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Défauts de vorticité dans un supraconducteur en présence d'impuretés

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à Claire ♡

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Introduction Générale

Avant-propos

L'intégralité de cette thèse est consacrée à l'étude de fonctionnelles de type Ginzburg-Landau sans champ magnétique. Elle se divise en trois parties. Dans la première partie, l'application d'un champ magnétique est remplacée par une prescription de degré topologique sur le bord du domaine et nous nous intéressons à l'existence de points critiques de l'énergie. Dans les deux parties suivantes, une condition de type Dirichlet est imposée et l'étude porte sur l'influence d'un terme de chevillage sur le comportement des minimiseurs.

1 Le modèle physique

Dans les années 50 les physiciens Vitali Lazarevitch Ginzburg et Lev Davidovitch Landau ont proposé la théorie éponyme de la supraconductivité.

Un supraconducteur $\mathcal{S} \subset \mathbb{R}^3$ (par exemple une bille ou une barre) est un matériau qui, au dessous d'une température critique T_c (T_c est généralement de l'ordre de quelques Kelvin ou dizaines de Kelvin), est susceptible de présenter des propriétés supraconductrices : absence de résistance et diamagnétisme parfait [(1)].

Nous nous intéressons par la suite essentiellement à des supraconducteurs de type II. Dans des supraconducteur de ce type, sous l'application d'un champ magnétique élevé, la supraconductivité est détruite dans certaines zones : on parle alors de défauts de vorticité (voir par exemple [69], [65] et des images sont disponibles dans [57]).

L'idée de Ginzburg et de Landau est :
L'état d'un supraconducteur $\mathcal{S} \subset \mathbb{R}^3$ soumis à un champ magnétique est déterminé par des données variationnelles, une fonction d'onde $\psi : \mathcal{S} \to \mathbb{C}$ et un potentiel magnétique $A : \mathcal{S} \to \mathbb{R}^3$, qui minimisent l'énergie de Ginzburg-Landau $E_{\mathcal{S}}$.

L'originalité des travaux de Ginzburg et de Landau est d'obtenir l'énergie de Ginzburg-Landau comme une troncature de l'énergie libre développée suivant les puissances de $|\psi|^2$ pour une température $T < T_c$ voisine de T_c.

L'énergie conjecturée est

$$E_{\mathcal{S}}(\psi, A) = G_0 + \int_{\mathbb{R}^3} \frac{|\text{curl}A - \mathbf{B}_\text{e}|^2}{8\pi} + \int_{\mathcal{S}} \left\{ \frac{1}{2m^*} |(\hbar\nabla - \frac{ie^*}{c}A)\psi|^2 + \alpha|\psi|^2 + \frac{\beta}{2}|\psi|^4 \right\},$$

où

- $\psi : \mathcal{S} \to \mathbb{C}$ est la fonction d'onde régissant l'appariement des électrons en paires de Cooper (selon la théorie BCS l'appariement des électrons en paires de Cooper est la source de la supraconductivité) ;

[1] Le diamagnétisme parfait (ou effet Meissner) est la propriété de repousser un champ magnétique appliqué en créant un contre champ opposé de même intensité.

- $\mathbf{B}_e \in \mathbb{R}^3$ est une constante représentant le champ magnétique appliqué au supraconducteur ;
- $A : \mathbb{R}^3 \to \mathbb{R}^3$ est le potentiel magnétique dont le rotationnel curl$A$ correspond au champ magnétique mesuré (ainsi si le champ $\mathbf{B}_e$ ne pénètre pas dans $\mathcal{S}$ on a curl$A = \mathbf{0}$) ;
- G_0 est une quantité indépendante de ψ et de A qui correspond à l'énergie du supraconducteur à l'état normal *i.e.* $\psi \equiv 0$ et curl$A = \mathbf{B}_e$;
- m^*, e^*, c et $\hbar$ sont des constantes universelles, m^* est la masse de deux électrons, $e^* = 2e$ avec e la charge électrique d'un proton, c est la vitesse de la lumière et $\hbar$ est la constante de Planck réduite ;
- $\alpha = \gamma(T - T_c)$ avec $\gamma > 0$ qui dépend uniquement du matériel supraconducteur et T est la température ambiante ;
- β est un paramètre dépendant uniquement du matériel supraconducteur [(2)].

Lorsque (ψ, A) est un état stable, $|\psi|^2 : \mathcal{S} \to [0, 1]$ représente la densité des paires de Cooper [(3)]. Si les conditions magnétiques ne sont pas trop intenses, les défauts de vorticité prennent la forme de fils fins traversant $\mathcal{S}$ de part en part. Ces zones peuvent être vues comme l'ensemble $\{|\psi|^2 \simeq 0\}$. De même, on peut dire que les zones où $\mathcal{S}$ est dans un état supraconducteur correspondent à $\{|\psi|^2 \simeq 1\}$ (voir par exemple [69]).

Les filaments de vorticité sont enveloppés par des courants supraconducteurs circulaires et orthogonaux au champ appliqué. Par induction, ces courants génèrent un contre champ empêchant la pénétration du champ extérieur de s'étendre au-delà d'un voisinage tubulaire de rayon de l'ordre de λ [(4)]. De même, les zones dans un état normal (résistance à un courant) s'assimilent à des tubes de rayon de l'ordre de ξ [(5)] entourant les zones de pénétration du champ (voir les figures 1 et 2(a)).

Afin d'éviter la pénétration du champ dans le matériel, des courants électriques, générant par induction un contre champ, apparaissent aussi sur le bord du supraconducteur ; ce sont les courants de Meissner (voir la figure 2(b)).

Il a été démontré (par exemple dans un supraconducteur cylindrique avec un champ appliqué constant et dirigé suivant son axe) qu'il y a une quantification énergétique des filaments de vorticité : chaque filament "coûte" une unité d'énergie (voir [69]) par unité de longueur. Il en est de même des courants d'écrantage circulant autour des filaments. Ainsi la quantité de filaments de vorticité est linéairement corrélée avec l'intensité du champ appliqué.

Bien que soumis à des phénomènes de répulsion entre défauts de vorticité, les filaments n'ont pas de positions bien déterminées dans le supraconducteur. Un des problèmes dans les applications multiples de la supraconductivité provient de la dynamique des défauts de vorticité. Dans la réalité (supraconducteur avec des impuretés, par exemple, des défauts du réseau cristallin) ou même en théorie (supraconducteur totalement homogène), les filaments de vorticité n'ont pas d'emplacements précis où se fixer. Ainsi ils se déplacent dans le

[2] Certains auteurs considèrent β comme étant dépendant de la température, cependant cette dépendance est négligeable lorsque T est proche de T_0.

[3] La quantité $|\psi|^2$ est une quantité observable car invariante par changement de jauge.

[4] $\lambda = \sqrt{\dfrac{\beta m^* c^2}{4\pi|\alpha|e^*}}$ est la longueur de pénétration de London, c'est l'épaisseur minimale afin d'atténuer la pénétration du champ magnétique d'un rapport e $\simeq 2,71$.

[5] $\xi = \dfrac{\hbar}{\sqrt{2m^*|\alpha|}}$ est la longueur de cohérence, c'est la distance minimale pour passer d'une zone supraconductrice à une zone normale.

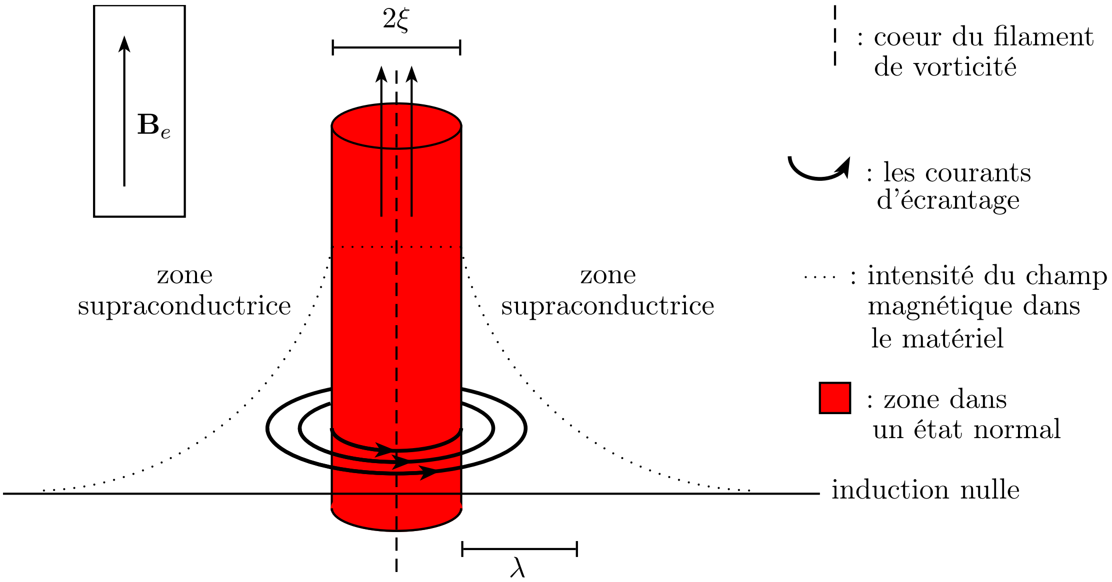
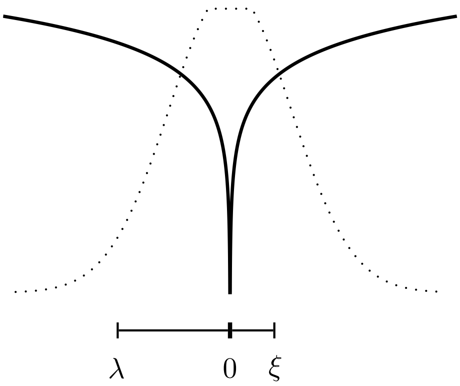
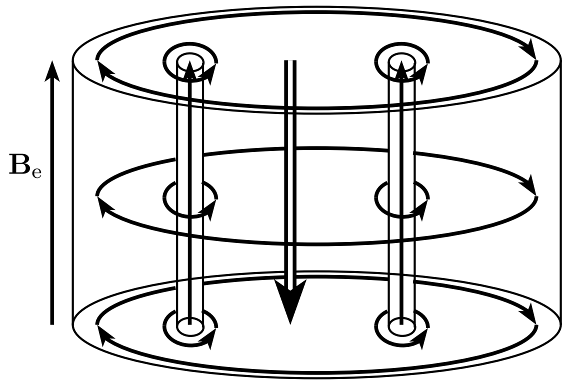


Figure 1 – Représentation d'un filament de vorticité mettant en évidence la longueur de pénétration λ, la longueur de cohérence ξ et les courants d'écrantage



(a) Diagramme représentant les variations du champ magnétique (en pointillé) et celles de la densité des paires de Cooper (en trait plein) dans un filament de vorticité centré en 0



(b) Une section d'un supraconducteur cylindrique dans un état mixte dans lequel deux lignes de vorticité sont représentées. Le contre champ est représenté en flèche double

Figure 2 – Un diagramme énergétique et un supraconducteur dans un état mixte

matériel durant l'utilisation du supraconducteur (voir [35]). Ces mouvements créent des perturbations de l'appariement en paires de Cooper des électrons et ainsi ils détruisent la supraconductivité sur leur passage.

Une des solutions pour remédier à ces mouvement anarchiques est d'offrir aux défauts de vorticité des emplacements dans lesquelles leur position est énergétiquement favorables. De plus on peut optimiser leur (absence de) déplacement en utilisant des zones dont les dimensions correspondent à celles des filaments de vorticité. Ces zones sont appelés les sites d'ancrage. Dans la pratique cela correspond soit à doper le supraconducteur par l'inclusion d'un autre matériel "moins supraconducteur" ou tout simplement conducteur ou bien encore à créer des zones avec des températures différentes.

Une autre motivation dans l'utilisation d'un site d'ancrage avec des inclusions supraconductrices ou des zones de température différente consiste en la détection de conditions détruisant la supraconductivité. Supposons que lors de l'utilisation d'un matériel supraconducteur on désire essentiellement préserver la supraconductivité dans une zone bien déterminée et que la destruction du caractère supraconducteur dans cette zone peut entrainer des défaillances ou des dégâts du système.

Afin de stopper le processus avant que la supraconductivité ne soit détruite dans la zone "utile", on peut placer dans des emplacements "sans grande importance pour le processus" des inclusions (de différente température ou d'un autre type de matériel supraconducteur) attirant les défauts de vorticité. L'idée est que tant que la supraconductivité n'est pas détruite dans les inclusions, le processus peut se dérouler sans problème. Par contre, dès que la supraconductivité commence à être détruite dans les sites d'ancrage, afin de sécuriser le processus, il est préférable d'arrêter l'utilisation du supraconducteur ou de réguler les conditions (magnétique ou de courant appliqué) créant la présence de défauts de vorticité.

2 Adimensionnalisation et variantes de l'énergie

2.1 Adimensionnalisation de l'énergie : les énergies classiques

Afin de simplifier l'étude de l'énergie $E_{\mathcal{S}}$, on normalise l'énergie libre en appliquant un changement d'échelle classique

$$u(x) = \sqrt{\frac{\beta}{|\alpha|}}\psi(\lambda x),\ \ \tilde{A}(x) = \frac{e^*}{\hbar c}\lambda A(\lambda x),\ \ \tilde{\mathbf{B}}_{\rm e} = \lambda^2\frac{e^*}{\hbar c}\mathbf{B}_{\rm e},\ \tilde{\Omega} = \frac{\mathcal{S}}{\lambda}.$$

En appliquant la normalisation précédente, on obtient

$$F(u,\tilde{A}) = G_0 + \frac{\hbar^2 c^2}{8\pi\lambda e^{*2}}\left[\int_{\mathbb{R}^3}|\text{curl}\tilde{A} - \tilde{\mathbf{B}}_{\rm e}|^2 + \int_{\tilde{\Omega}}\left\{|(\nabla - i\tilde{A})u|^2 + \frac{1}{2\varepsilon^2}(1-|u|^2)^2\right\}\right].$$

Ici, $\varepsilon = \frac{\xi}{\lambda}$ est un paramètre caractéristique du matériel supraconducteur utilisé. Puisque l'on considère uniquement des supraconducteur de type II, $\varepsilon < 1/\sqrt{2}$.

On voit ainsi clairement ressortir la fonctionnelle classique de Ginzburg-Landau avec champ magnétique comme étant l'unique quantité variationnelle dépendante de $u, \tilde{A}$:

$$GL_\varepsilon(u,\tilde{A}) = \frac{1}{2}\int_{\mathbb{R}^3}|\text{curl}\tilde{A} - \tilde{\mathbf{B}}_{\rm e}|^2 + \frac{1}{2}\int_{\tilde{\Omega}}\left\{|(\nabla - i\tilde{A})u|^2 + \frac{1}{2\varepsilon^2}(1-|u|^2)^2\right\}.$$

Une large littérature est consacrée à l'étude du cas spécial $\tilde{\Omega} = \Omega \times \mathbb{R}$ c'est à dire $\tilde{\Omega}$ est un cylindre infiniment long soumis à un champ magnétique constant et dirigé suivant son axe.

Dans cette situation l'étude se ramène à une situation bi-dimensionnelle :

$$GL_\varepsilon^{2D}(u,\tilde{A}) = \frac{1}{2}\int_{\mathbb{R}^2}|\text{curl}\tilde{A}-\tilde{h}_\text{e}|^2 + \frac{1}{2}\int_\Omega\left\{|(\nabla - i\tilde{A})u|^2 + \frac{1}{2\varepsilon^2}(1-|u|^2)^2\right\}, \qquad (1)$$

où $u \in H^1(\Omega,\mathbb{C})$, $\tilde{A}\in H^1(\Omega,\mathbb{R}^2)$ et $\tilde{h}_\text{e}\geq 0$ est l'intensité du champ appliqué.

Après cette réduction de la dimension, les filaments de vorticité sont remplacés par des vortex : des zéros isolés de u avec un degré topologique non nul (circulation des courants d'écrantage autour des filaments de vorticité).

Durant les quinze dernières années, l'étude de GL_ε^{2D} a été la source d'un nombre considérable de travaux. En particulier, dans une série d'articles, Sandier et Serfaty ont démontré rigoureusement l'existence de phénomènes observés par les physiciens tels que l'existence et le calcul de différents champs critiques, la quantification des vortex, les phénomènes de répulsion bord/vortex (apparition de vortex dans un sous domaine de Ω) et vortex/vortex (répartition uniforme dans le sous domaine), existence et quantification de la circulation autour des vortex...

Des versions simplifiées (sans champ magnétique) apparaissent au début des années 90 notamment dans des travaux de Bethuel, Brezis et Hélein ([17],[18]). Ils usent d'un artefact mathématique (le degré topologique) mimant l'effet d'un champ magnétique élevé en forçant la présence de vortex. Bethuel, Brezis et Hélein ont considéré (pour ε petit)

$$E_\varepsilon^0(u) = \frac{1}{2}\int_\Omega\left\{|\nabla u|^2 + \frac{1}{2\varepsilon^2}(1-|u|^2)^2\right\},\ u\in H^1(\Omega,\mathbb{C}).$$

Afin de modéliser l'effet d'un champ magnétique suffisant dans le but de créer des défauts de vorticité, ils ont minimisé E_ε^0 sous la contrainte $u=g$ sur $\partial\Omega$ avec $g\in C^\infty(\partial\Omega,\mathbb{S}^1)$, $\deg_{\partial\Omega}(g)\neq 0$.

L'un de leurs résultats le plus important tient dans l'analogie entre ce modèle simplifié et la réalité : leur approche met en évidence une réelle quantification de la vorticité (créée par la donnée au bord) à l'instar d'un potentiomètre magnétique utilisé par l'expérimentateur.

Chacun des vortex contient la même énergie et est entouré d'une circulation (quantifié elle aussi) analogue au courants d'écrantage. On observe aussi un phénomène de concentration de l'énergie dans les vortex. De plus il apparaît clairement un phénomène de répulsion lors d'une interaction vortex/vortex ou vortex/bord.

Bien que la condition imposée au bord (condition de type Dirichlet) ne soit pas physique (non invariante par changement de jauge), il est aujourd'hui communément admis que leur description des vortex est cohérente avec les problèmes étudiés en physique.

2.2 D'autre variante de l'énergie : le phénomène d'ancrage modélisé par un terme de chevillage

Le modèle avec un terme de chevillage est dû à Likharev (dans [46]). Il correspond à remplacer la non linéarité $(1-|u|^2)^2$ par une non linéarité plus générale $(a^2(x)-|u|^2)^2$.

En considérant une fonction étagée $a:\Omega\to\mathbb{R}_+^*$, cette modification permet de modéliser les impuretés dans un supraconducteur.

Plusieurs interprétations peuvent être faites d'un terme de chevillage étagé :

- un supraconducteur avec des zones de températures différentes ;
- un supraconducteur hétérogène ; une matrice supraconductrice avec des inclusions constituées d'un matériel supraconducteur distinct.

Par exemple, dans le cas où l'on considère un supraconducteur $\mathcal{S}$ de température critique T_c avec deux zones à températures distinctes, une température $T_1 < T_c$ dans un sous domaine $\mathcal{S}_1 \subset \mathcal{S}$ et $T_0 < T_c$ dans $\mathcal{S}_0 = \mathcal{S} \setminus \mathcal{S}_1$, en modifiant la normalisation faite Section 2.1 par

$$u(x) = \sqrt{\frac{\beta}{|\alpha_0|}}\psi(\lambda_0 x),\ \tilde{A}(x) = \frac{e^*}{\hbar c}\lambda_0 A(\lambda_0 x),\ \tilde{\mathbf{B}}_{\mathrm{e}} = {\lambda_0}^2\frac{e^*}{\hbar c}\mathbf{B}_{\mathrm{e}}$$

où

$$\lambda_0 = \sqrt{\frac{\beta m^* c^2}{4\pi|\alpha_0|e^*}},\ \xi_0 = \frac{\hbar}{\sqrt{2m^*|\alpha_0|}},$$

on obtient en notant $\varepsilon = \xi_0/\lambda_0$,

$$F(u,\tilde{A}) = F_0 + \frac{\hbar^2 c^2}{8\pi\lambda_0 e^{*2}}\left[\int_{\mathbb{R}^3}|\mathrm{curl}\tilde{A} - \tilde{\mathbf{B}}_{\mathrm{e}}|^2 + \int_{\mathcal{S}/\lambda_0}|(\nabla - i\tilde{A})u|^2 + \right.$$
$$\left. + \frac{1}{2\varepsilon^2}\int_{\mathcal{S}/\lambda_0}\left(\frac{|T-T_c|}{|T_0-T_c|} - |u|^2\right)^2\right].$$

Ici $T : \mathcal{S} \to \{T_0, T_1\}$ correspond à la fonction indiquant la température dans $\mathcal{S}$ et α_i au paramètre (dépendant de la température) α du matériel dans la zone $\mathcal{S}_i$.

Clairement, lorsque $\mathcal{S}/\lambda_0 = \Omega \times \mathbb{R}$, $\tilde{\mathbf{B}}_{\mathrm{e}} = (0,0,h_{\mathrm{e}})$ et $T_0 < T_1 < T_c$, en utilisant une réduction de la dimension du problème, la fonctionnelle simplifiée (qui ignore le champ magnétique) que l'on obtient prend la forme

$$E_\varepsilon(u) = \frac{1}{2}\int_\Omega\left\{|\nabla u|^2 + \frac{1}{2\varepsilon^2}(a^2 - |u|^2)^2\right\}.$$

Ici $a^2 : \Omega \to \left\{\frac{|T_1-T_c|}{|T_0-T_c|}, 1\right\}$ est une fonction étagée.

L'étude d'un terme de chevillage étagé a été menée dans un premier temps par Rubinstein (voir [58]) et ensuite poursuivie par André et Shafrir [6] et Lassoued et Mironescu [43]. Par la suite Aydi et Kachmar (voir [9]) ont traité le cas avec un champ magnétique.

Dans [43], Lassoued et de Mironescu ont développé une technique désormais classique dans le traitement d'une énergie de Ginzburg-Landau avec un terme de chevillage : remplacer la fonctionnelle avec un terme de chevillage par une fonctionnelle à poids.

Leur approche traite le cas de la minimisation d'une fonctionnelle du type

$$E_\varepsilon(u) = \frac{1}{2}\int_\Omega\left\{|\nabla u|^2 + \frac{1}{2\varepsilon^2}(a^2 - |u|^2)^2\right\},\ a \in L^\infty(\mathbb{R}^2, [b,1]),\ b \in (0,1)$$

avec Ω un domaine du plan, $u \in H^1_g := \{u \in H^1(\Omega,\mathbb{C})\,|\,\mathrm{tr}_{\partial\Omega}u = g\}$ et $g \in H^{1/2}(\Omega, \mathbb{S}^1)$.

La méthode de Lassoued et Mironescu consiste dans un premier temps à considérer **la** solution spéciale $U_\varepsilon \in H^1_1 \cap L^\infty(\Omega,[b,1])$ qui est l'unique minimiseur de E_ε avec la condition au bord $g \equiv 1$. Cette solution spéciale est l'outil fondamental dans la méthode. Elle permet de considérer le découplage

$$E_\varepsilon(U_\varepsilon v) = E_\varepsilon(U_\varepsilon) + F_\varepsilon(v),\ v \in H^1_g$$

avec

$$F_\varepsilon(v) = \frac{1}{2}\int_\Omega\left\{U_\varepsilon^2|\nabla v|^2 + \frac{U_\varepsilon^4}{2\varepsilon^2}(1 - |v|^2)^2\right\}.$$

Puisque $U_\varepsilon \equiv 1$ sur $\partial\Omega$, on a $v = U_\varepsilon v$ sur $\partial\Omega$. Ainsi, la minimisation de E_ε dans H^1_g est équivalente à celle de F_ε dans H^1_g dans le sens où $v_\varepsilon \in H^1_g$ minimise F_ε si et seulement si $u_\varepsilon = U_\varepsilon v_\varepsilon \in H^1_g$ minimise E_ε.

En utilisant le découplage de Lassoued-Mironescu, dans le cas où des vortex apparaissent ($d \neq 0$), en admettant qu'un phénomène de concentration de l'énergie apparaît dans un voisinage des vortex et que U_ε est une régularisation de a, on peut peut facilement se convaincre que notre terme de chevillage joue le rôle désiré : il offre un site bien défini pour l'ancrage des vortex en ses points de minimum.

Ce fait est clairement montré dans [58] et dans [43], Lassoued et Mironescu en fournissent un énoncé plus précis. Le but de la partie II réside dans l'étude de ce phénomène d'attraction de la vorticité lorsque le terme de chevillage dépend de ε et en particulier la situation où la taille des inclusions tend vers 0 (afin d'optimiser le piégeage des vortex).

3 Première partie : Existence de minimiseurs locaux dans un domaine multiplement connexe avec des conditions de type degrés (`publié : [36]`)

Pour $N \in \mathbb{N}^*$ et $N+1$ ouverts simplement connexes bornés et réguliers $\Omega, \omega_1, ..., \omega_N \subset \mathbb{R}^2$ tels que $\overline{\omega_i} \subset \Omega$ et $\overline{\omega_i} \cap \overline{\omega_j} = \emptyset$ pour $i \neq j$, on définit le domaine perforé $\mathcal{D} := \Omega \setminus \cup \overline{\omega_i}$.

On note

$$\mathcal{J} := \{u \in H^1(\mathcal{D}, \mathbb{C}) \,|\, |\mathrm{tr}_{\partial\mathcal{D}} u| = 1\}.$$

Pour $u \in \mathcal{J}$ et $U \in \{\Omega, \omega_1, ..., \omega_N\}$ on définit

$$\deg_{\partial U}(u) = \frac{1}{2\pi} \int_{\partial U} u \times \partial_\tau u \, \mathrm{d}\tau \in \mathbb{Z},$$

le degré de u par rapport à ∂U. Dans cette définition, τ est le vecteur unitaire tangent à U, c'est à dire que τ est tel que (ν, τ) soit une base orthonormée de $\mathbb{R}^2$ avec ν la normale extérieur à U. (Voir chapitre 1 pour plus de détails.)

Pour $\varepsilon > 0$, on considère la fonctionnelle de Ginzburg-Landau sur $\mathcal{D}$:

$$E_\varepsilon(u) = \frac{1}{2} \int_{\mathcal{D}} |\nabla u|^2 + \frac{1}{2\varepsilon^2}(1 - |u|^2)^2, \; u \in \mathcal{J}.$$

Dans [13] (voir aussi [14], [15]), Berlyand et Mironescu s'intéressent au problème d'existence de minimiseurs (globaux) de E_ε dans

$$\mathcal{J}_0 := \{u \in \mathcal{J} \,|\, \deg_{\partial\Omega}(u) = \deg_{\partial\omega_1}(u) = 1 \text{ et } \deg_{\partial\omega_i}(u) = 0 \text{ pour } i = 2, ..., N\}. \qquad (2)$$

Notons que l'on a $\mathcal{J}_0 \cap H^1(\mathcal{D}, \mathbb{S}^1) \neq \emptyset$ (voir [18]).

En raison de la non continuité du degré pour la convergence faible dans $H^1(\mathcal{D})$, l'existence de minimiseurs ne peut être obtenue par minimisation directe.

Dans [13], les auteurs montrent le lien entre l'existence d'un minimiseur global de E_ε dans $\mathcal{J}_0$ et une condition géométrique portant sur $\mathcal{D}$: la H^1-capacité de $\mathcal{D}$ notée $\mathrm{cap}(\mathcal{D})$.

Trois régimes se distinguent

- $\mathcal{D}$ est sur-critique : $\mathrm{cap}(\mathcal{D}) < \pi$,
- $\mathcal{D}$ est critique : $\mathrm{cap}(\mathcal{D}) = \pi$,
- $\mathcal{D}$ est sous-critique : $\mathrm{cap}(\mathcal{D}) > \pi$.

Berlyand et Mironescu ont montré dans [13] que dans les cas sous-critique et critique, pour tout $\varepsilon > 0$, E_ε admet un minimiseurs dans $\mathcal{J}_0$. Dans le cas sur-critique seulement l'une des trois possibilités suivantes peut avoir lieu (voir [13]) :

- pour tout $\varepsilon > 0$, E_ε admet un minimiseur dans $\mathcal{J}_0$,
- pour tout $\varepsilon > 0$, E_ε n'admet aucun minimiseur dans $\mathcal{J}_0$,
- il existe $\varepsilon_0 > 0$ tel que pour $\varepsilon < \varepsilon_0$, E_ε n'admette aucun minimiseur dans $\mathcal{J}_0$ et pour $\varepsilon > \varepsilon_0$, E_ε admet un minimiseur dans $\mathcal{J}_0$.

De plus une conjecture est établie pour le cas sur-critique : seulement la dernière des trois possibilités précédentes a lieu.

Finalement la conjecture de Berlyand et Mironescu est démontrée dans [12] lorsque $\mathcal{D}$ est de type annulaire et dans [15] pour $N \geq 2$.

Une question restait alors ouverte : à défaut d'existence de minimiseurs globaux (pour ε petit), peut-on obtenir l'existence de minimiseurs locaux de E_ε dans $\mathcal{J}_0$?

Berlyand et Rybalko ont répondu de manière affirmative dans [16] pour $N = 1$ et pour ε assez petit. Le chapitre 1 de cette thèse (publié : [36]) généralise le travail de Berlyand et Rybalko pour $N \in \mathbb{N}^*$.

En fait les résultats obtenus dans [16] et [36] ne sont pas spécifiques à des conditions de degré particulières du type $\deg_{\partial\Omega}(u) = \deg_{\partial\omega_1}(u) = 1$. Plus précisément en notant

$$\mathcal{J}_{\mathbf{p},q} = \{u \in \mathcal{J} \,|\, \deg_{\partial\Omega}(u) = q, \deg_{\partial\omega_1}(u) = p_1, ..., \deg_{\partial\omega_N}(u) = p_N\}$$

où $\mathbf{p} = (p_1, ..., p_N) \in \mathbb{Z}^N$, $q \in \mathbb{Z}$, on a le

Théorème : *Pour tout* $(\mathbf{p}, q) \in \mathbb{Z}^N \times \mathbb{Z}$ *et* $M \in \mathbb{N}^*$, *il existe* $\varepsilon_0(\mathcal{D}, \mathbf{p}, q, M) > 0$ *tel que pour* $0 < \varepsilon < \varepsilon_0$ *il existe au moins* M *minimiseurs locaux de* E_ε *dans* $\mathcal{J}_{\mathbf{p},q}$.

Le cas $N = 1$ est traité dans [16] et le cas $N \in \mathbb{N}^*$ dans [36].

La preuve du résultat principal du premier chapitre réside dans deux idées simples :

Idée 1. En observant l'asymptotique d'une suite $(u_n)_n \subset \mathcal{J}_{\mathbf{p},q}$ tel que $E_{\varepsilon_n}(u_n) \leq \Lambda$ ($\varepsilon_n \downarrow 0$), il est naturel de construire un "degré approché" stable pour la convergence faible dans H^1. À l'aide de cet outil, on définit des sous-ensembles *ad hoc* pour la minimisation.

Idée 2. En utilisant d'une part une bonne quantification de la perte de norme lors d'une convergence faible d'une suite minimisante et d'autre part un lemme technique quantifiant l'énergie nécessaire pour modifier le degré d'un élément de $\mathcal{J}$, on peut montrer que dans des sous-ensembles *ad hoc* de $\mathcal{J}_{\mathbf{p},q}$, sous certaines conditions, une suite minimisante de E_ε converge fortement.

4 Deuxième partie : La fonctionnelle de Ginzburg-Landau $2D$ avec un terme de chevillage

Dans cette partie, nous allons nous intéresser à la situation où le terme de chevillage dépend de ε et la donnée au bord est de type Dirichlet.

Le chapitre 2 consiste en l'étude d'une fonctionnelle de type Ginzburg-Landau avec un terme de chevillage étagé qui prend une valeur $b \in (0, 1)$ uniquement en un nombre fini d'inclusions dont la taille δ tend vers 0 avec ε. Le terme de chevillage est représenté Figure 3. Dans ce chapitre, on minimise l'énergie de type Ginzburg-Landau en imposant une condition de Dirichlet de degré non nul ; l'étude de la minimisation sous une condition de Dirichlet de degré nul est un cas particulier des résultats obtenus dans le chapitre 3.

Le but des chapitres 3 et 4 est le traitement d'une fonctionnelle de type Ginzburg-Landau avec un terme de chevillage périodique selon une grille $\delta \times \delta$, $\delta = \delta(\varepsilon) \underset{\varepsilon \to 0}{\to} 0$ et étagé dans chacune des cellules avec une condition au bord de Ω de type Dirichlet. Le

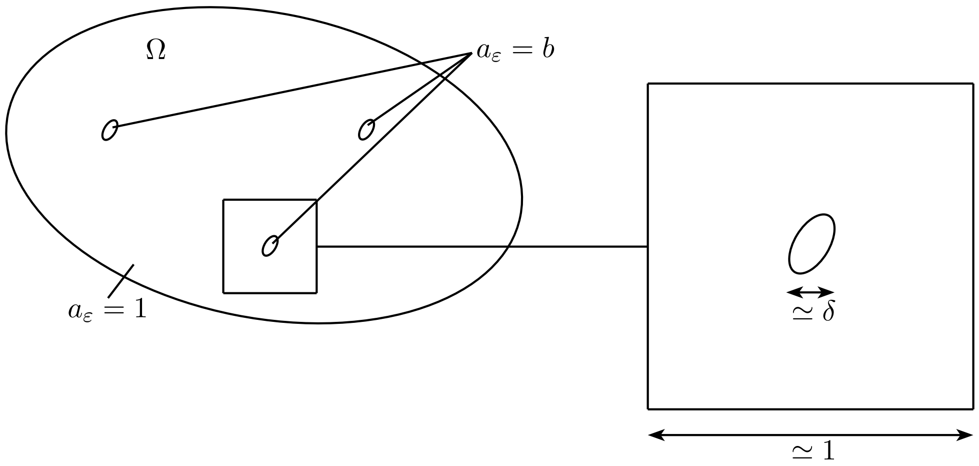


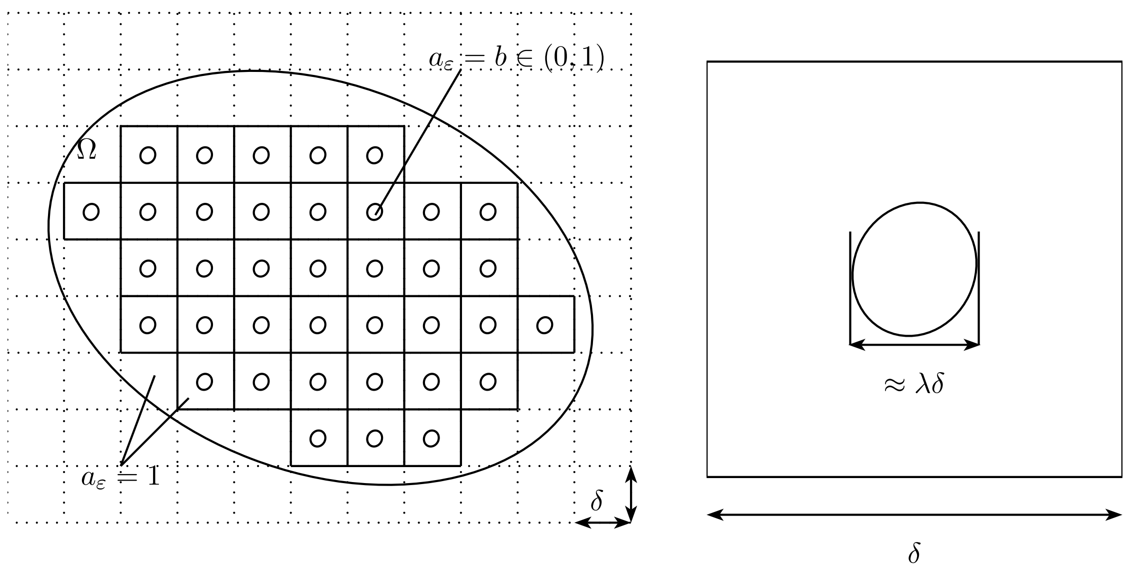
Figure 3 – Représentation d'un terme de chevillage avec $M = 3$ inclusions

chapitre 3 s'intéresse au cas où la donnée au bord est de degré nul et le chapitre 4 à la situation qui présente de la vorticité, le cas où la donnée au bord est de degré non nul.

La construction du terme de chevillage (périodique et étagé) que l'on considère utilise deux paramètres géométriques :

- le paramètre de période $\delta = \delta(\varepsilon) \underset{\varepsilon \to 0}{\to} 0$;
- la taille d'une inclusion dans une cellule : $\lambda = \lambda(\varepsilon) \in (0, 1]$.

Une construction rigoureuse est donnée Section 3.3, la figure 4 présente la géométrie du terme de chevillage.



(a) Le terme de chevillage est presque périodique sur une grille $\delta \times \delta$

(b) Le paramètre λ contrôle la taille de l'inclusion dans la cellule

Figure 4 – Le terme de chevillage presque périodique, étagé et rapidement oscillant

Dans chacun des trois chapitres de la deuxième partie, les informations les plus importantes que nous allons essayer d'obtenir sont :

- Quantification de la vorticité (nombre de zéros d'un minimiseur et multiplicité) ;
- Location de la vorticité (influence du terme de chevillage dans la position des zéros) ;
- Asymptotique d'une famille de minimiseurs lorsque $\varepsilon \to 0$.

4.1 Energie de Ginzburg-Landau avec un terme de chevillage étagé soumis à une condition de Dirichlet de degré non nul (En Collaboration avec Oleksandr Misiats)

Dans ce chapitre on considère un terme de chevillage étagé avec un nombre $M \geq 1$ d'inclusions identiques centrées autour de M points fixés et diluées selon un paramètre $\delta = \delta(\varepsilon) \to 0$. L'étude faite peut facilement être adaptée à la situation où les inclusions ne sont pas identiques. Le terme de chevillage considéré est représenté Figure 3.

On s'intéresse alors à une condition de Dirichlet de degré $d > 0$. On suppose aussi que g est régulière. (L'étude faite dans la première partie du chapitre 3 permet de traiter le cas où la donnée au bord est de degré nul.)

À l'aide d'un résultat d'η-ellipticité, on peut facilement obtenir que la vorticité est contenue dans un voisinage des inclusions. On montre que la vorticité est quantifiée et localisée : pour ε assez petit, un minimiseur v_ε de F_ε a exactement d zéros, tous ont un degré égal à 1 et sont contenus à l'intérieur des inclusions.

Deux cas sont distingués : $M \geq d$ et $M < d$. Dans la première situation, dans chaque inclusion, il y a au plus un zéro. De plus la sélection des inclusions contenant des racines de v_ε est régie par l'énergie renormalisée calculée par Bethuel, Brezis et Hélein dans [18].

Dans la deuxième situation ($M < d$), chaque inclusion contient soit $\lfloor \frac{d}{M} \rfloor$ soit $\lfloor \frac{d}{M} \rfloor + 1$ zéros et comme précédemment c'est l'énergie renormalisée de Bethuel, Brezis et Hélein qui détermine la configuration macroscopique des zéros.

Dans chacune des deux situations on établit la configuration limite que prennent les zéros à l'intérieur des inclusions : la position microscopique des zéros est indépendante de la donné au bords g.

On démontre que l'emplacement des zéros dans une inclusion ne dépend que de la géométrie des inclusions, de b et du nombre de zéros dans l'inclusion considérée.

Ce fait s'explique en considérant une estimation de l'énergie d'un minimiseur dans laquelle l'énergie renormalisée se découple en la somme de deux termes :

- Le premier terme régit la position macroscopique des zéros en sélectionnant les inclusions qui contiennent des zéros (c'est l'énergie renormalisée de Bethuel, Brezis et Hélein restreinte aux centres des inclusions).
- Le deuxième terme correspond à une information localisée dans des voisinages (fixes) des inclusions qui contiennent des zéros. Il se découple lui aussi en deux parties :
  - ⋆ une partie dépendant de la valeur prise par le minimiseur sur un cercle concentrique avec l'inclusion mais est indépendante de la position des zéros,
  - ⋆ une deuxième énergie qui dépend uniquement de la position des zéros, de b, de la géométrie de l'inclusion et du nombre de zéros qu'elle contient.

On établit aussi l'asymptotique d'un minimiseur ainsi que son comportement microscopique autour de l'inclusion.

4.2 Energie de Ginzburg-Landau avec un terme de chevillage rapidement oscillant et étagé soumis à une condition de Dirichlet de degré nul (`En Collaboration avec Petru Mironescu et Oleksandr Misiats, à paraitre : [37]`)

Dans le chapitre 3, on s'intéresse à la minimisation d'une énergie de type Ginzburg-Landau avec une condition de Dirichlet $g : \partial\Omega \to \mathbb{S}^1$ de degré nul. De par la simplicité du problème -en particulier la borne *a priori* sur l'énergie- on peut considérer la régularité la plus générale : $g \in H^{1/2}(\partial\Omega, \mathbb{S}^1)$.

Motivé par le découplage de Lassoued et Mironescu pour une énergie de Ginzburg-Landau avec un terme de chevillage, le chapitre 3 se décompose en deux sections :

- Etude d'une fonctionnelle de type Ginzburg-Landau avec des poids dépendants de ε ;
- Etude d'une fonctionnelle de Ginzburg-Landau avec un terme de chevillage illustré par la Figure 4.

On démontre dans un premier temps qu'en considérant une fonctionnelle à poids de type Ginzburg-Landau

$$\tilde{F}_\varepsilon(v) = \frac{1}{2}\int_\Omega \left\{ \alpha_\varepsilon |\nabla v|^2 + \frac{\beta_\varepsilon}{2\varepsilon^2}(1-|v|^2)^2 \right\}, \tag{3}$$

avec $\alpha_\varepsilon \in W^{1,\infty}(\Omega, [b,1])$, $\beta_\varepsilon \in L^\infty(\Omega, [b,1])$, $b \in (0,1)$ et v_ε un de ses minimiseurs dans H^1_g, lorsque $\varepsilon \to 0$, on a $|v_\varepsilon| \to 1$ dans $L^\infty(\overline{\Omega})$ et $H^1(\Omega)$.

Ainsi la section 3.2 répond clairement aux questions portant sur la vorticité : pour $a_\varepsilon \in L^\infty(\Omega, [b,1])$ et ε assez petit, un minimiseur ne développe pas de zéros.

Dans le cas particulier où pour une suite $\varepsilon_n \downarrow 0$, on a $\alpha_{\varepsilon_n} \to \kappa$ dans $L^2(\Omega)$, on démontre qu'une suite de minimiseurs de (3) dans H^1_g converge dans H^1 vers une limite identifiée (et indépendante de $(\beta_{\varepsilon_n})_n$).

La deuxième section consiste en l'étude des minimiseurs dans H^1_g d'une énergie de Ginzburg-Landau avec un terme de chevillage périodique $a_\varepsilon : \Omega \to \{b,1\}$, $b \in (0,1)$ représenté Figure 4.

Elle se divise en quatre parties qui correspondent chacune à une asymptotique particulière des paramètres δ, λ par rapport à ε. Précisément :
Section 3.3.1 : $\lambda \to 0$, la limite diluée ;
Section 3.3.2 : $\lambda = 1, \delta = \varepsilon$, le cas critique ;
Section 3.3.3 : $\lambda = 1, \varepsilon \ll \delta$, le cas physique ;
Section 3.3.4 : $\lambda = 1, \delta \ll \varepsilon$, le cas non physique.

Dans chacun des cas précédents, on détermine l'asymptotique de la solution spéciale U_ε et de v_ε un minimiseur de F_ε. Ainsi l'asymptotique d'un minimiseur $u_\varepsilon = U_\varepsilon v_\varepsilon$ de E_ε est identifiée.

4.3 Energie de Ginzburg-Landau avec un terme de chevillage presque périodique, rapidement oscillant et étagé soumis à une condition de Dirichlet de degré non nul

Dans ce chapitre on considère une fonctionnelle de Ginzburg-Landau avec un terme de chevillage décrit Figure 4 dans le régime physique ($\varepsilon \ll \delta$) soumis à une condition de Dirichlet de degré $d > 0$. On introduit aussi le paramètre de dilution $\lambda = \lambda(\varepsilon) \in (0,1]$ qui contrôle la taille de l'inclusion dans une cellule. Deux cas sont considérés : $\lambda \equiv 1$ ou $\lambda \to 0$.

Le chapitre se décompose en deux parties : on traite dans un premier temps trois problèmes auxiliaires que l'on applique ensuite à l'étude de la fonctionnelle de Ginzburg-Landau.

A l'instar des travaux de Bethuel, Brezis et Hélein, les problèmes auxiliaires concernent la minimisation de la fonctionnelle de Dirichlet avec un poids défini dans un domaine perforé par des trous circulaires dont le diamètre tend vers zéro. Le poids considéré dépend d'un paramètre ε, il est presque périodique et rapidement oscillant lorsque $\varepsilon \to 0$ (la période tend vers 0). La minimisation se fait parmi des applications unimodulaires avec différentes données au bord (de type Dirichlet ou de type degré).

En utilisant les problèmes auxiliaires, on démontre que pour ε assez petit et v_ε un minimiseur de la fonctionnelle de Ginzburg-Landau à poids F_ε, on a :

- La vorticité est quantifiée : v_ε admet exactement d zéros, tous ayant un degré égal à 1.
- Chacun des zéros de v_ε est capturé par une inclusion ; leur position (macroscopique) est soumise à des phénomènes de répulsion vortex/vortex et vortex/bord.
- Dans le cas dilué ($\lambda \to 0$), la position limite microscopique d'un zéro dans une inclusion est indépendante de g.
- Une limite homogénéisée de v_ε est obtenue. De plus le profil microscopique de v_ε autour de ses zéros correspond à la situation classique étudiée dans [28] et [54].

Un développement précis et explicite est démontré pour l'énergie d'un minimiseur dans l'asymptotique $\varepsilon \to 0$.

5 Troisième partie : La fonctionnelle de Ginzburg-Landau $3D$ avec un terme de chevillage étagé

Dans cette dernière partie on étudie la fonctionnelle de Ginzburg-Landau tridimensionnelle sans champ magnétique avec un terme de chevillage étagé indépendant de ε et présentant une seule inclusion ω :

$$E_\varepsilon(u) = \frac{1}{2}\int_\Omega \left\{ |\nabla u|^2 + \frac{1}{2\varepsilon^2}(a^2(x) - |u|^2)^2 \right\},\ a(x) = \begin{cases} b \in (0,1) & \text{si } x \in \omega \subset \Omega \\ 1 & \text{si } x \in \Omega \setminus \omega \end{cases}.$$

Afin de simplifier le travail on suppose que $\Omega \subset \mathbb{R}^3$ est convexe régulier et ω est strictement convexe et régulier.

Notre but est la description des filaments de vorticité d'un minimiseur u_ε de E_ε soumis à une condition de type Dirichlet $g \in H^{1/2}(\partial\Omega, \mathbb{S}^1) \setminus \overline{C^\infty(\partial\Omega, \mathbb{S}^1)}^{H^{1/2}}$.

Le cas d'un domaine Ω convexe et d'une énergie E_ε sans terme de chevillage a été étudié par Lin et Rivière dans [48], [49] et par Sandier dans [60]. Dans la situation où g admet un nombre fini de singularités, Sandier a montré que l'ensemble des filaments de vorticité tendent à être une connexion minimale des singularités de g pour la métrique euclidienne.

En suivant les mêmes arguments que Sandier, et sous différentes hypothèses sur Ω, ω et sur les singularités de g, on obtient des estimations de concentration d'énergie le long d'un ensemble de courbes qui est bien identifié : c'est une jonction minimale (pour une métrique bien déterminée) des singularités de g.

En utilisant le fait que l'énergie se concentre le long des filaments de vorticité, on déduit que ces derniers tendent à former une jonction minimale des singularités de g (pour une certaine métrique dépendant uniquement de ω, b).

Sous des hypothèses très fortes sur ω, Ω et g (comme la symétrie), la concentration de l'énergie permet d'identifier la limite et de localiser les défauts de vorticité des minimiseurs.

En général, ces résultats sont encore à obtenir. La difficulté majeure (η-ellipticité en présence d'un terme de chevillage) n'est pas encore levée (les travaux de Lin et Rivière

[48], [49], Bethuel, Brezis et Orlandi [19] ou bien ceux de Bethuel, Orlandi et Smets [20] ne semblent pas s'appliquer à notre situation).

On démontre aussi en suivant la technique développée dans [22] par Bourgain, Brezis et Mironescu (densité dans $H^{1/2}(\partial\Omega, \mathbb{S}^1)$ des applications avec un nombre fini de singularités) une estimation de l'énergie pour un minimiseur de E_ε avec une contrainte de type Dirichlet $g \in H^{1/2}(\partial\Omega, \mathbb{S}^1)$.

6 Perspectives

Bien que cette thèse réponde à un certain nombre de questions, d'autres interrogations naturelles restent ouvertes.

6.1 Partie I

Dans le premier chapitre de cette thèse, on démontre que pour ε assez petit, des minimiseurs locaux de la fonctionnelle de Ginzburg-Landau définie dans un domaine multiplement connexe du plan avec des conditions de degré existent toujours.

Qu'en est-il dans un domaine simplement connexe ? Peut-on, au moins, obtenir l'existence de points critiques ?

Lorsque le domaine est un disque, l'existence de points critiques peut facilement être obtenue (voir à la fin de la section 1.1, chapitre 1). Par contre, pour un domaine plus général, la question reste ouverte.

La technique développée dans le chapitre 1 ne peut pas être appliquée lorsque le domaine est simplement connexe. En effet, l'idée utilisée se scinde principalement en deux étapes : définir des sous-ensembles *ad hoc* $\tilde{\mathcal{I}}$ et minimiser l'énergie dans ces sous-ensembles.

La construction des ensembles $\tilde{\mathcal{I}}$ est basée sur l'observation du comportement asymptotique de fonctions $\tilde{u}_\varepsilon$ telles que $E_\varepsilon(\tilde{u}_\varepsilon)$ reste bornée indépendamment de ε.

La difficulté nouvelle dans le cas d'un domaine simplement connexe provient du fait qu'une éventuelle limite faible dans H^1 de fonctions $\tilde{u}_\varepsilon$ (à énergie borné indépendamment de ε) est nécessairement constante. Cette pauvreté dans l'ensemble des possibilités des limites faibles empêche la construction des ensembles *ad hoc*.

Ainsi, le cas où le domaine est simplement connexe nécessite une nouvelle approche.

6.2 Partie II

La taille des inclusions

L'étude d'une fonctionnelle de Ginzburg-Landau dans une situation présentant des défauts de vorticité, faite dans les chapitre 2 et chapitre 4, ne s'applique qu'a des inclusions dont la taille δ (ou $\lambda\delta$) est beaucoup plus grande que ε. Moralement, la condition imposée sur δ est un peu plus restrictive que de demander $\delta \gg \varepsilon^\alpha$ pour tout $\alpha \in (0, 1)$.

Comme expliqué dans la section 1, afin d'optimiser le piégeage des défauts de vorticité, il est intéressant de diminuer la taille des inclusions. Sans vouloir aller jusqu'à avoir δ de l'ordre de ε, il serait intéressant d'adapter la démarche afin d'arriver à considérer δ de l'ordre de ε^α pour un certain $\alpha \in (0, 1)$.

Terme de chevillage avec b petit

Dans le chapitre 4, les résultats principaux sont énoncés sous une hypothèse qui semble uniquement technique : $a_\varepsilon : \Omega \to \{b, 1\}$ avec $b^2 > \dfrac{1}{2}$.

L'hypothèse $b^2 > \dfrac{1}{2}$ sert essentiellement à mettre en évidence le phénomène répulsif bord du domaine/vortex. Moralement, il n'y a pas de raison pour que ce phénomène de répulsion soit astreint à une telle hypothèse.

Il est naturel d'essayer d'obtenir les principaux résultats du chapitre 4 pour tout $b \in (0, 1)$.

Emplacement de la vorticité dans les chapitres 2 et 4

Le chapitre 2 répond parfaitement à la question de la position macroscopique des vortex tandis que le chapitre 4 met uniquement en évidence les deux phénomènes répulsifs classiques.

Dans le chapitre 2 et dans le cas dilué du chapitre 4, on sait que la position microscopique des vortex dans les inclusions est indépendante de la donnée au bord. Dans la situation où les inclusions sont des disques, et lorsqu'une inclusion contient un seul vortex, il semble naturel de s'attendre au fait que le vortex tend à se situer au centre de l'inclusion. Ce résultat reste à prouver.

6.3 Partie III

Afin d'avoir une description complète des filaments de vorticité, le fait le plus important encore à obtenir est un résultat d'η-ellipticité en présence d'un terme de chevillage discontinu.

Il est établi dans le chapitre 5 que, sous certaines hypothèses portant sur la donnée au bord, une concentration de l'énergie a lieu le long d'un ensemble de courbes que nous supposons être proche des filaments de vorticité. Un résultat d'η-ellipticité permettrait d'affirmer que cet ensemble de courbes correspond à l'emplacement limite des défauts de vorticité lorsque $\varepsilon \to 0$.

Première partie

Minimisation locale de la fonctionnelle de Ginzburg-Landau avec des conditions de type degré

Chapter 1

Local minimization of the Ginzburg-Landau functional with prescribed degrees in a multiply connected domain

We consider, in a smooth bounded multiply connected domain $\mathcal{D} \subset \mathbb{R}^2$, the Ginzburg-Landau energy $E_\varepsilon(u) = \frac{1}{2}\int_{\mathcal{D}} \left\{|\nabla u|^2 + \frac{1}{2\varepsilon^2}(1-|u|^2)^2\right\}$ subject to prescribed degree conditions on each component of $\partial\mathcal{D}$. In general, minimal energy maps do not exist [13]. When $\mathcal{D}$ has a single hole, Berlyand and Rybalko [16] proved that for small ε local minimizers do exist. We extend the result in [16]: $E_\varepsilon(u)$ has, in domains $\mathcal{D}$ with $2, 3, ...$ holes and for small ε, local minimizers. Our approach is very similar to the one in [16]; the main difference stems in the construction of test functions with energy control.

Contents

1.1 Introduction

This chapter deals with the existence problem of local minimizers of the Ginzburg-Landau functional with prescribed degrees in a 2D perforated domain $\mathcal{D}$.

The domain we consider is of the form $\mathcal{D} = \Omega \setminus \cup_{i\in\mathbb{N}_N}\overline{\omega}_i$, where $N \in \mathbb{N}^*$, Ω and the ω_i's are simply connected, bounded and smooth open sets of $\mathbb{R}^2$.

We assume that $\overline{\omega_i} \subset \Omega$ and $\overline{\omega_i} \cap \overline{\omega_j} = \emptyset$ for $i, j \in \mathbb{N}_N := \{1, ..., N\}, i \neq j$.

The Ginzburg-Landau functional is

$$E_\varepsilon(u, \mathcal{D}) := \frac{1}{2}\int_{\mathcal{D}} \left\{ |\nabla u|^2 + \frac{1}{2\varepsilon^2}\left(1 - |u|^2\right)^2 \right\} \mathrm{d}x \tag{1.1}$$

with $u : \mathcal{D} \to \mathbb{C} \simeq \mathbb{R}^2$ and ε is a positive parameter (the inverse of κ, the Ginzburg-Landau parameter).

When there is no ambiguity we will write $E_\varepsilon(u)$ instead of $E_\varepsilon(u, \mathcal{D})$.

Functions we will consider belong to the class

$$\mathcal{J} = \left\{u \in H^1(\mathcal{D}, \mathbb{C}) \,|\, |u| = 1 \text{ on } \partial\mathcal{D}\right\}.$$

Clearly, $\mathcal{J}$ is closed under weak H^1-convergence.

This functional is a simplified version of the Ginzburg-Landau functional which arises in superconductivity (or superfluidity) to model the state of a superconductor submitted to a magnetic field (see, *e.g.*, [69] or [65]). The simplified version of the Ginzburg-Landau functional considered in (1.1) ignores the magnetic field. The issue we consider in this chapter is existence of local minimizers with prescribed degrees on $\partial\mathcal{D}$.

We next formulate rigorously the problem discussed in this chapter. To this purpose, we start by defining properly the degrees of a map $u \in \mathcal{J}$. For $\gamma \in \{\partial\Omega, ..., \partial\omega_N\}$ and $u \in \mathcal{J}$ we let

$$\deg_\gamma(u) = \frac{1}{2\pi}\int_\gamma u \times \partial_\tau u \,\mathrm{d}\tau.$$

Here:

- each γ is directly (counterclockwise) oriented,
- $\tau = \nu^\perp$, τ is the tangential vector of γ and ν the outward normal to Ω if $\gamma = \partial\Omega$ or ω_i if $\gamma = \partial\omega_i$,
- $\partial_\tau = \tau \cdot \nabla$, the tangential derivative and " $\cdot$ " stands for the scalar product in $\mathbb{R}^2$,
- " $\times$ " stands for the vectorial product in $\mathbb{C}$, $(z_1 + \imath z_2) \times (w_1 + \imath w_2) := z_1 w_2 - z_2 w_1$, $z_1, z_2, w_1, w_2 \in \mathbb{R}$,
- the integral over γ should be understood using the duality between $H^{1/2}(\gamma)$ and $H^{-1/2}(\gamma)$ (see, *e.g.*, [13] definition 1).

It is known that $\deg_\gamma(u)$ is an integer see [13] (the introduction) or [26].

We denote the (total) degree of $u \in \mathcal{J}$ in $\mathcal{D}$ by

$$\deg(u, \mathcal{D}) = \left(\deg_{\partial\omega_1}(u), ..., \deg_{\partial\omega_N}(u), \deg_{\partial\Omega}(u)\right) \in \mathbb{Z}^N \times \mathbb{Z}.$$

For $(\mathbf{p}, q) \in \mathbb{Z}^N \times \mathbb{Z}$, we are interested in the minimization of E_ε in

$$\mathcal{J}_{\mathbf{p},q} := \{u \in \mathcal{J} \,|\, \deg(u, \mathcal{D}) = (\mathbf{p}, q)\}.$$

There is an huge literature devoted to the minimization of E_ε. In a simply connected domain Ω, the minimization problem of E_ε with the Dirichlet boundary condition $g \in C^\infty(\partial\Omega, \mathbb{S}^1)$ is studied in detail in [18]. E_ε has a minimizer for each $\varepsilon > 0$. This minimizer need not to be unique. In this framework, when $\deg_{\partial\Omega}(g) \neq 0$, the authors studied the asymptotic behaviour of a sequence of minimizers (when $\varepsilon_n \downarrow 0$) and point out the existence (up to subsequence) of a finite set of singularities of the limit.

Other types of boundary conditions were studied, like Dirichlet condition $g \in C^\infty(\partial\Omega, \mathbb{C}\setminus\{0\})$ (in a simply connected domain Ω) in [7] and later for $g \in C^\infty(\partial\Omega, \mathbb{C})$ (see [8]).

If the boundary data is not $u_{|\partial\mathcal{D}}$, but a given set of degrees, then the existence of local minimizers is non trivial. Indeed, one can show that $\mathcal{J}_{\mathbf{p},q}$ is not closed under weak H^1-convergence (see next section), so that one cannot apply the direct method in the calculus of variations in order to derive existence of minimizers. Actually this is not just a technical difficulty, since in general the *infimum* of E_ε in $\mathcal{J}_{\mathbf{p},q}$ is **not** attained, we need more assumptions like the value of the H^1-capacity of $\mathcal{D}$ (see [12] and [13]).

Minimizers u of E_ε in $\mathcal{J}_{\mathbf{p},q}$, if they do exist, satisfy the equation

$$\begin{cases} -\Delta u = \dfrac{u}{\varepsilon^2}(1-|u|^2) & \text{in } \mathcal{D} \\ |u| = 1 & \text{on } \partial\mathcal{D} \\ u \times \partial_\nu u = 0 & \text{on } \partial\mathcal{D} \\ \deg(u,\mathcal{D}) = (\mathbf{p},q) & \end{cases} \tag{1.2}$$

where ∂_ν denotes the normal derivative, *i.e.*, $\partial_\nu = \dfrac{\partial}{\partial\nu} = \nu \cdot \nabla$.

Existence of local minimizers of E_ε is obtained following the same lines as in [16]. It turns out that, even if the *infimum* of E_ε in $\mathcal{J}_{\mathbf{p},q}$ is not attained, (1.2) may have solutions. This was established by Berlyand and Rybalko when $\mathcal{D}$ has a single hole, *i.e.*, when $N = 1$. Our main result is the following generalisation of the main result in [16]:

Theorem 1.1. *Let* $(\mathbf{p}, q) \in \mathbb{Z}^N \times \mathbb{Z}$ *and let* $M \in \mathbb{N}^*$*, there is* $\varepsilon_1(\mathbf{p}, q, M) > 0$ *s.t. for* $\varepsilon < \varepsilon_1$*, there are at least* M *locally minimizing solutions.*

Actually, we will prove a more precise form of Theorem 1.1 (see Theorem 1.4), whose statement relies on the notion of *approximate bulk degree* introduced in [16] and generalised in the next section.

The main difference with respect to [16] stems in the construction of the test functions with energy control in section 1.6. In a sense that will be explained in details in section 1.6, our construction is local, while the one in [16] is global. We also simplify and unify some proofs in [16].

We do not know whether the conclusion of theorem 1.1 still holds when $\mathcal{D}$ has no holes at all. That is, we do not know whether for a simply connected domain Ω, a given $d \in \mathbb{Z}^*$ and small ε, the problem

$$\begin{cases} -\Delta u = \dfrac{u}{\varepsilon^2}(1-|u|^2) & \text{in } \Omega \\ u \times \partial_\nu u = 0 & \text{on } \partial\Omega \\ |u| = 1 & \text{on } \partial\Omega \\ \deg_{\partial\Omega}(u) = d & \end{cases} \tag{1.3}$$

has solutions. Existence of a solution of (1.3) is clear when Ω is a disc, say $\Omega = D(0,R)$ (it suffices to consider a solution of $-\Delta u = \frac{u}{\varepsilon^2}(1-|u|^2)$ of the form $u(z) = f(|z|)\left(\dfrac{z}{|z|}\right)^d$ with $u_{|\partial\Omega} = \left(\dfrac{z}{|z|}\right)^d$). However, we do not know the answer when Ω is not radially symmetric anymore.

1.2 The approximate bulk degree

This section is a straightforward adaptation of [16].

Existence of (local) minimizers for E_ε in $\mathcal{J}_{\mathbf{p},q}$ is not straightforward since $\mathcal{J}_{\mathbf{p},q}$ is not closed under weak H^1-convergence. A typical example (see [13]) is a sequence $(M_n)_n$ s.t.

$$\begin{array}{rccc} M_n: & D(0,1) & \to & D(0,1) \\ & x & \mapsto & \dfrac{x-(1-1/n)}{(1-1/n)x-1} \end{array},$$

where $D(0,1)\subset\mathbb{C}$ is the open unit disc centered at the origin. Then $M_n \rightharpoonup 1$ in H^1, $\deg_{\mathbb{S}^1}(M_n)=1$ and $\deg_{\mathbb{S}^1}(1)=0$.

To obtain local minimizers, Berlyand and Rybalko (in [16]) devised a tool: the *approximate bulk degree*. We adapt this tool for a multiply connected domain.

We consider, for $i\in\mathbb{N}_N:=\{1,...,N\}$, V_i the unique solution of

$$\left\{\begin{array}{ll} -\Delta V_i=0 & \text{in } \mathcal{D} \\ V_i=1 & \text{on } \partial\mathcal{D}\setminus\partial\omega_i \\ V_i=0 & \text{on } \partial\omega_i \end{array}\right. . \tag{1.4}$$

For $u\in\mathcal{J}$, we set, noting $\partial_k u=\dfrac{\partial}{\partial x_k}u$

$$\text{abdeg}_i(u,\mathcal{D})=\frac{1}{2\pi}\int_{\mathcal{D}} u\times(\partial_1 V_i\,\partial_2 u-\partial_2 V_i\,\partial_1 u)\,\mathrm{d}x, \tag{1.5}$$

$$\text{abdeg}(u,\mathcal{D})=\left(\text{abdeg}_1(u,\mathcal{D})\,,...,\,\text{abdeg}_N(u,\mathcal{D})\,\right).$$

Following [16], we call $\text{abdeg}(u,\mathcal{D})$ the *approximate bulk degree* of u. $\text{abdeg}_i:\mathcal{J}\to\mathbb{R}$, in general, is not an integer (unlike the degree). However, we have

Proposition 1.2. *1) If $u\in H^1(\mathcal{D},\mathbb{S}^1)$, then* $\text{abdeg}_i(u,\mathcal{D})=\deg_{\partial\omega_i}(u)$*;*

2) Let $\Lambda,\varepsilon>0$ and $u,v\in\mathcal{J}$ s.t. $E_\varepsilon(u),E_\varepsilon(v)\le\Lambda$, then

$$|\text{abdeg}_i(u)-\text{abdeg}_i(v)|\le\frac{2}{\pi}\|V_i\|_{C^1(\mathcal{D})}\Lambda^{1/2}\|u-v\|_{L^2(\mathcal{D})}; \tag{1.6}$$

3) Let $\Lambda>0$ and $(u_\varepsilon)_{\varepsilon>0}\subset\mathcal{J}$ s.t. for all $\varepsilon>0$, $E_\varepsilon(u_\varepsilon)\le\Lambda$, then

$$\text{dist}(\text{abdeg}(u_\varepsilon),\mathbb{Z}^N)\to 0 \textit{ when } \varepsilon\to 0. \tag{1.7}$$

Proof of Proposition 1.2 is postponed to Appendix 1.B.

We define for $\mathbf{d}=(d_1,...,d_N)\in\mathbb{Z}^N$, $\mathbf{p}=(p_1,...,p_N)\in\mathbb{Z}^N$ and $q\in\mathbb{Z}$,

$$\mathcal{J}^{\mathbf{d}}_{\mathbf{p},q}=\mathcal{J}^{\mathbf{d}}_{\mathbf{p},q}(\mathcal{D}):=\left\{u\in\mathcal{J}_{\mathbf{p},q}\,|\,\|\text{abdeg}(u)-\mathbf{d}\|_\infty:=\max_{i\in\mathbb{N}_N}|d_i-\text{abdeg}_i(u)|\le\frac{1}{3}\right\}.$$

The following result states that $\mathcal{J}^{\mathbf{d}}_{\mathbf{p},q}$ in never empty for $(\mathbf{p},q,\mathbf{d})\in\mathbb{Z}^N\times\mathbb{Z}\times\mathbb{Z}^N$.

Proposition 1.3. *Let $(\mathbf{p},q,\mathbf{d})\in\mathbb{Z}^N\times\mathbb{Z}\times\mathbb{Z}^N$. Then $\mathcal{J}^{\mathbf{d}}_{\mathbf{p},q}\neq\emptyset$.*

Proof. For $i \in \{0, ..., N\}$, we denote $\mathbf{e}_i = (\delta_{i,1}, ..., \delta_{i,N}, \delta_{i,0}) \in \mathbb{Z}^{N+1}$ where

$$\delta_{i,k} = \begin{cases} 1 & \text{if } i = k \\ 0 & \text{otherwise} \end{cases} \quad \text{is the Kronecker symbol.}$$

For $i \in \{0, ..., N\}$, there is $M_n^i \in \mathcal{J}_{(p_i - d_i)\mathbf{e}_i}$ if $i \neq 0$ and $M_n^0 \in \mathcal{J}_{(q - \sum d_j)\mathbf{e}_0}$ s.t. $M_n^i \rightharpoonup 1$ in H^1 and $|M_n^i| \leq 1$ (Lemmas 6.1 and 6.2 in [13]). Let

$$E_{\mathbf{d}} := \left\{ u \in H^1(\mathcal{D}, \mathbb{S}^1) \,|\, \deg(u, \mathcal{D}) = (\mathbf{d}, d) \right\}, \mathbf{d} = (d_1, ..., d_N),\ d = \sum_{j=1}^{N} d_j.$$

We note that, $E_{\mathbf{d}} \neq \emptyset$, see, *e.g.*, [18]. Let $u \in E_{\mathbf{d}}$ and $u_n := u \prod_{i=0}^{N} M_n^i$. Then we will prove that, for large n, we have, up to subsequence, that $u_n \in \mathcal{J}_{\mathbf{p},q}^{\mathbf{d}}$. Indeed, up to subsequence,

$$u_n \rightharpoonup u \text{ in } H^1, \ \ u_n \in \mathcal{J}_{\mathbf{p},q}.$$

Using the fact that $\text{abdeg}(u) = \mathbf{d}$ and the weak H^1-continuity of the *approximate bulk degree*, we obtain for n sufficiently large, that $u_n \in \mathcal{J}_{\mathbf{p},q}^{\mathbf{d}}$. □

We denote $m_\varepsilon(\mathbf{p}, q, \mathbf{d})$ the *infimum* of E_ε on $\mathcal{J}_{\mathbf{p},q}^{\mathbf{d}}$, *i.e*,

$$m_\varepsilon(\mathbf{p}, q, \mathbf{d}) = \inf_{u \in \mathcal{J}_{\mathbf{p},q}^{\mathbf{d}}} E_\varepsilon(u)$$

and

$$I_0(\mathbf{d}, \mathcal{D}) = \inf_{u \in E_{\mathbf{d}}} \frac{1}{2} \int_{\mathcal{D}} |\nabla u|^2.$$

We may now state a refined version of Theorem 1.1.

Theorem 1.4. *Let $\mathbf{d} \in (\mathbb{N}^*)^N$. Then, for all $(p_1, ..., p_N, q) \in \mathbb{Z}^{N+1}$ s.t. $q \leq d$ and $p_i \leq d_i$, there is $\varepsilon_2 = \varepsilon_2(\mathbf{p}, q, \mathbf{d}) > 0$ s.t. for $0 < \varepsilon < \varepsilon_2$, $m_\varepsilon(\mathbf{p}, q, \mathbf{d})$ is attained.*

Moreover, we have the following estimate

$$m_\varepsilon(\mathbf{p}, q, \mathbf{d}) = I_0(\mathbf{d}, \mathcal{D}) + \pi\,(d_1 - p_1 + ... + d_N - p_N + d - q) - o_\varepsilon(1),\ \ o_\varepsilon(1) \underset{\varepsilon \to 0}{\to} 0.$$

For further use, a configuration of degrees $(\mathbf{p}, q, \mathbf{d}) \in \mathbb{Z}^N \times \mathbb{Z} \times (\mathbb{N}^*)^N$ s.t. $p_i \leq d_i$ and $q \leq \sum d_i$ will be called a "good configuration". Noting that, for $\mathbf{d} \neq \tilde{\mathbf{d}} \in \mathbb{Z}^N$ and $(\mathbf{p}, q) \in \mathbb{Z}^N \times \mathbb{Z}$, we have $\mathcal{J}_{\mathbf{p},q}^{\mathbf{d}} \cap \mathcal{J}_{\mathbf{p},q}^{\tilde{\mathbf{d}}} = \emptyset$, we are led to

Proof of Theorem 1.1: Let $(\mathbf{p}, q) \in \mathbb{Z}^N \times \mathbb{Z}$ and set for $k \in \mathbb{N}^*$,

$$d = \max\left\{ \max_i |p_i|, |q| \right\} \text{ and } \mathbf{d}_k = (d + k, ..., d + k).$$

We apply Theorem 1.4 to the class $\mathcal{J}_{\mathbf{p},q}^{\mathbf{d}_k}$. We obtain the existence of

$$\varepsilon_1(\mathbf{p}, q, M) = \min_{k \in \mathbb{N}_M} \varepsilon_2(\mathbf{p}, q, \mathbf{d}_k) > 0$$

s.t. for $\varepsilon < \varepsilon_1$, $k \in \mathbb{N}_M$, $m_\varepsilon(\mathbf{p}, q, \mathbf{d}_k)$ is achieved by u_ε^k.

Noting the continuity of the degree and of the *approximate bulk degree* for the strong H^1-convergence, there exists $V_\varepsilon^k \subset \mathcal{J}_{\mathbf{p},q}^{\mathbf{d}_k} \subset \mathcal{J}$ an open (for H^1-norm) neighbourhood of u_ε^k. It follows easily that

$$E_\varepsilon(u_\varepsilon^k) = \min_{u \in V_\varepsilon^k} E_\varepsilon(u).$$

Then $u_\varepsilon^k \in \mathcal{J}_{\mathbf{p},q}$ is a local minimizer of E_ε in $\mathcal{J}$ (for H^1-norm) for $0 < \varepsilon < \varepsilon_1(\mathbf{p}, q, M)$.

1.3 Basic facts of the Ginzburg-Landau theory

It is well known (*cf* [13], lemma 4.4 page 22) that the local minimizers of E_ε in $\mathcal{J}_{\mathbf{p},q}$ satisfy

$$-\Delta u = \frac{1}{\varepsilon^2} u(1-|u|^2) \text{ in } \mathcal{D}, \tag{1.8}$$

$$|u| = 1 \text{ and } u \times \partial_\nu u = 0 \text{ on } \partial\mathcal{D}. \tag{1.9}$$

Equation (1.8) and the Dirichlet condition on the modulus in (1.9) are classical. The Neumann condition on the phase in (1.9) is less standard but it is for example stated in [13].

Equation (1.8) combined with the boundary condition on $\partial\mathcal{D}$ implies, *via* a maximum principle, that

$$|u| \leq 1 \text{ in } \mathcal{D}. \tag{1.10}$$

One of the questions in the Ginzburg-Landau model is the location of the vortices of stable solutions (*i.e.*, local minimizers of E_ε). We will define *ad hoc* a vortex as an isolated zero x of u with nonzero degree on small circles around x.

The following result shows that, under energy bound assumptions on solutions of (1.8), vortices are expelled to the boundary when $\varepsilon \to 0$.

Lemma 1.5. *[53] Let $\Lambda > 0$ and let u be a solution of (1.8) satisfying (1.10) and the energy bound $E_\varepsilon(u) \leq \Lambda$. Then with C, C_k and ε_3 depending only on Λ, $\mathcal{D}$, we have, for $0 < \varepsilon < \varepsilon_3$ and $x \in \mathcal{D}$,*

$$1 - |u(x)|^2 \leq \frac{C\varepsilon^2}{\operatorname{dist}^2(x, \partial\mathcal{D})} \tag{1.11}$$

and

$$|D^k u(x)| \leq \frac{C_k}{\operatorname{dist}^k(x, \partial\mathcal{D})}. \tag{1.12}$$

When u is smooth in $\mathcal{D}$ and $\rho = |u| > 0$, the map $\dfrac{u}{\rho}$ admits a lifting θ , *i.e*, we may write

$$u = \rho e^{\imath\theta},$$

where θ is a smooth (and locally defined) real function on $\mathcal{D}$ and $\nabla\theta$ is a globally defined smooth vector field.

Using (1.8) and (1.9), we have

$$\begin{cases} \operatorname{div}(\rho^2 \nabla\theta) = 0 & \text{in } B \\ \partial_\nu \theta = 0 & \text{on } \partial\mathcal{D} \end{cases}, \tag{1.13}$$

$$\begin{cases} -\Delta\rho + |\nabla\theta|^2\rho + \dfrac{1}{\varepsilon^2}\rho(\rho^2 - 1) = 0 & \text{in } B \\ \rho = 1 & \text{on } \partial\mathcal{D} \end{cases}, \tag{1.14}$$

here, $B = \{x \in \mathcal{D} \,|\, u(x) \neq 0\}$.

We will need later the following.

Lemma 1.6. *[16] Let u be a solution of (1.8) and (1.9). Let $G \subset \mathcal{D}$ be an open Lipschitz set s.t. u does not vanish in $\overline{G}$. Write, in $\overline{G}$, $u = \rho v$ with $\rho = |u|$. Let $w \in H^1(G, \mathbb{C})$ be s.t. $|\operatorname{tr}_{\partial G} w| \equiv 1$. Then*

$$E_\varepsilon(\rho w, G) = E_\varepsilon(u, G) + L_\varepsilon(w, G),$$

with

$$L_\varepsilon(w, G) = \frac{1}{2}\int_G \rho^2 |\nabla w|^2 \, dx - \frac{1}{2}\int_G |w|^2 \rho^2 |\nabla v|^2 \, dx + \frac{1}{4\varepsilon^2}\int_G \rho^4 (1 - |w|^2)^2 \, dx.$$

For further use, we note that we may write, locally in $\overline{G}$, $u=\rho e^{\imath\theta}$, so that $v=e^{\imath\theta}$. It turns out that $\nabla\theta$ is smooth and globally defined in $\overline{G}$. In terms of $\nabla\theta$, we may rewrite

$$L_\varepsilon(w,G)=\frac{1}{2}\int_G\rho^2|\nabla w|^2\,\mathrm{d}x-\frac{1}{2}\int_G|w|^2\rho^2|\nabla\theta|^2\,\mathrm{d}x+\frac{1}{4\varepsilon^2}\int_G\rho^4(1-|w|^2)^2\,\mathrm{d}x.$$

For u a solution of (1.8) and (1.9), we can consider (see Lemma 7 in [16]) h the unique globally defined solution of

$$\begin{cases}\nabla^\perp h=u\times\nabla u & \text{in } \mathcal{D}\\ h=1 & \text{on } \partial\Omega\\ h=k_i & \text{on } \partial\omega_i\end{cases},\tag{1.15}$$

where k_i's are real constants uniquely defined by the first two equations in (1.15). Here

$\nabla^\perp h=\begin{pmatrix}-\partial_2 h\\ \partial_1 h\end{pmatrix}$ is the orthogonal gradient of h and $u\times\nabla u=\begin{pmatrix}u\times\partial_1 u\\ u\times\partial_2 u\end{pmatrix}$.

It is easy to show that

$$\begin{cases}\nabla h=-\rho^2\nabla^\perp\theta & \text{in } B\\ \operatorname{div}(\frac{1}{\rho^2}\nabla h)=0 & \text{in } B\\ \Delta h=2\partial_1 u\,\times\,\partial_2 u & \text{in } B\end{cases};\tag{1.16}$$

here, $B=\{x\in\mathcal{D}\,|\,u(x)\neq 0\}$.

In [18], Bethuel, Brezis and Hélein consider the minimization of $E(u)=\frac{1}{2}\int_{\mathcal{D}}|\nabla u|^2\,\mathrm{d}x$, the Dirichlet functional, in the class

$$E_{\mathbf{d}}=\{u\in H^1(\mathcal{D},\mathbb{S}^1)\,|\,\deg(u,\mathcal{D})=(\mathbf{d},d)\};$$

here, $d=\sum d_k$.

Theorem I.1 in [18] gives the existence of a unique solution (up to multiplication by an $\mathbb{S}^1$-constant) for the minimization of E in $E_{\mathbf{d}}$. We denote u_0 this solution. This u_0 is also a solution of

$$\begin{cases}-\Delta v=v|\nabla v|^2 & \text{in } \mathcal{D}\\ v\times\partial_\nu v=0 & \text{on } \partial\mathcal{D}\end{cases}.$$

Moreover, we have

$$I_0(\mathbf{d},\mathcal{D}):=\min_{u\in E_{\mathbf{d}}}E(u)=\frac{1}{2}\int_{\mathcal{D}}|\nabla h_0|^2\,\mathrm{d}x\tag{1.17}$$

with h_0 the unique solution of

$$\begin{cases}\Delta h_0=0 & \text{in } \mathcal{D}\\ h_0=1 & \text{on } \partial\Omega\\ h_0=\mathrm{Cst}_k & \text{on } \partial\omega_k,\ k\in\{1,...,N\}\\ \displaystyle\int_{\partial\omega_k}\partial_\nu h_0\,\mathrm{d}\sigma=2\pi d_k & \text{for } k\in\{1,...,N\}\end{cases}.\tag{1.18}$$

One may prove that h_0 is the (globally defined) harmonic conjugate of a local lifting of u_0.

1.4 Energy needed to change degrees

We denote

$$\text{æ} : \begin{array}{ccc} (\mathbb{Z}^N \times \mathbb{Z}) \times (\mathbb{Z}^N \times \mathbb{Z}) & \to & \mathbb{N} \\ ((\mathbf{d}, d), (\mathbf{p}, q)) & \mapsto & \sum_{i=1}^N |d_i - p_i| + |d - q| \end{array}.$$

The next result quantifies the energy needed to change degrees in the weak limit.

Lemma 1.7. *([13], Lemma 1) Let $(u_n)_n \subset \mathcal{J}_{\mathbf{p},q}$ be a sequence weakly converging in H^1 to u. Then*

$$\liminf_n E(u_n) \geq E(u) + \pi\text{æ}(\deg(u, \mathcal{D}), (\mathbf{p}, q)) \tag{1.19}$$

and for $\varepsilon > 0$

$$\liminf_n E_\varepsilon(u_n) \geq E_\varepsilon(u) + \pi\text{æ}(\deg(u, \mathcal{D}), (\mathbf{p}, q)). \tag{1.20}$$

The next lemma is proved in [16].

Lemma 1.8. *Let $\mathbf{d} = (d_1, ..., d_N), \mathbf{p} = (p_1, ..., p_N) \in \mathbb{Z}^N$, $q \in \mathbb{Z}$. There is $o_\varepsilon(1) \underset{\varepsilon\to 0}{\to} 0$ (depending of $(\mathbf{p}, q, \mathbf{d})$) s.t. for $u \in \mathcal{J}^{\mathbf{d}}_{\mathbf{p},q}$ we have*

$$E_\varepsilon(u) \geq I_0(\mathbf{d}, \mathcal{D}) + \pi\text{æ}((\mathbf{d}, d), (\mathbf{p}, q)) - o_\varepsilon(1). \tag{1.21}$$

Here, $d := \sum d_i$.

We present below a simpler proof than the original one in [16].

Proof. Let $(\mathbf{p}, q, \mathbf{d}) \in \mathbb{Z}^N \times \mathbb{Z} \times \mathbb{Z}^N$. We argue by contradiction and we suppose that there are $\delta > 0$, $\varepsilon_n \downarrow 0$ and $(u_n)_n \subset \mathcal{J}^{\mathbf{d}}_{\mathbf{p},q}$ s.t.

$$E_{\varepsilon_n}(u_n) \leq I_0(\mathbf{d}, \mathcal{D}) + \pi\text{æ}((\mathbf{d}, d), (\mathbf{p}, q)) - \delta. \tag{1.22}$$

Since $(u_n)_n$ is bounded in H^1, there is some u s.t., up to subsequence, $u_n \rightharpoonup u$ in H^1 and $u_n \to u$ in L^4. Using the strong convergence in L^4, (1.22) and Proposition 1.2, we have $u \in H^1(\mathcal{D}, \mathbb{S}^1) \cap \mathcal{J}^{\mathbf{d}}_{\mathbf{d},d} = E_{\mathbf{d}}$.

To conclude, we use (1.22) combined with Lemma 1.7

$$\begin{aligned} I_0(\mathbf{d}, \mathcal{D}) + \pi\text{æ}((\mathbf{d}, d), (\mathbf{p}, q)) - \delta &\geq \liminf_n E_{\varepsilon_n}(u_n) \\ &\geq \liminf_n E(u_n) \\ &\geq E(u) + \pi\text{æ}((\mathbf{d}, d), (\mathbf{p}, q)) \\ &\geq I_0(\mathbf{d}, \mathcal{D}) + \pi\text{æ}((\mathbf{d}, d), (\mathbf{p}, q)) \end{aligned}$$

which is a contradiction. □

One may easily proved (see Lemma 1.22 in Appendix 1.C) that for $\eta > 0$, $i \in \{0, ..., N\}$ and $u \in \mathcal{J}_{\deg(u,\mathcal{D})}$, there are $v_\pm \in \mathcal{J}_{\deg(u,\mathcal{D})\pm\mathbf{e}_i}$ s.t.

$$E_\varepsilon(v_\pm) \leq E_\varepsilon(u) + \pi + \eta.$$

The key ingredient is a sharper result which holds under two additional hypotheses. In order to unify the notations, we use the notation ω_0 for Ω. We may now state the main ingredient in the proof of Theorem 1.4.

Lemma 1.9. *Let $u \in \mathcal{J}_{\mathbf{p},q}$ be a solution of (1.8), (1.9).*

Assume that

$$\text{abdeg}_j(u) \in (d_j - \frac{1}{3}, d_j + \frac{1}{3}),\ \forall j \in \mathbb{N}_N. \tag{1.23}$$

Let $i \in \{0, ..., N\}$ and assume that there is some point $x^i \in \partial\omega_i$ s.t. $u \times \partial_\tau u(x^i) > 0$. Recall that τ is the direct tangent vector to $\partial\omega_i$.

Then there is $\tilde{u} \in \mathcal{J}_{(\mathbf{p},q)-\mathbf{e}_i}$ s.t.

$$E_\varepsilon(\tilde{u}) < E_\varepsilon(u) + \pi$$

and

$$\text{abdeg}_j(\tilde{u}) \in (d_j - \frac{1}{3}, d_j + \frac{1}{3}),\ \forall j \in \mathbb{N}_N.$$

The proof of Lemma 1.9 is postponed to section 1.6.

We also have an upper bound for $m_\varepsilon(\mathbf{p}, q, \mathbf{d})$.

Lemma 1.10. *Let $\varepsilon > 0$ and $(\mathbf{p}, q, \mathbf{d}) \in \mathbb{Z}^N \times \mathbb{Z} \times \mathbb{Z}^N$. Then*

$$m_\varepsilon(\mathbf{p}, q, \mathbf{d}) \le I_0(\mathbf{d}, \mathcal{D}) + \pi æ((\mathbf{d}, d), (\mathbf{p}, q)). \tag{1.24}$$

To prove Lemma 1.10, we need the following

Lemma 1.11. *Let $u \in \mathcal{J}$, $\varepsilon > 0$ and $\delta = (\delta_1, ..., \delta_N, \delta_0) \in \mathbb{Z}^{N+1}$. For all $\eta > 0$, there is $u_\eta^\delta \in \mathcal{J}_{\deg(u,\mathcal{D})+\delta}$ s.t.*

$$E_\varepsilon(u_\eta^\delta) \le E_\varepsilon(u) + \pi \sum_{i \in \{0,...,N\}} |\delta_i| + \eta \tag{1.25}$$

and

$$\|u - u_\eta^\delta\|_{L^2(\mathcal{D})} = o_\eta(1),\ o_\eta(1) \underset{\eta \to 0}{\to} 0. \tag{1.26}$$

The proof of Lemma 1.11 is postponed to Appendix 1.C.

Proof. We prove that for $\eta > 0$ small, we have

$$m_\varepsilon(\mathbf{p}, q, \mathbf{d}) \le I_0(\mathbf{d}, \mathcal{D}) + \pi æ((\mathbf{d}, d), (\mathbf{p}, q)) + \eta.$$

We denote $u_0 \in E_{\mathbf{d}}$ s.t. $E(u_0) = I_0(\mathbf{d}, \mathcal{D})$. Then $\text{abdeg}_i(u_0) = d_i$.

Using Lemma 1.11 with $\delta = (\mathbf{p}, q) - (\mathbf{d}, d)$, there is u_η s.t.

$u_\eta \in \mathcal{J}_{(\mathbf{p},q)}$ and $E_\varepsilon(u_\eta) \le E_\varepsilon(u_0) + \pi æ((\mathbf{d}, d), (\mathbf{p}, q)) + \eta = I_0(\mathbf{d}, \mathcal{D}) + \pi æ((\mathbf{d}, d), (\mathbf{p}, q)) + \eta.$

Furthermore, by (1.26), $\|u_0 - u_\eta\|_{L^2(\mathcal{D})} = o_\eta(1)$. For η small, by Proposition 1.2, we have $u_0 \in \mathcal{J}_{\mathbf{p},q}^{\mathbf{d}}$ which proves the lemma. □

1.5 A family with bounded energy converges

In this section we discuss:

1. the asymptotic behaviour of a sequence of solutions of (1.8), (1.9), $(u_{\varepsilon_n})_n \subset \mathcal{J}_{\mathbf{p},q}^{\mathbf{d}}$ $(\varepsilon_n \downarrow 0)$ with bounded energy , *i.e*, $E_{\varepsilon_n}(u_{\varepsilon_n}) \le \Lambda$,

2. the asymptotic behaviour of a minimizing sequence of E_ε in $\mathcal{J}_{\mathbf{p},q}^{\mathbf{d}}$,

3. a fundamental lemma.

Proposition 1.12. *Let $\varepsilon_n \downarrow 0$, $(u_{\varepsilon_n})_n \subset \mathcal{J}^{\mathbf{d}}_{\mathbf{p},q}$ with u_{ε_n} a solution of (1.8), (1.9), s.t. for $\Lambda > 0$, we have*

$$E_{\varepsilon_n}(u_{\varepsilon_n}) \leq \Lambda.$$

Then, denoting h_{ε_n} the unique solution of (1.15) with $u = u_{\varepsilon_n}$, we have

$$h_{\varepsilon_n} \rightharpoonup h_0 \text{ in } H^1(\mathcal{D}), \tag{1.27}$$

where h_0 is the unique solution of (1.18).
Up to subsequence, it holds

$$u_{\varepsilon_n} \rightharpoonup u_0 \text{ in } H^1(\mathcal{D}), \tag{1.28}$$

where $u_0 \in E_{\mathbf{d}}$ is the unique solution of (1.17) up to multiplication by an $\mathbb{S}^1$-constant.

Proof. Using the energy bound on u_{ε_n} and a Poincaré type inequality, we have, up to subsequence,

$$h_{\varepsilon_n} \rightharpoonup h \text{ in } H^1.$$

In order to establish (1.27), it suffices to prove that $h = h_0$.

The set $\mathcal{H} := \{h \in H^1(\mathcal{D}, \mathbb{R})\,;\, \partial_\tau h \equiv 0 \text{ on } \partial\mathcal{D} \text{ and } h_{|\partial\Omega} \equiv 1\}$ is closed convex in $H^1(\mathcal{D}, \mathbb{R})$. Since $(h_{\varepsilon_n})_n \subset \mathcal{H}$, we find that $h \in \mathcal{H}$.

By boundedness of $E_{\varepsilon_n}(u_{\varepsilon_n})$, Lemma 1.5 implies that u_{ε_n} is bounded in $C^2_{\text{loc}}(\mathcal{D}, \mathbb{R}^2)$. Therefore there is some $u \in C^1_{\text{loc}}(\mathcal{D}, \mathbb{C})$ s.t., up to subsequence, $u_{\varepsilon_n} \to u$ in $C^1_{\text{loc}}(\mathcal{D}, \mathbb{R}^2)$, $L^4(\mathcal{D}, \mathbb{R}^2)$ and weakly in $H^1(\mathcal{D}, \mathbb{R}^2)$.

Using the strong convergence in L^4 and the energy bound on u_{ε_n}, we find that $u \in H^1(\mathcal{D}, \mathbb{S}^1)$. It follows that $\partial_1 u \times \partial_2 u = 0$ in $\mathcal{D}$. On the other hand,

$$\Delta h_{\varepsilon_n} = 2\partial_1 u_{\varepsilon_n} \times \partial_2 u_{\varepsilon_n} \to 0 \text{ in } C^0_{\text{loc}}.$$

Therefore, h is a harmonic function in $\mathcal{D}$.

In order to show that $h = h_0$, it suffices to check that

$$\int_{\partial\omega_i} \partial_\nu h \,\mathrm{d}\sigma = 2\pi d_i.$$

To this end, we note that, since $u_{\varepsilon_n} \times (\partial_1 V_i \partial_2 u_{\varepsilon_n} - \partial_2 V_i \partial_1 u_{\varepsilon_n}) = \nabla V_i \cdot \nabla h_{\varepsilon_n}$, we have from (1.4)

$$2\pi \operatorname{abdeg}_i(u_{\varepsilon_n}) = \int_{\mathcal{D}} \nabla V_i \cdot \nabla h_{\varepsilon_n} \,\mathrm{d}x \xrightarrow[n\to\infty]{} \int_{\mathcal{D}} \nabla V_i \cdot \nabla h \,\mathrm{d}x = \int_{\partial\mathcal{D}\setminus\partial\omega_i} \partial_\nu h \,\mathrm{d}\sigma.$$

Noting that, by Proposition 1.2,

$$\begin{cases} \operatorname{abdeg}_i(u_{\varepsilon_n}) \xrightarrow[n\to\infty]{} \operatorname{abdeg}_i(u) = \deg_{\partial\omega_i}(u) \\ \operatorname{abdeg}_i(u_{\varepsilon_n}) \xrightarrow[n\to\infty]{} d_i \end{cases}$$

and that $0 = \displaystyle\int_{\mathcal{D}} \Delta h \,\mathrm{d}x = \int_{\partial\mathcal{D}} \partial_\nu h \,\mathrm{d}\sigma$, we obtain

$$\int_{\partial\mathcal{D}\setminus\partial\omega_i} \partial_\nu h \,\mathrm{d}\sigma = \int_{\partial\omega_i} \partial_\nu h \,\mathrm{d}\sigma = 2\pi\, d_i = 2\pi \deg_{\partial\omega_i}(u).$$

In the first integral, ν is the outward normal to $\mathcal{D}$, in the second, ν is the outward normal to ω_i.

This proves (1.27).

We next turn to (1.28). Let u_0 be s.t., up to subsequence, $u_{\varepsilon_n} \rightharpoonup u_0$ in $H^1(\mathcal{D})$. Since $|u_{\varepsilon_n}| \leq 1$, we find that

$$u_{\varepsilon_n} \times \nabla u_{\varepsilon_n} \rightharpoonup u_0 \times \nabla u_0 \text{ in } L^2(\mathcal{D}).$$

In view of (1.15) and (1.27), we have $u_0 \times \nabla u_0 = \nabla^\perp h_0$. Therefore,

$$E(u_0) = E(h_0) = I_0(\mathbf{d}, \mathcal{D}).$$

Proposition 1.2 implies that $u_0 \in E_{\mathbf{d}}$. Then u_0 is the unique, up to multiplication by an $\mathbb{S}^1$-constant, minimizer of E in $E_{\mathbf{d}}$. □

Proposition 1.13. *Let $(\mathbf{p}, q, \mathbf{d}) \in \mathbb{Z}^N \times \mathbb{Z} \times \mathbb{Z}^N$. For $\varepsilon > 0$, let $(u_n^\varepsilon)_{n\geq 0} \subset \mathcal{J}_{\mathbf{p},q}^{\mathbf{d}}$ be a minimizing sequence of E_ε in $\mathcal{J}_{\mathbf{p},q}^{\mathbf{d}}$. Then there is $\varepsilon_4(\mathbf{p}, q, \mathbf{d}) > 0$ s.t. for $0 < \varepsilon < \varepsilon_4$, up to subsequence, $u_n \rightharpoonup u$ in H^1 with u which minimizes E_ε in $\mathcal{J}_{\deg(u,\mathcal{D})}^{\mathbf{d}}$.*

Proof. For $\varepsilon > 0$, let $(u_n^\varepsilon)_n \subset \mathcal{J}_{\mathbf{p},q}^{\mathbf{d}}$ be a minimizing sequence of E_ε in $\mathcal{J}$. Up to subsequence, using Proposition 1.2,

$$u_n^\varepsilon \rightharpoonup u^\varepsilon \text{ in } H^1 \text{ with } u^\varepsilon \in \mathcal{J}_{\deg(u^\varepsilon,\mathcal{D})}^{\mathbf{d}}.$$

Using Lemmas 1.7 and 1.10, we see that $\{\deg(u^\varepsilon, \mathcal{D}), \varepsilon > 0\} \subset \mathbb{Z}^N \times \mathbb{Z}$ is a finite set and that $E_\varepsilon(u^\varepsilon)$ is bounded. Therefore, with Proposition 1.2, there is $\varepsilon_4 > 0$ s.t. $|\text{abdeg}_i(u^\varepsilon) - d_i| < \frac{1}{3}$ for all $i \in \mathbb{N}_N$ and $0 < \varepsilon < \varepsilon_4$.

We argue by contradiction and we assume that there is $\varepsilon < \varepsilon_4$ s.t.

$$E_\varepsilon(u^\varepsilon) = m_\varepsilon(\deg(u^\varepsilon, \mathcal{D}), \mathbf{d}) + 2\eta,\ \eta > 0.$$

Let $u \in \mathcal{J}_{\deg(u^\varepsilon,\mathcal{D})}^{\mathbf{d}}$ be s.t. $E_\varepsilon(u) \leq m_\varepsilon(\deg(u^\varepsilon, \mathcal{D}), \mathbf{d}) + \eta$.

Using Lemma 1.11 with $\delta = (\mathbf{p}, q) - \deg(u^\varepsilon, \mathcal{D})$, there is $v \in \mathcal{J}_{\mathbf{p},q}$ s.t.

$$E_\varepsilon(v) < E_\varepsilon(u) + \pi æ((\mathbf{p}, q), \deg(u^\varepsilon, \mathcal{D})) + \eta.$$

Furthermore, by (1.26), $\|u - v\|_{L^2}$ can be taken arbitrary small, so that we may further assume $v \in \mathcal{J}_{\mathbf{p},q}^{\mathbf{d}}$. To summarise we have

$$\begin{aligned} m_\varepsilon(\mathbf{p}, q, \mathbf{d}) &= \liminf_n E_\varepsilon(u_n^\varepsilon) \\ &\geq E_\varepsilon(u^\varepsilon) + \pi æ((\mathbf{p}, q), \deg(u^\varepsilon, \mathcal{D})) \\ &= m_\varepsilon(\deg(u^\varepsilon, \mathcal{D}), \mathbf{d}) + 2\eta + \pi æ((\mathbf{p}, q), \deg(u^\varepsilon, \mathcal{D})) \\ &\geq E_\varepsilon(u) + \pi æ((\mathbf{p}, q), \deg(u^\varepsilon, \mathcal{D})) + \eta \\ &> E_\varepsilon(v) \geq m_\varepsilon(\mathbf{p}, q, \mathbf{d}). \end{aligned}$$

This contradiction completes the proof.

□

The main tool requires the following lemma.

Lemma 1.14. *Let $(\mathbf{p}, q, \mathbf{d}) \in \mathbb{Z}^N \times \mathbb{Z} \times \mathbb{Z}^N$ and $\Lambda > 0$. There is $\varepsilon_5(\mathbf{p}, q, \mathbf{d}, \Lambda) > 0$ s.t. for $\varepsilon < \varepsilon_5$ and $u \in \mathcal{J}_{\mathbf{p},q}^{\mathbf{d}}$, a solution of (1.8) and (1.9) with $E_\varepsilon(u) \leq \Lambda$, if $d > 0$ (respectively $d_i > 0$), then there is $x^0 \in \partial\Omega$ (respectively $x^i \in \partial\omega_i$) s.t. $u \times \partial_\tau u(x^0) > 0$ (respectively $u \times \partial_\tau u(x^i) > 0$).*

Here τ is the direct tangent vector to $\partial\Omega$ (resp. $\partial\omega_i$).

Proof. We prove existence of $x^0 \in \partial\Omega$ under appropriate assumptions. Existence of x^i is similar.

We argue by contradiction. Assume that there are $\varepsilon_n \downarrow 0$, $(u_n) \subset \mathcal{J}^{\mathbf{d}}_{\mathbf{p},q}$ solutions of (1.8) and (1.9) with $E_{\varepsilon_n}(u_n) \le \Lambda$ s.t. $u_n \times \partial_\tau u_n \le 0$ on $\partial\Omega$.

Since $q = \dfrac{1}{2\pi}\displaystyle\int_{\partial\Omega} u_n \times \partial_\tau u_n$, we have $q \le 0$.

Up to subsequence, by Proposition 1.12, we can assume that

$$u_n \to u_0 \text{ a.e. with } u_0 \text{ the unique solution (up to } \mathbb{S}^1) \text{ of (1.17).}$$

Let $x_0 \in \partial\Omega$ and let $\gamma : \partial\Omega \to [0, \mathcal{H}^1(\partial\Omega)[=: I$ be s.t. γ^{-1} is the direct arc-length parametrization of $\partial\Omega$ with the origin at x_0.

We denote $\theta_n : I \to \mathbb{R}$ the smooth functions s.t.

$$\begin{cases} u_n(x) = \mathrm{e}^{\imath\theta_n[\gamma(x)]}\, \forall\, x \in \partial\Omega \\ 0 \le \theta_n(0) < 2\pi \end{cases}.$$

Then, for all n, θ_n is nonincreasing and $\theta_n \in [\theta_n(0) + 2\pi q, \theta_n(0)] \subset [2\pi q, 2\pi]$.

Using Helly's selection theorem, up to subsequence, we can assume that $\theta_n \to \theta$ everywhere on I with θ nonincreasing. Denote Ξ the set of discontinuity points of θ. Since θ is nonincreasing, Ξ is a countable set.

Using the monotonicity of θ, we can consider the following decomposition

$$\theta = \theta^c + \theta^\delta, \text{ with } \theta^c \text{ and } \theta^\delta \text{ are nonincreasing functions.}$$

θ^c is the continuous part of θ and θ^δ is the jump function. The set of discontinuity points of θ^δ is Ξ.

For $t \notin \Xi$,

$$\theta^\delta(t) = \sum_{0<s<t,\, s\in\Xi} \{\theta(s+) - \theta(s-)\}.$$

We obtain easily that $u_0(x) = \mathrm{e}^{\imath\theta[\gamma(x)]}$ a.e. $x \in \partial\Omega$. Since u_0, θ_n and γ have side limits at each points and $u_0 = \mathrm{e}^{\imath\theta\circ\gamma}$ a.e., we find that

$$u_0(x\pm) = \mathrm{e}^{\imath\theta[\gamma(x\pm)]} \text{ for } \textbf{each } x \in \partial\Omega.$$

Using the continuity of u_0, we obtain $\mathrm{e}^{\imath\theta[\gamma(x+)]} = \mathrm{e}^{\imath\theta[\gamma(x-)]}\, \forall\, x \in \partial\Omega$ which implies that

$$\theta[\gamma(x+)] - \theta[\gamma(x-)] \in 2\pi\mathbb{Z}\, \forall\, x \in \partial\Omega.$$

For $t \notin \Xi$,

$$\theta^\delta(t) = \sum_{0<s<t,\, s\in\Xi} \{\theta(s+) - \theta(s-)\} \in 2\pi\mathbb{Z}.$$

Then

$$u_0(x)\mathrm{e}^{-\imath\theta^c[\gamma(x)]} = \mathrm{e}^{\imath\theta^\delta[\gamma(x)]} = 1 \text{ a.e. } x \in \partial\Omega.$$

Finally, $u_0(x) = \mathrm{e}^{\imath\theta^c[\gamma(x)]}$ a.e. $x \in \partial\Omega$, which is equivalent (using the continuity of the functions) at $u_0 = \mathrm{e}^{\imath\theta^c\circ\gamma}$.

We have a contradiction observing that

$$0 < 2\pi \deg_{\partial\Omega}(u_0) = 2\pi d = \theta^c(\mathcal{H}^1(\partial\Omega)) - \theta^c(0)$$

and using the fact that θ^c is nonincreasing. □

1.6 Proof of Lemma 1.9

We prove only the part of the lemma concerning $\partial\Omega$. The proof for the other connected components of $\partial\mathcal{D}$ is similar.

For reader's convenience, we state the part of Lemma 1.9 that we will actually prove

Lemma . *Let $u \in \mathcal{J}_{\mathbf{p},q}$ be a solution of (1.8) and (1.9).*

Assume that

$$\text{abdeg}_j(u) \in (d_j - \frac{1}{3}, d_j + \frac{1}{3}),\ \ \forall\, j \in \mathbb{N}_N \tag{1.23}$$

and that there is some point $x^0 \in \partial\Omega$ s.t. $u \times \partial_\tau u(x^0) > 0$.

Then there is $\tilde{u} \in \mathcal{J}_{(\mathbf{p},q-1)}$ s.t.

$$E_\varepsilon(\tilde{u}) < E_\varepsilon(u) + \pi,$$

$$\text{abdeg}_j(\tilde{u}) \in (d_j - \frac{1}{3}, d_j + \frac{1}{3}),\ \forall\, j \in \mathbb{N}_N.$$

1.6.1 Decomposition of $\mathcal{D}$

By hypothesis, there is some $x^0 \in \partial\Omega$ s.t. $\partial_\nu h(x^0) > 0$. Without loss of generality, we may assume that $u(x^0) = 1$.

Then there is $\Upsilon \subset \overline{\mathcal{D}}$, a compact neighbourhood of x^0, simply connected and with nonempty interior, s.t.:

- $\gamma := \partial\Omega \cap \partial\Upsilon$ is connected with nonempty interior;
- x^0 is an interior point of γ;
- $|\nabla h| > 0$, $\rho > 0$, $h \leq 1$ in Υ;
- $\partial_\nu h > 0$ on γ (ν the outward normal to Ω).

It follows that, in Υ, θ, a lifting of $u/|u|$ is globally defined (we take the determination of θ which vanishes at x^0) .

Using the inverse function theorem, we may assume, by further restricting Υ, that there are some $0 < \eta, \delta < 1$ s.t.

$$\Upsilon = \{x \in \mathcal{D} \text{ s.t. } \operatorname{dist}(x, x^0) < \eta,\ 1 - \delta \leq h(x) \leq 1,\ -2\delta \leq \theta(x) \leq 2\delta\}.$$

We may further assume that, by replacing δ by smaller value if necessary and denoting $D_\delta := \overset{\circ}{\Upsilon}$ (see Figure 1.1), we have

(i) $$\begin{array}{rccl} \Theta := (\theta, h)_{|D_\delta} : & D_\delta & \to & (-2\delta, 2\delta) \times (1-\delta, 1) \quad \text{is a } C^1\text{-diffeomorphism,} \\ & x & \mapsto & (\theta, h) \end{array}$$

(ii) $\partial D_\delta \setminus (\{h = 1\} \cup \{h = 1 - \delta\}) = \partial D_\delta \cap (\{\theta = -2\delta\} \cup \{\theta = 2\delta\})$,

(iii) D_δ is a Lipschitz domain.

We consider $\delta_0 > 0$ s.t. for $\delta < \delta_0$, D_δ satisfies previous properties and

$$|D_\delta|^{1/2} < \frac{\pi\left|\|\text{abdeg}(u) - \mathbf{d}\|_\infty - \frac{1}{3}\right|}{6 \max_i \|V_i\|_{C^1(\mathcal{D})} \left(E_\varepsilon(u) + \pi\right)^{1/2}}. \tag{1.29}$$

Using Proposition 1.2 and (1.29), if $v \in H^1(\mathcal{D}, \mathbb{C})$ satisfies $u = v$ in $\mathcal{D} \setminus D_\delta$, $|v| \leq 2$ in $\mathcal{D}$ and $E_\varepsilon(v) < E_\varepsilon(u) + \pi$, then we have $\text{abdeg}_i(v) \in (d_i - 1/3, d_i + 1/3)$.
We let $\delta < \delta_0$ and we denote

$$D'_\delta := \Theta^{-1}\left[(-\delta, \delta) \times (1-\delta, 1)\right],$$
$$D^-_\delta := \Theta^{-1}\left[(-2\delta, -\delta) \times (1-\delta, 1)\right],$$
$$D^+_\delta := \Theta^{-1}\left[(\delta, 2\delta) \times (1-\delta, 1)\right],$$

so that D'_δ, D^-_δ and D^+_δ are Lipschitz domains (see Figure 1.1).

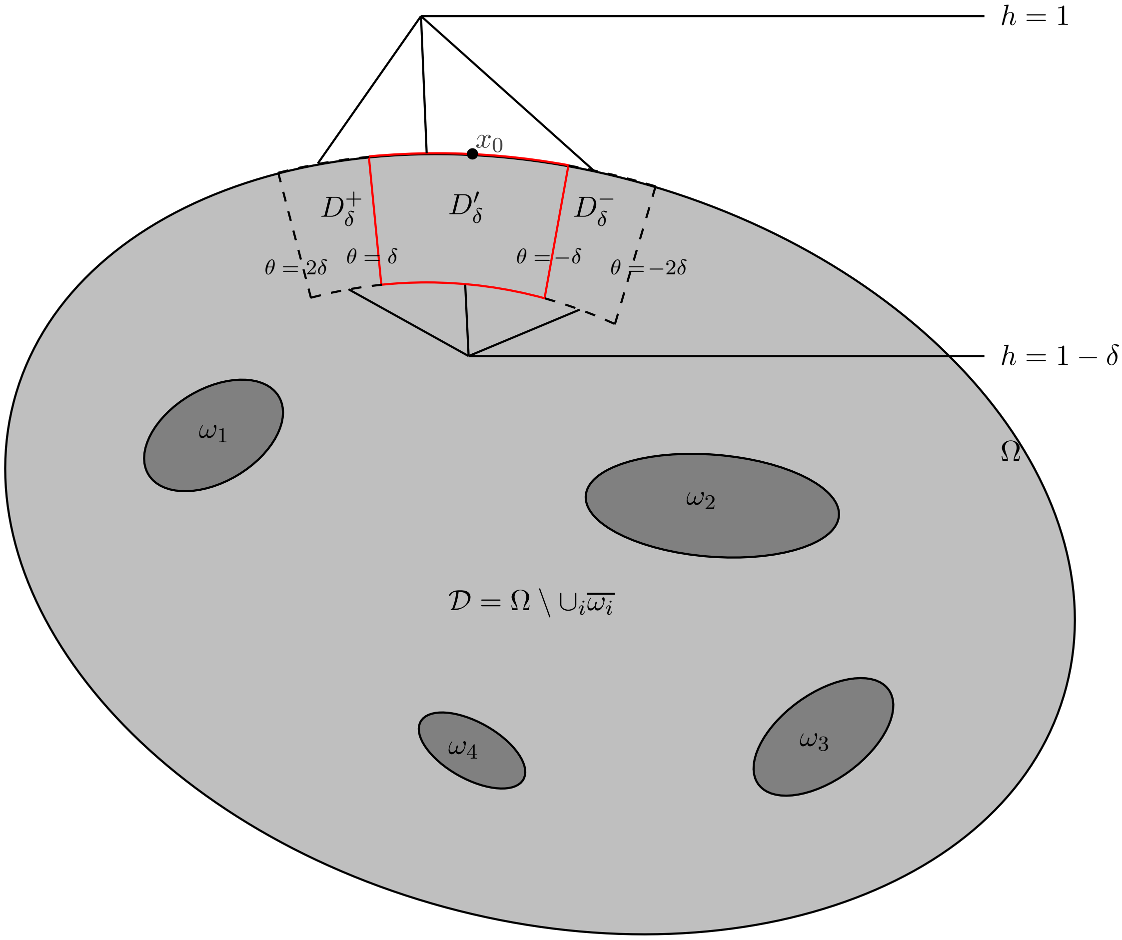


Figure 1.1: Decomposition of $\mathcal{D}$

1.6.2 Construction of the test function

We consider an application (with unknown expression in D_δ) $\psi_t : \mathcal{D} \to \mathbb{C}$ ($t > 0$ smaller than δ) s.t.

$$\psi_t(x) = \begin{cases} 1 & \text{in } \mathcal{D} \setminus D_\delta \\ \dfrac{\mathrm{e}^{-\imath\theta} - (1 - t\varphi(\theta))}{\mathrm{e}^{-\imath\theta}(1 - t\varphi(\theta)) - 1} & \text{on } \partial\Omega \cap \partial D_\delta \end{cases}, \tag{1.30}$$

with $0 \leq \varphi \leq 1$ a smooth, even and 2π-periodic function satisfying

$$\varphi_{|(-\delta/2, \delta/2)} \equiv 1 \text{ and } \varphi_{|[-\pi,\pi[\setminus(-\delta,\delta)} \equiv 0.$$

It is clear that $\psi_{t|\partial\mathcal{D}} \in C^\infty(\partial\mathcal{D})$ and

$$\deg_{\partial\omega_i}(\psi_t) = 0 \text{ for all } i \in \mathbb{N}_N. \tag{1.31}$$

Expanding in Fourier series, we have

$$\frac{\mathrm{e}^{-\imath\theta} - (1 - t\varphi(\theta))}{\mathrm{e}^{-\imath\theta}(1 - t\varphi(\theta)) - 1} = (1 - tb_{-1}(t)) + t\sum_{k\neq -1} b_k(t)\mathrm{e}^{-(k+1)\imath\theta}. \tag{1.32}$$

Noting that the real part of $\dfrac{\mathrm{e}^{-\imath\theta} - (1 - t\varphi(\theta))}{\mathrm{e}^{-\imath\theta}(1 - t\varphi(\theta)) - 1}$ is even and the imaginary part is odd, we obtain that $b_k(t) \in \mathbb{R}$ for all k, t.

The following lemma is proven in Appendix 1.B

Lemma 1.15. *We denote, for* $\mathrm{e}^{\imath\theta} \in \mathbb{S}^1$,

$$\Psi_t(\mathrm{e}^{\imath\theta}) = \frac{\mathrm{e}^{-\imath\theta} - (1 - t\varphi(\theta))}{\mathrm{e}^{-\imath\theta}(1 - t\varphi(\theta)) - 1} \qquad \text{and} \qquad \mathcal{F}_t(\mathrm{e}^{\imath\theta}) = \frac{\mathrm{e}^{-\imath\theta} - (1 - t)}{\mathrm{e}^{-\imath\theta}(1 - t) - 1}.$$

Then:

1) $|\Psi_t - \mathcal{F}_t| \leq C_\delta\, t$ *on* $\mathbb{S}^1$*;*

2) $\mathcal{F}_t(z) = \dfrac{\overline{z} - (1 - t)}{\overline{z}(1 - t) - 1} = (1 - tc_{-1}) + t\displaystyle\sum_{k\neq -1} c_k(t)\overline{z}^{k+1}$*, with*

$$c_k = \begin{cases} (t-2)(1-t)^k & \text{if } k \geq 0 \\ 0 & \text{if } k \leq -2\,; \\ 1 & \text{if } k = -1 \end{cases}$$

3) $|b_k(t) - c_k(t)| \leq C(n, \delta)\,(1 + |k|)^{-n}$, $\forall\, n > 0$ *with* $C(n, \delta)$ *independent of* t *sufficiently small.*

It is easy to see using Lemma 1.15 that, for t sufficiently small,

$$\deg_{\mathbb{S}^1}(\Psi_t) = \deg_{\mathbb{S}^1}(\mathcal{F}_t) = -1.$$

Using the previous equality and the fact that $\partial_\tau\theta > 0$ on γ, we find that

$$\deg_{\partial\Omega}(\psi_t) = -1. \tag{1.33}$$

It will be convenient to use h and θ as a shorthand for $h(x)$ and $\theta(x)$. With these notations, we will look for ψ_t of the form

$$\begin{aligned} \psi_t(x) &= \tilde{\psi}_t(h, \theta) \\ &= \begin{cases} (1 - tf_{-1}(h)b_{-1}(t)) + t\displaystyle\sum_{k\neq -1} b_k(t)f_k(h)\mathrm{e}^{-(k+1)\imath\theta} & \text{in } D'_\delta \\ \dfrac{\theta - \delta}{\delta} + \tilde{\psi}_t(h, \delta)\dfrac{2\delta - \theta}{\delta} & \text{in } D^+_\delta \\ -\dfrac{\theta + \delta}{\delta} + \tilde{\psi}_t(h, -\delta)\dfrac{2\delta + \theta}{\delta} & \text{in } D^-_\delta \end{cases} \end{aligned} \tag{1.34}$$

We impose $f_k(1 - \delta) = 0$ and $f_k(1) = 1$ for $k \in \mathbb{Z}$.

Our aim is to show that for $t > 0$ small and appropriate f_k's, the function ψ_t defined by (1.34) satisfies (1.30) and

$$L_\varepsilon(\psi_t e^{\imath\theta}, D_\delta) < \pi. \tag{1.35}$$

Here, L_ε is the functional defined in Lemma 1.6, so that

$$E_\varepsilon(\rho\psi_t e^{\imath\theta}, D_\delta) = E_\varepsilon(u, D_\delta) + L_\varepsilon(\psi_t e^{\imath\theta}, D_\delta).$$

Then, considering

$$\underline{\psi_t} = \begin{cases} \psi_t & \text{if } |\psi_t| \leq 2 \\ 2\dfrac{\psi_t}{|\psi_t|} & \text{if } |\psi_t| > 2 \end{cases}$$

and setting

$$\tilde{u} = \begin{cases} \rho w_t = \underline{\psi_t} u & \text{in } D_\delta \\ u & \text{in } \mathcal{D} \setminus D_\delta \end{cases},$$

in view of (1.35), it is straightforward that $\tilde{u}$ satisfies the conclusion of Lemma 1.9.

1.6.3 Upper bound for $L_\varepsilon(\cdot, D_\delta)$. An auxiliary problem

If we let $\tilde{w} : [1-\delta, 1] \times [-2\delta, 2\delta]$ be s.t. $\tilde{w}(h(x), \theta(x)) := w(x)$, then we have

$$\begin{aligned} |\nabla w|^2 = \sum_i |\partial_i w|^2 &= \sum_i |\partial_h \tilde{w}(h,\theta)\, \partial_i h + \partial_\theta \tilde{w}(h,\theta)\, \partial_i \theta|^2 \\ &= (\rho^4 |\partial_h \tilde{w}(h,\theta)|^2 + |\partial_\theta \tilde{w}(h,\theta)|^2)|\nabla\theta|^2. \end{aligned}$$

Therefore,

$$\begin{aligned} L_\varepsilon(w, D_\delta) &= \frac{1}{2}\int_{D_\delta} \Big\{ \big(\rho^4 |\partial_h \tilde{w}(h,\theta)|^2 + |\partial_\theta \tilde{w}(h,\theta)|^2 - |\tilde{w}(h,\theta)|^2\big) \rho^2 |\nabla\theta|^2 + \\ &\qquad\qquad + \frac{1}{2\varepsilon^2}\rho^4 (1 - |\tilde{w}(h,\theta)|^2)^2 \Big\}\, \mathrm{d}x \\ &\leq \frac{1}{2}\int_{D_\delta} \Big\{ |\partial_h \tilde{w}(h,\theta)|^2 + |\partial_\theta \tilde{w}(h,\theta)|^2 - |\tilde{w}(h,\theta)|^2 + \\ &\qquad\qquad + \lambda |e^{\imath\theta} - \tilde{w}(h,\theta)|^2 \Big\} \rho^2 |\nabla\theta|^2\, \mathrm{d}x \qquad (1.36) \\ &=: M_\lambda(w, D_\delta), \end{aligned}$$

provided that $|w| \leq 2$ in D_δ and $\lambda \geq \dfrac{9}{2\varepsilon^2 \inf_{D_\delta} |\nabla\theta|^2}$.

In order to simplify formulas, we will write, in what follows, the second integral in (1.36) as

$$\frac{1}{2}\int_{D_\delta} \Big\{ |\partial_h \tilde{w}|^2 + |\partial_\theta \tilde{w}|^2 - |\tilde{w}|^2 + \lambda |e^{\imath\theta} - \tilde{w}|^2 \Big\} \rho^2 |\nabla\theta|^2\, \mathrm{d}x.$$

The same simplified notation will be implicitly used for similar integrals.

Remark 1.16. If we replace w by $\underline{w} := \frac{w}{|w|}\min(|w|, 2)$, then M_λ does not increase. Furthermore replacing w by $\underline{w}$ does not affect the Dirichlet condition of (1.30). Therefore, by replacing w by $\underline{w}$ if necessary, we may assume $|w| \leq 2$.

We next state a lemma which allows us to give a new form of M_λ.

Lemma 1.17. *Let $f \in C^1(\mathbb{R}, \mathbb{R})$. Then, for $k \in \mathbb{Z}$, we have*

$$\int_{D'_\delta} f(h)\cos(k\theta)\rho^2|\nabla\theta|^2\,dx = \begin{cases} 2\delta \displaystyle\int_{1-\delta}^{1} f(s)\,ds & \text{if } k = 0 \\ \dfrac{2\sin(k\delta)}{k} \displaystyle\int_{1-\delta}^{1} f(s)\,ds & \text{if } k \neq 0 \end{cases},$$

$$\int_{D^\pm_\delta} f(h)\,\rho^2|\nabla\theta|^2\,dx = \delta \int_{1-\delta}^{1} f(s)\,ds.$$

Proof. This result is easily obtained by noting that the jacobian of the change of variable $x \mapsto (\theta(x), h(x))$ is exactly $\rho^2|\nabla\theta|^2$. □

For $w = w_t = \psi_t e^{\imath\theta}$ where ψ_t of the form given by (1.34), we have

$$M_\lambda(w, D_\delta) \quad = \quad \frac{1}{2}\int_{D_\delta} \left\{|\partial_h \tilde{w}|^2 + |\partial_\theta \tilde{w}|^2 - |\tilde{w}|^2 + \lambda|e^{\imath\theta} - \tilde{w}|^2\right\} \rho^2|\nabla\theta|^2\,\mathrm{d}x.$$

We next rewrite $M_\lambda(w, D'_\delta)$. Recalling that for a sequence $\{a_k\} \subset \mathbb{R}$, we have

$$\left|\sum_{k\in\mathbb{Z}} a_k e^{\imath k\theta}\right|^2 = \sum_{k\in\mathbb{Z}} a_k^2 + 2\sum_{\substack{k,l\in\mathbb{Z},\\ k>l}} a_k a_l \cos[(k-l)\theta].$$

Then we obtain

$$\begin{aligned} M_\lambda(w, D'_\delta) \quad = \quad & \int_{D'_\delta} \left\{ \frac{t^2}{2}\sum_{k\in\mathbb{Z}} {b_k}^2 \left[{f'_k}^2 + {f_k}^2(k^2 + \lambda - 1)\right] \right. \\ & - t\sum_{k\neq -1} b_k f_k (k+1)\cos[(k+1)\theta] \\ & - t^2 \sum_{k\neq -1} b_{-1} b_k [f'_{-1} f'_k - f_{-1} f_k (k - \lambda + 1)]\cos[(k+1)\theta] \\ & \left. + t^2 \sum_{\substack{k,l\neq -1\\ k-l>0}} b_k b_l [f'_k f'_l + (kl + \lambda - 1) f_k f_l]\cos[(k-l)\theta] \right\}\rho^2|\nabla\theta|^2. \quad (1.37) \end{aligned}$$

Using Lemma 1.17 and (1.37), we have

$$\begin{aligned} M_\lambda(w, D'_\delta) \quad = \quad & \delta t^2 \sum_{k\in\mathbb{Z}} b_k^2 \phi_k(f_k) - 2t\sum_{k\neq -1} b_k \sin[(k+1)\delta] \int_{1-\delta}^{1} f_k \\ & -2t^2 \sum_{k\neq -1} b_{-1} b_k \frac{\sin[(k+1)\delta]}{k+1} \int_{1-\delta}^{1} \{f'_{-1} f'_k - (k - \lambda + 1) f_{-1} f_k\} \\ & +2t^2 \sum_{\substack{k,l\neq -1\\ k-l>0}} b_k b_l \frac{\sin[(k-l)\delta]}{k-l} \int_{1-\delta}^{1} \{f'_k f'_l + (kl + \lambda - 1) f_k f_l\} \qquad (1.38) \\ = \quad & R_\lambda(w) - 2t \sum_{k\neq -1} b_k \sin[(k+1)\delta] \int_{1-\delta}^{1} f_k. \qquad (1.39) \end{aligned}$$

with

$$
\begin{aligned}
R_\lambda(w) &= \delta t^2 \sum_{k\in\mathbb{Z}} b_k^2 \phi_k(f_k) \\
&\quad -2t^2 \sum_{k\neq -1} b_{-1} b_k \frac{\sin[(k+1)\delta]}{k+1} \int_{1-\delta}^1 \left\{ f'_{-1} f'_k - (k-\lambda+1) f_{-1} f_k \right\} \\
&\quad +2t^2 \sum_{\substack{k,l\neq -1 \\ k-l>0}} b_k b_l \frac{\sin[(k-l)\delta]}{k-l} \int_{1-\delta}^1 \left\{ f'_k f'_l + (kl+\lambda-1) f_k f_l \right\},
\end{aligned}
$$

$$\phi_k(f) = \int_{1-\delta}^1 \left\{ f'^2 + \alpha_k^2 f^2 \right\}$$

and

$$\alpha_k = \sqrt{k^2+\lambda-1}.$$

We next establish a similar identity for $M_\lambda(w_t, D_\delta^\pm)$. Using (1.34), we have

$$
\begin{aligned}
M_\lambda(w_t, D_\delta^\pm) &= \frac{1}{2}\int_{D_\delta^\pm} \left\{ |\partial_h \tilde{w}(h,\theta)|^2 + |\partial_\theta \tilde{w}(h,\theta)|^2 - |w|^2 + \lambda |e^{\imath\theta} - w|^2 \right\} \rho^2 |\nabla\theta|^2 \\
&= \frac{1}{2}\int_{D_\delta^\pm} \Bigg\{ |\partial_h \tilde{\psi}_t(h,\pm\delta)|^2 \left(\frac{2\delta \mp \theta}{\delta} \right)^2 \\
&\quad +\delta^{-2}(1+\lambda(2\delta\mp\theta)^2)|\tilde{\psi}_t(h,\pm\delta)-1|^2 \mp 2\delta^{-1} \mathrm{Im}\ \tilde{\psi}_t(h,\pm\delta) \Bigg\} \rho^2|\nabla\theta|^2 \\
&= \frac{1}{2\delta^2}\int_{D_\delta^\pm} \Big\{ |\partial_h \tilde{\psi}_t(h,\pm\delta)|^2 (2\delta\mp\theta)^2 \\
&\qquad + (1+\lambda(2\delta\mp\theta)^2)|\tilde{\psi}_t(h,\pm\delta)-1|^2 \Big\} \rho^2|\nabla\theta|^2 \\
&\qquad\qquad + t \sum_{k\neq -1} b_k(t)\sin[(k+1)\delta] \int_{1-\delta}^1 f_k. \qquad (1.40)
\end{aligned}
$$

Here, Im ψ denotes the imaginary part of ψ. To obtain (1.40), we used the identity

$$|\partial_\theta(\psi e^{\imath\theta})|^2 = |\partial_\theta\psi|^2 + |\psi|^2 + 2\psi\times\partial_\theta\psi.$$

1.6.4 Choice of $w = \psi_t e^{\imath\theta}$

We take

$$f_k(h) = \frac{e^{\alpha_k(h-1)}}{1-e^{-2\alpha_k\delta}} + \frac{e^{-\alpha_k(h-1)}}{1-e^{2\alpha_k\delta}}. \qquad (1.41)$$

With this choice, by direct computations we have

$$\phi_k(f_k) = \alpha_k\left(1+\frac{2}{e^{2\alpha_k\delta}-1}\right), \qquad (1.42)$$

$$\int_{1-\delta}^1 f_k = \frac{1}{\alpha_k}\left(1-\frac{2}{e^{\alpha_k\delta}+1}\right) \qquad (1.43)$$

and for $k,l\in\mathbb{Z}$ s.t. $k\neq\pm l$,

$$\int_{1-\delta}^1 f_k f_l = \frac{1-e^{-2(\alpha_k+\alpha_l)\delta}}{(\alpha_k+\alpha_l)(1-e^{-2\alpha_k\delta})(1-e^{-2\alpha_l\delta})} - \frac{1-e^{-2(\alpha_k-\alpha_l)\delta}}{(\alpha_k-\alpha_l)(1-e^{-2\alpha_k\delta})(e^{2\alpha_l\delta}-1)}, \qquad (1.44)$$

$$\frac{1}{\alpha_k\alpha_l}\int_{1-\delta}^{1} f_k' f_l' = \frac{1-\mathrm{e}^{-2(\alpha_k+\alpha_l)\delta}}{(\alpha_k+\alpha_l)(1-\mathrm{e}^{-2\alpha_k\delta})(1-\mathrm{e}^{-2\alpha_l\delta})} + \frac{1-\mathrm{e}^{-2(\alpha_k-\alpha_l)\delta}}{(\alpha_k-\alpha_l)(1-\mathrm{e}^{-2\alpha_k\delta})(\mathrm{e}^{2\alpha_l\delta}-1)}. \tag{1.45}$$

Using (1.39)—(1.45), we may obtain the following estimate, whose proof is postponed to Appendix 1.B.

Lemma 1.18. *We have*

$$M_\lambda(w_t, D_\delta) \le \delta - 2\delta t + 4t^2 \sum_{k>l>0} c_k c_l \frac{\sin[(k-l)\delta]}{k-l}\frac{kl}{k+l} + o(t). \tag{1.46}$$

1.6.5 End of the proof of Lemma 1.9

We denote

$$S(\delta,t) := \sum_{k>l>0} c_k c_l \frac{\sin[(k-l)\delta]}{k-l}\frac{kl}{k+l}. \tag{1.47}$$

Setting $n = k-l$ and noting that $\frac{(n+l)l}{n+2l} = \frac{l}{2} + \frac{ln}{2(n+2l)}$, we have

$$\frac{2}{(t-2)^2}S(\delta,t) = \sum_{n>0}(1-t)^n\frac{\sin(n\delta)}{n}\sum_{l>0} l(1-t)^{2l} + \sum_{n,l>0}(1-t)^{n+2l}\sin(n\delta)\frac{l}{n+2l}.$$

Here, we have used the explicit formulae for the c_k's, given by Lemma 1.15.

Using Appendix 1.A (see Appendix 1.A.1) we find that for $0<t<\delta$, we have

$$S(\delta,t) = \frac{(1-t)^2}{2t^2}\left[\arctan\left(\frac{1-t-\cos\delta}{\sin\delta}\right) + \arctan\left(\frac{\cos\delta}{\sin\delta}\right)\right] + \frac{(1-t+\cos\delta)(2-t)}{8t\sin\delta} + \mathcal{O}(1). \tag{1.48}$$

We note that

$$\arctan\left(\frac{1-t-\cos\delta}{\sin\delta}\right) = \arctan\left(\frac{1-\cos\delta}{\sin\delta}\right) - \frac{t\sin\delta}{2(1-\cos\delta)} + \mathcal{O}(t^2) = \frac{\delta}{2} - \frac{t\sin\delta}{2(1-\cos\delta)} + \mathcal{O}(t^2) \tag{1.49}$$

and

$$\arctan\left(\frac{\cos\delta}{\sin\delta}\right) = \frac{\pi}{2} - \delta. \tag{1.50}$$

From (1.48)—(1.50) we infer

$$S(\delta,t) \le \frac{1}{4t^2}(\pi-\delta) + \frac{1}{t}\left[\frac{(1-t+\cos\delta)(2-t)}{8\sin\delta} - \frac{\sin\delta}{4(1-\cos\delta)}\right] + \mathcal{O}(1) \tag{1.51}$$

with

$$\frac{(1-t+\cos\delta)(2-t)}{8\sin\delta} - \frac{\sin\delta}{4(1-\cos\delta)} < \frac{(1+\cos\delta)}{4\sin\delta} - \frac{\sin\delta}{4(1-\cos\delta)} = 0. \tag{1.52}$$

From (1.46),

$$M_\lambda(w_t, D_\delta) \leq \delta - 2\delta t + 4t^2 S(\delta, t) + o(t). \tag{1.53}$$

Using (1.51) and (1.52),

$$4t^2 S(\delta, t) \leq \pi - \delta + o(t). \tag{1.54}$$

Finally, we have by combining (1.53) with (1.54),

$$M_\lambda(w_t, D_\delta) \leq \pi - 2\delta t + o(t) < \pi \text{ for } t \text{ small}. \tag{1.55}$$

We conclude that for t sufficiently small, $L^d_\varepsilon(\underline{w}_t, D_\delta) < \pi$.

1.6.6 Conclusion

$\tilde{u} := \underline{\psi} u$, with $\underline{\psi} = \psi_t \dfrac{\min(|\psi_t|, 2)}{|\psi_t|}$, satisfies the desired properties *i.e.*:

- $E_\varepsilon(\tilde{u}) < E_\varepsilon(u) + \pi$ (by (1.36) and (1.55)) ;
- $\tilde{u} \in \mathcal{J}^{\mathbf{d}}_{\mathbf{p},q-1}$ (by (1.29), (1.31) and (1.33)).

1.6.7 A direct consequence of Lemma 1.9

By applying Lemma 1.9 and next Lemma 1.11, one may easily obtain the following

Corollary 1.19. *Let $u \in \mathcal{J}_{\mathbf{p},q}$ be a solution of (1.8), (1.9).*
Assume that

$$\text{abdeg}_j(u) \in (d_j - \frac{1}{3}, d_j + \frac{1}{3}),\ \forall\, j \in \mathbb{N}_N.$$

Assume that there are $i_0 \in \{0, ..., N\}$ and $x_0 \in \partial\omega_{i_0}$ s.t. $u \times \partial_\tau u(x_0) > 0$.
Then for all $\delta = (\delta_1, ..., \delta_N, \delta_0) \in \mathbb{Z}^{N+1}$ s.t. $\delta_{i_0} > 0$, there is $\tilde{u}_\delta \in \mathcal{J}_{(\mathbf{p},q)-\delta}$ s.t.

$$E_\varepsilon(\tilde{u}_\delta) < E_\varepsilon(u) + \pi \sum_i |\delta_i|$$

and

$$\text{abdeg}_j(\tilde{u}_\delta) \in (d_j - \frac{1}{3}, d_j + \frac{1}{3}),\ \forall j \in \mathbb{N}_N.$$

1.7 Proof of Theorem 1.4

The energy estimate is obtained from Lemmas 1.8 and 1.10.
We call $(\mathbf{p}, q, \mathbf{d})$ a good configuration of degrees if

$$(\mathbf{p}, q, \mathbf{d}) \in \mathbb{Z}^N \times \mathbb{Z} \times (\mathbb{N}^*)^N,\ p_i \leq d_i \text{ and } q \leq \sum_i d_i =: d.$$

We first prove Theorem 1.4 when

$$\text{æ}((\mathbf{d}, d), (\mathbf{p}, q)) = |d_1 - p_1| + ... + |d_N - p_N| + |d - q| = 0 \Leftrightarrow \mathbf{p} = \mathbf{d} \text{ and } q = d.$$

For $\varepsilon > 0$, let $(u_n^\varepsilon)_n$ be a minimizing sequence of E_ε in $\mathcal{J}^{\mathbf{d}}_{\mathbf{d},d}$. For $\varepsilon < \varepsilon_4(\mathbf{d}, d, \mathbf{d})$, up to subsequence, using Proposition 1.13, $u_n^\varepsilon \to u_\varepsilon$ weakly in H^1 and strongly in L^4 and u_ε is a (global) minimizer of E_ε in $\mathcal{J}^{\mathbf{d}}_{\deg(u_\varepsilon, \mathcal{D})}$.

Applying Lemmas 1.7 and 1.8, for $\varepsilon<\varepsilon_2(\mathbf{d})\leq\varepsilon_4$ (here, ε_2 is s.t. the $o_\varepsilon(1)$ of Lemma 1.8 is lower than $\dfrac{\pi}{2}$),

$$\begin{aligned}I_0(\mathbf{d},\mathcal{D}) &\geq E_\varepsilon(u_\varepsilon)+\pi æ(\deg(u_\varepsilon,\mathcal{D}),(\mathbf{d},d))\\ &\geq I_0(\mathbf{d},\mathcal{D})-\frac{\pi}{2}+2\pi æ(\deg(u_\varepsilon,\mathcal{D}),(\mathbf{d},d)).\end{aligned}$$

It follows, $æ(\deg(u_\varepsilon,\mathcal{D}),(\mathbf{d},d))\leq\dfrac{1}{4}$ which implies $u_\varepsilon\in\mathcal{J}^{\mathbf{d}}_{\mathbf{d},d}$.

We now prove (following the same strategy) Theorem 1.4 for a good configuration $(\mathbf{p},q,\mathbf{d})$ s.t.

$$æ((\mathbf{p},q),(\mathbf{d},d))>0.$$

For $\varepsilon>0$ consider $(u_n^\varepsilon)_n$ a minimizing sequence of E_ε in $\mathcal{J}^{\mathbf{d}}_{\mathbf{p},q}$.

For $\varepsilon<\varepsilon_4(\mathbf{p},q,\mathbf{d})$, up to subsequence, using Proposition 1.13, $u_n^\varepsilon\to u_\varepsilon$ weakly in H^1 and strongly in L^4 and u_ε is a (global) minimizer of E_ε in $\mathcal{J}^{\mathbf{d}}_{\deg(u_\varepsilon,\mathcal{D})}$.

Let $\Lambda:=I_0(\mathbf{d},\mathcal{D})+æ((\mathbf{p},q),(\mathbf{d},d))\pi+1$, by Lemma 1.14, for $\varepsilon<\varepsilon_5(\mathbf{p},q,\mathbf{d},\Lambda)$, there is $x^0_\varepsilon\in\partial\Omega$ s.t. $(u_\varepsilon\times\partial_\tau u_\varepsilon)(x^0_\varepsilon)>0$.

The third assertion in Proposition 1.2 and the energy bound give the existence of $0<\varepsilon_2'(\mathbf{p},q,\mathbf{d},\Lambda)<\varepsilon_5(\mathbf{p},q,\mathbf{d},\Lambda)$ s.t. for $0<\varepsilon<\varepsilon_2'$,

$$\text{abdeg}_i(u_\varepsilon)\in(d_i-\frac{1}{3},d_i+\frac{1}{3}).$$

Fix $\varepsilon_2'(\mathbf{p},q,\mathbf{d})>\varepsilon_2(\mathbf{p},q,\mathbf{d})>0$ s.t. the $o_\varepsilon(1)$ in Lemma 1.8 is lower than $\dfrac{\pi}{2}$ (here ε_5 is defined in Lemma 1.14).

Using Lemmas 1.7, 1.8 and 1.10, we have for $\varepsilon<\varepsilon_2$

$$\begin{aligned}I_0(\mathbf{d},\mathcal{D})+\pi æ((\mathbf{p},q),(\mathbf{d},d)) &\geq \liminf E_\varepsilon(u_n^\varepsilon)\text{ (by Lemma 1.10 and the definition of }(u_n^\varepsilon)_n)\\ &\geq E_\varepsilon(u_\varepsilon)+\pi æ\left((\mathbf{p},q),\deg(u_\varepsilon,\mathcal{D})\right)\quad\text{(Lemma 1.7)}\\ &\geq I_0(\mathbf{d},\mathcal{D})+\pi\left[æ\left((\mathbf{p},q),\deg(u^\varepsilon,\mathcal{D})\right)\right.\\ &\qquad\left.+æ\left((\mathbf{d},d),\deg(u^\varepsilon,\mathcal{D})\right)\right]-\frac{\pi}{2}\quad\text{(Lemma 1.8)}\end{aligned}$$

It follows that

$$æ\left((\mathbf{p},q),\deg(u^\varepsilon,\mathcal{D})\right)+æ\left((\mathbf{d},d),\deg(u^\varepsilon,\mathcal{D})\right)=æ((\mathbf{p},q),(\mathbf{d},d)).\tag{1.56}$$

Thus

$$p_i\leq\deg_{\partial\omega_i}(u^\varepsilon)\leq d_i\text{ and }q\leq\deg_{\partial\Omega}(u^\varepsilon)\leq d.$$

Assume that there is $\varepsilon<\varepsilon_2$ s.t. $u_\varepsilon\notin\mathcal{J}^{\mathbf{d}}_{\mathbf{p},q}$. Then from Lemma 1.14 and (1.56), one may apply Corollary 1.19 to obtain the existence of $\tilde{u}_\varepsilon\in\mathcal{J}^{\mathbf{d}}_{\mathbf{p},q}$ s.t.

$$m_\varepsilon(\mathbf{p},q,\mathbf{d})\leq E_\varepsilon(\tilde{u}_\varepsilon)<E_\varepsilon(u_\varepsilon)+\pi æ\left((\mathbf{p},q),\deg(u_\varepsilon,\mathcal{D})\right)\leq\liminf E_\varepsilon(u_n^\varepsilon)=m_\varepsilon(\mathbf{p},q,\mathbf{d})$$

which is a contradiction.

Thus for $\varepsilon<\varepsilon_2$, $u_\varepsilon\in\mathcal{J}^{\mathbf{d}}_{\mathbf{p},q}$ and consequently u_ε is a minimizer of E_ε in $\mathcal{J}^{\mathbf{d}}_{\mathbf{p},q}$.

Appendix 1.A Results used in the proof of Lemma 1.9

1.A.1 Power series expansions

For $X\in\mathbb{C}$, $|X|<1$, we have

$$\sum_{k\geq1}\frac{|X|^k}{k}=-\ln(1-|X|),\tag{1.57}$$

$$\sum_{k\geq 0} X^k = \frac{1}{1-X}, \tag{1.58}$$

$$\sum_{k\geq 1} kX^k = \frac{X}{(1-X)^2}, \tag{1.59}$$

$$\sum_{k>0} \sin(k\delta)X^k = \frac{X\sin\delta}{1-2X\cos\delta+X^2}, \tag{1.60}$$

$$\sum_{k>0} \frac{\sin(k\delta)}{k}X^k = \arctan\left(\frac{X-\cos\delta}{\sin\delta}\right)+\arctan\left(\frac{\cos\delta}{\sin\delta}\right), \tag{1.61}$$

$$\sum_{n,l>0} \sin(n\delta)\frac{l}{n+2l}X^{n+2l} = \frac{X+\cos\delta}{4(1-X^2)\sin\delta}-\frac{1}{4\sin^2\delta}\arctan\left(\frac{X-\cos\delta}{\sin\delta}\right)+\mathrm{Cst}(\delta). \tag{1.62}$$

Proof: The first four identities are classical. We sketch the argument that leads to (1.61) and (1.62). The identity (1.61) follows from (1.60) by integration.

We next prove (1.62). Let

$$f(X) = \sum_{n,l>0} \sin(n\delta)\frac{l}{n+2l}X^{n+2l}.$$

On the one hand, by (1.59), (1.60),

$$f'(X) = \frac{1}{X}\sum_{n>0}\sin(n\delta)X^n\sum_{l>0} lX^{2l} = \frac{X^2\sin\delta}{(1-X^2)^2(1-2X\cos\delta+X^2)}.$$

On the other hand

$$\frac{\mathrm{d}}{\mathrm{d}X}\left(\frac{X+\cos\delta}{4\sin\delta(1-X^2)}-\frac{1}{4\sin^2\delta}\arctan\frac{X-\cos\delta}{\sin\delta}\right) = \frac{X^2\sin\delta}{(1-X^2)^2(1-2X\cos\delta+X^2)}.$$

1.A.2 Estimates for f_k and α_k

Recall that we defined, in section 1.6, f_k and α_k by

$$f_k(h) = \frac{\mathrm{e}^{\alpha_k(h-1)}}{1-\mathrm{e}^{-2\alpha_k\delta}}+\frac{\mathrm{e}^{-\alpha_k(h-1)}}{1-\mathrm{e}^{2\alpha_k\delta}},$$
$$\alpha_k = \sqrt{k^2+\lambda-1}.$$

In this part, we prove the following inequalities:

$$\alpha_k = |k| + \mathcal{O}\left(\frac{1}{|k|+1}\right), \tag{1.63}$$

$$|f_k(h)-\mathrm{e}^{-|k|(1-h)}| \leq \frac{C}{k^2}, \text{ with } C \text{ independent of } k\in\mathbb{Z}^*,\ h\in(1-\delta,1), \tag{1.64}$$

$$|f_k'(h)-|k|\mathrm{e}^{-|k|(1-h)}| \leq \frac{C}{|k|}, \text{ with } C \text{ independent of } k\in\mathbb{Z}^*,\ h\in(1-\delta,1). \tag{1.65}$$

Proof: The first assertion is obtained using a Taylor expansion.

Let $g_h(u) = \mathrm{e}^{u(h-1)}$, we have

$$\begin{aligned}|f_k(h)-\mathrm{e}^{-|k|(1-h)}| &\leq |g_h(\alpha_k)-g_h(|k|)|+\frac{C}{k^2}\\ &\leq \sup_{(|k|,\alpha_k)}|g_h'(u)||\alpha_k-|k||+\frac{C}{k^2}\leq\frac{1}{\mathrm{e}k}\frac{1}{2k}+\frac{C}{k^2}\leq\frac{C}{k^2}.\end{aligned}$$

The proof of (1.65) is similar, one uses $\tilde{g}_h(u) = u\mathrm{e}^{u(h-1)}$ instead of g_h

1.A.3 Further estimates on f_k and α_k

We have

$$0 \leq \int_{1-\delta}^{1} \left\{ {f_k'}^2 - \alpha_k^2 {f_k}^2 \right\} \leq \int_{1-\delta}^{1} \left\{ {f_k'}^2 - k^2 {f_k}^2 \right\} \leq \frac{C}{|k|+1}, \text{ with } C \text{ independent of } k \in \mathbb{Z}, \tag{1.66}$$

$$\left| \int_{1-\delta}^{1} f_k f_l \right| \leq \frac{C}{\max(|k|,|l|)}, \text{ with } C \text{ independent of } k,l \in \mathbb{Z}, \text{ s.t. } |k| \neq |l|, \tag{1.67}$$

$$\left| \int_{1-\delta}^{1} f_k' f_l' \right| \leq C\left(\min(|k|,|l|)+1\right), \text{ with } C \text{ independent of } k,l \in \mathbb{Z}, \text{ s.t. } |k| \neq |l|. \tag{1.68}$$

Proof: Actually (1.67), (1.68) still hold when $|k|=|l|$, but this will not used in the proof of Lemma 1.9 and requires a separate argument.

Since $\alpha_k \geq |k|$,

$$\int_{1-\delta}^{1} \left\{ {f_k'}^2 - \alpha_k^2 {f_k}^2 \right\} \leq \int_{1-\delta}^{1} \left\{ {f_k'}^2 - k^2 {f_k}^2 \right\}.$$

By direct computations,

$$0 \leq \int_{1-\delta}^{1} \left\{ {f_k'}^2 - \alpha_k^2 {f_k}^2 \right\} = \frac{4\delta\alpha_k^2}{(1-\mathrm{e}^{-2\alpha_k\delta})(\mathrm{e}^{2\alpha_k\delta}-1)} \leq \frac{C(\delta,n)}{k^n}, \qquad \forall n \in \mathbb{N}^*,$$

$$\int_{1-\delta}^{1} \left\{ {f_k'}^2 - k^2 {f_k}^2 \right\} = \int_{1-\delta}^{1} \left\{ {f_k'}^2 - \alpha_k^2 {f_k}^2 \right\} + (\lambda - 1)\int_{1-\delta}^{1} {f_k}^2,$$

$$\int_{1-\delta}^{1} {f_k}^2 = \frac{1}{2\alpha_k}\left(\frac{1}{1-\mathrm{e}^{-2\alpha_k\delta}} - \frac{1}{1-\mathrm{e}^{2\alpha_k\delta}} \right) + \mathcal{O}\left(\frac{1}{|k|+1}\right) = \mathcal{O}\left(\frac{1}{|k|+1}\right).$$

Which proves (1.66).
For $|k| \neq |l|$, we have

$$\begin{aligned} \left| \int_{1-\delta}^{1} f_k f_l \right| &= \left| \frac{1-\mathrm{e}^{-2(\alpha_k+\alpha_l)\delta}}{(\alpha_k+\alpha_l)(1-\mathrm{e}^{-2\alpha_k\delta})(1-\mathrm{e}^{-2\alpha_l\delta})} - \frac{1-\mathrm{e}^{-2(\alpha_k-\alpha_l)\delta}}{(\alpha_k-\alpha_l)(1-\mathrm{e}^{-2\alpha_k\delta})(\mathrm{e}^{2\alpha_l\delta}-1)} \right| \\ &\leq \frac{C}{\max(|k|,|l|)} + \left| \frac{1-\mathrm{e}^{-2(\alpha_k-\alpha_l)\delta}}{(\alpha_k-\alpha_l)(1-\mathrm{e}^{-2\alpha_k\delta})(\mathrm{e}^{2\alpha_l\delta}-1)} \right|. \end{aligned}$$

We assume that $|k| > |l|$ and we consider the two following cases: $\alpha_l < \alpha_k \leq 2\alpha_l$ and $\alpha_k > 2\alpha_l$. Noting that $\frac{1-\mathrm{e}^{-2x\delta}}{x}$ is bounded for $x \in \mathbb{R}_+^*$, we have

$$\left| \frac{1-\mathrm{e}^{-2(\alpha_k-\alpha_l)\delta}}{(\alpha_k-\alpha_l)(1-\mathrm{e}^{-2\alpha_k\delta})(\mathrm{e}^{2\alpha_l\delta}-1)} \right| \leq \frac{C}{\mathrm{e}^{2\alpha_l\delta}} \leq \frac{C}{\max(|k|,|l|)} \text{ if } \alpha_l < \alpha_k \leq 2\alpha_l,$$

$$\left| \frac{1-\mathrm{e}^{-2(\alpha_k-\alpha_l)\delta}}{(\alpha_k-\alpha_l)(1-\mathrm{e}^{-2\alpha_k\delta})(\mathrm{e}^{2\alpha_l\delta}-1)} \right| \leq \frac{C}{\alpha_k-\alpha_l} \leq \frac{C}{\max(|k|,|l|)} \text{ if } \alpha_k > 2\alpha_l.$$

This proves (1.67).
For $|k| \neq |l|$,

$$\int_{1-\delta}^{1} f_k' f_l' = \frac{\alpha_k\alpha_l\left(1-\mathrm{e}^{-2(\alpha_k+\alpha_l)\delta}\right)}{(\alpha_k+\alpha_l)(1-\mathrm{e}^{-2\alpha_k\delta})(1-\mathrm{e}^{-2\alpha_l\delta})} + \frac{\alpha_k\alpha_l\left(1-\mathrm{e}^{-2(\alpha_k-\alpha_l)\delta}\right)}{(\alpha_k-\alpha_l)(1-\mathrm{e}^{-2\alpha_k\delta})(\mathrm{e}^{2\alpha_l\delta}-1)}.$$

It is clear that,

$$\frac{\alpha_k\alpha_l(1-\mathrm{e}^{-2(\alpha_k+\alpha_l)\delta})}{(\alpha_k+\alpha_l)(1-\mathrm{e}^{-2\alpha_k\delta})(1-\mathrm{e}^{-2\alpha_l\delta})} \leq C\frac{\alpha_k\alpha_l}{\alpha_k+\alpha_l} \leq C\left[\min(|k|,|l|)+1\right]. \tag{1.69}$$

As in the proof of (1.67), we have

$$\left|\frac{\alpha_k\alpha_l(1-\mathrm{e}^{-2(\alpha_k-\alpha_l)\delta})}{(\alpha_k-\alpha_l)(1-\mathrm{e}^{-2\alpha_k\delta})(\mathrm{e}^{2\alpha_l\delta}-1)}\right| \leq \frac{C\alpha_k\alpha_l}{\max(|k|,|l|)} \leq C\left[\min(|k|,|l|)+1\right]. \tag{1.70}$$

Inequalities (1.68) follows from (1.69) and (1.70).

1.A.4 Two fundamental estimates

In this part, we let $k>l\geq 0$ and prove the following:

$$X_{k,l} := \frac{(\alpha_k\alpha_l+kl+\lambda-1)(1-\mathrm{e}^{-2(\alpha_k+\alpha_l)\delta})}{(\alpha_k+\alpha_l)(1-\mathrm{e}^{-2\alpha_k\delta})(1-\mathrm{e}^{-2\alpha_l\delta})} = \frac{2kl}{k+l}+\mathcal{O}\left(\frac{1}{l+1}\right), \tag{1.71}$$

$$Y_{k,l} := \frac{(\alpha_k\alpha_l+kl+\lambda-1)(1-\mathrm{e}^{-2(\alpha_k-\alpha_l)\delta})}{(\alpha_k-\alpha_l)(1-\mathrm{e}^{-2\alpha_k\delta})(\mathrm{e}^{2\alpha_l\delta}-1)} \leq C\mathrm{e}^{-\delta l}. \tag{1.72}$$

The computations are direct:

$$\begin{aligned}
X_{k,l}-\frac{2kl}{k+l} &= \frac{2kl}{(\alpha_k+\alpha_l)(1-\mathrm{e}^{-2\alpha_k\delta})(1-\mathrm{e}^{-2\alpha_l\delta})}-\frac{2kl}{k+l}+\mathcal{O}\left(\frac{1}{l+1}\right)\\
&= 2kl\frac{k+l-(\alpha_k+\alpha_l)(1-\mathrm{e}^{-2\alpha_k\delta})(1-\mathrm{e}^{-2\alpha_l\delta})}{(k+l)(\alpha_k+\alpha_l)(1-\mathrm{e}^{-2\alpha_k\delta})(1-\mathrm{e}^{-2\alpha_l\delta})}+\mathcal{O}\left(\frac{1}{l+1}\right)\\
&= \frac{\mathcal{O}\left(k+k^2l\mathrm{e}^{-l\delta/2}\right)}{(k+l)(\alpha_k+\alpha_l)(1-\mathrm{e}^{-2\alpha_k\delta})(1-\mathrm{e}^{-2\alpha_l\delta})}+\mathcal{O}\left(\frac{1}{l+1}\right)\\
&= \mathcal{O}\left(\frac{1}{l+1}\right).
\end{aligned}$$

We now turn to (1.72).

If $\alpha_k\geq 2\alpha_l$ (or equivalently, if $\alpha_k-\alpha_l\geq\frac{\alpha_k}{2}$), then

$$\frac{(\alpha_k\alpha_l+kl+\lambda-1)(1-\mathrm{e}^{-2(\alpha_k-\alpha_l)\delta})}{(\alpha_k-\alpha_l)(1-\mathrm{e}^{-2\alpha_k\delta})(\mathrm{e}^{2\alpha_l\delta}-1)} \leq C\frac{kl}{\alpha_k}\mathrm{e}^{-2\alpha_l\delta}\leq C\mathrm{e}^{-\delta l}.$$

If $\alpha_k<2\alpha_l$, then

$$\frac{(\alpha_k\alpha_l+kl+\lambda-1)(1-\mathrm{e}^{-2(\alpha_k-\alpha_l)\delta})}{(\alpha_k-\alpha_l)(1-\mathrm{e}^{-2\alpha_k\delta})(\mathrm{e}^{2\alpha_l\delta}-1)} \leq Cl^2\mathrm{e}^{-2\alpha_l\delta}\leq C\mathrm{e}^{-\delta l}.$$

Appendix 1.B Proof of Proposition 1.2 and of Lemma 1.15

1.B.1 Proof of Proposition 1.2

The proof of 1) is direct by noting that if $u\in H^1(\mathcal{D},\mathbb{S}^1)$, then $\partial_1 u$ and $\partial_2 u$ are pointwise proportional and $\deg_{\partial\Omega}(u)=\sum_i\deg_{\partial\omega_i}(u)$,

$$\begin{aligned}
\mathrm{abdeg}_i(u,\mathcal{D}) &= \frac{1}{2\pi}\sum_{k=1,2}(-1)^k\int_{\mathcal{D}}(u\times\partial_k u)\partial_{3-k}V_i\\
&= \frac{1}{2\pi}\int_{\partial\mathcal{D}}V_i\,u\times\partial_\tau u\,\mathrm{d}\tau = \deg_{\partial\Omega}(u)-\sum_{j\neq i}\deg_{\partial\omega_j}(u)=\deg_{\partial\omega_i}(u).
\end{aligned}$$

Proof of 2). Since V_i is locally constant on $\partial\mathcal{D}$, integrating by parts,

$$\int_{\mathcal{D}} v \times (\partial_1 u\, \partial_2 V_i - \partial_2 u\, \partial_1 V_i)\, \mathrm{d}x = \int_{\mathcal{D}} u \times (\partial_1 v\, \partial_2 V_i - \partial_2 v\, \partial_1 V_i)\, \mathrm{d}x.$$

Then

$$\begin{aligned}
2\pi|\mathrm{abdeg}_i(u) - \mathrm{abdeg}_i(v)| &= \Big|\int_{\mathcal{D}} (u - v) \times \Big[(\partial_1 V_i\, \partial_2 u - \partial_2 V_i\, \partial_1 u) \\
&\qquad + (\partial_1 V_i\, \partial_2 v - \partial_2 V_i\, \partial_1 v)\Big]\, \mathrm{d}x\Big| \\
&\le \sqrt{2}\|u - v\|_{L^2(\mathcal{D})}\|V_i\|_{C^1(\mathcal{D})}(\|\nabla u\|_{L^2(\mathcal{D})} + \|\nabla v\|_{L^2(\mathcal{D})}) \\
&\le 2\|u - v\|_{L^2(\mathcal{D})}\|V_i\|_{C^1(\mathcal{D})}[E_\varepsilon(u)^{1/2} + E_\varepsilon(v)^{1/2}] \\
&\le 4\|u - v\|_{L^2(\mathcal{D})}\|V_i\|_{C^1(\mathcal{D})}\Lambda^{1/2}.
\end{aligned}$$

We prove assertion 3) by showing that $\mathrm{dist}(\mathrm{abdeg}_i(u_\varepsilon), \mathbb{Z}) = o(1)$. Using the first and the second assertion, we have

$$\begin{aligned}
\mathrm{dist}(\mathrm{abdeg}_i(u_\varepsilon), \mathbb{Z}) &\le \inf_{v\in E_0^\Lambda} |\mathrm{abdeg}_i(u_\varepsilon) - \mathrm{abdeg}_i(v)| \\
&\le \frac{2}{\pi}\|V_i\|_{C^1(\mathcal{D})}\Lambda^{1/2} \inf_{v\in E_0^\Lambda} \|u_\varepsilon - v\|_{L^2(\mathcal{D})} \qquad (1.73)
\end{aligned}$$

where $E_0^\Lambda := \left\{u \in H^1(\mathcal{D}, \mathbb{S}^1) \text{ s.t. } \frac{1}{2}\int_{\mathcal{D}} |\nabla u|^2\, \mathrm{d}x \le \Lambda\right\} \ne \emptyset$.

Now, it suffices to show that $\inf_{v\in E_0^\Lambda} \|u_\varepsilon - v\|_{L^2(\mathcal{D})} \to 0$. We argue by contradiction and we assume that there is an extraction $(\varepsilon_n)_n \downarrow 0$ and $\delta > 0$ s.t. for all n, $\inf_{v\in E_0^\Lambda} \|u_{\varepsilon_n} - v\|_{L^2(\mathcal{D})} > \delta$.

We see that $(u_{\varepsilon_n})_n$ is bounded in H^1. Then, up to subsequence, u_n converges to $u \in H^1(\mathcal{D}, \mathbb{R}^2)$ weakly in H^1 and strongly in L^4.

Since $\||u_{\varepsilon_n}|^2 - 1\|_{L^2(\mathcal{D})} \to 0$, we have $u \in H^1(\mathcal{D}, \mathbb{S}^1)$ and by weakly convergence, $\|\nabla u\|^2_{L^2(\mathcal{D})} \le 2\Lambda$.

To conclude, we have $u \in E_0^\Lambda$ et $\|u_{\varepsilon_n} - u\|_{L^2} \to 0$, which is a contradiction.

1.B.2 Proof of Lemma 1.15

1) We see easily that, with $z = \mathrm{e}^{\imath\theta}$, we have

$$\frac{\Psi_t(\overline{z}) - \mathcal{F}_t(\overline{z})}{t} = \frac{(1 - \varphi(\theta))(1 - z^2)}{[z(1 - t) - 1]\,[z(1 - t\varphi(\theta)) - 1]} \equiv \frac{A(\theta, t)}{B(\theta, t)}. \qquad (1.74)$$

The modulus of the RHS of (1.74) can be bounded by noting that

- there is some $m > 0$ s.t. $|B(\theta, t)| \ge m$ for each t and each θ s.t. $|\theta| > \delta/2 \bmod 2\pi$;
- there is some $M > 0$ s.t. $|A(\theta, t)| \le M$ for each t and each θ s.t. $|\theta| > \delta/2 \bmod 2\pi$;
- if $|\theta| \le \delta/2$ (modulo 2π), then $(\Psi_t - \mathcal{F}_t)\, t^{-1} \equiv 0$.

2) This assertion is a standard expansion.

3) With a classical result relating regularity of $\Psi_t - \mathcal{F}_t$ to the asymptotic behaviour of its Fourier coefficients, we have

$$|b_k(t) - c_k(t)| \le \frac{2^{n+1}\pi\|\partial_\theta^n\, (\Psi_t - \mathcal{F}_t)\,\|_{L^\infty(\mathbb{S}^1)}}{t\,(1 + |k|)^n}.$$

Noting that, for $\partial_\theta^n\, (\Psi_t - \mathcal{F}_t)\, t^{-1} \equiv \dfrac{A_n(\theta, t)}{B_n(\theta, t)}$

- there is some $m_n > 0$ s.t. $|B_n(\theta,t)| \geq m_n$ for each t and each θ s.t. $|\theta| > \delta/2 \bmod 2\pi$;
- there is some $M_n > 0$ s.t. $|A_n(\theta,t)| \leq M_n$ for each t and each θ s.t. $|\theta| > \delta/2 \bmod 2\pi$;
- if $|\theta| \leq \delta/2$ (modulo 2π), then $(\Psi_t - \mathcal{F}_t)\, t^{-1} \equiv 0$.

Thus the result follows.

1.B.3 Proof of Lemma 1.18

The key argument to treat the energetic contribution of $D_\delta^\pm$ is the following lemma.

Lemma 1.20. *1.* $|\tilde{\psi}_t(h,\pm\delta) - 1| = \mathcal{O}(t)$*;*

2. $|\partial_h \tilde{\psi}_t(h,\pm\delta)| = \mathcal{O}(t|\ln t|)$.

Proof. (of Lemma 1.20)
Using Lemma 1.15, (1.58) and (1.64), we have

$$\begin{aligned} t^{-1}|\tilde{\psi}_t(h,\delta) - 1| &\leq \left| -c_{-1}f_{-1}(h) + \sum_{k\neq -1} c_k f_k(h) \mathrm{e}^{-\imath[(k+1)\delta]} \right| \\ &\quad + \left| -(b_{-1} - c_{-1}) f_{-1}(h) + \sum_{k\neq -1} (b_k - c_k) f_k(h) \mathrm{e}^{-\imath(k+1)\delta} \right| \\ &\leq C(\delta) \left\{ \left| \sum_{k\geq 0} \left((1-t)\mathrm{e}^{-(1-h)-\imath\delta} \right)^k \right| + 1 \right\} = \mathcal{O}(1). \end{aligned}$$

We prove that $|\partial_h \tilde{\psi}_t(h,\delta)| = \mathcal{O}(t|\ln t|)$. Using Lemma 1.15, (1.59) and (1.65),

$$\begin{aligned} t^{-1}|\partial_h \tilde{\psi}_t(h,\delta)| &\leq \left| -c_{-1}f'_{-1} + \sum_{k\neq -1} c_k f'_k \mathrm{e}^{-\imath(k+1)\delta} \right| \\ &\quad + \left| -(b_{-1} - c_{-1}) f'_{-1} + \sum_{k\neq -1} (b_k - c_k) f'_k \mathrm{e}^{-\imath(k+1)\delta} \right| \\ &\leq 2 \left| \sum_{k\geq 0} k \left[(1-t)\mathrm{e}^{-\imath\delta-(1-h)} \right]^k \right| + \mathcal{O}(|\ln t|) = \mathcal{O}(|\ln t|). \end{aligned}$$

□

Using (1.39), (1.40) and Lemma 1.20, we have (with the notation of section 1.6) that

$$M_\lambda(w_t, D_\delta) = R_\lambda(w_t) + o(t),$$

where

$$\begin{aligned} R_\lambda(w_t) &= \delta t^2 \sum_{k\in\mathbb{Z}} b_k^2 \phi_k(f_k) - 2t^2 \sum_{k\neq -1} b_{-1} b_k \frac{\sin[(k+1)\delta]}{k+1} \int_{1-\delta}^1 [f'_{-1} f'_k - (k - \lambda + 1) f_{-1} f_k] \\ &\quad + 2t^2 \sum_{\substack{k,l\neq -1 \\ k-l>0}} b_k b_l \frac{\sin[(k-l)\delta]}{k-l} \int_{1-\delta}^1 [f'_k f'_l + (kl + \lambda - 1) f_k f_l]. \end{aligned}$$

The proof of Lemma 1.20 is completed provided we establish the following estimate:

$$R_\lambda(w_t) \le \delta - 2\delta t + 4t^2 \sum_{\substack{k,l\ge 0\\ k-l>0}} c_k c_l \frac{\sin[(k-l)\delta]}{k-l}\frac{kl}{k+l} + o(t). \tag{1.75}$$

The remaining part of this appendix is devoted to the proof of (1.75).

We estimate the first term of R_λ:

Using (1.42) and Lemma 1.15, we have (with C independent of t)

$$\left|\sum_{k\in\mathbb{Z}} b_k^2\phi_k(f_k) - \sum_{k\in\mathbb{Z}} c_k^2\phi_k(f_k)\right| \le C. \tag{1.76}$$

With (1.42) and (1.63), we obtain

$$\phi_k(f_k) \quad = \quad \alpha(1+\frac{2}{\mathrm{e}^{2\alpha\delta}-1}) = |k| + \mathcal{O}\left(\frac{1}{|k|+1}\right) \text{ when } |k|\to\infty. \tag{1.77}$$

From (1.57), (1.59) and (1.77),

$$\begin{aligned} t^2\sum_{k\in\mathbb{Z}} c_k^2\phi_k(f_k) &= t^2\phi_{-1}(f_{-1}) + t^2(t-2)^2\sum_{k\ge 0}(1-t)^{2k}\phi_k(f_k) \\ &= t^2(t-2)^2\sum_{k>0} k(1-t)^{2k} + o(t) = 1-2t+o(t). \end{aligned} \tag{1.78}$$

We estimate the second term of R_λ:

Using Lemma 1.15, (1.67) and (1.68), we have (with C independent of t)

$$\left|\sum_{k\ne -1}(b_k - c_k)\frac{\sin[(k+1)\delta]}{k+1}\int_{1-\delta}^1 [f'_{-1}f'_k - (k-\lambda+1)f_{-1}f_k]\right| \le C.$$

Since $b_{-1}(t)$ is bounded by a quantity independent of t, in the order to estimate the third term of the RHS of (1.38), we observe that there is C independent of t s.t.

$$\begin{aligned} \left|\sum_{k\ge 0}(1-t)^k\frac{\sin[(k+1)\delta]}{k+1}\int_{1-\delta}^1 [f'_{-1}f'_k - (k-\lambda+1)f_{-1}f_k]\right| &\le C\left(\sum_{k\ge 1}\frac{(1-t)^k}{k}+1\right) \\ &= C(|\ln t|+1). \end{aligned}$$

Finally, using Lemma 1.15, (1.44) and (1.45), we have

$$\left|\sum_{k\ne -1} b_k\frac{\sin[(k+1)\delta]}{k+1}\int_{1-\delta}^1 [f'_{-1}f'_k - (k-\lambda+1)f_{-1}f_k]\right| \le C(|\ln t|+1). \tag{1.79}$$

We estimate the last term of R_λ:

First, we consider the case $k=-l>0$ (*i.e.*, $f_k = f_l$). Using (1.43), $0\le f_k\le 1$ and (1.66), we have (with C independent of t)

$$\left|\sum_{k>0} b_k b_{-k}\frac{\sin 2k\delta}{2k}\int_{1-\delta}^1 [f_k'^2 + (-k^2+\lambda-1)f_k^2]\right| \le C.$$

It remains to estimate the last sum in R_λ, considered only over the indices k and l s.t. $|k| \neq |l|$. We start with

$$\sum_{\substack{k,l\neq -1\\ k-l>0,k\neq -l}} (b_k b_l - c_k c_l)\frac{\sin[(k-l)\delta]}{k-l}\int_{1-\delta}^{1}[f_k' f_l' + (kl+\lambda-1)f_k f_l] \tag{1.80}$$
$$= \sum_{\substack{k,l\neq -1\\ k-l>0,k\neq -l}} [(b_k-c_k)(b_l-c_l)+c_k(b_l-c_l)+c_l(b_k-c_k)] *$$
$$* \frac{\sin[(k-l)\delta]}{k-l}\int_{1-\delta}^{1}[f_k' f_l' + (kl+\lambda-1)f_k f_l].$$

By Assertion 3) of Lemma 1.15, the first sum of the RHS of (1.80) is easily bounded by a quantity independent of t. By (1.67), (1.68) and Lemma 1.15,

$$\left|\sum_{\substack{k,l\neq -1\\ k-l>0,k\neq -l}} c_k(b_l-c_l)\frac{\sin[(k-l)\delta]}{k-l}\int_{1-\delta}^{1}[f_k' f_l' + (kl+\lambda-1)f_k f_l]\right|$$
$$\leq C \sum_{\substack{k\geq 0,l\neq -1\\ k-l>0,k\neq -l}} \frac{(1-t)^k|b_l-c_l||l|}{k-l} + C.$$

On the other hand (putting $n = k-l$),

$$\begin{aligned}
\sum_{\substack{k\geq 0,l\neq -1\\ k-l>0,k\neq -l}} \frac{(1-t)^k|b_l-c_l||l|}{k-l} &\leq \sum_{k>l\geq 0}\frac{(1-t)^k|b_l-c_l|l}{k-l} + \sum_{k\geq 0,l\leq -1}\frac{(1-t)^k|b_l-c_l||l|}{k+|l|}\\
&\leq \sum_{l\geq 0,n>0}\frac{(1-t)^n}{n}|b_l-c_l|l + \sum_{k>0,l\leq -1}\frac{(1-t)^k}{k}|b_l-c_l||l|\\
&= \mathcal{O}(|\ln t|).
\end{aligned}$$

Similarly, we may prove that

$$\left|\sum_{\substack{k,l\neq -1\\ k-l>0,k\neq -l}} c_l(b_k-c_k)\frac{\sin[(k-l)\delta]}{k-l}\int_{1-\delta}^{1}[f_k' f_l' + (kl+\lambda-1)f_k f_l]\right| = \mathcal{O}(|\ln t|).$$

We have thus proved that

$$\left|\sum_{\substack{k,l\neq -1\\ k-l>0,k\neq -l}} (b_k b_l - c_k c_l)\frac{\sin[(k-l)\delta]}{k-l}\int_{1-\delta}^{1}[f_k' f_l' + (kl+\lambda-1)f_k f_l]\right| = o(t^{-1}).$$

To finish the proof, it suffices to obtain

$$\sum_{\substack{k,l\neq -1\\ k-l>0,k\neq -l}} c_k c_l\frac{\sin[(k-l)\delta]}{k-l}\int_{1-\delta}^{1}[f_k' f_l' + (kl+\lambda-1)f_k f_l]$$
$$= 2\sum_{\substack{k,l\geq 0\\ k-l>0}} c_k c_l \frac{\sin[(k-l)\delta]}{k-l}\frac{kl}{k+l} + o(t^{-1}).$$

Since $c_m = 0$ for $m < -1$, it suffices to consider the case $k > l \geq 0$. Under these hypotheses, we have by (1.44), (1.45), (1.71) and (1.72),

$$\begin{aligned}\sum_{k>l\geq 0} c_k c_l \frac{\sin[(k-l)\delta]}{k-l}\int_{1-\delta}^{1}[f_k' f_l' + (kl+\lambda-1)f_k f_l] &= 2\sum_{k>l\geq 0} c_k c_l \frac{\sin[(k-l)\delta]}{k-l}\frac{kl}{k+l} \\ &\quad + \mathcal{O}\left(\sum_{k>l\geq 0}\frac{c_k c_l |\sin[(k-l)\delta]|}{k-l}\frac{1}{l+1}\right).\end{aligned}$$

We conclude by noting that

$$\left|\sum_{k>l\geq 0} c_k c_l \left|\frac{\sin[(k-l)\delta]}{(k-l)(l+1)}\right|\right| \leq C\left(1+\sum_{n>0}\frac{(1-t)^n}{n}\sum_{l>0}\frac{(1-t)^{2l}}{l}\right) \leq C(1+\ln^2 t).$$

Appendix 1.C Proof of Lemma 1.11

Lemma 1.21. *Let $0 < \eta, \delta < 1$, there is*

$$\begin{array}{rccc} M_{\eta,\delta}: & D(0,1) & \to & \mathbb{C} \\ & x & \mapsto & M_{\eta,\delta}(x) \end{array} \quad s.t.: \tag{1.81}$$

i) $|M_{\eta,\delta}| = 1$ *on* $\mathbb{S}^1$, $\deg_{\mathbb{S}^1}(M_{\eta,\delta}) = 1$,

ii) $\displaystyle\frac{1}{2}\int_{D(0,1)} |\nabla M_{\eta,\delta}|^2 \leq \pi + \eta$,

iii) $|M_{\eta,\delta}| \leq 2$

iv) *if* $|\theta| > \delta$ *mod* 2π, *then* $M_{\eta,\delta}(e^{\imath\theta}) = 1$.

Claim: Taking $\overline{M_{\eta,\delta}}$ instead of $M_{\eta,\delta}$, we obtain the same conclusions replacing the assertion *i)* by $\deg_{\mathbb{S}^1}(\overline{M_{\eta,\delta}}) = -1$.

Proof. As in section 1.6, let $\varphi \in C^\infty(\mathbb{R},\mathbb{R})$ be s.t.

- $0 \leq \varphi \leq 1$,
- φ is even and 2π-periodic,
- $\varphi_{|(-\delta/2,\delta/2)} \equiv 1$ and $\varphi_{|[-\pi,\pi[\setminus(-\delta,\delta)} \equiv 0$.

For $0 < t < \delta$, let $M_t = M$ be the unique solution of

$$\left\{\begin{array}{rcll} M(e^{\imath\theta}) &=& \dfrac{e^{\imath\theta}-(1-t\varphi(\theta))}{e^{\imath\theta}(1-t\varphi(\theta))-1} & \text{on } \partial D(0,1) \\ \Delta M &=& 0 & \text{in } D(0,1)\end{array}\right. .$$

It follows easily that M satisfies *i)*, *iii)* and *iv)*. We will prove that for t small *ii)* holds.

Using (1.32), we have

$$\frac{e^{\imath\theta}-(1-t\varphi(\theta))}{e^{\imath\theta}(1-t\varphi(\theta))-1} = (1-tb_{-1}(t)) + t\sum_{k\neq -1} b_k(t)e^{(k+1)\imath\theta}. \tag{1.82}$$

It is not difficult to see that

$$M(re^{\imath\theta}) = (1 - tb_{-1}(t)) + t\sum_{k\neq -1} b_k(t)r^{|k+1|}e^{(k+1)\imath\theta}. \tag{1.83}$$

From (1.83),

$$\begin{aligned}
\frac{1}{2}\int_{D(0,1)} |\nabla M|^2 &= t^2\int_0^{2\pi} d\theta\int_0^1 dr\sum_{k\neq -1} b_k^2(k+1)r^{2|k+1|-2}\\
&= \pi t^2\sum_{k\geq 0} b_k^2(k+1) + \pi t^2\sum_{k\leq -2}|k+1|b_k^2\\
&= \pi t^2\sum_{k\geq 0} c_k^2(k+1) + \mathcal{O}(t^2) \text{ (using Lemma 1.15)}\\
&= \pi(2-t)^2t^2\sum_{k\geq 0}(1-t)^{2k}(k+1) + \mathcal{O}(t^2) \text{ (using Lemma 1.15)}\\
&= \pi + \mathcal{O}(t^2) \text{ (using (1.58) and (1.59))}\\
&\leq \pi + \eta \text{ for } t \text{ small.}
\end{aligned}$$

We finish the proof taking, for t small, $M_{\eta,\delta} = M_t$. □

Lemma 1.22. *Let $u \in \mathcal{J}$, $i \in \{0, ..., N\}$ and $\varepsilon > 0$. For all $\eta > 0$, there is*

$$u_\eta^\pm \in \mathcal{J}_{\deg(u,\mathcal{D})\pm\mathbf{e}_i}$$

s.t.

$$E_\varepsilon(u_\eta^\pm) \leq E_\varepsilon(u) + \pi + \eta \tag{1.84}$$

and

$$\|u - u_\eta^\pm\|_{L^2(\mathcal{D})} = o_\eta(1),\ o_\eta(1) \underset{\eta\to 0}{\to} 0. \tag{1.85}$$

Proof. We prove that for $i = 0$, there is $u_\eta^+ \in \mathcal{J}_{\deg(u,\mathcal{D})+\mathbf{e}_i}$ satisfying (1.84) and (1.85). In the other cases the proof is similar.

Using the density of $C^0(\overline{\mathcal{D}}, \mathbb{C})\cap\mathcal{J}$ in $\mathcal{J}$ for the H^1-norm, we may assume $u \in C^0(\overline{\mathcal{D}}, \mathbb{C})\cap\mathcal{J}$.

It suffices to prove the result for $0 < \eta < \min\{10^{-3}, \varepsilon^2\}$.

Let $x^0 \in \partial\Omega$ and V_η be an open regular set of $\mathcal{D}$ s.t. :

- $\partial V_\eta \cap \partial\mathcal{D} \neq \emptyset$, $|V_\eta| \leq \eta^2$,
- x^0 is an interior point of $\partial\Omega \cap \partial V_\eta$,
- V_η is simply connected,
- $|u|^2 \leq 1 + \eta^2$ in V_η,
- $\|\nabla u\|_{L^2(V_\eta)} \leq \eta^2$.

Using the Carathéodory's theorem, there is

$$\Phi : \overline{V_\eta} \to \overline{D(0,1)},$$

a homeomorphism s.t. $\Phi_{|V_\eta} : V_\eta \to D(0,1)$ is a conformal mapping.

Without loss of generality, we may assume that $\Phi(x^0) = 1$. Let $\delta > 0$ be s.t. for $|\theta| \leq \delta$ we have $\Phi^{-1}(e^{\imath\theta}) \in \partial V_\eta \cap \partial\Omega$.

Let $N_\eta \in \mathcal{J}$ be defined by

$$N_\eta(x) = \begin{cases} 1 & \text{if } x \in \mathcal{D} \setminus V_\eta \\ M_{\eta^2,\delta}(\Phi(x)) & \text{otherwise} \end{cases}.$$

Here, $M_{\eta^2,\delta}$ is defined by Lemma 1.21. Using the conformal invariance of the Dirichlet functional, we have

$$\frac{1}{2}\int_{V_\eta} |\nabla N_\eta|^2 = \frac{1}{2}\int_{D(0,1)} |\nabla M_{\eta^2,\delta}|^2 \leq \pi + \eta^2. \tag{1.86}$$

It is not difficult to see that $u_\eta^+ := uN_\eta \in \mathcal{J}_{\deg(u,\mathcal{D})+\mathbf{e}_0}$. Since $|N_\eta| \leq 2$ and $\|N_\eta - 1\|_{L^2(\mathcal{D})} = o_\eta(1)$, using the Dominated convergence theorem, we may prove that $uN_\eta \to u$ in $L^2(\mathcal{D})$ when $\eta \to 0$. It follows that (1.85) holds.

From (1.86) and using the following formula,

$$|\nabla(uv)|^2 = |v|^2|\nabla u|^2 + |u|^2|\nabla v|^2 + 2\sum_{j=1,2}(v\partial_j u)\cdot(u\partial_j v)$$

we obtain

$$\begin{aligned} \frac{1}{2}\int_{V_\eta}|\nabla u_\eta^+|^2 &= \frac{1}{2}\int_{V_\eta}\left\{|N_\eta|^2|\nabla u|^2 + |u|^2|\nabla N_\eta|^2 + 2\sum_{j=1,2}(N_\eta\partial_j u)\cdot(u\partial_j N_\eta)\right\} \\ &\leq (1+\eta^2)(\pi+\eta^2) + 2\|\nabla u\|^2_{L^2(V_\eta)} + 4\sqrt{1+\eta^2}\|\nabla u\|_{L^2(V_\eta)}\|\nabla N_\eta\|_{L^2(V_\eta)} \\ &\leq \pi + \frac{\eta}{2}. \end{aligned} \tag{1.87}$$

Furthermore, we have

$$\frac{1}{4\varepsilon^2}\int_{V_\eta}(1-|u_\eta^+|^2)^2 \leq \frac{\eta^2}{4\varepsilon^2} \leq \frac{\eta}{2}. \tag{1.88}$$

From (1.87) and (1.88), it follows

$$E_\varepsilon(u_\eta^+, \mathcal{D}) = E_\varepsilon(u, \mathcal{D}\setminus V_\eta) + E_\varepsilon(u_\eta^+, V_\eta) \leq E_\varepsilon(u,\mathcal{D}) + \pi + \eta.$$

The previous inequality completes the proof. □

We may now prove Lemma 1.11. For the convenience of the reader, we recall the statement of the lemma.

Lemma . *Let $u \in \mathcal{J}$, $\varepsilon > 0$ and $\delta = (\delta_1, ..., \delta_N, \delta_0) \in \mathbb{Z}^{N+1}$. For all $\eta > 0$, there is $u_\eta^\delta \in \mathcal{J}_{\deg(u,\mathcal{D})+\delta}$ s.t.*

$$E_\varepsilon(u_\eta^\delta) \leq E_\varepsilon(u) + \pi\sum_{i\in\{0,\dots,N\}}|\delta_i| + \eta \tag{1.25}$$

and

$$\|u - u_\eta^\delta\|_{L^2(\mathcal{D})} = o_\eta(1),\ o_\eta(1) \underset{\eta\to 0}{\to} 0. \tag{1.26}$$

Proof. As in the previous lemma, it suffices to prove the proposition for $0 < \eta < \min\{10^{-3}, \varepsilon^2\}$ and $u \in C^0(\overline{\mathcal{D}}, \mathbb{C}) \cap \mathcal{J}$.

We construct u_η^δ in $\ell_1 = \sum_{i\in\{0,\dots,N\}}|\delta_i|$ steps. If $\ell_1 = 0$ (which is equivalent at $\delta = \mathbf{0}_{\mathbb{Z}^{N+1}}$) then, taking $u_\eta^\delta = u$, (1.25) and (1.26) hold.

Assume $\ell_1 \neq 0$. Let $\Gamma = \{i \in \{0, ..., N\} \,|\, \delta_i \neq 0\} \neq \emptyset$, $L = \operatorname{Card}\Gamma$ and $\mu = \dfrac{\eta}{\ell_1}$. We enumerate the elements of Γ in $(i_n)_{n\in\mathbb{N}_L}$ s.t. for $n \in \mathbb{N}_{L-1}$ we have $i_n < i_{n+1}$.

Let σ be the sign function *i.e.* for $x \in \mathbb{R}^*$, $\sigma(x) = \dfrac{x}{|x|}$.

For $n \in \mathbb{N}_L$ and $l \in \mathbb{N}_{|\delta_{i_n}|}$, we construct

$$v_n^l \in \mathcal{J}_{\deg(v_n^{l-1},\mathcal{D})+\sigma(\delta_i)\mathbf{e}_{i_n}}$$

s.t

$$\begin{array}{c} v_0^0 = u,\ v_n^0 = v_{n-1}^{|\delta_{i_{n-1}}|} \text{ with for } n=1,\ \delta_{i_0}=0, \\ v_n^{l+1} = \begin{cases} (v_n^l)_\mu^+ & \text{if } \delta_{i_n} > 0 \\ (v_n^l)_\mu^- & \text{if } \delta_{i_n} < 0 \end{cases},\ 0 \le l < |\delta_{i_n}| \end{array}.$$

Here, $(v_n^l)_\mu^\pm$ stands for $u_\mu^\pm$ defined by Lemma 1.22 taking $u = v_n^l$ and $\eta = \mu$.

It is clear that v_n^l is well defined and that for $n \in \mathbb{N}_L$, $v_n := v_n^{|\delta_{i_n}|} \in \mathcal{J}_{\deg(v_{n-1},\mathcal{D})+\delta_{i_n}\mathbf{e}_{i_n}}$ with $v_0 = u$.

Therefore, using (1.84), we have for $n \in \mathbb{N}_L$,

$$v_n \in \mathcal{J}_{\deg(u,\mathcal{D})+\sum_{k\in\mathbb{N}_n}\delta_{i_k}\mathbf{e}_{i_k}},\ E_\varepsilon(v_n) \le E_\varepsilon(u) + (\pi+\mu)\sum_{k\in\mathbb{N}_n}|\delta_{i_k}|.$$

Taking $n = L$, we obtain that

$$u_\eta^\delta = v_L \in \mathcal{J}_{\deg(u,\mathcal{D})+\delta},\ E_\varepsilon(u_\eta^\delta) \le E_\varepsilon(u) + \pi\sum_{i\in\{0,\dots,N\}}|\delta_i| + \eta.$$

Furthermore, u_η^δ is obtained from u multiplying by ℓ_1 factors N_l, $l \in \mathbb{N}_{\ell_1}$. Each N_l is bounded by 2 and converges to 1 in L^2-norm (when $\eta \to 0$). Using the Dominated convergence theorem, we may prove that u_η^δ satisfies (1.26).

□

Deuxième partie

Étude de la fonctionnelle de Ginzburg-Landau avec un terme de chevillage : le cas bi-dimensionnel

Chapter 2

The Ginzburg-Landau functional with a discontinuous pinning term. Finitely many dilute inclusions case

In Collaboration with Oleksandr Misiats

We consider a Ginzburg-Landau type energy with a piecewise constant pinning term a in the potential $(a^2 - |u|^2)^2$. The function a is different from 1 only on finitely many disjoint domains, called the *pinning domains*. These pinning domains model small impurities in a homogeneous superconductor and shrink to single points in the limit $\varepsilon \to 0$; here, ε is the inverse of the Ginzburg-Landau parameter. We study the energy minimization in a smooth simply connected domain $\Omega \subset \mathbb{C}$ with Dirichlet boundary condition g on $\partial\Omega$, with topological degree $\deg_{\partial\Omega}(g) = d > 0$. Our main result is that, for small ε, minimizers have d distinct zeros (vortices) which are inside the pinning domains and they have a degree equal to 1. The question of finding the locations of the pinning domains with vortices is reduced to a discrete minimization problem for a finite-dimensional functional. We also find the precise position of the vortices inside the pinning domains and we prove that, asymptotically, this position does not depend on the external boundary conditions.

Contents

2.1 Introduction and main results

In this work we study the minimizers of the Ginzburg-Landau type functional

$$E_{\varepsilon,\delta}(u) = \frac{1}{2}\int_\Omega \left\{ |\nabla u|^2 + \frac{1}{2\varepsilon^2}(a_\delta^2 - |u|^2)^2 \right\}, \tag{2.1}$$

where $\Omega \subset \mathbb{C}$ is a bounded, smooth, simply connected domain, ε is a positive parameter (the inverse of the Ginzburg-Landau parameter $\kappa = 1/\varepsilon$), $\delta = \delta(\varepsilon) > 0$ is a geometric parameter and u is a complex-valued map. In order to define the function a_δ, we need to introduce the notion of a *pinning domain.*

Fix $M > 0$ points $a_1, ..., a_M \in \Omega$. Let ω be an open subset such that $\overline{\omega} \subset B(0,1)$ and $0 \in \omega$. For $1 \le i \le M$ and for all $\delta > 0$ denote $\omega_\delta^i := a_i + \delta w$, *i.e.* the set ω scaled by δ and centered at a_i.
Definition. The set $\omega_\delta := \cup_{i=1}^M \omega_\delta^i$ is called a *pinning domain.*
For example, if $\omega = B(0, \frac{1}{2})$, then the pinning domain is $\omega_\delta = \cup_{i=1}^M B(a_i, \frac{\delta}{2})$.

We now define $a_\delta : \Omega \to \{b, 1\}$ as:

$$a_\delta(x) = \begin{cases} b & \text{if } x \in \omega_\delta \\ 1 & \text{if } x \in \Omega \setminus \omega_\delta \end{cases}. \tag{2.2}$$

The functionals of this type arise in models of superconductivity for composite superconductors. The experimental pictures suggest nearly 2D structure of parallel vortex tubes ([56], Fig I.4). Therefore, the domain Ω can be viewed as a cross-section of a multifilamentary wire with a number of thin superconducting filaments. Such multifilamentary wires are widely used in industry, including magnetic energy-storing devices, transformers and power generators [41], [38].

Another important practical issue in modeling superconductivity is to decrease the energy dissipation in superconductors. Here, the dissipation occurs due to currents associated with the motion of vortices ([47], [10]). This dissipation as well the thermomagnetic stability can be improved by *pinning* ("fixing the positions") of vortices. This, in turn, can be done by introducing impurities or inclusions in the superconductor. In the functional (2.1) the set ω_δ models the set of small impurities in a homogeneous superconductor. The size of the impurities in our model is characterized by the geometric parameter δ which goes to zero together with the material parameter ε. We assume henceforth that

$$\frac{|\ln \delta(\varepsilon)|^3}{|\ln \varepsilon|} \to 0. \tag{H}$$

For example, if $\varepsilon = 2^{-j}$ and $\delta(\varepsilon) = 2^{-k(j)}$, then (H) implies that $\dfrac{k(j)^3}{j} \to 0$.
Notation. In what follow:

- We consider a sequence $\varepsilon_n \downarrow 0$ and we write ε instead of ε_n; the dependence of ε on n is implicit.
- We simply write δ (instead of $\delta(\varepsilon)$); the dependence of δ on ε is implicit.

We study the minimization problem for the functional (2.1) in the class

$$H_g^1 := \{u \in H^1(\Omega, \mathbb{C}) \,|\, \mathrm{tr}_{\partial\Omega} u = g\}, \tag{2.3}$$

where $g \in C^\infty(\partial\Omega, \mathbb{S}^1)$ is such that $\deg_{\partial\Omega}(g) = d > 0$. Recall that the degree (winding number) of g is defined as

$$\deg_{\partial\Omega}(g) := \frac{1}{2\pi}\int_{\partial\Omega} g \times \partial_\tau g \, \mathrm{d}\tau.$$

Here "$\times$" stands for the vectorial product in $\mathbb{C}$, *i.e.* $z_1 \times z_2 = \mathrm{Im}(\overline{z_1} z_2)$, $z_1, z_2 \in \mathbb{C}$, and ∂_τ is the tangential derivative. The degree is an integer, and the condition $\deg_{\partial\Omega}(u) = d > 0$, $u \in H^1(\Omega, \mathbb{C})$ implies that u must have at least d zeros (counting multiplicity) inside Ω. The properties of the topological degree can be found, *e.g.*, in [26] or [13].

Minimization problems for Ginzburg-Landau type functionals have been extensively studied by a variety of authors. The pioneering work on modeling Ginzburg-Landau vortices is the work of Bethuel, Brezis and Hélein [18]. In this work the authors suggested to consider a simplified Ginzburg-Landau model (2.1) with $a \equiv 1$ in Ω (*i.e.* without pinning term), in which the physical source of vortices, the external magnetic field, is modeled via a Dirichlet boundary condition with a positive degree on the boundary (2.3). The analysis of full Ginzburg-Landau functional, with induced and applied magnetic fields, was later performed by Sandier and Serfaty in [65].

The functional (2.1) with non-constant $a(x)$ was proposed by Rubinstein in [58] as a model of pinning vortices for Ginzburg-Landau minimizers. Shortly after, André and Shafrir [6] studied the asymptotics of minimizers for a smooth (say C^1) a. One of the first works to consider a discontinuous pinning term, which models a composite two-phase superconductor, was [43]. In this work, a single inclusion described by a pinning term independent of the parameter ε was considered for a simplified Ginzburg-Landau functional with Dirichlet boundary condition g on $\partial\Omega$. Namely the pinning term is

$$a(x) = \begin{cases} 1 & \text{if } x \in \Omega \setminus \omega \\ b & \text{if } x \in \omega \end{cases},$$

here ω is a simply connected open set s.t. $\overline{\omega} \subset \Omega$.

The main objective of [43] was to establish that the vortices are attracted (pinned) by the inclusion ω, and their location inside ω can be obtained via minimization of certain finite-dimensional functional of renormalized energy. Full Ginzburg-Landau model with discontinuous pinning term was later considered by Aydi and Kachmar [9]. An ε-dependent but continuous pinning term $a_\varepsilon(x)$ was studied by Aftalion, Sandier and Serfaty in [1]. The work [5] studies the case of the smooth a with finite number of isolated zeros, and in [3] the pinning term a takes negative values in some regions of the domain Ω. The other works related to Ginzburg-Landau functional with pinning term include, *e.g.*, [47], [68].
In this work, we consider the minimization problem (2.1)-(2.3) with a discontinuous pinning term given by (2.2). We prove that despite the fact that $a_\varepsilon \to 1$ a.e. as $\varepsilon \to 0$, *i.e.* the pinning term disappears in the limit, the pinning domains ω_δ capture the vortices of Ginzburg-Landau minimizers of (2.1) for small ε.

The main difficulty in the analysis of this problem stems in the fact that the *a priori* Pohozhaev type estimate $\|1 - |v|^2\|^2_{L^2(\Omega)} \le C\varepsilon^2$ (on which the analysis in [18] and [43] is based) does not hold.

The situation we consider requires a different strategy reducing the study of the minimizers of (2.1) to the analysis of $\mathbb{S}^1$-valued maps via the uniform estimates on the modulus of minimizers away from the pinning domains (see Proposition 2.9 below).

Following [43], let U_ε be **the** unique global minimizer of E_ε in H^1 with $U_\varepsilon \equiv 1$ on $\partial\Omega$. This U_ε satisfies $b \le U_\varepsilon \le 1$. For $v \in H^1_g$ we define

$$F_\varepsilon(v) = F_\varepsilon(v,\Omega) := \frac{1}{2}\int_\Omega \left\{ U_\varepsilon^2|\nabla v|^2 + \frac{1}{2\varepsilon^2}U_\varepsilon^4(1-|v|^2)^2 \right\} \mathrm{d}x.$$

Using the Substitution Lemma of [43], we have that for $v \in H^1_g$,

$$E_\varepsilon(U_\varepsilon v) = E_\varepsilon(U_\varepsilon) + F_\varepsilon(v). \tag{2.4}$$

From the decomposition (2.4), we can reduce the minimization problem (2.1)-(2.3) to the minimization problem for F_ε in H^1_g.

In this study, two typical behaviors of the minimizers are distinguished.

First, outside a fixed neighborhood of $d' = \min\{d, M\}$ inclusions (centered at $\boldsymbol{a} = (a_{i_1}, ..., a_{i_{d'}})$), a minimizer v_ε is almost an $\mathbb{S}^1$-valued map. By minimality of v_ε, the selection of centers of inclusion containing its zeros and the degrees of v_ε around the a_i's is related to the minimization of the Bethuel-Brezis-Hélein renormalized energy W_g.

On the other hand, we prove that the vorticity is localized inside the inclusions and is quantized: all the zeros are isolated, lie in ω_δ and have a degree equal to 1. Moreover, the location of the zeros (vortices) inside each inclusion, depends only on d and M (see Theorem 2.23).

In particular:

- if $M = 3$ and $d = 2$, then the vorticity is contained in two distinct inclusions with exactly one zero inside each of both inclusions,

- if $M = 2$ and $d = 3$, then we have one zero inside an inclusion and two other zeros inside the remaining inclusion.

Depending on the relation between M (number of inclusions), and d (number of vortices), we distinguish two cases:
Case I: $M \ge d$,
Case II: $M < d$.

Our main result in Case I is the following:

Theorem 2.1. *Assume that $M \ge d$. Let v_ε be a minimizer of F_ε in $H^1_g(\Omega)$. For all sequence $\varepsilon_n \downarrow 0$, possibly after passing to a subsequence, there are*

- *d distinct points $\{a_{i_1}, ..., a_{i_d}\} \subset \{a_i, 1 \le i \le M\}$,*
- *$v^* \in H^1_{\mathrm{loc}}(\overline{\Omega} \setminus \{a_{i_1}, ..., a_{i_d}\}, \mathbb{S}^1)$,*

such that:

1. *v^* is a harmonic map,* i.e.

$$\begin{cases} -\Delta v^* = v^*|\nabla v^*|^2 & \text{in } \Omega \setminus \{a_{i_1}, ..., a_{i_d}\} \\ v^* = g & \text{on } \partial\Omega \end{cases}. \tag{2.5}$$

2. *We have $v_{\varepsilon_n} \to v^*$ strongly in $H^1_{\mathrm{loc}}(\overline{\Omega} \setminus \{a_{i_1}, ..., a_{i_d}\})$ and $v_{\varepsilon_n} \to v^*$ in $C^\infty_{\mathrm{loc}}(\Omega \setminus \{a_1, ..., a_M\})$.*

3. *v_{ε_n} has d distinct vortices $x_1^n, ..., x_d^n$ such that x_m^n is inside $\omega_\delta^{i_m}$, $m = 1, ..., d$ and for small fixed ρ, $\deg_{\partial B(x_i^n,\rho)}(v_{\varepsilon_n}) = 1$.*

4. *The following expansion holds*

$$F_\varepsilon(v_\varepsilon) = \pi d b^2|\ln\varepsilon| + \pi(1-b^2)d|\ln\delta| + W_g((a_{i_1},1),...,(a_{i_d},1)) + \tilde{W} + o_\varepsilon(1). \quad (2.6)$$

Here $\tilde{W} > 0$ is a local renormalized energy depending only on d, b and ω. Moreover, the d-subset $\{a_{i_1},...,a_{i_d}\} \subset \{a_1,...,a_M\}$ minimizes the Bethuel-Brezis-Hélein renormalized energy W_g among the d-subsets of $\{a_1,...,a_M\}$.

Remark 2.2. Here, W_g denotes the renormalized energy given by Theorem I.7 in [18] (with the degrees equal to 1 and the boundary data g). Its definition is recalled in Section 2.5.3.

The main result in Case II is

Theorem 2.3. *Assume that $M < d$. Let v_ε be a minimizer of F_ε in $H^1_g(\Omega)$. For all sequence $\varepsilon_n \downarrow 0$, possibly after passing to a subsequence, there are $v^* \in H^1_{\text{loc}}(\overline{\Omega} \setminus \{a_1,...,a_M\}, \mathbb{S}^1)$ which satisfies* (2.5) *in $\Omega \setminus \{a_1,...,a_M\}$, such that:*

1. *$v_{\varepsilon_n} \to v^*$ strongly in $H^1_{\text{loc}}(\overline{\Omega} \setminus \{a_1,...,a_M\})$ and $v_{\varepsilon_n} \to v^*$ in $C^\infty_{\text{loc}}(\Omega \setminus \{a_1,...,a_M\})$.*

2. *For $\rho > 0$ small, v_{ε_n} has exactly $d_i := \deg_{\partial B(a_i,\rho)}(v_{\varepsilon_n})$ zeros in $B(a_i,\rho)$. They are isolated, lie inside ω^i_δ and they have a degree equal to* 1.

3. $$\left[\frac{d}{M}\right] \le d_i \le \left[\frac{d}{M}\right] + 1, \text{ where } \left[\frac{d}{M}\right] \text{ is the integer part of } \frac{d}{M}. \quad (2.7)$$

   Moreover, if $\frac{d}{M} = m_0 \in \mathbb{N}$, then $d_i \equiv m_0$, $1 \le i \le M$. Otherwise, the configuration $\{(a_1,d_1),...,(a_M,d_M)\}$ minimizes the renormalized energy W_g among the configurations $\{(a_1,\tilde{d}_1),...,(a_M,\tilde{d}_M)\}$. Here $\{a_i \,|\, 1 \le i \le M\}$ are fixed and $\tilde{d}_i \in \mathbb{Z}$ are the subjects to the constraints (2.7) and $\sum_{i=1}^M \tilde{d}_i = d$.

4. *The following expansion holds when $\varepsilon \to 0$*

$$\inf_{H^1_g} F_\varepsilon = \pi d b^2|\ln\varepsilon| + \pi(\sum_{i=1}^M d_i^2 - db^2)|\ln\delta| + W_g(\{\boldsymbol{a},\boldsymbol{d}\}) + \tilde{W} + o_\varepsilon(1). \quad (2.8)$$

   Here, $\{\boldsymbol{a},\boldsymbol{d}\} = \{(a_1,d_1),...,(a_M,d_M)\}$ is a configuration given by the previous assertion and $\tilde{W}$ is local renormalized energy which depends only on ω, b, d and M.

In both cases, we prove that the asymptotical location of the vortices inside a pinning domain depends only on b, ω and on the number of zeros inside the inclusion (see Theorem 2.23): this location is independent of the boundary data g on $\partial\Omega$.

2.2 Main tools

In this section we establish:

- Estimates for U_ε,
- Upper bounds for the energy of minimizers in Case I and Case II,
- An η-ellipticity estimate for minimizers.

2.2.1 Properties of U_ε

Proposition 2.4 (Maximum principle for U_ε, [43] Proposition 1)**.** *The special solution U_ε satisfies $b \le U_\varepsilon \le 1$ in Ω.*

Proposition 2.5. *There are $C, c > 0$ (independent of ε) s.t. for any $R > \varepsilon$ we have*

$$|a_\varepsilon - U_\varepsilon| \le C\mathrm{e}^{-\frac{cR}{\varepsilon}} \text{ in } V_R := \{x \in \Omega \,|\, \mathrm{dist}(x, \partial\omega_\delta) \ge R\}, \tag{2.9}$$

$$|\nabla U_\varepsilon| \le \frac{C\mathrm{e}^{-\frac{cR}{\varepsilon}}}{\varepsilon} \text{ in } V_R. \tag{2.10}$$

The proof of the Proposition 2.5 is presented in the Appendix 2.A.

2.2.2 Upper Bounds

Proposition 2.6. *Let $\xi = \dfrac{\varepsilon}{\delta}$.*

1. *Upper bound in Case I: $M \ge d$*

   There is a constant C depending only on g, ω and Ω s.t. we have

$$\inf_{v \in H^1_g(\Omega)} F_\varepsilon(v, \Omega) \le \pi d b^2 |\ln \xi| + \pi d |\ln \delta| + C. \tag{2.11}$$

2. *Upper bound in Case II: $M < d$*

   There is a constant C depending only on g, ω and Ω s.t. for all $d_1, ..., d_M \in \mathbb{N}$ s.t. $\sum d_i = d$ we have

$$\inf_{v \in H^1_g(\Omega)} F_\varepsilon(v, \Omega) \le \pi d b^2 |\ln \xi| + \pi \sum_i d_i^2 |\ln \delta| + C. \tag{2.12}$$

The proof of Proposition 2.6 is given in Appendix 2.B.

2.2.3 Identifying bad discs

Lemma 2.7. *Let $g_\varepsilon, g_0 \in C^\infty(\partial\Omega, \mathbb{C})$ be s.t. $0 \le 1 - |g_\varepsilon| \le \varepsilon$ and $g_\varepsilon \to g_0$ in $C^1(\partial\Omega)$. Let also $\alpha_\varepsilon, \beta_\varepsilon \in L^\infty(\Omega, [b, 1])$.*

Consider the weighted Ginzburg-Landau functional

$$F_\varepsilon(v) = \frac{1}{2}\int_\Omega \left\{ \alpha_\varepsilon |\nabla v|^2 + \frac{\beta_\varepsilon}{\varepsilon^2}(1 - |v|^2)^2 \right\}.$$

Denote v_ε a minimizer of F_ε in $H^1_{g_\varepsilon}$. Then the following results hold:

1. *Let $\chi = \chi_\varepsilon \in (0, 1)$ be s.t. $\chi \to 0$. There are $\varepsilon_0 > 0$, $C > 0$ and $C_1 > 0$ depending only on $b, \chi, \Omega, \|g_0\|_{C^1(\partial\Omega)}$ s.t for $\varepsilon < \varepsilon_0$, if*

$$F_\varepsilon(v_\varepsilon, B(x, \varepsilon^{1/4}) \cap \Omega) \le \chi^2 |\ln \varepsilon| - C_1,$$

   then

$$|v_\varepsilon| \ge 1 - C\chi \text{ in } B(x, \varepsilon^{1/2}) \cap \Omega.$$

2. *Let $\mu \in (0, 1)$. Then there are $\varepsilon_0, C > 0$ depending only on $b, \mu, \Omega, \|g_0\|_{C^1(\partial\Omega)}$ s.t. for $\varepsilon < \varepsilon_0$, if*

$$F_\varepsilon(v_\varepsilon, B(x, \varepsilon^{1/4}) \cap \Omega) \le C |\ln \varepsilon|,$$

   then

$$|v_\varepsilon| \ge \mu \text{ in } B(x, \varepsilon^{1/2}) \cap \Omega.$$

Lemma 2.7 is proved in Appendix 2.C.

2.3 A model problem

By combining the results of Section 2.2, the proofs of both Theorem 2.1 and Theorem 2.3 are based on the analysis of two distinct problems:

1. A minimization problem of the Dirichlet functional among $\mathbb{S}^1$-valued map defined on a perforated domain.
2. The study of the minimizers v_ε around an inclusion.

This section is devoted to the second problem. More precisely, we fix $\rho > 0$ and we investigate the minimization problem of $F_\varepsilon(\cdot, B(a_i, \rho))$ under variable boundary conditions.

Fix $\rho > 0$ and let f_ε, $f_0 \in C^\infty(\partial B(0,\rho))$ be s.t. f_0 is $\mathbb{S}^1$-valued and s.t.

$$\|f_\varepsilon - f_0\|_{C^1(\partial B(0,\rho))} \to 0 \tag{2.13}$$

and

$$\||f_\varepsilon| - 1\|_{L^2(\partial B(0,\rho))} \leq C\varepsilon^2. \tag{2.14}$$

Assume also that $\deg_{\partial B(0,\rho)}(f_\varepsilon) = \deg_{\partial B(0,\rho)}(f) = d_0 > 0$.

For $i \in \{1, ..., M\}$ consider the minimization problem

$$F_\varepsilon(v, B(a_i,\rho)) := \frac{1}{2}\int_{B(a_i,\rho)} \left\{ U_\varepsilon^2 |\nabla v|^2 + \frac{1}{2\varepsilon^2} U_\varepsilon^4 (1-|v|^2)^2 \right\} \mathrm{d}x \tag{2.15}$$

in the class

$$H^1_{f_\varepsilon,i} := \{v \in H^1(B(a_i,\rho), \mathbb{C}) \,|\, \mathrm{tr}_{\partial B(a_i,\rho)} v(x) = f_\varepsilon(x - a_i)\}. \tag{2.16}$$

A suitable way to treat such a problem is to rescale the ball in order to fix the inclusion independently of ε.

Without loss of generality assume $a_i = 0$. Let v_ε be a minimizer of (2.15) in (2.16). Performing the change of variables $\hat{x} = \frac{x}{\delta}$ in (2.15), we have

$$F_\varepsilon(v_\varepsilon, B(0,\rho)) = \hat{F}_\xi(\hat{v}_\varepsilon, B(0, \frac{\rho}{\delta})) := \frac{1}{2}\int_{B(0,\frac{\rho}{\delta})} \left\{ \hat{U}_\varepsilon^2 |\nabla \hat{v}|^2 + \frac{1}{2\xi^2} \hat{U}_\varepsilon^4 (1-|\hat{v}|^2)^2 \right\} \mathrm{d}\hat{x}. \tag{2.17}$$

Here, for a map $w \in H^1(B(0,\rho))$, we denote $\hat{w}(\hat{x}) := w(\delta\hat{x})$ and $\xi = \dfrac{\varepsilon}{\delta}$. The class (2.16) under this change of variables becomes

$$\hat{H}^1_{f_\varepsilon} := \left\{ \hat{v} \in H^1(B(0,\frac{\rho}{\delta}), \mathbb{C}) \,|\, \mathrm{tr}_{\partial B(0,\frac{\rho}{\delta})} \hat{v}(\cdot) = f_\varepsilon(\delta \cdot) \right\}. \tag{2.18}$$

The asymptotic behavior of $\hat{v}_\varepsilon$ will be obtained in several steps:

- We first establish a bound for $|\hat{v}_\varepsilon|$. This bound will allow us to localize (roughly) the vortices of v_ε near the inclusion.
- We next establish sharp energy estimates (see Proposition 2.10) and use them to obtain the uniform convergence of solutions away from the inclusion. We establish the strong H^1 convergence of solutions away from the vortices and derive the equation satisfied by the limiting map.
- The last step is the location and quantization of the vorticity: for small ε, the minimizers admits exactly d_0 zeros, and all the zeros lie in the inclusion and have a degree equal to 1.

Following the same lines as for Proposition 2.6, one may prove

Proposition 2.8. *Let $\hat{v}_\varepsilon$ be a minimizer of $\hat{F}_\xi$ in (2.16). Then there is a constant C independent of ε s.t. we have*

$$F_\varepsilon(v_\varepsilon, B(0,\rho)) = \hat{F}_\xi(\hat{v}_\varepsilon, B(0,\frac{\rho}{\delta})) \leq \pi d_0 b^2 |\ln \xi| + \pi d_0^2 |\ln \delta| + C. \tag{2.19}$$

2.3.1 Uniform convergence of $|\hat{v}_\varepsilon|$ to 1 away from inclusions

Proposition 2.9. *Let $K \subset \mathbb{R}^2$ be a compact set such that $\omega \subset K$ and $\operatorname{dist}(\partial K, \omega) > 0$. Then there is $C > 0$ independent of ε s.t. for sufficiently small ε we have*

$$|\hat{v}_\varepsilon| \geq 1 - C|\ln \varepsilon|^{-1/3} \text{ in } B_{\frac{\rho}{\delta}} \setminus K.$$

Proof. Using Lemma 2.7 with $\chi = |\ln \varepsilon|^{-1/3}$, we find that there exist $C, C_1 > 0$ s.t. for $\varepsilon > 0$ small, if $F_\varepsilon(v_\varepsilon, B(x, \varepsilon^{1/4})) < |\ln \varepsilon|^{\frac{1}{3}} - C_1$ then $|v_\varepsilon| \geq 1 - C\chi$ in $B(x, \varepsilon^{1/2})$.

We argue by contradiction. Assume that there exists a compact K containing ω s.t. $\operatorname{dist}(\partial K, \omega) > 0$ and s.t., up to a subsequence, there is a sequence of points $\hat{x}_\varepsilon \in B(0, \frac{\rho}{\delta}) \setminus K$ s.t. $|\hat{v}_\varepsilon(\hat{x}_\varepsilon)| < 1 - C|\ln \varepsilon|^{-1/3}$ with C given by Lemma 2.7.

Note that $\hat{x}_\varepsilon \in B(0, \frac{\rho}{\delta}) \setminus K$ is the same as $x_\varepsilon \in B(0,\rho) \setminus (\delta \cdot K)$.

From Lemma 2.7 and Proposition 2.5

$$\frac{1}{2}\int_{B(x_\varepsilon, \varepsilon^{1/4})} \left\{ |\nabla v_\varepsilon|^2 + \frac{1}{2\varepsilon^2}(1 - |v_\varepsilon|^2)^2 \right\} \geq |\ln \varepsilon|^{1/3} - \mathcal{O}(1). \tag{2.20}$$

We claim that from the conditions (2.13), (2.14), we may extend v_ε (keeping the same notation for the extension) to a smooth map still denoted v_ε s.t.

$$\begin{cases} v_\varepsilon(x) = x^{d_0}/|x|^{d_0} \text{ in } B(0,3\rho) \setminus \overline{B(0,2\rho)} \\ \displaystyle\int_{B(0,3\rho)\setminus\overline{B(0,\rho)}} (1 - |v_\varepsilon|^2)^2 \leq C\varepsilon^2 \\ |\nabla v_\varepsilon| \leq C \text{ with } C > 0 \text{ is independent of } \varepsilon \end{cases}. \tag{2.21}$$

Indeed, set $\zeta \in C^\infty(\mathbb{R}^+, [0,1])$ s.t. $\zeta \equiv 0$ in $[0,\rho]$ and $\zeta \equiv 1$ in $[2\rho, 3\rho]$ and take

$$v_\varepsilon(s e^{\imath\theta}) = \Big[\zeta(s) + (1 - \zeta(s))|f_\varepsilon(\rho e^{\imath\theta})|\Big] e^{\imath[d_0\theta + (1-\zeta(s))\phi_\varepsilon(\rho e^{\imath\theta})]}.$$

Here $x = s e^{\imath\theta}$, $s > 0$ and $\phi_\varepsilon \in C^\infty(\partial B(0,\rho), \mathbb{R})$ s.t. $f_\varepsilon(\rho e^{\imath\theta}) = |f_\varepsilon| e^{\imath(d_0\theta + \phi_\varepsilon)}$. Consequently, as follows from (2.11) and (2.21), this map satisfies

$$\frac{1}{2}\int_{B(0,3\rho)} \left\{ |\nabla v_\varepsilon|^2 + \frac{1}{2\varepsilon^2}(1 - |v_\varepsilon|^2)^2 \right\} \leq C|\ln \varepsilon|.$$

Therefore, the map v_ε in $B(0,3\rho)$ agrees with the conditions of Theorem 4.1 in [65]. This theorem guarantees that:

- we may cover the set $\{x \in B(0, 3\rho - \varepsilon/b) \mid |v_\varepsilon(x)| < 1 - (\varepsilon/b)^{1/8}\}$ with a finite collection of disjoint balls $\mathcal{B}^\varepsilon := \{B_j^\varepsilon\}$;
- the radius of $\mathcal{B}^\varepsilon$, $\operatorname{rad}(\mathcal{B}^\varepsilon)$, which is defined as the sum of the radii of the balls B_j^ε, $\operatorname{rad}(\mathcal{B}^\varepsilon) := \sum_j \operatorname{rad}(B_j^\varepsilon)$, satisfies $\operatorname{rad}(\mathcal{B}^\varepsilon) \leq 10^{-2}\delta \cdot \operatorname{dist}(\omega, \partial K)$;
- denoting $d_j = \deg_{\partial B_j^\varepsilon}(v_\varepsilon)$ if $B_j^\varepsilon \subset B(0, 3\rho - \varepsilon/b)$ and $d_j = 0$ otherwise;

we have

$$\frac{1}{2}\int_{\mathcal{B}^\varepsilon}\left\{|\nabla v_\varepsilon|^2+\frac{b^2}{2\varepsilon^2}(1-|v_\varepsilon|^2)^2\right\}\geq \pi\sum_j |d_j|\ln\frac{\delta}{\varepsilon}-C. \tag{2.22}$$

Note that, by the construction of v_ε in $B(0,3\rho)\setminus\overline{B(0,\rho)}$, if we have $\deg_{\partial B_j^\varepsilon}(v_\varepsilon)\neq 0$ then $B_j^\varepsilon\subset B(0,5\rho/2)$. Thus $d_j=\deg_{\partial B_j^\varepsilon}(v_\varepsilon)$ for all j.

In order to obtain a lower bound for F_ε we use the identity

$$\begin{aligned} F_\varepsilon(v_\varepsilon,B(x_\varepsilon,\varepsilon^{1/4})\cup\mathcal{B}^\varepsilon) &= \frac{b^2}{2}\int_{B(x_\varepsilon,\varepsilon^{1/4})\cup\mathcal{B}^\varepsilon}\left\{|\nabla v_\varepsilon|^2+\frac{b^2}{2\varepsilon^2}(1-|v_\varepsilon|^2)^2\right\} \\ &+\frac{1}{2}\int_{B(x_\varepsilon,\varepsilon^{1/4})\cup\mathcal{B}^\varepsilon}\Big\{(U_\varepsilon^2-b^2)|\nabla v_\varepsilon|^2 \\ &\qquad+\frac{1}{2\varepsilon^2}(U_\varepsilon^4-b^4)(1-|v_\varepsilon|^2)^2\Big\}. \end{aligned}\tag{2.23}$$

The first integral in (2.23) is estimate via (2.22):

$$\begin{aligned}\frac{b^2}{2}\int_{B(x_\varepsilon,\varepsilon^{1/4})\cup\mathcal{B}^\varepsilon}\left\{|\nabla v_\varepsilon|^2+\frac{b^2}{2\varepsilon^2}(1-|v_\varepsilon|^2)^2\right\} &\geq \pi b^2\sum_j|\deg_{\partial B_j}(v_\varepsilon)|\ln\frac{\delta}{\varepsilon}-C \\ &\geq \pi b^2 d_0\ln\frac{\delta}{\varepsilon}-C_0. \end{aligned}\tag{2.24}$$

By combining (2.20) and Proposition 2.5, we have for small ε

$$\begin{aligned}&\frac{1}{2}\int_{B(x_\varepsilon,\varepsilon^{1/4})\cup\mathcal{B}^\varepsilon}\left\{(U_\varepsilon^2-b^2)|\nabla v_\varepsilon|^2+\frac{1}{2\varepsilon^2}(U_\varepsilon^4-b^4)(1-|v_\varepsilon|^2)^2\right\}\\ &\geq\frac{1-b^2}{2}\int_{B(x_\varepsilon,\varepsilon^{1/4})}\left\{|\nabla v_\varepsilon|^2+\frac{1}{2\varepsilon^2}(1-|v_\varepsilon|^2)^2\right\}-C\geq(1-b^2)|\ln\varepsilon|^{1/3}-C';\end{aligned}\tag{2.25}$$

here we rely on the assumption (H) on the behavior of $\delta(\varepsilon)$ as $\varepsilon\to 0$.

Substituting the bounds (2.24) and (2.25) in (2.23) we obtain a contradiction with (2.11). This completes the proof of Proposition 2.9. □

2.3.2 Distribution of Energy in $B(0,\frac{\rho}{\delta})$

Proposition 2.10. *The following estimates hold:*

$$\frac{1}{2}\int_{B(0,\rho/\delta)\setminus\overline{B(0,1)}}\hat{U}_\varepsilon^2|\nabla\hat{v}_\varepsilon|^2=\pi d_0^2|\ln\delta|+\mathcal{O}(1), \tag{2.26}$$

and (recall that $\xi=\dfrac{\varepsilon}{\delta}$*)*

$$\hat{F}_\xi(\hat{v}_\varepsilon,B(0,1))=\pi d_0 b^2|\ln\xi|+\mathcal{O}(1). \tag{2.27}$$

Proof. We start by proving that

$$\hat{F}_\xi(\hat{v}_\varepsilon,B(0,1))\geq\pi d_0 b^2|\ln\xi|-\mathcal{O}(1). \tag{2.28}$$

As before, we use Theorem 4.1 in [65]: for $0<r<r_0:=10^{-2}\cdot\operatorname{dist}(\omega,\partial B(0,1))$, there are $C>0$ and a finite covering by disjoint balls $B_1^\varepsilon,...,B_N^\varepsilon$ (with the sum of radii at most r) of the set $\{\hat{x}\in B(0,1-\xi/b)\,|\,1-|\hat{v}_\varepsilon(\hat{x})|\geq(\xi/b)^{1/8}\}$ s.t.

$$\frac{1}{2}\int_{\cup_j B_j^\varepsilon}\left\{|\nabla\hat{v}_\varepsilon|^2+\frac{b^2}{2\xi^2}(1-|\hat{v}_\varepsilon|^2)^2\right\}\geq\pi D|\ln\xi|-C, \tag{2.29}$$

with $D = \sum_j |d_j|$ and

$$d_j = \begin{cases} \deg_{\partial B_j^\varepsilon}(\hat{v}_\varepsilon) & \text{if } B_j^\varepsilon \subset B(0, 1-\xi/b) \\ 0 & \text{otherwise} \end{cases}.$$

From Proposition 2.9, for ε small, if $\deg_{\partial B_j^\varepsilon}(\hat{v}_\varepsilon) \neq 0$ then $B_j^\varepsilon \subset B(0, 1-r_0) \subset B(0, 1-\xi/b)$. It follows that $D \geq d_0$ and then (2.28) is a direct consequence of (2.29) and the bound $\hat{U}_\varepsilon \geq b$.

We next prove that there is $C > 0$ s.t.

$$\frac{1}{2}\int_{B(0,\frac{\rho}{\delta})\setminus\overline{B(0,1)}} \hat{U}_\varepsilon^2 |\nabla \hat{v}_\varepsilon|^2 \geq \pi d_0^2 |\ln \delta| - C. \tag{2.30}$$

By Proposition 2.9, $|v_\varepsilon| \geq 1/2$ in $B(0, \frac{\rho}{\delta}) \setminus \overline{B(0,1)}$, therefore, $\hat{w}_\varepsilon := \frac{\hat{v}_\varepsilon}{|\hat{v}_\varepsilon|}$ is well-defined in this domain. Observe that

$$\frac{1}{2}\int_{B(0,\frac{\rho}{\delta})\setminus\overline{B(0,1)}} |\nabla \hat{w}_\varepsilon|^2 \geq \frac{1}{2}\int_{B(0,\frac{\rho}{\delta})\setminus\overline{B(0,1)}} \left|\nabla \frac{z^{d_0}}{|z|^{d_0}}\right|^2 = \pi d_0^2 \ln \frac{\rho}{\delta}. \tag{2.31}$$

We claim that (2.30) holds with $C = \pi d_0^2 |\ln \rho| + 1$ (for small ε). By contradiction, assume (2.30) does not hold. Then, up to a subsequence, we have

$$\frac{1}{2}\int_{B(0,\frac{\rho}{\delta})\setminus\overline{B(0,1)}} \hat{U}_\varepsilon^2 |\nabla \hat{v}_\varepsilon|^2 < \pi d_0^2 \ln \frac{\rho}{\delta} - 1. \tag{2.32}$$

On the other hand, we have

$$|\nabla \hat{v}_\varepsilon|^2 = |\hat{v}_\varepsilon|^2 |\nabla \hat{w}_\varepsilon|^2 + |\nabla |\hat{v}_\varepsilon||^2$$

and therefore

$$\int_{B(0,\frac{\rho}{\delta})\setminus\overline{B(0,1)}} |\nabla \hat{v}_\varepsilon|^2 \geq \int_{B(0,\frac{\rho}{\delta})\setminus\overline{B(0,1)}} |\nabla \hat{w}_\varepsilon|^2 - \int_{B(0,\frac{\rho}{\delta})\setminus\overline{B(0,1)}} (1 - |\hat{v}_\varepsilon|^2) |\nabla \hat{w}_\varepsilon|^2. \tag{2.33}$$

Since $|\hat{v}_\varepsilon| \geq \frac{1}{2}$ in $B(0, \frac{\rho}{\delta})\setminus\overline{B(0,1)}$ we have $|\nabla \hat{w}_\varepsilon| \leq 2|\nabla \hat{v}_\varepsilon|$. Therefore, by (2.32), Proposition 2.9 and (H) we estimate the last term in (2.33):

$$\int_{B(0,\frac{\rho}{\delta})\setminus\overline{B(0,1)}} (1 - |\hat{v}_\varepsilon|^2) |\nabla \hat{w}_\varepsilon|^2 \leq C_2 |\ln \varepsilon|^{-\frac{1}{3}} \int_{B(0,\frac{\rho}{\delta})\setminus\overline{B(0,1)}} |\nabla \hat{v}_\varepsilon|^2 \leq C_3 \frac{|\ln \delta|}{|\ln \varepsilon|^{\frac{1}{3}}} \to 0. \tag{2.34}$$

Combining (2.31), (2.33) and (2.34), we find that

$$\int_{B(0,\frac{\rho}{\delta})\setminus\overline{B(0,1)}} |\nabla \hat{v}_\varepsilon|^2 \geq \pi d_0^2 \ln \frac{\rho}{\delta} - o_\varepsilon(1).$$

Since $|\hat{U}_\varepsilon - 1| \leq C\xi^4$ in $B_{\frac{\rho}{\delta}} \setminus \overline{B(0,1)}$ (see Proposition 2.5), we obtain a contradiction with (2.32), and (2.30) follows. Comparing the lower bounds (2.28) and (2.30) with the upper bound in Proposition 2.8, the Proposition 2.10 follows. □

Using exactly the same techniques as in the proof of Proposition 2.10, one may easily prove the following estimate.

Corollary 2.11. *For any* $R_2 > R_1 \geq 1$

$$F_\xi(\hat{v}_\varepsilon, B(0, R_2) \setminus \overline{B(0, R_1)}) = \mathcal{O}(1).$$

2.3.3 Convergence in $C^\infty(K)$ for a compact K s.t. $K \cap \overline{\omega} = \emptyset$

Proposition 2.12. *Let $K \subset \mathbb{R}^2 \setminus \overline{\omega}$ be a smooth compact set. Then we have*

$$\hat{v}_\varepsilon \text{ is bounded in } C^k(K) \text{ for all } k \geq 0 \tag{2.35}$$

and there is $C_K > 0$ s.t.

$$|\hat{v}_\varepsilon| \geq 1 - C_K \xi^2 \text{ in } K. \tag{2.36}$$

Proof. From Proposition 2.5

$$E_\xi(\hat{U}_\varepsilon, K) = \frac{1}{2}\int_K |\nabla \hat{U}_\varepsilon|^2 + \frac{1}{2\xi^2}(1 - \hat{U}_\varepsilon^2)^2 = \mathcal{O}(1). \tag{2.37}$$

As in [43], the following expansion holds

$$E_\xi(\hat{U}_\varepsilon \hat{v}_\varepsilon, K) = E_\xi(\hat{U}_\varepsilon, K) + \hat{F}_\xi(\hat{v}_\varepsilon, K) + \int_{\partial K} (|\hat{v}_\varepsilon|^2 - 1)\hat{U}_\varepsilon \partial_\nu \hat{U}_\varepsilon. \tag{2.38}$$

Using (2.10), we have

$$\int_{\partial K} (|\hat{v}_\varepsilon|^2 - 1)\hat{U}_\varepsilon \partial_\nu \hat{U}_\varepsilon = o_\varepsilon(1).$$

With (2.37) and (2.38), we conclude that $E_\xi(\hat{U}_\varepsilon \hat{v}_\varepsilon, K) = \mathcal{O}(1)$. Since $\hat{U}_\varepsilon$ and $\hat{U}_\varepsilon \hat{v}_\varepsilon$ satisfy the Ginzburg-Landau equation $-\Delta u = \frac{1}{\xi^2} u(1 - |u|^2)$ in K, as well as $|\hat{U}_\varepsilon| \leq 1$ and $|\hat{U}_\varepsilon \hat{v}_\varepsilon| \leq 1$. Theorem 1 in [53] implies that

$$\hat{U}_\varepsilon \text{ and } \hat{U}_\varepsilon \hat{v}_\varepsilon \text{ are bounded in } C^k(K) \text{ for all } k \geq 0.$$

It follows that $\hat{v}_\varepsilon$ is bounded in $C^k(K)$ for each $k \geq 0$. On the other hand, using the fact that $\hat{v}_\varepsilon$ is bounded in $C^k(K)$ together with the equation of $\hat{v}_\varepsilon$, we find that $1 - |\hat{v}_\varepsilon|^2 \leq C_K \xi^2$ in K. □

Corollary 2.13. *For $K \subset \mathbb{R}^2 \setminus \overline{\omega}$, up to a subsequence, there is some $v_0 \in C^\infty(K, \mathbb{S}^1)$ s.t. $\hat{v}_\varepsilon \to v_0$ in $C^\infty(K)$.*

We are now in position to bound the potential part of the energy.

Corollary 2.14. *There exists $C > 0$ independent of ε s.t.*

$$\frac{1}{\varepsilon^2}\int_{B(0,\rho)} (1 - |v_\varepsilon|^2)^2 = \frac{1}{\xi^2}\int_{B(0,\rho/\delta)} (1 - |\hat{v}_\varepsilon|^2)^2 \leq C. \tag{2.39}$$

Proof. Note that from Propositions 2.8, 2.10, we find that there is $C > 0$ s.t.

$$\frac{1}{\xi^2}\int_{B(0,\rho/\delta)\setminus\overline{B(0,1)}} (1 - |\hat{v}_\varepsilon|^2)^2 \leq C.$$

Thus it remains to prove the estimate in $B(0,1)$ for small ε. Using (2.35), $\text{tr}_{\partial B(0,1)} \hat{v}_\varepsilon$ is bounded in $C^1(\partial B(0,1))$ and $1 - |\hat{v}_\varepsilon|^2 \leq C\xi^2$ on $\partial B(0,1)$ (for small ε). These properties, allow us to construct a smooth extension $\tilde{v}_\varepsilon$ of $\text{tr}_{\partial B(0,1)} \hat{v}_\varepsilon$ into $B(0,2) \setminus \overline{B(0,1)}$, s.t. $h = \text{tr}_{\partial B(0,2)} \tilde{v}_\varepsilon$ is $\mathbb{S}^1$-valued and independent of ε, $1 - |\tilde{v}_\varepsilon|^2 \leq C\xi^2$ in $B(0,2) \setminus \overline{B(0,1)}$ and

$$\int_{B(0,2)\setminus\overline{B(0,1)}} \left\{ |\nabla \tilde{v}_\varepsilon|^2 + \frac{1}{2\xi^2}(1 - |\tilde{v}_\varepsilon|^2)^2 \right\} \leq C_0. \tag{2.40}$$

(For example, this construction is performed by mimicking (2.21))

Define w_ε as $w_\varepsilon = \hat{v}_\varepsilon$ in $B(0,1)$ and $w_\varepsilon = \tilde{v}_\varepsilon$ in $B(0,2) \setminus \overline{B(0,1)}$. Clearly, $w_\varepsilon \in H^1_h(B(0,2))$, w_ε is bounded in $L^2(B(0,2))$ and, thanks to Proposition 2.10 and (2.40),

$$\frac{1}{2}\int_{B(0,2)} \left\{ |\nabla w_\varepsilon|^2 + \frac{b^2}{2\xi^2}(1-|w_\varepsilon|^2)^2 \right\} \leq \pi d_0 |\ln \xi| + C_0.$$

We may now apply Proposition 0.1 in [33] to w_ε in $B(0,2)$ to conclude that $\displaystyle\frac{1}{\xi^2}\int_{B(0,2)}(1-|w_\varepsilon|^2)^2 \leq C_1$. Therefore the bound (2.39) holds. □

2.3.4 The bad discs

Consider a family of discs $(B(x_i, \varepsilon^{1/4}))_{i\in I}$ such that

$$\text{for all } i \in I \text{ we have } x_i \in \Omega,$$

$$B(x_i, \varepsilon^{1/4}/4) \cap B(x_i, \varepsilon^{1/4}/4) = \emptyset \text{ if } i \neq j,$$

$$\cup_{i\in I} B(x_i, \varepsilon^{1/4}) \supset \Omega.$$

For $\mu \in (1/2, 1)$, let $C = C(\mu)$, $\varepsilon_0 = \varepsilon_0(\mu)$ be defined as in the second part of Lemma 2.7. For $\varepsilon < \varepsilon_0$, we say that $B(x_i, \varepsilon^{1/4})$ is *μ-good disc* if

$$F_\varepsilon(v_\varepsilon, B(x_i, \varepsilon^{1/4}) \cap \Omega) \leq C(\mu)|\ln \varepsilon|$$

and $B(x_i, \varepsilon^{1/4})$ is *μ-bad disc* if

$$F_\varepsilon(v_\varepsilon, B(x_i, \varepsilon^{1/4}) \cap \Omega) > C(\mu)|\ln \varepsilon|. \tag{2.41}$$

Let $J_\varepsilon = J := \{i \in I \,|\, B(x_i, \varepsilon^{1/4}) \text{ is a } \mu\text{-bad disc}\}$.

Lemma 2.15. *There is an integer N, which depends only on g and μ, s.t.*

$$\operatorname{Card} J \leq N.$$

Proof. Since each point of Ω is covered by at most 16 discs $B(x_i, \varepsilon^{1/4})$, we have

$$\sum_{i\in I} F_\varepsilon(v_\varepsilon, B(x_i, \varepsilon^{1/4}) \cap \Omega) \leq 16 F_\varepsilon(v_\varepsilon, \Omega).$$

The previous assertion implies that Card $J \leq \frac{16C_0}{C_1(\mu)}$. □

The next result is a straightforward variant of Theorem IV.1 in [18].

Lemma 2.16. *Possibly after passing to a subsequence and relabeling I, we may choose $J' \subset J$ and a constant $\lambda \geq 1$ (independently of ε) s.t.*

$$J' = \{1, ..., N'\},\ N' = \text{Cst},$$

$$|x_i - x_j| \geq 8\lambda\varepsilon^{1/4} \text{ for } i, j \in J',\ i \neq j$$

and

$$\cup_{i\in J} B(x_i, \varepsilon^{1/4}) \subset \cup_{i\in J'} B(x_i, \lambda\varepsilon^{1/4}).$$

We will say that, for $i \in J'$, $B(x_i, \lambda\varepsilon^{1/4})$ are *separated μ-bad discs*. From now on, we work with separated μ-bad discs. Denote $\hat{x}_i = \dfrac{x_i}{\delta}$. By Proposition 2.9 we know that for small ε, we have $\hat{x}_i \in \overline{B_1}$. Clearly, up to a subsequence,

$$\left\{\begin{array}{c} \text{there are } \alpha_1, ..., \alpha_\kappa,\ \kappa \text{ distinct points in } \overline{B_1} \\ \text{and } \{\Lambda_1, ..., \Lambda_\kappa\} \text{ a partition (in non empty sets) of } J' \text{ s.t.} \\ \text{for } i \in J',\ \text{if } i \in \Lambda_k \text{ then } \hat{x}_i \to \alpha_k. \end{array}\right. \tag{2.42}$$

Note that for $i \in J'$, we have

$$y \in \{\alpha_1, ..., \alpha_\kappa\} \iff \begin{cases} \forall\, \eta > 0, \text{ for small } \varepsilon, \\ \text{there is a } \mu\text{-bad disc inside } B(y, \eta) \end{cases}. \tag{2.43}$$

2.3.5 Convergence in $H^1_{\text{loc}}(\mathbb{R}^2 \setminus \{\alpha_1, ..., \alpha_\kappa\})$

We have the following theorem.

Proposition 2.17. *Let $\alpha_1, ..., \alpha_\kappa$ be defined by* (2.42)*. Then we have:*

1. *The points $\alpha_1, ..., \alpha_\kappa$ belong to ω.*
2. *There exists $v_0 \in H^1_{\text{loc}}(\mathbb{R}^2 \setminus \{\alpha_1, ..., \alpha_\kappa\}, \mathbb{S}^1)$ s.t. (possibly after extraction)*
$$\hat{v}_\varepsilon \to v_0 \text{ in } H^1_{\text{loc}}(\mathbb{R}^2 \setminus \{\alpha_1, ..., \alpha_\kappa\}) \tag{2.44}$$
$$\hat{v}_\varepsilon \to v_0 \text{ in } C^0_{\text{loc}}(\mathbb{R}^2 \setminus \{\alpha_1, ..., \alpha_\kappa\}). \tag{2.45}$$
3. *There exists $\eta_0 > 0$ s.t. for all $0 < \eta < \eta_0$ and for sufficiently small ε we have*
$$\deg_{\partial B_\eta(\alpha_k)}(\hat{v}_\varepsilon/|\hat{v}_\varepsilon|) = \deg_{\partial B_{\eta_0}(\alpha_k)}(v_0) = 1.$$
4. *$\kappa = d_0$.*

Proof. **Step 1:** $\hat{v}_\varepsilon \rightharpoonup v_0$ in $H^1_{\text{loc}}(\mathbb{R}^2 \setminus \{\alpha_1, ..., \alpha_\kappa\})$, $v_0 \in H^1_{\text{loc}}(\mathbb{R}^2 \setminus \{\alpha_1, ..., \alpha_\kappa\}, \mathbb{S}^1)$ and $\alpha_k \in \omega$

Proposition 2.9 guarantees that $\alpha_1, ..., \alpha_\kappa \in \overline{\omega}$. Let

$$\eta_0 = \begin{cases} 10^{-2} \cdot \min_{k \neq k'} |\alpha_k - \alpha_{k'}| & \text{if } \kappa > 1 \\ 1 & \text{if } \kappa = 1 \end{cases}. \tag{2.46}$$

Applying Theorem 4.1 in [65] we have for all $0 < \eta < \eta_0$ and for small ε

$$\frac{1}{2} \int_{\cup_{k \in \{1,...,\kappa\}} B(\alpha_k, \eta)} \left\{ |\nabla \hat{v}_\varepsilon|^2 + \frac{b^2}{2\xi^2}(1 - |\hat{v}_\varepsilon|^2)^2 \right\} \geq \pi d_0 \ln \frac{\eta}{\xi} - C \tag{2.47}$$

with C independent of ε and η.

Combining (2.47) with (2.27) and Corollary 2.11 we obtain that $\hat{v}_\varepsilon$ is bounded in $H^1(K)$; here $K \subset \mathbb{R}^2 \setminus \{\alpha_1, ..., \alpha_\kappa\}$ is an arbitrary compact set. Therefore, there exists $v_0 \in H^1_{\text{loc}}(\mathbb{R}^2 \setminus \{\alpha_1, ..., \alpha_\kappa\})$ s.t. we have $\hat{v}_\varepsilon \rightharpoonup v_0$ in $H^1_{\text{loc}}(\mathbb{R}^2 \setminus \{\alpha_1, ..., \alpha_\kappa\})$ (possibly passing to a subsequence). Since $\|1 - |\hat{v}_\varepsilon|\|_{L^2(K)} \to 0$ for all compact sets $K \subset \mathbb{R}^2 \setminus \{\alpha_1, ..., \alpha_\kappa\}$, we find that v_0 is $\mathbb{S}^1$- valued.

Following the proof of Step 7 in Theorem C in [43], we can prove that $\alpha_1, ..., \alpha_\kappa \notin \partial\omega$, thus $\alpha_1, ..., \alpha_\kappa \in \omega$, and the first assertion follows.

Step 2: Proof of 2.

Adapting the techniques of [17] (Theorem 2, Step 1), we establish (2.44) and (2.45) in a ball $B = B(y, R_0)$ s.t. $\overline{B} \subset \mathbb{R}^2 \setminus \{\alpha_1, ..., \alpha_\kappa\}$.

Let $y \in \mathbb{R}^2$ and let $R' > R > 0$ be s.t. $B(y, R') \subset \mathbb{R}^2 \setminus \{\alpha_1, ..., \alpha_\kappa\}$. Since $\hat{F}_\xi(\hat{v}_\varepsilon, B(y, R'))$ is bounded independently on ε, there is $R_0 \in (R, R')$ (independent of ε) s.t., passing to a further subsequence if necessary we have

$$\int_{\partial B(y,R_0)} \left\{ |\partial_\tau \hat{v}_\varepsilon|^2 + \frac{1}{\xi^2}(1 - |\hat{v}_\varepsilon|^2)^2 \right\} \leq C \text{ with } C \text{ independent of } \varepsilon. \tag{2.48}$$

Indeed, for $r \in (R, R')$ denote

$$I_\varepsilon(r) = \int_{\partial B(y,r)} \left\{ |\nabla \hat{v}_\varepsilon|^2 + \frac{1}{\xi^2}(1 - |\hat{v}_\varepsilon|^2)^2 \right\}.$$

Using the Fubini theorem and the Fatou Lemma we have

$$0 \leq \int_R^{R'} \liminf_\varepsilon I_\varepsilon(r)\,\mathrm{d}r \leq \liminf_\varepsilon \int_R^{R'} I_\varepsilon(r)\,\mathrm{d}r \leq C'.$$

Consequently, $\liminf_\varepsilon I_\varepsilon(r) < \infty$ for almost all $r \in (R, R')$, so that (2.48) holds with $C = \dfrac{C'}{R' - R}$.

Let $g_\varepsilon = \mathrm{tr}_{\partial B} \hat{v}_\varepsilon$. Since $|v_\varepsilon| \geq 1/2$ in $B = B(y, R_0)$, we have $\deg_{\partial B}(g_\varepsilon) = 0$. The bound (2.48) implies that, up to choose a subsequence, g_ε is weakly convergent in $H^1(\partial B)$. Consequently there is $h \in H^1(\partial B, \mathbb{S}^1)$, $h = \mathrm{e}^{\imath\varphi}$, $\varphi \in H^1(\partial B, \mathbb{R})$ s.t.

$$g_\varepsilon \to h \text{ uniformly on } \partial B, \tag{2.49}$$

$$g_\varepsilon \to h \text{ in } H^{1/2}(\partial B). \tag{2.50}$$

Let $\eta_\varepsilon : B \to \mathbb{R}^+$ be the minimizer of $\displaystyle\int_B \left\{ |\nabla \eta|^2 + \frac{1}{\xi^2}(1 - \eta)^2 \right\}$ in $H^1_{|g_\varepsilon|}(B, \mathbb{R})$. Then η_ε satisfies

$$\begin{cases} -\xi^2 \Delta \eta_\varepsilon + \eta_\varepsilon = 1 & \text{in } B \\ \eta_\varepsilon = |g_\varepsilon| & \text{on } \partial B \end{cases}.$$

It follows from [17] that

$$\int_B \left\{ |\nabla \eta_\varepsilon|^2 + \frac{1}{\xi^2}(1 - \eta_\varepsilon)^2 \right\} \leq C\xi. \tag{2.51}$$

Using (2.49), there is $\varphi_\varepsilon \in H^1(\partial B, \mathbb{R})$, s.t. $g_\varepsilon = |g_\varepsilon|\mathrm{e}^{\imath\varphi_\varepsilon}$ and $\varphi_\varepsilon \to \varphi$ uniformly on ∂B. Following [17], denote by $\psi_\varepsilon \in H^1_{\varphi_\varepsilon}(B, \mathbb{R})$ the unique solution of $-\mathrm{div}(a^2 \nabla \psi_\varepsilon) = 0$. (Here $a = b$ in ω and $a = 1$ in $\mathbb{R}^2 \setminus \omega$.) From (2.50), $\psi_\varepsilon \to \psi$ in $H^1(B)$ where $\psi \in H^1_\varphi(B, \mathbb{R})$ is the unique solution of $-\mathrm{div}(a^2 \nabla \psi) = 0$. Since $\eta_\varepsilon \mathrm{e}^{\imath\psi_\varepsilon} \in H^1_{g_\varepsilon}(B)$, we have

$$\hat{F}_\xi(\hat{v}_\varepsilon, B) \leq \hat{F}_\xi(\eta_\varepsilon \mathrm{e}^{\imath\psi_\varepsilon}, B) \leq \frac{1}{2} \int_B \hat{U}_\varepsilon^2 |\nabla \psi_\varepsilon|^2 + C\xi \underset{\varepsilon \to 0}{\to} \frac{1}{2} \int_B a^2 |\nabla \psi|^2. \tag{2.52}$$

On the other hand, since $\hat{v}_\varepsilon \rightharpoonup v_0$ in $H^1(B)$, we have $v_0 = \mathrm{e}^{\imath\phi}$ with $\phi \in H^1_\varphi(B, \mathbb{R})$ and

$$\liminf_\varepsilon \hat{F}_\xi(\hat{v}_\varepsilon, B) \geq \liminf_\varepsilon \frac{1}{2} \int_B \hat{U}_\varepsilon^2 |\nabla \hat{v}_\varepsilon|^2 \geq \frac{1}{2} \int_B a^2 |\nabla v_0|^2 = \frac{1}{2} \int_B a^2 |\nabla \phi|^2. \tag{2.53}$$

(The last inequality follows from $\hat{U}_\varepsilon \to a$ in L^2, $|U_\varepsilon| \le 1$ and $\hat{v}_\varepsilon \rightharpoonup v_0$ in H^1.)

By combining (2.52), (2.53) and the fact that ψ minimizes $\displaystyle\int_B a^2|\nabla \cdot|^2$ in $H^1_\varphi(B,\mathbb{R})$, we find that (2.44) holds. Furthermore, the map ψ in (2.52) is the same as ϕ in (2.53).

Note that since

$$\frac{1}{2}\int_B \hat{U}_\varepsilon^2 \left|\nabla \frac{\hat{v}_\varepsilon}{|\hat{v}_\varepsilon|}\right|^2 - o_\varepsilon(1) \le \hat{F}_\xi(\hat{v}_\varepsilon, B),$$

by comparing (2.52) with (2.53), we also have

$$\int_K |\nabla|\hat{v}_\varepsilon||^2 + \frac{1}{\xi^2}(1-|\hat{v}_\varepsilon|^2)^2 \to 0. \tag{2.54}$$

In order to prove (2.45), it suffices to establish the convergence

$$\phi_\varepsilon \to \phi \text{ in } L^\infty(B) \text{ with } \phi_\varepsilon \in H^1_{\varphi_\varepsilon}(B,\mathbb{R}) \text{ and } \hat{v}_\varepsilon = |\hat{v}_\varepsilon|e^{\imath\phi_\varepsilon}, \tag{2.55}$$

and to use the fact that $|\hat{v}_\varepsilon| \to 1$ uniformly.

Proof of (2.55). If $\partial\omega \cap B = \emptyset$, then the argument is the same as in [17]. Assume next that $\partial\omega \cap B \neq \emptyset$, and let $\tilde{\psi} \in H^{3/2}(B,\mathbb{R})$ be the harmonic extension of φ. Since $\zeta := \phi - \tilde{\psi} \in H^1_0(B,\mathbb{R})$ satisfies $-\mathrm{div}(a^2\nabla\zeta) = \mathrm{div}(a^2\nabla\tilde{\psi})$, Theorem 1 in [52] implies that $\phi \in W^{1,p}(B,\mathbb{R})$ for some $p > 2$.

We next prove that, for some $q > 2$ and $\tilde{B} = B(y',\tilde{R})$ s.t. $B(y',2\tilde{R}) \subset B$, we have $\|\phi_\varepsilon - \phi\|_{W^{1,q}(\tilde{B})} \to 0$. (Once proved, this assertion will imply, via Sobolev embedding that (2.55) holds.)

Note that (up to a subsequence) $\phi_\varepsilon \to \phi$ in $L^2(B,\mathbb{R})$. Thus we have

$$\begin{cases} \mathrm{div}\left[\hat{U}^2|\hat{v}_\varepsilon|^2\nabla(\phi_\varepsilon - \phi)\right] = \mathrm{div}\left[(\hat{U}_\varepsilon^2|\hat{v}_\varepsilon|^2 - a^2)\nabla\phi\right] & \text{in } B \\ \|\phi_\varepsilon - \phi\|_{L^2(B)} \to 0 & \end{cases}.$$

From Theorem 2 in [52], there is $2 < q \le p$ and $C > 0$ s.t.

$$\|\nabla(\phi_\varepsilon - \phi)\|_{L^q(\tilde{B})} \le C\left(\tilde{R}^{-2+2/q}\|\phi_\varepsilon - \phi\|_{L^2(B)} + \|(\hat{U}_\varepsilon^2|\hat{v}_\varepsilon|^2 - a^2)\nabla\phi\|_{L^q(B)}\right) \underset{\varepsilon\to 0}{\to} 0.$$

Consequently, $\|\phi_\varepsilon - \phi\|_{W^{1,q}(\tilde{B})} \to 0$.

Step 3: We prove the third assertion

Let $\eta_0 > \eta > 0$, with η_0 defined by (2.46). Denote $d_k = \deg_{\partial B(\alpha_k,r)}(v_0)$. These integers do not depend on $r \in (\eta,\eta_0)$. Moreover, we have $\sum_k d_k = d_0$. For $r \in (\eta,\eta_0)$, we obtain that

$$2\pi|d_k| \le \int_{\partial B(\alpha_k,r)} |\partial_\tau v_0| \le \sqrt{2\pi r}\left(\int_{\partial B(\alpha_k,r)} |\partial_\tau v_0|^2\right)^{1/2},$$

and therefore

$$\frac{1}{2}\int_{B(\alpha_k,\eta_0)\setminus\overline{B(\alpha_k,\eta)}} |\nabla v_0|^2 \ge \pi d_k^2 \ln\frac{\eta_0}{\eta}.$$

Consequently, we have

$$\begin{aligned} \liminf \frac{1}{2}\int_{\cup_k B(\alpha_k,\eta_0)\setminus\overline{B(\alpha_k,\eta)}} |\nabla\hat{v}_\varepsilon|^2 &\ge \frac{1}{2}\int_{\cup_k B(\alpha_k,\eta_0)\setminus\overline{B(\alpha_k,\eta)}} |\nabla v_0|^2 \\ &\ge \pi\sum_k d_k^2 \ln\frac{\eta_0}{\eta}. \end{aligned} \tag{2.56}$$

By combining (2.47) and (2.56), we obtain the existence of C independent of ε and η s.t.

$$\begin{aligned}\frac{1}{2}\int_{\cup_k B(\alpha_k,\eta_0)}|\nabla\hat{v}_\varepsilon|^2 &\geq \pi\sum_k d_k^2\ln\frac{\eta_0}{\eta}+\pi d_0\ln\frac{\eta}{\xi}-C\\ &= \pi d_0\ln\frac{\eta_0}{\xi}+\pi(\sum_k d_k^2-d_0)\ln\frac{\eta_0}{\eta}-C.\end{aligned}$$

Therefore, d_k must be either 0 or 1. Otherwise, (2.27) cannot hold for small η. Applying the strong convergence result obtained Step 2 with $K=B(\alpha_k,\eta)\setminus\overline{B(\alpha_k,\frac{\eta}{2})}$, we obtain for small ε,

$$d_k=\deg_{\partial B(\alpha_k,\eta)}\left(\frac{\hat{v}_\varepsilon}{|\hat{v}_\varepsilon|}\right).$$

We next prove that $d_k=1$ for each k. We argue by contradiction. Assume that there is k_0 s.t. $d_{k_0}=0$. We may assume that $k_0=1$. From (2.43), there is a (separated) μ-bad disc $B(\hat{x}_0,\varepsilon^{1/4}/\delta)$ in $B(\alpha_1,\eta_0)$. Thus by (2.41), we have

$$\hat{F}_\xi(\hat{v}_\varepsilon,B(\hat{x}_0,\varepsilon^{1/4}/\delta))>C(\mu)|\ln\varepsilon|.$$

On the other hand, since in $|\hat{v}_\varepsilon|\geq 1/2$ in $B(\alpha_k,\eta_0)\setminus\overline{B(\alpha_k,\eta_0/2)}$, applying Theorem 4.1 in [65] in $B(\alpha_k,\eta_0)$, $k\in\{2,...,\kappa\}$, with $r=10^{-4}\cdot\eta_0$ we find that

$$\hat{F}_\xi(\hat{v}_\varepsilon,B(\alpha_k,\eta_0))\geq b^2|\deg_{\partial B(\alpha_k,\eta_0)}(\hat{v}_\varepsilon)||\ln\xi|-C,\ k=2,...,\kappa.$$

Since $\sum_{k=2}^{\kappa}\deg_{\partial B(\alpha_k,\eta_0)}(\hat{v}_\varepsilon)=d_0$, the above estimates yield

$$\hat{F}_\xi(\hat{v}_\varepsilon,B(0,\frac{\rho}{\delta}))\geq b^2 d_0|\ln\xi|+C(\mu)|\ln\varepsilon|-C$$

which is in contradiction with (H) and (2.11). Thus $d_k=1$ for $k\in\{1,...,\kappa\}$ and consequently, $\kappa=d_0$. □

Now we are in position to prove a rate for the uniform convergence of $|\hat{v}_\varepsilon|$ in $K\subset\mathbb{R}^2\setminus\{\alpha_1,...,\alpha_{d_0}\}$, K a compact set.

Corollary 2.18. *There is $C>0$ s.t. for $\eta_0>\eta>0$ and small ε we have*

$$|\hat{v}_\varepsilon|\geq 1-C|\ln\varepsilon|^{-1/3}\ \text{in}\ B(0,\frac{\rho}{\delta})\setminus\overline{B(\alpha_i,\eta)}.$$

Proof. Due to (2.36), it is sufficient to establish this result in $B(0,1)\setminus\overline{B(\alpha_i,\eta)}$. Combining Corollary 2.14 with (2.44), we obtain that

$$\hat{F}_\xi(\hat{v}_\varepsilon,B(0,2)\setminus\overline{B(\alpha_i,\eta/2)})\leq C(\eta).$$

Thus for all $x\in B(0,\rho)$ s.t. $B(\hat{x},\varepsilon^{1/4}/\delta)\subset B(0,2)\setminus\overline{B(\alpha_i,\eta/2)}$, for small ε we have

$$F_\varepsilon(v_\varepsilon,B(x,\varepsilon^{1/4}))\leq\hat{F}_\xi(\hat{v}_\varepsilon,B(0,2)\setminus\overline{B(\alpha_i,\eta/2)})<|\ln\varepsilon|^{1/3}.$$

From Lemma 2.7 (first assertion), we obtain the existence of $C>0$ (independent of ε and η) s.t. $|v_\varepsilon(x)|=|\hat{v}_\varepsilon(\hat{x})|\geq 1-C|\ln\varepsilon|^{-1/3}$. Finally, since for all $\hat{x}\in B(0,1)\setminus\overline{B(\alpha_i,\eta)}$ we have $B(\hat{x},\varepsilon^{1/4}/\delta)\subset B(0,2)\setminus\overline{B(\alpha_i,\eta/2)}$, Corollary 2.18 follows. □

2.3.6 Information about the limit v_0

Following [18] (Appendix IV, page 152) we have

Proposition 2.19. *For all $1 \leq p < 2$ and for any compact $K \subset \mathbb{R}^2$, $\hat{v}_\varepsilon$ is bounded in $W^{1,p}(K)$.*

Let θ_i be the main argument of $\frac{\hat{x}-\alpha_i}{|\hat{x}-\alpha_i|}$ and set $\theta = \theta_1 + ... + \theta_{d_0}$. Note that $\nabla\theta$ is smooth away from $\{\alpha_1, ..., \alpha_\kappa\}$ and $\Pi_i \frac{\hat{x}-\alpha_i}{|\hat{x}-\alpha_i|} = e^{\imath\theta}$. Let $\tilde{g} := \mathrm{tr}_{\partial B_1} v_0$ and $\varphi_0 \in C^\infty(\partial B_1, \mathbb{R})$ be s.t. $\tilde{g} = \Pi_i \frac{\hat{x}-\alpha_i}{|\hat{x}-\alpha_i|} e^{\imath\varphi_0} = e^{\imath(\theta+\varphi_0)}$ (see [25] for the existence of φ_0).

Proposition 2.20. *The limit v_0 satisfies $-\mathrm{div}\left(a^2\, v_0 \times \nabla v_0\right) = 0$ in $\mathcal{D}'(\mathbb{R}^2)$. Moreover we may write $v_0 = e^{\imath(\theta+\varphi_\star)}$. Here $\varphi_\star$ is the solution of*

$$\begin{cases} -\mathrm{div}\left[a^2 \nabla(\theta + \varphi_\star)\right] = 0 & \text{in } B_1 \\ \varphi_\star = \varphi_0 & \text{on } \partial B_1 \end{cases}. \tag{2.57}$$

Proof. Let $\phi \in \mathcal{D}(\mathbb{R}^2)$, and set $K = \mathrm{supp}(\phi)$. By Proposition 2.19, we have $\hat{U}_\varepsilon^2\, \hat{v}_\varepsilon \times \nabla\hat{v}_\varepsilon \rightharpoonup a^2\, v_0 \times \nabla v_0$ in $L^p(K)$ for $p < 2$. Multiplying the equation $-\mathrm{div}\left[\hat{U}_\varepsilon^2\, \hat{v}_\varepsilon \times \nabla\hat{v}_\varepsilon\right] = 0$ by ϕ and integrating by parts, we obtain

$$\begin{aligned} \int_K -\mathrm{div}\left[\hat{U}_\varepsilon^2\, \hat{v}_\varepsilon \times \nabla\hat{v}_\varepsilon\right]\phi &= \int_K \hat{U}_\varepsilon^2\, \hat{v}_\varepsilon \times \nabla\hat{v}_\varepsilon \cdot \nabla\phi \\ &\to \int_K a^2\, v_0 \times \nabla v_0 \cdot \nabla\phi = \int_K -\mathrm{div}\left(a^2\, v_0 \times \nabla v_0\right)\phi. \end{aligned}$$

Consequently $-\mathrm{div}\left(a^2\, v_0 \times \nabla v_0\right) = 0$ in $\mathcal{D}'(\mathbb{R}^2)$.

In order to prove that $-\mathrm{div}\left[a^2\nabla(\theta+\varphi_\star)\right] = 0$ in $B_1 \setminus \{\alpha_1, ..., \alpha_{d_0}\}$, we follow [43], step 12 of Theorem C.

Next, we prove that $\varphi_\star$ is harmonic in a neighborhood of α_k. Fix $\lambda > 0$ and $x_0 \in \omega$ s.t. $B(x_0, 2\lambda) \subset \omega \setminus \{\alpha_1, ..., \alpha_{d_0}\}$. As we established in Proposition 2.17, Step 2, $F_\xi(\hat{v}_\varepsilon, B(x_0, 2\lambda))$ is uniformly bounded in ε. Proceeding as in the Step 2, we conclude that exists $\lambda_0 \in (\lambda, 2\lambda)$ s.t., after passing to a further subsequence, we have

$$\int_{\partial B(x_0,\lambda_0)} \left\{|\nabla\hat{v}_\varepsilon|^2 + \frac{1}{2\xi^2}(1 - |\hat{v}_\varepsilon|^2)^2\right\} \leq C \tag{2.58}$$

with C and λ_0 independent of ε. Now, if $\hat{u}_\varepsilon$ minimizes

$$\hat{E}_\xi(\hat{u}) = \frac{1}{2}\int_{B(0,\frac{\rho}{\delta})} \left\{|\nabla\hat{u}|^2 + \frac{1}{2\xi^2}(a^2 - |\hat{u}|^2)^2\right\}$$

subject to $\hat{u}(x) = f_\varepsilon(\delta x)$ on $\partial B(0, \frac{\rho}{\delta})$, then $\hat{u}_\varepsilon$ minimizes $\hat{E}_\xi(\hat{u}, B(x_0, \lambda_0))$ with respect to its own boundary conditions. In other words, $\hat{w}_\varepsilon := \frac{\hat{u}_\varepsilon}{b}$ minimizes the classical energy

$$\frac{1}{2}\int_{B(x_0,\lambda_0)} \left\{|\nabla\hat{w}_\varepsilon|^2 + \frac{b^2}{2\xi^2}(1 - |\hat{w}_\varepsilon|^2)^2\right\}$$

among $w \in H^1(B(x_0, \lambda_0))$ such that $w = h_\varepsilon := \frac{\hat{u}_\varepsilon}{b}$ on $\partial B(x_0, \lambda_0)$. It follows from (2.58) and Proposition 2.5 that h_ε also satisfies

$$\int_{\partial B(x_0,\lambda_0)} \left\{|\partial_\tau h_\varepsilon|^2 + \frac{1}{2\xi^2}(1 - |h_\varepsilon|^2)^2\right\} \leq C + 1. \tag{2.59}$$

Note that by Proposition 2.5 we have

$$\|\hat{w}_\varepsilon\|_{L^\infty(B(x_0,\lambda_0))} \leq 1 + ce^{-c_0\xi}. \tag{2.60}$$

Using (2.60) and the uniform bound from Corollary 2.18, we may repeat the arguments of Theorem 2 in [17] and conclude that, up to a subsequence, there exists an $\mathbb{S}^1$-valued map w_0 s.t. for every compact $K \subset (\omega \setminus \{\alpha_1, ...\alpha_{d_0}\})$ we have

$$\hat{w}_\varepsilon \to w_0 \text{ in } C^\infty(K) \tag{2.61}$$

and

$$\frac{b^2(1-|\hat{w}_\varepsilon|^2)}{\xi^2} \to |\nabla w_0|^2 \text{ in } C^\infty(K). \tag{2.62}$$

Fix now $r < \min\left\{\frac{\min|\alpha_k - \alpha_j|}{8}, \frac{\text{dist}(\alpha_k, \partial\omega)}{8}\right\}$ and denote $\omega_r := \{x \in \omega, \text{dist}(x, \partial\omega) > r\}$. It follows from (2.61) that $\hat{w}_\varepsilon \to q_0 := \text{tr}_{\partial\omega_r} w_0$ in $C^\infty(\partial\omega_r)$. In view of Proposition 2.19, we have $w_0 \in W^{1,p}(\omega_r)$, $p < 2$. By Remark I.1 in [18], this implies that

$$w_0 = \tilde{w} \exp\left(i \sum_k c_k \ln|x - \alpha_k| + i\chi\right).$$

Here:

- $\tilde{w}$ is the *canonical harmonic map* (see [18], Sec. I.3.) having singularities $\{\alpha_k, k = 1, ..., d_0\}$ and equal to q_0 on $\partial\omega_r$;
- the c_k's are real coefficients;
- χ is the solution of

$$\begin{cases} \Delta\chi = 0 & \text{in } \omega_r \\ \chi(x) + \sum_k c_k \ln|x - \alpha_k| = 0 & \text{on } \partial\omega_r \end{cases}.$$

Repeating the argument of [18], Theorem VII.1, Step 2 (the key ingredients of this proof are (2.61), (2.62) and Corollary 2.14), we find that $c_k \equiv 0, k = 1, ..., d_0$, and, consequently, $w_0 \equiv \tilde{w}$ in ω_r. Finally, by [18], Corollary I.2., we know that the canonical harmonic map $\tilde{w}$ is of the form $\tilde{w} = e^{i(\theta+\varphi_\star)}$ with $\varphi_\star$ harmonic in ω_r.

□

2.3.7 Uniqueness of zeros

Proposition 2.21. *For ε sufficiently small, the minimizer $\hat{v}_\varepsilon$ has exactly d_0 zeros.*

Proof. It suffices to prove that for small ε there is a unique zero of $\hat{v}_\varepsilon$ in $B(\alpha_k, r)$, $k = 1, ..., d_0$, with r defined in the proof of Proposition 2.20.

Since $\hat{w}_\varepsilon = \frac{\hat{v}_\varepsilon \hat{U}_\varepsilon}{b}$, from Proposition 2.5 and Proposition 2.20 we see that $w_0 = v_0 = e^{i(\theta_k + H_k)}$ in $B(\alpha_k, r)$, where θ_k is the phase of $\frac{x-\alpha_k}{|x-\alpha_k|}$ and $H_k = \varphi_\star + \psi_k$ is harmonic in $B(\alpha_k, r)$. Using (2.61) and (2.62) and arguing as in the alternative proof of Theorem VII.4 in [18] (page 74) we obtain that $\nabla H_k(\alpha_k) = 0$.

Finally, we are now in position to obtain, as in Theorem IX.1 [18], that there is a unique zero of $\hat{w}_\varepsilon$ (and, therefore, of $\hat{v}_\varepsilon$) in $B(\alpha_k, r)$.

□

2.3.8 Summary

We have thus proved

Theorem 2.22. *Let $\varepsilon_n \downarrow 0$ and let $\hat{v}_{\varepsilon_n}$ be a minimizer for $\varepsilon = \varepsilon_n$ of* (2.17) *in* (2.18). *Up to a subsequence, there are d_0 distinct points $\alpha_1, ..., \alpha_{d_0} \in \omega$ and a function $v_0 \in H^1_{\text{loc}}(\mathbb{R}^2 \setminus \{\alpha_1, ..., \alpha_{d_0}\}, \mathbb{S}^1) \cap W^{1,p}_{\text{loc}}(\mathbb{R}^2, \mathbb{S}^1)$ $(p < 2)$ s.t.*

1. *$\hat{v}_{\varepsilon_n} \to v_0$ in $H^1_{\text{loc}}(\mathbb{R}^2 \setminus \{\alpha_1, ..., \alpha_{d_0}\})$ and $C^0_{\text{loc}}(\mathbb{R}^2 \setminus \{\alpha_1, ..., \alpha_{d_0}\})$,*
2. *$\hat{v}_{\varepsilon_n} \rightharpoonup v_0$ in $W^{1,p}_{\text{loc}}(\mathbb{R}^2)$ $(p < 2)$,*
3. *for $K \Subset \mathbb{R}^2 \backslash \{\alpha_1, ..., \alpha_{d_0}\}$, $|\hat{v}_{\varepsilon_n}| \geq 1 - |\ln \varepsilon_n|^{-1/4}$ in K and $\displaystyle\int_K |\nabla |\hat{v}_{\varepsilon_n}||^2 + \frac{1}{\xi^2}(1 - |\hat{v}_{\varepsilon_n}|^2)^2 \to 0$,*
4. *for $K \Subset \mathbb{R}^2 \setminus \overline{\omega}$, $\hat{v}_{\varepsilon_n} \to v_0$ in $C^\infty(K)$ and $1 - |\hat{v}_{\varepsilon_n}| \leq C_K \xi^2$,*
5. *$\hat{v}_{\varepsilon_n}$ has exactly d_0 zeros $x^n_1, ..., x^n_{d_0}$ and $x^n_i \to \alpha_i$,*
6. *v_0 satisfies $-\text{div}\left(a^2\, v_0 \times \nabla v_0\right) = 0$ in $\mathcal{D}'(\mathbb{R}^2)$.*

Let us summarize the proof of Theorem 2.22:

- Statement *1.* is established in Proposition 2.17,
- Statement *2.* follows from Propositions 2.19 and 2.20,
- Statement *3.* is a consequence of Corollary 2.18 and (2.54),
- Statement *4.* is Corollary 2.13,
- Statement *5.* is proved in Proposition 2.21,
- Statement *6.* is established in Proposition 2.20.

The proof of Theorem 2.22 is complete.

2.4 Renormalized energy for the model problem

In this section, we establish an expansion of $F_\varepsilon(v_\varepsilon, B(0, \rho)) = \hat{F}_\xi(\hat{v}_\varepsilon, B(0, \frac{\rho}{\delta}))$; more specifically we underline the specific form of the quantity

$$\lim_{\varepsilon} \left\{ \hat{F}_\xi(\hat{v}_\varepsilon, B(0, \frac{\rho}{\delta})) - \pi d_0^2 |\ln \delta| - \pi d_0 b^2 |\ln \xi| \right\}. \tag{2.63}$$

(We are going to prove that this limit exists).

In order to find an expression for (2.63), our strategy is the following:

- (Section 2.4.1) We first study minimal energies of the Dirichlet functional among $\mathbb{S}^1$-valued maps in annulars $B(0, \rho/\delta) \setminus \overline{B(0, 1)}$ under Dirichlet boundary conditions: $f^\delta(\delta \cdot)$ on $\partial B(0, \rho/\delta)$ and g^δ on $\partial B(0, 1)$. Here $f^\delta = \frac{f_\varepsilon}{|f_\varepsilon|}$ where f_ε is given by the model problem and $g^\delta, g^0 \in C^\infty(\partial B_1, \mathbb{S}^1)$ are s.t. $g^\delta \to g^0$ in C^1. We obtain the following expansion for a minimal energy: $\pi d_0^2 \ln(\rho/\delta) + \tilde{W}_0(f_0) + \tilde{W}_1(g^0) + o_\delta(1)$.

- (Sections 2.4.2, 2.4.3 and 2.4.4) In $B(0,1)$, we study weighted Ginzburg-Landau functional under a Dirichlet boundary condition g^δ on $\partial B(0,1)$. Making use of the previous bullet, one may obtain matching upper and lower bounds. From these estimates we complete the renormalized energy by its third term which depends on the limiting locations of the zeros $\boldsymbol{\beta}=(\beta_1,...,\beta_{d_0})\in\omega^{d_0}$ and on g^0. We establish that

$$\inf_{v\in H^1_{g^\delta}}\hat{F}_\xi(v,B(0,1))=\pi d_0b^2\ln\frac{b}{\xi}+d_0b^2\gamma+\tilde{W}_2(\boldsymbol{\beta},g^0)+o_\varepsilon(1).$$

- (Section 2.4.5) In order to conclude, we make a fundamental observation: the limiting function $g_0=\lim\mathrm{tr}_{\partial B_1}\hat{v}_\varepsilon$ and the points $\boldsymbol{\alpha}$ obtained by Theorem 2.22 form a minimal configuration for $W_1(g^0)+W_2(\boldsymbol{\beta},g^0)$. Thus we define

$$\tilde{W}(\boldsymbol{\beta})=\inf_{\substack{g^0\in C^\infty(\partial B_1,\mathbb{S}^1)\\ \text{with }\deg_{\partial B_1}(g)=d_0}}\left\{\tilde{W}_1(g^0)+\tilde{W}_2(\boldsymbol{\beta},g^0)\right\}.$$

  And we deduce that $\boldsymbol{\alpha}$ minimizes $\tilde{W}$.

In this section we prove the following theorem.

Theorem 2.23. *The following energy expansion holds when $\varepsilon\to0$*

$$\hat{F}_\xi(\hat{v}_\varepsilon,B(0,\frac{\rho}{\delta}))=\pi d_0b^2\ln\frac{b}{\xi}+\pi d_0^2\ln\frac{\rho}{\delta}+\tilde{W}_0(f_0)+\tilde{W}(\boldsymbol{\alpha})+d_0b^2\gamma+o_\varepsilon(1).\tag{2.64}$$

Here the points $\boldsymbol{\alpha}=(\alpha_1,...,\alpha_{d_0})$ are obtained from Theorem 2.22, $\gamma>0$ is an absolute constant and $\tilde{W}_0(f_0),\tilde{W}(\boldsymbol{\alpha})$ are renormalized energies:

- *$\tilde{W}_0$ is independent of the points $\alpha_1,...,\alpha_{d_0}$ and given by (2.72),*
- *$\tilde{W}$ is given by* (2.90)*, it is independent of f_0 and the limiting points $(\alpha_1,..,\alpha_{d_0})$ minimizes $\tilde{W}$.*

Remark 2.24. The renormalized energy in the expansion (2.64) decouples into the part that depends only on the external boundary conditions $\tilde{W}_0(f_0)$ and the part that depends only on the location of the vortices $\tilde{W}(\boldsymbol{\alpha})$. Since $\boldsymbol{\alpha}$ minimizes $\tilde{W}$, the external boundary data has no effect on the location of vortices inside the inclusion. This is a drastic difference with the results of [18] and [43], where the Dirichlet boundary data on the external boundary influences the location of the vortices.

2.4.1 Minimization among $\mathbb{S}^1$-valued maps away from the inclusion

Denote $B_\rho:=B(0,\rho)$. Let

$$(f^\delta)_{0<\delta<1}\subset C^\infty(\partial B_\rho,\mathbb{S}^1),\ f^0\in C^\infty(\partial B_\rho,\mathbb{S}^1)\text{ be s.t. }\begin{cases}f^\delta\underset{\delta\to0}{\to}f^0\text{ in }C^1(\partial B_\rho)\\ \deg_{\partial B_\rho}(f^\delta)=d_0\end{cases},$$

and

$$(g^\delta)_{0<\delta<1}\subset C^\infty(\partial B_1,\mathbb{S}^1),\ g^0\in C^\infty(\partial B_1,\mathbb{S}^1)\text{ be s.t. }\begin{cases}g^\delta\underset{\delta\to0}{\to}g^0\text{ in }C^1(\partial B_1)\\ \deg_{\partial B_1}(g^\delta)=d_0\end{cases}.$$

For $\delta\in(0,1)$, we denote $A_\delta=B_{\rho/\delta}\setminus\overline{B_1}$ and

$$W_\delta=\{u\in H^1(A_\delta,\mathbb{S}^1)\,|\,\mathrm{tr}_{\partial B_{\rho/\delta}}u(\cdot)=f^\delta(\delta\cdot)\text{ and }\mathrm{tr}_{\partial B_1}u=g^\delta\},$$

$$Y_\delta = \{u \in H^1(A_\delta, \mathbb{S}^1) \,|\, \mathrm{tr}_{\partial B_{\rho/\delta}} u(\cdot) = f^0(\delta \cdot) \text{ and } \mathrm{tr}_{\partial B_1} u = g^0\}.$$

Consider the following minimization problems:

$$I_\delta(f^\delta, g^\delta) = I_\delta = \inf_{u \in W_\delta} \frac{1}{2} \int_{A_\delta} |\nabla u|^2. \tag{P_δ}$$

$$J_\delta(f^0, g^0) = J_\delta = \inf_{u \in Y_\delta} \frac{1}{2} \int_{A_\delta} |\nabla u|^2. \tag{Q_δ}$$

Proposition 2.25. *For small ε, I_δ is close to J_δ, namely*

$$I_\delta = J_\delta + o_\delta(1). \tag{2.65}$$

Proof. In this subsection θ stands for the argument of z *i.e.* $\frac{z}{|z|} = \mathrm{e}^{\imath\theta}$. For $\delta \geq 0$, let $\phi_\delta \in C^\infty(\partial B_1, \mathbb{R})$ be s.t. $g^\delta = \mathrm{e}^{\imath(d_0\theta + \phi_\delta)}$ and $\zeta_\delta \in C^\infty(\partial B_\rho, \mathbb{R})$ be s.t. $f^\delta = \mathrm{e}^{\imath(d_0\theta + \zeta_\delta)}$. We may assume that $\phi_\delta \to \phi_0$ in $C^1(\partial B_1)$ and $\zeta_\delta \to \zeta_0$ in $C^1(\partial B_\rho)$. Note that

$$u \in W_\delta \Longleftrightarrow u = \mathrm{e}^{\imath(\varphi + d_0\theta)} \text{ with } \varphi \in w_\delta. \tag{2.66}$$

Here $w_\delta := \{\varphi \in H^1(A_\delta, \mathbb{R}) \,|\, \mathrm{tr}_{\partial B_{\frac{\rho}{\delta}}} \varphi(\cdot) = \zeta_\delta(\delta \cdot) \text{ and } \mathrm{tr}_{\partial B_1} \varphi = \phi_\delta\}$.

Since $\Delta\theta = 0$ in A_δ and $\partial_\nu \theta = 0$ on ∂A_δ, for $u \in W_\delta$ we have

$$\int_{A_\delta} |\nabla u|^2 = \int_{A_\delta} |\nabla(\varphi + d_0\theta)|^2 = d_0^2 \int_{A_\delta} |\nabla\theta|^2 + \int_{A_\delta} |\nabla\varphi|^2.$$

Consequently, the problem (P_δ) has a unique solution $u_\delta = \mathrm{e}^{\imath(d_0\theta + \varphi_\delta)}$, with φ_δ being the unique solution of

$$\begin{cases} -\Delta\varphi_\delta = 0 & \text{in } A_\delta \\ \varphi_\delta(\cdot) = \zeta_\delta\,(\delta \cdot) & \text{on } \partial B_{\frac{\rho}{\delta}} \\ \varphi_\delta = \phi_\delta & \text{on } \partial B_1 \end{cases}.$$

With the same argument, the problem (Q_δ) admits a unique solution $v_\delta = \mathrm{e}^{\imath(d_0\theta + \psi_\delta)}$ with ψ_δ being the unique solution of

$$\begin{cases} -\Delta\psi_\delta = 0 & \text{in } A_\delta \\ \psi_\delta(\cdot) = \zeta_0\,(\delta \cdot) & \text{on } \partial B_{\frac{\rho}{\delta}} \\ \psi_\delta = \phi_0 & \text{on } \partial B_1 \end{cases}.$$

Denote $\eta_\delta = \varphi_\delta - \psi_\delta$. Then η_δ is the unique solution of

$$\begin{cases} \Delta\eta_\delta = 0 & \text{in } A_\delta \\ \eta_\delta = \hat{\zeta}_\delta - \hat{\zeta}_0 & \text{on } \partial B_{\frac{\rho}{\delta}} \\ \eta_\delta = \phi_\delta - \phi_0 & \text{on } \partial B_1 \end{cases}.$$

(Here $\hat{\zeta}(x) := \zeta(\delta x)$).

One may prove that $\|\psi_\delta\|_{L^2(A_\delta)}$ is bounded and more precisely we have the following result.

Proposition 2.26.

$$\frac{1}{2} \int_{A_\delta} |\nabla\psi_\delta|^2 \to |\phi_0|^2_{H^{1/2}(\mathbb{S}^1)} + |\zeta_0|^2_{H^{1/2}(\partial B_\rho)}, \ \textit{as } \delta \to 0. \tag{2.67}$$

Proof. Let $(a_n)_{n\in\mathbb{Z}}, (b_n)_{n\in\mathbb{Z}} \subset \mathbb{C}$ be s.t.

$$\phi_0(e^{\imath\theta}) = \sum_{n\in\mathbb{Z}} a_n e^{\imath n\theta} \text{ and } \zeta_0(\rho e^{\imath\theta}) = \sum_{n\in\mathbb{Z}} b_n e^{\imath n\theta}.$$

We have

$$|\phi_0|^2_{H^{1/2}(\mathbb{S}^1)} = \sum_{\mathbb{Z}} |n||a_n|^2 \text{ and } |\zeta_0|^2_{H^{1/2}(\partial B_\rho)} = |\hat{\zeta}_0|^2_{H^{1/2}(\partial B_{\frac{\rho}{\delta}})} = \sum_{\mathbb{Z}} |n||b_n|^2.$$

From [13] (Appendix D.), denoting $R(\delta) = \frac{\rho}{\delta}$, we have

$$\begin{aligned}
\frac{1}{2\pi}\int_{A_\delta} |\nabla\psi_\delta|^2 &= \frac{|b_0 - a_0|^2}{\ln R(\delta)} + \sum_{n\neq 0} \frac{|n|}{R(\delta)^{2|n|}-1}\Big[(|a_n|^2+|b_n|^2)(R(\delta)^{2|n|}+1) \\
&\qquad - 2(\overline{a_n}b_n + a_n\overline{b_n})R(\delta)^{|n|}\Big] \\
&= |\phi_0|^2_{H^{1/2}(\partial B_1)} + |\psi|^2_{H^{1/2}(\partial B_\rho)} + \frac{|b_0-a_0|^2}{\ln R(\delta)} \\
&\qquad + \sum_{n\neq 0} \frac{2}{R(\delta)^{2|n|}-1}\Big[(|a_n|^2+|b_n|^2) - (\overline{a_n}b_n + a_n\overline{b_n})R(\delta)^{|n|}\Big] \\
&= |\phi_0|^2_{H^{1/2}(\partial B_1)} + |\zeta_0|^2_{H^{1/2}(\partial B_\rho)} + o_\delta(1).
\end{aligned}$$

Consequently, as $\delta \to 0$, we obtain (2.67). □

Following the same lines as Proposition 2.26 we obtain

$$\|\nabla\varphi_\delta\|_{L^2(A_\delta)} \le C \text{ with } C \text{ independent of } \delta, \tag{2.68}$$

and

$$\|\nabla\eta_\delta\|_{L^2(A_\delta)} \to 0. \tag{2.69}$$

It follows from (2.68) and (2.69) that

$$\begin{aligned}
I_\delta &= \frac{d_0^2}{2}\int_{A_\delta} |\nabla\theta|^2 + \frac{1}{2}\int_{A_\delta} |\nabla\varphi_\delta|^2 \\
&= \frac{d_0^2}{2}\int_{A_\delta} |\nabla\theta|^2 + \frac{1}{2}\int_{A_\delta} |\nabla\psi_\delta|^2 + \int_{A_\delta} \nabla\psi_\delta\cdot\nabla\eta_\delta + \frac{1}{2}\int_{A_\delta}|\nabla\eta_\delta|^2 \\
&= J_\delta + o_\delta(1).
\end{aligned} \tag{2.70}$$

□

From (2.70) and (2.67), we deduce that

$$I_\delta = J_\delta + o_\delta(1) = \pi d_0^2 \ln\frac{\rho}{\delta} + \tilde{W}_0(f^0) + \tilde{W}_1(g^0) + o_\delta(1) \tag{2.71}$$

with

$$\tilde{W}_0(f_0) = |\zeta_0|^2_{H^{1/2}(\partial B_\rho)} \text{ and } \tilde{W}_1(g^0) = |\phi_0|^2_{H^{1/2}(\partial B_1)}. \tag{2.72}$$

One of the main ingredients in the study of the renormalized energy is that the Dirichlet condition $f_{\min}(x) = \gamma_0\frac{x^{d_0}}{|x|^{d_0}}$, $\gamma_0 \in \mathbb{S}^1$ minimizes W_0. More precisely, for all $f_0 \in C^1(\partial B_1, \mathbb{S}^1)$ s.t. $\deg_{\partial B_1}(f_0) = d_0$, we have

$$W_0(f_{\min}) = 0 \le W_0(f_0). \tag{2.73}$$

2.4.2 Energy estimates for $\mathbb{S}^1$-valued maps around the inclusion

Let $g^0 \in C^\infty(\partial B_1, \mathbb{S}^1)$ be s.t. $\deg_{\partial B_1}(g^0) = d_0 > 0$, $\beta_1, ..., \beta_{d_0}$ are d_0 distinct points of ω,

$$\eta_0 := \frac{1}{4} \min_i \left\{ \operatorname{dist}(\beta_i, \partial\omega), \min_{j \neq i} |\beta_i - \beta_j| \right\}.$$

For $r \in (0, \eta_0)$, we define

$$\Omega_r := B_1 \setminus \cup_k \overline{B(\beta_k, r)},$$

$$\mathcal{E}_r := \{u \in H^1(\Omega_r, \mathbb{S}^1) \,|\, \operatorname{tr}_{\partial B_1} u = g^0 \text{ and } \deg_{\partial B(\beta_i, r)}(u) = 1\}$$

and

$$\mathcal{F}_r := \left\{ u \in H^1(\Omega_r, \mathbb{S}^1) \,|\, \operatorname{tr}_{\partial B_1} u = g^0 \text{ and there are } \gamma_i \in \mathbb{S}^1 \text{ s.t. } \operatorname{tr}_{\partial B(\beta_i, r)} u(x) = \gamma_i \frac{x - \beta_i}{|x - \beta_i|} \right\}.$$

Consider two minimization problems

$$K(r, g^0, \boldsymbol{\beta}) = K(r) = \inf_{u \in \mathcal{E}_r} \frac{1}{2} \int_{\Omega_r} a^2 |\nabla u|^2 \tag{R_r}$$

and

$$L(r, g^0, \boldsymbol{\beta}) = L(r) = \inf_{u \in \mathcal{F}_r} \frac{1}{2} \int_{\Omega_r} a^2 |\nabla u|^2, \ \boldsymbol{\beta} = \{\beta_1, ..., \beta_{d_0}\}. \tag{S_r}$$

We denote $\theta = \theta_1 + ... + \theta_{d_0}$ where θ_i is the main argument of $\frac{x - \beta_i}{|x - \beta_i|}$, *i.e.*, $\frac{x - \beta_i}{|x - \beta_i|} = e^{\imath \theta_i}$.

Let ψ_0 be the unique (up to an additive constant in $2\pi\mathbb{Z}$) solution of

$$\begin{cases} -\operatorname{div}\left[a^2(\nabla\psi_0 + \nabla\theta)\right] = 0 & \text{in } B_1 \\ e^{\imath(\theta + \psi_0)} = g^0 & \text{on } \partial B_1 \end{cases}. \tag{2.74}$$

Lemma 2.27. ([43], Appendix A.)

$$K(r) = \frac{1}{2} \int_{\Omega_r} a^2 |\nabla\theta + \nabla\psi_0|^2 + \mathcal{O}(r|\ln r|),$$

$$L(r) = \frac{1}{2} \int_{\Omega_r} a^2 |\nabla\theta + \nabla\psi_0|^2 + \mathcal{O}(r|\ln r|),$$

with

$$\frac{1}{2} \int_{\Omega_r} a^2 |\nabla\theta + \nabla\psi_0|^2 = \pi d_0 b^2 |\ln r| + \tilde{W}_2(\boldsymbol{\beta}, g^0) + \mathcal{O}(r^2). \tag{2.75}$$

In (2.75), $\tilde{W}_2(\boldsymbol{\beta}, g^0)$, whose explicit expression is given in [43], formula (106), depends only on $\boldsymbol{\beta}$ and g^0.

2.4.3 Upper bound for the energy

Lemma 2.28. *Fix $\rho > 0$ and let $f_\varepsilon \in C^\infty(\partial B_\rho)$, $f_0 \in C^\infty(\partial B_\rho, \mathbb{S}^1)$ be s.t. $f_\varepsilon \to f_0$ in $C^1(\partial B_\rho)$. Let $\boldsymbol{\beta} = (\beta_1, ..., \beta_{d_0}) \in \omega^{d_0}$ be s.t. $\beta_i \neq \beta_j$ for $i \neq j$. Then, for each $g^0 \in C^\infty(\partial B_1, \mathbb{S}^1)$, the following upper bound holds:*

$$\inf_{H^1_{f_\varepsilon}(B_\rho)} F_\varepsilon \leq \pi d_0 b^2 \ln \frac{b}{\xi} + \pi d_0^2 \ln \frac{\rho}{\delta} + \tilde{W}_0(f_0) + \tilde{W}_1(g^0) + \tilde{W}_2(\boldsymbol{\beta}, g^0) + d_0 b^2 \gamma + o_\varepsilon(1). \tag{2.76}$$

Here $\tilde{W}_0$, $\tilde{W}_1$ are defined by (2.72) and $\tilde{W}_2$ by (2.75).

Proof. We construct a test function $w_\varepsilon \in H^1_{\hat{f}_\varepsilon}(B_{\rho/\delta},\mathbb{C})$ which gives (2.76). Fix $0<r<\eta_0$. Let

$$u_\delta \text{ be the minimizer of } (P_\delta) \text{ with } g^\delta \equiv g^0 \text{ and } f^\delta = \frac{f_\varepsilon}{|f_\varepsilon|}$$

and

$$u_r \text{ be the minimizer of } (S_r).$$

Note that $f^\delta \to f_0 = \lim_\varepsilon f_\varepsilon$ in $C^1(\partial B_\rho)$. For each $i=1,...,d_0$ let $u_i^{\xi,r}$ be

- the global minimizer of the classic Ginzburg-Landau energy in $B(\beta_i,r)$
- with the parameter ξ/b
- and the boundary condition $u_i^{\xi,r}(x) = h_i^r(x) := \gamma_i\frac{x-\beta_i}{r}$ on $\partial B(\beta_i,r)$, $\gamma_i\in\mathbb{S}^1$ is given by u_r.

Denote

$$\begin{aligned} I(\xi/b,r) &:= \inf_{H^1_{h_i^r}(B(\beta_i,r))} \frac{1}{2}\int_{B(\beta_i,r)} \left\{|\nabla u|^2 + \frac{b^2}{2\xi^2}(1-|u|^2)^2\right\} \\ &= \frac{1}{2}\int_{B(\beta_i,r)} \left\{|\nabla u_i^{\xi,r}|^2 + \frac{b^2}{2\xi^2}(1-|u_i^{\xi,r}|^2)^2\right\}. \end{aligned} \tag{2.77}$$

Lemma IX.1 in [18] implies that

$$I(\xi/b,r) = \pi\ln\frac{br}{\xi} + \gamma + o_\xi(1). \tag{2.78}$$

We next extend the u_i's to $B_{\rho/\delta}$. For this purpose, we consider $\zeta\in C^\infty(\mathbb{R},[0,1])$ s.t. $\zeta=0$ in $\mathbb{R}^-$ and $\zeta=1$ in $[1,\infty)$ and set

$$\chi_\varepsilon(se^{\imath\theta}) = \zeta\left(s-\frac{\rho}{\delta}+1\right)\left[|f_\varepsilon|(\rho e^{\imath\theta})-1\right]+1.$$

In view of (2.14), we have $\|\chi_\varepsilon - 1\|_{L^2(B_{\rho/\delta})} \le C\varepsilon$. Consider the following test function

$$\tilde{u} = \begin{cases} \chi_\varepsilon u_\delta & \text{in } B_{\rho/\delta}\setminus\overline{B_1} \\ u_r & \text{in } B_1\setminus\cup\overline{B(\beta_i,r)} \\ u_i^{\xi,r} & \text{in } B(\beta_i,r) \end{cases}. \tag{2.79}$$

Clearly,

$$\inf_{H^1_{f_\varepsilon}} F_\varepsilon \le \hat{F}_\xi(w_\varepsilon) \le \pi d_0^2\ln\frac{\rho}{\delta} + \tilde{W}_1(g^0) + \tilde{W}_2(\boldsymbol{\beta},g^0) + \tilde{W}_0(f_0) + \pi d_0 b^2\ln\frac{b}{\xi} + d_0b^2\gamma + o_\varepsilon(1) + h(r)$$

with $h(r)=o_r(1)$. Thus, letting $r\to0$ as $\varepsilon\to0$ we obtain the result. □

2.4.4 Lower bound

We prove that the upper bound (2.76) is sharp by constructing the matching lower bound.

Lemma 2.29. *For $\varepsilon=\varepsilon_n\downarrow0$, up to a subsequence, one may consider $\hat{v}_\varepsilon$ and $\boldsymbol{\alpha}=(\alpha_1,...,\alpha_{d_0})\in\omega^{d_0}$ as in Theorem 2.22.*

Let also $g_0 := \lim \operatorname{tr}_{\partial B_1}\hat{v}_\varepsilon \in C^\infty(\partial B_1,\mathbb{S}^1)$. Then, the following lower bound holds:

$$F(\hat{v}_\varepsilon, B_{\frac{\rho}{\delta}}) \ge \pi d_0 b^2\ln\frac{b}{\xi} + \pi d_0^2\ln\frac{\rho}{\delta} + \tilde{W}_0(f_0) + \tilde{W}_1(g_0) + \tilde{W}_2(\boldsymbol{\alpha},g_0) + d_0b^2\gamma + o_\varepsilon(1). \tag{2.80}$$

Proof. As in the proof of Lemma 2.28, we split $B_{\frac{\varrho}{\delta}}$ into three parts: $B_{\frac{\varrho}{\delta}} \setminus \overline{B_1}$, $B_1 \setminus \cup \overline{B(\alpha_i, r)}$ and $\cup B(\alpha_i, r)$. Here, $0 < r < \eta_0$ is small.

In $B_{\frac{\varrho}{\delta}} \setminus \cup \overline{B(\alpha_i, r)}$, write $\hat{v}_\varepsilon = |\hat{v}_\varepsilon| w_\varepsilon$. Using Corollary 2.18 and (2.44) we have

$$\begin{aligned}
\hat{F}_\xi(\hat{v}_\varepsilon, B_{\frac{\varrho}{\delta}} \setminus \overline{B_1}) &= \frac{1}{2} \int_{B_{\frac{\varrho}{\delta}} \setminus \overline{B_1}} \left\{ \hat{U}_\varepsilon^2 |\hat{v}_\varepsilon|^2 |\nabla w_\varepsilon|^2 + \hat{U}_\varepsilon^2 |\nabla |\hat{v}_\varepsilon||^2 + \frac{\hat{U}_\varepsilon^4}{2\xi^2}(1 - |\hat{v}_\varepsilon|^2)^2 \right\} \\
&= \frac{1}{2} \int_{B_{\frac{\varrho}{\delta}} \setminus \overline{B_1}} \left\{ \hat{U}_\varepsilon^2 |\nabla w_\varepsilon|^2 + \hat{U}_\varepsilon^2 |\nabla |\hat{v}_\varepsilon||^2 + \frac{\hat{U}_\varepsilon^4}{2\xi^2}(1 - |\hat{v}_\varepsilon|^2)^2 \right\} + o_\varepsilon(1) \\
&\geq \frac{1}{2} \int_{B_{\frac{\varrho}{\delta}} \setminus \overline{B_1}} \hat{U}_\varepsilon^2 |\nabla w_\varepsilon|^2 + o_\varepsilon(1). \qquad (2.81)
\end{aligned}$$

We take $g^\delta = \dfrac{\mathrm{tr}_{\partial B_1} \hat{v}_\varepsilon}{|\mathrm{tr}_{\partial B_1} \hat{v}_\varepsilon|}$ and $f^\delta = \dfrac{\mathrm{tr}_{\partial B_\rho} v_\varepsilon}{|\mathrm{tr}_{\partial B_\rho} v_\varepsilon|}$. Note that with this choice of f^δ, g^δ one may apply the results of Sections 2.4.1 and 2.4.2.

From (2.81) we obtain the lower bound in $B_{\frac{\varrho}{\delta}} \setminus \overline{B_1}$:

$$\hat{F}_\xi(\hat{v}_\varepsilon, B_{\frac{\varrho}{\delta}} \setminus \overline{B_1}) \geq J_\delta + o_\varepsilon(1) \qquad (2.82)$$

with J_δ the energy associate to the minimization problem (Q_δ).

Let v_0 be defined by (2.44). Since we have $v_\varepsilon \to v_0$ in $H^1(B_1 \setminus \cup \overline{B(\alpha_i, r)})$ and $\hat{U}_\varepsilon \to a$ in $L^2(B_1 \setminus \cup \overline{B(\alpha_i, r)})$, we obtain from Proposition 2.20 and Lemma 2.27

$$\begin{aligned}
\hat{F}_\xi(\hat{v}_\varepsilon, B_1 \setminus \cup \overline{B(\alpha_i, r)}) &\geq \frac{1}{2} \int_{B_1 \setminus \cup \overline{B(\alpha_i, r)}} a^2 |\nabla v_0|^2 + o_\varepsilon(1) \\
&\geq \frac{1}{2} \int_{B_1 \setminus \cup \overline{B(\alpha_i, r)}} a^2 |\nabla \theta + \nabla \psi_0|^2 + o_\varepsilon(1) \\
&= K(r) + \mathcal{O}(r |\ln r|) + o_\varepsilon(1), \qquad (2.83)
\end{aligned}$$

here $K(r)$ is defined by (R_r).

In order to complete the proof of the lemma, we need to obtain a sharp lower bound in each ball $B(\alpha_i, r)$. Actually we will prove that

$$\hat{F}_\xi(\hat{v}_\varepsilon, B(\alpha_i, r)) \geq b^2 I(\xi/b, r) + o_r(1) + o_\varepsilon(1), \qquad (2.84)$$

here, $I(\xi/b, r)$ is defined in (2.77).

Estimate (2.84) is equivalent to

$$\hat{F}_\xi(\hat{v}_\varepsilon, B(\alpha_i, r)) \geq b^2 I(\xi/b, r + r^2) + o_r(1) + o_\varepsilon(1). \qquad (2.85)$$

Indeed by (2.78) we have $I(\xi, r + r^2) - I(\xi, r) = o_r(1)$.

The construction we use below was made by Lefter and Rădulescu in [45] and [44]. From Proposition 2.20, we know that $v_0 = \mathrm{e}^{\imath(\theta_i + \varphi_\star + \psi_i)}$ with $\varphi_\star$, ψ_i harmonic, and therefore smooth in $B(\alpha_i, \eta)$ ($\eta > r$ small). Set $\sigma_i = \varphi_\star + \psi_i$. Without loss of generality, we can assume that $\alpha_i = 0$ and $\sigma_i(0) = 0$. Consequently, $|\sigma_i(x)| \leq C|x|$ with C independent of η and $|x| \leq \eta$. Let

$$\hat{v}_\varepsilon = \lambda_\varepsilon \mathrm{e}^{\imath(\theta_i + \sigma_\varepsilon^i)} \text{ where } \lambda_\varepsilon := |\hat{v}_\varepsilon|.$$

From Proposition 2.17 and (2.54), we obtain that

$$\sigma_\varepsilon^i \to \sigma_i \text{ in } H^1(B_{r+r^2} \setminus \overline{B_r}), \qquad (2.86)$$

$$\lambda_\varepsilon \to 1 \text{ in } H^1(B_{r+r^2} \setminus \overline{B_r}) \text{ and } \frac{1}{\xi^2} \int_{B_{r+r^2} \setminus \overline{B_r}} (1-\lambda_\varepsilon)^2 \to 0. \tag{2.87}$$

Let

$$\beta_\varepsilon(se^{\imath\theta_i}) = \begin{cases} \hat{v}_\varepsilon(se^{\imath\theta_i}) & \text{if } s \in [0,r) \\ \left[\frac{1-\lambda_\varepsilon}{r^2}(s-r)+\lambda_\varepsilon\right] \exp\left\{\imath\left(\theta_i + \sigma_\varepsilon^i \frac{-s+r^2+r}{r^2}\right)\right\} & \text{if } s \in [r, r+r^2] \end{cases}.$$

Clearly, $\beta_\varepsilon \in H^1_{x/|x|}(B_{r+r^2})$. Consequently,

$$b^2 I(\xi/b, r+r^2) \leq \hat{F}_\xi(\hat{v}_\varepsilon, B_r) + \hat{F}_\xi(\beta_\varepsilon, B_{r+r^2} \setminus \overline{B_r}) + o_\varepsilon(1).$$

From (2.87), we easily obtain that

$$\int_{B_{r+r^2} \setminus \overline{B_r}} \left\{ |\nabla|\beta_\varepsilon||^2 + \frac{1}{\xi^2}(1-|\beta_\varepsilon|)^2 \right\} = o_\varepsilon(1).$$

It remains to estimate

$$\int_{B_{r+r^2} \setminus \overline{B_r}} \left| \nabla \left\{ \theta_i + \sigma_\varepsilon^i \frac{-s+r^2+r}{r^2} \right\} \right|^2.$$

From (2.86)

$$\int_{B_{r+r^2} \setminus \overline{B_r}} \left| \nabla \left\{ \theta_i + \sigma_\varepsilon^i \frac{-s+r^2+r}{r^2} \right\} \right|^2 = \int_{B_{r+r^2} \setminus \overline{B_r}} \left| \nabla \left\{ \theta_i + \sigma_i \frac{-s+r^2+r}{r^2} \right\} \right|^2 + o_\varepsilon(1).$$

Since $|\sigma_i(se^{\imath\theta})| \leq Cs$, $|\partial_s \sigma_i| \leq C$ and $|\partial_{\theta_i}\sigma_i| \leq Cs$ we have

$$\begin{aligned} \left| \nabla \left\{ \theta_i + \sigma_i \frac{-s+r^2+r}{r^2} \right\} \right|^2 &= \left| \partial_s \sigma_i \frac{-s+r^2+r}{r^2} - \frac{\sigma_i}{r^2} \right|^2 + \frac{1}{r^2} \left| 1 + \partial_{\theta_i}\sigma_i \frac{-s+r^2+r}{r^2} \right|^2 \\ &\leq C\left[(1+r^{-2}) + r^{-2}\right] = \mathcal{O}(r^{-2}). \end{aligned}$$

Since $|B_{r+r^2} \setminus \overline{B_r}| = \mathcal{O}(r^3)$ we find that

$$\int_{B_{r+r^2} \setminus \overline{B_r}} \left| \nabla \left\{ \theta_i + \sigma_\varepsilon^i \frac{-s+r^2+r}{r^2} \right\} \right|^2 = \mathcal{O}(r).$$

It follows that $\hat{F}_\xi(\beta_\varepsilon, B_{r+r^2} \setminus \overline{B_r}) = \mathcal{O}(r) + o_\varepsilon(1)$. Consequently, (2.85) holds and thus we obtain (2.84). Combining (2.82), (2.83) and (2.84), together with (2.71) and (2.75), we obtain

$$\begin{aligned} \hat{F}_\xi(\hat{v}_\varepsilon, B_{\frac{\rho}{\delta}}) &\geq I_\delta + K(r) + b^2 I(\xi/b, r) + o_\varepsilon(1) + o_r(1) \\ &= \pi d_0^2 \ln \frac{\rho}{\delta} + \pi d_0 b^2 \ln \frac{b}{\xi} + \tilde{W}_0(f_0) + \tilde{W}(\boldsymbol{\alpha}, g_0) + \\ &\qquad + d_0 b^2 \gamma + o_\varepsilon(1) + o_r(1). \end{aligned} \tag{2.88}$$

Letting $r \to 0$ as $\varepsilon \to 0$ we obtain the result. □

2.4.5 The function g_0 and the points $\{\alpha_1, ..., \alpha_{d_0}\}$ minimize the renormalized energy

In the previous section, we obtained an expansion for the energy of the model problem: $\hat{F}_\xi(\hat{v}_\varepsilon, B_{\frac{\rho}{\delta}})$.

Let us summarize: using (2.76) with (2.80) we obtain for $\varepsilon = \varepsilon_n \downarrow 0$, up to a subsequence, by Theorem 2.22, there are $g_0 = \lim \mathrm{tr}_{\partial B_1} \hat{v}_\varepsilon$ and $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_{d_0}) \in \omega^{d_0}$ s.t.

$$\hat{F}_\xi(\hat{v}_\varepsilon, B_{\frac{\rho}{\delta}}) = \pi d_0 b^2 \ln \frac{b}{\xi} + \pi d_0^2 \ln \frac{\rho}{\delta} + \tilde{W}(\boldsymbol{\alpha}, g_0) + \tilde{W}_0(f_0) + d_0 b^2 \gamma + o_\varepsilon(1), \tag{2.89}$$

with

$$\tilde{W}(\boldsymbol{\alpha}, g_0) = \tilde{W}_1(g_0) + \tilde{W}_2(\boldsymbol{\alpha}, g_0).$$

The goal of this section is to underline an important property of the points $\boldsymbol{\alpha}$: it minimizes the quantity $\inf_{g^0 \in C^\infty(\partial B_1, \mathbb{S}^1)} \tilde{W}(\cdot, g^0)$.

We have the following

Proposition 2.30. *Let $\boldsymbol{\beta} = (\beta_1, ..., \beta_{d_0}) \in \omega^{d_0}$ be a d_0-tuple of distinct points and let $g^0 \in C^\infty(\partial B_1, \mathbb{S}^1)$ be s.t. $\deg_{\partial B_1}(g^0) = d_0$. Then*

$$\tilde{W}(\boldsymbol{\alpha}, g_0) \leq \tilde{W}(\boldsymbol{\beta}, g^0).$$

Proof. Let $(\boldsymbol{\beta}, g^0)$ be as in Proposition 2.30.

Using the test function given by (2.79), we obtain that for all $\varepsilon > 0$ and $r > 0$ (small) there is $w_\varepsilon \in H^1_{\hat{f}_\varepsilon}(B_{\frac{\rho}{\delta}}, \mathbb{C})$ s.t.

$$\hat{F}_\xi(w_\varepsilon) = \pi d_0 b^2 \ln \frac{b}{\xi} + \pi d_0^2 \ln \frac{\rho}{\delta} + \tilde{W}(\boldsymbol{\beta}, g^0) + \tilde{W}_0(f_0) + d_0 b^2 \gamma + h^1_\varepsilon + h^2_r$$

here $h^1_\varepsilon = o_\varepsilon(1)$ and $h^2_r = \mathcal{O}(r)$.

On the other hand, for a sequence $\varepsilon = \varepsilon_n \downarrow 0$, by Theorem 2.22, from (2.89) we have

$$\tilde{W}(\boldsymbol{\beta}, g^0) \geq \tilde{W}(\boldsymbol{\alpha}, g_0) + o_\varepsilon(1) + h^2_r.$$

Previous estimate implies (letting $\varepsilon \to 0$ and $r \to 0$) that $\tilde{W}(\boldsymbol{\beta}, g^0) \geq \tilde{W}(\boldsymbol{\alpha}, g_0)$ which ends the proof. □

Thus, for $\boldsymbol{\beta} = (\beta_1, ..., \beta_{d_0}) \in \omega^{d_0}$ we define

$$\tilde{W}(\boldsymbol{\beta}) = \inf_{\substack{\tilde{g} \in C^\infty(\partial B_1, \mathbb{S}^1) \\ \deg_{\partial B_1}(\tilde{g}) = d_0}} \tilde{W}(\boldsymbol{\beta}, \tilde{g}) = \inf_{\substack{\tilde{g} \in C^\infty(\partial B_1, \mathbb{S}^1) \\ \deg_{\partial B_1}(\tilde{g}) = d_0}} \tilde{W}_1(\tilde{g}) + \tilde{W}_2(\boldsymbol{\beta}, \tilde{g}) \tag{2.90}$$

with $\tilde{W}_1$ and $\tilde{W}_2$ given by (2.72) and (2.75) respectively. It follows that for $\boldsymbol{\alpha}$ given by Theorem 2.22 and $g_0 = \mathrm{tr}_{\partial B_1} v_0$:

$$\tilde{W}(\boldsymbol{\alpha}) = \tilde{W}(\boldsymbol{\alpha}, g_0) \leq \tilde{W}(\boldsymbol{\beta}) \text{ for all } \boldsymbol{\beta} = (\beta_1, ..., \beta_{d_0}) \in \omega^{d_0}.$$

2.5 Proofs of Theorems 2.1 and 2.3

In this section v_ε is a minimizer of F_ε in $H^1_g(\Omega, \mathbb{C})$. We divide the proofs of Theorem 2.1 and 2.3 in three steps:

- (Section 2.5.1) Using estimates on $|v_\varepsilon|$, we first localize the vorticity in a neighborhood of a selection of inclusions. Next we give an energetic decomposition (with a bounded error term) dividing the domain Ω in two parts: outside a neighborhood of the selected inclusions and inside a neighborhood of the selected inclusions.

- (Section 2.5.2) We study the asymptotic behavior of v_ε, we prove that, for small ε, v_ε admits exactly d zeros and all the zeros have a degree equal to 1.

- (Section 2.5.3) We gives an expansion of $F_\varepsilon(v_\varepsilon)$ with an error term $o_\varepsilon(1)$ and we prove that the selection of inclusions which contain the vorticity is related to the renormalized energy of Bethuel, Brezis and Hélein.

2.5.1 Locating bad inclusions

The following result gives a uniform bound on the modulus of minimizers away from the inclusions.

Lemma 2.31. *There exists $C > 0$ s.t. for small ε we have*

1. $|v_\varepsilon| \geq 1 - C|\ln\varepsilon|^{-1/3}$ *in* $\Omega \setminus \cup_{i=1}^M \overline{B(a_i,\delta)}$,

2. *there are at most d points $a_{i_1}, ..., a_{i_{d'}}$ ($1 \leq d' = d'_\varepsilon \leq d$) s.t.* $\{|v_\varepsilon| < 1 - C|\ln\varepsilon|^{-1/3}\} \subset \cup_{k=1}^{d'} B(a_{i_k},\delta)$.

Proof. Using Lemma 2.7 with $\chi = |\ln\varepsilon|^{-1/3}$, we obtain that there exist $C, C_1 > 0$ s.t. for $\varepsilon > 0$ small,

$$\text{if } F_\varepsilon(v_\varepsilon, B(x,\varepsilon^{1/4})) < |\ln\varepsilon|^{\frac{1}{3}} - C_1 \text{ then } |v_\varepsilon| \geq 1 - C\chi \text{ in } B(x,\varepsilon^{1/2}).$$

We prove 1. by contradiction. Assume that, up to a subsequence, there is $x_\varepsilon \in \Omega \setminus \cup_{i=1}^M \overline{B(a_i,\delta)}$, s.t. $|v_\varepsilon(x_\varepsilon)| < 1 - C|\ln\varepsilon|^{-1/3}$ with C given by Lemma 2.7.

From Lemma 2.7 and Proposition 2.5

$$\frac{1}{2}\int_{B(x_\varepsilon,\varepsilon^{1/4})} \left\{ |\nabla v_\varepsilon|^2 + \frac{1}{2\varepsilon^2}(1-|v_\varepsilon|^2)^2 \right\} \geq |\ln\varepsilon|^{1/3} - \mathcal{O}(1). \tag{2.91}$$

Fix a bounded, simply connected domain Ω' such that $\overline{\Omega} \subset \Omega'$, and extend v_ε by a fixed smooth $\mathbb{S}^1$-valued map v in $\Omega' \setminus \overline{\Omega}$, s.t. $v = g$ on $\partial\Omega$.

In view of (2.11) for Case I or (2.12) for Case II, there exists $\tilde{C} > 0$ s.t. for small ε

$$\frac{1}{2}\int_{\Omega'} |\nabla v_\varepsilon|^2 + \frac{1}{2\varepsilon^2}(1-|v_\varepsilon|^2)^2 \leq \tilde{C}|\ln\varepsilon|.$$

Therefore, the map v_ε in Ω' satisfies the condition of Theorem 4.1 [65]. This theorem guarantees that

- there exists $\mathcal{B}^\varepsilon = \{B_j^\varepsilon\}$, a finite disjoint covering of the set

$$\{x \in \Omega' \,|\, \text{dist}(x,\partial\Omega') > \varepsilon/b \text{ and } |v_\varepsilon(x)| < 1 - (\varepsilon/b)^{1/8}\},$$

- such that $\text{rad}(\mathcal{B}^\varepsilon) := \sum_j \text{rad}(B_j^\varepsilon) \leq 10^{-2} \cdot \text{dist}(\omega, \partial B(0,1)) \cdot \delta$,

- and denoting $d_j = |\deg_{\partial B_j}(v_\varepsilon)|$ if $B_j^\varepsilon \subset \{\text{dist}(x,\partial\Omega') > \varepsilon/b\}$ and $d_j = 0$ otherwise we have

$$\begin{aligned}\frac{1}{2}\int_{\cup B_j^\varepsilon} |\nabla v_\varepsilon|^2 + \frac{b^2}{2\varepsilon^2}(1-|v_\varepsilon|^2)^2 &\geq \pi\sum_j d_j \ln\frac{\delta}{\varepsilon} - C \\ &= \pi\sum_j d_j|\ln\xi| - C, \end{aligned} \tag{2.92}$$

  with C independent of ε.

Note that since $|v_\varepsilon| \equiv 1$ in $\Omega' \setminus \overline{\Omega}$, if $d_j \neq 0$ then $B_j^\varepsilon \subset \{\text{dist}(x,\partial\Omega') > \varepsilon/b\}$. Consequently, we have $d_j = |\deg_{\partial B_j^\varepsilon}(v_\varepsilon)|$.

Assertion 1. follows as in the proof of Proposition 2.9 (use (2.91), (2.92) instead of (2.20) and (2.22)).

The proof of Assertion *2.* of Lemma 2.31 goes along the same lines. □

We next obtain some lower bounds for the energy.

Lemma 2.32. *For $k \in \{1,...,d'\}$, we denote $d_k = d_k^\varepsilon = \deg_{\partial B(a_{i_k},\delta)}(v_\varepsilon)$. There exist $C, \eta_0 > 0$ s.t. for small ε and $\rho \in [2\delta, \eta_0]$ we have*

$$\frac{1}{2}\int_{\Omega\setminus\cup_{k=1}^d \overline{B(a_{i_k},\rho)}} |\nabla v_\varepsilon|^2 \geq \pi\sum_{k=1}^d d_k^2|\ln\rho| - C \tag{2.93}$$

and

$$F_\varepsilon(v_\varepsilon, B(a_{i_k}, 2\delta)) \geq \pi|d_k|b^2|\ln\xi| - C. \tag{2.94}$$

Proof. Let $\eta_0 = 10^{-2}\min_i\{\text{dist}(a_i,\partial\Omega), \min_{j\neq i}|a_i - a_j|\}$ and $0 < \rho < \eta_0$.

We prove (2.93). By Lemma 2.31, $|v_\varepsilon| \geq 1/2$ in $\Omega\setminus\cup_{k=1}^d\overline{B(a_{i_k},\rho)}$, therefore, $w_\varepsilon = \frac{v_\varepsilon}{|v_\varepsilon|}$ is well-defined in this domain. From direct computations in $B(a_{i_k},\eta_0)\setminus\overline{B(a_{i_k},\rho)}$ we have

$$\frac{1}{2}\int_{\Omega\setminus\cup_{k=1}^d \overline{B(a_{i_k},\rho)}} |\nabla w_\varepsilon|^2 \geq \pi\sum_{i=1}^d d_i^2\ln\frac{\eta_0}{\rho}. \tag{2.95}$$

We claim that the bound (2.93) holds with $C = |\ln\eta_0| + 1$. Argue by contradiction: assume that up to a subsequence we have:

$$\frac{1}{2}\int_{\Omega\setminus\cup_{k=1}^d \overline{B(a_{i_k},\rho)}} |\nabla v_\varepsilon|^2 \leq \pi\sum_{i=1}^d d_i^2\ln\frac{\eta_0}{\rho} - 1. \tag{2.96}$$

On the other hand, we have

$$|\nabla v_\varepsilon|^2 = |v_\varepsilon|^2|\nabla w_\varepsilon|^2 + |\nabla|v_\varepsilon||^2$$

and therefore

$$\int_{\Omega\setminus\cup_{k=1}^d \overline{B(a_{i_k},\rho)}} |\nabla v_\varepsilon|^2 \geq \int_{\Omega\setminus\cup_{k=1}^d \overline{B(a_{i_k},\rho)}} |\nabla w_\varepsilon|^2 - (1-|v_\varepsilon|^2)|\nabla w_\varepsilon|^2. \tag{2.97}$$

Using the fact that $|v_\varepsilon| \geq \frac{1}{2}$ in $\Omega\setminus\cup_{k=1}^d\overline{B(a_{i_k},\rho)}$ we see that $|\nabla w_\varepsilon| \leq 2|\nabla v_\varepsilon|$. Therefore, by (2.96), (H) and Lemma 2.31 we estimate the last term in (2.97):

$$\int_{\Omega\setminus\cup_{k=1}^d \overline{B(a_{i_k},\rho)}} (1-|v_\varepsilon|^2)|\nabla w_\varepsilon|^2 \leq C|\ln\varepsilon|^{-\frac{1}{3}}\int_\Omega |\nabla v_\varepsilon|^2 \leq C\frac{|\ln\rho|}{|\ln\varepsilon|^{\frac{1}{3}}} \to 0. \tag{2.98}$$

By combining (2.95), (2.97) and (2.98), we see that (2.96) cannot hold for small ε; this implies (2.93).
We now prove (2.94). Performing the rescaling $\hat{x} = \frac{x-a_{i_k}}{\delta}$, we obtain

$$F_\varepsilon(v, B(a_{i_k}, 2\delta)) = \hat{F}_\xi(\hat{v}, B(0,2)) = \frac{1}{2}\int_{B(0,2)} \left\{ \hat{U}_\varepsilon^2 |\nabla \hat{v}|^2 + \frac{1}{2\xi^2}\hat{U}_\varepsilon^4 (1-|\hat{v}|^2)^2 \right\} \mathrm{d}\hat{x},$$

where, as in the model problem we set $\hat{v}(\hat{x}) = v(\delta\hat{x})$ and $\xi = \dfrac{\varepsilon}{\delta}$.

By Theorem 4.1 [65], for $r = 10^{-2}$ there are $C > 0$ and a finite covering by disjoint balls $B_1, ..., B_N$ (with the sum of radii at most r) of $\{\hat{x} \in B(0, 2-\xi/b) \,|\, 1 - |\hat{v}_\varepsilon(\hat{x})| \geq (\xi/b)^{1/8}\}$ and

$$\frac{1}{2}\int_{\cup_j B_j} \left\{ |\nabla \hat{v}_\varepsilon|^2 + \frac{b^2}{2\xi^2}(1-|\hat{v}_\varepsilon|^2)^2 \right\} \geq \pi D_k |\ln \xi| - C, \tag{2.99}$$

$D_k = \sum_j |m_j|$ and

$$m_j = \begin{cases} \deg_{\partial B_j}(\hat{v}_\varepsilon) & \text{if } \operatorname{dist}(B_j, \partial B(0,2)) \geq \xi/b \\ 0 & \text{otherwise} \end{cases}.$$

Since, by Lemma 2.31, $|\hat{v}_\varepsilon| \geq 1/2$ in $B(0,2) \setminus \overline{B(0,1)}$, $D_k \geq d_k$, and (2.94) follows from (2.99) and the estimate $U_\varepsilon \geq b$.

□

Corollary 2.33. *Assume that $M \geq d$. Then $d' = d$ and $d_k = 1$ for each k.*

Corollary 2.34. *Assume that $M < d$. Then $d' = M$ and $d_k \in \left\{ \left[\dfrac{d}{M}\right], \left[\dfrac{d}{M}\right] + 1 \right\}$ for each k.*

Proof of Coroallary 2.33 and 2.34. By combining (2.93) and (2.94) we obtain the lower bound for F_ε in Ω:

$$F_\varepsilon(v_\varepsilon) \geq \pi \sum_{i=1}^{M} \left\{ \deg_{\partial B(a_i,\delta)}(v_\varepsilon)^2 |\ln \delta| + b^2 |\deg_{\partial B(a_i,\delta)}(v_\varepsilon)| |\ln \xi| \right\} - C_1. \tag{2.100}$$

The conclusions of the above corollaries are obtained by solving the discrete minimization problem (2.100). □

As a direct consequence of Proposition 2.6 and Lemma 2.32, we have

Corollary 2.35. *There is $C > 0$ independent of ε s.t. for $1 > \rho > 2\delta$ we have*

$$\begin{aligned} \frac{1}{2}\int_{\Omega \setminus \cup_{k=1}^{d} \overline{B(a_{i_k}, \rho)}} |\nabla v_\varepsilon|^2 \,\mathrm{d}x &= \pi \sum_{k=1}^{d'} d_k^2 |\ln \rho| + \mathcal{O}(1) \\ &= \begin{cases} \pi d |\ln \rho| + \mathcal{O}(1) & \textit{in Case I} \\ \pi \min\limits_{\substack{\tilde{d}_1,...,\tilde{d}_M \in \mathbb{Z} \\ \tilde{d}_1+...+\tilde{d}_M = d}} \displaystyle\sum_{i=1}^{M} \tilde{d}_i^2 |\ln \rho| + \mathcal{O}(1) & \textit{in Case II} \end{cases}. \end{aligned}$$

2.5.2 Existence of limiting solution

We now return to the proof of Theorems 2.1 and 2.3.

Recall that $\{i_1^\varepsilon, ..., i_{d'}^\varepsilon\}$ is a set of distinct elements of $\{1, ..., M\}$. Since, up to a subsequence, this set does not depend on ε, we may simply denote it by $\{i_1, ..., i_{d'}\}$. We keep in mind the fact that we already selected the ε's s.t. i_j is independent of ε. In Case I, we have $d' = d$ and we may assume that $\{i_1, ..., i_{d'}\} = \{1, ..., d\}$. In Case II, we have $d' = M$.

Lemma 2.31 and Corollary 2.35 imply that for an appropriate extraction $\varepsilon = \varepsilon_n \downarrow 0$ and for a compact $K \subset \Omega \setminus \{a_{i_1}, ..., a_{i_{d'}}\}$, there is $C_K > 0$ s.t. for small ε we have

$$F_\varepsilon(v_\varepsilon, K) \leq C_K$$

and

$$|v_\varepsilon(x)| \geq 1 - C|\ln\varepsilon|^{-1/3} \text{ for all } x \in K.$$

Therefore, when $\varepsilon \to 0$, up to a subsequence, there exists $v^* \in H^1(\Omega \setminus \{a_{i_1}, ..., a_{i_{d'}}\}, \mathbb{S}^1)$ s.t. $v_\varepsilon \rightharpoonup v^* \in H^1_{\text{loc}}(\Omega \setminus \{a_{i_1}, ..., a_{i_{d'}}\})$.

We now fix such a sequence and a compact $K \subset \Omega \setminus \{a_{i_1}, ..., a_{i_{d'}}\}$. If $K \subset \Omega \setminus \{a_i, 1 \leq i \leq M\}$, then we have $K \cap \omega_\delta = \emptyset$ for small ε. By exactly the same argument as in Proposition 2.12 we deduce that v_ε is bounded in $C^k(K)$ for all $k \geq 0$ and $1 - |v_\varepsilon|^2 \leq C_K\varepsilon^2$ in K.

Consequently, up to subsequence we have for a compact set $K \subset \Omega \setminus \{a_1, ..., a_M\}$

$$v_\varepsilon \to v^* \text{ in } C^k(K) \text{ and } 1 - |v_\varepsilon|^2 \leq C_K\varepsilon^2. \tag{2.101}$$

Now, assume that K is s.t. $K \subset \Omega \setminus \{a_{i_1}, ..., a_{i_{d'}}\}$ but $K \cap \omega_\delta \neq \emptyset$ (then we are in Case I). Without loss of generality, assume $K = \overline{B(a_{k_0}, R)}$, where $a_{k_0} \in \{a_{d+1}, ..., a_M\}$ and $R > 0$ is sufficiently small in order to have $K \cap \{a_1, ..., a_M\} = \{a_{k_0}\}$.

Let $h_\varepsilon := \text{tr}_{\partial K} v_\varepsilon$. Since $\partial K \subset \Omega \setminus \{a_1, ..., a_M\}$, we have $h_\varepsilon \to h_0$ in $C^\infty(\partial K)$ (possibly after passing to a subsequence). Since $\deg(h_\varepsilon, \partial K) = 0$ we have $\deg(h_0, \partial K) = 0$ and consequently there is some $\varphi_0 \in C^\infty(\partial K, \mathbb{R})$ s.t. $h_0 = e^{i\varphi_0}$.

Let $\tilde{v}$ be a minimizer of $\displaystyle\int_K |\nabla v|^2$ in the class $H^1_{h_0}(K, \mathbb{S}^1)$. Clearly,

$$\int_K |\nabla \tilde{v}|^2 \leq \int_K |\nabla v^*|^2.$$

On the other hand, since $U_\varepsilon \leq 1$, we may construct (in the spirit of [17]) a test function in construction by interpolation and find that (see formula (93) in [17])

$$F_\varepsilon(v_\varepsilon, K) \leq \frac{1}{2}\int_K |\nabla\psi_\varepsilon|^2 + C\varepsilon, \tag{2.102}$$

where ψ_ε is the solution of

$$\begin{cases} \Delta\psi_\varepsilon = 0 & \text{in } K \\ \psi_\varepsilon = \varphi_\varepsilon & \text{on } \partial K \end{cases}.$$

Here, φ_ε is defined by

$$e^{i\varphi_\varepsilon} = \frac{h_\varepsilon}{|h_\varepsilon|} \text{ on } \partial K.$$

As $\varepsilon \to 0$, we have

$$\psi_\varepsilon \to \psi_0 \text{ strongly in } H^1(K), \text{ where } \begin{cases} \Delta\psi_0 = 0 & \text{in } K \\ \psi_0 = \varphi_0 & \text{on } \partial K \end{cases}. \tag{2.103}$$

From the fact that $v_\varepsilon \rightharpoonup v_*$ in $L^2(K)$, $U_\varepsilon \to 1$ in $L^2(K)$ and $|U_\varepsilon| \leq 1$ we have $U_\varepsilon^2 \nabla v_\varepsilon \rightharpoonup v^*$ in $L^2(K)$. Consequently, we obtain

$$\frac{1}{2}\int_K |\nabla v^*|^2 \leq \liminf_{\varepsilon\to 0} \frac{1}{2}\int_K U_\varepsilon^2 |\nabla v_\varepsilon|^2 \leq \liminf_{\varepsilon\to 0} F_\varepsilon(v_\varepsilon, K). \tag{2.104}$$

Combining (2.102), (2.103) and (2.104) we deduce that

$$\int_K |\nabla v^*|^2 \leq \int_K |\nabla \psi_0|^2 = \int_K |\nabla \tilde{v}|^2.$$

It follows that v^* minimizes the Dirichlet functional in

$$H^1_{h_0}(K, \mathbb{S}^1) := \{v \in H^1(K, \mathbb{S}^1), v = h_0 \text{ on } \partial K\}.$$

We find that hence $\tilde{v} = v^*$ in K. By a classic result of Morrey [55] (see also [17]), v^* satisfies (2.5). Moreover, as follows from weak lower semicontinuity of Dirichlet integral, (2.102), (2.103) and (2.104)

$$\frac{1}{2}\int_K |\nabla v^*|^2 \leq \liminf_{\varepsilon\to 0} \frac{1}{2}\int_K |\nabla v_\varepsilon|^2 \leq \limsup_{\varepsilon\to 0} F_\varepsilon(v_\varepsilon, K) \leq \frac{1}{2}\int_K |\nabla v^*|^2.$$

Therefore,

$$v_\varepsilon \text{ converges to } v^* \text{ strongly in } H^1(K). \tag{2.105}$$

From (2.101) and (2.105) we obtained that $v_\varepsilon \to v^*$ in $H^1_{\text{loc}}(\Omega \setminus \{a_1, ..., a_{d'}\})$. Convergence up to $\partial\Omega$ will be established in the next section.

In order to prove Assertion *3.* of Theorem 2.1 and Assertion *2.* of Theorem 2.3 , note that, for small $\rho > 0$, estimate (2.101) implies that $f_\varepsilon := \text{tr}_{\partial B(a_{i_k}, \rho)} v_\varepsilon$ satisfies the conditions (2.13) and (2.14) of Theorem 2.22. Thus leads to *3.* of Theorem 2.1 and *2.* of Theorem 2.3.

Assertion *3.* of Theorem 2.3 is is a consequence of Corollary 2.34.

2.5.3 The macroscopic position of vortices minimizes the Bethuel-Brezis-Hélein renormalized energy

Let us recall briefly the meaning of the renormalized energy $W_g((b_1, d_1), ..., (b_k, d_k))$ with

$$\begin{cases} g \in C^\infty(\partial\Omega, \mathbb{S}^1) \text{ s.t. } \deg_{\partial\Omega}(g) = d \\ b_1, ..., b_k \in \Omega,\ b_i \neq b_j \text{ for } i \neq j \\ d_i \in \mathbb{Z} \text{ and } \sum_i d_i = d \end{cases}.$$

For small $\rho > 0$, consider $\Omega_\rho = \Omega \setminus \cup_i \overline{B(b_i, \rho)}$ and the minimization problem

$$I_\rho((b_1, ..., b_k), (d_1, ..., d_k)) = \inf_{\substack{w \in H^1(\Omega_\rho, \mathbb{S}^1) \text{ s.t.} \\ w = g \text{ on } \partial\Omega \\ w(b_i + \rho e^{\imath\theta}) = \alpha_i e^{\imath d_i \theta},\, \alpha_i \in \mathbb{S}^1}} \frac{1}{2}\int_{\Omega_\rho} |\nabla w|^2.$$

Such a problem is studied in details in [18] (Chapter 1). In particular Bethuel, Brezis and Hélein proved that for small ρ, we have

$$I_\rho((b_1, ..., b_k), (d_1, ..., d_k)) = \pi d|\ln \rho| + W_g((b_1, d_1), ..., (b_k, d_k)) + o_\rho(1).$$

This equality plays an important role in the study done in [18]. In the minimization problem of the classical Ginzburg-Landau functional

$$\frac{1}{2}\int_\Omega \left\{|\nabla u|^2 + \frac{1}{2\varepsilon^2}(1-|u|^2)^2\right\},\ u \in H^1_g,$$

the vortices (with their degrees) of a minimizer tend to form (up to a subsequence) a minimal configuration for W_g.

We prove in this section that the (macroscopic) location of the vorticity of minimizers of F_ε is related to the minimization problem of $W_g((b_1, ..., b_k), (d_1, ..., d_k))$ with $b_1, ..., b_k \in \{a_1, ..., a_M\}$.

Here we treat Case I (Theorem 2.1). Case II settled along the same lines.

The proof of Assertion *4.* relies on two lemmas, providing sharp upper and lower bounds.

Lemma 2.36. *There exists $\rho_0 > 0$ s.t., for every $\rho < \rho_0$ and every $\varepsilon > 0$, we have*

$$F_\varepsilon(v_\varepsilon) \le \pi d|\ln \rho| + dJ(\varepsilon, \rho) + W_g((a_{i_1}, 1), ..., (a_{i_d}, 1)) + o_\rho(1), \tag{2.106}$$

where $J(\varepsilon, \rho) = \inf_{u \in H^1_{g_\rho}(B_\rho(0))} F_\varepsilon(u)$ with $g_\rho = \frac{z}{\rho}$ on $\partial B(0, \rho)$.

Proof. The proof, via construction of a test function, is the same as proof of Lemma VIII.1 in [18]. □

Lemma 2.37. *Let $\rho > 0$, $\rho < \rho_0$. Then for small ε we have*

$$F_\varepsilon(v_\varepsilon) \ge \pi d|\ln \rho| + dJ(\varepsilon, \rho) + W_g((a_{i_1}, 1), ..., (a_{i_d}, 1)) + o_\rho(1). \tag{2.107}$$

Proof. Split the domain Ω into two sub-domains: $\Omega \setminus \cup_i \overline{B(a_{k_i}, \rho)}$ and $\cup_i B(a_{k_i}, \rho)$. We start with the lower bound in the first sub-domain. By previous estimate v_ε weakly converges to v^* in $H^1(\Omega \setminus \cup_i \overline{B(a_{k_i}, \rho)})$. This implies that

$$\liminf \frac{1}{2}\int_{\Omega \setminus \cup_k \overline{B(a_{i_k}, \rho)}} U_\varepsilon^2 |\nabla v_\varepsilon|^2 \ge \frac{1}{2}\int_{\Omega \setminus \cup_k \overline{B(a_{i_k}, \rho)}} |\nabla v^*|^2.$$

Here, we used the fact that, since $U_\varepsilon \to 1$ in $L^2(\Omega)$, $|U_\varepsilon| \le 1$ and $\nabla v_\varepsilon \rightharpoonup \nabla v^*$ in $L^2(\Omega \setminus \cup_k \overline{B(a_{i_k}, \rho)})$, we have $U_\varepsilon \nabla v_\varepsilon \rightharpoonup \nabla v^*$ in $L^2(\Omega \setminus \cup_k \overline{B(a_{i_k}, \rho)})$.

Thus we deduce that, for small ε,

$$\frac{1}{2}\int_{\Omega \setminus \cup_k \overline{B(a_{i_k}, \rho)}} U_\varepsilon^2 |\nabla v_\varepsilon|^2 \ge \frac{1}{2}\int_{\Omega \setminus \cup_k \overline{B(a_{i_k}, \rho)}} |\nabla v^*|^2 - \rho^2. \tag{2.108}$$

On the other hand, as proved in [18],

$$\frac{1}{2}\int_{\Omega \setminus \cup_k \overline{B(a_{i_k}, \rho)}} |\nabla v^*|^2 \ge \pi d \ln \frac{1}{\rho} + W_g((a_{i_1}, 1), ..., (a_{i_d}, 1)) + o_\rho(1). \tag{2.109}$$

Thus, combining (2.108), (2.109) and using Proposition 2.5, for ε sufficiently small, we have

$$F_\varepsilon(v_\varepsilon, \Omega \setminus \cup_k \overline{B(a_{i_k}, \rho)}) \ge \pi d \ln \frac{1}{\rho} + W_g((a_{i_1}, 1), ..., (a_{i_d}, 1)) + o_\rho(1). \tag{2.110}$$

By Theorem 2.23 and Corollary 2.33 we have the following energy expansion:

$$F_\varepsilon(v_\varepsilon, B(a_{i_k},\rho)) = \pi\ln\rho + \pi b^2|\ln\varepsilon| + \pi(1-b^2)|\ln\delta| + \tilde{W}(\boldsymbol{\alpha}) + \tilde{W}_0(f_0) + b^2\gamma + o_\varepsilon(1). \quad (2.111)$$

Similarly, applying Theorem 2.23 to $J(\varepsilon,\rho)$ we obtain

$$J(\varepsilon,\rho) = \pi\ln\rho + \pi b^2|\ln\varepsilon| + \pi(1-b^2)|\ln\delta| + \tilde{W}(\boldsymbol{\alpha}) + \tilde{W}_0(z/|z|) + b^2\gamma + o_\varepsilon(1). \quad (2.112)$$

Here, the local renormalized energy $\tilde{W}(\boldsymbol{\alpha})$ is given by (2.90) and is the same in (2.111) and (2.112).

From (2.73), $\tilde{W}_0(f_0) \geq 0$ while $\tilde{W}_0(\frac{z}{|z|}) = 0$. Consequently, we have $F_\varepsilon(v_\varepsilon, B(a_{i_k},\rho)) - J(\varepsilon,\rho) \geq o_\varepsilon(1)$. Hence $\forall\rho > 0$ there exists $\varepsilon_\rho > 0$ s.t. for $\varepsilon < \varepsilon_\rho$ we have

$$F_\varepsilon(v_\varepsilon, B(a_{i_k},\rho)) \geq J(\varepsilon,\rho) - \rho^2$$

and thus

$$F_\varepsilon(v_\varepsilon, \cup_k B(a_{i_k},\rho)) \geq dJ(\varepsilon,\rho) - d\rho^2. \quad (2.113)$$

Which gives the lower bound in the second sub-domain. From (2.110) and (2.113) the bound (2.107) follows. □

Combining Lemma 2.36 and Lemma 2.37, we see that the points $\{a_{i_k}, 1 \leq k \leq d\}$ minimize W_g among $a_1, ..., a_M$. The expansion (2.6) follows from (2.106), (2.107) and (2.112).

We next turn to convergence of v_ε up to the boundary. It suffices to prove the H^1-convergence of v_ε in $\Omega_\rho = \Omega \setminus \cup_m \overline{B(a_{i_m},\rho)}$ (for small $\rho > 0$). We argue by contradiction and we assume that there are some $\rho_1 > 0$ and $\eta > 0$ s.t.

$$\liminf \frac{1}{2}\int_{\Omega_{\rho_1}} |\nabla v_\varepsilon|^2 \geq \frac{1}{2}\int_{\Omega_{\rho_1}} |\nabla v^*|^2 + \eta. \quad (2.114)$$

Note that for all $\rho \leq \rho_1$, (2.114) still holds in Ω_ρ.

If, in the proof of Lemma 2.37, we replace (2.108) by (2.114) (with ρ_1 replaced by ρ), then we obtain for small ρ a contradiction with Lemma 2.36. The proof of Theorem 2.1 is complete. The last assertion of Theorem 2.3 is obtained along the same lines.

Appendix 2.A Proof of Proposition 2.5

Let $x_0 \in V_R$ be s.t. $B_R = B(x_0,R) \subset \Omega \setminus \overline{\omega_\delta}$ and assume that $x_0 = 0$.

We follow the proof of Lemma 2 in [17].

In B_R, $\eta = 1 - U_\varepsilon$ satisfies

$$\begin{cases} -\varepsilon^2\Delta\eta + t\eta = -\eta(\eta^2 - 3\eta + 2 - t) & \text{in } B_R \\ \eta \leq 1 & \text{on } \partial B_R \end{cases},$$

here, t will be chosed later.

Since $\eta \in (0, 1-b)$, if we take $t = b(1+b)$, then we have

$$-\varepsilon^2\Delta\eta + t\eta \leq 0 \text{ in } B_R.$$

On the other hand, the function $w(x) = e^{\gamma(|x|^2 - R^2)}$ satisfies

$$\begin{cases} -\varepsilon^2\Delta w + tw = \left[-4\varepsilon^2\gamma(1+\gamma|x|^2) + t\right] w & \text{in } B_R \\ w = 1 & \text{on } \partial B_R \end{cases}.$$

A simple computation gives that

$$\begin{cases} -\varepsilon^2\Delta w + tw \geq 0 \text{ in } B_R \\ \gamma > 0 \end{cases} \Leftrightarrow 0 < \gamma \leq \frac{-\varepsilon + \sqrt{\varepsilon^2 + tR^2}}{2R^2\varepsilon}.$$

Take

$$\gamma = \frac{-\varepsilon + \sqrt{\varepsilon^2 + tR^2}}{2R^2\varepsilon} > 0.$$

Setting $v = \eta - w$, we have

$$\begin{cases} -\varepsilon^2\Delta v + tv \leq 0 & \text{in } B_R \\ v \leq 0 & \text{on } \partial B_R. \end{cases}$$

By the maximum principle, we have $v \leq 0$ in B_R. Therefore,

$$\eta(0) \leq \exp\left\{-\frac{-\varepsilon + \sqrt{\varepsilon^2 + tR^2}}{2\varepsilon}\right\} \leq Ce^{-\frac{\sqrt{t}R}{4\varepsilon}}.$$

Consequently, (2.9) holds in $\{x \in \Omega \,|\, \text{dist}(x, \partial\Omega) \geq R, \text{dist}(x, \omega_\delta) > R\}$. The estimate close to the $\partial\Omega$ is a direct consequence of $0 \leq U_\varepsilon \leq 1$, (2.9) holds in $\{x \in \Omega \,|\, \text{dist}(x, \partial\Omega) \geq R, \text{dist}(x, \omega_\delta) > R\}$ and the equation $-\Delta U_\varepsilon = \dfrac{1}{\varepsilon^2} U_\varepsilon(1 - |U_\varepsilon|^2)$ in $\{x \in \Omega \,|\, \text{dist}(x, \omega_\delta) > R\}$. Using a similar argument, we establish (2.9) in the case $V_R \cap \omega_\delta$. The proof of (2.9) is complete.

In order to prove (2.10), note that in $W_R := \{x \in \Omega \,|\, \text{dist}(x, \partial\omega_\delta) \geq R, \text{dist}(x, \partial\Omega) \geq R\}$ the function $\eta = a_\varepsilon - U_\varepsilon$ satisfies $\Delta\eta = \frac{U_\varepsilon}{\varepsilon^2}(a_\varepsilon^2 - U_\varepsilon^2)$. Thus, applying Lemma A.1 [17] to η in conjunction with (2.9) and the fact that $R \geq \varepsilon$, we obtain

$$|\nabla\eta| \leq \frac{C_1 e^{-\frac{cR}{\varepsilon}}}{\varepsilon} \text{ in } W_R.$$

Thus (2.10) holds far away from $\partial\Omega$ and the inclusions.

We next prove that the bound (2.10) holds near $\partial\Omega$.

Indeed, fix a smooth compact $K \subset \Omega$ s.t. for small δ we have $\omega_\delta \subset K$. Clearly, by (2.9), $0 \leq \eta_K := \text{tr}_{\partial K}\eta \leq Ce^{-\frac{cR}{\varepsilon}}$. In $\Omega \setminus K$, η satisfies

$$\begin{cases} \Delta\eta = \frac{1}{\varepsilon^2}U(1+U)\eta & \text{in } \Omega \setminus K \\ \eta = 0 & \text{on } \partial\Omega \\ \eta = \eta_K & \text{on } \partial K \end{cases}.$$

Let $\eta = \eta_1 + \eta_2$ be s.t. η_1 solves

$$\begin{cases} \Delta\eta_1 = \frac{1}{\varepsilon^2}U(1+U)\eta & \text{in } \Omega \setminus K \\ \eta_1 = 0 & \text{on } \partial\Omega \cup \partial K \end{cases}$$

and η_2 satisfies

$$\begin{cases} \Delta\eta_2 = 0 & \text{in } \Omega \setminus K \\ \eta_2 = 0 & \text{on } \partial\Omega \\ \eta_2 = \eta_K & \text{on } \partial K \end{cases}.$$

Note that $\|\eta_2\|_{L^\infty} \leq Ce^{-\frac{cR}{\varepsilon}}$ and thus $\|\eta_1\|_{L^\infty} \leq Ce^{-\frac{cR}{\varepsilon}}$.

Lemma A.2 in [17] implies the existence of a constant $C_{\Omega\setminus K} > 0$ s.t.

$$|\nabla \eta_1| \leq \frac{C_{\Omega\setminus K} e^{-\frac{cR}{\varepsilon}}}{\varepsilon} \text{ in } \Omega \setminus K.$$

In order to estimate $\nabla\eta_2$ near $\partial\Omega$, we express η_2 in terms of Green's function $G(x,y)$ in $\Omega \setminus K$: function, *i.e.*

$$\eta_2(x) = -\int_{\partial K} \eta_K(y) \frac{\partial G}{\partial \nu}(x,y) dS(y). \tag{2.115}$$

It follows from (2.115) and (2.9) that $|\nabla\eta_2| \leq C_0 e^{-\frac{cR}{\varepsilon}}$ away from ∂K. The estimate (2.10) is proved.

Appendix 2.B Proof of Proposition 2.6

This appendix is devoted to the proof of Proposition 2.6.

We prove the first assertion: when $M \geq d$ we have

$$\inf_{v\in H^1_g(\Omega)} F_\varepsilon(v,\Omega) \leq \pi d b^2 |\ln\xi| + \pi d|\ln\delta| + \mathcal{O}(1).$$

Fix first d distinct points-centers of inclusions $a_1, ..., a_d$. Let $\rho_0 := 10^{-2} \cdot \min(\mathrm{dist}(a_i, \partial\Omega), \min_{i\neq j} |a_i - a_j| > 0)$. Consider $\tilde{v}$ to be a smooth fixed function in $\Omega \setminus \cup_{i=1}^d \overline{B(a_i,\rho_0)}$, such that $|\tilde{v}| = 1$ in $\Omega \setminus \cup_{i=1}^d \overline{B(a_i,\rho_0)}$ and

$$\begin{cases} \tilde{v} = g & \text{on } \partial\Omega \\ \tilde{v}(x) = \frac{x-a_i}{|x-a_i|} & \text{on } \partial B(a_i,\rho_0) \end{cases}.$$

Such a function clearly exists since the compatibility condition $\deg_{\partial\Omega}(g) = \sum_{i=1}^d \deg_{\partial B(a_i,\rho_0)}(\tilde{v})$ is satisfied. Let $c_0 = 10^{-2} \cdot \mathrm{dist}(0, \partial\omega)$. For every $1 \leq i \leq M$, consider a disc $B(a_i, c_0\delta) \subset \omega^i_\delta$. By the choice of c_0, we have $\mathrm{dist}(\partial\omega_\delta, B(a_i, c_0\delta)) \geq c_0\delta$. Therefore, using Proposition 2.5

$$U_\varepsilon^2 - b^2 \leq C e^{-\frac{c\delta}{\varepsilon}} \text{ in } B(a_i, c_0\delta). \tag{2.116}$$

Consider the test function v_0^ε defined as

$$v_0^\varepsilon(x) = \begin{cases} \tilde{v}(x) & \text{for } x \in \Omega \setminus \cup_i \overline{B(a_i,\rho_0)} \\ \dfrac{x-a_i}{|x-a_i|} & \text{for } x \in B(a_i,\rho_0) \setminus \overline{B(a_i,\varepsilon)} \\ \dfrac{x-a_i}{\varepsilon} & \text{for } x \in B(a_i,\varepsilon) \end{cases}.$$

Using (2.116) and (H) we have

$$\begin{aligned} \inf_{v\in H^1_g(\Omega)} F_\varepsilon(v,\Omega) &\leq F_\varepsilon(v_0^\varepsilon) \\ &\leq \pi d b^2 |\ln\varepsilon| + \pi d(1-b^2)|\ln\delta| + C = \pi d b^2|\ln\xi| + \pi d|\ln\delta| + C. \end{aligned}$$

Now we prove the second assertion: when $M < d$ we have

$$\inf_{v\in H^1_g(\Omega)} F_\varepsilon(v,\Omega) \leq \pi d b^2 |\ln\xi| + \pi \sum_i d_i^2 |\ln\delta| + C.$$

Let $d_1, ..., d_M \in \mathbb{N}$ be s.t. $\sum d_i = d$. Set $c_0 = 10^{-2d} \cdot \mathrm{dist}(0,\partial\omega)$. For $i \in \{1, ..., M\}$ s.t. $d_i > 0$, fix $\alpha_{1,i}, ..., \alpha_{d_i,i} \in B(0, 10^d c_0) \subset \omega$ s.t.

$$\min\left(\min_{j\neq k} |\alpha_{j,i} - \alpha_{k,i}|, \mathrm{dist}(\alpha_{j,i}, \partial\omega)\right) > 4c_0.$$

Consider an ε-dependent map $\tilde{v}_0^\varepsilon \in H^1(\Omega \setminus \cup_{d_i>0}\overline{B(a_i, 10^d c_0\delta)}, \mathbb{S}^1)$ s.t.

$$\begin{cases} \tilde{v}_0^\varepsilon = g & \text{on } \partial\Omega \\ \tilde{v}_0^\varepsilon(x) = \dfrac{(x-a_i)^{d_i}}{|x-a_i|^{d_i}} & \text{on } \partial B(a_i, 10^d c_0\delta) \end{cases}$$

and satisfying

$$\int_{\Omega\setminus\cup_{d_i>0}\overline{B(a_i,10^d c_0\delta)}} |\nabla \tilde{v}_0^\varepsilon|^2 \le \pi \sum d_i^2 |\ln\delta| + C$$

with C depending only on Ω, ω and g.

(Such maps do exist, *e.g.*, consider the map introduced in [18], Remark I.5.)

For $i \in \{1, ..., M\}$ s.t. $d_i > 0$, we consider a map $v_i^\varepsilon \in H^1(B(0, 10^d c_0)\setminus\cup_{j=1}^{d_i}\overline{B(\alpha_{j,i}, \xi)}, \mathbb{S}^1)$ s.t.

- $v_i^\varepsilon(x) = x^{d_i}/|x|^{d_i}$ on $\partial B(0, 10^d c_0)$,
- $v_i^\varepsilon(x) = (x-\alpha_{j,i})/|x-\alpha_{j,i}|$ on $\partial B(\alpha_{j,i}, \xi)$,
- $\displaystyle\int_{B(0,10^d c_0)\setminus\cup_{j=1}^{d_i}\overline{B(\alpha_{j,i},\xi)}} |\nabla v_i^\varepsilon|^2 \le \pi d_i |\ln\xi| + C$ with C depending only on ω.

(For example, the map considered in Remark I.5 in [18] has these properties).

One obtains a test function satisfying the bound (2.12) by rescaling the v_i^ε's (in order to have maps defined in balls of size δ) and gluing the rescaled maps with $\tilde{v}_0^\varepsilon$.

Appendix 2.C Proof of the η-ellipticity Lemma

The main argument in the proof of the η-ellipticity result is the following convexity lemma

Lemma 2.38. *[Convexity Lemma]*

Let C be a chord in the closed unit disc, C different from a diameter. Let S be the smallest of two regions enclosed by the chord and the boundary of the disc.

Let O be a Lipschitz, bounded, connected domain and let $g \in C(\partial O, S)$.

Assume that v minimizes Ginzburg-Landau type energy

$$\tilde{F}(v) = \int_O \left\{ \tilde{\alpha}(x)|\nabla v|^2 + \tilde{\beta}(x)(1-|v|^2)^2 \right\} \mathrm{d}x$$

in $H_g^1(O)$, with $\tilde{\alpha}, \tilde{\beta} \in L^\infty(O, \mathbb{R})$ satisfying $\operatorname{essinf}\tilde{\alpha} > 0, \operatorname{essinf}\tilde{\beta} > 0$. *Then $v(O) \subset S$.*

Remark 2.39. This statement generalizes Lemma 8 in [15] (there $\tilde{\alpha} = 1, \tilde{\beta} = 1/(2\varepsilon^2)$). However, the proof in [15] does not apply directly to our situation.

Proof. Clearly, one may assume that O is connected.
We start by noting that v has the following properties:

- v is continuous in O (this relies on the equation satisfied by v, on Theorem 2 in [52] and on Sobolev embeddings).
- $|v| \le 1$. Indeed, consider the test function $v' = \begin{cases} v & \text{if } |v| \le 1 \\ \dfrac{v}{|v|} & \text{if } |v| > 1 \end{cases}$. Since v' has more energy than v, we find that $|v| \le 1$ a.e. and thus $|v| \le 1$.

Without loss of generality, we may assume that, for some $\mu \in (0,1)$, we have $C = \{z \in B_1(0) : \Re z = \mu\}$ and $S = \{z \in \overline{B_1(0)}; \Re z \geq \mu\}$.

The map $w := |\Re v| + i\Im v$ equals g on ∂O and has the same energy as v. Thus w minimizes $\tilde{F}$. In particular, w is continuous. Therefore, if we prove that $w(O) \subset S$, we will have $v(O) \subset S$. In conclusion, we reduced the problem to the case where $\Re v \geq 0$.

Let P be the orthogonal projection on S. When $z \in B_1(0) \cap \{\Re z \geq 0\}$, we have

$$P(z) = \begin{cases} z & \text{if } \Re z \geq \mu \\ \mu + i\Im z & \text{if } |\Im z| \leq \sqrt{1-\mu^2} \text{ and } \Re z < \mu \\ \mu + i(\text{sign } \Im z)\sqrt{1-\mu^2} & \text{if } |\Im z| > \sqrt{1-\mu^2} \text{ and } \Re z < \mu \end{cases} . \tag{2.117}$$

One may check easily that

$$|z| \leq |P(z)| \leq 1 \text{ for } z \in B_1(0) \cap \{\Re z \geq 0\}. \tag{2.118}$$

Set $\psi(z) := P(w(z))$, which equals g on ∂O. Since P is 1-Lipschitz, we have $|\nabla\psi| \leq |\nabla w|$. On the other hand, (2.118) implies $|w| \leq |\psi| \leq 1$. Consequently, $\tilde{F}(\psi) \leq \tilde{F}(w)$.

Since w is a minimizer, ψ is also a minimizer. Using the previous pointwise estimates and the equality of the energies, one may conclude that $|\psi(z)| = |w(z)|$ for each z (by continuity of w and ψ) and $|\nabla\psi| = |\nabla w|$ a.e.

By solving the equation $|z| = |P(z)|$, we see that $|\psi| = |w|$ implies that w takes values in $S \cup V$, where $V := \{z' \in \mathbb{S}^1 \,|\, 0 \leq \Re z' < \mu\}$.

We have to prove that $U := w^{-1}(V) = \emptyset$. We argue by contradiction and assume $U \neq \emptyset$. Then U is open, since $U = O \setminus w^{-1}(S)$ with S a closed set.

We first prove that w is locally constant in U. Indeed, in U, w satisfies $\text{div}(\tilde{\alpha}\nabla w) = 0$. Since $w \in H^1(U, \mathbb{S}^1)$, we may write, in U, $w = e^{\imath\varphi}$, where $\varphi \in H^1$ [21]. Let $\zeta \in C_c^\infty(U)$. If we multiply the equation $\text{div}(\tilde{\alpha}\nabla(\cos\varphi)) = 0$ by $\zeta\cos\varphi$ and the equation $\text{div}(\tilde{\alpha}\nabla(\sin\varphi)) = 0$ by $\zeta\sin\varphi$ and add the two results, we obtain $\displaystyle\int \tilde{\alpha}\zeta|\nabla\varphi|^2 = 0$, so that φ (and thus w) is locally constant in U.

Let $W \neq \emptyset$ be a connected component of U, so that $w \equiv s \in V$ in W. Consider the non empty set $Y := w^{-1}(\{s\})$. Then Y is open in O (since w is locally constant in U), and clearly Y is closed in O. Therefore, $Y = O$, *i. e.*, $w \equiv s$ in O. This contradicts the facts that $g : \partial O \to S$, $\text{tr}_{\partial O} w = g$ and $s \notin S$. □

We prove the first part of the lemma 2.7. Let $x \in \Omega$ be s.t. $\text{dist}(x, \partial\Omega) \geq \varepsilon^{1/4}$. We have

$$\begin{aligned} F_\varepsilon(v_\varepsilon, B(x,\varepsilon^{1/4})) &\geq \frac{b}{2}\int_{B(x,\varepsilon^{1/4})\setminus\overline{B(x,\varepsilon^{1/2})}} \left\{|\nabla v_\varepsilon|^2 + \frac{1}{\varepsilon^2}(1-|v_\varepsilon|^2)^2\right\} \\ &= \frac{b}{2}\int_{\varepsilon^{1/2}}^{\varepsilon^{1/4}} \frac{1}{r}\cdot r\int_{\partial B(x,r)} \left\{|\nabla v_\varepsilon|^2 + \frac{1}{\varepsilon^2}(1-|v_\varepsilon|^2)^2\right\}. \end{aligned}$$

By Mean Value theorem, exists $r \in (\varepsilon^{1/2}, \varepsilon^{1/4})$ s.t

$$r\int_{\partial B(x,r)} \left\{|\nabla v_\varepsilon|^2 + \frac{1}{\varepsilon^2}(1-|v_\varepsilon|^2)^2\right\} \leq \frac{\frac{2}{b}F_\varepsilon(v_\varepsilon, B(x,\varepsilon^{1/4}))}{\frac{1}{4}|\ln\varepsilon|}.$$

There is $C_2 = C_2(\chi, b) > 0$ s.t if $F_\varepsilon(v_\varepsilon, B(x,\varepsilon^{1/4})) \leq \chi^2|\ln\varepsilon|$, we have

$$\text{Var}(v_\varepsilon, \partial B(x,r)) \leq C_2\chi, \text{ where } \text{Var}\,(v_\varepsilon, \partial B(x,r)) := \int_{\partial B(x,r)} |\partial_\tau v_\varepsilon|. \tag{2.119}$$

It follows that

$$|v_\varepsilon|^2 \geq 1 - 3C_2\chi \text{ on } \partial B(x,r). \tag{2.120}$$

Indeed, arguing by contradiction, assume that there is $\varepsilon_n \downarrow 0$ and $y_n \in \partial B(x,r)$ s.t. $|v_{\varepsilon_n}(y_n)|^2 < 1 - 3C_2\chi$. Using (2.119) we obtain that

$$|v_{\varepsilon_n}|^2 \leq 1 - C_2\chi \text{ on } \partial B(x,r)$$

which implies that

$$\begin{aligned} 2\pi C_2^2\chi^2\varepsilon_n^{2(\frac{1}{2}-1)} &\leq \frac{2\pi C_2^2 r^2\chi^2}{\varepsilon_n^2} \\ &\leq \frac{r}{\varepsilon_n^2}\int_{\partial B(x,r)}(1-|v_{\varepsilon_n}|^2)^2 \\ &\leq \frac{\frac{2}{b}F_{\varepsilon_n}(v_{\varepsilon_n}, B(x,{\varepsilon_n}^{1/4}))}{\frac{1}{4}|\ln\varepsilon_n|} \leq \frac{8\chi^2}{b}. \end{aligned}$$

Clearly, the previous assertion gives contradiction.

From (2.119) and (2.120), there is $C = C(\chi, b) > 0$ and $\varepsilon_0 = \varepsilon_0(\chi) > 0$ s.t. for $\varepsilon < \varepsilon_0$,

$$v_\varepsilon : \partial B_r \to \{z \in B_1 \,|\, \Re z > 1 - C\chi\}.$$

Using Convexity Lemma (Lemma 2.38), we find that $|v_\varepsilon| \geq 1 - C\chi$ in $B(x,r) \supset B(x,\varepsilon^{1/2})$.

If $\text{dist}(x,\partial\Omega) < \varepsilon^{1/4}$, we denote $S_r = \Omega \cap \partial B(x,r)$, $r \in (\varepsilon^{1/2}, \varepsilon^{1/4})$. Clearly, we have

$$\frac{2}{b}F_\varepsilon(v_\varepsilon, B(x,\varepsilon^{1/4}) \setminus \overline{B(x,\varepsilon^{1/2})}) \geq \int_{\varepsilon^{1/2}}^{\varepsilon^{1/4}} \frac{1}{r}\cdot r\,\mathrm{d}r \int_{S_r}\left\{|\nabla v_\varepsilon|^2 + \frac{1}{\varepsilon^2}(1-|v_\varepsilon|^2)^2\right\}.$$

Using a mean value argument and the facts that $g_\varepsilon \to g_0$ in $C^1(\partial\Omega, \mathbb{S}^1)$ and that $0 \leq 1 - |g_\varepsilon| \leq \varepsilon$, there are $r \in (\varepsilon^{1/2}, \varepsilon^{1/4})$ and $C_1 = C_1(\|g_0\|_{C^1}, \Omega)$ s.t

$$r\int_{\partial(B(x,r)\cap\Omega)}\left\{|\partial_\tau v_\varepsilon|^2 + \frac{1}{\varepsilon^2}(1-|v_\varepsilon|^2)^2\right\} \leq \frac{\frac{2}{b}F_\varepsilon(v_\varepsilon, B(x,\varepsilon^{1/4})) + C_1}{\frac{1}{4}|\ln\varepsilon|}.$$

Using the same argument as before (taking $O = \Omega \cap B(x,r)$) we obtain the desired result.

We prove the second part of the lemma. Let $\mu \in (0,1)$ and $x \in \{\text{dist}(x,\partial\Omega) \geq \varepsilon^{1/4}\}$. Using mean value argument, there is $r \in (\varepsilon^{1/2}, \varepsilon^{1/4})$ s.t

$$r\int_{\partial B(x,r)}\left\{|\nabla v_\varepsilon|^2 + \frac{1}{\varepsilon^2}(1-|v_\varepsilon|^2)^2\right\} \leq \frac{\frac{2}{b}F_\varepsilon(v_\varepsilon, B(x,\varepsilon^{1/4}))}{\frac{1}{4}|\ln\varepsilon|}.$$

There exists $C_1 = C_1(\mu, b) > 0$ s.t if $F_\varepsilon(v_\varepsilon, B(x,\varepsilon^{1/4})) \leq C_1|\ln\varepsilon|$, we have

$$\text{Var}(v_\varepsilon, \partial B(x,r)) \leq \frac{1-\mu}{10} \text{ and } 1 - |v_\varepsilon| \leq \frac{1-\mu}{10} \text{ on } \partial B(x,r).$$

By Convexity Lemma $|v_\varepsilon| \geq \mu$ in $B(x,r) \supset B(x,\varepsilon^{1/2})$.

If $\text{dist}(x,\partial\Omega) < \varepsilon^{1/4}$, denote $S_r = \Omega \cap \partial B(x,r)$, $r \in (\varepsilon^{1/2}, \varepsilon^{1/4})$. Since

$$\frac{2}{b}F_\varepsilon(v_\varepsilon, B(x,\varepsilon^{1/4}) \setminus \overline{B(x,\varepsilon^{1/2})}) \geq \int_{\varepsilon^{1/2}}^{\varepsilon^{1/4}} \frac{1}{r}\cdot r\,\mathrm{d}r \int_{S_r}\left\{|\nabla v_\varepsilon|^2 + \frac{1}{\varepsilon^2}(1-|v_\varepsilon|^2)^2\right\}$$

and using the conditions on g_ε, by mean value argument there is $r \in (\varepsilon^{1/2}, \varepsilon^{1/4})$ s.t

$$r\int_{\partial(B(x,r)\cap\Omega)}\left\{|\partial_\tau v_\varepsilon|^2 + \frac{1}{\varepsilon^2}(1-|v_\varepsilon|^2)^2\right\} \leq \frac{\frac{2}{b}F_\varepsilon(v_\varepsilon, B(x,\varepsilon^{1/4})) + C(\|g_0\|_{C^1}, \Omega)}{\frac{1}{4}|\ln\varepsilon|}.$$

Using the same argument as before, the statement of the lemma follows.

Chapter 3

The Ginzburg-Landau functional with a discontinuous and rapidly oscillating pinning term. Part I: the zero degree case

`In Collaboration with Petru Mironescu and Oleksandr Misiats, to appear: [37]`

We consider minimizers of the Ginzburg-Landau energy with pinning term and zero degree Dirichlet boundary condition. Without any assumptions on the pinning term, we prove that these minimizers do not develop vortices in the limit $\varepsilon \to 0$. We next consider the specific case of a periodic discontinuous pinning term taking two values. In this setting, we determine the asymptotic behavior of the minimizers as $\varepsilon \to 0$.

Contents

3.1 Introduction

Let $\Omega \subset \mathbb{R}^2$ be a simply connected domain and let $a_\varepsilon : \Omega \to \mathbb{R}$ be a measurable function such that $0 < b \leq a_\varepsilon \leq 1$. We associate with a_ε a generalized Ginzburg-Landau type energy

$$E_\varepsilon(u) = \frac{1}{2}\int_\Omega \left\{ |\nabla u(x)|^2 + \frac{1}{2\varepsilon^2}(a_\varepsilon^2(x) - |u(x)|^2)^2 \right\} dx. \tag{3.1}$$

Here, $u \in H^1(\Omega, \mathbb{C})$ and $\varepsilon > 0$ is the inverse of the Ginzburg-Landau parameter.

This variant of the standard Ginzburg-Landau type energy (which corresponds to $a_\varepsilon \equiv 1$) is called Ginzburg-Landau functional with pinning term a_ε or pinned Ginzburg-Landau functional. We quote here few relevant papers among the vast literature concerning this energy functional.

- In [5], the authors consider the case where $a_\varepsilon = a \in C^\beta(\Omega)$ is independent of ε.
- [43] and [9] treat the case where $a_\varepsilon = a$ is independent of ε and takes the value b in ω and 1 outside ω, with ω smooth subset of Ω. The latter article considers the case of an applied magnetic field.
- In [1], a_ε depends on ε and is smooth. The oscillation rate of a_ε depends on ε.

The goal of this chapter is to study the pinned Ginzburg-Landau functional with a fast oscillating discontinuous pinning term a_ε. This may be viewed as a simplification of more realistic models which describe superconductivity phenomena for composite superconductors. (See the introduction of Chapter 2)

Our pinning term is periodic with respect to a $\delta \times \delta$ grid where $\delta = \delta(\varepsilon) \to 0$. As in [1], due to the fast oscillations, this problem is related to a periodic homogenization problem (depending on the relation between ε and δ).

The boundary condition we consider is the Dirichlet one. More specifically, we fix some $g \in H^{1/2}(\partial\Omega, \mathbb{S}^1)$. Our class of test functions is

$$H^1_g := \{u \in H^1(\Omega, \mathbb{C}) \,|\, u = g \text{ on } \partial\Omega\}. \tag{3.2}$$

We consider solutions u_ε of the minimization problem

$$\inf_{u \in H^1_g} E_\varepsilon(u). \tag{3.3}$$

In this chapter we will consider only the case where the boundary data g has zero degree. The case where the degree is not zero requires additional techniques and will be investigated in a forthcoming paper.

Recall that the degree (winding number) of g is defined as

$$\deg_{\partial\Omega}(g) = \frac{1}{2\pi}\int_{\partial\Omega} g \times \partial_\tau g \, \mathrm{d}\tau = 0,$$

where:

- For $z \in \mathbb{C}$, $\Re z$ denotes the real part of z and $\Im z$ denotes the imaginary part of z.
- "$\times$" stands for the "vectorial product" in $\mathbb{C}$, $z_1 \times z_2 = \Im(\overline{z_1} z_2)$, $z_1, z_2 \in \mathbb{C}$.
- τ is the unit and direct tangent vector at $\partial\Omega$, *i.e.*, denoting ν to be the unit outward normal to $\partial\Omega$, one has $\tau = \nu^\perp$.

- ∂_τ is the tangential derivative.

This degree is an integer. For a proof of this assertion and for more properties of the topological degree of g, see *e.g.* [26] or [13].

If u_ε is a minimizer of the problem (3.3), then it satisfies the Euler-Lagrange equation

$$\begin{cases} -\Delta u_\varepsilon = \dfrac{1}{\varepsilon^2} u_\varepsilon (a_\varepsilon^2 - |u_\varepsilon|^2) & \text{in } \Omega \\ u_\varepsilon = g & \text{on } \partial\Omega \end{cases}. \tag{3.4}$$

Following [43], one may prove that in the special case $g \equiv 1$ there is a unique minimizer U_ε. Moreover, this minimizer satisfies $b \leq U_\varepsilon \leq 1$. This U_ε plays an important role in the study of Ginzburg-Landau functional with pinning term. Indeed, define, for $u \in H_g^1$, a new map $v = \dfrac{u}{U_\varepsilon} \in H_g^1$. Then E_ε decouples as follows [43]

$$E_\varepsilon(u) = E_\varepsilon(U_\varepsilon v) = f(\varepsilon) + F_\varepsilon(v), \tag{3.5}$$

where

$$f(\varepsilon) := E_\varepsilon(U_\varepsilon), \quad F_\varepsilon(v) := \frac{1}{2}\int_\Omega \left\{ U_\varepsilon^2 |\nabla v|^2 + \frac{1}{2\varepsilon^2} U_\varepsilon^4 (1 - |v|^2)^2 \right\}. \tag{3.6}$$

Therefore, u minimizes E_ε in $H_g^1(\Omega)$ if and only if v minimizes F_ε in H_g^1. In what follows, we denote by v_ε a minimizer of F_ε in H_g^1.

Following again [43], we have $|v_\varepsilon| \leq 1$ and $|u_\varepsilon| \leq 1$ in Ω.

From (3.5) and (3.6) we see that the study of the pinned Ginzburg-Landau is reduced to the study of the weighted Ginzburg-Landau functional F_ε and to the study of the asymptotics of U_ε.

The plan of our work is the following: in Section 3.2 we prove a "clearing out" result (Theorem 3.1). More specifically, we prove that v_ε is "vortexless" for small ε, *i. e.*, that $|v_\varepsilon| \to 1$ uniformly in $\overline{\Omega}$ as $\varepsilon \to 0$. (Recall that $\deg_{\partial\Omega}(g) = 0$; this assumption is essential for our conclusion.) This result is true for any weighted Ginzburg-Landau functionals. Such general functionals are defined by formula (3.7) and do not require any assumption except uniform bounds on the weights. In particular, clearing out does not rely on any periodicity assumption. We believe that this result has its own interest.
The clearing out result reduces the study of the behavior of v_ε to the one of $\mathbb{S}^1$-valued maps. In other words, we will reduce the problem of minimizing F_ε in the class of all test functions to the one of minimizing F_ε in the class of $\mathbb{S}^1$-valued maps. The latter problem will be studied in detail in Section 3.3. There, the asymptotic analysis of minimizers of the F_ε among $\mathbb{S}^1$-valued maps, combined with an asymptotic analysis of U_ε (analysis performed at the beginning of Section 3.3), will allow us to conclude Section 3.3 by describing the behavior of u_ε as $\varepsilon \to 0$.

3.2 Clearing out for general weighted Ginzburg-Landau type functionals

Let $b \in (0,1)$ and let $\alpha_\varepsilon \in W^{1,\infty}(\Omega)$, $\beta_\varepsilon \in L^\infty(\Omega)$ be such that $b \leq \alpha_\varepsilon, \beta_\varepsilon \leq 1$. We associate to α_ε and β_ε the weighted Ginzburg-Landau type functional defined through the formula

$$\begin{array}{rccc} F_\varepsilon : & H^1(\Omega, \mathbb{C}) & \to & \mathbb{R}^+ \\ & v & \mapsto & \dfrac{1}{2}\displaystyle\int_\Omega \left\{ \alpha_\varepsilon |\nabla v|^2 + \frac{\beta_\varepsilon}{2\varepsilon^2}(1 - |v|^2)^2 \right\} \end{array}. \tag{3.7}$$

Let $g \in H^{1/2}(\partial\Omega, \mathbb{S}^1)$ be such that $\deg_{\partial\Omega}(g) = 0$. For $\varepsilon > 0$, we denote by v_ε a minimizer of F_ε in H^1_g. One may easily prove that v_ε satisfies

$$\begin{cases} -\mathrm{div}(\alpha_\varepsilon \nabla v_\varepsilon) = \dfrac{\beta_\varepsilon}{\varepsilon^2} v_\varepsilon(1 - |v_\varepsilon|^2) & \text{in } \Omega \\ v_\varepsilon = g & \text{on } \partial\Omega \end{cases}. \tag{3.8}$$

Since $\deg_{\partial\Omega}(g) = 0$, we have [17] $H^1_g(\Omega, \mathbb{S}^1) = \{v \in H^1_g \,|\, |v| = 1 \text{ in } \Omega\} \neq \emptyset$.
If we take any fixed map $v \in H^1_g(\Omega, \mathbb{S}^1)$ as a test function for F_ε, we find that there is C_0 depending only on g such that

$$\min_{v \in H^1_g(\Omega)} F_\varepsilon(v) = F_\varepsilon(v_\varepsilon) \leq C_0. \tag{3.9}$$

3.2.1 Uniform convergence of $|v_\varepsilon|$ to 1

This part is devoted to the proof of the following theorem.

Theorem 3.1. *When $\varepsilon \to 0$, we have $|v_\varepsilon| \to 1$ uniformly in $\overline{\Omega}$.*

For the convenience of the reader, we split the rather long proof of Theorem 3.1 into two parts.

Theorem 3.1 holds far away the boundary

We prove that, for sufficiently small ε, $|v_\varepsilon|$ is arbitrarly close to 1 outside an $2\sqrt{\varepsilon}$-neighborhood of $\partial\Omega$.

Proposition 3.2. *Let $\varepsilon_n \downarrow 0$ and $\{x_n\}_n \subset \Omega$ be such that $\mathrm{dist}(x_n, \partial\Omega) \geq 2\sqrt{\varepsilon_n}$. Then $|v_{\varepsilon_n}(x_n)| \to 1$.*

Proof. We write ε instead of ε_n. Let n be sufficiently large such that $\sqrt{\varepsilon} > \varepsilon$ and consider the circular annulus $B_{\sqrt{\varepsilon}}(x_n) \setminus B_\varepsilon(x_n)$.

From (3.9), we have, with $\mathcal{C}_r := \{|x - x_n| = r\}$,

$$\begin{aligned} C_0 &\geq \frac{b}{4} \int_{B_{\sqrt{\varepsilon}}(x_n) \setminus B_\varepsilon(x_n)} \left\{ |\nabla v_\varepsilon|^2 + \frac{1}{\varepsilon^2}(1 - |v_\varepsilon|^2)^2 \right\} \\ &= \frac{b}{4} \int_\varepsilon^{\sqrt{\varepsilon}} \frac{1}{r} \cdot r \int_{\mathcal{C}_r} \left\{ |\nabla v_\varepsilon|^2 + \frac{1}{\varepsilon^2}(1 - |v_\varepsilon|^2)^2 \right\}. \end{aligned}$$

By mean value argument, there are C_1 (depending only on g, Ω and b) and $r \in (\varepsilon, \sqrt{\varepsilon})$ such that

$$r \int_{\mathcal{C}_r} \left\{ |\nabla v_\varepsilon|^2 + \frac{1}{\varepsilon^2}(1 - |v_\varepsilon|^2)^2 \right\} \leq \frac{C_1}{|\ln \varepsilon|}. \tag{3.10}$$

Lemma 3.3. *Let $\delta > 0$. Then, for large n and for r as in (3.10), we have*

1. $\mathrm{Var}\,(v_\varepsilon, \mathcal{C}_r) \leq \delta$, *where* $\mathrm{Var}\,(v_\varepsilon, \mathcal{C}_r) := \displaystyle\int_{\mathcal{C}_r} |\partial_\tau v_\varepsilon|$;

2. $|v_\varepsilon| \geq 1 - 2\delta$ *on* $\mathcal{C}_r$.

Proof. Assertion *1.* is a direct consequence of the bound (3.10), which yields

$$\left(\int_{\mathcal{C}_r} |\partial_\tau v_\varepsilon|\right)^2 \leq \left(\int_{\mathcal{C}_r} |\nabla v_\varepsilon|\right)^2 \leq \int_{\mathcal{C}_r} 1 \int_{\mathcal{C}_r} |\nabla v_\varepsilon|^2 = 2\pi r \int_{\mathcal{C}_r} |\nabla v_\varepsilon|^2 \leq \frac{2\pi C_1}{|\ln \varepsilon|}.$$

It follows that, for large n, we have $|\mathrm{Var}\,(v_\varepsilon, \mathcal{C}_r)| \leq \delta$.

In order to prove *2.*, we argue by contradiction. Assume that there are $\delta > 0$, a subsequence $\{n_k\}_k$ and points $x_{n_k} \in \mathcal{C}_r$ such that $|v_\varepsilon(x_{n_k})| < 1 - 2\delta$ (here $\varepsilon = \varepsilon_{n_k}$).

From the estimate *1.* on $\mathrm{Var}\,(v_\varepsilon, \mathcal{C}_r)$, one has, for large k, $|v_\varepsilon| < 1 - \delta$ on $\mathcal{C}_r$.

Consequently, $r \displaystyle\int_{\mathcal{C}_r} (1 - |v_{\varepsilon_{n_k}}|^2)^2 \geq 2\pi r^2 \delta^2$. Since $r \geq \varepsilon$, this inequality contradicts the estimate (3.10) for small ε. □

So far, we proved the existence of a circle around x_n such that, on that circle, $|v_\varepsilon|$ is close to 1 and v_ε varies little. More specifically: if $0 < \gamma < 1$ then, for large n, there exists $S_\varepsilon \subset \overline{B_1(0)}$ such that

- $\mathrm{dist}(S_\varepsilon, 0) \geq 1 - \gamma$,
- S_ε is the smallest of the two regions delimited by a chord in the closed unit disc,
- $v_\varepsilon(\mathcal{C}_r) \subset S_\varepsilon$.

The following lemma implies that, under the above assumptions on S_ε and on r, we have, for large n, $|v_\varepsilon(x_n)| \geq 1 - \gamma$. This inequality completes the proof of Proposition 3.2, which is the first step in the proof of Theorem 3.1. □

Lemma 3.4. *Let C be a chord in the closed unit disc, C different from a diameter. Let S be the smallest of the two regions enclosed by the chord and the boundary of the disc.*
Let O be a Lipchitz bounded open set and let $g \in H^{1/2}(\partial O, S)$.
Let $\tilde\alpha, \tilde\beta \in L^\infty(O, \mathbb{R})$ satisfy $\operatorname{ess\,inf} \tilde\alpha > 0, \operatorname{ess\,inf} \tilde\beta > 0$.
If v minimizes Ginzburg-Landau type energy

$$\tilde F(v) = \int_O \left\{ \tilde\alpha(x)|\nabla v|^2 + \tilde\beta(x)(1 - |v|^2)^2 \right\}$$

in $H^1_g(O)$, then $v(O) \subset S$.

This lemma is proved in Chapter 2, Appendix 2.C (Lemma 2.38).

Theorem 3.1 holds close to the boundary

We prove that, inside an $o_\varepsilon(1)$-strip along $\partial\Omega$ and for sufficiently small ε, $|v_\varepsilon|$ is arbitrarily close to 1.

The key argument will be provided by the following lemma.

Lemma 3.5. *Let $(x_\varepsilon)_{\varepsilon>0} \subset \Omega$ be such that $r_\varepsilon := \mathrm{dist}(x_\varepsilon, \partial\Omega) \to 0$. Then we have, for all $C \geq 2$, $F_\varepsilon(v_\varepsilon, B_{Cr_\varepsilon}(x_\varepsilon)) \to 0$.*

Proof. Note that it suffices to prove the result for $C = 2$. (For larger values of C, it suffices to replace x_ε by the point at distance $\dfrac{C+1}{2} r_\varepsilon$ from x_ε and at distance $\dfrac{C+3}{2} r_\varepsilon$ from $\partial\Omega$.)

Let $\delta > 0$. We will prove that there is $\varepsilon_\delta > 0$ such that for $\varepsilon < \varepsilon_\delta$, we have $F_\varepsilon(v_\varepsilon, B_{2r_\varepsilon}(x_\varepsilon)) \leq \delta$. For the convenience of the reader, the proof is divided into four steps.

Step 1: Flattening of Ω and choice of a good triangle
Without loss of generality, we may assume that $\partial\Omega$ is flat near x_ε. The general case is obtained by flattening the boundary. This will affect the equation satisfied by v_ε and the energy associated with it, but not the conclusion of the proof below (which relies only on energy bounds and qualitative conclusions derived form the equation of v_ε). From now on, we assume that $\Omega \subset \mathbb{R}^2_+$ and $\partial\Omega \subset \mathbb{R}$ in a neighborhood of fixed size of x_ε. We also assume, without loss of generality, that $x_\varepsilon = (0, r_\varepsilon)$.

For $\ell > 0$, we set

$$T_\ell := \{(s,t)\,|\,t = s+\ell,\ s \in [-\ell, 0]\} \cup \{(s,t)\,|\,t = -s+\ell,\ s \in (0,\ell]\} \subset \mathbb{R}^2_+$$

(thus T_ℓ is the union of two segments).
Denote by ω_ℓ the (solid) triangle enclosed by T_ℓ and $\mathbb{R}$. Then we have $B(x_\varepsilon, 2r_\varepsilon)\cap\Omega \subset \omega_{5r_\varepsilon}$. Our goal is to construct, for an appropriate small ℓ (depending on x_ε and such that $\ell > 5r_\varepsilon$) a test function $h : \omega_\ell \to \mathbb{C}$ such that $\mathrm{tr}_{\partial\omega_\ell} h = \mathrm{tr}_{\partial\omega_\ell} v_\varepsilon$ and $F_\varepsilon(h, \omega_\ell) \to 0$. Since v_ε is a global minimizer of F_ε in $H^1_g(\Omega, \mathbb{C})$, it follows that v_ε is also a minimizer of F_ε in $H^1_{\mathrm{tr}_{\partial\omega_\ell} v}(\omega_\ell, \mathbb{C})$. Our goal is to prove that $F_\varepsilon(v_\varepsilon, \omega_\ell) \to 0$. Since $B_{2r_\varepsilon}(x_\varepsilon) \subset \omega_\ell$, the lemma will follow.

Let $\varepsilon_1 > 0$ be such that for $\varepsilon < \varepsilon_1$, $5r < \sqrt{r}$. Let w be the harmonic extension of g to Ω. We claim that

1. $\exists\, C_1 > 0$ (independent of ε) and $\exists\, \ell \in (5r, \sqrt{r})$ such that

$$\ell \int_{T_\ell} \left\{ |\nabla v_\varepsilon|^2 + \frac{1}{\varepsilon^2}(1 - |v_\varepsilon|^2)^2 + |\nabla w|^2 \right\} \leq \frac{C_1}{|\ln r|}, \tag{3.11}$$

2. $|v_\varepsilon(x)| \underset{x \in T_\ell,\, x \to \partial\Omega}{\longrightarrow} 1$,

3. $|v_\varepsilon| \geq 1/2$ on T_ℓ (for sufficiently small ε).

The claim 1. comes directly from (3.9) and a mean value argument.
Claim 2. is proved in Lemma 3.6 below, using an argument essentially due to Boutet de Monvel and Gabber [23].

In order to prove Claim 3., we start by noting that

$$\left(\int_{T_\ell} |\partial_\tau |v_\varepsilon|| \right)^2 \leq C\ell \int_{T_\ell} |\partial_\tau |v_\varepsilon||^2 \leq C\ell \int_{T_\ell} |\nabla v_\varepsilon|^2 \leq \frac{C'}{|\ln r|}. \tag{3.12}$$

Consequently, there exists $0 < \varepsilon_2 \leq \varepsilon_1$ such that, for $\varepsilon < \varepsilon_2$, the variation of $|v_\varepsilon|$ on T_ℓ is smaller than $1/2$. Since, by Lemma 3.6, we have $|v_\varepsilon| = 1$ at the endpoints of T_ℓ, we obtain that Claim 3. holds.

Lemma 3.6. *Let $\alpha \in W^{1,\infty}(\Omega)$, $\beta \in L^\infty(\Omega; \mathbb{R}_+)$ be such that $\inf \alpha > 0$. Let v be a critical point of $u \mapsto \int \alpha|\nabla u|^2 + \int \beta(1-|u|^2)^2$ in the class $H^1_g(\Omega)$, where $g \in H^{1/2}(\partial\Omega; \mathbb{S}^1)$. Then $|v| \in C(\overline{\Omega})$.*

Proof. We first note that $|v| \leq 1$ a. e. (by the maximum principle. This is obtained, e. g., by noting that $U := 1 - |v|^2$ satisfies $\begin{cases} -\mathrm{div}(\alpha\nabla U) + 4\beta|v|^2 U = 2\alpha|\nabla v|^2 & \text{in } \Omega \\ U = 0 & \text{on } \partial\Omega \end{cases}$, and consequently $U \geq 0$ in Ω.) We next split $v = v_1 + v_2$, where v_1 is the harmonic extension of g. It follows that v_2 satisfies $\begin{cases} -\Delta v_2 = \alpha^{-1}\nabla\alpha \cdot \nabla v + 2\alpha^{-1}\beta v(1 - |v|^2) & \text{in } \Omega \\ v_2 = 0 & \text{on } \partial\Omega \end{cases}$. Since $|v| \leq 1$ and $\alpha \in W^{1,\infty}$, we obtain $v_2 \in H^2(\Omega) \cap H^1_0 \subset C_0(\overline{\Omega})$. On the other hand, we have $v_1 \in C(\Omega)$ and $|v_1| \in C(\overline{\Omega})$ (the last point is essentially due to Boutet de Monvel and Gabber [23]; see also [29], Theorem A.3.2). Therefore, we have $|v| \in C(\overline{\Omega})$. □

Now that ℓ was properly chosen, we construct our test function h. This function will coincide with v_ε outside ω_ℓ. Therefore, we will only explain how to construct h inside ω_ℓ. In order to obtain a globally H^1-map, we will set h equal v_ε on T_ℓ. Let h be of the form $h=\rho e^{\imath\psi}$; in order to have $h=v_\varepsilon$ on T_ℓ, we will make sure that $\rho=|v|$ and $e^{\imath\psi}=\dfrac{v_\varepsilon}{|v_\varepsilon|}$ on T_ℓ. In Step 2, we construct ρ. In Step 3, we construct ψ. Finally, in Step 4 we estimate the energy of h and conclude.

Step 2 : Choice of the modulus ρ of the test function h

Let $\rho:\overline{\omega}_\ell\to[0,1]$ be defined by

$$\rho(s,t)=\begin{cases}\dfrac{t}{s+\ell}\left(|v_\varepsilon(s,s+\ell)|-1\right)+1 & \text{if } s<0\\ \dfrac{t}{-s+\ell}\left(|v_\varepsilon(s,-s+\ell)|-1\right)+1 & \text{if } s>0\end{cases}.$$

Clearly, $\rho\in H^1(\omega_\ell,[0,1])$, $\rho=|v_\varepsilon|$ on T_ℓ and $\rho=1$ on $\partial\omega_\ell\cap\partial\Omega$.

For further use, we estimate $\displaystyle\int_{\omega_\ell}\left\{|\nabla\rho|^2+\frac{1}{\varepsilon^2}(1-\rho^2)^2\right\}$. We denote $\omega_\ell^-=\{x=(s,t)\in\omega_\ell\,|\,s<0\}$ (this is the left half of the triangle ω_ℓ). We will estimate the quantity $\displaystyle\int_{\omega_\ell^-}\left\{|\nabla\rho|^2+\frac{1}{\varepsilon^2}(1-\rho^2)^2\right\}$. By symmetry, a similar estimate will hold in $\omega_\ell^+:=\omega_\ell\setminus\overline{\omega_\ell^-}$, and thus in ω_ℓ.

We have

$$\begin{aligned}\frac{1}{\varepsilon^2}\int_{\omega_\ell^-}(1-\rho^2)^2\leq&\frac{4}{\varepsilon^2}\int_{\omega_\ell^-}(1-\rho)^2\leq\frac{C}{\varepsilon^2}\int_{-\ell}^0\mathrm{d}s\int_0^{\ell+s}\frac{t^2}{(s+\ell)^2}\left(|v_\varepsilon(s,s+\ell)|-1\right)^2\mathrm{d}t\\ \leq&\frac{C\ell}{\varepsilon^2}\int_{-\ell}^0\left(|v_\varepsilon(s,s+\ell)|-1\right)^2\mathrm{d}s\leq\frac{C\ell}{\varepsilon^2}\int_{T_\ell}\left(|v_\varepsilon|-1\right)^2\mathrm{d}s\leq\frac{C}{|\ln r|}.\end{aligned}$$

(The last inequality comes from Claim 1.)

In order to estimate $\displaystyle\int_{\omega_\ell^-}|\nabla\rho|^2$, we start from the identity

$$\int_{\omega_\ell^-}|\nabla\rho|^2=\int_{-\ell}^0\mathrm{d}s\int_0^{\ell+s}\mathrm{d}t\left\{|\partial_s\rho|^2+|\partial_t\rho|^2\right\}.$$

On the one hand,

$$\begin{aligned}\int_{-\ell}^0\mathrm{d}s\int_0^{\ell+s}\mathrm{d}t|\partial_t\rho|^2=&\int_{-\ell}^0\frac{\left(|v_\varepsilon(s,s+\ell)|-1\right)^2}{s+\ell}\mathrm{d}s\ =\int_{-\ell}^0\frac{\mathrm{d}s}{s+\ell}\left(\int_{-\ell}^s\frac{\mathrm{d}}{\mathrm{d}k}\left[|v_\varepsilon|(k,k+\ell)\right]\right)^2\\ \leq&\sqrt{2}\ell\int_{T_\ell}|\nabla v_\varepsilon|^2\leq\frac{C}{|\ln r|}.\end{aligned}$$

On the other hand, we have

$$|\partial_s\rho|^2\leq2\left(\frac{t^2}{(s+\ell)^4}\left(|v_\varepsilon(s,s+\ell)|-1\right)^2+\frac{t^2}{(s+\ell)^2}\left(\nabla|v_\varepsilon|(s,s+\ell)\cdot(1,1)\right)^2\right)=2(A_1+A_2).$$

Since

$$\int_{-\ell}^0\int_0^{\ell+s}A_1\leq\int_{-\ell}^0\int_0^{\ell+s}\frac{1}{(s+\ell)^2}\left(|v_\varepsilon(s,s+\ell)|-1\right)^2=\int_{-\ell}^0\frac{1}{s+\ell}\left(|v_\varepsilon(s,s+\ell)|-1\right)^2\leq\frac{C}{|\ln r|}$$

and

$$\int_{-\ell}^{0}\int_{0}^{\ell+s} A_2 \le 2\ell \int_{T_\ell} |\nabla v_\varepsilon|^2 \le \frac{C}{|\ln r|},$$

we find that $\displaystyle\int_{\omega_\ell} |\nabla \rho|^2 \le \frac{C}{|\ln r|}$. In conclusion, the following estimate holds:

$$\int_{\omega_\ell} \left\{ |\nabla \rho|^2 + \frac{1}{\varepsilon^2}(1-\rho^2)^2 \right\} \le \frac{C}{|\ln r|}. \tag{3.13}$$

Step 3 : Construction of an auxiliary phase ψ

Recall that $|w(z)| \to 1$ uniformly as $z \to \partial\Omega$ [29]. Thus, there is some $0 < \varepsilon_3 \le \varepsilon_2$ such that for $\varepsilon < \varepsilon_3$ we have $|w| \ge 1/2$ in ω_ℓ. For $\varepsilon < \varepsilon_3$, we may write, in ω_ℓ, $w = |w|\mathrm{e}^{\imath\varphi}$ with $\varphi \in H^1(\omega_\ell, \mathbb{R})$. Note that, by choice of ℓ, we have $|v_\varepsilon| \ge 1/2$ on T_ℓ and $v_\varepsilon \in H^1(T_\ell)$. Therefore, we may write $v_\varepsilon = |v_\varepsilon|\mathrm{e}^{\imath\phi}$ on T_ℓ, with $1/2 \le |v_\varepsilon| \le 1$ and $\phi \in H^1(T_\ell)$.

Since $v_\varepsilon - w \in C(\overline{\Omega})$ (*cf* the proof of Lemma 3.6) and $v_\varepsilon = w$ on $\partial\Omega$, it follows that $\lim_{z\to\partial\Omega} (v_\varepsilon - w)(z) = 0$. Therefore, we have

$$\lim_{\substack{z\to\partial\Omega \\ z\in T_\ell\cap\partial\omega_\ell^-}} \mathrm{e}^{\imath(\phi(z)-\varphi(z))} = \lim_{\substack{z\to\partial\Omega \\ z\in T_\ell\cap\partial\omega_\ell^+}} \mathrm{e}^{\imath(\phi(z)-\varphi(z))} = 1.$$

Consequently, there are $k_+, k_- \in \mathbb{Z}$ such that

$$\lim_{\substack{z\to\partial\Omega \\ z\in T_\ell\cap\partial\omega_\ell^-}} \frac{\phi(z)-\varphi(z)}{2\pi} = k_- \text{ and } \lim_{\substack{z\to\partial\Omega \\ z\in T_\ell\cap\partial\omega_\ell^+}} \frac{\phi(z)-\varphi(z)}{2\pi} = k_+.$$

By (3.11) and the fact that $|v_\varepsilon|, |w| \ge \frac{1}{2}$ on T_ℓ, we obtain $\ell \displaystyle\int_{T_\ell} \left\{ |\nabla\phi|^2 + |\nabla\varphi|^2 \right\} \le \frac{C}{|\ln r|}$. Thus, for small ε, the variations of ϕ and φ are small on $\partial\omega_\ell \setminus \partial\Omega$ and consequently, there is $0 < \varepsilon_4 < \varepsilon_3$ such that for $\varepsilon < \varepsilon_4$, we have $k_- = k_+$. Without loss of generality, we may assume $k_- = k_+ = 0$.

Let $\psi : \overline{\omega}_\ell \to \mathbb{R}$ be defined by

1. $\mathrm{tr}_{\partial\omega_\ell}\psi = \mathrm{tr}_{\partial\omega_\ell}(\phi - \varphi)$,

2. $\psi(s,t) = \begin{cases} \dfrac{t}{\ell+s}\,[\phi(s,s+\ell) - \varphi(s,s+\ell)] & \text{if } s<0 \\ \dfrac{t}{\ell-s}\,\big[\phi(s,-s+\ell) - \varphi'(s,-s+\ell)\big] & \text{if } s>0 \end{cases}.$

For further use, we estimate the Dirichlet energy of ψ. It suffices to estimate the energy in ω_ℓ^-; a similar estimate holds in ω_ℓ.

We have

$$\int_{\omega_\ell^-} |\nabla\psi|^2 = \int_{-\ell}^{0} \mathrm{d}s \int_{0}^{\ell+s} \mathrm{d}t \left\{ |\partial_s\psi|^2 + |\partial_t\psi|^2 \right\} = B_1 + B_2.$$

First, we obtain, denoting $\xi = \phi - \varphi$,

$$B_1 = \int_{-\ell}^{0}\int_{0}^{\ell+s} |\partial_s\psi|^2 \le 2\int_{-\ell}^{0}\int_{0}^{\ell+s} \left\{ \left| \frac{\xi(s,s+\ell)}{\ell+s} \right|^2 + \left| \frac{\mathrm{d}}{\mathrm{d}s}\xi(s,s+\ell) \right|^2 \right\} = 2(B_{11} + B_{12}).$$

Now

$$B_{11} = \int_{-\ell}^{0} \frac{1}{\ell+s} |\xi(s,s+\ell)|^2 \leq \int_{-\ell}^{0} \frac{1}{\ell+s} \left| \int_{-\ell}^{s} \left| \frac{\mathrm{d}}{\mathrm{d}\alpha} \xi(\alpha,\alpha+\ell) \right| \mathrm{d}\alpha \right|^2$$

$$\leq C \int_{-\ell}^{0} \int_{T_\ell} |\mathrm{d}\xi|^2 \leq \ell \int_{T_\ell} |\mathrm{d}\xi|^2 \leq \frac{C}{|\ln r|}.$$

Next, we have

$$B_{12} = \int_{-\ell}^{0} \int_{0}^{\ell+s} |\mathrm{d}\xi|^2(s,s+\ell) \leq \ell \int_{T_\ell} |\mathrm{d}\xi|^2 \leq \frac{C}{|\ln r|}.$$

Similarly, we have $B_2 \leq \dfrac{C}{|\ln r|}$.

Finally, we find that

$$\int_{\omega_\ell} |\nabla\psi|^2 \leq \frac{C}{|\ln r|}. \tag{3.14}$$

Step 4: Conclusion (proof of Lemma 3.5 completed)

Consider the following test function

$$h := \begin{cases} v & \text{in } \Omega \setminus \omega_\ell \\ \rho e^{\imath(\varphi+\psi)} & \text{in } \omega_\ell \end{cases}.$$

Clearly $h \in H^1_g$ and

$$F_\varepsilon(v_\varepsilon, B_{2r_\varepsilon}(x_\varepsilon)) \leq F_\varepsilon(v_\varepsilon,\omega_\ell) \leq F_\varepsilon(h,\omega_\ell) \leq \frac{C}{|\ln r|} + 4\int_{\omega_\ell} |\nabla w|^2. \tag{3.15}$$

The last estimate follows by combining (3.13) with (3.14) and the fact that $|\nabla h|^2 = |\nabla\rho|^2 + \rho^2|\nabla(\varphi+\psi)|^2$.

Since $\displaystyle\int_{\omega_\ell} |\nabla w|^2 \to 0$ as $\varepsilon \to 0$, we find that $F_\varepsilon(v, B_{2r}(x)) < \delta$ for small ε. □

The next result completes the proof of Theorem 3.1.

Proposition 3.7. *Let* $\varepsilon_n \downarrow 0$ *and* $\{x_n\}_n \subset \Omega$ *be such that* $\operatorname{dist}(x_n,\partial\Omega) \to 0$. *Then* $|v_{\varepsilon_n}(x_n)| \to 1$.

Proof of Proposition 3.7. Let $\delta \in (0,1)$. Denote $d_n := \operatorname{dist}(x_n,\partial\Omega)$ and $v_n := v_{\varepsilon_n}$. Since there is $C_0 > 0$ such that $F_{\varepsilon_n}(v_n) \leq C_0$, we may choose $C_1 > 1$ and $r_n \in (d_n/C_1, d_n)$ such that

$$\frac{2\pi C_0}{\ln C_1} < \frac{\delta}{10^4} \tag{3.16}$$

and

$$r_n \int_{\mathcal{C}_n} \left\{ |\nabla v_n|^2 + \frac{1}{\varepsilon_n^2}(1-|v_n|^2)^2 \right\} \leq \frac{C_0}{\ln C_1}, \text{ with } \mathcal{C}_n = \{x \in \Omega \,|\, |x-x_n| = r_n\}. \tag{3.17}$$

As in the proof of *1.* in Lemma 3.3, we have

$$[\operatorname{Var}(v_n,\mathcal{C}_n)]^2 \leq \frac{2\pi C_0}{\ln C_1}. \tag{3.18}$$

Using (3.18) and the bound (3.16), we find that one of the two cases occurs:

1. $|v_n| \geq 1 - \dfrac{\delta}{10}$ on $\mathcal{C}_n$,

2. $|v_n| < 1 - \dfrac{\delta}{10^3}$ on $\mathcal{C}_n$.

In the first case, using (3.18) and Lemma 3.4, we obtain $|v_n(x_n)| \geq 1 - \delta$.

Assume that for infinitely many n the second case occurs. Up to subsequence, we may assume that it is true for each n.

For large n, let $y_n := \Pi_{\partial\Omega}(x_n)$ be the orthogonal projection of x_n on $\partial\Omega$ and let x'_n be the intersection point of the segment $[x_n, y_n]$ with $\mathcal{C}_n$. For large n and for all $z \in T_n := \left\{z \in \mathcal{C}_n \,|\, |x'_n - z| \leq \dfrac{r}{2}\right\}$ we have

$$|z - w_z| \leq 3d_n. \tag{3.19}$$

Here, w_z is the first intersection point with $\partial\Omega$ of the ray starting from x and passing through z.

Note that

$$z \in T_n \Leftrightarrow z = x_n + (x'_n - x_n)e^{\imath\theta} \text{ with } \theta \in [-\pi/6, \pi/6]. \tag{3.20}$$

For $\theta \in [-\pi/6, \pi/6]$ we denote $I_\theta := [z, w_z]$, where $z = z(\theta)$ is given by (3.20). Since $|v_n(z)| < 1 - \dfrac{\delta}{10^3}$ and $|v_n(w_z)| = 1$ we have

$$\frac{\delta^2}{10^6} \leq \left(\int_{I_\theta} \partial_\tau |v_n|\right)^2 \leq 3d_n \int_{I_\theta} |\partial_\tau v_n|^2. \tag{3.21}$$

Denote $A := \displaystyle\bigcup_{\theta\in[-\pi/6,\pi/6]} I_\theta$ and write each $x \in A$ as $x = x_n + s e^{\imath\theta}$ ($s \geq r_n$). By (3.19), (3.20) and (3.21) we have

$$\int_A |\nabla v_n|^2 \geq \int_{-\pi/6}^{\pi/6} \mathrm{d}\theta \int_{I_\theta} |\partial_\tau v_n|^2 s \,\mathrm{d}s \geq \frac{\pi}{3C_1} \inf_{z\in T_n} d_n \int_{I_\theta} |\partial_\tau v_n|^2 \geq \frac{\pi\delta^2}{9\cdot 10^6 \cdot C_1}.$$

Since C_1 is independent of n and $A \subset B_{3d_n}(x_n)$, the above estimate contradicts Lemma 3.5.

Hence, for sufficiently large n, we have $|v_n| \geq 1 - \dfrac{\delta}{10}$ on $\mathcal{C}_n$. This estimate together with Lemma 3.4 implies $|v_n(x_n)| \geq 1 - \delta$. □

3.2.2 A corollary of Theorem 3.1

From Theorem 3.1 one may easily prove that the contribution of the modulus is negligible. Indeed we have

Corollary 3.8. *The following hold.*

1. *We have* $\displaystyle\int_\Omega \left\{|\nabla|v_\varepsilon||^2 + \frac{1}{\varepsilon^2}(1 - |v_\varepsilon|^2)^2\right\} \to 0$ *as* $\varepsilon \to 0$.
   In particular, we have $|v_\varepsilon| \to 1$ *in* $H^1(\Omega)$.

2. *Assume that (possibly along some subsequence) we have* $\alpha_\varepsilon \to \kappa$ *in* $L^2(\Omega)$. *Write* $g = e^{\imath\varphi_0}$ *(see [17]), where* $\varphi_0 \in H^{1/2}(\partial\Omega, \mathbb{R})$. *Write, for small* ε, $v_\varepsilon = |v_\varepsilon|e^{\imath\varphi_\varepsilon}$, *where* $\varphi_\varepsilon \in H^1_{\varphi_0}(\Omega, \mathbb{R})$. *Then* $\varphi_\varepsilon \to \varphi^*$ *in* $H^1(\Omega)$, *where* φ^* *is the solution of* $\begin{cases} -\mathrm{div}(\kappa\nabla\varphi^*) = 0 & \text{in } \Omega \\ \varphi^* = \varphi_0 & \text{on } \partial\Omega \end{cases}.$

The above statement implicitly uses two results on lifting, for which we refer to [21, 22]. The first one is that each zero degree map $g \in H^{1/2}(\partial\Omega;\mathbb{S}^1)$ may be lifted as $g = e^{\imath\varphi_0}$ for some $\varphi_0 \in H^{1/2}(\partial\Omega;\mathbb{R})$. The second is that each map in $u \in H^1_g(\Omega;\mathbb{S}^1)$ may be written as $u = e^{\imath\varphi}$, with $\varphi \in H^1_{\varphi_0}(\Omega;\mathbb{R})$. Consequently, each map $u \in H^1_g(\Omega;\mathbb{R}^2)$ such that $0 <$*essinf* $|u| \leq$*esssup* $|u| < \infty$ may be written as $u = \rho e^{\imath\varphi}$, where $\rho = |u| \in H^1_1(\Omega;\mathbb{R}_+)$ and $\varphi \in H^1_{\varphi_0}(\Omega;\mathbb{R})$.

Proof. We start by noting that $b \leq \kappa \leq 1$.

Let v_ε be a minimizer of F_ε in H^1_g. By Theorem 3.1, we may write, for small ε , $v_\varepsilon = \rho_\varepsilon e^{\imath\varphi_\varepsilon}$, with $1/2 \leq \rho_\varepsilon := |v_\varepsilon| \leq 1$ and $\varphi_\varepsilon \in H^1_{\varphi_0}(\Omega,\mathbb{R})$.

Recall that $F_\varepsilon(v_\varepsilon) \leq C_0$ (with C_0 depending only on g,Ω and b). Thus, for small ε, we have $\displaystyle\int_\Omega |\nabla\varphi_\varepsilon|^2 \leq \frac{8C_0}{b}$.

If we set $w_\varepsilon := e^{\imath\varphi_\varepsilon} \in H^1_g$, then we have

$$F_\varepsilon(v_\varepsilon) = \frac{1}{2}\int_\Omega \left\{\alpha_\varepsilon(\rho_\varepsilon^2|\nabla\varphi_\varepsilon|^2 + |\nabla\rho_\varepsilon|^2) + \frac{\beta_\varepsilon}{2\varepsilon^2}(1-\rho_\varepsilon^2)^2\right\} \leq F_\varepsilon(w_\varepsilon) = \frac{1}{2}\int_\Omega \alpha_\varepsilon|\nabla\varphi_\varepsilon|^2.$$

Consequently,

$$\int_\Omega \left\{|\nabla\rho_\varepsilon|^2 + \frac{1}{\varepsilon^2}(1-\rho_\varepsilon^2)^2\right\} \leq \frac{2}{b}\int_\Omega (1-\rho_\varepsilon^2)|\nabla\varphi_\varepsilon|^2 \leq \frac{16C_0}{b^2}\|1-\rho_\varepsilon^2\|_{L^\infty(\Omega)} \underset{\varepsilon\to 0}{\to} 0.$$

We now prove *2.* We start by noting that $\varphi_\varepsilon - \varphi^*$ satisfies

$$\begin{cases} -\mathrm{div}[\alpha_\varepsilon\rho_\varepsilon^2\nabla(\varphi_\varepsilon - \varphi^*)] = \mathrm{div}[(\alpha_\varepsilon\rho_\varepsilon^2 - \kappa)\nabla\varphi^*] & \text{in } \Omega \\ \varphi_\varepsilon - \varphi^* = 0 & \text{on } \partial\Omega \end{cases}.$$

By the Lax-Milgram theorem, we find that

$$\|\nabla(\varphi_\varepsilon - \varphi^*)\|_{L^2} \leq C\|(\alpha_\varepsilon\rho_\varepsilon^2 - \kappa)\nabla\varphi^*\|_{L^2}. \tag{3.22}$$

We will next use the following simple fact: if $|f_n| \leq C$ and $f_n \to f$ in L^2 and if $g_n \to g$ in L^2, then $f_n g_n \to fg$ in L^2. This implies that $\alpha_\varepsilon\rho_\varepsilon^2 - \kappa \to 0$ in L^2 as $\varepsilon \to 0$. Finally, (3.22) implies that $\varphi_\varepsilon \to \varphi^*$ in H^1.

□

3.2.3 More on the convergence of v_ε

This part provides a more quantitative version of Theorem 3.1. Specifically, under some additional hypotheses on the boundary data g or on the behavior of the weight α_ε, we derive estimates on the rate of convergence of $|v_\varepsilon|$ to 1 or derive better convergence of the phase φ_ε of v_ε respectively.

In what follows, we assume that $g \in W^{1-1/q,q}(\partial\Omega,\mathbb{S}^1)$ for some $q > 2$. Let $\varphi_0 \in W^{1-1/q,q}(\partial\Omega,\mathbb{R})$ be such that $e^{\imath\varphi_0} = g$ (for the existence of φ_0, see, e. g., [21]). For a fixed measurable function $\kappa : \Omega \to [b,1]$, let $\varphi^* \in W^{1,q}(\Omega,\mathbb{R})$ be the solution of $\begin{cases} -\mathrm{div}(\kappa\nabla\varphi^*) = 0 & \text{in } \Omega \\ \varphi^* = \varphi_0 & \text{on } \partial\Omega \end{cases}.$

Proposition 3.9. *There is $p \in (2,q], \alpha \in (0,1), C > 0$ (depending only on q, b, Ω and g) such that, for $0 < \varepsilon < 1$ and v_ε a minimizer of F_ε in H^1_g, we have*

1. *$\{v_\varepsilon\}$ is bounded in $W^{1,p}$ by a constant C which depends only on g, b and Ω.*

2. *$\{v_\varepsilon\}$ is relatively compact in $C^{0,\alpha}(\overline{\Omega})$.*

3. *$1-|v_\varepsilon| \le C\varepsilon^\gamma$ and $\displaystyle\int_\Omega \left\{ |\nabla |v_\varepsilon||^2 + \frac{1}{\varepsilon^2}(1-|v_\varepsilon|^2)^2 \right\} \le C\varepsilon^\gamma$ with $\gamma = \dfrac{2\alpha}{2+\alpha}$.*

4. *Furthermore, if (possibly after passing to a subsequence) we have $\alpha_\varepsilon \to \kappa$ in L^2, then we have $\varphi_\varepsilon \to \varphi^*$ in $W^{1,p}$.*
   Here, we write, for small ε and in virtue of Theorem 3.1, $v_\varepsilon = \rho_\varepsilon e^{\imath\varphi_\varepsilon}$, with $\varphi_\varepsilon \in H^1_{\varphi_0}$, $\rho_\varepsilon := |v_\varepsilon| \in [1/2, 1]$.

Proof. Let φ be any fixed $W^{1,q}$-extension of φ_0. Then $\varphi_\varepsilon - \varphi$ satisfies

$$\begin{cases} -\mathrm{div}\left[\alpha_\varepsilon \rho_\varepsilon^2 \nabla(\varphi_\varepsilon - \varphi)\right] = \mathrm{div}(\alpha_\varepsilon \rho_\varepsilon^2 \nabla\varphi) & \text{in } \Omega \\ \varphi_\varepsilon - \varphi = 0 & \text{on } \partial\Omega \end{cases}. \tag{3.23}$$

Since

$$\|\alpha_\varepsilon \rho_\varepsilon^2 \nabla \varphi\|_{L^q(\Omega)} \le C,$$

it follows from Theorem 1 in [52] that there are $p_1 \in (2, q]$ and $C > 0$ (depending only on b and Ω) such that $\|\nabla(\varphi_\varepsilon - \varphi)\|_{L^{p_1}(\Omega)} \le C$. Thus $\{\varphi_\varepsilon\}$ is bounded in $W^{1,p_1}(\Omega)$.

We next prove that $\|1-\rho_\varepsilon\|_{L^{p_1/2}} \le C\varepsilon^2$. For this purpose, we start with the equation satisfied by ρ_ε:

$$\begin{cases} \mathrm{div}(\alpha_\varepsilon \nabla \rho_\varepsilon) + \dfrac{\beta_\varepsilon}{\varepsilon^2}\rho_\varepsilon(1-\rho_\varepsilon^2) = \alpha_\varepsilon \rho_\varepsilon |\nabla \varphi_\varepsilon|^2 & \text{in } \Omega \\ 1 - \rho_\varepsilon = 0 & \text{on } \partial\Omega \end{cases}. \tag{3.24}$$

Let $\eta_\varepsilon := 1 - \rho_\varepsilon$ and $p_1 > 2$ be as in the conclusion of Theorem 1 in [52]. Set $r := p_1/2$ and consider a sequence $\{\phi_k\} \subset C^\infty([0,1],[0,1])$ such that

$$\phi_k \text{ is nondecreasing, } \phi_k(0) = 0 \text{ and } \phi_k(s) \to |s|^{r-1} \text{ as } k \to \infty,\ \forall\, s \in [0,1].$$

Let $A_\varepsilon := \beta_\varepsilon \rho_\varepsilon(1+\rho_\varepsilon)$, which satisfies, for small ε, $3b/4 \le A_\varepsilon \le 2$. Set $B_\varepsilon := \alpha_\varepsilon \rho_\varepsilon |\nabla\varphi_\varepsilon|^2$, which is bounded in $L^{p_1/2}$. If we multiply (3.24) by $\phi_k(\eta_\varepsilon)$, we find that

$$\int_\Omega \alpha_\varepsilon |\nabla \eta_\varepsilon|^2 \phi_k'(\eta_\varepsilon) + \frac{1}{\varepsilon^2}\int_\Omega A_\varepsilon \eta_\varepsilon \phi_k(\eta_\varepsilon) = \int_\Omega B_\varepsilon \phi_k(\eta_\varepsilon).$$

Consequently, we have

$$\int_\Omega \eta_\varepsilon \phi_k(\eta_\varepsilon) \le C\varepsilon^2 \int_\Omega B_\varepsilon \phi_k(\eta_\varepsilon). \tag{3.25}$$

Note that, in (3.25), the constant C depends only on b. By letting $k \to \infty$, we obtain, with s being the conjugate exponent of r, that

$$\int_\Omega \eta_\varepsilon^r \le C\varepsilon^2 \int_\Omega B_\varepsilon \eta_\varepsilon^{r-1} \le C\varepsilon^2 \left(\int_\Omega \eta_\varepsilon^r\right)^{\frac{1}{s}} \|B_\varepsilon\|_{L^r}.$$

This implies that $\|1-\rho_\varepsilon\|_{L^{p_1/2}} \le C\varepsilon^2$ which we wanted to prove.

Going back to (3.24), we observe that η_ε satisfies $\mathrm{div}(\alpha_\varepsilon \nabla \eta_\varepsilon) = h_\varepsilon$, where h_ε is bounded in $L^{p_1/2}(\Omega)$. Using again [52], we find that there is some $p_2 > 2$ such that $\nabla \eta_\varepsilon$ is bounded in $L^{p_2}(\Omega)$.

It follows that v_ε is bounded in $W^{1,p}(\Omega)$, with $p := \min(p_1, p_2) > 2$.

We next prove that $|1-\rho_\varepsilon| \le C\varepsilon^\gamma$ and $\displaystyle\int_\Omega \left\{ |\nabla|v_\varepsilon||^2 + \frac{1}{\varepsilon^2}(1-|v_\varepsilon|^2)^2 \right\} \le C\varepsilon^\gamma$, where $\gamma := \dfrac{p-2}{p-1}$.

Indeed, let $\alpha := 1 - \dfrac{2}{p}$, so that v_ε is bounded in $C^\alpha(\overline{\Omega})$ and $\displaystyle\int_\Omega (1-\rho_\varepsilon) \le C\varepsilon^2$. Let $x_0 = x_0(\varepsilon)$ be a minimum point of ρ_ε in $\overline{\Omega}$. Since Ω is smooth, for $r > 0$ sufficiently small we have $|B_r(x_0) \cap \Omega| \ge Cr^2$. It follows that

$$C\varepsilon^2 \ge \int_{B_r(x_0)} (1-\rho_\varepsilon) \ge C(1-\rho_\varepsilon(x_0) - Cr^\alpha)r^2.$$

With $r := \varepsilon^{\frac{2}{\alpha+2}}$, we find that $1 - \rho_\varepsilon(x_0) = \sup\limits_{\overline{\Omega}}\{1-\rho_\varepsilon\} \le C\varepsilon^\gamma$.

The above estimate together with the inequality $F_\varepsilon(v_\varepsilon) \le F_\varepsilon(e^{\imath\varphi_\varepsilon})$ yield the bound on $\nabla\rho_\varepsilon$:

$$\int_\Omega \left\{\alpha_\varepsilon|\nabla\rho_\varepsilon|^2 + \frac{\beta_\varepsilon}{2\varepsilon^2}(1-\rho_\varepsilon^2)^2\right\} \le \int_\Omega \alpha_\varepsilon(1-\rho_\varepsilon^2)|\nabla\varphi_\varepsilon|^2 \le C\varepsilon^\gamma.$$

Finally, *4.* follows from the equation

$$\begin{cases} -\mathrm{div}\left[\alpha_\varepsilon\rho_\varepsilon^2\nabla(\varphi_\varepsilon - \varphi^*)\right] = \mathrm{div}\left[(\alpha_\varepsilon\rho_\varepsilon^2 - \kappa)\nabla\varphi^*\right] & \text{in } \Omega \\ \varphi_\varepsilon - \varphi^* = 0 & \text{on } \partial\Omega \end{cases}.$$

Indeed, since $(\alpha_\varepsilon\rho_\varepsilon^2 - \kappa)\nabla\varphi^* \to 0$ in $L^{p_3}(\Omega)$ for a suitable p_3 such that $\nabla\varphi^* \in L^{p_3}$, we obtain, using again [52], that $\varphi_\varepsilon \to \varphi^*$ in W^{1,p_4}, for a suitable $p_4 > 2$. We conclude by choosing $p := \min\{p_1, \dots, p_4\}$. □

3.3 The Ginzburg-Landau functional with a periodic pinning term

In this part, we apply the results obtained in the previous section to the study of a Ginzburg-Landau energy with a discontinuous periodic pinning term. Inside unit square $Y = [0,1)^2$, consider a smooth subset $\omega \prec Y$, which will play a role of inclusion (or impurity). The relative size of this inclusion (with respect to the size of the square) will be controlled by some parameter $\lambda > 0$ in the following way: for $x_0 \in \omega$, we set $\omega_\lambda = \lambda\omega + (1-\lambda)x_0$. We now define the pinning term $a = a(x,\lambda)$ so that it takes different constant values inside and outside of the inclusion:

$$a(x,\lambda) = \begin{cases} b & \text{if } x \in \omega_\lambda \\ 1 & \text{if } x \in Y \setminus \omega_\lambda \end{cases}, \tag{3.26}$$

where $b \in (0,1)$ is a fixed (material) parameter. We extend a to a periodic function in $\mathbb{R}^2$.

The analysis we develop here could apply to the more complicated situation where x_0 is allowed to depend on λ; however, we will not pursue in this direction here.

Let $\Omega \subset \mathbb{C}$ be a smooth, bounded, simply connected domain. For $1 > \delta > 0$, denote $\{C_n^\delta, n \ge 1\}$ a partition of $\mathbb{R}^2$ into squares with side δ; for simplicity, we suppose that the origin is an edge of one of the squares. We may assume, with no loss of generality, that the squares that lie inside Ω are labelled C_n^δ with $1 \le n \le N_\delta$. Denote $\Omega_\delta := \bigcup\limits_{n=1}^{N_\delta} C_n^\delta$.

We define the pinning term in Ω as

$$a_\varepsilon(x) = \begin{cases} a(x/\delta,\lambda) & \text{if } x \in \Omega_\delta \\ 1 & \text{if } x \in \Omega \setminus \Omega_\delta \end{cases};$$

the notation a_ε is justified by the fact that we will later let δ depend on the parameter ε. The values of a_ε are represented in Figure 4 in the introduction (Page xv).

The following energy will be associated with this pinning term:

$$E_\varepsilon(u) = \frac{1}{2}\int_\Omega \left\{ |\nabla u|^2 + \frac{1}{2\varepsilon^2}(a_\varepsilon^2 - |u|^2)^2 \right\}.$$

Under the Dirichlet condition $\mathrm{tr}_{\partial\Omega} u = 1$, one has the existence of the unique minimizer U_ε of E_ε [43].

The following lemma is straightforward.

Lemma 3.10. *There exists a constant C (independent of $\varepsilon \in (0,1)$) such that*

$$E_\varepsilon(U_\varepsilon) \le C\lambda \min\left(\frac{1}{\varepsilon\delta}, \frac{\lambda}{\varepsilon^2}\right)$$

and

$$|\nabla U_\varepsilon| \le \frac{C}{\varepsilon}.$$

When $\varepsilon < \lambda\delta$, the above lemma is obtained by considering as a test function an ε-regularization of a_ε. When $\varepsilon \ge \lambda\delta$, it suffices to estimate the energy of the test function 1.

As explained in [43], if u is of modulus 1 on $\partial\Omega$ and we set $v := u/U_\varepsilon$, then the energy E_ε decouples as follows:

$$E_\varepsilon(u) = E_\varepsilon(U_\varepsilon) + F_\varepsilon(v),$$

where

$$F_\varepsilon(v) := \frac{1}{2}\int_\Omega \left\{ U_\varepsilon^2 |\nabla v|^2 + \frac{U_\varepsilon^4}{2\varepsilon^2}(1 - |v|^2)^2 \right\}.$$

We next note that, by the maximum principle, we have $b \le U_\varepsilon \le 1$. Thus F_ε satisfies the assumptions of Theorem 3.1, Corollary 3.8 and Proposition 3.9. Therefore, if we let u_ε minimize E_ε in H^1_g, where $g : \partial\Omega \to \mathbb{S}^1$ is of zero degree, if U_ε minimizes E_ε in H^1_1 and if we decompose $u_\varepsilon = U_\varepsilon v_\varepsilon$, then the conclusions of these results apply to v_ε.

To be more specific, we fix $g \in H^{1/2}(\partial\Omega, \mathbb{S}^1)$ such that $\deg_{\partial\Omega}(g) = 0$. Then:

1. there is some $\varphi_0 \in H^{1/2}(\Omega, \mathbb{R})$ is such that $g = e^{i\varphi_0}$
2. we decompose a minimizer u_ε of E_ε in H^1_g as $u_\varepsilon = U_\varepsilon v_\varepsilon$, where U_ε minimizes E_ε in H^1_1 and v_ε minimizes F_ε in H^1_g
3. using Theorem 3.1 we have, for small ε, $|u_\varepsilon| \ge b/2$. Thus we may decompose, for small ε, $u_\varepsilon = |u_\varepsilon| \mathrm{e}^{\imath\varphi_\varepsilon}$ with $\varphi_\varepsilon \in H^1_{\varphi_0}(\Omega, \mathbb{R})$
4. consequently, for small ε we have $v_\varepsilon = |v_\varepsilon| \mathrm{e}^{\imath\varphi_\varepsilon}$ with $|u_\varepsilon| = U_\varepsilon |v_\varepsilon|$.

From Corollary 3.8, we know that $|v_\varepsilon| \to 1$ uniformly and in H^1. Consequently, we will obtain the asymptotics of u_ε from the one of U_ε and of φ_ε.

The remaining part of this section is devoted to the asymptotic analysis of U_ε and v_ε; as a byproduct, this will give the asymptotics of u_ε. It turns out that the analysis is governed by the relation between ε and δ, as well as by the size of λ. Possibly after passing to subsequences and rescaling, we may assume, with no loss of generality, that we are in one of the four following cases:
Section 3.3.1: $\lambda \to 0$, the dilute case,
Section 3.3.2: $\lambda = 1, \delta = \varepsilon$, the critical case,
Section 3.3.3: $\lambda = 1, \varepsilon \ll \delta$, the physical case,
Section 3.3.4: $\lambda = 1, \delta \ll \varepsilon$, the non-physical case.

3.3.1 The dilute limit $\lambda \to 0$

Behavior of U_ε

In this case, the energy bound given by Lemma 3.10 immediately implies

Proposition 3.11. *We have*

$$U_\varepsilon \to 1 \text{ in } L^2(\Omega). \tag{3.27}$$

Limit of φ_ε

Proposition 3.12. *Let φ_* be the harmonic extension of φ_0 in Ω. Then, as $\varepsilon \to 0$,*

1. *$\varphi_\varepsilon \to \varphi_*$ in H^1*
2. *if, in addition, there is some $q > 2$ such that $g \in W^{1-1/q,q}(\partial\Omega)$, then we have $\varphi_\varepsilon \to \varphi_*$ in $W^{1,p}$ for some suitable $p \in (2, q]$.*

Proof. The first part is a direct consequence of Corollary 3.8 and of Proposition 3.11. The second part is a direct consequence of Propositions 3.9 and 3.11. □

3.3.2 The case $\lambda = 1, \delta = \varepsilon$

Limit of U_ε

Recall that $Y := [0, 1)^2$. Let

$$H^1_{\text{per}}(Y, \mathbb{R}) = \{u \in H^1(Y, \mathbb{R}) \,|\, \text{the extension by } Y\text{-periodicity of } u \text{ in } \mathbb{R}^2 \text{ is in } H^1_{\text{loc}}(\mathbb{R}^2)\}.$$

We define similarly $H^1_{\text{per}}(Y, \mathbb{C})$. For simplicity, we ignore the reference to $\mathbb{R}$ or $\mathbb{C}$ when irrelevant.
Note that $u \in H^1(Y)$ extends to a Y-periodic H^1_{loc}-map if and only if

$$\text{tr}_{\{y_1=0\}} u(0, \cdot) = \text{tr}_{\{y_1=1\}} u(1, \cdot) \text{ and } \text{tr}_{\{y_2=0\}} u(\cdot, 0) = \text{tr}_{\{y_2=1\}} u(\cdot, 1)$$

$$\Leftrightarrow y_1(1 - y_1) \, [u(y_1, y_2) - u(y_1, 1 - y_2)] + y_2(1 - y_2) \, [u(y_1, y_2) - u(1 - y_1, y_2)] \in H^1_0(Y).$$

Using these characterizations of $H^1_{\text{per}}(Y)$, we find that $H^1_{\text{per}}(Y)$ is weakly H^1-closed. (For more properties of $H^1_{\text{per}}(Y)$, see, e. g., [31], part 3.4.)

It follows that there exists $\hat{u}$ which is a minimizer of

$$\mathcal{E}(u) = \frac{1}{2} \int_Y \left\{ |\nabla u|^2 + \frac{1}{2}(u^2 - a^2)^2 \right\} \text{ in the class } H^1_{\text{per}}(Y, \mathbb{R}).$$

Theorem 3.13. *The following hold:*

1. *The functional $\mathcal{E}$ has a unique (modulo multiplication by ± 1) minimizer $\hat{u}$ in $H^1_{per}(Y, \mathbb{R})$. Among the (exactly) two minimizers, one is positive, the other one negative*
2. *If $\hat{u}$ is the positive minimizer of $\mathcal{E}$ in $H^1_{per}(Y, \mathbb{R})$, then we have*

$$U_\varepsilon \rightharpoonup \int_Y \hat{u} \text{ in } L^2(\Omega) \text{ as } \varepsilon \to 0.$$

Proof. We first investigate property *1.* This is done via the following two lemmas.

Lemma 3.14. *The energy functional $\mathcal{E}$ admits a positive global minimizer in $H^1_{per}(Y,\mathbb{R})$. Furthermore, all global minimizer have constant sign and satisfy*

$$-\Delta\hat{u} = \hat{u}(a^2 - \hat{u}^2) \text{ in } Y, \tag{3.28}$$

$$b \le |\hat{u}| \le 1, \tag{3.29}$$

$$\partial_\nu\hat{u}(0,y_2) = -\partial_\nu\hat{u}(1,y_2) \text{ and } \partial_\nu\hat{u}(y_1,0) = -\partial_\nu\hat{u}(y_1,1). \tag{3.30}$$

Proof. (3.28) is clear. In order to prove (3.29), let $u \in H^1_{\text{per}}(Y,\mathbb{R})$ minimize $\mathcal{E}$. Let

$$v := \begin{cases} |u| & \text{if } b \le |u| \le 1 \\ 1 & \text{if } |u| > 1 \\ b & \text{if } |u| < b \end{cases}.$$

It is clear that $v \in H^1_{\text{per}}(Y,\mathbb{R})$. On the other hand, we have

$$\mathcal{E}(v) = \frac{1}{2}\int_{\{b\le|u|\le1\}} \left\{|\nabla u|^2 + \frac{1}{2}(a^2-u^2)^2\right\} + \frac{1}{4}\int_{\{|u|>1\}}(a^2-1)^2 + \frac{1}{4}\int_{\{|u|<b\}}(a^2-b^2)^2.$$

By the minimality of $\mathcal{E}(u)$, we find that $b \le |u| \le 1$ a. e. Noting that, if u is a minimizer, then u is continuous, we find that either u is either positive, or negative. In addition, either $b \le u \le 1$ or $-1 \le u \le -b$.

We next prove that minimizers $\hat{u}$ satisfy (3.30). Indeed, for all $\phi \in H^1_{\text{per}}(Y) \cap C(\overline{\Omega})$ we have

$$0 = \int_Y \nabla\hat{u}\cdot\nabla\phi - \hat{u}\phi(a^2-\hat{u}^2) = -\int_{\partial Y}\phi\ \partial_\nu\hat{u}. \tag{3.31}$$

We next note that

$$\begin{aligned} 0 = \int_{\partial Y}\phi\ \partial_\nu\hat{u} &= \int_0^1 (\partial_\nu\hat{u}(0,t) + \partial_\nu\hat{u}(1,t))\,\phi(0,t) + \int_0^1 (\partial_\nu\hat{u}(t,0) + \partial_\nu\hat{u}(t,1))\,\phi(t,0) \\ &= T_1(\phi_1(t)) + T_2(\phi_2(t)), \end{aligned}$$

with $\phi_1(t) = \phi(0,t)$ and $\phi_2(t) = \phi(t,0)$.

Since for each $\psi \in C_0^\infty((0,1),\mathbb{R})$ there is some $\phi \in H^1_{\text{per}}(Y,\mathbb{R})$ such that $\phi_1(t) = \psi(t)$ and $\phi_2 \equiv 0$, (3.31) implies that the map

$$\begin{aligned} T_1 :\ C_0^\infty((0,1),\mathbb{R}) &\to \mathbb{R} \\ \psi &\mapsto \int_0^1 (\partial_\nu\hat{u}(0,t) + \partial_\nu\hat{u}(1,t))\,\psi(t) \end{aligned}$$

is identically zero. It follows that $\partial_\nu\hat{u}(0,t) + \partial_\nu\hat{u}(1,t) = 0$. A similar argument leads to $\partial_\nu\hat{u}(t,0) + \partial_\nu\hat{u}(t,1) = 0$. □

Lemma 3.15. *The energy $\mathcal{E}$ has a unique positive minimizer in $H^1_{per}(Y,\mathbb{R})$.*

Proof. Let u, v be two positive minimizers and let $w := v/u \in H^1_{\text{per}}$. By the energy decoupling formula [43] (which adapts to the periodic case), we have

$$E_\varepsilon(u) = E_\varepsilon(v) = E_\varepsilon(u) + \frac{1}{2}\int\left\{u^2|\nabla w|^2 + \frac{1}{2}u^4(1-w^2)^2\right\}.$$

Thus $w \equiv 1$, which implies $u = v$. □

As a next (and rather long) step in the proof of Theorem 3.13, we examine the asymptotic behavior of the energy carried by U_ε.

Proposition 3.16. *We have* $\lim_{\varepsilon\to} \varepsilon^2 E_\varepsilon(U_\varepsilon) = |\Omega|\mathcal{E}(\hat{u})$.

Proof. We use the *unfolding operator* (see [30], definition 2.1). More specifically, we define, for $p \in (1,\infty)$,

$$\begin{array}{llll} \mathcal{T}_\varepsilon : & L^p(\Omega) & \to & L^p(\Omega \times Y) \\ & \phi & \mapsto & \mathcal{T}_\varepsilon(\phi)(x,y) = \begin{cases} \phi\left(\varepsilon\left[\dfrac{x}{\varepsilon}\right] + \varepsilon y\right) & \text{if } (x,y) \in \hat{\Omega}_\varepsilon \times Y \\ 0 & \text{if } (x,y) \in \Lambda_\varepsilon \times Y \end{cases}, \end{array}$$

$$\hat{\Omega}_\varepsilon := \bigcup_{\substack{Y_\varepsilon^K \subset \Omega \\ Y_\varepsilon^K = \varepsilon(K+Y),\, K\in\mathbb{Z}^2}} \overline{Y_\varepsilon^K},\ \Lambda_\varepsilon := \Omega \setminus \hat{\Omega}_\varepsilon \text{ and } \left[\frac{x}{\varepsilon}\right] := \left(\left[\frac{x_1}{\varepsilon}\right], \left[\frac{x_2}{\varepsilon}\right]\right).$$

Here, for $s \in \mathbb{R}$, $[s]$ is the integer part of s.

We will use the following results:

i) $\mathcal{T}_\varepsilon$ is linear and continuous, of norm at most 1 ([30], prop. 2.5);

ii) $\mathcal{T}_\varepsilon(uv) = \mathcal{T}_\varepsilon(u)\mathcal{T}_\varepsilon(v)$ and $\mathcal{T}_\varepsilon\left(\frac{u}{v}\right) = \frac{\mathcal{T}_\varepsilon(u)}{\mathcal{T}_\varepsilon(v)}\mathbb{1}_{\hat{\Omega}_\varepsilon\times Y}$ ([30], equation (2.2));

iii) "Unfolding criterion for integrals" (u. c. i., [30], prop. 2.6) : If $\phi_\varepsilon \in L^1(\Omega)$ is such that $\int_{\Lambda_\varepsilon} |\phi_\varepsilon| \to 0$, then we have

$$\int_\Omega \phi_\varepsilon - \int_{\Omega\times Y} \mathcal{T}_\varepsilon(\phi) \to 0;$$

iv) $\varepsilon\mathcal{T}_\varepsilon(\nabla\phi)(x,y) = \nabla_y \mathcal{T}_\varepsilon(\phi)(x,y)$ for $\phi \in W^{1,p}(\Omega)$ ([30], equation (3.1)).

As a first step in the proof of Proposition 3.16, we prove that $\limsup_\varepsilon \varepsilon^2 E_\varepsilon(U_\varepsilon) \le |\Omega|\mathcal{E}(\hat{u})$. Indeed, we consider the test function $H_\varepsilon \in H_1^1$ defined by

$$H_\varepsilon(x) := \rho_\varepsilon(x)\hat{u}\left(\left\{\frac{x}{\varepsilon}\right\}\right) + 1 - \rho_\varepsilon(x),$$

with

$$\rho_\varepsilon(x) := \min\left(1, \frac{\operatorname{dist}(x,\partial\Omega)}{\varepsilon}\right) \text{ and } \left\{\frac{x}{\varepsilon}\right\} = \frac{x}{\varepsilon} - \left[\frac{x}{\varepsilon}\right] \in Y.$$

Then we have

$$\mathcal{T}_\varepsilon(H_\varepsilon) \to \hat{u}(y) \text{ in } L^4(\Omega\times Y) \text{ and } \mathcal{T}_\varepsilon(\varepsilon\nabla H_\varepsilon)(x,y) \to \nabla_y\hat{u}(y) \text{ in } L^2(\Omega\times Y). \tag{3.32}$$

Indeed, the first convergence in (3.32) is a consequence of the fact that $\mathcal{T}_\varepsilon(H_\varepsilon) - \hat{u}(y)$ is bounded in $L^\infty(\Omega\times Y)$ and that its support is contained inside $\{x \in \Omega \,|\, \operatorname{dist}(x,\partial\Omega) < 3\varepsilon\} \times Y$. This implies at once that $\mathcal{T}_\varepsilon(H_\varepsilon) \to \hat{u}(y)$ in $L^4(\Omega\times Y)$.

In order to establish the second convergence in (3.32), we start from the identity

$$\begin{aligned} \mathcal{T}_\varepsilon(\varepsilon\nabla H_\varepsilon) &= \mathcal{T}_\varepsilon(\rho_\varepsilon)\mathcal{T}_\varepsilon\left[\varepsilon\nabla\left(\hat{u}\left(\left\{\frac{x}{\varepsilon}\right\}\right)\right)\right] + \mathcal{T}_\varepsilon(\varepsilon\nabla\rho_\varepsilon)\mathcal{T}_\varepsilon\left[\hat{u}\left(\left\{\frac{x}{\varepsilon}\right\}\right) - 1\right] \\ &= \nabla_y\hat{u}(y)\mathbb{1}_{\hat{\Omega}_\varepsilon}(x) + (\mathcal{T}_\varepsilon(\rho_\varepsilon) - 1)\nabla_y\hat{u}(y)\mathbb{1}_{\hat{\Omega}_\varepsilon}(x) + \nabla_y\mathcal{T}_\varepsilon(\rho_\varepsilon)\mathcal{T}_\varepsilon\left[\hat{u}\left(\left\{\frac{x}{\varepsilon}\right\}\right) - 1\right] \\ &\equiv \nabla_y\hat{u}(y)\mathbb{1}_{\hat{\Omega}_\varepsilon}(x) + R_\varepsilon. \end{aligned}$$

Since $\rho_\varepsilon \equiv 1$ in $\{x \in \Omega \,|\, \mathrm{dist}(x,\partial\Omega) > \varepsilon\}$ and since $\varepsilon|\nabla\rho_\varepsilon|$ is bounded in $L^\infty(\Omega)$, it is clear that the support of R_ε is included in $\{x \in \Omega \,|\, \mathrm{dist}(x,\partial\Omega) < 3\varepsilon\} \times Y$ and that R_ε is bounded in $L^\infty(\Omega \times Y)$. Thus $R_\varepsilon \to 0$ in $L^2(\Omega \times Y)$. It then suffices to note that $\nabla_y \hat{u}(y) \mathbb{1}_{\hat{\Omega}_\varepsilon}(x) \to \nabla_y \hat{u}(y)$ in $L^4(\Omega \times Y)$ in order to obtain the desired convergence result.

Similarly, we have $\mathcal{T}_\varepsilon(a_\varepsilon)(x,y) \to a(y)$ in $L^4(\Omega \times Y)$.

Finally,

$$\begin{aligned}
\limsup_\varepsilon \varepsilon^2 E_\varepsilon(U_\varepsilon) &\leq \lim_\varepsilon \varepsilon^2 E_\varepsilon(H_\varepsilon) = \lim_\varepsilon \frac{1}{2}\int_\Omega \left\{ |\varepsilon\nabla H_\varepsilon|^2 + \frac{1}{2}(H_\varepsilon^2 - a_\varepsilon^2)^2 \right\} \\
&= \left[\text{with } \phi = |\varepsilon\nabla H_\varepsilon|^2 + \frac{1}{2}(H_\varepsilon^2 - a_\varepsilon^2)^2 \right] \\
&= \lim_\varepsilon \frac{1}{2}\int_{\Omega\times Y} \mathcal{T}_\varepsilon(\phi) = [\text{here, we use u. c. i.}] \\
&= \lim_\varepsilon \frac{1}{2}\int_{\hat{\Omega}_\varepsilon\times Y} \left\{ |\nabla\hat{u}|^2 + \frac{1}{2}(\hat{u}^2 - \mathcal{T}_\varepsilon(a_\varepsilon)^2)^2 \right\} \\
&= \frac{1}{2}\int_{\Omega\times Y} \left\{ |\nabla\hat{u}(y)|^2 + \frac{1}{2}(\hat{u}(y)^2 - a(y)^2)^2 \right\} = |\Omega|\mathcal{E}(\hat{u}).
\end{aligned}$$

In order to complete the proof of Proposition 3.16, it suffices to establish the inequality

$$\liminf_\varepsilon \varepsilon^2 E_\varepsilon(U_\varepsilon) \geq |\Omega|\mathcal{E}(\hat{u}).$$

In order to obtain this estimate, we perform the following change of functions: for $u \in A := \{u \in H_1^1(\Omega) \text{ such that } b \leq u \leq 1\}$, we let $v := u^2$. We clearly have $v \in B := \{v \in H_1^1(\Omega) \text{ such that } b^2 \leq v \leq 1\}$. Both A and B are convex and closed in H_1^1. We have the following equivalences

$$\begin{aligned}
u \text{ minimizes } E_\varepsilon \text{ in } H_1^1(\Omega) &\Leftrightarrow u \text{ minimizes } E_\varepsilon \text{ in } \{u \in H_1^1(\Omega) \text{ such that } b \leq u \leq 1\} \\
&\Leftrightarrow u = \sqrt{v} \text{ minimizes } E_\varepsilon \text{ in } \{\sqrt{v} \in H_1^1(\Omega) \text{ such that } b^2 \leq v \leq 1\} \\
&\Leftrightarrow v = u^2 \text{ minimizes } G_\varepsilon \text{ in } \{v \in H_1^1(\Omega) \text{ such that } b^2 \leq v \leq 1\}
\end{aligned}$$

with

$$G_\varepsilon(v) := \frac{1}{4}\int_\Omega \left\{ \frac{|\nabla v|^2}{2v} + \frac{1}{\varepsilon^2}(a_\varepsilon^2 - v)^2 \right\}.$$

Let U_ε be the minimizer of E_ε in H_1^1. Then $V_\varepsilon := U_\varepsilon^2$ is the global minimizer of G_ε in $\{v \in H_1^1(\Omega) \text{ such that } b^2 \leq v \leq 1\}$. Let, for $v \in C := \{v \in H^1_{\mathrm{per}}(Y,\mathbb{R}) \text{ such that } v \geq b^2\}$,

$$\mathcal{G}(v) := \frac{1}{4}\int_Y \left\{ \frac{|\nabla v|^2}{2v} + (a^2 - v)^2 \right\}.$$

It is clear that $\mathcal{G}$ has a unique minimizer in C, namely $\hat{v} := \hat{u}^2$.

With these notations, we have

$$\begin{aligned}
\liminf_\varepsilon \varepsilon^2 E_\varepsilon(U_\varepsilon) &= \liminf_\varepsilon \varepsilon^2 G_\varepsilon(V_\varepsilon) \\
&= \liminf_\varepsilon \frac{1}{4}\int_\Omega \left\{ \frac{|\varepsilon\nabla V_\varepsilon|^2}{2V_\varepsilon} + (a_\varepsilon^2 - V_\varepsilon)^2 \right\} = \liminf_\varepsilon \frac{1}{4}\int_\Omega \tilde{\phi}_\varepsilon(V_\varepsilon),
\end{aligned}$$

where $\tilde{\phi}_\varepsilon(V_\varepsilon) := \dfrac{|\varepsilon\nabla V_\varepsilon|^2}{2V_\varepsilon} + (a_\varepsilon^2 - V_\varepsilon)^2$. Using the bound $|\nabla U_\varepsilon| \leq \dfrac{C}{\varepsilon}$ [17], we see that $\displaystyle\int_{\Lambda_\varepsilon} \tilde{\phi}_\varepsilon(V_\varepsilon) \to 0$. This property, together with the properties i)-iv) of the unfolding operator, imply

$$\liminf_{\varepsilon} \int_{\Omega} \tilde{\phi}_\varepsilon(V_\varepsilon) = \liminf_{\varepsilon} \int_{\Omega\times Y} \mathcal{T}_\varepsilon(\tilde{\phi}(V_\varepsilon)), \tag{3.33}$$

where

$$\mathcal{T}_\varepsilon(\tilde{\phi}_\varepsilon(V_\varepsilon)) = \begin{cases} \dfrac{|\nabla_y \mathcal{T}_\varepsilon(V_\varepsilon)|^2}{2\mathcal{T}_\varepsilon(V_\varepsilon)} + (\mathcal{T}_\varepsilon(V_\varepsilon) - \mathcal{T}_\varepsilon(a_\varepsilon)^2)^2 & \text{in } \hat{\Omega}_\varepsilon \times Y \\ 0 & \text{in } \Lambda_\varepsilon \times Y \end{cases} := \phi^y_\varepsilon(\mathcal{T}_\varepsilon(V_\varepsilon)).$$

For $W \in L^2(\Omega, H^1(Y))$ such that $W \geq b^2$ a.e. in $\hat{\Omega}_\varepsilon \times Y$, define

$$\phi^y_\varepsilon(W) := \left(\frac{|\nabla_y W|^2}{2W} + (W - \mathcal{T}_\varepsilon(a_\varepsilon)^2)^2 \right) 1\!\!1_{\hat{\Omega}_\varepsilon \times Y}.$$

Similarly, for $W \in L^2(\Omega, H^1(Y, \mathbb{R}))$ satisfying $W \geq b^2$ a.e. in $\Omega \times Y$, we denote

$$\phi^y(W) = \frac{|\nabla_y W(x,y)|^2}{2W(x,y)} + (W(x,y) - a(y)^2)^2.$$

One may prove that ϕ^y is a convex function of its argument W.

Using the strong convergence in $L^4(\Omega \times Y)$, as $\varepsilon \to 0$, of the family of $\mathcal{T}_\varepsilon(a_\varepsilon)$ to the map $(x,y) \mapsto a(y)$, it is not difficult to prove that the assumptions $W_\varepsilon \in L^2(\Omega, H^1(Y, \mathbb{R}))$, $W_\varepsilon \geq b^2$ a.e. in $\Omega \times Y$ and $|W_\varepsilon|, |\nabla_y W_\varepsilon| \leq C$ in $\Omega \times Y$ imply

$$\int_{\Omega\times Y} \{\phi^y_\varepsilon(W_\varepsilon) - \phi^y(W_\varepsilon)\} \to 0. \tag{3.34}$$

Since $\varepsilon \nabla V_\varepsilon$ is bounded in $L^\infty(\Omega)$ [17] and since V_ε is bounded in L^2, Corollary 3.2 in [30] implies that there exists some $\hat{V} \in L^2(\Omega, H^1_{\text{per}}(Y))$ such that, up to a subsequence, we have

$$\mathcal{T}_\varepsilon(V_\varepsilon) \rightharpoonup \hat{V} \text{ in } L^2(\Omega \times Y) \text{ and } \nabla_y(\mathcal{T}_\varepsilon(V_\varepsilon)) \rightharpoonup \nabla_y \hat{V} \text{ in } L^2(\Omega \times Y). \tag{3.35}$$

Let $W_\varepsilon := \mathcal{T}_\varepsilon(V_\varepsilon) + 1\!\!1_{\Lambda_\varepsilon \times Y}$, which satisfies the assumptions leading to (3.34) and, in addition, satisfies

$$W_\varepsilon - \mathcal{T}_\varepsilon(V_\varepsilon) \to 0 \text{ in } L^2(\Omega \times Y) \text{ and } \nabla_y W_\varepsilon = \nabla_y \mathcal{T}_\varepsilon(V_\varepsilon).$$

To resume, the definition of W_ε combined with (3.35) yields

$$\begin{cases} W_\varepsilon \rightharpoonup \hat{V} \text{ in } L^2(\Omega \times Y) \\ \nabla_y W_\varepsilon \rightharpoonup \nabla_y \hat{V} \text{ in } L^2(\Omega \times Y) \\ |W_\varepsilon|, |\nabla_y W_\varepsilon| \leq C \text{ and } W_\varepsilon \geq b^2 \end{cases}. \tag{3.36}$$

(Here, weak convergence is obtained after possibly passing to a subsequence)

We are now in position to prove that $\liminf_{\varepsilon} \varepsilon^2 E_\varepsilon(U_\varepsilon) \geq |\Omega| \mathcal{E}(\hat{u})$. Indeed, using the fact that $\mathcal{T}_\varepsilon(a_\varepsilon) \to a$ in $L^4(\Omega \times Y)$ and the convexity of ϕ^y, we obtain

$$\begin{aligned} \liminf_{\varepsilon} \varepsilon^2 E_\varepsilon(U_\varepsilon) &= [\text{from (3.33)}] = \liminf_{\varepsilon} \frac{1}{4} \int_{\Omega\times Y} \mathcal{T}_\varepsilon(\phi_\varepsilon(V_\varepsilon)) && (3.37) \\ &= [\text{since } W_\varepsilon = \mathcal{T}_\varepsilon(V_\varepsilon) \text{ in } \hat{\Omega}_\varepsilon \times Y] = \liminf_{\varepsilon} \frac{1}{4} \int_{\Omega\times Y} \phi^y_\varepsilon(W_\varepsilon) && (3.38) \\ &= [\text{using (3.34), (3.36)}] = \liminf_{\varepsilon} \frac{1}{4} \int_{\Omega\times Y} \phi^y(W_\varepsilon) && (3.39) \\ &\geq [\text{using (3.36) and the convexity of } \phi^y] \geq \frac{1}{4} \int_{\Omega\times Y} \phi^y(\hat{V}) && (3.40) \\ &= \int_\Omega \mathcal{G}(\hat{V}(x,\cdot))\mathrm{d}x \geq \int_\Omega \mathcal{G}(\hat{v})\mathrm{d}x = |\Omega| \mathcal{E}(\hat{u}). && (3.41) \end{aligned}$$

It follows that

$$\lim_{\varepsilon} \varepsilon^2 E_\varepsilon(U_\varepsilon) = |\Omega|\mathcal{E}(\hat{u}).$$

The proof of Proposition 3.16 is complete. □

We are now in position to complete the proof of Theorem 3.13, point *2.*, by identifying the weak limit of U_ε. From (3.37), it follows that, for a. e. $x \in \Omega$, $\hat{V}(x, \cdot)$ is a positive global minimizer of $\mathcal{G}$. For such x, we have $\hat{V}(x, \cdot) = \hat{v}(\cdot)$.

By combining the following facts:

$$\begin{cases} \lim_{\varepsilon} \dfrac{1}{4} \displaystyle\int_{\Omega\times Y} \mathcal{T}_\varepsilon(\tilde{\phi}_\varepsilon(V_\varepsilon)) = |\Omega|\mathcal{E}(\hat{u}) = |\Omega|\mathcal{G}(\hat{v}) \\ \mathcal{T}_\varepsilon(V_\varepsilon) \rightharpoonup \hat{v}, \ \nabla_y \mathcal{T}_\varepsilon(V_\varepsilon) \rightharpoonup \nabla_y \hat{v} \text{ in } L^2(\Omega \times Y) \end{cases},$$

we obtain

$$\lim_{\varepsilon} \int_{\Omega\times Y} (\mathcal{T}_\varepsilon(V_\varepsilon) - \mathcal{T}_\varepsilon(a_\varepsilon)^2)^2 = \lim_{\varepsilon} \int_{\Omega\times Y} (\hat{v} - a^2)^2.$$

The above equality implies

$$\lim_{\varepsilon} \int_{\Omega\times Y} \mathcal{T}_\varepsilon(V_\varepsilon)^2 = \lim_{\varepsilon} \int_{\Omega\times Y} \hat{v}^2,$$

which in turn implies $\mathcal{T}_\varepsilon(V_\varepsilon) \to \hat{v}$ in $L^2(\Omega \times Y)$. Since $\hat{v} = \hat{u}^2$ and $V_\varepsilon = U_\varepsilon^2$, we obtain

$$\int_{\Omega\times Y} (\mathcal{T}_\varepsilon(U_\varepsilon) - \hat{u})^2 \leq \frac{1}{4b^2} \int_{\Omega\times Y} (\mathcal{T}_\varepsilon(U_\varepsilon)^2 - \hat{u}^2)^2 \to 0,$$

that is, we find that $\mathcal{T}_\varepsilon(U_\varepsilon) \to \hat{u}$ in $L^2(\Omega \times Y)$. This fact combined with Proposition 2.9 iii) in [30] implies $U_\varepsilon \rightharpoonup \mathcal{M}_Y(\hat{u}) \equiv \int_Y \hat{u}(y)\mathrm{d}y$, which is the desired conclusion. □

Limit of v_ε in H^1

Recall that we are in the critical case $\lambda = 1$, $\delta = \varepsilon$.

In order to state the main result of this section we recall the following standard existence result (see, e. g., Theorem 4.27 in [31])

Proposition 3.17. *Let $f \in (H^1_{\text{per}}(Y))'$ have zero average. Then there exists an unique solution $h \in H^1_{per}(Y)$ of*

$$\operatorname{div}(\hat{u}^2 \nabla h) = f \text{ and } \mathcal{M}_Y(h) = 0.$$

In view of this proposition, let $\chi_j \in H^1_{\text{per}}(Y)$ be the unique solution of

$$\operatorname{div}(\hat{u}^2 \nabla \chi_j) = \partial_j(\hat{u}^2) \text{ and } \mathcal{M}_Y(\chi_j) = 0. \tag{3.42}$$

Recall that the homogenized matrix $\mathcal{A}$ of $\hat{u}^2\left(\dfrac{x}{\varepsilon}\right)\mathrm{Id}_{\mathbb{R}^2}$ is given by

$$\mathcal{A} = \int_Y \hat{u}^2 \begin{pmatrix} 1 - \partial_1\chi_1 & -\partial_1\chi_2 \\ -\partial_2\chi_1 & 1 - \partial_2\chi_2 \end{pmatrix} \tag{3.43}$$

(see, e. g., [40] chapter 1 or [31] chapter 6).

Proposition 3.18. *Let φ_* be the unique solution of*

$$\begin{cases} \mathrm{div}(\mathcal{A}\nabla\varphi_*) = 0 & \text{in } \Omega \\ \varphi_* = \varphi_0 & \text{on } \partial\Omega \end{cases}. \tag{3.44}$$

Let $g = e^{\imath\varphi_0}$. Also, for small ε, represent a minimizer u_ε of E_ε in H^1_g as $u_\varepsilon = U_\varepsilon\rho_\varepsilon e^{\imath\varphi_\varepsilon}$, where $\varphi_\varepsilon \in H^1_{\varphi_0}(\Omega)$.

Then $\varphi_\varepsilon \rightharpoonup \varphi_*$ in $H^1(\Omega)$ as $\varepsilon \to 0$.

Proof. This argument is an adaptation of the proof of Theorem 4 in [66].

First note that $\mathcal{T}_\varepsilon(U_\varepsilon^2)(x,y) \to \hat{u}^2(y)$ in $L^2(\Omega \times Y)$ and $|v_\varepsilon|^2 = \rho_\varepsilon^2 \to 1$ in $L^2(\Omega)$ imply that

$$\mathcal{T}_\varepsilon(\rho_\varepsilon^2 U_\varepsilon^2)(x,y) \to \hat{u}^2(y) \text{ in } L^2(\Omega \times Y).$$

Recalling that φ_ε is the solution of

$$\begin{cases} -\mathrm{div}(\rho_\varepsilon^2 U_\varepsilon^2 \nabla\varphi_\varepsilon) = 0 & \text{in } \Omega \\ \varphi_\varepsilon = \varphi_0 & \text{on } \partial\Omega \end{cases},$$

we find, using Proposition 3.16. iv) and the fact that $\rho_\varepsilon^2 U_\varepsilon^2 \nabla\varphi_\varepsilon \in H^1_{\mathrm{loc}}(\Omega)$, that

$$0 = \varepsilon\mathcal{T}_\varepsilon\left(-\mathrm{div}(\rho_\varepsilon^2 U_\varepsilon^2 \nabla\varphi_\varepsilon)\right)(x,y) = -\mathrm{div}_y\left(\mathcal{T}_\varepsilon(\rho_\varepsilon^2 U_\varepsilon^2)(x,y)\mathcal{T}_\varepsilon(\nabla\varphi_\varepsilon)(x,y)\right). \tag{3.45}$$

In order to prove that $\varphi_\varepsilon \rightharpoonup \varphi_*$ it suffices to prove that if, possibly up to a subsequence, we have $\varphi_\varepsilon \rightharpoonup \varphi^*$, then φ^* solves (3.44).

Using Theorem 3.5 in [30], we have the existence of $\hat{\varphi} \in L^2(\Omega, H^1_{\mathrm{per}}(Y))$ such that

$$\mathcal{T}_\varepsilon(\nabla\varphi_\varepsilon) \rightharpoonup \nabla\varphi^* + \nabla_y\hat{\varphi} \text{ in } L^2(\Omega \times Y) \text{ and } \mathcal{M}_Y(\hat{\varphi}) = 0. \tag{3.46}$$

By inserting (3.46) into (3.45) and passing to the weak limits in $L^2(\Omega, H^{-1}(Y))$, we obtain

$$-\mathrm{div}_y\left[\hat{u}^2(y)\left(\nabla\varphi^*(x) + \nabla_y\hat{\varphi}(x,y)\right)\right] = 0$$

which is equivalent to

$$-\mathrm{div}_y\left[\hat{u}^2(y)\nabla_y\hat{\varphi}(x,y)\right] = \nabla_y\hat{u}^2(y)\cdot\nabla\varphi^*(x).$$

This equality combined with (3.42) implies that

$$\hat{\varphi}(x,y) = -\chi_1\partial_{x_1}\varphi^* - \chi_2\partial_{x_2}\varphi^*.$$

Consequently, we have

$$\nabla\varphi^* + \nabla_y\hat{\varphi} = \begin{pmatrix} 1-\partial_1\chi_1 & -\partial_1\chi_2 \\ -\partial_2\chi_1 & 1-\partial_2\chi_2 \end{pmatrix}\nabla\varphi^*.$$

On the other hand, let $\xi \in \mathcal{D}(\Omega)$. Then, for sufficiently small ε we have (*cf* Proposition 2.5. (i) in [30])

$$\begin{aligned} 0 &= \lim_{\varepsilon\to 0}\int_\Omega \rho_\varepsilon^2 U_\varepsilon^2 \nabla\varphi_\varepsilon\cdot\nabla\xi = \lim_{\varepsilon\to 0}\int_{\Omega\times Y}\mathcal{T}_\varepsilon(\rho_\varepsilon^2 U_\varepsilon^2)\mathcal{T}_\varepsilon(\nabla\varphi_\varepsilon)\cdot\mathcal{T}_\varepsilon(\nabla\xi) \\ &= \int_\Omega\left\{\int_Y \hat{u}^2(y)(\nabla\varphi^* + \nabla_y\hat{\varphi})\right\}\cdot\nabla\xi = \int_\Omega \mathrm{div}_x\left\{\int_Y \hat{u}^2(y)(\nabla\varphi^* + \nabla_y\hat{\varphi})\right\}\xi. \end{aligned}$$

Therefore one has

$$\begin{aligned} \mathrm{div}_x\left[\int_Y \hat{u}^2(y)(\nabla\varphi^* + \nabla_y\hat{\varphi})\right] &= \mathrm{div}_x\left[\int_Y \hat{u}^2(y)\begin{pmatrix} 1-\partial_1\chi_1 & -\partial_1\chi_2 \\ -\partial_2\chi_1 & 1-\partial_2\chi_2 \end{pmatrix}\nabla\varphi^*\right] \\ &= \mathrm{div}_x\left(\mathcal{A}\nabla\varphi^*\right) = 0 \end{aligned}$$

and consequently φ^* solves (3.44). □

3.3.3 The case $\lambda = 1, \varepsilon \ll \delta$

Theorem 3.19. *Assume that $\lambda = 1$, $\delta \to 0$ and $\varepsilon/\delta \to 0$. Then, as $\varepsilon \to 0$, we have*

1. $\rho_\varepsilon = |u_\varepsilon| \rightharpoonup \mathcal{M}_Y(a)$ *in* $L^2(\Omega)$,

2. $\varphi_\varepsilon \rightharpoonup \varphi_*$ *in* $H^1(\Omega)$,

3. $\rho_\varepsilon^2 \nabla\varphi_\varepsilon \rightharpoonup \mathcal{A}\nabla\varphi_*$ *in* $L^2(\Omega)$,

where φ_* solves the homogenized problem

$$\begin{cases} \operatorname{div}(\mathcal{A}\nabla\varphi_*) = 0 & \text{in } \Omega \\ \varphi_* = \varphi_0 & \text{on } \partial\Omega \end{cases}. \tag{3.47}$$

Here, $\mathcal{A}$ is the homogenized matrix of $a^2\left(\dfrac{x}{\delta}\right)\mathrm{Id}_{\mathbb{R}^2}$.

Proof. Theorem 3.1 combined with Lemma 3.10 yields $\rho_\varepsilon - a_\varepsilon \to 0$ in $L^2(\Omega)$. On the other hand, we have $a_\varepsilon \to \mathcal{M}_Y(a)$ weakly in $L^2(\Omega)$ (see, e. g.,[31] Theorem 2.6), so that *1.* follows.

In order to prove *2.* and *3.*, we start from the equation

$$\begin{cases} \operatorname{div}(\rho_\varepsilon^2\nabla\varphi_\varepsilon) = 0 & \text{in}\,\Omega \\ \varphi_\varepsilon = \varphi_0 & \text{on } \partial\Omega \end{cases} \tag{3.48}$$

satisfied by φ_ε. In view of the fact that $\rho_\varepsilon - a_\varepsilon \to 0$ in $L^2(\Omega)$, it is natural to compare φ_ε to the solution $\hat\varphi_\varepsilon$ of

$$\begin{cases} \operatorname{div}(a_\varepsilon^2\nabla\hat\varphi_\varepsilon) = 0 & \text{in}\,\Omega \\ \hat\varphi_\varepsilon = \varphi_0 & \text{on } \partial\Omega \end{cases}. \tag{3.49}$$

The difference $\psi_\varepsilon := \hat\varphi_\varepsilon - \varphi_\varepsilon$ is solution of

$$\begin{cases} \operatorname{div}(a_\varepsilon^2\nabla\psi_\varepsilon) = \operatorname{div}\left[(\rho_\varepsilon^2 - a_\varepsilon^2)\nabla\varphi_\varepsilon\right] & \text{in}\,\Omega \\ \psi_\varepsilon = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.50}$$

We claim $\psi_\varepsilon \rightharpoonup 0$ in $H^1(\Omega)$. Indeed, we first note that, by (3.48), φ_ε is bounded in H^1. Using the fact that $b^2 \le a_\varepsilon^2 \le 1$ and (3.50) we obtain, via the Lax-Milgram theorem, that, with $C, C' > 0$ and $p < 2$ independent of ε, we have

$$\|\nabla\psi_\varepsilon\|_{L^2} \le C\|(\rho_\varepsilon^2 - a_\varepsilon^2)\nabla\varphi_\varepsilon\|_{L^2} \le C' < \infty$$

and (with $r := 2/(2-p)$)

$$\|\nabla\psi_\varepsilon\|_{L^p} \le C\|(\rho_\varepsilon^2 - a_\varepsilon^2)\nabla\varphi_\varepsilon\|_{L^p} \le C\|\rho_\varepsilon^2 - a_\varepsilon^2\|_{L^{rp}}\|\nabla\varphi_\varepsilon\|_{L^2}.$$

Consequently, ψ_ε is bounded in H_0^1 and converges strongly to 0 in $W^{1,p}(\Omega)$. It follows that $\psi_\varepsilon \rightharpoonup 0$ in $H^1(\Omega)$.

Now, using the classic periodic homogenization result (see, e. g., [40] chapter 1 or [31] chapter 6), we know that $\hat\varphi_\varepsilon \rightharpoonup \varphi_*$ in $H^1(\Omega)$ and $a_\varepsilon^2\nabla\hat\varphi_\varepsilon \rightharpoonup \mathcal{A}\nabla\varphi_*$ in $L^2(\Omega)$. These facts combined with the weak convergences $\psi_\varepsilon \rightharpoonup 0$ in $H^1(\Omega)$ and $(a_\varepsilon^2\nabla\hat\varphi_\varepsilon - \rho_\varepsilon^2\nabla\varphi_\varepsilon) \rightharpoonup 0$ in $L^2(\Omega)$ complete the proof of the theorem. □

3.3.4 The case $\lambda = 1, \delta \ll \varepsilon$

In this case, ε need not tend to 0. Up to subsequences, we may assume that either $\varepsilon = 1$ or $\varepsilon \to 0$.

Theorem 3.20. *The following hold.*

1. *Assume that $\varepsilon = 1$ and that $\delta \to 0$, and denote the energy by E_δ rather then E_ε. If u_δ is a minimizer of E_δ, then $u_\delta \rightharpoonup \hat{u}$ in $H^1(\Omega)$, where $\hat{u}$ solves*

$$\begin{cases} -\Delta \hat{u} = \hat{u}(\mathcal{M}_Y(a^2) - \hat{u}^2) & \text{in } \Omega \\ \hat{u} = g & \text{on } \partial\Omega \end{cases}. \tag{3.51}$$

2. *Assume that $\varepsilon \to 0$ and that $\delta/\varepsilon \to 0$. If $u_\varepsilon = \rho_\varepsilon e^{i\varphi_\varepsilon}$ is a minimizer of E_ε, then we have*

   (i) $\rho_\varepsilon \to \sqrt{\mathcal{M}_Y(a^2)}$ strongly in $L^2(\Omega)$,
   (ii) $\varphi_\varepsilon \to \varphi_*$ in $H^1(\Omega)$.

   Here, φ_* denotes the harmonic extension of φ_0.

Proof. In case *1.*, we start by noting that $\|u_\delta\|_{H^1(\Omega)}$ is uniformly bounded with respect to δ. Let $\hat{u}$ be such that, possibly after passing to a subsequence, u_δ weakly converges to $\hat{u}$ in H^1. In order to identify $\hat{u}$, we let $\delta \to 0$ in the weak form of the Ginzburg-Landau equation satisfied by u_δ, namely:

$$\int_\Omega \nabla u_\delta \cdot \nabla \psi \, dx = \int_\Omega u_\delta (a_\delta^2 - u_\delta^2) \psi \, dx, \; \forall \; \psi \in C_0^\infty(\Omega)$$

and find that (3.51) holds.

In order to prove *2.*, we consider a partition of $\mathbb{R}^2$ by a family $\{C_k^\varepsilon\}$ of $\delta \times \delta$ squares. We may assume that

$$\{C_k^\varepsilon \,|\, C_k^\varepsilon \subset \{x \in \Omega \,|\, \mathrm{dist}(x, \partial\Omega) > \varepsilon\}\} = \{C_k^\varepsilon \,|\, k \in \{1, ..., N_\varepsilon\}\}.$$

Clearly, we have $N_\varepsilon = |\Omega|\delta^{-2} + \mathcal{O}(\varepsilon\delta^{-2})$. Denote $\Omega'_\varepsilon := \bigcup_{k=1}^{N_\varepsilon} C_k^\varepsilon$.

For $C_0 > 0$ (independent of ε) consider

$$\mathcal{H}_\varepsilon^{C_0} = \{w \in H_g^1 \,|\, |\nabla w| \le \frac{C_0}{\varepsilon} \text{ in } \Omega'_\varepsilon \text{ and } |w| \le 1 \text{ in } \Omega\}.$$

Recall [17] that, for $\varepsilon < 1$ and a suitable C_0, each minimizer u_ε of E_ε in H_g^1 belongs to $\mathcal{H}_\varepsilon^{C_0}$.

For $w \in \mathcal{H}_\varepsilon^{C_0}$, we have

$$\int_\Omega (|w|^2 - a_\varepsilon^2)^2 = \int_\Omega (|w|^2 - \mathcal{M}_Y(a^2))^2 + |\Omega| \left[\mathcal{M}_Y(a^4) - \mathcal{M}_Y(a^2)^2\right] + H_\varepsilon(w). \tag{3.52}$$

Here, the reminder H_ε satisfies $|H_\varepsilon(w)| \le o_\varepsilon(1)$, with $o_\varepsilon(1)$ independent of w. Indeed, we have

$$\begin{aligned} \int_\Omega (|w|^2 - a_\varepsilon^2)^2 - \int_\Omega (|w|^2 - \mathcal{M}_Y(a^2))^2 \;\; &= \;\; \int_\Omega \left[a_\varepsilon^2 - \mathcal{M}_Y(a^2)\right] a_\varepsilon^2 \\ &\quad + \mathcal{M}_Y(a^2) \int_\Omega (a_\varepsilon^2 - \mathcal{M}_Y(a^2)) \\ &\quad - 2 \int_\Omega \left[a_\varepsilon^2 - \mathcal{M}_Y(a^2)\right] |w|^2. \end{aligned}$$

We next note the three following facts. First, we have

$$\begin{aligned}\int_\Omega \left[a_\varepsilon^2 - \mathcal{M}_Y(a^2)\right] a_\varepsilon^2 &= \sum_k \left\{\int_{C_\varepsilon^k} \left[a_\varepsilon^2 - \mathcal{M}_Y(a^2)\right] a_\varepsilon^2\right\} + \mathcal{O}(\varepsilon) \\ &= |\Omega| \left[\mathcal{M}_Y(a^4) - \mathcal{M}_Y(a^2)^2\right] + \mathcal{O}(\varepsilon).\end{aligned}$$

Next, it holds that

$$\int_\Omega (a_\varepsilon^2 - \mathcal{M}_Y(a^2)) = \mathcal{O}(\varepsilon) + \sum_k \int_{C_\varepsilon^k} (a_\varepsilon^2 - \mathcal{M}_Y(a^2)) = \mathcal{O}(\varepsilon).$$

Finally, we have

$$\begin{aligned}\left|\int_\Omega \left[a_\varepsilon^2 - \mathcal{M}_Y(a^2)\right] |w|^2\right| &\le \mathcal{O}(\varepsilon) + \sum_k \int_{C_\varepsilon^k} \left|a_\varepsilon^2 - \mathcal{M}_Y(a^2)\right| |w|^2 \\ &\le \mathcal{O}(\varepsilon) + \sum_k \int_{C_\varepsilon^k} \left|a_\varepsilon^2 - \mathcal{M}_Y(a^2)\right| = o_\varepsilon(1).\end{aligned}$$

Thus (3.52) holds. Consequently, for $u \in \mathcal{H}_\varepsilon^{C_0}$, one has

$$E_\varepsilon(u) = \frac{|\Omega|}{4\varepsilon^2}(\mathcal{M}_Y(a^4) - \mathcal{M}_Y(a^2)^2) + G_\varepsilon(u) + o\left(\frac{1}{\varepsilon^2}\right), \tag{3.53}$$

where

$$G_\varepsilon(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{1}{4\varepsilon^2}\int_\Omega (\mathcal{M}_Y(a^2) - |u|^2)^2.$$

We next claim that $\int_\Omega (|u_\varepsilon|^2 - \mathcal{M}_Y(a^2))^2\, dx \to 0$. Indeed, we consider a test function in the spirit of [43], more specifically we let $w_\varepsilon = |w_\varepsilon| e^{\imath\varphi_*}$, where φ_* is the harmonic extension of φ_0 and

$$|w_\varepsilon|(x) = \begin{cases} 1 - \dfrac{1 - \sqrt{\mathcal{M}_Y(a^2)}}{\varepsilon}\mathrm{dist}(x, \partial\Omega), & \text{if } \mathrm{dist}(x,\partial\Omega) < \varepsilon \\ \sqrt{\mathcal{M}_Y(a^2)}, & \text{otherwise}\end{cases}.$$

Note that, for a suitable C_0, we have $w_\varepsilon \in \mathcal{H}_\varepsilon^{C_0}$. A straightforward computation yields $G_\varepsilon[w_\varepsilon] \le \dfrac{C}{\varepsilon}$.

Consequently, we obtain

$$E_\varepsilon(u_\varepsilon) \le E_\varepsilon(w_\varepsilon) \le \frac{|\Omega|}{4\varepsilon^2}(\mathcal{M}_Y(a^4) - \mathcal{M}_Y(a^2)^2) + o(\varepsilon^{-2}).$$

This estimate combined with (3.53) implies that $|u_\varepsilon| \to \sqrt{\mathcal{M}_Y(a^2)}$ strongly in $L^2(\Omega)$.

Using the second part of Corollary 3.8, we obtain that $\varphi_\varepsilon \to \varphi_*$ in $H^1(\Omega)$ where φ_* is the harmonic extension of φ_0.

The proof of Theorem 3.20 is complete. $\square$

Chapter 4

The Ginzburg-Landau functional with a discontinuous and rapidly oscillating pinning term. Part II: the non-zero degree case

We consider minimizers of a Ginzburg-Landau energy with a discontinuous and rapidly oscillating pinning term, subject to a Dirichlet boundary condition of degree $d > 0$. We prove that minimizers have exactly d isolated zeros (vortices). These vortices are of degree 1 and pinned by the impurities. As in the standard case studied by Bethuel, Brezis and Hélein, the macroscopic location of vortices is governed by vortex/vortex and vortex/ boundary repelling effects. In addition, impurities affect the microscopic location of vortices.

Contents

4.1 Introduction and main results

This chapter is a follow up of the previous one. As there, we let $\Omega \subset \mathbb{R}^2$ be a simply connected domain and let $a_\varepsilon : \Omega \to \mathbb{R}$ be a measurable function s.t. $1 \geq a_\varepsilon \geq b > 0$. We associate to a_ε the pinned Ginzburg-Landau energy

$$E_\varepsilon(u) = \frac{1}{2}\int_\Omega \left\{ |\nabla u(x)|^2 + \frac{1}{2\varepsilon^2}\left(a_\varepsilon(x)^2 - |u(x)|^2\right)^2 \right\} \mathrm{d}x. \tag{4.1}$$

Here, $u \in H^1(\Omega, \mathbb{C})$ and $\varepsilon > 0$ is the inverse of the Ginzburg-Landau parameter.

In this chapter, our goal is to consider a discontinuous and rapidly oscillating pinning term (the same as in Section 3.3 in Chapter 3). Our pinning term is periodic with respect to a $\delta \times \delta$-grid with $\delta = \delta(\varepsilon) \to 0$ as $\varepsilon \to 0$. We are interested in the minimization of (4.1) in $H^1(\Omega, \mathbb{C})$ subject to a Dirichlet boundary condition: we fix $g \in C^\infty(\partial\Omega, \mathbb{S}^1)$ and thus the set of the test functions is

$$H^1_g = \{u \in H^1(\Omega, \mathbb{C}) \,|\, \mathrm{tr}_{\partial\Omega} u = g\}.$$

The situation where $d = \deg_{\partial\Omega}(g) = 0$ was studied in detail in Chapter 3. The non zero degree case ($d = \deg_{\partial\Omega}(g) > 0$) is the purpose of the present chapter.

Before going further, let us summarize two previous works in related directions [43], [1]. In these works, the role of the pinning term is identified: its points of *minimum* attract the vorticity defaults.

In [43], Lassoued and Mironescu considered the case where $a_\varepsilon \equiv a$. Here, the pinning term $a = \begin{cases} b & \text{in } \omega \\ 1 & \text{in } \Omega \setminus \omega \end{cases}$, $0 < b < 1$, and ω is a smooth inner domain of Ω. These authors proved that the vorticity defaults are localized in ω and that their position is governed by a renormalized energy (in the spirit of the [18]).

In [1], Aftalion, Sandier and Serfaty considered a smooth and ε-dependent pinning term a_ε. Their study allows to consider the case where the pinning term has fast oscillations: it is a perturbation of a fixed smooth function $\tilde{b} : \Omega \to [b, 1]$ s.t. $\tilde{b} \leq a_\varepsilon$.

They considered the following hypotheses on $a_\varepsilon, \tilde{b}$:

- $|\nabla a_\varepsilon| \leq C|\ln \varepsilon|$
- there is $\sigma_\varepsilon \in \mathbb{R}$ s.t. $\sigma_\varepsilon = o_\varepsilon\left((\ln|\ln\varepsilon|)^{-1/2}\right)$ and for all $x \in \Omega$, we have

$$\min_{B(x,\sigma_\varepsilon)} \left\{a_\varepsilon - \tilde{b}\right\} = 0.$$

These authors study a full Ginzburg-Landau energy GL_ε with a pinning term:

$$GL_\varepsilon(u, A) = \frac{1}{2}\int_\Omega \left\{ |\mathrm{curl} A - h_{\mathrm{ex}}|^2 + |(\nabla - iA)u|^2 + \frac{1}{2\varepsilon^2}(a_\varepsilon^2 - |u|^2)^2 \right\}.$$

We denoted by A the electromagnetic vector potential of the induced field and by h_{ex} the intensity of the applied magnetic field (see Section 2.1 in the introduction or [65] for more detail).

In the study of the full Ginzburg-Landau functional without pinning term GL^{2D}_ε (GL^{2D}_ε is defined by (1), page xi), the vorticity defaults appear for large apply magnetic field. They are characterized by two facts: the presence of isolated zeros x_i of a map u with a non zero degree around small circles centered in x_i and the existence of a magnetic field inside the domain. The nature of the superconductivity makes that both facts appear together.

Assuming that the intensity of the applied field $h_{\rm ex}$ depends on $0<\varepsilon<1$ and that $h_{\rm ex}/|\ln\varepsilon|\to\Lambda\in\mathbb{R}_+^*$, for the classical full Ginzburg-Landau energy, it is well known (see *e.g.* [65]) that there is an inner domain ω_Λ (non decreasing with Λ) s.t., when $\varepsilon\to 0$, the vorticity defaults are "uniformly located" by ω_Λ.

In [1], the authors proved the existence of ω_Λ, an inner set of Ω, where the penetration of the magnetic field is located. In contrast with the situation without pinning term, the presence of a_ε makes that, in general, the vortices are not uniformly located in ω_Λ. Although in the proofs of the main results of [1], the minimal points of $\tilde b$ seem play the role of a pinning site, this fact is not proved. They expect that the most favorable pinning sites should be close to the *minima* of $\tilde b$: ω_Λ should be located close to the points of *minimum* of $\tilde b$.

One of our goals is to prove that the minimum points of a rapidly oscillating and discontinuous pinning term attract the vorticity defaults.

For the convenience of the reader we recall the construction of our pinning term a_ε.

Consider $\delta=\delta(\varepsilon)\in(0,1)$, $\lambda=\lambda(\varepsilon)\in(0,1]$ and let $\omega\subset Y=(-1/2,1/2)^2$ be a smooth bounded and simply connected open set s.t. $(0,0)\in\omega$ and $\overline{\omega}\subset Y$. For $k,l\in\mathbb{Z}$ we denote

$$Y_{k,l}^\delta:=\delta\cdot Y+(\delta k,\delta l),\ \Omega_\delta^{\rm incl}=\bigcup_{Y_{k,l}^\delta\subset\Omega}\overline{Y_{k,l}^\delta},\ \omega^\lambda=\lambda\cdot\omega$$

$$\omega_{\rm per}^\lambda=\bigcup_{(k,l)\in\mathbb{Z}^2}\left\{\omega^\lambda+(k,l)\right\}\text{ and }\omega_\delta=\bigcup_{\substack{(k,l)\in\mathbb{Z}^2\text{ s.t.}\\ Y_{k,l}^\delta\subset\Omega}}\left\{\delta\cdot\omega^\lambda+(\delta k,\delta l)\right\}.$$

For $b\in(0,1)$, we define

$$a^\lambda:\ \begin{array}{ccc}\mathbb{R}^2&\to&\{b,1\}\\ x&\mapsto&\begin{cases}b&\text{if }x\in\omega_{\rm per}^\lambda\\ 1&\text{otherwise}\end{cases}\end{array}\quad\text{and}\quad a_\delta:\ \begin{array}{ccc}\mathbb{R}^2&\to&\{b,1\}\\ x&\mapsto&\begin{cases}b&\text{if }x\in\omega_\delta\\ 1&\text{otherwise}\end{cases}\end{array}.$$

The values of the pinning term are represented Figure 4 in the introduction (Page xv).

In the rest of this chapter $\lambda=\lambda(\varepsilon)$ and $\delta=\delta(\varepsilon)$ are functions of ε. We assume that $\delta\to 0$ as $\varepsilon\to0$. In addition, we assume that either $\lambda\equiv 1$, or $\lambda\to0$ as $\varepsilon\to0$. (The latter assumption is not restrictive, since it always holds up to some subsequence.) The case $\lambda\to 0$ is the "dilute case".

We make the assumption

$$\lim_\varepsilon\frac{|\ln(\lambda\delta)|^3}{|\ln\varepsilon|}=0.\tag{H}$$

Remark 4.1. • This is slightly more restrictive than asking that $\lambda\delta\gg\varepsilon^\alpha$ for some $0<\alpha\in(0,1)$.

- In [1] and in the situation where we have a bounded number of zeros, the smooth pinning term $a_\varepsilon^{\rm smooth}$ satisfies the condition $|\nabla a_\varepsilon^{\rm smooth}|\le C|\ln\varepsilon|$. In order to compare this assumption with (H), we may consider a regularization of our pinning term by a mollifier $\rho_t(x)=\rho(x/t)$. A suitable scale t to have a complete view of the variations of a_ε is $t=\lambda\delta$. Thus, $|\nabla(\rho_{\lambda\delta}*a_\varepsilon)|$ is of order $\dfrac{1}{\lambda\delta}$. Consequently, the condition (H) allows to consider a more rapidly oscillating than the condition in [1]. Indeed, we have $\ln|\nabla a_\varepsilon^{\rm smooth}|\le\ln|\ln\varepsilon|+C$ and on the other hand (H) is equivalent to $\ln|\nabla(\rho_{\lambda\delta}*a_\varepsilon)|\sim|\ln(\lambda\delta)|=o(|\ln\varepsilon|^{1/3})$.

We denote by U_ε **the** unique global minimizer of E_ε in H^1_1 (see [43]). Clearly, U_ε satisfies

$$\begin{cases} -\Delta U_\varepsilon = \dfrac{1}{\varepsilon^2} U_\varepsilon (a_\varepsilon^2 - U_\varepsilon^2) & \text{in } \Omega \\ U_\varepsilon = 1 & \text{on } \partial\Omega \end{cases}. \tag{4.2}$$

This special solution may be seen as a regularization of a_ε. For example, one may easily prove that U_ε is exponentially close to a_ε far away from $\partial\omega_\delta$. Namely, we have

Proposition 4.2. *There are $C, \alpha > 0$ independent of ε, R s.t.*

$$|a_\varepsilon - U_\varepsilon| \leq C\mathrm{e}^{-\frac{\alpha R}{\varepsilon}} \text{ in } V_R := \{x \in \Omega \,|\, \mathrm{dist}(x, \partial\omega_\delta) \geq R\}, \tag{4.3}$$

$$|\nabla U_\varepsilon| \leq \frac{C\mathrm{e}^{-\frac{\alpha R}{\varepsilon}}}{\varepsilon} \text{ in } W_R := \{x \in \Omega \,|\, \mathrm{dist}(x, \partial\omega_\delta), \mathrm{dist}(x, \partial\Omega) \geq R\}. \tag{4.4}$$

A similar result was proved in Chapter 2 (Proposition 2.5 in Appendix 2.A). The above proposition is proved using exactly the same arguments.

As in [43], we define

$$F_\varepsilon(v) = \frac{1}{2}\int_\Omega \left\{ U_\varepsilon^2 |\nabla v|^2 + \frac{1}{2\varepsilon^2} U_\varepsilon^4 (1 - |v|^2)^2 \right\}.$$

Then we have for all $v \in H^1_g$, (see [43])

$$E_\varepsilon(U_\varepsilon v) = E_\varepsilon(U_\varepsilon) + F_\varepsilon(v).$$

Therefore, u_ε is a minimizer of E_ε if and only if $u_\varepsilon = U_\varepsilon v_\varepsilon$ where v_ε is a minimizer of F_ε in H^1_g. Consequently, the study of a minimizer $u_\varepsilon = U_\varepsilon v_\varepsilon$ of E_ε in H^1_g (location of zeros and asymptotics) can be performed by combining the asymptotic of U_ε with one of v_ε.

The main results of this chapter are obtained under the conditions: $\lambda\delta$ satisfies (H) and $b^2 > 1/2$. They take the form of four theorems:

- The first theorem gives informations on the zeros of minimizers of $u_\varepsilon, v_\varepsilon$ (quantification and location).
- The second theorem establishes the asymptotics of v_ε.
- The third theorem establishes, under the additional hypothesis $\lambda \to 0$, that the microscopic position of the zeros is independent of the boundary condition g.
- The last theorem gives an expansion of $F_\varepsilon(v_\varepsilon)$.

Remark 4.3. The condition $b^2 > \dfrac{1}{2}$ is probably only a technical one. As we will see, this hypothesis on b will be used in order to prove the repelling effect of the boundary on the zeros.

Theorem 4.4. *Assume that $b^2 > 1/2$ and that λ, δ satisfy* (H)*. Then there is $\varepsilon_0 > 0$ s.t.:*

1. *for $0 < \varepsilon < \varepsilon_0$, v_ε has exactly d zeros $x_1^\varepsilon, ..., x_d^\varepsilon$,*
2. *there are $c > 0$ and $\eta_0 > 0$ s.t. for $\varepsilon < \varepsilon_0$, $B(x_i^\varepsilon, c\lambda\delta) \subset \omega_\delta$ and*

$$\min_i \left\{ \min_{j \neq i} |x_i^\varepsilon - x_j^\varepsilon|, \mathrm{dist}(x_i^\varepsilon, \partial\Omega) \right\} \geq \eta_0,$$

3. for $\rho=\rho(\varepsilon)\downarrow 0$ s.t. $|\ln\rho|/|\ln\varepsilon|\to 0$, we have for $\varepsilon<\varepsilon_0$,

$$|v_\varepsilon|\geq 1-C\sqrt{\frac{|\ln\rho|}{|\ln\varepsilon|}}\ \text{in }\Omega\setminus\cup B(x_i^\varepsilon,\rho).$$

Here C is independent of ε.

4. for $\varepsilon<\varepsilon_0$, $\deg_{\partial B(x_i^\varepsilon,\varepsilon)}(v_\varepsilon)=1$.

Theorem 4.5. *Assume that $b^2>1/2$ and that λ,δ satisfy (H). Let $\varepsilon_n\downarrow 0$, then, up to subsequence, we have the existence of $a_1,...,a_d\in\Omega$, d distinct points s.t. $x_i^{\varepsilon_n}\to a_i$ and*

$$|v_{\varepsilon_n}|\to 1\ \text{and}\ v_{\varepsilon_n}\rightharpoonup v_*\ \text{in}\ H^1_{\text{loc}}(\overline{\Omega}\setminus\{a_1,...,a_d\},\mathbb{S}^1)$$

where v_* solves

$$\begin{cases}-\text{div}(\mathcal{A}\nabla v_*)=(\mathcal{A}\nabla v_*\cdot\nabla v_*)v_* & \text{in }\Omega\setminus\{a_1,...,a_d\}\\ v_*=g & \text{on }\partial\Omega\end{cases}.$$

Here $\mathcal{A}$ is the homogenized matrix of $a^2\left(\frac{\cdot}{\delta}\right)\text{Id}_{\mathbb{R}^2}$ if $\lambda\equiv 1$ and $\mathcal{A}=\text{Id}_{\mathbb{R}^2}$ if $\lambda\to 0$.

In addition, for each $M>0$, $v'_{\varepsilon,i}(\cdot)=v_\varepsilon\left(x_i^\varepsilon+\frac{\varepsilon}{b}\cdot\right)$ converges, up to a subsequence, in $C^1(B(0,M))$ to $f(|x|)\frac{x}{|x|}\mathrm{e}^{\imath\theta_i}$ where $f:\mathbb{R}^+\to\mathbb{R}^+$ is the universal function defined in [54] and $\theta_i\in\mathbb{R}$.

Theorem 4.6. *Assume, in addition to the hypotheses of Theorem 4.5, that $\lambda\to 0$. Let $[x]=[(x_1,x_2)]=([x_1],[x_2])\in\mathbb{Z}^2$ denote the integer part of the point $x\in\mathbb{R}^2$.*

For x_i^ε zero of v_ε, let

$$\hat{\hat{x}}_i^\varepsilon=\frac{\frac{x_i^\varepsilon}{\delta}-[\frac{x_i^\varepsilon}{\delta}]}{\lambda}\in\omega.$$

Then, as $\varepsilon\to 0$, up to pass to a subsequence, we have $x_i^\varepsilon\to\tilde{a}_i\in\omega$. Here, $\tilde{a}_i$ is independent of g and minimizes a microscopic renormalized energy $\tilde{W}$ given in (2.90) in Chapter 2.

Theorem 4.7. *Assume that $b^2>1/2$ and that λ,δ satisfy (H). Then*

$$F_\varepsilon(v_\varepsilon)=J_{\varepsilon,\varepsilon}+db^2(\pi\ln b+\gamma)+o_\varepsilon(1)$$

where $J_{\varepsilon,\varepsilon}$ is defined in (4.8) and $\gamma>0$ is the universal constant defined in [18] Lemma IX.1.

This chapter is divided in two parts:

- In the first one we consider two auxiliary minimization problems for weighted Dirichlet functionals associated to $\mathbb{S}^1$-valued maps.

- The second part is devoted to the proofs of Theorems 4.4, 4.5, 4.6, 4.7. The main tool is an η-ellipticity result (Lemma 4.15). This lemma reduces (under the assumption that λ,δ satisfy (H)) the study of F_ε to the one of the auxiliary problems considered in Section 4.2.

4.2 Shrinking holes for weighted Dirichlet functionals

This section is devoted to the study of three minimization problems. The hypothesis (H) is not optimal for the statements in this section. For example, one may replace (H) by

$$\frac{\varepsilon^{1/4}}{\lambda\delta}\to 0.$$

The results we prove will be later used in the study of both cases $\lambda\equiv 1$ and $\lambda\to 0$.

4.2.1 Dirichlet Vs Degree Conditions in a fixed perforated domain

Let $g\in C^\infty(\partial\Omega,\mathbb{S}^1)$ be s.t. $\deg_{\partial\Omega}(g)=d>0$ and $10^{-5}\cdot 9^{-d^2}\text{diam}(\Omega)>\eta_{\text{stop}}>0$.

Consider $x_1,...,x_N\in\Omega$, $d\geq N\geq 1$ distinct points of Ω satisfying the condition $\eta_{\text{stop}}<10^{-3}\cdot 9^{-d^2}\min\text{dist}(x_i,\partial\Omega)$, and let $\rho>0$ be s.t. $\min\{\eta_{\text{stop}},\min_{i\neq j}|x_i-x_j|\}>8\rho$. Roughly speaking η_{stop} controls the distance between the points and $\partial\Omega$.

We denote $\Omega_\rho=\Omega\setminus\cup\overline{B(x_i,\rho)}$, $\mathbf{x}=(x_1,...,x_n)$ and for $\mathbf{d}=(d_1,...,d_N)\in(\mathbb{N}^*)^N$ s.t. $\sum_i d_i=d$ we define

$$\mathcal{I}_\rho(\mathbf{x},\mathbf{d})=\mathcal{I}_\rho=\left\{w\in H^1(\Omega_\rho,\mathbb{S}^1)\,|\,w=g\text{ on }\partial\Omega\text{ and }\deg_{\partial B(x_i,\rho)}(w)=d_i\right\}$$

and

$$\mathcal{J}_\rho(\mathbf{x},\mathbf{d})=\mathcal{J}_\rho=\left\{w\in H^1(\Omega_\rho,\mathbb{S}^1)\,|\,w=g\text{ on }\partial\Omega\text{ and }w(x)\frac{\rho^{d_i}}{(x-x_i)^{d_i}}=\text{Cst}_i\text{ on }\partial B(x_i,\rho)\right\}.$$

In this section, we compare the minimal energies corresponding to a weighted Dirichlet functional in the above sets.

Proposition 4.8. *Let $\alpha\in L^\infty(\Omega)$ be s.t. $b^2\leq\alpha\leq 1$. Consider the minimization problems*

$$\widehat{\mathcal{I}}_{\rho,\alpha}(\mathbf{x},\mathbf{d})=\inf_{w\in\mathcal{I}_\rho}\frac{1}{2}\int_{\Omega_\rho}\alpha|\nabla w|^2$$

and

$$\widehat{\mathcal{J}}_{\rho,\alpha}(\mathbf{x},\mathbf{d})=\inf_{w\in\mathcal{J}_\rho}\frac{1}{2}\int_{\Omega_\rho}\alpha|\nabla w|^2.$$

In both minimization problems the infima *are attained.*

Moreover, if $\alpha\in W^{1,\infty}(\Omega)$, then, denoting $w^{\deg}_{\rho,\alpha}$ (resp. $w^{\text{Dir}}_{\rho,\alpha}$) a global minimizer of $\frac{1}{2}\int_{\Omega_\rho}\alpha|\nabla\cdot|^2$ in $\mathcal{I}_\rho(\mathbf{x},\mathbf{d})$ (resp. in $\mathcal{J}_\rho(\mathbf{x},\mathbf{d})$) we have $w^{\deg}_{\rho,\alpha}\in H^2(\Omega_\rho,\mathbb{S}^1)$ (resp. $w^{\text{Dir}}_{\rho,\alpha}\in H^2(\Omega_\rho,\mathbb{S}^1)$) and

$$\begin{cases}-\text{div}(\alpha\nabla w^{\deg}_{\rho,\alpha})=\alpha|\nabla w^{\deg}_{\rho,\alpha}|^2w^{\deg}_{\rho,\alpha}\ \textit{in}\ \Omega_\rho\\ w^{\deg}_{\rho,\alpha}\in\mathcal{I}_\rho\,\text{and}\,w^{\deg}_{\rho,\alpha}\times\partial_\nu w^{\deg}_{\rho,\alpha}=0\ \textit{on}\ \partial B(x_i,\rho)\end{cases},\tag{4.5}$$

$$\begin{cases}-\text{div}(\alpha\nabla w^{\text{Dir}}_{\rho,\alpha})=\alpha|\nabla w^{\text{Dir}}_{\rho,\alpha}|^2w^{\text{Dir}}_{\rho,\alpha}\ \textit{in}\ \Omega_\rho\\ w^{\text{Dir}}_{\rho,\alpha}\in\mathcal{J}_\rho\,\text{and}\,\int_{\partial B(x_i,\rho)}\alpha w^{\text{Dir}}_{\rho,\alpha}\times\partial_\nu w^{\text{Dir}}_{\rho,\alpha}=0\end{cases}.\tag{4.6}$$

The proof of this standard result is postponed to Appendix 4.A.

In the special case $\alpha=U_\varepsilon^2$, we denote

$$\widehat{\mathcal{I}}_{\rho,\varepsilon}(\mathbf{x},\mathbf{d})=\inf_{w\in\mathcal{I}_\rho}\frac{1}{2}\int_{\Omega_\rho}U_\varepsilon^2|\nabla w|^2\text{ and }\widehat{\mathcal{J}}_{\rho,\varepsilon}(\mathbf{x},\mathbf{d})=\inf_{w\in\mathcal{J}_\rho}\frac{1}{2}\int_{\Omega_\rho}U_\varepsilon^2|\nabla w|^2.$$

The main result of this section is

Proposition 4.9. *There is $C_0 > 0$ depending only on $g, \Omega, \eta_{\rm stop}$ and b s.t. for $\alpha \in L^\infty(\Omega, [b^2, 1])$ we have*

$$\widehat{\mathcal{I}}_{\rho,\alpha}(\mathbf{x}, \mathbf{d}) \le \widehat{\mathcal{J}}_{\rho,\alpha}(\mathbf{x}, \mathbf{d}) \le \widehat{\mathcal{I}}_{\rho,\alpha}(\mathbf{x}, \mathbf{d}) + C_0.$$

The rigorous proof of Proposition 4.9 is presented in Appendix 4.B.
Here, we simply present the main lines of the proof.
Two situations are possible:

1. $N = 1$ or the points $x_1, ..., x_N$ are well separated: $\frac{1}{4}\min_{i\neq j}|x_i - x_j| > \eta_{\rm stop}$,
2. The points $x_1, ..., x_N$ are not well separated: $\frac{1}{4}\min_{i\neq j}|x_i - x_j| \le \eta_{\rm stop}$.

If the points are well separated (or $N = 1$), Proposition 4.9 can be easily proved: it is a direct consequence of Proposition 4.33 and Lemma 4.32 in Appendix 4.B.

Indeed the proof is made in three steps:

Step 1: Using Lemma 4.32, we obtain a constant C_1 (depending only on $g, \Omega, \eta_{\rm stop}$) s.t.

$$\widehat{\mathcal{J}}_{10^{-1}\eta_{\rm stop},\alpha}(\mathbf{x}, \mathbf{d}) \le C_1.$$

Step 2: With the help of Proposition 4.33, we obtain the existence of a constant C_2 (depending only on b) s.t. for $\tilde{d} \in \mathbb{N}$, denoting $A^i_\rho = B(x_i, 10^{-1}\eta_{\rm stop}) \setminus \overline{B(x_i, \rho)}$, we have

$$\inf_{\substack{w\in H^1(A^i_\rho,\mathbb{S}^1)\\ w(x_1+10^{-1}\eta_{\rm stop}e^{\imath\theta})={\rm Cst}_1e^{\imath\tilde{d}\theta}\\ w(x_1+\rho e^{\imath\theta})={\rm Cst}_2e^{\imath\tilde{d}\theta}}} \frac{1}{2}\int_{A^i_\rho}\alpha|\nabla w|^2 \le \inf_{\substack{w\in H^1(A^i_\rho,\mathbb{S}^1)\\ \deg_{\partial B(x_i,\rho)}=\tilde{d}}} \frac{1}{2}\int_{A^i_\rho}\alpha|\nabla w|^2 + C_2\tilde{d}^2.$$

Step 3: By extending a minimizer of $\widehat{\mathcal{J}}_{10^{-1}\eta_{\rm stop},\alpha}(\mathbf{x}, \mathbf{d})$ by the ones of $\frac{1}{2}\int_{A^i_\rho}\alpha|\nabla\cdot|^2$ with Dirichlet conditions, we can construct a map which proves the result taking $C_0 = C_1 + d^3C_2$.

4.2.2 Optimal perforated domains for the degree conditions

Consider $\Omega' \supset \Omega$ a smooth bounded domain s.t. ${\rm dist}(\partial\Omega', \Omega) > 0$ and a smooth $\mathbb{S}^1$-valued extension of g to $\Omega' \setminus \overline{\Omega}$ (still denoted by g).

In this section, we study the minimization problem

$$I_{\rho,\varepsilon} := \inf_{\substack{x_1,...,x_N\in\Omega\\ |x_i-x_j|\ge 8\rho\\ d_1,...,d_N>0,\sum d_i=d}} \inf_{\substack{w\in H^1_g(\Omega'_\rho,\mathbb{S}^1)\\ \deg_{\partial B(x_i,\rho)}(w)=d_i}} \frac{1}{2}\int_{\Omega'_\rho} U_\varepsilon^2|\nabla w|^2 \tag{4.7}$$

where

$$\Omega'_\rho = \Omega' \setminus \cup\overline{B(x_i, \rho)}$$

and

$$H^1_g(\Omega'_\rho, \mathbb{S}^1) = \left\{w \in H^1(\Omega'_\rho, \mathbb{S}^1) \,|\, w = g \text{ in } \Omega' \setminus \overline{\Omega \cup B(x_i, \rho)}\right\};$$

here, we extended U_ε with the value 1 outside Ω.

A first purpose of this section is the study of the behavior of $I_{\rho,\varepsilon}$ when $\rho = \rho(\varepsilon) \to 0$ as $\varepsilon \to 0$. In view of the application we have in mind we suppose that $\rho(\varepsilon) \ge \varepsilon$ but this is not crucial for our arguments.

A second objective of our study is to exhibit the behavior of almost minimizing configurations $(x_1^n, ..., x_N^n, d_1^n, ..., d_N^n)$.

For fixed ρ, ε, the existence of a minimizing configuration of points $\mathbf{x}_{\rho,\varepsilon}$ is the purpose of Proposition 4.13. In this section we consider only almost minimizing configurations.

For $\varepsilon_n \downarrow 0$, we say that $\{(x_1^n, ..., x_N^n), (d_1^n, ..., d_N^n)\}$ is an almost minimizing configuration for $\rho = \rho(\varepsilon_n) \downarrow 0$ when $x_1^n, ..., x_N^n \in \Omega$, $|x_i^n - x_j^n| \geq 8\rho$, $d_1^n, ..., d_N^n > 0, \sum d_i^n = d$ and there is $C > 0$ (independent of n) s.t.

$$\inf_{\substack{w \in H_g^1(\Omega'_\rho, \mathbb{S}^1) \\ \deg_{\partial B(x_i^n,\rho)}(w) = d_i^n}} \frac{1}{2}\int_{\Omega'_\rho} U_{\varepsilon_n}^2 |\nabla w|^2 - I_{\rho,\varepsilon_n} \leq C.$$

Roughly speaking, we prove in this section two repelling effects for the points: point/point and point/$\partial\Omega$; and an attractive effect for the points due to the inclusions ω_δ.

The main result of this section establishes that when $\varepsilon_n, \rho \downarrow 0$, an almost minimizing configuration $\{(x_1^n, ..., x_N^n), (d_1^n, ..., d_N^n)\}$ is s.t.

- the points x_i^n's cannot approach $\partial\Omega$,
- the points x_i^n's cannot be mutually close,
- the degrees d_i^n's are necessarily all equal to 1,
- if $\dfrac{\rho}{\lambda\delta} \to 0$, then there is $c > 0$ s.t., for large n, $B(x_i^n, c\lambda\delta) \subset \omega_\delta$ for all i.

These facts are expressed in the following proposition (whose proof is postponed to Appendix 4.C).

Proposition 4.10. *Assume that $b^2 > 1/2$. Let $\varepsilon_n \downarrow 0$, $\rho = \rho(\varepsilon_n) \downarrow 0$, $x_1^n, ..., x_N^n \in \Omega$ be s.t. $|x_i^n - x_j^n| \geq 8\rho$, $\rho \geq \varepsilon_n$ and let $d_1^n, ..., d_N^n \in \mathbb{N}^*$ be s.t. $\sum d_i^n = d$.*

1. *Assume that there is $i_0 \in \{1, ..., N\}$ s.t.* $\mathrm{dist}(x_{i_0}^n, \partial\Omega) \to 0$*. Then*
$$\inf_{\substack{w \in H_g^1(\Omega'_\rho, \mathbb{S}^1) \\ \deg_{\partial B(x_i^n,\rho)}(w) = d_i^n}} \left\{ \frac{1}{2}\int_{\Omega'_\rho} U_{\varepsilon_n}^2 |\nabla w|^2 - I_{\rho,\varepsilon_n} \right\} \to \infty.$$
2. *Assume that there is $i_0 \in \{1, ..., N\}$ s.t. $d_{i_0}^n \neq 1$ or that there are $i_0 \neq j_0$ s.t. $|x_{i_0}^n - x_{j_0}^n| \to 0$. Then*
$$\inf_{\substack{w \in H_g^1(\Omega'_\rho, \mathbb{S}^1) \\ \deg_{\partial B(x_i^n,\rho)}(w) = d_i^n}} \left\{ \frac{1}{2}\int_{\Omega'_\rho} U_{\varepsilon_n}^2 |\nabla w|^2 - I_{\rho,\varepsilon_n} \right\} \to \infty.$$
3. *Assume that $\dfrac{\rho}{\lambda\delta} \to 0$ and that there is i_0 s.t. $x_{i_0}^n \notin \omega_\delta$ or s.t. $x_{i_0}^n \in \omega_\delta$ and* $\dfrac{\mathrm{dist}(x_{i_0}^n, \partial\omega_\delta)}{\lambda\delta} \to 0$*. Then*
$$\inf_{\substack{w \in H_g^1(\Omega'_\rho, \mathbb{S}^1) \\ \deg_{\partial B(x_i^n,\rho)}(w) = d_i^n}} \left\{ \frac{1}{2}\int_{\Omega'_\rho} U_{\varepsilon_n}^2 |\nabla w|^2 - I_{\rho,\varepsilon_n} \right\} \to \infty.$$

A straightforward consequence of Proposition 4.10 is the following

Corollary 4.11. *1. Consider an almost minimal configuration* $(\mathbf{x}_{\rho,\varepsilon},\mathbf{d}_{\rho,\varepsilon})\in\Omega^N\times\mathbb{N}^{*N}$, i.e., *assume that there is* $w_{\rho,\varepsilon}\in H^1_g(\Omega'\setminus\cup\overline{B(x_i^{\rho,\varepsilon},\rho)},\mathbb{S}^1)$ *verifying*

$$\deg_{\partial B(x_i^{\rho,\varepsilon},\rho)}(w)=d_i^{\rho,\varepsilon}\text{ and }\frac{1}{2}\int_{\Omega'\setminus\cup\overline{B(x_i^{\rho,\varepsilon},\rho)}}U_\varepsilon^2|\nabla w|^2\le I_{\rho,\varepsilon}+C.$$

(Here, C is independent of ε.)

Then, there is some η_0 depending only on C s.t., for small ε, we have

$$|x_i^{\rho,\varepsilon}-x_j^{\rho,\varepsilon}|,\operatorname{dist}(x_i^{\rho,\varepsilon},\partial\Omega)\ge\eta_0\text{ and }d_i=1\text{ for all }i\neq j,\ i,j\in\{1,...,N\}.$$

In particular, we have $N=d$.

2. If, in addition, $\rho=\rho(\varepsilon)$ is s.t. $\rho\ge\varepsilon$ and $\dfrac{\rho}{\lambda\delta}\to0$, then there is $c>0$ depending only on ρ and C s.t., for small ε, we have $B(x_i^{\rho,\varepsilon},c\lambda\delta)\subset\omega_\delta$.

Proof. We prove the first part. Let $C>0$. We argue by contradiction and we assume that for all $n\in\mathbb{N}^*$ there are $0<\varepsilon_n\le\rho=\rho(\varepsilon_n)\le1/n$, $\mathbf{x}_n=\mathbf{x}_{\rho,\varepsilon_n}$, $(d_1,...,d_N)$ and $w_n=w_{\rho,\varepsilon_n}$ satisfying the hypotheses of Corollary 4.11 and s.t.

$$\min\left\{|x_i^n-x_j^n|,\operatorname{dist}(x_i^n,\partial\Omega)\right\}\to0\text{ or s.t. there is }i\in\{1,...,N\}\text{ for which we have }d_i\neq1.$$

By construction we have that $(\mathbf{x}_{\rho,\varepsilon_n},\mathbf{d})$ is an almost minimizing configuration for I_{ρ,ε_n} with $\rho=\rho(\varepsilon_n)\ge\varepsilon_n$. Clearly from Proposition 4.10 we find a contradiction.

The proof of the second part is similar. □

4.2.3 Existence of minimizing configurations of points

For $\eta>0$ sufficiently small (depending only on Ω, g) and $\rho<10^{-2}\eta$, we define

$$J^\eta_{\rho,\varepsilon}=J_{\rho,\varepsilon}:=\inf_{\substack{x_1,...,x_d\in\Omega\\|x_i-x_j|\ge8\rho\\\operatorname{dist}(x_i,\partial\Omega)\ge\eta}}\ \inf_{\substack{w\in H^1_g(\Omega_\rho,\mathbb{S}^1)\\w(x_i+\rho e^{\imath\theta})=e^{\imath(\theta+\theta_i)},\theta_i\in\mathbb{R}}}\frac{1}{2}\int_{\Omega_\rho}U_\varepsilon^2|\nabla w|^2.\tag{4.8}$$

First we prove that for η,ε,ρ sufficiently small, the conditions "$\operatorname{dist}(x_i,\partial\Omega)\ge\eta$" and "$|x_i-x_j|\ge8\rho$" are not saturated. Thus $J^\eta_{\rho,\varepsilon}$ may be defined independently of small η.

Lemma 4.12. *Let $\eta>0$. Then for sufficiently small ε,ρ, an almost minimizing configuration $(x_1,...,x_d)$ for $J_{\rho,\varepsilon}$ is an almost minimizing configuration for $I_{\rho,\varepsilon}$.*

Moreover, there is $C_0>0$ s.t. $J_{\rho,\varepsilon}\le I_{\rho,\varepsilon}+C_0$, C_0 is independent of ε and ρ.

Proof. Let $C\ge0$ and let $(x_1,...,x_d),(x'_1,...,x'_d)\in\Omega^d$ be s.t.

$$\hat{\mathcal{J}}_{\rho,\varepsilon}(x_1,...,x_d)\le J_{\rho,\varepsilon}+C$$

and

$$\hat{\mathcal{I}}_{\rho,\varepsilon}(x'_1,...,x'_d)\le I_{\rho,\varepsilon}+C.$$

From Corollary 4.11, there is $\eta_0=\eta_0(C)>0$ s.t. for $\varepsilon\le\rho\le\eta_0$, $\min_i\operatorname{dist}(x'_i,\partial\Omega)\ge\eta_0$. Using Proposition 4.9 we find the existence of C_0 s.t.

$$\begin{aligned}\hat{\mathcal{I}}_{\rho,\varepsilon}(x_1,...,x_d)\le\hat{\mathcal{J}}_{\rho,\varepsilon}(x_1,...,x_d)&\le J_{\rho,\varepsilon}+C\le\hat{\mathcal{J}}_{\rho,\varepsilon}(x'_1,...,x'_d)+C\\&\le\hat{\mathcal{I}}_{\rho,\varepsilon}(x'_1,...,x'_d)+C+C_0\\&\le I_{\rho,\varepsilon}+2C+C_0.\end{aligned}$$

□

Thus, from Corollary 4.11, there is $\eta_0 > 0$ s.t. for $\varepsilon \le \rho < \eta_0$, if $\mathbf{x} \in \Omega^d$ is a configuration of points s.t. $\hat{\mathcal{I}}_{\rho,\varepsilon}(\mathbf{x},(1,...,1)) \le I_{\rho,\varepsilon}+1$ or $\hat{\mathcal{J}}_{\rho,\varepsilon}(\mathbf{x},(1,...,1)) \le J_{\rho,\varepsilon}+1$ then $\operatorname{dist}(x_i,\partial\Omega) \ge \eta_0$ and $|x_i - x_j| \ge \eta_0$.

Therefore, for $0 < \varepsilon \le \rho \le \eta_0/8$ (η_0 defined above), $J^\eta_{\rho,\varepsilon}$ is independent of $\eta < \eta_0$.

We end this section with

Proposition 4.13. *For all $\varepsilon > 0$, there are $\mathbf{x}^{\rm deg}_{\rho,\varepsilon}, \mathbf{x}^{\rm Dir}_{\rho,\varepsilon} \in \Omega^d$ s.t. $\mathbf{x}^{\rm deg}_{\rho,\varepsilon}$ minimizes $I_{\rho,\varepsilon}$ and $\mathbf{x}^{\rm Dir}_{\rho,\varepsilon}$ minimizes $J_{\rho,\varepsilon}$.*

The proof of this result is in Appendix 4.D.

4.3 The pinned Ginzburg-Landau functional

In this section, we turn to the study of minimizers of (4.1) in H^1_g.

Recall that we fix $\delta = \delta(\varepsilon)$, $\delta \to 0$, $\lambda = \lambda(\varepsilon)$, $\lambda \equiv 1$ or $\lambda \to 0$ satisfying (H) and $b^2 \in (1/2, 1)$.

4.3.1 Sharp Upper Bound, η-ellipticity and Uniform Convergence

Sharp Upper Bound and an η-ellipticity result

We may easily prove the following upper bound.

Lemma 4.14. *There is a constant C independent of ε s.t., for $1 \ge \lambda\delta \ge \rho \ge \varepsilon > 0$, we have*

$$\inf_{v \in H^1_g(\Omega,\mathbb{C})} F_\varepsilon(v,\Omega) \le db^2\pi \ln\frac{\rho}{\varepsilon} + J_{\rho,\varepsilon} + C. \tag{4.9}$$

If, in addition, we assume that $\dfrac{\rho}{\lambda\delta} \to 0$, then we have for ε sufficiently small

$$\inf_{v \in H^1_g(\Omega,\mathbb{C})} F_\varepsilon(v,\Omega) \le db^2(\pi \ln\frac{b\rho}{\varepsilon} + \gamma) + J_{\rho,\varepsilon}, \tag{4.10}$$

where $\gamma > 0$ is a universal constant defined in [18], Lemma IX.1.

Proof. From Proposition 4.13, one may consider $(x_1^\varepsilon,...,x_d^\varepsilon) = \mathbf{x}^\varepsilon \in \Omega^d$, a minimizing configuration for $J_{\rho,\varepsilon}$.

Note that if $\dfrac{\rho}{\lambda\delta} \to 0$, then, for small ε, from Corollary 4.11 and Lemma 4.12, there are $\eta > 0$ and $c > 0$ s.t. $B(x_i^\varepsilon, c\lambda\delta) \subset \omega_\delta$ and $\min_i\{\min_{i\neq j}|x_i - x_j|, \operatorname{dist}(x_i,\partial\Omega)\} \ge \eta$.

Assume that $\dfrac{\rho}{\lambda\delta} \to 0$ and let w_ε be a minimizing map in $\mathcal{J}_{\rho,\varepsilon}(\mathbf{x}^\varepsilon,(1,...,1))$.

Consider $u_{\varepsilon/(b\rho)}$, the global minimizer of

$$E^0_{\varepsilon/(b\rho)}(u) = \frac{1}{2}\int_{B(0,1)} \left\{|\nabla u|^2 + \frac{b^2\rho^2}{2\varepsilon^2}(1-|u|^2)^2\right\},\ u \in H^1_{x/|x|}(B(0,1),\mathbb{C}).$$

We consider the test function

$$w_\varepsilon(x) = \begin{cases} w_\varepsilon & \text{in } \Omega_\rho \\ \alpha_i u_{\varepsilon/(b\rho)}\left(\dfrac{x - x_i^\varepsilon}{\rho}\right) & \text{in } B(x_i^\varepsilon,\rho) \end{cases}.$$

Estimate (4.10) is obtained by using the fact that $E^0_\varepsilon(u_\varepsilon) = \pi|\ln\varepsilon| + \gamma + o_\varepsilon(1)$ as $\varepsilon \to 0$ (see [18] Lemma IX.1) and Proposition 4.2.

In the situation where $\frac{\rho}{\lambda\delta} \nrightarrow 0$, we may assume that $\frac{\rho}{\lambda\delta} \geq C_0 > 0$. We can replace the minimal configuration $\mathbf{x}^\varepsilon$ by a configuration $\mathbf{y}^\varepsilon$ s.t. there is $C > 0$ independent of ε satisfying

$$y_i^\varepsilon \in \omega_\delta \cap \delta \cdot (\mathbb{Z} \times \mathbb{Z}) \text{ and } \hat{\mathcal{J}}_{\rho,\varepsilon}(\mathbf{x}, (1, ..., 1)) \leq J_{\rho,\varepsilon} + C.$$

We consider the test function

$$w_\varepsilon = \begin{cases} \text{a minimizer of } \mathcal{J}_{\rho,\varepsilon}(\mathbf{y}^\varepsilon, (1, ..., 1)) & \text{in } \Omega \setminus \cup \overline{B(y_i^\varepsilon, \rho)} \\ \alpha_i \dfrac{x - y_i^\varepsilon}{\rho} & \text{in } B(y_i^\varepsilon, \rho) \end{cases}.$$

A direct computation shows that (4.9) holds. □

Note that

$$I_{\rho,\varepsilon} \leq J_{\rho,\varepsilon} \leq \pi d |\ln \rho| + C. \tag{4.11}$$

We now turn to the η-ellipticity.

We denote by v_ε a global minimizer of F_ε in H^1_g. We extend $|v_\varepsilon|$ with the value 1 outside Ω.

One of the main ingredients in this work is the following result.

Lemma 4.15. *[η-ellipticity Lemma]*
Let $0 < \alpha < 1/2$. Then the following results hold:

1. *If for $\varepsilon < \varepsilon_0$*
$$F_\varepsilon(v_\varepsilon, B(x, \varepsilon^\alpha) \cap \Omega) \leq \chi^2 |\ln \varepsilon| - C_1,$$
then we have
$$|v_\varepsilon| \geq 1 - C\chi \text{ in } B(x, \varepsilon^{2\alpha}).$$
Here, $\chi_\varepsilon \in (0, 1)$ is s.t. $\chi_\varepsilon \to 0$ and $\varepsilon_0 > 0$, $C > 0$, $C_1 > 0$ depend only on $b, \alpha, \chi, \Omega, \|g\|_{C^1(\partial\Omega)}$.

2. *If for $\varepsilon < \varepsilon_0$*
$$F_\varepsilon(v_\varepsilon, B(x, \varepsilon^\alpha) \cap \Omega) \leq C|\ln \varepsilon|,$$
then we have
$$|v_\varepsilon| \geq \mu \text{ in } B(x, \varepsilon^{2\alpha}).$$
Here, $\mu \in (0, 1)$ and $\varepsilon_0, C > 0$ depend only on $b, \alpha, \mu, \Omega, \|g\|_{C^1(\partial\Omega)}$.

This result is a direct consequence of Lemma 2.7 (this lemma is proved in Chapter 2, Appendix 2.C).

Uniform convergence to 1 of $|\hat{v}_\varepsilon|$ in $\mathbb{R}^2 \setminus K$, K closed set, $\overline{\omega} \Subset K$

With the help of Lemma 4.15, we are in position to establish uniform convergence of $|v_\varepsilon|$ to 1 far away from $\overline{\omega_\delta}$.

Proposition 4.16. *Let $10^{-2} \cdot \text{dist}(\omega, \partial Y) > \mu > 0$ and $K_\varepsilon^\mu = \{x \in \Omega \,|\, \text{dist}(x, \omega_\delta) \geq \mu\lambda\delta\}$. Then, for sufficiently small ε, we have*

$$|v_\varepsilon| \geq 1 - C\sqrt{\frac{|\ln(\lambda\delta)|}{|\ln \varepsilon|}} \text{ in } K_\varepsilon^\mu.$$

Here C is independent of ε and μ.

Furthermore, if for some small ε*, we have* $|v_\varepsilon(x)| < 1 - C\sqrt{\frac{|\ln(\lambda\delta)|}{|\ln\varepsilon|}}$*, then*

$$F_\varepsilon(v_\varepsilon, B(x,\varepsilon^{1/4})) \geq \frac{\pi d+1}{b^2(1-b^2)}|\ln(\lambda\delta)|.$$

Proof. Using Lemma 4.15 Part 1. with $\alpha = 1/4$ and with $\chi = \sqrt{\frac{\pi d+1}{b^2(1-b^2)}\frac{|\ln(\lambda\delta)|}{|\ln\varepsilon|}}$, we obtain the existence of $C>0$ s.t. for $\varepsilon>0$ sufficiently small:

if $F_\varepsilon(v_\varepsilon, B(x,\varepsilon^{1/4})) < \frac{\pi d+1}{b^2(1-b^2)}|\ln(\lambda\delta)|$, then we have $|v_\varepsilon| \geq 1 - C\chi$ in $B(x,\varepsilon^{1/2})$.

In order to prove Proposition 4.16, we argue by contradiction. There are $\varepsilon_n \downarrow 0$, $\mu>0$ and $x_n \in K_\varepsilon^\mu$ s.t.

$$|v_{\varepsilon_n}(x_n)| < 1 - C\chi.$$

From (4.3), we find

$$|U_{\varepsilon_n} - 1| \leq C\mathrm{e}^{-\frac{\alpha\mu}{2\xi}} \text{ in } K_{\varepsilon_n}^{\mu/2}. \tag{4.12}$$

Consequently, Lemma 4.15, the definition of C and (4.12) imply that for large n,

$$\frac{1}{2}\int_{B(x_n,\varepsilon_n^{1/4})}\left\{|\nabla v_{\varepsilon_n}|^2 + \frac{1}{2\varepsilon_n^2}(1-|v_{\varepsilon_n}|^2)^2\right\} \geq \frac{\pi d+1}{b^2(1-b^2)}|\ln(\lambda\delta)| + o_\varepsilon(1). \tag{4.13}$$

We extend v_ε to $\Omega' := \Omega + B(0,1)$ with the help of a fixed smooth $\mathbb{S}^1$-valued map v s.t. $v = g$ on $\partial\Omega$. We also extend U_ε and a_ε with the value 1 outside Ω.

For n sufficiently large, we have

$$\frac{1}{2}\int_{\Omega'}\left\{|\nabla v_{\varepsilon_n}|^2 + \frac{1}{2\varepsilon_n^2}(1-|v_{\varepsilon_n}|^2)^2\right\} \leq C|\ln\varepsilon_n|.$$

Theorem 4.1 in [65] applied with $r = 10^{-2}\lambda\delta\mu$ and for large n, implies the existence of $\mathcal{B}^n = \{B_j^n\}$ a finite disjoint covering by balls of

$$\left\{x\in\Omega' \,\middle|\, \mathrm{dist}(x,\partial\Omega') > \frac{\varepsilon_n}{b} \text{ and } 1 - |v_{\varepsilon_n}(x)| \geq \left(\frac{\varepsilon_n}{b}\right)^{1/8}\right\}$$

s.t.

$$\mathrm{rad}\,(\mathcal{B}^n) \leq 10^{-2}\cdot\lambda\delta\mu$$

satisfying

$$\begin{aligned}\frac{1}{2}\int_{\cup B_j^n}\left\{|\nabla v_{\varepsilon_n}|^2 + \frac{b^2}{2\varepsilon_n^2}(1-|v_{\varepsilon_n}|^2)^2\right\} &\geq \pi\sum_j d_j^n(|\ln\varepsilon_n| - |\ln(\lambda\delta)|) - C\\ &= \pi\sum_j d_j^n|\ln\xi| - C.\end{aligned}$$

Here, $\mathrm{rad}\,(\mathcal{B}^n) = \sum_i \mathrm{rad}(B_j^n)$, $\mathrm{rad}(B)$ stands for the radius of the ball B, $\xi = \varepsilon_n/(\lambda\delta)$ and the integers d_j^n are defined by

$$d_j^n = \begin{cases}|\deg_{\partial B_j^n}(v_{\varepsilon_n})| & \text{if } B_j^n \subset \{x\in\Omega' \,|\, \mathrm{dist}(x,\partial\Omega') > \frac{\varepsilon_n}{b}\}\\ 0 & \text{otherwise}\end{cases}.$$

Since $B_j \subset \Omega + B_{1/2} \subset \{x \in \Omega' \,|\, \mathrm{dist}(x, \partial\Omega') > \frac{\varepsilon_n}{b}\}$, we obtain

$$\frac{1}{2}\int_{\cup B_j^n} \left\{ |\nabla v_{\varepsilon_n}|^2 + \frac{b^2}{2\varepsilon_n^2}(1-|v_{\varepsilon_n}|^2)^2 \right\} \geq \pi d |\ln \xi| - C. \tag{4.14}$$

From (4.12) and (H) we have

$$\begin{aligned} F_\xi(v_{\varepsilon_n}, \cup_j B_j \cup B(x_n, \varepsilon_n^{1/4})) \quad \geq \quad & \frac{b^2(1-b^2)}{2}\int_{B(x_n,\varepsilon_n^{1/4})} \left\{ |\nabla v_{\varepsilon_n}|^2 + \frac{1}{2\varepsilon_n^2}(1-|v_{\varepsilon_n}|^2)^2 \right\} + \\ & + \frac{b^2}{2}\int_{\cup_j B_j} \left\{ |\nabla v_{\varepsilon_n}|^2 + \frac{b^2}{2\varepsilon_n^2}(1-|v_{\varepsilon_n}|^2)^2 \right\} + o_n(1). \end{aligned} \tag{4.15}$$

By combining (4.9), (4.11), (4.13), (4.14) and (4.15), we find that

$$\begin{aligned} \pi d b^2 |\ln \xi| + \pi d |\ln(\lambda\delta)| \quad & \geq \quad F_{\varepsilon_n}(v_{\varepsilon_n}, \Omega') - \mathcal{O}_n(1) \\ & \geq \quad F_{\varepsilon_n}(v_{\varepsilon_n}, \cup_j B_j \cup B(x_n, \varepsilon_n^{1/4})) - \mathcal{O}_n(1) \\ & \geq \quad \pi d b^2 |\ln \xi| + (\pi d + 1)|\ln(\lambda\delta)| - \mathcal{O}_n(1), \end{aligned}$$

which is a contradiction. This completes the proof of Proposition 4.16. □

4.3.2 Bad discs

Construction and first properties of bad discs

Consider a family of discs $\big(B(x_i, \varepsilon^{1/4})\big)_{i\in I}$ s.t

$$x_i \in \Omega, \,\forall\, i \in I,$$

$$B(x_i, \varepsilon^{1/4}/4) \cap B(x_i, \varepsilon^{1/4}/4) = \emptyset \text{ if } i \neq j,$$

$$\cup_{i\in I} B(x_i, \varepsilon^{1/4}) \supset \Omega.$$

Let $C_0 = C_0(1/4, 7/8)$, $\varepsilon_0 = \varepsilon_0(1/4, 7/8)$ be defined by as in Lemma 4.15.2. For $\varepsilon < \varepsilon_0$, we say that $B(x_i, \varepsilon^{1/4})$ is a good disc if

$$F_\varepsilon(v_\varepsilon, B(x_i, \varepsilon^{1/4}) \cap \Omega) \leq C_0 |\ln \varepsilon|$$

and $B(x_i, \varepsilon^{1/4})$ is an initial bad disc if

$$F_\varepsilon(v_\varepsilon, B(x_i, \varepsilon^{1/4}) \cap \Omega) > C_0 |\ln \varepsilon|. \tag{4.16}$$

Define $J = J(\varepsilon) := \{i \in I \,|\, B(x_i, \varepsilon^{1/4}) \text{ is an initial bad disc}\}$.

Lemma 4.17. *There is an integer N which depends only on g and Ω s.t.*

$$\mathrm{Card}\, J \leq N.$$

Proof. Since each point of Ω is covered by at most $C > 0$ (universal constant) discs $B(x_i, \varepsilon^{1/4})$, we have

$$\sum_{i\in J} F_\varepsilon(v_\varepsilon, B(x_i, \varepsilon^{1/4}) \cap \Omega) \leq C F_\varepsilon(v_\varepsilon, \Omega).$$

The previous assertion implies that $\mathrm{Card}\, J \leq \dfrac{C\pi d}{C_0} + 1$. □

Let $\rho(\varepsilon) = \rho \downarrow 0$ be s.t.

$$\frac{\rho}{\lambda\delta} \to 0 \text{ and } \frac{|\ln\rho|^3}{|\ln\varepsilon|} \to 0. \tag{4.17}$$

Note that from Assumption (H), such a ρ exists, *e.g.*, $\rho = (\lambda\delta)^2$.

The following result is a straightforward variant of Theorem IV.1 in [18].

Lemma 4.18. *Let $\varepsilon_n \downarrow 0$. Then (possibly after passing to a subsequence and relabeling the indices), we may choose $J' \subset J$ and a constant κ independent of n s.t.*

$$J' = \{1, ..., N'\},\ N' = \mathrm{Cst},$$

$$|x_i - x_j| \geq 16\kappa\rho \text{ for } i,j \in J',\ i \neq j$$

and

$$\cup_{i\in J} B(x_i, \varepsilon_n^{1/4}) \subset \cup_{i\in J'} B(x_i, \kappa\rho).$$

For $i \in J'$, we say that $B(x_i, 2\kappa\rho)$ is a bad disc.

Proposition 4.19. *We have*

1. $\dfrac{\rho}{\mathrm{dist}(B(x_i, 2\kappa\rho), \partial\Omega)} \to 0.$
2. $\deg_{\partial B(x_i,2\kappa\rho)}(v_{\varepsilon_n}) > 0.$
3. $F_{\varepsilon_n}(v_{\varepsilon_n}, B(x_i, 2\kappa\rho)) \geq \pi b^2 \deg_{\partial B(x_i,2\kappa\rho)}(v_{\varepsilon_n}) \ln \dfrac{\rho}{\varepsilon_n} - \mathcal{O}(1).$
4. $|v_{\varepsilon_n}| \geq 1 - C\sqrt{\dfrac{|\ln\rho|}{|\ln\varepsilon_n|}}$ *in* $\Omega \setminus \cup_{i\in J'} B(x_i, 2\kappa\rho).$

Proof. We prove Assertions 1., 2. and 3.. Set

$$J'_0 := \{i \in J' \,|\, \deg_{\partial(B(x_i,2\kappa\rho)\cap\Omega)}(v_{\varepsilon_n}) > 0\}.$$

Since $|v_{\varepsilon_n}| \geq \frac{7}{8}$ in $\Omega \setminus \cup_{i\in J'} \overline{B(x_i, 2\kappa\rho)}$, we have

$$0 < d = \sum_{I\in J'} \deg_{\partial(B(x_i,2\kappa\rho)\cap\Omega)}(v_{\varepsilon_n}) \leq \sum_{I\in J'_0} \deg_{\partial(B(x_i,2\kappa\rho)\cap\Omega)}(v_{\varepsilon_n}). \tag{4.18}$$

Consequently $J'_0 \neq \emptyset$.

Up to a subsequence, we may assume that J'_0 is independent of n.

From Proposition 4.16, for all $i \in J'_0$, we have $\mathrm{dist}(B(x_i, \varepsilon^{1/4}), \partial\Omega) \gtrsim \delta$. Consequently, for $i \in J'_0$ we find

$$\frac{\mathrm{dist}(B(x_i, 2\kappa\rho), \partial\Omega)}{\rho} \to 0 \tag{4.19}$$

since $\dfrac{\rho}{\delta} \to 0$.

Assertions 1., 2. and 3. will follow from the estimate

$$F_{\varepsilon_n}(v_{\varepsilon_n}, B(x_i, 2\kappa\rho)) \geq b^2\pi \deg_{\partial B(x_i,2\kappa\rho)}(v_{\varepsilon_n}) \ln \frac{\rho}{\varepsilon_n} - \mathcal{O}(1), \tag{4.20}$$

valid for $i \in J'_0$. Indeed, assume for the moment that (4.20) holds.

Then, by combining (4.10), (4.11), (4.16), (4.17), (4.18) and (4.20), we find that $J'_0 = J'$, *i.e.*, 2. holds. Consequently, by combining Assertion 2. with (4.19), Assertion 1. yields and from Assertion 2. and (4.20), Assertion 3. holds.

We now turn to the proof of (4.20), which relies on Proposition 4.1 in [65]. We apply this proposition in the domain $B = B(0, 2\kappa)$, to the function $v'(x) = v_{\varepsilon_n}(\rho(x - x_i))$ and with the rescaled parameter $\xi_{\rm meso} = \dfrac{\varepsilon}{\rho}$.

Note that, from (4.17), $\varepsilon \ll \xi_{\rm meso} \ll \rho \ll \lambda\delta$ and $|\ln\varepsilon| \sim |\ln\xi_{\rm meso}| \gg |\ln(\lambda\delta)|$.

Clearly, v' satisfies

$$\begin{aligned}\int_B \left\{|\nabla v'|^2 + \frac{1}{\xi_{\rm meso}^2}(1-|v'|^2)^2\right\} &= \int_{B(x_i,2\kappa\rho)} \left\{|\nabla v_{\varepsilon_n}|^2 + \frac{1}{\varepsilon^2}(1-|v_{\varepsilon_n}|^2)^2\right\} \\ &= \mathcal{O}(|\ln\varepsilon|) = \mathcal{O}(|\ln\xi_{\rm meso}|).\end{aligned}$$

Hence, one may apply the following result of Serfaty and Sandier: there is $(B_j)_{j\in I}$, a finite covering of

$$\{x \in B(0, 2\kappa - \xi_{\rm meso}/b) \,|\, |v'(x)| \le 1 - (\xi_{\rm meso}/b)^{1/8}\}$$

with disjoint balls B_j of radius $r_j < 10^{-3}$ s.t.

$$\frac{1}{2}\int_{B\cap\cup B_j} \left\{|\nabla v'|^2 + \frac{b^2}{\xi_{\rm meso}^2}(1-|v'|^2)^2\right\} \ge \pi\sum_j d_j|\ln\xi_{\rm meso}| - \mathcal{O}(1);$$

here $d_j = \begin{cases} |\deg_{\partial B_j}(v')| & \text{if } B_j \subset B(0, 2\kappa - \xi_{\rm meso}/b) \\ 0 & \text{otherwise}\end{cases}.$

Note that from construction, $\{|v_{\varepsilon_n}| \le 7/8\} \subset \cup_J B(x_i, \varepsilon_n^{1/4}) \subset \cup_{J'} B(x_i, \kappa\rho)$. Consequently:

$$\text{if } \deg_{\partial(B_j\cap B(0,2\kappa-\xi_{\rm meso}/b))}(v') \neq 0, \text{ then we have } B_j \subset B(0, \frac{3}{2}\kappa).$$

Therefore, $\sum d_j = \deg_{\partial B(0,2\kappa)}(v') = \deg_{\partial B(x_i,2\kappa\rho)}(v_{\varepsilon_n})$ and

$$\begin{aligned}\frac{1}{2}\int_{B(x_i,2\kappa\rho)} \left\{|\nabla v_{\varepsilon_n}|^2 + \frac{1}{2\varepsilon^2}(1-|v_{\varepsilon_n}|^2)^2\right\} &\ge \pi\deg_{\partial B(x_i,2\kappa\rho)}(v_{\varepsilon_n})|\ln\xi_{\rm meso}| - \mathcal{O}(1) \\ &= \pi\deg_{\partial B(x_i,2\kappa\rho)}(v_{\varepsilon_n})\ln\frac{\rho}{\varepsilon} - \mathcal{O}(1).\end{aligned}$$

Thus (4.20) holds.

The last assertion is obtained using Lemmas 4.14 and 4.15. Indeed, note that the proof of (4.20) gives a more precise result

$$F_{\varepsilon_n}(v_{\varepsilon_n}, B(x_i, \frac{3}{2}\kappa\rho)) \ge b^2\pi\deg_{\partial B(x_i,2\kappa\rho)}(v_{\varepsilon_n})\ln\frac{\rho}{\varepsilon_n} - \mathcal{O}(1).$$

Let $x \in \Omega\setminus\cup_{J'}B(x_i, 2\kappa\rho)$ then $B(x, \varepsilon_n^{1/4}) \cap B(x_i, \frac{3}{2}\kappa\rho) = \emptyset$. Consequently, using Lemma 4.14 and the previous lower bound, we obtain:

$$F_{\varepsilon_n}(v_{\varepsilon_n}, B(x, \varepsilon_n^{1/4})) \le I_{2\kappa\rho,\varepsilon_n} + C_0 \le \pi d|\ln\rho| + C_0.$$

Therefore, from Lemma 4.15, there is $C > 0$, independent of x s.t. $|v_{\varepsilon_n}(x)| \ge 1 - C\sqrt{\dfrac{|\ln\rho|}{|\ln\varepsilon_n|}}$. □

Location and degree of bad discs

Let $w_n = \dfrac{v_{\varepsilon_n}}{|v_{\varepsilon_n}|} \in H^1(\Omega \setminus \cup_{J'} \overline{B(x_i, 2\kappa\rho)}, \mathbb{S}^1)$.

Proposition 4.20. *The map w_n is an almost minimizing function for $I_{2\kappa\rho,\varepsilon_n}$.*

Proof. Indeed, denote $K_n = \dfrac{1}{2}\displaystyle\int_{\Omega\setminus\cup_{J'}B(x_i,2\kappa\rho)} U_{\varepsilon_n}^2 |\nabla w_n|^2$, then we have

$$\begin{aligned}
K_n \quad &\leq \quad F_{\varepsilon_n}(v_{\varepsilon_n}, \Omega \setminus \cup_{J'} \overline{B(x_i, 2\kappa\rho)}) + \int_{\Omega\setminus\cup_{J'}\overline{B(x_i,2\kappa\rho)}} (1 - |v_{\varepsilon_n}|^2)|\nabla w_n|^2 \\
&= \quad F_{\varepsilon_n}(v_{\varepsilon_n}, \Omega) - F_{\varepsilon_n}(v_{\varepsilon_n}, \cup_{J'} B(x_i, 2\kappa\rho)) + \int_{\Omega\setminus\cup_{J'}\overline{B(x_i,2\kappa\rho)}} (1 - |v_{\varepsilon_n}|^2)|\nabla w_n|^2 \\
\leq (4.10), \text{ Prop } 4.19 &\leq \quad I_{2\kappa\rho,\varepsilon_n} + C\sqrt{\frac{|\ln\rho|}{|\ln\varepsilon_n|}} \int_{\Omega\setminus\cup_{J'}\overline{B(x_i,2\kappa\rho)}} |\nabla w_n|^2 + \mathcal{O}(1) \\
\leq (4.10), \text{ Prop } 4.19 &\leq \quad I_{2\kappa\rho,\varepsilon_n} + C\sqrt{\frac{|\ln\rho|}{|\ln\varepsilon_n|}} F_{\varepsilon_n}(v_{\varepsilon_n}, \Omega \setminus \cup_{J'} \overline{B(x_i, 2\kappa\rho)}) + \mathcal{O}(1) \\
\leq (4.9), (4.11) &\leq \quad I_{2\kappa\rho,\varepsilon_n} + C\sqrt{\frac{|\ln\rho|^3}{|\ln\varepsilon_n|}} + \mathcal{O}(1) \\
\leq (\mathrm{H}) &\leq \quad I_{2\kappa\rho,\varepsilon_n} + \mathcal{O}(1).
\end{aligned}$$

□

By combining Proposition 4.10 with Proposition 4.20, we obtain the following

Corollary 4.21. *The configuration $\{(x_1, ..., x_{N'}), (\deg_{\partial B(x_1,2\kappa\rho)}(v_{\varepsilon_n}), ..., \deg_{\partial B(x_{N'},2\kappa\rho)}(v_{\varepsilon_n}))\}$ is an almost minimizing configuration of $I_{2\kappa\rho,\varepsilon_n}$ and consequently, $N' = d$, $\deg_{\partial B(x_i,2\kappa\rho)}(v_{\varepsilon_n}) = 1$ for all i and there is $\eta_0 > 0$ independent of large n s.t.*

$$\min\left\{\min_{i\neq j} |x_i - x_j|, \min_i \operatorname{dist}(x_i, \partial\Omega)\right\} > 2\eta_0,$$

$$B(x_i, 2\eta_0\lambda\delta) \subset \omega_\delta.$$

4.3.3 H^1_{loc}-weak convergence

In order to keep notations simple, we replace from now on, $2\kappa\rho$ by $\rho/2$. In order to emphasize dependence on n, we write x_j^n rather than x_j.

Using Corollary 4.21, up to subsequence, there is $\{a_1, ..., a_d\} \subset \Omega$ s.t. possibly after passing to a subsequence, we have $x_n^i \to a_i$.

Let $\rho_0 > 0$ be defined as

$$\rho_0 = 10^{-2} \cdot \min_{k\neq l} \{\operatorname{dist}(a_k, \partial\Omega), |a_k - a_l|\}.$$

Proposition 4.22. *We have* $\displaystyle\int_\Omega \left\{|\nabla|v_{\varepsilon_n}||^2 + \frac{1}{\varepsilon_n^2}(1 - |v_{\varepsilon_n}|^2)^2\right\} = \mathcal{O}(1)$.

Proof. From (4.10), Proposition 4.19 (Assertion 1., 2. and 3.) and Proposition 4.20, we infer that

$$\int_{\Omega\setminus\cup_i B(x_i,\rho/2)} \left\{|\nabla|v_{\varepsilon_n}||^2 + \frac{1}{\varepsilon_n^2}(1 - |v_{\varepsilon_n}|^2)^2\right\} = \mathcal{O}(1).$$

Consequently it suffices to obtain a similar estimate in $B(x_i,\rho/2)$. Note that $B(x_i,\rho)\subset\omega_\delta$. Thus, if we set

$$u'(x)=\frac{u_{\varepsilon_n}(x_i+\rho x)}{b}:B(0,1)\to\mathbb{C},$$

then u' solves

$$-\Delta u'=\frac{1}{\left(\frac{\varepsilon_n}{b\rho}\right)^2}u'(1-|u'|^2)\text{ in }B(0,1).$$

From [20], we obtain

$$\frac12\int_{B(0,1/2)}\left\{\left|\nabla|u'|\right|^2+\frac{b^2\rho^2}{2\varepsilon_n^2}(1-|u'|^2)^2\right\}=\mathcal{O}(1).$$

This estimate is the subject of Theorem 1 for the potential part and Proposition 1 in [20] for the gradient of the modulus (see also Corollary 1 in [20]).

Set $K_n=\frac12\int_{B(0,1/2)}\left\{\left|\nabla|u'|\right|^2+\frac{b^2\rho^2}{2\varepsilon_n^2}(1-|u'|^2)^2\right\}$. Using Proposition 4.2, we obtain

$$\begin{aligned}K_n=\mathcal{O}(1)&=\frac{1}{2b^2}\int_{B(x_i,\rho/2)}\left\{|\nabla|U_{\varepsilon_n}v_{\varepsilon_n}||^2+\frac{b^4}{2\varepsilon_n^2}\left(1-\frac{|U_{\varepsilon_n}v_{\varepsilon_n}|^2}{b^2}\right)^2\right\}\\&=\frac12\int_{B(x_i,\rho/2)}\left\{|\nabla|v_{\varepsilon_n}||^2+\frac{b^2}{2\varepsilon_n^2}\left(1-|v_{\varepsilon_n}|^2\right)^2\right\}+o_n(1).\end{aligned}$$

Consequently, Proposition 4.22 holds. □

Proposition 4.23. *There is $C>0$ s.t. for (fixed) $0<\eta\le\rho_0$ and n sufficiently large we have*

$$\frac12\int_{\Omega'\setminus\cup\overline{B(a_i,\eta)}}U_{\varepsilon_n}^2|\nabla v_{\varepsilon_n}|^2-I_{\eta,\varepsilon_n}\le C. \tag{4.21}$$

Proof. We use results prove in Appendix 4.C, Section 4.C.1: Proposition 4.37 and Lemma 4.38.

Set

$$\mu_\varepsilon(B(x_0,R)\setminus\overline{B(x_0,r)},\tilde d)=\inf_{\substack{w\in H^1(B(x_0,R)\setminus\overline{B(x_0,r)},\mathbb{S}^1)\\ \deg_{\partial B(x_0,R)}(w)=\tilde d}}\frac12\int_{B(x_0,R)\setminus\overline{B(x_0,r)}}U_\varepsilon^2|\nabla w|^2.$$

Since the configuration $(x_1^n,...,x_d^n)$ is almost minimizing, from Proposition 4.37, we obtain that

$$\left|I_{\eta,\varepsilon_n}-\sum_i\mu_{\varepsilon_n}\left(B(x_i^n,\rho_0)\setminus\overline{B(x_i^n,\eta)}\right)\right|\le C; \tag{4.22}$$

here C is independent of η and large n. Using Lemma 4.38 we have:

- by Assertion 1. we find that

$$\frac12\int_{\cup B(x_n^i,5\eta)\setminus\overline{B(x_n^i,\eta/2)}}U_{\varepsilon_n}^2|\nabla w_n|^2\le C, \tag{4.23}$$

- by Assertion 2. we find that

$$\frac12\int_{\cup B(x_n^i,\rho_0)\setminus\overline{B(x_n^i,\eta)}}U_{\varepsilon_n}^2|\nabla w_n|^2-\sum_i\mu_{\varepsilon_n}\left(B(x_n^i,\rho_0)\setminus\overline{B(x_n^i,\eta)}\right)\le C, \tag{4.24}$$

• by Assertion 3. we obtain

$$\frac{1}{2}\int_{\Omega\setminus\cup\overline{B(x_n^i,\rho_0)}} U_{\varepsilon_n}^2|\nabla w_n|^2 \leq C; \tag{4.25}$$

here C is independent of η and large n.

Denote $K_{\eta,n} = \frac{1}{2}\int_{\Omega\setminus\cup\overline{B(a_i,\eta)}} U_{\varepsilon_n}^2|\nabla v_{\varepsilon_n}|^2 - I_{\eta,\varepsilon_n}$. We have for a fixed $\eta > 0$ and large n (s.t. $B(x_n^i,\eta/2) \subset B(a_i, 3\eta/4)$)

$$\begin{aligned} K_{\eta,n} \leq (4.22),\ \text{Prop. } 4.22 \leq & \frac{1}{2}\int_{\Omega\setminus\cup\overline{B(a_i,\eta)}} U_{\varepsilon_n}^2|\nabla w_n|^2 - \sum_i \mu_{\varepsilon_n}\left(B(x_i^n,\rho_0)\setminus\overline{B(x_i^n,\eta)}\right) + C' \\ \leq (4.23),(4.25) \leq & \frac{1}{2}\int_{\cup B(x_n^i,\rho_0)\setminus\overline{B(x_n^i,\eta)}} U_{\varepsilon_n}^2|\nabla w_n|^2 - \sum_i \mu_{\varepsilon_n}\left(B(x_n^i,\rho_0)\setminus\overline{B(x_n^i,\eta)}\right) + C'' \\ \leq (4.24) \leq & C. \end{aligned}$$

Here C', C'' and C are constants independent of η, n. □

Consequently, there is $v_* \in H^1_{\text{loc}}(\overline{\Omega}\setminus\{a_1,...,a_d\},\mathbb{S}^1)$ s.t. $v_{\varepsilon_n} \rightharpoonup v_*$ in $H^1_{\text{loc}}(\overline{\Omega}\setminus\{a_1,...,a_d\})$.

In order to obtain the expression of the homogenized problem, we use the *unfolding operator* (see [30], definition 2.1). More specifically, we define, for $\Omega_0 \subset \mathbb{R}^2$ an open set, $p \in (1,\infty)$ and $\delta > 0$,

$$\begin{array}{cccl} \mathcal{T}_\delta: & L^p(\Omega_0) & \to & L^p(\Omega_0\times Y) \\ & \phi & \mapsto & \mathcal{T}_\delta(\phi)(x,y) = \begin{cases} \phi\left(\delta\left[\frac{x}{\delta}\right]+\delta y\right) & \text{for } (x,y)\in\Omega_\delta^{\text{incl}}\times Y \\ 0 & \text{for } (x,y)\in\Lambda_\delta\times Y \end{cases}. \end{array}$$

Here, $[s]$ is the integer part of $s \in \mathbb{R}$ and

$$\Omega_\delta^{\text{incl}} := \bigcup_{\substack{Y_\delta^K\subset\Omega_0,\, K\in\mathbb{Z}^2 \\ Y_\delta^K=\delta\cdot(K+Y)}} \overline{Y_\delta^K},\ \Lambda_\delta := \Omega_0\setminus\Omega_\delta^{\text{incl}} \text{ and } \left[\frac{x}{\delta}\right] := \left(\left[\frac{x_1}{\delta}\right],\left[\frac{x_2}{\delta}\right]\right).$$

A straightforward adaptation of a result of Myrto Sauvageot ([66], Theorem 4) gives the following

Proposition 4.24. *Let $\Omega_0 \subset \mathbb{R}^2$ be a smooth bounded open set. Let $v_n \in H^2(\Omega_0,\mathbb{C})$ be s.t.*

1. $|v_n| \leq 1$ *and* $\int_{\Omega_0}(1-|v_n|^2)^2 \to 0$,
2. $v_n \rightharpoonup v_*$ *in* $H^1(\Omega_0)$ *for some* $v_* \in H^1(\Omega_0,\mathbb{S}^1)$,
3. *there are* $H_n \in W^{1,\infty}(\Omega_0,[b^2,1])$ *and* $\delta = \delta_n \downarrow 0$ *s.t.* $\mathcal{T}_\delta(H_n)(x,y) \to H_0(y)$ *in* $L^2(\Omega_0\times Y)$,
4. $-\text{div}(H_n\nabla v_n) = v_n f_n(x)$, $f_n \in L^\infty(\Omega_0,\mathbb{R})$.

Then v_* is the solution of

$$-\text{div}(\mathcal{A}\nabla v_*) = (\mathcal{A}\nabla v_*\cdot\nabla v_*)v_*$$

where $\mathcal{A}$ is the homogenized matrix of $H_0(\frac{\cdot}{\delta})\text{Id}_{\mathbb{R}^2}$.

The proof of Proposition 4.24 is postponed to Appendix 4.E.

We apply the above proposition to $\Omega_0 = \Omega \setminus \cup\overline{B(a_i, \eta)}$, $\delta = \delta_n \downarrow 0$ the sequence which defines a_{ε_n} and $H_n = U^2_{\varepsilon_n}$. By a straightforward application of Proposition 4.2, we obtain

$$\mathcal{T}_\delta(U^2_{\varepsilon_n})(x, y) \overset{L^2(\Omega_0\times Y)}{\rightarrow} \begin{cases} a^2(y) = 1 - (1 - b^2)\mathbb{1}_\omega(y) & \text{if } \lambda \equiv 1 \\ 1 & \text{if } \lambda \to 0 \end{cases}.$$

We find that v_* solves

$$\begin{aligned} -\text{div}(\mathcal{A}\nabla v_*) &= (\mathcal{A}\nabla v_* \cdot \nabla v_*)v_*, \quad &\text{if } \lambda \equiv 1, \\ -\Delta v_* &= |\nabla v_*|^2 v_*, &\text{if } \lambda \to 0. \end{aligned}$$

Here $\mathcal{A}$ is the homogenized matrix of $a^2(\frac{\cdot}{\delta})\text{Id}_{\mathbb{R}^2}$.

4.3.4 The small bad discs

Definition

From the global bound on the potential part (Proposition 4.22), one may construct bad discs of radius ε, in the following sense:

as in [18] (Theorem III.3), for $l \geq 2$, there are $\kappa_l, \mu_l > 0$ (depending only on Ω, g and l) s.t. for $x \in \Omega$, if

$$\frac{1}{\varepsilon^2}\int_{B(x,2\kappa_l\varepsilon)} (1 - |v_\varepsilon|^2)^2 \leq \mu_l$$

then

$$|v_\varepsilon| \geq 1 - \frac{1}{l^2} \text{ in } B(x, \kappa_l\varepsilon).$$

We fix $l \geq 2$ and we drop the subscript l. Let $(B(x_i, \kappa\varepsilon))_{i\in I}$ be a family of discs s.t

$$x_i \in \Omega, \forall\, i \in I,$$

$$B(x_i, \kappa\varepsilon/2) \cap B(x_j, \kappa\varepsilon/2) = \emptyset \text{ if } i \neq j,$$

$$\cup_{i\in I} B(x_i, \kappa\varepsilon) \supset \Omega.$$

We say that $B(x_i, \kappa\varepsilon)$ is a small good disc if

$$\frac{1}{\varepsilon^2}\int_{B(x_i,2\kappa\varepsilon)} (1 - |v|^2)^2 < \mu.$$

If $B(x_i, \kappa\varepsilon)$ is not a small good disc, then we call it a small bad disc. We denote $J \subset I$ the set of indices of small bad discs.

Following [18], using Proposition 4.22, there is $N_l = N > 0$ (depending only on Ω, g and l) s.t. $\text{Card}(J) \leq N$.

Using Lemma 4.31, for $\varepsilon_n \downarrow 0$, possibly after passing to a subsequence and relabeling the discs, there are $J' \subset J$ and $\kappa' \in \{\kappa, ..., 9^{N-1}\kappa\}$ s.t.

$$\{|v_{\varepsilon_n}| < 1 - 1/l^2\} \subset \cup_{i\in J} B(x_i, \kappa\varepsilon_n) \subset \cup_{i\in J'} B(x_i, \kappa'\varepsilon_n)$$

and

$$\frac{|x_i - x_j|}{\varepsilon_n} \geq 8\kappa' \text{ if } i, j \in J',\, i \neq j.$$

Separation of small bad discs

By a standard iterative procedure, we may assume that the small bad discs are mutually far away in the ε-scale.

Proposition 4.25. *Possibly after passing to a subsequence, we have, for large R and $J'' \subset J'$,*

$$\{|v_{\varepsilon_n}| < 1 - 1/l^2\} \subset \cup_{i \in J''} B(x_i^n, R\varepsilon_n),$$

where, for $i \neq j$,

$$\frac{|x_i^n - x_j^n|}{\varepsilon_n} \to \infty \text{ as } n \to \infty.$$

Each bad disc contains exactly one small bad disc

We already know that the separated small bad discs are covered by the ρ-bad discs defined in Lemma 4.18. We next prove that there are exactly d small bad discs and consequently, there is exactly one small bad discs per ρ-bad discs.

Proposition 4.26. *For large n and for all $i \in J''$, we have*

$$\deg_{\partial B(x_i^n, R\varepsilon_n)}(v_{\varepsilon_n}) = 1.$$

Proof. First we prove that, for large n and for all i, we have

$$\deg_{\partial B(x_i^n, R\varepsilon_n)}(v_{\varepsilon_n}) \neq 0.$$

We argue by contradiction and we assume that, up to a subsequence, there is i s.t. $\deg_{\partial B(x_i^n, R\varepsilon_n)}(v_{\varepsilon_n}) = 0$.

Set $M_n = \min\left(b \min_{i \neq j} \dfrac{|x_i^n - x_j^n|}{8R\varepsilon_n}, \delta^{-1}\right)$ and set

$$\begin{array}{rccc} u'_n : & B(0, M_n) & \to & \mathbb{C} \\ & x & \mapsto & \dfrac{u_{\varepsilon_n}(\frac{\varepsilon_n}{b}x + x_i^n)}{b} \end{array}.$$

Note that, $B(x_i^n, M\varepsilon_n) \subset \omega_\delta$ and by Proposition 4.25, we have $M_n \to \infty$.

It is easy to check that u'_n solves $-\Delta u'_n = u'_n(1 - |u'_n|^2)$. Following [28], up to a subsequence,

$$u'_n \to u_0 \text{ in } C^2_{\text{loc}}(\mathbb{R}^2); \tag{4.26}$$

here $u_0 : \mathbb{R}^2 \to \mathbb{C}$ solves $-\Delta u_0 = u_0(1 - |u_0|^2)$ in $\mathbb{R}^2$.

Then two cases occur: $\displaystyle\int_{\mathbb{C}} (1 - |u_0|^2)^2 < \infty$ or $\displaystyle\int_{\mathbb{C}} (1 - |u_0|^2)^2 = \infty$.

Assume first that $\displaystyle\int_{\mathbb{C}} (1 - |u_0|^2)^2 < \infty$. From [28], noting that the degree of u_0 on large circles centered in 0 is 0, we obtain that $u_0 = \text{Cst} \in \mathbb{S}^1$ and consequently $\displaystyle\int_{\mathbb{C}} (1 - |u_0|^2)^2 = 0$.

Since $u'_n \to u_0$ in $L^4(B(0, 2bR))$, we find that

$$\begin{aligned} \int_{B(0,2bR)} (1 - |u'_n|^2)^2 &= \frac{b^2}{\varepsilon_n^2} \int_{B(x_i^n, 2R\varepsilon_n)} (1 - |u_n/b|^2)^2 \\ &= \frac{b^2}{\varepsilon_n^2} \int_{B(x_i^n, 2R\varepsilon_n)} (1 - |v_{\varepsilon_n}|^2)^2 + o_n(1) \to 0. \end{aligned}$$

Noting that $B(x_i^n,\kappa\varepsilon_n)$ is a small bad disc and that $B(x_i^n,2\kappa\varepsilon_n)\subset B(x_i^n,2R\varepsilon_n)$, we have a contradiction.

Therefore $\int_{\mathbb{C}}(1-|u_0|^2)^2=\infty$. Consequently, there is $M>0$ s.t.

$$\int_{B(0,bM)}(1-|u_0|^2)^2\geq\sup_n\left\{\frac{4b^2}{\varepsilon_n^2}\int_\Omega(1-|v_{\varepsilon_n}|^2)^2\right\}.$$

Thus, for large n we have

$$\begin{aligned}\int_{B(0,bM)}(1-|u'_n|^2)^2 &= \frac{b^2}{\varepsilon_n^2}\int_{B(x_i^n,M\varepsilon_n)}(1-|u_{\varepsilon_n}/b|^2)^2\\ &= \frac{b^2}{\varepsilon_n^2}\int_{B(x_i^n,M\varepsilon_n)}(1-|v_{\varepsilon_n}|^2)^2+o_n(1)\\ &\geq \sup_n\left\{\frac{2b^2}{\varepsilon_n^2}\int_\Omega(1-|v_{\varepsilon_n}|^2)^2\right\},\end{aligned}$$

which is a contradiction with $B(x_i^n,M\varepsilon_n)\subset\Omega$.

Consequently we obtain that for large n, $\deg_{\partial B(x_i^n,R\varepsilon_n)}(v_{\varepsilon_n})\neq 0$.

Now we prove that

$$\deg_{\partial B(x_i^n,R\varepsilon_n)}(v_{\varepsilon_n})=1 \text{ for all } i \text{ and large } n. \tag{4.27}$$

Note that each small bad disc contains at least a zero of v_{ε_n}. Consequently, for ρ satisfying (4.17), all small bad discs are included in a ρ-bad disc $B(y,\rho)$ defined in Lemma 4.18. (For sake of simplicity we wrote $B(y,\rho)$ instead of $B(y,2\kappa\rho)$).

If $B(y,\rho)$ is a ρ-bad disc, we denote $\Lambda_y=\{i\in J''\,|\,x_i^n\in B(y,\rho)\}$. Clearly, if $\mathrm{Card}(\Lambda_y)=1$, then (4.27) holds.

We define

$$\text{æ}_n^y:=\begin{cases}10^{-2}\min_{i,j\in\Lambda_y,\,i\neq j}|x_i^n-x_j^n| & \text{if } \mathrm{Card}(\Lambda_y)>1\\ R\varepsilon_n & \text{otherwise}\end{cases}.$$

From Proposition 4.25, if $\mathrm{Card}(\Lambda_y)>1$ then $\text{æ}_n/\varepsilon_n\to\infty$.

For simplicity, we assume that $y=0$ and let

$$\tilde{B}=B(0,8)\setminus\cup_{i\in\Lambda_0}\overline{B\left(\frac{x_i}{\rho},\frac{\text{æ}_n^0}{\rho}\right)}.$$

Clearly, we are in position to apply Theorem 2 in [39] in the perforated domain $\tilde{B}$. After scaling, we find that

$$\frac{1}{2}\int_{B(y,8\rho)\setminus\cup\overline{B(x_i^n,\text{æ}_n^y)}}|\nabla v_{\varepsilon_n}|^2\geq\pi\left|\sum_{i\in\Lambda_y}\deg_{\partial B(x_i^n,R\varepsilon_n)}(v_{\varepsilon_n})\right|\ln\frac{\rho}{\text{æ}_n^y}-C=\pi\ln\frac{\rho}{\text{æ}_n^y}-C.$$

In order to prove (4.27), we observe the case where there is y s.t. $\mathrm{Card}(\Lambda_y)>1$. Note that if for all y centers of ρ-bad discs we have $\mathrm{Card}(\Lambda_y)=1$, then (4.27) holds. Moreover if $\mathrm{Card}(\Lambda_y)>1$, then we have

$$\sum_{i\in\Lambda_y}|\deg_{\partial B(x_i^n,R\varepsilon_n)}(v_{\varepsilon_n})|>1.$$

We obtain easily the following lower bound for $i\in\Lambda_y$:

$$\frac{1}{2}\int_{B(x_i^n,æ_n^y)\setminus\overline{B(x_i^n,R\varepsilon_n)}}|\nabla v_{\varepsilon_n}|^2\geq\pi\left|\deg_{\partial B(x_i^n,R\varepsilon_n)}(v_{\varepsilon_n})\right|\ln\frac{æ_n^y}{R\varepsilon_n}-C.$$

Summing for $i\in\Lambda_y$, we obtain that

$$\sum_{i\in\Lambda_y}\frac{1}{2}\int_{B(x_i^n,æ_n^y)\setminus\overline{B(x_i^n,R\varepsilon_n)}}|\nabla v_{\varepsilon_n}|^2\geq 2\pi\ln\frac{æ_n^y}{R\varepsilon_n}-C.$$

Consequently, we deduce that

$$\sum_y\frac{1}{2}\int_{B(y,8\rho)\setminus\cup\overline{B(x_i^n,R\varepsilon_n)}}|\nabla v_{\varepsilon_n}|^2\geq\pi d\ln\frac{\rho}{R\varepsilon_n}+\pi\sum_{y\text{ s.t. }\operatorname{Card}(\Lambda_y)>1}\ln\frac{æ_n^y}{R\varepsilon_n}-\mathcal{O}_n(1).$$

From Lemma 4.14 and Propositions 4.20 and 4.19, we deduce easily

$$\frac{1}{2}\int_{\bigcup B(y,8\rho)\setminus\cup\overline{B(x_i^n,R\varepsilon_n)}}U_{\varepsilon_n}^2|\nabla v_{\varepsilon_n}|^2=\pi db^2\ln\frac{\rho}{\varepsilon_n}+\mathcal{O}_n(1).$$

Combining the previous estimates, we obtain that

$$\{y\text{ center of }\rho\text{-bad discs}\,|\,\operatorname{Card}(\Lambda_y)>1\}=\emptyset,$$

and thus $\deg_{\partial B(x_i^n,R\varepsilon_n)}(v_{\varepsilon_n})=1$ for large n. □

Corollary 4.27. *For large n, there is a unique zero inside each separated small bad discs defined in Proposition 4.25.*

Proof. From Proposition 4.26, one may assume that $v_{\varepsilon_n}(x_i^n)=0$.

Let $i\in\{1,...,d\}$. In view of (4.26), if we denote

$$\begin{array}{rccc} u_n': & B(0,M_n) & \to & \mathbb{C} \\ & x & \mapsto & \dfrac{u_{\varepsilon_n}(\frac{\varepsilon_n}{b}x+x_i^n)}{b} \end{array},\tag{4.28}$$

then, up to a subsequence, $u_n'\to u_0$ in $C^1(\overline{B(0,bR)})$.

Using the main result of [54], we have the existence of a universal function $f:\mathbb{R}^+\to[0,1]$ s.t.

$$u_0(x)=f(|x|)\mathrm{e}^{\imath(\theta+\theta_i)}\text{ where }x=|x|\mathrm{e}^{\imath\theta},\ \theta_i\in\mathbb{R}\text{ and }f:\mathbb{R}^+\to\mathbb{R}^+\text{ is increasing.}\tag{4.29}$$

Therefore, we may apply Theorem 2.3 in [11] in order to obtain that, for large n, u_n' has a unique zero in $B(0,bR)$. Consequently, for large n, v_{ε_n} has a unique zero in $B(x_i^n,R\varepsilon_n)$. □

Corollary 4.28. *One may consider that R depends only on l (it is independent of the extraction we consider),* i.e, *for $l\geq 2$ there is $R_l>0$ s.t. for small ε, denoting $\{x_i^\varepsilon\,|\,i\in\{1,...,d\}\}$ the set of zeros of a minimizer v_ε, we have*

$$\{|v_\varepsilon|<1-1/l^2\}\subset\cup_i B(x_i^\varepsilon,R_l\varepsilon).$$

Proof. From Corollary 4.27, one may assume that $v_{\varepsilon_n}(x_i^n) = 0$.

Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be defined as in (4.29) and u_n' as in (4.28). For $l \geq 2$, consider $R_l > 0$ be s.t.

$$l \mapsto R_l \text{ is increasing and } f(bR_l) \geq 1 - \frac{1}{2l^2}.$$

Note that from [67], one may consider $R_l \simeq \sqrt{2}l/b$.

By uniqueness of f, the full sequence $|u_n'|$ converges to f in $L^\infty\left[B(0, b\max\{R, R_l\})\right]$. Consequently, for n sufficiently large, since f is not decreasing,

$$\{|v_{\varepsilon_n}| < 1 - 1/l^2\} \subset \cup_i B(x_i^n, R_l\varepsilon_n).$$

□

4.3.5 Asymptotic expansion of $F_\varepsilon(v_\varepsilon)$

Statement of the main result and corollaries

Our main result is

Proposition 4.29. *For all $\varepsilon_n \downarrow 0$, up to a subsequence, there is $\rho = \rho(\varepsilon_n)$ s.t. $\varepsilon_n \ll \rho \ll \lambda\delta$ and s.t. when $n \to \infty$ the following holds*

$$F_{\varepsilon_n}(v_{\varepsilon_n}) \geq J_{\rho,\varepsilon_n} + db^2(\pi \ln \frac{b\rho}{\varepsilon_n} + \gamma) + o_n(1), \tag{4.30}$$

where $J_{\rho,\varepsilon}$ is defined in (4.8) *and γ is a universal constant defined in [18], Lemma IX.1.*

Corollary 4.30. *Let $\varepsilon_n \downarrow 0, \rho$ be as in Proposition 4.29. Then we have*

$$J_{\varepsilon_n,\varepsilon_n} - J_{\rho,\varepsilon_n} = \pi db^2 \ln \frac{\rho}{\varepsilon_n} + o_n(1).$$

Proof of Corollary 4.30. Using Proposition 4.13, consider $(x_1, ..., x_d) \in \Omega^d$ a minimizing configuration of points for J_{ρ,ε_n}, *i.e.* s.t.

$$\hat{\mathcal{J}}_{\rho,\varepsilon_n}(x_1, ..., x_d) = J_{\rho,\varepsilon_n}.$$

Combining Lemma 4.12 with Proposition 4.10, we have the existence of $c > 0$ s.t. $B(x_i, c\lambda\delta) \subset \omega_\delta$.

Therefore, given a minimizing map w_n of $\hat{\mathcal{J}}_{\rho,\varepsilon_n}(x_1, ..., x_d)$, we may easily construct a map $\tilde{w}_n \in H^1(\Omega \setminus \cup_i \overline{B(x_i, \varepsilon_n)}, \mathbb{S}^1)$ s.t. $\tilde{w}_n \in \mathcal{J}_{\varepsilon_n}(x_1, ..., x_d)$ and

$$\begin{aligned} J_{\varepsilon_n,\varepsilon_n} &\leq \frac{1}{2}\int_{\Omega\setminus\cup\overline{B(x_i,\varepsilon_n)}} U_{\varepsilon_n}^2|\nabla\tilde{w}_n|^2 \\ &= \frac{1}{2}\int_{\Omega\setminus\cup\overline{B(x_i,\rho)}} U_{\varepsilon_n}^2|\nabla w_n|^2 + \frac{1}{2}\int_{\cup B(x_i,\rho)\setminus\overline{B(x_i,\varepsilon_n)}} U_{\varepsilon_n}^2|\nabla\tilde{w}_n|^2 \\ &= J_{\rho,\varepsilon_n} + db^2\pi\ln\frac{\rho}{\varepsilon_n} + o_n(1). \end{aligned} \tag{4.31}$$

On the other hand, Lemma 4.14 combined with Proposition 4.29 yield

$$J_{\rho,\varepsilon_n} + db^2(\pi \ln \frac{b\rho}{\varepsilon_n} + \gamma) + o_n(1) \leq F_{\varepsilon_n}(v_{\varepsilon_n}) \leq J_{\varepsilon_n,\varepsilon_n} + db^2(\pi \ln b + \gamma). \tag{4.32}$$

We conclude with the help of (4.31) and (4.32). □

Proof of Theorem 4.7

We are now in position to prove Theorem 4.7, *i.e.*, we are going to prove that

$$F_\varepsilon(v_\varepsilon) = J_{\varepsilon,\varepsilon} + db^2(\pi \ln b + \gamma) + o_\varepsilon(1).$$

Indeed, using Lemma 4.14, it suffices to prove that

$$F_\varepsilon(v_\varepsilon) \geq J_{\varepsilon,\varepsilon} + db^2(\pi \ln b + \gamma) + o_\varepsilon(1).$$

This estimate is equivalent to:

for all $\varepsilon_n \downarrow 0$, up to subsequence, we have $F_{\varepsilon_n}(v_{\varepsilon_n}) \geq J_{\varepsilon_n,\varepsilon_n} + db^2(\pi \ln b + \gamma) + o_n(1)$.

Let $\varepsilon_n \downarrow 0$. Then, up to a subsequence, there is $\rho = \rho_n$ given by Proposition 4.29 s.t.

$$F_{\varepsilon_n}(v_{\varepsilon_n}) \geq J_{\rho,\varepsilon_n} + db^2(\pi \ln \frac{b\rho}{\varepsilon_n} + \gamma) + o_n(1).$$

We deduce from Corollary 4.30 that

$$\begin{aligned} F_{\varepsilon_n}(v_{\varepsilon_n}) &\geq J_{\varepsilon_n,\varepsilon_n} - db^2 \ln \frac{\rho}{\varepsilon_n} + db^2(\pi \ln \frac{b\rho}{\varepsilon_n} + \gamma) + o_n(1) \\ &= J_{\varepsilon_n,\varepsilon_n} + db^2(\pi \ln b + \gamma) + o_n(1), \end{aligned}$$

which ends the proof of Theorem 4.7.

Proof of Proposition 4.29

In order to construct ρ, we first define a suitable extraction.

For $l \in \mathbb{N} \setminus \{0, 1\}$, consider R_l given by Corollary 4.28.

Using Proposition 4.26 and Corollary 4.27, for sufficiently large n, v_{ε_n} has exactly d zeros $x_1^n = x_1, ..., x_d^n = x_d$.

Clearly, these zeros are well separated and far from $\partial\Omega$ (independently of n).

Fix $i \in \{1, ..., d\}$ and consider

$$\begin{array}{rccc} u'_n: & B(0, \delta^2/\varepsilon_n) & \to & \mathbb{C} \\ & x & \mapsto & \dfrac{u_{\varepsilon_n}(\frac{\varepsilon_n}{b}x + x_i)}{b} \end{array}.$$

For simplicity, assume $x_i = 0$.

Up to a subsequence, one has, as in (4.29),

$$u'_n \to u_0 \text{ in } C^2_{\text{loc}}(\mathbb{R}^2, \mathbb{C}),\ u_0(x) = f(|x|)e^{\imath(\theta+\theta_i)}$$

where $x = |x|e^{\imath\theta}$, $\theta_i \in \mathbb{R}$ and $f : \mathbb{R}^+ \to \mathbb{R}^+$ is increasing.

Consequently, for $l \in \mathbb{N} \setminus \{0, 1\}$, one may construct an extraction $(n_l)_l$ s.t., denoting $u'_{n_l} = u'_l = |u'_l|e^{\imath(\theta+\phi'_l)}$ and $v_{\varepsilon_{n_l}} = v_l$,

$$\{|v_l| < 1 - 1/l^2\} \subset \cup_i B(x_i, R_l\varepsilon_{n_l}), \tag{4.33}$$

$$\rho_l := R_l\varepsilon_{n_l} \leq \frac{\lambda^2\delta^2}{l},$$

$$\left| \int_{B(0,bR_l)} |\nabla u'_l|^2 + \frac{1}{2}\left(1 - |u'_l|^2\right)^2 - \int_{B(0,bR_l)} |\nabla u_0|^2 + \frac{1}{2}\left(1 - |u_0|^2\right)^2 \right| \leq \frac{1}{l}, \tag{4.34}$$

and

$$\|\phi_l' - \theta_i\|_{C^1(B(0,bR_l))} \leq \frac{1}{l}. \tag{4.35}$$

Here $R_l \simeq \sqrt{2}l/b$ and is defined in Corollary 4.28.

Following the proof of Proposition 1, Step 2 in [28], one has

$$\int_{B(0,\frac{\lambda^2\delta^2}{\varepsilon_{n_l}})\setminus\overline{B(0,R_l)}} |\nabla\phi_l'|^2 \leq C \text{ independently of } l. \tag{4.36}$$

In $B(0,\lambda^2\delta^2)\setminus\overline{B(0,\varepsilon_{n_l})}$, we denote $v_{n_l} = v_l = |v_l|\mathrm{e}^{\imath(\theta+\phi_l)}$ ($\mathrm{e}^{\imath\theta} = x/|x|$). By conformal invariance, (4.35)implies that

$$\|\phi_l - \theta_i\|_{L^\infty(\partial B(0,\rho_l))} + |\phi_l|_{H^{1/2}(\partial B(0,\rho_l))} \leq \frac{C}{l}. \tag{4.37}$$

Denote $W_l = B(0,2\rho_l)\setminus\overline{B(0,\rho_l)}$ and consider $\psi_i^l \in H^{1/2}(\partial W_l,\mathbb{R})$ s.t.

$$\psi_l = \psi_i^l = \begin{cases} \phi_l - \theta_i & \text{on } \partial B(0,\rho_l) \\ 0 & \text{on } \partial B(0,2\rho_l) \end{cases}.$$

Using (4.37), it is clear that $\|\psi_l\|_{H^{1/2}} = \mathcal{O}(1/l)$. From this, it is straightforward that there exists a constant $C_0 > 0$ (independent of l) and $\Psi_i^l \in H^1(W_l,\mathbb{R})$ s.t.

$$\mathrm{tr}_{\partial W_l}\Psi_i^l = \psi_l = \psi_i^l \text{ and } \frac{1}{2}\int_{W_l}|\nabla\Psi_i^l|^2 \leq \frac{C_0}{l^2}.$$

Finally we define $\Psi_l \in H^1(\Omega\setminus\cup\overline{B(x_i,\rho_l)},\mathbb{R})$ by

$$\Psi_l = \begin{cases} \Psi_i^l(\cdot - x_i) & \text{in } x_i + W_l \\ 0 & \text{otherwise} \end{cases}$$

and

$$\tilde{w}_l = \frac{v_l}{|v_l|}\mathrm{e}^{-\imath\Psi_l} \in \mathcal{J}_{\rho_l}(x_1,...,x_d).$$

Therefore, denoting $w_l = \frac{v_l}{|v_l|} = \mathrm{e}^{\imath(\theta+\phi_l)}$, $U_l = U_{\varepsilon_{n_l}}$ and $\Omega_{\rho_l} = \Omega\setminus\overline{B(x_i,\rho_l)}$, we have

$$\hat{\mathcal{J}}_{\rho_l}(x_1,...,x_d) \leq \frac{1}{2}\int_{\Omega_{\rho_l}} U_l^2|\nabla\tilde{w}|^2 = \frac{1}{2}\int_{\Omega_{\rho_l}} U_l^2|\nabla w_l|^2 + 2U_l^2\nabla(\theta+\phi_l)\cdot\nabla\Psi_l + o_l(1).$$

From (4.36), we obtain easily that

$$\left|\int_{\Omega_{\rho_l}}\nabla(\theta+\phi_l)\cdot\nabla\Psi_l\right| = \sum_i\left|\int_{x_i+W_l}\nabla(\theta+\phi_l)\cdot\nabla\Psi_i^l(\cdot - x_i)\right| = o_n(1)$$

and consequently

$$\hat{\mathcal{J}}_{\rho_l}(x_1,...,x_d) \leq \frac{1}{2}\int_{\Omega_{\rho_l}} U_l^2|\nabla w_l|^2 + o_l(1). \tag{4.38}$$

On the other hand, from direct computations, one has

$$\frac{1}{2}\int_{\Omega_{\rho_l}} U_l^2|\nabla v_l|^2 \geq \frac{1}{2}\int_{\Omega_{\rho_l}} U_l^2|\nabla w_l|^2 + \frac{1}{2}\int_{\Omega_{\rho_l}} U_l^2(|v_l|^2-1)|\nabla(\theta+\phi_l)|^2.$$

Using the same argument as Mironescu in [54], one may obtain that

$$\frac{1}{2}\int_{\Omega_{\rho_l}} U_l^2(1-|v_l|^2)^{1/2}|\nabla\theta|^2 \le C \text{ with } C \text{ independent of } l. \tag{4.39}$$

From (4.39) and (4.33), we obtain

$$\frac{1}{2}\int_{\Omega_{\rho_l}} U_l^2|\nabla v_l|^2 \ge \frac{1}{2}\int_{\Omega_{\rho_l}} U_l^2|\nabla w_l|^2 - o_l(1).$$

Therefore, with (4.38),

$$F_{\varepsilon_{n_l}}(v_l,\Omega_{\rho_l}) + o_l(1) \ge \frac{1}{2}\int_{\Omega_{\rho_l}} U_l^2|\nabla v_l|^2 + o_l(1) \ge \hat{\mathcal{J}}_{\rho_l}. \tag{4.40}$$

In order to complete the proof of (4.30), it suffices to estimate the contribution of the discs $B(x_i,\rho_l)$.

One has (using (4.34))

$$\begin{aligned} F_{\varepsilon_{n_l}}(v_l,B(x_i,\rho_l)) &= \frac{b^2}{2}\int_{B(0,\rho_l)} \left|\nabla\left(\frac{u_l}{b}\right)\right|^2 + \frac{b^2}{2\varepsilon_{n_l}^2}\left(1-\left|\frac{u_l}{b}\right|^2\right)^2 + o_l(1) \\ &= \frac{b^2}{2}\int_{B(0,bR_l)} |\nabla u_l'|^2 + \frac{1}{2}\left(1-|u_l'|^2\right)^2 + o_l(1) \\ &= \frac{b^2}{2}\int_{B(0,bR_l)} |\nabla u_0|^2 + \frac{1}{2}\left(1-|u_0|^2\right)^2 + o_l(1). \end{aligned}$$

From Proposition 3.11 in [65], one has

$$\frac{1}{2}\int_{B(0,bR_l)} |\nabla u_0|^2 + \frac{1}{2}\left(1-|u_0|^2\right)^2 = \pi\ln(bR_l) + \gamma + o_l(1),$$

hence

$$F_{\varepsilon_{n_l}}(v_l,B(x_i,\rho_l)) = b^2[\pi\ln(bR_l)+\gamma] + o_l(1). \tag{4.41}$$

By combining (4.40) with (4.41), we obtain (4.30) with $\rho_l = R_l\varepsilon_{n_l}$.

4.3.6 Proof of Theorems 4.4 and 4.5

We prove Theorem 4.4:

- The existence of exactly d zeros is a direct consequence of Corollary 4.27.
- The fact that they are well included in ω_δ and that v_ε has a degree equal to 1 on small circles around the zeros are obtained by Proposition 4.26 and Corollary 4.27.
- The lower bound for $|v_\varepsilon|$ is given by Proposition 4.19.

Theorem 4.5 is obtained by combining:

- The weak H^1-convergence of v_{ε_n} to v_* which is a direct consequence of Proposition 4.24 (this is explained right after Proposition 4.24).
- The behavior in an ε-neighborhood of the zeros of v_{ε_n}, given by (4.26) and Theorem 4.4 (combined with the main result of [54]).

- In the case where $\lambda \to 0$, the fact that we may localize the zeros inside the inclusions (this is obtained via Theorem 2.23 in Chapter 2).

  Indeed we take $f_n(x) = \mathrm{tr}_{\partial B((k_n,l_n),\delta/2)} v_{\varepsilon_n}((k_n,l_n)+\delta x)$ with $(k_n,l_n) \in \delta \cdot \mathbb{Z}^2$ is a center of a cell containing a zero of v_{ε_n}. Using the main result of [53], one may easily prove that f_n satisfies the conditions 2.13 and 2.14. Thus we can apply Theorem 2.23 and infer that the location of the zero inside the inclusion is governed by a renormalized energy which is independent of the boundary condition.

Appendix 4.A Proof of Proposition 4.8

We prove the existence of minimal map in $\mathcal{I}_\rho$ and in $\mathcal{J}_\rho$. The main ingredient is the fact that these sets are closed under H^1-weak convergence (see [42]). Thus, considering a minimizing sequence for $\frac{1}{2}\int_{\Omega_\rho} \alpha|\nabla\cdot|^2$ in above sets, we obtained the result.

We fix $\theta_0, \theta_i : \Omega_\rho \to \mathbb{R}$ some multivalued functions with smooth gradient s.t. $e^{\imath\theta_0} = \Pi_i \left(\frac{x-x_i}{|x-x_i|}\right)^{d_i}$ and $e^{\imath\theta_i} = \frac{x-x_i}{|x-x_i|}$. Here $d_i \in \mathbb{N}^*$, and they are given by the definition of $\mathcal{I}_\rho$ or if we are considering the minimization in $\mathcal{J}_\rho$, then we have $d_i = 1$.

From Lemma 11 in [25], there is $\phi_0 \in C^\infty(\partial\Omega, \mathbb{R})$ s.t. $g e^{-\imath\theta_0} = e^{\imath\phi_0}$.

Note that

$$w \in \mathcal{I}_\rho \iff w = e^{\imath(\theta_0+\phi)} \text{ with } \phi \in H^1(\Omega_\rho,\mathbb{R}) \text{ and } \mathrm{tr}_{\partial\Omega}\phi = \phi_0, \tag{4.42}$$

$$w \in \mathcal{J}_\rho \iff \left\{ \begin{array}{c} w = e^{\imath(\theta_0+\phi)} \text{ with } \phi \in H^1(\Omega_\rho,\mathbb{R}), \\ \displaystyle\sum_{j\neq i} \theta_i + \phi = \mathrm{Cst}_i \text{ on } \partial B(x_i,\rho) \text{ and } \mathrm{tr}_{\partial\Omega}\phi = \phi_0 \end{array} \right. . \tag{4.43}$$

Clearly, from (4.42) and (4.43), $\mathcal{I}_\rho$ and in $\mathcal{J}_\rho$ are H^1-weakly closed.

We now prove the second part of Proposition 4.8.

One may easily obtain that for some $\lambda : \Omega_\rho \to \mathbb{R}$, denoting $w = e^{\imath(\theta_0+\phi)}$, $\phi \in H^1(\Omega_\rho, \mathbb{R})$ (and thus $w \in \mathcal{I}_\rho$), we have

$$-\mathrm{div}(\alpha\nabla w) = \lambda w \iff \left\{-\mathrm{div}\left[\alpha\nabla(\theta_0+\phi)\right] = 0 \text{ and } \lambda = \alpha|\nabla w|^2\right\}. \tag{4.44}$$

This observation is a direct consequence of the following identity

$$-\mathrm{div}\left[\alpha\nabla e^{\imath(\theta_0+\phi)}\right] = -\mathrm{div}\left[\alpha\nabla(\theta_0+\phi)\right]\imath e^{\imath(\theta_0+\phi)} + \alpha|\nabla(\theta_0+\phi)|^2 e^{\imath(\theta_0+\phi)}.$$

Note that under these notations one has $|\nabla w| = |\nabla(\theta_0+\phi)|$. Thus w is a minimizer in $\mathcal{I}_\rho$ or $\mathcal{J}_\rho$ if and only if $\theta_0+\phi$ minimizes the weighted Dirichlet functional under the condition fixed by the RHS of (4.42) or (4.43).

Consequently, we find that $\theta+\phi$ minimizes the weighted Dirichlet functional under its Dirichlet boundary condition.

Therefore, we obtain easily that $-\mathrm{div}\left[\alpha\nabla(\theta_0+\phi)\right] = 0$. The identity $\nabla(\theta_0+\phi) = w \times \nabla w$ yields $-\mathrm{div}(\alpha\nabla w) = \lambda w$.

Hence, the Euler-Lagrange equations in (4.5),(4.6) are direct consequences of (4.44).

The condition on the boundary for $w_{\rho,\alpha}^{\mathrm{deg}}$ (resp. $w_{\rho,\alpha}^{\mathrm{Dir}}$) follows from multiplying the equation satisfied by $\theta + \phi_{\rho,\alpha}^{\mathrm{deg}}$, $w_{\rho,\alpha}^{\mathrm{deg}} = e^{\imath(\theta+\phi_{\rho,\alpha}^{\mathrm{deg}})}$ (resp. $\theta + \phi_{\rho,\alpha}^{\mathrm{Dir}}$, $w_{\rho,\alpha}^{\mathrm{Dir}} = e^{\imath(\theta+\phi_{\rho,\alpha}^{\mathrm{Dir}})}$) by $\psi \in \mathcal{D}(\Omega,\mathbb{R})$ (resp. $\psi \in \mathcal{D}(\Omega,\mathbb{R})$ s.t $\psi \equiv \mathrm{Cst}_i$ in $B(x_i,\rho)$).

Since α is sufficiently smooth, we can rewrite the Euler-Lagrange equation as

$$-\Delta\phi = \frac{\nabla\alpha\cdot\nabla(\phi+\theta)}{\alpha} \text{ with } \frac{\nabla\alpha\cdot\nabla(\phi+\theta)}{\alpha} \in L^2(\Omega_\rho).$$

So, by elliptic regularity $\phi_{\rho,\alpha}^{\mathrm{deg}}, \phi_{\rho,\alpha}^{\mathrm{Dir}} \in H^2(\Omega_\rho,\mathbb{R})$, and consequently $w_{\rho,\alpha}^{\mathrm{deg}}, w_{\rho,\alpha}^{\mathrm{Dir}} \in H^2(\Omega_\rho,\mathbb{S}^1)$.

Appendix 4.B Proof of Proposition 4.9

As explained in Section 4.2.1, Proposition 4.9 is easily established when either $N = 1$ or when the points are well separated. It remains to consider the case where $N \geq 2$ and there are $i \neq j$ s.t. $|x_i - x_j| \leq 4\eta_{\text{stop}}$.

4.B.1 The separation process

We assume that $N \geq 2$ and that the points are not well separated. Our purpose is to compare the energy of $\hat{\mathcal{J}}_{\rho,\alpha}$ to the energy of $\hat{\mathcal{I}}_{\rho,\alpha}$. To this purpose, we decompose Ω_ρ into several regions and we compare energies in each regions. These regions are constructed recursively using the following version of Theorem IV.1 in [18].

Lemma 4.31. *Let $N \geq 2$, $x_1, ..., x_N \in \mathbb{R}^2$ and $\eta > 0$. There are $\kappa \in \{9^0, ..., 9^{N-1}\}$ and $\{y_1, ..., y_{N'}\} \subset \{x_1, ..., x_N\}$ s.t.*

$$\cup_{i=1}^N B(x_i, \eta) \subset \cup_{i=1}^{N'} B(y_i, \kappa\eta)$$

and

$$|y_i - y_j| \geq 8\kappa\eta \text{ for } i \neq j.$$

We let $x_1^0, ..., x_N^0$ denote the initial points $x_1, ..., x_N$. For $k \geq 1$ (here, k is an iteration in the construction of the regions), we let N_k denote the number of points selected at Step k, and denote the points we select by $x_1^k, ..., x_{N_k}^k$.

The recursive construction is made in such a way that $N_k > N_{k+1}$ and $N_k \geq 1$ for all $k \geq 1$.

The process will stop at the end of Step k if and only if one of the following conditions yields

Rule 1: there is a unique point in the selection (*i.e.* $N_k = 1$),

Rule 2: $\min_{i \neq j} |x_i^k - x_j^k| > 4\eta_{\text{stop}}$.

Step k, $k \geq 1$: Let $\eta'_k = \frac{1}{4} \min_{i \neq j} |x_i^{k-1} - x_j^{k-1}|$.

Using Lemma 4.31, there are

$$\kappa_k \in \{9^1, ..., 9^{N_{k-1}-1}\} \text{ and } \{x_1^k, ..., x_{N_k}^k\} \subset \{x_1^{k-1}, ..., x_{N_{k-1}}^{k-1}\}$$

s.t.

$$\cup_i B(x_i^{k-1}, \eta'_k) \subset \cup_j B(x_j^k, \kappa_k \eta'_k) \text{ and } |x_i^k - x_j^k| \geq 8\kappa_k \eta'_k \text{ for } i \neq j.$$

We denote $\eta_k = 2\kappa_k \eta'_k$. We stop the construction if $N_k = 1$ (Rule 1) or if $\frac{1}{4} \min |x_i^{k-1} - x_j^{k-1}| > \eta_{\text{stop}}$ (Rule 2).

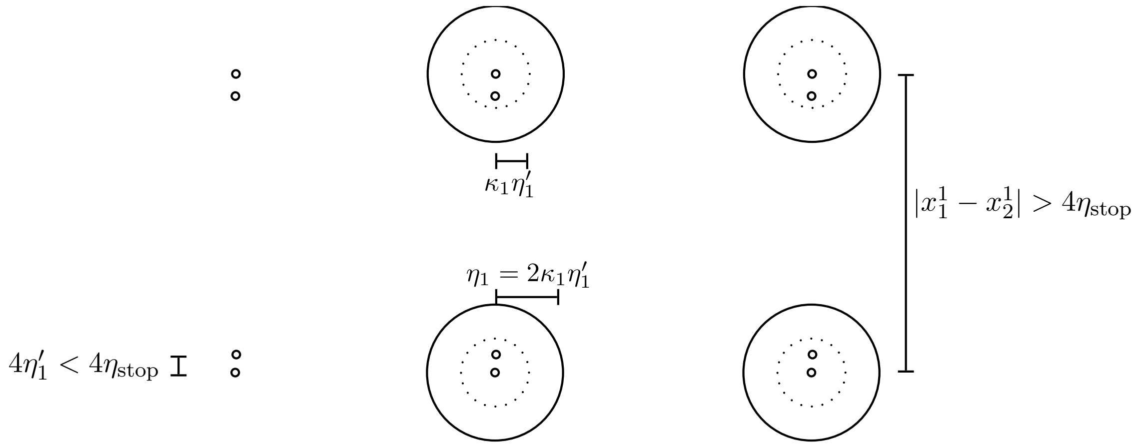
In Figure 4.1 and 4.2 both stop conditions are presented.

Claim:

i. From the definitions of η'_k and η_k, we have $N_k < N_{k-1}$ and $\eta_{k-1} \leq \eta'_k < \eta_k$.

ii. The balls $B(x_j^k, 2\eta_k)$ are disjoint.

iii. Denoting $\Lambda_j^k \subset \{1, ..., N_{k-1}\}$ the set of indices i s.t. $x_i^{k-1} \in B(x_j^k, \kappa_k \eta'_k)$, then for $i \in \Lambda_j^k$ we have $B(x_i^{k-1}, \eta'_k) \subset B(x_j^k, \kappa_k \eta'_k)$. Furthermore, by construction, $|x_i^{k-1} - x_j^{k-1}| \geq 4\eta'_k$.

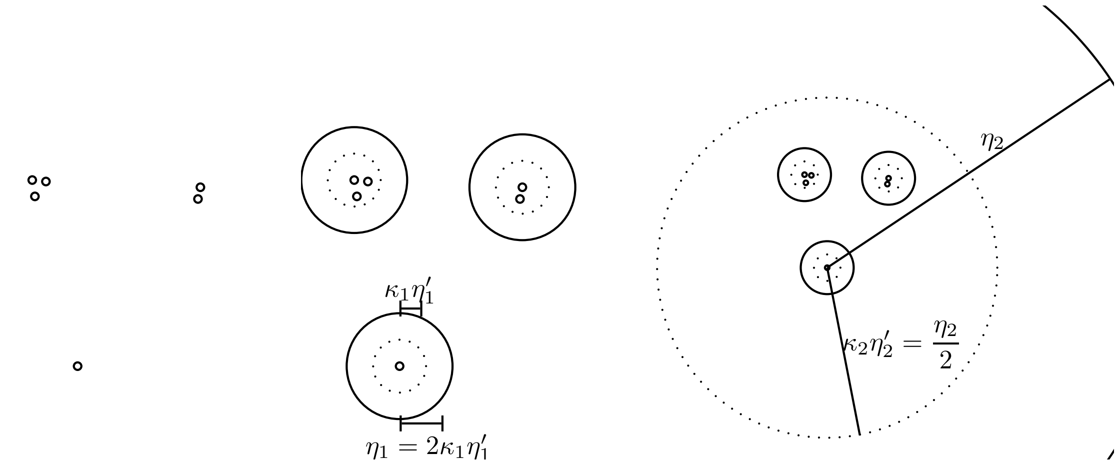


(a) The initial balls

(b) The first step: a selection of two centers

(c) The process stops at the end of the first step since there are two well separated balls.

Figure 4.1: The process stops when we obtain well separated balls



(a) The initial balls

(b) The first step: a selection of three centers

(c) The second step: it remains a unique ball (the picture is at scale 1/2)

Figure 4.2: The process stops when we obtain a unique ball

4.B.2 The separation process gives a natural partition of Ω

Let Ω, g, $x_1, ..., x_N$, $\mathbf{d}$ and $\rho, \eta_{\rm stop}$ like in Section 4.2.1 with $N \geq 2$ and s.t. the points are not well separated.

We apply the separation process. The process stops after K steps, $1 \leq K \leq N-1$.

We denote

$$\{y_1, ..., y_{N'}\} \subset \{x_1, ..., x_N\} \text{ the selection that we obtain, } i.e., \ x_j^K = y_j \text{ and } N' = N_K,$$

$$\eta = \begin{cases} 9^N \cdot \eta_{\rm stop} & \text{if } N' = 1 \\ \min\left\{9^N \cdot \eta_{\rm stop}\,,\, \frac{1}{4}\min|y_i - y_j|\right\} & \text{if } N' > 1 \end{cases}, \text{ so } \eta \geq \max(\eta_K, \eta_{\rm stop}), \tag{4.45}$$

$$\Lambda_j = \{i \in \{1, ..., N\} \,|\, x_i \in B(y_j, \eta)\}.$$

We denote

$$D_{j,k} = B(x_j^k, \eta_k) \setminus \cup_{x_i^{k-1} \in B(x_j^k, \eta_k)} B(x_i^{k-1}, \eta_k'),\ k \in \{1, ..., K\},\ j \in \{1, ..., N_k\}, \tag{4.46}$$

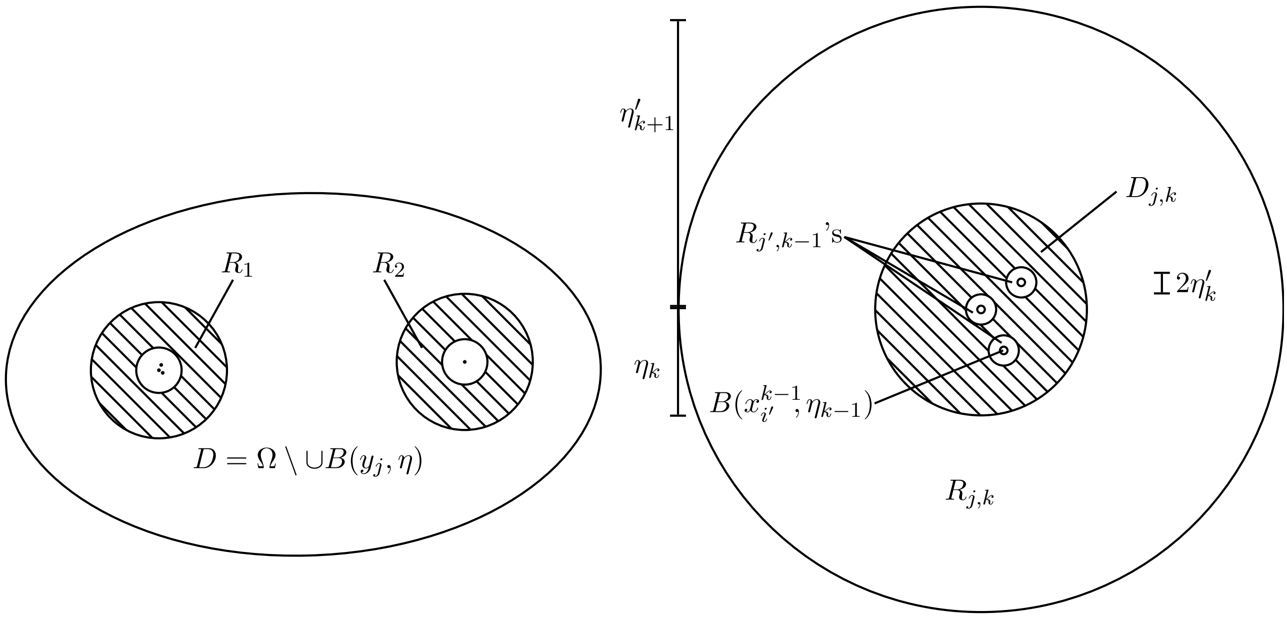
$$R_{j,k} = B(x_j^k, \eta'_{k+1}) \setminus B(x_j^k, \eta_k),\ k \in \{0, ..., K-1\},\ j \in \{1, ..., N_k\}, \tag{4.47}$$

$$R_j = B(y_j, \eta) \setminus B(y_j, \eta_K),\ j \in \{1, ..., N'\} \tag{4.48}$$

and

$$D = \Omega \setminus \cup_{j\in\{1,...,N'\}} B(y_j, \eta).$$

Note that by construction of η'_k, η_k and x_i^k the following properties are satisfied:



(a) The macroscopic perforated domain and the first mesoscopic rings

(b) A mesoscopic ring and a mesoscopic perforated domain

$$\text{the balls } B(x_i^{k-1}, 2\eta'_k) \text{ are disjoint} \tag{4.49}$$

and

$$2 \cdot 9\eta'_k \le \eta_k \le 9^N \eta'_k. \tag{4.50}$$

Therefore

$$\Omega_\rho = D \bigcup \cup_{j,k} D_{j,k} \bigcup \cup_{j,k} R_{j,k} \bigcup \cup_j R_j \text{ with disjoint unions.} \tag{4.51}$$

Construction of test functions in D and $D_{j,k}$

Lemma 4.32. *1. Let $\eta > 0$. There is $C_1(\eta, \Omega, g) > 0$ s.t. if $x_1, ..., x_N \in \Omega$ satisfy $\min_{i\neq j} |x_i - x_j|, \min_i \operatorname{dist}(x_i, \partial\Omega) > 4\eta$ and $d_1, ..., d_N \in \mathbb{N}^*$ are s.t. $\sum d_i = d$ then there is $w \in H^1_g(\Omega_\eta, \mathbb{S}^1)$ s.t. $w(x) = \frac{(x-x_i)^{d_i}}{\eta^{d_i}}$ on $\partial B(x_i, \eta)$ and*

$$\int_{\Omega_\eta} |\nabla w|^2 \le C_1(\eta).$$

Moreover C_1 can be considered decreasing with η.

2. *Let* $\eta>0,\kappa\geq 8$, $d_0,d_1,...,d_N\in\mathbb{N}^*$ *be s.t.* $\sum_{1\leq i\leq N}d_i=d_0$. *Then, there is* $C_2(\kappa,d_0)$ *s.t.* $x_1,...,x_N\in B(0,\kappa\eta)$ *satisfying* $\min_{i\neq j}|x_i-x_j|\geq 4\eta$ *we can associate a map to each family* $w\in H^1(B(0,2\kappa\eta)\setminus\cup\overline{B(x_i,\eta)},\mathbb{S}^1)$ *s.t.*

$$w(x)=\begin{cases}\dfrac{x^{d_0}}{(2\kappa\eta)^{d_0}} & \text{on } \partial B(0,2\kappa\eta)\\ \dfrac{(x-x_i)^{d_i}}{\eta^{d_i}} & \text{on } \partial B(x_i,\eta)\end{cases}$$

and

$$\int_{B(0,2\kappa\eta)\setminus\cup\overline{B(x_i,\eta)}}|\nabla w|^2\leq C_2(\kappa,d_0).$$

Moreover C_2 *can be considered increasing with* κ, d_0.

Proof. In order to prove 1., we consider, *e.g.*, the test function

$$w=\mathrm{e}^{\imath H}\Pi_i\frac{(x-x_i)^{d_i}}{|x-x_i|^{d_i}}\text{ with } H \text{ s.t. }\begin{cases}H:\Omega_\eta\to\mathbb{R}\\ H\equiv 0\text{ in }\left\{\text{dist}\left[x,\partial\left(\Omega\setminus\overline{B(x_i,\eta)}\right)\right]\geq\eta\right\}\\ -\Delta H=0\text{ in }\left\{\text{dist}\left[x,\partial\left(\Omega\setminus\overline{B(x_i,\eta)}\right)\right]<\eta\right\}\\ w\in H^1_g(\Omega_\eta,\mathbb{S}^1)\text{ and } w(x)=\frac{(x-x_i)^{d_i}}{\eta^{d_i}}\text{ on }\partial B(x_i,\eta)\end{cases}.$$

Assertion 2. was essentially established in [39], Section 3. We adapt here the argument in [39]. By conformal invariance, we may assume that $\eta=1$. We let

$$w(x)=\begin{cases}\Pi_i\dfrac{\left[x+2x_i\left(\frac{|x|}{\kappa}-2\right)\right]^{d_i}}{\left|x+x_i\left(\frac{|x|}{\kappa}-2\right)\right|^{d_i}} & \text{in } B(0,2\kappa)\setminus\overline{B(0,\frac{3\kappa}{2})}\\ \Pi_i\dfrac{(x-x_i)^{d_i}}{|x-x_i|^{d_i}} & \text{in } B(0,\frac{3\kappa}{2})\setminus\cup\overline{B(x_i,3/2)}\\ \dfrac{(x-x_i)^{d_i}}{|x-x_i|^{d_i}}\mathrm{e}^{\imath(2|x-x_i|-2)\varphi_i} & \text{in } B(x_i,3/2)\setminus\cup\overline{B(x_i,1)}\end{cases};$$

here $\varphi_i\in C^\infty(B(x_i,3/2)\setminus\cup\overline{B(x_i,1)},\mathbb{R})$ is defined by $\mathrm{e}^{\imath\varphi_i}=\Pi_{j\neq i}\dfrac{(x-x_j)^{d_j}}{|x-x_j|^{d_j}}$. Clearly $\|\varphi_i\|_{H^1(B(x_i,3/2)\setminus\cup\overline{B(x_i,1)})}$ is bounded by a constant which depends only on d_0. □

By (4.45) and Lemma 4.32, part 1., one may find a map $w_0\in H^1(D,\mathbb{S}^1)$ s.t.

$$w_0=\begin{cases}g & \text{on }\partial\Omega\\ w_0(x)=\dfrac{(x-y_j)^{\tilde{d}_j}}{\eta^{\tilde{d}_j}} & \text{on }\partial B(y_j,\eta)\end{cases}\ (\text{ where }\tilde{d}_j=\sum_{x_i\in B(y_j,\eta)}d_i)$$

satisfying in addition

$$\int_D|\nabla w_0|^2\leq C_1(\eta)\leq C_1(\eta_{\text{stop}}).\tag{4.52}$$

For each $D_{j,k}$, combining (4.46), (4.49), (4.50) and using Lemma 4.32, part 2, there exists a map $w_{j,k}\in H^1(D_{j,k},\mathbb{S}^1)$ s.t.

$$w_{j,k}(x)=\begin{cases}\dfrac{(x-x_j^k)^{\tilde{d}_{j,k}}}{\eta_k^{\tilde{d}_{j,k}}} & \text{for } x\in\partial B(x_j^k,\eta_k)\\ \dfrac{(x-x_i^{k-1})^{\tilde{d}_{i,k-1}}}{\eta_k'^{\tilde{d}_{i,k-1}}} & \text{for } x\in\partial B(x_i^{k-1},\eta'_k)\end{cases}.$$

Here,

$$\tilde{d}_{j,k} = \sum_{x_i \in B(x_j^k, \eta_k)} d_i$$

and

$$\int_{D_{j,k}} |\nabla w_{j,k}|^2 \leq C_2(2\kappa_k, d_{j,k}) \leq C_2(2\cdot 9^{d-1}, d). \tag{4.53}$$

Construction of test functions in R_j's and $R_{j,k}$'s

For $R > r > 0$, $x_0 \in \mathbb{R}^2$ and $\alpha \in L^\infty(\mathbb{R}^2, [b^2, 1])$, we define

$$\mu_\alpha(B(x_0,R)\setminus\overline{B(x_0,r)}, \tilde{d}) = \inf_{\substack{w \in H^1(B(x_0,R)\setminus\overline{B(x_0,r)}, \mathbb{S}^1)\\ \deg_{\partial B(x_0,R)}(w) = \tilde{d}}} \frac{1}{2}\int_{B(x_0,R)\setminus\overline{B(x_0,r)}} \alpha|\nabla w|^2 \tag{4.54}$$

and

$$\mu_\alpha^{\text{Dir}}(B(x_0,R)\setminus\overline{B(x_0,r)}, \tilde{d}) = \inf_{\substack{w \in H^1(B(x_0,R)\setminus\overline{B(x_0,r)}, \mathbb{S}^1)\\ w(x_0+Re^{\imath\theta}) = e^{\imath\tilde{d}\theta}\\ w(x_0+re^{\imath\theta})e^{-\imath\tilde{d}\theta} = \text{Cst}}} \frac{1}{2}\int_{B(x_0,R)\setminus\overline{B(x_0,r)}} \alpha|\nabla w|^2. \tag{4.55}$$

In the special case $\alpha = U_\varepsilon^2$, we denote

$$\mu_\varepsilon(B(x_0,R)\setminus\overline{B(x_0,r)}, \tilde{d}) = \mu_{U_\varepsilon^2}(B(x_0,R)\setminus\overline{B(x_0,r)}, \tilde{d})$$

and

$$\mu_\varepsilon^{\text{Dir}}(B(x_0,R)\setminus\overline{B(x_0,r)}, \tilde{d}) = \mu_{U_\varepsilon^2}^{\text{Dir}}(B(x_0,R)\setminus\overline{B(x_0,r)}, \tilde{d}).$$

Note that the minimization problems (4.54) and (4.55) admit solutions; this is obtained by adapting the proof of Proposition 4.8.

If α is Lipschitz, then the solutions of (4.54) and (4.55) are in H^2.

We present an adaptation of a result of Sauvageot, Theorem 2 in [66].

Proposition 4.33. *There is $C_3 > 0$ depending only on $b \in (0,1)$ s.t. for $R > r > 0$ and $\alpha \in L^\infty(\mathbb{R}^2, \mathbb{R})$ satisfying $1 \geq \alpha \geq b^2$, we have*

$$\mu_\alpha^{\text{Dir}}(B(x_0,R)\setminus\overline{B(x_0,r)}, \tilde{d}) \leq \mu_\alpha(B(x_0,R)\setminus\overline{B(x_0,r)}, \tilde{d}) + \tilde{d}^2 C_3.$$

Proof. This result was obtained by Sauvageot with $\alpha \in W^{1,\infty}(\mathbb{R}^2, [b^2, 1])$. We may extend this estimate to $\alpha \in L^\infty(\mathbb{R}^2, [b^2, 1])$.

Indeed, let $(\rho_t)_{1>t>0}$ be a classical mollifier, namely $\rho_t(x) = t^{-2}\rho(x/t)$ with $\rho \in C^\infty(\mathbb{R}^2, [0,1])$, $\operatorname{Supp}\rho \subset B(0,1)$ and $\int_{\mathbb{R}^2}\rho = 1$.

Set $\alpha_t = \alpha * \rho_t \in W^{1,\infty}(B(x_0,R), [b^2, 1])$. We have

$$\lim_{t\to 0} \mu_{\alpha_t}(B(x_0,R)\setminus\overline{B(x_0,r)}, \tilde{d}) = \mu_\alpha(B(x_0,R)\setminus\overline{B(x_0,r)}, \tilde{d}) \tag{4.56}$$

and

$$\lim_{t\to 0} \mu_{\alpha_t}^{\text{Dir}}(B(x_0,R)\setminus\overline{B(x_0,r)}, \tilde{d}) = \mu_\alpha^{\text{Dir}}(B(x_0,R)\setminus\overline{B(x_0,r)}, \tilde{d}). \tag{4.57}$$

We prove (4.56), Equality (4.57) follows with the same lines.

Let w be a minimizer of $\mu_\alpha(B(x_0,R)\setminus\overline{B(x_0,r)}, \tilde{d})$. By using Dominated convergence theorem, since $\alpha_t \to \alpha$ in $L^1(B(x_0,R))$, we obtain that $\alpha_t|\nabla w|^2 \to \alpha|\nabla w|^2$ in $L^1(B(x_0,R)\setminus\overline{B(x_0,r)})$ as $t \to 0$. Consequently

$$\lim_{t\to 0} \mu_{\alpha_t}(B(x_0,R)\setminus\overline{B(x_0,r)}, \tilde{d}) \leq \mu_\alpha(B(x_0,R)\setminus\overline{B(x_0,r)}, \tilde{d}).$$

On the other hand, let w_t be a minimizer of $\mu_\alpha(B(x_0,R)\setminus\overline{B(x_0,r)},\tilde{d})$ and let $t_n\downarrow 0$. Up to a subsequence, $w_{t_n}\rightharpoonup w_0$ in $H^1(B(x_0,R)\setminus\overline{B(x_0,r)})$ as $n\to\infty$ and $\sqrt{\alpha_{t_n}}\nabla w_{t_n}\rightharpoonup\sqrt{\alpha}\nabla w_0$ in $L^2(B(x_0,R)\setminus\overline{B(x_0,r)})$.

Since the class $\mathcal{I}:=\{w\in H^1(B(x_0,R)\setminus\overline{B(x_0,r)},\mathbb{S}^1)\,|\,\deg_{B(x_0,R)}(w)=\tilde{d}\}$ is closed under the H^1-weak convergence (see Appendix 4.A or [42]), we obtain that $w_0\in\mathcal{I}$. Consequently, we have

$$\liminf_{t\to 0}\mu_{\alpha_t}(B(x_0,R)\setminus\overline{B(x_0,r)},\tilde{d})\geq\mu_\alpha(B(x_0,R)\setminus\overline{B(x_0,r)},\tilde{d}).$$

Thus the proof of (4.56) is complete.

Therefore, without loss of generality, we may assume that α is Lipschitz.

One may easily prove that if $R\leq 4r$, then $\mu_\alpha^{\rm Dir}(B(x_0,R)\setminus\overline{B(x_0,r)},\tilde{d})\leq 2\tilde{d}^2\pi\ln 4$. Thus we assume that $R>4r$. Clearly, it suffices to obtain the result for $\tilde{d}=1$ and $x_0=0$.

Let w be a global minimizer of $\mu_\alpha(B(x_0,R/2)\setminus\overline{B(x_0,2r)},1)$. As explained in Section 4.A, denoting $x/|x|={\rm e}^{\imath\theta}$, one may write $w={\rm e}^{\imath(\theta+\phi)}$ for some $\phi\in H^2(B(x_0,R/2)\setminus\overline{B(x_0,2r)},\mathbb{R})$. Now we switch to polar coordinates.

Consider

$$I=\left\{\rho\in[2r,R/2]\,|\,\int_0^{2\pi}\alpha|\nabla(\theta+\phi)|^2(\rho,\theta)\,{\rm d}\theta\leq\frac{1}{\rho^2}\int_0^{2\pi}\alpha(\rho,\theta)\,{\rm d}\theta\right\}.$$

Then I is closed (since $\phi\in H^2$). On the other hand, I is non empty, by the mean value theorem.

Let $r_1=\min I$ and $r_2=\max I$. We may assume that $\phi(r_2,0)=0$ and $\phi(r_1,0)=\theta_0$. We construct a test function:

$$\phi'(\rho,\theta)=\begin{cases}0 & \text{if } 2r_2\leq\rho\leq R\\ \dfrac{2r_2-\rho}{r_2}\phi(r_2,\theta) & \text{if } r_2\leq\rho\leq 2r_2\\ \phi(r,\theta) & \text{if } r_1\leq\rho\leq r_2\\ \dfrac{2\rho-r_1}{r_1}\phi(r_1,\theta)+2\dfrac{r_1-\rho}{r_1}\theta_0 & \text{if } r_1/2\leq r\leq r_1\\ \theta_0 & \text{if } r\leq\rho\leq r_1/2\end{cases}.$$

As explained in [66], there is C depending only on b s.t.

$$\frac{1}{2}\int_{B(0,R/2)\setminus\overline{B(0,2r)}}\alpha\left(|\nabla(\theta+\phi')|^2-|\nabla(\theta+\phi)|^2\right)\leq C.$$

Thus the result follows. □

For $\alpha\in L^\infty(\Omega,[b^2,1])$, using Proposition 4.33, there is C_3 depending only on $b\in(0,1)$ s.t. for all $k\in\{1,...,K-1\}$, $j\in\{1,...,N_k\}$, there is $w_{\varepsilon,j,k}\in H^1(R_{j,k},\mathbb{S}^1)$ s.t.

$$w_{\alpha,j,k}(x)=\begin{cases}\dfrac{(x-x_j^k)^{\tilde{d}_{j,k}}}{{\eta'}_{k+1}^{\tilde{d}_{j,k}}} & \text{for } x\in\partial B(x_j^k,\eta'_{k+1})\\ \gamma_{\alpha,j,k}\dfrac{(x-x_j^k)^{\tilde{d}_{j,k}}}{\eta_k^{\tilde{d}_{j,k}}} & \text{for } x\in\partial B(x_j^k,\eta_k) \text{ where } \gamma_{\alpha,j,k}\in\mathbb{S}^1\end{cases}$$

and s.t. for all $w\in H^1(R_{j,k},\mathbb{S}^1)$ satisfying $\deg_{\partial B(x_j^k,\eta_k)}(w)=\tilde{d}_{j,k}$ one has

$$\int_{R_{j,k}}\alpha|\nabla w_{\alpha,j,k}|^2\leq\int_{R_{j,k}}\alpha|\nabla w|^2+C_3\tilde{d}_{j,k}^2\leq\int_{R_{j,k}}\alpha|\nabla w|^2+C_3d^2. \tag{4.58}$$

Now we consider the rings R_j. For $j \in \{1, ..., N'\}$, we denote

$$\tilde{d}_j = \sum_{x_i \in B(y_j,\eta)} d_i.$$

Using Proposition 4.33, for $j \in \{1, ..., N'\}$, we obtain $w_{\alpha,j} \in H^1(R_j, \mathbb{S}^1)$ s.t.

$$w_{\alpha,j}(x) = \begin{cases} \dfrac{(x-y_j)^d}{\eta^d} & \text{for } x \in \partial B(y_j,\eta) \\ \gamma_{\alpha,j}\dfrac{(x-y_j)^d}{\eta_K^d} & \text{for } x \in \partial B(y_j,\eta_K) \text{ where } \gamma_{\alpha,j} \in \mathbb{S}^1 \end{cases}$$

and s.t. for all $w \in H^1(R_j, \mathbb{S}^1)$ satisfying $\deg_{\partial B(y_j,\eta)}(w) = \tilde{d}_j$ one has

$$\int_{R_j} \alpha|\nabla w_{\alpha,j}|^2 \le \int_{R_j} \alpha|\nabla w|^2 + C_3 d^2. \tag{4.59}$$

4.B.3 Proof of Proposition 4.9

Note that there are at most d^2 regions $D_{j,k}$, at most d^2 rings $R_{j,k}$ and at most d rings R_j. Consequently, denoting

$$C_4 = C_4(g,\Omega,b,\eta_{\text{stop}}) = C_1(\eta_{\text{stop}},g,\Omega) + d^2C_2(2\cdot 9^{d-1},d) + 2d^2C_3(b)d^2$$

and using (4.51), (4.52), (4.53), (4.58), (4.59), one may construct a test function $w_\alpha \in \mathcal{J}_\rho$ (up to multiply by some $\mathbb{S}^1$-Constants each function previously constructed) s.t. for all $w \in \mathcal{I}_\rho$, one has

$$\int_{\Omega_\rho} \alpha|\nabla w_\alpha|^2 \le \int_{\Omega_\rho} \alpha|\nabla w|^2 + C_4. \tag{4.60}$$

Clearly, (4.60) allows us to prove Proposition 4.9 with $C_0 = C_4/2$.

Appendix 4.C Proof of Proposition 4.10

4.C.1 Upper bound for $I_{\rho,\varepsilon}$. Behavior of almost minimizers of $I_{\rho,\varepsilon}$

Upper bound for $I_{\rho,\varepsilon}$

We start with some preliminary facts (Proposition 4.34 and 4.37 below).

Proposition 4.34. *Let $\alpha \in L^\infty(\mathbb{R}^2,[b,1])$, $R > r_1 > r > 0$, $\tilde{d} \in \mathbb{Z}$ and $x_0 \in \mathbb{R}^2$, then we have*

1. $\mu_\alpha(B(x_0,R) \setminus \overline{B(x_0,r)}, \tilde{d}) = \tilde{d}^2\mu_\alpha(B(x_0,R) \setminus \overline{B(x_0,r)}, 1)$,
2. $b^2\pi \ln \dfrac{R}{r} \le \mu_\alpha(B(x_0,R) \setminus \overline{B(x_0,r)}, 1) \le \pi \ln \dfrac{R}{r}$,
3. $\mu_\alpha(B(x_0,R)\setminus\overline{B(x_0,r)},1) \le \mu_\alpha(B(x_0,R)\setminus\overline{B(x_0,r_1)},1) + \mu_\alpha(B(x_0,r_1)\setminus\overline{B(x_0,r)},1) + 2C_3$ *where C_3 is given by Proposition 4.33 and depending only on b.*

Assertion 1 and 2 are direct. The third one is a consequence of Proposition 4.33.

Lemma 4.35. *Assume that $b^2 \ge \frac{1}{3}$. Then U_ε is closed to being $\delta\cdot(\mathbb{Z}\times\mathbb{Z})$-periodic in $\Omega_\delta^{\text{incl}}$ in the sense that*

$$|U_\varepsilon(x) - U_\varepsilon[x + (\delta k, \delta l)]| \le C e^{-\frac{\gamma}{\xi}} \text{ if } x, x + (\delta k, \delta l) \in \Omega_\delta^{\text{incl}},\ \xi = \varepsilon/\delta \text{ and } k, l \in \mathbb{Z}.$$

Here, $\gamma > 0$ is an appropriate constant.

Recall that $\Omega_\delta^{\rm incl}$ is the union of the cells $[\delta k,\delta(k+1))\times[\delta l,\delta(l+1))\subset\mathbb{R}^2$ $(k,l\in\mathbb{Z})$ that are include in Ω.

Proof. For $W\subset\mathbb{R}^2$, we denote $\hat W=\dfrac{W}{\delta}=\{\hat x\in\mathbb{R}^2\,|\,\delta\hat x\in W\}$ and for $w\in H^1(W)$, we define $\hat w\in H^1(\hat W)$ s.t. $\hat w(\hat x)=w(\delta\hat x)$.

It suffices to prove that there is $V_\xi:Y=(-\frac12,\frac12)\times(-\frac12,\frac12)\to[b,1]$ s.t. for $x\in Y$ and $x+(k,l)\in\widehat{\Omega_\delta^{\rm incl}}$ we have

$$|V_\xi(x)-\hat U_\varepsilon\,[x+(k,l)]\,|\le C\mathrm{e}^{-\frac{\gamma}{\xi}}.$$

Let

$$\begin{array}{rccc} a_\lambda: & Y & \to & \{b,1\} \\ & x & \mapsto & \begin{cases} b & \text{if } x\in\omega^\lambda=\lambda\cdot\omega \\ 1 & \text{otherwise}\end{cases}\end{array}.$$

We consider V_ξ the unique minimizer of

$$E_\xi^{a_\lambda}(V,Y)=\frac12\int_Y|\nabla V|^2+\frac{1}{2\xi^2}(a_\lambda^2-V^2)^2,\ V\in H_1^1(Y,\mathbb{R}).$$

Denote $W(x)=V_\xi(x)-\hat U_\varepsilon[x+(k,l)]$ which satisfies (using (4.3))

$$\begin{cases}-\Delta W=\dfrac{W}{\xi^2}\left[a_\lambda^2-(V^2+UV+U^2)\right] & \text{in } Y\\ 0\le W\le C\mathrm{e}^{-\frac{\gamma}{\xi}} & \text{on } \partial Y\end{cases}.$$

Since $b^2\ge\dfrac13$, using the weak maximum principle, we find that $W\ge0$ in Y. Consequently, since W is subharmonic, we deduce that $W\le C\mathrm{e}^{-\frac{\gamma}{\xi}}$. □

From Lemma 4.35 we obtain the next result.

Lemma 4.36. *For all $1\ge R>r\ge\varepsilon$, $x,x_0\in\mathbb{R}^2$ s.t. $B(x_0,R)\subset\Omega_\delta^{\rm incl}$ and $x-x_0\in\delta\cdot\mathbb{Z}^2$, we have*

$$\mu_\varepsilon(B(x,R)\setminus\overline{B(x,r)},1)\ge\mu_\varepsilon(B(x_0,R)\setminus\overline{B(x_0,r)},1)-o_\varepsilon(1).$$

Adding the condition that $B(x,R)\subset\Omega_\delta^{\rm incl}$, we have

$$\left|\mu_\varepsilon(B(x,R)\setminus\overline{B(x,r)},1)-\mu_\varepsilon(B(x_0,R)\setminus\overline{B(x_0,r)},1)\right|\le o_\varepsilon(1).$$

Moreover the $o_\varepsilon(1)$ may be considered independent of x,x_0,R,r.

Lemma 4.36 implies the following

Proposition 4.37. *Let $\eta>0$ and $\eta>\rho\ge\varepsilon$. Then there is $C=C(\Omega,\Omega',g,\eta)>0$ s.t. for $x_0\in\mathbb{R}^2$ we have*

$$I_{\rho,\varepsilon}\le d\,\mu_\varepsilon(B(x_0,\eta)\setminus\overline{B(x_0,\rho)},1)+C(\eta),$$

where $C(\eta)$ is a constant independent of x_0.

Estimates for almost minimizers

Lemma 4.38. *1. Let $x \in \mathbb{R}^2$, $0 < r < R$, $\alpha \in L^\infty(\mathbb{R}^2, [b^2, 1])$, $C_0 > 0$ and a map $w \in H^1(B(x,R) \setminus \overline{B(x,r)}, \mathbb{S}^1)$ s.t. $\deg_{\partial B(x,R)}(w) = 1$ and*

$$\frac{1}{2}\int_{B(x,R)\setminus\overline{B(x,r)}} \alpha|\nabla w|^2 - \mu_\alpha(B(x,R) \setminus \overline{B(x,r)}, 1) \leq C_0.$$

Then for all r', R' s.t. $r < r' < R' < R$ one has

$$\frac{1}{2}\int_{B(x,R')\setminus\overline{B(x,r')}} \alpha|\nabla w|^2 - \mu_\alpha(B(x,R') \setminus \overline{B(x,r')}, 1) \leq 4C_3 + C_0,$$

where C_3 depends only on b and is given by Proposition 4.33.

2. Let $x_1, ..., x_d \in \Omega$, $d_i = 1$, $\varepsilon < \rho < 10^{-2}\eta$, $\eta := 10^{-2} \cdot \min\{|x_i - x_j|, \mathrm{dist}(x_i, \partial\Omega)\}$, $C_0 > 0$ and $w \in H^1(\Omega'_\rho, \mathbb{S}^1)$ s.t.

$$\frac{1}{2}\int_{\Omega'_\rho} U_\varepsilon^2|\nabla w|^2 \leq I_{\rho,\varepsilon} + C_0.$$

Then for $\rho \leq r < R < \eta$ one has for all i

$$\frac{1}{2}\int_{B(x_i,R)\setminus\overline{B(x_i,r)}} U_\varepsilon^2|\nabla w|^2 - \mu_\varepsilon(B(x_i,R) \setminus \overline{B(x_i,r)}, 1) \leq C_0 + C(\eta);$$

here $C(\eta)$ depends only on b, g, Ω, Ω' and η.

3. Under the hypotheses of 2., we also have for $\eta > \rho_0 > \rho$

$$\frac{1}{2}\int_{\Omega'_{\rho_0}} U_\varepsilon^2|\nabla w|^2 \leq C(\rho_0, C_0);$$

here $C(\eta, C_0)$ depends only on $b, g, \Omega, \Omega', C_0, \rho_0$ and η.

Proof. Using the third part of Proposition 4.34, we have

$$\begin{aligned}\frac{1}{2}\int_{B(x,R)\setminus\overline{B(x,r)}} \alpha|\nabla w|^2 \;\leq\; & \mu_\alpha(B(x,R) \setminus \overline{B(x,R')}, 1) + \mu_\alpha(B(x,R') \setminus \overline{B(x,r')}, 1) \\ & + \mu_\alpha(B(x,r') \setminus \overline{B(x,r)}, 1) + 4C_3 + C_0.\end{aligned}$$

We easily obtain

$$\begin{aligned}\frac{1}{2}\int_{B(x,R)\setminus\overline{B(x,r)}} \alpha|\nabla w|^2 \;\geq\; & \mu_\alpha(B(x,R) \setminus \overline{B(x,R')}, 1) + \frac{1}{2}\int_{B(x,R')\setminus\overline{B(x,r')}} \alpha|\nabla w|^2 \\ & + \mu_\alpha(B(x,r') \setminus \overline{B(x,r)}, 1)\end{aligned}$$

which proves the first assertion.

The second assertion is obtained by using the same argument combined with Proposition 4.37.

Last assertion is a straightforward consequence of Proposition 4.37 and both previous assertions. □

Lemma 4.39. *Let $x_n \in \mathbb{R}^2$, $\alpha_n \in L^\infty(\mathbb{R}^2, [b^2, 1])$ and $0 < r_n < R_n < 1$ satisfying $\frac{r_n}{R_n} \to 0$.*

Consider $w_n \in H^2(W_n, \mathbb{S}^1)$ where $W_n = B(x_n, R_n) \setminus \overline{B(x_n, r_n)}$, $\deg_{\partial B(x_n,R_n)}(w_n) = 1$. Assume that there exists $C_0 > 0$ s.t.

$$\frac{1}{2}\int_{W_n} \alpha_n |\nabla w_n|^2 \leq \mu_{\alpha_n}(W_n, 1) + C_0.$$

Let $2\pi \geq \theta_0 > 2\pi(1 - b^2)$ and let K_n be a compact cone with vertex x_n and aperture θ_0. Then it holds:

$$\frac{1}{2}\int_{W_n \cap K_n} |\nabla w_n|^2 \underset{n\to\infty}{\to} \infty.$$

Proof. We argue by contradiction and we assume that up to subsequence there is $C_1 > 0$ s.t.

$$\frac{1}{2}\int_{W_n \cap K_n} |\nabla w_n|^2 \leq C_1. \tag{4.61}$$

We drop the subscript n.

For $\rho \in (r, R)$, we denote

$$C_\rho = \{y \,|\, |y - x| = \rho\},\ C_\rho^+ = C_\rho \cap K \text{ and } C_\rho^- = C_\rho \setminus K.$$

We use the polar coordinate centered in x, *i.e.*, $y = x + \rho e^{\imath\theta}$, $\rho > 0$, $\theta \in [0, 2\pi)$. One may assume that $I_+ = [0, \theta_0]$ (resp. $I_- = (\theta_0, 2\pi)$) is the set of angles θ which defines C_ρ^+ (resp. C_ρ^-).

The argument is based on the variation of $w = e^{\imath\varphi}$ on $C_\rho^\pm$. Here, φ is locally defined in W, its gradient is globally defined and lies in $H^1(W, \mathbb{R}^2)$.

We denote

$$A(\rho) = \text{Var}(w, C_\rho) = \int_{C_\rho} |\mathrm{d}w| = \int_0^{2\pi} |\partial_\theta \varphi(\rho, \theta)|,$$

$$A^\pm(\rho) = \text{Var}(w, C_\rho^\pm) = \int_{I^\pm} |\partial_\theta \varphi(\rho, \theta)|.$$

Using the Cauchy-Schwarz inequality, we deduce

$$\frac{A^\pm(\rho)^2}{\text{long}(I_\pm)} \leq \int_{I_\pm} |\partial_\theta \varphi(\rho, \theta)|^2, \tag{4.62}$$

$$\frac{A(\rho)^2}{2\pi} \leq \int_I |\partial_\theta \varphi(\rho, \theta)|^2. \tag{4.63}$$

Note that from (4.61), (4.62) and (4.63), we obtain

$$\int_r^R \frac{\mathrm{d}\rho}{\rho} A^+(\rho)^2 \leq 2C_1\theta_0 \text{ and } \int_r^R \frac{\mathrm{d}\rho}{\rho} A(\rho)^2 \leq C \ln\frac{R}{r}. \tag{4.64}$$

From (4.62), (4.63) and (4.64), we have

$$\begin{aligned}\int_W |\nabla w|^2 &\geq \int_r^R \frac{\mathrm{d}\rho}{\rho}\int_0^{2\pi} |\partial_\theta\varphi(\rho,\theta)|^2\mathrm{d}\theta\\ &= \int_r^R \frac{\mathrm{d}\rho}{\rho}\left\{\int_{I_+} |\partial_\theta\varphi(\rho,\theta)|^2\mathrm{d}\theta + \int_{I_-} |\partial_\theta\varphi(\rho,\theta)|^2\mathrm{d}\theta\right\}\\ &\geq \int_r^R \frac{\mathrm{d}\rho}{\rho}\left\{\frac{A^+(\rho)^2}{\theta_0}+\frac{A^-(\rho)^2}{2\pi-\theta_0}\right\}\\ &\geq \int_r^R \frac{\mathrm{d}\rho}{\rho}\left\{\frac{A^+(\rho)^2}{\theta_0}+\frac{A(\rho)^2-2A^+(\rho)A(\rho)+A^+(\rho)^2}{2\pi-\theta_0}\right\}\\ &\geq \int_r^R \frac{\mathrm{d}\rho}{\rho}\frac{A(\rho)^2}{2\pi-\theta_0} - \mathcal{O}\left(\sqrt{\ln\frac{R}{r}}\right). \end{aligned} \tag{4.65}$$

Noting that $A(\rho)\geq 2\pi$, $\theta_0 > 2\pi(1-b^2)$ and using (4.65), we finally obtain

$$\frac{1}{2}\int_W \alpha|\nabla w|^2 \geq \frac{2b^2\pi^2}{2\pi-\theta_0}\ln\frac{R}{r} - \mathcal{O}\left(\sqrt{\ln\frac{R}{r}}\right) = \pi\ln\frac{R}{r} + H_n \geq \mu_\alpha(W,1)+H_n,$$

with $H_n\to\infty$. This contradiction ends the proof. □

4.C.2 Proof of the first part of Proposition 4.10

Let $x_1^n,...,x_{N_n}^n\in\Omega$ s.t. $|x_i^n-x_j^n|\geq 8\rho$ and $d_1^n,...,d_{N_n}^n>0, \sum d_i^n = d$. Note that, up to subsequence, one may assume d_i^n and N_n are independent of n.

We fix $x_0 = x_0^n \in \omega_\delta\cap[\delta\cdot(\mathbb{Z}\times\mathbb{Z})]$ s.t. $B(x_0,r)\subset\Omega$ where $r>0$ is a constant which depends only on Ω.

Assume that there is $i_0\in\{1,...,N\}$ s.t. $\mathrm{dist}(x_{i_0}^n,\partial\Omega)\to 0$.

It suffices to prove that, up to a subsequence,

$$\inf_{\substack{w\in H^1_g(\Omega'_\rho,\mathbb{S}^1)\\ \deg_{\partial B(x_i^n,\rho)}(w)=d_i}} \frac{1}{2}\int_{\Omega'_\rho} U^2_{\varepsilon_n}|\nabla w|^2 - I_{\rho,\varepsilon_n}\to\infty.$$

Up to a subsequence, there are $a_1,...,a_M\in\overline{\Omega}$ and $\{\Lambda_1,...,\Lambda_M\}$ a partition of $\{1,...,N\}$ s.t.

$$i\in\Lambda_l \iff x_i^n\to a_l.$$

By hypothesis, we may assume that $a_1\in\partial\Omega$. Let

$$\eta_{\text{stop}} = 10^{-8}\cdot 9^{-d^4}\cdot\min\left\{\min_{l\neq m}|a_l-a_m|, \min_{a_l\notin\partial\Omega}\mathrm{dist}(a_l,\partial\Omega), \mathrm{dist}(\partial\Omega,\partial\Omega')\right\},\ \rho_0 = 10^3\cdot 9^{d^2}\eta_{\text{stop}},$$

where

$$\begin{cases}\min_{l\neq m}|a_l-a_m| = +\infty & \text{if } M=1\\ \min_{a_l\notin\partial\Omega}\mathrm{dist}(a_l,\partial\Omega)=+\infty & \text{if } a_l\in\partial\Omega,\ l\in\{1,...,M\}\end{cases}.$$

For $l\in\{1,...,M\}$, we denote

$$\Omega_l^n = B(a_l,2\rho_0)\setminus\cup_{i\in\Lambda_l}\overline{B(x_i,\rho)} \text{ and } \tilde{d}_l = \sum_{i\in\Lambda_l} d_i.$$

We consider four cases:

1. $|\Lambda_l| = 1$ and $a_l \notin \partial\Omega$,
2. $|\Lambda_l| = 1$ and $a_l \in \partial\Omega$,
3. $|\Lambda_l| > 1$ and $a_l \notin \partial\Omega$,
4. $|\Lambda_l| > 1$ and $a_l \in \partial\Omega$.

We first treat the cases where $|\Lambda_l| = 1$.

Assume that $a_l \notin \partial\Omega$ (thus we are in Case 1). From Proposition 4.34 and Lemma 4.36, we have

$$\frac{1}{2}\int_{\Omega_l^n} U_{\varepsilon_n}^2|\nabla w|^2 \geq \tilde{d}_l\,\mu_{\varepsilon_n}\left(B(x_0,\rho_0)\setminus\overline{B(x_0,\rho)},1\right) - \mathcal{O}(1). \tag{4.66}$$

Assume that $a_l \in \partial\Omega$ (thus we are in Case 2) and let $\eta = \max(4|a_l - x_i|, 16\rho)$ with $i \in \Lambda_l$.

Clearly

$$\frac{1}{2}\int_{\Omega_l^n} U_{\varepsilon_n}^2|\nabla w|^2 \geq \frac{1}{2}\int_{B(a_l,2\rho_0)\setminus\overline{B(a_l,2\eta)}} U_{\varepsilon_n}^2|\nabla w|^2 + \frac{1}{2}\int_{B(x_i,\eta)\setminus\overline{B(x_i,\rho)}} U_{\varepsilon_n}^2|\nabla w|^2. \tag{4.67}$$

As direct consequence of (4.67) we have

$$\frac{1}{2}\int_{\Omega_l^n} U_{\varepsilon_n}^2|\nabla w|^2 \geq \frac{b^2}{2}\int_{B(a_l,2\rho_0)\setminus\overline{B(a_l,2\eta)}} |\nabla w|^2 + \mu_{\varepsilon_n}\left(B(x_i,\eta)\setminus\overline{B(x_i,\rho)},\tilde{d}_l\right).$$

From Lemma VI.1 in [18], since $a_l \in \partial\Omega$ and $w = g$ in $\Omega'\setminus\overline{\Omega}$, we obtain

$$\frac{1}{2}\int_{B(a_l,2\rho_0)\setminus\overline{B(a_l,2\eta)}} |\nabla w|^2 \geq 2\pi\tilde{d}_l|\ln\eta| - C \text{ with } C \text{ independent of } n.$$

Consequently, since $b^2 > 1/2$, there is $H_n \to \infty$ s.t.

$$\frac{b^2}{2}\int_{B(a_l,2\rho_0)\setminus\overline{B(a_l,2\eta)}} |\nabla w|^2 \geq \pi\tilde{d}_l|\ln\eta| + H_n + \mathcal{O}(1) \geq \tilde{d}_l\,\mu_{\varepsilon_n}\left(B(x_i,2\rho_0)\setminus\overline{B(x_i,\eta)},1\right) + H_n.$$

From Proposition 4.34, Lemma 4.36 and (4.67) we obtain

$$\frac{1}{2}\int_{\Omega_l^n} U_{\varepsilon_n}^2|\nabla w|^2 \geq \tilde{d}_l\,\mu_{\varepsilon_n}\left(B(x_0,\rho_0)\setminus\overline{B(x_0,\rho)},1\right) + \tilde{H}_n,\ \tilde{H}_n \to \infty. \tag{4.68}$$

We now consider the remaining cases: $|\Lambda_l| > 1$.

We apply in Ω_l^n the separation process defined in Section 4.B.1.

Since for $i, j \in \Lambda_l$ we have $|x_i - x_j| \ll \eta_{\text{stop}}$, in the end of the process (after K steps), we obtain a unique $x_1^K = y_l \in \{x_i \,|\, i \in \Lambda_l\}$ in the final selection of points and $\eta_K \to 0$.

For $k \in \{1, ..., K\}$ we denote $\{x_1^k, ..., x_{N_k}^k\}$ the selection of points made in Step k, η_k the radius of the final balls in Step k and η'_k the radius of the intermediate balls. Note that $\eta_0 = \rho$.

From (4.47) and (4.48), the following rings are mutually disjoint

$$R_{j,k} = B(x_j^k,\eta'_{k+1})\setminus\overline{B(x_j^k,\eta_k)},\ \tilde{d}_{j,k} = \sum_{x_i\in B(x_j^k,\eta'_{k+1})} d_i \text{ with } k \in \{0,...,K-1\},\ j \in \{1,...,N_k\},$$

$$R_0^l = B(y_l,\rho_0)\setminus\overline{B(y_l,\eta_K)}.$$

So, one has for $w \in H^1_g(\Omega'_\rho, \mathbb{S}^1)$ (setting $L_n = \frac{1}{2}\int_{\Omega_l^n} U^2_{\varepsilon_n}|\nabla w|^2$):

$$\begin{aligned}
L_n &\geq \frac{1}{2}\int_{R_0^l} U^2_{\varepsilon_n}|\nabla w|^2 + \sum_{k=0}^{K-1}\sum_{j=1}^{N_k}\frac{1}{2}\int_{R_{j,k}} U^2_{\varepsilon_n}|\nabla w|^2 \\
&\geq \frac{1}{2}\int_{R_0^l} U^2_{\varepsilon_n}|\nabla w|^2 + \sum_{k=0}^{K-1}\sum_{j=1}^{N_k}\mu_{\varepsilon_n}(B(x_j^k, \eta'_{k+1}) \setminus \overline{B(x_j^k, \eta_k)}, \tilde{d}_{j,k}) \\
\text{(Lem 4.36)} &\geq \frac{1}{2}\int_{R_0^l} U^2_{\varepsilon_n}|\nabla w|^2 + \sum_{k=0}^{K-1}\sum_{j=1}^{N_k}\tilde{d}_{j,k}\,\mu_{\varepsilon_n}(B(x_0, \eta_{k+1}) \setminus \overline{B(x_0, \eta_k)}, 1) - \mathcal{O}(1) \\
\text{((4.50), Prop. 4.34)} &\geq \frac{1}{2}\int_{R_0^l} U^2_{\varepsilon_n}|\nabla w|^2 + \tilde{d}_l\,\mu_{\varepsilon_n}(B(x_0, \eta_K) \setminus \overline{B(x_0, \rho)}, 1) - \mathcal{O}(1). \qquad (4.69)
\end{aligned}$$

Here we used the fact that $\sum_{j=1}^{N_k}\tilde{d}_{j,k} = \tilde{d}_l$.

If $a_l \notin \partial\Omega$, then we obtain

$$\frac{1}{2}\int_{\Omega_l^n} U^2_{\varepsilon_n}|\nabla w|^2 \geq \tilde{d}_l\,\mu_{\varepsilon_n}(B(x_0, \rho_0) \setminus \overline{B(x_0, \rho)}, 1) - \mathcal{O}(1). \qquad (4.70)$$

If $a_l \in \partial\Omega$, we have to consider two cases: $|a_l - y_l| < 4\eta_K$ and $|a_l - y_l| \geq 4\eta_K$.

Let $\eta := \max(|a_l - y_l|, 4\eta_K)$. Note that $\eta \to 0$. For $w \in H^1_g(\Omega'_\rho, \mathbb{S}^1)$, we have

$$\begin{aligned}
\frac{1}{2}\int_{R_0^l} U^2_{\varepsilon_n}|\nabla w|^2 &= \frac{1}{2}\int_{B(y_l,\rho_0)\setminus\overline{B(a_l,\rho_0/2)}} U^2_{\varepsilon_n}|\nabla w|^2 + \frac{1}{2}\int_{B(a_l,\rho_0/2)\setminus\overline{B(a_l,16\eta)}} U^2_{\varepsilon_n}|\nabla w|^2 \\
&\quad + \frac{1}{2}\int_{B(a_l,16\eta)\setminus\overline{B(y_l,\eta)}} U^2_{\varepsilon_n}|\nabla w|^2 + \frac{1}{2}\int_{B(y_l,\eta)\setminus\overline{B(y_l,\eta_K)}} U^2_{\varepsilon_n}|\nabla w|^2 \\
&\geq \frac{1}{2}\int_{B(a_l,\rho_0/2)\setminus\overline{B(a_l,16\eta)}} U^2_{\varepsilon_n}|\nabla w|^2 + \\
&\qquad + \tilde{d}_l\,\mu_{\varepsilon_n}(B(x_0, \eta) \setminus \overline{B(x_0, \eta_K)}, 1) - \mathcal{O}(1). \qquad (4.71)
\end{aligned}$$

From Lemma VI.1 in [18], since $a_l \in \partial\Omega$ and $w = g$ in $\Omega' \setminus \overline{\Omega}$ we find

$$\frac{1}{2}\int_{B(a_l,\rho_0/2)\setminus\overline{B(a_l,16\eta)}} |\nabla w|^2 \geq 2\pi\tilde{d}_l|\ln\eta| - C \text{ with } C \text{ independent of } n. \qquad (4.72)$$

Consequently, using Proposition 4.34, (4.69) and (4.71) and (4.72): there is $H_n \to \infty$ s.t.

$$\frac{1}{2}\int_{\Omega_l^n} U^2_{\varepsilon_n}|\nabla w|^2 \geq \tilde{d}_l\,\mu_{\varepsilon_n}(B(x_0, \rho_0/2) \setminus \overline{B(x_0, \rho)}, 1) + H_n - \mathcal{O}(1). \qquad (4.73)$$

Summing over l the lower bounds given by (4.66), (4.68),(4.70), (4.73) and using Proposition 4.37, we obtain the result since by assumption, (4.68) or (4.73) occurs.

4.C.3 Proof of the second part of Proposition 4.10

Let $x_1^n, ..., x_N^n \in \Omega$ s.t. $|x_i^n - x_j^n| \geq 8\rho$ and $d_1, ..., d_N > 0, \sum d_i = d$ (up to subsequence the degrees may be considered independent of n).

Assume that there is $i_0 \in \{1, ..., N\}$ s.t. $d_{i_0} \neq 1$ or that there are $i \neq j$ s.t. $|x_i^n - x_j^n| \to 0$.

Up to a subsequence, there are $a_1, ..., a_M \in \overline{\Omega}$ and $\{\Lambda_1, ..., \Lambda_M\}$ a partition of $\{1, ..., N\}$ s.t.

$$i \in \Lambda_l \iff x_i^n \to a_l.$$

Note that since $d_i > 0$, the hypotheses of the second part of Proposition 4.10 are equivalent to

$$\text{there exists } l_0 \in \{1, ..., M\} \text{ s.t. } \tilde{d}_{l_0} = \sum_{i\in\Lambda_{l_0}} d_i > 1.$$

We argue as in the previous section and with the same notations. There are two cases:

1. $\mathrm{Card}(\Lambda_l) > 1$,
2. $\mathrm{Card}(\Lambda_l) = 1$.

In the first case, we apply the separation process in $\Omega_l^n = B(a_l, 2\rho_0) \setminus \cup_{i\in\Lambda_l}\overline{B(x_i,\rho)}$.

For $w \in H^1_g(\Omega'_\rho, \mathbb{S}^1)$ we have

$$\begin{aligned}\frac{1}{2}\int_{\Omega_l^n} U_{\varepsilon_n}^2|\nabla w|^2 &\geq \frac{1}{2}\int_{R_0^l} U_{\varepsilon_n}^2|\nabla w|^2 + \sum_{k=0}^{K-1}\sum_{j=1}^{N_k}\frac{1}{2}\int_{R_{j,k}} U_{\varepsilon_n}^2|\nabla w|^2\\ &\geq \frac{1}{2}\int_{R_0^l} U_{\varepsilon_n}^2|\nabla w|^2 + \sum_{k=0}^{K-1}\sum_{j=1}^{N_k}\mu_{\varepsilon_n}(B(x_j^k,\eta'_{k+1})\setminus\overline{B(x_j^k,\eta_k)},\tilde{d}_{j,k})\\ &\geq \tilde{d}_l^2\,\mu_{\varepsilon_n}(B(x_0,\rho_0)\setminus\overline{B(x_0,\eta_K)},1)+\\ &\qquad+\sum_{k=0}^{K-1}\sum_{j=1}^{N_k}\tilde{d}_{j,k}\,\mu_{\varepsilon_n}(B(x_0,\eta_{k+1})\setminus\overline{B(x_0,\eta_k)},1)-\mathcal{O}(1)\\ &\geq \tilde{d}_l\,\mu_{\varepsilon_n}(B(x_0,\rho_0)\setminus\overline{B(x_0,\rho)},1)+(\tilde{d}_l^2-\tilde{d}_l)\pi b^2|\ln\eta_K|-\mathcal{O}(1).\end{aligned}\tag{4.74}$$

In the second case the computations are direct

$$\begin{aligned}\frac{1}{2}\int_{\Omega_l^n} U_{\varepsilon_n}^2|\nabla w|^2 &\geq \frac{1}{2}\int_{B(x_i,\rho_0)\setminus\overline{B(x_i,\rho)}} U_{\varepsilon_n}^2|\nabla w|^2\\ &\geq \tilde{d}_l\,\mu_{\varepsilon_n}(B(x_0,\rho_0)\setminus\overline{B(x_0,\rho)},1)+(\tilde{d}_l^2-\tilde{d}_l)\pi b^2|\ln\rho|-\mathcal{O}(1).\end{aligned}\tag{4.75}$$

Summing the lower bounds (4.74) and (4.75) over l and applying Proposition 4.37, we obtain the result since $\eta \to 0$, $\eta \in \{\eta_K, \rho\}$ and $\tilde{d}_{l_0} > 1$.

4.C.4 Proof of the third part of Proposition 4.10

From the first and the second assertion, one may consider $x_1^n, ..., x_d^n \in \Omega$ s.t.

$$\min\left\{\min_{i\neq j}|x_i^n - x_j^n|, \min_i \mathrm{dist}(x_i^n, \partial\Omega)\right\} \geq 10^2\cdot\eta_0 > 0 \text{ and } d_1, ..., d_d = 1.$$

We divide the proof into two steps:

Step 1. If the configuration is almost minimizing, then for all i, we have $x_i^n \in \overline{\omega_\delta}$.

Step 2. If the configuration is almost minimizing, then for all i, we have $\liminf_n \dfrac{\mathrm{dist}(x_i^n, \partial\omega_\delta)}{\lambda\delta} > 0$.

We now prove Step 1.

Assume that there exist $C_0 > 0$, sequences $\varepsilon_n, \rho \downarrow 0$, $\rho = \rho(\varepsilon_n) \geq \varepsilon_n$ and distinct points $x_1^n, ..., x_d^n$ (well separated and far from $\partial\Omega$) s.t. $\rho/(\lambda\delta) \to 0$, $x_1^n \notin \overline{\omega_\delta}$ and

$$\inf_{\substack{w\in H^1_g(\Omega'_\rho,\mathbb{S}^1)\\ \deg_{\partial B(x_i,\rho)}(w)=1}} \frac{1}{2}\int_{\Omega'_\rho} U_{\varepsilon_n}^2|\nabla w|^2 - I_{\rho,\varepsilon_n} \leq C_0. \tag{4.76}$$

Denote w_n a minimizer for $\widehat{\mathcal{I}}_{\rho,\varepsilon_n}(\{(x_1^n,...,x_d^n),(1,...,1)\})$ (see Proposition 4.8). Using Lemma 4.38 Part 2, for $\rho \le r < R < \eta_0$, one has

$$\int_{B(x_1^n,R)\setminus\overline{B(x_1^n,r)}} U_{\varepsilon_n}^2|\nabla w_n|^2 - \mu_{\varepsilon_n}(B(x_1^n,R)\setminus\overline{B(x_1^n,r)},1) \le C_0 + C(\eta_0).$$

There are two cases to consider:

i. up to a subsequence, we have $\dfrac{\operatorname{dist}(x_1^n,\omega_\delta)}{\lambda\delta} \to c \in (0,\infty]$,

ii. up to a subsequence, we have $\dfrac{\operatorname{dist}(x_1^n,\omega_\delta)}{\lambda\delta} \to 0$.

The first case is the easiest. Indeed, let $\kappa \in (0, 10^{-2}\cdot c)$ be s.t. $B(0,2\kappa)\subset\omega\subset Y$ and $y_n \in \delta\cdot(\mathbb{Z}\times\mathbb{Z})$ s.t. $x_1^n, y_n \in \overline{Y_{k,l}^\delta}$. Note that using (4.3), $U_{\varepsilon_n} = 1 + V_n$ in $B(x_1^n,\kappa\lambda\delta)$ and $U_{\varepsilon_n} = b + V_n$ in $B(y_n,\kappa\lambda\delta)$, $\|V_n\|_{L^\infty} = o(\varepsilon_n^2)$. Therefore we have

$$\begin{cases} \mu_{\varepsilon_n}(B(x_1^n,\kappa\lambda\delta)\setminus\overline{B(x_1^n,\rho)},1) = \pi\ln\dfrac{\lambda\delta}{\rho} + \mathcal{O}(1) \\ \mu_{\varepsilon_n}(B(y_n,\kappa\lambda\delta)\setminus\overline{B(y_n,\rho)},1) = b^2\pi\ln\dfrac{\lambda\delta}{\rho} + \mathcal{O}(1) \end{cases}. \tag{4.77}$$

With $\mathbf{1} = (1,...,1)$, we have

$$\begin{aligned}\widehat{\mathcal{I}}_{\rho,\varepsilon_n}(\{(x_1^n,...,x_d^n),\mathbf{1}\}) \ge\ & \mu_{\varepsilon_n}(B(x_i^n,\eta_0)\setminus\overline{B(x_i^n,\kappa\lambda\delta)},1) \\ & + \textstyle\sum_i \mu_{\varepsilon_n}(B(x_i^n,\kappa\lambda\delta)\setminus\overline{B(x_i^n,\rho)},1)\end{aligned} \tag{4.78}$$

and

$$\begin{aligned}\widehat{\mathcal{I}}_{\rho,\varepsilon_n}(\{(y_n,x_2^n,...,x_d^n),\mathbf{1}\}) =\ & \mu_{\varepsilon_n}(B(y_n,\kappa\lambda\delta)\setminus\overline{B(y_n,\rho)},1) \\ & + \textstyle\sum_{i=2}^d \mu_{\varepsilon_n}(B(x_i^n,\kappa\lambda\delta)\setminus\overline{B(x_i^n,\rho)},1) \\ & + \textstyle\sum_{i=1}^d \mu_{\varepsilon_n}(B(x_i^n,\eta_0)\setminus\overline{B(x_i^n,\kappa\lambda\delta)},1) + \mathcal{O}(1).\end{aligned} \tag{4.79}$$

From (4.77), (4.78) and (4.79), we obtain

$$\widehat{\mathcal{I}}_{\rho,\varepsilon_n}(\{(x_1^n,...,x_d^n),\mathbf{1}\}) - \widehat{\mathcal{I}}_{\rho,\varepsilon_n}(\{(y_n,x_2^n,...,x_d^n),\mathbf{1}\}) \to \infty;$$

this contradicts (4.76).

We now turn to case ii.. Arguing as in case i., it suffices to prove that

$$\mu_{\varepsilon_n}(B(x_1^n,\kappa\lambda\delta)\setminus\overline{B(x_1^n,\rho)},1) - \mu_{\varepsilon_n}(B(y_n,\kappa\lambda\delta)\setminus\overline{B(y_n,\rho)},1) \to \infty \text{ for some fixed } \kappa. \tag{4.80}$$

In contrast with case i., we cannot rely on (4.77) anymore.

Let $\kappa > 0$ depending only on ω be s.t.

$$\kappa < 10^{-2}\cdot\operatorname{dist}(\omega,\partial Y) \text{ and } B(0,10^2\cdot\kappa)\subset\omega.$$

We have $U_{\varepsilon_n} = 1 + V_n$ in $W_n \cap K_n$ where, $\|V_n\|_{L^\infty} = o(\varepsilon_n^2)$,

$$W_n = B(x_n,R_n)\setminus\overline{B(x_n,r_n)},\ R_n = \kappa\lambda\delta,\ r_n = \max(\varepsilon_n^{1/4},\rho),\ x_n = x_1^n$$

and K_n is the cone of vertex x_n and aperture $\theta_0 = \dfrac{3\pi}{2} - b^2\pi$ which admits the line $(x_n, \Pi_{\partial\omega_\delta}x_n)$ for symmetry axis and s.t. $K_n\cap\omega_\delta\cap W_n = \emptyset$. Here $\Pi_{\partial\omega_\delta}(x_n)$ is the orthogonal projection of x_n on $\partial\omega_\delta$. Since $b^2 > 1/2$, we have $\pi > \theta_0 > 2\pi(1-b^2)$.

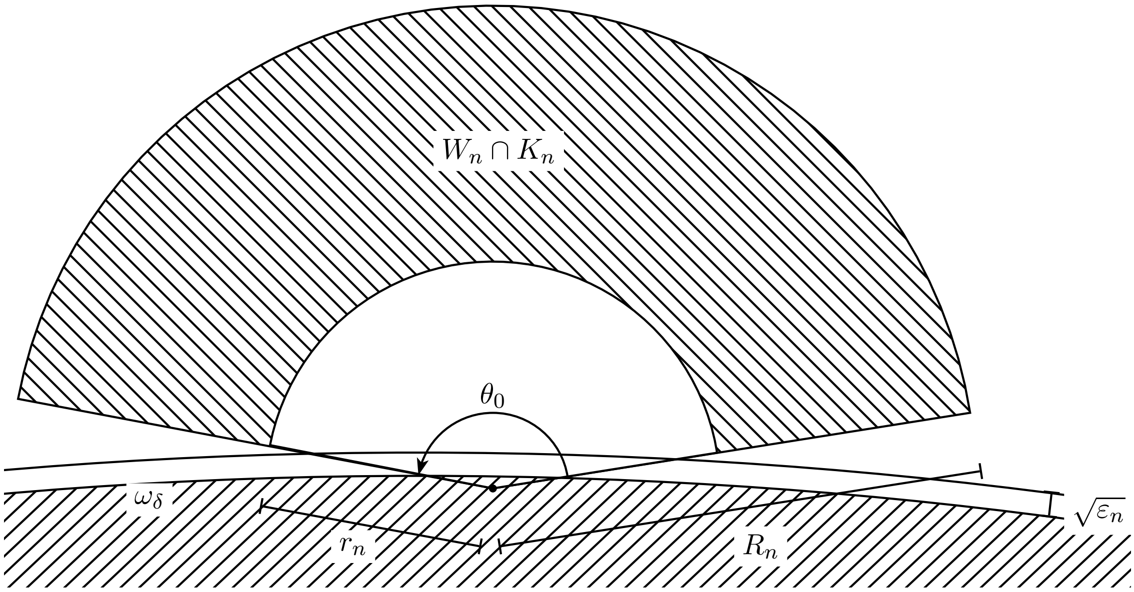


Figure 4.3: The domain $W_n \cap K_n$

Note that since $\dfrac{\text{dist}(x_n, \omega_\delta)}{\lambda\delta} \to 0$, for large n and small κ (independently of n), by smoothness of ω, K_n is well defined.

In Figure 4.3 we have represented the domain $W_n \cap K_n$ used in the proof of Step 2 ; in the situation Step 1 Case ii., the construction is similar except the fact that the vertex of the cone is outside ω_δ.

Applying Lemmas 4.38 and 4.39, for w_n a minimizer of $\widehat{\mathcal{I}}_{\rho,\varepsilon_n}(\{(x_1^n, ..., x_d^n), \mathbf{1}\})$,

$$\int_{W_n \cap K_n} |\nabla w_n|^2 \to \infty. \tag{4.81}$$

By smoothness of ω, for large n, $\text{dist}(W_n \cap K_n, \partial\omega_\delta) > \sqrt{\varepsilon}_n$. Consequently, using (4.3), $U_{\varepsilon_n} = 1 + V_n$, $\|V_n\|_{L^\infty} = o(\varepsilon_n^2)$ in $W_n \cap K_n$.

Let $y_n \in \delta \cdot (\mathbb{Z} \times \mathbb{Z})$ be s.t. $x_1^n, y_n \in \overline{Y_{k,l}^\delta}$. We set $\tilde{W_n} = B(y_n, R_n) \setminus \overline{B(y_n, r_n)}$, $\tilde{K_n} = K_n + y_n - x_n$ and $\tilde{w}_n(\cdot) = w_n(\cdot - y_n + x_n)$. We have

$$\begin{aligned}
\mu_{\varepsilon_n}(W_n, 1) + \mathcal{O}(1) &= \frac{1}{2}\int_{W_n} U_{\varepsilon_n}^2 |\nabla w_n|^2 + \mathcal{O}(1) \\
&\geq \frac{1}{2}\int_{W_n \cap K_n} |\nabla w_n|^2 + \frac{b^2}{2}\int_{W_n \setminus K_n} |\nabla w_n|^2 \\
&\geq \frac{1}{2}\int_{\tilde{W_n} \cap \tilde{K_n}} |\nabla \tilde{w}_n|^2 + \frac{b^2}{2}\int_{\tilde{W_n} \setminus \tilde{K_n}} |\nabla \tilde{w}_n|^2 \\
&\geq \frac{1}{2}\int_{\tilde{W_n}} U_{\varepsilon_n}^2 |\nabla \tilde{w}_n|^2 + \frac{1-b^2}{2}\int_{\tilde{W_n} \cap \tilde{K_n}} |\nabla \tilde{w}_n|^2 + \mathcal{O}(1) \\
&\geq \mu_{\varepsilon_n}(\tilde{W_n}, 1) + \frac{1-b^2}{2}\int_{W_n \cap K_n} |\nabla w_n|^2 + \mathcal{O}(1).
\end{aligned} \tag{4.82}$$

By combining (4.81) and (4.82), we obtain

$$\mu_{\varepsilon_n}(W_n, 1) - \mu_{\varepsilon_n}(\tilde{W_n}, 1) \to \infty. \tag{4.83}$$

On the other hand, we have

$$\mu_{\varepsilon_n}(B(y_n,r_n)\setminus\overline{B(y_n,\rho)},1)\le\mu_{\varepsilon_n}(B(x_n,r_n)\setminus\overline{B(x_n,\rho)},1)+\mathcal{O}(1).$$

By Proposition 4.34 Assertion 3, Estimate (4.80) follows by combining this inequality with (4.83). Step 1 is complete.

We now prove Step 2. Assume that $x_1^n\in\overline{\omega_\delta}$ and $\dfrac{\text{dist}(x_1^n,\partial\omega_\delta)}{\lambda\delta}\to 0$. Denote $x_1^n=x_n$.

Let $R_n=\kappa\lambda\delta$ (κ small and independent of n), $r_n=\max(\rho,\varepsilon_n^{1/4},\sqrt{\lambda\delta\cdot\text{dist}(x_n,\partial\omega_\delta)})$ and let D_n be the line passing through x_n and $\Pi_{\partial\omega_\delta}(x_n)$. (If $x_n\in\partial\omega_\delta$, D_n is the line orthogonal to $\partial\omega_\delta$ at x_n)

As in Step 1, we denote $W_n=B(x_n,R_n)\setminus\overline{B(x_n,r_n)}$. We let K_n be the cone with vertex x_n and aperture $\theta_0=\dfrac{3\pi}{2}-b^2\pi$ which admits the line D_n for symmetry axis and s.t. $W_n\cap K_n\cap\omega_\delta=\emptyset$. (We represent $W_n\cap K_n$ in Figure 4.3.)

Clearly, for large n, K_n is well defined. Applying Lemmas 4.38 and 4.39 for w_n a minimizer of $\mathcal{I}_{\rho,\varepsilon_n}(\{(x_n^1,...,x_d^n),\mathbf{1}\})$, we obtain

$$\int_{W_n\cap K_n}|\nabla w_n|^2\to\infty.$$

Note that since $\text{dist}(W_n\cap K_n,\omega_\delta)>\varepsilon^{1/2}$, we have $U_{\varepsilon_n}=1+V_n$, $\|V_n\|_{L^\infty(W_n\cap K_n)}=o_n(\varepsilon_n^2)$. Using this fact, we complete Step 2 by arguing as in Step 1 Case ii.

Appendix 4.D Proof of Proposition 4.13

We prove a more general form of Proposition 4.13.

For $\alpha\in L^\infty(\mathbb{R}^2,[b^2,1])$ we define

$$I_{\rho,\alpha}:=\inf_{\substack{x_1,\dots,x_N\in\Omega\\|x_i-x_j|\ge8\rho\\d_1,\dots,d_N>0,\sum d_i=d}}\ \inf_{\substack{w\in H^1_g(\Omega'_\rho,\mathbb{S}^1)\\\deg_{\partial B(x_i,\rho)}(w)=d_i}}\frac12\int_{\Omega'_\rho}\alpha|\nabla w|^2$$

and

$$J^\eta_{\rho,\alpha}:=\inf_{\substack{x_1,\dots,x_d\in\Omega\\|x_i-x_j|\ge8\rho\\\text{dist}(x_i,\partial\Omega)\ge\eta}}\ \inf_{\substack{w\in H^1_g(\Omega_\rho,\mathbb{S}^1)\\w(x_i+\rho e^{\imath\theta})=e^{\imath(\theta+\theta_i)},\theta_i\in\mathbb{R}}}\frac12\int_{\Omega_\rho}\alpha|\nabla w|^2.$$

Here $\Omega'_\rho=\Omega'\setminus\cup\overline{B(x_i,\rho)}$.

We prove the existence of a minimizing configuration $\{\mathbf{x},\mathbf{d}\}=\{(x_1,...,x_N),(d_1,...,d_n)\}$ for $I_{\rho,\alpha}$.

Let $(\{\mathbf{x}_n,\mathbf{d}_n\})_n$ be a minimizing sequence of configuration of $I_{\rho,\alpha}$, *i.e.*,

$$\inf_{\substack{w\in H^1(\Omega^n_\rho,\mathbb{S}^1)\text{ s.t.}\\w=g\text{ in }\Omega'\setminus\overline{\Omega\bigcup\cup B(x_i^n,\rho)}\\\deg_{\partial B(x_i^n,\rho)}(w)=d_i^n\text{ for all }i}}\frac12\int_{\Omega^n_\rho}\alpha|\nabla w|^2\to I_{\rho,\alpha};$$

here $\Omega^n_\rho=\Omega'\setminus\cup\overline{B(x_i^n,\rho)}$.

Up to a subsequence, we have $N_n=N=\text{Cst}$, $\mathbf{d}_n=\mathbf{d}=\text{Cst}$ and $\mathbf{x}_n\to\mathbf{x}$ with $\mathbf{x}=(x_1,...,x_N)$ s.t. $\min_{i\neq j}|x_i-x_j|\ge8\rho$.

Consider $w_n\in\mathcal{I}_\rho(\mathbf{x}_n,\mathbf{d})$ a minimizing map. Since w_n is bounded independently of n in $H^1(\Omega^n_\rho)$, up to a subsequence, we have $w_n\rightharpoonup w_0$ in $H^1_{\text{loc}}(\Omega^0_\rho)$, $\Omega^0_\rho=\Omega'\setminus\cup\overline{B(x_i,\rho)}$.

Clearly the following properties hold:

- $w_0 \in H^1_{\rm loc}(\Omega^0_\rho, \mathbb{S}^1)$ and $w_0 = g$ in $\Omega^0_\rho \setminus \overline{\Omega}$.
- For all compact $K \subset \Omega^0_\rho$ we have $\displaystyle\frac{1}{2}\int_K \alpha|\nabla w_0|^2 \leq \liminf \frac{1}{2}\int_K \alpha|\nabla w_n|^2 \leq I_{\rho,\alpha}$.

Thus $w_0 \in H^1_g(\Omega^0_\rho, \mathbb{S}^1)$ and $\displaystyle\int_{\Omega^0_\rho} \alpha|\nabla w_0|^2 \leq I_{\rho,\alpha}$.

Now, it suffices to check that $\deg_{\partial B(x_i,\rho)}(w_0) \in \mathbb{N}^*$ for all i. Since w_0 is $\mathbb{S}^1$-valued, this fact is equivalent to $\deg_{\partial B(x_i,\rho')}(w_0) \in \mathbb{N}^*$ for all i and for all $\rho' \in (\rho, 2\rho)$.

In view of the fact that for $\rho' \in (\rho, 2\rho)$ we have $w'_n = w_{n|\Omega'\setminus\cup\overline{B(x_i,\rho')}} \rightharpoonup w'_0 = w_{0|\Omega'\setminus\cup\overline{B(x_i^n,\rho')}}$ and on the other hand the set

$$\mathcal{I}' := \{w' \in H^1(\Omega' \setminus \cup\overline{B(x_i,\rho')}, \mathbb{S}^1) \,|\, \deg_{\partial B(x_i,\rho')}(w') = d_i \text{ for all } i \in \{1,...,N\}\}$$

is closed under the H^1-weak convergence (see Appendix 4.A or [42]), since $w'_n \in \mathcal{I}'$, we obtain that $w'_0 \in \mathcal{I}'$. Therefore $\{\mathbf{x}, \mathbf{d}\} = \{(x_1,...,x_N), (d_1,...,d_n)\}$ is a minimizing configuration for $I_{\rho,\alpha}$.

Now we prove the existence of a minimizing configuration for $J^\eta_{\rho,\alpha}$.

Let $(\mathbf{x}_n)_n$ be a minimizing sequence of configuration for $J^\eta_{\rho,\alpha}$, *i.e.*,

$$\hat{\mathcal{J}}_{\rho,\alpha}(\mathbf{x}_n, \mathbf{1}) \to J^\eta_{\rho,\alpha}.$$

Up to a subsequence, one may assume that there is $\mathbf{x} = (x_1,...,x_d) \in \Omega^d$ s.t. $x_i^n \to x_i$, $|x_i - x_j| \geq 8\rho$ and $\operatorname{dist}(x_i, \partial\Omega) \geq \eta$.

Let $\eta_n = 8\max|x_i^n - x_i|$. There is a smooth diffeomorphism $\phi_n : \mathbb{R}^2 \to \mathbb{R}^2$ satisfying

$$\begin{cases} \phi_n \equiv \mathrm{Id}_{\mathbb{R}^2} & \text{in } \mathbb{R}^2 \setminus \cup\overline{B(x_i^n, \rho + \eta_n^{1/2})} \\ \phi_n\left[x_i + (1+\eta_n)x\right] = x_i^n + x & \text{for } x \in B(0,\rho) \\ \|\phi_n - \mathrm{Id}_{\mathbb{R}^2}\|_{C^1(\mathbb{R}^2)} = o_n(1) & \end{cases}.$$

For example we can consider $\phi_n = \mathrm{Id}_{\mathbb{R}^2} + H_n$ with

$$\begin{cases} H_n \equiv 0 & \text{in } \mathbb{R}^2 \setminus \cup\overline{B(x_i^n, \rho + \eta_n^{1/2})} \\ H_n\left[x_i + (1+\eta_n)x\right] = (1 - \psi_n(|x|))(x_i^n - x_i - \eta_n x) & \text{for } x \in B(0, \dfrac{\rho + \eta_n^{1/2}}{1+\eta_n}) \end{cases}.$$

Here $\psi_n : \mathbb{R}^+ \to [0,1]$ is a smooth function satisfying

$$\psi_n(r) = \begin{cases} 0 & \text{if } r \leq \rho \\ 1 & \text{if } r \geq \rho + \eta_n^{1/2}/2 \end{cases} \text{ and } |\psi'_n| = \mathcal{O}(\eta_n^{-1/2}).$$

For $w_n \in \mathcal{J}_\rho(\mathbf{x}_n, \mathbf{1})$ a minimizing map, we consider

$$\begin{array}{rccc} \tilde{w}_n : & \Omega \setminus \cup_i\overline{B(x_i, (1+\eta_n)\rho)} & \to & \mathbb{S}^1 \\ & x & \mapsto & w_n\left[\phi_n(x)\right] \end{array}.$$

Clearly $\tilde{w}_n$ is well defined and we have

$$\int_{\Omega\setminus\cup_i\overline{B(x_i,(1+\eta_n)\rho)}} \alpha|\nabla\tilde{w}_n|^2 = \int_{\Omega\setminus\cup_i\overline{B(x_i^n,\rho)}} \alpha|\nabla w_n|^2 + o_n(1),$$

$$\tilde{w}_n\left[x_i + (1+\eta_n)\rho e^{\imath\theta}\right] = w_n\left[\phi(x_i + (1+\eta_n)\rho e^{\imath\theta})\right] = w_n\left[x_i^n + \rho e^{\imath\theta}\right] = e^{\imath(\theta+\theta_i)}.$$

We can extend $\tilde{w}_n$ in $\cup_i B(x_i,(1+\eta_n)\rho)\setminus\overline{B(x_i,\rho)}$ by $\tilde{w}_n(x_i+re^{\imath\theta})=e^{\imath(\theta+\theta_i)}$, $\rho<r<(1+\eta_n)\rho$.

Clearly, we have $\tilde{w}_n\in\mathcal{J}_{\rho,\alpha}(\mathbf{x},\mathbf{1})$ and $\frac{1}{2}\int_{\Omega\setminus\cup_i\overline{B(x_i,\rho)}}\alpha|\nabla\tilde{w}_n|^2=J_{\rho,\alpha}+o_n(1)$.

Thus considering $w\in\mathcal{J}_{\rho,\alpha}(\mathbf{x},\mathbf{1})$ a minimizer of $\frac{1}{2}\int_{\Omega\setminus\cup_i\overline{B(x_i,\rho)}}\alpha|\nabla\cdot|^2$, we obtain

$$\frac{1}{2}\int_{\Omega\setminus\cup_i\overline{B(x_i,\rho)}}\alpha|\nabla w|^2\le\frac{1}{2}\int_{\Omega\setminus\cup_i\overline{B(x_i,\rho)}}\alpha|\nabla\tilde{w}_n|^2=J_{\rho,\alpha}+o_n(1).$$

Letting $n\to\infty$ we deduce that the configuration $\mathbf{x}=(x_1,...,x_d)$ is minimizing.

Appendix 4.E Proof of Proposition 4.24

We use the *unfolding operator* (see [30], definition 2.1). We define, for $\Omega_0\subset\mathbb{R}^2$ an open set, $p\in(1,\infty)$ and $\delta>0$:

$$\begin{array}{llll}\mathcal{T}_\delta: & L^p(\Omega_0) & \to & L^p(\Omega_0\times Y)\\ & \phi & \mapsto & \mathcal{T}_\delta(\phi)(x,y)=\begin{cases}\phi\left(\delta\left[\frac{x}{\delta}\right]+\delta y\right) & \text{for } (x,y)\in\Omega_\delta^{\text{incl}}\times Y\\ 0 & \text{for } (x,y)\in\Lambda_\delta\times Y\end{cases}\end{array}$$

and

$$\Omega_\delta^{\text{incl}}:=\bigcup_{\substack{Y_\delta^K\subset\Omega_0\\ Y_\delta^K=\delta(K+Y),\,K\in\mathbb{Z}^2}}\overline{Y_\delta^K},\ \Lambda_\delta:=\Omega_0\setminus\Omega_\delta^{\text{incl}}\text{ and }\left[\frac{x}{\delta}\right]:=\left(\left[\frac{x_1}{\delta}\right],\left[\frac{x_2}{\delta}\right]\right)\in\mathbb{Z}^2.$$

Here, for $s\in\mathbb{R}$, $[s]$ is the integer part of s.

We will use the following results:

$$\mathcal{T}_\delta\text{ is linear and continuous, of norm at most 1 ([30], Proposition 2.5),}\tag{4.84}$$

$$\mathcal{T}_\delta(\phi\psi)=\mathcal{T}_\delta(\phi)\mathcal{T}_\delta(\psi)\text{ ([30], equation (2.2)),}\tag{4.85}$$

$$\delta\mathcal{T}_\delta(\nabla\phi)(x,y)=\nabla_y\mathcal{T}_\delta(\phi)(x,y)\text{ for }\phi\in W^{1,p}(\Omega_0)\text{ ([30], equation (3.1)),}\tag{4.86}$$

$$\text{for }\phi\in L^1(\Omega_0),\ \text{ we have }\int_{\Omega_\delta^{\text{incl}}}\phi=\int_{\Omega_0\times Y}\mathcal{T}_\delta(\phi_\delta)\text{ ([30], Proposition. 2.5 (i)).}\tag{4.87}$$

If $\phi_\delta\in H^1(\Omega_0)$ is such that $\phi_\delta\rightharpoonup\phi_0$ in H^1, then, up to subsequence, there exists $\hat{\phi}\in L^2(\Omega_0,H^1_{\text{per}}(Y))$ s.t.:

$$\mathcal{T}_\delta(\phi_\delta)\to\phi_0\text{ and }\mathcal{T}_\delta(\nabla\phi_\delta)\rightharpoonup\nabla\phi_0+\nabla_y\hat{\phi}\text{ in }L^2(\Omega_0\times Y)\text{ ([30], Theorem 3.5).}\tag{4.88}$$

Here $H^1_{\text{per}}(Y)$ stands for the set of functions $\phi\in H^1(Y)$ s.t. the extending of ϕ by Y-periodicity is in $H^1_{\text{loc}}(\mathbb{R}^2)$ (see [31], section 3.4).

In order to define properly the homogenized matrix $\mathcal{A}$ we recall a classical result (see Theorem 4.27 in [31]).

Proposition 4.40. *Let $H_0\in L^\infty(Y,[b^2,1])$. For all $f\in(H^1_{\text{per}}(Y))'$ s.t. f annihilates the constants there exists a unique solution $h\in H^1_{\text{per}}(Y)$ of*

$$\text{div}(H_0\nabla_y h)=f\ \textit{and}\ \mathcal{M}_Y(h)=\int_Y h=0.$$

Using the previous theorem we denote $\chi_j \in H^1_{\text{per}}(Y)$ the unique solution of

$$\text{div}(H_0 \nabla_y \chi_j) = \partial_{y_j}(H_0) \text{ and } \mathcal{M}_Y(\chi_j) = 0. \tag{4.89}$$

With these auxiliary functions, we can give an explicit expression of $\mathcal{A}$ the homogenized matrix of $H_0(\frac{\cdot}{\delta})\text{Id}_{\mathbb{R}^2}$ (see Theorem 6.1 in [31]):

$$\mathcal{A} = \int_Y H_0 \begin{pmatrix} 1 - \partial_{y_1}\chi_1 & -\partial_{y_1}\chi_2 \\ -\partial_{y_2}\chi_1 & 1 - \partial_{y_2}\chi_2 \end{pmatrix} = \int_Y H_0(\text{Id}_{\mathbb{R}^2} - \nabla_y \chi),\ \chi = (\chi_1, \chi_2).$$

For the convenience of the reader we restate, in larger detail, Proposition 4.24.

Proposition. *Let $\Omega_0 \subset \mathbb{R}^2$ be a smooth bounded open set and let $v_n \in H^2(\Omega_0, \mathbb{C})$ be s.t.*

1. $|v_n| \leq 1$ *and* $\int_{\Omega_0} (1 - |v_n|^2)^2 \to 0$,
2. $v_n \rightharpoonup v_*$ *in* $H^1(\Omega_0)$ *and* $v_* \in H^1(\Omega_0, \mathbb{S}^1)$,
3. *there is* $H_n \in W^{1,\infty}(\Omega_0, [b^2, 1])$ *and* $\delta_n \downarrow 0$ *s.t.* $\mathcal{T}_{\delta_n}(H_n) \to H_0$ *in* $L^2(\Omega_0 \times Y)$ *with* H_0 *independent of* $x \in \Omega_0$,
4. $-\text{div}(H_n \nabla v_n) = v_n f_n(x)$, $f_n \in L^\infty(\Omega_0, \mathbb{R})$.

Then v_* is the solution of

$$-\text{div}(\mathcal{A} \nabla v_*) = (\mathcal{A} \nabla v_* \cdot \nabla v_*) v_*.$$

Here $\mathcal{A}$ is the homogenized matrix of $H_0(\frac{\cdot}{\delta})\text{Id}_{\mathbb{R}^2}$ given by

$$\mathcal{A} = \int_Y H_0 \begin{pmatrix} 1 - \partial_{y_1}\chi_1 & -\partial_{y_1}\chi_2 \\ -\partial_{y_2}\chi_1 & 1 - \partial_{y_2}\chi_2 \end{pmatrix}.$$

Proof. In order to keep notations simple, we write, in what follows, δ rather than δ_n.

Since f_n is real valued, we have that $\text{div}(H_n \nabla v_n) \times v_n = 0$. From (4.84) and (4.85), we obtain

$$\text{div}_y \left[\mathcal{T}_\delta(H_n)(x,y)\mathcal{T}_\delta(\nabla v_n)(x,y)\right] \times \mathcal{T}_\delta(v_n)(x,y) = 0 \text{ in } \Omega_0 \times Y. \tag{4.90}$$

Note that from the assumptions and (4.84),(4.88), passing to a subsequence, there is $\hat{w} \in L^2(\Omega_0, H^1_{\text{per}}(Y))$ s.t.

$$\mathcal{T}_\delta(v_n)(x,y) \to v_*(x),\ \mathcal{T}_\delta(\nabla v_n)(x,y) \rightharpoonup \nabla v_*(x) + \nabla_y \hat{v}(x,y) \text{ in } L^2(\Omega_0 \times Y)$$

and

$$\mathcal{T}_\delta(H_n)(x,y) \to H_0(y) \text{ in } L^2(\Omega_0 \times Y).$$

Thus we obtain the convergence:

$$\text{div}_y \left[\mathcal{T}_\delta(H_n)(x,y)\mathcal{T}_\delta(\nabla v_n)(x,y)\right] \times \mathcal{T}_\delta(v_n)(x,y) \rightharpoonup \text{div}_y \left[H_0(\nabla v_* + \nabla_y \hat{v})\right] \times v_* \text{ in } L^2(\Omega_0 \times H^{-1}(Y)).$$

Consequently,

$$\text{div}_y \left[H_0(\nabla v_* + \nabla_y \hat{v})\right] \times v_* = 0.$$

Since v_* is independent of $y \in Y$, the previous assertion is equivalent to

$$-\mathrm{div}_y \left[H_0 \nabla_y(\hat{v} \times v_*)\right] = (\nabla_y H_0 \cdot \nabla v_*) \times v_*,$$

which in turn is equivalent to

$$-\mathrm{div}_y \left[H_0 \nabla_y(\hat{v} \times v_*)\right] = \sum_i \partial_{y_i} H_0 (\partial_i v_* \times v_*).$$

Hence, from Proposition 4.40 and (4.89), we obtain

$$\hat{v} \times v_* = -\sum_i \chi_i(\partial_i v_* \times v_*) = -\chi \cdot (\nabla v_* \times v_*),\ \chi = (\chi_1, \chi_2)\,. \tag{4.91}$$

Let $\psi \in \mathcal{D}(\Omega_0)$ and n sufficiently large s.t. $\mathrm{Supp}(\psi) \subset \Omega_\delta^{\mathrm{incl}}$. Since $-\mathrm{div}\left[H_n \nabla v_n \times v_n\right] = 0$, we have

$$\int_{\Omega_\delta^{\mathrm{incl}}} H_n \nabla v_n \times v_n \cdot \nabla \psi = 0.$$

This identity combined with (4.87) implies that

$$\int_{\Omega_0 \times Y} \mathcal{T}_\delta[H_n(\nabla v_n \times v_n) \cdot \nabla \psi] = 0.$$

Therefore, using (4.86) and (4.88), we obtain:

$$\begin{aligned} 0 = \int_{\Omega_0 \times Y} \mathcal{T}_\delta\left[H_n(\nabla v_n \times v_n) \cdot \nabla \psi\right] &= \int_{\Omega_0 \times Y} \mathcal{T}_\delta(H_n)\mathcal{T}_\delta(\nabla v_n) \times \mathcal{T}_\delta(v_n) \cdot \mathcal{T}_\delta(\nabla \psi) \\ &\underset{n\to\infty}{\longrightarrow} \int_{\Omega_0 \times Y} H_0\left[\nabla v_* \times v_* + \nabla_y(\hat{v} \times v_*)\right] \cdot \nabla \psi. \end{aligned}$$

Finally, for all $\psi \in \mathcal{D}(\Omega_0)$, using (4.91), we have

$$\begin{aligned} 0 = \int_{\Omega_0 \times Y} H_0 \nabla v_* \times v_* \left[\mathrm{Id}_{\mathbb{R}^2} - \nabla_y \chi\right] \cdot \nabla \psi &= \int_{\Omega_0} \left(\left\{\int_Y H_0\left[\mathrm{Id}_{\mathbb{R}^2} - \nabla_y \chi\right]\right\} \nabla v_* \times v_*\right) \nabla \psi \\ &= -\int_{\Omega_0} -\mathrm{div}\left(\mathcal{A} \nabla v_* \times v_*\right) \psi. \end{aligned}$$

Here $\mathcal{A} = \int_Y H_0\left(\mathrm{Id}_{\mathbb{R}^2} - \nabla_y \chi\right)$.

Thus $-\mathrm{div}\left(\mathcal{A} \nabla v_* \times v_*\right) = 0$. Note that, since H_0 and χ are independent of x, $\mathcal{A}$ is a constant matrix. This fact combined with the equation $-\mathrm{div}\left(\mathcal{A} \nabla v_* \times v_*\right) = 0$ implies that v_* satisfies

$$-\mathrm{div}(\mathcal{A} \nabla v_*) = (\mathcal{A} \nabla v_* \cdot \nabla v_*) v_*. \tag{4.92}$$

Indeed, we can always consider φ_* which is locally defined in Ω_0 and whose gradient is globally defined and in $L^2(\Omega_0, \mathbb{R}^2)$ s.t. $v_* = \mathrm{e}^{\imath \varphi_*}$.

Since $v_* \times \nabla v_* = \nabla \varphi_*$ we obtain that $\mathrm{div}(\mathcal{A} \nabla \varphi_*) = 0$. Identity (4.92) follows from the equation of φ_* and the fact that $|\nabla \varphi_*|^2 = |\nabla v_*|^2$.

□

Troisième partie

Étude de la fonctionnelle de Ginzburg-Landau avec un terme de chevillage : le cas tri-dimensionnel

Chapter 5

Study of a Ginzburg-Landau functional with a discontinuous pinning term: the three-dimensional case

In a convex domain $\Omega \subset \mathbb{R}^3$, we consider the minimization of a $3D$-Ginzburg-Landau type energy with a discontinuous pinning term among $H^1(\Omega, \mathbb{C})$-maps subject to a boundary Dirichlet condition $g \in H^{1/2}(\partial\Omega, \mathbb{S}^1)$. The pinning term $a : \mathbb{R}^3 \to \mathbb{R}^*_+$ takes a constant value $b \in (0,1)$ in ω, an inner strictly convex subdomain of Ω, and 1 outside ω. We prove energy estimates with various error terms depending on our assumptions on Ω, ω and g. In some special cases, we identify the vorticity lines. We also establish the concentration of the energy along the vorticity lines.

Contents

5.1 Introduction

In a convex domain $\Omega \subset \mathbb{R}^3$, we consider the minimization of a $3D$-Ginzburg-Landau type energy with a discontinuous pinning term among $H^1(\Omega, \mathbb{C})$-maps subject to a boundary Dirichlet condition $g \in H^{1/2}(\partial\Omega, \mathbb{S}^1)$. The pinning term $a : \mathbb{R}^3 \to \mathbb{R}_+^*$ takes a constant value $b \in (0, 1)$ in ω, an inner strictly convex subdomain of Ω, and 1 outside ω. The strict convexity of ω is not necessary but it allows to make a simpler description of the technics used in this chapter.

Our Ginzburg-Landau type energy is

$$E_\varepsilon(u) = \frac{1}{2}\int_\Omega \left\{ |\nabla u(x)|^2 + \frac{1}{2\varepsilon^2}(a(x)^2 - |u(x)|^2)^2 \right\} \mathrm{d}x. \tag{5.1}$$

In (5.1), $u \in H^1_g := \{u \in H^1(\Omega, \mathbb{C}) \,|\, \mathrm{tr}_{\partial\Omega} u = g\}$.

Let U_ε be **the** unique minimizer of E_ε in H^1_1. If $v \in H^1(\Omega, \mathbb{C})$ and $|v| \equiv 1$ on $\partial\Omega$, then [43]

$$E_\varepsilon(U_\varepsilon v) = E_\varepsilon(U_\varepsilon) + F_\varepsilon(v), \text{ where } F_\varepsilon(v) = \frac{1}{2}\int_\Omega \left\{ U_\varepsilon^2 |\nabla v|^2 + \frac{U_\varepsilon^4}{2\varepsilon^2}(1 - |v|^2)^2 \right\}.$$

Consequently the study of minimizers of E_ε in H^1_g is related to the study of minimizers of F_ε in H^1_g.

Our technics are directly inspired from whose initially developed by Sandier in [60] (whose purpose was to give, in some special situations, a simple proof of the $3D$ analysis of the Ginzburg-Landau equation, by Lin and Riviere [48]), and by their adaptations in [22].

We prove energy estimates with various error terms depending on our assumptions on Ω and g. In some special cases, we identify the vorticity lines. We also establish the concentration of the energy along the vorticity lines. At the end of this chapter, we will present a strategy which could lead to the localization of the vortex lines.

The results we present are a first step towards a more precise description of the vorticity defaults and of the asymptotic of minimizers.

Before stating our own results, we start by recalling the asymptotic expansion of the energy in the standard Ginzburg-Landau model in $3D$.

For $g \in H^{1/2}(\partial\Omega, \mathbb{S}^1)$, if we let

$$E^0_\varepsilon(u) = \frac{1}{2}\int_\Omega \left\{ |\nabla u|^2 + \frac{1}{2\varepsilon^2}(1 - |u|^2)^2 \right\},$$

then we have

$$\inf_{H^1_g} E^0_\varepsilon = C(g)|\ln\varepsilon| + o(|\ln\varepsilon|). \tag{5.2}$$

Moreover, $\dfrac{C(g)}{\pi}$ is given by the length of a minimal connection connecting the singularities of g (in the spirit of Brezis, Coron, Lieb [27]). (See [48], [49] and [60] and [22]).

For special g's and for a convex domain Ω, (5.2) was obtained by Lin and Rivière [48] (see also [49]) and Sandier [60]. The case of a general data $g \in H^{1/2}(\partial\Omega, \mathbb{S}^1)$ and a simply connected Ω is due to Bourgain, Brezis and Mironescu [22].

The above articles are our main references in this work. One of our main results is the analog of (5.2) for the minimization of F_ε (Theorem 5.4). This result is first proved when g is in a dense set $\mathcal{H} \subset H^{1/2}(\partial\Omega, \mathbb{S}^1)$ and then extended by density. The upper bound is obtained directly using the technics developed in [60] and [22]. The lower bound needs an adaptation in the argument of Sandier [60]. The main ingredient used to obtain a lower bound in Sandier [60] is the existence of a "structure function" adapted to the singularities of g. In the spirit of [60], we prove, under suitable assumptions on Ω, ω and g, the existence of structure functions adapted to our situation. We presented below constructions (in the spirit of Sandier) of structure functions under restrictive hypotheses on the geometries of Ω, ω and on the singularities of g (see Corollaries 5.8, 5.11 and Proposition 5.12).

In our situation, when g admits a finite number of singularities, the constant $\dfrac{C(g)}{\pi}$ is the length of a minimal connection between the singularities of g. This minimal connection is computed with respect to a metric d_{a^2} depending only on a (see (5.9)). (This generalizes the case of the standard potential $(1-|u|^2)^2$, where the distance is the euclidean one.)

When g has a finite number of singularities, one may prove a concentration of the energy along the vorticity lines (See Theorems 5.5 and 5.6). As in [48] and [60], we obtain, after normalization, that the energy of minimizer is uniform along the vorticity lines (See Theorem 5.5). These vorticity lines are identified: they are geodesic segments associated to d_{a^2}.

The goal of this work is to explain how the vorticity lines are modified under the effect of a pinning term. Although from the theorems below we have an idea on the form of the vorticity lines, in order to have a complete description of the defaults, we need an η-ellipticity results in the spirit of [19] for the minimizers of F_ε. Namely: fix $r>0$ then for small ε and v a minimizer of F_ε

$$\textit{if, in a ball } B(x,r), \textit{ the quantity } \frac{F_\varepsilon(v,B(x,r))}{|\ln\varepsilon|} \textit{ is small, then } |v(x)|\simeq 1.$$

It seems that an η-ellipticity result cannot be obtained by the standard method, which relies on a monoticity formula obtained from a Pohozaev identity. The oscillating behavior of U_ε yields impossible the direct application of monotonicity formulae. When U_ε does not oscillate, it is possible to derive η-ellipticity (see *e.g.* [50]). In our case, η-ellipticity would require a uniform control of the Lipschitz norm of U_ε; this does not hold in our situation.

5.2 Description of the special solution U_ε

Let $\overline{\omega}\subset\Omega\subset\mathbb{R}^3$ be two smooth bounded open sets s.t. Ω is convex and ω is strictly convex. For $b\in(0,1)$ we define

$$\begin{array}{rccl} a: & \mathbb{R}^3 & \to & \{b,1\} \\ & x & \mapsto & \begin{cases} b & \text{if } x\in\omega \\ 1 & \text{otherwise}\end{cases} \end{array}.$$

We denote E_ε the Ginzburg-Landau functional with a as pinning term, namely

$$E_\varepsilon(u)=\frac12\int_\Omega\left\{|\nabla u(x)|^2+\frac{1}{2\varepsilon^2}(a(x)^2-|u(x)|^2)^2\right\}\mathrm{d}x.$$

For $\varepsilon>0$, we denote U_ε **the** unique global minimizer of E_ε in

$$H^1_1:=\{u\in H^1(\Omega,\mathbb{C})\,|\,\mathrm{tr}_{\partial\Omega}u\equiv 1\}.$$

In the following, we will denote also $U_\varepsilon\in H^1_{\mathrm{loc}}(\mathbb{R}^3,\mathbb{C})$ the extension by 1 of the unique global minimizer of E_ε in H^1_1.

Proposition 5.1. *The following assertions are true*

1. $U_\varepsilon : \mathbb{R}^3 \to [b,1]$ *(from [43])*,

2. $-\Delta U_\varepsilon = \dfrac{1}{\varepsilon^2} U_\varepsilon (a^2 - U_\varepsilon^2)$ *in* Ω,

3. $E_\varepsilon(U_\varepsilon) \underset{\varepsilon\to 0}{\sim} \dfrac{1}{\varepsilon^2}\displaystyle\int_\Omega (a^2 - U_\varepsilon^2)^2 \underset{\varepsilon\to 0}{\sim} \dfrac{1}{\varepsilon}$ *(same argument as in [43])*,

4. *There are* $C, \gamma > 0$ *s.t. for* $x \in \Omega$ *we have (same proof as in Chapter 2, Appendix 2.A)*
$$|U_\varepsilon(x) - a(x)| \le C\mathrm{e}^{-\gamma \mathrm{dist}(x,\partial\omega)/\varepsilon}, \tag{5.3}$$

5. *If* $v \in \mathcal{J} := \{v \in H^1(\Omega,\mathbb{C}) \,|\, |\mathrm{tr}_{\partial\Omega} v| = 1\}$ *then* $E_\varepsilon(U_\varepsilon v) = E_\varepsilon(U_\varepsilon) + F_\varepsilon(v)$ *(same proof as [43]) with*
$$F_\varepsilon(v) = \frac{1}{2}\int_\Omega \left\{ U_\varepsilon^2 |\nabla v|^2 + \frac{U_\varepsilon^4}{2\varepsilon^2}(1 - |v|^2)^2 \right\}, \tag{5.4}$$

6. *If* v *minimises* F_ε *in* $H^1_g := \{v \in H^1(\Omega,\mathbb{C}) \,|\, \mathrm{tr}_{\partial\Omega} v = g\}$ *then* $|v| \le 1$ *in* Ω *(same proof as [43])*.

5.3 Minimal connections, geodesic links

In this section we define the main geometrical objects which appear in the description of the vorticity lines.

5.3.1 Length of a minimal connection of a map $g \in H^{1/2}(\partial\Omega, \mathbb{S}^1)$

For $g \in H^{1/2}(\partial\Omega, \mathbb{S}^1)$, following [22], one may one may associate to g a continuous linear form
$$T_g : (\mathrm{Lip}(\partial\Omega,\mathbb{R}), \|\cdot\|_{\mathrm{Lip}}) \to \mathbb{R}.$$
Here $\|\varphi\|_{\mathrm{Lip}} = \|\varphi\|_{L^\infty} + \sup\limits_{\substack{x,y\in\partial\Omega \\ x\neq y}} \dfrac{|\varphi(x) - \varphi(y)|}{|x-y|}$ with $|x-y| = d_{\mathrm{eucl}}(x,y)$ is the euclidean distance in $\mathbb{R}^3$ between x and y.

The map T_g is defined by the following way: let $\varphi \in \mathrm{Lip}(\partial\Omega,\mathbb{R})$ and $g \in H^{1/2}(\partial\Omega,\mathbb{R})$;

- fix $u \in H^1_g$ and consider $H = 2(\partial_2 u \times \partial_3 u\,,\, \partial_3 u \times \partial_1 u\,,\, \partial_1 u \times \partial_2 u)$;
- fix $\phi \in \mathrm{Lip}(\Omega,\mathbb{R})$ s.t. $\phi = \varphi$ on $\partial\Omega$;

then
$$\begin{array}{rccc} T_g : & \mathrm{Lip}(\partial\Omega,\mathbb{R}) & \to & \mathbb{R} \\ & \varphi & \mapsto & \displaystyle\int_\Omega H\cdot\nabla\phi \end{array}$$
is independent of the choice of u and ϕ.

Following [22], we denote, for $g \in H^{1/2}(\partial\Omega, \mathbb{S}^1)$ and d an equivalent distance with d_{eucl} on $\partial\Omega$,
$$L(g,d) := \sup\{T_g(\varphi) \,|\, |\varphi|_d \le 1\} = \max\{T_g(\varphi) \,|\, |\varphi|_d \le 1\} \tag{5.5}$$
with
$$|\varphi|_d = \sup_{\substack{x\neq y \\ x,y\in\partial\Omega}} \frac{|\varphi(x) - \varphi(y)|}{d(x,y)}.$$

Note that $L(g, d)$ is finite, since $T_g : (\mathrm{Lip}(\partial\Omega, \mathbb{R}), \|\cdot\|_{\mathrm{Lip}}) \to \mathbb{R}$ is continuous and d, d_{eucl} are equivalent on $\partial\Omega$.

A special subset of $H^{1/2}(\partial\Omega, \mathbb{S}^1)$ is

$$\mathcal{H} = \left\{ g \in \bigcap_{1 \le p < 2} W^{1,p}(\partial\Omega, \mathbb{S}^1) \;\middle|\; \begin{array}{c} g \text{ is smooth outside a finite set } \mathcal{C}, \\ \forall M \in \mathcal{C} \text{ we have for } x \text{ close to } M: \\ |\nabla g(x)| \le C/|x - M|, \\ \exists\, R_M \in \mathcal{O}(3) \text{ s.t. } \left| g(x) - R_M\left(\frac{x-M}{|x-M|}\right)\right| \le C|x - M|. \end{array} \right\}.$$

Here we considered $\mathbb{S}^1 \simeq \{0\} \times \mathbb{S}^1 \subset \mathbb{S}^2$.

One may define $\deg(u, M)$, the topological degree of u with respect to M: if $R_M \in \mathcal{O}(3)^+$ then $\deg(u, M) = 1$ otherwise $\deg(u, M) = -1$.

In order to justify the term of "degree", assume that in a neighborhood of $M \in \mathcal{C}$, $\partial\Omega$ is flat. Then, for $r > 0$ sufficiently small, $C = \partial B(M, r) \cap \partial\Omega$ is a circle centered in M. This circle has a natural orientation induced by $B(M, r) \cap \Omega$. Thus $g_{|C} \in C^\infty(C, \mathbb{S}^1)$ admits a well defined topological degree (see *e.g.* [26]), and this degree does not depend on small r.

We consider

$$P = \{M \in \mathcal{C} \,|\, \deg(u, M) = 1\} \text{ and } N = \{M \in \mathcal{C} \,|\, \deg(u, M) = -1\}.$$

One may also consider for $g \in \mathcal{H}$ the degree of g with respect to ∂U for U a non empty smooth open set of $\partial\Omega$ s.t. ∂U does not contain any singularities of g. This degree is defined as

$$\deg(g, \partial U) = \mathrm{Card}(\{p \in P \,|\, p \in U\}) - \mathrm{Card}(\{n \in N \,|\, n \in U\}). \tag{5.6}$$

From [22], we have the following

Proposition 5.2. *Let $g, h \in H^{1/2}(\partial\Omega, \mathbb{S}^1)$, then we have*

1. $T_{gh} = T_g + T_h$ *and* $T_{\overline{g}} = -T_g$ *(Lemma 9),*
2. $|(T_g - T_h)(\varphi)| \le C|g - h|_{H^{1/2}}(|g|_{H^{1/2}} + |h|_{H^{1/2}})|\varphi|_{d_{\mathrm{eucl}}}$, $\varphi \in \mathrm{Lip}(\partial\Omega, \mathbb{R})$ *(Eq. (1.6)),*
3. $\mathcal{H}$ *is dense in* $H^{1/2}(\partial\Omega, \mathbb{S}^1)$ *(Lemma B.1),*
4. *if* $u \in \mathcal{H}$, *then* $\mathrm{Card}(P) = \mathrm{Card}(N)$ *and* $T_g = 2\pi \sum_{p \in P} \delta_p - 2\pi \sum_{n \in N} \delta_n$ *(Lemma 2),*
5. *if* $u \in \mathcal{H}$, *then* $L(g, d) = \min_{\sigma \in S_k} \sum_i d(p_i, n_{\sigma(i)})$ *where* d *is a distance equivalent with* d_{eucl} *on* $\partial\Omega$ *(Theorem 1).*

5.3.2 Minimal connections, minimal length and geodesic links

In the last assertion of Proposition 5.2, we used the notion of length of a minimal connection. Namely, consider d a distance on $\mathcal{C} = P \cup N$, $P, N \subset \mathbb{R}^3$ two sets of k distinct points s.t. $P \cap N = \emptyset$, $P = \{p_1, ..., p_k\}$ and $N = \{n_1, ..., n_k\}$.

We denote by $L(\mathcal{C}, d)$ the length of a minimal connection of $\mathcal{C}$ in $(\mathcal{C}, d)$, *i.e.*,

$$L(\mathcal{C}, d) = \min_{\sigma \in S_k} \sum_{i=1}^{k} d(p_i, n_{\sigma(i)}). \tag{5.7}$$

In [27] (Lemma 4.2), the authors proved that

$$L(\mathcal{C},d)=\max\left\{\sum_{i=1}^{k}\{\varphi(p_i)-\varphi(n_i)\}\mid\varphi:\mathcal{C}\to\mathbb{R},\,|\varphi|_d^{\mathcal{C}}\leq 1\right\}\tag{5.8}$$

with

$$|\varphi|_d^{\mathcal{C}}=\sup_{\substack{x\neq y\\ x,y\in\mathcal{C}}}\frac{|\varphi(x)-\varphi(y)|}{d(x,y)}.$$

A permutation σ s.t. $\sum_i d(p_i,n_{\sigma(i)})=L(P\cup N,d)$ is called a minimal connection of $(P\cup N,d)$.

In the following we will consider a special form of distance d on $\partial\Omega$: the geodesic distance in $\overline{\Omega}$ equipped with a metric we will describe below.

Let us first introduce some notations. Let $f:\mathbb{R}^3\to[b^2,1]$ be a Borel function and let $\Gamma\subset\overline{\Omega}$ be Lipschitz curve. We denote by $\mathrm{long}_f(\Gamma)$ the length of Γ in the metric fh (here h is the euclidean metric in $\mathbb{R}^3$), *i.e.*,

$$\mathrm{long}_f(\Gamma):=\int_0^1 f(\gamma(s))\,|\gamma'(s)|\mathrm{d}s,\ \gamma:[0,1]\to\Gamma \text{ is a admissible parametrization of }\Gamma.$$

In this paper, when we consider a curve (or arc) Γ, it will be implicitly that it is a Lipschitz one.

We define d_f as the geodesic distance in fh (h is the euclidean metric in $\mathbb{R}^3$).

Thus, for $x,y\in\mathbb{R}^3$, $x\neq y$, we have

$$d_f(x,y)=\inf_{\substack{\Gamma\text{ Lipschitz arc}\\ \text{with endpoints }x,y}}\mathrm{long}_f(\Gamma).\tag{5.9}$$

In the special case $f=a^2$, one may easily prove the following proposition

Proposition 5.3. *Let $x,y\in\mathbb{R}^3$, $x\neq y$. The following assertions are true*

1. *In* (5.9) *the* infimum *is attained.*

   We denote by Γ_0 a minimal curve in (5.9).

2. *If $x,y\in\overline{\Omega}$ then a geodesic Γ_0 is included in $\overline{\Omega}$.*

3. *A geodesic $\Gamma_0=\cup_{i=1}^k S_i$ is a union of at most three line segments.*

4. *These line segments are such that*

   a. *if $x,y\in\omega$ then $k=1$,*
   b. *if $k=2$ then $S_1\cap S_2\subset\partial\omega$,*
   c. *if $k=3$ then $x,y\in\mathbb{R}^3\setminus\overline{\omega}$ and S_2 is a chord of ω,*
   d. *if $[x,y]\cap\overline{\omega}=\{z\}$ then $k\in\{2,3\}$.*

In the case $d=d_{a^2}$ and $\mathcal{C}=P\cup N\subset\partial\Omega$, we say that $\cup_i\Gamma_i$ is a geodesic link when σ is a minimal connexion in $(\mathcal{C},d_{a^2})$ and Γ_i is a geodesic joining p_i to $n_{\sigma(i)}$. In Figure 5.1, we have represented a geodesic link for $k=2$ and a certain $b\in(0,1)$.

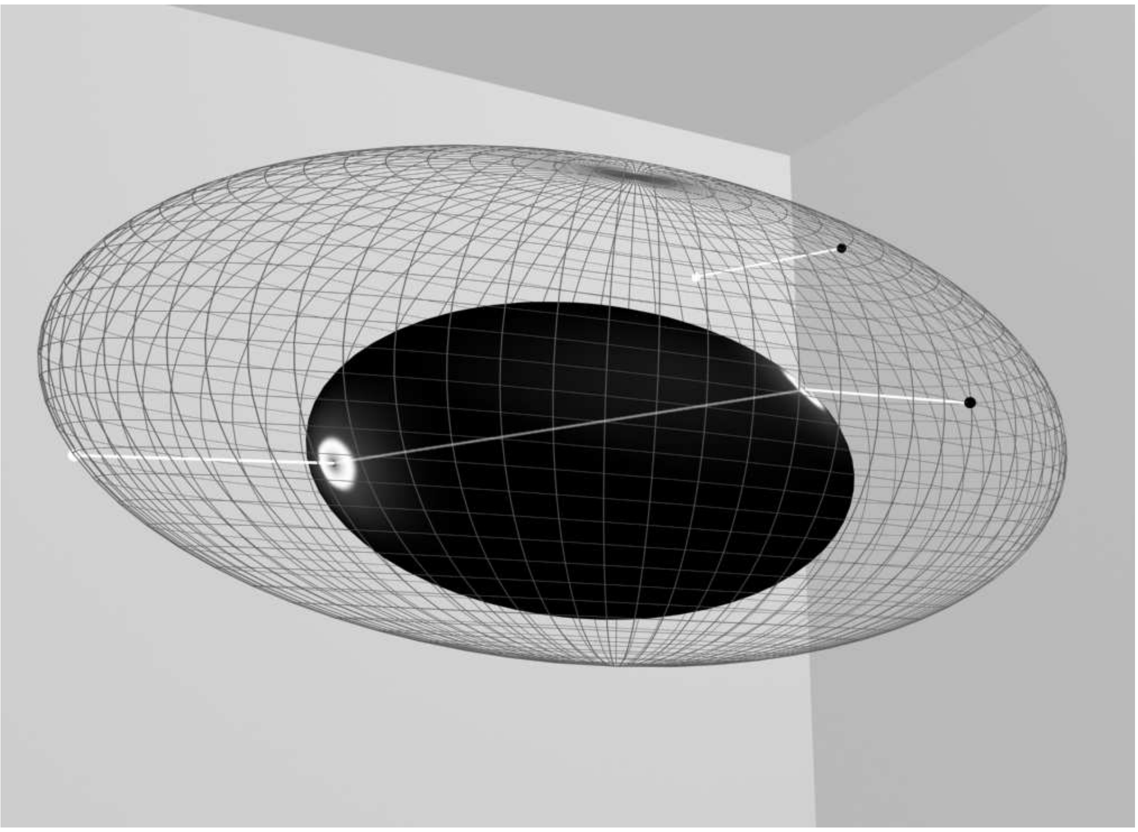


Figure 5.1: Illustration of a geodesic link with $k = 2$: the boundary of Ω is in wire, the one of ω is black filled, the positive points are white, the negative ones are black and a geodesic link is represented in white. The shaded off on the two penetration points gives indications about the $3D$-geometry of the geodesic link and of the inclusion. (Courtesy of Alexandre Marotta)

5.3.3 The main results

Theorem 5.4. *Let $g \in H^{1/2}(\partial\Omega, \mathbb{S}^1)$. Then we have*

$$\inf_{v \in H^1_g} F_\varepsilon(v) = \pi L(g, d_{a^2})|\ln\varepsilon| + o(|\ln\varepsilon|).$$

Theorem 5.5. *Let $g \in \mathcal{H}$ be s.t. $(\mathcal{C} = P \cup N, d_{a^2})$ admits a unique geodesic link which is denoted $\cup_i \Gamma_i$.*

Let v_ε be a minimizer of F_ε in H^1_g. Then the normalized energy density

$$\mu_\varepsilon = \frac{\dfrac{U_\varepsilon^2}{2}|\nabla v_\varepsilon|^2 + \dfrac{U_\varepsilon^4}{4\varepsilon^2}(1-|v_\varepsilon|^2)^2}{|\ln\varepsilon|}\mathscr{H}^3 \text{ weakly converges in } \Omega \text{ in the sense of the measure to } \pi a^2 \mathscr{H}^1_{|\cup_i\Gamma_i}.$$

Here $\mathscr{H}^3$ is the 3-dimensional Hausdorff measure and $\mathscr{H}^1_{|\cup_i\Gamma_i}$ is the one dimensional Hausdorff mesure on $\cup_i\Gamma_i$.

In other words

$$\forall \phi \in C^0_0(\Omega, \mathbb{R}) \cap L^\infty(\Omega, \mathbb{R}) \text{ we have } \int_\Omega \phi \, \mathrm{d}\mu_\varepsilon \to \pi \int_{\cup_i\Gamma_i} \phi a^2 \, \mathrm{d}\mathscr{H}^1.$$

Note that this result gives an (uniform) energy concentration property of the minimizers along the geodesic link. Namely, for all compact K s.t. $K \cap \cup_i \Gamma_i = \emptyset$, we have $F_\varepsilon(v_\varepsilon, K) = o(|\ln \varepsilon|)$.

In order to obtain a more precise statement we assume that $\Omega = B(0,1)$ and $\omega = B(0, r_0)$, $r_0 \in (0,1)$, $g \in \mathcal{H}$ is s.t. $\mathcal{C} = \{p, n\}$ with $p = -n$. Under these hypotheses we have

Theorem 5.6. *The following estimation holds*

$$\inf_{H^1_g} F_\varepsilon = \pi d_{a^2}(p,n)|\ln \varepsilon| + \mathcal{O}(1).$$

Moreover, for all $\eta > 0$, there is $C_\eta > 0$ s.t. denoting $V_\eta = \{x \in \Omega \,|\, \mathrm{dist}(x, [p,n]) \geq \eta\}$ and v_ε a minimizer of F_ε in H^1_g, we have

$$F_\varepsilon(v_\varepsilon, V_\eta) \leq C_\eta.$$

5.4 Outline of the proofs

The proofs of the above theorems strongly rely on the technics developed in [60]. The proofs of theorems 5.4, 5.6 consist essentially into two parts devoted to obtaining respectively lower and upper bounds.

The upper bound is obtained by the construction of a test function. The test function was obtained by Sandier in [60] in the situation where there is a geodesic link in $(\mathcal{C}, d_{a^2})$ which is a union of line segments. In this special case, one may obtain (see Section 5.5.1):

$$\inf_{v \in H^1_g} F_\varepsilon(v) \leq \pi L(g, d_{a^2})|\ln \varepsilon| + \mathcal{O}(1). \tag{5.10}$$

For the general case, when the geodesic links are not unions of line segments, in [22], Bourgain, Brezis and Mironescu adapted the construction of Sandier. In our case this leads to the bound:

$$\inf_{v \in H^1_g} F_\varepsilon(v) \leq \pi L(g, d_{a^2})|\ln \varepsilon| + o(|\ln \varepsilon|). \tag{5.11}$$

(See Section 5.5.2.)

The lower bounds are obtained as in [60]. The key ingredient is the construction of a "structure function" $\xi : \mathbb{R}^3 \to \mathbb{R}$ (see Section 5.6 for a precise definition). Due to the fact that for $M \in \mathbb{R}^3$, $x \to \psi_M(x) = d_{a^2}(x, M)$ is not C^1 (its gradient is not continuous on $\partial\omega$ since $|\nabla \psi_M| = a^2$ in $\mathbb{R}^3 \setminus \partial\omega$), we cannot obtain ξ with exactly the same properties as in [60] (see Corollary 5.8). The consequence of this lack of smoothness for the distance function implies that our best lower bound is

$$\inf_{v \in H^1_g} F_\varepsilon(v) \geq \pi L(g, d_{a^2})|\ln \varepsilon| - o(|\ln \varepsilon|). \tag{5.12}$$

However, under strong symmetry hypotheses, namely, $\Omega = B(0,1), \omega = B(0, r_0)$ and $\mathcal{C} = \{p, n = -p\}$, the structure function ξ enjoys additional properties (see Proposition 5.12). In this symmetric case, one may obtain the sharper bound

$$\inf_{v \in H^1_g} F_\varepsilon(v) \geq \pi L(g, d_{a^2})|\ln \varepsilon| - \mathcal{O}(1). \tag{5.13}$$

The estimate on $\inf_{H^1_g} F_\varepsilon$ in Theorem 5.4 (resp. Theorem 5.6) is a direct consequence of (5.11), (5.12) (resp. (5.10) and (5.13)) and of the density of $\mathcal{H}$ in $H^{1/2}(\partial\Omega, \mathbb{S}^1)$ (see Section 5.8) .

Theorem 5.5 is proved along the main lines in [60].

Roughly speaking, under the hypotheses of Theorem 5.5, for all $x \in \Omega$, there is $\rho_x > 0$ s.t. for $K = \overline{B(x, \rho_x)}$, one may consider a structure function ξ adapted to $\mathcal{C}$ which is constant in K (see Section 5.6.2). Arguing as in [60], if K does not intersect the geodesic link, then we obtain that in K, a minimizer of F_ε has its energy of order $o(|\ln\varepsilon|)$ (see (5.27)). Thus μ, the weak limit of μ_ε (which exists up to subsequence), is supported in $\overline{\Omega} \setminus K$. Therefore, one may prove that the support of μ is included in the geodesic link.

Otherwise, if x is on the geodesic link, as explain in [60], then we obtain for v_ε a minimizer and ρ sufficiently small that

$$\limsup \frac{F_\varepsilon(v_\varepsilon, K)}{|\ln\varepsilon|} \leq \pi \text{long}_{a^2}(K \cap \cup_i \Gamma_i).$$

Theorem 5.5 is obtained by comparing μ to $\pi a^2 \mathscr{H}^1_{|\cup_i \Gamma_i}$.

5.5 The upper bounds for $\inf_{H^1_g} F_\varepsilon$, $g \in \mathcal{H}$

5.5.1 The case where $\mathcal{C}$ admits a geodesic link in $(\mathbb{R}^3, d_{a^2})$ which is a union of lines

Assume that there is $\Gamma = \cup \Gamma_i$, a geodesic link of $\mathcal{C}$ in $(\mathbb{R}^3, d_{a^2})$ s.t. Γ_i is a line segment for all i. One may assume that the minimal connection associated to Γ is the identity.

In this situation, we may mimic the construction of the test function made in Section 1 of Sandier [60].

The test function is a fixed (independent of ε) $\mathbb{S}^1$-valued function outside $\mathcal{V}_\eta$, an η-tubular neighborhood of Γ.

Inside each tubular neighborhood $\mathcal{V}_{\eta,i}$ of a geodesic $p_i\Gamma_i n_i$, the test function takes the form (in the basis $\{p_i, (\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)\}$ where $n_i = (0, 0, |p_i - n_i|)$)

$$v_\varepsilon(x,y,z) = \begin{cases} \alpha \dfrac{(x,y)}{|(x,y)|} \text{ if } \eta < z < |p_i - n_i| - \eta \text{ and } \varepsilon < |(x,y)| < \eta \\ \alpha \dfrac{(x,y)}{\varepsilon} \text{ if } \eta < z < |p_i - n_i| - \eta \text{ and } |(x,y)| < \varepsilon \\ \dfrac{1}{2} \displaystyle\int_{\tilde{\mathcal{V}}_{\eta,i}} \left\{ |\nabla v_\varepsilon|^2 + \frac{1}{2\varepsilon^2}(1 - |v_\varepsilon|^2)^2 \right\} \leq 2\eta\pi |\ln\varepsilon| \\ \text{with } \tilde{\mathcal{V}}_{\eta,i} = \mathcal{V}_{\eta,i} \cap \{0 < z < \eta, |p_i - n_i| - \eta < z < |p_i - n_i|\} \end{cases}. \tag{5.14}$$

Here $\alpha \in \mathbb{S}^1$ is a fixed constant.

From the strict convexity of ω, for all line $D \subset \mathbb{R}^3$, we have

$$\text{long}(D \cap \{x \in \mathbb{R}^3 \,|\, \text{dist}(x, \partial\omega) \leq \sqrt{\varepsilon}\} \leq C\sqrt{\varepsilon} \text{ with } C > 0 \text{ is independent of } \varepsilon.$$

Thus one may obtain from Proposition 5.1 Assertion *4.* that (5.10) holds.

5.5.2 The general case

One may adapt the above construction to the more general situation where the geodesic links are not unions of line segments.

For the standard Ginzburg-Landau energy, this has been done in [22]; there, Ω is not supposed convex. Roughly speaking, the argument there consists in replacing in Sandier's proof, line segments by curves.

Their construction begins with the modification of Ω (flattening $\partial\Omega$ close to the singularities) and, for $\eta > 0$, by the construction of an approximate (smooth) geodesic link $\Gamma_\eta \subset \Omega$ s.t. $\text{long}_{a^2}(\Gamma_\eta) \leq \text{long}_{a^2}(\Gamma) + \eta$. Here Γ is a geodesic link.

In order to be applicable to our situation, this construction requires the additional property $\mathscr{H}^1(\Gamma_\eta \cap \{\text{dist}(x, \partial\omega) < \sqrt{\varepsilon}\}) \sim \sqrt{\varepsilon}$; we can clearly find Γ_η satisfying this property.

By adapting the construction of v_ε in (5.14), one may construct a test function v_ε^η having Γ_η as set of zeroes and satisfying, for each $\eta > 0$, the estimate

$$\frac{\inf_{v \in H^1_g} F_\varepsilon(v)}{|\ln \varepsilon|} \leq \frac{F_\varepsilon(v_\varepsilon^\eta)}{|\ln \varepsilon|} + o_\varepsilon(1) = \pi \text{long}_{a^2}(\Gamma_\eta) + o_\varepsilon(1) \leq \pi \text{long}_{a^2}(\Gamma) + \eta + o_\varepsilon(1).$$

In order to obtain this estimate we rely on the formula of v_ε^η, Proposition 5.1 Assertion *4.* and the assumption $\mathscr{H}^1(\Gamma_\eta \cap \{\text{dist}(x, \partial\omega) < \sqrt{\varepsilon}\}) \sim \sqrt{\varepsilon}$.

Consequently we deduce that (5.11) holds.

5.6 The structure functions

For $g \in \mathcal{H}$, we will construct a suitable structure function adapted to the singularities of g.

Roughly speaking, a structure function ξ is a smooth map which almost minimizes (5.5). More qualitative properties of ξ will be describe in Corollaries 5.8, 5.11 and Proposition 5.12.

We present below three constructions of structure functions, corresponding to three different settings.

Throughout this section, we fix $\mathcal{C} = P \cup N$, $\text{Card}(P) = \text{Card}(N) = k \in \mathbb{N}^*$, $P \cap N = \emptyset$.

Let $\delta_0 = 10^{-2} \cdot \text{dist}(\partial\omega, \partial\Omega)$. For $0 < \delta < \delta_0$, we define $\omega_\delta := \omega + B(0, \delta)$, $\alpha_0 = a^2$ and

$$\begin{array}{rccc} \alpha_\delta : & \mathbb{R}^3 & \to & \{1, b^2\} \\ & x & \mapsto & \begin{cases} b^2 & \text{if } x \in \omega_\delta \\ 1 & \text{otherwise} \end{cases} \end{array}.$$

For $x, y \in \mathbb{R}^3$ and $0 \leq \delta < \delta' \leq \delta_0$, we have

$$d_{\alpha_{\delta'}}(x, y) \leq d_{\alpha_\delta}(x, y) \leq d_{\alpha_{\delta'}}(x, y) + \mathcal{O}(\delta' - \delta). \tag{5.15}$$

The first inequality is a direct consequence of $\alpha_{\delta'} \leq \alpha_\delta$. We prove the second inequality. Consider $x, y \in \mathbb{R}^3$ s.t. $d_{\alpha_{\delta'}}(x, y) < d_{\alpha_\delta}(x, y)$. We obtain that if Γ is geodesic joining x and y in $(\mathbb{R}^3, d_{\alpha_{\delta'}})$, then we have $\Gamma \cap \partial\omega_{\delta'} \neq \emptyset$.

Note that by Proposition 5.3, we have $\text{Card}(\Gamma \cap \partial\omega_{\delta'}) \in \{1, 2\}$.

Assume that $\Gamma \cap \partial\omega_{\delta'} = \{x', y'\}$ with $d_{\alpha_{\delta'}}(x, x') < d_{\alpha_{\delta'}}(y, x')$. The situation where $\Gamma \cap \partial\omega_{\delta'} = \{z\}$ is similar.

Consider $x'' = \Pi_{\overline{\omega_\delta}}(x')$ and $y'' = \Pi_{\overline{\omega_\delta}}(y')$. Here $\Pi_{\overline{\omega_\delta}}$ stands for the orthogonal projection on $\overline{\omega_\delta}$. By the definition of x'' and y'' we have $d_{\text{eucl}}(x', x'') = d_{\text{eucl}}(y', y'') = \delta$. By Proposition 5.3, we deduce that $d_{\alpha_{\delta'}}(x'', y'') = d_{\alpha_\delta}(x'', y'')$.

Since $x', y' \in \Gamma$, we have

$$\begin{array}{rcl} d_{\alpha_{\delta'}}(x, y) & = & d_{\alpha_{\delta'}}(x, x') + d_{\alpha_{\delta'}}(x', y') + d_{\alpha_{\delta'}}(y', y) \\ & \geq & d_{\alpha_{\delta'}}(x, x') + d_{\alpha_{\delta'}}(x'', y'') + d_{\alpha_{\delta'}}(y', y) - 2b^2\delta \\ & \geq & d_{\alpha_\delta}(x, x') + d_{\alpha_\delta}(x'', y'') + d_{\alpha_\delta}(y', y) - 2b^2\delta \\ & \geq & d_{\alpha_\delta}(x, x') + d_{\alpha_\delta}(x', y') + d_{\alpha_\delta}(y', y) - 2(1 + b^2)\delta \\ & \geq & d_{\alpha_\delta}(x, y) - 2(1 + b^2)\delta. \end{array}$$

Consequently, (5.15) holds.

Thus, for $\mathcal{C} = P \cup N$ as defined above, we obtain that

$$L(\mathcal{C}, d_{\alpha_\delta}) = L(\mathcal{C}, d_{\alpha_{\delta'}}) + \mathcal{O}(\delta' - \delta). \tag{5.16}$$

5.6.1 First step in the proof of Theorem 5.4: construction of a structure function

We have the following proposition

Proposition 5.7. *For $\eta > 0$ there is $\delta_\eta > 0$ s.t. for $\delta_\eta > \delta > 0$ there are $C_{\eta,\delta} > 0$, $E_{\eta,\delta} \subset \mathbb{R}$ and $\xi_{\eta,\delta} \in C^\infty(\mathbb{R}^3, \mathbb{R})$ s.t.*

1. $|\nabla \xi_{\eta,\delta}| \le \alpha_\delta$ *in* $\mathbb{R}^3$
2. $\sum_{i \in \mathbb{N}_k} \{\xi_{\eta,\delta}(p_i) - \xi_{\eta,\delta}(n_i)\} \ge L(\mathcal{C}, d_{\alpha_\delta}) - \eta$
3. $\mathscr{H}^1(E_{\eta,\delta}) \le \eta$ *and for all* $t \in \mathbb{R} \setminus E_{\eta,\delta}$, $\{\xi_{\eta,\delta} = t\}$ *is a closed two dimensional surface with its second fundamental form which is bounded by* $C_{\eta,\delta}$.

Proof. We construct $\xi_{\eta,\delta}$ in five steps.

Let $\eta > 0$ and $0 < \delta < \delta' < \delta_0$. We denote $\alpha = \alpha_\delta$ and $\alpha' = \alpha_{\delta'}$. Assume that $P = \{p_1, ..., p_k\}$ and $N = \{n_1, ..., n_k\}$ are s.t. $\sigma = \mathrm{Id}$ is a minimal connection in $(\mathcal{C}, d_{\alpha'})$.

Step 1: There is $\xi_0 : \mathcal{C} \to \mathbb{R}$ s.t. ξ_0 is 1-Lipschitz in $(\mathcal{C}, d_{\alpha'})$ and $\xi_0(p_i) - \xi_0(n_i) = d_{\alpha'}(p_i, n_i)$

This step is a direct consequence of Lemma 4.2 in [27] (see also Lemma 2.2 in [60] or Lemma 2 in [24]).

Step 2: We extend ξ_0 to $\mathbb{R}^3$: there is some $\xi_1 \in \mathrm{Lip}(\mathbb{R}^3, \mathbb{R})$ s.t. $|\nabla \xi_1| = \alpha'$ and $\xi_{1|\mathcal{C}} \equiv \xi_0$

Although the argument is the same as in [60], for the convenience of the reader, we recall the construction.

Consider

$$\xi_1(x) = \max_i \{\xi_0(p_i) - d_{\alpha'}(x, p_i)\}, \ x \in \mathbb{R}^3.$$

Then we have

- $\xi_{1|\mathcal{C}} \equiv \xi_0$: let $M \in \mathcal{C}$ and i be s.t. $M \in \{p_i, n_i\}$ and $j \ne i$, it is clear that

$$\xi_0(p_i) - d_{\alpha'}(M, p_i) - \xi_0(p_j) + d_{\alpha'}(M, p_j) = \begin{cases} \xi_0(p_i) - \xi_0(p_j) + d_{\alpha'}(p_i, p_j) \ge 0 & \text{if } M = p_i \\ \xi_0(n_i) - \xi_0(p_j) + d_{\alpha'}(n_i, p_j) \ge 0 & \text{if } M = n_i \end{cases}.$$

- $|\nabla \xi_1| = \alpha'$: for all i we have

$$|\nabla [\xi_0(p_i) - d_{\alpha'}(x, p_i)]| = |\nabla d_{\alpha'}(x, p_i)| = \alpha' \text{ in } L^\infty(\mathbb{R}^3).$$

Step 3: We construct a smooth approximation: $\xi_2 \in C^\infty(\mathbb{R}^3, \mathbb{R})$ is s.t. $|\nabla \xi_2| \le \lambda\alpha$ ($\lambda < 1$) and

$$\sum_{i \in \mathbb{N}_k} \{\xi_2(p_i) - \xi_2(n_i)\} \ge L(\mathcal{C}, d_\alpha) - \eta/2 \tag{5.17}$$

Let $\delta > \beta > 0$ and let $(\rho_t)_{\delta > t > 0}$ be a classical mollifier, namely $\rho_t(x) = t^{-3}\rho(x/t)$ with $\rho \in C^\infty(\mathbb{R}^3, [0,1])$, $\mathrm{Supp}\,\rho \subset B(0,1)$ and $\int_{\mathbb{R}^3} \rho = 1$.

Consider

$$\xi_2(x) := (1-\beta)\xi_1 * \rho_t(x).$$

Condition (5.17) is clearly satisfied when t and β are small. On the other hand, the point estimate $|\nabla\xi_2(x)| \leq (1-\beta)\|\nabla\xi_1\|_{L^\infty(B(x,t))}$ implies that $|\nabla\xi_2| \leq \lambda\alpha$ for appropriate $\lambda < 1$, provided t is sufficiently small.

Step 4: Let $\tilde{\Omega}$ be a neighborhood of $\overline{\Omega}$. We approximate ξ_2 by $\xi_{\eta,\delta}$ s.t. we have $\xi_{\eta,\delta} \in C^\infty(\mathbb{R}^3,\mathbb{R})$ and

$$\|\xi_{\eta,\delta} - \xi_2\|_{L^\infty(\tilde{\Omega})} \leq \eta/(4k),$$
$$|\nabla\xi_{\eta,\delta}| \leq \alpha,$$
$$\xi_{\eta,\delta} \text{ is a Morse function,}$$
$$\exists R = R(\eta,\delta) > 0 \text{ s.t. in } \mathbb{R}^3\setminus B(0,R),\ \xi_{\eta,\delta} = |x|/2$$

Clearly $\xi_{\eta,\delta}$ satisfies *1.* et *2.* of Proposition 5.7.

Step 5: We follow [60]. We construct $E_{\eta,\delta}$

Let $\{x_1,...,x_l\}$ be the set of the critical points of $\xi_{\eta,\delta}$. Then there is $C = C(\eta,\delta) > 0$ s.t.:

$$\inf_{B(0,R)\setminus\cup_i B(x_i,\rho)} |\nabla\xi_{\eta,\delta}| \geq \frac{\rho}{C} \text{ since the critical points are not degenerate}$$

and

$$\mathscr{H}^1\left[\xi_{\eta,\delta}(\cup_i B(x_i,\rho))\right] \leq C\rho^2.$$

We consider $\rho > 0$ s.t. $C\rho^2 \leq \eta$ and set $E_{\eta,\delta} = \xi_{\eta,\delta}(\cup_i B(x_i,\rho))$.

For $t \notin E_{\eta,\delta}$, we have

- if $x \in \{\xi_{\eta,\delta} = t\}\setminus B(0,R)$, then the second fundamental form of $\{\xi_{\eta,\delta} = t\}$ in x is bounded,
- if $x \in \{\xi_{\eta,\delta} = t\}\cap B(0,R)$, then the second form is bounded by $C_{\eta,\delta} = \dfrac{C\sup_{B(0,R)}|D^2\xi_{\eta,\delta}|}{\inf_{B(0,R)\setminus\cup_i B(x_i,\rho)}|\nabla\xi_{\eta,\delta}|}$.

We find that the second fundamental form is globally bounded. □

Our next result provides a sharper estimate on the gradient of structure functions.

Corollary 5.8. *For all $\eta > 0$, there is $C_\eta > 0$, $E_\eta \subset \mathbb{R}$, $\xi_\eta \in C^\infty(\mathbb{R}^3,\mathbb{R})$ and $\varepsilon_\eta > 0$ s.t. for $0 < \varepsilon < \varepsilon_\eta$,*

1. $|\nabla\xi_\eta| \leq \min(a^2, U_\varepsilon^2 + \varepsilon^4)$ *in* $\mathbb{R}^3$,
2. $\sum_{i\in\mathbb{N}_k}\{\xi_\eta(p_i) - \xi_\eta(n_i)\} \geq L(\mathcal{C}, d_{a^2}) - \eta$,
3. $\mathscr{H}^1(E_\eta) \leq \eta$ *and for all* $t \in \mathbb{R}\setminus E_\eta$, $\{\xi_\eta = t\}$ *is a closed hypersurface whose second fundamental form is bounded by* C_η.

Proof. Let $\eta > 0$ and fix $0 < \delta < \delta_\eta$ (δ_η given by Proposition 5.7) s.t.

$$L(\mathcal{C}, d_{\alpha_\delta}) + \frac{\eta}{2} \geq L(\mathcal{C}, d_{a^2}).$$

Consider $\varepsilon_\eta > 0$ s.t. for $0 < \varepsilon < \varepsilon_\eta$ we have

$$C\mathrm{e}^{-\gamma\delta/\varepsilon} < \varepsilon^4 \text{ (} C \text{ and } \gamma \text{ are given by (5.3)).}$$

We take $\xi_\eta = \xi_{\eta/2,\delta}$ obtained from Proposition 5.7.

Clearly, ξ_η satisfies *2.* and *3.* with $E_\eta = E_{\eta/2,\delta}$ and $C_\eta = C_{\eta/2,\delta}$.

It is direct to obtain that

$$|\nabla \xi_\eta| - U_\varepsilon^2 \leq \alpha_\delta - U_\varepsilon^2 \leq \begin{cases} b^2 - U_\varepsilon^2 \leq 0 & \text{if } \operatorname{dist}(x,\omega) < \delta \\ \varepsilon^4 & \text{otherwise} \end{cases}.$$

It follows that ξ_η satisfies *1* since $\alpha_\delta \leq a^2$. □

5.6.2 First step in the proof of Theorem 5.5: construction of a structure function

Definition and properties of a special pseudometric

Let $f : \mathbb{R}^3 \to [b^2, 1]$ be a Borel function and let $K \subset \mathbb{R}^3$ be a smooth compact set. We define

$$d_f^K(x,y) = \min\left\{d_f(x,y), d_f(x,K) + d_f(y,K)\right\}.$$

Here $d_f(x,K) = \min_{y\in K} d_f(x,y)$.

Then d_f^K is a pseudometric in $\mathbb{R}^3$. If, in addition $K \cap \mathcal{C} = \emptyset$, then d_f^K is a distance in $\mathcal{C}$. Therefore the minimal connection of $\mathcal{C}$ and the length of a minimal connection $L(\mathcal{C}, d_f^K)$ with respect to d_f^K make sense.

Clearly, if $x, y \in \mathbb{R}^3$, then we have $d_f^K(x,y) = 0 \Leftrightarrow x = y$ or $x, y \in K$. One may easily prove that

$$d_f^K(x,y) \leq d_f(x,y) \leq d_f^K(x,y) + \operatorname{diam}(K).$$

We are interested in the special case $K = \overline{B(x_0,r)}$ for some $x_0 \in \Omega$ and $f = \alpha_\delta$ with $\delta \in [0, \delta_0]$.

Note that we have a similar estimate to (5.16), namely for $0 \leq \delta < \delta' < \delta_0$

$$L(\mathcal{C}, d_{\alpha_\delta}^K) = L(\mathcal{C}, d_{\alpha_{\delta'}}^K) + \mathcal{O}(|\delta' - \delta|). \tag{5.18}$$

Definition. For $y \notin K$ and $x \in \mathbb{R}^3$, we say that

- Γ is a K-curve joining x, y if Γ is a finite union of curves included in $\mathbb{R}^3 \setminus K$ s.t. their endpoints are either x or y or an element of ∂K,
- Γ is a minimal K-curve joining x, y if $\Gamma = \cup_i \Gamma_i$ is a K-curve joining x, y, where the Γ_i's are disjoint curves and $\sum_i \operatorname{long}_{a^2}(\gamma_i) = d_{a^2}^K(x,y)$.

We next sum up the main properties of $d_{a^2}^K$.

Proposition 5.9. *Let $x_0 \in \mathbb{R}^3$, $r > 0$ and $K = \overline{B(x_0,r)}$. Then:*

1. *If $y \notin K$ then for all $x \in \mathbb{R}^3$ there is a minimal K-curve joining x, y. Moreover, a minimal K-curve is the union of at most two geodesics in $(\mathbb{R}^3, d_{a^2})$.*
2. *For $x_0, x, y \in \mathbb{R}^3$, $x \neq y$ and $x_0 \neq x, y$, we have:*
   - *i. If $x_0 \in \mathbb{R}^3 \setminus \partial\omega$ and x_0 is on a geodesic joining x, y in $(\mathbb{R}^3, d_{a^2})$, then there is $r_{x_0,x,y} > 0$ s.t. for all $r < r_{x_0,x,y}$, $d_{a^2}^K(x,y) = d_{a^2}(x,y) - 2a^2(x_0)r$,*
   - *ii. If $x_0 \in \partial\omega$ and x_0 is on a geodesic joining x, y in $(\mathbb{R}^3, d_{a^2})$, then there is $r_{x_0,x,y} > 0$ s.t. for all $r < r_{x_0,x,y}$, $d_{a^2}^K(x,y) = d_{a^2}(x,y) - (1 + b^2)r$,*
   - *iii. If x_0 is not on a geodesic joining x, y in $(\mathbb{R}^3, d_{a^2})$, then there is $r_{x_0,x,y} > 0$ s.t. for all $r < r_{x_0,x,y}$, $d_{a^2}^K(x,y) = d_{a^2}(x,y)$.*

Proof. We prove the first assertion. There are two cases to consider: $x \in K$ and $x \notin K$.

If $x \in K$ and $y \notin K$, then we have the existence of a unique point $y_0 \in K$ which minimizes $d_{a^2}(y,z)$ among the points $z \in K$. Clearly considering Γ a geodesic in $(\mathbb{R}^3, d_{a^2})$ joining y with y_0, by definition of y_0, $\Gamma \cap K = \emptyset$. Thus Γ is a minimal K-curve according to the definition given above.

If $x, y \notin K$, then we consider Γ a geodesic joining x, y in $(\mathbb{R}^3, d_{a^2})$ and, for $z \in \{x, y\}$, let Γ_z be a minimal curve in $(\mathbb{R}^3, d_{a^2})$ joining z with K.

If $\text{long}_{a^2}(\Gamma_x) + \text{long}_{a^2}(\Gamma_y) < \text{long}_{a^2}(\Gamma \setminus K) \le d_\alpha(x,y)$, then one may consider $\Gamma_x \cup \Gamma_z$ as a minimal K-curve. Indeed, in this situation, $d_{a^2}^K(x,y) < d_{a^2}(x,y)$ which implies that a minimizing sequence of K-curves $\tilde{\Gamma}_n$ satisfies for large n that $\tilde{\Gamma}_n$ contains curves with an endpoint on ∂K. More precisely, by definition, there are Γ_x^n, Γ_z^n two connected components of $\tilde{\Gamma}_n$ s.t. for $z \in \{x, y\}$, Γ_z^n has z and z'_n for endpoints with $z'_n \in \partial K$. Therefore

$$\text{long}_{a^2}(\Gamma_x) + \text{long}_{a^2}(\Gamma_y) \le \text{long}_{a^2}(\tilde{\Gamma}_n).$$

Otherwise, $\text{long}_{a^2}(\Gamma_x) + \text{long}_{a^2}(\Gamma_y) \ge d_{a^2}(x,y)$. Consequently, denoting Γ a geodesic in $(\mathbb{R}^3, d_{a^2})$ joining x with y, $\Gamma \setminus K$ is a K-curve and has a minimal length.

It remains to prove that Γ, a minimal K-curve, is a union of at most two geodesics in $(\mathbb{R}^3, d_{a^2})$. If Γ is connected, then, by the definition of a K-curve, $\Gamma \cap K = \emptyset$. Thus Γ is a geodesic joining x, y.

Otherwise, assume that Γ is not connected. By the definition of a K-curve and by the minimality of Γ, for $z \in \{x, y\}$, there are $z' \in \partial K$ and Γ_z a connected component of Γ s.t. z, z' are the endpoints of Γ_z. Thus, by minimality of Γ, Γ_z is a geodesic joining z, z' and $\Gamma = \Gamma_x \cup \Gamma_y$.

Now we prove the second assertion. First, we assume that $x_0 \notin \partial\omega$ and that x_0 is on a geodesic curve joining x, y in $(\mathbb{R}^3, d_{a^2})$.

Consider $r_{x_0,x,y} = 10^{-2} \min\{|x - x_0|, |y - x_0|, \text{dist}(x_0, \partial\omega)\}$. Then, for $r < r_{x_0,x,y}$, considering the K-curve $\Gamma \setminus K$ where Γ a geodesic joining x, y in $(\mathbb{R}^3, d_{a^2})$ and containing x_0, we obtain that

$$d_{a^2}^K(x,y) \le d_{a^2}(x,y) - 2a^2(x_0)r. \tag{5.19}$$

This comes from the fact that $\Gamma \cap K$ is a diameter of K and that this diameter is contained in the same connected component of $\mathbb{R}^3 \setminus \partial\omega$ as x_0. To obtain the reverse estimate, it suffices to consider Γ, a minimal K-curve joining x, y. From (5.19), we know that Γ as exactly two connected components: Γ_x, Γ_y with Γ_z has z, z' for endpoints with $z \in \{x, y\}$ and $z' \in \partial K$. Thus it suffices to complete Γ by the line segments $[x_0, x']$ and $[x_0, y']$ to obtain the reverse inequality. (Note that in this situation, $[x', y']$ is a diameter of K)

If $x_0 \in \partial\omega$, then the argument is similar taking $0 < r_{x_0,x,y} < 10^{-2}\min\{|x - x_0|, |y - x_0|\}$ sufficiently small s.t.:

- $B(x_0, r_{x_0,x,y}) \setminus \partial\omega$ has exactly two connected components,
- For all geodesic Γ joining x, y in $(\mathbb{R}^3, d_{a^2})$, if $x_0 \in \Gamma$ then $(\Gamma \cap K) \setminus \partial\omega$ has exactly two connected components: one in ω and the other in $\mathbb{R}^3 \setminus \overline{\omega}$.

Note that from Proposition 5.3, Assertion *4.d.*, $r_{x_0,x,y}$ is well defined.

Now we prove the last assertion arguing by contradiction. Assume that there is $r_n \downarrow 0$ s.t. denoting $K_n = \overline{B(x_0, r_n)}$, we have $d_{a^2}^{K_n}(x,y) < d_{a^2}(x,y)$. Consequently there are $x_n, y_n \in \partial K_n$ and $\Gamma_n = \Gamma_x^n \cup \Gamma_y^n$ where Γ_z^n is a geodesic joining z and z_n in $(\mathbb{R}^3, d_{a^2})$, $z \in \{x, y\}$. Consequently, for $z \in \{x, y\}$, one may complete Γ_z^n by the line segment $[z'_n, x_0]$ whose length in $(\mathbb{R}^3, d_{a^2})$ is at most r_n. We denote $\tilde{\Gamma}_z^n$ this curve. Clearly $d_{a^2}(z, x_0) \le \text{long}_{a^2}(\tilde{\Gamma}_z^n) \le \text{long}_{a^2}(\Gamma_z^n) + r_n$.

It suffices to claim that in a metric space (X, d) which admits geodesic curves we have for x_0, x, y three distinct points in X

$$x_0 \text{ is on a geodesic joining } x, y \Longleftrightarrow d(x,y) = d(x,x_0) + d(x_0,y).$$

Since x_0 is not on a geodesic curve joining x, y in $(\mathbb{R}^3, d_{a^2})$, there is $\eta > 0$ s.t. $d_{a^2}(x,y) + \eta < d_{a^2}(x,x_0) + d_{a^2}(x_0,y)$ and thus

$$\text{long}_{a^2}(\Gamma_x^n) + \text{long}_{a^2}(\Gamma_y^n) = d_{a^2}^{K_n}(x,y) < d_{a^2}(x,y) \le \text{long}_{a^2}(\Gamma_x^n) + \text{long}_{a^2}(\Gamma_y^n) + 2r_n - \eta.$$

Clearly we obtain a contradiction for n sufficiently large s.t. $r_n < \eta/2$. □

Let $x_0 \in \Omega$ and $\mathcal{C} \subset \partial\Omega$ as above. If for all minimal connexion σ of $\mathcal{C}$ and for $i \in \{1, ..., k\}$, we have that x_0 is not on a geodesic joining $p_i, n_{\sigma(i)}$ in $(\mathbb{R}^3, d_{a^2})$, then there is $r_{x_0,\mathcal{C}} > 0$ s.t. for all $r < r_{x_0,\mathcal{C}}$, we have

$$L(\mathcal{C}, d_{a^2}^K) = L(\mathcal{C}, d_{a^2}). \tag{5.20}$$

Construction of a structure function

Proposition 5.10. *Let $K = \overline{B(x_0, r)}$ be s.t. $\overline{B(x_0, 2r)} \subset \mathbb{R}^3 \setminus \mathcal{C}$ and $\eta > 0$. Then there is $\delta_{\eta,K} > 0$ s.t. for $0 < \delta < \delta_{\eta,K}$ there are $C_{\eta,K,\delta}$, $E_{\eta,K,\delta} \subset \mathbb{R}$ and $\xi_{\eta,K,\delta} \in C^\infty(\mathbb{R}^3, \mathbb{R})$ satisfying*

1. *$|\nabla \xi_{\eta,K,\delta}| \le \alpha_\delta$ in $\mathbb{R}^3$ and $\xi_{\eta,K,\delta}$ is constant in K,*
2. *$\sum_{i \in \mathbb{N}_k} \{\xi_{\eta,K,\delta}(p_i) - \xi_{\eta,K,\delta}(n_i)\} \ge L(\mathcal{C}, d_{\alpha_\delta}^K) - \eta$,*
3. *$\mathscr{H}^1(E_{\eta,K,\delta}) \le \eta$ and for $t \in \mathbb{R} \setminus E_{\eta,K,\delta}$, $\{\xi_{\eta,K,\delta} = t\}$ is a closed hypersurface whose second fundamental form is bounded by $C_{\eta,K,\delta}$.*

Proof. The main point is that we require that $\xi_{\eta,K,\delta}$ is constant in K. All the other requirements are satisfied by the map $\xi_{\eta,\delta}$ constructed in Proposition 5.7.

For $\delta < r/2$, let $K_1 = \overline{B(0, r + 2\delta)}$ and $K_2 = K + \overline{B(0, r + \delta)}$. We denote $\alpha = \alpha_\delta$ et $\alpha' = \alpha_{2\delta}$.

Step 1: As in the proof of Proposition 5.7, there is a function $\xi_0 : \mathcal{C} \to \mathbb{R}$, 1-Lipschitz function with respect to $d_{\alpha'}^{K_1}$ and s.t. $\xi_0(p_i) - \xi_0(n_i) = d_{\alpha'}^{K_1}(p_i, n_i)$.

Step 2: We extend ξ_0 to a map $\xi_1 : \mathbb{R}^3 \to \mathbb{R}$, 1-Lipschitz and constant in K_1

For example, we may take

$$\xi_1(x) = \max_i \xi_0(p_i) - d_{\alpha'}^{K_1}(x, p_i).$$

As in the proof of Proposition 5.7, $\xi_{1|\mathcal{C}} = \xi_0$ and $|\nabla \xi_1| \le \alpha'$. Moreover, ξ_1 is constant in K_1. Indeed, for all $x \in K_1$, we have $\xi_0(x) = \max_i \xi_0(p_i) - d_{\alpha'}^{K_1}(x, p_i) = \max_i \xi_0(p_i) - d_{\alpha'}(p_i, K_1)$.

Step 3: We approximate ξ_1 by $\xi_2 \in C^\infty(\mathbb{R}^3, \mathbb{R})$ satisfying $|\nabla \xi_2| \le \lambda\alpha$ $(\lambda < 1)$, $\sum_{i \in \mathbb{N}_k} \{\xi_2(p_i) - \xi_2(n_i)\} \ge L(\mathcal{C}, d_\alpha^K) - \eta/2$ (for δ sufficiently small), and s.t. ξ_2 is constant in K_2

The approximation ξ_2 is obtain (as in Proposition 5.7) by regularization using a mollifier and noting that

$$L(\mathcal{C}, d_\alpha^K) \ge L(\mathcal{C}, d_{\alpha'}^K) \ge L(\mathcal{C}, d_{\alpha'}^{K_2}) \ge L(\mathcal{C}, d_{\alpha'}^{K_1}) \ge L(\mathcal{C}, d_{\alpha'}^K) - \mathcal{O}(\delta) = L(\mathcal{C}, d_\alpha^K) - \mathcal{O}(\delta).$$

Step 4: Let $\tilde{\Omega}$ be a neighborhood of $\overline{\Omega}$. We approximate ξ_2 by ξ_3 where $\xi_3 \in C^\infty(\mathbb{R}^3, \mathbb{R})$ satisfies

$$\|\xi_3 - \xi_2\|_{L^\infty(\tilde{\Omega})} < \eta^2\delta^2,$$

$$|\nabla \xi_3| \leq \frac{1+\lambda}{2}\alpha,$$

$$\xi_3 \text{ is a Morse function,}$$

$$\exists R > 0 \text{ s.t. in } \mathbb{R}^3 \setminus B(0,R),\ \xi_3 = |x|/2.$$

Step 5: We modify ξ_3 in order to have $\xi_{\eta,K,\delta} \equiv C_0$ in K

By construction, there is $C_0 \in \xi_3(K)$ s.t. $\|\xi_3 - C_0\|_{L^\infty(K_2)} < \eta^2\delta^2$. Noting that $\text{dist}(\partial K_2, K) = \delta$, one may construct $\xi_{\eta,K,\delta} \in C^\infty(\mathbb{R}^3)$ s.t.

$$\begin{cases} \xi_{\eta,K,\delta} = \xi_3 \text{ in } \mathbb{R}^3 \setminus K_2,\ \xi_{\eta,K,\delta} \equiv C_0 \text{ in } K, \\ \|\xi_{\eta,K,\delta} - C_0\|_{L^\infty(K_2)} < \eta^2\delta^2 \text{ and } |\nabla \xi_{\eta,K,\delta}| \leq b^2 \text{ in } K_2. \end{cases}$$

Clearly $\xi_{\eta,K,\delta}$ satisfies *1.* and *2.* in Proposition 5.10.

Step 6: We construct $E_{\eta,K,\delta}$

For $\rho > 0$, we consider $E^1_{\eta,K,\delta} = \xi_{\eta,K,\delta}(\cup_i B(x_i,\rho))$ where $\{x_1, ..., x_l\}$ is the set of the critical points of $\xi_{\eta,K,\delta}$ in $B(0,R) \setminus K_2$.

For the same reasons as in Proposition 5.7, we have $\mathscr{H}^1(E^1_{\eta,K,\delta}) \leq C\rho$.

We also define $E^2_{\eta,K,\delta} = \xi_{\eta,K,\delta}(K_2)$. By construction, we have $\mathscr{H}^1(E^2_{\eta,K,\delta}) \leq 2\eta^2\delta^2$.

Thus it suffices to consider δ, ρ s.t. $C\rho + 2\eta^2\delta^2 \leq \eta$ and to set $E_{\eta,K,\delta} = E^1_{\eta,K,\delta} \cup E^2_{\eta,K,\delta}$. □

In the spirit of Corollary 5.8, we have

Corollary 5.11. *Let $x_0 \in \mathbb{R}^3$ be s.t. x_0 does not belong any minimal link of $\mathcal{C}$ in $(\mathbb{R}^3, d_{a^2})$. There is $r_{x_0} > 0$ s.t. denoting $K = \overline{B(x_0, r_{x_0})}$, for $\eta > 0$ there are $\xi_{\eta,K} \in C^\infty(\mathbb{R}^3, \mathbb{R})$, $E_{\eta,K} \subset \mathbb{R}$, $C_{\eta,K} > 0$ and $\varepsilon_{\eta,K} > 0$ s.t. for $0 < \varepsilon < \varepsilon_{\eta,K}$,*

1. $|\nabla \xi_{\eta,K}| \leq \min(a^2, U_\varepsilon^2 + \varepsilon^4)$ *in* $\mathbb{R}^3$
2. $\sum_{i \in \mathbb{N}_k} \{\xi_{\eta,K}(p_i) - \xi_{\eta,K}(n_i)\} \geq L(\mathcal{C}, d_{a^2}) - \eta$
3. $\mathscr{H}^1(E_{\eta,K}) \leq \eta$ *and for all* $t \in \mathbb{R} \setminus E_{\eta,K}$, $\{\xi_{\eta,K} = t\}$ *is a closed hypersurface whose second fundamental form is bounded by* $C_{\eta,K}$.

Proof. Assume that $\sigma = \text{Id}$ is a minimal connexion in $(\mathcal{C}, d_{a^2})$.

Let $r_{x_0,\mathcal{C}} > 0$ (given by Proposition 5.9, Assertion *2.iii*) be s.t. for $K = \overline{B(x_0, r_{x_0,\mathcal{C}}/2)}$, we have $d^K_{a^2}(p_i, n_i) = d_{a^2}(p_i, n_i)$. Consequently, $L(\mathcal{C}, d_{a^2}) = L(\mathcal{C}, d^K_{a^2})$.

Now we apply Proposition 5.10: there is $\delta_{\eta/2,K} > 0$ s.t. for $0 < \delta < \delta_{\eta/2,K}$, there are $C_{\eta/2,K,\delta} > 0$, $E_{\eta/2,K,\delta} \subset \mathbb{R}$ and $\xi_{\eta/2,K,\delta} \in C^\infty(\mathbb{R}^3, \mathbb{R})$ satisfying the conclusions of Proposition 5.10.

From (5.20), one may fix $0 < \delta < \delta_{\eta/2,K}$ s.t.

$$L(\mathcal{C}, d_{a^2}) - L(\mathcal{C}, d^K_{\alpha_\delta}) < \eta/2.$$

Consequently, considering $\varepsilon_{\eta,K} > 0$ s.t. for $0 < \varepsilon < \varepsilon_{\eta,K}$, we have $Ce^{-\gamma\delta/\varepsilon} < \varepsilon^4$ (C and γ are given by (5.3)).

We obtain the result taking $C_{\eta,K} = C_{\eta/2,K,\delta}$, $E_{\eta,K} = E_{\eta/2,K,\delta}$ and $\xi_{\eta,K} = \xi_{\eta/2,K,\delta}$. □

5.6.3 A structure function in presence of symmetries

In this section we assume that $\Omega = B(0,1)$ and that $\omega = B(0,r_0)$, with $r_0 \in (0,1)$.

Consider $\mathcal{C} = \{(1,0,0),(-1,0,0)\} = \{p,n\}$, $p = (1,0,0)$. It is clear that in this situation, the line segment $[p,n]$ is the unique geodesic between p and n in $(\mathbb{R}^3, d_{a^2})$.

The main result of this section is

Proposition 5.12. *Let $M \in \Omega \setminus [p,n]$. Then there is $\mathcal{V}$, an open neighbourhood of M s.t. for $\varepsilon > 0$, there is $\xi_\varepsilon : \mathbb{R}^3 \to \mathbb{R}$ a Lipschitz function s.t.*

1. $\xi_\varepsilon(p) - \xi_\varepsilon(n) = d_{U_\varepsilon^2}(p,n)$,
2. $|\nabla \xi_\varepsilon| \le U_\varepsilon^2$,
3. $\xi_\varepsilon \equiv 0$ *in* $\mathcal{V}$,
4. $\forall\, t \in \xi_\varepsilon(\mathbb{R}^3) \setminus \{0, \xi_\varepsilon(p), \xi_\varepsilon(n)\}$, $\{\xi_\varepsilon = t\}$ *is a sphere whose radius is at least* 1.

Using the spherical symmetry of Ω, ω and the minimality of U_ε, one may easily prove the following proposition.

Proposition 5.13. *The unique minimizer U_ε of E_ε in H_1^1, is radially symmetric and non decreasing.*

Proposition 5.12 is a particular case of the following lemma.

Lemma 5.14. *[The dumbbell lemma]*

Let $U : \mathbb{R}^3 \to [b,1]$ be a radially symmetric and non decreasing Borel function. Fix $p,n \in \mathbb{S}^2$, $p = -n$ and let $M \in \Omega \setminus [p,n]$.

Then there are $\xi : \mathbb{R}^3 \to \mathbb{R}$ and B^+, B^- two distinct open balls, B^+, B^- are exteriorly tangent and independent of U s.t.

1. $\xi(p) - \xi(n) = d_{U^2}(p,n)$,
2. $|\nabla \xi| \le U^2$,
3. $\xi \equiv 0$ *in* $\mathcal{V} := \mathbb{R}^3 \setminus (B^+ \cup B^-)$,
4. $M \in T$ *with T which is the common tangent plan of B^+ and B^-,*
5. B^+ *is centered in* $2p$, B^- *is centered in* $2n$,
6. *denoting $\tilde{B}^+$ (resp. $\tilde{B}^-$) the ball centered in $2p$ (resp. $2n$) with radius 1, ξ is locally constant in $\tilde{B}^+ \cup \tilde{B}^-$,*
7. $\forall t \in \xi(\mathbb{R}^3) \setminus \{0, \xi(p), \xi(n)\}$, $\{\xi = t\}$ *is a sphere centered in $2p$ or $2n$.*

Using the symmetry of the situation, the function ξ is represented in the Figure 5.2.

Proof. Let $p,n \in \partial\Omega$, $p = -n$ and $\{0,(e_1,e_2,e_3)\}$ an orthonormal and direct coordinate system of $\mathbb{R}^3$ s.t. $p = (1,0,0)$ et $n = (-1,0,0)$. Let $M(x_0,y_0,z_0) \in \Omega \setminus [p,n]$.

Step 1: We construct $\xi_0 : [-1,1] \to \mathbb{R}$ s.t. $\xi_0(1) - \xi_0(-1) = d_{U^2}(p,n)$, ${\xi_0}'(s) = U^2(s,0,0)$ and $\xi_0(x_0) = 0$

It suffices to consider

$$\xi_0(s) = \int_{x_0}^{s} U^2(t,0,0)\,\mathrm{d}t.$$

Step 2: We construct $\xi : \mathbb{R}^3 \to \mathbb{R}$

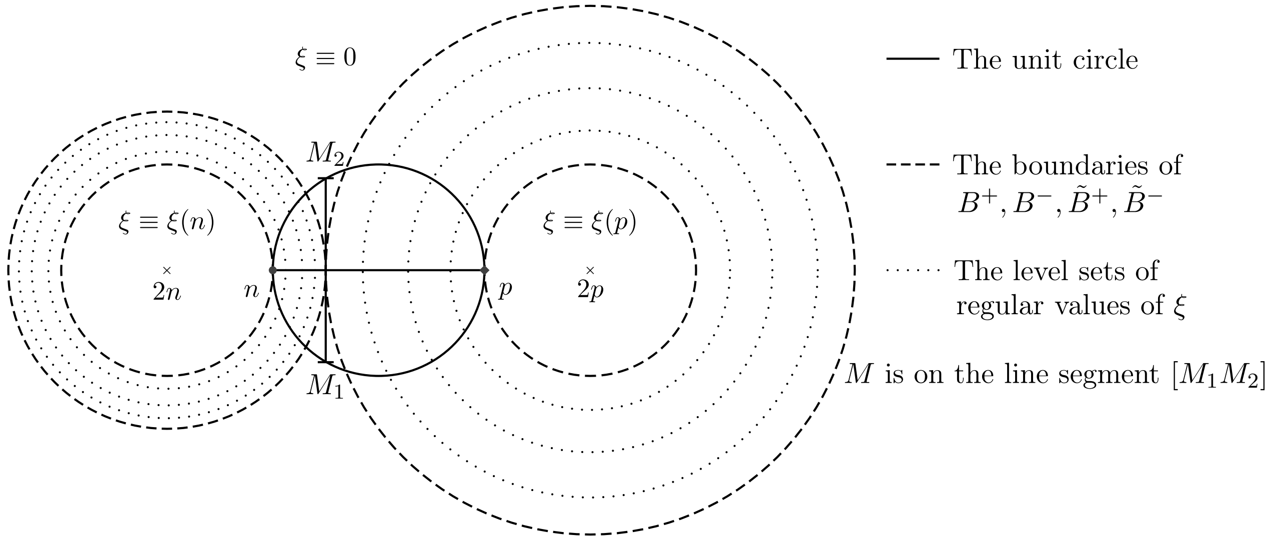


Figure 5.2: The geometry of the level sets of ξ (intersected with the plane defined by p, n, M)

We denote

$$\Sigma_r^+ = \partial B((2,0,0), r) \text{ for } r \in (1, 2 - x_0)$$

and

$$\Sigma_r^- = \partial B((-2,0,0), r) \text{ for } r \in (1, 2 + x_0).$$

We define $\xi : \mathbb{R}^3 \to \mathbb{R}$ by its level sets:

$$\xi = \begin{cases} \xi_0(2-r) & \text{on } \Sigma_r^+,\ r \in (1, 2-x_0) \\ \xi_0(r-2) & \text{on } \Sigma_r^-,\ r \in (1, 2+x_0) \\ \xi_0(-1) & \text{in } \overline{B((-2,0,0),1)} \\ \xi_0(1) & \text{in } \overline{B((2,0,0),1)} \\ 0 & \text{otherwise} \end{cases}.$$

Step 3: ξ satisfies the properties of Lemma 5.14

The assertion *1.* is easily satisfied since $\xi(p) = \xi_0(1)$, $\xi(n) = \xi_0(-1)$ and $\xi_0(1) - \xi_0(-1) = d_{U^2}(p, n)$.

We take $B^+ = B((2,0,0), 2 - x_0)$ et $B^- = B((-2,0,0), 2 + x_0)$.

Clearly the assertions *3.*, *4.*, *5.*, *6.* and *7.* hold.

We check *2.*. Since ξ is locally constant in

$$V := [\mathbb{R}^3 \setminus (B^+ \cup B^-)] \cup \overline{\tilde{B}^+ \cup \tilde{B}^-},$$

it suffices to prove that $|\nabla \xi| \le U^2$ in $\mathbb{R}^3 \setminus V$.

The key argument is the fact that for $Q, Q' \in \mathbb{R}^3$, $Q \ne Q'$ and $0 < r < |Q - Q'|$ we have $\text{dist}(Q, \partial B(Q', r)) = |Q - Q'| - r = |Q - Q_0|$ where $[Q, Q'] \cap \partial B(Q', r) = \{Q_0\}$. This is obvious if we draw a picture and may be easily justified. Indeed, if Q_0 is a minimal point, then line segment $[Q, Q_0]$ is orthogonal to $\partial B(Q', r)$. Only two points on $\partial B(Q', r)$ satisfy this condition and one of them is clearly not minimal.

Consequently, taking $Q = 0$ and $Q' \in \{2p, 2n\}$ we have that

$$\min_{Q_0 \in \Sigma_r^{\pm}} |Q_0| = |(\pm(2-r), 0, 0)|.$$

Note that U is radially symmetric and non decreasing. Since in each connected components of

$$(B^+ \cup B^-) \setminus \overline{\tilde{B^+} \cup \tilde{B^-}},$$

ξ admits a spherical symmetry, we have

$$\begin{aligned} |\nabla \xi(x)| &= \begin{cases} |\xi_0{}'(2-r)| = U^2(2-r,0,0) = \min_{\Sigma_r^+} U^2 & \text{if } x \in \Sigma_r^+ \\ |\xi_0{}'(r-2)| = U^2(r-2,0,0) = \min_{\Sigma_r^-} U^2 & \text{if } x \in \Sigma_r^- \end{cases} \\ &\le U^2(x). \end{aligned}$$

□

5.7 Lower bound for $F_\varepsilon(v_\varepsilon)$ when $g \in \mathcal{H}$: the argument of Sandier

In the computation of a sharp lower bound for $F_\varepsilon(v_\varepsilon)$, one of the main ingredients is Proposition 3.5 in [60]. For the convenience of the reader, we recall this result.

Proposition 5.15. *Let $\tilde{\Sigma}$ be a closed and oriented hypersurface in $\mathbb{R}^3$ whose second fundamental form is bounded by K. We denote by $d(\cdot,\cdot)$ the Euclidean distance restricted to $\tilde{\Sigma}$.*

Consider $\Sigma \subset \tilde{\Sigma}$, a bounded open set and $v : \Sigma \to \mathbb{C}$ s.t. there is $0 < \alpha < 1$ satisfying

$$\mathrm{dist}(x, \partial\Sigma) < \alpha \Rightarrow |v(x)| \ge 1/2.$$

Then we have the existence of $C > 0$ depending only on K and $\deg(v, \partial\Sigma)$ s.t.

$$\frac{1}{2}\int_\Sigma \left\{ |\nabla v|^2 + \frac{1}{2\varepsilon^2}(1-|v|^2)^2 \right\} \ge \pi|\deg(v,\partial\Sigma)| \ln\frac{\alpha}{\varepsilon} - C.$$

This section is devoted to the proof the following propostion.

Proposition 5.16. *Let $g \in \mathcal{H}$ and $\mathcal{C} = P \cup N$ the set of its singularity.*

1) We have

$$\liminf_{\varepsilon \to 0} \frac{F_\varepsilon(v_\varepsilon)}{|\ln \varepsilon|} \ge \pi L(g, d_{a^2}). \tag{5.21}$$

2) We denote $< \Gamma >$ the union of all minimal links of $\mathcal{C}$ in $(\mathbb{R}^3, d_{a^2})$ and for $\mu > 0$, $K_\mu := \{x \in \Omega \,|\, \mathrm{dist}(x, < \Gamma >) \ge \mu\}$. Then we have

$$\liminf_{\varepsilon \to 0} \frac{F_\varepsilon(v_\varepsilon, \Omega \setminus K_\mu)}{|\ln \varepsilon|} \ge \pi L(g, d_{a^2}). \tag{5.22}$$

3) Moreover, if we are in the symmetric case of Section 5.6.3, then there is $C_\mu > 0$ s.t.

$$F_\varepsilon(v_\varepsilon, \Omega \setminus K_\mu) \ge \pi d_{a^2}(p,n)|\ln \varepsilon| - C_\mu. \tag{5.23}$$

Theorem 5.4 for $g \in \mathcal{H}$, as well as Theorems 5.5, 5.6, are straightforward consequences of Proposition 5.16 combined with the upper bounds (5.10), (5.11).

We prove in detail (5.21), and we will sketch the proofs of (5.22),(5.23) which are, as explain in [60], obtained exactly in the same way as (5.21).

We prove that for all $\tilde{\eta} := \eta(8k^2 + 3k + 1) > 0$, the following holds

$$\liminf_{\varepsilon \to 0} \frac{F_\varepsilon(v_\varepsilon)}{|\ln \varepsilon|} \geq \pi L(g, d_{a^2}) - \tilde{\eta}. \tag{5.24}$$

Let $\eta > 0$, $\varepsilon_n \downarrow 0$, let $(v_n)_n \subset H^1_g$ be a sequence of minimizers of F_{ε_n} in H^1_g and let ξ_η, C_η, E_η be given by Corollary 5.8 (for n sufficiently large).

Let $0 < \rho < \eta$ and set

$$\Omega_\rho := \{x \in \mathbb{R}^3 \,|\, \mathrm{dist}(x, \Omega) < \rho \text{ et } \mathrm{dist}(x, \mathcal{C}) > \rho\}.$$

One may assume that ρ is sufficiently small s.t. in $\Omega_\rho \setminus \Omega$, $\Pi_{\partial\Omega}$, the orthogonal projection on $\partial\Omega$, is well defined and smooth.

Then we extend v_n (we use the same notation for the extension) by letting

$$v_n : \Omega_\rho \to \mathbb{R}^2,\ x \mapsto \begin{cases} v_n(x) & \text{if } x \in \Omega \\ g(\Pi_{\partial\Omega}(x)) & \text{if } x \in \Omega_\rho \setminus \Omega \end{cases}.$$

Since $g \in \mathcal{H}$ and $v_{n|\Omega_\rho \setminus \Omega}$ does not depend on n and takes its values in $\mathbb{S}^1$, we obtain the existence of $C(\rho)$ depending only on ρ, Ω, g s.t.

$$F_{\varepsilon_n}(v_n, \Omega) \geq F_{\varepsilon_n}(v_n, \Omega_\rho) - C(\rho)$$

If we define $F = F_{\eta,\rho} := E_\eta \cup [\xi_\eta(\mathcal{C}) - 2\rho, \xi_\eta(\mathcal{C}) + 2\rho]$, then we have

$$\mathscr{H}^1(F) \leq 8k\rho + \eta \leq (8k+1)\eta.$$

If $t \in \mathbb{R} \setminus F$, we denote by $\tilde{\Sigma}_t = \{\xi_\eta = t\}$. We construct for almost all $t \in \mathbb{R} \setminus F$ a closed submanifold $\Sigma_t \subset \tilde{\Sigma}_t$.

Note that for $t \in \mathbb{R} \setminus F$, we have $\mathrm{dist}(t, \xi_\eta(\mathcal{C})) \geq 2\rho$. Consequently, for $t \in \mathbb{R} \setminus F$, we obtain that $\tilde{\Sigma}_t \cap \{\Omega + B(0,\rho)\} = \tilde{\Sigma}_t \cap \Omega_\rho$.

Since $t \in \mathbb{R} \setminus F$ is not a critical value of ξ_η, the connected components W's of $\tilde{\Sigma}_t = \partial\{\xi_\eta \geq t\} = \{\xi_\eta = t\}$ have no boundary. If such W intersects Ω_ρ, then we distinguish two cases:

a) $W \cap \partial\Omega_\rho = \emptyset$

b) $W \cap \partial\Omega_\rho \neq \emptyset$.

Denote by W_a, resp. W_b, the set of the connected components satisfying a), resp. b).

If $W_b = \emptyset$, then we define $\Sigma_t = \tilde{\Sigma}_t \cap \Omega_\rho = \{\xi_\eta = t\} \cap \Omega_\rho$.

Thus it remains to construct Σ_t when $W_b \neq \emptyset$. Consider

$$\begin{array}{rccc} f: & \Omega + B(0,\rho) & \to & \mathbb{R}^2 \\ & x & \mapsto & (\xi_\eta(x), \mathrm{dist}[x, \partial(\Omega + B(0,\rho))]) \end{array}.$$

Using the Constant Rank Theorem (see Theorem 4.3.2, page 91 in [34]), the set $f^{-1}(\{t\} \times [r, \infty))$ $(r \in (0, \rho/2))$ is a manifold with boundary when

- t is a regular value of ξ_η,
- (t, r) is a regular value of f.

Thus, using Sard's Lemma, for almost all $t \in \mathbb{R} \setminus F$ s.t. $W_b \neq \emptyset$, there is $r = r(t) \in (0, \rho/2)$ s.t. $\Sigma_t = f^{-1}(\{t\} \times [r, \infty]) \subset \tilde{\Sigma_t}$ is a closed submanifold with boundary. Moreover, we have $\partial\Sigma_t \subset \partial\{\Omega + B(0, \rho - r)\} \cap \Omega_\rho$.

We denote by G the set

$$G := \{t \in \mathbb{R} \setminus F \,|\, W_b = \emptyset \text{ or } W_b \neq \emptyset \text{ and } \Sigma_t = f^{-1}(\{t\} \times [r, \infty)) \text{ with } r \in (0, \rho/2)\}.$$

For $t \in G$ we have

$$\operatorname{dist}(\partial\Sigma_t, \Omega) \geq \rho/2. \tag{5.25}$$

Let $x \in \Sigma_t$ be s.t. $\operatorname{dist}(x, \partial\Sigma_t) < \rho/2$. Using (5.25), we have $x \in \Omega_\rho \setminus \Omega$ and therefore $|v_n(x)| = 1$.

Finally, we are in position to apply Proposition 5.15 :

$$\frac{1}{2}\int_{\Sigma_t} \left\{ |\nabla v_n|^2 + \frac{b^2}{2\varepsilon_n^2}(1 - |v_n|^2)^2 \right\} \geq \pi |\deg(v_n, \partial\Sigma_t)| \ln \frac{\rho}{\varepsilon_n} - C(\deg(v_n, \partial\Sigma_t)). \tag{5.26}$$

For $M \in \mathcal{C}$ and for $t \in G$ we denote $M^t \in \partial(\Omega + B(\rho - r(t))$ s.t. $\Pi_{\partial\Omega}(M^t) = M$. Here we set $r(t) = 0$ when $W_b = \emptyset$, *i.e.*, when $\Sigma_t = \tilde{\Sigma}_t \cap \Omega_\rho$. It is clear that M^t is uniquely defined.

Since $d(n, t) = \deg(v_n, \partial\Sigma_t) = \operatorname{Card}(\{p_i^t \in \{\xi_\eta \geq t\}\}) - \operatorname{Card}(\{n_i^t \in \{\xi_\eta \geq t\}\})$ takes at most $2k$ values, one may assume that $C(\deg(v_n, \partial\Sigma_t))$ is uniformly bounded in n and t. Note that $d(n, t)$ is defined for almost all t.

The key argument in this proof is the way to pass from lower bounds on hypersurfaces to a lower bound in Ω. We have the following lemma.

Lemma 5.17. *The following lower bound holds*

$$\int_{\mathbb{R}} d(n, t)\, \mathrm{d}t \geq L(g, d_{a^2}) - \eta(2k + 1).$$

Proof. Let $m = \inf_{\Omega_\rho} \xi_\eta$, then we have

$$\begin{aligned}
\int_{\mathbb{R}} d(n, t) &= \int_{\mathbb{R}} |\{p_i^t \in \{\xi_\eta \geq t\}\}| - |\{n_i^t \in \{\xi_\eta \geq t\}\}| \\
&\geq \int_{\mathbb{R}} |\{p_i \in \{\xi_\eta \geq t + \rho\}\}| - |\{n_i \in \{\xi_\eta \geq t - \rho\}\}| \\
&\geq \sum_{i=1}^{k} \int_m^\infty \{\mathbb{I}_{\xi_\eta(p_i) > t + \rho} - \mathbb{I}_{\xi_\eta(n_i) > t - \rho}\} \\
&\geq \sum_{i=1}^{k} \{\xi_\eta(p_i) - \xi_\eta(n_i)\} - 2k\rho \geq L(g, d_{a^2}) - \eta(2k + 1).
\end{aligned}$$

□

With the help of Lemma 5.17, we have

$$\begin{aligned}
F_{\varepsilon_n}(v_n, \Omega_\rho) \geq (\text{Corollary 5.8}) &\geq \frac{1}{2}\int_\Omega (|\nabla\xi_\eta| - \varepsilon_n^4) \left[|\nabla v_n|^2 + \frac{U_{\varepsilon_n}^2}{2\varepsilon_n^2}(1 - |v_n|^2)^2 \right] \\
\geq (5.11) &\geq \frac{1}{2}\int_{\mathbb{R}\setminus F} \frac{1}{2}\int_{\Sigma_t} \left\{ |\nabla v_n|^2 + \frac{b^2}{2\varepsilon_n^2}(1 - |v_n|^2)^2 \right\} \mathrm{d}t - C_0 \\
\geq (5.26) &\geq \pi(\ln \frac{\rho}{\varepsilon_n} - C) \int_{\mathbb{R}\setminus F} |d(t)| - C_0 \\
\geq (\text{Lemma 5.17}) &\geq \pi(\ln \frac{\rho}{\varepsilon_n} - C) \left[L(g, d_{a^2}) - \eta(2k + 1) - k\mathscr{H}^1(F) \right] - C \\
&\geq \pi |\ln \varepsilon_n| \left[L(g, d_{a^2}) - \eta(8k^2 + 3k + 1) \right] - C.
\end{aligned}$$

It follows that

$$\liminf_n \frac{F_{\varepsilon_n}(v_n,\Omega)}{|\ln \varepsilon_n|} \geq \pi L(g, d_{a^2}) - \eta(8k^2+3k+1),\ \forall\, \eta > 0.$$

Proposition 5.16.1) is obtained by letting $\eta \to 0$ in the above estimate.

We now briefly sketch the arguments leading to (5.22) and (5.23). The fundamental ingredient is a lower bound for $F_\varepsilon(v_\varepsilon, \Omega \setminus K_\mu)$. Without loss of generality, we may assume that $K_\mu = K = \overline{B(x, r_x)}$ for some x which does not belong to a geodesic link between the singularities of g; here, $r_x > 0$ is some small number.

In order to prove (5.22), we use Corollary 5.11.

Following the same lines of proof of lower bound as in Proposition 5.16.1), we find that

$$\liminf_n \frac{F_{\varepsilon_n}(v_n,\Omega\setminus K)}{|\ln \varepsilon_n|} \geq \pi L(g, d_{a^2}).$$

Combining this lower bound with (5.11), we obtain

$$F_{\varepsilon_n}(v_n, K) = o(|\ln \varepsilon_n|). \tag{5.27}$$

In the symmetric case, using Proposition 5.12, we obtain the existence of r_x s.t., with $K = \overline{B(x, r_x)}$, we have

$$F_{\varepsilon_n}(v_n, \Omega \setminus K) \geq \pi d_{a^2}(p,n)|\ln \varepsilon| - C_K.$$

Consequently from the upper bound (5.10), we deduce

$$F_{\varepsilon_n}(v_n, K) \leq C'_K.$$

5.8 Extension by density of Theorem 5.4

From (5.22) and (5.11), we obtain that Theorem 5.4 holds for $g \in \mathcal{H}$. This section is devoted to the extension of Theorem 5.4 to the general case $g \in H^{1/2}(\partial\Omega, \mathbb{S}^1)$.

For $g \in H^{1/2}(\partial\Omega, \mathbb{S}^1)$, we denote

$$f_{\varepsilon,g} = \min_{v \in H^1_g} F_\varepsilon(v).$$

Using exactly the same argument as in [22], we have

Proposition 5.18. *1. Let $\delta \in (0,1)$ Then there is $C(\delta) > 0$ s.t. for $g_1, g_2 \in H^{1/2}(\partial\Omega, \mathbb{S}^1)$, we have ((5.1),(5.2) in [22])*

$$(1-\delta) f_{\varepsilon,g_1} - C(\delta) f_{\varepsilon,g_2} \leq f_{\varepsilon,g_1 g_2} \leq (1+\delta) f_{\varepsilon,g_1} + C(\delta) f_{\varepsilon,g_2}. \tag{5.28}$$

2. There is $C > 0$ depending only on Ω s.t. for $g \in H^{1/2}(\partial\Omega, \mathbb{S}^1)$ we have ((5.4) in [22])

$$f_{\varepsilon,g} \leq C|g|^2_{H^{1/2}(\partial\Omega)}(1 + |\ln \varepsilon|). \tag{5.29}$$

3. If $(g_n)_n \subset \mathcal{H}$ is s.t. $g_n \to g$ in $H^{1/2}(\partial\Omega)$ then Lemma 17 in [22] applied with $u_n = g_n/g$ and $v = g$ yields

$$\left|\frac{g_n}{g}\right|_{H^{1/2}(\partial\Omega)} \to 0. \tag{5.30}$$

4. There is $C>0$ depending only on Ω and on a s.t. for $g_1, g_2 \in H^{1/2}(\partial\Omega,\mathbb{S}^1)$ we have ((2.6) in [22])

$$|L(g_1,d_{a^2})-L(g_2,d_{a^2})|\leq C|g_1-g_2|_{H^{1/2}(\partial\Omega)}\left(|g_1|_{H^{1/2}(\partial\Omega)}+|g_2|_{H^{1/2}(\partial\Omega)}\right). \quad (5.31)$$

Using this proposition, Theorem 5.4 is proved as follows.

Let $g\in H^{1/2}(\partial\Omega,\mathbb{S}^1)$. By Proposition 5.2 (the third assertion), there is $(g_n)_n\subset\mathcal{H}$ s.t. $g_n\to g$ in $H^{1/2}(\partial\Omega)$.

Let $\varepsilon\in(0,1)$ and $\delta>0$. Then, by (5.28), we have

$$(1-\delta)\frac{f_{\varepsilon,g_n}}{|\ln\varepsilon|}-C(\delta)\frac{f_{\varepsilon,g/g_n}}{|\ln\varepsilon|}\leq\frac{f_{\varepsilon,g}}{|\ln\varepsilon|}\leq(1+\delta)\frac{f_{\varepsilon,g_n}}{|\ln\varepsilon|}+C(\delta)\frac{f_{\varepsilon,g/g_n}}{|\ln\varepsilon|}.$$

From (5.29) and the fact that Theorem 5.4 holds for g_n, we have

$$\begin{aligned}(1-\delta)\pi L(g_n,d_{a^2})-C'(\delta)|g/g_n|_{H^{1/2}} &\leq \liminf_{\varepsilon}\frac{f_{\varepsilon,g}}{|\ln\varepsilon|}\\ &\leq \limsup_{\varepsilon}\frac{f_{\varepsilon,g}}{|\ln\varepsilon|}\\ &\leq (1-\delta)\pi L(g_n,d_{a^2})+C'(\delta)|g/g_n|_{H^{1/2}}. \end{aligned}\quad(5.32)$$

Using (5.31), we obtain that $L(g_n,d_{a^2})\to L(g,d_{a^2})$. If, in (5.32), we first let $n\to\infty$, we use (5.30) and we next let $\delta\to0$, we obtain that

$$\lim_{\varepsilon}\frac{f_{\varepsilon,g}}{|\ln\varepsilon|}=\pi L(g,d_{a^2}).$$

The proof of Theorem 5.4 is complete.

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Résumé

Cette thèse est consacrée à l'étude mathématique de quelques modèles suggérés par la théorie de la supraconductivité. Plus spécifiquement, nous étudions le modèle simplifié (sans champ magnétique) en présence de condition de type Dirichlet ou du type degrés prescrits.

Dans une première partie nous traitons le problème d'existence de minimiseurs locaux dans un domaine multiplement connexe du plan pour des conditions de type degrés prescrits: pour toute combinaison de degrés, il existe toujours des minimiseurs locaux parmi les applications avec les degrés prescrits. Ce travail est une généralisation de travaux de Berlyand et Rybalko qui ont démontrés le même résultat pour un domaine de type annulaire.

La deuxième partie traite l'effet d'un terme de chevillage dans l'énergie de Ginzburg-Landau bi-dimensionnelle en imposant une condition de type Dirichlet. Cette partie se décompose en trois chapitres.

On commence par l'étude d'un terme de chevillage (dépendant du paramètre de Ginzburg-Landau) qui est étagé et prend une valeur différente de 1 uniquement en un nombre fixe de sous domaines (aussi appelés inclusions) dont la taille tend vers zéro. On montre alors qu'en considérant une donnée de type Dirichlet avec un degré non nul, la vorticité est quantifiée et localisée dans les inclusions. On exhibe une énergie renormalisée régissant la localisation de la vorticité de manière macroscopique (sous l'influence de la donnée au bord) et de manière microscopique.

Dans le chapitre suivant, nous considérons le cas d'un terme de chevillage sans hypothèse de structure particulière dans le cas où la donnée au bord est de degré nul. Par un argument de Lassoued et Mironescu, l'étude de la fonctionnelle de Ginzburg-Landau avec un terme de chevillage se ramène à celle d'une fonctionnelle de type Ginzburg-Landau à poids. Une fois cette réduction faite, nous démontrons que, sous des conditions raisonnables et peu restrictives portant sur les poids, les modules des minimiseurs (globaux) d'une fonction de type Ginzburg-Landau avec poids convergent uniformément vers 1.

On applique ensuite ce résultat à un terme de chevillage étagé et uniformément distribué. A l'aide de techniques classiques d'homogénéisation, nous mettons en évidence différentes asymptotiques des minimiseurs suivant la relation entre le paramètre de Ginzburg-Landau et la taille de la période du terme de chevillage.

Dans le dernier chapitre de la deuxième partie, nous traitons le cas d'un terme de chevillage étagé et uniformément distribué avec une condition de type Dirichlet de degré non nul. On montre que la vorticité est quantifiée et localisée dans les inclusions. On prouve que l'emplacement macroscopique de la vorticité est régi, à l'instar des travaux de Bethuel, Brezis et Hélein, par un problème auxiliaire de minimisation d'une fonctionnelle de Dirichlet à poids dans un domaine perforé parmi des applications unimodulaires avec condition de Dirichlet. On démontre aussi que si nous rajoutons un paramètre de dilution pour les inclusions alors la position microscopique des zéros dans les inclusions est indépendante de la donnée au bord.

La dernière partie s'intéresse à l'effet d'un terme de chevillage étagé (indépendant du paramètre de Ginzburg-Landau) dans un domaine tridimensionnel avec une condition de Dirichlet. Les résultats préliminaires que nous présentons permettent d'appréhender la manière dont les filaments de vorticité sont "tordus" par l'effet du terme de chevillage.

Abstract

This thesis is devoted to the mathematical study of some models suggested by the theory of the superconductivity. More specifically, we consider the simplified model of Ginzburg-Landau (without magnetic field) in presence of a Dirichlet or a degree condition.

In the first part we treat the existence problem of local minimizers in a multiply connected domain of the plan with prescribed degrees conditions: for each configuration of degrees, there exist local minimizers among maps with the prescribed degrees. This work is a generalization of the work of Berlyand and Rybalko who proved the same result for annular type domains.

In the second part, we discuss the effect of a pinning term in the two-dimensional Ginzburg-Landau functional. This part is divided in three chapters.

We first consider the situation of a pinning term (depending on the Ginzburg-Landau parameter) which is a simple function and takes a value different to 1 only in a fixed number of subdomains (also called inclusions) whose size tends to zero. We prove that, considering a Dirichlet condition with a non zero degree, the vorticity is quantized and localized inside the inclusions. We exhibit a renormalized energy which governs the macroscopic position of the vorticity (with an effect of the boundary data) and another renormalized energy which controls the location of the vortices inside the inclusions.

In the second chapter, we consider the situation of a pinning term without specific structure. We imposed a Dirichlet boundary condition with a null degree. By an argument of Lassoued and Mironescu, the study of the pinned Ginzburg-Landau functional is related to the one of a weighted Ginzburg-Landau functional. Once this reduction is done, we prove that under mild assumptions on the weights, the modulus of the (global) minimizer of weighted Ginzburg-Landau functionals converges uniformly to 1.

We then apply this result to a simple and uniformly distributed pinning term. With the help of classical homogenization technics, we exhibit various asymptotic behaviors of the minimizers, according to the relation between the Ginzburg-Landau and the period of the pinning term parameters.

In the last chapter of the second part, we deal with the case of a simple and uniformly distributed pinning term. We impose a Dirichlet boundary condition with a non zero degree. We prove that the vorticity is quantized and localized inside the inclusions. We prove that the macroscopic position of the vortices is governed by an auxiliary minimization problem of weighted Dirichlet functional in a perforated domain among $\mathbb{S}^1$-valued maps with Dirichlet conditions. We establish also that, if we add a dilution parameter on the inclusions, then the microscopic location of the vortices inside the inclusions is independent of the boundary data.

The last part deals with the effect of a simple pinning term (independent of the Ginzburg-Landau parameter) in the three-dimensional Ginzburg-Landau functional. The preliminary results we present allow to understand how the vorticity lines are bent under the effect of the pinning term.