \tag{4.21}$$

Then, we can now describe the curve solutions of this Hamiltonian system (4.21) [Protogenov, Breitenlohner]. To this end, we draw in the phase-plane $\big(\Phi,\,\Psi\big)$, the vector field $\big(\dot{\Phi},\,\dot{\Psi}\big)$, representing the velocity of each phase-point. Hence, the orbit solutions of the dynamical system lie on curves tangent to the velocity vector field. However, we restrict our investigation to finite orbits which are the only solutions physically consistent. We first search for critical points, which can be considered as orbits degenerated to a point, by solving the constraints

$$\begin{cases} \Psi = 0 \,, \\[1ex] \Psi + \Phi\big(\Phi^2 - 1\big) = 0 \,. \end{cases} \tag{4.22}$$

A simple algebra leads to the three critical points $\big(\Phi,\,\Psi\big) = \big\{\big(0,\,0\big)\,;\,\big(\pm 1,\,0\big)\big\}$, to which we characterize the equilibrium by analysing the eigenvalues and the eigenvectors of the stability matrix,

$$\Delta = -\begin{pmatrix} -\dfrac{\partial^2 H}{\partial \Phi\,\partial \Psi} & -\dfrac{\partial^2 \mathcal{H}}{\partial \Psi^2} \\[2ex] \dfrac{\partial^2 \mathcal{H}}{\partial \Phi^2} + \dfrac{\partial^2 \mathcal{R}}{\partial \Phi\, d\Psi} & \dfrac{\partial^2 \mathcal{H}}{\partial \Phi\,\partial \Psi} + \dfrac{\partial^2 \mathcal{R}}{\partial \Psi^2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\[1ex] 3\Phi^2 - 1 & 1 \end{pmatrix} . \tag{4.23}$$

- For the fixed point $\big(\Phi,\,\Psi\big) = \big(0,\,0\big)$, the eigenvalues of the stability matrix read,

$$\lambda_\pm = 1/2 \pm i\sqrt{3}/2 \,.$$

As the complex conjugated eigenvalues $\lambda_\pm$ have both a positive real part, then the orbits are spiraling out with respect to the focus $\big(0,\,0\big)$. Hence, the critical point $\big(0,\,0\big)$ is unstable and is considered to be a negative attractor.

- For the fixed points $\big(\Phi,\,\Psi\big) = \big(\pm 1,\,0\big)$, the eigenvalues of the stability matrix Δ read

$$\lambda_+ = 2 \quad \text{and} \quad \lambda_- = -1 \,.$$

The fixed points are saddle points with the stable direction given by λ_- and the unstable direction given by λ_+ .

• We also consider the two bounded solutions, noted $\Phi_\mp$, represented by the curves joining the negative attractor (0, 0) to the saddle points (∓ 1, 0). The corresponding curves solution are represented in the phase portrait below by Φ_- in pink line and Φ_+ in cyan.

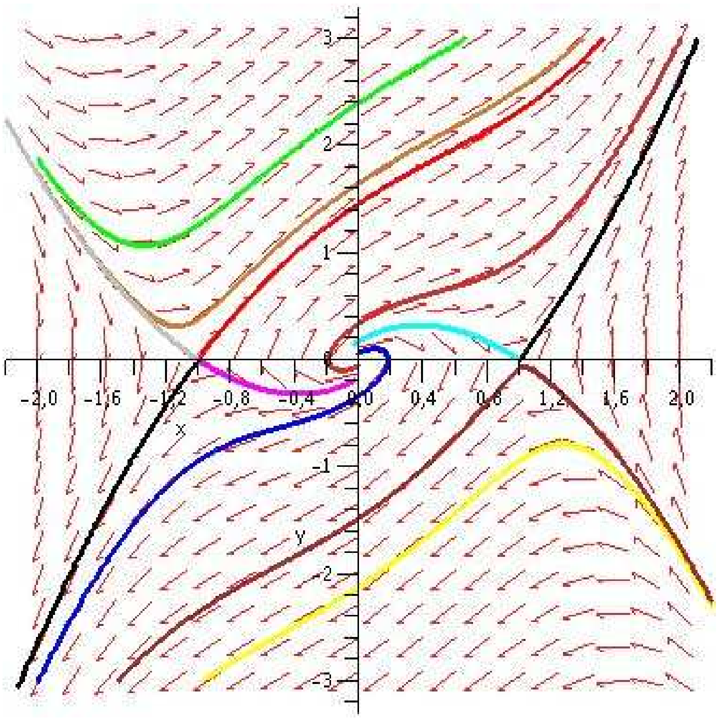

Figure 4.1: Phase portrait of the Hamiltonian system (4.21).

Let us now discuss the bounded solutions of the non-linear differential equation (4.14). For $r \in \mathbb{R}^+$, we have the five orbits solution,

$$\Phi(r) = \{-1,\, 1,\, 0,\, \Phi_-,\, \Phi_+\}\,, \tag{4.24}$$

that we introduce in the Wu-Yang forms (4.8) and (4.12) to obtain the shape of the gauge field.

• The degenerate orbit solution $\Phi = -1$ leads to the pure gauge field

$$A_i^a = 2g\varepsilon_{iaj}\,\frac{x^j}{r^2} \quad \text{since} \quad F_{0i}^a = F_{jk}^a = 0\,. \tag{4.25}$$

• The choice $\Phi = 1$ corresponds to the null gauge potential,

$$A_i^a = 0 \quad \text{with} \quad F_{0i}^a = F_{jk}^a = 0\,. \tag{4.26}$$

It is worth noting that the two previous gauge potentials, with both vanishing field strength, can be transformed the one by the other using a suitable gauge transformation. They are therefore gauge equivalent.

• We consider the two curves solution Φ_- and Φ_+, admitting the asymptotic limits,

$$\Phi_\pm = \begin{cases} \Phi_\pm^\infty & \text{when} \quad r \gg 1 \\ \Phi_\pm^0 & \text{when} \quad r \ll 1 \end{cases} \qquad \text{with} \qquad \begin{cases} \lim_{r\to\infty} \Phi_\pm^\infty = \pm 1 \\ \lim_{r\to 0} \Phi_\pm^0 = 0\,. \end{cases} \tag{4.27}$$

Taking into account the behavior of the curves solution in the neighborhood of the origin, we can neglect the cubic term of the non-linear equation (4.14). Thus, we deduce that $\Phi_\pm^0$ satisfy the differential equation,

$$r^2 \frac{d^2\Phi_\pm^0(r)}{dr^2} + \Phi_\pm^0(r) = 0\,, \tag{4.28}$$

which provides us with the non-analytic solution,

$$\Phi_\pm^0 = \pm\alpha\sqrt{r}\cos\left(\frac{\sqrt{3}}{2}\ln\frac{r}{r_0}\right)\,, \quad \text{with} \quad \big(\alpha,\, r_0 \ll 1\big) \in \mathbb{R}^\star \times \mathbb{R}^{+\star}\,. \tag{4.29}$$

In the case where $r \gg 1$, the equation (4.14) reduces to the simple form

$$\frac{d^2\Phi_\pm^\infty(r)}{dr^2} = \frac{\text{const}}{r^3} + O\left(\frac{1}{r^4}\right)\,, \tag{4.30}$$

so, we derive the behavior of the radial functions $\Phi_\pm$ at infinity as

$$\Phi_\pm^\infty(r) = \pm 1 \mp \frac{\gamma}{r} + O\left(\frac{1}{r^2}\right)\,, \quad \text{where} \quad \gamma > 0\,. \tag{4.31}$$

After the complete description of the asymptotic behavior of the functions $\Phi_\pm$, our business now is to fill the gap between these two limit cases. Here we investigate the "solution" K_+ but the case K_- is not more complicated. Thus, we integrate numerically the non-linear equation (4.14) from a point r_0, located in the neighborhood of the origin so that Φ_+ is approximated by Φ_+^0, till a sufficiently great value of r so that Φ_+ can be approximated by Φ_+^∞. Following the usual procedure, let us begin the numerical integration by adding some analytic correction terms, a_i, into the early expression (4.29) of Φ_+^0. Thus, we express the numerical lower bound as

$$\tilde{\Phi}_+^0(r) = \pm\alpha\sqrt{r}\cos\left(\frac{\sqrt{3}}{2}\ln\frac{r}{r_0}\right) + \sum_{i=0}^{4} a_i\big(\alpha,\, r_0\big)\, r^i\,, \tag{4.32}$$

where the coefficients a_i depend on the values of $r_0 \ll 1$ and the fixed parameter α. For the fixed values of the integration parameters,

$$r_0 = 0.007873997658\,, \qquad \alpha = -0.8873554901\,, \tag{4.33}$$

and with the initial conditions of the numerical integration given by

$$\left(\tilde{\Phi}^0_{\pm}(r_0),\ \frac{d}{dr}\tilde{\Phi}^0_{\pm}(r_0)\right)\ ,$$

we obtain, for $r \in [r_0, 14]$, the curve solution $\Phi_+(r)$ of the non-linear equation (4.14). For $e = g = 1$, we draw the field strength intensity of the usual $SU(2)$ Dirac monopole (in circling dashes) together with the intensity of the field strength solution (4.12), namely $B_{\pm}(r) = \frac{g}{r^2}\left(1 - \Phi_+^2\right)$, carrying the branch Φ_+ (in heavy line).

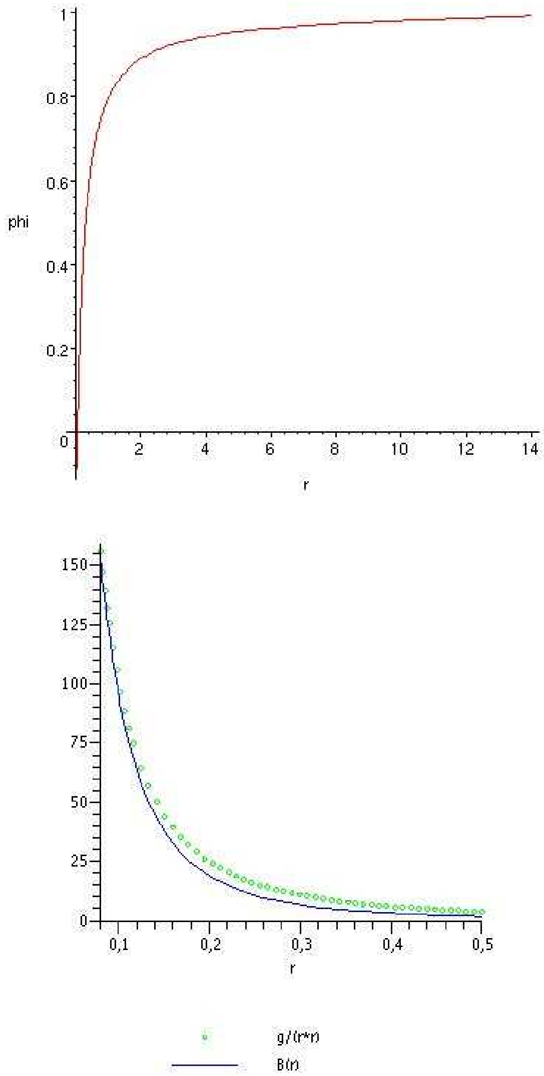

Figure 4.2: In the top side we plot the curve solution Φ_+; and in the bottom side we compare the field strength intensities of B_+ with the imbedded Dirac monopole.

It is worth mentioning that, in the asymptotic limits, the two curves coincide. The latter is due to the fact that when the radius tends to zero, $B_\pm(r)$ corresponds to the length of a Dirac monopole field; and when the radius tends to infinity, we get the null length of the vacuum. In the case of intermediary values of the radius, the two curves become quite different. This is a consequence of the radial function Φ_+ which takes non-quantized but continuous values between zero and one. (See Figure 4.2).

• Let us now investigate the last bounded solution, $\Phi_\pm = 0$. The comparison of the intensities of field strength made above provides us with strong assumption on the nature of the solution when $\Phi_\pm = 0$. Let us analyze further by injecting the solution $\Phi(r) = 0$ into the Wu-Yang ansatz (4.8). In that event, we obtain the gauge field

$$A_i^{a\,WY} = g\varepsilon_{iaj}\frac{x^j}{r^2}\,, \tag{4.34}$$

which implies that the field strength (4.12) reduces to that of a Wu-Yang monopole,

$$F_{ij}^{a\,WY} = g\varepsilon_{ijk}\frac{x^k x^a}{r^4}\,. \tag{4.35}$$

The energy density,

$$\mathcal{E}(r) = \frac{1}{4}\,F_{ij}^{a\,WY}\,F^{a\,ij\,WY} = \frac{g^2}{2r^4}\,, \tag{4.36}$$

is singular at the origin $r = 0$, so the Wu-Yang monopole possesses an infinite magnetic field energy.

Moreover we can prove, as suggested by the figure in the bottom of (4.2), that the non-Abelian Wu-Yang monopole can be viewed as an imbedded Dirac monopole. Indeed, we consider the trivial imbedding of Dirac monopole field $A_\mu^{D\,U(1)}$ into $SU(2)$,

$$A_\mu^{D\,U(1)} \longrightarrow A_\mu^{D\,SU(2)} = A_\mu^{a\,D}\,\tau_a\,, \quad \big(\tau_a = \sigma_a/2\big)\,, \tag{4.37}$$

where the Abelian gauge potential reads

$$A_\mu^{a\,D} = \begin{cases} 0 \quad \text{for} \quad a = 1,2\,, \\ A_\mu^{3\,D} = \pm\, g\,(1 \mp \cos\theta)\,\partial_\mu\phi\,. \end{cases} \tag{4.38}$$

Thus, a short algebra yields the imbedded gauge potential,

$$A_\mu^{D\,SU(2)} = \begin{cases} A_0^{D\,SU(2)} = 0 \quad \text{(temporal gauge)}\,, \\ A_i^{D\,SU(2)} = \pm\, g\,\dfrac{(1 \mp \cos\theta)}{r\sin\theta}\,(-\sin\phi,\, \cos\phi,\, 0)\,\tau_3\,. \end{cases} \tag{4.39}$$

In order to avoid the Dirac string singularity, we apply the singular "hedgehog" gauge transformation,

$$A_\mu^{D\,SU(2)} \longrightarrow U\left(A_\mu^{D\,SU(2)} + \frac{i}{e}\partial_\mu\right)U^{-1} = A_\mu^{WY}\,, \tag{4.40}$$

which rotates the unit vector on the sphere S^2 to the third axis in isospace. Thus, U is charaterized by the unitary matrix,

$$U\big(\theta,\phi\big) = \begin{pmatrix} \cos\dfrac{\theta}{2} & -\sin\dfrac{\theta}{2}\,\exp\left(-in\phi\right) \\ \sin\dfrac{\theta}{2}\exp\left(in\phi\right) & \cos\dfrac{\theta}{2} \end{pmatrix}, \quad n \in \mathbb{N}^{\star}. \tag{4.41}$$

Applying (4.40) with (4.41), it is straightforward to derive the gauge equivalent potential,

$$\begin{cases} A_0^{a\,WY} = 0 \qquad (\text{temporal gauge}), \\ A_i^{1\,WY} = \dfrac{n}{e}\cos\theta\sin\theta\cos\big(n\phi\big)\partial_i\phi + \dfrac{1}{e}\sin\big(n\phi\big)\partial_i\theta\,, \\ A_i^{2\,WY} = \dfrac{n}{e}\cos\theta\sin\theta\sin\big(n\phi\big)\partial_i\phi - \dfrac{1}{e}\cos\big(n\phi\big)\partial_i\theta\,, \\ A_i^{3\,WY} = -\dfrac{n}{e}\sin^2\theta\,\partial_i\phi\,. \end{cases} \tag{4.42}$$

When taking into account the Dirac quantization relation [3], $eg = n = 1$, and the following algebraic relations,

$$\partial_i\theta = \frac{1}{r^2}\left(z\cos\phi,\ z\sin\phi,\ -\frac{x}{\cos\phi}\right) \quad \text{and} \quad \partial_i\phi = \frac{1}{r^2\sin^2\theta}\left(-y,\ x,\ 0\right), \tag{4.43}$$

the gauge potential (4.42) transforms into the cartesian form as

$$\begin{cases} A_0^{a\,WY} = 0 \qquad (\text{temporal gauge}), \\ A_i^{1\,WY} = \dfrac{g}{r^2}\big(0,\ z,\ -y\big), \\ A_i^{2\,WY} = \dfrac{g}{r^2}\big(-z,\ 0,\ x\big), \\ A_i^{3\,WY} = \dfrac{g}{r^2}\big(y,\ -x,\ 0\big). \end{cases} \tag{4.44}$$

By compacting (4.44), we hence recover the exact expression [see (4.34) and (4.35)] of the non-Abelian Wu-Yang gauge potential with a “hedgehog” magnetic field. It is now clear that the non-Abelian Wu-Yang monopole field can be obtained by imbedding the Abelian Dirac monopole field into an $SU(2)$ gauge theory.

Let us now inquire about conserved quantities. To do this, we first Identify the su(2) Lie algebra of the non-Abelian generator, τ_a, with $\mathbb{R}^3$. Thus, we make the replacement,

$$\tau_a \longrightarrow \mathcal{I}_a\,, \quad \text{with the internal index } a = 1,2,3\,. \tag{4.45}$$

Here the non-Abelian variable $\mathcal{I}_a$ represents the isospin vector which satisfies the Poisson-

[3] See the formula (3.27) in section 3.1.

bracket algebra,

$$\{\mathcal{I}^a, \mathcal{I}^b\} = -\epsilon^{abc}\mathcal{I}_c\,. \tag{4.46}$$

Hence, we consider an isospin-carrying particle [Balachandran 1977, Duval 1978, Duval 1980, Duval 1982] moving in a Wu-Yang monopole field [Schechter, Boulware 1976, Stern 1977, Wipf 1986], augmented by a scalar potential [Schonfeld, Fehér 1984], described by the gauge covariant Hamiltonian,

$$\mathcal{H} = \frac{\vec{\pi}^2}{2} + V(\vec{x}, \mathcal{I}^a)\,, \qquad \vec{\pi} = \vec{p} - e\mathcal{I}^a \vec{A}^{a\,WY}\,. \tag{4.47}$$

We define the covariant Poisson-brackets as

$$\{f, g\} = D_j f \frac{\partial g}{\partial \pi_j} - \frac{\partial f}{\partial \pi_j} D_j g + e\,\mathcal{I}^a F^{a\,WY}_{jk} \frac{\partial f}{\partial \pi_j}\frac{\partial g}{\partial \pi_k} - \epsilon_{abc}\frac{\partial f}{\partial \mathcal{I}^a}\frac{\partial g}{\partial \mathcal{I}^b}\mathcal{I}^c\,, \tag{4.48}$$

where D_j is the covariant derivative,

$$D_j f = \partial_j f - e\epsilon_{abc}\mathcal{I}^a A^{b\,WY}_j \frac{\partial f}{\partial \mathcal{I}^c}. \tag{4.49}$$

Thus, the commutator of the covariant derivatives is recorded as

$$[D_i, D_j] = -\epsilon_{abc}\mathcal{I}^a F^{b\,WY}_{ij} \frac{\partial}{\partial \mathcal{I}^c}\,. \tag{4.50}$$

Following van Holten's recipe [van Holten 2007], conserved quantities $\mathcal{Q}(\vec{x}, \vec{\mathcal{I}}, \vec{\pi})$ can conveniently be sought for in the form of an expansion into powers of the covariant momentum,

$$\mathcal{Q}(\vec{x}, \vec{\mathcal{I}}, \vec{\pi}) = C(\vec{x}, \vec{\mathcal{I}}) + C_i(\vec{x}, \vec{\mathcal{I}})\pi_i + \frac{1}{2!}C_{ij}(\vec{x}, \vec{\mathcal{I}})\pi_i\pi_j + \cdots \tag{4.51}$$

Requiring $\mathcal{Q}$ to Poisson-commute with the Hamiltonian, $\{\mathcal{Q}, \mathcal{H}\} = 0$, provides us with the set of constraints to be satisfied,

$$\begin{aligned}
&C_i D_i V + \epsilon_{abc}\mathcal{I}^a \frac{\partial C}{\partial \mathcal{I}^b}\frac{\partial V}{\partial \mathcal{I}^c} = 0, && o(0)\\
&D_i C = e\mathcal{I}^a F^{a\,WY}_{ij} C_j + C_{ij} D_j V + \epsilon_{abc}\mathcal{I}^a \frac{\partial C_i}{\partial \mathcal{I}^b}\frac{\partial V}{\partial \mathcal{I}^c}, && o(1)\\
&D_i C_j + D_j C_i = e\mathcal{I}^a (F^{a\,WY}_{ik} C_{kj} + F^{a\,WY}_{jk} C_{ki}) + C_{ijk} D_k V + \epsilon_{abc}\mathcal{I}^a \frac{\partial C_{ij}}{\partial \mathcal{I}^b}\frac{\partial V}{\partial \mathcal{I}^c}, && o(2)\\
&D_i C_{jk} + D_j C_{ki} + D_k C_{ij} = e\mathcal{I}^a (F^{a\,WY}_{il} C_{ljk} + F^{a\,WY}_{jl} C_{lki} + F^{a\,WY}_{kl} C_{lij}) && \\
&\qquad\qquad + C_{ijkl} D_l V + \epsilon_{abc}\mathcal{I}^a \frac{\partial C_{ijk}}{\partial \mathcal{I}^b}\frac{\partial V}{\partial \mathcal{I}^c}, && o(3)\\
&\vdots \qquad\qquad\qquad \vdots && \vdots
\end{aligned} \tag{4.52}$$

To start, we search for zeroth-order conserved quantity. Thus $C_i = C_{ij} = \cdots = 0$, so that the series of constraints (4.52) reduces to

$$\begin{cases} \epsilon_{abc}\mathcal{I}^a \dfrac{\partial C}{\partial \mathcal{I}^b}\dfrac{\partial V}{\partial \mathcal{I}^c} = 0 & \mathrm{o}(0) \\ D_i C = 0\,. & \mathrm{o}(1) \end{cases} \tag{4.53}$$

The zeroth-order equation of (4.53) is identically satisfied for rotationally invariant potentials with respect to $\vec{x}$ and $\vec{\mathcal{I}}$. Applying for the consistency condition,

$$[D_i, D_j]\, C = -\epsilon_{abc}\mathcal{I}^a F_{ij}^{b\,WY} \frac{\partial C}{\partial \mathcal{I}^c} = 0\,, \tag{4.54}$$

we get the shape of the derivative of C along the isospin variable,

$$\frac{\partial C}{\partial \mathcal{I}^c} = f\big(r,\mathcal{I}\big)x_c + h\big(r,\mathcal{I}\big)\mathcal{I}_c\,. \tag{4.55}$$

Injecting (4.55) into the first-order constraint of (4.53) implies, for an arbitrary function $h\big(r,\mathcal{I}\big)$, that

$$\partial_i C = f\big(r,\mathcal{I}\big)\left(\mathcal{I}_i - \frac{\vec{x}\cdot\vec{\mathcal{I}}}{r^2}x_i\right)\,, \quad [\text{with } eg = 1]\,. \tag{4.56}$$

From the commutation rule,

$$[\,\partial_i\,,\,\partial_j\,]\,C = 0 \quad\Longrightarrow\quad \frac{1}{r}\left(\frac{df}{dr} + \frac{f}{r}\right) = 0\,, \tag{4.57}$$

we derive the exact form of the function $f(r,\mathcal{I})$,

$$f\big(r,\mathcal{I}\big) = \frac{\beta}{r}\,, \quad \beta = \mathrm{const} \in \mathbb{R}\,. \tag{4.58}$$

Taking into account the result (4.58), the equations (4.56) and (4.55) lead to the system of equtions,

$$\begin{cases} \partial_i C = \beta\left(\dfrac{\mathcal{I}_i}{r} - \dfrac{\vec{x}\cdot\vec{\mathcal{I}}}{r^3}x_i\right)\,, \\ \dfrac{\partial C}{\partial \mathcal{I}^a} = \beta\dfrac{x_a}{r} + h\big(r,\mathcal{I}\big)\mathcal{I}_a\,, \end{cases} \tag{4.59}$$

which is solved uniquely by the covariantly constant charge,

$$\mathcal{Q}(\vec{x},\vec{\mathcal{I}}) = \beta\,\frac{\vec{x}\cdot\vec{\mathcal{I}}}{r} + \gamma\,\mathcal{Q}\big(\mathcal{I}\big)\,, \quad \gamma = \mathrm{const} \in \mathbb{R}\,. \tag{4.60}$$

This charge can be viewed as a linear combination of two quantities separately conserved along the particle's motion since β and γ are arbitrary real numbers. Note that the first term on the right-hand side of (4.60), namely,

$$\mathcal{Q}_0 = \frac{\vec{x}\cdot\vec{\mathcal{I}}}{r}\,, \tag{4.61}$$

can be seen as a conserved electric charge; and its conservation admits a nice interpretation in terms of fiber bundles [Horváthy 12/1984, Horváthy 06/1985]. The $SU(2)$ gauge field is a connection form defined on a bundle over the 3-dimensional space so that for Wu-Yang monopole, the su(2) connection living on the (trivial) bundle reduces to the $U(1)$ Dirac monopole bundle. This is the reason why the electric charge is conserved in the Wu-Yang case: the latter is, as already seen, an imbedded Abelian Dirac monopole.

Next, we study conserved quantities which are linear in π_i, $C_{ij} = C_{ijk} = \cdots = 0$. We therefore have to solve the constraints,

$$\begin{cases} C_i D_i V + \epsilon_{abc} \mathcal{I}^a \dfrac{\partial C}{\partial \mathcal{I}^b} \dfrac{\partial V}{\partial \mathcal{I}^c} = 0\,, & o(0) \\ D_i C = e\mathcal{I}^a F_{ij}^{a\,WY} C_j + \epsilon_{abc} \mathcal{I}^a \dfrac{\partial C_i}{\partial \mathcal{I}^b} \dfrac{\partial V}{\partial \mathcal{I}^c}\,, & o(1) \\ D_i C_j + D_j C_i = 0\,. & o(2) \end{cases} \tag{4.62}$$

When the potential is invariant with respect to joint rotation of $\vec{x}$ and $\vec{\mathcal{I}}$, inserting the Killing vector generating the spatial rotations,

$$\vec{C} = \vec{n} \times \vec{x}\,, \tag{4.63}$$

into the series of constraints (4.171) yields

$$C = -\mathcal{Q}_0 \frac{\vec{n} \cdot \vec{x}}{r}\,. \tag{4.64}$$

Collecting the two previous results provides us with the conserved angular momentum,

$$\vec{J} = \vec{x} \times \vec{\pi} - \mathcal{Q}_0 \frac{\vec{x}}{r}\,. \tag{4.65}$$

Let us now turn to quadratic conserved quantities, $C_{ijk} = C_{ijkl} = \cdots = 0$. The set of constraints (4.52) reduces to

$$\begin{cases} C_i D_i V + \epsilon_{abc} \mathcal{I}^a \dfrac{\partial C}{\partial \mathcal{I}^b} \dfrac{\partial V}{\partial \mathcal{I}^c} = 0, & o(0) \\ D_i C = e\mathcal{I}^a F_{ij}^{a\,WY} C_j + C_{ij} D_j V + \epsilon_{abc} \mathcal{I}^a \dfrac{\partial C_i}{\partial \mathcal{I}^b} \dfrac{\partial V}{\partial \mathcal{I}^c}, & o(1) \\ D_i C_j + D_j C_i = e\mathcal{I}^a (F_{ik}^{a\,WY} C_{kj} + F_{jk}^{a\,WY} C_{ki}) + \epsilon_{abc} \mathcal{I}^a \dfrac{\partial C_{ij}}{\partial \mathcal{I}^b} \dfrac{\partial V}{\partial \mathcal{I}^c}, & o(2) \\ D_i C_{jk} + D_j C_{ki} + D_k C_{ij} = 0\,. & o(3) \end{cases} \tag{4.66}$$

We observe that the rank-2 Killing tensor generating the Kepler-type dynamical symmetry has the property,

$$C_{ij} = 2\delta_{ij}\, \vec{n} \cdot \vec{x} - (n_i x_j + n_j x_i)\,. \tag{4.67}$$

Inserting (4.67) into (4.66), from the 2nd-order equation we find, therefore,

$$\vec{C} = \mathcal{Q}_0 \frac{\vec{n} \times \vec{x}}{r} \,. \tag{4.68}$$

The first-order equation requires in turn

$$D_i C = \left\{ \mathcal{Q}_0^2 \, (\vec{n} \cdot \vec{x}) \frac{x_i}{r^4} - \mathcal{Q}_0^2 \, \frac{n_i}{r^2} \right\} + 2 \, \vec{n} \cdot \vec{x} \, D_i V - n_i \, \vec{x} \cdot \vec{D} V - x_i \, \vec{n} \cdot \vec{D} V \,. \tag{4.69}$$

Restricting ourselves to potentials falling off at infinity and invariant by rotations with respect to $\vec{x}$ and $\vec{\mathcal{I}}$, we hence obtain

$$V = \frac{\mathcal{Q}_0^2}{2r^2} + \frac{\alpha}{r} + \beta \quad \text{and} \quad C = \alpha \, \frac{\vec{n} \cdot \vec{x}}{r}, \tag{4.70}$$

where α and β are arbitrary constants. Note that the inverse-square term in the previous potential is fixed by the requirement of canceling the bracketed term on the right-hand side of (4.69). Collecting our results yields,

$$\vec{K} = \vec{\pi} \times \vec{J} + \alpha \frac{\vec{x}}{r} \,, \tag{4.71}$$

which is indeed a conserved Runge-Lenz vector for an isospin-carrying particle in the Wu-Yang monopole field, combined with the potential (4.70) [Horváthy 1991]. Let us emphasize that the fine-tuned inverse-square term is necessary to overcome the obstruction in solving the constraint equation [Ngome 02/2009]; without it, no Runge-Lenz vector would exist. The expression (4.71) is actually not surprising since the Wu-Yang monopole field, as we proved it, corresponds to an imbedded Dirac monopole field.

The importance of the conserved quantities $\vec{J}$ and $\vec{K}$ is understood by noting that they determine the trajectory : multiplying the conserved angular momentum by the position, $\vec{x}$, yields

$$\vec{J} \cdot \frac{\vec{x}}{r} = -\mathcal{Q}_0, \tag{4.72}$$

so that the particle moves, as always in the presence of a monopole, on the surface of a cone of half opening angle,

$$\varphi = \arccos(|\mathcal{Q}_0|/J) \quad \text{where} \quad J = |\vec{J}| \,. \tag{4.73}$$

On the other hand, the projection of the position onto the vector $\vec{N}$, given by,

$$\vec{N} = \vec{K} + (\alpha/\mathcal{Q}_0)\vec{J}, \qquad \vec{N} \cdot \vec{x} = J^2 - \mathcal{Q}_0^2 = \text{const}, \tag{4.74}$$

implying that the trajectory lies on a plane perpendicular to $\vec{N}$. The motion is, therefore, a *conic section*. Careful analysis would show that the trajectory is an ellipse, a parabola, or a hyperbola, depending on the energy, being smaller, equal or larger than β [Fehér 1984, Fehér 1985, Fehér 1986, Fehér 1987, Cordani 1990]. In particular, for sufficiently low energies, the motion remains bounded.

The conserved vectors $\vec{J}$ and $\vec{K}$ satisfy, furthermore, the commutation relations

$$\{J_i, J_j\} = \epsilon_{ijk} J_k\,, \quad \{J_i, K_j\} = \epsilon_{ijk} K_k\,, \quad \{K_i, K_j\} = 2(\beta - \mathcal{H})\epsilon_{ijk} J_k\,, \tag{4.75}$$

with the following Casimir relations,

$$\vec{J} \cdot \vec{K} = -\alpha\, \mathcal{Q}_0, \qquad K^2 = 2(\mathcal{H} - \beta)(J^2 - \mathcal{Q}_0^2) + \alpha^2. \tag{4.76}$$

Normalizing $\vec{K}$ by $[2(\beta - \mathcal{H})]^{1/2}$, we get, therefore an $\mathrm{SO}(3)/\mathrm{E}(3)/\mathrm{SO}(3,1)$ dynamical symmetry, depending on the energy being smaller/equal/larger than β [4].

We remark that although our investigations have been purely classical, there would be no difficulty to extend them to a quantum particle. In the self-dual Wu-Yang case, the $\mathrm{SO}(4)/\mathrm{SO}(3,1)$ dynamical symmetry allows, in particular, to derive the bound-state spectrum and the Scattering-matrix group theoretically, using the algebraic relations (4.75) and (4.76) [Fehér 1984, Cordani 1988, Fehér 1988, Fehér 1989].

[4] The dynamical symmetry actually extends to an isospin-dependent representation of $\mathrm{SO}(4,2)$ [Horváthy 1991].

4.2 The Berry phase - general theory

The Berry phase arises for systems which can be conveniently described in terms of two sets of degrees of freedom [Berry 1984]. The one is "*fast*" moving with large differences between excitation levels, and the other is "*slow*" with small associated energy differences. This decomposition is extensively used in molecular physics through the adiabatic or Born-Oppenheimer approximation. In a molecule, for instance, the electronic motion is described by the "*fast*" variables $\vec{r}$, and the nuclear motion by the "*slow*" degrees of freedom $\vec{R}$.

We first deal with the "*fast*" degrees of freedom, keeping the "*slow*" as approximately fixed. In that event, we simply solve the stationary Schrödinger equation for the "*fast*" variables, with the "*slow*" variables appearing parametrically,

$$\mathcal{H}(\vec{R})\Psi_m(\vec{r},\vec{R}) = \mathcal{E}_m(\vec{R})\Psi_m(\vec{r},\vec{R})\,.$$

Next, we complete the analysis by allowing slow variations in time for the previously fixed variables. The (adiabatic) assumption is that the slowly varying degrees of freedom $\vec{R}$ do not change quickly enough to induce transitions from one $\mathcal{E}_n$ level to another. Thus, the system starting in an initial eigenstate remains in this state in response to the slow change of the variables $\vec{R}$ appearing parametrically.

As a consequence of this effective dynamics, an external vector potential called the Berry connection is induced. It has been argued that the "feed-back" coming from the Berry phase modifies the (semi)classical dynamics [Chang 1995, Niu]. The associated magnetic-like field is the Berry curvature, and the line integral of the connection is the "celebrated" Berry phase [Berry 1984, Simon]. See also [Aitchison].

For better understanding, let us study deeper the way to obtain the Berry gauge potentials. To this end, we separate the following in Abelian and non-Abelian cases.

The Abelian gauge potential : non-degenerate states

Following Berry's original paper [Berry 1984], let us point that an $U(1)$ gauge field may appear when a single non-degenerate level is subject to adiabatically varying external parameter. To this end, we consider a physical system described by a Hamiltonian which depends on time through the vector $\vec{R}(t)$,

$$\mathcal{H} = \mathcal{H}(\vec{R})\,, \quad \vec{R} = \vec{R}(t)\,. \tag{4.77}$$

Here $\vec{R}(t)$ denotes a set of m classical parameters,

$$\vec{R}(t) = \big(R_1(t), R_2(t), \cdots, R_m(t)\big)\,,$$

slowly varying along a closed path $\mathcal{C}$ in the parameter space, since the system is assumed to evolve adiabatically. We introduce an instantaneous orthonormal basis constructed with the eigenstates of $\mathcal{H}(\vec{R}(t))$ at each value of the parameter $\vec{R}$. The eigenvalue equation reads

$$\mathcal{H}(\vec{R}(t))|m\,,\vec{R}(t)\rangle = \mathcal{E}_m(\vec{R}(t))|m\,,\vec{R}(t)\rangle\,. \tag{4.78}$$

The basis eigenfunctions $|m\,,\vec{R}(t)\rangle$ are not completely determined by (4.78). Indeed, without loss of generality, while normalizing the eigenfunctions,

$$\langle m\,,\vec{R}(t)|m\,,\vec{R}(t)\rangle = 1\,,$$

this implies that eigenfunctions are unique up to multiplication by a phase factor. Moreover, for a slowly varying Hamiltonian, the quantum adiabatic theorem states that a system initially prepared to be in one of its eigenstate $|m\,,\vec{R}(0)\rangle$, at $t=0$, remains in this instantaneous eigenstate along the cyclic process $\mathcal{C}$. Consequently, the quantum state at time t can be written as,

$$|\Psi_m(t)\rangle = \exp\big(ia_m(t)\big)|m\,,\vec{R}(t)\rangle\,, \tag{4.79}$$

where the exponential term in (4.79) is the only degree of freedom we can have in the quantum state. Substituting the expression (4.79) into the Schrödinger equation,

$$i\hbar\frac{\partial}{\partial t}|\Psi_m(t)\rangle = \mathcal{H}(\vec{R}(t))|\Psi_m(t)\rangle\,, \tag{4.80}$$

and projecting both sides of the equation onto $\langle n\,,\vec{R}(t)|$, yields the equation for the phase,

$$a_n = i\oint_{\mathcal{C}}\langle m\,,\vec{R}\,|\,\vec{\nabla}_{\vec{R}}|\,m\,,\vec{R}\rangle\cdot d\vec{R} - \frac{1}{\hbar}\int_0^T \mathcal{E}_n(\vec{R}(t'))dt'\,. \tag{4.81}$$

Thus, in addition to the usual dynamical phase,

$$-\frac{1}{\hbar}\int_0^T \mathcal{E}_n(\vec{R}(t'))dt'\,, \tag{4.82}$$

the quantum state acquires an additional geometric phase during the evolution through a closed path in the external parameter space [Berry 1984, Simon],

$$\gamma_n(\mathcal{C}) = i\oint_{\mathcal{C}}\langle n\,,\vec{R}\,|\,\vec{\nabla}_{\vec{R}}\,|\,n\,,\vec{R}\rangle\cdot d\vec{R} = \oint_{\mathcal{C}}\vec{A}_n(\vec{R})\cdot d\vec{R}\,. \tag{4.83}$$

The path integral in the parameter space, $\gamma_n(\vec{R})$, represents the "celebrated" Berry phase and the integrand,

$$\vec{A}_n(\vec{R}) = i\langle n\,,\vec{R}\,|\,\vec{\nabla}_{\vec{R}}\,|\,n\,,\vec{R}\rangle\,, \tag{4.84}$$

is a vector-valued function called the Berry connection or the Berry vector potential.

Let us note that the Berry connection transforms as a gauge vector field. Indeed, the gauge transformation

$$|\,n\,,\vec{R}\rangle \quad\longrightarrow\quad \exp\big(i\xi(\vec{R})\big)|\,n\,,\vec{R}\rangle\,, \tag{4.85}$$

with $\xi(\vec{R})$ being an arbitrary smooth function, acts on the Berry connection as

$$\vec{A}_n(\vec{R}) \quad \longrightarrow \quad \vec{A}_n(\vec{R}) - \vec{\nabla}_{\vec{R}}\xi(\vec{R})\,. \tag{4.86}$$

Consequently the Berry vector potential $\vec{A}_n(\vec{R})$ transforms as an $U(1)$ gauge potential.

In analogy to the electrodynamics, the gauge field tensor derived from the Berry vector potential reads

$$F^n_{ij} = \frac{\partial}{\partial R^i} A^n_j - \frac{\partial}{\partial R^j} A^n_i\,, \tag{4.87}$$

and is known as the Berry curvature. Using Stokes's theorem, we can express the Berry phase as an integral of the Berry curvature throughout the surface $\mathcal{S}$ enclosed by the path $\mathcal{C}$,

$$\gamma^n = \frac{1}{2}\int_{\mathcal{S}} dR^i \wedge dR^j\, F^n_{ij}\,. \tag{4.88}$$

It is worth noting that the Berry curvature can be viewed as a $U(1)$ gauge-invariant magnetic field in the parameter space. It is therefore observable.

The non-Abelian gauge potential : degenerate states

The Berry phase admits a non-Abelian generalization when the energy levels of the Hamiltonian are degenerate [Wilczek 1984].

Thus, we now consider a quantum system described by an Hamiltonian $\mathcal{H}(\vec{R}(t))$ with k-fold degenerate ground states for all values of the external parameter $\vec{R}(t)$. For simplicity, we fix $k=2$ such that the energy levels are two-fold degenerate and the Hamiltonian has two independent eigenstates, $|n^a,\,\vec{R}(t)\rangle$, $a=1,2$. The eigenvalue equation now reads as

$$\mathcal{H}(\vec{R}(t))|n^a,\,\vec{R}(t)\rangle = \mathcal{E}_n(\vec{R}(t))|n^a,\,\vec{R}(t)\rangle\,, \tag{4.89}$$

where, without loss of generality, the eigenstates are chosen such that

$$\langle n_a,\,\vec{R}(t)\,|\,n^b,\,\vec{R}(t)\rangle = \delta^b_a\,, \quad a,b=1,2\,. \tag{4.90}$$

Assuming that the system starts in one of its eigenstate $|n^a\,,\vec{R}(0)\rangle$, the adiabatic approximation implies that the system stays in its initial instantaneous eigenstate, after a cyclic tour through the space of parameters. The eigenfunctions of the system are

$$|\Psi^a_m(t)\rangle = |m^b\,,\vec{R}(t)\rangle\, U^a_b(\vec{R})\,, \tag{4.91}$$

where $U^a_b(\vec{R}) \in SU(2)$ is an unitary matrix. Let us now substitute the expression (4.91) into the time-dependent Schrödinger equation,

$$i\hbar\frac{\partial}{\partial t}|\Psi^a_m(t)\rangle = \mathcal{H}(\vec{R}(t))|\Psi^a_m(t)\rangle\,, \tag{4.92}$$

and multiply it from the left by $\langle n^a\,, \vec{R}(t)|$, one finds

$$\frac{\partial \vec{R}}{\partial t}\,\vec{A}_c^b(\vec{R})\,U_b^a(\vec{R}) + i\frac{\partial U_c^a(\vec{R})}{\partial t} - \frac{1}{\hbar}\mathcal{E}_n\,U_c^a(\vec{R}) = 0\,, \quad a,b,c = 1,2\,, \tag{4.93}$$

where we introduced the notation,

$$\vec{A}_c^b(\vec{R}) = i\langle n_c\,, \vec{R}(t)|\vec{\nabla}_{\vec{R}}|n^b\,, \vec{R}(t)\rangle\,. \tag{4.94}$$

with $a,b,c = 1,2$ denoting matrix indices. Hence, the vector potential $\vec{A}(\vec{R})$ is a $\big(2\times 2\big)$ anti-Hermitian matrix lying in the $su(2)$ Lie algebra. Indeed, under the non-Abelian gauge transformation,

$$|m'^a\,, \vec{R}(t)\rangle = |m^b\,, \vec{R}(t)\rangle\,U_b^a(\vec{R})\,, \tag{4.95}$$

the field (4.94) transforms as

$$\vec{A}_c^b(\vec{R}) \quad\longrightarrow\quad U^{-1}\left(\vec{A}_c^b(\vec{R}) - \frac{\partial}{\partial\vec{R}}\right)U\,, \tag{4.96}$$

and therefore defines a $su(2)$ valued Berry vector potential. The associated non-Abelian Berry curvature then reads as

$$F_{ij}^a = \frac{\partial}{\partial R^i}A_j^a - \frac{\partial}{\partial R^j}A_i^a + i\left[A_i^b,\, A_j^c\right]\,. \tag{4.97}$$

Writting the solution of (4.93) in terms of the path-ordered integrals,

$$U = P\exp\left(\oint \vec{A}_c^b(\vec{R})\cdot d\vec{R}\right)\times\exp\left(-\frac{1}{\hbar}\int_0^T dt'\mathcal{E}_n(\vec{R}(t'))\right)\,. \tag{4.98}$$

It is worth remarking that the system undergoes a $SU(2)$ rotation which depends on the path taken,

$$\gamma_n = P\exp\left(\oint \vec{A}_c^b(\vec{R})\cdot d\vec{R}\right)\,. \tag{4.99}$$

The latter defines the non-Abelian generalization of the Berry phase factor known as the Wilson loop.

Compared to the Berry phase which is always associated with a closed path, the Berry curvature is truly a local quantity. It provides a local description of the geometric properties of the parameter space while the Berry phase can be identified with the holonomy of the fiber bundle [Simon].

Also, the Berry curvature also plays the role of a (non-Abelian) magnetic-like field, which affects the particle dynamics in his neighborhood. A relevant example is provided with the "Berry" non-Abelian monopole-like fields arising in diatomic molecule systems [Wilczek 1986], see the next section.

4.3 Monopole-like fields in the diatomic molecule

As first investigated by Moody, Shapere and Wilczek [Wilczek 1986], a "truly" non-Abelian gauge fields mimicking monopole-like fields can arise in a diatomic molecule system. These come from the non-Abelian generalization [Wilczek 1984] of the Berry gauge potentials. In this case, we consider sets of levels, k-fold degenerate, subjected to adiabatically varying external parameters. For $k = 1$, a single level, we recover the $U(1)$ gauge fields discussed by Berry and Simon [Berry 1984, Simon]. For $k \geq 2$, the effective gauge fields take a "truly" non-Abelian form.

The latter can be extended to systems where the slow dynamical variables are no longer external but are themselves quantized. This is the case, in particular, for the diatomic molecule where the quantized parameters define nuclear coordinates [Wilczek 1986].

To see this, let us consider a diatom with two atomic nuclei and one or more gravitating electrons. The study of this system amounts to investigating a many-body problem which reduces, in the simplest case, to a three-body problem. Neglecting the spin degree of freedom and the relativistic effects, the Hamiltonian employed in the diatomic molecule system reads [in units $\hbar = 1$],

$$\begin{aligned}\mathcal{H}(\vec{X}_i,\, \vec{x}_k) = &\left\{ -\frac{1}{2m_k}\sum_{k=1}^{n} \vec{\nabla}^2_{x_k} - \sum_{i=1}^{2} \frac{1}{2M_i}\vec{\nabla}^2_{X_i} \right\} \\ &- \sum_{i=1}^{2}\sum_{k=1}^{n} \frac{Z_i\, e}{|\vec{X}_i - \vec{x}_k|} + \frac{Z_1 Z_2}{|\vec{X}_1 - \vec{X}_2|} + \sum_{j=1}^{n}\sum_{k>j}^{n} \frac{e^2}{|\vec{x}_j - \vec{x}_k|}\,.\end{aligned} \tag{4.100}$$

Here the atomic number Z_j corresponds to the electric charge of the jth nucleus and the positions $\vec{X}_i$ and $\vec{x}_k$ denote the nuclei and the electrons coordinates, respectively. The bracketed terms in the Hamiltonian are the kinetic energy of the electron of mass m plus the kinetic energy of the nuclei of mass M_i, with $\vec{\nabla}_{x_i}$ and $\vec{\nabla}_{X_k}$ referring to the Laplacians of the ith electron and of the kth nucleus, respectively. The two following terms in (4.100) define the classical Coulomb electron-nuclei interaction and the nucleus-nucleus interaction, respectively. The remaining term represents the electron-electron interactions.

From now on we consider the simple configuration of the molecular ion H_2^+ which possesses only one gravitating electron and two identical hydrogen nuclei. Then, the electric charge of the nuclei are the same, $Z_1 = Z_2 = Z$, and electron-electron interactions vanish since only one electron is considered in the present context. The total non-relativistic wave function $\Psi(\vec{x},\, \vec{X}_i)$ of this diatomic molecule system is a solution of the stationary Schrödinger equation,

$$\mathcal{H}\,\Psi(\vec{x},\, \vec{X}_i) = \mathcal{E}\,\Psi(\vec{x},\, \vec{X}_i)\,. \tag{4.101}$$

The description of the diatomic molecule properties is commonly made using the Born-Oppenheimer approximation. Indeed, the Born-Oppenheimer or adiabatic approximation is applied to separate, in an appropriate way, the electronic motion and the slower nuclei's degrees of freedom that couple to it, since $M_i \gg m$. To investigate electronic motions, we first assume that the nuclei positions $\vec{X}_1$ and $\vec{X}_2$ are fixed and correspond to infinite nuclear masses $M_1 = M_2 = \infty$. Thus for a fixed nuclear configuration, we obtain the electronic Hamiltonian, $\mathcal{H}_{el}$, carrying a parametrical dependence on the nuclear relative coordinate $\vec{X}_{12} = \vec{X}_1 - \vec{X}_2$,

$$\begin{aligned} \mathcal{H}_{el}\big(\vec{x}, \vec{X}_{12}\big) &= -\frac{1}{2m}\vec{\nabla}^2_{\vec{x}} + V\big(\vec{x}, \vec{X}_{12}\big)\,, \\ V\big(\vec{x}, \vec{X}_{12}\big) &= -\frac{Ze}{|\vec{X}_1 - \vec{x}|} - \frac{Ze}{|\vec{X}_2 - \vec{x}|} + \frac{Z^2}{|\vec{X}_{12}|}\,. \end{aligned} \tag{4.102}$$

Since $\vec{X}_{12}$ is just a parameter, then the last term in the previous potential is a constant and shifts the eigenvalues only by some constant amount. In the context of the Born-Oppenheimer approximation, we consider also

$$\Psi(\vec{x},\, \vec{X}_i) \approx \Psi(\vec{x},\, \vec{X}_{12})\,, \tag{4.103}$$

where the molecular wave function, $\Psi(\vec{x},\, \vec{X}_{12})$, can be expanded into a combination of the electronic wave function $\varphi_m(\vec{x})$ and the nuclear wave function $\chi_m(\vec{X}_{12})$,

$$\Psi(\vec{x},\, \vec{X}_{12}) = \sum_m \varphi_m(\vec{x}, \vec{X}_{12})\, \chi_m(\vec{X}_{12})\,. \tag{4.104}$$

Note that the electronic eigenfunction depends implicitly on the nuclear relative coordinate, $\vec{X}_{12}$, and the summation index m denote the eventual energy's degeneracy of the electronic eigenstate. Hence, the electronic eigenfunction obeys to the electronic stationary Schrödinger equation,

$$\mathcal{H}_{el}\, \varphi_m(\vec{x}) = \mathcal{E}_{el,m}\, \varphi_m(\vec{x})\,, \tag{4.105}$$

and form a complete set. While when investigating the nuclear motions, we have to consider the electron as remaining in the same quantum eigenstate so that the nuclear wave function is a solution of the Schrödinger equation with an effective potential generated by the electron,

$$\left(-\frac{1}{2M}\sum_{i=1}^{2}\vec{\nabla}^2_{X_i} + \mathcal{H}_{el} + \frac{Z^2}{|\vec{X}_{12}|}\right)\chi_k(\vec{X}_{12}) = E\,\chi_k(\vec{X}_{12})\,. \tag{4.106}$$

Let us first investigate the eigenvalues equation (4.105) for the molecular ion H_2^+ with the nuclei located on the orthogonal z-axis [see Figure 4.3]. We use the spherical coordinates $\big(r,\theta,\phi\big)$ [see Figure 4.3] to rewrite the electronic Hamiltonian (4.102) in this coordinate system,

$$\begin{aligned} \mathcal{H}_{el}(r,\theta,\phi) &= -\frac{1}{2m}\frac{1}{r}\frac{\partial^2}{\partial r^2}r + \frac{L^2}{2m\,r^2} + V\left(r,\,\theta,\,\phi\right)\,, \\ V\big(r,\theta,\phi\big) \equiv V\big(r,\theta\big) &= -\frac{Ze}{\sqrt{r^2 + R_2^2 + 2r\,R_2\cos\theta}} - \frac{Ze}{\sqrt{r^2 + R_1^2 - 2r\,R_1\cos\theta}}\,. \end{aligned}$$

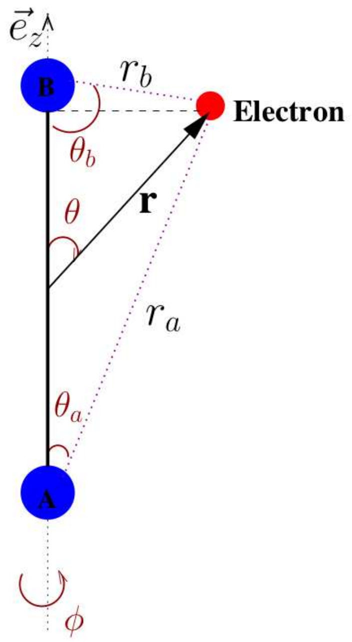

Figure 4.3: Molecular ion H_2^+ with the set (r, θ, ϕ) representing spherical coordinates and $\{\theta_a,\, \theta_b,\, R_1,\, R_2,\, r_a,\, r_b\}$ providing us with elliptic coordinate system on the plan.

The Casimir L^2 and the projection of the electronic orbital angular momentum L_z read

$$L^2 = -\frac{\partial^2}{\partial\theta^2} - \frac{1}{\tan\theta}\frac{\partial}{\partial\theta} - \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\,, \qquad L_z = -i\frac{\partial}{\partial\phi}\,. \tag{4.107}$$

Note that the potential, $V(r,\,\theta)$, which does not depend on the azimuthal angle ϕ, is rotationally symmetric around the axis of the nuclei,

$$[L_z,\, V\,(r,\,\theta)] = 0\,. \tag{4.108}$$

Since the component of the angular momentum L_z also satisfies,

$$\left[L_z,\, L^2\right] = 0\,, \tag{4.109}$$

then, (4.108) and (4.109) yields the conservation of L_z along the electronic motion,

$$[L_z,\, \mathcal{H}_{el}] = 0\,. \tag{4.110}$$

The eigenvalues of the quantized quantity L_z are given by the eigenvalues equation,

$$L_z\,\varphi_m(r,\,\theta,\,\phi) = m\,\varphi_m(r,\,\theta,\,\phi)\,. \tag{4.111}$$

The diatomic molecule therefore possesses a privileged direction carried by the axis of the nuclei [the z-axis]. Thus, L_z, generates an $SO(2)$ symmetry group. Moreover the configuration of the nuclei is also invariant under spatial inversion. Consequently the electronic Hamiltonian, $\mathcal{H}_{el}$, admits the same symmetry group $\mathcal{G}$ as the nuclei's configuration,

$$\mathcal{G} = SO(2) \times \big(\text{Parity}\big) \,. \tag{4.112}$$

Taking into account the $SO(2)$ symmetry of the diatomic molecule, we can separate the electronic wave functions, φ_m, under the form of the product,

$$\varphi_m\big(r,\, \theta,\, \phi\big) = g_m\big(\phi\big) f\big(r,\, \theta\big) \,. \tag{4.113}$$

Injecting the expanded form (4.113) into the eigenvalues equation (4.111), it is straightforward to obtain the normalized eigenfunction g_m,

$$g_{\pm m}\,(\phi) = \frac{1}{\sqrt{2\pi}}\, \exp\left(\pm i\, m\phi\right) \,, \quad m \,\in\, \mathbb{Z} \,. \tag{4.114}$$

Now, our task is to investigate the function, $f\big(r,\, \theta\big)$, appearing as a part of the electronic wave function $\varphi_m\big(r,\, \theta,\, \phi\big)$ in (4.113). To this, we switch to elliptic coordinates, $(r,\, \theta) \longrightarrow (\xi,\, \eta)$, see Figure 4.3,

$$\begin{cases} \xi = \dfrac{r_a + r_b}{R} \,, \;\; \eta = \dfrac{r_a - r_b}{R} \,, \\ R = R_2 + R_1 \,, \;\; \xi \,\in\, [1,\, \infty[\;, \;\; \eta \,\in\, [-1,\, 1] \,. \end{cases} \tag{4.115}$$

We can now express the potential and the Laplacian operator in elliptic coordinates,

$$\begin{aligned} V\big(\xi, \eta\big) &= -\frac{4eZ}{R}\frac{\xi}{\xi^2 - \eta^2} \,, \\ \vec{\nabla}^2 &= \frac{4}{R^2\left(\xi^2 - \eta^2\right)} \bigg\{ \left(\xi^2 - 1\right) \frac{\partial^2}{\partial \xi^2} - \left(\eta^2 - 1\right) \frac{\partial^2}{\partial \eta^2} \\ &\quad + 2\xi \frac{\partial}{\partial \xi} - 2\eta \frac{\partial}{\partial \eta} + \left(\frac{1}{\xi^2 - 1} - \frac{1}{\eta^2 - 1}\right) \frac{\partial^2}{\partial \phi^2} \bigg\} \,, \end{aligned} \tag{4.116}$$

so that the electronic Schrödinger equation (4.105) which takes, in elliptic coordinates, the form

$$\mathcal{H}_{el}\, g_{\pm\, m}\big(\phi\big) f\big(\xi,\, \eta\big) = \mathcal{E}_{el}\, g_{\pm\, m}\big(\phi\big) f\big(\xi,\, \eta\big) \,, \tag{4.117}$$

separates into

$$\begin{aligned} &\bigg\{ -\frac{2}{mR^2} \left(\big(\xi^2 - 1\big) \frac{\partial^2}{\partial \xi^2} + 2\xi \frac{\partial}{\partial \xi} - \frac{m^2}{\xi^2 - 1} \right) - \xi^2 \mathcal{E}_{el} - \frac{4eZ}{R}\xi \\ &+ \frac{2}{mR^2} \left(\big(\eta^2 - 1\big) \frac{\partial^2}{\partial \eta^2} + 2\eta \frac{\partial}{\partial \eta} - \frac{m^2}{\eta^2 - 1} \right) + \eta^2 \mathcal{E}_{el} \bigg\}\, g_{\pm m}\big(\phi\big) f\big(\xi,\, \eta\big) = 0 \,. \end{aligned} \tag{4.118}$$

Let us note that the dependence on the azimuthal angle has disappeared from the bracketed terms in (4.118) and is replaced by the parameter m^2. Consequently, the electronic energy spectrum depend on m^2,

$$\mathcal{H}_{el}\,\varphi_{\pm m} = \mathcal{H}_{el}\,\varphi_{\mp m} \quad \Longleftrightarrow \quad \mathcal{E}_{el,\pm m} = \mathcal{E}_{el,\mp m}\,, \tag{4.119}$$

so that, for $m \neq 0$, each electronic level is doubly degenerated. Moreover, it is now clear that the variables ξ and η separate in the eigenfunctions $f(\xi,\,\eta)$ as

$$f\,(\xi,\,\eta) = f_0\,(\xi)\,f_1\,(\eta)\;, \tag{4.120}$$

where f_0 and f_1 are solutions of the following spheroidal wave equations [5],

$$\begin{cases} \left(\left(\xi^2-1\right)\dfrac{\partial^2}{\partial\xi^2} + 2\xi\,\dfrac{\partial}{\partial\xi} + \left(\alpha + \gamma\,\xi - p^2\,\xi^2 - \dfrac{m^2}{\xi^2-1}\right)\right) f_0(\xi) = 0 \\ \left(\left(\eta^2-1\right)\dfrac{\partial^2}{\partial\eta^2} + 2\eta\,\dfrac{\partial}{\partial\eta} + \left(-\alpha + p^2\,\eta^2 - \dfrac{m^2}{\eta^2-1}\right)\right) f_1(\eta) = 0\,. \end{cases} \tag{4.121}$$

Here m^2 and α are the separation constants of the differential equation, with

$$\gamma = 2Rm\,e^2\,, \quad p^2 = -\frac{R^2}{2}m\,\mathcal{E}_{el}\,. \tag{4.122}$$

The bound states are solutions of the Schrödinger equation (4.105) associated with quantized negative energies, $\{E_{el,m}\}_{m\in\mathbb{N}}$.

Collecting our results (4.114) and (4.121) provides us with the complete electronic wave functions,

$$\varphi_{\pm m}\,(\xi,\eta,\phi) = \langle(\xi,\eta,\phi)\,|\pm m\rangle = \frac{1}{\sqrt{2\pi}}\,f_0\,(\xi)\,f_1\,(\eta)\exp\left(\pm im\phi\right), \quad m\,\in\,\mathbb{Z}\,. \tag{4.123}$$

Let us recall that the integer m which corresponds to the eigenvalue of the component of the angular momentum along the axis of symmetry z is a “good” quantum number,

$$L_z\,\varphi_m = \pm\Lambda_0\,\varphi_m\,, \quad \text{with} \quad \Lambda_0 = |\,m\,|\,. \tag{4.124}$$

When introducing the electronic spin degree of freedom, S_z, a non-vanishing additional term can be included, at each nuclear configuration, in the electronic Hamiltonian of the diatom H_2^+, namely

$$\mathcal{H}_{so} = \mu L_z\cdot S_z\,, \quad \mu\,\in\,\mathbb{R}\,, \tag{4.125}$$

corresponding to the spin-orbit effects. In that event, we can show that S_z and L_z are separately conserved,

$$\begin{aligned} [S_z,\,\mathcal{H}_{el}] &= [S_z,\,L_z\cdot S_z] = 0 \\ [L_z,\,\mathcal{H}_{el}] &= [L_z,\,L_z\cdot S_z] = 0\,. \end{aligned} \tag{4.126}$$

The projection of the total angular momentum, $J_z = L_z + S_z$, onto the z-axis is therefore

[5] Spheroidal wave equations are generalization of Mathieu differential equations.

quantized,

$$J_z\,\varphi_{\pm k}\,(\xi,\,\eta,\,\phi) = k\,\varphi_{\pm k}\,(\xi,\,\eta,\,\phi)\;,\quad k=\pm\Lambda\,, \tag{4.127}$$

so that for fixed quantum number Λ_0, the eigenvalue, Λ, associated with J_z takes the two half-integer values,

$$\Lambda = \Lambda_0 - \frac{1}{2},\quad \Lambda_0 + \frac{1}{2}\,. \tag{4.128}$$

Consequently, the introduction of the electron spin degree of freedom does not modify the double degeneracy of the electron system. The novelty here is that even the ground level $\Lambda_0 = 0$ remains doubly degenerated. Thus, the electronic wave functions are now characterized by the quantum number Λ,

$$\varphi_{\pm k}\,(\xi,\eta,\phi) = \langle(\xi,\eta,\phi)\,|\pm k\rangle = \frac{1}{\sqrt{2\pi}} f_0\,(\xi)\, f_1\,(\eta)\exp\left(\pm i\,k\phi\right),\quad k=\pm\Lambda\,. \tag{4.129}$$

Following the Born-Oppenheimer approximation, after describing the electronic wave functions labeled by the index k with the energy eigenvalues $\mathcal{E}_{el,k}$ which are parametric function of the relative internuclear coordinate, we are now interested on the nuclear motions described by the Schrödinger equation (4.106). Let us sandwich (4.106) between electronic eigenstates,

$$\begin{aligned} &-\int d\vec{x}\varphi_n^\star(\vec{x},\vec{X}_{12})\Big(\sum_{i=1}^{2}\frac{1}{2M}\vec{\nabla}^2_{X_i} - \mathcal{H}_{el} - \frac{Z^2}{|\vec{X}_{12}|}\Big)\sum_m \varphi_m(\vec{x},\vec{X}_{12})\chi_m(\vec{X}_{12}) \\ &= E\int d\vec{x}\varphi_n^\star(\vec{x},\vec{X}_{12})\sum_m \varphi_m(\vec{x},\vec{X}_{12})\chi_m(\vec{X}_{12})\,. \end{aligned} \tag{4.130}$$

Thus, this provides us with

$$\begin{aligned} &-\sum_{i=1}^{2}\frac{1}{2M}\int d\vec{x}\varphi_n^\star(\vec{x},\vec{X}_{12})\vec{\nabla}^2_{X_i}\sum_m \varphi_m(\vec{x},\vec{X}_{12})\chi_m(\vec{X}_{12}) \\ &+\mathcal{E}_{el}\chi_n(\vec{X}_{12}) + \frac{Z^2}{|\vec{X}_{12}|}\chi_n(\vec{X}_{12}) = E\chi_n(\vec{X}_{12})\,, \end{aligned} \tag{4.131}$$

where the first term can be expanded using the Leibniz rule on differentiation so that we obtain for $|\varphi_m\rangle = |m\rangle$,

$$-\frac{1}{2M}\Big(\sum_m\langle n|\vec{\nabla}^2_{\vec{X}_{12}}|m\rangle\chi_m + 2\sum_m\langle n|\vec{\nabla}_{\vec{X}_{12}}|m\rangle\vec{\nabla}\chi_m + \vec{\nabla}^2_{\vec{X}_{12}}\chi_n\Big) + \big(\mathcal{E}_{el} + \frac{Z^2}{|\vec{X}_{12}|}\big)\chi_n = E\chi_n\,.$$

Hence, we derive the effective Hamiltonian describing the nuclear motions,

$$H_{nm} = -\frac{1}{2M}\sum_k\big(\vec{\nabla}_{\vec{X}_{12}} + \langle\,n\,|\,\vec{\nabla}_{\vec{X}_{12}}\,|\,k\,\rangle\big)\big(\vec{\nabla}_{\vec{X}_{12}} + \langle\,k\,|\,\vec{\nabla}_{\vec{X}_{12}}\,|m\rangle\big) + \big(\frac{Z^2}{|\vec{X}_{12}|} + \mathcal{E}_{el}\big)\delta_{nm}\,.$$

In the adiabatic approximation, where the nuclei move slowly when compare to the electronic motion, the electron has to be considered to remain in the same 2-fold degenerate nth level. Consequently, the off-diagonal transition terms are neglected, implying the relevant effective nuclear Hamiltonian,

$$\begin{aligned} & H = -\frac{1}{2M}\left(\vec{\nabla}_{\vec{X}_{12}} - i\vec{A}(\vec{X}_{12})\right)^2 + V(\vec{X}_{12}) \,, \\ & \text{with} \quad \vec{A} = i\langle\, n \,|\, \vec{\nabla}_{\vec{X}_{12}} \,|\, n \,\rangle \quad \text{and} \quad V(\vec{X}_{12}) = \frac{Z^2}{|\vec{X}_{12}|} + \mathcal{E}_{el} \,. \end{aligned} \tag{4.132}$$

Here V acts as an effective scalar potential for nuclear motion and the induced gauge potential $\vec{A}$ is a (2×2) matrix, since the state $|\, n\,\rangle$ belongs to a 2-fold degenerate level, see (4.119) and (4.128). Hence, $\vec{A}$ transforms as a $U(2)$ gauge potential.

For the nuclear axis in the initial direction given by the polar and azimuthal angles $\theta = \phi = 0$,

$$|\, n(\vec{e}_z)\,\rangle = |\, n(0,0)\,\rangle \,, \tag{4.133}$$

we can generate a set of eigenstates adapted to nuclei pointing toward (θ,ϕ) by rotating the initial eigenstate. Then, the Wigner theorem provides us with the two possible parametrizations,

$$\begin{aligned} |n(\theta,\phi)\rangle &= \exp(iJ_3\phi)\exp(iJ_1\theta)\exp(-iJ_3\phi)|n(0,0)\rangle, \quad \text{for} \quad \theta \neq \pi \\ |\widetilde{n(\theta,\phi)}\rangle &= \exp(iJ_3\phi)\exp(iJ_1\theta)\exp(iJ_3\phi)|n(0,0)\rangle, \quad \text{for} \quad \theta \neq 0 \,. \end{aligned} \tag{4.134}$$

Note that the two previous parametrizations are linked by

$$\widetilde{|\, n\,\rangle} = \exp(2in\phi)|\, n\,\rangle \,. \tag{4.135}$$

Thus, the $U(2)$ gauge potentials, which are defined on the space spanned by the electronic eigenstate, depend on the geometry of the 2-fold degenerate eigenstate space so that

$$\begin{cases} A_r = i\langle\, n(r,\theta,\phi)\,|\,\partial_r\,|\, n(r,\theta,\phi)\rangle \\ A_\theta = i\langle\, n(r,\theta,\phi)\,|\,\partial_\theta\,|\, n(r,\theta,\phi)\rangle \\ A_\phi = i\langle\, n(r,\theta,\phi)\,|\,\partial_\phi\,|\, n(r,\theta,\phi)\rangle \,. \end{cases} \tag{4.136}$$

Performing the calculation in the case of the $\theta \neq \pi$ parametrization [6] leads to

$$\begin{cases} A_r = 0 \,, \\ A_\theta = \langle\, n\,(0,0)\,|\, -\cos\phi\, J_1 + \sin\phi\, J_2\,|\, n\,(0,0)\,\rangle \,, \\ A_\phi = \langle\, n\,(0,0)\,|\, (1-\cos\theta)\, J_3 + \sin\theta\,(\sin\phi\, J_1 + \cos\phi\, J_2)\,|\, n\,(0,0)\,\rangle \,. \end{cases} \tag{4.137}$$

We posit,

$$J_1 = a\sigma_1, \quad J_2 = b\sigma_2, \quad J_3 = c\sigma_3 \,, \tag{4.138}$$

[6] The procedure is exactly the same for the parametrization with $\theta \neq 0$.

and we obtain by direct calculation the shape of the non-vanishing gauge potentials induced by nuclear rotations,

$$\begin{cases} A_\theta = -a\cos\phi\,\sigma_1 + b\sin\phi\,\sigma_2\,, \\ A_\phi = \sin\theta\,(a\sin\phi\,\sigma_1 + b\cos\phi\,\sigma_2) + c\,(1-\cos\theta)\,\sigma_3\,. \end{cases} \tag{4.139}$$

The corresponding field strength, $F_{\theta\phi}$, reads

$$F_{\theta\phi} = \alpha\sin\theta\,\sigma_3 + (\cos\theta - 1)\,(\beta\cos\phi\,\sigma_2 + \gamma\sin\phi\,\sigma_1)\;, \tag{4.140}$$

where α, β and γ satisfy the relations,

$$\alpha = c - 2ab\,, \quad \beta = b - 2ac\,, \quad \gamma = a - 2bc\,. \tag{4.141}$$

Let us now inquire about the real nature of the $U(2)$ gauge potentials (4.139) induced by nuclear motions. Are these imbedded Abelian gauge fields into $U(2)$? or not? To respond to this question, let us recall that a field strength,

$$\mathcal{F}'_{\theta\phi} = m\sigma_1 + n\sigma_2 + p\sigma_3\,, \tag{4.142}$$

can always be gauge-transformed so that it points in one single direction σ_1, σ_2 or σ_3 say. In the present context, we search for gauge transformations which rotate the field strength (4.142) in the "Abelian" direction σ_3,

$$\mathcal{F}_{\theta,\phi} = f\sigma_3\,, \quad f \neq 0\,. \tag{4.143}$$

In the limiting case where $f = 0$, i.e. in the null field strength configuration, the gauge potentials are pure gauge. Then, the gauge potentials must be gauge equivalent to that of the vacuum.

We are looking for matrices U taking values in $U(2)$,

$$U = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{4.144}$$

so that $\mathcal{F}_{\theta\phi} = U\mathcal{F}'_{\theta\phi}U^{-1}$. Consequently, we derive from (4.3) the series of constraints to be solved

$$(S):\begin{cases} (p-f)\,A + (m+in)\,B = 0 \\ (m-in)\,A - (f+p)\,B = 0 \\ (m-in)\,C + (f-p)\,D = 0 \\ (f+p)\,C + (m+in)\,D = 0\,. \end{cases} \tag{4.145}$$

The constraints (S) can be solved provided its determinant vanished,

$$\det(S) = 0 \quad \Longleftrightarrow \quad f^2 = m^2 + n^2 + p^2 \,. \tag{4.146}$$

We then obtain an equivalence between the length of $\mathcal{F}_{\theta\phi}$ and $\mathcal{F}'_{\theta\phi}$ which express as conservation of the length of the field strength under a gauge transformation . Solving the constraints (S) , for $f \neq 0$, yields

$$U = \sqrt{\frac{f+p}{2f}} \begin{pmatrix} \exp\left(i\mu\right) & \dfrac{m-in}{f+p}\exp\left(i\mu\right) \\ -\dfrac{m+in}{f+p}\exp\left(i\nu\right) & \exp\left(i\nu\right) \end{pmatrix}$$

with arbitrary real constants

$$\mu = arg\big(A\big) \,, \quad \nu = arg\big(D\big) \,. \tag{4.147}$$

Applying the previous gauge transformation to (4.140) and (4.139) we must take

$$\begin{cases} m = \gamma \sin\phi\big(\cos\theta - 1\big) \\ n = \beta\cos\phi\big(\cos\theta - 1\big) \\ p = \alpha\sin\theta \,, \end{cases} \tag{4.148}$$

so that the length of the field strength reads

$$f^2\big(\theta,\phi\big) = \alpha^2\sin^2\theta + \big(\cos\theta - 1\big)^2\big(\gamma^2\sin^2\phi + \beta^2\cos^2\phi\big) \,. \tag{4.149}$$

Without loss of generality, we can choose [Wilczek 1986, Rho 1992],

$$\alpha = \frac{1}{2}\big(1-\kappa^2\big) \,, \quad \beta = \gamma = 0^{\,7} \quad \text{with} \quad \kappa \in \mathbb{R}^{\,8} \,, \tag{4.150}$$

so that applying the gauge transformation on the Berry potentials (4.139),

$$\begin{cases} \widetilde{A}_\theta = U\big(A_\theta + i\partial_\theta\big)U^{-1} \\ \widetilde{A}_\phi = U\big(A_\phi + i\partial_\phi\big)U^{-1} \,, \end{cases} \tag{4.151}$$

provides us with the gauge-equivalent potentials,

$$\widetilde{A}_\theta = \mp\frac{|\kappa|}{2}\begin{pmatrix} 0 & e^{i\left(\mu-\nu+\phi\right)} \\ e^{i\left(\nu-\mu-\phi\right)} & 0 \end{pmatrix} \,, \tag{4.152}$$

[7] This choice implies that $a = b = \langle + | J_1 | - \rangle = \pm\frac{1}{2}|\kappa|$ and $c = \frac{1}{2}$.

[8] Since the electronic eigenstates are not eigenfunctions of angular momentum, but only of J_3 , κ can take any real value.

and

$$\widetilde{A}_\phi = \begin{pmatrix} \frac{1}{2}(1-\cos\theta) & \mp\frac{i}{2}|\kappa|\sin\theta\, e^{i(\mu-\nu+\phi)} \\ \pm\frac{i}{2}|\kappa|\sin\theta\, e^{i(\nu-\mu-\phi)} & -\frac{1}{2}(1-\cos\theta) \end{pmatrix}. \tag{4.153}$$

It is now clear that the Berry gauge potentials (4.139) or (4.152) and (4.153) become *"Abelianized" gauge potential* for $\kappa = 0$. In that event, they represent a Dirac monopole field of unit charge imbedded into $U(2)$,

$$\widetilde{A}_\theta = 0\,, \qquad \widetilde{A}_\phi = \frac{1}{2}(1-\cos\theta)\,\sigma_3\,. \tag{4.154}$$

For $|\kappa| \neq 0$, we obtain the *truly non-Abelian case*, where the off-diagonal terms can not be eliminated [9],

$$\begin{cases} \widetilde{A}_\theta = \mp\frac{|\kappa|}{2}\big(\cos\phi\,\sigma_1 - \sin\phi\,\sigma_2\big)\,, \\ \widetilde{A}_\phi = \pm\frac{|\kappa|}{2}\sin\theta\big(\sin\phi\,\sigma_1 + \cos\phi\,\sigma_2\big) + \frac{1}{2}(1-\cos\theta)\sigma_3\,. \end{cases} \tag{4.155}$$

The corresponding field strength is

$$\widetilde{F}_{\theta\phi} = \frac{1}{2}\big(1-\kappa^2\big)\sin\theta\sigma_3\,. \tag{4.156}$$

The field strength (4.156) superficially resembles to that of a monopole field but the interpretation is quite different. Indeed, $|\kappa| \neq 0$ is not quantized here and the gauge fields induced by nuclear motions of the diatomic molecule are *truly non-Abelian* [Wilczek 1986]. See also [Zygelman 1990].

Note that when $\kappa = \pm 1$, the field strength vanishes and (4.155) is a gauge transform of the vacuum.

Our next step is to present the monopole-like field (4.155) in a more convenient "hedgehog" form. This can be achieved, by applying a suitable gauge transformations [Jackiw 1986] to the diatomic molecule gauge potential (4.155). Finally, the Berry gauge potential mimics the structure of a non-Abelian monopole ['t Hooft 1974, Polyakov 1974],

$$\widetilde{A}_i^a = (1-\kappa)\epsilon_{iaj}\,\frac{x^j}{r^2}\,, \quad \widetilde{F}_{ij}^a = (1-\kappa^2)\epsilon_{ijk}\frac{x^k x^a}{r^4}\,. \tag{4.157}$$

Note *the presence of the unquantized constant factor* $(1-\kappa^2)$ *in the above magnetic field.*

Classical dynamics and conserved quantities

Now we turn to investigating the symmetries of an isospin-carrying particle, with unit charge, evolving in the monopole-like field of the diatom (4.157) plus a scalar potential.

[9] Here we fixed $\mu = \nu$.

The Hamiltonian describing the dynamics of this particle is expressed as

$$\mathcal{H} = \frac{\vec{\pi}^2}{2} - \big(g/4\big)\epsilon_{ijk}\widetilde{F}^a_{ij}\,S^k + V(\vec{x},\,\mathcal{I}^a)\,, \quad \pi_i = p_i - \widetilde{A}^a_i\,\mathcal{I}^a\,, \tag{4.158}$$

where the spin-rotation coupling disappears when we study particle carrying null gyromagnetic ratio, $g = 0$. The resulting Hamiltonian has the same form of that of a scalar particle [10] evolving in the same magnetic field. We define the covariant Poisson-brackets as

$$\big\{M, N\big\} = D_jM\frac{\partial N}{\partial \pi_j} - \frac{\partial M}{\partial \pi_j}D_jN + \mathcal{I}^a\widetilde{F}^a_{jk}\frac{\partial M}{\partial \pi_j}\frac{\partial N}{\partial \pi_k} - \epsilon_{abc}\frac{\partial M}{\partial \mathcal{I}^a}\frac{\partial N}{\partial \mathcal{I}^b}\mathcal{I}^c\,, \tag{4.159}$$

where D_j is the covariant derivative,

$$D_jf = \partial_jf - \epsilon_{abc}\mathcal{I}^a\widetilde{A}^b_j\,\frac{\partial f}{\partial \mathcal{I}^c}\,. \tag{4.160}$$

The commutator of the covariant derivatives is recorded as

$$[D_i, D_j] = -\epsilon_{abc}\mathcal{I}^a\widetilde{F}^b_{ij}\,\frac{\partial}{\partial \mathcal{I}^c}\,. \tag{4.161}$$

The non-vanishing brackets are

$$\{x^i, \pi_j\} = \delta^i_j\,, \quad \{\pi_i, \pi_j\} = \mathcal{I}^a\widetilde{F}^a_{ij}\,, \quad \{\mathcal{I}^a, \mathcal{I}^b\} = -\epsilon_{abc}\mathcal{I}^c\,, \tag{4.162}$$

and the equations of motion governing an isospin-carrying particle in the static non-Abelian gauge field (4.157) read

$$\begin{cases} \ddot{x}_i - \mathcal{I}^a\widetilde{F}^a_{ij}\,\dot{x}^j + D_iV = 0\,, \\ \dot{\mathcal{I}}^a + \epsilon_{abc}\,\mathcal{I}^b\left(\widetilde{A}^c_j\,\dot{x}^j - \dfrac{\partial V}{\partial \mathcal{I}^c}\right) = 0\,. \end{cases} \tag{4.163}$$

The first equation in (4.163) describes the 3D real motion implying a generalized Lorentz force plus an interaction with the scalar potential; while the second equation is the Kerner-Wong equation augmented with a scalar field interaction. The latter describes, as expected, the isospin classical motion.

Let us now recall the van Holten procedure yielding the conserved quantities. The constants of the motion are expanded in powers of the momenta,

$$\mathcal{Q}\big(\vec{x}, \vec{\mathcal{I}}, \vec{\pi}\big) = C(\vec{x}, \vec{\mathcal{I}}) + C_i(\vec{x}, \vec{\mathcal{I}})\pi_i + \frac{1}{2!}C_{ij}(\vec{x}, \vec{\mathcal{I}})\pi_i\pi_j + \cdots\,, \tag{4.164}$$

and we require $\mathcal{Q}$ to Poisson-commute with the Hamiltonian,

$$\{\mathcal{Q},\ \mathcal{H} = \frac{\vec{\pi}^2}{2} + V(\vec{x},\,\mathcal{I}^a)\} = 0\,. \tag{4.165}$$

[10] i.e. particle without spin.

We therefore get the set of constraints which have to be solved,

$$\begin{aligned}
&C_i D_i V + \epsilon_{abc}\mathcal{I}^a \frac{\partial C}{\partial \mathcal{I}^b}\frac{\partial V}{\partial \mathcal{I}^c} = 0, && o(0)\\
&D_i C = \mathcal{I}^a \widetilde{F}^a_{ij} C_j + C_{ij} D_j V + \epsilon_{abc}\mathcal{I}^a \frac{\partial C_i}{\partial \mathcal{I}^b}\frac{\partial V}{\partial \mathcal{I}^c}, && o(1)\\
&D_i C_j + D_j C_i = \mathcal{I}^a (\widetilde{F}^a_{ik} C_{kj} + \widetilde{F}^a_{jk} C_{ki}) + C_{ijk} D_k V + \epsilon_{abc}\mathcal{I}^a \frac{\partial C_{ij}}{\partial \mathcal{I}^b}\frac{\partial V}{\partial \mathcal{I}^c}, && o(2)\\
&D_i C_{jk} + D_j C_{ki} + D_k C_{ij} = \mathcal{I}^a (\widetilde{F}^a_{il} C_{ljk} + \widetilde{F}^a_{jl} C_{lki} + \widetilde{F}^a_{kl} C_{lij})\\
&\qquad\qquad + C_{ijkl} D_l V + \epsilon_{abc}\mathcal{I}^a \frac{\partial C_{ijk}}{\partial \mathcal{I}^b}\frac{\partial V}{\partial \mathcal{I}^c}, && o(3)\\
&\vdots \qquad\qquad\qquad \vdots && \vdots
\end{aligned} \tag{4.166}$$

Turning to the zeroth-order conserved charge, we note that, for $\kappa \neq 0$, the used-to-be electric charge,

$$Q = \frac{\vec{x}\cdot\vec{\mathcal{I}}}{r}\,, \tag{4.167}$$

is *not more covariantly conserved* in general,

$$\{Q, \mathcal{H}\} = \vec{\pi}\cdot\vec{D}Q, \quad D_j Q = \frac{\kappa}{r}\left(\mathcal{I}^j - Q\frac{x_j}{r}\right). \tag{4.168}$$

An exception occurs when the isospin is aligned into the radial direction, as seen from (4.168). A detailed calculation shows that the equation $D_j Q = 0$ can only be solved, for imbedded Abelian monopole field, when $\kappa = 0, \pm 1$.

Nor is Q^2 conserved,

$$\{Q^2, \mathcal{H}\} = 2\kappa Q(\vec{\pi}\cdot\vec{D}Q)\,. \tag{4.169}$$

Note for further reference that, unlike Q^2, the length of the isospin, $\mathcal{I}^2$, *is* conserved,

$$\{\mathcal{H}, \mathcal{I}^2\} = 0\,.$$

The monopole-like gauge field (4.157) is rotationally symmetric and an isospin-carrying particle moving in it admits a conserved angular momentum [Wilczek 1986, Jackiw 1986]. Its form is, however, somewhat unconventional, and we re-derive it, therefore, in detail [Ngome 02/2009].

1) We start our investigation with conserved quantities which are linear in the covariant momentum. We have therefore

$$C_{ij} = C_{ijk} = \cdots = 0\,, \tag{4.170}$$

so that the series of constraints (4.166) reduce to

$$\begin{cases} C_i D_i V + \epsilon_{abc} \mathcal{I}^a \dfrac{\partial C}{\partial \mathcal{I}^b} \dfrac{\partial V}{\partial \mathcal{I}^c} = 0\,, & o(0) \\ D_i C = \mathcal{I}^a \widetilde{F}^a_{ij} C_j + \epsilon_{abc} \mathcal{I}^a \dfrac{\partial C_i}{\partial \mathcal{I}^b} \dfrac{\partial V}{\partial \mathcal{I}^c}\,, & o(1) \\ D_i C_j + D_j C_i = 0\,. & o(2) \end{cases} \tag{4.171}$$

We use the Killing vector generating spatial rotations,

$$\vec{C} = \vec{n} \times \vec{x}\,. \tag{4.172}$$

Choosing $V = V(r)$, we see that, again due to the non-conservation of Q, $D_j V \neq 0$ in general. The zeroth-order condition $\vec{C} \cdot \vec{D} V = 0$ in (4.171) is, nevertheless, satisfied when V is a radial function independent of $\vec{\mathcal{I}}$, since then $\vec{D} V = \vec{\nabla} V$, which is perpendicular to infinitesimal rotations, $\vec{C}$.

Evaluating the right hand side of the first-order constraint of (4.171) with $\widetilde{F}^a_{jk}$ as given in (4.157), the equation to be solved becomes

$$D_i C = (1-\kappa^2) \frac{Q}{r} \left((\vec{n} \cdot \frac{\vec{x}}{r}) \frac{x_i}{r} - n_i \right). \tag{4.173}$$

In the Wu-Yang case, $\kappa = 0$, this equation was solved by $C = -\vec{n} \cdot Q \dfrac{\vec{x}}{r}$. But for $\kappa \neq 0$, the electric charge, Q, is not conserved, and using (4.168), (4.173), as well as the relations

$$\begin{cases} D_i (\vec{\mathcal{I}} \cdot \vec{n}) & = (1-\kappa) \left(\dfrac{Q}{r} n_i - \dfrac{\vec{n} \cdot \hat{r}}{r} \mathcal{I}_i \right)\,, \\ D_i \left(Q\, \vec{n} \cdot \dfrac{\vec{x}}{r} \right) & = \dfrac{Q}{r} \left(n_i - (1+\kappa)(\vec{n} \cdot \hat{r}) \dfrac{x_i}{r} \right) + \dfrac{\kappa}{r} (\vec{n} \cdot \hat{r}) \mathcal{I}_i \\ \mathcal{I}^a F^a_{ij} & = (1-\kappa^2)\, Q\, \dfrac{\epsilon_{ijk} x_k}{r^3}\,, \end{cases} \tag{4.174}$$

we find,

$$-(1-\kappa) D_i \left(Q\, \vec{n} \cdot \frac{\vec{x}}{r} \right) = \kappa D_i (\vec{\mathcal{I}} \cdot \vec{n}) + D_i C.$$

This allows us to infer that

$$C = -\Big((1-\kappa)\, Q\, \frac{\vec{x}}{r} + \kappa \vec{\mathcal{I}} \Big) \cdot \vec{n}\,. \tag{4.175}$$

The conserved angular momentum is, therefore,

$$\begin{cases} \vec{J} = \vec{x} \times \vec{\pi} - \vec{\Psi}\,, \\ \vec{\Psi} = (1-\kappa)\, Q\, \dfrac{\vec{x}}{r} + \kappa \vec{\mathcal{I}} \ = Q\, \dfrac{\vec{x}}{r} + \kappa \left(\dfrac{\vec{x}}{r} \times \vec{\mathcal{I}} \right) \times \dfrac{\vec{x}}{r}\,, \end{cases} \tag{4.176}$$

consistently with the results in [Jackiw 1986, Rho 1992]. Moody, Shapere and Wilczek [Wilczek 1986] found the correct expression, (4.176), for $\kappa = 0$ but, as they say it, "they are not aware of a canonical derivation when $\kappa \neq 0$". Our construction here is an alternative to that of Jackiw [Jackiw 1986], who obtained it using the method of Reference [Jackiw-Manton 1980]. In his approach, based on the study of symmetric gauge fields [Forgács-Manton 1980], each infinitesimal rotation, (4.172), is a symmetry of the monopole in the sense that it changes the potential by a surface term.

It is worth noting that comparison with the Wu-Yang case yields the "replacement rule",

$$Q\,\frac{\vec{x}}{r} \to \vec{\Psi}. \tag{4.177}$$

For $\kappa = 0$ we recover the Wu-Yang expression (4.65). Eliminating $\vec{\pi}$ in favor of $\vec{p} - \vec{A} = \vec{\pi}$ allows us to rewrite the total angular momentum as

$$\vec{J} = \vec{x} \times \vec{p} - \vec{\mathcal{I}}\,, \tag{4.178}$$

making manifest the celebrated "spin from isospin term" [Jackiw 1976].

Alternatively, a direct calculation, using the same formulae (4.168)-(4.174), allows us to confirm that $\vec{J}$ commutes with the Hamiltonian, $\{J_i, \mathcal{H}\} = 0$.

Multiplying (4.176) by $\dfrac{\vec{x}}{r}$ yields, once again, the relation (4.72) i.e.,

$$\vec{J} \cdot \frac{\vec{x}}{r} = -Q\,, \tag{4.179}$$

the same as in the Wu-Yang case. This is, however, less useful as before, since Q is not a constant of the motion so that the angle between $\vec{J}$ and the radius vector, $\vec{x}(t)$, is not more a constant. The components of the angular momentum (4.176) close, nevertheless, to so(3),

$$\big\{J_i, J_j\big\} = \epsilon_{ijk} J_k\,. \tag{4.180}$$

In addition of $\vec{J}$, it is worth mentioning that the Casimir

$$J^2 = \big(\vec{x} \times \vec{\pi}\big)^2 + \big(1 - \kappa\big)^2 Q^2 - \kappa^2 \vec{\mathcal{I}}^2 - 2\kappa \vec{J} \cdot \vec{\mathcal{I}} \tag{4.181}$$

is obviously conserved since the angular momentum of the diatom is conserved (4.176).

2) Returning to the van Holten algorithm, quadratic conserved quantities are sought by taking

$$C_{ijk} = C_{ijkl} \cdots = 0\,. \tag{4.182}$$

Consequently (4.166) reduces to

$$\begin{cases} C_iD_iV+\epsilon_{abc}\mathcal{I}^a\dfrac{\partial C}{\partial \mathcal{I}^b}\dfrac{\partial V}{\partial \mathcal{I}^c}=0, & o(0)\\ D_iC=\mathcal{I}^a\widetilde{F}^a_{ij}C_j+C_{ij}D_jV+\epsilon_{abc}\mathcal{I}^a\dfrac{\partial C_i}{\partial \mathcal{I}^b}\dfrac{\partial V}{\partial \mathcal{I}^c}, & o(1)\\ D_iC_j+D_jC_i=\mathcal{I}^a(\widetilde{F}^a_{ik}C_{kj}+\widetilde{F}^a_{jk}C_{ki})+\epsilon_{abc}\mathcal{I}^a\dfrac{\partial C_{ij}}{\partial \mathcal{I}^b}\dfrac{\partial V}{\partial \mathcal{I}^c}, & o(2)\\ D_iC_{jk}+D_jC_{ki}+D_kC_{ij}=0\,. & o(3)\end{cases} \tag{4.183}$$

We consider the rank- 2 Killing tensor,

$$C_{ij}=2\delta_{ij}x^2-2x_ix_j\,, \tag{4.184}$$

which satisfies the third-order constraint of (4.183). Injecting (4.184) into the second-order constraint yields,

$$D_iC_j+D_jC_i=0\,, \tag{4.185}$$

which can be solved by taking $C_i=0$. For radial potentials independent of $\vec{\mathcal{I}}$, it is straightforward to satisfy the first- and the zeroth-order constraints of (4.183) with $C=0$. Thus, we obtain the conserved Casimir,

$$L^2=\big(\vec{x}\times\vec{\pi}\big)^2=x^2\vec{\pi}^2-\big(\vec{x}\cdot\vec{\pi}\big)^2, \tag{4.186}$$

which is the square of the non-conserved orbital angular momentum, $\vec{L}=\vec{x}\times\vec{\pi}$.

Since J^2 and L^2 are both conserved, it is now straightforward to identify the charge,

$$\Gamma=J^2-L^2=\big(1-\kappa\big)^2Q^2-\kappa^2\vec{\mathcal{I}}^2-2\kappa\vec{J}\cdot\vec{\mathcal{I}}\,, \tag{4.187}$$

which is conserved along the motion in the monopole-like field of diatomic molecule. *It is worth noting that the charge Γ corresponds, in the Abelian limit with $\kappa=0$, to the square of the electric charge.* As the constants of the motion $\vec{J}$, J^2 and L^2, the charge Γ is conserved for any radially symmetric potential, $V(r)$.

Note that Γ can also be obtained by using the Killing vector,

$$\vec{C}=2\kappa\big(\vec{x}\times\vec{\mathcal{I}}\big)\,, \tag{4.188}$$

into the van Holten algorithm (4.171).

Let us now decompose the covariant momentum, into radial and transverse components, with the vector identity,

$$(\vec{\pi})^2=(\vec{\pi}\cdot\frac{\vec{x}}{r})^2+(\vec{\pi}\times\frac{\vec{x}}{r})^2=\pi_r^2+\frac{L^2}{r^2}\,. \tag{4.189}$$

This hence allows us to express the diatomic molecule Hamiltonian (4.158) as

$$\mathcal{H} = \frac{1}{2}(\vec{\pi}\cdot\frac{\vec{x}}{r})^2 + \frac{J^2}{2r^2} - \left\{\frac{(1-\kappa)^2 Q^2 - \kappa^2 \mathcal{I}^2 - 2\kappa \vec{J}\cdot\vec{\mathcal{I}}}{2r^2}\right\} + V(r)\,. \tag{4.190}$$

Suggesting that the charge takes the fixed value $Q^2 = \mathcal{I}^2 = 1/4$, Jackiw found a similar decomposition as (4.190) [Jackiw 1986], but this is, however, only legitimate when $\kappa = 0$, since Q^2 is not conserved for $\kappa \neq 0$.

For $\kappa \neq 0$, the "good" approach is to recognize the fixed charge Γ, which yields the nice decomposition,

$$\mathcal{H} = \frac{1}{2}(\vec{\pi}\cdot\frac{\vec{x}}{r})^2 + \frac{J^2}{2r^2} - \frac{\Gamma}{2r^2} + V(r)\,. \tag{4.191}$$

Let us underline that the effective field of a diatomic molecule provides us with an interesting generalization of the Wu-Yang monopole. For $\kappa \neq 0, \pm 1$, it is truly non-Abelian, i.e., *not reducible* to one on an $U(1)$ bundle. No covariantly constant direction field, and, therefore, *no conserved electric charge* does exist in this case.

The field is nevertheless radially symmetric, but the conserved angular momentum (4.176) has a non-conventional form.

In bundle terms, the action of a symmetry generator can be lifted to the bundle so that it preserves the connection form which represents the potential. But the group structure may not be conserved; this requires another, consistency condition [Jackiw-Manton 1980], which may or may not be satisfied. In the diatomic case, it is not satisfied when $\kappa \neq 0, \pm 1$.

Is it possible to redefine the "lift" so that the group structure be preserved ? In the Abelian case, the answer can be given in cohomological terms [Duval 1982]. If this obstruction does not vanish, it is only a *central extension* that acts on the bundle.

In the truly non-Abelian case, the consistency condition involves the covariant, rather than ordinary derivative and covariantly constant sections only exist in exceptional cases – namely when the bundle is reducible. Thus, only some (non-central extension) acts on the bundle.

It is worth noting that for $\kappa \neq 0$ the configuration (4.157) does not satisfy the vacuum Yang-Mills equations. It only satisfies indeed with a suitable conserved current [Jackiw 1986],

$$\mathcal{D}_i F_{ik} = j_k, \quad \vec{\jmath} = \frac{\kappa(1-\kappa^2)}{r^4}\,\vec{x}\times\vec{T}\,, \tag{4.192}$$

Interestingly, this current can also be produced by a hedgehog Higgs field,

$$j_k = \big[\mathcal{D}_k\Phi, \Phi\big], \quad \Phi^a = \frac{\sqrt{1-\kappa^2}}{r}\,\frac{x_a}{r}\,. \tag{4.193}$$

For $\kappa = 0$, it is straightforward to derive the conserved Runge-Lenz vector since this case is exactly equivalent to the Wu-Yang case, an imbedded Abelian monopole. For

$\kappa \neq 0\,,\pm1$, we derived a new conserved charge, namely Γ, which has an unconventional form, see (4.187). In the limit case $\kappa = 0$, this conserved charge reduces to $\Gamma = Q^2$; while for $\kappa = \pm1$, we obtain $\Gamma \sim \vec{L}\cdot\vec{\mathcal{I}}$.

Let us emphasize that the derivation of the non-Abelian field configuration (4.157) from molecular physics [Wilczek 1986] indicates that our analysis may not be of purely academic interest. The situation could well be analogous to what happened before with the non-Abelian Aharonov-Bohm experiment, first put forward and studied theoretically in [Wu Yang 1975, Horváthy 04/1985], but which became recently accessible experimentally, namely by applying laser beams to cold atoms [Öhberg 2005, Öhberg 2007, Dalibard *et al.*]. A similar technique can be used to create non-Abelian monopole-type fields [Dalibard *et al.*].

Chapter 5

Supersymmetric extension of the van Holten algorithm

Contents

We investigate the super and dynamical symmetries of a fermion in external magnetic fields using a SUSY extension of the van Holten framework, based on Grassmann-valued Killing tensors.

5.1 Supersymmetry of the monopole

In this section, we investigate the super- and the dynamical symmetries of fermions in a D-dimensional monopole background. Following an interesting result of D'Hoker and Vinet [D'Hoker 1984], a non-relativistic spin-$\frac{1}{2}$ charged particle with gyromagnetic ratio $g=2$ interacting with a point magnetic monopole, admits an $\mathfrak{osp}(1|2)$ supersymmetry. This was also seen in the following papers [Gibbons 1993, DeJonghe 1995, Rietdijk, Horváthy 2000, Plyushchay 2000, Plyushchay 04/2000, Leiva 2003, Horváthy 2005].

Later, Fehér [Fehér 1987] has shown that a $g=2$ spin-particle in a monopole field does not admit a Runge-Lenz type dynamical symmetry.

Another, surprising, result of D'Hoker and Vinet [D'Hoker 01/1985, D'Hoker 09/1985, D'Hoker 04/1986] says, however, that a non-relativistic spin-$\frac{1}{2}$ charged particle with *anomalous gyromagnetic ratio* $g=4$, interacting with a point magnetic monopole plus a Coulomb plus a fine-tuned inverse-square potential, does have such a dynamical symmetry. This is to be compared with the one about the O(4) symmetry of a scalar particle in such a combined field [Mcintosh 1970, Zwanziger 1968]. Replacing the scalar particle by a spin 1/2 particle with gyromagnetic ratio $g=0$, one can prove that two anomalous systems, the one with $g=4$ and the one with $g=0$ are, in fact, superpartners [Fehér 1988]. Note that for both particular g-values, one also has an additional o(3) "spin" symmetry.

On the other hand, it has been shown by Spector [Spector] that the $\mathcal{N}=1$ supersymmetry only allows $g=2$ and no scalar potential. Runge-Lenz and SUSY appear, hence, inconsistent.

We study the bosonic as well as supersymmetries of the Pauli-type Hamiltonian,

$$\mathcal{H}_g = \frac{\vec{\Pi}^2}{2} - \frac{eg}{2}\,\vec{S}\cdot\vec{B} + V(r)\,, \quad \vec{\Pi} = \vec{p} - e\,\vec{A}\,, \tag{5.1}$$

which describes the motion of a fermion with spin $\vec{S}$ and electric charge e, in the combined magnetic field, $\vec{B}$, plus a spherically symmetric scalar field $V(r)$, which also includes a Coulomb term (a "dyon" in what follows). In the Hamiltonian (5.1), $\vec{\Pi}$ denotes the gauge covariant momentum and the constant parameter g represents the gyromagnetic ratio of the spinning particle.

Let us first describe the Hamiltonian dynamics, defined by (5.1), of the charged spin-$\frac{1}{2}$ particle, moving in the flat manifold $\mathcal{M}^{D+d}$. Note that $\mathcal{M}^{D+d}$ is the extension of the bosonic configuration space $\mathcal{M}^{D}$ by a d-dimensional internal space carrying the fermionic degrees of freedom [Cariglia]. The $(D+d)$-dimensional space $\mathcal{M}^{D+d}$ is described by the local coordinates $(x^{\mu},\, \psi^{a})$ where $\mu = 1, \cdots, D$ and $a = 1, \cdots, d$. The motion of the spin-particle is, therefore, described by the curve

$$\tau \to (\, x(\tau),\, \psi(\tau)) \,\in\, \mathcal{M}^{D+d}\,. \tag{5.2}$$

We choose $D = d = 3$ and we focus our attention to the spin-$\frac{1}{2}$ charged particle interacting with the static $U(1)$ monopole background,

$$\vec{B} = \vec{\nabla}\times\vec{A} = \frac{q}{e}\frac{\vec{x}}{r^3}\,, \tag{5.3}$$

so that the system is defined by the Hamiltonian (5.1). We introduce the covariant hamiltonian formalism extending van Holten's framework to fermions. The basic phase-space reads $\left(x^j, \Pi_j, \psi^a\right)$, where the variables ψ^a transform as tangent vectors and satisfy the Grassmann algebra,

$$\psi^i\psi^j + \psi^j\psi^i = 0\,. \tag{5.4}$$

The internal angular momentum of the particle can also be described in terms of vector-like Grassmann variables,

$$S^j = -\frac{i}{2}\epsilon^j{}_{kl}\psi^k\,\psi^l\,. \tag{5.5}$$

Defining the covariant Poisson-Dirac brackets for functions f and h of the phase-space as

$$\{f,h\} \;=\; \partial_j f\,\frac{\partial h}{\partial \Pi_j} - \frac{\partial f}{\partial \Pi_j}\,\partial_j h + eF_{ij}\,\frac{\partial f}{\partial \Pi_i}\,\frac{\partial h}{\partial \Pi_j} + i(-1)^{a^f}\,\frac{\partial f}{\partial \psi^a}\,\frac{\partial h}{\partial \psi_a}\,, \tag{5.6}$$

where $a^f = (0,1)$ is the Grassmann parity of the phase-space function f and the magnetic field reads $B_i = (1/2)\epsilon_{ijk}F_{jk}$. It is straightforward to obtain the non-vanishing fundamental brackets,

$$\left\{x^i,\,\Pi_j\right\} = \delta^i_j,\quad \left\{\Pi_i,\,\Pi_j\right\} = e\,F_{ij},\quad \left\{\psi^i,\,\psi^j\right\} = -i\,\delta^{ij}\,, \tag{5.7}$$

$$\left\{S^i,\,G^j\right\} = \epsilon_k^{\ ij}\,G^k \quad \text{with} \quad G^k = \psi^k,\,S^k\,. \tag{5.8}$$

It follows that, away from the monopole's location, the Jacobi identities are verified [Jackiw 1985, Chaichian 2009]. Thus, the equations of motion can be obtained in this covariant Hamiltonian framework [1],

$$\dot{\vec{G}} = \frac{eg}{2}\,\vec{G}\times\vec{B}\,, \tag{5.9}$$

$$\dot{\vec{\Pi}} = e\,\vec{\Pi}\times\vec{B} - \vec{\nabla}V(r) + \frac{eg}{2}\,\vec{\nabla}\left(\vec{S}\cdot\vec{B}\right). \tag{5.10}$$

Equation (5.9) shows that the fermionic vectors $\vec{S}$ and $\vec{\psi}$ are conserved when the spin and the magnetic field are uncoupled, i.e. for *vanishing gyromagnetic ratio*, $g = 0$. Note that, in addition to the magnetic field term, the Lorentz equation (5.10) also involves a potential term augmented with a spin-field interaction term (Stern and Gerlach term).

We now proceed by deducing, in a classical framework, the supersymmetries and conservation laws of the system (5.1), using the SUSY extension of the van Holten algorithm [Ngome 03/2010] developed in section 2.2. What is new here is that the generators of SUSY are Grassmann-valued Killing tensors. We expand the phase-space function, associated with one (super)symmetry, in powers of the covariant momenta,

$$\mathcal{Q}\left(\vec{x},\,\vec{\Pi},\,\vec{\psi}\right) = C(\vec{x},\,\vec{\psi}) + \sum_{k=1}^{n-1}\frac{1}{k!}\,C^{i_1\cdots i_k}(\vec{x},\,\vec{\psi})\,\Pi_{i_1}\cdots\Pi_{i_k}\,. \tag{5.11}$$

Note the dependence on Grassmann variables of the tensors $C(\vec{x},\,\vec{\psi})$. Requiring that $\mathcal{Q}$ Poisson-commutes with the Hamiltonian, $\left\{\mathcal{H}_g,\,\mathcal{Q}\right\} = 0$, implies the series of constraints,

$$\begin{aligned}
&C_i\partial_i V + \frac{ieg}{4}\psi^l\psi^m C_j\partial_j F_{lm} - \frac{eg}{2}\psi^m\frac{\partial C}{\partial\psi^a}F_{am} = 0, && \mathrm{o}(0)\\
&\partial_j C = C_{jk}\partial_k V + eF_{jk}C_k + \frac{ieg}{4}\psi^l\psi^m C_{jk}\partial_k F_{lm} - \frac{eg}{2}\psi^m\frac{\partial C_j}{\partial\psi^a}F_{am}, && \mathrm{o}(1)\\
&\partial_{(j}\,C_{k)} = C_{jkm}\partial_m V + e\left(F_{jm}C_{mk} + F_{km}C_{mj}\right)\\
&\qquad\qquad + \frac{ieg}{4}\psi^l\psi^m C_{ijk}\partial_i F_{lm} - \frac{eg}{2}\psi^m\frac{\partial C_{jk}}{\partial\psi^a}F_{am}, && \mathrm{o}(2)\\
&\partial_{(j}\,C_{kl)} = C_{jklm}\partial_m V + e\left(F_{jm}C_{mkl} + F_{lm}C_{mjk} + F_{km}C_{mlj}\right)\\
&\qquad\qquad + \frac{ieg}{4}\psi^m\psi^n C_{ijkl}\partial_i F_{mn} - \frac{eg}{2}\psi^m\frac{\partial C_{jkl}}{\partial\psi^a}F_{am}\,, && \mathrm{o}(3)\\
&\ \vdots \qquad\qquad \vdots && \vdots
\end{aligned} \tag{5.12}$$

[1] The dot means derivative w.r.t. the evolution parameter, $\frac{d}{d\tau}$.

This series of constraint can be truncated at a finite order n provided *the higher order constraint becomes a Killing equation*. The zeroth-order equation can be interpreted as a *consistency condition between the potential and the (super)invariant*. Apart from the zeroth-order constants of the motion, i.e., such that do not depend on the momentum, all other order-n (super)invariants are deduced by the systematic method (5.12) implying rank-n Killing tensors. Each Killing tensor solves the higher order constraint of (5.12) and can therefore generate a (super)invariant.

We focus our attention on searching for conserved quantities which are linear or quadratic in the covariant momenta. Thus, we have to determine generic Grassmann-valued Killing tensors of rank-one and rank-two.

• Let us first consider the Killing equation,

$$\partial_j C^k(\vec{x}, \vec{\psi}) + \partial_k C^j(\vec{x}, \vec{\psi}) = 0\,. \tag{5.13}$$

Following Berezin and Marinov [Berezin], any tensor which takes its values in the Grassmann algebra may be represented as a finite sum of homogeneous monomials,

$$C^i(\vec{x}, \vec{\psi}) = \sum_{k\geq 0} \mathcal{C}^i_{a_1\cdots a_k}(\vec{x})\psi^{a_1}\cdots\psi^{a_k}\,, \tag{5.14}$$

where the coefficients tensors, $\mathcal{C}^i_{a_1\cdots a_k}$, are completely anti-symmetric in the fermionic indices $\{a_k\}$. The tensors (5.14) satisfy (5.13), from which we deduce that their (tensor) coefficients satisfy

$$\Big(\partial_j \mathcal{C}^k_{a_1\cdots a_m}(\vec{x}) + \partial_k \mathcal{C}^j_{a_1\cdots a_m}(\vec{x})\Big)\,\psi^{a_1}\cdots\psi^{a_m} = 0 \quad\Longrightarrow\quad \partial_i\partial_j \mathcal{C}^k_{a_1\cdots a_m}(\vec{x}) = 0\,, \tag{5.15}$$

providing us with the most general rank-1 Grassmann-valued Killing tensor

$$C^i(\vec{x}, \vec{\psi}) = \sum_{k\geq 0} \big(M^{ij}\,x^j + N^i\big)_{a_1\cdots a_k}\psi^{a_1}\cdots\psi^{a_k}\,, \quad M^{ij} = -M^{ji}\,, \tag{5.16}$$

where N^i and the antisymmetric M^{ij} define constant tensors.

• Let us now construct the rank-2 Killing tensors which solve the Killing equation,

$$\partial_j C^{kl}(\vec{x}, \vec{\psi}) + \partial_l C^{jk}(\vec{x}, \vec{\psi}) + \partial_k C^{lj}(\vec{x}, \vec{\psi}) = 0\,. \tag{5.17}$$

Considering the expansion in terms of Grassmann degrees of freedom [Berezin] of the Killing tensor $C^{jk}(\vec{x}, \vec{\psi})$, we get the coefficients tensors $\mathcal{C}^{ij}_{a_1\cdots a_k}$ which are constructed as symmetrized products [Gibbons 1987] of Yano-type Killing tensors, $\mathcal{C}^i_Y(\vec{x})$, associated with the rank-1 Killing tensors $\mathcal{C}^i(\vec{x})$ obtained by (5.15),

$$\mathcal{C}^{ij}_{a_1\cdots a_k}(\vec{x}) = \frac{1}{2}\Big(\mathcal{C}^i_Y\widetilde{\mathcal{C}}^{jY} + \widetilde{\mathcal{C}}^i_Y\mathcal{C}^{jY}\Big)_{a_1\cdots a_k}\,. \tag{5.18}$$

It is worth noting that the Killing tensor defined in (5.18) is symmetric in its bosonic indices and anti-symmetric in the fermionic indices. Thus, we obtain

$$\begin{aligned} C^{ij}(\vec{x},\,\vec{\psi}) = \sum_{k\geq 0}\Big(&M^{(i}_{\;ln}\widetilde{M}^{j)\,n}_{\;m}x^l x^m + M^{(i}_{\;ln}\widetilde{N}^{j)\,n}x^l \\ &+N^{(i}_{\;n}\widetilde{M}^{j)\,n}_{\;m}x^m + N^{(i}_{\;n}\widetilde{N}^{j)\,n}\Big)_{a_1\cdots a_k}\psi^{a_1}\cdots\psi^{a_k}\,, \end{aligned} \tag{5.19}$$

where $M^{ij}_{\;\;k}$, $\widetilde{M}^{ij}_{\;\;k}$, $N^j_{\;k}$ and $\widetilde{N}^j_k$ are skew-symmetric constants tensors. Then one can verify with direct calculations that (5.16) and (5.19) satisfy Killing equations.

Having constructed the generic Killing tensors (5.16) and (5.19) generating constants of the motion, we can now describe the supersymmetries of the Pauli-like Hamiltonian (5.1). To start, we search for momentum-independent invariants, i.e. which are not derived from a Killing tensor, $C^i = C^{ij} = \cdots = 0$. In that event, the system of equations (5.12) reduces to the two constraints,

$$\begin{cases} g\psi^m \dfrac{\partial \mathcal{Q}_c(\vec{x},\vec{\psi})}{\partial \psi^a}\,F_{am} = 0\,, & \text{o}(0) \\ \partial_i \mathcal{Q}_c(\vec{x},\vec{\psi}) = 0\,. & \text{o}(1) \end{cases} \tag{5.20}$$

For $g = 0$, which means no spin-gauge field coupling, it is straightforward to see that the spin vector, in particular, and all arbitrary functions $f\big(\vec{\psi}\,\big)$ which depend only on the Grassmann variables are conserved along the motion.

For nonvanishing gyromagnetic ratio g, only the "chiral" charge

$$\mathcal{Q}_c = \vec{\psi}\cdot\vec{S} \tag{5.21}$$

remains conserved. The "chiral" charge $\mathcal{Q}_c$ can be considered as the projection of the internal angular momentum, $\vec{S}$, onto the internal trajectory $\psi(\tau)$. Thus, $\mathcal{Q}_c$ can be viewed as the internal analogue of the projection of the angular momentum, in bosonic sector, onto the classical trajectory $x(\tau)$.

Let us now construct superinvariants linear in the covariant momentum. $C^{ij} = \cdots = 0$ such that (5.12) becomes

$$\begin{cases} C^i\,\partial_i V + \dfrac{ieg}{4}\,\psi^l\psi^m C^j(\vec{x},\vec{\psi})\,\partial_j F_{lm} - \dfrac{eg}{2}\psi^m \dfrac{\partial C(\vec{x},\vec{\psi})}{\partial\psi^a}\,F_{am} = 0\,, & \text{o}(0) \\ \partial_j C(\vec{x},\vec{\psi}) = eF_{jk}C^k(\vec{x},\vec{\psi}) - \dfrac{eg}{2}\psi^m \dfrac{\partial C^j(\vec{x},\vec{\psi})}{\partial\psi^a}\,F_{am}\,, & \text{o}(1) \\ \partial_j C^k(\vec{x},\vec{\psi}) + \partial_k C^j(\vec{x},\vec{\psi}) = 0\,. & \text{o}(2) \end{cases} \tag{5.22}$$

Choosing the non-vanishing term $N^j_a = \delta^j_a$, in the general rank-1 Killing tensor (5.16), provides us with the rank-1 Killing tensor generating the supersymmetry transformation,

$$C^j(\vec{x},\vec{\psi}) = \delta^j_a\,\psi^a\,. \tag{5.23}$$

By substitution of this Grassmann-valued Killing tensor into the first-order equation of (5.22) we get

$$\vec{\nabla}C(\vec{x},\vec{\psi}) = \frac{q}{2}\,(g-2)\,\frac{\vec{x}\times\vec{\psi}}{r^3}\,. \tag{5.24}$$

Consequently, a solution $C(\vec{x},\vec{\psi}) = 0$ of (5.24) is only obtained for a fermion with ordinary gyromagnetic ratio

$$g = 2\,. \tag{5.25}$$

Thus we obtain, for $V(r) = 0$, the *Grassmann-odd supercharge* generating the $\mathcal{N} = 1$ supersymmetry of the spin-monopole field system,

$$\mathcal{Q} = \vec{\psi}\cdot\vec{\Pi}\,, \qquad \big\{\mathcal{Q},\,\mathcal{Q}\big\} = -2i\mathcal{H}_2\,. \tag{5.26}$$

For nonvanishing potential, $V(r) \neq 0$, the zeroth-order consistency condition of (5.22) is expressed as [2]

$$V'(r)\frac{\vec{\psi}\cdot\vec{x}}{r} = 0\,. \tag{5.27}$$

Consequently, adding *any* spherically symmetric potential $V(r)$ breaks the supersymmetry generated by the Killing tensor $C^j = \delta^j_a\,\psi^a$: $\mathcal{N} = 1$ SUSY requires an ordinary gyromagnetic factor, and no additional radial potential is allowed [Spector].

Another Killing tensor deduced from (5.16) is obtained by considering the particular case with the non-null tensor $N^j_{a_1a_2} = \epsilon^j_{\ a_1a_2}$. This leads to the rank-1 Killing tensor,

$$C^j(\vec{x},\vec{\psi}) = \epsilon^j_{\ ab}\psi^a\psi^b\,. \tag{5.28}$$

In this case, the first-order constraint of (5.22) is solved by $C(\vec{x},\vec{\psi}) = 0$, provided the gyromagnetic ratio takes the value $g = 2$. For vanishing potential, it is straightforward to verify the zeroth-order consistency constraint and therefore to obtain *the Grassmann-even supercharge*,

$$\mathcal{Q}_1 = \vec{S}\cdot\vec{\Pi}\,, \tag{5.29}$$

defining the "helicity" of the spinning particle. As expected, the consistency condition of superinvariance under (5.29) is again violated for $V(r) \neq 0$, breaking the supersymmetry of the Hamiltonian $\mathcal{H}_2$, in (5.26).

Let us now consider the rank-1 Killing vector,

$$C^j(\vec{x},\vec{\psi}) = \big(\vec{S}\times\vec{x}\big)^j\,, \tag{5.30}$$

obtained by putting $M^{ij}_{a_1a_2} = (i/2)\epsilon^{kij}\,\epsilon_{ka_1a_2}$ into the generic rank-1 Killing tensor (5.16). The first-order constraint is satisfied with $C(\vec{x},\vec{\psi}) = 0$, provided the particle carries gyromagnetic ratio $g = 2$. Thus, we obtain the supercharge,

$$\mathcal{Q}_2 = (\vec{x}\times\vec{\Pi})\cdot\vec{S}\,, \tag{5.31}$$

[2] We use the identity $S^kG^j\partial_jB^k = \psi^l\psi^m\,G^j\partial_jF_{lm} = 0\,.$

which, just like those in (5.26) and (5.29) only appears when the potential is absent, $V = 0$.

We consider the SUSY given when $M_a^{ij} = \epsilon_a{}^{ij}$ so that the Killing tensor (5.16) reduces to

$$C^j(\vec{x},\vec{\psi}) = -\epsilon^j{}_{ka} x^k \psi^a \,. \tag{5.32}$$

The first-order constraint of (5.22) is solved with $C(\vec{x},\vec{\psi}) = \frac{q}{2}(g-2)\frac{\vec{\psi}\cdot\vec{x}}{r}$. The zeroth-order consistency condition is, in this case, identically satisfied for an arbitrary radial potential. We have thus constructed the Grassmann-odd supercharge,

$$\mathcal{Q}_3 = (\vec{x}\times\vec{\Pi})\cdot\vec{\psi} + \frac{q}{2}(g-2)\frac{\vec{\psi}\cdot\vec{x}}{r}\,, \tag{5.33}$$

which is still conserved for a particle carrying an arbitrary gyromagnetic ratio g. Note, that this supercharge generalizes the one obtained in the restricted case with $g = 2$ [DeJonghe 1995]. See also [Horváthy 2000].

Now we turn to invariants which are quadratic in the covariant momentum. For this, we solve the reduced series of constraints,

$$\begin{cases} C^i\partial_i V + \frac{ieg}{4}\psi^l\psi^m C^j\partial_j F_{lm} - \frac{eg}{2}\psi^m\frac{\partial C}{\partial\psi^a}F_{am} = 0, & \text{o}(0)\\ \partial_j C = C^{jk}\partial_k V + eF_{jk}C^k + \frac{ieg}{4}\psi^l\psi^m\, C^{jk}\partial_k F_{lm} - \frac{eg}{2}\psi^m\frac{\partial C^j}{\partial\psi^a}F_{am}, & \text{o}(1)\\ \partial_j C^k + \partial_k C^j = e\Big(F_{jm}C^{mk} + F_{km}C^{mj}\Big) - \frac{eg}{2}\psi^m\frac{\partial C^{jk}}{\partial\psi^a}F_{am}\,, & \text{o}(2)\\ \partial_j C^{km} + \partial_m C^{jk} + \partial_k C^{mj} = 0\,. & \text{o}(3) \end{cases} \tag{5.34}$$

We first observe that $C^{ij}(\vec{x},\vec{\psi}) = \delta^{ij}$ is a constant Killing tensor. Solving the second- and the first-order constraints of (5.34), we obtain

$$C^j(\vec{x},\vec{\psi}) = 0 \quad \text{and} \quad C(\vec{x},\vec{\psi}) = V(r) - \frac{eg}{2}\vec{S}\cdot\vec{B}\,, \tag{5.35}$$

respectively. The zeroth-order consistency condition is identically satisfied so we obtain the conserved energy of the spinning particle,

$$\mathcal{E} = \frac{1}{2}\vec{\Pi}^2 - \frac{eg}{2}\vec{S}\cdot\vec{B} + V(r)\,. \tag{5.36}$$

Next, we introduce the nonvanishing constants tensors, $M^{ijk} = \epsilon^{ijk}$, $\widetilde{N}_a^{ij} = -\epsilon^{ij}{}_a$, into (5.19) in order to derive the rank-2 Killing tensor with the property,

$$C^{jk}(\vec{x},\vec{\psi}) = 2\,\delta^{jk}(\vec{x}\cdot\vec{\psi}) - x^j\psi^k - x^k\psi^j\,. \tag{5.37}$$

Injecting the Killing tensor (5.37) into (5.34), we satisfy the second-order constraints with

$$\vec{C}(\vec{x},\vec{\psi}) = \frac{q}{2}\,(2-g)\,\frac{\vec{\psi}\times\vec{x}}{r}\,. \tag{5.38}$$

To deduce the integrability condition of (5.34), we require, in the first-order constraint, the vanishing of the commutator,

$$[\partial_i,\,\partial_j]\,C(\vec{x}) = 0 \quad \Longrightarrow \quad \Delta\left(V(r) - (2-g)^2\,\frac{q^2}{8r^2}\right) = 0\,. \tag{5.39}$$

Then the Laplace equation (5.39) provides us with the *most general form of the potential admitting a Grassmann-odd charge quadratic in the velocity*, namely with

$$V(r) = (2-g)^2\,\frac{q^2}{8r^2} + \frac{\alpha}{r} + \beta\,. \tag{5.40}$$

Consequently, we solve the first-order constraint with

$$C(\vec{x},\vec{\psi}) = \left(\frac{\alpha}{r} - eg\vec{S}\cdot\vec{B}\right)\vec{x}\cdot\vec{\psi}\,, \tag{5.41}$$

so that the zeroth-order consistency constraint is identically satisfied. Collecting our results leads to the Grassmann-odd conserved charge quadratic in the velocity [Ngome 03/2010],

$$\mathcal{Q}_4 = \left(\vec{\Pi}\times(\vec{x}\times\vec{\Pi})\right)\cdot\vec{\psi} + \frac{q}{2}\,(2-g)\,\frac{\vec{x}\times\vec{\Pi}}{r}\cdot\vec{\psi} + \left(\frac{\alpha}{r} - eg\vec{S}\cdot\vec{B}\right)\vec{x}\cdot\vec{\psi}\,. \tag{5.42}$$

Let us underline that the conserved charge $\mathcal{Q}_4$ which is *not* a square root of the Hamiltonian $\mathcal{H}_g$ remains conserved without restriction on the gyromagnetic factor, g. We can also remark that for $g=0$, this charge coincides with the scalar product of the *separately conserved Runge-Lenz vector* [3] [Mcintosh 1970, Zwanziger 1968] *by the Grassmann-odd vector*:

$$\mathcal{Q}_4\big|_{g=0} = \vec{K}_{s=0}\cdot\vec{\psi}\,. \tag{5.43}$$

The supercharges $\mathcal{Q}$ and $\mathcal{Q}_j$ with $j=0,\cdots,3$, previously determined, form together, for ordinary gyromagnetic ratio, the classical superalgebra,

$$\begin{aligned}
&\{\mathcal{Q}_0,\,\mathcal{Q}_0\} = \{\mathcal{Q}_0,\,\mathcal{Q}_1\} = \{\mathcal{Q},\,\mathcal{Q}_1\} = \{\mathcal{Q}_1,\,\mathcal{Q}_1\} = \{\mathcal{Q}_2,\,\mathcal{Q}_2\} = 0\,,\\
&\{\mathcal{Q}_0,\,\mathcal{Q}\} = i\mathcal{Q}_1\,, \quad \{\mathcal{Q}_0,\,\mathcal{Q}_2\} = \{\mathcal{Q}_2,\,\mathcal{Q}_3\} = 0\,,\\
&\{\mathcal{Q}_0,\,\mathcal{Q}_3\} = i\mathcal{Q}_2\,, \quad \{\mathcal{Q},\,\mathcal{Q}\} = -2i\mathcal{H}_2\,,\\
&\{\mathcal{Q},\,\mathcal{Q}_2\} = \{\mathcal{Q}_1,\,\mathcal{Q}_3\} = \mathcal{Q}_4\,,\\
&\{\mathcal{Q},\,\mathcal{Q}_3\} = 2i\mathcal{Q}_1\,, \quad \{\mathcal{Q}_1,\,\mathcal{Q}_2\} = i\mathcal{Q}_3\mathcal{Q}\,, \quad \{\mathcal{Q}_3,\,\mathcal{Q}_3\} = i\,(2\mathcal{Q}_2 - \mathcal{Q}_5)\,,
\end{aligned} \tag{5.44}$$

[3] The case of spinning particle with null gyromagnetic ratio, $g=0$, coincides with a spinless particle.

where $\mathcal{Q}_5$ is a bosonic supercharge that we will construct below [5.54]. From (5.44) it follows that the linear combination $\mathcal{Q}_Y = \mathcal{Q}_3 - 2\mathcal{Q}_0$ has the special property that its bracket with the standard supercharge $\mathcal{Q}$ vanishes:

$$\left\{\mathcal{Q}_Y, \mathcal{Q}\right\} = 0. \tag{5.45}$$

Indeed, $\mathcal{Q}_Y$ is precisely the Killing-Yano supercharge constructed by De Jonghe, Macfarlane, Peeters and van Holten [DeJonghe 1995].

Let us now investigate the bosonic symmetries of the Pauli-like Hamiltonian (5.1). We use the generic Killing tensors previously constructed [cf. (5.16) and (5.19)] to derive the associated bosonic constants of the motion.

Firstly, we describe the rotationally invariance of the system by solving the reduced series of constraints (5.22). For this, we consider the Killing vector provided by the replacement, $M^{ij} = -\epsilon^{ij}{}_k n^k$ into (5.16). Thus for any unit vector $\vec{n}$, we obtain the generator of space rotations around $\vec{n}$,

$$\vec{C}(\vec{x}, \vec{\psi}) = \vec{n} \times \vec{x}\,. \tag{5.46}$$

Inserting the previous Killing vector in the first-order equation of (5.22) yields

$$C(\vec{x}, \vec{\psi}) = c(\vec{\psi}) - q\frac{\vec{n}\cdot\vec{x}}{r}\,. \tag{5.47}$$

Moreover the zeroth-order consistency condition of (5.22) requires for arbitrary radial potential,

$$c(\vec{\psi}) = \vec{S}\cdot\vec{n}\,. \tag{5.48}$$

Collecting our results provides us with the total angular momentum, which is plainly conserved for arbitrary gyromagnetic ratio,

$$\vec{J} = \vec{L} + \vec{S} = \vec{x}\times\vec{\Pi} - q\,\frac{\vec{x}}{r} + \vec{S}\,. \tag{5.49}$$

In addition to the typical monopole term, the conserved angular momentum also involves the spin vector, $\vec{S}$. It generates an $\mathrm{o}(3)_{rotations}$ bosonic symmetry algebra,

$$\left\{J^i, J^j\right\} = \epsilon^{ijk}J^k\,. \tag{5.50}$$

In the particular case of vanishing gyromagnetic factor $g = 0$, the usual monopole angular momentum $\vec{L}$ and the internal spin angular momentum $\vec{S}$ are separately conserved involving an

$$\mathrm{o}(3)_{rotations} \oplus \mathrm{o}(3)_{spin} \tag{5.51}$$

symmetry algebra.

We turn into invariants which are quadratic in the covariant momenta. Then, we have to solve the series of constraints (5.34). We first observe that for $M^{jmk} = \widetilde{M}^{jmk} = \epsilon^{jmk}$, the Killing tensor (5.19) reduces to the rank-2 Killing-Stäckel tensor,

$$C^{ij}(\vec{x},\vec{\psi}) = 2\delta^{ij}\,\vec{x}^{\,2} - 2x^i x^j\,. \tag{5.52}$$

Inserting (5.52) into the second- and in the first-order constraints of (5.34), we get for any gyromagnetic factor and for any arbitrary radial potential,

$$\vec{C}(\vec{x},\vec{\psi}) = 0 \quad \text{and} \quad C(\vec{x},\vec{\psi}) = -gq\,\frac{\vec{x}\cdot\vec{S}}{r}\,. \tag{5.53}$$

Hence, we obtain the Casimir

$$\mathcal{Q}_5 = \vec{J}^2 - q^2 + (g-2)\,\vec{J}\cdot\vec{S} - g\mathcal{Q}_2\,. \tag{5.54}$$

The bosonic supercharge $\mathcal{Q}_5$ is, as expected, *the square of the total angular momentum, augmented with another, separately conserved charge* [Ngome 03/2010],

$$(g-2)\,\vec{J}\cdot\vec{S} - g\mathcal{Q}_2\,. \tag{5.55}$$

• Indeed, for $g=0$, (5.55) directly implies that the product, $\vec{J}\cdot\vec{S}$, and hence the spin vector, $\vec{S}$, are separately conserved.

• For $g=2$, we recover the conservation of the supercharge $\mathcal{Q}_2$ [cf. (5.31)].

• For the anomalous gyromagnetic ratio $g=4$, we obtain that $\vec{J}\cdot\vec{S} - 2\mathcal{Q}_2$ is a constant of the motion.

Now we are interested in the hidden symmetry generated by a conserved Laplace-Runge-Lenz-type vector. Then, we introduce into the algorithm (5.34) the generator,

$$C^{ij}(\vec{x},\vec{\psi}) = 2\,\delta^{ij}\,\vec{n}\cdot\vec{x} - n^i x^j - n^j x^i\,, \tag{5.56}$$

easily obtained by choosing the non-vanishing, $\widetilde{N}^{ij} = \epsilon^{imj} n^m$ and $M^{ijm} = \epsilon^{ijm}$, into the generic rank-2 Killing tensor (5.19). Inserting (5.56) into the second-order constraint of (5.34), we get

$$\vec{C}(\vec{x},\vec{\psi}) = q\frac{\vec{n}\times\vec{x}}{r} + \vec{C}(\vec{\psi})\,. \tag{5.57}$$

We solve the first-order constraint of (5.34) by expanding $C(\vec{x},\,\vec{\psi})$ in terms of Grassmann variables [Berezin],

$$C(\vec{x},\,\vec{\psi}) = C(\vec{x}) + \sum_{k\geq 1} C_{a_1\cdots a_k}(\vec{x})\psi^{a_1}\cdots\psi^{a_k}\,. \tag{5.58}$$

Consequently, the first- and the zeroth-order equations of (5.34) can be classified order-by-order in Grassmann-odd variables. Thus, inserting (5.57) in the first-order equation, and requiring again the vanishing of the commutator,

$$[\partial_i, \partial_j]\, C(\vec{x}) = 0 \implies \Delta\left(V(r) - \frac{q^2}{2r^2}\right) = 0\,, \tag{5.59}$$

we deduce the most general radial potential admitting a conserved Laplace-Runge-Lenz vector in the fermion-monopole interaction, namely

$$V(r) = \frac{q^2}{2r^2} + \frac{\mu}{r} + \gamma\,, \quad \mu\,,\gamma \in \mathbb{R}\,. \tag{5.60}$$

Investigating the first term on the right-hand side of (5.58), we obtain

$$C(\vec{x}) = \mu\frac{(\vec{n}\cdot\vec{x})}{r}\,. \tag{5.61}$$

Introducing (5.57) and (5.60) into the first-order constraint of (5.34), on one hand, provides us with

$$\vec{C}(\vec{\psi}\,) = -\frac{g}{2}\vec{n}\times\vec{S}\,, \tag{5.62}$$

and on the other hand with

$$\sum_{k\geq 1} C_{a_1\cdots a_k}(\vec{x})\psi^{a_1}\cdots\psi^{a_k} = -\frac{eg}{2}\left(\vec{S}\cdot\vec{B}\right)(\vec{n}\cdot\vec{x}) - \frac{gq}{2}\left(1-\frac{g}{2}\right)\frac{\vec{n}\cdot\vec{S}}{r} + C(\vec{\psi})\,, \qquad \text{with} \quad g\big(g-4\big) = 0\,. \tag{5.63}$$

Let us precise that the zeroth-order consistency condition of (5.34) is only satisfied for

$$C(\vec{\psi}) = \frac{\mu}{q}\vec{S}\cdot\vec{n}\,. \tag{5.64}$$

Collecting our results, (5.56), (5.57), (5.60) and (5.63), we obtain a conserved Runge-Lenz vector if and only if

$$g = 0 \qquad \text{or} \qquad g = 4\,; \tag{5.65}$$

we get namely

$$\vec{K}_g = \vec{\Pi}\times\vec{J} + \mu\,\frac{\vec{x}}{r} + \left(1-\frac{g}{2}\right)\vec{S}\times\vec{\Pi} - \frac{eg}{2}\left(\vec{S}\cdot\vec{B}\right)\vec{x} - \frac{gq}{2}\left(1-\frac{g}{2}\right)\frac{\vec{S}}{r} + \frac{\mu}{q}\vec{S}\,. \tag{5.66}$$

Note that the spin angular momentum which generates the extra "spin" symmetry for vanishing gyromagnetic ratio is no more separately conserved for $g = 4$. Then, an interesting question is to know if the extra "spin" symmetry of $g = 0$ is still present for the anomalous superpartner $g = 4$ in some "hidden" way.

Let us consider the "spin" transformation generated by the rank-2 Killing tensor with the property,

$$C^{mk}(\vec{x},\vec{\psi}) = 2\delta^{mk}\big(\vec{S}\cdot\vec{n}\big) - \frac{g}{2}\big(S^m n^k + S^k n^m\big)\,. \tag{5.67}$$

The rank-2 Killing tensor (5.67) which can be separated as $C^{mk} = C^{mk}_{+} + C^{mk}_{-}$ is obtained by putting

$$\begin{aligned} N^{jk}_{+} &= \frac{g}{2}\epsilon^{jk}_{l}\, n^l\,, \qquad \widetilde{N}^{jk}_{+\ a} = -\frac{i}{2}\epsilon^{jk}_{\ \ m}\,\epsilon^{m}_{\ a_1a_2}\,,\\ N^{jkl}_{-} &= (1-\frac{g}{2})\epsilon^{jkl}\,, \qquad \widetilde{N}^{jkl}_{-\ a} = -\frac{i}{4}\epsilon^{jkl}\, n_m\,\epsilon^{m}_{\ a_1a_2}\,, \end{aligned} \tag{5.68}$$

into the general rank-2 Killing tensor (5.19). Inserting (5.67) into the second-order constraint of (5.34) leads to

$$\vec{C}(\vec{x},\vec{\psi}) = -\frac{qg}{2}\frac{(\vec{S}\times\vec{n})}{r} + \vec{C}(\psi) \quad \text{and} \quad g(g-4)=0\,. \tag{5.69}$$

We use the potential (5.60) to solve the first-order equation of (5.34),

$$\begin{aligned} C(\vec{x},\vec{\psi}) &= \left(2V(r) - \frac{q^2g^2}{8r^2} - \frac{\mu g^2}{4r}\right)\vec{S}\cdot\vec{n} + c(\psi)\,,\\ \vec{C}(\psi) &= \frac{\mu g}{2q}\vec{n}\times\vec{S} \qquad \text{and} \qquad g(g-4)=0\,. \end{aligned} \tag{5.70}$$

The zeroth-order consistency condition is satisfied with

$$c(\psi) = -\frac{g^2}{8}\frac{\mu^2}{q^2}\vec{S}\cdot\vec{n}\,, \tag{5.71}$$

so that collecting our results provides us with the conserved “spin" vector,

$$\begin{aligned} \vec{\Omega}_g = &\left(\vec{\Pi}^2 + (2-\frac{g^2}{4})V(r)\right)\vec{S} - \frac{g}{2}(\vec{\Pi}\cdot\vec{S})\vec{\Pi} + \frac{g}{2}(\frac{q}{r}+\frac{\mu}{q})\vec{S}\times\vec{\Pi}\\ &-\frac{g^2}{4}(\frac{\mu^2}{2q^2}-\gamma)\vec{S} \qquad \text{with} \qquad g(g-4)=0\,. \end{aligned} \tag{5.72}$$

In conclusion, the additional $\mathrm{o}(3)_{spin}$ “spin” symmetry is recovered in the same particular cases of anomalous gyromagnetic ratios 0 and 4 [cf. (5.65)].

- For $g=0$, in particular,

$$\vec{\Omega}_0 = 2\mathcal{E}\,\vec{S}\,. \tag{5.73}$$

- For $g=4$, we find an expression equivalent to that of D’Hoker and Vinet [D’Hoker 09/1985], namely

$$\vec{\Omega}_4 = \left(\vec{\Pi}^2 - 2V(r)\right)\vec{S} - 2(\vec{\Pi}\cdot\vec{S})\,\vec{\Pi} + 2(\frac{q}{r}+\frac{\mu}{q})\vec{S}\times\vec{\Pi} - 4\left(\frac{\mu^2}{2q^2}-\gamma\right)\vec{S}\,. \tag{5.74}$$

Note that this extra symmetry is generated by a *Killing tensor*, rather than a Killing vector, as for “ordinary” angular momentum. Thus, for sufficiently low energy, the motions are bounded and the conserved vectors $\vec{J}$, $\vec{K}_g$ and $\vec{\Omega}_g$ generate an

$$\mathrm{o}(4)\oplus\mathrm{o}(3)_{spin} \tag{5.75}$$

bosonic symmetry algebra.

So far we have seen that, for a spinning particle with a single Grassmann variable, SUSY and dynamical symmetry are inconsistent, since they require different values for the g-factor. Now, adapting the idea of D'Hoker and Vinet to our framework, we show that the two contradictory conditions can be conciliated by doubling the odd degrees of freedom. The systems with $g = 0$ and $g = 4$ will then become superpartners inside a unified $\mathcal{N} = 2$ SUSY system [Fehér 1988].

We consider, hence, a charged spin-$\frac{1}{2}$ particle moving in a flat manifold $\mathcal{M}^{D+2d}$, interacting with a static magnetic field $\vec{B}$. The fermionic degrees of freedom are now carried by a $2d$-dimensional internal space [Bellucci, Kochan, Gonzales, Avery 2008]. This is to be compared with the d-dimensional internal space sufficient to describe the $\mathcal{N} = 1$ SUSY of the monopole. In terms of Grassmann-odd variables $\psi_{1,2}$, the local coordinates of the fermionic extension $\mathcal{M}^{2d}$ read $\left(\psi_1^a,\, \psi_2^b\right)$ with $a, b = 1, \cdots, d$. The system is still described by the Pauli-like Hamiltonian (5.1). Choosing $d = 3$, we consider the fermion ξ_α which is a two-component spinor, $\xi_\alpha = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$, and whose conjugate is $\bar{\xi}^\alpha$ [Salomonson]. Thus, we have a representation of the spin angular momentum,

$$S^k = \frac{1}{2}\bar{\xi}^\alpha\, \sigma^{k\ \beta}_{\alpha}\, \xi_\beta \quad \text{with} \quad \alpha, \beta = 1, 2\,, \tag{5.76}$$

where the $\sigma^{k\ \beta}_{\alpha}$ with $k = 1, 2, 3$ define the standard Pauli matrices. Defining the covariant Poisson-Dirac brackets as

$$\begin{aligned} \left\{f, h\right\} = \partial_j f \frac{\partial h}{\partial \Pi_j} - \frac{\partial f}{\partial \Pi_j}\partial_j h + e\epsilon_{ijk}B^k \frac{\partial f}{\partial \Pi_i}\frac{\partial h}{\partial \Pi_j} \\ + i(-1)^{a^f}\left(\frac{\partial f}{\partial \xi_\alpha}\frac{\partial h}{\partial \bar{\xi}^\alpha} + \frac{\partial f}{\partial \bar{\xi}^\alpha}\frac{\partial h}{\partial \xi_\alpha}\right), \end{aligned} \tag{5.77}$$

we deduce the non-vanishing fundamental brackets,

$$\begin{aligned} &\left\{x^i, \Pi_j\right\} = \delta^i_j, \quad \left\{\Pi_i, \Pi_j\right\} = e\, \epsilon_{ijk}B^k, \quad \left\{\xi_\alpha, \bar{\xi}^\beta\right\} = -i\delta_\alpha^{\ \beta}, \\ &\left\{S^k, S^l\right\} = \epsilon^{kl}_{\ \ m}S^m, \quad \left\{S^k, \bar{\xi}^\beta\right\} = -\frac{i}{2}\bar{\xi}^\mu \sigma^{k\ \beta}_{\mu}, \quad \left\{S^k, \xi_\beta\right\} = \frac{i}{2}\sigma^{k\ \nu}_{\beta}\xi_\nu\,. \end{aligned} \tag{5.78}$$

We also introduce an auxiliary scalar field, $\Phi(r)$, satisfying *the "self-duality" or " Bogomolny" relation*[4],

$$\left\{\Pi^k, \Phi(r)\right\} = \pm eB^k\,. \tag{5.79}$$

This auxiliary scalar field also defines a square root of the external potential of the system so that

$$\frac{1}{2}\Phi^2(r) = V(r)\,. \tag{5.80}$$

[4] See [Fehér 1988] to justify terminology.

As an illustration we obtain the potential [5] defined in (5.60) by considering the auxiliary field

$$\Phi(r) = \pm\left(\frac{q}{r} + \frac{\mu}{q}\right). \tag{5.81}$$

In order to investigate the $\mathcal{N}=2$ supersymmetry of the Pauli-like Hamiltonian (5.1), we outline the algorithm developed we use to construct supercharges linear in the gauge covariant momentum,

$$\begin{cases} \mp e\Phi(r)\, B^j C^j + \dfrac{ieg}{4} B^k \left(\bar\xi^\mu \sigma_\mu^{k\,\nu} \dfrac{\partial C}{\partial \bar\xi^\nu} - \dfrac{\partial C}{\partial \xi_\mu}\sigma_\mu^{k\,\nu}\xi_\nu\right) - \dfrac{eg}{4}\,\bar\xi^\mu \sigma_\mu^{k\,\nu}\xi_\nu\, C^j \partial_j B^k = 0\,, & \text{o}(0) \\ \partial_m C = e\,\epsilon_{mjk} B^k C^j + i\dfrac{eg}{4} B^k \left(\bar\xi^\mu \sigma_\mu^{k\,\nu}\dfrac{\partial C^m}{\partial\bar\xi^\nu} - \dfrac{\partial C^m}{\partial \xi_\mu}\sigma_\mu^{k\,\nu}\xi_\nu\right), & \text{o}(1) \\ \partial_j C^k(x,\xi,\bar\xi) + \partial_k C^j(x,\xi,\bar\xi) = 0\,. & \text{o}(2) \end{cases} \tag{5.82}$$

Let us first consider the Killing spinor,

$$C^j_\beta = \frac{1}{2}\sigma_\beta^{j\,\alpha}\,\xi_\alpha\,. \tag{5.83}$$

Inserting this Killing spinor into the first-order equation of (5.82) provides us with

$$\partial_m C_\beta = -\frac{i}{2} e\, B_m\, \xi_\beta \quad \text{and} \quad g = 4\,, \tag{5.84}$$

which can be solved using *the self-duality relation (5.79)*. Thus, we get

$$C_\beta(\vec{x},\vec{\xi}) = \pm\frac{i}{2}\Phi(r)\,\xi_\beta\,, \tag{5.85}$$

provided the anomalous gyromagnetic factor is $g=4$. The zeroth-order constraint of (5.82) is identically satisfied, so that collecting our results provides us with the supercharge,

$$\mathcal{Q}_\beta = \frac{1}{2}\Pi_j\,\sigma_\beta^{j\,\alpha}\,\xi_\alpha \pm \frac{i}{2}\Phi(r)\xi_\beta\,. \tag{5.86}$$

To obtain the supercharge conjugate to (5.86), we consider the conjugate Killing spinor,

$$\bar{C}^{k\,\beta} = \frac{1}{2}\bar\xi^\alpha\,\sigma_\alpha^{k\,\beta}\,. \tag{5.87}$$

In the case of anomalous value of the gyromagnetic ratio $g=4$, the first-order equation of (5.82) is solved by using the Bogomolny equation (5.79). This leads to the conjugate

$$\bar{C}^\beta(\vec{x},\vec{\xi}) = \mp\frac{i}{2}\Phi(r)\bar\xi^\beta\,. \tag{5.88}$$

[5]The constant is $\gamma = \dfrac{\mu^2}{2q^2}$.

The zeroth-order consistency constraint is still satisfied, so we obtain the odd-supercharge,

$$\bar{\mathcal{Q}}^{\beta} = \frac{1}{2}\bar{\xi}^{\alpha}\,\sigma_{\alpha}^{k\,\beta}\Pi_k \mp \frac{i}{2}\Phi(r)\,\bar{\xi}^{\beta}\,. \tag{5.89}$$

The supercharges $\mathcal{Q}_\beta$ and $\bar{\mathcal{Q}}^\beta$ are, both, square roots of the Pauli-like Hamiltonian $\mathcal{H}_4$,

$$\left\{\bar{\mathcal{Q}}^{\beta}, \mathcal{Q}_{\beta}\right\} = -i\mathcal{H}_4\,\mathbb{1}\,, \tag{5.90}$$

and therefore *generate the $\mathcal{N}=2$ supersymmetry of the spin-monopole field system.* It is worth noting that defining the rescaled,

$$\bar{\mathcal{U}}^{\beta} = \bar{\mathcal{Q}}^{\beta}\frac{1}{\sqrt{\mathcal{H}_4}} \qquad \text{and} \qquad \mathcal{U}_{\beta} = \frac{1}{\sqrt{\mathcal{H}_4}}\mathcal{Q}_{\beta}\,, \tag{5.91}$$

it is straightforward to get,

$$\mathcal{H}_0 = \bar{\mathcal{U}}^{\beta}\,\mathcal{H}_4\,\mathcal{U}_{\beta}\,, \tag{5.92}$$

which make manifest the fact that the two anomalous cases $g=0$ and $g=4$ can be viewed as superpartners [6], see [Fehér 1988]. Moreover, in our enlarged system, the following bosonic charges

$$\begin{aligned}
\vec{J} &= \vec{x}\times\vec{\Pi} - q\,\frac{\vec{x}}{r} + \vec{S}\,,\\
\vec{K} &= \vec{\Pi}\times\vec{J} + \mu\,\frac{\vec{x}}{r} - \vec{S}\times\vec{\Pi} - 2e\left(\vec{S}\cdot\vec{B}\right)\vec{x} + 2q\frac{\vec{S}}{r} + \frac{\mu}{q}\vec{S}\,,\\
\vec{\Omega} &= \bar{\mathcal{Q}}^{\beta}\,\vec{\sigma}_{\beta}^{\,\alpha}\,\mathcal{Q}_{\alpha} = \frac{1}{2}\left(\Phi^2(r) - \vec{\Pi}^2\right)\vec{S} + (\vec{\Pi}\cdot\vec{S})\vec{\Pi} \mp \Phi(r)\,\vec{S}\times\vec{\Pi},
\end{aligned} \tag{5.93}$$

remain conserved such that they form, together with the supercharges $\mathcal{Q}_\beta$ and $\bar{\mathcal{Q}}^\beta$, the classical symmetry superalgebra [D'Hoker 09/1985, Fehér 1988],

$$\begin{aligned}
&\left\{\bar{\mathcal{Q}}^{\beta}, \mathcal{Q}_{\beta}\right\} = -i\mathcal{H}_4\,\mathbb{1}\,, \quad \left\{\bar{\mathcal{Q}}^{\beta}, \bar{\mathcal{Q}}^{\beta}\right\} = \left\{\mathcal{Q}_{\beta}, \mathcal{Q}_{\beta}\right\} = 0\,, \quad \left\{\bar{\mathcal{Q}}^{\beta}, J^k\right\} = \frac{i}{4}\bar{\mathcal{Q}}^{\alpha}\sigma_{\alpha}^{k\,\beta}\,,\\
&\left\{\mathcal{Q}_{\beta}, J^k\right\} = -\frac{i}{4}\sigma_{\beta}^{k\,\alpha}\mathcal{Q}_{\alpha}\,, \quad \left\{\bar{\mathcal{Q}}^{\beta}, K^j\right\} = -\frac{i}{4}\,\frac{\mu}{q}\,\bar{\mathcal{Q}}^{\alpha}\sigma_{\alpha}^{j\,\beta}\,, \quad \left\{\mathcal{Q}_{\beta}, K^j\right\} = \frac{i}{4}\,\frac{\mu}{q}\,\sigma_{\beta}^{j\,\alpha}\mathcal{Q}_{\alpha}\,,\\
&\left\{\bar{\mathcal{Q}}^{\beta}, \Omega^k\right\} = -i\mathcal{H}_4\,\bar{\mathcal{Q}}^{\alpha}\sigma_{\alpha}^{k\,\beta}\,, \quad \left\{\mathcal{Q}_{\beta}, \Omega^k\right\} = i\mathcal{H}_4\,\sigma_{\beta}^{k\,\alpha}\mathcal{Q}_{\alpha}\,, \quad \left\{\Omega^i, K^j\right\} = \frac{\mu}{q}\epsilon^{ijk}\,\Omega^k\,,\\
&\left\{K^i, K^j\right\} = \epsilon^{ijk}\left[\left(\frac{\mu^2}{q^2} - 2\mathcal{H}_4\right)J^k + 2\Omega^k\right]\,, \quad \left\{\Omega^i, \Omega^j\right\} = \epsilon^{ijk}\,\mathcal{H}_4\,\Omega^k\,,\\
&\left\{J^i, \Lambda^j\right\} = \epsilon^{ijk}\Lambda^k \quad \text{with} \quad \Lambda^l = J^l, K^l, \Omega^l\,.
\end{aligned}$$

We have shown, in this section, that the Runge-Lenz-type dynamical symmetry and the

[6] With The scalar $\bar{\xi}^{\beta}\xi_{\beta} = 2\,.$

additional extra “spin" symmetry both require instead an anomalous gyromagnetic ratio,

$$g = 0 \quad \text{or} \quad g = 4\,. \tag{5.94}$$

These particular values of the g-factor come from the effective coupling of the form $F_{ij} \mp \epsilon_{ijk}D_k\Phi$, which add or cancel for self-dual fields [Fehér 1988],

$$F_{ij} = \epsilon_{ijk}D_k\Phi\,. \tag{5.95}$$

Moreover, the super- and the bosonic symmetry can be combined in this enlarged fermionic space and provides us with an $\mathcal{N} = 2$ SUSY, as proposed by D'Hoker and Vinet [D'Hoker 09/1985]. See also [Fehér 1988, Fehér 1989, Fehér 02/1989, Bloore].

At last, let us remark that confining the spinning particle onto a sphere of fixed radius ρ implies the set of constraints [DeJonghe 1995],

$$\vec{x}^2 = \rho^2\,, \quad \vec{x}\cdot\vec{\psi} = 0 \quad \text{and} \quad \vec{x}\cdot\vec{\Pi} = 0\,. \tag{5.96}$$

This freezes the radial potential to a constant, and we recover the $\mathcal{N} = 1$ SUSY described by the supercharges $\mathcal{Q}$, $\mathcal{Q}_1$ and $\mathcal{Q}_2$ for ordinary gyromagnetic factor $g = 2$.

5.2 $\mathcal{N}=2$ SUSY in the plane

The planar system consisting of a spinning particle interacting with a static magnetic field in the plane exhibits more symmetries as its higher-dimensional counterpart. Indeed, the $\mathcal{N}=2$ supersymmetry, here, is realized without doubling the Grassmann-variable of the internal space as it was the case in three-dimensional space system, see section 5.1. Such an "exotic" supersymmetry, which is realized in two different ways, is only possible in two spatial dimensions [Duval 1993, Duval 1995, Duval 2008]. This is one more indication of the particular status of two-dimensional physics.

To see this, we investigate the two dimensional model given by the Pauli-like Hamiltonian,

$$\mathcal{H}=\frac{1}{2}\,\Pi^2-\frac{eg}{2}\,SB+V(r)\,, \tag{5.97}$$

where the magnetic field simplifies into [7]

$$F_{ij}=\varepsilon_{ij}B=\partial_iA_j-\partial_jA_i\,, \tag{5.98}$$

and the spin tensor is actually a scalar

$$S=-\frac{i}{2}\,\varepsilon_{ij}\psi^i\psi^j. \tag{5.99}$$

The fundamental brackets remain the same as in (5.6), and the spatial and the internal motions of the particle are governed by the following equations,

$$\begin{aligned}\ddot{x}_k&=\frac{eg}{2}S\partial_kB+eB\varepsilon_{kj}\dot{x}^j+\partial_kV\,,\\ \dot{\psi}_i&=\frac{eg}{2}B\varepsilon_{ij}\psi^j\,,\quad \dot{S}=0\,.\end{aligned} \tag{5.100}$$

Observe the conservation of the spin S along the particle motion and let us recall that all quantities quadratic in the Grassmann variables are proportional to S.

We search for dynamical quantities which are constants of the motion, for the planar system, by solving the series of constraints:

$$\begin{array}{ll}C_i\partial_i\mathcal{H}+i\dfrac{\partial\mathcal{H}}{\partial\psi_i}\dfrac{\partial C}{\partial\psi_i}=0\,, & \mathrm{o}(0)\\ \partial_iC=eF_{ij}C_j+i\dfrac{\partial\mathcal{H}}{\partial\psi_j}\dfrac{\partial C_i}{\partial\psi_j}+C_{ij}\partial_j\mathcal{H}\,, & \mathrm{o}(1)\\ \partial_iC_j+\partial_jC_i=e\left(F_{ik}C_{kj}-C_{ik}F_{kj}\right)+i\dfrac{\partial\mathcal{H}}{\partial\psi_k}\dfrac{\partial C_{ij}}{\partial\psi_k}+C_{ijk}\partial_k\mathcal{H}\,, & \mathrm{o}(2)\\ \partial_iC_{jk}+\partial_jC_{ki}+\partial_kC_{ij}=C_{ijkl}\partial_l\mathcal{H}+(\text{terms linear in }C_{lmn})\,. & \mathrm{o}(3)\\ \vdots \qquad\qquad \vdots & \vdots\end{array} \tag{5.101}$$

[7] We dropped the irrelevant third z-direction.

Using the equality

$$i\frac{\partial\mathcal{H}}{\partial\psi^i}=-\frac{eg}{2}\,F_{ij}\psi^j=-\frac{eg}{2}\,B\varepsilon_{ij}\psi^j\,, \tag{5.102}$$

the zeroth-order constraint in (5.101) becomes

$$\frac{eg}{2}\,B\varepsilon_{ij}\psi_j\frac{\partial C}{\partial\psi_i}=C_i\left(\partial_iV-\frac{eg}{2}\,S\,\partial_iB\right), \tag{5.103}$$

complemented by the first-order equation of (5.101)

$$\partial_iC=eB\left(\varepsilon_{ij}C_j+\frac{g}{2}\,\varepsilon_{jk}\psi_j\frac{\partial C_i}{\partial\psi_k}\right)+C_{ij}\left(\partial_jV-\frac{eg}{2}\,S\partial_jB\right). \tag{5.104}$$

Similarly the second and higher-order equations take the form

$$\partial_{(i}C_{j)}=eB\left(\varepsilon_{ik}C_{kj}+\varepsilon_{jk}C_{ki}+\frac{g}{2}\,\varepsilon_{jk}\psi_j\frac{\partial C_i}{\partial\psi_k}\right)+C_{ijk}\left(\partial_kV-\frac{eg}{2}\,S\partial_kB\right). \tag{5.105}$$

For radial functions $V(r)$ and $B(r)$,

$$\partial_iV=\frac{x_i}{r}V'\,,\qquad \partial_iB=\frac{x_i}{r}B'\,, \tag{5.106}$$

hence

$$\left(\partial_jV-\frac{eg}{2}\,S\partial_jB\right)C_{i\ldots j}=\frac{x_j}{r}\left(V'-\frac{eg}{2}\,SB'\right)C_{i\ldots j}. \tag{5.107}$$

Let us now consider some specific cases. To this, we introduce the universal generalized Killing vectors,

$$C_i=\left\{\gamma_i,\ \varepsilon_{ij}x^j,\ \psi_i,\ \varepsilon_{ij}\psi^j\right\}\,, \tag{5.108}$$

where γ_i denotes a constant vector.

• A constant Killing vector γ_i gives a constant of the motion only if we can find solutions for the equations

$$\partial_iC=eB\varepsilon_{ij}\gamma_j,\qquad B\varepsilon_{ji}\psi_i\frac{\partial C}{\partial\psi_j}=\gamma_i\left(\frac{2}{eg}\,\partial_iV-S\partial_iB\right). \tag{5.109}$$

Now for a Grassmann-even function

$$C=c_0+c_2S\,, \tag{5.110}$$

the left-hand side of the second equation in (5.109) vanishes, therefore we must require B and V to be constant. This leads to the solution

$$C=-eB\varepsilon_{ij}\gamma^ix^j,\qquad V=\text{const},\qquad B=\text{const}\,. \tag{5.111}$$

The corresponding constant of the motion, ζ, is identified with the *“magnetic translations”*

[Hughes 1986],

$$\zeta = \gamma^i P_i \qquad \text{with} \qquad P_i = \Pi_i - eB\varepsilon_{ij}x^j\,. \tag{5.112}$$

• Next we consider the linear Killing vector $C_i = \varepsilon_{ij}x^j$, with all higher-order coefficients $C_{ij\ldots} = 0$. Again for Grassmann-even C the left-hand side of equation (5.103) vanishes, and we get the condition

$$\varepsilon_{ij}x_i\partial_j B = \varepsilon_{ij}x_i\partial_j V = 0, \tag{5.113}$$

which is automatically satisfied for radial functions $B(r)$ and $V(r)$. Therefore we only have to solve the equation (5.104):

$$\partial_i C = -eBx_i = -\frac{ex_i}{r}\,(rB). \tag{5.114}$$

We infer that $C(r)$ is a radial function, with

$$C' = -erB\,. \tag{5.115}$$

Therefore C is given by *the magnetic flux through the disk D_r centered at the origin with radius r*:

$$C = -\frac{e}{2\pi}\int_{D_r} B(r)d^2x \equiv -\frac{e}{2\pi}\,\Phi_B(r). \tag{5.116}$$

We then find the constant of the motion representing the angular momentum [Ngome 03/2010],

$$L = \varepsilon_{ij}x_i\Pi_j + \frac{e}{2\pi}\,\Phi_B(r), \tag{5.117}$$

associated with the $\mathrm{o}(2)_{rotations}$ symmetry group.

• There are two Grassmann-odd Killing vectors, the first one being $C_i = \psi_i$. With this Ansatz, we get for the scalar contribution to the constant of the motion the constraints

$$\frac{eg}{2}B\,\varepsilon_{ij}\psi_j\,\frac{\partial C}{\partial\psi_i} = \psi_i\partial_i V \qquad \text{and} \qquad \partial_i C = \frac{eB}{2}\,(2-g)\,\varepsilon_{ij}\psi_j\,. \tag{5.118}$$

It follows that either $g = 2$ and $\big(C,\,V\big)$ are constant, in which case one may take $C = V = 0$, or $g \neq 2$ and C is of the form

$$C = \varepsilon_{ij}K_i(r)\psi_j \quad \text{with} \quad \partial_i V = -\frac{eg}{2}\,BK_i\,, \qquad \partial_i K_j = \frac{(2-g)eB}{2}\,\delta_{ij}. \tag{5.119}$$

This is possible only if B is constant and

$$K_i = \frac{eB(2-g)}{2}\,x_i \equiv \kappa x_i\,, \qquad V(r) = \frac{g(g-2)}{8}\,e^2B^2r^2 = -\frac{eg\kappa}{4\pi}\,\Phi_B(r). \tag{5.120}$$

It follows that we have a conserved supercharge of the form,

$$Q = \psi^i\left(\Pi_i - \kappa\varepsilon_{ij}x^j\right). \tag{5.121}$$

The bracket algebra of this supercharge takes the form

$$i\{Q,Q\} = 2\mathcal{H} + (2-g)eBJ, \qquad J = L + S. \tag{5.122}$$

Of course, as S and L are separately conserved, J is a constant of the motion as well. It is now easy to see that for ordinary g-factor the supercharge $\mathcal{Q}$ in (5.121) is a square root of the Hamiltonian.

• Let us remark that for the anomalous gyromagnetic ratio $g = 1$, we construct the conserved conformal supercharge,

$$\mathcal{S} = \vec{x}\cdot\vec{\psi} - t\mathcal{Q}\,, \tag{5.123}$$

obtained by using the internal equation of the motion in (5.100).

• Finally we consider the dual Grassmann-odd Killing vector $C_i = \varepsilon_{ij}\psi_j$. Then the constraints (5.103) and (5.104) become

$$\frac{eg}{2}B\frac{\partial C}{\partial\psi_i} = \partial_i V, \qquad \partial_i C = \frac{(g-2)eB}{2}\psi_i, \tag{5.124}$$

implying that

$$C = N_i(x)\psi_i \quad \text{and} \quad \frac{eg}{2}B\,N_i = \partial_i V, \qquad \partial_i N_j = \frac{(g-2)eB}{2}\delta_{ij}. \tag{5.125}$$

As before, the magnetic field B must be constant and the potential is identical to (5.120),

$$N_i = -\kappa x_i, \qquad V = -\frac{eg\kappa}{4\pi}\Phi_B(r) = \frac{g(g-2)}{8}e^2B^2r^2. \tag{5.126}$$

Thus, we find the dual conserved supercharge [8] [Horváthy 2005],

$$\mathcal{Q}^\star = \varepsilon_{ij}\psi_i\left(\Pi_j - \kappa\varepsilon_{jk}x_k\right) = \psi_i\left(\varepsilon_{ij}\Pi_j + \kappa x_i\right)\,, \tag{5.127}$$

corresponding, in the case of ordinary gyromagnetic ratio, to the "twisted" supercharge used by Jackiw [Jackiw 1984] to describe the Landau states in a constant magnetic field. Moreover $\mathcal{Q}^\star$ satisfies the bracket relations

$$i\{\mathcal{Q}^\star,\mathcal{Q}^\star\} = 2\mathcal{H} + (2-g)eBJ, \qquad i\{Q,\mathcal{Q}^\star\} = 0. \tag{5.128}$$

Thus the harmonic potential (5.120) with constant magnetic field B allows a classical $\mathcal{N}=2$ supersymmetry with supercharges $\big(Q,\mathcal{Q}^\star\big)$, whilst the special conditions $g=2$ and $V=0$ allows for $\mathcal{N}=2$ supersymmetry for any $B(r)$.

• As a consequence of the conservation of the "twisted" supercharge, we construct for

[8] The cross product of two planar vectors, $\vec{a}\times\vec{b} = \varepsilon_{ij}a^ib^j$, again defines a scalar.

$g=1$, the associate conserved conformal supercharge

$$\mathcal{S}^\star = \vec{x}\times\vec{\psi} + t\mathcal{Q}^\star\,. \tag{5.129}$$

Thus, for the non-ordinary gyromagnetic ratio $g=1$, the supercharges $\mathcal{Q}$, $\mathcal{Q}^\star$, $\mathcal{S}$ and $\mathcal{S}^\star$ extend the $o(2,1)$ algebra into an $\mathfrak{osp}(1,1)$ superalgebra and satisfy the commutation relations,

$$\begin{aligned}
&\big\{\mathcal{Q},\mathcal{Q}\big\} = \big\{\mathcal{Q}^\star,\mathcal{Q}^\star\big\} = -i\big(2\mathcal{H}+eBJ\big)\,, \quad \big\{\mathcal{S}^\star,\mathcal{S}^\star\big\} = -it^2\big(2\mathcal{H}+eBJ\big)\,,\\
&\big\{\mathcal{Q}^\star,\mathcal{S}^\star\big\} = -\big\{\mathcal{Q},\mathcal{S}\big\} = -it\big(2\mathcal{H}+eBJ\big) + i\vec{x}\cdot\vec{\Pi}\,, \quad \big\{\mathcal{Q},\mathcal{Q}^\star\big\} = 0\,,\\
&\big\{\mathcal{Q}^\star,\mathcal{S}\big\} = \big\{\mathcal{Q},\mathcal{S}^\star\big\} = -i\big(L+2S\big)\,, \quad \big\{\mathcal{S},\mathcal{S}^\star\big\} = 2it\big(L+2S\big)\,,\\
&\big\{\mathcal{S},\mathcal{S}\big\} = -it^2\big(2\mathcal{H}+eBJ\big) + 2it\,\vec{x}\cdot\vec{\Pi} - ir^2\,.
\end{aligned} \tag{5.130}$$

The van Holten recipe is therefore relevant to study planar fermions in an arbitrary planar magnetic field, i.e. one perpendicular to the plane. As an illustration, we have shown, for ordinary gyromagnetic factor, that in addition to the usual supercharge (5.121) generating the supersymmetry, the system also admits another square root of the Pauli Hamiltonian $\mathcal{H}$ [Horváthy 2005]. This happens due to the existence of a dual Killing tensor generating the “twisted” supercharge.

Chapter 6

Non-commutative models

Contents

A non-commutative oscillator with no kinetic term but with a certain momentum-dependent potential is constructed. The classical trajectories followed by a non-commutative particle in this oscillator field lie on (arcs of) ellipses.

6.1 Non-commutative oscillator with Kepler-type dynamical symmetry

In recent years, a remarkable non-commutative model was derived in the context of solid state physics by Chang and Niu [Chang 1995]. They stated that the semiclassical analysis of a Bloch electron in a three-dimensional crystal lattice reveals an extra "Berry phase" term, $\vec{\Theta}$, which can take a monopole-like form in the band structure. The study of the wave-packet dynamics of this Bloch electron, under perturbations slowly varying in space and in time, leads to the equations of the motion in the mth band [in units $\hbar = 1$],

$$\dot{\vec{k}} = -e\vec{E} - e\dot{\vec{x}} \times \vec{B}\big(\vec{x}\big)\,, \quad \dot{\vec{x}} = \frac{\partial \mathcal{E}_m\big(\vec{k}\big)}{\partial \vec{k}} - \dot{\vec{k}} \times \vec{\Theta}(\vec{k})\,. \tag{6.1}$$

Here $\mathcal{E}_m\big(\vec{k}\big)$, $\vec{x}$ and $\vec{k}$ denote the Bloch electron's band energy, the intracell position and the quasi-momentum, respectively. Note that in the right hand side equation of (6.1), the electron velocity gains an anomalous velocity term, $\dot{\vec{k}} \times \vec{\Theta}(\vec{k})$, which is the mechanical counterpart of the anomalous current.

In a magnetic field-free theory [with $\vec{B} = \vec{0}$], the equations (6.1) can also be deduced using the symplectic closed two-form,

$$\Omega = dp_i \wedge dx_i + \frac{1}{2}\epsilon_{ijk}\Theta^i dp_j \wedge dp_k\,, \tag{6.2}$$

where the "extra" term induced by the Berry phase yields the position coordinates non-commutative [Chang 1995, Niu],

$$\{x_i, x_j\} = \epsilon_{ijk}\Theta_k = \Theta_{ij}, \qquad \{x_i, p_j\} = \delta_{ij}, \qquad \{p_i, p_j\} = 0\,. \tag{6.3}$$

Applying the Jacobi identities to the coordinates, we get

$$\begin{cases} 0 = \{p_i, \{x_j, x_k\}\}_{cyclic} = -\epsilon_{jkm} \dfrac{\partial \Theta^m}{\partial x_i}\,, \\ 0 = \{x_i, \{x_j, x_k\}\}_{cyclic} = \dfrac{\partial \Theta^{ij}}{\partial p_k} + \dfrac{\partial \Theta^{jk}}{\partial p_i} + \dfrac{\partial \Theta^{ki}}{\partial p_j}\,. \end{cases} \tag{6.4}$$

Then, the vector field $\vec{\Theta}$ has the property [1] to be only momentum-dependent [Bérard 2004],

$$\Theta_i = \Theta_i(\vec{p})\,, \tag{6.5}$$

and also requires the consistency condition

$$\vec{\nabla}_{\vec{p}} \cdot \vec{\Theta}(\vec{p}) = 0\,, \tag{6.6}$$

which can be interpreted as a field Maxwell equation in the dual momentum space. Choosing, for example, the non-commutative vector aligned in the third direction,

$$\Theta_i = \theta \delta_{i3}\,, \quad \theta = \text{const}\,, \tag{6.7}$$

the 3-dimensional theory reduces to the planar mechanics based on "exotic" Galilean symmetry [Lukierski 1997, Duval 2000, Duval 2001, Chaichian 2001, Nair 2001, Scholtz 2005, Scholtz 2009, Horváthy 2010]. As an application of (6.7), some interesting results, including perihelion point precession of the planetary orbit, can be derived [Romero 2003] when taking into account the Kepler potential,

$$V(r) \propto r^{-1}\,. \tag{6.8}$$

Other applications of (6.7) concern, for example, the Quantum Hall Effect [Dunne 1990, Duval 2000, Horváthy 2002].

Such a choice only allows for axial symmetry, though. In our theory, however, we restore the full rotational symmetry by choosing instead $\vec{\Theta}$ to be a "monopole in $\vec{p}$-space" [Bérard 2004],

$$\Theta_i = \theta\, \frac{p_i}{p^3}\,, \qquad \theta = \text{const}\,, \tag{6.9}$$

where $p = |\vec{p}|$. Indeed, away from the origin, the dual monopole (6.9) is the only spherically symmetric possibility consistent with the Jacobi identities [2]. Let us mention that the $\vec{p}$-monopole form in (6.9) has already been observed experimentally by Fang et al. in the context of anomalous Hall effect in the metallic ferromagnet SrRuO3 [Fang 2003].

As expected, (6.9) corresponds to extra, "monopole" term in the symplectic structure (6.2) which is in fact that of a mass-zero spin-θ coadjoint orbit of the Poincaré group. The orbit is indeed that of the $o(4,2)$ conformal group [Penrose 1972, Penrose 1977,

[1] For a more general theory which also includes magnetic fields, see, e.g., [Chang 1995, Niu, Duval 2000]. For simplicity, the mass has been chosen unity.

[2] See the equivalent demonstration in real $\vec{x}$-space in section 3.1.

Cordani 1990].

We can now study the $3D$ mechanics with non-commutativity (6.9), augmented with the Hamiltonian,

$$\mathcal{H} = \frac{p^2}{2} + V(\vec{x}, \vec{p})\,, \tag{6.10}$$

where we allowed that the potential may also depend on the momentum variable, $\vec{p}$ [3].

The equations of motion of the system read

$$\dot{x}_i = p_i + \frac{\partial V}{\partial p_i} + \theta \epsilon_{ijk} \frac{p_k}{p^3} \frac{\partial V}{\partial x_j}, \qquad \dot{p}_i = -\frac{\partial V}{\partial x_i}\,, \tag{6.11}$$

where, in the first relation, the “anomalous velocity terms” is due to our assumptions (6.9).

We are particularly interested in finding conserved quantities. This task is conveniently achieved by using van Holten's covariant framework [van Holten 2007], which amounts to searching for an expansion into integer powers of the momentum,

$$Q = C_0(\vec{x}) + C_i(\vec{x})p_i + \frac{1}{2!}C_{ij}(\vec{x})p_i p_j + \frac{1}{3!}C_{ijk}(\vec{x})p_i p_j p_k + \dots\,. \tag{6.12}$$

Requiring Q to Poisson-commute with the Hamiltonian yields an infinite series of constraints. However, the expansion can be truncated at a finite order n, provided to satisfy the Killing equation, $D_{(i_1}C_{i_2\dots i_n)} = 0$, when we can set $C_{i_1\dots i_{n+1}\dots} = 0$.

Let us assume that the potential has the form $V\big(|\vec{x}|, |\vec{p}|\big)$, and try to find the conserved angular momentum, associated with the Killing vector $\vec{C} = \vec{n} \times \vec{x}$, which represents space rotations around $\vec{n}$. An easy calculation shows that the procedure fails to work, however, owing to the $\vec{p}$-monopole term. We propose, therefore, to work instead in a “dual” framework [Ngome 06/2010], i.e. in momentum space, and search for conserved quantities expanded rather into powers of the position,

$$Q = C_0(\vec{p}) + C_i(\vec{p})x_i + \frac{1}{2!}C_{ij}(\vec{p})x_i x_j + \frac{1}{3!}C_{ijk}(\vec{p})x_i x_j x_k \dots\,. \tag{6.13}$$

[3] Note that momentum-dependent potentials are frequently used in nuclear physics and correspond to non-local interactions.

Then, the covariant van Holten algorithm, presented in section 2.2, is replaced by

$$\begin{array}{lrl}
C_i\left(p_i+\dfrac{\partial V}{\partial p_i}\right) & = 0 & \text{o}(0)\\
\dfrac{1}{r}\dfrac{\partial V}{\partial r}\left(\theta\epsilon_{ijk}\dfrac{p_k}{p^3}C_i-\dfrac{\partial C}{\partial p_j}\right)+C_{ij}\left(p_i+\dfrac{\partial V}{\partial p_i}\right) & = 0 & \text{o}(1)\\
\dfrac{1}{r}\dfrac{\partial V}{\partial r}\left(\theta\dfrac{p_m}{p^3}\left(\epsilon_{ijm}C_{ik}+\epsilon_{ikm}C_{ij}\right)-(\dfrac{\partial C_k}{\partial p_j}+\dfrac{\partial C_j}{\partial p_k})\right)+C_{ijk}\left(p_i+\dfrac{\partial V}{\partial p_i}\right) & = 0 & \text{o}(2)\\
\dfrac{1}{r}\dfrac{\partial V}{\partial r}\left(\theta\dfrac{p_m}{p^3}\left(\epsilon_{lim}C_{ljk}+\epsilon_{ljm}C_{lki}+\epsilon_{lkm}C_{lij}\right)-(\dfrac{\partial C_{ij}}{\partial p_k}+\dfrac{\partial C_{jk}}{\partial p_i}+\dfrac{\partial C_{ki}}{\partial p_j})\right)+ & & \\
\qquad C_{lijk}\left(p_l+\dfrac{\partial V}{\partial p_l}\right) & = 0 & \text{o}(3)\\
\vdots \qquad\qquad \vdots & \vdots &
\end{array}$$

where $r=|\vec{x}|$. The expansion (6.13) can again be truncated at a finite order n, provided the higher order constraint of the previous series of constraints transforms into a dual Killing equation,

$$\partial_{(p_{i_1}}C_{p_{i_2}\cdots p_{i_n})}=0\,. \tag{6.14}$$

Then, for linear conserved quantities, $Q=C_0(\vec{p})+C_i(\vec{p})x_i$, we can set $C_{ij}=C_{ijk}=\ldots 0$. The dual algorithm therefore reduces to

$$\begin{cases}
C_i\left(p_i+\dfrac{\partial V}{\partial p_i}\right)=0\,, & \text{o}(0)\\
\theta\epsilon_{ijk}\dfrac{p_k}{p^3}C_i-\dfrac{\partial C}{\partial p_j}=0\,, & \text{o}(1)\\
\dfrac{\partial C_k}{\partial p_j}+\dfrac{\partial C_j}{\partial p_k}=0\,. & \text{o}(2)
\end{cases} \tag{6.15}$$

Introducing the dual Killing vector

$$\vec{C}=\vec{n}\times\vec{p}$$

into the previous algorithm provides us with

$$C=\theta\,\vec{n}\cdot\hat{p}\,,\qquad \hat{p}=\frac{\vec{p}}{p}\,. \tag{6.16}$$

Thus, we obtain the conserved angular momentum,

$$\vec{J}=\vec{L}-\theta\,\hat{p}=\vec{x}\times\vec{p}-\theta\,\hat{p}\,, \tag{6.17}$$

which is what one would expect, due to the "monopole in $\vec{p}$-space", whereas the non-commutative parameter, θ, behaves as the "monopole charge" [Cortes 1996].

The next step is to inquire about second order conserved quantities. Then, the series of constraints which has to be solve read

$$\begin{cases} C_i\left(p_i+\dfrac{\partial V}{\partial p_i}\right)=0\,, & \text{o}(0)\\ \dfrac{1}{r}\dfrac{\partial V}{\partial r}\left(\theta\epsilon_{ijk}\dfrac{p_k}{p^3}C_i-\dfrac{\partial C}{\partial p_j}\right)+C_{ij}\left(p_i+\dfrac{\partial V}{\partial p_i}\right)=0\,, & \text{o}(1)\\ \theta\dfrac{p_m}{p^3}\left(\epsilon_{ijm}C_{ik}+\epsilon_{ikm}C_{ij}\right)-\left(\dfrac{\partial C_k}{\partial p_j}+\dfrac{\partial C_j}{\partial p_k}\right)=0\,, & \text{o}(2)\\ \dfrac{\partial C_{ij}}{\partial p_k}+\dfrac{\partial C_{jk}}{\partial p_i}+\dfrac{\partial C_{ki}}{\partial p_j}=0\,. & \text{o}(3) \end{cases} \tag{6.18}$$

Remark that usually the Runge-Lenz vector is generated by the rank-2 Killing tensor $C_{ij}=2\delta_{ij}\vec{n}\cdot\vec{x}-n_ix_j-n_jx_i$ where $\vec{n}$ is some fixed unit vector [van Holten 2007]. Not surprisingly, the original procedure fails once again. The dual procedure works, though. The dual two-tensor

$$C_{ij}=2\delta_{ij}\vec{n}\cdot\vec{p}-n_ip_j-n_jp_i\,, \tag{6.19}$$

verifies the dual Killing equation of order 3 in (6.18). Then the order-2 equation yields

$$\vec{C}=\theta\frac{\vec{n}\times\vec{p}}{p}\,. \tag{6.20}$$

Inserting into the first-order constraint of (6.18) and assuming $\partial_rV\neq0$, the constraint is satisfied with

$$C=\alpha\,\vec{n}\cdot\hat{p} \tag{6.21}$$

α being an arbitrary constant, *provided* the momentum-dependent potential and the Hamiltonian take the form

$$V=\frac{\vec{x}^2}{2}-\frac{p^2}{2}+\frac{\theta^2}{2p^2}+\frac{\alpha}{p}\quad\text{and}\quad\mathcal{H}=\frac{\vec{x}^2}{2}+\frac{\theta^2}{2p^2}+\frac{\alpha}{p}\,, \tag{6.22}$$

respectively. Then the dual algorithm provides us with the Runge-Lenz-type vector

$$\vec{K}=\vec{x}\times\vec{J}-\alpha\hat{p}\,. \tag{6.23}$$

Its conservation can also be checked by a direct calculation, using the equations of the motion,

$$\dot{\vec{x}}=\theta\,\frac{\vec{x}\times\vec{p}}{p^3}-\left(\frac{\theta^2}{p^4}+\frac{\alpha}{p^3}\right)\vec{p}\,,\qquad\dot{\vec{p}}=-\vec{x}\,, \tag{6.24}$$

where the anomalous velocity term in the first relation is transversal.

Note that the $(-p^2/2)$ term in the potential cancels the usual kinetic term, and our system describes a *non-relativistic, non-commutative particle with no mass term in an oscillator field, plus some momentum-dependent interaction.*

Writing the Hamiltonian as

$$\mathcal{H} = \frac{\vec{x}^2}{2} + \frac{\theta^2}{2}\left(\frac{1}{p} + \frac{\alpha}{\theta^2}\right)^2 - \frac{\alpha^2}{2\theta^2} \tag{6.25}$$

shows, moreover, that $\mathcal{H} \geq -\dfrac{\alpha^2}{2\theta^2}$ with equality only attained when $p = -\dfrac{\theta^2}{\alpha}$, which plainly requires $\alpha < 0$.

It is easy to understand the reason why our modified algorithm did work : calling

$$\begin{cases} \vec{p} \longrightarrow \vec{R} & \text{“position”} \\ -\vec{x} \longrightarrow \vec{P} & \text{“momentum”}, \end{cases} \tag{6.26}$$

the system can also be interpreted as an “ordinary” (i.e. massive and commutative) *non-relativistic charged particle in the field of a Dirac monopole of strength* θ*, augmented with an inverse-square plus a Newtonian potential.* This is the well-known “McIntosh-Cisneros – Zwanziger” (MICZ) system [Mcintosh 1970, Zwanziger 1968], for which the fine-tuned inverse-square potential is known to cancel the effect of the monopole, allowing for a Kepler-type dynamical symmetry [Mcintosh 1970, Zwanziger 1968]. The angular momentum, (6.17), and the Runge-Lenz vector, (6.23), are, in particular, that of the MICZ problem [Mcintosh 1970, Zwanziger 1968] in “dual” momentum-space.

The conserved quantities provide us with valuable information on the motion. Mimicking what is done for the MICZ case, we note that

$$\vec{J} \cdot \hat{p} = -\theta \tag{6.27}$$

implies that the vector $\vec{p}$ moves on a cone of opening angle $\arccos\big(-\theta/J\big)$. On the other hand, defining the conserved vector

$$\vec{N} = \alpha\vec{J} - \theta\vec{K}\,, \tag{6.28}$$

we construct the constant,

$$\vec{N} \cdot \vec{p} = \theta(J^2 - \theta^2) = \theta L^2\,, \tag{6.29}$$

so that the $\vec{p}$-motion lies on the plane perpendicular to $\vec{N}$. *The trajectory in p-space belongs therefore to a conic section.*

For the MICZ problem, this is the main result, but for us here our main interest lies in finding the real space trajectories, $\vec{x}(t)$. By (6.24), this amounts to find the [momentum-] *“hodograph”* of the MICZ problem. Curiously, while the hodograph of the Kepler problem is well-known, it is actually a circle or a circular arc, we could not find the corresponding result in the vast literature of MICZ system.

Returning to our notations, we note that due to

$$\vec{N} \cdot \vec{x} = 0\,, \tag{6.30}$$

the $\vec{x}(t)$-trajectories also belongs to an oblique plane, whose normal is $\vec{N} = \alpha\vec{J} - \theta\vec{K}$. We can thus conveniently study the problem in an adapted coordinate system. One proves indeed that

$$\begin{aligned} \Big\{\hat{\imath},\hat{\jmath},\hat{k}\Big\} &= \Big\{\frac{1}{|\epsilon L|}\vec{K}\times\vec{J},\ \frac{1}{|\lambda\epsilon|}(2\theta\mathcal{H}\vec{J}+\alpha\vec{K}),\ \frac{1}{|\lambda L|}(\alpha\vec{J}-\theta\vec{K})\Big\} \\ \text{with}\quad \lambda^2 &= \alpha^2+2\mathcal{H}\theta^2\,,\quad \epsilon^2=\alpha^2+2\mathcal{H}J^2 \quad\text{and}\quad L^2=J^2-\theta^2\,, \end{aligned} \tag{6.31}$$

is a convenient orthonormal basis to study the $\vec{x}$-trajectories. Here we recognize, in $\hat{k}$, $\vec{N}/N$ in particular.

- Firstly, projecting onto these axis,

$$\begin{cases} p_z = \vec{p}\cdot\hat{k} = \theta L/|\lambda| = \text{const}\,, \\ p_x = \vec{p}\cdot\hat{\imath}\,, \\ p_y = \vec{p}\cdot\hat{\jmath}\,, \end{cases} \tag{6.32}$$

we find the equation

$$\frac{\left(p_y+\dfrac{|\epsilon|\alpha}{2|\lambda|\mathcal{H}}\right)^2}{\lambda^2/4\mathcal{H}^2} - \frac{p_x^2}{L^2/2\mathcal{H}} = 1, \tag{6.33}$$

which is the equation of a hyperbola or of an ellipse in momentum space, depending on the sign of $\mathcal{H}$, positive or negative. For vanishing $\mathcal{H}$ one gets a parabola. This confirms what is known for the MICZ problem [Mcintosh 1970, Zwanziger 1968], and is consistent with what we deduced geometrically.

- Next, projecting the $\vec{x}$-motion onto the orthonormal basis (6.31) yields

$$X = \vec{x}\cdot\hat{\imath} = -\frac{2|L|}{|\epsilon|}(\mathcal{H}-\frac{\alpha}{2p})\,,\qquad Y=\vec{x}\cdot\hat{\jmath} = -\frac{|\lambda|}{|\epsilon|}\frac{\vec{x}\cdot\vec{p}}{p}\,,\qquad Z=\vec{x}\cdot\hat{k}=0\,. \tag{6.34}$$

An easy calculation leads to the equation

$$\big(X+\frac{|\epsilon|L}{J^2}\big)^2 + \frac{\alpha^2L^2}{\lambda^2J^2}Y^2 = \frac{L^2\alpha^2}{J^4} \tag{6.35}$$

which always describes an ellipse or an arc of ellipse, since

$$\lambda^2 = \alpha^2+2\mathcal{H}\theta^2 \geq 0\,. \tag{6.36}$$

The center has been shifted along the axis $\hat{\imath}$ by the quantity $\big(-|\epsilon|L/J^2\big)$ and the major axis is directed along $\hat{\jmath}$. Note that, unlike as in $\vec{p}$-space, the $\vec{x}$-trajectories are always bounded.

When the energy is negative, $\mathcal{H}<0$, which is only possible when the Newtonian potential is attractive, $\alpha<0$, the $\vec{x}$-trajectories are full ellipses. The origin is inside the ellipse :

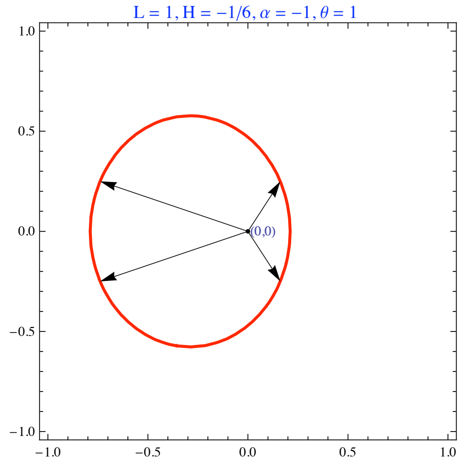

Figure 6.1: $\mathcal{H} < 0$ and the Newtonian potential is attractive $\alpha < 0$, so the trajectories describe a whole ellipse.

When $\mathcal{H} > 0$, which is the only possibility in the repulsive case $\alpha > 0$, the origin is outside the ellipse so that only the right arc [denoted with the heavy line in the left side figure of (6.2)] between the tangents drawn from the origin is obtained. However, positive hamiltonian $\mathcal{H} > 0$, is also allowed for attractive Newtonian potential $\alpha < 0$ but in that event the origin is again outside the ellipse so that the $\vec{x}$-trajectories are confined on the left arc of the ellipse [denoted with the heavy line in the right side figure of (6.2)] :

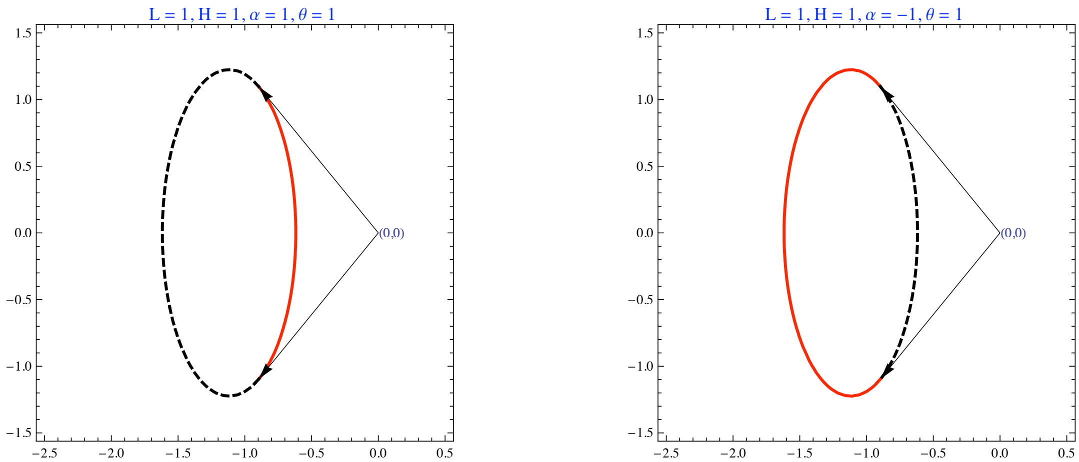

Figure 6.2: The left side figure represents a right arc of an ellipse spanned by the $\vec{x}$-trajectories for $\mathcal{H} > 0$ and $\alpha > 0$. While the right side figure represents a left arc of an ellipse spanned by the $\vec{x}$-trajectories for $\mathcal{H} > 0$ and $\alpha < 0$.

For $\mathcal{H} = 0$, the origin lies on the ellipse, and “motion” reduces to this single point :

When the non-commutativity is turned off, $\theta \to 0$, the known circular hodographs of

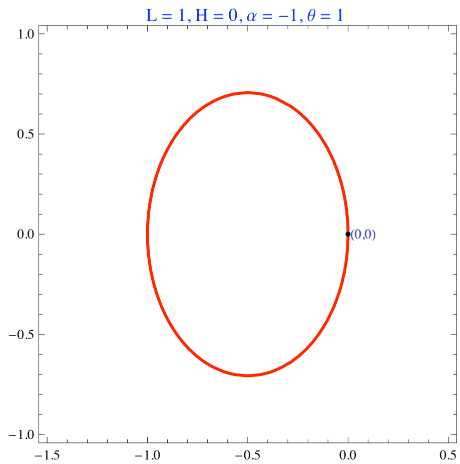

Figure 6.3: $\vec{x}$-trajectories degenerate to one single point for $\mathcal{H} = 0$.

the dual Kepler problem are recovered. As $\alpha \to 0$, the trajectory becomes unbounded, and follows the y-axis.

So far, we only discussed classical mechanics. Quantization is now straightforward using the known group theoretical properties of the MICZ problem in dual space. The non-commutativity, alias monopole charge, θ has to be an integer or half integer. This is indeed the first indication about the quantization of the non-commutative parameter. The wave functions should be chosen in the momentum representation, $\psi(\vec{p})$. The angular momentum, $\vec{J}$, and the rescaled Runge-Lenz vector, $\vec{K}/\sqrt{2|\mathcal{H}|}$, close into $o(3,1)/o(4)$ depending on the sign of the energy. In the last case, the representation theory provides us with the discrete energy spectrum, see (3.56), [in units $\hbar = 1$]

$$E_n = -\frac{\alpha^2}{2n^2}, \qquad n = n_r + \frac{1}{2} + (l + \frac{1}{2})\sqrt{1 + \frac{4\theta^2}{(2l+1)^2}}\,, \tag{6.37}$$

where $n = 0, 1, \ldots$, $l = 0, 1, \ldots$, with degeneracy

$$n^2 - \theta^2 = \big(n - \theta\big)\big(n + \theta\big)\,.$$

Note that the degeneracy always takes integer or half-integer value, as it should, since n and θ are simultaneously integer or half-integer. The same result can plainly be derived directly by solving the Schrödinger equation in $\vec{p}$-space [Mcintosh 1970, Zwanziger 1968]. Also related to the MICZ system, calculation of energy levels of hydrogen atom using NC QED theory is discussed in [Chaichian 2001].

Moreover, the symmetry extends to the conformal $o(4,2)$ symmetry, due to the fact that the massless Poincaré orbits with helicity θ are in fact orbits of the conformal group, cf. [Cordani 1990].

Let us observe that in most approaches one studies the properties (like trajectories, symmetries, etc.) of some given physical system. Here we followed the reverse direction: after positing the fundamental commutation relations, we were looking for potentials with remarkable properties. This leads us to the momentum-dependent potentials (6.22), realizing a McIntosh-Cisneros-Zwanziger system [Mcintosh 1970, Zwanziger 1968] in dual space. Unlike as in a constant electric field [Horváthy 2006], the motions lie in an (oblique) plane. The particle is confined to bounded trajectories, namely to (arcs of) ellipses.

The best way to figure our motions is to think of them as analogs of the circular hodographs of the Kepler problem to which they indeed reduce when the non-commutativity is turned off. For $\mathcal{H} < 0$, for example, the dual motions are bound, and the velocity turns around the whole ellipse; for $H > 0$ instead, the motion along a finite arc, starting from one extreme point and tending to the other one at the end of the arc, corresponds to the variation of the velocity in the course of a hyperbolic motion of a comet, or in Rutherford scattering, but in dual space.

Our system, with monopole-type non-commutativity (6.9), has some remarkable properties :

Momentum-dependent potentials are widely used in nuclear physics, namely in the study of heavy ion collisions, where they correspond to non-local interactions [Gale 1987, Das 2003, Das 2004]. Remarkably, in non-commutative field theory, a $1/p^2$ contribution to the propagator emerges from UV-IR mixing.

The absence of a mass term should not be thought of as the system being massless; it is rather reminiscent of "Chern-Simons dynamics" [Dunne 1990].

One can be puzzled how the system would look like in configuration space. Trying to eliminate the momentum from the phase-space equations (6.24) in the usual way, which amounts to deriving $\dot{\vec{x}}$ with respect to time and using the equations for $\dot{\vec{p}}$, fails, however, owing to the presence of underived $\vec{p}$ in the resulting equation. This reflects the non-local character of the system.

One can, instead, eliminate $\vec{x}$ using the same procedure, but in dual space. This yields in fact the equations of the motion of MICZ in dual momentum space,

$$\ddot{p} = \frac{J^2}{p^3} + \frac{\alpha}{p^2}\,, \qquad \ddot{\vec{p}} = \frac{\alpha}{p^3}\,\vec{p} - \frac{\theta}{p^3}\,\vec{J}\,. \tag{6.38}$$

Are these equations related to a theory with higher-order derivatives of the type [Lukierski 1997, Lukierski 2003] ? The answer is yes and no. The clue is that *time is not a "good" parameter* for Kepler-type problems, owing to the impossibility of expressing it from the Kepler equation [Cordani 2003]. This is also the reason for which we describe the *shape* of the trajectories, but we do not integrate the equations of the motion. A "better" parameter can be found along the lines indicated by Souriau [Souriau 1982, Bates 1989] and then, deriving with respect to the new parameter, transforms (6.38) into a fourth-order linear matrix differential equation, which can be solved.

It is, however, not clear at all if these equations derive from some higher-order Lagrangian, and if they happen to do, what would be the physical meaning of the latter.

The fourth-order equations do certainly *not* come from one of the type stated in Ref. [Lukierski 1997, Lukierski 2003] : the latter lives in fact in two space dimensions and has constant scalar non-commutativity θ, while our system is 3-dimensional and has a momentum-dependent vector $\vec{\Theta}(\vec{p})$, given in (6.9).

It is tempting to ask if the relation to the "closest physical theory" with a momentum-dependent potential, namely nuclear physics, can be further developed and if similar (super)symmetries can be found also in nuclear physics. Once again, the answer seems to be negative, though : while dynamical symmetries *do* play a role in nuclear physics [Iachello 1993], those used so far do not seem to be of a momentum-dependent Keplerian type.

Chapter 7

Concluding remarks

In this thesis, we developed a systematic method to search for hidden and (super)symmetries of several physics system. In some cases, like in the SUSY of the monopole, our recipe needs to be extended to fermionic degrees of freedom. In the case of the momentum-space monopole, we needed to adapt our recipe to the non-commutative structure by interchanging the role of positions and momenta. In these models, as expected, the hidden symmetry of the Kepler-type is always related to the addition into the system of a fine-tuned inverse-square potential. This requirement appears clearly, in the van Holten algorithm, to be a consistency condition on the existence of a conserved Runge-Lenz-type vector.

Having introduced the Abelian Dirac magnetic monopole-field; we studied, in particular, the classical geodesic motion of a particle in Kaluza-Klein-type monopole spaces and its generalization: the Gibbons-Hawking space. We derived the conditions under which the Killing tensors imply the existence of conserved quantities on the dimensionally reduced curved manifold. We observed that the Killing tensor generating the Runge-Lenz-type vector, preserved by the geodesic motion, can be lifted to an extended manifold, namely, (3.143) and (3.149) [Duval 1991]. As an illustration, we have treated, in detail, the generalized Taub-NUT metric, for which we derived the most general additional scalar potential so that the combined system admits a Runge-Lenz vector [Gibbons 09/2006]. Another example considered is the multi-center metric where we have found a conserved Runge-Lenz-type scalar (3.140), in the special case of motions confined onto a particular 2-sphere. Moreover, from the Theorem 3.3.1 we deduced, for $N > 2$, that no Runge-Lenz vector does exist in the case of N-center metrics. It is worth mentioning that apart from the generic importance of constructing constants of motion, namely in the confinement of particle to conic sections; the existence, in particular, of quadratic conserved quantity like Runge-Lenz vector yields the separability of the Hamilton-Jacobi equation for the generalized Taub-NUT metric and for the two-center metric.

In the case of isospin-carrying particle in a non-Abelian Wu-Yang monopole field, we found the most general scalar potential such that the combined system admits a conserved Runge-Lenz vector. Indeed, it generalizes the fine-tuned inverse-square plus Coulomb potential [Mcintosh 1970, Zwanziger 1968], for a charged particle in the field of a Dirac monopole. Following Fehér, the result is interpreted as describing motion in the asymptotic field of a self-dual Prasad-Sommerfield monopole [Fehér 1984, Fehér 1985, Fehér 1986].

We also treated the case of the effective "truly" non-Abelian monopole-like field generates by nuclear motion in a diatomic molecule. This system is due to Moody, Shapere and Wilczek where *despite the non-conservation of the electric charge (4.168)*, we surprisingly constructed, in addition to the "unusual" angular momentum (69), a new conserved charge (4.187).

We remarked that Runge-Lenz-type vector plays a role also in SUSY. Indeed, we investigated the bosonic symmetries as well as the supersymmetries of a spinning particle coupled to a magnetic monopole field. The gyromagnetic ratio determines the type of (super)symmetry the system can admit : for the Pauli-like hamiltonian (5.1) $\mathcal{N}=1$ SUSY only arises for gyromagnetic ratio $g=2$ and with no external potential, $V=0$, confirming Spector's observation [Spector]. We also derived additional supercharges, which are not square roots of the Hamiltonian of the system, though. A Runge-Lenz-type dynamical symmetry requires instead an anomalous gyromagnetic ratio,

$$g=0 \quad \text{or} \quad g=4\,,$$

with the additional bonus of an extra "spin" symmetry. These particular values of gyroratio come from the effective coupling of the form $F_{ij} \mp \epsilon_{ijk} D_k \Phi$, which add or cancel for self-dual fields, $F_{ij} = \epsilon_{ijk} D_k \Phi$ [Fehér 1988]. We found that the super- and the bosonic symmetry can be combined, but the price to pay is, however, to enlarge the fermionic space. This provides us with an $\mathcal{N}=2$ SUSY.

We also applied the van Holten algorithm to a planar fermion in any planar magnetic field, i.e. one perpendicular to the plane. We shown, for ordinary gyromagnetic, that in addition to the usual supercharge (5.121) generating the supersymmetry, the system also admits another square root of the Pauli Hamiltonian [Horváthy 2005] happening due to the existence of a dual Killing tensor.

A three-dimensional non-commutative oscillator with no mass term but with a certain momentum-dependent potential is obtained when studying the hidden symmetry of a monopole-type non-commutativity [Ngome 06/2010]. This oscillator system exhibits a conserved Runge-Lenz-type vector derived from the dual description in momentum space. The latter corresponds, but in dual space, to a Dirac monopole with a fine-tuned inverse-square plus Newtonian potential, introduced by McIntosh, Cisneros, and by Zwanziger some time ago. The resulting additional Kepler-type symmetry leads to the confinement of the particle's trajectories to bounded trajectories, namely to (arcs of) ellipses. When the non-commutativity is turned off, i.e. in the commutative limit, the motions reduce to the circular hodographs of the Kepler problem. It is worth mentioning that the momentum-dependent potentials which are rather unusual in high-energy physics, however, are widely used in nuclear physics, namely in the study of heavy ion collisions; they correspond to non-local interactions [Gale 1987, Das 2003, Das 2004]. Moreover, in non-commutative field theory, it is remarkable that a $1/p^2$ contribution to the propagator emerges from the UV-IR mixing. See in [Gubser 2001]. The absence of a mass term in the Hamiltonian describing this non-commutative oscillator should not be thought of as the system being massless; it is rather reminiscent of "Chern-Simons dynamics" [Dunne 1990].

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