Modèles de Hubbard unidimensionels généralisés

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Modèles de Hubbard unidimensionels généralisés

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Résumé :

Cette thèse est consacrée à l'étude du modèle de Hubbard unidimensionnel et à ses généralisations. Le modèle de Hubbard est un modèle fondamental de la physique de la matière condensée, décrivant des électrons en interaction sur un réseau. Il a une très riche structure physique. Malgré la simplicité de sa construction, le modèle a été appliqué dans différents problèmes comme la supraconductivité à haute température, le magnétisme et la transition métal-isolant. A une dimension, le modèle de Hubbard est un modèle intégrable très étudié qui a servi de 'laboratoire' pour la physique de la matière condensée. Récemment, les systèmes intégrables quantiques d'une facon générale, et le modèle de Hubbard en particulier, sont apparus d'une manière surprenante dans le contexte de la correspondance AdS/CFT. Le point de contact entre ces domaines est les équations de Bethe : celles de nouveaux modèles intégrables et de modèles existants généralisés sont à priori significatifs dans l'application en dualité AdS/CFT.

Dans la premiere partie de la thèse, les notions de base sur l'intégrabilité quantique sont présentées : formalisme de la matrice R, équation de Yang-Baxter, chaînes de spin intégrables. Dans la deuxième partie, certaines résultats fondamentaux concernant le modèle de Hubbard sont passés en revue : la solution par l'Ansatz de Bethe coordonnée, les solutions réelles des équations de Lieb-Wu etc. De plus, l'application dans la correspondance AdS/CFT est considérée. Cependant, on trouve que certaines modifications du modèle de Hubbard sont nécessaires pour reproduire les résultats de cette correspondance. Cela est une des motivations principales d'étude de modèles de Hubbard généralisés. La quatrième partie est consacrée aux généralisations du modèle de Hubbard, en se concentrant sur les cas supersymétriques. La chapitre cinq expose les résultats obtenus dans le cadre de cette thèse sur les modèles de Hubbard généralisés, en particulier, l'Ansatz de Bethe coordonnée ainsi que les solutions réelles des équations de Bethe obtenues dans la limite thermodynamique. Les équations de Bethe obtenues sont différentes de celle de Lieb et Wu par des phases dont la manifestation est un signe encourageant pour l'application en AdS/CFT contexte. Les applications possibles, notamment dans le domaine de la physique de la matière condensée, sont également considérées.

Abstract :

This thesis is devoted to the one-dimensional integrable Hubbard model and its generalizations. The Hubbard model is one of the fundamental models in condensed matter physics which describes interacting electrons on the lattice. It has very rich physical structure. Inspite of its construction simplicity it has been applied to diverse problems as high-T_c superconductivity, band magnetism and the metal-insulator transition. In one dimension the Hubbard model is an integrable model which has been intensively studied and served as a theoretical laboratory for the condensed matter physics. Recently, the integrable systems and in particular the Hubbard model, have surprisingly appeared in the AdS/CFT correspondence context. The intersection point between the fields is the Bethe equations: the ones of new integrable models and the generalizations of existing ones are relevant in the application in AdS/CFT duality.

In the first part of the thesis, we present basis notions of the quantum integrability: the R-matrix formalism, the Yang–Baxter equation, integrable spin chains etc. In the second part we review several fundamental results of the one-dimensional Hubbard model: the Coordinate Bethe Ansatz solution, real solutions of the Lieb-Wu equations etc. Moreover, applications in the AdS/CFT duality are considered. However, it turns out that certain modifications of the integrable Hubbard model are necessary to reproduce the correct results of the AdS/CFT context. This is one of the main motiva-

tions of the studies of generalized Hubbard models. The fourth chapter is devoted to generalizations of the Hubbard model and we focus our attention on supersymmetric ones. The fifth chapter contains the results obtained in the framework of this thesis on the supersymmetric generalizations of the Hubbard models. Namely, the Coordinate Bethe Ansatz solution and real solutions of the Bethe equations in the thermodynamic limit are exposed. We point out that the obtained Bethe equations differs from the Lieb-Wu ones by phases which appearance is encouraging sign for the application in the AdS/CFT context. We also discuss possible applications in the AdS/CFT duality and in condensed matter physics.

Keyswords : Integrable systems, Hubbard model, Coordinate Bethe ansatz

Contents

Chapter 1

Introduction

The one-dimensional Hubbard model and its generalizations as the subject of this thesis are some examples of solvable models in the field of integrable systems which is a part of mathematical physics.

What are the integrable systems. One knows that the classical N body problem in three dimensions is solvable only in the case $N = 2$ and is not solvable for interacting systems with $N \geq 3$. However, certain one-dimensional systems appears to be solvable for any N, i.e. one can integrate the equations of motions for classical systems or find the spectrum and the eigenfunctions for quantum systems. The integrable systems are bi-dimensional models which can be solvable. The origin of the term "integrable systems" is related to the studies of dynamical systems in the classical mechanics. In the work of J.Liouville [6], the notions of integrable Hamiltonian systems were formulated. The consequence of the integrability for classical systems is that there is a canonical transformation which allows easily to solve the model.

The integrable systems started their development about eighty years ago (depending on the counting manner) and nowadays it is a highly investigated field of mathematical physics. This part of physics is reputed by words of B.Sutherland to be very mathematical to be physics and not enough rigorous to be 'real' mathematics. However, the integrable systems serve as not only a 'playground' for real physical models, they provide the non-perturbative information about systems, but also they can be used as methods applicable in other physical fields, e.g. the AdS/CFT duality. There exists a huge amount of literature on different aspects of integrable systems starting from the classical integrability of nonlinear equations, different statistical models, up to various quantum models, etc. We provide a modest list of references on the subject: [1], [2], [3], [10], [12], [13],[14], [15] etc.

The Yang–Baxter equation. The solvable models have a similar property that being low-dimensional systems their inner structure is related to the so-called Yang–Baxter equation, classical or quantum.

Speaking about the classical integrable systems, one always considers a theory of classical completely integrable differential equations. The latter one admits the Hamiltonian interpretation. In other words any Hamiltonian system is related to a system of equations of motion in the case of models with a finite number of degrees of freedom, e.g. Toda model on the lattice. Similarly, a Hamiltonian system is related to one equation of motion in the case of field theories, e.g. KdV equation, Sine-Gordon equation, nonlinear Schrodinger equation etc. The integrability in the Liouville sense implies the existence of independent conserverved quantities by its definition. The number of such quantities is finite for the lattice models and countably infinite for the field theories. The theorem formulated by J.Liouville says that there is a canonical transformation of variables which leads to the solvability of the model. However, the explicit construction of such transformation is not trivial. Different approaches exist in the literature in order to solve the classical models, the classical inverse scattering method is one of the techniques which allows to calculate the conserved charges

and find the solutions of the equations of motion. The Lax pair, two operators L and M, is primary in the method. Moreover, the involution property of the conserved charges is insured by an objet called classical r-matrix r which satisfies the classical Yang–Baxter equation:

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0 \tag{1.0.1}$$

More details on the classical integrability can be found in [10].

Quantum integrable systems with a finite number of degrees of freedom, namely, the spin chains Hamiltonians, are systems with one spatial dimension and one internal degree of freedom. These models are given by the Hamiltonian, such that we can construct the sufficient number of independent quantities commuting with the Hamiltonian. It was found that the Hamiltonian and its conserved charges can be deriven from an object called, the R-matrix $\mathcal{R}$. The R-matrix obeys the Yang–Baxter equation:

$$\mathcal{R}_{12}(u_1, u_2)\mathcal{R}_{13}(u_1, u_3)\mathcal{R}_{23}(u_2, u_3) = \mathcal{R}_{23}(u_2, u_3)\mathcal{R}_{13}(u_1, u_3)\mathcal{R}_{12}(u_1, u_2) \tag{1.0.2}$$

The spin chains as isotropic or anisotropic Heisenberg model, the Hubbard model and their generalizations are examples of such integrable systems. We will consider several spin chains more in details in the next chapter.

In 1+1 dimensional quantum field theories, the integrablility, similarly to the systems with a finite number of degrees of freedom, should be provided by a countably infinite number of charges. The conserved charges are related to the symmetry of the model and thus, integrable field theories have inifinite dimensional algebras behind. In the field theory, the key object is the so-called S-matrix which incodes the interaction between asymptotic incoming and outcoming states. The integrable quantum field theories, e.g. Affine Toda Field Theories etc, have a multiparticle S-matrix factorisable into two-particle S-matrix, i.e. any process can be considered as consecutive interaction between two particles. Two-particle S-matrix satisfies the braided version of the Yang–Baxter equation, e.g. [4]. The two-particle S-matrix gives the mapping of an incoming asymptotic two-particle state into an outgoing asymptotic state: $\mathcal{S}_{ab}(\theta - \theta') : V_a(\theta) \otimes V_b(\theta') \to V_b(\theta') \otimes V_a(\theta)$ and it satisfies

$$\mathcal{S}_{23}(\theta_1 - \theta_2)\mathcal{S}_{12}(\theta_1 - \theta_3)\mathcal{S}_{23}(\theta_2 - \theta_3) = \mathcal{S}_{12}(\theta_2 - \theta_3)\mathcal{S}_{23}(\theta_1 - \theta_3)\mathcal{S}_{12}(\theta_1 - \theta_2) \tag{1.0.3}$$

In addition, these models have an infinite quantum group symmetry which determines the corresponding two-particle S-matrix up to a factor.

The statistical models are systems on a two-dimensional lattice in equilibrium (no time dimension), such as vertex models: 6-vertex model (equiv. to XXZ spin chain), 8-vertex model (equiv. to XYZ spin chain). They are totally described by the partition function $\mathcal{Z}$ whereas the latter consists of all possible configurations of the so-called Boltzmann weights W_V of a vertex V: $\mathcal{Z} = \sum_C \prod_V W_V$. The Boltzmann weights in their turn satisfy again the Yang–Baxter equation, e.g. [2], [13].

Integrable systems with boundaries. The above mentioned models are considered under periodic conditions for spin chains and statistical models and on the infinite line for field theories. Furthermore, there exist integrable systems with boundary conditions. In addition to the Yang–Baxter equation there is the so-called *Reflection equation* which asserts the integrability, e.g. [5]. The Reflection equation relates the boundary matrices $\mathcal{K}^{\pm}$ (one matrix at one boundary) and the R-matrix $\mathcal{R}$, or in other words it singles out the boundary matrices such that the system remains integrable:

$$\mathcal{R}_{12}(u_1 - u_2)\mathcal{K}_1^-(u_1)\mathcal{R}_{21}(u_1 + u_2)\mathcal{K}_2^-(u_2) = \mathcal{K}_2^-(u_2)\mathcal{R}_{12}(u_1 + u_2)\mathcal{K}_1^-(u_1)\mathcal{R}_{21}(u_1 - u_2) \tag{1.0.4}$$

In this thesis we deal with periodic spin chains.

Methods in quantum integrable systems. The Yang–Baxter equation and the Reflection equation are mathematical "restrictions" imposed on the system to be integrable, but they do not provide the

methods how to solve the system. There are many different techniques proposed from the very begining in order to obtain the solution of the systems. The pioneering paper of H.Bethe [11] in 1931 on the resolution of the anisotropic Heisenberg spin chain on the one-dimensional periodic lattice gave rise to the method - *Coordinate Bethe ansatz*. This method was highly expoited for other 'toy' models as one-dimensional system of bosons with repulsive delta-function interaction by E.H.Lieb and W.Liniger [34], similar model of particles with spin by C.N.Yang [35] and same system for fermions by M.Gaudin [37] etc. The one-dimensional Hubbard model was firstly solved using this method by E.H.Lieb and F.Y.Wu [48]. Another technique proposed later in 80s by L.D.Faddeev and collaborators generalizes the previous method and gives more obvious relations with the R-matrix and the Yang–Baxter equation, it is called the *Algebraic Bethe Ansatz* or Quantum Inverse Scattering Method (QISM), e.g. [68]. This powerful method allows to solve a large number of spin chains, i.e. find the spectrum and derive the so-called *Bethe equations*. Moreover in the frame of this method the correlation functions can be calculated what is, by the way, one of the most challenging topics in the integrable systems. Some other methods which allows to derive the Bethe equations are, for example, *Analytic Bethe ansatz* (see [32]) and *Separation of variables method* or Functional Bethe ansatz [31]. In the thesis we will discuss in details the Coordinate Bethe ansatz and the Algebraic Bethe ansatz.

As we previously said the field of integrable systems is widely developing and it contacts with other physical fields. The recent experimental progress in the realization of one-dimensional systems highly stimulates the interest in Bethe ansatz integrable models of interacting bosons and multicomponent fermion. From mathematical point of view these models are solved through Bethe Ansatz technique long ago, but this opens up an oportunity to experimental studies of such models which exhibit new quantum effects peculiar to one-dimensional systems. Another interesting application of integrable systems has recently found in the context of the super Yang-Mills theories and the AdS/CFT correspondence. In this thesis we will review how the one-dimensional Hubbard model is related to the $\mathcal{N} = 4$ super Yang-Mills theory.

Hubbard model and generalizations. The Hubbard model is known in the condensed matter physics as a model describing the electrons' behaviour in the transition and rare-earth metals. The model, on the lattice, it consists of a electron's hopping term on the nearest-neighbour sites and the on-site Coulomb interaction. Despite of its simplicity of the definition, the Hubbard model exhibits different effects relevant in condensed matter physics: metal-insulator transition, ferromagnetism, superconductivity etc. In one dimension, the Hubbard model is integrable and, hence, theoretical studies of non-perturbative effects are available. Due to refinement of experimental techniques, it is not only a 'toy' model, but a framework of experimental relevance for strongly correlated electron systems.

The one-dimensional Hubbard model was solved by E.H.Lieb and F.Y.Wu using the Coordinate Bethe ansatz in 1968. The Bethe equations (Lieb-Wu equations) were obtained and solved in a certain limit allowing to calculate the ground state energy. In further developments, all solutions of the Lieb-Wu equations in the thermodynamic limit, infinite lattice, were proposed by M.Takahashi and some thermodynamic properties of the Hubbard model were obtained. The proof of the integrability, the R-matrix corresponding to the Hubbard model was found much later by B.S.Shastry in 1986 and the proof of the Yang–Baxter equation in 1995 by M.Shiroishi and M.Wadati. The Hubbard model, therefore, entered in the paradigm of the integrable models, however being a rather special model in comparing with basic examples as the XXX or XXZ models.

The Hubbard model as a model of strongly correlated electrons is relevant to condensed matter physics. Its generalizations: t-J, EKS, supersymmetric U, etc were widely studied in the literature. Multiparticle generalizations of the Hubbard model, similarly as the extensions of $gl(2)$ XXX model to $gl(n)$ one, were proposed by Z.Maassarani. These generalizations are important in the construction of the supersymmetric extentions of Shastry's R-matrix which are considered in this thesis.

Another aspect of applications of the Hubbard model is its connection with the AdS/CFT duality. The latter is a new highly investigating topic on the conjecture of the duality between Conformal Field Theories and String theories. The integrability methods are succesfully used in the tests of the duality and integrable models are recognized in its studies. The Hubbard model was also obtained in certain limit: the half-filled band limit of the Hubbard model reproduces the dilatation operator of the $\mathcal{N} = 4$ super Yang-Mills theory up to three-loop order. Several modifications in the Bethe equations of the dilation operator appear at four-loop order. Thus, certain modifications of the Hubbard model are needed in order to recognize the integrable model behind the $\mathcal{N} = 4$ super Yang-Mills theory in the considered sector.

In this thesis we considered the supersymmetric extentions of Shastry's R-matrix and associated spin chains. We obtained the corresponding Bethe equations and found real solutions, similarly to the Lieb-Wu approach to the Hubbard model. We compare our Bethe equations with the AdS/CFT duality results, and we conclude that further investigations on the modifications of the Hubbard model are needed.

Thesis plan

The work is divided into three parts:

- The second chapter (2) is devoted to the quantum integrable systems. We give an introduction to quantum integrable systems, discussing the R-matrix formalism from "physical" and "mathematical" points of view. In the end of the first chapter the basic examples of integrable spin chains are considered.

- In the third chapter (3) the one-dimensional Hubbard model is considered. The main goal of the part is a pedagogical introduction to the Hubbard model. We review its origin in condensed matter physics and integrability aspects. In more details we describe the Coordinate Bethe Ansatz for the Hubbard model and real solutions of the Lieb-Wu equations. In the end of the chapter we review certain impacts of the one-dimensional Hubbard with $\mathcal{N} = 4$ super Yang-Mills Theory. We discuss also the importance to include a phase in the Lieb-Wu equations.

- The fourth chapter (4) deals with one-dimensional Hubbard-like models. At first we review several important models existing in the literature related to the Hubbard model and then we consider more in details supersymmetric extensions of the Hubbard model [123],[124].

- In the fifth chapter the solution via the Coordinate Bethe Ansatz of the supersymmetric Hubbard-like models are considered, [126],[127]. Then, the obtained Bethe equations are then considered in the thermodynamic limit. This chapter consists of the personal contribution to the subject.

Chapter 2

Quantum Integrable systems and spin chains

In this chapter an introduction to quantum integrable systems and spin chains will be given. In the following sections we will discuss the R-matrix formalism (R matrix, the Yang–Baxter equation, transfer matrix, associated Hamiltonian, Hopf algebra) and give some examples of integrable spin chains.

Quantum integrable systems are basically $1+1$ or $2+0$ dimensional systems, field theories or spin chains. They share common features which single out them as whole class of models. We speak about the integrability. The latter was firstly formulated for the classical systems with a finite number of degrees of freedom. The notion of the classical integrability was introduced in the works of J.Liouville [6] where its definition and the theorem of integrability were formulated. The theorem of integrability states 'roughly' that any integrable system is solvable via the so-called "action-angle" transformation. In other words, the model is called integrable if there exists a sufficient finite number of independent conserved charges. The "action-angle" transformation is a canonical transformation which allows one to construct these conserved charges. Later, the notion of the classical integrability was extended on the $1+1$ field theories demanding the existence of a countably infinite number of independent conserved charges. The construction of these charges is, however, a nontrivial question. Historically, several methods were used in order to obtain the 'soliton' solutions of such nonlinear equations as Korteweg-de Vries, Sine-Gordon, Toda, nonlinear Schrodinger etc, [8], [7]. They gave rise to the so-called classical inverse scattering method (CISM) which became, nowadays, a well developed branch of mathematical physics. It allows to solve a wide class of nonlinear differential equations. The Hamiltonian interpretation of the nonlinear equations in the framework of the CISM provides an opportunity for quasiclassical quantization. The quantum theory of solitons was constructed and it was shown that after quantization, the classical solutions - solitons appear as elementary particles in the spectrum of the Hamiltonian. More information about classical integrability can be found, for example, in [10].

The history of the exactly solvable (integrable) quantum systems goes back to H. Bethe's 1931 article [11] on the spin-1/2 Heisenberg chain in the early days of quantum theory. H.Bethe constructed the many-body wavefunctions (Coordinate Bethe Ansatz) and reduced the problem of calculating the spectrum of the Hamiltonian to the problem of solving a set of coupled algebraic equations (Bethe ansatz equations). However, the integrability was not yet proven at that time. The generalization of Bethe's ansatz to models with spin degrees of freedom proved to be very hard, because the scattering involves certain changes of the internal states of the scatterers. This problem was eventually solved by C.N.Yang and M.Gaudin by means of what is nowadays called 'nested Bethe ansatz'. The con-

dition for the applicability of the nested Bethe ansatz is the consistent factorization of multi-particle scattering processes into two-particle ones. The consistency requires the two-particle scattering matrix to fulfill certain algebraic equations, the 'Yang–Baxter Equation'.

The role of the Yang–Baxter equation as a defining structure of the integrable models was emphasized by L. D. Faddeev, E. K. Sklyanin and L. Takhtajan and other members of the St. Petersburg branch of the Steklov Mathematical Institute. They established a relation between quantum many-body models solved by Bethe's ansatz and classical integrable evolution equations. Building on this connection, they initiated a systematic search for solutions of the Yang–Baxter equation and developed a program for the solution of integrable models, called the 'Quantum Inverse Scattering Method' (Algebraic Bethe Ansatz). An important element of this method is the algebraization of the construction of eigenstates of the transfer matrix. For the references, we cite several books and lectures on the subject [12],[68].

The role of the Yang–Baxter equation goes beyond the theory of dynamical systems. It is very important in the theory of knots and quantum groups. For an introduction to the mathematical aspects of the Yang–Baxter equation see e.g. [15], [13].

2.1 R-matrix formalism in general

In this section we speak about the role of R-matrix in the integrable systems. At first we consider it in the physical point of view as a tool which lies in the base of the integrability of models. Next, we review its connection with algebras. Originally formulated as a condition for the integrability of the model, the Yang–Baxter equation consists a hidden symmetry. This symmetry is the new concept in mathematics, the quantum group, which unifies the framework of two-dimensional exact models.

I The Hamiltonian H of a certain quantum model with the number of degrees of freedom n is integrable if there exists n independent conserved charges Q_i such that they commute with the Hamiltonian: $[H, Q_i] = 0$ with $i = 1, 2, ..., n$.

The construction of these independent conserved charges and the diagonalization of the one-dimensional Hamiltonian were proposed by L.D.Faddeev and collaborators, as we mentioned before, in the Algebraic Bethe Ansatz approach. In this method the crucial role is played by the R-matrix. The R-matrix allows one to derive the Hamiltonian H and the conserved charges Q_i. Below we consider it in details.

The notations we use here are following: E^{ij} is a $\mathfrak{n} \times \mathfrak{n}$ matrix with 1 at the intersection of line i and column j and 0 elsewhere. They acts on a vector space V. Similarly, E_a^{ij} is a matrix E^{ij} which acts non-trivially on the vector space V_a of the tensor product of spaces $V_1 \otimes V_2 \otimes ... \otimes V_N$. The permutation matrix P_{ab} acts non-trivially on the tensor product of two vector spaces $V_a \otimes V_b$ as

$$\begin{aligned} P_{ab}: \qquad & V_a \otimes V_b \to V_b \otimes V_a \\ & v \otimes u \to u \otimes v \end{aligned}$$

If $\dim V_a = \dim V_b = \mathfrak{n}$, then it can be represented in the matrix form as

$$P_{ab} = \sum_{i,j=1}^{\mathfrak{n}} E_a^{ij} E_b^{ji} \tag{2.1.1}$$

Now we introduce the R-matrix. It is a matrix (in more pragmatic point of view) which acts on the tensor product of two spaces, $V_a \otimes V_b$, and which depends on some complex parameter u. We will denote it by $R_{ab}(u)$. In the 'matrix form' it can be written as:

$$R_{ab}(u) = \sum_{i,j,k,l=1}^{n} R_{ij}^{kl}(u) E_a^{ij} E_b^{kl} \tag{2.1.2}$$

It satisfies generally the following conditions:

1. Unitarity: $R_{ab}(u)R_{ba}(-u) = \rho(u)\ \mathbb{I}_a \otimes \mathbb{I}_b$
2. Regularity: $R_{ab}(0) = P_{ab}$

The most important property which it has to satisfy is the so-called **Yang–Baxter equation**:

$$R_{ab}(u-v)R_{ac}(u)R_{bc}(v) = R_{bc}(v)R_{ac}(u)R_{ab}(u-v) \tag{2.1.3}$$

As the most simple example, one can verify that the permutation matrix P_{ab} satisfies the Yang–Baxter equation. Another example one can consider is the R-matrix of the XXX spin chain:

$$R_{ab}(u) = u\mathbb{I}_a \otimes \mathbb{I}_b + P_{ab} \tag{2.1.4}$$

This matrix is unitary, satisfies the regularity condition and the Yang–Baxter equation.

The Yang–Baxter equation (2.1.3) can be considered as an intertwining relation for matrices $R_{ac}(u)$, $R_{bc}(v)$. It can be generalized introducing a new object called *monodromy matrix* $T_0(u)$, which acts on the tensor product of spaces $V_0 \otimes V_1 \otimes ... \otimes V_N$. The space V_0 is called the auxiliary space and the remaining V_a are "physical" spaces (these notions will become clear later). Thus, we define $T_0(u)$ as

$$T_0(u) = R_{01}(u)...R_{0N}(u) \tag{2.1.5}$$

It can be easily verified using (2.1.3) that $T_0(u)$ also satisfies the equation similar to (2.1.3):

$$R_{00'}(u-v)T_0(u)T_{0'}(v) = T_{0'}(v)T_0(v)R_{00'}(u-v) \tag{2.1.6}$$

which is also called the RTT relations.

If one considers $T_0(u)$ at $u = 0$, and using the regularity property of the R-matrix, one can see that $T_0(0)$ is a cyclic permutation matrix: $T_0(0) = P_{01}...P_{0N}$ such that

$$T_0(0)A_a = \begin{cases} a = 0, & A_N T_0(0) \\ a \neq 0, & A_{a-1} T_0(0) \end{cases} \tag{2.1.7}$$

If now one takes the partial trace of $T_0(u)$ (i.e. in the space 0) denoted as $\tau(u) = tr_0(T_0(u))$, using the Yang–Baxter equation for $T_0(u)$ one can show that

$$[\tau(u), \tau(v)] = 0, \tag{2.1.8}$$

where we used $\tau(u)\tau(v) = tr_{0,0'}(T_0(u)T_{0'}(v)) = tr_{0,0'}(R_{00'}^{-1}(u-v)R_{00'}(u-v)T_0(u)T_{0'}(v))$ and the cyclicity of the trace $tr_{0,0'}(A_{00'}B_{00'}) = tr_{0,0'}(B_{00'}A_{00'})$.

Therefore, we see that $\tau(u)$, called the *transfer matrix*, provides mutually commuting quantities: $\tau(u) = \sum_{i=0}^{N} u^i H_i$ such that $[H_i, H_j] = 0$. We can associate the coefficient H_1 with the physical Hamiltonian and due to (2.1.8), H_1 has the common basis of eigenvectors with $\tau(u)$ and all remaining H_i. Thus, the Hamiltonian H_1 can be written as

$$H_1 = \frac{d}{du}\Big(\log(\tau(u))\Big)_{u\to 0} = \tau^{-1}(0) \cdot \frac{d\tau(u)}{du}\bigg|_{u\to 0} \tag{2.1.9}$$

In the case when the R-matrix satisfies the regularity property (2), the Hamiltonian can be showed to be local. More explicitly, one has

$$H_1 = \sum_{a=1}^{N} \left(P_{aa+1} R'_{aa+1}(0) \right), \quad N+1 \equiv 1 \tag{2.1.10}$$

where $R'_{aa+1}(0) = \frac{d}{du}[R_{aa+1}(u)]_{u\to 0}$. The Hamiltonian here acts on the "physical" spaces $V_1 \otimes ... \otimes V_N$. The notation $N+1 \equiv 1$ means that we have the periodic boundary conditions: for $a = N$ we have $P_{aa+1} R'_{aa+1}(0) = P_{N,1} R'_{N,1}(0)$.

II From the mathematical point of view, the R-matrix of the system has a more deep connections with its algebraic structure. It was shown independently by V.G.Drinfeld and M.Jimbo [16] that the Yang–Baxter equation (2.1.3) and the RTT relations (2.1.6) are related to the Hopf algebra structures and to the deformations of universal enveloping Lie algebras. The R-matrix $R_{12}(u)$ and the monodromy matrix $T_0(u)$ appear to be certain representations of some universal object called the universal R-matrix, which verifies the Yang–Baxter equation. Below we will write some statements of the Hopf algebra to make a more clear connection[1].

At the first step we should introduce the notions of associative algebras. Thus, the associative algebra $\mathcal{A}$ over a complex field $\mathbb{C}$ is a vector space with the multiplication operation $\mathfrak{m}$ and the unit operation i. The multiplication $\mathfrak{m}$ such that $\mathfrak{m} : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ is linear and associative: for any $X, Y, Z, U \in \mathcal{A}$ and $a, b, c, d \in \mathbb{C}$

$$\begin{aligned} \mathfrak{m} \circ (\mathrm{id} \otimes \mathfrak{m})(X \otimes Y \otimes Z) &= \mathfrak{m}(X, \mathfrak{m}(Y, Z)) = \mathfrak{m}(\mathfrak{m}(X, Y), Z) = \mathfrak{m} \circ (\mathfrak{m} \otimes \mathrm{id})(X \otimes Y \otimes Z) \\ \mathfrak{m}(aX + bY, cZ + dU) &= ac\,\mathfrak{m}(X, Z) + bc\,\mathfrak{m}(Y, Z) + ad\,\mathfrak{m}(X, U) + bd\,\mathfrak{m}(Y, U) \end{aligned}$$

here $\circ$ means the consecutive action of operations and id means $\mathrm{id}(\mathcal{A}) = \mathcal{A}$.

The unit operation $i : \mathbb{C} \to \mathcal{A}$ has properties: for any $X \in \mathcal{A}$

$$\mathfrak{m} \circ (\mathrm{id} \otimes i)(X \otimes c) = \mathfrak{m}(X \otimes cI) = c\mathfrak{m}(X, I) = cX = \mathfrak{m}(cI \otimes X) = \mathfrak{m} \circ (i \otimes \mathrm{id})(c \otimes X)$$

where I is the unit element of $\mathcal{A}$ and $c \in \mathbb{C}$.

The commutative algebra is such that $\mathfrak{m}(X, Y) = \mathfrak{m}(Y, X)$. If we also introduce the permutation operator σ such that $\sigma(X \otimes Y) = Y \otimes X$, the commutativity condition can be rewritten as $\mathfrak{m} \circ \sigma \equiv \mathfrak{m}^o = \mathfrak{m}$.

Then one can define 'dual' operations: the comultiplication (or coproduct) operation Δ on the algebra $\mathcal{A}$ such that $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ and the counit ϵ such that $\epsilon : \mathcal{A} \to \mathbb{C}$. The comultiplication is "coassociative" and with the counit they satisfy similar relations as $\mathfrak{m}$ and i:

$$(\mathrm{id} \otimes \Delta) \circ \Delta\mathcal{A} = (\Delta \otimes \mathrm{id}) \circ \Delta\mathcal{A}$$

and

$$\begin{aligned} (\mathrm{id} \circ \epsilon) \circ \Delta(\mathcal{A}) = (\mathrm{id} \otimes \epsilon)\mathcal{A} \otimes \mathcal{A} = \mathcal{A} \otimes \mathbb{C} \cong \mathcal{A} \\ \cong \mathbb{C} \otimes \mathcal{A} = (\epsilon \otimes \mathrm{id})\mathcal{A} \otimes \mathcal{A} = (\epsilon \otimes \mathrm{id}) \circ \Delta(\mathcal{A}) \end{aligned}$$

[1] For more details on Hopf algebras reader is referred to [17] or [13]

For example, let $\Delta(X) = X \otimes I + I \otimes X$ and then the associativity means that

$$(\mathrm{id} \otimes \Delta) \circ \Delta(X) = (\mathrm{id} \otimes \Delta)(X \otimes I + I \otimes X) = X \otimes I \otimes I + I \otimes (X \otimes I + I \otimes X) =$$
$$= (X \otimes I \otimes I + I \otimes X \otimes I) + I \otimes I \otimes X = (\Delta \otimes \mathrm{id})(X \otimes I + I \otimes X) = (\Delta \otimes \mathrm{id}) \circ \Delta(X)$$

The algebra with a coassociative comultiplication Δ and a counit ϵ satisfying previous relations is called a coalgebra. Similarly, the cocommutative coalgebra (as commutative algebras) is a coalgebra with the comultiplication which satisfies $\sigma \circ \Delta \equiv \Delta^o = \Delta$.

Now returning to the Hopf algebra, the latter has algebra and coalgebra structures, therefore we can write the **definition of the Hopf algebra**:

The Hopf algebra $\mathcal{A}$ over $\mathbb{C}$ is a vector space such that $\mathcal{A}$ is an algebra and a coalgebra over $\mathbb{C}$ with the operations: multiplication $\mathfrak{m}$, unit i, comultiplication Δ and counit ϵ. The comultiplication Δ and the counit ϵ are homomorphisms of algebras (i.e. preserves the algebraic structure). Similarly, the multiplication $\mathfrak{m}$ and the unit i are homomorphisms of coalgebras. The Hopf algebra has also a bijective antimorphism, called antipod, $S : \mathcal{A} \to \mathcal{A}$ such that

$$\mathfrak{m} \circ (\mathrm{id} \otimes S) \circ \Delta(A) = \mathfrak{m} \circ (S \otimes \mathrm{id}) \circ \Delta(A) = i \circ \epsilon(A)$$

Futhermore one can define the *almost cocommutative Hopf algebra*: if there exists an invertible element $\mathcal{R} \in \mathcal{A} \otimes \mathcal{A}$ such that for any $X \in \mathcal{A}$

$$\mathcal{R}\Delta(X) = \Delta^o(X)\mathcal{R} \tag{2.1.11}$$

and the *almost cocommutative quasitriangular Hopf algebra*: if $\mathcal{R}$ satisfies

$$(\mathrm{id} \otimes \Delta)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{12} \tag{2.1.12}$$
$$(\Delta \otimes \mathrm{id})\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{23} \tag{2.1.13}$$

It follows from the above equations that $\mathcal{R}$ satisfies the Yang–Baxter equation:

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12} \tag{2.1.14}$$

The object $\mathcal{R}$ is called the universal R-matrix and it has an immmediate connection with the R-matrix considered above. Once one can find the universal R-matrix as the solution of the intertwining equation (2.1.11), hence this universal R-matrix in a certain representation gives the matrix R:

$$R_{12}(u_1 - u_2) = \hat{\pi}_1 \otimes \hat{\pi}_2(\mathcal{R}) \tag{2.1.15}$$

the parameters u_1, u_2 associated with evaluation representations: $\hat{\pi}_i = \rho \circ \mathrm{ev}_i$, where ρ is a representation of algebra $\mathcal{A}$, mapping from $\mathcal{A}$ to $\mathrm{End}(V)$ (V is a vector space), and ev_i is an algebra homomorphism, called the evaluation map, usually from the infinite $\mathcal{A}$ to a finite algebra. We remark that $\hat{\pi}_1, \hat{\pi}_2$ in general can be different evaluation representations.

As simple examples of the Hopf algebra we can consider the universal enveloping algebra $U(sl(2))$ of the Lie algebra $sl(2)$ and the quantum deformation of the universal enveloping algebra $U_q(sl(2))$:

A Lie algebra $sl(2)$ in the Serre-Chevalley basis consists of the generators $\{e, h, f\}$ subject to the following commutation relations:

$$[h, e] = 2e, \quad [h, f] = -2f \quad \text{and} \quad [e, f] = h \tag{2.1.16}$$

The universal enveloping algebra $U(sl(2))$ can be viewed as a set of generators t being powers of the generators e, f, h: $t = \sum_{i,j,k} c_{i,j,k} e^i h^j f^k$. The multiplication $\mathfrak{m}$ is such that for any $t', t'' \in U(sl(2))$, it can be defined as

$$\mathfrak{m}(t', t'') = \mathfrak{m}\left(\sum_{i,j,k} c'_{i,j,k} e^i h^j f^k, \sum_{i,j,k} c''_{i,j,k} e^i h^j f^k\right) = \sum_{i,j,k} c'''_{i,j,k} e^i h^j f^k$$

with $c'''_{i,j,k}$ is defined by $c'_{i,j,k}$ and $c''_{i,j,k}$.

This algebra has the following Hopf structure: for any $x, y \in U(sl(2))$

$$\Delta(x) = x \otimes I + I \otimes x, \qquad S(x) = -x, \qquad \epsilon(x) = 0$$

$$\Delta(xy) = \Delta(x)\Delta(y), \qquad S(xy) = S(y)S(x), \quad \epsilon(xy) = \epsilon(x)\epsilon(y)$$

Thus one can verify that for example Δ is a homomorphism of the algebra such that

$$[\Delta(h), \Delta(e)] = 2\Delta(e), \quad [\Delta(h), \Delta(f)] = -2\Delta(f) \quad \text{and} \quad [\Delta(e), \Delta(f)] = \Delta(h)$$

It is obvious that $\Delta^o = \Delta$, hence the universal R-matrix $\mathcal{R}$ is trivial, i.e. it reduces to the identity operator.

B Quantum deformation of the universal enveloping algebra $U_q(sl(2))$ is an associative algebra with a unit generated by $\{e, f, k, k^{-1}\}$, subject to the following relations:

$$kk^{-1} = k^{-1}k = \mathbb{I}, \tag{2.1.17}$$

$$kek^{-1} = q^2 e, \qquad kfk^{-1} = q^{-2} f, \qquad [e, f] = \frac{k - k^{-1}}{q - q^{-1}} \tag{2.1.18}$$

The $U_q(sl(2))$ algebra in the limit $q \to 1$ reduces to the universal enveloping algebra $U(sl(2))$ if we represent $k = q^h$. We understand that $q^h = I + \log(q)h + \frac{1}{2!}\log(q)^2 h^2 + ...$ with $q \in \mathbb{C}$ (not being the root of unity). Then, the commutation relations rewrite as

$$[h, e] = 2e, \quad [h, f] = -2f \quad \text{and} \quad [e, f] = \frac{q^h - q^{-h}}{q - q^{-1}} \tag{2.1.19}$$

The Hopf structure of $U_q(sl(2))$ is following:

$$\Delta(e) = e \otimes q^{-h/2} + q^{h/2} \otimes e, \qquad \Delta(f) = f \otimes q^{-h/2} + q^{h/2} \otimes f, \quad \Delta(q^h) = q^h \otimes q^h \tag{2.1.20}$$

$$S(h) = -h, \qquad S(e) = -q^{-h} e, \quad S(f) = -f q^h \text{ and } \epsilon(x) = 0 \tag{2.1.21}$$

$$\Delta(xy) = \Delta(x)\Delta(y), \qquad S(xy) = S(y)S(x), \quad \epsilon(xy) = \epsilon(x)\epsilon(y)$$

for any $x, y \in U_q(sl(2))$.

It is possible to verify that Δ,S and ϵ are homomorphisms of algebra. In this case $\Delta^o \neq \Delta$ and this provides a non-trivial universal R-matrix. Its simplest representation will be considered in the next section.

The solutions of the Yang–Baxter equation plays a central role in the theory of quantum integrable models. These solutions arise as the intertwiners of the quantum affine algebras $U_q(\hat{g})$, deformations

of affine Kac-Moody algebras, e.g. see [16]. The R-matrices in the fundamental representation as well as the universal R-matrices are found almost for all (super)algebras e.g. see [20]-[23] and [24] (or see [25] for basic examples) and the references therein. The higher spin representations of the R-matrix can be constructed using the fusion procedure [26]. In the next section we consider few basic examples of the R-matrices.

2.2 Few examples of R-matrices and associated spin chain models

In this section we will consider several examples of the R-matrices and associated with them Hamiltonians. Especially, we will write the explicit form of the R-matrices corresponding to the XXX spin 1/2 model, XXZ model and XX model. The latter model will be used in the construction of the R-matrix of the Hubbard model.

XXX model. We consider the R-matrix of the XXX spin chain, already mentioned in (2.1.4). It is a 4×4 matrix acting on $V_1 \otimes V_2$ which satisfies the Yang–Baxter equation, unitarity and regularity conditions:

$$R_{ab}(u) = u\mathbb{I}_a \otimes \mathbb{I}_b + P_{ab} \tag{2.2.1}$$

and

$$\begin{aligned} R_{ab}(u-v)R_{ac}(u)R_{bc}(v) &= R_{bc}(v)R_{ac}(u)R_{ab}(u-v) && (2.2.2)\\ R_{ab}(u)R_{ba}(-u) &= (1-u^2)\mathbb{I}_a \otimes \mathbb{I}_b && (2.2.3)\\ R_{ab}(0) &= P_{ab} && (2.2.4) \end{aligned}$$

Using the technique described in the previous section one can derive the Hamiltonian H_1 (2.1.10) corresponding to this R-matrix:

$$H_1 = \sum_{a=1}^{N} (P_{aa+1}), \quad N+1 \equiv 1 \tag{2.2.5}$$

The Hamiltonian can be rewritten in a different form using the Pauli matrices: $\sigma_1 = E^{12} + E^{21}$, $\sigma_2 = -iE^{12} + iE^{21}$ and $\sigma_3 = E^{11} - E^{22}$ and introducing the spin operators $S^i = \sigma_i/2$. Thus, the Hamiltonian takes a more familiar physical form:

$$H_1 = 2\sum_{a=1}^{N} \left(\mathbf{S}_a \mathbf{S}_{a+1} - \frac{1}{4} \right), \quad N+1 \equiv 1 \tag{2.2.6}$$

Indeed, H_1 is equivalent to the Hamiltonian of the Heisenberg spin chain with the nearest neighbours interaction in one dimension with the coupling constants $J_x = J_y = J_z$ equal 2: $H = \sum_{<i,j>}(J_x S_i^x S_j^x + J_y S_i^y S_j^y + J_z S_i^z S_j^z)$.

The R-matrix of the XXX model provides the integrability for this one-dimensional isotropic Heisenberg model. Furthermore one can construct the eigenvectors and the eigenvalues of the transfer matrix $\tau(u)$ via the Algebraic Bethe Ansatz. Hence, one finds the spectrum and the eigenfunctions of the Hamiltonian H_1. This method will be explained in one of the next chapters.

One of the properties of the R-matrix is that it has the $gl(2)$ algebra symmetry: $R_{ab}(u)\Delta_{ab}(g) = \Delta_{ab}(g)R_{ab}(u)$ with $g \in gl(2)$ and $\Delta_{ab}(g) = \pi_a(g) \otimes I_b + I_a \otimes \pi_b(g)$ where $\pi(g)$ is the fundamental

matrix representation of $gl(2)$. From this point of view, one can generalize the R-matrix on the $gl(n)$ algebra demanding accordingly the $gl(n)$ symmetry of the R-matrix.

Therefore, the $gl(n)$ R-matrix has the similar form as in (2.2.1) but it is now a $n^2 \times n^2$ matrix.

$$R_{ab}^{gl(n)}(u) = u\mathbb{I}_a \otimes \mathbb{I}_b + \sum_{i,j=1}^{n} E_a^{ij} E_b^{ji}$$

The Hamiltonian derived from this R-matrix has also the same form. It is the permutation operator as the previous one but its physical interpretation should be somehow different.

$$H_1^{gl(n)} = \sum_{a=1}^{N} \left(\sum_{i,j=1}^{n} E_a^{ij} E_{a+1}^{ji} \right), \quad N+1 \equiv 1 \tag{2.2.7}$$

Before considering another examples of R-matrices, we say few words about the diagonalization of above models. The spectrum and the eigenvector of these $gl(2)$ and $gl(n)$ spin chain Hamiltonians can be found using different techniques, such as Coordinate Bethe Ansatz, Algebraic Bethe Ansatz. The $gl(2)$ Hamiltonian was firstly diagonalized by H.Bethe using the so-called Coordinate Bethe ansatz method as we already mentioned. The algebraic approach developed by L.D.Faddeev and collaborators is applicable for any $gl(n)$ spin chains, e.g. see [27], and for any simple Lie algebra [28]. We will present the Algebraic Bethe Ansatz for the $gl(2)$ spin chain in one of the next sections on the Hubbard model 3.3.2. We remark also that the above R-matrices are constructed using the fundamental representation of the $gl(2)$ and $gl(n)$ algebras respectively. The R-matrices for higher spin representations and thus corresponding spin chains can be constructed using other representations of their universal R-matrices, or the fusion procedure [26]. For $gl(2)$ case, see [29], the Hamiltonian for any spin s is written as

$$H_s = J \sum_{a=1}^{N} Q_{2s}(\mathbf{S}_a \mathbf{S}_{a+1}), \quad N+1 \equiv 1 \tag{2.2.8}$$

where $\mathbf{S}$ is a spin s matrix and $Q_{2s}(x)$ is a polynomial function. For example, for spin $s = 1$ we have $Q_2(x) = \frac{1}{2}\left(x - x^2\right)$.

XXZ model. This example is also one of the basic solutions of the Yang–Baxter equation. As it can be seen from the title of the paragraph we will write the R-matrix which provides integrability for the one-dimensional anisotropic Heisenberg model ($J_x = J_y \neq J_z$).

In this case the R-matrix is the solution of the intertwining equation (2.1.11) for the q-deformed affine algebra $U_q(\hat{sl}(2))$. It has two parameters: q is connected with the algebraic symmetry and $u \in \mathbb{C}$ is a spectral parameter. It is also a 4×4 matrix acting on $V_1 \otimes V_2$ which satisfies the Yang–Baxter equation, unitarity and the regularity conditions:

$$R_{ab}(u) = (qu - \frac{1}{qu}) \sum_{i=1}^{2} (E_a^{ii} E_b^{ii}) + (u - \frac{1}{u}) \sum_{1 \leq i \neq j \leq 2} (E_a^{ii} E_b^{jj}) + (q - \frac{1}{q}) \sum_{1 \leq i \neq j \leq 2} (u^{\mathrm{sign}(i-j)} E_a^{ij} E_b^{ji}) \tag{2.2.9}$$

and

$$R_{ab}(u/v) R_{ac}(u) R_{bc}(v) = R_{bc}(v) R_{ac}(u) R_{ab}(u/v) \tag{2.2.10}$$

$$R_{ab}(u) R_{ba}(\frac{1}{u}) = \left(qu - \frac{1}{qu}\right)\left(\frac{q}{u} - \frac{u}{q}\right) \mathbb{I}_a \otimes \mathbb{I}_b \tag{2.2.11}$$

$$R_{ab}(1) = \left(q - \frac{1}{q}\right) P_{ab} \tag{2.2.12}$$

Here the R-matrix is written in the rational form which explains the appearance of u/v in the Yang–Baxter equation and the limit $u \to 1$ in the regularity condition. One can pass to the trigonometric form using the transformation $u = \exp(i\lambda)$ and $q = \exp(i\gamma)$. In addition one can find the symmetric R-matrix performing the gauge transformation: $R^s_{ab}(u) = U_a(\phi_a)U_b(\phi_b)R_{ab}(u)U_b^{-1}(\phi_b)U_a^{-1}(\phi_a)$ with $U(\phi) = E^{11} + \exp(i\phi)E^{22}$ and $\phi_a - \phi_b = i\log(u)$. Therefore, explicitly we have the trigonometric symmetric R-matrix (we renormalize the expression of $R^s_{ab}(u)$ by $2i$):

$$R^s_{12}(\lambda) = \begin{pmatrix} \sin(\lambda+\gamma) & 0 & 0 & 0 \\ 0 & \sin(\lambda) & \sin(\gamma) & 0 \\ 0 & \sin(\gamma) & \sin(\lambda) & 0 \\ 0 & 0 & 0 & \sin(\lambda+\gamma) \end{pmatrix} \tag{2.2.13}$$

The XXZ Hamiltonian is derived from this symmetric trigonometric R-matrix using (2.1.10):

$$H_1 = \frac{1}{2}\sum_{a=1}^{N}\left(\sigma^1_a\sigma^1_{a+1} + \sigma^2_a\sigma^2_{a+1} + \cos(\gamma)\sigma^3_a\sigma^3_{a+1}\right) + \frac{N}{2}\cos(\gamma), \quad N+1 \equiv 1 \tag{2.2.14}$$

The origin of this R-matrix can be viewed as a direct solution of (2.1.11) for the fundamental representation of the q-deformed affine algebra $U_q(\hat{sl}(2))$. Moreover, one can generalize the expression (2.2.9) of the R-matrix to the $U_q(\hat{sl}(n))$ algebra. Thus, the $gl(n)$ XXZ R-matrix is

$$R^{gl(n)}_{ab}(u) = (qu - \frac{1}{qu})\sum_{i=1}^{n}(E^{ii}_aE^{ii}_b) + (u - \frac{1}{u})\sum_{1\le i\ne j\le n}(E^{ii}_aE^{jj}_b) + (q - \frac{1}{q})\sum_{1\le i\ne j\le n}(u^{\mathrm{sign}(i-j)}E^{ij}_aE^{ji}_b) \tag{2.2.15}$$

Corresponding XXZ-like $gl(n)$ Hamiltonian is derived from the $gl(n)$ R-matrix using (2.1.10):

$$H^{gl(n)}_1 = \sum_{a=1}^{N}\left((q+1/q)\sum_{i=1}^{n}E^{ii}_aE^{ii}_{a+1} + 2\sum_{1\le i\ne j\le n}E^{ij}_aE^{ji}_{a+1} + (q-1/q)\sum_{1\le i\ne j\le n}(-1)^{\mathrm{sign}(i-j)}E^{ii}_aE^{jj}_{a+1}\right), \quad N+1 \equiv 1 \tag{2.2.16}$$

The diagonalization of above models can be performed by the Algebraic Bethe Ansatz [30] which is a powerful method for solving the spin chains in fundamental and any spin representation. In general, the XXX and XXZ spin chains based on $gl(n)$ and $gl(n|m)$ algebras can be all treated in the same schema of Algebraic Bethe Ansatz, e.g. see [33].

XX model. The last example of the R-matrix in this section is closely connected with the construction of Shastry's R-matrix (the R-matrix of the Hubbard model). Here we consider only the $gl(2)$ case, other XX models will be treated in the next sections related to the Hubbard model and its generalizations. The model which we will obtain from the R-matrix in this example also enters in the class of the one-dimensional anisotropic Heisenberg models, more precisely $J_x = J_y, J_z = 0$. The R-matrix of the XX model satisfying the Yang–Baxter equation, unitarity and the regularity conditions is

$$R_{ab}(u) = \Sigma_{ab}P_{ab} + \Sigma_{ab}\sin u + (P_{ab} - \Sigma_{ab}P_{ab})\cos u \tag{2.2.17}$$

where P_{ab} is the permutation matrix and $\Sigma_{ab} = E^{11}_aE^{22}_b + E^{22}_aE^{11}_b$.

$$R_{ab}(u-v)R_{ac}(u)R_{bc}(v) = R_{bc}(v)R_{ac}(u)R_{ab}(u-v) \tag{2.2.18}$$

$$R_{ab}(u)R_{ba}(-u) = \cos^2(u)\mathbb{I}_a \otimes \mathbb{I}_b \tag{2.2.19}$$

$$R_{ab}(0) = P_{ab} \tag{2.2.20}$$

The Hamiltonian corresponding to this R-matrix is:

$$H_1 = \sum_{a=1}^{N} \left(\Sigma_{aa+1} P_{aa+1}\right), \quad N+1 \equiv 1 \tag{2.2.21}$$

In terms of the Pauli matrices the Hamiltonian can be rewritten as

$$H_1 = \frac{1}{2} \sum_{a=1}^{N} \left(\sigma_a^1 \sigma_{a+1}^1 + \sigma_a^2 \sigma_{a+1}^2\right), \quad N+1 \equiv 1 \tag{2.2.22}$$

This Hamiltonian represent a spin chain without z-direction interaction, the XX spin chain, but on the other hand it can be transformed to the Hamiltonian of free electrons using the so-called Jordan-Wigner transformation (see below). Thus, the Hamiltonian H_1 can be written as

$$H_1 = \sum_{a=1}^{N-1} \left(c_a^\dagger c_{a+1} + c_{a+1}^\dagger c_a\right) + \text{"twist"}, \tag{2.2.23}$$

where $c, c^\dagger$ are the fermionic operators, $\{c_a^\dagger, c_b\} = \delta_{ab}$. The term marked as "twist" is related to the periodic boundary conditions of the initial Hamiltonian and the non-locality property of the Jordan-Wigner transformation of bosons to fermions.

The Jordan-Wigner transformation is a map between the $n \times n$ matrices E^{ij} and anticommuting operators. The matrices E^{ij} can have bosonic or fermionic gradation. In our case we want to map 2×2 Pauli matrices into the fermionic operators. It is a non-local mapping due to the bosonic structure of the Pauli matrices:

$$c_a^\dagger = \sigma_a^- \prod_{k=a+1}^{N} \sigma_k^3, \quad c_a = \sigma_a^+ \prod_{k=a+1}^{N} \sigma_k^3 \tag{2.2.24}$$

$$\sigma_a^+ = c_a \prod_{k=a+1}^{N} (1 - 2n_k), \quad \sigma_a^- = c_a^\dagger \prod_{k=a+1}^{N} (1 - 2n_k) \tag{2.2.25}$$

with $\sigma^\pm = \frac{1}{2}(\sigma^1 \pm i\sigma^2)$ and $n_k = c_k^\dagger c_k$.

At the same time, the Jordan-Wigner transformation preserves the gradation in the sense that the matrices E^{ij} with the fermionic gradation are locally mapped on the anticommuting operators.

In general case, the Jordan-Wigner transformation is very important and is highly exploited in the construction of the Hubbard model. We will give more details on it in the following chapters.

Another property one can remark is that the XX model and its R-matrix can be derived from the XXZ model and its symmetric trigonometric R-matrix taking the limit $q \to i$ (or $\gamma \to \pi/2$ in (2.2.13)).

Chapter 3

Hubbard model

This chapter is devoted to the one-dimensional Hubbard model - integrable model. In the begining we review its origin in condensed matter physics as a model of interacting electrons on the lattice in three dimensions and then we restrict our consideration on the one-dimensional model. The one-dimensional integrable Hubbard model and its R-matrix being a particular object in the hierarchy of R-matrices are considered in the section 3.2. The Coordinate Bethe Ansatz and real solutions of the Lieb-Wu equations are presented in the sections 3.3 and 3.4. In the end of the chapter we review several applications of the one-dimensional Hubbard model in the AdS/CFT duality.

3.1 Origin of the Hubbard model

In condensed matter physics, at its early development much attention has been given to the theory of correlation effects in the free electron gas which served as a model for conduction bands of metals and alloys. However, transition and rare-earth metals have in addition to their conduction bands also partly filled d- or f-bands. The latter gives rise to the characteristic properties of these metals. Therefore the correlation effects of these metals are highly influenced by these d- and f- bands and it appeared that the free gas approximation did not provide a good model for them. Thus, one required another theory which could take into account the atomistic nature of solid. Moreover, it was found experimentally that d-electrons exhibit behaviours characteristic of both the ordinary band model and the atomic model.

Starting from 1963 J.Hubbard in a series of papers [39] introduced an approximate model for the interaction of electrons in the narrow energy bands. This model is the simplest generalization beyond the free electrons theory. It contains a minimum of necessary properties in order to get the band and the atomic descriptions at the same time. The introduced model allows to capture many physical features of solids.

In the following we will show how to derive the Hubbard Hamiltonian [1]. The starting point in order to derive the Hubbard model Hamiltonian is to consider the electrons on the lattice of ions with the Coulomb repulsion between the electrons. We assume a static three-dimensional lattice, static is due to the fact that the ions are much heavier than the electrons. Thus, we consider N electrons in the periodic lattice potential $V_0(\mathbf{x})$ with the Coulomb interaction $V_C(\mathbf{x}) = e^2/\mathbf{x}^2$. Therefore, the dynamics can be described by the Hamiltonian:

[1] The description is based mostly on [55]

$$H = \sum_{i=1}^{N} \left(\frac{\mathbf{p}_i^2}{2m} + V_0(\mathbf{x}_i) \right) + \sum_{1 \le i < j \le N} V_C(\mathbf{x}_i - \mathbf{x}_j) \tag{3.1.1}$$

This Hamiltonian as a many-body system with interactions is very difficult to solve exactly. Thus, one needs to use some approximations in order to achieve some results. Mean field theory approach gives the first step in the deriving of the Hubbard Hamiltonian. The main idea is to replace all interactions to one-body average or effective interactions. More precisely the approximation is based on adding an one-particle auxiliary potential $V_a(\mathbf{x})$ such that it modifies the one-particle Hamiltonian and reduces the multi-particle interaction:

$$H = \sum_{i=1}^{N} h(\mathbf{x}_i, \mathbf{p}_i) + \sum_{1 \le i < j \le N} U(\mathbf{x}_i, \mathbf{x}_j) \tag{3.1.2}$$

with $h(\mathbf{x}, \mathbf{p}) = \frac{\mathbf{p}^2}{2m} + V_0(\mathbf{x}) + V_a(\mathbf{x})$ and $U(\mathbf{x}_i, \mathbf{x}_j) = V_C(\mathbf{x}_i - \mathbf{x}_j) - \frac{1}{N-1}(V_a(\mathbf{x}_i) + V_a(\mathbf{x}_j))$ [2].

Mean field approximation means to choose appropriately $V_a(\mathbf{x})$ such that matrix elements of $U(\mathbf{x}_i, \mathbf{x}_j)$ on the eigenstates of $h(\mathbf{x}, \mathbf{p})$ become 'small'. The expression (3.1.2) for the moment contains no approximations.

Now using the "second-quantification" formalism one can derive the Hamiltonian in terms of the creation/annihilation operators $c^\dagger_{\alpha i,\sigma}$ and $c_{\alpha i,\sigma}$. These operators create and destroy the electron in the Wannier state[3] with the spin σ on the lattice site i and in the band α. Therefore, we can rewrite (3.1.2) as

$$H = \sum_{\alpha,i,j,\sigma} t^{\alpha}_{ij} c^\dagger_{\alpha i,\sigma} c_{\alpha j,\sigma} + \frac{1}{2} \sum_{\alpha,\beta,\gamma,\delta} \sum_{\substack{i,j,k,l \\ \sigma,\tau}} U^{\alpha\beta\gamma\delta}_{ijkl} c^\dagger_{\alpha i,\sigma} c^\dagger_{\beta j,\tau} c_{\gamma k,\tau} c_{\delta l,\sigma} \tag{3.1.3}$$

where hopping matrix elements t^{α}_{ij} and interaction parameters $U^{\alpha\beta\gamma\delta}_{ijkl}$ are written as some integrals of the operators $h(\mathbf{x}, \mathbf{p})$ and $U(\mathbf{x}_i, \mathbf{x}_j)$ respectively on the Wannier states. Their expressions are not important in the following and we do not write them.

We point out again that the Hamiltonian (3.1.3) is completely equivalent to (3.1.2). We can obtain the Hubbard model taking into account several approximations:

1. small range of Coulomb interaction $\Leftrightarrow U^{\alpha\beta\gamma\delta}_{ijkl} \Longrightarrow U^{\alpha\beta\gamma\delta}_{iiii} = U$
2. one band interation $\Leftrightarrow \alpha = 1$
3. isotropic nearest neighbours approximation $\Leftrightarrow t^{\alpha}_{<i,j>} = -t$, the rest is zero

Finally, the Hamiltonian (3.1.3) reduces to

$$H = -t \sum_{<i,j>} \sum_{\sigma=\uparrow,\downarrow} c^\dagger_{i,\sigma} c_{j,\sigma} + U \sum_i n_{i,\uparrow} n_{i,\downarrow} \tag{3.1.4}$$

with $n_{i,\sigma} = c^\dagger_{i,\sigma} c_{i,\sigma}$ and here the symbol $< i, j >$ means the ordered summation over the nearest neighbours particles.

[2] It can be easily seen that $\sum_{1 \le i < j \le N} (v_i + v_j) = (N-1) \sum_{i=1}^{N} v_i$

[3] Wannier states are a complete set of orthogonal functions used in solid-state physics. Its most common definition is as follows: let $\psi_{\mathbf{k}}(\mathbf{r})$ be the Bloch state of a single band in a crystal, then the Wannier state is defined by $\phi_{\mathbf{R}}(\mathbf{r}) = \frac{1}{\sqrt{N}} \sum_{\mathbf{k}} e^{-i\mathbf{k}\cdot\mathbf{R}} \psi_{\mathbf{k}}(\mathbf{r})$. Here $\mathbf{R}$ is any lattice vector (e.g. Bravais lattice vector), N is a number of primitive cells in the crystal, the sum on $\mathbf{k}$ is for all $\mathbf{k}$ in the Brillouin zone.

The Hamiltonian (3.1.4) is known as the Hubbard model Hamiltonian. As one can see there are four possible electronic configurations on the site:

- $|0>$ - vacuum state (no electrons),
- $|\uparrow>$ - electron with spin up,
- $|\downarrow>$ - electron with spin down and
- $|\updownarrow>$ - double occupied site.

The potential term of the Hamiltonian in this base can be represented as a diagonal matrix, it counts how many double occupied sites are in the eigenfunction. Similarly, the kinetic term move the electrons from one site to a nearest one on the lattice without changing the spin.

In the limit $U \to 0$ the Hamiltonian corresponds to the free electrons model. Whereas at the half-filling (the number of electrons N equals the number of sites in the lattice) if $U \to \infty$ one can get the Heisenberg spin chain.

The Hubbard model plays an essential role in several topics of condensed matter physics. In spite of its more or less simple definition, the Hubbard model exhibit various phenomena including the metal-insulator transition, ferrimagnetism, ferromagnetism, Tomonaga-Luttinger liquid, and the superconductivity (see [38, 40, 41, 42, 47, 56, 57]). It also plays a role in the chemistry of aromatic compounds e.g. benzene [45, 46].

There exists also a more global point of view[4] such that the importance of the Hubbard model may be understood from the philosophy of "universality". It is believed that a non-trivial physical phenomena found in a suitable idealized model can also be found in other systems in the same "universality class" as the idealized model. A similar situation is for the Ising model for classical spin systems which is too simple to be a realistic model of magnetic materials, for example. But it has turned out to be extremely important and useful in developing various notions and techniques in statistical physics of many degrees of freedom. It is expected that the universality class is often large and rich enough to contain various realistic systems with complicated details which are ignored in the idealized model. As for strongly interacting electron systems, the Hubbard model is regarded as one of the most promising candidates for an idealized model to be used in the search of possible universality classes.

However, in spite of all fascinating results discovered in the Hubbard model, few **exact** results are known for the two or three-dimensional Hubbard model. Nevertheless, these models are still actively investigated. In contrast, the one-dimensional Hubbard model has a distinctive feature: it is *integrable*. In the next section we will treat the one-dimensional Hubbard model and discuss in details its integrability.

3.2 1D Hubbard model as an integrable system

Over the years the one-dimensional Hubbard model has become very important in several topics of condensed matter physics including the one-dimensional conductors, the high-T_c superconductivity, but also in mathematical physics. Despite of its appealing conceptual simplicity, rigorous results for the Hubbard model are rare. The dimension of the underlying lattice is a crucial parameter. However, there are two important theorems which are valid for an arbitrary lattice dimension, due to Y.Nagaoka [43] and to E.H.Lieb [47]. Some simplifications occur in the limit of the infinite lattice

[4] mostly based on [57]

dimension [44]. Nevertheless, exact results have been obtained only for the one-dimensional lattice. The complete set of eigenfunctions of the Hubbard Hamiltonian is known only for this case. The one-dimensional Hubbard model was solved by E.H.Lieb and F.Y.Wu [48].

The Hubbard Hamiltonian in one dimension under the periodic conditions is written as

$$\mathcal{H}_{Hub} = -\sum_{i=1}^{L}\sum_{\sigma=\uparrow,\downarrow}\left(c^{\dagger}_{i,\sigma}c_{i+1,\sigma} + c^{\dagger}_{i+1,\sigma}c_{i,\sigma}\right) + \frac{U}{4}\sum_{i=1}^{L}(1-2n_{i,\uparrow})(1-2n_{i,\downarrow}), \quad L+1 \equiv 1 \tag{3.2.1}$$

We remark that in the literature the Hubbard Hamiltonian sometimes appears in a slightly modified form: $H = -\sum_{i=1}^{L}\sum_{\sigma=\uparrow,\downarrow}\left(c^{\dagger}_{i,\sigma}c_{i+1,\sigma} + c^{\dagger}_{i+1,\sigma}c_{i,\sigma}\right) + U\sum_{i=1}^{L} n_{i,\uparrow}n_{i,\downarrow}$. These Hamiltonians mutually commute and thus have the same eigenvectors. Here we will consider the Hubbard Hamiltonian in the form (3.2.1).

In the next subsections we present the integrability structure of the Hubbard model and its symmetries.

3.2.1 Shastry's R-matrix

Shastry's R-matrix (the R-matrix of the Hubbard model) was firstly introduced by B.S.Shastry [58] and E.Olmedilla et al. [60] (by coupling two R-matrices of the XX model with U-interaction term). The proof of the Yang–Baxter relation was given later by M.Shiroishi and M.Wadati [61].

Notations we use here are following: E^{ij}_a are $gl(1|1)$ elementary matrices acting non-trivially on the graded vector space V_a in the tensor space $V_1 \otimes ... \otimes V_N$. The gradation is given by $[E^{ij}] = [i] + [j]$

$$[j] = \begin{cases} 0 & \text{for} \quad j = 1 \\ 1 & \text{for} \quad j = 2. \end{cases} \tag{3.2.2}$$

The matrices E^{ij}_a satisfy the commutation relations:

$$\left[E^{ij}_a, E^{kl}_b\right]_g = \delta_{ab}\left(\delta_{jk}E^{il}_a - (-1)^{([i]+[j])([k]+[l])}\delta_{il}E^{kj}_a\right) \tag{3.2.3}$$

with the graded commutator defined as $[A,B]_g = AB - (-1)^{[A][B]}BA$

One should also define the multiplication for the tensor product by $(a\otimes b)(c\otimes d) = (-1)^{[b][c]}(ac\otimes bd)$. For example, introducing the graded permutation operator

$$P_{ab} = \sum_{i,j=1,2}(-1)^{[j]}E^{ij}_a E^{ji}_b,$$

one can verify that $P_{ab}P_{ab} = \mathbb{I}_a \otimes \mathbb{I}_b$. Thus, hereafter the multiplication is considered with the gradation factor.

$gl(1|1)$ XX model. Before speaking about Shastry's R-matrix it is useful to introduce the R-matrix of $gl(1|1)$ XX model. We have already considered $gl(2)$ XX model and its R-matrix in previous chapter (2.2.17), but in this case we should add the gradation to matrices. Hence, the R-matrix of $gl(1|1)$ XX model acting on the tensor product $V_1 \otimes V_2$ is given by

$$R_{12}(\lambda) = \Sigma_{12}\,P_{12} + \Sigma_{12}\,\sin\lambda + (P_{12} - \Sigma_{12}P_{12})\,\cos\lambda \tag{3.2.4}$$

where P_{12} is the graded permutation operator, $\Sigma_{ab} = E_a^{11}E_b^{22} + E_a^{22}E_b^{11}$ and $\lambda \in \mathbb{C}$ is the spectral parameter.

This R-matrix obeys the graded Yang–Baxter equation, is unitary and regular :

$$\begin{aligned} R_{ab}(u-v)R_{ac}(u)R_{bc}(v) &= R_{bc}(v)R_{ac}(u)R_{ab}(u-v) && (3.2.5)\\ R_{ab}(u)R_{ba}(-u) &= \cos^2(u)\mathbb{I}_a \otimes \mathbb{I}_b && (3.2.6)\\ R_{ab}(0) &= P_{ab} && (3.2.7) \end{aligned}$$

Furthermore one can introduce a matrix C

$$C = E^{11} - E^{22} \tag{3.2.8}$$

There are several connections between C and Σ matrices which will be exploited in generalizations of the XX model's R-matrix:

$$\Sigma_{ab} = \frac{1}{2}\left(\mathbb{I}_{ab} - C_a C_b\right). \tag{3.2.9}$$

There are some intertwining properties with $R_{ab}(u)$:

$$\begin{aligned} R_{ab}(-u)C_b &= C_a R_{ab}(u) && (3.2.10)\\ R_{ab}(u)C_aC_b &= C_aC_bR_{ab}(u) && (3.2.11) \end{aligned}$$

In addition to the graded Yang–Baxter equation we have also the so-called decorated Yang–Baxter equation:

$$R_{ab}(u+v)C_aR_{ac}(u)R_{bc}(v) = R_{bc}(v)R_{ac}(u)C_aR_{ab}(u+v) \tag{3.2.12}$$

The Hamiltonian which corresponds to this R-matrix is similar to the one in (2.2.21) and in terms of E^{ij} super-matrices it is written as

$$H_{XX} = \sum_{a=1}^{N}\left(-E_a^{12}E_{a+1}^{21} + E_a^{21}E_{a+1}^{12}\right), \quad N+1 \equiv 1 \tag{3.2.13}$$

Due to the anticommuting properties of E^{ij} super-matrices we get the free electrons model Hamiltonian: $E_a^{21} = c_a^\dagger$ and $E_a^{12} = c_a$

$$H_{XX} = \sum_{a=1}^{N}\left(c_a^\dagger c_{a+1} + c_{a+1}^\dagger c_a\right), \quad N+1 \equiv 1 \tag{3.2.14}$$

where $c, c^\dagger$ are the fermionic operators, $\{c_a^\dagger, c_b\} = \delta_{ab}$.

This finishes the preliminary part and we are ready to consider the Hubbard model.

Hubbard model. The R-matrix of the Hubbard model is obtained by coupling the R-matrices $R_{12}^{\uparrow}(\lambda)$ and $R_{12}^{\downarrow}(\lambda)$ of two independent XX models, the coupling function is related to the constant U of the Hubbard model under consideration. More details on this construction can be found in [58],[55]. The R-matrix of the Hubbard model acts on the tensor product of four graded vector spaces: $V_{1\uparrow} \otimes V_{1\downarrow} \otimes V_{2\uparrow} \otimes V_{2\downarrow}$ and it can be written in the following form

$$R_{12}^{\uparrow\downarrow}(\lambda_1,\lambda_2) = R_{12}^{\uparrow}(\lambda_{12})\,R_{12}^{\downarrow}(\lambda_{12}) + \frac{\sin(\lambda_{12})}{\sin(\lambda'_{12})}\,\tanh(h'_{12})\,R_{12}^{\uparrow}(\lambda'_{12})\,C_{\uparrow 1}\,R_{12}^{\downarrow}(\lambda'_{12})\,C_{\downarrow 1} \tag{3.2.15}$$

where $\lambda_{12} = \lambda_1 - \lambda_2$ and $\lambda'_{12} = \lambda_1 + \lambda_2$. In the same way, $h'_{12} = h(\lambda_1) + h(\lambda_2)$ and the function $h(\lambda)$ is such that

$$\sinh(2h) = -\frac{U}{4}\,\sin(2\lambda)\,. \tag{3.2.16}$$

we point out here that our definition of the function $h(\lambda)$ is slightly different from the standard one: $\sinh(2h) = U\,\sin(2\lambda)$.

The definition of the R-matrix (3.2.15) as a coupling of two fermionic XX models is similar to the R-matrix of bosonic Hubbard model introduced by B.S.Shastry in [58], [59]. However, the fermionic Hubbard model R-matrix also appears in a different form introduced in [60] or see the appendix of [69]. These R-matrices are related by gauge transformations. Hereafter we use only the definition of the R-matrix given in (3.2.15).

Note that the site 1 for $R_{12}^{\uparrow\downarrow}$ is composed from the tensor product of the site "$1\uparrow$" appearing in the matrix $R_{12}^{\uparrow}$ by the site "$1\downarrow$" which is in the matrix $R_{12}^{\downarrow}$. This is obviously the same for any site we will consider in the following. It means explicitly that the spin chain is composed as $1\uparrow, 1\downarrow, 2\uparrow, 2\downarrow$ $...L\uparrow, L\downarrow$.

The R-matrix (3.2.15) is symmetric, regular and satisfies the unitary relation:

$$R_{12}^{\uparrow\downarrow}(\lambda_1,\lambda_2)P_{12}^{\uparrow\downarrow}R_{12}^{\uparrow\downarrow}(\lambda_2,\lambda_1)P_{12}^{\uparrow\downarrow} = \Big(\cos^4\lambda_{12} - \frac{\sin^2\lambda_{12}}{\sin^2\lambda'_{12}}\tanh^2 h'_{12}\Big)\mathbb{I}_{12}^{\uparrow\downarrow} \tag{3.2.17}$$

where $P_{12}^{\uparrow\downarrow} = P_{12}^{\uparrow}P_{12}^{\downarrow} \in V_{1\uparrow}\otimes V_{1\downarrow}\otimes V_{2\uparrow}\otimes V_{2\downarrow}$ is the permutation operator.

Moreover, when the relation (3.2.16) holds, the R-matrix (3.2.15) satisfies the graded Yang–Baxter equation:

$$R_{12}^{\uparrow\downarrow}(\lambda_1,\lambda_2)\,R_{13}^{\uparrow\downarrow}(\lambda_1,\lambda_3)\,R_{23}^{\uparrow\downarrow}(\lambda_2,\lambda_3) \;\;=\;\; R_{23}^{\uparrow\downarrow}(\lambda_2,\lambda_3)\,R_{13}^{\uparrow\downarrow}(\lambda_1,\lambda_3)\,R_{12}^{\uparrow\downarrow}(\lambda_1,\lambda_2)\,. \tag{3.2.18}$$

Being equipped with an R-matrix with all required properties, we can proceed to define the corresponding quantum integrable system, by performing the following steps: monodromy matrix, transfer matrix and the Hamiltonian.

The L-site monodromy matrix is given

$$T_{a<b_1\ldots b_L>}(\lambda) = R_{ab_1}^{\uparrow\downarrow}(\lambda,0)\ldots R_{ab_L}^{\uparrow\downarrow}(\lambda,0) \tag{3.2.19}$$

and its transfer matrix is the (super)trace in the auxiliary space:

$$t(\lambda) = tr_a T_{a<b_1\ldots b_L>}(\lambda)\,. \tag{3.2.20}$$

Then the Hubbard Hamiltonian reads

$$H = -\frac{d}{d\lambda}\ln t(\lambda)\Big|_{\lambda=0} = -\sum_{x=1}^{L} H_{x,x+1}, \quad L+1\equiv 1 \tag{3.2.21}$$

with

$$H_{x,x+1} = -(\Sigma P)_{\uparrow\, x,x+1} - (\Sigma P)_{\downarrow\, x,x+1} + \frac{U}{4}\,C_{\uparrow x}\,C_{\downarrow x}\,, \tag{3.2.22}$$

The notation $\mathcal{O}_{\uparrow\, x,x+1}$ means that the operator $\mathcal{O}$ acts non-trivially in the parts $x\uparrow$ and $(x+1)\uparrow$ only. It acts as identity on all the sites different from x and $x+1$ and also on the parts $x\downarrow$ and $(x+1)\downarrow$ of sites x and $x+1$. Explicitly, one has

$$(\Sigma P)_{\alpha\, x,x+1} = E^{12}_{\alpha\, x}E^{21}_{\alpha\, x+1} + E^{21}_{\alpha\, x}E^{12}_{\alpha\, x+1} \tag{3.2.23}$$

with $\alpha = \uparrow$ or $\downarrow$.

Due to the fermionic structure of $E^{12}_{\alpha\, x}$ and $E^{21}_{\alpha\, x}$ matrices, we can replace them directly by the fermionic operators $c^{\dagger}_{\alpha\, x}$ and $c_{\alpha\, x}$, therefore we get (3.2.1).

3.2.2 Symmetries of the Hubbard Hamiltonian

The Hamiltonian (3.2.1) preserves the number of involved particles and therefore has a symmetry: $[\mathcal{H}_{Hub}, \sum_{i=1}^{L} n_{\alpha\, i}] = 0$. In addition it was found also that Hubbard model for even L has $so(4)$ algebra symmetry realized in terms of the fermionic operators $c^\dagger$,c: first $su(2)$ Lie algebra is written as

$$\zeta = \sum_{i=1}^{L} c^\dagger_{\uparrow i} c_{\downarrow i}, \quad \zeta^\dagger = \sum_{i=1}^{L} c^\dagger_{\downarrow i} c_{\uparrow i}, \quad \zeta_z = \frac{1}{2}\sum_{i=1}^{L}(n_{\downarrow i} - n_{\uparrow i}), \tag{3.2.24}$$

$$[\zeta, \zeta^\dagger] = -2\zeta_z, \quad [\zeta_z, \zeta] = -\zeta, \quad [\zeta_z, \zeta^\dagger] = \zeta^\dagger. \tag{3.2.25}$$

and so-called η-pairing symmetry:

$$\eta = \sum_{i=1}^{L}(-1)^i c_{\uparrow i} c_{\downarrow i}, \quad \eta^\dagger = \sum_{i=1}^{L}(-1)^i c^\dagger_{\downarrow i} c^\dagger_{\uparrow i}, \quad \eta_z = \frac{1}{2}\sum_{i=1}^{L}(n_{\downarrow i} + n_{\uparrow i}) - \frac{L}{2}, \tag{3.2.26}$$

$$[\eta, \eta^\dagger] = -2\eta_z, \quad [\eta_z, \eta] = -\eta, \quad [\eta_z, \eta^\dagger] = \eta^\dagger. \tag{3.2.27}$$

For the odd number of sites L the η-pairing symmetry disappears and the Hamiltonian has only $su(2)$ symmetry. The $so(4)$ symmetry plays an important role in the proof of the completeness of the Bethe ansatz solution for the Hubbard model.

Moreover in [66][5] it was found that the Hamiltonian in the limit of the infinite chain $L \to \infty$ has the infinite extra symmetry $Y(su(2)) \oplus Y(su(2))$. It was verified that Yangian generators (in Drinfeld's second realization) written in terms of the creation and annihilation operators $c^\dagger$,c commute with the infinite chain Hamiltonian. The Yangian algebra [6] can be presented by generators Q_0^m and Q_1^m where the generators Q_0^m form the finite Lie algebra and Q_1^m generates the extention to the Yangian algebra. These generators should satisfy the commutation relations:

$$[Q_0^n, Q_0^m] = f^{nmk} Q_0^k, \quad [Q_0^n, Q_1^m] = f^{nmk} Q_1^k \tag{3.2.28}$$

where f^{nmk} - structure constants of the finite algebra. Moreover, the generators also should satisfy certain additional closing relations. We will not write them (one can find them in [17] for example)

For the Hubbard model, the role of Q_0^k of $Y(su(2)) \oplus Y(su(2))$ plays two $su(2)$ symmetry generators $\zeta^\dagger$,ζ and ζ_z and $\eta^\dagger$,η and η^z. In notation of [66] the Yangian generators $Y(su(2))$: level zero generators are

$$E_0 = \sum_i \mathcal{E}_i^0, \quad F_0 = \sum_i \mathcal{F}_i^0, \quad H_0 = \sum_i \mathcal{H}_i^0, \tag{3.2.29}$$

here the sums $\sum_i$ are infinite and the notations are $\mathcal{E}_i^n = c^\dagger_{\uparrow i} c_{\downarrow i+n}$, $\mathcal{F}_i^n = c^\dagger_{\downarrow i} c_{\uparrow i+n}$ and $\mathcal{H}_i^n = c^\dagger_{\uparrow i} c_{\uparrow i+n} - c^\dagger_{\downarrow i} c_{\downarrow i+n}$.

Then, the level one generators are:

$$\begin{aligned} E_1 &= \sum_i \left(\mathcal{E}_i^1 - \mathcal{E}_i^{-1}\right) - \frac{U}{2}\sum_{i<j}\left(\mathcal{E}_i^0 \mathcal{H}_j^0 - \mathcal{E}_j^0 \mathcal{H}_i^0\right) \\ F_1 &= \sum_i \left(\mathcal{F}_i^1 - \mathcal{F}_i^{-1}\right) + \frac{U}{2}\sum_{i<j}\left(\mathcal{F}_i^0 \mathcal{H}_j^0 - \mathcal{F}_j^0 \mathcal{H}_i^0\right) \\ H_1 &= \sum_i \left(\mathcal{H}_i^1 - \mathcal{H}_i^{-1}\right) + U\sum_{i<j}\left(\mathcal{E}_i^0 \mathcal{F}_j^0 - \mathcal{E}_j^0 \mathcal{F}_i^0\right) \end{aligned} \tag{3.2.30}$$

[5] On the Yangian symmetry of the Hubbard model R-matrix see [67] and [80]

[6] for more details on Yangians see [17], [64] or [65] and the references therein.

The second Yangian generators $Y(su(2))$ E_i',F_i',H_i' with $i = 0, 1$ can be obtained from the previous ones by transformation:

$$c_{\downarrow i} \to c_{\downarrow i}, \quad c_{\uparrow i} \to c^\dagger_{\uparrow i}, \quad U \to -U$$
$$c^\dagger_{\downarrow i} \to c^\dagger_{\downarrow i}, \quad c^\dagger_{\uparrow i} \to c_{\uparrow i}$$

they commute with the first Yangian generators $Y(su(2))$.

The generators (3.2.29) and (3.2.30) satisfy the commutation relations (3.2.28) with the constants f^{nmk7} such as

$$f^{EFH} = -f^{FEH} = 1, \quad f^{HEE} = -f^{EHE} = -f^{HFF} = f^{FHF} = 2 \text{ and the rest is zero} \tag{3.2.31}$$

and they are constrained by deformed Serre relations which can be found in [66].

Thus, it can be verified that the generators E_i, F_i, H_i and E_i', H_i', F_i' with $i = 0, 1$ form the $Y(su(2)) \oplus Y(su(2))$ symmetry and commutes with the Hubbard Hamiltonian (3.2.1) when both boundaries are sent to infinity, denoted by $\mathcal{H}^{(\infty)}_{Hub}$. The sums $\sum_{i=1}^{L}$ in (3.2.1) should be replaced by $\sum_{i=-L}^{L}$ with $L \to \infty$.

$$[\mathcal{H}^{(\infty)}_{Hub}, \mathcal{J}_i] = 0 \quad \text{for} \quad \mathcal{J}_i = E_i, F_i, H_i, E_i', H_i', F_i' \tag{3.2.32}$$

3.3 Coordinate Bethe Ansatz (CBA) solution

In 1968 the exact solution of the one-dimensional Hubbard model by the Coordinate Bethe ansatz was presented in the work of E.H.Lieb and F.Y.Wu [48]. The origin of the method goes back to the work of H.Bethe [11], the seminal articles of C.N.Yang [35] and M.Gaudin [37]. In the work [48], E.H.Lieb and F.Y.Wu derived a system of non-linear equations (Bethe equations) which are often called in the literature the Lieb-Wu equations and calculated the ground state energy of the system. They showed also that the model at the half-filling limit is an insulator for an arbitrary positive value of the coupling U. In other words, the half-filled model undergoes a Mott transition at the critical coupling U = 0. The solutions of the Lieb-Wu equations parametrize the spectrum and the eigenvectors of the Hubbard model, encoding the complete information about the model. However, in the general case on the finite lattice they are not explicitly known.

In this section we reproduce in details some results of E.H.Lieb and F.Y.Wu using the Coordinate Bethe ansatz method. We divide the "nested" Bethe ansatz in two parts - "levels", in the first part (level 1) we look for the eigenfunctions and the eigenvalues of the Hubbard Hamiltonian and in the second part (level 2) we, contralily to E.H.Lieb and F.Y.Wu, use the Algebraic Bethe Ansatz approach for the auxiliary problem and we diagonalize the transfer matrix which generates the level 2 Hamiltonians. The Algebraic Bethe Ansatz was developed by the Leningrad school in 80-s by L.D.Faddeev and collaborators. Lectures on the Algebraic Bethe Ansatz [68] can be a good support in its studies.

Once again, we start considering the Hubbard Hamiltonian written as

$$\mathcal{H}_{Hub} = -\sum_{i=1}^{L} \sum_{\sigma=\uparrow,\downarrow} \left(c^\dagger_{i,\sigma} c_{i+1,\sigma} + c^\dagger_{i+1,\sigma} c_{i,\sigma} \right) + \frac{U}{4} \sum_{i=1}^{L} (1 - 2n_{i,\uparrow})(1 - 2n_{i,\downarrow}), \quad L + 1 \equiv 1 \tag{3.3.1}$$

we obtain this Hamiltonian directly from the R-matrix formalism (3.2.21).

[7] Here the structure constants f^{nmk} are different from the ones corresponding to the orthogonal generators Q^n_0. It implies several changes in the deformed Serre relations.

Before we start the description of the Coordinate Bethe Ansatz, we give some general analysis for the Hubbard model. We are interested in the solutions of the equation:

$$\mathcal{H}_{Hub}\phi_N(\overline{\sigma}) = E_N(\overline{\sigma})\phi_N(\overline{\sigma}) \tag{3.3.2}$$

where N denotes the total number of particles spin up and down, $\overline{\sigma}$ denotes the configuration of particles' spins $(\sigma_1, \sigma_2, ..., \sigma_N)$. Moreover, we assume that there are M spin down particles.

Due to the form of the Hamiltonian there is a symmetry between particles and holes, thus using the appropriate transformation ($c_{i,\sigma} \to c^\dagger_{i,\sigma}$) we get the identity:

$$E_N(\overline{\sigma}) = E_{2L-N}(\overline{\sigma}') \tag{3.3.3}$$

with $\overline{\sigma}'$ denotes a new configuration such that $N' = 2L - N$ and $M' = L - M$. Therefore, if $N > L$ we have the results for $N' < L$ and thus we can restrict the consideration to

$$N \leq L \tag{3.3.4}$$

In addition, due to the spin up and spin down symmetry, we can consider only

$$M \leq \frac{N}{2} \tag{3.3.5}$$

3.3.1 CBA. Level 1

We use the Coordinate Bethe ansatz to find the eigenvalues and the eigenvectors of the Hubbard Hamiltonian (3.3.1). The content of the model as we already know consists of the electrons spin up and spin down: for example if $w_{i,\sigma}$ denotes an empty site of spin σ then $c^\dagger_{i,\sigma}w_{i,\sigma}$ represent an electron on the site i with spin σ. Evidently, the Hubbard Hamiltonian preserves the number of particles involved in dynamics, thus the number of electrons spin up and down are the "good" quantum numbers to characterize the eigenfunctions.

First of all we define the reference state (pseudo-vacuum) ϕ_0 as the empty chain: $\phi_0 = \prod_{i=1}^{L} w_{i,\uparrow}w_{i,\downarrow}$ and

$$c_{i,\sigma}\phi_0 = 0, \quad \forall i, \sigma \tag{3.3.6}$$

and the corresponding eigenvalue is given by:

$$\mathcal{H}_{Hub}\phi_0 = \frac{UL}{4}\,\phi_0\,. \tag{3.3.7}$$

One excitation. Next we consider one electron excitation with any spin σ over the vacuum:

$$\phi_1[\sigma] = \sum_{x\in[1,L]} \Psi[x,\sigma]\, c^\dagger_{x,\sigma}\phi_0 \tag{3.3.8}$$

here the coefficients $\Psi[x,\sigma]$ are to be determined.

We apply the Hamiltonian on this one-excitation function $\mathcal{H}_{Hub}\phi_1[\sigma] = E\phi_1[\sigma]$ and it transforms to the Schrodinger equation on the coefficients $\Psi[x,\sigma]$ when we project it on the $c^\dagger_{x,\sigma}\phi_0$ vector:

$$\phi_0^* c_{x,\sigma}\mathcal{H}_{Hub}\phi_1[\sigma] = E\phi_1[\sigma] \tag{3.3.9}$$

with ϕ_0^* defined as $\phi_0^*\phi_0 = 1$. After some simplifications we arrive to

$$-(\Psi[x+1,\sigma] + \Psi[x-1,\sigma]) + \Big(\frac{U}{4}(L-2) - E\Big)\Psi[x,\sigma] = 0, \quad \text{for} \quad x \neq 1, L \tag{3.3.10}$$

$$-(\Psi[2,\sigma] + \Psi[L,\sigma]) + \Big(\frac{U}{4}(L-2) - E\Big)\Psi[1,\sigma] = 0, \quad \text{for} \quad x = 1 \tag{3.3.11}$$

$$-(\Psi[1,\sigma] + \Psi[L-1,\sigma]) + \Big(\frac{U}{4}(L-2) - E\Big)\Psi[L,\sigma] = 0, \quad \text{for} \quad x = L \tag{3.3.12}$$

The first equation gives the dynamics of one-excitation function on the chain (without boundary conditions) whereas the second and the third ones express the periodic boundary conditions of the model. The solution of the first equation can be given by $\Psi[x,\sigma] = e^{ikx}$,[8] where k is *à priori* a complex parameter to be determined. Indeed, k play the role of the particle's momentum but it is often called a *Bethe root* in the CBA context. Inserting this solution $\Psi[x,\sigma] = e^{ikx}$ into the first equation we get the energy $E = -2\cos(k) + U(L-2)/4$. From the second and the third equations, using the expressions for $\Psi[x,\sigma]$ and energy E, one gets the ordinary for condensed matter physics periodic boundary conditions:

$$e^{ikL} = 1 \tag{3.3.13}$$

The real parameter k can be defined between $-\pi$ and π and thus $k = 2\pi n/L$ with $n = -L/2$,..., $L/2-1$. We see that the boundary conditions give the "quantification" of the Bethe root. The obtained equation on k is the simplest version of the so-called *Bethe equations*.

Two excitations. Now we treat the case of two electrons excitation with any spin σ_1 and σ_2 over the reference state. The eigenfunction writes in the form:

$$\phi_2[\overline{\sigma}] = \sum_{\mathbf{x}\in[1,L]} \Psi[\mathbf{x},\overline{\sigma}] \prod_{i=1}^{2} c^\dagger_{x_i,\sigma_i}\phi_0, \tag{3.3.14}$$

with some coefficients $\Psi[\mathbf{x},\overline{\sigma}]$ to be determined. We noted $\overline{\sigma} = (\sigma_1,\sigma_2)$, $\mathbf{x} = (x_1,x_2)$ and by $\prod_{i=1}^2 c^\dagger_{x_i,\sigma_i}$ is considered the ordered product $c^\dagger_{x_1,\sigma_1}c^\dagger_{x_2,\sigma_2}$.

Once again we apply the Hamiltonian (3.3.1) on this two-excitations function, $H\phi_2[\overline{\sigma}] = E\phi_2[\overline{\sigma}]$, and project the result on the vector $\prod_{i=1}^2 c^\dagger_{x_i,\sigma_i}\phi_0$. Thus, we get the Schrodinger equation on the coefficients $\Psi[\mathbf{x},\overline{\sigma}]$: for $\mathbf{x} \neq 1, L$

$$\begin{aligned} &-\sum_{m=1}^{2}(\Psi[\mathbf{x}+\mathbf{e}_m,\overline{\sigma}]\Delta_m^+ + \Psi[\mathbf{x}-\mathbf{e}_m,\overline{\sigma}]\Delta_m^-) + \\ &+\Delta^3\Big(\frac{U}{4}(L-4) + U\sum_{\substack{l,m=1\\ l<m}}^{2}\delta(x_l - x_m)\delta(\sigma_l \neq \sigma_m) - E\Big)\Psi[\mathbf{x},\overline{\sigma}] = 0, \end{aligned} \tag{3.3.15}$$

where $\mathbf{e}_m$ is an elementary vector in $\mathbb{C}^N$ (here $N = 2$) with entry 1 on the m^{th} position and 0 elsewhere:

[8] The fact that there is no reversed mode e^{-ikx} is connected with the fact that we have periodic boundary conditions and somehow there is no reflections. It is not the case for the open spin chains.

$\mathbf{e}_k = (0,...0,\overset{k}{\underset{\downarrow}{1}},0,...0)$. Also we denoted

$$\Delta_m^{\pm} = \prod_{\substack{l=1\\ l\neq m}}^{2}\prod_{\substack{n=l+1\\ n\neq m}}^{2}\delta^{\updownarrow}(x_l \neq x_n)\prod_{\substack{l=1\\ l\neq m}}^{2}\delta^{\updownarrow}(x_l \neq x_m)\delta^{\updownarrow}(x_l \neq x_m \pm 1), \tag{3.3.16}$$

$$\Delta^3 = \prod_{l=1}^{2}\prod_{n=l+1}^{2}\delta^{\updownarrow}(x_l \neq x_n), \tag{3.3.17}$$

$$\delta^{\updownarrow}(x_l \neq x_n) = 1 - \delta(x_l - x_n)\delta(\sigma_l - \sigma_n)\,. \tag{3.3.18}$$

These symbols mean that there is no particles with the same spin on the same and neighbouring sites with some conditions corresponding to each symbol (exclusion principle). Otherwise, the Schrodinger equation (3.3.15) is identically zero.

There are also the equations due to the periodic boundary conditions, which are almost similar to (3.3.15). Let us take $x_k = L$ (or respectively $x_k = 1$) for any $k = 1,2$, then we understand by $x_k + 1 = L + 1 \equiv 1$ (or respectively $x_k - 1 = 0 \equiv L$). Thus, the Schrodinger equations with the periodic boundary conditions are written as: for $k = 1,2$

$$-\sum_{\substack{m=1\\ m\neq k}}^{2}(\Psi[\mathbf{x}+\mathbf{e}_m,\overline{\sigma}]\Delta_m^{+} + \Psi[\mathbf{x}-\mathbf{e}_m,\overline{\sigma}]\Delta_m^{-}) - (\Psi[\mathbf{x}+\mathbf{e}_k,\overline{\sigma}]\Delta_k^{+} + \Psi[\mathbf{x}-\mathbf{e}_k,\overline{\sigma}]\Delta_k^{-})$$
$$+\Delta^3\Big(\frac{U}{4}(L-4) + U\sum_{\substack{l,m=1\\ l<m}}^{2}\delta(x_l - x_m)\delta(\sigma_l \neq \sigma_m) - E\Big)\Psi[\mathbf{x},\overline{\sigma}] = 0, \quad \text{for } x_k = 1, L \tag{3.3.19}$$

In order to solve all these equations we use the so-called *Bethe ansatz* which originally was proposed by H.Bethe in one of his works. The idea is that we look for the solution as a product of free particles multiplied by certain coefficients. More precisely, let us divide the coordinate space (x_1, x_2) into 2 sectors: $x_1 \leq x_2$ and $x_1 > x_2$. For sector Q: $x_{Q(1)} < x_{Q(2)}$ we define $\Psi[\mathbf{x},\overline{\sigma}]$ as

$$\Psi_Q[\mathbf{x},\overline{\sigma}] = \sum_{P\in\mathfrak{S}_2}(-1)^{[P]}\Phi_{\overline{\sigma}}(PQ,P^{-1})e^{iP\mathbf{k}\cdot\mathbf{x}} \tag{3.3.20}$$

here Q and P are elements of the permutation group $\mathfrak{S}_2$. Also it is noted $\hat{P}\mathbf{k}\mathbf{x} = \sum_{i=1}^{2} k_{\hat{P}(i)}x_i$, k_1,k_2 are unequal complex numbers, Bethe roots. Symbol $[P]$ stands for the signature of the P-permutation: for example let Π_{ii+1} be the permutation of the elements i,$i+1$ then we have: $[P\Pi_{ii+1}] = [P] + 1$.

One should remark that when we introduce the Bethe root k_1,k_2 two electron's eigenfunction ϕ_2 and the coefficients Ψ_Q become indirectly dependent of these Bethe roots. Thus, we have $\phi_2[\overline{\sigma}] = \phi_2[\overline{\sigma},\mathbf{k}]$ and $\Psi_Q[\mathbf{x},\overline{\sigma}] = \Psi_Q[\mathbf{x},\overline{\sigma},\mathbf{k}]$.

The last property for any $\overline{\sigma}$ we can derive is the energy E. We consider the Schrodinger equation (3.3.15) for $\mathbf{x} \neq 1, L$ and in the sector $x_{Q(1)} << x_{Q(2)}$ such that all symbols $\Delta_m^{\pm}, \Delta^3$ equal 1 and there is also no potential term. Thus we have

$$-\sum_{m=1}^{2}(\Psi_Q[\mathbf{x}+\mathbf{e}_m,\overline{\sigma}] + \Psi_Q[\mathbf{x}-\mathbf{e}_m,\overline{\sigma}]) + \Big(\frac{U}{4}(L-4) - E\Big)\Psi_Q[\mathbf{x},\overline{\sigma}] = 0, \tag{3.3.21}$$

and inserting the ansatz (3.3.20) in this equation we get the energy:

$$E = -2\sum_{m=1}^{2}\cos(k_m) + U(L-4)/4 \tag{3.3.22}$$

Now we start to treat separately different cases of $\overline{\sigma}$. Indeed, physics depend on the spin variable and the wavefunctions differ drastically. There are two different cases: two particles are identical or non-identical. In both cases we will manage to find the relations between unknown coefficients $\Phi_{\overline{\sigma}}(PQ, P^{-1})$.

Explicitly, the functions $\Psi_Q[\mathbf{x}, \overline{\sigma}]$ are given by

$$\Psi_{id}[(x_1, x_2), \overline{\sigma}] = \Phi_{\overline{\sigma}}(id, id)e^{i(k_1x_1+k_2x_2)} - \Phi_{\overline{\sigma}}(\Pi_{12}, \Pi_{12})e^{i(k_2x_1+k_1x_2)},$$

for identical particles: $\sigma_1 = \sigma_2$ and there is only 1 sector $x_1 < x_2$, and

$$\Psi_Q[(x_1, x_2), \overline{\sigma}] = \begin{cases} Q = id, & \Phi_{\overline{\sigma}}(id, id)e^{i(k_1x_1+k_2x_2)} - \Phi_{\overline{\sigma}}(\Pi_{12}, \Pi_{12})e^{i(k_2x_1+k_1x_2)} \\ Q = \Pi_{12}, & \Phi_{\overline{\sigma}}(\Pi_{12}, id)e^{i(k_1x_1+k_2x_2)} - \Phi_{\overline{\sigma}}(id, \Pi_{12})e^{i(k_2x_1+k_1x_2)}, \end{cases}$$

for non-identical particles: $\sigma_1 \neq \sigma_2$, there are 2 sectors: $Q = id,\ x_1 < x_2$ and $Q = \Pi_{12},\ x_2 < x_1$.

1. Let us firstly consider two electrons with the same spin. In this case one can not distinguish the particles and we have $\sigma_{Q(1)} = \sigma_{Q(2)}$. It is also evident that the wavefunction in this case should be antisymmetric due to the anticommutation properties of $c^\dagger$ operators. Therefore we can consider only one sector and only one function $\Psi_Q[\mathbf{x}, \overline{\sigma}]$ such that $\Psi_{\Pi_{Q(1)Q(2)}Q}[\Pi_{Q(1)Q(2)}\mathbf{x}, \overline{\sigma}] = -\Psi_Q[\mathbf{x}, \overline{\sigma}]$, (we denoted $\Pi_{ij}\mathbf{x} = (..., \overset{i}{\underset{\downarrow}{x_j}}, ..., \overset{j}{\underset{\downarrow}{x_i}}, ...)$). It implies that :

$$\Phi_{\overline{\sigma}}(PQ, P^{-1}) = \Phi_{\overline{\sigma}}(PQ, (P\Pi_{Q(1)Q(2)})^{-1}) \tag{3.3.23}$$

where Π_{ab} is the permutation of objects a and b.

Let us now consider the sector $x_{Q(1)} < x_{Q(2)}$ with $x_{Q(1)} = x_{Q(2)} - 1$. In this case $\Delta^-_{Q(2)} = 0$ and $\Delta^+_{Q(1)} = 0$, all other symbols in (3.3.18) are equal to 1. Thus, equation (3.3.15) becomes

$$-\Big(\Psi_Q[\mathbf{x} - \mathbf{e}_{Q(1)}, \overline{\sigma}] + \Psi_Q[\mathbf{x} + \mathbf{e}_{Q(2)}, \overline{\sigma}]\Big) = \big(E - \frac{U}{4}(L+4)\big)\Psi_Q[\mathbf{x}, \overline{\sigma}] \tag{3.3.24}$$

Using the expression for the energy E we find that $\Psi_Q[(x, x), \overline{\sigma}] = 0$ and inserting the ansatz (3.3.20) we get

$$\Phi_{\overline{\sigma}}(PQ, P^{-1}) = \Phi_{\overline{\sigma}}(PQ\Pi_{12}, (P\Pi_{Q(1)Q(2)})^{-1}) \tag{3.3.25}$$

Combining (3.3.23) and (3.3.23) one gets

$$\Phi_{\overline{\sigma}}(\Pi_{ab}PQ, P^{-1}) = \Phi_{\overline{\sigma}}(PQ, P^{-1}) \tag{3.3.26}$$

with $a = PQ(1), b = PQ(2)$.

Moreover one can verify that the total eigenfunction of the Hubbard Hamiltonian is antisymmetric, $\phi_2[(\sigma_2, \sigma_1), (k_2, k_1)] = -\phi_2[(\sigma_1, \sigma_2), (k_1, k_2)]$:

$$\phi_2[(\sigma_2, \sigma_1), (k_2, k_1)] = 2 \sum_{x_1 < x_2 \in [1,L]} \Psi_{id}[\mathbf{x}, \overline{\sigma}, (k_2, k_1)] \prod_{i=1}^{2} c^\dagger_{x_i,\sigma_i}\phi_0 =$$

$$= 2 \sum_{x_1 < x_2 \in [1,L]} \Phi_{\overline{\sigma}}(id, id)(e^{i\mathbf{k}\cdot\mathbf{x}} - e^{i\mathbf{k}'\cdot\mathbf{x}}) \prod_{i=1}^{2} c^\dagger_{x_i,\sigma_i}\phi_0 = -\phi_2[(\sigma_1, \sigma_2), (k_1, k_2)]$$

where defined $\mathbf{k}' \cdot \mathbf{x} = k_2x_1 + k_1x_2$ and we used (3.3.20) and (3.3.25) passing from the first line to the second one.

2. Now let us consider two electrons with different spins, $\sigma_{Q(1)} \neq \sigma_{Q(2)}$. We have the continuity condition for $\Psi_Q[\mathbf{x}, \overline{\sigma}]$: $\Psi_Q[\mathbf{x}, \overline{\sigma}] = \Psi_{Q\Pi_{12}}[\mathbf{x}, \overline{\sigma}]$ with $x_{Q(1)} = x_{Q(2)}$, what leads to the following relation between $\Phi_{\overline{\sigma}}(PQ, P^{-1})$:

$$\Phi_{\overline{\sigma}}(PQ, P^{-1}) - \Phi_{\overline{\sigma}}(PQ\Pi_{12}, (P\Pi_{Q(1)Q(2)})^{-1}) = \Phi_{\overline{\sigma}}(PQ\Pi_{12}, P^{-1}) - \Phi_{\overline{\sigma}}(PQ, (P\Pi_{Q(1)Q(2)})^{-1}) \tag{3.3.27}$$

Next we consider the Schrodinger equation (3.3.15) when two electrons are on the same site: $x_{Q(1)} = x_{Q(2)}$. In this case all $\Delta^{\pm}_m$,Δ^3 equal 1 and hence the equation (3.3.15) becomes

$$-\Big(\Psi_Q[\mathbf{x} - \mathbf{e}_{q_1}, \overline{\sigma}] + \Psi_{Q\Pi_{12}}[\mathbf{x} + \mathbf{e}_{q_1}, \overline{\sigma}] + \Psi_{Q\Pi_{12}}[\mathbf{x} - \mathbf{e}_{q_2}, \overline{\sigma}] + \\ + \Psi_Q[\mathbf{x} + \mathbf{e}_{q_2}, \overline{\sigma}]\Big) + \left(\frac{U}{4}(L-4) + U - E\right)\Psi_Q[\mathbf{x}, \overline{\sigma}] = 0$$

and again using the expression for the energy E we get an intermediate result:

$$\Psi_{Q\Pi_{12}}[\mathbf{x} + \mathbf{e}_{q_1}, \overline{\sigma}] + \Psi_{Q\Pi_{12}}[\mathbf{x} - \mathbf{e}_{q_2}, \overline{\sigma}] - \Big(\Psi_Q[\mathbf{x} + \mathbf{e}_{q_1}, \overline{\sigma}] + \Psi_Q[\mathbf{x} - \mathbf{e}_{q_2}, \overline{\sigma}] + U\Psi_Q[\mathbf{x}, \overline{\sigma}]\Big) = 0\,.$$

and finally

$$(e^{ik_{PQ(1)}} + e^{-ik_{PQ(2)}} + U)\Phi_{\overline{\sigma}}(PQ, P^{-1}) + (e^{-ik_{PQ(1)}} + e^{ik_{PQ(2)}})\Phi_{\overline{\sigma}}(PQ, (P\Pi_{Q(1)Q(2)})^{-1} = \\ = (e^{ik_{PQ(1)}} + e^{-ik_{PQ(2)}})\Phi_{\overline{\sigma}}(PQ\Pi_{12}, P^{-1}) + (e^{-ik_{PQ(1)}} + e^{ik_{PQ(2)}} + U)\Phi_{\overline{\sigma}}(PQ\Pi_{12}, (P\Pi_{Q(1)Q(2)})^{-1}) \tag{3.3.28}$$

The above equation combining with (3.3.27) allows to express $\Phi_{\overline{\sigma}}(PQ\Pi_{12}, (P\Pi_{Q(1)Q(2)})^{-1})$ and $\Phi_{\overline{\sigma}}(PQ\Pi_{12}, P^{-1})$ as the functions of $\Phi_{\overline{\sigma}}(PQ, P^{-1})$ and $\Phi_{\overline{\sigma}}(PQ, (P\Pi_{Q(1)Q(2)})^{-1})$. We write it in the matrix form:

$$\begin{pmatrix} \Phi_{\overline{\sigma}}(\Pi_{ab}PQ, P^{-1}) \\ \Phi_{\overline{\sigma}}(\Pi_{ab}PQ, (\Pi_{ab}P)^{-1}) \end{pmatrix} = \begin{pmatrix} t_{ab} & r_{ab} \\ r_{ab} & t_{ab} \end{pmatrix} \begin{pmatrix} \Phi_{\overline{\sigma}}(PQ, P^{-1}) \\ \Phi_{\overline{\sigma}}(PQ, (\Pi_{ab}P)^{-1}) \end{pmatrix} \tag{3.3.29}$$

with $a = PQ(1)$, $b = PQ(2)$ and

$$t_{ab} = -\frac{2i(\lambda_a - \lambda_b)}{U - 2i(\lambda_a - \lambda_b)}, \quad r_{ab} = \frac{U}{U - 2i(\lambda_a - \lambda_b)}, \quad \lambda_a = \sin k_a \tag{3.3.30}$$

Thus, we obtained the relations between different coefficients $\Phi_{\overline{\sigma}}(PQ, P^{-1})$ for any case of the spin configuration σ. These relations can be gathered in a compact matrix form. Let us define a vector, for $P' \equiv PQ \in \mathfrak{S}_2$ and $Q' \equiv P^{-1} \in \mathfrak{S}_2$,

$$\hat{\Phi}(P') \equiv \sum_{Q', \overline{\sigma}} \Phi_{\overline{\sigma}}(P', Q') \prod_{i=1}^{2} e_i^{\sigma_{Q'(i)}} \tag{3.3.31}$$

where the summation is a direct sum over all types of excitations and all corresponding sectors. The vector $\prod_{i=1}^{2} e^2_{i,\sigma_{Q'(i)}}$ belongs to $V_1 \otimes V_2$, where V is spanned by $\{e^{\uparrow}, e^{\downarrow}\}$[9] and represents one type of 2

[9]One can see $e^{\uparrow} = e^1$ and $e^{\downarrow} = e^2$, where e^i is an elementary vector with 1 on i place and 0 the rest.

excitations.To clarify the notation we write it explicitly:

$$\begin{aligned}\hat{\Phi}(P') &= \Phi_{(\uparrow,\uparrow)}(P', id)\, e_1^{\uparrow} e_2^{\uparrow} + \Phi_{(\downarrow,\downarrow)}(P', id)\, e_1^{\downarrow} e_2^{\downarrow} \\ &+ \left(\Phi_{(\uparrow,\downarrow)}(P', id)\, e_1^{\uparrow} e_2^{\downarrow} + \Phi_{(\uparrow,\downarrow)}(P', \Pi_{12})\, e_1^{\downarrow} e_2^{\uparrow} \right) = \\ &= \left(\Phi_{(\uparrow,\uparrow)}(P', id), \Phi_{(\uparrow,\downarrow)}(P', id), \Phi_{(\uparrow,\downarrow)}(P', \Pi_{12}), \Phi_{(\downarrow,\downarrow)}(P', id) \right)^T \end{aligned} \tag{3.3.32}$$

Then, all relations between the coefficients $\Phi_{\overline{\sigma}}(PQ, P^{-1})$ can be expressed in a compact form:

$$\hat{\Phi}(\Pi_{12} P) = S_{12}^{(1)}(k_1, k_2) \hat{\Phi}(P) \tag{3.3.33}$$

where $S_{12}^{(1)}(k_1, k_2)$ (simply denoted by $S_{12}^{(1)}$ in the following) acts on elementary vectors $e_1^{\sigma_1} e_2^{\sigma_2}$. In the matrix form it means:

$$\begin{pmatrix} \Phi_{(\uparrow,\uparrow)}(\Pi_{12}P, Q) \\ \Phi_{(\uparrow,\downarrow)}(\Pi_{12}P, Q) \\ \Phi_{(\uparrow,\downarrow)}(\Pi_{12}P, Q\Pi_{12}) \\ \Phi_{(\downarrow,\downarrow)}(\Pi_{12}P, Q) \end{pmatrix} = \begin{pmatrix} 1 & & & \\ & t_{12} & r_{12} & \\ & r_{12} & t_{12} & \\ & & & 1 \end{pmatrix} \begin{pmatrix} \Phi_{(\uparrow,\uparrow)}(P, Q) \\ \Phi_{(\uparrow,\downarrow)}(P, Q) \\ \Phi_{(\uparrow,\downarrow)}(P, Q\Pi_{12}) \\ \Phi_{(\downarrow,\downarrow)}(P, Q) \end{pmatrix} \tag{3.3.34}$$

we omitted zeros in the $S_{12}^{(1)}(k_1, k_2)$ matrix for a more clear vision.

In the case of N-particles we will see that $S_{12}^{(1)}(k_1, k_2)$ matrix can be factorized and it satisfies the Yang–Baxter equation, what is a good sign for the integrability.

We have almost finished the case of two electrons excitation, however we did not used the periodic boundary conditions in order to determine the Bethe roots k_1, k_2. It is not hard to verify that from the Schrodinger equation (3.3.19) the periodic boundary conditions are equivalent to

$$\Psi_{QC_2}[\mathbf{x} - \mathbf{e}_{Q(2)} L, \overline{\sigma}] = \Psi_Q[\mathbf{x}, \overline{\sigma}]\,. \tag{3.3.35}$$

with $C_2 = \Pi_{21}$ being a cyclic permutation in $\mathfrak{S}_2$.

In terms of $\hat{\Phi}(P)$ this yields the condition

$$\hat{\Phi}(PC_2) = e^{ik_{P(2)}L} \hat{\Phi}(P)\,. \tag{3.3.36}$$

If we choose $P = C_2^{2-j}$ with $j = 1, 2$, we can derive a system of equations on the coefficients $\hat{\Phi}(id)$ which is called the "auxiliary problem":

$$S_{12}^{(1)}\, \hat{\Phi}(id) = e^{-ik_1 L} \hat{\Phi}(id), \quad j = 1 \tag{3.3.37}$$

$$S_{12}^{(1)}\, \hat{\Phi}(id) = e^{ik_2 L} \hat{\Phi}(id), \quad j = 2 \tag{3.3.38}$$

$$\tag{3.3.39}$$

where we omitted the arguments of the S-matrices, $S_{ab}^{(1)} \equiv S_{ab}^{(1)}(k_a, k_b)$.

The solution of this simple matrix equation can be found by taking the determinant. Thus we have different solutions: for identical particles

$$e^{i(k_1+k_2)L} = 1, \tag{3.3.40}$$

$$e^{i(k_1 L)} = 1 \tag{3.3.41}$$

and for particles with different spins

$$e^{i(k_1+k_2)L} = 1, \tag{3.3.42}$$

$$e^{-i(k_1L)} = t_{12} - r_{12} = \frac{\sin(k_1) - \sin(k_2) - iU/2}{\sin(k_1) - \sin(k_2) + iU/2}, \quad \text{or} \tag{3.3.43}$$

$$e^{i(k_1L)} = t_{12} + r_{12} = 1, \tag{3.3.44}$$

we remind that the Bethe roots can not be equal: $k_1 \neq k_2$.

Physical interpretation of the obtained results is following: in the first case we have two electrons with the same spin, therefore the wavefunction ϕ_2 is antisymmetric and their momenta k_i are "quantized" as for one excitation case. In the second case two electrons with different spins form also an antisymetric wavefunction ϕ_2. The wavefunction is composed of the spin part in $\Phi_{\overline{\sigma}}[PQ, P^{-1}]$ and the coordinate part in $\Psi[\mathbf{x}, \overline{\sigma}]$ which could be or (symmetric, antisymmetric) or (antisymmetric, symmetric). If the spin part is symmetric then the coordinate part will be antisymmetric and it implies however two electrons can not be at the same site and effectively they looks like free electrons. That is why one of the Bethe equations in this case is similar to the free electrons "quantification". Whereas, when the coordinate part is symmetric we see that the "quantification" is more involved comparing to other cases.

Now we generalize the obtained results to the case of the N electrons excitation. Indeed, the periodic boundary conditions become more complicated and it gives rise to the second level of CBA.

N excitations. Let we have the N electrons excitation with the spin configuration $\overline{\sigma} = (\sigma_1, ..., \sigma_N)$ over the pseudo-vacuum:

$$\phi_N[\overline{\sigma}] = \sum_{\mathbf{x}\in[1,L]} \Psi[\mathbf{x}, \overline{\sigma}] \prod_{i=1}^{N} c^{\dagger}_{x_i,\sigma_i} \phi_0, \tag{3.3.45}$$

the coefficients $\Psi[\mathbf{x}, \overline{\sigma}]$ are to be determined. We noted $\mathbf{x} = (x_1, ..., x_N)$ and by $\prod_{i=1}^{N} c^{\dagger}_{x_i,\sigma_i}$ it is considered the ordered product $c^{\dagger}_{x_1,\sigma_1} ... c^{\dagger}_{x_N,\sigma_N}$.

The Schrodinger equation on the coefficients $\Psi[\mathbf{x}, \overline{\sigma}]$ is for $x \neq 1, L$

$$\begin{aligned} &- \sum_{m=1}^{N} (\Psi[\mathbf{x} + \mathbf{e}_m, \overline{\sigma}]\Delta_m^{+} + \Psi[\mathbf{x} - \mathbf{e}_m, \overline{\sigma}]\Delta_m^{-}) + \\ &+\Delta^3 \Big(\frac{U}{4}(L - 2N) + U \sum_{\substack{l,m=1 \\ l<m}}^{N} \delta(x_l - x_m)\delta(\sigma_l \neq \sigma_m) - E \Big) \Psi[\mathbf{x}, \overline{\sigma}] = 0, \end{aligned} \tag{3.3.46}$$

the notations are the same as in the case $N = 2$.

In addition, there are the equations due to the periodicity. For any $k = 1, ..., N$ we suppose that $x_k = L$ or $x_k = 1$, then we understand by $x_k + 1 = L + 1 \equiv 1$ or $x_k - 1 = 0 \equiv L$, respectively. Therefore, we can write the Schrodinger equations with the periodic boundary conditions in the following form: for $k = 1, ..., N$

$$\begin{aligned} &- \sum_{\substack{m=1 \\ m\neq k}}^{N} (\Psi[\mathbf{x} + \mathbf{e}_m, \overline{\sigma}]\Delta_m^{+} + \Psi[\mathbf{x} - \mathbf{e}_m, \overline{\sigma}]\Delta_m^{-}) - (\Psi[\mathbf{x} + \mathbf{e}_k, \overline{\sigma}]\Delta_k^{+} + \Psi[\mathbf{x} - \mathbf{e}_k, \overline{\sigma}]\Delta_k^{-}) \\ &+\Delta^3 \Big(\frac{U}{4}(L - 2N) + U \sum_{\substack{l,m=1 \\ l<m}}^{N} \delta(x_l - x_m)\delta(\sigma_l \neq \sigma_m) - E \Big) \Psi[\mathbf{x}, \overline{\sigma}] = 0, \quad \text{for } x_k = 1, L \end{aligned} \tag{3.3.47}$$

The *Bethe ansatz* can be written similarly: introducing Q and P from $\mathfrak{S}_N$ we divide the coordinate space $(x_1, ..., x_N)$ into $N!$ sectors: for the sector Q: $x_{Q(1)} < ... < x_{Q(N)}$ we define $\Psi[\mathbf{x}, \overline{\sigma}]$ as

$$\Psi_Q[\mathbf{x}, \overline{\sigma}] = \sum_{P \in \mathfrak{S}_N} (-1)^{[P]} \Phi_{\overline{\sigma}}(PQ, P^{-1}) e^{iP\mathbf{k}\cdot\mathbf{x}} \tag{3.3.48}$$

where k_1,...,k_N are unequal complex numbers, the Bethe roots, and $\hat{P}\mathbf{kx} = \sum_{i=1}^{N} k_{\hat{P}(i)} x_i$. The symbol $[P]$ stands for the signature of the P-permutation: for example $[P\Pi_{ii+1}] = [P] + 1$.

In the non-interacting case: $x_{Q(1)} << ... << x_{Q(N)}$ we get immediately the energy:

$$E = -2 \sum_{m=1}^{N} \cos(k_m) + U(L - 2N)/4 \tag{3.3.49}$$

Next we consider different cases in order to find the relations between $\Phi_{\overline{\sigma}}(PQ, P^{-1})$: 1) two identical particles on the neighbouring sites, 2) two electrons with different spins on the same site.

1. Let among the electrons, for certain integers $1 \leq i, j \leq N$, we have $\sigma_{Q(i)} = \sigma_{Q(j)}$. The wavefunction in this case should be antisymmetric with respect to this particles. It implies that :

$$\Phi_{\overline{\sigma}}(PQ, P^{-1}) = \Phi_{\overline{\sigma}}(PQ, (P\Pi_{Q(i)Q(j)})^{-1}) \tag{3.3.50}$$

where Π_{ab} is the permutation of objects a and b.

Let us now consider the sector $x_{Q(1)} << ... << x_{Q(i)} < x_{Q(i+1)} << ... << x_{Q(N)}$ with $x_{Q(i)} = x_{Q(i+1)} - 1$, similarly we get

$$\Phi_{\overline{\sigma}}(PQ, P^{-1}) = \Phi_{\overline{\sigma}}(PQ\Pi_{ii+1}, (P\Pi_{Q(i)Q(i+1)})^{-1}) \tag{3.3.51}$$

The combination of these two relations leads to

$$\Phi_{\overline{\sigma}}(\Pi_{ab}PQ, P^{-1}) = \Phi_{\overline{\sigma}}(PQ, P^{-1}) \tag{3.3.52}$$

with $a = PQ(i), b = PQ(i+1)$.

2. The case of two electrons with different spins, $\sigma_{Q(i)} \neq \sigma_{Q(i+1)}$. We have the continuity condition for $\Psi_Q[\mathbf{x}, \overline{\sigma}]$: $\Psi_Q[\mathbf{x}, \overline{\sigma}] = \Psi_{Q\Pi_{ii+1}}[\mathbf{x}, \overline{\sigma}]$ with $x_{Q(i)} = x_{Q(i+1)}$, what leads to the following relation between the cooeficients $\Phi_{\overline{\sigma}}(PQ, P^{-1})$:

$$\Phi_{\overline{\sigma}}(PQ, P^{-1}) - \Phi_{\overline{\sigma}}(PQ\Pi_{ii+1}, (P\Pi_{Q(i)Q(i+1)})^{-1}) = \Phi_{\overline{\sigma}}(PQ\Pi_{ii+1}, P^{-1}) - \Phi_{\overline{\sigma}}(PQ, (P\Pi_{Q(i)Q(i+1)})^{-1}) \tag{3.3.53}$$

There is also a relation coming from the Schrodinger equation with $x_{Q(i)} = x_{Q(i+1)}$:

$$\begin{aligned}&(e^{ik_{PQ(i)}} + e^{-ik_{PQ(i+1)}} + U)\Phi_{\overline{\sigma}}(PQ, P^{-1}) + (e^{-ik_{PQ(i)}} + e^{ik_{PQ(i+1)}})\Phi_{\overline{\sigma}}(PQ, (P\Pi_{Q(i)Q(i+1)})^{-1} = \\ &= (e^{ik_{PQ(i)}} + e^{-ik_{PQ(i+1)}})\Phi_{\overline{\sigma}}(PQ\Pi_{ii+1}, P^{-1}) + (e^{-ik_{PQ(i)}} + e^{ik_{PQ(i+1)}} + U)\Phi_{\overline{\sigma}}(PQ\Pi_{ii+1}, (P\Pi_{Q(i)Q(i+1)})^{-1})\end{aligned} \tag{3.3.54}$$

We can combine it with (3.3.27) and we have the relations between the coefficients $\Phi_{\overline{\sigma}}(PQ, P^{-1})$. In the matrix form it is

$$\begin{pmatrix} \Phi_{\overline{\sigma}}(\Pi_{ab}PQ, P^{-1}) \\ \Phi_{\overline{\sigma}}(\Pi_{ab}PQ, (\Pi_{ab}P)^{-1}) \end{pmatrix} = \begin{pmatrix} t_{ab} & r_{ab} \\ r_{ab} & t_{ab} \end{pmatrix} \begin{pmatrix} \Phi_{\overline{\sigma}}(PQ, P^{-1}) \\ \Phi_{\overline{\sigma}}(PQ, (\Pi_{ab}P)^{-1}) \end{pmatrix} \tag{3.3.55}$$

with $a = PQ(i)$, $b = PQ(i+1)$ and

$$t_{ab} = -\frac{2i(\lambda_a - \lambda_b)}{U - 2i(\lambda_a - \lambda_b)}, \quad r_{ab} = \frac{U}{U - 2i(\lambda_a - \lambda_b)}, \quad \lambda_a = \sin k_a \tag{3.3.56}$$

All these relations can be collected in a compact matrix form if we define a vector, for $P' \equiv PQ \in \mathfrak{S}_N$ and $Q' \equiv P^{-1} \in \mathfrak{S}_N$,

$$\hat{\Phi}(P') \equiv \sum_{Q',\overline{\sigma}} \Phi_{\overline{\sigma}}(P', Q') \prod_{i=1}^{N} e_i^{\sigma_{Q'(i)}}, \tag{3.3.57}$$

here the summation is also a direct sum over all types of excitations and all corresponding sectors. The vector $\prod_{i=1}^{N} e_i^{\sigma_{Q'(i)}}$ belongs to the tensor product $V_1 \otimes ... \otimes V_N$, where the space V is spanned by $\{e^{\uparrow}, e^{\downarrow}\}$ (see footnote 8) and represents one type of the N excitations. The ordering of the particles is chosen such that, for $Q' = id$, the vector $\prod_{i=1}^{N} e_i^{\sigma_{Q'(i)}}$ is taken as

$$\prod_{i=1}^{N} e_i^{\sigma_i} = \overbrace{e^{\uparrow} \otimes ... \otimes e^{\uparrow}}^{N_{\uparrow}} \otimes \overbrace{e^{\downarrow} \otimes ... \otimes e^{\downarrow}}^{N_{\downarrow}} \tag{3.3.58}$$

with N_σ is the number of spin σ particles.

Thus, the relations between the coefficients $\Phi_{\overline{\sigma}}(PQ, P^{-1})$ can be expressed in a compact form:

$$\hat{\Phi}(\Pi_{ab} P) = S_{ab}^{(1)}(k_a, k_b) \hat{\Phi}(P) \tag{3.3.59}$$

where $S_{ab}^{(1)}(k_a, k_b) \in \mathrm{End}(V_1) \otimes \mathrm{End}(V_2) \otimes ... \otimes \mathrm{End}(V_N)$ (simply denoted by $S_{ab}^{(1)}$ in the following) acts non-trivially on the elementary vectors $e_a^{\sigma_{Q'(a)}} e_b^{\sigma_{Q'(b)}}$ under the conditions $(P')^{-1}(a) - (P')^{-1}(b) = -1$. In the matrix form $S_{ab}^{(1)}(k_a, k_b)$ is given by:

$$\begin{pmatrix} \Phi_{(\uparrow,\uparrow)}(\Pi_{ab} P, Q) \\ \Phi_{(\uparrow,\downarrow)}(\Pi_{ab} P, Q) \\ \Phi_{(\uparrow,\downarrow)}(\Pi_{ab} P, Q \Pi_{ab}) \\ \Phi_{(\downarrow,\downarrow)}(\Pi_{ab} P, Q) \end{pmatrix} = \begin{pmatrix} 1 & & & \\ & t_{ab} & r_{ab} & \\ & r_{ab} & t_{ab} & \\ & & & 1 \end{pmatrix} \begin{pmatrix} \Phi_{(\uparrow,\uparrow)}(P, Q) \\ \Phi_{(\uparrow,\downarrow)}(P, Q) \\ \Phi_{(\uparrow,\downarrow)}(P, Q \Pi_{ab}) \\ \Phi_{(\downarrow,\downarrow)}(P, Q) \end{pmatrix} \tag{3.3.60}$$

hence if $\sigma_{Q'(a)} = \sigma_{Q'(b)}$ one has relations between Φ's given by the 1st and the 4th lines, whereas if $\sigma_{Q'(a)} \neq \sigma_{Q'(b)}$ - the relations come from the 2nd and the 3rd lines.

If we take k_a equal k_b we have the regularity property of the matrix $S_{ab}^{(1)}(k_a, k_b)$:

$$S_{ab}^{(1)}(k_a, k_a) = P_{ab} \tag{3.3.61}$$

where P_{ab} is the permutation matrix acting non-trivially on the spaces V_a and V_b.

More generally using the definitions of t_{ab} and r_{ab} (3.3.56) and their property $t_{ab} + r_{ab} = 1$ one can write the matrix $S_{ab}^{(1)}(k_a, k_b)$ as

$$S_{ab}^{(1)}(k_a, k_b) = t_{ab}(k_a, k_b) \mathbb{I}_{ab} + r_{ab}(k_a, k_b) P_{ab} \tag{3.3.62}$$

here we explicitly noted the dependence on spectral parameters.

Thus, the action of this S-matrix can be presented also as

$$S_{ab}^{(1)}(k_a, k_b) e_a^{\sigma} e_b^{\sigma} = e_a^{\sigma} e_b^{\sigma}, \quad \text{for any } \sigma, \tag{3.3.63}$$

$$S_{ab}^{(1)}(k_a, k_b) e_a^{\sigma} e_b^{\tau} = t_{ab}(k_a, k_b) e_a^{\sigma} e_b^{\tau} + r_{ab}(k_a, k_b) e_a^{\tau} e_b^{\sigma}, \quad \text{for } \sigma \neq \tau \tag{3.3.64}$$

This matrix is unitary:

$$S^{(1)}_{ab}(k_a,k_b)S^{(1)}_{ba}(k_b,k_a)=\mathbb{I}_{ab} \tag{3.3.65}$$

One of the important properties of $S^{(1)}_{ab}(k_a,k_b)$ that it satisfies the Yang–Baxter equation:

$$S^{(1)}_{12}(k_1,k_2)S^{(1)}_{13}(k_1,k_3)S^{(1)}_{23}(k_2,k_3)=S^{(1)}_{23}(k_2,k_3)S^{(1)}_{13}(k_1,k_3)S^{(1)}_{12}(k_1,k_2)\,. \tag{3.3.66}$$

The last condition which $\Psi_Q[\mathbf{x},\overline{\sigma}]$ should satisfy is the periodicity condition (3.3.47). One can show that they are similar to the case $N=2$ (3.3.35) and for any N are written in the form:

$$\Psi_{QC_N}[\mathbf{x}-\mathbf{e}_{Q(N)}L,\overline{\sigma}]=\Psi_Q[\mathbf{x},\overline{\sigma}]\,. \tag{3.3.67}$$

where $C_N=\Pi_{N1}...\Pi_{NN-1}$ is the cyclic permutation in $\mathfrak{S}_N$.

In terms of $\hat{\Phi}(P)$ one gets

$$\hat{\Phi}(PC_N)=e^{ik_{P(N)}L}\hat{\Phi}(P)\,. \tag{3.3.68}$$

On the one hand we have $PC_N=\Pi_{P(1)P(N)}\Pi_{P(2)P(N)}...\Pi_{P(N-1)P(N)}P$ and we want to use the S-matrix relations (3.3.59). It is easy to prove that $a=P(1)$ and $b=P(N)$ satisfy $\tilde{P}^{-1}(a)-\tilde{P}^{-1}(b)=-1$ with $\tilde{P}=\Pi_{P(2)P(N)}...\Pi_{P(N-1)P(N)}P$. Hence, this allows to use (3.3.59) and we obtain:

$$\hat{\Phi}(\Pi_{P(1)P(N)}\Pi_{P(2)P(N)}...\Pi_{P(N-1)P(N)}P)=S^{(1)}_{P(1)P(N)}\hat{\Phi}(\Pi_{P(2)P(N)}...\Pi_{P(N-1)P(N)}P) \tag{3.3.69}$$

Similarly, one can find that for $a=P(i)$ and $b=P(N)$ the relation $\tilde{P}^{-1}(a)-\tilde{P}^{-1}(b)=-1$ is also satisfied with now $\tilde{P}=\Pi_{P(i+1)P(N)}...\Pi_{P(N-1)P(N)}P$ for any $i=1...N-1$. Therefore, we get

$$\hat{\Phi}(\Pi_{P(1)P(N)}\Pi_{P(2)P(N)}...\Pi_{P(N-1)P(N)}P)=S^{(1)}_{P(1)P(N)}S^{(1)}_{P(2)P(N)}...S^{(1)}_{P(N-1)P(N)}\hat{\Phi}(P) \tag{3.3.70}$$

On the other hand, when we choose $P=C_N^{N-j}$ with $j=1,2,...N$ and use the recurrence coming from (3.3.68) we obtain:

$$\hat{\Phi}(C_N^{N-j+1})=e^{ik_jL}\hat{\Phi}(C_N^{N-j})=e^{ik_jL}e^{ik_{j+1}L}\hat{\Phi}(C_N^{N-j-1})=...=e^{ik_jL}\prod_{m=1}^{N-j}e^{ik_{j+m}L}\hat{\Phi}(id)\,. \tag{3.3.71}$$

indeed, we conclude that $\hat{\Phi}(C_N^{N-j})=\prod_{m=1}^{N-j}e^{ik_{j+m}L}\hat{\Phi}(id)$. Moreover, the following relations could be useful later $C_N^{N-j}(m)=\mod(m+j-N,N)$ for any $m,j=1,...,N$.

Thus, combining two results we obtain again a system of equations on the coefficients $\hat{\Phi}(id)$ which is called the "auxiliary problem":

$$\begin{aligned}&H^{(1)}_j\hat{\Phi}(id)=e^{ik_jL}\hat{\Phi}(id),\quad j=1,...,N\\ &\text{with}\quad H^{(1)}_j=S^{(1)}_{j+1,j}S^{(1)}_{j+2,j}...S^{(1)}_{N,j}S^{(1)}_{1,j}...S^{(1)}_{j-1,j}\end{aligned} \tag{3.3.72}$$

where we omitted the arguments of the S-matrices, $S^{(1)}_{ab}\equiv S^{(1)}_{ab}(k_a,k_b)$.

The auxiliary problem is a set of equations which represent a new eigenvalue and eigenvector problem. These new Hamiltonians $H^{(1)}_j$ can be interpreted as the particle j scatters with all the rest particles on the ring. The spectra Λ_j of these new Hamiltonians $H^{(1)}_j$ will give the Bethe equations $\Lambda_j=e^{ik_jL}$. The eigenvector will give new relations between the coefficients $\hat{\Phi}(id)$ which allows in the end to find the eigenvector of the Hubbard model. In the next subsection we will see that these new Hamiltonians commute and, thus, one can diagonalize only one of them. We will review also the Algebraic Bethe Ansatz method to perform this diagonalization.

Example of $N = 3$ excitations

In this subsection we give explicit details on the construction of $\hat{\Phi}(P)$. At first reading this description can be skipped, reader can go directly to the next subsection.

In the case when $N = 3$, the Bethe ansatz is given by

$$\Psi_Q[\mathbf{x}, \overline{\sigma}] = \sum_{P \in \mathfrak{S}_3} (-1)^{[P]} \Phi_{\overline{\sigma}}(PQ, P^{-1}) \prod_{m=1}^{3} e^{ik_{P(i)}x_i} \tag{3.3.73}$$

with $\overline{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$.

For any $\sigma = \uparrow, \downarrow$ there are 4 possible cases $\{(\uparrow,\uparrow,\uparrow); (\uparrow,\uparrow,\downarrow); (\uparrow,\downarrow,\downarrow); (\downarrow,\downarrow,\downarrow)\}$ and only 2 basic cases exists: $\{(\uparrow,\uparrow,\uparrow); (\uparrow,\uparrow,\downarrow)\}$. The rest is connected by the symmetry between the spin up and spin down.

Case $\overline{\sigma} = (\uparrow,\uparrow,\uparrow)$*:* we want to find how many independent coefficients $\Phi_{\overline{\sigma}}(PQ, P^{-1})$ exist in the Bethe ansatz. At first we remark that there is only one sector $Q = id$: $x_1 < x_2 < x_3$, thus we have only $\Phi_{\overline{\sigma}}(P, P^{-1})$ coefficients. Secondly, one should use the relations (3.3.50):

$$\Phi_{\overline{\sigma}}(P, P^{-1}) = \Phi_{\overline{\sigma}}(P, P^{-1}\Pi_{P(i),P(j)}) \tag{3.3.74}$$

It is possible to show using the fact that any permutation element P can be written as a product of permutations Π_{ij}, hence one can find using the above expression that all coefficients $\Phi_{\overline{\sigma}}(P, P^{-1}) = \Phi_{\overline{\sigma}}(P, id)$. For $N = 3$, it is not hard to verify this statement.

Thus, the Bethe ansatz can be written intermediately as

$$\Psi_Q[\mathbf{x}, \overline{\sigma}] = \sum_{P \in \mathfrak{S}_3} (-1)^{[P]} \Phi_{\overline{\sigma}}(P, id) \prod_{m=1}^{3} e^{ik_{P(i)}x_i}$$

In the next step we will use the relations (3.3.51):

$$\Phi_{\overline{\sigma}}(\Pi_{ab}P, P^{-1}\Pi_{ab}) = \Phi_{\overline{\sigma}}(P, P^{-1}) \tag{3.3.75}$$

with $a = P(i)$, $b = P(i+1)$ to constrain more the coefficients $\Phi_{\overline{\sigma}}(P, id)$

By similar arguments on the structure of P one can prove that for any P using two previous relation we get $\Phi_{\overline{\sigma}}(P, id) = \Phi_{\overline{\sigma}}(id, id)$. For $N = 3$ we can give an example: for $P = \Pi_{23}\Pi_{12}$

$$\Phi_{\overline{\sigma}}(\Pi_{23}\Pi_{12}, id) = \Phi_{\overline{\sigma}}(\Pi_{13}\Pi_{23}, id) = \Phi_{\overline{\sigma}}(\Pi_{23}, id) = \Phi_{\overline{\sigma}}(id, id) \tag{3.3.76}$$

where we used 1) $\Pi_{23}\Pi_{12} = \Pi_{13}\Pi_{23}$ then 2) supposing that $P = \Pi_{23}$ and $P(1) = 1$, $P(2) = 3$ we use (3.3.51): $\Phi_{\overline{\sigma}}(\Pi_{P(1),P(2)}P, id) = \Phi_{\overline{\sigma}}(P, id)$. And 3) similarly, let $P = id$ and $P(2) = 2$, $P(3) = 3$ we use again (3.3.51): $\Phi_{\overline{\sigma}}(\Pi_{P(2),P(3)}P, id) = \Phi_{\overline{\sigma}}(P, id)$.

Finally, we find the Bethe ansatz in this case:

$$\Psi_{id}[\mathbf{x}, \overline{\sigma}] = \Phi_{\overline{\sigma}}(id, id)\left(e^{i\mathbf{k}.\mathbf{x}} - e^{i\Pi_{12}\mathbf{k}.\mathbf{x}} - e^{i\Pi_{23}\mathbf{k}.\mathbf{x}} + e^{i\Pi_{12}\Pi_{23}\mathbf{k}.\mathbf{x}} + e^{i\Pi_{23}\Pi_{12}\mathbf{k}.\mathbf{x}} - e^{i\Pi_{13}\mathbf{k}.\mathbf{x}}\right) \tag{3.3.77}$$

with $\overline{\sigma} = (\uparrow,\uparrow,\uparrow)$ and $\Pi_{ij}\mathbf{k} = (..., \overset{i}{\underset{\downarrow}{k_j}}, ..., \overset{j}{\underset{\downarrow}{k_i}}, ...)$.

Case $\overline{\sigma} = (\uparrow,\uparrow,\downarrow)$*:* again we want to find how many independent coefficients $\Phi_{\overline{\sigma}}(PQ, P^{-1})$ exist in the Bethe ansatz. There are 36 coefficients related to 6 sectors Q and 6 permutations P. For identical particles one should use again the relations (3.3.50):

$$\Phi_{\overline{\sigma}}(PQ, P^{-1}) = \Phi_{\overline{\sigma}}(PQ, P^{-1}\Pi_{PQ(i),PQ(j)}) \tag{3.3.78}$$

when $\sigma_{Q(i)} = \sigma_{Q(j)}$.

Using this relation one can prove that for $\overline{\sigma} = (\uparrow, \uparrow, \downarrow)$:

$$\Phi_{\overline{\sigma}}(Q, \Pi_{12}) = \Phi_{\overline{\sigma}}(Q, id), \quad \Phi_{\overline{\sigma}}(\Pi_{23}Q, \Pi_{12}\Pi_{23}) = \Phi_{\overline{\sigma}}(\Pi_{23}Q, \Pi_{23}) \tag{3.3.79}$$

and

$$\Phi_{\overline{\sigma}}(\Pi_{13}Q, \Pi_{23}\Pi_{12}) = \Phi_{\overline{\sigma}}(\Pi_{13}Q, \Pi_{13}) \tag{3.3.80}$$

thus we have for the moment 18 independant coefficients. A similar situation appears in general case.

Moreover, we should use also the relations (3.3.52) and (3.3.55): for $\sigma_{Q(i)} = \sigma_{Q(i+1)}$

$$\Phi_{\overline{\sigma}}(\Pi_{ab}PQ, P^{-1}) = \Phi_{\overline{\sigma}}(PQ, P^{-1}) \tag{3.3.81}$$

and for $\sigma_{Q(i)} \neq \sigma_{Q(i+1)}$

$$\Phi_{\overline{\sigma}}(\Pi_{ab}PQ, P^{-1}) = t_{ab}\Phi_{\overline{\sigma}}(PQ, P^{-1}) + r_{ab}\Phi_{\overline{\sigma}}(PQ, P^{-1}\Pi_{ab}), \tag{3.3.82}$$

$$\Phi_{\overline{\sigma}}(\Pi_{ab}PQ, P^{-1}\Pi_{ab}) = r_{ab}\Phi_{\overline{\sigma}}(PQ, P^{-1}) + t_{ab}\Phi_{\overline{\sigma}}(PQ, P^{-1}\Pi_{ab}). \tag{3.3.83}$$

with $a = PQ(i)$, $b = PQ(i+1)$.

It can be shown that all coefficients $\Phi_{\overline{\sigma}}(PQ, P^{-1})$ can be generated by $\Phi_{\overline{\sigma}}(id, id)$, $\Phi_{\overline{\sigma}}(id, \Pi_{23})$ and $\Phi_{\overline{\sigma}}(id, \Pi_{13})$. For example:

$$\Phi_{\overline{\sigma}}(\Pi_{23}\Pi_{12}, \Pi_{13}) = t_{13}\Phi_{\overline{\sigma}}(\Pi_{23}, id) + r_{13}\Phi_{\overline{\sigma}}(\Pi_{23}, \Pi_{13}) \tag{3.3.84}$$

it can be seen if supposing $PQ = \Pi_{23}$, $P^{-1} = \Pi_{13}$ then $\Pi_{PQ(1),PQ(2)} = \Pi_{13}$ and $\sigma_{Q(1)} \neq \sigma_{Q(2)}$ we use one of the above relations and get the result.The coefficients $\Phi_{\overline{\sigma}}(\Pi_{23}, id)$ and $\Phi_{\overline{\sigma}}(\Pi_{23}, \Pi_{13})$ can be decomposed:

$$\Phi_{\overline{\sigma}}(\Pi_{23}, id) = t_{23}\Phi_{\overline{\sigma}}(id, id) + r_{23}\Phi_{\overline{\sigma}}(id, \Pi_{23}) \tag{3.3.85}$$

with $PQ = id$, $P^{-1} = id$, $\Pi_{PQ(2),PQ(3)} = \Pi_{23}$ and $\sigma_{Q(2)} \neq \sigma_{Q(3)}$ and

$$\Phi_{\overline{\sigma}}(\Pi_{23}, \Pi_{13}) = \Phi_{\overline{\sigma}}(id, \Pi_{13}) \tag{3.3.86}$$

with $PQ = id$, $P^{-1} = \Pi_{13}$, $\Pi_{PQ(2),PQ(3)} = \Pi_{23}$ and $\sigma_{Q(2)} = \sigma_{Q(3)}$.

Vector $\hat{\Phi}(P)$ and S-matrix. Here we will write explicitly the vector $\hat{\Phi}(P)$ and the matrix $S^{(1)}_{ab}(k_a, k_b)$ for $N = 3$. Following the definition of $\hat{\Phi}(P)$ given by:

$$\hat{\Phi}(P') \equiv \sum_{Q', \overline{\sigma}} \Phi_{\overline{\sigma}}(P', Q') \prod_{i=1}^{N} e_i^{\sigma_{Q'(i)}} \tag{3.3.87}$$

here the summation is a direct sum is over all types of excitations and all corresponding sectors. The vector $\prod_{i=1}^{N} e_i^{\sigma_{Q'(i)}}$ belongs to $V_1 \otimes ... \otimes V_N$, where V is spanned by $\{e^{\uparrow}, e^{\downarrow}\}$ and represents one type of N excitations. One can represent $e^{\uparrow} = e^1$ and $e^{\downarrow} = e^2$ with e^i - elementary vector, thus $\prod_{i=1}^{3} e_i^{\sigma_{Q'(i)}}$ is vector of size 8 with 1 on some place determined by $\sigma_{Q'(i)}$ and 0 on the rest. For example, $e_1^{\uparrow}e_2^{\downarrow}e_3^{\uparrow}$ is a vector $(0,0|1,0|0,0|0,0)^T$ and $e_1^{\downarrow}e_2^{\downarrow}e_3^{\uparrow}$ is a vector $(0,0|0,0|0,0|1,0)^T$.

Therefore, the vector $\hat{\Phi}(P)$ is

$$\begin{aligned}
\hat{\Phi}(P) \;=\; & e_1^{\uparrow}e_2^{\uparrow}e_3^{\uparrow}\Phi_{(\uparrow,\uparrow,\uparrow)}(P, id) + e_1^{\uparrow}e_2^{\uparrow}e_3^{\downarrow}\Phi_{(\uparrow,\uparrow,\downarrow)}(P, id) + e_1^{\uparrow}e_2^{\downarrow}e_3^{\uparrow}\Phi_{(\uparrow,\uparrow,\downarrow)}(P, \Pi_{23}) + \\
& + e_1^{\uparrow}e_2^{\downarrow}e_3^{\downarrow}\Phi_{(\uparrow,\downarrow,\downarrow)}(P, id) + e_1^{\downarrow}e_2^{\uparrow}e_3^{\uparrow}\Phi_{(\uparrow,\uparrow,\downarrow)}(P, \Pi_{13}) + e_1^{\downarrow}e_2^{\uparrow}e_3^{\downarrow}\Phi_{(\uparrow,\downarrow,\downarrow)}(P, \Pi_{12}) \\
& + e_1^{\downarrow}e_2^{\downarrow}e_3^{\uparrow}\Phi_{(\uparrow,\downarrow,\downarrow)}(P, \Pi_{13}) + e_1^{\downarrow}e_2^{\downarrow}e_3^{\downarrow}\Phi_{(\downarrow,\downarrow,\downarrow)}(P, id) =
\end{aligned}$$

$$= \Big(\Phi_{(\uparrow,\uparrow,\uparrow)}(P,id), \Phi_{(\uparrow,\uparrow,\downarrow)}(P,id) \,|\quad \Phi_{(\uparrow,\uparrow,\downarrow)}(P,\Pi_{23}), \Phi_{(\uparrow,\downarrow,\downarrow)}(P,id) \quad |$$
$$\Phi_{(\uparrow,\uparrow,\downarrow)}(P,\Pi_{13}), \Phi_{(\uparrow,\downarrow,\downarrow)}(P,\Pi_{12}) \,|\quad \Phi_{(\uparrow,\downarrow,\downarrow)}(P,\Pi_{13}), \Phi_{(\downarrow,\downarrow,\downarrow)}(P,id)\Big)^T$$

Now we will show in the case of $N = 3$ how to derive the relation (3.3.59):

$$\hat{\Phi}(\Pi_{ab}P) = S^{(1)}_{ab}(k_a,k_b)\hat{\Phi}(P) \tag{3.3.88}$$

where $S^{(1)}_{ab}(k_a,k_b) \in \mathrm{End}(V_1)\otimes\mathrm{End}(V_2)\otimes ...\otimes\mathrm{End}(V_N)$ (simply denoted by $S^{(1)}_{ab}$ in the following) acts non-trivially on the elementary vectors $e_a^{\sigma_{Q'(a)}} e_b^{\sigma_{Q'(b)}}$ with the conditions $(P')^{-1}(a) - (P')^{-1}(b) = -1$. Here P' and Q' are the elements of the permutation group and prime denotes that there are relations with P and Q introduced in the Bethe ansatz: $Q' = P^{-1}$ and $P' = PQ$. Later we will denote by P and Q any permutation.

Firstly, we can write the relations (3.3.52) and (3.3.55) in general:

$$\Phi_{\overline{\sigma}}(\Pi_{ab}P,Q) = \hat{t}_{ab}\Phi_{\overline{\sigma}}(P,Q) + \hat{r}_{ab}\Phi_{\overline{\sigma}}(P,Q\Pi_{ab}), \tag{3.3.89}$$

with

$$\hat{t}_{ab}(Q,\overline{\sigma}) = \begin{cases} 1, & \text{if } \sigma_{Q(a)} = \sigma_{Q(a)} \\ t_{ab}, & \text{if } \sigma_{Q(a)} \neq \sigma_{Q(a)} \end{cases} \quad \text{and} \quad \hat{r}_{ab}(Q,\overline{\sigma}) = \begin{cases} 0, & \text{if } \sigma_{Q(a)} = \sigma_{Q(a)} \\ r_{ab}, & \text{if } \sigma_{Q(a)} \neq \sigma_{Q(a)} \end{cases} \tag{3.3.90}$$

Then we can rewrite $\hat{\Phi}(\Pi_{ab}P)$ as

$$\begin{aligned}\hat{\Phi}(\Pi_{ab}P) &= \sum_{\overline{\sigma},Q}\Phi_{\overline{\sigma}}(\Pi_{ab}P,Q)\prod_{m=1}^{3} e_m^{\sigma_{Q(m)}} = \sum_{\overline{\sigma},Q}\Phi_{\overline{\sigma}}(\Pi_{ab}P,Q)(...e_a^{\sigma_{Q(a)}}...e_b^{\sigma_{Q(b)}}...) = \\ &= \sum_{\overline{\sigma},Q}(\hat{t}_{ab}\Phi_{\overline{\sigma}}(P,Q) + \hat{r}_{ab}\Phi_{\overline{\sigma}}(P,Q\Pi_{ab}))(...e_a^{\sigma_{Q(a)}}...e_b^{\sigma_{Q(b)}}...)\end{aligned}$$

then we separate the sums into two and shift the summation variable Q by $Q\Pi_{ab}$ in the second term. Indeed, the parameters $\hat{t}_{ab}$ and $\hat{r}_{ab}$ are invariant by this shift and we can write then

$$\hat{\Phi}(\Pi_{ab}P) = \sum_{\overline{\sigma},Q}\Phi_{\overline{\sigma}}(P,Q)\Big(\hat{t}_{ab}(...e_a^{\sigma_{Q(a)}}...e_b^{\sigma_{Q(b)}}...) + \hat{r}_{ab}(...e_a^{\sigma_{Q(b)}}...e_b^{\sigma_{Q(a)}}...)\Big) \equiv S^{(1)}_{ab}(k_a,k_b)\hat{\Phi}(P).$$

where the S-matrix acts on $e_a^{\sigma_{Q(b)}}, e_b^{\sigma_{Q(a)}}$ as

$$S^{(1)}_{ab}(k_a,k_b)(...e_a^{\sigma_{Q(a)}}...e_b^{\sigma_{Q(b)}}...) = \hat{t}_{ab}(...e_a^{\sigma_{Q(a)}}...e_b^{\sigma_{Q(b)}}...) + \hat{r}_{ab}(...e_a^{\sigma_{Q(b)}}...e_b^{\sigma_{Q(a)}}...) \tag{3.3.91}$$

and it shows that $S^{(1)}_{ab}(k_a,k_b) \in \mathrm{End}(V_1)\otimes\mathrm{End}(V_2)\otimes\mathrm{End}(V_3)$ acts non-trivially only on a and b particles. This result can be similarly done for any N.

We will give an example: $\hat{\Phi}(\Pi_{12}) = S^{(1)}_{12}(k_1,k_2)\hat{\Phi}(id)$. Hence, in the matrix form it can be written as

$$\begin{pmatrix} \Phi_{(\uparrow,\uparrow,\uparrow)}(\Pi_{12},id) \\ \Phi_{(\uparrow,\uparrow,\downarrow)}(\Pi_{12},id) \\ \hline \Phi_{(\uparrow,\uparrow,\downarrow)}(\Pi_{12},\Pi_{23}) \\ \Phi_{(\uparrow,\downarrow,\downarrow)}(\Pi_{12},id) \\ \hline \Phi_{(\uparrow,\uparrow,\downarrow)}(\Pi_{12},\Pi_{13}) \\ \Phi_{(\uparrow,\downarrow,\downarrow)}(\Pi_{12},\Pi_{12}) \\ \hline \Phi_{(\uparrow,\downarrow,\downarrow)}(\Pi_{12},\Pi_{13}) \\ \Phi_{(\downarrow,\downarrow,\downarrow)}(\Pi_{12},id) \end{pmatrix} = \left(\begin{array}{cc|cc|cc|cc} 1 & & & & & & & \\ & 1 & & & & & & \\ \hline & & t_{12} & & r_{12} & & & \\ & & & t_{12} & & r_{12} & & \\ \hline & & r_{12} & & t_{12} & & & \\ & & & r_{12} & & t_{12} & & \\ \hline & & & & & & 1 & \\ & & & & & & & 1 \end{array}\right) \begin{pmatrix} \Phi_{(\uparrow,\uparrow,\uparrow)}(id,id) \\ \Phi_{(\uparrow,\uparrow,\downarrow)}(id,id) \\ \hline \Phi_{(\uparrow,\uparrow,\downarrow)}(id,\Pi_{23}) \\ \Phi_{(\uparrow,\downarrow,\downarrow)}(id,id) \\ \hline \Phi_{(\uparrow,\uparrow,\downarrow)}(id,\Pi_{13}) \\ \Phi_{(\uparrow,\downarrow,\downarrow)}(id,\Pi_{12}) \\ \hline \Phi_{(\uparrow,\downarrow,\downarrow)}(id,\Pi_{13}) \\ \Phi_{(\downarrow,\downarrow,\downarrow)}(id,id) \end{pmatrix} \tag{3.3.92}$$

and we see that $S^{(1)}_{12}$ acts on non-trivially on the 1st and the 2nd spaces.

3.3.2 Level 2. Algebraic Bethe ansatz method

In this section we describe a method how to diagonalize the auxiliary problem's Hamiltonian. We briefly recall that considering the Hubbard Hamiltonian, the periodic boundary conditions lead to the system of equations (3.3.72). The solution of this system provide the Bethe equations for the Hubbard model.

The idea is that, at first, one can connect $H_j^{(1)}$ to a more general object $\tau(u)$ and using that it is easy to show that all $H_j^{(1)}$ mutually commute for any j. Then, using a technique one can diagonalize $\tau(u)$ and find its eigenvalues and eigenvectors. This method is called *Algebraic Bethe Ansatz*.

The new Hamiltonians $H_j^{(1)}$ are given by the expression (3.3.72):

$$H_j^{(1)} = S_{j+1,j}^{(1)} S_{j+2,j}^{(1)} ... S_{N,j}^{(1)} S_{1,j}^{(1)} ... S_{j-1,j}^{(1)} \tag{3.3.93}$$

At first, we recall one of the properties of $S_{i,j}^{(1)}(k_i, k_j)$: $S_{i,j}^{(1)}(k_i, k_i) = P_{ab}$ is the permutation operator. Now we consider the transfer matrix $\tau(u)$ defined by

$$\tau(u) = Tr_0\Big(S_{j,0}^{(1)}(k_j, u) S_{j+1,0}^{(1)}(k_{j+1}, u) ... S_{N,0}^{(1)}(k_N, u) S_{1,0}^{(1)}(k_1, u) ... S_{j-1,0}^{(1)}(k_{j-1}, u)\Big) \tag{3.3.94}$$

where the trace is taken over some "auxiliary" space 0.

Indeed, one can see that $\tau(k_j)$ is related to $H_j^{(1)}$: using $Tr_0(P_{0,j}) = \mathbb{I}_j$

$$\begin{aligned} \tau(k_j) \quad &= Tr_0\Big(P_{0,j} S_{j+1,0}^{(1)}(k_{j+1}, k_j) ... S_{N,0}^{(1)}(k_N, k_j) S_{1,0}^{(1)}(k_1, k_j) ... S_{j-1,0}^{(1)}(k_{j-1}, k_j)\Big) = \\ &= Tr_0\Big(S_{j+1,j}^{(1)}(k_{j+1}, k_j) ... S_{N,j}^{(1)}(k_N, k_j) S_{1,j}^{(1)}(k_1, k_j) ... S_{j-1,j}^{(1)}(k_{j-1}, k_j) P_{0,j}\Big) = H_j^{(1)} \end{aligned}$$

Next, one can show that $\tau(u)$ commute with $\tau(v)$ for any u and v. One should use the property that $S_{ij}^{(1)}(k_i, k_j, U) = S_{ji}^{(1)}(k_j, k_i, -U)$, where we noted also the dependence on the Hubbard coupling constant U. Hence, we can see that the Hamiltonians $H_j^{(1)}$ commute.

Algebraic Bethe ansatz. As we have just seen the Hamiltonians of the auxiliary problem are related to the transfer matrix $\tau(u)$ which represent the XXX spin chain with inhomogeneities. Indeed, the matrix $S_{0,a}^{(1)}(u, k_b)$ plays the role of the R-matrix and the parameters k_i with $i = 1, .., N$ are inhomogeneity parameters and u is the spectral parameter. Contrarily to the homogeneous case, the Hamiltonian can not be obtained taking the derivation on the spectral parameter of the logarithm of the transfer matrix in the special point. The object $T_0(u, \{k\}) = \tilde{S}_{0,j}^{(1)}(u, k_j) \tilde{S}_{0,j+1}^{(1)}(u, k_{j+1}) ... \tilde{S}_{0,N}^{(1)}(u, k_N)$ $\tilde{S}_{0,1}^{(1)}(u, k_1) ... \tilde{S}_{0,j-1}^{(1)}(u, k_{j-1})$ with

$$\tilde{S}_{0,i}^{(1)}(u, k_i) = S_{0,i}^{(1)}(u, k_i)|_{U \to -U}$$

corresponds to the monodromy matrix. Sometimes it can be presented in the matrix form as

$$T_0(u, \{k\}) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix} \tag{3.3.95}$$

The entries A, B, C and D are huge $2^N \times 2^N$ matrices which represent the spin chain structure. However, in general, A, B, C and D consist of some algebra generators.

As in the R-matrix formalism it can be easily proven that the monodromy matrix $T_0(u,\{k\})$ satisfies the RTT relation:

$$\tilde{S}^{(1)}_{0,0'}(u,v)T_0(u,\{k\})T_{0'}(v,\{k\}) = T_{0'}(v,\{k\})T_0(u,\{k\})\tilde{S}^{(1)}_{0,0'}(u,v) \tag{3.3.96}$$

In the following we review the Algebraic Bethe Ansatz method how to diagonalize the transfer matrix. This method is a general approach in solving the integrable models and it is applicable for a lot of spin chains. That is why we prefer to use it in this case. At first step as in the Coordinate Bethe ansatz we choose the reference state as the tensor product of local pseudo-vacuums:

$$\Omega = \prod_{i=1}^{N} \omega_i \tag{3.3.97}$$

with $\omega_i = e_i^{\uparrow}$ and defined as

$$\tilde{S}^{(1)}_{0,i}(u,k_i)\omega_i = \begin{pmatrix} 1\omega_i & \star \\ 0 & \tilde{t}_{0i}\omega_i \end{pmatrix} \tag{3.3.98}$$

where by $\star$ we denote the action of corresponding part of matrix $S^{(1)}_{i,0}$ on "vacuum" ω_i. Here also noted $\tilde{t}_{0i} = t_{i0}(k_i,u)$.

The monodromy matrix acting on the reference state gives the following

$$T_0(u,\{k\})\Omega = \begin{pmatrix} \Omega & \star \\ 0 & d(u)\Omega \end{pmatrix} \tag{3.3.99}$$

where again we denoted $B(u)\Omega = \star$ and $d(u)$ here is the eigenvalue of $D(u)$ acting on the reference state, $d(u) = \prod_{i=1}^{N} \tilde{t}_{0i}(u,k_i)$. Hence, we see that the reference state is the eigenvector of the transfer matrix $\tau(u) = A(u) + D(u)$ and its eigenvalue equals to

$$\tau(u)\Omega = (1 + \prod_{i=1}^{N} \tilde{t}_{0i})\Omega \tag{3.3.100}$$

and therefore $H_j^{(1)}\Omega = \Omega$. Indeed, that $\prod_{i=1}^{N} \tilde{t}_{0i} = 0$ due to $t_{jj}(k_j,k_j) = 0$.

We can remark that $B(u)$ and $C(u)$ play the role of the excitation creation and annihilation operators. Thus, the M-excitations function is given by

$$\Phi_M(\{u_i\}) = B(u_1)...B(u_M)\Omega \tag{3.3.101}$$

When the transfer matrix $\tau(u)$ acts on $\Phi_M(\{u_i\})$ one should use the commutation relations between $A(u)$ and $B(v)$ (similarly between $D(u)$ and $B(v)$) in order to let $A(u)$ and $D(u)$ pass through the creation operators $B(u_i)$. Now we stop for the moment our discussion and find these commutation relations. Apparently, they comes from the RTT relation for the monodromy matrix (3.3.96). For a moment we will denote the entries of $T_0(u,\{k\})$, the elements $A(u), B(u), C(u)$ and $D(u)$ by $T^{11}(u), T^{22}(u), T^{33}(u)$ and $T^{44}(u)$ correspondingly, therefore from the RTT relation one gets

$$(\tilde{t}_{00'}E_0^{ii}E_{0'}^{jj} + \tilde{r}_{00'}E_0^{ij}E_{0'}^{ji})(T^{mn}(u)E_0^{mn})(T^{kl}(v)E_0^{kl}) = (T^{kl}(v)E_0^{kl})(T^{mn}(u)E_0^{mn})(\tilde{t}_{00'}E_0^{ii}E_{0'}^{jj} + \tilde{r}_{00'}E_0^{ij}E_{0'}^{ji})$$

where E_a^{ij} is an elementary matrix on the site a.

Expanding the terms and projecting the results on the $E_0^{ij}E_{0'}^{kl}$ one can get:

$$[T^{ij}(u), T^{kl}(v)] = -\frac{\tilde{r}_{00'}(u,v)}{\tilde{t}_{00'}(u,v)}\Big(T^{kj}(u)T^{il}(v) - T^{kj}(v)T^{il}(u)\Big) \tag{3.3.102}$$

what is the Yangian commutation relations.

Thus, the relations we are interested in can be deduced from above:

$$A(u)B(v) = \frac{1}{\tilde{t}_{0'0}(v,u)}B(v)A(u) + \frac{\tilde{r}_{00'}(u,v)}{\tilde{t}_{00'}(u,v)}B(u)A(v) \tag{3.3.103}$$

and

$$D(u)B(v) = \frac{1}{\tilde{t}_{00'}(u,v)}B(v)D(u) - \frac{\tilde{r}_{00'}(u,v)}{\tilde{t}_{00'}(u,v)}B(u)D(v) \tag{3.3.104}$$

We use these commutation relations to calculate the result of $\tau(u)$ on $\Phi_M(\{u_i\})$. Indeed, when $A(u)$ and $D(u)$ pass throught $B(u_i)$, each of them produce 2 terms. We will have one so-called "wanted" term proportional to $\Phi_M(\{u_i\})$ and plenty of other terms. The latter is called "unwanted" terms and they should vanish due to the corresponding Bethe equations. Thus, we have

$$\tau(u)\Phi_M(\{u_i\}) = \Lambda(u,\{u_i\})B(u_1)...B(u_M)\Omega + \sum_{m=1}^{M} K_m(u,\{u_i\})\, B(u_1)...B(u_m = u)...B(u_M)\Omega \tag{3.3.105}$$

We want to determine more precisely the eigenvalue of $\tau(u)$ with $u = k_j$ and this implies that $d(k_j) = 0$. It means that the contribution to wanted term coming from the $D(k_j)$ operator is equal to zero. Thus, in this case due to (3.3.103) we have the eigenvalue $\Lambda(u,\{u_i\})$ equals

$$\Lambda(k_j,\{u_i\}) = \prod_{i=1}^{M}\tilde{t}_{i0}^{-1}(u_i,k_j) = \prod_{i=1}^{M}\frac{a_i - \sin(k_j) - iU/4}{a_i - \sin(k_j) + iU/4} \tag{3.3.106}$$

where we changed the notations for the new Bethe roots u_i: $\sin(u_i) = a_i + iU/4$.

The coefficients $K_m(k_j,\{u_i\})$ can be calculated using the original arguments of the paper on the Quantum Inverse Scattering Method by L.Faddeev and L.Takhtajan [68] or see the book of V.E. Korepin, N.M. Bogoliubov, A.G. Izergin, [12]. The property that the operators $B(u)$ and $B(v)$ commute yields the 'symmetry' between the coefficients $K_m(u,\{u_i\})$. Let us rewrite the eigenfunction $\Phi_M(\{u_i\})$ in the form:

$$\Phi_M(\{u_i\}) = B(u_m)\prod_{\substack{k=1\\k\neq m}}^{M} B(u_k)\Omega \tag{3.3.107}$$

When the operator $A(k_j)$ (or $D(k_j)$) passes $B(u_m)$ in $\Phi_M(\{u_i\})$ we get the unwanted term in the form:

$$A(k_j)\Phi_M(\{u_i\}) = \left(\frac{\tilde{r}_{0m}(k_j,u_m)}{\tilde{t}_{0m}(k_j,u_m)}\right) B(k_j)A(u_m)\prod_{\substack{k=1\\k\neq m}}^{M} B(u_k)\Omega \tag{3.3.108}$$

$$D(k_j)\Phi_M(\{u_i\}) = -\left(\frac{\tilde{r}_{0m}(k_j,u_m)}{\tilde{t}_{0m}(k_j,u_m)}\right) B(k_j)A(u_m)\prod_{\substack{k=1\\k\neq m}}^{M} B(u_k)\Omega \tag{3.3.109}$$

In order to obtain the coefficient $K_m(k_j,\{u_i\})$ the operator $A(u_m)$ should pass through $\prod B(u_k)\Omega$ without producing the unwanted terms, otherwise the unwanted term would not correspond to $K_m(k_j,\{u_i\})$. Thus, we obtain

$$A(k_j)\Phi_M(\{u_i\}) = \left(\frac{\tilde{r}_{0m}(k_j,u_m)}{\tilde{t}_{0m}(k_j,u_m)}\right)\prod_{\substack{l=1\\l\neq m}}^{M}\left(\frac{1}{\tilde{t}_{lm}(u_l,u_m)}\right) B(k_j)\prod_{\substack{k=1\\k\neq m}}^{M} B(u_k)\Omega \tag{3.3.110}$$

$$D(k_j)\Phi_M(\{u_i\}) = -d(u_m)\left(\frac{\tilde{r}_{0m}(k_j,u_m)}{\tilde{t}_{0m}(k_j,u_m)}\right)\prod_{\substack{l=1\\l\neq m}}^{M}\left(\frac{1}{\tilde{t}_{ml}(u_m,u_l)}\right) B(k_j)\prod_{\substack{k=1\\k\neq m}}^{M} B(u_k)\Omega \tag{3.3.111}$$

and therefore the coefficients $K_m(k_j, \{u_i\})$ for any $m = 1, ...M$

$$K_m(k_j, \{u_i\}) = \left(\frac{\tilde{r}_{0m}(k_j, u_m)}{\tilde{t}_{0m}(k_j, u_m)}\right) \Big(\prod_{\substack{l=1 \\ l\neq m}}^{M} \left(\frac{1}{\tilde{t}_{lm}(u_l, u_m)}\right) - d(u_m) \prod_{\substack{l=1 \\ l\neq m}}^{M} \left(\frac{1}{\tilde{t}_{ml}(u_m, u_l)}\right) \Big) \tag{3.3.112}$$

Finally, we obtain the Bethe equations for $i = 1, ..., M$

$$\prod_{j=1}^{N} \frac{a_i - \sin(k_j) + iU/4}{a_i - \sin(k_j) - iU/4} = \prod_{l\neq i}^{M} \frac{a_i - a_l + iU/2}{a_i - a_l - iU/2} \tag{3.3.113}$$

One should mention that number of excitations M is restricted by condition $M \leq N/2$ due to "highest weight" properties of vectors $\Phi_M(\{u_i\})$. Namely, at first one can easily see that $S^{(1)}_{ij}$ has $su(2)$ symmetry, e.g commute with the represented coproduct $\Delta_{ij}(g)$ where $g \in su(2)$: $[S^{(1)}_{ij}, g_i + g_j] = 0$. Then, one can find that $[T_0(u, \{k\}), g_0 + g_1 ... + g_N] = 0$. The generators of $su(2)$ algebra can be written as $s^+ = \sum_{i=1}^{N} E^{12}_i$, $s^- = \sum_{i=1}^{N} E^{21}_i$ and $s^3 = \sum_{i=1}^{N}(E^{11}_i - E^{22}_i)$, they acts on the tensor product of N vector spaces. Due to the highest weight properties of the vacuum Ω, one finds that $s^+\Omega = 0$. Moreover, it can be proven that $s^+\Phi_M(\{u_i\}) = 0$ using the commutation relations from $[B(u), s^+] = D(u) - A(u)$ and the Bethe equations (3.3.113). The latter result means that $\Phi_M(\{u_i\})$ are the highest weight vectors with non-negative values of s^3: $s^3\Phi_M(\{u_i\}) = (N - 2M)\Phi_M(\{u_i\})$. All eigenvectors can be obtained by acting on $\Phi_M(\{u_i\})$ with s^-.

These results conclude the resolution of the level 2 auxiliary problem. Thus, we have found the eigenvalues Λ_j and the eigenfunctions $\Phi_M(\{u_i\})$ of the matrix $\tau(k_j)$ which is directly related to the auxiliary Hamiltonians $H^{(1)}_j = \tau(k_j)$:

$$H^{(1)}_j \Phi_M(\{u_i\}) = \Lambda(k_j, \{u_i\})\Phi_M(\{u_i\}) \quad \text{with} \tag{3.3.114}$$

$$\Phi_M(\{u_i\}) = B(u_1)...B(u_M)\Omega \tag{3.3.115}$$

under conditions (3.3.113).

The connection between $\hat{\Phi}(id)$ given in (3.3.72) and $\Phi_M(\{u_i\})$ can be established writing explicitly the operator $B(u)$ as a matrix and associating Ω with $\Phi_{(\uparrow,...,\uparrow)}(id, id)\ \prod_{i=1}^{N} e^{\uparrow}_i$. For more details on the construction of the eigenvectors see [51, 55].

Results for the Hubbard model. In this paragraph we summarize the obtained results for the Hubbard model: the Bethe equations and the energy. The Hubbard Hamiltonian is

$$\mathcal{H}_{Hub} = -\sum_{i=1}^{L} \sum_{\sigma=\uparrow,\downarrow} \left(c^{\dagger}_{i,\sigma} c_{i+1,\sigma} + c^{\dagger}_{i+1,\sigma} c_{i,\sigma}\right) + \frac{U}{4} \sum_{i=1}^{L} (1 - 2n_{i,\uparrow})(1 - 2n_{i,\downarrow}), \quad L + 1 \equiv 1 \tag{3.3.116}$$

Its eigenvectors $\phi_N[\overline{\sigma}]$ are parametrized by the Bethe roots k_i for $i = 1, ..., N$ and a_l for $l = 1, ..., M$, however the energy depends only on one type of the Bethe roots and is given by

$$E = -2 \sum_{m=1}^{N} \cos(k_m) + U(L - 2N)/4 \tag{3.3.117}$$

The quantifications of the Bethe roots is given by the following system of equations (the Lieb-Wu equations):

$$e^{ik_jL} = \prod_{i=1}^{M} \frac{a_i - \sin(k_j) - iU/4}{a_i - \sin(k_j) + iU/4}, \tag{3.3.118}$$

$$\prod_{j=1}^{N} \frac{a_i - \sin(k_j) + iU/4}{a_i - \sin(k_j) - iU/4} = \prod_{l\neq i}^{M} \frac{a_i - a_l + iU/2}{a_i - a_l - iU/2} \tag{3.3.119}$$

for $j = 1, ..., N$ and $i = 1, ..., M$.

Indeed, the obtained results in this point do not end the resolution of the Hubbard model, but we stop here for a moment to discuss different approaches to get the same results.

On algebraic methods for Hubbard model

As it was said in very begining, the Bethe equations for the Hubbard model were firstly derived by E.H.Lieb and F.Y.Wu in 1968 using the Coordinate Bethe ansatz technique. However at that time it was not possible to say whether the Hubbard model is integrable or not, the R-matrix formalism and Shasty's R-matrix have appeared later. The Algebraic Bethe ansatz was developed in 80s and it allows to find the spectrum and the eigenfunctions of a more general object which consists the Hamiltonian. Regardless its complexity this approach is considered to be more general than the Coordinate Bethe ansatz due to the "algebraic foundation" behind the R-matrix. Indeed, in general *RTT relations* define the commutation relations for the generators T of an Hopf algebra, then choosing the representation of the generators T, one can construct a matrix object which consists of the Hamiltonian. Indeed, the choice of the representation defines different models of the same "class" of models. Comparing with the level 2 of the nested Bethe ansatz approach we used in the previous subsection, the RTT relations in that case define the Yangian algebra. Then, we used the fundamental representation of the Yangian algebra and constructed the generators T as a product of the R-matrices with additional parameters.

Inspite of the progress made in the resolution of various models via the Algebraic Bethe ansatz, the Hubbard model has stayed aside until the work of M.J.Martins and P.B.Ramos [69]. Almost at the same time a similar method using analytical ansatz was proposed by T.Deguchi and R.Yue in [70]. In [69], the Algebraic method was used to derive the Bethe equations. Due to complexity of the commutation relations the main difficulty is to write properly all "wanted" and "unwanted" terms in the general case of N excitations.

3.4 Solutions of Lieb-Wu equations

The Lieb-Wu equations arise in the resolution via the Coordinate Bethe ansatz of the one-dimensional Hubbard model. The solutions of these equations allows to calculate the eigenvectors and the values of the spectrum. One can also give the answer on the completeness of the ansatz. However, commonly in the treatment of the Bethe equations, the thermodynamic limit and the string hypothesis are used to find the solutions. In 1972 the classification of the solutions of the Lieb-Wu equations was proposed by M.Takahashi [49], which is referred as "Takahashi's string hypothesis". Using this hypothesis it is possible to obtain a set of nonlinear integral equations that determines the thermodynamics of the Hubbard model. Solving these equations in some limiting cases, M.Takahashi was able to calculate the low temperature specific heat in [50].

Further analysis of the Lieb-Wu equations was done by F.Woynarovich in 80's in [51]. He resumed the study of the excitation spectrum of the Hubbard model, gave a detailed analysis of the charge excitations at half-filling and he presented the explicit form of the Bethe ansatz wavefunction.

The proof of completeness of the Bethe ansatz was given in [52] and it is based on Takahashi's string hypothesis and the $so(4)$ symmetry. More precisely, it was found that the eigenfunction in the Bethe ansatz $\Psi(N_{e\uparrow}, N_{e\uparrow})$ is the lowest weight state of the $so(4)$ symmetry (3.2.24),(3.2.24): $\zeta\Psi(N_{e\uparrow}, N_{e\uparrow}) = 0$, $\eta\Psi(N_{e\uparrow}, N_{e\uparrow}) = 0$. The action of the raising operators $\zeta^\dagger$ and $\eta^\dagger$ on $\Psi(N_{e\uparrow}, N_{e\uparrow})$ leads to new states which are not predicted by the Lieb-Wu equations. Therefore, the combinatorics due to the Lieb-Wu equations with Takahashi's string hypothesis and the $so(4)$ symmetry gives the necessary result 4^L. We remark that in the case when L is odd one can introduce some operators which allows together with the remaining $su(2)$ symmetry to obtain all states from those predicted by the Bethe ansatz, see i.g. [71].

In the following sections we consider Lieb and Wu original approach to solve the Lieb-Wu equations. We suppose all Bethe roots to be real and take the thermodynamic limit $L \to \infty$, infinite lattice with constant particle densities. In that limit the Lieb-Wu equations can be rewritten in form of integrals. At the half-filing limit when number of spin up particles equals the number of spin down particles and is a half of the lenght of the lattice, we will write the ground state energy and the root densities.

3.4.1 Real solutions of the Lieb-Wu equations

The Lieb-Wu equations obtained in (3.3.119) are written as

$$\begin{aligned} e^{ik_jL} &= \prod_{i=1}^{M} \frac{a_i - \sin(k_j) - iU/4}{a_i - \sin(k_j) + iU/4}, \\ \prod_{j=1}^{N} \frac{a_i - \sin(k_j) + iU/4}{a_i - \sin(k_j) - iU/4} &= -\prod_{l=1}^{M} \frac{a_i - a_l + iU/2}{a_i - a_l - iU/2} \end{aligned} \tag{3.4.1}$$

for $j = 1, ..., N$ and $i = 1, ..., M$.

We suppose that the Bethe roots k_i and a_j are reals and ordered: $k_1 < k_2 < ... < k_N$ and $a_1 < a_2 ... < a_M$. In addition, k_i are defined in the sector between $-\pi$ and π whereas a_j are not restricted.

Firstly, we define a new function $\theta(x)$ such that

$$\theta(x) \equiv -2\arctan(\frac{2x}{U}) \tag{3.4.2}$$

here arctan is considered to be bounded by $-\pi/2 \leq \arctan(x) \leq \pi/2$.

This function $\theta(x)$ can be related to the log of a complex function.

$$\frac{1}{i}\log(\frac{x+ic}{x-ic}) = \pi - 2\arctan(\frac{x}{c}) \tag{3.4.3}$$

with $x, c > 0$.

Now we can analyze the Lieb-Wu equations. We take the logarithm and use (3.4.3) with the

introduced function $\theta(x)$:

1. $$k_j L = 2\pi I'_j + \sum_{i=1}^{M} \frac{1}{i} \log\left(\frac{a_i - \sin k_j - iU/4}{a_i - \sin k_j + iU/4}\right) = 2\pi\Big(I'_j + \frac{M}{2}\Big) + \sum_{i=1}^{M} \theta(2\sin k_j - 2a_i)$$

2. $$\sum_{j=1}^{N} \frac{1}{i} \log\left(\frac{a_i - \sin k_j - iU/4}{a_i - \sin k_j + iU/4}\right) = 2\pi J'_i + \pi + \sum_{l=1}^{M} \frac{1}{i} \log\left(\frac{a_i - a_l - iU/2}{a_i - a_l + iU/2}\right) \to$$
$$\sum_{j=1}^{N} \theta(2\sin k_j - 2a_i) = 2\pi\Big(J'_i - \frac{N}{2} + \frac{M+1}{2}\Big) - \sum_{l=1}^{M} \theta(a_i - a_l)$$

where I'_j, J'_i are integers.
Thus, finally we get

$$k_j L = 2\pi I_j + \sum_{i=1}^{M} \theta(2\sin k_j - 2a_i) \tag{3.4.4}$$

$$\sum_{j=1}^{N} \theta(2\sin k_j - 2a_i) = 2\pi J_i - \sum_{l=1}^{M} \theta(a_i - a_l) \tag{3.4.5}$$

where we introduced variables I_j, J_i. After redefinition we see that: *i)* I_j is integer if M is even and half-integer if M is odd; *ii)* J_i is integer if $N - M + 1$ is even and half-integer if $N - M + 1$ is odd.

The integers I_j, J_i count different solutions for k_j, a_i of the logarithmic Lieb-Wu equations (3.4.4), (3.4.5)[10] and their permitted values can be determined by the procedure introduced by C.N.Yang and C.P.Yang in [36]. Introducing the counting functions $I(k)$ and $J(a)$ defined by

$$I(k) = k - \frac{1}{L} \sum_{i=1}^{M} \theta(2\sin k - 2a_i), \tag{3.4.6}$$

$$J(a) = \frac{1}{L} \sum_{j=1}^{N} \theta(2\sin k_j - 2a) + \frac{1}{L} \sum_{l=1}^{M} \theta(a - a_l) \tag{3.4.7}$$

and evaluated in the points $I(k_j) = 2\pi I_j/L$ and $J(a_i) = 2\pi J_i/L$. The crucial property of $I(k)$ and $J(a)$ is that they are monotonically increasing functions of their arguments[11].

The range of I_j can be found using the property that the logarithmic Lieb-Wu equations are 'invariant' shifting $k_j \to k_j + 2\pi$, which implies $I(k + 2\pi) - I(k) = 2\pi$. For k defined in $-\pi$ to π we get $I(\pi) - I(-\pi) = 2\pi$ and finally, one gets

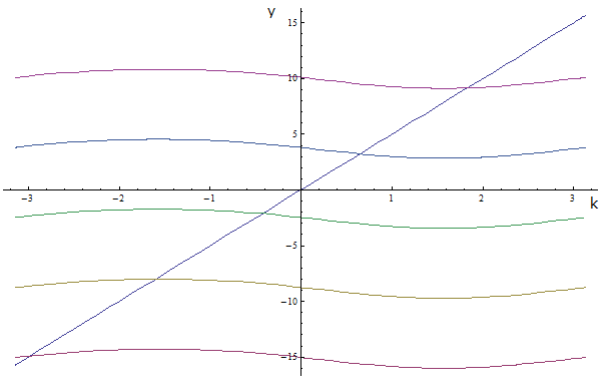
$$-\frac{L}{2} \leq I_j < \frac{L}{2} \tag{3.4.8}$$

A similar analysis can be performed for the second equation: namely $J_i < \frac{L}{2\pi} \lim_{a\to\infty}(J(a))$ and $J_i > \frac{L}{2\pi} \lim_{a\to-\infty}(J(a))$:

$$J(a)|_{a\to\infty} = \frac{1}{L}\Big(2\pi N - 2\pi(M-1)\Big)$$
$$J(a)|_{a\to-\infty} = \frac{1}{L}\Big(-2\pi N + 2\pi(M-1)\Big)$$

[10]For detailed analysis of the Lieb-Wu equations and correspondence with the logarithmic ones reader is refered to [54] or [55].

[11]We assume here that $LU > 8$.



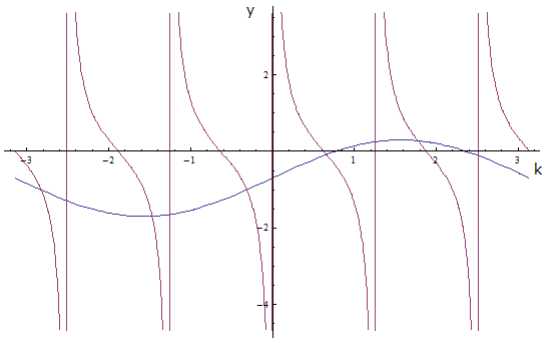


Figure 3.1: Left. Plot of the first logarithmic Lieb-Wu equation: $y = kL$ and $y = 2\pi I + \theta(2\sin(k) - 2a)$ for $L = 5$, $N = 2$ and $M = 1$, the root $a = 0.7$ and $U = 4$. The 5 curves correspond to the values $I = -\frac{5}{2}, -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}$ of $y = 2\pi I + \theta(2\sin(k) - 2a)$.

Figure 3.2: Right. Plot of $y = \sin k_j - a$ and $y = \frac{U}{4}\cot(\frac{k_j L}{2})$ for $L = 5$, the root $a = 0.7$ and $U = 4$.

and thus we get

$$-\frac{N-M+1}{2} < J_i < \frac{N-M+1}{2} \tag{3.4.9}$$

The fact that I_j, J_i count different solutions for k_j, a_i one can see from taking the most simplest example: $M = 0$ which implies $k_j L = 2\pi I_j$ and the solutions are $k_j = 2\pi I_j/L$ with the condition $\pi \leq k_j < \pi$, thus one value of I_j corresponds one value of k_j. Similarly, one can consider the situation $M = 1$ and $N = 2$. There is only one possible value of $J_1 = 0$, hence there is only one real solution for a_i. Considering roots k_1 and k_2, they are distributed between $-\pi$ and π such that $k_1 \neq k_2$. The integers I_j take half-integer values in between $-L/2$ and $L/2$, what leads to $L!/(2!(L-2)!)$ solutions for all different roots k_1 and k_2. One can justify that ploting the first equation of the logarithmic Lieb-Wu equations (3.4.4) and taking a_i as a parameter. For example, $L = 5$ we have $I_j \in \{-\frac{5}{2}, -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}\}$ and comparing with figure 3.1 there 5 graphical solutions for k_j. Using this example we can remark a property of the counting functions $I(k)$ and $J(a)$. We have 5 solutions $\{q_1, q_2, ..., q_5\}$ of equation $kL = 2\pi I + \theta(2\sin(k) - 2a)$, thus the quantity $2\pi(I(q_i) - I(q_j))/L$ with $q_i > q_j$ gives the number of possible solutions k in the interval $[q_j, q_i[$.

A similar analysis of the logarithmic Lieb-Wu equations (3.4.4),(3.4.5) can be done for $M = 1$ and any number N. We can rewrite the Lieb-Wu equations (3.4.1) in the form: $i)$ we take the product on all $j = 1, ..., N$ of the first equation and use the second equation, then $ii)$ we solve directly the first equation for $\sin k_j - a_1$, and finally we get

$$\sum_{j=1}^{N} k_j = 2\pi\frac{m}{L}, \quad m = 0, ..., L-1 \tag{3.4.10}$$

$$\sin k_j - a_1 = \frac{U}{4}\cot(\frac{k_j L}{2}), \quad j = 1, ..., N \tag{3.4.11}$$

The last equation can be easily solved graphically for the variable k_j as a function of a_1, see figure 3.2. Considering carefully the behavior of the curves depending on paramaters U and L, there is a

condition, namely $LU > 8$, should be satisfied. This condition comes from the requirement to have the unique intersection with each branch of the cotangent.

The total momentum $P = \sum_{j=1}^{N} k_j$ can be determined by the integers I_j and J_i using the logarithmic Lieb-Wu equations (3.4.4), (3.4.5) and the antisymmetric property $\theta(-x) = -\theta(x)$, thus:

$$P = \frac{2\pi}{L}\sum_{j=1}^{N} I_j + \frac{1}{L}\sum_{j=1}^{N}\sum_{i=1}^{M}\theta(2\sin k_j - 2a_i) = \frac{2\pi}{L}\sum_{j=1}^{N} I_j + \frac{2\pi}{L}\sum_{i=1}^{M} J_i \tag{3.4.12}$$

Due to the complexity in solving the logarithmic equations in general case, next step will be to take the thermodynamic limit.

Thermodynamic limit. We consider the thermodynamic limit $L, N, M \to \infty$ such that the densities $\frac{N}{L}$, $\frac{M}{L}$ are constant. In this limit the real roots k_j and a_i become close to each other: $k_j - k_{j-1} \to 0$ and $a_i - a_{i-1} \to 0$ with $L \to \infty$. It can be seen from the following: considering (3.4.4) for k_j and k_{j-1}, one can subtract these two equations, then the absolute value is

$$|k_j - k_{j-1}| = \frac{2\pi|I_j - I_{j-1}|}{L} + \frac{1}{L}\sum_{i=1}^{M}|\theta(2\sin k_j - a_i) - \theta(2\sin k_{j-1} - a_i)| \to 0, \quad \text{with } L \to \infty \tag{3.4.13}$$

where the second term $\frac{1}{L}\sum_{i=1}^{M}|\theta(2\sin k_j - a_i) - \theta(2\sin k_{j-1} - a_i)|$ can be bounded by $\frac{2\pi}{L}$ which goes to zero in the limit $L \to \infty$. The first term similarly vanishes in the limit.

Considering (3.4.5), one can develop all terms consisting a_i in series on $U >> 1$: for example $\theta(2\sin k_j - 2a_i) \approx -\frac{2\sin k_j - 2a_i}{U}$ then we get

$$|a_i - a_{i-1}| = \frac{2\pi|J_i - J_{i-1}|}{L}\frac{U}{2(N/L) - (M/L) + 1/L} \to 0, \quad \text{with } L \to \infty \tag{3.4.14}$$

Therefore, the roots k_j and a_i become continious parameters k and a, distributed in the sectors between $-Q$,Q and respectively between $-B$,B. Here Q is the maximal value of k such that $Q \leq \pi$ and B is the maximal value of a with $B < \infty$.

Now, we can define the root densities $\rho(k)$ and $\sigma(a)$ such that they give the number of roots k_j and a_i in a small interval dk and da.

$$L\rho(k)dk = \text{number of solutions } k \text{ in } dk$$
$$L\sigma(a)da = \text{number of solutions } a \text{ in } da$$

and they are normalized as

$$\int_{-Q}^{Q}\rho(k)dk = \frac{N}{L} \quad \text{and} \quad \int_{-B}^{B}\sigma(a)da = \frac{M}{L} \tag{3.4.15}$$

On the other hand, we can relate these densities to the counting functions $I(k)$ and $J(a)$. More precisely, for a given set of solutions of the logarithmic Lieb-Wu equations (3.4.4), (3.4.5) to each set $\{k_j\}_{j=1,\dots,N}$ ($\{a_i\}_{i=1,\dots,M}$) corresponds a set of values $\{I_j\}_{j=1,\dots,N}$ (respectively $\{J_i\}_{i=1,\dots,M}$). Whereas $\{I_j\}_{j=1,\dots,N}$ are particular points in the interval where all interegers I_j's are defined. In the literature, speaking about particular values $\{I_j\}_{j=1,\dots,N}$ one often mentions the notion of "particles". The "non-occupied" values in the interval where all I_j's are defined are refered as "holes". Thus, the particle's distribution $\rho(k)$ is for k between $-Q$ and Q. Now, using the counting functions, by their definitions:

i) $I(k+dk)-I(k)$ gives the number of all possible solutions k of (3.4.4) in the interval dk and *ii)* $J(a+da)-J(a)$ gives the number of possible solutions a of (3.4.5) in the interval da, it implies that

$$\frac{dI(k)}{dk}=2\pi\rho(k) \quad \text{and} \quad \frac{dJ(a)}{da}=2\pi\sigma(a) \tag{3.4.16}$$

All these introductions are necessary to transform the logarithmic Lieb-Wu equations (3.4.4), (3.4.5) into the integral equations on the densities $\rho(k)$ and $\sigma(a)$, which now play the role of the Bethe roots k and a. Therefore, in the continuum limit the sums can be replaced by integrals:

$$\frac{1}{L}\sum_{j=1}^{N} f(k_j)\to\int_{-Q}^{Q} f(k)\rho(k)dk \quad \text{and} \quad \frac{1}{L}\sum_{i=1}^{M} f(a_i)\to\int_{-B}^{B} f(a)\sigma(a)da$$

The equations (3.4.4), (3.4.5) in the continuum limit become

$$k=I(k)+\int_{-B}^{B}\theta(2\sin k-2a)\sigma(a)da \tag{3.4.17}$$

$$\int_{-Q}^{Q}\theta(2\sin k-2a)\rho(k)dk=J(a)-\int_{-B}^{B}\theta(a-a')\sigma(a')da' \tag{3.4.18}$$

then one can take the derivatives with respect to the variables k and a and obtain

$$1=2\pi\rho(k)-2\cos k\int_{-B}^{B}\theta'(2\sin k-2a)\sigma(a)da \tag{3.4.19}$$

$$-2\int_{-Q}^{Q}\theta'(2\sin k-2a)\rho(k)dk=2\pi\sigma(a)-\int_{-B}^{B}\theta'(a-a')\sigma(a')da' \tag{3.4.20}$$

where $\theta'(x)=\frac{d\theta(x)}{dx}=-\frac{4/U}{1+(2x/U)^2}$.

The energy in the thermodynamic limit writes as

$$E=-2L\int_{-Q}^{Q}\cos(k)\,\rho(k)dk+U(L-2N)/4 \tag{3.4.21}$$

Thus, we have the integral equations on the densities $\rho(k)$ and $\sigma(a)$ to be determined with their normalizations (3.4.15). The latter means that the normalization conditions determine values of Q and B. However, even in this case one should use some special configurations of the model in order to get solutions. Namely, it is possible to solve these equations in the half-filled limit.

Half-filled limit and the ground state. The "half-filled band" limit is the case when particles spin up and down fill the whole lattice and the number of spin up particles equals the number of spin down particles:

$$N=L \quad \text{and} \quad 2M=N \tag{3.4.22}$$

Moreover, there is a theorem given by E.H.Lieb in [47] which states that the ground state is unique for the half-filled band limit and L is even. In thermodynamic limit the last condition is not important.

In the half-filled band limit it is possible to get the exact values of Q and B. We will not rigorously prove certain statements referring to [48], however we show how to get the values. Firstly, we obtain $Q=\pi$. Considering the normalization relations for the densities (3.4.15) and inserting the relation (3.4.16) one finds:

$$1=\int_{-Q}^{Q}\rho(k)dk=\frac{1}{2\pi}\Big(I(Q)-I(-Q)\Big) \tag{3.4.23}$$

Next, we take (3.4.18) with the values $k = Q$, $k = -Q$ and perform the subtraction using the previous results for $I(Q) - I(-Q)$:

$$2Q = 2\pi + \int_{-B}^{B} da\sigma(a)\Big(\theta(2\sin Q - 2a) + \theta(2\sin Q + 2a)\Big)$$

Assuming that $\sigma(a)$ is positive and decreasing for the all values of a, the unique solution of the above equation is $Q = \pi$.

Now we consider the equation (3.4.20) and we integrate it over a from $-\infty$ to $+\infty$ using $\int_{-\infty}^{\infty} \theta'(x)dx = -2\pi$:

$$\int_{-Q}^{Q} \rho(k)dk = \Big(\int_{-\infty}^{\infty} + \int_{-B}^{B}\Big)\sigma(a)da \tag{3.4.24}$$

Next using the normalization equation (3.4.15) we obtain:

$$\frac{N-2M}{L} = \Big(\int_{-\infty}^{-B} + \int_{B}^{\infty}\Big)\sigma(a)da \tag{3.4.25}$$

The RHS is positive due to properties of $\sigma(a)$ and equal zero when $B \to \infty$. Thus, if $B < \infty$ then $N > 2M$ what contradicts to the half-filled band limit and thus the only solution we have is $B = \infty$.

Hence, in the half-filled band limit $Q = \pi$ and $B = \infty$ then the equations (3.4.19), (3.4.20) become:

$$1 = 2\pi\rho(k) - 2\cos k \int_{-\infty}^{\infty} \theta'(2\sin k - 2a)\sigma(a)da \tag{3.4.26}$$

$$-2\int_{-\pi}^{\pi} \theta'(2\sin k - 2a)\rho(k)dk = 2\pi\sigma(a) - \int_{-\infty}^{\infty} \theta'(a-a')\sigma(a')da' \tag{3.4.27}$$

Inserting $\rho(k)$ of the first equation into the second we get the equation on the density $\sigma(a)$:

$$-2\int_{-\pi}^{\pi} \theta'(2\sin k - 2a)\frac{1}{2\pi}dk = 2\pi\sigma(a) - \int_{-\infty}^{\infty} \theta'(a-a')\sigma(a')da' \tag{3.4.28}$$

the second term in RHS of (3.4.26) produce the integral of type $\int_{-\pi}^{\pi} f(\sin k)\cos k dk$ which is equal zero.

The above equation can be solved by the Fourier transform:

$$\hat{\sigma}(\omega) = \int_{-\infty}^{\infty} e^{i\omega a}\sigma(s)da \tag{3.4.29}$$

hence the equation becomes

$$\begin{aligned} -2\int_{-\infty}^{\infty} da e^{i\omega a}\int_{-\pi}^{\pi} \theta'(2\sin k - 2a)\frac{1}{2\pi}dk &= 2\pi\hat{\sigma}(\omega) - \int_{-\infty}^{\infty} da e^{i\omega a}\int_{-\infty}^{\infty} \theta'(a-a')\sigma(a')da' \to \\ \int_{-\pi}^{\pi} dk e^{-|\omega|U/4 + i\omega\sin k} &= 2\pi\hat{\sigma}(\omega)\Big(1 + e^{-|\omega|U/2}\Big) \end{aligned}$$

where $\int_{-\infty}^{\infty} dx \frac{e^{i\omega x}}{x^2+1} = \pi e^{-|\omega|}$.

The LHS of the above equation can be identified with the Bessel function $J_0(\omega) = \frac{1}{\pi}\int_0^{\pi} \cos(\omega\sin x)dx$ therefore we obtain

$$\hat{\sigma}(\omega) = \frac{J_0(\omega)}{2\cosh(|\omega|U/4)} \tag{3.4.30}$$

and the inverse Fourier transformation gives:

$$\sigma(a) = \frac{1}{2\pi}\int_0^\infty d\omega \frac{\cos(\omega\, a) J_0(\omega)}{2\cosh(\omega U/4)} \tag{3.4.31}$$

and respectively $\rho(k)$ can be calculated from (3.4.26):

$$\rho(k) = \frac{1}{2\pi} + 2\cos k \int_{-\infty}^{\infty} \theta'(2\sin k - 2a)\sigma(a)da = \frac{1}{2\pi} + \frac{\cos k}{\pi}\int_0^\infty d\omega \frac{J_0(\omega)\cos(\omega \sin k)}{e^{\omega U/2}+1} \tag{3.4.32}$$

We substitute these results into the expression of the energy (3.4.21) and it yields the ground state energy:

$$E = -2L\int_{-Q}^{Q}\cos(k)\,\rho(k)dk + U(L-2N)/4 = -4L\int_0^\infty d\omega \frac{J_0(\omega)J_1(\omega)}{\omega\left(1+e^{\omega U/2}\right)} + U(L-2N)/4 \tag{3.4.33}$$

where $J_1(\omega) = \frac{\omega}{\pi}\int_0^\infty dx \cos(\omega \sin x)\cos^2 x$ is the Bessel function of order one.

One can simplify the expressions of the densities $\rho(k)$ and $\sigma(a)$ considering some limits: $i)$ no interaction $U = 0$: $\rho(k) = 1/\pi$, $\sigma(a) = (1/2\pi)(1-a^2)^{-1/2}$ for $a < 1$ and zero otherwise. The ground state energy is equal to $E = -4L/\pi$. $ii)$ limit $U = \infty$: $\rho(k) = \frac{1}{2\pi}$, $\sigma(a) = 0$ and the ground state energy is $E = 0$.

Among the solutions of the Lieb-Wu equations, however there are not only real solutions, there exist also complex solutions which can be arranged in the so-called "string" solutions in the thermodynamic limit.

3.5 Applications in the AdS/CFT duality

In this section we give a brief review of applications of the one-dimensional Hubbard model in the AdS/CFT duality[12]. Namely, the Hubbard Hamiltonian in the half-filled band limit was identified with the dilatation operator of the $\mathcal{N} = 4$ super Yang-Mills theory in a certain approximation, [75], [77], and Shastry's R-matrix plays the role of the S-matrix in a subsector of the $\mathcal{N} = 4$ super Yang-Mills theory, [79]. However, certain discrepancies of the Hubbard model and more complete results on the dilatation operator in the field theory have appeared,[77], what partly motivates the studies of generalized Hubbard models. The latter will be considered in the next chapters.

Recent developments in quantum field theories has opened new unexpected directions for the integrable systems. Starting from the paper of J.Minahan and K.Zarembo [73], in the test of the AdS/CFT duality the integrability has made its appearance in the $\mathcal{N} = 4$ super Yang-Mills theory. This observation appeared to be a first hint at a very deep connection between the intergrability and the gauge/string correspondence. We should also mention that integrable spin chains have arised earlier in the perturbative analysis of Regge scattering in the large-N QCD where the Bethe ansatz techniques were extensively applied, [74].

[12]The AdS/CFT duality - the conjecture of the correspondence between the four-dimensional conformal field theory $\mathcal{N} = 4$ super Yang-Mills and the string theory on the $AdS_5 \times S^5$ geometry (the abbreviation AdS_5 refers to an anti-de Sitter space in five dimensions, S^5 refers to a five-dimensional sphere). This conjectured duality is the prime example of a more general assumption which states that certain four-dimensional quantum gauge theories can be alternatively described in terms of closed strings moving in a ten-dimensional curved spacetime. The latter one was proposed by J.Maldacena in 1997 [72] and since then it became an important subject of research in gauge and string theories.

There is a huge amount of literature on the AdS/CFT duality which can be used as reviews on the subject [90] - [96]. In the following we will make a brief introduction to the AdS/CFT duality and we speak about the $\mathcal{N}=4$ super Yang-Mills theory. The connection between the integrability and the $\mathcal{N}=4$ super Yang-Mills theory is established through the so-called dilatation operator. The latter is mapped to integrable spin chains in some subsectors and approxiamations. Below we review some details on the $\mathcal{N}=4$ super Yang-Mills theory and the dilatation operator.

$\mathcal{N}=4$ super Yang-Mills theory. This is a maximally supersymmetric non-abelian gauge theory like the quantum chromodynamics theory (QCD) in 3+1 dimensions with $SU(N)$ gauge group.

The matter content of the theory includes

i) six scalar fields Φ^a, $a=1,...,6$.

ii) four fermionic fields Ψ^A_α and $\bar{\Psi}^{\dot{A}}_{\dot{\alpha}}$ with $A,\dot{A}=1,...,4$ and the spinor indices $\alpha,\dot{\alpha}=1,2$.

iii) gauge fields $\mathcal{A}_\mu$ with the Lorentz vector index $\mu=0,1,2,3$.

The matter content is connected with the global symmetry group which is $PSU(2,2|4)$. The bosonic part of this symmetry group consists of $SO(4,2)\times SO(6)$ group: conformal group in four-dimensions which includes Lorentz group $SO(3,1)$ as a subgroup multiplied by so-called R-symmetry group [13]. In addition, all fields are in the adjoint representation of $SU(N)$ group. The fields transform under the local transformations $U(x)\in SU(N)$ as $W\to U^\dagger W U$ where W denotes Φ^a, Ψ^A_α, $\bar{\Psi}^{\dot{A}}_{\dot{\alpha}}$ and the covariant derivative $\mathcal{D}_\mu=\partial_\mu - ig\mathcal{A}_\mu$.

The Lagrangian of the $\mathcal{N}=4$ super Yang-Mills theory has two free parameters: gauge group parameter N and the coupling constant g. It writes as

$$\mathcal{L}=Tr_N\Big(\frac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}+\frac{1}{2}\mathcal{D}_\mu\Phi^a\mathcal{D}^\mu\Phi_a-\frac{1}{4}g^2[\Phi^a,\Phi^b][\Phi_a,\Phi_b]+\text{fermions}\Big) \tag{3.5.1}$$

where we do not specify the fermionic terms and we denoted the field strength tensor as $\mathcal{F}_{\mu\nu}=ig^{-1}[\mathcal{D}_\mu,\mathcal{D}_\nu]$.

Conformal symmetry. In the global symmetry of the $\mathcal{N}=4$ super Yang-Mills theory there is a conformal group. The conformal group[14] consists of the all transformations which leave the metric tensor $g_{\mu\nu}$ invariant up to a factor: $g'_{\mu\nu}(x')=\Lambda(x)g(x)_{\mu\nu}$. In the case when the dimension of spacetime equal two, the conformal group is an infinite group, but in this case it is composed of the translation and rotation transformations (Lorentz subgroup) and in addition of the dilation and the so-called special conformal transformations:

$$\begin{aligned}
\text{translations}: &\quad x'^\mu=x^\mu+a^\mu\\
\text{rotations}: &\quad x'^\mu=M^\mu_\nu x^\nu\\
\text{dilation}: &\quad x'^\mu=\alpha x^\nu\\
\text{SCF}: &\quad \frac{x'^\mu}{x'^2}=\frac{x^\mu}{x^2}-b^\mu
\end{aligned}$$

There are generators associated with these transformations which form the conformal algebra $so(4,2)$. Together with the bosonic generators of the R-symmetry and the fermionic generators the conformal algebra is enlarged to the $psu(2,2|4)$ algebra.

[13] R-symmetry is the symmetry transforming different supercharges in the theory into each other.

[14] For more details on conformal group and conformal field theories the reader is refered to [97] and the references therein.

Figure 3.3: Common in the literature pictures for a planar diagram (left) and a non-planar diagram (right)

Classical fields of the theory transform under the dilation operator as $D: \quad W(x_\mu) \to \alpha^{\Delta_0} W(\alpha x_\mu)$, where Δ_0 is the mass dimension of the field: $i)$ $\Delta_0 = 1$ for the scalars Φ^a; $ii)$ $\Delta_0 = 3/2$ for the fermions Ψ, $\bar{\Psi}$ and $\Delta_0 = 2$ for the field strength tensor $\mathcal{F}_{\mu\nu}$.

The super Yang-Mills theory as ordinary Yang-Mills or QCD with massless quarks is conformally invariant on the classical level. However, the QCD and the Yang-Mills theories are no more conformally invariant on the quantum level, the invariance under the dilatation transformation is broken. The main consequence of this breaking is the non-vanishing β-function: $\beta(g) < 0$ which leads to a running coupling constant g as in the QCD. The super Yang-Mills, nevertheless, preserve the conformal invariance on the quantum level and has $\beta(g) = 0$, therefore the coupling constant does not renormalize.

The conformal invariance on the quantum level implies huge constraints on the correlation functions of the local operators $\hat{\mathcal{O}}(x)$. The Lorentz invariance requires that the two-point correlation function should be

$$\left\langle \hat{\mathcal{O}}_N(x_1)\hat{\mathcal{O}}_M(x_1)\right\rangle = f_{NM}(x_1 - x_2) \tag{3.5.2}$$

where f_{NM} is an arbitrary function. However, the conformal invariance constrains more:

$$\left\langle \hat{\mathcal{O}}^{ren}_N(x_1)\hat{\mathcal{O}}^{ren}_M(x_1)\right\rangle = \frac{C_{NM}(g)}{(x_1 - x_2)^{\Delta_N(g)+\Delta_M(g)}} \tag{3.5.3}$$

Inspite of $\beta(g) = 0$, there is a renormalization due to the wavefunctions of fields and it implies that the scaling dimensions of the operators obtain quantum corrections: $\Delta(g) = \Delta_0 + \gamma(g)$ where $\gamma(g)$ is an anomalous part. One should remark that the dilatation operator is the generator whose eigenvalues corresponds to the scaling dimensions:

$$D\hat{\mathcal{O}}^{ren}(x) = \Delta(g)\hat{\mathcal{O}}^{ren}(x) \tag{3.5.4}$$

The renormalized operator $\hat{\mathcal{O}}^{ren}(x)$ can mix under the renormalization: $\hat{\mathcal{O}}^{ren}_M(x) = Z_{MN}\hat{\mathcal{O}}^{bare}_N(x)$, where Z_{MN} depends on the cutoff $\log \Lambda$. Equivalently, the dilatation generator D can be represented as $D = D_0 + \delta D$. The diagonal part D_0 which gives the classical mass dimension and the non-diagonal matrix δD is such that its eigenvalues correspond to the anomalous dimensions $\gamma(g)$.

Planar limit. There is a limit proposed by G.'t Hooft as an alternative expansion scheme which leads to important simplifications. Comparing with the QCD, the low energy physics such as the quark confinement is non-perturbative with respect to the coupling constant g. However, the limit when $SU(3)$, the gauge group of the QCD, is replaced by $SU(N)$ and $N \to \infty$ with $\lambda = g^2 N$ fixed, this limit makes sense. The quantity λ is known as the 't Hooft coupling. The same limit can be taken in the $\mathcal{N} = 4$ super Yang-Mills which drastically simplifies the "topology" of Feynman diagrams.

Any Feynman diagram in the $\mathcal{N}=4$ super Yang-Mills can be thought roughly speaking as a two-dimensional surfaces. There are diagrams which can be put on the plane and the others can be put only on some curved surface, see figure 3.3. Hence, in the planar limit there are only planar diagrams which contribute and, thus, it reduces the number of diagrams to consider in the perturbation theory.

Integrability in the $\mathcal{N}=4$ super Yang-Mill theory. As we already mentioned in the begining of the section the integrability has made its first appearance in the $\mathcal{N}=4$ super Yang-Mills theory in the paper of J.Minahan and K.Zarembo [73]. They computed the one-loop anomalous dimensions of an infinite set of single traces for all scalar operators. Thus, the $so(6)$ spin chain Hamiltonian arised in their calculations. Here we will consider only the $su(2)$ sector for simplicity, what means that we have only two complex scalar fields X and Y. They are related to previously introduced scalar fields Φ^a: $X=\Phi^5+i\Phi^6$ and $Y=\Phi^3+i\Phi^4$. Thus, the Heisenberg spin chain will appear.

We are interested in the anomalous dimensions δD of all operators of type: $\hat{\mathcal{O}} \propto Tr_N[X^{J_1}Y^{J_2}]$ as for example $Tr_N[X\,X\,X\,X\,Y]$ or $Tr_N[X\,X\,X\,Y\,X]$ etc. The correspondence with the spin chain is following:

1. Number of sites of the spin chain L is equal J_1+J_2.
2. Spin up (down) on the site i corresponds to the field $X(Y)$ on the place i in the trace.
3. Periodicity of the spin chain is due to the cyclicity of the trace.

The matrix elements of the anomalous dimension δD are determined by the renormalization structure $Z_{MN}(\log\Lambda)$: $\delta D_{MN}=\frac{d\log Z_{MN}}{d\log\Lambda}$.

The calculation of all Z_{MN} corresponding to different diagrams in the one-loop approximation and in the planar limit lead to the following result:

$$\delta D \propto \sum_{i=1}^{L}(\mathbb{I}_{ii+1}-P_{ii+1}) \tag{3.5.5}$$

where P_{ii+1} is the permutation operator which exchanges two spins.

Thus, the diagonalization of the anomalous part of the dilatation operator is transformed into the diagonalization of the spin chain Hamiltonian. The latter one is the prime subject of the integrable systems. As the result, one gets the spectrum of the Hamiltonian, the eigenfunctions and the Bethe equations on the parameters introduced by the diagonalization procedure.

In [98] further development was done on the integrability for the $psu(2,2|4)$ algebra with the construction of the R-matrix corresponding to the one-loop dilatation operator. For higher loops the dilatation operator and the corresponding spin chain Hamiltonian are still unknown. The all-loop conjecture of the Bethe equations (assymtotic all-loop Bethe equations) was given in [99]. The basic ingredient for these equations is the two-body scattering S-matrix, which can be constrained by the supersymmetry up to a scalar phase.

3.5.1 Hubbard model and super Yang-Mills theory

In this subsection we speak about the correspondence between the one-dimensional Hubbard model and the dilatation operator in the $su(2)$ sector of the $\mathcal{N}=4$ super Yang-Mills theory. This observation was firstly reported in [75] and here we briefly review the result.

After the works [73] and [98] where the Hamiltonians in the sector $su(2)$ and for the whole algebra $psu(2,2|4)$ were derived in the one-loop approximation, the higher loop Bethe equations for the dilatation operator were conjectured. Namely, in the $su(2)$ sector the so-called BDS-conjecture of the

Bethe equations for the dilatation operator up to four-loop order [76] has a very compact form:

$$e^{ip_k L} = \prod_{\substack{j=1 \\ j\neq k}}^{M} \frac{u_k - u_j + i}{u_k - u_j - i}, \quad k = 1, ..., M \tag{3.5.6}$$

where the rapidities $u_k = u(p_k)$ are given by

$$u(p) = \frac{1}{2}\cot(p/2)\sqrt{1 + 8g^2 \sin^2(p/2)} \tag{3.5.7}$$

and the energy of the spin chain Hamiltonian is

$$E(g) = -\frac{M}{g^2} + \frac{1}{g^2}\sum_{k=1}^{M}\sqrt{1 + 8g^2 \sin^2(p_k/2)} \tag{3.5.8}$$

This is related to the anomalous dimensions $\Delta(g) = L + g^2 E(g)$ of the operators type $Tr_N(X^{L-M}Y^M)$ and all possible dispositions of X and Y. At one-loop order the above conjecture reduces to the Bethe equations and the energy of the Heisenberg spin chains.

In [75] it was found that the BDS long-range spin chain is the strong coupling limit of the Hubbard model. Several methods were proposed to show the correspondence. The solution for the ground state density of the real Bethe roots and the energy in the thermodynamic limit matches the results of Lieb-Wu. Secondly, the perturbative solutions of the Lieb-Wu equations at half filling in expansion on the coulpling constant U reproduce the spectrum of the BDS spin chain and the Hubbard Hamiltonian corresponds to BDS Hamiltonian. Here we review the first correspondence.

Considering the BDS Bethe equations (3.5.6), we assume that the Bethe roots u_i are all real and we take the logarithm:

$$p_k L = 2\pi I_k - 2\sum_{l=1}^{M}\arctan(u_k - u_l) \tag{3.5.9}$$

In the thermodynamic limit when $L \to \infty$, the Bethe roots become closer $u_{k+1} - u_k \to 0$ and one can introduce the density of the root $\rho(u)$: $2\pi\rho(u) = \frac{dI(u)}{du}$ where $I(u)$ is a counting function ($I(u_k) = (2\pi/L)I_k$). Therefore, the above equation in the thermodynamic limit rewrites as

$$p(u) = I(u) - 2\int_{-\infty}^{\infty} du'\rho(u')\arctan(u' - u) \tag{3.5.10}$$

and taking the derivative on u we obtain the equation on root density $\rho(u)$:

$$\frac{dp(u)}{du} = 2\pi\rho(u) + 2\int_{-\infty}^{\infty}\frac{\rho(u')}{1 + (u - u')^2} \tag{3.5.11}$$

The term $\frac{dp(u)}{du}$ can be calculated using the notations common in the super Yang-Mills theory: $x(u)$ such that $ip(u) = \log(x^+(u)/x^-(u))$ with $x^\pm = x(u \pm i/2)$ and $x(u) = \frac{u}{2}\left(1 + \sqrt{1 - 2g^2/u^2}\right)$. Hence,

$$\frac{dp(u)}{du} = i\Big((u - i/2)^2 - 2g^2\Big)^{-1/2} - i\Big((u + i/2)^2 - 2g^2\Big)^{-1/2} \tag{3.5.12}$$

The solution for the density $\rho(u)$ can be found by the Fourier transformation. In the limit $g = 0$, one get the result for the Heisenberg spin chain: $\rho_0(u) = \frac{1}{2\cosh(\pi u)}$ and for any g one gets:

$$\rho(u) = \frac{1}{2\pi}\int_0^{\infty} d\omega \frac{\cos(\omega u)J_0(\sqrt{2}g\omega)}{\cosh(\omega/2)} \tag{3.5.13}$$

and the energy

$$E(g) = \frac{4L}{\sqrt{2}g}\int_0^\infty d\omega \frac{J_0(\sqrt{2}g\omega)J_1(\sqrt{2}g\omega)}{\omega\left(1+e^\omega\right)} \tag{3.5.14}$$

These result coincides up to a constant with the Lieb-Wu results (3.4.31) and (3.4.33).

The mapping between the Lieb-Wu equations and BDS equations is realized considering the $k-\Lambda$-string solutions[15] of the Lieb-Wu equations at the half-filling band limit. It should be, however, noted that the Hubbard model may not be a candidate for the higher loops spin chain because of the so-called "dressing" factor appeared in [77]:

$$e^{ip_kL} = \prod_{\substack{j=1\\ j\neq k}}^{M} \frac{u_k - u_j + i}{u_k - u_j - i} e^{2i\theta(u_k,u_j)}, \quad k = 1, ..., M \tag{3.5.15}$$

where $\theta(u_k, u_j)$ is some transcendental function.

The Hubbard model gives a trivial factor $\theta = 1$ what contradicts also the explicit perturbation calculation. This dressing phase starts to appear from the fourth order only. It is likely that a some suitable deformation of the Hubbard model would lead to new "dressed" BDS equations.

3.5.2 Shastry's R-matrix and the centrally extended $psu(2|2)$ algebra

In the previous subsection we discussed the correpondence between the Hubbard Hamiltonian and the dilatation operator in the context of the AdS/CFT duality and it was shown that it predicts the correct result up to the fourth order in the perturbation theory. However, it is not the only correspondence, Shastry's R-matrix was found in the $\mathcal{N} = 4$ super Yang-Mills theory and later in the string theory as a scattering matrix, [79],[83].

By the S-matrix in the $\mathcal{N} = 4$ super Yang-Mills theory [78] it is considered the two-body interaction S-matrix which appears in the first level of the Coordinate Bethe Ansatz[16]. One of its important properties is their symmetry which often helps to construct the S-matrix. The full symmetry algebra of the $\mathcal{N} = 4$ super Yang-Mills theory is $psu(2,2|4)$ algebra, however, it was found that the S-matrix is decomposed into the tensor product of two centrally extended $psu(2|2)$ algebras: $S_{psu(2,2|4)} = S_0 \cdot S_{psu(2|2)} \otimes S_{psu(2|2)}$ where S_0 is a scalar function. The form of the $S_{psu(2|2)}$ was derived in [79] and it can be mapped on Shastry's R-matrix. Further developments were done in this direction and the q-deformed R-matrix with the q-deformed Hubbard model were proposed in [82]. In addition, the Yangian symmetry of $S_{psu(2|2)}$ was discovered in [80].

In this section we will review the main steps in the derivation of the S-matrix $S_{psu(2|2)}$ and discuss its relation with the Hubbard model. The key point is the Hopf algebra of the centrally extended algebra $psu(2|2)$. One considers the fundamental representation and, thus, it allows to find the S-matrix which commutes with the coproduct of the algebra, constrained up to a scalar function. The obtained S-matrix satisfies the Yang–Baxter equation. A similar approach to find the solutions of the Yang–Baxter equation was highly investigated by G.Delius, M.Gould, Y.-Z.Zhang and collaborators, see [21],[22] and the references therein. In other words, the importance of [79] is that the centrally extended $psu(2|2)$ algebra reproduces Shastry's R-matrix.

[15] $k-\Lambda$-string solution is a type of solutions of the Lieb-Wu equations proposed by Takahashi's string hypothesis, [49].

[16] An example of such S-matrix is appeared in the CBA of the Hubbard model, (3.3.62)

Centrally extended $psu(2|2)$ algebra. This algebra is the unique finite superalgebra[17] which admits 3 central extensions, [63]. It consists of 'ordinary' $psu(2|2)$ algebra and 3 central charges. However, the Hopf structure of the centrally extended algebra is different from the $psu(2|2)$ algebra. The complete Hopf stucture of this algebra is discussed in [81], but here we are interested only in the coproduct of the generators. We remark that centrally extended $psu(2|2)$ algebra is not a quantum algebra and it would be naturally assume the trivial coproduct for the generators as for the 'ordinary' $psu(2|2)$ algebra: $\Delta(g) = g \otimes \mathbb{I} + \mathbb{I} \otimes g$. But the central charges play an important role in the appearance of the so-called "braiding" factor which modifies the coproduct. Thus, denoting any generator of the centrally extended $psu(2|2)$ algebra by $\mathcal{J}$, the coproduct is given by

$$\Delta(\mathcal{J}) = \mathcal{J} \otimes \mathbb{I} + \mathcal{U}^{[\mathcal{J}]} \otimes \mathcal{J} \tag{3.5.16}$$

where $\mathcal{U}$ is an abelian generator. The coefficients $[\mathcal{J}]$ are non-zero for fermionic part of the algebra and the central charges.

Due to the Hopf structure there are certain conditions between the braiding factor and central charges such that $\mathcal{U}$ is zero if central charges are removed.

S-matrix and Shastry's R-matrix. In general, the derivation of the S-matrix is closely related to the symmetry of the system. Two-particle scatering S-matrix acts on the tensor product of two multiplets, thus the symmetry of the S-matrix is related to the Hopf structure of the symmetry algebra. Namely, the 'represented' coproduct $\Delta_{12}(g) \equiv (\pi_1 \otimes \pi_2)\Delta(g)$ similarly acts on the tensor product of two vector spaces. If the commutator $[S_{12}, \Delta_{12}(g)] = 0$, the S-matrix has the symmetry and in addition it satisfies the Yang–Baxter equation. Thus, one can find a S-matrix based on the algebra. One should remark that the spectral dependence of such S-matrix usually appears when one considers an evaluation representation of the affine algebras (see [21],[22]). Moreover, the spectral parameter dependence of the S-matrix $S_{12}(u, v)$ with affine algebra symmetry is $S_{12}(u, v) = S_{12}(u - v)$. That is why, in the case of centrally extended algebra, introduced "braiding" generator $\mathcal{U}$ plays a crucial role in the Hopf structure of the considered algebra. We remark that it disappears ($\mathcal{U} = 0$) when the central extensions are removed and the S-matrix is a trivial one. But when the braiding generator $\mathcal{U}$ is not zero, the symmetry leads to the S-matrix with two spectral without difference depedence: $S_{12}(u, v) \neq S_{12}(u - v)$.

Hence, the S-matrix symmetry implies

$$[S_{12}, \Delta_{12}(\mathcal{J})] = 0, \quad \text{for all } \mathcal{J} \tag{3.5.17}$$

where $\Delta_{12}(\mathcal{J})$ is a 'represented' coproduct $\Delta_{12}(\mathcal{J}) \equiv (\pi_1 \otimes \pi_2)\Delta(\mathcal{J})$ and π is a four-dimensional representation of the centrally extended $psu(2|2)$ algebra.

The invariance with $su(2) \oplus su(2)$ subalgebra (bosonic part of the algebra) leads to 10 non-zero

[17] More details on superalgebras and particularly on $gl(2|2)$ and its representation theory, reader is refered to see [62],[23] and the references therein

coefficients in the 16×16 matrix S_{12}:

$$S_{12} = \begin{pmatrix}
A & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{A+B}{2} & 0 & 0 & \frac{A-B}{2} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{F}{2} & 0 & 0 & -\frac{F}{2} & 0 \\
0 & 0 & H & 0 & 0 & 0 & 0 & 0 & -L & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & H & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -L & 0 & 0 & 0 \\
0 & \frac{A-B}{2} & 0 & 0 & \frac{A+B}{2} & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{F}{2} & 0 & 0 & \frac{F}{2} & 0 \\
0 & 0 & 0 & 0 & 0 & A & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & H & 0 & 0 & -L & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & H & 0 & 0 & 0 & 0 & 0 & -L & 0 & 0 \\
0 & 0 & G & 0 & 0 & 0 & 0 & 0 & -K & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & G & 0 & 0 & -K & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -D & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{C}{2} & 0 & 0 & -\frac{C}{2} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{E+D}{2} & 0 & 0 & \frac{1}{2}(D-E) & 0 \\
0 & 0 & 0 & G & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -K & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & G & 0 & 0 & 0 & 0 & 0 & -K & 0 & 0 \\
0 & -\frac{C}{2} & 0 & 0 & \frac{C}{2} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(D-E) & 0 & 0 & \frac{E+D}{2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -D
\end{pmatrix} \tag{3.5.18}$$

These coefficients can be fixed up to a scalar function using the commutator with the fermionic generators. Therefore, the coefficients $A, B, C, ...$ become functions of the spectral parameters $x^{\pm}_{1,2}$ and constants g, $\gamma_{1,2}$. The latter ones are usual notations in gauge/string duality literature and they parametrize the central charges, the braiding factor and fermionic generators of the centrally extended algebra. The table of these coefficients $A, B, C, ...$ can be found in [79].

Relations with the Hubbard model. The S-matrix $S_{12}(x^{\pm}_1, x^{\pm}_2)$ satisfying $[S_{12}, \Delta_{12}(\mathcal{J})] = 0$, for all $\mathcal{J}$, is a solution of the braided version of the graded Yang–Baxter equation. It can be, thus, called the $\check{R}$-matrix using the R-matrix formalism language. It is related to the R-matrix by $R_{12}(x^{\pm}_1, x^{\pm}_2) = P^g_{12} S_{12}(x^{\pm}_1, x^{\pm}_2)$, where P^g_{12} is the graded permutation. This R-matrix R_{12} satisfies the graded Yang–Baxter equation:

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \quad \text{with} \quad R_{ij} = R_{ij}(x^{\pm}_i, x^{\pm}_j) \tag{3.5.19}$$

An interesting feature of this R-matrix is that it can be mapped on Shastry's R-matrix. The mapping is following: the coefficients g, γ_i, $x^{\pm}_i$ are related to trigonometric functions of Shastry's R-matrix depending on two spectral parameters $\lambda_{1,2}$

$$g = \frac{1}{U}, \quad x^{\pm}_i = \left(\frac{i \sin \lambda_i}{\cos \lambda_i}\right)^{\pm 1} e^{2h(\lambda_i)}, \quad \gamma_i = \sqrt{\alpha} \frac{e^{h(\lambda_i)}}{\cos \lambda_i} \tag{3.5.20}$$

There is a constraint between $x^{\pm}_i$ which transforms into the condition on the coupling function of Shastry's R-matrix $h(\lambda)$:

$$4 \sinh(2h(\lambda_i)) = U \sin(2\lambda_i) \tag{3.5.21}$$

Here we will not write explicitly the passage between the R-matrices. We mentioned that the R-matrix $R_{12}(\lambda_1, \lambda_2)$, e.g. [79], [82], can be related to Shastry's R-matrix given in [69] via the transformation given in [87]. In [87] a similar observation is investigated from the string theory side.

Moreover, the Yangian symmetry was discovered for the AdS/CFT S-matrix in [80] and investigated in [88], see also [85]. In [88] an additional "hidden" symmetry was found. This fact could possibly enlarge the Yangian symmetry of the centrally extended $psu(2|2)$ algebra to a larger algebra.

The connection between the AdS/CFT S-matrix of the centrally extended algebra and the Hubbard model opens new directions as well as for integrable systems and for the AdS/CFT duality

itself. The works on the R-matrix in other atypical representations and mixing with the typical representation have been already done in the string theory side. Indeed, these new R-matrices would be fundamental for the integrability of new Hubbard-like models family based on these R-matrices. On the other hand, the integrable generalizations of the Hubbard model could find a priori applications in the AdS/CFT duality.

3.5.3 On the dressing phase in the AdS/CFT duality

As we discussed in the begining of the previous subsection, the S-matrix of the $\mathcal{N}=4$ super Yang-Mills theory exhibits the centrally extended $psu(2|2)\oplus psu(2|2)$ symmetry: $S_{psu(2,2|4)} = S_0 \cdot S_{psu(2|2)} \otimes S_{psu(2|2)}$. Remarkably, the symmetry determines the S-matrix up to a scalar factor S_0. The latter is a function of the momenta and the coupling. This factor, the dressing phase, was constructed in [77] using different approaches. However, despite of the success in its determination, the clear understanding of the scalar factor is still mising. The procedures in [77] do not explain why the scalar factor should exhibit its particular structure:

$$S_0(u_k,u_j) = \frac{x_k^- - x_j^+}{x_k^+ - x_j^-}\frac{1-g^2(x_j^-/x_k^+)}{1-g^2(x_j^+/x_k^-)}e^{2i\theta(u_k,u_j)} \tag{3.5.22}$$

where $2\theta(u_k,u_j)$ is called the dressing phase. We will write it in the integral form:

$$2\theta(u_k,u_j) = 2ig^2\int_{-\infty}^{\infty}dt e^{itu_k-|t|/2}\int_{-\infty}^{\infty}dt' e^{it'u_j-|t'|/2}\left(\hat{K}_d(2gt,2gt')-\hat{K}_d(2gt',2gt)\right), \tag{3.5.23}$$

where $\hat{K}_d(t,t') = 8g^2\int_0^\infty dt''\hat{K}_1(t,2gt'')\frac{t''}{e^{t''}-1}\hat{K}_0(2gt'',t')$ and the constituent kernels are given by the Bessel functions.

In integrable field theories, the construction of the S-matrix can be performed by the so-called factorized bootstrap program proposed by Al.Zamolodchikov and A.Zamolodchikov in [4]. The two-particle S-matrix satisfies the unitarity condition, the braided Yang–Baxter equation and the crossing symmetry. These conditions constrain the form of the S-matrix up to the CDD (Castillejo-Dalitz-Dyson) ambiguity which can be removed imposing some additional requirements. It was shown that the scalar factor S_0 is compatible with the crossing symmetry constraint. There exists another approach to determine the phase considering the R-matrix of the model and writing the explicit S-matrix between physical excitations on the physical vacuum, e.g. [89].

For the AdS/CFT correspondence, the determination of the scalar factor is important in two aspects, [89]: firstly, it is the last missing element for the systematic construction of the spectrum of the scaling dimension/energy on the Yang-Mills/string side. Secondly, the identification of the scalar factors on both sides serves as a strong quantitative check of the AdS/CFT duality.

Furthermore, the dressing phase can be implemented in the all-loops Bethe equations, [77]. In particular, it appears in the $su(2)$ sector as it was discussed in 3.5.1. From the integrability side, the Hubbard model describing the dilatation operator until three-loops, apparently, does not predict the Bethe equations with the dressing phase. Thus, a suitable modification of the Hubbard model is needed for this purpose.

In the next chapter we review several generalized Hubbard models which exist in the literature. Most of them are related to the Hubbard model but do not include it in the limit. However, it is desirable that the generalized model includes the Hubbard model in some limit case.

Chapter 4

Generalized Hubbard models

The Hubbard model is a basic example of the models describing strongly correlated electrons. The study of strongly correlated electrons on a lattice is an important tool in theoretical condensed matter physics in general, and in the study of high-T_c superconductivity in particular. On the other hand, the integrable generalizations of the Hubbard model can be important not only for such applications, but they form classes of integrable models sharing similar universal features.

Recent applications of the integrability, in particular of the Hubbard model, in the AdS/CFT duality also suggest the studies of generalized integrable Hubbard models. The phase which appears in the Bethe equations in the $su(2)$ sector of $\mathcal{N} = 4$ super Yang-Mills leads to the discrepancy with the Hubbard model. Several modifications of the Hubbard model are necessary in order to predict the form of the dilation operator. Moreover, it is needless to say, that the integrability in the application in AdS/CFT could also bring interesting results from both sides, e.g. the relations between Shastry's R-matrix and centrally extended $psu(2|2)$ algebra. Thus, for that purposes it motivates our studies of Hubbard-like models.

In the first section of this chapter we review several 'famous' generalized Hubbard models and related models. In the second section (4.2), we present superalgebraic extensions of Shastry's R-matrix and associated Hubbard-like models, [123],[124]. The last section is devoted to physical applications of certain superalgebra extended Hubbard-like models.

4.1 On generalised Hubbard models and related models

In this section we aim to present several important models of strongly correlated electrons from an inexhaustible list of solvable one-dimensional electronic models existing in the literature. All models we present here are related to the Hubbard model or its strong coupling limit so-called t-J model. Several of them share mostly the same properties or a similar integrability structure. The main interest in these models of strongly correlated electrons is motivated by the understanding of fundamental aspects of statistical mechanics. They are relevant to many realistic physical systems such as high-T_c superconductors. Moreover, an important feature of these models is that they are integrable and therefore they provide a non-perturbative information concerning physical properties.

In the following, we briefly review the models:

- i) t-J model and its variants. [101],[102]
- ii) EKS model. [103],[104]
- iii) supersymmetric U models and its quantum deformation. [106],[108],[111],[110]

- *iv*) Extended Hubbard models. [112]

t-J model. The t-J model is a lattice model where the occurrence of two electrons on the same site is forbidden. This restriction corresponds to an implicitly infinite on-site Coulomb repulsion $U \to \infty$ of the Hubbard model. Similarly to the Hubbard model, there are two types of interactions between electrons on nearest-neighbours sites: a charge interaction of strength V and a spin-exchange interaction J. The Hamiltonian of the extended version of the t-J model has the form:

$$H_{t-J} = -\sum_{j,\sigma} P_S \left(c^{\dagger}_{j,\sigma} c_{j+1,\sigma} + c^{\dagger}_{j+1,\sigma} c_{j,\sigma} \right) P_S + J \sum_{\substack{j,\sigma,\sigma' \\ \sigma \neq \sigma'}} c^{\dagger}_{j,\sigma} c_{j,\sigma'} \, c^{\dagger}_{j+1,\sigma'} c_{j+1,\sigma} + \sum_{j,\sigma,\sigma'} V_{\sigma,\sigma'} c^{\dagger}_{j,\sigma} c_{j,\sigma} \, c^{\dagger}_{j+1,\sigma'} c_{j+1,\sigma'} \quad (4.1.1)$$

where j runs from 1 to L, $\sigma = 1, ..., 2S+1$ - spin component of the particles and P_S is a projector on the subspace of non-double occupied states. The matrix $V_{\sigma,\sigma'}$ reproduces the anisotropy in the charge interactions.

In the isotropic case $V_{\sigma,\sigma'} = V$ the Hamiltonian H_{t-J} corresponds to the traditional t-J model which was exactly solved by the Bethe-ansatz method at the supersymmetric point $(V = -J = 1)$ for the case $S = 1/2$, [101]. The anisotropic generalization of the t - J model has been constructed in [102] and it was shown that the model is solvable for the arbitrary spin S and special values of the coupling J and $V_{\sigma,\sigma'}$: $V_{\sigma,\sigma'} = J\Big((1+\epsilon_\sigma) \cosh \gamma \, \delta_{\sigma\sigma'} + e^{\mathrm{sign}(\sigma-\sigma')\gamma} (1 - \delta_{\sigma\sigma'}) \Big)$ with $\gamma > 0$ - parameter of anisotropy.

On the other hand, the perturbative approach to the Hubbard model in the strong limit $U >> 1$ leads in the second order to the t-J isotropic Hamiltonian , see for details [55]:

$$H'_{t-J} = -\sum_{j=1}^{L} \sum_{\sigma=\uparrow,\downarrow} P_{1/2} \left(c^{\dagger}_{j,\sigma} c_{j+1,\sigma} + c^{\dagger}_{j+1,\sigma} c_{j,\sigma} \right) P_{1/2} + \frac{1}{U} \sum_{j=1}^{L} \left(S^{\alpha}_j S^{\alpha}_{j+1} - \frac{n_j n_{j+1}}{4} \right) P_{1/2} \quad (4.1.2)$$

with the projector $P_{1/2} = \prod_{i=1}^{L} (1 - n_{j,\uparrow} n_{j,\downarrow})$, spin operators S^{α}_j: $S^{+}_j = c^{\dagger}_{j,\uparrow} c_{j,\downarrow}$, $S^{-}_j = c^{\dagger}_{j,\downarrow} c_{j,\uparrow}$ and $S^{z}_j = \left(n_{j,\uparrow} - n_{j,\downarrow} \right)/2$ and $n_j = n_{j,\uparrow} + n_{j,\downarrow}$.

This Hamiltonian is equivalent to the isotropic H_{t-J} with $V = -J = 1/U$. At the half-filled band limit: $n_j \phi_{h.f.b} = \phi_{h.f.b}$, the first term of the t-J Hamiltonian $P_{1/2} \left(c^{\dagger}_{j,\sigma} c_{j+1,\sigma} + c^{\dagger}_{j+1,\sigma} c_{j,\sigma} \right) P_{1/2} = 0$ and the Hamiltonian becomes the Heisenberg spin chain:

$$H'_{xxx} = \frac{1}{U} \sum_{j=1}^{L} \left(S^{\alpha}_j S^{\alpha}_{j+1} - \frac{1}{4} \right) \quad (4.1.3)$$

EKS model. This lattice model as an extended Hubbard model was firstly proposed by F.H.L.Essler, V.E.Korepin and K.Schoutens in [103] with the main motivation to study the high-T_c superconductivity. It is a solvable model in one dimension which combines and extends interesting features of the Hubbard model and the t-J model.

Similarly to the Hubbard model, there are electrons of spin up and down, created on the site j by the fermionic operator $c^{\dagger}_{j,\sigma}$. The EKS Hamiltonian is

$$H_{EKS} = -\sum_{j=1}^{L} T_{j,j+1} + U \sum_{j=1}^{L} (2n_{j,\uparrow} - 1)(2n_{j,\downarrow} - 1), \quad L+1 \equiv L \quad (4.1.4)$$

with the kinetic term $T_{j,k}$

$$T_{j,k} = \sum_{\sigma=\uparrow,\downarrow}\Big(c^{\dagger}_{j,\sigma}c_{k,\uparrow} - c^{\dagger}_{k,\sigma}c_{j,\sigma}\Big)(1-n_{j,\bar{\sigma}}-n_{k,\bar{\sigma}}) + \frac{1}{2}(n_j-1)(n_k-1) - \frac{1}{2}(n_{j,\uparrow}-n_{j,\downarrow})(n_{k,\uparrow}-n_{k,\downarrow}) + c^{\dagger}_{j,\uparrow}c^{\dagger}_{j,\downarrow}c_{k,\downarrow}c_{k,\uparrow} + c_{j,\downarrow}c_{j,\uparrow}c^{\dagger}_{k,\uparrow}c^{\dagger}_{k,\downarrow} - c^{\dagger}_{j,\downarrow}c_{j,\uparrow}c^{\dagger}_{k,\uparrow}c_{k,\downarrow} - c^{\dagger}_{j,\uparrow}c_{j,\downarrow}c^{\dagger}_{k,\downarrow}c_{k,\uparrow} + \sum_{a=j,k}(n_{a,\uparrow}-\frac{1}{2})(n_{a,\downarrow}-\frac{1}{2}) \tag{4.1.5}$$

where $n_j = n_{j,\uparrow} + n_{j,\downarrow}$ and $\bar{\sigma}$ is the opposite spin value of σ, such that if $\sigma = \uparrow$ then $\bar{\sigma} = \downarrow$ and vice versa.

This Hamiltonian contains the Hubbard model Coulomb interaction and t-J model interaction terms. Moreover, the Hamiltonian H_{EKS} have $u(2|2)$ symmetry realized as spin $su(2)$ algebra with η-paring $su(2)$ symmetry and 8 fermionic generators and central elements:

$$su(2): \quad S^{+} = \sum_{j=1}^{L} c^{\dagger}_{j,\downarrow}c_{j,\uparrow}, \quad S^{-} = \sum_{j=1}^{L} c^{\dagger}_{j,\uparrow}c_{j,\downarrow}, \quad S^{z} = \sum_{j=1}^{L}\frac{1}{2}(n_{j,\uparrow}-n_{j,\downarrow})$$
$$su(2): \quad \eta^{+} = \sum_{j=1}^{L} c^{\dagger}_{j,\downarrow}c^{\dagger}_{j,\uparrow}, \quad \eta^{-} = \sum_{j=1}^{L} c_{j,\uparrow}c_{j,\downarrow}, \quad \eta^{z} = -\sum_{j=1}^{L}\frac{1}{2}(n_j-1)$$

and the fermions

$$Q_{\sigma} = \sum_{j=1}^{L} c_{j,\sigma}(1-n_{j,\bar{\sigma}}), \quad Q^{\dagger}_{\sigma} = \sum_{j=1}^{L} c^{\dagger}_{j,\sigma}(1-n_{j,\bar{\sigma}}),$$
$$\tilde{Q}_{\sigma} = \sum_{j=1}^{L} c_{j,\sigma}n_{j,\bar{\sigma}}, \quad \tilde{Q}^{\dagger}_{\sigma} = \sum_{j=1}^{L} c^{\dagger}_{j,\sigma}n_{j,\bar{\sigma}},$$

with central elements: $\sum_{j=1}^{L} 1$ and $X = \sum_{j=1}^{L}(n_{j,\uparrow}-\frac{1}{2})(n_{j,\downarrow}-\frac{1}{2})$. Note that unlike for the case of the Hubbard model treated in previous section there is no factor of $(-1)^j$ in the definition of η generators.

In addition, it can be shown that the EKS Hamiltonian is related to a graded permutation P_{ij} of $gl(2|2)$ which permutes tensor product of 4 possible states (v - vacuum, $c^{\dagger}_{\uparrow}v$, $c^{\dagger}_{\downarrow}v$ and $c^{\dagger}_{\uparrow}c^{\dagger}_{\downarrow}v$):

$$H_{EKS} = -\sum_{j=1}^{L} P_{j,j+1} + U\sum_{j=1}^{L}(2n_{j,\uparrow}-1)(2n_{j,\downarrow}-1) \tag{4.1.6}$$

The solution of the EKS model by the algebraic Bethe ansatz and thermodynamic analysis with study of the spectrum of low-lying excitations have been obtained in [104]. Concerning the integrability of the model, the R-matrix for the permutation part of the Hamiltonian is given by the Yangian $gl(2|2)$ R-matrix.

We remark also that the integrability paradigm go beyond and in the work of M.Gould, Y.-Z.Zhang and collaborators [105] the R-matrices and new integrable electronic models with twisted affine q-deformed superalgebra $U_q(sl(2|2)^{(2)})$ and $U_q(osp(2|2))$ symmetry are proposed. The work [105] is a continuation of derivation $U_q(\hat{g})$ invariant represented R-matrices proposed in [21],[22] using the so-called tensor product graph (TPG) method. In the limit $q \to 1$ the $U_q(osp(2|2))$ model reduces to EKS model which has a larger, $u(2|2)$, symmetry.

Supersymmetric U model and its extensions. A new direction for new lattice models of correlated electrons was proposed by [106]. The main features of these models that they have the same

4^L-dimensional Hilbert space as the Hubbard model and a free parameter as a coupling constant. However, the symmetry of the Hamiltonian is different from the Hubbard model one. The origin for the construction of this model is the one-parameter family of inequivalent typical 4-dimensional irreducible representations of $gl(2|1)$ algebra. At first, we review the model and then we will speak about the integrability and the R-matrix formalism.

Similarly to the Hubbard model, there are electrons of spin up and down, created on the site j by the fermionic operator $c^{\dagger}_{j,\sigma}$. The supersymmetric U model Hamiltonian is

$$\begin{aligned} H_{U\,ss} &= H_{Hub}(u) + \frac{u}{2}\sum_{j=1}^{L}\sum_{\sigma=\uparrow,\downarrow}(c^{\dagger}_{j,\sigma}c^{\dagger}_{j,\bar{\sigma}}c_{j+1,\bar{\sigma}}c_{j+1,\sigma} + \text{h.c.}) + (u+2)\sum_{j=1}^{L} n_j + \\ &+\Big(1+\frac{u}{|u|}\sqrt{u+1}\Big)\sum_{j=1}^{L}\sum_{\sigma=\uparrow,\downarrow}(c^{\dagger}_{j,\sigma}c_{j+1,\sigma} + c^{\dagger}_{j+1,\sigma}c_{j,\sigma})(n_{j,\bar{\sigma}} + n_{j+1,\bar{\sigma}}) - \\ &-\Big(1+\frac{u}{|u|}\sqrt{u+1}\Big)^2\sum_{j=1}^{L}\sum_{\sigma=\uparrow,\downarrow}(c^{\dagger}_{j,\sigma}c_{j+1,\sigma} + c^{\dagger}_{j+1,\sigma}c_{j,\sigma})n_{j,\bar{\sigma}}n_{j+1,\bar{\sigma}} \end{aligned} \tag{4.1.7}$$

where $H_{Hub}(u) = \mathcal{H}_{Hub}(U \to 2u)$ with $\mathcal{H}_{Hub}(U)$ defined in (3.3.1), $n_j = n_{j,\uparrow} + n_{j,\downarrow}$ and $\bar{\sigma}$ is the opposite spin value of σ, e.g if $\sigma =\uparrow$ then $\bar{\sigma} =\downarrow$.

This Hamiltonian $H_{U\,ss}$ has $gl(2|1)$ symmetry which is closely related to the integrability of the model. The R-matrix corresponding to this Hamiltonian can be considered as a "classical" limit ($q \to 1$) of the $U_q(gl(2|1))$ invariant R-matrix in the four-dimensional representation with one free parameter.

The solution of the supersymmetric U model using the Algebraic Bethe Ansatz and the Coordinate Bethe Ansatz are presented in [108] and in [109].

An extension of the supersymmetric U model, a q-deformed version with the quantum superalgebra $U_q(gl(2|1))$ supersymmetry, was also proposed in [110]. Futhermore, the eight-state supersymmetric U model of strongly correlated electrons with the Lie superalgebra $gl(3|1)$ symmetry, and the two-parameter (q-deformed) eight-state supersymmetric fermion model with the quantum superalgebra $U_q(gl(3|1))$ symmetry, were introduced [111].

Extended Hubbard models. We also mention [112] where a wide class of model (extended Hubbard models) was proposed. In this class the symmetries are adjustable and extended models in several limits can be mapped on the EKS model or the $gl(2|1)$ supersymmetric U model. Moreover, so-called Polynomial R-matrix Technique (PRT) is proposed to find solutions of the Yang–Baxter equation.

An extended electron's model invariant under the spin-flip and which conserves the total number of electrons and the magnetization is considered and it is written as

$$\begin{aligned} H_{EHM} &= -\sum_{<j,k>,\sigma}\Big(t - X(n_{j,\bar{\sigma}} + n_{k,\bar{\sigma}}) + \tilde{X}n_{j,\bar{\sigma}}n_{k,\bar{\sigma}}\Big)c^{\dagger}_{j,\sigma}c_{k,\sigma} + U\sum_j n_{j,\uparrow}n_{j,\downarrow} + \frac{V}{2}\sum_{<j,k>} n_j n_k + \\ &+\frac{W}{2}\sum_{\substack{<j,k>\\ \sigma,\sigma'}} c^{\dagger}_{j,\sigma}c^{\dagger}_{k,\sigma'}c_{j,\sigma'}c_{j,\sigma} + Y\sum_{<j,k>} c^{\dagger}_{j,\uparrow}c^{\dagger}_{j,\downarrow}c_{k,\downarrow}c_{k,\uparrow} + P\sum_{<j,k>} n_{j,\uparrow}n_{j,\downarrow}n_k + \\ &+\frac{Q}{2}\sum_{<j,k>} n_{j,\uparrow}n_{j,\downarrow}n_{k,\uparrow}n_{k,\downarrow} + \mu\sum_{j,\sigma} n_{j,\sigma} \end{aligned}$$

where the spin variable $\sigma =\uparrow,\downarrow$, the sums over $< j,k >$ - nearest neighbours on the lattice of L sites and similarly to previous notations: $n_j = n_{j,\uparrow} + n_{j,\downarrow}$ and $\bar{\sigma}$ is the opposite spin value of σ, e.g if $\sigma =\uparrow$ then $\bar{\sigma} =\downarrow$.

This Extended Hubbard Hamiltonian contains 10 coefficients t, X, $\tilde{X}$, U, V, W, Y, P, Q, μ such that t is a hopping constant, U - on-site Coulomb interaction as in the ordinary Hubbard model, V is the neighboring site charge interaction, X - the bond-charge interaction, W the exchange term, and Y the pair-hopping term. X correlates hopping with on-site occupation number, and P and Q describe three and four-electron interactions. Finally μ is the chemical potential. Similar terms as ones with V, W, Y are in EKS model Hamiltonian (4.1.5) and supersymmetric U model (4.1.7).

The choice of these parameters $t, X, \tilde{X}$ etc implies the Hamiltonian symmetry. We will not consider all the cases of the parameters, we just mention that there are $i)$ $su(2) \oplus u(1) \oplus u(1)$ symmetry, $ii)$ 2 different $so(4)$ symmetries, $iii)$ $gl(2|1)$ supersymmetry, $iv)$ $so(5)$ symmetry and $v)$ $u(2|2)$ symmetry. In the cases $iii)$ and $v)$ the Hamiltonian H_{EHM} is mapped on the EKS model and supersymmetric U model. The case $ii)$ for some values of parameters can be related to a model proposed by Bariev and Alcaraz in [113].

The integrability of this model is related to so-called Polynomial R-matrix Technique (PRT) and to the so-called *generalized permutator* introduced in [112]. Once for a given model the R-matrix is known, its spectrum is obtained within the Algebraic Bethe Ansatz approach by diagonalizing the corresponding transfer matrix. The Coordinate Bethe ansatz for the $so(4)$ symmetric EHM Hamiltonian was done in the frame of the model proposed by R.Z.Bariev and F.C.Alcaraz in [113]. Due to the form of the Hamiltonian acting as a permutator, the investigations become simplier and such kind of similar models were already treated in [114]. The ground state and thermodynamics for certian limits are derived in the third reference of [112].

There are certainly a lot of other models which can be related to the one-dimensional Hubbard model that we did not mentioned above. First of all, we point out that the above mentioned integrable models do not contain the Hubbard model as a limiting case, what can be probably explained by the non-trivial spectral parameter dependence of Shastry's R-matrix. However, several extensions of Shastry's R-matrix to the $su(n)$ case were proposed by Z.Maassarani in [115], [116]. We will review a slightly more general construction in the next section. Another extension of the Hubbard model was recently appeared from the AdS/CFT context. Due to the centrally extended $psu(2|2)$ symmetry of the Hubbard model's R-matrix, a q-deformation of this R-matrix was done in [82] and, thus, a q-deformed version of the Hubbard model was obtained, which contains certain case of Alcaraz-Bariev model, [113], and the Hubbard model in the limit $q = 1$. There are also the so-called variable range hopping models and the one related to the Hubbard model was invetigated in [120].

4.2 R-matrix formalism for generalized Hubbard models

In this section we present in details another type of generalized Hubbard models proposed in [123], [124]. These new models are superalgebraic extensions of the $su(N)$ Hubbard models proposed in [115], [116] and in the limiting case include the ordinary Hubbard model.

The idea of the construction of Hubbard-like models is to generalize Shastry's R-matrix based on $gl(1|1)$ algebra on larger algerbras. It appears that the principal ingredient of this construction is the C-matrix which was defined for the Hubbard model in (3.2.8). Generalizations of this C-matrix lead to new XX spin chains and Hubbard-like R-matrices similar to [116]. Below we review the R-matrix construction of generalized XX and Hubbard-like models.

Generalized XX models. Considering the XX model based on the algebra $gl(\mathfrak{n}|\mathfrak{m})$ we firstly define the projectors: π and $\bar{\pi}$ such that

$$\pi : V \to W \subset V \quad \text{and} \quad \pi + \bar{\pi} = \mathbb{I} \tag{4.2.1}$$

$$\pi = \sum_{i=1}^{\mathfrak{p}} E^{ii} \quad \text{and} \quad \bar{\pi} = \sum_{i=\mathfrak{p}+1}^{\mathfrak{n}+\mathfrak{m}} E^{ii} \tag{4.2.2}$$

where V is a graded vector space, number $\mathfrak{p}$ is such that $1 \leq \mathfrak{p} \leq \mathfrak{n}+\mathfrak{m}$ and E_a^{ij} denotes the elementary super-matrices acting in the a^{th} copy of $\mathrm{End}(\mathbb{C}^{\mathfrak{n}|\mathfrak{m}})$. These matrices form a representation of the $gl(\mathfrak{n}|\mathfrak{m})$ algebra and satisfy the following commutation relations:

$$\left[E_a^{ij}, E_b^{kl}\right]_g = \delta_{ab}\Big(\delta_{jk}E_a^{il} - (-1)^{([i]+[j])([k]+[l])}\delta_{il}E_a^{kj}\Big)$$

with the graded commutator defined as $[A,B]_g = AB - (-1)^{[A][B]}BA$. The gradation is defined as usually: $[1] = ... = [\mathfrak{n}] = 0$ and $[\mathfrak{n}+1] = ... = [\mathfrak{n}+\mathfrak{m}] = 1$. One should also define the multiplication for the tensor products by $(a \otimes b)(c \otimes d) = (-1)^{[b][c]}(ac \otimes bd)$. We also recall the definition of graded permutation:

$$P_{12} = \sum_{i,j=1}^{\mathfrak{n}+\mathfrak{m}} (-1)^{[j]} E_1^{ij} E_2^{ji} \tag{4.2.3}$$

Let us introduce for simplicity the sets of integers:

$$\mathcal{N} = \{1, 2, ..., \mathfrak{p}\}\,, \; \overline{\mathcal{N}} = \{\mathfrak{p}+1, ..., \mathfrak{n}+\mathfrak{m}\} \tag{4.2.4}$$

Let us then define the C-matrix as

$$C_a = \sum_{j \in \mathcal{N}} E_a^{jj} - \sum_{\bar{\jmath} \in \overline{\mathcal{N}}} E_a^{\bar{\jmath}\bar{\jmath}} \quad \in \mathrm{End}(\mathbb{C}^{\mathfrak{n}|\mathfrak{m}}) \tag{4.2.5}$$

comparing with the one for the ordinary Hubbard model (3.2.8).

The R-matrix of the generalized XX model writes as:

$$R_{12}(u) = \Sigma_{12}\, P_{12} + \Sigma_{12}\, \sin u + (\mathbb{I} \otimes \mathbb{I} - \Sigma_{12})\, P_{12}\, \cos u \tag{4.2.6}$$

where $u \in \mathbb{C}$ is the spectral parameter, matrix Σ_{12} is defined by the C-matrix as $\Sigma_{12} = \frac{1}{2}\Big(1 - C_1 C_2\Big)$:

$$\Sigma_{12} = \sum_{j \in \mathcal{N}} \sum_{\bar{\jmath} \in \overline{\mathcal{N}}} \Big(E_1^{jj} E_2^{\bar{\jmath}\bar{\jmath}} + E_1^{\bar{\jmath}\bar{\jmath}} E_2^{jj}\Big), \tag{4.2.7}$$

The generalized XX-model R-matrix (4.2.6) obeys the Yang–Baxter equation, is unitary and regular:

$$R_{ab}(u-v)R_{ac}(u)R_{bc}(v) = R_{bc}(v)R_{ac}(u)R_{ab}(u-v) \tag{4.2.8}$$

$$R_{ab}(u)R_{ba}(-u) = \cos^2(u)\mathbb{I}_a \otimes \mathbb{I}_b \tag{4.2.9}$$

$$R_{ab}(0) = P_{ab} \tag{4.2.10}$$

The Hamiltonian which corresponds to this R-matrix is similar to the ordinary XX model in (2.2.21) and in terms of E^{ij} super-matrices it is written as

$$H_{gen\,XX} = \sum_{a=1}^{N} H_{aa+1}, \quad N+1 \equiv 1 \tag{4.2.11}$$

$$H_{aa+1} = \sum_{j \in \mathcal{N}} \sum_{\bar{\jmath} \in \overline{\mathcal{N}}} \Big((-1)^{[\bar{\jmath}]} E_a^{j\bar{\jmath}} E_{a+1}^{\bar{\jmath}j} + (-1)^{[j]} E_a^{\bar{\jmath}j} E_{a+1}^{j\bar{\jmath}}\Big)\,, \tag{4.2.12}$$

We recall that for $gl(1|1)$ and $\mathfrak{p} \neq 2$, the above Hamiltonian describes the model of free electrons: $E_a^{21} = c_a^\dagger$. Similarly, considering the "particle" content of this model, for $gl(2|1)$ and $\mathfrak{p} = 2$ as example, the Hamiltonian is $H_{aa+1} = E_a^{31}E_{a+1}^{13} + E_a^{32}E_{a+1}^{23} + h.c.$. We see that E^{31} plays a similar role of the fermion creation operator and the vector $e^3 = E^{31}e^1$ corresponds to the electron particle, whereas the particle $e^2 = E^{21}e^1$ resembles more to the vacuum state e^1. We note that e^3 particle is related to $\bar{\pi}$ projector: $\bar{\pi}e^3 = e^3$ and $\pi e^3 = 0$. Similarly, e^1, e^2 correspond to π projector: $\pi e^i = e^i$ and $\bar{\pi}e^i = 0$ for $i = 1, 2$. In general case, we have several particles corresponding to π projector, in the following we will call them π-particles or "heavy" particles, and the particles which correspond to $\bar{\pi}$ projector, we will call them $\bar{\pi}$-particles or electron-like particles. Thus, in general case, we have $\mathfrak{p}$ π-particles and $\mathfrak{n} + \mathfrak{m} - \mathfrak{p}$ $\bar{\pi}$-particles. However, If $\mathfrak{p} < \mathfrak{n}$ then among $\bar{\pi}$-particles there are $\mathfrak{n} - \mathfrak{p}$ particles with bosonic gradation (called hard core bosons) and $\mathfrak{m}$ particle with fermionic gradation.

The Hamiltonian has $gl(\mathfrak{p}) \oplus gl(\mathfrak{n} - \mathfrak{p}|\mathfrak{m})$ for $\mathfrak{p} \leq \mathfrak{n}$ and $gl(\mathfrak{n}|\mathfrak{p} - \mathfrak{n}) \oplus gl(\mathfrak{m} - \mathfrak{p} + \mathfrak{n})$ for $\mathfrak{p} > \mathfrak{n}$ symmetry. It can be seen from the symmetry with R-matrix:

$$[\check{R}_{12}(u), E_1^{jj'} + E_2^{jj'}] = 0 \quad \text{and} \quad [\check{R}_{12}(u), E_1^{\bar{j}\bar{j}'} + E_2^{\bar{j}\bar{j}'}] = 0 \tag{4.2.13}$$

for $j, j' = 1, ..., \mathfrak{p}$ and $\bar{\jmath}, \bar{\jmath}' = \mathfrak{p} + 1, ..., \mathfrak{n} + \mathfrak{m}$.

Before speaking about the Hubbard-like models, we remark that the parameter $\mathfrak{p}$ lead to $\mathfrak{n} + \mathfrak{m} - 1$ different non-trivial models. However, it is possible to introduce one more parameter $\mathfrak{p}'$ such that we split the graded vector space $V = V_0 \oplus V_1$ in four parts:

$$\pi = \textstyle\sum_{i=1}^{\mathfrak{p}} E^{ii}, \qquad \bar{\pi} = \sum_{i=\mathfrak{p}+1}^{\mathfrak{n}} E^{ii} \quad \text{and} \quad (\pi + \bar{\pi})V_0 = V_0 \tag{4.2.14}$$

$$\pi' = \textstyle\sum_{i=\mathfrak{n}+1}^{\mathfrak{n}+\mathfrak{p}'} E^{ii}, \quad \bar{\pi}' = \sum_{i=\mathfrak{n}+\mathfrak{p}'+1}^{\mathfrak{n}+\mathfrak{m}} E^{ii} \quad \text{and} \quad (\pi' + \bar{\pi}')V_1 = V_1 \tag{4.2.15}$$

with $\mathfrak{p} < \mathfrak{n}$ and $\mathfrak{p}' < \mathfrak{m}$. This construction adds more different non-trivial models but we will restrict ourself only to one parameter models.

Generalized Hubbard models. The definition of the (generalized) Hubbard R-matrix uses as basic ingredient the R-matrix of the XX model (or its generalization), which are coupled *à la Shastry*, the coupling constant being related to the potential U of the Hubbard model. The two underlying XX models may be based on two different superalgebras $gl(\mathfrak{n}|\mathfrak{m}')_\uparrow$ and $gl(\mathfrak{n}'|\mathfrak{m}')_\downarrow$ and depend on two different integers $\mathfrak{p}$ and $\mathfrak{q}$, [124]. Let us introduce similarly sets of integers

$$\mathcal{N}_\uparrow = \{1, 2, ..., \mathfrak{p}\}\,, \overline{\mathcal{N}}_\uparrow = \{\mathfrak{p} + 1, ..., \mathfrak{n} + \mathfrak{m}\} \quad \text{and} \quad \mathcal{N}_\downarrow = \{1, 2, ..., \mathfrak{q}\}\,, \overline{\mathcal{N}}_\downarrow = \{\mathfrak{q} + 1, ..., \mathfrak{n}' + \mathfrak{m}'\}\,. \tag{4.2.16}$$

The projectors are defined for the spin up and spin down vector spaces $V_\uparrow$, $V_\downarrow$ in the same way as for the generalized XX model:

$$\pi_\sigma = \sum_{j \in \mathcal{N}_\sigma} E_\sigma^{jj} \quad \text{and} \quad \bar{\pi} = \sum_{\bar{\jmath} \in \overline{\mathcal{N}}_\sigma} E_\sigma^{\bar{\jmath}\bar{\jmath}}, \quad \sigma = \uparrow, \downarrow \tag{4.2.17}$$

and they determine the basic ingredient of the construction - the matrices C_σ:

$$C_\sigma = \sum_{j \in \mathcal{N}_\sigma} E_\sigma^{jj} - \sum_{\bar{\jmath} \in \overline{\mathcal{N}}_\sigma} E_\sigma^{\bar{\jmath}\bar{\jmath}} \tag{4.2.18}$$

The R-matrix of the Hubbard-like model is based on the pair $(gl(\mathfrak{n}|\mathfrak{m})\,, \mathfrak{p}\,;\, gl(\mathfrak{n}'|\mathfrak{m}')\,, \mathfrak{q})$ or noted by $\left(\begin{smallmatrix}\mathfrak{n}|\mathfrak{m}\\ \mathfrak{p}\end{smallmatrix}\,;\,\begin{smallmatrix}\mathfrak{n}'|\mathfrak{m}'\\ \mathfrak{q}\end{smallmatrix}\right)$ and it acts on the tensor product $V_{1\uparrow} \otimes V_{1\downarrow} \otimes V_{2\uparrow} \otimes V_{2\downarrow}$:

$$R_{12}^{\uparrow\downarrow}(\lambda_1, \lambda_2) = R_{12}^{\uparrow}(\lambda_{12})\, R_{12}^{\downarrow}(\lambda_{12}) + \frac{\sin(\lambda_{12})}{\sin(\lambda'_{12})} \tanh(h'_{12})\, R_{12}^{\uparrow}(\lambda'_{12})\, C_{\uparrow 1}\, R_{12}^{\downarrow}(\lambda'_{12})\, C_{\downarrow 1} \tag{4.2.19}$$

where $\lambda_{12} = \lambda_1 - \lambda_2$, $\lambda'_{12} = \lambda_1 + \lambda_2$. The coupling $h'_{12} = h(\lambda_1) + h(\lambda_2)$ is based on the function $h(\lambda)$ such that

$$\sinh(2h) = \frac{U}{4}\sin(2\lambda)\,, \tag{4.2.20}$$

we note that this definition of the coupling fuction $h(\lambda)$ is slightly different from the (3.2.16).

We remark that the structure of the above R-matrix is almost similar to Shastry's R-matrix one (3.2.15), but here $R^{\sigma}_{12}(\lambda_{12})$ are the generalized XX model R-matrices (4.2.6).

The R-matrix (4.2.19) is symmetric, regular and satisfies the unitary relation. Moreover, when the relation (4.2.20) holds, the R-matrix (4.2.19) satisfies the Yang–Baxter equation:

$$R^{\uparrow\downarrow}_{12}(\lambda_1,\lambda_2)\,R^{\uparrow\downarrow}_{13}(\lambda_1,\lambda_3)\,R^{\uparrow\downarrow}_{23}(\lambda_2,\lambda_3) \;\;=\;\; R^{\uparrow\downarrow}_{23}(\lambda_2,\lambda_3)\,R^{\uparrow\downarrow}_{13}(\lambda_1,\lambda_3)\,R^{\uparrow\downarrow}_{12}(\lambda_1,\lambda_2)\,. \tag{4.2.21}$$

Using the R-matrix formalism, one can define the corresponding quantum integrable system, by performing the following steps: monodromy matrix, transfer matrix and Hamiltonian. The L-site monodromy matrix is given

$$T_{a<b_1\ldots b_L>}(\lambda) = R^{\uparrow\downarrow}_{ab_1}(\lambda,0)\ldots R^{\uparrow\downarrow}_{ab_L}(\lambda,0) \tag{4.2.22}$$

and its transfer matrix is the (super)trace in the auxiliary space:

$$t(\lambda) = tr_a(T_{a<b_1\ldots b_L>}(\lambda))\,. \tag{4.2.23}$$

Then the generalized Hubbard Hamiltonian reads

$$\mathcal{H}_{gen\,Hub} = \frac{d}{d\lambda}\ln t(\lambda)\Big|_{\lambda=0} = \sum_{x=1}^{L}\Big((\Sigma P)_{\uparrow\,x,x+1} + (\Sigma P)_{\downarrow\,x,x+1} + \frac{U}{4}\,C_{\uparrow x}\,C_{\downarrow x}\Big), \quad L+1\equiv 1 \tag{4.2.24}$$

with

$$(\Sigma P)_{\sigma\,x,x+1} = \sum_{i\in\mathcal{N}_\sigma}\sum_{j\in\overline{\mathcal{N}}_\sigma}\Big((-1)^{[j]}E^{ij}_{\sigma\,x}E^{ji}_{\sigma\,x+1} + (-1)^{[i]}E^{ji}_{\sigma\,x}E^{ij}_{\sigma\,x+1}\Big)\,, \tag{4.2.25}$$

$$C_{\sigma\,x} = \sum_{i\in\mathcal{N}_\sigma}E^{ii}_{\sigma\,x} - \sum_{j\in\overline{\mathcal{N}}_\sigma}E^{jj}_{\sigma\,x}, \quad \sigma = \uparrow,\downarrow\,. \tag{4.2.26}$$

In the Hamiltonian (4.2.24) we recognise two generalized XX model Hamiltonians (4.2.12) for spin up and spin down particles. The generalized Hubbard-like model $(gl(\mathfrak{n}|\mathfrak{m}),\mathfrak{p};\,gl(\mathfrak{n}'|\mathfrak{m}'),\mathfrak{q})$ or a $\left(\begin{smallmatrix}\mathfrak{n}|\mathfrak{m}\\ \mathfrak{p}\end{smallmatrix};\begin{smallmatrix}\mathfrak{n}'|\mathfrak{m}'\\ \mathfrak{q}\end{smallmatrix}\right)$-model contains four different types of particles $\pi\uparrow$, $\bar\pi\uparrow$, $\pi\downarrow$ and $\bar\pi\downarrow$, each type being 'colored': the $\pi\uparrow$-particles have 'colors' $1\uparrow, 2\uparrow, ..., \mathfrak{p}\uparrow$; the $\bar\pi\uparrow$-particles have 'colors' $(\mathfrak{p}+1)\uparrow, ..., \mathfrak{n}+\mathfrak{m}\uparrow$, while the 'colors' for $\pi\downarrow$ and $\bar\pi\downarrow$-particles are $1\downarrow, ..., \mathfrak{q}\downarrow$ and $(\mathfrak{q}+1)\downarrow, ..., \mathfrak{n}'+\mathfrak{m}'\downarrow$ respectively.

The usual Hubbard model is the $\left(\begin{smallmatrix}1|1\\ 1\end{smallmatrix};\begin{smallmatrix}1|1\\ 1\end{smallmatrix}\right)$-model and the bosonic Hubbard model corresponds to $\left(\begin{smallmatrix}2\\ 1\end{smallmatrix};\begin{smallmatrix}2\\ 1\end{smallmatrix}\right)$-model. Generalizations to the superalgebra case with additional two parameters as done in [124] will be noted $\left(\begin{smallmatrix}\mathfrak{m}|\mathfrak{n}\\ \mathfrak{p}|\mathfrak{q}\end{smallmatrix};\begin{smallmatrix}\mathfrak{m}'|\mathfrak{n}'\\ \mathfrak{p}'|\mathfrak{q}'\end{smallmatrix}\right)$-models with obvious notation. Hereafter, we will consider the case of $\left(\begin{smallmatrix}\mathfrak{n}\\ \mathfrak{p}\end{smallmatrix};\begin{smallmatrix}\mathfrak{m}\\ \mathfrak{q}\end{smallmatrix}\right)$-models. Note that the $\left(\begin{smallmatrix}\mathfrak{n}\\ \mathfrak{n}-\mathfrak{p}\end{smallmatrix};\begin{smallmatrix}\mathfrak{m}\\ \mathfrak{m}-\mathfrak{q}\end{smallmatrix}\right)$-model is equivalent to the $\left(\begin{smallmatrix}\mathfrak{n}\\ \mathfrak{p}\end{smallmatrix};\begin{smallmatrix}\mathfrak{m}\\ \mathfrak{q}\end{smallmatrix}\right)$-model. The symmetry algebra of the $\left(\begin{smallmatrix}\mathfrak{n}\\ \mathfrak{p}\end{smallmatrix};\begin{smallmatrix}\mathfrak{m}\\ \mathfrak{q}\end{smallmatrix}\right)$-model is a $gl(\mathfrak{p})\oplus gl(\mathfrak{n}-\mathfrak{p})\oplus gl(\mathfrak{q})\oplus gl(\mathfrak{m}-\mathfrak{q})$ algebra.

4.3 Applications

In this section we consider several examples of models presented in the previous section. Considering the cases of small superalgebras, the Hamiltonians have 'multi-particle' forms similar to Hubbard-like models as EKS or supersymmetric U models. We suppose that they can exhibit interesting properties for condensed matter physics. At first we review the Jordan-Wigner transformation which will be useful for writting the Hubbard-like Hamiltonians in terms of fermionic generators. We also give the eigenfunction and the spectrum for several Hamiltonian, although the solution via the Coordinate Bethe Ansatz is provided in the next chapter. In the last part of this section we compare our results with the ones of the AdS/CFT duality.

4.3.1 Jordan-Wigner transformation

The Jordan–Wigner transformation [125] or [55],[123] essentially consists of the construction of a mapping

$$E^{ij} \in gl(2^{K-1}|2^{K-1}) \leftrightarrow \{c^{\dagger}, c\ ;\ d^{\dagger}, d\ ;\ e^{\dagger}, e\ ;\ ...\}, \tag{4.3.1}$$

where $c, d, e, ...$ are fermionic operators.

To present this construction, let us consider K sets of fermionic operators: $c^{(a)\dagger}_{i,\sigma}$,$c^{(a)}_{i,\sigma}$ with $i = 1, ..., L$, $a = 1, ..., K$ and $\sigma = \uparrow, \downarrow$ such that they satisfy the usual anticommutation relations:

$$\{c^{(a)\dagger}_{i,\sigma}, c^{(b)}_{j,\sigma'}\} = \delta_{ij}\delta_{ab}\delta_{\sigma,\sigma'}, \quad \{c^{(a)}_{i,\sigma}, c^{(b)}_{j,\sigma'}\} = 0 \tag{4.3.2}$$

Then, let us define a matrix $X^{(a)}_i$ (without spin for instance):

$$X^{(a)}_i = \begin{pmatrix} 1 - n^{(a)}_i & c^{(a)}_i \\ c^{(a)\dagger}_i & n^{(a)}_i \end{pmatrix} \tag{4.3.3}$$

with $n^{(a)}_i = c^{(a)\dagger}_i c^{(a)}_i$. The components of the matrix will be denoted by $X^{(a)}_{i;\alpha\beta}$ where $\alpha, \beta = 1, 2$. Indeed, the components $X^{(a)}_{i;11}$, $X^{(a)}_{i;22}$ have bosonic gradation and $X^{(a)}_{i;12}$, $X^{(a)}_{i;21}$ - fermionic.

This matrix can be easily mapped on $gl(1|1)$ or $gl(2)$ algebra:

$$gl(1|1): \quad \mathbb{E}_i = \left(E^{kl}_i\right)_{k,l=1,2} = \begin{pmatrix} X^{(1)}_{i;11} & X^{(1)}_{i;12} \\ X^{(1)}_{i;21} & X^{(1)}_{i;22} \end{pmatrix}, \tag{4.3.4}$$

$$gl(2): \quad \mathbb{E}_i = \begin{pmatrix} X^{(1)}_{i;11} & X^{(1)}_{i;12} Z^{(1)}_{i+1,L} \\ X^{(1)}_{i;21} Z^{(1)}_{i+1,L} & X^{(1)}_{i;22} \end{pmatrix} \tag{4.3.5}$$

for $i = 1, ..., L$ and where $Z^{(a)}_{i,L} = \prod_{k=i}^{L}(X^{(a)}_{k;11} - X^{(a)}_{k;22})$ is introduced in order to E^{12}_i commute with E^{21}_k for $k \neq i$.

We see that the Jordan-Wigner transformation preserves the gradation for the superalgebra case. It is a local transformation and comparing with the ordinary algebra case. The Jordan-Wigner transformation for Lie algebra is non-local transformation. The last point is crucial when one considers a "bosonic" model with the periodic boundary conditions. The Jordan-Wigner transformation is non compatible with the periodic boundary conditions and it induces a twisting term on the boundaries, see for example $gl(2)$ XX model in (2.2).

Moreover, if we consider the Jordan-Wigner transformation for $E^{ij}_{p,\uparrow} \in gl(1|1)$ and $E^{ij}_{q,\downarrow} \in gl(2)$ and these matrices commutes: $[E^{ij}_{p,\uparrow}, E^{kl}_{q,\downarrow}] = 0$, then the transformation is

$$gl(1|1)_\uparrow : \quad \mathbb{E}_{i,\uparrow} = \begin{pmatrix} X^{(1)}_{i,\uparrow;11} & X^{(1)}_{i,\uparrow;12} Z^{(1)}_{1,L;\downarrow} \\ X^{(1)}_{i,\uparrow;21} Z^{(1)}_{1,L;\downarrow} & X^{(1)}_{i,\uparrow;22} \end{pmatrix} \tag{4.3.6}$$

$$gl(2)_\downarrow : \quad \mathbb{E}_{i,\downarrow} = \begin{pmatrix} X^{(1)}_{i,\downarrow;11} & X^{(1)}_{i,\downarrow;12} Z^{(1)}_{i+1,L;\downarrow} \\ X^{(1)}_{i,\downarrow;21} Z^{(1)}_{i+1,L;\downarrow} & X^{(1)}_{i,\downarrow;22} \end{pmatrix} \tag{4.3.7}$$

with similarly $Z^{(a)}_{i,L;\sigma} = \prod_{k=i}^{L} (X^{(a)}_{k,\sigma;11} - X^{(a)}_{k,\sigma;22})$ is introduced in order to generators $E^{ij}_{p,\uparrow}$ and $E^{kl}_{q,\downarrow}$ commute.

For larger algebras as $gl(2^{K-1}|2^{K-1})$ the mapping is following:

$$E^{ij} \leftrightarrow (-1)^s X^{(1)}_{\alpha_1\beta_1} \otimes X^{(2)}_{\alpha_2\beta_2} \otimes ... \otimes X^{(K)}_{\alpha_K\beta_K} \tag{4.3.8}$$

where to every value i and $j \in [1, 2^K]$ the LHS is associated with an element $\{\alpha_1,\ \alpha_2,\ ...\ , \alpha_K\}$ and $\{\beta_1,\ \beta_2,\ ...\ , \beta_K\}$ respectively with $\alpha_i, \beta_i = 1$ or 2. Total grading is given by

$$s = \sum_{i=2}^{K} [\alpha_i] \left(\sum_{j=1}^{i-1} ([\alpha_j] + [\beta_j]) \right) \tag{4.3.9}$$

For example, let $E^{ij}_p \in gl(2|2)$, then we associate for $i = 1$ the pair $(\alpha_1, \alpha_2) = 11$, $i = 2 \quad \rightarrow \quad (\alpha_1, \alpha_2) = 22$, $i = 3 \quad \rightarrow \quad (\alpha_1, \alpha_2) = 12$ and $i = 4 \quad \rightarrow \quad (\alpha_1, \alpha_2) = 21$. It means for instance that $E^{12} = (-1)^s X^{(1)}_{12} \otimes X^{(2)}_{12}$ etc. Thus, the transformation is

$$\mathbb{E}_a \equiv \begin{pmatrix} (1-n^c_a)(1-n^d_a) & -c_a d_a & (1-n^c_a) d_a & -c_a(1-n^d_a) \\ c^\dagger_a d^\dagger_a & n^c_a n^d_a & c^\dagger_a n^d_a & n^c_a d^\dagger_a \\ (1-n^c_a) d^\dagger_a & n^d_a c_a & (1-n^c_a) n^d_a & c_a d^\dagger_a \\ -c^\dagger_a (1-n^d_a) & n^c_a d_a & d_a c^\dagger_a & n^c_a (1-n^d_a) \end{pmatrix}, \tag{4.3.10}$$

for $\mathbb{E} = (E^{ij})_{i,j=1,...,4} \in gl(2|2)$. The operators c, d are fermionic generators associated with $X^{(1)}$ and $X^{(2)}$.

Similarly, for the matrices $E^{ij}_{p,\sigma}$ with spin $\sigma = \uparrow, \downarrow$ which commutes: $[E^{ij}_{p,\uparrow}, E^{kl}_{q,\downarrow}] = 0$, the Jordan-Wigner transformation is such that the fermionic generator $E^{ij}_{p,\uparrow}$ should be multiplied by $Z_{1,L;\downarrow}$:

$$\mathbb{E}_{a,\uparrow} = \begin{pmatrix} B_\uparrow & B_\uparrow & F_\uparrow Z_{1,L;\downarrow} & F_\uparrow Z_{1,L;\downarrow} \\ B_\uparrow & B_\uparrow & F_\uparrow Z_{1,L;\downarrow} & F_\uparrow Z_{1,L;\downarrow} \\ F_\uparrow Z_{1,L;\downarrow} & F_\uparrow Z_{1,L;\downarrow} & B_\uparrow & B_\uparrow \\ F_\uparrow Z_{1,L;\downarrow} & F_\uparrow Z_{1,L;\downarrow} & B_\uparrow & B_\uparrow \end{pmatrix} \text{ and } \mathbb{E}_{a,\downarrow} = \begin{pmatrix} B_\downarrow & B_\downarrow & F_\downarrow & F_\downarrow \\ B_\downarrow & B_\downarrow & F_\downarrow & F_\downarrow \\ F_\downarrow & F_\downarrow & B_\downarrow & B_\downarrow \\ F_\downarrow & F_\downarrow & B_\downarrow & B_\downarrow \end{pmatrix} \tag{4.3.11}$$

We also remark that the mapping is not unique and it is always possible to perform the transformations as $c^{(a)}_{i,\sigma} \rightarrow \pm c^{(a)\dagger}_{i,\sigma}$.

If we consider the algebras $gl(\mathfrak{n}|\mathfrak{m})$ with $\mathfrak{n} \neq \mathfrak{m}$ then the Jordan-Wigner transformation of the elements $E^{ij} \in gl(\mathfrak{n}|\mathfrak{m})$ is embedded in $gl(2^{p-1}|2^{p-1})$ transformation without certain row and lines. The parameter p defined such that $2^{p-2} < \max(\mathfrak{n}, \mathfrak{m}) < 2^{p-1}$.

For example, we take $gl(2|1)$ and $gl(1|2)$ algebras. The Jordan-Wigner transformation is obviously embedded in $gl(2|2)$ algebra one. Therefore, the transformation for $E^{ij}_p \in gl(2|1)$ is (4.3.10) without 3rd row and 3rd line and similarly, the transformation for $E^{ij}_p \in gl(1|2)$ is (4.3.10) without 2nd row and 2nd line:

$$gl(2|1): \qquad \mathbb{E}_a = \begin{pmatrix} (1-n^c_a)(1-n^d_a) & -c_a d_a & -c_a(1-n^d_a) \\ c^\dagger_a d^\dagger_a & n^c_a n^d_a & n^c_a d^\dagger_a \\ -c^\dagger_a(1-n^d_a) & n^c_a d_a & n^c_a(1-n^d_a) \end{pmatrix}, \tag{4.3.12}$$

$$gl(1|2): \qquad \mathbb{E}_a = \begin{pmatrix} (1-n^c_a)(1-n^d_a) & (1-n^c_a)d_a & -c_a(1-n^d_a) \\ (1-n^c_a)d^\dagger_a & (1-n^c_a)n^d_a & c_a d^\dagger_a \\ -c^\dagger_a(1-n^d_a) & d_a c^\dagger_a & n^c_a(1-n^d_a) \end{pmatrix}, \tag{4.3.13}$$

4.3.2 Examples of Hubbard-like Hamiltonians

In this subsection we consider certain examples of the Hubbard-like models based on small algebras. Namely, we consider the models based on the $gl(2|2) \oplus gl(2)$ and $gl(2|1) \oplus gl(2)$ algebras. Then we consider the model based on the $gl(1|2) \oplus gl(1|2)$ algebra, we consider different projectors π and $\bar{\pi}$. In the final part we discuss the applications in the AdS/CFT context. These Hamiltonians also seem to be interesting to condensed matter physics. The following results are taken from [126], [127].

$\left(\begin{smallmatrix}2|2\\1\end{smallmatrix}; \begin{smallmatrix}2|0\\1\end{smallmatrix}\right)$ **model.** The following model is based on the $gl(2|2) \oplus gl(2)$ algebra. The projectors π_σ and $\bar{\pi}_\sigma$ are chosen to be trivial: $\pi_\uparrow = E^{11}_\uparrow$, $\pi_\downarrow = E^{11}_\downarrow$ and $\bar{\pi}_\uparrow = E^{22}_\uparrow + E^{33}_\uparrow + E^{44}_\uparrow$, $\bar{\pi}_\downarrow = E^{22}_\downarrow$. Hence, the particle content of the model is such that we have only $\bar{\pi}$-particles on the reference state. The Hamiltonian (4.2.24) in terms of matrix $E^{ij}_{p,\sigma}$ is written as

$$\begin{aligned} \mathcal{H}_{gl(2|2)\oplus gl(2)} &= \sum_{x=1}^{L} \Big(\sum_{\bar{j}=2}^{4} \Big\{ (-1)^{[\bar{j}]} E^{1\bar{j}}_{\uparrow\, x} E^{\bar{j}1}_{\uparrow\, x+1} + E^{\bar{j}1}_{\uparrow\, x} E^{1\bar{j}}_{\uparrow\, x+1} \Big\} + \Big\{ E^{12}_{\downarrow\, x} E^{21}_{\downarrow\, x+1} + E^{21}_{\downarrow\, x} E^{12}_{\downarrow\, x+1} \Big\} + \\ &+ \frac{U}{4} (E^{11}_{\uparrow x} - \sum_{\bar{j}=2}^{4} E^{\bar{j}\bar{j}}_{\uparrow\, x})(E^{11}_{\downarrow x} - E^{22}_{\downarrow\, x}) \Big), \quad L+1 \equiv 1 \end{aligned} \tag{4.3.14}$$

where the grading is $[1] = [2] = 0$ and $[3] = [4] = 1$.

Using the Jordan-Wigner transformation, the Hamiltonian of the model can be rewritten as

$$\begin{aligned} \mathcal{H}_{gl(2|2)\oplus gl(2)} &= -\mathcal{H}_{Hub}(c, c^\dagger, -U) + \sum_{x=1}^{L} (c^\dagger_{\uparrow\, x+1} c_{\uparrow\, x} + c^\dagger_{\uparrow\, x} c_{\uparrow\, x+1})(n^d_{x\uparrow} n^d_{x+1\uparrow} - n^d_{x\uparrow} - n^d_{x+1\uparrow}) \\ &+ \sum_{x=1}^{L} (d^\dagger_{\uparrow\, x+1} d_{\uparrow\, x} + d^\dagger_{\uparrow\, x} d_{\uparrow\, x+1})(n^c_{x\uparrow} n^c_{x+1\uparrow} - n^c_{x\uparrow} - n^c_{x+1\uparrow}) \\ &+ \sum_{x=1}^{L} \Big(c^\dagger_{\uparrow\, x+1} c_{\uparrow\, x} d^\dagger_{\uparrow\, x+1} d_{\uparrow\, x} + c^\dagger_{\uparrow\, x} c_{\uparrow\, x+1} d^\dagger_{\uparrow\, x} d_{\uparrow\, x+1} \Big) \\ &- \frac{U}{4} \sum_{x=1}^{L} (1 - 2n^c_{x\,\downarrow})(1 - n^c_{x\,\uparrow}) 2n^d_{x\,\uparrow}, \quad L+1 \equiv 1 \end{aligned} \tag{4.3.15}$$

with

$$\mathcal{H}_{Hub}(c, c^\dagger, U) = -\sum_{\substack{x=1 \\ \alpha=\uparrow,\downarrow}}^{L} [c^\dagger_{\alpha\, x+1} c_{\alpha\, x} + c^\dagger_{\alpha\, x} c_{\alpha\, x+1}] + \frac{U}{4} \sum_{x=1}^{L} (1 - 2n^c_{x\,\downarrow})(1 - 2n^c_{x\,\uparrow}). \quad L+1 \equiv 1 \tag{4.3.16}$$

The eigenfunctions for this Hamiltonian are made of the creator operators $c^\dagger_\uparrow, d^\dagger_\uparrow$ and $c^\dagger_\downarrow$. They can be written in the following form and correspond to the solutions which can be found in the next section:

$$\begin{aligned}\Phi^{(n)}_{N_1,N_2,N_3} &= \sum_{\substack{\vec{z}\\ z_k\neq z_l}}\sum_{\substack{\vec{y}\\ y_k\neq y_l}}\sum_{\substack{\vec{x}\\ x_i\neq x_j, y_k}} \Psi'(\vec{x},\vec{y},\vec{z}) \prod_{j=1}^{N_1-n} c^\dagger_{x_j\uparrow} \prod_{j=N_1+1}^{N_1+N_2-n} d^\dagger_{x_j\uparrow} \prod_{k=1}^{n} c^\dagger_{y_k\uparrow} d^\dagger_{y_k\uparrow} \prod_{j=1}^{N_3} c^\dagger_{z_j\downarrow}|0> \\ &= (-1)^{N_1+N_3}\phi[\bar{A}]\,, \quad \text{with} \quad n=0,...,\min(N_1,N_2) \qquad (4.3.17)\end{aligned}$$

$$\text{and} \qquad \bar{A} = (\overbrace{4\uparrow,...,4\uparrow}^{N_1-n},\overbrace{3\uparrow,...,3\uparrow}^{N_2-n},\overbrace{2\uparrow,...,2\uparrow}^{n},\overbrace{2\downarrow,...,2\downarrow}^{N_3})\,. \qquad (4.3.18)$$

$\phi[\bar{A}]$ is the eigenfunction given in (5.1.10). Remark that, in addition to the particles $c^\dagger_\uparrow$, $d^\dagger_\uparrow$ and $c^\dagger_\downarrow$ (corresponding to $4\uparrow$, $3\uparrow$ and $2\downarrow$ resp.), we have a doublet $c^\dagger_\uparrow d^\dagger_\uparrow$ corresponding to $2\uparrow$. The particles $c^\dagger_\uparrow$ and $c^\dagger_\downarrow$ can be identified with a (spin up and down) electron, while $d^\dagger_\uparrow$ can be viewed as a spin 0 fermion that can form bound state with the spin up electron.

The energy of the excited state $\Phi^{(n)}_{N_1,N_2,N_3}$ reads

$$E^{(n)} = \frac{U}{4}(L-2(N_1+N_2+N_3-n)) + 2\sum_{l=1}^{N_1+N_2+N_3-n}\cos k_l\,,$$

where the parameters k_l are Bethe roots defined by equations given in the next section 5.1.1 with $\mathfrak{n}=\mathfrak{m}=2$.

$gl(2|1)\oplus gl(2)$ **model.** This model is based on the $gl(2|1)\oplus gl(2)$ algebra with the trivial choice of the projectors π_σ and $\bar{\pi}_\sigma$: $\pi_\uparrow = E^{11}_\uparrow$, $\pi_\downarrow = E^{11}_\downarrow$ and $\bar{\pi}_\uparrow = E^{22}_\uparrow + E^{33}_\uparrow$, $\bar{\pi}_\downarrow = E^{22}_\downarrow$. Again we have only $\bar{\pi}$-particles on the reference state. The Hamiltonian (4.2.24) in terms of matrices $E^{ij}_{p,\sigma}$ is written as

$$\begin{aligned}\mathcal{H}_{gl(2|1)\oplus gl(2)} &= \sum_{x=1}^{L}\Big(\sum_{\bar{j}=2,3}\Big\{(-1)^{[\bar{j}]}E^{1\bar{j}}_{\uparrow\,x}E^{\bar{j}1}_{\uparrow\,x+1} + E^{\bar{j}1}_{\uparrow\,x}E^{1\bar{j}}_{\uparrow\,x+1}\Big\} + \Big\{E^{12}_{\downarrow\,x}E^{21}_{\downarrow\,x+1} + E^{21}_{\downarrow\,x}E^{12}_{\downarrow\,x+1}\Big\} + \\ &\quad + \frac{U}{4}(E^{11}_{\uparrow x} - \sum_{\bar{j}=2,3}E^{\bar{j}\bar{j}}_{\uparrow\,x})(E^{11}_{\downarrow x} - E^{22}_{\downarrow\,x})\Big), \quad L+1\equiv 1 \qquad (4.3.19)\end{aligned}$$

where the grading is defined by $[1]=[2]=0$ and $[3]=1$.

In terms of the fermionic operators the Hamiltonian of the model becomes

$$\begin{aligned}\mathcal{H}_{gl(2|1)\oplus gl(2)} &= -\mathcal{H}_{Hub}(c,c^\dagger,-U) + \sum_{x=1}^{L}(c^\dagger_{\uparrow\,x+1}c_{\uparrow\,x} + c^\dagger_{\uparrow\,x}c_{\uparrow\,x+1})(n^d_{\uparrow\,x}n^d_{x+1\uparrow} - n^d_{\uparrow\,x} - n^d_{\uparrow\,x+1}) \\ &\quad + \sum_{x=1}^{L}\Big(c^\dagger_{\uparrow\,x+1}c_{\uparrow\,x}d^\dagger_{\uparrow\,x+1}d_{\uparrow\,x} + c^\dagger_{\uparrow\,x}c_{\uparrow\,x+1}d^\dagger_{\uparrow\,x}d_{\uparrow\,x+1}\Big) \\ &\quad - \frac{U}{4}\sum_{x=1}^{L}(1-2n^c_{x\,\downarrow})(1-n^c_{x\,\uparrow})n^d_{\uparrow\,x}, \quad L+1\equiv 1 \qquad (4.3.20)\end{aligned}$$

where $\mathcal{H}_{Hub}(c,c^\dagger,U)$ has been given in (4.3.16). Again, the eigenfunctions for this Hamiltonian cor-

respond to the solutions from the next sections. They have the form

$$\begin{aligned}\Phi_{N_1,N_2,N_3} &= \sum_{\substack{\vec{z}\\ z_k\neq z_l}}\sum_{\substack{\vec{y}\\ y_k\neq y_l}}\sum_{\substack{\vec{x}\\ x_i\neq x_j,y_k}} \Psi(\vec{x},\vec{y},\vec{z})\prod_{l=1}^{N_1} c^\dagger_{x_l\uparrow}\prod_{l=1}^{N_2} c^\dagger_{y_l\uparrow}d^\dagger_{y_l\uparrow}\prod_{l=1}^{N_3} c^\dagger_{z_l\downarrow}|0> \\ &= (-1)^{N_1+N_3}\phi[\bar{A}]\,, \quad \text{with} \quad \bar{A} = (\overbrace{3\uparrow,...,3\uparrow}^{N_1},\overbrace{2\uparrow,...,2\uparrow}^{N_2},\overbrace{2\downarrow,...,2\downarrow}^{N_3}) \end{aligned} \tag{4.3.21}$$

$\phi[\bar{A}]$ has been defined in (5.1.10) and the corresponding eigenvalue reads

$$E = \frac{U}{4}(L-2(N_1+N_2+N_3)) + 2\sum_{l=1}^{N_1+N_2+N_3}\cos k_l$$

with Bethe roots k_l obeying the equations given in the next section 5.1.1 with $\mathfrak{n}=2$, $\mathfrak{m}=1$.

$\left(\begin{smallmatrix}1|2\\1\end{smallmatrix};\begin{smallmatrix}1|2\\1\end{smallmatrix}\right)$**-model.** The next example we consider here is $\left(\begin{smallmatrix}1|2\\1\end{smallmatrix};\begin{smallmatrix}1|2\\1\end{smallmatrix}\right)$-model which is based on $gl(1|2)\oplus gl(1|2)$ superalgebra. The projectors π_σ and $\bar{\pi}_\sigma$ are chosen such that $\pi_\sigma = E^{11}_\sigma$ and $\bar{\pi}_\sigma = E^{22}_\sigma + E^{33}_\sigma$. Therefore, there are only two $\bar{\pi}_\uparrow$ and two $\bar{\pi}_\downarrow$ particles. The Hamiltonian (4.2.24) in terms of matrix $E^{ij}_{\sigma\, p} \in gl(1|2)$ is written as

$$\begin{aligned}\mathcal{H}_{gl(1|2)\oplus gl(1|2)} &= \sum_{x=1}^{L}\Big(\sum_{\bar{\jmath}=2,3}\Big\{-E^{1\bar{\jmath}}_{\uparrow\, x}E^{\bar{\jmath}1}_{\uparrow\, x+1}+E^{\bar{\jmath}1}_{\uparrow\, x}E^{1\bar{\jmath}}_{\uparrow\, x+1}\Big\}+\sum_{\bar{\jmath}=2,3}\Big\{-E^{1\bar{\jmath}}_{\downarrow\, x}E^{\bar{\jmath}1}_{\downarrow\, x+1}+E^{\bar{\jmath}1}_{\downarrow\, x}E^{1\bar{\jmath}}_{\downarrow\, x+1}\Big\}+ \\ &+\frac{U}{4}(E^{11}_{\uparrow x}-\sum_{\bar{\jmath}=2,3}E^{\bar{\jmath}\bar{\jmath}}_{\uparrow\, x})(E^{11}_{\downarrow x}-\sum_{\bar{\jmath}=2,3}E^{\bar{\jmath}\bar{\jmath}}_{\downarrow\, x})\Big),\quad L+1\equiv 1\end{aligned} \tag{4.3.22}$$

Using the Jordan-Wigner transformation (4.3.13) for $E^{ij}_{\uparrow\, x}$ and $E^{ij}_{\downarrow\, x}$ matrices we obtain the Hamiltonian in terms of fermionic operators $c_{x,\sigma}$ and $d_{x,\sigma}$ with $\sigma=\uparrow,\downarrow$:

$$\mathcal{H}_{gl(1|2)\oplus gl(1|2)} = -\mathcal{H}_{Hub}(c,c^\dagger,-U) - \mathcal{H}_{Hub}(d,d^\dagger,-U) - V_{int}(c,c^\dagger,d,d^\dagger,-U) \tag{4.3.23}$$

$$\mathcal{H}_{Hub}(c,c^\dagger,U) = -\sum_{k=1}^{L}\sum_{\sigma=\uparrow,\downarrow}(c^\dagger_{k\sigma}c_{k+1\sigma}+c^\dagger_{k+1\sigma}c_{k\sigma}) + \frac{U}{4}\sum_{k=1}^{L}(1-2n^c_{k\uparrow})(1-2n^c_{k\downarrow}) \tag{4.3.24}$$

under the periodicity conditions $L+1\equiv 1$. The potential term $V_{int}(c,c^\dagger,d,d^\dagger,U)$ is

$$\begin{aligned}V_{int}(c,c^\dagger,d,d^\dagger,U) &= -\sum_{x=1}^{L}\sum_{\sigma=\uparrow,\downarrow}\Big((c^\dagger_{x\sigma}c_{x+1\sigma}+c^\dagger_{x+1\sigma}c_{x\sigma})(n^d_{x,\sigma}n^d_{x+1,\sigma}-n^d_{x,\sigma}-n^d_{x+1,\sigma}) \\ &+ (d^\dagger_{x\sigma}d_{x+1\sigma}+d^\dagger_{x+1\sigma}d_{x\sigma})(n^c_{x,\sigma}n^c_{x+1,\sigma}-n^c_{x,\sigma}-n^c_{x+1,\sigma})\Big) \\ &+ \frac{U}{4}\sum_{x=1}^{L}\Big((1-2n^c_{x\uparrow})(1-2n^d_{x\uparrow})+(1-2n^d_{x\uparrow})(1-2n^c_{x\uparrow}) \\ &- 2(1-n^c_{x\uparrow}-n^d_{x\uparrow})(1+2n^c_{x\downarrow}n^d_{x\downarrow})-2(1+2n^c_{x\uparrow}n^d_{x\uparrow})(1-n^c_{x\downarrow}-n^d_{x\downarrow}) \\ &+ (1+2n^c_{x\uparrow}n^d_{x\uparrow})(1+2n^c_{x\downarrow}n^d_{x\downarrow})\Big)\end{aligned} \tag{4.3.25}$$

The Hamiltonian (4.3.23) can be interpreted as two periodic lines on which the electrons (described by $c, c^\dagger$ for the first line, and by $d, d^\dagger$ for the second one) interact via the usual Hubbard Hamiltonian (4.3.24), plus a term of interaction V_{int} between the two lines. Thus, one gets a 'double-row' Hubbard Hamiltonian, and considering more involved $\left(\begin{smallmatrix}\mathtt{n}\\ \mathtt{p}\end{smallmatrix};\begin{smallmatrix}\mathtt{m}\\ \mathtt{q}\end{smallmatrix}\right)$-models, one can construct multileg-Hubbards models. In this way, we construct an almost two-dimensional Hubbard model that is still integrable. See [121] and the references therein for the information about ladder models.

The eigenfunctions for this Hamiltonian are made of creator operators $c^\dagger_\uparrow$,$c^\dagger_\downarrow$,$d^\dagger_\uparrow$ and $d^\dagger_\downarrow$. They can be written in the following form and correspond to the solutions constructed in the section 5.2.2:

$$\begin{aligned}\Phi_{N_1,N_2,N_3,N_4} \;=\; & \sum_{\substack{\vec{x}^{(1)}\\ x^{(1)}_k\neq x^{(1)}_l}} \sum_{\substack{\vec{x}^{(2)}\\ x^{(2)}_k\neq x^{(2)}_l, x^{(1)}_l}} \sum_{\substack{\vec{x}^{(3)}\\ x^{(3)}_i\neq x^{(3)}_j}} \sum_{\substack{\vec{x}^{(4)}\\ x^{(4)}_i\neq x^{(4)}_j, x^{(3)}_l}} \Psi'(\vec{x}^{(1)},\vec{x}^{(2)},\vec{x}^{(3)},\vec{x}^{(4)}) \\ & \prod_{j=1}^{N_1} c^\dagger_{x^{(1)}_j\uparrow} \prod_{j=1}^{N_2} c^\dagger_{x^{(2)}_j\downarrow} \prod_{j=1}^{N_3} d^\dagger_{x^{(3)}_j\uparrow} \prod_{j=1}^{N_4} d^\dagger_{x^{(4)}_j\downarrow} |0>= (-1)^{N_1+N_2}\phi[\bar{A}] \qquad (4.3.26)\end{aligned}$$

$$\text{and} \qquad \bar{A} = (\overbrace{3\uparrow,...,3\uparrow}^{N_1},\overbrace{3\downarrow,...,3\downarrow}^{N_2},\overbrace{2\uparrow,...,2\uparrow}^{N_3}\overbrace{2\downarrow,...,2\downarrow}^{N_4}). \qquad (4.3.27)$$

where $\phi[\bar{A}]$ is the eigenfunction given in (5.2.15). The particles $c^\dagger_\uparrow$,$c^\dagger_\downarrow$, $d^\dagger_\uparrow$ and $d^\dagger_\downarrow$ correspond to $3\uparrow$, $3\downarrow$, $2\uparrow$ and $2\downarrow$ respectively. The particles $c^\dagger_\sigma$ and $d^\dagger_\sigma$ can be identified with a (spin up and down) electrons.

The energy of the excited state Φ_{N_1,N_2,N_3,N_4} reads

$$E = \frac{U}{4}\Big(L-2N\Big) + 2\sum_{l=1}^{N}\cos k_l \quad \text{with} \quad N = N_1+N_2+N_3+N_4$$

where the parameters k_l are Bethe roots defined by the Bethe equations (5.2.1):

$$e^{ik_jL} = \prod_{l=1}^{N_1+N_3}\frac{\lambda_l+\sin k_j - iU/4}{\lambda_l+\sin k_j+iU/4}\prod_{i=1}^{N_4} b_i\,, \quad \text{for} \quad j\in[1,N] \qquad (4.3.28)$$

$$b_i^{N_2+N_4} = 1, \;\; arg(b_i) < arg(b_{i+1}), \quad \text{with} \quad i\in[1,N_4], \qquad (4.3.29)$$

$$\prod_{j=1}^{N}\frac{\lambda_l+\sin k_j-iU/4}{\lambda_l+\sin k_j+iU/4} = e^{\frac{2\pi i}{N_1+N_3}\sum_{i=1}^{N_3}\bar{n}_i}\prod_{i=1}^{N_4} b_i \prod_{\substack{m=1\\ m\neq l}}^{N_1+N_3}\frac{\lambda_l-\lambda_m-iU/2}{\lambda_l-\lambda_m+iU/2}, \qquad (4.3.30)$$
$$\text{with} \quad 1\le\bar{n}_1<...<\bar{n}_{N_3}\le N_1+N_3, \quad \text{for} \quad l\in[1,N_1+N_3]$$

Another example which is based also on the $gl(1|2)$ superalgebra is $\left(\begin{smallmatrix}1|2\\ 2\end{smallmatrix};\begin{smallmatrix}1|2\\ 1\end{smallmatrix}\right)$-model. Now the projectors π_σ and $\bar{\pi}_\sigma$ are chosen such that $\pi_\uparrow = E^{11}_\uparrow + E^{22}_\uparrow$, $\pi_\downarrow = E^{11}_\downarrow$ and $\bar{\pi}_\uparrow = E^{33}_\uparrow$, $\bar{\pi}_\downarrow = E^{22}_\downarrow + E^{33}_\downarrow$. The particle content is different from the previous case: one $\pi_\uparrow$, one $\bar{\pi}_\uparrow$ particles and two $\bar{\pi}_\downarrow$ particles. The Hamiltonian (4.2.24) in terms of matrix $E^{ij}_{p,\sigma}\in gl(2|2)$ is written as

$$\begin{aligned}\mathcal{H}'_{gl(1|2)\oplus gl(1|2)} \;=\; & \sum_{x=1}^{L}\Big(-\sum_{J=1,2}\Big\{E^{J3}_{\uparrow\,x}E^{3J}_{\uparrow\,x+1}+(-1)^{[J]}E^{3J}_{\uparrow\,x}E^{J3}_{\uparrow\,x+1}\Big\}+\sum_{\bar{J}=2,3}\Big\{-E^{1\bar{J}}_{\downarrow\,x}E^{\bar{J}1}_{\downarrow\,x+1}+E^{\bar{J}1}_{\downarrow\,x}E^{1\bar{J}}_{\downarrow\,x+1}\Big\}+ \\ & +\frac{U}{4}\Big(\sum_{J=1,2}E^{JJ}_{\uparrow\,x}-E^{33}_{\uparrow x}\Big)\Big(E^{11}_{\downarrow x}-\sum_{\bar{J}=2,3}E^{\bar{J}\bar{J}}_{\downarrow\,x}\Big)\Big), \;\; L+1\equiv 1 \qquad (4.3.31)\end{aligned}$$

We use the Jordan-Wigner transformation (4.3.13) for $E^{ij}_{\uparrow\, x}$ and $E^{ij}_{\downarrow\, x}$ matrices we obtain the Hamiltonian in terms of fermionic operators $c_{x,\sigma}$ and $d_{x,\sigma}$ with $\sigma =\uparrow,\downarrow$:

$$\mathcal{H}'_{gl(1|2)\oplus gl(1|2)} = -\mathcal{H}_{Hub}(c,c^\dagger,-U) + \sum_{x=1}^{L}\left(d^\dagger_{x\downarrow}d_{x+1\downarrow} + d^\dagger_{x+1\downarrow}d_{x\downarrow}\right) - V'_{int}(c,c^\dagger,d,d^\dagger,-U) \quad (4.3.32)$$

with

$$\begin{aligned} V'_{int}(c,c^\dagger,d,d^\dagger,U) &= -\sum_{x=1}^{L}\sum_{\sigma=\uparrow,\downarrow}\left((c^\dagger_{x\sigma}c_{x+1\sigma} + c^\dagger_{x+1\sigma}c_{x\sigma})(n^d_{x,\sigma}n^d_{x+1,\sigma} - n^d_{x,\sigma} - n^d_{x+1,\sigma})\right) \\ &+ \sum_{x=1}^{L}\left((d^\dagger_{x\downarrow}d_{x+1\downarrow} + d^\dagger_{x+1\downarrow}d_{x\downarrow})(n^c_{x,\downarrow}n^c_{x+1,\downarrow} - n^c_{x,\downarrow} - n^c_{x+1,\downarrow})\right) \\ &+ \sum_{x=1}^{L}\left(c^\dagger_{x\uparrow}c_{x+1\uparrow}d^\dagger_{x+1\uparrow}d_{x\uparrow} + c^\dagger_{x+1\uparrow}c_{x\uparrow}d^\dagger_{x\uparrow}d_{x+1\uparrow}\right) \\ &+ \frac{U}{4}\sum_{x=1}^{L}\left(2n^c_{x\uparrow}n^d_{x\uparrow}(1-2n^c_{x\downarrow}) - 2n^d_{x\downarrow}(1-2n^c_{x\uparrow})(1-2n^c_{x\downarrow}) - 4n^c_{x\uparrow}n^d_{x\uparrow}n^d_{x\downarrow}(1-2n^c_{x\downarrow})\right) \end{aligned} \quad (4.3.33)$$

We will not write the Bethe equations and the form of the eigenvectors in this case.

Comparison with AdS/CFT

The (energy) spectrum of the (usual) Hubbard model has been shown [75] to reproduce correctly the spectrum of the dilation operator in the $su(2)$ subsector up to three loops. The perturbation theory is done at $g \to 0$ on the SYM-side, while it is done at $U \to \infty$ on the Hubbard's side. Starting at fourth loop, the corrections [77] differ with the Hubbard model. The Bethe equations (for the $su(2)$ subsector) read

$$\left(\frac{x(u_k+\frac{i}{2})}{x(u_k-\frac{i}{2})}\right)^L = \prod_{\substack{j=1\\ j\neq k}}^{M}\frac{u_k-u_j+i}{u_k-u_j-i}\,e^{i\theta(u_k,u_j)} \quad \text{with} \quad \begin{cases} u_j = \frac{1}{2}\text{cotan}(\frac{1}{2}p_j) \\ x(u) = \frac{u}{2}\left(1+\sqrt{1-\frac{2g^2}{u^2}}\right) \end{cases} \quad (4.3.34)$$

where $\theta(u_k,u_j)$ is the so-called dressing phase mention in (3.5.1).

In the Bethe equations of $\left(\begin{smallmatrix}\mathfrak{n}\\ \mathfrak{p}\end{smallmatrix};\begin{smallmatrix}\mathfrak{m}\\ \mathfrak{q}\end{smallmatrix}\right)$-models as we will see in the next section (5.2.13), a phase occurs, that depends on some Bethe parameters. Unfortunately, for the present models, the Bethe parameters involved in the phase seem not to be of the type u_j. However, we believe that the present construction could be a first step for the construction of an integrable model possessing such a dependence. In the context of super Yang-Mills theories, one should stick to the single circle interpretation, with electrons possessing some internal degrees of freedom. Then, to get a true $su(2)$ model, one would have to integrate over the internal degrees of freedom, to get an effective model.

To be more appealing, we reformulate the Bethe equations (4.3.28)-(4.3.30) with the notation and half-filling constraints:

$$U = -\frac{\sqrt{2}}{g} \quad ; \quad \lambda_\ell = -\frac{u_\ell}{\sqrt{2}g} \quad ; \quad N = N_1+N_2+N_3+N_4 = L \quad ; \quad N_1+N_3 = M \quad (4.3.35)$$

Forgetting for a while the state multiplicity, eq. (4.3.29), we also set

$$\prod_{i=1}^{N_4} b_i = e^{\frac{2i\pi}{L-M}\theta_{24}} \quad \text{and} \quad e^{\frac{2\pi i}{N_1+N_3}\sum_{i=1}^{N_3}\bar{n}_i} = e^{\frac{2i\pi}{M}\theta_{13}} \tag{4.3.36}$$

Then, we get:

$$e^{ik_jL} = e^{\frac{2i\pi}{L-M}\theta_{24}} \prod_{\ell=1}^{M} \frac{u_\ell - g\sqrt{2}\,\sin k_j - \frac{i}{2}}{u_\ell - g\sqrt{2}\,\sin k_j + \frac{i}{2}} \quad \text{for} \quad j = 1, \ldots, L \tag{4.3.37}$$

$$\prod_{j=1}^{L} \frac{u_l - g\sqrt{2}\,\sin k_j - \frac{i}{2}}{u_l - g\sqrt{2}\,\sin k_j + \frac{i}{2}} = e^{\frac{2i\pi}{M}\theta_{13}+\frac{2i\pi}{L-M}\theta_{24}} \prod_{\substack{m=1 \\ m\neq l}}^{M} \frac{u_l - u_m - i}{u_l - u_m + i} \quad \text{for} \quad l = 1, .., M \tag{4.3.38}$$

The first equation can be used to determine the the Bethe roots k_j, while the second one has to be compared with (4.3.34).

Chapter 5

Coordinate Bethe ansatz solution of Hubbard-like models

In this chapter we discuss the Coordinate Bethe Ansatz solution of the generalized Hubbard models introduced in (4.2). In the section (5.1) we present the solution of $gl(\mathfrak{n}|\mathfrak{m}) \oplus gl(2)$ model, [126]. We review, then, several examples of these models based on the $gl(2|2) \oplus gl(2)$ and $gl(2|1) \oplus gl(2)$ algebras. We write the Hamiltonians in terms of the fermionic operators. The section (5.2) is devoted to the solution of $gl(\mathfrak{n}) \oplus gl(\mathfrak{m})$ models. The thermodynamic limit of the Bethe equations and examples of physical Hamiltonians are considered in next subsections. The results of these sections are taken from [127]. This chapter consists of the personal contribution.

The generalizations of the Hubbard model based on the $su(N)$ algebras, [115], [116], or model introduced in (4.2) based on the superalgebras, were introduced several time ago. The solution of the Hubbard model was done firstly by E.H.Lieb and F.Y.Wu by the Coordinate Bethe Ansatz, then by M.Martins and P.B.Ramos using the Algebraic Bethe Ansatz. However, the solution of these generalized Hubbard models, supersymmetric or not supersymmetric, was not done since the introduction[1]. One of the obstacles is that the spin chain appearing in the auxiliary problem contains the permutation between particles with the same spin. Thus, the Algebraic Bethe Ansatz seems not to work in this case. In the Coordinate Bethe Ansatz, however, it is possible to deal with this obstacle.

In the construction of the generalized Hubbard models (4.2), it was shown that one can use different algebras for the spin up chain and the spin down chain. Therefore, at the first step one can consider the models based on the $gl(\mathfrak{n}|\mathfrak{m})_{\uparrow} \oplus gl(2)_{\downarrow}$ algebras. Here we explicitly show that the spin up chain contains $\mathfrak{n} + \mathfrak{m} - 1$ particles and the reference state. The spin down chain is similar to the Hubbard model, there is only one spin down particle. In this model, the auxiliary problem is simplier than the general case due to the fact that the permutation arises only in the next level of the auxiliary problem. Hence, we firstly consider this kind of models and we show how to use the Coordinate Bethe Ansatz for the auxiliary problem. Next we deal with the general case of the $gl(\mathfrak{n}) \oplus gl(\mathfrak{m})$ Hubbard models with π-particles.

5.1 $gl(\mathfrak{n}|\mathfrak{m}) \oplus gl(2)$ model

In this section we show how to generalize the Coordinate Bethe Ansatz for the $gl(\mathfrak{n}|\mathfrak{m}) \oplus gl(2)$ Hubbard model from (4.2). This model is an example before considering more general cases, however the

[1]There are papers on the solution of the $su(N)$ generalized Hubbard models, [117], but the obtained results seem not to be correct.

technique we use in this case can be borrowed for subcases of the general $gl(\mathfrak{n}) \oplus gl(\mathfrak{n})$ model. We point out that here we do not have the π-particles introduced in (4.2).

We briefy outline the strategy of the Coordinate Bethe Ansatz for the generalized $gl(\mathfrak{n}|\mathfrak{m}) \oplus gl(2)$ Hubbard model. The model consists of $\mathfrak{n}+\mathfrak{m}-1$ spin up and one spin down excitations on the reference state. The first level of the Coordinate Bethe Ansatz is almost similar to the one used in the solution of the Hubbard model (3.3.1). Apparently, the modifications arise in the structure of the S-matrix. The auxiliary problem is presented by the Hamiltonians composed of the product of the S-matrices. We proceed to solve the auxiliary problem using again the Coordinate Bethe Ansatz. This method in application to the auxiliary problem of the Hubbard model is described in the lectures on the Hubbard model by B.Sutherland in [122]. We show that it can be generalized to the auxiliary problem of the $gl(\mathfrak{n}|\mathfrak{m}) \oplus gl(2)$ Hubbard model and finally we arrive to the third level of the nested Coordinate Bethe Ansatz which is represented by the auxiliary problem, level 3. The Hamiltonian of the latter appears to be a product of permutation operators, what can be solved easily.

In the next subsection we present the model and write the Bethe equations with the spectrum of the Hamiltonian. Then, we present the Coordinate Bethe Ansatz in the following subsection.

5.1.1 Results for $gl(\mathfrak{n}|\mathfrak{m}) \oplus gl(2)$ model

The Hamiltonian of the $gl(\mathfrak{n}|\mathfrak{m})_\uparrow \oplus gl(2)_\downarrow$ or $\left(\begin{smallmatrix}\mathfrak{n}|\mathfrak{m}\\1\end{smallmatrix};\begin{smallmatrix}2\\1\end{smallmatrix}\right)$ Hubbard model is given by the expression

$$\mathcal{H}_{gl(\mathfrak{n}|\mathfrak{m})\oplus gl(2)} = \sum_{x=1}^{L}\left[(\Sigma P)_{\uparrow\, x,x+1} + (\Sigma P)_{\downarrow\, x,x+1}\right] + \frac{U}{4}\sum_{x=1}^{L}\left(C_{\uparrow\, x}C_{\downarrow\, x}\right), \quad L+1 \equiv 1 \tag{5.1.1}$$

with the notation:

$$(\Sigma P)_{\uparrow\, x,x+1} = \sum_{a=2}^{\mathfrak{n}+\mathfrak{m}}\left(E^{a1}_{\uparrow\, x}E^{1a}_{\uparrow\, x+1} + (-1)^{[a]}E^{1a}_{\uparrow\, x}E^{a1}_{\uparrow\, x+1}\right) \tag{5.1.2}$$

$$(\Sigma P)_{\downarrow\, x,x+1} = E^{12}_{\downarrow\, x}E^{21}_{\downarrow\, x+1} + E^{21}_{\downarrow\, x}E^{12}_{\downarrow\, x+1} \tag{5.1.3}$$

$$C_{\uparrow\, x} = E^{11}_{\uparrow\, x} - \sum_{a=2}^{\mathfrak{n}+\mathfrak{m}} E^{aa}_{\uparrow\, x} \quad ; \quad C_{\downarrow\, x} = E^{11}_{\downarrow\, x} - E^{22}_{\downarrow\, x} \tag{5.1.4}$$

and we use the grading given by $[1] = ...[\mathfrak{n}] = 0$ and $[\mathfrak{n}+1] = ...[\mathfrak{n}+\mathfrak{m}] = 1$.

The Bethe equations of the model $gl(\mathfrak{n}|\mathfrak{m}) \oplus gl(2)$ can be written down as

$$e^{ik_jL} = (-1)^{K+N+1}\prod_{m=1}^{K}\frac{i\sin k_j + ia_m + \frac{U}{4}}{i\sin k_j + ia_m - \frac{U}{4}}, \quad j = 1,\dots,N \tag{5.1.5}$$

$$(-1)^N\prod_{j=1}^{N}\frac{i\sin k_j + ia_m + \frac{U}{4}}{i\sin k_j + ia_m - \frac{U}{4}} = \Lambda(\vec{n}^{(3)})\prod_{\substack{l=1\\l\neq m}}^{K}\frac{ia_m - ia_l + \frac{U}{2}}{ia_m - ia_l - \frac{U}{2}}, \quad m = 1,\dots,K \tag{5.1.6}$$

$$\Lambda(\vec{n}^{(3)}) = \exp\left(\frac{2i\pi}{K}|\vec{n}^{(3)}|\right), \quad |\vec{n}^{(3)}| = \sum_{i=1}^{M} n_i^{(3)}$$

$$0 \le M \le K \quad \text{and} \quad 1 \le n_1^{(3)} < n_2^{(3)} < ... < n_M^{(3)} \le K \tag{5.1.7}$$

where L is the number of sites considered in the Hubbard-like model, N is the total number of $e^{2\downarrow}$,$e^{2\uparrow}$, $e^{3\uparrow}$,...,$e^{(\mathfrak{n}+\mathfrak{m})\uparrow}$ "particles". K counts the total number of excitations from $e^{2\uparrow}$ to $e^{(\mathfrak{n}+\mathfrak{m})\uparrow}$ and finaly M numbers the $e^{3\uparrow}$,...,$e^{(\mathfrak{n}+\mathfrak{m})\uparrow}$ particles.

There are Bethe parameters $n_i^{(k)}$, $3 < k \leq \mathfrak{m}+\mathfrak{n}$, for each particle $e^{k\uparrow}$, but they do not show up in the Bethe equations. In the section (5.2.4), it is shown more precisely how all these remaining parameters (that are quantized) appear in the Bethe ansatz construction.

The energies associated to these states are given by

$$E = \frac{U}{4}(L-2N) + 2\sum_{j=1}^{N}\cos k_j \tag{5.1.8}$$

and their momentum reads

$$\mathfrak{p} = \sum_{j=1}^{N} k_j \,. \tag{5.1.9}$$

Let us note that the Bethe equations for $gl(\mathfrak{n}|\mathfrak{m}) \oplus gl(2)$ are very close to the ones obtained for the Hubbard model (3.3.119).

5.1.2 CBA for $gl(\mathfrak{n}|\mathfrak{m}) \oplus gl(2)$ models

We solve this model via the Coordinate Bethe Ansatz. In this model we have $\mathfrak{n}+\mathfrak{m}+2$ types of different 'particles' denoted by $e^{1\uparrow}, e^{2\uparrow}, ..., e^{(\mathfrak{n}+\mathfrak{m})\uparrow}$ and $e^{1\downarrow}, e^{2\downarrow}$. We choose the vacuum as

$$\phi_0 = \prod_{x=1}^{L} e_x^{1\uparrow} e_x^{1\downarrow} \,.$$

The excitations above the vacuum state are given by

$$\phi[\overline{A}] = \sum_{\mathbf{x}\in[1,L]} \Psi[\mathbf{x},\overline{A}] e_{x_1}^{A_1} ... e_{x_N}^{A_N} \phi_0 \tag{5.1.10}$$

with indices $A_j \in \{2\uparrow, 3\uparrow, ..., (\mathfrak{n}+\mathfrak{m})\uparrow, 2\downarrow\}$ corresponding to vectors $e^{2\uparrow}, e^{3\uparrow}, ..., e^{(\mathfrak{n}+\mathfrak{m})\uparrow}, e^{2\downarrow}$. We assume that we have the numbers $N_{\uparrow 2}$, $N_{\uparrow 3}$, ..., $N_{\uparrow \mathfrak{n}+\mathfrak{m}}$, $N_{\downarrow 2}$ of $e^{2\uparrow}$, $e^{3\uparrow}$, ..., $e^{(\mathfrak{n}+\mathfrak{m})\uparrow}$, $e^{2\downarrow}$ particles respectively. The sum over $\mathbf{x}$ is again considered without points where two "particles" with the same spin are on the same site. N describes the number of all excitations and goes from 1 to L. In (5.1.10), the sites carrying vectors $e_j^{1\uparrow} e_j^{1\downarrow}$, not associated to any excitation, have been omitted.

Now we assume the *Bethe ansatz* for $\Psi(\mathbf{x})$ and follow the steps detailed in section (3.3). We divide the coordinate space $(x_1, x_2, .., x_N)$ into $N!$ sectors. For $x_{Q(1)} < x_{Q(2)} < .. < x_{Q(N)}$, we have

$$\Psi_Q(\mathbf{x}) = \sum_P (-1)^{[sg(Q)]} \Phi(P, QP^{-1}) e^{iP\mathbf{k}\cdot Q\mathbf{x}} \tag{5.1.11}$$

where $P = [P(1), P(2), .., P(N)]$ and $Q = [Q(1), Q(2), .., Q(N)]$ are two permutations of the integers $1, 2, .., N$ and $P\mathbf{k}\cdot Q\mathbf{x} = \sum_i k_{P(i)} x_{Q(i)}$. The symbol $(-1)^{[sg(Q)]}$ stands for the signature of the Q-permutation when restricted to fermionic particles $e^{(\mathfrak{n}+1)\uparrow}, ..., e^{(\mathfrak{n}+\mathfrak{m})\uparrow}$. For instance, we have the property (valid for any permutation Q and any permutation Π_{ii+1}): $(-1)^{[sg(Q\Pi_{ii+1})]} = (-1)^{[sg(Q)]+[A_i][A_{i+1}]}$. We recall that $\Psi(\mathbf{x})$ and accordingly $\Phi(P, QP^{-1})$ both depend on the type of excitations $\overline{A}$.

The first level of the Coordinate Bethe Ansatz almost repeats all basic steps of the one of the Hubbard model, (3.3.1). The difference between both models is in the S-matrix. However, the procedure of the derivation of the S-matrix is practically the same. We will not repeat it here. One should also take into account that only the fermionic particles of spin up and spin down are considered in the Hubbard model. Whereas in this case we have bosons and fermions with spin up and only bosons

with spin down. The difference of these cases arises in the coefficients r_{12} and t_{12} of the S-matrix. For more details we suggest to see (3.3.1).

We gather all the coefficients $\Phi[P, QP^{-1}, \overline{A}]$ in a vector

$$\hat{\Phi}(P) \equiv \sum_{Q,\overline{A}} \Phi[P, QP^{-1}, \overline{A}] \prod_{i=1}^{N} e_i^{A_{Q(i)}}$$

where the sum is done over all possible types $\overline{A}$ and all corresponding sectors Q. A vector $\prod_{i=1}^{N} e_i^{A_{Q(i)}}$ represents one state of N possible excitations and belongs to $V_1 \otimes ... \otimes V_N$ (where $V = \text{span}\{2\downarrow; 2\uparrow; 3\uparrow; 4\uparrow; ...; (\mathfrak{n}+\mathfrak{m})\uparrow\}$). The order in the vector is chosen such that, for $Q = id$, the vector $\prod_{i=1}^{N} e_i^{A_i}$ is

$$\prod_{i=1}^{N} e_i^{A_i} \equiv \overbrace{e^{2\uparrow} \otimes ... \otimes e^{2\uparrow}}^{N_{\uparrow 2}} \otimes ... \otimes \overbrace{e^{\mathfrak{n}+\mathfrak{m}\uparrow} \otimes ... \otimes e^{\mathfrak{n}+\mathfrak{m}\uparrow}}^{N_{\uparrow \mathfrak{n}+\mathfrak{m}}} \otimes \overbrace{e^{2\downarrow} \otimes ... \otimes e^{2\downarrow}}^{N_{\downarrow 2}} \tag{5.1.12}$$

In the case of two particles ($N = 2$) we can introduce the S-matrix:

$$\hat{\Phi}(\Pi_{12} P) = S_{12}^{(1)}(\lambda_1 - \lambda_2)\hat{\Phi}(P)\,.$$

Here, $S_{12}^{(1)}(\lambda_1 - \lambda_2) \equiv S_{12}^{(1)}$ acts on elementary vectors $e^{A_1} \otimes e^{A_2}$ as

$$S_{12}^{(1)} e^{A_1} \otimes e^{A_2} = -e^{A_2} \otimes e^{A_1}, \qquad \text{for} \quad A_1, A_2 = 2\uparrow, ..., (\mathfrak{n}+\mathfrak{m})\uparrow \quad \text{or both} \;\; 2\downarrow \tag{5.1.13}$$

$$S_{12}^{(1)} e^{A_1} \otimes e^{A_2} = t_{12}\, e^{A_1} \otimes e^{A_2} + r_{12}\, e^{A_2} \otimes e^{A_1}, \tag{5.1.14}$$
$$\text{for} \;\; A_{1(2)} = 2\uparrow, ..., (\mathfrak{n}+\mathfrak{m})\uparrow, \;\; A_{2(1)} = 2\downarrow$$

where

$$t_{12} = \frac{2i(\lambda_1 - \lambda_2)}{U + 2i(\lambda_1 - \lambda_2)}, \;\; r_{12} = \frac{-U}{U + 2i(\lambda_1 - \lambda_2)}, \quad \lambda_i = \sin k_i \tag{5.1.15}$$

For an arbitrary number of excitation N we have

$$\hat{\Phi}(\Pi_{ab} P) = S_{ab}^{(1)}(\lambda_a - \lambda_b)\, \hat{\Phi}(P),$$

where the matrix $S_{ab}^{(1)}$ acts non-trivially only on $V_a \otimes V_b$ vector spaces. It can be shown that the S-matrix $S_{ab}^{(1)}$ satisfies the Yang-Baxter equation. In the special point $\lambda_a = \lambda_b$, the S-matrix is proportional to the permutation operator: $S_{ab}^{(1)}(0) = -P_{ab}$.

The periodic boundary condition on the function $\Psi(\mathbf{x})$ writes as

$$\Psi_{QC}(\mathbf{x} + \mathbf{e}_{Q(1)} L) = \Psi_Q(\mathbf{x}) \quad \text{with} \quad C = \Pi_{N1} ... \Pi_{NN-1}$$

and it leads to the auxiliary problem:

$$\mathfrak{h}_j \hat{\Phi}(id) = e^{ik_j L} \hat{\Phi}(id), \quad \text{with} \quad \mathfrak{h}_j = S_{j+1j}^{(1)} ... S_{Nj}^{(1)} S_{1j}^{(1)} ... S_{j-1j}^{(1)} \qquad j = 1, \ldots, N\,. \tag{5.1.16}$$

We already mentioned that the S-matrix satisfies the Yang-Baxter equation. Due to this, it is possible to show that the auxiliary Hamiltonians $\mathfrak{h}_j$ mutually commute.

Auxiliary problem, level 2

In the section (3.3) devoted to the solution of the Hubbard model we used the Algebraic Bethe Ansatz in order to solve the auxiliary problem (3.3.72). In this case we will use the Coordinate Bethe Ansatz to diagonalize the auxiliary Hamiltonian $\mathfrak{h}_j$:

$$\mathfrak{h}_j = S^{(1)}_{j+1j}...S^{(1)}_{Nj}S^{(1)}_{1j}...S^{(1)}_{j-1j} \tag{5.1.17}$$

Firstly, we transform slightly the S-matrix $S^{(1)}_{12} \to -S^{(1)}_{12} \equiv S_{12}$ to simplify the calculations. At this step we have $\mathfrak{n}+\mathfrak{m}$ types of different excitations: $e^{2\downarrow}, e^{2\uparrow}, e^{3\uparrow}, \ldots, e^{(\mathfrak{n}+\mathfrak{m})\uparrow}$. Choosing the "new" vacuum as

$$\phi^{(1)}_0 = \prod_{k=1}^{N} e^{2\downarrow}_k, \tag{5.1.18}$$

the state of K excitations of any type is written as

$$\phi^{(1)}[\bar{A}] = \sum_{\mathbf{x}\in[1,N]} \Psi^{(1)}(\mathbf{x},\bar{A}) \prod_{i=1}^{K} e^{A_i}_{x_i}\phi^{(1)}_0 \tag{5.1.19}$$

where the sum is done over all coordinates x_i without coinciding points $x_i = x_k$ for any i,k (exclusion principle). $\bar{A}$ is a vector $(A_1,..,A_K)$ with $A_i = 2\uparrow, 3\uparrow, ..., (\mathfrak{n}+\mathfrak{m})\uparrow$ (corresponding to $e^{2\uparrow}, e^{3\uparrow}$ $...e^{(\mathfrak{n}+\mathfrak{m})\uparrow}$). Again, sites carrying $e^{2\downarrow}$ (no excitation) have been omitted in (5.1.19).

The Bethe ansatz for the coefficients $\Psi(\mathbf{x},\bar{A})$, in the sector $x_{Q(1)} < x_{Q(2)} < ... < x_{Q(K)}$ where $Q = [Q(1),...,Q(K)]$ is the permutation of the integers $1,2,...,K$, is given by

$$\Psi_Q(\mathbf{x},\bar{A}) = \sum_P \Phi_{\bar{A}}(P, QP^{-1}) \prod_{i=1}^{K} f_{x_{Q(i)}}(a_{P(i)})\,. \tag{5.1.20}$$

where $f_x(a)$ is the one-particle solution (for any type of excitation A) with rapidity a:

$$f_x(a) = \prod_{m=1}^{x-1}\left(-\frac{i\lambda_m + ia + \frac{U}{4}}{i\lambda_{m+1} + ia - \frac{U}{4}}\right). \tag{5.1.21}$$

The eigenvalue corresponding to this state $\phi(\bar{A})$ takes the form

$$\Lambda_j = \sigma_j(a_1)...\sigma_j(a_K) \tag{5.1.22}$$

where $\sigma_j(a)$ is the eigenvalue of the one-particle solution

$$\sigma_j(a) = -\frac{i\lambda_j + ia + \frac{U}{4}}{i\lambda_j + ia - \frac{U}{4}}\,. \tag{5.1.23}$$

How to obtain the one-particle function $f_x(a)$. The idea is similar as in the first level of the Coordinate Bethe Ansatz. One should take the case $K=1$ and apply the auxiliary Hamiltonian $\mathfrak{h}_j$ on the function (5.1.19). The application of the whole Hamiltonian $\mathfrak{h}_j$ on the one-particle function can be calculated recursively. We can define a recursive Hamiltonian as

$$S_{j-k,j}\ldots S_{j-1,j}\phi^{(1)}[\bar{A}] \equiv \sum_{\mathbf{x}\in[1,N]} \hat{H}^{(k)}_j \Psi_Q[\mathbf{x},\bar{A}] \prod_{n=1}^{K} e^{A_n}_{x_n}\phi^{(1)}_0 \tag{5.1.24}$$

There are the recursive relations between the coefficients $\hat{H}_j^{(k)}\Psi_Q[\mathbf{x},\bar{A}]$ and $\hat{H}_j^{(k-1)}\Psi_Q[\mathbf{x},\bar{A}]$ which can be found easily using the S-matrix relations (5.1.13),(5.1.14):

$$\hat{H}_j^{(k)}\Psi_Q[\mathbf{x};\bar{A}] = \hat{H}_j^{(k-1)}\Psi_Q[\mathbf{x};\bar{A}], \quad \text{for } \mathbf{x} \neq j, j-k \tag{5.1.25}$$

$$\hat{H}_j^{(k)}\Psi_Q[\mathbf{x}', \overset{x_n}{\underset{\downarrow}{j}};\bar{A}] = -t_{j-k,j}\hat{H}_j^{(k-1)}\Psi_Q[\mathbf{x}', \overset{x_n}{\underset{\downarrow}{j}};\bar{A}] - r_{j-k,j}\hat{H}_j^{(k-1)}\Psi_Q[\mathbf{x}', \overset{x_n}{\underset{\downarrow}{j-k}};\bar{A}], \tag{5.1.26}$$

$$\hat{H}_j^{(k)}\Psi_Q[\mathbf{x}', \overset{x_n}{\underset{\downarrow}{j-k}};\bar{A}] = -t_{j-k,j}\hat{H}_j^{(k-1)}\Psi_Q[\mathbf{x}', \overset{x_n}{\underset{\downarrow}{j-k}};\bar{A}] - r_{j-k,j}\hat{H}_j^{(k-1)}\Psi_Q[\mathbf{x}', \overset{x_n}{\underset{\downarrow}{j}};\bar{A}] \tag{5.1.27}$$

$$\hat{H}_j^{(k)}\Psi_Q[\mathbf{x}', \overset{x_n}{\underset{\downarrow}{j}}, \overset{x_m}{\underset{\downarrow}{j-k}};\bar{A}] = \hat{H}_j^{(k-1)}\Psi_Q[\mathbf{x}', \overset{x_n}{\underset{\downarrow}{j-k}}, \overset{x_m}{\underset{\downarrow}{j}};\bar{A}] \tag{5.1.28}$$

where the notation $\mathbf{x}', \overset{x_n}{\underset{\downarrow}{j}}$ means that among all $\mathbf{x} = (x_1, ..., x_K)$ there is $x_n = j$ and the rest is not equal to j nor $j-k$. Similarly, the notation $\mathbf{x}', \overset{x_n}{\underset{\downarrow}{j}}, \overset{x_m}{\underset{\downarrow}{j-k}}$ means that among all $\mathbf{x} = (x_1, ..., x_K)$ there are $x_n = j$ and $x_m = j-k$. The rest is not equal to j nor $j-k$.

In the case $K = 1$, $\Psi_Q[\mathbf{x};\bar{A}] = f_x(a)$, one can solve these relations and find the function $f_x(a)$. The parameter a is a Bethe parameter. We point out that for all $A_i = 2\uparrow, ..., \mathfrak{n}+\mathfrak{m}\uparrow$ there is one commun eigenfunction. We will see in the next section that it is not the case for the $gl(\mathfrak{n}) \oplus gl(\mathfrak{m})$ Hubbard-like models. Thus, the recursive relations for $K = 1$ are

$$\hat{H}_j^{(k)} f_j(a) = -t_{j-k,j}\hat{H}_j^{(k-1)} f_j(a) - r_{j-k,j}\hat{H}_j^{(k-1)} f_{j-k}(a) \tag{5.1.29}$$

$$\hat{H}_j^{(k)} f_{j-k}(a) = -t_{j-k,j}\hat{H}_j^{(k-1)} f_{j-k}(a) - r_{j-k,j}\hat{H}_j^{(k-1)} f_j(a) = \sigma_j f_{j-k}(a) \tag{5.1.30}$$

by the last equality we impose that for any k the function $f_x(\lambda)$ have the eigenvalue σ_j.

The second equation can be solved expressing $\hat{H}_j^{(k-1)} f_j(a)$ and using the fact that $\hat{H}_j^{(k-1)} f_{j-k}(\lambda) = f_{j-k}(\lambda)$:

$$\hat{H}_j^{(k-1)} f_j(a) = -\frac{(\sigma_j + t_{j-k,j}) f_{j-k}(a)}{r_{j-k,j}} \tag{5.1.31}$$

and $k \to k+1$ we obtain

$$\hat{H}_j^{(k)} f_j(a) = -\frac{(\sigma_j + t_{j-k-1,j}) f_{j-k-1}(a)}{r_{j-k-1,j}} \tag{5.1.32}$$

Then we can insert these results in (5.1.29) and we find

$$-\frac{(\sigma_j + t_{j-k-1,j}) f_{j-k-1}(a)}{r_{j-k-1,j}} = t_{j-k,j}\frac{(\sigma_j + t_{j-k,j}) f_{j-k}(a)}{r_{j-k,j}} - r_{j-k,j} f_{j-k}(a)$$

what can be solved using (5.1.15) as

$$\frac{f_{j-k-1}}{f_{j-k}} = -\frac{\left((\sigma_j + t_{j-k,j}) t_{j-k,j} - r_{j-k,j}^2\right) r_{j-k-1,j}}{(\sigma_j + t_{j-k-1,j}) r_{j-k,j}} = -\frac{i\sin k_{j-k} + ia - \frac{U}{4}}{i\sin k_{j-k-1} + ia + \frac{U}{4}}$$

where we introduce a Bethe root a as

$$\sigma_j(a) = -\frac{i\sin(k_j) + ia + \frac{U}{4}}{i\sin(k_j) + ia - \frac{U}{4}}$$

Thus, finally, we obtain the solution for one-excitation function $f_x(a)$:

$$f_x(a) = \prod_{m=1}^{x-1}\left(-\frac{i\sin k_m + ia + \frac{U}{4}}{i\sin k_{m+1} + ia - \frac{U}{4}}\right)$$

with the eigenvalue $\sigma_j(a)$ defined above.

We write several convinient formulae which are useful in the following description:

$$\hat{H}_j^{(N-1)} f_x(a) = \sigma_j(a) f_x(a), \quad \text{for} \quad x \neq j \tag{5.1.33}$$

$$\hat{H}_j^{(N-1)} f_j(a) = \prod_{l=1, l\neq j}^{N} \sigma_l^{-1}(a) f_j(a) \tag{5.1.34}$$

Two-particle and K-particle eigenfunction and eigenvalue. In the case $K = 2$ the eigenfunction is given by the Bethe ansatz (5.1.20). The coefficients $\Phi_{\bar{A}}(P, QP^{-1})$ gives the information about the scattering of the particles.

Firstly, we impose the relation for the indentical particles: let $A_i = A_j$, then $\Psi_Q[\mathbf{x}; \bar{A}] = \Psi_{Q\Pi_{ij}}[\Pi_{ij}\mathbf{x}; \bar{A}]$ and it implies

$$\frac{\Phi_{\bar{A}}(P\Pi_{ij}, QP^{-1})}{\Phi_{\bar{A}}(P, QP^{-1})} = 1 \tag{5.1.35}$$

for any $i, j = 1, ..., K$. Furthermore, the action of the recursive Hamiltonian $\hat{H}_j^{(k)}$ allows to determine additional ratios between the coefficients $\Phi_{\bar{A}}(P, QP^{-1})$.

Let us consider two non-identical particles and we want to calculate the coefficient $\hat{H}_j^{(k)}\Psi_Q[x_1, x_2; \bar{A}]$ for any $\mathbf{x} = (x_1, x_2)$. If x_1 is 'out of reach' of the recursive Hamiltonian, i.e. $x_1 < j - k$ and x_2 is such that $j - k \leq x_2 \leq j$ then two-particle function is effectively like one-particle function. Hence we have

$$\hat{H}_j^{(k)}\Psi_Q[x_1, x_2; \bar{A}] = \sum_P \Phi_{\bar{A}}(P, QP^{-1}) \prod_{i=1,2} \hat{H}_j^{(k)} f_{x_{Q(i)}}(a_{P(i)}) \tag{5.1.36}$$

The functions $\hat{H}_j^{(k)} f_{x_i}(a_{P(i)})$ are such that: $\hat{H}_j^{(k)} f_{x_1}(a_{P(1)}) = f_{x_1}(a_{P(1)})$ and

$$\hat{H}_j^{(k)} f_{x_2}(a_{P(2)}) = \sigma_j(a_{P(2)}) f_{x_2}(a_{P(2)}) \tag{5.1.37}$$

for $x_2 \neq j$ and

$$\hat{H}_j^{(k)} f_{x_2}(a_{P(2)}) = \sigma_{j-1}^{-1}(a_{P(2)}) ... \sigma_{j-k}^{-1}(a_{P(2)}) f_{x_2}(a_{P(2)}) \tag{5.1.38}$$

for $x_2 = j$.

When the recursive Hamiltonian $\hat{H}_j^{(k)}$ acts on two particles, we use the relations (5.1.28):

$$\hat{H}_j^{(k)}\Psi_Q[\overset{x_1}{\underset{\downarrow}{j}}, \overset{x_2}{\underset{\downarrow}{j-k}}; \bar{A}] = \hat{H}_j^{(k-1)}\Psi_{Q'}[j-k, j; \bar{A}] = \sum_P \Phi_{\bar{A}}(P, Q'P^{-1}) \prod_{i=1,2} \hat{H}_j^{(k-1)} f_{x_{Q'(i)}}(a_{P(i)}),$$

where $Q' = Q\Pi_{12}$.

We assume that (5.1.36) is valable for $j-k \leq x_1, x_2 < j$:

$$\hat{H}_j^{(k)} \Psi_Q[x_1, x_2; \bar{A}] \equiv \sum_P \Phi_{\bar{A}}(P, QP^{-1}) \prod_{i=1,2} \hat{H}_j^{(k)} f_{x_{Q(i)}}(a_{P(i)}) \tag{5.1.39}$$

This hypothesis seems natural if we want to have the eigenvalue to be the product of the one-particle eigenvalues, i.e. (5.1.22). We point out that the point $x_i = j$ is not included in the hypothesis and we need to determine it separately. On the one hand, we make an assumption of the form of the coefficient $\hat{H}_j^{(k)} \Psi_Q[x_1, x_2; \bar{A}]$, on the other hand, we can calculate it recursively. Therefore, it leads to the relations between the coefficients $\Phi_{\bar{A}}(P, QP^{-1})$.

Let us continue to work with the coefficient $\hat{H}_j^{(k)} \Psi_Q[\overset{x_1}{\downarrow} j, j \overset{x_2}{\downarrow} -k; \bar{A}]$. We can see from the above relation that it can not be reduced to the form depending on $\hat{H}_j^{(k)} f_{x_{Q(i)}}(a_{P(i)})$. Indeed, the relations on the functions $f_x(a)$ (5.1.29) are not sufficient to write the coefficient $\hat{H}_j^{(k)} \Psi_Q[j, j-k; \bar{A}]$ proportional to $\sum_P < ... > \prod_{i=1,2} \hat{H}_j^{(k)} f_{x_{Q(i)}}(a_{P(i)})$. Nevertheless, we can determine this coefficient from another point. The coefficient $\hat{H}_j^{(k+1)} \Psi_{Q'}[j-k-1, j-k; \bar{A}]$ with $Q' = Q\Pi_{12}$ can be expanded due to (5.1.26) as

$$\hat{H}_j^{(k+1)} \Psi_{Q'}[j-k-1, j-k; \bar{A}] = -t_{j-k-1,j} \hat{H}_j^{(k)} \Psi_{Q'}[j-k-1, j-k; \bar{A}] - r_{j-k-1,j} \hat{H}_j^{(k+1)} \Psi_Q[j, j-k; \bar{A}] \tag{5.1.40}$$

The coefficient $\hat{H}_j^{(k)} \Psi_{Q'}[j-k-1, j-k; \bar{A}]$ is determined by (5.1.36) with the terms $\Phi_{\bar{A}}(P, Q'P^{-1})$ and the initial coefficient $\hat{H}_j^{(k+1)} \Psi_{Q'}[j-k-1, j-k; \bar{A}]$ is determined by the hypothesis (5.1.39) with the terms $\Phi_{\bar{A}}(P, Q'P^{-1})$, thus we see that it is possible to satisfy the relation if we impose $\hat{H}_j^{(k+1)} \Psi_Q[j, j-k; \bar{A}]$ to be

$$\hat{H}_j^{(k)} \Psi_Q[\overset{x_1}{\downarrow} j, j \overset{x_2}{\downarrow} -k; \bar{A}] \equiv \sum_P \Phi_{\bar{A}}(P, Q\Pi_{12}P^{-1}) \prod_{i=1,2} \hat{H}_j^{(k)} f_{x_{Q\Pi_{12}(i)}}(a_{P(i)})$$

This relation can be compared with (5.1.39). It leads to the relations between $\Phi_{\bar{A}}(P, QP^{-1})$:

$$\sum_P \Phi_{\bar{A}}(P, Q\Pi_{12}P^{-1})(\hat{H}_j^{(k)} f_j(a_{P(1)}) \hat{H}_j^{(k)} f_{j-k}(a_{P(2)}) - \hat{H}_j^{(k-1)} f_{j-k}(a_{P(1)}) \hat{H}_j^{(k-1)} f_j(a_{P(2)}) = 0 \tag{5.1.41}$$

which can be solved expressing the ratio $\Phi_{\bar{A}}(P, QP^{-1}) / \Phi_{\bar{A}}(P\Pi_{12}, Q(P\Pi_{12})^{-1})$:

$$\frac{\Phi_{\bar{A}}(P, QP^{-1})}{\Phi_{\bar{A}}(P\Pi_{12}, Q(P\Pi_{12})^{-1})} = \frac{ia_{P(1)} - ia_{P(2)} + U/2}{ia_{P(1)} - ia_{P(2)} - U/2} \tag{5.1.42}$$

The same result can be found for any K number of excitation: for $i = 1, ..., K$

$$\frac{\Phi_{\bar{A}}(P, QP^{-1})}{\Phi_{\bar{A}}(P\Pi_{ii+1}, Q(P\Pi_{ii+1})^{-1})} = \frac{ia_{P(i)} - ia_{P(i+1)} + U/2}{ia_{P(i)} - ia_{P(i+1)} - U/2} \tag{5.1.43}$$

The coefficient $\hat{H}_j^{(k)} \Psi_Q[\mathbf{x}', j; \bar{A}]$ is defined by similar expression. Let us suppose that $x_{Q(1)}$ up to a certain $x_{Q(K-n-1)}$ are strictly less than $j-k$, i.e they are 'spectators' and are 'out of reach' of the recursive Hamiltonian, and $x_{Q(K-n)}$ up to $x_{Q(K)} = j$ are, thus, in the sector between $j-k$ and j. Then, we have

$$\hat{H}_j^{(k)} \Psi_Q[\mathbf{x}', j; \bar{A}] \equiv \sum_P \Phi_{\bar{A}}(P, Q'P^{-1}) \prod_{i=1}^{K} \hat{H}_j^{(k)} f_{x_{Q'(i)}}(a_{P(i)}) \tag{5.1.44}$$

where Q' is such that: $Q' = Q\Pi_{K-nK}...\Pi_{K-1K}$. The product is ordered as following:

$$\prod_{i=1}^{K} \hat{H}_j^{(k)} f_{x_{Q'(i)}}(a_{P(i)}) = \overbrace{f_{x_{Q(1)}}(a_{P(1)})...f_{x_{Q(K-n-1)}}(a_{P(K-n-1)})}^{\text{spectators}} f_j(a_{P(K-n)}) f_{x_{Q(K-n)}}(a_{P(K-n+1)})...f_{x_{Q(K-1)}}(a_{P(K)}) \tag{5.1.45}$$

The eigenvalue can be calculated considering the coefficient $\hat{H}_j^{(N-1)}\Psi_Q[\mathbf{x};\bar{A}]$ for $\mathbf{x} \neq j$ and the hypothesis (5.1.39). Hence, we obtain

$$\hat{H}_j^{(N-1)}\Psi_Q[\mathbf{x};\bar{A}] = \Lambda_j\Psi_Q[\mathbf{x};\bar{A}] = \prod_{i=1}^{K}\sigma_j(a_i)\Psi_Q[\mathbf{x};\bar{A}] \tag{5.1.46}$$

The periodic boundary conditions are already satisfied if we consider the coefficient $\hat{H}_j^{(N-1)}$ $\Psi_Q[\mathbf{x}', j;\bar{A}] = \Lambda_j\Psi_Q[\mathbf{x}', j;\bar{A}]$. Using the expressions (5.1.44) and (5.1.45) for $k = N-1$ (recursive Hamiltonian acts on the whole chain, hence there are no 'spectators') and skipping the calculations we obtain

$$\Phi_{\bar{A}}(PC_K, QP^{-1}) = \prod_{i=1}^{N}\sigma_i(a_{P(1)})\Phi_{\bar{A}}(P, QP^{-1}) \tag{5.1.47}$$

where C_K is a cyclic permutation given by $C_K = \Pi_{K1}...\Pi_{KK-1}$

The S-matrix. Next we gather all the coefficients $\Phi(P, QP^{-1}, \bar{A})$ in a vector

$$\hat{\Phi}(P) \equiv \sum_{Q,\overline{A}} \Phi(P, QP^{-1}, \overline{A}) \prod_{i=1}^{K} e_i^{A_{Q(i)}} \tag{5.1.48}$$

where the sum is over all types of excitations and all corresponding sectors. The vector $\prod_{i=1}^{K} e_i^{A_{Q(i)}}$ belongs to $V_1 \otimes ... \otimes V_K$ (where $V = \text{span}\{e^{2\uparrow}; e^{3\uparrow}; ...; e^{(\mathfrak{n}+\mathfrak{m})\uparrow}\}$) and represents one combination of K excitations.

When we apply the Hamiltonian (5.1.16) on the excited state $\phi(\bar{A})$, there are two types of conditions imposed on the coefficients $\hat{\Phi}(P)$: (5.1.43) and (5.1.47). From the first condition, in the case of two "particles" ($K = 2$), we can introduce a S-matrix corresponding to this auxiliary problem. The second type of conditions implies the periodicity condition. The S-matrix is defined by

$$\hat{\Phi}(\Pi_{12}P) = S_{12}^{(2)}(a_1 - a_2)\hat{\Phi}(P) \tag{5.1.49}$$

with

$$S_{12}^{(2)}(a_1 - a_2) = \alpha_{12}P_{12}, \text{ and } \alpha_{12} = \frac{ia_1 - ia_2 + \frac{u}{2}}{ia_1 - ia_2 - \frac{u}{2}}. \tag{5.1.50}$$

For an arbitrary number K of excitations we have

$$\hat{\Phi}(\Pi_{ij}P) = S_{ij}^{(2)}(a_i - a_j)\hat{\Phi}(P), \tag{5.1.51}$$

where $p^{-1}(j) - p^{-1}(i) = 1$ and the permutation $S_{ij}^{(2)}(a_i - a_j)$ acts non-trivially only on the tensor product $V_i \otimes V_j$. The matrix $S_{ij}^{(2)}(a_i - a_j)$ satisfies the Yang–Baxter equation since it is the permutation matrix.

The periodic boundary conditions on $\hat{\Phi}(P)$ can be written in the following form:

$$S_{m+1m}^{(2)}...S_{Km}^{(2)}S_{1m}^{(2)}...S_{m-1m}^{(2)}\hat{\Phi}(id) = \prod_{k=1}^{N}\sigma_k(a_m)\hat{\Phi}(id) \quad \text{for} \quad m = 1, ..., K, \tag{5.1.52}$$

where the S-matrix arguments were omitted for simplicity.

Permutation problem, level 3

Thus, we arrive to the third step of nested coordinate Bethe ansatz. We briefly recall the procedure: *i*) The eigenvector of the $gl(\mathfrak{n}|\mathfrak{m}) \oplus gl(2)$ Hubbard-like Hamiltonian is given by the ansatz (5.1.10). The periodicity conditions lead to the auxiliary problem, (5.1.2). *ii*) We diagonalize the auxiliary Hamiltonian. The periodic boundary conditions, similarly, lead us to another auxiliary problem, level 3.

Here, we have the auxiliary problem with the particle content: $e^{2\uparrow}$, $e^{3\uparrow}$, ... ,$e^{(\mathfrak{n}+\mathfrak{m})\uparrow}$. The Hamiltonian Γ is represented by a cyclic permutation, thus is independent from j.

$$\Gamma\,\phi = P_{j+1j}...P_{Kj}P_{1j}...P_{j-1j}\,\phi = \Lambda\,\phi\,. \tag{5.1.53}$$

The eigenvalues of the cyclic permutation is set of phases depending on the size of the chain K: $\exp(2i\pi n/K)$. However, one variable n is not enough to characterize completely the eigenvector ϕ. In the section (5.2.4) we show how to solve such auxiliary problem in general case and get the complete set of the eigenfunctions.

On this point we finish the description of the Coordinate Bethe Ansatz. The results: the Bethe equations, the spectrum are presented in the begining of the section, (5.1.1).

5.2 $gl(\mathfrak{n}) \oplus gl(\mathfrak{m})$ model

In the chapter (4.2) the generalized Hubbard Hamiltonians (4.2.24) based on $gl(\mathfrak{n}|\mathfrak{m})_\uparrow \oplus gl(\mathfrak{n}'|\mathfrak{m}')_\downarrow$ superalgebra or $\left(\begin{smallmatrix}\mathfrak{n}|\mathfrak{m}\\ \mathfrak{p}\end{smallmatrix};\begin{smallmatrix}\mathfrak{n}'|\mathfrak{m}'\\ \mathfrak{q}\end{smallmatrix}\right)$-model were obtained using the R-matrix formalism. There are two parameters $\mathfrak{p}$ and $\mathfrak{q}$ which give the number of types of π_σ particles involved in the model.

In this section we present the solution of the subcase of such models: $\left(\begin{smallmatrix}\mathfrak{n}\\ \mathfrak{p}\end{smallmatrix};\begin{smallmatrix}\mathfrak{m}\\ \mathfrak{q}\end{smallmatrix}\right)$-model. At first we write the results for the model: the Bethe equations, the energy of the Hamiltonian. Then, in the following subsections, we describe the Coordinate Bethe ansatz method. This description is more general than we presented in the previous section (5.1). We finish this section by considering the thermodynamic limit of the obtained Bethe equations.

As for the supersymmetric $\left(\begin{smallmatrix}\mathfrak{n}|\mathfrak{m}\\ \mathfrak{p}\end{smallmatrix};\begin{smallmatrix}\mathfrak{n}'|\mathfrak{m}'\\ \mathfrak{q}\end{smallmatrix}\right)$-models. We remark that the ansatz solution for the wavefunction can be suggested from the ansatz of $\left(\begin{smallmatrix}\mathfrak{n}\\ \mathfrak{p}\end{smallmatrix};\begin{smallmatrix}\mathfrak{m}\\ \mathfrak{q}\end{smallmatrix}\right)$-models with a factor correponding to the fermions as in (5.1.11) for the $\left(\begin{smallmatrix}\mathfrak{n}|\mathfrak{m}\\ 1\end{smallmatrix};\begin{smallmatrix}2\\ 1\end{smallmatrix}\right)$-models.

5.2.1 Results for $gl(\mathfrak{n}) \oplus gl(\mathfrak{m})$ model

The Hamiltonian of $\left(\begin{smallmatrix}\mathfrak{n}\\ \mathfrak{p}\end{smallmatrix};\begin{smallmatrix}\mathfrak{m}\\ \mathfrak{q}\end{smallmatrix}\right)$-model writes as

$$\mathcal{H} = \sum_{x=1}^{L}\sum_{\sigma=\uparrow,\downarrow}\sum_{\substack{i\in\mathcal{N}_\sigma\\ j\in\overline{\mathcal{N}}_\sigma}}\left(E^{ij}_{\sigma\,x}E^{ji}_{\sigma\,x+1} + E^{ji}_{\sigma\,x}E^{ij}_{\sigma\,x+1}\right) + \frac{U}{4}\sum_{x=1}^{L}(1-2\sum_{j\in\overline{\mathcal{N}}_\uparrow}E^{jj}_{\uparrow\,x})(1-2\sum_{j\in\overline{\mathcal{N}}_\downarrow}E^{jj}_{\downarrow\,x}),\quad L+1\equiv 1 \tag{5.2.1}$$

with the sets $\mathcal{N}_\sigma, \overline{\mathcal{N}}_\sigma$ are defined as

$$\mathcal{N}_\uparrow = \{1,2,...,\mathfrak{p}\}\,,\ \overline{\mathcal{N}}_\uparrow = \{\mathfrak{p}+1,...,\mathfrak{n}\}\quad\text{and}\quad \mathcal{N}_\downarrow = \{1,2,...,\mathfrak{q}\}\,,\ \overline{\mathcal{N}}_\downarrow = \{\mathfrak{q}+1,...,\mathfrak{m}\}\,. \tag{5.2.2}$$

The derivation of the Bethe Ansatz equations of the model is based on the use of the Coordinate Bethe Ansatz. The diagonalization of the Hamiltonian (5.2.1), which involves $\mathfrak{n}+\mathfrak{m}$ types of

particles, is done in two main steps. The Hamiltonian eigenfunctions are written as linear combinations of plane waves, whose coefficients are found to be solutions of a new eigenvalue problem, the Hamiltonian of which shows up as a chain of S-matrices. This is the first auxiliary problem.

In order to diagonalize this auxiliary Hamiltonian, the corresponding wavefunction is decomposed as excitations above a suitably chosen pseudo-vacuum with some coefficients. Using a recursive representation of the auxiliary Hamiltonian, recursive relations between these coefficients can be found. The resolution of these relations depend on the effective structure of the Bethe roots, since the choice of the pseudo-vacuum breaks the symmetry between spin up and spin down particles. The different cases are then considered.

Bethe equations. We write immediately the main result for these models (5.2.1), namely the Bethe Ansatz Equations. The details of the computation are presented in the following.

The spectrum of the generalized $\left(\begin{smallmatrix}n\\p\end{smallmatrix};\begin{smallmatrix}m\\q\end{smallmatrix}\right)$-Hubbard model is

$$E = \frac{U}{4}(L - 2N_{\bar{\pi}}) + 2\sum_{l\in\mathcal{M}_{\pi\uparrow}\cup\mathcal{M}_{\bar{\pi}\downarrow}} \cos(k_l) \tag{5.2.3}$$

where the Bethe roots k_i are parameters solution of the Bethe Ansatz equations (see below) and L is the number of sites. To present them, we introduce integers $0 \leq K \leq N \leq L$, $N_{\uparrow\pi}, N_{\uparrow\bar{\pi}}, N_{\downarrow\pi}, N_{\downarrow\bar{\pi}} \geq 0$ such that $K = N_{\uparrow\pi} + N_{\uparrow\bar{\pi}} + N_{\downarrow\pi} + N_{\downarrow\bar{\pi}}$, and sets of integers

$$\begin{aligned}
\mathcal{M}_{\pi_\uparrow} &= \{1,\dots,N_{\uparrow\pi}\}\,, \quad \mathcal{M}_{\bar{\pi}_\uparrow} = \{N_{\uparrow\pi}+1,\dots,N_{\uparrow\pi}+N_{\uparrow\bar{\pi}}\}\,, && (5.2.4)\\
\mathcal{M}_{\pi_\downarrow} &= \{N_{\uparrow\pi}+N_{\uparrow\bar{\pi}}+1,\dots,N_{\uparrow\pi}+N_{\uparrow\bar{\pi}}+N_{\downarrow\pi}\}\,, && (5.2.5)\\
\mathcal{M}_{\bar{\pi}_\downarrow} &= \{N_{\uparrow\pi}+N_{\uparrow\bar{\pi}}+N_{\downarrow\pi}+1,\dots,K\}, && (5.2.6)\\
\mathbb{A} &= \mathbb{A}_{\pi\uparrow}\cup\mathbb{A}_{\pi\downarrow} = \{a_1,a_2,\dots,a_{N_{\uparrow\pi}}\}\cup\{a_{N_{\uparrow\pi}+N_{\uparrow\bar{\pi}}+1}, a_{N_{\uparrow\pi}+N_{\uparrow\bar{\pi}}+2},\dots,a_{N_{\uparrow\pi}+N_{\uparrow\bar{\pi}}+N_{\downarrow\pi}}\} && (5.2.7)
\end{aligned}$$

The integers a_j are such that $a_i \neq a_j$ for $i \in \mathcal{M}_{\pi\uparrow}$, $j \in \mathcal{M}_{\pi\downarrow}$ and

$$1 \leq a_1 < a_2 < ... < a_{N_{\uparrow\pi}} \leq N \quad , \quad 1 \leq a_{N_{\uparrow\pi}+N_{\uparrow\bar{\pi}}+1} < ... < a_{N_{\uparrow\pi}+N_{\uparrow\bar{\pi}}+N_{\downarrow\pi}} \leq N \tag{5.2.8}$$

Then, Bethe Ansatz equations are

$$e^{ik_j(L-N_{\uparrow\pi})} = (-1)^{N_{\uparrow\pi}-1} e^{2\pi i \frac{m_{\uparrow\pi}}{N_{\uparrow\pi}}}, \;\; m_{\uparrow\pi} = 1, ..., N_{\uparrow\pi} \quad \text{for} \quad j \in \mathbb{A}_{\pi\uparrow}, \tag{5.2.9}$$

$$e^{ik_j(L-N_{\downarrow\bar{\pi}}-N_{\downarrow\mathfrak{m}})} = (-1)^{N_{\downarrow\pi}-1} e^{2\pi i \frac{m_{\downarrow\pi}}{N_{\downarrow\pi}}}, \;\; m_{\downarrow\pi} = 1, ..., N_{\downarrow\pi} \quad \text{for} \quad j \in \mathbb{A}_{\pi\downarrow}, \tag{5.2.10}$$

$$e^{ik_jL} = (-1)^{N_{\downarrow\pi}+N_{\downarrow\mathfrak{m}}-1} \prod_{m\in\mathcal{M}_{\pi\downarrow}} e^{-ik_{a_m}} \prod_{l\in\mathcal{M}_{\bar{\pi}\uparrow}} \frac{i\sin k_j + i\lambda_l + \frac{U}{4}}{i\sin k_j + i\lambda_l - \frac{U}{4}} \prod_{m'\in\mathcal{M}_{\bar{\pi}\downarrow}} b_{m'} \,, \tag{5.2.11}$$
$$\text{for} \quad j \in [1,N] \setminus \mathbb{A}$$

$$b_l^{N_{\downarrow\pi}+N_{\downarrow\mathfrak{m}}} = e^{\frac{2\pi i}{N_{\downarrow\pi}} \sum_{j=1}^{N_{\downarrow\pi}-N_{\downarrow(\mathfrak{m}-1)}} n_j}, \;\; l \in \mathcal{M}_{\bar{\pi}\downarrow}, \;\; arg(b_l) < arg(b_{l+1}) \tag{5.2.12}$$
$$\text{with} \quad 1 \le n_1 < ... < n_{N_{\downarrow\pi}-N_{\downarrow(\mathfrak{m}-1)}} \le N_{\downarrow\bar{\pi}},$$

$$\Lambda \prod_{\substack{l\in\mathcal{M}_{\bar{\pi}\uparrow}\\ l\neq m}} \frac{i\lambda_m - i\lambda_l + \frac{U}{2}}{i\lambda_m - i\lambda_l - \frac{U}{2}} = \prod_{\substack{l=1\\ l\notin\mathbb{A}}}^{N} \frac{i\sin k_l + i\lambda_m + \frac{U}{4}}{i\sin k_l + i\lambda_m - \frac{U}{4}}, \quad \text{for} \quad m \in \mathcal{M}_{\uparrow\bar{\pi}} \tag{5.2.13}$$

$$\Lambda = (-1)^{N-N_{\pi\uparrow}-N_{\pi\downarrow}} \prod_{m\in\mathcal{M}_{\pi\downarrow}} e^{ik_{a_m}} \prod_{l\in\mathcal{M}_{\pi\downarrow}} b_l^{-1} \prod_{m'\in\mathcal{M}_{\pi\uparrow}} e^{-ik_{a_{m'}}} e^{\frac{2\pi i}{N_{\uparrow\bar{\pi}}} \sum_{l'=1}^{N_{\uparrow\bar{\pi}}-N_{\uparrow\mathfrak{n}}} \bar{n}_{l'}}, \tag{5.2.14}$$
$$\text{with} \quad 1 \le \bar{n}_1 < ... < \bar{n}_{N_{\uparrow\bar{\pi}}-N_{\uparrow\mathfrak{n}}} \le N_{\uparrow\bar{\pi}}$$

We have chosen $e^{1\uparrow} \otimes e^{1\downarrow}$ as the reference state at the first level of the Bethe ansatz (see next subsection), so that all states with $e^{2\uparrow}$,...,$e^{\mathfrak{n}\uparrow}$ and $e^{2\downarrow}$,...,$e^{\mathfrak{m}\downarrow}$ appear as excitations ("particles") above the reference state. N is total number of such particles.

The parameter $N_{\uparrow i}$ is the number of $i\uparrow$ particles for $i = 2, ..., \mathfrak{n}$ and $N_{\downarrow j}$ is the number of $j\downarrow$ particles for $j = 2, ..., \mathfrak{m}$. $N_{\uparrow\pi} = \sum_{i=2}^{\mathfrak{p}} N_{\uparrow i}$ and $N_{\downarrow\pi} = \sum_{i=2}^{\mathfrak{q}} N_{\downarrow i}$ count the number of π-particles with spin up and spin down respectively. $N_{\uparrow\bar{\pi}} = \sum_{i=\mathfrak{p}+1}^{\mathfrak{n}} N_{\uparrow i}$ counts the number of spin up $\bar{\pi}$-particles, while $N_{\downarrow\bar{\pi}} = \sum_{i=\mathfrak{q}+1}^{\mathfrak{m}-1} N_{\downarrow i}$ counts the number of spin down $\bar{\pi}$-particles *that are not of type* $\mathfrak{m}\downarrow$. The reason for this latter choice will become clear in the following. In the same way, $K = N_{\uparrow\pi} + N_{\uparrow\bar{\pi}} + N_{\downarrow\pi} + N_{\downarrow\bar{\pi}}$ is the total number of particles that are not of type $\mathfrak{m}\downarrow$.

For given integers $N_{\uparrow\pi}$, $N_{\uparrow\bar{\pi}}$, $N_{\downarrow\pi}$ and $N_{\downarrow\bar{\pi}}$, the phases b_j and the integers n_j and $\bar{n}_j$ correspond to the different "colors" that can have particles of a given type ($\uparrow\pi$, $\uparrow\bar{\pi}$, $\downarrow\pi$ or $\downarrow\bar{\pi}$ types).

The integers a_j (entering in the sets $\mathbb{A}_{\pi\uparrow}$ and $\mathbb{A}_{\pi\downarrow}$) define the order between momenta of the $\uparrow\pi$ and $\downarrow\pi$ particles. This order is preserved (up to a cyclic permutation) by the action of the Hamiltonian on the wavefunction.

5.2.2 CBA for $gl(\mathfrak{n}) \oplus gl(\mathfrak{m})$ models

The states corresponding to N excitations are written as

$$\phi[\overline{A}] = \sum_{\mathbf{x}\in[1,L]} \Psi[\mathbf{x}, \overline{A}] \prod_{i=1}^{N} e_{x_i}^{A_i} \phi_0 \,, \tag{5.2.15}$$

with $\bar{A} = (\overbrace{2\uparrow, ..., 2\uparrow}^{N_{\uparrow 2}}, , ..., \overbrace{\mathfrak{n}\uparrow, ..., \mathfrak{n}\uparrow}^{N_{\uparrow\mathfrak{n}}}, \overbrace{2\downarrow, ..., 2\downarrow}^{N_{\downarrow 2}}, ..., \overbrace{\mathfrak{m}\downarrow, ..., \mathfrak{m}\downarrow}^{N_{\downarrow\mathfrak{m}}})$. The sum is done on $\mathbf{x} = (x_1, x_2, ..., x_N)$ without points where two particles with the same spin coincide.

We point out that the coordinate x_i is associated with the type of particle in the sens that if $i \in \mathcal{M}_{\pi\sigma}$ or $\mathcal{M}_{\bar{\pi}\sigma}$ then x_i is a coordinate of a $\pi\sigma$-particle or $\bar{\pi}\sigma$-particle, respectively.

The vacuum is chosen as

$$\phi_0 = \prod_{k=1}^{L} e_k^{1\uparrow} e_k^{1\downarrow}. \tag{5.2.16}$$

Acting with the Hamiltonian (5.2.1) on the excited state, we get the eigenvalue equation for the Ψ function:

$$\sum_{l\in\mathcal{M}_{\bar{\pi}\uparrow}\cup\mathcal{M}_{\bar{\pi}\downarrow}} \big(\Psi[\mathbf{x}-\mathbf{e}_l+\delta_1^-\mathbf{e}_\pi,\overline{A}]\Delta_l^- + \Psi[\mathbf{x}+\mathbf{e}_l-\delta_1^+\mathbf{e}_\pi,\overline{A}]\Delta_l^+\big) + \Big(\frac{U}{4}(L-2N_{\bar{\pi}}) + U\sum_{l\in\mathcal{M}_{\bar{\pi}_\uparrow}}\sum_{n\in\mathcal{M}_{\bar{\pi}_\downarrow}}\delta(x_l=x_n) - E\Big)\Psi[\mathbf{x},\overline{A}]\Delta^3 = 0 \tag{5.2.17}$$

where $\mathbf{e}_m$ is an elementary vector in $\mathbb{C}^N$ with 1 on the m position and 0 elsewhere. The notation $\Psi[\mathbf{x}-\mathbf{e}_l+\delta_1^\pm\mathbf{e}_\pi]$ means that, if under the Hamiltonian's action some $\bar{\pi}$-particle (at position x_l with some spin α_l) is moved to a place already occupied by some other π-particle with same spin, then they exchange their places. The symbols $\Delta_l^\pm, \Delta^3$ represent the exclusion principle for $\bar{\pi}$-particles: they are equal to zero if two such particles with the same spin coincide on the same site. Explicitly, they read:

$$\Delta_m^\pm = \prod_{\substack{l\in\mathcal{M}_{\bar{\pi}\uparrow}\cup\mathcal{M}_{\bar{\pi}\downarrow}\\ l\neq m}} \prod_{\substack{n\in\mathcal{M}_{\bar{\pi}\uparrow}\cup\mathcal{M}_{\bar{\pi}\downarrow}\\ n>l, n\neq m}} \delta^\updownarrow(x_l\neq x_n) \prod_{\substack{l\in\mathcal{M}_{\bar{\pi}\uparrow}\cup\mathcal{M}_{\bar{\pi}\downarrow}\\ l\neq m}} \delta^\updownarrow(x_l\neq x_m)\delta^\updownarrow(x_l\neq x_m\pm 1), \tag{5.2.18}$$

$$\Delta^3 = \prod_{l\in\mathcal{M}_{\bar{\pi}\uparrow}\cup\mathcal{M}_{\bar{\pi}\downarrow}} \prod_{\substack{n\in\mathcal{M}_{\bar{\pi}\uparrow}\cup\mathcal{M}_{\bar{\pi}\downarrow}\\ n>l}} \delta^\updownarrow(x_l\neq x_n), \tag{5.2.19}$$

where

$$\delta^\updownarrow(x_l\neq x_n) = 1-\delta(x_l-x_n). \tag{5.2.20}$$

We assume the Bethe hypothesis for the general solution of $\Psi[\mathbf{x},\overline{A}]$. Dividing the coordinate space $(x_1,x_2,..,x_N)$ into $N!$ sectors, we write for $x_{Q(1)}<x_{Q(2)}<..<x_{Q(N)}$,

$$\Psi_Q^{P_{\pi\bar{\pi}}}[\mathbf{x},\overline{A}] = \sum_{P'=P_\pi P_{\bar{\pi}}} \Phi_{\overline{A}}(\hat{P}Q,\hat{P}^{-1})e^{i\hat{P}\mathbf{k}\mathbf{x}}, \quad \hat{P}=P_{\pi\bar{\pi}}P' \tag{5.2.21}$$

where $k_1,k_2,..,k_N$ are unequal numbers (the *Bethe roots*), $Q=[Q(1),Q(2),..,Q(N)]$ is an element of the permutation group $\mathfrak{S}_N$ and $\hat{P}\mathbf{k}\mathbf{x}=\sum_i k_{\hat{P}(i)}x_i$. We consider the permutation $\hat{P}$ in its factorized form $\hat{P}=P_{\pi\bar{\pi}}P_\pi P_{\bar{\pi}}$, where $P_{\pi\bar{\pi}}$ is a global and fixed permutation of π and $\bar{\pi}$ particles, while the terms P_π and $P_{\bar{\pi}}$ permute π and $\bar{\pi}$ particles separately.

The energy depends on the "global" $P_{\pi\bar{\pi}}$ permutation:

$$E^{P_{\pi\bar{\pi}}} = 2\sum_{l\in\mathcal{M}_{\bar{\pi}\uparrow}\cup\mathcal{M}_{\bar{\pi}\downarrow}} \cos(k_{P_{\pi\bar{\pi}}(l)}) + \frac{U}{4}(L-2N_{\bar{\pi}}). \tag{5.2.22}$$

The coefficients $\Phi_{\overline{A}}(\hat{P}Q,\hat{P}^{-1})$ in (5.2.21) are not all independent. Indeed, using the symmetry of the wavefunction and the application of the Hamiltonian represented in (5.2.17), it is possible to reduce their number in several cases:

1. For identical particles of any kind (π or $\bar{\pi}$), the wavefunction satisfies the following symmetry property

$$\Psi_Q^{P_{\pi\bar{\pi}}}[\mathbf{x}, \overline{A}] = \Psi_{Q\Pi_{ii+1}}^{P_{\pi\bar{\pi}}}[\Pi_{Q(i)Q(i+1)}\mathbf{x}, \Pi_{Q(i)Q(i+1)}\overline{A}] \tag{5.2.23}$$

that leads to the relation

$$\Phi_{\overline{A}}(\hat{P}Q, \hat{P}^{-1}) = \Phi_{\overline{A}}(\hat{P}Q, (\hat{P}\Pi_{Q(i)Q(i+1)})^{-1}) \tag{5.2.24}$$

where Π_{ab} is the permutation of objects a and b, the indices $Q(i), Q(i+1)$ correspond to identical particles.

2. For particles with the same spin we also impose some kind of weak "exclusion principle" such that the wavefunction vanishes if two particles with the same spin coincide (*but the particles can differ by their π or $\bar{\pi}$ type*). This principle is verified for π-particles if we use (5.2.17) with two particles at positions $x_{Q(i)} = x_{Q(i+1)} - 1$ (in the sector $x_{Q(1)} \ll .. \ll x_{Q(i)} < x_{Q(i+1)} \ll x_{Q(N)}$, the notation $\ll$ means that the particles are far enough from each other). We generalize this condition to any kind of particles. It leads to:

$$\Psi_Q^{P_{\pi\bar{\pi}}}[\mathbf{x}, \overline{A}]_{x_{Q(i)}=x_{Q(i+1)}} = 0\ , \quad A_{Q(i)}, A_{Q(i+1)} \in \mathcal{N}_\uparrow \cup \mathcal{N}_\downarrow \text{ or } \overline{\mathcal{N}}_\uparrow \cup \overline{\mathcal{N}}_\downarrow \tag{5.2.25}$$

that is to say

$$\Phi_{\overline{A}}(\hat{P}Q, \hat{P}^{-1}) = -\Phi_{\overline{A}}(\hat{P}Q\Pi_{ii+1}, (\hat{P}\Pi_{Q(i)Q(i+1)})^{-1}) \tag{5.2.26}$$

for any i such that $A_{Q(i)}, A_{Q(i+1)} \in \mathcal{N}_\uparrow \cup \mathcal{N}_\downarrow$ or $\overline{\mathcal{N}}_\uparrow \cup \overline{\mathcal{N}}_\downarrow$.

3. For particles with different spins, there is no exclusion principle and they can be at the same site. Thus, let us consider the case: $x_{Q(1)} \ll .... \ll x_{Q(i)} = x_{Q(i+1)} \ll x_{Q(N)}$ and ask for the continuity of the wavefunction on the boundary of the two sectors Q and $Q\Pi_{ii+1}$:

$$\Psi_Q^{P_{\pi\bar{\pi}}}[\mathbf{x}, \overline{A}]_{x_{Q(i)}=x_{Q(i+1)}} = \Psi_{Q\Pi_{ii+1}}^{P_{\pi\bar{\pi}}}[\mathbf{x}, \overline{A}]_{x_{Q(i)}=x_{Q(i+1)}}\ . \tag{5.2.27}$$

For two $\bar{\pi}$-particles, it implies the relation, for $A_{Q(i)} \in \overline{\mathcal{N}}_{\uparrow(\downarrow)}$ and $A_{Q(i+1)} \in \overline{\mathcal{N}}_{\downarrow(\uparrow)}$,

$$\begin{aligned}\Phi_{\overline{A}}(\hat{P}Q, \hat{P}^{-1}) + \Phi_{\overline{A}}(\hat{P}Q\Pi_{ii+1}, (\hat{P}\Pi_{Q(i)Q(i+1)})^{-1}) = \\ \Phi_{\overline{A}}(\hat{P}Q, (\hat{P}\Pi_{Q(i)Q(i+1)})^{-1}) + \Phi_{\overline{A}}(\hat{P}Q\Pi_{ii+1}, \hat{P}^{-1})\end{aligned} \tag{5.2.28}$$

while for two π-particles it leads, for $A_{Q(i)} \in \mathcal{N}_{\uparrow(\downarrow)}$ and $A_{Q(i+1)} \in \mathcal{N}_{\downarrow(\uparrow)}$, to

$$\Phi_{\overline{A}}(\hat{P}Q\Pi_{ii+1}, \hat{P}^{-1}) = \Phi_{\overline{A}}(\hat{P}Q, \hat{P}^{-1})\,. \tag{5.2.29}$$

When one particle is of type π and the other one of type $\bar{\pi}$, we find, for $A_{Q(i)} \in \mathcal{N}_{\uparrow(\downarrow)}$ and $A_{Q(i+1)} \in \overline{\mathcal{N}}_{\downarrow(\uparrow)}$,

$$\Phi_{\overline{A}}(\hat{P}Q\Pi_{ii+1}, \hat{P}^{-1}) = \Phi_{\overline{A}}(\hat{P}Q, \hat{P}^{-1})\,. \tag{5.2.30}$$

There is another condition for $\bar{\pi}$-particles when using (5.2.17). It is a relation on the coefficient $\Phi_{\overline{A}}(\hat{P}Q, \hat{P}^{-1})$ which one obtains in the $\left(\begin{smallmatrix}2\\1\end{smallmatrix}; \begin{smallmatrix}2\\1\end{smallmatrix}\right)$ Hubbard model for spin up and spin down interacting electrons:

$$\begin{aligned}\sum_{m \neq Q_i, Q_{i+1}} \left(\Psi_Q^{P_{\pi\bar{\pi}}}[\mathbf{x} - \mathbf{e}_m, \overline{A}] + \Psi_Q^{P_{\pi\bar{\pi}}}[\mathbf{x} + \mathbf{e}_m, \overline{A}]\right) + (\frac{U}{4}(L - 2N_{\bar{\pi}}) + U - E)\Psi_Q^{P_{\pi\bar{\pi}}}[\mathbf{x}, \overline{A}] + \\ \Psi_Q^{P_{\pi\bar{\pi}}}[\mathbf{x} - \mathbf{e}_{Q_i}, \overline{A}] + \Psi_{Q\Pi_{ii+1}}^{P_{\pi\bar{\pi}}}[\mathbf{x} + \mathbf{e}_{Q_i}, \overline{A}] + \Psi_{Q\Pi_{ii+1}}^{P_{\pi\bar{\pi}}}[\mathbf{x} - \mathbf{e}_{Q_{i+1}}, \overline{A}] + \Psi_Q^{P_{\pi\bar{\pi}}}[\mathbf{x} + \mathbf{e}_{Q_{i+1}}, \overline{A}] = 0\end{aligned}$$

Skipping the intermediate calculations and combining the results with (5.2.28), we can write the conditions on $\Phi(\hat{P}Q, \hat{P}^{-1})$ in a matrix form:

$$\begin{pmatrix} \Phi(\Pi_{ab}\hat{P}Q, \hat{P}^{-1}) \\ \Phi(\Pi_{ab}\hat{P}Q, (\Pi_{ab}\hat{P})^{-1}) \end{pmatrix} = \begin{pmatrix} t_{ab} & r_{ab} \\ r_{ab} & t_{ab} \end{pmatrix} \begin{pmatrix} \Phi(\hat{P}Q, \hat{P}^{-1}) \\ \Phi(\hat{P}Q, (\Pi_{ab}\hat{P})^{-1}) \end{pmatrix} \tag{5.2.31}$$

with $a = \hat{P}Q(i)$, $b = \hat{P}Q(i+1)$ and

$$t_{ab} = \frac{2i(\lambda_a - \lambda_b)}{U + 2i(\lambda_a - \lambda_b)}, \quad r_{ab} = \frac{-U}{U + 2i(\lambda_a - \lambda_b)}, \quad \lambda_a = \sin k_a \tag{5.2.32}$$

These equations hold for any type of $\bar{\pi}$-excitations (any value of A_{Q_i},$A_{Q_{i+1}}$ being $\bar{\pi}$-particles).

4. Finally we consider the interaction between π and $\bar{\pi}$ particles with the same spin. Let $A_{Q(i)}$ be a $\bar{\pi}$-particle, $A_{Q(i+1)}$ be a π-particle, with coordinates $x_{Q(i)} = x_{Q(i+1)} - 1$, thus $\delta_1^+ = 1$ and $\delta_1^- = 0$. Using the equation (5.2.17) we derive the relation:

$$\Psi^{P_{\pi\bar{\pi}}}_{Q\Pi_{ii+1}}[\mathbf{x} + \mathbf{e}_{Q(i)} - \mathbf{e}_{Q(i+1)}, \overline{A}] - \Psi^{P_{\pi\bar{\pi}}}_{Q}[\mathbf{x} + \mathbf{e}_{Q(i)}, \overline{A}] = 0 \tag{5.2.33}$$

that implies the following condition on $\Phi_{\overline{A}}(\hat{P}Q, \hat{P}^{-1})$:

$$\Phi_{\overline{A}}(\Pi_{\hat{P}Q(i)\hat{P}Q(i+1)}\hat{P}Q, \hat{P}^{-1}) = e^{ik_{\hat{P}Q(i+1)}}\Phi_{\overline{A}}(\hat{P}Q, \hat{P}^{-1})\,. \tag{5.2.34}$$

We rewrite now all obtained conditions (5.2.24), (5.2.26)-(5.2.31) and (5.2.34) in a more compact form. Let us introduce, for $P' \equiv \hat{P}Q \in \mathfrak{S}_N$ and $Q' \equiv \hat{P}^{-1} \in \mathfrak{S}_N$,

$$\hat{\Phi}(P') \equiv \sum_{Q', \overline{A}} \Phi_{\overline{A}}(P', Q') \prod_{i=1}^{N} e_i^{A_{Q'(i)}} \tag{5.2.35}$$

where the summation is over all types of excitations and all corresponding sectors. The vector $\prod_{i=1}^{N} e_i^{A_{Q'(i)}}$ belongs to $V_1 \otimes ... \otimes V_N$, where V is spanned by $\{e^{2\uparrow}, e^{3\uparrow}, ..., e^{\mathfrak{n}\uparrow}, e^{2\downarrow}, e^{3\downarrow}, ..., e^{\mathfrak{m}\downarrow}\}$ and represents one type of N excitations. The ordering of the particles is chosen such that, for $Q' = id$, the vector $\prod_{i=1}^{N} e_i^{A_i}$ is taken as in (5.2.15):

$$\prod_{i=1}^{N} e_i^{A_i} = \overbrace{e^{2\uparrow} \otimes ... \otimes e^{2\uparrow}}^{N_{\uparrow 2}} \otimes ... \otimes \overbrace{e^{\mathfrak{n}\uparrow} \otimes ... \otimes e^{\mathfrak{n}\uparrow}}^{N_{\uparrow \mathfrak{n}}} \otimes \overbrace{e^{2\downarrow} \otimes ... \otimes e^{2\downarrow}}^{N_{\downarrow 2}} \otimes ... \otimes \overbrace{e^{\mathfrak{m}\downarrow} \otimes ... \otimes e^{\mathfrak{m}\downarrow}}^{N_{\downarrow \mathfrak{m}}} \tag{5.2.36}$$

To clarify the notation for $\hat{\Phi}(P')$, we write an example for the case $N = 2$:

$$\begin{aligned} \hat{\Phi}(P') \;=\; & \sum_{j=2}^{\mathfrak{n}} \Phi_{(j\uparrow,j\uparrow)}(P', id)\, e_1^{j\uparrow} e_2^{j\uparrow} + \sum_{j=2}^{\mathfrak{m}} \Phi_{(j\downarrow,j\downarrow)}(P', id)\, e_1^{j\downarrow} e_2^{j\downarrow} \\ & + \sum_{2 \le j < k \le \mathfrak{n}} \left(\Phi_{(j\uparrow,k\uparrow)}(P', id)\, e_1^{j\uparrow} e_2^{k\uparrow} + \Phi_{(j\uparrow,k\uparrow)}(P', \Pi_{12})\, e_1^{k\uparrow} e_2^{j\uparrow} \right) \\ & + \sum_{2 \le j < k \le \mathfrak{m}} \left(\Phi_{(j\downarrow,k\downarrow)}(P', id)\, e_1^{j\downarrow} e_2^{k\downarrow} + \Phi_{(j\downarrow,k\downarrow)}(P', \Pi_{12})\, e_1^{k\downarrow} e_2^{j\downarrow} \right) \\ & + \sum_{j=2}^{\mathfrak{n}} \sum_{k=2}^{\mathfrak{m}} \left(\Phi_{(j\uparrow,k\downarrow)}(P', id)\, e_1^{j\uparrow} e_2^{k\downarrow} + \Phi_{(j\uparrow,k\downarrow)}(P', \Pi_{12})\, e_1^{k\downarrow} e_2^{j\uparrow} \right) \end{aligned} \tag{5.2.37}$$

Then, all the relations can be expressed in a matrix form. For $N = 2$, it reads

$$\hat{\Phi}(\Pi_{12}P) = S_{12}^{(1)}(k_1, k_2)\hat{\Phi}(P) \tag{5.2.38}$$

where $S_{12}^{(1)}(k_1, k_2)$ (simply denoted by $S_{12}^{(1)}$ in the following) acts on elementary vectors $e_1^{A_1}e_2^{A_2}$ as

1) for π-particles:

$$\begin{aligned} &S_{12}^{(1)}\ e_1^{A_1}e_2^{A_2} = -e_1^{A_2}e_2^{A_1},\ \text{ for }\ A_1, A_2 \in \mathcal{N}_{\uparrow(\downarrow)} \\ &S_{12}^{(1)}\ e_1^{A_1}e_2^{A_2} = e_1^{A_1}e_2^{A_2},\ \text{ for }\ A_1 \in \mathcal{N}_{\uparrow(\downarrow)},\ A_2 \in \mathcal{N}_{\downarrow(\uparrow)} \end{aligned} \tag{5.2.39}$$

2) for $\bar{\pi}$-particles:

$$\begin{aligned} &S_{12}^{(1)}\ e_1^{\bar{A}_1}e_2^{\bar{A}_2} = -e_1^{\bar{A}_2}e_2^{\bar{A}_1},\ \text{ for }\ \bar{A}_1, \bar{A}_2 \in \overline{\mathcal{N}}_{\uparrow(\downarrow)} \\ &S_{12}^{(1)}\ e_1^{\bar{A}_1}e_2^{\bar{A}_2} = t_{12}\, e_1^{\bar{A}_1}e_2^{\bar{A}_2} + r_{12}\, e_1^{\bar{A}_2}e_2^{\bar{A}_1},\ \text{ for }\ \bar{A}_1 \in \overline{\mathcal{N}}_{\uparrow(\downarrow)},\ \bar{A}_2 \in \overline{\mathcal{N}}_{\downarrow(\uparrow)} \end{aligned} \tag{5.2.40}$$

3) for mixed π and $\bar{\pi}$-particles:

$$\begin{aligned} &S_{12}^{(1)}\ e_1^{A_1}e_2^{\bar{A}_2} = e^{-ik_1}e_1^{A_1}e_2^{\bar{A}_2} \text{ and } S_{12}^{(1)}\ e_1^{\bar{A}_2}e_2^{A_1} = e^{ik_2}e_1^{\bar{A}_2}e_2^{A_1},\ \text{ for }\ A_1 \in \mathcal{N}_{\uparrow(\downarrow)},\ \bar{A}_2 \in \overline{\mathcal{N}}_{\uparrow(\downarrow)} \\ &S_{12}^{(1)}\ e_1^{A_1}e_2^{\bar{A}_2} = e_1^{A_1}e_2^{\bar{A}_2} \text{ and } S_{12}^{(1)}\ e_1^{\bar{A}_2}e_2^{A_1} = e_1^{\bar{A}_2}e_2^{A_1},\ \text{ for }\ A_1 \in \mathcal{N}_{\uparrow(\downarrow)},\ \bar{A}_2 \in \overline{\mathcal{N}}_{\downarrow(\uparrow)} \end{aligned} \tag{5.2.41}$$

The parameters t_{12}, r_{12} are defined in (5.2.32) and $\mathcal{N}_{\uparrow(\downarrow)}$, $\overline{\mathcal{N}}_{\uparrow(\downarrow)}$ are given in (5.2.2).
The notation $A \in \mathcal{N}_{\uparrow(\downarrow)}$ means that the excitation A can be equal to any value in $\mathcal{N}_{\uparrow(\downarrow)}$ except the value $1\uparrow(\downarrow)$ since it was chosen as the vacuum at first level.

We illustrate the explicit form of the S-matrix in the case of the $\left(\begin{smallmatrix}3\\2\end{smallmatrix};\begin{smallmatrix}3\\2\end{smallmatrix}\right)$ model. One gets

$$S_{12}^{(1)}(k_1, k_2)e_1^{A_1}e_2^{A_2} =$$

$$\left(\begin{array}{c|c|c|c}
-1 & & & \\
& e^{-ik_1} & & \\
& & 1 & \\
& & & 1 \\
\hline
& e^{ik_2} & & \\
& & -1 & \\
& & & 1 \\
& & & t_{12} \\
\hline
& & 1 & \\
& & & 1 \\
& & & -1 \\
& & & e^{-ik_1} \\
\hline
& & & 1 \\
& r_{12} & & \\
& & t_{12} & \\
& & & e^{ik_2} \\
& & & -1
\end{array}\right)
\left(\begin{array}{c}
e_1^{2\uparrow}e_2^{2\uparrow} \\ e_1^{2\uparrow}e_2^{3\uparrow} \\ e_1^{2\uparrow}e_2^{2\downarrow} \\ e_1^{2\uparrow}e_2^{3\downarrow} \\ \hline
e_1^{3\uparrow}e_2^{2\uparrow} \\ e_1^{3\uparrow}e_2^{3\uparrow} \\ e_1^{3\uparrow}e_2^{2\downarrow} \\ e_1^{3\uparrow}e_2^{3\downarrow} \\ \hline
e_1^{2\downarrow}e_2^{2\uparrow} \\ e_1^{2\downarrow}e_2^{3\uparrow} \\ e_1^{2\downarrow}e_2^{2\downarrow} \\ e_1^{2\downarrow}e_2^{3\downarrow} \\ \hline
e_1^{3\downarrow}e_2^{2\uparrow} \\ e_1^{3\downarrow}e_2^{3\uparrow} \\ e_1^{3\downarrow}e_2^{2\downarrow} \\ e_1^{3\downarrow}e_2^{3\downarrow}
\end{array}\right)$$

For an arbitrary N excitations number, we write

$$\hat{\Phi}(\Pi_{ab}P) = S_{ab}^{(1)}(k_a, k_b)\hat{\Phi}(P), \tag{5.2.42}$$

where the matrix $S^{(1)}_{ab}(k_a,k_b)$ acts non-trivially only on the $V_a \otimes V_b$ vector space. It can be written using permutations and projectors:

$$S^{(1)}_{ab}(k_a,k_b) \;=\; -\sum_{\sigma=\uparrow,\downarrow}\left(P^{\mathcal{N}_\sigma,\mathcal{N}_\sigma}_{ab}+P^{\overline{\mathcal{N}}_\sigma,\overline{\mathcal{N}}_\sigma}_{ab}\right)+\sum_{\sigma=\uparrow,\downarrow}\left(e^{-ik_a}Id^{\mathcal{N}_\sigma,\overline{\mathcal{N}}_\sigma}_{ab}+e^{ik_b}Id^{\overline{\mathcal{N}}_\sigma,\mathcal{N}_\sigma}_{ab}\right)+$$
$$+\sum_{\sigma=\uparrow,\downarrow}\left(Id^{\mathcal{N}_\sigma,\mathcal{N}_{\bar\sigma}}_{ab}+Id^{\mathcal{N}_\sigma,\overline{\mathcal{N}}_{\bar\sigma}}_{ab}+Id^{\overline{\mathcal{N}}_\sigma,\mathcal{N}_{\bar\sigma}}_{ab}\right)+\sum_{\sigma=\uparrow,\downarrow}\left(t_{ab}Id^{\overline{\mathcal{N}}_\sigma,\overline{\mathcal{N}}_{\bar\sigma}}_{ab}+r_{ab}P^{\overline{\mathcal{N}}_\sigma,\overline{\mathcal{N}}_{\bar\sigma}}_{ab}\right) \tag{5.2.43}$$

where $\bar\sigma$ denote the opposite value of σ and

$$Id^{\mathcal{L},\mathcal{L}'}_{ab}=\sum_{i\in\mathcal{L}}\sum_{j\in\mathcal{L}'}E^{ii}_aE^{jj}_b \quad\text{and}\quad P^{\mathcal{L},\mathcal{L}'}_{ab}=\sum_{i\in\mathcal{L}}\sum_{j\in\mathcal{L}'}E^{ij}_aE^{ji}_b\,,\quad \mathcal{L},\mathcal{L}'=\mathcal{N}_\uparrow,\mathcal{N}_\downarrow,\overline{\mathcal{N}}_\uparrow,\overline{\mathcal{N}}_\downarrow. \tag{5.2.44}$$

The matrix $S^{(1)}_{ab}(k_a,k_b)$ satisfies the Yang–Baxter equation (4.2.21).

$$S^{(1)}_{12}(k_1,k_2)S^{(1)}_{13}(k_1,k_3)S^{(1)}_{23}(k_2,k_3)=S^{(1)}_{23}(k_2,k_3)S^{(1)}_{13}(k_1,k_3)S^{(1)}_{12}(k_1,k_2) \tag{5.2.45}$$

Now we can write the periodic boundary conditions. Let $C_N=\Pi_{N1}...\Pi_{NN-1}$ be a cyclic permutation. The sites $L+1$ and 1 being identified, we have the periodicity conditions:

$$\Psi^{P_{\pi\pi}}_{QC_N}[\mathbf{x}-\mathbf{e}_{Q(N)}L,\overline{A}]=\Psi^{P_{\pi\pi}}_{Q}[\mathbf{x},\overline{A}]\,. \tag{5.2.46}$$

In terms of $\hat\Phi(P)$ this yields the condition

$$\hat\Phi(PC_N)=e^{ik_{P(N)}L}\hat\Phi(P)\,. \tag{5.2.47}$$

If we choose $P=C_N^{N-j}$ with $j=1,\dots,N$, we can derive a system of equations on the coefficients $\hat\Phi(id)$ which is called the "auxiliary problem":

$$\mathfrak{h}_j\,\hat\Phi(id)=e^{ik_jL}\hat\Phi(id)\quad\text{with}\quad \mathfrak{h}_j=S^{(1)}_{j+1,j}...S^{(1)}_{Nj}\,S^{(1)}_{1j}...S^{(1)}_{j-1,j}\,,\qquad j=1,...,N \tag{5.2.48}$$

where we omitted the arguments of the S-matrices, $S^{(1)}_{ab}\equiv S^{(1)}_{ab}(k_a,k_b)$.
The Yang–Baxter equation for the S matrix implies that $[\mathfrak{h}_j\,,\,\mathfrak{h}_k]=0$ for all j,k, so that the new Hamiltonians $\mathfrak{h}_j$ can be simultaneously diagonalized: we do it in the folowing section.

Auxiliary problem. Level 2

In order to simplify the calculations, we change the sign of the S-matrix: $S^{(1)}_{12}\to -S^{(1)}_{12}\equiv S_{12}$. The auxiliary problem, thus, reads

$$S_{j+1,j}...S_{Nj}S_{1j}...S_{j-1,j}\phi=\Lambda_j\phi\,. \tag{5.2.49}$$

We use again the Coordinate Bethe ansatz. At this level we have $\mathfrak{n}+\mathfrak{m}-2$ types of different excitations: $e^{2\uparrow},...,e^{\mathfrak{n}\uparrow},e^{2\downarrow},...,e^{\mathfrak{m}\downarrow}$. We choose as the reference state (pseudovacuum):

$$\phi^{(1)}_0=\prod_{k=1}^{N}e^{\mathfrak{m}\downarrow}_k \tag{5.2.50}$$

The eigenvectors for this auxiliary problem are given by

$$\begin{aligned}
\phi^{(1)}[\bar{A}] &= \sum_{\mathbf{x}\in[1,N]} \Psi^{(1)}[\mathbf{x},\bar{A}] \prod_{n=1}^{K} e_{x_n}^{A_n} \phi_0^{(1)} \qquad (5.2.51)\\
K &= \sum_{i=2}^{\mathfrak{p}} N_{\uparrow i} + \sum_{i=2}^{\mathfrak{q}} N_{\downarrow i} + \sum_{i=\mathfrak{p}+1}^{\mathfrak{n}} N_{\uparrow i} + \sum_{i=\mathfrak{q}+1}^{\mathfrak{m}-1} N_{\downarrow i}\,,\\
\bar{A} &= (\underbrace{2\uparrow,...,2\uparrow}_{N_{\uparrow 2}},...,\underbrace{\mathfrak{n}\uparrow,...,\mathfrak{n}\uparrow}_{N_{\uparrow \mathfrak{n}}},\underbrace{2\downarrow,...,2\downarrow}_{N_{\downarrow 2}},...,\underbrace{(\mathfrak{m}-1)\downarrow,...,(\mathfrak{m}-1)\downarrow}_{N_{\downarrow(\mathfrak{m}-1)}})\,.
\end{aligned}$$

The ansatz for $\Psi^{(1)}[\mathbf{x},\bar{A}] \equiv \Psi_Q[\mathbf{x},\bar{A}]$ can be written as, for $x_{Q(1)} < x_{Q(2)} < ... < x_{Q(K)}$ and $Q \in \mathfrak{S}_K$,

$$\Psi_Q[\mathbf{x},\bar{A}] = \sum_P \Phi_{\bar{A}}(PQ,P^{-1}) \prod_{i\in\mathcal{M}_{\pi\uparrow}} h_{x_i}(a_{P(i)}) \prod_{i\in\mathcal{M}_{\bar{\pi}\uparrow}} f_{x_i}(\lambda_{P(i)}) \prod_{i\in\mathcal{M}_{\pi\downarrow}} \bar{h}_{x_i}(a_{P(i)}) \prod_{i\in\mathcal{M}_{\bar{\pi}\downarrow}} g_{x_i}(b_{P(i)}), \quad (5.2.52)$$

where $P = P_{\pi\uparrow}P_{\bar{\pi}\uparrow}P_{\pi\downarrow}P_{\bar{\pi}\downarrow}$ is a factorized permutation such that $P_{\pi\uparrow}$, $P_{\bar{\pi}\uparrow}$, $P_{\pi\downarrow}$ and $P_{\bar{\pi}\downarrow}$ are the sets of permutations in $\mathfrak{S}_K$ permuting only particles of type $\pi\uparrow$, $\bar{\pi}\uparrow$, $\pi\downarrow$ and $\bar{\pi}\downarrow$ respectively, e.g. $P_A(i) = i$ for $i \notin \mathcal{M}_A$ and $P_A(j) \in \mathcal{M}_A$ for $j \in \mathcal{M}_A$ with $A = \pi\uparrow, \bar{\pi}\uparrow, \pi\downarrow, \bar{\pi}\downarrow$. We recall that the sets $\mathcal{M}_{\pi_\uparrow}$, $\mathcal{M}_{\bar{\pi}_\uparrow}$, $\mathcal{M}_{\pi_\downarrow}$ and $\mathcal{M}_{\bar{\pi}_\downarrow}$ are defined in section 5.2.1. The eigenfunctions $f_x(\lambda)$, $g_x(b)$, $h_x(a)$ and $\bar{h}_x(a)$ correspond to one-particle solutions. Below we will show how to derive them.

The action of the auxiliary Hamiltonian (5.2.48) on the wavefunction can be calculated using the relations given in the previous section. We define a recursive representation of the Hamiltonian:

$$S_{j-k,j}\dots S_{j-1,j}\phi^{(1)}[\bar{A}] \equiv \sum_{Q,\mathbf{x}\in Q} \hat{H}_j^{(k)} \Psi_Q[\mathbf{x},\bar{A}] \prod_{n=1}^{K} e_{x_n}^{A_n} \phi_0^{(1)} \qquad (5.2.53)$$

Using this definition one can write the following recursive relations between the coefficients $\hat{H}_j^{(k)}\Psi_Q[\mathbf{x},\bar{A}]$ for decreasing k. Non-trivial relations occur when $\hat{H}_j^{(k)}$ acts on $\Psi_Q[\mathbf{x},\bar{A}]$ with some coordinate x_p is equal to j and/or $j-k$. One gets two sets of relations: the first one reads

$$\hat{H}_j^{(k)}\Psi_Q[\mathbf{x}', \overset{x_p}{\underset{\downarrow}{j}};\bar{A}] = \begin{cases} -t_{j-k,j}\hat{H}_j^{(k-1)}\Psi_Q[\mathbf{x}', \overset{x_p}{\underset{\downarrow}{j}};\bar{A}] - r_{j-k,j}\hat{H}_j^{(k-1)}\Psi_{Q'}[\mathbf{x}', \overset{x_p}{\underset{\downarrow}{j-k}};\bar{A}] & \text{for } p \in \mathcal{M}_{\bar{\pi}\uparrow} \\ \hat{H}_j^{(k-1)}\Psi_{Q'}[\mathbf{x}', \overset{x_p}{\underset{\downarrow}{j-k}};\bar{A}] & \text{for } p \in \mathcal{M}_{\bar{\pi}\downarrow} \\ -\hat{H}_j^{(k-1)}\Psi_Q[\mathbf{x}', \overset{x_p}{\underset{\downarrow}{j}};\bar{A}] & \text{for } p \in \mathcal{M}_{\pi\uparrow} \\ -e^{ik_j}\hat{H}_j^{(k-1)}\Psi_Q[\mathbf{x}', \overset{x_p}{\underset{\downarrow}{j}};\bar{A}] & \text{for } p \in \mathcal{M}_{\pi\downarrow} \end{cases} \qquad (5.2.54)$$

The above relations are invariant if we exchange j with $j-k$ except the last relation which becomes:

$$\hat{H}_j^{(k)}\Psi_Q[\mathbf{x}', \overset{x_p}{\underset{\downarrow}{j-k}};\bar{A}] = -e^{-ik_{j-k}}\hat{H}_j^{(k-1)}\Psi_Q[\mathbf{x}', \overset{x_p}{\underset{\downarrow}{j-k}};\bar{A}] \quad \text{for} \quad p \in \mathcal{M}_{\pi\downarrow} \qquad (5.2.55)$$

The second set of relations is

$$\hat{H}_j^{(k)}\Psi_Q[\mathbf{x}', \overset{x_{p_1}}{\underset{\downarrow}{j}}, \overset{x_{p_2}}{\underset{\downarrow}{j-k}}; \bar{A}] = \begin{cases} \hat{H}_j^{(k-1)}\Psi_{Q'}[\mathbf{x}', j-k, j; \bar{A}], & \text{for } A_{p_1}, A_{p_2} \in \mathcal{M}_{\pi\sigma}, \mathcal{M}_{\bar{\pi}\sigma} \\ -t_{j-k,j}\hat{H}_j^{(k-1)}\Psi_Q[\mathbf{x}', j, j-k; \bar{A}] - r_{j-k,j}\hat{H}_j^{(k-1)}\Psi_{Q'}[\mathbf{x}', j-k, j; \bar{A}], & \\ \qquad\qquad \text{for } p_1 \in \mathcal{M}_{\bar{\pi}\uparrow(\downarrow)},\ p_2 \in \mathcal{M}_{\bar{\pi}\downarrow(\uparrow)} & \\ -e^{-ik_{j-k}}\hat{H}_j^{(k-1)}\Psi_Q[\mathbf{x}', j, j-k; \bar{A}], & \text{for } p_1 \in \mathcal{M}_{\bar{\pi}\uparrow(\downarrow)},\ p_2 \in \mathcal{M}_{\pi\uparrow(\downarrow)} \\ -e^{ik_j}\hat{H}_j^{(k-1)}\Psi_Q[\mathbf{x}', j, j-k; \bar{A}], & \text{for } p_1 \in \mathcal{M}_{\pi\uparrow(\downarrow)},\ p_2 \in \mathcal{M}_{\bar{\pi}\uparrow(\downarrow)} \\ -\hat{H}_j^{(k-1)}\Psi_Q[\mathbf{x}', j, j-k; \bar{A}], & \text{for } p_1 \in \mathcal{M}_{\bar{\pi}\uparrow(\downarrow)},\ p_2 \in \mathcal{M}_{\pi\downarrow(\uparrow)} \\ -\hat{H}_j^{(k-1)}\Psi_Q[\mathbf{x}', j, j-k; \bar{A}], & \text{for } p_1 \in \mathcal{M}_{\pi\uparrow(\downarrow)},\ p_2 \in \mathcal{M}_{\bar{\pi}\downarrow(\uparrow)} \\ -\hat{H}_j^{(k-1)}\Psi_Q[\mathbf{x}', j, j-k; \bar{A}], & \text{for } p_1 \in \mathcal{M}_{\pi\downarrow(\uparrow)},\ p_2 \in \mathcal{M}_{\pi\uparrow(\downarrow)} \end{cases} \tag{5.2.56}$$

In eqs. (5.2.54)–(5.2.56), the vector $\mathbf{x}'$ corresponds to the vector $\mathbf{x}$ without the components x_p or x_{p_1}, x_{p_2} and does not contain any position $x_q = j$ or $j-k$. The notation Q' means that the coefficient $\Psi_{Q'}$ is not in the same sector as Ψ_Q. We will write later the explicit relation between Q' and Q.

The eigenfunctions $f_x(\lambda)$, $g_x(b)$, $h_x(a)$ and $\bar{h}_x(a)$ and are

$$f_x(\lambda) = \prod_{m=1}^{x-1}\left(-\frac{i\sin k_m + i\lambda + \frac{U}{4}}{i\sin k_{m+1} + i\lambda - \frac{U}{4}}\right) \quad ; \quad g_x(b) = b^x \quad ; \quad \bar{h}_x(a) = h_x(a) = \delta(x-a). \tag{5.2.57}$$

we show how to obtain it in the following paragraphs.

$\bar{\pi}_\uparrow$ particle case. We consider that $K = 1$ and e^{A_1} is a $\bar{\pi}_\uparrow$ particle ($A_1 = \mathfrak{p}+1\ \uparrow, ..., \mathfrak{n}\uparrow$), it yields $\Psi[x; A_1] = f_x(\lambda)$. Then the recursive relations become:

$$\hat{H}_j^{(k)} f_j(\lambda) = -t_{j-k,j}\hat{H}_j^{(k-1)} f_j(\lambda) - r_{j-k,j}\hat{H}_j^{(k-1)} f_{j-k}(\lambda) \tag{5.2.58}$$

$$\hat{H}_j^{(k)} f_{j-k}(\lambda) = -t_{j-k,j}\hat{H}_j^{(k-1)} f_{j-k}(\lambda) - r_{j-k,j}\hat{H}_j^{(k-1)} f_j(\lambda) = \sigma_j f_{j-k}(\lambda) \tag{5.2.59}$$

the last equality means that for any k the function $f_x(\lambda)$ has an eigenvalue σ_j.

We can solve the second equation expressing $\hat{H}_j^{(k-1)} f_j(\lambda)$ and using the fact that $\hat{H}_j^{(k-1)} f_{j-k}(\lambda) = f_{j-k}(\lambda)$:

$$\hat{H}_j^{(k-1)} f_j(\lambda) = -\frac{(\sigma_j + t_{j-k,j}) f_{j-k}(\lambda)}{r_{j-k,j}} \tag{5.2.60}$$

we shift $k \to k+1$ and we obtain

$$\hat{H}_j^{(k)} f_j(\lambda) = -\frac{(\sigma_j + t_{j-k-1,j}) f_{j-k-1}(\lambda)}{r_{j-k-1,j}} \tag{5.2.61}$$

Inserting these results in (5.2.58) we find

$$-\frac{(\sigma_j + t_{j-k-1,j}) f_{j-k-1}(\lambda)}{r_{j-k-1,j}} = t_{j-k,j}\frac{(\sigma_j + t_{j-k,j}) f_{j-k}(\lambda)}{r_{j-k,j}} - r_{j-k,j} f_{j-k}(\lambda) \tag{5.2.62}$$

what can be solved using (5.2.32) as

$$\frac{f_{j-k-1}}{f_{j-k}} = -\frac{\Big((\sigma_j + t_{j-k,j})t_{j-k,j} - r^2_{j-k,j}\Big)r_{j-k-1,j}}{(\sigma_j + t_{j-k-1,j})r_{j-k,j}} = -\frac{i\sin k_{j-k} + i\lambda - \frac{U}{4}}{i\sin k_{j-k-1} + i\lambda + \frac{U}{4}} \tag{5.2.63}$$

where we introduced a Bethe root λ such that

$$\sigma_j(\lambda) = -\frac{i\sin(k_j) + i\lambda + \frac{U}{4}}{i\sin(k_j) + i\lambda - \frac{U}{4}} \tag{5.2.64}$$

Finally, we get the solution for one-excitation function $f_x(\lambda)$:

$$f_x(\lambda) = \prod_{m=1}^{x-1}\left(-\frac{i\sin k_m + i\lambda + \frac{U}{4}}{i\sin k_{m+1} + i\lambda - \frac{U}{4}}\right) \tag{5.2.65}$$

with the eigenvalue $\sigma_j(\lambda)$ defined above.

$\bar{\pi}_\downarrow$ particle case. Similarly we consider $K = 1$ and e^{A_1} is a $\bar{\pi}_\downarrow$ particle ($A_1 = \mathfrak{q}+1\ \downarrow, ..., \mathfrak{m}-1\ \downarrow$), it yields $\Psi[x; A_1] = g_x(b)$. Again from the recursive relations we obtain:

$$\hat{H}^{(k)}_j g_j(b) = \hat{H}^{(k-1)}_j g_{j-k}(b) \tag{5.2.66}$$

$$\hat{H}^{(k)}_j g_{j-k}(b) = \hat{H}^{(k-1)}_j g_j(b) = \bar{\sigma}_j g_{j-k}(b) \tag{5.2.67}$$

In this case we can similarly solve the second equation on $\hat{H}^{(k-1)}_j g_j(b)$ and rescale $k \to k+1$ and using the fact that $\hat{H}^{(k-1)}_j g_{j-k}(b) = g_{j-k}(b)$ we obtain:

$$\bar{\sigma}_j g_{j-k-1}(b) = g_{j-k}(b) \tag{5.2.68}$$

We see that $\bar{\sigma}_j$ here play the role of the Bethe root, hence we identify $\bar{\sigma}_j = b$ and finally, we get the solution for one excitation function $g_x(b)$:

$$g_x(b) = b^x \tag{5.2.69}$$

with the eigenvalue $\bar{\sigma}_j(b) = b$.

π_σ particle case. These two cases can be treated in the same manner. Let $K = 1$ and e^{A_1} is a π_σ particle ($A_1 = 1\sigma, ..., \substack{\mathfrak{p}\uparrow \\ \mathfrak{q}\downarrow}$). The wavefunctions are $\Psi[x; A_1] = h_x(a)$ for $\sigma = \uparrow$ and $\Psi[x; A_1] = \bar{h}_x(a)$ for $\sigma = \downarrow$. The recursive relations are trivial:

$$\hat{H}^{(k)}_j h_j(a) = -\hat{H}^{(k-1)}_j h_j(a) = \Lambda_j h_j(a) \tag{5.2.70}$$

$$\hat{H}^{(k)}_j h_{j-k}(a) = -\hat{H}^{(k-1)}_j h_{j-k}(a) = \Lambda_{j-k} h_{j-k}(a) \tag{5.2.71}$$

and

$$\hat{H}^{(k)}_j \bar{h}_j(a) = -e^{ik_j}\hat{H}^{(k-1)}_j \bar{h}_j(a) = \bar{\Lambda}_j \bar{h}_j(a) \tag{5.2.72}$$

$$\hat{H}^{(k)}_j \bar{h}_{j-k}(a) = -e^{-ik_{j-k}}\hat{H}^{(k-1)}_j \bar{h}_{j-k}(a) = \bar{\Lambda}_{j-k} \bar{h}_{j-k}(a) \tag{5.2.73}$$

We remark that the equations are not coupled and can be solved easily. In both cases eigenfunctions on site j and on the site $j-k$ lead to different eigenvalues. For whole chain when $k = N-1$ we have

$$\Lambda_j = (-1)^{N-1} \text{ and } \Lambda_{j-k} = -1 \tag{5.2.74}$$

$$\bar{\Lambda}_j = (-1)^{N-1} e^{ik_j(N-1)}, \text{ and } \bar{\Lambda}_{j=k} = -e^{ik_{j-k}} \tag{5.2.75}$$

Moreover, the fact that the initial Hamiltonians $S_{j+1,j}...S_{Nj}S_{1j}...S_{j-1,j}$ commute and thus have the common base of eigenfunctions, it implies that the one-particle solutions for π_σ particles should be independent of the index j:

$$\bar{h}_x(a) = h_x(a) = \delta(x-a). \tag{5.2.76}$$

Although the functions h and $\bar{h}$ are identical, they will be associated to different kinds of particles ($\pi\uparrow$ and $\pi\downarrow$ respectively). Since they will lead to different Hamiltonian eigenvalues (see below (5.2.78)), we have to distinguish them in (5.2.52).

We can remark that the Bethe roots a_i for $i \in \mathcal{M}_{\pi_\downarrow}$ or $\mathcal{M}_{\pi_\uparrow}$ corresponding to $\pi_\downarrow$ and $\pi_\uparrow$ particles are already quantized on the small chain. In order to have a number of independent eigenvectors (5.2.51), one should take the following conditions into account:

$$a_i < a_{i+1} \text{ for } i \in \mathcal{M}_{\pi_{\uparrow(\downarrow)}} \tag{5.2.77}$$

that leads to the fact that in (5.2.52), $P_{\pi\uparrow}$ and $P_{\pi\downarrow}$ are fixed: $P_{\pi_{\uparrow(\downarrow)}}Q(j) = \text{const}$ (see below).

Using the relations (5.2.54)–(5.2.56), we can then apply the whole product of S-matrices (5.2.49) on the one-excitation functions. We get for $x \neq j$

$$\begin{aligned} &\hat{H}_j^{(N-1)} f_x(\lambda) = \sigma_j(\lambda) f_x(\lambda) \quad \text{and} \quad \hat{H}_j^{(N-1)} g_x(b) = b g_x(b) \\ &\hat{H}_j^{(N-1)} h_x(a) = -h_x(a) \quad \text{and} \quad \hat{H}_j^{(N-1)} \bar{h}_x(a) = -e^{-ik_x} \bar{h}_x(a) \end{aligned} \tag{5.2.78}$$

and for $x = j$

$$\begin{aligned} &\hat{H}_j^{(N-1)} f_j(\lambda) = \prod_{l=1, l\neq j}^{N} \sigma_l^{-1}(\lambda) f_j(\lambda) \quad \text{and} \quad \hat{H}_j^{(N-1)} g_j(b) = b^{1-N} g_j(b) \\ &\hat{H}_j^{(N-1)} h_j(a) = (-1)^{N-1} h_j(a) \quad \text{and} \quad \hat{H}_j^{(N-1)} \bar{h}_j(a) = (-1)^{N-1} e^{ik_j(N-1)} \bar{h}_j(a) \end{aligned} \tag{5.2.79}$$

where

$$\sigma_j(\lambda) = -\frac{i\sin(k_j) + i\lambda + \frac{U}{4}}{i\sin(k_j) + i\lambda - \frac{U}{4}} \tag{5.2.80}$$

Now we will consider the K-excitations eigenvector (5.2.51) with the ansatz (5.2.52). There are three different possible cases:

I) there exists a Bethe root $a_l = j$ for some $l \in \mathcal{M}_{\pi\uparrow}$, with j being the index in (5.2.48),

II) there exists another Bethe root $a_m = j$ for some $m \in \mathcal{M}_{\pi\downarrow}$,

III) there is no such Bethe roots.

We detail the calculations for each of these cases.

I) There exists a Bethe root $a_l = j$ for some $l \in \mathcal{M}_{\pi\uparrow}$. In this first case, let us introduce a set of integers $\{\alpha_i \in [1,K]\}_{i\in\mathcal{M}_{\pi\uparrow}}$ such that we have $Q(\alpha_i) \in \mathcal{M}_{\pi\uparrow}$. The set is ordered: $\alpha_i < \alpha_{i+1}$. We also define α_l such that $x_{Q(\alpha_l)} = j$.

The Hamiltonian acting on the wavefunction gives the following result:

$$\begin{aligned}
&\hat{H}_j^{(N-1)}\Psi_Q(\mathbf{x}; x_{Q(\alpha_1)},...,x_{Q(\alpha_{l-1})}, \overset{x_{Q(\alpha_l)}}{\overset{\downarrow}{j}}, x_{Q(\alpha_{l+1})},...,x_{Q(\alpha_{N_{\uparrow\pi}})}) \\
&= (-1)^{N-1-N_{\uparrow\pi}} e^{ik_j N_{\uparrow\bar{\pi}}}\, \Psi_{QC_{\pi\uparrow}}(\mathbf{x}; x_{Q(\alpha_2)},..., \overset{x_{Q(\alpha_l)}}{\overset{\downarrow}{j}}, x_{Q(\alpha_{l+1})}, x_{Q(\alpha_{l+2})},..., x_{Q(\alpha_1)}) \\
&= \Lambda_j \Psi_Q(\mathbf{x}; x_{Q(\alpha_1)},..., x_{Q(\alpha_{l-1})}, j, x_{Q(\alpha_{l+1})},..., x_{Q(\alpha_{N_{\uparrow\pi}})})
\end{aligned} \tag{5.2.81}$$

and finally we get the condition on $\Phi_{\bar{A}}(PQ, P^{-1})$:

$$\frac{\Phi_{\bar{A}}(P'QC_{\pi\uparrow}, P'^{-1})}{\Phi_{\bar{A}}(PQ, P^{-1})} = \Lambda_j (-1)^{N-1-N_{\uparrow\pi}} e^{-ik_j N_{\uparrow\bar{\pi}}}\,. \tag{5.2.82}$$

In equality (5.2.82), we have introduced $C_{\pi\uparrow} = \Pi^{\alpha_1}_{\alpha_{N_{\uparrow\pi}}} ... \Pi^{\alpha_{N_{\uparrow\pi}-1}}_{\alpha_{N_{\uparrow\pi}}}$ and for $P = P_{\pi\uparrow}P_{\pi\downarrow}P_{\bar{\pi}\uparrow}P_{\bar{\pi}\downarrow}$, $P' = P'_{\pi\uparrow}P_{\pi\downarrow}P_{\bar{\pi}\uparrow}P_{\bar{\pi}\downarrow}$ where $P'_{\pi\uparrow} = \Pi^{N_{\uparrow\pi}-1}_{N_{\uparrow\pi}} ... \Pi^1_{N_{\uparrow\pi}} P_{\pi\uparrow}$.

If we change the sector $Q \to Q\Pi_{ab}$, which means that we interchange two identical particles $x_{Q(a)}$ and $x_{Q(b)}$, the coefficient $\Phi_{\bar{A}}(PQ\Pi_{ab}, P^{-1})$ should remain invariant:

$$\Phi_{\bar{A}}(P'Q\Pi_{ab}, P'^{-1}) = \Phi_{\bar{A}}(PQ, P^{-1}) \quad \text{with} \quad Q(a), Q(b) \in \mathcal{M}_{\pi\uparrow} \tag{5.2.83}$$

where $P' = P'_{\pi\uparrow}P_{\pi\downarrow}P_{\bar{\pi}\uparrow}P_{\bar{\pi}\downarrow}$ and $P'_{\pi\uparrow}$ is such that $P'_{\pi\uparrow} = P_{\pi\uparrow}\Pi_{Q(a)Q(b)}$.
The relation (5.2.82) can be simplified to:

$$\frac{\Phi_{\bar{A}}(PQ, P^{-1}\tilde{C}_{\pi\uparrow})}{\Phi_{\bar{A}}(PQ, P^{-1})} = \Lambda_j (-1)^{N-N_{\uparrow\pi}} e^{-ik_j N_{\uparrow\bar{\pi}}} \tag{5.2.84}$$

with $\tilde{C}_{\pi\uparrow} = \Pi^1_{N_{\uparrow\pi}} ... \Pi^{N_{\uparrow\pi}-1}_{N_{\uparrow\pi}}$. Now, taking the product of this equation by itself $N_{\uparrow\pi}$ times and changing $P \to \tilde{C}_{\pi\uparrow}P \to \tilde{C}^2_{\pi\uparrow}P \to \ldots \to \tilde{C}^{N_{\uparrow\pi}}_{\pi\uparrow}P$, we obtain

$$\Lambda_j (-1)^{N-N_{\uparrow\pi}} e^{-ik_j N_{\uparrow\bar{\pi}}} = e^{2\pi i \frac{m_{\uparrow\pi}}{N_{\uparrow\pi}}} \qquad (m_{\uparrow\pi} = 1,..., N_{\uparrow\pi}) \tag{5.2.85}$$

It determines the eigenvalue of the auxiliary problem for spin up π-particles. We also recall that the result has to be multiplied by $(-1)^{N-1}$ in order to come back to the initial problem due to the sign change done at the beginning of the section, see (5.2.49).

II) There exists another Bethe root $a_m = j$ for some $m \in \mathcal{M}_{\pi\downarrow}$. Following the same steps as in the first case, we obtain

$$\frac{\Phi_{\bar{A}}(PQ, P^{-1}\tilde{C}_{\pi\downarrow})}{\Phi_{\bar{A}}(PQ, P^{-1})} = \Lambda_j (-1)^{N-N_{\downarrow\pi}} e^{-ik_j (N_{\downarrow\bar{\pi}} + N_{\downarrow\mathfrak{m}})} \tag{5.2.86}$$

where $P = P_{\pi\uparrow}P_{\pi\downarrow}P_{\bar{\pi}\uparrow}P_{\bar{\pi}\downarrow}$, $P' = P_{\pi\uparrow}P'_{\pi\downarrow}P_{\bar{\pi}\uparrow}P_{\bar{\pi}\downarrow}$ and $\tilde{C}_{\pi\downarrow} = \Pi^1_{N_{\downarrow\pi}} ... \Pi^{N_{\downarrow\pi}-1}_{N_{\downarrow\pi}}$.
Finally using the same trick for the product of equations, we obtain

$$\Lambda_j (-1)^{N-N_{\downarrow\pi}} e^{-ik_j (N_{\downarrow\bar{\pi}} + N_{\downarrow\mathfrak{m}})} = e^{2\pi i \frac{m_{\downarrow\pi}}{N_{\downarrow\pi}}}, \quad m_{\downarrow\pi} = 1,..., N_{\downarrow\pi} \tag{5.2.87}$$

It allows to determine the eigenvalue of the auxiliary problem for $\pi\downarrow$ particles. Again, this result should be multiplied by a factor $(-1)^{N-1}$ to come back to the initial problem.

For identical particles we have the relation:

$$\Phi_{\bar{A}}(P'Q\Pi_{ab}, P'^{-1}) = \Phi_{\bar{A}}(PQ, P^{-1}), \text{ with } Q(a), Q(b) \in \mathcal{M}_{\pi\uparrow} \tag{5.2.88}$$

where $P' = P_{\pi\uparrow}P'_{\pi\downarrow}P_{\bar{\pi}\uparrow}P_{\bar{\pi}\downarrow}$ and $P'_{\pi\downarrow}$ is such that $P'_{\pi\downarrow} = P_{\pi\downarrow}\Pi_{Q(a)Q(b)}$.

III) There is no such Bethe root. In this last case, we have π-particles with spin up or down distributed anywhere on the chain excluding the site j. We act with the Hamiltonian (5.2.48) on the wavefunction (5.2.52) and we get

$$\hat{H}_j^{(N-1)}\Psi_Q[\mathbf{x}, \bar{A}] = (-1)^{N_{\pi\uparrow}+N_{\pi\downarrow}} \prod_{l\in\mathcal{M}_{\pi\downarrow}} e^{-ik_{a_l}} \prod_{i\in\mathcal{M}_{\pi\uparrow}} \sigma_j(\lambda_i) \prod_{i\in\mathcal{M}_{\bar{\pi}\downarrow}} b_i\, \Psi_Q[\mathbf{x}, \bar{A}], \text{ if } \mathbf{x} \neq j \tag{5.2.89}$$

with the following conditions on the coefficients $\Phi_{\bar{A}}(PQ, P^{-1})$:

1) for all $PQ(i)$,$PQ(i+1)$ in $\mathcal{M}_{\bar{\pi}\uparrow}$

$$\frac{\Phi_{\bar{A}}(PQ\Pi_{ii+1}, (P\Pi_{Q(i)Q(i+1)})^{-1})}{\Phi_{\bar{A}}(PQ, P^{-1})} = \frac{i\lambda_{PQ(i)} - i\lambda_{PQ(i+1)} - \frac{U}{2}}{i\lambda_{PQ(i)} - i\lambda_{PQ(i+1)} + \frac{U}{2}} \equiv \alpha^{-1}_{PQ(i)PQ(i+1)} \tag{5.2.90}$$

2) for all $PQ(i)$,$PQ(i+1)$ in $\mathcal{M}_{\bar{\pi}\downarrow}$ we impose

$$\frac{\Phi_{\bar{A}}(PQ\Pi_{ii+1}, (P\Pi_{Q(i)Q(i+1)})^{-1})}{\Phi_{\bar{A}}(PQ, P^{-1})} = 1 \tag{5.2.91}$$

3) for all $PQ(i) \in \mathcal{M}_{\bar{\pi}\downarrow}$ and $PQ(i+1) \in \mathcal{M}_{\bar{\pi}\uparrow}$

$$\frac{\Phi_{\bar{A}}(PQ\Pi_{ii+1}, P^{-1})}{\Phi_{\bar{A}}(PQ, P^{-1})} = b^{-1}_{PQ(i)} \tag{5.2.92}$$

the same relation holds for all $PQ(i) \in \mathcal{M}_{\bar{\pi}\uparrow}$, $PQ(i+1) \in \mathcal{M}_{\bar{\pi}\downarrow}$ and changing $Q \to Q\Pi_{ii+1}$.

4) for all $PQ(i) \in \mathcal{M}_{\pi\uparrow}$ and $PQ(i+1) \in \mathcal{M}_{\bar{\pi}\uparrow}$

$$\frac{\Phi_{\bar{A}}(PQ\Pi_{ii+1}, P^{-1})}{\Phi_{\bar{A}}(PQ, P^{-1})} = e^{-ik_{a_{PQ(i)}}}\sigma_{a_{PQ(i)}}(\lambda_{PQ(i+1)}) \tag{5.2.93}$$

5) for all $PQ(i) \in \mathcal{M}_{\pi\downarrow}$ and $PQ(i+1) \in \mathcal{M}_{\bar{\pi}\uparrow}$

$$\frac{\Phi_{\bar{A}}(PQ\Pi_{ii+1}, P^{-1})}{\Phi_{\bar{A}}(PQ, P^{-1})} = e^{ik_{a_{PQ(i)}}}\sigma_{a_{PQ(i)}}(\lambda_{PQ(i+1)}) \tag{5.2.94}$$

6) for all $PQ(i) \in \mathcal{M}_{\pi\uparrow}$ and $PQ(i+1) \in \mathcal{M}_{\bar{\pi}\downarrow}$

$$\frac{\Phi_{\bar{A}}(PQ\Pi_{ii+1}, P^{-1})}{\Phi_{\bar{A}}(PQ, P^{-1})} = b_{PQ(i+1)} \tag{5.2.95}$$

7) for all $PQ(i) \in \mathcal{M}_{\pi\downarrow}$ and $PQ(i+1) \in \mathcal{M}_{\bar{\pi}\downarrow}$

$$\frac{\Phi_{\bar{A}}(PQ\Pi_{ii+1}, P^{-1})}{\Phi_{\bar{A}}(PQ, P^{-1})} = b_{PQ(i+1)} \tag{5.2.96}$$

In addition, for all $PQ(i)$,$PQ(i+1)$ in $\mathcal{M}_{\bar{\pi}\uparrow}$ or $\mathcal{M}_{\bar{\pi}\downarrow}$ we impose the condition for identical particles:

$$\frac{\Phi_{\bar{A}}(PQ,(P\Pi_{Q(i)Q(i+1)})^{-1})}{\Phi_{\bar{A}}(PQ,P^{-1})}=1 \tag{5.2.97}$$

The ratios (5.2.90),(5.2.92)-(5.2.96) can be calculated applying the recursive Hamiltonian $\hat{H}_j^{(k)}$ on the wavefunction.

Scattering between particles. *Two $\bar{\pi}\uparrow$-particles.* The scattering between $\bar{\pi}\uparrow$-particles, relation (5.2.90), can be obtained in the same way as it was done in the section on the auxiliary problem of the $gl(\mathfrak{n}|\mathfrak{m})\oplus gl(2)$ Hubbard model (5.1.2). We will not repeat the derivation, but we recall briefly the method. We assume that the coefficient $\hat{H}_j^{(k)}\Psi_Q[\mathbf{x};\bar{A}]$ can be written as a product of one-particle solutions for any $\mathbf{x}\neq j$:

$$\hat{H}_j^{(k)}\Psi_Q[\mathbf{x};\bar{A}]=\sum_P \Phi_{\bar{A}}(PQ,P^{-1})...\prod_{i\in\mathcal{M}_{\bar{\pi}\uparrow}}\hat{H}_j^{(k)}f_{x_i}(\lambda_{P(i)})... \tag{5.2.98}$$

where we omitted the other particles except $\bar{\pi}\uparrow$-particles. This hypothesis yields the form of the coefficients $\hat{H}_j^{(k)}\Psi_Q[\mathbf{x}',j;\bar{A}]$ 'after' the interaction where a certain $\bar{\pi}\uparrow$-particle is on the site $x=j$. However, the relation (5.2.56) allows to calculate the later one 'before' the interaction. The compatibility of both results leads to the relation (5.2.90).

Two $\bar{\pi}\downarrow$-particles. In the case of $\bar{\pi}\downarrow$-particles the similar method does not allow to obtain the ratio between the coefficients $\Phi_{\bar{A}}(PQ,P^{-1})$. When one applies the similar reasoning the ratio is undetermined, the coefficients $\hat{H}_j^{(k)}\Psi_Q[\mathbf{x}',j;\bar{A}]$ 'before' the interaction and 'after' are the same. However, if one imposes the condition (5.2.91) one obtains the necessary multiplicity of the wavefunction in $\bar{\pi}\downarrow$-particles sector.

$\bar{\pi}\uparrow$-particle and $\bar{\pi}\downarrow$-particle. We assume similarly that the coefficient $\hat{H}_j^{(k)}\Psi_Q[\mathbf{x}',j-k,j-k+1;\bar{A}]$ with $x_{Q(i)}=j-k$ and $x_{Q(i+1)}=j-k+1$ being $\bar{\pi}\uparrow$-particle and $\bar{\pi}\downarrow$-particle can be written as

$$\hat{H}_j^{(k)}\Psi_Q[\mathbf{x}',j-k,j-k+1;\bar{A}]=\sum_P \Phi_{\bar{A}}(PQ,P^{-1})...\hat{H}_j^{(k)}f_{j-k}(\lambda_{PQ(i)})\hat{H}_j^{(k)}g_{j-k+1}(b_{PQ(i+1)})... \tag{5.2.99}$$

we neglect other particles.

The relation (5.2.56) gives a rule how to expand this coefficient $\hat{H}_j^{(k)}\Psi_Q[\mathbf{x}',j-k,j-k+1;\bar{A}]$ in terms of $\hat{H}_j^{(k-1)}\Psi_Q[\mathbf{x}',j-k,j-k+1;\bar{A}]$ and $\hat{H}_j^{(k-1)}\Psi_{Q'}[\mathbf{x}',j,j-k+1;\bar{A}]$ with $Q'=Q\Pi_{ii+1}$. The first coefficient can be calculated as a $\bar{\pi}\downarrow$ one-particle function, the $\bar{\pi}\uparrow$-particle become a 'spectator' or $\hat{H}_j^{(k-1)}f_{j-k}(\lambda_{PQ(i)})=f_{j-k}(\lambda_{PQ(i)})$:

$$\hat{H}_j^{(k-1)}\Psi_Q[\mathbf{x}',j-k,j-k+1;\bar{A}]=\sum_P \Phi_{\bar{A}}(PQ,P^{-1})...\underbrace{\hat{H}_j^{(k-1)}f_{j-k}(\lambda_{PQ(i)})}_{\text{spectator}}\hat{H}_j^{(k-1)}g_{j-k+1}(b_{PQ(i+1)})... \tag{5.2.100}$$

Thus, in order to get $\hat{H}_j^{(k)}\Psi_Q[\mathbf{x}',j-k,j-k+1;\bar{A}]$ in the written above form, it imposes the second coefficient 'after' the interaction to be:

$$\hat{H}_j^{(k-1)}\Psi_{Q'}[\mathbf{x}',j,j-k+1;\bar{A}]\equiv\sum_P \Phi_{\bar{A}}(PQ,P^{-1})...\hat{H}_j^{(k-1)}f_j(\lambda_{PQ(i)})\hat{H}_j^{(k-1)}g_{j-k+1}(b_{PQ(i+1)})... \tag{5.2.101}$$

On the other hand we can calculate this coefficient 'before' the interaction using (5.2.56):

$$\hat{H}_j^{(k-1)}\Psi_{Q'}[\mathbf{x}', j, j-k+1; \bar{A}] = -t_{j-k+1,j}\hat{H}_j^{(k-2)}\Psi_{Q'}[\mathbf{x}', j, j-k+1; \bar{A}] - r_{j-k+1,j}\hat{H}_j^{(k-2)}\Psi_{Q'}[\mathbf{x}', j-k+1, j; \bar{A}] \tag{5.2.102}$$

We see that the particle on the site $j-k+1$ become a spectator and thus both coefficient can be calculated as one-particle functions:

$$\hat{H}_j^{(k-2)}\Psi_{Q'}[\mathbf{x}', j, j-k+1; \bar{A}] = \sum_P \Phi_{\bar{A}}(PQ', P^{-1})...\hat{H}_j^{(k-2)} f_j(\lambda_{PQ(i)})\hat{H}_j^{(k-2)} g_{j-k+1}(b_{PQ(i+1)})...$$
$$\hat{H}_j^{(k-2)}\Psi_{Q'}[\mathbf{x}', j-k+1, j; \bar{A}] = \sum_P \Phi_{\bar{A}}(PQ, P^{-1})...\hat{H}_j^{(k-2)} f_{j-k+1}(\lambda_{PQ(i)})\hat{H}_j^{(k-2)} g_j(b_{PQ(i+1)})...$$

The coefficient 'before' the interaction should be compatible with the one 'after' the interaction, hence we impose that

$$\Phi_{\bar{A}}(PQ\Pi_{ii+1}, P^{-1})\hat{H}_j^{(k-2)} g_{j-k+1}(b_{PQ(i+1)}) \equiv \Phi_{\bar{A}}(PQ, P^{-1})\hat{H}_j^{(k-1)} g_{j-k+1}(b_{PQ(i+1)}) \tag{5.2.103}$$

The coefficients $\hat{H}_j^{(k-2)} g_{j-k+1}(b_{PQ(i+1)})$ and $\hat{H}_j^{(k-1)} g_{j-k+1}(b_{PQ(i+1)})$ can be calculated using (5.2.78) and (5.2.79). Finally, it yields the relation (5.2.92) if one changes $Q \to Q\Pi_{ii+1}$.

One can repeat the same method in order to obtain the other ratios (5.2.93)-(5.2.96) but we will skip the derivation.

Auxiliary problem. Level 3

For the perodicity conditions, there are again two different subcases:

III-A) a $\bar{\pi}\downarrow$-particle is on the j site,

III-B) a $\bar{\pi}\uparrow$-particle is on the j site.

Each case leading to slightly different conditions, we treat them separately.

III-A) A $\bar{\pi}\downarrow$-particle is on the j site. The periodic boundary condition on the coefficients $\Phi_{\bar{A}}(PQ, P^{-1})$ comes from the relations

$$\hat{H}_j^{(N-1)}\Psi_Q[\mathbf{x}, \overset{\bar{\pi}\downarrow}{\underset{\downarrow}{j}}; \bar{A}] = (-1)^{N_{\pi\uparrow}+N_{\pi\downarrow}} \prod_{l\in\mathcal{M}_{\pi\downarrow}} e^{-ik_{a_l}} \prod_{i\in\mathcal{M}_{\pi\uparrow}} \sigma_j(\lambda_i) \prod_{i\in\mathcal{M}_{\pi\downarrow}} b_i\, \Psi_Q[\mathbf{x}, \overset{\bar{\pi}\downarrow}{\underset{\downarrow}{j}}; \bar{A}], \text{ and } \mathbf{x} \neq j$$

Using the relations (5.2.90)-(5.2.97) we can calculate the coefficient $\hat{H}_j^{(N-1)}\Psi_Q[\mathbf{x}, j; \bar{A}]$ and we get the condition

$$\frac{\Phi_{\bar{A}}(PQC_K, P^{-1})}{\Phi_{\bar{A}}(PQ, P^{-1})} = [b_{PQ(K)}]^N \quad \text{with} \quad C_K = \Pi_K^1 ... \Pi_K^{K-1}\,. \tag{5.2.104}$$

We start with $Q = Q_{\bar{\pi}\downarrow}$ such that

$$Q_{\bar{\pi}\downarrow}(i) = \begin{cases} i, & \text{for } i \notin \mathcal{M}_{\bar{\pi}\downarrow} \\ j \in \mathcal{M}_{\bar{\pi}\downarrow}, & \text{for } i \in \mathcal{M}_g \end{cases} \quad \Rightarrow \quad PQ(i) = \begin{cases} m \equiv \hat{P}(i) \in [1,K] \cap \mathcal{M}_{\bar{\pi}\downarrow}, & i \notin \mathcal{M}_{\bar{\pi}\downarrow} \\ P_{\bar{\pi}\downarrow}Q_{\bar{\pi}\downarrow}(i) \equiv \tilde{P}(i) \in \mathcal{M}_{\bar{\pi}\downarrow}, & i \in \mathcal{M}_{\bar{\pi}\downarrow} \end{cases} \tag{5.2.105}$$

and

$$PQC_K = \Pi_{\tilde{P}(K)}^{\hat{P}(1)}\Pi_{\tilde{P}(K)}^{\hat{P}(2)}...\Pi_{\tilde{P}(K)}^{\hat{P}(K-N_{\downarrow\bar{\pi}})}\Pi_{\tilde{P}(K)}^{\tilde{P}(K-N_{\downarrow\bar{\pi}}+1)}...\Pi_{\tilde{P}(K)}^{\tilde{P}(K-1)} PQ\,. \tag{5.2.106}$$

We use the relations (5.2.95), (5.2.96) and "conjugated" (5.2.92) when $Q \to Q\Pi_{ii+1}$ and the coefficient $\Phi_{\bar{A}}(PQC_K, P^{-1})$ simplifies to:

$$\begin{aligned}\Phi_{\bar{A}}(PQC_K, P^{-1}) &= \Phi_{\bar{A}}(\Pi_{\tilde{P}(K)}^{\tilde{P}(1)}\Pi_{\tilde{P}(K)}^{\tilde{P}(2)}...\Pi_{\tilde{P}(K)}^{\tilde{P}(K-N_{\downarrow\bar{\pi}})}\Pi_{\tilde{P}(K)}^{\tilde{P}(K-N_{\downarrow\bar{\pi}}+1)}....\Pi_{\tilde{P}(K)}^{\tilde{P}(K-1)}PQ, P^{-1}) \\ &= \left(b_{\tilde{P}(K)}\right)^{N_{\uparrow\pi}+N_{\uparrow\bar{\pi}}+N_{\downarrow\pi}}\Phi_{\bar{A}}(\Pi_{\tilde{P}(K)}^{\tilde{P}(K-N_{\downarrow\bar{\pi}}+1)}...\Pi_{\tilde{P}(K)}^{\tilde{P}(K-1)}PQ, P^{-1}). \end{aligned} \tag{5.2.107}$$

Finally we have the equation

$$\Phi_{\bar{A}}(\Pi_{\tilde{P}(K)}^{\tilde{P}(K-N_{\downarrow\bar{\pi}}+1)}...\Pi_{\tilde{P}(K)}^{\tilde{P}(K-1)}PQ, P^{-1}) = \left(b_{\tilde{P}(K)}\right)^{N-(N_{\uparrow\pi}+N_{\uparrow\bar{\pi}}+N_{\downarrow\pi})}\Phi_{\bar{A}}(PQ, P^{-1}) \tag{5.2.108}$$

In the next step of the procedure, we introduce the vector $\hat{\Phi}_{\bar{\pi}\downarrow}(P)$:

$$\hat{\Phi}_{\bar{\pi}\downarrow}(P) \equiv \sum_{\bar{A}\in G}\sum_{Q'\in\mathfrak{S}_{N_{\downarrow\bar{\pi}}}}\Phi_{\bar{A}}(P, Q')\prod_{i\in\mathcal{M}_{\bar{\pi}\downarrow}} e_i^{\bar{A}_{Q'(i)}} \tag{5.2.109}$$

where $G = \text{span}\{(\mathfrak{q}+1)\downarrow, ..., (\mathfrak{m}-1)\downarrow\}$. The summation on $\bar{A} \in G$ is done on $\bar{A}_i$ (defined in (5.2.51)) for $i \in \mathcal{M}_{\bar{\pi}\downarrow}$ equal to all possible values in G. The particle order is chosen such that for $Q' = id$, the vector $\prod_{i\in\mathcal{M}_{\bar{\pi}\downarrow}} e_i^{\bar{A}_i}$ is equal to $\overbrace{e^{(\mathfrak{q}+1)\downarrow}\otimes ...\otimes e^{(\mathfrak{q}+1)\downarrow}}^{N_{\downarrow(\mathfrak{q}+1)}}\otimes ...\otimes\overbrace{e^{(\mathfrak{m}-1)\downarrow}\otimes ...\otimes e^{(\mathfrak{m}-1)\downarrow}}^{N_{\downarrow(\mathfrak{m}-1)}}$. The permutation Q' is the image of $Q_{\bar{\pi}\downarrow}$ when one shifts by $K - N_{\downarrow\bar{\pi}}$ the space of integers on which the permutation group $\mathfrak{S}_{N_{\downarrow\bar{\pi}}}$ acts.

For example if $\pi\downarrow$ particle is only $2\downarrow$ particle ($\mathfrak{q} = 2$) and number of $\bar{\pi}\downarrow$ particles types $\mathfrak{m} - \mathfrak{q} = 3$ thus we have $G = \text{span}\{3\downarrow, 4\downarrow, 5\downarrow\}$ than we have

$$\begin{aligned}\hat{\Phi}_{\bar{\pi}\downarrow}(P) &= e_{X+1}^{3\downarrow}e_{X+2}^{3\downarrow}e_{X+3}^{3\downarrow}\Phi_{3\downarrow,3\downarrow,3\downarrow}(P, Id) + e_{X+1}^{3\downarrow}e_{X+2}^{3\downarrow}e_{X+3}^{4\downarrow}\Phi_{3\downarrow,3\downarrow,4\downarrow}(P, Id) + \\ &\quad e_{X+1}^{3\downarrow}e_{X+2}^{3\downarrow}e_{X+3}^{5\downarrow}\Phi_{3\downarrow,3\downarrow,5\downarrow}(P, Id) + e_{X+1}^{3\downarrow}e_{X+2}^{4\downarrow}e_{X+3}^{3\downarrow}\Phi_{3\downarrow,3\downarrow,4\downarrow}(P, \Pi_{X+3}^{X+2}) + ... \\ &\quad e_{X+1}^{4\downarrow}e_{X+2}^{3\downarrow}e_{X+3}^{3\downarrow}\Phi_{3\downarrow,3\downarrow,4\downarrow}(P, \Pi_{X+1}^{X+3}) + e_{X+1}^{4\downarrow}e_{X+2}^{3\downarrow}e_{X+3}^{4\downarrow}\Phi_{3\downarrow,4\downarrow,4\downarrow}(P, \Pi_{X+2}^{X+1}) + ...\end{aligned} \tag{5.2.110}$$

where $X = K - N_{\downarrow\bar{\pi}}$.

The relations (5.2.91) and (5.2.97) can be gathered in $\hat{\Phi}_{\bar{\pi}\downarrow}(P)$:

$$\hat{\Phi}_{\bar{\pi}\downarrow}(\Pi_{ab}P) = \mathcal{P}_{ab}\hat{\Phi}_{\bar{\pi}\downarrow}(P) \tag{5.2.111}$$

where $\mathcal{P}_{ab}$ is the permutation acting on the particles located at the a and b positions in $\hat{\Phi}_{\bar{\pi}\downarrow}(P)$. The periodicity problem can then be rewritten in the following form

$$\hat{\Phi}_{\bar{\pi}\downarrow}\left(\Pi_{P(K)}^{P(K-N_{\downarrow\bar{\pi}}+1)}...\Pi_{P(K)}^{P(K-1)}P\right) = \left(b_{P(K)}\right)^{N-K+N_{\downarrow\bar{\pi}}}\hat{\Phi}_{\bar{\pi}\downarrow}(P) \tag{5.2.112}$$

which can be simplified to

$$\mathcal{P}_{P(K-N_{\downarrow\bar{\pi}}+1)P(K)}...\mathcal{P}_{P(K-1)P(K)}\hat{\Phi}_{\bar{\pi}\downarrow}(P) = \left(b_{P(K)}\right)^{N-K+N_{\downarrow\bar{\pi}}}\hat{\Phi}_{\bar{\pi}\downarrow}(P) \tag{5.2.113}$$

Choosing $P = C_{\bar{\pi}\downarrow}^{-m}$ with $C_{\bar{\pi}\downarrow} = \Pi_K^{K-N_{\downarrow\pi}+1}...\Pi_K^{K-1}$, we get the following Bethe equation

$$\left(b_m\right)^{N-K+N_{\downarrow\bar{\pi}}} = \exp\left(\frac{2\pi i}{N_{\downarrow\bar{\pi}}}\sum_{j=1}^{N_{\downarrow\bar{\pi}}-N_{\downarrow(\mathfrak{m}-1)}} n_j\right),\ m \in \mathcal{M}_{\bar{\pi}\downarrow} \tag{5.2.114}$$

$$\text{with} \qquad 1 \le n_1 < ... < n_{N_{\downarrow\bar{\pi}}-N_{\downarrow(\mathfrak{m}-1)}} \le N_{\downarrow\bar{\pi}} \tag{5.2.115}$$

In order to get all quantum numbers which characterize the eigenfunction and obtain the right number of states, one should solve the permutation problem. We will take the result from the section (5.2.4) on the solution of the permutation problem and we obtain additional sets of integers

$$1 \leq n_1^{(k)} < n_2^{(k)} < ... < n^{(k)}_{N_{\downarrow(\mathfrak{q}+1)}+...+N_{\downarrow(k-1)}} \leq N_{\downarrow(\mathfrak{q}+1)} + ... + N_{\downarrow k}\,, \quad k = \mathfrak{q}+2, ..., \mathfrak{m}-2\,. \tag{5.2.116}$$

One can remark that if we set $k = \mathfrak{m}-1$ in the above condition, we recover the relation (5.2.115). This result ends the first subcase.

III-B) A $\bar{\pi}$ $\uparrow$-particle is on the j site. The periodic boundary conditions on the coefficients $\Phi_{\bar{A}}(PQ, P^{-1})$ can be written similarly to the previous case

$$\frac{\Phi_{\bar{A}}(PQC_K, P^{-1})}{\Phi_{\bar{A}}(PQ, P^{-1})} = \prod_{l=1}^{N} \sigma_l(\lambda_{PQ(K)}) \quad \text{for} \quad Q \quad \text{such that} \quad Q(K) \in \mathcal{M}_{\bar{\pi}\uparrow}\,. \tag{5.2.117}$$

We proceed as in the case III-A, but instead we choose $Q = Q_{\bar{\pi}\uparrow} C_K^{K-N_{\uparrow\pi}-N_{\uparrow\bar{\pi}}}$. Here $Q_{\bar{\pi}\uparrow}$ defined similarly to $Q_{\bar{\pi}\downarrow}$: $Q_{\bar{\pi}\uparrow}(i) \in \mathcal{M}_{\bar{\pi}\uparrow}$ if $i \in \mathcal{M}_{\bar{\pi}\uparrow}$ and $Q_{\bar{\pi}\uparrow}(i) = i$ for the remaining indices. The cyclic permutation is chosen such that all π and $\bar{\pi}$ down particles are moved to the begining from their original ordering for $Q = id$, that is $C_K^{K-N_{\uparrow\pi}-N_{\uparrow\bar{\pi}}}(i) = (i + N_{\uparrow\pi} + N_{\uparrow\bar{\pi}}) \mod K$ for any i. This choice implies in particular the following relation:

$$\begin{aligned} PQC_K \;=\; & \Big(\overrightarrow{\prod_{i\in\mathcal{M}_{\pi\downarrow}}} \Pi^i_{\tilde{P}(N_{\uparrow\pi}+N_{\uparrow\bar{\pi}})}\Big)\Big(\overrightarrow{\prod_{i\in\mathcal{M}_{\bar{\pi}\downarrow}}} \Pi^{P_{\pi\downarrow}(i)}_{\tilde{P}(N_{\uparrow\pi}+N_{\uparrow\bar{\pi}})}\Big)\Big(\overrightarrow{\prod_{i\in\mathcal{M}_{\pi\uparrow}}} \Pi^i_{\tilde{P}(N_{\uparrow\pi}+N_{\uparrow\bar{\pi}})}\Big) \times \\ & \times \Big(\Pi^{\tilde{P}(N_{\uparrow\pi}+1)}_{\tilde{P}(N_{\uparrow\pi}+N_{\uparrow\bar{\pi}})} ... \Pi^{\tilde{P}(N_{\uparrow\pi}+N_{\uparrow\bar{\pi}}-1)}_{\tilde{P}(N_{\uparrow\pi}+N_{\uparrow\bar{\pi}})}\Big) PQ \end{aligned} \tag{5.2.118}$$

where $\tilde{P} = P_{\bar{\pi}\uparrow} Q_{\bar{\pi}\uparrow}$ and $PQ = P_{\bar{\pi}\downarrow} \tilde{P} C_K^{K-N_{\uparrow\pi}-N_{\uparrow\bar{\pi}}}$. Here, due to the choice of Q, $P_{\pi\downarrow}$ and $P_{\pi\uparrow}$ are equal to identity. The symbol $\overrightarrow{\prod_{i\in\mathcal{M}_A}} \Pi^i_{\tilde{P}(N_{\uparrow\pi}+N_{\uparrow\bar{\pi}})}$ denotes the ordered product of permutation: $...\Pi^i_{\tilde{P}(N_{\uparrow\pi}+N_{\uparrow\bar{\pi}})} \Pi^{i+1}_{\tilde{P}(N_{\uparrow\pi}+N_{\uparrow\bar{\pi}})} ...$

Next, we use the relations (5.2.92), (5.2.93) and (5.2.94) to simplify the coefficient $\Phi^{PQC_K}_{P^{-1}}$:

$$\begin{aligned} \Phi_{\bar{A}}(PQC_K, P^{-1}) \;=\; & \prod_{i\in\mathcal{M}_{\pi\downarrow}} e^{ik_{a_i}} \sigma_{a_i}(\lambda_{\tilde{P}(N_{\uparrow\pi}+N_{\uparrow\bar{\pi}})}) \prod_{i\in\mathcal{M}_{\bar{\pi}\downarrow}} b_i^{-1} \prod_{i\in\mathcal{M}_{\pi\uparrow}} e^{-ik_{a_i}} \sigma_{a_i}(\lambda_{\tilde{P}(N_{\uparrow\pi}+N_{\uparrow\bar{\pi}})}) \times \\ & \times \Phi_{\bar{A}}\Big(\Pi^{\tilde{P}(N_{\uparrow\pi}+1)}_{\tilde{P}(N_{\uparrow\pi}+N_{\uparrow\bar{\pi}})} ... \Pi^{\tilde{P}(N_{\uparrow\pi}+N_{\uparrow\bar{\pi}}-1)}_{\tilde{P}(N_{\uparrow\pi}+N_{\uparrow\bar{\pi}})} PQ, P^{-1}\Big) \end{aligned} \tag{5.2.119}$$

and finally we get

$$\begin{aligned} & \prod_{i\in\mathcal{M}_{\pi\downarrow}} e^{ik_{a_i}} \sigma_{a_i}(\lambda_{\tilde{P}(N_{\uparrow\pi}+N_{\uparrow\bar{\pi}})}) \prod_{i\in\mathcal{M}_{\bar{\pi}\downarrow}} b_i^{-1} \prod_{i\in\mathcal{M}_{\pi\uparrow}} e^{-ik_{a_i}} \sigma_{a_i}(\lambda_{\tilde{P}(N_{\uparrow\pi}+N_{\uparrow\bar{\pi}})}) \times \\ & \Phi_{\bar{A}}\Big(\Pi^{\tilde{P}(N_{\uparrow\pi}+1)}_{\tilde{P}(N_{\uparrow\pi}+N_{\uparrow\bar{\pi}})} ... \Pi^{\tilde{P}(N_{\uparrow\pi}+N_{\uparrow\bar{\pi}}-1)}_{\tilde{P}(N_{\uparrow\pi}+N_{\uparrow\bar{\pi}})} PQ, P^{-1}\Big) \;=\; \prod_{l=1}^{N} \sigma_l(\lambda_{\tilde{P}(N_{\uparrow\pi}+N_{\uparrow\bar{\pi}})}) \Phi_{\bar{A}}(PQ, P^{-1}) \end{aligned} \tag{5.2.120}$$

Now we introduce similarly as in previous case the vector $\hat{\Phi}_{\bar{\pi}\uparrow}(P)$:

$$\hat{\Phi}_{\bar{\pi}\uparrow}(P) \equiv \sum_{\bar{A}\in F} \sum_{Q'\in\mathfrak{S}_{N_{\bar{\pi}\uparrow}}} \Phi_{\bar{A}}(P, Q') \prod_{i\in\mathcal{M}_{\bar{\pi}\uparrow}} e_i^{\bar{A}_{Q'(i)}} \tag{5.2.121}$$

where $F = \text{span}\{(\mathfrak{p}+1)\uparrow, ..., \mathfrak{n}\uparrow\}$.
The relations (5.2.90) and (5.2.97) can be gathered in $\hat{\Phi}_{\bar{\pi}\uparrow}(P)$:

$$\hat{\Phi}(\Pi_{ab}P) = \alpha_{ab}^{-1}\mathcal{P}_{ab}\hat{\Phi}(P), \quad \text{with} \quad P^{-1}(a) - P^{-1}(b) = -1 \tag{5.2.122}$$

where $\mathcal{P}_{ab}$ is the permutation acting on the particles situated on a and b positions in $\hat{\Phi}_{\bar{\pi}\uparrow}(P)$.
Therefore, the periodicity problem can be rewritten as follows:

$$X_m \mathcal{P}_{\tilde{P}(N_{\uparrow\pi}+1)m} ... \mathcal{P}_{\tilde{P}(N_{\uparrow\pi}+N_{\uparrow\bar{\pi}}-1)m} \hat{\Phi}_{\bar{\pi}\uparrow}(\tilde{P}) = \prod_{l=1}^{N} \sigma_l(\lambda_m) \hat{\Phi}_{\bar{\pi}\uparrow}(\tilde{P}), \tag{5.2.123}$$

with $m = \tilde{P}(N_{\uparrow\pi} + N_{\uparrow\bar{\pi}})$ and

$$X_m = \prod_{l \in \mathcal{M}_{\bar{\pi}\uparrow}, l \neq m} \alpha_{ml} \prod_{i \in \mathcal{M}_{\pi\downarrow}} e^{ik_{a_i}} \sigma_{a_i}(\lambda_m) \prod_{i \in \mathcal{M}_{\bar{\pi}\downarrow}} b_i^{-1} \prod_{i \in \mathcal{M}_{\pi\uparrow}} e^{-ik_{a_i}} \sigma_{a_i}(\lambda_m) \tag{5.2.124}$$

Choosing $\tilde{P} = C_{\bar{\pi}\uparrow}^{-m}$ with $C_{\bar{\pi}\uparrow} = \Pi_{N_{\uparrow\pi}+N_{\uparrow\bar{\pi}}}^{N_{\uparrow\pi}+1} ... \Pi_{N_{\uparrow\pi}+N_{\uparrow\bar{\pi}}}^{N_{\uparrow\pi}+N_{\uparrow\bar{\pi}}-1}$, we find the Bethe equations:

$$\prod_{l \in \mathcal{M}_{\bar{\pi}\uparrow}, l \neq m} \alpha_{ml} \prod_{i \in \mathcal{M}_{\pi\downarrow}} e^{ik_{a_i}} \sigma_{a_i}(\lambda_m) \prod_{i \in \mathcal{M}_{\bar{\pi}\downarrow}} b_i^{-1} \prod_{i \in \mathcal{M}_{\pi\uparrow}} e^{-ik_{a_i}} \sigma_{a_i}(\lambda_m)\, e^{\frac{2\pi i}{N_{\uparrow\bar{\pi}}} \sum_{i=1}^{N_{\uparrow\bar{\pi}}-N_{\uparrow\mathfrak{n}}} \bar{n}_i} = \prod_{l=1}^{N} \sigma_l(\lambda_m)$$
$$\text{with} \quad m \in \mathcal{M}_{\uparrow\bar{\pi}} \quad \text{and} \quad 1 \leq \bar{n}_1 < ... < \bar{n}_{N_{\uparrow\bar{\pi}}-N_{\uparrow\mathfrak{n}}} \leq N_{\uparrow\bar{\pi}} \tag{5.2.125}$$

The problem of the complete characterization of the eigenfunction and of state counting arises in a similar way as in case III-A. We get additional sets of integers, taking the results from the section (5.2.4) on the solution of the permutation problem:

$$1 \leq n_1^{(k)} < n_2^{(k)} < ... < n_{N_{\uparrow(\mathfrak{p}+1)}+...+N_{\uparrow(k-1)}}^{(k)} \leq N_{\uparrow(\mathfrak{p}+1)} + ... + N_{\uparrow k}\,, \quad k = \mathfrak{p}+2, ..., \mathfrak{n}-1\,. \tag{5.2.126}$$

Again, we recover relation (5.2.125) if we take $k = \mathfrak{n}$ in the above condition.

5.2.3 Thermodynamic limit

The Bethe equations given in the section 5.2.1 differ from the (usual) Hubbard model's ones by some phases. In this section, we study our Bethe equations in more detail and find real solutions in the thermodynamic limit $L \to \infty$ and at the half-filled band limit, similarly to (3.4).

Simplification of Bethe equations The Bethe equations (5.2.9)-(5.2.13) can be partly solved for the Bethe roots k_j with $j \in \mathbb{A}_{\pi\uparrow} \cup \mathbb{A}_{\pi\downarrow}$ and b_l with $l \in \mathcal{M}_{\bar{\pi}\downarrow}$:

$$\begin{aligned} k_j &= \frac{2\pi}{L - N_{\uparrow\bar{\pi}}} \left(\frac{N_{\uparrow\pi} - 1}{2} + \frac{m_{\uparrow\pi}}{N_{\uparrow\pi}} + I_j^{\uparrow\pi} \right), \quad \text{for} \quad j \in \mathbb{A}_{\pi\uparrow}, \qquad (5.2.127) \\ & \quad m_{\uparrow\pi} = 1, ..., N_{\uparrow\pi} \quad \text{and} \quad 1 \leq I_1^{\uparrow\pi} < ... < I_{N_{\uparrow\pi}}^{\uparrow\pi} \leq L - N_{\uparrow\bar{\pi}} \\ k_j &= \frac{2\pi}{L - N_{\downarrow\bar{\pi}} - N_{\downarrow\mathfrak{m}}} \left(\frac{N_{\downarrow\pi} - 1}{2} + \frac{m_{\downarrow\pi}}{N_{\downarrow\pi}} + I_j^{\downarrow\pi} \right), \quad \text{for} \quad j \in \mathbb{A}_{\pi\downarrow}, \qquad (5.2.128) \\ & \quad m_{\downarrow\pi} = 1, ..., N_{\downarrow\pi} \quad \text{and} \quad 1 \leq I_1^{\downarrow\pi} < ... < I_{N_{\downarrow\pi}}^{\downarrow\pi} \leq L - N_{\downarrow\bar{\pi}} - N_{\downarrow\mathfrak{m}} \\ \ln b_l &= \frac{2\pi i}{N_{\downarrow\bar{\pi}} + N_{\downarrow\mathfrak{m}}} \left(\sum_{j=1}^{N_{\downarrow\bar{\pi}} - N_{\downarrow\mathfrak{m}-1}} \frac{n_j}{N_{\downarrow\bar{\pi}}} + I_l^{\downarrow\bar{\pi}} \right), \quad l \in \mathcal{M}_{\bar{\pi}\downarrow}, \qquad (5.2.129) \\ & \quad 1 \leq n_1 < ... < n_{N_{\downarrow\bar{\pi}} - N_{\downarrow(\mathfrak{m}-1)}} \leq N_{\downarrow\bar{\pi}} \ ; \ 1 \leq I_1^{\downarrow\bar{\pi}} < ... < I_{N_{\downarrow\bar{\pi}}}^{\downarrow\bar{\pi}} \leq N_{\downarrow\bar{\pi}} + N_{\downarrow\mathfrak{m}} \\ e^{ik_j L} &= e^{2\pi i \Phi} \prod_{l \in \mathcal{M}_{\bar{\pi}\uparrow}} \frac{i \sin k_j + i\lambda_l + \frac{U}{4}}{i \sin k_j + i\lambda_l - \frac{U}{4}}, \quad \text{for} \quad j \in [1, N] \setminus \mathbb{A} \qquad (5.2.130) \end{aligned}$$

$$\prod_{\substack{l=1 \\ l \notin \mathbb{A}}}^{N} \frac{i \sin k_l + i\lambda_m + \frac{U}{4}}{i \sin k_l + i\lambda_m - \frac{U}{4}} = e^{2\pi i \Psi} \prod_{\substack{l \in \mathcal{M}_{\bar{\pi}\uparrow} \\ l \neq m}} \frac{i\lambda_m - i\lambda_l + \frac{U}{2}}{i\lambda_m - i\lambda_l - \frac{U}{2}}, \quad \text{for} \quad m \in \mathcal{M}_{\bar{\pi}\uparrow} \qquad (5.2.131)$$

where the phases Φ and Ψ are defined by

$$\begin{aligned} \Phi &\equiv \frac{N_{\downarrow\bar{\pi}} + N_{\downarrow\mathfrak{m}} - 1}{2} - \left(\frac{N_{\downarrow\pi} - 1}{2} + \frac{m_{\downarrow\pi}}{N_{\downarrow\pi}} \right) \frac{N_{\downarrow\pi}}{L - N_{\downarrow\bar{\pi}} - N_{\downarrow\mathfrak{m}}} \\ & \quad - \sum_{j=1}^{N_{\downarrow\pi}} \frac{I_j^{\downarrow\pi}}{L - N_{\downarrow\bar{\pi}} - N_{\downarrow\mathfrak{m}}} + \sum_{j=1}^{N_{\downarrow\pi} - N_{\downarrow(\mathfrak{m}-1)}} \frac{n_j}{N_{\downarrow\bar{\pi}} + N_{\downarrow\mathfrak{m}}} + \sum_{j=1}^{N_{\downarrow\bar{\pi}}} \frac{I_j^{\downarrow\bar{\pi}}}{N_{\downarrow\bar{\pi}} + N_{\downarrow\mathfrak{m}}} \qquad (5.2.132) \end{aligned}$$

and

$$\begin{aligned} \Psi &\equiv \sum_{\sigma=\uparrow,\downarrow} \left((-1)^{\delta_{\sigma,\downarrow}} \left(\frac{N_{\sigma\pi} - 1}{2} + \frac{m_{\sigma\pi}}{N_{\sigma\pi}} \right) \frac{N_{\sigma\pi}}{L - N_{\sigma\bar{\pi}} - \delta_{\sigma,\downarrow} N_{\downarrow\mathfrak{m}}} + \sum_{j=1}^{N_{\sigma\pi}} \frac{I_j^{\sigma\pi}}{L - N_{\sigma\bar{\pi}} - \delta_{\sigma,\downarrow} N_{\downarrow\mathfrak{m}}} \right) + \\ & \quad + \frac{N_{\uparrow\bar{\pi}} + N_{\downarrow\bar{\pi}} + N_{\downarrow\mathfrak{m}}}{2} - \sum_{j=1}^{N_{\downarrow\bar{\pi}} - N_{\downarrow\mathfrak{m}-1}} \frac{n_j}{N_{\downarrow\bar{\pi}} + N_{\downarrow\mathfrak{m}}} + \sum_{j=1}^{N_{\downarrow\bar{\pi}}} \frac{I_j^{\downarrow\bar{\pi}}}{N_{\downarrow\bar{\pi}} + N_{\downarrow\mathfrak{m}}} + \sum_{j=1}^{N_{\uparrow\bar{\pi}} - N_{\uparrow\mathfrak{n}}} \frac{\bar{n}_j}{N_{\uparrow\bar{\pi}}} \qquad (5.2.133) \end{aligned}$$

with $1 \leq \bar{n}_1 < ... < \bar{n}_{N_{\uparrow\bar{\pi}} - N_{\uparrow\mathfrak{n}}} \leq N_{\uparrow\bar{\pi}}$.

We recall that

$$\mathbb{A} = \mathbb{A}_{\pi\uparrow} \cup \mathbb{A}_{\pi\downarrow} = \{a_1, a_2, \ldots, a_{N_{\uparrow\pi}}\} \cup \{a_{N_{\uparrow\pi}+N_{\uparrow\bar{\pi}}+1}, a_{N_{\uparrow\pi}+N_{\uparrow\bar{\pi}}+2}, \ldots, a_{N_{\uparrow\pi}+N_{\uparrow\bar{\pi}}+N_{\downarrow\pi}}\} \qquad (5.2.134)$$

where the integers a_i are ordered according to the inequalities

$$1 \leq a_1 < a_2 < ... < a_{N_{\uparrow\pi}} \leq N \text{ and } 1 \leq a_{N_{\uparrow\pi}+N_{\uparrow\bar{\pi}}+1} < a_{N_{\uparrow\pi}+N_{\uparrow\bar{\pi}}+2} < ... < a_{N_{\uparrow\pi}+N_{\uparrow\bar{\pi}}+N_{\downarrow\pi}} \leq N \qquad (5.2.135)$$

and $a_i \neq a_j$ for $i \in \mathcal{M}_{\pi\uparrow}$, $j \in \mathcal{M}_{\pi\downarrow}$.

The ground state of the model is given by real values of the Bethe roots k_j and λ_i.

Following E.H.Lieb and F.Y.Wu [48] or (3.4.1), we take the logarithm of the Bethe equations (5.2.130) and (5.2.131):

$$k_j L = 2\pi(\Phi + I_j) + \sum_{i=1}^{N_{\uparrow\bar{\pi}}} \vartheta\Big(\frac{\lambda_i + \sin k_j}{U/4}\Big), \quad j \in [1,N] \setminus \mathbb{A} \tag{5.2.136}$$

$$\sum_{\substack{j=1 \\ j\notin\mathbb{A}}}^{N} \vartheta\Big(\frac{\lambda_i + \sin k_j}{U/4}\Big) = 2\pi(J_i + \Psi) + \sum_{k=1}^{N_{\uparrow\bar{\pi}}} \vartheta\Big(\frac{\lambda_i - \lambda_k}{U/2}\Big), \quad i \in \mathcal{M}_{\bar{\pi}\uparrow} \tag{5.2.137}$$

where $\vartheta(x) = 2\arctan(x) \in]-\pi,\pi]$ and we take the cut off for the logarithm as $\frac{1}{i}\log\Big(\frac{x+i}{x-i}\Big) = \pi - \vartheta(x) \in [0,2\pi[$.

The quantum number I_j is integer or half-integer depending whether $N_{\uparrow\bar{\pi}} + N_{\downarrow\bar{\pi}} + N_{\downarrow\mathfrak{m}} - 1$ is even or odd, similarly J_i is integer (half-integer) for $N_{\uparrow\bar{\pi}} + 1$ even (odd).

Thermodynamic limit We consider the thermodynamic limit $L \to \infty$ where the particle densities $\frac{N_{\sigma\bar{\pi}}}{L}$, $\frac{N_{\sigma\pi}}{L}$ are kept fixed ($\sigma = \uparrow,\downarrow$). Considering the phases Φ and Ψ in the thermodynamic limit the ratios $\frac{\Phi}{L}$ and $\frac{\Psi}{L}$ do not vanish and depend on the particle densities.

In this limit, the real numbers k_j and λ_i become continous variables: $k_{j+1} - k_j \to 0$, $\lambda_{i+1} - \lambda_i \to 0$ with $L \to \infty$. They are distributed between $-Q(\Phi)$ and $Q(\Phi) \leq \pi$ and $-B(\Psi)$ and $B(\Psi) < \infty$ for some $Q(\Phi)$ and $B(\Psi)$. In the small intervals dk and $d\lambda$, the numbers of k_j and λ_i are $L\rho(k)dk$ and $L\sigma(\lambda)d\lambda$ respectively, where $\rho(k)$ and $\sigma(\lambda)$ are density functions to be determined. They are normalized as follows:

$$\int_{-Q(\Phi)}^{Q(\Phi)} \rho(k)dk = \frac{N - N_{\uparrow\pi} - N_{\downarrow\pi}}{L} \quad \text{and} \quad \int_{-B(\Psi)}^{B(\Psi)} \sigma(\lambda)d\lambda = \frac{N_{\uparrow\bar{\pi}}}{L} \tag{5.2.138}$$

The counting functions $I(k)$ and $J(\lambda)$ are defined in the continuum limit as following:

$$I(k) = k - \frac{2\pi\Phi}{L} - \int_{-B(\Psi)}^{B(\Psi)} d\lambda\sigma(\lambda)\vartheta\Big(\frac{\lambda + \sin k}{U/4}\Big), \tag{5.2.139}$$

$$J(\lambda) = \int_{-Q(\Phi)}^{Q(\Phi)} dk\rho(k)\vartheta\Big(\frac{\lambda + \sin k}{U/4}\Big) - \frac{2\pi\Psi}{L} - \int_{-B(\Psi)}^{B(\Psi)} d\lambda'\sigma(\lambda')\vartheta\Big(\frac{\lambda - \lambda'}{U/2}\Big) \tag{5.2.140}$$

These functions are such that $I(k_j) = 2\pi I_j/L$ and $J(\lambda_i) = 2\pi J_i/L$.

Since $I(k+dk) - I(k)$ counts the number of k values between k and $k+dk$, we have $\frac{dI(k)}{dk} = 2\pi\rho(k)$, and similarly, $\frac{dJ(\lambda)}{d\lambda} = 2\pi\sigma(\lambda)$.

Now taking the derivatives of (5.2.137), and considering the fact that the phases Φ and Ψ do not depend on the Bethe roots k or λ, we get the equations on densities, which are the same as Lieb-Wu ones in [48]:

$$1 = 2\pi\rho(k) + \frac{\cos k}{U/4}\int_{-B(\Psi)}^{B(\Psi)} d\lambda\sigma(\lambda)\vartheta'\Big(\frac{\lambda + \sin k}{U/4}\Big) \tag{5.2.141}$$

$$\int_{-Q(\Phi)}^{Q(\Phi)} dk\frac{\rho(k)}{U/4}\vartheta'\Big(\frac{\lambda + \sin k}{U/4}\Big) = 2\pi\sigma(\lambda) + \int_{-B(\Psi)}^{B(\Psi)} d\lambda'\frac{\sigma(\lambda)}{U/2}\vartheta'\Big(\frac{\lambda - \lambda'}{U/2}\Big) \tag{5.2.142}$$

where $k \in [-Q(\Phi), Q(\Phi)]$, $\lambda \in [-B(\Psi), B(\Psi)]$ and $\vartheta'(x) = d\vartheta(x)/dx = 2/(1+x^2)$.

Ground state. We consider the "half-filled band" limit, defined as

$$N - N_{\uparrow\pi} - N_{\downarrow\pi} = L \quad \text{and} \quad 2N_{\uparrow\bar{\pi}} = N - N_{\uparrow\pi} - N_{\downarrow\pi}\,. \tag{5.2.143}$$

Using the same arguments as in 3.4.1, we obtain $Q(\Phi) = \pi$ and $B(\Psi) = \infty$. This can be seen as follows. Taking the normalization relations for $\rho(k)$ and $\sigma(\lambda)$ (5.2.138) and inserting the relations between the counting functions and the densities, we get:

$$I(Q) - I(-Q) = N - N_{\uparrow\pi} - N_{\downarrow\pi}, \quad J(B) - J(-B) = N_{\uparrow\bar{\pi}} \tag{5.2.144}$$

Inserting the definitions of the counting functions (5.2.140) in these equations, the following conditions arise: if $Q \to \pi$ then $N - N_{\uparrow\pi} - N_{\downarrow\pi} = L$ and if $B \to \infty$ then $2N_{\uparrow\bar{\pi}} = N - N_{\uparrow\pi} - N_{\downarrow\pi}$.

This limit allows us to find the solution for the densities $\rho(k)$ and $\sigma(\lambda)$ by Fourier transform:

$$\sigma_0(\lambda) = \frac{1}{2\pi}\int_0^\infty dw\, \frac{\cos(w\lambda)}{\cos(wU/4)}\, J_0(w)\,, \quad \lambda \in [-\infty, \infty] \tag{5.2.145}$$

$$\rho_0(k) = \frac{1}{2\pi} + \frac{\cos k}{\pi}\int_0^\infty dw\, \frac{\cos(w\sin k)}{e^{Uw/2}+1}\, J_0(w)\,, \quad k \in [-\pi, \pi] \tag{5.2.146}$$

with zeroth order Bessel function $J_0(x) = \dfrac{1}{\pi}\displaystyle\int_0^\infty dw \cos(x\sin w)$.
The ground state energy is then equal:

$$E = \frac{U}{4}(L - 2N_{\bar{\pi}}) + 2\sum_{l\in\mathcal{M}_{\pi\uparrow}\cup\mathcal{M}_{\pi\downarrow}} \cos(k_l) = \frac{U}{4}(L - 2N_{\bar{\pi}}) + 4L\int_0^\infty dw\, \frac{J_1(w)J_0(w)}{w(e^{Uw/2}+1)} \tag{5.2.147}$$

with order one Bessel function $J_1(x) = \dfrac{x}{\pi}\displaystyle\int_0^\infty dw\ \cos(x\sin w)\cos^2 w$.

On this point we finish the discussion of the generalized Hubbard models. We point out that the treatment of the obtained Bethe equations (5.2.9)-(5.2.13) is not accomplished and a string hypothesis incluiding Takahashi's one, due to new appeared phases, is needed. From the AdS/CFT duality side, the phases in our Bethe equations are trivial to be compared with the "dressing" factor of [77], (3.5.15). Further modifications of the R-matrix are needed, such as the introduction of free parameters using the twist of the underlying Hopf algebra e.g Reshetikhin twist or higher spin generalizations etc.

Finally, we think that our models can be used in condensed matter physics to define multi-leg Hubbard models.

5.2.4 Permutation problem

In this section we review the solution of permutation problem. This solution allows to obtain all necessary quantum number (inner level Bethe roots) in order to fully characterize the coefficients $\Phi_{\bar{A}}(PQ, P^{-1})$ in the ansatz (5.2.52) or the coefficients $\Phi_{\bar{A}}(P, QP^{-1})$ in the ansatz (5.1.2). Due to the fact that such kind of problem arises two times for different sectors of "Auxiliary problem. Level 3" (5.2.2) and in (5.1.2), therefore we will show the solution in general case.

Let we have $e^{2\sigma}$, $e^{3\sigma}$, ... ,$e^{\hbar\sigma}$ "particles" with any spin σ that move 'freely', the Hamiltonian Γ being constructed on permutations only:

$$\Gamma\,\phi = P_{j+1j}...P_{Kj}P_{1j}...P_{j-1j}\,\phi = \Lambda\,\phi\,. \tag{5.2.148}$$

Note that Γ is a cyclic permutation, and is independent from j.

We choose the "particle" $e^{2\sigma}$ as the vacuum state:

$$\phi_{M=0} = \prod_{i=1}^{K} e_i^{2\sigma}\,, \quad \Lambda = 1\,, \tag{5.2.149}$$

and introduce the function

$$\phi_M^{(3)}(\bar{A}) = \sum_{\vec{x}} \Psi(\vec{x}) \prod_{i=1}^{M} e_{x_i}^{A_i}, \;\; A_i = 3\sigma, 4\sigma, ..., \mathfrak{h}\sigma\,. \tag{5.2.150}$$

It describes a state with M excitations above the vacuum state $\phi_{M=0}$.

The coefficients $\Psi(\vec{x})$ are defined in the sector $1 \leq x_{Q(1)} < x_{Q(2)} < ... < x_{Q(M)} \leq K$, with $Q \in \mathfrak{S}_M$, by

$$\Psi_Q(\vec{x}) = \sum_{P \in \mathfrak{S}_M} \Phi^{(3)}(P, QP^{-1}) \prod_{i=1}^{M} g_{x_{Q(i)}}(a_{P(i)}^{(3)})\,, \qquad g_x(a) = a^x\,. \tag{5.2.151}$$

One can verify that $\phi_M^{(3)}(\vec{A})$ is an eigenfunction with the following eigenvalue

$$\Lambda = \prod_{i=1}^{M} a_i^{(3)} \tag{5.2.152}$$

if some conditions, which we precise below, are satisfied.

Application of the Hamiltonian gives only the periodicity condition on the coefficients $\Phi^{(3)}(P, QP^{-1})$:

$$\frac{\Phi^{(3)}(PC, QP^{-1})}{\Phi^{(3)}(P, QP^{-1})} = [a_{P(M)}^{(3)}]^K\,, \qquad C = \Pi_{1M}...\Pi_{M-1M}\,. \tag{5.2.153}$$

We assume some relations on the coefficients $\Phi^{(3)}(P, QP^{-1})$, but, now, the form of these relations depend on whether the particles are identical or not. If, for a given i, x_i and x_{i+1} correspond to identical particles we impose

$$\Phi^{(3)}(\Pi_{P(i)P(i+1)}P, QP^{-1}) = \Phi^{(3)}(P, QP^{-1}), \tag{5.2.154}$$

while, otherwise, we set

$$\Phi^{(3)}(\Pi_{P(i)P(i+1)}P, QP^{-1}\Pi_{P(i)P(i+1)}) = \Phi^{(3)}(P, QP^{-1})\,. \tag{5.2.155}$$

As we can see, there is a sector changing in the relations above, and we proceed recursively using the same methods as above. We introduce

$$\hat{\Phi}^{(3)}(P) \equiv \sum_{Q,\bar{A}} \Phi^{(3)}[P, Q, \bar{A}]|A_{q_1}, .., A_{q_M}> \tag{5.2.156}$$

where the sum is over possible types $\bar{A}$ and all corresponding sectors $Q \in \mathfrak{S}_M$. The vector $|A_{q_1}, ..., A_{q_N}>$ represents one state with M excitations and belongs to $V_1 \otimes ... \otimes V_N$ (where $V = \text{span}\{e^{3\sigma}, e^{4\sigma}, ..., e^{\mathfrak{h}\sigma}\}$). Then, relations (5.2.154) and (5.2.155) can be rewritten in the following form

$$\hat{\Phi}^{(3)}(\Pi_{ab}P) = S_{ab}^{(3)}\hat{\Phi}^{(3)}(P)\,, \qquad S_{ab}^{(3)} = P_{ab}\,. \tag{5.2.157}$$

Therefore, the periodic boundary conditions on $\hat{\Phi}(P)$ implied by the action of the chain of permutations is written as

$$P_{m+1m}...P_{Mm}P_{1m}...P_{m-1m}\hat{\Phi}^{(3)}(id) = [a_m^{(3)}]^K\hat{\Phi}^{(3)}(id)\,, \quad m = 1, ..., M \tag{5.2.158}$$

but here we have already only $e^{3\sigma}$, ... ,$e^{\mathfrak{h}\sigma}$ "particles" involved in the calculations. Thus, we arrive to the next level of nested Coordinate Bethe ansatz, with, again, an Hamiltonian built on permutations only, and a new chain of length M.

Using the previous considerations, we repeat the same method and we "eliminate" one by one the "particles" $e^{3\sigma}$, ... up to $e^{\mathfrak{h}-1\,\sigma}$, choosing it as the vacuum state at each nested level.

We suppose that we have M_3 "particles" of type $e^{3\sigma}$, M_4 of type $e^{4\sigma}$,..., $M_\mathfrak{h}$ of type $e^{\mathfrak{h}\sigma}$, so that $M_3 + M_4 + ... + M_\mathfrak{h} = M$. At each level $k = 3, ..., \mathfrak{h} - 1$, we have particles $e^{(k+1)\sigma}, ..., e^{\mathfrak{h}\sigma}$ as different types of excitations above the vacuum state built on $e^{k\sigma}$. The eigenvector $\hat{\Phi}^{(k)}(id)$ can be written in the same form as in (5.2.150) and (5.2.151), with the set of Bethe roots $\{a_i^{(k+1)}\}_{i=1}^{M_{k+1}+...+M_\mathfrak{h}}$ and the coefficients $\Phi^{(k+1)}(P, QP^{-1})$ with $P, Q \in \mathfrak{S}_{M_{k+1}+...+M_\mathfrak{h}}$. These coefficients are used to write the vector $\hat{\Phi}^{(k+1)}(id)$ that obeys the periodicity condition:

$$P_{m+1,m}...P_{M_{k+1}+...+M_\mathfrak{h},m}P_{1m}...P_{m-1,m}\hat{\Phi}^{(k+1)}(id) = \left[a_m^{(k+1)}\right]^{M_k+...+M_\mathfrak{h}}\hat{\Phi}^{(k+1)}(id), \tag{5.2.159}$$

for $m = 1, ..., (M_{k+1} + ... + M_\mathfrak{h})$.

Also, we find the periodicity condition of the previous level (when we pass from level $k-1$ to level k)

$$\left[a_m^{(k)}\right]^{M_{k-1}+...+M_\mathfrak{h}} = \prod_{i=1}^{M_{k+1}+...+M_\mathfrak{h}} a_i^{(k+1)} \tag{5.2.160}$$

for $m = 1, ..., (M_{k+1} + ... + M_\mathfrak{h})$ and $k = 3, ..., \mathfrak{h} - 1$ with $M_2 = K - (M_3 + ... + M_\mathfrak{h})$.

At last level, we have only one type of excitations $e^{\mathfrak{h}\sigma}$ on the vacuum state $e^{\mathfrak{h}-1\,\sigma}$. Thus, using the relation (5.2.160) for $k = \mathfrak{h} - 1$, we find the following Bethe equations which link the case of one type excitation $e^{\mathfrak{h}\sigma}$ and the previous level with two types of excitations $\{e^{\mathfrak{h}-1\,\sigma}, e^{\mathfrak{h}\,\sigma}\}$:

$$[a_i^{(\mathfrak{h}-1)}]^{M_{\mathfrak{h}-2}+M_{\mathfrak{h}-1}+M_\mathfrak{h}} = \prod_{j=1}^{M_\mathfrak{h}} a_j^{(\mathfrak{h})}, \quad i = 1, \ldots, M_{\mathfrak{h}-1} + M_\mathfrak{h}\,, \tag{5.2.161}$$

together with the result for one type excitation on the vacuum

$$a_j^{(\mathfrak{h})} = e^{2\pi i\frac{n_j^{(\mathfrak{h})}}{M_{\mathfrak{h}-1}+M_\mathfrak{h}}}, \quad j = 1, \ldots, M_\mathfrak{h} \tag{5.2.162}$$

$$1 \leq n_1^{(\mathfrak{h})} < n_2^{(\mathfrak{h})} < ... < n_{M_\mathfrak{h}}^{(\mathfrak{h})} \leq M_{\mathfrak{h}-1} + M_\mathfrak{h}\,. \tag{5.2.163}$$

In the same way, we can write the Bethe equations corresponding to the transition between level with $\{e^{\mathfrak{h}-1\,\sigma}, e^{\mathfrak{h}\,\sigma}\}$ and the previous one

$$[a_i^{(\mathfrak{h}-2)}]^{M_{\mathfrak{h}-3}+...+M_\mathfrak{h}} = \prod_{j=1}^{M_{\mathfrak{h}-1}+M_\mathfrak{h}} a_j^{(\mathfrak{h}-1)}, \;\; i = 1, \ldots, M_{\mathfrak{h}-2} + ... + M_\mathfrak{h} \tag{5.2.164}$$

and we can continue this recurrence up to $a^{(3)}$.

Solutions for every set of Bethe roots $a^{(k)}$ can be computed as following. For $k = 3, ..., \mathfrak{h}$ with $M_2 = K - (M_3 + ... + M_{\mathfrak{h}})$ we have

$$a_j^{(k)} = e^{\frac{2\pi i}{M_{k-1}+...+M_{\mathfrak{h}}}\left[n_j^{(k)} + \frac{\sum_{m=k+1}^{\mathfrak{h}} \sum_{i=1}^{M_m+...+M_{\mathfrak{h}}} n_i^{(m)}}{M_k+...+M_{\mathfrak{h}}}\right]}, \quad j = 1, \ldots, (M_k + ... + M_{\mathfrak{h}})$$
$$1 \leq n_1^{(k)} < n_2^{(k)} < ... < n_{M_k+...+M_{\mathfrak{h}}}^{(k)} \leq M_{k-1} + ... + M_{\mathfrak{h}}\,. \tag{5.2.165}$$

Therefore, the function is the eigenvector of the initial permutation problem (5.2.148)

$$\phi_M^{(3)}(\bar{A}) = \sum_{Q\in\mathfrak{S}_M} \sum_{\vec{x}\in Q} \sum_{P\in\mathfrak{S}_M} \Phi^{(3)}(P, QP^{-1}) \prod_{i=1}^{M} [a_{P(i)}^{(3)}]^{x_{Q(i)}} e_{x_i}^{A_i}\, \phi_0(2\sigma) \tag{5.2.166}$$

where we denoted explicitely the 'empty sites' (with no excitation) as $\phi_0(2\sigma)$.

The coefficients $\Phi^{(3)}(id, Q)$ gathered in $\hat{\Phi}^{(3)}(id)$ are connected with the next level coefficients $\Phi^{(4)}(id, Q)$. To see it, we fix the value of M_3 and consider vectors $\bar{B}$ which characterizes excitations of the form

$$\bar{B} = (\overbrace{\mathfrak{h}\sigma, ..., \mathfrak{h}\sigma, ..., 4\sigma, ..., 4\sigma}^{M-M_3}, \overbrace{3\sigma, ..., 3\sigma}^{M_3})\,. \tag{5.2.167}$$

Then, we consider the restriction of the relation (5.2.156) to

$$\hat{\Phi}^{(3)}(id)\Big|_{restric.} = \sum_{Q,\bar{B}} \Phi^{(3)}[id, Q, \bar{B}]|B_{q_1}, .., B_{q_M}> \ . \tag{5.2.168}$$

One can recognize in this term the $\phi_{M-M_3}^{(4)}(\bar{B})$ coefficient (here we take only first $M - M_3$ values $\bar{B}$):

$$\sum_{Q} \Phi^{(3)}[id, Q, \bar{B}] e_{q^{-1}(1)}^{B_1} ... e_{q^{-1}(M-M_3)}^{B_{M-M_3}} \phi_0(3\sigma) = \sum_{\vec{y}\in[1,M]} \Psi(\vec{y}) \prod_{i=1}^{M-M_3} e_{y_i}^{B_i} \phi_0(3\sigma) \equiv \phi_{M-M_3}^{(4)}(\bar{B}), \tag{5.2.169}$$

with $y_i = q^{-1}(i)$ for $i = 1, ..., M - M_3$. In the left hand side of the equation $e_{q^{-1}(i)}^{B_i}$ are the operators which create the corresponding excitations B_i on the site $q^{-1}(i)$ of the chain of particles $e^{3\sigma}$.

Therefore, using the same ansatz as in (5.2.151) for $\Psi(\vec{y})$ in $\phi_{M-M_3}^{(4)}(\bar{B})$ we can identify the coefficients $\Phi^{(3)}(id, Q)$ as

$$\Phi^{(3)}(id, Q, \bar{B}) = \sum_{P\in\mathfrak{S}_{M-M_3}} \Phi^{(4)}(P, Q'P^{-1}, \bar{B}) \prod_{i=1}^{M-M_3} [a_{P(i)}^{(4)}]^i, \tag{5.2.170}$$

with $Q' \in \mathfrak{S}_{M-M_3}$ defined by $Q'(i) = Q(i)$ for $i = 1, ..., M - M_3$ and

$$B_i = 4\sigma, ..., \mathfrak{h}\,\sigma, \quad i = 1, ..., M - M_3\,. \tag{5.2.171}$$

In the general case, the coefficients $\Phi^{(k)}(P, Q)$ are defined by the same relations: for $k = 3, ..., \mathfrak{h} - 2$, we have

$$\Phi^{(k)}(id, Q, \bar{B}) = \sum_{P\in\mathfrak{S}_{M_{k+1}+...+M_{\mathfrak{h}}}} \Phi^{(k+1)}(P, Q'P^{-1}, \bar{B}) \prod_{i=1}^{M_{k+1}+...+M_{\mathfrak{h}}} [a_{P(i)}^{(k+1)}]^i, \tag{5.2.172}$$

where $Q' \in \mathfrak{S}_{M_{k+1}+...+M_{\mathfrak{h}}}$ is defined by $Q'(i) = Q(i)$ for $i = 1, ..., M_{k+1} + ... + M_{\mathfrak{h}}$,

$$B_i = (k+1)\sigma, ..., \mathfrak{h}\,\sigma, \quad i = 1, ..., M_{k+1} + ... + M_{\mathfrak{h}}$$

and there are relations similar to (5.2.154) and (5.2.155).

At last, when $k = \mathfrak{h} - 1$ we get

$$\Phi^{(\mathfrak{h}-1)}(id, Q, \bar{B}) = \sum_{P \in \mathfrak{S}_{M_{\mathfrak{h}-1}+M_{\mathfrak{h}}}} \prod_{i=1}^{M_{\mathfrak{h}-1}+M_{\mathfrak{h}}} [a^{(\mathfrak{h})}_{P(i)}]^i. \tag{5.2.173}$$

Equations of the type (5.2.170) and (5.2.172) together with relations (5.2.154) and (5.2.155) allow us to derive all the coefficients $\Phi^{(3)}(P, Q)$. The eigenvalue reads

$$\Lambda(\vec{n}^{(3)}, ..., \vec{n}^{(\mathfrak{h})}) \equiv \Lambda(\vec{n}^{(3)}) = \prod_{i=1}^{M} a_i^{(3)} = \exp\left(\frac{2\pi i}{K} \sum_{i=1}^{M} n_i^{(3)}\right) = \exp\left(\frac{2\pi i}{K} |\vec{n}^{(3)}|\right). \tag{5.2.174}$$

The Bethe parameters $\vec{n}^{(k)}$, $k > 3$, ensure the correct multiplicity of eigenfunctions. Indeed, the total number of states $\phi_M(\vec{n}^{(3)}, \vec{n}^{(4)}, ..., \vec{n}^{(\mathfrak{h})})$ is

$$\sum_{\substack{\vec{n}^{(3)}, \vec{n}^{(4)}, ..., \vec{n}^{(\mathfrak{h})} \\ 1 \le n_i^{(k)} < n_{i+1}^{(k)} \le J_k}} 1 \quad = \quad \frac{K!}{M_3! ... M_{\mathfrak{h}}! (K - M)!}, \quad \text{where} \quad J_k = \sum_{\ell=k-1}^{\mathfrak{m}+\mathfrak{n}} M_\ell, \tag{5.2.175}$$

which shows that the ansatz is complete.

Therefore, we apply these results in "Auxiliary problem. Level 3" (5.2.2).

Chapter 6

Conclusion and outlook

The one-dimensional Hubbard model is an example of integrable spin chains. It is a less 'trivial' example than basic spin chains as XXX or XXZ and there is a lot of important results have been obtained for this model.

In this thesis we considered the Hubbard model and its supersymmetric extensions. In details we described the Coordinate Bethe Ansatz method in order to derive the Bethe equations for considered models. The Bethe equations in the case of our supersymmetric extended Hubbard models are similar to the Lieb-Wu equations up to certain phases depending on inner level Bethe roots. We reviewed the solution of the Bethe equations for the Hubbard model (Lieb-Wu equations) and for our supersymmetric extended models in the case of real roots in the thermodynamic limit. We reproduced the results for the ground state energy in the half-filled band limit for these models. We found that our phases do not contribute to the ground state energy in the half-filled band limit. The string hypothesis for our supersymmetric extended Hubbard models can be based on the Takahashi's one but, however, it is needed to be enlarged due to the phases appeared in the Bethe equations.

The studies of the Hubbard model and its generalizations, were motivated by its applications in the $\mathcal{N} = 4$ super Yang-Mills theory (SYM). Although it appeared that the one-dimensional Hubbard model is not a proper answer due to the transcendental contributions to the anomalous dimensions (in the su(2) subsector of the theory), one may find new directions in this field by studying integrable extensions of the one-dimensional Hubbard model.

Unfortunately, the phases appeared in the Bethe equations of our supersymmetric extended Hubbard models do not depend on the 1st and 2nd level Bethe roots as in the AdS/CFT context, but we believe that our construction can be used as a first step in the construction of the integrable model that would reproduce the expected phase. An open problem is thus to look for an amendment of the construction to provide k-dependent phases for the AdS/CFT.

Another aspect lies in the possibility of applications to condensed matter physics, particularly when dealing with small rank algebras. Our models can be used in condensed matter physics probably to define multi-leg Hubbard models. However, this direction still need to be exploited.

Further researches on generalized Hubbard model can be based on the introduction of free parameters in Shastry's R-matrix. An example of this is related to the so-called Reshetikhin twist. Such construction were already used for basic spin chains and certain generalizations of the Hubbard model as U model. For the Hubbard model, a kind of twist is necessary to consider in order to obtain free parameters appearing in the Bethe equations.

Another interesting question is to generalize the spin degree of freedom. The Hubbard model and its generalizations contains particles of spin 1/2. It was also argued by Z.Maassarani that the R-matrix corresponding to higher spins can be constructed by the fusion procedure. However, even

in the simplest case it become difficult to operate with the R-matrix. The generalization of the spin degree of freedom to $sl(3)$ or $sl(n)$ algebra seems still an open question. Probably, the construction of the R-matrix is similar to the one of the Bariev spin chain.

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