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Le temps, qui atténue les souvenirs, aggrave celui du Zahir. Autrefois, je me représentais l'avers, puis le revers; maintenant, je les vois simultanément tous les deux. Les choses ne se passent pas comme si le Zahir était en cristal, car une face ne se superpose pas à l'autre; mais plutôt comme si la vue offerte était sphérique et que le Zahir se présentât au milieu. Ce qui n'est pas le Zahir me parvient tamisé et comme lointain: la dédaigneuse image de Teodelina, la douleur physique. Tennyson a dit que si nous pouvions comprendre une seule fleur nous saurions qui nous sommes et ce qu'est le monde. Il a peut-être voulu dire qu'il n'y a aucun fait, si humble soit-il, qui n'implique l'histoire universelle et son enchaînement infini d'effets et de causes. Il a peut-être voulu dire que le monde visible nous est donné tout entier en chaque représentation, de même que la volonté, selon Schopenhauer, nous est donnée toute entière en chaque sujet. Les cabalistes opinèrent que l'homme est un microcosme, un miroir symbolique de l'univers; tout le serait d'après Tennyson. Tout, même l'intolérable Zahir.

Jorge Luis Borges, *Le Zahir* in *L'Aleph*

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Comme probablement pour une partie non négligeable des thésards, tout ceci a commencé il y a déjà quelques temps par une volonté farouche et un brin destructrice de « démonter des trucs pour voir comment ça marche ». L'apparente abstraction des objets qu'elle met en jeu ne doit pas cacher le fait que cette thèse est également le fruit de joyeux bricolages qui sont le pendant de ce désir enfantin. Après tout, pour paraphraser Vladimir Arnold, les mathématiques forment aussi une part des sciences de la nature : celle où les expériences sont peu coûteuses. J'ai conscience que ce manuscrit peut paraître austère, pour ne pas dire légèrement rébarbatif. Il témoigne pourtant de l'immense plaisir que j'ai eu à plonger pour la première fois dans des rouages encore partiellement mystérieux, pour poser les mains sur de délicats mecanismes et, cette fois « pour de vrai », essayer de comprendre un peu comment tout ça marche. Le document qui suit marque en quelque sorte le début de mon parcours de mathématicien, ma première contribution. Sa simple existence met en lumière la dette que j'ai contractée envers de nombreuses personnes, à qui je souhaite exprimer ici ma sincère gratitude.

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Introduction

Le théorème de Kohno–Drinfeld est un résultat relié à la *quantification* de certaines structures algébriques, qui établit une équivalence entre deux familles de représentations du groupe de tresses d'Artin [**Ar**] qui interviennent naturellement dans des questions de physique mathématique. La première provient de la monodromie du système différentiel de Knizhnik–Zamolodchikov [**KZ**], la seconde d'une construction explicite reliée à la théorie des représentations des groupes quantiques. Ce résultat a motivé l'étude de la monodromie de différents analogues de la connexion KZ, pour lesquels des généralisations du théorème de Kohno–Drinfeld ont été démontrées [**EG**, **GL**, **To**].

L'objet principal de cette thèse est la généralisation de ce théorème à la monodromie de variantes « cyclotomiques » du système KZ, qui fournissent des représentations du groupe de tresses de type de Coxeter B.

Le groupe de tresses pures PB_n est défini comme le groupe fondamental de l'espace des configurations Y_n de n particules dans un plan

$$Y_n = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid i \neq j \Rightarrow z_i \neq z_j\}$$

et le groupe de tresses B_n comme le groupe fondamental de l'espace de configuration de n particules non distinguables $X_n = Y_n/\mathfrak{S}_n$. Au choix d'une algèbre de Lie simple $\mathfrak{g}$, d'un $\mathfrak{g}$-module V de dimension finie et d'un 2-tenseur symétrique $\mathfrak{g}$-invariant t, on associe un système différentiel introduit par Knizhnik et Zamolodchikov qui induit une connexion sur le fibré trivial de base Y_n et de fibre $V^{\otimes n}[[\hbar]]$ où $\hbar$ est une variable formelle. Étant $\mathfrak{S}_n$-équivariante, elle induit une connexion sur le fibré $(Y_n \times V^{\otimes n}[[\hbar]])/\mathfrak{S}_n$. Étant plate, elle fournit un morphisme de monodromie

$$\rho_{\mathrm{KZ}} : B_n \to GL(V^{\otimes n}[[\hbar]]).$$

La connexion KZ est plate parce que t satisfait une équation algébrique étroitement liée à l'équation de Yang-Baxter classique, qui est une version infinitésimale (une « approximation au premier ordre ») des relations définissant le groupe de tresses.

Du côté algébrique, on s'intéresse à une construction qui s'inscrit dans le cadre de la *quantification par déformation*, telle que formulée par Bayen *et al* dans les années 80 [**BFF+**], qui tire son inspiration de la physique : la quantification d'un système physique « classique » est la donnée d'un système physique « quantique » qui se réduit au système classique de départ si on néglige la constante de Planck. Mathématiquement, cette idée se traduit par la construction d'objets dépendant du paramètre formel $\hbar$ qui se spécialisent en un objet classique donné à la limite $\hbar \to 0$. Drinfeld [**Dr1**] et Jimbo [**Ji**] ont ainsi associé à toute algèbre de Lie simple $\mathfrak{g}$ un *groupe quantique*, c'est-à-dire une $\mathbb{C}[[\hbar]]$-algèbre de Hopf $U_\hbar(\mathfrak{g})$ qui est une déformation de l'algèbre enveloppante universelle $U(\mathfrak{g})$ de $\mathfrak{g}$.

L'une des premières motivations pour introduire ces objets est qu'ils permettent de construire des solutions universelles de l'équation de Yang-Baxter quantique, ce qui implique en particulier qu'ils permettent d'obtenir des représentations de B_n. Il est en effet bien connu que le produit tensoriel de $\mathfrak{g}$-modules est commutatif, en ce sens que pour tous $\mathfrak{g}$-modules V_1, V_2 l'isomorphisme naturel de $\mathbb{C}$-espaces vectoriels

$$P : V_1 \otimes V_2 \longrightarrow V_2 \otimes V_1$$

commute avec l'action de $\mathfrak{g}$. Autrement dit, $U(\mathfrak{g})$ est naturellement une algèbre de Hopf cocommutative. On a donc pour chaque $\mathfrak{g}$-module V une action de $\mathfrak{S}_n$ sur $V^{\otimes n}$ par $\mathfrak{g}$-automorphismes. En suivant la philosophie de la quantification par déformation, on s'attend à ce que ce résultat soit « presque » vrai dans le cas de modules sur le groupe quantique $U_\hbar(\mathfrak{g})$. Ce « presque » se traduit formellement par le fait qu'il faut déformer l'action de $\mathfrak{S}_n$ en une action du groupe de tresses B_n, qui apparait donc comme une façon de contrôler la non commutativité du produit tensoriel de $U_\hbar(\mathfrak{g})$-modules. C'est ce qui a motivé l'introduction par Drinfeld de la notion d'algèbre de Hopf quasi-triangulaire, dont les groupes quantiques forment les principaux exemples : c'est une paire $(A, \mathcal{R})$ où A est une algèbre de Hopf et $\mathcal{R}$ un élément inversible de $A^{\otimes 2}$ satisfaisant une série d'axiomes. Ces axiomes impliquent en particulier que A est presque cocommutative (c'est-à-dire cocommutative à conjugaison par $\mathcal{R}$ près), et que $\mathcal{R}$ est solution de l'équation de Yang-Baxter quantique

$$\mathcal{R}^{1,2}\mathcal{R}^{1,3}\mathcal{R}^{2,3} = \mathcal{R}^{2,3}\mathcal{R}^{1,3}\mathcal{R}^{1,2}$$

Ainsi, pour tout A-module $\tilde{V}$, l'application

$$\sigma_i \longmapsto \left(P \circ \mathcal{R}_{|\tilde{V}\otimes\tilde{V}}\right)^{i,i+1}$$

induit une action de B_n sur $\tilde{V}^{\otimes n}$. En appliquant ce résultat au cas de $U_\hbar(\mathfrak{g})$, on obtient une représentation algébrique et explicite de B_n

$$\rho_\hbar : B_n \longrightarrow GL(V_\hbar^{\otimes n})$$

pour tout $U_\hbar(\mathfrak{g})$-module $V_\hbar$.

Drinfeld a prouvé l'existence d'un isomorphisme d'algèbres $U(\mathfrak{g})[[\hbar]] \to U_\hbar(\mathfrak{g})$ et donc d'une équivalence de catégories entre $\mathfrak{g}$-**mod** et $U_\hbar(\mathfrak{g})$-**mod**. Ainsi, pour tout $\mathfrak{g}$-module V, l'action de $\mathcal{R}_\hbar$ sur le $U_\hbar(\mathfrak{g})$ module correspondant à V induit une action de B_n sur $V^{\otimes n}[[\hbar]]$, ce qui permet de donner un sens à l'énoncé :

Théorème (Kohno [**Ko**], Drinfeld [**Dr3**, **Dr4**]). *Les représentations $\rho_\hbar$ et ρ_{KZ} sont équivalentes.*

Il s'agit d'un résultat de quantification en ce sens que les représentations qui entrent en jeu sont respectivement associées à une solution de l'équation de Yang-Baxter classique et à une solution de l'équation de Yang-Baxter quantique. Drinfeld obtient cette équivalence comme corollaire d'un résultat plus fort. Il commence par introduire la notion d'algèbre de quasi-Hopf quasi-triangulaire (QTQHA) en affaiblissant les axiomes d'algèbres de Hopf : c'est un triplet $(A, \mathcal{R}, \Phi)$ satisfaisant une série d'axiomes. Ces structures induisent également des représentations de B_n. Les QTQHA peuvent être modifiées par une opération appelée « twist » qui préserve les classes d'équivalence de représentations de B_n correspondantes. Drinfeld construit alors à l'aide du système KZ un *associateur* $\Phi_{\mathrm{KZ}} \in U(\mathfrak{g})^{\otimes 3}[[\hbar]]$ qui exprime universellement l'holonomie de la connexion KZ le long d'un chemin joignant les singularités de Y_n correspondant aux générateurs de B_n. Il prouve alors que $(U(\mathfrak{g})[[\hbar]], \mathcal{R}_{\mathrm{KZ}} = \exp(\hbar t/2), \Phi_{\mathrm{KZ}})$ est une QTQHA. Du côté quantique, $(U_\hbar(\mathfrak{g}), \mathcal{R}_\hbar)$ est vue comme une QTQHA en posant $\Phi = 1$. Finalement, en utilisant une interprétation cohomologique des équations satisfaites par Φ et des arguments de théorie de la déformation, il prouve que ces deux QTQHA sont twist-équivalentes ce qui implique le résultat.

Ce résultat donne une justification *a posteriori* au petit miracle qu'est l'existence de $U_\hbar(\mathfrak{g})$. Il a donc ouvert la voie à une stratégie de quantification par twist qui utilise de manière cruciale le langage des associateurs (citons par exemple le théorème de quantification d'Etingof-Kazhdan [**En1**, **EK**]).

On s'intéresse dans cette thèse à une variante cyclotomique du système KZ qui apparaît notamment dans [**En2**, **EE2**]. On se donne, en plus de $\mathfrak{g}$ et t, un automorphisme σ de $\mathfrak{g}$ tel que $\sigma^N = \mathrm{Id}$ et tel que $\mathfrak{h} = \mathfrak{g}^\sigma$ soit une sous-algèbre de Cartan. Au triplet $(\mathfrak{g}, t, \sigma)$ et au choix d'un $\mathfrak{g}$-module V et d'un $\mathfrak{h}$-module W, on associe une connexion sur le fibré trivial de base

$$X_{n,N} = \{(z_1, \dots, z_n) \in (\mathbb{C}^\times)^n \mid i \neq j \Rightarrow \forall \zeta \in \mu_N, z_i \neq \zeta z_j\}$$

(où μ_N est le groupe des racines Nièmes de l'unité) et de fibre $W \otimes V^{\otimes n}[[\hbar]]$. Cette connexion est $(\mathbb{Z}/N\mathbb{Z})^n \rtimes \mathfrak{S}_n$-équivariante, et induit donc une représentation de monodromie

$$\rho_{\mathrm{KZ}} : B_n^1 \cong \pi_1(X_{n,N}/((\mathbb{Z}/N\mathbb{Z})^n \rtimes \mathfrak{S}_n) \longrightarrow GL(W \otimes V^{\otimes n}[[\hbar]])$$

où B_n^1 est le groupe de tresses généralisé associé au groupe de Coxeter de type B.

On commence par montrer que si A est une algèbre, $\mathcal{R} \in (A^{\otimes 2})^\times$ une solution de l'équation de Yang–Baxter quantique, C une sous algèbre de A et E, K des éléments de $(C \otimes A)^\times$ satisfaisant certains axiomes, les représentations de B_n obtenues grâce au couple $(A, \mathcal{R})$ s'étendent en des représentations de B_n^1, l'image du générateur d'Artin additionnel étant défini par

$$\tau \longmapsto \prod_{i=1}^{n} (\mathcal{R}^{1,i})^{-1} \prod_{i=1}^{n} K^{1,i} E^{0,1} \qquad (\star)$$

De tels éléments peuvent être obtenus de la façon suivante (voir section 2.3) : on pose $(A,\mathcal{R}) = (U_\hbar(\mathfrak{g}) \rtimes_\sigma \mathbb{Z}, \mathcal{R}_\hbar)$, $C = U_\hbar(\mathfrak{h})$,

$$K = \exp(\hbar t_\mathfrak{h}/2) \qquad E = \exp(\hbar(t_\mathfrak{h} + \frac{1}{2}t_\mathfrak{h}^{2,2}))(1\otimes\tilde{\sigma}_\hbar)$$

L'isomorphisme de Drinfeld se restreint en un isomorphisme d'algèbre $U_\hbar(\mathfrak{h}) \cong U(\mathfrak{h})[[\hbar]]$, induisant une équivalence de catégorie entre $\mathfrak{h}$ **-mod** et $U_\hbar(\mathfrak{h})$ **-mod**. Via cette équivalence, les formules précédentes définissent donc une représentation

$$\rho_\hbar : B_n^1 \longrightarrow GL(W\otimes V^{\otimes n}[[\hbar]])$$

pour tout $U(\mathfrak{g}) \rtimes_\sigma \mathbb{Z}$-module V et tout $\mathfrak{h}$-module W. Le principal résultat de cette thèse est le suivant :

Théorème. *Les représentations $\rho_\hbar$ et ρ_{KZ} de B_n^1 sont équivalentes.*

La preuve de ce résultat, largement inspirée par celle de Drinfeld, repose aussi sur la twist-équivalence de deux structures algébriques. Enriquez a introduit dans [**En2**] la notion d'algèbre de quasi-réflexion (QRA) sur une QTQHA c'est une version « quasi » de la notion d'algèbre comodule sur une bialgèbre, à laquelle on ajoute une notion de tressage. Une QRA est donc un triplet (B,Ψ,E) où B est une algèbre et Ψ, E des éléments satisfaisant certains axiomes (notamment le pentagone mixte et l'octogone). Si (B,Ψ,E) est une QRA sur une QTQBA (A,R,Φ), chaque couple $(\tilde{V},\tilde{W}) \in A$ **-mod** $\times B$ **-mod** fournit une représentation de B_n^1 sur $\tilde{W}\otimes\tilde{V}^{\otimes n}$ compatible avec les représentations de B_n sur $\tilde{V}^{\otimes n}$. Là encore les QRA peuvent être modifiées par un twist qui préserve les classes d'équivalence de représentations de B_n^1.

La connexion KZ cyclotomique permet de construire une QRA

$$(U(\mathfrak{h})[[\hbar]], E_{\text{KZ}}, \Psi_{\text{KZ}})$$

sur la QTQHA provenant du système KZ. On s'attellera donc dans un premier temps à la construction d'une QRA sur le groupe quantique $(U_\hbar(\mathfrak{g}),\mathcal{R})$. L'équation du pentagone mixte est une version algébrique d'une équation fonctionnelle dont les solutions sont appelées twists dynamiques dans la littérature. Les auteurs de [**ABRR**, **Ba**, **ESS**, **ES2**] construisent des twists dynamiques à partir d'une équation linéaire (l'équation ABRR). D'un autre côté, les auteurs de [**EE1**, **EEM**] construisent un twist dynamique algébrique à partir d'un analogue quantique de la forme de Shapovalov. Dans ce dernier cas, le twist dynamique s'identifie à un élément d'une localisation appropriée de $U_\hbar(\mathfrak{h})\otimes U_\hbar(\mathfrak{g})^{\otimes 2}$. On montre que le twist dynamique ainsi obtenu satisfait une variante algébrique de l'équation ABRR. En appliquant un décalage à sa partie fonctionnelle, on obtient un twist dynamique qui satisfait une version modifiée d'ABRR faisant intervenir l'automorphisme σ, à partir de laquelle on donne une preuve directe des équations du pentagone mixte et de l'octogone. On prouve que le décalage permet d'éviter les pôles du twist dynamique et donc de l'identifier à un élément de l'algèbre non localisée $U_\hbar(\mathfrak{h})\otimes U_\hbar(\mathfrak{g})^{\otimes 2}$. On obtient finalement une QRA sur $U_\hbar(\mathfrak{g})$ dont les représentations de B_n^1 coïncident avec celles données par ($\star$).

En appliquant le twist du Théorème de Kohno–Drinfeld original à cette construction, on obtient une deuxième QRA sur $(U(\mathfrak{g})[[\hbar]],\mathcal{R}_{\text{KZ}},\Phi_{\text{KZ}})$. Dans la seconde partie de ce travail, on prouve qu'elle est twist-équivalente à la QRA provenant du système KZ cyclotomique. La preuve repose sur une classification des twists dynamiques basée sur des arguments cohomologiques et des résultats d'Etingof–Varchenko sur les r-matrices dynamiques [**EV**], ainsi que sur des arguments de rigidité imposée par l'équation de l'octogone.

Organisation de ce document

Dans le premier chapitre, nous rappelons un certain nombre de résultats et de constructions en esquissant la preuve du théorème de Kohno–Drinfeld. Le deuxième chapitre contient la définition du principal objet d'étude de cette thèse, à savoir la connexion KZ cyclotomique, ainsi que les principaux résultats obtenus. Le court chapitre 3 donne une preuve directe que les formules ($\star$) donnent effectivement des représentations de B_n^1. Dans le quatrième chapitre, nous rappelons la notion d'algèbre de quasi-réflexion ainsi que le lien avec la connexion KZ cyclotomique. Le chapitre 5 est consacré à la construction d'une QRA quantique, en établissant des liens explicites entre ABRR, l'octogone et l'équation du twist dynamique. Enfin, le principal résultat du chapitre 6 est une classification, modulo quelques conditions de non dégénérescence, des solutions du pentagone mixte associé à $\mathfrak{h}$ et Φ_{KZ}. Dans l'appendice, on clarifie la relation entre la connexions KZ cyclotomique et les groupes de tresses complexes. Les chapitres 2 à 6 sont essentiellement tirés de la prépublication [**Bro**].

English introduction

The Kohno–Drinfeld theorem is a result related to the *quantization* of some algebraic structures, which leads to an equivalence between two families of representations of the Artin braid group [**Ar**] occurring in mathematical physics topics. The first one comes from the monodromy of the Knizhnik–Zamolodchikov differential system [**KZ**], the second one from an explicit algebraic construction involving quantum groups. This result has motivated the study of the monodromy of different analogs of the KZ connexion, leading to generalizations of the Kohno–Drinfeld theorem [**EG**, **GL**, **To**].

The main goal of this thesis is the generalization of this theorem for the monodromy of "cyclotomic" analogs of the KZ system, leading to representations of the braid group of Coxeter type B.

The pure braid group P_n is defined as the fundamental group of the configuration space Y_n of n particles in the plane

$$Y_n = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid i \neq j \Rightarrow z_i \neq z_j\}$$

and the full braid group B_n as the fundamental group of the configuration space of n indistinguishable particles $X_n = Y_n/\mathfrak{S}_n$. To the data of a simple Lie algebra $\mathfrak{g}$, a finite dimensional $\mathfrak{g}$-module V and a symmetric $\mathfrak{g}$-invariant 2-tensor t, one associates a differential system leading to a connection on the trivial bundle with base space Y_n and fiber $V^{\otimes n}[[\hbar]]$, where $\hbar$ is a formal variable. Being $\mathfrak{S}_n$-equivariant, it induces a connection on the bundle $(Y_n \times V^{\otimes n}[[\hbar]])/\mathfrak{S}_n$. Being flat, it gives rise to a monodromy morphism

$$\rho_{\mathrm{KZ}} : B_n \to GL(V^{\otimes n}[[\hbar]]).$$

The KZ connection is flat because t satisfies an algebraic equation closely related to the Classical Yang–Baxter equation, which can be thought of as an infinitesimal version (a "first order approximation") of the defining relations of B_n.

On the algebraic side, we consider a construction arising in the framework of the *deformation quantization* theory, as formulated by Bayen *et al* in the 80's [**BFF$^+$**], which has its origin in physics: the quantization of a "classical" physic system is a "quantum" physic system which reduces to the classical one if the Planck constant is neglected. Mathematically, one consider objects depending on a formal parameter $\hbar$ which reduces to a given classical object at the limit $\hbar \to 0$. Drinfeld [**Dr1**] and Jimbo [**Ji**] associated to any simple Lie algebra $\mathfrak{g}$ a *quantum group*, that is a $\mathbb{C}[[\hbar]]$-Hopf algebra $U_\hbar(\mathfrak{g})$ which is a deformation of the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$.

One of the first motivation for the definition of quantum groups is the fact that they leads to universal solution of the Quantum Yang–Baxter Equation, which in particular implies that they give rise to representations of B_n. Indeed, the tensor product of $\mathfrak{g}$-modules is commutative in the following sense: if V_1, V_2 are two finite dimensional $\mathfrak{g}$-modules, the natural vector space isomorphism

$$P : V_1 \otimes V_2 \longrightarrow V_2 \otimes V_1$$

commutes with the action of $\mathfrak{g}$. In other words, $U(\mathfrak{g})$ has a natural *cocommutative* Hopf algebra structure. Hence, one get for each $\mathfrak{g}$-module V an action of $\mathfrak{S}_n$ on $V^{\otimes n}$ by $\mathfrak{g}$-automorphisms. Following the quantization deformation philosophy, it is expected that this results remains "almost true" in the case of modules over $U_\hbar(\mathfrak{g})$. Indeed, the tensor product of $U_\hbar(\mathfrak{g})$-modules is almost commutative in the sense that one has to replace the $\mathfrak{S}_n$ action by an action of B_n. Indeed, quantum groups are the main examples of Quasi-Triangular Hopf algebras, a notion introduced by Drinfeld: this is a pair $(A, \mathcal{R})$ where A is a Hopf algebra and $\mathcal{R}$ an invertible element of $A^{\otimes 2}$ satisfying a set of axioms. These axioms imply that A is almost cocommutative (i.e. cocommutative up to conjugation by $\mathcal{R}$), and that $\mathcal{R}$ satisfies the quantum Yang–Baxter equation

$$\mathcal{R}^{1,2}\mathcal{R}^{1,3}\mathcal{R}^{2,3} = \mathcal{R}^{2,3}\mathcal{R}^{1,3}\mathcal{R}^{1,2}$$

Therefore, the map

$$\sigma_i \longmapsto \left(P \circ \mathcal{R}_{|\tilde{V}\otimes\tilde{V}}\right)^{i,i+1}$$

induces an action of B_n on $\tilde{V}^{\otimes n}$ for each A-module $\tilde{V}$. In particular, one get an algebraic, explicit representation of B_n

$$\rho_\hbar : B_n \longrightarrow GL(V_\hbar^{\otimes n})$$

for each $U_\hbar(\mathfrak{g})$-module $V_\hbar$.

Drinfeld proved the existence of an algebra isomorphism $U(\mathfrak{g})[[\hbar]] \to U_\hbar(\mathfrak{g})$, hence of an equivalence between the categories $\mathfrak{g}$ **-mod** and $U_\hbar(\mathfrak{g})$ **-mod**. Therefore, if V is the $\mathfrak{g}$-module corresponding to $U_\hbar(\mathfrak{g})$, the representation $\rho_\hbar$ can be viewed as a representation on $V^{\otimes n}[[\hbar]]$. The Kohno–Drinfeld theorem is the following:

Theorem (Kohno [**Ko**], Drinfeld [**Dr3**, **Dr4**])**.** *The two representations $\rho_\hbar$ et ρ_{KZ} are equivalent.*

It is a quantization result because it relates representations of B_n associated to a solution of the classical and the quantum Yang-Baxter equation respectively. In Drinfeld's proof, it follows from a stronger result. He first define the notion of Quasi-Triangular Quasi-Hopf algebra (QTQHA) by weakening the defining axioms of Hopf algebras: it is a tuple $(A, \mathcal{R}, \Phi)$ satisfying a set of axioms. These structures also leads to representations of B_n. QTQHA's can be modified by an operation called "twist", which preserves the equivalence classes of representations of B_n. Drinfeld construct an *associator* Φ_{KZ} using the KZ system which is the universal holonomy along a path joining the singularities of Y_n corresponding to the generators of B_n. He proves that $(U(\mathfrak{g})[[\hbar]], \mathcal{R}_{\mathrm{KZ}} = \exp(\hbar t/2), \Phi_{\mathrm{KZ}})$ is a QTQHA. On the quantum side, $(U_\hbar(\mathfrak{g}), \mathcal{R}_\hbar)$ is seen as a QTQHA by setting $\Phi = 1$. Finally, using a cohomological interpretation of the equations satisfied by Φ and deformation theory arguments, he proves that these two QTQHA are twist equivalent, implying the result.

This results gives an "explanation" of the existence of $U_\hbar(\mathfrak{g})$. Hence, it leads to a general strategy of "twist-quantization" of algebraic structure using the language of associators (one of the main example is the Etingof–Kazhdan quantization theorem [**En1**, **EK**]).

We study in this thesis a cyclotomic analog of the KZ connection which was defined in [**En2**, **EE2**]. One first choose an automorphism σ of $\mathfrak{g}$ satisfying $\sigma^N = \mathrm{Id}$ and such that $\mathfrak{h} = \mathfrak{g}^\sigma$ is a Cartan subalgebra. To the data of $(\mathfrak{g}, t, \sigma)$, a $\mathfrak{g}$-module V and a $\mathfrak{h}$-module W, is associated a connection on the trivial fiber bundle with base space

$$Y_{n,N} = \{(z_1, \ldots, z_n) \in (\mathbb{C}^\times)^n \mid i \neq j \Rightarrow \forall \zeta \in \mu_N, z_i \neq \zeta z_j\}$$

(where μ_N is the group of Nth roots of) and fiber $W \otimes V^{\otimes n}[[\hbar]]$. This connection is $(\mathbb{Z}/N\mathbb{Z})^n \rtimes \mathfrak{S}_n$-equivariant, hence gives rise to a monodromy representation

$$\rho_{\mathrm{KZ}} : B_n^1 \cong \pi_1(Y_{n,N}/((\mathbb{Z}/N\mathbb{Z})^n \rtimes \mathfrak{S}_n) \longrightarrow GL(W \otimes V^{\otimes n}[[\hbar]])$$

where B_n^1 is the generalized braid group of Coxeter type B.

We first show that if A is an algebra, $\mathcal{R} \in (A^{\otimes 2})^\times$ a solution of the quantum Yang–Baxter equation, C is a subalgebra of A and E, K are elements of $(C \otimes A)^\times$ satisfying a set of axioms, the representations of B_n coming from the pair $(A, \mathcal{R})$ are extended by setting

$$\tau \longmapsto \prod_{i=1}^{n} (\mathcal{R}^{1,i})^{-1} \prod_{i=1}^{n} K^{1,i} E^{0,1} \tag{$\star$}$$

where τ is the additional Artin generator of B_n^1. Such data can be constructed as follows: $(A, \mathcal{R}) = (U_\hbar(\mathfrak{g}) \rtimes \mathbb{Z}, \mathcal{R}_\hbar)$, $C = U_\hbar(\mathfrak{h})$,

$$K = e^{\hbar t_{\mathfrak{h}}/2} \qquad\qquad E = e^{\hbar(t_{\mathfrak{h}} + \frac{1}{2} t_{\mathfrak{h}}^{2,2})}(1 \otimes \tilde{\sigma}_\hbar)$$

The Drinfeld isomorphism induces an algebra isomorphism $U_\hbar(\mathfrak{h}) \cong U(\mathfrak{h})[[\hbar]]$, and therefore an equivalence of categories between $\mathfrak{h}$**-mod** and $U_\hbar(\mathfrak{h})$**-mod**, denoted by $W \mapsto W_\hbar$. Again, $W_\hbar \cong W[[\hbar]]$ as a $\mathbb{C}[[\hbar]]$-module. Thus, any pair $(W, V) \in \mathfrak{h}$**-mod** $\times \mathfrak{g}$**-mod** gives rise to a representation of B_n^1 in $W \otimes V^{\otimes n}[[\hbar]]$.

The main result of the thesis is the following generalization of the Kohno–Drinfeld theorem:

Theorem. *These two representations of B_n^1 in $W \otimes V^{\otimes n}[[\hbar]]$ are equivalent.*

The proof of this result also follows from the twist equivalence between two algebraic structures. Enriquez introduced in [**En2**] the notion of Quasi-Reflection Algebra (QRA): this is a "quasi" version of the notion of comodule algebra over a bialgebra, together with a notion of braiding. Hence, a QRA is a tuple (B,Ψ,E) where B is an algebra and Ψ,E are invertible elements satisfying suitable axioms (the octagon and the mixed pentagon). If (B,Ψ,E) is a QRA over a QTQBA (A,R,Φ), each $(\tilde{V},\tilde{W})\in A\text{-}\mathbf{mod}\times B\text{-}\mathbf{mod}$ gives rise to a representation of B_n^1 in $\tilde{W}\otimes\tilde{V}^{\otimes n}$ which is compatible with the representation of B_n on $\tilde{V}^{\otimes n}$. As before, one defines a "twist" operation for QRAs over QTQBAs, which does not change the equivalence classes of representations of B_n^1.

The cyclotomic KZ connection allows one to construct a QRA

$$(U(\mathfrak{h})[[\hbar]],E_{\mathrm{KZ}},\Psi_{\mathrm{KZ}})$$

over the QTQHA $(U(\mathfrak{g})[[\hbar]],\mathcal{R}_{\mathrm{KZ}},\Phi_{\mathrm{KZ}})$. Therefore, our first goal will be the construction of a QRA over the quantum group $U_\hbar(\mathfrak{g})$. The mixed pentagon equation is an algebraic version of a functional equation, the solutions of which are called dynamical twists in the literature. The authors of [**ABRR**, **Ba**, **ESS**, **ES2**] construct a dynamical twist by using a specific linear equation (the ABRR equation). On the other hand, the authors of [**EE1**, **EEM**] construct an algebraic dynamical twist using a quantum analog of the Shapovalov form. It is an element of a suitable localization of $U_\hbar(\mathfrak{h})\otimes U_\hbar(\mathfrak{g})^{\otimes 2}$. We first show that it satisfies an algebraic analog of the ABRR equation. By applying a shift to its functional part, one obtains a dynamical twist satisfying a modified ABRR equation involving the automorphism σ. We use this modified ABRR relation to give a direct proof of the mixed pentagon and the octagon equation. We show that the shift allows to avoid the poles of the dynamical twist, which is thus identified to an element of the *non localized* algebra $U_\hbar(\mathfrak{h})\otimes U_\hbar(\mathfrak{g})^{\otimes 2}$. Finally, we obtain a quantum QRA over $U_\hbar(\mathfrak{g})$, and we show that the corresponding representations of B_n^1 are actually given by ($\star$).

Applying the twist of the original Kohno–Drinfeld Theorem, on get another QRA over $(U(\mathfrak{g})[[\hbar]],\mathcal{R}_{\mathrm{KZ}},\Phi_{\mathrm{KZ}})$. In the second part of this work, we prove that the latter is twist equivalent to the QRA arising from the cyclotomic KZ connection. The proof is based on a cohomological interpretation of the mixed pentagon equation, on the classification of dynamical r-matrices by Etingof–Varchencko [**EV**], and on rigidity arguments related to the octagon equation.

Outline of this document

In the first chapter, we recall some results and constructions by sketching the proof of the Kohno–Drinfeld theorem. The second chapter contains the definition of the main object of study of this thesis, that is the cyclotomic KZ connection, and the statements of the main results. The short third chapter provides a direct proof that the formulae ($\star$) gives a representation of B_n^1. In the fourth chapter, we recall the notion of quasi-reflection algebra as well as its relationship with the cyclotomic KZ connection. The chapter 5 is devoted to the construction of a quantum QRA, by establishing explicit relations between ABRR, the octagon and the dynamical twist equation. Finally, the main result of the chapter 6 is a classification of the solutions of the mixed pentagon equation associated to $\mathfrak{h}$ and Φ_{KZ}. In the appendix, we clarify the relationship between the cyclotomic KZ connection and complex braid groups. The chapters 2 to 6 are mostly taken from the preprint [**Bro**].

CHAPTER 1

Knizhnik–Zamolodchikov connection and the Kohno–Drinfeld Theorem

In this chapter, we sketch the proof of the original Kohno–Drinfeld Theorem and recall some related definitions and results which will be used in the sequel. We refer the reader to the original papers of Drinfeld [Dr1, Dr2, Dr3, Dr4] as well as to [CP, ES1, Kas, SS] for detailed proof. As many proofs in Chapter 5 are based on quite explicit computations, hence requiring precise normalizations, let us also mention that we follow the conventions of [CP] for the definition of $U_\hbar(\mathfrak{g})$.

1.1. Notations and conventions

Let $\mathfrak{g}$ be a simple $\mathbb{C}$-Lie algebra, $\mathfrak{h}$ a Cartan subalgebra. Recall that $\mathfrak{g}$ admits the following decomposition:

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\lambda \in \mathfrak{h}^* - \{0\}} \mathfrak{g}_\lambda$$

where $\mathfrak{g}_\lambda = \{x \in \mathfrak{g},\ \forall h \in \mathfrak{h},\ [h,x] = \lambda(h)x\}$. Let R be the root system associated to $\mathfrak{h}$, that is, the finite set

$$\{\alpha \in \mathfrak{h}^* \mid \mathfrak{g}_\alpha \neq \{0\}\}$$

For each root $\alpha \in R$, denote by h_α the unique element of $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subset \mathfrak{h}$ such that $\alpha(h_\alpha) = 2$. Let $(\alpha_i)_{i=1\ldots r}$ be a choice of simple roots, and set $h_i = h_{\alpha_i}$. For each positive root α, let e_α be a nonzero element of $\mathfrak{g}_\alpha$ and $e_{-\alpha}$ be the unique element of $\mathfrak{g}_{-\alpha}$ such that $[e_\alpha, e_{-\alpha}] = h_\alpha$. We set $e_i^\pm = e_{\pm\alpha_i}$. Recall that $\{e_i^\pm, h_i\}$, $i = 1\ldots r$ is a generating set of $\mathfrak{g}$, and that $\{e_\alpha, h_i\}$, $i = 1 \ldots r$, $\alpha \in R$ is a basis of $\mathfrak{g}$. Let $\mathfrak{n}^\pm$ be the Lie subalgebra generated by $(e_i^\pm)_{i=1\ldots r}$.

A simple Lie algebra $\mathfrak{g}$ is completely determined by its Cartan matrix $A = (a_{i,j})$ where $a_{i,j} = \alpha_j(h_i)$. The matrix A is symmetrizable in the following sense: there exists a unique set $\{d_1, \ldots, d_r\}$ of coprime positive integers such that the matrix $B = (d_i a_{i,j})$ is symmetric.

There exists a non-degenerate symmetric $\mathfrak{g}$-invariant bilinear form on $\mathfrak{g}$, unique up to scalar multiplication. It can be normalized in such a way that $(h_i, h_j) = d_j^{-1} a_{i,j}$.

Let t be a symmetric $\mathfrak{g}$-invariant element of $\mathfrak{g}^{\otimes 2}$. It will be convenient for explicit computations to fix $t = 2(\ ,\ )^{-1}$. Let $t_{\mathfrak{h}}$ be the image of t in $S^2(\mathfrak{h})$ through the projection $\mathfrak{g} \to \mathfrak{h}$ relative to the above decomposition.

By an "algebra", we always mean an associative algebra with unit, over the field $\mathbb{C}$ or the ring of formal power series $\mathbb{C}[[\hbar]]$. Recall that a $\mathbb{C}[[\hbar]]$-module V is called *topologically free* if it is isomorphic to $V_0[[\hbar]]$ for some $\mathbb{C}$-vector space V_0. We will always assume that modules (and in particular $\mathbb{C}[[\hbar]]$-algebras) are topologically free.

If V, W are two topologically free $\mathbb{C}[[\hbar]]$-modules respectively isomorphic to $V_0[[\hbar]]$ and $W_0[[\hbar]]$, define the completed tensor product in the $\hbar$-adic topology by:

$$V \hat{\otimes} W := \lim_{\leftarrow} (V \otimes_{\mathbb{C}[[\hbar]]} W)/(\hbar^n (V \otimes_{\mathbb{C}[[\hbar]]} W)).$$

The module $V \hat{\otimes} W$ is also topologically free. More precisely:

$$V \hat{\otimes} W \cong (V_0 \otimes_{\mathbb{C}} W_0)[[\hbar]]$$

If A is a (Lie) algebra over $\mathbb{C}$, we denote by A**-mod** the category of its finite dimensional modules. If A is defined over $\mathbb{C}[[\hbar]]$, then A**-mod** is the category of A-modules which are topologically free $\mathbb{C}[[\hbar]]$-modules of finite type.

1.2. Braids, the KZ system and monodromy

1.2.1. Braids and configuration spaces. Let Y_n be the topological space

$$Y_n = \mathbb{C}^n - \bigcup_{1 \le i,j \le n} H_{i,j}$$

where $H_{i,j}$ is the hyperplane $\{(z_1, \ldots, z_n) | z_i = z_j\}$. The pure braid group P_n is the fundamental group of Y_n. There is a natural action of the symmetric group $\mathfrak{S}_n$ on Y_n by permutation of the coordinates. The braid group B_n is defined as the fundamental group of $Y_n/\mathfrak{S}_n$. The natural map $Y_n \to Y_n/\mathfrak{S}_n$ is a Galois covering, hence induces a short exact sequence :

$$1 \to P_n \to B_n \xrightarrow{p} \mathfrak{S}_n \to 1$$

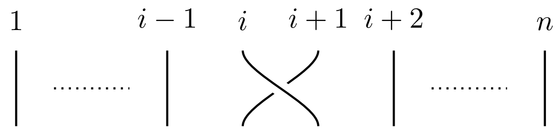
A presentation of B_n is obtained from that of $\mathfrak{S}_n$ by removing the torsion relations:

Theorem 1.1 (Artin)**.**

$$B_n = \langle \sigma_1, \ldots, \sigma_{n-1} \mid \sigma_i \sigma_j = \sigma_j \sigma_i \qquad \textit{if } |i-j| > 1$$
$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \rangle$$

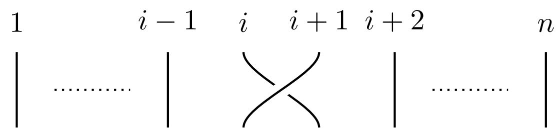
The projection p sends the generator σ_i to the transposition $(i, i+1)$. The generators can be depicted as follows:

$\sigma_i =$



and

$\sigma_i^{-1} =$



1.2.2. The KZ differential system. Let $\mathfrak{g}$ be a (semi)simple $\mathbb{C}$-Lie algebra and $V_1, \ldots, V_n$ be finite dimensional $\mathfrak{g}$-modules. The most general form of the KZ differential system is:

$$\frac{\partial F(x_1, \ldots, z_n)}{\partial z_i} = \frac{\hbar}{2\pi\sqrt{-1}} \left(\sum_{\substack{j=1 \\ j \neq i}}^{n} r^{i,j}(z_i - z_j) \right) F(z_1, \ldots, z_n)$$

where F is a function on Y_n with values in $V_1 \otimes \ldots \otimes V_n$. Here, $r(z)$ is a holomorphic function from $\mathbb{C}^\times$ to $\mathfrak{g} \otimes \mathfrak{g}$ and $r^{i,j}(z)$ is seen as a function from $\mathbb{C}^\times$ to $End(V_1 \otimes \ldots V_n)$. The solutions of the KZ system are the sections of the connection

$$\nabla = d - \frac{\hbar}{2\pi\sqrt{-1}} \sum_{i<j} r^{i,j}(z_i - z_j)(dz_i - dz_j)$$

on the trivial vector bundle $Y_n \times (V_1 \otimes \ldots V_n)[[\hbar]]$.

Proposition 1.2. *The connection ∇ is flat if and only if $r(z)$ is a solution of the Classical Yang–Baxter Equation (CYBE) with spectral parameter:*

$$[r^{1,2}(z_1 - z_2), r^{1,3}(z_1 - z_3) + r^{2,3}(z_2 - z_3)] + [r^{1,3}(z_1 - z_3), r^{2,3}(z_2 - z_3)] = 0$$

The simplest solution of the CYBE is $r(z) = \frac{t}{z}$ where t is an element of $S^2(\mathfrak{g})^{\mathfrak{g}}$. Recall that such an element is uniquely determined up to multiplication by a scalar. The corresponding system is:

$$\frac{\partial F(x_1, \ldots, z_n)}{\partial z_i} = \frac{\hbar}{2\pi\sqrt{-1}} \left(\sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{t^{i,j}}{z_i - z_j} \right) F(z_1, \ldots, z_n) \qquad (\mathrm{KZ}_n)$$

If $\gamma : [0,1] \to Y_n$ is a smooth path, the connection ∇ allows to lift γ uniquely to a vector space isomorphism (the holonomy along γ) M_γ from the fiber over $\gamma(0)$ to the fiber over $\gamma(1)$:

$$M_\gamma : (V_1 \otimes \ldots \otimes V_n[[\hbar]])_{\gamma(0)} \longrightarrow (V_1 \otimes \ldots \otimes V_n[[\hbar]])_{\gamma(1)}$$

This construction is compatible with the composition of paths, and the flatness of the connection implies that M_γ depends only on the homotopy class of γ. Hence, it leads to a group morphism

$$\begin{array}{ccc} P_n & \longrightarrow & GL(V_1 \otimes \ldots \otimes V_n[[\hbar]]) \\ [\gamma] & \longmapsto & M_{[\gamma]} \end{array}$$

called the monodromy morphism.

Assuming moreover that $V_1 = \cdots = V_n = V$, one gets an action of $\mathfrak{S}_n$ on each fiber. It is easily seen that the connection associated to (KZ$_n$) is $\mathfrak{S}_n$-equivariant, meaning that it leads to a connection on the (non-trivial) bundle $(Y_n \times V^{\otimes n}[[\hbar]])/\mathfrak{S}_n$. We obtain this way a group morphism

$$\rho_{\mathrm{KZ}} : B_n \longrightarrow GL(V^{\otimes n}[[\hbar]]).$$

1.3. Quantized enveloping algebras and representations of B_n

1.3.1. Bialgebras, deformations. Recall that a bialgebra H over $\mathbb{C}$ is a $\mathbb{C}$-algebra, equipped with algebra morphisms

$$\Delta_H : H \longrightarrow H \otimes H$$

(the coproduct) and

$$\epsilon_H : H \longrightarrow \mathbb{C}$$

(the counit) such that:

- $(\Delta_H \otimes \mathrm{Id}_H) \circ \Delta_H = (\mathrm{Id}_H \otimes \Delta_H) \circ \Delta_H$ (coassociativity)
- $(\epsilon_H \otimes \mathrm{Id}_H) \circ \Delta_H = (\mathrm{Id}_H \otimes \epsilon_H) \circ \Delta_H = \mathrm{Id}_H$

Topological Hopf algebras over $\mathbb{C}[[\hbar]]$ are defined in a similar fashion, the only difference being that the coproduct takes values in the completed tensor product $H \hat{\otimes} H$. A bialgebra is called cocommutative if $\Delta^{2,1} = \Delta$.

A Hopf algebra is a bialgebra H together with an algebra anti-morphism (the antipode)

$$S_H : H \longrightarrow H$$

such that

$$\forall h \in H,\ (S_H \otimes \mathrm{Id}_H) \circ \Delta_H(h) = (\mathrm{Id}_H \otimes S_H) \circ \Delta_H(h) = \epsilon_H(h) 1_H.$$

Example 1.3. Let $\mathfrak{g}$ be a $\mathbb{C}$-Lie algebra. Its universal enveloping algebra is generated as an algebra by the image of $\mathfrak{g}$ through the natural map $\mathfrak{g} \to U(\mathfrak{g})$. For $x \in \mathfrak{g}$, define:

- $\Delta_0(x) = x \otimes 1 + 1 \otimes x$
- $\epsilon(x) = 0$
- $S_0(x) = -x$

Extending algebraically these operations to the whole algebra turns $U(\mathfrak{g})$ into a Hopf algebra.

A short definition of an Hopf algebra is that it is a *group object* in the category of vector spaces. The coproduct replaces the diagonal map and the antipode is the analog of the inversion map. From a representation theoretic point of view, it essentially means that the tensor product over the base field of two H-modules, and the dual of a H-module are again H-modules.

Hence, it is not surprising that the first examples of Hopf algebras are related with (Lie) group theory. The basic intuition is thus to replace a Lie group (actually, the universal enveloping algebra of its Lie algebra) by an Hopf algebra infinitely close to it, hence the name *quantum group.*

Definition 1.4. *Let A be a $\mathbb{C}$-(bi)algebra and $\hbar$ be a formal parameter. A* flat deformation *of A is a (topological) $\mathbb{C}[[\hbar]]$-(bi)algebra $A_\hbar$ such that:*

- *there exists a (bi)algebra isomorphism $A_\hbar/\hbar A_\hbar \longrightarrow A$*

- *$A_\hbar$ is isomorphic to $A[[\hbar]]$ as a $\mathbb{C}[[\hbar]]$-module.*

Two deformations of A are equivalent if they are related by a (bi)algebra isomorphism whose restriction to A is the identity. The deformations of a (bi)algebra are governed by a generalization of the Hochschild cohomology [**CP**, Chap. 6.1]. It leads to the following fundamental results concerning the algebra structure of deformations of $U(\mathfrak{g})$:

Theorem 1.5 (Drinfeld)**.** *Let $\mathfrak{g}$ be a simple Lie algebra. Every deformation of $U(\mathfrak{g})$ is equivalent* as an algebra *to the trivial deformation $U(\mathfrak{g})[[\hbar]]$.*

It shows that what does really matter is the bialgebra structure. Moreover, it can be shown that every cocommutative deformation of $U(\mathfrak{g})$ is equivalent as a bialgebra to the trivial deformation.

1.3.2. Quantized universal enveloping algebras. The results of the previous section do not say much about the *existence* of a nontrivial deformation of $U(\mathfrak{g})$. This question was also settled by Drinfeld who gave an explicit presentation of a "standard" quantization.

Definition 1.6. *Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$, $A = (a_{i,j})$ be its Cartan matrix, d_i, $i = 1 \dots r$ be the unique coprime integers such that $d_i a_{ij} = d_j a_{ji}$. Let $U_\hbar(\mathfrak{g})$ the $\mathbb{C}[[\hbar]]$-algebra topologically generated by $\{e_i^\pm, h_i\}_{i=1\dots r}$ with relations:*

$$[h_i, e_j^\pm] = \pm a_{ij} e_j^\pm \qquad [h_i, h_j] = 0$$

$$[e_i^\pm, e_j^\mp] = \delta_{ij} \frac{e^{\hbar d_i h_i} - e^{-\hbar d_i h_i}}{e^{\hbar d_i} - e^{-\hbar d_i}}$$

together with the quantum analogs of Serre's relations:

$$\sum_{k=0}^{1-a_{ij}} \frac{(-1)^k}{[k]_q! [1 - a_{ij} - k]_q!} (e_i^\pm)^{1-a_{ij}-k} e_j^\pm (e_i^\pm)^k = 0$$

where

$$[x]_q = \frac{e^{\hbar x} - e^{-\hbar x}}{e^\hbar - e^{-\hbar}}$$

and

$$[x]_q! = \prod_{i=1}^{x} [i]_q, \ x \in \mathbb{N}$$

Proposition 1.7. *The algebra morphism*

$$\Delta_\hbar : U_\hbar(\mathfrak{g}) \to U_\hbar(\mathfrak{g}) \hat{\otimes} U_\hbar(\mathfrak{g})$$

defined by:

$$\Delta_\hbar(h_i) = h_i \otimes 1 + 1 \otimes h_i$$

$$\Delta_\hbar(e_i^+) = e_i^+ \otimes e^{\hbar d_i h_i} + 1 \otimes e_i^+ \qquad \Delta_\hbar(e_i^-) = e_i^- \otimes 1 + e^{-\hbar d_i h_i} \otimes e_i^-$$

and the algebra anti-morphism

$$S_\hbar(h_i) = -h_i, \ S_\hbar(e_i^+) = -e_i^+ e^{-\hbar d_i h_i}, \ S_\hbar(e_i^-) = -e^{\hbar d_i h_i} e_i^-$$

turn $(U_\hbar(\mathfrak{g}), \Delta_\hbar, S_\hbar)$ into a topological Hopf algebra which is a flat deformation of $(U(\mathfrak{g}), \Delta_0, S_0)$.

1.3.3. Some properties of $U_\hbar(\mathfrak{g})$.

1.3.3.1. *Triangular decomposition.* Let $U_\hbar(\mathfrak{n}^\pm)$ and $U_\hbar(\mathfrak{h})$ be the subalgebras of $U_\hbar(\mathfrak{g})$ generated by $(e_i^\pm)_{i=1\dots r}$ and $(h_i)_{i=1\dots r}$ respectively.

Proposition 1.8. *The multiplication induces an isomorphism of $\mathbb{C}[[\hbar]]$-modules*

$$U_\hbar(\mathfrak{n}^+)\hat{\otimes}U_\hbar(\mathfrak{h})\hat{\otimes}U_\hbar(\mathfrak{n}^-) \longrightarrow U_\hbar(\mathfrak{g})$$

1.3.3.2. *Rational form.* Let q be a variable and $U_q(\mathfrak{g})$ be the $\mathbb{C}(q)$-algebra generated by $(e_i^\pm, k_i^{\pm 1})$ for $i = 1 \dots r$, with relations

$$k_ik_j = k_ik_j \qquad k_i^{\pm 1}k_i^{\mp 1} = 1$$

$$k_je_i^\pm k_j^{-1} = q^{\pm a_{ji}}e_i \qquad [e_i^+, e_j^-] = \delta_{ij}\frac{k_i^{d_i} - k_i^{-d_i}}{q^{d_i} - q^{-d_i}}$$

and the quantum Serre's relations where $e^\hbar$ is replaced by q. The same formulas as for $U_\hbar(\mathfrak{g})$ turn $U_q(\mathfrak{g})$ into a Hopf algebra. Let $U_q(\mathfrak{n}^+)$ and $U_q(\mathfrak{h})$ be the subalgebras of $U_q(\mathfrak{g})$ generated by $e_i^\pm$ and k_i, k_i^{-1} respectively. We summarize the basic properties of $U_q(\mathfrak{g})$ in the following

Proposition 1.9.

- *The multiplication induces an isomorphism of $\mathbb{C}(q)$-vector spaces*

$$U_q(\mathfrak{n}^+) \otimes U_q(\mathfrak{h}) \otimes U_q(\mathfrak{n}^-) \longrightarrow U_q(\mathfrak{g})$$

- $U_q(\mathfrak{h})$ *is isomorphic as an algebra to* $\mathbb{C}(q)[k_i^{\pm 1}, i = 1 \dots r]$.
- *There is a field morphism $\mathbb{C}(q) \to \mathbb{C}[[\hbar]][\hbar^{-1}]$ given by $q \mapsto e^\hbar$, and a compatible injective Hopf algebra morphism*

$$U_q(\mathfrak{g}) \to U_\hbar(\mathfrak{g})[\hbar^{-1}]$$

given by

$$e_i^\pm \mapsto e_i^\pm \qquad k_i \mapsto e^{\hbar d_i h_i}$$

1.3.4. Braided monoidal categories. A (braided) monoidal category is the categorical analog of a (commutative) monoid, where associativity (and commutativity) holds only up to isomorphism. More precisely, a monoidal category is a category $\mathcal{C}$ together with:

- a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$
- a natural isomorphism $\alpha : (- \otimes -) \otimes - \longrightarrow - \otimes (- \otimes -)$ (the associativity constraint)

such that the following diagram commutes for any objects A, B, C of $\mathcal{C}$:

$$\begin{array}{ccccc}
 & & (A \otimes (B \otimes C)) \otimes D & & \\
 & \overset{\alpha_{A,B,C} \otimes \mathrm{Id}_D}{\nearrow} & & \overset{\alpha_{A,B\otimes C,D}}{\searrow} & \\
((A \otimes B) \otimes C) \otimes D & & & & A \otimes ((B \otimes C) \otimes D) \\
\downarrow \alpha_{A\otimes B,C,D} & & & & \downarrow \mathrm{Id}_A \otimes \alpha_{B,C,D} \\
(A \otimes B) \otimes (C \otimes D) & & \xrightarrow[\alpha_{A,B,C\otimes D}]{} & & A \otimes (B \otimes (C \otimes D))
\end{array}$$

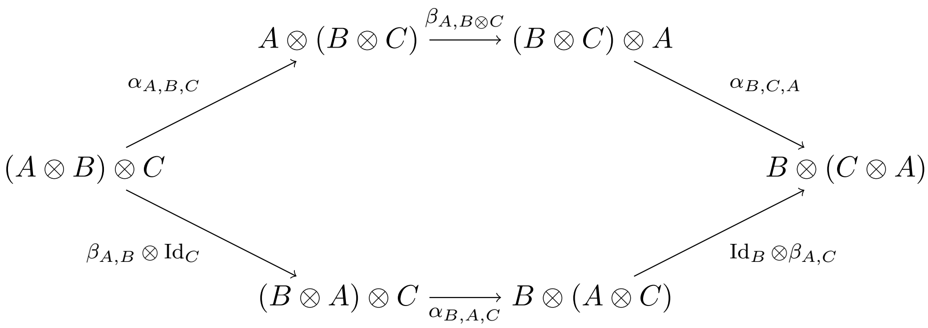
A strict monoidal category is a monoidal category for which $\alpha = \mathrm{Id}$. The axioms for the coproduct of a bialgebra imply the following:

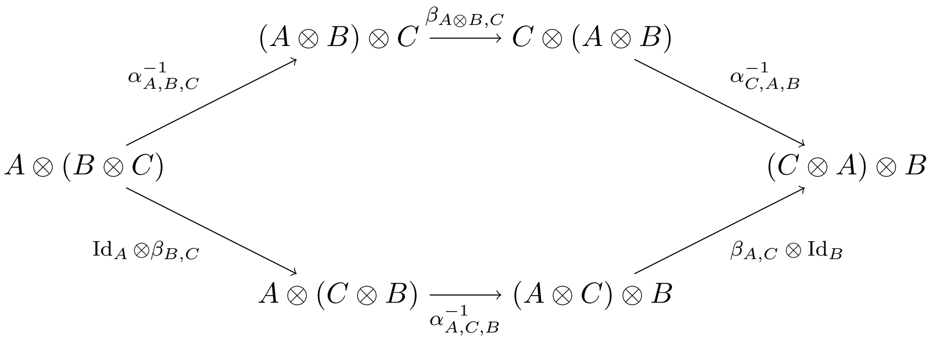
Proposition 1.10. *Let H be a bialgebra, V_1, V_2 be two H-modules. The following action*

$$\forall h \in H, \forall v_1 \otimes v_2 \in V_1 \otimes V_2, h \cdot (v_1 \otimes v_2) := \Delta(h) \cdot v_1 \otimes v_2$$

turns $V_1 \otimes V_2$ into a H-module. Equipped with this operation, the category H-**mod** *of finite dimensional H-modules is a strict monoidal category.*

If H is cocommutative, the tensor product in H-**mod** is commutative in the obvious sense. Relaxing this notion of commutativity leads to the following definition: a braided monoidal category is a monoidal category $(\mathcal{C}, \alpha)$ together with a natural isomorphism $\beta : - \otimes - \to - \otimes^{op} -$ such that the following diagrams commute:





The name "braided" is justified by the following fact:

Proposition 1.11. *Let $(\mathcal{C}, \alpha, \beta)$ be a braided monoidal category, $V_1, \ldots, V_n$ be n copy of the same object V of $\mathcal{C}$ and set*

$$V^{\otimes n} := (\ldots((V_1 \otimes V_2) \otimes V_3) \ldots \otimes V_n).$$

There exists a unique group morphism

$$B_n \longrightarrow \mathrm{Aut}_{\mathcal{C}}(V^{\otimes n})$$

given by

$$\sigma_1 \longmapsto \beta_{V_1,V_2}$$
$$\sigma_i \longmapsto \alpha^{-1}_{V^{\otimes i-1},V_i,V_{i+1}} \circ \beta_{V_i,V_{i+1}} \circ \alpha_{V^{\otimes i-1},V_i,V_{i+1}}$$

In other words, there exists a functor

$$\mathcal{C} \longrightarrow B_n\text{-}\mathbf{mod}$$

Roughly speaking, σ_i acts by:

1) changing the bracketing in such a way that V_i and V_{i+1} are enclosed in the same bracket
2) applying the braiding of the category to $V_i \otimes V_{i+1}$
3) changing the bracketing back to the starting position.

REMARK 1.12. The axioms of a braided monoidal category implies something stronger than the existence of the above group morphism. Namely, there are group morphisms $B_n \to B_m$ obtained by adding strands on the left or or the right, or by "doubling" a given strand. The two hexagonal diagrams are sufficient conditions to guarantee the compatibility with these doubling operations.

It is natural to ask what the algebraic structure corresponding to non-strict (braided) monoidal category is. It leads to the following definition:

Definition 1.13. *A Quasi-Bialgebra (QBA) is a tuple*

$$(H, \Delta_H, \Phi_H)$$

where H is a $\mathbb{C}$-algebra, $\Delta_H : H \to H^{\otimes 2}$ an algebra morphism and Φ_H an invertible element of $H^{\otimes 3}$ such that

$$\Phi_H^{2,3,4}\Phi_H^{1,23,4}\Phi_H^{1,2,3} = \Phi_H^{1,2,34}\Phi_H^{12,3,4} \tag{1.1}$$

(the pentagon equation) and

$$(\mathrm{Id}_H \otimes \Delta_H) \circ \Delta_H(h) = \Phi_H \left[(\Delta_H \otimes \mathrm{Id}_H) \circ \Delta_H(h)\right] \Phi_H^{-1}, \ \forall h \in H \tag{1.2}$$

Definition 1.14. *A Quasi-Triangular Quasi-Bialgebra (QTQBA) is a tuple $(H, \Delta_H, \mathcal{R}_H, \Phi_H)$ where (H, Δ_H, Φ_H) is a QBA and $\mathcal{R}_H$ is an invertible element of $H^{\otimes 2}$ such that*[1]

$$\mathcal{R}_H \Delta_H(h) = \Delta_H^{2,1}(h)\mathcal{R}_H, \ \forall h \in H$$

and

$$\mathcal{R}_H^{12,3} = \Phi_H^{3,2,1}\mathcal{R}_H^{1,3}(\Phi_H^{1,3,2})^{-1}\mathcal{R}_H^{2,3}\Phi_H^{1,2,3} \tag{1.3a}$$

$$\mathcal{R}_H^{1,23} = (\Phi_H^{2,3,1})^{-1}\mathcal{R}_H^{1,3}\Phi_H^{2,1,3}\mathcal{R}_H^{1,2}(\Phi_H^{1,2,3})^{-1} \tag{1.3b}$$

(the two hexagon equations).

These algebraic axioms are nothing but translations of the defining diagrams of braided monoidal categories:

Proposition 1.15. *Let $(H, \Delta_H, \mathcal{R}_H, \Phi_H)$ be a QTQBA and V_1, V_2, V_3 be three H-modules. Define natural isomorphisms*

$$\alpha_H := (\Phi_H)_{|V_1 \otimes V_2 \otimes V_3}$$

and

$$\beta_H := P \circ (\mathcal{R}_H)_{|V_1 \otimes V_2}$$

where $P(v_1 \otimes v_2) = v_2 \otimes v_1$. Then $(H\text{-}\mathbf{mod}, \alpha_H, \beta_H)$ is a braided monoidal category.

---

[1] Here $X^{2,3,4} = 1 \otimes X$, $X^{1,23,4} = (\mathrm{Id}_H \otimes \Delta_H \otimes \mathrm{Id}_H)(X)$ etc. . .

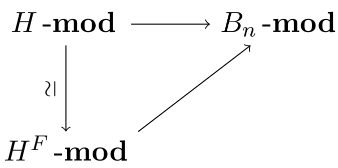
Hence, if $(H, \mathcal{R}_H)$ is a quasi-triangular bialgebra (i.e. if $\Phi_H = 1$), $\mathcal{R}_H$ satifies the Quantum Yang-Baxter Equation (QYBE):

$$\mathcal{R}^{1,2}\mathcal{R}^{1,3}\mathcal{R}^{2,3} = \mathcal{R}^{2,3}\mathcal{R}^{1,3}\mathcal{R}^{1,2} \tag{1.4}$$

1.3.5. Twists. If $H = (H, \Delta_H, \Phi_H, \mathcal{R}_H)$ is a QTQBA, the twist of H by an element $F \in (H^{\otimes 2})^\times$ is the QTQBA $H^F = (H, \Delta_H^F, \Phi_H^F, \mathcal{R}_H^F)$ where

- $\Delta_H^F(h) = F^{-1}\Delta_H(h)F$, $h \in H$
- $\mathcal{R}_H^F = F^{2,1}\mathcal{R}_H F^{-1}$
- $\Phi_H^F = F^{2,3}F^{1,23}\Phi_H(F^{12,3})^{-1}(F^{1,2})^{-1}$

As the underlying algebra structure is preserved, the categories H **-mod** and H^F **-mod** identify canonically. The main property of the twist operation is that the following diagram commutes:



1.3.6. A Quasi-Triangular structure on $U_\hbar(\mathfrak{g})$. All the definitions of the previous section would be of little interest without the following result:

Theorem 1.16 (Drinfeld)**.** *There exists $\mathcal{R}_\hbar \in U_\hbar(\mathfrak{g})^{\hat{\otimes}2}$ such that $(U_\hbar(\mathfrak{g}), \mathcal{R}_\hbar)$ is a QTBA.*

Let $K = e^{\hbar t_{\mathfrak{h}}/2}$ and $\bar{\mathcal{R}} = K^{-1}\mathcal{R}$. Then $\bar{\mathcal{R}} \in U_\hbar(\mathfrak{n}^+)\hat{\otimes}U_\hbar(\mathfrak{n}^-)$. The rational form $U_q(\mathfrak{g})$ is not quite quasi-triangular because $\mathcal{R} \notin U_q(\mathfrak{g})^{\otimes 2}$. However, $\bar{\mathcal{R}}$ can be identified to an element of a suitable degree completion of $U_q(\mathfrak{n}^+) \otimes U_q(\mathfrak{n}^-)$ and one can define an automorphism of $U_q(\mathfrak{g})^{\otimes 2}$ modeled on the action of $\mathrm{Ad}(K)$. Therefore, the category $U_q(\mathfrak{g})$ **-mod** is still braided monoidal, meaning that the algebraic representations of B_n coming from the universal R-matrix $\mathcal{R}$ extend to this setup.

1.4. The Kohno–Drinfeld Theorem

In order to compare these representations of B_n, let us first choose an algebra isomorphism

$$\alpha : U(\mathfrak{g})[[\hbar]] \longrightarrow U_\hbar(\mathfrak{g})$$

the existence of which is ensured by the rigidity Theorem 1.5. It induces an equivalence of categories between $\mathfrak{g}$ **-mod** and $U_\hbar(\mathfrak{g})$ **-mod** which we denote by $V \mapsto V_\hbar$. Therefore, for each $\mathfrak{g}$-module V, the representation of B_n on $V_\hbar^{\otimes n}$ coming from the quasi-triangular structure of $U_\hbar(\mathfrak{g})$ leads to a representation

$$\tilde{\rho}_\hbar : B_n \longrightarrow GL(V^{\otimes n}[[\hbar]])$$

Theorem (Kohno,Drinfeld)**.** *The representations ρ_{KZ} and $\tilde{\rho}_\hbar$ are equivalent.*

1.4.1. Algebraic monodromy of the KZ connection. One drawback of the definition of the monodromy morphism is its dependence on the basepoint, which makes it non-canonical. Hence, the strategy of Drinfeld was to work with a whole set of basepoints "at infinity" (asymptotic zones) which can be constructed in a canonical way, and behave well under the natural action of S_n on Y_n. These are basepoints infinitely close to the hyperplanes $H_{i,i+1}$ corresponding to the generators of B_n, so that the corresponding monodromy reduces to that of KZ_2 which is given by the action of a very simple element $\mathcal{R}_{\mathrm{KZ}} \in U(\mathfrak{g})^{\otimes 2}[[\hbar]]$. Then, Drinfeld constructs an associator using KZ_3, that is an element $\Phi_{\mathrm{KZ}} \in U(\mathfrak{g})^{\otimes 3}[[\hbar]]$ whose action expresses the holonomy along a path joining two asymptotic zones in a simply connected subspace of Y_n. He shows that the monodromy morphism of KZ_n depends algebraically on Φ_{KZ} and $\mathcal{R}_{\mathrm{KZ}}$. Finally, he proves that these elements turn $U(\mathfrak{g})[[\hbar]]$ into a QTQBA whose induced representations of B_n coincide with that of the monodromy of KZ_n.

The space $Y_n \cap \mathbb{R}^n$ is made of $n!$ connected component corresponding to elements σ of $\mathfrak{S}_n$. Let

$$K_n^\sigma = \{(x_1, \dots, x_n) \in \mathbb{R}^n,\ x_{\sigma(1)} < x_{\sigma(2)} < \cdots < x_{\sigma(n)}\}$$

To each complete bracketing[2] of the word $x_{\sigma(1)}x_{\sigma(2)} \dots x_{\sigma(n)}$ is associated an asymptotic zone whose elements are the tuples $(x_1, \dots, x_n) \in K_n^\sigma$ satisfying the following rule: if i, j, k are three indices such that there is a pair of bracket containing $x_{\sigma(i)}$ and $x_{\sigma(j)}$ but not $x_{\sigma(k)}$ then

$$|x_{\sigma(i)} - x_{\sigma(j)}| \ll |x_{\sigma(i)} - x_{\sigma(k)}|$$

Here "$\ll$" means "much less than". Although this symbol does not seems to have a precise meaning in the above definition, we will use it for the sake of clarity, as this construction can be made rigorous [**Kap**, **DCP**]. Figure 1.1 illustrates asymptotic zones in X_3. Therefore, instead of comparing solutions of KZ_n in a neighbourhood of an arbitrary basepoint, we will compare the asymptotic behaviour of these solutions at the limit where the z_i's are going very close one from each other. It makes sense because the KZ system is invariant under transformations of the form $z_i \mapsto z_i + b$ for $b \in \mathbb{R}$, meaning that the solutions depends only on the differences $z_i - z_j$. Hence, an asymptotic zone tells precisely in which direction the limit as to be considered.

We will work with the semi-universal KZ equation by letting $V = U(\mathfrak{g})$. Although $U(\mathfrak{g})$ is not finite dimensional, there is still a well defined (see for instance [**Ma**, Appendix 1]) monodromy morphism:

$$\rho_{\mathrm{KZ}} : B_n \longrightarrow (U(\mathfrak{g})^{\otimes n}[[\hbar]])^\times$$

The monodromy of KZ_2 is very explicit and quite easily computed: the image of the unique generator σ_1 of B_2 is $(1,2)\mathcal{R}_{\mathrm{KZ}}$ where

$$\mathcal{R}_{\mathrm{KZ}} = e^{\hbar t^{1,2}/2} \in U(\mathfrak{g})^{\otimes 2}[[\hbar]].$$

[2]Recall that a complete bracketing is defined by the following recursive rule: the only complete bracketing of a single letter is the empty one. If w_1, w_2 are completely bracketed words, then so is $(w_1 w_2)$.

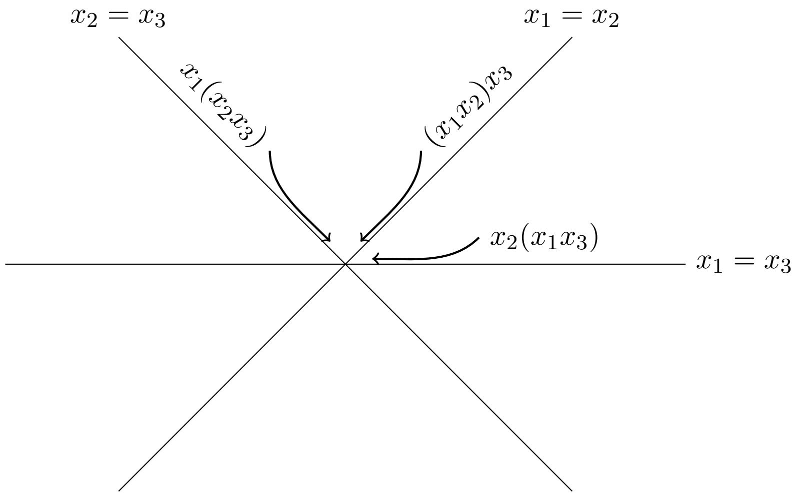


FIGURE 1.1. Some asymptotic zones for KZ_3

The Drinfeld associator Φ_{KZ} is defined as the holonomy of KZ_3 along a path in the simply connected space K_3^{Id} from $(x_1x_2)x_3$ to $x_1(x_2x_3)$. To be more concrete, note that a solution of KZ_3 in K_3^{Id} is of the form

$$F(x_1, x_2, x_3) = (x_3 - x_1)^{\hbar(t^{1,2}+t^{1,3}+t^{2,3})} G(\frac{x_2 - x_1}{x_3 - x_1})$$

where $G(x)$ is a solution of the following equation

$$G'(x) = \frac{\hbar}{2\pi\sqrt{-1}} \left(\frac{t^{1,2}}{x} + \frac{t^{2,3}}{x-1} \right) G(x) \tag{1.5}$$

for $0 < x < 1$. The two above asymptotic zones correspond to the limits $x \to 0^+$ and $x \to 1^-$ respectively.

Proposition 1.17. *There exist unique solutions G_0, G_1 of (1.5) such that $G_0(x) \sim_{0^+} x^{\frac{\hbar}{2\pi\sqrt{-1}}t^{1,2}}$ and $G_1(x) \sim_{1^-} (1-x)^{\frac{\hbar}{2\pi\sqrt{-1}}t^{2,3}}$.*

Here $G_0(x) \sim_{0^+} x^{\frac{\hbar}{2\pi\sqrt{-1}}t^{1,2}}$ means that $\tilde{G}_0(x) := G_0(x)x^{-\frac{\hbar}{2\pi\sqrt{-1}}t^{1,2}}$ is analytic on $]-1,1[$ and $\tilde{G}_0(0) = 1$. Set $\Phi_{KZ} = G_1(x)^{-1}G_0(x) \in U(\mathfrak{g})^{\otimes 3}[[\hbar]]$.

Theorem 1.18 (Drinfeld)**.** *$A_{KZ} = (U(\mathfrak{g})[[\hbar]], \Delta_0, \mathcal{R}_{KZ}, \Phi_{KZ})$ is a QTQBA.*

Recall that the $t^{i,j}$'s are $\mathfrak{g}$-invariant. Hence, so is Φ_{KZ}, implying equation (1.2). The proof of the pentagon equation (1.1) follows from the fact that the two paths

$$((x_1x_2)x_3)x_4 \to (x_1x_2)(x_3x_4) \to x_1(x_2(x_3x_4))$$

and

$$((x_1x_2)x_3)x_4 \to (x_1(x_2x_3))x_4 \to x_1((x_2x_3))x_4) \to x_1(x_2(x_3x_4))$$

are homotopic in K_4^{Id}.

For the first hexagon equation (1.3a), one has to extend the solutions of (1.5) by analytic continuation to the simply connected space

$$\{z \in \mathbb{P}^1(\mathbb{C}) - \{0, 1, \infty\}, \text{Im}(z) \geq 0\}.$$

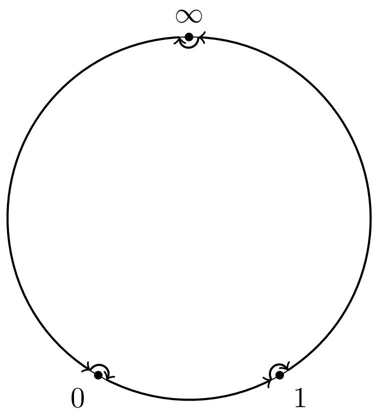


FIGURE 1.2. Path for the hexagon equation

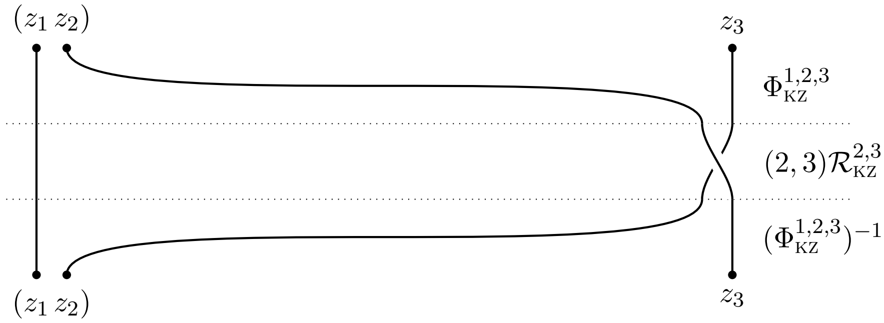


FIGURE 1.3. A loop representing σ_2

Then, the relation follows from the triviality of the path of Figure 1.2. The second hexagon equation (1.3b) is proved in a similar fashion.

Proposition 1.19. *The monodromy morphism ρ_{KZ} coincides with the representation of B_n coming from A_{KZ}.*

This is true essentially by construction: choosing the asymptotic zone

$$(\ldots((z_1 z_2) z_3) \ldots z_n)$$

as a basepoint, every loop in $X_n = Y_n/S_n$ can be decomposed into moves between asymptotic zones (changes of bracketing) and small loops around the singularities (application of the braiding operator) as in Figure 1.3. Hence ρ_{KZ} can be expressed algebraically using Φ_{KZ} and $\mathcal{R}_{\mathrm{KZ}}$ exactly by the same formula as in Proposition 1.11.

1.4.2. Twist equivalence. Now the Kohno–Drinfeld Theorem is a consequence of the following

Theorem 1.20 (Drinfeld)**.** *There exists an invertible element $F \in U(\mathfrak{g})^{\otimes 2}[[\hbar]]$ such that*

$$(U(\mathfrak{g})[[\hbar]], \Delta_0, \Phi_{\mathrm{KZ}}, \mathcal{R}_{\mathrm{KZ}}) = \alpha((U_\hbar(\mathfrak{g}), \Delta_\hbar, 1, \mathcal{R}_\hbar))^F$$

The proof of this result in [**Dr4**] is divided into the following steps:

(a) One first shows by cohomological arguments that there exists an invertible element $F_1 \in U(\mathfrak{g})^{\otimes 2}[[\hbar]]$ such that for all $x \in U_\hbar(\mathfrak{g})$, $(\alpha \otimes \alpha)(\Delta_\hbar(x)) = F_1 \Delta_0(\alpha(x)) F_1^{-1}$. Twisting by F_1 leads to a QTQBA with non-trivial associator

$$(U(\mathfrak{g})[[\hbar]], \Delta_0, \mathcal{R}_1, \Phi_1)$$

The coassociativity of Δ_0 implies that Φ_1 is $\mathfrak{g}$-invariant.

(b) Twisting by $F_2 = (\mathcal{R}_1(\mathcal{R}_1^{2,1}\mathcal{R})^{\frac{1}{2}})^{\frac{1}{2}}$, one shows that $\mathcal{R}_2 = F_2^{2,1}\mathcal{R}_1 F^{-1}$ is symmetric, that is satisfies $\mathcal{R}_2^{2,1} = \mathcal{R}_2$. The defining properties of $\mathcal{R}_1$ imply that F_2 is $\mathfrak{g}$-invariant, which ensure that Δ_0 is not modified. Then, it can be shown that this implies that $\mathcal{R}_2 = \mathcal{R}_{\mathrm{KZ}}$. We obtain this way a QTQBA

$$(U(\mathfrak{g})[[\hbar]], \Delta_0, \mathcal{R}_{\mathrm{KZ}}, \Phi_2)$$

(c) Finally, using the vanishing of some cohomology group of a subcomplex of the coHochschild cohomology [**Car**], one shows that Φ_{KZ} is unique up to twist by a $\mathfrak{g}$-invariant invertible element $F \in U(\mathfrak{g})^{\otimes 2}[[\hbar]]$ satisfying $F^{2,1} = F$. This last condition guarantees that the resulting R-matrix is again $\mathcal{R}_{\mathrm{KZ}}$.

CHAPTER 2

Definitions and statement of results

In this chapter, we first recall from [En2] the construction of the cyclotomic KZ connections and of the corresponding analytic representations of B_n^1, the braid group of Coxeter type B. We give a set of axioms which leads to a general algebraic construction of representations of B_n^1. We show that such representations can be constructed from the quantized enveloping algebra of any simple Lie algebra. Finally, we state our main result.

2.1. Cyclotomic KZ connection, analytic representations of B_n^1

Let $N \geq 2$ be an integer and $\mu_N \subset \mathbb{C}^\times$ be the group of Nth roots of unity. For each $1 \leq i, j \leq n$ and each $\zeta \in \mu_N$, define the hyperplane $D_{i,j,\zeta} = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid z_i = \zeta z_j\} \subset \mathbb{C}^n$. Let $Y_{n,N}$ be the configuration space

$$(\mathbb{C}^\times)^n - \bigcup_{\substack{1 \leq i,j \leq n \\ \zeta \in \mu_N}} D_{i,j,\zeta}$$

The pure braid group associated to $Y_{n,N}$ is $P_{n,N} = \pi_1(Y_{n,N}, z^*)$ where $z^* = (z_1^*, \ldots, z_n^*) \in \mathbb{R}^n$ is such that $0 < z_1^* < \cdots < z_n^*$.

Let now $H_{i,j}$ be the hyperplane $\{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid z_i = z_j\} \subset \mathbb{C}^n$ and X_n^1 be the configuration space

$$\Big\{(\mathbb{C}^\times)^n - \bigcup_{1 \leq i,j \leq n} H_{i,j}\Big\}/\mathfrak{S}_n$$

The fundamental group $\pi_1(X_n^1, \mathfrak{S}_n z^*)$ is the Coxeter type B braid group $B_n^1 \cong B_{n+1} \times_{\mathfrak{S}_{n+1}} \mathfrak{S}_n$, where $\mathfrak{S}_n$ is the subgroup of the group of permutations of $\{0, \ldots, n\}$ which fix 0. The canonical map

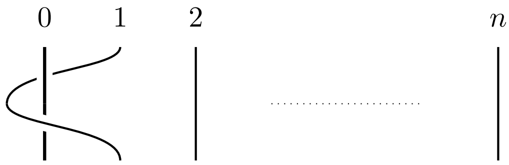
$$\begin{array}{ccc} Y_{n,N} & \longrightarrow & X_n^1 \\ (z_1, \ldots, z_n) & \longmapsto & \big[(z_1^N, \ldots, z_n^N)\big] \end{array}$$

is the covering corresponding to the group morphism $\phi_{n,N} : B_n^1 \to (\mathbb{Z}/N\mathbb{Z})^n \rtimes \mathfrak{S}_n$. We have then $P_{n,N} \cong \ker \phi_{n,N}$.

Proposition 2.1 ([Bri]). *The group B_n^1 admits the following presentation:*

$$B_n^1 = \langle \tau, \sigma_1, \ldots, \sigma_{n-1} \mid \tau\sigma_1\tau\sigma_1 = \sigma_1\tau\sigma_1\tau$$
$$\tau\sigma_i = \sigma_i\tau \text{ if } i > 1$$
$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \ \forall i \in \{1, \ldots, n-2\}$$
$$\sigma_i\sigma_j = \sigma_j\sigma_i \text{ if } |i-j| \geq 2\rangle$$

The braid group B_n is identified with the subgroup of B_n^1 generated by $\sigma_1, \ldots, \sigma_{n-1}$. The condition that $z_i \neq 0$ in X_n^1 can be visualized diagrammatically by adding a fixed strand on the left. Therefore, the additional Artin generator τ can be represented by the following diagram:



Let $\mathfrak{g}$ be a simple Lie algebra, $t \in S^2(\mathfrak{g})^{\mathfrak{g}}$. Choose an automorphism σ of $\mathfrak{g}$ such that

(i) $\sigma^N = \mathrm{Id}_{\mathfrak{g}}$
(ii) $\mathfrak{h} = \mathfrak{g}^\sigma$ is a Cartan subalgebra
(iii) $\sigma = \mathrm{Ad}(X)$ for some $X \in H$, where H is the simply connected Lie group whose Lie algebra is $\mathfrak{h}$

Hence, there exists a set of Nth roots of unity $\{\omega_1, \ldots, \omega_r\}$ such that

$$\sigma(e_i^{\pm}) = \omega_i^{\pm 1} e_i^{\pm}$$

Let $t_{\mathfrak{h}}$ be the image of t in $S^2(\mathfrak{h})$ through the projection $\mathfrak{g} \to \mathfrak{h}$ relative to the triangular decomposition $\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{n}^- \oplus \mathfrak{h}$.

Let $A_{\mathrm{KZ}} = U(\mathfrak{g})[[\hbar]] \rtimes \mathbb{Z}$ where $\tilde{\sigma} \cong 1 \in \mathbb{Z}$ acts by σ. Let V be a finite dimensional A_{KZ}-module and W be a finite dimensional $\mathfrak{h}$-module. If m is the multiplication of $U(\mathfrak{h})$, let $t_{\mathfrak{h}}^{i,i}$ be equal to $m(t_{\mathfrak{h}}) \in U(\mathfrak{h})$ viewed as an element of $End(V)$ acting on the ith component of $W \otimes V^{\otimes n}$ (here W has index 0). In the same way, $t_{\mathfrak{h}}^{0,i}$ is viewed as an element of $End(W \otimes V)$ acting on the 0th and the ith component of $W \otimes V^{\otimes n}$, and $t^{i,j}$ is defined similarly. Then, the cyclotomic KZ differential system is

$$\frac{\partial H}{\partial z_i} = \frac{\hbar}{2\pi\sqrt{-1}} \left(\frac{N(t_{\mathfrak{h}}^{0,i} + \frac{1}{2}t_{\mathfrak{h}}^{i,i})}{z_i} + \sum_{j \neq i, j=1}^{n} \sum_{a \in \mathbb{Z}/N\mathbb{Z}} \frac{(\sigma^a \otimes 1)(t)^{i,j}}{z_i - \zeta_N^a z_j} \right) H$$

where H is a function $H : Y_{n,N} \to W \otimes V^{\otimes n}[[\hbar]]$ and $\zeta_N = e^{2\sqrt{-1}\pi/N}$. This system is compatible, and thus defines a flat connection over $Y_{n,N}$ with fiber $W \otimes V^{\otimes n}[[\hbar]]$. It follows that it induces a monodromy morphism

$$\pi_1(Y_{n,N}) \longrightarrow GL(W \otimes V^{\otimes n}[[\hbar]])$$

There is a natural action of $G_{n,N} = (\mathbb{Z}/N\mathbb{Z})^n \rtimes \mathfrak{S}_n$ on $Y_{n,N}$, and the action of σ on V induces an action of $G_{n,N}$ on $W \otimes V^{\otimes n}$ (it acts trivially on W). The KZ connection is $G_{n,N}$-equivariant with respect to these actions, which implies that it also induces a monodromy representation of B_n^1. Moreover, as both t and $t_{\mathfrak{h}}$ are $\mathfrak{h}$-invariant, so is the KZ system.

REMARK 2.2. $G_{n,N}$ is a complex reflection group, denoted by $G(N, 1, n)$ in the Shephard-Todd classification [ST] (see appendix A). In particular, $G_{n,2}$ is the Coxeter group of type B.

2.2. Algebraic representations of B_n^1

Theorem 2.3. *Let A be an associative algebra, C a commutative subalgebra of A and elements $E, K \in (C^{\otimes 2})^\times$, $R \in (A^{\otimes 2})^\times$ such that:*

(a) R is a solution of the Quantum Yang-Baxter equation (1.4)
(b) $E^{1,2}E^{1,3}(K^{2,3})^2$ commutes with $(2,3)R^{2,3}$
(c) $K^{1,2}K^{1,3}$ commutes with $R^{2,3}$
(d) $K^{1,2} = K^{2,1}$

There exists a unique group morphism $\rho : B_n^1 \to (C \otimes A^{\otimes n} \rtimes \mathfrak{S}_n)^\times$ given by[1]:

$$\tau \longmapsto \prod_{i=2}^{n} (R^{1,i})^{-1} \prod_{i=2}^{n} K^{1,i} E^{0,1}$$
$$\sigma_i \longmapsto (i, i+1)R^{i,i+1}$$

We also have the following particular case:

Lemma 2.4. *If (A, R) is a QTQBA, C is a commutative sub-bialgebra of A and $E, K \in (C^{\otimes 2})^\times$ satisfy*

- $(\mathrm{Id}_A \otimes \Delta_A)(E) = E^{1,2}E^{1,3}(K^{2,3})^2$
- $(\mathrm{Id}_A \otimes \Delta_A)(K) = K^{1,2}K^{1,3}$
- $K^{1,2} = K^{2,1}$

then (A, C, R, K, E) verifies the assumptions of Theorem 2.3.

2.3. Representations of B_n^1 attached to $(\mathfrak{g}, t, \sigma)$

Let $(\mathfrak{g}, t, \sigma)$ be as in section 2.1. Extend the automorphism σ to an automorphism $\sigma_\hbar$ of $U_\hbar(\mathfrak{g})$ by setting:

$$\sigma_\hbar(e_i^\pm) = \omega_i^\pm e_i^\pm \qquad\qquad \sigma_\hbar(h_i) = 0$$

Denotes by $A_{alg} = (U_\hbar(\mathfrak{g}) \rtimes \mathbb{Z}$ the semi-direct product in which $\tilde{\sigma}_\hbar \cong 1 \in \mathbb{Z}$ acts by $\sigma_\hbar$. The coproduct of $U_\hbar(\mathfrak{g})$ is extended to A_{alg} by setting $\Delta_\hbar(\tilde{\sigma}_\hbar) = \tilde{\sigma}_\hbar \otimes \tilde{\sigma}_\hbar$. Since $\sigma_\hbar^{\otimes 2}(\mathcal{R}_\hbar) = \mathcal{R}_\hbar$, $(A_{alg}, \mathcal{R}_\hbar)$ is a quasi-triangular Hopf algebra.

Theorem 2.5. *Let $E_{\hbar,\sigma} = e^{\hbar(t_\mathfrak{h} + \frac{1}{2}t_\mathfrak{h}^{2,2})}(1 \otimes \tilde{\sigma}_\hbar)$ and $K = e^{\hbar t_\mathfrak{h}/2}$. There exists a unique representation of B_n^1 in $(U_\hbar(\mathfrak{h}) \otimes A_{alg}^{\otimes n} \rtimes \mathfrak{S}_n)^\times$ given by:*

$$\tau \longmapsto \prod_{i=2}^{n} (\mathcal{R}_\hbar^{1,i})^{-1} \prod_{i=2}^{n} K^{1,i} E_{\hbar,\sigma}^{0,1}$$
$$\sigma_i \longmapsto (i, i+1)\mathcal{R}_\hbar^{i,i+1}$$

[1] Here, $\prod_{i=1}^n a_i$ is the ordered product $a_1 a_2 \ldots a_n$

Proof. According to Lemma 2.4, it is enough to prove that

$$(E_{\hbar,\sigma})^{1,23} = (E_{\hbar,\sigma})^{1,2}(E_{\hbar,\sigma})^{1,3}(K^{2,3})^2 \tag{2.1}$$

Indeed:

$$\begin{aligned}(1\otimes\Delta_\hbar)(q^{t_{\mathfrak{h}}^{1,2}+\frac{1}{2}t_{\mathfrak{h}}^{2,2}}(1\otimes\sigma_\hbar)) &= q^{(t_{\mathfrak{h}}^{1,2}+t_{\mathfrak{h}}^{1,3})+\frac{1}{2}t_{\mathfrak{h}}^{2,2}+\frac{1}{2}t_{\mathfrak{h}}^{3,3}+t_{\mathfrak{h}}^{2,3}}(1\otimes\tilde{\sigma}_\hbar\otimes\tilde{\sigma}_\hbar)\\ &= q^{t_{\mathfrak{h}}^{1,2}+\frac{1}{2}t_{\mathfrak{h}}^{2,2}}(1\otimes\tilde{\sigma}_\hbar\otimes 1)q^{t_{\mathfrak{h}}^{1,3}+\frac{1}{2}t_{\mathfrak{h}}^{3,3}}(1\otimes 1\otimes\tilde{\sigma}_\hbar)q^{t_{\mathfrak{h}}^{2,3}}\\ &= (E_{\hbar,\sigma})^{1,2}(E_{\hbar,\sigma})^{1,3}(K^{2,3})^2\end{aligned}$$

□

Remark 2.6. More generally, the above formulas lead to representations of B_n^1 for any automorphism $\sigma_\hbar$ of $U_\hbar(\mathfrak{g})$ which satisfies $\sigma_\hbar^{\otimes 2}(\mathcal{R}_\hbar) = \mathcal{R}_\hbar$, for example:

- Cartan automorphisms, that is of the form $\sigma_\hbar(e_i^\pm) = \lambda_i^{\pm 1} e_i^\pm$ where $(\lambda_i)_{i=1\ldots n}$ is a family of invertible elements of $\mathbb{C}[[\hbar]]$
- diagram automorphisms [ESS].

2.4. Equivalence of representations

It follows from the previous sections that for any finite dimensional $U(\mathfrak{g}) \rtimes \mathbb{Z}$-module V and any finite dimensional $\mathfrak{h}$-module W, one can construct two representations $\rho_\hbar$ and ρ_{KZ} of B_n^1 in $W \otimes V^{\otimes n}[[\hbar]]$. The rest of this work is devoted to a proof of the following theorem:

Theorem. *The representations $\rho_\hbar$ and ρ_{KZ} are equivalent.*

CHAPTER 3

Algebraic construction of representations of B_n^1

The goal of this section is to prove Theorem 2.3 by a direct computation.

The relation between the $\rho(\sigma_i)$s are satisfied thanks to the QYBE (1.4). For $i > 1$ the relation

$$\rho(\tau)\rho(\sigma_i) = \rho(\sigma_i)\rho(\tau)$$

follows from the QYBE and axiom (c). Thus, it remains to check the relation

$$\rho(\tau)\rho(\sigma_1)\rho(\tau)\rho(\sigma_1) = \rho(\sigma_1)\rho(\tau)\rho(\sigma_1)\rho(\tau).$$

Axiom (c) implies that $K^{1,2}K^{1,3}$ commutes with $(R^{2,3})^{-1}$. Therefore, $K^{1,2}K^{1,i}$ commutes with $(R^{2,i})^{-1}$ for $i \in \{3, \ldots, n\}$. As $K^{1,j}$ obviously commutes with $(R^{2,i})^{-1}$ if $i \neq j$, it follows that

$$\prod_{i=3}^{n}(R^{2,i})^{-1}\prod_{i=2}^{n}K^{1,i} = \prod_{i=2}^{n}K^{1,i}\prod_{i=3}^{n}(R^{2,i})^{-1} \tag{3.1}$$

Then, $\rho(\sigma_1)\rho(\tau)\rho(\sigma_1)\rho(\tau)$ is equal to:

$$\begin{aligned}
&(1,2)R^{1,2}\prod_{i=2}^{n}(R^{1,i})^{-1}\prod_{i=2}^{n}K^{1,i}E^{0,1}(1,2)R^{1,2}\prod_{i=2}^{n}(R^{1,i})^{-1}\prod_{i=2}^{n}K^{1,i}E^{0,1}\\
=&(1,2)\prod_{i=3}^{n}(R^{1,i})^{-1}\prod_{i=2}^{n}K^{1,i}E^{0,1}(1,2)\prod_{i=3}^{n}(R^{1,i})^{-1}\prod_{i=2}^{n}K^{1,i}E^{0,1}\\
=&\prod_{i=3}^{n}(R^{2,i})^{-1}\prod_{i=3}^{n}(R^{1,i})^{-1}\prod_{i=2}^{n}K^{1,i}\prod_{i=3}^{n}K^{2,i}K^{1,2}E^{0,1}E^{0,2}
\end{aligned}$$

where the last step follows from (3.1) and the commutativity of C.

On the other hand, $\rho(\tau)\rho(\sigma_1)\rho(\tau)\rho(\sigma_1)$ is equal to

$$\begin{aligned}
&\prod_{i=2}^{n}(R^{1,i})^{-1}\prod_{i=2}^{n}K^{1,i}E^{0,1}(1,2)R^{1,2}\prod_{i=2}^{n}(R^{1,i})^{-1}\prod_{i=2}^{n}K^{1,i}E^{0,1}(1,2)R^{1,2}\\
=&\prod_{i=2}^{n}(R^{1,i})^{-1}\prod_{i=2}^{n}K^{1,i}E^{0,1}\prod_{i=3}^{n}(R^{2,i})^{-1}(1,2)\prod_{i=2}^{n}K^{1,i}E^{0,1}(1,2)R^{1,2}\\
=&\prod_{i=2}^{n}(R^{1,i})^{-1}\prod_{i=3}^{n}(R^{2,i})^{-1}\prod_{i=2}^{n}K^{1,i}E^{0,1}(1,2)\prod_{i=2}^{n}K^{1,i}E^{0,1}(1,2)R^{1,2}\\
=&\prod_{i=2}^{n}(R^{1,i})^{-1}\prod_{i=3}^{n}(R^{2,i})^{-1}\prod_{i=3}^{n}K^{1,i}\prod_{i=3}^{n}K^{2,i}(K^{1,2})^2E^{0,1}E^{0,2}R^{1,2}
\end{aligned}$$

Axioms (c) and (d) implies that $K^{1,i}K^{2,i}$ commutes with $R^{1,2}$. Using axiom (b), it follows that:

$$\begin{aligned}
&\rho(\tau)\rho(\sigma_1)\rho(\tau)\rho(\sigma_1) = \\
&\quad (R^{1,2})^{-1}\prod_{i=3}^{n}(R^{1,i})^{-1}\prod_{i=3}^{n}(R^{2,i})^{-1}R^{1,2}\prod_{i=3}^{n}K^{1,i}\prod_{i=3}^{n}K^{2,i}(K^{1,2})^2E^{0,1}E^{0,2}
\end{aligned}$$

If $i \neq j$, then $(R^{1,i})^{-1}$ commutes with $(R^{2,j})^{-1}$, implying that

$$\prod_{i=3}^{n}(R^{1,i})^{-1}\prod_{i=3}^{n}(R^{2,i})^{-1} = \prod_{i=3}^{n}(R^{2,i}R^{1,i})^{-1}$$

Finally, the QYBE implies that

$$(R^{1,2})^{-1}\prod_{i=3}^{n}(R^{2,i}R^{1,i})^{-1}R^{1,2} = \prod_{i=3}^{n}(R^{1,i}R^{2,i})^{-1}$$

which concludes the proof.

CHAPTER 4

Quasi-Reflection algebras and the cyclotomic KZ connection

4.1. Quasi-Reflection Algebras

Recall that a braided monoidal category $(\mathcal{C}, \alpha, \beta)$ is the categorical analog of a commutative monoid. Letting $\mathcal{C}$ act on another category leads to the notion of (right) module category over a monoidal category: this is a category $\mathcal{M}$ together with a bifunctor $\otimes : \mathcal{M} \times \mathcal{C} \to \mathcal{M}$ and a natural isomorphism $\gamma_{M,V,W} : (M \otimes V) \otimes W \longrightarrow M \otimes (V \otimes W)$ for $M \in \mathcal{M}$ and $V, W \in \mathcal{C}$. As for monoidal categories, we impose a coherence condition on γ: the following diagram commutes:

$$
\begin{array}{ccccc}
 & & (M \otimes (U \otimes V)) \otimes W & & \\
 & \overset{\gamma_{M,U,V} \otimes \mathrm{Id}_W}{\nearrow} & & \overset{\gamma_{M,U\otimes V,W}}{\searrow} & \\
((M \otimes U) \otimes V) \otimes W & & & & M \otimes ((U \otimes V) \otimes W) \\
\Big\downarrow {\scriptstyle \gamma_{M\otimes U,V,W}} & & & & \Big\downarrow {\scriptstyle \mathrm{Id}_M \otimes \alpha_{U,V,W}} \\
(M \otimes U) \otimes (V \otimes W) & & \xrightarrow[\gamma_{M,U,V\otimes W}]{} & & M \otimes (U \otimes (V \otimes W))
\end{array}
$$

A braided module category over a braided monoidal category $(\mathcal{C}, \alpha, \beta)$ is a module category $(\mathcal{M}, \gamma)$ over $(\mathcal{C}, \alpha)$ together with a natural automorphism η of the functor $\otimes : \mathcal{M} \times \mathcal{C} \to \mathcal{M}$ such that the following diagram commutes for $M \in \mathcal{M},\ V, W \in \mathcal{C}$:

$$
\begin{array}{ccc}
(M \otimes V) \otimes W & \xrightarrow{\eta_{M\otimes V,W}} & (M \otimes V) \otimes W \\
\Big\downarrow {\scriptstyle \gamma_{M,V,W}} & & \Big\uparrow {\scriptstyle (\gamma_{M,V,W})^{-1}} \\
M \otimes (V \otimes W) & & M \otimes (V \otimes W) \\
\Big\downarrow {\scriptstyle \mathrm{Id}_M \otimes \beta_{V,W}} & & \Big\uparrow {\scriptstyle \mathrm{Id}_M \otimes \beta_{W,V}} \\
M \otimes (W \otimes V) & & M \otimes (W \otimes V) \\
\Big\downarrow {\scriptstyle (\gamma_{M,W,V})^{-1}} & & \Big\uparrow {\scriptstyle \gamma_{M,W,V}} \\
(M \otimes W) \otimes V & \xrightarrow[\eta_{M,W} \otimes \mathrm{Id}_V]{} & (M \otimes W) \otimes V
\end{array}
$$

Now, Proposition 1.11 can be extended in the following way:

Proposition 4.1. *Let $(\mathcal{M},\gamma,\eta)$ be a braided module category over a braided monoidal category $(\mathcal{C},\alpha,\beta)$, $M\in\mathcal{M}$, $V_1,\dots,V_n$ be n copies of the same object $V\in\mathcal{C}$ and set*

$$M\otimes V^{\otimes n} := M\otimes(\dots(V_1\otimes V_2)\otimes V_3)\dots\otimes V_n)\in\mathcal{M}$$

and let

$$\tilde{\alpha} := \alpha_{V_1,V_2\otimes\dots\otimes V_{n-1},V_n}\circ\dots\alpha_{V_1,V_2\otimes V_3,V_4}\circ\alpha_{V_1,V_2,V_3}$$

There exists a unique group morphism

$$B_n^1 \longrightarrow \mathrm{Aut}_{\mathcal{M}}(M\otimes V^{\otimes n})$$

given by

$$\begin{aligned}\tau &\longmapsto \tilde{\alpha}^{-1}\circ\gamma_{M,V_1,V_2\otimes\dots V_n}\circ\eta_{M,V_1}\circ\gamma^{-1}_{M,V_1,V_2\otimes\dots V_n}\circ\tilde{\alpha}\\ \sigma_1 &\longmapsto \beta_{V_1,V_2}\\ \sigma_i &\longmapsto \alpha^{-1}_{V^{\otimes i-1},V_i,V_{i+1}}\circ\beta_{V_i,V_{i+1}}\circ\alpha_{V^{\otimes i-1},V_i,V_{i+1}}\end{aligned}$$

Remark 4.2. The choice of the initial bracketing in the above proposition may not seem to be the most natural one. Actually, it is chosen in such a way that the images of the σ_i's are exactly the same as in Proposition 1.11. It is a basic property of braided module categories that different choices of initial bracketing leads to equivalent representations of B_n^1.

The algebraic counterpart of braided module categories is the notion of Quasi-Reflection Algebra, introduced by Enriquez [**En2**]. This is a natural generalization of the notion of QTQBA.

Definition 4.3. *A* dynamical pseudo-twist *(DPT) over a QBA H is a tuple (B,Δ_B,Ψ_B) where B is an associative algebra, Δ_B is an algebra morphism $B\to B\otimes H$ and Ψ_B is an element in $(B\otimes H^{\otimes 2})^\times$ such that:*

$$(\mathrm{Id}_B\otimes\Delta_H)\circ\Delta_B(b) = \Psi_B\left[(\Delta_B\otimes\mathrm{Id}_H)\circ\Delta_B(b)\right]\Psi_B^{-1},\ \forall b\in B \tag{4.1}$$

$$\Psi_B^{1,2,34}\Psi_B^{12,3,4} = \Phi_H^{2,3,4}\Psi_B^{1,23,4}\Psi_B^{1,2,3} \tag{4.2}$$

The last relation is called the mixed pentagon relation.

Remark 4.4. This is the "quasi" version of the notion of comodule-algebra over a bialgebra.

Definition 4.5 ([**En2**])**.** *A* Quasi-Reflection Algebra *(QRA) over a QTQBA H is a tuple (B,Δ_B,Ψ_B,E_B) such that:*

- *(B,Δ_B,Ψ_B) is a dynamical pseudo-twist over H*
- *$E_B\in(B\otimes H)^\times$ satisfies the octagon equation*

$$(\Delta_B\otimes\mathrm{Id})(E_B) = \Psi_B^{-1}\mathcal{R}_H^{3,2}\Psi_B^{1,3,2}E_B^{1,3}(\Psi_B^{1,3,2})^{-1}\mathcal{R}_H^{2,3}\Psi_B \tag{4.3}$$

and

$$\Delta_B(b)E_B = E_B\Delta_B(b),\ \forall b\in B \tag{4.4}$$

As expected, we have the

Proposition 4.6 ([**En2**])**.** *If (B,Ψ_B,E_B) is a QRA over a QTQBA $(H,\Phi_H,\mathcal{R}_H)$, then B-**mod** is a braided module category over the braided monoidal category H-**mod**.*

Then, if $B = (B, \Delta_B, \Psi_B, E_B)$ is a QRA over H, the twist of (H, B) by (F, G) where $G \in (B \otimes H)^\times$ is the QRA $B^{(F,G)} = (B, \Delta_B^G, \Psi_B^{(F,G)}, E_B^G)$ over H^F, where

- $\Delta_B^{(F,G)}(b) = G^{-1}\Delta_B(b)G$, $b \in B$
- $\Psi_B^{(F,G)} = F^{2,3}G^{1,23}\Psi_B(G^{12,3})^{-1}(G^{1,2})^{-1}$
- $E_B^G = G^{-1}E_B G$

The categories H**-mod** $\times B$**-mod** and H^F**-mod** $\times B^{(F,G)}$**-mod** identify canonically, because the underlying algebras are the same. As before, the main property of the twist operation is that the following diagram commutes:

$$\begin{array}{ccc} H\textbf{-mod}\times B\textbf{-mod} & \longrightarrow & B_n^1\textbf{-mod} \\ \downarrow \wr & \nearrow & \\ H^F\textbf{-mod}\times B^{(F,G)}\textbf{-mod} & & \end{array}$$

It follows that the corresponding representations of B_n^1 are equivalent.

4.2. A QRA arising from the KZ equation

Recall that the monodromy of the KZ connection can be expressed algebraically in the framework of QTQBAs. In the same way, one constructs a QRA using the $n = 1$ and $n = 2$ cyclotomic KZ system, whose induced representations of B_n^1 coincide with the monodromy morphism of the cyclotomic KZ system [**En2**]. First, note that

$$\begin{aligned}\sum_{i=1}^{n} \frac{z_i \partial F}{\partial z_i} &= \frac{\hbar}{2\pi\sqrt{-1}}\left(N\sum_i (t_{\mathfrak{h}}^{0,i} + t_{\mathfrak{h}}^{i,i}) + \sum_{i\neq j}\sum_{a\in\mathbb{Z}/N\mathbb{Z}} \frac{(\sigma^a\otimes \mathrm{Id})(t)^{i,j}}{1-\zeta^a\frac{z_j}{z_i}}\right)F \\ &= \frac{\hbar}{2\pi\sqrt{-1}}\left(N\sum_i (t_{\mathfrak{h}}^{0,i} + t_{\mathfrak{h}}^{i,i})\right. \\ &\qquad \left. + \sum_{i<j}\sum_{a\in\mathbb{Z}/N\mathbb{Z}}\left(\frac{(\sigma^a\otimes \mathrm{Id})(t)^{i,j}}{1-\zeta^a\frac{z_j}{z_i}} + \frac{(\sigma^{-a}\otimes \mathrm{Id})(t)^{j,i}}{1-\zeta^{-a}\frac{z_i}{z_j}}\right)\right)F \\ &= \frac{\hbar}{2\pi\sqrt{-1}}\left(N\sum_i (t_{\mathfrak{h}}^{0,i} + t_{\mathfrak{h}}^{i,i}) + \sum_{i<j}\sum_{a\in\mathbb{Z}/N\mathbb{Z}} (\sigma^a\otimes \mathrm{Id})(t)^{i,j}\right)F\end{aligned}$$

where we used the fact that $(\sigma\otimes\mathrm{Id})(t)^{i,j} = (\sigma^{-1}\otimes\mathrm{Id})(t)^{j,i}$. As for the original KZ equation, it means that solutions are homogoneous, that is satisfy the relation

$$F(uz_1, \ldots, uz_n) = u^{\frac{\hbar}{2\pi\sqrt{-1}}T_n} F(z_1, \ldots, z_n)$$

where

$$T_n := N\sum_{i=1}^{n}(t_{\mathfrak{h}}^{0,i} + t_{\mathfrak{h}}^{i,i}) + \sum_{1\leq i<j\leq n}\sum_{a\in\mathbb{Z}/N\mathbb{Z}} (\sigma^a\otimes \mathrm{Id})(t)^{i,j}$$

on any simply connected subspace of $Y_{n,N}$. Hence, let $u_i = z_i/z_n$, $i = 1 \dots n-1$, it follows that

$$F(z_1, \dots, z_n) = z_n^{\frac{\hbar}{2\pi\sqrt{-1}}T_n} F(u_1, \dots, u_{n-1}, 1)$$

Therefore, F is a solution of the KZ system if and only if $G(u_1, \dots, u_{n-1}) := F(u_1, \dots, u_{n-1}, 1)$ is a solution of the following system:

$$\frac{\partial G}{\partial u_i} = \frac{\hbar}{2\pi\sqrt{-1}} \left(\frac{N(t_{\mathfrak{h}}^{0,i} + \frac{1}{2} t_{\mathfrak{h}}^{i,i})}{u_i} + \sum_{\substack{j \neq i \\ 1 \leq j \leq n-1}} \sum_{a \in \mathbb{Z}/N\mathbb{Z}} \frac{(\sigma^a \otimes \mathrm{Id})(t)^{i,j}}{u_i - \zeta^a u_j} + \sum_{a \in \mathbb{Z}/N\mathbb{Z}} \frac{(\sigma^a \otimes \mathrm{Id})(t)^{i,n}}{u_i - \zeta^a} \right) G$$

Hence, for $n = 1$, choose a path from $z = 1$ to $\sigma^{-1}(z) = \zeta$ which rotate around 0 counterclokwise. The holonomy along this path is

$$z^{-\frac{\hbar}{2\pi\sqrt{-1}} N(t_{\mathfrak{h}}^{0,1} + \frac{1}{2} t_{\mathfrak{h}}^{1,1})} (\zeta z)^{\frac{\hbar}{2\pi\sqrt{-1}} N(t_{\mathfrak{h}}^{0,1} + \frac{1}{2} t_{\mathfrak{h}}^{1,1})} = e^{\hbar(t_{\mathfrak{h}}^{0,1} + \frac{1}{2} t_{\mathfrak{h}}^{1,1})}$$

The image of the generator τ is thus $E_{\mathrm{KZ}} = e^{\hbar(t_{\mathfrak{h}}^{0,1} + \frac{1}{2} t_{\mathfrak{h}}^{1,1})}(1 \otimes \sigma)$.

Let $n = 2$ and define Ψ_{KZ} as the renormalized holonomy from 0 to 1 of the following equation:

$$G'(x) = \frac{\hbar}{2\pi\sqrt{-1}} \left(\frac{N(t_{\mathfrak{h}}^{0,1} + \frac{1}{2} t_{\mathfrak{h}}^{1,1})}{x} + \sum_{a \in \mathbb{Z}/N\mathbb{Z}} \frac{(\sigma^a \otimes \mathrm{Id})(t)^{1,2}}{x - \zeta^a} \right) G(x)$$

Assuming that $x \in]0,1[$, there exists unique solutions G_0, G_1 with asymptotic behaviour

$$G_0(x) \sim_{0^+} z^{\frac{\hbar}{2\pi\sqrt{-1}} N(t_{\mathfrak{h}}^{0,1} + \frac{1}{2} t_{\mathfrak{h}}^{1,1})}$$

and

$$G_1(x) \sim_{1^-} (1 - z)^{\frac{\hbar}{2\pi\sqrt{-1}} t^{1,2}}$$

then $\Psi_{\mathrm{KZ}} = G_1(x)^{-1} G_0(x)$. Hence, Ψ_{KZ} is the holonomy of the cyclotomic KZ system from the asymptotic zones $(0z_1)z_2$ to $0(z_1 z_2)$.

Theorem 4.7 (Enriquez). *The elements Ψ_{KZ} and E_{KZ} turn $U(\mathfrak{h})[[\hbar]]$ into a QRA over the QTQBA $(U(\mathfrak{g})[[\hbar]], \Phi_{\mathrm{KZ}}, \mathcal{R}_{\mathrm{KZ}})$.*

Indeed, in any asymptotic zone for which $(u_1, \dots, u_{n-1}) \in \mathbb{R}^n_{>0}$ and $|u_i - u_j| \ll u_i$ for all $1 \leq i \neq j \leq n-1$, the monodromy of the cyclotomic KZ system reduces to that of the original KZ equation. In particular, the holonomy along a path from $0((u_1 u_2)1)$ to $0(u_1(u_2 1))$ is $\Phi_{\mathrm{KZ}}^{1,2,3}$. Hence, the mixed pentagon equation follows from the fact that the following paths

$$((0u_1)u_2)1 \to (0u_1)(u_2 1) \to 0(u_1(u_2 1))$$

and

$$((0u_1)u_2)1 \to (0(u_1 u_2))1 \to 0((u_1 u_2))1) \to 0(u_1(u_2 1))$$

are homotopic. The octagon equation follows from the triviality of the path of Figure 4.1 in the simply connected space

$$\{z \in \mathbb{P}^1(\mathbb{C}) - \{0, \infty, \zeta^a, 0 \leq a \leq N-1\} \mid 0 \leq \arg(z) \leq 2\pi/N\}.$$

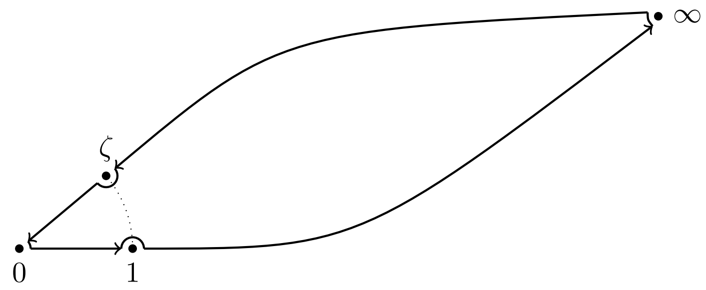


FIGURE 4.1. The path for the octagon equation

In the same way as in section 1.4.1, one proves that the representations of B_n^1 coming from the QRA $(U(\mathfrak{h})[[\hbar]], \Psi_{\mathrm{KZ}}, E_{\mathrm{KZ}})$ are equivalent to those coming from the monodromy of the cyclotomic KZ connection.

CHAPTER 5

A quantum Quasi-Reflection Algebra

In this chapter, we give an explicit construction of a QRA over the "quantum group" $U_\hbar(\mathfrak{g})$. We show that the resulting representations identify with those of Theorem 2.5.

For this, we recall the construction of a dynamical twist based on the quantum Shapovalov form [**ES2**]. Whereas the approach of [**ES2**] is "functional" (i.e. the twist is a map $J : \mathfrak{h}^* \to U_q(\mathfrak{g})^{\otimes 2}$), ours is purely algebraic: the twist is a family $\{\Psi_{q,m}, m \in \mathbb{N}^r\}$ lying in a localization $U_q(\mathfrak{h}) \otimes U_q(\mathfrak{g})^{\otimes 2}[1/D^{12}]$ of $U_q(\mathfrak{h}) \otimes U_q(\mathfrak{g})^{\otimes 2}$. To establish the properties of this twist (namely the ABRR equation (Theorem 5.10) and the mixed pentagon equation (Theorem 5.12)), we use an injection

$$U_q(\mathfrak{h}) \otimes U_q(\mathfrak{g})^{\otimes 2}[1/D^{12}] \hookrightarrow U'_{\hbar,loc}(\mathfrak{h}) \tilde{\otimes} U_\hbar(\mathfrak{g})^{\hat{\otimes} 2}[\hbar^{-1}]$$

(see Lemma 5.5). We then apply to the dynamical twist an algebraic analog of a shift of the functional variable which allows one to avoid the poles of $J(\lambda)$. This leads to a family $\{\Psi^{\sigma_\nu}_{q,m}, m \in \mathbb{N}^r\}$ lying in a new localized algebra $U_q(\mathfrak{h}) \otimes U_q(\mathfrak{g})^{\otimes 2}[1/D^{12}_\nu]$. This algebra injects into the *non-localized* algebra $U'_\hbar(\mathfrak{h}) \hat{\otimes} U_\hbar(\mathfrak{g})^{\hat{\otimes} 2}$, which leads to an element $\Psi_{\hbar,\sigma_\nu}$ in this algebra satisfying the mixed pentagon equation and a modified ABRR equation (Theorem 5.16). The fact that $\Psi_{\hbar,\sigma_\nu}$ is well defined relies on a study of its $\hbar$-adic valuation (Lemma 5.15). This is illustrated in the following diagram:

$$\begin{array}{ccc}
\Psi_{q,m} \in U_q(\mathfrak{h}) \otimes U_q(\mathfrak{g})^{\otimes 2}[1/D^{12}] & \xrightarrow[\text{Shift}]{\simeq} & U_q(\mathfrak{h}) \otimes U_q(\mathfrak{g})^{\otimes 2}[1/D^{12}_\nu] \ni \Psi^{\sigma_\nu}_{q,m} \\
\big\downarrow & & \big\downarrow \\
\Psi_\hbar \in U'_{\hbar,loc}(\mathfrak{h}) \tilde{\otimes} U_\hbar(\mathfrak{g})^{\hat{\otimes} 2}[\hbar^{-1}] & & U'_\hbar(\mathfrak{h}) \hat{\otimes} U_\hbar(\mathfrak{g})^{\hat{\otimes} 2}[\hbar^{-1}] \ni \Psi_{\hbar,\sigma_\nu}
\end{array}$$

Then, we prove general results relating the ABRR equation and the octagon equation (Theorem 5.18). Using the objects constructed in the first section, this gives rise to an "algebraic" QRA (Theorem 5.19). Finally, we give an explicit form of the representations of B^1_n attached to this QRA.

5.1. Algebraic solutions of the ABRR and the mixed pentagon equation

Conventions. The Killing form on $\mathfrak{g}$ is normalized in such a way that $(h_i, h_j) = d_j^{-1} a_{ij}$ and from now on we assume that $t = 2(\ ,\ )^{-1}$. It follows that for all $h \in \mathfrak{h}$, $\alpha_j(h) = (d_j h_j, h)$. Let $\check{h}_i \in \mathfrak{h}$ be the unique element such that $(h_i, \check{h}_j) = \delta_{ij}$, then $t_{\mathfrak{h}} = 2\sum_i \check{h}_i \otimes h_i$. Finally, let $\rho \in \mathfrak{h}$ be the unique element such that for $j = 1\dots r$, $\alpha_j(\rho) = 2d_j$. If $m \in \mathbb{N}^r$, set $\underline{mdh} = \sum_i m_i d_i h_i$ and $\underline{md} = \sum_j m_j d_j$.

5.1.1. Quantum groups and localization.

5.1.1.1. Recall that $U_\hbar(\mathfrak{b}^\pm)$ is the subalgebra of $U_\hbar(\mathfrak{g})$ generated by $(e_i^\pm, h_i)_{i=1\dots r}$. $U_q(\mathfrak{b}^\pm)$ is defined in a similar fashion. Let $U_q(\mathfrak{n}^\pm)$ be the subalgebra of $U_q(\mathfrak{g})$ generated by $(e_i^\pm)_{i=1\dots r}$ and $U_q(\mathfrak{h})$ be the subalgebra generated by $(k_i^{\pm 1})_{i=1\dots r}$.

Define a $\mathbb{Z}^r$-grading on $U_q(\mathfrak{g})$ by setting $|e_i^\pm| = \pm\delta_i$ and $|k_i^{\pm 1}| = 0$ where δ_i is the ith basis vector of $\mathbb{Z}^r$. Recall that the element $\bar{\mathcal{R}} = e^{-t_{\mathfrak{h}}/2}\mathcal{R} \in U_\hbar(\mathfrak{g})^{\otimes 2}$ can be identified with an invertible element of $U_q(\mathfrak{n}^+)\hat{\otimes} U_q(\mathfrak{n}^-)$ where $\hat{\otimes}$ is the completed tensor product over $\mathbb{C}(q)$ with respect to the $\mathbb{Z}^r$-grading.

If $\beta = \sum_{i=1}^r n_i\alpha_i$, $n_i \in \mathbb{Z}$, set $k_\beta = \prod_{i=1}^r k_i^{n_i}$.

5.1.1.2. In the following, tensor products are understood over $\mathbb{C}(q)$. Recall that $U_q(\mathfrak{g})$ is isomorphic as a $\mathbb{C}(q)$-vector space to

$$U_q(\mathfrak{h}) \otimes U_q(\mathfrak{n}^+) \otimes U_q(\mathfrak{n}^-)$$

and that $U_q(\mathfrak{h})$ is isomorphic as an algebra to $\mathbb{C}(q)[k_i^{\pm 1}, i = 1\dots r]$. $\mathbb{Z}^r$ acts on $U_q(\mathfrak{h}) \cong \mathbb{C}(q)[k_i^{\pm 1}, i = 1\dots r]$ by $\delta_j \cdot k_i = q^{a_{i,j}} k_i$. Extend this action to an action of $(\mathbb{Z}^r)^n$ on $U_q(\mathfrak{h})^{\otimes n+1}$, the first copy of $U_q(\mathfrak{h})$ being acted on trivially. Then, the following relation holds for any $x \in U_q(\mathfrak{h})^{\otimes n+1}$ and any homogeneous element $y \in U_q(\mathfrak{h}) \otimes U_q(\mathfrak{g})^{\otimes n}$:

$$yx = (|y| \cdot x)y \tag{5.1}$$

Let A be an algebra without zero divisor, recall that a left denominator set for A is a multiplicative subset D of $A - \{0\}$ such that for each $d \in D$, $a \in A$, there exists $a' \in A$, $d' \in D$ satisfying the Ore condition [**MR**, Chap. 2]:

$$a'd = d'a$$

Then, we have the following generalization of the localization of commutative rings:

Theorem 5.1 (Ore). *If D is a left denominator set for an algebra A without zero divisor, then there exists a left quotient ring $A[1/D]$ of A in the following sense:*

- *there exists an injective algebra morphism $\phi : A \to A[1/D]$ such that for all $d \in D$, $\phi(d)$ is invertible in $A[1/D]$*
- *if B is any algebra and θ is a morphism $\theta : A \to B$ such that $\theta(D) \subset B^\times$, then there exists a unique morphism $\theta' : A[1/D] \to B$ such that $\theta = \theta' \circ \phi$.*

Moreover, it is easily seen that if θ is injective, then so is θ'. Indeed, if ad^{-1} is an element of the kernel of θ', then $(ad^{-1})d = a$ belongs to $\ker\theta' \cap A = \ker\theta = \{0\}$, meaning that $a = 0$ and therefore that $\ker\theta' = \{0\}$. In order to apply the Ore's Theorem to $U_q(\mathfrak{g})$, we need the following:

Lemma 5.2. *Let $A = U_q(\mathfrak{h}) \otimes \bigotimes_{i=1}^n U_i$ where U_i is either $U_q(\mathfrak{g})$ or $U_q(\mathfrak{b}^\pm)$, and D be a multiplicative subset of $U_q(\mathfrak{h})^{\otimes n+1} - \{0\}$ which is stable under the action of $(\mathbb{Z}^r)^n$ on it last n components. Then, D is a left denominator set for A.*

Proof. If x is an homogeneous element of A, the Ore condition follows from (5.1) and from the fact that $|x| \cdot d \in D$ because D is stable under the action of $(\mathbb{Z}^r)^n$. If x is the sum of two homogeneous elements x_1, x_2 and $d \in D$, let

$$x' = (|x_2| \cdot d)x_1 + (|x_1| \cdot d)x_2$$

As for $i = 1, 2$, $|(|x_i| \cdot d)| = 0$ it follows that:

$$\begin{aligned} x'd &= ((|x_1| + |x_2|) \cdot d)x_1 + ((|x_1| + |x_2|) \cdot d)x_2 \\ &= ((|x_1| + |x_2|) \cdot d)x \end{aligned}$$

The same argument can obviously be applied in the general case of an arbitrary sum of n homogeneous elements. □

For $\beta \in R^+, i \in \mathbb{N}$ let

$$D_{\beta,i} = 1 - k_\beta q^{-\frac{i}{2}(\beta,\beta)} \in U_q(\mathfrak{h})$$

and let

$$D^{12} = \big\langle (\{0\} \times \mathbb{Z}^r) \cdot \{\Delta_q(D_{\beta,i}), \beta \in R^+, i \in \mathbb{N}\} \big\rangle \subset U_q(\mathfrak{h})^{\otimes 2}$$

and

$$D^{123} = \big\langle (\{0\} \times (\mathbb{Z}^r)^2) \cdot \{(\mathrm{Id} \otimes \Delta_q) \circ \Delta_q(D_{\beta,i}), \beta \in R^+, i \in \mathbb{N}\} \big\rangle \subset U_q(\mathfrak{h})^{\otimes 3}$$

where $\langle X \rangle$ denotes the multiplicative subset spanned by X.

Lemma 5.3. *There are algebra morphisms*

$$U_q(\mathfrak{h}) \otimes U_q(\mathfrak{b}^+) \otimes U_q(\mathfrak{n}^-)[1/D^{12}] \to U_q(\mathfrak{h}) \otimes U_q(\mathfrak{b}^+) \otimes U_q(\mathfrak{g}) \otimes U_q(\mathfrak{n}^-)[1/D^{12}, 1/D^{123}]$$

induced by

$$\begin{aligned} X &\mapsto X^{0,1,2} & X &\mapsto X^{0,12,3} \\ X &\mapsto X^{0,1,23} & X &\mapsto X^{01,2,3} \end{aligned}$$

Proof. The image of D^{12} by each of these maps is a subset either of D^{12} or of D^{123}, which contains only invertible elements by construction. Thus, the proof follows from the universal property of the localization. □

5.1.1.3. Let $U'_\hbar(\mathfrak{h})$ be the Quantum Formal Series Hopf Algebra [Ga] associated to $U_\hbar(\mathfrak{h})$, that is the subalgebra of $U_\hbar(\mathfrak{h})$ generated by $\hbar h_i$, $i = 1 \dots r$. It is a flat deformation of the algebra $\widehat{S(\mathfrak{h})}$ of formal functions on $\mathfrak{h}$. Indeed, there is an algebra isomorphism

$$\begin{array}{ccc} U'_\hbar(\mathfrak{h}) & \longrightarrow & \mathbb{C}[[u_1, \dots, u_r, \hbar]] \\ \hbar h_i & \longmapsto & u_i \end{array}$$

For $\beta = \sum_{i=1}^r \beta_i \alpha_i \in R^+$, let

$$\ell_\beta = \sum_{i=1}^r \beta_i d_i u_i$$

and set

$$U'_{\hbar,loc}(\mathfrak{h}) = \mathbb{C}[[u_1, \dots, u_r]][\frac{1}{\ell_\beta}, \beta \in R^+][[\hbar]]$$

Let A be a topologically free $\mathbb{C}[[\hbar]]$-algebra. Recall that it means that there exists a $\mathbb{C}$-vector space A_0 such that A is isomorphic to $A_0[[\hbar]]$ as a $\mathbb{C}[[\hbar]]$-module. We define the completed tensor product

$$U'_{\hbar,loc}(\mathfrak{h}) \tilde{\otimes} A = A_0[[u_1, \dots, u_r]][\frac{1}{\ell_\beta}, \beta \in R^+][[\hbar]]$$

Lemma 5.4. *The coproduct $\Delta_\hbar : U_\hbar(\mathfrak{h}) \to U_\hbar(\mathfrak{h})^{\hat{\otimes} 2}$ induces an algebra morphism*

$$U'_{\hbar,loc}(\mathfrak{h}) \longrightarrow U'_{\hbar,loc}(\mathfrak{h}) \tilde{\otimes} U_\hbar(\mathfrak{h})$$

Proof. Under the identification $U'_\hbar(\mathfrak{h}) \cong \mathbb{C}[[u_1, \dots, u_n, \hbar]]$, the coproduct of $U_\hbar(\mathfrak{h})$ becomes an algebra morphism $\mathbb{C}[[u_1, \dots, u_n, \hbar]] \to \mathbb{C}[[u_1, \dots, u_n, \hbar]] \hat{\otimes} U_\hbar(\mathfrak{h})$ induced by

$$u_i \longmapsto u_i \otimes 1 + 1 \otimes \hbar h_i$$

which induces an algebra morphism $\mathbb{C}[[u_1, \dots, u_n, \hbar]] \to \mathbb{C}[[u_1, \dots, u_n]][\frac{1}{\ell_\beta}][[\hbar]] \tilde{\otimes} U_\hbar(\mathfrak{h})$. As the image of ℓ_β ($\beta = \sum_i \beta_i \alpha_i$) through this map is invertible, its inverse being

$$\sum_{k \geq 0} \hbar^k (-1)^k \frac{1}{\ell_\beta^{k+1}} \otimes (\sum_i (\beta_i d_i h_i))^k,$$

it extends to an algebra morphism

$$U'_{\hbar,loc}(\mathfrak{h}) \longrightarrow U'_{\hbar,loc}(\mathfrak{h}) \tilde{\otimes} U_\hbar(\mathfrak{h})$$

□

5.1.1.4. There is a field morphism $\mathbb{C}(q) \to \mathbb{C}[[\hbar]][\hbar^{-1}]$ given by $q \mapsto e^\hbar$, and a compatible injective Hopf algebra morphism

$$U_q(\mathfrak{g}) \to U_\hbar(\mathfrak{g})[\hbar^{-1}]$$

given by

$$e_i^\pm \mapsto e_i^\pm \qquad\qquad k_i \mapsto e^{\hbar d_i h_i}$$

Lemma 5.5. *There are injective algebra morphisms*

$$U_q(\mathfrak{h}) \otimes U_q(\mathfrak{b}^+) \otimes U_q(\mathfrak{n}^-)[1/D^{12}] \to U'_{\hbar,loc}(\mathfrak{h}) \tilde{\otimes} (U_\hbar(\mathfrak{b}^+) \hat{\otimes} U_\hbar(\mathfrak{n}^-))[\hbar^{-1}]$$

and

$$U_q(\mathfrak{h}) \otimes U_q(\mathfrak{b}^+) \otimes U_q(\mathfrak{g}) \otimes U_q(\mathfrak{n}^-)[1/D^{12}, 1/D^{123}] \\ \to U'_{\hbar,loc}(\mathfrak{h}) \tilde{\otimes} (U_\hbar(\mathfrak{b}^+) \hat{\otimes} U_\hbar(\mathfrak{g}) \hat{\otimes} U_\hbar(\mathfrak{n}^-))[\hbar^{-1}]$$

which commute with the insertion-coproduct morphisms of Lemma 5.3.

Proof. The image of D^{12} in $U'_\hbar(\mathfrak{h}) \hat{\otimes} U_\hbar(\mathfrak{b}^+) \hat{\otimes} U_\hbar(\mathfrak{n}^-)[\hbar^{-1}]$ is the multiplicative subset spanned by

$$\left\{ 1 - \left(\exp(\ell_\beta) \otimes \exp(\hbar \sum_i \beta_i d_i h_i) \otimes 1 \right) \right. \\ \left. \times \exp\left(-\hbar \left(\frac{i}{2}(\beta, \beta) + \sum_{j,k} \beta_j m_k a_{jk} \right) \right) \mid \beta \in R^+, i \in \mathbb{N} \right\} \quad (5.2)$$

These elements actually belong to $U'_\hbar(\mathfrak{h}) \hat{\otimes} U_\hbar(\mathfrak{b}^+) \hat{\otimes} U_\hbar(\mathfrak{n}^-)$ and are invertible in this algebra, as they reduce to elements of the form $1 - \exp(\ell_\beta)$ modulo $\hbar$, the latter being invertible in $\mathbb{C}[[u_1, \ldots, u_r]][\frac{1}{\ell_\beta}]$ by construction. Hence elements of (5.2) are invertible, which implies that the above morphism factors through $U_q(\mathfrak{h}) \otimes U_q(\mathfrak{b}^+) \otimes U_q(\mathfrak{n}^-)[1/D^{12}]$. The same argument apply to the second case, and the compatibility with the insertion-coproduct maps is obvious. □

5.1.1.5. *Shift of a localization.* For $\nu = (\nu_1, \ldots, \nu_r) \in (\mathbb{C}^\times)^r$ define the automorphism Sh_ν of $U_q(\mathfrak{h})$ by

$$\mathrm{Sh}_\nu(k_i^{\pm 1}) = \nu_i^{\pm 1} k_i^{\pm 1}$$

and set $D^{12}_\nu = (\mathrm{Sh}_\nu \otimes \mathrm{Id})(D^{12})$. This is again a Ore set for $U_q(\mathfrak{h}) \otimes U_q(\mathfrak{b}^+) \otimes U_q(\mathfrak{n}^-)$ thanks to Lemma 5.2. By construction, $\mathrm{Sh}_\nu \otimes \mathrm{Id}^{\otimes 2}$ is an algebra isomorphism

$$U_q(\mathfrak{h}) \otimes U_q(\mathfrak{b}^+) \otimes U_q(\mathfrak{n}^-)[1/D^{12}] \to U_q(\mathfrak{h}) \otimes U_q(\mathfrak{b}^+) \otimes U_q(\mathfrak{n}^-)[1/D^{12}_\nu]$$

The following lemma shows that by applying an appropriate shift, the above localized "rational" algebra can be injected into the *non-localized* "formal" algebra:

Lemma 5.6. *Assume that $\nu_\beta \neq 1$ for any $\beta \in R^+$, then the natural map*

$$U_q(\mathfrak{h}) \otimes U_q(\mathfrak{b}^+) \otimes U_q(\mathfrak{n}^-) \to U'_\hbar(\mathfrak{h}) \hat{\otimes} U_\hbar(\mathfrak{b}^+) \hat{\otimes} U_\hbar(\mathfrak{n}^-)[\hbar^{-1}]$$

extends to an injective algebra morphism

$$U_q(\mathfrak{h}) \otimes U_q(\mathfrak{b}^+) \otimes U_q(\mathfrak{n}^-)[1/D^{12}_\nu] \to U'_\hbar(\mathfrak{h}) \hat{\otimes} U_\hbar(\mathfrak{b}^+) \hat{\otimes} U_\hbar(\mathfrak{n}^-)[\hbar^{-1}]$$

Proof. The set D^{12}_ν is multiplicatively generated by the

$$\left\{ 1 - \nu_\beta q^{-\frac{i}{2}(\beta,\beta)} (k_\beta \otimes k_\beta) q^{\langle z, \beta \rangle}, \beta \in R^+, i \in \mathbb{N}, z \in \mathbb{Z}^r \right\}$$

whose image in $U'_\hbar(\mathfrak{h}) \otimes U_\hbar(\mathfrak{b}^+) \otimes U_\hbar(\mathfrak{n}^-)[\hbar^{-1}]$ is

$$\left\{ 1 - \nu_\beta q^{-\frac{i}{2}(\beta,\beta)} (e^{u_\beta} \otimes e^{\hbar d_\beta h_\beta}) q^{\langle z, \beta \rangle}, \beta \in R^+, i \in \mathbb{N}, z \in \mathbb{Z}^r \right\}$$

The reduction modulo $\hbar$ of each element of the latter set is of the form

$$1 - \nu_\beta e^{u_\beta} \otimes 1 \otimes 1$$

which is invertible in $\mathbb{C}[[u_1, \dots, u_r]]\tilde{\otimes} U(\mathfrak{b}^+) \otimes U(\mathfrak{n}^-)$ because $1 - \nu_\beta \neq 0$. Hence the Lemma follows from the universal property of localization.

□

5.1.2. Construction of the dynamical twist. Following [**EE2**, **EEM**], we will construct a dynamical twist using a quantum analog of the Shapovalov form, prove that it satisfies the ABRR equation [**ABRR**] and give a direct proof that it satisfies the mixed pentagon equation.

For $m \in \mathbb{Z}^r$, set $\mu(m) = \sum_{i,j} m_i m_j d_i a_{ij}$. While $q^{t_\mathfrak{h}}$ (resp. $q^{t_\mathfrak{h}^{1,1}}$) does not belong to $U_q(\mathfrak{g})^{\otimes 2}$ (resp. $U_q(\mathfrak{g})$) or to one of its completion, conjugation by these elements still make sense in the "rational" setup. Namely, there are automorphisms $\underline{\mathrm{Ad}(q^{t_\mathfrak{h}})}$ of $U_q(\mathfrak{g})^{\otimes 2}$ and $\underline{\mathrm{Ad}(q^{t_\mathfrak{h}^{1,1}/2})}$ of $U_q(\mathfrak{g})$ uniquely defined by

$$\underline{\mathrm{Ad}(q^{t_\mathfrak{h}})}(e_j^\pm \otimes 1) = e_j^\pm \otimes k_j^{\pm 2} \qquad \underline{\mathrm{Ad}(q^{t_\mathfrak{h}})}(1 \otimes e_j^\pm) = k_j^{\pm 2} \otimes e_j^\pm$$

$$\underline{\mathrm{Ad}(q^{t_\mathfrak{h}})}(X) = X \qquad \forall X \in U_q(\mathfrak{h})^{\otimes 2}, j \in \{1, \dots, r\}$$

and for any $x_m \in U_q(\mathfrak{g})[m], m \in \mathbb{Z}^r$

$$\underline{\mathrm{Ad}(q^{t_\mathfrak{h}^{1,1}/2})}(x_m) = x_m k^{2\underline{m}} q^{\mu(m)}$$

where $k^{\underline{m}} = \prod_j k_j^{m_j}$ and $U_q(\mathfrak{g})[m]$ is the degree m part of $U_q(\mathfrak{g})$.

5.1.2.1. *Quantum Shapovalov form.* Let H be the unique linear map $U_q(\mathfrak{g}) \to U_q(\mathfrak{h})$ such that for all $x_\pm \in U_q(\mathfrak{n}^\pm)$, $x_0 \in U_q(\mathfrak{h})$

$$H(x_- x_0 x_+) = \epsilon(x_-)\epsilon(x_+)x_0$$

It follows from the definition that H is $U_q(\mathfrak{h})$-linear (on the left and on the right). Moreover, if x_m, y_n are homogeneous elements of $U_q(\mathfrak{n}^+)$ and $U_q(\mathfrak{n}^-)$ of degrees m and n respectively, then $H(x_m y_n) = 0$ if $m \neq -n$. Thus, it induces a family of pairings

$$\begin{array}{ccc} U_q(\mathfrak{b}^+)[m] \times U_q(\mathfrak{b}^-)[-m] & \longrightarrow & U_q(\mathfrak{h}) \\ (x_m, y_{-m}) & \longmapsto & H(x_m y_{-m}) \end{array}$$

which restrict to a family of pairings $U_q(\mathfrak{n}^+)[m] \times U_q(\mathfrak{n}^-)[-m] \to U_q(\mathfrak{h})$.

Let K_m be the inverse element of this form, that is[1]: $K_m = K_m^{[1]} \otimes K_m^{[2]} \otimes K_m^{[3]}$ is the unique element of $U_q(\mathfrak{n}^-)[-m] \otimes U_q(\mathfrak{n}^+)[m] \otimes \mathbb{C}(q, k_i^{\pm 1})$ such that for all $x_\pm \in U_q(\mathfrak{b}^\pm)[\pm m]$,

$$H(x_+ x_-) = H(x_+ K_m^{[1]}) K_m^{[3]} H(K_m^{[2]} x_-) \tag{5.3}$$

Set

$$\bar{\mathcal{R}} = q^{-t_\mathfrak{h}/2}\mathcal{R} = \sum_{m \in \mathbb{N}^r} \bar{\mathcal{R}}_m = \sum_{m \in \mathbb{N}^r} \bar{\mathcal{R}}_m^{[1]} \otimes \bar{\mathcal{R}}_m^{[2]} \in U_q(\mathfrak{n}^+) \otimes U_q(\mathfrak{n}^-)$$

and

$$\bar{\mathcal{R}}^{-1} = \sum_{m \in \mathbb{N}^r} (\bar{\mathcal{R}}^{-1})_m = \sum_{m \in \mathbb{N}^r} r_m^{[1]} \otimes r_m^{[2]}.$$

[1]If $V_1, \dots, V_k$ are vector spaces or modules, we abbreviate a general element $a = \sum_{i \in I} a_1(i) \otimes \dots \otimes a_k(i)$ of $V_1 \otimes \dots \otimes V_k$ by $a = a^{[1]} \otimes \dots \otimes a^{[k]}$

The following result of De Concini and Kac will be crucial for this section:

Theorem 5.7 ([DCK]). *Up to some invertible element of $\mathbb{C}(q)$ depending on the choice of basis of $U_q(\mathfrak{n}^+)[m]$ and $U_q(\mathfrak{n}^-)[-m]$, the determinant of H restricted to $U_q(\mathfrak{n}^+)[m] \times U_q(\mathfrak{n}^-)[-m]$ (viewed as a matrix with coefficient in $\mathbb{C}(q, k_1, \dots, k_r)$) is*

$$\det_m = \prod_{\beta \in R^+} \prod_{i \in \mathbb{N}} (1 - k_\beta q^{-(\beta,\rho) - \frac{i}{2}(\beta,\beta)})^{P(\sum_j m_j \alpha_j - i\beta)}$$

where $P(\beta)$ is the Kostant partition function, that is the number of ways one can write β as a integral linear combination of the positive roots with non-negative coefficients.

As a consequence, K_m actually belongs to

$$U_q(\mathfrak{n}^-) \otimes U_q(\mathfrak{n}^+) \otimes U_q(\mathfrak{h})[(1 - k_\beta q^{-(\beta,\rho) - \frac{i}{2}(\beta,\beta)})^{-1}, \beta \in R^+, i \in \mathbb{N}].$$

Proposition 5.8. *The elements $\{K_m, m \in \mathbb{N}^r\}$ satisfy the following relation:*

$$K_m = \sum_{m' + m'' = m} (\bar{\mathcal{R}}^{[2]}_{m'} \otimes 1 \otimes 1)(S^2 \otimes \mathrm{Id} \otimes \mathrm{Id})(K_{m''})(1 \otimes \bar{\mathcal{R}}^{[1]}_{m'} \otimes 1)$$
$$(1 \otimes 1 \otimes k^{-2\underline{m''}} q^{\mu(m'')} k^{-\underline{m'}} q^{\sum_{i,j} d_i m'_i m''_j a_{ij}}) \quad (5.4)$$

Proof. In the following, we identify K_m with its image through the injective algebra morphism

$$U_q(\mathfrak{n}^-) \otimes U_q(\mathfrak{n}^+) \otimes U_q(\mathfrak{h})[(1 - k_\beta q^{-(\beta,\rho) - \frac{i}{2}(\beta,\beta)})^{-1}, \beta \in R^+, i \in \mathbb{N}]$$
$$\to U_\hbar(\mathfrak{n}^-) \hat{\otimes} U_\hbar(\mathfrak{n}^+) \tilde{\otimes} U'_{\hbar,loc}(\mathfrak{h})[\hbar^{-1}]$$

the existence of which follows as in Lemma 5.5 from the fact that $1 - k_\beta q^{-(\beta,\rho) - \frac{i}{2}(\beta,\beta)} \mapsto 1 - e^{\ell_\beta} + O(\hbar)$.

By construction there are commuting squares

$$\begin{CD} U_q(\mathfrak{g}) @>{\mathrm{Ad}(q^{t^{1,1}_{\mathfrak{h}}})}>> U_q(\mathfrak{g}) \\ @VVV @VVV \\ U_\hbar(\mathfrak{g})[\hbar^{-1}] @>>{\mathrm{Ad}(e^{\hbar t^{1,1}_{\mathfrak{h}}})}> U_\hbar(\mathfrak{g})[\hbar^{-1}] \end{CD}$$

and

$$\begin{CD} U_q(\mathfrak{g})^{\otimes 2} @>{\mathrm{Ad}(q^{t_{\mathfrak{h}}})}>> U_q(\mathfrak{g})^{\otimes 2} \\ @VVV @VVV \\ U_\hbar(\mathfrak{g})^{\tilde{\otimes} 2}[\hbar^{-1}] @>>{\mathrm{Ad}(e^{\hbar t_{\mathfrak{h}}})}> U_\hbar(\mathfrak{g})^{\tilde{\otimes} 2}[\hbar^{-1}] \end{CD}$$

Set $u = \sum_m r_m^{[1]} q^{t_{\mathfrak{h}}^{1,1}/2} S^{-1}(r_m^{[2]})$, according to [Dr2] $q^\rho S^{-1}(u)$ is a central element of $U_\hbar(\mathfrak{g})$. Let $x_\pm \in U_\hbar(\mathfrak{n}^\pm)$, we compute $H(x_+ S^{-1}(u) x_-)$ in two different ways. On the one hand,

$$\begin{aligned} H(x_+ S^{-1}(u) x_-) &= H(x_+ q^{-\rho} x_- q^\rho S^{-1}(u)) \\ &= H(x_+ q^{-\rho} x_- q^\rho q^{t_{\mathfrak{h}}^{1,1}/2}) \end{aligned}$$

because $S^{-1}(u)$ is of the form $q^{t_{\mathfrak{h}}^{1,1}/2} + U_\hbar(\mathfrak{b}^-)[<0] U_\hbar(\mathfrak{b}^+)[>0]$. Thus,

$$\begin{aligned} H(x_+ S^{-1}(u) x_-) &= \sum_m H(x_+ q^{-\rho} K_m^{[1]}) K_m^{[3]} H(K_m^{[2]} x_-) q^\rho q^{t_{\mathfrak{h}}^{1,1}/2} \\ &= \sum_m H(x_+ q^{-\rho} K_m^{[1]} q^\rho) K_m^{[3]} q^{t_{\mathfrak{h}}^{1,1}/2} H(K_m^{[2]} x_-) \end{aligned}$$

On the other hand,

$$\begin{aligned} H(x_+ S^{-1}(u) x_-) &= \sum_{m'} H(x_+ S^{-2}(r_{m'}^{[2]}) q^{t_{\mathfrak{h}}^{1,1}/2} S^{-1}(r_{m'}^{[1]}) x_-) \\ &= \sum_{m',m''} H(x_+ S^{-2}(r_{m'}^{[2]}) q^{t_{\mathfrak{h}}^{1,1}/2} K_{m''}^{[1]}) K_{m''}^{[3]} H(K_{m''}^{[2]} S^{-1}(r_{m'}^{[1]}) x_-) \end{aligned}$$

Recall that $(S^{-1} \otimes \mathrm{Id})(\mathcal{R}^{-1}) = \mathcal{R}$, thus

$$\begin{aligned} (S^{-1} \otimes \mathrm{Id})(\bar{\mathcal{R}}^{-1} q^{-t_{\mathfrak{h}}/2}) &= q^{t_{\mathfrak{h}}/2} \bar{\mathcal{R}} \\ \sum_m (1 \otimes r_m^{[2]}) q^{t_{\mathfrak{h}}/2} (S^{-1}(r_m^{[1]}) \otimes 1) &= q^{t_{\mathfrak{h}}/2} \sum_m \bar{\mathcal{R}}_m^{[1]} \otimes \bar{\mathcal{R}}_m^{[2]} \\ S^{-1}(r_m^{[1]}) \otimes r_m^{[2]} &= q^{-\underline{dmh}} \bar{\mathcal{R}}_m^{[1]} \otimes \bar{\mathcal{R}}_m^{[2]} \end{aligned}$$

Under the normalization chosen at the beginning of this section, the following relation hold:

$$\begin{aligned} [t_{\mathfrak{h}}/2, e_j^\pm \otimes 1] &= \pm e_j^\pm \otimes \sum_i \alpha_j(\check{h}_i) h_i \\ &= \pm e_j^\pm \otimes d_j h_j \end{aligned}$$

Lemma 5.9. *Let $x_m^\pm$ be an homogeneous element of degree $\pm m$ in $U_\hbar(\mathfrak{n}^\pm)$, then*

$$\begin{aligned} [t_{\mathfrak{h}}^{1,1}/2, x_m] &= x_m \sum_i m_i d_i (\pm 2 h_i + \sum_j m_j a_{ij}) \\ &= x_m (\pm 2 \underline{mdh} + \mu(m)) \end{aligned}$$

PROOF. It follows from a direct computation:

$$\begin{aligned}\sum_i h_i\check{h}_i x_m^{\pm} &= \sum_i h_i x_m^{\pm}(\check{h}_i \pm \sum_j m_j\alpha_j(\check{h}_i))\\ &= \sum_i h_i x_m^{\pm}(\check{h}_i \pm m_i d_i))\\ &= \sum_i x_m^{\pm}(h_i \pm \sum_j m_j\alpha_j(h_i))(\check{h}_i \pm m_i d_i)\\ &= \sum_i x_m^{\pm} h_i\check{h}_i + \sum_i \pm(m_i d_i h_i + \check{h}_i\sum_j m_j\alpha_j(h_i)) + \sum_i (m_i d_i \sum_j m_j a_{ij})\\ &= \sum_i x_m^{\pm} h_i\check{h}_i + \sum_i m_i d_i(\pm 2h_i + \sum_j m_j a_{ij})\end{aligned}$$

□

Together with the fact that for any element $x \in U_\hbar(\mathfrak{g})$, $S^{-2}(x) = \mathrm{Ad}(q^{-\rho})(x)$, it implies that

$$\begin{aligned}H(x_+S^{-1}(u)x_-) &= \sum_{m',m''} H(x_+q^{-\rho}\bar{\mathcal{R}}^{[2]}_{m'}q^{\rho}q^{t_{\mathfrak{h}}^{1,1}/2}K^{[1]}_{m''})K^{[3]}_{m''}H(K^{[2]}_{m''}q^{-\underline{dm'h}}\bar{\mathcal{R}}^{[1]}_{m'}x_-)\\ &= \sum_{m',m''} H(x_+q^{-\rho}\bar{\mathcal{R}}^{[2]}_{m'}q^{\rho}K^{[1]}_{m''})K^{[3]}_{m''}q^{t_{\mathfrak{h}}^{1,1}/2}q^{-2\underline{dm''h}}q^{\mu(m'')}\\ &\qquad\times q^{-\underline{dm'h}}q^{\sum_{i,j}d_i m'_i m''_j a_{ij}}H(K^{[2]}_{m''}\bar{\mathcal{R}}^{[1]}_{m'}x_-)\end{aligned}$$

By the uniqueness of K_m, it follows that

$$\begin{aligned}K^{[1]}_m\otimes K^{[2]}_m\otimes K^{[3]}_m = \sum_{m'+m''=m} &\bar{\mathcal{R}}^{[2]}_{m'}q^{\rho}K^{[1]}_{m''}q^{-\rho}\otimes K^{[2]}_{m''}\bar{\mathcal{R}}^{[1]}_{m'}\\ &\otimes K^{[3]}_{m''}q^{-2\underline{dm''h}}q^{\mu(m'')}q^{-\underline{dm'h}}q^{\sum_{i,j}d_i m'_i m''_j a_{ij}}\end{aligned}$$

□

5.1.2.2. *The dynamical twist.* Let $\rho^* \in \mathfrak{h}^*$ be the dual element of ρ with respect to the Killing form. Then,

$$(\frac{1}{2}\rho^*\otimes\mathrm{Id})(t_{\mathfrak{h}}) = \rho. \tag{5.5}$$

Define an algebra automorphism $\mathrm{sh}_{\rho^*}: U_q(\mathfrak{h})\to U_q(\mathfrak{h})$ by:

$$\mathrm{sh}_{\rho^*}(k_i^{\pm 1}) = q^{\pm\frac{1}{2}\rho^*(d_i h_i)}k_i^{\pm 1}$$

For any $m\in\mathbb{N}^r$, let

$$J_m = S(K^{[3](1)}_m)\otimes S(K^{[2]}_m)S(K^{[3](2)}_m)\otimes K^{[1]}_m$$

The map sh_{ρ^*} induces an isomorphism

$$\begin{aligned}\mathrm{sh}_{\rho^*}\otimes\mathrm{Id}^{\otimes 2}: U_q(\mathfrak{h})\otimes U_q(\mathfrak{b}^+)\otimes U_q(\mathfrak{n}^-)[1/(\mathrm{sh}_\rho^{-1}\otimes\mathrm{Id})(D^{12})]\\ \to U_q(\mathfrak{h})\otimes U_q(\mathfrak{b}^+)\otimes U_q(\mathfrak{n}^-)[1/D^{12}]\end{aligned}$$

Thus define[2]

$$\Psi_{q,m} = (\mathrm{sh}_{\rho^*} \otimes \mathrm{Id}^{\otimes 2}) \circ (\underline{\mathrm{Ad}(q^{-t_{\mathfrak{h}}^{1,2}/2})} \otimes \mathrm{Id})(J_m)$$

Theorem 5.10. *The elements* $\{\Psi_{q,m}, m \in \mathbb{N}^r\}$ *are* $\mathfrak{h}$*-invariant elements of* $U_q(\mathfrak{h}) \otimes U_q(\mathfrak{b}^+) \otimes U_q(\mathfrak{n}^-)[1/D^{12}]$ *and satisfy the ABRR equation*

$$\Psi_{q,m} = \sum_{m'+m''=m} (\bar{\mathcal{R}}^{-1})^{2,3}_{m'} \underline{\mathrm{Ad}(q^{t_{\mathfrak{h}}^{1,2}+t_{\mathfrak{h}}^{2,2}/2})}(\Psi_{q,m''}) \tag{5.6}$$

Proof. The first statement is true because

$$(\mathrm{sh}_{\rho^*} \otimes \mathrm{Id}) \circ \underline{\mathrm{Ad}(q^{-t_{\mathfrak{h}}^{1,2}/2})}(\det{}^{12}_m) \in D^{12}$$

by construction. Thus, $\Psi_{q,m}$ can be identified with an element of

$$U'_{\hbar,loc}(\mathfrak{h}) \tilde{\otimes} U_\hbar(\mathfrak{b}^+) \hat{\otimes} U_\hbar(\mathfrak{n}^-)[\hbar^{-1}]$$

thanks to Lemma 5.5, and we will prove the above equation in the latter algebra. Set $J_m = J_m^{[1]} \otimes J_m^{[2]} \otimes J_m^{[3]}$. Applying the transformation from K_m to J_m to (5.4), one has:

$$J_m^{[1]} \otimes J_m^{[2]} \otimes J_m^{[3]} = \sum_{m'+m''=m} J_m^{[1]} q^{2\underline{dm''h}} q^{\underline{dm'h}} q^{\mu(m'')} q^{\sum_{i,j} d_i m'_i m''_j a_{ij}} \otimes S(\bar{\mathcal{R}}^{[1]}_{m'}) J^{[2]}_{m''} q^{2\underline{dm''h}} q^{\underline{dm'h}} \otimes \bar{\mathcal{R}}^{[2]}_{m'} q^\rho J^{[3]}_{m''} q^{-\rho}$$

that is

$$J_m^{[1]} \otimes J_m^{[2]} \otimes J_m^{[3]} = \sum_{m'+m''=m} J_m^{[1]} q^{2\underline{dm''h}} q^{\underline{dm'h}} q^{\mu(m'')} \otimes S(\bar{\mathcal{R}}^{[1]}_{m'}) q^{\underline{dm'h}} J^{[2]}_{m''} q^{2\underline{dm''h}} \otimes \bar{\mathcal{R}}^{[2]}_{m'} q^\rho J^{[3]}_{m''} q^{-\rho}$$

As

$$\sum_m q^{-\underline{dmh}} \bar{\mathcal{R}}^{[1]}_m \otimes \bar{\mathcal{R}}^{[2]}_m = \sum_m S^{-1}(r^{[1]}_m) \otimes r^{[2]}_m$$

it follows that

$$\sum_m S(\bar{\mathcal{R}}^{[1]}_m) q^{\underline{dmh}} \otimes \bar{\mathcal{R}}^{[2]}_m = \sum_m r^{[1]}_m \otimes r^{[2]}_m$$

Hence,

$$\begin{aligned} J_m^{[1]} \otimes J_m^{[2]} \otimes J_m^{[3]} &= \sum_{m'+m''=m} J_m^{[1]} q^{2\underline{dm''h}} q^{\underline{dm'h}} q^{\mu(m'')} \otimes r^{[1]}_m J^{[2]}_{m''} q^{2\underline{dm''h}} \otimes r^{[2]}_{m'} q^\rho J^{[3]}_{m''} q^{-\rho} \\ &= \sum_{m'+m''=m} q^{t_{\mathfrak{h}}^{1,2}/2} (1 \otimes r^{[1]}_{m'} \otimes r^{[2]}_{m'}) q^{-t_{\mathfrak{h}}^{1,2}/2} q^{t_{\mathfrak{h}}^{1,2}+t_{\mathfrak{h}}^{2,2}/2} \\ &\qquad \times (J^{[1]}_{m''} \otimes J^{[2]}_{m''} \otimes q^\rho J^{[3]}_{m''} q^{-\rho}) q^{-t_{\mathfrak{h}}^{1,2}-t_{\mathfrak{h}}^{2,2}/2} \end{aligned}$$

[2]According to [**EE2, EEM**], the map $a \otimes b \otimes c \mapsto S(c^{(1)}) \otimes S(b) S(c^{(2)}) \otimes a$ applied to K already leads to a dynamical twist. The two other transformations are needed here because we want a specific form of the ABRR equation.

Multiplying the two sides of the last line by $q^{t_{\mathfrak{h}}^{1,2}/2}$ on the right and using the $\mathfrak{h}$-invariance of J, the following relation holds

$$q^{-t_{\mathfrak{h}}^{1,2}/2} J_m q^{t_{\mathfrak{h}}^{1,2}/2} = \sum_{m=m'+m''} (\bar{\mathcal{R}}_{m'}^{-1})^{2,3} q^{t_{\mathfrak{h}}^{1,2}+t_{\mathfrak{h}}^{2,2}/2-\rho^{(2)}} (q^{-t_{\mathfrak{h}}^{1,2}/2} J_{m''} q^{t_{\mathfrak{h}}^{1,2}/2}) q^{-t_{\mathfrak{h}}^{1,2}-t_{\mathfrak{h}}^{2,2}/2+\rho^{(2)}}$$

that is, thanks to relation (5.5),

$$\Psi_{q,m} = \sum_{m=m'+m''} (\bar{\mathcal{R}}^{-1})_{m'}^{2,3} q^{t_{\mathfrak{h}}^{1,2}+t_{\mathfrak{h}}^{2,2}/2} \Psi_{q,m''} q^{-t_{\mathfrak{h}}^{1,2}-t_{\mathfrak{h}}^{2,2}/2}$$

as wanted. □

Corollary 5.11. *Let $\Psi_{\hbar,m}$ be the image of $\Psi_{q,m}$ in*

$$U_{\hbar,loc}(\mathfrak{h}) \tilde{\otimes} U_\hbar(\mathfrak{b}^+) \hat{\otimes} U'_\hbar(\mathfrak{n}^-)[\hbar^{-1}],$$

the sum

$$\Psi_\hbar = \sum_{m\in\mathbb{N}^r} \Psi_{\hbar,m}$$

is convergent in the $\hbar$-adic topology and has nonnegative $\hbar$-adic valuation. In other words, $\Psi_\hbar \in U'_{\hbar,loc}(\mathfrak{h}) \tilde{\otimes} U_\hbar(\mathfrak{b}^+) \hat{\otimes} U_\hbar(\mathfrak{n}^-)$.

PROOF. Set

$$A_\hbar = U'_{\hbar,loc}(\mathfrak{h}) \tilde{\otimes} U_\hbar(\mathfrak{b}^+) \hat{\otimes} U_\hbar(\mathfrak{n}^-)$$

We will show that for any $m \in \mathbb{N}^r$, $\Psi_{\hbar,m} \in \hbar^{|m|} A_\hbar$ where $|m| = \sum_i m_i$. This will be proved by induction on $\omega = |m|$. If $\omega = 0$, this is true because $\Psi_{\hbar,0} = 1$. Assume that this holds for any m such that $|m| < \omega$. Let m_0 be such that $|m_0| = \omega$, according to the ABRR equation

$$(\mathrm{Id} - \mathrm{Ad}(q^{t_{\mathfrak{h}}^{1,2}+\frac{1}{2}t_{\mathfrak{h}}^{2,2}}))(\Psi_{\hbar,m_0}) = \sum_{\substack{m'+m''=m_0\\(m',m'')\neq(0,0)}} (\bar{\mathcal{R}}^{-1})_{m'}^{2,3} \mathrm{Ad}(q^{t_{\mathfrak{h}}^{1,2}+\frac{1}{2}t_{\mathfrak{h}}^{2,2}})(\Psi_{\hbar,m''})$$

that is

$$\Psi_{\hbar,m_0}(1 - e^{2\underline{dm_0 u}} \otimes q^{\mu(m_0)} q^{2\underline{dm_0 h}} \otimes 1) = \sum_{\substack{m'+m''=m_0\\(m',m'')\neq(0,0)}} (\bar{\mathcal{R}}^{-1})_{m'}^{2,3} \Psi_{\hbar,m''}(e^{2\underline{dm'' u}} \otimes q^{\mu(m'')} q^{2\underline{dm'' h}} \otimes 1) \quad (5.7)$$

It is well known (see for instance [**CP**]) that $(\bar{\mathcal{R}}^{-1})_{m'} \in \hbar^{|m'|} A_\hbar$, and by assumption $\Psi_{\hbar,m''} \in \hbar^{|m''|} A_\hbar$ implying that the right hand side of the above equation belongs to $\hbar^{|m_0|} A_\hbar$.

Let v be the $\hbar$-adic valuation of $\Psi_{\hbar,m_0}$, that is $\hbar^{-v}\Psi_{\hbar,m_0} \in A_\hbar \backslash \hbar A_\hbar$. Assume that $v < |m_0|$ and let $\psi = (\hbar^{-v}\Psi_{\hbar,m_0} \mod \hbar)$ which is a non-zero element of $U(\mathfrak{b}^+) \otimes U(\mathfrak{n}^-)[[u_1, \dots, u_r]][\frac{1}{\ell_\beta}]$, equation (5.7) implies that

$$\psi(1 - e^{2\underline{dm_0 u}} \otimes q^{\mu(m_0)} q^{2\underline{dm_0 h}} \otimes 1) = \psi((1 - e^{2\underline{dm_0 u}}) \otimes 1 \otimes 1) + O(\hbar) \in \hbar^{|m_0|-v} A_\hbar$$

meaning that

$$\psi((1 - e^{2\underline{dm_0 u}}) \otimes 1 \otimes 1) = 0$$

It leads to a contradiction because the linear map

$$\begin{array}{rcl} U(\mathfrak{b}^+)\otimes U(\mathfrak{n}^-)[[u_1,\dots,u_r]][\frac{1}{\ell_\beta}] & \to & U(\mathfrak{b}^+)\otimes U(\mathfrak{n}^-)[[u_1,\dots,u_r]][\frac{1}{\ell_\beta}] \\ x & \mapsto & x((1-e^{2\underline{dmu}})\otimes 1\otimes 1) \end{array}$$

is injective for $m\neq 0$. Hence, $\Psi_{\hbar,m_0}\in \hbar^{|m_0|}A_\hbar$ as required. □

5.1.2.3. *The mixed pentagon equation.*

Theorem 5.12. *The set $\{\Psi_{q,m}, m\in\mathbb{N}^r\}$ satisfies the mixed pentagon system of equations*

$$\forall m\in\mathbb{N}^r,\ \sum_{m=m'+m''}\Psi^{1,2,34}_{q,m'}\Psi^{12,3,4}_{q,m''}=\sum_{m=m'+m''}\Psi^{1,23,4}_{q,m'}\Psi^{1,2,3}_{q,m''}$$

in $U_q(\mathfrak{h})\otimes U_q(\mathfrak{b}^+)\otimes U_q(\mathfrak{g})\otimes U_q(\mathfrak{n}^-)[1/D^{12},1/D^{123}]$.

Proof. Again each $\Psi_{q,m}$ is identified with its image

$$\Psi_{\hbar,m}\in U'_{\hbar,loc}(\mathfrak{h})\tilde{\otimes}U_\hbar(\mathfrak{b}^+)\hat{\otimes}U_\hbar(\mathfrak{n}^-)[\hbar^{-1}].$$

Let

$$\Psi_\hbar=\sum_{m\in\mathbb{N}^r}\Psi_{\hbar,m}\in U'_{\hbar,loc}(\mathfrak{h})\tilde{\otimes}U_\hbar(\mathfrak{b}^+)\hat{\otimes}U_\hbar(\mathfrak{n}^-)$$

(see Corollary 5.11).

The ABRR equation implies that $\Psi^{1,2,34}_\hbar$ satisfies

$$\Psi^{1,2,34}_\hbar=(\mathcal{R}^{2,3})^{-1}(\mathcal{R}^{2,4})^{-1}q^{t_\mathfrak{h}^{2,3}/2+t_\mathfrak{h}^{2,4}/2}q^{t_\mathfrak{h}^{1,2}+t_\mathfrak{h}^{2,2}/2}\Psi^{1,2,34}_\hbar q^{-t_\mathfrak{h}^{1,2}-t_\mathfrak{h}^{2,2}/2}$$

Hence:

$$\Psi^{1,2,34}_\hbar\Psi^{12,3,4}_\hbar=(\mathcal{R}^{2,3})^{-1}(\mathcal{R}^{2,4})^{-1}q^{t_\mathfrak{h}^{2,3}/2+t_\mathfrak{h}^{2,4}/2}q^{t_\mathfrak{h}^{1,2}+t_\mathfrak{h}^{2,2}/2}\Psi^{1,2,34}_\hbar\Psi^{12,3,4}_\hbar q^{-t_\mathfrak{h}^{1,2}-t_\mathfrak{h}^{2,2}/2}$$

Similarly, $\Psi^{1,23,4}_\hbar$ satisfies

$$\Psi^{1,23,4}_\hbar=(\mathcal{R}^{3,4})^{-1}(\mathcal{R}^{2,4})^{-1}q^{t_\mathfrak{h}^{3,4}/2+t_\mathfrak{h}^{2,4}/2}q^{t_\mathfrak{h}^{1,2}+t_\mathfrak{h}^{1,3}+t_\mathfrak{h}^{2,2}/2+t_\mathfrak{h}^{3,3}/2+t_\mathfrak{h}^{2,3}}\Psi^{1,23,4}_\hbar q^{-t_\mathfrak{h}^{1,2}-t_\mathfrak{h}^{1,3}-t_\mathfrak{h}^{2,2}/2-t_\mathfrak{h}^{3,3}/2-t_\mathfrak{h}^{2,3}}$$

By the $\mathfrak{h}$-invariance of $\Psi_\hbar$:

$$\Psi^{1,23,4}_\hbar\Psi^{1,2,3}_\hbar=(\mathcal{R}^{23,4})^{-1}q^{t_\mathfrak{h}^{23,4}/2}q^{t_\mathfrak{h}^{1,2}+t_\mathfrak{h}^{1,3}+t_\mathfrak{h}^{2,2}/2+t_\mathfrak{h}^{3,3}/2+t_\mathfrak{h}^{2,3}}\Psi^{1,23,4}_\hbar\Psi^{1,2,3}_\hbar q^{-t_\mathfrak{h}^{1,2}-t_\mathfrak{h}^{1,3}-t_\mathfrak{h}^{2,2}/2-t_\mathfrak{h}^{3,3}/2-t_\mathfrak{h}^{2,3}} \quad (5.8)$$

Let A_r, A_l be the linear endomorphisms of $U'_{\hbar,loc}(\mathfrak{h})\tilde{\otimes}U_\hbar(\mathfrak{b}^+)\hat{\otimes}U_\hbar(\mathfrak{g})\hat{\otimes}U_\hbar(\mathfrak{n}^-)$ defined by:

$$A_r(X)=(\mathcal{R}^{2,3})^{-1}(\mathcal{R}^{2,4})^{-1}q^{t_\mathfrak{h}^{2,3}/2+t_\mathfrak{h}^{2,4}/2}q^{t_\mathfrak{h}^{1,2}+t_\mathfrak{h}^{2,2}/2}Xq^{-t_\mathfrak{h}^{1,2}-t_\mathfrak{h}^{2,2}/2}$$

and

$$A_l(X)=(\mathcal{R}^{3,4})^{-1}(\mathcal{R}^{2,4})^{-1}q^{t_\mathfrak{h}^{3,4}/2+t_\mathfrak{h}^{2,4}/2}q^{t_\mathfrak{h}^{1,2}+t_\mathfrak{h}^{1,3}+t_\mathfrak{h}^{2,2}/2+t_\mathfrak{h}^{3,3}/2+t_\mathfrak{h}^{2,3}}X q^{-t_\mathfrak{h}^{1,2}-t_\mathfrak{h}^{1,3}-t_\mathfrak{h}^{2,2}/2-t_\mathfrak{h}^{3,3}/2-t_\mathfrak{h}^{2,3}}$$

Lemma 5.13. *The operators A_l and A_r commute.*

PROOF. It reduces to show the equality:

$$(\mathcal{R}^{2,34})^{-1}q^{t_{\mathfrak{h}}^{2,34}/2}q^{t_{\mathfrak{h}}^{1,2}+t_{\mathfrak{h}}^{2,2}/2}(\mathcal{R}^{23,4})^{-1}q^{t_{\mathfrak{h}}^{23,4}/2}q^{t_{\mathfrak{h}}^{1,23}+t_{\mathfrak{h}}^{2,2}/2+t_{\mathfrak{h}}^{3,3}/2+t_{\mathfrak{h}}^{2,3}}$$
$$= (\mathcal{R}^{23,4})^{-1}q^{t_{\mathfrak{h}}^{23,4}/2}q^{t_{\mathfrak{h}}^{1,23}+t_{\mathfrak{h}}^{2,2}/2+t_{\mathfrak{h}}^{3,3}/2+t_{\mathfrak{h}}^{2,3}}(\mathcal{R}^{2,34})^{-1}q^{t_{\mathfrak{h}}^{2,34}/2}q^{t_{\mathfrak{h}}^{1,2}+t_{\mathfrak{h}}^{2,2}/2} \quad (5.9)$$

It is done by a direct computation:

$$(\mathcal{R}^{2,34})^{-1}q^{t_{\mathfrak{h}}^{2,34}/2}q^{t_{\mathfrak{h}}^{1,2}+t_{\mathfrak{h}}^{2,2}/2}(\mathcal{R}^{23,4})^{-1}q^{t_{\mathfrak{h}}^{23,4}/2}q^{t_{\mathfrak{h}}^{1,23}+t_{\mathfrak{h}}^{2,2}/2+t_{\mathfrak{h}}^{3,3}/2+t_{\mathfrak{h}}^{2,3}/2+t_{\mathfrak{h}}^{2,3}/2}$$
$$=(\mathcal{R}^{2,34})^{-1}(\mathcal{R}^{3,4})^{-1}q^{t_{\mathfrak{h}}^{2,3}/2+t_{\mathfrak{h}}^{2,4}/2}q^{t_{\mathfrak{h}}^{1,2}+t_{\mathfrak{h}}^{2,2}/2}(\mathcal{R}^{2,4})^{-1}$$
$$\times q^{t_{\mathfrak{h}}^{2,34}/2}q^{t_{\mathfrak{h}}^{3,24}/2}q^{t_{\mathfrak{h}}^{1,2}}q^{t_{\mathfrak{h}}^{1,3}}q^{t_{\mathfrak{h}}^{2,2}/2+t_{\mathfrak{h}}^{3,3}/2}$$
$$=(\mathcal{R}^{23,4})^{-1}(\mathcal{R}^{2,3})^{-1}q^{t_{\mathfrak{h}}^{2,3}/2+t_{\mathfrak{h}}^{24}/2}q^{t_{\mathfrak{h}}^{1,2}+t_{\mathfrak{h}}^{2,2}/2+t_{\mathfrak{h}}^{3,3}/2}$$
$$\times q^{t_{\mathfrak{h}}^{3,24}/2}q^{t_{\mathfrak{h}}^{1,3}}(\mathcal{R}^{2,4})^{-1}q^{t_{\mathfrak{h}}^{2,34}/2}q^{t_{\mathfrak{h}}^{1,2}}q^{t_{\mathfrak{h}}^{2,2}/2}$$
$$=(\mathcal{R}^{23,4})^{-1}(\mathcal{R}^{2,3})^{-1}q^{t_{\mathfrak{h}}^{23,4}/2}q^{t_{\mathfrak{h}}^{1,23}}q^{t_{\mathfrak{h}}^{2,2}/2+t_{\mathfrak{h}}^{3,3}/2+t_{\mathfrak{h}}^{2,3}}(\mathcal{R}^{2,4})^{-1}q^{t_{\mathfrak{h}}^{2,34}/2}q^{t_{\mathfrak{h}}^{1,2}}q^{t_{\mathfrak{h}}^{2,2}/2}$$
$$=(\mathcal{R}^{23,4})^{-1}q^{t_{\mathfrak{h}}^{23,4}/2}q^{t_{\mathfrak{h}}^{1,23}}q^{t_{\mathfrak{h}}^{2,2}/2+t_{\mathfrak{h}}^{3,3}/2+t_{\mathfrak{h}}^{2,3}}(\mathcal{R}^{2,3})^{-1}(\mathcal{R}^{2,4})^{-1}q^{t_{\mathfrak{h}}^{2,34}/2}q^{t_{\mathfrak{h}}^{1,2}}q^{t_{\mathfrak{h}}^{2,2}/2}$$

□

The above Lemma implies that

$$A_r(A_l(\Psi_\hbar^{1,2,34}\Psi_\hbar^{12,3,4})) = A_l(\Psi_\hbar^{1,2,34}\Psi_\hbar^{12,3,4})$$

Any $\mathfrak{h}$-invariant solution of the equation $A_r(X) = X$ is uniquely determined by its degree zero part with respect to the gradation on its last component. Hence,

$$A_l(\Psi_\hbar^{1,2,34}\Psi_\hbar^{12,3,4}) = \Psi_\hbar^{1,2,34}\Psi_\hbar^{12,3,4}$$

Then, both $\Psi_\hbar^{1,2,34}\Psi_\hbar^{12,3,4}$ and $\Psi_\hbar^{1,23,4}\Psi_\hbar^{1,2,3}$ are $\mathfrak{h}$-invariant solutions of the equation $A_l(X) = X$, and their degree zero parts with respect to the last component are both $\Psi_\hbar^{1,2,3}$, which implies that they are equal. □

5.1.3. The shifted dynamical twist. Let $\nu = (\nu_1, \ldots, \nu_r) \in \mathbb{C}^r$ and assume that for any $\beta = \sum_i \beta_i\alpha_i \in R^+$, $\prod_i \nu_i^{\beta_i} \neq 1$. Define an automorphism of $U_q(\mathfrak{g})$ by

$$\forall i \in \{1, \ldots, r\},\ \sigma_\nu(e_i^\pm) = \nu_i^{\pm 1}e_i^\pm, \quad \sigma_\nu(k_i^{\pm 1}) = k_i^{\pm 1}$$

For all $m \in \mathbb{N}^r$, define $\Psi_{q,m}^{\sigma_\nu} = (Sh_\nu \otimes \mathrm{Id}^{\otimes 2})(\Psi_{q,m})$ which belongs to $U_q(\mathfrak{h}) \otimes U_q(\mathfrak{b}^+) \otimes U_q(\mathfrak{n}^-)[1/D_\nu^{12}]$ according to Lemma 5.6.

Theorem 5.14. *(a) The set $\{\Psi_{q,m}^{\sigma_\nu}, m \in \mathbb{N}^r\}$ satisfies the system of mixed pentagon equation (see Theorem 5.12)*

(b) The set $\{\Psi_{q,m}^{\sigma_\nu}, m \in \mathbb{N}^r\}$ satisfies the modified ABRR system

$$\forall m \in \mathbb{N}^r, \Psi_{q,m}^{\sigma_\nu} = \sum_{m=m'+m''} (\bar{\mathcal{R}}^{-1})_{m'}^{2,3}\underline{\mathrm{Ad}(q^{t_{\mathfrak{h}}^{1,2}+t_{\mathfrak{h}}^{2,2}/2})} \circ (\mathrm{Id} \otimes \sigma_\nu \otimes \mathrm{Id})(\Psi_{q,m''}^{\sigma_\nu}) \quad (5.10)$$

PROOF. Part (a) follows from the fact that Sh_ν commutes with the various insertion-coproduct morphisms. For proving (b) it is enough to show that the following diagram commutes:

$$\begin{array}{ccc} U_q(\mathfrak{h})\otimes U_q(\mathfrak{b}^+) & \xrightarrow{\underline{\mathrm{Ad}(q^{t_\mathfrak{h}})}} & U_q(\mathfrak{h})\otimes U_q(\mathfrak{b}^+) \\ \downarrow{\scriptstyle \mathrm{Sh}_\nu\otimes\mathrm{Id}} & & \downarrow{\scriptstyle \mathrm{Sh}_\nu\otimes\mathrm{Id}} \\ U_q(\mathfrak{h})\otimes U_q(\mathfrak{b}^+) & \xrightarrow[\underline{\mathrm{Ad}(q^{t_\mathfrak{h}})}\circ\sigma_\nu^{(2)}]{} & U_q(\mathfrak{h})\otimes U_q(\mathfrak{b}^+) \end{array}$$

As the restriction of $\underline{\mathrm{Ad}(q^{t_\mathfrak{h}})}\circ\sigma_\nu^{(2)}$ and $\underline{\mathrm{Ad}(q^{t_\mathfrak{h}})}$ to $U_q(\mathfrak{h})^{\otimes 2}$ is the identity, it suffices to prove it for the $1\otimes e_i, i=1\ldots r$. Indeed, on the one hand

$$\begin{aligned}\mathrm{Sh}_\nu\circ\underline{\mathrm{Ad}(q^{t_\mathfrak{h}})}(1\otimes e_i) &= \mathrm{Sh}_\nu(k_i^2\otimes e_i)\\ &= \nu_i k_i^2\otimes e_i\end{aligned}$$

on the other hand

$$\begin{aligned}\underline{\mathrm{Ad}(q^{t_\mathfrak{h}})}\circ\sigma_\nu^{(2)}\circ\mathrm{Sh}_\nu(1\otimes e_i) &= \underline{\mathrm{Ad}(q^{t_\mathfrak{h}})}\circ\sigma_\nu^{(2)}(1\otimes e_i)\\ &= \nu_i k_i^2\otimes e_i\end{aligned}$$

□

Using Lemma 5.6 and by the assumption on ν, $\Psi_{q,m}^{\sigma_\nu}\in U_q(\mathfrak{h})\otimes U_q(\mathfrak{b}^+)\otimes U_q(\mathfrak{n}^-)[1/D_\nu^{12}]$ can be identified with an element $(\Psi_{\hbar,\sigma_\nu})_m\in U'_\hbar(\mathfrak{h})\hat\otimes U_\hbar(\mathfrak{b}^+)\hat\otimes U_\hbar(\mathfrak{n}^-)[\hbar^{-1}]$. Let us first prove:

Lemma 5.15. *The sum*

$$\Psi_{\hbar,\sigma_\nu}=\sum_{m\in\mathbb{N}^r}(\Psi_{\hbar,\sigma_\nu})_m$$

is convergent in the $\hbar$-adic topology and has non-negative $\hbar$-adic valuation, hence defines an element of $U'_\hbar(\mathfrak{h})\hat\otimes U_\hbar(\mathfrak{b}^+)\hat\otimes U_\hbar(\mathfrak{n}^-)$ of the form $1+O(\hbar)$.

PROOF. We show by induction on $|m|$ that for all $m\in\mathbb{N}^r$,

$$(\Psi_{\hbar,\sigma_\nu})_m\in\hbar^{|m|}(U'_\hbar(\mathfrak{h})\hat\otimes U_\hbar(\mathfrak{b}^+)\hat\otimes U_\hbar(\mathfrak{n}^-)).$$

The modified ABRR equation leads to the relation

$$\begin{aligned}&(\Psi_{\hbar,\sigma_\nu})_{m_0}(1-\nu^{\underline{m_0}}e^{2\underline{dm_0u}}\otimes q^{\mu(m_0)}q^{2\underline{dm_0h}}\otimes 1)\\ &=\sum_{\substack{m'+m''=m_0\\(m',m'')\neq(0,0)}}(\bar{\mathcal{R}}^{-1})^{2,3}_{m'}(\Psi_{\hbar,\sigma_\nu})_{m''}(\nu^{\underline{m''}}e^{2\underline{dm''u}}\otimes q^{\mu(m'')}q^{2\underline{dm''h}}\otimes 1)\end{aligned}\qquad(5.11)$$

Then, as in the proof of Corollary 5.11 one shows that the $\hbar$-adic valuation of $(\Psi_{\hbar,\sigma})_m$ is at least $|m|$ by remarking that the map

$$\begin{array}{rcl}U(\mathfrak{b}^+)\otimes U(\mathfrak{n}^-)[[u_1,\ldots,u_r]] & \to & U(\mathfrak{b}^+)\otimes U(\mathfrak{n}^-)[[u_1,\ldots,u_r]]\\ x & \mapsto & x((1-\nu^{\underline{m}}e^{2\underline{dmu}})\otimes 1\otimes 1)\end{array}$$

is injective for $m\neq 0$. The last statement follows from the fact that $(\Psi_{\hbar,\sigma_\nu})_0=1$.

□

We finally get the desired result:

Theorem 5.16. *The element $\Psi_{\hbar,\sigma_\nu}$ satisfies the mixed pentagon equation*

$$(\Psi_{\hbar,\sigma_\nu})^{1,2,34}(\Psi_{\hbar,\sigma_\nu})^{12,3,4} = (\Psi_{\hbar,\sigma_\nu})^{1,23,4}(\Psi_{\hbar,\sigma_\nu})^{1,2,3}$$

in $U_\hbar(\mathfrak{h})\hat{\otimes}U_\hbar(\mathfrak{g})^{\hat{\otimes}3}$ and the modified ABRR equation

$$\Psi_{\hbar,\sigma_\nu} = (\bar{\mathcal{R}}^{-1})^{2,3}\,\mathrm{Ad}(q^{t_{\mathfrak{h}}^{1,2}+t_{\mathfrak{h}}^{2,2}/2}) \circ (\mathrm{Id}\otimes\sigma_\nu\otimes\mathrm{Id})(\Psi_{\hbar,\sigma_\nu})$$

in $U_\hbar(\mathfrak{h})\hat{\otimes}U_\hbar(\mathfrak{g})^{\hat{\otimes}2}$.

Proof. For a given $m \in \mathbb{N}^r$,

$$\sum_{m'+m''=m} (\Psi_{\hbar,\sigma_\nu})^{1,2,34}_{m'}(\Psi_{\hbar,\sigma_\nu})^{12,3,4}_{m''}$$

and

$$\sum_{m'+m''=m} (\Psi_{\hbar,\sigma_\nu})^{1,23,4}_{m'}(\Psi_{\hbar,\sigma_\nu})^{1,2,3}_{m''}$$

are equal according to Theorem 5.14, and both belong to $\hbar^{|m|}U'_\hbar(\mathfrak{h})\hat{\otimes}U_\hbar(\mathfrak{b}^+)\hat{\otimes}U_\hbar(\mathfrak{n}^-)$. Hence the sums over m of these expressions are convergent in the $\hbar$-adic topology and:

$$\sum_{m\in\mathbb{N}^r}\sum_{m'+m''=m} (\Psi_{\hbar,\sigma_\nu})^{1,2,34}_{m'}(\Psi_{\hbar,\sigma_\nu})^{12,3,4}_{m''} = \sum_{m\in\mathbb{N}^r}\sum_{m'+m''=m} (\Psi_{\hbar,\sigma_\nu})^{1,23,4}_{m'}(\Psi_{\hbar,\sigma_\nu})^{1,2,3}_{m''}$$

which implies the mixed pentagon equation. A similar argument implies the second statement. □

5.2. ABRR and the octagon equation

Let $\nu = (\nu_1, \dots, \nu_r) \in (\mathbb{C}^\times)^r$, $\sigma = \sigma_\nu$ (as in section 5.1.3). Let $A_{alg} = U_\hbar(\mathfrak{g}) \rtimes_\sigma \mathbb{Z}$ (the generator of $\mathbb{Z}$ is denoted by $\tilde{\sigma}$) be equipped with the coproduct $\Delta_\hbar$ extended to A_{alg} by setting $\Delta_\hbar(\tilde{\sigma}) = \tilde{\sigma}\otimes\tilde{\sigma}$. Let $B_{alg} = U_\hbar(\mathfrak{h}) \subset A_{alg}$ and set $\Delta_{B_{alg}} = (\Delta_\hbar)_{|B_{alg}}$. Let $\mathcal{R} \in U_\hbar(\mathfrak{g})^{\hat{\otimes}2}$ be the R-matrix of $U_\hbar(\mathfrak{g})$ and $K = q^{t_{\mathfrak{h}}/2}$. Recall that $\bar{\mathcal{R}} = K^{-1}\mathcal{R} \in U_\hbar(\mathfrak{n}^+)\hat{\otimes}U_\hbar(\mathfrak{n}^-)$.

Proposition 5.17. *Let $\Psi \in (B_{alg}\hat{\otimes}U_\hbar(\mathfrak{b}^+)\hat{\otimes}U_\hbar(\mathfrak{n}^-))^\times$ and $E \in (U_\hbar(\mathfrak{h})\hat{\otimes}(U_\hbar(\mathfrak{h})\rtimes_\sigma \mathbb{Z}))^\times$. Then the octagon equation (4.3) for $(E, \Psi, \mathcal{R})$ is equivalent to the following system of equations:*

$$\bar{\mathcal{R}}^{3,2}\Psi^{1,3,2}E^{1,3} = E^{1,3}\Psi^{1,3,2} \tag{5.12a}$$

$$\Psi E^{12,3} = E^{12,3}\bar{\mathcal{R}}^{2,3}\Psi \tag{5.12b}$$

$$E^{12,3} = (K^{2,3})^2E^{1,3} \tag{5.12c}$$

Proof. Rewrite the octagon equation as

$$\Psi E^{12,3}\Psi^{-1}(\mathcal{R}^{2,3})^{-1} = \mathcal{R}^{3,2}\Psi^{1,3,2}E^{1,3}(\Psi^{1,3,2})^{-1}$$

The left-hand side belongs to $U_\hbar(\mathfrak{h})\hat{\otimes}U_\hbar(\mathfrak{b}^+)\hat{\otimes}(U_\hbar(\mathfrak{b}^-)\rtimes_\sigma\mathbb{Z})$, and the right-hand side to $U_\hbar(\mathfrak{h})\hat{\otimes}U_\hbar(\mathfrak{b}^-)\hat{\otimes}(U_\hbar(\mathfrak{b}^+)\rtimes_\sigma\mathbb{Z})$. Thus, each side has to coincide with its degree (0,0) part, and their degree (0,0) parts have to be equal as well (here $U_\hbar(\mathfrak{b}^\pm)\rtimes_\sigma\mathbb{Z}$ is graded by $|e_i^\pm| = \delta_i, |h_i| = |\tilde{\sigma}| = 0$). Denoting by

$\tilde{\Psi}$ the image of Ψ under this projection, the octagon equation is equivalent to:

$$\begin{aligned} \Psi E^{12,3}\Psi^{-1}(\bar{\mathcal{R}}^{2,3})^{-1}(K^{2,3})^{-1} &= \tilde{\Psi}E^{12,3}\tilde{\Psi}^{-1}(K^{2,3})^{-1} \\ K^{2,3}\bar{\mathcal{R}}^{3,2}\Psi^{1,3,2}E^{1,3}(\Psi^{1,3,2})^{-1} &= K^{2,3}\tilde{\Psi}^{1,3,2}E^{1,3}(\tilde{\Psi}^{1,3,2})^{-1} \\ \tilde{\Psi}E^{12,3}\tilde{\Psi}^{-1}(K^{2,3})^{-1} &= K^{2,3}\tilde{\Psi}^{1,3,2}E^{1,3}(\tilde{\Psi}^{1,3,2})^{-1} \end{aligned}$$

Then, the commutativity of $U_\hbar(\mathfrak{h}) \rtimes_\sigma \mathbb{Z}$ implies that $\tilde{\Psi}$ and $\tilde{\Psi}^{-1}$ cancel out, which leads to the system (5.12). □

We make the system (5.12) explicit in a particular case.

Theorem 5.18. *Set $E_{\hbar,\sigma} = q^{t_{\mathfrak{h}}^{1,2}+\frac{1}{2}t_{\mathfrak{h}}^{2,2}}(1\otimes\tilde{\sigma})$. For any solution $\Psi \in (U'_\hbar(\mathfrak{h})\otimes U_\hbar(\mathfrak{b}^+)\otimes U_\hbar(\mathfrak{b}^-))^{\mathfrak{h}}$ of the linear equation*

$$\Psi^{1,2,3}(E_\hbar^\sigma)^{1,2} = (\bar{\mathcal{R}}^{2,3})^{-1}(E_\hbar^\sigma)^{1,2}\Psi^{1,2,3}, \tag{5.13}$$

the triple $(E_{\hbar,\sigma}, \Psi, \mathcal{R})$ satisfies the octagon equation.

Proof. Let us first check (5.12c):

$$\begin{aligned} (\Delta_\hbar \otimes \mathrm{Id})(q^{t_{\mathfrak{h}}^{1,2}+\frac{1}{2}t_{\mathfrak{h}}^{2,2}}(1\otimes\tilde{\sigma})) &= q^{t_{\mathfrak{h}}^{1,3}+t_{\mathfrak{h}}^{2,3}+\frac{1}{2}t_{\mathfrak{h}}^{3,3}}(1\otimes 1\otimes\tilde{\sigma}) \\ &= q^{t_{\mathfrak{h}}^{2,3}}q^{t_{\mathfrak{h}}^{1,3}+\frac{1}{2}t_{\mathfrak{h}}^{3,3}}(1\otimes 1\otimes\tilde{\sigma}) \\ &= (K^{2,3})^2(E_{\hbar,\sigma})^{1,3}. \end{aligned}$$

Recall from equation (2.1), section 2.3 that $E_{\hbar,\sigma}$ satisfies:

$$(E_{\hbar,\sigma})^{1,23} = (E_{\hbar,\sigma})^{1,2}(E_{\hbar,\sigma})^{1,3}(K^{2,3})^2 \tag{5.14}$$

Equation (5.13) is the same as (5.12a), thus, starting from (5.12a), permuting the two last component and multiplying by $(E_{\hbar,\sigma})^{1,3}(K^{2,3})^2$ on the right, one gets

$$\bar{\mathcal{R}}^{2,3}\Psi^{1,2,3}(E_{\hbar,\sigma})^{1,2}(E_{\hbar,\sigma})^{1,3}(K^{2,3})^2 = (E_{\hbar,\sigma})^{1,2}\Psi^{1,2,3}(E_{\hbar,\sigma})^{1,3}(K^{2,3})^2$$

Then, using equation (2.1)

$$\bar{\mathcal{R}}^{2,3}\Psi(E_{\hbar,\sigma})^{1,23} = (E_{\hbar,\sigma})^{1,2}\Psi(E_{\hbar,\sigma})^{1,3}(K^{2,3})^2 \tag{5.15}$$

By the $\mathfrak{h}$-invariance of $\bar{\mathcal{R}}$ and Ψ and the commutativity of $U_\hbar(\mathfrak{h})$, $\bar{\mathcal{R}}^{2,3}\Psi$ commutes with $(E_{\hbar,\sigma})^{1,23}$, and $(E_{\hbar,\sigma})^{1,3}$ commutes with $(K^{2,3})^2$. Using Equation (5.12c), equation (5.15) then implies

$$(E_{\hbar,\sigma})^{1,23}\bar{\mathcal{R}}^{2,3}\Psi = (E_{\hbar,\sigma})^{1,2}\Psi(E_{\hbar,\sigma})^{12,3} \tag{5.16}$$

Equations (5.14) and (5.12c) together with the commutativity of $U_\hbar(\mathfrak{h})$ implies that $(E_{\hbar,\sigma})^{1,23} = (E_{\hbar,\sigma})^{1,2}(E_{\hbar,\sigma})^{12,3}$. Equation (5.16) then implies

$$(E_{\hbar,\sigma})^{1,2}(E_{\hbar,\sigma})^{12,3}\bar{\mathcal{R}}^{2,3}\Psi = (E_{\hbar,\sigma})^{1,2}\Psi(E_{\hbar,\sigma})^{12,3}$$

which implies

$$(E_{\hbar,\sigma})^{12,3}\bar{\mathcal{R}}^{2,3}\Psi = \Psi(E_{\hbar,\sigma})^{12,3}$$

The last line is (5.12b). Then, by Proposition 5.18, $(E_{\hbar,\sigma}, \Psi, \mathcal{R})$ satisfies the octagon equation. □

5.3. A QRA over $U_\hbar(\mathfrak{g})$

We are now in a position to state the main Theorem of this section:

Theorem 5.19. *$B_{alg,\sigma} = (U_\hbar(\mathfrak{h}), \Delta_\hbar, \Psi_{\hbar,\sigma}, E_{\hbar,\sigma})$ is a QRA over $A_{alg} = U_\hbar(\mathfrak{g}) \rtimes_\sigma \mathbb{Z}$.*

Proof. The relation (4.4) is clear and the relation (4.1) is the $\mathfrak{h}$-invariance of $\Psi_{\hbar,\sigma}$. The element $\Psi_{\hbar,\sigma}$ belongs to $U_\hbar(\mathfrak{h})\hat{\otimes}U_\hbar(\mathfrak{g})^{\hat{\otimes}2}$ and satisfies the mixed pentagon equation (4.2) with trivial associator according to Theorem 5.16. Equation (5.13) of Theorem 5.18 is nothing but the ABRR equation (5.10) of Theorem 5.16, meaning that $\Psi_{\hbar,\sigma}$ satisfies the octagon equation with $(E_{\hbar,\sigma}, \mathcal{R})$. □

5.4. An explicit formula for the representations of B_n^1

Being a QRA, $B_{alg,\sigma}$ induces a group morphism $\rho_\hbar : B_n^1 \to (B_{alg,\sigma} \otimes A_{alg}^{\otimes n})^\times$.

Theorem 5.20. *The morphism $\rho_\hbar$ coincides with the morphism given in Theorem 2.5.*

Proof. According to Section 4.1, the image of the generator τ is

$$(\Psi_{\hbar,\sigma}^{0,1,2\ldots n}) E_{\hbar,\sigma}^{0,1} (\Psi_{\hbar,\sigma}^{0,1,2\ldots n})^{-1}$$

that is

$$(\mathrm{Id}_{U_\hbar(\mathfrak{h})} \otimes \mathrm{Id}_{U_\hbar(\mathfrak{g})} \otimes \Delta_\hbar^{(n-1)} (\Psi_{\hbar,\sigma} E_{\hbar,\sigma}^{0,1} \Psi_{\hbar,\sigma}^{-1})$$

Thanks to (5.13), this is equal to

$$(\mathrm{Id}_{U_\hbar(\mathfrak{h})} \otimes \mathrm{Id}_{U_\hbar(\mathfrak{g})} \otimes \Delta_\hbar^{(n-1)})((\mathcal{R}^{1,2})^{-1} K^{1,2} E_{\hbar,\sigma}^{0,1})$$

We then use the fact the $K^{1,23} = K^{1,2}K^{1,3}$ and $\mathcal{R}^{1,23} = \mathcal{R}^{1,3}\mathcal{R}^{1,2}$ to identify this with the image of τ in Theorem 2.5. □

CHAPTER 6

Equivalence

6.1. Preliminaries

Let $U(\mathfrak{h})[[\hbar]]'$ be the QFSHA [Ga] associated to the Quantized Enveloping Algebra $U(\mathfrak{h})[[\hbar]]$. Under the identification $U(\mathfrak{h}) \cong \mathbb{C}[h_1, \ldots, h_r]$, $U(\mathfrak{h})[[\hbar]]' \cong \mathbb{C}[[\hbar h_1, \ldots, \hbar h_r, \hbar]] \subset \mathbb{C}[h_1, \ldots, h_r][[\hbar]]$.

We have constructed two QRA's:

(i) $B_{alg} = (U_\hbar(\mathfrak{h}), \Delta_\hbar, \Psi_{\hbar,\sigma}, E_{\hbar,\sigma})$ over $A_{alg} = (U_\hbar(\mathfrak{g}) \rtimes_{\sigma_\hbar} \mathbb{Z}, \Delta_\hbar, \Phi = 1, \mathcal{R})$, where $\Psi_{\hbar,\sigma}$ belongs to $U'_\hbar(\mathfrak{h}) \hat{\otimes} U_\hbar(\mathfrak{g})^{\hat{\otimes} 2}$

(ii) $B_{\mathrm{KZ}} = (U(\mathfrak{h})[[\hbar]], \Delta_0, \Psi_{\mathrm{KZ},\sigma}, E_{\mathrm{KZ},\sigma})$ over $A_{\mathrm{KZ}} = (U(\mathfrak{g})[[\hbar]] \rtimes_\sigma \mathbb{Z}, \Delta_0, \Phi_{\mathrm{KZ}}, \mathcal{R}_{\mathrm{KZ}})$, where $\Psi_{\mathrm{KZ},\sigma}$ belongs to $U(\mathfrak{h})[[\hbar]]' \hat{\otimes} U(\mathfrak{g})^{\hat{\otimes} 2}[[\hbar]]$ according to [EE2, Prop. 4.7].

Theorem 6.1. *These two QRA's are twist equivalent.*

Let V be a $\mathfrak{g}$-module, $V_\mathfrak{h}$ the corresponding $U_\hbar(\mathfrak{g})$-module, W an $\mathfrak{h}$-module and $W_\hbar = W[[\hbar]]$ the corresponding $U_\hbar(\mathfrak{h})$-module. From the data of $(\mathfrak{g}, t, \sigma)$, one can construct:

- a morphism $\rho_{\mathrm{KZ}} : B_n^1 \to GL(W \otimes V^{\otimes n}[[\hbar]])$ using the KZ equation
- a morphism $\rho_\hbar : B_n^1 \to GL(W_\hbar \otimes V_\hbar^{\otimes n})$ using the QRA B_{alg}.

Corollary 6.2. *The representations ρ_{KZ} and $\rho_\hbar$ are equivalent.*

This chapter is devoted to the proof of this Theorem. We first apply a twist bringing A_{alg} to A_{KZ}, the subalgebra B_{alg} to B_{KZ} and E_{alg} to E_{KZ}. Then, both $\Psi = \Psi_{\mathrm{KZ},\sigma}$ and the image $\tilde{\Psi}$ of $\Psi_{\hbar,\sigma}$ satisfy the mixed pentagon equation with Φ_{KZ}. Using deformation theory arguments, we then prove that Ψ and $\tilde{\Psi}$ are related by an infinitesimal functional shift and an $\mathfrak{h}$-invariant twist. The fact that both Ψ and the twisted $\tilde{\Psi}$ satisfy the octagon equation with E_{KZ} and $\mathcal{R}_{\mathrm{KZ}}$ implies that the shift is actually trivial.

Let us first recall the following

Theorem 6.3 ([Dr3, Dr4])**.** *There exists an algebra isomorphism $\alpha : U_\hbar(\mathfrak{g}) \to U(\mathfrak{g})[[\hbar]]$ and a twist $F \in U(\mathfrak{g})^{\otimes 2}[[\hbar]]$ such that:*

(a) $(U(\mathfrak{g})[[\hbar]], \Delta_0, \Phi_{\rm KZ}, \mathcal{R}_{\rm KZ}) = \alpha((U_\hbar(\mathfrak{g}), \Delta_\hbar, 1, \mathcal{R}))^F$ *as QTQHA*
(b) α *restricts to the canonical isomorphism* $U_\hbar(\mathfrak{h}) \to U(\mathfrak{h})[[\hbar]]$
(c) F *is* $\mathfrak{h}$*-invariant*

Corollary 6.4. *The isomorphism* α *extends to an algebra isomorphism* $\alpha : A_{alg} \to A_{\rm KZ}$ *and* $A_{\rm KZ} = \alpha(A_{alg})^F$ *as QTQHA.*

Proof. Set $\alpha(\tilde{\sigma}_\hbar) = \tilde{\sigma}$. Both $U_\hbar(\mathfrak{g})$ and $U(\mathfrak{g})[[\hbar]]$ are direct sums of their weight subspaces, and α preserves these decompositions thanks to property (b). Since $\sigma_\hbar$ acts on a given weight subspace of $U_\hbar(\mathfrak{g})$ as σ does on the corresponding weight subspace of $U(\mathfrak{g})[[\hbar]]$, α intertwines the action of these automorphisms. This implies the first statement.

Being $\mathfrak{h}$-invariant, F belongs to $(\bigoplus_{m\in\mathbb{Z}^r} U(\mathfrak{g})[m]\otimes U(\mathfrak{g})[-m])[[\hbar]]$, hence $(\sigma\otimes\sigma)(F) = F$. Therefore,

$$\begin{aligned} F\left(\alpha^{\otimes 2}\circ\Delta_\hbar\circ\alpha^{-1}(\tilde{\sigma})\right)F^{-1} &= F(\tilde{\sigma}\otimes\tilde{\sigma})F^{-1} \\ &= F\sigma^{\otimes 2}(F^{-1})(\tilde{\sigma}\otimes\tilde{\sigma}) \\ &= \Delta_0(\tilde{\sigma}) \end{aligned}$$

which implies the second statement. □

We then apply the twist $(F, 1)$ to the QRA B_{alg}. The restriction of α leads to the algebra isomorphism $U_\hbar(\mathfrak{h}) \to U(\mathfrak{h})[[\hbar]]$ given by $h_i \mapsto h_i$. Hence, it maps the subalgebra $U'_\hbar(\mathfrak{h})$ to $U(\mathfrak{h})[[\hbar]]'$ and $E_{\hbar,\sigma} = q^{t_\mathfrak{h}+\frac{1}{2}t_\mathfrak{h}^{2,2}}(1\otimes\tilde{\sigma}_\hbar)$ to $E_{\rm KZ} = e^{\hbar(t_\mathfrak{h}+\frac{1}{2}t_\mathfrak{h}^{2,2})}(1\otimes\tilde{\sigma})$. It leads to a QRA

(iii) $B = (U(\mathfrak{h})[[\hbar]], \Delta_0, E_{\rm KZ}, \Psi')$ over $A_{\rm KZ}$

Hence, it remains to prove that Ψ and Ψ' are related by a twist. We first show that they are related by a twist and a "shift" (Theorem 6.5), and then that the shift actually vanishes (Proposition 6.15).

6.2. Gauge transformations and shifts

Following [EE1], let $\mathcal{G}$ be the subgroup of $(U(\mathfrak{h})[[\hbar]]'\hat{\otimes}U(\mathfrak{g})^\mathfrak{h})^\times$ of elements of the form

$$G = 1 + \sum_{n\geq 1}\hbar^n g_n, g_n \in U(\mathfrak{h})\otimes U(\mathfrak{g})$$

such that the image of G through the tensor product of the reduction maps $U(\mathfrak{h})[[\hbar]]' \to \widehat{S(\mathfrak{h})}$ and $U(\mathfrak{g})[[\hbar]] \to U(\mathfrak{g})$ is of the form $\exp(q)$ where $q \in \widehat{S(\mathfrak{h})}\otimes\mathfrak{g}$. If $\Psi \in 1 + \hbar(U(\mathfrak{h})[[\hbar]]'\hat{\otimes}U(\mathfrak{g})^{\otimes 2}[[\hbar]])^\mathfrak{h}$ is a solution of the mixed pentagon equation, define the *gauge transformation* of Ψ by G by

$$G\star\Psi := G^{1,23}\Psi(G^{12,3})^{-1}(G^{1,2})^{-1}.$$

Then, $G\star\Psi$ belongs to $1+\hbar(U(\mathfrak{h})[[\hbar]]'\hat{\otimes}U(\mathfrak{g})^{\otimes 2}[[\hbar]])^\mathfrak{h}$ and satisfies the mixed pentagon equation. Indeed, a gauge transformation is a particular case of the notion of twist of a QRA as defined in section 4.1.

The notion of shift is defined in the following way: for $\mu\in\mathfrak{h}^*[[\hbar]]$, define an automorphism of $U(\mathfrak{h})[[\hbar]]$ by $h_i \mapsto h_i + \mu(h_i)$. This restricts to an automorphism sh_μ of $U(\mathfrak{h})[[\hbar]]' = \mathbb{C}[[u_1,\ldots,u_r,\hbar]])$ (we set $u_i = \hbar h_i$) given by $u_i \mapsto u_i + \hbar\mu(h_i)$. For $X \in U(\mathfrak{h})[[\hbar]]'\hat{\otimes}U(\mathfrak{g})^{\otimes n}[[\hbar]]$ set

$$X_\mu = (\mathrm{sh}_\mu\otimes\mathrm{Id}^{\otimes n})(X)$$

The action of $\mathfrak{h}^*[[\hbar]]$ restricts to an action on $\mathcal{G}$, and if $G \in \mathcal{G}$ and $\mu \in \mathfrak{h}^*[[\hbar]]$, one has

$$(G^{1,23}\Psi(G^{12,3})^{-1}(G^{1,2})^{-1})_\mu = G^{1,23}_\mu \Psi_\mu (G^{12,3}_\mu)^{-1}(G^{1,2}_\mu)^{-1}.$$

Hence, there is a well defined action of $\mathcal{G} \rtimes (\mathfrak{h}^*[[\hbar]])$ on the set $1+\hbar(U(\mathfrak{h})[[\hbar]]'\hat{\otimes}U(\mathfrak{g})^{\otimes 2}[[\hbar]])^{\mathfrak{h}}$.

6.3. Classification of dynamical pseudo twists

Let $\mathfrak{m} = \mathfrak{n}^+ \oplus \mathfrak{n}^- \subset \mathfrak{g}$, and for any $\alpha \in R^+$, choose basis elements $e_{\pm\alpha}$ of the root subspaces $\mathfrak{g}_{\pm\alpha}$ such that $(e_\alpha, e_{-\alpha}) = 1$. Hence,

$$t_{\mathfrak{m}} = t - t_{\mathfrak{h}} = 2 \sum_{\alpha \in R^+} (e_\alpha \otimes e_{-\alpha} + e_{-\alpha} \otimes e_\alpha) \in S^2(\mathfrak{m})$$

. In this section, we will prove the following:

Theorem 6.5. *Let $\Psi, \Psi' \in 1 + \hbar(U(\mathfrak{h})[[h]]'\hat{\otimes}U(\mathfrak{g})^{\otimes 2}[[\hbar]])^{\mathfrak{h}}$ be two solutions of the mixed pentagon equation*

$$\Psi^{1,2,34}\Psi^{12,3,4} = \Phi_{\mathrm{KZ}}^{2,3,4}\Psi^{1,23,4}\Psi^{1,2,3},$$

such that:

(a) the image of $\frac{\Psi-1}{\hbar}$ through the reduction map

$$U(\mathfrak{h})[[\hbar]]'\hat{\otimes}U(\mathfrak{g})^{\otimes 2}[[\hbar]] \subset U(\mathfrak{h}) \otimes U(\mathfrak{g})^{\otimes 2}[[\hbar]] \longrightarrow U(\mathfrak{h}) \otimes U(\mathfrak{g})^{\otimes 2}$$

is of the form $1 \otimes \rho_0$ where $\rho_0 \in (\wedge^2\mathfrak{m})^{\mathfrak{h}}$.

(b) if $\rho_0 = \sum_{\alpha \in R^+} \rho_\alpha (e_\alpha \wedge e_{-\alpha})$, where $\rho_\alpha \in \mathbb{C}$ then for any $\alpha \in R^+$, $\rho_\alpha \neq \pm\frac{1}{2}$.

(c)

$$\Psi - \Psi' \in \hbar^2 U(\mathfrak{h}) \otimes U(\mathfrak{g})^{\otimes 2}[[\hbar]]$$

Then there exists $(G, \mu) \in \mathcal{G} \rtimes \mathfrak{h}^*[[\hbar]]$ such that the relation

$$\Psi' = G \star \Psi_\mu$$

holds in $U(\mathfrak{h})[[\hbar]]'\hat{\otimes}U(\mathfrak{g})^{\otimes 2}[[\hbar]]$.

Proof. Let us explain the strategy of the proof: we will first prove the following:

Proposition 6.6. *If $\Psi, \Psi' \in 1+\hbar(U(\mathfrak{h})[[\hbar]]'\hat{\otimes}U(\mathfrak{g})^{\otimes 2}[[\hbar]])^{\mathfrak{h}}$ are two solutions of the mixed pentagon equation such that for some $n \geq 2$*

$$\Psi - \Psi' \in \hbar^n(U(\mathfrak{h}) \otimes U(\mathfrak{g})^{\otimes 2}[[\hbar]])$$

then there exists $(G, \mu) \in \mathcal{G} \rtimes \mathfrak{h}^*[[\hbar]]$ such that $G - 1 \in \hbar^n U(\mathfrak{h}) \otimes U(\mathfrak{g})[[\hbar]]$, $\mu \in \hbar^{n-2}\mathfrak{h}^*[[\hbar]]$ and

$$\Psi' - G \star \Psi_\mu \in \hbar^{n+1}(U(\mathfrak{h}) \otimes U(\mathfrak{g})^{\otimes 2}[[\hbar]]).$$

Then we construct an infinite sequence $(G_1, \mu_1), (G_2, \mu_2), \ldots, G_n \in 1 + \hbar^{n+1}(U(\mathfrak{h}) \otimes U(\mathfrak{g})[[\hbar]])^{\mathfrak{h}}$, $\mu_n \in \hbar^{n-1}\mathfrak{h}^*[[\hbar]]$, such that if $\Psi^{(0)} = \Psi$ and

$$\Psi^{(n)} = G_n \star \Psi^{(n-1)}_{\mu_n}$$

then $\Psi^{(n)} \to \Psi'$ in the $\hbar$-adic topology of $U(\mathfrak{h}) \otimes U(\mathfrak{g})^{\otimes 2}[[\hbar]]$. We construct this sequence recursively by applying Proposition 6.6 to the pair $(\Psi^{(n-1)}, \Psi')$.

Hence, thanks to the properties of the (G_n, μ_n), the product

$$\bar{G} = \prod_{n\geq 1}^{>} (G_n)_{\sum_{k>n}\mu_k}$$

converges in $U(\mathfrak{h}) \otimes U(\mathfrak{g})[[\hbar]]$, belongs to $U(\mathfrak{h})[[\hbar]]' \hat{\otimes} U(\mathfrak{g})[[\hbar]]$ (since this is a closed subspace of the latter for the $\hbar$-adic topology) and is invertible and $\mathfrak{h}$-invariant, and the sum

$$\bar{\mu} = \sum_{k\geq 1} \mu_k$$

converges in $\mathfrak{h}^*[[\hbar]]$. Finally, a direct computation shows that

$$\Psi' = \bar{G} \star \Psi_{\bar{\mu}}$$

implying the conclusion of the Theorem.

Let us recall some facts about the cohomological structure associated to the mixed pentagon equation. We define the cochain complex

$$C_1 = \bigoplus_{n\geq 0} (U(\mathfrak{h}) \otimes U(\mathfrak{g})^{\otimes n})^{\mathfrak{h}}$$

with differential

$$\begin{array}{rccl} d_1^{n,n+1}: & U(\mathfrak{h}) \otimes U(\mathfrak{g})^{\otimes n} & \longrightarrow & U(\mathfrak{h}) \otimes U(\mathfrak{g})^{\otimes n+1} \\ & x & \longmapsto & x^{1,2,\dots,n+1} + \sum_{i=1}^{n+1} (-1)^i x^{1,2,\dots,ii+1,\dots,n+2} \end{array}$$

Let $C_2 = \bigoplus (S(\mathfrak{m})^{\otimes n})^{\mathfrak{h}}$ together with the usual coHochschild differential

$$\begin{array}{rccl} d_2^{n,n+1}: & S(\mathfrak{m})^{\otimes n} & \longrightarrow & S(\mathfrak{m})^{\otimes n+1} \\ & x & \longmapsto & x^{1,2,\dots,n} + \sum_{i=1}^{n} (-1)^i x^{1,2,\dots,ii+1,\dots,n+1} + (-1)^{n+1} x^{2,3,\dots,n+1} \end{array} \tag{6.1}$$

The inclusion $\mathfrak{m} \subset \mathfrak{g}$ extend to an inclusion $S(\mathfrak{m}) \subset S(\mathfrak{g})$. Recall that $S(\mathfrak{g})$ and $U(\mathfrak{g})$ are isomorphic as coalgebras. Thus, as d_2 only involves the coalgebra structures, (C_2, d_2) can be embedded into (C_1, d_1). The cohomology of the complex (C_1, d_1) was computed by D. Calaque in the general case of a reductive pair (which is the case here):

Theorem 6.7 ([Cal]). *(a) There exists an $\mathfrak{h}$-equivariant linear map $P : U(\mathfrak{g}) \to S(\mathfrak{m})$ restricting to the projection $\mathfrak{g} \to \mathfrak{m}$ along $\mathfrak{h}$ and such that*

$$\epsilon \otimes P^{\otimes n} : (U(\mathfrak{h}) \otimes U(\mathfrak{g})^{\otimes n})^{\mathfrak{h}} \longrightarrow (S(\mathfrak{m})^{\otimes n})^{\mathfrak{h}}$$

is a morphism of complex $(C_1, d_1) \to (C_2, d_2)$ inducing an isomorphism at the level of cohomology.

(b) Moreover, there exists a linear map $\kappa : C_1 \to C_1[-1]$ such that:
- $\kappa d_1 + d_1 \kappa = \mathrm{Id} - (\epsilon \otimes P^{\otimes n})$
- $\kappa(U(\mathfrak{h})_{\leq k} \otimes U(\mathfrak{g})^{\otimes n}) \subset U(\mathfrak{h})_{\leq k+1} \otimes U(\mathfrak{g})^{\otimes n-1}$

Here $U(\mathfrak{h})_{\leq n}$ is the subspace of $U(\mathfrak{h})$ of elements of degree at most n in the generators $h_1, \dots, h_r$.

Corollary 6.8. *The nth cohomology group* $H^n(C_1, d_1)$ *is isomorphic to* $(\wedge^n \mathfrak{m})^{\mathfrak{h}}$*.*

If V is any vector space, define the operator $\mathrm{Alt}_n : V^{\otimes n} \to \wedge^n V \subset V^{\otimes n}$ by

$$\mathrm{Alt}_n(v_1 \otimes \ldots \otimes v_n) = \sum_{\sigma \in \mathfrak{S}_n} \mathrm{sgn}(\sigma)(v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(n)})$$

The following Corollary will also be useful:

Corollary 6.9. *The linear map* $\mathrm{Alt}_n \circ (\epsilon \otimes P^{\otimes n}) : U(\mathfrak{h}) \otimes U(\mathfrak{g})^{\otimes n} \to S(\mathfrak{m})^{\otimes n}$ *maps* $\ker d_1^{n,n+1}$ *onto* $(\wedge^n \mathfrak{m})^{\mathfrak{h}}$ *and* $\mathrm{Im}\, d_1^{n-1,n}$ *onto 0.*

Proof. According to [Dr4] the Alt_n operator induces a surjective linear map

$$\mathrm{Alt}_n : \ker d_2^{n,n+1} \longrightarrow (\wedge^n \mathfrak{m})^{\mathfrak{h}}$$

mapping $\mathrm{Im}\, d_2^{n-1,n}$ to 0, and according to Theorem 6.7, the map $\epsilon \otimes P^{\otimes n}$ induces a surjective linear map

$$\epsilon \otimes P^{\otimes n} : \ker d_1^{n,n+1} \longrightarrow \ker d_2^{n,n+1}$$

mapping $\mathrm{Im}\, d_1^{n-1,n}$ into $\mathrm{Im}\, d_2^{n-1,n}$. Hence, the composition of these two morphism maps $\ker d_1^{n,n+1}$ onto $(\wedge^n \mathfrak{m})^{\mathfrak{h}}$ and $\mathrm{Im}\, d_1^{n-1,n}$ onto 0. □

Let $\mathrm{CYB}(x, y)$ be the bilinear map $\mathfrak{g}^{\otimes 2} \times \mathfrak{g}^{\otimes 2} \to \mathfrak{g}^{\otimes 3}$

$$\mathrm{CYB}(x, y) = [x^{1,2}, y^{1,3}] + [x^{1,3}, y^{2,3}] + [x^{1,2}, y^{2,3}] + [y^{1,2}, x^{1,3}] + [y^{1,3}, x^{2,3}] + [y^{1,2}, x^{2,3}]$$

and define $\mathrm{CYB}(x) = \frac{1}{2}\mathrm{CYB}(x, x)$.

Lemma 6.10. *There exists a bilinear map*

$$\overline{\mathrm{CYB}} : (\wedge^2 \mathfrak{m})^{\mathfrak{h}} \times (\wedge^2 \mathfrak{m})^{\mathfrak{h}} \to (\wedge^3 \mathfrak{m})^{\mathfrak{h}}$$

fitting in the following diagram

$$\begin{array}{ccccc}
\wedge^2 \mathfrak{g} \times \wedge^2 \mathfrak{g} & \hookleftarrow & (\wedge^2 \mathfrak{g})^{\mathfrak{h}} \times (\wedge^2 \mathfrak{g})^{\mathfrak{h}} & \twoheadrightarrow & (\wedge^2 \mathfrak{m})^{\mathfrak{h}} \times (\wedge^2 \mathfrak{m})^{\mathfrak{h}} \\
\downarrow \mathrm{CYB} & & \downarrow \mathrm{CYB} & & \downarrow \overline{\mathrm{CYB}} \\
\wedge^3 \mathfrak{g} & \hookleftarrow & (\wedge^3 \mathfrak{g})^{\mathfrak{h}} & \twoheadrightarrow & (\wedge^3 \mathfrak{m})^{\mathfrak{h}}
\end{array}$$

Proof. The commutativity of the left square is clear.

CYB maps $(\wedge^2 \mathfrak{g})^{\mathfrak{h}} \times (\mathfrak{h} \wedge \mathfrak{g})^{\mathfrak{h}}$ to $(\mathfrak{h} \wedge \bigwedge^2 \mathfrak{g})^{\mathfrak{h}}$, meaning that CYB induces a bilinear map

$$\frac{(\wedge^2 \mathfrak{g})^{\mathfrak{h}}}{(\mathfrak{h} \wedge \mathfrak{g})^{\mathfrak{h}}} \times \frac{(\wedge^2 \mathfrak{g})^{\mathfrak{h}}}{(\mathfrak{h} \wedge \mathfrak{g})^{\mathfrak{h}}} \to \frac{(\wedge^3 \mathfrak{g})^{\mathfrak{h}}}{(\mathfrak{h} \wedge \bigwedge^2 \mathfrak{g})^{\mathfrak{h}}}$$

As the decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ is reductive,

$$\wedge^2 \mathfrak{g} \cong \wedge^2 \mathfrak{h} \oplus (\mathfrak{h} \otimes \mathfrak{m}) \oplus \wedge^2 \mathfrak{m}. \tag{6.2}$$

Hence,

$$\frac{(\wedge^2 \mathfrak{g})^{\mathfrak{h}}}{(\mathfrak{h} \wedge \mathfrak{g})^{\mathfrak{h}}} \cong (\wedge^2 \mathfrak{m})^{\mathfrak{h}}$$

In the same way,

$$\frac{(\wedge^3\mathfrak{g})^{\mathfrak{h}}}{(\mathfrak{h}\wedge\bigwedge^2\mathfrak{g})^{\mathfrak{h}}} \cong (\wedge^3\mathfrak{m})^{\mathfrak{h}}$$

Composing (6.2) with these isomorphisms, we obtain a map

$$\overline{\mathrm{CYB}} : (\wedge^2\mathfrak{m})^{\mathfrak{h}} \times (\wedge^2\mathfrak{m})^{\mathfrak{h}} \longrightarrow (\wedge^3\mathfrak{m})^{\mathfrak{h}}$$

which makes the right diagram commutes. □

Proof of Prop. 6.6. We first reduce the classification of solutions of the mixed pentagon equation to the classification of infinitesimal deformations of dynamical r-matrices (Lemma 6.13). Then we prove that under the assumptions of Theorem 6.5, such deformations are induced by infinitesimal shifts of the variable (Proposition 6.14).

Let

$$\Psi = \sum_{k\geq 0}\hbar^k\Psi_k,\ \ \Psi_k \in U(\mathfrak{h})\otimes U(\mathfrak{g})^{\otimes 2}$$

and

$$\Psi' = \sum_{k\geq 0}\hbar^k\Psi'_k,\ \ \Psi'_k \in U(\mathfrak{h})\otimes U(\mathfrak{g})^{\otimes 2}$$

Assume that there exists $n>1$ such that for all $1\leq k<n$ $\Psi_k=\Psi'_k$. The mixed pentagon equation (4.2) implies that $d_1^{2,3}(\Psi_n-\Psi'_n)=0$. Hence, there exists $a\in(\wedge^2\mathfrak{m})^{\mathfrak{h}}$ and $\mathfrak{g}\in(U(\mathfrak{h})\otimes U(\mathfrak{g}))^{\mathfrak{h}}$ such that

$$\Psi_n-\Psi'_n = 1\otimes a + d_1^{1,2}(g)$$

Both Ψ_n and Ψ'_n belongs to $U(\mathfrak{h})_{\leq n-1}\otimes U(\mathfrak{g})^{\otimes 2}$ by assumption, then so does $d_1^{1,2}(g)$. Hence, according to Theorem 6.7, g can be chosen is such a way that it belongs to $U(\mathfrak{h})_{\leq n}\otimes U(\mathfrak{g})$. Moreover, the fact that $d_1^{1,2}(g)\in U(\mathfrak{h})_{\leq n-1}\otimes U(\mathfrak{g})^{\otimes 2}$ implies that the $U(\mathfrak{h})$-degree n part g_n of g satisfies

$$g_n^{1,23}-g_n^{1,2}-g^{1,3}=0$$

because $g_n^{12,3} = g_n^{1,3}+\{\text{lower degree terms w.r.t. the } U(\mathfrak{h}) \text{ component}\}$. Hence, g_n actually belongs to $U(\mathfrak{h})\otimes\mathfrak{g}$, meaning that $G=\exp(\hbar^n g)\in\mathcal{G}$. Thus, let

$$\Psi'' = G\star\Psi'.$$

By construction, $\Psi-\Psi''\in\hbar^{n-1}U(\mathfrak{h})\otimes U(\mathfrak{g})^{\otimes 2}$ and

$$\Psi_n-\Psi''_n = 1\otimes a \tag{6.3}$$

Set

$$\bar{r} = \mathrm{Alt}_2(\rho_0) = 2\rho_0 \in (\wedge^2\mathfrak{m})^{\mathfrak{h}}$$

Lemma 6.11.

$$\overline{\mathrm{CYB}}(\bar{r},a) = 0 \in (\wedge^3\mathfrak{m})^{\mathfrak{h}}$$

PROOF. Expanding the mixed pentagon equation up to order $n+1$ leads to:

$$d_1^{2,3}(\Psi_{n+1}-\Psi''_{n+1}) = \rho_0^{2,34}a^{3,4}+a^{2,34}\rho_0^{3,4}-\rho_0^{23,4}a^{2,3}-a^{23,4}\rho_0^{2,3} \in U(\mathfrak{h})\otimes U(\mathfrak{g})^{\otimes 3}$$

Set $\psi_{n+1} = (\epsilon\otimes \mathrm{Alt}_3)(d_1^{2,3}(\Psi_{n+1}-\Psi''_{n+1}))$ we have:

$$\mathrm{Alt}_3(\rho_0^{1,23}a^{2,3}+a^{1,23}\rho_0^{2,3}-\rho_0^{12,3}a^{1,2}-a^{12,3}\rho_0^{1,2}) = \psi_{n+1}$$

Each component of a and ρ_0 is primitive, that is

$$a^{12,3} = a^{1,3}+a^{2,3} \tag{6.4}$$

$$a^{1,23} = a^{1,2}+a^{1,3} \tag{6.5}$$

$$\rho_0^{12,3} = \rho_0^{1,3}+\rho_0^{2,3} \tag{6.6}$$

$$\rho_0^{1,23} = \rho_0^{1,2}+\rho_0^{1,3} \tag{6.7}$$

It follows that:

$$\begin{aligned}
&\rho_0^{1,2}a^{2,3}+\rho_0^{1,3}a^{2,3}+a^{1,2}\rho_0^{2,3}+a^{1,3}\rho_0^{2,3}-\rho_0^{1,3}a^{1,2}-\rho_0^{2,3}a^{1,2}-a^{1,3}\rho_0^{1,2}-a^{2,3}\rho_0^{1,2}\\
&-\rho_0^{2,1}a^{1,3}-\rho_0^{2,3}a^{1,3}-a^{2,1}\rho_0^{1,3}-a^{2,3}\rho_0^{1,3}+\rho_0^{2,3}a^{2,1}+\rho_0^{1,3}a^{2,1}+a^{2,3}\rho_0^{2,1}+a^{1,3}\rho_0^{2,1}\\
&-\rho_0^{3,2}a^{2,1}-\rho_0^{3,1}a^{2,1}-a^{3,2}\rho_0^{2,1}-a^{3,1}\rho_0^{2,1}+\rho_0^{3,1}a^{3,2}+\rho_0^{2,1}a^{3,2}+a^{3,1}\rho_0^{3,2}+a^{2,1}\rho_0^{3,2}\\
&-\rho_0^{1,3}a^{3,2}-\rho_0^{1,2}a^{3,2}-a^{1,3}\rho_0^{3,2}-a^{1,2}\rho_0^{3,2}+\rho_0^{1,2}a^{1,3}+\rho_0^{3,2}a^{1,3}+a^{1,2}\rho_0^{1,3}+a^{3,2}\rho_0^{1,3}\\
&+\rho_0^{2,3}a^{3,1}+\rho_0^{2,1}a^{3,1}+a^{2,3}\rho_0^{3,1}+a^{2,1}\rho_0^{3,1}-\rho_0^{2,1}a^{2,3}-\rho_0^{3,1}a^{2,3}-a^{2,1}\rho_0^{2,3}-a^{3,1}\rho_0^{2,3}\\
&+\rho_0^{3,1}a^{1,2}+\rho_0^{3,2}a^{1,2}+a^{3,1}\rho_0^{1,2}+a^{3,2}\rho_0^{1,2}-\rho_0^{3,2}a^{3,1}-\rho_0^{1,2}a^{3,1}-a^{3,2}\rho_0^{3,1}-a^{1,2}\rho_0^{3,1}\\
&=\psi_{n+1}
\end{aligned}$$

As $a^{2,1} = -a^{1,2}$, it implies that:

$$\begin{aligned}
\psi_{n+1} = {}&[\rho_0^{1,2},a^{2,3}]+[\rho_0^{1,3},a^{2,3}]+[a^{1,2},\rho_0^{2,3}]+[a^{1,3},\rho_0^{2,3}]-[\rho_0^{1,3},a^{1,2}]-[a^{1,3},\rho_0^{1,2}]\\
&-[\rho_0^{2,1},a^{1,3}]+[a^{2,3},\rho_0^{2,1}]+[\rho_0^{3,2},a^{1,2}]+[\rho_0^{3,1},a^{1,2}]-[\rho_0^{3,1},a^{2,3}]-[a^{1,3},\rho_0^{3,2}]
\end{aligned}$$

that is:

$$\begin{aligned}
\psi_{n+1} = {}&[\rho_0^{1,2}-\rho_0^{2,1},a^{2,3}]+[\rho_0^{1,3}-\rho_0^{3,1},a^{2,3}]+[a^{1,2},\rho_0^{2,3}-\rho_0^{3,2}]\\
&+[a^{1,3},\rho_0^{2,3}-\rho_0^{3,2}]+[a^{1,2},\rho_0^{1,3}-\rho_0^{3,1}]+[\rho_0^{1,2}-\rho_0^{2,1},a^{1,3}]\\
={}&\mathrm{CYB}(\bar{r},a)
\end{aligned}$$

because $\bar{r} = \rho_0^{1,2}-\rho_0^{2,1}$. The projection of ψ_{n+1} in $(\wedge^3\mathfrak{m})^{\mathfrak{h}}$ is 0 by Corollary 6.9. Hence, as both $\bar{r}$ and a belong to $(\wedge^2\mathfrak{m})^{\mathfrak{h}}$, one has:

$$P^{\otimes 3}(\mathrm{CYB}(\bar{r},a)) = \overline{\mathrm{CYB}}(\bar{r},a) = 0$$

by Lemma 6.10. □

Let $J \in \widehat{S(\mathfrak{h})}\otimes U(\mathfrak{g})^{\otimes 2}$ be the image of $\frac{\Psi-1}{\hbar}$ through the tensor product of the reduction maps $U(\mathfrak{h})[[\hbar]]' \to \widehat{S(\mathfrak{h})}$ and $U(\mathfrak{g})[[\hbar]] \to U(\mathfrak{g})$. The coproduct of $U(\mathfrak{h})[[\hbar]]$ induces an algebra map

$$\Delta : \widehat{S(\mathfrak{h})}[[\hbar]] \to \widehat{S(\mathfrak{h})}\otimes U(\mathfrak{g})[[\hbar]]$$

defined by

$$u_i \longmapsto u_i\otimes 1 + 1\otimes \hbar h_i$$

The reduction of the mixed pentagon equation implies that J satisfies

$$J^{1,3,4}+J^{1,2,34}-J^{1,23,4}-J^{1,2,3}=0$$

because the reduction of $\frac{1\otimes\Phi_{\mathrm{KZ}}-1}{\hbar}$ in $\widehat{S(\mathfrak{h})}\otimes U(\mathfrak{g})^{\otimes 3}$ is 0, and using the fact that

$$J^{12,3,4}=J^{1,3,4}+O(\hbar).$$

It means that $(\mathrm{Id}\otimes b^{2,3})(J)=0$ where b is the differential of the coHochschild cochain complex $\bigoplus_{n\geq 0}U(\mathfrak{g})^{\otimes n}$ defined by the same formula as in (6.1). According to [**Dr4**, Prop. 2.2], the linear operator $\mathrm{Alt}_n: U(\mathfrak{g})^{\otimes n}\to U(\mathfrak{g})^{\otimes n}$ maps cocycles into $\wedge^n\mathfrak{g}$. Hence,

$$\rho=(\mathrm{Id}\otimes\mathrm{Alt}_2)(J)\in\widehat{S(\mathfrak{h})}\otimes\wedge^2\mathfrak{g}$$

Let $Z=\mathrm{Alt}_3(\frac{\Phi_{\mathrm{KZ}}-1}{\hbar^2})\mod\hbar\in(\wedge^3\mathfrak{g})^{\mathfrak{g}}$ (recall [**Dr4**] that $\Phi_{\mathrm{KZ}}=1+O(\hbar^2)$). A direct consequence of the mixed pentagon equation is the following [**Fe**, **Xu**]:

Proposition 6.12. *ρ satisfies the modified Dynamical Yang-Baxter Equation (mCDYBE)*

$$(\mathrm{Id}\otimes\mathrm{CYB})(\rho)+(\mathrm{Id}\otimes\widetilde{\mathrm{Alt}})(d\rho)=1\otimes Z\in\widehat{S(\mathfrak{h})}\otimes(\wedge^3\mathfrak{g})^{\mathfrak{g}}\tag{6.8}$$

where $d:\widehat{S(\mathfrak{h})}\otimes\mathfrak{g}^{\otimes 2}\to\widehat{S(\mathfrak{h})}\otimes\mathfrak{h}\otimes\mathfrak{g}^{\otimes 2}$ is the formal de Rham differential and

$$\widetilde{\mathrm{Alt}}(x)=x^{1,2,3}-x^{2,1,3}+x^{3,1,2}$$

As a consequence [**ES3**], $\bar{r}$ satisfies the modified Classical Yang-Baxter Equation

$$\overline{\mathrm{CYB}}(\bar{r})=\overline{Z}\in\wedge^3\mathfrak{m}\tag{6.9}$$

where $x\mapsto\overline{x}$ is the projection $(\wedge^n\mathfrak{g})^{\mathfrak{h}}\to(\wedge^n\mathfrak{m})^{\mathfrak{h}}$. Let ε be a formal variable, we have the following:

Lemma 6.13. *$\overline{r}+\varepsilon a$ is an infinitesimal deformation of $\overline{r}$, i.e.*

$$\overline{\mathrm{CYB}}(\overline{r}+\varepsilon a)=\bar{Z}\in(\wedge^3\mathfrak{m})^{\mathfrak{h}}[\varepsilon]/(\varepsilon^2)$$

Proof. As

$$\overline{\mathrm{CYB}}(r+\varepsilon a)=\overline{\mathrm{CYB}}(r)+2\varepsilon\overline{\mathrm{CYB}}(r,a)\mod\varepsilon^2$$

it follows from lemma 6.11 and from the fact that

$$\overline{\mathrm{CYB}}(\overline{r})=\overline{Z}$$

that $\overline{\mathrm{CYB}}(\overline{r}+\varepsilon a)=\overline{Z}\mod\varepsilon^2$. □

Solutions of the mCDYBE in the (semi-)simple case were classified by Etingof–Varchenko [**EV**, Theorem 3.10]. Their work implies that ρ is actually the Taylor expansion around the origin of a holomorphic function

$$r:D\to(\wedge^2\mathfrak{g})^{\mathfrak{h}}$$

where $D\subset\mathfrak{h}^*$ is a neighbourhood of 0. Moreover, if

$$r(\lambda)=r_{\mathfrak{h}}(\lambda)+\sum_{\alpha\in R}\phi_\alpha(\lambda)e_\alpha\otimes e_{-\alpha}$$

the assumption (b) of Theorem 6.5 means that $\phi_\alpha(0)\neq\pm 1$ which implies that there exists $\nu\in\mathfrak{h}^*$ such that

$$\forall\alpha\in R,\ \phi_\alpha(\lambda)=\coth(2(\alpha,\lambda-\nu))$$

An infinitesimal shift of $r(\lambda)$ by an element μ of $\mathfrak{h}^*$ induces an infinitesimal deformations of it as a solution of the mCDYBE:

$$r(\lambda+\varepsilon\mu) = r(\lambda) + \varepsilon(\mu\otimes \mathrm{Id}^{\otimes 2})(dr(\lambda)) + O(\varepsilon^2)$$

Evaluating the above expression in 0 and projecting in $(\wedge^2\mathfrak{m})^{\mathfrak{h}}$, it follows that μ also induces an infinitesimal deformation of $\bar{r}$ as a solution of the (modified) Classical Yang-Baxter equation in $(\wedge^2\mathfrak{m})^{\mathfrak{h}}$. This defines a linear map $\theta : \mathfrak{h}^* \to \mathcal{T}_{def}(\bar{r})$ by

$$\mu \mapsto \overline{(\mu\otimes \mathrm{Id}\otimes \mathrm{Id})(dr(0))}$$

where $\mathcal{T}_{def}(\bar{r})$ is the vector space[1] of infinitesimal deformation of $\bar{r}$, that is:

$$\mathcal{T}_{def}(\bar{r}) = \{x \in (\wedge^2\mathfrak{m})^{\mathfrak{h}} \mid \overline{\mathrm{CYB}}(\bar{r},x)=0\}$$

Proposition 6.14. *The linear map θ is an isomorphism.*

Proof. Let us first prove that $\dim \mathcal{T}_{def}(\bar{r}) \leq \dim \mathfrak{h}^*$. For $\alpha \in R$, let $r_\alpha = \phi_\alpha(0)$, by construction

$$\bar{r} = \sum_{\alpha\in R} r_\alpha e_\alpha \otimes e_{-\alpha}$$

and $r_{-\alpha} = -r_\alpha$ by antisymmetry. The only nontrivial parts of $\overline{\mathrm{CYB}}(\bar{r},a)$ are those belonging to $\mathfrak{g}_\alpha\otimes\mathfrak{g}_\beta\otimes\mathfrak{g}_\gamma$ with $\alpha+\beta+\gamma=0$, because $[e_\alpha,e_{-\alpha}]=0$ in $\mathfrak{m}=\mathfrak{g}/\mathfrak{h}$.

Write

$$a = \sum_{\alpha\in R} a_\alpha e_\alpha \otimes e_{-\alpha}$$

with $a_{-\alpha} = -a_\alpha$. The fact that $\overline{\mathrm{CYB}}(\bar{r},a)=0$ leads to the following relation:

$$a_{\alpha+\beta}(r_\alpha+r_\beta) = a_\alpha(r_\beta - r_{\alpha+\beta}) + a_\beta(r_\alpha - r_{\alpha+\beta}) \tag{6.10}$$

Following again [EV], the mCDYBE for $r(\lambda)$ implies the following relation in $\mathfrak{g}_\alpha\otimes\mathfrak{g}_\beta\otimes\mathfrak{g}_{-\alpha-\beta}$, $\forall \alpha,\beta\neq 0, \alpha+\beta\in R$:

$$r_\alpha r_\beta - r_\alpha r_{\alpha+\beta} - r_\beta r_{\alpha+\beta} + 1 = 0$$

Hence, if $r_\alpha + r_\beta$ were equal to 0, it would imply that $r_\alpha^2 = 1$, hence that $r_\alpha = \pm 1$ contradicting assumption (b) of Theorem 6.5. Thus, if $a_\alpha = 0$ for any $\alpha\in\Pi$, then $a_\alpha = 0$ for all $\alpha \in R$. It means that the linear map

$$\mathcal{T}_{def} \to \mathbb{C}^{\dim \mathfrak{h}^*}$$

defined by

$$(a_\alpha)_{\alpha\in R^+} \longmapsto (a_\alpha)_{\alpha\in\Pi}$$

is injective. Hence, $\dim \mathcal{T}_{def}(\bar{r}) \leq \dim \mathfrak{h}^*$.

Observe now that for $\alpha\in R$, $d\phi_\alpha(0)$ is the linear map $\mathfrak{h}^*\to\mathbb{C}$ defined by

$$\mu \longmapsto -2(\alpha,\mu)\,\mathrm{csch}^2(2(\alpha,\nu))$$

where $\mathrm{csch}(z) = \frac{2}{e^z - e^{-z}}$ is the hyperbolic cosecant. Hence, let

$$\bar{r}(\lambda) = \sum_{\alpha\in R} \phi_\alpha(\lambda) e_\alpha\otimes e_{-\alpha}$$

[1] It is a vector space because the map $a \mapsto \mathrm{CYB}(\bar{r},a)$ is a linear map.

be the function $D \to (\wedge^2\mathfrak{m})^{\mathfrak{h}}$ induced by $r(\lambda)$. Let $\mu_0 \in \mathfrak{h}^*$ be such that $(\mu_0 \otimes \mathrm{Id}^{\otimes 2})(d\bar{r}(0)) = 0$. As $\mathrm{csch}^2(z) \neq 0$ for all $z \in \mathbb{C}$, it implies that

$$\forall \alpha \in R^+,\ (\alpha, \mu_0) = 0$$

and thus that $\mu_0 = 0$ because $(\ ,\ )$ is non-degenerate. It follows that the map θ is injective, hence an isomorphism. □

Hence, let $\mu_a \in \mathfrak{h}^*$ be such that $\theta(\mu_a) = a$ and set $\tilde{\mu}_a = \hbar^{n-2}\mu_a \in \mathfrak{h}^*[[\hbar]]$. By definition,

$$(\mathrm{Id} \otimes \mathrm{Alt}_2)(\Psi_2) = dr(0) + 1 \otimes U(\mathfrak{g})^{\otimes 2}$$

Hence, $(\Psi_{\tilde{\mu}_a})_k = (\Psi)_k$ for $k < n$ and as shifts act trivially on $1 \otimes U(\mathfrak{g})^{\otimes 2}$,

$$\mathrm{Alt}_2 \circ (\epsilon \otimes P^{\otimes 2})(\Psi_{\tilde{\mu}_a})_n = \mathrm{Alt}_2 \circ (\epsilon \otimes P^{\otimes 2})(\Psi_n) + a$$

Using equation (6.3), it means that

$$\mathrm{Alt}_2 \circ (\epsilon \otimes P^{\otimes 2})((\Psi_{\tilde{\mu}_a})_n - \Psi''_n) = 0$$

and thus that $(\Psi_{\tilde{\mu}_a})_n$ and Ψ'_n are cohomologous. Hence, there exists $g \in (U(\mathfrak{h}) \otimes U(\mathfrak{g}))^{\mathfrak{h}}$ such that

$$\Psi'_n = (\Psi_{\tilde{\mu}_a})_n + d_1(g)$$

One shows as before that $G = \exp(\hbar^n g) \in \mathcal{G}$, and by construction

$$\Psi' = G \star \Psi_{\tilde{\mu}_a} \mod \hbar^{n+1}$$

The Proposition is proved. □

The proof of the Theorem then follows from an induction on n. □

6.4. Twist equivalence

Proposition 6.15. *Under the assumptions of Theorem 6.5, if Ψ, Ψ' satisfy the octagon equation with $E_{\mathrm{KZ},\sigma}$, then they are actually twist equivalents.*

PROOF. According to Theorem 6.5, we can assume that there exists $\mu \in \mathfrak{h}^*[[\hbar]]$ such that

$$\Psi = \Psi'_\mu$$

Let E_μ be the image of $E_{\mathrm{KZ},\sigma}$ by the shift by μ. Hence, $E_\mu = e^{\hbar\check{\mu}^{(2)}} E_{\mathrm{KZ}}$ where $\check{\mu} = (\mu \otimes \mathrm{Id})(t_{\mathfrak{h}}) \in \mathfrak{h}[[\hbar]]$. Then, on the one hand, Ψ satisfies the octagon equation with E_{KZ} by definition. On the other hand, as $\Psi = \Psi'_\mu$ and because Ψ'_μ satisfies the octagon equation with E_μ, Ψ has to satisfies the octagon equation with E_μ too. Hence,

$$\Psi^{-1} e^{-\hbar t^{2,3}/2} \Psi^{1,3,2} = (E^{12,3}_{\mathrm{KZ},\sigma})^{-1} \Psi^{-1} e^{\hbar t^{2,3}/2} \Psi^{1,3,2} E^{1,3}_{\mathrm{KZ},\sigma}$$

and

$$\Psi^{-1} e^{-\hbar t^{2,3}/2} \Psi^{1,3,2} = (E^{12,3}_{\mu})^{-1} \Psi^{-1} e^{\hbar t^{2,3}/2} \Psi^{1,3,2} E^{1,3}_{\mu}$$

Therefore, the right hand sides of these equations are equal, meaning that $e^{\hbar\check{\mu}}$ satisfies:

$$\Psi^{-1} e^{\hbar t^{2,3}/2} \Psi^{1,3,2} = e^{\hbar\check{\mu}^{(3)}} \Psi^{-1} e^{\hbar t^{2,3}/2} \Psi^{1,3,2} e^{-\hbar\check{\mu}^{(3)}}$$

which implies that

$$[\check{\mu}^{(3)}, \Psi^{1,3,2}_1 - \Psi_1 + 1 \otimes t/2] = 0$$

that is

$$[\check{\mu}^{(3)}, \Psi_1^{1,3,2} - \Psi_1 + 1 \otimes t_{\mathfrak{m}}/2] = 0 \tag{6.11}$$

because

$$[\check{\mu}^{(3)}, t_{\mathfrak{h}}] = 0$$

Write

$$\Psi_1^{1,3,2} - \Psi_1 + 1 \otimes t/2 = 1 \otimes \sum_{\alpha \in R} \lambda_\alpha e_\alpha \otimes e_{-\alpha}$$

and recall that

$$t_{\mathfrak{m}}/2 = \sum_{\alpha \in R^+} (e_\alpha \otimes e_{-\alpha} + e_{-\alpha} \otimes e_\alpha)$$

Hence, as $\Psi_1^{1,3,2} - \Psi_1$ is antisymmetric, $\lambda_\alpha + \lambda_{-\alpha} = 1$. It means that at least one of $\lambda_\alpha, \lambda_{-\alpha}$ is non-zero. By (6.11):

$$0 = [\check{\mu}^{(3)}, \Psi_1^{1,3,2} - \Psi_1 + 1 \otimes t_{\mathfrak{m}}/2] = 1 \otimes \sum_{\alpha \in R} -\lambda_\alpha \alpha(\check{\mu}) e_\alpha \otimes e_{-\alpha}$$

Therefore, $\alpha(\check{\mu}) = 0$ for all $\alpha \in R$, and thus that $\check{\mu} = 0$. Therefore, $\mu = 0$. □

Let us now check that the dynamical pseudo-twists Ψ, Ψ' associated to the QRAs B_{KZ}, B satisfies the assumptions of Theorem 6.5. The quasi-classical limit of $\Psi_{\text{KZ},\sigma}$ was computed in [**EE2**], and the quasi-classical limit of $\Psi_\hbar$ can be computed by considering the quasi-classical limit of the modified ABRR equation. Actually, these two results can be deduced from the following more general result:

Lemma 6.16. *Let $\Psi \in 1 + \hbar(U(\mathfrak{h})[[\hbar]]' \hat{\otimes} U(\mathfrak{g})^{\otimes 2}[[\hbar]])^{\mathfrak{h}}$ be a solution of the mixed pentagon equation which satisfies the octagon equation with $E = E_{\text{KZ},\sigma}$. Then there exists $G \in \mathcal{G}$ such that*

$$\frac{G \star \Psi - 1}{\hbar} \mod \hbar = \frac{1}{2}\left(\operatorname{Id} \otimes \frac{\operatorname{Id} + \sigma}{\operatorname{Id} - \sigma}\right)((t - t_{\mathfrak{h}})/2) \in \wedge^2 \mathfrak{m}$$

Proof. The quasi-classical limit of the octagon equation is

$$\tilde{\sigma}^{(3)}(t_{\mathfrak{h}}^{1,3} + t_{\mathfrak{h}}^{2,3} + \frac{1}{2} t_{\mathfrak{h}}^{3,3}) = (\rho_0^{1,3,2} - \rho_0 + t^{2,3}/2)\tilde{\sigma}^{(3)} + t_{\mathfrak{h}}^{1,3} + \frac{1}{2} t_{\mathfrak{h}}^{3,3} + \tilde{\sigma}^{(3)}(\rho_0 - \rho_0^{1,3,2} + t^{2,3}/2)$$

where

$$\rho_0 := \frac{\Psi - 1}{\hbar} \mod \hbar$$

Hence,

$$\begin{aligned}
\tilde{\sigma}^{(3)} t_{\mathfrak{h}}^{2,3} &= (\rho_0^{1,3,2} - \rho_0 + t^{2,3}/2)\tilde{\sigma}^{(3)} + \tilde{\sigma}^{(3)}(\rho_0 - \rho_0^{1,3,2} + t^{2,3}/2) \\
t_{\mathfrak{h}}^{2,3} &= (\operatorname{Id} - \sigma^{(3)})(\rho_0 - \rho_0^{1,3,2}) + (\operatorname{Id} + \sigma^{(3)})(t^{2,3}/2)
\end{aligned}$$

Finally

$$\begin{aligned}
\rho_0 - \rho_0^{1,3,2} &= (\operatorname{Id} - \sigma^{(3)})^{-1}((\operatorname{Id} + \sigma^{(3)})(t^{2,3}/2) - t_{\mathfrak{h}}^{2,3}) \\
&= (\operatorname{Id} - \sigma^{(3)})^{-1}((\operatorname{Id} + \sigma^{(3)})((t^{2,3} - t_{\mathfrak{h}}^{2,3})/2) + t_{\mathfrak{h}}^{2,3} - t_{\mathfrak{h}}^{2,3}) \\
&= \frac{\operatorname{Id} + \sigma^{(3)}}{\operatorname{Id} - \sigma^{(3)}}(t_{\mathfrak{m}}^{2,3}/2)
\end{aligned}$$

as $(\mathrm{Id}+\sigma^{(2)})(t_{\mathfrak{h}}/2) = t_{\mathfrak{h}}$. Thus,

$$r_0 := (\epsilon \otimes \mathrm{Alt}_2)(\rho_0) = \frac{\mathrm{Id}+\sigma^{(2)}}{\mathrm{Id}-\sigma^{(2)}}(t_{\mathfrak{m}}/2) = \frac{\mathrm{Id}+\sigma^{(2)}}{\mathrm{Id}-\sigma^{(2)}}(\sum_{\alpha\in R} e_\alpha \otimes e_{-\alpha} + e_{-\alpha}\otimes e_\alpha)$$

The fact that $r_0 \in (\wedge^2\mathfrak{m})^{\mathfrak{h}}$ can be checked directly, but follows more generally from the fact that ρ_0 is a cocycle in C_1, as implied by the mixed pentagon equation. Then according to Theorem 6.7 it means that there exists $g \in U(\mathfrak{h})_{\leq 1} \otimes U(\mathfrak{g})$ such that

$$\rho_0 = \frac{1}{2} r_0 - d_1^{1,2}(g)$$

Therefore, $G = \exp(\hbar g) \in \mathcal{G}$ and

$$\frac{G \star \Psi - 1}{\hbar} \mod \hbar = \frac{1}{2} r_0$$

as required. □

Proof of Theorem 6.1. Lemma 6.16 implies that there exists $G, G' \in \mathcal{G}$ such that $\tilde{\Psi} = G \star \Psi$ and $\tilde{\Psi}' = G' \star \Psi'$ satisfy the assumptions of Theorem 6.5. Consequently, there exists $(\tilde{G}, \mu) \in \mathcal{G} \rtimes \mathfrak{h}^*[[\hbar]]$ such that

$$\tilde{\Psi} = \tilde{G} \star \tilde{\Psi}'_\mu$$

Note that $(U(\mathfrak{h}) \otimes U(\mathfrak{g})[[\hbar]])^{\mathfrak{h}} = U(\mathfrak{h}) \otimes (U(\mathfrak{g})^{\mathfrak{h}})[[\hbar]]$, which implies that $\mathfrak{h}$-invariant twists actually commutes with $E_{\mathrm{KZ},\sigma}$.

It follows that $\tilde{\Psi}$ and $\tilde{G}\star\tilde{\Psi}'$ both satisfy the octagon equation with $E_{\mathrm{KZ},\sigma}$. Then, Proposition 6.15 implies that $\mu = 0$, and hence that they are equal. Finally

$$\Psi = (G^{-1}\tilde{G}G) \star \Psi'$$

implying that the QRAs B and B_{KZ} are twist-equivalent. The Theorem is proved. □

APPENDIX $\mathcal{A}$

Complex braid groups and monodromy

In this appendix, we explain the relationship between the cyclotomic KZ connection and the theory of complex braid groups, which are generalizations of the Artin braid group associated to complex reflection groups. We focus on the infinite family $G(de, e, n)$ of complex reflection groups for which $d > 1$ (which we describe below). As mentioned in section 2.1, B_n^1 is the braid group associated to the sub family $G(N, 1, n)$. If turns out that the cyclotomic KZ connection attached to $(\mathfrak{g}, t, \sigma)$, $\sigma^N = \mathrm{Id}$ induces a flat connection over the underlying topological space of the braid group associated to $G(de, e, n)$ for each proper divisor e of N. We show that our main result extends to this setting.

A.1. Complex braid groups

Complex reflection groups are generalizations of reflection groups, which are generated by pseudo (or complex) reflections.

Definition A.1. *A pseudo-reflection is a finite order element f of $GL_n(\mathbb{C})$ such that $\ker(f - \mathrm{Id})$ is of codimension 1. A complex reflection group is a finite subgroup of $GL_n(\mathbb{C})$ generated by pseudo-reflections.*

Let G be a complex reflection group in $GL_n(\mathbb{C})$. To G is associated an hyperplane arrangement in $\mathbb{C}^n$

$$\mathcal{A}_G := \{\ker(f - \mathrm{Id}),\ f \text{ is a pseudo-reflection in } G\}$$

Let

$$X_G = \mathbb{C}^n - \bigcup_{H \in \mathcal{A}_G} H$$

Definition A.2. *The pure braid group P_G associated to G is the fundamental group of X_G.*

The action of G on $\mathbb{C}^n$ permutes the elements of $\mathcal{A}_G$, hence it restricts to an action of G on X_G.

Definition A.3. *The full braid group associated to G is the fundamental group of X_G/G.*

Complex reflection groups were classified by Shephard and Todd [**ST**]. These are product of irreducible complex reflection groups which, apart from 34 exceptional cases, all belongs to an infinite family $G(de,e,n)$ defined as follows:

Definition A.4. *Let e,d,n be positive integers. $G(de,e,n)$ is the subgroup of $GL_n(\mathbb{C})$ whose elements are*

$$[\underline{a},\sigma]:(z_1,\dots,z_n)\longmapsto(a_1z_{\sigma(1)},\dots,a_nz_{\sigma(n)})$$

where $\sigma\in\mathfrak{S}_n$ and $\underline{a}=(a_1,\dots,a_n)$ is a n-tuple of $(de)-$th roots of unity satisfying $(\prod_{i=1}^n a_i)^d=1$.

It turns out that if $d>1$, the hyperplane arrangement of $G(de,e,n)$, and therefore the corresponding pure braid group, depends only on the product de and on n:

Proposition A.5 ([**BMR**])**.** *Assume that $d>1$. The complement in $\mathbb{C}^n$ of the hyperplane arrangement associated to $G(de,e,n)$ is*

$$Y_{n,de}=\{(z_1,\dots,z_n)\in(\mathbb{C}^*)^n,\ i\neq j\Rightarrow\forall\zeta\in\mu_{de},z_i\neq\zeta z_j\}$$

On the other hand, the quotient space

$$X_n^e=Y_{n,de}/G(de,e,n)$$

depends only on e and n, meaning that for all $d>1$, $G(de,e,n)$ and $G(2e,e,n)$ share the same full braid group denoted by B_n^e.

Moreover, it is easily seen that $G(de,e,n)$ is a normal subgroup of index e of $G(de,1,n)$. Therefore, there is a natural covering map

$$Y_{n,de}/G(de,e,n)\longrightarrow Y_{n,de}/G(de,1,n)$$

inducing an injective group morphism $B_n^e\hookrightarrow B_n^1$. Indeed, B_n^e can be identified with the subgroup of B_n^1 generated by $\tau_e=\tau^e,\tau'=\tau\sigma_1\tau^{-1},\sigma_1,\dots,\sigma_{n-1}$.

A.2. The monodromy morphism

Let $(\mathfrak{g},t,\sigma)$ be as in section 2.1 and e be a divisor of N such that $e\neq N$. The connection over $Y_{n,N}$ attached to $(\mathfrak{g},t,\sigma)$ is $G(de,e,n)$-equivariant, hence induces a flat connection over X_n^e and thus a monodromy morphism

$$B_n^e\longrightarrow(U(\mathfrak{h})\otimes U(\mathfrak{g})^{\otimes n}[[\hbar]])^\times.$$

It is clear that the above morphism is the restriction to B_n^e of the monodromy representation of B_n^1. It can thus be expressed algebraically using $(\Psi_{\mathrm{KZ},\sigma},E_{\mathrm{KZ},\sigma})$. Recall from theorem 5.20 that the image of τ through the representation of B_n^1 associated to the quantum QRA is

$$T_{n,\sigma}=\prod_{i=2}^n(\mathcal{R}_\hbar^{1,i})^{-1}\prod_{i=2}^n K^{1,i}E_{\hbar,\sigma}^{0,1}$$

Hence, we get the following

Corollary A.6. *The above representation of B_n^e is equivalent to the representation $B_n^e \to (U_\hbar(\mathfrak{h})) \otimes U_\hbar(\mathfrak{g})^{\otimes n})^\times$ given by*

$$\tau_e \longmapsto (T_{n,\sigma})^e$$

$$\tau' \longmapsto T_{n,\sigma}(1,2)\mathcal{R}_\hbar^{1,2}(T_{n,\sigma})^{-1}$$

$$\sigma_i \longmapsto (i, i+1)\mathcal{R}_\hbar^{i,i+1}$$

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